INEQUALITIES FOR THE HODGE NUMBERS OF IRREGULAR COMPACT KÄHLER MANIFOLDS

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Abstract. Based on work of R. Lazarsfeld and M. Popa, we use the derivative complex associated to the bundle of the holomorphic \( p \)-forms to provide inequalities for all the Hodge numbers of a special class of irregular compact Kähler manifolds. For 3-folds and 4-folds we give an asymptotic bound for all the Hodge numbers in terms of the irregularity. As a byproduct, via the BGG correspondence, we also bound the regularity of the exterior cohomology modules of bundles of holomorphic \( p \)-forms.

1. Introduction

Given an irregular compact Kähler manifold \( X \), a problem one tries to understand is under which hypotheses there exist relations between its various Hodge numbers \( h^{p,q}(X) \). Along these lines, one can ask whether there exist formulas for the \( h^{p,q}(X) \)'s in terms of the fundamental invariant \( q(X) = h^{1,0}(X) \), the irregularity of \( X \). A classical result in this direction is the Castelnuovo-De Franchis inequality \( h^{0,2}(X) \geq 2q(X) - 3 \) (see [BHPV] IV.5.2), which holds for surfaces that do not carry any fibrations onto a smooth curve of genus \( g \geq 2 \). This was generalized by F. Catanese to higher dimensional manifolds as follows: if a manifold \( X \) does not admit any higher irrational pencil \(^1\) then \( h^{0,k}(X) \geq k(q(X) - k) + 1 \) for all \( k \) (cf. [Cat]), by means of sophisticated arguments involving the exterior algebra of holomorphic forms. Another generalization of the Castelnuovo-De Franchis inequality is provided by the work of G. Pareschi and M. Popa. Theorem A in [PP2] states that if \( X \) is of maximal Albanese dimension and does not carry any higher irrational pencil then \( \chi(\omega_X) \geq q(X) - \dim X \). Their inequality is deduced using Generic Vanishing Theory for irregular varieties and the Evans-Griffith Syzygy Theorem.

New techniques for the study of this problem were introduced recently by R. Lazarsfeld and M. Popa in [LP]. Their approach relies on the study of a global version of the derivative complex associated to the structure sheaf \( O_X \) and on the theory of vector bundles on projective spaces. Their inequalities mainly involve Hodge numbers of type \( h^{0,k}(X) \).

In this paper we extend the methods of [LP] to the bundles of holomorphic \( p \)-forms \( \Omega^p_X \). In this way we get inequalities for all the Hodge numbers of a special class of irregular compact Kähler manifolds. The main idea behind the inequalities of [LP] and the ones of

\(^1\)A higher irrational pencil is a surjective map with connected fibers \( f : X \to Y \) having the property that any resolution singularities of \( Y \) is of maximal Albanese dimension and with non-surjective Albanese map.
this paper goes as follows. Via cup product any element \( 0 \neq v \in H^1(X, \mathcal{O}_X) \) defines a complex of vector spaces
\[
0 \rightarrow H^0(X, \Omega^p_X) \xrightarrow{\cup v} H^1(X, \Omega^p_X) \xrightarrow{\cup v} \ldots \xrightarrow{\cup v} H^d(X, \Omega^p_X) \rightarrow 0.
\]
Denoting by \( \mathbf{P} = \mathbf{P}_{\text{sub}}(H^1(X, \mathcal{O}_X)) \) the projective space of one dimensional linear subspaces of \( H^1(X, \mathcal{O}_X) \), we can arrange all of these, as \( v \) varies, into a complex of locally free sheaves on \( \mathbf{P} \):
\[
0 \rightarrow \mathcal{O}_\mathbf{P}(-d) \otimes H^0(X, \Omega^p_X) \rightarrow \mathcal{O}_\mathbf{P}(-d + 1) \otimes H^1(X, \Omega^p_X) \rightarrow \ldots
\]
\[
\ldots \rightarrow \mathcal{O}_\mathbf{P}(-1) \otimes H^{d-1}(X, \Omega^p_X) \rightarrow \mathcal{O}_\mathbf{P} \otimes H^d(X, \Omega^p_X) \rightarrow 0,
\]
whose fiber at a point \([v] \in \mathbf{P}\) is the complex (1). The complex (1) is known as the \textit{derivative complex associated to the bundle} \( \Omega^p_X \) \textit{with respect to the vector} \( v \). It was introduced for the first time by M. Green and R. Lazarsfeld in [GL1] and [GL2] in their study of Generic Vanishing Theorems for irregular compact Kähler manifolds. At this point it is important to study the exactness of the complex (2). In [LP] the case \( p = 0 \) is analyzed. For this case we have that if \( X \) does not carry any irregular fibrations, i.e. morphisms \( f : X \rightarrow Y \) with connected fibers having the property that any smooth model of \( Y \) is of maximal Albanese dimension, then the complex (2) is everywhere exact except possibly at the last term and furthermore all the involved maps are of constant rank. In general, for \( p > 0 \), in Proposition 2.1 we show that the exactness of the complex (2) depends on the non-negative integer \( m(X) \), the least codimension of the zero-locus of a non-zero holomorphic one-form, i.e.
\[
m(X) = \min \{ \text{codim } Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega^1_X) \},
\]
with the convention \( m(X) = \infty \) if every non-zero holomorphic 1-form is everywhere non-vanishing. For instance, it was observed in [GL1] Remark on p. 405 and in [La] Proposition 6.3.10, examples of varieties with \( m(X) = \dim X \) are the smooth subvarieties of an abelian variety with ample normal bundle (hence all smooth subvarieties of a simple abelian variety). In particular we show that if \( m(X) > p \) then the complex (2) is exact at the first \((m(X) - p)\)-steps from the left with the first \( m(X) - p \) maps being of constant rank. This is enough to ensure that the cokernel of the map
\[
\mathcal{O}_\mathbf{P}(m(X) - d - p - 1) \otimes H^{m(X)-p-1}(X, \Omega^p_X) \rightarrow \mathcal{O}_\mathbf{P}(m(X) - d - p) \otimes H^{m(X)-p}(X, \Omega^p_X)
\]
is a locally free sheaf. This in turn leads to inequalities for the Hodge numbers by the Evans-Griffith Theorem and by the fact that the Chern classes of a globally generated locally free sheaf are non-negative.

