Is a Nonclassical Symmetry a Symmetry?

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Various versions of the definition of nonclassical symmetries existing in the literature are analyzed. Comparing properties of Lie and nonclassical symmetries leads to the conclusion that in fact a nonclassical symmetry is not a symmetry in the usual sense. Hence the term “reduction operator” is suggested instead of the name “operator of nonclassical symmetries”. It is shown that in contrast to the case of single partial differential equations a satisfactory definition of nonclassical symmetries for systems of such equations has not been proposed up to now. Moreover, the cardinality of essential nonclassical symmetries is discussed, taking into account equivalence relations on the entire set of nonclassical symmetries.

1 Introduction

The “nonclassical” method of finding similarity solutions was introduced by Bluman and Cole in 1969 [3]. In fact, the method was first appeared in [2] in terms of “nonclassical group” but the terminology was changed in [3]. Over the years the “nonclassical” method began to be associated with the term nonclassical symmetry [13] (also called Q-conditional [8] or, simply, conditional symmetry [6, 10]). In the past two decades, the theoretical background of nonclassical symmetry was intensively investigated and nonclassical symmetry techniques were effectively applied to finding exact solutions of many partial differential equations arising in physics, biology, financial mathematics, etc. See, e.g., the review on investigations of nonclassical symmetries in [18].

Here we mention only works which are directly connected with the subject of our paper. In the pioneering paper [3] the “nonclassical” method was described by means of the example of the (1 + 1)-dimensional linear heat equation. It was emphasized that any solution of the corresponding (nonlinear) determining equations gives the coefficients of an operator such that an ansatz based on it reduces the heat equation to an ordinary differential equation. A veritable surge of interest in nonclassical symmetry was triggered by the papers [16, 17, 9]. In [16] the “nonclassical” method was considered in the course of a comprehensive analysis of a wide range of methods for constructing exact solutions. The concept of weak symmetry of a system of partial differential equations, generalizing the “nonclas-
sical” method, was introduced in [17], where also the reduction procedure was discussed. Moreover, fundamental identities [17, eq. (23)] crucially important for the theory of nonclassical symmetries were derived (see Myth 3 below). The first version of the conditional invariance criterion explicitly taking into account differential consequences was proposed in [9]. Generalizing results of [9, 7] and other previous papers, in [6] Fushchych introduced the notion of general conditional invariance. From the collection of papers containing [6] it becomes apparent that around this time a number of authors began to regularly use the terms “conditional invariance” and “Q-conditional invariance” in connection with the method of Bluman and Cole. The direct (ansatz) method closely related to this method was explicitly formulated in [4]. To the best of our knowledge, the name “nonclassical symmetry” was first used in [13]. Before this, there was no special name for operators calculated in this approach and the existing terminology on the subject emphasized characteristics of the method or invariance. The involution condition for families of operators was first considered in the formulation of the conditional invariance criterion in [10, 27]. The relations between nonclassical symmetries, reduction and formal compatibility of the combined system consisting of the initial equation and the invariant surface equation were discovered in [23] and were also studied in [15].

The problem of the algorithmization of calculating nonclassical symmetries was posed in [5]. Furthermore, the equivalence of the non-classical (conditional symmetry) and direct (ansatz) approaches to the reduction of partial differential equations was established in general form in [28], making use of the precise definition of reduction of differential equations.

In spite of the long history of nonclassical symmetry and the encouraging results in its applications, a number of basic problems of this theory are still open. Moreover, there exists a variety of non-rigorous definitions of related key notions and heuristic results on fundamental properties of nonclassical symmetry in the literature, which are used up to now and form what we would like to call the “mythology” of nonclassical symmetry. These definitions and results require particular care and presuppose the tacit assumption of a number of conventions in order to correctly apply them. Otherwise, certain contradictions and inaccurate statements may be obtained. Note that mythology interpreted in the above sense is an unavoidable and necessary step in the development of any subject.

Basic myths on nonclassical symmetries presented in the literature are discussed in this paper. We try to answer, in particular, the following questions.

- Is a nonclassical symmetry a Lie symmetry of the united system of the initial equation and the corresponding invariant surface condition? Can a nonclassical symmetry be viewed as a conditional symmetry of the initial equation if the corresponding invariant surface condition is taken as the additional constraint? Is nonclassical symmetry a kind of symmetry in general? Does there exist a more appropriate name for this notion?

- What is a rigorous definition of nonclassical symmetry for systems of differential equations? Can such a definition be formulated as a straightforward
extension of the definition of nonclassical symmetry for single partial differential equations?

• Is the number of nonclassical symmetries essentially greater than the number of classical symmetries?

2 Definition of nonclassical symmetry

Following [9, 10, 22, 28], in this section we briefly recall some basic notions and results on nonclassical (conditional) symmetries of partial differential equations. This will form the basis for our discussion of myths in the next sections.

