Localized cohomology and some applications of Popa’s cocycle superrigidity theorem

Asger Törnquist

July 4, 2009

Abstract

We prove that orbit equivalence of measure preserving ergodic a.e.
free actions of a countable group with the relative property (T) is a
complete analytic equivalence relation.

§1. Introduction

In [17], S. Popa introduced the notion of quotients of Bernoulli shifts in
order to obtain an infinite family of measure preserving ergodic a.e.
free orbit-inequivalent actions of a countable group $\Gamma$ with the relative property
($T$) over an infinite normal subgroup $\Lambda$. These actions are defined as follows:
Let $A$ be a countable Abelian group, and let $\hat{A}$ be its dual (character) group,
equipped with the normalized Haar measure. Let $X = \hat{A} \Gamma$, equipped with
the product measure. Then the (left) shift-action of $\Gamma$ on $\hat{A} \Gamma$ commutes with
the action of $\hat{A}$, and we obtain a measure preserving a.e. free ergodic action
$\sigma^{\hat{A}}$ of $\Gamma$ on the quotient $\hat{A} \Gamma / \hat{A}$.

Popa proved in [17] that $(\sigma^{\hat{A}} : A$ is torsion free countable abelian) is a
family of $\Gamma$-actions that are orbit equivalent precisely for isomorphic groups,
when $\Gamma$ is a group with the relative property ($T$) over an infinite normal sub-
group $\Lambda$. The first aim of this paper is to prove this without any normality
assumption on the subgroup $\Lambda$:

Theorem 1. Suppose $\Gamma$ is a countable discrete group with the relative
property ($T$) over an infinite subgroup $\Lambda$, and $A$ and $A'$ are countably infinite
Abelian groups. Then $\sigma^{\hat{A}}$ and $\sigma^{\hat{A}'}$ are orbit equivalent iff $A \simeq A'$. 
The result relies on Popa’s cocycle superrigidity Theorem, [18]. Specifically, we exploit the “local” untwisting Theorem, [18, Theorem 5.2], to obtain information about the action of the subgroup \( \Lambda \leq \Gamma \) in relation to the action of the ambient group \( \Gamma \).

Theorem 1 has an interesting consequence for the complexity of orbit equivalence for groups with the relative property (T). Namely, in \( \S 5 \) we will show that the family \( \langle \sigma^A \rangle \) is Borel with respect to the parameter \( A \). More precisely, let \( \mathcal{A}(\Gamma, [0, 1]) \) denote the natural Polish space of measure preserving actions of \( \Gamma \) on \([0, 1]\), and let \( \text{ABEL}_{\aleph_0} \) be the natural Polish space of countably infinite abelian groups. Let \( \mathcal{A}^*_e(\Gamma, [0, 1]) \) be the subspace of \( \mathcal{A}(\Gamma, [0, 1]) \) consisting of ergodic a.e. free \( \Gamma \)-actions. We will show that there is a Borel \( f : \text{ABEL}_{\aleph_0} \to \mathcal{A}^*_e(\Gamma, [0, 1]) \) with the property that \( f(A) \) and \( f(A') \) are orbit equivalent if and only if \( A \) and \( A' \) are isomorphic. That is, there is a Borel reduction of the isomorphism relation for countably infinite Abelian groups to orbit equivalence for m.p. ergodic a.e. free actions of a countable \( \Gamma \) with the relative property (T). It is known by [6] that the isomorphism relation in \( \text{ABEL}_{\aleph_0} \) is complete analytic, from which we obtain:

**Theorem 2.** Suppose \( \Gamma \) is a countable discrete group with the relative property (T) over an infinite subgroup \( \Lambda \). Then orbit equivalence, regarded as a subset of \( \mathcal{A}^*_e(\Gamma, [0, 1]) \times \mathcal{A}^*_e(\Gamma, [0, 1]) \), is a complete analytic set. In particular, it is not Borel.

We also obtain in Corollary 5.6 the same result for conjugacy in \( \mathcal{A}^*_e(\Gamma, [0, 1]) \):

Under the assumptions of Theorem 2, conjugacy is analytic, but not Borel.

**Organization:** In \( \S 2 \) we introduce the notion of “localized cohomology” which is the central tool used to distinguish the actions \( \sigma^A \) up to orbit equivalence. In \( \S 3 \) we do a preliminary analysis that proves that a countable group \( \Gamma \) with the relative property (T) has continuum many orbit inequivalent actions. In \( \S 4 \) we refine this analysis to prove Theorem 1 above. Theorem 2 is proved in \( \S 5 \).

Research for this paper was supported in part by the Danish Natural Science Research Council grant no. 272-06-0211.

\( \S 2. \) **Localized cohomology**

2
Let $\Gamma$ be a countable group and $\sigma$ a probability measure preserving (p.m.p.) $\Gamma$-action on standard Borel probability space $(X, \mu)$. Recall that a 1-cocycle for $\sigma$ is a measurable map $\alpha : \Gamma \times X \to \mathbb{T}$ such that

$$\alpha(\gamma_1 \gamma_2, x) = \alpha(\gamma_1, \sigma(\gamma_2)(x)) \alpha(\gamma_2, x) \quad (\gamma_1, \gamma_2 \in \Gamma, \mu\text{-a.e. } x \in X).$$

The set of all such cocycles is denoted $Z^1(\sigma)$, and forms a Polish group under pointwise multiplication, when given the subspace topology inherited from $L^\infty(X, \mathbb{T})^\Gamma$. A 1-coboundary is a cocycle $\beta \in Z^1(\sigma)$ of the form

$$\gamma_f(g, x) = f(x)^* f(\sigma(g)(x)),$$

where $f : X \to \mathbb{T}$ is a measurable map. The coboundaries form a subgroup denoted $B^1(\sigma)$. The 1st cohomology group is then defined as

$$H^1(\sigma) = Z^1(\sigma)/B^1(\sigma).$$

We now introduce the notion of a localized coboundary:

2.1. **Definition.** Suppose $\Lambda < \Gamma$ is a subgroup. We say that $\beta \in Z^1(\sigma)$ is a $\Lambda$-local coboundary if there is a measurable $f : X \to \mathbb{T}$ such that

$$(\forall \gamma \in \Lambda) \alpha(\gamma, x) = f(x)^* f(\sigma(\gamma)(x)),$$

i.e. if $\beta|\Lambda$ is a 1-coboundary for $\sigma|\Lambda$. We denote by $B^1_\Lambda(\sigma)$ the group of $\Lambda$-local coboundaries. The $\Lambda$-localized 1st cohomology group is defined as

$$H^1_\Lambda(\sigma) = Z^1(\sigma)/B^1_\Lambda(\sigma).$$

We can make $H^1_\Lambda(\sigma)$ into a topological group by giving it the quotient topology.

The following relativization of a result of Schmidt’s in [19], [20], was already noted in [17] 1.6.2, though not stated in this form. See also [7] Theorem 4.2 for a more general result along these lines.

2.2. **Proposition.** If $\Gamma$ is a countable group with the relative property (T) over an infinite subgroup $\Lambda < \Gamma$ and $\sigma$ is a p.m.p. action of $\Gamma$ on a standard Borel probability space $(X, \mu)$ such that $\sigma|\Lambda$ is ergodic, then $B^1_\Lambda(\sigma)$ is an open subgroup of $Z^1(\sigma)$, and $H^1_\Lambda(\sigma)$ is discrete in the quotient topology.

