First occurrences of square-free gaps and an algorithm for their computation

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1 Abstract

This paper reports the results of a search for first occurrences of square-free gaps using an algorithm based on the sieve of Eratosthenes. Using $Qgap(L)$ to denote the starting number of the first gap having exactly the length $L$, the following values were found since August 1999: $Qgap(10) = 262,315,467$, $Qgap(12) = 47,255,689,915$, $Qgap(13) = 82,462,576,220$, $Qgap(14) = 1,043,460,553,364$, $Qgap(15) = 79,180,770,078,548$, $Qgap(16) = 3,215,226,335,143,218$, $Qgap(17) = 23,742,453,640,900,972$ and $Qgap(18) = 125,781,000,834,058,568$. No gaps longer than 18 were found up to $N = 125,870,000,000,000,000$.

2 Introduction

2.1 Square-free numbers

A number is said to be square-free if its prime decomposition contains no repeated factors. For example, 30 is square-free since its prime decomposition $2 \times 3 \times 5$ contains no repeated factors. However, 18 is not square-free since the factor 3 appears twice in its prime decomposition $2 \times 3 \times 3$.

The first few square-free numbers give the sequence: 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 46, 47, 51, 53, 55, 57, 58, 59, 61, 62, 65, 66, 67, 69, 70, 71, 73, 74, 77, 78, 82, 83, 85, 86, 87, 89, 91, 93, 94, 95, 97, etc. (Sloane’s A005117 1.)

More details on square-free numbers can be found at mathworld.wolfram.com/Squarefree.html. If a square-free number is used as the argument of the Möbius function2, a non-zero value (+1 or −1) is obtained.

2.2 Gaps between square-free numbers

A square-free gap is a series of $L$ consecutive numbers missing from the sequence of square-free numbers. The first square-free gap in the sequence of square-free numbers starts at $N = 4$ and has a length of one. The next gap starts at $N = 8$ and has a length $L = 2$ (since 8 and 9 are non-square-free). The following table lists the first few gaps and their lengths.

| Gap starts at $N$ | 4 | 8 | 12 | 16 | 18 | 20 | 24 | 27 | 32 | 36 | 40 | 44 | 48 | 52 | 54 | 56 |
|------------------|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Length of gap $L$ | 1 | 2 | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 2 | 3 | 1 | 1 | 1 |

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1 oeis.org/A005117
2 mathworld.wolfram.com/MoebiusFunction.html
For any $L$, it can be shown that there exist infinitely many gaps of length greater than $L$ in the sequence of square-free numbers. Longer lists of square-free gaps and recent results are available in Appendices B, C and D. These gaps are series of consecutive squareful numbers (Sloane’s A013929).

Note that the term “squarefull” sometimes denotes a positive integer $n$ such that if $p$ is a prime dividing $n$, then $p^2$ divides $n$. (See the selected Preprints of Michael Filaseta.)

The smallest integer of the first gap having exactly the length $L$ is denoted here as $Qgap(L)$ (for “Quadratfrei”, or squarefree). Thus $Qgap(1) = 4$, $Qgap(2) = 8$, $Qgap(3) = 48$, etc.

### 2.3 Upper limits for $Qgap(16)$ to $Qgap(24)$

Erick Bryce Wong found upper limits for $Qgap(L)$ for $L = 16$ to 24. His idea was to find them by prescribing a repeated prime factors for each term and using the Chinese Remainder Theorem to obtain a number. More precisely, he prescribed all but five of the moduli and then tested the last moduli, up to the first 1000 primes, to check if the number is squarefree. He tried this over millions of permutations. His impressive results are:

| $Qgap(L)$ | Upper limit | Found on |
|-----------|-------------|----------|
| $Qgap(16)$ | $46\,717\,595\,829\,767\,167$ | Feb. 17th 2000 |
| $Qgap(17)$ | $23\,742\,453\,640\,900\,972$ | Feb. 21st 2000 |
| $Qgap(18)$ | $125\,781\,000\,834\,058\,568$ | June 26th 2000 |
| $Qgap(19)$ | $31\,310\,794\,237\,768\,728\,712$ | July 18th 2000 |
| $Qgap(20)$ | $148\,372\,453\,443\,663\,297\,638\,331$ | July 10th 2000 |
| $Qgap(21)$ | $321\,362\,101\,382\,225\,854\,472$ | Feb. 17th 2000 |
| $Qgap(22)$ | $213\,922\,449\,434\,979\,698\,424\,416$ | Aug. 4th 2000 |
| $Qgap(23)$ | $687\,445\,369\,966\,391\,012\,821\,156\,868$ | July 18th 2000 |
| $Qgap(24)$ | $28\,548\,715\,276\,566\,524\,078\,226\,797\,585\,011$ | Sept. 4th 2000 |

### 2.4 First occurrence of a gap of length $L$

$Qgap(L)$ is listed in the following table for $L$ up to 18. The third column gives the prime factors that are repeated for each number in the gap. The values of $Qgap(L < 10)$ and $Qgap(11)$ have been confirmed by different sources. (See for example, “Sloane’s On-Line Encyclopedia of Integer Sequences” sequence A045882.) The sequence $Qgap(L)$ is listed under A051681 in the Encyclopedia of Integer sequences.
2.4.1 Basic algorithm: the sieve of Eratosthenes

The square-free gaps can be calculated by finding consecutive numbers that are not square-free. A simple method to show that \( N \) is not squarefree is to find a prime factor of \( N \) whose square divides \( N \). By trying every prime up to the square root of \( N \), one can establish whether \( N \) is square-free or not. However, this is a very inefficient way to test billions of numbers.

A faster algorithm is used by “Mathematica” to determine if a number is square-free. The method is quite interesting\(^{14}\).

However, to determine which of many consecutive numbers are square-free, an algorithm based on to the sieve of Eratosthenes\(^{15}\) is much faster. It uses a list of numbers from which each composite number is removed. Once the process is finished, only the prime numbers are in the list.

To find the square-free numbers using a sieve, a similar technique is used but the algorithm eliminates numbers that are not square-free. Starting with a list of integers, first cross out the multiples of 4:

\[
\begin{array}{c|c|c|c}
N & Q_{\text{gap}}(N) & \text{Repeated prime factors of each number in the gap} & \text{Gap reported by} \\
--- & --- & --- & --- \\
1 & 4 & 2 & E. Friedman \\
2 & 8 & 2, 3 & E. Friedman \\
3 & 48 & 2, 7, 5 & E. Friedman \\
4 & 242 & 11, 3, 2, 7 & E. Friedman \\
5 & 844 & 2, 13, 3, 11, 2 & E. Friedman \\
6 & 22020 & 2, 19, 11, 3, 2, 5 & E. Friedman \\
7 & 217070 & 7, 3, 2, 113, 11, 5, 2 & E. Friedman \\
8 & 1092747 & 19, 2, 7, 5, 11, 2, 3, 13 & E. Friedman \\
9 & 8870024 & 2, 5, 11, 29, 2, 7, 31, 3, 2 & P. De Geest \\
10 & 262315467 & 3, 2, 29, 2957, 79, 2, 7, 17, 5, 2 \times 3 & D. Bernier \\
11 & 221167422 & 3, 31, 2, 5, 37, 13, 2, 7, 11, 3, 2 & P. De Geest \\
12 & 47255689915 & 7, 2, 3, 103, 43, 2, 29, 17, 13, 2, 5, 3 & L. Marmet \\
13 & 82462576220 & 2, 3, 13, 23, 2, 5, 17, 41, 2, 19, 3, 7, 2 & L. Marmet \\
14 & 104346055364 & 2, 3, 7, 19, 2, 13 \times 59, 67, 43, 2, 181, 3, 5, 2, 11 & L. Marmet \\
15 & 7918077078548 & 2, 3, 5, 29, 2, 13, 17, 53, 2, 19, 3, 41, 2, 31, 67 & L. Marmet \\
16 & 3215226335143218 & 11, 23, 2, 3, 269, 53, 2, 5, 17, 163, 2, 101, 3, 19, 2, 137 & Z. McGregor-Dorsey \\
17 & 23742453640900972 & 2, 11 \times 23, 127, 5, 2, 3, 53, 37, 2, 7, 13, 17, 2, 19, 3, 29, 2 & E. Wong \\
18 & 125781000834058568 & 2, 3, 37, 31, 2, 19, 29, 5, 2, 7 \times 23, 3, 139, 2, 11, 17, 13, 2, 199 & L. Marmet \\
\end{array}
\]

The first gaps reported in this work were found on the following dates.

\[
\begin{array}{c|c|c|c}
Q_{\text{gap}}(10) & = & 262315467 & \text{August 1999} & \text{D. Bernier}, \\
Q_{\text{gap}}(12) & = & 47255689915 & \text{October 19th 1999} & \text{L. Marmet}, \\
Q_{\text{gap}}(13) & = & 82462576220 & \text{October 20th 1999} & \text{L. Marmet}, \\
Q_{\text{gap}}(14) & = & 104346055364 & \text{October 25th 1999} & \text{L. Marmet}, \\
Q_{\text{gap}}(15) & = & 7918077078548 & \text{November 29th 1999} & \text{L. Marmet}, \\
Q_{\text{gap}}(16) & = & 3215226335143218 & \text{July 22nd 2000} & \text{Z. McGregor-Dorsey et al.}, \\
Q_{\text{gap}}(17) & = & 23742453640900972 & \text{July 8th 2001} & \text{E. Wong et al.}, \\
Q_{\text{gap}}(18) & = & 125781000834058568 & \text{September 9th 2005} & \text{L. Marmet et al.} \\
\end{array}
\]

\(^{14}\)reference.wolfram.com/mathematica/ref/SquareFreeQ.html

\(^{15}\)mathworld.wolfram.com/SieveofEratosthenes.html
then the multiples of 9, 25, etc., up to the last number in the list:

1 2 3 X 5 6 7 X X 10 11 X 13 14 15 X 17 X 19 X 21 22 23 X X 26 ...

The remaining numbers are square-free numbers; the gaps are indicated by the series of consecutive “X”.

2.4.2 Improvements of the algorithm

The following improvements were implemented in a computer program and are presented in the same order they were added to the program.

2.4.3 Improvement I

“Lists of squared-primes and the next non-square-free number use less memory.”

To implement this algorithm on a computer, it is not necessary to keep the entire list of integers in memory. An improvement of the algorithm uses instead two shorter arrays to calculate the next non-square-free number after \( N \):

- the first array, called \( p2 \), gives the squares of the prime numbers up to the largest number to be tested \( N_{\text{max}} \).
- the second array, called \( \text{nsqf} \), gives for each \( p2[i] \) the next non-square-free number, that is, the smallest number larger than \( N \) that is a multiple of \( p2[i] \). This array can easily be calculated using modulo arithmetic.

