SUB-RIEMANNIAN CURVATURE OF CARNOT GROUPS WITH RANK-TWO DISTRIBUTIONS

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Abstract. The notion of curvature discussed in this paper is a far going generalization of the Riemannian sectional curvature. It was first introduced by Agrachev, Barilari and Rizzi in [2], and it is defined for a wide class of optimal control problems: a unified framework including geometric structures such as Riemannian, sub-Riemannian, Finsler, and sub-Finsler structures. In this work we study the generalized sectional curvature of Carnot groups with rank-two distributions. In particular, we consider the Cartan group and Carnot groups with horizontal distribution of Goursat-type. In these Carnot groups we characterize ample and equiregular geodesics. For Carnot groups with horizontal Goursat distribution we show that their generalized sectional curvatures depend only on the Engel part of the distribution. This family of Carnot groups contains naturally the three-dimensional Heisenberg group, as well as the Engel group. Moreover, we also show that in the Engel and Cartan groups there exist initial covectors for which there is an infinite discrete set of times at which the corresponding ample geodesics are not equiregular.

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1. Introduction

In sub-Riemannian geometry there is no canonical connection such as the miraculous Levi-Civita connection in Riemannian geometry. Despite this obstacle, several groups of mathematicians in recent years have been trying to define a notion of curvature in the sub-Riemannian setting. The curvature we consider in this paper was introduced in [2], and it is a generalization of the sectional curvature from Riemannian geometry.

Let $M$ be an $n$-dimensional Riemannian manifold with Riemannian distance $d$ and let $\gamma_v(t), \gamma_w(s)$ be two arc-length parametrized geodesics with

$$\gamma_v(0) = \gamma_w(0) = x_0 \in M \quad \text{and} \quad \dot{\gamma}_v(0) = v, \dot{\gamma}_w(0) = w \in T_{x_0}M.$$ 

The geodesic cost associated with $\gamma_v$ is

$$c_t(x) = -\frac{1}{2t}d^2(x, \gamma_v(t)).$$

Consider the function

$$C(t, s) = -tc_t(\gamma_w(s)).$$

The sectional curvature of a Riemannian manifold can be recover from the asymptotic expansion of the function $C$. It can be shown that $C$ is smooth at $(0, 0)$. The following formula, which is due to Loeper, see [32], holds true for the Taylor expansion of $C(t, s)$ at $(0, 0)$

$$C(t, s) = \frac{1}{2}\left( t^2 + s^2 - 2g(v, w)ts \right) - \frac{1}{6}g(R(v, w)v, w) t^2 s^2 + \mathcal{O}(t^4)$$

where $g$ denotes the Riemannian inner product and $R$ is the Riemannian curvature tensor. From the expansion of $C(t, s)$, we easily get $d_{x_0}c_t(\cdot) = g(\cdot, v)$. Now, if we let $\dot{c}_t = \frac{\partial}{\partial t}c_t$, then $d_{x_0}\dot{c}_t = 0$. Hence, the Hessian $d_{x_0}^2\dot{c}_t$ is a well-defined quadratic form on $T_{x_0}M$. From the Taylor expansion of $C(t, s)$, we finally get

$$d_{x_0}^2\dot{c}_t(w) = \frac{1}{t^2}g(w, w) + \frac{1}{3}g(R(v, w)v, w) + \mathcal{O}(t).$$
The derivative $c_t$ of the geodesic cost has a very nice geometric interpretation. Let $W^t_{x,\gamma(t)} \in T_{\gamma(t)}M$ be the final tangent vector of the unique minimizer connecting $x$ with $\gamma(t)$ in time $t$. From [2, Appendix H] we have:

\begin{equation}
(1.2) \quad c_t(x) = \frac{1}{2}||W^t_{x,0,\gamma(t)} - W^t_{x,\gamma(t)}||^2 - \frac{1}{2}||W^t_{x,0,\gamma(t)}||^2.
\end{equation}

The “curvature” at the initial point $x_0$ is hidden in the behavior of this function for small $t$ and $x$ close to $x_0$. For instance, in a positively (resp. negatively) curved Riemannian manifold, for which Eq. (1.2) still holds, then the two tangent vectors, compared at $\gamma(t)$, are more (resp. less) divergent w.r.t. $t$.

Despite the fact that the sub-Riemannian distance has a more complicated behavior than the Riemannian one, Agrachev, Barilari and Rizzi wrote in [2] an expansion for the square of the curvature-type invariants for a wide class of optimal control problems. One must mention here that formula (1.2) is also valid in the sub-Riemannian setting, see [2].

We will now describe some results of the paper [2]. Let $(\mathcal{D},g)$ be a sub-Riemannian structure on a smooth manifold $M$, $\gamma$ be a fixed normal geodesic, with $\gamma(0) = x_0$ and initial covector $\lambda_0$. Let

\begin{equation}
\tilde{c}_t(x) = -\frac{1}{2t}d^2(x,\gamma(t)), \quad t > 0,
\end{equation}

be the geodesic cost function. If $\gamma$ is also strongly normal, see Definition 2.7, then the function $c_t(x)$ is smooth in $t$ and $x$ for small $t > 0$ and $x$ close to $x_0$. Moreover, $d_{x_0}c_t = \lambda_0$, and hence, $x_0$ is a critical point for $\tilde{c}_t = \partial_t c_t$.

The second differential of $\tilde{c}_t$ is well defined at $x_0$, and we can associate with the family of quadratic forms $d^2_{x_0}\tilde{c}_t|_{\mathcal{D}_{x_0}} : \mathcal{D}_{x_0} \to \mathbb{R}$, a family of symmetric operators $Q_{\lambda_0}(t)$, via the formula

\begin{equation}
d^2_{x_0}\tilde{c}_t(v) = g(Q_{\lambda_0}(t)v, v), \quad t > 0 \text{ and } v \in \mathcal{D}_{x_0}.
\end{equation}

If we also assume that $\gamma$ is ample, see Definition 2.7, then we have the following Laurent expansion for the family of symmetric operators $Q_{\lambda_0}(t) : \mathcal{D}_{x_0} \to \mathcal{D}_{x_0}$,

\begin{equation}
Q_{\lambda_0}(t) = \frac{1}{t^2}I_{\lambda_0} + \frac{1}{3}R_{\lambda_0} + O(t) \quad t > 0.
\end{equation}

Moreover, in [2] the authors showed that, if the geodesic $\gamma$ is also equiregular, see Definition 2.7, the explicit expression of the invariants $I_{\lambda_0}$ and $R_{\lambda_0}$ can be computed in terms of the symplectic invariants of the so-called Jacobi curve. The invariant $R_{\lambda_0}$ is called the curvature operator. The Jacobi curve arises naturally from the geometric interpretation of the second derivative of the geodesic cost. These symplectic invariants can be computed via an algorithm which is quite hard to implement. Despite the enormous difficulty of this algorithm we are able to provide in this paper the explicit expression of the invariants $I_{\lambda_0}$ and $R_{\lambda_0}$ in Carnot groups with horizontal distribution of Goursat-type.

We consider in $\mathbb{R}^n$, $n \geq 3$, a system of vector fields $\{X_1, \ldots, X_n\}$ such that

\begin{equation}
[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad \ldots, \quad [X_1, X_{n-1}] = X_n,
\end{equation}

and the other brackets assume to be trivial. Let the $\mathfrak{g}$ be the nilpotent stratified Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_{n-1}$, with $\mathfrak{g}_1 = \text{span} \{X_1, X_2\}$, $\mathfrak{g}_i = \text{span} \{X_{i+1}\}$ for $i = 2, \ldots, n-1$. There exists a unique simply connected Lie group $J^n$ such that $\mathfrak{g}$ is its Lie algebra of left-invariant vector fields. The Heisenberg group $(n = 3)$ and the Engel group $(n = 4)$ are very important examples of this family. The group $J^n$ is a Carnot group with horizontal distribution of Goursat-type.

The name Goursat distribution is related to the work [24], in which Goursat popularized these distributions. Goursat’s predecessors were Engel and Cartan. These distributions are quite important since they are examples of sub-Riemannian manifolds of step higher than two. Goursat distributions of dimension $n$ can be obtained by the $(n-2)$-fold prolongation of a two-dimensional surface, see [33].

A particular physical problem described by the distribution (1.4) is the motion of electric charges in certain static inhomogeneous magnetic fields, see for instance [8]. A different physical problem which is also related to Goursat distributions is the so-called $N$-trailer problem, which consists of steering a robot with a number $N$ of trailers, see [22], [25], [26], [27], [28], [29], [30], [31], [23], [36], [37], [38] and [39], among many others.
Let us now describe the main results of this paper. For \( i = 1, \ldots, n \), let \( h_i \) be the linear on fiber function given by \( h_i(\lambda) = \langle \lambda, X_i \rangle \), with \( \lambda \in T^*\mathbb{R}^n 
). 

**Theorem 1.1.** Let \( \gamma : [0, T] \to \mathbb{R}^n \) be an ample and equiregular geodesic in \( \mathbb{J}^n \), \( n \geq 3 \), with \( \gamma(0) = x_0 \) and initial covector \( \lambda(0) = \lambda_0 \in T_{x_0}^*\mathbb{R}^n \). Then, we have the following explicit expansion of \( d_{x_0}^2 \dot{c}_t \) as \( t \to 0 \),

\[
\tag{1.5}
\frac{d^2}{dt^2} c_t |_{\mathcal{D}} \quad = \quad \frac{1}{t^2} \mathcal{L}_{\lambda_0} + \frac{1}{3} \mathcal{R}_{\lambda_0} + O(t)
\]

\[
= \quad \frac{1}{t^2} \left[ (n-1)^2 0 \quad 0 \right] + \frac{1}{3} \left[ \frac{3(n-1)}{4(n-1)^2-1} R_{11}(\lambda_0) \quad 0 \right] + O(t),
\]

in a suitable orthonormal basis of \( \mathcal{D}_{x_0} \), where

\[
R_{11}(\lambda_0) \quad = \quad -\frac{1}{6} (n-1) (12 + n (4n - 17)) \left( h_3^2 + h_2 h_4 \right)
\]

\[
- (n-1)(n-2)(n-3) h_3^2 h_2^2,
\]

\[\tag{1.6}
\lambda
\]

Here we use the convention \( h_4 \equiv 0 \) when \( n = 3 \).

In the Heisenberg group all geodesics are ample and equiregular. The invariant \( \mathcal{R}_{\lambda_0} \) in this group, computed for the first time in the works \([2], [10]\), is given by

\[
\mathcal{R}_{\lambda_0} = \frac{2}{3} \left( \begin{array}{cc} h_3^2(\lambda_0) & 0 \\ 0 & 0 \end{array} \right).
\]

For \( \mathbb{J}^n \), \( n \geq 4 \), we obtain from Theorem 1.1 that \( \mathcal{R}_{\lambda_0} \) depends only on the Engel part of the distribution and on the dimension \( n \). Furthermore, notice that contrary to the Heisenberg group case, when \( n \geq 4 \), the quantity \( R_{11}(\lambda_0) \) changes sign and has a singularity when \( h_1(\lambda_0) = 0 \). In Theorem 4.2 we show, for instance, that an ample geodesic \( \gamma = \pi(\lambda) \) is equiregular at \( t \) if and only if \( h_1(\lambda(t)) \neq 0 \). Equiregular geodesics can be thought of as the microlocal counterpart of equiregular distributions.

The operator \( \mathcal{L}_{\lambda_0} \) carries important geometric information. More precisely, the trace of \( \mathcal{L}_{\lambda_0} \) is equal to the geodesic dimension of \( \mathbb{J}^n \). The geodesic dimension in \( \mathbb{J}^n \), for instance, is defined as follows. Let \( \Omega \subset \mathbb{R}^n \) be a bounded and measurable subset of positive volume and let \( \Omega_{x_0,t} \), for \( 0 \leq t \leq 1 \), be a family of subsets obtained from \( \Omega \) by the homothety of \( \Omega \) with respect to a fixed point \( x_0 \) along the shortest geodesics connecting \( x_0 \) with the points of \( \Omega \), so that \( \Omega_{x_0,0} = \{ x_0 \}, \Omega_{x_0,1} = \Omega \). The volume of \( \Omega_{x_0,t} \) has order \( t^{N_{x_0}} \), where \( N_{x_0} \) is the geodesic dimension at \( x_0 \) (see \([2], \text{Section 5.6}\) and also \([3], [4]\) for a more intrinsic and general definition of the geodesic dimension, which is also valid for general metric measure spaces).

The next natural candidate to study would be the Cartan group \( \mathfrak{E} \), with underlying manifold \( \mathbb{R}^5 \). The Lie group \( \mathfrak{E} \) is the free nilpotent group with five-dimensional nilpotent Lie algebra \( \mathfrak{L} = \text{span} \{ X_1, \ldots, X_5 \} \), whose generators satisfy

\[
[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5
\]

Notice that the distribution of \( \mathfrak{E} \) is not of Goursat-type.

For the Cartan group we show that the generalized sectional curvature is bounded from above by the energy integral of the pendulum, which is a first integral of the normal Hamiltonian system. For \( i = 1, \ldots, 5 \), let \( h_i \) be the linear on fiber function given by \( h_i(\lambda) = \langle \lambda, X_i \rangle \), with \( \lambda \in T^*\mathbb{R}^5 \). Then, the energy integral of the pendulum is given by

\[
E = \frac{h_2^2}{2} + h_1 h_5 - h_2 h_4.
\]

**Theorem 1.2.** Let \( \gamma : [0, T] \to \mathbb{R}^5 \) be an ample and equiregular geodesic, with \( \gamma(0) = x_0 \) and initial covector \( \lambda(0) = \lambda_0 \in T_{x_0}^*\mathbb{R}^5 \). We have the following explicit expansion of \( d_{x_0}^2 \dot{c}_t \) as \( t \to 0 \),

\[
\frac{d^2}{dt^2} c_t |_{\mathcal{D}} \quad = \quad \frac{1}{t^2} \mathcal{L}_{\lambda_0} + \frac{1}{3} \mathcal{R}_{\lambda_0} + O(t)
\]

\[
= \quad \frac{1}{t^2} \left[ 16 \quad 0 \quad 0 \right] + \frac{1}{3} \left[ \frac{1}{27} R_{11}(\lambda_0) \quad 0 \right] + O(t),
\]

\[
\mathcal{R}_{\lambda_0} = \frac{2}{3} \left( \begin{array}{cc} h_3^2(\lambda_0) & 0 \\ 0 & 0 \end{array} \right).
\]
in a suitable orthonormal basis of \( \mathcal{D}_{x_0} \), where

\[
R_{11}(\lambda_0) = 6E - \frac{8}{h_3^2} (h_1 h_4 + h_2 h_5)^2.
\]

In Proposition 6.1, we show that for an ample geodesic \( \gamma = \pi \circ \lambda \) the times \( t \) where \( h_3(\lambda(t)) = 0 \) are times of loss of equiregularity.

The function \( R_{11}(\lambda) \) in Eq. (1.6) and (1.7) is a symplectic invariant of the so-called Jacobi curve. Several authors have used the symplectic invariants associated with a Jacobi curve to get important new results in sub-Riemannian geometry. This set of invariants was first introduced by Agrachev and Gamkrelidze in [4], Agrachev and Zelenko in [7] and successively extended by Zelenko and Li in [41]. In the works [5] and [6], Agrachev and Lee made extensive use of these invariants to establish for three-dimensional Sasakian manifolds a generalized measure-contraction property, from which it follows: a volume doubling property, a local Poincaré inequality, a Harnack inequality for positive harmonic functions and a Liouville property. They were also able to prove Bishop and Laplacian comparison theorems on such three-dimensional manifolds. One has to mention here that the three-dimensional Heisenberg group is a very important example of a Sasakian manifold. Furthermore, in a recent paper [10], Barilari and Rizzi employed the invariants of the Jacobi curve to establish comparison theorems for conjugate points in sub-Riemannian manifolds.

