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of Discrete Groups

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EXACTNESS AND UNIFORM EMBEDDABILITY OF DISCRETE GROUPS

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ABSTRACT. We will describe a numerical invariant of discrete groups whose values interpolate between uniform embeddability in a Hilbert space and exactness.

1. INTRODUCTION

Gromov introduced the notion of uniform embeddability of metric spaces [4]. Recall that a uniform embedding of one metric space \((X, d_X)\) into another \((Y, d_Y)\) is a function \(f : X \to Y\) for which there exist non-decreasing functions \(\rho_\pm : [0, \infty) \to \mathbb{R}\) such that \(\lim_{r \to \infty} \rho_\pm(r) = \infty\) (in other words, \(\rho_\pm\) are proper) and such that for all \(x, y \in X\)
\[
\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)).
\]

In an appendix we collect the known facts about the relation between uniform embeddings and coarse geometry notions along with their elementary proofs.

Gromov raised the question of whether a finitely presented group that is uniformly embeddable in a Hilbert space satisfies the Novikov Conjecture [3]. This was answered affirmatively by Yu:

**Theorem** ([16, 15]). Let \(\Gamma\) be a finitely presented group which is uniformly embeddable in Hilbert space. Then \(\Gamma\) satisfies both the Novikov Conjecture and the Coarse Baum-Connes Conjecture.

Gromov has constructed a countable discrete group \(\Gamma\) which is not uniformly embeddable embeddable in a Hilbert space, [5]. However, Higson has observed that the groups constructed by Gromov satisfy the Novikov Conjecture, although whether they satisfy the Coarse Baum-Connes Conjecture is not known, [9].

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Kirchberg and Wassermann extensively studied the notion of exactness of a countable discrete group $\Gamma$ [10, 11]. Recall that a countable discrete group is \emph{exact} if $C^*_r(\Gamma)$ is an exact $C^*$-algebra, i.e. if taking minimal tensor product with $C^*_r(\Gamma)$ on each of the terms in a short exact sequence of $C^*$-algebras preserves the exactness of the sequence.

Uniform embeddability is a geometric property of a group and exactness is more closely related to the harmonic analysis of $\Gamma$. The connection between these types of properties is related to the Baum-Connes Conjecture. Recently, it was shown that exactness of a countable discrete group $\Gamma$ implies its uniform embeddability in a Hilbert space.

\textbf{Theorem} ([8, 12]). \textit{If $C^*_r(\Gamma)$ is an exact $C^*$-algebra, then $\Gamma$ is uniformly embeddable in a Hilbert space.}

One may ask to what extent the converse of this result holds. The question of whether a uniformly embeddable group is exact has been studied from various perspectives, (c.f. [6, 7]). In the present paper we introduce a numerical invariant $R(\Gamma)$ of a finitely generated discrete group $\Gamma$ which can be viewed as parametrizing the difference between the group being exact and being uniformly embeddable in a Hilbert space.

\section{The definition of $R(\Gamma)$}

In this section we will give the definition of $R(\Gamma)$ and verify some of its basic properties. We start by introducing large scale Lipschitz maps, [4].

\textit{Definition.} A function $f : X \to Y$ is \textit{large scale Lipschitz} if there exists $C > 0$ and $D \geq 0$ such that
\[ d_Y(f(x), f(y)) < C d_X(x, y) + D \]

This is the same as the function $\rho_+$ in the definition of uniform embeddability being linear. In general, this is properly weaker than being Lipschitz as the following example shows.

Let $X = \{ (n, 1/n), (n, 0) \} \subset \mathbb{R}^2$ with the induced metric; $X$ is a quasi-geodesic metric space. Define $f : X \to \mathbb{R}^2$ by $f(n, 1/n) = (n, 1)$, $f(n, 0) = (n, 0)$. Then $f$ is a both a uniform embedding and a large-scale Lipschitz map; it is, however, not a Lipschitz map.

In the case of countable groups with a word length metric one has that a uniform embedding is in fact Lipschitz. This is easy to verify, but we present a generalization of this fact which will be of use later.

\textit{Definition.} A discrete metric space $X$ is a \textit{quasi-geodesic space} if there exist $\delta > 0$ and $\lambda \geq 1$ such that for all $x$ and $y \in X$ there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ of elements of
such that
\[ d_X(x, y) \leq \lambda \sum_{i=1}^{n} d_X(x_{i-1}, x_i) \]
\[ d_X(x_{i-1}, x_i) \leq \delta, \quad \text{for all } 1 \leq i \leq n. \]

Note that a discrete group with a word length metric is quasi-geodesic. We have the following result.

**Proposition 2.1.** Let \( X \) and \( Y \) be discrete metric spaces, and assume that \( X \) is a quasi-geodesic metric space. Let \( f : X \to Y \) be a function for which there exists a \( \rho_+ \) so that
\[ d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)). \]
Then \( f \) is a large-scale Lipschitz map.

**Proof.** Let \( \lambda \geq 1 \) and \( \delta > 0 \) be the constants supplied by the fact that \( X \) is a quasi-geodesic space. We must show that there exist constants \( C > 0, D \geq 0 \) such that
\[ d_Y(f(x), f(y)) \leq Cd_X(x, y) + D, \quad \text{for all } x, y \in X. \]

Let \( x, y \in X \) and let \( x_0, \ldots, x_n \) be a sequence of elements of \( X \) guaranteed by the fact that \( X \) is a quasi-geodesic space. Extract a subsequence \( x_{i_1}, \ldots, x_{i_m} \) as follows: \( i_0 = 0 \) and, assuming \( i_0, \ldots, i_j \) are already defined,
\[ i_{j+1} = \begin{cases} \text{the smallest integer } k \text{ such that } d_X(x_{i_j}, x_k) \geq \delta/2, \\ \text{if such exists; if no such } k \text{ exists put } m = j. \end{cases} \]

The subsequence has the following properties:

