Equivariant Kähler Geometry and Localization in the $G/G$ Model

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Abstract

We analyze in detail the equivariant supersymmetry of the $G/G$ model. In spite of the fact that this supersymmetry does not model the infinitesimal action of the group of gauge transformations, localization can be established by standard arguments. The theory localizes onto reducible connections and a careful evaluation of the fixed point contributions leads to an alternative derivation of the Verlinde formula for the $G_k$ WZW model.

We show that the supersymmetry of the $G/G$ model can be regarded as an infinite dimensional realization of Bismut’s theory of equivariant Bott-Chern currents on Kähler manifolds, thus providing a convenient cohomological setting for understanding the Verlinde formula.

We also show that the supersymmetry is related to a non-linear generalization ($q$-deformation) of the ordinary moment map of symplectic geometry in which a representation of the Lie algebra of a group $G$ is replaced by a representation of its group algebra with commutator $[g,h]=gh-hg$. In the large $k$ limit it reduces to the ordinary moment map of two-dimensional gauge theories.

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A Aspects of Localization in Yang-Mills Theory

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# 1 Introduction

In [1] we showed how to obtain the Verlinde formula [2] for the dimension of the space of conformal blocks of the $G_k$ Wess-Zumino-Witten model by explicit evaluation of the partition function of the $G_k/G_k$ model using Abelianization, i.e. a functional integral version of the Weyl integral formula for compact Lie groups. This Abelianization could alternatively be regarded as a localization of the
path integral, although the supersymmetric structure of equivariant cohomology usually responsible for such a localization was not manifest in [1].

On the other hand, in [3, 4] it was pointed out that the $G/G$ model has a supersymmetric extension analogous to that (and useful for the cohomological interpretation) of Yang-Mills theory [3, 4]. While the supersymmetry $\delta$ is somewhat unusual in that it does not square to infinitesimal gauge transformations and hence does not model the action of the gauge group on the space of fields, we want to emphasize that in principle the localization theorems also apply in this situation and can be used to evaluate the partition function. The reason for this is that if the action contains a term of the form $\delta f$ with $\delta^2 f = 0$ then the functional integral is (formally) independent of the coefficient of this term and hence the addition of such terms to the action can be used to localize the integral without changing its value. If $\delta$ happens to square to infinitesimal gauge transformations, then such functionals $f$ are easy to find (the requirement $\delta^2 f = 0$ amounting to the gauge invariance of $f$), while in general this may be more difficult. In the $G/G$ model, though, one does not have to look very far as one is in the fortunate situation where the classical action itself already contains a term of this type whose coefficient can then be varied to establish localization.

That the Verlinde formula should have an interpretation as a fixed point formula had been suggested long ago [1] on the basis of its algebraic structure, and here we find a manifestation of this at the path integral level. The link with the method used in [4] is provided by the observation that localization with respect to this supersymmetry essentially abelianizes the theory in the sense that it localizes to reducible connections. The detailed path integral argument for localization turns out to be slightly more complicated than a simple stationary phase approximation argument would suggest, which is why we present it in some detail here both for the $G/G$ model and, in an appendix, for Yang-Mills theory. But although there are also some subtleties related e.g. to obstructions to the global diagonalizability of group valued maps, explored from a mathematical point of view in [7], that part of the story is nevertheless quite straightforward (a rough, although not quite correct and to the point, sketch of the argument having already been given in [3]) and, all by itself, not terribly enlightening.

What we want to mainly draw attention to in this paper is that this supersymmetry actually encodes a much richer and more interesting structure, both from the complex holomorphic and the equivariant symplectic point of view, than would be required for localization alone. First of all, although the supersymmetry is not
nilpotent, satisfying e.g.

\[ \delta^2 A_z = A_z^g - A_z, \quad \delta^2 \bar{A}_z = A_{\bar{z}} - A_{\bar{z}}^g, \]

it can be split into a sum of two nilpotent operators \( Q \) and \( \bar{Q} \),

\[ \delta = Q + \bar{Q}, \quad Q^2 = \bar{Q}^2 = 0. \]

These operators can be regarded as equivariant Dolbeault operators with respect to a \((g\text{-dependent})\) holomorphic Killing vector field \( X \) on the space \( \mathcal{A} \) of gauge fields,

\[ Q = \partial_A + i(X^{(0,1)}), \quad \bar{Q} = \bar{\partial}_A + i(X^{(1,0)}). \]

Although the \( G/G \) action itself is not manifestly topological, it splits naturally into a \( Q\bar{Q} \)-exact part and a cohomologically non-trivial term, the latter being the manifestly topological gauged Wess-Zumino term. Formally, the theory should then be independent of the coefficient of the former and in this way we recover the one-parameter family of deformations of the \( G/G \) model discussed by Witten \cite{8}. The supersymmetric extension we consider here automatically keeps track of the required quantum corrections to ensure the constancy of this one-parameter family of theories also at the quantum level.

Modulo the usual one-loop determinant, the calculation of the partition function then reduces essentially to the evaluation of the gauged Wess-Zumino term \( \Gamma(g, A) \) for reducible configurations \( A^g = A \). For fixed \( g \), \( \Gamma(g, A) \) turns out to be independent of \( A \) and related to the generalized winding numbers introduced in \cite{7}. This relates the Verlinde formula to Chern classes of torus bundles and provides another manifestation of the Abelianization inherent in the Verlinde formula.

Interestingly, the gauge field functional integral of the supersymmetric extension of the \( G/G \) model is precisely of the form of the integrals studied by Bismut \cite{9} in his investigations of the relations among complex equivariant cohomology, Ray-Singer torsion, anomaly formulae for Quillen metrics and equivariant Bott-Chern currents. Here we make this analogy precise in the belief that it provides a convenient, and from other points of view not completely obvious, cohomological setting for understanding the Verlinde formula. In particular, it identifies the above winding numbers as equivariant cohomology classes on the space of connections.

What is still missing to complete the picture is a direct demonstration that the \( G/G \) functional integral represents the Riemann-Roch integral over the moduli space of flat connections for the dimension of the space of conformal blocks (or holomorphic sections of some power of the determinant line bundle). In particular,
both in the approach pursued in this paper and in the one based on Abelianization, localization onto flat connections is conspicuously absent at every stage of the calculation. One possibility would be to try to find a cohomological topological field theory which has the same relation to the \( G/G \) model that 2d Donaldson theory has to BF (topological Yang-Mills) theory \([5]\). Finding such an alternative localization should also provide one directly with a finite dimensional integral which yields the Verlinde formula via some fixed point theorem or localization formula, but our attempts in this direction have as of yet been unsuccessful.

All this is more or less analogous to the situation in mathematics where such a direct proof of the Verlinde formula is also still missing (see \([10]\) for an up-to-date account of the mathematical status of the Verlinde formula), while Szenes \([11]\) has indicated how it would follow from a proof \([12]\) of the Witten conjectures \([5]\) on the cohomology of the moduli space of flat connections.

The somewhat unusual supersymmetry of the \( G/G \) model also leads to a modification of the underlying symplectic geometry. The \( g \)-dependent vector fields \( X = X(g) \) on \( \mathcal{A} \) satisfy the algebra

\[
[X(g), X(h)] = X(gh) - X(hg)
\]

In particular, therefore, they do not provide a representation of the Lie algebra of the gauge group on \( \mathcal{A} \), but rather of its group algebra equipped with the Lie bracket \( [g, h] = gh - hg \). These vector fields are Hamiltonian, the Hamiltonian (or moment map) being the \( G/G \) action \( S(g, A) \) itself. This moment map is equivariant in the sense that the Lie bracket relation among the vector fields can be lifted to the Poisson algebra of function(al)s on \( \mathcal{A} \) - in fact, equivariance turns out to be equivalent to the Polyakov-Wiegmann identity and this fixes the \( \mathcal{A} \)-independent part of the Hamiltonian \( S(g, A) \) up to a natural ambiguity. Hence the above translates into the Poisson bracket relation

\[
\{S(g, A), S(h, A)\} = S(gh, A) - S(hg, A)
\]

for the \( G/G \) action. This moment map with its generalized equivariance, the Lie algebra having been replaced by the group algebra, is a deformation of the ordinary equivariant moment map of two-dimensional gauge theories in the sense that it reduces to it in the \( k \to \infty \) limit where the \( G/G \) action at level \( k \) becomes the BF action. The latter is nothing other than the generator of ordinary gauge transformations on the space of gauge fields.

This paper is organized as follows. In section 2 we discuss various aspects of the supersymmetric extension of the \( G/G \) model and its one-parameter family of
deformations. The following two sections can then be read fairly independently of each other. In section 3 we first describe the relevant aspects of Bismut’s theory of Bott-Chern currents as well as the localization theorem for their integrals. We then investigate in some detail the path integral argument leading to the localization of the partition function of the $G/G$ model to the ‘classical’ set of reducible configurations. The corresponding argument for Yang-Mills theory as well as some alternative strategies are discussed in Appendix A. At this point the intermediate expression for the partition function one obtains is identical to that arrived at in [1] upon Abelianization and we therefore only sketch briefly how everything can be put together to obtain the Verlinde formula, referring to [1, 4] for details. We begin section 4 with a brief review of ordinary Hamiltonian group actions, show that the $G/G$ action can be interpreted as a moment map satisfying the above generalized equivariance condition, discuss the $k \to \infty$ limit and finally extract from the preceding discussion the basic structure of generalized Hamiltonian group actions and the relation with the standard theory.

2 The Supersymmetry of the $G/G$ Model

We begin with a brief review of those aspects of the $G/G$ model which are of relevance to us. The action of the $G/G$ model at level $k \in \mathbb{Z}$ is

$$kS_{G/G}(g, A) = kS_{G}(g, A) - ik\Gamma(g, A),$$

$$S_{G}(g, A) = -\frac{1}{8\pi} \int_{\Sigma} g^{-1} dAg \ast g^{-1} dAg,$$

$$\Gamma(g, A) = \frac{1}{12\pi} \int_{N} (g^{-1} dg)^3 - \frac{1}{4\pi} \int_{\Sigma} (Adg g^{-1} + AA^g).$$

Here $g \in \mathcal{G} = \text{Map}(\Sigma, G)$ is a (smooth) group valued field on a two-dimensional closed surface $\Sigma$ (with an extension to a bounding three manifold $N$ in the Wess-Zumino term $\Gamma(g) = \Gamma(g, A = 0)$). Not aiming for maximal generality, we will assume that $G$ is simply connected. $A$ is a gauge field for the diagonal $G$ subgroup of the $G_L \times G_R$ symmetry of the ungauged WZW action $S_{G}(g, A = 0)$. The covariant derivative is $d_{Ag} = dg + [A, g]$, $A^g = g^{-1}Ag + g^{-1}dg$ is the gauge transform of $A$, and $\ast$ is the Hodge duality operator with respect to some metric on $\Sigma$. Acting on one-forms, $\ast$ is conformally invariant so that the action only depends on a complex structure on $\Sigma$. In the above formulae and in the following, integrals of Lie algebra valued forms are understood to include a trace. We will occasionally find it convenient to split this action into its $A$-independent and
A-dependent part as \( S_{G/G}(g, A) = S_G(g) + S_G(g, A) \).