**Proof.** It suffices to show that $B^1_\Lambda(\sigma)$ contains a neighbourhood of the identity. Let $Q \subseteq \Gamma$ be a finite subset and $\epsilon > 0$ such that if $\pi$ is a unitary
representation of $\Gamma$ with $(Q, \varepsilon)$-invariant vectors, then it has a non-zero $\Lambda$-invariant vector. Suppose now that $\alpha \in Z^1(\sigma)$ is such that

$$\|\alpha(\gamma, x) - 1\|_\infty < \varepsilon^2$$

for all $\gamma \in Q$. Consider the unitary representation $\pi$ of $\Gamma$ on $L^2(X)$ given by

$$\pi(\gamma)(f)(x) = \alpha(\gamma^{-1}, x)^{-1} f(\gamma^{-1} \cdot \sigma x).$$

Then the constant 1 function is $(Q, \varepsilon)$-invariant. Hence there is a $\Lambda$-invariant non-zero $f \in L^2(X)$. Invariance amounts to

$$\alpha(\gamma^{-1}, x)^{-1} f(\gamma^{-1} \cdot \sigma x) = f(x),$$

for all $\gamma \in \Lambda$, which is equivalent to

$$f(\gamma \cdot \sigma x) = \alpha(\gamma, x) f(x).$$

By the ergodicity of $\sigma|\Lambda$ we have that $f(x) \neq 0$ almost everywhere. Since we also have

$$|f(\gamma \cdot \sigma x)| = |f(\gamma \cdot \sigma x) \alpha(\gamma^{-1}, x)^{-1}| = |f(x)|$$

it follows that if $\psi(x) = f(x)/|f(x)|$ then $\psi : X \to \mathbb{T}$ and

$$\alpha(\gamma, x) = \psi(x)^* \psi(\gamma \cdot \sigma x)$$

for all $\gamma \in \Lambda$, thus proving that $B^1_{1,\Lambda}(\sigma)$ is open in $Z^1(\sigma)$. \hfill \Box

2.3. Reduced cohomology. Along with the localized cohomology group we also introduce the reduced localized cohomology group, $H^1_{1,\Lambda,r}(\sigma)$ as follows: Let $B^1_{1,r}(\sigma)$ consist of all $\alpha \in Z^1(\sigma)$ of the form

$$\alpha(g, x) = f(g \cdot x) \beta(g, x) f(x)^*$$

where $\beta|\Lambda \times X$ is a character (does not depend on $x \in X$). The reduced localized cohomology group is defined as

$$H^1_{1,r}(\sigma) = Z^1(\sigma)/B^1_{1,r}(\sigma).$$

It is clear that if we let

$$C^1_{\Lambda}(\sigma) = \{ \beta \in Z^1(\sigma) : (\exists \chi \in \text{Char}(\Lambda))\beta(g, x) = \chi(g) \text{ a.e.} \}$$

then

$$B^1_{1}(\sigma) = C^1_{\Lambda}(\sigma) B^1(\sigma).$$

Further, we have:
2.4. Lemma. If $\sigma|\Lambda$ is weakly mixing then $B^1(\sigma) \cap C_\Lambda(\sigma) = \{1\}$.

Proof. It follows that for $g \in \Lambda$ we have

$$f(g \cdot x) = \beta(g)f(x).$$

Hence $f$ is a $\Lambda$-eigenfunction. Since the $\Lambda$-action is weakly mixing, we must have $f = 1$. \qed

2.5. Local untwisting. The notion of local untwisting of cocycles is, of course, the crux of Popa's construction in [18]. Much of the point of the present paper is that local untwisting suffices for certain applications.

Let $\Gamma$ be a countable discrete group and $\Lambda$ a subgroup, and suppose that $\sigma$ is a p.m.p. action of $\Gamma$ on $(X,\mu)$. We will now consider cocycles with target group $H$, which is assumed to be in Popa’s class of Polish groups of finite type, i.e. realizable as a closed subgroup of the unitary group of a finite countably generated von Neumann algebra. For our purposes the reader can assume that $H$ is either countable discrete, or is the circle group $\mathbb{T}$.

Recall from [17], [18] that an action $\sigma$ on $(X,\mu)$ is malleable if the flip-automorphism on $X \times X$ is in the (path) connected component of the identity in the commutator of the product action $\sigma \times \sigma$ on $X \times X$. We will now state a “local” cocycle superrigidity theorem, which was proven by Popa in [18, Theorem 5.2]. It plays a key role in the arguments in this paper.

2.6. Theorem. ("Local" superrigidity, S. Popa [18].) Suppose $\Lambda$ is an infinite subgroup of $\Gamma$ such that $(\Gamma,\Lambda)$ has property (T). Suppose $\sigma$ is a malleable p.m.p. action of $\Gamma$ and that $\sigma|\Lambda$ is weakly mixing. If $\alpha : \Gamma \times X \to H$ is a measurable cocycle with target group in Popa's class, then there is a homomorphism $\rho : \Lambda \to H$ and $\psi : X \to H$ measurable such that

$$(\forall g \in \Lambda)\psi(g \cdot x)\alpha(g,x)\psi(x)^{-1} = \rho(g).$$

Remark. In [18], Popa shows that under various additional algebraic “weak normality” assumptions on the group $\Lambda < \Gamma$, the untwisting can be continued to the whole group, thus giving a classical type superrigidity theorem.
2.7. Corollary. Under the assumptions of the previous theorem, if \( \alpha \in Z^1(\sigma) \) then \( \alpha|\Lambda \) is cohomologous to a character \( \chi : \Lambda \to \mathbb{T} \).

2.8. Corollary. Under the assumptions of the previous theorem, \( H^1_\Lambda(\sigma) \) is isomorphic to a countable subgroup of \( \text{Char}(\Lambda) \), and \( H^1_{\Lambda,r}(\sigma) = \{1\} \).

Proof. Clear from Proposition 2.2 and the previous Corollary and the definition of the reduced localized cohomology group.

We end this section by noting a fact about localized cohomology and how the relative property (T) “transfers” when we have local untwisting of cocycles, as in Theorem 2.6. This will play a crucial role in our arguments:

2.9. Proposition. Let \( \Gamma \) be a countable discrete group and \( \Lambda \leq \Gamma \) a subgroup. Suppose \( \Gamma \) acts by p.m.p. transformations on \( (X, \mu) \) and that \( \alpha : \Gamma \times X \to H \) is a measurable cocycle, and there is a homomorphism \( \rho : \Lambda \to H \) such that \( \alpha|\Lambda = \rho \). If \( (\Gamma, \Lambda) \) has property (T) then \( (H, \rho(\Lambda)) \) has property (T).

