Subgap Conductivity of a Superconducting-Normal Tunnel Interface

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At low temperatures, the transport through a superconducting-normal tunnel interface is due to tunneling of electrons in pairs. The probability for this process is shown to depend on the layout of the electrodes near the tunnel junction, rather than on properties of the tunnel barrier. This dependence is due to interference of the electron waves on a space scale determined by the coherence length, either in the normal or the superconducting metal. The approach developed allows us to evaluate the subgap current for different layouts of interest.

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It is well-known that the charge transport through a tunnel NS interface between a normal metal and a superconductor is strongly suppressed at voltages lower than $\Delta/e$, $\Delta$ being the superconducting energy gap [1]. Indeed, energy conservation forbids the transfer of a normal electron with an energy below the gap to the superconductor, since it would have been converted into a quasiparticle with an energy larger than $\Delta$.

Experimentally, some residual conductivity has been observed at subgap voltages even at very low temperatures. There is a tendency to ascribe this either to imperfections in the tunnel barrier or to normal inclusions in the superconductor. Another mechanism of the subgap conductivity is the so-called two-electron tunneling [2]. The point is that two normal electrons can be converted into a Cooper pair, thus this transfer may cost no energy. The current will be proportional to the fourth power of tunnel matrix elements; therefore it is much smaller than the one-electron current.

The problem was previously treated under the simple assumption that the electron wavefunctions in both metals are just plane waves. In this case one can consider a barrier of arbitrary transparency in order to describe the crossover from a tunnel to a perfectly conducting interface [3]. But some important physics may be missed under this assumption. Let us compare the two realizations of the NS interface depicted in Fig. 1. In Fig. 1a the electron transmitted through the interface does not experience any scattering in the metals and never gets back to the junction. The plane wave picture seems to be relevant for such a geometry. An alternative situation is shown in Fig. 1b. This case is usually realized when the tunnel junction is formed by imposing two thin metal films. The transmitted electron gets back to the junction many times before leaving the junction region. Thus the tunneling occurs between electron states of very complex structure which emerges from interference between scattered waves.

This interference has no effect on one-electron transport, since the average one-electron density of states does not depend on the stucture of the wave function. However, it matters for two-electron tunneling, since two electrons penetrating the barrier will interfere. Such an interference occurs at a space scale corresponding to the energy difference between the two
electron states. It makes the probability of two-electron tunneling dependent on the system layout at the corresponding mesoscopic space scale. Our aim is to evaluate this interference effect for an arbitrary given layout. The rapid progress of nanotechnology makes it possible to fabricate numerous relevant structures, so it is worthwhile to be able to give guidelines to a designer. As we will see below, the subgap conductivity is strongly enhanced if the interference effect is essential.

We first review shortly the two-electron tunneling through a superconducting-normal interface as it has been discussed by Wilkins \[2\] and more recently by Hekking \textit{et al.} \[4\]. The total Hamiltonian can be written as $\hat{H} = \hat{H}_N + \hat{H}_S + \hat{H}_T$. The subscripts $N$ and $S$ refer to the normal and the superconducting electrode respectively; the transfer of electrons through the tunnel interface is described by the tunnel Hamiltonian $\hat{H}_T$. The latter is expressed in terms of quasiparticle operators $\hat{\gamma}$, $\hat{\gamma}^\dagger$ for the superconductor, and electron operators $\hat{a}$, $\hat{a}^\dagger$ for the normal metal:

\[\hat{H}_T = \sum_{k,p,\sigma} \{ t_{kp} \hat{a}_{k,\sigma} \hat{\gamma}_{p,\sigma} + v_{p,\sigma} \hat{\gamma}^\dagger_{-p,-\sigma} - \lambda_{p,\sigma} \hat{\gamma}^\dagger_{p,\sigma} \hat{a}_{k,\sigma} \}.\]

Here, $t_{kp}$ are the tunnel matrix elements which we take to be spin-independent, and $u_{p,\sigma}, v_{p,\sigma}$ are the BCS coherence factors \[1\]; the sum is taken over momenta $k, p$ and spin $\sigma = \uparrow, \downarrow$.

