Characterization of the Linear Failure Rate Distribution by General Progressively Type-II Right Censored Order Statistics

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Abstract: In this article, we establish recurrence relations among single moments and among product moments of general progressively Type-II right censored order statistics and characterization for linear failure rate distribution using recurrence relations of single moments and product moments of general progressively Type-II right censored order statistics. Further, the results are specialized to the progressively Type-II right censored order statistics.

Keywords: Characterization, General Progressively Type-II Right Censored, Linear Failure Rate, Order Statistics, Recurrence Relations, Single and Product Moments

1. Introduction

In failure data analysis, it is common that some individuals cannot be observed for the full time to failure. The progressive Type-II right censored is a useful and more general scheme in which a specific fraction of individuals at risk may be removed from the study at each of several ordered failure times. Progressively censored samples have been considered, among others, by Balakrishnan et al. [6] and Davis and Feldstein [8], Bain [5] derived analysis for the Linear Failure-Rate life-testing distribution. Aggarwala and Balakrishnan [3] derived recurrence relations for single and product moments of progressive Type-II right censored order statistics from exponential, Pareto and power function distributions and their truncated forms. Abd El-Aty and MarwaMohie El-Din [1] derived recurrence relations for single and double moments of GOS from the inverted linear exponential distribution and any continuous function. Mokhlis et al. [13] derived recurrence relations for moments of GOS from Marshall-Olkin-Extended Burr XII distribution. Mohie El-Din, and Kotb [12] derived recurrence relations for single and product moments of generalized order statistics for modified Burr XII-Geometric distribution and characterization. Mohie El-Din et al. [11] derived Statistical Inference and Characterizations from Independent and Identical Exponential-Bernoulli Mixture Distribution. Athar et al. [4] discussed some new moments of progressively Type-II right censored order statistics from Lindley distribution. Saran and Pushkarna [9] derived moments of progressive Type-II right censored order statistics from a general class of doubly truncated continuous distributions. Abd El-Hamid et al. [2] derived inference and optimal design based on step-partially accelerated life tests for the generalized Pareto distribution under progressive Type-I censoring.

This scheme of censoring was generalized by Balakrishnan and Sandhu [7] as follows: at time $T_0 \equiv 0$, $n$ units are placed on test; the first $r$ failure times, $X_1, \ldots, X_r$, are not observed; at time $X_1 + 0$, where $X_i$ is the $i^{th}$ ordered failure time $(i = r + 1, \ldots, m - 1)$, $R_i$ units are removed from the test randomly, so prior to the $(i + 1)^{th}$ failure there are $n_i = n - i - \sum_{j=r+1}^{i} R_j$ units on test; finally, at the time of the $m^{th}$ failure, $X_m$, the experiment is terminated, i.e., the remaining $R_m$ units are removed from the test. The $R_i$’s, $m$
and \( r \) are prespecified integers which must satisfy the conditions: 
\[ 0 \leq r < m \leq n, 0 \leq R_i \leq n_i-1 \text{ for } i = r + 1, \ldots, m - 1 \text{ with } n_r = n - r \text{ and } R_m = n - m - 1. \]

The joint probability density function of the general progressively Type-II right censored order statistics failure times \( X_{r+1;m;n} \) is given by

\[
f_{X_{r+1};m;n}(x_{r+1}, \ldots, x_m) = A_{(n,m-1)} \prod_{i=r+1}^{n} f(x_i, \theta)[1 - F(x_i, \theta)]^{R_i}
\]

where,
\[
A_{(n,m-1)} = \frac{n!}{r!(n-r)!} \left( \sum_{j=0}^{n-r} n_j \right), \quad n_i = n - i - \sum_{j=r+1}^{i} R_j,
\]

\( i = r + 1, \ldots, m - 1. \)

In this paper, we shall introduce recurrence relations among single and product moments of general progressively Type-II right censored order statistics. Characterization for linear failure rate distribution using recurrence relations of Type-II right censored order statistics. Characterization for among single and product moments of general progressively Type-II right censored order statistics from linear failure rate distribution.

For any continuous distribution, we shall denote the single and product moments of general progressively Type-II right censored order statistics. Let \( X_{r+1;R_2;\ldots;R_m} \) be the \( m \) ordered observed failure times in a sample of size \( (n - r) \) under general progressively Type-II right censored order statistics from the linear failure rate distribution with probability density function (pdf) given by

\[
f(x, \theta) = [\alpha + \theta x] e^{-\alpha x - \theta x^2} / \alpha, \theta > 0, x \geq 0 \quad (2)
\]

The corresponding cumulative distribution function (cdf) is given by

\[
F(x, \theta) = 1 - e^{-\alpha x - \theta x^2} \quad (3)
\]

It may be noted that from (2) and (3) the relation between pdf and cdf is given by,

\[
f(x) = [\alpha + \theta x][1 - F(x)] \quad (4)
\]