These arrays will have approximately \( 2\sqrt{N_{\text{max}}}/\ln N_{\text{max}} \) elements\(^{16}\).

To find square-free gaps, one finds sequences of non-square-free numbers. The following example shows the arrays used to find gaps starting from \( N = 20 \)\(^{17}\).

| Index | i   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|-----|---|---|---|---|---|---|---|---|---|
| Squared prime | \( p2[i] \) | 4 | 9 | 25 | 49 | 121 | 169 | 289 | 361 | 529 |
| Next non-square-free | \( \text{nsqf}[i] \) | 24 | 27 | 25 | 49 | 121 | 169 | 289 | 361 | 529 |

Using this table, it is easy to find the next non-square-free number: it is the smallest number in the array \( \text{nsqf} \), that is, 24. We set \( N = 24 \) and recalculate the array. This is easy since the only needed operation is to add the corresponding squared-prime to the multiple: \( 24 + 4 = 28 \). The following table is obtained:

| Index | i   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|-----|---|---|---|---|---|---|---|---|---|
| Squared prime | \( p2[i] \) | 4 | 9 | 25 | 49 | 121 | 169 | 289 | 361 | 529 |
| Next non-square-free | \( \text{nsqf}[i] \) | 28 | 27 | 25 | 49 | 121 | 169 | 289 | 361 | 529 |

Again, the next non-square-free number is the smallest \( \text{nsqf}[i] \). By repeating this procedure, \( N \) takes the values of all the non-square-free numbers. It is advantageous to sort \( \text{nsqf} \) in increasing order at each step. This way, the smallest is always \( \text{nsqf}[0] \). We set \( N = 25 \) and recalculate the array to obtain:

\(^{16}\text{www.utm.edu/research/primes/howmany.shtml}\)

\(^{17}\text{The notation used in the programming language C is used here, where the index of an array starts at 0.}\)
The order of the array $p^2[i]$ has also been changed so that each number $p^2[i]$ always corresponds to its multiple $nsqf[i]$. Repeating the procedure will generate the non-square-free numbers $N = 27, 28, 32, 36, 40, \ldots$. Note that special care has to be taken when some numbers in $nsqf$ are equal - each of these has to be increased by the value of its corresponding $p^2[i]$.

The sort is relatively efficient since after $nsqf[0]$ is given its new value, $nsqf[1], nsqf[2]$ and the following elements are still in increasing order. The new value is moved up the array until its proper place is found.

### 2.4.4 Improvement II

"Many non-square-free numbers can be skipped."

If gaps of a given length $L_{min}$ or more are searched, some $nsqf[i]$ can be skipped. To show this, one finds first the minimum number of squared-primes $NP2_{min}$ \(\text{18}\) required in a gap of length $L$:

| Gap length $L$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | ...
|---------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|...
| $NP2_{min}[L]$ | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 7 | 7 | 8 | 9 | 9 | 10 | 11 | 12 | 12 | ...

For example if we choose $L = 7$, $NP2_{min}[L] = 6$ prime factors are required for the gap starting at $N = 217070$ (in this case, the prime factors are 2, 3, 5, 7, 11 and 113).

Continuing with the example given above, we now specifically search gaps with length $L_{min} = 7$ or longer. There is no gap of length $L_{min} = 7$ in the interval starting at $nsqf[0]$ and ending at $nsqf[5]$, if $nsqf[5] > nsqf[0] + 7$. (In general, there is no gap of length $L_{min}$ in the interval starting at $nsqf[0]$ and ending at $nsqf[NP2_{min}[L_{min} - 1]]$, if $nsqf[NP2_{min}[L_{min} - 1]] > nsqf[0] + L_{min}$.)

With $N = 40$, we have:

| Index | i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ...
|-------|---|---|---|---|---|---|---|---|---|---|...
| Squared prime $p^2[i]$ | 4 | 9 | 49 | 25 | 121 | 169 | 289 | 361 | 529 | ...
| Next non-square-free $nsqf[i]$ | 44 | 45 | 49 | 50 | 121 | 169 | 289 | 361 | 529 | ...

Since $nsqf[5] = 169 > nsqf[0] + 7 = 51$, there is no gap of length 7 in the interval starting at 44 and ending at 169. We can therefore safely set $N = nsqf[5] - L_{min} + 1 = 163$ and recalculate the following table:

| Index | i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ...
|-------|---|---|---|---|---|---|---|---|---|---|...
| Squared prime $p^2[i]$ | 4 | 169 | 9 | 25 | 49 | 121 | 289 | 361 | 529 | ...
| Next non-square-free $nsqf[i]$ | 164 | 169 | 171 | 175 | 196 | 242 | 289 | 361 | 529 | ...

This cuts down on the number of non-square-free that have to be tested and the speed of the calculation is increased. A factor five in speed was obtained when this was implemented in the program which was used to find $Qgap(14)$ and $Qgap(15)$.

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\(\text{18}\)\url{http://oeis.org/A107079}
2.4.5 Improvement III

"The smallest squared-primes are not needed to calculate the sieve."

This variation on Improvement II was suggested by Joseph Wetherell. It turns out to be more efficient when it is combined with Improvements IV and V. The trick is to consider the smallest squared-primes separately from the large squared-primes. Most of the time in the algorithm is spent on the process of taking the smallest elements off of \( n_{sqf} \) and sorting them back into the array. If one can reduce the number of non-square-free numbers which are tested, the speed of the algorithm will improve.

This is actually possible since the smallest squared-primes are not needed to calculate the sieve. If we have a gap of, say, \( L = L_{min} = 7 \) non-square-free numbers, then at least \( N P_{2_{min}}[7] = 6 \) different primes are found in the gap. This means that the smallest \( k_1 = N P_{2_{min}}[L_{min}] - 1 = 5 \) squared-primes can be left out of the calculation. If we search for the 6th squared-prime, every gap of length \( L = 7 \) (or more) will be found.

This method therefore separates the small squared-primes from the large ones, creating a base with \( k_1 \) elements. The table for \( N = 163 \) would now look like this:

| Index | \( i \) | 0 | 1 | 2 | 3 | 4 | \( k_1 \) | 6 | 7 | 8 | ...
|-------|--------|---|---|---|---|---|--------|---|---|---|-------|
| Squared prime | \( p_2[i] \) | 4 | 9 | 25 | 49 | 121 | 169 | 289 | 361 | 529 | ...
| Next non-square-free | \( n_{sqf}[i] \) | -- | -- | -- | -- | -- | 169 | 289 | 361 | 529 | ...

with the base shown as -- for the values of \( n_{sqf}[i] \). From the table, one sees that one must test for a gap having \( L_{min} = 7 \) around \( N = 169 \). If none is found, then \( n_{sqf}[k_1] \) is increased by 169 and sorted back into the array of large squared-primes to get:

| Index | \( i \) | 0 | 1 | 2 | 3 | 4 | \( k_1 \) | 6 | 7 | 8 | ...
|-------|--------|---|---|---|---|---|--------|---|---|---|-------|
| Squared prime | \( p_2[i] \) | 4 | 9 | 25 | 49 | 121 | 289 | 169 | 361 | 529 | ...
| Next non-square-free | \( n_{sqf}[i] \) | -- | -- | -- | -- | -- | 289 | 338 | 361 | 529 | ...

The sort is faster since it is only done on the large squared-primes. The process is then continued at \( N = 289 \).

2.4.6 Improvement IV

"The values of the small modulos can be computed ahead of time."

To test if there is a gap of \( L_{min} \) around \( N \), one must still know about multiples of the \( k_1 \) small squared-primes near \( N \). As suggested by Joseph Wetherell, this can be done by trial division; even if trial division is slow, it is faster than resorting the base array. One can also optimize the trial divisions, because the trial divisions for, say, \( N + 1 \) and \( N + 2 \) are related to each other. For each small prime \( p \), compute \( N \% p_2 \) and store it in a list \( \text{mod} \) (the "%" symbol is the modulo function in the C language). Now to see if \( p_2 \) divides \( N + 1 \), we just test if this stored value is \(-1 \mod p_2\). To see if \( p_2 \) divides \( N + 2 \), we test if this stored value is \(-2 \mod p_2\). (We also precompute the value of \(-1 \mod p_2\), \(-2 \mod p_2\), etc., for the small set of values which we will possibly need to test.) Note that it is also necessary to test \( N - 1, N - 2, \) etc. For \( N = 289 \), we have the following arrays:

| Index | \( i \) | 0 | 1 | 2 | 3 | 4 | \( k_1 \) | 6 | 7 | 8 | ...
|-------|--------|---|---|---|---|---|--------|---|---|---|-------|
| Squared prime | \( p_2[i] \) | 4 | 9 | 25 | 49 | 121 | 289 | 169 | 361 | 529 | ...
| Next non-square-free | \( n_{sqf}[i] \) | -- | -- | -- | -- | -- | 289 | 338 | 361 | 529 | ...
| \( N \% p_2[i] \) | \( \text{mod}[i] \) | 1 | 1 | 14 | 44 | 47 | ... |
We see that \( p_2[0] \) and \( p_2[1] \) divide \( N - 1 = 288 \), but this is the only other non-square-free number for this gap of length 2. We can therefore increase \( \text{nsqf}[k1] \) by 289, sort it back into the array (reordering \( p_2 \) accordingly) and continue with \( N = 338 \):

| Index | i   | 0  | 1  | 2  | 3  | 4  | k1 | 6  | 7  | 8  | ...
|-------|-----|----|----|----|----|----|----|----|----|----|-------
| Squared prime | \( p_2[i] \) | 4  | 9  | 25 | 49 | 121 | 169 | 361 | 529 | 289 | ...
| Next non-square-free | \( \text{nsqf}[i] \) | -- | -- | -- | -- | -- | -- | 338 | 361 | 529 | 578 | ...
| \( N \% p_2[i] \) | \( \text{mod}[i] \) | 2  | 5  | 13 | 44 | 96 |

2.4.7 Improvement V

"Two large primes-squared that are too far cannot result in a gap."

If we include \( p_2[4] \) in the array of large squared-primes, (so there are only \( k2 = \text{NP2}_{min}[L_{min}] - 2 = 4 \) elements in the base), then we know that a number \( N = \text{nsqf}[k2] \) we are testing can be part of a gap only if the next number in \( \text{nsqf} \) is close to \( N \), that is, if \( \text{nsqf}[k2+1] \) is not larger than \( N + L_{min} - 1 \). Based on this suggestion by Joseph Wetherell, the arrays become with \( N = 338 \):

| Index | i   | 0  | 1  | 2  | 3  | 4  | k2 | 5  | 6  | 7  | 8  | ...
|-------|-----|----|----|----|----|----|----|----|----|----|----|-------
| Squared prime | \( p_2[i] \) | 4  | 9  | 25 | 49 | 169 | 361 | 121 | 529 | 289 | ...
| Next non-square-free | \( \text{nsqf}[i] \) | -- | -- | -- | -- | -- | 338 | 361 | 363 | 529 | 578 | ...
| \( N \% p_2[i] \) | \( \text{mod}[i] \) | -- | -- | -- | -- | -- |

Since \( \text{nsqf}[k2+1] = 361 \) is larger than \( N + L_{min} - 1 = 344 \), we can skip to \( N = 361 \). Since \( N \) does not pass the closeness test, we do not have to do any computations with the base, saving us a lot of time.