We turn to a more detailed presentation of our results. To simplify notation we only present the case \( m(X) = \dim X \) and we refer to Theorem 3.1 and Theorem 3.2 for general statements where all possible values of \( m(X) \) are considered. Fix an integer \( 1 \leq p \leq d \) and for any \( 1 \leq i \leq q - 1 \) define \( \gamma_i(X, \Omega^p_X) \) to be the coefficient of \( t^i \) in the formal power series:
\[
\gamma_i(X, \Omega^p_X; t) \overset{\text{def}}{=} \prod_{j=1}^p (1 - jt)^{-1} h^{p,d-p+i+j} \in \mathbb{Z}[\![t]\!],
\]
where \( h^{i,j} = h^{i,j}(X) \) are the Hodge numbers of \( X \).

**Theorem 1.1.** Let \( X \) be a compact Kähler manifold of dimension \( d \), irregularity \( q \geq 1 \) and with \( m(X) = d \). Then

(i). Any Schur polynomial of weight \( \leq q - 1 \) in the \( \gamma_i(X, \Omega^p_X) \) is non-negative. In particular

\[
\gamma_i(X, \Omega^p_X) \geq 0.
\]

(ii). If \( q > \max \{p, d - p - 1\} \) then

\[
\sum_{j=d-p}^d (-1)^{d-p+j} h^{p,j} \geq q - p.
\]

For instance, the \( \gamma_1(X, \Omega^p_X) \)'s are non-negative linear polynomials in the variables \( h^{p,j} \), and in the case of surfaces the inequalities above become \( h^{0,2} \geq 2q - 3 \) and \( h^{1,1} \geq 2q - 1 \), well-known inequalities true for surfaces which do not admit any irregular pencils of genus \( \geq 2 \) (cf. [BHPV] IV.5.4). In higher dimension the polynomials \( \gamma_i(X, \Omega^p_X) \), of degree \( i \) in the variables \( h^{p,j} \), give new inequalities involving Hodge numbers of \( X \). In addition to the methods for \( O_X \), for \( p > 0 \) Serre Duality offers a way to see until which step the complex \( \mathcal{E} \) is exact counting from the right. This trick leads to further inequalities for the Hodge numbers and, in the special case when \( m(X) = \dim X \) and \( q(X) > \dim X \geq 2 \), also to get a bound on the Euler characteristic for the bundles \( \Omega^p_X \):

\[
|\chi(\Omega^1_X)| \geq 2 \quad \text{and} \quad |\chi(\Omega^p_X)| \geq 1 \quad \text{for} \quad p = 2, \ldots, \dim X - 2
\]

(cf. Corollary [2.3]). In section IV we list the inequalities coming from Theorem 1.1 for manifolds of dimension three, four and five. Finally for threefolds and fourfolds with \( m(X) = \dim X \) we are able to give asymptotic bounds for all the Hodge numbers in terms of the irregularity \( q \). In the case of threefolds we obtain

\[
h^{0,2} \geq 4q, \quad h^{0,3} \geq 4q, \quad h^{1,1} \geq 2q + \sqrt{2q}, \quad h^{1,2} \geq 5q + \sqrt{2q}
\]

and for the case of fourfolds we get

\[
h^{0,2} \geq 4q, \quad h^{0,3} \geq 5q + \sqrt{2q}, \quad h^{0,4} \geq 4q \quad \text{and} \quad h^{1,1} \geq 2q, \quad h^{1,2} \geq 8q + 2\sqrt{2q}, \quad h^{1,3} \geq 12q + 3\sqrt{2q}, \quad h^{2,2} \geq 8q + 4\sqrt{2q}.
\]

Asymptotic inequalities for Hodge numbers of type \( h^{0,j} \) were already established in [LP] for manifolds which do not carry any irregular fibrations.

Setting \( E = \bigwedge^* H^1(X, O_X) \) for the graded exterior algebra over \( H^1(X, O_X) \), in the last section, we use the Bernstein-Gel’fand-Gel’fand (BGG) correspondence and Generic Vanishing Theorems for bundles of holomorphic \( p \)-forms to bound the regularity of the \( E \)-modules \( \bigoplus_i H^i(X, O^p_X) \) for any value of \( p \). We refer to Section V for the definition of regularity for finitely generated graded modules over an exterior algebra and for references about the BGG correspondence and the Generic Vanishing Theorems used. The case \( p = \)}
dim $X$ has been studied in [LP]. If we denote by $k$ for the dimension of the general fiber of the Albanese map $\text{alb}_X : X \rightarrow \text{Alb}(X)$, then the $E$-module $\bigoplus_i H^i(X, \omega_X)$ is $k$-regular but not $(k - 1)$-regular. In general, for all the others values of $p$, we are not able to determine the regularity of the $E$-module $\bigoplus_i H^i(X, \Omega^p_X)$ but only to give a bound in terms of the minimal and maximal dimensions of the fibers of the Albanese map $\text{alb}_X : X \rightarrow \text{Alb}(X)$.

**Theorem 1.2.** Let $X$ be a compact Kähler manifold of dimension $d$ and irregularity $q \geq 1$. Let $k$ be the dimension of the general fiber of $\text{alb}_X : X \rightarrow \text{Alb}(X)$ and $f$ be the maximal dimension of a fiber of $\text{alb}_X$. Let $0 \leq p \leq d$ be an integer and set $l = \max\{k, f - 1\}$. If $p > l$ then the $E$-module $\bigoplus_i H^i(X, \Omega^p_X)$ is $(d - p + l)$-regular.

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2. Exactness of the complex $L^p_X$

Let $X$ be a compact Kähler manifold of dimension $d$. The *irregularity* of $X$ is the non-negative integer $q(X) \text{ def }= h^1(X, \mathcal{O}_X)$. The manifold $X$ is said to be *irregular* if $q(X) > 0$. We aim to study the exactness of the complex (2) in terms of the non-negative integer

$$m = m(X) := \min\{\text{codim} Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega^1_X)\},$$

with the convention $m(X) = \infty$ if every non-zero holomorphic 1-form is nowhere vanishing. Refer also to Proposition 5.1 for another study of the exactness in terms of different invariants.

**Proposition 2.1.** Let $X$ be an irregular compact Kähler manifold of dimension $d$ and $0 \leq p \leq d$ be an integer.

