Effect of quantum fluctuations on soliton regimes in microlasers

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Abstract. We present a theoretical investigation of effect of quantum fluctuations on laser solitons. Derivation of the stochastic equation, linearized with respect to quantum perturbations is carried out and the solutions are found. Explicit expressions are obtained for the time dependence of the soliton coordinates and momentum dispersion (variance) for perturbations averaged over the reservoir. It is shown that the dispersion of the soliton momentum becomes constant. It is shown that the dispersion of quantum perturbations tends to infinity near the Andronov-Hopf bifurcation threshold. The magnitude of quantum perturbations near the threshold of the appearance of hysteresis is estimated. It is shown that quantum perturbations do not significantly noise the soliton profile even with a very low intensity tending to zero. The number of photons in such solitons without supporting radiation, can reach unity.

1. Introduction
The effect of quantum fluctuations on the position and spectral characteristics of solitons in microlasers is one of the topical problems in set-up experiments [1, 2], the purpose of which is to use them in optical memory circuits, [3]. These efforts are focused on the use of conservative solitons with Kerr nonlinearity. However, the properties of conservative solitons, even in the classical limit, imply a drift of both spectral and spatial characteristics for an arbitrarily small perturbation, not necessarily quantum. Recently, a general theory has been developed that describes the effect of quantum fluctuations on the position and spectrum of laser solitons, the equilibrium and stability of which is associated with dissipative nonlinearity at the centre of the lasing line, see [4, 5]. The development of alternative schemes for controlling solitons based on dissipative nonlinearity, which makes it possible to more accurately control the discrete spectrum of their characteristics, can also provide the prospect of their miniaturization. In our works [6, 7], direct numerical simulation of two-dimensional laser solitons was carried out taking into account quantum pumping fluctuations. It is shown that quantum fluctuations do not significantly change the stability region of laser solitons, and the use of a supporting beam is possible down to very low intensities, six orders of magnitude less than the saturation intensity of the passive medium, which makes it possible to control the drift of the soliton position and phase. Controlling the frequency and amplitude of the small supporting radiation allows you to switch between different states of stable vortex combinations. Direct numerical simulation within the framework of solving the stochastic equation shows the stability of two-dimensional solitons against quantum pumping perturbations, including the stability of the trajectories of the classical motion of a soliton under the condition of the presence of coherent sustaining radiation.
In this communication, we analyse the spectrum and dynamics of the dispersion of quantum pump fluctuations. Magnitude of quantum perturbations near the threshold of the onset of hysteresis is estimated, which makes it possible to consider soliton states with a minimum number of photons.

2. The model

The starting point is the master Heisenberg-Langevin equation derived in [8]. Direct numerical simulation of quantum fluctuations was carried out using the stochastic equation for the averaged radiation field in the cavity, \( E \) in the c-number representation of dimensionless variables, [6]:

\[
\frac{\partial}{\partial t} E - (i + d) \Delta E = E_{in} + i\theta_{in} E + f(I) E + \chi_g Q_E,
\]

(1)

where the time is normalized to linear losses in the resonator, \( \kappa \). Transverse coordinates are in units \( k_0^2 \), \( v_g \) is the group velocity, \( d \) is the effective diffusion coefficient reflecting the angular selectivity of radiation losses in the resonator, \( 0 < d \ll 1 \). \( E_{in} \) is the amplitude of the classical field of supporting radiation, \( \theta_{in} \) is the mismatch between the frequencies of the supporting radiation and the longitudinal mode of the resonator, \( I = |E|^2 \) is the radiation intensity normalized to the intensity of absorption saturation \( I_a \). An active medium with a gain \( g_0 \) and a passive medium with absorption \( a_0 \) participate in the formation of nonlinearity in (1):

\[
f(I) = -1 + \frac{g_0}{1 + I/b} - \frac{a_0}{1 + I}, \quad b = I_g / I_a.
\]

The amplitude of the dimensionless stochastic force \( \chi_g Q_E \) is specified by the reservoir-averaged values of the correlators \( \langle Q_E^2 \rangle \), [8], which are proportional to the intensity, and by the dimensionless constant \( \chi_g = \sqrt{\kappa k_0 I_a v_g} \), the value of which determines the magnitude of the quantum perturbations of a pump. Taking into account the “atomic” definitions of the saturation intensity [5] the expression for the constant is represented in the form:

\[
\chi_g \approx 2 \times 10^{-5} \varepsilon_f \varepsilon_a \left[ \mu_a \text{Deby} \right] = 6 \times 10^{-4} \varepsilon_f \varepsilon_a.
\]

