Recurrence and Ergodicity of Switching Diffusions with Past-Dependent Switching Having A Countable State Space

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Abstract

This work focuses on recurrence and ergodicity of switching diffusions consisting of continuous and discrete components, in which the discrete component takes values in a countably infinite set and the rates of switching at current time depend on the value of the continuous component over an interval including certain past history. Sufficient conditions for recurrence and ergodicity are given. Moreover, the relationship between systems of partial differential equations and recurrence when the switching is past-independent is established under suitable conditions.

Keywords. Switching diffusion, past-dependent switching, recurrence, ergodicity.

Subject Classification. 60H10, 60J60, 60J75, 37A50.

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1 Introduction

Emerging and existing applications in wireless communications, queueing networks, biological models, ecological systems, financial engineering, and social networks demand the mathematical modeling, analysis, and computation of hybrid systems in which continuous dynamics and discrete events coexist. Switching diffusions are one of such hybrid models. A switching diffusion is a two-component process \((X(t), \alpha(t))\), a continuous component and a discrete component taking values in a set consisting of isolated points. When the discrete component takes a value \(\alpha\) (i.e., \(\alpha(t) = \alpha\)), the continuous component \(X(t)\) evolves according to the diffusion process whose drift and diffusion coefficients depend on \(\alpha\). Such processes have received growing attention recently because of their ability to a wide range of applications; see [11, 13, 20, 26] and the references therein.

In the comprehensive treatment of hybrid switching diffusions in [14], it was assumed that \(\alpha(t)\) is a continuous-time and homogeneous Markov chain independent of the Brownian motion and that the generator of the Markov chain is a constant matrix. For broader impact on applications, considering the two components jointly, the work [25] extended the study to the Markov process \((X(t), \alpha(t))\) by allowing the generator of \(\alpha(t)\) to depend on the current state \(X(t)\). Until very recently, most of the works treat \(\alpha(t)\) as a process taking values in a finite set. Even when \(\alpha(t)\) is allowed to take values in a countable state space, almost all works required the systems being memoryless. That is, the switching depends on the continuous state, with the dependence on the current continuous state only, no delays are involved; see, for example, [14, 19, 20, 25] and references therein. To be able to treat more complex models and to broaden the applicability, we have undertaken the task of investigating the dynamics of \((X(t), \alpha(t))\) in which \(\alpha(t)\) has a countable state space and its switching intensities depend on the history of the continuous component \(X(t)\). As a first attempt, this type of switching diffusion was considered in [16], which was motivated by queueing and control systems applications. In particular, the evolution of two interacting species was considered in the aforementioned reference. One of the species is micro described by a logistic differential equation perturbed by a white noise, and the other is macro. Let \(X(t)\) be the density of the micro species and \(\alpha(t)\) the population of the macro species. The reproduction process of \(\alpha(t)\) is non-instantaneous, resulting in past-dependent switching. In [16], we gave precise formulation of the process \((X(t), \alpha(t))\) and established the existence and uniqueness of solutions together with such properties as Markov-Feller property and Feller property of function-valued stochastic processes associated with our processes under suitable conditions.

Many real-world systems are in operation for a long period of time. Similar to their diffusion counterpart, longtime behavior of switching diffusion systems is very important. When the switching is independent of past, a number of results have been obtained in [11, 19, 20, 21, 26]. Nevertheless, when past-dependent switching is considered, not much of the desired asymptotic properties are known to the best of our knowledge. Motivated by the practical needs, this paper studies recurrence and ergodicity of switching diffusion processes with past-dependent switching having a countable state space. Such systems are more difficult to handle. To begin, the pair \((X(t), \alpha(t))\) is no longer a Markov process because of past-dependence of the switching process. Most of the arguments based on Markov property cannot be used for the aforementioned pair, so different approaches have to be taken.
The next idea is to treat the process \((X_t, \alpha(t))\), where \(X_t\) is the so-called segment process associated with \(X(t)\). For such processes, although we have Markov properties, the systems that we face become infinite dimensional. As will be seen in the subsequent sections, the problems require much more attention and careful consideration.

The rest of the paper is organized as follows. The formulation of switching diffusions with past-dependent switching and countably many possible switching locations is given in Section 2. Also some relevant results, including the functional Itô formula, are recalled briefly. These results play a crucial role. Section 3 provides certain sufficient conditions for recurrence, positive recurrence, and ergodicity of the related Markov process associated with our switching diffusions. In Section 4 we characterize the recurrence of switching diffusion that are past independent. Furthermore, the relationship between recurrence and the associated systems of partial differential equations is established. Finally, we provide the proofs of some technical results in an appendix.

2 Formulation

Denote by \(C([a, b], \mathbb{R}^n)\) the set of \(\mathbb{R}^n\)-valued continuous functions defined on \([a, b]\). Let \(r\) be a fixed positive number. In what follows, we mainly work with \(C([-r, 0], \mathbb{R}^n)\), and simply denote \(C := C([-r, 0], \mathbb{R}^n)\). For \(\phi \in C\), we use the norm \(\|\phi\| = \sup\{|\phi(t)| : t \in [-r, 0]\}\).

For \(t \geq 0\), we denote by \(y_t\) the so-called segment function (or memory segment function) \(y_t = \{y(t+s) : -r \leq s \leq 0\}\). For \(x \in \mathbb{R}^n\), denote by \(|x|\) the Euclidean norm of \(x\). Let \(\mathfrak{B}(C)\) and \(\mathfrak{B}(C \times \mathbb{Z}_+)\) be the \(\sigma\)-algebras on \(C\) and \(C \times \mathbb{Z}_+\) respectively. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space with the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual condition, i.e., it is increasing and right continuous while \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets. Let \(W(t)\) be an \(\mathcal{F}_t\)-adapted and \(\mathbb{R}^d\)-valued Brownian motion. Suppose \(b(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^n\) and \(\sigma(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}^{n \times d}\). Consider the two-component process \((X(t), \alpha(t))\) where \(\alpha(t)\) is a pure jump process taking value in \(\mathbb{Z}_+ = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}\), the set of positive integers, and \(X(t)\) satisfies

\[
\frac{dX(t)}{dt} = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dW(t).
\]

We assume that the switching intensity of \(\alpha(t)\) depends on the trajectory of \(X(t)\) in the interval \([t-r, t]\), that is, there are functions \(q_{ij}(\cdot) : C \to \mathbb{R}\) for \(i, j \in \mathbb{Z}_+\) satisfying \(q_{ij}(\phi) \geq 0\) \(\forall i \neq j\) and \(\sum_{j=1, j \neq i}^{\infty} q_{ij}(\phi) = q_i(\phi)\) for all \(\phi \in C\), which means that \(\alpha(t)\) is conservative. If \(q_i(\phi)\) is uniformly bounded in \((\phi, i) \in C \times \mathbb{Z}_+\), and \(q_i(\cdot)\) and \(q_{ij}(\cdot)\) are continuous, then we have the following interpretation

\[
\begin{align*}
\mathbb{P}\{\alpha(t + \Delta) = j | \alpha(t) = i, X_s, \alpha(s), s \leq t\} &= q_{ij}(X_t)\Delta + o(\Delta) \text{ if } i \neq j \text{ and } \\
\mathbb{P}\{\alpha(t + \Delta) = i | \alpha(t) = i, X_s, \alpha(s), s \leq t\} &= 1 - q_i(X_t)\Delta + o(\Delta).
\end{align*}
\]

[Note that in (2.2), in contrast to [25], in lieu of \(X(t)\), \(X_i\) is used.] For more general \(q_i(\cdot)\) and \(q_{ij}(\cdot)\), the process \(\alpha(t)\) can be defined rigorously as the solution to a stochastic differential equation with respect to a Poisson random measure. For each function \(\phi : [-r, 0] \to \mathbb{R}^n\), and \(i \in \mathbb{Z}_+\), let \(\Delta_{ij}(\phi), j \neq i\) be the consecutive left-closed, right-open intervals of the real
line, each having length \( q_{ij}(\phi) \). That is

\[
\Delta_{i1}(\phi) = [0, q_{11}(\phi)), \\
\Delta_{ij}(\phi) = \left[ \sum_{k=1, k \neq i}^{j-1} q_{ik}(\phi), \sum_{k=1, k \neq i}^{j} q_{ik}(\phi) \right], \quad j > 1 \text{ and } j \neq i.
\]