For any continuous distribution, we shall denote the \( i^{th} \) single and the \( j^{th} \) and \( j^{th} \) product moments as \( \mu_{q;m:n}^{(R_{r+1};R_{r+2};\ldots;R_m)} \) and \( \mu_{q;m:n}^{(R_{r+1};R_{r+2};R_{r+2};\ldots;R_m)} \) for single moment of general progressively Type-II right censored order statistics in view of Eq. (1) as

\[
\mu_{q;m:n}^{(R_{r+1};R_{r+2};\ldots;R_m)} = E \left[ \left( \frac{f(x_{r+1}, \theta)}{f(x_{r+1}; \theta)} \right)^{R_{r+1}} \cdots \left( \frac{f(x_m, \theta)}{f(x_m; \theta)} \right)^{R_m} \right] = A_{(n,m-1)} \prod_{i=r+1}^{n} \frac{f(x_i, \theta)[1 - F(x_i, \theta)]^{R_i}}{f(x_i; \theta)[1 - F(x_i; \theta)]^{R_i}}
\]

and the \( i^{th} \) and \( j^{th} \) product moments as

\[
\mu_{q;m:n}^{(R_{r+1};R_{r+2};\ldots;R_m)} = E \left[ \left( \frac{f(x_{r+1}, \theta)}{f(x_{r+1}; \theta)} \right)^{R_{r+1}} \cdots \left( \frac{f(x_m, \theta)}{f(x_m; \theta)} \right)^{R_m} \right] = A_{(n,m-1)} \prod_{i=r+1}^{n} \frac{f(x_i, \theta)[1 - F(x_i, \theta)]^{R_i}}{f(x_i; \theta)[1 - F(x_i; \theta)]^{R_i}}
\]

2. Recurrence Relations for Single and Product Moments

In this section we introduce the recurrence relation for single and product moments of general progressively Type-II right censored order statistics from linear failure rate distribution. In the next theorem we introduce the recurrence relation for single moment of general progressively Type-II right censored order statistics.

Theorem 2.1

For \( r + 2 \leq q \leq m - 1, m \leq n \) and \( i \geq 0 \), then

\[
\mu_{q;m:n}^{(R_{r+1};R_{r+2};\ldots;R_m)} = \left( \frac{\alpha(i + 2)}{\theta(i + 1)} \right) \mu_{q;m:n}^{(R_{r+1};R_{r+2};\ldots;R_m)} + \left( \frac{\alpha(i + 2)}{\theta(i + 1)} \right) \mu_{q;m:n}^{(R_{r+1};R_{r+2};R_{r+2};\ldots;R_m)}
\]

\[
+ \frac{(n - R_{r+1} - \ldots - R_{q-1} - q)}{(R_q + 1)} \left[ \mu_{q-1;m-1:n}^{(R_{r+1};R_{r+2};\ldots;R_m)}(i+2) \right] + \frac{(n - R_{r+1} - \ldots - R_{q-1})}{(R_q + 1)} \left[ \mu_{q;m-1:n}^{(R_{r+1};R_{r+2};\ldots;R_m)}(i+2) \right]
\]

\[
- \frac{(n - R_{r+1} - \ldots - R_{q-1} - q)}{(R_q + 1)} \left[ \mu_{q-1;m-1:n}^{(R_{r+1};R_{r+2};\ldots;R_m)}(i+2) \right] + \frac{(n - R_{r+1} - \ldots - R_{q-1} - q)}{(R_q + 1)} \left[ \mu_{q;m-1:n}^{(R_{r+1};R_{r+2};\ldots;R_m)}(i+2) \right]
\]
Proof
From Eq. (4) and Eq. (5), we get

\[ \mu_{q; m:n}^{(R_{r+1}, \ldots, R_{m})} = A_{(n, m-1)} \int \cdots \int_{0 < x_{r+1} < \cdots < x_{q-1} < x_{q+1} < \cdots < x_m < \infty} I_1(x_{q-1}, x_{q+1}) [F(x_{r+1})]^r \times f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \cdots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} \cdots f(x_{m})[1 - F(x_{m})]^{R_m} \times dx_{r+1} \cdots dx_{q-1} \times dx_{q+1} \cdots dx_m \]

(8)

where

\[ I_1(x_{q-1}, x_{q+1}) = \int_{x_{q-1}}^{x_{q+1}} x_4^2 (\alpha + \theta x_4) [1 - F(x_4)]^{R_{q+1}} dx_4 \]

(9)

Now, integrating by parts gives

\[ \int x_4^2 (\alpha + \theta x_4) [1 - F(x_4)]^{R_{q+1}} dx_4 = \frac{\alpha [1 - F(x_4)]^{R_{q+1}}}{\theta} \left[ \frac{d}{dx_4} \left( x_4 S(x_4) \right) \right]_{x_4=x_{q-1}}^{x_4=x_{q+1}} \]

Substituting Eq. (10) in Eq. (8) and simplifying, yields Eq. (7).

This completes the proof.