Using second order perturbation theory in $\hat{H}_T$ one can calculate the amplitude for the transfer of two electrons from the normal to the superconducting electrode:

\[A_{k+k'} = \sum_p \{ t_{kp}^* t_{k'p}^* u_p v_p \left( \frac{1}{\xi_k - \varepsilon_p} + \frac{1}{\xi_{k'} - \varepsilon_p} \right) \}.\]

Here the spin dependence of the coherence factors was dropped after using the relation $v_{p,\uparrow} = -v_{-p,\downarrow}$. We define electron energies $\xi_k$ and $\xi_p$ for the normal and the superconducting electrode respectively, and quasiparticle energies $\varepsilon_p = \sqrt{\Delta^2 + \xi_p^2}$. The denominators in (2) reflect the fact that a virtual state is formed when the first electron enters the superconductor as a quasiparticle. The second electron couples to this quasiparticle, thus forming a Cooper pair. The corresponding rate $\Gamma(V)$ as a function of the voltage $V$ applied across the junction
can be found by using Fermi’s Golden Rule

\[ \Gamma(V) = \frac{4\pi}{\hbar} \sum_{k,k'} |A_{k \uparrow k' \downarrow}|^2 f(\xi_k) f(\xi_{k'}) \delta(\xi_k + \xi_{k'} + 2eV). \]  

(3)

It contains the Fermi functions \( f \) for electrons with energies \( \xi_k, \xi_{k'} \) in the normal metal. We recall that the normal conductance of the junction is determined by the rate \( \gamma(V) \) for usual electron tunneling, which is proportional to \( |t_{kp}|^2 \): \( \gamma(V) = |t_{kp}|^2 f(\xi_k) (1 - f_r(\zeta_p)) \delta(\xi_k - \zeta_p + eV) \).

The calculation of \( |A_{k \uparrow k' \downarrow}|^2 \) in Eq. (3) involves summations over momentum of a product of four tunnel matrix elements. It therefore requires an assumption about the dependence of the \( t_{kp} \) on the wave vectors \( k \) and \( p \). This dependence is strongly related to the nature of the electron motion in the electrodes, as we discussed above. Following [2] we assume first that plane electron waves propagating in the electrodes are transmitted specularly by a rectangular tunnel barrier with, say, a length \( L_b \) and a height \( U \) (See Fig. 1a). The area of the junction will be denoted by \( S \). Specular scattering implies that the components of momentum \( k_\parallel \) and \( p_\parallel \) parallel to the barrier plane are conserved. If \( S \) is of the order of \( \lambda_F^2 \) (with \( \lambda_F \) the Fermi wavelength of the electrons) the values of \( k_\parallel \) and \( p_\parallel \) are quantized, leading to discrete transport channels [3]. The corresponding quantum numbers are equal: \( n_k = n_p \); the effective number of channels \( N_{eff} \) contributing to the transport will be calculated below. The magnitude of \( t_{kp} \) decreases exponentially with decreasing squared component \( k_\perp^2 \) perpendicular to the barrier. This results in the assumption \( t_{kp} \propto \delta_{n_k,n_p} \exp -k_\parallel^2 \lambda^2 \), with \( \lambda = \hbar L_b / \sqrt{8mU} \), where \( m \) is the electron mass. The calculation of the rate (3) is easily performed using this model for \( t_{kp} \), by averaging products of these matrix elements over directions of momentum. As a result we find that \( \Gamma \propto 1/N_{eff}^3 \). Similarly we obtain \( \gamma \propto 1/N_{eff} \). Comparing \( \Gamma \) with \( \gamma^2 \) we find the effective number of transport channels penetrating the barrier, \( N_{eff} \):

\[ \frac{\Gamma}{\gamma^2} \propto \langle |\langle t_{kp} | t_{k'p} \rangle_p|^2 \rangle_{kk'}/\langle |t_{kp}|^2 \rangle_{kp}^2 = 4\pi \lambda^2 / S = 1/N_{eff}. \]