### Symmetries and Equations of Motion of the \( G/G \) Model

We now list some properties of the \( G/G \) model we will make use of below. First of all, by construction, the action is invariant under the local gauge transformations

\[
g \to g^h \equiv h^{-1}gh, \quad A \to A^h \equiv h^{-1}Ah + h^{-1}dh.
\]

(2.4)

The variation of the action \( S_{G/G} \) with respect to the gauge fields is

\[
\delta S_{G/G}(g, A) = \frac{1}{2\pi} \int_{\Sigma} (J_z \delta A \bar{z} - J_{\bar{z}} \delta A_z),
\]

(2.5)

where \( J_z \) and \( J_{\bar{z}} \) are the covariantized versions

\[
J_z = g^{-1}D_zg = A^g_z - A_z, \quad J_{\bar{z}} = D_{\bar{z}}g^{-1} = A_{\bar{z}} - A^g_{\bar{z}}^{-1},
\]

(2.6)

of the currents \( j_z = g^{-1}\partial_zg \) and \( j_{\bar{z}} = \partial_{\bar{z}}g^{-1} \) generating the Kac-Moody symmetry of the WZW model \( S_G(g) \). Since they are gauge currents, they are set to zero by the equations of motion of the gauge fields. An equivalent way of expressing the vanishing of the current \( J = J_z dz + J_{\bar{z}} d\bar{z} \) is

\[
J_z = J_{\bar{z}} = 0 \iff d_A g = 0 \iff A^g = A.
\]

(2.7)

The remaining equation of motion can then be cast into the form \( F_A = 0 \) so that classical configurations are gauge equivalence classes of pairs \( (A, g) \) where \( A \) is flat and \( g \) is a symmetry of \( A \). This is very reminiscent of the phase space of Chern-Simons theory on a three-manifold of the form \( \Sigma \times \mathbb{R} \) and even more of that of two-dimensional non-Abelian BF theory (see [4] for a detailed comparison of these two theories). This already suggests that the \( G/G \) model is a topological field theory and this can indeed be established, either by showing directly that the variation of the partition function with respect to the metric is zero [13] or by referring to the equivalence of the \( G/G \) model with Chern-Simons theory on \( \Sigma \times S^1 \) established in [1]. Yet another argument will follow from our considerations below, concerning the relation between the \( G/G \) model and the manifestly topological theory with action the gauged WZ term \( \Gamma(g, A) \).

For later use, we note here a cocycle identity satisfied by the action of the \( G/G \) model. It is a generalization of the Polyakov-Wiegmann identity

\[
S_G(gh) = S_G(g) + S_G(h) - \frac{1}{2\pi} \int_{\Sigma} j_z(g)j_{\bar{z}}(h)
\]

(2.8)
for the WZW action and reads

\[ S_{G/G}(gh, A) = S_{G/G}(g, A) + S_{G/G}(h, A) - \frac{1}{2\pi} \int_{\Sigma} J_z(g)J_z(h) . \]

(2.9)

This ends our review of the \( G/G \) model and now we turn to those aspects of the theory related to supersymmetry and the (equivariant) Kähler geometry on the space of fields.

The Supersymmetric Extension of the \( G/G \) Action

In the case of BF theory and 2d Yang-Mills theory it was found \cite{3} that the geometric interpretation of the theory was greatly facilitated by adding to the original bosonic action a term \( \sim \int_{\Sigma} \psi_z \psi_{\bar{z}} \) quadratic in the Grassmann odd variables \( \psi \) and representing the symplectic form \( \sim \int_{\Sigma} \delta A \delta A \) on the space \( A \) of gauge fields on \( \Sigma \). The resulting theory turned out to be supersymmetric and the supersymmetry could be interpreted as a representation of equivariant cohomology with respect to the infinitesimal action of the gauge group on \( A \). Something analogous is also possible (and turns out to be useful) here, and we just want to mention in passing that a similar supersymmetry can also be shown to exist in Chern-Simons theory on \( \Sigma \times S^1 \).

As a consequence of (2.5) the combined action

\[ S(g, A, \psi) = S_{G/G}(g, A) - \Omega(\psi) \]

(2.10)

\[ \Omega(\psi) = \frac{1}{2\pi} \int_{\Sigma} \psi_z \psi_{\bar{z}} \]

(2.11)

is invariant under the (supersymmetry) transformations

\[ \delta A_z = \psi_z , \quad \delta \psi_z = J_z , \]

\[ \delta A_{\bar{z}} = \psi_{\bar{z}} , \quad \delta \psi_{\bar{z}} = J_{\bar{z}} , \]

(2.12)

supplemented by \( \delta g = 0 \). Note that this is a complex transformation, \( \psi_z \) and \( \psi_{\bar{z}} \) not transforming as complex conjugates of each other. That such a transformation can nevertheless be a symmetry of the action is due to the fact that in Euclidean space the action of the (gauged) WZW model is itself complex, the imaginary part being given by the (gauged) WZ term.

What is interesting about this supersymmetry is that, unlike its Yang-Mills counterpart, it does not square to infinitesimal gauge transformations but rather to ‘global’ or ‘large’ gauge transformations,

\[ \delta^2 A_z = A^g_z - A_z , \quad \delta^2 \psi_z = \psi^g_z - \psi_z \equiv g^{-1}\psi_zg - \psi_z , \]

\[ \delta^2 A_{\bar{z}} = A_{\bar{z}}^g - A_{\bar{z}}^{-1} , \quad \delta^2 \psi_{\bar{z}} = \psi_{\bar{z}}^g - \psi_{\bar{z}}^{-1} \equiv \psi_{\bar{z}} - g\psi_{\bar{z}}g^{-1} . \]

(2.13)
In particular, this implies that, in addition to (infinitesimal) gauge invariance, the purely bosonic action $S_{G/G}$ has another infinitesimal invariance $\Delta$ given by

$$\Delta A = J \Rightarrow \Delta S_{G/G}(g, A) = 0 . \quad (2.14)$$

It is, however, a rather trivial symmetry from another point of view as it is simply proportional to the classical equations of motion and as such a symmetry present in any action: if $S(\Phi^k)$ is a functional of the fields $\Phi^k$ and one defines a variation of $\Phi^k$ by

$$\Delta \Phi^k = \epsilon^{kl} \frac{\delta S}{\delta \Phi^l} \quad (2.15)$$

where $\epsilon^{kl}$ is antisymmetric, then the action is invariant,

$$\Delta S = \epsilon^{kl} \frac{\delta S}{\delta \Phi^k} \frac{\delta S}{\delta \Phi^l} = 0 . \quad (2.16)$$

Actually, also the supersymmetry (2.12) itself can be regarded as such a trivial symmetry of the extended action (2.10) as $\psi_z$ acts as a source for $\psi\bar{Z}$ and vice-versa. From this point of view it is perhaps even more surprising that this supersymmetry is nevertheless a useful symmetry to consider. Partly this is due to the fact that, while the bosonic symmetry is only an infinitesimal symmetry (its exponentiated version involving higher derivatives of the action), its fermionic version is a full symmetry of the action as it stands.

In the case of Yang-Mills theory and BF theory, the two infinitesimal symmetries, gauge transformations and $\Delta$, coincide whereas here the supersymmetry does not model the standard equivariant cohomology on the space of gauge fields. While this does not preclude localization (which can, after all, be established for any Killing vector field on a symplectic manifold [14]), it does lead to certain unusual features and some care has to be exercised when adapting the usual arguments establishing localization of the partition function to the present case. In particular, it suggests that some global (or rather, as we will see, $q$-deformed) counterpart of ordinary infinitesimal equivariant cohomology could provide the right interpretational framework for this model - an issue that appears to merit further investigation. In section 4 we will explore some of the symplectic geometry involved. In the following, however, we will focus on another geometrical framework, related to the theory of equivariant Bismut-Bott-Chern currents [9], a framework which seems to be particularly well adapted to the study of the $G/G$ model and within which localization can be established along similar lines as in
the case of ordinary equivariant cohomology.

The Supersymmetry and Holomorphic Killing Vector Fields on $A$

We start by rewriting the supersymmetry in a slightly more familiar form. The supersymmetry operator $\delta$ can be written as the sum of two nilpotent Dolbeault like operators $Q$ and $\bar{Q}$,

$$\delta = Q + \bar{Q} , \quad Q^2 = \bar{Q}^2 = 0 ,$$

where e.g.

$$QA_z = \psi_z , \quad QA_{\bar{z}} = 0 ,$$

$$Q\psi_z = 0 , \quad Q\psi_{\bar{z}} = J_{\bar{z}} .$$

The action $S_{G/G}$ is also separately $Q$ and $\bar{Q}$ invariant. We should perhaps properly write $\delta = \delta(g)$ and $Q = Q(g)$ and we will occasionally do this when we find it necessary to emphasize that there is not just one but rather a whole $G$’s worth of these derivations on $A$.

To gain some insight into the geometrical meaning of this supersymmetry, we introduce the $g$-dependent vector field

$$X_A(g) = J_z(g)\frac{\delta}{\delta A_z} + J_{\bar{z}}(g)\frac{\delta}{\delta A_{\bar{z}}} \equiv X_A(g)^{(1,0)} + X_A(g)^{(0,1)}$$

on $A$ (actually a section of the complexified tangent bundle of $A$). Note that the infinitesimal action of this vector field generates a global chiral gauge transformation, in the sense that e.g. $X_A(g)A_z = A_z^g - A_z$, so that the exponentiated action takes the form of an iterated (chiral) gauge transformation if $A_z$ and $A_{\bar{z}}$ are treated as independent fields.

Denoting the exterior derivative on $A$ by $d_A$ and contraction by a vectorfield $Y$ by $i(Y)$ we can represent the supersymmetry $\delta$ as the equivariant exterior derivative on $A$ with respect to $X_A(g)$,

$$\delta(g) = d_A + i(X_A(g)) ,$$

and $Q(g)$ and $\bar{Q}(g)$ by

$$Q(g) = \partial_A + i(X_A(g)^{(0,1)}) , \quad Q(g) = \bar{\partial}_A + i(X_A(g)^{(1,0)}) .$$

Within this context, the nilpotency of $Q$, $Q^2 = 0$, expresses the holomorphicity of the vector field $X_A(g)$ on the Kähler manifold $A$. On $X_A(g)$-invariant forms one
also has $Q(g)\bar{Q}(g) = -\bar{Q}(g)Q(g)$. Furthermore, on the fixed point set of $X_A(g)$, $\delta(g)$ reduces to the ordinary exterior derivative.

It can be checked directly that $X_A(g)$ is also a Killing vector field (for the Kähler metric on $A$). Alternatively this follows from the supersymmetry invariance of the action which implies that $X_A(g)$ is symplectic,

$$\delta(g)S(g, A, \psi) = 0 \Rightarrow i(X_A(g))\Omega(\psi) = d_A S_{G/G}(g, A) \ ,$$

with Hamiltonian the $G/G$ action itself - see section 4. Hence, since $X_A(g)$ is holomorphic and $A$ Kähler, $X_A(g)$ is Killing. We are thus precisely in the setting of a Kähler manifold with a holomorphic Killing vector field considered by Bismut \[9\] (albeit for a single vector field and not a whole family of them).

### Splitting the Action of the $G/G$ Model

Before explaining the relation between the $G/G$ model and Bismut’s theory of equivariant Bott-Chern currents, we will look at some more down-to-earth consequences of the supersymmetry of the $G/G$ model. These will eventually lead us to the localization argument of the next section. We will show that the $G/G$ action $S(g, A, \psi)$ can be split into a $Q\bar{Q}$-exact part and a cohomologically non-trivial piece. This will allow us to understand from a slightly different point of view the one-parameter family of deformations of the $G/G$ model already considered by Witten in \[8\].

We first rewrite the kinetic term $S_G(g, A)$ of the $G/G$ action as

$$S_G(g, A) = -\frac{1}{8\pi} \int g^{-1} d_A g \ast g^{-1} d_A g = -\frac{1}{4\pi} \int g^{-1} D_z g \ g^{-1} D_{\bar{z}} g = -\frac{1}{4\pi} \int J_z(g)(J_{\bar{z}}(g))^g \ .$$

Since $\bar{Q}\psi_z = J_z$ and $Q\psi_{\bar{z}} = J_{\bar{z}}$, this term is actually $Q\bar{Q}$-exact (modulo terms involving $\psi$). The complete $G/G$ action $S(g, A, \psi)$ can, in fact, be written as

$$S(g, A, \psi) = -\frac{1}{4\pi} QQ \int \psi_z \psi_{\bar{z}} - \Gamma(g, A, \psi) \ ,$$

where

$$\Gamma(g, A, \psi) = i\Gamma(g, A) + \frac{1}{4\pi} \int (\psi_z \psi_{\bar{z}} + \psi_{\bar{z}} \psi_z) \ .$$

The twisted symplectic form

$$\Omega^g(\psi) = \frac{1}{2\pi} \int \psi_z \psi_{\bar{z}}^g$$
appearing in (2.24) is equivariant,

$$Q\bar{Q} \int_{\Sigma} \psi z \psi^g = -\bar{Q}Q \int_{\Sigma} \psi z \psi^g .$$  \hspace{1cm} (2.27)

\(\Omega^g(\psi)\) is almost as natural a symplectic form to consider on \(\mathcal{A}\) as \(\Omega(\psi)\). In particular, it is easily verified that the vector field \(X_A(g)\) is also Hamiltonian with respect to \(\Omega^g(\psi)\), the corresponding Hamiltonian being \(-S_{G/G}(g^{-1}, A)\) instead of \(S_{G/G}(g, A)\) for the untwisted symplectic form.