Here small parameters of the problem [1] are used: \( \varepsilon_a \approx \sqrt{\gamma_1 / \gamma_2} \), \( \varepsilon_f = \kappa / \gamma_1 \), taking into account the adiabatic approximation, and \( v_g \approx c \). \( \gamma_{1,2} \) are relaxation rates of the lower and upper levels, and \( \gamma_1 \ll \gamma_2 \) for an effective absorber. Here we have substituted for estimation \( \varepsilon_0 \approx 2 \), \( \lambda = 600 \text{ nm} \), \( k_0 = 2\pi / \lambda \), and the dipole moment \( \mu_a = 30 \text{ Deby} \) for carriers in CdSe quantum dots. In the most optimistic case \( \varepsilon_f \varepsilon_a \approx 10^{-2} \), which gives an estimate for quantum fluctuations \( \chi_g \sim 10^{-5} \).

3. Dispersion or variance of quantum fluctuations

A sufficiently small value of quantum perturbations allows one to obtain analytically the time dependence of the variance of the coordinate and momentum of a soliton. For this, the initial stochastic equation was linearized, and its solution was averaged over the reservoir of perturbations. The temporal asymptotes of the variances are different. If for the soliton coordinate it grows linearly (which indicates the presence of uncompensated Brownian motion of the soliton as a whole), then the dispersion of the momentum becomes constant. The latter circumstance can be used to realize the effect of quantum compression of the momentum perturbations of a dissipative soliton, see [8] and [9]. We consider weak fluctuations of the field in the vicinity of the classical soliton: \( E(t, x) = e_s(x) + \delta e(t, x) \). Linearization of equation (1) with respect to \( \delta e(t, x) \) leads to an inhomogeneous linear stochastic equation, which in matrix form has the form
\[ i(\partial / \partial t + \gamma)U(t,x) + LU(t,x) = iW(t,x), \]

where is a vector of functions, and a vector of stochastic sources:

\[ U(t,x) = \{\delta e(t,x), \delta e^*(t,x)\}^T, \quad W(t,x) = \chi_e \{Q_E(t,x), Q'_E(t,x)\}^T, \]

dissipative damping factor and detuning in the linear approximation:

\[ \gamma = -\Re f(0) - \Re F(I_H), \quad \theta = \theta_a + \Im f(0) + \Im F(I_H). \]

Finally, the matrix of differential operators depends from soliton intensity \( I_S(x) \):

\[ L = \begin{bmatrix} 1 - i\alpha \frac{\partial^2}{\partial x^2} + \theta + \Im F_1 - i\Re F_1, & -iF_2^* \\ -iF_2, & -(1 + i\alpha) \frac{\partial^2}{\partial x^2} - \theta - \Im F_1 - i\Re F \end{bmatrix}, \]

\[ F_1 = F_1(x) = \delta F(I_S(x)) + I_1 F'(I_S(x)), \quad F_2 = F_2(x) = F'(I_S(x))e^2(x), \quad F' = dF / dI. \]

The fundamental difference between the type of optical nonlinearity in the considered laser and the case of an interferometer with Kerr nonlinearity lies in the dissipative character of the media with nonlinear laser amplification and absorption. In view of this, in our case, a number of symmetries are absent, indicated in [9] for an interferometer with nonlinearity of only the refractive index. At the same time, symmetry (2) is preserved to coordinate inversion \( x \rightarrow -x \).

Following [8], we will seek solution (2) by expanding in eigenfunctions of the homogeneous equation corresponding to (2). An eigenfunction with an eigenvalue has the form

\[ U(t,x) = U_\lambda(x)e^{-\gamma t} + U_\lambda^*(x)e^{-\gamma t}, \quad U_\lambda^*(x) = \sigma_3 U_\lambda^*(x), \]

\[ U(t,x) = \{\delta e(t,x), \delta e^*(t,x)\}^T, \quad U_\lambda(x) = \{\varphi_\lambda(x), \varphi_\lambda^*(x)\}^T. \]

