Convergence analysis of a Crank-Nicolson Galerkin method for an inverse source problem for parabolic systems with boundary observations

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Abstract. This work is devoted to an inverse problem of identifying a source term depending on both spatial and time variables in a parabolic equation from single Cauchy data on a part of the boundary. A Crank-Nicolson Galerkin method is applied to the least squares functional with quadratic stabilizing penalty term. The convergence of finite dimensional regularized approximations to the sought source as measurement noise levels and mesh sizes approach to zero with appropriate regularization parameter is proved. Moreover, under a suitable source condition, an error bound and corresponding convergence rates are proved. Finally, two numerical experiments are presented to illustrate the efficiency of the theoretical findings.

Key words and phrases. Inverse source problem, Tikhonov regularization, Crank-Nicolson Galerkin method, source condition, convergence rates, ill-posedness, parabolic problem

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1 Introduction

The problem of identifying a source in a heat transfer or diffusion process has got attention of many researchers during last years. This problem leads to determining a term in the right hand side of parabolic equations from some observations of the solution which is well known to be ill-posed. For surveys on the subject, we refer the reader to the books \[ 5, 13, 24, 25, 31 \], the recent papers \[ 19, 32 \] and the references therein.

Although there have been many papers devoted to the source identification problems with observations in the whole domain or at the final moment, those with boundary observations are quite few. Furthermore, the sought term depends either on the spatial variable as in \[ 5, 7, 8, 9, 10, 11, 14, 16, 17, 18, 41, 46, 47 \], or only on the time variable as in \[ 20 \]. In this paper, we consider the problem of determining the right hand side depending on both spatial and time variables by a variational method. We also treat the case when the sought term depends either on the spatial or time variable. Indeed, let \( \Omega \) be an open bounded connected set of \( \mathbb{R}^d, d \geq 2 \) with boundary \( \partial \Omega \) and \( T > 0 \) be a given constant. We investigate the problem of identifying the source term \( f = f(x,t) \) in the Robin boundary value problem for the parabolic equation

\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) + Lu(x,t) &= f(x,t) \text{ in } \Omega_T := \Omega \times (0,T], \\
\frac{\partial u(x,t)}{\partial n} + \sigma(x,t)u(x,t) &= g(x,t) \text{ on } \mathcal{S} := \partial \Omega \times (0,T], \\
u(x,0) &= q(x) \text{ in } \Omega
\end{align*}
\tag{1.1}
\]

from a partial boundary measurement \( z_\delta := z_\delta(x,t) \in L^2(\Sigma) \) of the solution \( u(x,t) \) on the surface \( \Sigma := \Gamma \times (0,T) \subset \mathcal{S} \) satisfying

\[
\|Z - z_\delta\|_{L^2(\Sigma)} \leq \delta, 
\tag{1.2}
\]

where \( Z = u|_\Sigma, \Gamma \text{ is a relatively open subset of } \partial \Omega \text{ and the positive constant } \delta \text{ stands for the measurement error.} \)

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In \([1.1]\) \(L\) is a time-dependent, second order self-adjoint elliptic operator of the form
\[
L u(x,t) := - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x,t) \frac{\partial u(x,t)}{\partial x_j} \right) + b(x,t) u(x,t),
\]
where \(A := (a_{ij})_{1 \leq i,j \leq d} \in C(\overline{\Omega_T})^{d \times d}\) is a symmetric diffusion matrix satisfying the uniformly elliptic condition
\[
A(x,t) \xi \cdot \xi = \sum_{i,j=1}^{d} a_{ij}(x,t) \xi_i \xi_j \geq a \sum_{i,j=1}^{d} |\xi_i|^2 \quad \text{in} \quad \overline{\Omega_T}
\]
for all \(\xi = (\xi_i)_{1 \leq i \leq d} \in \mathbb{R}^d\) with some constant \(a > 0\) and \(b(x,t) \in C(\overline{\Omega_T})\) is a non-negative function. The vector \(\vec{n} := \vec{n}(x,t)\) is the unit outward normal on \(S\) and
\[
\frac{\partial u(x,t)}{\partial \vec{n}} := A(x,t) \nabla u(x,t) \cdot \vec{n}
\]
with \(\nabla u(x,t) := \nabla u(x,t) = \left( \frac{\partial u(x,t)}{\partial x_1}, \ldots, \frac{\partial u(x,t)}{\partial x_d} \right)\). In addition, the functions \(q \in H^1(\Omega), g \in C(\overline{S})\) and \(\sigma \in C(\overline{S})\) with \(\sigma(x,t) \geq 0\) in \(S\) are assumed to be given. The source term \(f = f(x,t)\) is sought in the space \(L^2(\Omega_T)\).

The contents of this paper are as follows: For any fixed \(f \in L^2(\Omega_T)\) let \(u = u(f) \in W(0,T)\) denote the unique weak solution of the system \([1.1]\), see Section 2 for the definition of related functional spaces. Adopting the output least squares method combined with the Tikhonov regularization, we consider the (unique) minimizer of the minimization problem
\[
\min_{f \in L^2(\Omega_T)} J_{\rho,\delta}(f) \quad \text{with} \quad J_{\rho,\delta}(f) := \|u(f) - z_\delta\|_{L^2(\Omega)}^2 + \rho \|f - f^*\|_{L^2(\Omega_T)}^2 \quad (P_{\rho,\delta})
\]
as a reconstruction, where \(\rho \in (0,1)\) is the regularization parameter and \(f^*\) is an a priori estimate of the true source which is identified. It is shown that the cost functional \(J_{\rho,\delta}(\cdot)\) is Fréchet differentiable and for each \(f \in L^2(\Omega_T)\) the \(L^2\)-gradient of \(J_{\rho,\delta}(\cdot)\) at \(f\) is given by
\[
\nabla J_{\rho,\delta}(f) = 2p(f) + 2\rho(f - f^*),
\]
i.e. there holds the relation
\[
J'_{\rho,\delta}(f) \xi = (2p(f) + 2\rho(f - f^*),\xi)_{L^2(\Omega_T)}
\]
for all \(\xi \in L^2(\Omega_T)\), where \(p(f)\) is the adjoint state of \(u(f)\) that is discussed in the detail in the next section.

For discretization we employ the Crank-Nicolson Galerkin method, where the finite dimensional space \(V^1_h\) of piecewise linear, continuous finite elements is used to discretize the state with respect to the spatial variable. Further, to discretize the state with respect to the time variable, we divide the time interval \((0,T)\) into \(M\) equal subintervals and introduce a time step \(\tau := T/M\) together with time levels
\[
t^n := n\tau \quad \text{with} \quad n \in I_0 := \{0,1,\ldots,M\}.
\]
As a result, the state \(u(f)\) is then approximated by the finite sequence \((U^n_{h,\tau}(f))_n\) in which for each \(n \in I_0\) the element
\[
U^n_{h,\tau}(f) \in V^1_h := \left\{ \varphi_h \in C(\overline{\Omega}) \mid \varphi_{h,T} \in P_1(T) \quad \text{for all} \quad T \in \mathcal{T}_h \right\}.
\]
With these notions at hand, we examine the discrete regularized problem corresponding to \((P_{\rho,\delta})\) i.e. the following strictly convex minimization problem
\[
\min_{f \in L^2(\Omega_T)} J_{\rho,\delta,h,\tau}(f) \quad \text{with} \quad J_{\rho,\delta,h,\tau}(f) := \sum_{n=1}^{M} \int_{t^{n-1}}^{t^n} \|U^n_{h,\tau}(f) - z_\delta\|_{L^2(T)}^2 dt + \rho \|f - f^*\|_{L^2(\Omega_T)}^2 \quad (P_{\rho,\delta,h,\tau})
\]
which admits a unique solution \(f_{\rho,\delta,h,\tau}\) obeying the relation (Section 3)
\[
f_{\rho,\delta,h,\tau}|_{\Omega \times (t^{n-1},t^n]} = f^* - \frac{1}{\rho} P_{h,\tau}^{-1}(f_{\rho,\delta,h,\tau})
\]
(1.4)
for any $n \in I := \{1, \ldots, M\}$, where $(P^n_{\rho,\delta,h,\tau})_{n=0}^M$ is the approximation of the adjoint state $p(f)$. Using the variational discretization concept introduced in [21], the minimizer automatically belongs to the finite dimensional space

$$V_{h,\tau}^{1,0} := \{ \Phi \in L^2(0,T;V_h^1) \mid \Phi|_{\Omega \times \{t_n-1,t_n\}} := \varphi_h^n \in V_h^1 \text{ for all } n \in I \}$$

provided an a priori estimate $f^* \in V_{h,\tau}^{1,0}$ and hence a discretization of the admissible set $L^2(\Omega_T)$ can be avoided. Furthermore, due to the equation (1.4) the $L^2(\Omega_T)$-norm solution of the above (IP) with $\rho = \rho(\delta,h,\tau) \to 0$, we show in Section 4 that the whole sequence $(f_{\rho,\delta,h,\tau})_{\rho > 0}$ converges in the $L^2(\Omega_T)$-norm to the unique $f^*$-minimum-norm solution $f^1$ of the identification problem defined by

$$f^1 = \arg \min_{f \in \{ f \in L^2(\Omega_T) \mid u(f) = Z \} } \| f - f^* \|_{L^2(\Omega_T)}. \quad (IP)$$

The corresponding state sequence then converges in the $L^2(0,T;H^1(\Omega))$-norm to the exact state of the problem (1.1).

Section 5 is devoted to convergence rates for the discretized problem, where we first show that if $f \in \{ f \in L^2(\Omega_T) \mid u(f) = Z \}$ and there exists a function $w \in L^2(\Sigma)$ such that $f = F(w) + f^*$, where $F(w)$ is the unique weak solution of the parabolic system

$$-\frac{\partial F}{\partial t}(x,t) + LF(x,t) = 0 \text{ in } \Omega_T, \quad \frac{\partial F(x,t)}{\partial n} + \sigma(x,t)F(x,t) = w\chi_{\Sigma} \text{ on } \Sigma, \quad F(x,T) = 0 \text{ in } \Omega$$

with $\chi_{\Sigma}$ being the characteristic function of $\Sigma \subset \Sigma$, then $f = f^1$, i.e. it is the unique $f^*$-minimum-norm solution of the above (IP). Furthermore, if the data appearing in the system (1.1) are regular enough the convergence rate

$$\| f_{\rho,\delta,h,\tau} - f^1 \|_{L^2(\Omega_T)}^2 \leq C \left( h^3 \rho^{-1} + \tau^2 h^{-1} \rho^{-1} + \delta + \rho + \delta^2 \rho^{-1} \right)$$

is established, where $f_{\rho,\delta,h,\tau}$ is the unique minimizer of $(P_{\rho,\delta,h,\tau})$.

For the numerical solution of the discrete regularized problem $(P_{\rho,\delta,h,\tau})$ we in Section 6 utilize a conjugate gradient algorithm. Numerical studies are presented for two cases where the sought source is smooth and discontinuous as well, that illustrates the efficiency of our theoretical findings.

In some practical situations the source term has the form

$$F_1(x,t)f(x,t) + F_2(x,t). \quad (1.5)$$

Motivating by this reason we in Section 7 present briefly some related results for the problem of identifying the part $f(x,t)$ in the source term expressed by (1.5), where the functions $F_1(x,t)$ and $F_2(x,t)$ are known.

To conclude this introduction we wish to mention that to the best of our knowledge, although there have been many papers devoted to source identification problems for parabolic equations, we however have not yet found investigations on the discretization analysis for those with boundary observations — which is more realistic from the practical point of view, a fact that motivated the research presented in the paper. Concerning the identification problem in elliptic equations utilizing boundary measurements, we here would like to comment briefly some previously published works. In [32, 44] the authors used finite element methods
to numerically recover the fluxes on the inaccessible boundary $\Gamma_i$ from measurement data of the state on the accessible boundary $\Gamma_a$, while the problem of identifying the Robin coefficient on $\Gamma_i$ is also investigated in [43]. Recently, authors of [22, 23] have adopted the variational approach of Kohn and Vogelius to the source term and scalar diffusion coefficient identification, respectively, using observations available on the whole boundary. Finite element analysis for the reaction coefficient identification problem from partial observations is carried out in [35], while a survey of the problem of simultaneously identifying the source term and coefficients from distributed observations can be found in [34].

Throughout the paper we use the standard notion of Sobolev spaces $H^1(\Omega)$, $H^2(\Omega)$, $W^{k,p}(\Omega)$, etc. from, for example, [1, 39].

2 Problem setting and preliminaries

To formulate the identification problem, we first give some notations [40]. Let $(\mathcal{X}, \| \cdot \|_\mathcal{X})$ be a Banach space, we denote by

$$C([0, T], \mathcal{X}) := \left\{ w : [0, T] \to \mathcal{X} \mid w \text{ is continuous on } [0, T], \text{ i.e. } \lim_{\tau \to t} \| w(\tau) - w(t) \|_\mathcal{X} = 0 \text{ for all } t \in [0, T] \right\}$$

which is also a Banach space with respect to the norm

$$\| w \|_{C([0, T], \mathcal{X})} := \max_{0 \leq t \leq T} \| w(t) \|_\mathcal{X}.$$ 

We define for $1 \leq p \leq \infty$ the Banach space

$$L^p(0, T; \mathcal{X}) := \left\{ w : [0, T] \to \mathcal{X} \mid \| w \|_{L^p(0, T; \mathcal{X})} < \infty \right\},$$

where

$$\| w \|_{L^p(0, T; \mathcal{X})} := \left( \int_0^T \| w(t) \|^p_{\mathcal{X}} dt \right)^{1/p} \quad \text{if } 1 \leq p < \infty \text{ and } \| w \|_{L^\infty(0, T; \mathcal{X})} := \text{ess sup}_{0 \leq t \leq T} \| w(t) \|_{\mathcal{X}}.$$ 

Let $H^1(\Omega)^*$ be the dual space of $H^1(\Omega)$, we use the notation

$$\mathcal{W}(0, T) := \left\{ w \in L^2(0, T; H^1(\Omega)) \mid \frac{\partial w}{\partial t} \in L^2(0, T; H^1(\Omega)^*) \right\}.$$ 

It is a Banach space equipped with the norm

$$\| w \|_{\mathcal{W}(0, T)} := \left( \| w \|^2_{L^2(0, T; H^1(\Omega))} + \| \frac{\partial w}{\partial t} \|^2_{L^2(0, T; H^1(\Omega)^*)} \right)^{1/2}. \quad (2.1)$$

We note that, since $\mathcal{W}(0, T)$ with respect to the norm (2.1) is a closed subspace of the reflexive space $L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega)^*)$, it is itself reflexive. We now quote the following useful result.