Consider an involutive family

$$Q = \{Q^1, \ldots, Q^l\}$$

of $$l \leq n$$ first order differential operators (vector fields)

$$Q^s = \xi^s(x, u)\partial_i + \eta^s(x, u)\partial_u, \quad s = 1, \ldots, l,$$

in the space of the variables $$x$$ and $$u$$, satisfying the condition $$\text{rank} \|\xi^s(x, u)\| = l$$.

Here and in what follows $$x$$ denotes the $$n$$-tuple of independent variables $$(x_1, \ldots, x_n)$$, $$n > 1$$, and $$u$$ is treated as the unknown function. The indices $$i$$ and $$j$$ run from 1 to $$n$$, the indices $$s$$ and $$\sigma$$ run from 1 to $$l$$, and we use the summation convention for repeated indices. Subscripts of functions denote differentiation with respect to the corresponding variables, $$\partial_i = \partial/\partial x_i$$ and $$\partial_u = \partial/\partial u$$. Any function is considered as its zero-order derivative. All our considerations are in the local setting.

The requirement of involution for the family $$Q$$ means that the commutator of any pair of operators from $$Q$$ belongs to the span of $$Q$$ over the ring of smooth functions of the variables $$x$$ and $$u$$, i.e.,

$$\forall s, s' \exists \zeta^{ss'}(x, u): [Q^s, Q^{s'}] = \zeta^{ss'} Q^\sigma.$$  

The set of such families will be denoted by $$\Omega^l$$.

Consider an $$r$$th-order differential equation $$L$$ of the form

$$L[u] := L(x, u(r)) = 0$$

for the unknown function $$u$$ of the independent variables $$x$$. Here, $$u(r)$$ denotes the set of all derivatives of the function $$u$$ with respect to $$x$$ of order not greater than $$r$$, including $$u$$ as the derivative of order zero. Within the local approach the equation $$L$$ is treated as an algebraic equation in the jet space $$J^r$$ of the order $$r$$ and is identified with the manifold of its solutions in $$J^r$$. Denote this manifold by the same symbol $$L$$ and the manifold defined by the set of all the differential consequences of the characteristic system $$Q[u] = 0$$ in $$J^r$$ by $$Q_{(r)}$$, i.e.,

$$Q_{(r)} = \{(x, u_{(r)}) \in J^r \mid D_1^{\alpha_1} \cdots D_n^{\alpha_n} Q^s[u] = 0, \quad \alpha_i \in \mathbb{N} \cup \{0\}, \quad |\alpha| < r\}$$

where $$D_i = \partial_{x_i} + u_{\alpha + \delta} \partial_{u_\alpha}$$ is the operator of total differentiation with respect to the variable $$x_i$$, $$Q^s[u] := \eta^s - \xi^s u_i$$ is the characteristic of the operator $$Q$$, $$\alpha = (\alpha_1, \ldots, \alpha_n)$$ is an arbitrary multi-index, $$|\alpha| := \alpha_1 + \cdots + \alpha_n$$, $$\delta_i$$ is the multiindex whose $$i$$th entry equals 1 and whose other entries are zero. The variable $$u_\alpha$$ of the jet space $$J^r$$ corresponds to the derivative $$\partial^{|\alpha|} u/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$$.
Definition 1. The differential equation \( \mathcal{L} \) is called \textit{conditionally invariant} with respect to the involutive family \( Q \) if the relation

\[
Q^s_{(r)} L(x, u_{(r)}) \big|_{\mathcal{L} \cap Q_{(r)}} = 0
\]

(1)

holds, which is called the \textit{conditional invariance criterion}. Then \( Q \) is called an \textit{involutive family of conditional symmetry} (or \( Q \)-conditional symmetry, nonclassical symmetry, etc.) operators of the equation \( \mathcal{L} \).

Here the symbol \( Q^s_{(r)} \) stands for the standard \( r \)th prolongation of the operator \( Q^s \):

\[
Q^s_{(r)} = Q^s + \sum_{0 < |\alpha| \leq r} (D_1^{\alpha_1} \ldots D_n^{\alpha_n} Q^s[u] + \xi^{s_i} u_{\alpha + \delta_i}) \partial u_{\alpha}.
\]

3 Myths on name and definition

We restrict our consideration mainly to the case of families consisting of single operators (\( l = 1 \)) for simplicity and since mostly this case is investigated in the literature. Then the involution condition degenerates to an identity and we can omit the words “involutive family” and talk only about operators.

Myth 1. A nonclassical symmetry operator \( Q \) of an equation \( \mathcal{L} \) is a vector field \( Q \) which is a Lie symmetry operator of the united system of the equation \( \mathcal{L} \) and the invariant surface condition \( Q[u] = 0 \) corresponding to \( Q \).