It follows from (2.27) and the supersymmetry of the action that \(\Gamma(g, A, \psi)\) is both \(Q\)- and \(\bar{Q}\)-closed (but not exact),

$$Q\Gamma(g, A, \psi) = \bar{Q}\Gamma(g, A, \psi) = 0 .$$  \hspace{1cm} (2.28)

In particular, therefore, \(\Gamma(g, A, \psi)\) defines an equivariant cohomology class on \(\mathcal{A}\). We will see later that it represents winding numbers or Chern classes associated to reducible connections.

A One-Parmater Family of Deformations of the \(G/G\) Model

Since we have split the action of the \(G/G\) model into a \(Q\bar{Q}\)-exact piece and a rest, the theory should be independent of the coefficient of the former which can then be used to localize the functional integral. Let us therefore consider the one-parameter family of theories given by

$$S^s_{G/G}(g, A) = sS_G(g, A) - i\Gamma(g, A) .$$  \hspace{1cm} (2.29)

Actually, including the level \(k\) (an integer) of the (gauged) WZW model, we have a two-parameter family of theories, but \(k\) will play no role in the discussions of this section.

In [8], Witten argued that classically this one-parameter family of theories is constant as a variation with respect to \(s\) is proportional to the classical equations of motion \(J(g) = 0\) of the undeformed model. Furthermore, the classical equations of motion following from the variation of (2.29) are equivalent to those of the undeformed model. In fact, varying \(A_z\) and \(A_{\bar{z}}\) one finds

$$g^{-1}D_z g - \lambda D_{\bar{z}} g g^{-1} = 0 ,$$  \hspace{1cm} (2.30)

$$D_{\bar{z}} g g^{-1} - \lambda g^{-1}D_z g = 0 ,$$  \hspace{1cm} (2.31)

where

$$\lambda = \frac{s - 1}{s + 1} .$$  \hspace{1cm} (2.32)
For $0 < s < \infty$ one has $-1 < \lambda < 1$. Since $|\text{Ad}(g)| \leq 1$ and the equations of motion can be written as

\begin{align}
(1 - \lambda\text{Ad}(g^{-1}))D_{\bar{z}}(g) &= 0 , \\
(1 - \lambda\text{Ad}(g))D_{\bar{z}}(g) &= 0 ,
\end{align}

they are equivalent to the equations of motion $D_{\bar{z}}g = D_{\bar{z}}g = 0$ of the $G/G$ model. Likewise the equation $F_{\bar{z}\bar{z}} = 0$ is unaffected, as a variation of $g$ in $\text{(2.29)}$ leads to

\begin{equation}
D_{\bar{z}}(g^{-1}D_{\bar{z}}g) + \lambda D_{\bar{z}}(g^{-1}D_{\bar{z}}g) + F_{\bar{z}\bar{z}} = 0 ,
\end{equation}

which, by $D_{\bar{z}}g = D_{\bar{z}}g = 0$, implies $F_{\bar{z}\bar{z}} = 0$.

While this establishes the classical constancy of the one-parameter family of theories $S^s_{G/G}$, Witten suggests that quantum mechanically the invariance under $s \to s + \delta s$ is broken, because in the path integral the change in $s$ can only be compensated by a field redefinition of $A$ which involves $A$ itself, leading to a Jacobian which needs to be regularised. Thus, quantum mechanically the $G/G$ model (for any value of $s$) should be equivalent to the manifestly topological theory at $s = 0$, perturbed by quantum corrections of the kind calculated in [1]. For the purposes of localization we will be interested in the opposite limit $s \to \infty$.

On the other hand, the supersymmetric extension of $\text{(2.29)}$, which will also lead to an $s$ dependence of the term quadratic in the $\psi$'s (see $\text{(2.37)}$ below), will automatically keep track of these determinants. We will check below that, formally, the ratio of determinants arising from the terms quadratic in $A$ and $\psi$ is $s$-independent. Just as in [1], their regularization will give rise to quantum corrections to the $G/G$ action, in particular to the shift $k \to k + h$ of the level, confirming Witten’s argument concerning the relation between the theories for different values of $s$.

Consider now the supersymmetric extension of $\text{(2.29)}$, given by

\begin{equation}
S^s(g, A, \psi) = -\frac{s}{4\pi} Q\bar{Q} \int_{\Sigma} \psi_{\bar{z}}\psi_{\bar{z}} - \Gamma(g, A, \psi) ,
\end{equation}

where $\Gamma(g, A, \psi)$ was defined in $\text{(2.25)}$. The $\psi$'s enter in this action in the form

\begin{equation}
\psi_{\bar{z}}[(1 + s) + (1 - s)\text{Ad}(g)]\psi_{\bar{z}} = \psi_{\bar{z}}[(1 + s)(1 - \lambda\text{Ad}(g))]\psi_{\bar{z}} ,
\end{equation}

while the term quadratic in the gauge fields is

\begin{equation}
A_{\bar{z}}[2 + (1 - s)\text{Ad}(g) - (1 + s)\text{Ad}(g^{-1})]A_{\bar{z}} .
\end{equation}

This can be factorized as

\begin{equation}
A_{\bar{z}}[((1 + s) + (1 - s)\text{Ad}(g))(1 - \text{Ad}(g^{-1}))]A_{\bar{z}} .
\end{equation}
Thus formally the ratio of determinants is indeed $s$-independent and given by
the inverse square root of the determinant of the operator $(1 - \text{Ad}(g^{-1}))$ acting on
one-forms. This determinant, restricted to the normal bundle of the fixed point
locus, will arise upon localization as the equivariant Euler class of the normal
bundle, as in the stationary phase formulae of Duistermaat-Heckman [15] and
Berline-Vergne [14]. It can also be checked that no $s$-dependence is reintroduced
into the action through the source terms coupling to $A$ and $\psi$.

3 Localization of the $G/G$ Model

In this section we will show how the above considerations concerning supersym-
metry and deformations of the $G/G$ model can be used to localize the $G/G$ func-
tional integral and to therefore provide an alternative derivation of the Verlinde
formula in terms of equivariant Kähler geometry. This localization could be car-
ried out directly on the basis of what we have established so far. Nevertheless we
find it interesting and instructive that precisely the structure of the $G/G$ model
described in the previous section (the supersymmetry related to a holomorphic
Killing vector field, the action of the form $Q\bar{Q}$ (symplectic form) plus a cohomolog-
ically non-trivial piece) appears in the work of Bismut [9] on the relation between
complex equivariant cohomology, Ray-Singer torsion, Quillen metrics, and Bott-
Chern currents. We therefore start with a brief description of what we believe
is the appropriate mathematical setting for the $G/G$ model before working out
the details of the localization. Ideally this setting should allow one to establish
directly that the $G/G$ action represents the Riemann-Roch-Hirzebruch integrand
for the Verlinde formula in equivariant cohomology on $A$ but so far we have been
unable to show that.

The Mathematical Setting: Equivariant Bismut-Bott-Chern Currents

We will have to introduce some notation. Let $(M, \Omega)$ be a compact Kähler mani-
fold and $X$ a holomorphic Killing vector field on $M$ so that $L(X)\Omega = 0$. Denote
by $M_X$ the zero locus of $X$ (this is also a Kähler manifold), by $N_X$ the normal
bundle to $M_X$ in $M$ and by $J_X$ the skew-adjoint endomorphism of $N$ given by the
infinitesimal action of $X$ in $N_X$. Let $d_X = d + i(X)$ be the equivariant exterior
derivative and $\partial_X$ and $\bar{\partial}_X$ the equivariant Dolbeault operators

$$\partial_X = \partial + i(X^{(0,1)}) \quad \text{and} \quad \bar{\partial}_X = \bar{\partial} + i(X^{(1,0)}) \quad (3.1)$$
satisfying the relations

\[ (\partial_X)^2 = (\bar{\partial}_X)^2 = 0 \ , \tag{3.2} \]
\[ (\partial_X + \bar{\partial}_X)^2 = \partial_X \bar{\partial}_X + \bar{\partial}_X \partial_X = L(X) \ , \tag{3.3} \]
\[ \partial_X \bar{\partial}_X \Omega = -\bar{\partial}_X \partial_X \Omega \ . \tag{3.4} \]

In [9], Bismut studies integrals of the form

\[ \int_M \exp(-is \partial_X \bar{\partial}_X \Omega - i\Gamma) \tag{3.5} \]

where \( s \) is a real parameter and \( \exp(-i\Gamma) \) is some smooth (inhomogeneous) differential form on \( M \) which, in the cases of interest, is equivariantly closed with respect to both \( \partial_X \) and \( \bar{\partial}_X \),

\[ \partial_X \exp(-i\Gamma) = \bar{\partial}_X \exp(-i\Gamma) = 0 \ . \tag{3.6} \]

These integrals are finite dimensional analogues of integrals which appear in the loop space integral approach to index theorems and in the study by Bismut, Gillet and Soulé [10] of Quillen metrics on holomorphic determinant bundles. In the infinite-dimensional case, \( M \) would be the loop space of a Kähler manifold and \( X \) the canonical vector field on the loop space generating rigid rotations of the loops. The equivariant cohomology of operators like \( \partial_X \) has been investigated in [17].

At this point we can clarify the relationship of these considerations with the formulae encountered in our discussion of the \( G/G \) model: \( M \) is \( A \), \( X \) is \( X_A \), (3.1), (3.2) and (3.4) correspond to (2.21), (2.17) and (2.27) respectively, (3.6) is the counterpart of (2.28), and the integral (3.5) corresponds to the functional integral (over \( A \)) of the \((s\text{-deformed}) G/G \) action (2.36).

As a consequence of (3.6), the integrand in (3.5) has the property that its \( s \)-derivative is

\[ \frac{\partial}{\partial s} \exp(-is \partial_X \bar{\partial}_X \Omega - i\Gamma) = -\partial_X \bar{\partial}_X [i\Omega \exp(-is \partial_X \bar{\partial}_X \Omega - i\Gamma)] \ , \tag{3.7} \]

so that, in particular, the integral is \( s \)-independent. The first part of the exponent can be written in the form

\[ i\partial_X \bar{\partial}_X \Omega = \frac{1}{2}d_X(\bar{\partial}_X - \partial_X)i\Omega = \frac{1}{2}(d + i(X))X' \ , \tag{3.8} \]

where \( X' \) is the metric dual of \( X \). Hence one is in a position to apply the standard localization theorems of Duistermaat-Heckman [15] and Berline-Vergne [14].
essence of these theorems is that an equivariantly closed form \( \mu \), 
\((d + i(X))\mu = 0\) for \( X \) a Killing vector field is equivariantly exact away from the zeros of \( X \). To see this, define the (inhomogeneous) differential form \( \nu \) on the complement of a neighbourhood of \( M_X \) in \( M \) by
\[
\nu = \frac{\alpha}{1 + d\alpha} \mu ,
\]
where \( \alpha \) is the normalized metric dual of \( X \),
\[
\alpha = X'/||X||^2 .
\]
As a consequence of the easily verified identities
\[
L(X)\alpha = i(X)d\alpha = 0 \quad (3.11)
\]
one finds that
\[
(d + i(X))\mu = 0 \Rightarrow \mu = (d + i(X))\nu \quad \text{on } M \setminus M_X . \quad (3.12)
\]
In particular, therefore, the top-form component of \( \mu \) is exact on \( M \setminus M_X \) and the integral \( \int_M \mu \) is determined by an infinitesimal neighbourhood of \( M_X \). Explicitly, the integral (3.5) is
\[
\int_M \exp(-is\partial_X \bar{\partial}_X \Omega - i\bar{\Gamma}) = \int_{M_X} \exp(-i\bar{\Gamma})E^{-1}(N_X) , \quad (3.13)
\]
where \( E(N_X) \) is the equivariant Euler class of \( N_X \), represented in terms of \( J_X \) and the curvature form \( R(N_X) \) of \( N_X \) by
\[
E(N_X) = \det[\frac{i}{2\pi}(J_X + R(N_X))] . \quad (3.14)
\]
This can also be thought of as the square root of the determinant of the operator acting on the underlying real bundle - a point of view more natural in gauge theories.