Proof. We use Jolissaint’s characterization of relative property (T), see [13]. Let \( (Q, \varepsilon) \) be a Kazhdan pair for \( (\Gamma, \Lambda) \) such that any \( (Q, \varepsilon) \)-invariant vector is within \( \frac{1}{10} \) of a \( \Lambda \)-invariant vector. Let \( Q' \subseteq H \) be a finite set such that

\[
\mu(\{ x \in X : \alpha(Q, x) \subseteq Q' \}) > 1 - \frac{\varepsilon^2}{8}.
\]

We claim that \( (Q', \varepsilon/\sqrt{2}) \) is a Kazhdan pair for \( (H, \rho(\Lambda)) \). To see this, let \( \pi : H \to U(\mathcal{H}) \) be a unitary representation on a Hilbert space \( (\mathcal{H}, \|\cdot\|) \) and suppose \( \xi \in \mathcal{H} \) is a \( (Q', \varepsilon/\sqrt{2}) \)-invariant unit vector. Define a representation \( \pi^\alpha \) of \( \Gamma \) on \( L^2(X, \mathcal{H}) \) by

\[
\pi^\alpha(g)(f)(x) = \pi(\alpha(g^{-1}, x)^{-1})(f(g^{-1} \cdot x)).
\]

Then

\[
\pi^\alpha(g_1g_2)(f)(x) = \pi(\alpha(g_1^{-1}g_2^{-1}, x)^{-1}f(g_1^{-1}g_2^{-1} \cdot x))
= \pi(\alpha(g_1^{-1}, x)^{-1}\alpha(g_2^{-1}, g_1^{-1} \cdot x)^{-1}f(g_2^{-1}g_1^{-1} \cdot x))
= \pi^\alpha(g_1)(\pi^\alpha(g_2)(f))(x).
\]
Let \( f_\xi(x) = \xi \) for all \( x \in X \). Then for \( g \in Q \) we have
\[
\|\pi^\alpha(g)(f_\xi) - f_\xi\|^2_{L^2(X,\mathcal{F})} = \int \|\pi(\alpha(g^{-1}, x)^{-1})(f_\xi(g^{-1} \cdot x)) - \xi\|^2 d\mu(x)
\]
\[
= \int_{\{x: \alpha(g,x) \in Q'\}} \|\pi(\alpha(g^{-1}, x)^{-1})(f_\xi(g^{-1} \cdot x)) - \xi\|^2 d\mu(x)
\]
\[
+ \int_{\{x: \alpha(g,x) \notin Q'\}} \|\pi(\alpha(g^{-1}, x)^{-1})(f_\xi(g^{-1} \cdot x)) - \xi\|^2 d\mu(x)
\]
\[\leq \varepsilon^2 + 4 \varepsilon^2 8 = \varepsilon^2.
\]

It follows that there is a \( \Lambda \)-invariant unit vector \( f_0 \in L^2(X,\mathcal{F}) \) such that \( \|f_0 - f_\xi\|_{L^2(X,\mathcal{F})} \leq \frac{1}{10} \). Let \( V_{\mathcal{F}} \) be the subspace of \( L^2(X,\mathcal{F}) \) consisting of constant functions. Since \( \|f_0 - f_\xi\| \leq \frac{1}{10} \), the projection of \( f_0 \) unto \( V_{\mathcal{F}} \) is not 0, so let \( f = \text{proj}_{V_{\mathcal{F}}}(f_0)/\|\text{proj}_{V_{\mathcal{F}}}(f_0)\|\) and suppose \( f = f_{\xi_0} \) for some \( \xi_0 \in \mathcal{F} \). Note that \( V_{\mathcal{F}} \) is a \( \Lambda \) invariant subspace. Since \( \pi^\alpha \) is a unitary representation, we must then have for \( h \in \Lambda \) that
\[
\pi^\alpha(h)(\text{proj}_{V_{\mathcal{F}}} f_0) = \text{proj}_{V_{\mathcal{F}}} (\pi^\alpha(h)f_0) = \text{proj}_{V_{\mathcal{F}}}(f_0).
\]

Hence \( f \) is \( \Lambda \)-invariant, and so \( \pi(\rho(h))(\xi_0) = \xi_0 \). This shows that \((H, \rho(\Lambda))\) has property (T).

\[\Box\]

§3. ORBIT EQUIVALENCE

We consider the following set-up: \( \Gamma \) is a countably infinite group, \( \sigma : \Gamma \curvearrowright (X,\mu) \) is a p.m.p. malleable action of \( \Gamma \) and \( \Lambda \leq \Gamma \) is an infinite subgroup such that \( \sigma|\Lambda \) is weakly mixing. Additionally, there is a compact 2nd countable group \( K \) acting in a measure preserving way on \((X,\mu)\), the action of which commutes with \( \sigma \). The action of \( K \) gives rise to a factor \((Y,\nu)\) consisting of \( K \)-equivalence classes, and we have the factor map
\[
\theta : x \to [x]_K.
\]

The measure \( \nu \) is the push-forward measure of \( \mu \). Note that \((Y,\nu)\) is standard because \( K \) is assumed to be compact. \( \Gamma \) acts on \((Y,\nu)\) in a p.m.p. way, and we denote this action \( \sigma^K \). (The action of \( K \) will always be implicit.)

The quotients of Bernoulli shifts \( \sigma^A \) discussed in §1 is an example of this situation. We note the following easy fact about \( \sigma^A \):

7
3.1. Lemma. If $\Lambda$ is an infinite subgroup of $\Gamma$, then $\sigma^A|\Lambda$ is mixing.

Proof. Let $B \subseteq \hat{A}^F$ be Borel and $\hat{A}$-invariant. Since the Bernoulli shift $\sigma$ is mixing on all infinite subgroups it holds for all $\varepsilon > 0$ that the set of $\gamma \in \Lambda$ such that $|\mu(\sigma(\gamma)(B) \cap B) - \mu(B)^2| \geq \varepsilon$ is finite. Hence $\sigma^A|\Lambda$ is mixing. □

The following Lemma is certainly implicit in [17]:

3.2. Lemma. Let $\Gamma$ be a countable group with the relative property (T) over an infinite subgroup $\Lambda$ and let $A$ be a countably infinite abelian group. Suppose $\sigma : \Gamma \acts (X,\mu)$ is a p.m.p. action with $\sigma|\Lambda$ weakly mixing, and that $\hat{A} = \text{Char}(A)$ acts on $(X,\mu)$ in a free, measure preserving way commuting with the action of $\Gamma$. Let $(Y,\nu)$ be the corresponding factor, $\theta : X \to Y$ the factor map and let $\sigma^A$ be the quotient action. Then $H^1_{\Lambda,r}(\sigma) = \{1\}$ implies that $H^1_{\Lambda,r}(\sigma^A) = A$.

Proof. For each $\alpha \in Z^1(\sigma^A)$, let $\alpha' \in Z^1(\sigma)$ be

$$\alpha'(g,x) = \alpha(g,\theta(x)).$$

By assumption we can find $f : X \to \mathbb{T}$ and $\beta \in C_\Lambda(\sigma)$ such that

$$\alpha'(g,x) = f(g \cdot x)\beta(g,x)f(x)^*.$$

**Claim 1.** There is a character $\chi : \hat{A} \to \mathbb{T}$ such that $(\forall a \in \hat{A})f(a \cdot x) = \chi(a)f(x)$.