This is the conventional non-rigorous way in order to quickly define nonclassical symmetry (see, e.g., [10, 11]). It becomes rigorous only after a special interpretation of the notions of system of differential equations and Lie symmetry. Otherwise, using the empiric definition leads to a number of inconsistencies.

A closer look reveals that the above definition is a tautology. Indeed, the invariant surface condition \( Q[u] = 0 \) means that the function \( u \) is a fixed point of the one-parametric local group \( G_Q \) of local transformations generated by the operator \( Q \). Therefore, we can reformulate the definition in the following way.

Reformulation. If the set of those solutions of the equation \( \mathcal{L} \) which are fixed points of \( G_Q \), is invariant with respect to \( G_Q \), then \( Q \) is called a nonclassical symmetry operator \( Q \) of the equation \( \mathcal{L} \).

The tautology of the reformulation is obvious. If each element of the set is invariant then the whole set is necessarily invariant. The definition of nonclassical symmetry according to Myth 1 leads to the conclusion that \textit{any differential equation is invariant, in the nonclassical sense, with respect to any vector field in the corresponding space of dependent and independent variables.}

The case when the equation \( \mathcal{L} \) has no \( Q \)-invariant solutions fits well into the non-rigorous approach in the sense that the empty set is a particularly symmetric set.
Therefore, uncritically following the non-rigorous approach, we would get no effective methods for constructing exact solutions and no information on the partial differential equations under consideration.

There exist a number reformulations of Myth 1 in the literature in different terms. The first one is in terms of conditional symmetry.

**Myth 2.** A nonclassical symmetry operator $Q$ of an equation $\mathcal{L}$ is a conditional symmetry operator of the equation $\mathcal{L}$ under the auxiliary condition $Q[u] = 0$.

The association of nonclassical symmetries (under the name $Q$-conditional symmetries) with conditional ones can be traced back to [7] (see also [8] and earlier papers of the same authors). Here the term conditional symmetry is understood in the following sense [6] (it can easily be defined for the case of a general system of differential equations).

**Definition 2.** A vector field $Q$ is called a conditional symmetry operator of a system $\mathcal{L}$ of differential equations under an auxiliary condition $\mathcal{L}'$ (which is another system of differential equations in the same variables) if $Q$ is a Lie symmetry operator of the united system of $\mathcal{L}$ and $\mathcal{L}'$.

Conditional symmetries defined in this way essentially differ from nonclassical symmetries. In particular, auxiliary conditions for conditional symmetries do not involve any associated conditional symmetry operators. The conditional symmetry operators of a system $\mathcal{L}$ under an auxiliary condition $\mathcal{L}'$ form a Lie algebra. Conditional symmetry indeed is a kind of symmetry and can be applied to generate new solutions from known ones. At the same time, in contrast to the case of nonclassical symmetries, finding auxiliary conditions associated with nontrivial conditional symmetries is an art rather than an algorithmic procedure. This is why sometimes nonclassical symmetries are called either $Q$-conditional symmetries, where the prefix “$Q$” is used to emphasize the differences between nonclassical and conditional symmetries, or conditional symmetries without any connection with Definition 2.

The second reformulation of Myth 1 is in infinitesimal terms. Note that infinitesimal criteria lie at the basis of Lie symmetry theory since they allow one to study linear problems for infinitesimal transformations instead of nonlinear problems for finite transformations.

**Myth 3.** The conditional invariance criterion for an equation $\mathcal{L}$ and an operator $Q$ coincides with the infinitesimal Lie invariance criterion for the united system $\{\mathcal{L}, Q[u] = 0\}$ with respect to the same operator, i.e.,

$$Q_{(r)}L[u] = 0 \quad \text{if} \quad L[u] = 0 \quad \text{and} \quad Q[u] = 0.$$  

The infinitesimal Lie invariance criterion for the invariant surface condition $Q[u] = 0$ with respect to the operator $Q$ is identically satisfied as an algebraic consequence of this condition since

$$Q_{(r)}Q[u] = Q_{(1)}Q[u] = (\eta u - \xi^j u_j)Q[u] \equiv 0 \quad \text{if} \quad Q[u] = 0.$$
We also have
\[ Q(r) L[u] = \xi^i D_i L[u] + \sum_{|\alpha| \leq r} L_{u_{\alpha}}[u] D_{1}^{\alpha_1} \cdots D_{n}^{\alpha_n} Q[u], \] (2)
i.e., the equation \( Q(r) L[u] = 0 \) is a differential consequence of the equations \( L[u] = 0 \) and \( Q[u] = 0 \) and, therefore, becomes an identity on the set of their common solutions. This tautology was first observed in \([17]\).