In the \( G/G \) model this formula can now be applied to (or derived from) the functional integral over the gauge fields. The main difference is, of course, that in the \( G/G \) model we are dealing with a family \( \{X_A(g)\} \) of holomorphic Killing vector fields, indexed by \( g \in \mathcal{G} \), as well as with a family of symplectic forms on \( \mathcal{A} \) (the twisted symplectic forms \( \Omega^g(\psi) \)) with respect to which the action takes the form (3.3). Hence, for each \( g \in \mathcal{G} \) the gauge field functional integral will reduce to an integral over the zero locus of \( X_A(g) \), i.e. the connections satisfying \( A^g = A \) and this still needs to be integrated over \( \mathcal{G} \).
Formulae like (3.4) are very reminiscent of formulae characterizing Bott-Chern forms (or currents). In fact, Bott-Chern forms [18] are holomorphic analogues of the Chern-Simons secondary characteristic classes of differential geometry. The latter typically express the independence (in cohomology) of certain Chern-Weil characteristic classes $\Phi_{CW}(F_A)$ by transgression formulae like

$$\Phi_{CW}(F_A) - \Phi_{CW}(F_{A'}) = d\Phi_{CS}(A, A') \quad ,$$

(3.15)

where $A$ and $A'$ are two connections on the same bundle. In the holomorphic context one seeks analogous formulae with the exterior derivative $d$ on the right hand side replaced by $\partial \bar{\partial}$ so that one is dealing with a double transgression. For example, let $E$ be a holomorphic vector bundle over a complex manifold $M$ and denote by $\nabla^h$ the unique holomorphic Hermitian connection on $E$ associated to the Hermitian structure $h$ on $E$ and by $F^h$ its curvature. Consider the (scaled) Chern character

$$\text{ch}(\nabla^h) = \text{Tr}[\exp-(\nabla^h)^2] \quad .$$

(3.16)

Then the main results of Bott and Chern (see [18] and [16]) are that under a variation of $h$ one has

$$\delta \text{Tr} \exp[-(\nabla^h)^2] = \partial \bar{\partial} \text{Tr}[h^{-1}\delta h \exp-(\nabla^h)^2]$$

(3.17)

(this is to be regarded as the analogue of (3.7)) and that this can be ‘integrated’ to give an explicit expression for the Bott-Chern class $\Phi_{BC}^{}(h, h')$ satisfying

$$\text{Tr}[\exp-(\nabla^h)^2] - \text{Tr}[\exp-(\nabla^{h'})^2] = \partial \bar{\partial}\Phi_{BC}^{}(h, h') \quad .$$

(3.18)

We hope that these analogies between the $G/G$ functional integral and integrals of Bott-Chern currents will eventually lead to a better cohomological understanding of the $G/G$ action.

Preliminary Remarks on Localization and the Fixed Point Locus

It follows either from the above arguments (formally extended to functional integrals) or from considering the $s \to \infty$ limit of the gauge field functional integral

$$Z_{G/G}(g, \psi) = \int_A D[A] \exp[-S^s(g, A, \psi)]$$

$$= \int_A D[A] \exp[-\frac{s}{4\pi}Q\bar{Q} \int_{\Sigma} \psi_{\bar{z}}\bar{\psi}_{\bar{z}} - \Gamma(g, A, \psi)] \quad ,$$

(3.19)

that $Z_{G/G}(g, \psi)$ localizes onto the minima $g^{-1}d_A g = 0$ of the kinetic term, i.e. onto the zero locus of $X_A(g)$. A rough (but not quite correct) path integral argument
for this would run roughly as follows. First one decomposes the gauge field \( A \) and
and the group valued field \( g \) into their ‘classical’ and ‘quantum’ parts as

\[
A = A_c + A_q , \quad g = g_c g_q , \quad A^q_c = A_c , \tag{3.20}
\]

and the quantum parts are taken to be orthogonal to the classical configurations
so that the quadratic form for the quantum fields is non-degenerate. Then one
scales the quantum fields by \( 1/\sqrt{s} \),

\[
A_q \rightarrow A_q/\sqrt{s} , \quad g_q = \exp \phi \rightarrow \exp(\phi/\sqrt{s}) , \tag{3.21}
\]

so that the quadratic term is \( s \)-independent. Then, in the limit \( s \rightarrow \infty \) only the
determinant arising from the integral over the quantum fields and the classical
action \( \Gamma(g_c, A_c) \) survive - which is just what (3.13) expresses. Actually, in the
case at hand we will have to be a little bit more careful. The zero locus is still
infinite-dimensional and the quadratic form for \( A_c \) is provided by part of the
quantum field \( g_q \) (in fact, for fixed \( g_c \), \( \Gamma(g_c, A_c) \) turns out to be independent of
\( A_c \)) which should hence not be scaled away.

The reason for the occurrence of this problem is the fact that the condition for
\( A \) to be a critical point of the vector field \( X_A(g) \) is a condition on both \( A \) and \( g \)
while e.g. the localization theorem of the previous section only applies (formally
at least) to the \( A \)-part of the integral for fixed \( g \). Thus one should be careful
to implement this \( g \)-dependent localization correctly. This can be achieved by
choosing a parametrization for \( g \) in terms of ‘classical’ and ‘quantum’ fields which
is more explicit than the one used above. In particular we will see that localization
can be achieved by treating the \( g \)-integral exactly, using the \( s \)-independence only
to massage the gauge field integral.

As this complication is already present in the large \( k \) limit of the \( G/G \) model, i.e.
in BF theory, and its origin as well as the way to handle it are somewhat easier
to understand in that example, we discuss it in some detail in the appendix. Here
we will instead present a streamlined version of the argument adapted to the \( G/G \)
model.

Let us first take a closer look at the space

\[
\mathcal{A}(g) = \{ A \in \mathcal{A} : A^g = A \} \tag{3.22}
\]

onto which the theory eventually localizes. While usually the reducibility condi-
tion is regarded as an equation for \( g \) for a fixed \( A \), here instead \( g \) is fixed and
one is looking for gauge fields for which \( g \) is contained in their isotropy group.
Nevertheless, this will turn out to be a condition on certain components of \( A \) as
well as on $g$ since for most $g$ there will be no $A$ whatsoever satisfying $A^g = A$. For instance, by multiplying by powers of $g$ and taking traces, one finds that

$$A^g = A \Rightarrow d \text{Tr} \, g^n = 0, \forall n \in \mathbb{Z}, \quad (3.23)$$

so that, essentially, $g$ is conjugate to a constant matrix.

In order to obtain some more information on $A(g)$, we will use the method of diagonalization introduced in [1] to calculate the $G/G$ functional integral. Thus assume that $g$ can be written as $g = h t h^{-1}$, where $t \in \text{Map}(\Sigma, T)$ takes values in the maximal torus $T$ of $G$. Pointwise this can of course always be achieved, and the global issues have been analyzed in detail in [7]. In particular, one finds that if $g$ is regular, i.e. if at every point $x \in \Sigma$ the dimension of the centralizer of $g(x)$ in $G$ is equal to the rank of $G$, then $t$ can be chosen to be smooth globally. Moreover, the torus component of the transformed gauge field $A^h$ is a connection on a possibly non-trivial $T$ bundle over $\Sigma$, indicating that $h$ will in general not be smooth globally. The $T$ bundle in question turns out to be the pull-back of $G \rightarrow G/T$ to $\Sigma$ via the $G/T$-part of a lift of $g$ to $G/T \times T$. Thus, for regular maps the reducibility condition can be written as

$$A^g = A \iff A^h = A^{ht}$$

$$\iff (A^h)^t = (A^h)^t + t^{-1} dt$$

$$\iff (A^h)^{g/t} = t^{-1} (A^h)^{g/t}$$

$$\iff dt = 0 \text{ and } (A^h)^{g/t} = 0 \quad (3.26)$$

We therefore see that the localization essentially abelianizes the theory and at this point the analysis can proceed more or less as in [1]. In particular, for a regular $g$ with $h^{-1}gh = t$ constant, the space $A(g)$ is isomorphic to the space of gauge fields on a torus bundle over $\Sigma$ and hence

$$dt \neq 0 \Rightarrow A(g) = \emptyset$$

$$dt = 0 \Rightarrow A(g) \sim \Omega^1(\Sigma, t) \quad (3.27)$$

We want to draw attention to the fact that there is no condition on the torus gauge field $(A^h)^t$, so that that part of the gauge field functional integral is not localized and needs to be calculated directly.

At the other extreme, when $g$ is the identity matrix, there is no localization at all, $A(g) \sim A$, and the functional integral (3.19) is hopelessly divergent as the action is then identically zero. In general, some regularization prescription has to be adopted to deal with highly non-regular elements of $G$, for all of which the
quadratic form for the gauge fields in the $G/G$ action is in some sense degenerate. The results of [1, 4] suggest that any reasonable prescription should be tantamount to integrating only over regular maps and discarding the non-regular maps. This is what we will henceforth do.

**Evaluation of the Action on the Fixed Point Locus and Winding Numbers**

On $A(g)$, the action $S^g_{G/G}(g, A)$ reduces to $-i\Gamma(g, A)$. But, since

$$\Gamma(g, A) = \Gamma(g) - \frac{1}{4\pi} \int_{\Sigma} Adg g^{-1} + AA^g,$$

one finds that this simplifies to

$$\Gamma(g, A)|_{A(g)} = \Gamma(g) - \frac{1}{4\pi} \int_{\Sigma} Adg g^{-1} = \Gamma(g) - \frac{1}{4\pi} \int_{\Sigma} Ag^{-1}dg \equiv W(g, A)$$

(where the second line follows from $A^g = A$ and $f_\Sigma(g^{-1}dg)^2 = 0$). $W(g, A)$ is precisely the cocycle,

$$W(gh, A) = W(g, A^{h^{-1}}) + W(h, A),$$

implementing the lift of the $G$ action to the prequantum line bundle of Chern-Simons theory, i.e. to the line bundle over $A$ with curvature form equal to $(k$ times) the basic symplectic form $\Omega(\psi)$.

$W(g, A)$ has some more or less obvious properties which suggest that it is a topological invariant associated with $g$. First of all, on $A(g)$ it is of course invariant under smooth gauge transformations,

$$A^g = A \Rightarrow W(g^h, A^h) = W(g, A).$$

What is more interesting, however, is that it is independent of $A \in A(g)$,

$$W(g, A) = W(g, A') \forall A, A' \in A(g).$$

The easiest way to see that is to use the representation $g = hth^{-1}$ to write $A^h = a$ and $g^h = t$ where $a$ is a torus gauge fields and $t$ is constant. Then one finds

$$\int_{\Sigma} Ag^{-1}dg = \int_{\Sigma} (hah^{-1} - dh h^{-1})(ht^{-1}h^{-1}dhth^{-1} - dh h^{-1})$$

$$= \int_{\Sigma} t^{-1}h^{-1}dhth^{-1}dh.$$

As this is independent of $A$, the claim follows. Hence, as mentioned above, a quadratic form for the $A_c$-integration will have to be provided by those parts of $g_q$ which couple to $A_c$. We will discuss this in more detail below.
Another property of $W(g, A)$, which follows directly from the cocycle identity (3.30), using $A^g = A$, is

$$W(g^n, A) = nW(g, A) \quad \forall \ n \in \mathbb{Z} \ .$$  \hspace{1cm} (3.34)

These observations taken together strongly suggest that $W(g, A)$ is related to the winding numbers for regular maps $g \in \text{Map}(\Sigma, G_r)$ introduced in [7]. These winding numbers and winding number sectors exist in $\text{Map}(\Sigma, G_r)$ because, although the non-regular elements of $G$ are of codimension three in $G$, maps from a two-manifold into $G$ which pass through a non-regular element somewhere are actually of codimension one. Hence, in contrast to $\text{Map}(\Sigma, G)$, $\text{Map}(\Sigma, G_r)$ is not connected but turns out to consist of a $\mathbb{Z}^r$'s worth of connected components, where $r$ is the rank of $G$,

$$\pi_0(\text{Map}(\Sigma, G_r)) = \mathbb{Z}^r .$$ \hspace{1cm} (3.35)

It is no coincidence that this is the same as $\pi_2(G/T)$. Explicitly, these winding numbers of $g$ can be written as

$$n^l(g) = \frac{1}{4\pi} \int_{\Sigma} \alpha^l [h^{-1}dh, h^{-1}dh] \quad l = 1, \ldots, r ,$$ \hspace{1cm} (3.36)

where the $\alpha^l$ are simple roots of $G$. To see how they are related to $W(g, A)$, let us write $t$ as $t = \exp \phi$ where $\phi = \alpha^l \phi_l$ is constant. Then the relationship between $W(g, A)$ and $n^k(g)$ is

$$W(g, A) = \phi_l n^l(g) .$$ \hspace{1cm} (3.37)

Thus on the fixed point set $\mathcal{A}(g)$, the action of the $G/G$ model simply reduces to a linear combination of the winding numbers (3.36),

$$kS_{G/G}(g, A)|_{\mathcal{A}(g)} = ik\phi_l n^l(g) .$$ \hspace{1cm} (3.38)

These winding numbers are also the Chern classes of the connection $(A^h)^t$ [7, Corollary 4]. We will obtain both these results in the next section when discussing how the action of the $G/G$ model reduces to that on the fixed point locus analyzed here.

