Hitting minors on bounded treewidth graphs.
II. Single-exponential algorithms

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Abstract

For a finite collection of graphs \( F \), the \( F\text{-M-Deletion} \) (resp. \( F\text{-TM-Deletion} \)) problem consists in, given a graph \( G \) and an integer \( k \), decide whether there exists \( S \subseteq V(G) \) with \( |S| \leq k \) such that \( G \setminus S \) does not contain any of the graphs in \( F \) as a minor (resp. topological minor). We are interested in the parameterized complexity of both problems when the parameter is the treewidth of \( G \), denoted by \( \text{tw} \), and specifically in the cases where \( F \) contains a single connected planar graph \( H \). We present algorithms running in time \( 2^{O(\text{tw})} \cdot n^{O(1)} \), called \textit{single-exponential}, when \( H \) is either \( P_3 \), \( P_4 \), \( C_4 \), the paw, the chair, and the banner for both \( \{H\}\text{-M-Deletion} \) and \( \{H\}\text{-TM-Deletion} \), and when \( H = K_{1,i} \), with \( i \geq 1 \), for \( \{H\}\text{-TM-Deletion} \). Some of these algorithms use the rank-based approach introduced by Bodlaender et al. [Inform Comput, 2015]. This is the second of a series of articles on this topic, and the results given here together with other ones allow us, in particular, to provide a tight dichotomy on the complexity of \( \{H\}\text{-M-Deletion} \) in terms of \( H \).

Keywords: parameterized complexity; graph minors; treewidth; hitting minors; topological minors; dynamic programming; Exponential Time Hypothesis.

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1 Introduction

Let \( \mathcal{F} \) be a finite non-empty collection of non-empty graphs. In the \( \mathcal{F} \)-M-Deletion (resp. \( \mathcal{F} \)-TM-Deletion) problem, we are given a graph \( G \) and an integer \( k \), and the objective is to decide whether there exists a set \( S \subseteq V(G) \) with \( |S| \leq k \) such that \( G \setminus S \) does not contain any of the graphs in \( \mathcal{F} \) as a minor (resp. topological minor). These problems have a big expressive power, as instantiations of them correspond to several well-studied problems. For instance, the cases \( \mathcal{F} = \{ K_2 \} \), \( \mathcal{F} = \{ K_3 \} \), and \( \mathcal{F} = \{ K_5, K_{3,3} \} \) of \( \mathcal{F} \)-M-Deletion (or \( \mathcal{F} \)-TM-Deletion) correspond to Vertex Cover, Feedback Vertex Set, and Vertex Planarization, respectively. For the sake of readability, we use the notation \( \mathcal{F} \)-Deletion in statements that apply to both \( \mathcal{F} \)-M-Deletion and \( \mathcal{F} \)-TM-Deletion.

We are interested in the parameterized complexity of \( \mathcal{F} \)-Deletion when the parameter is the treewidth of the input graph. Courcelle’s theorem \cite{Courcelle} implies that \( \mathcal{F} \)-Deletion can be solved in time \( O^*(f(tw)) \) on graphs with treewidth at most \( tw \), where \( f \) is some computable function\(^1\). Our objective is to determine, for a fixed collection \( \mathcal{F} \), which is the smallest such function \( f \) that one can (asymptotically) hope for, subject to reasonable complexity assumptions.

This line of research has recently attracted some attention in the parameterized complexity community. For instance, Vertex Cover is easily solvable in time \( O^*(2^{O(tw)}) \), called single-exponential, by standard dynamic-programming techniques, and no algorithm with running time \( O^*(2^{o(tw)}) \) exists, unless the Exponential Time Hypothesis (ETH)\(^2\) fails \cite{Cai}. For Feedback Vertex Set, standard dynamic programming techniques give a running time of \( O^*(2^{O(tw \cdot \log tw)}) \), while the lower bound under the ETH \cite{Cai} is again \( O^*(2^{o(tw)}) \). This gap remained open for a while, until Cygan et al. \cite{Cygan} presented an optimal algorithm running in time \( O^*(2^{O(tw)}) \), introducing the celebrated Cut&Count technique, which produces randomized algorithms. This article triggered several other techniques to obtain single-exponential deterministic algorithms for so-called connectivity problems on graphs of bounded treewidth, mostly based on algebraic tools \cite{Bodlaender, Fomin}. We refer the reader to \cite{FlumGrohe} for a more detailed discussion about related work. In particular, in this article we make use of one of the techniques presented by Bodlaender et al. \cite{Bodlaender}, called rank-based approach. It is worth mentioning that this approach has been recently applied to dense graph classes, namely those with structured neighborhoods \cite{Bodlaender2}.

Our results and techniques. We provide several single-exponential algorithms when \( \mathcal{F} \) contains a single connected planar graph \( H \). Namely, we show that if \( \mathcal{F} \in \{ \{P_3\}, \{P_4\}, \{K_{1,1}\} \),

\(^1\)The notation \( O^*(\cdot) \) suppresses polynomial factors depending on the size of the input graph.

\(^2\)The ETH states that 3-SAT on \( n \) variables cannot be solved in time \( 2^{o(n)} \); see \cite{Cai} for more details.
\{C_4, \text{paw}, \{\text{chair}\}, \{\text{banner}\}\}$ (see Figure 1 for an illustration of these graphs), then $F$-TM-Deletion can be solved in single-exponential time. Note that all these graphs have maximum degree at most three, except $K_{1,i}$ for $i \geq 4$, and therefore the corresponding algorithms also apply to the $F$-M-Deletion problem. Indeed, for graphs $H$ with maximum degree at most three, containing $H$ as a minor is equivalent to containing $H$ as a topological minor. The fact that we are not able to provide single-exponential algorithms for $\{K_{1,i}\}$-M-Deletion with $i \geq 4$ seems to be unavoidable: we prove in [7] that there is no algorithm in time $O^*(2^{o \cdot (tw \cdot \log tw)})$ for these cases, unless the ETH fails. This exhibits, to the best of our knowledge, the first difference between the computational complexity of both problems.

The single-exponential algorithms presented in this article are ad hoc, some being easier than others. All of them exploit a structural characterization of the graphs that exclude that particular graph $H$ as a (topological) minor; cf. for instance Lemmas 2 and 9. Intuitively, the “complexity” of this characterization is what determines the difficulty of the corresponding dynamic programming algorithm, and is also what makes the difference between being solvable in single-exponential time or not.

More precisely, the algorithms for $\{P_3\}$-Deletion, $\{P_4\}$-Deletion, and $\{K_{1,i}\}$-TM-Deletion use standard (but non-trivial) dynamic programming techniques on graphs of bounded treewidth, exploiting the simple structure of graphs that do not contain these particular graphs as a topological minor (or as a subgraph, which in these cases is equivalent). The algorithms for $\{P_3\}$-Deletion and $\{K_{1,i}\}$-TM-Deletion are quite simple, while the one for $\{P_4\}$-Deletion is slightly more technical.

The algorithms for $\{C_4\}$-Deletion and $\{\text{paw}\}$-Deletion are more involved, and use the rank-based approach introduced by Bodlaender et al. [9], exploiting again the structure of graphs that do not contain $C_4$ or the paw as a minor (cf. Lemma 5 and 7, respectively). It might seem counterintuitive that this technique works for $C_4$, and stops working for $C_i$ with $i \geq 5$. A possible reason for that is that the only cycles of a $C_i$-minor-free graph are triangles and each triangle must be contained in a bag of a tree decomposition. This property, which is not true anymore for $C_i$-minor-free graphs with $i \geq 5$, permits to keep track of the structure of partial solutions with tables of small size. The algorithm for $\{\text{paw}\}$-Deletion combines classical dynamic programming techniques and the rank-based approach.

Finally, the algorithms for $\{\text{chair}\}$-Deletion and $\{\text{banner}\}$-Deletion are a combination of the above ones, the latter one using again the rank-based approach. Given the large amount of labels that we need in the tables and the similarity with other algorithms for which we provide all the details, we only present a sketch of these two algorithms.
Results in other articles of the series and discussion. In the first article of this series [6], we show, among other results, that for every collection $\mathcal{F}$ containing at least one planar graph (resp. subcubic planar graph), $\mathcal{F}$-M-Deletion (resp. $\mathcal{F}$-TM-Deletion) can be solved in time $O^*(2^{O(tw \cdot \log tw)})$. In the third article of this series [7], we focus on lower bounds under the ETH. Namely, we prove that for any connected $\mathcal{F}$, $\mathcal{F}$-Deletion cannot be solved in time $O^*(2^{o(tw)})$, even if the input graph $G$ is planar, and we provide superexponential lower bounds for a number of collections $\mathcal{F}$. In particular, we prove a lower bound of $O^*(2^{o(tw \cdot \log tw)})$ when $\mathcal{F}$ contains a single connected graph that is either $P_5$ or is not a minor of the banner, with the exception of $K_{1,i}$ for the topological minor version. These lower bounds, together with the ad hoc single-exponential algorithms given in this article and the general algorithms described in [6], cover all the cases of $\mathcal{F}$-M-Deletion where $\mathcal{F}$ contains a single connected planar graph $H$, yielding a dichotomy in terms of $H$. In the fourth article of this series [5] (whose full version is [4]), we presented an algorithm for $\mathcal{F}$-M-Deletion in time $O^*(2^{O(tw \cdot \log tw)})$ for any collection $\mathcal{F}$, yielding together with the lower bounds in [7] and the results of the current article a dichotomy for $\mathcal{F}$-M-Deletion where $\mathcal{F}$ consists of a single connected (non-necessarily planar) graph $H$. Namely, as stated in [4], if $H$ is a connected graph on at least two vertices, then the $\{H\}$-M-Deletion problem can be solved in time

- $O^*(2^{O(tw)})$, if $H$ is a contraction of the chair or the banner, and
- $O^*(2^{O(tw \cdot \log tw)})$, otherwise.

In the above statements, we use the $\Theta$-notation to indicate that these algorithms are optimal under the ETH. Note that the first item is equivalent to $H$ being a minor of the banner that is different from $P_5$. This dichotomy is depicted in Figure 1, containing all connected graphs $H$ with $2 \leq |V(H)| \leq 5$; note that if $|V(H)| \geq 6$, then $H$ is not a contraction of the chair or the banner, and therefore the second item above applies. Note also that $K_4$ and the diamond are the only graphs on at most four vertices for which the problem is solvable in time $O^*(2^{O(tw \cdot \log tw)})$ and that the chair and the banner are the only graphs on at least five vertices for which the problem is solvable in time $O^*(2^{O(tw)})$. Note also that the cases $\mathcal{F} = \{P_2\}$ [13, 19], $\mathcal{F} = \{P_3\}$ [1, 23], and $\mathcal{F} = \{C_3\}$ [9, 14] were already known.

The crucial role played by the banner and $P_5$ (or equivalently, the chair and the banner) in the complexity dichotomy may seem surprising at first sight. In fact, we realized a posteriori that the “easy” cases can be succinctly described in terms of the banner and $P_5$ by taking a look at Figure 1. Nevertheless, there is some intuitive reason for which excluding the banner constitutes the horizon on the existence of single-exponential

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3 A connected collection $\mathcal{F}$ is a collection containing only connected graphs.
algorithms (forgetting about the “exception” $\mathcal{F} = \{P_5\}$). Namely, every connected component of a graph that excludes the banner as a (topological) minor is either a cycle (of any length) or a tree in which some vertices have been replaced by triangles; both such types of components can be maintained by a dynamic programming algorithm in single-exponential time. It appears that if the characterization of the allowed connected components is enriched in some way, such as restricting the length of the allowed cycles or forbidding certain degrees, the problem becomes inherently more difficult.

Figure 1: Classification of the complexity of $\{H\}$-M-DELETION for all connected simple graphs $H$ with $2 \leq |V(H)| \leq 5$: for the nine graphs on the left (resp. 21 graphs on the right, and all the larger ones), the problem is solvable in time $2^{\Theta(tw)} \cdot n^{O(1)}$ (resp. $2^{\Theta(tw \cdot \log tw)} \cdot n^{O(1)}$). For $\{H\}$-TM-DELETION, $K_{1,4}$ should be on the left. This figure also appears in [7].
Organization of the paper. In Section 2 we give some preliminaries. We deal with $P_3$, $P_4$, $K_{1,n}$, $C_4$, the paw, the chair, and the banner in Sections 3, 4, 5, 6, 7, 8, and 9, respectively. We conclude in Section 10 with some questions for further research.

2 Preliminaries

In this section we provide some preliminaries to be used in the following sections.

Sets, integers, and functions. We denote by $\mathbb{N}$ the set of every non-negative integer and we set $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Given two integers $p$ and $q$, the set $[p,q]$ refers to the set of every integer $r$ such that $p \leq r \leq q$. Moreover, for each integer $p \geq 1$, we set $\mathbb{N}_{\geq p} = \mathbb{N} \setminus [0,p-1]$.

We use $\emptyset$ to denote the empty set and $\varnothing$ to denote the empty function, i.e., the unique subset of $\emptyset \times \emptyset$. Given a function $f : A \to B$ and a set $S$, we define $f|_S = \{(x,f(x)) \mid x \in S \cap A\}$. Moreover if $S \subseteq A$, we set $f(S) = \bigcup_{s \in S} \{f(s)\}$. Given a set $S$, we denote by $\binom{S}{2}$ the set containing every subset of $S$ that has cardinality two.