In the works mentioned above the symplectic invariants have been computed explicitly only for sub-Riemannian manifolds of step two. In the present paper we provide for the first time explicit expressions of some of these invariants for sub-Riemannian manifolds of step higher than two. The importance of these expressions is that they might allow us to test whether or not results which are true in step two generalize to step three or higher.

Before describing the organization of the paper let us mention that in [15] Baudoin and Garofalo introduced a notion of \textit{curvature}, quite different from the one that appears here, for sub-Riemannian manifolds with \textit{transverse symmetries}. More precisely, they found a generalization of the curvature-dimension inequality from Riemannian geometry which is appropriate for a rich class of sub-Riemannian manifolds. In [20] Baudoin and Wang extended this notion of curvature to non-symmetric contact manifolds. In another recent paper, [19], Baudoin, Kim and Wang established a curvature-dimension inequality on Riemannian foliations with totally geodesic leaves. The interested reader might also consult the following list of references, among many others, for applications of the generalized curvature-dimension inequality: [12], [13], [14], [16], [17] and [18].

The organization of the paper is as follows. In sections 2 and 3 we recall several deep results of the manuscript [2]. For instance, in these sections we recall the construction of the \textit{curvature operator} and the concept of \textit{Jacobi curves}. In Section 4 we compute explicitly the symplectic invariant \( R_{aa,11} \) of the Jacobi curve in Carnot groups with horizontal Goursat distributions, and therefore, by a crucial result in [2] we obtain the explicit expression of the invariants \( I_3 \) and \( R_3 \). In Section 5 we analyze ample and equiregular geodesics in the Engel group. In Section 6 we investigate the quantities \( I_3 \) and \( R_3 \) in the Cartan group.

### 2. The Curvature Operator

In this section we recall the concept of \textit{curvature operator} in sub-Riemannian manifolds. This operator, as we have already mentioned, was recently introduced by Agrachev, Barilari and Rizzi in the paper [2].

**Definition 2.1.** Let \( M \) be a connected, smooth \( n \)-dimensional manifold. A sub-Riemannian structure on \( M \) is a pair \((U, f)\) where:

1. \( U \) is a smooth rank \( k \) Euclidean vector bundle with base \( M \) and fiber \( U_x \), i.e. for every \( x \in M \), \( U_x \) is a \( k \)-dimensional vector space endowed with an inner product.
2. \( f : U \to TM \) is a smooth linear morphism of vector bundles, i.e. \( f \) is linear on fibers and the following diagram is commutative:

\[
\begin{array}{ccc}
U & \xrightarrow{f} & TM \\
\pi_U & \searrow & \pi \\
& \downarrow & \\
& M & \\
\end{array}
\]

The maps \( \pi_U \) and \( \pi \) are the canonical projections of the vector bundles \( U \) and \( TM \), respectively.
Definition 2.2. The distribution \( \mathcal{D} \subset TM \) is the family of subspaces
\[ \mathcal{D} = \{ \mathcal{D}_x \}_{x \in M}, \quad \text{where} \quad \mathcal{D}_x = f(U_x) \subset T_x M. \]
The family of horizontal vector fields \( \mathcal{D} \subset \text{Vec}(M) \) is
\[ \mathcal{D} = \text{span}\{ f \circ \sigma, \sigma : M \to U \text{ is a smooth section of } U \}. \]

The sub-Riemannian geodesic is an admissible curve
\[ \langle \lambda, T \cdot \rangle \text{ denotes the action of the covector } \lambda \text{ on vectors}. \]

The sub-Riemannian Hamiltonian is written as
\[ H(\lambda) = \frac{1}{2} \sum_{i=1}^{k} \langle \lambda, X_i \rangle^2 \quad \forall \lambda \in T^* M, \]
where \( \langle \cdot , \cdot \rangle \) denotes the action of the covector \( \lambda \) on vectors. We recall that the definition of \( H \) in (2.2) is intrinsic and does not depend on the frame \( \{ X_1, \ldots, X_k \} \). For \( \lambda \in T^* M, T_\lambda (T^* M) \) is a symplectic vector space with the canonical symplectic form \( \sigma_\lambda \), defined as the differential of the Liouville form. Recall that the tautological (or Liouville) 1-form on \( T^* M \) is \( s \in \Lambda^1(T^* M) \), and it is defined as follows:
\[ s : \lambda \to s_\lambda \in T_\lambda^* (T^* M), \quad \langle s_\lambda, w \rangle = \langle \lambda, \pi_* w \rangle, \quad \lambda \in T^* M, w \in T_\lambda T^* M, \]
where \( \pi : T^* M \to M \) denotes the canonical projection.

Denote by \( \mathcal{F} \) the Hamiltonian vector field on \( T^* M \) associated with a function \( a \in C^\infty(T^* M) \). The vector field \( \mathcal{F} \) is given by the formula \( da = \sigma(\cdot, \mathcal{F}) \). For \( i = 1, \ldots, k \), let \( h_i \in C^\infty(T^* M) \) be the linear-on-fiber functions defined by \( h_i(\lambda) = \langle \lambda, X_i \rangle \). Notice that
\[ \mathcal{H} = \sum_{i=1}^{k} h_i h_i. \]

We recall now a weak version of the Pontryagin Maximum Principle (PMP) in the sub-Riemannian setting (see [28]). This principle tell us that trajectories minimizing the distance
between two points are solutions of first-order necessary conditions for optimality. For an elementary proof, the reader can consult the reference [1].

**Theorem 2.3.** Let \( \gamma : [0,T] \to M \) be a sub-Riemannian geodesic associated with a non-zero control \( u \in L^\infty([0,T],\mathbb{R}^k) \). Then there exists a Lipschitz curve \( \lambda : [0,T] \to T^*M \), such that \( \pi \circ \lambda = \gamma \), and only one of the following conditions holds for a.e. \( t \in [0,T] \):

1. \( \dot{\lambda}(t) = \vec{H}|_{\lambda(t)} \), with \( h_i(\lambda(t)) = u_i(t) \).
2. \( \dot{\lambda}(t) = \sum_{i=1}^k u_i(t) \vec{h}_i|_{\lambda(t)} \), \( h_i(\lambda(t)) = 0 \).

If \( \lambda : [0,T] \to M \) is a solution of (1) (resp. (2)) it is called a normal (resp. abnormal) extremal. It is well known that if \( \lambda(t) \) is a normal extremal, then its projection \( \gamma(t) := \pi(\lambda(t)) \) is a smooth geodesic. This does not hold in general for abnormal extremals. Notice that extremals satisfying (1) are simply integral lines of the Hamiltonian field \( \vec{H} \). Thus, let \( \lambda(t) = e^{t\vec{H}}(\lambda_0) \) denote the integral line of \( \vec{H} \), with initial condition \( \lambda(0) = \lambda_0 \). On the other hand, a geodesic can be at the same time normal and abnormal, namely it admits distinct extremals, satisfying (1) and (2). In the Riemannian setting there are no abnormal extremals.

**Definition 2.4.** A normal extremal trajectory \( \gamma : [0,T] \to M \) is called strictly normal if it is not abnormal. Moreover, if for all \( s \in [0,T] \) the restriction \( \gamma|_{[0,s]} \) is strictly normal, then \( \gamma \) is called strongly normal.

2.1. **Geodesic Flag and Young Diagram.** Let \( \gamma : [0,T] \to M \) be a smooth admissible curve such that \( \gamma(0) = x_0 \). By definition, this means that \( \dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)} \) for all \( t \in [0,T] \). Consider a smooth horizontal extension of the tangent vector, namely a horizontal vector field \( T \in \mathcal{D} \) such that \( T|_{\gamma(t)} = \dot{\gamma}(t) \).

For each smooth admissible curve, we consider a family of subspaces, which is related with a micro-local characterization of the sub-Riemannian structure along the trajectory itself.

**Definition 2.5 ([2]).** The flag of the admissible curve \( \gamma \) is the sequence of subspaces

\[ \mathcal{F}_i^\gamma(t) = \text{span}\{ \mathcal{L}_T(X)|_{\gamma(t)} : X \in \mathcal{D}, j \leq i - 1 \} \subset T_{\gamma(t)}M, \]

where \( \mathcal{L}_T \) denotes the Lie derivative in the direction of \( T \).

Notice that, by definition, this is a filtration of \( T_{\gamma(t)}M \), i.e. \( \mathcal{F}_i^\gamma(t) \subset \mathcal{F}_{i+1}^\gamma(t) \), for all \( i \geq 1 \). Moreover, \( \mathcal{F}_1^\gamma(t) = \mathcal{D}_{\gamma(t)} \).

**Definition 2.6 ([2]).** Let \( k_i(t) = \dim \mathcal{F}_i^\gamma(t) \). The growth vector of the admissible curve \( \gamma \) is the sequence of integers \( \mathcal{G}_\gamma(t) = \{ k_1(t), k_2(t), \ldots \} \).

**Definition 2.7 ([2]).** Let \( \gamma : [0,T] \to M \) be an admissible curve, with growth vector \( \mathcal{G}_\gamma(t) \). We say that the geodesic is:

1. Equiregular at \( t \) if its growth vector is locally constant at \( t \).
2. Ample at \( t \) if there exists an integer \( m = m(t) \) such that \( \mathcal{F}_m^\gamma(t) = T_{\gamma(t)}M \). We call the minimal \( m(t) \) such that the curve is ample the step at \( t \) of the admissible curve.

Finally, an admissible curve is ample (resp. equiregular) if it is ample (resp. equiregular) at each \( t \in [0,T] \).

We stress that equiregular (resp. ample) geodesics are the microlocal counterpart of equiregular (resp. bracket-generating) distributions.

Let

\[ d_i = \dim \mathcal{F}_i^\gamma(t) - \dim \mathcal{F}_{i-1}^\gamma(t), \]

for \( i \geq 1 \), be the increment of dimension of the flag of the geodesic at each step (with the convention \( k_0 \equiv 0 \)). For an equiregular geodesic we have \( d_1 \geq d_2 \geq \ldots \geq d_m \). This result was proved in [2, Appendix E]. Moreover, if the geodesic is also ample we have \( \sum_{i=1}^m d_i = n \).

For any ample and equiregular geodesic \( \gamma : [0,T] \to M \), we draw a Young diagram \( D \) with \( d_i \) blocks in the \( i \)-th column, with \( i \geq 1 \), and we define \( n_1, \ldots, n_k \) as the lengths of its rows (that may depend on \( \gamma \)).
2.2. Curvature operator. In order to carry out the construction of the curvature operator done in [2], we recall the definitions of geodesic cost and of second differential of a function.

**Definition 2.8.** The geodesic cost associated with a strongly normal geodesic \( \gamma : [0,T] \rightarrow M \) with \( \gamma(0) = x_0 \) is the family of functions
\[
c_t(x) = -\frac{1}{2t}d^2(x, \gamma(t)), \quad x \in M, t > 0.
\]

The following theorem, whose proof can be found in [3], gives us the regularity of the geodesic cost, and more importantly, it will allow us to define its Hessian.

**Theorem 2.9.** Let \( x_0 \in M \) and \( \gamma : [0,T] \rightarrow M \) be a strongly normal geodesic with \( \gamma(0) = x_0 \) and initial covector \( \lambda_0 \). Then, there exists \( \epsilon > 0 \) and an open set \( U \subset (0, \epsilon) \times M \) such that
\begin{enumerate}
  \item \( (t,x_0) \in U \) for all \( t \in (0, \epsilon) \),
  \item the function \( (t,x) \rightarrow c_t(x) \) is smooth on \( U \),
  \item For any \( (t,x) \in U \), the covector \( \lambda_t = d_xc_t \) is the initial covector of the unique minimizing geodesic connecting \( x \) with \( \gamma(t) \) in time \( t \).
\end{enumerate}

In particular, \( d_{x_0}c_t = \lambda_0 \) and \( x_0 \) is a critical point for the function \( c_t = \frac{d}{dt}c_t \) for every \( t \in (0, \epsilon) \).

Now, consider a smooth function \( f : M \rightarrow \mathbb{R} \). Its first differential at a point \( x \in M \) is the linear map \( d_xf : T_xM \rightarrow \mathbb{R} \). The second differential of \( f \) is well defined only at a critical point, i.e., at those points \( x \) such that \( d_xf = 0 \). Indeed, in this case the map
\[
d_x^2f : T_xM \times T_xM \rightarrow \mathbb{R}, \quad d_x^2f(v,w) = V(W)(x),
\]
where \( V,W \) are vector fields such that \( V(x) = v \) and \( W(x) = w \), respectively, is a well defined symmetric bilinear form that does not depend on the choice of the extensions. The associated quadratic form, that we denote by the same symbol \( d_x^2f : T_xM \rightarrow \mathbb{R} \), is defined by
\[
d_x^2f(v) = \frac{d^2}{dt^2}f(\alpha(t)), \quad \text{where} \quad \alpha(0) = x \text{ and } \dot{\alpha}(0) = v.
\]

By Theorem [2.9] for every \( t \in (0, \epsilon) \), the function \( x \rightarrow c_t(x) \) has a critical point at \( x_0 \). Remember that \( \lambda_0 \) is the initial covector of the geodesic \( \gamma \). Now, using the inner product \( g \) on \( \mathcal{D}_x \) we can associated with \( d_{x_0}^2\dot{c}_t(v) \) a family of symmetric operators on the distribution \( \mathcal{Q}_{\lambda_0}(t) \) defined by the identity
\[
d_{x_0}^2\dot{c}_t(v) = g(\mathcal{Q}_{\lambda_0}(t)v,v)\lambda_0, \quad t \in (0, \epsilon), v \in \mathcal{D}_{x_0}.
\]

The following theorems, namely Theorem 2.10, Theorem 2.12 and Theorem 2.13 are among the main results in the already mentioned paper [2]. These important results are valid for a wide class of optimal control problems.

**Theorem 2.10.** Let \( \gamma : [0,T] \rightarrow M \) be an ample geodesic at \( t = 0 \), with initial covector \( \lambda_0 \in T_x^*M \). The map \( t \rightarrow t^2\mathcal{Q}_{\lambda_0}(t) \) can be extended to a smooth family of operators on \( \mathcal{D}_{x_0} \) for small \( t \geq 0 \), symmetric with respect to \( g \). Moreover,
\[
\mathcal{I}_{\lambda_0} \doteq \lim_{t \rightarrow 0^+} t^2 \mathcal{Q}_{\lambda_0}(t) \geq I > 0
\]
as operators on \( (\mathcal{D}_{x_0}, g_{x_0}) \). Finally,
\[
\frac{d}{dt} \bigg|_{t=0} t^2 \mathcal{Q}_{\lambda_0}(t) = 0.
\]

**Definition 2.11.** The curvature operator \( \mathcal{R}_{\lambda_0} : \mathcal{D}_{x_0} \rightarrow \mathcal{D}_{x_0} \) at \( \lambda_0 \in T_x^*M \) is defined by
\[
\mathcal{R}_{\lambda_0} \doteq \frac{3}{2} \frac{d^2}{dt^2} \bigg|_{t=0} t^2 \mathcal{Q}_{\lambda_0}(t).
\]

Moreover, the Ricci curvature at \( \lambda_0 \in T_x^*M \) is defined by \( \text{Ric}(\lambda_0) \doteq \text{tr} \mathcal{R}_{\lambda_0} \).

