Deterministic parameterized connected vertex cover

Marek Cygan
Institute of Informatics, University of Warsaw, Poland
cygan@mimuw.edu.pl

Abstract. In the Connected Vertex Cover problem we are given an undirected graph $G$ together with an integer $k$ and we are to find a subset of vertices $X$ of size at most $k$, such that $X$ contains at least one end-point of each edge and moreover $X$ induces a connected subgraph. For this problem we present a deterministic algorithm running in $O(2^k n^{O(1)})$ time and polynomial space, improving over previously best $O(2^{0.4882k} n^{O(1)})$ deterministic algorithm and $O(2^k n^{O(1)})$ randomized algorithm. Furthermore, when usage of exponential space is allowed, we present an $O(2^k k(n + m))$ time algorithm that solves a more general variant with arbitrary real weights.

Finally, we show that in $O(2^k k(n + m))$ time and $O(2^k k)$ space one can count the number of connected vertex covers of size at most $k$, which can not be improved to $O((2^k - \varepsilon)^k n^{O(1)})$ for any $\varepsilon > 0$ under the Strong Exponential Time Hypothesis, as shown by Cygan et al. [CCC'12].

1 Introduction

In the classical vertex cover problem we are asked whether there exists a set of at most $k$ vertices, containing at least one end-point of each edge. As a basic problem in the graph theory Vertex Cover is extensively studied, together with its natural variants. One of the generalizations of Vertex Cover is the Connected Vertex Cover problem, where a vertex cover is called a connected vertex cover if it induces a connected subgraph.

As Connected Vertex Cover is NP-complete we can not hope for polynomial time solutions, however it is possible to efficiently solve the problem for small values of $k$. Obviously, for any fixed $k$, we can solve the problem in polynomial time, by trying all $n^k$ possible subsets of vertices. In the parameterized complexity setting we are interested in finding algorithms of $f(k)n^{O(1)}$ running time, for some computable function $f$, that is polynomial for each fixed value of $k$, but where the degree of the polynomial is independent of $k$. 

| Connected Vertex Cover |
|---|
| **Input:** An undirected graph $G = (V, E)$ and an integer $k$. |
| **Parameter:** $k$ |
| **Question:** Does there exist a connected vertex cover of $G$ of cardinality at most $k$? |
A few fixed-parameter algorithms were designed for the CONNECTED VERTEX COVER problem during the last years. The fastest deterministic algorithm is due to Binkele-Raible [1] running in $O^*(2.4882^k)$ time, while the fastest (randomized) algorithm is due to Cygan et al. [4] running in $O^*(2^k)$ time, where by $O^*$ we denote the standard big $O$ notation, with polynomial factors omitted. In Table 1 we summarize the history of parameterized algorithms for CONNECTED VERTEX COVER.

Table 1. Summary of parameterized algorithms for CONNECTED VERTEX COVER.

| Algorithm             | Authors          |
|-----------------------|------------------|
| $O^*(6^k)$            | Guo et al. [11]  |
| $O^*(3.2361^k)$       | Mölle et al. [12]|
| $O^*(2.9316^k)$       | Fernau et al. [9]|
| $O^*(2.7606^k)$       | Mölle et al. [13]|
| $O^*(2.4882^k)$       | Binkele-Raible [1]|
| $O^*(2^k)$ (randomized) | Cygan et al. [4] |
| $O^*(2^k)$            | this paper       |

Our results: The main result of this paper is a deterministic algorithm solving CONNECTED VERTEX COVER in $O^*(2^k)$ time. Moreover, when we allow exponential space, in the same running time we can solve weighted and counting versions of the CONNECTED VERTEX COVER problem, which was not possible with the previously fastest randomized algorithm of [4].

⊕-CONNECTED VERTEX COVER (⊕-CVC)
Input: An undirected graph $G = (V, E)$, an integer $k$.
Parameter: $k$
Goal: Find the number of connected vertex covers of cardinality at most $k$.

WEIGHTED CONNECTED VERTEX COVER (WCVC)
Input: An undirected graph $G = (V, E)$, a weight function $\omega : V \to \mathbb{R}_+$ and an integer $k$.
Parameter: $k$
Goal: Find a minimum weight connected vertex cover of cardinality at most $k$.

Theorem 1. WEIGHTED CONNECTED VERTEX COVER can be solved in $O(2^k(|V| + |E|))$ time and $O(2^k)$ space.