2.4.8 Improvement VI

"A chained list is faster for the sort."

A chained list can be built with a set of numbers that specify an order for the elements of an array, as suggested by Joseph Wetherell. If we use a chained list represented by the array called \( \text{next} \) such that \( \text{nsqf}[\text{next}[i]] \geq \text{nsqf}[i] \), we can sort the array \( \text{nsqf} \) without moving any data within the arrays \( \text{nsqf} \) or \( p_2 \). For a reason that will become obvious later, we choose to have \( p_2 \) sorted in increasing order. With \( N = 361 \), the arrays would be:

| Index | i   | 0  | 1  | 2  | 3  | 4  | k2 | 5  | 6  | 7  | 8  | ...
|-------|-----|----|----|----|----|----|----|----|----|----|----|-------
| Chained list | \( \text{next}[i] \) | -- | -- | -- | -- | k2 | 5  | 8  | 9  | 4  | 6  | ...
| Squared prime | \( p_2[i] \) | 4  | 9  | 25 | 49 | 121 | 169 | 289 | 361 | 529 | ...
| Next non-square-free | \( \text{nsqf}[i] \) | -- | -- | -- | -- | 363 | 507 | 578 | 361 | 529 | ...
| \( N \% p_2[i] \) | \( \text{mod}[i] \) | 1  | 1  | 11 | 18 |

We use an additional variable, \( \text{head} \), which points to the smallest item in the array \( \text{nsqf} \). For the table above, we have \( \text{head} = 7 \) (\( \text{next[head]} \) is highlighted). Since we only need to change two values in the array next to perform a sort, this method is faster. Searching through the array \( \text{nsqf} \) now consists of going through the data in the following order:

\[
\text{for (i=head; tempnsqf>=nsqf[\text{next}[i]]; i=\text{next}[i])...}
\]
There is another advantage to this method: since the array \( p2 \) is sorted in increasing order, we know that if we have to sort item \( nsqf[\text{head}] \), then \( nsqf[\text{head}-1] \leq nsqf[\text{head}] + p2[\text{head}] \). This means we can jump immediately to \( \text{head}-1 \) and start searching from there. In general, this cuts the search in half! Searching through the list now consists of:

for (\( \text{head}-1; \text{tempnsqf} = nsqf[\text{head}] + p2[\text{head}]; \text{head} = \text{head}-1 \) ) ...

We continue the example with the above table. Since \( nsqf[\text{next[head]}] \) = 363 is not larger than \( N + L_{\text{min}} - 1 = 367 \), we calculated the modulus. Clearly, there is only a gap of length \( L = 2 \) starting at 360. We therefore set \( N = 363 \) and sort \( \text{tempnsqf} = nsqf[\text{head}] = nsqf[\text{head}] + p2[\text{head}] = 722 \). We start with \( \text{head} = 6 \) and the sort requires only one comparison! The following arrays are obtained:

| Index | \( i \) | 0 | 1 | 2 | 3 | 5 | 6 | 7 | 8 |
|-------|--------|---|---|---|---|---|---|---|---|
| Chained list | next[\( i \)] | -- | -- | -- | -- | \( k2 \) | 5 | 8 | 7 | 9 | 6 | ... |
| Squared prime | \( p2[i] \) | 4 | 9 | 25 | 49 | 121 | 169 | 289 | 361 | 529 | ... |
| Next non-square-free | \( nsqf[\( i \)] \) | -- | -- | -- | -- | 363 | 507 | 578 | 722 | 529 | ... |
| \( N \% p2[i] \) | mod[\( i \)] | -- | -- | -- | -- | ... |

with \( \text{head} \) now equal to 4.

A factor three in speed was obtained when this algorithm was implemented in the program.

### 2.4.9 Improvement VII

“Look for the largest spacing between two of three large squared-primes.”

Instead of having \( k2 \) elements in the base and look for two large squared-primes that are not too far apart, we can use a base with \( k3 = NP_{2_{\text{min}}}[L_{\text{min}}] - 3 = 3 \) squared-primes, but look to see if \( nsqf[\text{head}] \) and \( nsqf[\text{next[next[head]]}] \) are not more than \( L_{\text{min}} \) apart.

| Index | \( i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 6 | ... |
|-------|--------|---|---|---|---|---|---|---|---|---|---|---|---|
| Chained list | next[\( i \)] | -- | -- | -- | -- | \( k3 \) | 5 | 8 | 7 | 9 | 6 | ... |
| Squared prime | \( p2[i] \) | 4 | 9 | 25 | 49 | 121 | 169 | 289 | 361 | 529 | ... |
| Next non-square-free | \( nsqf[\( i \)] \) | -- | -- | -- | -- | 392 | 363 | 507 | 578 | 722 | 529 | ... |
| \( N \% p2[i] \) | mod[\( i \)] | -- | -- | -- | -- | ... |

In this example, we look at the difference between \( nsqf[\text{head}] = 363 \) and \( nsqf[\text{next[next[head]]}] = 507 \). Since the difference between the two values is larger than \( L_{\text{min}} - 1 \), there is no gap of length \( L_{\text{min}} \) or longer.

This improvement gives the program a 37% speed increase with \( L_{\text{min}} = 14 \).

The program becomes slower if four or more large squared-primes are considered (this was confirmed in tests for \( L > 13, N = 10^{14} \) to \( 10^{14} + 10^{9} \)).

### 2.4.10 Evaluation of order of algorithm

This evaluation applies to the first improvement of the algorithm. It was found empirically that for a given value of \( L_{\text{min}} \), the other improvements increased the speed of the calculation by a constant factor.

To calculate if \( N \) is a square-free number, the algorithm takes advantage of the known remainders for \( N - 1 \). Each time \( N \) is tested, a new non-square-free number is calculated using \( nsqf[0] = nsqf[0] + p2[0] \). This new value has to be moved up the list to keep \( nsqf \) in increasing order. It is that operation that requires most
of the computation time. To evaluate the speed of the algorithm, it is necessary to find the average number of moves \( m \) that will be required to bring the new value \( \text{nsqf}[0] \) to its correct position in the list, above the number \( \text{nsqf}[m] \). This is done by evaluating, for every \( i \), the probability that \( \text{nsqf}[0] > \text{nsqf}[i] \), and then summing over \( i \).

First, consider the case when \( p^2[0] = 4 \) which occurs with a probability of \( 1/4 \). Since \( \text{nsqf}[0] \) has been increased by 4, there is a probability of \( 4/9 \) that it will have to be moved above \( \text{nsqf}[j] \) (if \( p^2[j] = 9 \)). There is an additional probability of \( 4/25 \) that \( \text{nsqf}[0] \) will have to be moved above \( \text{nsqf}[k] \) (if \( p^2[k] = 25 \)), etc. We therefore get the average number of steps required to place the new \( \text{nsqf}[0] \) to its correct position:

\[
S(1) = 1/4 \times (4/9 + 4/25 + 4/49 + 4/121 + ...) = \sum_{i=2}^{\infty} \frac{1}{p^2(i)}
\]

where \( p(i) \) is the ith prime number (\( p(1) = 2, p(2) = 3, p(3) = 5, \) etc.) and \( p^2(i) = p(i) \times p(i) \).

In the case when \( p^2[0] = 9 \) (which occurs with a probability of \( 1/9 \)), one move is always necessary to bring \( \text{nsqf}[0] \) above \( \text{nsqf}[j] \) (if \( p^2[j] = 4 \)). There is a probability of \( 9/25 \) that \( \text{nsqf}[0] \) will have to be moved above \( \text{nsqf}[k] \) (if \( p^2[k] = 25 \)), etc. We therefore get:

\[
S(2) = 1/9 \times (1 + 9/25 + 9/49 + 9/121 + ...) = \frac{1}{p^2(2)} + \sum_{i=3}^{\infty} \frac{1}{p^2(i)}
\]

In general, when \( p^2[0] = p^2(m) \), the average number of moves is:

\[
S(m) = \frac{m-1}{p^2(m)} + \sum_{i=m+1}^{\infty} \frac{1}{p^2(i)}
\]

The sum over all the \( S(m) \) gives the average number of moves required to place \( \text{nsqf}[0] \) to its correct position in the list:

\[
\sum_{m=1}^{\infty} S(m) = 2 \times \sum_{i=2}^{\infty} \frac{i-1}{p^2(i)}
\]

This series converges, as determined with a convergence test\(^{19}\). It converges very slowly to approximately 1.30... Therefore, given the remainders for \( N-1 \), the number of operations required to find out if \( N \) is square-free is independent of the value of \( N \).

### 2.5 Acknowledgements

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Contributed to find \( Qgap(16) \): Zach McGregor-Dorsey, Louis Marmet, Joe Wetherell, Gunnard Engebret, D. Bernier, Erick Wong, Alan Simpson and Nicolas Marmet.

\(^{19}\)Suppose that \( f(x) \) is a positive decreasing function and that

\[
\lim_{k \to \infty} \frac{e^k f(e^k)}{f(k)} = q
\]

for natural \( k \). If \( q < 1 \), the series \( \sum_{k=1}^{\infty} f(k) \) converges. If \( q > 1 \), this series diverges. (Ermakov)” Equation 0.224, “Table of Integrals, Series, and Products,” Gradshteyn and Ryzhik (Academic Press, Inc., p. 5)
Contributed to find Qgap(17): E. Wong, Z. McGregor-Dorsey, L. Marmet, Jean-Pierre Bernier, D. Bernier, Nancy Robertson, N. Marmet, Charles Ward and G. Engebreth.

Contributed to find Qgap(18): D. Bernier, L. Marmet, E. Wong, J. Wetherell, Z. McGregor-Dorsey, G. Engebreth, A. Simpson, N. Marmet, N. Robertson, J.-P. Bernier, C.R. Ward, Bruno Le Tual and Horand Gassmann.

This project started from an idea that was initially suggested to me by David Bernier.

2.6 Related web pages and references

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A translation of this paper is available in Belorussian at www.webhostinghub.com/support/by/edu/index-marmet-be.

This article was first published at www.marmet.org/louis/sqfgap/index.html.