Special case

\[ \mu_{q; m:n}^{(R_{r+1}, \ldots, R_{m})} = \frac{(i + 2)}{\theta (R_q + 1)} \mu_{q; m:n}^{(R_{r+1}, \ldots, R_{m})} \]

(10)

\[ + \frac{(n - R_1 - \cdots - R_{q-1} - q + 1)}{(R_q + 1)} \left[ \mu_{q; m:n}^{(R_{r+1}, \ldots, R_{m})} \right]^{(i+2)} + \frac{\alpha (i + 2)}{\theta (i + 1)} \mu_{q; m:n}^{(R_{r+1}, \ldots, R_{m})} \]

(11)

In the next two theorems, we introduce recurrence relations for product moments of general progressively Type-II right censored order statistics from the general progressively Type-II right censored order statistics when \( r = 0 \),

\[ \mu_{q; m:n}^{(R_{r+1}, \ldots, R_{m})} = \frac{(i + 2)}{\theta (R_q + 1)} \mu_{q; m:n}^{(R_{r+1}, \ldots, R_{m})} \]

(12)

Theorem (2.1) will be valid for the progressively Type-II right censored order statistics as a special case from the general progressively Type-II right censored order statistics distribution.

Proof
From Eq. (6), we get

\[ A_{(n, m-1)} \int \cdots \int_{0 < x_{r+1} < \cdots < x_{q-1} < x_{q+1} < \cdots < x_m < \infty} x_4^2 I_2(x_{q-1}, x_{q+1}) [F(x_{r+1})]^r \times f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \cdots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} \cdots f(x_{m})[1 - F(x_{m})]^{R_m} \times dx_{r+1} \cdots dx_{q-1} \times dx_{q+1} \cdots dx_m \]

\[ \times f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \cdots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} \cdots f(x_{m})[1 - F(x_{m})]^{R_m} \times \]

(13)

\[ \times f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \cdots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} \cdots f(x_{m})[1 - F(x_{m})]^{R_m} \times \]

(14)
Substituting Eq. (10) in Eq. (12) and simplifying, yields

\[ f(x_m)[1 - F(x_m)]^m dx_{r+1} dx_{r+2} ... dx_{q-1} dx_{q+1} ... dx_m \]  

(12)

This completes the proof.

**Special case**

\[
\frac{\mu^{(i+j+2)}}{R_{q_m:n}} = \frac{(i+2)}{\theta(R_q + 1)} \frac{(i+j+2)}{\mu^{(i+j+1)}} \frac{\alpha(i+2)}{\theta(i+1)} \frac{(R_{i+2} - R_{i+1})}{\mu^{(i+j+2)}} 
\]

\[
\frac{(n - R_{r+1} - ... - R_{q-1} - q + 1)}{R_q + 1} \left[ \frac{(R_{i+2} - R_{i+1} - R_{q+2} + R_{q+1})}{\mu^{(i+j+1)}} \right] 
\]

Theorem 2.3

For \( r + 1 \leq q < s \leq m - 1, m \leq n \) and \( i, j \geq 0 \), then

\[
\frac{(n - R_{r+1} - ... - R_{q-1} - s + 1)}{R_s + 1} \left[ \frac{(R_{i+2} - R_{i+1} - R_{q+2} + R_{q+1})}{\mu^{(i+j+1)}} \right] 
\]

**Proof**

From Eq. (4) and Eq. (6), we get

\[
\mu^{(i+j+2)} = A(n-m-1) \int \frac{x_i}{x_q} I_2(x_{s-1} x_{s+1}) \frac{f(x_{r+1})}{1 - F(x_{r+1})} \frac{f(x_{s-1})}{1 - F(x_{s-1})} \frac{f(x_{s+1})}{1 - F(x_{s+1})} \frac{R_{s+1}}{R_{s+1}} \frac{R_{s+1}}{R_{s+1}} dx_{s-1} dx_{s+1} 
\]

(14)

where

\[
I_2(x_{s-1} x_{s+1}) = \int_{x_{s-1}}^{x_{s+1}} x^{\alpha + \theta x_s} \frac{f(x_s)}{1 - F(x_s)} \frac{R_{s+1}}{R_{s+1}} dx_s 
\]

(15)

Now, integrating by parts we obtain

\[
\left\{ \begin{array}{l}
\frac{d}{dx} [x^{j+2} f(x)] = \frac{d}{dx} [x^{j+2} f(x)]
\end{array} \right\} 
\]

(16)

Substituting Eq. (16) in Eq. (14) and simplifying, yields Eq. (13).

This completes the proof.

**Special case**

\[
\frac{(n - R_{r+1} - ... - R_{q-1} - s + 1)}{R_s + 1} \left[ \frac{(R_{i+2} - R_{i+1} - R_{q+2} + R_{q+1})}{\mu^{(i+j+1)}} \right] 
\]

**Theorem (2.2)** will be valid for the progressively Type-II right censored order statistics as a special case from the general progressively Type-II right censored order statistics when \( r = 0 \),
3. Characterization for Single and Product Moments

In this section, we introduce the characterization of the linear failure rate distribution using the relation between pdf and cdf and using recurrence relation for single and product moments of general progressively Type-II right censored order statistics from linear failure rate distribution.

In the next theorem, we introduce the characterization of the linear failure rate distribution using relation between pdf and cdf.

**Theorem 3.1**

Let \( X \) be a continuous random variable with pdf \( f(\cdot) \), cdf \( F(\cdot) \) and survival function \( [1 - F(\cdot)] \). Then \( X \) has linear failure rate distribution iff

\[
f(x) = [\alpha + \theta x][1 - F(x)], \quad x \geq 0 \tag{17}
\]

**Proof**

**Necessity:**

From Eq. (2) and Eq. (3) we can easily obtain Eq. (17).