Theorem 2. ⊕-CONNECTED VERTEX COVER can be solved in $O(2^k(|V| + |E|))$ time and $O(2^k)$ space.

Recently Cygan et al [3] have shown that unless the Strong Exponential Time Hypothesis (SETH) fails, it is not possible to count the number of connected
vertex covers of size at most \( k \) in \( O^*((2 - \varepsilon)^k) \) time, for any constant \( \varepsilon > 0 \). Consequently our counting algorithm is tight under SETH, which is an example of few parameterized problems with nontrivial solutions for which there exists an evidence of optimality.

When restricted to polynomial space, we prove that the weighted variant can still be solved in \( O^*(2^k) \) running time, assuming weights are polynomially bounded integers.

**Theorem 3.** Weights Connected Vertex Cover with polynomially bounded integer weights can be solved in \( O(2^k n^{O(1)}) \) time and polynomial space.

**Related work** Vertex Cover is one of the longest studied problem in the parameterized complexity. The currently fastest known parameterized algorithm for the Vertex Cover problem is due to Chen et al., running in \( O(1.2738^k + kn) \) time [2]. Recently, new parameterizations of Vertex Cover are considered, when the parameter is \( k - |M| \) [15], where \( M \) is a maximum cardinality matching, or \( k - \text{LP} \), where \( \text{LP} \) is the optimum value of a natural linear programming relaxation [7,17].

A notion very close to fixed parameter tractability, or even a subfield of it, is kernelization. We call a polynomial time preprocessing routine a kernel, if given an instance \( I \) with parameter \( k \) the algorithm produces a single instance \( I' \) with parameter \( k' \), such that \( I' \) is a YES-instance iff \( I \) is a YES-instance, and moreover \( |I'| + k' \leq g(k) \). It is well known that a problem admits a kernel if and only if it is kernelizable, however we are mostly interested in kernelization algorithms with the function \( g \) being a polynomial. Unfortunately, for Connected Vertex Cover no polynomial kernel exists as shown by Dom et al. [8], unless \( \text{NP} \subseteq \text{coNP/poly} \).

**Organization** In Section 2 we prove Theorem 1. For the sake of presentation we describe small differences needed to solve the counting variant, that is to prove Theorem 2 in separate Section 3. Next, in Section 4 we prove Theorem 3 and finally, we finish the article with conclusions and open problems in Section 5.

**Notation.** We use standard graph notation. For a graph \( G \), by \( V(G) \) and \( E(G) \) we denote its vertex and edge sets, respectively. When it is clear which graph we are describing we use \( n \) as the number of its vertices and \( m \) as the number of its edges. For \( v \in V(G) \), its neighborhood \( N(v) \) is defined as \( N(v) = \{ u : vw \in E(G) \} \), and \( N[v] = N(v) \cup \{ v \} \) is the closed neighborhood of \( v \). We extend this notation to subsets of vertices: \( N[X] = \bigcup_{v \in X} N[v] \) and \( N(X) = N[X] \setminus X \). For a set \( X \subseteq V(G) \) by \( G[X] \) we denote the subgraph of \( G \) induced by \( X \). For a set \( X \) of vertices or edges of \( G \), by \( G \setminus X \) we denote the graph with the vertices or edges of \( X \) removed; in case of vertex removal, we remove also all the incident edges. For two subsets of vertices \( X, Y \subseteq V \) by \( E(X,Y) \) we denote the set of edges with one endpoint in \( X \) and the other in \( Y \). In particular by \( E(X,X) \) we denote the set of edges with both endpoints in \( X \).
2 Algorithm

In this section we prove Theorem 1. As the starting point we use the iterative compression technique in Section 2.1. As a consequence we are left with a problem, where additionally each instance is equipped with a connected vertex cover \( Z \) of size at most \( k + 2 \). In Section 2.2 we show how to take advantage of the set \( Z \) by showing a natural algorithm, solving a bipartite Steiner tree problem as a subroutine (described in Section 2.4). The key part of the proof of Theorem 1 is the time complexity analysis of the presented algorithm, which relies on a combinatorial lemma proved in Section 2.3.

2.1 Iterative compression

We start with a standard technique in the design of parameterized algorithms, that is, iterative compression, introduced by Reed et al. [16]. Iterative compression was also the first step of the Monte Carlo algorithm for CONNECTED VERTEX COVER [4].