Graphs. All the graphs that we consider in this paper are undirected, finite, and without loops or multiple edges. We use standard graph-theoretic notation, and we refer the reader to [15] for any undefined terminology. Given a graph $G$, we denote by $V(G)$ the set of vertices of $G$ and by $E(G)$ the set of the edges of $G$. We call $|V(G)|$ the size of $G$. A graph is the empty graph if its size is zero. We also denote by $L(G)$ the set of the vertices of $G$ that have degree exactly one. If $G$ is a tree (i.e., a connected acyclic graph) then $L(G)$ is the set of the leaves of $G$. A vertex labeling of $G$ is some injection $\rho : V(G) \to \mathbb{N}^+$. Given a vertex $v \in V(G)$, we define the neighborhood of $v$ as $N_G(v) = \{u \mid u \in V(G), \{u,v\} \in E(G)\}$ and the closed neighborhood of $v$ as $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, then we write $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$. The degree of a vertex $v$ in $G$ is defined as $\deg_G(v) = |N_G(v)|$. A graph is called subcubic if all its vertices have degree at most three.

A subgraph $H = (V_H, E_H)$ of a graph $G = (V,E)$ is a graph such that $V_H \subseteq V(G)$ and $E_H \subseteq E(G) \cap \binom{V_H}{2}$. If $S \subseteq V(G)$, the subgraph of $G$ induced by $S$, denoted $G[S]$, is the graph $(S, E(G) \cap \binom{S}{2})$. We also define $G \setminus S$ to be the subgraph of $G$ induced by $V(G) \setminus S$. If $S \subseteq E(G)$, we denote by $G \setminus S$ the graph $(V(G), E(G) \setminus S)$.

If $s, t \in V(G)$, an $(s,t)$-path of $G$ is any connected subgraph $P$ of $G$ with maximum degree two and where $s, t \in L(P)$. We say that two vertices $s$ and $t$ are connected in $G$ if $G$ contains an $(s,t)$-path as a subgraph. We finally denote by $\mathcal{P}(G)$ the set of all paths of $G$. Given $P \in \mathcal{P}(G)$, we say that $v \in V(P)$ is an internal vertex of $P$ if $\deg_P(v) = 2$. Given an integer $i$ and a graph $G$, we say that $G$ is $i$-connected if for each
\{u, v\} \in \binom{V(G)}{2}, \text{ there exists a set } Q \subseteq \mathcal{P}(G) \text{ of } (u, v)\text{-paths of } G \text{ such that } |Q| = i \text{ and for each } P_1, P_2 \in Q \text{ such that } P_1 \neq P_2, V(P_1) \cap V(P_2) = \{u, v\}. \text{ We denote by } K_r, P_r, \text{ and } C_r, \text{ the complete graph, the path, and the cycle on } r \text{ vertices, respectively.}

**Minors and topological minors.** Given two graphs \(H\) and \(G\) and two functions \(\phi : V(H) \to V(G)\) and \(\sigma : E(H) \to \mathcal{P}(G)\), we say that \((\phi, \sigma)\) is a *topological minor model of \(H \text{ in } G\)* if

- for every \(\{x, y\} \in E(H), \sigma(\{x, y\})\) is an \((\phi(x), \phi(y))\)-path in \(G\) and
- if \(P_1, P_2\) are two distinct paths in \(\sigma(E(H))\), then none of the internal vertices of \(P_1\) is a vertex of \(P_2\).

The branch vertices of \((\phi, \sigma)\) are the vertices in \(\phi(V(E))\), while the subdivision vertices of \((\phi, \sigma)\) are the internal vertices of the paths in \(\sigma(E(H))\).

We say that \(G\) contains \(H\) as a *topological minor*, denoted by \(H \lesssim_{\text{tm}} G\), if there is a topological minor model \((\phi, \sigma)\) of \(H\) in \(G\).

Given two graphs \(H\) and \(G\) and a function \(\phi : V(H) \to 2^{V(G)}\), we say that \(\phi\) is a *minor model of \(H\) in \(G\)* if

- for every \(x \in V(H), G[\phi(x)]\) is a connected non-empty graph and
- for every \(\{x, y\} \in E(H), \text{ there exist } x' \in \phi(x) \text{ and } y' \in \phi(y) \text{ such that } \{x', y'\} \in E(G)\).

We say that \(G\) contains \(H\) as a *minor*, denoted by \(H \lesssim_m G\), if there is a minor model \(\phi\) of \(H\) in \(G\).

**Graph collections.** Let \(\mathcal{F}\) be a collection of graphs. From now on instead of “collection of graphs” we use the shortcut “collection”. If \(\mathcal{F}\) is a collection that is finite, non-empty, and all its graphs are non-empty, then we say that \(\mathcal{F}\) is a *proper collection*. For any proper collection \(\mathcal{F}\), we define \(\text{size}(\mathcal{F}) = \max\{|V(H)| \mid H \in \mathcal{F}\} \cup \{|\mathcal{F}|\}\). Note that if the size of \(\mathcal{F}\) is bounded, then the size of the graphs in \(\mathcal{F}\) is also bounded. We say that \(\mathcal{F}\) is a *planar collection* (resp. planar subcubic collection) if it is proper and at least one of the graphs in \(\mathcal{F}\) is planar (resp. planar and subcubic). We say that \(\mathcal{F}\) is a *connected collection* if it is proper and all the graphs in \(\mathcal{F}\) are connected. We say that \(\mathcal{F}\) is an (topological) minor antichain if no two of its elements are comparable via the (topological) minor relation.

Let \(\mathcal{F}\) be a proper collection. We extend the (topological) minor relation to \(\mathcal{F}\) such that, given a graph \(G\), \(\mathcal{F} \lesssim_{\text{tm}} G\) (resp. \(\mathcal{F} \lesssim_m G\)) if and only if there exists a graph \(H \in \mathcal{F}\) such that \(H \lesssim_{\text{tm}} G\) (resp. \(H \lesssim_m G\)). We also denote \(\text{ex}_{\text{tm}}(\mathcal{F}) = \{G \mid \mathcal{F} \not\lesssim_{\text{tm}} G\}, \text{ and } \text{ex}_m(\mathcal{F}) = \{G \mid \mathcal{F} \not\lesssim_m G\}\).
i.e., \( \text{ex}_m(\mathcal{F}) \) is the class of graphs that do not contain any graph in \( \mathcal{F} \) as a topological minor. The set \( \text{ex}_m(\mathcal{F}) \) is defined analogously.

**Tree decompositions.** A *tree decomposition* of a graph \( G \) is a pair \( \mathcal{D} = (T, \mathcal{X}) \), where \( T \) is a tree and \( \mathcal{X} = \{ X_t \mid t \in V(T) \} \) is a collection of subsets of \( V(G) \) such that:

- \( \bigcup_{t \in V(T)} X_t = V(G) \),
- for every edge \( \{u, v\} \in E \), there is a \( t \in V(T) \) such that \( \{u, v\} \subseteq X_t \), and
- for each \( \{x, y, z\} \subseteq V(T) \) such that \( z \) lies on the unique path between \( x \) and \( y \) in \( T \), \( X_x \cap X_y \subseteq X_z \).

We call the vertices of \( T \) *nodes* of \( \mathcal{D} \) and the sets in \( \mathcal{X} \) *bags* of \( \mathcal{D} \). The *width* of a tree decomposition \( \mathcal{D} = (T, \mathcal{X}) \) is \( \max_{t \in V(T)} |X_t| - 1 \). The *treewidth* of a graph \( G \), denoted by \( \text{tw}(G) \), is the smallest integer \( w \) such that there exists a tree decomposition of \( G \) of width at most \( w \). For each \( t \in V(T) \), we denote by \( E_t \) the set \( E(G[X_t]) \).

We need to introduce nice tree decompositions, which will make the presentation of the algorithms much simpler.

**Nice tree decompositions.** Let \( \mathcal{D} = (T, \mathcal{X}) \) be a tree decomposition of \( G \), \( r \) be a vertex of \( T \), and \( \mathcal{G} = \{ G_t \mid t \in V(T) \} \) be a collection of subgraphs of \( G \), indexed by the vertices of \( T \). We say that the triple \((\mathcal{D}, r, \mathcal{G})\) is a *nice tree decomposition* of \( G \) if the following conditions hold:

- \( X_r = \emptyset \) and \( G_r = G \),
- each node of \( \mathcal{D} \) has at most two children in \( T \),
- for each leaf \( t \in V(T) \), \( X_t = \emptyset \) and \( G_t = (\emptyset, \emptyset) \). Such \( t \) is called a *leaf node*,
- if \( t \in V(T) \) has exactly one child \( t' \), then either
  - \( X_t = X_{t'} \cup \{ v_{\text{insert}} \} \) for some \( v_{\text{insert}} \not\in X_{t'} \) and \( G_t = G[V(G_{t'}) \cup \{ v_{\text{insert}} \}] \). The node \( t \) is called *introduce vertex* node and the vertex \( v_{\text{insert}} \) is the *insertion vertex* of \( X_t \),
  - \( X_t = X_{t'} \setminus \{ v_{\text{forget}} \} \) for some \( v_{\text{forget}} \in X_{t'} \) and \( G_t = G_{t'} \). The node \( t \) is called *forget vertex* node and \( v_{\text{forget}} \) is the *forget vertex* of \( X_t \).
- if \( t \in V(T) \) has exactly two children \( t' \) and \( t'' \), then \( X_t = X_{t'} = X_{t''} \), \( E(G_{t'}) \cap E(G_{t''}) = E(G[X_t]) \), and \( G_t = (V(G_{t'}) \cup V(G_{t''}), E(G_{t'}) \cup E(G_{t''})) \). The node \( t \) is called a *join node*. 

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For each \( t \in V(T) \), we denote by \( V_t \) the set \( V(G_t) \). As discussed in [20], given a tree decomposition, it is possible to transform it in polynomial time to a nice new one of the same width. Moreover, by Bodlaender et al. [10] we can find in time \( 2^{O(tw)} \cdot n \) a tree decomposition of width \( O(tw) \) of any graph \( G \). Hence, since in this section we focus on single-exponential algorithms, we may assume that a nice tree decomposition of width \( w = O(tw) \) is given with the input.

We also need the following simple observation that will be implicitly used in the algorithms of Sections 3, 4, and 6.

**Observation 1.** Let \( G \) be a graph and \( h \) be a positive integer. Then the following assertions are equivalent.

- \( G \) contains \( P_h \) as a topological minor.
- \( G \) contains \( P_h \) as a minor.
- \( G \) contains \( P_h \) as a subgraph.

Moreover, the following assertions are also equivalent.

- \( G \) contains \( C_h \) as a topological minor.
- \( G \) contains \( C_h \) as a minor.

**Parameterized complexity.** We refer the reader to [13,16] for basic background on parameterized complexity, and we recall here only some very basic definitions. A parameterized problem is a language \( L \subseteq \Sigma^* \times \mathbb{N} \). For an instance \( I = (x,k) \in \Sigma^* \times \mathbb{N} \), \( k \) is called the parameter. A parameterized problem is fixed-parameter tractable (FPT) if there exists an algorithm \( A \), a computable function \( f \), and a constant \( c \) such that given an instance \( I = (x,k) \), \( A \) (called an FPT algorithm) correctly decides whether \( I \in L \) in time bounded by \( f(k) \cdot |I|^c \).

**Main ingredients of the rank-based approach.** We are now going to restate the tools introduced by Bodlaender et al. [9] that we need for our purposes.

Let \( U \) be a set. We define \( \Pi(U) \) to be the set of all partitions of \( U \). Given two partitions \( p \) and \( q \) of \( U \), we define the coarsening relation \( \sqsubseteq \) such that \( p \sqsubseteq q \) if for each \( S \in q \), there exists \( S' \in p \) such that \( S \subseteq S' \). \( (\Pi(U), \sqsubseteq) \) defines a lattice with minimum element \( \{\{U\}\} \) and maximum element \( \{\{x\} \mid x \in U\} \). On this lattice, we denote by \( \sqcap \) the meet operation and by \( \sqcup \) the join operation.

Let \( p \in \Pi(U) \). For \( X \subseteq U \) we denote by \( p_{\downarrow X} = \{S \cap X \mid S \in p, S \cap X \neq \emptyset\} \in \Pi(X) \) the partition obtained by removing all elements not in \( X \) from \( p \), and analogously for
Given a set \( \mathcal{A} \subseteq \Pi(U) \times \mathbb{N} \), we define the partition \( \Pi(\mathcal{A}) = \{(p, w) \in \mathcal{A} \mid \forall(p', w') \in \mathcal{A} : p' = p \Rightarrow w = w' \} \).

We now define some operations on weighted partitions. Let \( U \) be a set and \( \mathcal{A} \subseteq \Pi(U) \times \mathbb{N} \).

**Union.** Given \( \mathcal{B} \subseteq \Pi(U) \times \mathbb{N} \), we define \( \mathcal{A} \cup \mathcal{B} = \text{rmc}(\mathcal{A} \cup \mathcal{B}) \).

**Insert.** Given a set \( X \) such that \( X \cap U = \emptyset \), we define \( \text{ins}(X, \mathcal{A}) = \{(p_{\cup X}, w) \mid (p, w) \in \mathcal{A} \} \).

**Shift.** Given \( w' \in \mathbb{N} \), we define \( \text{shft}(w', \mathcal{A}) = \{(p, w + w') \mid (p, w) \in \mathcal{A} \} \).

**Glue.** Given a set \( S \), we define \( \hat{U} = U \cup S \) and \( \text{glue}(S, \mathcal{A}) \subseteq \Pi(\hat{U}) \times \mathbb{N} \) as
\[
\text{glue}(S, \mathcal{A}) = \text{rmc}(\{(\hat{U}[S] \cap p_{\hat{U}}, w) \mid (p, w) \in \mathcal{A} \})
\]

Given \( w : \hat{U} \times \hat{U} \to \mathcal{N} \), we define \( \text{glue}_{\mathcal{w}}(\{u, v\}, \mathcal{A}) = \text{shft}(w(u, v), \text{glue}(\{u, v\}, \mathcal{A})) \).