**Sufficiency:**

Suppose that \( X \) is a continuous random variable with pdf \( f(\cdot) \) and cdf \( F(\cdot) \). Suppose, also, that equation Eq. (17) is true. Then we have:

\[
-\frac{d[1 - F(x)]}{1 - F(x)} = [\alpha + \theta x]dx
\]

\[
\mu_{q:m:n}^{(R_{r+1}...R_m)} = \frac{(i + 2)}{\theta(R_q + 1)} \mu_{q:m:n}^{(R_{r+1}...R_m)} + \frac{\alpha(i + 2)}{\theta(i + 1)} \mu_{q:m:n}^{(R_{r+1}...R_m)}\tag{18}
\]

On integrating, we get

\[
-\ln[1 - F(x)] = ax + \frac{\theta}{2}x^2 + C,
\]

where \( C \) is an arbitrary constant.

Now, since \( [1 - F(0)] = 1 \), then putting \( x = 0 \) in this equation, we get \( C = 0 \).

Therefore,

\[
-\ln[1 - F(x)] = ax + \frac{\theta}{2}x^2,
\]

Hence,

\[
F(x) = 1 - e^{-ax-\frac{\theta}{2}x^2}.
\]

That is the distribution function of linear failure rate distribution.

This completes the proof.

In the next theorem, we introduce the characterization of the linear failure rate distribution using recurrence relation for single moment of general progressively Type-II right censored order statistics has introduced in the following theorems.

**Theorem 3.2**

Let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{m:n} \) be the order statistic of a random sample of size \( n - r \). Then \( X \) has linear failure rate distribution iff, for \( r + 2 \leq q \leq m - 1, m \leq n \) and \( i \geq 0 \),

\[
\mu_{q:m:n}^{(R_{r+1}...R_m)} = \frac{\theta(R_q + 1)}{(i + 2)} \mu_{q:m:n}^{(R_{r+1}...R_m)} + \frac{\alpha(R_q + 1)}{(i + 1)} \mu_{q:m:n}^{(R_{r+1}...R_m)}
\]

\[
+ \frac{(n - R_{r+1} - \ldots - R_q - q + 1)}{(R_q + 1)} \left[ \frac{(R_{r+1}...R_q-2,R_{r+1}...R_{q+1})}{\mu_{q:m:n}^{(R_{r+1}...R_m)}} + \frac{\alpha(R_{r+1}...R_q-2,R_{r+1}...R_{q+1})}{(i + 1)} \mu_{q:m:n}^{(R_{r+1}...R_m)} \right] + \ldots + \frac{\alpha(R_{r+1}...R_q-2,R_{r+1}...R_{q+1})}{(i + 1)} \mu_{q:m:n}^{(R_{r+1}...R_m)}\tag{19}
\]

**Proof**

**Necessity:**

Theorem 2.1 proved the necessary part of this theorem.

\[
\mu_{q:m:n}^{(R_{r+1}...R_m)} = \frac{\theta(R_q + 1)}{(i + 2)} \mu_{q:m:n}^{(R_{r+1}...R_m)} + \frac{\alpha(R_q + 1)}{(i + 1)} \mu_{q:m:n}^{(R_{r+1}...R_m)}
\]

\[
+ \frac{(n - R_{r+1} - \ldots - R_q - q)}{(i + 2)} \mu_{q:m:n}^{(R_{r+1}...R_m)} + \frac{\alpha(R_{r+1}...R_q-2,R_{r+1}...R_{q+1})}{(i + 1)} \mu_{q:m:n}^{(R_{r+1}...R_m)}\tag{19}
\]

From Eq. (4) and Eq. (5), we get
\[
\mu_{q;m:n}^{(R_{r+1},\ldots,R_{m})^{(i+1)}} = A_{(n,m-1)} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_{m} < \infty} I_3(x_{q-1}, x_{q+1}) [F(x_{r+1})]^r \times
\]

\[
f(x_{r+1}) [1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1}) [1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+1}) [1 - F(x_{q+1})]^{R_{q+1}} \ldots \times
\]

\[
f(x_m) [1 - F(x_m)]^{R_m} dx_{r+1} \ldots dx_{q-1} dx_{q+1} \ldots dx_m
\]

Equation (20)

Integrating by parts, we obtain

\[
I_3(x_{q-1}, x_{q+1}) = \frac{1}{R_{q+1}} x_{q+1}^{i+1} f(x_q) [1 - F(x_q)]^{R_{q+1}} dx_q
\]

Equation (21)