(i) \( x_{i_0} = x, \)
(ii) \( d_X(x_{i_j}, x_{i_{j+1}}) \leq 3\delta/2, \)
(iii) \( d_X(x_{i_j}, x_{i_{j+1}}) \geq \delta/2, \) and
(iv) \( d_X(x_{i_m}, y) \leq \delta/2. \)

We have the following simple estimates:
\[ d_Y(f(x), f(y)) \leq \sum_{j=1}^{m} d_Y(f(x_{i_j-1}), f(x_{i_j})) + d_Y(f(x_{i_m}, y) \leq m\rho_+(3\delta/2) + \rho_+(\delta/2), \]
\[ m(\delta/2) \leq \sum_{j=1}^{m} d_X(x_{i_j-1}, x_{i_j}) \leq \sum_{i=1}^{n} d_X(x_{i-1}, x_i) \leq \lambda d_X(x, y). \]

From the second we conclude that \( m \leq 2\delta^{-1}\lambda d_X(x, y) \) which, combined with the first yields
\[ d_Y(f(x), f(y)) \leq 2\delta^{-1}\lambda \rho_+(3\delta/2) d_X(x, y) + \rho_+(\delta/2). \]
According to the proposition, when studying uniform embeddings of quasi-geodesic metric spaces there is no loss of generality in assuming all maps are large-scale Lipschitz. Moreover, to simplify notation in the rest of the paper, and since we will mainly be considering only maps, \( f : \Gamma \to \mathcal{H} \), from a discrete group to a Hilbert space, we will denote the metric in the domain by \( d(x, y) \). Following Gromov [4], we define the \textit{compression} \( \rho_f \) of a large scale Lipschitz map \( f : \Gamma \to \mathcal{H} \) to be

\[
\rho_f(r) = \inf_{d(x, y) \geq r} ||f(x) - f(y)||.
\]  

The compression function \( \rho_f \) is a non-decreasing, non-negative real-valued function with the property that if \( \rho_- \) is a function satisfying the inequality in (1) then \( \rho_- \leq \rho_f \). Consequently, \( f \) is a uniform embedding if and only if \( \rho_f(r) \) is proper.

Let \( \text{Lip}^b(X, \mathcal{H}) \) denote the set of large-scale Lipschitz maps from \( X \) into \( Y \).

We define a real valued invariant for a quasi-geodesic metric space \( X \) as follows.

\textbf{Definition.} The \textit{asymptotic compression} \( R_f \) of a large scale Lipschitz embedding \( f \in \text{Lip}^b(X, \mathcal{H}) \) is

\[
R_f = \liminf_{n \to \infty} \frac{\log \rho_f(n)}{\log n}.
\]  

The \textit{Hilbert space distortion} of \( X \) is

\[
R(X) = \sup \{ R_f : f \in \text{Lip}^b(X, \mathcal{H}) \}.
\]  

Our primary interest is the case when \( X \) is a finitely generated discrete group \( \Gamma \) equipped with the left invariant metric associated to a proper length function. The word length function associated to a symmetric generating set is one such choice, although it will turn out that the value \( R(\Gamma) \) is independent of the choice of proper length function.

\textbf{Proposition 2.2.} The Hilbert space distortion, \( R(\Gamma) \), of a finitely generated discrete group \( \Gamma \) is at most 1. That is, \( R(\Gamma) \leq 1 \).

\textit{Proof.} This follows from the observation that the asymptotic compression of a large-scale Lipschitz embedding is at most 1. Indeed, let \( C > 0 \) and \( D \geq 0 \) be the constants provided by definition of large-scale Lipschitz. Then we have \( \rho_f(n) = \inf_{d(x, y) \geq n} ||f(x) - f(y)|| \leq Cn + D \) and, thus,

\[
\liminf_{n \to \infty} \frac{\log \rho_f(n)}{\log n} \leq \liminf_{n \to \infty} \frac{\log(Cn + D)}{\log(n)} = 1.
\]

\( \square \)
Example 2.3. If a finitely generated discrete group $\Gamma$ admits an isometric embedding into a Hilbert space then its Hilbert space distortion is 1. In particular, $R(\mathbb{Z}^n) = 1$.

This follows since the asymptotic distortion of an isometric embedding into Hilbert space is 1. Thus, if $\Gamma$ admits an isometric embedding into a Hilbert space $R(\Gamma) \geq 1$. But, by the previous example we know that $R(\Gamma) = 1$. In fact, we shall see later that the same conclusion follows from the existence of a quasi-isometric embedding of $\Gamma$ into a Hilbert space, (see Thm. 2 below).

Example 2.4. The Hilbert space distortion of any of the groups constructed by Gromov which are not uniformly embeddable in a Hilbert space is zero.

This will follow from results in later sections.

Thus, the extreme values of 0 and 1 of the invariant $R(\Gamma)$ can be realized.

We next establish the quasi-isometry invariance of $R(\Gamma)$. Recall that a map $\varphi: X \to Y$ is a quasi-isometry if there exists $C > 0$ and $D \geq 0$ such that

$$C^{-1}d(x, y) - D \leq d(\varphi(x), \varphi(y)) \leq C d(x, y) + D, \quad \text{for all } x, y \in X.$$ 

If, further, the range of $\varphi$ is $K$-dense, for some $K$, in the sense that

$$\forall x' \in Y \exists x \in X \text{ such that } d(\varphi(x), x') \leq K.$$

then we say that $X$ and $Y$ are quasi-isometric. Note that a quasi-isometry $\varphi$ establishes $X$ is quasi-isometric to $Y$ if and only if there exists a quasi-isometry $\psi: Y \to X$ with the property that

$$d(\varphi \psi(x), x) \leq K, \quad \text{for all } x \in X$$

$$d(\psi \varphi(y), y) \leq K, \quad \text{for all } y \in Y.$$

Theorem. Let $X$ and $Y$ be quasi-geodesic metric spaces. If there exists a quasi-isometry $\varphi: X \to Y$, then $R(X) \geq R(Y)$.