Lemma 2.1 ([15, 48]). (i) The embedding $\mathcal{W}(0, T) \hookrightarrow C([0, T]; L^2(\Omega))$ is continuous, meanwhile the one $\mathcal{W}(0, T) \hookrightarrow L^2([0, T]; L^2(\Omega)) = L^2(\Omega_T)$ is compact.

(ii) Let $v \in \mathcal{W}(0, T)$. The mapping $t \mapsto \| v(t) \|^2_{L^2(\Omega)}$ is absolutely continuous and

$$\frac{d}{dt} \| v(t) \|^2_{L^2(\Omega)} = 2 \left\langle \frac{dv(t)}{dt}, v(t) \right\rangle_{(H^1(\Omega)^*, H^1(\Omega))} \quad (2.2)$$

for a.e. $t \in [0, T]$.

(iii) For all $v, w \in \mathcal{W}(0, T)$ and $[\alpha, \beta] \subset [0, T]$ the equation

$$\int_\alpha^\beta \left\langle \frac{\partial v}{\partial t}(t), w(t) \right\rangle_{(H^1(\Omega)^*, H^1(\Omega))} dt + \int_\alpha^\beta \left\langle \frac{\partial w}{\partial t}(t), v(t) \right\rangle_{(H^1(\Omega)^*, H^1(\Omega))} dt = \int_\Omega v(\beta)w(\beta)dx - \int_\Omega v(\alpha)w(\alpha)dx \quad (2.3)$$

holds.
2.1 Direct and inverse problems

For considering the problem \([1.1]\), we set
\[
a(t; v, w) := \int_{\Omega} A(x, t) \nabla v(x) \cdot \nabla w(x) dx + \int_{\Omega} b(x, t) v(x) w(x) dx + \int_{\partial \Omega} \sigma(x, t) v(x) w(x) dx
\]
\[
:= a(v, w),
\]
where \(t \in (0, T]\) and \(v, w \in H^1(\Omega)\). Then, for each \(f \in L^2(\Omega_T)\) the Robin boundary value problem \([1.1]\) defines a unique weak solution \(u := u(x, t; f) := u(f)\) in the sense that \(u(f) \in W(0, T)\) with \(u(x, 0) = q(x)\) for a.e. \(x \in \Omega\) and the following variational equation is satisfied (cf. \([29, 33]\))
\[
\left\{ \frac{\partial u}{\partial t} + \varphi \right\}_{(H^1(\Omega), H^1(\Omega))} + a(u, \varphi) = (f, \varphi)_{L^2(\Omega)} + (g, \varphi)_{L^2(\Omega T)} \quad \forall \varphi \in H^1(\Omega), \text{ a.e. } t \in (0, T].
\]
Furthermore, the estimate
\[
\|u\|_{W(0, T)} \leq C_R (\|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(S)} + \|q\|_{L^2(\Omega)})
\]
holds, where \(C_R\) is a positive constant independent of \(f, g\) and \(q\). To emphasize the dependence, we sometimes write \(u(x, t, f, q, g)\) or \(u(f)\) if there is no confusion.

Therefore, we can define the source-to-state operator
\[
u : L^2(\Omega_T) \to W(0, T)
\]
which maps each \(f \in L^2(\Omega_T)\) to the unique weak solution \(u := u(f)\) of the problem \([1.1]\). The inverse problem is stated as follows:

**Given the boundary data** \(Z := u|_\Sigma\) **of the exact solution** \(u\), **find an element** \(f \in L^2(\Omega_T)\) **such that** \(u(f)|\Sigma = Z\).

2.2 Variational method

In practice only the observation \(z_\delta \in L^2(\Sigma)\) of the exact \(Z\) with an error level
\[
\|Z - z_\delta\|_{L^2(\Sigma)} \leq \delta, \quad \delta > 0
\]
is available. Hence, our problem is to reconstruct an element \(f \in L^2(\Omega_T)\) in \([1.1]\) from noisy observation \(z_\delta\) of \(Z\). For this purpose we use the standard least squares method with Tikhonov regularization, i.e. we consider a minimizer of the minimization problem
\[
\min_{f \in L^2(\Omega_T)} J_{\rho, \delta}(f) \quad \text{with} \quad J_{\rho, \delta}(f) := \|u(f) - z_\delta\|_{L^2(\Sigma)}^2 + \rho \|f - f^*\|_{L^2(\Omega_T)}^2
\]
as a reconstruction.

**Remark 2.2.** In case \(\inf_{(x, t) \in \Omega_T} b(x, t) > 0\) or \(\inf_{(x, t) \in S} \sigma(x, t) > 0\) the expression \(a(t; v, w)\) in \([2.4]\) generates a scalar product on the space \(H^1(\Omega)\) equivalent to the usual one, i.e. there exist positive constants \(C_1, C_2\) such that (cf. \([29, 33]\))
\[
C_1 \|\varphi\|_{H^1(\Omega)}^2 \leq a(t; \varphi, \varphi) \leq C_2 \|\varphi\|_{H^1(\Omega)}^2
\]
for all \(\varphi \in H^1(\Omega)\) and \(t \in (0, T]\).

Now we assume that \(b = \sigma = 0\). A change of the variable \(u = e^t v\), the system \([1.1]\) has the form
\[
\frac{\partial v}{\partial t}(x, t) - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x, t) \frac{\partial v(x, t)}{\partial x_j} \right) + v(x, t) = e^{-t} f(x, t) \quad \text{in } \Omega_T,
\]
\[
\frac{\partial v(x, t)}{\partial n} = e^{-t} g(x, t) \quad \text{on } S,
\]
\(v(x, 0) = q(x)\) in \(\Omega\).
Therefore, in the sequel we consider the case \( \inf (x,t) \in \Omega T \, b(x,t) > 0 \) or \( \inf (x,t) \in \Sigma \sigma (x,t) > 0 \) only. All results in present paper are still valid for the case \( b = \sigma = 0 \).

Now we summarize some useful properties of the source-to-state operator \( u = u(f) \).

**Lemma 2.3.** The source-to-state operator \( u : L^2(\Omega T) \to \mathcal{W}(0,T) \) is Fréchet differentiable. For any fixed \( f \in L^2(\Omega T) \) the differential \( u'(f) \xi \) in the direction \( \xi \in L^2(\Omega T) \) is the unique weak solution in \( \mathcal{W}(0,T) \) to the problem

\[
\begin{align*}
\frac{\partial \hat{u}}{\partial t} (x,t) + \mathcal{L} \hat{u} (x,t) &= \xi (x,t) \text{ in } \Omega T, \\
\frac{\partial \hat{u} (x,t)}{\partial n} + \sigma (x,t) \hat{u} (x,t) &= 0 \text{ on } \Sigma, \\
\hat{u} (x,0) &= 0 \text{ in } \Omega. 
\end{align*}
\tag{2.8}
\]

Furthermore, there holds the estimate

\[
\| \hat{u} \|_{\mathcal{W}(0,T)} \leq C_R \| \xi \|_{L^2(\Omega T)}.
\]

**Proof.** We have \( u(x,0; f) = q(x) \) for a.e. \( x \in \Omega \) and along with \( \langle 2.3 \rangle \) also get for all \( \varphi \in H^1(\Omega) \) and a.e. \( t \in (0,T] \) that

\[
\begin{align*}
&\left\langle \frac{\partial u (f + \xi)}{\partial t}, \varphi \right\rangle_{(H^1(\Omega)),(H^1(\Omega))} + a(u(f + \xi), \varphi) = (f + \xi, \varphi)_{L^2(\Omega)} + (g, \varphi)_{L^2(\partial \Omega)}, \\
&\left\langle \frac{\partial \hat{u} (\xi)}{\partial t}, \varphi \right\rangle_{(H^1(\Omega)),(H^1(\Omega))} + a(\hat{u} (\xi), \varphi) = (\xi, \varphi)_{L^2(\Omega)} 
\end{align*}
\tag{2.9}
\]

and \( u(x,0; f + \xi) = q(x), \, \hat{u} (x,0; \xi) = 0 \) for a.e. \( x \in \Omega \). Let \( \theta := u(f + \xi) - u(f) - \hat{u} (\xi) \) satisfy \( \theta (x,0) = 0 \) for a.e. \( x \in \Omega \). Further, combining \( \langle 2.9 \rangle \) with \( 2.3 \), we arrive at

\[
\left\langle \frac{d}{dt} \theta, \varphi \right\rangle_{(H^1(\Omega)),(H^1(\Omega))} + a(\theta, \varphi) = 0
\]

for all \( \varphi \in H^1(\Omega) \). Taking \( \varphi = \theta (t) \in H^1(\Omega) \) and using \( \langle 2.2 \rangle \), we have

\[
\left\langle \frac{d}{dt} \theta, \varphi \right\rangle_{(H^1(\Omega)),(H^1(\Omega))} = \frac{1}{2} \frac{d}{dt} \| \theta(t) \|^2_{L^2(\Omega)} = \frac{d}{dt} \| \theta(t) \|_{L^2(\Omega)}
\]

which together with \( \langle 2.7 \rangle \) yield \( \frac{d}{dt} \| \theta(t) \|_{L^2(\Omega)} \leq 0 \). Thus, by the Gronwall’s inequality, we get \( \| \theta(t) \|_{L^2(\Omega)} \leq \| \theta(0) \|_{L^2(\Omega)} \) for a.e. \( t \in (0,T] \), which finishes the proof.

Together with the problems \( \langle 1.1 \rangle \) and \( 2.8 \), we consider the problem

\[
\begin{align*}
\frac{\partial \hat{u}}{\partial t} (x,t) + \mathcal{L} \hat{u} (x,t) &= 0 \text{ in } \Omega T, \\
\frac{\partial \hat{u} (x,t)}{\partial n} + \sigma (x,t) \hat{u} (x,t) &= g(x,t) \text{ on } \Sigma, \\
\hat{u} (x,0) &= q(x) \text{ in } \Omega. 
\end{align*}
\tag{2.10}
\]

Then we see that

\[
u(f) = \hat{u}(f) + \check{u} \text{ for all } f \in L^2(\Omega T),
\tag{2.11}
\]

where \( \check{u}(f) \) comes from \( 2.8 \) by using \( f \) in the right hand side instead of \( \xi \), which depends linearly on \( f \). Further, for each \( f \in L^2(\Omega T) \) we consider the adjoint problem

\[
\begin{align*}
- \frac{\partial p}{\partial t} (x,t) + \mathcal{L} p(x,t) &= 0 \text{ in } \Omega T, \\
\frac{\partial p (x,t)}{\partial n} + \sigma (x,t) p(x,t) &= (u(x,t; f) - z_0 (x,t)) \chi_{\Sigma} \text{ on } \Sigma, \\
p(x,T) &= 0 \text{ in } \Omega, 
\end{align*}
\tag{2.12}
\]
where \( \chi_\Sigma \) is the characteristic function of \( \Sigma \subset \mathcal{S} \), i.e. \( \chi_\Sigma(x, t) = 1 \) if \((x, t) \in \Sigma\) and equals to zero otherwise. A function \( p \in \mathcal{W}(0, T) \) is said to be a weak solution to this problem, if \( p(x, T) = 0 \) for a.e. \( x \in \Omega \) and

\[
- \left\langle \frac{\partial p}{\partial t}, \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} + a(p, \varphi) = \int_T \left( u(x, t; f) - z\delta(x, t) \right) \varphi(x) dx \quad \forall \varphi \in H^1(\Omega), \text{ a.e. } t \in (0, T]. \tag{2.13}
\]

Since \( u \in \mathcal{W}(0, T) \), the boundary value \((u(x, t; f) - z\delta(x, t)) \chi_\Sigma \) belongs to \( L^2(\Sigma) \) and by changing the time direction we see that (2.12) attains a unique weak solution \( p(x, t; f) := p(f) \in \mathcal{W}(0, T) \).

**Lemma 2.4.** Let us denote by

\[
J_0(f) := \|u(f) - z\delta\|^2_{L^2(\Sigma)}
\]

with \( f \in L^2(\Omega_T) \). Then the Fréchet derivative of \( J_0 \) is given by

\[
\nabla J_0(f) = 2p(f). \tag{2.14}
\]

**Proof.** In view of Lemma 2.3, for each \( f \in L^2(\Omega_T) \) the action of the Fréchet derivative \( J_0'(f)\xi \) in the direction \( \xi \in L^2(\Omega_T) \) is given by

\[
\frac{1}{2} J_0'(f) \xi = (u(f) - z\delta, u'(f) \xi)_{L^2(\Sigma)} = (u(f) - z\delta, \tilde{u}(\xi))_{L^2(\Sigma)}
\]

\[
= \int_0^T \left\langle \frac{\partial u(f)}{\partial t}, \tilde{u}(\xi) \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + \int_0^T a(p(f), \tilde{u}(\xi)) dt
\]

\[
= \int_0^T \left\langle \frac{\partial \tilde{u}(\xi)}{\partial t}, p(f) \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + \int_\Omega p(x, 0; f) \tilde{u}(x, 0; \xi) dx
\]

\[
- \int_\Omega p(x, T; f) \tilde{u}(x, T; \xi) dx + \int_0^T a(\tilde{u}(\xi), p(f)) dt
\]

\[
= \int_0^T \left\langle \frac{\partial \tilde{u}(\xi)}{\partial t}, p(f) \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + \int_0^T a(\tilde{u}(\xi), p(f)) dt
\]

\[
= \int_{\Omega_T} \xi p(f) dx dt,
\]

here we used (2.3). \( \square \)

Before going farther we state the following result.

**Lemma 2.5.** Assume that the sequence \((f_k) \subset L^2(\Omega_T) \) weakly converges in \( L^2(\Omega_T) \) to an element \( f \). Then the sequence \((u(f_k))\) weakly converges in \( \mathcal{W}(0, T) \) (and strongly in \( L^2(\mathcal{S}) \)) to \( u(f) \).