Substituting in Eq. (20), we get

\[
\mu_{q;m:n}^{(R_{r+1},\ldots,R_{m})^{(i+1)}} = \frac{i + 1}{R} A_{(n,m-1)} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_{m} < \infty} \left[ F(x_{r+1}) \right]^r \times
\]

\[
f(x_{r+1}) [1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1}) [1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+1}) [1 - F(x_{q+1})]^{R_{q+1}} \ldots \times \]

\[
f(x_m) [1 - F(x_m)]^{R_m} dx_{r+1} \ldots dx_{q-1} dx_{q+1} \ldots dx_m
\]

\[
+ \frac{A_{(n,m-1)}}{R_q + 1} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_{m} < \infty} x_{q+1}^{i+1} [F(x_{r+1})]^r \times
\]

\[
f(x_{r+1}) [1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1}) [1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+1}) [1 - F(x_{q+1})]^{R_{q+1}} \ldots \times \]

\[
f(x_m) [1 - F(x_m)]^{R_m} dx_{r+1} \ldots dx_{q-1} dx_{q+1} \ldots dx_m
\]

\[
- \frac{A_{(n,m-1)}}{R_q + 1} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_{m} < \infty} x_{q+1}^{i+1} [F(x_{r+1})]^r \times
\]

\[
f(x_{r+1}) [1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1}) [1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+1}) [1 - F(x_{q+1})]^{R_{q+1}} \ldots \times \]

\[
f(x_m) [1 - F(x_m)]^{R_m} dx_{r+1} \ldots dx_{q-1} dx_{q+1} \ldots dx_m
\]

\[
= A_{(n,m-1)} \frac{i + 1}{R_q + 1} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_{m} < \infty} \left[ F(x_{r+1}) \right]^r \times
\]

\[
f(x_{r+1}) [1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1}) [1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+1}) [1 - F(x_{q+1})]^{R_{q+1}} \ldots \times \]

\[
f(x_m) [1 - F(x_m)]^{R_m} dx_{r+1} \ldots dx_{q-1} dx_{q+1} \ldots dx_m + \frac{(n - R_{r+1} - \ldots - R_q - q)}{R_q + 1} \mu_{q;m:n}^{(R_{r+1},\ldots,R_{m})^{(i+1)}} + \frac{(n - R_{r+1} - \ldots - R_{q+1} - q + 1)}{R_{q+1}} \mu_{q;m:n}^{(R_{r+1},\ldots,R_{m})^{(i+1)}}
\]

Equation (23)
Now by substituting for $\mu_{q:n}^{(R_{r+1},...R_m)^{(i+2)}}$ and $\mu_{q:n}^{(R_{r+1},...R_m)^{(i+2)}}$ from Eq. (23) and Eq. (24) in Eq. (19), we get

$$
\mu_{q:m}^{(R_{r+1},...R_m)^{(i)}} = A_{(n,m-1)} \frac{1}{R + 1} \int \cdots \int_{0 < x_{r+1} < \cdots < x_m < 0} \left[ F(x_{r+1}) \right]^\alpha \times
\left[ 1 - F(x_q) \right]^{R_1} \cdots \left[ 1 - F(x_{r+1}) \right]^{R_{r+1}} \times
f(x_q) \left[ 1 - F(x_{r+1}) \right]^{R_{r+1}} \cdots \left[ 1 - F(x_{m-n+2}) \right]^{R_{m-n+2}} \times
f(x_{m-n+1})^{R_{m-n+1}} \cdots f(x_m) \left[ 1 - F(x_{m-n+1}) \right]^{R_{m-n+1}} \cdots dx_{m-n+1} \times
f(x_m) \left[ 1 - F(x_{m-n+1}) \right]^{R_{m-n+1}} \cdots dx_m = 0 \tag{25}
$$

We get

$$
A_{(n,m-1)} \frac{1}{R + 1} \int \cdots \int_{0 < x_{r+1} < \cdots < x_m < 0} \left[ F(x_{r+1}) \right]^\alpha \times
\left[ 1 - F(x_q) \right]^{R_1} \cdots \left[ 1 - F(x_{r+1}) \right]^{R_{r+1}} \times
f(x_q) \left[ 1 - F(x_{r+1}) \right]^{R_{r+1}} \cdots \left[ 1 - F(x_{m-n+2}) \right]^{R_{m-n+2}} \times
f(x_{m-n+1})^{R_{m-n+1}} \cdots f(x_m) \left[ 1 - F(x_{m-n+1}) \right]^{R_{m-n+1}} \cdots dx_{m-n+1} \times
f(x_m) \left[ 1 - F(x_{m-n+1}) \right]^{R_{m-n+1}} \cdots dx_m = 0 \tag{25}
$$

Using Muntz-Szasz theorem, (See, Hwang and Lin [10]), we get

$$
f(x_q) = (\alpha + \theta x_q) \left[ 1 - F(x_q) \right]
$$

Using Theorem 3.1, we get

$$
F(x) = 1 - e^{-ax - \frac{\theta x^2}{2}}
$$

That is the distribution function of linear failure rate distribution.

This completes the proof.

In the next two theorems, we introduce the characterize the linear failure rate distribution using recurrence relation for product moment of general progressively Type-II right censored order statistics.