Proof. Let $\varphi: X \to Y$ be a quasi-isometry. In particular, $\varphi$ is large-scale Lipschitz and hence composition with $\varphi$ provides a map $\text{Lip}^\text{ls}(Y, \mathcal{H}) \to \text{Lip}^\text{ls}(X, \mathcal{H})$:

$$f \mapsto f \circ \varphi: \text{Lip}^\text{ls}(Y, \mathcal{H}) \to \text{Lip}^\text{ls}(X, \mathcal{H}).$$

It suffices to show that

$$R_f \leq R_{f \circ \varphi}, \quad \text{for all } f \in \text{Lip}^\text{ls}(Y, \mathcal{H}).$$
This inequality will follow from the fact that the compression function of a quasi-isometry is linear.

Let $C$ and $D$ be constants given by the fact that $\varphi$ is a quasi-isometry, so that,

$$d(\varphi(x), \varphi(y)) \geq C^{-1}d(x, y) - D, \quad \text{for all } x, y \in X.$$  

Let $f \in \text{Lip}^{ls}(Y, \mathcal{H})$ and note that

$$\rho_{f \circ \varphi}(n) = \inf \left\{ \|f \circ \varphi(x) - f \circ \varphi(y)\| : d(x, y) \geq n \right\}$$

$$\geq \inf \left\{ \|f(x') - f(y')\| : d(x', y') \geq nC^{-1} - D \right\}$$

$$= \rho_f(nC^{-1} - D).$$

Further,

$$R_{f \circ \varphi} = \liminf_{n \to \infty} \left\{ \frac{\log \rho_{f \circ \varphi}(n)}{\log n} \right\}$$

$$\geq \liminf_{n \to \infty} \left\{ \frac{\log \rho_f(nC^{-1} - D)}{\log(nC^{-1} - D)} \right\} \left\{ \frac{\log(nC^{-1} - D)}{\log n} \right\}$$

$$= \liminf_{n \to \infty} \left\{ \frac{\log \rho_f(nC^{-1} - D)}{\log(nC^{-1} - D)} \right\}$$

$$= R_f$$

The behavior of $R(X)$ under quasi-isometries is a special case of its behavior under large-scale Lipschitz embeddings. Indeed, let $\varphi : X \to Y$ be a large scale Lipschitz map and consider the asymptotic compression $R_\varphi$ of $\varphi$. As in the proposition, composition with $\varphi$ maps $\text{Lip}^{ls}(Y, \mathcal{H}) \to \text{Lip}^{ls}(X, \mathcal{H})$ and we have

$$\rho_{f \circ \varphi} \geq \rho_f \circ \rho_\varphi$$

$$R_{f \circ \varphi} \geq R_f \cdot R_\varphi, \quad \text{for all } f \in \text{Lip}^{ls}(Y).$$

Consequently, $R(X) \geq R(Y) \cdot R_\varphi$.

The behavior of $R(X)$ under quasi-isometries has a few immediate corollaries.

**Corollary 2.5.** If $X$ and $Y$ are quasi-isometric then $R(X) = R(Y)$.

**Corollary 2.6.** Let $\Gamma$ be a finitely generated discrete group. Then the invariant $R(\Gamma)$ is independent of the proper length function used to define the metric on $\Gamma$. If two finitely generated discrete groups, $\Gamma$ and $\Gamma'$, are quasi-isometric then $R(\Gamma) = R(\Gamma')$. 
Proof. If $d$ and $d'$ the left invariant metrics associated to two proper length functions on $\Gamma$ then it is straightforward to verify that the identity map is a quasi-isometry $(\Gamma, d) \rightarrow (\Gamma, d')$.

3. Uniform embeddings and exactness

In this section we will relate the Hilbert space compression of a metric space $X$ to uniform embeddability, and, in the case of a finitely generated discrete group, to exactness.

Proposition 3.1. Let $X$ be a metric space. If the Hilbert space compression of $X$ is nonzero then $X$ is uniformly embeddable in Hilbert space.

Proof. Let $X$ be given with $R(X) > 0$. Then, from the definition (4) of $R(\Gamma)$ we see that there exists $\varepsilon > 0$ and a large scale Lipschitz map $f \in \text{Lip}^s(X, \mathcal{H})$ with asymptotic compression greater than $\varepsilon$:

$$R_f = \liminf_{r \rightarrow \infty} \frac{\log \rho_f(r)}{\log r} > \varepsilon.$$ 

In particular, for all sufficiently large $r$, we have $\log \rho_f(r) \geq \frac{\varepsilon}{2} \log r$, hence $\rho_f(r) \geq r^{\varepsilon/2}$. Consequently $\rho_f$ is proper and $f$ is a uniform embedding.

Theorem 3.2. Let $\Gamma$ be a finitely generated discrete group. If the Hilbert space compression of $\Gamma$ is greater than $1/2$ then $\Gamma$ is exact.

We pause briefly to give an outline of the proof of the theorem, which rests on the following characterization of exactness inspired by [8]:

Proposition 3.3 ([12]). Let $\Gamma$ be a finitely generated discrete group, equipped with word length and metric associated to a finite, symmetric set of generators. Then $\Gamma$ is exact if and only if there exists a sequence $u_n$ of normalized positive definite functions $\Gamma \times \Gamma \rightarrow \mathbb{R}$ satisfying the convergence condition

$$\forall C > 0, \ u_n \rightarrow 1 \text{ uniformly on the strip } \{(s, t) : d(s, t) \leq C\}$$

and the support condition

$$\forall n, \exists R > 0 \text{ such that } u_n(s, t) = 0 \text{ if } d(s, t) \geq R.$$

A kernel $\Gamma \times \Gamma \rightarrow \mathbb{R}$ satisfying the support condition will be called of finite width.