**Proof.** Since the sequence \((f_k) \subset L^2(\Omega_T) \) is weakly convergent, it is a bounded sequence in the \( L^2(\Omega_T) \)-norm. Due to (2.6), the sequence \((u(f_k))\) is bounded in the reflexive space \( \mathcal{W}(0, T) \). Hence, there exists a subsequence of it denoted the same symbol such that \((u(f_k))\) weakly converges to an element \( u \) in \( \mathcal{W}(0, T) \). For all \( \phi \in L^2(0, T; H^1(\Omega)) \) and \( k \in \mathbb{N} \) we have that

\[
\int_0^T \left\langle \frac{\partial u(f_k)}{\partial t}, \phi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + \int_0^T a(u(f_k), \phi) dt = \int_0^T (f_k, \phi)_{L^2(\Omega)} dt + \int_0^T (g, \phi)_{L^2(\partial \Omega)} dt. \tag{2.15}
\]

Sending \( k \) to \( \infty \), we thus obtain that

\[
\left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} + a(u, \varphi) = (f, \varphi)_{L^2(\Omega)} + (g, \varphi)_{L^2(\partial \Omega)} \tag{2.16}
\]

for all \( \varphi \in H^1(\Omega) \) and a.e. \( t \in (0, T] \). We show that \( u(x, 0) = q(x) \) for a.e. \( x \in \Omega \). In fact, let \( \Phi \in C^1([0, T]; H^1(\Omega)) \) be arbitrary with \( \Phi(T) := \Phi(x, T) = 0 \) for a.e. \( x \in \Omega \). By (2.3), we have from (2.15) for
all \( k \in \mathbb{N} \) that
\[
- \int_0^T \left( \frac{\partial \Phi}{\partial t}, u(f_k) \right)_{L^2(\Omega)} dt + \int_0^T a(u(f_k), \Phi) dt \\
= \int_0^T (f_k, \Phi)_{L^2(\Omega)} dt + \int_0^T (g, \Phi)_{L^2(\partial \Omega)} dt + \int_\Omega u(x,0; f_k) \Phi(x,0) dx.
\]
Noting \( u(x,0; f_k) = q(x) \) for a.e. \( x \in \Omega \), sending \( k \) to \( \infty \) in the last equation, we get
\[
- \int_0^T \left( \frac{\partial \Phi}{\partial t}, u \right)_{L^2(\Omega)} dt + \int_0^T a(u, \Phi) dt \\
= \int_0^T (f, \Phi)_{L^2(\Omega)} dt + \int_0^T (g, \Phi)_{L^2(\partial \Omega)} dt + \int_\Omega q(x) \Phi(x,0) dx.
\]
Likewise, using (2.3) and (2.16), we deduce
\[
- \int_0^T \left( \frac{\partial \Phi}{\partial t}, u \right)_{L^2(\Omega)} dt + \int_0^T a(u, \Phi) dt \\
= \int_0^T (f, \Phi)_{L^2(\Omega)} dt + \int_0^T (g, \Phi)_{L^2(\partial \Omega)} dt + \int_\Omega u(x,0) \Phi(x,0) dx.
\]
We thus obtain from (2.17), (2.18) that \( \int_\Omega (u(x,0) - q(x)) \Phi(x,0) dx = 0 \), where \( \Phi(\cdot,0) \) is arbitrary. This results that \( u(x,0) = q(x) \) for a.e. \( x \in \Omega \) and so \( u = u(f) \). Since \( u(f) \) is unique, we get that the whole sequence \( (u(f_k)) \) weakly converges in \( \mathcal{V}(0,T) \) to \( u(f) \).

Further, since the trace operator \( \gamma : H^1(\Omega) \mapsto L^2(\partial \Omega) \) is compact (cf. [28, pp. 31]), so \( L^2(0,T; H^1(\Omega)) \mapsto L^2(0,T; L^2(\partial \Omega)) = L^2(S) \) is also compact (cf. [18, pp. 18]). We therefore conclude that \( (u(f_k)) \) strongly converges to \( u(f) \) in \( L^2(S) \), that finishes the proof.

Now, we are in a position to prove the main result of this section.

**Theorem 2.6.** The minimization problem \( (P_{\rho,\delta}) \) attains a unique minimizer \( f_{\rho,\delta} \) which satisfies the equation
\[
f_{\rho,\delta} = f^* - \frac{1}{\rho} p(f_{\rho,\delta}),
\]
where \( p(f_{\rho,\delta}) \) is the adjoint state defined by (2.12).

**Proof.** Due to Lemma 2.5 as \( f_k \) weakly converges to \( f \) in \( L^2(\Omega_T) \) the sequence \( u(f_k)_{\Sigma} \rightarrow u(f)_{\Sigma} \) in \( L^2(\Sigma) \). And since the \( L^2 \)-norm is weakly lower semi-continuous, is so the functional \( J_{\rho,\delta}(f) \). Furthermore, it is clear that \( J_{\rho,\delta}(f) \) is strictly convex, hence it attains a unique minimizer.

Next, for all \( \xi \in L^2(\Omega_T) \) we have from the optimality of \( f_{\rho,\delta} \) that \( J'_{\rho,\delta}(f_{\rho,\delta})\xi = 0 \). By (2.14), we get
\[
\frac{1}{2} J'_{\rho,\delta}(f_{\rho,\delta})\xi = (p(f_{\rho,\delta}),\xi)_{L^2(\Omega_T)} + \rho(f_{\rho,\delta} - f^*,\xi)_{L^2(\Omega_T)} = (p(f_{\rho,\delta}) + \rho(f_{\rho,\delta} - f^*))\xi_{L^2(\Omega_T)},
\]
which finishes the proof.

### 3 Crank-Nicolson Galerkin discretization

In this section we present the **Crank-Nicolson Galerkin** method (see, e.g., [28]) to discretize the regularized minimization problem in finite dimensional spaces.

Let \( (T_h)_{0 < h < 1} \) be a family of regular and quasi-uniform triangulations of the domain \( \Omega \) with the mesh size \( h \). For the definition of the discretization space of the state functions with respect to the spatial variable let us denote by
\[
Y^1_h := \left\{ \varphi_h \in C(\overline{\Omega}) \mid \varphi_h|_T \in P_1(T) \text{ for all } T \in T_h \right\}
\]
with \( P_1 \) consisting all polynomial functions of degree at most 1.

We here recall the interpolation operator

\[
\Pi_h : L^1(\Omega) \rightarrow V^1_h
\]

which satisfies the following properties

\[
\lim_{h \to 0} \| v - \Pi_h v \|_{H^k(\Omega)} = 0 \quad \text{for all} \quad k \in \{0, 1\} \quad (3.2)
\]

and

\[
\| v - \Pi_h v \|_{H^k(\Omega)} \leq C h^{l-k} \| v \|_{H^l(\Omega)} \quad (3.3)
\]

for \( 0 \leq k \leq l \leq 2 \) (see [13, 2, 3, 37]). We also mention that for all \( v \in V^1_h \), the inverse inequality (cf. e.g., [4, 12])

\[
\| v \|_{H^1(\Omega)} \leq C h^{-1} \| v \|_{L^2(\Omega)} \quad (3.4)
\]

holds true. Further, for all \( v \in H^1(\Omega) \) and \( T \in T_h \) there holds the local estimate (cf. [30])

\[
\| v \|_{L^2(\partial T)} \leq C \left( h^{-1/2} \| v \|_{L^2(T)} + h^{1/2} \| \nabla v \|_{L^2(T)} \right). \quad (3.5)
\]

To discretize the state functions with respect to the time variable, we divide the time interval \((0, T)\) into \( M \) equal subintervals and introduce a time step \( \tau := T/M \) together with time levels

\[
t^n := n \tau \quad \text{with} \quad n \in I.
\]

For a continuous function \( \zeta : [0, T] \rightarrow \mathbb{R} \) and \( t^n \in [0, T] \), we denote by

\[
\zeta^n := \zeta(t^n).
\]

Then, we set

\[
a^n(v, w) := \int_{\Omega} A^n(x) \nabla v(x) \cdot \nabla w(x) dx + \int_{\Omega} b^n(x)v(x)w(x)dx + \int_{\partial\Omega} \sigma^n(x)v(x)w(x)dx, \quad \forall v, w \in H^1(\Omega).
\]

Linking to the above partition of \((0, T)\), we introduce the constant piecewise, discontinuous interpolation operator

\[
\pi_\tau : L^2(0, T) \rightarrow V^0_\tau := \{ \varphi \in L^2(0, T) \mid \varphi|_{(t^n-1, t^n]} = \text{constant}, \forall n \in I \}
\]

which is defined for each \( \eta \in L^2(0, T) \) by

\[
\pi_\tau \eta(t)|_{(t^n-1, t^n]} := \eta_\tau(t)|_{(t^n-1, t^n]} := \frac{1}{\tau} \int_{t^n-1}^{t^n} \eta(s)ds := \eta^n, \quad \forall n \in I.
\]

By Proposition 9 of [36, pp. 129], we get the limit

\[
\lim_{\tau \to 0} \| \eta - \eta_\tau \|_{L^2(0, T)} = 0 \quad (3.7)
\]

and the estimate (cf. [27] Proposition 5.1)

\[
\| \eta - \eta_\tau \|_{L^2(0, T)} \leq C_\tau \left\| \frac{d\eta}{dt} \right\|_{L^2(0, T)} \quad (3.8)
\]

in case \( \eta \in H^1(0, T) \).

For a sequence \((w^k) \in L^2(\Omega)\) we respectively introduce the backward difference quotient and the mean as follows

\[
\partial w^k := \frac{w^k - w^{k-1}}{\tau} \quad \text{and} \quad \overline{\partial w^k} := \frac{w^k + w^{k-1}}{2}.
\]
With the above notations, the Crank-Nicolson Galerkin method applied to (1.1) reads: Let \( q_h := \Pi_h q \in V_h^1 \), find \( U^n(f) := U^n_{q^h,T}(f) \in V_h^1 \) such that

\[
(\partial U^n(f), \varphi_h)_{L^2(\Omega)} + a^n(\nabla U^n, \varphi_h) = \left( f^n, \varphi_h \right)_{L^2(\Omega)} + \left( g^n, \varphi_h \right)_{L^2(\partial \Omega)}, \quad \forall \varphi_h \in V_h^1, \quad n \in I
\]

(3.9)

If \( n \in I \) and \( U^{n-1}(f) \) is given, then \( U^n(f) \) can be defined from the elliptic variational problem

\[
\hat{a}(U^n(f), \varphi_h) = \hat{l}(\varphi_h), \quad \forall \varphi_h \in V_h^1
\]

(3.10)

with

\[
\hat{a}(U^n(f), \varphi_h) := \frac{1}{2} a^n(U^n(f), \varphi_h) + \frac{1}{\tau} \int_{\Omega} U^n(f) \varphi_h \, dx,
\]

\[
\hat{l}(\varphi_h) := \left( f^n, \varphi_h \right)_{L^2(\Omega)} + \left( g^n, \varphi_h \right)_{L^2(\partial \Omega)} + \frac{1}{\tau} \int_{\Omega} U^{n-1}(f) \varphi_h \, dx - \frac{1}{2} a^n(U^{n-1}(f), \varphi_h).
\]

Since \( U^0(f) = q_h \) is known, we can compute \( U^1(f) \), and so on \( U^2(f), \ldots, U^M(f) \).

We now introduce the discrete optimization problem related to the continuous \((P_{\rho,\delta})\):

\[
\min_{f \in L^2(\Omega_T)} J_{\rho,\delta,h,T}(f) \quad \text{with} \quad J_{\rho,\delta,h,T}(f) := \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \left( \| U^n(f) - z^n \|^2_{L^2(\Gamma)} + \rho \| f - f^* ||^2_{L^2(\Omega_T)} \right). \tag{P_{\rho,\delta,h,T}}
\]

We emphasize at this point that the admissible space \( L^2(\Omega_T) \) of the identified source in the minimization problem \((P_{\rho,\delta,h,T})\) is not discretized. However, due to Theorem \(3.5\) below we will show that any solution to \((P_{\rho,\delta,h,T})\) automatically belongs to the finite dimensional space \( V_{h,T}^1 \) defined by

\[
V_{h,T}^1 := \{ \Phi \in L^2(0,T;V_h^1) \mid \Phi|_{t \in [t_{n-1},t_n]} := \varphi^n_h \in V_h^1 \text{ for all } n \in I \}.
\]

(3.11)

Thus a discretization of \( L^2(\Omega_T) \) can be avoided.

To begin, we recall the discrete Gronwall inequality.

**Lemma 3.1.** Assume that \((Y_n)_n\), \((X_n)_n\), and \((\alpha_n)_n\) are non-negative sequences such that

\[ Y_n \leq X_n + \sum_{m=0}^{n-1} \alpha_m X_m \quad \text{for all } n \geq 0. \]

Then

\[ Y_n \leq X_n + \sum_{m=0}^{n-1} \alpha_m X_m e^{\sum_{s=m+1}^{n-1} \alpha_s} \]

for all \( n \geq 0. \)

**Lemma 3.2.** (i) There holds the estimate

\[
\max_{n \in I_0} \| U^n(f) \|^2_{H^1(\Omega)} \leq C \left( \| q \|^2_{H^1(\Omega)} + \| g \|^2_{L^\infty(0,T;L^2(\partial \Omega))} + \| f \|^2_{L^2(\Omega_T)} \right) \tag{3.12}
\]

as \( \tau \sim h^2. \)

(ii) The inequalities

\[
\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \left( \| \partial U^n(f), \theta \|_{L^2(\Omega)} \right) \, dt \leq C \left( \| q \|_{H^1(\Omega)} + \| g \|_{L^\infty(0,T;L^2(\partial \Omega))} + \| f \|_{L^2(\Omega_T)} \right) \| \theta \|_{L^2(0,T;H^1(\Omega))} \tag{3.13}
\]

for all \( \theta \in L^2(0,T;H^1(\Omega)) \) and

\[
\sum_{n=1}^{M} \tau \sup_{\| \varphi \|_{H^1(\Omega)} \leq 1} \left( \| \partial U^n(f), \varphi \|_{L^2(\Omega)} \right)^2 \leq C \left( \| q \|^2_{H^1(\Omega)} + \| g \|^2_{L^\infty(0,T;L^2(\partial \Omega))} + \| f \|^2_{L^2(\Omega_T)} \right). \tag{3.14}
\]
hold true.