Here \( \varphi_\lambda^*(x) = \varphi_{-\lambda}(x) \). Then for the operator \( L \) and the Hermitian conjugate operator \( L^* = \sigma_3 L \sigma_3 \)

\[ (L + \lambda)U_\lambda = 0, \quad (L - \lambda^*)U_\lambda^* = 0, \quad (L^* + \lambda^*)U_\lambda^* = 0. \]

Whence it follows that for each eigenvalue \( \lambda \) there is a second eigenvector:

\[ U_\lambda^* = \{\varphi_\lambda^*(x), \varphi_\lambda^*(x)\}^T = \{\varphi_{-\lambda}(x), \varphi_{-\lambda}(x)\}^T = U_{-\lambda}(x). \]

When \( \Re \lambda = 0 \) these two vectors coincide, \( U_\lambda^* = U_\lambda \). Unlike [9], where the medium possessed only “conservative” Kerr nonlinearity, the operator \( \sigma_3 L \) is not Hermitian, therefore its eigenvectors are not orthogonal (here the Pauli matrices \( \sigma_{1,3} \) are used).

Figure 1 shows the spatial distributions of the amplitude (figure 1a-d) and phase (figure 1e-h) of the classical fundamental soliton (figure 1a, e) and the components of an odd eigenfunction with eigenvalues. It can be seen from figure 1a that the sustaining radiation weakly changes the amplitude profile in the central region of the soliton. The eigenfunctions of the discrete spectrum are localized and exponentially decrease at the periphery of the soliton, while the phase in this region increases linearly with distance from the center. For a laser with its dissipative nonlinearity, the relationship between the eigenvalues of the operators \( L \) and \( L^* \), which is valid for an interferometer with Kerr nonlinearity, is violated. This circumstance decreases the rate of damping of perturbations of a stable soliton, and when crossing the boundary of the region of its stability due to changes in the parameters of the scheme, the growth of perturbations with simultaneous oscillations with frequency \( \omega_s \) serves as a sign of Andronov-Hopf bifurcation. The approach developed in [8] makes it possible to find the dynamic changes of the perturbation of the coordinates of the center \( \delta x \) and momentum \( \delta p \) of a soliton, which can be defined as follows:

\[ \delta x(t) = i^{-1} \left\{ \frac{1}{\alpha} (U_{\lambda}^* + U_{\lambda}^* \sigma_3) \right\} U(t), \quad \delta p(t) = i^{-1} \left\{ U_{\lambda}^* \sigma_3 \right\} U(t). \]
Finally, we obtain statistically averaged squares of fluctuations of these quantities:

$$\langle \langle \delta \hat{X}^2(t) \rangle \rangle = 4 \chi_0^2 U_{in}^2 \left[ u_1 + \frac{\lambda}{\kappa} \left( 1 - e^{-2(\gamma - \gamma'_1)} \right) \right] + \chi_0^2 U_{in}^2 \left[ \cos \theta_1 - \cos(\theta_1 - 2\omega t) e^{-2(\gamma - \gamma'_1)} \right] + 8 \chi_0^2 U_{in} U_{im} \left[ \sin \theta_1 - \sin(\theta_1 - \omega t) e^{2(\gamma - \gamma'_1)} \right],$$

$$\langle \langle \delta \hat{p}^2(t) \rangle \rangle = 4 \chi_0^2 |U_{01}|^2 \left[ u_1 e^{-2(\gamma - \gamma'_1)} + \frac{\lambda}{\kappa} \left( 1 - e^{-2(\gamma - \gamma'_1)} \right) + \frac{\lambda}{\kappa} \left[ \cos \theta_1 - \cos(\theta_1 - 2\omega t) e^{-2(\gamma - \gamma'_1)} \right] \right].$$

(9)

When deriving (9), (10), we used definitions for scalar products of eigenvectors, which are no longer orthogonal to each other

$$U_{11} = \langle U_{11}^\dagger \sigma_3 U_{11} \rangle = \int dx \left[ |\varphi^*| - |\varphi| \right], U_{01} = \langle U_{01}^\dagger \sigma_3 U_{01} \rangle = \int dx \left[ \varphi_0^* \varphi_1 - \varphi_0 \varphi_1^* \right], U_{im} = \text{Im} U_{01}$$

and designations $\theta_1 = \arg \frac{\lambda}{\kappa_1}$, $\theta_{01} = \arg \frac{\lambda}{\kappa_1}$, $\dot{\theta}_{01} = \arg U_{01}$, $\theta_x = \theta_1 + 2\theta_{01}$; and $u_1$ and $u_\rho$ are parameters specifying the initial values of the fluctuations of the coordinate and momentum.

