Anti-deSitter black holes, perfect fluids, and holography

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We consider asymptotically anti-de Sitter black holes in \( d \)-spacetime dimensions in the thermodynamically stable regime. We show that the Bekenstein-Hawking entropy and its leading order corrections due to thermal fluctuations are reproduced by a weakly interacting fluid of bosons and fermions (‘dual gas’) in \( \Delta = \alpha(d - 2) + 1 \) spacetime dimensions, where the energy-momentum dispersion relation for the constituents of the fluid is assumed to be \( \epsilon = \kappa p^\alpha \). We examine implications of this result for entropy bounds and the holographic hypothesis.

I. INTRODUCTION

It is believed that black holes are thermodynamical objects with entropy given by one-quarter their horizon areas \( A_H \) (in Planck units) [1]

\[
S_{BH} = \frac{A_H}{4\ell_{Pl}^d},
\]

where \( \ell_{Pl} = (G\hbar/c^3)^{1/(d-2)} \) is the Planck length and \( G_d \) the Newton’s constant in \( d \)-spacetime dimensions. They satisfy the two laws of ‘black hole thermodynamics’

\[
\begin{align*}
S_{BH} &= \frac{A_H}{4\ell_{Pl}^d} + \text{work terms} & (2) \\
\Delta S_{BH} &\geq 0 & (3)
\end{align*}
\]

where \( T_H \) is the Hawking temperature. These laws are semi-classical, and apply to large black holes with \( A_H \gg \ell_{Pl}^d \).

The idea that black holes are thermodynamic systems in equilibrium has led to two related areas of study. The first constitutes suggestions for the underlying microscopic degrees of freedom. These are postulated based on one’s views of what are the fundamental degrees of freedom arising from a quantum theory of gravity. The goal is to reproduce the entropy formula (1) by tracing a density matrix describing a black hole over unobserved quantum degrees of freedom. A related approach involves calculating the number of microstates associated with macroscopic parameters of a black hole, such as its mass and charge [2–4].

The second area of study arises from the observation that the entropy of a confined non-black hole system is proportional to spatial volume, rather than bounding area. Reconciling this with the fact that black holes are believed to be the most entropic systems leads to a viewpoint known as the holographic hypothesis. This hypothesis states that a \( d \)-dimensional theory may be encoded exactly in a \((d-1)\)-dimensional theory. What remains is the significant task of providing the details of the encoding, the “holographic map.” A part of this map consists in matching the thermodynamics of black holes with a suitable system in one lower dimension.

In this paper we focus on the second area of study motivated by black holes, namely holography. Specifically we address the following question: What are the criteria under which the thermodynamics of the AdS-Schwarzchild black hole in spacetime dimension \( d \) is reproduced by a weakly interacting gas in spacetime dimension \( \Delta \)? This question is useful to ask for at least two reasons: (i) From the viewpoint of “physical” holography, one would like to see whether a bulk physical system in a bounded region (here the black hole) can be described by a gas in one lower dimension, and (ii) whether their exist thermodynamic dualities more general than holography, where the difference in dimension is different from one. We answer the question by showing that the thermodynamics properties of the AdS black hole can be encoded in a gas of free bosons and fermions, such that the dimension of the dual gas depends on the dimension of the black hole and on the dispersion relation of its constituents. We also show that the matching of thermodynamics for these two systems extends to thermal fluctuations.

In the next section, we review recent work on thermal fluctuation corrections to entropy and its application to black holes. In section (III) we discuss the thermodynamics of asymptotically anti-de Sitter black holes and thermal fluctuation corrections to entropy. In section (IV), we show that a perfect fluid of bosons and fermions captures the thermodynamics of these black holes, including thermal fluctuation corrections. We examine the implications of our results for entropy bounds [5–8] in section (VI), and conclude in section (VII) with a summary and list of open questions.

II. THERMAL FLUCTUATIONS AND ENTROPY

Before we give a systematic derivation of leading order corrections to entropy of a system due to small thermal fluctuations, we present a heuristic argument showing how logarithmic corrections to entropy arise for a black hole.

Consider Wheeler’s “it from bit” idea, which ascribes one bit of information (equivalently a spin \( 1/2 \)) to each
Planck area on the black hole’s horizon [5,9]. This idea also incorporates “holography” in that degrees of freedom are associated with area rather than volume. If the area of the black hole horizon is $A$, the number of microstates is $\Omega = 2^A$, with $n = A/\ell_p^d-2$, so that entropy $S_{BH} = \ln \Omega$ is proportional to horizon area. This model may be obviously generalised to associate a spin $s$ with each Planck area without changing the basic result. The number of microstates is now $\Omega = (2s + 1)^n$, which gives entropy proportional to area with a different coefficient.