(i). If $p < m \leq d$ then the complex $L^p_X$ is exact at the first $m - p$ steps from the left, and the first $m - p$ maps are of constant rank.

(ii). If $d - p < m \leq d$ then the complex $L^p_X$ is exact at the first $m - d + p$ steps from the right, and the last $m - d + p$ maps are of constant rank.

(iii). If $m = \infty$ then the whole complex $L^p_X$ is exact and all the involved maps are of constant rank.

**Proof.** Under the Hodge conjugate-linear isomorphism $H^i(X, \Omega^p_X) \cong H^j(X, \Omega^1_X)$, the fiber at a point $[v] \in \mathcal{P}$ of the complex $L^p_X$ is identified with the complex

$$0 \longrightarrow H^p(X, \mathcal{O}_X) \overset{\omega}{\longrightarrow} H^p(X, \Omega^1_X) \overset{\omega}{\longrightarrow} \ldots \overset{\omega}{\longrightarrow} H^p(X, \Omega^d_X) \longrightarrow 0,$$

where $\omega$ is the conjugate-linear isomorphism.
where \( \omega \in H^0(X, \Omega_X^1) \) is the holomorphic 1-form conjugate to \( v \in H^1(X, \mathcal{O}_X) \). For every non-zero holomorphic one-form \( \omega \) the complex (3) is exact at the first \( m - p \) steps from the left by [GL1] Proposition 3.4. Hence the complex \( \mathcal{L}^p_X \) is itself exact since exactness can be checked at the level of fibers. This also shows that the first \( m - p \) maps of the complex \( \mathcal{L}^p_X \) are of constant rank.

For point (ii), using Serre Duality and thinking of the spaces \( H^p(X, \Omega_X^q) \) as the \((p,q)\)-Dolbeault cohomology, we have a diagram

\[
\begin{array}{cccccccc}
\cdots & \xrightarrow{\omega} & H^{d-p}(X, \Omega^i_X) & \xrightarrow{\omega} & H^{d-p}(X, \Omega^{i+1}_X) & \xrightarrow{\omega} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots & & H^p(X, \Omega^{d-i+1}_X) & & H^p(X, \Omega^{d-i}_X) & & \cdots
\end{array}
\]

where the bottom complex is the dual complex of (3). This diagram commutes up to sign and hence, if \( m > d - p \), the upper complex (and therefore also the bottom one) is exact at the first \( m - d + p \) steps from the left. Finally dualizing again the bottom complex we have that (3) is exact at the first \( m - d + p \) steps from the right.

The case \( m = \infty \) follows as the complexes (3) are now everywhere exact. \( \square \)

In the special case when the zero-set of every non-zero holomorphic 1-form consists of a finite number of points, i.e. when \( m(X) = \dim X \), Proposition 2.1 implies that the complex \( \mathcal{L}^p_X \) is everywhere exact except at most at one step. This allows us to give a bound on the Euler characteristic of the bundles of holomorphic \( p \)-forms in the case \( q(X) > \dim X \). We recall from the introduction that examples of manifolds with \( m(X) = \dim X \) are provided by smooth subvarieties of an abelian varieties having ample normal bundle. Before stating the bounds, we prove a simple Lemma which will be useful in the sequel.

**Lemma 2.2.** Let \( e \geq 2, t \geq 1, q \geq 1 \) and \( a \) be integers. For \( i = 1, \ldots, e + 1 \) and \( s = 1, \ldots, t \) let \( V_i \) and \( Z_s \) be complex vector spaces of positive dimension.

(i). If a complex of locally free sheaves on \( \mathbb{P} = \mathbb{P}^{d-1} \) of length \( e + 1 \) of the form

\[
0 \to V_{e+1} \otimes \mathcal{O}_P(-a) \to V_e \otimes \mathcal{O}_P(-a+1) \to \cdots \to V_1 \otimes \mathcal{O}_P(-a+e) \to 0
\]

is exact, then \( q \leq e \).

(ii). Let \( k_s \geq -a + e \) be integers. If a complex of locally free sheaves on \( \mathbb{P} = \mathbb{P}^{d-1} \) of length \( e + 2 \) of the form

\[
0 \to V_{e+1} \otimes \mathcal{O}_P(-a) \to V_e \otimes \mathcal{O}_P(-a+1) \to \cdots \to V_1 \otimes \mathcal{O}_P(-a+e) \to \bigoplus_{s=1}^t (Z_s \otimes \mathcal{O}_P(k_s)) \to 0
\]

is exact, then \( q \leq e + 1 \).

**Proof.** If \( q = 1 \) then clearly \( q \leq e \), therefore we can assume \( q > 1 \). If \( e = 2 \) then \( q = 2 \), since line bundles on projective spaces have no intermediate cohomology and so we can suppose
$e > 2$. After having twisted the complex (4) by $\mathcal{O}_P(-e + a)$ we get the complex

$$0 \rightarrow V_{e+1} \otimes \mathcal{O}_P(-e) \xrightarrow{f_1} V_e \otimes \mathcal{O}_P(-e+1) \rightarrow \ldots$$

$$\ldots \rightarrow V_4 \otimes \mathcal{O}_P(-3) \xrightarrow{f_{e-2}} V_3 \otimes \mathcal{O}_P(-2) \rightarrow V_2 \otimes \mathcal{O}_P(-1) \rightarrow V_1 \otimes \mathcal{O}_P \rightarrow 0.$$

Set $W_j = \text{coker } f_j$ for $j = 1, \ldots, e - 2$. If $q > e$, we would have that $H^{e-1-j}(P, W_j) \neq 0$ for every $j = 1, \ldots, e - 2$ and hence that $H^{e-1}(P, V_{e+1} \otimes \mathcal{O}_P(-e)) \neq 0$. This yields a contradiction and then $q \leq e$. To prove (ii) we can use the same argument used to prove (i).

**Corollary 2.3.** Let $X$ be a compact Kähler manifold of dimension $d \geq 2$ and irregularity $q(X) > d$. If $m(X) = d$ then

$$(−1)^{d−1} \chi(Ω^1_X) \geq 2,$$

and

$$(−1)^{d−p} \chi(Ω^p_X) \geq 1$$

for any $p = 2, \ldots, d − 2$.