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Appendix A

The following graph shows an estimation of how large we can expect $Q_{gap}(L)$ to be.

![Graph of $Q_{gap}(L)$ for L=1 to 25](image)

Figure 1: Known values of $Q_{gap}(L)$ (squares) and estimated values (empty circles).

The empirical estimation, based on the calculated values for $L < 16$, uses an approximation of the probability of obtaining the minimum number of primes required to produce a gap with length $L$. The upper limits for $Q_{gap}(L > 16)$ were obtained by E. Wong. The values of $Q_{gap}(L)$ lie within the ranges indicated by the gray lines.
Appendix D

Square-free gaps and their length ≥ 16, up to 125,870,000,000,000,000

| Gap Length | Initial Gap | Length | Gap Length | Final Gap | Length |
|------------|------------|--------|------------|-----------|--------|
| 16         | 3215226335143218: | 16    | 50374378394286240: | 17    |
| 17         | 23742453640900972: | 17    | 52806946967186660: | 17    |
| 16         | 28696958943616635: | 16    | 55039310568335610: | 16    |
| 16         | 31401976920688950: | 16    | 58042999008997036: | 17    |
| 17         | 36985881099122836: | 17    | 58511100456350360: | 17    |
| 16         | 46717595829767167: | 16    | 63057303299988150: | 16    |
| 16         | 48772582754041310: | 16    | 64287162889072035: | 16    |
| 16         | 49428341049041863: | 16    | 65191494685146343: | 16    |
| 17         | 50011847799468448: | 17    | 79132612264348838: | 16    |
Appendix E

Completed ranges computed on 44 different processors by 14 different users.