Corrections to entropy arise in this picture if the component of total spin along a given axis is fixed. Let $x^i$ of the $n$ spins have a state $s^i$ chosen from the $(2s + 1)$ projections along a fixed axis. (The total spin component along that axis is then $\sum_i x_i(2s/(2s + 1))s_i - s$.) The number of possible states $\mathcal{N}$ subject to $\sum_i x_i = n$ is given by the multinomial distribution function (assuming that the probability for finding any spin in state $i$ is independent of $i$, and equal to $1/(2s + 1)$):

$$\mathcal{N} = \frac{n!}{x_1!x_2!\cdots x_{2s+1}!}. \quad (4)$$

The entropy $S = \ln \mathcal{N}$ may be evaluated using Stirling’s formula, assuming $x^i > 1$ to give

$$S = an - s \ln n - bS_{BH} - s \ln(S_{BH}) \quad (5)$$

where $S_{BH} = A \ln(2s+1)/\ell_p^2$ is the entropy with no restriction on total spin component, $a \equiv \sum_i k_i \ln k_i^{-1}$ and $b = a/\ln(2s+1)$ are of order unity, and $k_i \equiv x_i/n$. This is a generalisation of the $s = 1/2$ case considered in [10]. Thus, the leading corrections are logarithmic, with the exact coefficient depending on the spin of the building blocks of the black hole.

We now discuss some generalities of thermodynamic systems in equilibrium and their thermal fluctuations, before proceeding to AdS black holes and the free boson/fermion gas. This is a review of work [11–14] which shows that logarithmic corrections to entropy arise from quite general considerations.

The canonical partition function for any thermodynamic system in equilibrium is

$$Z(\beta) = \int_0^\infty \rho(E) e^{-\beta E} dE. \quad (6)$$

The density of states $\rho(E)$ can be written as an inverse Laplace transform of the partition function,

$$\rho(E) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(\beta) e^{\beta E} d\beta = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{S(\beta)} d\beta, \quad (7)$$

where

$$S(\beta) = \ln Z + \beta E \quad (8)$$

is the ‘entropy function’. A saddle-point approximation of $S(\beta)$ around the equilibrium temperature $\beta_0^{-1}$ gives

$$\rho(E) = \frac{e^{S_0}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{1/2(\beta-\beta_0)^2 S''_0} d\beta, \quad (9)$$

where $S_0 = S(\beta_0)$ and $S''_0 = S_{\beta_0}^{\beta_0}$. By substituting $\beta - \beta_0 = ix$, and choosing an appropriate contour, the integral can be evaluated exactly to give

$$\rho(E) = \frac{e^{S_0}}{\sqrt{2\pi S''_0}}. \quad (10)$$

The corrected entropy is

$$S := \ln \rho(E) = S_0 - \frac{1}{2} \ln S''_0 + \text{ (smaller terms)}. \quad (11)$$

That the quantity $S''_0$ is indeed a measure of fluctuations of the system can be seen from the relation

$$S''(\beta) = \frac{1}{Z} \left( \frac{\partial^2 Z(\beta)}{\partial \beta^2} \right) - \frac{1}{Z^2} \left( \frac{\partial Z}{\partial \beta} \right)^2 \quad (12)$$

$$= <E^2> - <E>^2, \quad (13)$$

where we have used the definitions

$$<E> = - \left( \frac{\partial \ln Z}{\partial \beta} \right)_{\beta=\beta_0} \quad <E^2> = \frac{1}{Z} \left( \frac{\partial^2 Z}{\partial \beta^2} \right)_{\beta=\beta_0}. \quad (14)$$

Using the fact that the specific heat of a thermodynamic system in equilibrium can be written as

$$C \equiv \left( \frac{\partial E}{\partial T} \right)_{T_0} = \frac{1}{T^2} \left[ \frac{1}{Z} \left( \frac{\partial^2 Z}{\partial \beta^2} \right)_{\beta=\beta_0} - \frac{1}{Z^2} \left( \frac{\partial Z}{\partial \beta} \right)_{\beta=\beta_0}^2 \right]$$

$$= \frac{S''_0}{T^2}, \quad (15)$$

gives

$$S''_0 = CT^2, \quad (16)$$

and hence from (11) that

$$S = S_0 - \frac{1}{2} \ln \left( CT^2 \right) + \cdots. \quad (17)$$

This formula is applicable to all thermodynamic systems. In particular it may be applied to black holes by setting $S_0 = S_{BH}$, $T = T_{BH}$ and $C = C_{BH}$ for the specific black hole under consideration. It is understood that the quantity within the logarithm is divided by $k_B^2$, the square of the Boltzmann constant.