**Theorem 3.3**

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{m:n}$ be the order statistics of a random sample of size $(n-r)$. Then $X$ has linear failure rate distribution iff, for $r + 1 \leq q < s \leq m - 1$, $m \geq n$ and $i, j \geq 0,

$$
\mu_{q:m}^{(R_{r+1},...R_m)^{(i+2)}} = A_{(n,m-1)} \frac{1}{R + 1} \int \cdots \int_{0 < x_{r+1} < \cdots < x_m < 0} \left[ F(x_{r+1}) \right]^\alpha \times
\left[ 1 - F(x_q) \right]^{R_1} \cdots \left[ 1 - F(x_{r+1}) \right]^{R_{r+1}} \times
f(x_q) \left[ 1 - F(x_{r+1}) \right]^{R_{r+1}} \cdots f(x_{m-n+1})^{R_{m-n+1}} \times
f(x_{m-n+1})^{R_{m-n+1}} \cdots f(x_m) \left[ 1 - F(x_{m-n+1}) \right]^{R_{m-n+1}} \cdots dx_m = 0 \tag{26}
$$

Using Muntz-Szasz theorem, (See, Hwang and Lin [10]), we get

$$
f(x_q) = (\alpha + \theta x_q) \left[ 1 - F(x_q) \right]
$$

Using Theorem 3.1, we get

$$
F(x) = 1 - e^{-ax - \frac{\theta x^2}{2}}
$$

That is the distribution function of linear failure rate distribution.

This completes the proof.
Proof

Necessity:
Theorem 2.2 proved the necessary part of this theorem

Sufficiency:
Assuming that equation (26) holds, then we have:

\[
\mu_{q,s:m:n}^{(R_{r+1} \ldots R_m)} = \frac{\theta (R_q + 1)}{(i + 2)} \mu_{q,s:m:n}^{(R_{r+1} \ldots R_m)}(i+2) + \frac{\alpha (R_q + 1)}{(i + 1)} \mu_{q,s:m:n}^{(R_{r+1} \ldots R_m)}(i+1) \\
+ (n - R_{r+1} - \ldots - R_{q-1} - q) \left[ \frac{\theta (R_{r+1} \ldots R_{q-1} - (R_q + R_{q+1} + 1), R_{q+2} \ldots R_m)}{(i + 2)} \mu_{q,s-1:m-1:n}^{(R_{r+1} \ldots R_{q-1} - (R_q + R_{q+1} + 1), R_{q+2} \ldots R_m)}(i+2) \\
+ \frac{\alpha}{(i + 1)} \mu_{q,s-1:m-1:n}^{(R_{r+1} \ldots R_{q-2} - (R_q + R_{q+1} + 1), R_{q+2} \ldots R_m)}(i+1) \right] \\
- (n - R_{r+1} - \ldots - R_q - q + 1) \left[ \frac{\theta (R_{r+1} \ldots R_{q-2} - (R_q + R_{q+1} + 1), R_{q+2} \ldots R_m)}{(i + 2)} \mu_{q,s-1:m-1:n}^{(R_{r+1} \ldots R_{q-2} - (R_q + R_{q+1} + 1), R_{q+2} \ldots R_m)}(i+2) \\
+ \frac{\alpha}{(i + 1)} \mu_{q,s-1:m-1:n}^{(R_{r+1} \ldots R_{q-2} - (R_q + R_{q+1} + 1), R_{q+2} \ldots R_m)}(i+1) \right] \\
\] (27)

where

\[
\mu_{q,s:m:n}^{(R_{r+1} \ldots R_m)} = A_{n(m-1)} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_m < 0} [F(x_{r+1})]^r \times \\
l_1(x_{q-1}, x_{q+1}) \prod_{j=q}^{n} [1 - F(x_{q+j})]^{R_{q+j}} \ldots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} \times \\
f(x_{q+j})[1 - F(x_{q+j})]^{R_{q+j}} \ldots f(x_m)[1 - F(x_m)]^{R_m} d \tau_{r+1} \ldots d x_{q-1} d x_{q+1} \ldots d x_m \] (28)

Substituting by Eq. (22) in Eq. (28), we get

\[
\mu_{q,s:m:n}^{(R_{r+1} \ldots R_m)} = A_{n(m-1)} \frac{i + 1}{R_q + 1} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_m < 0} x_q^j [F(x_{r+1})]^r \times \\
f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} \int_{x_{q-1}}^{x_{q+j}} x_q^j [1 - F(x_q)]^{R_{q+j}} d x_q \times \\
f(x_{q+j})[1 - F(x_{q+j})]^{R_{q+j}} \ldots f(x_m)[1 - F(x_m)]^{R_m} d \tau_{r+1} \ldots d x_{q-1} d x_{q+1} \ldots d x_m \\
+ \frac{A_{n(m-1)}}{R_q + 1} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_m < 0} x_q^{j+1} x_q^j [F(x_{r+1})]^r \times \\
f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+j})[1 - F(x_{q+j})]^{R_{q+j}} \times \\
\ldots f(x_m)[1 - F(x_m)]^{R_m} d \tau_{r+1} \ldots d x_{q-1} d x_{q+1} \ldots d x_m \\
- \frac{A_{n(m-1)}}{R_q + 1} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_m < 0} x_q^{j+1} x_q^j [F(x_{r+1})]^r \times \\
f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \ldots f(x_{q-1})[1 - F(x_{q-1})]^{R_{q-1}} f(x_{q+j})[1 - F(x_{q+j})]^{R_{q+j}} \times \\
\ldots f(x_m)[1 - F(x_m)]^{R_m} d \tau_{r+1} \ldots d x_{q-1} d x_{q+1} \ldots d x_m \\
= A_{n(m-1)} \frac{i + 1}{R_q + 1} \int \ldots \int_{0 < x_{r+1} < \ldots < x_{q-1} < x_{q+1} < \ldots < x_m < 0} x_q^j [F(x_{r+1})]^r \times 
\]
Now by substituting for $\mu_{q,s;m:n}^{(i+2,f)}$ from Eq. (29) and Eq. (30) in Eq. (27), we get