Under the assumption on $R(\Gamma)$ we will construct a sequence of positive definite kernels on $\Gamma \times \Gamma$ satisfying the convergence and support conditions in the proposition. If $f : \Gamma \rightarrow \mathcal{H}$ is
a uniform embedding, then
\[ u_\kappa(s, t) = e^{-\kappa \|f(s) - f(t)\|^2}, \quad \kappa > 0, \ s, t \in \Gamma; \]
are normalized, (in the sense that \( u_\kappa(s, s) = 1 \)), positive definite kernels. It is easy to see that the \( u_\kappa \) satisfy the convergence condition, but they may not satisfy the support condition. However, if the uniform embedding \( f \) satisfies that its asymptotic compression is sufficiently large, then each \( u_\kappa \) may be approximated uniformly as a function on \( \Gamma \times \Gamma \), by a positive definite kernel \( \hat{u}_\kappa \) that is of finite width. This approximation is achieved in two steps: first, the condition that \( R_f > 1/2 \) implies that the kernels \( u_\kappa \) define operators in \( UC^*(\Gamma) \), the uniform Roe algebra of \( \Gamma \). Recall that the uniform Roe algebra is the \( C^* \)-subalgebra of bounded operators on \( l^2(\Gamma) \) generated by the operators \( \text{Op}(k) \), where \( k \) is a bounded finite width kernel. Second, these operators are then approximated by finite width positive operators which are represented by required kernels \( \hat{u}_\kappa \).

Before proving the theorem we require a few preliminaries. Given a complex-valued kernel \( k : \Gamma \times \Gamma \to \mathbb{C} \), define an operator \( \text{Op}(k) \) by the convolution formula:
\[ \text{Op}(k)\xi(x) = \sum_{y \in \Gamma} k(x, y)\xi(y), \quad \xi \in l^2(\Gamma). \] (5)

Despite the notation \( \text{Op}(k) \) does not always define an operator on the Hilbert space \( l^2(\Gamma) \). There are two criteria we will make use of in the proof.

The first is that if the kernel \( k \) is bounded and has finite width then \( \text{Op}(k) \) defines a bounded operator on \( l^2(\Gamma) \). The second is a special case of the Schur test, (c.f. [13]). Let \( k \) be a non-negative real-valued kernel with the following property: there exists \( C > 0 \) such that
\[ \sum_{s \in \Gamma} k(s, t) \leq C, \quad \text{for all } t \in \Gamma \]
\[ \sum_{t \in \Gamma} k(s, t) \leq C, \quad \text{for all } s \in \Gamma. \] (6)

Then (5) defines a bounded operator \( \text{Op}(k) \) on \( l^2(\Gamma) \) and \( \| \text{Op}(k) \| \leq C \).

**Proof of Thm. 3.2.** Let \( \Gamma \) be a finitely generated discrete group provided with word length metric associated to a finite symmetric generating set. Assuming that \( R(\Gamma) > 1/2 \) and arguing as in Proposition 3.1 one concludes that there exists a large scale Lipschitz map \( f \in \text{Lip}_b(\Gamma, \mathcal{H}) \), an \( \varepsilon > 0 \) and an \( r_0 > 0 \) such that
\[ \rho_f(r) \geq r^{(1+\varepsilon)/2}, \quad \text{for all } r \geq r_0. \] (7)
Define, for \( k \geq 1 \), a function \( u_k : \Gamma \times \Gamma \to \mathbb{R} \) by
\[
 u_k(s, t) = e^{-\|f(s) - f(t)\|^2 k^{-1}}, \quad \text{for all } s, t \in \Gamma.
\]
Since the function \( \|f(s) - f(t)\|^2 \) is of negative type [2], each \( u_k \) is positive definite by Schoenberg's theorem [1], and is also normalized. Further, since \( f \) is large scale Lipschitz, the sequence \( u_k \) satisfies the convergence condition. Instead of the support condition holding, they possess a weaker decay property. The remainder of the proof will be devoted to approximating the \( u_k \) uniformly by finite width positive definite kernels.

**Lemma 3.4.** The operators \( \text{Op}(u_k) \in UC^*(\Gamma) \), for all \( k \geq 1 \).

**Proof.** We show that for every \( \kappa > 0 \) the kernel \( u : \Gamma \times \Gamma \to \mathbb{C} \) defined by
\[
 u(s, t) = e^{-\|f(s) - f(t)\|^2 \kappa}, \quad s, t \in \Gamma
\]
defines an element \( \text{Op}(u) \in UC^*(\Gamma) \). To do so truncate \( u \) by defining for \( n \in \mathbb{N} \)
\[
k_n(s, t) = \begin{cases} u(s, t), & \text{if } d(s, t) > n \\ 0, & \text{otherwise.} \end{cases}
\]
Note that \( u - k_n \) is a bounded finite width kernel so that \( \text{Op}(u - k_n) \in UC^*(\Gamma) \). Since \( \text{Op}(u) = \text{Op}(u - k_n) + \text{Op}(k_n) \) as operators on compactly supported elements of \( l^2(\Gamma) \), it suffices to show that \( \|\text{Op}(k_n)\| \to 0 \) as \( n \to \infty \).

We proceed using the Schur test. Since the \( k_n \) are non-negative real-valued and symmetric, it is sufficient to check either one of the inequalities in (6). For this, we will show that there exists a sequence \( C_n \to 0 \) such that
\[
 \sum_{t \in \Gamma} k_n(s, t) = \sum_{m > n} \sum_{d(s, t) = m} u(s, t) \leq C_n, \quad \text{for all } s \in \Gamma.
\]
This, in turn, follows from the assertion that there exists \( C \) such that
\[
 \sum_{n \geq 0} \sum_{d(s, t) = n} u(s, t) \leq C, \quad \text{for all } s \in \Gamma.
\]
To obtain \( C \), let \( \sigma \) be the spherical growth function of \( \Gamma \) defined by \( \sigma(n) = \text{card}\{ t \in \Gamma : d(t, e) = n \} \). Observe that \( \sigma(n) \leq (\text{card } S)^n \). Combining (7) and (8) see that if \( d(s, t) = n \geq r_0 \) then \( n^{(1+\epsilon)/2} \leq \rho_f(n) \leq \|f(s) - f(t)\| \), and also \( u(s, t) \leq e^{-\kappa n^{1+\epsilon}} \). Let \( m \geq r_0 \) be
sufficiently large such that \( \text{card}(S) < e^{\kappa n^x} \). We calculate:

\[
\sum_{n \geq 0} \sum_{d(s,t)=n} u(s,t) = \sum_{n \leq m} \sum_{d(s,t)=n} u(s,t) + \sum_{n > m} \sum_{d(s,t)=n} u(s,t)
\]

\[
\leq \sum_{n \leq m} \sigma(n) + \sum_{n > m} \sum_{d(s,t)=n} e^{-\kappa n^{1+e}}
\]

\[
\leq \sum_{n \leq m} \sigma(n) + \sum_{n > m} \sigma(n) e^{-\kappa n^{1+e}}
\]

\[
\leq \sum_{n \leq m} \sigma(n) + \sum_{n > m} \left\{ \frac{\text{card}(S)}{e^{\kappa n^x}} \right\}
\]

\[
\leq \sum_{n \leq m} \sigma(n) + \sum_{n > m} \left\{ \frac{\text{card}(S)}{e^{\kappa n^x}} \right\},
\]

which is both finite and independent of \( s \in \Gamma \). We set \( C \) equal to the right hand side of the inequality. \( \square \)

We now complete the proof of the theorem. Since \( u_k \) is normalized we have \( \| \text{Op}(u_k) \| \geq 1 \).

A simple calculation shows that since the \( u_k \) are positive definite kernels the \( \text{Op}(u_k) \) are positive operators. Let \( V_k \in UC^*(\Gamma) \) be the positive square root of \( \text{Op}(u_k) \) and let \( W_k \in UC^*(\Gamma) \) be operators represented by finite width kernels and such that \( \| V_k - W_k \| \rightarrow 0 \).

Define kernels \( \hat{u}_k \) by

\[
\hat{u}_k(s,t) = \langle W_k \delta_t, W_k \delta_s \rangle, \quad s, t \in \Gamma.
\]

The \( \hat{u}_k \) are positive definite kernels and, since the \( W_k \) are represented by finite width kernels, the \( \hat{u}_k \) are themselves finite width kernels. Finally,

\[
|u_k(s,t) - \hat{u}_k(s,t)| = |\langle (\text{Op}(u_k) - W_k^* W_k) \delta_t, \delta_s \rangle|
\]

\[
\leq \| V_k^* V_k - W_k^* W_k \|
\]

\[
\leq \| V_k - W_k \| (\| V_k \| + \| W_k \|)
\]

\[
\leq \| V_k - W_k \| (2\| V_k \| + \| V_k - W_k \|),
\]

which tends to zero as \( k \rightarrow \infty \). Consequently \( u_k - \hat{u}_k \rightarrow 0 \) uniformly on \( \Gamma \times \Gamma \) and since the \( u_k \) satisfy the convergence condition so do the \( \hat{u}_k \). \( \square \)

There is an interesting consequence of this result.

**Theorem 3.5.** Let \( f : \Gamma \rightarrow H \) be a uniform embedding of a finitely generated group into a Hilbert space. Suppose that \( f(\Gamma) \subseteq H \) is a quasi-geodesic space with the induced metric. Then \( C^*_r(\Gamma) \) is an exact \( C^* \)-algebra.
Proof. Since $f$ is a uniform embedding one has that $\Gamma$ is coarsely equivalent to $f(\Gamma)$. Since both are quasi-geodesic spaces, it follows that $\Gamma$ is quasi-isometric to $f(\Gamma)$. But the latter is isometrically embedded in a Hilbert space, so $R(f(\Gamma)) = 1$. By quasi-isometry invariance of the invariant, we get $R(\Gamma) = 1$, and hence, by Theorem 3.3, $C^*_e(\Gamma)$ is exact. \hfill \qed

Remark. The above proof uses the fact that it is always the case that $R(f(\Gamma)) = 1$. Unfortunately, the argument in Theorem 3.3 does not apply to $f(\Gamma)$ because the spherical growth of $f(\Gamma)$ with its induced metric may be too great for the computations in (9) to carry over.

4. Behavior of $R(\Gamma)$ under direct sums and free products

Let $X$ and $Y$ be metric spaces. Let $X \times Y$ be the cartesian product with the metric

$$d_{X \times Y}((x,y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

We will obtain a formula for the Hilbert space distortion $R(X \times Y)$ in terms of $R(X)$ and $R(Y)$. The general case of $R(X \times Y; Z)$ will be discussed afterward.

Proposition 4.1. For metric spaces $X$ and $Y$ we have

$$R(X \times Y) = \min \{ R(X), R(Y) \}.$$ 

Proof. First note that, for fixed $y_0 \in Y$, the map $x \mapsto (x, y_0)$ provides an isometry $X \to X \times Y$. Applying Thm. 2 we conclude that $R(X) \geq R(X \times Y)$. Similarly $R(Y) \geq R(X \times Y)$ so one obtains $\min \{ R(X), R(Y) \} \geq R(X \times Y)$.

We must prove the reverse inequality. Assume that $R(X) \leq R(Y)$. Let $\varepsilon > 0$ be given. We will show that there exists a large scale Lipschitz map $h: X \times Y \to \mathcal{H}$ such that $R_h \geq R(X) - \varepsilon$. From this one obtains

$$R(X \times Y) \geq R_h \geq R(X) - \varepsilon = \min \{ R(X), R(Y) \} - \varepsilon,$$

and the desired inequality will follow.