**Project.** Given \( X \subseteq U \), we define \( \overline{X} = U \setminus X \) and \( \text{proj}(X, \mathcal{A}) \subseteq \Pi(\overline{X}) \times \mathbb{N} \) as
\[
\text{proj}(X, \mathcal{A}) = \text{rmc}(\{(p_{\overline{X}}, w) \mid (p, w) \in \mathcal{A} \forall e \in X : \forall e' \in \overline{X} : p \subseteq U[e'e'] \})
\]

**Join.** Given a set \( U', \mathcal{B} \subseteq \Pi(U') \times \mathbb{N} \), and \( \hat{U} = U \cup U' \), we define \( \text{join}(\mathcal{A}, \mathcal{B}) \subseteq \Pi(\hat{U}) \times \mathbb{N} \) as
\[
\text{join}(\mathcal{A}, \mathcal{B}) = \text{rmc}(\{(p_{\hat{U}} \cap q_{\hat{U}}, w_1 + w_2) \mid (p, w_1) \in \mathcal{A}, (q, w_2) \in \mathcal{B} \})
\]

**Proposition 1** (Bodlaender et al. [9]). *Each of the operations union, insert, shift, glue, and project can be carried out in time \( s \cdot |U|^{O(1)} \), where \( s \) is the size of the input of the operation. Given two weighted partitions \( \mathcal{A} \) and \( \mathcal{B} \), \( \text{join}(\mathcal{A}, \mathcal{B}) \) can be computed in time \( |\mathcal{A}| \cdot |\mathcal{B}| \cdot |U|^{O(1)} \).

**Proposition 2** (Bodlaender et al. [9]). The union, insert, shift, glue, project, and join operations preserve representation.
Theorem 3 (Bodlaender et al. [9]). There exists an algorithm \textit{reduce} that, given a set of weighted partitions \( A \subseteq \Pi(U) \times \mathbb{N} \), outputs in time \(|A| \cdot 2^{(\omega-1)|U|} \cdot |U|^O(1)\) a set of weighted partitions \( A' \subseteq A \) such that \( A' \) represents \( A \) and \(|A'| \leq 2^{|U|}\), where \( \omega \) denotes the matrix multiplication exponent.

Definition of the problems. Let \( \mathcal{F} \) be a proper collection. We define the parameter \( \text{tm}_{\mathcal{F}} \) as the function that maps graphs to non-negative integers as follows:

\[
\text{tm}_{\mathcal{F}}(G) = \min \{|S| \mid S \subseteq V(G) \land G \setminus S \in \text{ex}_{\text{tm}}(\mathcal{F})\}.
\]

The parameter \( m_{\mathcal{F}} \) is defined analogously. The main objective of this paper is to study the problem of computing the parameters \( \text{tm}_{\mathcal{F}} \) and \( m_{\mathcal{F}} \) for graphs of bounded treewidth under several instantiations of the collection \( \mathcal{F} \). The corresponding decision problems are formally defined as follows.

| \( \mathcal{F} \)-TM-Deletion | \( \mathcal{F} \)-M-Deletion |
|--------------------------------|--------------------------------|
| **Input:** A graph \( G \) and an integer \( k \in \mathbb{N} \). | **Input:** A graph \( G \) and an integer \( k \in \mathbb{N} \). |
| **Parameter:** The treewidth of \( G \). | **Parameter:** The treewidth of \( G \). |
| **Output:** Is \( \text{tm}_{\mathcal{F}}(G) \leq k \)? | **Output:** Is \( m_{\mathcal{F}}(G) \leq k \)? |

Note that in both above problems, we can always assume that \( \mathcal{F} \) is an antichain with respect to the considered relation. Indeed, this is the case because if \( \mathcal{F} \) contains two graphs \( H_1 \) and \( H_2 \) where \( H_1 \preceq_{\text{tm}} H_2 \), then \( \text{tm}_{\mathcal{F}}(G) = \text{tm}_{\mathcal{F'}}(G) \) where \( \mathcal{F'} = \mathcal{F} \setminus \{H_2\} \) (similarly for the minor relation).

Throughout the article, we let \( n \) and \( \text{tw} \) be the number of vertices and the treewidth of the input graph of the considered problem, respectively. We will also use \( w \) to denote the width of a (nice) tree decomposition that is given together with the input graph (which, based on [10], will differ from \( \text{tw} \) by at most a factor five).

3 A single-exponential algorithm for \( \{P_3\}\)-TM-Deletion

It should be noted that a single-exponential algorithm for \( \{P_3\}\)-TM-Deletion is already known. Indeed, Tu et al. [23] presented an algorithm running in time \( O^*(4^{\text{tw}}) \), and very recently Bai et al. [1] improved it to \( O^*(3^{\text{tw}}) \). Nevertheless, for completeness we present in this section a simpler algorithm, but involving a greater constant than \([1, 23]\).

We first give a simple structural characterization of the graphs that exclude \( P_3 \) as a topological minor.

**Lemma 1.** Let \( G \) be a graph. \( P_3 \preceq_{\text{tm}} G \) if and only if each vertex of \( G \) has degree at most one.
Proof. Let $G$ be a graph. If $G$ has a connected component of size at least three, then clearly it contains a $P_3$. This implies that, if $P_3 \not\preceq_{\text{tm}} G$, then each connected component of $G$ has size at most two and so, each vertex of $G$ has degree at most one. Conversely, if each vertex of $G$ has degree at most one, then, as $P_3$ contains a vertex of degree two, $P_3 \not\preceq_{\text{tm}} G$.

We present an algorithm using classical dynamic programming techniques over a tree decomposition of the input graph. Let $G$ be an instance of $\{P_3\}$-TM-Deletion and let $((T, \mathcal{X}), r, G)$ be a nice tree decomposition of $G$.

We define, for each $t \in V(T)$, the set $I_t = \{(S, S_0) \mid S, S_0 \subseteq X_t, S \cap S_0 = \emptyset\}$ and a function $r_t : I_t \rightarrow \mathbb{N}$ such that for each $(S, S_0) \in I_t$, $r(S, S_0)$ is the minimum $\ell$ such that there exists a set $\hat{S} \subseteq V(G_t)$, called the witness of $(S, S_0)$, that satisfies:

- $|\hat{S}| \leq \ell$,
- $\hat{S} \cap X_t = S$,
- $P_3 \not\preceq_{\text{tm}} G_t \setminus \hat{S}$, and
- $S_0$ is the set of vertices of $X_t$ of degree 0 in $G_t \setminus S$.

Note that with this definition, $\text{tm}_F(G) = r_r(\emptyset, \emptyset)$. For each $t \in V(T)$, we assume that we have already computed $r_{t'}$ for each children $t'$ of $t$, and we proceed to the computation of $r_t$. We distinguish several cases depending on the type of node $t$.

**Leaf.** $I_t = \{((\emptyset, \emptyset))\}$ and $r_t(\emptyset, \emptyset) = 0$.

**Introduce vertex.** If $v$ is the insertion vertex of $X_t$ and $t'$ is the child of $t$, then for each $(S, S_0) \in I_t$,

\[
r_t(S, S_0) = \min \left( \{ r_{t'}(S', S_0') + 1 \mid (S', S_0') \in I_{t'}, S = S' \cup \{v\} \} \right.
\]

\[
\cup \{ r_{t'}(S, S_0') \mid (S, S_0') \in I_{t'}, S = S_0' \cup \{v\}, N_{G_t[X_t]}(v) \setminus S = \emptyset \}
\]

\[
\cup \{ r_{t'}(S, S_0') \mid (S, S_0') \in I_{t'}, S = S_0' \setminus \{u\}, u \in S_0, N_{G_t[X_t]}(v) \setminus S = \{u\} \} \}.
\]

**Forget vertex.** If $v$ is the forget vertex of $X_t$ and $t'$ is the child of $t$, then for each $(S, S_0) \in I_t$,

\[
r_t(S, S_0) = \min \{ r_{t'}(S', S_0') \mid (S', S_0') \in I_{t'}, S = S' \setminus \{v\}, S_0 = S_0' \setminus \{v\} \}
\]

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Join. If \( t' \) and \( t'' \) are the children of \( t \), then for each \((S, S_0) \in \mathcal{I}_t,\)
\[
\mathbf{r}(S, S_0) = \min \{ \mathbf{r}(S', S'_0) + \mathbf{r}(S'', S''_0) - |S' \cap S''|, \ (S', S'_0) \in \mathcal{I}_{t'}, (S'', S''_0) \in \mathcal{I}_{t''}, \ S = S' \cup S'', S_0 = S'_0 \cap S''_0, X_t \setminus S \subseteq S'_0 \cup S''_0 \}.
\]

Let us analyze the running time of this algorithm. As, for each \( t \in V(T), S \) and \( S_0 \)
are disjoint subsets of \( X_t \), we have that \( |\mathcal{I}_t| \leq 3^{|X_t|} \). Note that if \( t \) is a leaf, then \( \mathbf{r}_t \) can be computed in time \( O(1) \), if \( t \) is an introduce vertex or a forget vertex node, and \( t' \) is
the child of \( t \), then \( \mathbf{r}_t \) can be computed in time \( O(|\mathcal{I}_{t'}| \cdot |X_t|) \), and if \( t \) is a join node, and \( t' \) and \( t'' \) are the two children of \( t \), then \( \mathbf{r}_t \) can be computed in time \( O(|\mathcal{I}_{t'}| \cdot |\mathcal{I}_{t''}| \cdot |X_t|) \).

We now show that for each \( t \in V(T) \), the function \( \mathbf{r}_t \) is correctly computed by the
algorithm.

Leaf. This follows directly from the definition of \( \mathbf{r}_t \).

Introduce vertex. Let \( v \) be the insertion vertex of \( X_t \). As \( v \) is the insertion vertex, we
have that \( N_{G_t[V]}(v) = N_{G_t}(v) \), and so for each value we add to the set, we can
find a witness of \((S, S_0)\) of size bounded by this value.

Conversely, let \((S, S_0) \in \mathcal{I}_t \) and let \( \mathcal{S} \) be a witness. If \( v \in S \), then \( (S \setminus \{v\}, S_0) \in \mathcal{I}_{t'} \)
and \( \mathbf{r}(S \setminus \{v\}, S_0) \leq |\mathcal{S}| - 1, \) if \( v \in S \) then \( (S, S_0 \setminus \{v\}) \in \mathcal{I}_{t'} \) and \( \mathbf{r}(S, S_0 \setminus \{v\}) \leq |\mathcal{S}|, \) and if \( v \in X_t \setminus (S \cup S_0) \), then by definition \( v \) has a unique neighbor, say \( u \), in \( G_t \setminus \mathcal{S} \),
moreover \( u \in X_t \setminus (S \cup S_0) \), \( v \) is the unique neighbor of \( u \) in \( G_t \setminus \mathcal{S}, (S, S_0 \cup \{u\}) \in \mathcal{I}_{t'}, \) and \( \mathbf{r}(S, S_0 \cup \{u\}) \leq |\mathcal{S}|. \)

Forget vertex. This also follows directly from the definition of \( \mathbf{r}_t. \)

Join. Let \((S', S'_0) \in \mathcal{R}_{t'} \) and let \((S'', S''_0) \in \mathcal{I}_{t''} \) with witnesses \( \mathcal{S}' \) and \( \mathcal{S}'' \), respectively.
If \( S = S' \cup S'' \) and \( S'_0 \cap S''_0 = X_t \setminus S \), then the condition \( X_t \setminus S \subseteq S'_0 \cup S''_0 \) ensures that \( G_t \setminus (\mathcal{S}' \cup \mathcal{S}'') \) has no vertex of degree at least two and so \( \mathcal{S}' \cup \mathcal{S}'' \) is a witness
of \((S, S'_0 \cap S''_0) \in \mathcal{I}_t \) of size at most \( \mathbf{r}_{t'}(S', S'_0) + \mathbf{r}_{t''}(S'', S''_0) - |S' \cap S''|. \)

Conversely, let \((S, S_0) \in \mathcal{I}_t \) with witness \( \mathcal{S} \). If \( \mathcal{S}' = \mathcal{S} \cap V(G_{t'}) \) and \( \mathcal{S}'' = \mathcal{S} \cap V(G_{t''}) \),
then by definition of \( \mathcal{S}, \mathcal{S}' \) is a witness of some \((S', S'_0) \in \mathcal{I}_{t'}, \) and \( \mathcal{S}'' \) is a witness
of some \((S'', S''_0) \in \mathcal{I}_{t''} \) such that \( S = S' \cup S'', S'_0 \cup S''_0 = X_t \setminus S \), and \( S_0 = S'_0 \cap S''_0 \),
and we have \( \mathbf{r}_{t'}(S', S'_0) + \mathbf{r}_{t''}(S'', S''_0) - |S| \leq |\mathcal{S}|. \)

The following theorem summarizes the above discussion.

**Theorem 4.** If a nice tree decomposition of \( G \) of width \( w \) is given, \( \{P_3\} \)-TM-DELETION can be solved in time \( O(9^w \cdot w \cdot n) \).
4 A single-exponential algorithm for \(\{P_4\}\)-TM-Deletion

Similarly to what we did for \(\{P_3\}\)-TM-Deletion, we start with a structural definition of the graphs that exclude \(P_4\) as a topological minor.

**Lemma 2.** Let \(G\) be a graph. \(P_4 \not\preceq_{\text{tm}} G\) if and only if each connected component of \(G\) is either a \(C_3\) or a star.

**Proof.** First note that if each connected component of \(G\) is either a \(C_3\) or a star, then \(P_4 \not\preceq_{\text{tm}} G\). Conversely, assume that \(P_4 \not\preceq_{\text{tm}} G\). Then each connected component of \(G\) of size at least 4 should contain at most 1 vertex of degree at least 2, hence such component is a star. On the other hand, the only graph on at most 3 vertices that is not a star is \(C_3\). The lemma follows.

As we did for \(\{P_3\}\)-TM-Deletion, we present an algorithm using classical dynamic programming techniques over a tree decomposition of the input graph. Let \(G\) be an instance of \(\{P_4\}\)-Deletion, and let \(((T, \mathcal{X}), r, G)\) be a nice tree decomposition of \(G\).