We define a compression problem, where the input additionally contains a connected vertex cover \( Z \subseteq V \). The name compression might be misleading in our case, since in the problem definition below we are not explicitly interested in compressing the solution, but we want to find a minimum weight connected vertex cover using the size of \( Z \) as our structural parameter. In particular not only we use the fact that \( Z \) is a vertex cover (which ensures that \( V \setminus Z \) is an independent set), but also we use the fact that \( G[Z] \) is connected, which is crucial in the time complexity analysis of our algorithm.

**Compression Weighted Connected Vertex Cover (Comp-WCVC)**

**Input:** An undirected graph \( G = (V,E) \), a weight function \( \omega : V \to \mathbb{R}_+ \), an integer \( k \) and a connected vertex cover \( Z \subseteq V \) of \( G \).

**Parameter:** \( |Z| \)

**Goal:** Find a minimum weight connected vertex cover of cardinality at most \( k \).

In Section 2.2 we prove the following lemma providing a parameterized algorithm for the above compression problem.

**Lemma 4.** Comp-WCVC can be solved in \( O(2^{|Z|}k(|V|+|E|)) \) time and \( O(2^{|Z|}k) \) space. Moreover, when the weight function is uniform, we can solve the problem in \( O(2^{|Z|}(|V|+|E|)) \) time and \( O(2^{|Z|}) \) space.

Having the above lemma we show how to efficiently find a connected vertex cover of size at most \( k \) (if it exists).

**Lemma 5.** Given an undirected graph \( G = (V,E) \) and an integer \( k \) one can find a connected vertex cover of size at most \( k \), or verify that it does not exist, in \( O(2^k k(|V|+|E|)) \) time and \( O(2^k) \) space.
Proof. First, let us assume that $G$ does not contain isolated vertices, since we can remove them. Moreover we can assume that $G$ is connected, since if $G$ contains at least two connected components (and no isolated vertices) then it can not admit a connected vertex cover of any size. Therefore, let $V = \{v_1, \ldots, v_n\}$ be an ordering of vertices, such that for each $1 \leq i \leq n$, the graph $G[V_i]$ is connected, where $V_i = \{v_i, \ldots, v_n\}$. For $1 \leq i \leq n$ let $G_i$ be the graph $G$, with vertices of $V_i$ identified to a single vertex. Alternatively, we can say that $G_i$ comes from a contraction of the set of edges of a spanning tree of $G[V_i]$. Since Connected Vertex Cover is closed under edge contractions, we infer that if there is no connected vertex cover of size at most $k$ in $G_i$, then clearly there is no connected vertex cover of size at most $k$ in $G$.

We are going to construct a sequence of sets $X_i \subseteq V(G_i)$ of size at most $k$, such that $X_i$ is a connected vertex cover of $G_i$. First, observe that the set $X_1 = \emptyset$ is a connected vertex cover of $G_1$ of size at most $k$. Next, let us consider each value of $i = 2, \ldots, n$ one by one. Observe that there is an edge $e$ in $E(G_i)$, such that the graph $G_{i-1}$ is exactly the graph $G_i$ with the edge $e$ contracted. In particular as $e$ we may take any edge between $v_{i-1}$ and $V_i$. Let $x$ be the vertex in $G_{i-1}$ which corresponds to the set $V_{i-1}$ and let $y$ be the vertex in $G_i$ corresponding to the set $V_i$. We claim that $Z = (X_{i-1} \setminus \{x\}) \cup \{v_i, y\}$ is a connected vertex cover of $G_i$ of size at most $k + 2$. Since $|X_{i-1}| \leq k$ the bound on the size of $Z$ holds. Moreover, since $X_{i-1}$ is a vertex cover of $G_{i-1}$, the set $Z$ is a vertex cover of $G_i$. Finally, $G_i[Z]$ is connected, because either $x$ is contained in $X_{i-1}$, or a neighbour of $x$ belongs to $X_{i-1}$, or $x$ is an isolated vertex which means that $i = 2$ and then $Z = V(G_2)$ induces a connected subgraph.