In the local approach to group analysis of differential equations, a system of differential equations is associated with the infinite tuple of systems of algebraic equations defined by this system and its differential consequences in the infinite tower of the corresponding jet spaces. The exclusion of the differential consequence \( Q(r) L[u] \) when considering the system \( L[u] = 0 \) and \( Q[u] = 0 \) seems unnatural from the viewpoint of group analysis.

A variation of Myth 3 is to replace, due to the Hadamard lemma, the “invariance condition” holding on the solution set of the system \( L[u] = 0 \) and \( Q[u] = 0 \) by the associated multiplier-condition, to be satisfied on the entire jet space \( J^r \).

**Myth 4.** An operator \( Q \) is a nonclassical symmetry of an equation \( L \) if there exist \( \lambda^1 \) and \( \lambda^2 \) such that
\[ Q(r) L[u] = \lambda^1 L[u] + \lambda^2 Q[u]. \] (3)
The problem is to precisely define the nature of the multipliers \( \lambda^1 \) and \( \lambda^2 \). A number of different conditions on the multipliers have been put forward in the literature. The simplest version is to prescribe no conditions at all on \( \lambda^1 \) and \( \lambda^2 \), which is obviously unacceptable.

Sometimes \( \lambda^1 \) and \( \lambda^2 \) are assumed to be differential functions. This condition is natural for \( \lambda^1 \) but overly restrictive for \( \lambda^2 \). In fact, if only such \( \lambda^2 \) are allowed, the equivalence relation of nonclassical symmetries up to nonvanishing functional multipliers will be broken. Moreover, in this case the associated invariance criterion will become merely a sufficient condition for an ansatz constructed with the operator \( Q \) to reduce the equation \( L \). As a result, a number of well-defined reductions may be lost.

On the other hand, requiring that both the multipliers \( \lambda^1 \) and \( \lambda^2 \) are polynomials of total differentiation operators with respect to the independent variables, whose coefficients are differential functions, is too weak an assumption. It arises from the association of nonclassical symmetries with conditional symmetries for which such multipliers are admissible. If we choose
\[ \lambda^1 = \xi^i D_i \quad \text{and} \quad \lambda^2 = \sum_{|\alpha| \leq r} L_{u_{\alpha}}[u] D_{1}^{\alpha_1} \cdots D_{n}^{\alpha_n}, \]
condition (3) obviously becomes an identity for any operator \( Q \). In other words, condition (3) reduces to the tautology (2) if both \( \lambda^1 \) and \( \lambda^2 \) are treated as differential operators of the above kind.
Comparing Definition [1] and Myth [4] shows that $\lambda^1$ should be a differential function (i.e., a zeroth order operator) and $\lambda^2$ should be an order $(r - 1)$ operator. These conditions for the multipliers can be weakened. Thus, bounding the order of total differentiations in $\lambda^2$ is not essential. If $\lambda^1$ is a differential function, condition [4] implies that $\lambda^2$ cannot include total differentiations of orders greater than $r - 1$. At the same time, explicitly prescribing the bound allows one to fix the order of the jet space under consideration.

**Myth 5** (The main myth of the theory). *Nonclassical symmetry is a kind of symmetry of differential equations.*

Any kind of symmetry of differential equations (Lie, contact, hidden, conditional, approximate, generalized, potential, nonlocal etc.) has the invariance property, i.e., symmetries transform solutions to solutions in an appropriate sense.

The basic prerequisite of the definition of nonclassical symmetry is the consideration of only the set of solutions invariant under the associated finite transformations. It is impossible to use nonclassical symmetries in order to generate new solutions from known ones. A nonclassical symmetry operator $Q$ of $L$ represents only a symmetry of

- each $Q$-invariant solution of $L$ (as a weak symmetry [17]) and
- the manifold $L \cap Q(r)$ in $J^r$, where $r = \text{ord} L$.

The manifold $L \cap Q(r)$ is properly related to the joint system $L[u] = 0$ and $Q[u] = 0$ of differential equations only if the operator $Q$ and the equation $L$ satisfy the conditional invariance criterion.

At the same time, properties of the set of nonclassical symmetries and properties of the set of $Q$-invariant solutions for each nonclassical symmetry operator $Q$ characterize the equation $L$.

Since a nonclassical symmetry is not in fact a kind of symmetry of differential equations, it is of utmost importance to discuss possibilities for replacing the name by one not involving the word “symmetry”.

## 4 Nonclassical symmetry, compatibility and reduction

To understand the real nature of nonclassical symmetry, we discuss properties and applications of Lie symmetries and single out those of them which carry over to nonclassical symmetries.

**Properties of Lie symmetries:**

*Invariance.* Any Lie symmetry (in the form of a parameterized family of finite transformations) locally maps the solution set of the corresponding system of differential equations onto itself. This is the main characteristic of any kind of symmetry. It gives rise to the possibility of generating new solutions from known ones.
Formal compatibility. Attaching the invariant surface conditions associated with a Lie invariance algebra to the initial system of differential equations results in a system having no nontrivial differential consequences. In other words, the invariant surface conditions forms a class of proper universal differential constraints and, therefore, is appropriate for finding subsets of solutions of the initial system.