**Proof.** To begin with, we note that $h^d(X, Ω^p_X) \neq 0$ so that the complex $L^p_X$ is non-zero. By Proposition 2.1 (ii), the assumption $m(X) = d$ implies that the non-zero complex $L^d_X$ is exact at the first $d$ steps from the right. If we had $h^d(X, Ω^p_X) = h^p(X, ω_X) = 0$, then $L^d_X$ would induce an exact complex of length $\leq d$ whose terms are sums of line bundles all of the same degree, and by Lemma 2.2 we would have a contradiction.

By Proposition 2.1 the complex $L^p_X$ is exact at the first $d - p$ steps from the left and at the first $p$ steps from the right. Therefore we get two exact sequences:

$$0 \rightarrow \mathcal{O}_P(-d) \otimes H^0(X, Ω^p_X) \rightarrow \ldots \rightarrow \mathcal{O}_P(-p - 1) \otimes H^{d-p-1}(X, Ω^p_X) \xrightarrow{f}$$

$$\xrightarrow{f} \mathcal{O}_P(-p) \otimes H^{d-p}(X, Ω^p_X) \rightarrow F \rightarrow 0,$$

where the locally free sheaf $F$ is the cokernel of the map $f$, and

$$0 \rightarrow G \rightarrow \mathcal{O}_P(-p) \otimes H^{d-p}(X, Ω^p_X) \xrightarrow{g}$$

$$\xrightarrow{g} \mathcal{O}_P(-p + 1) \otimes H^{d-p+1}(X, Ω^p_X) \rightarrow \ldots \rightarrow \mathcal{O}_P \otimes H^d(X, Ω^p_X) \rightarrow 0,$$

where the locally free sheaf $G$ is the kernel of the map $g$. We also get an induced map of locally free sheaves $h : F \rightarrow \mathcal{O}_P(-p + 1) \otimes H^{d-p+1}(X, Ω^p_X)$, which is of constant rank. Denoting by $E$ the kernel of $h$ we obtain a new exact sequence of locally free sheaves

$$0 \rightarrow E \rightarrow F \rightarrow \mathcal{O}_P(-p + 1) \otimes H^{d-p+1}(X, Ω^p_X) \rightarrow \ldots \rightarrow \mathcal{O}_P \otimes H^d(X, Ω^p_X) \rightarrow 0$$

from which we can read $\text{rank } E = (−1)^{d−p} \chi(Ω^p_X)$. If the locally free sheaf $E$ were the zero sheaf then the complex $L^p_X$ would be an exact complex of length $\leq d + 1$ whose terms are sum of line bundles all of the same degree, which is impossible by Lemma 2.2 (i). Thus

$$\text{rank } E = (−1)^{d−p} \chi(Ω^p_X) \geq 1.$$


If \( p = d - 1 \) we can improve our bound. In this case the complex \( L^{d-1}_X \) is exact at the first \( d - 1 \) steps from the right, and hence we get a short exact sequence
\[
0 \longrightarrow \mathcal{O}_P(-d) \otimes H^0(X, \Omega^{d-1}_X) \xrightarrow{h'} G \longrightarrow E' \longrightarrow 0,
\]
where \( E' \) is the cokernel of the map \( h' \). The locally free sheaf \( E' \) is non-zero again by Lemma 2.2. If the rank of \( E' \) were one, then \( E' \) would be a line bundle, \( E' = \mathcal{O}_P(k) \) for some integer \( k \), and \( G \in \text{Ext}^1(\mathcal{O}_P(k), \mathcal{O}_P(-d) \otimes H^0(X, \Omega^{d-1}_X)) = H^1(P, \mathcal{O}_P(d-k) \otimes H^0(X, \Omega^{d-1}_X)) = 0 \). Hence \( G \) would split as a sum of line bundles and by Lemma 2.2 (ii) this is not possible. Therefore
\[
\text{rank } E' = (-1)^{d-1} \chi(\Omega^1_X) \geq 2.
\]

\[\Box\]

### 3. Inequalities for the Hodge Numbers

After having studied the exactness of the complex (2) we can derive inequalities for the Hodge numbers by using well-known results for locally free sheaves on projective spaces: the Evans-Griffith Theorem and the non negativity of the Chern classes for globally generated locally free sheaves.

Throughout this section we fix integers \( d \geq 1, \ q \geq 1, 0 \leq p \leq d \) and \( 0 < m \leq d \). We denote by \( h^{p,q} = h^{p,q}(X) = \dim H^q(X, \Omega^p_X) \) the Hodge numbers of \( X \) and by \( q = q(X) \) the irregularity of \( X \).

Before stating the results we need to introduce some notation. If \( d - p < m \leq d \), for \( 1 \leq i \leq q - 1 \) we define \( \gamma_i(X, \Omega^p_X) \) to be the coefficient of \( t^i \) in the formal power series:
\[
\gamma(X, \Omega^p_X; t) \overset{\text{def}}{=} \prod_{j=1}^{m-d+p} (1 - jt)^{(-1)^j h^{p,2d-m-p+j}} \in \mathbb{Z}[[t]].
\]

If \( p < m \leq d \), for \( 1 \leq i \leq q - 1 \) we define \( \delta_i(X, \Omega^p_X) \) to be the coefficient of \( t^i \) in the formal power series:
\[
\delta(X, \Omega^p_X; t) \overset{\text{def}}{=} \prod_{j=1}^{m-p} (1 - jt)^{(-1)^j h^{p,m-p-j}} \in \mathbb{Z}[[t]].
\]

If \( m = \infty \), for \( i = 1, \ldots q - 1 \) we define \( \varepsilon_i(X, \Omega^p_X) \) to be the coefficient of \( t^i \) in the formal power series:
\[
\varepsilon(X, \Omega^p_X; t) \overset{\text{def}}{=} \prod_{j=1}^{d} (1 - jt)^{(-1)^j h^{p,d-j}} \in \mathbb{Z}[[t]].
\]

Also consider the following pieces of the Euler characteristic of the bundle \( \Omega^p_X \). If \( d - p < m \leq d \) define
\[
\chi^{\geq 2d-m-p}(\Omega^p_X) \overset{\text{def}}{=} \sum_{j=2d-m-p}^{d} (-1)^{2d-m-p+j} h^{p,j}
\]
and if \( p < m \leq d \) define
\[
\chi^{m-p}(\Omega^p_X) \overset{def}{=} \sum_{j=0}^{m-p} (-1)^{m-p+j} h^p j.
\]

**Theorem 3.1.** Let \( X \) be a compact Kähler manifold of dimension \( d \) and irregularity \( q \geq 1 \). Let \( m = m(X) = \min \{ \text{codim} Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega^1_X) \} \) and let \( 0 \leq p \leq d \) be an integer.