This result is obtained by assuming ballistic motion of the electrons in the electrodes. This assumption is correct only if the scattering of the electron is negligible. Scattering
may occur at the boundaries of the electrodes or at impurities inside the electrodes. Both processes can be characterized by a space scale \( l_e \), which corresponds to the distance the electron traverses before undergoing the first scattering event. Interference occurs on a space scale \( \xi_{\text{cor}} \). The ballistic picture is valid if the typical size \( \sqrt{S} \) of the junction or \( \xi_{\text{cor}} \) is smaller than \( l_e \). When these lengths are of the same order, we expect a cross-over to different behaviour. In the opposite limit the electron moves diffusively in the junction region. Due to interference between incoming and backscattered electron waves \( N_{\text{eff}} \) will decrease, thereby increasing the rate \( \Gamma \), and hence the conductance due to two-electron tunneling.

Now we will present a method to describe two-electron tunneling in the diffusive transport regime, employing the quasiclassical approximation. This enables us to evaluate the tunnel matrix elements and express the rate \( \Gamma \) in terms of quasiclassical diffusion propagators. The method is similar to the one presented in Ref. [7]. We start by rewriting the matrix elements \( t_{kp} = \int drdr' \psi_k^*(r) \psi_p(r')t(r,r') \), where \( \psi_p(r) \) forms a complete set of one-electron wave functions in the electrodes, and \( t(r,r') \) describes the tunneling from a point \( r' \) in the superconductor to a point \( r \) in the normal metal (primed space arguments refer to the superconductor). We also define a propagator from \( r_2 \) to \( r_1 \) by

\[
K_\xi(r_1,r_2) = \sum_k \delta(\xi - \xi_k) \psi_k(r_1) \psi_k^*(r_2)
\]

With these definitions it is possible to rewrite Eq. (3) as

\[
\Gamma(V) = \frac{4\pi}{\hbar} \int d\xi d\xi' d\zeta d\zeta' F(\zeta;\xi,\xi')F(\zeta';\xi,\xi')\Xi(\zeta,\zeta';\xi,\xi')f(\xi)f(\xi')\delta(\xi + \xi' + 2eV) \tag{4}
\]

with \( F(\zeta;\xi,\xi') = u(\zeta)v(\zeta) \{(\xi - \varepsilon)^{-1} + (\xi' - \varepsilon)^{-1}\} \) where \( \varepsilon = \sqrt{\Delta^2 + \zeta^2} \), and

\[
\Xi(\zeta,\zeta';\xi,\xi') = \int d^3r_1...d^3r_4 \int d^3r'_1...d^3r'_4 t^*(r_1,r'_1)t(r_2,r'_2)t(r_3,r'_3)t(r_4,r'_4) \times K_\xi(r_1,r_3)K_{\xi'}(r_2,r_4)K_\zeta(r'_2,r'_3)K_{\zeta'}(r'_1,r'_4) \tag{5}
\]

The physical meaning of Eq. (4) can be understood easily by depicting the integrand of Eq. (5) diagrammatically, as has been done in Fig. 2. We see two electrons that propagate in the normal electrode with energy \( \xi \) and \( \xi' \). The first electron reaches the barrier at \( r_1 \) and tunnels to \( r'_1 \), the second electron tunnels from \( r_2 \) to \( r'_2 \); both change their energy to \( \zeta \). In the superconductor they form a Cooper pair. Since tunneling occurs only between
neighboring positions, we have in addition \( r_i \approx r'_i \). The diagram expresses a probability, and therefore is completed by adding the time-reversed process.