We define, for each \(t \in T\), the set \(I_t\) to be the set of each tuple \((S, S_{1+}, S_{1-}, S_*, S_{3+}, S_{3-})\) such that \(\{S, S_{1+}, S_{1-}, S_*, S_{3+}, S_{3-}\}\) is a partition of \(X_t\) and the function \(r : I_t \rightarrow \mathbb{N}\) such that, for each \((S, S_{1+}, S_{1-}, S_*, S_{3+}, S_{3-}) \in I_t\), \(r_t(S, S_{1+}, S_{1-}, S_*, S_{3+}, S_{3-})\) is the minimum \(\ell\) such that there exists a triple \((\tilde{S}, \tilde{S}_*, \tilde{S}_{3-}) \subseteq V(G_t) \times V(G_t) \times V(G_t)\), called the witness of \((S, S_{1+}, S_{1-}, S_*, S_{3+}, S_{3-})\), which satisfies the following properties:

- \(\tilde{S}, \tilde{S}_*, \tilde{S}_{3-}\) are pairwise disjoint,
- \(\tilde{S} \cap X_t = S, \tilde{S}_* \cap X_t = S_*, \text{ and } \tilde{S}_{3-} \cap X_t = S_{3-}\),
- \(|\tilde{S}| \leq \ell\),
- \(P_4 \not\preceq_{\text{tm}} G_t \setminus \tilde{S}\),
- \(S_{1+}\) is a set of vertices of degree 0 in \(G_t \setminus \tilde{S}\),
- each vertex of \(S_{1-}\) has a unique neighbor in \(G_t \setminus \tilde{S}\) and this neighbor is in \(\tilde{S}_*\),
- each connected component of \(G_t[\tilde{S}_{3-}]\) is a \(C_3\),
- there is no edge in \(G_t \setminus \tilde{S}\) between a vertex of \(\tilde{S}_{3-}\) and a vertex of \(V(G_t) \setminus (\tilde{S} \cup \tilde{S}_{3-})\),
- there is no edge in \(G_t \setminus \tilde{S}\) between a vertex of \(S_{3+}\) and a vertex of \(V(G_t) \setminus (\tilde{S} \cup S_{3+})\),
- and
- there is no edge in \(G_t \setminus \tilde{S}\) between two vertices of \(S_*\).
Intuitively, $\hat{S}$ corresponds to a partial solution in $G_t$. Note that, by Lemma 2, each component of $G_t \setminus \hat{S}$ must be either a star or a $C_3$. With this in mind, $\hat{S}_*$ is the set of vertices that are centers of a star in $G_t \setminus \hat{S}$, $S_{1+}$ is the set of leaves of a star that are not yet connected to a vertex of $\hat{S}$, $S_{1-}$ is the set of leaves of a star that are already connected to a vertex of $\hat{S}$, $\hat{S}_{3-}$ is the set of vertices that induce $C_3$'s in $G_t$, and $S_{3+}$ is a set of vertices that will induce $C_3$'s when further edges will appear.

Note that with this definition, $\text{tm}_F(G) = r_t(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$. For each $t \in V(T)$, we assume that we have already computed $r_{t'}$ for each children $t'$ of $t$, and we proceed to the computation of $r_t$. We distinguish several cases depending on the type of node $t$.

**Leaf.** $I_t = \{(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset)\}$ and $r_t(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset) = 0$.

**Introduce vertex.** If $v$ is the insertion vertex of $X_t$ and $t'$ is the child of $t$, then, for each $(S, S_{1+}, S_{1-}, S_*, S_{3+}, S_{3-}) \in I_t$,
\( r_t(S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) = \min \left( \{ r'_t(S', S_1^+, S_1^-, S_*, S_3^+, S_3^-) + 1 \right. \\
| (S', S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in R'_t, \ S = S' \cup \{v\} \right) \\
\cup \{ r'_t(S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \right. \\
| (S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in R'_t, \\
S_1^+ = S_1' \cup \{v\}, \ G_{[X \setminus S](v)}(v) = 0 \left. \} \right) \\
\cup \{ r'_t(S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \right. \\
| (S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in R'_t, \\
S_1^- = S_1' \cup \{v\}, \ z \in S_*, \ G_{[X \setminus S](v)}(v) = \{z\} \left. \} \right) \\
\cup \{ r'_t(S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \right. \\
| (S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in R'_t, \\
S_3^+ = S_3' \cup \{v\}, \ N_{G_{[X \setminus S]}(v)}(v^*) \subseteq S_1^+ \left. \} \right) \\
\cup \{ r'_t(S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \right. \\
| (S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in R'_t, \\
S_3^- = S_3' \cup \{v\}, \ N_{G_{[X \setminus S]}(v)}(v) = \emptyset \left. \} \right) \\
\cup \{ r'_t(S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \right. \\
| (S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in R'_t, \\
S_3^+ = S_3' \setminus \{z, z'\}, \ S_3^- = S_3' \cup \{z, z', v\}, \ z, z' \in S_3' \left. \} \right) \\
N_{G_{[X \setminus S]}(v)}(v) = \{z, z'\}, \ N_{G_{[X \setminus S]}(v)}(z) = \{v, z\} \left. \} \right). \\

**Forget vertex.** If \( v \) is the forget vertex of \( X_t \) and \( t' \) is the child of \( t \), then, for each \( (S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in \mathcal{I}_t \),

\( r_t(S, S_1^+, S_1^-, S_*, S_3^+, S_3^-) = \min \{ r'_t(S', S_1^+, S_1^-, S_*, S_3^+, S_3^-) \right. \\
| (S', S_1^+, S_1^-, S_*, S_3^+, S_3^-) \in \mathcal{I}_t', \\
S = S' \setminus \{v\}, \ S_1^- = S_1' \setminus \{v\}, \ S_3^+ = S_3' \setminus \{v\}, \ S_3^- = S_3' \setminus \{v\} \left. \} \right). 

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**Join.** If \( t' \) and \( t'' \) are the children of \( t \), then for each \((S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \in \mathcal{I}_t\), 
\[ r_t(S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \] 
\[
\min\{r'_t(S, S'_{1+}, S'_{1-}, S_s, S'_{3+}, S_{3-}) + r''_t(S, S''_{1+}, S''_{1-}, S_s, S''_{3+}, S_{3-}) - |S| 
| (S, S'_{1+}, S'_{1-}, S_s, S'_{3+}, S_{3-}) \in \mathcal{I}_t', \ (S, S''_{1+}, S''_{1-}, S_s, S''_{3+}, S_{3-}) \in \mathcal{I}_t'' \}, 
(S'_{1+} \cup S'_{1-}) \cap (S''_{3+} \cup S''_{3-}) = (S'_{1+} \cup S'_{1-}) \cap (S''_{3+} \cup S''_{3-}) = \emptyset, 
\forall v \in S'_{1-} \cap S''_{1-}, \exists z \in S_s : N_{G_t[X_t]}(v) = \left\{ z \right\}, 
S_{1-} = (S'_{1-} \cup S''_{1-}) \cup S_s, \ S_{1+} = S'_{1+} \cup S''_{1+}, 
\forall v \in S'_{3-} \cap S''_{3-}, \exists z, z' \in S'_{3-} \cup S''_{3-} : v, z, z' \text{ induce a } C_3 \text{ in } G_t[X_t \setminus S], 
S_{3-} = (S'_{3-} \cup S''_{3-}) \cup S_s, \ S_{3+} = S'_{3+} \cup S''_{3+}. 
\]

Let us analyze the running time of this algorithm. As, for each \( t \in V(T), \ S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-} \) form a partition of \( X_t \), we have that \( |\mathcal{I}_t| \leq 6^{|X_t|} \). Note that if \( t \) is a leaf, then \( r_t \) can be computed in time \( O(1) \), if \( t \) is an introduce vertex or a forget vertex node, and \( t' \) is the child of \( t \), then \( r_t \) can be computed in time \( O(|\mathcal{I}_t'| \cdot |X_t|) \), and if \( t \) is a join node, and \( t' \) and \( t'' \) are the two children of \( t \), then \( r_t \) can be computed in time \( O(|\mathcal{I}_t'| \cdot |\mathcal{I}_t''| \cdot |X_t|) \).

We now show that for each \( t \in V(T), \ r_t \) is correctly computed by the algorithm. For each \((S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \in \mathcal{I}_t \), it can be easily checked that each value \( \ell \) we compute respects, \( r_t(S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \) for each witness \((\hat{S}, \hat{S}_s, \hat{S}_{3-}) \) of \((S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \) satisfies \( \ell \leq |\hat{S}| \). We again distinguish the type of node \( t \).

**Leaf.** This follows directly from the definition of \( r_t \).

**Introduce vertex.** Let \( v \) be the insertion vertex of \( X_t \), let \((S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \in \mathcal{R}_t \), and let \((\hat{S}, \hat{S}_s, \hat{S}_{3-}) \) be a witness.

- If \( v \in S \), then \((S \setminus \{v\}, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \in \mathcal{I}_t \) and 
  \[ r'_t(S \setminus \{v\}, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) \leq |\hat{S}| - 1. \]

- If \( v \in S_{1+} \), then \( v \) is of degree 0 in \( G_t[\hat{S}] \), hence \((S \setminus \{v\}, S_{1-}, S_s, S_{3+}, S_{3-}) \in \mathcal{I}_t \) and 
  \[ r'_t(S \setminus \{v\}, S_{1-}, S_s, S_{3+}, S_{3-}) \leq |\hat{S}|. \]

- If \( v \in S_{1-} \), then \( v \) has a unique neighbor that is in \( \hat{S}_s \). As \( v \) is the insertion vertex of \( X_t \), it implies that \( N_{G_t}(v) \subseteq S_s \), and so \((S, S_{1+}, S_{1-} \setminus \{v\}, S_s, S_{3+}, S_{3-}) \in \mathcal{I}_t \) and 
  \[ r'_t(S, S_{1+}, S_{1-} \setminus \{v\}, S_s, S_{3+}, S_{3-}) \leq |\hat{S}|. \]
• If \( v \in S_+ \), then every neighbor of \( v \) is in \( S_1^- \) and has degree 1 in \( G_t \setminus \hat{S} \). Thus, \( (S, S_1^+ \cup N_{G_t \setminus \hat{S}}(v), S_1^- \setminus N_{G_t \setminus \hat{S}}(v), S_+ \setminus \{v\}, S_{3+} \setminus S_{3-}) \in I_t \) and \( r_{t'}(S, S_1^+ \cup N_{G_t \setminus \hat{S}}(v), S_1^- \setminus N_{G_t \setminus \hat{S}}(v), S_+ \setminus \{v\}, S_{3+} \setminus S_{3-}) \leq |\hat{S}| \).

• If \( v \in S_{3+} \), then \( (S, S_1^+, S_{1-}, S_+ \setminus \{v\}, S_{3+} \setminus \{v\}, S_{3-}) \in I_t \) and \( r_{t'}(S, S_1^+, S_{1-}, S_+ \setminus \{v\}, S_{3+} \setminus \{v\}, S_{3-}) \leq |\hat{S}| \).

• If \( v \in S_{3-} \), then there exist \( z \) and \( z' \) in \( S_{3-} \) such that \( \{v, z, z'\} \) induce a \( C_3 \) in \( G_t \setminus \hat{S} \) and there is no edge in \( G_t \setminus \hat{S} \) between a vertex of \( \{v, z, z'\} \) and a vertex of \( V(G_t \setminus \hat{S}) \setminus \{x, z, z'\} \). So \( (S, S_1^+, S_{1-}, S_+ \cup \{z, z'\}, S_+ \setminus \{x, z, z'\}) \in I_t \) and \( r_{t'}(S, S_1^+, S_{1-}, S_+ \cup \{z, z'\}, S_+ \setminus \{x, z, z'\}) \leq |\hat{S}| \).