(i). If \( d - p < m \leq d \) then any Schur polynomial of weight \( \leq q - 1 \) in the \( \gamma_i(X, \Omega^p_X) \) is non-negative. In particular
\[
\gamma_i(X, \Omega^p_X) \geq 0
\]
for every \( 1 \leq i \leq q - 1 \). Moreover, if \( i \) is an index with \( \chi^{2d-m-p}(\Omega^p_X) < i < q \), then \( \gamma_i(X, \Omega^p_X) = 0 \).

(ii). If \( p < m \leq d \) then any Schur polynomial of weight \( \leq q - 1 \) in the \( \delta_i(X, \Omega^p_X) \) is non-negative. In particular
\[
\delta_i(X, \Omega^p_X) \geq 0
\]
for every \( 1 \leq i \leq q - 1 \). Moreover, if \( i \) is an index with \( \chi^{m-p}(\Omega^p_X) < i < q \), then \( \delta_i(X, \Omega^p_X) = 0 \).

(iii). If \( m = \infty \) then
\[
\varepsilon_i(X, \Omega^p_X) = 0
\]
for every \( i = 1, \ldots, q - 1 \).

**Proof.** If \( m > d - p \) then by Proposition 2.1(ii) the complex \( \mathbb{L}_X^p \) is exact at the first \( m-d+p \) steps from the right, and hence we get the exact sequence
\[
0 \rightarrow G \rightarrow \mathcal{O}_p(d - m - p) \otimes H^{2d-m-p}(X, \Omega^p_X) \overset{g}{\rightarrow}
\]
where \( G \) is the kernel of the map \( g \). Tensoring (5) by \( \mathcal{O}_p(d - m - p) \) and then dualizing it, we note that the polynomial \( \gamma(X, \Omega^p_X; t) \) is the Chern polynomial of the locally free sheaf \( G^\vee(m - d + p) \). Then its Chern classes \( c_i(G^\vee(m - d + p)) \), as well as the Schur polynomials in these, are non-negative since \( G^\vee(m - d + p) \) is globally generated. In particular we get
\[
\gamma_i(X, \Omega^p_X) = \deg c_i(G^\vee(m - d + p)) \geq 0.
\]
The last statement of (i) follows from the fact that \( c_i(G) = 0 \) for \( i > \text{rank} \ G = \chi^{2d-m-p}(\Omega^p_X) \).

The proof of (ii) is analogous to the proof of the previous point. If \( m > p \) then by Proposition 2.1(i) the complex \( \mathbb{L}_X^p \) is exact at the first \( m-p \) steps from the left and induces the following exact complex
\[
0 \rightarrow \mathcal{O}_p(-d) \otimes H^0(X, \Omega^p_X) \rightarrow \ldots \rightarrow \mathcal{O}_p(-d + m - p - 1) \otimes H^{m-p-1}(X, \Omega^p_X) \overset{f}{\rightarrow}
\]
where \( \mathcal{O}_p(-d + m - p) \otimes H^{m-p}(X, \Omega^p_X) \rightarrow F \rightarrow 0 \).
where $F$ is the cokernel of the map $f$. Tensoring (10) by $\mathcal{O}_P(d - m + p)$ we get that the locally free sheaf $F(d - m + p)$ is globally generated and moreover that its Chern polynomial is the polynomial $\delta(X, \Omega^p_X; t)$. Now we conclude as in (i).

If $m = \infty$ then the complex $L^p_X$ is everywhere exact and the polynomial $\varepsilon(X, \Omega^p_X; t)$ is just the Chern polynomial of the zero sheaf. Thus its Chern classes satisfy $\varepsilon_i(X, \Omega^p_X) = 0$, for every $i = 1, \ldots, q - 1$. □

Under the assumption of Theorem 3.1 we also have

**Theorem 3.2.**

(i). If $d - p < m \leq d$ and $q > \max\{m - d + p, d - p - 1\}$ then

$$\chi^{\geq 2d - m - p}(\Omega^p_X) \geq q + d - m - p$$

and

$$h^{d - p, 1} \geq h^{d - p, 0} + q - 1.$$

(ii). If $p < m \leq d$ and $q > \max\{m - p, p - 1\}$ then

$$\chi^{\leq m - p}(\Omega^p_X) \geq q - m + p$$

and

$$h^{p, 1} \geq h^{p, 0} + q - 1.$$

**Proof.** (i). By Proposition 2.1 the complex $L^p_X$ is exact at the first $m - d + p$ steps from the right. Since $q > d - p - 1$ we can prove, with an argument similar to the one used in Corollary 2.3 that $h^d(X, \Omega^p_X)$ is non-zero and hence that the complex $L^p_X$ is non-zero as well (this observation allow us to use Lemma 2.2). If $q = m - d + p + 1$ then it is enough to prove that the rank of the locally free sheaf $G$ in (5) is at least one. By Lemma 2.2 the locally free sheaf $G$ is neither zero nor splits as a sum of line bundles. Then by the Evans-Griffith Theorem (see [La] p. 92)

$$\text{rank } G = \chi^{\geq 2d - m - p}(\Omega^p_X) \geq q + d - m - p.$$

For the inequality $h^{d - p, 1} \geq h^{d - p, 0} + q - 1$, it is enough to note that the complex $L^p_X$ induces a surjection $H^{d - 1}(X, \Omega^p_X) \otimes \mathcal{O}_P \rightarrow H^d(X, \Omega^p_X) \otimes \mathcal{O}_P(1) \rightarrow 0$ and, since $h^d(X, \Omega^p_X) \neq 0$, we can conclude thanks to Example 7.2.2 in [La].