Reduction. Each Lie invariance algebra satisfying the infinitesimal transversality condition leads to an ansatz reducing the initial system to a system with a smaller number of independent variables, i.e., the reduced system is more easily solvable than the initial one.

Conditional compatibility. There exists a bijection between solutions of the initial system which satisfy the invariant surface conditions, and solutions of the corresponding reduced system. This means that all solutions of the initial system invariant with respect to a Lie invariance algebra, can be constructed via solving the corresponding reduced system.

For nonclassical symmetries, the property of invariance is broken but the other properties (formal compatibility, reduction, conditional compatibility) are preserved. In fact, the conditional invariance criterion is the condition of formal compatibility of the joint system \( L[u] = 0 \) and \( Q[u] = 0 \)\(^1\). We can identify nonclassical symmetries of \( L \) with first-order quasilinear differential constraints which are formally compatible with \( L \).

**Definition 3.** The differential equation \( L \) is called **conditionally invariant** with respect to the involutive family of operators \( Q \) if the joint system of \( L \) with the characteristic system \( Q[u] = 0 \) is formally compatible.

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\(^1\)In fact, this claim and Definition are not entirely rigorous. Their precise formulation depends on what definition of formal compatibility is used. Consider, e.g., the definition presented in \([24, 25]\). We temporarily use notations compatible with these references, hence slightly different from the rest of the paper.

Let \( L_r \) be a system of \( r \) differential equations \( L^1[u] = 0, \ldots, L^r[u] = 0 \) in \( n \) independent variables \( x = (x_1, \ldots, x_n) \) and \( m \) dependent variables \( u = (u^1, \ldots, u^m) \), which involves derivatives of \( u \) up to order \( r \). The system \( L_r \) is interpreted as a system of algebraic equations in the jet space \( J^r \) and defines a manifold in \( J^r \), which is also denoted by \( L_r \). The \( s \)th order prolongation \( \mathcal{L}_{r+s} \) of the system \( L_r \), \( s \in \mathbb{N} \), is the system in \( J^{r+s} \) consisting of the equations \( D_1^{s_1} \cdots D_n^{s_n} L^k[u] = 0, k = 1, \ldots, l, \, |\alpha| \leq s \). The projection of the corresponding manifold on \( J^{r+s-q} \), where \( q \in \mathbb{N} \) and \( q < s \), is denoted by \( \mathcal{L}_{r+s-q}^{(q)} \). The system \( L_r \) is called **formally integrable** (or **formally compatible**) if \( \mathcal{L}_{r+s}^{(s)} \) is for any \( s \in \mathbb{N} \) \([24, 25]\).

The first obstacle in the harmonization of the above definition of formal compatibility and the definition of nonclassical symmetry is that the equations \( L[u] = 0 \) and \( Q[u] = 0 \) have, as a rule, different orders. Therefore, differential consequences of the equation \( Q[u] = 0 \) should be attached to the joint system \( L[u] = 0 \) and \( Q[u] = 0 \) before testing its compatibility. The second obstacle is that the order of \( L[u] \) may be lowered on the manifold \( Q_{(r)} \) if \( Q \) is a singular vector field for the equation \( L[u] = 0 \). Hence instead of the equation \( L[u] = 0 \) we should use the equation \( L_s[u] = 0 \), where \( L_s \) is a differential function which coincides with \( \lambda L \) on \( Q_{(r)} \) for some nonvanishing differential function \( \lambda \) and whose order \( r_s \) is minimal among differential functions possessing this property. Finally, we arrive at the following definition: The differential equation \( L \) is called **conditionally invariant** with respect to the involutive family of operators \( Q \) if the system \( L_s[u] = 0, D_1^{s_1} \cdots D_n^{s_n} Q[u] = 0, |\alpha| < r_s \), is formally compatible.
What is the main property that adequately represents the essence of nonclassical symmetry?

The fact that the characteristic equations \( Q^s[u] = 0 \) are quasilinear and of first order implies the possibility of integrating them explicitly, i.e., an ansatz associated with the characteristic system \( Q[u] = 0 \) can be constructed. In view of the Frobenius theorem, the involution and transversality conditions for the family \( Q \) (together with the fact that the operators from \( Q \) are of first order) imply that the ansatz involves one new unknown function of \( n - l \) new independent variables. Then the formal compatibility of the joint system \( L[u] = 0 \) and \( Q[u] = 0 \) guarantees the reduction of \( L \) by the ansatz to a single differential equations \( L' \) in \( n - l \) independent variables. Thus, the number of dependent variables and equations are preserved under the reduction with \( Q \) and the number of independent variables decreases by the cardinality of \( Q \), i.e., similarly to Lie symmetries nonclassical symmetries lead to the conventional reduction of the number of independent variables.