Define \( h : \mathcal{C} \times \mathbb{Z}_+ \times \mathbb{R} \mapsto \mathbb{R} \) by \( h(\phi, i, z) = \sum_{j=1, j \neq i}^{\infty} (j-i)1_{\{z \in \Delta_{ij}(\phi)\}} \). The process \( \alpha(t) \) can be defined as the solution to

\[
d\alpha(t) = \int_{\mathbb{R}} h(X_t, \alpha(t^-), z)p(dt, dz),
\]

where \( a(t^-) = \lim_{s \to t^-} \alpha(s) \) and \( p(dt, dz) \) is a Poisson random measure with intensity \( dt \times m(dz) \) and \( m \) is the Lebesgue measure on \( \mathbb{R} \) such that \( p(dt, dz) \) is independent of the Brownian motion \( W(\cdot) \). The pair \((X(t), \alpha(t))\) is therefore a solution to

\[
\begin{aligned}
&dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dW(t) \\
&d\alpha(t) = \int_{\mathbb{R}} h(X_t, \alpha(t^-), z)p(dt, dz). 
\end{aligned} \tag{2.3}
\]

With initial data \((\xi, i_0)\) being a \( \mathcal{C} \times \mathbb{Z}_+ \)-valued and \( \mathcal{F}_0 \)-measurable random variable, a strong solution to \((2.3)\) on \([0, T]\), is an \( \mathcal{F}_t \)-adapted process \((X(t), \alpha(t))\) such that

- \( X(t) \) is continuous and \( \alpha(t) \) is cadlag (right continuous with left limits) with probability 1 w.p.1.
- \( X(t) = \xi(t) \) for \( t \in [-r, 0] \) and \( \alpha(0) = i_0 \)
- \((X(t), \alpha(t))\) satisfies \((2.3)\) for all \( t \in [0, T] \) w.p.1.

To ensure the existence and uniqueness of solutions, we assume that one of the following two assumptions holds through this paper (see [16]).

**Assumption 2.1.** The following conditions hold.

(i) For each \( H > 0 \), \( i \in \mathbb{Z}_+ \), there is a positive constant \( L_{H,i} \) such that

\[
|b(x, i) - b(y, i)| + |\sigma(x, i) - \sigma(y, i)| \leq L_{H,i}|x - y|, \quad \forall |x|, |y| \leq H, i \in \mathbb{Z}_+.
\]

(ii) For each \( i \in \mathbb{Z}_+ \), there exists a twice continuously differentiable function \( V_i(x) \) and a constant \( C_i > 0 \) such that

\[
\lim_{R \to \infty} \left( \inf_{|x| \geq R} \{V_i(x) : |x| \geq R\} \right) = \infty \quad \text{and} \quad \mathcal{L}_i V_i(x) \leq C_i (1 + V_i(x)) \forall x \in \mathbb{R}^n,
\]

where \( \mathcal{L}_i \) is the generator of the diffusion at the fixed state \( i \), that is,

\[
\mathcal{L}_i V(x) = \nabla V(x)b(x, i) + \frac{1}{2} \text{tr} \left( \nabla^2 V(x)A(x, i) \right) \tag{2.4}
\]

for \( V(\cdot) \in C^2(\mathbb{R}^n) \), where \( A(x, i) = \sigma(x, i)\sigma^T(x, i) \).
(iii) \( q_{ij}(\phi) \) and \( q_i(\phi) \) are continuous in \( \phi \in \mathcal{C} \) for each \( i, j \in \mathbb{Z}_+ \), \( i \neq j \). Moreover,
\[
M := \sup_{\phi \in \mathcal{C}, i \in \mathbb{Z}_+} \{ q_i(\phi) \} < \infty.
\]

**Assumption 2.2.** The following conditions hold.

(i) For each \( H > 0, i \in \mathbb{Z}_+ \), there is a positive constant \( L_{H,i} \) such that
\[
|b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \leq L_{H,i}|x - y|, \quad \forall |x|, |y| \leq H, \ i \in \mathbb{Z}_+.
\]

(ii) There exists a twice continuously differentiable function \( V(x) \) and a constant \( C > 0 \) independent of \( i \in \mathbb{Z}_+ \) such that
\[
\lim_{R \to \infty} \left( \inf \{ V(x) : |x| \geq R \} \right) = \infty \quad \text{and} \quad \mathcal{L}_i V(x) \leq C(1 + V(x)), \ \forall x \in \mathbb{R}^n, \ i \in \mathbb{Z}_+.
\]

(iii) \( q_{ij}(\phi) \) and \( q_i(\phi) \) are continuous in \( \phi \in \mathcal{C} \) for each \( i, j \in \mathbb{Z}_+, \ i \neq j \). Moreover, for any \( H > 0, \)
\[
M_H := \sup_{\phi \in \mathcal{C}, \|\phi\| \leq H, i \in \mathbb{Z}_+} \{ q_i(\phi) \} < \infty.
\]

It is proved in \([16]\) that under either of the above two assumptions, the process \((X_t, \alpha(t))\) satisfying (2.3) is a Markov-Feller process. Now we state the functional Itô formula for our process (see \([5]\) for more details). Let \( \mathbb{D} \) be the space of cadlag functions \( f : [-r, 0) \to \mathbb{R}^n \). For \( \phi \in \mathbb{D} \), we define horizontal and vertical perturbations for \( h \geq 0 \) and \( y \in \mathbb{R}^n \) as
\[
\phi_h(s) = \begin{cases} 
\phi(s + h) & \text{if } s \in [-r, -h], \\
\phi(0) & \text{if } s \in [-h, -0], 
\end{cases}
\]
and
\[
\phi^y(s) = \begin{cases} 
\phi(s) & \text{if } s \in [-r, 0), \\
\phi(0) + y, & \text{if } s \in [-h, -0], 
\end{cases}
\]
respectively. Let \( V : \mathbb{D} \times \mathbb{Z}_+ \to \mathbb{R} \). The horizontal derivative at \((\phi, i)\) and vertical partial derivative of \( V \) are defined as
\[
V_i(\phi, i) = \lim_{h \to 0} \frac{V(\phi_h, i) - V(\phi)}{h} \tag{2.5}
\]
and
\[
\partial_i V(\phi, i) = \lim_{h \to 0} \frac{V(\phi^h e_i, i) - V(\phi)}{h} \tag{2.6}
\]
if these limits exist. In \([2,6]\), \( e_i \) is the standard unit vector in \( \mathbb{R}^n \) whose \( i \)-th component is 1 and other components are 0. Let \( \mathbb{F} \) be the family of function \( V(\cdot, \cdot) : \mathbb{D} \times \mathbb{Z}_+ \to \mathbb{R} \) satisfying that
\begin{itemize}
  \item \( V \) is continuous, that is, for any \( \varepsilon > 0, (\phi, i) \in \mathbb{D} \times \mathbb{Z}_+ \), there is a \( \delta > 0 \) such that \( |V(\phi, i) - V(\phi', i)| < \varepsilon \) as long as \( \|\phi - \phi'\| < \delta \).
\end{itemize}
• The derivatives \( V_t, V_x = (\partial_k V), \) and \( V_{xx} = (\partial_{kl} V) \) exist and are continuous.

• \( V_t, V_x = (\partial_k V) \) and \( V_{xx} = (\partial_{kl} V) \) are bounded in each \( B_R := \{(\phi, i) : \|\phi\| \leq R, i \leq R\}, R > 0. \)

Remark 2.1. Recently, a functional Itô formula was developed in [6], which encouraged subsequent development (for example, [5, 18]). We briefly recall the main idea in what follows. Consider functions of the form

\[
V(\phi, i) = f_1(\phi(0), i) + \int_{-r}^0 g(t, i) f_2(\phi(t), i) \, dt,
\]

where \( f_2(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \rightarrow \mathbb{R} \) is a continuous function and \( f_1(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \rightarrow \mathbb{R} \) is a function that is twice continuously differentiable in the first variable and \( g(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{R} \) be a continuously differentiable function in the first variable. Then at \( (\phi, i) \in \mathcal{C} \times \mathbb{Z}_+ \) we have (see [18] for the detailed computations)

\[
V_t(\phi, i) = g(0, i) f_2(\phi(0), i) - g(-r, i) f_2(\phi(-r), i) - \int_{-r}^0 f_2(\phi(t), i) \, dg(t, i),
\]

\[
\partial_k V(\phi, i) = \frac{\partial f_1}{\partial x_k}(\phi(0), i), \quad \partial_{kl} V(\phi, i) = \frac{\partial^2 f_1}{\partial x_k \partial x_l}(\phi(0), i).
\]

Let \( V(\cdot, \cdot) \in \mathcal{F} \), we define the operator

\[
\mathcal{L}V(\phi, i) = V_t(\phi, i) + V_x(\phi, i) b(\phi(0), i) + \frac{1}{2} \text{tr} \left( V_{xx}(\phi, i) A(\phi(0), i) \right) + \sum_{j=1, j \neq i}^{\infty} q_{ij}(\phi) [V(\phi, j) - V(\phi, i)]
\]

\[
= V_t(\phi, i) + \sum_{k=1}^{n} b_k(\phi(0), i) V_k(\phi, i) + \frac{1}{2} \sum_{k,l=1}^{n} a_{kl}(\phi(0), i) V_{kl}(\phi, i)
\]

\[
+ \sum_{j=1, j \neq i}^{\infty} q_{ij}(\phi) [V(\phi, j) - V(\phi, i)]
\]

for any bounded stopping time \( \tau_1 \leq \tau_2 \), we have the functional Itô formula:

\[
\mathbb{E}V(X_{\tau_2}, \alpha(\tau_2)) = \mathbb{E}V(X_{\tau_1}, \alpha(\tau_1)) + \mathbb{E} \int_{\tau_1}^{\tau_2} \mathcal{L}V(X_s, \alpha(s)) \, ds \tag{2.8}
\]

if the expectations involved exist. Equation (2.8) is obtained by applying the functional Itô formula for general semimartingales given in [4, 5] specialized to our processes.