To analyze expression (5), it is important to consider the scale of separation of the coordinates \( r_1 \approx r'_1, \ldots, r_4 \approx r'_4 \) lying on the interface. In the ballistic transport regime, these coordinates are separated only by a few Fermi wavelengths. In this case the contribution depends on properties of the tunnel barrier only. Below we will concentrate on contributions to (5) which arise when the region of integration is defined by coordinates which are pairwise close, but with the pairs separated by a distance much larger than the Fermi wavelength. These contributions contain averaged products of two propagators \( K \), which are known to decay on a mesoscopic scale in the diffusive transport regime \( 6 \). These products correspond to the semiclassical motion of electrons from one point on the interface back to another point on this interface. They describe the interference between scattered waves. In the diffusive regime these contributions dominate; that is why we concentrate on them.

There are three types of contributions, which are depicted in Fig 3. Fig. 3a corresponds to the case \( r_1 \approx r_2 \) and \( r_3 \approx r_4 \). The interference contribution originates from the normal electrode. Fig. 3b describes the opposite situation with interference occurring in the superconducting electrode. Here, \( r_1 \approx r_3 \) and \( r_2 \approx r_4 \). Finally, in Fig. 3c, interference occurs both in the normal and in the superconducting electrode, when \( r_1 \approx r_4 \) and \( r_2 \approx r_3 \). However, we estimated this contribution to be less important, compared to the previous diagrams. Therefore, the total effect can be represented as the sum of the interference contributions from the superconducting and the normal metal.

As an example we will discuss the contribution of Fig. 3b to the rate (4) in some detail. The averaged product of the propagators in the superconductor determines the semiclassical conditional probability \( P(r'_1, r'_2; n, n'; t) \) that an electron with position \( r'_2 \) and momentum direction \( n' \) at time \( t = 0 \) has position \( r'_1 \) and momentum direction \( n \) at time \( t \). Since the tunnel amplitude \( t(r, r') \) is nonzero only when \( r \) and \( r' \) are close to the junction interface, we can restrict spatial integrations to planar integrations over the junction surface. It is
possible to show that

\[ \Xi_S = \Xi_S(\zeta - \zeta') = \frac{\hbar}{8\pi^3 e^4 \nu_S} \int d^2 r_1' d^2 r_2' \int d^2 n d^2 n' g(n,r')g(n',r_2) \times \]

\[ \int dt e^{i(\zeta - \zeta')n/\hbar} P(r_1',r_2';n,n';t) \]  

(6)

where \( \nu_S \) is the density of states for the superconductor for two spin directions and \( \int d^2 n \) denotes integration over directions of momentum. The function \( g(n,r) \) defines the normal conductance of the junction: \( G_T = \int d^2 r \int d^2 n g(n,r) \). An expression similar to (6) can be obtained for Fig. 3a, by replacing subscript \( S \rightarrow N \), energies \( \zeta \rightarrow \xi \), and primed space arguments by unprimed ones.

Let us start our consideration of concrete layouts with the simplest geometry of an infinite uniform junction between a normal and a superconducting film (Fig. 4a). We assume that the film thickness is much less than the coherence length in the superconductor. Then we can exploit the picture of two-dimensional electron diffusion. The probability function we need is given by

\[ P(r_1, r_2; t) = \frac{1}{4\pi D d t} \exp\left(-|r_1 - r_2|^2/4Dt\right), \]  

(7)

d being the thickness of either the superconducting or the normal metal film. Taking the Fourier transform of this function with respect to time and integrating (6) over coordinates we obtain

\[ \Xi_S(\zeta - \zeta') = (4\pi^2 e^4 S \nu_S d_S)^{-1}\delta(\zeta - \zeta'); \Xi_N(\xi - \xi') = (4\pi^2 e^4 S \nu_N d_N)^{-1}\delta(\xi - \xi'); \]  

(8)