(iii) The limit

$$\lim_{h,\tau \to 0} \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (U^n(f) - U^{n-1}(f), \theta)_{H^1(\Omega)} dt = 0$$  \hspace{1cm} (3.15)$$

is satisfied for all \( \theta \in L^2 (0, T; H^1(\Omega)) \).

Proof. (i) Taking \( \phi_h = 2\tau^2 \partial U^n(f) \) in (3.9), we have

$$2\|U^n(f) - U^{n-1}(f)\|_{L^2(\Omega)}^2 + \tau a^n(U^n(f), U^n(f))$$

$$= \tau a^n(U^{n-1}(f), U^{n-1}(f)) + 2\tau (\bar{T}^n, U^n(f) - U^{n-1}(f))_{L^2(\Omega)} + 2\tau (g^n, U^n(f) - U^{n-1}(f))_{L^2(\partial \Omega)}.$$

By (2.7), we thus get

$$2\|U^n(f) - U^{n-1}(f)\|_{L^2(\Omega)}^2 + C_1 \tau \|U^n(f)\|_{H^1(\Omega)}^2$$

$$\leq C_2 \tau \|U^{n-1}(f)\|_{H^1(\Omega)}^2 + 2\tau \|\bar{T}^n\|_{L^2(\Omega)} \|U^n(f) - U^{n-1}(f)\|_{L^2(\Omega)}$$

$$+ 2\tau \|g^n\|_{L^2(\partial \Omega)} \|U^n(f) - U^{n-1}(f)\|_{L^2(\partial \Omega)}.$$

For an arbitrary \( \epsilon > 0 \), an application of Young’s inequality yields that

$$2\tau \|\bar{T}^n\|_{L^2(\Omega)} \|U^n(f) - U^{n-1}(f)\|_{L^2(\Omega)} \leq \epsilon \|U^n(f) - U^{n-1}(f)\|_{L^2(\Omega)}^2 + \frac{\tau^2}{\epsilon} \|\bar{T}^n\|_{L^2(\Omega)}^2$$

$$= \epsilon \|U^n(f) - U^{n-1}(f)\|_{L^2(\Omega)}^2 + \frac{\tau}{\epsilon} \int_{t_{n-1}}^{t_n} \|f\|_{L^2(\Omega)}^2 dt.$$  

Meanwhile, using (3.5) and (3.4), we have

$$2\tau \|g^n\|_{L^2(\partial \Omega)} \|U^n(f) - U^{n-1}(f)\|_{L^2(\partial \Omega)}$$

$$\leq C h^{-1/2} \tau \|U^n(f) - U^{n-1}(f)\|_{L^2(\partial \Omega)} \|g^n\|_{L^2(\partial \Omega)}$$

$$\leq \epsilon \|U^n(f) - U^{n-1}(f)\|_{L^2(\Omega)}^2 + \frac{C h^{-1} \tau^2}{\epsilon} \|g\|_{L^\infty (t_{n-1}, t_n; L^2(\partial \Omega))}^2.$$  

We thus arrive at

$$\|U^n(f) - U^{n-1}(f)\|_{H^1(\Omega)}^2 + \tau \|U^n(f)\|_{H^1(\Omega)}^2$$

$$\leq C \tau \left( \|U^{n-1}(f)\|_{H^1(\Omega)}^2 + \int_{t_{n-1}}^{t_n} \|f\|_{L^2(\Omega)}^2 dt + \|g\|_{L^\infty (t_{n-1}, t_n; L^2(\partial \Omega))}^2 \right)$$  \hspace{1cm} (3.16)$$

as \( \epsilon < 1/2 \) and \( \tau \sim h^2 \) with \( C \) independent of \( n \). Since \( \|U^n(f)\|_{H^1(\Omega)} = \|\Pi_h q\|_{H^1(\Omega)} \leq C \|q\|_{H^1(\Omega)} \), we deduce from (3.16) that

$$\|U^n(f)\|_{H^1(\Omega)}^2 \leq C \left( \|q\|_{H^1(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 \right) + \sum_{m=0}^{n-1} \alpha_m \|U^n(f)\|_{H^1(\Omega)}^2, \forall n \geq 1,$$

where \( \alpha_0 = \ldots = \alpha_{n-2} = 0 \) and \( \alpha_{n-1} = C \). Therefore, an application of the discrete Gronwall inequality implies (3.12) for all \( n \geq 0 \).

(ii) By (3.9), for all \( \theta \in L^2 (0, T; H^1(\Omega)) \) we rewrite

$$\langle \partial U^n(f), \theta \rangle_{L^2(\Omega)} = \langle \partial U^n(f), \Pi_h \theta \rangle_{L^2(\Omega)} + \langle \partial U^n(f), \theta - \Pi_h \theta \rangle_{L^2(\Omega)}$$

$$= (\bar{T}^n, \Pi_h \theta)_{L^2(\Omega)} + (g^n, \Pi_h \theta)_{L^2(\partial \Omega)} - a^n (\bar{U}^n(f), \Pi_h \theta) + (\partial U^n(f), \theta - \Pi_h \theta)_{L^2(\Omega)}.$$  \hspace{1cm} (3.17)$$

We have that

$$\sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (\bar{T}^n, \Pi_h \theta)_{L^2(\Omega)} dt \leq \sum_{n=1}^{M} \left( \int_{t_{n-1}}^{t_n} \|f\|_{L^2(\Omega)}^2 dt \right)^{1/2} \left( \int_{t_{n-1}}^{t_n} \|\Pi_h \theta\|_{L^2(\Omega)}^2 dt \right)^{1/2}$$

$$\leq C \|f\|_{L^2(\Omega_T)} \|\theta\|_{L^2(0, T; H^1(\Omega))}.$$  

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and
\[ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (g^n, \Pi_h \theta)_{L^2(\Omega)} dt \leq \sum_{n=1}^{M} \left( \tau \| g \|_{L^2(\Omega)}^2 \right)^{1/2} \left( \int_{t_{n-1}}^{t_n} \| \Pi_h \theta \|_{L^2(\Omega)}^2 dt \right)^{1/2} \leq C \| g \|_{L^\infty(0,T;L^2(\Omega))} \| \theta \|_{L^2(0,T;H^1(\Omega))}. \]

Using (3.12), we further get that
\[ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} a^n (\partial U^n(f), \Pi_h \theta) dt \]
\[ \leq C \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \left( \| U^n(f) \|_{H^1(\Omega)} + \| U^{n-1}(f) \|_{H^1(\Omega)} \right) \| \Pi_h \theta \|_{H^1(\Omega)} dt \]
\[ \leq C \sum_{n=1}^{M} \left( \tau \| U^n(f) \|_{H^1(\Omega)}^2 \right)^{1/2} \left( \int_{t_{n-1}}^{t_n} \| \theta \|_{H^1(\Omega)}^2 dt \right)^{1/2} \]
\[ \leq C \left( \sum_{n=1}^{M} \tau \| U^n(f) \|_{H^1(\Omega)}^2 \right)^{1/2} \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \| \theta \|_{H^1(\Omega)}^2 dt \right)^{1/2} \]
\[ \leq C \left( \| g \|_{H^1(\Omega)} + \| g \|_{L^\infty(0,T;L^2(\Omega))} + \| f \|_{L^2(\Omega)} \right) \| \theta \|_{L^2(0,T;H^1(\Omega))}. \]

Next, we deduce from (3.10) that
\[ \sum_{n=1}^{M} \| U^n(f) - U^{n-1}(f) \|_{L^2(\Omega)}^2 \leq C \left( \| g \|_{H^1(\Omega)}^2 + \| g \|_{L^\infty(0,T;L^2(\Omega))}^2 + \| f \|_{L^2(\Omega)}^2 \right). \] (3.18)

Utilizing (3.3), we have that
\[ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (\partial U^n(f), \theta - \Pi_h \theta)_{L^2(\Omega)} dt \]
\[ \leq \left( \sum_{n=1}^{M} \tau \| \partial U^n(f) \|_{L^2(\Omega)}^2 \right)^{1/2} \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \| \theta - \Pi_h \theta \|_{L^2(\Omega)}^2 dt \right)^{1/2} \]
\[ \leq C \left( \sum_{n=1}^{M} \tau \| \partial U^n(f) \|_{L^2(\Omega)}^2 \right)^{1/2} \| \theta \|_{L^2(0,T;H^1(\Omega))} \]
\[ \leq C \left( \| g \|_{H^1(\Omega)} + \| g \|_{L^\infty(0,T;L^2(\Omega))} + \| f \|_{L^2(\Omega)} \right) \| \theta \|_{L^2(0,T;H^1(\Omega))}. \]

by (3.18) and \( \tau \sim h^2 \). Therefore, (3.13) follows from (3.17) and the above estimates. Furthermore, in the same manner we also get (3.14).

(iii) We consider the piecewise constant function with respect to \( t \) defined as follows
\[ \Phi_{h,\tau} := U^n(f) - U^{n-1}(f) \quad \text{for all} \quad n \in I. \]

Due to (3.12), the sequence \( \Phi_{h,\tau} \) is bounded in the \( L^2(0,T;H^1(\Omega)) \)-norm. Therefore, there exist a subsequence of it denoted the same symbol and an element \( \Phi \in L^2(0,T;H^1(\Omega)) \) such that
\[ \lim_{h,\tau \to 0} \int_0^T (\Phi_{h,\tau}(\theta)_{H^1(\Omega)} ) dt = \int_0^T \Phi_{H^1(\Omega)} dt \quad \text{and} \quad \lim_{h,\tau \to 0} \| \Phi_{h,\tau} - \Phi \|_{L^2(\Omega_T)} = 0 \]
for all \( \theta \in L^2(0,T;H^1(\Omega)) \), since the embedding \( L^2(0,T;H^1(\Omega)) \to L^2(\Omega_T) \) is compact. On the other hand, it follows from (3.18) that
\[ \| \Phi_{h,\tau} \|_{L^2(\Omega_T)}^2 = \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \| \Phi_{h,\tau} \|_{L^2(\Omega)}^2 = \sum_{n=1}^{M} \tau \| U^n(f) - U^{n-1}(f) \|_{L^2(\Omega)}^2 \to 0 \]
as \( h, \tau \to 0 \), which finishes the proof.

**Lemma 3.3.** Assume that the sequence \((f_k) \subset L^2(\Omega_T)\) weakly converges in \(L^2(\Omega_T)\) to an element \(f\). Then for any fixed \( n \in I_0 \), the sequence \((U^n(f_k))\) converges to \(U^n(f)\) in the \(H^1(\Omega)\)-norm.

**Proof.** We show the assertion by induction on \( n \in I_0 \). First, for all \( k \in \mathbb{N} \), since \( U^0(f_k) = U^0(f) = q_h \), it holds true with \( n = 0 \). We now assume that the statement is valid for \( n - 1 \), where \( n \in I \). To complete, we need to show it for \( n \).

In fact, for all \( n \in I \), by (3.12), the sequence \((U^n(f_k))\) is bounded in the finite dimensional space \(V^n_{\Omega} := \mathcal{V}_h \times \mathcal{V}_h \times \ldots \times \mathcal{V}_h\). Then, there are the subsequences of it denoted the same symbol and an element \( \theta^n \in V^n_{\Omega} \) such that \((U^n(f_k))_k\) converges to \( \theta^n \) in the \(H^1(\Omega)\)-norm. For all \( k \in \mathbb{N} \) and \( \varphi_h \in V^n_{\Omega} \) we have from (3.10) that

\[
\frac{1}{\tau} \int_\Omega U^n(f_k)\varphi_h dx + \frac{1}{2} a^n(U^n(f_k), \varphi_h) = (\mathcal{T}_k^n, \varphi_h)_{L^2(\Omega)} + (g^n, \varphi_h)_{L^2(\Omega)} + \frac{1}{\tau} \int_\Omega U^{n-1}(f_k)\varphi_h dx - \frac{1}{2} a^n(U^{n-1}(f_k), \varphi_h). \tag{3.19}
\]

Set \( \Phi_h := \varphi_h\chi_{(\tau^{-1}, t^n]} \). We get

\[
(\mathcal{T}_k^n, \varphi_h)_{L^2(\Omega)} = \frac{1}{\tau} \int_\Omega \left( \frac{1}{\tau} \int_{t^{n-1}}^{t^n} f_k(x,t) dt \right) \varphi_h(x) dx
\]

\[
= \frac{1}{\tau} \int_{(\tau^{-1}, t^n]} f_k(x,t) \varphi_h(x) dx dt
\]

\[
= \frac{1}{\tau} \int_\Omega f_k \Phi_h dx dt
\]

\[
= \frac{1}{\tau} \int_\Omega f \Phi_h dx dt
\]

\[
= (\mathcal{T}^n, \varphi_h)_{L^2(\Omega)}
\]

as \( k \to \infty \). Further, note that \( \lim_{k \to \infty} a^n(U^n(f_k) - \theta^n, \varphi_h) = 0 \). Thus, sending \( k \to \infty \) in (3.19), we get

\[
\frac{1}{\tau} \int_\Omega \theta^n \varphi_h dx + \frac{1}{2} a^n(\theta^n, \varphi_h) = (\mathcal{T}^n, \varphi_h)_{L^2(\Omega)} + (g^n, \varphi_h)_{L^2(\Omega)} + \frac{1}{\tau} \int_\Omega U^{n-1}(f)\varphi_h dx - \frac{1}{2} a^n(U^{n-1}(f), \varphi_h)
\]

\[
= \frac{1}{\tau} \int_\Omega U^n(f)\varphi_h dx + \frac{1}{2} a^n(U^n(f), \varphi_h)
\]

which implies

\[
\frac{1}{\tau} \int_\Omega (U^n(f) - \theta^n)\varphi_h dx + \frac{1}{2} a^n(U^n(f) - \theta^n, \varphi_h) = 0.
\]

Taking \( \varphi_h = U^n(f) - \theta^n \), we get \( U^n(f) = \theta^n \), and thus finish the proof.

We are now in a position to state the main result of this section. In the space \( (V^n_{\Omega})^M := V^n_{\Omega} \times V^n_{\Omega} \times \ldots \times V^n_{\Omega} \) \( M \)-times

we use the inner product

\[
(F, G)_{(V^n_{\Omega})^M} := \sum_{n=1}^M (F^n, G^n)_{L^2(\Omega)}
\]

for all \( F = (F^1, \ldots, F^M), G = (G^1, \ldots, G^M) \in (V^n_{\Omega})^M \).