It can be seen from (9) and (10) that the variance of fluctuations increases indefinitely when approaching the stability boundary of a soliton, when $\gamma'_1 \to \gamma$. In the region of stability of a soliton $\gamma'_1 < \gamma$. As in the case of an interferometer, the square of fluctuations of the coordinate of the center of the soliton predominantly increases linearly with time (consideration is limited to the times while the perturbation remains weak). A new aspect is the appearance, along with linear growth, and exponential decay of oscillations of dispersion with frequency $2\omega$. This is caused by the above-mentioned splitting of the eigenvalue of the discrete mode: $-i\lambda_{z1} = \gamma'_1 \mp i\omega_1$.

Thus, a linear analysis of quantum fluctuations near a classical laser soliton shows both common features with the case of solitons in an interferometer with Kerr nonlinearity and new features (temporal damped oscillations) caused by the essentially dissipative nature of the optical nonlinearity of the laser medium.

![Figure 1.](image)

Figure 1. (a, e) - Profile of the amplitude $A_x$ and phase $\Phi_x$ of a classical soliton at $E_{in} = 0$ (solid line) and $0.02$ (dashed line); inset at (a) shows the peripheral area. (b, f) is the real amplitude for the components of the eigenvector of the operator $L$. The designation $\varphi_1$ corresponds to the functions of an eigenvector with an eigenvalue $\lambda = \lambda_1$. $E_{in} = 0$; (b, c, g). $0.02$ (f, d, h). (c, g), (d, h) are the amplitude (c, d) and phase (g, h) of the eigenfunctions of the homogeneous equation corresponding to (2). $\delta e_x = \delta e(0, x)$ (curves 1 and 2) corresponds to the functions $\lambda = \lambda_{z1}$.
4. The number of photons in a soliton near the bistability threshold

Let us analyze the possibility of realizing solitons with a small number of photons. This possibility appears if the value of the absorption coefficient of a weak signal is chosen near the threshold of the appearance of hysteresis in a “free” laser without a supporting beam. In this case, the bistability interval of two homogeneous solutions (without generation and with the intensity of stable generation) is compressed to a point and the intensity of stable generation tends to zero. Bistability threshold: 

\[a_0 > a_{thr} = \frac{1}{b - 1},\]

where \(b = I_g / I_s > 1\) is the ratio of saturation intensities of two media. It is convenient to normalize the absorption relative threshold by introducing \(\delta a_0 = a_{thr} (1 + \delta a_0)\).

Then the hysteresis of uniform modes exists in the gain interval \(g_{dn} < g_0 < g_{up}\), and the lasing intensity varies over the interval \(I_{dn} < I_b < I_{up}\). Threshold expressions are

\[I_{dn} = \sqrt{1 + \delta a_0} - 1, \quad I_{up} = \delta a_0, \quad g_{dn} = g_{dn} - I_{dn}^2 / b, \quad g_{up} = g_{up} - I_{up}^2 / b, \quad g_{dn} = g_{dn} + I_{dn}^2 / b, \quad g_{up} = g_{up} + I_{up}^2 / b,
\]

where \(g_{dn}\) is the threshold value of gain (hysteresis exists only at npu \(a_0 > a_{thr}, \quad g_0 > g_{thr}\)). If positive value \(\delta a_0 \rightarrow 0\), we are approaching the point where hysteresis occurs. Moreover, the lasing intensities at both ends of the hysteresis interval are small \(I_{dn,up} \rightarrow 0\). The soliton is stable in the interval within the hysteresis region; therefore, estimating the number of photons in the soliton, we can assume that its intensity is \(I_{sol} \approx (I_{dn} + I_{up}) / 2 \approx 3\delta a_0 / 4\). The soliton width is of the order of several Fresnel zones, \(x_0 = \sqrt{\frac{v_g}{\kappa k_0}}\). Saturation intensity is entered as two-dimensional photon density \(\left[ I_a \right] = [n] = [\text{cm}^{-2}]\). Therefore, the number of photons in a soliton is \(n_{sol} = I_a I_{sol} \chi_{sol}^2 = I_{sol} / \chi_{sol}^2\), taking into account that \(\chi_{sol}^2 = 1 / \left(I_a \chi_0^2\right)\). Thus, the dimensionless constant, which determines the scale of the correlators of quantum fluctuations of the pump, \(\chi = \chi_{sol}^2\), simultaneously determines the scale of the dimensionless intensity of the soliton in the number of photons. One can proceed from a given number of photons in a soliton, for example \(n_{sol} \approx 0.7\delta a_0 / \chi_{sol}^2 = 10^{-100}\), and determine the accuracy with which it is necessary to approach the hysteresis threshold in order to obtain so many photons: \(\delta a_0 \approx 1.2\chi_{sol}^2 n_{sol} \sim 10^{-9}, 10^{-8}\), where we substituted the above estimate of the dimensionless constant \(\chi_{sol} \sim 10^{-5}\) (the dimensionless absorption coefficient \(\delta a_0\) is measured relative to the bistability threshold).