If, for a fixed $i$, we use Lemma 4 for the Comp-WCVC instance $(G_i, \omega, k, Z)$, with $\omega$ being a uniform unit weight function, then in $O(2^{|Z|}(n + m)) = O(2^k(n + m))$ time and $O(2^{|Z|}) = O(2^k)$ space we can find a set $X_i$, which is a connected vertex cover of $G_i$ of cardinality at most $k$, or verify that no connected vertex cover of cardinality at most $k$ in the graph $G$ exists. Since $G_n = G$, the set $X_n$ is a connected vertex cover of $G$ of size at most $k$, which we can find in $O(2^k n(n + m))$ time, because we use Lemma 4 exactly $n - 1$ times. In order to reduce the polynomial factor from $n(n + m)$ to $k(n + m)$ observe, that if we order the set $V$, such that the set $\{v_{i-\ell+1}, \ldots, v_n\}$ forms a connected vertex cover of the graph $G$, then as the set $X_{i-\ell+1}$ we can set a singleton set containing the vertex corresponding to $V_{i-\ell+1}$ and reduce the number of rounds in the inductive process from $n$ to $\ell$. However, a simple $O(n + m)$ time 2-approximation of the Connected Vertex Cover problem is known [10], which just takes as the solution the set of internal nodes of a depth first search tree of the given graph. Therefore, assuming a vertex cover of size at most $k$ exists, we can find a connected vertex cover of size at most $2k$ in $O(n + m)$ time and consequently reduce the number of rounds of the inductive process to at most $2k$, which leads to $O(2^k k(n + m))$ time complexity. 

\[1\] For the sake of completeness in Appendix A we present a proof of correctness of this algorithm.
By Lemma 5 we can find a connected vertex cover $Z$ of size at most $k$, if it exists. Consequently we can use Lemma 4 which proves Theorem 1.

2.2 Compression algorithm

In this section we present a proof of Lemma 4. The advantage we have while solving \textsc{Comp-WCVC} instead of \textsc{WCVC} is the additional set $Z$, which forms a connected vertex cover of $G$ and the size of $Z$ is our new parameter. We show how to use the set $Z$ as an insight into the structure of the graph and solve compression problem efficiently. The algorithm itself is straightforward, but the crucial part of its time complexity analysis lies in the following combinatorial bound, which we prove in Section 2.3.

**Lemma 6.** For any connected graph $G = (V,E)$ we have

$$\sum_{V_1 \subseteq V \atop E(G[V[V_1]] = \emptyset)} 2^{\#cc(G[V_1])} \leq 3 \cdot 2^{|V|-1},$$

where by $cc(H)$ we denote the set of connected components of a graph $H$.

Observe, that in the above lemma we sum over all sets $V_1$, that form a vertex cover of $G$. The second tool we use in the proof of Lemma 4 is the following lemma solving the node-weighted Steiner tree problem in bipartite graphs, where both the terminals and non-terminals form independent sets. The proof of it can be found in Section 2.4.

**Lemma 7.** Let $G = (V,E)$ be a bipartite graph and $T \subseteq V$ be a set of terminals, such that $T$ and $V \setminus T$ are independent sets. For a given weight function $\omega : V \setminus T \to \mathbb{R}_+$ and an integer $k$ in $O(2^{|T|}k(|V| + |E|))$ time and $O(2^{|T|}k)$ space we can find a minimum weight subset $X \subseteq V \setminus T$ of cardinality at most $k$, such that $G[T \cup X]$ is connected, or verify that such a set does not exist. Moreover for a uniform weight function $\omega$ we improve the running time to $O(2^{|T|}(|V| + |E|))$ and space usage to $O(2^{|T|})$.

Having Lemmas 6 and 7 we can prove Lemma 4.

**Proof (of Lemma 4).** Similarly as in the proof of Lemma 5 we may assume that the graph $G$ is connected. We start with guessing, by trying all $2^{|Z|}$ possibilities, a subset $Z_1$ of $Z$ that is a part of a connected vertex cover and denote $Z_0 = Z \setminus Z_1$.

First, let us consider a special case, that is $Z_1 = \emptyset$. Then we need to take the whole set $V \setminus Z$ to cover the edges $E(Z_1, V \setminus Z)$, since each vertex of $V \setminus Z$ has at least one neighbour in $Z$ (otherwise it would be isolated). It is easy to verify whether $(V \setminus Z)$ is a connected vertex cover of size at most $k$.

Therefore, we assume that $Z_1 \neq \emptyset$ and moreover $E(Z_0, Z_0) = \emptyset$, since otherwise there is no vertex cover disjoint with $Z_0$. Let us partition the set $V \setminus Z$ into $V_1 = (V \setminus Z) \cap N(Z_0)$ and $V_0 = (V \setminus Z) \setminus V_1$. Less formally, we split the vertices of $V \setminus Z$ depending on whether they have a neighbour in $Z_0$ or not. Since we need
to cover the edges adjacent to $Z_0$, any vertex cover disjoint with $Z_0$ contains all the vertices of $V_1$.