(ii). The hypothesis $p < m \leq d$ implies that the complex $L^p_X$ is exact at the first $m - p$ steps from the left. Since $q > p - 1$ we have that $h^0(X, \Omega^p_X) = h^d(X, \Omega^{d - p}_X) \neq 0$ as in (i), and therefore that the complex $L^p_X$ is non-zero as well. After having noted that rank $F = \chi^{\leq m - p}(\Omega^p_X)$ we can argue as in the previous point. □
3.1. **The case** $m(X) = \dim X$. When $X$ is an irregular compact Kähler manifold with $m(X) = \dim X$ further inequalities hold thanks to Catanese’s work [Cat]. Let $\text{alb}_X : X \to \text{Alb}(X)$ be the Albanese map of $X$. We say that $X$ is of maximal Albanese dimension if $\dim \text{alb}_X(X) = \dim X$. Following Catanese’s terminology we say that $X$ is of Albanese general type if $q(X) > \dim X$ and if it is of maximal Albanese dimension. A higher irrational pencil is a surjective map with connected fibers $f : X \to Y$ onto a normal lower dimensional variety $Y$ and such that any smooth model of $Y$ is of Albanese general type.

**Lemma 3.3.** If $X$ is an irregular compact Kähler manifold with $m(X) = \dim X$, then $X$ does not carry any higher irrational pencils.

**Proof.** We proceed by contradiction. Suppose a higher irrational pencil $f : X \to Y$ exists and let $\text{alb}_Y : Y \to \text{Alb}(Y)$ be the Albanese map of $Y$, which is well defined since $Y$ is normal. The map $\text{alb}_Y$ is not surjective, hence following an idea contained in the proof of [EL] Proposition 2.2, one can show that given a general point $y \in Y$ there exists a holomorphic 1-form $\omega$ of $\text{Alb}(Y)$ whose restriction to $\text{alb}_Y(Y)$ vanishes at $\text{alb}_Y(y)$. Pulling back $\omega$ to $X$, we get a holomorphic 1-form which vanishes along some fibers of $f$ which are of positive dimension, this contradicting the hypothesis $m(X) = \dim X$. The form $\omega$ can be constructed as follows. Let $z$ be a smooth point of the Albanese image $\text{alb}_Y(Y) \subset \text{Alb}(Y)$. The coderivative map $T_z^* \text{Alb}(Y) \to T_z^* \text{alb}_Y(Y)$ is surjective with non trivial kernel. Then take $\omega$ to be the extension to a holomorphic 1-form on $\text{Alb}(Y)$ of any non-zero form belonging to this kernel. 

In the following Proposition we collect inequalities for Hodge numbers in the case $m(X) = \dim X$, which will be used to give asymptotic bounds for Hodge numbers in terms of the irregularity for manifolds of dimension three and four (cf. Corollary 4.1 and Corollary 4.2). These are essentially extracted from [LP] Remark 3.3 and the references therein together with Lemma 3.3.

**Proposition 3.4.** Let $X$ be an irregular compact Kähler manifold with $m(X) = \dim X$. Then

\begin{equation}
\label{eq:inequality1}
h^{0,k} \geq k(q(X) - k) + 1
\end{equation}

for any $k = 0, \ldots, \dim X$. If $\dim X \geq 3$ then

\begin{equation}
\label{eq:inequality2}
h^{0,2} \geq 4q(X) - 10.
\end{equation}

If $q(X) \geq \dim X$, and for any value of $\dim X$, then

\begin{equation}
\label{eq:inequality3}
\chi(\omega_X) \geq q(X) - \dim X.
\end{equation}

**Proof.** One can show that if $\omega_1, \ldots, \omega_k$ are linearly independent holomorphic 1-forms then $\omega_1 \wedge \ldots \wedge \omega_k \neq 0$. This can be done by induction on $k$ and by using the above Lemma 3.3, Theorem 1.14 and Lemma 2.20 in [Cat]. This fact can be reformulated as saying that the natural maps

\[ \phi_k : \bigwedge^k H^1(X, \mathcal{O}_X) \to H^k(X, \mathcal{O}_X) \]
are injective on primitive forms $\omega_1 \wedge \ldots \wedge \omega_k$. The set of such forms is the cone over the image of the Plücker embedding, i.e. over the Grassmannian $G(k, H^1(X, \mathcal{O}_X))$, and by comparing the dimensions we have the bounds. For the inequality (8) we can apply the same argument used in [LP] Remark 3.3 and for the inequality (9) we note that when $q(X) \geq \dim X$ then $X$ is of maximal Albanese dimension. In fact by [GL1] Remark on p. 405, the cohomological support loci $V_i(\omega_X) = \{ \alpha \in \text{Pic}^0(X) | H^i(X, \omega_X \otimes \alpha) \neq 0 \}$ consist of at most a finite set of points and, by [LP] Remark 1.4 or by [BLNP] Proposition 2.7, $X$ is of maximal Albanese dimension. Now the inequality $\chi(\omega_X) \geq q(X) - \dim X$ follows by [PP2] Corollary 4.2 or by [LP] Theorem 3.1. □

4. Examples and asymptotic bounds for threefolds and fourfolds

In this section we list concrete inequalities coming from Theorems 3.1 and 3.2 in the most interesting case $m(X) = \dim X$, $q(X) \geq \dim X$, and for $\dim X = 3, 4, 5$. Moreover for threefolds and fourfolds we list asymptotic bounds in terms of the irregularity $q(X)$ for all the Hodge numbers. We also point out that some of the inequalities are still valid for some values of $q(X)$ smaller than $\dim X$ and that other inequalities hold for different values of $m(X)$. Set $q = q(X)$ for the irregularity and $h^{p,q} = h^{p,q}(X)$ for the Hodge numbers.

Let us start with Theorem 3.1. We get a first set of inequalities by imposing the conditions $\gamma_1(X, \Omega^k_X) \geq 0$ for $k = 0, 1, 2$. Hence

$$h^{0,2} \geq 2q - 3, \quad h^{1,1} \geq 2q$$

for $\dim X = 3$

$$h^{1,2} \geq 2h^{1,1} - 4q, \quad h^{1,3} \geq 2h^{1,2} - 3q + 4$$

for $\dim X = 4$

$$h^{0,4} \geq 4q - 3h^{0,2} + 2h^{0,3} - 5, \quad h^{1,4} \geq 4h^{1,1} - 3h^{1,2} + 2h^{1,3}, \quad h^{2,2} \geq 2h^{1,2} - 3h^{0,2}$$

for $\dim X = 5$

Finer inequalities are obtained by solving $\gamma_2(X, \Omega^k_X) \geq 0$. Then for $\dim X = 3$ we have