For the main operating point in direct numerical simulation \(I_{sol} \approx 9\) and the number of photons in a soliton \(n_{sol} \sim 10^{11}\). Let us estimate the variance of the number of photons in a soliton, calculated using the linearized equation. According to the results of [8], the scale of the dispersion of the number of photons, without taking into account the phase relations of the correlators, and for the normalized eigenfunction \(\varphi_0(r)\) depending on dimensionless coordinates \(r\), is mainly determined by the term:

\[\bar{\delta n}^2 \sim \chi_{sol}^2 \bar{\varphi_0^2} - \chi_{sol}^2 \bar{\varphi_0^2} \gamma - \gamma_1, \quad \bar{\varphi_0}^2 = \int_{-\infty}^{\infty} d^2 r \left[ \varphi_0 \varphi_0^* \right] \left[ Q_E \right] (r) + \text{Re} \int_{-\infty}^{\infty} d^2 r \left[ \varphi_0 \varphi_0^* \right] \left[ Q_E \right] (r),\]

where \(\gamma\) is the linear damping factor in the dissipative system, \(\chi_{sol}\) is the real part of the eigenvalue in the discrete spectrum of the boundary value problem in operator form. When \(\gamma_1 > \gamma\) the soliton is unstable due to the Andronov-Hopf bifurcation. Taking into account that \(\bar{Q}_0^2 \sim I_{sol}\), and \(\bar{Q}_0^2 = 2g_0\) at \(I_{sol} \rightarrow 0\), we find that the variance of fluctuations remains constant near the threshold of the onset of hysteresis, despite the fact that the intensity of the soliton decreases to zero:
Thus, the relative amplitude of the noisiness of the soliton increases at $I_{\text{sol}} \to 0$, as well as at the boundary of soliton stability. The total noisiness of the soliton corresponds to $\sqrt{\overline{\delta n^2}} / n_{\text{sol}} \sim 1$, i.e. $I_{\text{sol}} \sim \chi_g \sqrt{\frac{2g_0}{\gamma - \gamma_1}}$, $n_{\text{sol}} \sim \chi_g \sqrt{\frac{2g_0}{\gamma - \gamma_1}}$. This means that the complete noisiness of a soliton is not achieved up to solitons consisting of several photons, since $\chi_g \ll 1$. Only in the region of vanishing proximity to the stability boundary, when $\gamma_1 = \gamma - 2g_0 \chi_g / n_{\text{sol}}$ the soliton consisting of $n_{\text{sol}}$ photons is completely noisy.

5. Discussion
Quantum pumping perturbations in a laser soliton are small, but still lead to Brownian motion of its centre of inertia. For self-propulsion of a soliton in the classical limit, due to the inhomogeneous energy balance, quantum perturbations lead to only a small dispersion of the classical trajectory. The average phase of the soliton also shifts along the trajectory of the Brownian motion. However, the introduction of a small supporting beam is sufficient to stabilize these trajectories. On the contrary, the dispersion of the total momentum of the soliton is stabilized, which can be used for quantum compression of the energy spectrum [9].

Maintaining the exact ratio of gain and absorption near the threshold of the appearance of hysteresis of uniform distributions of the generation field allows one to go over to the regime of low-energy solitons with a very small number of photons. Quantum perturbations of pump do not critically affect the possibility of detecting such solitons, although their characteristics change significantly. First of all, the transverse size of solitons greatly increases, which leads to almost complete filling of the limited laser aperture by such a soliton.

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Disclosures
The authors declare no conflicts of interest.

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