Observe, that if there exists a vertex $v \in V_1$, such that $N(v) \subseteq Z_0$, no vertex cover disjoint with $Z_0$ is connected, since the vertex $v$ can not be in the same connected component as any vertex of $Z_1$, meaning that this choice of $Z_0$ is invalid (see Fig. 1). Consequently each vertex in $V_1$ has at least one neighbour in $Z_1$. Moreover, $Z_1 \cup V_1$ forms a vertex cover of the graph $G$, as $V_0 \cup Z_0$ is an independent set. Hence we want to investigate how $Z_1 \cup V_1$ can be complemented with vertices of $V_0$, to make the vertex cover induce a connected subgraph. Let $G'$ be the graph $G[Z_1 \cup V_0 \cup V_1]$ with connected components of $G[Z_1 \cup V_1]$ contracted to single vertices. Denote the vertices corresponding to contracted components of $G[Z_1 \cup V_1]$ as $T$. Note that $G'$ is bipartite, since $G[V_0]$ is an independent set. By Lemma 7 we can find a minimum weight set $X \subseteq V_0$ of cardinality at most $(k - |Z_1| - |V_1|)$, such that $G'[T \cup X]$ is connected, which is equivalent to $G[Z_1 \cup V_1 \cup X]$ being connected. Observe, that the size of the set $T$ is upper bounded by the number of connected components of the induced subgraph $G[Z_1]$, as each vertex of $V_1$ has at least one neighbour in $Z_1$. Therefore, by Lemma 7 for a fixed choice of $Z_1$ we can find the set $X$ in $O(2^{|T|}(k - |Z_1| - |V_1|)((|V(G)| + |E(G)|)) = O(2^{2^{cc(G[Z_1])}k(|V(G)| + |E(G)|)))$ time and $O(2^{|T|}(k - |Z_1| - |V_1|))) = O(2^k)$ space. Moreover for a uniform weight function, by Lemma 7 the running time is $O(2^{2^{cc(G[Z_1])}(|V(G)| + |E(G)|))$ and space usage is $O(2^k)$.

![Fig. 1. An example of invalid choice of $Z_0$, since a vertex of $V_1$ has neighbours in $Z_1$.](image)

Summing up the running time over all the choices of $Z_1$, for which $Z_0$ is an independent set, by Lemma 6 applied to the graph $G[Z]$ we prove the total running time of our algorithm is $O(2^k(|V(G)| + |E(G)|))$ for a general weight function and $O(2^k(|V(G)| + |E(G)|))$ for a uniform weight function. 

2.3 Combinatorial bound

Now we prove Lemma 6 where we reduce the trivial $3^{|V|}$ bound to $3 \cdot 2^{|V|-1}$, by using a similar idea, as was previously used for Bandwidth [5,6] and Connected Vertex Cover [4].
Proof (of Lemma 7). Note, that we may rewrite the sum we want to bound as follows:

$$\sum_{\substack{V_1 \subseteq V \\ E(G[V \setminus V_1]) = \emptyset}} 2^{\left| \text{cc}(G[V_1]) \right|} = \left| \{ (V_1, \mathcal{E}) : V_1 \subseteq V, \mathcal{E} \subseteq \text{cc}(G[V_1]), E[G[V \setminus V_1]] = \emptyset \} \right| .$$

That is we count the number of pairs \((V_1, \mathcal{E})\), such that \(V_1\) forms a vertex cover of \(G\) and \(\mathcal{E}\) is any subset of connected components of the subgraph induced by \(V_1\). Denote the set of all pairs \((V_1, \mathcal{E})\) we are counting as \(S\). Observe, that we can easily construct an injection \(\phi\) from \(S\) to \(\{\text{ii}, \text{io}, \text{o}\}^{|V_1|}\), where for a pair \((V_1, \mathcal{E})\) as \(\phi((V_1, \mathcal{E}))(v)\) we set:

- \text{ii} (in-in) when \(v \in V_1\) and the connected component of \(G[V_1]\) containing \(v\) belongs to \(\mathcal{E}\),
- \text{io} (in-out) when \(v \in V_1\) and the connected component of \(G[V_1]\) containing \(v\) does not belong to \(\mathcal{E}\),
- \text{o} (out) when \(v \notin V_1\).