**Proof of Claim 1:** To see this, note that since $\alpha'$ is $\hat{A}$-invariant we have for all $a \in \hat{A}$ and $g \in \Lambda$ that

$$f(g \cdot x)\beta(g,x)f(x)^* = f(g \cdot a \cdot x)\beta(g,a \cdot x)f(a \cdot x)^*.$$

Using that $\beta(g,x)$ does not depend on $x$ for $g \in \Lambda$, this gives us

$$f(g \cdot x)f(x)^* = f(g \cdot a \cdot x)f(a \cdot x)^*,$$

and using that the $\Gamma$ and $\hat{A}$ actions commute this in turn gives us

$$f(a \cdot g \cdot x)^*f(g \cdot x) = f(a \cdot x)^*f(x).$$

Hence $f(a \cdot x)^*f(x)$ is $\Lambda$-invariant, and since the $\Lambda$-action is weakly mixing this means it must be constant. Thus

$$f(a \cdot x) = c_a f(x)$$

for some constant $c_a$. Let $\chi(a) = c_a$. □
It is easy to check now that if we define
\[ \gamma_f(g, x) = f(g \cdot x) f(x)^* \]
then this also defines a 1-cocycle of \( \sigma \hat{A} \), since \( \gamma_f(g, x) \) is \( \hat{A} \)-invariant. Moreover, \( \beta(g, x) \) is also \( \hat{A} \)-invariant, since
\[ \beta(g, x) = \alpha'(g, x) f(g \cdot x)^* f(x). \]
Hence \( \beta(g, x) \) is a \( \sigma \hat{A} \) 1-cocycle in \( C_\Lambda(\sigma \hat{A}) \). Let \( E \subseteq Z^1(\sigma \hat{A}) \) denote the subgroup of all 1-cocycles \( \delta \) satisfying
\[ \delta(g, \theta(x)) = f(g \cdot x) f(x)^* \]
for some \( \hat{A} \)-eigenfunction \( f : X \to T \). By the above we have \( Z^1(\sigma \hat{A}) = E C_\Lambda(\sigma \hat{A}) \), and by Lemma 2.4 we also have \( E \cap C_\Lambda(\sigma \hat{A}) = \{1\} \), and so it follows that
\[ H^1_{\Lambda, r}(\sigma \hat{A}) = E C_\Lambda(\sigma \hat{A}) / B^1(\sigma \hat{A}) C_\Lambda(\sigma \hat{A}) = E / B^1(\sigma \hat{A}). \]
Since \( \hat{A} \) is compact and acts freely on \( X \) it is possible for each character \( \chi : \hat{A} \to T \) to find a measurable function \( f : X \to T \) such that \( f(a \cdot x) = \chi(a) f(x) \) a.e. Hence
\[ H^1_{\Lambda, r}(\sigma \hat{A}) = E / B^1(\sigma \hat{A}) \simeq \text{Char}(\hat{A}) = A. \]

\[ \square \]

Recall that if \( E \) is a measure preserving equivalence relation then \( \text{Inn}(E) \) is the group of measure preserving transformations \( T \in \text{Aut}(X, \mu) \) such that \( x ET(x) \) a.e. Then we have:

3.3. LEMMA. Suppose \( \sigma \) and \( \tau \) a.e. free p.m.p. actions of a countable group \( \Gamma \) on \( (X, \mu) \) generating the same orbit equivalence relation \( E_\sigma = E_\tau = E \). Suppose \( \Lambda \subseteq \Gamma \) is a subgroup and that there is \( T \in \text{Inn}(E) \) such that \( T \sigma T^{-1} | \Lambda = \tau | \Lambda \). Then \( H^1_{\Lambda, r}(\sigma) \simeq H^1_{\Lambda, r}(\tau) \).

Proof. We may assume that \( \sigma | \Lambda = \tau | \Lambda \). Let \( \alpha : \Gamma \times X \to \Gamma \) be the cocycle defined by \( \tau(\alpha(g, x))(x) = \sigma(g)(x) \). Then \( \alpha | \Lambda = \text{Id} \). For each \( \beta \in Z^1(\tau) \) define
\[ \tilde{\beta}(g, x) = \beta(\alpha(g, x), x). \]
Then $\beta \mapsto \tilde{\beta}$ is an isomorphism $Z^1(\tau) \to Z^1(\sigma)$, since

$$
\tilde{\beta}(gg', x) = \beta(\alpha(gg', x), x)
= \beta(\alpha(g, \sigma(g')(x))\alpha(g'), x, x)
= \beta(\alpha(g, \sigma(g')(x)), \tau(\alpha(g', x))(\beta(\alpha(g', x), x)
= \beta(\alpha(g, \sigma(g')(x)), \sigma(g')(x))\beta(\alpha(g', x), x)
= \tilde{\beta}(g, \sigma(g')(x))\tilde{\beta}(g', x).
$$

Moreover, for $\gamma \in \Lambda$ we have

$$
\tilde{\beta}(\gamma, x) = \beta(\alpha(\gamma, x), x) = \beta(\gamma, x).
$$

Hence $\beta \mapsto \tilde{\beta}$ maps $B^1_{\Lambda, r}(\tau)$ isomorphically onto $B^1_{\Lambda, r}(\sigma)$, and so it follows that $H^1_{\Lambda, r}(\tau) \simeq H^1_{\Lambda, r}(\sigma)$.

Before stating the next Lemma, we recall various basic notions from [21]. Let $E$ be a measure preserving equivalence relation. We will say that two actions $\sigma$ and $\tau$ of a countable group $\Gamma$ with $E_\sigma, E_\tau \subseteq E$ such as in the previous Lemma are $E$-inner conjugate on $\Lambda$ if there is $T \in \text{Inn}(E)$ such that

$$
T\sigma|\Lambda T^{-1} = \tau|\Lambda.
$$

Following [21], we will say that a p.m.p. action $\sigma$ of the group $\Gamma$ is ergodic on $\Lambda$ (resp. weakly mixing on $\Lambda$), where $\Lambda \subseteq \Gamma$, just in case $\sigma|\Lambda$ is ergodic (resp. weakly mixing) as a $\Lambda$ action. The following was proved in [21], Lemma 4.1:

3.4. Lemma. Suppose $\Gamma$ has the relative property (T) over an infinite subgroup $\Lambda \subseteq \Gamma$ and let $E$ be a measure preserving countable equivalence relation. Then there are at most countably many ergodic on $\Lambda$ p.m.p. $\Gamma$ actions $E_\sigma \subseteq E$ that are not $E$-inner conjugate on $\Lambda$.

With this in hand we now have:

3.5. Theorem. If $\Gamma$ is a countable group with the relative property (T) over an infinite subgroup $\Lambda$, then $\Gamma$ has uncountably many orbit inequivalent a.e. free p.m.p. actions on a standard Borel probability space.

Proof. Suppose for a contradiction that there are uncountably many non-isomorphic countably infinite groups $\langle A_\xi : \xi < \omega_1 \rangle$ such that $\sigma^{\hat{A}_\xi}$ (as defined in Lemma 3.2) are orbit equivalent for all $\xi < \omega_1$. We can assume that all
The actions $\sigma^A\xi_1$ and $\sigma^A\xi_2$ generate the same orbit equivalence relation $E$. By the previous Lemma it follows that there are $\xi_1, \xi_2 < \omega$, $\xi_1 \neq \xi_2$, such that $\sigma^A\xi_1$ and $\sigma^A\xi_2$ are $E$-inner conjugate on $\Lambda$. But by Lemma 3.3 it then follows that

$$A_{\xi_1} \simeq H_{A,r}^1(\sigma^A\xi_1) = H_{A,r}^1(\sigma^A\xi_2) \simeq A_{\xi_2}$$

contradicting that $A_{\xi_1}$ and $A_{\xi_2}$ are not isomorphic. □

§4. A finer analysis

We now aim to refine Theorem 3.5 to show that in fact the actions $\sigma^A\xi$ are orbit inequivalent for non-isomorphic $A$. We start by noting a general lemma which is interesting in its own right:

4.1. Lemma. Suppose $\Gamma$ is a countable group with the relative property (T) over $\Lambda \trianglerighteq \Gamma$. Suppose $\sigma : \Gamma \ltimes (X,\mu)$ is an a.e. free p.m.p. malleable action which is weakly mixing on all infinite subgroups of $\Lambda$. Suppose $G$ is a countable group and $\tau : G \ltimes (X,\mu)$ is an a.e. free p.m.p. action which is ergodic on all infinite subgroups and such that $E_\sigma = E_\tau$ a.e. Then there is a homomorphism $\rho : \Lambda \rightarrow G$ such that $(G,\rho(\Lambda))$ has the relative property (T), $H_{\rho(\Lambda),r}(\tau) \simeq H_{\Lambda,r}(\sigma)$ and $H_{\rho(\Lambda),r}(\tau) \simeq H_{\rho(\Lambda),r}(\sigma) = \{1\}$.