In particular, we have the following Laurent expansion for the family of symmetric operators \( \mathcal{Q}_{\lambda_0}(t) : \mathcal{D}_{x_0} \rightarrow \mathcal{D}_{x_0} \)
\[
\mathcal{Q}_{\lambda_0}(t) = \frac{1}{t^2} \mathcal{I}_{\lambda_0} + \frac{1}{3} \mathcal{R}_{\lambda_0} + O(t) \quad t > 0.
\]

**Theorem 2.12.** Let \( \gamma : [0,T] \rightarrow M \) be an ample and equiregular geodesic. Then, the symmetric operator \( \mathcal{I}_{\lambda_0} : \mathcal{D}_{x_0} \rightarrow \mathcal{D}_{x_0} \) satisfies
\begin{enumerate}
  \item \text{spec} \( \mathcal{I}_{\lambda_0} = \{ n_1^2, \ldots, n_k^2 \} \),
  \item \text{tr} \( \mathcal{I}_{\lambda_0} = \sum_{i=1}^{k} n_i^2 \).
\end{enumerate}
where \( n_1, \ldots, n_k \) are the lengths of the rows of the associated Young diagram.

Let \( \Delta_{\mu} \) be the sub-Laplacian associated with a smooth volume form \( \mu \). The next result is an explicit expression for the asymptotic of the sub-Laplacian of the squared distance from a geodesic, computed at the initial point \( x_0 \) of the geodesic \( \gamma \). Let \( \hat{f}_t() = \frac{1}{2}d^2(\cdot, \gamma(t)) \).

**Theorem 2.13.** Let \( \gamma \) be an ample and equiregular geodesic with initial covector \( \lambda_0 \in T^*_x M \). Assume also that \( \dim \mathcal{D} \) is constant in a neighborhood of \( x_0 \). Then there exists a smooth n-form \( \omega \) defined along \( \gamma \), such that for any volume form \( \mu \) on \( M \), \( \mu_{\gamma(t)} = e^{g(t)} \omega_{\gamma(t)} \), we have

\[
\Delta_{\mu} \hat{f}_t|_{x_0} = \text{tr} \mathcal{L}_\gamma - \dot{g}(0)t - \frac{1}{3} \text{Ric}(\lambda_0)^2 + O(t^2).
\]

In the following section we recall how to find the explicit expression of the curvature operator \( \mathcal{R}_{\lambda_0} \) when the geodesic \( \gamma \) is also equiregular. In [2] it was shown that in order to obtain such expression of the operator we need to compute certain symplectic invariants associated with the Jacobi curve.

3. **The Curvature Operator: Jacobi Curves**

In this section we recall the concept of Jacobi curve associated with a normal geodesic, which is a curve of Lagrangian subspaces in a symplectic vector space. We also introduce a key technical tool, the so-called canonical frame, associated with a monotone, ample, equiregular Jacobi curve.

Let \( (\Sigma, \sigma) \) be a \( 2n \)-dimensional symplectic vector space. A subspace \( \Lambda \subset \Sigma \) is called Lagrangian if it has dimension \( n \) and \( \sigma|_{\Lambda} = 0 \). The Lagrange Grassmannian \( L(\Sigma) \) is the set of all \( n \)-dimensional Lagrangian subspaces of \( \Sigma \).

Fix now \( \Lambda \in L(\Sigma) \). The tangent space \( T_{\Lambda}L(\Sigma) \) to the Lagrange Grassmannian at the point \( \Lambda \) can be canonically identified with the set of quadratic forms on the space \( \Lambda \) itself, namely

\[
T_{\Lambda}L(\Sigma) \simeq Q(\Lambda).
\]

Indeed, consider a smooth curve \( \Lambda(\cdot) \) in \( L(\Sigma) \) such that \( \Lambda(0) = \Lambda \), and denote by \( \dot{\Lambda} \in T_{\Lambda}L(\Sigma) \) its tangent vector. For any point \( z \in \Lambda \) and any smooth extension \( z(t) \in \Lambda(t) \), we define the quadratic form

\[
\dot{\Lambda} \equiv z \rightarrow \sigma(z, \dot{z}),
\]

where \( \dot{z} \equiv \dot{z}(0) \). A simple check shows that the definition does not depend on the extension \( z(t) \).

Let \( J(\cdot) \in L(\Sigma) \) be a smooth curve in the Lagrange Grassmannian. For \( i \in \mathbb{N} \), consider

\[
J^{(i)}(t) = \text{span} \left\{ \frac{d^{j}}{dt^{j}}\ell(t) \mid \ell(t) \in J(t), \ell(t) \text{ smooth, } 0 \leq j \leq i \right\} \subset \Sigma, \quad i \geq 0.
\]

**Definition 3.1.** The subspace \( J^{(i)}(t) \) is the \( i \)-th extension of the curve \( J(\cdot) \) at \( t \). The flag

\[
J(t) = J^{(0)}(t) \subset J^{(1)}(t) \subset J^{(2)}(t) \subset \ldots \subset \Sigma,
\]

is the associated flag of the curve at the point \( t \). The curve \( J(\cdot) \) is called:

1. **equiregular at** \( t \) **if** \( \text{dim} J^{(i)}(\cdot) \) **is locally constant at** \( t \), **for all** \( i \in \mathbb{N} \),
2. **ample at** \( t \) **if there exists** \( N \in \mathbb{N} \) **such that** \( J^{(N)}(t) = \Sigma \),
3. **monotone increasing (resp. decreasing) at** \( t \) **if** \( J(t) \) **is non-negative (resp. non-positive) as a quadratic form.**

The step of the curve at \( t \) is the minimal \( N \in \mathbb{N} \) such that \( J^{(N)}(t) = \Sigma \).

The Jacobi curve arises from the geometric interpretation of the second derivative of the geodesic cost. Thus, in order to define the Jacobi curve we need to recall the notion of second differential of a function \( f \in C^\infty(M) \) at non-critical points.

**Definition 3.2.** Let \( f \in C^\infty(M) \), and

\[
df : T^*M \to M, \quad df : x \to dx(f).
\]

Fix \( x \in M \), and let \( \lambda = dx(f) \in T^*M \). The second differential of \( f \) at \( x \in M \) is the linear map

\[
d^2_x f = dx(df) : T_x M \to T_{\lambda}(T^*M), \quad d^2_x f : v \to \left. \frac{d}{ds}\right|_{s=0} d_{\gamma(s)} f,
\]

where \( \gamma(\cdot) \) is a curve on \( M \) such that \( \gamma(0) = x \), and \( \dot{\gamma}(0) = v \).
Moreover, there exists a smooth normal moving frame distinct row but with same length, i.e. if and only if cally nonincreasing ample and equiregular curve of length set of all the boxes $a_i$ of the diagram. We employ letters from the beginning of the alphabet to denote superboxes. Notice that two boxes $a_i$ and $a_j$ are called levels of the Young diagram. The size of the level is called the size of the level. The flag of the Jacobi curve associated with $\alpha$ is ample of step $k$.

\begin{equation}
J(\lambda_0)(t) = \alpha(t), \quad (i.e. \lambda(t) \in T_\gamma(t)(M)).
\end{equation}

For any $\lambda \in T^*M$, $\pi(\lambda) = x$, we denote with the symbol $V_\lambda \equiv T_\lambda(T^*_x M)$ the vertical subspace at the point $\lambda \in T^*M$, i.e. the tangent space at $\lambda$ to the fiber $T^*_x M$. Observe that, if $\pi : T^*M \to M$ is the bundle projection, $V_\lambda = \ker \pi_*|T_\lambda(T^*_x M)$.

**Definition 3.3.** The Jacobi curve associated with $\gamma$ is the smooth curve $J(\lambda_0) : [0, T] \to L(T(\lambda_0)(T^*M))$ defined by

\begin{equation}
J(\lambda_0)(t) = d^2_{x_0}c (T_{x_0} M),
\end{equation}

for $t \in (0, T]$, and $J(\lambda_0)(0) = V_\lambda$.

Now, let $v \in T_{x_0}M$ and $\alpha$ a smooth curve such that $\alpha(0) = x_0$ and $\alpha(0) = v$. For s small enough, $d_{\alpha(s)}c (T_{\alpha(s)} M)$ is the initial covector of the unique normal geodesic that connects $\alpha(s)$ and $\gamma(t)$ in time $t$, or in other words, $\pi \circ e^{tH} \circ d_{\alpha(s)}c (T_{\alpha(s)} M) = \gamma(t)$. Then,

\begin{equation}
\pi_* \circ e^{tH} \circ d^2_{x_0}c(v) = \frac{d}{ds}|_{s=0} \pi \circ e^{tH} \circ d_{\alpha(s)}c (T_{\alpha(s)} M) = 0.
\end{equation}

Therefore, one can actually write

\begin{equation}
J(\lambda_0)(t) = e^{-tH}V_\lambda(t).
\end{equation}

Moreover, from Proposition 6.12 in [2], we have that the Jacobi curve $J(\lambda)$ is monotone decreasing for every $\lambda \in T^*M$.

The following proposition provides the connection between the flag of a normal geodesic and the flag of the associated Jacobi curve, see Proposition 6.15 in [2].

**Proposition 3.4.** Let $\gamma(t) = (\pi \circ \alpha)(t)$ be a normal geodesic associated with the initial covector $\lambda_0$. The flag of the Jacobi curve $J(\lambda_0)$ projects to the flag of the geodesic $\gamma$ at $t = 0$, namely

\begin{equation}
\pi_* J(\lambda_0)(0) = F^1(0), \quad \forall i \in \mathbb{N}.
\end{equation}

Moreover, $\dim J(\lambda_0)(t) = n - \dim F^1(t)$. Therefore $\gamma$ is ample of step $m$ (resp. equiregular) if and only if $J(\lambda_0)$ is ample of step $m$ (resp. equiregular).

In order to compute explicitly the expansion of the operator $Q(t)$ we need to recall the concept of canonical frame along the curve $J(\lambda_0)$. This frame generalizes the concept of parallel transport from Riemannian geometry to (sufficiently regular) sub-Riemannian extremals. The canonical frame was first introduced by Agrachev and Gamkrelidze in [3], Agrachev and Zelenko in [2] and successively extended by Zelenko and Li in [41].

Consider an ample, equiregular geodesic, with Young diagram $D$, with $k$ rows, of length $n_1, \ldots, n_k$. Indeed $n_1 + \ldots + n_k = n$. The Moving frame we are going to introduce is indexed by the boxes of the Young diagram, so we need to fix some terminology first. Each box is labelled “ai”, where $a = 1, \ldots, k$ is the row index, and $i = 1, \ldots, n_a$ is the progressive box number, starting from the left, in the specified row. Briefly, the notation $a_i \in D$ denotes the generic box of the diagram. We employ letters from the beginning of the alphabet $a, b, c, \ldots$ for rows, and letters from the middle of the alphabet $i, j, h, \ldots$ for the position of the box in the row.

We collect the rows with the same length in $D$, and we call them levels of the Young diagram. In particular, a level is the union of $r$ rows $D_1, \ldots, D_r$, and $r$ is called the size of the level. The set of all the boxes $a_i \in D$ that belong to the same column and the same level of $D$ is called superbox. We use greek letters $\alpha, \beta, \ldots$ to denote superboxes. Notice that two boxes $a_i, b_j$ are in the same superbox if and only if $a_i$ and $b_j$ are in the same column of $D$ and in possibly distinct row but with same length, i.e. if and only if $i = j$ and $n_a = n_b$.

**Theorem 3.5.** There exists a smooth normal moving frame $\{E_{a_i}(t), F_{a_i}(t)\}_{ai \in D}$ of a monotonically nonincreasing ample and equiregular curve $J(\cdot)$ with given Young diagram $D$, with $k$ rows, of length $n_a$, for $a = 1, \ldots, k$, such that

1. $J(t) = \text{span} \{E_{a_i}(t)\}_{ai \in D}$ for any $t$. 

(2) It is a Darboux basis, namely,

\[ \sigma(E_{ai}, E_{bj}) = \sigma(F_{ai}, F_{bj}) = \sigma(E_{ai}, F_{bj}) = \delta_{ab}\delta_{ij} = 0, \]

for every \( ai, bj \in D \).

(3) There exists a one-parametric family of \( n \times n \), with \( n_1 + \ldots + n_k = n \), symmetric matrices \( R(t) \), with components \( R_{ab,ij}(t) = R_{ba,ji}(t) \), indexed by the boxes of the Young diagram \( D \) such that the moving frame satisfies the structural equations

\[
\begin{align*}
\dot{E}_{ai} &= E_{a(i-1)}(t), & a = 1, \ldots, k, i = 2, \ldots, n_a, \\
\dot{E}_{a1} &= -F_{a1}(t), & a = 1, \ldots, k, \\
\dot{F}_{ai} &= \sum_{b=1}^{k} \sum_{j=1}^{n_b} R_{ab,ij}(t) E_{bj}(t) - F_{a(i+1)}(t), & a = 1, \ldots, k, i = 1, \ldots, n_a - 1, \\
\dot{F}_{an_a} &= \sum_{b=1}^{k} \sum_{j=1}^{n_b} R_{ab,n_a}(t) E_{bj}(t), & a = 1, \ldots, k.
\end{align*}
\]

The matrix \( R(t) \) is normal in the sense of [41]. See also [2, Appendix F], where the normal condition for \( R(t) \) is written explicitly in this notation.

Properties (1)-(3) uniquely define the frame up to orthogonal transformation that preserve the Young diagram. More precisely, if \( \{E_{ai}, F_{ai}\}_{ai \in D} \) is another smooth moving frame along \( \lambda(t) \) satisfying (1)-(3), for some family \( \tilde{R}(t) \), then for any superbox \( \alpha \) of size \( r \) there exists an orthogonal (constant) \( r \times r \) matrix \( O^\alpha \) such that

\[
\begin{align*}
\dot{E}_{ai} &= \sum_{bj \in \alpha} O^\alpha_{ai,bj} E_{bj}, & \tilde{F}_{ai} &= \sum_{bj \in \alpha} O^\alpha_{ai,bj} F_{bj}, & ai \in \alpha.
\end{align*}
\]

Any canonical Darboux frame \( \{E_{ai}, F_{ai}\}_{ai \in D} \) defines a Lagrangian splitting \( \Sigma_{\lambda_0} = V_{\lambda_0} \oplus H_{\lambda_0} \), where

\[
\begin{align*}
V_{\lambda_0} &= \text{span}\{E_{ai}(0)\}_{ai \in D}, \quad H_{\lambda_0} = \text{span}\{F_{ai}(0)\}_{ai \in D}.
\end{align*}
\]

Observe that \( V_{\lambda_0} = J_{\lambda_0}(0) = \ker \pi_\star |_{T_\lambda(T^*M)} \), and \( \pi_\star \) induces an isomorphism between \( H_{\lambda_0} \) and \( T_{x_0}M \). Now, the curve \( J_{\lambda_0}(t) \) is the graph of a linear map \( S(t) : V_{\lambda_0} \rightarrow H_{\lambda_0} \) for small \( t \geq 0 \). Equivalently, by [2, Lemma 6.3], for \( 0 < t < \varepsilon, J_{\lambda_0}(t) \) is the graph of \( S(t)^{-1} : H_{\lambda_0} \rightarrow V_{\lambda_0} \). We stress that the function \( S(t)^{-1} \) is defined only for \( t > 0 \) sufficiently small.