3 Recurrence and Ergodicity

First, we need some conditions for irreducibility of the process \( \{(X_t, \alpha(t)) : t \geq 0\} \).
(H1) (a) For any $i \in \mathbb{Z}_+$, $A(x, i)$ is elliptic uniformly on each compact set, that is, for any $R > 0$, there is a $\theta_{R,i} > 0$ such that
\[ y^\top A(x, i)y \geq \theta_{R,i}|y|^2 \quad \forall |x| \leq R, \; y \in \mathbb{R}^d. \tag{3.1} \]

(b) There is an $i^*$ satisfying that for any $i \in \mathbb{Z}_+$, there exist $i_1, \ldots, i_k \in \mathbb{Z}_+$ and $\phi_1, \ldots, \phi_{k+1} \in C$ such that $q_{i_1,i}(\phi_1) > 0$, $q_{i_l,i_{l+1}}(\phi_{l+1}) > 0, l = 1, \ldots, k - 1$, and $q_{i_1,i^*(\phi(k+1))} > 0$.

(H2) There exists an $i^* \in \mathbb{Z}_+$ such that

(a) $A(x, i^*)$ is elliptic uniformly on each compact set;

(b) for any $(\phi, i) \in C \times \mathbb{Z}_+$, there exist positive integers $i = i_1, \ldots, i_k = i^*$ satisfying $\nu_{i_1,i}(\phi) > 0, l = 1, \ldots, k - 1$.

Let $(X^{\phi,i}, \alpha^{\phi,i}(t))$ be the solution to (2.3) with initial data $(\phi, i) \in C \times \mathbb{Z}_+$. To simplify the notation, we denote by $P_{\phi,i}$ the probability measure conditioned on the initial data $(\phi, i)$, that is, for any $t > 0$,
\[ P_{\phi,i}\{(X_t, \alpha(t)) \in \cdot\} = P\{(X_t^{\phi,i}, \alpha^{\phi,i}(t)) \in \cdot\}, \]
and $E_{\phi,i}$ the expectation associated with $P_{\phi,i}$. To proceed, we state some auxiliary lemmas.

**Lemma 3.1.** Let $\phi \in C$ and $q_{ij}(\phi) > 0$. For any $\varepsilon > 0$, there is a $\delta > 0$ such that
\[ \inf_{\psi \in C:||\psi - \phi|| < \delta} P_{\psi,i}\{|X_\delta - \phi| < \varepsilon, \alpha(\delta) = j\} > 0. \]

**Lemma 3.2.** For any $i > 0$, $R > 0$ and $\varepsilon > 0$, there is a compact set $A \in C$ such that
\[ \inf_{||\phi|| \leq R} P_{\phi,i}\{X_r \in A, \alpha(r) = i\} > 0. \tag{3.2} \]
Moreover, if (3.1) holds for $i$, then for any $k > 0$, there is a $T = T(k, i, R) > 0$ such that
\[ \inf_{||\phi|| \leq R} P_{\phi,i}\{|X_t| > k \text{ for some } t \in [0, T]\} > 0. \tag{3.3} \]

**Lemma 3.3.** Assume that either (H1) or (H2) is satisfied. There is a nontrivial measure $\nu(\cdot)$ on $B(C)$ such that $\nu(D) > 0$ if $D$ is a nonempty open subset of $C$ and that for any $R > 0, T > r$, there is a $d_{R,T} > 0$ satisfying
\[ P_{\phi,i^*}\{X_T \in B \text{ and } \alpha(T) = i^*\} \geq d_{R,T} \nu(B), B \in B(C) \text{ given that } ||\phi|| \leq R. \tag{3.4} \]

The three lemmas above will be proved in the appendix.

**Lemma 3.4.** Assume that either (H1) or (H2) holds. For any $i \in \mathbb{Z}_+$, there is a $T_i > 0$ such that for any $T > T_i$ and any open set $B \subset C$, we have
\[ P_{\phi,i}\{X_T \in B, \alpha(T) = i^*\} > 0, \phi \in C \]
where $i^*$ is as in (H1) or (H2) accordingly.
Proof. Suppose that (H1) holds with \( i = i_1, \ldots, i_k = i^* \in \mathbb{Z}_+ \) and \( \phi_1, \ldots, \phi_{k+1} \in \mathcal{C} \) such that \( q_{i_1,i_{l+1}}(\phi_{l+1}) > 0, l = 1, \ldots, k - 1 \). Since \( q_{i_1,i_{l+1}}(\phi_{l+1}) > 0 \), it follows from Lemma 3.1 that

\[
P_{\psi,i} \{ \|X_{\varepsilon_l} - \phi_{l+1}\| < 1, \alpha(\varepsilon_l) = i_{l+1} \} > 0 \quad \text{if} \quad \|\psi - \phi_{l+1}\| < \varepsilon_l
\]  

for some \( \varepsilon_l \in (0, 1) \). In view of Lemma 3.3,

\[
P_{\psi,i} \{ \|X_{1+r} - \phi_{l+1}\| < \varepsilon_l, \alpha(1 + r) = i_l \} > 0 \quad \text{for any} \quad \psi \in \mathcal{C},
\]  

and

\[
P_{\psi,i} \{ \|X_{1+r+T'} - \phi_{l+1}\| < \varepsilon_l, \alpha(1 + r + T') = i^* \} > 0 \quad \text{for any} \quad \psi \in \mathcal{C}, T' \geq 0.
\]

By (3.5), (3.6), and the Markov property of \((X_t, \alpha(t))\), we have

\[
P_{\psi,i} \{ \|X_{1+r+\varepsilon_l} - \phi_l\| < 1, \alpha(1 + r + \varepsilon_l) = i_{l+1} \} > 0 \quad \text{for any} \quad \psi \in \mathcal{C}.
\]

Using (3.7), (3.8), and applying the Kolmogorov-Chapman equation again, we obtain

\[
P_{\psi,i} \left\{ X_{k(1+r)+\sum \varepsilon_l+T'} \in \mathcal{B}, \alpha \left( k(1 + r) + \sum \varepsilon_l + T' \right) = i^* \right\} > 0 \quad \text{for any} \quad \psi \in \mathcal{C}.
\]

The lemma is proved with \( T_i = k(2 + r) \geq k(1 + r) + \sum \varepsilon_l \).

Now, suppose that (H2) holds, it follows from Lemma 3.1 and the Kolmogorov-Chapman equation that

\[
P_{\phi,i} \{ \|X_{\varepsilon} - \phi\| < 1, \alpha(\varepsilon) = i^* \} > 0
\]

for sufficiently small \( \varepsilon \). The desired result follows from (3.7) and (3.10) with \( T_i = 2 + r \).

Lemma 3.5. Assume that either (H1) or (H2) holds. Let

\[
\eta_k = \inf \{ t > 0 : \|X_t\| \vee \alpha(t) > k \}.
\]

Then for any \((\phi, i) \in \mathcal{C} \times \mathbb{Z}_+\), we have \( \mathbb{P}_{\phi,i} \{ \eta_k < \infty \} = 1, \forall k \in \mathbb{Z}_+ \).