According to the definition of $R(X)$ and $R(Y)$ there exists $f \in \text{Lip}^b(X, \mathcal{H}_X)$ and $g \in \text{Lip}^b(Y, \mathcal{H}_Y)$ such that

$$R_f \geq R(X) - \varepsilon$$

$$R_g \geq R(Y) - \varepsilon \geq R(X) - \varepsilon.$$

Define $h: X \times Y \to \mathcal{H} = \mathcal{H}_X \oplus \mathcal{H}_Y$ by $h(x, y) = f(x) \oplus g(y)$. From the inequality

$$\frac{\alpha + \beta}{\sqrt{2}} \leq (\alpha^2 + \beta^2)^{1/2} \leq \alpha + \beta, \quad \text{for all } \alpha, \beta \geq 0.$$  \hfill (10)
one obtains that \( h \in \text{Lip}^h(X \times Y, \mathcal{H}) \). It remains to estimate the compression of \( h \), again using (10). We have,

\[
\|h(x, y) - h(x', y')\| = \|f(x) - f(x') \oplus g(y) - g(y')\| \\
\geq \frac{1}{\sqrt{2}} \{\|f(x) - f(x')\| + \|g(y) - g(y')\|\}
\]

If \( d_{X \times Y}((x, y), (x', y')) \geq r \) then at least one of \( d_X(x, x') \) or \( d_Y(y, y') \geq r/2 \). Consequently,

\[
\rho_h(r) = \inf \{\|h(x, y) - h(x', y')\| : d_{X \times Y}((x, y), (x', y')) \geq r\} \\
\geq \frac{1}{\sqrt{2}} \inf \{\|f(x) - f(x')\| + \|g(y) - g(y')\| : d_{X \times Y}((x, y), (x', y')) \geq r\} \\
\geq \frac{1}{\sqrt{2}} \min \left\{ \rho_f \left( \frac{r}{2} \right), \rho_g \left( \frac{r}{2} \right) \right\}
\]

It follows that,

\[
R_h = \liminf_{r \to \infty} \log \frac{\rho_h(r)}{\log r} \\
\geq \liminf_{r \to \infty} \min \left\{ \frac{\log \rho_f \left( \frac{r}{2} \right)}{\log r}, \frac{\log \rho_g \left( \frac{r}{2} \right)}{\log r} \right\} \\
= \min \{R_f, R_g\} \geq R(X) - \varepsilon. \quad \square
\]

**Remark.** The above methods will also yield the inequality,

\[
R(X \times Y, Z \times Z) \geq \min \{ R(X, Z), R(Y, Z) \} \geq R(X \times Y, Z),
\]

but the analogous formula to that in Proposition 4.1 would require that \( Z \times Z \) be isometric to \( Z \).

Next, we will consider free products of groups. This requires a new technique for scaling uniform embeddings.

**Proposition 4.2.** Let \( \mathbb{F}_2 \) be the free group on two generators. We have \( R(\mathbb{F}_2) = 1 \).

**Proof.** Let \( X = (V, E) \) be the Cayley graph of \( \mathbb{F}_2 \), \( V \cong \mathbb{F}_2 \) being the set of vertices and \( E \) the set of edges. Define

\[
f : \mathbb{F}_2 \to \mathcal{H}, \quad f(s) = \delta_{e_1(s)} + \cdots + \delta_{e_k(s)},
\]

where \( \delta_e \) is the Dirac function of the edge \( e \) and \( e_1(s), \ldots, e_k(s) \) are the edges on the unique path in the Cayley graph from \( s \in \mathbb{F}_2 \) to the identity \( 1 \in \mathbb{F}_2 \). In particular \( k = d(s, 1) \) so
that \( \|f(s)\| = \sqrt{d(s, 1)} \). Indeed, it is not difficult to see that the asymptotic compression of \( f \) is 1/2.

Our strategy for proving the proposition is to produce, by placing appropriate weights into the above formula for \( f \), a family of (large scale) Lipschitz embeddings \( f_\varepsilon \in \text{Lip}^s(\mathbb{F}_2, \mathcal{H}) \), for \( 0 < \varepsilon < 1/2 \), such that \( R_{f_\varepsilon} \to 1 \) as \( \varepsilon \to 1/2 \). Denote \( \xi_\varepsilon(x) = x^\varepsilon \) and define weights by \( c_{\varepsilon,n} = \xi_\varepsilon(n) = n^\varepsilon \), for \( n \in \mathbb{N} \). Define \( f_\varepsilon: \mathbb{F}_2 \to l^2(E) \) by

\[
f_\varepsilon(s) = c_{\varepsilon,1} \delta_{e_1(s)} + \cdots + c_{\varepsilon,k} \delta_{e_k(s)},
\]

where \( k \) and \( e_1(s), \ldots, e_k(s) \) are as above.

In order to show that \( f_\varepsilon \) is a (large scale) Lipschitz map it suffices to show that there exists \( C > 0 \) such that

\[
d(s, t) = 1 \implies \|f_\varepsilon(s) - f_\varepsilon(t)\|^2 \leq C, \quad \text{for all } s, t \in \mathbb{F}_2.
\]

Let \( s, t \in \mathbb{F}_2 \) be such that \( d(s, t) = 1 \). Denote by \( k \) the length of \( s \) and, without loss of generality, \( k + 1 \) the length of \( t \). We have

\[
\|f_\varepsilon(s) - f_\varepsilon(t)\|^2 = c_{\varepsilon,1}^2 + (c_{\varepsilon,2} - c_{\varepsilon,1})^2 + \cdots + (c_{\varepsilon,k+1} - c_{\varepsilon,k})^2
\]

so that the desired inequality follows from the elementary fact that \( \sum_{j=2}^\infty (c_{\varepsilon,j} - c_{\varepsilon,j-1})^2 \) is finite. Indeed,

\[
\sum_{j=2}^\infty (c_{\varepsilon,j} - c_{\varepsilon,j-1})^2 = \sum_{j=2}^\infty \left( \int_{j-1}^j \xi_\varepsilon(x) \, dx \right)^2 \leq \sum_{j=2}^\infty \int_{j-1}^j (\xi_\varepsilon'(x))^2 \, dx \\
= \int_1^\infty \varepsilon^2 x^{2\varepsilon-2} \, dx = \frac{\varepsilon^2}{1-2\varepsilon}.
\]