Forget vertex. Let \( v \) be the forget vertex of \( X_t \), let \( (S, S_1^+, S_{1-}, S_+ \setminus \{v\}, S_{3+}, S_{3-}) \in I_t \), and let \( (\hat{S}, \hat{S}_+, \hat{S}_{3+}) \) be a witness. If \( v \) has degree 0 in \( G_t \setminus \hat{S} \), then \( (S, S_1^+, S_{1-}, S_+ \cup \{v\}, S_{3+}, S_{3-}) \in I_t \) and \( r_{t'}(S, S_1^+, S_{1-}, S_+ \cup \{v\}, S_{3+}, S_{3-}) \leq |\hat{S}| \). If \( v \) has degree at least 1 in \( G_t \setminus \hat{S} \), then \( N_{G_t \setminus \hat{S}}(v) \cap S_{3+} = \emptyset \), as otherwise there would be an edge in \( G_t \setminus \hat{S} \) between a vertex of \( S_{3+} \) and a vertex of \( V(G_t \setminus \hat{S}) \cup S_{3+} \). So, one of the following case occurs:

• \( v \in \hat{S}_+, (S \cup \{v\}, S_+ \setminus \{v\}, S_{3+} \setminus S_{3-}) \in I_t \), and \( r_{t'}(S \cup \{v\}, S_+ \setminus \{v\}, S_{3+} \setminus S_{3-}) \leq |\hat{S}| \),

• \( v \in \hat{S}_+, (S, S_1^+, S_{1-} \cup \{v\}, S_{3+} \setminus S_{3-}) \in I_t \), and \( r_{t'}(S, S_1^+, S_{1-} \cup \{v\}, S_{3+} \setminus S_{3-}) \leq |\hat{S}| \),

• \( N_{G_t \setminus \hat{S}}(v) \subseteq \hat{S}_+, (S, S_1^+, S_{1-} \cup \{v\}, S_+ \setminus S_{3-}) \in I_t \), and \( r_{t'}(S, S_1^+, S_{1-} \cup \{v\}, S_+ \setminus S_{3-}) \leq |\hat{S}| \), or

• \( v \in \hat{S}_{3-}, (S, S_1^+, S_{1-} \cup \{v\}, S_{3+} \setminus S_{3-}) \in I_t \), and \( r_{t'}(S, S_1^+, S_{1-} \cup \{v\}, S_{3+} \setminus S_{3-}) \leq |\hat{S}| \)

Join. Let \( (S, S_1^+, S_{1-}, S_+ \setminus \{v\}, S_{3+} \setminus S_{3-}) \in I_t \), and let \( (\hat{S}, \hat{S}_+, \hat{S}_{3+}) \) be a witness. Let \( t' \) and \( t'' \) be the two children of \( t \). We define \( \hat{S}' = \hat{S} \cap V(G_{t'}) \), \( \hat{S}'' = \hat{S} \cap V(G_{t''}) \), \( \hat{S}'_3 = \hat{S} \cap V(G_{t'}) \), \( \hat{S}''_3 = \hat{S} \cap V(G_{t''}) \), \( \hat{S}'_{3-} \subseteq \hat{S}_{3-} \cap V(G_{t'}) \), and \( \hat{S}''_{3-} \subseteq \hat{S}_{3-} \cap V(G_{t''}) \), such that each connected component of \( G_t[\hat{S}'_{3-}] \) (resp. \( G_t[\hat{S}''_{3-}] \)) is a \( C_3 \) and \( G_{t'} \setminus (\hat{S}' \cup \hat{S}_{3-}') \) (resp. \( G_{t''} \setminus (\hat{S}'' \cup \hat{S}_{3-}'') \)) is a forest. Then we define

• \( S' = \hat{S}' \cap X_t \),

• \( S'_{1+} = S_{1+} \cup \{v \in S_{1-} \mid N_{G_t \setminus \hat{S}}(v) \subseteq \hat{S}'_{3-} \} \),

• \( S'_{1-} = \{v \in S_{1-} \mid N_{G_t \setminus \hat{S}}(v) \subseteq \hat{S}'_{3-} \} \),

• \( S'_s = S_s \cap V(G_{t'}) \),

• \( S'_{3-} = \hat{S}'_{3-} \cap X_t \), and
• $S_{3+}' = S_{3+} \cup (S_{3-} \setminus S_{3-}')$.

Note that $(S', S_{1+}', S_{1-}', S_s', S_{3+}', S_{3-}') \in \mathcal{I}'$. We define $(S'', S_{1+}', S_{1-}', S_s', S_{3+}', S_{3-}') \in \mathcal{I}''$ similarly. Moreover we can easily check that

- $S = S' = S''$, $S_s = S_s' = S_s''$,
- $(S_{1+}' \cup S_{1-}') \cap (S_{3+}'' \cup S_{3-}'') = (S_{1+}'' \cup S_{1-}'') \cap (S_{3+}' \cup S_{3-}') = \emptyset$,
- $\forall v \in S_{1-}' \cap S_{1-}'', \exists z \in S_s : N_{G_t[X_t \setminus S]}(v) = \{z\}$,
- $\forall v \in S_{3-}' \cap S_{3-}'', \exists z, z' \in S_{3-}' \cap S_{3-}' : v, z, z'$ induce a $C_3$ in $G_t[X_t \setminus S]$,
- $(S, S_{1+}, S_{1-}, S_s, S_{3+}, S_{3-}) = (S, S_{1+}' \cap S_{1+}'', S_{1-}' \cup S_{1-}'', S_s', S_{3+}' \cap S_{3+}'', S_{3-}' \cup S_{3-}'')$, and
- $r'(S', S_{1+}', S_{1-}', S_s', S_{3+}', S_{3-}') + r''(S'', S_{1+}, S_{1-}'', S_s'', S_{3+}'', S_{3-}'') - |S| \leq |\hat{S}|$.

This concludes the proof of correctness of the algorithm. The following theorem summarizes the above discussion.

**Theorem 5.** If a nice tree decomposition of $G$ of width $w$ is given, $\{P_4\}$-DELETION can be solved in time $O(36^w \cdot w \cdot n)$.

## 5 Single-exponential algorithms for $\{K_{1,s}\}$-TM-DELETION

Similarly to what we did before, we start with a (trivial) structural characterization of the graphs that exclude $K_{1,s}$, for some fixed integer $s$, as a topological minor.

**Lemma 3.** Let $s$ be a positive integer. A graph $G$ contains $K_{1,s}$ as a topological minor if and only if it contains a vertex of degree at least $s$.

**Proof.** Let $s$ be a fixed integer. If a graph $G$ contains a vertex $v$ of degree at least $s$, then $G$ contains $K_{1,s}$ as a subgraph and so, as a topological minor. If $G$ contains $K_{1,s}$ as a topological minor, then it implies that there exist in $G$ a vertex $v$ and $s$ paths of size at least two such that the intersection of any two of these paths contains precisely $v$. Thus $v$ has degree at least $s$. \hfill $\Box$

Given a fixed integer $s \geq 1$, the $s$-BOUNDDED-DEGREE VERTEX DELETION problem asks, given a graph $G$ and an integer $k$, whether one can remove at most $k$ vertices from $G$ such that the remaining graph has maximum degree at most $s$. Lemma 3 implies that for every positive integer $s$, $\{K_{1,s}\}$-TM-DELETION is exactly $(s-1)$-BOUNDDED-DEGREE VERTEX DELETION. For completeness, we provide a simple single-exponential algorithm parameterized by treewidth that solves $s$-BOUNDDED-DEGREE VERTEX DELETION for any fixed integer $s \geq 1$. 

Let \( s \geq 1 \) be a fixed integer, let \( G \) be an instance of \( s \)-Bounded-degree Vertex Deletion, and let \( ((T, \mathcal{X}), r, \mathcal{G}) \) be a nice tree decomposition of \( G \). We define, for each \( t \in V(T) \), the set \( \mathcal{I}_t = \{ (S, f) \mid S \subseteq X_t, f : X_t \setminus S \to [0, s - 1]\} \) and a function \( r_t : \mathcal{I}_t \to \mathbb{N} \) such that for each \((S, f) \in \mathcal{I}_t, r(S, f)\) is the minimum \( \ell \) such that there exists a set \( \hat{S} \subseteq V(G_t) \), called the witness of \((S, f)\), that satisfies:

- \( |\hat{S}| \leq \ell \),

- \( \hat{S} \cap X_t = S \), and

- for each \( v \in X_t \setminus S \), \( \deg_{G_t \setminus \hat{S}}(v) = f(v) \).

Note that with this definition, \( \text{tm}_{F}(G) = r_r(\emptyset, \emptyset) \). For each \( t \in V(T) \), we assume that we have already computed \( r_{t'} \) for each children \( t' \) of \( t \), and we proceed to the computation of \( r_t \). We distinguish several cases depending on the type of node \( t \).

**Leaf.** \( \mathcal{I}_t = \{ (\emptyset, \emptyset) \} \) and \( r_t(\emptyset, \emptyset) = 0 \).

**Introduce vertex.** If \( v \) is the insertion vertex of \( X_t \) and \( t' \) is the child of \( t \), then for each \((S, f) \in \mathcal{I}_t, r_t(S, f) = \min \{ r_{t'}(S', f') + 1 \mid (S', f') \in \mathcal{I}_{t'}, S = S' \cup \{v\} \}
\cup \{ r_{t'}(S, f') \mid (S, f') \in \mathcal{I}_{t'}, f(v) = \deg_{G_t \setminus X_t}(v),
\forall v' \in N_{G_t \setminus X_t}(v), f'(v') = f'(v') + 1,
\forall v' \in X_{t'} \setminus (S \cup N_{G_t \setminus X_t}(v)), f'(v') = f'(v') \} \).

**Forget vertex.** If \( v \) is the forget vertex of \( X_t \) and \( t' \) is the child of \( t \), then for each \((S, f) \in \mathcal{I}_t, r_t(S, f) = \min \{ r_{t'}(S', f') \mid (S', f') \in \mathcal{I}_{t'}, S = S' \setminus \{v\}, \forall v' \in X_t \setminus S, f'(v') = f'(v') \} \).

**Join.** If \( t' \) and \( t'' \) are the children of \( t \), then for each \((S, f) \in \mathcal{I}_t, r(S, f) = \min \{ r(S', f') + r(S, f'') - |S| \mid (S, f') \in \mathcal{I}_{t'}, (S, f'') \in \mathcal{I}_{t''}, \forall v \in X_t \setminus S, f(v) = f'(v) + f''(v) - \deg_{G_t \setminus X_t}(v) \} \).

One can check that for each \( t \in V(T) \), the set \( \mathcal{I}_t \) is of size at most \((s + 1)^{|X_t|}\): for each vertex in \( X_t \setminus S \) there are \( s \) possible values for its degree, together with the choice of belonging to \( S \) or not for each vertex in \( X_t \). Using the same argumentation as in the previous algorithms, we obtain the following theorem.

**Theorem 6.** Let \( s \geq 1 \) be a fixed integer. If a nice tree decomposition of \( G \) of width \( w \) is given, \( \{K_{1,s}\}\)-TM-Deletion can be solved in time \( O((s + 1)^{2w} \cdot w \cdot n) \).
6 A single-exponential algorithm for $\{C_4\}$-TM-DELETION

As discussed before, in this section we use the dynamic programming techniques introduced by Bodlaender et al. [9] to obtain a single-exponential algorithm for $\{C_4\}$-TM-DELETION. It is worth mentioning that the $\{C_4\}$-TM-DELETION problem has been studied in digraphs from a non-parameterized point of view [22]. The algorithm we present solves the decision version of $\{C_4\}$-TM-DELETION: the input is a pair $(G, k)$, where $G$ is a graph and $k$ is an integer, and the output is the boolean value $\text{tm}_F(G) \leq k$.

We give some definitions that will be used for the following algorithm. Given a graph $G$, we denote by $n(G) = |V(G)|$, $m(G) = |E(G)|$, $c_3(G)$ the number of $C_3$’s that are subgraphs of $G$, and $\text{cc}(G)$ the number of connected components of $G$. We say that $G$ satisfies the $C_4$-condition if the following conditions hold:

- $G$ does not contain the diamond as a subgraph, and
- $n(G) - m(G) + c_3(G) = \text{cc}(G)$.

As in the case of $P_3$ and $P_4$, we state in Lemma 5 a structural characterization of the graphs that exclude $C_4$ as a (topological) minor. We first need an auxiliary lemma.

Lemma 4. Let $n_0$ be a positive integer. Assume that for each graph $G'$ such that $1 \leq n(G') \leq n_0$, $C_4 \not\subseteq_{\text{tm}} G'$ if and only if $G'$ satisfies the $C_4$-condition. If $G$ is a graph that does not contain a diamond as a subgraph and such that $n(G) = n_0$, then $n(G) - m(G) + c_3(G) \leq \text{cc}(G)$.

Proof. Let $n_0$ be a positive integer, and assume that for each graph $G'$ such that $1 \leq n(G') \leq n_0$, $C_4 \not\subseteq_{\text{tm}} G'$ if and only if $G'$ satisfies the $C_4$-condition. Let $G$ be a graph that does not contain a diamond as a subgraph and such that $n(G) = n_0$. Let $S \subseteq E(G)$ such that $C_4 \not\subseteq_{\text{tm}} G \setminus S$ and $\text{cc}(G \setminus S) = \text{cc}(G)$ (note that any minimal feedback edge set satisfies these conditions). We have, by hypothesis, that $G \setminus S$ satisfies the $C_4$-condition, so $n(G \setminus S) - m(G \setminus S) + c_3(G \setminus S) = \text{cc}(G \setminus S)$. Moreover, as $G$ does not contain a diamond as a subgraph, each edge of $G$ participates in at most one $C_3$, and thus $c_3(G) - c_3(G \setminus S) \leq |S|$. As by definition $n(G) = n(G \setminus S)$ and $m(G) - m(G \setminus S) = |S|$, we obtain that $n(G) - m(G) + c_3(G) \leq \text{cc}(G \setminus S) = \text{cc}(G)$.

Lemma 5. Let $G$ be a non-empty graph. $C_4 \not\subseteq_{\text{tm}} G$ if and only if $G$ satisfies the $C_4$-condition.

Proof. Let $G$ be a non-empty graph, and assume first that $C_4 \not\subseteq_{\text{tm}} G$. This directly implies that $G$ does not contain the diamond as a subgraph. In particular, any two cycles of $G$, which are necessarily $C_3$’s, cannot share an edge. Let $S$ be a set containing
an arbitrary edge of each $C_3$ in $G$. By construction, $G \setminus S$ is a forest. As in a forest $F$, we have $n(F) - m(F) = \text{cc}(F)$, and $S$ is defined such that $|S| = c_3(G)$ because each edge of $G$ participates in at most one $C_3$, we obtain that $n(G) - m(G) + c_3(G) = \text{cc}(G)$. Thus, $G$ satisfies the $C_1$-condition.

Conversely, assume now that $G$ satisfies the $C_4$-condition. We prove that $C_4 \not\unlhd_{\text{tm}} G$ by induction on $n(G)$. If $n(G) \leq 3$, then $n(G) < n(C_4)$ and so $C_4 \not\unlhd_{\text{tm}} G$. Assume now that $n(G) \geq 4$, and that for each graph $G'$ such that $1 \leq n(G') < n(G)$, if $G'$ satisfies the $C_4$-condition, then $C_4 \not\unlhd_{\text{tm}} G'$.

We prove that this last implication is also true for $G$. Note that, as two $C_3$ cannot share an edge in $G$, we have that $c_3(G) \leq \frac{m(G)}{3}$. This implies that the minimum degree of $G$ is at most 2. Indeed, if each vertex of $G$ had degree at least 3, then $m(G) \geq \frac{3}{2}n(G)$, which together with the relations $c_3(G) \leq \frac{m(G)}{3}$ and $n(G) - m(G) + c_3(G) = \text{cc}(G)$ would imply that $\text{cc}(G) \leq 0$, a contradiction. Let $v \in V(G)$ be a vertex with minimum degree. We distinguish two cases according to the degree of $v$.