**Definition 3.4.** The mapping \( u_{h, \tau} : L^2(\Omega_T) \to (V^n_{\Omega})^M \) defined for each \( f \in L^2(\Omega_T) \) by

\[
u_{h, \tau}(f) := (U^1(f), \ldots, U^M(f)) \in (V^n_{\Omega})^M
\]

is called the discrete source-to-state operator.
The operator \( u_{h,\tau} \) is Fréchet differentiable on \( L^2(\Omega_T) \). For each \( f \in L^2(\Omega_T) \) in the direction \( \xi \in L^2(\Omega_T) \) its differential \( u'_{h,\tau}(f)\xi \) is \( \left( \tilde{U}^1(\xi), \ldots, \tilde{U}^M(\xi) \right) \), where \( \tilde{U}^n(\xi) := \tilde{U}^n_{h,\tau}(\xi) \in \mathcal{V}_h^1 \) is defined by the variational equation

\[
\left( \partial \tilde{U}^n(\xi), \varphi_h \right)_{L^2(\Omega)} + a^n(\tilde{\vartheta}^n(\xi), \varphi_h) = (\xi^n, \varphi_h)_{L^2(\Omega)}, \quad \forall \varphi_h \in \mathcal{V}_h^1, \ n \in I
\]

(3.20)

In fact, we have from (3.9) and (3.20) that

\[
(\partial \theta^n, \varphi_h)_{L^2(\Omega)} + a^n(\tilde{\vartheta}^n, \varphi_h) = 0, \quad \forall \varphi_h \in \mathcal{V}_h^1, \ n \in I,
\]

where \( \theta^n := U^n(f + \xi) - U^n(f) - \tilde{U}^n(\xi) \) and \( \theta^0 = 0 \). Taking \( \varphi_h = \tilde{\vartheta}^n \), summing the resulting equalities over \( n = 1, 2, \ldots, m \) with \( 1 \leq m \leq M \), utilizing (2.7), we get

\[
0 = \frac{1}{2\tau} \| \theta^n \|_{L^2(\Omega)}^2 + \sum_{n=1}^M a^n(\tilde{\vartheta}^n, \tilde{\vartheta}^n) \geq \frac{1}{2\tau} \| \theta^n \|_{L^2(\Omega)}^2
\]

and thus obtain \( \theta^n = 0 \).

The adjoint problem (2.12) is discretized via the process that for \( n \in I_0 \) the element \( P^n(f) := P^n_{h,\tau}(f) \in \mathcal{V}_h^1 \) satisfies the system

\[
-\tau (\partial P^n(f), \varphi_h)_{L^2(\Omega)} + \tau a^n(\tilde{\vartheta}^n, \varphi_h) = \int_{t_{n-1}}^{t_n} (U^n(f) - z_\delta, \varphi_h)_{L^2(\Gamma)} dt, \quad \forall \varphi_h \in \mathcal{V}_h^1
\]

(3.21)

\[
P^M(f) = 0.
\]

Note that from \( P^M(f) = 0 \) we can compute \( P^{M-1}(f) \) due to (3.21), and so \( P^{M-2}(f), \ldots, P^1(f), P^0(f) \).

**Theorem 3.5.** The problem \( (P_{\rho,\delta,h,\tau}) \) attains a unique solution \( f := f_{\rho,\delta,h,\tau} \) satisfying the equation

\[
f|_{\Omega \times (t^{n-1},t^n)} = f^* - \frac{1}{\rho} P^{n-1}(f)
\]

(3.22)

for any \( n \in I \).

**Proof.** In virtue of Lemma 3.3, the proof of the existence and uniqueness of a solution \( f \) to \( (P_{\rho,\delta,h,\tau}) \) is similar to that of Theorem 2.6. We now show the equation (3.22). We have \( J'_{\rho,\delta,h,\tau}(f)\xi = 0 \) for all \( \xi \in L^2(\Omega_T) \). By (3.20)–(3.21), we get

\[
J'_{\rho,\delta,h,\tau}(f)\xi = 2 \sum_{n=1}^M \int_{t_{n-1}}^{t_n} (U^n(f) - z_\delta, U^n(f)\xi)_{L^2(\Gamma)} dt + 2\rho(f - f^*, \xi)_{L^2(\Omega_T)}
\]

\[
= 2 \sum_{n=1}^M \int_{t_{n-1}}^{t_n} (U^n(f) - z_\delta, \tilde{U}^n(\xi))_{L^2(\Gamma)} dt + 2\rho(f - f^*, \xi)_{L^2(\Omega_T)}
\]

\[
= 2\tau \sum_{n=1}^M - (\partial P^n(f), \tilde{U}^n(\xi))_{L^2(\Omega)} + 2\tau \sum_{n=1}^M a^n(\tilde{\vartheta}^n(f), \tilde{U}^n(\xi)) + 2\rho(f - f^*, \xi)_{L^2(\Omega_T)}.
\]

Using the identities

\[
\sum_{n=1}^M (\alpha^n - \alpha^{n-1}) \beta^n = \alpha^M \beta^M - \alpha^0 \beta^0 - \sum_{n=1}^M (\beta^n - \beta^{n-1}) \alpha^{n-1},
\]

(3.23)

\[
\sum_{n=1}^M (\alpha^n + \alpha^{n-1}) \beta^n = \alpha^M \beta^M - \alpha^0 \beta^0 + \sum_{n=1}^M (\beta^n + \beta^{n-1}) \alpha^{n-1}
\]
In view of the identity (3.24), the equality holds true for all $(i)$ For all $\phi \in L^2(\Omega \times (t^{n-1}, t^n))$ we consider $\xi := \varphi_\Omega \chi_{(t^{n-1}, t^n)} \in L^2(\Omega_T)$ and then have $\widehat{\xi}^k = 0$ as $k \neq n$ and
\[
\tau \sum_{k=1}^{M} (\rho P^{k-1}(f), \xi^k)_{L^2(\Omega)} = \tau (P^{n-1}(f), \widehat{\xi}^n)_{L^2(\Omega)} \]
\[
= \tau \int_{\Omega} \left( P^{n-1}(x; f) - \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \varphi(x, t) dt \right) dx \]
\[
= \int_{t^{n-1}}^{t^n} (P^{n-1}(f), \varphi)_{L^2(\Omega)} dt \]
as well as
\[
(f - f^*, \xi)_{L^2(\Omega_T)} = \int_{t^{n-1}}^{t^n} (f - f^*, \varphi)_{L^2(\Omega)} dt. \]
Thus, we arrive at
\[
\int_{t^{n-1}}^{t^n} (P^{n-1}(f) + \rho(f - f^*), \varphi)_{L^2(\Omega)} dt = 0, \]
where $\varphi \in L^2(\Omega \times (t^{n-1}, t^n))$ is arbitrary. This implies (3.22). The proof is finished. \hfill \square

**Remark 3.6.** For any fixed $f \in L^2(\Omega_T)$, denote by
\[
\mathcal{G}_f(x, t; f)_{|\Omega \times \{t^n\}} := P^{n-1}(x, t; f) \quad \text{with} \quad n \in I. \]
In view of the identity (3.24), the $L^2$-gradient of the cost functional at $f$ is given by
\[
\nabla J_{\rho, \delta, h, \tau}(x, t; f) = 2\mathcal{G}_f(x, t; f) + 2\rho(f - f^*) \quad \text{(3.25)}
\]
i.e. the equality
\[
J'_{\rho, \delta, h, \tau}(f)\xi = (\nabla J_{\rho, \delta, h, \tau}(f), \xi)_{L^2(\Omega_T)}
\]
holds true for all $\xi \in L^2(\Omega_T)$.

## 4 Convergence of finite dimensional approximations

The aim of this section is to show finite dimensional approximations, i.e. solutions of $(P_{\rho, \delta, h, \tau})$, converge to the sought source. To do so, we state some auxiliary results.

**Lemma 4.1.** (i) For all $\phi \in L^2(0, T; H^k(\Omega))$ with $k \in \{0, 1\}$ there hold
\[
\lim_{h \to 0} \| \phi - \Pi_h \phi \|_{L^2(0,T; H^k(\Omega))} = 0, \quad \lim_{\tau \to 0} \| \phi - \phi_{\tau} \|_{L^2(0,T; H^k(\Omega))} = 0 \quad \text{and}
\]
\[
\lim_{h, \tau \to 0} \| \phi - \Pi_h \phi_{\tau} \|_{L^2(0,T; H^k(\Omega))} = 0. \quad \text{(4.1)}
\]
(ii) Assume that $\phi \in L^\infty(\Omega_T)$, then
\[
\lim_{\tau \to 0} \int_0^T \int_{\Omega} (\phi_{\tau} - \phi) uv^t dx dt = 0 \quad \text{(4.2)}
\]
for all $u \in L^2(\Omega_T)$ and any bounded sequence $(v^t)_{\tau}$ in $L^2(\Omega_T)$. 

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Proof. (i) The first and second statements of (4.1) follow directly from (3.2) and (3.7), respectively, and the Lebesgue’s dominated convergence theorem. Meanwhile, for a.e in $t \in (0, T)$, since
\[
\|\phi - \Pi_h \phi_r\|_{H^1(\Omega)} \leq \|\phi - \Pi_h \phi\|_{H^1(\Omega)} + \|\Pi_h (\phi - \phi_r)\|_{H^1(\Omega)} \leq \|\phi - \Pi_h \phi\|_{H^1(\Omega)} + C\|\phi - \phi_r\|_{H^1(\Omega)},
\]
the third assertion follows from the first and second ones.

For (ii) we take an arbitrary $\epsilon > 0$ and $u^\epsilon \in C(\overline{\Omega_T})$ such that $\|u - u^\epsilon\|_{L^2(\Omega_T)} < \epsilon$. Then we have
\[
\left|\int_0^T \int_\Omega (\phi_r - \phi) u^\epsilon v^\tau dxdt\right| \leq \int_0^T \int_\Omega |\phi_r - \phi| |u^\epsilon| |v^\tau| dxdt + \int_0^T \int_\Omega |\phi_r - \phi| |u - u^\epsilon| |v^\tau| dxdt
\leq \|u^\epsilon\|_{C(\overline{\Omega_T})} \|v^\tau\|_{L^2(\Omega_T)} \|\phi_r - \phi\|_{L^2(\Omega_T)} + (2\|\phi\|_{L^\infty(\Omega_T)} \|v^\tau\|_{L^2(\Omega_T)}) \epsilon.
\]
Sending $\tau$ to zero, we thus have that $\lim_{\tau \to 0} \left|\int_0^T \int_\Omega (\phi_r - \phi) u^\epsilon v^\tau dxdt\right| \leq C\epsilon$ for all $\epsilon > 0$, which yields (4.2).

The proof is completed.

Lemma 4.2. Assume that $\lim_{k \to \infty} h_k = \lim_{k \to \infty} \tau_k = 0$ and the sequence $(z^\delta_k) \subset L^2(\Sigma)$ converge to $z_\delta$ in the $L^2(\Sigma)$-norm. Let the sequence $(f_k) \subset L^2(\Omega_T)$ weakly converge in $L^2(\Omega_T)$ to $f$. Then,
\[
\lim_{k \to \infty} \sum_{n=1}^{M_k} \int_{t_n-1}^{t_n} \|U^\delta_{h_k, \tau_k}(f_k) - z^\delta_k\|_{L^2(\Gamma)}^2 dt = \|u(f) - z_\delta\|_{L^2(\Sigma)}^2,
\]
where $M_k = T/\tau_k$ and $U^\delta_{h_k, \tau_k}(f_k)$ defined by (3.9).

Proof. For convenience of exposition we denote by $U^n_k := U^n_{h_k, \tau_k}(f_k)$. Let $\Phi_k = \Phi_k(x, t)$ be the piecewise linear, continuous interpolation of $(U^n_k)_{n=0, \ldots, M_k}$ with respect to $t$, i.e.
\[
\Phi_k(x, t) := (t - t_n^{-1}) \partial U^n_k + U^n_k
\]
with $\partial U^n_k = \tau_k^{-1}(U^n_k - U^{n-1}_k)$ and $(x, t) \in \Omega \times (t^{n-1}, t^n)$, $n = 1, \ldots, M_k$. We first note that for all $t \in (t^{n-1}, t^n)$
\[
\frac{\partial \Phi_k}{\partial t} = \partial U^n_k \quad \text{and} \quad \int_{t_n-1}^{t_n} \Phi_k dt = \tau_k \partial U^n_k.
\]
Thus, $(\Phi_k) \subset H^1(0, T; \mathcal{V}^\delta_k) \subset H^1(0, T; H^1(\Omega)) \subset \mathcal{W}(0, T)$. Further, the inequalities (3.12) and (3.14) yield that the sequence $(\Phi_k)$ is bounded in the reflexive space $\mathcal{W}(0, T)$. There exists a subsequence of $(\Phi_k)$ denoted again by $(\Phi_k)$ and an element $u \in \mathcal{W}(0, T)$ such that $(\Phi_k)$ weakly converges in $\mathcal{W}(0, T)$ to $u$.