There exist integrable differential constraints which are not formally compatible with the initial system. Differential constraints can be formally compatible with the initial system and, at the same time, non-integrable in an explicit form. An ansatz constructed with a general integrable differential constraint may involve a number of new unknown functions depending on different variables. Therefore, only all the above properties combined (first order, quasilinearity, formal compatibility, transversality and involution) result in the classical reduction procedure.

The conditional invariance of the equation \( L \) with respect to the family \( Q \) is equivalent to the ansatz constructed with this family reducing \( L \) to a differential equation with \( n - l \) independent variables \[28\]. Moreover, reducing the number of independent variables in partial differential equations is the main goal in the study of nonclassical symmetries. Since the reduction by the associated ansatz is the quintessence of nonclassical symmetries, it was proposed in \[21, 22, 26\] to call involutive families of nonclassical symmetry operators families of reduction operators of \( L \).

Another important property holding for Lie symmetries is broken for nonclassical symmetries. Let the equation \( L \) be of order \( r \) and

\[ L_{(k)} = \{D_1^{\alpha_1} \ldots D_n^{\alpha_n} L[u] = 0, \ |\alpha| \leq k - r \}. \]

Denote by \( L_{(k)} \) a maximal set of algebraically independent differential consequences of \( L \) that have, as differential equations, orders not greater than \( k \). We identify \( L_{(k)} \) with the corresponding system of algebraic equations in \( J^k(x|u) \) and

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\[2\] Extended notions of reduction are also used. Thus, weak symmetries imply reductions decreasing the number of independent variables, preserving the number of unknown functions and increasing the number of equations \[17\]. The reduced system can be much more overdetermined than the initial one. The reductions associated with higher-order nonclassical symmetries preserve the determinacy type of systems, simultaneously increasing the numbers of unknown functions and equations \[15\].
associate it with the manifold $\mathcal{L}_k$ determined by this system. For Lie symmetries we have the following properties.

1. If $Q$ is a Lie symmetry operator of $\mathcal{L}(r)$ then $Q$ is a Lie symmetry operator of $\mathcal{L}(\rho)$ for any $\rho > r$.

2. If $Q$ is a Lie symmetry operator of $\mathcal{L}(\rho)$ for some $\rho > r$ then $Q$ is a Lie symmetry of $\mathcal{L}(r)$.

The first of these properties extends to nonclassical symmetries but this is not the case for the second one. In fact:

1. If $Q$ is a Lie symmetry operator of $\mathcal{L}(r) \cap Q(r)$ then $Q$ is a Lie symmetry operator of $\mathcal{L}(\rho) \cap Q(\rho)$ for any $\rho > r$.

2. The fact that $Q$ is a Lie symmetry operator of $\mathcal{L}(\rho) \cap Q(\rho)$ for some $\rho > r$ does not imply that $\mathcal{L}(r) \cap Q(r)$ admits the operator $Q$.

**Example 1.** Let $L[u] = u_t + u_{xx} + tu_{xx}$, $\mathcal{L}$: $L[u] = 0$ and $Q = \partial_t$. Then the manifold $\mathcal{L}(\rho) \cap Q(\rho)$ is determined in $J^2$ by the equations $u_t = u_{tt} = u_{tx} = 0$ and $u_{xx} = -t u_{xx}$. Since

$$Q(\rho)L_{\mathcal{L}(\rho) \cap Q(\rho)} = u_x \neq 0,$$

the operator $\partial_t$ is not a reduction operator of $\mathcal{L}$. Substituting the corresponding ansatz $u = \varphi(\omega)$, where the invariant independent variable is $\omega = x$, into $\mathcal{L}$ results in the equation $\varphi_{\omega \omega} + t \varphi_{\omega} = 0$, in which the “parametric” variable $t$ cannot be excluded via multiplying by a nonvanishing differential function. As expected, the ansatz does not reduce the equation $\mathcal{L}$.

Consider the same operator $Q$ and the first prolongation $\mathcal{L}(3)$ of $\mathcal{L}$, which is determined by the equations $L[u] = 0$, $D_t L[u] = 0$ and $D_x L[u] = 0$. The manifold $\mathcal{L}(3) \cap Q(3)$ is singled out from $J^3$ by the equations

$$u_t = u_{tt} = u_{tx} = u_{ttt} = u_{txx} = 0, \quad u_x = u_{xx} = u_{xxx} = 0.$$