The application of (17) to black holes was considered in detail in [13,14], where it was shown that it provides a general approach to understanding corrections to black hole entropy computed in various models in the literature. For related and other approaches, see [15–30]. For other applications of (17), see [31].
III. ASYMPTOTICALLY ANTI-DE SITTER BLACK HOLES

Here we review thermodynamic properties of the Bañados-Teitelboim-Zanelli (BTZ) \cite{32} and AdS-Schwarzschild black holes, and compute entropy corrections using the method discussed above. This method does not apply to Schwarzschild black holes which have negative specific heat.

For the BTZ black holes

\[ ds^2 = - \left( \frac{r^2}{\ell^2} - \frac{8G_3 M}{c^2} \right) c^2 dt^2 + \left( \frac{r^2}{\ell^2} - \frac{8G_3 M}{c^2} \right)^{-1} dr^2 + r^2 d\Omega^2 , \]

(18)

the entropy, temperature and specific heat are

\[ S_{\text{BH}} = \frac{2\pi r_+}{4\ell p_1} \left( \frac{\ell \pi c^2 \sqrt{2}}{\hbar \sqrt{G_3}} \right) \sqrt{M} \]

(19)

\[ T_H = \frac{\hbar c r_+}{2\pi \ell^2} = \frac{\hbar \sqrt{2G_3 M}}{\pi \ell} = \left( \frac{\hbar \sqrt{G_3}}{\pi \ell c} \right)^2 S_{\text{BH}} \]

(20)

\[ C_{\text{BH}} = \frac{dM c^2}{dT_H} = S_{\text{BH}} , \]

(21)

where \( r_+ = \ell \sqrt{8G_3 M}/c \) is the horizon radius. Substituting in (17) gives

\[ S = S_{\text{BH}} - \frac{1}{2} \ln \left( S_{\text{BH}} S_{\text{BH}}^2 \right) + \cdots \]

\[ = S_{\text{BH}} - \frac{3}{2} \ln (S_{\text{BH}}) + \cdots . \]

(22)

This agrees with correction computed using conformal field theory by Carlip \cite{20}, including the coefficient \(-3/2\) in front of the logarithm.

Similarly, for AdS-Schwarzschild black holes in \( d \) dimensions, with a cosmological constant

\[ \Lambda = -(d - 1)(d - 2)/2\ell^2 , \]

the metric is

\[ ds^2 = - \left( 1 - \frac{16\pi G_d M}{(d - 2) \Omega_{d-2} c^2 r^{d-3}} + \frac{r^2}{\ell^2} \right) c^2 dt^2 + \left( 1 - \frac{16\pi G_d M}{(d - 2) \Omega_{d-2} c^2 r^{d-3}} + \frac{r^2}{\ell^2} \right)^{-1} dr^2 + r^2 d\Omega_{d-2} . \]

(23)

The entropy, temperature and specific heat are given by

\[ S_{\text{BH}} = \frac{A_{d-2}}{4\ell^{d-2} p_1} = \frac{\Omega_{d-2} r_{+}^{d-2}}{4\ell^{d-2} p_1} \]

(24)

\[ T_H = \frac{\hbar c}{4\pi \ell^2 r_+} (d - 1)r_+^2 + (d - 3)\ell^2 \]

(25)

\[ C_{\text{BH}} = (d - 2) \left[ \frac{(d - 1)r_+^2}{\ell^2} + (d - 3) \right] S_{\text{BH}} , \]

(26)

where \( \Omega_{d-2} \) is the area of an unit \( S_{d-2} \). In the so-called ‘high-temperature limit’ \( r_+ >> \ell \), a regime in which the specific heat is positive, the corrected entropy using (17) is

\[ S = S_{\text{BH}} - \frac{1}{2} \ln \left( S_{\text{BH}} S_{\text{BH}}^2 \right) + \cdots \]

\[ = S_{\text{BH}} - \frac{d}{2(d - 2)} \ln (S_{\text{BH}}) + \cdots . \]

(27)

It is curious to note that on substituting \( d = 3 \) in the above, the BTZ result (22) is recovered, although the metric of the latter is not a special case of (23).

The free energy of the black hole in the limit \( r_+ >> \ell \) may be computed from Eq.(23-25):

\[ F_{\text{BH}} = Mc^2 - T_H S_0 = -c_1 T_H^{d-1} , \]

(28)

where

\[ c_1 = \frac{\Omega_{d-2}}{4(d - 1)\ell^{d-2} p_1} \left( \frac{4\pi \ell^2}{\hbar c(d - 1)} \right)^{d-2} . \]

(29)

The entropy may be expressed as

\[ S_{\text{BH}} = -\frac{\partial F_{\text{BH}}}{\partial T_{\text{BH}}} = c_1(d - 1) T_H^{d-2} . \]

(30)

IV. DUAL GAS

We now attempt to model the thermodynamic properties of asymptotically AdS black holes using free bosons and fermions at temperature \( T \) in spacetime dimension \( \Delta \), and with the dispersion relation

\[ \epsilon = \kappa p^\alpha . \]