Having any function \(f : V \to \{\text{ii}, \text{io}, \text{o}\}\) we can reconstruct a pair \((V_1, \mathcal{E})\) (if it exists), such that \(\phi((V_1, \mathcal{E})) = f\). However, the injection \(\phi\) is not a surjection, for at least two reasons. Consider any \(s \in \phi(S)\). Firstly, for any edge \(uv \in E\), we have \(f(u) \in \{\text{ii}, \text{io}\}\) or \(f(v) \in \{\text{ii}, \text{io}\}\), since otherwise \(V_1\) is not a vertex cover of \(G\). Secondly, for any edge \(uv \in E\), if we have \(f(u) \in \{\text{ii}, \text{io}\}\), then either \(f(v) = \text{o}\) or \(f(u) = f(v) = f(u)\), because if both \(u\) and \(v\) belong to \(V_1\), then they are are part of exactly the same connected component \(C\) of \(G[V_1]\), and therefore knowing \(f(u)\) we can infer whether \(C \in \mathcal{E}\) or \(C \notin \mathcal{E}\).

Let us formalize the intuition above, to prove that for almost each vertex we have at most two, instead of three possibilities. Consider a spanning tree \(T\) of \(G\) and root it in an arbitrary vertex \(r\). We construct the following function \(\phi' : S \to \{\text{ii}, \text{io}, \text{o}\} \times \{a, b\}^{V \setminus \{r\}}\). For a given pair \((V_1, \mathcal{E}) \in S\) we set \(\phi'((V_1, \mathcal{E}))(r) = (\phi((V_1, \mathcal{E}))(r), f)\), where the function \(f : V \setminus \{r\} \to \{a, b\}\) is defined in a top-down manner, regarding the tree \(T\), as follows. Let \(v \in V \setminus \{r\}\) and denote \(p \in V\) as the parent of \(v\) in \(T\).

- If \(p \in V_1\), then if \(v \in V_1\), we set \(f(v) = a\) and otherwise (if \(v \notin V_1\)), we set \(f(v) = b\).
- If \(p \notin V_1\), then we have \(v \in V_1\) (since otherwise \(V_1\) would not be a vertex cover), and if the connected component of \(G[V_1]\) containing \(v\) belongs to \(\mathcal{E}\), then \(f(v) = a\), otherwise \(f(v) = b\).

Since \(\phi'\) is also a surjection, we have \(|S| \leq 3 \cdot 2^{|V| - 1}\), and the lemma follows. An example showing both functions \(\phi, \phi'\) is depicted in Fig. 2. \(\square\)

### 2.4 Bipartite Steiner Tree

Here we prove Lemma 7 which concerns the following bipartite variant of the node-weighted Steiner tree problem.
Weighted Bipartite Steiner Tree

**Input:** An undirected bipartite graph $G = (V, E)$, a weight function $\omega : V \to \mathbb{R}^+$, an integer $k$ and a set of terminals $T \subseteq V$, such that both $T$ and $V \setminus T$ are independent sets in $G$.

**Parameter:** $|T|$

**Goal:** Find a minimum weight subset $X \subseteq V \setminus T$ of size at most $k$, such that $G[T \cup X]$ is connected.

**Proof (of Lemma 7).** By a dynamic programming routine, for each subset $T_0 \subseteq T$ and integer $0 \leq j \leq k$ we compute the value $t(T_0, j)$, defined as the minimum weight of a subset $X \subseteq V \setminus T$, satisfying:

- $|X| = j$,
- $N(X) = T_0$,
- $G[T_0 \cup X]$ is connected.

Less formally, the value $t(T_0, j)$ is the minimum weight of a set $X$ of cardinality exactly $j$, such that $G[T_0 \cup X]$ induces a connected subgraph, and there is no edge from $X$ to $T \setminus T_0$. Observe, that $\min_{1 \leq j \leq k} t(T, j)$ is the minimum weight solution for the Weighted Bipartite Steiner Tree problem, therefore in the rest of the proof we describe how to compute all the $(k + 1)2^{|T|}$ values $t$ efficiently.

