EXISTENCE, BLOW-UP AND EXPONENTIAL DECAY ESTIMATES FOR A SYSTEM OF NONLINEAR VISCOELASTIC WAVE EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. This paper is devoted to the study of a system of nonlinear viscoelastic wave equations with nonlinear boundary conditions. Based on the Faedo-Galerkin method and standard arguments of density corresponding to the regularity of initial conditions, we first establish two local existence theorems of weak solutions. By the construction of a suitable Lyapunov functional, we next prove a blow up result and a decay result of global solutions.

1. Introduction. In this paper, we consider the initial-boundary value problem for the following system of nonlinear viscoelastic wave equations

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
    u_{tt} - \frac{\partial}{\partial x} \left( \mu_1(x,t)u_x \right) + \lambda_1 |u_t|^{r_1-2} u_t = f_1(u,v) + F_1(x,t), & 0 < x < 1, \ 0 < t < T, \\
    v_{tt} - \frac{\partial}{\partial x} \left( \mu_2(x,t)v_x \right) + \lambda_2 |v_t|^{r_2-2} v_t + \int_0^t g(t-s) \frac{\partial}{\partial x} \left( \bar{\mu}_2(x,s)v_x(x,s) \right) ds = f_2(u,v) + F_2(x,t), & 0 < x < 1, \ 0 < t < T, \\
    \mu_1(0,t)u_x(0,t) = \lambda_1 |u_t(0,t)|^{r_1-2} u_t(0,t) - g_0(t), \\
    u(1,t) = v(0,t) = v(1,t) = 0 
\end{array} \right.
\end{aligned}
\]

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\[
\begin{aligned}
\left\{
\begin{array}{l}
u(x,0) = \tilde{u}_0(x), \quad \nu_t(x,0) = \tilde{u}_1(x), \\
\nu(x,0) = \tilde{v}_0(x), \quad \nu_t(x,0) = \tilde{v}_1(x),
\end{array}
\right.
\end{aligned}
\] (3)

where \(\lambda_i > 0, r_i \geq 2 (i = 1, 2), \bar{\lambda}_1 > 0, \bar{r}_1 \geq 2\) are given constants and \(\tilde{u}_i, \tilde{v}_i (i = 0, 1), F_i, f_i, \mu_i (i = 1, 2), \bar{\mu}_2, G, g, g_0\) are given functions satisfying conditions specified later.

Problems of this type with many results concerning local existence, global existence, decay, and blow-up of solutions for a system of wave equations have been extensively studied by many authors. For example, we refer to \([1]-[6], [8, 9], [11]-[30]\), and the references given therein.

In \([11]\), the general decay and blow-up of solutions for a system of viscoelastic equations of Kirchhoff type with strong damping were considered. Here, two blow-up results with nonpositive initial energy as well as positive initial energy by exploiting the convexity technique were established and a decay result of global solutions by the perturbed energy method under a weaker assumption on the relaxation functions were proved.

In \([17]\), Messaoudi established a blow up result for solutions with negative initial energy and a global existence result for arbitrary initial data of a nonlinear viscoelastic wave equation associated with initial and Dirichlet boundary conditions.

It is also well known that the single viscoelastic wave equation of the form

\[
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u ds + h(u_t) = f(u), \quad (x,t) \in \Omega \times \mathbb{R}_+,
\]

with initial and boundary conditions, where \(\Omega \subset \mathbb{R}^n\) is bounded domains with a smooth boundary \(\partial \Omega\), has been extensively studied and many results concerning existence, nonexistence, exponential decay and blow up in finite time have been proved. For example, we refer to the series works of Thomas C Sideris, Zhen Lei and references therein \([8, 9], [25]-[27]\) about classical solutions to the equations of elasticity and viscoelasticity.

In \([26]\), Sideris considered the equations of motion for the displacement of an isotropic, homogeneous, hyperelastic material of the form of a quasilinear hyperbolic system in three space dimensions and the author proved that for certain classes of materials, small initial disturbances gave rise to global smooth solutions. These special materials were distinguished by a null condition imposed on the quadratic portion of the nonlinearity.

In \([27]\), the existence of global classical solutions to the Cauchy problem in nonlinear elastodynamics was studied. The unbounded elastic medium was assumed to be homogeneous, isotropic, and hyperelastic. As in the theory of 3D nonlinear wave equations in three space dimensions, global existence hinges on two basic assumptions. First, the initial deformation must be a small placement from equilibrium, in this case a prestressed homogeneous dilation of the reference configuration, and equally important, the nonlinear must obey a type of nonresonance or null condition. The omission of either of these assumptions can lead to the breakdown of solutions.

In \([8]\), the long-time behavior of elastic waves for isotropic incompressible materials was studied in 2-D. The equations of incompressible elastodynamics displayed a linear degeneracy in the isotropic case; i.e., the equation inherently satisfied a null condition. The authors proved that the 2-D incompressible isotropic nonlinear elastic system was almost globally well-posed for small initial data. More precisely,
the authors proved that for initial data of the form $u_0$, there exists a unique solution for a time interval $[0, \exp(T(u_0)/\varepsilon)]$, where $T(u_0)$ depends only on a certain weighted Sobolev norm of the $u_0$.

M. M. Cavalcanti et al. [4] studied the existence of global solutions and the asymptotic behavior of the energy related to a degenerate system of wave equations with boundary conditions of memory type. By the construction of a suitable Lyapunov functional, the authors proved that the energy decays exponentially. The same method was also used in [24] to study the asymptotic behavior of the solutions to a coupled system of wave equations having integral convolutions as memory terms. The author showed that the solution of that system decays uniformly in time, with rates depending on the rate of decay of the kernel of the convolutions.

In [16, 28], the existence, regularity, blow up and exponential decay estimates of solutions for nonlinear wave equations associated with two-point boundary conditions have been established. In which, the main tools to obtain existence results are the Galerkin method associated to a priori estimates, weak convergence, compactness techniques. On the other hand, a suitable Lyapunov functional was constructed to obtain the blow up and exponential decay results.

The above mentioned works lead to the study of the existence, blow-up and exponential decay estimates for a system of nonlinear wave equations (1)-(3). It consists of four sections.

Section 2 is devoted to the presentation of the existence results based on Faedo-Galerkin method and standard arguments of density corresponding to the regularity of initial conditions. In this section, problem (1)-(3) is dealt with two cases of $(\tilde{r}_1, G)$: (i) $\tilde{r}_1 = 2$ and $G \in C^1(\mathbb{R})$ or (ii) $\tilde{r}_1 > 2$ and $G \in C^2(\mathbb{R})$ and $r_1, r_2 \geq 2$. In the cases $\tilde{r}_1 = 2$ and $G \in C^1(\mathbb{R})$ or $\tilde{r}_1 > 2$ and $G \in C^2(\mathbb{R})$ such that $G''(z) \leq \zeta_{\max} < \mu_{1*}$, $\forall z \in \mathbb{R}$, with $\mu_1(x, t) \geq \mu_{1*} > 0$ for all $(x, t) \in [0, 1] \times \mathbb{R}_+$, the solution obtained here is unique. Because of the difficulties arising in passing to the limit in the nonlinear terms imposed the value on boundary, we need to use the Sobolev imbedding theorem suitably combined with the technical estimates; thus, the problem here is considered in the cases of $(\tilde{r}_1, G)$ and it is still an open problem in higher-dimensional case.

In Sections 3, 4, problem (1)-(3) is considered with $r_1 = r_2 = \tilde{r}_1 = 2$ and $\mu_2(x, t) = \tilde{\mu}_2(x)$. Under some suitable conditions, by applying techniques as in [28] with some necessary modifications, we prove that the solution of (1)-(3) blows up in finite time. We also prove that the solution $(u(t), v(t))$ will exponential decay if the initial energy is positive and small.

2. The existence and uniqueness of a weak solution. The notations we use in this paper is standard and can be found in Lions’ book [10], with $\| \cdot \|$ for the norm in $L^2$ and $\| \cdot \|_{H^1}$ for the norm in $H^1$. Using the norm $\| v \|_{H^1} = (\| v \|^2 + \| v_x \|^2)^{1/2}$, we have the following lemma, it is a known property.

Lemma 2.1. The imbedding $H^1 \hookrightarrow C^0([0, 1])$ is compact and

$$\| v \|_{C^0([0, 1])} \leq \sqrt{2} \| v \|_{H^1} \text{ for all } v \in H^1.$$  

Remark 2.2. On $V = \{ v \in H^1 : v(1) = 0 \}$, two norms $v \mapsto \| v \|_{H^1}$ and $v \mapsto \| v_x \|$ are equivalent. Furthermore, $\| v \|_{C^0([0, 1])} \leq \| v_x \|$ for all $v \in V$. 


Remark 2.3. The weak formulation of the initial boundary valued problem (1)-(3) can be given in the following manner: Find \((u, v) \in W = \{(u, v) \in L^\infty(0, T; (V \cap \mathcal{H}^2) \times (H_0^1 \cap \mathcal{H}^2)); (u', v') \in L^\infty(0, T; \mathcal{V} \times H_0^1); (u'', v'') \in L^\infty(0, T; L^2 \times L^2)\}, \) such that \((u, v)\) satisfies the following variational equation

\[
\begin{aligned}
&\langle u''(t), w \rangle + a_1(t; u(t), w) + \lambda_1 \langle \Psi_{r_1}(u'(t)), w \rangle + \lambda_1 \Psi_{r_1}(u'(0), t)w(0) \\
&= G(u, 0, t) + g_0(t)w(0) + \langle f_1(u(t), v(t)), w \rangle + \langle F_1(t), w \rangle,
\end{aligned}
\]

\[
\begin{aligned}
&\langle v''(t), \phi \rangle + a_2(t; v(t), \phi) + \lambda_2 \langle \Psi_{r_2}(v'(t)), \phi \rangle - \int_0^t g(t-s)\tilde{a}_2(s; v(s), \phi)ds \\
&= \langle f_2(u(t), v(t)), \phi \rangle + \langle F_2(t), \phi \rangle
\end{aligned}
\]

for all \((w, \phi) \in V \times H_0^1\), together with the initial conditions

\[
\begin{aligned}
&u(0), u'(0) = (\tilde{u}_0, \tilde{u}_1), \quad v(0), v'(0) = (\tilde{v}_0, \tilde{v}_1),
\end{aligned}
\]

where \(\Psi_{r}(z) = |z|^{r-2}z, r \in \{r_1, r_2, r_1\}, \) with \(r_1, r_2 \geq 2, i = 1, 2\) are given constants, 
\(\{a_1(t; \cdot, \cdot)\}_{0 \leq t \leq T}, \{a_2(t; \cdot, \cdot)\}_{0 \leq t \leq T}\) are the families of symmetric bilinear forms on \(V \times V\) or \(H_0^1 \times H_0^1\) are defined by

\[
\begin{aligned}
a_1(t; u, w) = \langle \mu_1(t)u_x, w_x \rangle \quad \text{for all } u, w \in V, \quad 0 \leq t \leq T,
\end{aligned}
\]

and

\[
\begin{aligned}
a_2(t; v, \phi) = \langle \mu_2(t)v_x, \phi_x \rangle, \quad \tilde{a}_2(t; v, \phi) = \langle \tilde{\mu}_2(t)v_x, \phi_x \rangle \quad \text{for all } v, \phi \in H_0^1, \quad 0 \leq t \leq T.
\end{aligned}
\]

In this section, we shall consider problem (4) with \(\lambda_1, \lambda_2, \tilde{\lambda}_1 > 0, r_1, r_2, r_1 \geq 2\) and make the following assumptions:

\((H_1)\) \((\tilde{u}_0, \tilde{v}_0) \in (V \cap \mathcal{H}^2) \times (H_0^1 \cap \mathcal{H}^2), \ (\tilde{u}_1, \tilde{v}_1) \in V \times H_0^1; \)

\((H_2)\) \(\mu_1, \mu_2, \tilde{\mu}_2 \in C^1(\mathbb{R}^2; L^\infty), \mu_1', \mu_2', \tilde{\mu}_2' \in L^1(0, T; L^\infty)\) and there exist the positive constants \(\mu_1, \mu_2, \tilde{\mu}_2\) such that

\((i)\) \(\mu_1(x, t) \geq \mu_1 > 0 \ \forall (x, t) \in [0, 1] \times \mathbb{R}_+; \)

\((ii)\) \(\tilde{\mu}_2(x, t) \geq \tilde{\mu}_2 > 0 \ \forall (x, t) \in [0, 1] \times \mathbb{R}_+; \)

\((H_3)\) there exists \(F \in C^2(\mathbb{R}^2; \mathbb{R})\) such that

\((i)\) \(D_1F(u, v) = f_1(u, v), \) \(D_2F(u, v) = f_2(u, v), \)

\((ii)\) there exist the constants \(\alpha, \beta, \alpha_1, \beta_1, \tilde{C}_1 > 0,\) such that

\[
F(u, v) \leq \tilde{C}_1 \left(1 + |u|^{\alpha} + |u|^{\alpha_1} |v|^{\beta_1} + |v|^\beta\right) \quad \forall u, v \in \mathbb{R};
\]

\((H_4)\) \(G \in C^1(\mathbb{R})\) and there exist the constants \(\gamma > 1, \tilde{C}_2 > 0,\) such that

\((i)\) \(gG(y) > 0 \ \forall y \in \mathbb{R}, \ y \neq 0,\)

\((ii)\) \(\int_0^y G(z)dz \leq \tilde{C}_2 (1 + |y|^\gamma) \ \forall y \in \mathbb{R};\)

\((H_5)\) \(g \in W^{2,1}(0, T); \)

\((H_6)\) \(f_1, f_2 \in L^1(0, T; L^2)\) such that \(f_1', f_2' \in L^1(0, T; L^2);\)

\((H_7)\) \(g_0 \in W^{2,1}(0, T);\)

\((H_1^*)\) \((\tilde{u}_0, \tilde{v}_0) \in V \times H_0^1, \ (\tilde{u}_1, \tilde{v}_1) \in L^2 \times L^2; \)

\((H_2^*)\) \(g \in H^1(0, T); \)

\((H_6^*)\) \(F_1, F_2 \in L^2(Q_T), \ Q_T = \Omega \times (0, T); \)

\((H_7^*)\) \(g_0 \in H^1(0, T). \)

Remark 2.4. Below we present an example such that the functions \(f_1, f_2, G\) satisfy the assumptions \((H_3), \) \((H_4),\) respectively. In this example, \(f_1, f_2\) are more general than the functions given in [19, 21].

Example 2.5. Consider the following functions

\[
f_1(u, v) = \gamma_1 \left(\alpha \ln^{k_1}(e + u^2) + 2k_1 \frac{u^2 \ln^{k_1-1}(e + u^2)}{e + u^2}\right) u^{a-2}u
\]

(8)
such that
\[
2 f(u, v) = \gamma_2 \left( \beta \ln k_2 (e + v^2) + 2k_2 \frac{v^2 \ln k_2 - 1 (e + v^2)}{e + v^2} \right) |v|^{\beta_2 - 2} v
\]
\[+
\gamma_3 \left( \beta_1 \ln k_3 (e + u^2 + v^2) + 2k_3 \frac{v^2 \ln k_3 - 1 (e + u^2 + v^2)}{e + u^2 + v^2} \right) |u|^{\alpha_1 - 2} u |v|^{\beta_1 - 2} v,
\]
\[G(y) = \tilde{g} |y|^{-2} \ln k_4 (e + y^2),
\]
\[\alpha, \beta, \alpha_1, \beta_1, \gamma > 2, k_1, k_2, k_3, k_4 > 1, \gamma_1, \gamma_2, \gamma_3, \tilde{g} \text{ are positive constants.}
\]
It is obvious that \((H_3, (i))\) holds, because there exists a \(F \in C^2(\mathbb{R}^2; \mathbb{R})\) with
\[\mathcal{F}(u, v) = \gamma_1 |u|^\alpha \ln k_1 (e + u^2) + \gamma_2 |v|^\beta \ln k_2 (e + v^2) + \gamma_3 |u|^{\alpha_1} |v|^{\beta_1} \ln k_3 (e + u^2 + v^2)
\](9)
such that
\[
D_1 \mathcal{F}(u, v) = f_1(u, v), \quad D_2 \mathcal{F}(u, v) = f_2(u, v),
\]
\[\mathcal{F}(u, v) \leq C_1 \left( 1 + |u|^{N_1} + |v|^{N_1} + |u|^{N_1} |v|^{N_1} \right)
\]
where
\[C_1 = 7 \max \{ \gamma_1 2^{k_1-1}, \gamma_2 2^{k_2-1}, \gamma_3 2^{k_3-1} \}, \]
\[N_1 = \max \{ \alpha + 2k_1, \beta + 2k_2, \alpha_1 + 2k_3, \beta_1 + 2k_3 \}.
\]
We check (10)_2 as below. Using the inequalities \(\ln(1 + x) \leq x, (a + b)^k \leq 2^{k-1}(a^k + b^k) \forall x, a, b \geq 0, \forall k \geq 1,\) it is clear that
\[
\ln k_1 (e + u^2) = \left[ 1 + \ln(1 + \frac{u^2}{e}) \right]^{k_1} \leq \left[ 1 + \frac{u^2}{e} \right]^{k_1}
\]
\[
\leq 2^{k_1 - 1} \left( 1 + \frac{|u|^{2k_1}}{e^{k_1}} \right) \leq 2^{k_1 - 1} \left( 1 + |u|^{2k_1} \right),
\]
\[\ln k_2 (e + v^2) \leq 2^{k_2 - 1} \left( 1 + |v|^{2k_2} \right),
\]
\[\ln k_3 (e + u^2 + v^2) \leq 2^{k_3 - 1} \left[ 1 + \left( \frac{u^2 + v^2}{e} \right)^{k_3} \right]
\]
\[
\leq 2^{k_3 - 1} \left( 1 + \frac{2^{k_3 - 1}}{e^{k_3}} \left( |u|^{2k_3} + |v|^{2k_3} \right) \right)
\]
\[\leq 2^{k_3 - 1} \left( 1 + |u|^{2k_3} + |v|^{2k_3} \right).
\]
By \(x^k \leq 1 + x^{N_1} \forall x \geq 0, \forall k \in (0, N_1),\) (9) and (12) give
\[
\mathcal{F}(u, v) \leq \gamma_1 2^{k_1 - 1} |u|^{\alpha} \left( 1 + |u|^{2k_1} \right) + \gamma_2 2^{k_2 - 1} |v|^\beta \left( 1 + |v|^{2k_2} \right)
\]
\[
+ \gamma_3 2^{k_3 - 1} |u|^{\alpha_1} |v|^{\beta_1} \left( 1 + |u|^{2k_3} + |v|^{2k_3} \right)
\]
\[
\leq C_1 \left( |u|^{\alpha} + |u|^{\alpha + 2k_1} + |v|^\beta + |v|^\beta + 2k_2 + |u|^{\alpha_1} |v|^{\beta_1} \right)
\]
\[
+ C_1 \left( |u|^{\alpha_1 + 2k_3} |v|^{\beta_1} + |u|^{\beta_1 + 2k_3} |u|^{\alpha_1} \right)
\]
\[
\leq C_1 \left[ 2 + 2 |u|^{N_1} + 2 |v|^{N_1} + 3 \left( 1 + |u|^{N_1} \right) \left( 1 + |v|^{N_1} \right) \right]
\]
that for (15) there exists a local weak solution or

Suppose that

satisfy (\( \hat{\gamma}_1 \hat{2}^{k_1-1}, \hat{\gamma}_2 2^{k_2-1}, \hat{\gamma}_3 2^{k_3-1} \)). Thus \((H_3, (ii))\) is true.