Proof. Suppose that \( p_0 = \mathbb{P} \{ \eta_k < \infty \} < 1 \). Since \( A(x, i^*) \) is elliptic, in view of (3.3), there is a \( T > 0 \) such that

\[
P_{\phi',i'} \{ \eta_k < T \} > 0, \forall k > 1, \|\phi'\| \leq 1.
\]

In view of Lemma 3.3 and (3.12), \( \forall (\psi, j) \in \mathcal{C} \times \mathbb{Z}_+ \) there are \( T_{\psi,j} \), \( p_{\psi,j} > 0 \) such that

\[
P_{\psi,j} \{ \eta_k < T_{\psi,j} \} > 2p_{\psi,j}.
\]

Due to the Feller property of \((X_t, \alpha(t))\), there exists a \( \delta_{\psi,j} > 0 \) such that

\[
P_{\phi,i} \{ \eta_k \leq T_{\phi,i^*} \} > p_{\psi,j}, \forall \psi' \in \mathcal{C}, \|\psi - \psi'\| < \delta_{\psi,j}.
\]

Since \( \sigma(\cdot, i) \) and \( b(\cdot, i) \) are locally compact for each \( i \in \mathbb{Z}_+ \), similar to [16] Lemma 4.6, we can show that there is an \( h_k > 0 \) such that for any \( t > 0 \),

\[
P_{\phi,i} \left\{ \frac{|X(s) - X(s')|}{|s - s'|^{0.25}} \leq h_k, \forall 0 \lor (\eta_k \land t - r) \leq s' < s < \eta_k \land t \right\} > \frac{1 + p_0}{2}.
\]
Since the set
\[ A_k = \left\{ \psi \in \mathcal{C} : |\psi| \leq k, \frac{|\psi(s) - \psi(s')|}{|s - s'|^{0.25}} \leq h_k, \forall -r \leq s' < s \leq 0 \right\} \]
is compact in \( \mathcal{C} \), we have from (3.14) that there exist \( T_k \) and \( \tilde{p}_k \) such that
\[ P_{\psi,j}\{\eta_n < T_k\} > \tilde{p}_k > 0, \forall \psi \in A_k, j \leq k. \tag{3.16} \]
Since \( \lim_{t \to \infty} P_{\phi,i}\{\eta_t < t\} = p_0 < 1 \), there is a \( T' > 0 \) such that
\[ p_0 \geq P_{\phi,i}\{\eta_k \leq T'\} \geq p_0 - \frac{1 - p_0}{2} \tilde{p}_k. \tag{3.17} \]
In view of (3.15) and (3.17), we have \( P_{\phi,i}\{T' < \eta_k, X_{T'} \in A_k\} > \frac{1 - p_0}{2}. \) By the Markov property and (3.16),
\[ P_{\phi,i}\{T' < \eta_k < \infty\} \geq P_{\phi,i}\{X_{T'} \in A_k, T' < \eta_k\} \geq E_{\phi,i}\{1_{\{T' < \eta_k, X_{T'} \in A_k\}} P_{X_{T'}, \alpha(T')}\{\eta_k < \infty\}\} \geq 1 - p_0 - \frac{1 - p_0}{2} \tilde{p}_k. \tag{3.18} \]
We have from (3.17) and (3.18) that
\[ p_0 = P_{\phi,i}\{\eta_k < \infty\} = P_{\phi,i}\{\eta_k \leq T'\} + P_{\phi,i}\{T' < \eta_k < \infty\} > p_0 - \frac{1 - p_0}{2} \tilde{p}_k + \frac{1 - p_0}{2} \tilde{p}_k = p_0, \]
which is a contradiction. Thus \( p_0 = 1. \)

**Definition 3.1.** The process \( \{(X_t, \alpha(t)) : t \geq 0\} \) is said to be recurrent (resp., positive recurrent) relative to a measurable set \( \mathcal{E} \subseteq \mathcal{C} \times \mathbb{Z}_+ \) if
\[ P_{\phi,i}\{(X_t, \alpha(t)) \in \mathcal{E} \text{ for some } t \geq 0\} = 1 \]
(resp. \( E_{\phi,i}\{\inf\{t > 0 : (X_t, \alpha(t)) \in \mathcal{E}\} < \infty\} \))
for any \( (\phi, i) \in \mathcal{C} \times \mathbb{Z}_+ \).

**Theorem 3.1.** Suppose that either hypothesis (H1) or (H2) holds. Let \( \mathcal{D} \) be a bounded open subset of \( \mathcal{C} \) and \( N \) be a finite subset of \( \mathbb{Z}_+ \). If \( (X_t, \alpha(t)) \) is recurrent relative to \( \mathcal{D} \times N \) then \( (X_t, \alpha(t)) \) is recurrent relative to \( \mathcal{D}' \times \mathbb{N}' \) for any open set \( \mathcal{D}' \subseteq \mathcal{C} \) and a finite set \( \mathbb{N}' \subseteq \mathbb{Z}_+ \) containing \( i^* \) with \( i^* \) given in either (H1) or (H2) according to which hypothesis is satisfied.

**Proof.** Let \( (\phi_0, i_0) \in \mathcal{C} \times \mathbb{Z}_+ \). In view of Lemma 3.2, there exists a compact set \( A_\mathcal{D} \subseteq \mathcal{C} \) such that
\[ \inf\{(\psi,j) \in \mathcal{D} \times N\} P_{\phi,j}\{X_r \in A_\mathcal{D}, \alpha(r) \in N\} := p_{\mathcal{D},N} > 0. \tag{3.19} \]
Since \( i^* \in \mathbb{N}' \), by Lemma 3.4, the Feller property of \( (X_t, \alpha(t)) \) and the compactness of \( A_\mathcal{D} \), there is a \( T > 0 \) such that
\[ \inf\{(\psi,j) \in A_\mathcal{D} \times \mathbb{N}\} P_{\psi,j}\{X_T \in \mathcal{D}' \times \mathbb{N}'\} \geq \varepsilon_0. \tag{3.20} \]
Define the stopping times \( \vartheta_0 = 0, \vartheta_{k+1} = \inf\{t > \vartheta_k + T : X_{\eta_k+1} \in D \times N\} \). By the hypothesis of the theorem,

\[ \mathbb{P}_{\varphi_0,i_0} \{\vartheta_k < \infty\} = 1, \forall k \in \mathbb{Z}_+. \]

On the other hand, it follows from (3.19) and (3.20) that

\[ \mathbb{P}_{\psi,j} \{(X_T, \alpha(T)) \in D' \times N'\} \geq p_{D',N|0}, \text{ for all } (\psi, j) \in D \times N. \quad (3.21) \]

Consider the events

\[ A^k = \{(X_{\vartheta_k + T} \notin D' \times N'), k \in \mathbb{Z}_+. \]

By the strong Markov property of \((X_t, \alpha(t))\), we have

\[ \mathbb{P}_{\varphi_0,i_0} \left( \bigcap_{k' = k}^{\infty} A^{k'} \right) = \lim_{l \to \infty} \mathbb{P}_{\varphi_0,i_0} \left( \bigcap_{k' = k}^{l} A^{k'} \right) \leq \lim_{l \to \infty} (1 - p_{D,N|0})^{l-k} = 0. \]

Thus

\[ \mathbb{P}_{\varphi_0,i_0} \left( \bigcap_{k' = k}^{\infty} A^{k'} \right) = 0. \]

It indicates that the event \( \{(X_{\vartheta_k + T}, \alpha(\vartheta_k + T)) \in D' \times N\} \) must occur with probability 1.

\[ \square \]

**Theorem 3.2.** Suppose that either hypothesis \( [H1] \) or \( [H2] \) holds. Let \( V(\cdot, \cdot) \in \mathbb{F} \) such that

\[ \lim_{n \to \infty} \inf \left\{ V(\phi, i) : |\phi(0)| \vee i \geq n \right\} = \infty. \]

Suppose further that there are positive constants \( C \) and \( H \) such that

\[ \mathcal{L}V(\phi, i) \leq C1_{\{V(\phi, i) \leq H\}}. \quad (3.22) \]

Then the process \((X_t, \alpha(t))\) is recurrent relative to \( D \times N \), where \( D \) is any open bounded subset of \( C \) and \( N \subset \mathbb{Z}_+ \) contains \( i^* \).