Proof. Let $E = E_\sigma = E_\tau$. Let $\alpha : \Gamma \times X \rightarrow G$ be the cocycle defined by

$$\tau(\alpha(\gamma,x))(x) = \sigma(\gamma)(x).$$

Since $\sigma$ fulfills the hypothesis of the local superrigidity Theorem 2.6, we can find $\psi : X \rightarrow G$ and a homomorphism $\rho : \Lambda \rightarrow G$ such that

$$(\forall \gamma \in \Lambda)\psi(\gamma \cdot_\sigma x)\alpha(\gamma, x)\psi(x)^{-1} = \rho(\gamma).$$

Define $\Psi(x) = \psi(x) \cdot_\tau x$. Then $\Psi \subseteq E$ and for all $\gamma \in \Lambda$ we have

$$\Psi(\gamma \cdot_\sigma x) = \psi(\gamma \cdot_\sigma x) \cdot_\tau (\gamma \cdot_\sigma x) = (\psi(\gamma \cdot_\sigma x)\alpha(\gamma, x)) \cdot_\tau x = (\rho(\gamma)\psi(x)) \cdot_\tau x = \rho(\gamma) \cdot_\tau \Psi(x).$$

Thus $\Psi$ conjugates the $\Lambda$ and $\rho(\Lambda)$ actions via $\rho$, that is

$$\left(\forall \gamma \in \Lambda\right) \Psi(\gamma \cdot_\sigma x) = \rho(\gamma) \cdot_\tau \Psi(x). \quad (1)$$
Claim 1: \(|\ker(\rho)| < \infty.\)

Proof of Claim 1: Suppose not. The map \(\Psi\) is \(\ker(\rho)\) invariant by (1) and so since \(\sigma\) is ergodic on \(\ker(\rho)\) by assumption, we have that \(\Psi\) is constant on a measure 1 set. But this contradicts that \(\Psi \subseteq E\).

It follows that \(\rho(\Lambda)\) is infinite. Since moreover \(\Psi(X)\) is \(\rho(\Lambda)\) invariant, it follows by the ergodicity assumptions for the \(G\) action that \(\Psi(X)\) has full measure. Let \(\Psi'\) be a Borel right inverse of \(\Psi\), i.e. \(\Psi(\Psi'(y)) = y\). Then \(\Psi'\) is 1-1 and \(\Psi' \subseteq E\), and so \(\Psi'\) is measure preserving (see [15], proposition 2.1.) Thus \(\mu(\Psi'(X)) = 1\) and so \(\Psi\) is in fact a measure preserving transformation, with \(\Psi = \Psi^{-1}\). Note that it now follows that \(\ker(\rho) = \{1\}\) so that \(\rho(\Lambda)\) is in fact isomorphic to \(\Lambda\). Moreover, \((G, \rho(\Lambda))\) has property (T) by Proposition 2.9.

Claim 2: \(H^1_{\rho(\Lambda)}(\tau) \simeq H^1_{\Lambda}(\sigma)\) and \(H^1_{\rho(\Lambda),r}(\tau) \simeq H^1_{\Lambda,r}(\sigma)\).

Proof of Claim 2: The proof is similar to Lemma 3.3. After conjugating the \(G\)-action with \(\Psi\), we can assume that
\[
(\forall \gamma \in \Lambda) \sigma(\gamma)(x) = \tau(\rho(\gamma))(x) \quad (\text{a.e.})
\]
Note that since \(\Psi\) is inner, we still have that \(E_\sigma = E_\tau\). Let \(\alpha_0 : \Gamma \times X \to G\) be the corresponding cocycle defined by \(\tau(\alpha_0(\gamma, x))(x) = \sigma(\gamma)(x)\). Then for \(\gamma \in \Lambda\) we have \(\alpha_0(\gamma, x) = \rho(\gamma)\). Now we can proceed exactly as in Lemma 3.3 by defining an isomorphism \(Z^1(\tau) \to Z^1(\sigma) : \beta \mapsto \tilde{\beta}\) by
\[
\tilde{\beta}(\gamma, x) = \beta(\alpha_0(\gamma, x), x)
\]
and verify that \(\beta \mapsto \tilde{\beta}\) maps \(B^1_{\rho(\Lambda)}(\tau)\) isomorphically onto \(B^1_{\Lambda}(\sigma)\), and \(B^1_{\rho(\Lambda),r}(\tau)\) isomorphically onto \(B^1_{\Lambda,r}(\sigma)\).

Finally \(H^1_{\Lambda,r}(\sigma) = \{1\}\) follows from Corollary 2.8.

We now prove the “quotient” version of the previous Lemma:

4.2. Lemma   Suppose \(\Gamma, \Lambda\) and \(\sigma : \Gamma \acts (X, \mu)\) are as in the previous Lemma. Suppose \(A\) is a countably infinite Abelian group, \(\hat{A} = \text{Char}(A)\) its dual, that \(\hat{A}\) acts in a free, measure preserving way on \((X, \mu)\), and that the action of \(\hat{A}\) and \(\sigma\) commute. Let \((Y, \nu)\) be the corresponding quotient, \(\theta : X \to Y\) the quotient map, \(\sigma^\hat{A}\) the quotient action. Then if \(G\) is a countable group and \(\tau : G \acts (Y, \nu)\) is a p.m.p. a.e. free action of \(G\) which
is ergodic on all infinite subgroups and $E_\tau = E_{\sigma^A}$, then there is a subgroup $K \leq G$ such that $(G, K)$ has property $(T)$ and $H_{K,r}^1(\tau) = A$.

**Proof.** Since $E_{\sigma^A} = E_\tau$ and $\sigma^A$ and $\tau$ are a.e. free, we have a measurable cocycle $\alpha : G \times Y \to \Gamma$ such that $\tau(g)(x) = \sigma^A(\alpha(g, x))(x)$. Let $\alpha' : G \times X \to \Gamma$ be the lifted cocycle defined by $\alpha'(\gamma, x) = \alpha(\gamma, \theta(x))$. Note that $\alpha'$ determines an a.e. free p.m.p. action $\tau'$ of $G$ on $X$ by

$$
\tau'(g)(x) = \sigma(\alpha'(g, x))(x) = \sigma(\alpha(g, \theta(x)))(x).
$$

Namely, by this definition $\theta(\tau'(g)(x)) = \sigma^A(\alpha(g, \theta(x)))(\theta(x)) = \tau(g)(\theta(x))$ (2) for all $g \in G$. Thus we have

$$
\tau'(g_1 g_2)(x) = \sigma(\alpha'(g_1 g_2, x))(x) = \sigma(\alpha(g_1, \tau(g_2)(\theta(x)))\alpha(g_2, \theta(x)))(x)
= \sigma(\alpha(g_1, \theta(\tau'(g_2)(x)))]\alpha(g_2, \theta(x)))(x)
= \sigma(\alpha(g_1, \theta(\tau'(g_2)(x)))]\tau'(g_2)(x)
= \tau'(g_1)\tau'(g_2)(x).
$$

By the previous Lemma, we now have that $G$ has property $(T)$ over some infinite subgroup $K \leq G$, and that $H_{K,r}^1(\sigma^A) \simeq A$. But then by the Lemma 3.2 and (2) we have that $H_{K,r}^1(\tau) = A$, since $\tau$ and the action of $A$ commute.