By \( yG(y) = \tilde{g} |y|^\gamma \ln^{k_4}(e + y^2) > 0 \ \forall y \neq 0, \ (H_4, (i))\) is also true.

We continue to check the condition \((H_4, (ii))\). Using integration by parts as below

\[
\int_0^y G(z)dz = \frac{\tilde{g}}{\gamma} \int_0^y \frac{z^2 dz}{(e + z^2) \ln(e + z^2)} (14)
\]

\[
= \frac{\tilde{g}}{\gamma} |y|^\gamma \ln^{k_4}(e + y^2) - 2k_4 \int_0^y \frac{z^2 dz}{(e + z^2) \ln(e + z^2)}
\]

in which

\[
K(y) = \int_0^y G(z)dz \geq \tilde{g} \gamma \ln^{k_4}(e + y^2) \quad \gamma \geq 0 \ \forall y \in \mathbb{R}. \quad (15)
\]

By the inequalities (12) and \(|y|^\gamma \leq 1 + |y|^{\gamma + 2k_4} \ \forall y \in \mathbb{R},\) we deduce from (14) and (15) that

\[
\int_0^y G(z)dz \leq \frac{\tilde{g}}{\gamma} |y|^\gamma \ln^{k_4}(e + y^2) \quad (16)
\]

\[
\leq \frac{\tilde{g}}{\gamma} |y|^\gamma 2^{k_4-1} \left( 1 + |y|^{2k_4} \right) \leq \frac{2k_4 \tilde{g}}{\gamma} \left( 1 + |y|^{\gamma + 2k_4} \right).
\]

Thus \((H_4, (ii))\) is true. We note more that the above functions \(f_1, f_2, G\) also satisfy \((\bar{H}_2), (\bar{H}_4), (\bar{H}_3), (\bar{H}_4)\) given in Sections 3, 4.

We now state the first theorem about the existence of a “strong solution”.

**Theorem 2.6.** Suppose that \((H_1) - (H_7)\) hold and the initial data satisfy the compatibility conditions

\[
\mu_1(0, 0) \tilde{u}_0(0) = -G(\tilde{u}_0(0)) + \lambda_1 |\tilde{u}_1(0)|^{\bar{r}_1-2} \tilde{u}_1(0) - g_0(0). \quad (17)
\]

If either of the following cases is valid

(i) \( \bar{r}_1 = 2 \) and \( G \in C^1(\mathbb{R}) \), or (ii) \( \bar{r}_1 > 2 \) and \( G \in C^2(\mathbb{R}) \),

then there exists a local weak solution \((u, v)\) of problem (1)-(3) such that

\[
\begin{align*}
(u, v) & \in L^\infty(0, T_*; (V \cap H^2) \times (H_0^1 \cap H^2)), \\
(u', v') & \in L^\infty(0, T_*; V \times H_0^1), \\
(u'', v'') & \in L^\infty(0, T_*; L^2 \times L^2),
\end{align*}
\]

\[
|u'|^{\bar{r}_1-1}u', |v'|^{\bar{r}_1-1}v' \in H^1(Q_{T_*}), |u'(0, \cdot)|^{\bar{r}_1-1}u'(0, \cdot) \in H^1(0, T_*)
\]

for \( T_* > 0 \) small enough.

Furthermore, if either of the following cases is valid

(i) \( \bar{r}_1 = 2 \) and \( G \in C^1(\mathbb{R}) \),

or (ii) \( \bar{r}_1 > 2 \) and \( G \in C^2(\mathbb{R}) \), there exists a constant \( \zeta_{\text{max}} > 0 \) with \( \frac{\zeta_{\text{max}}}{\bar{r}_1} < 1 \) such that

\[
G'(z) \leq \zeta_{\text{max}}, \ \forall z \in \mathbb{R} \quad (21)
\]

with \( \mu_{\text{eq}} \) as in \((H_2, (i))\), then the solution is unique.
Remark 2.7. The regularity obtained by (19) shows that problem (1)-(3) has a
strong solution

\[
\begin{cases}
(u, v) \in L^\infty(0, T_\ast; (V \cap H^2) \times (H^1_0 \cap H^2)) \cap C^0([0, T_\ast]; V \times H^1_0) \\
\cap C^1([0, T_\ast]; L^2 \times L^2), \\
(u', v') \in L^\infty(0, T_\ast; V \times H^1_0) \cap C^0([0, T_\ast]; L^2 \times L^2), \\
(u'', v'') \in L^\infty(0, T_\ast; L^2 \times L^2).
\end{cases}
\tag{22}
\]

With less regular initial data, we have the second theorem about the existence of a weak solution.

Theorem 2.8. Suppose that \((H^1_0), (H^2) - (H_4), (H^2_\ast), (H^2_\ast)\) hold.

If either of the cases given in (18) is valid, then there exists a local weak solution
\((u, v)\) of problem (1)-(3) such that

\[
\begin{cases}
(u, v) \in C([0, T_\ast]; [V \times H^2_0]) \cap C^1([0, T_\ast]; L^2 \times L^2), \\
(u', v') \in L^r_1(Q_{T_\ast}) \times L^r_2(Q_{T_\ast}), \\
u(0, \cdot) \in W^{1, r_1}(0, T_\ast)
\end{cases}
\tag{23}
\]

for \(T_\ast > 0\) small enough.

Furthermore, if either of the cases given in (20), (21) is valid, then the solution
is unique.

Proof of Theorem 2.2. The proof consists of four steps.

Step 1. The Faedo-Galerkin approximation. Let \(\{(w_1, \phi_j)\}\) be a denumerable base
of \((V \cap H^2) \times (H^1_0 \cap H^2)\). Consider the approximate solution of problem (1)-(3) in the form

\[
u_m(t) = \sum_{j=1}^m c_{mj}(t)w_j, \quad \nu_m(t) = \sum_{j=1}^m d_{mj}(t)\phi_j
\tag{24}
\]

with the coefficient functions \((c_{mj}, d_{mj})\) defined via the following system

\[
\begin{cases}
\langle u_m''(t), w_j \rangle + a_1(t; u_m(t), w_j) + \lambda_1 \langle \Psi_{r_1}(u_m(t)), w_j \rangle + \bar{\lambda}_1 \Psi_{r_1}(u_m(0, t))w_j(0) = \langle G(u_m(0, t)) + g_0(t) \rangle w_j(0) + \langle f_1(u_m(t), \nu_m(t)), w_j \rangle + \langle F_1(t), w_j \rangle, \\
\langle v_m''(t), \phi_j \rangle + a_2(t; v_m(t), \phi_j) + \lambda_2 \langle \Psi_{r_2}(v_m(t)), \phi_j \rangle = \langle f_2(u_m(t), \nu_m(t)), \phi_j \rangle + \langle F_2(t), \phi_j \rangle, \\
u_m(0), v_m(0) = (\tilde{u}_0, \tilde{v}_0), \quad (\nu_m(0), v_m(0) = (\tilde{\nu}_0, \tilde{v}_0).
\end{cases}
\tag{25}
\]

By the assumptions of Theorem 2.2, it is obviously that system (25) has a solution
\((u_m(t), \nu_m(t))\) on an interval \([0, T_m] \subset [0, T]\).

Step 2. The first estimate. Multiplying the \(j\)th equation of (25) by \((c_{mj}(t), d_{mj}(t))\) and summing with respect to \(j\), and afterwards integrating with respect to
the time variable from \(0\) to \(t\), we get after some rearrangements

\[
S_m(t) = S_m(0) - 2 \int_0^{\tilde{u}_0(0)} G(z)dz - 2 \int_0^t F(\tilde{u}_0(x), \tilde{v}_0(x))dx
\tag{26}
\]

\[
+ 2 \int_0^t g_0(s)u_m''(0, s)ds + \int_0^t ds \int_0^t \left[ \mu_1'(x, t)u_m^2(x, s) + \mu_2'(x, s)v_m^2(x, s) \right] dx
+ 2 \int_0^t g(t-s)\tilde{a}_2(s; v_m(s), v_m(t))ds - 2g(0) \int_0^t \| \tilde{\mu}_2(s)v_m^2(s) \|^2 ds
+ 2 \int_0^t d\tau \int_0^\tau g'(\tau-s)\tilde{a}_2(s; v_m(s), v_m(t))ds
\]

\[
+ 2 \int_0^t \left[ (F_1(s), u_m'(s)) + (F_2(s), v_m'(s)) \right] ds
\]

The text is well-structured and the mathematical expressions are clear. The document discusses the regularity of solutions to a particular problem and presents a theorem about the existence of a weak solution under certain conditions. The proof is divided into four steps, with the first step focusing on the Faedo-Galerkin approximation. The document includes mathematical notation and equations that are typical of a technical or academic text on partial differential equations.
Lemma 2.9. Let

\[ S_m(0) - 2 \int_0^{\tilde{u}_0(0)} G(z) dz - 2 \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{v}_0(x)) dx = \sum_{i=1}^8 I_i \]

where

\[ S_m(t) = \|u'_m(t)\|^2 + \|v'_m(t)\|^2 + \left\| \sqrt{\mu_1(t)} u_{mx}(t) \right\|^2 + \left\| \sqrt{\mu_2(t)} v_{mx}(t) \right\|^2 \]

\[ + 2\lambda_1 \int_0^t \|u'_m(s)\|_{L^2}^2 ds + 2\lambda_2 \int_0^t \|v'_m(s)\|_{L^2}^2 ds + 2\lambda_1 \int_0^t |u'_m(0, s)|^{q_1} ds. \]

By (25) and (27), we obtain

\[ S_m(0) - 2 \int_0^{\tilde{u}_0(0)} G(z) dz - 2 \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{v}_0(x)) dx \]

\[ = \|\tilde{u}_1\|^2 + \|\tilde{v}_1\|^2 + \left\| \sqrt{\mu_1(0)} \tilde{u}_{0x} \right\|^2 + \left\| \sqrt{\mu_2(0)} \tilde{v}_{0x} \right\|^2 - 2 \int_0^{\tilde{u}_0(0)} G(z) dz \]

\[ - 2 \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{v}_0(x)) dx \]

\[ \equiv C_0 \text{ for all } m. \]

\[ \square \]

We need the following lemma.

**Lemma 2.9.** Let \( r > 1 \) and \( \delta > 0 \). Then

\[ ab \leq \delta a^r + C_\delta b^{r'} \text{ for all } a, b \geq 0 \]

where

\[ C_\delta = C_\delta(r) = (1 - \frac{1}{r}) (r\delta)^{-1/(r-1)}, \quad r' = \frac{r}{r-1}. \]

The proof is straightforward and we omit the details.

We now to estimate the terms of (26) as belows.

Using the inequalities (29) and

\[ x^q \leq 1 + x^N \forall x \geq 0, \forall q \in (0, N] \]

with

\[ N = \max \left\{ 1, \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma (\gamma - 1)\bar{r}_1}{2(\bar{r}_1 - 1)} \right\} \]

we obtain

\[ I_1 = 2 \int_0^t g_0(s) u'_m(0, s) ds \]

\[ \leq 2 \int_0^t \left[ \delta \left| \bar{r}_1 \bar{\lambda}_1 \right|^{1/\bar{r}_1} u'_m(0, s) \right|^{\bar{r}_1} + C_\delta \left| \bar{r}_1 \bar{\lambda}_1 \right|^{-1/\bar{r}_1} g_0(s) \right|^{\bar{r}_1} ds \]

\[ \leq 2\bar{r}_1 \bar{\lambda}_1 \delta \int_0^t |u'_m(0, s)|^{\bar{r}_1} ds + 2C_\delta \left( \bar{r}_1 \bar{\lambda}_1 \right)^{-\bar{r}_1/\bar{r}_1} |g_0|^{\bar{r}_1} \left| L^{\bar{r}_1}(0, T) \right| \]

\[ \leq \delta S_m(t) + C_T(\delta) \]

where we note that \( C_T(\delta) \) always indicates a bound depending on \( T \) and \( \delta \).
Note that
\[ S_m(t) \geq \mu_* \|u_{mx}(t)\|^2 + \mu_2 \|v_{mx}(t)\|^2 \geq \mu_* \left( \|u_{mx}(t)\|^2 + \|v_{mx}(t)\|^2 \right) \tag{34} \]
where \( \mu_* = \min\{\mu_1, \mu_2\} \), it follows from (H2), (31), (32), that
\[
I_2 = \int_0^t ds \int_0^1 \left[ \mu'_1(x,s)u_{mx}^2(x,s) + \mu'_2(x,s)v_{mx}^2(x,s) \right] dx \\
\leq \frac{1}{\mu_*} \max\{\|\mu'_1\|_{C^0([0,T];L^\infty)}, \|\mu'_2\|_{C^0([0,T];L^\infty)}\} \int_0^t S_m(s)ds \\
\leq C_T \int_0^t \left[ 1 + S_m^N(s) \right] ds
\]
where \( C_T \) also indicates a bound depending on \( T \).

By (H2), (H5), (H6), (31), (32), (34), we continue to estimate the other terms in the right-hand side of (14) as follows
\[
I_3 = 2 \int_0^t g(t-s)\tilde{a}_2(s;v_m(s),v_m(t))ds \\
\leq \delta S_m(t) + \frac{1}{\delta \mu_*^2} \|\bar{\mu}_2\|_{C^0([0,T];L^\infty)} \|g\|_{L^2(0,T)}^2 \int_0^t S_m(s)ds \\
\leq \delta S_m(t) + C_T(\delta) \int_0^t \left[ 1 + S_m^N(s) \right] ds;
\]
\[
I_4 = -2g(0) \int_0^t \|\sqrt{\bar{\mu}_2(s)v_{mx}(s)}\|^2 \tag{36} ds \\
\leq \frac{2}{\mu_*} |g(0)| \|\bar{\mu}_2\|_{C^0([0,T];L^\infty)} \int_0^t S_m(s)ds \\
\leq C_T \int_0^t \left[ 1 + S_m^N(s) \right] ds;
\]
\[
I_5 = -2 \int_0^t dt \int_0^\tau g'(\tau-s)\tilde{a}_2(s;v_m(s),v_m(\tau))ds \\
\leq \frac{2}{\mu_*} \|\bar{\mu}_2\|_{C^0([0,T];L^\infty)} \sqrt{T} \|g'\|_{L^2(0,T)} \int_0^t S_m(s)ds \\
\leq C_T \int_0^t \left[ 1 + S_m^N(s) \right] ds;
\]
\[
I_6 = 2 \int_0^t \left[ \langle F_1(s),u_m'(s) \rangle + \langle F_2(s),v_m'(s) \rangle \right] ds \\
\leq \|F_1\|^2_{L^2(Q_T)} + \|F_2\|^2_{L^2(Q_T)} + \int_0^t S_m(s)ds \\
\leq C_T + \int_0^t \left[ 1 + S_m^N(s) \right] ds
\]
for all \( \delta > 0 \).