**Proof.** Let \( v_H = \inf\{t > 0 : V(X_t, \alpha(t)) \leq H\} \) and \( \eta_k \) be defined as in Lemma 3.5. In view of Lemma 3.5, \( \mathbb{P}_{\psi,j} \{\eta_k < \infty\} = 1, \forall k \in \mathbb{Z}_+ \). Let \( t > 0 \). By Itô’s formula

\[ \mathbb{E}_{\psi,j} V(X_{t \wedge v_H \wedge \eta_k}, \alpha(t \wedge v_H \wedge \eta_k)) \leq V(\psi, j). \]

Letting \( t \to \infty \), we obtain

\[ \mathbb{E}_{\psi,j} V(X_{v_H \wedge \eta_k}, \alpha(v_H \wedge \eta_k)) \leq V(\psi, j), \]

which implies

\[ \mathbb{P}_{\psi,j} \{v_H > \eta_k\} \leq \frac{V(\psi, j)}{\inf\{V(\phi, i) : |\phi(0)| \vee i \geq k\}}. \]

Letting \( k \to \infty \) yields \( \mathbb{P}_{\psi,j} \{v_H > \eta_k\} \to 0 \). Thus,

\[ \mathbb{P}_{\psi,j} \{v_H < \infty\} = 1, \forall (\psi, j) \in C \times \mathbb{Z}_+. \quad (3.23) \]
Now, let $k_0 > 0$ such that $\inf \{ V(\phi, i) : |\phi(0)| \forall i \geq k_0 \} \geq 2(H+C+r)$. For any $(\psi, j) \in C \times \mathbb{Z}_+$ satisfying $V(\psi, j) \leq H$. We have from (3.22) and Itô’s formula that

$$E_{\psi,j} V(X_{r \wedge \eta_0}, \alpha(r \wedge \eta_0)) \leq H + C + r,$$

which implies

$$\mathbb{P}_{\psi,j} \{ \eta_0 < r \} \leq \frac{H + C + r}{2(H + C + r)} \leq \frac{1}{2} \quad (3.24)$$

Thus,

$$\mathbb{P}_{\psi,j} \{ \| X_r \| < n_0, \alpha(r) < n_0 \} \geq \mathbb{P}_{\psi,j} \{ \eta_0 > r \} > \frac{1}{2} \quad \text{provided } V(\psi, j) \leq H. \quad (3.25)$$

Now, fix $(\phi_0, i_0) \in C \times \mathbb{Z}_+$. By (3.23) and Lemma 3.5, we can define almost surely finite stopping times

$$\zeta_1 = \inf \{ t \geq 0 : V(X_t, \alpha(t)) \leq H \},$$

$$\zeta_{2k} = \inf \{ t \geq \zeta_{2k-1} + r : \| X_t \| \wedge \alpha(t) \geq n_0 \},$$

$$\zeta_{2k+1} = \inf \{ t \geq \zeta_{2k} : V(X_t, \alpha(t)) \leq H \}.$$

Define events $B_k = \{ \| X_{\zeta_{2k+1}+r} \| \wedge \alpha(\zeta_{2k+1}+r) < n_0 \}$. In view of (3.25) and the strong Markov property of $(X_t, \alpha(t))$, we can use standard arguments in Theorem 3.1 to show that

$$\mathbb{P}_{\phi_0,i_0} \{ B_k \text{ occurs for some } k \} = 1.$$

Thus, $(X_t, \alpha(t))$ is recurrent relative to $\{ (\phi, i) : \| \phi \| \forall i \leq n_0 \}$. Combining this with Theorem 3.1 yields the desired result. \hfill \Box

**Example 3.1.** Let

$q_{12}(\phi) = 1, q_{1j}(\phi) = 0$ for $j \geq 3$;

$q_{i,j-1}(\phi) = C_i + (1 + \| \phi \|)^{-1}, q_{i,j+1}(\phi) = C_i + (1 + \| \phi \|)^{-1}$ for $i \geq 2, C_i \geq 0$;

$q_{ij}(\phi) = 0$ for $i \geq 2, j \notin \{ i-1, i, i+1 \}$.

Suppose the switching diffusion is given by

$$dX(t) = \sigma(X(t), \alpha(t))dW(t) - X(t)b(X(t), \alpha(t))dt$$

where $b(x, i) > 0, \sigma(x, i)$ are locally Lipchitz in $x$ and uniformly bounded in $K \times \mathbb{Z}_+$ for each compact set $K \in \mathbb{R}$. Let $f(x)$ be twice continuously differentiable such that $f(x) > 0$ and $f(x) = |x|$ if $|x| \geq 1$. Let

$$\kappa := \sup_{|x| \leq 1, i \in \mathbb{Z}_+} \left| - \left[ \frac{df}{dx}(x) \right] xb(x, i) + \frac{1}{2} \left[ \frac{d^2f}{dx^2}(x) \right] \sigma^2(x, i) \right| < \infty. \quad (3.27)$$

Let

$$V(\phi, i) = f(\phi(0)) + 2\kappa i.$$
Direct computation leads to

$$\mathcal{L}V(\phi, i) = \begin{cases} -\left[\frac{df}{dx}(\phi(0))\right] & \phi(0)b(\phi(0), i) + \frac{\sigma^2(i)}{2} \left[\frac{df}{dx}(\phi(0))\right] - 2\kappa(1 + ||\phi||)^{-1} \text{ if } i > 1, \\ -\left[\frac{df}{dx}(\phi(0))\right] & \phi(0)b(\phi(0), i) + \left[\frac{df}{dx}(\phi(0))\right] + 2\kappa \text{ if } i = 1. \end{cases}$$

(3.28)

In view of (3.27) and the fact that \(\frac{df}{dx}(x) = \text{sgn}(x), \frac{df}{dx^2}(x) = 0\) for \(|x| \geq 1, i > 1\) we have

$$\mathcal{L}V(\phi, i) \leq 0 \forall \phi \in \mathcal{C}, i > 1.$$  

(3.29)

By (3.28), if we assume further \(\lim_{x \to \infty} |x|b(x, 1) = \infty\), then we can verify that

$$\mathcal{L}V(\phi, 1) \leq \tilde{C}_11_{\{\phi(0) < \tilde{H}\}} - \tilde{C}_2, \forall \phi \in \mathcal{C},$$

(3.30)

where \(\tilde{C}_1, \tilde{C}_2, \tilde{H}\) are some positive constants. In view of (3.29) and (3.30), we can easily check that (3.22) holds in this example, for \(V(\phi, i)\) defined above and suitable \(C, H\). Thus, if there exists \(i^* \in \mathbb{Z}_+\) such that \(\sigma(x, i^*) \neq 0\) for any \(x \in \mathbb{R}\), then the conclusion of Theorem 3.2 holds for this example.

To proceed, let us recall some technical concepts and results needed to prove the main theorem. Let \(\Phi = (\Phi_0, \Phi_1, \ldots)\) be a discrete-time Markov chain on a general state space \((E, \mathcal{E})\), where \(\mathcal{E}\) is a countably generated \(\sigma\)-algebra. Denote by \(\mathcal{P}\) the Markov transition kernel for \(\Phi\). If there is a non-trivial \(\sigma\)-finite positive measure \(\varphi\) on \((E, \mathcal{E})\) such that for any \(A \in \mathcal{E}\) satisfying \(\varphi(A) > 0\) we have

$$\sum_{n=1}^{\infty} \mathcal{P}^n(x, A) > 0, \ x \in E$$

where \(\mathcal{P}^n\) is the \(n\)-step transition kernel of \(\Phi\) then the Markov chain \(\Phi\) is called \(\varphi\)-irreducible. It can be shown (see [17]) that if \(\Phi\) is \(\varphi\)-irreducible, then there exists a positive integer \(d\) and disjoint subsets \(E_0, \ldots, E_{d-1}\) such that for all \(i = 0, \ldots, d - 1\) and all \(x \in E_i\) we have

$$\mathcal{P}(x, E_j) = 1 \text{ where } j = i + 1 \text{ (mod } d)$$

and

$$\varphi \left( E \setminus \bigcup_{i=0}^{d-1} E_i \right) = 0.$$  

The smallest positive integer \(d\) satisfying the above is called the period of \(\Phi\). An aperiodic Markov chain is a chain with period \(d = 1\). A set \(C \in \mathcal{E}\) is called petite if there exists a non-negative sequence \((a_n)_{n \in \mathbb{Z}_+}\) with \(\sum_{n=1}^{\infty} a_n = 1\) and a nontrivial positive measure \(\nu\) on \((E, \mathcal{E})\) satisfying that

$$\sum_{n=1}^{\infty} a_n \mathcal{P}^n(x, A) \geq \nu(A), \ x \in C, A \in \mathcal{E}.$$

**Lemma 3.6.** Assume either [H1] or [H2] holds. The Markov chain \(\{(X_k, \alpha(k)) : k \in \mathbb{Z}_+\}\) is irreducible and aperiodic. Moreover, for every bounded set \(D \in \mathcal{C}\) and a finite set \(N \in \mathbb{Z}_+\), the set \(D \times N\) is petite for \(\{(X_k, \alpha(k)) : k \in \mathbb{Z}_+\}\).
Proof. Similar to (3.31), there are \( k_0 \in \mathbb{Z}_+ \), \( k_0 > r \), \( \tilde{d}_{D,N} > 0 \) such that
\[
\mathbb{P}_{\phi,i} \{ X_t \in \mathcal{D}, \alpha_{k_0} = i^* \} \geq \tilde{d}_{D,N} \text{ for all } (\phi,i) \in \mathcal{D} \times X.
\] (3.31)