If $v$ has degree 0 or 1, then the graph $G \setminus \{v\}$ satisfies the $C_4$-condition as well, implying that $C_4 \not\unlhd_{\text{tm}} G \setminus \{v\}$. As $v$ has degree at most one, it cannot be inside a cycle, hence $C_4 \not\unlhd_{\text{tm}} G$.

Assume that $v$ has degree two and participates in a $C_3$. As $G$ does not contain a diamond as a subgraph, $C_4 \not\unlhd_{\text{tm}} G$ if and only if $C_4 \not\unlhd_{\text{tm}} G \setminus \{v\}$. Moreover $n(G \setminus \{v\}) = n(G) - 1$, $m(G \setminus \{v\}) = m(G) - 2$, $c_3(G \setminus \{v\}) = c_3(G) - 1$, and $\text{cc}(G \setminus \{v\}) = \text{cc}(G)$. This implies that $G \setminus \{v\}$ satisfies the $C_4$-condition, hence $C_4 \not\unlhd_{\text{tm}} G \setminus \{v\}$, and therefore $C_4 \not\unlhd_{\text{tm}} G$.

Finally, assume that $v$ has degree two and does not belong to any $C_3$. Using the induction hypothesis and Lemma 4, we have that $n(G \setminus \{v\}) - m(G \setminus \{v\}) + c_3(G \setminus \{v\}) \leq \text{cc}(G \setminus \{v\})$. As $n(G \setminus \{v\}) = n(G) - 1$, $m(G \setminus \{v\}) = m(G) - 2$, $c_3(G \setminus \{v\}) = c_3(G)$, $v$ has degree two in $G$, and $G$ satisfies the $C_4$-condition, we obtain that $\text{cc}(G \setminus \{v\}) = \text{cc}(G) - 1$. This implies that $G \setminus \{v\}$ satisfies the $C_4$-condition, and thus $C_4 \not\unlhd_{\text{tm}} G \setminus \{v\}$. Since $v$ disconnects one of the connected components of $G$ it cannot participate in a cycle of $G$, hence $C_4 \not\unlhd_{\text{tm}} G$. 

\textbf{Lemma 6.} If $G$ is a non-empty graph such that $C_4 \not\unlhd_{\text{tm}} G$, then $m(G) \leq \frac{3}{2}(n(G) - 1)$.

\textbf{Proof.} As $C_4 \not\unlhd_{\text{tm}} G$, by Lemma 5 $G$ satisfies the $C_4$-condition. It follows that $c_3(G) \leq \frac{1}{3}m(G)$. Moreover, as $G$ is non-empty, we have that $1 \leq \text{cc}(G)$. The lemma follows by using these inequalities in the equality $n(G) - m(G) + c_3(G) = \text{cc}(G)$. 

We now have all the tools needed to describe our algorithm. Recall that the basic ingredients of the rank-based approach of Bodlaender et al. [9] were given in Section 2. Let $G$ be a graph and $k$ be an integer. The algorithm we describe solves the decision
version of \( C_4 \)-TM-Deletion. This algorithm is based on the one given in [9, Section 3.5] for Feedback Vertex Set.

We define a new graph \( G_0 = (V(G) \cup \{v_0\}, E(G) \cup E_0) \), where \( v_0 \) is a new vertex and \( E_0 = \{(v_0, v) \mid v \in V(G)\} \). The role of \( v_0 \) is to artificially guarantee the connectivity of the solution graph, so that the machinery of Bodlaender et al. [9] can be applied. In the following, for each subgraph \( H \) of \( G_0 \), for each \( Z \subseteq V(H) \), and for each \( Z_0 \subseteq E_0 \cap E(H[Z]) \), we denote by \( H[Z, Z_0] \) the graph \((Z, Z_0 \cup E(H[Z] \setminus \{v_0\}))\).

Given a nice tree decomposition of \( G \) of width \( w \), we define a nice tree decomposition \(((T, \mathcal{X}), r, \mathcal{G}) \) of \( G_0 \) of width \( w + 1 \) such that the only empty bags are the root and the leaves and for each \( t \in T \), if \( X_t \neq \emptyset \) then \( v_0 \in X_t \). Note that this can be done in linear time. For each bag \( t \), each integers \( i, j, \) and \( \ell \), each function \( s : X_t \to \{0, 1\} \), each function \( s_0 : \{v_0\} \times s^{-1}(1) \to \{0, 1\} \), each function \( r : E(G_t(s^{-1}(1), s_0^{-1}(1))) \to \{0, 1\} \), and for each partition \( p \in \Pi(s^{-1}(1)) \), if \( C_4 \preceq_{tm} G_t(s^{-1}(1), s_0^{-1}(1)) \), we define:

\[
\mathcal{E}_t(p, s, s_0, r, i, j, \ell) = \{(Z, Z_0) \mid (Z, Z_0) \in 2^{V_t} \times 2^{E_0 \cap E(G_t)} \wedge |Z| = i, \ |E(G_t(Z, Z_0))| = j, \ c_3(G_t(Z, Z_0)) = \ell, \ G_t(Z, Z_0) \text{ does not contain the diamond as a subgraph,} \ Z \cap X_t = s^{-1}(1), \ Z_0 \cap (X_t \times X_t) = s_0^{-1}(1), \ v_0 \in X_t \Rightarrow s(v_0) = 1, \ \forall u \in Z \setminus X_t : \text{either } t \text{ is the root or} \ \exists u' \in s^{-1}(1) : \text{u and } u' \text{ are connected in } G_t(Z, Z_0), \ \forall v_1, v_2 \in s^{-1}(1) : p \subseteq V_t[\{v_1, v_2\}] \iff v_1 \text{ and } v_2 \text{ are connected in } G_t(Z, Z_0), \ \forall e \in E(G_t(Z, Z_0)) \cap \binom{s^{-1}(1)}{2} : r(e) = 1 \iff e \text{ is an edge of a } C_3 \text{ in } G_t(Z, Z_0)\}.
\]

\[
\mathcal{A}_t(s, s_0, r, i, j, \ell) = \{p \mid p \in \Pi(s^{-1}(1)), \ \mathcal{E}_t(p, s, s_0, r, i, j, \ell) \neq \emptyset\}.
\]

Otherwise, i.e., if \( C_4 \preceq_{tm} G_t(s^{-1}(1), s_0^{-1}(1)) \), we define

\[
\mathcal{A}_t(s, s_0, r, i, j, \ell) = \emptyset.
\]

Note that we do not need to keep track of partial solutions if \( C_4 \preceq_{tm} G_t(s^{-1}(1), s_0^{-1}(1)) \), as we already know they will not lead to a global solution. Moreover, if \( C_4 \not\preceq_{tm} \)
\( G_t(s^{-1}(1), s_0^{-1}(1)) \), then by Lemma 6, \( m(G_t(s^{-1}(1), s_0^{-1}(1))) \leq \frac{3}{2}(n(G_t(s^{-1}(1), s_0^{-1}(1)) - 1).

Using the definition of \( A_r \), Lemma 5, and Lemma 6, we have that \( t_m\{C_4\}(G) \leq k \) if and only if for some \( i \geq |V(G) \cup \{v_0\}| - k \) and some \( j \leq \frac{3}{2}(i - 1) \), we have \( A_r(\emptyset, \emptyset, i, j, 1 + j - i) \neq \emptyset \). For each \( t \in V(T) \), we assume that we have already computed \( A_r \) for each children \( t' \) of \( t \), and we proceed to the computation of \( A_t \). As usual, we distinguish several cases depending on the type of node \( t \).

**Leaf.** By definition of \( A_t \) we have \( A_t(\emptyset, \emptyset, 0, 0, 0) = \{\emptyset\} \).

**Introduce vertex.** Let \( v \) be the insertion vertex of \( X_t \), let \( t' \) be the child of \( t \), let \( s, s_0, \) and \( r \) the functions defined as before, let \( H = G_t(s^{-1}(1), s_0^{-1}(1)) \), and let \( d_3 \) be the number of \( C_3 \)'s of \( H \) that contain the vertex \( v \).

- If \( C_4 \preceq t_m \) \( H \) or if \( v = v_0 \) and \( s(v_0) = 0 \), then by definition of \( A_t \) we have that \( A_t(s, s_0, r, i, j, \ell) = \emptyset \).
- Otherwise, if \( s(v) = 0 \), then, by definition of \( A_t \), it holds that \( A_t(s, s_0, r, i, j, \ell) = A_{t'}(s|_{X_{t'}}, s_0|_{E_{t'}}, r|_{E_{t'}}, i, j, \ell) \).
- Otherwise, if \( v = v_0 \), then by construction of the nice tree decomposition, we know that \( t' \) is a leaf of \( T \) and so \( s = \{(v_0, 1)\} \), \( s_0 = \emptyset \), \( j = \ell = i - 1 = 0 \) and \( A_t(s, s_0, r, i, j, \ell) = \text{ins}(\{v_0\}, A_{t'}(\emptyset, \emptyset, 0, 0, 0)) \).
- Otherwise, we know that \( v \neq v_0 \), \( s(v) = 1 \), \( v_0 \in N_{G(s^{-1}(1))}(v) \), and \( C_4 \preceq t_m \) \( H \). As \( s(v) = 1 \), we have to insert \( v \) and we have to make sure that all vertices of \( N_H[v \setminus \{v_0\}] \) are in the same connected component of \( H \). The only remaining choice is either we insert the edge \( \{v, v_0\} \) or not. This is handle by the value of \( s_0(\{v_0, v\}) \). So, by adding \( v \), we add one vertex, \( |N_H(v)| \) edges, and \( d_3 \) \( C_3 \)'s. We also have to take care not to introduce a diamond. For this, the function \( r \) should be such that, for every edge \( e \) contained in a \( C_3 \)'s of \( H \) that contains the vertex \( v \), \( r(e) = 1 \). We define \( r': E(H[X_{t'}]) \to \{0, 1\} \) such that for every edge \( e \in E(H[X_{t'}]) \) contained in a \( C_3 \)'s of \( H \) that contains the vertex \( v \), \( r'(e) = 0 \), and for each other edge \( e \) of \( H[X_{t'}] \), \( r'(e) = r(e) \). Therefore, we have that

\[
A_t(s, s_0, r, i, j, \ell) =
\text{glue}(N_H[v], \text{ins}(\{v\}, A_{t'}(s|_{X_{t'}}, s_0|_{E_{t'}}, r', i - 1, j - |N_H(v)|, \ell - d_3))).
\]

**Forget vertex.** Let \( v \) be the forget vertex of \( X_t \), let \( t' \) be the child of \( t \), and let \( s, s_0, \) and \( r \) the functions defined as before. For each function, we have a choice on how
it can be extended in \( t' \), and we potentially need to consider every possible such extension. Note the number of vertices, edges, or \( C_3 \)'s is not affected. We obtain that

\[
\mathcal{A}_t(s, s_0, r, i, j, \ell) = A^v_\ell(s \cup \{(v, 0)\}, s_0, r, i, j, \ell)
\]

\[
\bigcup \text{proj}(\{v\}, A^v(s', s_0', r', i, j, \ell)).
\]

\[
s' : X_{t' \rightarrow \{0, 1\}}, s'|_{X_{t}} = s, s'(v) = 1
\]

\[
s_0'|_{\{v_0\} \times s'^{-1}(1) \rightarrow \{0, 1\}}, s_0'|_{X_{t}} = s_0
\]

\[
r' : E(G_t(s'^{-1}(1), s_0'^{-1}(1)) \rightarrow \{0, 1\}, r'|_{X_{t}} = r
\]

**Join.** Let \( t' \) and \( t'' \) be the two children of \( t \), let \( s \), \( s_0 \), and \( r \) be the functions defined as before, let \( H = G_t(s'^{-1}(1), s_0'^{-1}(1)) \), and let \( S \subseteq E(H) \) be the set of edges that participate in a \( C_3 \) of \( H \).