(31)

Although this relation is rather unusual, we will see below that it leads, via standard statistical mechanics, to the perfect fluid equation of state

\[ P = \left( \frac{\alpha}{\Delta - 1} \right) \rho , \]

(32)

relating pressure \( P \) and energy density \( \rho \). This shows, independent of all other results presented here, that the dispersion relation (31) describes the microscopics of the \( P = \kappa \rho \) perfect fluid commonly used in general relativity. It is interesting that the dominant energy condition (positive energy density, and timelike or null energy fluxes), places a restriction on \( \alpha \) and \( \Delta \):

\[ 0 < \alpha \leq \Delta - 1 . \]

(33)

This shows for example that for \( \Delta = 4 \), all \( \alpha \) such that \( 0 < \alpha \leq 3 \) give physically acceptable fluids. In the following we derive (32), and compute other thermodynamical
quantities, for subsequent comparison with black holes variables.

The thermodynamic potential $\Omega$ is [33]
\[
\Omega = \pm T \sum_i \ln \left[ 1 \mp e^{(\mu - \epsilon_i)/T} \right]
\] (34)
where $+$ and $-$ signatures refer to bosons and fermions respectively, and $\mu$ is the chemical potential. In the continuum limit, partial integration gives
\[
\Omega = \pm \frac{V_{\Delta-1}\Omega_{\Delta-2}}{(2\pi\hbar)\Delta-1} \int_0^\infty \ln \left[ 1 \mp e^{(\mu - \epsilon)/T} \right] e^{\alpha (\Delta-1)/\alpha} \times \\
\times \int_0^\infty \frac{e^{(\Delta-1)/\alpha} de}{e^{(\mu - \epsilon)/T} \pm 1}.
\] (35)
The energy of the gas is
\[
E = \int_0^\infty \epsilon \; d\Gamma\!(\epsilon)
= \frac{V_{\Delta-1}\Omega_{\Delta-2}}{\alpha(2\pi\hbar)\Delta-1} \int_0^\infty \frac{e^{(\Delta-1)/\alpha} de}{e^{(\mu - \epsilon)/T} \pm 1}.
\] (36)
Comparing with (35) and using the relation $\Omega = -PV_{\Delta-1}$ gives
\[
E = \left( \frac{\Delta-1}{\alpha} \right) P V_{\Delta-1}.
\] (37)
This can be written in terms of the energy density $\rho = E/V_{\Delta-1}$ as
\[
P = k\rho,
\] (38)
where
\[
k = \frac{\alpha}{\Delta-1}.
\] (39)
This relates spacetime dimension $\Delta$ and dispersion relation power $\alpha$ to the coefficient $k$ in the equation of state.
The free energy of the gas is
\[
F_{\text{gas}} = -T \ln z = \pm T \sum_i \ln(1 \mp e^{-\beta \epsilon_i}).
\] (40)
The temperature dependence is obtained by partial integration after taking the continuum limit:
\[
F_{\text{gas}} = \pm \frac{T V_{\Delta-1}\Omega_{\Delta-2}}{(2\pi\hbar)\Delta-1} \int_0^\infty \ln(1 \mp e^{-\beta \epsilon}) dp \; p^{\Delta-2}
= -c_2 V_{\Delta-1} T \frac{\Delta-1}{\alpha} + 1,
\] (41)
where $V_{\Delta-1}$ is the volume of the gas, and
\[
c_2 = \frac{\Omega_{\Delta-2}}{(\Delta-1)\kappa(\Delta-1)/\alpha(2\pi\hbar)(\Delta-1)} \times \\
\times \zeta\left( \frac{\Delta-1}{\alpha} + 1 \right) \Gamma\left( \frac{\Delta-1}{\alpha} + 1 \right) \times \\
\times \left( n_B + n_F - \frac{n_F}{2(\Delta-1)/\alpha} \right).
\] (42)
Here $n_B(n_F)$ is the total number of bosonic (fermionic) degrees of freedom for the fluid. A similar temperature scaling for the energy density in four spacetime dimensions was discussed in [34]. The entropy is
\[
S_{\text{gas}} = -\frac{\partial F_{\text{gas}}}{\partial T} = c_2 \left( \frac{\Delta-1}{\alpha} + 1 \right) V_{\Delta-1} T^{\frac{\Delta-1}{\alpha}}.
\] (43)

V. MATCHING THERMODYNAMICS

At least two approaches may be taken for comparing thermodynamics of any two physical systems. A strong duality might involve equating partition functions, and hence free energies. This necessarily leads to the matching of all thermodynamical quantities.

A weaker duality might involve matching only the entropy. Holographic ideas arising from black hole entropy considerations suggest only the weaker version. Indeed, the holographic hypothesis at its basic level is concerned with matching information, which is kinematical, versus stronger dualities which involve dynamical comparisons as well.