**Theorem 1.** Suppose $\Gamma$ is a countably infinite group with the relative property $(T)$ over $\Lambda \leq \Gamma$ and $G$ is any countably infinite group. Let $A, A'$ be countably infinite abelian groups and let $\sigma^A$ and $\sigma^{A'}$ be quotients of classical Bernoulli shifts of $\Gamma$ and $G$ respectively. Then if $\sigma^A$ and $\sigma^{A'}$ are orbit equivalent, then $A$ and $A'$ are isomorphic.

**Proof.** We apply the previous Lemma to $\sigma^A$ and $\sigma^{A'}$. Then it follows that $G$ has the relative property $(T)$ over some infinite subgroup $K \leq G$ and that $A' \simeq H_{K,r}^1(\sigma^{A'}) \simeq A$.

4.3. **Corollary.** Let $\Gamma, A, A'$ be as in Theorem 1, and let $\sigma^A$ and $\sigma^{A'}$ be quotients of $\Gamma$-Bernoulli shifts. Then $\sigma^A$ and $\sigma^{A'}$ are orbit equivalent if and only if $A$ is isomorphic to $A'$.

**Proof.** By Theorem 1, it suffices to note that if $A$ is isomorphic to $A'$ then clearly $\sigma^A$ and $\sigma^{A'}$ are conjugate, so they are in particular orbit equivalent.
Let $\Gamma$ be a countable group, $(X, \mu)$ a standard Borel probability space. We denote by $\text{Aut}(X, \mu)$ the group of all $\mu$-measure preserving transformations of $X$, and equip it with the weak topology (see [8].) We let

$$\mathcal{A}(\Gamma, X) = \{\sigma \in \text{Aut}(X, \mu)^\Gamma : (\forall \gamma_1, \gamma_2 \in \Gamma)\sigma(\gamma_1 \gamma_2) = \sigma(\gamma_1)\sigma(\gamma_2)\}.$$ 

Since this set is closed in the product topology it is Polish, and we naturally identify $\mathcal{A}(\Gamma, X)$ with the space of all measure preserving actions of $\Gamma$ on $X$. There are various natural subspaces, namely, the a.e. free actions which we denote by $\mathcal{A}^*(\Gamma, X)$ and the ergodic a.e. free actions, denoted $\mathcal{A}^*_e(\Gamma, X)$.

It is natural consider the relations of conjugacy and orbit equivalence in $\mathcal{A}(\Gamma, X)$, or $\mathcal{A}^*_e(\Gamma, X)$. We denote them by $\simeq$ and $\simeq_{oe}$, respectively. It is easy to see that conjugacy is, prima facie, an analytic equivalence relation induced by the natural conjugation action of $\text{Aut}(X, \mu)$ on $\mathcal{A}(\Gamma, X)$. It can be shown (see below) that orbit equivalence is also an analytic equivalence relation. However, Dye’s Theorem implies that orbit equivalence has only one class in $\mathcal{A}^*_e(\mathbb{Z}, X)$, so it is certainly not just analytic here, it is Borel.

The main goal of this section is to prove:

**Theorem 2, (v.1).** Let $\Gamma$ be a countably infinite group with the relative property (T). Then orbit equivalence, considered as an equivalence relation in $\mathcal{A}^*_e(\Gamma, X)$, is complete analytic, and so in particular is not Borel.

5.1. **Borel reducibility.** To prove Theorem 2, we will utilize the theory of Borel reducibility of equivalence relations that has been developed extensively in descriptive set theory. Let $X, Y$ be Polish spaces and $E, F$ be equivalence relations on $X, Y$, respectively. (We do not assume that $X$ and $Y$ have any other structure than their Polish topology, and we do not assume anything about $E$ and $F$ for the moment, other than they are equivalence relations.) Then $E$ is said to be Borel reducible to $F$, written $E \leq_B F$, if there is a Borel $f : X \rightarrow Y$ such that

$$xEy \iff f(x)Ff(y).$$

A quick introduction to the significance of this notion is given in the introduction of [21]. Here it suffices to say that $\leq_B$ gives a degree theory for the complexity of equivalence relations.
Let $\text{ABEL}_{\aleph_0}$ denote the space of countably infinite Abelian groups, $\simeq_{\text{ABEL}_{\aleph_0}}$ the isomorphism relation among such groups. $\text{ABEL}_{\aleph_0}$ can be identified with the following Polish space:

$$\text{ABEL}_{\aleph_0} = \{(\cdot,e) \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N} : ((\forall i,j,k \in \mathbb{N})(i \cdot j) \cdot k = i \cdot (j \cdot k) \land ((\forall j \in \mathbb{N})e \cdot j = j) \land ((\forall k \in \mathbb{N})(\exists l \in \mathbb{N})k \cdot l = e) \land (\forall i,j \in \mathbb{N})i \cdot j = j \cdot i) \}. $$

This is clearly a closed set in the product topology, and so it is Polish. Note that $\simeq_{\text{ABEL}_{\aleph_0}}$ is induced by the natural action of the infinite symmetric group $S_\infty$ on $\mathbb{N}$. For notational convenience, if $G \in \text{ABEL}_{\aleph_0}$ then we will write $\cdot_G$ for multiplication in $G$ and $e_G$ for the identity in $G$, i.e. $G = (\cdot_G, e_G)$.

It is known by Theorem 6 of [6] that the isomorphism relation for Abelian $p$-groups is complete analytic. Hence Theorem 2 version 1 will follow at once from Theorem 2 version 2 below, which itself is a consequence of Theorem 1. Note that per the usual convention in descriptive set theory, $\simeq_{\text{TF A}_{\aleph_0}}(\Gamma, X)$ denotes the restriction of $\simeq_{\text{oe}}$ to $A_0^*(\Gamma, X)$.

**Theorem 2 (v.2).** If $\Gamma$ is a countably infinite group with the relative property (T) and $(X, \mu)$ is a standard Borel probability space then $\simeq_{\text{ABEL}_{\aleph_0}}$ is Borel reducible to $\simeq_{\text{TF A}_{\aleph_0}}(\Gamma, X)$.

**Remark.** Let $\text{TF A}_{\aleph_0}$ denotes the subset of $\text{ABEL}_{\aleph_0}$ consisting of torsion free Abelian groups. It was shown by Hjorth in [11] that if $E$ is an equivalence relation on a Polish space $X$ and $\simeq_{\text{TF A}_{\aleph_0}} \leq_B E$ then $E$ cannot be Borel. Using Hjorth’s technique, Downey and Montalban have recently shown in [3] that in fact $\simeq_{\text{TF A}_{\aleph_0}}$ is a complete analytic subset of $\text{TF A}_{\aleph_0}^2$. Theorem 2, v.2 clearly shows that $\simeq_{\text{TF A}_{\aleph_0}} \leq_B \simeq_{\text{TF A}_{\aleph_0}}(\Gamma, X)$ and so the result of Downey and Montalban gives another reason why $\simeq_{\text{TF A}_{\aleph_0}}(\Gamma, X)$ is complete analytic.