We join every compatible entries \( \mathcal{A}_t(s', s_0', r', i', j', \ell') \) and \( \mathcal{A}_{t''}(s'', s_0'', r'', i'', j'', \ell'') \). For two such entries being compatible, we need \( s' = s'' = s \) and \( s_0' = s_0'' = s_0 \). Moreover, we do not want the solution graph to contain a **diamond** as a subgraph, and for this we need \( r'^{-1}(1) \cap r''^{-1}(1) = S \). Indeed, either \( H \) contains the **diamond** as a subgraph, and then \( A^v(s', s_0', r', i', j', \ell') = A^v(s'', s_0'', r'', i'', j'', \ell'') = \emptyset \), or the **diamond** is created by joining two \( C_3 \)'s, one from \( t' \) and the other one from \( t'' \), sharing a common edge. This is possible only if \( (r'^{-1}(1) \cap r''^{-1}(1)) \setminus S \neq \emptyset \). For the counters, we have to be careful in order not to count some element twice. We obtain that

\[
\mathcal{A}_t(s, s_0, r, i, j, \ell) = \bigcup_{r', r'' : E(H) \rightarrow \{0, 1\}, r'^{-1}(1) \cap r''^{-1}(1) = S}
\]

\[
\text{join}(A^v(s, s_0, r', i', j', \ell'), A^v(s, s_0, r'', i'', j'', \ell'')).
\]

**Theorem 7.** \( \{C_4\} \)-TM-DELETION can be solved in time \( 2^{O(tw)} \cdot n^7 \).

**Proof.** The algorithm works in the following way. For each node \( t \in V(T) \) and for each entry \( M \) of its table, instead of storing \( \mathcal{A}_t(M) \), we store \( \mathcal{A}'_t(M) = \text{reduce}(\mathcal{A}_t(M)) \) by using Theorem 3. As each of the operation we use preserves representation by Proposition 2, we obtain that for each node \( t \in V(T) \) and for each possible entry \( M \), \( \mathcal{A}'_t(M) \) represents \( \mathcal{A}_t(M) \). In particular, we have that \( \mathcal{A}'_t(M) = \text{reduce}(\mathcal{A}_t(M)) \) for each possible entry \( M \). Using the definition of \( \mathcal{A}_r \), Lemma 5, and Lemma 6, we have that \( \text{tm}_{i}(C_4)(G) \leq k \) if and only if for some \( i \geq |V(G) \cup \{v_0\}| - k \) and some \( j \leq \frac{i}{3}(i - 1) \), we have \( \mathcal{A}_t'(\emptyset, \emptyset, \emptyset, i, j, 1 + j - i) \neq \emptyset \).
We now focus on the running time of the algorithm. The size of the intermediate sets of weighted partitions, for a leaf node and for an introduce vertex node are upper-bounded by $2^{|s| - 1}$. For a forget vertex node, as in the big union operation we take into consideration a unique extension of $s$, at most two possible extensions of $s_0$, and at most $2^{|s| - 1}$ possible extensions for $r$, we obtain that the intermediate sets of weighted partitions have size at most $2^{|s| - 1} + 2 \cdot 2^{|s| - 1} \cdot 2^{|s| - 1} \leq 2^{2|s| - 1} + 2$. For a join node, as in the big union operation we take into consideration at most $2^{|E(H)|}$ possible functions $r'$ and as many functions $r''$, at most $\frac{3}{2}(n - 1) + |E(H)|$ choices for $i'$ and $i''$, and at most $\frac{3}{2}(n - 1) + |E(H)|$ choices for $\ell'$ and $\ell''$, we obtain that the intermediate sets of weighted partitions have size at most $2^{|E(H)|} \cdot 2^{|E(H)|} \cdot (n + |s| - 1) \cdot (\frac{3}{2}(n - 1) + |E(H)|) \cdot (\frac{3}{2}(n - 1) + \frac{1}{3}|E(H)|) \cdot 4^{|s| - 1}$. As each time we can check the condition $C_4 \not\preceq_{tm} H$, by Lemma 6 $m(H) \leq \frac{3}{2}(n - H - 1)$, so we obtain that the intermediate sets of weighted partitions have size at most $6 \cdot n^3 \cdot 2^{5|s| - 1}$. Moreover, for each node $t \in V(T)$, the function reduce will be called as many times as the number of possible entries, i.e., at most $2^{O(w)} \cdot n^3$ times. Thus, using Theorem 3, $A'_t$ can be computed in time $2^{O(w)} \cdot n^6$. The theorem follows by taking into account the linear number of nodes in a nice tree decomposition.

7 A single-exponential algorithm for \{paw\}-TM-DELETION

Again, we start with a simple structural characterization of the simple graphs that exclude the paw as a topological minor; recall the paw graph in Figure 2.

![Figure 2: The paw graph.](image)

**Lemma 7.** Let $G$ be a simple graph. paw $\not\preceq_{tm} G$ if and only if each connected component of $G$ is either a cycle or a tree.

**Proof.** It is easy to see that neither a cycle nor a tree contain the paw as a topological minor. Let $G$ be a graph such that paw $\not\preceq_{tm} G$. Let us assume w.l.o.g. that $G$ is connected. If $G$ does not contain a cycle, then it is a tree. Otherwise, let $C$ be a chordless cycle in $G$. If $G$ contains a vertex $v$ that is not in $C$ then, as $G$ is connected, there exists a path from $v$ to $C$ containing at least two vertices. This is not possible, as it would imply that $G$ contains the paw as a topological minor. As $C$ is chordless and $G$ is simple, we obtain that $G$ is exactly the cycle $C$, and the lemma follows. \qed
We present an algorithm that solves the decision version of \{paw\}-TM-Deletion. As the algorithm that we presented for \{C_4\}-TM-Deletion in Section 6, this algorithm is based on the one given in [9, Section 3.5] for Feedback Vertex Set. Let $G$ be a graph and $k$ be an integer. The idea of the following algorithm is to partition $V(G)$ into three sets. The first one will be the solution set $S$, the second one will be a set $F$ of vertices that induces a forest, and the third one will be a set $C$ of vertices that induces a collection of cycles. If we can partition our graph into three such sets $(S, F, C)$ such that there is no edge between a vertex of $F$ and a vertex of $C$ and such that $|S| \leq k$, then, using Lemma 7, we know that $\text{tm}_{\{\text{paw}\}}(G) \leq k$. On the other hand, if such a partition does not exist, we know that $\text{tm}_{\{\text{paw}\}}(G) > k$. The main idea of this algorithm is to combine classical dynamic programming techniques in order to verify that $C$ induces a collection of cycles, and the rank-based approach in order to verify that $F$ induces a forest.

As for \{C_4\}-TM-Deletion, we define a new graph $G_0 = (V(G) \cup \{v_0\}, E(G) \cup E_0)$, where $v_0$ is a new vertex and $E_0 = \{\{v_0, v\} \mid v \in V(G)\}$. We recall that for each subgraph $H$ of $G_0$, for each $Z_1 \subseteq V(H)$, and for each $Y \subseteq E_0 \cap E(H[Z_1])$, we denote by $H(Z_1, Y)$ the graph $(Z_1, Y \cup E(H[Z_1 \setminus \{v_0\}]))$.

Given a nice tree decomposition of $G$ of width $w$, we define a nice tree decomposition $((T, \mathcal{X}), r, \mathcal{G})$ of $G_0$ of width $w + 1$ such that the only empty bags are the root and the leaves and for each $t \in T$, if $X_t \neq \emptyset$ then $v_0 \in X_t$. Note that this can be done in linear time. For each bag $t$, each integers $i$, $j$, and $\ell$, each function $s : X_t \to \{0, 1, 2_0, 2_1, 2_2\}$, each function $s_0 : \{v_0\} \times s^{-1}(1) \to \{0, 1\}$, and each partition $p \in \Pi(s^{-1}(1))$, we define:
\(\mathcal{E}_t(p, s, s_0, i, j, \ell) = \{(Z_1, Z_2, Y) \mid (Z_1, Z_2, Y) \in 2^V \times 2^V \times 2^{E_0 \cap E(G_t)}, Z_1 \cap Z_2 = \emptyset, |Z_1| = i, |Z_2| = \ell, |E(G_t[Z_1 \setminus \{v_0\}]) \cup Y| = j, \forall e \in E_0 \cap E_t, s_0(e) = 1 \iff e \in Y, \forall v \in Z_2 \cap X_t, s(v) = 2, \exists z = \deg_{G_t[Z_2]}(v), \forall v \in Z_2 \setminus X_t, \deg_{G_t[Z_2]}(v) = 2, Z_1 \cap X_t = s^{-1}(1), v_0 \in X_t \Rightarrow s(v_0) = 1, G_t(s^{-1}(1), s_0^{-1}(1)) \text{ is a forest, } \forall u \in Z_1 \setminus X_t: \text{ either } t \text{ is the root or } \exists u' \in s^{-1}(1): u \text{ and } u' \text{ are connected in } G_t(Z_1, Y), \forall v_1, v_2 \in s^{-1}(1): p \subseteq V_t[{\{v_1, v_2\}}] \iff v_1 \text{ and } v_2 \text{ are connected in } G_t(Z_1, Y), \forall (u, v) \in (Z_1 \setminus \{v_0\}) \times Z_2, \{u, v\} \not\subseteq E(G_t)\).

\[\mathcal{A}_t(s, s_0, i, j, \ell) = \{p \mid p \in \Pi(s^{-1}(1)), \mathcal{E}_t(p, s, s_0, i, j, \ell) \neq \emptyset\} .\]

In the definition of \(\mathcal{E}_t\), the sets \(Z_1\) (resp. \(Z_2\)) correspond to the set \(F\) (resp. \(C\)) restricted to \(G_t\). The vertex \(v_0\) and the set \(Y\) exist to ensure that \(F\) will be connected.

By Lemma 7, we have that the given instance of \(\{\text{paw}\}\text{-TM-Deletion}\) is a \(\text{Yes}\)-instance if and only if for some \(i\) and \(\ell\), \(i + \ell \geq |V(G) \cup \{v_0\}|-k\) and \(\mathcal{A}_r(\emptyset, \emptyset, i, i-1, \ell) \neq \emptyset\). For each \(t \in V(T)\), we assume that we have already computed \(\mathcal{A}_{t'}\) for every children \(t'\) of \(t\), and we proceed to the computation of \(\mathcal{A}_t\). As usual, we distinguish several cases depending on the type of node \(t\).

**Leaf.** By definition of \(\mathcal{A}_t\), we have \(\mathcal{A}_t(\emptyset, \emptyset, 0, 0, 0) = \emptyset\).

**Introduce vertex.** Let \(v\) be the insertion vertex of \(X_t\), let \(t'\) be the child of \(t\), let \(s: X_t \rightarrow \{0, 1, 2_0, 2_1, 2_2\}, s_0: \{v_0\} \times s^{-1}(1) \rightarrow \{0, 1\}\), and let \(H = G_t(s^{-1}(1), s_0^{-1}(1))\).

- If \(v = v_0\) and \(s(v_0) \in \{0, 2_0, 2_1, 2_2\}\) or if \(H\) contains a cycle, then by definition of \(\mathcal{A}_t\) we have that \(\mathcal{A}_t(s, s_0, i, j, \ell) = \emptyset\).
- Otherwise, if \(v = v_0\), then by construction of the nice tree decomposition, we know that \(t'\) is a leaf of \(T\) and so \(s = \{(v_0, 1)\}\), \(j = \ell = i - 1 = 0\) and \(\mathcal{A}_t(s, s_0, i, j, \ell) = \text{ins}(\{v_0\}, \mathcal{A}_{t'}(\emptyset, \emptyset, 0, 0, 0))\).
• Otherwise, if \( s(v) = 0 \), then, by definition of \( A_t \), it holds that \( A_t(s, s_0, i, j, \ell) = A_{t'}(s|_{X_{t'}}, s|_{E_{t'}}, i, j, \ell) \).

• Otherwise, if \( s(v) = 2z_i, z \in \{0, 1, 2\} \), then let \( Z_2' = N_{G_t[X_t]}(v) \setminus s^{-1}(0) \). If \( Z_2' \not\subset s^{-1}(\{2_1, 2_2\}) \) or \( |Z_2'| \not= z \) then \( A_t(s, s_0, i, j, \ell) = \emptyset \). Otherwise \( Z_2' \subset s^{-1}(\{2_1, 2_2\}) \) and \( |Z_2'| = z \), and with \( s' : X_{t'} \to \{0, 1, 2_0, 2_1, 2_2\} \) defined such that \( \forall v' \in X_{t'} \setminus Z_2', s'(v') = s(v') \) and for each \( v' \in Z'_2 \) such that \( s(v') = 2z_i', z' \in \{1, 2\}, s'(v') = 2z'_{i-1} \). It holds that \( A_t(s, s_0, i, j, \ell) = A_{t'}(s', s_0, i, j, \ell - 1) \).

• Otherwise, we know that \( v \neq v_0 \), \( s(v) = 1 \), and \( v_0 \in N_{G_{[s^{-1}(1)]}}(v) \). First, if \( N_{G_{[X_t]}(v)} \setminus s^{-1}(0) \not\subset s^{-1}(1) \), then \( A_t(s, s_0, i, j, \ell) = \emptyset \). Indeed, this implies that the cycle part and the forest part are connected. As \( s(v) = 1 \), we have to insert \( v \) in the forest part and we have to make sure that all vertices of \( N_H(v) \) are in the same connected component of \( H \). The only remaining choice is to insert the edge \( \{v, v_0\} \) or not. Again, this is handled by the function \( s_0 \). By adding \( v \), we add one vertex and \( |N_H(v)| \) edges in the forest part. Therefore, we have that

\[
A_t(s, s_0, i, j, \ell) = \text{glue}(N_H[v], \text{ins}(\{v\}, A_{t'}(s|_{X_{t'}}, s_0, i - 1, j - |N_H(v)|, \ell))).
\]

**Forget vertex.** Let \( v \) be the forget vertex of \( X_t \), let \( t' \) be the child of \( t \), and let \( s : X_t \to \{0, 1, 2_0, 2_1, 2_2\} \). As a vertex from the collection of cycles can be removed only if it has exactly two neighbors, we obtain that

\[
A_t(s, s_0, i, j, \ell) = A_{t'}(s \cup \{(v, 0)\}, s_0, i, j, \ell)
\]

\[\cup \text{proj}(\{v\}, A_{t'}(s \cup \{(v, 1)\}, s_0 \cup \{(v_0, v, 0)\}, i, j, \ell))\]

\[\cup \text{proj}(\{v\}, A_{t'}(s \cup \{(v, 1)\}, s_0 \cup \{(v_0, v, 1)\}, i, j, \ell))\]

\[\cup A_{t'}(s \cup \{(v, 2_2)\}, s_0, i, j, \ell).\]

**Join.** Let \( t' \) and \( t'' \) be the two children of \( t \), let \( s : X_t \to \{0, 1, 2_0, 2_1, 2_2\}, s_0 : \{v_0\} \times s^{-1}(1) \to \{0, 1\} \), and let \( H = G_t(s^{-1}(1), s_0^{-1}(1)) \). Given three functions \( s^*, s', s'' : X_t \to \{0, 1, 2_0, 2_1, 2_2\} \), we say that \( s^* \equiv s' \oplus s'' \) if for each \( v \in s^{-1}(\{0, 1\}) \), \( s^*(v) = s'(v) = s''(v) \), and for each \( v \in X_t \) such that \( s^*(v) = 2z_i, z \in \{0, 1, 2\} \), there exist \( z', z'' \in \{0, 1, 2\} \) such that \( s'(v) = 2z_i', s''(v) = 2z''_i, z = z' + z'' - \deg_{G_t[X_t \setminus s^{-1}(0)]}(v) \).