Path Integral Derivation of the Localization

Having analyzed the ‘classical’ action of the $G/G$ model, we will still have to establish how and in which sense localization reduces the path integral to an integral over the classical fields. Above we sketched a rough (albeit wrong) path integral argument for localization in the $G/G$ model. We will now present a more
careful argument which has the virtue of being correct. It is the exact counterpart of the method (more precisely, method (2)) used in the appendix for solving Yang-Mills theory, and we refer to the appendix for a more detailed discussion in that case.

We begin by writing the action of the (deformed) G/G model more explicitly in terms of the ‘classical’ and ‘quantum’ fields. It follows from the above that a general regular map $g$ can be written in the form

$$g = h t_c \bar{t} h^{-1} = (ht_c h^{-1})(h \bar{t} h^{-1}) ,$$  \hspace{1cm} (3.39)

where the classical field $t_c$ is constant and $\bar{t}$ contains no constant mode. Note that (3.39) is invariant under $h \rightarrow h \tau$ for $\tau \in \text{Map}(\Sigma, T)$, while $h \rightarrow \gamma h$ for $\gamma \in \text{Map}(\Sigma, G)$ generates the adjoint (gauge) transformation on $g$. The second equality in (3.39) represents the improved and refined version of the decomposition $g = g_c g_q$ used in (3.20). If we plug this form of $g$ into the deformed bosonic G/G action (2.29), it is clear that by gauge invariance the first term can alternatively be written as

$$S_G(g, A) = S_G(t_c \bar{t}, A^h) .$$  \hspace{1cm} (3.40)

Clearly, $A^h$ is invariant under gauge transformations. It is, however, not invariant under the ‘parametrization symmetry’ $h \rightarrow h \tau$. On the other hand, this is certainly an invariance of the action as the fields appearing on the left hand side of (3.40) are inert under this transformation.

Decomposing $A^h$ into its $t$- and $g/t$-components, $A^h = (A^h)^t + (A^h)_{g/t}$, we can see what $h \rightarrow h \tau$ implies for $A^h$ explicitly:

$$(A^h)^t \rightarrow (A^h)^t + \tau^{-1} d\tau ,$$

$$(A^h)_{g/t} \rightarrow \tau^{-1} (A^h)_{g/t} \tau .$$  \hspace{1cm} (3.41)

Thus $(A^h)^t$ transforms as and hence is a connection on some $T$ bundle, while the $g/t$-component is a section of a bundle associated to it via the adjoint action of $T$ on $g/t$.

In terms of this decomposition of $A^h$, the action (3.40) becomes

$$S_G(g, A) = -\frac{1}{8\pi} \int_{\Sigma} d\bar{t} \ast d\bar{t} + (1 - \text{Ad}(t_c \bar{t})) (A^h)_{g/t} \ast (1 - \text{Ad}(t_c \bar{t})) (A^h)_{g/t} .$$  \hspace{1cm} (3.42)

The gauged WZ term requires a little bit more care. Technically the reason for this is that, in contrast to $S_G(g, A)$, $\Gamma(g, A)$ is not invariant under arbitrary, possibly discontinuous, gauge transformations, the integrand transforming homogenously under gauge transformations only up to a total derivative on $\Sigma$. Hence one cannot
invoke gauge invariance to (falsely) conclude that $\Gamma(g, A) = \Gamma(t_c \bar{t} h^{-1}, A)$ because $h$ may not be continuous. We will instead calculate $\Gamma(ht_c \bar{t} h^{-1}, A)$ directly.

Let us start with the WZ term $\Gamma(g)$. We have to find an extension of $g = ht_c \bar{t} h^{-1}$ to some bounding three-manifold $N$. First of all we choose $N$ to be $N = \Sigma \times [0, 1]$ with $\partial N = \Sigma \times \{1\} - (\Sigma \times \{0\})$. We will now extend $g$ to $N$ in such a way that $g|_{\Sigma \times \{0\}} = 1$ so that there are no contributions to the action from that part of the boundary. Writing $t = t_c \bar{t}$ as $t_c \bar{t} = \exp \phi \equiv \exp(\phi_c + \bar{\phi})$, we choose this extension to be simply

$$g(x, s) = h(x) \exp(s\phi) h^{-1}(x) ,$$

which has the desired properties

$$g(x, 0) = 1 , \quad g(x, 1) = g(x) ,$$

as well as preservation of the right $T$-invariance of $h$. It is then a matter of straightforward calculation to determine $\Gamma(g)$:

$$\Gamma(g) = \frac{1}{2\pi} \int_{\Sigma} \int_{0}^{1} ds \left( g(x, s)^{-1} dg(x, s) \right)^3$$

$$= \frac{1}{4\pi} \int_{\Sigma} \phi [h^{-1} dh, h^{-1} dh]$$

$$+ \frac{1}{4\pi} \int_{\Sigma} \int_{0}^{1} ds \frac{d}{ds} \left[ \exp(-s\phi) h^{-1} dh \exp(s\phi) h^{-1} dh \right]$$

$$= (\phi_c) m^l (g) + \frac{1}{4\pi} \int_{\Sigma} \phi [h^{-1} dh, h^{-1} dh] + \frac{1}{4\pi} \int_{\Sigma} t^{-1} h^{-1} dh \bar{h} dh .$$

We see here the emergence of the winding number term (3.36,3.37) anticipated in the previous section. Determining the remaining terms of $\Gamma(g, A)$ is straightforward and, putting everything together, the gauged WZ term can be written as

$$\Gamma(g, A) = \frac{1}{4\pi} \int_{\Sigma} \phi [h^{-1} dh, h^{-1} dh] + \frac{1}{2\pi} \int_{\Sigma} h^{-1} dh d\bar{\phi}$$

$$- \frac{1}{4\pi} \int_{\Sigma} (A^h)^{\mathfrak{g}/t} \text{Ad}(t_c \bar{t})(A^h)^{\mathfrak{g}/t} - \frac{1}{2\pi} \int_{\Sigma} (A^h)^{t} d\bar{\phi} .$$

This expression is not yet particularly transparent. In particular, as $(A^h)^{t}$ may be a connection on a non-trivial torus bundle, it is not clear that (3.46) is even well defined. However, because of the interplay between winding numbers and Chern classes this is indeed the case. In particular, although integrating by parts the last term is illegal, the first, second and fourth terms combine to give

$$\Gamma(g, A) = \frac{1}{4\pi} \int_{\Sigma} \phi F (A^h)^{\mathfrak{g}/t} - \frac{1}{4\pi} \int_{\Sigma} (A^h)^{\mathfrak{g}/t} \text{Ad}(t_c \bar{t})(A^h)^{\mathfrak{g}/t}$$

$$- \frac{1}{2\pi} \int_{\Sigma} d(\bar{\phi} h^{-1} Ah) .$$

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This makes it manifest that (3.46) is globally well defined. The last term could be non-zero only because of possible winding modes of $\bar{\phi}$ but we will see immediately that this term is actually not there at all.

One can now decompose $(A^h)^t$ into the sum of a background connection $A_0$ and a one-form $a^t$. Integrating over the latter imposes the condition that $d\bar{\phi} = 0$ and hence $\bar{\phi} = 0$ as $\bar{\phi}$ has no constant modes. Hence $\bar{t}$ disappears from $S_G(g, A)$ while (3.46) and (3.47) reduce to

$$\Gamma(g, A) \to \frac{1}{4\pi} \int_{\Sigma} \phi_c [h^{-1} dh, h^{-1} dh] - \frac{1}{4\pi} \int_{\Sigma} (A^h)^{g/t} \text{Ad}(t_c)(A^h)^{g/t}$$

(3.48)

and

$$\Gamma(g, A) \to -\frac{1}{2\pi} \text{Tr} \phi_c \int_{\Sigma} F_{A_0} - \frac{1}{4\pi} \int_{\Sigma} (A^h)^{g/t} \text{Ad}(t_c)(A^h)^{g/t}$$

(3.49)

respectively. This establishes among other things the relation between the winding numbers $n^l(g)$ of (3.36) and the Chern classes of the corresponding torus connection $(A^h)^t$,

$$n^l(g) = -\frac{1}{2\pi} \int_{\Sigma} \omega^t F_{(A^h)^t} \equiv n^l((A^h)^t) .$$

(3.50)

There are now several ways to calculate the path integral over the remaining fields (and hence the partition function of the $G/G$ model). One possibility is to choose the gauge $h = 1$. This is essentially what we did in [1] and is the Abelianization approach to the evaluation of the path integral which we will not repeat here. We just mention that for $s = 1$ the terms from (3.42) and (3.47) involving $(A^h)^{g/t} = A^h$ combine to give the chiral quadratic form $A^g/t (1 - \text{Ad}(t^{-1}))A^g/t$ whose determinant formally cancels against the Faddev-Popov determinant up to zero modes.

Alternatively one can now solve the theory via localization. To that end we scale $(A^h)^{g/t}$ and its superpartner by

$$(A^h)^{g/t} \to s^{-1/2}(A^h)^{g/t} , \quad (\psi^h)^{g/t} \to s^{-1/2}(\psi^h)^{g/t} ,$$

(3.51)

(as the $\psi$’s are Grassmann odd, this does not introduce any $s$-dependence in the measure) and take the limit $s \to \infty$. In this limit, (3.42) reduces to

$$-\frac{1}{4\pi} \int_{\Sigma} (A^h)^{g/t} (1 - \text{Ad}(t_c)^2)(A^h)^{g/t} ,$$

(3.52)

only the first (topological) term of (3.49) survives, and the fermionic part (2.37) becomes

$$-\frac{1}{4\pi} \int_{\Sigma} (\psi^h)^{g/t} (1 - \text{Ad}(t_c)(\psi^h)^{g/t} - \frac{1}{4\pi} \int_{\Sigma} (\psi^h)^t(\psi^h)^t .$$

(3.53)
Note that the action is still completely gauge invariant under
\[ A \rightarrow A^{\tilde{g}}, \quad h \rightarrow \tilde{g}^{-1} h \]  
(3.54)
(the latter corresponding to \( g \rightarrow \tilde{g}^{-1}g\tilde{g} \)), as the gauge fields only appear in the manifestly gauge invariant St"uckelberg combination \( A^h \)).

Changing variables from \( A \) and \( \psi \) to \( A^h \) and \( \psi^h \) the \( h \)-integral (gauge volume) factors out. Performing the integral over \( A^{g/t} \) and \( \psi^{g/t} \), one obtains the functional determinant

\[
\text{Det}^{-1/2}[1 - \text{Ad}(t_c)]|_{\Omega^1(\Sigma, g/t)} .
\]  
(3.55)

Here we have indicated explicitly that this is a functional determinant on the space of \( g/t \)-valued one-forms on \( \Sigma \). On the other hand, the Jacobian from the change of variables \( g \rightarrow (h, t) \) is

\[
\text{Det} [1 - \text{Ad}(t)]|_{\Omega^0(\Sigma, g/t)} .
\]  
(3.56)

Obviously these determinants almost cancel. It has been shown in [1, 4] that this ratio is (up to a phase) a finite-dimensional determinant, arising from the unmatched harmonic modes between zero- and one-forms. Using a regularization that preserves the \( T \) gauge symmetry, one explicitly finds that the regularized product of (3.55) and (3.56) is

\[
\exp(ih\phi \eta^l(g)) \det^{\chi(\Sigma)/2}(1 - \text{Ad}(t_c))|_{g/t} ,
\]  
(3.57)

where \( h \) is the dual Coxeter number of \( G \) and \( \chi(\Sigma) \) the Euler number of \( \Sigma \). The remaining action is then simply the linear combination of Chern classes appearing in (3.38) or (3.49), i.e. the \( G/G \) action evaluated on the classical configurations (the zero locus of the vector field \( X_A(g) \)) \( A^g = A \), the net effect of the phase in (3.57) being to shift the coefficient of this term from the level \( k \) to \( k + h \).