| User Name          | (Computer Name)            | Range             | DateCompleted       |
|--------------------|----------------------------|-------------------|---------------------|
| L. Marmet          | (neurone5 Linux .8GHz)     | $125.7 \times 10^{15}$ to $125.8 \times 10^{15}$ | September 12<sup>th</sup>, 2005 |
| H. Gassmann        | (Gus’s beast)              | $125.3 \times 10^{15}$ to $125.7 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| L. Marmet          | (neurone3 .8GHz)           | $125.1 \times 10^{15}$ to $125.3 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.51)               | $124.7 \times 10^{15}$ to $125.1 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.35)               | $124.4 \times 10^{15}$ to $124.7 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.141)              | $124.1 \times 10^{15}$ to $124.4 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.142.Brain1)       | $123.8 \times 10^{15}$ to $124.1 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| N. Robertson       | (Laika.Droica)             | $123.4 \times 10^{15}$ to $123.8 \times 10^{15}$ | August 20<sup>th</sup>, 2005 |
| L. Marmet          | (neurone2 .8GHz)           | $122.7 \times 10^{15}$ to $123.4 \times 10^{15}$ | August 10<sup>th</sup>, 2005 |
| L. Marmet          | (neurone1 .8GHz)           | $122.1 \times 10^{15}$ to $122.7 \times 10^{15}$ | August 7<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.94)               | $121.6 \times 10^{15}$ to $122.1 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.63)               | $121.3 \times 10^{15}$ to $121.6 \times 10^{15}$ | August 19<sup>th</sup>, 2005 |
| L. Marmet          | (neurone5 Linux .8GHz)     | $120.9 \times 10^{15}$ to $121.3 \times 10^{15}$ | July 26<sup>th</sup>, 2005 |
| N. Robertson       | (Laika.Droica)             | $120.5 \times 10^{15}$ to $120.9 \times 10^{15}$ | July 16<sup>th</sup>, 2005 |
| L. Marmet          | (neurone4 .8GHz)           | $119.3 \times 10^{15}$ to $120.5 \times 10^{15}$ | August 13<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.142.Brain2)       | $118.0 \times 10^{15}$ to $119.3 \times 10^{15}$ | September 8<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.107)              | $116.3 \times 10^{15}$ to $118.0 \times 10^{15}$ | September 2<sup>nd</sup>, 2005 |
| N. Marmet          | (Poisson.31)               | $115.7 \times 10^{15}$ to $116.3 \times 10^{15}$ | August 29<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.30)               | $114.9 \times 10^{15}$ to $115.7 \times 10^{15}$ | August 28<sup>th</sup>, 2005 |
| J.-P. Bernier      | (Pentium 600MHz)           | $114.0 \times 10^{15}$ to $114.9 \times 10^{15}$ | August 24<sup>th</sup>, 2005 |
| L. Marmet          | (neurone0 W95 .233GHz)     | $113.5 \times 10^{15}$ to $114.0 \times 10^{15}$ | August 16<sup>th</sup>, 2005 |
| L. Marmet          | (neurone5 Linux .8GHz)     | $113.1 \times 10^{15}$ to $113.5 \times 10^{15}$ | June 30<sup>th</sup>, 2005 |
| N. Robertson       | (Laika.Droica)             | $112.7 \times 10^{15}$ to $113.1 \times 10^{15}$ | June 24<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.29)               | $111.5 \times 10^{15}$ to $112.7 \times 10^{15}$ | August 28<sup>th</sup>, 2005 |
| H. Gassmann        | (Gus’s beast)              | $111.0 \times 10^{15}$ to $111.5 \times 10^{15}$ | July 27<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.44)               | $110.4 \times 10^{15}$ to $111.0 \times 10^{15}$ | August 24<sup>th</sup>, 2005 |
| N. Marmet          | (Poisson.37)               | $109.6 \times 10^{15}$ to $110.4 \times 10^{15}$ | September 1<sup>st</sup>, 2005 |
| N. Marmet          | (Poisson.34)               | $108.8 \times 10^{15}$ to $109.6 \times 10^{15}$ | September 3<sup>rd</sup>, 2005 |
| J.-P. Bernier      | (Athlon 2000)              | $106.3 \times 10^{15}$ to $108.8 \times 10^{15}$ | August 5<sup>th</sup>, 2005 |
| B. Le Tual         | (Celeron 2.4GHz)           | $105.3 \times 10^{15}$ to $106.3 \times 10^{15}$ | July 16<sup>th</sup>, 2005 |
| User Name     | (Computer Name)                     | Range            | Date Completed |
|--------------|-------------------------------------|------------------|----------------|
| B. Le Tual   | (Celeron .9GHz)                     | $104.9 \times 10^{15}$ to $105.3 \times 10^{15}$ | July 16th, 2005 |
| L. Marmet    | (neurone2 .8GHz)                    | $103.2 \times 10^{15}$ to $104.9 \times 10^{15}$ | July 8th, 2005  |
| B. Le Tual   | (Celeron 2.4GHz)                    | $102.7 \times 10^{15}$ to $103.2 \times 10^{15}$ | May 13th, 2005  |
| L. Marmet    | (neurone4 .8GHz)                    | $101.1 \times 10^{15}$ to $102.7 \times 10^{15}$ | June 25th, 2005 |
| L. Marmet    | (neurone1 .8GHz)                    | $99.5 \times 10^{15}$ to $101.1 \times 10^{15}$ | July 2nd, 2005  |
| L. Marmet    | (Poisson.63)                        | $98.8 \times 10^{15}$ to $99.5 \times 10^{15}$ | July 12th, 2005 |
| H. Gassmann  | (Gus’s beast)                       | $98.4 \times 10^{15}$ to $98.8 \times 10^{15}$ | June 1st, 2005  |
| N. Robertson | (Laika.Droica)                      | $98.0 \times 10^{15}$ to $98.4 \times 10^{15}$ | May 9th, 2005   |
| J.-P. Bernier| (Athlon 2000)                       | $96.2 \times 10^{15}$ to $98.0 \times 10^{15}$ | May 24th, 2005  |
| L. Marmet    | (neurone0 W95 .233GHz)              | $95.7 \times 10^{15}$ to $96.2 \times 10^{15}$ | June 10th, 2005 |
| N. Marmet    | (Poisson.142.Brain2)                | $94.2 \times 10^{15}$ to $95.7 \times 10^{15}$ | June 13th, 2005 |
| N. Marmet    | (Poisson.94)                        | $92.7 \times 10^{15}$ to $94.2 \times 10^{15}$ | July 12th, 2005 |
| J.-P. Bernier| (Pentium 600MHz)                    | $91.7 \times 10^{15}$ to $92.7 \times 10^{15}$ | June 12th, 2005 |
| N. Marmet    | (Poisson.142.Brain1)                | $89.7 \times 10^{15}$ to $91.7 \times 10^{15}$ | July 21st, 2005 |
| N. Marmet    | (Poisson.34)                        | $88.9 \times 10^{15}$ to $89.7 \times 10^{15}$ | May 30th, 2005  |
| N. Marmet    | (Poisson.107)                       | $86.8 \times 10^{15}$ to $88.9 \times 10^{15}$ | June 13th, 2005 |
| N. Marmet    | (Poisson.29)                        | $85.5 \times 10^{15}$ to $86.8 \times 10^{15}$ | June 2nd, 2005  |
| N. Marmet    | (Poisson.31)                        | $84.7 \times 10^{15}$ to $85.5 \times 10^{15}$ | June 13th, 2005 |
| N. Marmet    | (Poisson.51)                        | $82.4 \times 10^{15}$ to $84.7 \times 10^{15}$ | July 25th, 2005 |
| N. Marmet    | (Poisson.30)                        | $81.4 \times 10^{15}$ to $82.4 \times 10^{15}$ | May 13th, 2005  |
| N. Marmet    | (Poisson.45)                        | $80.8 \times 10^{15}$ to $81.4 \times 10^{15}$ | May 30th, 2005  |
| N. Marmet    | (Poisson.37)                        | $80.0 \times 10^{15}$ to $80.8 \times 10^{15}$ | May 30th, 2005  |
| L. Marmet    | (neurone5 Linux .8GHz)              | $79.2 \times 10^{15}$ to $80.0 \times 10^{15}$ | June 9th, 2005  |
| L. Marmet    | (neurone5 Linux .8GHz)              | $78.8 \times 10^{15}$ to $79.2 \times 10^{15}$ | April 21st, 2005|
| J.-P. Bernier| (Athlon 2000)                       | $78.1 \times 10^{15}$ to $78.8 \times 10^{15}$ | April 9th, 2005 |
| N. Marmet    | (Poisson.94)                        | $77.6 \times 10^{15}$ to $78.1 \times 10^{15}$ | April 5th, 2005 |
| N. Marmet    | (Poisson.142.Brain2)                | $77.1 \times 10^{15}$ to $77.6 \times 10^{15}$ | April 5th, 2005 |
| N. Marmet    | (Poisson.142.Brain1)                | $76.6 \times 10^{15}$ to $77.1 \times 10^{15}$ | March 31st, 2005|
| N. Marmet    | (Poisson.107)                       | $76.1 \times 10^{15}$ to $76.6 \times 10^{15}$ | March 30th, 2005|
| N. Marmet    | (Poisson.51)                        | $75.6 \times 10^{15}$ to $76.1 \times 10^{15}$ | March 29th, 2005|
| N. Marmet    | (Poisson.63)                        | $75.1 \times 10^{15}$ to $75.6 \times 10^{15}$ | April 26th, 2005|
| N. Marmet    | (Poisson.30)                        | $74.9 \times 10^{15}$ to $75.1 \times 10^{15}$ | March 28th, 2005|
| N. Marmet    | (Poisson.31)                        | $74.7 \times 10^{15}$ to $74.9 \times 10^{15}$ | March 30th, 2005|
| N. Marmet    | (Poisson.34)                        | $74.5 \times 10^{15}$ to $74.7 \times 10^{15}$ | March 30th, 2005|
| N. Marmet    | (Poisson.37)                        | $74.4 \times 10^{15}$ to $74.5 \times 10^{15}$ | March 23rd, 2005|
| N. Marmet    | (Poisson.45)                        | $74.3 \times 10^{15}$ to $74.4 \times 10^{15}$ | March 25th, 2005|
| N. Robertson | (Laika.Droica)                      | $73.8 \times 10^{15}$ to $74.3 \times 10^{15}$ | April 5th, 2005 |
| H. Gassmann  | (Gus’s beast)                       | $73.5 \times 10^{15}$ to $73.8 \times 10^{15}$ | April 27th, 2005|
| L. Marmet    | (neurone4 .8GHz)                    | $71.9 \times 10^{15}$ to $73.5 \times 10^{15}$ | April 30th, 2005|
| L. Marmet    | (neurone2 .8GHz)                    | $70.2 \times 10^{15}$ to $71.9 \times 10^{15}$ | May 4th, 2005   |
| User Name   | (Computer Name)                  | Range                  | Date Completed |
|-------------|----------------------------------|------------------------|----------------|
| L. Marmet   | (neurone1 .8GHz)                 | $6.8 \times 10^{15}$ to $7.0 \times 10^{15}$ | April 28<sup>th</sup>, 2005 |
| J.-P. Bernier | (Pentium 600MHz)               | $6.8 \times 10^{15}$ to $6.8 \times 10^{15}$ | April 3<sup>rd</sup>, 2005 |
| B. Le Tual  | (Celeron .9GHz)                  | $6.8 \times 10^{15}$ to $6.8 \times 10^{15}$ | May 15<sup>th</sup>, 2005 |
| B. Le Tual  | (Celeron 2.4GHz)                 | $6.7 \times 10^{15}$ to $6.8 \times 10^{15}$ | May 1<sup>st</sup>, 2005 |
| J.-P. Bernier | (Athlon 2000)                   | $6.7 \times 10^{15}$ to $6.8 \times 10^{15}$ | March 17<sup>th</sup>, 2005 |