By (27)-(29), (31), (32) and (34), we obtain
\[
|u_m(0,t)|^\gamma = |\bar{u}_0(0)|^\gamma + \gamma \int_0^t |u_m(0,s)|^{\gamma-2} u_m(0,s)u_m'(0,s)ds \tag{37}
\]
\[
\begin{align*}
\leq |\tilde{u}_0(0)|^\gamma &+ \gamma \int_0^t \left[C_\delta \left(\frac{r_1 \lambda_1}{\gamma C_2}\right)^{-1/\tilde{r}_i} |u_m(0, s)|^{-2} u_m(0, s) \right]^{\tilde{r}_i} \\
&\quad + \delta \left(\frac{r_1 \lambda_1}{\gamma C_2}\right)^{1/\tilde{r}_i} |u'_m(0, s)|^{\tilde{r}_i} \right] ds \\
\leq |\tilde{u}_0(0)|^\gamma &+ \gamma C_\delta \left(\frac{r_1 \lambda_1}{\gamma C_2}\right)^{-r_i/\tilde{r}_i} \int_0^t \left[1 + S_m^N(s) \right] ds \\
&\quad + \frac{\delta}{C_2^{\tilde{r}_i}} \int_0^t |u'_m(0, s)|^{\tilde{r}_i} ds \\
\leq |\tilde{u}_0(0)|^\gamma &+ \gamma C_\delta \int_0^t \left[1 + S_m^N(s) \right] ds + \delta S_m(t).
\end{align*}
\]

Thus
\[
I_7 = 2 \int_{0}^{u_{m}(0, t)} G(z) dz \leq 2\tilde{C}_2 \left(1 + |u_m(0, t)|^\gamma \right) + \hat{C}_0 + C_T(\delta) \int_0^t \left[1 + S_m^N(s) \right] ds + \delta S_m(t).
\]

Using the assumption \(H_5, (ii)\), we have
\[
I_8 = 2 \int_{0}^{1} F(u_m(x, t), v_m(x, t)) dx \leq 2\tilde{C}_1 \left(1 + \|u_m(t)\|_{L^\alpha}^\beta + \|v_m(t)\|_{L^\beta}^\beta + \int_0^1 |u_m(x, t)|^\alpha |v_m(x, t)|^\beta dx \right).
\]

By \(27, 31, 32, \) and \(34, \) we have
\[
\|u_m(t)\|_{L^\alpha}^\alpha - \|\tilde{u}_0\|_{L^\alpha}^\alpha = \alpha \int_0^t \left|u_m(s)\right|^{-2} u_m(s) ds \leq \alpha \sqrt{\mu_s^{-1}} \int_0^t S_m^{\alpha/2}(s) ds \\
\leq \alpha \sqrt{\mu_s^{-1}} \int_0^t \left[1 + S_m^N(s) \right] ds.
\]

Similarly
\[
\|v_m(t)\|_{L^\beta}^\beta - \|\tilde{v}_0\|_{L^\beta}^\beta \leq \beta \sqrt{\mu_s^{-1}} \int_0^t \left[1 + S_m^N(s) \right] ds.
\]

In order to estimates the term \(\int_0^1 |u_m(x, t)|^\alpha |v_m(x, t)|^\beta dx, \) we write
\[
\int_0^1 |u_m(x, t)|^\alpha |v_m(x, t)|^\beta dx \leq \int_0^1 |\tilde{u}_0(x)|^\alpha |\tilde{v}_0(x)|^\beta dx
\]
\begin{equation}
\alpha_1 \int_0^1 dx \int_0^t \left| u_m(x,s) \right|^{\alpha_1 - 2} u_m(x,s) \left| v_m(x,s) \right|^{\beta_1} v_m'(x,s)ds \\
\beta_1 \int_0^1 dx \int_0^t \left| u_m(x,s) \right|^{\alpha_1} \left| v_m(x,s) \right|^{\beta_1 - 2} v_m(x,s)v_m'(x,s)ds \\
= \int_0^1 \left| \tilde{u}_0(x) \right|^{\alpha_1} \left| \tilde{v}_0(x) \right|^{\beta_1} dx + \tilde{I}_1 + \tilde{I}_2.
\end{equation}

By (27), (29) and (34) we have
\begin{equation}
\tilde{I}_1 = \alpha_1 \int_0^1 dx \int_0^t \left| u_m(x,s) \right|^{\alpha_1 - 2} u_m(x,s) \left| v_m(x,s) \right|^{\beta_1} v_m'(x,s)ds
\end{equation}
\begin{equation}
\leq \alpha_1 \int_0^1 dx \int_0^t \left( \frac{S_m(s)}{\mu_s} \right)^{\alpha_1 + \beta_1 - 1} \left| u_m'(x,s) \right| ds
\end{equation}
\begin{equation}
\leq \alpha_1 \int_0^1 dx \int_0^t \left[ \delta \left( \frac{\lambda_1}{\alpha_1 C_1} \right)^{-1/r_1} \left( \frac{S_m(s)}{\mu_s} \right)^{\alpha_1 + \beta_1 - 1} r_1 \right]
\end{equation}
\begin{equation}
= \frac{\alpha_1 C_\delta \left( \frac{\lambda_1}{\alpha_1 C_1} \right)^{-r_1/r_1}}{(\mu_s)^{(\alpha_1 + \beta_1 - 1)r_1}} \int_0^t \left( \frac{S_m(s)}{\mu_s} \right)^{(\alpha_1 + \beta_1 - 1)r_1} ds + \frac{\lambda_1 \delta}{C_1} \int_0^t \left| u_m'(s) \right|^{r_1} ds.
\end{equation}

Similarly
\begin{equation}
\tilde{I}_2 = \beta_1 \int_0^1 dx \int_0^t \left| u_m(x,s) \right|^{\alpha_1} \left| v_m(x,s) \right|^{\beta_1 - 2} v_m(x,s)v_m'(x,s)ds
\end{equation}
\begin{equation}
\leq \frac{1}{2} C_T(\delta) \int_0^t \left[ 1 + S_m^N(s) \right] ds + \frac{\lambda_2 \delta}{C_1} \int_0^t \left| v_m'(s) \right|^{2r_2} ds.
\end{equation}

It follows from (40)-(44) that
\begin{equation}
\int_0^1 \left| u_m(x,t) \right|^{\alpha_1} \left| v_m(x,t) \right|^{\beta_1} dx \leq \hat{C}_0 + C_T(\delta) \int_0^t \left[ 1 + S_m^N(s) \right] ds + \frac{\delta}{2C_1} S_m(t).
\end{equation}

Hence, by (25), (31), (32), (39)-(41) and (45), it follows that
\begin{equation}
I_s = 2 \int_0^1 \mathcal{F}(u_m(x,t), v_m(x,t))dx
\end{equation}
\begin{equation}
\leq \hat{C}_0 + C_T(\delta) \int_0^t \left[ 1 + S_m^N(s) \right] ds + \delta S_m(t) \text{ for all } \delta > 0.
\end{equation}

Choosing \( \delta = \frac{1}{8} \), it follows from (26), (28), (33), (35), (36), (38) and (46), that
\begin{equation}
S_m(t) \leq \tilde{C}_T^{(1)} + \tilde{C}_T^{(2)} \int_0^t S_m(s)ds
\end{equation}

where \( \tilde{C}_T^{(1)} \) and \( \tilde{C}_T^{(2)} \) are the constants depending on \( T \).
Then, by solving a nonlinear Volterra integral inequality (based on the methods in [7]), we get the following lemma.

**Lemma 2.10.** There exists a constant $T_0 > 0$ depending on $T$ (independent of $m$) such that

$$S_m(t) \leq M_T \forall m \in \mathbb{N}, \forall t \in [0, T_0]$$

(48)

where $M_T$ is a constant depending only on $T$ as above.

Lemma 2.5 allows one to take constant $T_m = T_0$ for all $m$.

The second estimate.

First of all, we estimate $\|u_m''(0)\|^2 + \|u_m''(0)\|^2$. Letting $t \to 0^+$ in (25), multiplying the result by $c_m''(0)$ and using the compatibility (17), we have

$$\|u_m''(0)\|^2 - \left(\frac{\partial}{\partial x} (\mu_1(0)\tilde{u}_{0x}) + \mu_1''(0)\right) + \lambda_1 \left|\tilde{u}_1\right|^2 \|u_m''(0)\| = \left\langle f_1(\tilde{u}_0, \tilde{v}_0), u_m''(0) \right\rangle + \left\langle F_1(0), u_m''(0) \right\rangle$$

(49)

This gives

$$\|u_m''(0)\| \leq \left\| \frac{\partial}{\partial x} (\mu_1(0)\tilde{u}_{0x}) \right\| + \lambda_1 \left|\tilde{u}_1\right|^2 + \|f_1(\tilde{u}_0, \tilde{v}_0)\| + \|F_1(0)\| = \mathcal{C}_{01}$$

(50)

where $\mathcal{C}_{01}$ is a constant depending only on $r_1$, $\lambda_1$, $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{u}_1$, $\lambda_1$, $f_1$, $F_1$.

Similarly, we have

$$\|v_m''(0)\| \leq \left\| \frac{\partial}{\partial x} (\mu_2(0)\tilde{v}_{0x}) \right\| + \lambda_2 \left|\tilde{v}_1\right|^2 + \|f_2(\tilde{u}_0, \tilde{v}_0)\| + \|F_2(0)\| = \mathcal{C}_{02}$$

(51)

where $\mathcal{C}_{02}$ is a constant depending only on $r_2$, $\lambda_2$, $\tilde{u}_0$, $\tilde{v}_0$, $\tilde{v}_1$, $\mu_2$, $f_2$, $F_2$.

Now differentiating (25) with respect to $t$, it leads to

$$\left\langle u_m'''(t), w_j \right\rangle + \left\langle \mu_1(t)u_m'''(t), w_j \right\rangle + \left\langle \mu_1'(t)u_m''(t), w_j \right\rangle + \lambda_1 \Psi_1 (\mu_1(0), t)u_m'''(0)w_j(0) = \left\langle G'(u_m(0), t)u_m'''(0), w_j \right\rangle + \left\langle D_1 f_1(u_m(t), v_m(t))v_m'''(t), w_j \right\rangle + \left\langle F_1''(0), w_j \right\rangle,

\left\langle v_m'''(t), \phi_j \right\rangle + \lambda_2 \Psi_2 (v_m(0), t)v_m'''(0)\phi_j(0) = \left\langle G'(u_m(0), t)v_m'''(0), \phi_j \right\rangle + \left\langle D_2 f_2(u_m(t), v_m(t))v_m'''(t), \phi_j \right\rangle + \left\langle F_2''(0), \phi_j \right\rangle$$

(52)

for all $1 \leq j \leq m$.

Multiplying the $j^{th}$ equation of (52) by $(c_m''(j), d_m''(j))$ and summing with respect to $j$, and afterwards integrating with respect to the time variable from 0 to $t$, we get after some rearrangements

$$X_m(t) = X_m(0) + 2 \left\langle \mu_1'(0)\tilde{u}_{0x}, \tilde{u}_1 \right\rangle + 2 \left\langle \mu_2'(0)\tilde{v}_{0x}, \tilde{v}_1 \right\rangle - 2g_0(0)\tilde{u}_0\tilde{v}_0 - 2g_0(0)\tilde{u}_1\tilde{v}_1$$

(53)

$$+ 2 \int_0^t \left\langle D_1^2 F(u_m, v_m)u_m'(s) + D_2 D_1 F(u_m, v_m)v_m'(s), u_m'''(s), v_m'''(s) \right\rangle ds$$

$$+ 2 \int_0^t \left\langle D_1 D_2 F(u_m, v_m)u_m'(s) + D_2^2 F(u_m, v_m)v_m'(s), u_m'''(s), v_m'''(s) \right\rangle ds$$
where we also note that by (25),

\[ u_m(s), v_m(s) \]

and (50), (51), (54), we get

\[ 1 K t + 8 v(0) + 2 T, t \int t \int t (0) \langle v_m(x, s), v_m(t) \rangle + 2 \int_0^t g'(t-s)\overline{a_2}(s; v_m(s), v_m(t))ds \]

\[ 2 \int_0^t ds \int_0^t (\mu_1'(s)u_m(x, s)) ds \]

\[ 2 \mu_1'(t)u_m(t, t) - 2 \mu_2'(t)v_m(t, t) \]

\[ + 2(\mu_0'(0)v_0(t, t) \overline{a_2}(s; v_m(s), v_m(t))ds + 2g_0(t)u_m'(0, t) \]

\[ = X_m(0) + 2(\mu_1'(0)\overline{u}_0x, \overline{u}_1x) + 2(\mu_2'(0)\overline{v}_0x, \overline{v}_1x) - 2g(0) \langle \mu_2(0)\overline{v}_0x, \overline{v}_1x \rangle - 2g_0(0)\overline{a}_1(0) + \sum_{j=1}^{15} J_j \]

\[ X_m(t) = \| u_m(t) \|^2 + \| u_m'(t) \|^2 + \| v_m(t) \|^2 + \| v_m'(t) \|^2 + \| \sqrt{\mu_1(t)}u_m(x, s) \|^2 + \| \sqrt{\mu_2(t)}v_m(x, s) \|^2 \]  

\[ + \frac{8\lambda_1(r_1-1)}{r_1^2} \int_0^t \left\| \frac{\partial}{\partial s} (\| u_m(x, s) \|^{2-1} u_m(x, s)) \right\|^2 ds \]

\[ + \frac{8\lambda_2(r_2-1)}{r_2^2} \int_0^t \left\| \frac{\partial}{\partial s} (\| v_m(x, s) \|^{2-1} v_m(x, s)) \right\|^2 ds \]

\[ + \frac{8\lambda_3(r_3-1)}{r_3^2} \int_0^t \left\| \frac{\partial}{\partial s} (\| v_m'(0, s) \|^{2-1} u_m'(0, s)) \right\|^2 ds. \]  

By (25), (50), (51), (54), we get

\[ X_m(0) + 2(\mu_1'(0)\overline{u}_0x, \overline{u}_1x) + 2(\mu_2'(0)\overline{v}_0x, \overline{v}_1x) - 2g(0) \langle \mu_2(0)\overline{v}_0x, \overline{v}_1x \rangle \]

\[ - 2g_0(0)\overline{a}_1(0) \leq \tilde{C}_0, \text{ for all } m \]

where we also note that \( \tilde{C}_0 \) always indicates a positive constant depending only on \( \overline{u}_0, \overline{v}_0, \overline{u}_1, \overline{v}_1, f_1, f_2, F_1, F_2, \mu_1, \mu_2, G, g, \rho_1, \rho_2, r_1, r_2, \lambda_1, \lambda_2. \)

Put

\[ K_2(T, F) = \sum_{|\alpha|=2} \sup \left\{ |D^\alpha F(y, z)| : |y|, |z| \leq \sqrt{\frac{MT}{\mu_s}} \right\}. \]

We shall estimate all terms \( J_j, j = 1, \cdots, 15 \) in the right-hand side of (53).

From (27), (48), (54), (56) and the Cauchy-Schwartz inequality, it is clear that

\[ J_1 = 2 \int_0^t \langle D_1^2 F(u_m, v_m), u'_m(s) \rangle ds + D_2 D_1 \langle u_m, v_m \rangle \int_0^t u''_m(s) ds \]

\[ \leq 2K_2(T, F) \int_0^t \left( \| u'_m(s) \|^2 + \| v'_m(s) \|^2 \right) ds + \int_0^t \| u''_m(s) \|^2 ds \]
\[ \leq 2K^2(T,F) \int_0^t S_m(s)ds + \int_0^t X_m(s)ds \leq C_T + \int_0^t X_m(s)ds. \]

Similarly,
\[
J_2 = 2 \int_0^t \langle D_1 D_2 F(u_m, v_m) u'_m(s), v''_m(s) \rangle ds \tag{58}
\]
\[ \leq C_T + \int_0^t X_m(s)ds. \]

Using the assumption \((H_6)\), we deduce from the Cauchy-Schwartz inequality that
\[
J_3 = 2 \int_0^t \langle [F'_1(s), u''_m(s)] + [F'_2(s), v''_m(s)] \rangle ds \tag{59}
\]
\[ \leq \|F'_1\|_{L^1(0,T;L^2)} + \|F'_2\|_{L^1(0,T;L^2)} + \int_0^t (\|F'_1(s)\| + \|F'_2(s)\|) X_m(s)ds \]
\[ \leq C_T + \int_0^t (\|F'_1(s)\| + \|F'_2(s)\|) X_m(s)ds. \]

By \((H_2), (H_5), (H_7)\) and \((27), (48), (54)\), we estimate the terms \(J_i, i = 4, 5, \ldots, 14\), as follows
\[
J_4 = 3 \int_0^t ds \int_0^1 \left( \mu_1'(x,s) |u'_m(x,s)|^2 + \mu_2'(x,s) |v''_m(x,s)|^2 \right) dx \tag{60}
\]
\[ \leq \frac{3}{\mu_*} \max\{\|\mu_1'\|_{C^0([0,T];L^\infty)}, \|\mu_2'\|_{C^0([0,T];L^\infty)}\} \int_0^t X_m(s)ds \]
\[ \leq C_T \int_0^t X_m(s)ds; \]
\[
J_5 = 2 \int_0^t \langle \mu_1''(s) u''_m(s), u_m'(s) \rangle + \langle \mu_2''(s) v''_m(s), v_m'(s) \rangle ds \leq C_T + \int_0^t (\|\mu_1''(s)\|_{L^\infty} + \|\mu_2''(s)\|_{L^\infty}) X_m(s)ds; \]
\[
J_6 = -2g(0) \int_0^t \left\| \sqrt{\mu_2(s)} v''_m(s) \right\|^2 ds \]
\[ \leq \frac{2}{\mu_*} |g(0)| \|\mu_2\|_{C^0([0,T];L^\infty)} \int_0^t X_m(s)ds \leq C_T \int_0^t X_m(s)ds; \]
\[
J_7 = -2g(0) \int_0^t \langle \mu_2'(s) v_m'(s), v''_m(s) \rangle ds \]
\[ \leq \frac{1}{\mu_*} g^2(0) \|\mu_2'\|_{C^0([0,T];L^\infty)}^2 M_T + \int_0^t X_m(s)ds \leq C_T + \int_0^t X_m(s)ds; \]
\[
J_8 = -2 \langle \mu_1'(t) u_m'(t), u''_m(t) \rangle \]
\[ \leq \frac{1}{\delta_1 \mu_*^2} \|\mu_1'\|_{C^0([0,T];L^\infty)}^2 M_T + \delta_1 X_m(t) \leq \frac{1}{\delta_1} C_T + \delta_1 X_m(t); \]
\[
J_9 = -2 \langle \mu_2'(t) v_m'(t), v''_m(t) \rangle \leq \frac{1}{\delta_1} C_T + \delta_1 X_m(t); \]
\[
J_{10} = 2g(0) \langle \mu_2(t) v_m'(t), v''_m(t) \rangle \]
\[ \leq \frac{1}{\delta_1 \mu_*^2} |g(0)| \|\mu_2\|_{C^0([0,T];L^\infty)}^2 M_T + \delta_1 X_m(t) \leq \frac{1}{\delta_1} C_T + \delta_1 X_m(t); \]
\[ J_{11} = 2 \int_0^t g'(t - s)\bar{a}_2(s; v_m(s), v'_m(t))ds \]
\[ \leq \frac{1}{\delta_1 \mu_*^2} \| \bar{\mu}_2 \|^2_{C^0([0,T]_L; L^\infty)} M_T^2 \left( \int_0^T |g'(s)| ds \right)^2 + \delta_1 X_m(t) \]
\[ \leq \frac{1}{\delta_1} C_T + \delta_1 X_m(t); \]
\[ J_{12} = -2 \int_0^t \tau \int_0^\tau g''(\tau - s)\bar{a}_2(s; v_m(s), v'_m(\tau))ds \]
\[ \leq \frac{2}{\mu_*} \| \bar{\mu}_2 \|^2_{C^0([0,T]_L; L^\infty)} \sqrt{MT} \| g'' \|_{L^1(0,T)} \int_0^t \sqrt{|X_m(s)|} ds \]
\[ \leq C_T + \int_0^t X_m(s)ds; \]
\[ J_{13} = 2g_0(t)u''_m(0,t) \leq \frac{1}{\delta_1 \mu_*} \| g_0 \|^2_{C^0([0,T])} + \delta_1 X_m(t) \leq \frac{1}{\delta_1} C_T + \delta_1 X_m(t); \]
\[ J_{14} = -2 \int_0^t g''(s)u'_m(0,s)ds \leq \frac{1}{\mu_*} \| g'' \|_{L^1(0,T)} + \int_0^t |g''(s)| X_m(s)ds \]
\[ \leq C_T + \int_0^t |g''(s)| X_m(s)ds \]

for all \( \delta_1 > 0 \).