We follow a weak duality approach, and derive the consequences of imposing the condition
\[
S_{\text{BH}} = S_{\text{gas}}.
\] (44)
This weak condition does not automatically lead to matching of other thermodynamical quantities, such as the specific heat. Therefore it is still necessary, for example, to compare entropy corrections due to thermal fluctuations (17), which depend on temperature and specific heat. If temperature matching is imposed in addition to (44), it is a stronger duality, and leads to entropy corrections matching automatically. We consider below both the weak and strong cases.

Matching powers of temperature in the entropy formulas (30) and (43) gives our first result relating spacetime dimensions and $\alpha$:
\[
\Delta = \alpha(d-2) + 1.
\] (45)
Thus, given an anti-de Sitter Schwarzschild black hole in $d$-spacetime dimensions, there is a dual gas in $\Delta$-spacetime dimensions which captures thermodynamic information of the black hole.

Eliminating $\alpha$ in (33) using (45) gives the relation
\[
0 < \frac{1}{d-2} \leq 1
\] (46)
for $\Delta \neq 1$, which holds for all $d > 2$, and is independent of $\Delta$. Thus (33) and (45) are consistent. ($\Delta = 1$ is disallowed by (33) in any case.)
Next we compare the coefficients of the power of temperature in (44). There are two ways to do this depending on the point of view taken on holography: (i) the holographic degrees of freedom associated with a black hole reside on a surface \( r = r_0 \) in the black hole spacetime, or (ii) the holographic degrees of freedom do not reside on any bounding surface in the black hole background, but rather are defined on their own flat background spacetime.

According to (i), the temperature \( T \) of the gas must be taken as the red-shifted black hole temperature

\[
T = \frac{T_H}{\sqrt{-g_{00}}} = \frac{\ell T_H}{r_0}
\]

(47)

The black hole entropy in terms of \( T \) is

\[
S_{BH} = c_1(d - 1) \left( \frac{r_0}{T} \right)^{d-2} T^{d-2}.
\]

(48)

Matching coefficients of powers of \( T \) in (44) now gives

\[
c_1 = c_2 \Omega_{\Delta - 1} \ell^{d - 2} r_0^{\alpha(d - 2)},
\]

(49)

where we have used the fact that the gas lives in a subspace of the surface \( r = r_0 \) so that its volume is \( V_{\Delta - 1} = \Omega_{\Delta - 1} r_0^{\Delta - 1} \).

A special case of the relation (49) arises in the context of the AdS/CFT conjecture. For \( d = 5 \), \( \alpha = 1 \), \( \Delta = 4 \) and \( n_B = n_F = 8(N^2 - 1) \), it was shown \([35,36]\), with the additional AdS/CFT relation

\[
\pi \ell^3 / G_5 = 2N^2,
\]

that Eq.(49) is satisfied up to a factor of 4/3.

According to the point of view (ii), the temperature of the gas is equated to the black hole temperature without any red-shift factors. This leads to the following relation between coefficients:

\[
c_1 = c_2 V_{\Delta - 1} \ell^{d - 2}
\]

(50)

Note that in either case (i) or (ii), the holographic dimension \( \Delta \) is given by Eq.(45).

From (45), it follows that for the special case \( \alpha = 1 \),

\[
\Delta = d - 1,
\]

(51)

which is normally assumed in the context of holography \([5,6]\). Another indication that \( \alpha = 1 \) may be ‘preferred’ lies in the fact that the \( r_0 \)-dependence drops out of the relation (49) only for this value of \( \alpha \).

For near-extremal stringy black holes, the near horizon geometry is BTZ \([37]\). Thus, to describe the thermodynamics of these black holes, one substitutes \( d = 3 \) in (51). This gives a \((1 + 1)\)-dimensional gas, which is known to reproduce the entropy and Hawking radiation rates of near extremal black holes in string theory \([38-40]\). Some other thermodynamic properties of BTZ black holes have also been shown to follow form an effective one dimensional gas \([41]\).

We now compare the leading order entropy corrections of the black hole and gas. Since the correction term is proportional to \( \ln C_{T^2} \) by Eqn. (17), this comparison is trivial if both entropy and temperature are matched as for case (ii) discussed above. However for case (i), the temperatures are not equated exactly due to the red-shift factor associated with the radial location of the gas. Therefore there are additional subleading entropy corrections for this case.