From the definition of second differential we have that if \( \alpha(\cdot) \) is a smooth arc with \( \alpha(0) = x_0 \) and \( \alpha'(0) = v \in T_{x_0}M \), then

\[
\pi_\star d^2\alpha \xi_Q = \frac{d}{ds} \bigg|_{s=0} \pi_\star \circ d\alpha(s) \xi_Q = \frac{d}{ds} \bigg|_{s=0} \alpha(s) = v.
\]

Fix \( v \in T_{x_0}M \) and let \( \tilde{v} \in H_{\lambda_0} \) be the unique horizontal lift such that \( \pi_\star \tilde{v} = v \). Hence, for \( t > 0 \) we have

\[
d_{x_0}^2 \xi_Q = S(t)^{-1} \tilde{v} + \tilde{v}.
\]

Then, by the standard identification \( V_{\lambda_0} \simeq T_{x_0}^*M \) and the fact that, for \( \xi \in V_\lambda \) and for any \( X \in T_\lambda(T^*M) \) then \( \sigma(\xi, X) = \langle \xi, \pi_\star X \rangle \), we finally get

\[
(3.3) \quad g(Q_{\lambda_0}(t)v, v) = \frac{d}{dt} \sigma(S(t)^{-1} \tilde{v}, \tilde{v}), \quad v \in \mathscr{D}_{x_0} \text{ and } t > 0.
\]

Since \( J_{\lambda_0}(0) = V_{\lambda_0} \), it follows that \( S^{-1}(t) \) is singular at \( t = 0 \). Now, let \( X_0 = \pi_\star F_{a1}(0) \in T_{x_0}M \). Then, from [2, Lemma 7.9] the set \( \{X_a\}_{a=1}^{k} \) is an orthonormal basis for \( (\mathscr{D}_{x_0}, g) \). Hence, if \( v = \sum_{a=1}^{k} v_a X_a \in \mathscr{D}_{x_0} \), we have \( \tilde{v} = \sum_{a=1}^{k} v_a F_{a1}(0) \). In Young diagram notation, we have

\[
S(t) = S(t)_{ab,ij}.
\]

Thus, we get from (3.3) that

\[
g(Q_{\lambda_0}(t)v, v) = \frac{d}{dt} \sum_{a,b=1}^{k} S(t)^{-1}_{ab,11} v_a v_b, \quad t > 0.
\]

For convenience, introduce for \( t > 0 \) the smooth family of \( k \times k \) matrices \( S^0(t)^{-1} \) defined by

\[
S^0(t)^{-1} = [S(t)^{-1}]_{ab,11}, \quad t > 0.
\]

Then, the quadratic form associated with the operator \( Q_{\lambda_0} : \mathscr{D}_{x_0} \rightarrow \mathscr{D}_{x_0} \) via the Hamiltonian inner product is represented by the matrix \( \frac{d}{dt} S^0(t)^{-1} \). The following crucial result, see [2, Corollary 7.5], connects the curvature operator with the invariants of the Jacobi curve, since it gives
the asymptotic expansion of $S^g(t)^{-1}$ in terms of the symplectic invariants $R(t)$ of the canonical frame.

**Theorem 3.6.** Let $\gamma(\cdot)$ be an ample and equiregular geodesic with a given Young diagram $D$ with $k$ rows, of length $n_a$, for $a = 1, \ldots, k$. Then, for $0 < |t| < \varepsilon$

\begin{equation}
S^g(t)^{-1} = -\delta_{ab} \frac{n_a^2}{t} + R_{ab,11}(0) \Omega(n_a,n_b) t + O(t^2),
\end{equation}

where

\begin{equation}
\Omega(n_a,n_b) = \begin{cases} 
0, & |n_a - n_b| \geq 2 \\
\frac{1}{4(n_a+n_b)}, & |n_a - n_b| = 1 \\
\frac{4n_a^2 - 1}{4n_a^2 - T^2}, & n_a = n_b.
\end{cases}
\end{equation}

4. **Carnot groups with horizontal distribution of Goursat type**

In the following, we will compute the part of the canonical curvature $R(t)$ that is relevant for computing the curvature operator for Carnot groups $\mathbb{J}^n, n \geq 3$, with horizontal distribution of Goursat-type. These groups are $n$-dimensional Carnot groups of $(n-1)$-step with two dimensional horizontal sub-bundle.

In $\mathbb{R}^n$ with coordinates $(x, y_0, y_1, \ldots, y_{n-2})$, we consider the vector fields

\[
X_1 = \frac{\partial}{\partial x}, \quad X_{i+2} = \sum_{j=1}^{n-2} \frac{x^{j-i}}{(j-i)!} \frac{\partial}{\partial y_j}, \quad i = 0, \ldots, n-2.
\]

The vector fields $X_1$ and $X_2$ are called horizontal and satisfy the Hörmander condition, i.e. they generate the whole tangent bundle by their commutators:

\[
[X_1, X_i] = X_{i+1}, \quad \text{for } i = 2, \ldots, n-1,
\]

and all other commutators being zero.

We can introduce a unique Lie group structure (law of multiplication) in $\mathbb{R}^n$, making a Lie group of $\mathbb{R}^n$, so that $X_1, \ldots, X_n$ become basic left-invariant fields on this Lie group. We denote this Lie group by $\mathbb{J}^n$. We have that $\mathcal{D}|h = \{X_1, X_2\}|_h$, for all $h \in \mathbb{J}^n$, is a distribution left-invariant by the action $L_g : \mathbb{J}^n \to \mathbb{J}^n$. Any left-invariant scalar product $g$ on $\mathcal{D}|_h$ induces a left-invariant sub-Riemannian structure $(\mathcal{D}, g)$ on $\mathbb{J}^n$. Since any two different choices give rise to isometric sub-Riemannian structures, we choose without loss of generality $g$ such that $X_1$ and $X_2$ are orthonormal.

Now let $\{v_1, \ldots, v_n\}$ be the dual frame of $\{X_1, \ldots, X_n\}$. This dual frame induces coordinates $\{h_1, \ldots, h_n\}$ in each fiber of $T^* \mathbb{R}^n$,

\[
\lambda = (h_1, \ldots, h_n) \iff \lambda = h_1 v_1 + \cdots + h_n v_n,
\]

where $h_i(\lambda) = \langle \lambda, X_i \rangle$ are the linear-on-fibers functions associated with $X_i$, for $i = 1, \ldots, n$.

Let $\vec{h}_i \in \text{Vec}(T^* \mathbb{R}^n)$ be the Hamiltonian vector fields associated with $h_i \in C^\infty(T^* \mathbb{R}^n)$ for $i = 1, \ldots, n$, respectively. Consider the vertical vector fields $\partial_{h_i} \in \text{Vec}(T^* \mathbb{R}^n)$, for $i = 1, \ldots, n$. The vector fields

\[
\vec{h}_1, \ldots, \vec{h}_n, \partial_{h_1}, \ldots, \partial_{h_n},
\]

are a local frame of vector fields of $T^* \mathbb{R}^n$. Equivalently, we can introduce the cylindrical coordinates $\rho, \theta, h_3, \ldots, h_n$ on each fiber of $T^* \mathbb{R}^n \setminus \{0\}$ by

\[
h_1 = \rho \cos \left( \theta + \frac{\pi}{2} \right), \quad h_2 = \rho \sin \left( \theta + \frac{\pi}{2} \right),
\]

with $\rho \in (0, +\infty)$ and $\theta \in (-\pi, \pi]$, and employ instead the local frame

\[
\vec{h}_1, \ldots, \vec{h}_n, \partial_{\theta}, \partial_{h_3}, \ldots, \partial_{h_n},
\]

where $\partial_{\theta} \equiv h_1 \partial_{h_2} - h_2 \partial_{h_1}$. Finally, let the Euler vector field be given by

\[
\epsilon \equiv \sum_{i=1}^{n} h_i \partial_{h_i} = \rho \partial_{\rho} + \sum_{i=3}^{n} h_i \partial_{h_i}.
\]

Notice that $\epsilon$ is a vertical vector field on $T^* \mathbb{R}^n$, i.e. $\pi_* \epsilon = 0$, and is the generator of the dilations $\lambda \mapsto e^\varepsilon \lambda$ along the fibers of $T^* \mathbb{R}^n$. 
Notice that the symplectic form $\sigma$ in the cylindric coordinates $\rho, \theta, h_3, \ldots, h_n$ has the following expression:

$$
\sigma = \rho d\rho \wedge d\theta - \rho^2 d\theta \wedge d\nu_0 + \sum_{i=3}^n dh_i \wedge \nu_i + \rho^2 h_3 \nu_0 \wedge d\nu_0 + \sum_{i=4}^n h_i (h_2 \nu_{i-1} \wedge \nu_0 + h_1 \nu_{i-1} \wedge \nu_0).
$$

where $\{\nu_0, \nu_0\}$ is the dual co-frame associated with the frame $\{X_\theta, X_\bar{\theta}\}$, where $X_\bar{\theta} = h_1 X_1 + h_2 X_2$ and $X_\theta = h_2 X_1 - h_1 X_2$. If we use the formula $dH = \sigma(\cdot, \bar{H})$, we can write the explicit expression for the Hamiltonian vector field $\bar{H}$ as follows:

$$(4.2) \quad \bar{H} = X_\bar{\theta} + h_3 \partial_\theta + \sum_{i=3}^{n-1} h_i h_{i+1} \partial_{h_i}.$$

The following proposition will be useful for the characterization of ample/equiregular geodesics in $\mathbb{J}^n$ and the Cartan group $\mathcal{C}$. For a proof see [2] Proposition 3.12 and [21] Section 1.3.

**Proposition 4.1.** For any smooth geodesic $\gamma : [0, T] \to M$, on a real-analytic structure, such as Carnot groups, we have that the following properties are equivalent: ample at 0, ample at $t$, and suppose that it is given by $\gamma(0) = 0$, then $h_1 \equiv h_3 \equiv 0$. In this case, the geodesic is not ample, and we are in the case of an abnormal geodesic.

**Theorem 4.2.** Let $\gamma = \pi \circ \lambda : [0, T] \to \mathbb{R}^n, n \geq 4$, be a normal geodesic. Then,

1. If there is no time $t$ such that $h_1(t)$ and $h_3(t)$ are both zero, then $\gamma$ is equiregular at a time $t \in [0, T]$ if and only if $h_1(t) \neq 0$. Moreover, $\gamma$ is ample and the geodesic growth vector is given by

$$(4.3) \quad \mathcal{G}_\gamma(t) = \begin{cases} (2, 3, 4, \ldots, j, j + 1, \ldots, n) & \text{if } h_1(t) \neq 0, \\ (2, 3, 4, 4, \ldots, j, j, \ldots, n) & \text{if } h_1(t) = 0. \end{cases}$$

2. If there is a point $l \in [0, T]$ such that $h_1(l) = h_3(l) = 0$, then $h_1 \equiv h_3 \equiv 0$. In this case, the geodesic is not ample, and we are in the case of an abnormal geodesic.

**Proof.** Recall that the geodesic equations in the group $\mathbb{J}^n$ are:

$$(4.4) \quad \dot{h}_1 = -h_2 h_3, \quad \dot{h}_i = h_1 h_{i+1}, \quad i = 2, \ldots, n - 1, \quad \dot{h}_n = 0.$$

(1) Let us now consider the case when there is no time $t$ such that $h_1(t)$ and $h_3(t)$ are both zero. Recall that an admissible extension of $\dot{\gamma}_1$, where

$$
\dot{\gamma}_1(t) = h_1(t) X_1|_{\gamma(t)} + h_2(t) X_2|_{\gamma(t)},
$$

is a smooth vector field $T$, of the form $T = v_1 X_1 + v_2 X_2$, where $v_i \in C^\infty(\mathbb{R}^n)$ with $v_i(\gamma(t)) = h_i(t)$ for $i = 1, 2$. Let $X \in \mathcal{G}$ and suppose that it is given by $X = w_1 X_1 + w_2 X_2$, where $w_i \in C^\infty(\mathbb{R}^n)$ for $i = 1, 2$. It is easy to see that for $j \geq 1$,

$$
\mathcal{L}_{T}^j(X)|_{\gamma(t)} = w_1(\gamma(t)) \mathcal{L}_{T}^1(X_1)|_{\gamma(t)} + w_2(\gamma(t)) \mathcal{L}_{T}^2(X_2)|_{\gamma(t)} \mod \mathcal{F}_\gamma(t).
$$

A simple computation gives

$$(4.5) \quad \mathcal{L}_{T}(X_1)|_{\gamma(t)} = -h_2 X_3 \mod \mathcal{G}_\gamma(t),$$

$$(4.6) \quad \mathcal{L}_{T}(X_2)|_{\gamma(t)} = h_1 X_3 \mod \mathcal{G}_\gamma(t).$$

(a) Assume that $h_1(t) \neq 0$. By using the geodesic equations (4.4) together with (4.5) and (4.6), we can easily prove by induction that

$$
\mathcal{L}_{T}^j(X_1)|_{\gamma(t)} = -h_1^{j-1}(t) h_2(t) X_{j+2}|_{\gamma(t)} \mod \mathcal{F}_\gamma(t)
$$

and

$$
\mathcal{L}_{T}^j(X_2)|_{\gamma(t)} = h_1^j(t) X_{j+2}|_{\gamma(t)} \mod \mathcal{F}_\gamma(t),
$$

where

$$
\mathcal{F}_\gamma(t) = \text{span}\{X_1, \ldots, X_{j+1}\},
$$

and $1 \leq j \leq n - 2$. Therefore, $\mathcal{F}_{\gamma}^{n-1}(t) = T_{\gamma(t)} \mathbb{R}^n$, and the geodesic growth vector for the case $h_1(t) \neq 0$ is given by

$$
\mathcal{G}_\gamma(t) = (2, 3, 4, \ldots, j, j + 1, \ldots, n).
$$
(b) Assume that \( h_1(t) = 0 \). By equation (4.3) we have that
\[
\mathcal{R}_\gamma^2(t) = \text{span}\{X_1, X_2, X_3\}.
\]
In this case we obtain from the geodesic equations (4.4) the following formulas:
\[
\begin{align*}
\mathcal{L}_T^j(X_1)|_{\gamma(t)} &= (-1)^j h_1(t) h_2(t) h_3^{-1}(t) X_{j+3} |_{\gamma(t)} \mod \mathcal{R}_\gamma^2(t) \\
\mathcal{L}_T^{j+1}(X_1)|_{\gamma(t)} &= (-1)^{j+1} h_2(t) h_3(t) X_{j+3} |_{\gamma(t)} \mod \mathcal{R}_\gamma^{j+1}(t),
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{L}_T^j(X_2)|_{\gamma(t)} &= (-1)^j h_2^2(t) h_3(t) X_{j+2} |_{\gamma(t)} \mod \mathcal{R}_\gamma^j(t) \\
\mathcal{L}_T^{j+1}(X_2)|_{\gamma(t)} &= (-1)^j h_1(t) h_2(t) h_3(t) X_{j+3} |_{\gamma(t)} \mod \mathcal{R}_\gamma^{j+1}(t),
\end{align*}
\]
where
\[
\mathcal{R}_\gamma^j(t) = \text{span}\{X_1, \ldots, X_{j+2}\}, \quad \mathcal{R}_\gamma^{j+1}(t) = \mathcal{R}_\gamma^j(t),
\]
and \( j \geq 1 \). From these formulas, which are easily proved by induction, we obtain that the growth vector of the geodesic at times \( t \), with \( h_1(t) = 0 \) and \( h_3(t) \neq 0 \), is
\[
\mathcal{G}_\gamma(t) = (2, 3, 3, 4, 4, \ldots, j, \ldots, n).
\]
Notice that, if there is no time \( \bar{t} \) such that \( h_1(\bar{t}) = h_3(\bar{t}) = 0 \) then, by the geodesic equations, the set of times at which \( h_1(t) = 0 \) is discrete, and \( \gamma \) loses equiregularity precisely at these times. Hence, \( \gamma \) is equiregular at \( t \in [0, T] \) if and only if \( h_1(t) \neq 0 \). From (4.3) we also have that the geodesic is ample.