In order to estimate the term \( J_{15} = 2 \int_0^t G'(u_m(0,s))u'_m(0,s)u''_m(0,s)ds \), we need the following lemma.

**Lemma 2.11.** Let \( \tilde{r}_1 = 2 \) and \( G \in C^1(\mathbb{R}) \) or \( \tilde{r}_1 > 2 \) and \( G \in C^2(\mathbb{R}) \), we have

(i) \( J_{15} \leq C_T(\delta_1) + \delta_1 X_m(t) + C_T(\delta_1) \int_0^t X_m(s)ds \forall \delta_1 > 0, \) \hspace{1cm} (61)

if \( \tilde{r}_1 = 2 \) or \( \tilde{r}_1 \geq 3 \),

(ii) \( J_{15} \leq C_T(\delta_1) + \delta_1 X_m(t) + C_T \int_0^t (X_m(s))^{3/r_1} ds \forall \delta_1 > 0 \), if \( 2 < \tilde{r}_1 < 3 \).

**Proof of Lemma 2.6.** We consider two cases for \( \tilde{r}_1 \) and \( G \).

**Case 1.** \( \tilde{r}_1 = 2 \) and \( G \in C^1(\mathbb{R}) \): Put \( D_1 = \sup \left\{ \left| G'(z) \right| : |z| \leq \sqrt{\frac{M}{\mu_*}} \right\} \), we have

\[ X_m(t) \geq 2\bar{\lambda}_1 \int_0^t |u''_m(0,s)|^2 ds \] \hspace{1cm} (62)

and

\[ J_{15} = 2 \int_0^t G'(u_m(0,s))u'_m(0,s)u''_m(0,s)ds \]
\[ \leq \frac{2}{\sqrt{\mu_*}} D_1 \int_0^t \sqrt{|X_m(s)|} |u''_m(0,s)| ds \]
\[ \leq \frac{1}{2\bar{\lambda}_1 \delta_1 \mu_*} \tilde{D}_1^2 \int_0^t X_m(s)ds + 2\delta_1 \bar{\lambda}_1 \int_0^t |u''_m(0,s)|^2 ds \]
\[ \leq \frac{1}{\delta_1} C_T \int_0^t X_m(s)ds + \delta_1 X_m(t) \]
we deduce from (66) that
\[ u \leq C_T(\delta_1) + \delta_1 X_m(t) + C_T(\delta_1) \int_0^t X_m(s) ds \] for all \( \delta_1 > 0 \).

Case 2. \( \bar{r}_1 > 2 \) and \( G \in C^2(\mathbb{R}) \): Put \( \tilde{D}_2 = \sup \left\{ |G'(z)| + |G''(z)| : |z| \leq \sqrt{\frac{M_T}{m}} \right\} \), by using integration by parts, we have
\[
J_{15} = 2 \int_0^t G'(u_m(0, s)) u_m'(0, s) u_m''(0, s) ds 
= G'(u_0(0, t)) |u_m'(0, t)|^2 - G'(\tilde{u}_0(0)) \tilde{u}_1^2(0) 
- \int_0^t G''(u_m(0, s)) |u_m'(0, s)|^2 u_m'(0, s) ds 
\leq |G'(\tilde{u}_0(0))| \tilde{u}_1^2(0) + \tilde{D}_2 \left( |u_m'(0, t)|^2 + \int_0^t |u_m'(0, s)|^3 ds \right).
\]

Note that
\[
|u_m'(0, t)|^{\bar{r}_1} u_m'(0, t) = |\tilde{u}_1(0)|^{\bar{r}_1} \tilde{u}_1(0) + \int_0^t \frac{\partial}{\partial s} \left( |u_m'(0, s)|^{\bar{r}_1} u_m'(0, s) \right) ds.
\]
Using the inequality \((a + b)^2 \leq 2a^2 + 2b^2\) for all \( a, b \geq 0 \), it follows from (65) that
\[
|u_m'(0, t)|^{\bar{r}_1} \leq 2 |\tilde{u}_1(0)|^{\bar{r}_1} + 2t \int_0^t \frac{\partial}{\partial s} \left( |u_m'(0, s)|^{\bar{r}_1} u_m'(0, s) \right) ds 
\leq 2 |\tilde{u}_1(0)|^{\bar{r}_1} + \frac{\bar{r}_1^2 T}{4 \lambda_1(\bar{r}_1 - 1)} X_m(t)
\equiv 2 |\tilde{u}_1(0)|^{\bar{r}_1} + \tilde{D}_T X_m(t).
\]
By (29), (66), we obtain
\[
|u_m'(0, t)|^2 \leq \delta_1 \left( \tilde{D}_T \right)^{-2/\bar{r}_1} |u_m'(0, t)|^{\frac{\bar{r}_1}{2}} + C_{\delta_1} \left( \tilde{D}_T \right)^{2/\bar{r}_1} \left| \frac{\bar{r}_1}{2} \right|
\leq \frac{\delta_1}{\tilde{D}_T} |u_m'(0, t)|^{\bar{r}_1} + C_{\delta_1} \left( \tilde{D}_T \right)^{2/\bar{r}_1} \left| \frac{\bar{r}_1}{2} \right|
\leq \frac{2 \delta_1}{\tilde{D}_T} |\tilde{u}_1(0)|^{\bar{r}_1} + C_{\delta_1} \left( \tilde{D}_T \right)^{2/\bar{r}_1} \left| \frac{\bar{r}_1}{2} \right|
= \tilde{C}_{\delta_1}(\tilde{D}_T) + \frac{\delta_1}{\tilde{D}_T} X_m(t) \text{ for all } \delta_1 > 0.
\]
Using the inequalities
\[
(a + b)^{1/r} \leq a^{1/r} + b^{1/r},
(a + b)^3 \leq 4(a^3 + b^3) \text{ for all } a, b \geq 0, r \geq 1,
\]
we deduce from (66) that
\[
|u_m'(0, t)|^3 \leq 4 \left( \sqrt[3]{8} |\tilde{u}_1(0)|^3 + \sqrt[3]{\tilde{D}_T^3} (X_m(t))^{3/\bar{r}_1} \right)
\equiv \tilde{C}_0 + C_T (X_m(t))^{3/\bar{r}_1}.
\]
It follows from (64), (67) and (69), that
\[
J_{15} \leq |G'(\tilde{u}_0(0))| \tilde{u}_1^2(0) + \hat{D}_2 \tilde{C}_T^1(\delta_1) + T\tilde{C}_0\hat{D}_2
\]
\[+ \delta_1 X_m(t) + \hat{D}_2 \tilde{C}_T \int_0^t (X_m(s))^{3/\tilde{r}_1} ds
\]
\[= \tilde{C}_T(\delta_1) + \delta_1 X_m(t) + C_T \int_0^t (X_m(s))^{3/\tilde{r}_1} ds.
\]

If \( \tilde{r}_1 = 3 \), we have (61) holds. If \( \tilde{r}_1 > 3 \), by (29), we have
\[
(X_m(s))^{3/\tilde{r}_1} \leq \delta_1 (X_m(s))^{3/\tilde{r}_1} \tilde{F}_1 + C_{\delta_1} = \delta_1 X_m(s) + C_{\delta_1}.
\]

Thus (61) holds.

Lemma 2.6 is proved completely. \( \Box \)

(i) Case 1: \( \tilde{r}_1 = 2 \) or \( \tilde{r}_1 \geq 3 \). Choosing \( \delta_1 = \frac{1}{12} \), it follows from (53), (55) and (57)-(60) and (61), that
\[
X_m(t) \leq D^{(1)}_T + \int_0^t D^{(3)}_T(s)X_m(s)ds, \quad 0 \leq t \leq T_*
\]
with
\[
D^{(3)}_T(s) = D^{(2)}_T + 2 (\|F'_1(s)\| + ||F'_2(s)|| + ||\mu''_1(s)||_{L^\infty} + \|\mu''_2(s)\|_{L^\infty} + |g''_0(s)|)
\]
where \( D^{(1)}_T \) and \( D^{(2)}_T \) are the constants depending on \( T \) and \( D^{(3)}_T \in L^1(0,T) \).

Using Gronwall’s Lemma, (71) leads to
\[
X_m(t) \leq D^{(1)}_T \exp \left( \int_0^T D^{(3)}_T(s)ds \right) \leq \bar{M}_T
\]
for all \( t \in [0,T_*] \), for all \( m \in \mathbb{N} \).

(ii) Case 2: \( 2 < \tilde{r}_1 < 3 \). Choosing \( \delta_1 = \frac{1}{12} \), it follows from (53), (55) and (57)-(60) and (61), that
\[
X_m(t) \leq \hat{D}^{(1)}_T + \int_0^t \hat{D}^{(3)}_T(s)X_m(s)ds + C_T \int_0^t (X_m(s))^{3/\tilde{r}_1} ds, \quad 0 \leq t \leq T_*
\]
where
\[
\hat{D}^{(3)}_T(s) = \hat{D}^{(2)}_T + 2 (\|F'_1(s)\| + ||F'_2(s)|| + ||\mu''_1(s)||_{L^\infty} + \|\mu''_2(s)\|_{L^\infty} + |g''_0(s)|),
\]
\[
\hat{D}^{(3)}_T \in L^1(0,T)
\]
in which \( \hat{D}^{(1)}_T \), \( \hat{D}^{(2)}_T \) and \( C_T \) are the constants depending on \( T \).

By solving a nonlinear Volterra integral inequality (74), there exists a constant \( T_{**} > 0 \) depending on \( T_* \) (independent of \( m \)) such that
\[
X_m(t) \leq \bar{M}_T \quad \forall m \in \mathbb{N}, \quad \forall t \in [0,T_{**}]
\]
where \( \bar{M}_T \) is a constant depending only on \( T \).

In what follows, we will denote \( T_* \) for \( T_{**} \) or \( T_* \).
Step 3. Limiting process. From (27), (48), (54), (73) or (76), we deduce the existence of a subsequence of \(\{(u_m, v_m)\}\) (with the same denotation), such that

\[
\begin{align*}
(u_m, v_m) &\to (u, v) \quad \text{in} \ L^\infty(0, T_*; V \times H_0^1) \quad \text{weak*}, \\
(u_m', v_m') &\to (u', v') \quad \text{in} \ L^\infty(0, T_*; V \times H_0^1) \quad \text{weak*}, \\
(u_m, v_m) &\to (u_m', v_m') \quad \text{in} \ L^\infty(0, T_*; L^2 \times L^2) \quad \text{weak*}, \\
(u_m', v_m') &\to (u', v') \quad \text{in} \ L^r(Q_T, x) \times L^r(Q_T, y) \quad \text{weakly}, \\
v_m(0, \cdot) &\to u(0, \cdot) \quad \text{in} \ W^{1,r}(0, T) \quad \text{weakly}, \\
v_m'(0, \cdot) &\to u'(0, \cdot) \quad \text{in} \ L^{r}(0, T) \quad \text{weakly}, \\
|u_m'(0, \cdot)|^{\frac{1}{r}} - u_m'(0, \cdot) &\to \chi_0 \quad \text{in} \ H^1(0, T) \quad \text{weakly}, \\
\frac{\partial}{\partial t} \left( |u_m'|^{\frac{1}{r}} - u_m' \right) &\to \chi_1 \quad \text{in} \ L^2(Q_T) \quad \text{weakly}, \\
\frac{\partial}{\partial t} \left( |v_m'|^{\frac{1}{r}} - v_m' \right) &\to \chi_2 \quad \text{in} \ L^2(Q_T) \quad \text{weakly}.
\end{align*}
\]

By the compactness lemma of Lions ([10], p.57) and the embeddings \(H^1(0, T_*) \hookrightarrow C^0([0, T_*])\) and \(W^{1,r}(0, T_* \hookrightarrow C^0([0, T_*])\), we can deduce from (77) the existence of a subsequence still denoted by \(\{(u_m, v_m)\}\) such that

\[
\begin{align*}
(u_m, v_m) &\to (u, v) \quad \text{strongly in} \ L^2(Q_T) \times L^2(Q_T) \quad \text{and a.e. in} \ Q_T, \\
(u_m', v_m') &\to (u', v') \quad \text{strongly in} \ L^2(Q_T) \times L^2(Q_T) \quad \text{and a.e. in} \ Q_T, \\
v_m(0, \cdot) &\to u(0, \cdot) \quad \text{strongly in} \ C^0([0, T_*]), \\
|u_m'(0, \cdot)|^{\frac{1}{r}} - u_m'(0, \cdot) &\to \chi_0 \quad \text{strongly in} \ C^0([0, T_*]).
\end{align*}
\]

By (56), (73) or (76), we have

\[
\|f_1(u_m, v_m) - f_1(u, v)\|_{L^2(Q_T)} \leq K_2(T_*, x) \left( \|u_m - u\|_{L^2(Q_T)} + \|v_m - v\|_{L^2(Q_T)} \right).
\]

We deduce from (78) and (79) that

\[
f_1(u_m, v_m) \to f_1(u, v) \quad \text{strongly in} \ L^2(Q_T).
\]

Similarly

\[
f_2(u_m, v_m) \to f_1(u, v) \quad \text{strongly in} \ L^2(Q_T).
\]

By the following inequality

\[
|\Psi_r(x) - \Psi_r(y)| \leq (r - 1) R^{-r} |x - y| \quad \text{for all} \ x, y \in [-R, R],
\]

\(R > 0, r \geq 2,\) it follows from (54), (73) or (76) that

\[
\begin{align*}
\Psi_r(u_m') &\to \Psi_r(u') \quad \text{strongly in} \ L^2(Q_T), \\
\Psi_r(v_m') &\to \Psi_r(v') \quad \text{strongly in} \ L^2(Q_T).
\end{align*}
\]

By the continuity of \(G\), it follows from (78) that

\[
G(u_m(0, \cdot)) \to G(u(0, \cdot)) \quad \text{strongly in} \ C^0([0, T_*]).
\]

Put \(\eta_m = |u_m'(0, \cdot)|^{\frac{1}{r}} - u_m'(0, \cdot)\), from (78) we obtain

\[
\eta_m \to \chi_0 \quad \text{strongly in} \ C^0([0, T_*]).
\]

It follows from (85) that

\[
u_m'(0, \cdot) = |\eta_m|^{\frac{1}{r}} - \eta_m \to |\chi_0|^{\frac{1}{r}} - \chi_0 \quad \text{strongly in} \ C^0([0, T_*]).
\]

By (77) and (86), it yields

\[
|\chi_0|^{\frac{1}{r}} - \chi_0 = u'(0, \cdot).
\]
Because of the continuity of $\Psi_{r_i}$, it follows from (86) and (87) that

$$\Psi_{r_i}(u_i(0,\cdot)) \to \Psi_{r_i}(u'(0,\cdot)) \text{ strongly in } C^0([0, T_\ast]).$$

(88)

Passing to the limit in (25) by (77)1,2,3, (78), (80), (81), (83), (84) and (88), we verify $(u, v)$ satisfying the problem

$$
\begin{align*}
&\langle u''(t), w \rangle + a_2(t; u(t), w) + \lambda_1 (\Psi_{r_i}(u''(t)), w) + \lambda_1 \Psi_{r_i}(u'(0, t))w(0) \\
&= \langle G(u(0, t)) + g_0(t) \rangle w(0) + \langle f_1(u(t), v(t)), w \rangle + \langle F_1(t, w), w \rangle,
\end{align*}
$$

(89)

for all $(w, \phi) \in V \times H^1_0$, together with the initial conditions

$$
(u(0), u'(0)) = (\tilde{u}_0, \tilde{u}_1), \quad (v(0), v'(0)) = (\tilde{v}_0, \tilde{v}_1).
$$

(90)

On the other hand, it follows from $(H_2)$, $(H_3)$, $(H_6)$, and (77)1,2,3, that

$$
u_{xx} = \frac{1}{\mu_1} \left[ -\mu_1 u_x + u_{tt} + \lambda_1 |u_x|^{r_i-2} u_t - f_1(u, v) - F_1 \right] \equiv G_1 \in L^\infty(0, T_\ast; L^2). \quad (91)
$$

Thus $u \in L^\infty(0, T; V \cap H^2)$. Furthermore

$$
\begin{align*}
\mu_2(t) v_{xx}(t) - \int_0^t g(t - s) \tilde{\mu}_2(s) v_{xx}(s) ds \\
= \int_0^t g(t - s) \tilde{\mu}_2(s) v_x(s) ds - \mu_2(x)v_x + v_{tt} + \lambda_2 |v_t|^{r_i-2} v_t - f_2(u, v) - F_2(t) \\
\equiv G_2(t) \in L^\infty(0, T_\ast; L^2),
\end{align*}
$$

so

$$
\begin{align*}
\mu_2 \|v_{xx}(t)\| &\leq \|\mu_2(t)v_{xx}(t)\| \\
&\leq \|G_2\|_{L^\infty(0, T_\ast; L^2)} + \|\tilde{\mu}_2\|_{C^0([0, T]; L^\infty)} \int_0^t g(t - s) \|v_{xx}(s)\| ds.
\end{align*}
$$

(93)

Therefore

$$
\begin{align*}
\|v_{xx}(t)\|^2 &\leq \frac{2}{\mu_2} \|G_2\|^2_{L^\infty(0, T; L^2)} + \frac{2}{\mu_2} \|\tilde{\mu}_2\|^2_{C^0([0, T]; L^\infty)} \|g\|^2_{L^2(0, T)} \int_0^t \|v_{xx}(s)\|^2 ds.
\end{align*}
$$

(94)

By using Gronwall’s Lemma, (94) gives

$$
\begin{align*}
\|v_{xx}(t)\|^2 &\leq \frac{2}{\mu_2} \|G_2\|^2_{L^\infty(0, T; L^2)} \exp \left( \frac{2T}{\mu_2} \|\tilde{\mu}_2\|^2_{C^0([0, T]; L^\infty)} \|g\|^2_{L^2(0, T)} \right)
\end{align*}
$$

(95)

for all $t \in [0, T_\ast]$.