Initially, for each $t_0 \in T$ we set $t(\{t_0\}, 0) := 0$, while all other values in the table $t$ are set to $\infty$. Next, consider all the subsets $T_0 \subseteq T$ in the order of their
increasing cardinality, and for each integer $0 \leq j < k$ and each vertex $v \in N(T_0)$ do

$$t(T_0 \cup N(v), j + 1) := \min(t(T_0 \cup N(v), j + 1), t(T_0, j) + \omega(v)).$$

Note, that the assumption $v \in N(T_0)$ ensures, that vertices $N(v) \setminus T_0$ get connected to the vertices of $T_0$.

With this simple dynamic programming routine we compute all the values $t(T_0, j)$ in $O(2^{|T|}k(|V(G)| + |E(G)|)$ time and $O(2^{|T|}k)$ space. Note, that by standard methods we can reconstruct a set $X$ corresponding to the value $t(T, j)$ in the same running time. Moreover, if the weight function is uniform, than the second dimension of our dynamic programming table is unnecessary, since the cardinality and weight of a set are equal. This observation reduces both the running time and space usage by a factor of $k$.

3 Counting

In this Section we present a proof of Theorem 2, which is similar to the proof of Theorem 1.

Proof (of Theorem 2). Similarly as in the proof of Theorem 2, by using Lemma 5 in $O(2^{|T|}k(|V| + |E|))$ time we construct a set $Z$, which is a connected vertex cover of $G$ of size at most $k$, or verify that such a set does not exist.

Next, we proceed as in the proof of Lemma 4, however we have to justify the assumption that $G$ is a connected graph. When $G$ contains at least two connected components containing at least two vertices each, then there is no connected vertex cover in the graph $G$. If there is one connected component containing at least two vertices, then no connected vertex cover contains any of the isolated vertices, hence we can remove them. Finally, when the graph contains only isolated vertices, then it admits an empty connected vertex cover and $|V|$ connected vertex covers containing a single vertex only.

The rest of the proof of Lemma 4 remains unchanged and what we are left with is to show an $O(2^{|T|}k(|V| + |E|))$ running time algorithm for the following ⊕-Bipartite Steiner Tree problem.

⊠-Bipartite Steiner Tree

Input: An undirected bipartite graph $G = (V,E)$, an integer $k$ and a set of terminals $T \subseteq V$, such that both $T$ and $V \setminus T$ are independent sets in $G$.

Parameter: $|T|$

Goal: Find the number of subsets $X \subseteq V \setminus T$ of size at most $k$, such that $G[T \cup X]$ is connected.

We do it similarly as in the proof of Lemma 4, that is for each $T_0 \subseteq T$ and each $0 \leq j \leq k$ we define the value $t(T_0, j)$, which is equal to the number of subsets $X \subseteq V \setminus T$ of size exactly $j$, such that $N(X) \subseteq T_0$ and $G[T_0 \cup X]$ is connected. We leave the details of the dynamic programming routine to the reader.
4 Polynomial space

The only place in our algorithm, where we use exponential space is when solving the Bipartite Steiner Tree problem. If, instead of using Lemma 7, we use the algorithm of Nederlof [14], running in $O(2^{|T|}n^{O(1)})$ time, we obtain an $O(2^k n^{O(1)})$ time and polynomial space algorithm for the Connected Vertex Cover problem. The algorithm by Nederlof solves also the weighted case, but only when the weights are polynomially bounded integers, which is enough to prove Theorem 3. Unfortunately, we are not aware of an algorithm which counts the number of solutions to the Bipartite Steiner Tree problem in $2^{|T||V|^{O(1)}}$ time and polynomial space (note that the algorithm of [14] counts the number of branching walks, not the number of subsets of vertices inducing a solution).

5 Conclusions and open problems

In [4] Cygan et al. we have shown a randomized $O(3^k n^{O(1)})$ algorithm for the Feedback Vertex Set problem, where we want to make the graph acyclic by removing at most $k$ vertices. Is it possible to design a deterministic algorithm of the same running time?

The Cut&Count technique presented in [4] does not allow neither to count the number of solution nor to solve problems with arbitrary real weights. Nevertheless, for the Connected Vertex Cover problem we were able to solve both the weighted and counting variants in the same running time. Is it possible to design $cw n^{O(1)}$ time algorithms for counting or weighted variants of the connectivity problems parameterized by treewidth for which the Cut&Count technique can be applied?

Finally, we know that it is not possible to count the number of connected vertex covers of size at most $k$ in $O((2-\epsilon)^k n^{O(1)})$ time, unless SETH fails. Can we prove that we can not solve the decision version of the problem as well in such running time?