By the Markov property, we deduce from (3.4) and (3.31) that for any nonempty bounded set in \( \{ \mathcal{C} \times \mathbb{Z}_+ \} \), irreducible and every nonempty open set of \( \mathcal{B} \), \( \mathcal{B} \in \mathfrak{B}(\mathcal{C}) \), \( (\phi,i) \in \mathcal{D} \times X \). Then (3.32) and (3.33) can be rewritten as
\[
\mathbb{P}_{\phi,i} \{ X_k : \alpha(k) = \alpha(k_0) \in \mathcal{E} \} \geq \tilde{d}_{D,N,k} \mathbb{E}(\mathcal{E}), \mathcal{E} \in \mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+), (\phi,i) \in \mathcal{D} \times X.
\] (3.33)

It can be checked that (3.33) implies that the Markov chain \( (X_k, \alpha(k)) : k \in \mathbb{Z}_+ \) is \( \mathbb{E} \)-irreducible and every nonempty bounded set in \( \mathcal{C} \times \mathbb{Z}_+ \) is petite. Moreover, suppose that \( (X_k, \alpha(k)) \) is not aperiodic. Then, there are disjoint set \( \mathcal{E}_0, \ldots, \mathcal{E}_{d-1} \), \( d > 1 \) such that
\[
\mathbb{P}_{\phi,i} \{ (X_1, \alpha(1)) \in \mathcal{E}_j \} = 1 \text{ if } j = j' + 1 \text{ (mod } d \text{) if } (\phi,i) \in \mathcal{E}_{j'},
\]
which results in
\[
\mathbb{P}_{\phi,i} \{ (X_m, \alpha(m)) \in \mathcal{E}_j \} = \begin{cases} 1 & \text{where } m = j' + 1 \text{ (mod } d \text{) if } (\phi,i) \in \mathcal{E}_{j'}; \\ 0 & \text{otherwise} \end{cases}
\] (3.35)

In view of (3.32), for any \( m > k_0 + r \) and \( (\phi,i) \in \mathcal{C} \times \mathbb{Z}_+ \), there is a \( \tilde{p}_{\phi,i,k} > 0 \) such that
\[
\mathbb{P}_{\phi,i} \{ (X_m, \alpha(m)) \in \mathcal{E} \} \geq \tilde{p}_{\phi,i,k} \mathbb{E}(\mathcal{E})
\] (3.36)

for any measurable set \( \mathcal{E} \in \mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+) \). As a result of (3.35) and (3.36), we have that
\[
\mathbb{E}(\mathcal{E}_j) = 0 \text{ for any } j = 0, \ldots, d-1. \text{ Thus,}
\]
\[
\mathbb{E} \left( \mathcal{C} \times \mathbb{Z}_+ \setminus \bigcup_{j=0}^{d-1} \mathcal{E}_j \right) = \mathbb{E}(\mathcal{C} \times \mathbb{Z}_+) > 0,
\] (3.37)
which contradicts (3.34). This contradiction implies that \( (X_k, \alpha(k)) \) is aperiodic. \( \square \)

**Theorem 3.3.** Suppose that either [H1] or [H2] holds. Let \( V(\cdot, \cdot) \in \mathbb{P} \) such that
\[
\lim \inf_{n \to \infty} \{ V(\phi,i) : |\phi(0)| \vee i \geq n \} = \infty.
\] (3.38)

Suppose further that there are positive constants \( C_1, C_2 \) and \( H \) such that
\[
\mathcal{L}V(\phi,i) \leq -C_1 + C_2 \mathbb{1}_{\{V(\phi,i) \geq H\}}.
\] (3.39)

Then, \( (X_t, \alpha(t)) \) is positive recurrent relative to any set of the form \( \mathcal{D} \times N \) where \( \mathcal{D} \) is a nonempty open set of \( \mathcal{C} \) and \( N \ni i^* \) with \( i^* \) given in either (H1) and (H2). Moreover, there is a unique invariant probability measure \( \mu^* \), and for any \( (\phi,i) \in \mathcal{C} \times \mathbb{Z}_+ \)
\[
\lim_{t \to \infty} \| P(t, (\phi,i), \cdot) - \mu^* \|_{TV} = 0.
\]
Proof. Let \( v_H = \inf \{ t \geq 0 : V(X_t, \alpha(t)) \leq H \} \). In view of the functional Itô formula,

\[
\mathbb{E}_{\phi,i}V(X_{1 \wedge v_H}, \alpha(1 \wedge v_H)) = V(\phi, i) + \mathbb{E}_{\phi,i} \int_0^{1 \wedge v_H} \mathcal{L}V(X_s, \alpha(s))ds \\
\leq V(\phi, i) - C_1\mathbb{E}_{\phi,i}1 \wedge v_H \\
\leq V(\phi, i) - C_1\mathbb{P}_{\phi,i}\{v_H \geq 1\}.
\]

(3.40)

For any \( t \leq 1 \) and \( V(\phi, i) \leq H \), we have

\[
\mathbb{E}_{\phi,i}V(X_t, \alpha(t)) = V(\phi, i) + \mathbb{E}_{\phi,i} \int_0^t \mathcal{L}V(X_s, \alpha(s))ds \\
\leq V(\phi, i) + C_2t \\
\leq H + C_2.
\]

(3.41)

It follows from (3.41) and the strong Markov property of \((X_t, \alpha(t))\) that

\[
\mathbb{E}_{\phi,i}\left[1_{\{v_H < 1\}}V(X_1, \alpha(1))\right] \leq (H + C_2)\mathbb{P}_{\phi,i}\{v_H < 1\} \\
\leq 2(H + C_2) - (H + C_2)\mathbb{P}_{\phi,i}\{v_H < 1\}.
\]

(3.42)

Let \( \mathcal{C}_V := \{(\psi', j') : V(\psi, j) \leq 2(H + C_2)\} \). In view of (3.40) and (3.42),

\[
\mathbb{E}_{\phi,i}V(X_1, \alpha(1)) \leq \mathbb{E}_{\phi,i}\left[1_{\{v_H < 1\}}V(X_1, \alpha(1))\right] + \mathbb{E}_{\phi,i}V(X_{1 \wedge v_H}, \alpha(1 \wedge v_H)) \\
\leq V(\phi, i) - \min\{C_1, H + C_2\} + 2(H + C_2)1_{\{\phi, i\in \mathcal{C}_V\}}.
\]

(3.43)

Let \( n_0 \in \mathbb{Z}_+ \) such that

\[
V(\phi, i) \geq 2(2H + 2C_2 + C_2r) \quad \text{for any} \quad \|\phi\| \vee i \geq n_0,
\]

(3.44)

and define \( \hat{\zeta}_V = \inf\{t \geq 0 : V(X_t, \alpha(t)) \geq 2(2H + 2C_2 + C_2r)\} \). Similar to (3.24), we have

\[
\mathbb{P}_{\phi,i}\{\hat{\zeta}_V \leq r\} \leq \frac{1}{2} \quad \text{for} \quad (\phi, i) \in \mathcal{C}_V.
\]

(3.45)

Thus,

\[
\mathbb{P}_{\phi,i}\{\|X_r\| \vee \alpha(r) \leq n_0\} \geq \mathbb{P}_{\phi,i}\{V(X_r, \alpha(r)) \geq H + C_2r + 1\} \\
\geq 1 - \mathbb{P}_{\phi,i}\{\hat{\zeta}_V \leq r\} = \frac{1}{2}, \quad (\phi, i) \in \mathcal{C}_H.
\]

(3.46)

In view of (3.33) and (3.46),

\[
\mathbb{P}_{\phi,i}\{(X_{k+k_0}, \alpha(k + k_0)) \in \mathcal{E}\} \geq \tilde{d}_{H,k}\hat{\nu}(\mathcal{E}), \quad \mathcal{E} \in \mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+), \quad \text{if} \quad V(\phi, i) \leq H, \quad k > r
\]

(3.47)

for some \( \tilde{d}_{H,k} > 0 \). Thus, the set \( \{(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+ : V(\phi, i) \leq H\} \) is petite for \( \{(X_k, \alpha(k)) : k \in \mathbb{Z}_+\} \). Using this and (3.47), it follows from [23, Theorem 2.1] (or [15]) that

\[
\lim_{n \to \infty} \|P(n, (\phi, i), \cdot) - \mu^*\|_{TV} = 0
\]

where \( P(t, (\phi, i), \cdot) \) is the transition probability of \((X_t, \alpha(t))\) and \( \mu^* \) is an invariant probability measure of the Markov chain \( \{X_k, \alpha(k), k \in \mathbb{Z}_+\} \). It is easy to show that \( \mu^* \) is also an
invariant probability measure of the process \(\{(X_t, \alpha(t))\}\). Thus \(\|P(t, (\phi, i), \cdot) - \mu^*\|_{TV}\) is decreasing in \(t\), which leads to

\[
\lim_{t \to \infty} \|P(t, (\phi, i), \cdot) - \mu^*\|_{TV} = 0.
\]

Now we show that the process \((X_t, \alpha(t))\) is positive recurrent. Similar to (3.40), we deduce from the functional Itô formula that

\[
\mathbb{E}_{\phi, i} \varphi_H \leq C^{-1}_1 \mathbb{V}(\phi, i).
\]

Owing to this and (3.47), we can use the arguments in the proof of [26, Lemma 3.6] to show that \((X_t, \alpha(t))\) is positive recurrent.

