The Static Potential with Hypercubic Blocking

A. Hasenfratz\textsuperscript{a}, R. Hoffmann\textsuperscript{a} and F. Knechtli\textsuperscript{a,†}

\textsuperscript{a}Department of Physics, University of Colorado, Campus Box 390, Boulder, CO. 80309, USA

We measure the static potential from Wilson loops constructed using hypercubic blocked (HYP) links. The HYP potential agrees with the potential measured using thin links for distances $r/a \geq 2$. We calculated the lowest order perturbative expansion of the lattice Coulomb potential of HYP links. These results are used in analyzing the static potential both on quenched and dynamical lattices. The statistical accuracy of the potential with HYP links improves by about an order of magnitude, giving a reliable scale even with limited statistics both on quenched and dynamical lattices.

1. Introduction

Recently Hasenfratz and Knechtli introduced the hypercubic blocking transformation (HYP) and used it to study dynamical staggered fermions with greatly improved flavor symmetry. Here we use the HYP blocking as an operator to measure the static potential from Wilson loops. The HYP smearing mixes gauge links within hypercubes attached to the original link only, therefore it has less impact on the short-distance properties of the gauge configurations than repeated levels of APE blocking. The statistical accuracy of the potential measured with HYP links improves by about an order of magnitude.

We calculated the lowest order perturbative expansion of the lattice Coulomb potential of HYP links. The distortion effect of the smearing is visible at distances $r/a \lesssim 2$ but is negligible beyond that. By removing lattice artifacts using the perturbative Coulomb potential we could fit the static potential with the continuum form even from distance $r/a \approx 1$ with low $\chi^2$. This, combined with the improved statistical accuracy at large distances makes it possible to get reliable values for both $r_0$ and $\sqrt{\sigma}$ even with limited statistics.

2. Constructing the HYP link

A single level of HYP smearing consists of three levels of modified APE smearing, where the staples are restricted in such a way that only links within hypercubes attached to the original link are used. The construction equations can be found in \cite{1}. There are three free APE-blocking parameters $\alpha_i$ ($i=1\ldots3$). These were optimized non-perturbatively to suppress fluctuations at the plaquette level. The perturbative cancellation of the flavor symmetry violating terms at tree level of the staggered HYP action gives consistent values for the blocking parameters \cite{2}.

The figure shows a schematic representation of the construction in 3D: The fat link is built from the four double-lined staples (there are six in 4D) (a), which in turn are constructed from the two staples (four in 4D) that stay within the hypercubes attached to the original link (b). The last step in the construction is not doable in three dimensions.

3. Perturbative potential

Let the gauge fields for the thin links be $A_\mu(x)$ and those for the HYP links $B_\mu(x)$. In Fourier space they are related as:

$$B_\mu(p) = \sum_\nu h_{\mu\nu}(p) A_\nu(p) + \mathcal{O}(A^2),$$

where

$$h_{\mu\nu}(p) = \delta_{\mu\nu} \left[ 1 - \frac{\alpha_1}{6} \sum_\rho \hat{p}_\rho^2 \Omega_{\mu\nu}(p) \right] + \frac{\alpha_2}{6} \hat{p}_\mu \hat{p}_\nu \Omega_{\mu\nu}(p),$$

\textsuperscript{*}currently at Universität Regensburg
\textsuperscript{†}currently at Humboldt Universität Berlin
\( \Omega_{\mu\nu}(p) = 1 + \alpha_2(1 + \alpha_3) - \frac{\alpha_2^3}{4}(1 + 2\alpha_3)(p^2 - \hat{p}_\mu^2 - \hat{p}_\nu^2) + \frac{\alpha_2\alpha_3^2}{4} \prod_{\eta \neq \mu, \nu} \hat{p}_\eta^2 \)
and \( \hat{p}_\mu = 2 \sin(p_\mu/2) \)

Using this relation one can calculate the Coulomb part of the static quark potential measured with HYP links in lowest order perturbation theory. For a Wilson loop made from HYP links with a spatial extension \( a \cdot n \) the potential is:

\[
aV_{\text{pert}}(an) = -\frac{4}{3} g^2 \int_{-\pi}^{\pi} \frac{d^3p}{(2\pi)^3} \frac{\cos(n \cdot p) \times \text{[smearing factor]}}{\sum_{i=1}^{3} \hat{p}_i^2} \]

where: 
\[
\text{[smearing factor]} = \begin{cases} 
1 - \frac{\alpha_2}{6} \sum_{i=1}^{3} \hat{p}_i^2 \Omega_{i0}^2 & \text{for the HYP potential} \\
1 & \text{for the thin potential}
\end{cases}
\]

Here we use the Wilson plaquette gauge action. The generalization to other gauge actions modifies the propagator in the above formula.