| N. Robertson | (Poisson.29)                   | $6.6 \times 10^{15}$ to $6.7 \times 10^{15}$ | March 30<sup>th</sup>, 2005 |
| J.-P. Bernier | (Athlon 2000)                   | $6.6 \times 10^{15}$ to $6.6 \times 10^{15}$ | January 1<sup>st</sup>, 2004 |
| J.-P. Bernier | (Pentium 600MHz)               | $6.6 \times 10^{15}$ to $6.6 \times 10^{15}$ | January 12<sup>th</sup>, 2004 |
| B. Le Tual  | (Celeron 2.4GHz)                 | $6.5 \times 10^{15}$ to $6.6 \times 10^{15}$ | January 4<sup>th</sup>, 2004 |
| B. Le Tual  | (Celeron .9GHz)                  | $6.5 \times 10^{15}$ to $6.5 \times 10^{15}$ | January 9<sup>th</sup>, 2004 |
| J.-P. Bernier | (Athlon 2000)                   | $6.5 \times 10^{15}$ to $6.5 \times 10^{15}$ | December 19<sup>th</sup>, 2003 |
| J.-P. Bernier | (Pentium 600MHz)               | $6.4 \times 10^{15}$ to $6.5 \times 10^{15}$ | December 14<sup>th</sup>, 2003 |
| J.-P. Bernier | (Athlon 2000)                   | $6.4 \times 10^{15}$ to $6.4 \times 10^{15}$ | December 4<sup>th</sup>, 2003 |
| B. Le Tual  | (Celeron 2.4GHz)                 | $6.4 \times 10^{15}$ to $6.4 \times 10^{15}$ | December 9<sup>th</sup>, 2003 |
| L. Marmet   | (neurone0 W95 .233GHz)           | $6.3 \times 10^{15}$ to $6.4 \times 10^{15}$ | April 11<sup>th</sup>, 2005 |
| J.-P. Bernier | (Athlon 2000)                   | $6.3 \times 10^{15}$ to $6.3 \times 10^{15}$ | November 13<sup>th</sup>, 2003 |
| B. Le Tual  | (Celeron .9GHz)                  | $6.3 \times 10^{15}$ to $6.3 \times 10^{15}$ | December 5<sup>th</sup>, 2003 |
| J.-P. Bernier | (Pentium 600MHz)               | $6.3 \times 10^{15}$ to $6.3 \times 10^{15}$ | November 24<sup>th</sup>, 2003 |
| D. Bernier  | (Gecko)                         | $6.2 \times 10^{15}$ to $6.2 \times 10^{15}$ | December 27<sup>th</sup>, 2003 |
| J.-P. Bernier | (Athlon 2000)                   | $6.2 \times 10^{15}$ to $6.2 \times 10^{15}$ | November 1<sup>st</sup>, 2003 |
| N. Robertson | (Laika.Droica)                  | $6.1 \times 10^{15}$ to $6.2 \times 10^{15}$ | March 9<sup>th</sup>, 2005 |
| L. Marmet   | (neurone5 Linux .8GHz)           | $6.1 \times 10^{15}$ to $6.1 \times 10^{15}$ | March 27<sup>th</sup>, 2005 |
| J.-P. Bernier | (Athlon 2000)                   | $6.0 \times 10^{15}$ to $6.1 \times 10^{15}$ | October 15<sup>th</sup>, 2003 |
| L. Marmet   | (Riyadh)                        | $6.0 \times 10^{15}$ to $6.0 \times 10^{15}$ | November 9<sup>th</sup>, 2003 |
| J.-P. Bernier | (Pentium 600MHz)               | $6.0 \times 10^{15}$ to $6.0 \times 10^{15}$ | October 31<sup>st</sup>, 2003 |
| B. Le Tual  | (Celeron 2.4GHz)                 | $6.0 \times 10^{15}$ to $6.0 \times 10^{15}$ | November 12<sup>th</sup>, 2003 |
| B. Le Tual  | (Celeron .9GHz)                  | $5.9 \times 10^{15}$ to $6.0 \times 10^{15}$ | November 1<sup>st</sup>, 2003 |
| N. Robertson | (Laika.Droica)                  | $5.9 \times 10^{15}$ to $5.9 \times 10^{15}$ | October 1<sup>st</sup>, 2003 |
| B. Le Tual  | (Celeron .9GHz)                  | $5.8 \times 10^{15}$ to $5.9 \times 10^{15}$ | September 28<sup>th</sup>, 2003 |
| B. Le Tual  | (Celeron 2.4GHz)                 | $5.8 \times 10^{15}$ to $5.8 \times 10^{15}$ | October 5<sup>th</sup>, 2003 |
| J.-P. Bernier | (Pentium 600MHz)               | $5.8 \times 10^{15}$ to $5.8 \times 10^{15}$ | October 7<sup>th</sup>, 2003 |
| L. Marmet   | (Riyadh)                        | $5.8 \times 10^{15}$ to $5.8 \times 10^{15}$ | October 7<sup>th</sup>, 2003 |
| D. Bernier  | (Gecko)                         | $5.7 \times 10^{15}$ to $5.8 \times 10^{15}$ | October 29<sup>th</sup>, 2003 |
| N. Robertson | (Maya)                          | $5.7 \times 10^{15}$ to $5.7 \times 10^{15}$ | September 19<sup>th</sup>, 2003 |
| N. Robertson | (Laika.Droica)                  | $5.7 \times 10^{15}$ to $5.7 \times 10^{15}$ | September 10<sup>th</sup>, 2003 |
| J.-P. Bernier | (Pentium 600MHz)               | $5.6 \times 10^{15}$ to $5.7 \times 10^{15}$ | September 10<sup>th</sup>, 2003 |
| N. Robertson | (Maya)                          | $5.6 \times 10^{15}$ to $5.6 \times 10^{15}$ | August 4<sup>th</sup>, 2003 |
| N. Robertson | (Laika.Droica)                  | $5.6 \times 10^{15}$ to $5.6 \times 10^{15}$ | July 29<sup>th</sup>, 2003 |
| L. Marmet   | (Riyadh)                        | $5.5 \times 10^{15}$ to $5.6 \times 10^{15}$ | August 13<sup>th</sup>, 2003 |
| J.-P. Bernier | (Pentium 600MHz)               | $5.5 \times 10^{15}$ to $5.5 \times 10^{15}$ | August 4<sup>th</sup>, 2003 |
| User Name       | (Computer Name)          | Range               | Date Completed |
|----------------|--------------------------|---------------------|---------------|
| J.-P. Bernier  | (Pentium 600MHz)         | 5.53 x 10^15 to 5.56 x 10^15 | July 10th, 2003 |
| J.-P. Bernier  | (Pentium 600MHz)         | 5.50 x 10^15 to 5.53 x 10^15 | June 15th, 2003 |
| L. Marmet      | (Riyadh)                 | 5.47 x 10^15 to 5.50 x 10^15 | June 28th, 2003 |
| N. Robertson   | (Maya)                   | 5.44 x 10^15 to 5.47 x 10^15 | June 10th, 2003 |
| N. Robertson   | (Laika.Droica)           | 5.41 x 10^15 to 5.44 x 10^15 | July 9th, 2003  |
| J.-P. Bernier  | (Pentium 600MHz)         | 5.38 x 10^15 to 5.41 x 10^15 | May 14th, 2003  |
| D. Bernier     | (Gecko)                  | 5.33 x 10^15 to 5.38 x 10^15 | August 23rd, 2003 |
| N. Robertson   | (Maya)                   | 5.30 x 10^15 to 5.33 x 10^15 | May 12th, 2003  |
| N. Robertson   | (Laika.Droica)           | 5.27 x 10^15 to 5.30 x 10^15 | May 7th, 2003   |
| L. Marmet      | (Riyadh)                 | 5.24 x 10^15 to 5.27 x 10^15 | May 16th, 2003  |
| J.-P. Bernier  | (Pentium 600MHz)         | 5.21 x 10^15 to 5.24 x 10^15 | April 23rd, 2003 |
| N. Robertson   | (Maya)                   | 5.18 x 10^15 to 5.21 x 10^15 | March 31st, 2003 |
| L. Marmet      | (Riyadh)                 | 5.15 x 10^15 to 5.18 x 10^15 | April 6th, 2003 |
| N. Robertson   | (Laika.Droica)           | 5.12 x 10^15 to 5.15 x 10^15 | March 22nd, 2003 |
| D. Bernier     | (Gecko)                  | 5.11 x 10^15 to 5.12 x 10^15 | April 20th, 2003|
| D. Bernier     | (Gecko)                  | 5.07 x 10^15 to 5.11 x 10^15 | April 2nd, 2003 |
| J.-P. Bernier  | (Pentium 600MHz)         | 5.04 x 10^15 to 5.07 x 10^15 | March 24th, 2003|
| D. Bernier     | (Gecko)                  | 5.01 x 10^15 to 5.04 x 10^15 | February 26th, 2003 |
| L. Marmet      | (Riyadh)                 | 4.98 x 10^15 to 5.01 x 10^15 | March 9th, 2003 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.95 x 10^15 to 4.98 x 10^15 | February 25th, 2003 |
| L. Marmet      | (Riyadh)                 | 4.92 x 10^15 to 4.95 x 10^15 | January 31st, 2003 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.90 x 10^15 to 4.92 x 10^15 | January 25th, 2003 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.88 x 10^15 to 4.90 x 10^15 | December 26th, 2002 |
| L. Marmet      | (Riyadh)                 | 4.85 x 10^15 to 4.88 x 10^15 | December 22nd, 2002 |
| N. Robertson   | (Maya)                   | 4.82 x 10^15 to 4.85 x 10^15 | November 28th, 2002 |
| N. Robertson   | (Laika.Droica)           | 4.79 x 10^15 to 4.82 x 10^15 | January 11th, 2003 |
| N. Robertson   | (Arthurus)               | 4.78 x 10^15 to 4.79 x 10^15 | February 13th, 2003 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.75 x 10^15 to 4.78 x 10^15 | December 6th, 2002 |
| D. Bernier     | (Pentium 500MHz)         | 4.72 x 10^15 to 4.75 x 10^15 | November 6th, 2002 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.69 x 10^15 to 4.72 x 10^15 | November 7th, 2002 |
| D. Bernier     | (Pentium 500MHz)         | 4.66 x 10^15 to 4.69 x 10^15 | October 14th, 2002 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.63 x 10^15 to 4.66 x 10^15 | October 13th, 2002 |
| L. Marmet      | (Riyadh)                 | 4.60 x 10^15 to 4.63 x 10^15 | October 15th, 2002 |
| N. Robertson   | (Maya)                   | 4.57 x 10^15 to 4.60 x 10^15 | November 3rd, 2002 |
| N. Robertson   | (Laika.Droica)           | 4.54 x 10^15 to 4.57 x 10^15 | October 30th, 2002 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.51 x 10^15 to 4.54 x 10^15 | September 12th, 2002 |
| D. Bernier     | (Pentium 500MHz)         | 4.48 x 10^15 to 4.51 x 10^15 | September 15th, 2002 |
| J.-P. Bernier  | (Pentium 600MHz)         | 4.45 x 10^15 to 4.48 x 10^15 | August 13th, 2002 |
| D. Bernier     | (Pentium 500MHz)         | 4.42 x 10^15 to 4.45 x 10^15 | August 12th, 2002 |
| L. Marmet      | (Riyadh)                 | 4.39 x 10^15 to 4.42 x 10^15 | August 7th, 2002 |
| User Name (Computer Name) | Range            | Date Completed |
|---------------------------|------------------|----------------|
| N. Robertson (Maya)       | $43.6 \times 10^{15}$ to $43.9 \times 10^{15}$ | July 31<sup>st</sup>, 2002 |
| N. Robertson (Laika.Droica) | $43.3 \times 10^{15}$ to $43.6 \times 10^{15}$ | July 27<sup>th</sup>, 2002 |
| J.-P. Bernier (Pentium 600MHz) | $43.0 \times 10^{15}$ to $43.3 \times 10^{15}$ | July 23<sup>rd</sup>, 2002 |
| D. Bernier (Pentium 500MHz) | $42.7 \times 10^{15}$ to $43.0 \times 10^{15}$ | July 11<sup>th</sup>, 2002 |
| L. Marmet (Riyadh)        | $42.4 \times 10^{15}$ to $42.7 \times 10^{15}$ | July 11<sup>th</sup>, 2002 |
| J.-P. Bernier (Pentium 600MHz) | $42.1 \times 10^{15}$ to $42.4 \times 10^{15}$ | June 24<sup>th</sup>, 2002 |
| N. Robertson (Laika.Droica) | $41.8 \times 10^{15}$ to $42.1 \times 10^{15}$ | June 17<sup>th</sup>, 2002 |
| L. Marmet (Riyadh)        | $41.5 \times 10^{15}$ to $41.8 \times 10^{15}$ | June 14<sup>th</sup>, 2002 |
| N. Robertson (Maya)       | $41.2 \times 10^{15}$ to $41.5 \times 10^{15}$ | June 9<sup>th</sup>, 2002 |
| N. Robertson (Laika.Droica) | $40.9 \times 10^{15}$ to $41.2 \times 10^{15}$ | May 28<sup>th</sup>, 2002 |
| J.-P. Bernier (Pentium 600MHz) | $40.6 \times 10^{15}$ to $40.9 \times 10^{15}$ | June 5<sup>th</sup>, 2002 |
| D. Bernier (Pentium 500MHz) | $40.3 \times 10^{15}$ to $40.6 \times 10^{15}$ | May 26<sup>th</sup>, 2002 |