Example 3.2. In Example 3.1 if we assume further that

\[
\lim_{|x| \to \infty} \inf_{i \in \mathbb{Z}+} \{|x| b(x, i)\} > 0 \quad (3.48)
\]

then it follows from (3.27) and (3.28) that

\[
\mathcal{L} \mathbb{V}(\phi, i) \leq -\hat{C} \quad \text{for} \ \phi \in \mathcal{C}, i \geq 2,
\]

for some positive constant \(\hat{C}\). This combined with (3.30) shows that (3.39) holds for the switching diffusion \((X(t), \alpha(t))\) and the function \(\mathbb{V}(\phi, i)\) in Example 3.1. Thus the conclusion of Theorem 3.3 holds for the switching diffusion in Example 3.1 with the additional condition (3.48).

**Theorem 3.4.** Suppose that either (H1) or (H2) holds. Let \(V(\cdot, \cdot) \in \mathbb{F}\) such that

\[
\lim_{n \to \infty} \inf_{\phi(0) \geq i \geq n} \{V(\phi, i) : \phi(0) \geq i \geq n\} = \infty. \quad (3.49)
\]

Suppose further that there are \(C_1\) and \(C_2 > 0\) such that

\[
\mathcal{L} \mathbb{V}(\phi, i) \leq -C_1 \mathbb{V}(\phi, i) + C_2. \quad (3.50)
\]

Then, there is a unique invariant probability measure \(\mu^*\) and \(\theta > 0\) such that for any \((\phi, i) \in \mathcal{C} \times \mathbb{Z}+\)

\[
\lim_{t \to \infty} \exp(\theta t)\|P(t, (\phi, i), \cdot) - \mu^*\|_{TV} = 0. \quad (3.51)
\]

**Proof.**

\[
\mathbb{E}_{\phi, i} \exp\{C_1(\eta_k \wedge t)\} V(X_{\eta_k \wedge t}, \alpha(\eta_k \wedge t))
= V(\phi, i) + \mathbb{E}_{\phi, i} \int_0^{\eta_k \wedge t} e^{C_1 s} [\mathcal{L} V(X_s, \alpha(s)) + C_1 \mathbb{V}(X_s, \alpha(s))] ds
\leq V(\phi, i) + C_2 \mathbb{E}_{\phi, i} \int_0^{\eta_k \wedge t} e^{C_1 s} ds
\leq V(\phi, i) + C_1^{-1} C_2 e^{C_1 t}.
\]
Letting \( k \to \infty \), we obtain
\[
\mathbb{E}_{\phi,i} V(X_t, \alpha(t)) \leq e^{-C_1 t} V(\phi, i) + C_1^{-1} C_2
\]
(3.53)

Let \( \gamma_1 = e^{-C_1} \) and \( \gamma_2 \in (\gamma_1, 1) \). It follows from (3.53) and (3.42) that
\[
\mathbb{E} V(X_1, \alpha(1)) \leq \gamma_1 V(\phi, i) + C_1^{-1} C_2
\]
\[
= \gamma_2 V(\phi, i) + \left[ C_1^{-1} C_2 - (\gamma_2 - \gamma_1) V(\phi, i) \right]
\]
\[
\leq \gamma_2 V(\phi, i) + [C_1^{-1} C_2] 1_{\{V(\phi, i) \leq H'\}}
\]
(3.54)

where \( H' = C_1^{-1} C_2 (\gamma_2 - \gamma_1)^{-1} \). Similar to (3.47), the set \( \{(\phi, i) \in C \times \mathbb{Z}_+ : V(\phi, i) \leq H'\} \) is petite, which combined with (3.54) implies the existence of \( \gamma_3 \in (0, 1) \) such that
\[
\lim_{n \to \infty} \gamma_3^n \|P(n, (\phi, i), \cdot) - \mu^*\|_{TV} = 0
\]
due to a well-known theorem (see, e.g., [15]). Then (3.51) follows from (3.54) and the decreasing property of \( \|P(t, (\phi, i), \cdot) - \mu^*\|_{TV} \) in \( t \).

**Example 3.3.** Suppose that
\[
q_{12}(\phi) = 1, q_{ij}(\phi) = 0 \text{ for } j \geq 3;
\]
\[
q_{i,i}(\phi) = 2 \int_0^0 |\phi(s)| ds, q_{i,i+1}(\phi) = i \int_{-r}^0 |\phi(s)| ds \text{ for } i \geq 2
\]
\[
q_{ij}(\phi) = 0 \text{ for } i \geq 2, j \not\in \{1, i, i + 1\}.
\]

and that the equation for the diffusion part is
\[
dX(t) = \sigma(X(t), \alpha(t))dW(t) - b(X(t), \alpha(t))X(t)dt
\]
where \( \sigma(x, i), b(x, i) \) are locally Lipchitz in \( x \) and uniformly bounded in \( K \times \mathbb{Z}_+ \) for each compact set \( K \in \mathbb{R} \). Let \( V(\phi, i) \) be defined as in Example (3.1). Similar to Examples 3.1 and 3.2 under the assumption that \( b := \inf_{(x,i) \in \mathbb{R} \times \mathbb{Z}_+} \{b(x, i)\} > 0 \), one can show that (3.50) holds in this example with this function \( V \). Thus, the conclusion of Theorem 3.2 holds for this example if there exists \( i^* \in \mathbb{Z}_+ \) such that \( \sigma(x, i^*) \neq 0 \) for any \( x \in \mathbb{R} \).

**Example 3.4.** Let
\[
q_{12}(\phi) = 1, q_{ij}(\phi) = 0 \text{ for } j \geq 3;
\]
\[
q_{i,i-1}(\phi) = C_i + 2|\phi(0)|, q_{i,i+1} = C_i + |\phi(-r)| \text{ for } i \geq 2, C_i \geq 0;
\]
\[
q_{ij}(\phi) = 0 \text{ for } i \geq 2, j \not\in \{i - 1, i, i + 1\}.
\]

Consider the general equation for diffusion (2.1), where \( \sigma(x, i), b(x, i) \) are locally Lipchitz in \( x \) in \( K \times \mathbb{Z}_+ \) for each compact set \( K \in \mathbb{R} \). Suppose there is a function \( U(x) : \mathbb{R}^n \mapsto \mathbb{R}_+ \) satisfying

- \( U(x) \) is twice continuously differentiable in \( x \).
• \( \lim_{|x| \to \infty} U(x, i) = \infty. \)

• There are positive constants \( C_1, C_2, H \) such that
  \[
  \mathcal{L}_i U(x) \leq -C_1 U(x) + C_2. \tag{3.55}
  \]

Let \( V(x, i) = U(x) + i + \int_0^t \exp\left\{ \frac{\ln 2}{r} (s + r) \right\} ds \). By Remark 2.1,

\[
\mathcal{L} V(\phi, i) = \begin{cases} 
\mathcal{L}_i U(x) - i - \frac{\ln 2}{r} \int_0^t \exp\left\{ \frac{\ln 2}{r} (s + r) \right\} ds + 2 & \text{if } i > 1 \\
\mathcal{L}_i U(x) - \frac{\ln 2}{r} \int_0^t \exp\left\{ \frac{\ln 2}{r} (s + r) \right\} ds + 1 & \text{if } i = 1
\end{cases} \tag{3.56}
\]

As a consequence of (3.55) and (3.56), there are \( C_3 \) and \( C_4 > 0 \) such that

\[
\mathcal{L} V(\phi, i) \leq -C_3 V(\phi, i) + C_4 \text{ for } (\phi, i) \in \mathcal{C} \times \mathbb{Z}^+. \]

Thus, the conclusion of Theorem 3.2 holds for this example if there exists \( i^* \in \mathbb{Z}^+ \) such that \( A(x, i^*) \) is elliptic.

### 4 Recurrence of Past-Independent Switching Diffusions

This section is devoted mainly to characterizing the recurrence of \((X(t), \alpha(t))\) using the corresponding system of partial differential equations when the switching intensities of \( \alpha(t) \) depends only on the current state of \( X(t) \), that is \( q_{ij}(\cdot), i, j \in \mathbb{Z}^+ \) are functions on \( \mathbb{R}^n \) rather than on \( \mathcal{C} \). To simplify the presentation, throughout this section, we set \( q_{ii}(x) = 0 \) for \((x, i) \in \mathbb{R}^n \times \mathbb{Z}^+ \). Thus, \( q_i(x) = -\sum_{j \in \mathbb{Z}^+} q_{ij}(x) \). In this section, we use the following assumption.