(2) Assume that there is a point \( \bar{t} \in [0, T] \) such that \( h_1(\bar{t}) = h_3(\bar{t}) = 0 \). From the geodesic equations (4.4), the solution \( \tilde{\lambda}(t) \) of \( \tilde{\lambda} = \tilde{H}(\tilde{\lambda}) \), with initial condition
\[
\tilde{\lambda}(0) = (0, h_2(\bar{t}), h_4(\bar{t}), \ldots, h_n(\bar{t})),
\]
is constant \( \tilde{\lambda}(t) \equiv \tilde{\lambda}(0) \). Since the flag of the admissible curve \( \gamma \) does not depend on the extension \( T \), we can assume that \( T = v_2 X_2 \), with \( v_2 \equiv \text{const} \). Once again from the geodesic equations (4.4) we obtain that
\[
\begin{align*}
\mathcal{L}_T^j(X_1)|_{\gamma(t)} &\in \mathcal{R}_\gamma^j(t) = \text{span}\{X_1, X_2, X_3\}, \\
\mathcal{L}_T^j(X_2)|_{\gamma(t)} &\in \mathcal{R}_\gamma(t),
\end{align*}
\]
for \( j \geq 2 \). Therefore, since \( n \geq 4 \), the geodesic \( \gamma \) is not ample, and hence abnormal.

Let \( \lambda_0 \) be the initial covector associated to an ample and equiregular unit-speed geodesic \( \gamma \) with Young diagram \( D \), which by Theorem 4.2 is
\[
D = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_a} \\ b_{n_a} & \end{bmatrix}
\]
with \( n_a = n - 1 \) and \( n_b = 1 \). For such Young diagram, a canonical frame is a smooth family
\[
\{E_{a_1}, E_{a_2}, \ldots, E_{a_{n_a}}, E_{b_1}, F_{a_1}, F_{a_2}, \ldots, F_{a_{n_a}}, E_{b_1}\} \in T_{\lambda_0} (T^* \mathbb{R}^n),
\]
with the following properties:

(1) It is attached to the Jacobi curve, namely
\[
J_{\lambda_0}(t) = \text{span}\{E_{a_1}(t), E_{a_2}(t), \ldots, E_{a_{n_a}}(t), E_{b_1}(t)\}.
\]

(2) From [11] Lemma 5.7 we have:
\[
E_{b_1}(t) = e^{-t\tilde{H}} \epsilon = \epsilon - t\tilde{H},
\]
as a consequence all curvatures \( R_{ab,a} \), (where * is any other index) vanish.

(3) In this “easy” Young diagram case, the normal condition means that the matrix \([R_{ab,ij}])_{i,j=1,\ldots,n_a}
is diagonal.
(4) They satisfy the structural equations:

\[ \dot{E}_{ai} = E_{a(i-1)}(t), \quad i = 2, \ldots, n_a, \]

\[ \dot{E}_{a1} = -F_{a1}(t), \]

\[ \dot{E}_{b1} = -F_{b1}(t), \]

\[ \dot{F}_{ai} = R_{aa,ai}(t)E_{ai}(t) - F_{ai(i+1)}(t), \quad i = 1, \ldots, n_a - 1, \]

\[ \dot{F}_{a_{na}} = R_{aa,n_a,n_a}(t)E_{a_{na}}(t), \]

\[ \dot{F}_{b1} = 0. \]

Finally, Eq. (4.8) follows easily.

and, by choosing the positive sign, is given by

\[ \dot{F}_{ai} = R_{aa,ai}(t)E_{ai}(t) - F_{ai(i+1)}(t), \quad i = 1, \ldots, n_a - 1, \]

\[ \dot{F}_{a_{na}} = R_{aa,n_a,n_a}(t)E_{a_{na}}(t), \]

\[ \dot{F}_{b1} = 0. \]

We compute the canonical frame following the general algorithm in [41]. The computation is presented through a series of lemmas. We start by computing some very useful identities.

**Lemma 4.3.** The following identities hold true:

(4.8) \[ [\vec{H}, X_{\theta}] = -X_3 + h_3 X_{\theta}, \]

(4.9) \[ [\vec{H}, \partial_\theta] = \begin{cases} X_{\theta} & \text{for } n = 3, \\ X_{\theta} + h_2 \sum_{i=3}^{n-1} h_i \partial_{h_i} & \text{for } n \geq 4, \end{cases} \]

(4.10) \[ [\vec{H}, \partial_{h_3}] = -\partial_{h_3}, \]

(4.11) \[ [\vec{H}, \partial_{h_i}] = -h_i \partial_{h_{i-1}} & \text{for } i = 4, \ldots, n, \]

(4.12) \[ [\vec{H}, \epsilon] = -\vec{H}. \]

Proof. Let us begin with Eq. (4.9) when \( n \geq 4 \). Recall that \( \partial_{\theta} = h_1 \partial_{h_2} - h_2 \partial_{h_1} \). By Eq. (4.2), we obtain

\[ [\vec{H}, \partial_\theta] = -[h_1 \partial_{h_2} - h_2 \partial_{h_1}, h_1 X_1 + h_2 X_2] - \partial_{h_1} \sum_{i=3}^{n-1} h_{i+1} \partial_{h_i} \]

\[ = h_2 X_1 - h_1 X_2 + h_2 \sum_{i=3}^{n-1} h_{i+1} \partial_{h_i}. \]

For Eq. (4.10) we use once again the explicit expression of \( \vec{H} \) of Eq. (4.2) to obtain

\[ [\vec{H}, \partial_{h_3}] = [-h_2 h_3 \partial_{h_3} + h_1 h_3 \partial_{h_2}, \partial_{h_3}] = h_2 \partial_{h_3} - h_1 \partial_{h_2} = -\partial_{h_3}. \]

Formula (4.11) follows in similar fashion. For Eq. (4.12), we have

\[ [\vec{H}, \epsilon] = [h_1 \vec{h}_1 + h_2 \vec{h}_2, \sum_{i=1}^{n} h_i \partial_{h_i}] \]

\[ = -h_1 \vec{h}_1 - h_2 \vec{h}_2 + \sum_{i=1}^{n} h_i \partial_{h_i} \]

\[ = -h_1 \vec{h}_1 - h_2 \vec{h}_2. \]

Finally, Eq. (4.8) follows easily. \( \square \)

With this lemma in hand we can start computing the canonical frame. For convenience we employ the following notation:

\[ f^{(j)}(t) = \frac{d^{j}}{dt^{j}} f(t). \]

**Lemma 4.4.** \( E_{a_{na}}(t) \) is uniquely specified (up to a sign) by the following conditions

1. \( E_{a_{na}}(t) \in J_{a_{na}}(t), \)
2. \( E_{a_{na}}^{(i)}(t) \in J_{a_{na}}(t), \) for \( i = 1, \ldots, n_a - 1, \)
3. \( \sigma_{\lambda} \left( E_{a_{na}}^{(n-1)}, E_{a_{na}}^{(n-2)} \right) = 1, \)

and, by choosing the positive sign, is given by

\[ E_{a_{na}}(t) = e^{-it\vec{h}_1^2 - n\vec{h}_{n_a} \partial_{h_{n_a}}}. \]
Proof. Condition (1) and the definition of Jacobi curve $J_{\lambda_0}(t) = e^{-t\mathcal{H}}V_{\lambda(t)}$ imply that

$$E_{an_n}(t) = e^{-t\mathcal{H}} \left( \sum_{i=1}^{n} f_i(t)\partial_{h_i} \right),$$

for some smooth functions $f_i(t)$, with $i = 1, \ldots, n$. Consider for $i = 1, 2, \ldots, n$, the vector fields $V_i(t) = e^{-t\mathcal{H}} f_i(t)\partial_{h_i}$ and the vector spaces $V_i = \text{span}\{\partial_{h_1}, \ldots, \partial_{h_n}\}$. It is easy to see that for $i \geq 4$ and $0 \leq j \leq i - 3$, we can write by Eq. (4.13) in Lemma 4.3

$$(4.13) \quad V_i^{(j)}(t) = e^{-t\mathcal{H}} \left( (-1)^j f_i(t) h_1^{i-j} \partial_{h_{i-j}} \mod V_{i-j+1} \right).$$

Condition (2) is equivalent to $\pi_* \circ e^{t\mathcal{H}} \dot{E}_{an_n}(t) = 0$. Since the vector fields $\partial_{h_1}, \ldots, \partial_{h_n}$ are vertical, namely $\pi_*\partial_{h_i} = 0$ for $l = 1, \ldots, n$, we obtain the following two equations:

$$0 = \pi_* \circ e^{t\mathcal{H}} V_1(t) = -f_1 X_1 \quad \text{and} \quad 0 = \pi_* \circ e^{t\mathcal{H}} V_2(t) = -f_2 X_2.$$  

From this it immediately follows that $f_1 = f_2 \equiv 0$. Hence, $E_{an_n}(t) = \sum_{i=3}^{n} V_i(t)$. If we use Eq. (4.13) with $j = i - 3$ for $4 \leq i \leq n - 1$ together with the identities in Lemma 4.3 we get

$$\frac{d^2}{dt^2} V_i^{(i-3)}(t) = e^{-t\mathcal{H}} \left( (-1)^{i-2} f_i(t) h_1^{i-3} \partial_{h_3} \mod V_3 \right).$$

Hence, Condition (2) implies that $E_{an_n} = V_n = f_n \partial_{h_n}$, for some function $f_n$.

From Eq. (4.13) with $i = n$ and $j = n - 4$, we have after some computations

$$E_{an_n}^{(n-2)}(t) = e^{-t\mathcal{H}} \left( (-1)^{n-2} f_n(t) h_1^{n-3} \partial_{h_3} \mod V_3 \right),$$

and

$$E_{an_n}^{(n-1)}(t) = e^{-t\mathcal{H}} \left( (-1)^{n-2} f_n(t) h_1^{n-3} \partial_{h_3} \mod V_3 \right).$$

We rewrite Condition (3) as

$$1 = \sigma_{\lambda_0} \left( E_{an_n}^{(n-2)}(t), -E_{an_n}^{(n-1)}(t) \right) = \sigma_{\lambda(t)} \left( e^{t\mathcal{H}} E_{an_n}^{(n-2)}(t), -e^{t\mathcal{H}} E_{an_n}^{(n-1)}(t) \right) \equiv -f^2 h_1^{2n-6} \sigma_{\lambda(t)} \left( \partial_{h_3}, X_3 \right) = f^2 h_1^{2n-6}.$$  

By taking the positive sign, we obtain

$$E_{an_n}(t) = e^{-t\mathcal{H}} h_1^{2-n} \partial_{h_n}. \quad \square$$

The following proposition is an extension of Eq. (4.13).

**Proposition 4.5.** Let $E_{an_n}^{(i)}(t) = e^{-t\mathcal{H}} \left( \sum_{j=0}^{i} a_{ij}(t) \partial_{h_{n-j}} \right), \text{for} \ i = 0, \ldots, n-3, \ \text{with} \ n \geq 3$. We have the following formulas for the coefficients $a_{ij}$’s:

$$a_{ii} = (-1)^i h_1^{2-n+2},$$

$$a_{ii-1} = (-1)^{i-1} \sum_{k=0}^{i-1} h_1^{k} \mathcal{H} \left( h_1^{2-n+1-i-k} \right), \quad i \geq 1,$$

$$a_{ii-2} = (-1)^{i-2} \sum_{k=0}^{i-2} h_1^{k} \mathcal{H} \left( \sum_{l_k=0}^{i-2-k} h_1^{l_k} \left( h_1^{2-n+1-i-2-k-l_k} \right) \right), \quad i \geq 2.$$  

**Proof.** It is easy to see that the formulas are valid for $i = 0, 1, 2$. Assume that the formulas are valid for some $2 \leq i < n - 3$. Now, if we take the derivative of $E_{an_n}^{(i)} = \sum_{j=0}^{i} a_{ij} \partial_{h_{n-j}}$, we have

$$E_{an_n}^{(i+1)}(t) = e^{-t\mathcal{H}} \left( -h_1 a_{ii} \partial_{h_{n-i-1}} + (\dot{a}_{ii} - h_1 a_{ii-1}) \partial_{h_{n-i}} + (\dot{a}_{ii-1} - h_1 a_{ii-2}) \partial_{h_{n-i+1}} \mod V_{n-i+2} \right).$$
where $V_{n-i+2} = \text{span}\{\partial_{h_{n-i+2}}, \ldots, \partial_{h_n}\}$. Hence,
\[
\begin{align*}
\alpha_{i+1i+1} & = -h_1\alpha_{ii} \\
& = (-1)^{i+1} h_1^{2-n_a+i+1}; \\
\alpha_{i+1i} & = \dot{\alpha}_{ii} - h_1\alpha_{ii-1} \\
& = (-1)^i h_1^{2-n_a+i} + (-1)^i \sum_{k=0}^{i-1} h_1^{k+1} h_1^{2-n_a+(i-1)-k} \\
& = (-1)^i h_1^{2-n_a+i} + (-1)^i \sum_{l=1}^{i} h_1^l h_1^{2-n_a+i-l} \quad (l = k + 1) \\
& = (-1)^i \sum_{l=0}^{i} h_1^l h_1^{2-n_a+i-l}.
\end{align*}
\]

For the last term we have
\[
\begin{align*}
\alpha_{i+1i-1} & = \dot{\alpha}_{ii-1} - h_1\alpha_{ii-2} \\
& = (-1)^{i-1-1} h_1^{i+1} \left( \sum_{l=0}^{i-1} h_1^l h_1^{2-n_a+(i-1)-l} \right) \\
& \quad + (-1)^{i-1-2} \sum_{k=0}^{i-2} h_1^{k+1} h_1^{2-n_a+(i-2)-k-l_k} \\
& = (-1)^{i-1} h_1^{2-n_a+(i-1)-l} \\
& \quad + (-1)^{i-1} \sum_{j=1}^{i-1} h_1^j h_1^{2-n_a+(i-1)-j-l_j} \quad (j = k + 1) \\
& = (-1)^{i-1} \sum_{j=0}^{i-1} h_1^j h_1^{2-n_a+(i-1)-j-l_j}.
\end{align*}
\]

For future computations we need explicit formulas for $F_{a1}(0)$ and its derivative $\dot{F}_{a1}(0)$. By Proposition \[\ref{prop:explicit_formulas}\] with $i = n - 3$, we have:

- For $n = 3$,
  \[E_{a_3}(0) = \partial_{h_3}.\]

- For $n = 4$,
  \[E_{a_4}(0) = -\partial_{h_3} + \frac{(n - 2)(n - 3)}{2} h_3 h_2 \partial_{h_4}.\]

- For $n \geq 5$,
  \[E_{a_{n-3}}^{(n-3)}(0) = (-1)^{n-4} \left(-\partial_{h_3} + \sum_{k=0}^{n-4} h_1^k h_1^{2-n_a+k} \partial_{h_4} \right) \\
  + (-1)^{n-5} \left( \sum_{k=0}^{n-5} h_1^k h_1^{2-n_a+k-l_k} \right) \partial_{h_5} \mod V_6 \]
  \[= (-1)^{n-4} \left(-\partial_{h_3} + \frac{(n - 3)(n - 2)}{2} h_2 h_3 \partial_{h_4} \right) \\
  + (-1)^{n-5} \left( A_1(n) \frac{h_2^2 h_3^2}{h_1^4} + A_2(n) \frac{h_2^3 + h_2 h_4}{h_1^4} \right) \partial_{h_5} \mod V_6,\]

where $A_1(n)$ and $A_2(n)$ are given by

$$
A_1(n) = \sum_{k=0}^{n-5} (3 + k) \left( \sum_{j_k=0}^{n-5-k} (k + j_k + 2) \right)
= \frac{1}{8} (n - 4) \left( (n - 4) \left( n^2 + 10n - 27 \right) - (n - 5) (8n - 18) \right)
$$

(4.14)

$$
A_2(n) = \sum_{k=0}^{n-5} \left( \sum_{j_k=0}^{n-5-k} (k + j_k + 2) \right)
= \frac{1}{6} (n - 4) \left( 3(n - 4)(n - 1) - n(n - 5) \right)
$$

(4.15)

After some simple computations using the identities in Lemma 4.3 we obtain the following formulas for $n \geq 3,$

$$
F_{a1}(0) = (-1)^{n-1} \left( X_\theta + \frac{(n-2)(n-3)}{2} \frac{h_2 h_3}{h_1} \partial_\theta \right)
$$

(4.16)

$$
F_{\dot{a}1}(0) = (-1)^{n-1} \left( -X_3 + h_3 X_\dot{\theta} + \frac{(n-2)(n-3)}{2} \frac{h_2 h_3}{h_1} X_\theta \right)
$$

(4.17)

and

$$
\dot{F}_{a1}(0) = (-1)^{n-1} \left( -X_3 + h_3 X_\dot{\theta} + \frac{(n-2)(n-3)}{2} \frac{h_2 h_3}{h_1} X_\theta \right)
$$

(4.16)

$$
\dot{F}_{\dot{a}1}(0) = (-1)^{n-1} \left( -X_3 + h_3 X_\dot{\theta} + \frac{(n-2)(n-3)}{2} \frac{h_2 h_3}{h_1} X_\theta \right)
$$

(4.17)

with the convention $h_4 \equiv 0$ for $n = 3.$ Notice that for $n = 3, 4,$ the functions $A_1, A_2$ are defined and, moreover, they vanish.