We conclude the proof by showing that \( R_{f_\varepsilon} \geq 1/2 + \varepsilon \). In view of the definition (3) of the asymptotic compression it suffices to show that there exists a constant \( C_\varepsilon > 0 \), depending only on \( \varepsilon \), such that

\[
\|f_\varepsilon(s) - f_\varepsilon(t)\|^2 \geq C_\varepsilon r^{1+2\varepsilon}, \quad \text{for all } s, t \in \mathbb{F}_2 \text{ with } d(s, t) \geq r.
\]

Indeed, it follows from this that \( \rho_{f_\varepsilon}(r) \geq \sqrt{C_\varepsilon} r^{1/2 + \varepsilon} \) for all \( r \geq 1 \) and hence that \( R_{f_\varepsilon} = \lim \inf_{r \to \infty} \frac{\log \rho_{f_\varepsilon}(r)}{\log r} \geq 1/2 + \varepsilon \). To prove the inequality let \( s, t \in \mathbb{F}_2 \) be such that \( d(s, t) \geq r \) and assume, without loss of generality, that \( d(1, s) \leq d(1, t) \). We distinguish two cases: first \( s \) lies on the unique path from \( t \) to 1 and second it does not. In either case, denoting the smallest integer greater than \( r/2 \) by \( \#(r/2) \), one checks easily that the edges \( e_1(t), \ldots, e_{\#(r/2)}(t) \)
appear in the expression for $f_\varepsilon(t)$, but do not appear in that of $f_\varepsilon(s)$. In particular,
\[ \|f_\varepsilon(s) - f_\varepsilon(t)\|^2 \geq c_{\varepsilon,1}^2 + \cdots + c_{\varepsilon,\#(r/2)}^2 \geq \int_0^{r/2} \xi^2(x) \, dx = \frac{r^{2\varepsilon+1}}{(2\varepsilon+1)(2\varepsilon+1)}. \]

The following result can be obtained by similar, but more complicated, methods.

**Theorem 4.3.** Let $\Gamma = \Gamma_1 \ast \Gamma_2$ be the free product of the finitely generated discrete groups $\Gamma_1$ and $\Gamma_2$. Suppose that both admit an isometric embedding into a Hilbert space. Then $R(\Gamma) = 1$.

Note that this would be a consequence of the expected formula $R(\Gamma_1 \ast \Gamma_2) = \min\{ R(\Gamma_1), R(\Gamma_2) \}$.

5. The equivariant case

We adapt the previous definitions and results to the equivariant case. Let $\Gamma$ be a countable discrete group and let $X$ be a metric space on which $\Gamma$ acts by isometries. We define the equivariant Hilbert space compression of $X$ by restricting our attention to $\Gamma$-equivariant large scale Lipschitz embeddings of $X$ into Hilbert spaces equipped with actions of $\Gamma$ by affine isometries. Precisely, define

\[ \text{Lip}_{ls}^\Gamma(X, \mathcal{H}) = \left\{ \text{$\Gamma$-equivariant large scale Lipschitz maps } f : X \to \mathcal{H}, \mathcal{H} \text{ a } \Gamma \text{-Hilbert space} \right\} \quad (11) \]

The definition of the compression and asymptotic compression of $f \in \text{Lip}_{ls}^\Gamma(X, \mathcal{H})$ are the same as in the non-equivariant case (see (2) and (3), respectively); the $\Gamma$-equivariant Hilbert space compression of $X$ is defined by (compare (4))

\[ R_\Gamma(X) = \sup \{ R_f : f \in \text{Lip}_{ls}^\Gamma(X, \mathcal{H}) \}. \]

With these definitions in place the following analogs of Thm. 2 and its corollaries are proved in the same manner.

**Theorem 5.1.** Let $X$ and $Y$ be metric spaces on which the countable discrete group $\Gamma$ acts by isometries. If there exists an equivariant quasi-isometry $X \to Y$ then $R_\Gamma(X) \geq R_\Gamma(Y)$.  

**Corollary 5.2.** Let $\Gamma$ be a finitely generated discrete group. The invariant $R_\Gamma(\Gamma)$ is independent of the finite symmetric generating set used to define the length function and metric on $\Gamma$.  

Let $\Gamma$ be a countable discrete group. Recall that an affine isometric action of $\Gamma$ on a Hilbert space $\mathcal{H}$ consists of an orthogonal representation $t \mapsto \pi_t$ of $\Gamma$ on $\mathcal{H}$ and a function $b : \Gamma \to \mathcal{H}$ satisfying the \textit{cocycle identity}

$$ b(st) = \pi_s(b(t)) + b(s). $$

The cocycle identity insures that $t \mapsto \pi_t + b(t)$ defines a homomorphism from $\Gamma$ into the group of affine isometries of $\mathcal{H}$. An affine isometric action of $\Gamma$ on a Hilbert space $\mathcal{H}$ is \textit{metrically proper} if for every bounded set $B \subseteq \mathcal{H}$

$$ \{ s \in \Gamma : s \cdot B \cap B \neq \emptyset \} $$

or equivalently if the cocycle $b$ is \textit{proper} in the sense that for every $C > 0$ the set $\{ s \in \Gamma : \|b(s)\| \leq C \}$ is finite. A countable discrete group $\Gamma$ has the \textit{Haagerup property} if it admits a metrically proper affine isometric action on a Hilbert space. The first part of the next theorem is analogous to Prop. 3.1; the second part is analogous to Thm. 3.2.