Using \( C_{gas} = d(F_{gas} + T S_{gas})/dT \), the dimension matching formula Eq.(17), the corrected entropy of the gas is

\[
S_{gas}^{corr} = S_{gas} - \frac{d}{2(d-2)} \ln S_{gas} - \frac{1}{d-2} \ln (c_2 V_{\Delta - 1}).
\]

(52)

Since \( S_{BH} = S_{gas} \), we see that the leading logarithmic term agrees precisely with that on the black hole side, Eq.(27), for all spacetime dimensions, and for any value of \( \alpha \) in the dispersion relation (31). The last term in (52) depends on the volume of the gas, as well the number of species, and can be interpreted as a finite size effect. These are considered sub-dominant so long as the entropy \( S_{BH} \) remains large. Thus, the exact details of the boundary theory are seen to be irrelevant for all the leading order corrections to match. (Finite size effects were also considered in \([42]\).)

The issue of log corrections for \( d = 5 \) and \( \alpha = 1 \) were first analysed in \([43]\). However, there the authors had ignored the factor of \( T_H^2 \) in (17), by setting the ‘scale’ of the logarithm to \( T_H \) itself. This scale is actually the Boltzmann constant \( k_B \), which has been set to unity here. Consequently, the coefficient in front of the log term was incorrect there, both for the black hole and the gas. In addition, the finite-size effect terms were missed.

For \( AdS_5/CFT_4 \) correspondence, when \( d = 5 \), \( \alpha = 1 \), \( \Delta = 4 \), \( n_B = n_F = 8(N^2 - 1) \), the coefficient of the log corrections is 5/6, agreeing perfectly with its black hole counterpart, Eq.(27).

VI. ENTROPY BOUNDS WITH ADS BLACK HOLES

In this section we consider the Bekenstein and spherical bounds for asymptotically anti-de Sitter black holes, and examine the effects of entropy corrections on these bounds.

The Bekenstein bound states that the entropy of matter in a closed region of linear dimension \( R \) and energy \( E \) is bounded above by the inequality \([7,8]\)

\[
S_{matter} \leq \frac{2\pi E R}{\hbar c}.
\]

(53)
One way to arrive at this result is to consider a ‘Ge-roch process,’ in which a matter system in a box is lowered slowly from an asymptotically flat region, and then dropped into the black hole where the box just touches the horizon. The argument assumes that the energy of a floating box near the horizon is added to the black hole, which increases its entropy. The generalised second law $\Delta S_{total} = S_{BH}^{final} - (S_{BH}^{initial} + S_{matter}) \geq 0$ then leads to the desired inequality. The drop off point occurs where the center of mass of the matter system is a distance $R$ from the horizon, where the energy of the matter acquires a factor $\sqrt{-g_{00}}$ relative to its energy at infinity.

This argument may be applied to an AdS-Schwarzschild black hole background, where the energy of the system is assumed to be $E$ at a large radius $r \gg r_+$. The energy of the matter a proper distance $R$ from the horizon is

$$E = \sqrt{-g_{00}} R \ E = \left[ \frac{(d-1)r_+}{2l^2} \right] \ ER.$$  \hfill (54)

The energy gain of the black hole is

$$\delta E \leq \left[ \frac{(d-1)r_+}{2l^2} \right] \ ER,$$  \hfill (55)

and the corresponding entropy gain is

$$\delta S_{BH} = \frac{\partial S_{BH}}{\partial E} \delta E.$$  \hfill (56)

Now, in order to find the corrected Bekenstein bound, we use (27) to compute the entropy derivative. This gives a modified Hawking temperature $T'_H = \partial S/\partial M_{BH}$ given by

$$T'_H = T_H + \frac{d(d-1)\hbar c e^{d-2}}{\pi (d-2)\Omega_{d-2} l^2 r_+^{d-3}}.$$  \hfill (57)

Substituting this into (56) gives

$$\delta S_{BH} = \left[ 1 - \frac{2\ell^{d-2}}{\Omega_{d-2}(d-2)c^{d-2}} \right] \frac{2\pi E R}{\hbar c}.$$  \hfill (58)

Imposing the generalised 2nd. law gives

$$S_{matter} \leq \left[ 1 - \frac{2\ell^{d-2}}{\Omega_{d-2}(d-2)c^{d-2}} \right] \frac{2\pi E R}{\hbar c}.$$  \hfill (59)

Notice that the leading term is identical to the Schwarzschild case [8]. The correction, which depends on the horizon radius, reduces the bound. The correction term attains its maximum when the radius is of the order of Planck length, which gives

$$S_{matter} \leq \left[ 1 - \frac{2}{\Omega_{d-2}(d-2)} \right] \frac{2\pi E R}{\hbar c}.$$  \hfill (60)

This equation may be regarded as the modified Bekenstein bound. (Corrections to the Bekenstein bound from finite volume corrections were analysed in [44].)