Thus $v \in L^\infty(0, T; H^1_0 \cap H^2)$ and the existence is proved completely.

Step 4. Uniqueness of the solution.

Let $(u_i, v_i), i = 1, 2$ be two weak solutions of problem (1)-(3) such that

$$
\begin{align*}
\{ (u_i, v_i) &\in L^\infty(0, T_\ast; (V \cap H^2) \times (H^1_0 \cap H^2)), \\
(u'_i, v'_i) &\in L^\infty(0, T_\ast; V \times H^1_0) \cap (L^{r_i}(Q_T) \times L^2(Q_T)), \\
(u''_i, v''_i) &\in L^\infty(0, T_\ast; L^2 \times L^2), \\
|u'_i|^\frac{r_i}{2}-1 u'_i, |v'_i|^\frac{r_i}{2}-1 v'_i &\in H^1(Q_T,)] |u''(0, \cdot)|^\frac{r_i}{2}-1 u'(0, \cdot) \in H^1(0, T_\ast), \quad i = 1, 2.
\end{align*}
$$

(96)
Then \((u, v) = (u_1 - u_2, v_1 - v_2)\) satisfy the variational problem
\[
\begin{align*}
\langle u''(t), w \rangle + a_1(t; u(t), w) + \lambda_1 \langle \Psi_r(u_1'(t)) - \Psi_r(u_2'(t)), w \rangle \\
+ \lambda_1 \langle \Psi_r(u_1'(0)) - \Psi_r(u_2'(0)), w \rangle = [G(u_1(0, t)) - G(u_2(0, t))] w(0) \\
+ \langle f_1(u_1, v_1) - f_1(u_2, v_2), w \rangle,
\end{align*}
\]
\[
\langle v''(t), \phi \rangle + a_2(t; v(t), \phi) + \lambda_2 \langle \Psi_r(v_1'(t)) - \Psi_r(v_2'(t)), \phi \rangle \\
- \int_0^t g(t-s)\bar{a}_2(s; v(s), \phi) ds = \langle f_2(u_1, v_1) - f_2(u_2, v_2), \phi \rangle \text{ for all } (w, \phi) \in V \times H_0^1,
\]
\[
u(0) = v(0) = u'(0) = v'(0) = 0.
\]
We take \((w, \phi) = (u', v')\) in (97)\textsubscript{1,2}, and integrating with respect to \(t\), we obtain
\[
\sigma(t) = 2 \int_0^t \left[ \langle f_1(u_1, v_1) - f_1(u_2, v_2), u'(s) \rangle + \langle f_2(u_1, v_1) - f_2(u_2, v_2), v'(s) \rangle \right] ds 
\]
\[
+ \int_0^t ds \int_0^t \left[ \mu_1'(x, s)u_1^2(x, s) + \mu_2'(x, s)v_2^2(x, s) \right] dx \\
- 2g(0) \int_0^t \left\| \sqrt{\mu_2(s)}v_2(s) \right\|^2 ds - 2 \int_0^t ds \int_0^\tau g'(\tau-s)\bar{a}_2(s; v(s), v(\tau)) ds \\
+ 2 \int_0^t g(t-s)\bar{a}_2(s; v(s), v(t)) ds + 2 \int_0^t [G(u_1(0, s)) - G(u_2(0, s))] u'(0, s) ds
\]
\[
\equiv Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6
\]
where
\[
\sigma(t) = \left\| u'(t) \right\|^2 + \left\| v'(t) \right\|^2 + \left\| \sqrt{\mu_1(t)}u_1(t) \right\|^2 + \left\| \sqrt{\mu_2(t)}v_2(t) \right\|^2
\]
\[
+ 2\lambda_1 \int_0^t \langle \Psi_r(u_1'(s)) - \Psi_r(u_2'(s)), u'(s) \rangle ds
\]
\[
+ 2\lambda_2 \int_0^t \langle \Psi_r(v_1'(s)) - \Psi_r(v_2'(s)), v'(s) \rangle ds
\]
\[
+ 2\lambda_1 \int_0^t \langle \Psi_r(u_1'(0, s)) - \Psi_r(u_2'(0, s)), u'(0, s) \rangle ds.
\]
Using the following inequality
\[
\forall r \geq 2, \exists C_r > 0 : \langle \Psi_r(x) - \Psi_r(y) \rangle (x - y) \geq C_r |x - y|^r \quad \forall x, y \in \mathbb{R},
\]
it follows from (99) that
\[
\sigma(t) \geq \left\| u'(t) \right\|^2 + \left\| v'(t) \right\|^2 + \mu_* \left( \left\| u_1(t) \right\|^2 + \left\| v_2(t) \right\|^2 \right) 
\]
\[
+ 2C_r \lambda_1 \int_0^t \left\| u'(s) \right\|^r_{L^{r_1}} ds + 2C_r \lambda_2 \int_0^t \left\| v'(s) \right\|^r_{L^{r_2}} ds \\
+ 2C_r \lambda_1 \int_0^t \left\| u'(0, s) \right\|^r_{L^{r_1}} ds.
\]
Put
\[
R = \max_{1 \leq i \leq 2} \left( \left\| u_{ix} \right\|_{L^\infty(0,T;L^2)} + \left\| v_{ix} \right\|_{L^\infty(0,T;L^2)} \right),
\]
\[
L_R(f) = \sum_{i=1}^2 \sup_{|y|, |z| \leq R} \left( \left| \frac{\partial f_i}{\partial y} (y, z) \right| + \left| \frac{\partial f_i}{\partial z} (y, z) \right| \right).
\]
We estimate all the integrals \(Z_1, \cdots, Z_6\) on the right-hand side of (98) as follows:
Apply the Cauchy-Schwartz inequality, (100), (101) give

$$Z_1 = 2 \int_0^t [(f_1(u_1, v_1) - f_1(u_2, v_2), u'(s)) + (f_2(u_1, v_1) - f_2(u_2, v_2), v'(s))] ds$$

$$\leq 2L_R(f) \int_0^t (\|u(s)\| + \|v(s)\|)(\|u'(s)\| + \|v'(s)\|) ds$$

$$\leq \frac{2L_R(f)}{\sqrt{\mu_*}} \int_0^t \sigma(s) ds.$$

With the integrals $Z_2, Z_3, Z_4$, it is clear to see that

$$Z_2 = \int_0^t ds \int_0^t [\mu'_1(x, s)u_x^2(x, s) + \mu'_2(x, s)v_x^2(x, s)] dx$$

$$\leq \frac{1}{\mu_*} \max_{i=1,2} \|\mu'_i\|_{C^0([0,T];L^\infty)} \int_0^t \sigma(s) ds;$$

$$Z_3 = -2g(0) \int_0^t \left\| \sqrt{\mu_2(s)}v_x(s) \right\|^2 ds \leq \frac{2}{\mu_2(s)} \|g(0)\| \|\mu_2\|_{C^0([0,T];L^\infty)} \int_0^t \sigma(s) ds;$$

$$Z_4 = -2 \int_0^t d\tau \int_0^\tau g'(\tau - s)\bar{a}_2(s; v(s), v(s)) ds$$

$$\leq \frac{2}{\mu_*} \|\mu_2\|_{C^0([0,T];L^\infty)} \sqrt{T} \|g'\|_{L^2(0,T)} \int_0^t \sigma(s) ds;$$

$$Z_5 = 2 \int_0^t g(t - s)\bar{a}_2(s; v(s), v(t)) ds$$

$$\leq \frac{1}{\delta \mu_*} \|\mu_2\|_{C^0([0,T];L^\infty)}^2 \|g\|_{L^2(0,T)}^2 \int_0^t \sigma(s) ds + \delta \sigma(t)$$

$\forall \delta > 0.$

About the integral $Z_6 = 2 \int_0^t [G(u_1(0, s)) - G(u_2(0, s))] u'(0, s) ds$, we consider two cases for $\bar{r}_1 = 2$ and $G \in C^1(\mathbb{R})$ or $\bar{r}_1 > 2$ and $G \in C^2(\mathbb{R})$.

Case 1. $\bar{r}_1 = 2$ and $G \in C^1(\mathbb{R})$: Put $L_R(G) = \sup_{|y| \leq R} |G'(y)|$, we have

$$\sigma(t) \geq 2\lambda_1 \int_0^t |u'(0, s)|^2 ds$$

(104)

and

$$Z_6 = 2 \int_0^t [G(u_1(0, s)) - G(u_2(0, s))] u'(0, s) ds$$

(105)

$$\leq 2L_R(G) \int_0^t |u(0, s)| |u'(0, s)| ds$$

$$\leq \frac{1}{2\lambda_1 \mu_* \delta} L_R^2(G) \int_0^t \sigma(s) ds + \delta \sigma(t) \text{ for all } \delta > 0.$$

Choosing $\delta = \frac{1}{4}$, it follows from (98), (102), (103) and (105), that

$$\sigma(t) \leq \tilde{K}_T \int_0^t \sigma(s) ds$$

(106)
where
\[ \tilde{K}_T = 4 \left[ \frac{L_R(f)}{\sqrt{\mu_*}} + \frac{L_R^2(G)}{\lambda_1 \mu_*} + \frac{1}{2 \mu_* \max_{i=1,2} \| \mu'_i \|_{C^0([0,T];L^\infty)}} \right] \]
\[ + 4 \left[ \frac{|g(0)|}{\mu_{2*}} + \sqrt{T} \frac{\| g' \|_{L^2(0,T)}}{\mu_*} + \frac{2}{\mu_{2*}^2} \frac{\| g \|_{L^2(0,T)} \| \bar{\mu}_2 \|_{C^0([0,T];L^\infty)}} \right] \| \bar{\mu}_2 \|_{C^0([0,T];L^\infty)} \cdot \]

By Gronwall’s lemma, (106) leads to \( \sigma(t) \equiv 0 \), i.e., \( u = u_1 - u_2 \equiv 0 \), \( v = v_1 - v_2 \equiv 0 \).

Case 2. \( \bar{r}_1 > 2 \) and \( G \in C^2(\mathbb{R}) \) with \( G'(z) \leq \zeta_{\max} < \mu_{\ast} \) for all \( z \in \mathbb{R} \): From the following formula
\[ G(u_1(0,s)) - G(u_2(0,s)) = u(0,s) \int_0^1 G'(u_2(0,s) + \theta u(0,s))d\theta, \]
by using integration by parts, we have
\[ Z_0 = 2 \int_0^1 \left[ G(u_1(0,s)) - G(u_2(0,s)) \right] u'(0,s)ds \]
\[ = 2 \int_0^1 \left[ u(0,s) \int_0^1 G'(u_2(0,s) + \theta u(0,s))d\theta \right] u'(0,s)ds \]
\[ = \int_0^1 d\theta \int_0^1 \left[ G'(u_2(0,s) + \theta u(0,s)) \right] \frac{d}{ds} u^2(0,s)ds \]
\[ = \left( \int_0^1 G_1(\theta,t)d\theta \right) u^2(0,t) - \int_0^t u^2(0,s)ds \int_0^1 G_2(\theta,s)d\theta \]
where
\[ G_1(\theta,t) = G'(u_2(0,t) + \theta u(0,t)), \]
\[ G_2(\theta,s) = [G''(u_2(0,s) + \theta u(0,s)) \right) (u'_2(0,s) + \theta u'(0,s)). \]

Put \( \bar{L}_R(G) = \sup_{|y| \leq R} |G''(y)| \), we have
\[ G_1(\theta,t) = G'(u_2(0,t) + \theta u(0,t)) \leq \zeta_{\max}, \]
\[ |G_2(\theta,s)| \leq \bar{L}_R(G)(|u'_2(0,s)| + |u'_2(0,s)|) \leq \bar{L}_R(G)(|u'_2|_{L^\infty([0,T];L^2)} + |u'_2|_{L^\infty([0,T];L^2)}) = \tilde{K}_T^{(1)}. \]

Hence, it follows from (107), (109) that
\[ Z_0 \leq \zeta_{\max}u^2(0,t) + \tilde{K}_T^{(1)} \int_0^t u^2(0,s)ds \leq \frac{\zeta_{\max}}{\mu_{1*}} \sigma(t) + \frac{\tilde{K}_T^{(1)}}{\mu_{1*}} \int_0^t \sigma(s)ds. \]

Combining (98), (102), (103) and (110) to obtain
\[ \left( 1 - \frac{\zeta_{\max}}{\mu_{1*}} - \delta \right) \sigma(t) \leq \tilde{K}_T^{(2)}(\delta) \int_0^t \sigma(s)ds \]
where
\[ \tilde{K}_T^{(2)}(\delta) = \frac{2L_R(f)}{\sqrt{\mu_*}} + \tilde{K}_T^{(1)} \frac{1}{\mu_{1*}} \frac{1}{\max_{i=1,2} \| \mu'_i \|_{C^0([0,T];L^\infty)}} \]
\[ + 2 \left( \frac{|g(0)|}{\mu_{2*}} + \sqrt{T} \frac{\| g' \|_{L^2(0,T)}}{\mu_*} + \frac{\| \bar{\mu}_2 \|_{C^0([0,T];L^\infty)} \| g \|_{L^2(0,T)}^2}{2\delta \mu_{2*}^2} \right) \| \bar{\mu}_2 \|_{C^0([0,T];L^\infty)} . \]
By $1 - \frac{\zeta_{\text{max}}}{\mu_{\text{tr}}} > 0$, we can choose $\delta > 0$ such that $1 - \frac{\zeta_{\text{max}}}{\mu_{\text{tr}}} - \delta > 0$. By Gronwall’s lemma, (111) leads to $\sigma(t) \equiv 0$, i.e., $u = u_1 - u_2 \equiv 0, \ b = v_1 - v_2 \equiv 0$.

Theorem 2.2 is proved completely.

Proof of Theorem 2.3. For the existence of a weak solution, we use standard arguments of density. The proof is similar to the arguments in [20], so we omit the details. The uniqueness of a weak solution is obtained by using the well-known regularization procedure due to Lions, see for example Ngoc et al. [23]. □

Remark 2.12. Let $N = \max \left\{ 1, \frac{\alpha}{\alpha}, \frac{\beta}{\beta}, \frac{\gamma}{\gamma}, \frac{(\gamma - 1)r_1}{(r_1 - 1)}, \frac{(\alpha + \beta_1 - 1)r_1}{2(r_1 - 1)}, \frac{(\alpha_1 + \beta_1 - 1)r_2}{2(r_2 - 1)} \right\} \leq 1$.

By solving nonlinear Volterra integral inequalities as in (47), it ensures that a global estimate will be obtained. However, this global estimate is not enough to pass to the limit in all nonlinear terms, specially passing to the limit the term $|u_t(0, t)|^{\gamma-2} u_t(0, t)$ imposed the value at boundary point $x = 0$, only if $r_1 = r_2 = \tilde{r}_1 = 2$. Otherwise, we need the second estimate, by solving nonlinear Volterra integral inequalities as in (71), this leads immediately to the global existence of solutions for problem (1)-(3) when $\tilde{r}_1 \geq 3$.

Because of the difficulties arising in passing to the limit in the nonlinear terms imposed the value on boundary, the Sobolev imbedding theorem can not be chosen to apply appropriately; thus, it is still an open problem in higher-dimensional case.

3. Blow up result. In this section, Prob. (1)-(3) is considered with $r_1 = r_2 = \tilde{r}_1 = 2, \bar{\mu}_2(x, t) = \bar{\mu}_2(x), F_1(x, t) = F_2(x, t) = g_0(t) = 0$, as follows

\[
\begin{cases}
  u_{tt} - \frac{\partial}{\partial x} (\mu_1(x, t) u_x) + \lambda_1 u_t = f_1(u, v), \quad 0 < x < 1, \ 0 < t < T, \\
  v_{tt} - \frac{\partial}{\partial x} (\mu_2(x, t) v_x) + \lambda_2 v_t + \int_0^t g(t-s) \frac{\partial}{\partial x} (\bar{\mu}_2(x) v(x, s)) \ ds = f_2(u, v), \quad 0 < x < 1, \ 0 < t < T, \\
  \mu_1(0, t) u_x(0, t) = -G(u(0, t)) + \lambda_1 u_t(0, t), \\
  (u(1, t) = v(1, t) = 0, \\
  (u(x, 0), v(x, 0)) = (\bar{u}_0(x), \bar{v}_0(x)), \quad (u_t(x, 0), v_t(x, 0)) = (\bar{u}_1(x), \bar{v}_1(x))
\end{cases}
\]

where $\lambda_1, \lambda_2, \bar{\lambda}_1 > 0$ are given constants and $\bar{u}_i, \bar{v}_i \ (i = 0, 1), f_i, \mu_i \ (i = 1, 2), \bar{\mu}_2, G, g$ are given functions satisfying conditions specified later.

Suppose that $(H_1^1), (H_2^1) - (H_4^1), (H_5^1)$ and $\bar{\mu}_2(x, t) = \bar{\mu}_2(x)$. Let $(u, v)$ be a weak solution of (112) satisfying

\[
(u, v) \in C \left( [0, T_*]; V \times H^1_0 \right) \cap C^1 \left( [0, T_*]; L^2 \times L^2 \right), \ u(0, \cdot) \in H^1(0, T_*)
\]

for $T_* > 0$ small enough.

First, we make the following assumptions.