We show that $u = u(f)$. In fact, for all $\phi \in L^2(0, T; H^1(\Omega))$ we have
\[
\int_0^T \left\langle \frac{\partial \Phi_k}{\partial t}, \phi \right\rangle_{(H^1(\Omega), H^1(\Omega))} dt + \int_0^T a(\Phi_k, \phi) dt
= \sum_{n=1}^{M_k} \int_{t_n-1}^{t_n} (\partial U^n_k, \phi)_{L^2(\Omega)} dt + \int_0^T \int_\Omega A \nabla \Phi_k \cdot \nabla \phi dxdt + \int_0^T \int_\Omega b \Phi_k \phi dxdt + \int_0^T \int_{\partial \Omega} \sigma \Phi_k \phi dxdt.
\]
For all $t \in (t^{n-1}, t^n)$, we have
\[
(\partial U^n_k, \phi)_{L^2(\Omega)} = (\partial U^n_k, \Pi_h \phi^n)_{L^2(\Omega)} + (\partial U^n_k, \phi - \Pi_h \phi^n)_{L^2(\Omega)}
\]
and, by (3.13) and (4.1),
\[
\sum_{n=1}^{M_k} \int_{t_n-1}^{t_n} (\partial U^n_k, \phi - \Pi_h \phi^n)_{L^2(\Omega)} \leq C\|\phi - \Pi_h \phi_{\tau_k}\|_{L^2(\Omega)} \to 0
\]
as \( k \to \infty \). Using the estimates (3.12) and (3.13) as well as the equalities (4.1), (4.2), we decompose the remainder in the right hand side of (4.5) as follows

\[
\int_0^T \int_\Omega A \nabla \Phi_k \cdot \nabla \phi dxdt + \int_0^T \int_\Omega b \Phi_k \phi dxdt + \int_0^T \int_{\partial \Omega} \sigma \Phi_k \phi dxdt
\]

\[
= \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \int_\Omega A^n \nabla \Phi_k \cdot \nabla \Pi_{h_k} \phi^n dxdt + \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \int_\Omega b^n \Phi_k \Pi_{h_k} \phi^n dxdt
\]

\[
+ \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \int_{\partial \Omega} \sigma^n \Phi_k \Pi_{h_k} \phi^n dxdt + R_k
\]

\[
= \sum_{n=1}^{M_k} \tau_k a^n (\mathcal{D} U^n_k, \Pi_{h_k} \phi^n) + R_k, \quad (4.8)
\]

by (4.4), where

\[
R_k := \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} (A - A^n) \nabla \Phi_k \cdot \nabla \phi dxdt + \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} A^n \nabla \Phi_k \cdot \nabla (\phi - \Pi_{h_k} \phi^n) dxdt
\]

\[
+ \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} (b - b^n) \nabla \Phi_k \cdot \nabla \phi dxdt + \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} b^n \nabla \Phi_k \cdot \nabla (\phi - \Pi_{h_k} \phi^n) dxdt
\]

\[
+ \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} (\sigma - \sigma^n) \nabla \Phi_k \cdot \nabla \phi dxdt + \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \sigma^n \nabla \Phi_k \cdot \nabla (\phi - \Pi_{h_k} \phi^n) dxdt.
\]

We remark that due to the continuity of data and (4.1), the relation

\[
\lim_{k \to \infty} R_k = 0 \quad (4.9)
\]

holds true. Therefore, we obtain from (4.5)–(4.9) that

\[
\int_0^T \left\langle \frac{\partial u}{\partial t}, \phi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + \int_0^T a(u, \phi) dt
\]

\[
= \lim_{k \to \infty} \left( \int_0^T \left\langle \frac{\partial \Phi_k}{\partial t}, \phi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + \int_0^T a(\Phi_k, \phi) dt \right)
\]

\[
= \lim_{k \to \infty} \sum_{n=1}^{M_k} \tau_k \left( (\mathcal{D} U^n_k, \Pi_{h_k} \phi^n)_{L^2(\Omega)} + a^n (\mathcal{D} U^n_k, \Pi_{h_k} \phi^n) \right)
\]

\[
= \lim_{k \to \infty} \sum_{n=1}^{M_k} \tau_k \left( (f^n, \phi)_{L^2(\partial \Omega)} + (g^n, \Pi_{h_k} \phi^n)_{L^2(\partial \Omega)} \right)
\]

\[
= (f, \phi)_{L^2(0,T; L^2(\Omega))} + (g, \phi)_{L^2(0,T; L^2(\partial \Omega))},
\]

here we used (3.9). Further, the proof of Lemma 2.5 included an argument which can be used to show that \( u(x, 0) = q(x) \). Thus, we get that the sequence \( (\Phi_k) \) weakly converges in \( \mathcal{W}(0,T) \) to \( u(f) \) and strongly in \( L^2 \left( 0, T; L^2(\Gamma) \right) \), i.e.

\[
\lim_{k \to \infty} \left\| \Phi_k - u(f) \right\|_{L^2(\Sigma)} = 0. \quad (4.10)
\]

Next, for each \( \theta \in C^1 \left( [0,T]; H^1(\Omega) \right) \) we will show that

\[
\lim_{k \to \infty} \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left\langle \Phi_k - U^n_k, \theta \right\rangle_{H^1(\Omega)} dt = 0, \quad (4.11)
\]
which then holds true for each $\theta \in L^2 \left(0, T; H^1(\Omega)\right)$, by the density argumentation. We have that

$$
\int_{t_{n-1}}^{t_n} (\Phi_k - U^n_k, \theta)_{H^1(\Omega)} dt
= \frac{1}{2} \int_{t_{n-1}}^{t_n} (\partial U^n_k, \theta)_{H^1(\Omega)} d(t - t^{n-1} - \tau_k)^2
= \frac{\tau_k^2}{2} \partial U^n_k, \theta(t^{n-1})\right)_{H^1(\Omega)} - \frac{1}{2} \int_{t_{n-1}}^{t_n} (t - t^{n-1} - \tau_k)^2 \left(\partial U^n_k, \frac{\partial \theta}{\partial t}\right)_{H^1(\Omega)} dt
$$

and so

$$
\left| \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} (\Phi_k - U^n_k, \theta)_{H^1(\Omega)} \right|
\leq \frac{1}{2} \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left| U^n_k - U^{n-1}_k, \theta(t^{n-1})\right|_{H^1(\Omega)} dt + C\tau_k \left\| \theta \right\|_{C^1([0, T]; H^1(\Omega))} \sum_{n=1}^{M_k} \tau_k^2 \left\| \partial U^n_k \right\|_{H^1(\Omega)}
\rightarrow 0 \text{ as } k \rightarrow \infty,
$$

by (3.15) and (3.12). Using the compactness of the embedding $L^2 \left(0, T; H^1(\Omega)\right) \hookrightarrow L^2 \left(0, T; L^2(\partial \Omega)\right)$, again we obtain from (4.11) that

$$
\lim_{k \rightarrow \infty} \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left\| \Phi_k - U^n_k \right\|_{L^2(\Gamma)}^2 dt = 0.
$$

Combining this with (4.10), we arrive at $\lim_{k \rightarrow \infty} \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left\| U^n_k - u(f) \right\|_{L^2(\Gamma)}^2 dt = 0$ and conclude that

$$
\sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left| U^n_k - z_{\delta_k} \right|_{L^2(\Gamma)}^2 dt - \left| u(f) - z_{\delta} \right|_{L^2(\Sigma)}^2
\leq \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left| U^n_k - z_{\delta_k} \right|_{L^2(\Gamma)}^2 - \left| u(f) - z_{\delta} \right|_{L^2(\Gamma)}^2 dt
\leq C \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left| U^n_k - z_{\delta_k} - u(f) + z_{\delta} \right|_{L^2(\Gamma)}^2 dt
\leq C \left( \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left| U^n_k - u(f) \right|_{L^2(\Gamma)}^2 dt + \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left| z_{\delta_k} - z_{\delta} \right|_{L^2(\Gamma)}^2 dt \right)
\leq C \left( \left( \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \left| U^n_k - u(f) \right|_{L^2(\Gamma)}^2 dt \right)^{1/2} + \left| z_{\delta_k} - z_{\delta} \right|_{L^2(\Sigma)}^2 \right)
\rightarrow 0
$$

as $k \rightarrow \infty$, which finishes the proof. 

Next we introduce the notion of the $f^*$-minimum-norm solution of the identification problem.

**Lemma 4.3.** The problem

$$
\min_{f \in Z(\Sigma)} \| f - f^* \|_{L^2(\Omega_T)} \tag{IP}
$$

attains a unique solution, which is called the $f^*$-minimum-norm solution of the identification problem, where

$$
Z(\Sigma) := \left\{ f \in L^2(\Omega_T) \mid u(f)_{|\Sigma} = Z \right\}.
\tag{4.12}
$$

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Proof. Due to Lemma 2.5, \( I(Z) \neq \emptyset \) is a close subset of \( L^2(\Omega_T) \). Furthermore, it is a convex set. In fact, let \( f_1, f_2 \in I(Z) \) and \( c_1, c_2 \in [0,1] \) with \( c_1 + c_2 = 1 \), it follows from the relation (2.11) that

\[
u(c_1 f_1 + c_2 f_2) = \hat{\nu}(c_1 f_1 + c_2 f_2) + \bar{\nu} = c_1 \hat{\nu}(f_1) + c_2 \hat{\nu}(f_2) + \bar{\nu} = c_1 \nu(f_1) + c_2 \nu(f_2) = (c_1 Z + c_2 Z) = Z.
\]

Therefore, the minimization problem has a unique solution, which finishes the proof.

We now show the main result of this section on the convergence of finite dimensional approximations \( f_{p,\delta,h,\tau} \) of \( (P_{\rho,\delta,h,\tau}) \) to the \( f^* \)-minimizing-norm solution of the identification problem \( (IP) \). For any fixed \( f \in L^2(\Omega_T) \), let \( u(x,t) := u(f) \) and \( U_{h,\tau}^n(f) := U^n(f) \) define by 2.5 and 3.9, respectively. We recall the convergence of the Crank-Nicolson Galerkin method for linear parabolic problems

\[
\lim_{h,\tau \to 0} \omega_{h,\tau}(f) = 0 \quad \text{with} \quad \omega_{h,\tau}(f) := \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h,\tau}^n(f) - u(f) \|_{H^1(\Omega)}^2 \, dt.
\]

**Theorem 4.4.** Let \( f^+ \) be the unique \( f^* \)-minimum-norm solution of the identification problem \( (IP) \). Let \( (h_k^2) \sim (\tau_k) \), \( (\delta_k) \) and \( (\rho_k) \) be any positive sequences such that

\[
h_k \to 0 \quad \rho_k \to 0, \quad \frac{\delta_k^2}{\rho_k} \to 0, \quad \frac{\omega_{h_k,\tau_k}(f^+)}{\rho_k} \to 0
\]

as \( k \to \infty \). Furthermore, assume that \( (z_{\delta_k}) \subset L^2(\Sigma) \) is a sequence satisfying

\[
\| z_{\delta_k} - Z \|_{L^2(\Sigma)} \leq \delta_k
\]

and \( f_k \) denotes the unique minimizer of \( (P_{\rho_k,\delta_k,h_k,\tau_k}) \) for each \( k \in \mathbb{N} \). Then:

(i) The sequence \( (f_k) \) converges to \( f^+ \) in the \( L^2(\Omega_T) \)-norm.

(ii) The following equality holds true

\[
\lim_{k \to \infty} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h_k,\tau_k}^n(f_k) - u(f^+)(t) \|_{H^1(\Omega)}^2 \, dt = 0.
\]

Proof. For \( n = 1, \ldots, M_k \), we write \( U_{h_k,\tau_k}^n \) instead \( U_{h_k,\tau_k}^n(f_k) \) and \( U_{h_k,\tau_k}^n(f^+) \) for short, respectively. Since \( f_k \) is the solution of \( (P_{\rho_k,\delta_k,h_k,\tau_k}) \), we have

\[
\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h_k,\tau_k}^n - z_{\delta_k} \|_{L^2(\Gamma)}^2 dt + \rho_k \| f_k - f^+ \|_{L^2(\Omega_T)}^2
\leq \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h_k,\tau_k}^n - f^+ \|_{L^2(\Omega_T)}^2 dt - \rho_k \| f_k - f^+ \|_{L^2(\Omega_T)}^2.
\]

We get

\[
\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h_k,\tau_k}^n - z_{\delta_k} \|_{L^2(\Gamma)}^2 dt = \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h_k,\tau_k}^n - u(f^+) \|_{L^2(\gamma)}^2 dt + \rho_k \| f_k - f^+ \|_{L^2(\Omega_T)}^2
\leq 2 \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h_k,\tau_k}^n - u(f^+) \|_{L^2(\gamma)}^2 dt + \rho_k \| f_k - f^+ \|_{L^2(\Omega_T)}^2 dt
\leq 2 (\omega_{h_k,\tau_k}(f^+) + \delta_k^2).
\]

It follows from (4.15) and (4.16) that

\[
\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \| U_{h_k,\tau_k}^n - z_{\delta_k} \|_{L^2(\Gamma)}^2 dt + \rho_k \| f_k - f^+ \|_{L^2(\Omega_T)}^2 \leq 2 (\omega_{h_k,\tau_k}(f^+) + \delta_k^2) + \rho_k \| f_k - f^+ \|_{L^2(\Omega_T)}^2.
\]
Therefore, by (4.14), we have

\[
\lim_{k \to \infty} \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \| U_k^n - z_{\delta_k} \|^2_{L^2(\Gamma)} dt = 0 \tag{4.18}
\]

and

\[
\limsup_{k \to \infty} \| f_k - f^* \|_{L^2(\Omega_T)} \leq \| f^1 - f^* \|_{L^2(\Omega_T)}. \tag{4.19}
\]

Applying Lemma 4.2 we deduce from the boundedness of \( (f_k) \) due to (4.19) that there are a subsequence of it denoted by the same symbol and an element \( \hat{f} \in L^2(\Omega) \) such that

\[
f_k - f^* \rightharpoonup \hat{f} - f^* \quad \text{weakly in} \quad L^2(\Omega_T) \quad \text{and} \quad \| \hat{f} - f^* \|_{L^2(\Omega_T)} \leq \liminf_{k \to \infty} \| f_k - f^* \|_{L^2(\Omega_T)},
\]

\[
\lim_{k \to \infty} \sum_{n=1}^{M_k} \int_{t_{n-1}}^{t_n} \| U_k^n - z_{\delta_k} \|^2_{L^2(\Gamma)} dt = \| u(\hat{f}) - Z \|^2_{L^2(\Sigma)}. \tag{4.20}
\]

We thus obtain from (4.18) and (4.20) that

\[ u(\hat{f})_{|\Sigma} = Z \quad \text{or} \quad \hat{f} \in \mathcal{I}(Z). \]

Further, combining (4.19) with (4.20), we also obtain

\[
\| \hat{f} - f^* \|_{L^2(\Omega_T)} \leq \liminf_{k \to \infty} \| f_k - f^* \|_{L^2(\Omega_T)} \leq \limsup_{k \to \infty} \| f_k - f^* \|_{L^2(\Omega_T)} \leq \| f^1 - f^* \|_{L^2(\Omega_T)}
\]

and so, by the uniqueness of the \( f^* \)-minimum-norm solution \( f^1 \),

\[ \hat{f} = f^1 \quad \text{and} \quad \lim_{k \to \infty} \| f_k - f^1 \|_{L^2(\Omega_T)} = 0. \]

Next, by (3.9), for all \( \varphi_{h_k} \in \mathcal{V}_{h_k}^1 \) and \( n \in I \) we have

\[
(\partial U^n_k - \partial U^n_k, \varphi_{h_k})_{L^2(\Omega)} + a^n(\overline{D}U^n_k - \overline{D}U^n_k, \varphi_{h_k}) = (\overline{f}_k^n - f^1_k, \varphi_{h_k})_{L^2(\Omega)}, \tag{4.21}
\]