It is interesting to see what happens to this bound at the expected saturation point where the system $(E, R)$ is an AdS-Schwarzschild black hole. The entropy bound becomes

$$S_{matter} \leq \left( \frac{d-2}{2} \right) \left[ 1 - \frac{2}{(d-2)\Omega_{d-2}} \right] \left( \frac{r_+^2}{\ell^2} \right) S_{BH}.$$  \hfill (61)

Since $r_+ / \ell \gg 1$, the right side is much larger than the black hole entropy. That is, the black hole entropy is far from saturating the Bekenstein bound, implying that the latter is rather weak in this case. This is a significant departure from the asymptotically flat case where the upper limit is provided by the black hole entropy.

The spherical entropy bound similarly undergoes the modification

$$S_{matter} \leq \frac{A_H}{4l^{d-2}} - \frac{d}{2(d-2)} \ln \frac{A_H}{4l^{d-2}}.$$  \hfill (62)

Of course in this case, the black hole saturates this bound, by definition. From (61) and (62) it follows that the two bounds are not of the same order for asymptotically anti-de Sitter black holes, and that the Bekenstein bound is much weaker than the spherical bound. (After this work was done, we became aware of a recent paper where corrections to Bekenstein and spherical entropy bounds have been proposed [45]. The correction to the spherical bound proposed in that paper agrees with our's, while that to the Bekenstein bound does not.)

Finally we point out the AdS black hole analogues of two related entropy bounds previously discussed for Schwarzschild black holes [5]. These are bounds on the entropy of a collection of black holes, and on the entropy of matter contained in a closed spatial region.

We ask if the spherical entropy bound is satisfied for a collection of AdS-Schwarzschild black holes with masses $M_i$ in a bounded region of area $A$. To do so let us first compare the sum of the entropies $S_{total} = \Sigma S_i$ of a collection of black holes with the entropy $S_M$ of a black hole of mass $M = \Sigma M_i$:

$$S_{total} = \frac{\Omega_{d-2}}{4\ell^{d-2}} \left[ \frac{16\pi G_d \ell^2}{(d-2)\Omega_{d-2} c^2} \right]^{(d-2)/(d-1)} \times \sum_i M_i^{(d-2)/(d-1)},$$  \hfill (63)

and

$$S_M = \frac{\Omega_{d-2}}{4\ell^{d-2}} \left[ \frac{16\pi G_d \ell^2}{(d-2)\Omega_{d-2} c^2} \right]^{(d-2)/(d-1)} \times \left( \sum_i M_i \right)^{(d-2)/(d-1)},$$  \hfill (64)
The relation
\[ \sum_i M_i^{(d-2)/(d-1)} > \left( \sum_i M_i \right)^{(d-2)/(d-1)} \]  
(65)
for powers \((d-2)/(d-1) < 1\) gives
\[ S_{\text{total}} > S_M \]
(66)
This is the opposite of the Schwarzschild black hole result, and indicates that if \(A\) is taken to be the horizon area of the composite black hole, then \(S_{\text{total}} > A/4\ell_{\text{Pl}}^2\).

Thus an upper bound on the entropy of a collection of AdS black holes cannot be derived form this argument.

To obtain a specific example of an entropy bound on a matter system derived using AdS black holes, we use the gas discussed in the last section. The energy and entropy of the gas, consisting of \(Z\) species, and with the dispersion relation \((31)\), follows from Eq.\((41)\), with the replacement \(\Delta \rightarrow d\) (also see \([5,8,46]\)):
\[
E = Z c_2 \frac{d-1}{\alpha} V_{d-1} T^{(d-1)/\alpha+1},
\]
\[
S = Z c_2 \frac{d-1}{\alpha} + 1 V_{d-1} T^{(d-1)/\alpha}.
\]
(67)
(68)
With \(V_{d-1} = \Omega_{d-1} R^{d-1}\) we can write the entropy \(S(Z,E,R)\) as
\[ S = c_3 Z^{\alpha/(d-1+\alpha)} R^{\alpha/(d-1+\alpha)} E^{(d-1)/(d-1+\alpha)}, \]
(69)
where
\[ c_3 = c_2^{\alpha/(d-1+\alpha)} \Omega_{d-1}^{\alpha/(d-1+\alpha)} \left( \frac{\alpha}{d-1} \right)^{(d-1)/(d-1+\alpha)}. \]
(70)
An entropy bound arises by requiring that the gas is outside its AdS-Schwarzschild radius. This is obtained from the expression for the mass of the Schwarzschild-AdS black hole in the \(r \gg \ell\) limit:
\[ M = \left[ \frac{(d-2)\Omega_d-2c^2}{16\pi G_d d^2} \right] r_+^{d-1}. \]
(71)
As a result the entropy \((69)\) satisfies the inequality:
\[ S < c_4 Z^{\alpha/(d-1+\alpha)} A^{(d-1)/(d-2)} \ell^{2(d-1)/(d-1+\alpha)}, \]
(72)
where
\[ c_4 = c_3 \left[ \frac{(d-2)\Omega_d-2c^2}{16\pi G_d} \right]^{(d-1)/(d-1+\alpha)}. \]
(73)
Note that the exponent of \(A\) in \((72)\) is greater than unity, unlike the asymptotically flat case, again resulting in a weaker bound. Eq.\((72)\) can be written as:
\[ S < c_5 \left( \frac{\ell_{\text{Pl}}}{\lambda} \right)^{(4-\Delta)/(d-1+\alpha)} \left( \frac{\ell_{\text{Pl}}}{\ell} \right)^{(d-1)/(d-1+\alpha)} r_+ S_{\text{BH}}, \]
(74)
where \(S_{\text{BH}}\) is the black hole entropy for the same horizon area, and
\[ \lambda^{\alpha-1} = \frac{\ell h^{\alpha-1}}{c^\alpha}. \]
(75)
\[ c_5 = 4\alpha^{d-1} \Omega_d \pi^{d-1} \frac{d-2}{16\pi} \frac{d-1}{d-1+\alpha} \left[ \zeta \left( \frac{d-1}{\alpha} + 1 \right) \Gamma \left( \frac{d-1}{\alpha} + 1 \right) \right] \times \left[ (n_B + n_F - \frac{n_F}{2(d-1)/\alpha}) \right]^{d-1+\alpha}. \]
(76)