$$A_{(n,m-1)} \int \int \int \int_0<\text{for } x_q \leqslant x_m \int \int \int \int x_q^s x_m^t F(x_r+1)^{\gamma} \times $$

$$f(x_r+1) (1 - F(x_r+1))^R_r \int \int \int \int f(x_m) (1 - F(x_m))^R_m \text{d}x_r \text{d}x_m $$

Then

$$A_{(n,m-1)} \int \int \int \int_0<\text{for } x_q \leqslant x_m \int \int \int \int x_q^s x_m^t \left( \left[ 1 - F(x_q)^{\gamma} \int \int \int \int \left( 1 - F(x_q) \right)^{R_q+1} \times f(x_q) \right] \right) $$

Using Muntz-Szasz theorem, (See, Hwang and Lin [10]), we get

$$f(x_q) = \left( \alpha + \theta x_q \right) \left[ 1 - F(x_q) \right]$$

Using Theorem 3.1, we get

$$F(x) = 1 - e^{-ax \theta x^2 \over 2}$$

That is the distribution function of linear failure rate distribution.

This completes the proof.  

\text{Theorem 3.4}  

Let $X_{r+1} \leqslant X_{r+2} \leqslant \ldots \leqslant X_{m}$ be the order statistics of a random sample of size $(n-r)$. Then $X$ has linear failure rate distribution iff, for $r + 1 \leq q < s \leq m - 1$, $m \leq n$ and $i,j \geq 0$,  

$$\mu_{q,s;m:n}^{(R_{r+1}, \ldots, R_m) (i+2,f)} \times \mu_{q,s;m:n}^{(R_{r+1}, \ldots, R_m) (i+1)}$$
where

\[
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\]

Proof:
Necessity:
Theorem 2.3 proved the necessary part of this theorem.

Sufficiency:
Assuming that equation (32) holds, then we have:

\[
\left( R_{s} + 1 \right) \mu_{q, s ; m ; n}^{(l, j)} = \theta \left( R_{s} + 1 \right) + \alpha \left( R_{s} + 1 \right) \mu_{q, s ; m ; n}^{(l, j+1)}
\]

\[
\left( n - R_{s+1} - \cdots - R_{s} - s + 1 \right) \left[ \frac{\theta \left( R_{s} + 1 \right) + \alpha \left( R_{s} + 1 \right)}{\left( j + 2 \right)} \mu_{q, s ; m ; n}^{(l, j+2)} \right] + \left( n - R_{s+1} - \cdots - R_{s} - s + 1 \right) \left[ \frac{\alpha \left( R_{s} + 1 \right)}{\left( j + 1 \right)} \mu_{q, s ; m ; n}^{(l, j+1)} \right]
\]

From Eq. (4) and Eq. (6), we get

\[
I_{4} \left( x_{s-1}, x_{s+1} \right) f \left( x_{s+1} \right) \left[ 1 - F \left( x_{s+1} \right) \right]^{R_{s+1}} f \left( x_{s-1} \right) \left[ 1 - F \left( x_{s-1} \right) \right]^{R_{s-1}}
\]

\[
f \left( x_{s+1} \right) \left[ 1 - \sum_{s=1}^{m} d_{s-1} x_{s-1} \right] f \left( x_{s} \right) \left[ 1 - \sum_{s=1}^{m} d_{s} x_{s} \right]
\]

where

\[
I_{4} \left( x_{s-1}, x_{s+1} \right) = \int_{x_{s-1}}^{x_{s+1}} \frac{f \left( x \right) \left[ 1 - F \left( x \right) \right]^{R_{s} d_{s}}}{x_{s}} dx
\]

Integrating by parts, we have obtain

\[
I_{4} \left( x_{s-1}, x_{s+1} \right) = -\frac{1}{R_{s} + 1} x_{s+1}^{j+1} \left[ 1 - F \left( x_{s+1} \right) \right]^{R_{s+1}} + \frac{x_{s-1}}{R_{s} + 1} \left[ 1 - F \left( x_{s-1} \right) \right]^{R_{s-1}}
\]

\[
+ \frac{j+1}{R_{s} + 1} \int_{x_{s-1}}^{x_{s+1}} x_{s}^{j} \left[ 1 - F \left( x_{s} \right) \right]^{R_{s} d_{s}} dx_{s}
\]