$$h^{0,2} \geq 2q - \frac{7}{2} + \frac{\sqrt{8q - 23}}{2}, \quad h^{1,1} \geq 2q - \frac{1}{2} + \frac{\sqrt{8q + 1}}{2}$$

and for $\dim X = 4$ we get

$$h^{0,3} \geq 2h^{0,2} - 3q + \frac{7}{2} + \frac{\sqrt{8h^{0,2} - 24q + 49}}{2}$$

$$h^{1,2} \geq 2h^{1,1} - 3q + \sqrt{4h^{1,1} - 9q}$$

$$h^{1,3} \geq 2h^{0,2} - \frac{1}{2} + \frac{\sqrt{8h^{0,2} + 1}}{2}$$

where the quantity $4h^{1,1} - 9q$ is non-negative by the second inequality of Theorem 3.2 (i) and the quantity $8h^{0,2} - 24q + 49$ is non-negative by inequality (8). Finally for $\dim X = 5$
we get

\[
\begin{align*}
    h^{0,4} &\geq 4q - 3h^{0,2} + 2h^{0,3} - \frac{11}{2} + \frac{\sqrt{48q - 24h^{0,2} + 8h^{0,3} - 79}}{2} \\
h^{1,4} &\geq 2h^{1,3} + 4h^{1,1} - 3h^{1,2} - \frac{1}{2} + \frac{\sqrt{48h^{1,1} - 24h^{1,2} + 8h^{1,3} + 1}}{2} \\
h^{2,2} &\geq 2h^{1,2} - 3h^{0,2} - \frac{1}{2} + \frac{\sqrt{8h^{1,2} - 24h^{0,2} + 1}}{2}
\end{align*}
\]

which hold as long as the quantity under the square root are non-negative.

Applying Theorem 3.2 with \(m(X) = \dim X\) and \(q(X) \geq \dim X\), we get for \(\dim X = 3\)

\[
\chi(\omega_X) \geq q - 3, \quad h^{1,1} \geq 2q - 1, \quad h^{1,2} \geq h^{1,1} - 2, \quad h^{1,2} \geq h^{0,2} + q - 1,
\]

for \(\dim X = 4\)

\[
\begin{align*}
    \chi(\omega_X) &\geq q - 4, \quad h^{2,2} \geq h^{1,2} - h^{0,2} + q - 2, \quad h^{1,3} \geq h^{1,2} - h^{1,1} + 2q - 3 \\
h^{1,1} &\geq 2q - 1, \quad h^{1,2} \geq h^{2,0} + q - 1, \quad h^{1,3} \geq h^{0,3} + q - 1
\end{align*}
\]

and for \(\dim X = 5\)

\[
\begin{align*}
    \chi(\omega_X) &\geq q - 5, \quad h^{1,4} \geq h^{1,3} - h^{1,2} + h^{1,1} - 4, \quad h^{1,1} \geq 2q - 1, \\
2h^{1,2} &\geq h^{2,2} + h^{0,2} + q - 3, \quad h^{1,2} \geq h^{0,2} + q - 1, \quad h^{1,2} \geq h^{1,3} - h^{0,3} + q - 2, \\
h^{1,3} &\geq h^{0,3} + q - 1, \quad h^{1,4} \geq h^{0,4} + q - 1.
\end{align*}
\]

We select the strongest of the inequalities above in dimension three and four in statements formulated asymptotically for simplicity:

**Corollary 4.1.** Let \(X\) be an irregular compact Kähler threefold with \(m(X) = 3\). Then asymptotically

\[
\begin{align*}
    h^{0,2} &\geq 4q, \quad h^{0,3} \geq 4q, \quad h^{1,1} \geq 2q + \sqrt{2q}, \quad h^{1,2} \geq 5q + \sqrt{2q}.
\end{align*}
\]

**Proof.** The inequality (8) gives \(h^{0,2} \geq 4q\). The inequality \(\chi(\omega_X) \geq q - 3\) of Theorem 3.2 implies the inequality \(h^{0,3} \geq h^{0,2} - 2\) and therefore asymptotically \(h^{0,3} \geq 4q\). The asymptotic bound for \(h^{1,1}\) follows by (10). Since by Corollary 2.3 \(\chi(\Omega_X^1) \geq 2\) we also get the bound for \(h^{1,2}\). \(\square\)

**Corollary 4.2.** Let \(X\) be an irregular compact Kähler fourfold with \(m(X) = 4\). Then asymptotically

\[
\begin{align*}
    h^{0,2} &\geq 4q, \quad h^{0,3} \geq 5q + \sqrt{2q}, \quad h^{0,4} \geq 4q \\
h^{1,1} &\geq 2q, \quad h^{1,2} \geq 8q + 2\sqrt{2q}, \quad h^{1,3} \geq 12q + 3\sqrt{2q}, \quad h^{2,2} \geq 8q + 4\sqrt{2q}.
\end{align*}
\]

**Proof.** The asymptotic bounds for \(h^{0,2}, h^{0,3}\) and \(h^{0,4}\) follow by (8), (11) and (7) respectively. Using the second inequality of Theorem 3.2 (i) we get \(h^{1,1} \geq 2q\), and by inequality (13) we get the bound for \(h^{1,2}\). By Corollary 2.3 we have \(\chi(\Omega_X^1) \leq 2\) and \(\chi(\Omega_X^2) \geq 1\) which imply the bounds for \(h^{1,3}\) and \(h^{2,2}\). \(\square\)
5. Regularity of the cohomology modules

In this section we give the proof of Theorem 1.2 from the Introduction.

Let $V$ be a complex vector space and $E = \bigwedge^* V$ be the graded exterior algebra over $V$. A finitely generated graded $E$-module $P = \bigoplus_{j \geq 0} P_j$ is called $m$-regular if it is generated in degrees $0$ up to $-m$, and if its minimal free resolution has at most $m + 1$ linear strands. Equivalently, $P$ is $m$-regular if and only if $\text{Tor}^E_i(P, C)_{-i-j} = 0$ for all $i \geq 0$ and all $j \geq m + 1$. The dual over $E$ of the module $P$ is defined to be the $E$-module $\hat{P} = \bigoplus_j P^*_{-j}$ (cf. [Eis], [EFS], [LP]).