Modulo the modifications brought about the fact that the \( G/G \) partition function includes an integral over \( g \), this result agrees exactly with the predictions of Bismut’s localization formula (3.13) for the integral (3.5). For example, it is easy to see that the determinant (3.55) is precisely the equivariant Euler class (3.14) of the normal bundle to \( A(g) \) appearing in (3.13). The only thing to note is that the normal bundle \( N(g) \) to \( A(g) \) in \( A \) is trivial, as \( A(g) \) is contractible (it need of course not be equivariantly trivial) and that \( N(g) \) has also got vanishing curvature in terms of the connection it inherits from the (flat) Levi-Civita connection on \( A(g) \). Thus the equivariant curvature \( J_X + R(N_X) \) is given entirely by the scalar part \( J_X \sim J(g) \) which acts as

\[
J(g)Y = (Y^g_z - Y_z)dz + (Y^g_{\bar{z}} - Y_{\bar{z}})d\bar{z} ,
\]  
(3.58)
leading to the above determinants.

Putting Everything Together: The Verlinde Formula

One can now follow exactly the same steps as in [1, section 7] to complete the evaluation of the partition function. As the path integral derivation of the Verlinde formula has been explained in detail in [1, 4], we will be rather brief in this section. We will collect the results obtained above and then only summarize the main steps of the evaluation. The interested reader is referred to [1, 4].

As a consequence of what we have learnt so far, we already know that the $G/G$ partition function

$$
Z_{G/G}(\Sigma, k) = \int_g D[g] \int_A D[A] \exp(-kS_{G/G}(g, A)) \quad (3.59)
$$

reduces to an expression involving only an infinite sum (arising from the sum over all isomorphism classes of torus bundles on $\Sigma$) and a finite dimensional integral over $T$,

$$
Z_{G/G}(\Sigma, k) = \sum_{(n^l) \in \mathbb{Z}^r} \int_T \prod_{l=1}^r d\phi^l \exp(i(k + h)\phi^l n^l) \det \chi(\Sigma) / 2 |g/t \cdot (1 - \text{Ad}(\exp \alpha^l \phi_l))|_{g/t} . \quad (3.60)
$$

Now the infinite sum is a periodic delta function giving rise to a quantization condition on the torus fields $\phi_l$. The allowed values of $\phi_l$ are

$$
\sum_{(n^l)} \Rightarrow \phi_l = \frac{2\pi m_l}{k + h} , \quad m_l \in \mathbb{Z} . \quad (3.61)
$$

This turns the integral over $T$ itself into a sum. As the $\phi_l$ are compact scalar fields, only a finite number of the discrete values for $\phi$ are allowed and hence this sum is finite. By restricting the sum to be over regular elements of $T$ only and by eliminating the residual Weyl group invariance, this sum can be shown to be a sum over the integrable representations of the group $G$ at level $k$. For example, for $G = SU(n)$ one finds

$$
\phi_l = \frac{2\pi m_l}{k + n} , \quad m_l > 0 , \quad \sum m_l < k + h \quad (3.62)
$$

(the values 0 and $k + h$ have been excluded because they correspond to non-regular values of $t$). The range of the $m_l$ is precisely the range labelling the integrable representations of $SU(n)$. To be even more concrete, let us consider the case $G = SU(2)$. Using $\det(1 - \text{Ad}(t)) \propto \sin^2 \phi/2$, one finds that

$$
Z_{SU(2)/SU(2)}(\Sigma, k) \propto \sum_{l=1}^{k+1} \left( \sin \frac{\pi l}{k + 2} \right) \chi(\Sigma) . \quad (3.63)
$$
Up to a normalization factor \(((k + 2)/2)^{\chi(\Sigma)}\) (which can also be determined - see \([1]\)) this is indeed the \(SU(2)\) Verlinde formula. Analogously one obtains the Verlinde formula for other compact groups. We refer to \([1, 4]\) for further details concerning e.g. the range of \(\phi\) and the role of the action of the Weyl group and to \([7, 11]\) for what happens in the case of non-simply connected groups and/or non-trivial \(G\)-bundles.

4 The \(G/G\) Action as a Generalized Moment Map

In this section we want to uncover the symplectic geometry underlying the supersymmetry of the \(G/G\) model. As already hinted at above, the structure that we will find is not that of ordinary Hamiltonian group actions on symplectic manifolds together with their infinitesimal moment maps but rather a globalization thereof in which the role played by the Lie algebra in the usual setting is played by the group (or rather, as we will see, by its group algebra) instead. We will find that the \(G/G\) action can be interpreted as such a generalized moment map for the group action on \(A\) generated by the vector fields \(X_A(g)\). Furthermore the (generalized) equivariance of this moment map turns out to be equivalent to the Polyakov-Wiegmann identity and hence determines the \(A\)-independent part of the action to be the WZW action \(S_G(g)\).

A Brief Review of Hamiltonian Group Actions

Let \((M, \Omega)\) be a symplectic manifold and \(H\) be a group acting by diffeomorphisms on \(M\). Denote by \(X_M\) the vector field on \(M\) corresponding to \(X \in h = \text{Lie}_H\) so that one has

\[ [X_M, Y_M] = [X, Y]_M \ . \tag{4.1} \]

The action on \(M\) is said to be symplectic if each vector field \(X_M\) leaves the symplectic form invariant,

\[ L(X_M)\Omega \equiv (d + i(X_M))^2 \Omega = 0 \quad \forall X \in h \ . \tag{4.2} \]

As \(\Omega\) is closed this is equivalent to \(di(X_M)\Omega = 0\). If \(i(X_M)\Omega\) is not only closed but actually exact,

\[ i(X_M)\Omega = dF(X) \tag{4.3} \]

for some function \(F(X)\) on \(M\), then the action is said to be Hamiltonian with \(X_M \equiv V_{F(X)}\) the corresponding Hamiltonian vector field. Note that this defines
\( F(X) \) only up to the addition of an \( X \)-dependent constant \( c(X) \). It follows that the inhomogenous form \( F(X) - \Omega \) is equivariantly closed,

\[
(d + i(V_{F(X)}))(F(X) - \Omega) = 0 .
\]  

(4.4)

Introducing a basis \( \{X_a\} \) of \( h \) such that \([X_a, X_b] = f_{ab}^c X_c \) and \( X = \phi^a X_a \), and denoting the corresponding Hamiltonian and Hamiltonian vector field by \( F_a \) and \( V_a \) respectively, this can also be written as

\[
(d + \phi^a i(V_a))(\phi^a F_a - \Omega) = 0 .
\]  

(4.5)

The operator on the left hand side is nilpotent on \( \mathbf{H} \)-invariant forms and its cohomology can be used to define the \( \mathbf{H} \)-equivariant cohomology \( H^*_{\mathbf{H}}(M) \) of \( M \). For more on the relation between this (Cartan) and other models of equivariant cohomology see [19].

The collection of functions \( \{F(X)\} \) can equivalently be thought of as either a map from \( h \) to \( C^\infty(M) \) or as a map \( J \) from \( M \) to the dual \( h^* \) of the Lie algebra of \( \mathbf{H} \). These two pictures are related by

\[
F : h \to C^\infty(M) \quad \text{(4.6)}
\]

\[
J : M \to h^* \quad \text{(4.7)}
\]

\[
J(m)(X) = F(X)(m) . \quad \text{(4.8)}
\]

\( J \) is called the moment map of the Hamiltonian group action.

If one defines the Poisson bracket of two functions \( F(X) \) and \( F(Y) \) in the usual way by

\[
\{F(X), F(Y)\} = L(V_{F(X)})F(Y) = i(V_{F(X)})i(V_{F(Y)})\Omega ,
\]  

(4.9)

then it follows straightforwardly from the definitions that

\[
V_{\{F(X), F(Y)\}} = [V_{F(X)}, V_{F(Y)}] = V_{F([X,Y])} .
\]  

(4.10)

Because of the non-degeneracy of the symplectic form this implies that

\[
d\{F(X), F(Y)\} = dF([X,Y]) ,
\]  

(4.11)

Hence the Poisson bracket \( \{F(X), F(Y)\} \) differs from the Hamiltonian \( F([X,Y]) \) only by a constant \( c(X,Y) \). As a consequence of the Jacobi identity, \( c(.,.) \) defines a two-cocycle on \( h \). If this two-cocycle is trivial, the constants \( c(X) \) can be adjusted in such a way that

\[
\{F(X), F(Y)\} = F([X,Y]) \quad \forall X,Y \in h ,
\]  

(4.12)
i.e. such that

\[ \{ F_a, F_b \} = f^c_{ab} F_c . \]  

(4.13)

Then the assignment \( X \rightarrow F(X) \) defines a Lie algebra morphism from \( \mathfrak{h} \) to the (Poisson) Lie algebra \( C^\infty(M, \Omega) \). In that case the moment map \( J \) intertwines the \( \mathfrak{h} \)-action on \( M \) and the coadjoint action on \( \mathfrak{h}^\ast \) and is said to be equivariant. If either the second Lie algebra cohomology group of \( \mathfrak{h} \) is trivial, \( H^2(\mathfrak{h}) = 0 \), or \( M \) is compact, equivariance can always be achieved (in the latter case one can fix the constants \( c(X) \) by demanding that \( f_M d\mu F(X) = 0 \) where \( d\mu \) is the Liouville measure on \( M \)). Furthermore, if \( H^1(\mathfrak{h}) = 0 \) (i.e. if \([\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}\)), equivariance fixes the moment map uniquely (otherwise one can, without violating (4.12), add any functional \( c(.) \) to \( F \) which vanishes on commutators, i.e. \( c \in (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}])^\ast \)).

This sort of structure occurs naturally in e.g. 2d Yang-Mills theory or BF theory. The moment map is the generator of gauge transformations on the symplectic space \( \mathcal{A} \) of 2d connections and the action is of the form \( \phi^a F_a - \Omega \) (plus a term quadratic in \( \phi \) for Yang-Mills theory), where now \( \phi^a \) is a Lie algebra valued scalar field and \( F_a \) is the curvature two-form. In this context (4.5) expresses the (equivariant) supersymmetry of the theory - see [5] or [4] for more information.

**Interpretation of the \( G/G \) Action as a Generalized Moment Map**

The structure one finds in the \( G/G \) model is to a large extent analogous to the one discussed above, the main difference being that the infinitesimal group action on \( \mathcal{A} \) is not paramterized by elements of the Lie algebra of the gauge group \( \mathcal{G} \) but rather (see (2.19,2.20)) by the elements of \( \mathcal{G} \) itself. This then presents a departure from the standard theory of Hamiltonian group actions and some of the concepts (like the equivariance condition) will have to be modified accordingly. We will show first how this structure arises in the \( G/G \) model and then extract from it the general features in analogy with what we did above in the case of ordinary Hamiltonian group actions.

As a first step we rewrite the supersymmetry \( \delta S(g, A, \psi) = 0 \) of the action (2.10) as

\[ i(X_A(g))\Omega(\psi) = d_A S_{G/G}(g, A) , \]  

(4.14)

where \( 2\pi \Omega(\psi) = \int_\Sigma \psi \bar{\psi} \) denotes the symplectic form on \( \mathcal{A} \). Comparing this with (4.3) we are tempted to interpret the action of the \( G/G \) model as the ‘moment map’ for the action generated by \( X_A(g) \) on \( \mathcal{A} \), the crucial difference being that this moment map now depends non-linearly on \( g \in \mathcal{H} = \mathcal{G} \) rather than linearly on \( X \in \mathfrak{h} \). There exists an exact analogue of the first description (4.6) of the
moment map by regarding $F(g) \equiv S_{G/G}(g,.)$ as a function(al) on $\mathcal{A}$,

\begin{align*}
F : \mathcal{H} = \mathcal{G} &\to C^\infty(\mathcal{A}) \quad (4.15) \\
F(g)(A) &= S_{G/G}(g,A) \quad . (4.16)
\end{align*}

The counterpart of the second description (the moment map $J$ as a map from the symplectic manifold to the space $\mathfrak{h}^*$ of linear functions on $\mathfrak{h}$) can be obtained by replacing $\mathfrak{h}^*$ by a space $\mathcal{F}(\mathcal{G})$ of function(al)s on $\mathcal{G}$,

\begin{align*}
J : \mathcal{A} &\to \mathcal{F}(\mathcal{G}) \quad (4.17) \\
J(A)(g) &= S_{G/G}(g,A) \quad . (4.18)
\end{align*}

Infinitesimally, of course, these correspond to linear functions on Lie $\mathcal{G}$, $F$ inducing a linear map $T_1 F$ (the derivative at the identity element) from Lie $\mathcal{G}$ to $C^\infty(\mathcal{A})$.