| N. Robertson (Maya)       | $40.0 \times 10^{15}$ to $40.3 \times 10^{15}$ | March 6<sup>th</sup>, 2003 |
| J.-P. Bernier (Pentium 600MHz) | $39.7 \times 10^{15}$ to $40.0 \times 10^{15}$ | May 2<sup>nd</sup>, 2002 |
| D. Bernier (Pentium 500MHz) | $39.4 \times 10^{15}$ to $39.7 \times 10^{15}$ | May 1<sup>st</sup>, 2002 |
| N. Robertson (Maya)       | $39.1 \times 10^{15}$ to $39.4 \times 10^{15}$ | April 23<sup>rd</sup>, 2002 |
| N. Robertson (Laika.Droica) | $38.8 \times 10^{15}$ to $39.1 \times 10^{15}$ | April 18<sup>th</sup>, 2002 |
| L. Marmet (Riyadh)        | $38.5 \times 10^{15}$ to $38.8 \times 10^{15}$ | April 26<sup>th</sup>, 2002 |
| N. Robertson (Maya)       | $38.2 \times 10^{15}$ to $38.5 \times 10^{15}$ | March 25<sup>th</sup>, 2002 |
| N. Robertson (Laika.Droica) | $37.9 \times 10^{15}$ to $38.2 \times 10^{15}$ | March 20<sup>th</sup>, 2002 |
| L. Marmet (Riyadh)        | $37.6 \times 10^{15}$ to $37.9 \times 10^{15}$ | March 27<sup>th</sup>, 2002 |
| J.-P. Bernier (Pentium 600MHz) | $37.3 \times 10^{15}$ to $37.6 \times 10^{15}$ | March 6<sup>th</sup>, 2002 |
| D. Bernier (Pentium 500MHz) | $37.0 \times 10^{15}$ to $37.3 \times 10^{15}$ | March 12<sup>th</sup>, 2002 |
| N. Robertson (Maya)       | $36.7 \times 10^{15}$ to $37.0 \times 10^{15}$ | March 4<sup>th</sup>, 2002 |
| N. Robertson (Laika.Droica) | $36.4 \times 10^{15}$ to $36.7 \times 10^{15}$ | February 28<sup>th</sup>, 2002 |
| D. Bernier (Pentium 500MHz) | $36.2 \times 10^{15}$ to $36.4 \times 10^{15}$ | June 17<sup>th</sup>, 2002 |
| N. Robertson (Arthurus)   | $36.0 \times 10^{15}$ to $36.2 \times 10^{15}$ | October 2<sup>nd</sup>, 2002 |
| L. Marmet (Riyadh)        | $35.7 \times 10^{15}$ to $36.0 \times 10^{15}$ | February 18<sup>th</sup>, 2002 |
| C.R. Ward (Cosmos)        | $35.4 \times 10^{15}$ to $35.7 \times 10^{15}$ | March 10<sup>th</sup>, 2002 |
| J.-P. Bernier (Pentium 600MHz) | $35.1 \times 10^{15}$ to $35.4 \times 10^{15}$ | February 5<sup>th</sup>, 2002 |
| D. Bernier (Pentium 500MHz) | $34.8 \times 10^{15}$ to $35.1 \times 10^{15}$ | February 2<sup>nd</sup>, 2002 |
| L. Marmet (Riyadh)        | $34.5 \times 10^{15}$ to $34.8 \times 10^{15}$ | January 6<sup>th</sup>, 2002 |
| N. Robertson (Maya)       | $34.2 \times 10^{15}$ to $34.5 \times 10^{15}$ | December 30<sup>th</sup>, 2001 |
| N. Robertson (Laika.Droica) | $33.9 \times 10^{15}$ to $34.2 \times 10^{15}$ | December 25<sup>th</sup>, 2001 |
| J.-P. Bernier (Pentium 600MHz) | $33.6 \times 10^{15}$ to $33.9 \times 10^{15}$ | January 10<sup>th</sup>, 2002 |
| C.R. Ward (Cosmos)        | $33.3 \times 10^{15}$ to $33.6 \times 10^{15}$ | January 16<sup>th</sup>, 2002 |
| L. Marmet (Riyadh)        | $33.0 \times 10^{15}$ to $33.3 \times 10^{15}$ | December 8<sup>th</sup>, 2001 |
| N. Robertson (Maya)       | $32.7 \times 10^{15}$ to $33.0 \times 10^{15}$ | December 4<sup>th</sup>, 2001 |
| N. Robertson (Laika.Droica) | $32.4 \times 10^{15}$ to $32.7 \times 10^{15}$ | November 27<sup>th</sup>, 2001 |
| D. Bernier (Pentium 500MHz) | $32.1 \times 10^{15}$ to $32.4 \times 10^{15}$ | December 26<sup>th</sup>, 2001 |
| User Name       | (Computer Name)             | Range            | DateCompleted |
|-----------------|------------------------------|------------------|---------------|
| J.-P. Bernier   | (Pentium 600MHz)            | $3.18 \times 10^{15}$ to $3.21 \times 10^{15}$ | December 6th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $3.15 \times 10^{15}$ to $3.18 \times 10^{15}$ | November 1st, 2001 |
| C.R. Ward       | (Cosmos)                    | $3.12 \times 10^{15}$ to $3.15 \times 10^{15}$ | November 25th, 2001 |
| L. Marmet       | (Riyadh)                    | $3.09 \times 10^{15}$ to $3.12 \times 10^{15}$ | November 2nd, 2001 |
| N. Robertson    | (Maya)                      | $3.06 \times 10^{15}$ to $3.09 \times 10^{15}$ | October 26th, 2001 |
| N. Robertson    | (Laika.Droica)              | $3.03 \times 10^{15}$ to $3.06 \times 10^{15}$ | October 20th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $3.00 \times 10^{15}$ to $3.03 \times 10^{15}$ | September 29th, 2001 |
| N. Robertson    | (Maya)                      | $2.97 \times 10^{15}$ to $3.00 \times 10^{15}$ | September 18th, 2001 |
| N. Robertson    | (Laika.Droica)              | $2.94 \times 10^{15}$ to $2.97 \times 10^{15}$ | September 16th, 2001 |
| L. Marmet       | (Riyadh)                    | $2.91 \times 10^{15}$ to $2.94 \times 10^{15}$ | September 19th, 2001 |
| N. Robertson    | (Rosette.Droica)            | $2.89 \times 10^{15}$ to $2.91 \times 10^{15}$ | December 12th, 2001 |
| N. Robertson    | (Laika.Droica)              | $2.86 \times 10^{15}$ to $2.89 \times 10^{15}$ | August 28th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $2.83 \times 10^{15}$ to $2.86 \times 10^{15}$ | September 2nd, 2001 |
| N. Robertson    | (Maya)                      | $2.80 \times 10^{15}$ to $2.83 \times 10^{15}$ | August 28th, 2001 |
| C.R. Ward       | (Cosmos)                    | $2.77 \times 10^{15}$ to $2.80 \times 10^{15}$ | September 22nd, 2001 |
| L. Marmet       | (Riyadh)                    | $2.74 \times 10^{15}$ to $2.77 \times 10^{15}$ | August 22nd, 2001 |
| N. Robertson    | (Maya)                      | $2.71 \times 10^{15}$ to $2.74 \times 10^{15}$ | August 7th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $2.68 \times 10^{15}$ to $2.71 \times 10^{15}$ | August 8th, 2001 |
| D. Bernier      | (Pentium 500MHz)            | $2.65 \times 10^{15}$ to $2.68 \times 10^{15}$ | April 6th, 2002 |
| C.R. Ward       | (Cosmos)                    | $2.62 \times 10^{15}$ to $2.65 \times 10^{15}$ | April 29th, 2002 |
| D. Bernier      | (Pentium 500MHz)            | $2.59 \times 10^{15}$ to $2.62 \times 10^{15}$ | November 2nd, 2001 |
| N. Robertson    | (Maya)                      | $2.56 \times 10^{15}$ to $2.59 \times 10^{15}$ | July 16th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $2.53 \times 10^{15}$ to $2.56 \times 10^{15}$ | July 15th, 2001 |
| L. Marmet       | (Riyadh)                    | $2.50 \times 10^{15}$ to $2.53 \times 10^{15}$ | July 23rd, 2001 |
| C.R. Ward       | (Cosmos)                    | $2.47 \times 10^{15}$ to $2.50 \times 10^{15}$ | August 2nd, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $2.44 \times 10^{15}$ to $2.47 \times 10^{15}$ | April 8th, 2002 |
| Z. McGregor-Dorsey | (Abzug)        | $2.41 \times 10^{15}$ to $2.44 \times 10^{15}$ | July 7th, 2001 |
| N. Robertson    | (Arthurus)                  | $2.39 \times 10^{15}$ to $2.41 \times 10^{15}$ | November 30th, 2001 |
| L. Marmet       | (Riyadh)                    | $2.36 \times 10^{15}$ to $2.39 \times 10^{15}$ | June 14th, 2001 |
| N. Robertson    | (Maya)                      | $2.33 \times 10^{15}$ to $2.36 \times 10^{15}$ | June 18th, 2001 |
| Z. McGregor-Dorsey | (Hayduke)          | $2.30 \times 10^{15}$ to $2.33 \times 10^{15}$ | July 7th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $2.27 \times 10^{15}$ to $2.30 \times 10^{15}$ | June 16th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $2.24 \times 10^{15}$ to $2.27 \times 10^{15}$ | May 26th, 2001 |
| D. Bernier      | (Pentium 500MHz)            | $2.21 \times 10^{15}$ to $2.24 \times 10^{15}$ | June 19th, 2001 |
| Z. McGregor-Dorsey | (Castalia)       | $2.18 \times 10^{15}$ to $2.21 \times 10^{15}$ | June 5th, 2001 |
| C.R. Ward       | (Cosmos)                    | $2.15 \times 10^{15}$ to $2.18 \times 10^{15}$ | June 7th, 2001 |
| J.-P. Bernier   | (Pentium 600MHz)            | $2.12 \times 10^{15}$ to $2.15 \times 10^{15}$ | May 8th, 2001 |
| L. Marmet       | (Riyadh)                    | $2.09 \times 10^{15}$ to $2.12 \times 10^{15}$ | May 16th, 2001 |
| D. Bernier      | (Pentium 500MHz)            | $2.06 \times 10^{15}$ to $2.09 \times 10^{15}$ | May 3rd, 2001 |
| L. Marmet       | (Fontaine)                  | $2.03 \times 10^{15}$ to $2.06 \times 10^{15}$ | May 8th, 2001 |
| User Name        | (Computer Name)     | Range            | Date/Completed |
|------------------|---------------------|------------------|----------------|
| J.-P. Bernier    | (Pentium 600MHz)    | 20.0 x 10^{15} to 20.3 x 10^{15} | April 20^{th}, 2001 |
| C.R. Ward        | (Cosmos)            | 19.8 x 10^{15} to 20.0 x 10^{15} | April 20^{th}, 2001 |
| L. Marmet        | (Riyadh)            | 19.6 x 10^{15} to 19.8 x 10^{15} | April 18^{th}, 2001 |
| Z. McGregor-Dorsey | (Abzug)          | 19.4 x 10^{15} to 19.6 x 10^{15} | May 27^{th}, 2001 |
| Z. McGregor-Dorsey | (Hayduke)         | 19.2 x 10^{15} to 19.4 x 10^{15} | May 10^{th}, 2001 |
| D. Bernier       | (Pentium 500MHz)    | 19.0 x 10^{15} to 19.2 x 10^{15} | April 11^{th}, 2001 |
| L. Marmet        | (Fontaine)          | 18.8 x 10^{15} to 19.0 x 10^{15} | April 7^{th}, 2001 |
| D. Bernier       | (Pentium 500MHz)    | 18.6 x 10^{15} to 18.8 x 10^{15} | April 4^{th}, 2001 |
| J.-P. Bernier    | (Pentium 600MHz)    | 18.4 x 10^{15} to 18.6 x 10^{15} | March 27^{th}, 2001 |
| L. Marmet        | (Riyadh)            | 18.2 x 10^{15} to 18.4 x 10^{15} | April 1^{st}, 2001 |
| D. Bernier       | (Pentium 500MHz)    | 18.0 x 10^{15} to 18.2 x 10^{15} | March 18^{th}, 2001 |
| L. Marmet        | (Fontaine)          | 17.8 x 10^{15} to 18.0 x 10^{15} | March 21^{st}, 2001 |
| L. Marmet        | (Riyadh)            | 17.6 x 10^{15} to 17.8 x 10^{15} | March 15^{th}, 2001 |
| J.-P. Bernier    | (Pentium 600MHz)    | 17.4 x 10^{15} to 17.6 x 10^{15} | March 14^{th}, 2001 |
| D. Bernier       | (Pentium 500MHz)    | 17.2 x 10^{15} to 17.4 x 10^{15} | March 5^{th}, 2001 |
| J.-P. Bernier    | (Pentium 600MHz)    | 17.0 x 10^{15} to 17.2 x 10^{15} | March 2^{nd}, 2001 |
| L. Marmet        | (Fontaine)          | 16.8 x 10^{15} to 17.0 x 10^{15} | March 3^{rd}, 2001 |
| L. Marmet        | (Riyadh)            | 16.6 x 10^{15} to 16.8 x 10^{15} | February 26^{th}, 2001 |
| D. Bernier       | (Pentium 500MHz)    | 16.4 x 10^{15} to 16.6 x 10^{15} | February 19^{th}, 2001 |
| J.-P. Bernier    | (Pentium 600MHz)    | 16.2 x 10^{15} to 16.4 x 10^{15} | February 17^{th}, 2001 |