Proof of Theorem 1.1. By direct inspection, the orthonormal basis $\{X_a, X_b\}$ for $\mathcal{D}_{x_0}$ obtained by projection of the canonical frame is

$$
X_a = \pi_* F_{a1}(0) \quad X_b = \pi_* F_{b1}(0).
$$

Recall that in the coordinates associated to the splitting $\Sigma = \mathcal{V}_{x_0} \oplus \mathcal{H}_{x_0}$ we have

$$
\mathcal{Q}_{x_0}(t) = \frac{d}{dt} S^h(t)^{-1}.
$$

From Eqs. (3.4), (3.5) in Theorem 3.6 the formula for $S^h(t)^{-1}$ in the basis $\{X_a, X_b\}$ is given by

$$
S^h(t)^{-1} = -\frac{1}{t} \begin{pmatrix} (n-1)^2 & 0 \\ 0 & 1 \end{pmatrix} + \frac{t}{3} \begin{pmatrix} 3(n-1) & 0 \\ 0 & 4(n-1)^2-1 \end{pmatrix} R_{aa,11}(0) \begin{pmatrix} 0 \\ 0 \end{pmatrix} + O(t^2),
$$

since $R_{ab,11} = R_{ba,11} = R_{bb,11} \equiv 0$ by [11 Lemma 5.7]. Therefore, the curvature operator has the following expression

$$
\mathcal{R}_{x_0} = \frac{3(n-1)}{4(n-1)^2-1} \begin{pmatrix} R_{aa,11}(0) \\ 0 \\ 0 \end{pmatrix}.
$$

(4.18)
From Eq. (4.16) and Eq. (4.17) we obtain the following explicit expression for the term $R_{aa,11}$ with $n \geq 3$,

\begin{equation}
R_{aa,11}(0) = \frac{\sigma_\lambda(\dot{F}_a(0), F_a(0))}{2} \\
= \frac{1}{2} (2 - 5(n - 2)(n - 3) - 4A_2(n))(h_3^2 + h_2h_4) \\
+ \frac{1}{4} ((n - 2)^2(n - 3)^2 - 14(n - 2)(n - 3) - 8A_1(n)) \frac{h_3^2}{h_1^2} \\
= -\frac{1}{6} (n - 1)(12 + 4(n - 17))(h_3^2 + h_2h_4) \\
- (n - 1)(n - 2)(n - 3)h_3^2 \tan^2 \left( \theta + \frac{\pi}{2} \right),
\end{equation}

where $A_1(n)$ and $A_2(n)$ are given by Eq. (4.14) and Eq. (4.15) for $n \geq 3$. Here we use again the convention $h_4 \equiv 0$ for $n = 3$.

We show next how the complete set of invariants, $R(t)$, in principle, might be obtained from formulas (4.16), (4.17) and (4.19). Since the matrix $R(t)$ is normal, the sub-matrix $R_{aa} = [R_{aa,ij}]$ is diagonal, and hence, from the structural equations we easily get

\begin{equation}
F_{ai} = \sum_{j=1}^{n-1} (-1)^j \frac{d}{dt} (R_{i-j}E_{ai-j}) + (-1)^{i-1} F_{ai-1}.
\end{equation}

Therefore, using Eq. (4.20) and the fact that $R_{aa,ii} = \sigma(\dot{F}_a, F_a)$, we obtain for $1 \leq i \leq n - 1$,

\begin{equation*}
R_{aa,ii}(t) = \sum_{j=1}^{i-1} (-1)^{i+j-2} \sigma_{\lambda_0} \left( \frac{d}{dt} (R_{a,i-j}E_{ai-j})(t), F_{ai-1}^{i-1}(t) \right) \\
+ \sum_{k_1,k_2=1}^{i-1} (-1)^{k_1+k_2-2} \sigma_{\lambda_0} \left( \frac{d}{dt} (R_{a,i-k}E_{ai-k})(t), \frac{d}{dt} (R_{a,i-k}E_{ai-k})(t) \right) \\
+ \sigma_{\lambda_0} \left( F_{ai}^{i-1}(t), F_{ai}^{i-1}(t) \right).
\end{equation*}

5. Engel group

In this section we analyze the geodesics in the Engel group, $n = 4$. In order to do this we limit ourselves to the level surface $\{ H = \frac{1}{2} \}$ and consider the coordinate system $(\theta, c, \alpha)$:

\begin{align*}
h_1 &= \cos \left( \theta + \frac{\pi}{2} \right), & h_2 &= \sin \left( \theta + \frac{\pi}{2} \right), & h_3 &= c, & h_4 &= \alpha.
\end{align*}

In the variables $(\theta, c, \alpha)$ the Hamiltonian system assumes the following form:

\begin{align*}
\dot{\theta} &= c, \\
\dot{c} &= -\alpha \sin \theta, \\
\dot{\alpha} &= 0.
\end{align*}

Note that this system for the costate variables reduces to the pendulum equation

\begin{equation}
\ddot{\theta} = -\alpha \sin \theta, \quad \dot{\alpha} = 0.
\end{equation}

Let us introduce the energy integral of the pendulum (5.2):

\begin{equation*}
E = \frac{c^2}{2} - \alpha \cos \theta \in [-|\alpha|, +\infty), \quad \dot{E} = 0.
\end{equation*}

Let $\gamma : [0, T] \to \mathbb{R}^4$ be a normal geodesic and $T$ be an admissible extension of $\gamma$. From Theorem (4.2) we get that if for the initial covector $\lambda$ we have $h_1(\lambda) = h_3(\lambda) = 0$, then the geodesic is not ample at $t = 0$, and hence not ample for all $t$. On the other hand, from Theorem (4.2) we also have that if there is no time $t$ such that $h_1(\lambda(t))$ and $h_3(\lambda(t))$ are both zero, the geodesic growth vector is

\begin{equation}
G_{\gamma}(t) = \begin{cases} 
(2,3,4) & \text{if } h_1(\lambda(t)) \neq 0, \\
(2,3,4) & \text{if } h_1(\lambda(t)) = 0.
\end{cases}
\end{equation}
We will use extensively the results in [9] to study the set of times of loss of equiregularity of a geodesic. The family of normal extremal trajectories can be parametrized by points in the cylinder

\[ C = T^*_{x_0} \mathbb{R}^4 \cap \{ H = \frac{1}{2} \} = \{(h_1, h_2, h_3, h_4) \in \mathbb{R}^4 : h_1^2 + h_2^2 = 1 \} \]

\[ = \{ (\theta, c, \alpha) : \theta \in S^1, c, \alpha \in \mathbb{R} \}. \]

Following [9], we partition \( C \) into subsets corresponding to different types of pendulum trajectories:

\[ C = \bigcup_{i=1}^{7} C_i, \quad C_i \cap C_j = \emptyset, \quad i \neq j, \quad \lambda = (\theta, c, \alpha), \]

\[ C_1 = \{ \lambda \in C : \alpha \neq 0, E \in (-|\alpha|, |\alpha|) \}, \]
\[ C_2 = \{ \lambda \in C : \alpha \neq 0, E \in (|\alpha|, +\infty) \}, \]
\[ C_3 = \{ \lambda \in C : \alpha \neq 0, E = |\alpha|, c \neq 0 \}, \]
\[ C_4 = \{ \lambda \in C : \alpha \neq 0, E = -|\alpha| \}, \]
\[ C_5 = \{ \lambda \in C : \alpha \neq 0, E = |\alpha|, c = 0 \}, \]
\[ C_6 = \{ \lambda \in C : \alpha = 0, c \neq 0 \}, \]
\[ C_7 = \{ \lambda \in C : \alpha = c = 0 \}. \]

The sets \( C_i, i = 1, \ldots, 5 \), are further subdivided into subsets depending on the sign of \( \alpha \):

\[ C_i^+ = C_i \cup \{ \alpha > 0 \}, \quad C_i^- = C_i \cup \{ \alpha < 0 \}, \quad i \in \{1, \ldots, 5\}. \]

Here and throughout, \( \text{sn}, \text{cn}, \text{dn} \) are Jacobian elliptic functions (see [40]). Following [9], in order to calculate the extremal trajectories from the subsets \( C_1, C_2 \) and \( C_3 \) we introduce coordinates \((\varphi, k, \alpha)\) in which the system (5.1) is straightened out. Since the general case \( \alpha \neq 0 \) can be obtained from the special case \( \alpha > 0 \), see [9], we will describe these coordinates assuming that \( \alpha > 0 \).

In the domain \( C_1^+ \), we set

\[ k = \sqrt{\frac{E + \alpha}{2\alpha}} = \sqrt{\frac{c^2}{4\alpha} + \sin^2 \frac{\theta}{2}} \in (0, 1), \]
\[ \sin \frac{\theta}{2} = k \text{sn} (\sqrt{\alpha} \varphi), \quad \cos \frac{\theta}{2} = \text{dn} (\sqrt{\alpha} \varphi), \quad \frac{c}{2} = k \sqrt{\alpha} \text{cn} (\sqrt{\alpha} \varphi), \quad \varphi \in [0, 4K], \]

where \( 4K \) is the period of the Jacobian elliptic functions \( \text{sn} \) and \( \text{cn} \).

In the domain \( C_2^+ \), we set

\[ k = \sqrt{\frac{2\alpha}{E + \alpha}} = \frac{1}{\sqrt{c^2/(4\alpha) + \sin^2(\theta/2)}} \in (0, 1), \]
\[ \sin \frac{\theta}{2} = \text{sgn} c \frac{\sqrt{\alpha} \varphi}{k}, \quad \cos \frac{\theta}{2} = \frac{\sqrt{\alpha} \varphi}{k}, \quad \frac{c}{2} = \text{sgn} c \frac{\sqrt{\alpha} \varphi}{k} \]
\[ \text{dn} \frac{\sqrt{\alpha} \varphi}{k}, \quad \varphi \in [0, 2kK], \]

where \( \varphi \in [0, 2kK] \).

In the domain \( C_3^+ \), we set

\[ k = 1, \]
\[ \sin \frac{\theta}{2} = \text{sgn} c \tanh(\sqrt{\alpha} \varphi), \quad \cos \frac{\theta}{2} = \frac{1}{\cosh(\sqrt{\alpha} \varphi)}, \quad \frac{c}{2} = \text{sgn} c \frac{\sqrt{\alpha}}{\cosh(\sqrt{\alpha} \varphi)}, \]

and \( \varphi \in (-\infty, \infty) \).

Immediate differentiation shows that in these coordinates the subsystem for the costate variables (5.1) takes the following form:

\[ \dot{\varphi} = 1, \quad \dot{k} = 0, \quad \dot{\alpha} = 0. \]

so that it has solutions

\[ \varphi(t) = \varphi_0 + \varphi + t, \quad k = \text{const}, \quad \alpha = \text{const}. \]

The elliptic coordinate \( \varphi \) is the time of movement along trajectories of the pendulum equation and \( k \) is a parameter that distinguishes trajectories with different energies.
Proposition 5.1. For \( \lambda \in C_1, C_2, C_6 \) the geodesic is ample and has an infinite and discrete set of times of loss of equiregularity. If \( \lambda \in C_3 \), the geodesic is ample and has an unique time of loss of equiregularity. For \( \lambda \in C_4, C_5 \) the geodesic is not ample and hence abnormal. If \( \lambda \in C_7 \) then it depends: usually the geodesic is ample and equiregular, but there are some non-ample cases (corresponding to straight lines).

**Proof.** We have, by the symmetries of the Hamiltonian system (5.1), namely dilations and reflections, that for \( \lambda \in C_1, C_2, C_3 \), the solution to the system (5.1) for the case \( \alpha \neq 0 \) can be recovered from the special case \( \alpha = 1 \), see [9] for details. Hence, things do not qualitatively change if we assume that for \( \lambda \in C_1, C_2, C_3 \) we have \( \alpha = 1 \).

1. Let \( \lambda \in C_1 \) with \( \alpha = 1 \). In this case we have
   \[
   h_1(t) = -2k \, \text{sn} \, \varphi_t \, \text{dn} \, \varphi_t, \quad k = \sqrt{\frac{E + 1}{2}},
   \]
   and \( \varphi_t = \varphi + t \), with \( \varphi \in [0, 4K] \). Hence, the geodesic has a infinite and discrete set of times of loss of equiregularity.

2. For \( \lambda \in C_2 \) with \( \alpha = 1 \), we have
   \[
   h_1(t) = -2 \, \text{sgn} \, c \, \text{sn} \, \psi_t \, \text{cn} \, \psi_t, \quad k = \sqrt{\frac{2}{E + 1}},
   \]
   and \( \psi_t = \frac{\varphi t}{k} \), with \( \varphi \in [0, 2kK] \). As in the previous case, the geodesic has a infinite and discrete set of times of loss of equiregularity.

3. Assume that \( \lambda \in C_3 \) and \( \alpha = 1 \). Then, we have
   \[
   h_1(t) = -2 \, \text{sgn} \, c \, \text{tanh} \, \varphi_t \, \text{cosh} \, \varphi_t,
   \]
   where \( \varphi_t = \varphi + t \) and \( \varphi \in \mathbb{R} \). Hence, \( h_1(t) = 0 \) for \( \varphi + t = 0 \).

4. Suppose that \( \lambda \) is in \( C_4 \) or in \( C_5 \). In either case, we will have that \( h_1(0) = h_3(0) = 0 \).

5. Therefore, the geodesic is not ample, and hence abnormal.

6. Let \( \lambda \in C_6 \). By the geodesic equations we have that \( \dot{\theta} = 0 \), so that \( \theta(t) = ct + \theta \), where \( c = \text{const} \neq 0 \) and \( \theta = \text{const} \). Therefore,
   \[
   h_1(t) = \cos(\theta(t)) = 0 \quad \text{if and only if} \quad ct + \theta = (2k + 1)\frac{\pi}{2},
   \]
   where \( k \in \mathbb{Z} \).

7. In this case we also have \( \dot{\theta} = 0 \), so \( \theta(t) \equiv \theta \), where \( \theta \) is constant. Therefore, if \( \theta \neq (2k + 1)\frac{\pi}{2} \), with \( k \in \mathbb{Z} \), then \( h_1(t) \equiv \text{const} \neq 0 \). Moreover, if \( \theta = (2k + 1)\frac{\pi}{2} \), then \( h_1(t) = h_3(t) \equiv 0 \), so the geodesic is abnormal.