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We join every compatible entries \( A_t'(s', s_0', i', j', \ell') \) and \( A_t''(s'', s_0'', i'', j'', \ell'') \). For two such entries being compatible, we need \( s' \oplus s'' \) to be defined and \( s_0' = s_0'' \). We obtain that
\[
A_t(s, s_0, i, j, \ell) = \bigcup_{s', s'' : X_1 \rightarrow \{0, 1, 2, 3\}} \text{join}(A_t'(s', s_0', i', j', \ell'), A_t''(s'', s_0'', i'', j'', \ell'')).
\]

Theorem 8. \( \{\text{paw}\}\) - \( \text{TM-Deletion} \) can be solved in time \( 2^{O(tw)} \cdot n^7 \).

Proof. The algorithm works in the following way. For each node \( t \in V(T) \) and for each entry \( M \) of its table, instead of storing \( A_t(M) \), we store \( A'_t(M) = \text{reduce}(A_t(M)) \) by using Theorem 3. As each of the operations we use preserves representation by Proposition 2, we obtain that for each node \( t \in V(T) \) and for each possible entry \( M \), \( A'_t(M) \) represents \( A_t(M) \). In particular, we have that \( A'_t(M) = \text{reduce}(A_t(M)) \) for each possible entry \( M \).

Using the definition of \( A_t \) and Lemma 7, we have that \( \text{tm}(paw) \leq k \) if and only if for some \( i \) and \( \ell \), \( i + \ell \geq |V(G) \cup \{v_0\}| - k \) and \( A'_t(\emptyset, \emptyset, i, i - 1, \ell) \neq \emptyset \).

We now focus on the running time of the algorithm. The size of the intermediate sets of weighted partitions for a leaf node and for a introduce vertex node, are upper-bounded by \( 2|s^{(1)}| \). For a forget vertex node, we take the union of four sets of size \( 2|s^{(1)}| \), so the intermediate sets of weighted partitions have size at most \( 4 \cdot 2|s^{(1)}| \). For a join node, as in the big union operation we take into consideration at most \( 5^{|X_t|} \) possible functions \( s' \), as many functions \( s'' \), at most \( n + |s^{(1)}| \) choices for \( i' \) and \( i'' \), at most \( n + |s^{(1)}| \) choices for \( j' \) and \( j'' \) (as \( H \) is always a forest), and at most \( n + |s^{(1)}| \cdot \{2, 21, 22\} \) choices for \( \ell' \) and \( \ell'' \), we obtain that the intermediate sets of weighted partitions have size at most \( 25^{|X_t|} \cdot (n + |s^{(1)}|)^2 \cdot (n + |s^{(1)}| \cdot \{2, 21, 22\}) \cdot 4|s^{(1)}| \). We obtain that the intermediate sets of weighted partitions have size at most \( (n + |X_t|)^3 \cdot 100|X_t| \). Moreover, for each node \( t \in V(T) \), the function \( \text{reduce} \) will be called as many times as the number of possible entries, i.e., at most \( 2^{O(w)} \cdot n^3 \) times. Thus, using Theorem 3, \( A'_t \) can be computed in time \( 2^{O(w)} \cdot n^6 \). The theorem follows by taking into account the linear number of nodes in a nice tree decomposition. \( \square \)

8 A single-exponential algorithm for \( \{\text{chair}\}\) - \( \text{TM-Deletion} \)

As in the previous cases that we solved in single-exponential time, we start with a structural characterization of the graphs that exclude the \( \text{chair} \) as a topological minor (hence, as a minor as well).
Lemma 8. Let $G$ be a graph. $\text{chair} \not\preceq \text{tm} G$ if and only if every connected component of $G$ of size at least five is a path, a cycle, or a star.

Proof. It is straightforward to check that a path, a cycle, a star, or a graph of size at most 4 do not contain the chair as a topological minor. Conversely, let $G$ be a connected graph of size at least five that excludes the chair as a topological minor. Let $P$ be a longest path of $G$. We denote by $p_1$ and $p_2$ the endpoints of this path $P$. It is easy to see that if $|V(P)| \leq 3$, then $G$ has to be a star. Assume now that $|V(P)| \geq 4$. Then we have that $V(P) = V(G)$. Indeed, assume that $V(P) \neq V(G)$ and let $v \in V(G) \setminus V(P)$ be a vertex adjacent with a vertex of $V(P)$ in $G$. This vertex should exist by the connectivity of $G$. By maximality of $P$, $v$ cannot be adjacent to $p_1$ or $p_2$ and as $\text{chair} \not\preceq \text{tm} G$, $v$ cannot be adjacent to an internal vertex of the path $P$. Thus, $|V(P)| = |V(G)| \geq 5$. Moreover, as $\text{chair} \not\preceq \text{tm} G$, neither $p_1$ nor $p_2$ can be adjacent to an internal vertex of the path $P$, and so, $E(P) \subseteq E(G) \subseteq E(P) \cup \{p_1, p_2\}$. Thus, $G$ is either a path or a cycle. \hfill \Box

With Lemma 8 at hand, an algorithm for $\{\text{chair}\}$-TM-Deletion running in time $2^{O(tw)} \cdot n^{O(1)}$ can be obtained using standard dynamic programming techniques. For this, as we did for $\{P_3\}$-TM-Deletion, $\{P_4\}$-TM-Deletion, and $\{K_{1,s}\}$-TM-Deletion, we label the vertices on each bag. Each label carries two types of information. On the one hand, it indicates whether a vertex is in a collection of paths or cycles, a collection of stars, a clique of size 4, a paw, or a diamond; note that these are all the possible graphs on at most 4 vertices (see Figure 1). On the other hand, it also indicates in which “state” a vertex is with regard to the already computed graph, which will become clear below.

As the number of distinct labels needed for the algorithm for $\{\text{chair}\}$-TM-Deletion is quite large and the algorithm itself is not complicated, we will avoid the formal description of it. Namely, as we will discuss, we need 17 distinct labels on the vertices and three distinct labels on the edges. Each label carries two types of information. On the one hand, it indicates whether a vertex is in a collection of paths or cycles, a collection of stars, a clique of size 4, a paw, or a diamond; note that these are all the possible graphs on at most 4 vertices (see Figure 1). On the other hand, it also indicates in which “state” a vertex is with regard to the already computed graph, which will become clear below.

Let us proceed to the description of the labels. First, we use one label to indicate whether a vertex belongs in the solution or not. For the collection of paths or cycles, we use three labels 0, 1, and 2, corresponding to the current degree of this vertex. For a collection of stars, we use three labels, $c$, 0, 1 where $c$ labels a vertex that is a center of a star, 0 labels a vertex that is leaf of a star that is not connected to a center yet, and 1 labels a leaf of a star that is already connected to a center. For the clique of size 4, we only need two labels 0 and 1, where 0 means that the vertex should be in a $K_4$ but we do not know which one yet, and 1 means that we already found in which $K_4$ the vertex
is. The crucial argument for this is the fact that if four vertices induce a $K_4$, then by the properties of a tree decomposition there exists a bag that contains the four vertices. For the **paw**, we use six labels $c_0$, $c_1$, $d_f$, $d_w$, $\ell_0$, and $\ell_1$. We call leaf the only vertex of the **paw** of degree 1. The label $\ell_0$ (resp. $\ell_1$) corresponds to a leaf of a **paw** that has degree 0 (resp. 1) in the currently processed graph. The label $c_0$ (resp. $c_1$) corresponds to a vertex of the cycle of a **paw** that is not (resp. is) connected to a leaf and for which we do not know yet the three vertices of the cycle. The label $d_f$ corresponds to a vertex of the cycle of a **paw** for which we already know the three vertices of the cycle and that is full, i.e., it cannot be connected to a leaf anymore. The label $d_w$ corresponds to a vertex of the cycle of a **paw** for which we know the three vertices of the cycle and that is waiting for a leaf to be connected to.

Dealing with the **diamond** is a bit more complicated. We see the **diamond** as two $C_3$ glued by an edge called chord. The crucial argument is that for each $C_3$, there exists a bag that contains the three vertices of the cycle and we “only” need to remember which edge of the cycle is the chord of the **diamond**. We use two labels for the vertices $a_0$ and $a_1$, and, this time, we also use three labels on the edges $b_0$, $b_1$, and $b_2$. Note that in a **diamond**, the number of edges is linear in the number of vertices and so, we can afford to label the edges. Namely, $a_0$ is used for vertices that can still be in a $C_3$ and $a_1$ is used for vertices that cannot be in a new $C_3$ anymore. The label $b_0$ is used for edges that are not in a $C_3$ yet, the label $b_1$ is used for chords for which we only found one of the two $C_3$’s, and the label $b_2$ is used for edges that cannot be used anymore. Note that the endpoints of a chord belong to two $C_3$’s, so when detecting the first one, these endpoints will remain labeled $a_0$ but the chord will be now labeled $b_1$.

Using all these labels, and updating them in a bottom-up fashion in a tree decomposition in a standard way, we obtain the following theorem.

**Theorem 9.** If a nice tree decomposition of $G$ of width $w$ is given, $\{\text{chair}\}$-Deletion can be solved in time $2^{O(w)} \cdot n$.

**9 A single-exponential algorithm for $\{\text{banner}\}$-TM-Deletion**

Similarly as before, we start with a structural characterization of the graphs that exclude the **banner** as a (topological) minor.

**Lemma 9.** Let $G$ be a graph with at least five vertices. $\text{banner} \not\subseteq_{\text{tm}} G$ if and only if $G$ is a cycle or $C_4 \not\subseteq_{\text{tm}} G$.

**Proof.** Let $G$ be a connected graph with $|V(G)| \geq 5$. It is straightforward to see that if $G$ is a cycle or if $C_4 \not\subseteq_{\text{tm}} G$, then $\text{banner} \not\subseteq_{\text{tm}} G$. Conversely, assume that $G$ excludes the
banner as a minor and contains a cycle $C$ of size at least four. As we can assume that $G$ is connected and excludes the banner as a minor, we have that $V(G) = V(C)$. Indeed, if there exists a vertex $v \in V(G) \setminus V(C)$, then there exists a vertex $v' \in V(G') \setminus V(C)$ that is a neighbor of a vertex of $V(C)$, and so the graph $G[V(C) \cup \{v'\}]$ contains the banner as a topological minor. By assumption, we have that $|V(C)| = |V(G)| \geq 5$. Assume now for contradiction that $G$ is not a cycle, that is, that $E(G) \setminus E(C)$ contains an edge $e$. Then, as $|V(C)| \geq 5$, there exists a cycle $C'$ containing $e$, such that $4 \leq |V(C')| < |V(C)|$, and so $G$ contains the banner as a minor, a contradiction. The lemma follows.

The idea of the algorithm is, as we did for $\{\text{paw}\}$-TM-Deletion (see Section 7), to use the structural properties given by Lemma 9 and to combine the rank-based approach and the standard dynamic programming techniques. As for $\{\text{chair}\}$-TM-Deletion (see Section 8), we will need, in particular, to label vertices that appear in components of size at most 4 and in components that are cycles. As for $\{C_4\}$-TM-Deletion (see Section 6), we also use one label for the solution and another label for the components that exclude $C_4$ as a minor. This means that, for this algorithm, we need 15 labels for the vertices and three labels for the edges. Because of this, again we do not provide the full formal description of the algorithm, and instead we provide a high-level description of how it works, reusing what we presented previously.

Namely, we explain how, starting from the algorithm for $\{C_4\}$-TM-Deletion given in Section 6, we obtain the desired algorithm for $\{\text{banner}\}$-TM-Deletion. As we did for $\{\text{paw}\}$-TM-Deletion, we modify the function $s : X_t \to \{0,1\}$ to a function $s : X_t \to \{0,1\} \cup L$, where $L$ corresponds to the set of labels needed for detecting if a component is of size at most 4 or a cycle. We also add a function $r_d$ for the labeling of the edges that appear in a diamond. Then, as we did for $\{\text{paw}\}$-TM-Deletion, when doing the dynamic programming operations, we use the rank-based approach for the elements of $s^{-1}(1)$ and standard dynamic programming operations for the elements of $s^{-1}(L)$. Doing this, we obtain the following theorem.

**Theorem 10.** If a nice tree decomposition of $G$ of width $w$ is given, $\{\text{banner}\}$-Deletion can be solved in time $2^{O(w)} \cdot n^{O(1)}$.

### 10 Conclusions and further research

We presented single-exponential algorithms for $\{H\}$-TM-Deletion taking as parameter the treewidth of the input graph, when $H \in \{P_3, P_4, K_{1,i}, C_4, \text{paw}, \text{chair}, \text{banner}\}$. These algorithms, combined with the results of [4–7], settle completely the complexity of $\{H\}$-M-Deletion when $H$ is connected; see Figure 1 for an illustration.
Concerning the topological minor version, in order to establish a dichotomy for \( \{H\}\)-\(\text{TM-Deletion}\) when \( H \) is planar and connected, it remains to obtain algorithms in time \( O^*(2^{O(tw \log tw)}) \) for the graphs \( H \) with maximum degree at least four, like the \text{gem} or the \text{dart} (see Figure 1), as for those graphs the algorithm in time \( O^*(2^{O(tw \log tw)}) \) given in [6] cannot be applied.

Our algorithms for \( \{C_i\}\)-\text{Deletion} given here and in [6] may also be used to devise approximation algorithms for hitting or packing long cycles in a graph (in the spirit of [11] for other patterns), by using the fact that cycles of length at least \( i \) satisfy the \text{Erdős-Pósa property} [17]. We did not focus on optimizing the degree of the polynomials or the constants involved in our algorithms. Concerning the latter, one could use the framework by Lokshtanov et al. [21] to prove lower bounds based on the \text{Strong ETH}.

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