Now by substituting in Eq. (34), we get

\[
\mu_{q, s ; m ; n}^{(R_{s+1}, \cdots, R_{m}) (j+1)} = A_{(n, m-1)} \int_{0 < S_{x} < \cdots < S_{s-1} < S_{s+1} < \cdots < S_{m}} \left[ 1 - F \left( S_{x} \right) \right]^{R_{s} d_{s}} \times
\]

\[
f \left( x_{s+1} \right) \left[ 1 - \sum_{s=1}^{m} d_{s-1} x_{s-1} \right] f \left( x_{s} \right) \left[ 1 - \sum_{s=1}^{m} d_{s} x_{s} \right]
\]

\[
+ \frac{A_{(n, m-1)}}{R_{s} + 1} \int_{0 < S_{x} < \cdots < S_{s-1} < S_{s+1} < \cdots < S_{m}} \left[ 1 - F \left( S_{x} \right) \right]^{R_{s} d_{s}} \times
\]

\[
f \left( x_{s+1} \right) \left[ 1 - \sum_{s=1}^{m} d_{s-1} x_{s-1} \right] f \left( x_{s} \right) \left[ 1 - \sum_{s=1}^{m} d_{s} x_{s} \right]
\]

\[
f \left( x_{m} \right) \left[ 1 - \sum_{s=1}^{m} d_{s} x_{s} \right]
\]
\[-\frac{A_{(n,m-1)}}{R_s + 1} \int \cdots \int _{0 < x_{r+1} < \cdots < x_{s-1} < x_{s+1} < \cdots < x_m < \infty} x_{s+1}^{j+1} x_q^i [F(x_{r+1})]^r \times \]

\[f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \cdots f(x_{s-1})[1 - F(x_{s-1})]^{R_{s-1}} f(x_{s+1})[1 - F(x_{s+1})]^{1 + R_s + R_{s+1}} \cdots \times \]

\[f(x_m)[1 - F(x_m)]^{R_m} d x_{r+1} \cdots d x_{s-1} d x_{s+1} \cdots d x_m \]

\[= A_{(n,m-1)} \frac{j + 1}{R_s + 1} \int \cdots \int _{0 < x_{r+1} < \cdots < x_{s-1} < x_{s+1} < \cdots < x_m < \infty} [F(x_{r+1})]^r \times \]

\[f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \cdots f(x_{s-1})[1 - F(x_{s-1})]^{R_{s-1}} \cdots f(x_{s+1})[1 - F(x_{s+1})]^{R_{s+1}} d x_s \]

\[f(x_s)[1 - F(x_s)]^{R_s} \cdots f(x_m)[1 - F(x_m)]^{R_m} d x_{r+1} \cdots d x_{s-1} d x_{s+1} \cdots d x_m + \]

\[\frac{(n - R_{r+1} - \cdots - R_{s-1} - s + 1)}{R_s + 1} \frac{\mu^{(R_{r+1} - \cdots - R_{s-1} - R_s + 1, R_{s+1} + \cdots - R_m)^{(j+1)}}_{q,s-1,m-1,n}}{R_s + 1} \]

\[\mu_{q,s,m:n}^{(R_{r+1} - \cdots - R_{s-1} - R_s + 1, R_{s+1} + \cdots - R_m)^{(j+1)}} \]

(37)

Now by substituting for $\mu_{q,s,m:n}^{(R_{r+1} - \cdots - R_{s-1} - R_s + 1, R_{s+1} + \cdots - R_m)^{(j+2)}}$ and $\mu_{q,s,m:n}^{(R_{r+1} - \cdots - R_{s-1} - R_s + 1, R_{s+1} + \cdots - R_m)^{(j+2)}}$ from Eq. (37) and Eq. (38) in Eq. (33), we get

\[\mu_{q,s,m:n}^{(R_{r+1} - \cdots - R_{s-1} - R_s + 1, R_{s+1} + \cdots - R_m)^{(j+1)}} \]

(39)

We get

\[A_{(n,m-1)} \int \cdots \int _{0 < x_{r+1} < \cdots < x_m < \infty} x_q^i x_s^j [F(x_{r+1})]^r [f(x_q) - (\alpha + \theta x_q)[1 - F(x_q)] \times \]

\[1 - F(x_s)^{R_s} f(x_{r+1})[1 - F(x_{r+1})]^{R_{r+1}} \cdots f(x_{s-1})[1 - F(x_{s-1})]^{R_{s-1}} f(x_s)[1 - F(x_s)]^{R_s} \times \]

\[f(x_{s+1})[1 - F(x_{s+1})]^{R_{s+1}} \cdots f(x_m)[1 - F(x_m)]^{R_m} d x_{r+1} \cdots d x_m = 0 \]

Using Muntz-Szasz theorem, (See, Hwang and Lin [10]), we get

\[f(x) = (\alpha + \theta x)[1 - F(x)] \]

Using Theorem 3.1 above, we get

\[F(x) = 1 - e^{-\eta x - \frac{\theta x^2}{2}} \]

That is the distribution function of linear failure rate distribution.

This completes the proof.
4. Conclusion

We derived recurrence relations among single and product moments of general progressively Type-II right censored order statistics (Theorem 2.1, 2.2 and 2.3). Characterization for the random variable $X$ following the linear failure rate distribution is obtained using the previous recurrence relations (Theorems 3.2, 3.3 and 3.4) and distribution function (Theorem 3.1). For future work, estimation for the scale and location parameters could be obtained by applying the best linear unbiased estimation to the previous results.

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