The proof of Theorem 2, v.2, involves an amount of coding since the measure preserving actions we used to prove Theorem 1 are defined on different probability spaces. We need a few general lemmata to deal with this. The reader should know that we rely heavily on the results in [14], chapters 4.F, 12 and 17 and 28; it is indeed the correct reference for all the descriptive set theory needed here.
5.2. Lemma. Suppose $X$ is a Polish space and let $P_c(X)$ denote the Polish space of continuous probability measures on $X$. Then there is a Borel map $f : P_c(X) \times X \to [0, 1]$ such that for all $\mu \in P_c(X)$ the map $f(\mu, \cdot) = f_\mu$ is a $\mu$-measure preserving bijection from a set of full $\mu$-measure in $X$ onto a set of full measure in $([0, 1], m)$, where $m$ is Lebesgue measure.

Proof. We may assume that $X = [0, 1]$. Then we can go ahead as in the proof of [13, 17.41], and define

$$f(\mu, x) = \mu([0, x]).$$

By [13, 17.25], this is Borel. Exactly as in the proof of [13, 17.41], we have that $f_\mu$ is a measure preserving bijection between sets of full measure, so $f$ is as promised. \qed

Let

$$C = \{(G, x) \in \text{ABEL}_{\mathbb{N}_0} \times T^n : (\forall g_1, g_2 \in \mathbb{N})x(g_1 \cdot G, g_2) = x(g_1)x(g_2)\}.$$ 

Then for each $G \in \text{ABEL}_{\mathbb{N}_0}$ the set $C_G$ is exactly the set of characters on the group $(\mathbb{N}, \cdot, e_G)$. Since $C_G$ is compact we have by [13, 28.8] that the map $\text{Char} : \text{ABEL}_{\mathbb{N}_0} \to K(T^n)$ where $\text{Char}(G) = C_G$ is Borel, where $K(T^n)$ denotes the compact hyperspace of $T^n$ as defined in [13, 4F]. We now have

5.3. Lemma. The map $H : \text{ABEL}_{\mathbb{N}_0} \to P(T^n)$, which assigns to $G \in \text{ABEL}_{\mathbb{N}_0}$ the Haar measure on $\text{Char}(G)$, is Borel.

Proof. Let $(O_n)$ be a countable basis for the topology on $T$. Let $P$ be the set of all finite partial functions $f$ with $\text{dom}(f) \subseteq \mathbb{N}$ and $\text{ran}(f) \subseteq \mathbb{N}$. For each such $f$, let

$$U_f = \{x \in T^n : (\forall i \in \text{dom}(f))x(i) \in O_{f(i)}\}.$$ 

Then $(U_f)_{f \in P}$ forms a countable basis for the product topology on $T^n$, which is invariant under the action of the full permutation group $S_\infty$ of $\mathbb{N}$ on $T^n$. Let $F_f = T^n \setminus U_f$. Define

$$H = \{(G, \mu) \in \text{ABEL}_{\mathbb{N}_0} \times P(T^n) : \mu(\text{Char}(G)) = 1 \land (\forall f \in P)(\forall g \in \mathbb{N})\mu(F_f) = \mu(F_{g \cdot f})\}$$

where $g \cdot f(i) = j \iff f(g^{-1} \cdot i) = j$. By [13, 17.29] $H$ is Borel, and by definition we have $(G, \mu) \in H$ precisely when $\mu$ is the Haar measure on $\text{Char}(G)$. By the uniqueness of the Haar measure and [13, 14.12] it follows that $H$ defines a Borel function $\text{ABEL}_{\mathbb{N}_0} \to P(T^n)$ as required. \qed
If $f : X \to Y$ is Borel, $X$, $Y$ Polish spaces, and $\mu$ a measure on $X$, then we denote by $f[\mu]$ the push-forward measure on $Y$. (Note that our notation differs from [14] here, but is in line with [21]):

5.4. **Lemma.** If $f : X \times Y \to Z$ is a Borel map then there is a Borel $f^* : X \times P(Y) \to P(Z)$ such that $f^*(x, \mu) = f_x[\mu]$, where $f_x : Y \to Z : y \mapsto f(x, y)$.

**Proof.** By [14] 17.27 and 17.40 the map $X \times P(Y) \to P(X \times Y) : (x, y) \mapsto \delta_x \times \mu$ is Borel. So by [14] 17.28 we have that the map $X \times P(Y) \to P(Z) : (x, \mu) \mapsto f[\delta_x \times \mu]$ is Borel. Now note that $f[\delta_x \times \mu] = f_x[\mu]$. \qed

**Proof of Theorem 2, v. 2.** Let $\Gamma$ be a fixed countably infinite group and let $X = (\mathbb{T}^\mathbb{N})^\Gamma$. Consider $K(X)$, the space of compact subsets of $X$. Note that $\Gamma$ acts on $K(X)$ since it acts on $X$ by left-shift, and for each $G \in \text{ABEL}_{\mathbb{N}_0}$, $\text{Char}(G) = C_G$ acts naturally on $X$. Consider the map $f : \text{ABEL}_{\mathbb{N}_0} \times X \to K(X)$ defined by

$$f(G, x) = [x]_{C_G}.$$ 

The map $f$ is Borel since if we fix Borel $d_n : K(X) \to X$ and $d'_n : K(\mathbb{T}^\mathbb{N}) \to \mathbb{T}^\mathbb{N}$ as in [14] 12.13, with $(d_n(K))_{n \in \mathbb{N}}$ dense in $K$ for all $K \in K(X)$ and $(d'_n(K'))_{n \in \mathbb{N}}$ dense in $K'$ for all $K' \in K(\mathbb{T}^\mathbb{N})$, then

$$f(G, x) = K \iff (\forall n)(\exists \chi \in C_G) \chi \cdot x = d_n(K) \land (\forall m)d'_n(C_G) \cdot K = K$$

gives an analytic definition of the graph of $f$, which suffices by [14] 14.12. We identify the space $C_G^T/C_G$ with the range of $f_G = f(G, \cdot)$.