We close this section by noticing that if we use the energy integral and Eq. (1.19) we immediately obtain the following bound:

\[
R_{aa,11}(\lambda) = -6c^2 \csc^2 \theta + 4E \leq 4E.
\]

6. **Cartan group**

We now turn our attention to the Cartan group and compute its curvature operator. In \( \mathbb{R}^5 \) with coordinates \((x, y, z, v, w)\), we consider the vector fields

\[
X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} - \frac{x^2 + y^2}{2} \frac{\partial}{\partial w}, \quad X_2 = \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z} + \frac{x^2 + y^2}{2} \frac{\partial}{\partial v}.
\]

The vector fields \( X_1 \) and \( X_2 \) are called horizontal and satisfy the Hörmander condition, i.e. they generate the whole tangent bundle by their commutators:

\[
X_3 = [X_1, X_2] = \frac{\partial}{\partial z} + x \frac{\partial}{\partial v} + y \frac{\partial}{\partial w},
\]

\[
X_4 = [X_1, X_3] = \frac{\partial}{\partial v},
\]

\[
X_5 = [X_2, X_3] = \frac{\partial}{\partial w},
\]

and all other commutators being zero. Notice that the distribution of the Cartan group is not of Goursat-type.
We can introduce a unique Lie group structure (law of multiplication) in $\mathbb{R}^5$, making a Lie group of $\mathbb{R}^5$, so that $X_1, \ldots, X_5$, become basic left-invariant fields on this Lie group. We denote this Lie group by $\mathfrak{c}$. We have that $\mathscr{D}|_h = \text{span } \{X_1, X_2\}|_h$, for all $h \in \mathfrak{c}$, is a distribution left-invariant by the action $L_g : \mathfrak{c} \to \mathfrak{c}$. Any left-invariant scalar product $g$ on $\mathscr{D}|_h$ induces a left-invariant sub-Riemannian structure $(\mathscr{D}, g)$ on $\mathfrak{c}$. Since any two different choices give rise to isometric sub-Riemannian structures, we choose without loss of generality $g$ such that $X_1$ and $X_2$ are orthonormal.

Let $\{\nu_1, \ldots, \nu_5\}$ be the dual frame of $\{X_1, \ldots, X_5\}$. This dual frame induces coordinates $\{h_1, \ldots, h_5\}$ in each fiber of $T^*\mathbb{R}^5$,

$$\lambda = (h_1, \ldots, h_5) \iff \lambda = h_1 \nu_1 + \ldots + h_5 \nu_5$$

where $h_i(\lambda) = (\lambda, X_i)$ are the linear-on-fibers functions associated with $X_i$, for $i = 1, \ldots, 5$.

Let $h_i \in \text{Vec}(T^*\mathbb{R}^5)$ be the Hamiltonian vector fields associated with $h_i \in C^\infty(T^*\mathbb{R}^5)$ for $i = 1, \ldots, 5$, respectively. Consider the vertical vector fields $\partial_{h_i} \in \text{Vec}(T^*\mathbb{R}^5)$, for $i = 1, \ldots, 5$. The vector fields

$$\vec{h}_1, \ldots, \vec{h}_5, \partial_{h_1}, \ldots, \partial_{h_5},$$

are a local frame of vector fields of $T^*\mathbb{R}^5$. Equivalently, we can introduce the cylindrical coordinates $\rho, \theta, h_3, h_4, h_5$ on each fiber of $T^*\mathbb{R}^5$ by

$$h_1 = \rho \cos \theta, \quad h_2 = \rho \sin \theta,$$

with $\rho \in (0, +\infty)$ and $\theta \in (-\pi, \pi]$, and employ instead the local frame

$$\vec{h}_1, \ldots, \vec{h}_5, \partial_\theta, \partial_\rho, \partial_{h_3}, \partial_{h_4}, \partial_{h_5},$$

where $\partial_\theta = h_1 \partial_{h_2} - h_2 \partial_{h_1}$.

The Euler vector field is given by

$$\varepsilon = \sum_{i=1}^5 h_i \partial_{h_i} = \rho \partial_\rho + \sum_{i=3}^5 h_i \partial_{h_i}.$$  

Recall that $\varepsilon$ is a vertical vector field on $T^*\mathbb{R}^5$, i.e. $\pi_\ast \varepsilon = 0$, and is the generator of the dilations $\lambda \mapsto e^\varepsilon \lambda$ along the fibers of $T^*\mathbb{R}^5$. The sub-Riemannian Hamiltonian is

$$H = \frac{1}{2} (h_1^2 + h_2^2),$$

and, therefore, the Hamiltonian vector field $\vec{H}$ is given by

$$\vec{H} = h_1 \vec{h}_1 + h_2 \vec{h}_2.$$  

Let $\{\nu_3, \nu_5\}$ be the dual co-frame associated with the frame $\{X_3, X_5\}$, given by $X_3 = h_1 X_1 + h_2 X_2$ and $X_5 = h_3 X_1 - h_1 X_2$. The symplectic form in the cylindr coordinates $\rho, \theta, h_3, h_4, h_5$ has the following expression:

$$\sigma = \rho d\rho \land \nu_3 - \rho^2 d\theta \land \nu_5 + \rho^2 h_3 \nu_3 \land \nu_5 + (h_2 h_4 - h_1 h_5) \nu_3 \land \nu_5$$

$$+ (h_1 h_4 + h_2 h_5) \nu_3 \land \nu_5 + \sum_{i=3}^5 d h_i \land \nu_i.$$  

The Hamiltonian vector field $\vec{H}$ can be written as

$$\vec{H} = X_\theta + h_3 \partial_\theta + (h_1 h_4 + h_2 h_5) \partial_{h_3}.$$  

6.1. Ample geodesics in the Cartan group. Observe that $h_4$ and $h_5$ are first integrals of the Hamiltonian system: $\vec{H} h_4 = \vec{H} h_5 = 0$. Another important first integral of the system is the so-called energy integral:

$$E = \frac{h_4^2}{2} + h_1 h_5 - h_2 h_4.$$  

It is not difficult to see that $\vec{H} E = 0$. In the coordinates

$$h_1 = \cos \theta, \quad h_2 = \sin \theta, \quad c = h_3, \quad h_4 = \alpha \sin \beta, \quad h_5 = -\alpha \cos \beta,$$

on the fibres of the unit cotangent bundle, where $\rho = 1$, the energy has the form

$$E = \frac{c^2}{2} - \alpha \cos(\theta - \beta) \in [-\alpha, \infty).$$

Proposition 6.1. Let $\gamma = \pi \circ \lambda : [0, T] \to \mathbb{R}^5$ be a normal unit-speed geodesic in $\mathfrak{c}$. Then,
(1) If \( h_3 \) is not identically equal to zero, then \( \gamma \) is equiregular at \( t \in [0,T] \) if and only if \( h_3(t) \neq 0 \). Moreover, \( \gamma \) is ample and the geodesic growth vector is given by

\[
\mathcal{G}_{\gamma}(t) = \begin{cases} 
(2,3,4,5), & \text{if } h_3(t) \neq 0, \\
(2,3,4,4,5), & \text{if } h_3(t) = 0.
\end{cases}
\]

(2) If \( h_3 \equiv 0 \), the geodesic is not ample, and we are in the case of an abnormal geodesic.

**Proof.** Let \( \gamma : [0,T] \to \mathbb{R}^5 \) be a normal geodesic and \( T = v_1X_1 + v_2X_2 \) be an admissible extension of \( \gamma \). Recall the geodesic equations for the Cartan Group:

\[
\dot{h}_1 = -h_3h_3, \quad \dot{h}_2 = h_1h_3, \quad \dot{h}_3 = h_1h_4 + h_2h_5, \quad \dot{h}_4 = \dot{h}_5 = 0.
\]

(1) Let us compute the growth vector when \( h_3 \) is not identically equal to zero. For \( \mathcal{F}_2 \) we have

\[
\mathcal{L}_T(X_1) = \begin{cases} 
\mathcal{L}_T(X_2) = h_1X_3 \mod \mathcal{F}_1,
\end{cases}
\]

Hence, we obtain \( \mathcal{F}_2 = \text{span}\{X_1, X_2, X_3\} \) for all \( t \). Now, observe that

\[
\mathcal{L}_T^3(X_1) = -h_2(h_1X_4 + h_2X_5) \mod \mathcal{F}_2,
\]

\[
\mathcal{L}_T^3(X_2) = h_1(h_1X_4 + h_2X_5) \mod \mathcal{F}_2.
\]

Therefore, \( \mathcal{F}_3 = \text{span}\{X_1, X_2, X_3, h_1X_4 + h_2X_5\} \), and \( k_3 = \dim \mathcal{F}_3 = 4 \), for all \( t \).

For the computation of \( \mathcal{F}_4 \), we have

\[
\mathcal{L}_T^3(X_1) = h_2h_3(h_2X_4 - h_1X_5) \mod \mathcal{F}_3,
\]

\[
\mathcal{L}_T^3(X_2) = -h_1h_3(h_2X_4 - h_1X_5) \mod \mathcal{F}_3.
\]

Then, if \( h_3(t) \neq 0 \), we obtain

\[
\mathcal{F}_4 = \text{span}\{X_1, X_2, X_3, h_1X_4 + h_2X_5, h_2X_4 - h_1X_5\}
\]

and \( k_4 = \dim \mathcal{F}_4 = 5 \), since \( h_1^2 + h_2^2 = 1 \).

If \( h_3(\bar{t}) = 0 \), then \( k_4(\bar{t}) = 4 \). In this case, we need to compute \( \mathcal{F}_5 \). Notice that

\[
\mathcal{L}_T^4(X_1) = h_2(h_1h_4 + h_2h_5)(h_2X_4 - h_1X_5) \mod \mathcal{F}_4,
\]

\[
\mathcal{L}_T^4(X_1) = -h_1(h_1h_4 + h_2h_5)(h_2X_4 - h_1X_5) \mod \mathcal{F}_4.
\]

From the geodesic equations (6.2), if \( h_1(\bar{t})h_4(\bar{t}) + h_2(\bar{t})h_5(\bar{t}) = 0 \), then \( h_3 \equiv 0 \), which contradicts the hypothesis. Hence, we have that the geodesic growth vector for a geodesic with \( h_3 \) not identically equal to zero is given by (6.1).

(2) Now, if \( h_3 \equiv 0 \), then \( h_1(t) \) and \( h_2(t) \) are constant. Hence, from the computations of case (1), it follows that in this case the geodesic is not ample, and hence the geodesic is abnormal.

\[\square\]

The family of normal extremal trajectories can be parametrized by points in the cylinder

\[
C = T_{x_0}^*\mathbb{R}^5 \cap \{ H = \frac{1}{2} \} = \{(h_1,h_2,h_3,h_4,h_5) \in \mathbb{R}^5 : h_1^2 + h_2^2 = 1 \}
\]

\[
= \{ (\theta,c,\alpha,\beta) : \theta, \beta \in S^1, c, \alpha \in \mathbb{R} \}
\]

Following [35], we partition \( C \) into subsets corresponding to different types of pendulum trajectories:

\[
C = \bigcup_{i=1}^{7} C_i, \quad C_i \cap C_j = \emptyset, \quad i \neq j, \quad \lambda = (\theta,c,\alpha,\beta),
\]

\[
\begin{align*}
C_1 & = \{ \lambda \in C : \alpha \neq 0, E \in (-\alpha,\alpha) \}, \\
C_2 & = \{ \lambda \in C : \alpha \neq 0, E \in (\alpha, +\infty) \}, \\
C_3 & = \{ \lambda \in C : \alpha \neq 0, E = \alpha, \theta - \beta \neq \pi \}, \\
C_4 & = \{ \lambda \in C : \alpha \neq 0, E = -\alpha \}, \\
C_5 & = \{ \lambda \in C : \alpha \neq 0, E = \alpha, \theta - \beta = \pi \}, \\
C_6 & = \{ \lambda \in C : \alpha = 0, c \neq 0 \}, \\
C_7 & = \{ \lambda \in C : \alpha = c = 0 \}.
\end{align*}
\]
The elliptic coordinate
For initial covectors
Proposition 6.2.
and $k$ equiregularity. For $\lambda \in C_1$ we have that ample geodesics belong to $C_1, C_2, C_3$, and $C_6$. We now analyze the equiregularity of normal geodesics with initial values in these subsets of the cylinder $C$. We will use extensively the results in [35].

Following [35], we introduce elliptic coordinates $(k, \varphi, \alpha, \beta)$ in the subsets $C_1, C_2$ and $C_3$ of the cylinder $C$ as following.

For $\lambda \in C_1$, we set:

$$k = \sqrt{\frac{E + \alpha}{2\alpha}} = \sqrt{\frac{\sin^2 \frac{\theta - \beta}{2} + \frac{c^2}{4\alpha}}{\varphi \in [0, 4K]},}$$

$$\begin{cases}
\sin \frac{\theta - \beta}{2} = k \sin(\sqrt{\alpha}\varphi); \\
\hat{\varphi} = k\sqrt{\alpha} \cos(\sqrt{\alpha}\varphi);
\end{cases}$$

For $\lambda \in C_2$, we set:

$$k = \sqrt{\frac{2\alpha}{E + \alpha}} = \frac{1}{\sqrt{\sin^2 ((\theta - \beta)/2) + c^2/(4\alpha)}} \in (0, 1),$$

$$\varphi \in [0, 2kK],$$

$$\begin{cases}
\sin \frac{\theta - \beta}{2} = \pm k \sin \frac{\sqrt{\alpha}\varphi}{k}; \\
\hat{\varphi} = \pm \sqrt{\alpha} \cos(\sqrt{\alpha}\varphi); \\
\pm = \text{sgn } c
\end{cases}$$

Here $4K$ is the period of the Jacobian elliptic functions $sn$ and $cn$.

For $\lambda \in C_3$, we set

$$\alpha \neq 0, \quad E = \alpha, \quad \theta - \beta \neq \pi,$$

$$k = 1,$$

$$\varphi \in (-\infty, \infty),$$

$$\begin{cases}
\sin \frac{\theta - \beta}{2} = \pm \tanh(\sqrt{\alpha}\varphi); \\
\hat{\varphi} = \pm \sqrt{\alpha} \cosh(\sqrt{\alpha}\varphi); \\
\pm = \text{sgn } c.
\end{cases}$$

In the elliptic coordinates $(k, \varphi, \alpha, \beta)$ on $\cup_{i=1}^3 C_i$ the vertical part of the normal Hamiltonian system

$$\dot{\theta} = c, \quad \dot{\varphi} = -\alpha \sin(\theta - \beta), \quad \dot{\alpha} = \dot{\beta} = 0,$$

simplifies to

$$\dot{\varphi} = 1, \quad \dot{k} = \dot{\alpha} = \dot{\beta} = 0.$$

The elliptic coordinate $\varphi$ is the time of movement along trajectories of the pendulum equation and $k$ is a parameter that distinguishes trajectories with different energies.

**Proposition 6.2.** For initial covectors $\lambda$ in $C_1$, there exists an infinite set of times of loss of equiregularity. For $\lambda \in C_2, C_3$ and $C_6$, the geodesic is equiregular for all times. Moreover, for $\lambda \in C_4, C_5, C_7$, the corresponding geodesic is not ample, and thus abnormal.

**Proof.**

1. Let $\lambda \in C_1$. Then, $c^2 = 4k^2\alpha \cos^2(\sqrt{\alpha}\varphi)$, with $\varphi_t = \varphi + t$ and $k = \sqrt{\frac{E + 1}{2}}$. In this case the geodesic has a infinite and discrete set of times of loss of equiregularity.

2. Let $\lambda \in C_2$. Then, we have $c^2 = \frac{4\alpha}{k^2} \cos^2(\sqrt{\alpha}\varphi)$, with $\psi(t) = \frac{\varphi + t}{k}$ and $k = \sqrt{\frac{2}{c + 1}}$.

Hence, the geodesic is equiregular.

3. Let $\lambda \in C_3$. Then, $c^2 = \frac{4\alpha}{\cos^2(\sqrt{\alpha}\varphi)}$, with $\varphi_t = \varphi + t$. In this case the geodesic is also equiregular.