The current is given by a sum of two terms \((eV \approx \Delta \gg T)\):

\[ I(V) = I_N + I_S; \]

\[ I_N = \frac{2G_T^2 \hbar}{e^3 \nu_S d_S}; \]

\[ I_S = \frac{G_T^2 \hbar}{e^3 S \nu_S d_S} \frac{eV}{\pi \Delta \sqrt{1 - eV/\Delta}} \]  

(9)

It is plotted in Fig. 5. The part emerging from the interference in the normal metal does not depend on voltage. So the current sharply jumps at zero voltage, provided \( T = 0 \). The jump is smoothened at voltages of the order of the temperature:
\[ I(V, T) = I_N \tanh(eV/2T) \quad (10) \]

The other contribution diverges near the threshold voltage indicating the necessity to make use of higher order terms in tunneling amplitudes to describe the crossover between two-electron and one-electron tunneling.

It is worthwhile to compare the magnitude of the result with the one we derived assuming ballistic motion. The order of the ratio at voltages of the order of \( \Delta \) is \( I_{int}/I_{ball} \approx \xi_{\text{clean}}/d \), \( \xi_{\text{clean}} \) being the coherence length in the pure superconductor. Therefore the interference term dominates under the assumptions we made.

The coherence along a normal or superconducting film is characterized by the coherence length \( \xi_{\text{cor}}^{N,S} = \sqrt{D/eV}, \sqrt{D/\Delta} \), respectively. The relations (9) are valid, provided the junction size is much larger than these lengths. In the opposite limit of small junctions, the subgap conductivity will be determined by the junction surroundings, rather than by the junction itself. Let us illustrate this by considering the geometry in Fig. 4b, where a normal electrode is connected to a superconducting sheet by the tunnel junction. In this case we find

\[ \Xi_S(\zeta - \zeta') = \frac{\hbar}{e^2} \frac{R_S G_T^2}{8\pi^4} \ln \frac{\hbar}{(\zeta - \zeta')\tau}; \Xi_N(\xi - \xi') = \frac{\hbar}{e^2} \frac{R_N G_T^2}{8\pi^4} \ln \frac{\hbar}{(\xi - \xi')\tau} \quad (11) \]

The time \( \tau \) is of the order of \( S/D \), the time spent in the junction area, and provides a lower cut-off for the Fourier integral. The sheet resistance of the normal (superconducting) film is given by \( R_{\Box}^{N(S)} = (e^2\nu D_{N(S)} d_{N(S)})^{-1} \). Indeed does the result not only depend on the properties of the tunnel barrier itself (through the dependence on \( G_T \)), but also on properties of its surroundings through the dependence on \( R_{\Box} \). Moreover, the dependence on the precise geometry of the layout enters through numerical prefactors. If, e.g., the tunneling would occur towards an infinite superconducting sheet instead of a superconducting half plane, the semiclassical probability \( P \) would be twice smaller, thus decreasing \( \Xi \), and hence the rate \( \Gamma \), by a factor of 2. The current is again given by a sum of two terms \( I_N \) and \( I_S \):

\[ I_N = \frac{2V}{\pi} R_{\Box,N} G_T^2 \ln \frac{\hbar}{eV\tau}; I_S = \frac{2V}{\pi} R_{\Box,S} G_T^2 \ln \frac{\hbar}{\Delta\tau} \quad (12) \]
Note that, in contrast to Eq. (9), the subgap conductivity depends only weakly on the junction area through the cut-off time $\tau$.