Denoting by \( e^n_k := U^n_k - U^n_{k+1} \) and taking \( \varphi_{h_k} = 2\tau_k^2\partial e^n_k \) in the above equation, we can estimate the left hand side of (4.21) by

\[
(\partial U^n_k - \partial U^n_k, \varphi_{h_k})_{L^2(\Omega)} + a^n(\overline{D}U^n_k - \overline{D}U^n_k, \varphi_{h_k})
= 2 \left( e^n_k - e^n_{k-1}, e^n_k - e^n_{k-1} \right)_{L^2(\Omega)} + \tau_k a^n \left( e^n_k, e^n_{k-1}, e^n_k - e^n_{k-1} \right)
\geq 2 \left\| e^n_k - e^n_{k-1} \right\|^2_{L^2(\Omega)} + C_1 \tau_k \left\| e^n_k \right\|^2_{H^1(\Omega)} - C_2 \tau_k \left\| e^n_{k-1} \right\|^2_{H^1(\Omega)} \tag{4.22}
\]

and the right hand side by

\[
(\overline{f}_k^n - f^1_k, \varphi_{h_k})_{L^2(\Omega)} = \int_\Omega \left( \frac{1}{\tau_k} \int_{t_{n-1}}^{t_n} (f_k(t) - f^1(t)) dt \right) (2\tau_k (e^n_k - e^n_{k-1})) dx
\]

\[
= 2 \int_\Omega \left( \int_{t_{n-1}}^{t_n} (f_k(t) - f^1(t)) dt \right) (e^n_k - e^n_{k-1}) dx
\]

\[
\leq 2 \left( \int_\Omega \left( \int_{t_{n-1}}^{t_n} (f_k(t) - f^1(t)) dt \right)^2 dx \right)^{1/2} \left( \int_\Omega (e^n_k - e^n_{k-1})^2 dx \right)^{1/2}
\]

\[
\leq \epsilon \left\| e^n_k - e^n_{k-1} \right\|^2_{L^2(\Omega)} + \frac{C \tau_k}{\epsilon} \int_\Omega \int_{t_{n-1}}^{t_n} (f_k(t) - f^1(t))^2 dt dx \tag{4.23}
\]

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for any $\epsilon > 0$. We then get from \[4.21\]-\[4.23\]
\[
2 \| e^n_k - e^{n-1}_k \|^2_{L^2(\Omega)} + C_1 \tau_k \| e^n_k \|^2_{H^1(\Omega)}
\leq \epsilon \| e^n_k - e^{n-1}_k \|^2_{L^2(\Omega)} + C \frac{\tau_k}{\epsilon} \int_0^T \int_{\Omega} (f_k(t) - f^*(t))^2 \, dt \, dx + C_2 \tau_k \| e^{n-1}_k \|^2_{H^1(\Omega)}
\]
that yields
\[
\| e^n_k \|^2_{H^1(\Omega)} \leq \int_0^T \int_{\Omega} C(f_k(t) - f^*(t))^2 \, dt \, dx + \sum_{m=0}^{n-1} \alpha_m \| e^m_k \|^2_{H^1(\Omega)},
\]
where $\alpha_0 = \ldots = \alpha_{n-2} = 0$ and $\alpha_{n-1} = C$. Since $e^n_k = 0$, applying Gronwall’s inequality, we obtain
\[
\| e^n_k \|^2_{H^1(\Omega)} \leq C \int_0^T \int_{\Omega} (f_k(t) - f^*(t))^2 \, dt \, dx + C \int_0^T \int_{\Omega} (f_k(t) - f^*(t))^2 \, dt \, dx
\]
and so that
\[
\sum_{k=1}^{M_k} \tau_k \| e^n_k \|^2_{H^1(\Omega)} \leq C \| f_k - f^* \|^2_{L^2(\Omega)} \to 0
\]
as $k \to \infty$. Therefore, we in view of \[4.13\] conclude that
\[
\sum_{n=1}^{M_k} \int_{t^n}^{t^{n-1}} \| U^n_k - u(f^*) \|^2_{H^1(\Omega)} \, dt \leq 2 \sum_{n=1}^{M_k} \int_{t^n}^{t^{n-1}} \| U^n_k - U^1_k \|^2_{H^1(\Omega)} \, dt + 2 \sum_{n=1}^{M_k} \int_{t^n}^{t^{n-1}} \| U^1_k - u(f^*) \|^2_{H^1(\Omega)} \, dt \to 0
\]
as $k \to \infty$, which finishes the proof.

5 Convergence rates

To obtain convergence rates, we assume that the sought source term $f^*$ and the data appearing in the system \[2.5\] are regular enough such that the following error bound of the Crank-Nicolson Galerkin method for linear parabolic problems is fulfilled, see, e.g., \[38\].

Lemma 5.1. Let $u(f^*)$ and $U^n(f^*)$ be the solutions of \[2.5\] and \[3.9\], respectively. Then, the estimate
\[
\| U^n(f^*) - u^n(f^*) \|^2_{H^1(\Omega)} \leq C h^{2-s} \left( \| q \|^2_{H^2(\Omega)} + \int_0^T \| u_t(f^*) \|^2_{H^2(\Omega)} \, dt \right) + C \tau^2 h^{-s} \int_0^T \left( \| u_{tt}(f^*) \|^2_{L^2(\Omega)} + \| \Delta u_{tt}(f^*) \|^2_{L^2(\Omega)} \right) \, dt
\]
holds true for all $n \in I$, where $s \in \{0,1\}$.

Theorem 5.2. Assume that $\tilde{f} \in I(Z)$ and there exists a function $w \in L^2(\Sigma)$ such that $\tilde{f} = F(w) + f^*$, where $F(w)$ is the unique weak solution of the parabolic system
\[
-\frac{\partial F(x,t)}{\partial t} + LF(x,t) = 0 \quad in \, \Omega_T,
\frac{\partial F(x,t)}{\partial \eta} + \sigma(x,t)F(x,t) = w \chi^\Sigma \quad on \, S,
\]
\[
F(x,T) = 0 \quad in \, \Omega.
\]
Then: (i) $\tilde{f} = f^*$, i.e. it is the unique $f^*$-minimum-norm solution of the identification problem (IP).

(ii) The estimate
\[
\| f - f^* \|^2_{L^2(\Omega^\Delta)} \leq C \left( h^3 \rho^{-1} + \tau^2 h^{-1} \rho^{-1} + \delta + \rho + \delta^2 \rho^{-1} \right)
\]
holds, where $f$ denotes the unique minimizer of \( P_{\rho,\delta,h,\tau} \).
Proof. (i) For all $\theta \in \mathcal{I}(\mathcal{Z})$ we rewrite
\[
\|\theta - f^*\|_{L^2(\Omega_T)}^2 - \|\tilde{f} - f^*\|_{L^2(\Omega_T)}^2 = \|\theta - \tilde{f}\|^2_{L^2(\Omega_T)} + 2(\tilde{f} - f^*, \theta - \tilde{f})_{L^2(\Omega_T)} \geq 2(\tilde{f} - f^*, \theta - \tilde{f})_{L^2(\Omega_T)}
\]
and so need to show that $(\tilde{f} - f^*, \theta - \tilde{f})_{L^2(\Omega_T)} = 0$. In fact, by (2.5), we have
\[
(\tilde{f} - f^*, \theta - \tilde{f})_{L^2(\Omega_T)} = \int_0^T \int_\Omega \theta F(w) dxdt + \int_0^T \int_\Omega gF(w) dxdt - \int_0^T \int_\Omega \tilde{f} F(w) dxdt - \int_0^T \int_\Omega gF(w) dxdt
\]
\[
= \int_0^T \left( \partial \left( u(\theta) - u(\tilde{f}) \right) \right)_{(H^1(\Omega), H^1(\Omega))} dt + \int_0^T a \left( u(\theta) - u(\tilde{f}), F(w) \right) dt.
\]
Since $F(x, T; w) = u(x, 0; \theta) - u(x, 0; \tilde{f}) = 0$, we thus have from (2.3) that
\[
(\tilde{f} - f^*, \theta - \tilde{f})_{L^2(\Omega_T)} = -\int_0^T \int_\Omega \partial \left( u(\theta) - u(\tilde{f}) \right)_{(H^1(\Omega), H^1(\Omega))} dt + \int_0^T a \left( F(w), u(\theta) - u(\tilde{f}) \right) dt
\]
\[
= \int_\Sigma w(u(\theta) - u(\tilde{f})) dxdt = 0,
\]
by $\tilde{f}, \theta \in \mathcal{I}(\mathcal{Z})$.

(ii) By the optimality of $f$, we get that
\[
\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f) - z_\delta\|^2_{L^2(\Gamma)} dt + \rho \|f - f^*\|^2_{L^2(\Omega_T)} \leq \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - z_\delta\|^2_{L^2(\Gamma)} dt + \rho \|f^1 - f^*\|^2_{L^2(\Omega_T)}
\]
with
\[
\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - z_\delta\|^2_{L^2(\Gamma)} dt = \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1) + Z - z_\delta\|^2_{L^2(\Gamma)} dt \leq C \left( \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1)\|^2_{L^2(\Gamma)} dt + \delta^2 \right).
\]
Furthermore, by (3.5), we get that
\[
\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1)\|^2_{L^2(\Gamma)} dt \\
\leq Ch^{-1} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1)\|^2_{L^2(\Omega)} dt + Ch \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1)\|^2_{H^1(\Omega)} dt.
\]
Due to (5.1), the first term in the right hand side of (5.4) is bounded by
\[
\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1)\|^2_{L^2(\Omega)} dt \\
\leq Ch^{-1} \left( \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1)\|^2_{L^2(\Omega)} dt + \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|u(f^1) - u(f^1)\|^2_{L^2(\Omega)} dt \right) \\
\leq Ch^{-1} \left( h^4 + \tau^4 + \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|u(f^1) - u(f^1)\|^2_{L^2(\Omega)} dt \right) \\
\leq Ch^{-1} \left( h^4 + \tau^4 + \tau^4 \|u(f^1)\|_{L^2(0,T;L^2(\Omega))} \right) \\
\leq C(h^3 + \tau^4 h^{-1} + \tau^2 h^{-1}).
\]
Likewise, we have that
\[ h \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|U^n(f^1) - u(f^1)\|_{H^1(\Omega)}^2 dt \leq C h^3 + \tau^2 h^{-2} + \tau^2 \|u_1(f^1)\|_{L^2(0,T;H^1(\Omega))} \]
\[ \leq C(h^3 + \tau^4 h^{-1} + \tau^2 h). \quad (5.6) \]

We deduce from (3.2)–(5.6) that
\[ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|U^n(f) - z_\delta\|_{L^2(\Gamma')} dt + \rho \|f - f^1\|_{L^2(\Omega_T)}^2 \leq C (h^3 + \tau^2 h^{-1} + \delta^2) + 2 \rho (f^1 - f^*, f^1 - f)_{L^2(\Omega_T)}. \quad (5.7) \]

Since \( f^1 - f^* = F(w) \), we have
\[ (f^1 - f^*, f^1 - f)_{L^2(\Omega_T)} \]
\[ = \int_\Sigma w(u(f^1) - u(f)) dx dt 
= \int_\Sigma w(u(f^1) - z_\delta) dx dt + \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} w(z_\delta - U^n(f)) dt \right) dx 
+ \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} w(U^n(f) - u(f)) dt \right) dx. \]

We bound for each term in the right hand side of the above equation. First, we get
\[ \int_\Sigma w(u(f^1) - z_\delta) dx dt \leq \|w\|_{L^2(\Sigma)} \|u(f^1) - z_\delta\|_{L^2(\Sigma)} \]
\[ \leq \|w\|_{L^2(\Sigma)} \delta \quad (5.9) \]
and
\[ \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} w(z_\delta - U^n(f)) dt \right) dx \]
\[ \leq \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} w^2 dt \right)^{1/2} \left( \int_{t_{n-1}}^{t_n} (z_\delta - U^n(f))^2 dt \right)^{1/2} dx 
\leq \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} w^2 dt \right)^{1/2} \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (z_\delta - U^n(f))^2 dt \right)^{1/2} dx 
\leq \left( \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} w^2 dt \right) dx \right)^{1/2} \left( \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (z_\delta - U^n(f))^2 dt \right) dx \right)^{1/2} 
\leq \rho \|w\|_{L^2(\Sigma)} + \frac{1}{4 \rho} \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|U^n(f) - z_\delta\|_{L^2(\Gamma')}^2 dt. \quad (5.10) \]

Further, as the above estimates (5.4)–(5.6), we get
\[ \int_\Gamma \left( \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} w(U^n(f) - u(f)) dt \right) dx \leq C (h^3 + \tau^2 h^{-1}) . \quad (5.11) \]

It follows from (5.8)–(5.11) that
\[ 2 \rho (f^1 - f^*, f^1 - f)_{L^2(\Omega_T)} \leq C (h^3 + \tau^2 h^{-1} + \rho \delta + \rho^2) + \frac{1}{2} \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|U^n(f) - z_\delta\|_{L^2(\Gamma')}^2 dt. \quad (5.12) \]

We therefore conclude from (5.7) and (5.12)
\[ \frac{1}{2} \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|U^n(f) - z_\delta\|_{L^2(\Gamma')}^2 \leq C (h^3 + \tau^2 h^{-1} + \rho \delta + \rho^2 + \delta^2) , \]
which completes the proof. \( \square \)
6 Numerical examples

In this section, we present two numerical examples. In the first model we work with data derived from smooth source functions, while in the second one the source function is discontinuous.

After fully discretizing the problem as described in Section 3, we need to minimize the cost function \( (P_{\rho,\delta,h,\tau}) \).