We continue to denote by $X$ an irregular compact Kähler manifold of dimension $d$ and irregularity $q$. Set $V = H^1(X, \mathcal{O}_X)$ and $E = \bigwedge^* V$. In [LP] the authors determined the regularity of the graded $E$-module $Q_X = \bigoplus_i H^i(X, \omega_X)$ by studying the exactness of the complex associated to its dual module $P_X = \bigoplus_i H^i(X, \mathcal{O}_X)$ via the BGG correspondence. Fore references on the BGG correspondence see [BGG], [EFS] and Chapter 7B of [Eis]. By applying their same technique and by using Generic Vanishing Theorems for bundles of holomorphic $p$-forms (see [PP1] and [CH]) we give a bound for the regularity of the $E$-modules $\bigoplus_i H^i(X, \Omega^p_X)$ for any $p$.

Fix an integer $p = 0, \ldots, d$. Via cup product consider the graded $E$-module

$$P^p_X = \bigoplus_i H^i(X, \Omega^{d-p}_X)$$

where the graded piece $H^i(X, \Omega^{d-p}_X)$ has degree $d - i$. The dual module over $E$ of $P^p_X$ is the module

$$Q^p_X = \bigoplus_i H^i(X, \Omega^p_X)$$

where the graded piece $H^i(X, \Omega^p_X)$ has degree $-i$.

Let $W = V^*$ be the dual vector spicasas for linuxce of $V$ and $S = \text{Sym}(W)$ be the symmetric algebra over $W$. Also denote by $P = P_{\text{sub}}(V)$ the projective space of dimension $q - 1$ over $V$. By applying the functor $\Gamma_*$ to the complex $L^p_X$, we get the complex $L^p_X$ of $S$-graded modules in homological degree 0 to $d$

$$L^p_X : \quad 0 \longrightarrow S \otimes C H^0(X, \Omega^p_X) \longrightarrow S \otimes C H^1(X, \Omega^p_X) \longrightarrow \ldots \longrightarrow S \otimes C H^d(X, \Omega^p_X) \longrightarrow 0.$$

(see [Ha] p. 118 for the definition of $\Gamma_*$). To bound the regularity of the module $Q^p_X = \bigoplus_i H^i(X, \Omega^p_X)$ is enough to understand until which step the complex $L^{d-p}_X$ is exact. In fact, by following the proof of [LP] Lemma 2.3, one can show that the complex $L^{d-p}_X$ is identified with the complex associated to the module $P^p_X = \bigoplus_i H^i(X, \Omega^{d-p}_X)$ via the BGG correspondence and one can use Proposition 2.2 (cf. loc. cit.) to bound the regularity of $Q^p_X$. At this point Theorem 1.2 follows by the previous discussion together with point (i) of the following Proposition.

**Proposition 5.1.** Let $X$ be an irregular compact Kähler manifold of dimension $d$. Let $k$ be the dimension of the generic fiber of the Albanese map $\text{alb}_X : X \longrightarrow \text{Alb}(X)$ and the let $f$ be the maximal dimension of a fiber of $\text{alb}_X$. Set $l = \max\{k, f - 1\}$. 
(i) If \( d - p > l \) then the complexes \( L^p_X \) and \( L^p_X \) are exact in the first \( d - p - l \) steps from the left.

(ii) If \( p > l \) then the complexes \( L^p_X \) and \( L^p_X \) are exact in the first \( p - l \) steps from the right.

Proof. We follow [LP] Proposition 1.1. Let \( A = \text{Spec}(\text{Sym}(W)) \) be the affine space corresponding to \( V \) viewed as an affine algebraic variety. Consider the following complex \( K_p \) of trivial locally free sheaves on \( A \):

\[
0 \to O_A \otimes H^0(X, \Omega^p_X) \to O_A \otimes H^1(X, \Omega^p_X) \to \ldots \to O_A \otimes H^d(X, \Omega^p_X) \to 0,
\]

with maps given at each point of \( A \) by cupping with the corresponding element of \( V \). Since \( \Gamma(A, O_A) = S \) one sees that \( L^p_X = \Gamma(A, K_p) \) i.e. \( L^p_X \) is obtained from \( K_p \) by applying the global sections functor. On the other hand the complex \( L^p_X \) is obtained by \( L^p_X \) via sheafification which is an exact functor, and hence its exactness is implied by the exactness of the complex \( K_p \), and consequently by the exactness of \( K_p \) since the functor global sections is exact on affine spaces. Let \( V \) be the vector space \( V \) considered as a complex manifold. By GAGA, the exactness of \( K_p \) is equivalent to the exactness of \( K^\text{an}_p \), where \( K^\text{an}_p \) is the complex

\[
0 \to O_V \otimes H^0(X, \Omega^p_X) \to O_V \otimes H^1(X, \Omega^p_X) \to \ldots \to O_V \otimes H^d(X, \Omega^p_X) \to 0
\]

of analytic sheaves on \( V \). If \( d - p > l \) then it is enough to check the exactness of \( K^\text{an}_p \) at the first \( d - p - l \) steps from the left, or equivalently the vanishing of the cohomologies \( \mathcal{H}^i(K^\text{an}_p) \) for any \( i < d - p - l \). Since the differentials of the complex \( K^\text{an}_p \) scale linearly in radial directions through the origin, it is then enough to check the vanishing of its stalks at the origin 0, i.e.

\[
\mathcal{H}^i(K^\text{an}_p)_0 = 0
\]

for \( i < d - p - l \).

Let now \( p_1 : X \times \text{Pic}^0(X) \to X \) and \( p_2 : X \times \text{Pic}^0(X) \to \text{Pic}^0(X) \) be the projections onto the first and second factor respectively and let \( \mathcal{P} \) be a normalized Poincaré line bundle on \( X \times \text{Pic}^0(X) \). Then Theorem 6.2 in [CH] gives an identification

\[
\mathcal{H}^i(K^\text{an}_p)_0 \simeq R^ip_{2*}(p_1^*\Omega^p_X \otimes \mathcal{P})_0,
\]

via the exponential map \( \exp : V \to \text{Pic}^0(X) \). At this point Theorem 5.11 (1) and Theorem 3.7 in [PP1] imply that \( R^ip_{2*}(p_1^*\Omega^p_X \otimes \mathcal{P}) = 0 \) for any \( i < d - p - l \).

The proof of (ii) is analogous to the proof in Proposition 2.1. □

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