Global Equivariance of the Moment Map and the Polyakov-Wiegmann Identity

One other thing worth noting about (4.14) is that it only determines the $A$-dependent part $S_{G/G}(g,A)$ of $S_{G/G}(g,A)$ and that any functional of the form $F(g) = S_{G/G}(g,.) + C(g)$ will also satisfy (4.14). This is the analogue of the ambiguity $F(X) \to F(X) + c(X)$ we discussed above. There this ambiguity could be fixed by demanding equivariance of the moment map. It is thus natural to ask whether a similar criterion can be used here to determine $C(g)$ to be the WZW action $S_G(g)$. This turns out to be the case. To get an idea of what the analogue of the equivariance condition (4.12) should be, we shall first determine the counterpart of (4.10) and then try to lift that to an equation at the level of moment maps.

By straightforward calculation one finds that the Lie bracket of two vector fields $X_A(g)$ and $X_A(h)$ is

\begin{equation}
[X_A(g), X_A(h)] = X_A(gh) - X_A(hg) \quad . (4.19)
\end{equation}

Hence the equivariance condition one expects the moment map to satisfy is

\begin{equation}
\{F(g), F(h)\} = F(gh) - F(hg) \quad , (4.20)
\end{equation}

which is the (not completely obvious, but natural) counterpart of $\{F(X), F(Y)\} = F([X,Y])$ (4.14). The interpretation of this equivariance condition and its relation with (4.12) will be discussed below. We will now show that with the choice $F(g)(A) = S_{G/G}(g,A)$ (i.e. with $C(g) = S_G(g)$) this equation is satisfied.

It follows from the generalized Polyakov-Wiegmann identity (2.3) that the right hand side of (4.20) is

\begin{equation}
S_{G/G}(gh,A) - S_{G/G}(hg,A) = \frac{1}{2\pi} \int_\Sigma J_z(h)J_{\bar{z}}(g) - J_z(g)J_{\bar{z}}(h) \quad . (4.21)
\end{equation}
The Poisson bracket on the left hand side can be calculated by contracting $\Omega(\psi)$ with $X_A(g)$ and $X_A(h)$ and one finds

$$\{S_{G/G}(g, A), S_{G/G}(h, A)\} = i(X_A(g))i(X_A(h))\Omega(\psi)$$

$$= \frac{1}{2\pi} i(X_A(g)) \int_{\Sigma} J_z(h) \psi_z - J_z(h) \psi_z$$

$$= \frac{1}{2\pi} \int_{\Sigma} J_z(h) J_z(g) - J_z(g) J_z(h),$$  \hspace{1cm} (4.22)

so that one indeed has the rather remarkable equation

$$\{S_{G/G}(g, A), S_{G/G}(h, A)\} = S_{G/G}(gh, A) - S_{G/G}(hg, A) \hspace{1cm} (4.23)$$

satisfied by the $G/G$ action with respect to the Poisson bracket on the space $\mathcal{A}$ of gauge fields.

However, demanding that (4.20) holds still does not fix $F(g)$ uniquely to be the $G/G$ action. It is clear from the above that any functional of the form $S_{G/G}(g, .) + C(g)$ with $C(.)$ a class function will also satisfy (4.20), as then $C(gh) = C(hg) \forall \, g, h$. In particular, this allows us to add to the $G/G$ action any of the observables of the $G/G$ model like traces of $g$ as well as possible quantum corrections which are also of this form \cite{1,8} without losing the underlying equivariant geometry.

Deformations of the Generator of Gauge Transformations and the $k \to \infty$ Limit

We now want to show that the level $k$ $G/G$ action and its equivariance relation (4.23) can be regarded as a deformation of the ordinary generator of gauge transformations on $\mathcal{A}$, the BF action

$$S_{BF}(\phi, A) = \frac{1}{2\pi} \int_{\Sigma} \phi F_A$$  \hspace{1cm} (4.24)

(with $\phi \in \text{LieG}$), and its standard equivariance condition

$$\{S_{BF}(\phi, A), S_{BF}(\phi', A)\} = S_{BF}([\phi, \phi'], A) \hspace{1cm} ,$$  \hspace{1cm} (4.25)

to which it reduces in the $k \to \infty$ limit. That the level $k$ plays the role of a deformation parameter in the $G/G$ model can also be seen from several other points of view. For instance, while the partition function of BF (and Yang-Mills) theory is given by a sum over all unitary irreps of $G$, in the $G/G$ model at level $k$ only the level $k$ integrable representations appear so that the finiteness of $k$ effectively provides a cutoff on the representations contributing to the partition function. These integrable representations are also known to be related to the
representations of quantum groups \(G_q\) for \(q\) a root of unity, \(q = \exp(i\pi/k + h)\), with \(q \to 1\) for \(k \to \infty\). This suggests that the correct cohomological interpretation of the supersymmetry and the localization could be in terms of (a yet to be developed) \(G_q\) (or rather \(G_q = \text{Map}(\Sigma, G_q)\)) equivariant cohomology on the space of gauge fields.

Moreover, it follows from the Riemann-Roch formula (or from standard arguments concerning the semi-classical limit of a quantum theory) that the large \(k\) limit of the Verlinde formula, or of the partition function of the \(G/G\) model, calculates the volume of the moduli space of flat connections, in agreement with the fact that this is what the BF theory calculates.

To keep track of the \(k\)-dependence, we rewrite (4.23) in terms of the \(G/G\) action at level \(k\) and the Poisson bracket

\[
\{.,.\}_k = k^{-1}\{.,.\}_1 \equiv k^{-1}\{.,.\}
\]

of the corresponding symplectic form \(k\Omega(\psi)\),

\[
\{kS_{G/G}(g, A), kS_{G/G}(h, A)\}_k = kS_{G/G}(gh, A) - kS_{G/G}(hg, A) .
\]

Let us now parametrize \(g\) as \(g = \exp \phi/k\). Then in the \(k \to \infty\) limit the action becomes the BF action \(S_{BF}(\phi, A)\),

\[
\lim_{k \to \infty} kS_{G/G}(g, A) = \frac{1}{2\pi} \int_\Sigma \phi F_A + O(k^{-1}) ,
\]

the kinetic term \(S_G(g, A)\) being of order \(O(k^{-2})\) and therefore not contributing in the limit. Therefore, (4.27) becomes

\[
\{S_{BF}(\phi, A), S_{BF}(\phi', A)\} = \\
\lim_{k \to \infty} k^2(S_{G/G}(\exp \phi/k \exp \phi'/k, A) - S_{G/G}(\exp \phi'/k \exp \phi/k, A)) .
\]

Calculating either the left or the right hand side of (4.29), one finds (4.25), which is precisely the ordinary equivariance (4.12) of the generator of gauge transformations on \(A\).

The Basic Structure of Generalized Hamiltonian Group Actions

Let us now briefly, and at the risk of being repetitive, extract from the above the basic structure we have found in the \(G/G\) model characterizing the generalized Hamiltonian group actions and compare it with the standard theory.

First of all, there is an assignment of vector fields \(X(g)\) on a symplectic manifold \((M, \Omega)\) to elements \(g\) of a group \(H\). These vector fields satisfy

\[
[X(g), X(h)] = X(gh) - X(hg) .
\]
This should be thought of as the generalization of the condition (4.1) expressing the fact that the vector fields $X_M$ provide a realization of the Lie algebra $\mathfrak{h}$ of $\mathbf{H}$ on $M$. (4.30) can be interpreted as follows. By linearity one can extend the assignment of vector fields to elements of $\mathbf{H}$ to elements of the group algebra $\mathbb{C} \mathbf{H}$ of $\mathbf{H}$ so that we can write the right hand side of (4.30) as $X(gh - hg)$. The group algebra can be equipped with a Lie algebra structure by defining the commutator to be $[g, h] = gh - hg$. Then (4.30) can be read as expressing the fact that the vector fields $X(g)$ provide a representation of the Lie algebra $(\mathbb{C} \mathbf{H}, [~,~])$ on $M$,

$$[X(g), X(h)] = X([g, h]) \quad (4.31)$$

Next we demand that these vector fields are Hamiltonian, i.e. that there exist functions $F(g)$ on $M$ such that

$$i(X(g))\Omega = dF(g) \quad (4.32)$$

We then write $X(g) = V_{F(g)}$. It follows from (4.30) and (4.32) that the Hamiltonian vector field corresponding to the Poisson bracket of two functions $F(g)$ and $F(h)$ is

$$V_{\{F(g), F(h)\}} = V_{F(gh)} - V_{F(hg)} \quad (4.33)$$

this being the analogue of (4.10). We then have a generalized moment map

$$F : \mathbf{H} \to C^\infty(M) \quad (4.34)$$

which can also be considered as a map

$$J : M \to C^\infty(\mathbf{H}) \quad (4.35)$$

via

$$(J(m))(h) \equiv (F(h))(m) \quad (4.36)$$

(see (1.6-1.8). Perhaps it will turn out to be more convenient to regard $J$ as a map into the distributions on $\mathbf{H}$. Either way it is natural to say that the moment map is equivariant if (4.31) or (4.33) can be lifted to hold at the level of Hamiltonian functions, i.e. if one has a representation of the Lie algebra $(\mathbb{C} \mathbf{H}, [~,~])$ in the Poisson algebra of functions on $M$,

$$\{F(g), F(h)\} = F(gh) - F(hg) \equiv F([g, h]) \quad (4.37)$$

(in writing the second equality we have extended functions on $\mathbf{H}$ to functions on $\mathbb{C} \mathbf{H}$ by linearity).
That this is indeed a reasonable generalization of the ordinary equivariance condition can be seen by noting that (4.37) implies that the first moments of $F(g)$, defined by

$$F^{(1)}(X) = \frac{d}{dt} F(\exp(tX))|_{t=0}, \quad (4.38)$$

satisfy the ordinary equivariance condition (4.12),

$$\{F^{(1)}(X), F^{(1)}(Y)\} = F^{(1)}([X,Y]). \quad (4.39)$$

This is the counterpart of the $k \to \infty$ argument we gave in the case of the $G/G$ action and the above argument could have alternatively been phrased in similar terms.

The converse, that ordinary equivariance (4.12) implies (4.37), need however not be true as the generalized equivariance condition implies a whole hierarchy of conditions on the higher moments

$$F^{(n)}(X) = (\frac{d}{dt})^n F(\exp(tX))|_{t=0} \quad (4.40)$$

which are necessary in order for (4.12) to exponentiate to (4.37).

There is another way of relating (4.37) to ordinary moment maps, pointed out to us by V. Fock. Namely, let us associate to a function $F(g) \in C^\infty(M)$ a function $\hat{F}(\mu,X) \in C^\infty(M)$, where $\mu \in R(H)$ labels an irreducible unitary $d_\mu$-dimensional representation $V_\mu$ of $H$ and $X \in \text{Lie}H$, by

$$\hat{F}(\mu,X) = -d_\mu \int dg F(g) \text{Tr}_\mu(Xg). \quad (4.41)$$

It then follows from (4.37) and the orthogonality of the traces that the $\hat{F}(\mu,X)$ satisfy the Poisson bracket relations

$$\{\hat{F}(\mu,X), \hat{F}(\nu,Y)\} = \delta_{\mu\nu} \hat{F}(\mu, [X,Y]). \quad (4.42)$$

Thus $\hat{F}$ can be thought of as an equivariant moment map (in the ordinary sense) for the direct sum of Lie algebras

$$\bigoplus_{\mu \in R(H)} \mu(\text{Lie}H) \subset \bigoplus_{\mu \in R(H)} \text{End} V_\mu. \quad (4.43)$$

Modulo analytical problems, $F(g)$ can be recovered from the functions $\hat{F}$.