To this end, we use the conjugate gradient (CG) method for finite dimensional problems, e.g., [26]. The main point of this method is, starting with an initial guess \( f_0 \), to iteratively update the minimizer

\[
f_{k+1} = f_k + \alpha_k d_k, \quad k = 0, 1, \ldots
\]

where \( d_k \) is the update direction and \( \alpha_k \) is the update step size until it meets a stopping criterion of the form

\[
\|\nabla J_{\rho,\delta,h,\tau}(f_k)\| \leq \tau_a + \tau_r \|\nabla J_{\rho,\delta,h,\tau}(f_0)\|.
\]

To compute \( \alpha_k \), we consider the quadratic minimization problem

\[
\arg \min_{\alpha \geq 0} J_{\rho,\delta,h,\tau}(f_k + \alpha d_k).
\]

It is shown that the unique minimizer of \( (6.3) \) is given by

\[
\alpha_k = -\frac{\sum_{n=1}^{M} \int_{t_n-1}^{t_n} (U^n_0(d_k), U^n(f_k) - z_n)_{L^2(\Gamma)} dt + \rho(d_k, f_k - f^*)_{L^2(\Omega_T)}}{\sum_{n=1}^{M} \tau_{n} \|U^n_0(d_k)\|^2_{L^2(\Gamma)} + \rho \|d_k\|^2_{L^2(\Omega_T)}} (6.4)
\]

where for each \( n \in I \), \( U^n_0(d_k) \in V^k_1 \) is the unique solution of the system

\[
(\partial U^n(f), \varphi)_{L^2(\Omega)} + a^n(\partial U_0^n(f), \varphi) = (\overline{a} - L^2(\Omega) + \beta_k d_{k-1})
\]

while \( (U^n(f))_{n=1}^{M} \) is defined by the system \( (3.9) \). The update direction is computed as (cf. \( (3.25) \))

\[
d_k = \begin{cases} -\nabla J_{\rho,\delta,h,\tau}(f_k) & \text{if } k = 0, \\ -\nabla J_{\rho,\delta,h,\tau}(f_k) + \beta_k d_{k-1} & \text{if } k > 0. \end{cases} (6.5)
\]

The coefficient \( \beta_k \) in \( (6.5) \) is computed by using either the Fletcher-Reeves formula

\[
\beta_k = \frac{\|\nabla J_{\rho,\delta,h,\tau}(f_k)\|^2_{L^2(\Omega_T)}}{\|\nabla J_{\rho,\delta,h,\tau}(f_{k-1})\|^2_{L^2(\Omega_T)}} (6.6)
\]

or the Polak-Ribière formula

\[
\beta_k = \frac{(\nabla J_{\rho,\delta,h,\tau}(f_k), \nabla J_{\rho,\delta,h,\tau}(f_k) - \nabla J_{\rho,\delta,h,\tau}(f_{k-1}))_{L^2(\Omega_T)}}{\|\nabla J_{\rho,\delta,h,\tau}(f_{k-1})\|^2_{L^2(\Omega_T)}} (6.7)
\]

Practical steps can be summarized in Algorithm. 1

**Algorithm 1** CG method for minimizing the cost functional \( (P_{\rho,\delta,h,\tau}) \)

**Require:** \( f_0, \tau_a, \tau_r, k_{\text{max}} \)

**Ensure:** An approximate of the unique minimizer to \( (P_{\rho,\delta,h,\tau}) \)

1: Calculate \( \nabla J_{\rho,\delta,h,\tau}(f_0) \) and set \( d_0 = -\nabla J_{\rho,\delta,h,\tau}(f_0) \).
2: Compute \( \alpha_0 \) using \( (6.4) \).
3: Update \( f_1 = f_0 + \alpha_0 d_0 \).
4: Set \( k = 1 \) and compute \( \nabla J_{\rho,\delta,h,\tau}(f_k) \).
5: while \( k \leq k_{\text{max}} \) and \( (6.2) \) does not hold do
6: Compute \( \beta_k \) using \( (6.6) \) or \( (6.7) \).
7: Update the direction, using \( (6.5) \).
8: Compute \( \alpha_k \), using \( (6.4) \).
9: Update the minimizer, using \( (6.1) \).
10: Set \( k = k + 1 \) and compute \( \nabla J_{\rho,\delta,h,\tau}(f_k) \).
11: end while
6.1 Smooth source

We consider a two-dimensional case of (1.1) with
\[
A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = 1, \quad \sigma = 1
\] (6.8)
on the domain
\[
(x, y, t) \in \Omega_T = (-1, 1) \times (-1, 1) \times (0, 1].
\] (6.9)
We discretize \( \Omega \) using the FEM method with mesh size \( h = 0.03 \) and the time domain \([0, 1]\) by the equidistant grid with \( \tau = 0.02 \). With the analytic solution
\[
u(x, y, t) = \cos t(y - x^2),
\] one can easily determine the boundary condition \( g(x, y, t) \) and the source function
\[
f(x, y, t) = \cos t(6 + y - x^2) - \sin t(y - x^2).
\]
First, we denote by \( f_{\text{mean}} \) the mean value of \( f \) on \( \Omega_T \). For predicted source, we choose \( f^* \) as a large perturbation of \( f \) by
\[
f^* = f + 0.2(f - f_{\text{mean}}).
\]
The initial guess is chosen as the constant
\[
f_0 = f_{\text{mean}}.
\]
The observation is performed on the boundary part (Fig. 1)
\[
\Sigma = [-1, 1] \times \{-1\} \times [0, 1].
\]
Then, \( z_\delta \) is derived by randomly perturbing the true solution
\[
z_\delta = u(f)|_{\Sigma} + \delta \cdot z_{\text{random}}
\]to stand for the noise measurement. The constant \( \delta \) is chosen such that the measurement error level \( \|u(f) - z_\delta\|_{L^2(\Sigma)} \) is 1%, 0.1% and 0.01%. Other parameters are chosen as follows
\[
\rho = 5 \cdot 10^{-4}, \quad \tau_r = 5 \cdot 10^{-7}, \quad \tau_a = 5 \cdot 10^{-9}.
\] (6.10)

![Figure 1: Smooth source: location of observation and two test points](image-url)
To assess the quality of the recovered source function, first we consider it at two test points $P_1 (-0.4, -0.55)$ and $P_2 (0.6, 0.2)$, see Fig. 1 and plot its variation along the time in Fig. 2. We can see that both of cases give good approximation while the one at $P_1$ seems to be slightly better than that at $P_2$. Moreover, a close look shows that the recovered source function is less sensitive to measurement error at $P_2$ than at $P_1$ which may be logical since $P_2$ is further from the observation than $P_1$. In Fig. 3, we compare the recovered source function from different measurements with the true source function at time $t = 0.5$. It is quite realizable that with less measurement error, the recovered source looks more like the true source.
6.2 Discontinuous source

In this test, we use the same setting as (6.8) and (6.9) and the same discretization. Instead of specifying the exact solution a priori, we are given the right hand side, the initial and boundary conditions and the corresponding solution is numerically computed using the MATLAB. We will refer to this as true solution. The noise measurement $z_\delta$ is generated in same way as in previous example but with the numerical solution.

To this end, let the right side be given as

$$f(x,y,t) = \begin{cases} 
-1, & \text{if } (x, y, t) \in [-1, 0] \times [-1, 1] \times [0, 0.5], \\
1, & \text{if } (x, y, t) \in [0, 1] \times [-1, 1] \times [0, 0.5], \\
0, & \text{otherwise}.
\end{cases}$$

The initial and boundary conditions of the problem are constant functions defined by $u(\cdot, \cdot, 0) = 0.4, g = 0.4$. The predicted source $f^*$ is chosen as the constant 0.2 while $f_0$ is taken by 0. The regularization parameter $\rho$ and the constants $(\tau_r, \tau_a)$ in (6.2) are chosen as

$$\rho = 5 \cdot 10^{-4}, \quad \tau_r = 10^{-4}, \quad \tau_a = 10^{-6}.$$ 

We would like to note that due to the discontinuity of the right hand side, the state changes rapidly near
discontinuous area. Practically, it is more difficult than the previous model. Therefore, we perform the observation on the whole boundary $\Sigma = \partial \Omega \times [0, T]$.

To generate the noise data, we simulate the forward problem with the aforementioned data but on a finer grid with $h_{\text{fine}} = 0.02$ to prevent the so-called inverse-crime phenomenon. The derived solution is then interpolated on the grid $h = 0.03$ and perturbed in the same way as in the previous example. In Fig. 4, we plot the true and the recovered sources at $t = 0.4$. Observation shows that the proposed scheme can somewhat recover the behavior of the true source with various measurement error levels.

![Figure 4: Discontinuous source: comparison at time $t = 0.4$ of the true source (a) and recovered source from measurements with errors 0.01% (b), 0.1% (c), and 1% (d).](image)

7 Appendix

In this section we present briefly the problem of identifying the function $f(x, t)$ in the system

$$\begin{align*}
\frac{\partial u}{\partial t}(x, t) + L u(x, t) &= F_1(x, t)f(x, t) + F_2(x, t) \quad \text{in } \Omega_T := \Omega \times (0, T], \\
\frac{\partial u(x, t)}{\partial n} + \sigma(x, t)u(x, t) &= g(x, t) \quad \text{on } S := \partial \Omega \times (0, T], \\
u(x, 0) &= q(x) \quad \text{in } \Omega
\end{align*}$$

(7.1)
The discrete cost functional is now given by

\[ W \in \mathcal{W}(f) \] for all \( f \in L^2(\Omega_T) \), where for all \( u \in \mathcal{U}(\xi) \)

Theorem 7.1. We conclude this section by stating the following result.

Let \( u : L^2(\Omega_T) \to \mathcal{W}(0, T) \) with \( f \to u(f) \) be the source-to-state operator, that maps each \( f \in L^2(\Omega_T) \) to the unique weak solution \( u \) of the problem [7.1]. This operator is Fréchet differentiable on \( L^2(\Omega_T) \). For each \( f \in L^2(\Omega_T) \) the Fréchet differential \( u'(f) \xi \) in the direction \( \xi \in L^2(\Omega_T) \) is the unique weak solution in \( \mathcal{W}(0, T) \) to the problem

\[
\frac{\partial \hat{u}}{\partial t}(x, t) + \mathcal{L}\hat{u}(x, t) = F_1(x, t)\xi(x, t) \quad \text{in} \quad \Omega_T,
\]

\[
\frac{\partial \hat{u}}{\partial n}(x, t) + \sigma(x, t)\hat{u}(x, t) = 0 \quad \text{on} \quad \mathcal{S},
\]

\( \hat{u}(x, 0) = 0 \) in \( \Omega \).

Further, there holds the relation

\[ u(f) = \hat{u}(f) + \hat{u} \]

for all \( f \in L^2(\Omega_T) \), where \( \hat{u} \) is independent of \( f \) and defined as the unique weak solution in \( \mathcal{W}(0, T) \) to the problem

\[
\frac{\partial \hat{u}}{\partial t}(x, t) + \mathcal{L}\hat{u}(x, t) = F_2(x, t) \quad \text{in} \quad \Omega_T,
\]

\[
\frac{\partial \hat{u}}{\partial n}(x, t) + \sigma(x, t)\hat{u}(x, t) = g(x, t) \quad \text{on} \quad \mathcal{S},
\]

\( \hat{u}(x, 0) = 0 \) in \( \Omega \).

Then the Crank-Nicolson Galerkin method applied to [7.1] reads: Find \( U^n(f) := U^n_{\rho, \delta, h, \tau}(f) \in \mathcal{V}_h^1 \) such that

\[
(\partial U^n(f), \varphi_h)_{L^2(\Omega)} + a^n(\partial U^n(f), \varphi_h) = (F^n_1, F^n_2, \varphi_h)_{L^2(\Omega)} + (g^n, \varphi_h)_{L^2(\mathcal{S})}, \quad \forall \varphi_h \in \mathcal{V}_h^1, \quad n \in I
\]

\( U^0(f) = q_h \).

The discrete cost functional is now given by

\[
\min_{f \in L^2(\Omega_T)} J_{\rho, \delta, h, \tau}(f) \quad \text{with} \quad J_{\rho, \delta, h, \tau}(f) := \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \| U^n(f) - z_\delta \|^2_{L^2(\Omega_T)} dt + \rho \| f - f^* \|^2_{L^2(\Omega_T)}. \quad (P_{\rho, \delta, h, \tau})
\]

The mapping \( u_{h, \tau} : L^2(\Omega_T) \to (\mathcal{V}_h^1)^M \) defined for each \( f \in L^2(\Omega_T) \) by

\[
U^n_{\rho, \delta, h, \tau}(f) := (U^n_1(f), \ldots, U^n_M(f)) \in (\mathcal{V}_h^1)^M
\]

is called the discrete source-to-state operator. For all \( f, \xi \in L^2(\Omega_T) \) we have that

\[
U^n_{\rho, \delta, h, \tau}(\xi) = \left( \tilde{U}_1^n(\xi), \ldots, \tilde{U}_M^n(\xi) \right),
\]

where \( \tilde{U}_n^n(\xi) \) is defined by

\[
\left( \partial \tilde{U}_n^n(\xi), \varphi_h \right)_{L^2(\Omega)} + a^n(\partial \tilde{U}_n^n(\xi), \varphi_h) = (F^n_1\xi^n, \varphi_h)_{L^2(\Omega)}, \quad \forall \varphi_h \in \mathcal{V}_h^1, \quad n \in I
\]

\( \tilde{U}_0^n(\xi) = 0 \).

We conclude this section by stating the following result.

Theorem 7.1. (i) The problem \((P_{\rho, \delta, h, \tau})\) attains a unique solution \( f := f_{\rho, \delta, h, \tau} \) satisfying

\[
f_{[\Omega \times [t_{n-1}, t_n]]} = f^* - \frac{1}{\rho} P^{n-1}(f),
\]

where for all \( n = 1, \ldots, M \)

\[
-\tau (\partial P^n(f), \varphi_h)_{L^2(\Omega)} + \tau a^n(\partial P^n(f), \varphi_h) = \int_{t_{n-1}}^{t_n} (U^n(f) - z_\delta, \varphi_h)_{L^2(\Omega)} dt, \quad \forall \varphi_h \in \mathcal{V}_h^1.
\]

\( P^M(f) = 0 \).
(ii) The $L^2$-gradient of the cost functional $J_{\rho,\delta,h,\tau}$ of the problem $P_{\rho,\delta,h,\tau}$ at $f$ is given by

$$\nabla J_{\rho,\delta,h,\tau}(f)_{\Omega \times [t_n-1,t_n]} = 2P^{n-1}(f) + 2\rho(f-f^*)_{\Omega \times [t_n-1,t_n)}$$

for all $n = 1, \ldots, M$.

Furthermore, we note that the results on convergence and convergence rates stated in Section 4 and Section 5, respectively, are still valid for the identification problem in this section.

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