4. For $\lambda \in C_6$ we have $\alpha = 0$, and $c = \text{const} \neq 0$. Therefore, the geodesic $\gamma$ is equiregular.

5. For $\lambda \in C_4, C_5, C_7$, we have that $c \equiv 0$. Hence, from Proposition 6.1 the corresponding geodesic is not ample, hence abnormal.

$\square$
6.2. The canonical frame. Let $\lambda$ be the initial covector associated to an ample, equiregular, unit-speed geodesic $\gamma : [0, T] \to \mathbb{R}^5$, with $\gamma(0) = x_0$. The associated Young diagram is given by

$$D = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 \end{bmatrix}$$

For such Young diagram, a canonical frame is a smooth family

$$\{E_{a_1}, \ldots, E_{a_4}, E_{b_1}, F_{a_1}, \ldots, F_{a_4}, F_{b_1}\} \in T_{x_0} (T^*\mathbb{R}^5),$$

with the following properties:

(1) It is attached to the Jacobi curve, namely

$$J_{x_0}(t) = \text{span}\{E_{a_1}(t), E_{a_2}(t), E_{a_3}(t), E_{a_4}(t), E_{b_1}(t)\}.$$ 

(2) From [11, Lemma 5.7] we have:

$$E_{b_1}(t) = e^{-it\bar{H}}e = e - t\bar{H},$$ 

as a consequence all curvatures $R_{a,*,1}$, (where $*$ is any other index) vanish.

(3) The family of symmetric matrices $R(t)$ is normal in the sense of [11]. In this “easy” Young diagram case, the normal condition means that the matrix $[R_{aa,ij}]_{i,j=1,\ldots,n_a}$ is diagonal.

(4) They satisfy the structural equations:

$$\dot{E}_{ai} = E_{a(i-1)}(t), \quad i = 2, 3, 4,$$

$$\dot{E}_{a1} = -F_{a1}(t),$$

$$\dot{F}_{b1} = -F_{b1}(t),$$

$$\dot{F}_{ai} = R_{a,i}E_{a_i}(t) - F_{a(i+1)}(t), \quad i = 1, 2, 3,$$

$$\dot{F}_{a,na} = R_{a,na}E_{an}(t),$$

$$\dot{F}_{b1} = 0.$$ 

The proof of the next lemma follows the lines of the proof of Lemma 4.3. It is, therefore, left to the interested reader.

**Lemma 6.3.** The following identities hold true:

(6.3) $[\bar{H}, X_3] = -X_3 + h_3 X_\theta,$

(6.4) $[\bar{H}, \partial_{h_5}] = -h_2 \partial_{h_3},$

(6.5) $[\bar{H}, \partial_{h_4}] = -h_1 \partial_{h_3},$

(6.6) $[\bar{H}, \partial_{h_3}] = -\partial_{\theta},$

(6.7) $[\bar{H}, \partial_{\theta}] = X_\theta + (h_2 h_4 - h_1 h_5) \partial_{h_3}.$

We can start now the computation of the canonical frame.

**Lemma 6.4.** $E_{a4}(t)$ is uniquely specified (up to a sign) by the following conditions

(1) $E_{a4}(t) \in J_{x_0}(t),$

(2) $E_{a4}^{(i)}(t) \in J_{x_0}(t)$, for $i = 1, \ldots, 3,$

(3) $\sigma_\lambda \left( E_{a4}^{(4)}(t), E_{a4}^{(3)}(t) \right) = 1,$

and, by choosing the positive sign, is given by

$$E_{an_a}(t) = e^{-it\bar{H}} \left( \frac{h_2}{h_3} \partial_{h_4} - \frac{h_1}{h_3} \partial_{h_5} \right).$$

**Proof.** Condition (1) and the definition of Jacobi curve $J_{x_0}(t) = e^{-it\bar{H}} \mathcal{Y}(t)$ imply that

$$E_{an_a} = e^{-it\bar{H}} \left( \sum_{i=1}^{5} f_i(t) \partial_{h_i} \right),$$
for some smooth functions $f_i(t)$, with $i = 1, \ldots, n$. We compute the derivative:

$$\dot{E}_{a4}(t) = e_x^{-i\bar{H}} \left( \sum_{i=1}^{5} f_i(t)[\bar{H}, \partial_{h_i}] + \dot{f}_i(t) \partial_{h_i} \right).$$

Condition (2) implies $\pi_x \circ e_x^{i\bar{H}} \dot{E}_{a4}(t) = 0$. Since $\pi_x \partial_{h_i} = 0$, we obtain from Lemma 6.3

$$0 = \pi_x \sum_{i=1}^{5} f_i(t)[\bar{H}, \partial_{h_i}] = -f_1(t)X_1 - f_2(t)X_2.$$ 

From this we obtain $f_1 = f_2 \equiv 0$. Then $E_{a4}(t)$ must be of the form

$$E_{a4}(t) = e_x^{-i\bar{H}} (f_3(t)\partial_{h_3} + f_4(t)\partial_{h_4} + f_5(t)\partial_{h_5}).$$

After some computations we obtain

$$e_x^{i\bar{H}} \dot{E}_{a4} = -f_3 \partial_{h_3} + \left( \bar{f}_3 - h_1 f_4 - h_2 f_5 \right) \partial_{h_3} + \bar{f}_4 \partial_{h_4} + \bar{f}_5 \partial_{h_5},$$

$$e_x^{i\bar{H}} \ddot{E}_{a4} = -f_3 X_0 - \left( 2 \bar{f}_3 - h_1 f_4 - h_2 f_5 \right) \partial_{\theta} + \left( \bar{H} \left( \bar{f}_3 - h_1 f_4 - h_2 f_5 \right) - f_3 (h_2 h_4 - h_1 h_5) - h_1 \bar{f}_4 - h_2 \bar{f}_5 \right) \partial_{h_3} + \bar{f}_4 \partial_{h_4} + \bar{f}_5 \partial_{h_5}.$$ 

Condition (2) is equivalent to $\pi_x \circ e_x^{i\bar{H}} \ddot{E}_{a4}(t) = 0$. Since $\pi_x \partial_{h_i} = 0$, we obtain once again from Lemma 6.3

$$0 = \pi_x \circ e_x^{i\bar{H}} \ddot{E}_{a4}(t) = -f_3 (h_2 X_1 - h_1 X_2).$$

This gives $f_3(t) \equiv 0$. Therefore, $E_{a4}(t)$ must be of the form

$$E_{a4}(t) = e_x^{-i\bar{H}} (f_4(t)\partial_{h_4} + f_5(t)\partial_{h_5}).$$

Finally,

$$e_x^{i\bar{H}} \dddot{E}_{a4} = (h_1 f_4 + h_2 f_5) X_0 + \left( 2 \bar{H} (h_1 f_4 + h_2 f_5) + h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) \partial_{\theta}$$

$$- \left( (h_1 f_4 + h_2 f_5) (h_1 h_5 - h_2 h_4) + \bar{H}^2 (h_1 f_4 + h_2 f_5) + \bar{H} \left( h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) + h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) \partial_{h_3}$$

$$+ \bar{f}_4 \partial_{h_4} + \bar{f}_5 \partial_{h_5}.$$ 

Since $\pi_x \circ e_x^{i\bar{H}} \dddot{E}_{a4} = 0$, we must have

(6.8) \hspace{1cm} h_1 f_4 + h_2 f_5 = 0.

Therefore,

$$e_x^{i\bar{H}} \dddot{E}_{a4} = \left( h_1 f_4 + h_2 f_5 \right) \partial_{\theta} - \left( \bar{H} \left( h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) + h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) \partial_{h_3}$$

$$+ \bar{f}_4 \partial_{h_4} + \bar{f}_5 \partial_{h_5}.$$ 

and

$$e_x^{i\bar{H}} E_{a4}^{(4)}(t) = \left( h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) X_0 + \left( 2 \bar{H} \left( h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) + h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) \partial_{\theta}$$

$$- \left( \bar{H}^2 \left( h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) + \bar{H} \left( h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) + h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) \partial_{h_3}$$

$$+ \left( h_1 \bar{f}_4 + h_2 \bar{f}_5 \right) (h_2 h_4 - h_1 h_5) \partial_{h_3} + \bar{f}_4^{(4)} \partial_{h_4} + \bar{f}_5^{(4)} \partial_{h_5}.$$ 

Then,

$$1 = \sigma_\lambda \left( E_{a4}^{(4)}(t), \dddot{E}_{a4}(t) \right) = \sigma_\lambda(t) \left( e_x^{i\bar{H}} E_{a4}^{(4)}(t), e_x^{i\bar{H}} \dddot{E}_{a4}(t) \right)$$

$$= \left( h_1 \bar{f}_4(t) + h_2 \bar{f}_5(t) \right)^2.$$ 

By choosing the positive sign, using equation (6.8), and the geodesic equations, we have that

(6.9) \hspace{1cm} 1 = h_1 \bar{f}_4 + h_2 \bar{f}_5 = h_3 (h_2 f_4 - h_1 f_5). 

The solution to the system of equations given by (6.8) and (6.9) is

$$f_4 = \frac{h_2}{h_3} \quad f_5 = -\frac{h_1}{h_3}.$$
If we take the derivative of Eq. (6.9), we obtain
\[ h_1 \ddot{f}_4 + h_2 \ddot{f}_5 = -\dot{h}_1 \dot{f}_4 - \dot{h}_2 \dot{f}_5 = \frac{h_3}{h_3} \left( h_2 \dot{f}_4 - h_1 \dot{f}_5 \right). \]

A straightforward computation gives us
\[ \ddot{f}_4 = h_1 - \frac{h_2}{h_3} \left( h_1 h_4 + h_2 h_5 \right), \]
\[ \ddot{f}_5 = h_2 + \frac{h_1}{h_3} \left( h_1 h_4 + h_2 h_5 \right). \]

Therefore
\[ h_1 \ddot{f}_4 + h_2 \ddot{f}_5 = -\frac{h_1 h_4 + h_2 h_5}{h_3}. \]

Furthermore, we have
\[ h_1 \dddot{f}_4 + h_2 \dddot{f}_5 = \dddot{H} \left( h_1 \ddot{f}_4 + h_2 \ddot{f}_5 \right) - \dot{h}_1 \dddot{f}_4 - \dot{h}_2 \dddot{f}_5. \]

After some computations, we have
\[ \dddot{f}_4 = -h_2 h_3 - \frac{1}{h_3^2} \left( h_3^2 \left( 2h_1 h_2 h_5 + h_4 \left( h_1^2 - h_2^2 \right) \right) - 2h_2 \left( h_1 h_4 + h_2 h_5 \right)^2 \right), \]
\[ \dddot{f}_5 = h_1 h_3 + \frac{1}{h_3^2} \left( h_3^2 \left( -2h_1 h_2 h_4 + h_5 \left( h_1^2 - h_2^2 \right) \right) - 2h_1 \left( h_1 h_4 + h_2 h_5 \right)^2 \right), \]

and
\[ \dddot{H} \left( h_1 \ddot{f}_4 + h_2 \ddot{f}_5 \right) = h_2 h_4 - h_1 h_5 + \frac{\left( h_1 h_4 + h_2 h_5 \right)^2}{h_3}. \]

Hence,
\[ h_1 \dddot{f}_4 + h_2 \dddot{f}_5 = \left( h_3^2 + (h_1 h_5 - h_2 h_4) - \frac{2}{h_3} \left( h_1 h_4 + h_2 h_5 \right)^2 \right). \]

Therefore
\[ h_1 \dddot{f}_4 + h_2 \dddot{f}_5 = -\left( h_3^2 + 2 \left( h_1 h_5 - h_2 h_4 \right) - \frac{3}{h_3} \left( h_1 h_4 + h_2 h_5 \right)^2 \right). \]

Finally, we obtain the following expressions for \( F_{a1} \) and \( \dot{F}_{a1} \):
\[ F_{a1}(0) = -\left( h_1 \dddot{f}_4 + h_2 \dddot{f}_5 \right) X_\theta - \left( h_1 \dddot{f}_4 + h_2 \dddot{f}_5 \right) \partial_{h_3} + \left( \dddot{H} \left( h_1 \ddot{f}_4 + h_2 \ddot{f}_5 \right) + h_1 \dddot{f}_4 + h_2 \dddot{f}_5 \right) \partial_{h_3} \]
\[ - \left( h_1 \dddot{f}_4 + h_2 \dddot{f}_5 \right) \left( h_2 h_4 - h_1 h_5 \right) \partial_{h_3} - f_4^{(4)} \partial_{h_4} - f_5^{(4)} \partial_{h_5} \]
\[ = -X_\theta + \frac{h_1 h_4 + h_2 h_5}{h_3} \partial_{h_3} - \left( h_3^2 + 2 \left( h_1 h_5 - h_2 h_4 \right) - \frac{4}{h_3^2} \left( h_1 h_4 + h_2 h_5 \right)^2 \right) \partial_{h_3} \] \( \text{mod } \mathcal{V}_4 \),

and
\[ \dot{F}_{a1}(0) = X_3 - h_3 X_\theta + \frac{h_1 h_4 + h_2 h_5}{h_3} X_\theta + \left( h_3^2 + \frac{3}{h_3} \left( h_1 h_5 - h_2 h_4 \right) - \frac{5}{h_3} \left( h_1 h_4 + h_2 h_5 \right)^2 \right) \partial_\theta \] \( \text{mod } \mathcal{V}_3 \),

where \( \mathcal{V}_3 = \text{span} \{ \partial_{h_3}, \partial_{h_4}, \partial_{h_5} \} \) and \( \mathcal{V}_4 = \text{span} \{ \partial_{h_4}, \partial_{h_5} \} \).

Proof of Theorem 1.2. If we use formulas (6.11) and (6.12), we obtain
\[ R_{a_1} = \sigma x_0 \left( \dot{F}_{a1}(0), F_{a1}(0) \right) = 3h_3^2 + 6 \left( h_1 h_5 - h_2 h_4 \right) - \frac{8}{h_3^2} \left( h_1 h_4 + h_2 h_5 \right)^2. \]

In terms of the first integral
\[ E = \frac{h_3^2}{2} + h_1 h_5 - h_2 h_4. \]
and the coordinates \((\theta, c, \alpha, \beta)\) introduced in Section 6.1, we can write
\[
R_{aa,11} = 6E - 8\frac{\alpha^2}{c^2} \sin^2(\theta - \beta).
\]
\[
\leq 6E.
\]
By direct inspection, the orthonormal basis \(\{X_a, X_b\}\) for \(\mathcal{D}_{x_0}\) obtained by the projection of the canonical frame is
\[
X_a = \pi_* F_{a1}(0), \quad X_b = \pi_* F_{b1}(0).
\]
In the coordinates associated to the splitting \(\Sigma_\lambda = \mathcal{V}_{\lambda_0} \oplus \mathcal{H}_{\lambda_0}\) we have
\[
\mathcal{Q}_{\lambda_0}(t) = \frac{d}{dt} S^\theta(t)^{-1},
\]
where \(S^\theta(t)^{-1}\), in the basis \(\{X_a, X_b\}\), is given by:
\[
S^\theta(t)^{-1} = -\frac{1}{t} \left[ 16 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \frac{4}{63} \left( \begin{pmatrix} R_{aa,11} & 0 \\ 0 & 0 \end{pmatrix} \right) t + O(t^2) \right].
\]
Therefore, the curvature operator has the following expression
\[
\mathcal{R}_{\lambda_0} = \left( \begin{pmatrix} \frac{4}{21} R_{aa,11}(0) & 0 \\ 0 & 0 \end{pmatrix} \right),
\]
where \(R_{aa,11} = 6E - 8\frac{\alpha^2}{c^2} \sin^2(\theta - \beta)\).

\[\square\]

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