The conditional invariance criterion is satisfied for the prolonged system $\mathcal{L}(3)$ and the operator $Q$:

$$Q(\rho)L_{\mathcal{L}(\rho) \cap Q(\rho)} = Q(\rho)D_t L_{\mathcal{L}(\rho) \cap Q(\rho)} = Q(\rho)D_x L_{\mathcal{L}(\rho) \cap Q(\rho)} = 0,$$

i.e., $Q$ is a nonclassical symmetry operator of the system $\mathcal{L}(3)$ and the above ansatz reduces $\mathcal{L}(3)$ to the system of three ordinary differential equations $\varphi_{\omega} = 0$, $\varphi_{\omega \omega} = 0$ and $\varphi_{\omega \omega \omega} = 0$ since

$$
\begin{pmatrix}
\varphi_{\omega \omega} + t \varphi_{\omega} \\
\varphi_{\omega} \\
\varphi_{\omega \omega \omega} + t \varphi_{\omega \omega}
\end{pmatrix} =
\begin{pmatrix}
t & 1 & 0 \\
1 & 0 & 0 \\
0 & t & 1
\end{pmatrix}
\begin{pmatrix}
\varphi_{\omega} \\
\varphi_{\omega \omega} \\
\varphi_{\omega \omega \omega}
\end{pmatrix} = 0 \quad \text{and} \quad
\begin{pmatrix}
t & 1 & 0 \\
1 & 0 & 0 \\
0 & t & 1
\end{pmatrix}
\neq 0.
$$

**Note 1.** In general, for any system $\mathcal{L}$ and any involutive family $Q$ there exists an order $r$ such that $\mathcal{L}(r) \cap Q(r)$ is invariant with respect to $Q(r)$. This gives the theoretical background of the notion of weak symmetry [17].
5 Definition of nonclassical symmetries for systems

Myth 6. The definition of nonclassical symmetry for systems of differential equations is a simple extension of the definition of nonclassical symmetry for single partial differential equations to the case of systems.

Example [1] and Note [1] indicate problems arising in attempts of defining nonclassical symmetries for systems of partial differential equations.

Let $L$ denote a system $L(x, u(r)) = 0$ of $l$ differential equations $L^l = 0$ for $m$ unknown functions $u = (u^1, \ldots, u^m)$ of $n$ independent variables $x = (x_1, \ldots, x_n)$. It is always assumed that the set of differential equations forming the system under consideration canonically represents this system and is minimal. The minimality of a set of equations means that no equation from this set is a differential consequence of the other equations. For $L(k)$ we will denote a maximal set of algebraically independent differential consequences of $L$ that have, as differential equations, orders not greater than $k$. We identify $L(k)$ with the corresponding system of algebraic equations in the jet space $J^k$ and associate it with the manifold $L(k)$ determined by this system. Let $L(r) = \{\hat{L}^\nu, \nu = 1, \ldots, \hat{l}\}$.

What is the correct conditional invariance criterion for the system $L$?

\[
Q(r)L^\mu|_{L \cap Q(r)} = 0, \quad \mu = 1, \ldots, l?
\]
\[
Q(r)L^\mu|_{L(r) \cap Q(r)} = 0, \quad \mu = 1, \ldots, l?
\]
\[
Q(r)\hat{L}^\nu|_{L(r) \cap Q(r)} = 0, \quad \nu = 1, \ldots, \hat{l}?
\]

All of the above candidates for the criterion are not satisfactory. The second candidate is not a good choice since it neglects the equations having lower orders than the order of the whole system. Taking the third candidate, we obtain nonclassical symmetries of a prolongation of the system. As shown by Example [1], these may be weakly related to nonclassical symmetries of the system. It is not well understood what differential consequences are really essential. Thus, elements of $L(r)$ whose trivial differential consequences also belong to $L(r)$ are neglected by this candidate.

Although all operators satisfying the first of the above criteria give proper reductions, it is overly restrictive and in fact is only a sufficient condition for nonclassical symmetries. Even Lie symmetries can be lost when employing it.

The above discussion is illustrated by the following example.

Example 2. Consider the system

\[
\ddot{u}_t + (\dddot{u} \cdot \nabla)\dddot{u} - \Delta \dddot{u} + \nabla p + \vec{x} \times \nabla \text{div} \dddot{u} = \vec{0}, \quad \text{div} \dddot{u} = 0.
\]

which is obviously equivalent to the system of Navier–Stokes equations describing the motion of an incompressible fluid. (The additional term $\vec{x} \times \nabla (\text{div} \dddot{u})$ vanishes if $\text{div} \dddot{u} = 0$.) If we do not take into account differential consequences of
system (4), we derive the unnatural claim that this system is not conditionally invariant with respect to translations of the space variables \( x_i \). At the same time, the infinitesimal generators of these translations belong to the maximal Lie invariance algebra of the Navier–Stokes equations. A maximal set \( L(2) \) of algebraically independent differential consequences of \( L \) that have, as differential equations, orders not greater than 2 is formed by the equations

\[
\begin{align*}
\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} - \Delta \vec{u} + \nabla p &= \vec{0}, \\
\text{div } \vec{u}_t &= 0, \\
\nabla \text{div } \vec{u} &= 0, \\
\vec{u}_i u^i_j + \Delta p &= 0.
\end{align*}
\]

Here the indices \( i \) and \( j \) run from 1 to 3. The equation \( Q(2) \) \( \text{div } \vec{u} = \vec{0} \) is identically satisfied on the set \( L(2) \cap Q(2) \). Therefore, the application of the second or third candidate for the conditional invariance criterion to the equation \( \text{div } \vec{u} = \vec{0} \) gives no determining equations for nonclassical symmetries of the system (4).