Acknowledgements

We thank Daniel Lokshtanov, Marcin Pilipczuk and Michal Pilipczuk for helpful discussions.

References

1. Daniel Binkele-Raible. Amortized Analysis of Exponential Time and Parameterized Algorithms: Measure and Conquer and Reference Search Trees. PhD thesis, University of Trier, 2010.
2. Jianer Chen, Iyad A. Kanj, and Ge Xia. Improved upper bounds for vertex cover. Theor. Comput. Sci., 411(40-42):3736–3756, 2010.
3. Marek Cygan, Holger Dell, Daniel Lokshtanov, Dániel Marx, Jesper Nederlof, Yoshio Okamoto, Ramamohan Paturi, Saket Saurabh, and Magnus Wahlström. On problems as hard as CNF-SAT. In CCC, page to appear, 2012.
4. Marek Cygan, Jesper Nederlof, Marcin Pilipczuk, Michal Pilipczuk, Johan M. M. van Rooij, and Jakub Onufry Wojtaszczyk. Solving connectivity problems parameterized by treewidth in single exponential time. In Rafail Ostrovsky, editor, FOCS, pages 150–159. IEEE, 2011.
5. Marek Cygan and Marcin Pilipczuk. Faster exact bandwidth. In WG, pages 101–109, 2008.
6. Marek Cygan and Marcin Pilipczuk. Exact and approximate bandwidth. Theor. Comput. Sci., 411(40-42):3701–3713, 2010.
7. Marek Cygan, Marcin Pilipczuk, Michal Pilipczuk, and Jakub Onufry Wojtaszczyk. On multiway cut parameterized above lower bounds. In IPEC, page to appear, 2011.
8. Michael Dom, Daniel Lokshtanov, and Saket Saurabh. Incompressibility through colors and IDs. In ICALP (1), pages 378–389, 2009.
9. Henning Fernau and David Manlove. Vertex and edge covers with clustering properties: Complexity and algorithms. J. Discrete Algorithms, 7(2):149–167, 2009.
10. Sudipto Guha and Samir Khuller. Approximation algorithms for connected dominating sets. Algorithmica, 20(4):374–387, 1998.
11. Jiong Guo, Rolf Niedermeier, and Sebastian Wernicke. Parameterized complexity of generalized vertex cover problems. In WADS, pages 36–48, 2005.
12. Daniel Mölle, Stefan Richter, and Peter Rossmanith. Enumerate and expand: Improved algorithms for connected vertex cover and tree cover. In CSR, pages 270–280, 2006.
13. Daniel Mölle, Stefan Richter, and Peter Rossmanith. Enumerate and expand: Improved algorithms for connected vertex cover and tree cover. Theory Comput. Syst., 43(2):234–253, 2008.
14. Jesper Nederlof. Fast polynomial-space algorithms using mobius inversion: Improving on Steiner Tree and related problems. Algorithmica, page to appear.
15. Igor Razgon and Barry O’Sullivan. Almost 2-SAT is fixed-parameter tractable. J. Comput. Syst. Sci., 75(8):435–450, 2009.
16. Bruce A. Reed, Kaleigh Smith, and Adrian Vetta. Finding odd cycle transversals. Oper. Res. Lett., 32(4):299–301, 2004.
17. Ramanujan M. S., Venkatesh Raman, Saket Saurabh, and Narayanaswamy N. S. LP can be a cure for parameterized problems. In STACS, page to appear, 2012.
A Approximation

In this section we show a simple algorithm, providing a 2-approximation for the CONNECTED VERTEX COVER problem, as observed by Guha and Khuller [10].

Lemma 8. Let \( G \) be a connected graph and let \( T \) be its depth first search tree. The set of internal nodes of \( T \) forms a connected vertex cover of \( G \) of cardinality at most twice the size of a minimum connected vertex cover of \( G \).

Proof. Let \( X \) be the set of internal nodes of \( T \). Clearly \( X \) is a connected vertex cover of \( G \), since there are not cross edges in any DFS tree. We prove that there is matching of size at least \( |X|/2 \) in \( G \), proving that there is no connected vertex cover (even no vertex cover) of size smaller than \( |X|/2 \).

If the number of internal nodes at odd levels is at least the number of internal nodes at even levels in \( T \), then we match each internal node on an odd level with its arbitrary child. Otherwise we match each internal node on an even level with its arbitrary child. In this way we show a matching of size at least \( |X|/2 \) and the lemma follows.