$(\tilde{H}_1) \ \mu_1, \mu_2 \in C^1 ([0, 1] \times \mathbb{R}_+), \bar{\mu}_2 \in C^0 ([0, 1])$ and there exist the positive constants $\mu_{i*} \ (i = 1, 2)$ such that

(i) $\mu_i'(x, t) \leq \mu_i \leq \mu_i(x, t) \ \forall (x, t) \in [0, 1] \times \mathbb{R}_+$

(ii) $0 < \mu_{i*} = \min_{0 \leq x \leq 1} \mu_i(x) \leq \bar{\mu}_2(x) \leq \max_{0 \leq x \leq 1} \bar{\mu}_2(x)$

$(\tilde{H}_2) \ \exists \mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants $\alpha, \beta, \hat{\alpha}, \hat{\beta}, \tilde{\alpha}_1, \tilde{\beta}_1, d_1 > 2, \tilde{d}_1, d_2 > 0$ such that

(i) $D_1 F(u, v) = f_1(u, v), \ D_2 F(u, v) = f_2(u, v)$ $\forall u, v \in \mathbb{R}$

(ii) $u f_1(u, v) + v f_2(u, v) \geq d_1 F(u, v) \ \forall u, v \in \mathbb{R}$

(iii) $\tilde{d}_1 \left( |u|^{\alpha} + |v|^{\beta} \right) \leq F(u, v) \leq \tilde{d}_2 \left( 1 + |u|^2 + |v|^2 \right) |v|^{\alpha_1} + |v|^{\beta_1}$ $\forall u, v \in \mathbb{R}$;
\( \hat{H}_4 \) \( G \in C^0(\mathbb{R}) \) and there exist the constants \( \gamma, \bar{\gamma}, \bar{g}_1 > 2, \bar{g}_2 > 0 \), such that

- (i) \( yG(y) \geq g_1 \int_0^y G(z)dz \forall y \in \mathbb{R} \)
- (ii) \( \bar{g}_1 |y|^\bar{\gamma} \leq \int_0^y G(z)dz \leq \bar{g}_2 (1 + |y|^\bar{\gamma}) \forall y \in \mathbb{R} \);

\( \hat{H}_5 \) \( g \in H^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \) with the following properties

- (i) \( g'(t) \leq 0 \forall t \geq 0 \)
- (ii) \( 0 < \int_0^\infty g(s)ds < \frac{\beta^2}{\alpha^2} (d_* - 2) \)
- (iii) \( d_* = \min\{d_1, g_1\} > 2 \) with \( d_1, g_1 \) as in (\( \hat{H}_3 \)) and (\( \hat{H}_4 \)).

**Remark 3.1.** In the following, we will show that the functions \( f_1, f_2, G \) in the above example satisfy (\( \hat{H}_3 \)), (\( \hat{H}_4 \)).

Consider

\[
\begin{aligned}
f_1(u, v) &= D_1F(u, v), \quad f_2(u, v) = D_2F(u, v), \quad G(y) = \bar{g} |y|^{\gamma-2} y \ln k^4(e + y^2)
\end{aligned}
\]

where

\[
\begin{aligned}
\mathcal{F}(u, v) &= \gamma_1 |u|^\alpha \ln k^1(e + u^2) + \gamma_2 |v|^\beta \ln k^2(e + v^2) + \gamma_3 |u|^{\alpha_1} |v|^{\beta_1} \\
&= \gamma_1 \left[ \alpha \ln k^1(e + u^2) + 2k_1 \frac{u^2 \ln k^{1-1}(e + u^2)}{e + u^2} \right] |u|^\alpha \\
&\quad + \gamma_2 \left[ \beta \ln k^2(e + v^2) + 2k_2 \frac{v^2 \ln k^{2-1}(e + v^2)}{e + v^2} \right] |v|^\beta \\
&\quad + \gamma_3 \left[ (\alpha_1 + \beta_1) \ln k^3(e + u^2 + v^2) + 2k_3 \frac{(u^2 + v^2) \ln k^{3-1}(e + u^2 + v^2)}{e + u^2 + v^2} \right] |u|^{\alpha_1} |v|^{\beta_1}.
\end{aligned}
\]

By

\[
\begin{aligned}
|u|^{\alpha} |v|^\beta \geq \gamma_1 \alpha |u|^\alpha \ln k^1(e + u^2) + \gamma_2 \beta |v|^\beta \ln k^2(e + v^2) + \gamma_3 (\alpha_1 + \beta_1) |u|^{\alpha_1} |v|^{\beta_1} \\
&\geq d_1 F(u, v) \forall u, v \in \mathbb{R}
\end{aligned}
\]

where \( d_1 = \min\{\alpha, \beta, \alpha_1 + \beta_1\} \). Then, (\( \hat{H}_3, (ii) \)) holds.

With the condition (\( \hat{H}_3, (iii) \)):

It is obvious that \( \ln k^1(e + u^2) \geq 1, \ln k^2(e + v^2) \geq 1 \), this yields

\[
\begin{aligned}
\mathcal{F}(u, v) \geq \gamma_1 |u|^\alpha + \gamma_2 |v|^\beta \geq \min\{\gamma_1, \gamma_2\} \left( |u|^\alpha + |v|^\beta \right) \equiv \bar{d}_1 \left( |u|^\alpha + |v|^\beta \right) \forall u, v \in \mathbb{R},
\end{aligned}
\]

so (\( \hat{H}_3, (iii) \)) holds.

With the condition (\( \hat{H}_4, (i) \)):

By (14) and (15), we have

\[
\int_0^y G(z)dz \leq \bar{g} \gamma |y|^\gamma \ln k^4(e + y^2) = \frac{1}{\gamma} yG(y),
\]

(\( \hat{H}_4, (i) \)) holds with \( g_1 = \gamma \).
With the condition \((\hat{H}_4, (ii))\) : Note that, we deduce from (15) that
\[
K(y) = \int_0^y G(z) \frac{z^2 dz}{(e + z^2) \ln(e + z^2)} \leq \int_0^y G(z) dz \quad \forall y \in \mathbb{R}. \tag{119}
\]
Hence, it follows from (14), (119) that
\[
\int_0^y G(z) dz \geq \frac{\bar{g}}{\gamma + 2k_4} |y|^\gamma \ln^k(t(e + y^2)) - \frac{2k_4}{\gamma} \int_0^y G(z) dz \quad \forall y \in \mathbb{R}. \tag{120}
\]
This implies that
\[
\int_0^y G(z) dz \geq \frac{\bar{g}}{\gamma + 2k_4} |y|^\gamma \ln^k(t(e + y^2)) \geq \frac{\bar{g}}{\gamma + 2k_4} |y|^\gamma \quad \forall y \in \mathbb{R}. \tag{121}
\]
Thus \((\hat{H}_4, (i))\) holds with \(\bar{g}_1 = \frac{\bar{g}}{\gamma + 2k_4} \).

With \(\bar{\mu}_2 = \bar{\mu}_2(x)\), we put
\[
\bar{a}_2(v, \phi) = \langle \bar{\mu}_2 v_x, \phi_x \rangle, \quad \|v\|_{\bar{a}_2} = \|\sqrt{\bar{\mu}_2} v_x\| = \sqrt{\bar{a}_2(v, v)} \quad \text{for all } v, \phi \in H_0^1.
\]
We note that
\[
a_1(t; u, u) = \|\sqrt{\mu_1(t)} u_x\|^2, \quad a_2(t; v, v) = \|\sqrt{\mu_2(t)} v_x\|^2 \quad \text{for all } u \in V, \quad v \in H_0^1, \quad t \geq 0.
\]
Put
\[
H(0) = -\frac{1}{2} \left(\|\bar{\mu}_1\|^2 + \|\bar{\mu}_1\|^2\right) - \frac{1}{2} \left(\|\sqrt{\mu_1(0)} \bar{u}_{0x}\|^2 + \|\sqrt{\mu_2(0)} \bar{v}_{0x}\|^2\right) + \int_0^{\bar{a}_0(0)} G(z) dz + \int_0^1 F(\bar{u}_0(x), \bar{v}_0(x)) dx.
\]
Then we obtain the theorem.

**Theorem 3.2.** Let the assumptions \((\hat{H}_2) - (\hat{H}_5)\) hold. Then, for any \((\bar{u}_0, \bar{v}_0) \in V \times H_0^1, (\bar{u}_1, \bar{v}_1) \in L^2 \times L^2\) such that \(H(0) > 0\), the weak solution \((u, v)\) of Prob. \((112)\) blows up in finite time.

**Proof of Theorem 3.1.** It consists of three steps.

**Step 1. Constructing Lyapunov functional.**

We denote by \(E(t)\) the energy associated to the solution \((u, v)\), defined by
\[
E(t) = \frac{1}{2} \left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{1}{2} (g \odot v)(t) \tag{122}
\]
\[
+ \frac{1}{2} \left(\|\sqrt{\mu_1(t)} u_x(t)\|^2 + \|\sqrt{\mu_2(t)} v_x(t)\|^2\right) - \|\sqrt{\mu_2(t)} v_x(t)\|^2 \int_0^t g(s) ds
\]
\[
- \int_0^{u(t)} G(z) dz - \int_0^t F(u(x, t), v(x, t)) dx
\]
where
\[
(g \odot v)(t) = \int_0^t g(t - s) \|v(t) - v(s)\|_{\bar{a}_2}^2 ds \tag{123}
\]
and we put
\[
H(t) = -E(t). \tag{124}
\]

On the other hand, by multiplying \((112)_{1,2}\) by \((u'(x, t), v'(x, t))\) and integrating over \([0, 1]\), we get
\[
H'(t) = \lambda_1 \|u'(t)\|^2 + \lambda_2 \|v'(t)\|^2 + \lambda_1 |u'(0, t)|^2 - \frac{1}{2} (g' \odot v)(t) \tag{125}
\]
\(-\frac{1}{2} \int_0^1 \mu_1'(x,t)u_1^2(x,t) \, dx - \frac{1}{2} \int_0^1 \mu_2'(x,t)u_2^2(x,t) \, dx + \frac{1}{2} g(t) \| \sqrt{\mu_2}v_1(t) \|^2 \geq 0 \forall t \in [0, T_*)\)

for any regular solution \((u, v)\). We also can extend (125) to weak solutions by using density arguments.

Hence, we can deduce from (125) and \(H(0) > 0\) that
\[ 0 < H(0) \leq H(t) \forall t \in [0, T_*) \tag{126} \]
or
\[ 0 < H(0) \leq H(t) = -\frac{1}{2} \left( \| u'(t) \|^2 + \| v'(t) \|^2 \right) - \frac{1}{2} (g \circ v)(t) \tag{127} \]
\[ -\frac{1}{2} \left( \| \sqrt{\mu_1(t)}u_x(t) \|^2 + \| \sqrt{\mu_2(t)}v_x(t) \|^2 \right) - \| \sqrt{\mu_2}v_1(t) \|^2 \int_0^t g(s) \, ds \]
\[ + \int_0^{u_0(t)} G(z) \, dz + \int_0^1 \mathcal{F}(u(x, t), v(x, t)) \, dx \forall t \in [0, T_*) \]
which implies that
\[ 0 < H(0) \leq H(t) \leq \int_0^{u_0(t)} G(z) \, dz + \int_0^1 \mathcal{F}(u(x, t), v(x, t)) \, dx, \tag{128} \]
\[ \| u'(t) \|^2 + \| v'(t) \|^2 + (g \circ v)(t) + \| \sqrt{\mu_1(t)}u_x(t) \|^2 \]
\[ + \| \sqrt{\mu_2(t)}v_x(t) \|^2 - \| \sqrt{\mu_2}v_1(t) \|^2 \int_0^t g(s) \, ds \]
\[ \leq 2 \int_0^{u_0(t)} G(z) \, dz + 2 \int_0^1 \mathcal{F}(u(x, t), v(x, t)) \, dx \forall t \in [0, T_*) \] Now, we define the functional
\[ L(t) = H^{1-\eta}(t) + \varepsilon \Phi(t) \tag{129} \]
where
\[ \Phi(t) = \langle u(t), u'(t) \rangle + \langle v(t), v'(t) \rangle + \frac{\lambda_1}{2} \| u(t) \|^2 + \frac{\lambda_2}{2} \| v(t) \|^2 + \frac{\lambda_1}{2} u^2(0, t) \tag{130} \]
for \(\varepsilon\) small enough and
\[ 0 < 2\eta < 1, \ 2/(1-2\eta) \leq \min \{\alpha, \beta, \gamma\}. \tag{131} \]
We remark that, there exists a constant \(\bar{t}_1 > 0\) such that
\[ L'(t) \geq \bar{t}_1 \left[ H(t) + \| u'(t) \|^2 + \| v'(t) \|^2 + \| u_x(t) \|^2 \right. \]
\[ + \| v_x(t) \|^2 + \| u(t) \|^2_{L^\alpha} + \| v(t) \|^2_{L^\beta} + |u(0, t)|^\gamma \] Indeed, multiplying \((112)_{1,2}\) by \(\langle u(x, t), v(x, t) \rangle\) and integrating over \([0, 1]\), it yields
\[ \Phi'(t) = \| u'(t) \|^2 + \| v'(t) \|^2 - \| \sqrt{\mu_1(t)}u_x(t) \|^2 - \| \sqrt{\mu_2(t)}v_x(t) \|^2 \]
\[ + u(0, t)G(0) + \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \]
\[ + \int_0^t g(t-s) \tilde{a}_2(v(s), v(t)) \, ds. \]
Taking a derivative of (129) and using (133) to obtain
\[ L'(t) = (1 - \eta)H^{-\eta}(t)H'(t) + \varepsilon \Phi'(t) \geq \varepsilon \Phi'(t). \] (134)

By means of (\(\tilde{H}_3\), (\(\tilde{H}_4\)), we get
\[
\begin{cases}
\langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle 
\geq d_1 \int_0^1 F(u(x, t), v(x, t)) \, dx, \\
u(0, t)G(u(0, t)) \geq g_1 \int_0^{u(0, t)} G(z) \, dz, \\
\|\sqrt{\mu_2} v_x(t)\| \geq \frac{\bar{\mu}_2}{\mu_2} \|\sqrt{\mu_2} v_x(t)\|, \\
\int_0^t \|g(t-s)\bar{a}_2(v(s), v(t))\| \geq \frac{1}{4} (g \circ v)(t).
\end{cases}
\]

Hence, combining (122), (124), (133) and (135) give
\[ \Phi'(t) \geq \|u'(t)\|^2 + \|v'(t)\|^2 - \left[ \|\sqrt{\mu_1(t)} u_x(t)\|^2 + \|\sqrt{\mu_2(t)} v_x(t)\|^2 \right] \]
\[ + \left( g_1 \int_0^{u(0, t)} G(z) \, dz + d_1 \int_0^1 F(u(x, t), v(x, t)) \, dx \right) \]
\[ \geq \left( 1 + \frac{d_1}{2} (1 - \delta_1) \right) \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) + \frac{1}{2} (d_* - \frac{1}{2} - \delta_1 d_*) (g \circ v)(t) \]
\[ + d_* (1 - \delta_1) H(t) + d_* \delta_1 \left( \int_0^{u(0, t)} G(z) \, dz + \int_0^1 F(u(x, t), v(x, t)) \, dx \right) \]
\[ + \frac{1}{2} \left( d_* - 2 - \delta_1 d_* \right) \frac{\|\mu_1(t) u_x(t)\|^2}{\|\mu_2(t) v_x(t)\|^2} \]
\[ + \frac{d_* \bar{\mu}_2}{\mu_2} \left[ \left( \frac{d_* - 2}{d_*} \right) \frac{\mu_2}{\bar{\mu}_2} - \int_0^\infty g(s) \, ds \right] \]
\[ + \delta_1 \left( \int_0^\infty g(s) \, ds - \frac{\mu_2}{\bar{\mu}_2} \right) \frac{\|\mu_2(t) v_x(t)\|^2}{\|\mu_2(t) v_x(t)\|^2}. \]

By \(d_* = \min\{d_1, g_1\} > 2\) and \(\left( \frac{d_* - 2}{d_*} \right) \frac{\mu_2}{\bar{\mu}_2} - \int_0^\infty g(s) \, ds\), we can choose \(\delta_1 \in (0, 1)\) such that
\[ d_* - 2 - \delta_1 d_* > 0 \quad \text{and} \quad \left( \frac{d_* - 2}{d_*} \right) \frac{\mu_2}{\bar{\mu}_2} - \int_0^\infty g(s) \, ds + \delta_1 \left( \int_0^\infty g(s) \, ds - \frac{\mu_2}{\bar{\mu}_2} \right) > 0. \] (137)

Note that
\[
\int_0^1 F(u(x, t), v(x, t)) \, dx \geq \bar{d}_1 \left( \|u(t)\|_{L^a_x}^\alpha + \|v(t)\|_{L^b_x}^\beta \right),
\int_0^{u(0, t)} G(z) \, dz \geq \bar{g}_1 |u(0, t)|^\gamma,
\] (138)
by using the inequalities (134), (136), (138), we obtain (132) with choosing \( \bar{l}_1 > 0 \) small enough.

From the formula of \( L(t) \) and (132), we can choose \( \varepsilon \) small enough such that
\[
L(t) \geq L(0) > 0 \quad \forall t \in [0, T_*).
\]

Using the inequality
\[
\left( \sum_{i=1}^{6} x_i^r \right)^r \leq 6^{r-1} \sum_{i=1}^{6} x_i^r, \quad \text{for all } r > 1 \text{ and } x_1, \ldots, x_6 \geq 0,
\]
we deduce from (129)-(131) that
\[
L^{1/(1-\eta)}(t) \leq \text{Const} \left( H(t) + |\langle u(t), u'(t) \rangle|^{1/(1-\eta)} + |\langle v(t), v'(t) \rangle|^{1/(1-\eta)} \right.
\]
\[
+ \|u(t)\|^{2/(1-\eta)} + \|v(t)\|^{2/(1-\eta)} + |u(0, t)|^{2/(1-\eta)} \right).
\]

**Step 2. The estimates.**

Using Young’s inequality, we have
\[
|\langle u(t), u'(t) \rangle|^{1/(1-\eta)} \leq \text{Const} \left( \|u(t)\|^s_{L^\infty} + \|u'(t)\|^2 \right),
\]
\[
|\langle v(t), v'(t) \rangle|^{1/(1-\eta)} \leq \text{Const} \left( \|v(t)\|^s_{L^\infty} + \|v'(t)\|^2 \right)
\]
where \( s = 2/(1 - 2\eta) \leq \min \{\alpha, \beta, \gamma\} \) by (131).

