A REMARK ON THE TOPOLOGICAL STABILITY OF SYMPLECTOMORPHISMS

MÁRIO BESSA AND JORGE ROCHA

Abstract. We prove that the $C^1$ interior of the set of all topologically stable $C^1$ symplectomorphisms is contained in the set of Anosov symplectomorphisms.

1. Introduction: Basic Definitions and Statement of the Results

Let $M$ be a $2d$-dimensional smooth manifold endowed with a symplectic form, say a closed and nondegenerate 2-form $\omega$. The pair $(M, \omega)$ is called a symplectic manifold which is also a volume manifold by Liouville’s theorem.

Let $\mu$ be the so-called Lebesgue measure associated to the volume form wedging $\omega$ $d$-times, i.e., $\omega^d = \omega \wedge \cdots \wedge \omega$. On $M$ we also fix a Riemannian structure which induces a norm $\| \cdot \|$ on the tangent bundle $T_x M$. We will use the canonical norm of a bounded linear map $A$ given by $\|A\| = \sup_{\|v\| = 1} \|Av\|$. By Darboux theorem (see e.g. [9, Theorem 1.18]) there exists an atlas $\{h_j : U_j \to \mathbb{R}^{2d}\}$ satisfying $h_j^* \omega_0 = \omega$ with

$$\omega_0 = \sum_{i=1}^{d} dy_i \wedge dy_{d+i}.$$  

A diffeomorphism $f : (M, \omega) \to (M, \omega)$ is called a symplectomorphism if $f^* \omega = \omega$. Observe that, since $f^* \omega^d = \omega^d$, a symplectomorphism $f : M \to M$ preserves the Lebesgue measure $\mu$.

Symplectomorphisms arise naturally in the classical and analytical mechanics formalism as first return maps of hamiltonian vector fields. Thus, it has long been one of the most interesting research fields in mathematical physics. We suggest the reference [9] for more details on general hamiltonian and symplectic theories.

Let $(\text{Symp}_1^1(M), C^1)$ denote the set of all symplectomorphisms of class $C^1$ defined on $M$, topologized with the usual $C^1$ Whitney topology.

We say that $f \in \text{Symp}_1^1(M)$ is Anosov if, there exist $\lambda \in (0,1)$ and $C > 0$ such that the tangent vector bundle over $M$ splits into two $Df$-invariant subbundles $TM = E^u \oplus E^s$, with $\|Df^n|_{E^s}\| \leq C\lambda^n$ and $\|Df^{-n}|_{E^u}\| \leq C\lambda^n$. We observe that there are plenty Anosov diffeomorphisms which are not sympletic.
We say that \( f \in \mathcal{F}_1^\omega(M) \) if there exists a neighborhood \( V \) of \( f \) in \( \text{Symp}_1^\omega(M) \) such that any \( g \in V \) has all the periodic orbits hyperbolic. We will use the following weighty result, which is a direct consequence of a theorem of Newhouse (see [10, Theorem 1.1]).

**Theorem 1.1.** If \( f \in \mathcal{F}_1^\omega(M) \) then \( f \) is Anosov.

Let us recall that a periodic point \( p \) of period \( \pi \) is said to be 1-elliptic if the tangent map \( Df^\pi(p) \) has two (non-real) norm one eigenvalues and the other eigenvalues have norm different from one. Actually, unfolding homoclinic tangencies of symplectomorphisms \( C^1 \)-far from the Anosov ones, Newhouse was able to prove that, \( C^1 \)-generically, symplectomorphisms are either Anosov or else the 1-elliptic periodic points of \( f \) are dense in the whole manifold. We observe that the existence of 1-elliptic periodic points is a sufficient condition to guarantee that the system is not structurally stable.

Given \( f, g \in \text{Symp}_1^\omega(M) \) we say that \( g \) is semiconjugated to \( f \) if there exists a continuous and onto map \( h : M \to M \) such that, for all \( x \in M \), we have the following conjugacy relation \( h(g(x)) = f(h(x)) \).

We say that \( f \) is topologically stable in \( \text{Symp}_1^\omega(M) \) if, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( g \in \text{Symp}_1^\omega(M) \) \( \delta \)-\( C^0 \)-close to \( f \), there exists a semiconjugacy from \( g \) to \( f \), i.e., there exists \( h : M \to M \) satisfying \( h(g(x)) = f(h(x)) \) and \( d(h(x), x) < \epsilon \), for all \( x \in M \). Once again we emphasize that our definition of topological stability is restricted to the symplectomorphism setting and not to the broader space of volume-preserving (or even dissipative) diffeomorphisms. Let us denote by \( \tau_{S_\omega}(M) \) the subset of \( \text{Symp}_1^\omega(M) \) formed by the topologically stable symplectomorphisms.

The notion of topological stability was first introduced by Walters in ([13]) proving that Anosov diffeomorphisms are topologically stable. Afterwards, in ([11]), Nitecki proved that topological stability was a necessary condition to get Axiom A plus strong transversality. Later, in ([14]), Robinson proved that Morse-Smale flows are topologically stable. We point out that Hurley obtained necessary conditions for topological stability (see [4, 5, 6]). In the early nineties Moriyasu ([7]) proved that the \( C^1 \)-interior of the set of all topologically stable diffeomorphisms is characterized as the set of all \( C^1 \)-structurally stable diffeomorphisms. A few years ago Moriyasu, Sakai and Sumi (see [8]) proved that, if \( X^t \) is a flow in the \( C^1 \) interior of the set of topologically stable flows, then \( X^t \) satisfies the Axiom A and the strong transversality condition. Recently, in [2], the authors proved a version of [8] for the class of incompressible flows and also for volume-preserving diffeomorphisms.

The result in this note is a generalization of the theorems in [8, 2] for symplectomorphisms.

Given a set \( A \subset \text{Symp}_1^\omega(M) \) let \( \text{int}_{C^1}(A) \) denote the interior of \( A \) in \( \text{Symp}_1^\omega(M) \) with respect to the \( C^1 \)-topology.

**Theorem 1.** If \( f \in \text{int}_{C^1}(\tau_{S_\omega}(M)) \) then \( f \) is Anosov.
It is well known that Anosov diffeomorphisms impose severe topological restrictions to the manifold where they are supported. In fact the known examples of Anosov diffeomorphisms are supported in infranilmanifolds. An old conjecture of Smale states that any Anosov diffeomorphism is conjugated to an Anosov automorphism defined on an infranilmanifold. We end the introduction with this simple consequence of Theorem 1.

Corollary 1.2. If $M$ does not support an Anosov diffeomorphism, then

$$\text{int}_{C^1}(\tau_{S^\omega}(M)) = \emptyset.$$
small neighborhood of the orbit of $q$ and the same cannot occur for $f_2$ because $q$ is a hyperbolic periodic orbit for $f_2$.

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