Definition 3 can also not be extended to the case of systems in an easy way. The problem again is to define what set of differential consequences of the initial system should be chosen for testing formal compatibility with the appropriate characteristic system.

The notions of nonclassical symmetry and reduction are strongly related in the case of single partial differential equations. It therefore seems natural for these notions to also be closely related in the case of systems. Hence the problem of rigorously defining nonclassical symmetries for systems is additionally complicated by the absence of a canonical extension of the classical reduction to the case of systems. A chain of simple examples can be presented to illustrate possible features of such an extension.

6 Myths on number of nonclassical symmetries

**Myth 7.** The number of nonclassical symmetries is essentially greater than the number of classical symmetries.

At first sight this statement seems obviously true. There exist classes of partial differential equations whose maximal Lie invariance algebra is zero and which admit large sets of reduction operators. This is the case, e.g., for general \( (1+1) \)-dimensional evolution equations. At the same time, certain circumstances significantly reduce the number of essential nonclassical symmetries. We briefly list them below.

- The usual equivalence of families of reduction operators. Involutive families \( Q \) and \( \tilde{Q} \) of \( l \) operators are called equivalent if \( \tilde{Q}^s = \lambda^s \sigma Q^\sigma \) for some \( \lambda^s \sigma = \lambda^s \sigma(x, u) \) with \( \det \| \lambda^s \sigma \| \neq 0 \).
- Nonclassical symmetries equivalent to Lie symmetries.
The equivalence of nonclassical symmetries with respect to Lie symmetry groups of single differential equations \cite{13, 20} and equivalence groups of classes of such equations \cite{22}.

No-go cases. The problem of finding certain wide subsets of reduction operators may turn out to be equivalent to solving the initial equation \cite{12, 21}.

Non-Lie reductions leading to Lie invariant solutions.

Thus, the existence of a wide Lie symmetry group for a partial differential equation $L$ complicates, in a certain sense, finding nonclassical symmetries of $L$. Indeed, any subalgebra of the corresponding maximal Lie invariance algebra, satisfying the transversality condition, generates a class of equivalent Lie families of reduction operators. If a non-Lie family of reduction operators exists, the action of symmetry transformations on it results in a series of non-Lie families of reduction operators, which are inequivalent in the usual sense. Therefore, for any fixed value of $l$ the system of determining equations for the coefficients of operators from the set $\Omega^l(L)$ of families of $l$ reduction operators is not sufficiently overdetermined to be completely integrated in an easy way, even after factorizing with respect to the equivalence relation in $\Omega^l(L)$. To produce essentially different non-Lie reductions, one has to exclude the solutions of the determining equations which give Lie families of reduction operators and non-Lie families which are equivalent to others with respect to the Lie symmetry group of $L$. As a result, the ratio of efficiency of such reductions to the expended efforts can become vanishingly small.

7 Conclusion

Although the name “nonclassical symmetry” and other analogous names for reduction operators, which refer to symmetry or invariance, do not reflect actual properties of these objects, the usage of such names is justified by historical conventions and additionally supported by the terminology of related fields of group analysis of differential equations. It is a quite common situation for different fields of human activity that a modifier completely changes the meaning of the initial notion (think of terms like “negative growth”, “military intelligence”, etc.). Empiric definitions of nonclassical symmetry can be used in a consistent way if all involved terms and notions are properly interpreted. Nevertheless, as we have argued, the term reduction operator more adequately captures the underlying mathematical content.

In this paper we discussed certain basic myths of the theory of nonclassical symmetries, pertaining to different versions of their definition and the estimation of their cardinality. Over and above these, there are a number of more sophisticated myths concerning, among others, the factorization of sets of nonclassical symmetries, involutive families of reduction operators in the multidimensional case, and singular sets of reduction operators. A discussion of such myths requires a careful theoretical analysis substantiated by nontrivial examples and will be the subject of a forthcoming paper.
Acknowledgements

ROP thanks the members of the local organizing committee for the nice conference and for their hospitality. MK was supported by START-project Y237 of the Austrian Science Fund. The research of ROP was supported by the Austrian Science Fund (FWF), project P20632.

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