Furthermore, we have the following property.

**Lemma 3.3.** Let \( s = 2/(1 - 2\eta) \leq \min \{\alpha, \beta, \gamma\} \), we obtain
\[
\begin{align*}
i) \|u\|^s_{L^\infty} + \|u\|^{2/(1-\eta)} + |u(0)|^{2/(1-\eta)} & \leq 3 \left( \|u_x\|^2 + \|u\|^2_{L^\infty} + |u(0)|^{2}\right), \\
ii) \|v\|^s_{L^\infty} + \|v\|^{2/(1-\eta)} & \leq 2 \left( \|v_x\|^2 + \|v\|^2_{L^\infty} \right)
\end{align*}
\]
for any \((u, v) \in V \times H_0^1\).

Proof of Lemma 3.2 is straightforward, so we omit the details.

**Step 3. Blow up.**

It follows from (141)-(144) that
\[
L^{1/(1-\eta)}(t) \leq \text{Const} \left( H(t) + \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2 \right.
\]
\[
+ \|u(t)\|^s_{L^\infty} + \|v(t)\|^s_{L^\infty} + |u(0, t)|^{2} \right) \forall t \in [0, T_*).
\]

Using (132), (145) yields
\[
L'(t) \geq \bar{l}_2 L^{1/(1-\eta)}(t) \forall t \in [0, T_*)
\]
where \( \bar{l}_2 \) is a positive constant. By integrating (146) over \((0, t)\), it gives
\[
L^{n/(1-\eta)}(t) \geq \frac{1}{L^{-n/(1-\eta)}(0) - \frac{\bar{l}_2}{1-n} t}, \quad 0 \leq t < \frac{1}{\bar{l}_2 n} (1 - \eta) L^{-n/(1-\eta)}(0).
\]

Therefore, \( L(t) \) blows up in a finite time given by \( T_* = \frac{1}{\bar{l}_2 n} (1 - \eta) L^{-n/(1-\eta)}(0) \).

Theorem 3.1 is proved completely. \(\Box\)
4. Decay result. In this section, Prob. (1)-(3) is considered with $r_1 = r_2 = r_1 = 2$, $\mu_2(x, t) = \mu(x)$, as follows

$$
\begin{aligned}
&\begin{cases}
  u_{tt} - \frac{\partial}{\partial x} \left( \mu_1(x,t)u_x \right) + \lambda_1 u_t = f_1(u,v) + F_1(x,t), \quad 0 < x < 1, \; 0 < t < T, \\
  u_{tt} - \frac{\partial}{\partial x} \left( \mu_2(x,t)v_x \right) + \lambda_2 v_t + \int_0^t g(t-s) \frac{\partial}{\partial x} (\mu_2(x,v_x(x,s))) \, ds \\
  = f_2(u,v) + F_2(x,t), \quad 0 < x < 1, \; 0 < t < T, \\
  \mu_1(0,t)u_x(0,t) = -G(u(0,t)) + \lambda_1 u_t(0,t) - g_0(t), \\
  u(1,t) = v(0,t) = v(1,t) = 0, \\
  (u(x,0), v(x,0)) = (\bar{u}_0(x), \bar{v}_0(x)), \quad (u_t(x,0), v_t(x,0)) = (\bar{u}_1(x), \bar{v}_1(x))
\end{cases}
\end{aligned}
$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$ are given constants and $\bar{u}_i, \bar{v}_i$ ($i = 0, 1$), $F_i, f_i, \mu_i$ ($i = 1, 2$), $\mu_2, G, g, g_0$ are given functions satisfying conditions specified later.

First, in order to obtain the decay result, we make the following assumptions.

$$(\bar{H}_2) \; \mu_1, \mu_2 \in C^1([0,1] \times \mathbb{R}^+), \; \mu_2 \in C^0([0,1]) \text{ and there exist the positive constants } \mu_1, (i = 1, 2) \text{ together with the nonnegative functions } \mu_i \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+^\infty) \text{ such that}

\begin{enumerate}
  \item $\mu_i(x,t) \geq \mu_i > 0 \; \forall (x,t) \in [0,1] \times \mathbb{R}_+$,
  \item $\mu_i(x,t) \leq \mu_i(t) \; \forall (x,t) \in [0,1] \times \mathbb{R}_+$,
  \item $0 < \mu_2, = \min_{0 \leq x \leq 1} \mu_2(x) \leq \mu_2 \leq \max_{0 \leq x \leq 1} \mu_2(x) \; \forall x \in [0,1]$;
\end{enumerate}

$$(\bar{H}_3) \; \text{There exist } F \in C^2(\mathbb{R}^2; \mathbb{R}) \text{ and the constants } \alpha, \beta, d_2 > 2, \alpha, \beta, d_2 > 0 \text{ such that}

\begin{enumerate}
  \item $D_1 F(u,v) = f_1(u,v), \; D_2 F(u,v) = f_2(u,v) \; \forall u, v \in \mathbb{R}$,
  \item $u f_1(u,v) + v f_2(u,v) \leq d_2 F(u,v) \; \forall u, v \in \mathbb{R}$,
  \item $F(u,v) \leq d_2 \left( \|u\|^\alpha + |v|^\beta + |u|^{a_1} |v|^{b_1} \right) \left( 1 + |u|^{\alpha} + |v|^\beta \right) \; \forall u, v \in \mathbb{R}$;
\end{enumerate}

$$(\bar{H}_4) \; G \in C^0(\mathbb{R}) \text{ and there exist the constants } \gamma > 2, \gamma, g_2, g_2 > 0 \text{ such that}

\begin{enumerate}
  \item $y G(y) \leq g_2 \int_0^y G(z) \, dz \; \forall y \in \mathbb{R}$,
  \item $\int_0^y G(z) \, dz \leq g_2 |y|\gamma \left( 1 + |y|\gamma \right) \; \forall y \in \mathbb{R}$;
\end{enumerate}

$$(\bar{H}_5) \; g \in H^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \text{ and there exists a constant } \zeta_0 > 0 \text{ such that}

\begin{enumerate}
  \item $g(t) \leq -\zeta_0 g(t) < 0 \; \forall t \geq 0$,
  \item $\mu_2 - \bar{\mu}_2 \int_{0}^{\infty} g(s) \, ds > 0 \text{ with } \mu_2, \bar{\mu}_2 \text{ as in } (\bar{H}_2)$;
\end{enumerate}

$$(\bar{H}_6) \; F_1, F_2 \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2), \; g_0 \in L^2(\mathbb{R}_+) \text{ and there exist the constants } \bar{C}_0, \bar{\gamma}_0 > 0 \text{ such that}

$$
\|F_1(t)\|^2 + \|F_2(t)\|^2 + g_0(t) \leq \bar{C}_0 e^{-\bar{\gamma}_0 t} \; \forall t \geq 0.
$$

Remark 4.1. The functions $f_1, f_2, G$ given in the example of section 2 also satisfy $(\bar{H}_3), (\bar{H}_4)$. Indeed, we consider

$$
\begin{aligned}
f_1(u,v) = D_1 F(u,v), \; f_2(u,v) = D_2 F(u,v), \; \bar{G}(y) = \bar{g} |y|^{\gamma - 2} y \ln k\varepsilon(e + y^2)
\end{aligned}
$$

where

$$
\bar{F}(u,v) = \bar{\gamma}_1 |u|^\alpha \ln h\varepsilon(e + u^2) + \bar{\gamma}_2 |v|^{\beta} \ln k\varepsilon(e + u^2)
$$

$$
+ \bar{\gamma}_3 |u|^{a_1} |v|^{b_1} \ln k\varepsilon(e + u^2 + v^2)
$$

with $\alpha, \beta, \alpha_1, \beta_1, \gamma > 2, k_1, k_2, k_3, k_4 > 1, \bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{g}$ are positive constants.

Obviously, $(\bar{H}_3, (i))$ holds. We check the condition $(\bar{H}_3, (ii))$:

By the following inequalities

$$
0 \leq \frac{u^2 \ln k^{i-1}(e + u^2)}{e + u^2} \leq \ln k\varepsilon(e + u^2),
$$
0 \leq \frac{v^2 \ln k^2 - 1 (e + v^2)}{e + v^2} \leq \ln k^2 (e + v^2),

0 \leq \frac{(u^2 + v^2) \ln k^2 - 1 (e + u^2 + v^2)}{e + u^2 + v^2} \leq \ln k^2 (e + u^2 + v^2),

we deduce from (116) that

\begin{equation}
uf_1 (u, v) + vf_2 (u, v) 
\leq \gamma_1 (\alpha + 2k_1) |u|^\alpha \ln k^1 (e + u^2) + \gamma_2 (\beta + 2k_2) |v|^{\beta} \ln k^2 (e + v^2) 
+ \gamma_3 (\alpha_1 + \beta_1 + 2k_3) |u|^{\alpha_1} |v|^{\beta_1} \ln k^3 (e + u^2 + v^2) 
\leq d_2 \mathcal{F}(u, v) \forall u, v \in \mathbb{R}
\end{equation}

where \( d_2 = \max \{\alpha + 2k_1, \beta + 2k_2, \alpha_1 + \beta_1 + 2k_3\} \).

Thus (\( H_3, (ii) \)) holds.

With the condition (\( H_3, (iii) \)):

Using the inequalities \( \ln (1 + x) \leq x, (a + b)^k \leq 2^{k-1}(a^k + b^k), x^q \leq 1 + x^{N_2} \forall x, \)
a, b \geq 0, \forall k \geq 1, \forall q \in (0, N_2] \) with \( N_2 = \max\{2k_1, 2k_2, 2k_3\} \), it follows from (12) that

\begin{equation}
\ln k^1 (e + u^2) \leq 2^{k_1 - 1} \left( 1 + |u|^{2k_1} \right) \leq 2^{k_1} \left( 2 + |u|^{N_2} \right)
\leq 2^{k_1} \left( 1 + |u|^{N_2} + |v|^{N_2} \right),
\end{equation}

\begin{equation}
\ln k^2 (e + v^2) \leq 2^{k_2 - 1} \left( 1 + |v|^{2k_2} \right) \leq 2^{k_2} \left( 2 + |v|^{N_2} \right)
\leq 2^{k_2} \left( 1 + |u|^{N_2} + |v|^{N_2} \right),
\end{equation}

\begin{equation}
\ln k^3 (e + u^2 + v^2) \leq 2^{k_3 - 1} \left( 1 + |u|^{2k_3} + |v|^{2k_3} \right) \leq 2^{k_3} \left( 1 + |u|^{N_2} + |v|^{N_2} \right).
\end{equation}

This implies that

\begin{equation}
\mathcal{F}(u, v) \leq d_2 \left( |u|^\alpha + |v|^\beta + |u|^{\alpha_1} |v|^{\beta_1} \right) \left( 1 + |u|^{N_2} + |v|^{N_2} \right)
\end{equation}

where \( d_2 = \max \{\gamma_1 2^{k_1}, \gamma_2 2^{k_2}, \gamma_3 2^{k_3+1} \} \).

Thus (\( H_3, (iii) \)) holds.

With the condition (\( H_4, (i) \)):

By (120), we obtain

\begin{equation}
yG(y) = \tilde{g} |y|^\gamma \ln k^4 (e + y^2) \leq (\gamma + 2k_4) \int_0^y G(z)dz \forall y \in \mathbb{R}.
\end{equation}

Thus (\( H_4, (i) \)) holds with \( g_2 = \gamma + 2k_4 \).

With the condition (\( H_4, (ii) \)):

By (16), we obtain

\begin{equation}
\int_0^y G(z)dz \leq \frac{\tilde{g}}{\gamma} |y|^\gamma \ln k^4 (e + y^2) \leq \frac{\tilde{g} 2^{k_4 - 1}}{\gamma} |y|^\gamma \left( 1 + |y|^{2k_4} \right).
\end{equation}

Thus (\( H_4, (ii) \)) holds with \( \tilde{g}_2 = \frac{\tilde{g} 2^{k_4 - 1}}{\gamma}, \gamma = 2k_4 \).

We shall show that each solution \((u, v)\) of Prob. (1)-(3) is global and exponential decay provided that \( f(0) = ||\sqrt{\mu_1(0)}u_0||^2 + ||\sqrt{\mu_2(0)}v_0||^2 - p \int_0^{\delta(u)(0)} G(z)dz - p \int_0^1 \mathcal{F}(\tilde{u}_0(x), \tilde{v}_0(x))dx > 0 \) and \( E(0) \) is small enough, where \( p > d^* = \max\{d_2, g_2\} \) with \( d_2, g_2 \) given in (\( H_3 \)) and (\( H_4 \)). Let \((u, v)\) be a weak solution of Prob. (1)-(3) satisfying (113).
In order to obtain the decay result, we construct the functional
\[ \mathcal{L}(t) = E(t) + \delta \Phi(t) \] (157)
where \( \delta > 0; E(t) \) and \( \Phi(t) \) are defined as in (122), (130), respectively.

We rewrite \( E(t) \) as follows
\[ E(t) = \frac{1}{2} \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) + \frac{1}{2} (g \otimes v)(t) \]
\[ + \left( \frac{1}{2} - \frac{1}{p} \right) \left( \| \sqrt{\mu_1(t)} u_x(t) \|^2 \right. \]
\[ + \left. \| \sqrt{\mu_2(t)} v_x(t) \|^2 - \| \sqrt{\mu_2(t)} v_x(t) \|^2 \int_0^t g(s) ds \right) + \frac{1}{p} I(t) \]
where
\[ I(t) = \| \sqrt{\mu_1(t)} u_x(t) \|^2 + \| \sqrt{\mu_2(t)} v_x(t) \|^2 - \| \sqrt{\mu_2(t)} v_x(t) \|^2 \int_0^t g(s) ds \]
\[ - p \left( \int_0^1 \mathcal{F} (u(x,t), v(x,t)) dx + \int_0^{\rho u(0,t)} G(z) dz \right). \]

Then we have the following theorem.

**Theorem 4.2.** Assume that \( \tilde{H}_2 \), \( \tilde{H}_3 \) - \( \tilde{H}_6 \) hold. Let \( (\tilde{u}_0, \tilde{v}_0) \in V \times H_0^1 \), \( (\tilde{u}_1, \tilde{v}_1) \in L^2 \times L^2 \) such that \( I(0) > 0 \) and the initial energy \( E(0) \) satisfy
\[ \eta^* = \mu^* - p[\tilde{d}_2(1 + R_3^2 + R_5^2)(R_4^3 - 2 + \frac{1}{2} R_4^2 + \beta_1 - 2)] > 0 \]
\[ + \tilde{g}_2(1 + R_4^2) R_4^2 \] (160)

where
\[ R_3^2 = \frac{2pE_+}{(p - 2) \mu^*}, \quad \mu^* = \min\{\mu_1^*, \mu_2^*, L_\gamma \}, \quad L_\gamma = \mu_2^* - \frac{p}{2} \int_0^\infty g(s) ds > 0, \]
\[ E_+ = \left( E(0) + \frac{1}{2} \rho_1 \right) \exp(\rho_2), \]
\[ \rho_1 = \sum_{i=1}^2 \| F_i \|^2_{L^1_\gamma(R^+;L^2)} + \frac{1}{\lambda_1} \| g_0 \|^2_{L^2_\gamma(R^+)} \],
\[ \rho_2 = \sum_{i=1}^2 \| F_i \|^2_{L^1_\gamma(R^+;L^2)} + \frac{p \mu^*}{p - 2} \sum_{i=1}^2 \| \mu_i \|^2_{L^1_\gamma(R^+)} \].

Assume that
\[ \| F_1(t) \|^2 + \| F_2(t) \|^2 + g_0^2 (t) \leq \tilde{C}_0 e^{-\tilde{\gamma}_0 t} \forall t \geq 0 \] (161)
where \( \tilde{C}_0, \tilde{\gamma}_0 \) are two positive constants.

Then, for \( \sum_{i=1}^2 \| \mu_i \|^2_{L^\infty(R^+)} \) sufficiently small, there exist positive constants \( \tilde{C}, \tilde{\gamma} \) such that
\[ \| u'(t) \|^2 + \| v'(t) \|^2 + \| u_x(t) \|^2 + \| v_x(t) \|^2 \leq \tilde{C} \exp(-\tilde{\gamma} t) \text{ for all } t \geq 0. \] (162)

**Proof of Theorem 4.1.** It consists of three steps.

1. The estimate of \( E'(t) \).
The functional $E'(t)$ satisfies

\[ E'(t) \leq \frac{1}{2\lambda_1} g_0^2(t) + \frac{1}{2} \sum_{i=1}^2 \|F_i(t)\|^2 + \frac{1}{2} \sum_{i=1}^2 \|F_i(t)\| \left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{1}{2} \sum_{i=1}^2 \mu_i(t) \left(\|u_x(t)\|^2 + \|v_x(t)\|^2\right), \]  \hspace{1cm} (163)

for all $\varepsilon_1 > 0$ where $F(t) = \|F_1(t)\|^2 + \|F_2(t)\|^2 + g_0^2(t)$.

Indeed, multiplying (148) by $(u'(x, t), v'(x, t))$ and integrating over $[0, 1]$, we get

\[ E'(t) = -\lambda_1 \|u'(t)\|^2 - \lambda_2 \|v'(t)\|^2 - \lambda_1 (\|u'(0, t)\|^2 - \frac{1}{2} (g' \circ v)(t)) + \frac{1}{2} \int_0^1 \mu_1(x, t) u_x^2(x, t) dx + \frac{1}{2} \int_0^1 \mu_2(x, t) v_x^2(x, t) dx - \frac{1}{2} g(t) \|\mu v_x(t)\|^2 + \langle F_1(t), u'(t) \rangle + \langle F_2(t), v'(t) \rangle + g_0(t) u'(0, t). \] \hspace{1cm} (164)

On the other hand

\[ \langle F_1(t), u'(t) \rangle \leq \frac{1}{2} \|F_1(t)\| + \frac{1}{2} \langle F_1(t), \|u'(t)\|^2, \] \hspace{1cm} (165)

\[ \langle F_2(t), v'(t) \rangle \leq \frac{1}{2} \|F_2(t)\| + \frac{1}{2} \left(\|F_2(t)\| \|v'(t)\|^2, \right. \] \hspace{1cm} (166)

\text{Thus}

\[ g_0(t) u'(0, t) \leq \frac{1}{2\lambda_1} g_0^2(t) + \frac{\lambda_1}{2} |u'(0, t)|^2, \] \hspace{1cm} (167)

\[ g'(t) \leq -\zeta_1 g(t) < 0, \mu'_i(x, t) \leq \mu_i(t) (i = 1, 2, \forall t \geq 0). \]
By (164) and (167), (163) (ii) is valid.