| N. Robertson     | (Arthurus)          | 16.0 x 10^{15} to 16.2 x 10^{15} | May 27^{th}, 2001 |
| L. Marmet        | (Fontaine)          | 15.8 x 10^{15} to 16.0 x 10^{15} | February 11^{th}, 2001 |
| Z. McGregor-Dorsey | (Castalia)         | 15.6 x 10^{15} to 15.8 x 10^{15} | April 25^{th}, 2001 |
| Z. McGregor-Dorsey | (Abzug)            | 15.4 x 10^{15} to 15.6 x 10^{15} | April 17^{th}, 2001 |
| L. Marmet        | (Riyadh)            | 15.2 x 10^{15} to 15.4 x 10^{15} | February 8^{th}, 2001 |
| Z. McGregor-Dorsey | (Hayduke)         | 15.0 x 10^{15} to 15.2 x 10^{15} | April 5^{th}, 2001 |
| J.-P. Bernier    | (Pentium 600MHz)    | 14.8 x 10^{15} to 15.0 x 10^{15} | February 4^{th}, 2001 |
| L. Marmet        | (Riyadh)            | 14.6 x 10^{15} to 14.8 x 10^{15} | January 22^{nd}, 2001 |
| Z. McGregor-Dorsey | (Castalia)         | 14.4 x 10^{15} to 14.6 x 10^{15} | April 3^{rd}, 2001 |
| Z. McGregor-Dorsey | (Abzug)            | 14.2 x 10^{15} to 14.4 x 10^{15} | March 19^{th}, 2001 |
| Z. McGregor-Dorsey | (Hayduke)          | 14.0 x 10^{15} to 14.2 x 10^{15} | March 19^{th}, 2001 |
| L. Marmet        | (Strontium)         | 13.8 x 10^{15} to 14.0 x 10^{15} | December 17^{th}, 2000 |
| L. Marmet        | (Riyadh)            | 13.6 x 10^{15} to 13.8 x 10^{15} | January 5^{th}, 2001 |
| L. Marmet        | (Strontium)         | 13.4 x 10^{15} to 13.6 x 10^{15} | December 11^{th}, 2000 |
| L. Marmet        | (Fontaine)          | 13.2 x 10^{15} to 13.4 x 10^{15} | December 13^{th}, 2000 |
| L. Marmet        | (Strontium)         | 13.0 x 10^{15} to 13.2 x 10^{15} | December 4^{th}, 2000 |
| L. Marmet        | (Riyadh)            | 12.8 x 10^{15} to 13.0 x 10^{15} | December 9^{th}, 2000 |
| L. Marmet        | (Strontium)         | 12.6 x 10^{15} to 12.8 x 10^{15} | November 26^{th}, 2000 |
| L. Marmet        | (Fontaine)          | 12.4 x 10^{15} to 12.6 x 10^{15} | November 24^{th}, 2000 |
| Z. McGregor-Dorsey | (Abzug)            | 12.2 x 10^{15} to 12.4 x 10^{15} | January 21^{st}, 2001 |
| User Name     | (Computer Name) | Range                | Date Completed         |
|---------------|-----------------|----------------------|------------------------|
| L. Marmet     | (Riyadh)        | $12.0 \times 10^{15}$ to $12.2 \times 10^{15}$ | November 22\textsuperscript{nd}, 2000 |
| L. Marmet     | (Strontium)     | $11.8 \times 10^{15}$ to $12.0 \times 10^{15}$ | November 17\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Castalia)   | $11.6 \times 10^{15}$ to $11.8 \times 10^{15}$ | January 22\textsuperscript{nd}, 2001 |
| L. Marmet     | (Riyadh)        | $11.4 \times 10^{15}$ to $11.6 \times 10^{15}$ | November 7\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $11.2 \times 10^{15}$ to $11.4 \times 10^{15}$ | January 21\textsuperscript{st}, 2001 |
| L. Marmet     | (Riyadh)        | $11.0 \times 10^{15}$ to $11.2 \times 10^{15}$ | November 1\textsuperscript{st}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $10.8 \times 10^{15}$ to $11.0 \times 10^{15}$ | December 10\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Castalia)  | $10.6 \times 10^{15}$ to $10.8 \times 10^{15}$ | December 10\textsuperscript{th}, 2000 |
| L. Marmet     | (Riyadh)        | $10.4 \times 10^{15}$ to $10.6 \times 10^{15}$ | October 13\textsuperscript{th}, 2000 |
| N. Robertson  | (Arthurus)      | $10.2 \times 10^{15}$ to $10.4 \times 10^{15}$ | January 27\textsuperscript{th}, 2001 |
| N. Marmet     | (Computer)      | $10.0 \times 10^{15}$ to $10.2 \times 10^{15}$ | October 21\textsuperscript{st}, 2000 |
| L. Marmet     | (Riyadh)        | $9.8 \times 10^{15}$ to $10.0 \times 10^{15}$ | September 27\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Castalia)  | $9.6 \times 10^{15}$ to $9.8 \times 10^{15}$ | November 16\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $9.4 \times 10^{15}$ to $9.6 \times 10^{15}$ | January 26\textsuperscript{th}, 2001 |
| L. Marmet     | (Fontaine)      | $9.2 \times 10^{15}$ to $9.4 \times 10^{15}$ | January 26\textsuperscript{th}, 2001 |
| Z. McGregor-Dorsey | (Castalia)  | $9.0 \times 10^{15}$ to $9.2 \times 10^{15}$ | October 24\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Castalia)  | $8.8 \times 10^{15}$ to $9.0 \times 10^{15}$ | September 27\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $8.6 \times 10^{15}$ to $8.8 \times 10^{15}$ | September 11\textsuperscript{th}, 2000 |
| L. Marmet     | (Riyadh)        | $8.4 \times 10^{15}$ to $8.6 \times 10^{15}$ | September 11\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Castalia)  | $8.2 \times 10^{15}$ to $8.4 \times 10^{15}$ | September 3\textsuperscript{rd}, 2000 |
| Z. McGregor-Dorsey | (Abzug)      | $8.0 \times 10^{15}$ to $8.2 \times 10^{15}$ | October 16\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $7.8 \times 10^{15}$ to $8.0 \times 10^{15}$ | October 16\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Abzug)      | $7.6 \times 10^{15}$ to $7.8 \times 10^{15}$ | September 27\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $7.4 \times 10^{15}$ to $7.6 \times 10^{15}$ | August 26\textsuperscript{th}, 2000 |
| N. Marmet     | (Computer)      | $7.2 \times 10^{15}$ to $7.4 \times 10^{15}$ | September 17\textsuperscript{th}, 2000 |
| L. Marmet     | (Riyadh)        | $7.0 \times 10^{15}$ to $7.2 \times 10^{15}$ | August 25\textsuperscript{th}, 2000 |
| G. Engebreth  | (Computer)      | $6.8 \times 10^{15}$ to $7.0 \times 10^{15}$ | August 28\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $6.6 \times 10^{15}$ to $6.8 \times 10^{15}$ | August 17\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Abzug)      | $6.4 \times 10^{15}$ to $6.6 \times 10^{15}$ | September 2\textsuperscript{nd}, 2000 |
| Z. McGregor-Dorsey | (Abzug)      | $6.2 \times 10^{15}$ to $6.4 \times 10^{15}$ | August 26\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Abzug)      | $6.0 \times 10^{15}$ to $6.2 \times 10^{15}$ | August 7\textsuperscript{th}, 2000 |
| G. Engebreth  | (Computer)      | $5.8 \times 10^{15}$ to $6.0 \times 10^{15}$ | August 10\textsuperscript{th}, 2000 |
| L. Marmet     | (Riyadh)        | $5.6 \times 10^{15}$ to $5.8 \times 10^{15}$ | August 6\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Abzug)      | $5.4 \times 10^{15}$ to $5.6 \times 10^{15}$ | August 26\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $5.2 \times 10^{15}$ to $5.4 \times 10^{15}$ | July 28\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Castalia)   | $5.0 \times 10^{15}$ to $5.2 \times 10^{15}$ | August 16\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Castalia)   | $4.8 \times 10^{15}$ to $5.0 \times 10^{15}$ | July 28\textsuperscript{th}, 2000 |
| N. Marmet     | (Computer)      | $4.6 \times 10^{15}$ to $4.8 \times 10^{15}$ | August 15\textsuperscript{th}, 2000 |
| E. Wong       | (Computer)      | $4.4 \times 10^{15}$ to $4.6 \times 10^{15}$ | September 27\textsuperscript{th}, 2000 |
| Z. McGregor-Dorsey | (Haydude)    | $4.2 \times 10^{15}$ to $4.4 \times 10^{15}$ | July 14\textsuperscript{th}, 2000 |
| User Name          | Computer Name | Range         | Date Completed |
|--------------------|---------------|---------------|----------------|
| L. Marmet         | Riyadh        | $4 \times 10^{15}$ to $4.2 \times 10^{15}$ | July 20th, 2000 |
| Z. McGregor-Dorsey| Abzug         | $3.8 \times 10^{15}$ to $4.0 \times 10^{15}$ | July 18th, 2000 |
| Z. McGregor-Dorsey| Castalia      | $3.6 \times 10^{15}$ to $3.8 \times 10^{15}$ | July 10th, 2000 |
| Z. McGregor-Dorsey| Abzug         | $3.4 \times 10^{15}$ to $3.6 \times 10^{15}$ | July 10th, 2000 |
| Z. McGregor-Dorsey| Hayduke       | $3.2 \times 10^{15}$ to $3.4 \times 10^{15}$ | July 1st, 2000  |
| L. Marmet         | Riyadh        | $3.0 \times 10^{15}$ to $3.2 \times 10^{15}$ | June 29th, 2000 |
| Z. McGregor-Dorsey| Castalia      | $2.8 \times 10^{15}$ to $3.0 \times 10^{15}$ | June 18th, 2000 |
| G. Engebrith      | Computer      | $2.6 \times 10^{15}$ to $2.8 \times 10^{15}$ | July 22nd, 2000 |
| Z. McGregor-Dorsey| Castalia      | $2.4 \times 10^{15}$ to $2.6 \times 10^{15}$ | June 8th, 2000  |
| N. Marmet         | Computer      | $2.2 \times 10^{15}$ to $2.4 \times 10^{15}$ | July 5th, 2000  |
| L. Marmet         | Riyadh        | $2.0 \times 10^{15}$ to $2.2 \times 10^{15}$ | June 9th, 2000  |
| Z. McGregor-Dorsey| Abzug         | $1.8 \times 10^{15}$ to $2.0 \times 10^{15}$ | May 27th, 2000  |
| A. Simpson        | Computer      | $1.6 \times 10^{15}$ to $1.8 \times 10^{15}$ | June 30th, 2000 |
| Z. McGregor-Dorsey| Hayduke       | $1.4 \times 10^{15}$ to $1.6 \times 10^{15}$ | May 13th, 2000  |
| L. Marmet         | Riyadh        | $1.2 \times 10^{15}$ to $1.4 \times 10^{15}$ | May 16th, 2000  |
| G. Engebrith      | Computer      | $1.1 \times 10^{15}$ to $1.2 \times 10^{15}$ | June 1st, 2000  |
| E. Wong           | Computer      | $1.0 \times 10^{15}$ to $1.1 \times 10^{15}$ | May 31st, 2000  |
| L. Marmet         | Lion          | $9 \times 10^{14}$ to $10 \times 10^{14}$   | June 13th, 2000 |
| L. Marmet         | Riyadh        | $8 \times 10^{14}$ to $9 \times 10^{14}$    | April 5th, 2000 |
| Z. McGregor-Dorsey| Castalia      | $7 \times 10^{14}$ to $8 \times 10^{14}$    | May 1st, 2000   |
| L. Marmet         | Fontaine      | $6 \times 10^{14}$ to $7 \times 10^{14}$    | April 22nd, 2000|
| E. Wong           | Computer      | $5 \times 10^{14}$ to $6 \times 10^{14}$    | April 8th, 2000 |
| L. Marmet         | Fontaine      | $4 \times 10^{14}$ to $5 \times 10^{14}$    | March 5th, 2000 |
| D. Bernier        | Pentium 500MHz| $3 \times 10^{14}$ to $4 \times 10^{14}$    | March 24th, 2000|
| L. Marmet         | Riyadh        | $2 \times 10^{14}$ to $3 \times 10^{14}$    | February 4th, 2000|
| D. Bernier        | Pentium 500MHz| $1.5 \times 10^{14}$ to $2.0 \times 10^{14}$| January 24th, 2000|
| L. Marmet         | Lion          | $4 \times 10^{9}$ to $1500 \times 10^{11}$  | December 20th, 1999|