VII. DISCUSSIONS

We have shown that the thermodynamics of black holes can be reproduced by a dual (or holographic) gas with a generalised dispersion relation. Specifically we have the general result that for AdS-Schwarzsschild black holes in \(d\)-spacetime dimensions, the thermodynamics can be encoded in a gas of free bosons and fermions in \(\Delta = \alpha(d-2) + 1\) spacetime dimensions. Thus for a given \(d\), a variety of \(\Delta\) can serve our purpose, depending on \(\alpha\). We have also seen that some results in the AdS/CFT context arise as special cases of the thermodynamic matching we have used.

We have also derived corrections to entropy bounds, and discussed specific examples of the bounds, using Schwarzschild-AdS black holes. An interesting result here is the lack of a spherical bound for a collection of these black holes.

The dominant energy condition \((33)\) and the matching condition \((45)\) do not in themselves imply that \(\Delta < d\), which is an intuitive expectation from holography. For example, these equations permit \(d = 5\) and \(\Delta = 7\) with \(\alpha = 2\). Thus, a condition over and above entropy matching is required for this. If \(\Delta < d\) is imposed by hand, it gives the stronger constraint
\[ \alpha < \frac{d-1}{d-2} \]
(77)
on the coefficient \(\alpha\) in the dispersion relation. This is consistent with both \((33)\) and \((45)\). Thus it appears that entropy matching alone does not necessarily imply dimensional reduction.

Particles with \(\alpha = 1\) do not seem to be necessary in the holographic mapping, although they are sufficient. It is
interesting to note however, that for $\alpha = 1$, the relations that map the entropies of the black hole and gas become independent of $r_0$, the ‘location’ of the dual gas in the black hole spacetime. Thus, this may be an additional reason to attach a preferred status to particles satisfying the relativistic dispersion relation.

There are several open questions which would be interesting to pursue. From the integral in (9), it is clear that the next-to-leading-order corrections are difficult, if not impossible to compute analytically. It may be possible to do this numerically to find the dependence of the corrections on horizon area. As noted earlier, the corrected entropy is always less than the uncorrected one, signifying a reduction in the number of accessible states, when entropy is always less than the uncorrected one, signifies a positive specific heat [49]. It would also be interesting to pursue. From the integral in (9), it is clear that the relativistic dispersion relation.

An important generalisation of the current formalism would be to calculate entropy corrections for asymptotically flat black holes. One way would be to enclose them in a finite box, such that there is one black hole solution with positive specific heat [49]. It would also be interesting to compare these corrections with those coming from other sources, such as quantum spacetime fluctuations [50].

A related problem concerns de-Sitter black holes. Here the temperature and specific heat can be obtained from the corresponding expressions for asymptotically anti-de Sitter black holes [Eqs.(25-26)], by substituting $\ell^2 \to -\ell^2$.

$$T_H = \frac{hc}{4\pi\ell^2 r_+} \frac{-(d-1)r^2_+ + (d-3)\ell^2}{(d-2)} = \frac{4\pi\ell^2 r_+}{(d-1)r^2_+ / \ell^2 - (d-3)} S_{BH}.$$  

Here in the regime in which Hawking temperature is positive, the specific heat is negative, again signaling an apparent breakdown of the approach. However, since it has been claimed that the energy of these black holes is negative [51–53], one may use a modified definition of specific heat, namely $C = d(-E)/dT$. This gives a positive $C$. Corrections to entropy would then be identical to that for the AdS case, Eq.(27), and it appears that the mapping of thermodynamics can also be done.

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