**Theorem 5.3.** Let $\Gamma$ be a finitely generated discrete group. If $R_\Gamma(\Gamma) > 0$ then $\Gamma$ has the Haagerup property. If $R_\Gamma(\Gamma) > \frac{1}{2}$, then $\Gamma$ is amenable. \hfill $\Box$

Put in other terms the theorem states that if a finitely generated discrete group $\Gamma$ has an orthogonal representation on a Hilbert space that admits a cocycle $b$ of sufficiently rapid growth then $\Gamma$ is amenable. Indeed, suppose that $\pi$ is an orthogonal action of $\Gamma$ on $\mathcal{H}$. A cocycle $b$ for $\pi$ is an element of $\text{Lip}^b(\Gamma, \mathcal{H})$ where we view $\Gamma$ as acting on $\mathcal{H}$ by the affine isometric action $\alpha_t = \pi_t + b(t)$ and on itself by multiplication on the left; the required equivariance follows from the cocycle identity and it is easy to verify that $b$ is large-scale Lipschitz. Similarly, one has

$$ \|b(s) - b(t)\| = \|\pi_t(b(t^{-1}s))\| = \|b(t^{-1}s)\| $$

from which follows that

$$ \rho_b(r) = \inf \{ \|b(s) - b(t)\| : d(s, t) \geq r \} = \inf \{ \|b(s)\| : d(s, e) \geq r \}. $$

According to Theorem 5.3 if an orthogonal action of $\Gamma$ on a Hilbert space $\mathcal{H}$ admits a cocycle $b$ for which

$$ R_b = \liminf_{r \to \infty} \frac{\log \inf \{ \|b(s)\| : d(s, e) \geq r \}}{\log r} \geq \frac{1}{2} $$

then $\Gamma$ is amenable. In particular, this is the case if the cocycle $b$ satisfies $\|b(s)\| \geq (d(s, e))^{1/2 + \varepsilon}$ for some $\varepsilon > 0$. 

As an illustration consider once again $\Gamma = \mathbb{F}_2$, the free group on two generators. As in the proof of Prop. 4.2, let $X = (V, E)$ be the Cayley graph of $\mathbb{F}_2$, $V \cong \mathbb{F}_2$ being the set of vertices and $E$ the set of edges. The Hilbert space $H = l^2(E)$ is equipped with an orthogonal action $\lambda$ of $\mathbb{F}_2$, and the function

$$b : \mathbb{F}_2 \to l^2(E), \quad b(s) = \begin{cases} \text{characteristic function of the set of} \\ \text{edges on the unique path from } s \text{ to} \\ \text{the identity} \end{cases}$$

satisfies the cocycle identity $b(st) = \lambda_s(b(t)) + b(s)$. Consequently,

$$\alpha : \mathbb{F}_2 \to \text{Isom}(l^2(E)), \quad \alpha_s = \lambda_s + b(s)$$

defines an affine isometric action of $\mathbb{F}_2$ on $l^2(E)$. Equivalently, $b : \mathbb{F}_2 \to l^2(E)$ is an equivariant map, where $\mathbb{F}_2$ acts on itself by left multiplication and on $l^2(E)$ affine isometrically via $\alpha$. Further,

$$\|b(s) - b(t)\| = \sqrt{d(s, t)}, \quad \text{for all } s, t \in \mathbb{F}_2.$$

Consequently the asymptotic compression of $b$ is $R_b = \liminf_{r \to \infty} \frac{\log \rho_b(r)}{\log r} = 1/2$, and the equivariant Hilbert space compression of $\mathbb{F}_2$ satisfies $R_{l^2}(\mathbb{F}_2) \geq 1/2$. On the other hand, since $\mathbb{F}_2$ is not amenable we have $R_{l^2}(\mathbb{F}_2) \leq 1/2$. Hence $R_{l^2}(\mathbb{F}_2) = 1/2$. This should be compared to Proposition 4.2 which states that $R(\mathbb{F}_2) = 1$.

6. Appendix

In this appendix we will review the known relations between uniform embeddings and the notions of coarse geometry, [14]. We will include the elementary proofs for the convenience of the reader.

Definition. Let $X$ and $Y$ be metric spaces. A coarse map is a function $f : X \to Y$ satisfying

i) for all $R > 0$ there exists an $S > 0$ such that $d_X(x, x') \leq R$ implies that $d_Y(f(x), f(x')) \leq S$,

ii) if $B \subseteq Y$ is bounded, then $f^{-1}(B)$ is bounded.

Definition. A coarse map $f : X \to Y$ is a coarse equivalence if there is a coarse map $g : Y \to X$ satisfying that there exists a $K > 0$ such that $d_X(g \circ f(x), x) \leq K$ for all $x \in X$, and $d_Y(f \circ g(y), y) \leq K$ for all $y \in Y$.

Proposition 6.1. Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is a uniform embedding if and only if it satisfies
i) for all $R > 0$ there exists an $S > 0$ such that $d_X(x, x') \leq R$ implies that $d_Y(f(x), f(x')) \leq S$,

ii) for all $S > 0$ there exists an $R > 0$ such that $d_X(x, x') \geq R$ implies that $d_Y(f(x), f(x')) \geq S$.

Condition (ii) in Proposition 6.1 implies condition (ii) in the preceding definition. However, a coarse map need not be a uniform embedding as is shown by the existence of a proper Lipschitz map of a Gromov group into a Hilbert space.

Proposition 6.2. Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is a uniform embedding if and only if it is a coarse equivalence of $X$ with $f(X) \subseteq Y$ with the induced metric.

Proposition 6.3. Let $X$ and $Y$ be quasi-geodesic metric spaces. Then a function $f : X \to Y$ is a uniform embedding if and only if it is a quasi-isometric equivalence.

Proposition 6.4. Let $X$ be a quasi-geodesic metric space. Then a uniform embedding $f : X \to Y$ is a quasi-isometry if and only if $f(X)$ is a quasi-geodesic subspace of $Y$ with the induced metric.

The inclusion of a finitely generated group as a subgroup in another finitely generated group is a uniform embedding, but its range need not be a quasi-geodesic metric space with the induced metric. For example the inclusion of $\mathbb{Z}$ in the discrete 3-dimensional Heisenberg group is such.

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