The moment map in the $G/G$ model has a further property, namely its gauge invariance. This property, however, is linked with a second action of the group $H = G$ on $M = A$ (namely via gauge transformations), and in the general context would take the form $F(g)(m) = F(h^{-1}gh)(h.m)$, $h.m$ denoting this extra action.
of $h \in H$ on $m \in M$. However, there seems to be no reason to demand some such property to hold in general, and we thus take the conditions (4.30, 4.32) (and (4.37)) to define what we mean by a generalized (equivariant) Hamiltonian group action.

Clearly much remains to be understood about the properties of these generalized group actions, primarily of course whether this is at all an interesting structure to consider in general.

Acknowledgements

We would like to thank A. Alekseev, F. Delduc, V. Fock, L. Jeffrey and A. Weinstein for discussions and helpful remarks, and the people at the École Normale in Lyon and the CPT II in Marseille, where part of this work was carried out, for their hospitality.

A Aspects of Localization in Yang-Mills Theory

In this section we illustrate the subtleties we encountered in section 3 when adapting the usual localization arguments to the $G/G$ model in the simpler case of Yang-Mills theory (or, actually, its topological limit, BF theory). We will freely make use of the results established in [7] concerning the global issues involved when diagonalizing Lie algebra or group valued maps without drawing attention to it every time, as these are only of secondary importance in the issue at stake.

The action we will consider is

$$S = \int \phi F_A - \frac{1}{2} \int \psi \psi ,$$

which has the equivariant supersymmetry

$$\delta A = \psi , \quad \delta \psi = d_A \phi .$$

This supersymmetry can be used in various ways to localize the theory to reducible configurations, i.e. to solutions $(A_c, \phi_c)$ of the equation

$$d_{A_c} \phi_c = 0 .$$

One way of seeing this is to add to the action a $\delta$-exact term enforcing this localization in some limit, e.g.

$$S^* = S - s \delta \int \psi * d_A \phi$$

$$= \int (\phi F_A - s d_A \phi * d_A \phi) - \int (\frac{1}{2} \psi \psi + s \psi * [\psi, \phi]) .$$
as $s$ tends to infinity. This is precisely the large $k$ limit of the deformed $G/G$ action \[ (2.29, 2.36) \]. If one were to invoke localization naively, however, one would conclude that the action of the theory reduces to $\int \phi_c F_{A_c}$. For fixed $\phi_c$, this integral is independent of $A_c$,

$$d_{A_c} \phi_c = 0 \text{ and } d_{A_c+X} \phi_c = 0 \Rightarrow \int \phi_c F_{A_c} = \int \phi_c F_{A_c+X} , \quad (A.5)$$

so that the $A_c$ integral would not be damped and the naive stationary phase approximation to the path integral diverges. This is the counterpart of the observation made in section 3 that the gauged WZ term $\Gamma(g_c, A_c)$ is independent of $A_c$. In order to correctly separate the classical from the quantum fields, one needs a convenient parametrization of the classical fields, i.e. the space of solutions to \[(A.3)\]. Assuming that only the main branch of solutions to these equations is relevant, up to possibly singular gauge transformations the classical solutions can be parametrized by pairs $(a_0, \phi_c)$ where $a_0$ and $\phi_c$ are $t$-valued and $\phi_c$ is constant. Notice that the condition for $A$ to be a critical point of the vector field

$$X_A(\phi) = d_A \phi \frac{\delta}{\delta A} , \quad (A.6)$$

is also a condition on $\phi$. This is a case not covered directly by the traditional localization theorems (which tell us nothing about the $\phi$-integral), and it is then not surprising that a naive application of localization to the joint $(A, \phi)$ system may lead one astray. Notice also that there is no other condition on the torus gauge field $a_0$, so that localization does nothing there. This reflects the fact that localization is empty once one is left with an Abelian (quadratic) action.

Now a general (generic) field $\phi$ can always be written in the form $\phi = h \phi^t h^{-1}$ for some $t$-valued field $\phi^t$. One is thus led to the decomposition

$$\phi = h \phi^t h^{-1} = h(\phi_c + \bar{\phi}) h^{-1} , \quad (A.7)$$

where $\bar{\phi}$ has no constant mode. This is the correct form of the naive classical-quantum decomposition $\phi = \phi_c + \phi_q$, disentangling at the same time localization and gauge invariance. Changing variables from $\phi$ to $(h, \phi_c, \bar{\phi})$, the bosonic part of the action becomes

$$S_{BF}^s = \int (\phi_c + \bar{\phi}) F_{A^h} - s d_{A^h} (\phi_c + \bar{\phi}) \ast d_{A^h} (\phi_c + \bar{\phi}) , \quad (A.8)$$

This change of variables also leads to a Jacobian which we write as

$$\text{Det} [\text{ad}(\phi_c + \bar{\phi})]_{\Omega^p(\Sigma, g/t)} , \quad (A.9)$$

the subscripts indicating that this is a functional determinant on $g/t$-valued zero-forms.
Note that this action is manifestly gauge invariant under

\[ A \rightarrow A^g, \quad h \rightarrow g^{-1}h \]  

(A.10)

(the latter corresponding to \( \phi \rightarrow \phi^g \)), as the gauge field \( A \) only appears in the gauge invariant St"uckelberg combination \( A^h \). It is convenient to split this gauge field into its \( t \)- and \( g/t \)-components, \( A^h = (A^h)^t + (A^h)^{g/t} \), so that one has

\[
S^{\text{BF}}_{\text{eff}} = \int (\phi_c + \bar{\phi})(F(A^h)^t + \frac{1}{2}[(A^h)^{g/t}, (A^h)^{g/t}]) \\
- s \int d\bar{\phi} \wedge d\phi + [(A^h)^{g/t}, \phi_c + \bar{\phi}] \ast [(A^h)^{g/t}, \phi_c + \bar{\phi}] .
\]  

(A.11)

At this point the second problem with the naive localization argument is apparent. Namely, what appears to be the quadratic form for the ‘quantum field’ \( \bar{\phi} \) can be absorbed into the first term of the action by a shift of \( (A^h)^t \). Thus, if one were to scale this quantum field by \( \sqrt{s} \) to eliminate the \( s \)-dependence from the quadratic term, one would simultaneously kill the kinetic term for \( \bar{\phi} \) and \( (A^h)^t \) in the limit \( s \rightarrow \infty \) (which is, as we have seen, essentially what a straightforward implementation of localization would lead one to believe).

The crux of the matter is of course that, as argued above, localization applies \textit{a priori} only to the gauge field integral. But as this localization is \( \phi \) dependent, one needs a good parametrization of the \( \phi \)'s to implement this localization correctly. Arguments based on \( s \)-independence alone are nevertheless fine as long as one makes sure that one keeps quadratic forms for all the fields involved. It is precisely to ensure this and to avoid pitfalls like the above that it is helpful to use an explicit parametrization of the (gauge orbits of) classical configurations.

We now split \( (A^h)^t \) into a (possibly non-trivial) background gauge field \( A^0 \) such that the components of \( dA^0 \) are harmonic, and a \( t \)-valued one-form \( a^t \) and shift \( a^t \) by \( s \ast d\bar{\phi} \). This decouples \( (A^0, \phi_c) \) from \( (a^t, \bar{\phi}) \) and the sole effect of integrating over \( a^t \) is now to set \( \bar{\phi} \) to a constant and hence to zero as, by assumption, \( \bar{\phi} \) has no constant mode. Reintroducing the fermionic fields, one is thus left with the action

\[
S^{\text{eff}}_{\text{eff}} = 2\pi \phi_c \cdot n + \int (\frac{1}{2} \phi_c[(A^h)^{g/t}, (A^h)^{g/t}]) - s [(A^h)^{g/t}, \phi_c] \ast [(A^h)^{g/t}, \phi_c] \\
- \int (\frac{1}{2} \psi^h \psi^h + s \psi^h \ast \psi^h, \phi_c) ,
\]  

(A.12)

the first term representing the pairing between the constant field \( \phi_c \) and the \( r \)-tuple of integers characterizing the first Chern class \([dA^0]\) of \( (A^h)^t \) in \( H^2(\Sigma, \mathbb{Z}^r) \sim \mathbb{Z}^r \).

Note again that this action is still gauge invariant and that no localization or approximation has entered into the derivation of (A.12). One can now proceed in
a number of ways to evaluate the partition function, each one of them also being more or less readily available in the $G/G$ model. It is here that one has the choice between solving the theory by Abelianization or by localization, but the following discussion should make it clear that at this point the distinction between the two methods is rather artificial. This illustrates once more the main point we wanted to make in section 3 in the context of the $G/G$ model, namely that localization abelianizes the theory (the converse having already been established in [1, 4]).

1. As everything is still independent of $s$ and well defined for $s = 0$, one can simply set $s$ equal to zero. One is then just left with the original theory, expressed in terms of $\phi_c$ and $(A^h)^{g/t}$, $(A^h)^{t}$ and $\bar{\phi}$ having been integrated out. The group valued field $h$ just represents the gauge degrees of freedom and has to be dealt with in some way:

(a) Performing the change of variables $A \rightarrow A^h$, the $h$-integral becomes the gauge volume and factors out. The integral over $A^{g/t}$ produces the functional determinant

$$\text{Det}^{-1/2}[\text{ad}\phi_c]|_{\Omega(\Sigma,g/t)} \ . \quad (A.13)$$

Combined with the Jacobian $(A.9)$ from the change of variables, this gives the residual finite-dimensional determinant

$$\text{det} \chi(\Sigma)^{1/2}[\text{ad}\phi_c]|_{g/t} \ , \quad (A.14)$$

($\chi(\Sigma)$ denoting the Euler number of $\Sigma$) leading to the standard result for the partition function of Yang-Mills theory upon summation over all topological sectors and performing the finite-dimensional integral over $\phi_c$ [4].

(b) One can also choose the gauge $h = 1$ (this is Abelianization). This obviously has the same effect as the above change of variables.

(c) Lastly, one can of course choose any other gauge condition as well, e.g. a covariant gauge, and still do all the integrals explicitly. The integrals over $h$, the ghosts and the Lagrange multiplier enforcing the gauge condition combine to give 1 (by running the Faddeev-Popov trick backwards), reducing one to possibility (a).

2. Alternatively, one can consider the limit $s \rightarrow \infty$ (localization). To that end one scales the quantum fields $(A^h)^{g/t}$ and their superpartners $(\psi^h)^{g/t}$ as

$$(A^h)^{g/t} \rightarrow s^{-1/2}(A^h)^{g/t} \ , \quad (\psi^h)^{g/t} \rightarrow s^{-1/2}(\psi^h)^{g/t} \ . \quad (A.15)$$
In the limit $s \to \infty$, the terms coming from the original BF theory and involving $(A^h)^{g/t}$ disappear and one obtains

$$S_{s \to \infty}^{s_{eff}} = 2\pi \phi_c n - \int \Sigma [(A^h)^{g/t}, \phi_c] * [(A^h)^{g/t}, \phi_c]$$

$$- \int \Sigma (\psi^h)^{g/t} * (\psi^h)^{g/t}, \phi_c] + \frac{1}{2} \psi^t \psi^t . \tag{A.16}$$

Again the $h$-integral can be dealt with in several ways. For simplicity we will follow option 1(a) above and perform the change of variables $A \to A^h$. Then one finds that the integrals over $A^{g/t}$ and $\psi^{g/t}$ give

$$\text{Det}^{-1/2}[(\text{ad} \phi_c)^2]_{\Omega^1(\Sigma, g/t)} \tag{A.17}$$

and

$$\text{Det}^{1/2}[\text{ad} \phi_c]_{\Omega^1(\Sigma, g/t)} \tag{A.18}$$

respectively, combining to give the net contribution (A.13), in agreement with the result obtained in 1(a).

3. In this example it is also straightforward to work out what happens for finite values of $s$. Once again with $A \to A^h$ for simplicity, one finds that the quadratic terms in $A^{g/t}$ and $\psi^{g/t}$ are of the form

$$A^{g/t} \text{ad} \phi_c (1 - 2s \text{ ad} \phi_c *) A^{g/t} + \psi^{g/t} (1 - 2s \text{ ad} \phi_c *) \psi^{g/t} .$$

Evidently this also leads to the same net determinant (A.13), establishing explicitly the $s$-independence of the theory.

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