### 4. Potential measurements

The improvement in statistics due to the HYP smearing is best seen by comparing the static quark potential measured from Wilson loops using thin and HYP links on the same configurations. The set consists of 240 \( 8^3 \times 24 \) configurations generated with the Wilson plaquette action at \( \beta = 5.7 \). The thin link potential has been shifted by an unphysical constant to match the HYP potential for \( r/a > 2.5 \):

For distances \( r/a \geq \sqrt{2} \) the HYP and thin potential agree well and for larger distances the statistical error in the HYP potential is extremely reduced by the smearing transformation. This is especially important for a precise measurement of the string tension. The solid line is a perturbatively improved fit that will be discussed in the following section.

### 5. Improved fit

The deviation of \( V_{\text{pert}} \) from the continuum Coulomb potential \( V_C(r) = -\frac{e^2}{4\pi r} \) is best seen in an \( r \) vs. \( r \cdot V \) plot:

The perturbative calculation captures the short distance distortion of the HYP smearing and the improved rotational symmetry for larger distances.

With this perturbative data one can make a four parameter fit to the lattice data. If we assume that most of the lattice artifacts are captured by \( V_{\text{pert}} \) we can write \( V_{\text{lat}} \) as

\[
V_{\text{lat}} = V_{\text{cont}} + \Delta V_{\text{lat}} \quad \text{where}
\]

\[
V_{\text{cont}}(r) = \frac{c_0}{r} + c_1 + c_2 r
\]

is the continuum potential and

\[
\Delta V_{\text{lat}}(r) = \tilde{c}_0 \left( V_{\text{pert}}(r) - V_C(r) \right)
\]

represents the lattice artifacts.

\( \tilde{c}_0, c_0, c_1 \) and \( c_2 \) are the four parameters of the improved fit and from \( c_0 \) and \( c_2 \) the string tension \( \sigma \) and Sommer scale \( r_0 \) can be obtained in the usual way.
6. Results from the improved fit

For both thin and HYP potentials the following plot shows the actual potential data and the "corrected" data, where the lowest order lattice correction is removed, namely: \( V_{lat} - \Delta V_{lat} \). The continuum part of the fit, \( V_{cont} \), is shown as a solid line. The fit uses all data points in the range \( r/a \in (0.9, 5) \):

![Potential data and fit](image)

After removing the lattice artifacts and adjusting an overall constant the thin and HYP measurements match very closely.

The perturbatively corrected potential shows very good rotational symmetry and closely follows the continuum form over the whole range of \( r/a \). The \( \chi^2 \) per degree of freedom decreases by about a factor of 10 if we include the perturbative correction in the fit. The correction of the lattice artifacts is so effective that one can use all the points, even \( r/a = 1 \), for fitting. As an example the following plot shows the dependence of \( \sqrt{\sigma} \) on the lower limit of the fit \( (r_{\text{min}}) \):

![Potential dependence on lower limit](image)

The results for the potential parameters of the \( \beta = 5.7 \) quenched lattices from standard jackknife analysis are (together with data from some large scale simulations at the same coupling value):

| \( r_0/a \) | \( a\sqrt{\sigma} \) | \( N_{\text{conf}} \) | type of link |
|------------|-----------------|----------------|-------------|
| 2.93(2)    | 0.398(8)        | 240            | HYP         |
| 2.922(9)   | -               | 1000           | multi-hit   |
| 2.990(24)  | 0.3879(34)      | 4000           | thin        |

7. Summary

We have measured the static potential using HYP smeared links.

- The HYP potential has greatly reduced statistical errors and shows improved rotational symmetry.
- The short distance potential is distorted only at \( r/a < 2 \).
- The perturbative HYP Coulomb potential describes most of these distortions.
- With the perturbative correction potential data even at \( r/a < 2 \) can be used for fitting.

The improved statistical accuracy and the perturbative correction make the HYP potential measurement very effective for the analysis of both quenched and dynamical configurations. We would like to emphasize that at present other refined techniques, like multi-hit or multi-level, are not applicable with dynamical fermions in contrast to the HYP smearing, with which we were able to obtain reliable scale determinations on dynamical lattices even with limited statistics.

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