Let $f^* : \text{ABEL}_{\mathbb{N}_0} \times P(X) \to P(K(X))$ be as in Lemma 5.4. Let $H$ be as in Lemma 5.3; then we have a map $H^T : \text{ABEL}_{\mathbb{N}_0} \to P((\mathbb{T}^\mathbb{N})^\Gamma)$ such that $H^T(G)$ the product measure $H(G)^T$ and this map is Borel by (the obvious generalization of) [14] 17.40. Note that $f^*(G, H^T(G))$ is the push-forward measure on $C_G^T/C_G$ of the measure $H^T(G)$ under the map $f_G$. Now fix a map $f_0 : P(K(X)) \times K(X) \to [0, 1]$ as in Lemma 5.2. Define

$$\theta : \text{ABEL}_{\mathbb{N}_0} \times K(X) \to [0, 1] : \theta(G, K) = f_0(f^*(G, H^T(G)), K).$$

Then for each $G \in \text{ABEL}_{\mathbb{N}_0}$ the map $\theta_G = \theta(G, \cdot)$ defines a measure preserving bijection between co-null subsets of $(C_G^T/C_G, f^*(G, H^T(G))$ and $([0, 1], m)$. Define $\Theta : \text{ABEL}_{\mathbb{N}_0} \times \Gamma \times [0, 1] \to [0, 1]$ by

$$\Theta(G)(g)(x) = y \iff (\exists K \in K(X)) \theta_G(K) = x \land \theta_G(g \cdot K) = y.$$
Since the measure quantifiers preserve analyticity (see [14] p. 233) Θ is a Borel function, and by construction Θ_G is a measure preserving Γ-action on [0, 1] which is conjugate with the action of Γ on C^1_{CG}/C_G, for all G ∈ ABEL_{ℵ₀}. Corollary 4.3 now guarantees that G ↦→ Θ_G is a Borel reduction of ≃_{ABEL_{ℵ₀}} to orbit equivalence in $A^*_e(Γ, [0, 1])$. □

In order to verify Theorem 2, v.1, it suffices to prove the following easy lemma.

5.5. LE M M A. If Γ is a countable group then ≃_{oe} is an analytic subset of $A(Γ, X, μ) \times A(Γ, X, μ)$.

Proof. As proved in Lemma 3 in [22], there is a Borel relation $E ⊆ Aut(X, μ) \times X \times X$ such that for each $S ∈ Aut(X, μ)$ we have that

$\tilde{S}(x) = y \iff E(S, x, y)$

defines a measure preserving Borel function $\tilde{S}$ a.e. such that $\tilde{S} ∈ S$. Define

$R(σ, x, y) \iff (∃g ∈ Γ)E(σ(g), x, y)$.

Then

$(∀μ x, y)xE_σy \iff R(σ, x, y)$

and thus

$σ ≃_{oe} τ \iff (∃T ∈ Aut(X, μ))(∀μ x, y)R(σ, x, y) \iff R(TτT^{-1}, x, y)$,

which proves that ≃_{oe} is analytic, since the measure quantifiers preserve analyticity. □

Remark 1. Clearly the proof also gives a Borel reduction of ≃_{TFA_{ℵ₀}} to conjugacy of measure preserving actions. We explicitly note that the following corollary, which should be compared with the result of a similar nature for Z-actions, due to Foreman, Rudolph and Weiss, in [4]:

5.6. C O R O L L A R Y. If Γ is a countably infinite group with the relative property (T) then the conjugacy relation for ergodic, a.e. free p.m.p. actions of Γ on [0, 1] is analytic, but not Borel.
Remark 2. The results of [21] imply that under fairly general conditions, if a countably infinite group \( \Gamma \) has the relative property (T), then both conjugacy and orbit equivalence of p.m.p. ergodic a.e. free actions of \( \Gamma \) is not classifiable by “countable structures” (as defined in [10]), which in particular implies that it is not possible to Borel reduce conjugacy and orbit equivalence in this setting to \( \text{ABEL}_{\aleph_0} \). Thus we have the following:

5.7. Corollary. If \( \Gamma \) has the relative property (T) over an infinite subgroup which either contains an infinite abelian subgroup, or is normal in \( \Gamma \), then \( \simeq_{\text{ABEL}_{\aleph_0}} <_B \simeq_{\text{oe}}^{A_{e}}(\Gamma, [0,1]) \). The same holds for the conjugacy relation in \( A_{e}^{*}(\Gamma, [0,1]) \).

The normality condition in Corollary 5.6 can be replaced with the technically weaker notion of being index stable; we refer the reader to the last section of [21] for details.

Note: Since the appearance of this paper, Ioana, Kechris and Tsankov have shown that if \( \Gamma \) is any non-amenable countable discrete group, then orbit equivalence of its measure preserving ergodic (indeed mixing) actions are not classifiable by countable structures, see [12].

References

[1] H. Becker, A. Kechris, The descriptive set theory of Polish group actions, London Mathematical Society lecture notes, vol. 232 (1996), Cambridge University Press.

[2] B. Bekka, P. de la Harpe, A. Valette, Kazhdan’s Property (T), Cambridge University Press, (2008).

[3] R. Downey, A. Montalban, The isomorphism problem for torsion free abelian groups is analytic complete, J. Algebra vol. 320 (2008), no. 6, pp. 2291–2300.

[4] M. Foreman, D. Rudolph, B. Weiss On the conjugacy relation in ergodic theory, Comptes Rendus Mathematique, vol. 343 (2006), Issue 10, pp. 653–656.

[5] M. Foreman, B. Weiss, An anti-classification theorem for ergodic measure preserving transformations, J. Eur. Math. Soc. 6 (2004), no. 3, pp. 277–292.

[6] H. Friedman, Lee Stanley, A Borel reducibility theory for classes of countable structures, Journal of Symbolic Logic, vol. 54, no. 3 (1989), pp. 894-914.

[7] A. Furman, On Popa’s cocycle superrigidity theorem, Int. Math. Res. Not. IMRN 2007, no. 19, Art. ID rnm073, 46 pp.
[8] P. Halmos, *Lectures on ergodic theory*, Chelsea Publishing Co., New York (1956).

[9] G. Hjorth, *A converse to Dye’s theorem*, Trans. Amer. Math. Soc., 357 (2005) no. 8, pp. 3083–3103.

[10] G. Hjorth, *Classification and Orbit Equivalence Relations*, Mathematical Surveys and Monographs 75, American Mathematical Society (2000).

[11] G. Hjorth, *The isomorphism relation on countable torsion free abelian groups*, Fundamenta Mathematicae, 175, (2002), pp. 241–257.

[12] A. Ioana, A. Kechris, T. Tsankov, *Subequivalence relations and positive-definite functions*, to appear in Groups, Geometry and Dynamics, (2009).

[13] P. Jolissaint, *On property (T) for pairs of topological groups*, Enseign. Math., (2) 51 (2005), no. 1-2, pp. 31–45.

[14] A. Kechris, *Classical descriptive set theory*, Springer Verlag, New York, (1995).

[15] A. Kechris, *Lectures on costs and equivalence relations and groups*, Lecture notes, Caltech, (2001).

[16] A. Kechris, *Global aspects of ergodic group actions and equivalence relations*, to appear in the series ”Mathematical Surveys and Monographs” of the AMS.

[17] Popa, S. *Some computations of 1-cohomology groups and construction of non-orbit-equivalent actions*, J. Inst. Math. Jussieu 5 (2006), no. 2, pp. 309–332

[18] Popa, S. *Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups*, to appear in Invent.Math. [math.GR/0512646]

[19] K. Schmidt, *Asymptotically invariant sequences and an action of SL(2, Z) on the sphere*, Israel Journal of Mathematics, vol. 37 (1980), pp. 193–208.

[20] K. Schmidt, *Amenability, Kazhdan’s property T, strong ergodicity and invariant means for ergodic group-actions*, Ergodic Theory and Dynamical Systems 1 (1981), pp. 223–236.

[21] A. Törnquist, *Conjugacy, orbit equivalence and classification for measure preserving group actions*, Ergodic Theory And Dynamical Systems (electronic, 2008).

[22] A. Törnquist, *Orbit equivalence and actions of $\mathbb{F}_n$*, Journal of Symbolic Logic, vol. 71, no. 1 (2006), pp. 265–282.