Step 2. The estimate of \( I(t) \).

By the continuity of \( I(t) \) and \( I(0) > 0 \), there exists \( T_1 > 0 \) such that

\[
I(t) \geq 0 \quad \forall t \in [0, T_1],
\]

(168)

this implies that

\[
E(t) \geq \frac{1}{2} \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) + \left( \frac{1}{2} - \frac{1}{p} \right) \left[ \mu_1 \|u_x(t)\|^2 + \left( \mu_2 - \bar{\mu}_2 \int_0^\infty g(s) ds \right) \|v_x(t)\|^2 \right]
\]

\[
\geq \frac{1}{2} \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) + \frac{p-2}{2p} \mu_* \left[ \|u_x(t)\|^2 + \|v_x(t)\|^2 \right] \quad \forall t \in [0, T_1].
\]

(169)

Combining (163)i, (169) and using Gronwall’s inequality to obtain

\[
\|u_x(t)\|^2 + \|v_x(t)\|^2 \leq \frac{2pE(t)}{(p-2) \mu_*} \leq \frac{2pE_*}{(p-2) \mu_*} \equiv R^2 \quad \forall t \in [0, T_1]
\]

(170)

where \( E_* \) as in (160).

Hence, it follows from \( \tilde{H}_3, (iii), \tilde{H}_4, (ii), (160), (170) \) that

\[
p \left( \int_0^1 \mathcal{F}(u(x,t), v(x,t)) \, dx + \int_0^1 \mathcal{G}(z) \, dz \right) \leq \tilde{p}d_2 \left( \|u(t)\|_{L^\alpha} + \|v(t)\|_{L^\beta} + \int_0^1 |u(x,t)|^\alpha |v(x,t)|^\beta \, dx \right)
\]

\[
\cdot \left( 1 + \|u_x(t)\|^\gamma + \|v_x(t)\|^\gamma \right) + \tilde{p}g_2 |u(0,t)|^\gamma (1 + |u(0,t)|^\gamma)
\]

\[
\leq \tilde{p}d_2 \left( \|u_x(t)\|_{L^\alpha} + \|v_x(t)\|_{L^\beta} + \|u_x(t)\|_{L^\alpha} |v_x(t)|_{L^\beta} \right) \left( 1 + R^2_* + R^2_\gamma \right)
\]

\[
+ \tilde{p}g_2 \|u_x(t)\|^\gamma (1 + \|u_x(t)\|^\gamma)
\]

\[
\leq \tilde{p}d_2 \left( 1 + R^2_* + R^2_\gamma \right) \left( R_*^{\alpha-2} + R_*^{\beta-2} + \frac{1}{2} R_*^{\alpha_1 + \beta_1 - 2} \right) \left( \|u_x(t)\|^2 + \|v_x(t)\|^2 \right)
\]

\[
+ \tilde{p}g_2 \left( 1 + R^2_\gamma \right) R_*^{\gamma-2} \left( \|u_x(t)\|^2 + \|v_x(t)\|^2 \right)
\]

for all \( t \in [0, T_1] \).

Combining (159), (171), to obtain

\[
I(t) \geq \mu_1 \|u_x(t)\|^2 + L_\gamma \|v_x(t)\|^2
\]

(172)

\[
- p \left[ \tilde{d}_2 \left( 1 + R^2_* + R^2_\gamma \right) \left( R_*^{\alpha-2} + R_*^{\beta-2} + \frac{1}{2} R_*^{\alpha_1 + \beta_1 - 2} \right) + \tilde{d}_2 \left( 1 + R^2_* \right) R_*^{\gamma-2} \right]
\]

\[
\times \left( \|u_x(t)\|^2 + \|v_x(t)\|^2 \right)
\]

\[
\geq \eta^* \left( \|u_x(t)\|^2 + \|v_x(t)\|^2 \right)
\]

where \( \eta^* \) as in (160).

Consequently, \( I(t) > 0 \ \forall t \in [0, T_1] \). Now, we put

\[
T_\infty = \sup \{ T > 0 : I(t) > 0 \ \forall t \in [0, T] \}.
\]
If $T_\infty < +\infty$ then the continuity of $I(t)$ leads to $I(T_\infty) \geq 0$. By the same arguments as above, we can deduce that there exists $T' > T_\infty$ such that $I(t) > 0$ for all $t \in [0, T']$.

Hence, we conclude that $I(t) > 0$ for all $t \geq 0$.

**Step 3. Decay result.**

At first, we note that if $\beta_1 > 0$ is small enough and

$$E_1(t) = \|u'(t)\|^2 + \|v'(t)\|^2 + \|\sqrt{\mu_1(t)}u_x(t)\|^2 + \|\sqrt{\mu_2(t)}v_x(t)\|^2$$

then there exist the positive constants $\tilde{\beta}_1$, $\tilde{\beta}_2$ such that

$$\tilde{\beta}_1 E_1(t) \leq \mathcal{L}(t) \leq \tilde{\beta}_2 E_1(t) \quad \forall t \geq 0,$$

with $\delta > 0$ is small enough. (174)

Indeed, it is not difficult to see that

$$\mathcal{L}(t) = \frac{1}{2} \left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{1}{2} (g \circ v)(t)$$

+ \left(\frac{1}{2} - \frac{1}{p}\right) \left(\|\sqrt{\mu_1(t)}u_x(t)\|^2 + \|\sqrt{\mu_2(t)}v_x(t)\|^2 - \nu \int_0^t g(s)ds\right)

+ \frac{1}{p} I(t) + \delta |\langle u(t), u'(t) \rangle + \langle v(t), v'(t) \rangle|$$

+ $\frac{\delta \lambda_1}{2} \|u(t)\|^2 + \frac{\delta \lambda_2}{2} \|v(t)\|^2 + \frac{\delta \lambda_1}{2} u^2(0,t)$.

On the other hand

$$\langle u(t), u'(t) \rangle \leq \frac{1}{2 \mu_1}\left(\|\sqrt{\mu_1(t)}u_x(t)\|^2 + \frac{1}{2} \|u'(t)\|^2\right),$$

$$\langle v(t), v'(t) \rangle \leq \frac{1}{2 \mu_2}\left(\|\sqrt{\mu_2(t)}v_x(t)\|^2 + \frac{1}{2} \|v'(t)\|^2\right).$$

This implies that

$$\mathcal{L}(t) \geq \frac{1 - \delta}{2} \left(\|u'(t)\|^2 + \|v'(t)\|^2\right) + \frac{1}{2} (g \circ v)(t) + \frac{1}{p} I(t)$$

+ $\frac{1}{2 \mu_1}\left[\frac{(p-2) \mu_1}{p}\delta \right] \left(\|\sqrt{\mu_1(t)}u_x(t)\|^2\right)$

+ $\frac{1}{2 \mu_2}\left[\frac{(p-2) L^*}{p}\delta \right] \left(\|\sqrt{\mu_2(t)}v_x(t)\|^2\right)$

$\geq \tilde{\beta}_1 E_1(t)$

where $\delta > 0$ is small enough and

$$\tilde{\beta}_1 = \min \left\{ \frac{1 - \delta}{2}; \frac{1}{p}; \frac{1}{2 \mu_1}\left[\frac{(p-2) \mu_1}{p}\delta \right]; \frac{1}{2 \mu_2}\left[\frac{(p-2) L^*}{p}\delta \right] \right\} > 0,$$

$0 < \delta < \min \left\{ 1; \frac{(p-2) \mu_1}{p}; \frac{(p-2) L^*}{p} \right\}$. (178)
Similarly, we can verify that
\[
\mathcal{L}(t) \leq \frac{1 + \delta}{2} \left( \|u'(t)\|^2 + \|v'(t)\|^2 \right) + \frac{1}{2} (g \otimes v)(t) + \frac{1}{p} f(t) + \frac{p - 2}{2p} \left( \frac{\delta (1 + \lambda_1 + \lambda_1)}{2\mu_1} \right) \left\| \sqrt{\mu_1(t)} u_x(t) \right\|^2 \\
+ \left( \frac{p - 2}{2p} + \frac{\delta (1 + \lambda_2)}{2\mu_2} \right) \left\| \sqrt{\mu_2(t)} v_x(t) \right\|^2 \\
\leq \beta_2 E_1(t)
\]
where \( \beta_2 = \max \left\{ \frac{1 + \delta}{2}, \frac{p - 2}{2p} + \frac{\delta (1 + \lambda_1 + \lambda_1)}{2\mu_1}, \frac{p - 2}{2p} + \frac{\delta (1 + \lambda_2)}{2\mu_2} \right\} \).

Next, we show that the functional \( \Phi(t) \) satisfies
\[
\Phi'(t) \leq \|u'(t)\|^2 + \|v'(t)\|^2 + \frac{1}{2\varepsilon_3} (g \otimes v)(t) - \left( \frac{d^*}{p} - \frac{\varepsilon_2}{\eta^*} \right) I(t) + \frac{1}{2\varepsilon_2} \tilde{F}(t)
\]
for all \( \varepsilon_2, \varepsilon_3 > 0 \). The proof is as follows.

By multiplying (184) by \((u(x,t), v(x,t))\) and integrating over \([0, 1]\), we get
\[
\Phi'(t) = \|u'(t)\|^2 + \|v'(t)\|^2 - \left\| \sqrt{\mu_1(t)} u_x(t) \right\|^2 - \left\| \sqrt{\mu_2(t)} v_x(t) \right\|^2 \\
+ \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle + u(0, t)G(u(0, t)) \\
+ \langle F_1(t), u(t) \rangle + \langle F_2(t), v(t) \rangle + g_0(t)u(0, t) + \int_0^t g(t - s)\tilde{a}_2(v(s), v(t)) \, ds.
\]

On the other hand
\[
\langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle + u(0, t)G(u(0, t))
\]
\[
\leq d_2 \int_0^1 F(u(x,t), v(x,t)) \, dx + g_2 \int_0^1 G(z) \, dz \\
\leq d^* \left[ \int_0^1 F(u(x,t), v(x,t)) \, dx + \int_0^1 G(z) \, dz \right] \\
= \frac{d^*}{p} \left[ \left\| \sqrt{\mu_1(t)} u_x(t) \right\|^2 + \left\| \sqrt{\mu_2(t)} v_x(t) \right\|^2 - \|\sqrt{\mu_2(t)} v_x(t)\|^2 \int_0^t g(s) \, ds - I(t) \right]
\]
with \( d^* = \max\{d_2, g_2\} < p \).

We also have
\[
\langle F_1(t), u(t) \rangle + \langle F_2(t), v(t) \rangle + g_0(t)u(0, t)
\]
\[
\leq \frac{1}{2\varepsilon_2} \left( \|F_1(t)\|^2 + \|F_2(t)\|^2 + g_0^2(t) \right) + \frac{\varepsilon_2}{2} \left( \|u(t)\|^2 + \|v(t)\|^2 + u^2(0,t) \right) \\
\leq \frac{1}{2\varepsilon_2} \tilde{F}(t) + \frac{\varepsilon_2}{\eta^*} I(t).
\]

We continue to estimate the last term in the right-hand side of (181) as follows.
\[
\int_0^t g(t - s)\tilde{a}_2(v(s), v(t)) \, ds
\]
\[
\begin{align*}
&\int_0^t g(t-s)\tilde{a}_2 (v(s) - v(t), v(t)) ds + \|\sqrt{\mu_2 v_x(t)}\|^2 \int_0^t g(s) ds \\
&\leq \frac{1}{2\varepsilon_3} (g \circ v)(t) + \left(1 + \frac{\varepsilon_3}{2}\right) \|\sqrt{\mu_2 v_x(t)}\|^2 \int_0^t g(s) ds
\end{align*}
\]
for all \(\varepsilon_3 > 0\).

Hence, it follows from (181)-(184) that
\[
\Phi'(t) \leq \|u'(t)\|^2 + \|v'(t)\|^2 + \frac{1}{2\varepsilon_3} (g \circ v)(t) - \left(\frac{d^*}{p} - \frac{\varepsilon_2}{\eta^*}\right) I(t) + \frac{1}{2\varepsilon_2} \hat{F}(t) \quad \text{(185)}
\]
\[
- \left(1 - \frac{d^*}{p}\right) \left[\|\sqrt{\mu_1(t) u_x(t)}\|^2 + \|\sqrt{\mu_2(t) v_x(t)}\|^2\right] - \|\sqrt{\mu_2 v_x(t)}\|^2 \int_0^t g(s) ds
\]
\[
+ \frac{\varepsilon_3}{2} \|\sqrt{\mu_2 v_x(t)}\|^2 \int_0^t g(s) ds.
\]
Note that
\[
\begin{align*}
\left[\|\sqrt{\mu_1(t) u_x(t)}\|^2 + \|\sqrt{\mu_2(t) v_x(t)}\|^2\right] &\geq \|\sqrt{\mu_1(t) u_x(t)}\|^2 - \|\sqrt{\mu_2 v_x(t)}\|^2 \int_0^t g(s) ds \quad \text{(186)} \\
&\geq \frac{L_*}{\mu_2} \left[\|\sqrt{\mu_1(t) u_x(t)}\|^2 + \|\sqrt{\mu_2(t) v_x(t)}\|^2\right] ;
\end{align*}
\]
\[
\frac{\varepsilon_3}{2} \|\sqrt{\mu_2 v_x(t)}\|^2 \int_0^t g(s) ds \leq \frac{\varepsilon_3 \hat{\mu}^*}{2\mu_2} \int_0^\infty g(s) ds \left(\|\sqrt{\mu_1(t) u_x(t)}\|^2 + \|\sqrt{\mu_2(t) v_x(t)}\|^2\right) \quad \text{(187)}.
\]
Combining (185), (186) and (187), we obtain (180).

Based on the estimates \((163)\) and (180), we have
\[
\begin{align*}
\mathcal{L}'(t) &\leq -\left(\lambda_1 - \frac{\varepsilon_1}{2} - \delta\right) \|u'(t)\|^2 - \left(\lambda_2 - \frac{\varepsilon_1}{2} - \delta\right) \|v'(t)\|^2 \\
&- \left(\lambda_1 - \frac{\varepsilon_1}{2}\right) |u'(0,t)|^2 \\
&- \frac{1}{2} \left(\frac{\varepsilon_1}{\varepsilon_3}\right) (g \circ v)(t) - \delta \left(\frac{d^*}{p} - \frac{\varepsilon_2}{\eta^*}\right) I(t) + \frac{1}{2} \left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right) \hat{F}(t) \\
&- \frac{1}{2\mu_*} \left[\frac{2\mu_\delta}{\mu_2} \left(1 - \frac{d^*}{p}\right) L_* - \frac{\varepsilon_2 \hat{\mu}^*}{2} \int_0^\infty g(s) ds\right] - \sum_{i=1}^2 \|\hat{\mu}_i\|_{L^\infty(\mathbb{R}^+)\} \\
&\times \left(\|\sqrt{\mu_1(t) u_x(t)}\|^2 + \|\sqrt{\mu_2(t) v_x(t)}\|^2\right)
\end{align*}
\]
for all \(\delta, \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0\). By \(p > d^* = \max\{d_2, g_2\}\), we can choose \(\varepsilon_3 > 0\) as follows
\[
\sigma_1 = \left(1 - \frac{d^*}{p}\right) L_* - \frac{\varepsilon_3 \hat{\mu}^*}{2} \int_0^\infty g(s) ds > 0.
\]

Next, we can choose \(\varepsilon_2 > 0\) and \(\varepsilon_1 > 0\) such that
\[
\sigma_2 = \frac{d^*}{p} - \frac{\varepsilon_2}{\eta^*} > 0, \quad 0 < \frac{\varepsilon_1}{2} < \min\{\lambda_1, \lambda_2, \lambda_1\}.
\]
Then, we choose \(\delta > 0\) as in (178) such that
\[
\sigma_3 = \lambda_1 - \frac{\varepsilon_1}{2} - \delta, \quad \sigma_4 = \lambda_2 - \frac{\varepsilon_1}{2} - \delta > 0, \quad \sigma_5 = \frac{\varepsilon_1}{\varepsilon_3} > 0.
\]
Hence, if
\[ \sum_{i=1}^{2} ||\hat{\mu}_i||_{L^\infty(\mathbb{R}^+)} \leq \frac{2\mu_*\delta}{\mu_{2*}} \sigma_1 \leq \frac{2\mu_*\delta}{\mu_{2*}} \left(1 - \frac{d^*}{p}\right)L_* - \frac{\varepsilon_3\mu_2^*}{2} \int_0^\infty g(s)ds \right), \] (192)
then, we deduce from (188)-(192) that
\[ \mathcal{L}'(t) \leq -\beta_3 E_1(t) + \tilde{C}_1 e^{-\gamma_0 t} \leq -\beta_3 E_1(t) \] (193)
where \( \beta_3 = \min\{\sigma_3; \sigma_4; \sigma_2; \delta \sigma_1; \frac{1}{\mu_{2*}} \left(2\mu_*\delta \sigma_1 - \sum_{i=1}^{3} ||\hat{\mu}_i||_{L^\infty(\mathbb{R}^+)})\}, 0 < \gamma < \min\{\beta_3, \gamma_0\}\].
Furthermore, we have
\[ \mathcal{L}(t) \geq \beta_1 \min\{1, \mu_{1*}, \mu_{2*}\} \left[ ||u'(t)||^2 + ||v'(t)||^2 + ||u_x(t)||^2 + ||v_x(t)||^2 \right]. \] (194)
Combining (193) and (194), we get (162). Theorem 4.1 is proved.

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