We finally consider the geometry depicted in Fig. 4c. It consists of a small island (length $L_S$, thickness $d_S$), coupled to two macroscopic leads by tunnel barriers. The grain is linked capacitively to the leads. Electron transport through such a system, characterized by a small electric capacitance $C$, has been studied both experimentally and theoretically in great detail during the past years [8]. The key point is that variations of the charge of the island in the course of electron tunneling increase the electrostatic energy, typically by an amount $E_C = e^2/2C$. This is why electron tunneling through a small grain is suppressed (Coulomb blockade). The case of a superconducting island connected to two normal electrodes (NSN geometry), was studied recently in Refs. [4,9,10]. Our method to include interference effects can also be applied to this case. The charging energy $E_C$ will enter our results explicitly via the virtual state denominators in (4) [4], resulting in a dependence of the function $F$ on $E_C$. We will restrict ourselves to the situation in which $E_C$ is smaller than the superconducting gap: $E_C < \Delta$. In order to calculate the contribution due to interference on the island we assume that the time $\hbar/(\Delta - E_C)$ spent by the virtual electron on the island is much longer than the classical diffusion time $L_S^2/D$. If $\Delta \lesssim E_C$, the size of the island is smaller than $\xi_S^{\text{cor}}$. In this case, the electron motion covers the whole island and the probability $P$ is constant: $P = 1/(L_S^2 d_S)$. As a result we find:

$$\Xi_S(\zeta - \zeta') = \frac{\hbar^2 w_S G_T^2}{4\pi^2 e^4} \delta(\zeta - \zeta')$$

(13)

where $w_S = 1/(\nu_S L_S^2 d_S)$ denotes the level spacing of the island, which shows once more that the rate (4) is not only determined by properties of the tunnel barrier. The corresponding current $I_S$ reads

$$I_S = V \frac{\Delta}{\pi e^2 G_T} \frac{w_S \Delta}{E_C^2} \times$$

$$\left[ \frac{\pi}{2} - \frac{2\Delta}{\sqrt{\Delta^2 - E_C^2}} \left\{ 1 - \frac{E_C^2}{\Delta^2 - E_C^2} \right\} \arctan \sqrt{\frac{\Delta - E_C}{\Delta + E_C} + \frac{\Delta E_C}{\Delta^2 - E_C^2}} \right]$$

(14)
When interference in the normal electrode is taken into account, we find \( \Xi_N(\xi - \xi') = (\hbar/e^2)(R_N^N G_T^2/4\pi^4) \ln[h/(\xi - \xi')\tau] \), like in Eq. (11), however larger by a factor of 2, due to the difference in geometry. The resulting current reads:

\[
I_N = \frac{4V}{\pi^3} R_N^N G_T^2 \ln \frac{\hbar}{eV\tau} \left\{ \frac{4\Delta}{\sqrt{\Delta^2 - E_C^2}} \arctan \left[ \frac{\Delta + E_C^2}{\Delta - E_C^2} \right] \right\}^2
\]  

(15)

In conclusion, we evaluated the low-voltage supgap conductivity of NS boundary interface. In many interesting cases it was shown to be determined by the conditions of electron motion in the electrodes rather than the properties of the tunnel barrier. Therefore it depends on the system layout on the mesoscopic scale. We presented an approach which gives exact results for any layout given.

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FIGURES

FIG. 1. Two typical realizations of a tunnel junction between a normal (N) and a superconducting (S) electrode. In (a) the electron moves ballistically, in (b) diffusively in the junction region.

FIG. 2. Diagram corresponding to Eq. (5). Electrons propagate (solid lines) with energy $\xi, \xi'$ in the normal (N) and energy $\zeta, \zeta'$ in the superconducting (S) electrode. They tunnel through the barrier (shaded region) at positions 1,...,4, marked by crosses.

FIG. 3. Contributions to the subgap conductivity due to interference in (a) the normal electrode, (b) the superconducting electrode, and (c) both electrodes.

FIG. 4. Various layouts discussed in the text: (a) infinite uniform junction, (b) normal electrode connected to a superconducting halfplane, and (c) a superconducting island connected to two normal electrodes.

FIG. 5. $I-V$-characteristics for an infinite junction. The curves (from bottom to top) represent $I_S$, $I_N$, and $I(V) = I_S + I_N$. 