**Assumption 4.1.** Suppose that

1. either Assumption 2.1 or Assumption 2.2 holds with \( \phi \in \mathcal{C} \) replaced by \( x \in \mathbb{R}^n \);
2. for each \( i \in \mathbb{Z}^+ \), \( A(x, i) \) is uniformly elliptic in each compact set;
3. for any \( x \in \mathbb{R}^n \), there are \( \tilde{q} = \tilde{q}(x) > 0 \) and \( n_{\tilde{q}} = n_{\tilde{q}}(x) > 0 \) such that

\[
\sum_{j \leq n_{\tilde{q}}} q_{ij}(x) \geq \tilde{q} \text{ for any } i > n_{\tilde{q}}. \tag{4.1}
\]

**Remark 4.1.** We note the following facts.

- Part 3 of Assumption 4.1 stems from a familiar condition for uniform ergodicity of the Markov chain having a countable state space. In other word, if (4.1) holds, for each \( x \in \mathbb{R}^n \), the Markov chain \( \tilde{\alpha}(t) \) with generator \( Q(x) \) has a property that

\[
\sup_{i \in \mathbb{Z}^+} \mathbb{E}_i \zeta < \infty
\]

where \( \zeta \) is the first time the process \( \tilde{\alpha}(t) \) jumps to \( \{1, \ldots, n_0\} \) and \( \mathbb{E}_i \) is the expectation with condition \( \tilde{\alpha}(0) = i \).
• Since \( q_{ij}(x) \) is continuous in \( x \in \mathbb{R}^n \), with the use of the Heine-Borel covering theorem, it is easy to show that for any bounded set \( D \subseteq \mathbb{R}^n \), there is \( \varepsilon_0 = \varepsilon_0(D) \) and \( n_0 = n_0(D) \) such that

\[
\sum_{j \leq n_0} q_{ij}(x) \geq \varepsilon_0 \quad \text{for any} \quad i > n_0, x \in D. \tag{4.2}
\]

For an open set \( D \subset \mathbb{R}^n \), define

\[
\tau_D = \inf\{t \geq 0 : X(t) \notin D\},
\]
and \( W^{2,\beta}_{\text{loc}}(D) \) is the set of functions \( u : \overline{D} \mapsto \mathbb{R} \) that has generalized derivatives \( D^\beta u \) for any multiple-index \( \beta = (\beta_1, \ldots, \beta_n) \) with \( |\beta| = \sum \beta_i \leq 2 \) satisfying \( D^\beta u \in L^p_{\text{loc}}(D) \) if \( |\beta| \leq 2 \). Let \( H^p(D) \) be the set of functions \( u(x,i) \) in \( \overline{D} \times \mathbb{Z}_+ \) satisfying that

• For each \( i \in \mathbb{Z}_+ \), \( u(\cdot,i) \in W^{2,\beta}_{\text{loc}}(D) \) and \( u(\cdot,i) \) is continuous in the closure \( \overline{D} \) of \( D \).

• For any compact set \( K \subset \mathbb{R}^n \), \( \sup_{(x,i) \in (K \cap \overline{\tau}_D \times \mathbb{Z}_+)} \{u(x,i)\} < \infty \).

Let \( \mathcal{L}_i \) be defined as in \( (4.1) \). We state the two main results of this section.

**Theorem 4.1.** Suppose that Assumption \[4.1\] holds. Let \( D_1 \) be a bounded open set of \( \mathbb{R}^n \) with \( \partial D_1 \subset C^2 \) and \( D = \overline{D}_1^c \) be the complement of \( \overline{D}_1 \). Let \( p > n \). The process \( X(t) \) is recurrent relative to \( D_1 \), if and only if the Dirichlet problem

\[
\begin{aligned}
&\mathcal{L}_i u(x,i) - q_i(x) u(x,i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) u(x,j) = 0 \quad \text{in} \ D \times \mathbb{Z}_+ \\
&u(x,i) = f(x,i) \quad \text{on} \ \partial D \times \mathbb{Z}_+. 
\end{aligned}
\tag{4.3}
\]

has a unique solution in \( H^p(D) \) given that \( f(x,i) \) is continuous in \( \partial D \) and bounded in \( \partial D \times \mathbb{Z}_+ \).

**Theorem 4.2.** Suppose that Assumption \[4.1\] holds. Let \( D_1 \) be a bounded open set of \( \mathbb{R}^n \) with boundary \( \partial D_1 \subset C^2 \) and \( D = \overline{D}_1^c \) be its complement. Let \( p > n \). Suppose further that for each compact set \( K \subset \mathbb{R}^n \), the function \( u(x,i) = E_{x,i} \tau_D \) is bounded in \( K \times \mathbb{Z}_+ \). Then \( \{u(x,i)\} \in H^p(D) \), \( p > n \) is a strong solution to

\[
\begin{aligned}
&\mathcal{L}_i u(x,i) - q_i(x) u(x,i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) u(x,j) = -1 \quad \text{in} \ D \times \mathbb{Z}_+ \\
&u(x,i) = 0 \quad \text{on} \ \partial D \times \mathbb{Z}_+. 
\end{aligned}
\tag{4.4}
\]

The solution is unique in \( H^p(D) \), \( p > n \).

**Lemma 4.1.** Let \( D \subset \mathbb{R}^n \) be an open bounded set with \( \partial D \subset C^2 \). The Dirichlet problem

\[
\begin{aligned}
&\mathcal{L}_i u(x,i) - q_i(x) u(x,i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) u(x,j) = f(x,i) \quad \text{in} \ D \times \mathbb{Z}_+ \\
&u(x,i)|_{\partial D} = \phi(x,i) \quad \text{on} \ \partial D \times \mathbb{Z}_+. 
\end{aligned}
\tag{4.5}
\]

has a unique strong solution \( \{u(x,i)\} \in H^p(D) \), \( p > n \) if \( \phi(x,i) \) and \( f(x,i) \) are continuous and bounded on \( \partial D \times \mathbb{Z}_+ \) and \( D \times \mathbb{Z}_+ \) respectively.
Proof. The proof is motivated by that of [4, Proposition A]. However, because there are infinitely many equations, significant modification is needed. Let $p > n$ and

$$
\hat{M} = \sup_{(x,i)\in\partial D \times \mathbb{Z}^+} \{|\phi(x,i)|\} + \sup_{(x,i)\in D \times \mathbb{Z}^+} \{|f(x,i)|\} < \infty. \quad (4.6)
$$

By [24, Theorem 9.1.5], for each $i \in \mathbb{Z}^+$, there exists a strong solution $u_0(x,i) \in W^{2,p}_{\text{loc}}(D) \cap C(D)$ to

$$
\begin{cases}
\mathcal{L}_t u_0(x,i) - q_t(x) u(x,i) = 0 & \text{in } D \times \mathbb{Z}^+ \\
u_0(x,i) \big|_{\partial D} = \phi(x,i) & \text{on } \partial D \times \mathbb{Z}^+.
\end{cases} \quad (4.7)
$$

Let $Y^{x,i}(t)$ be the solution to

$$
dY(t) = b(Y(t),i)dt + \sigma(Y(t),i)dW(t), \ t \geq 0
$$

with initial condition $x$ and $\tau^x_i = \inf\{t \geq 0 : Y^{x,i}(t) \notin D\}$. In view of the Feynman-Kac formula for diffusion processes

$$
u_0(x,i) = \mathbb{E}_{x,i} \left[ \phi(Y(\tau_D)),i \right] \exp \left( - \int_0^{\tau_D} q_t(Y(s))ds \right) - \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_s(Y(s))ds \right) f(Y(t),i)dt. \quad (4.9)
$$

Note that in (4.9) and what follows, we drop the superscripts $x$ and $i$ in $Y^{x,i}$ and $\tau^x_i$ whenever the expectation $\mathbb{E}_{x,i}$ or probability $\mathbb{P}_{x,i}$ is used. By part (2) of Assumption 4.1, $\sup_{x\in D} \mathbb{E}_{x,i}\tau_D < \infty$ for any $i \in \mathbb{Z}^+$. In view of (4.9), we have

$$
|u_0(x,i)| \leq \sup_{x\in \partial D} \{|\phi(x,i)|\} + \sup_{x\in D} \{|f(x,i)|\} \sup_{x\in D} \{\mathbb{E}_{x,i}\tau_D\}. \quad (4.10)
$$

Let $n_0$ and $\varepsilon_0$ satisfy (4.2). In particular, for $i > n_0$, $q_t(x) \geq \varepsilon_0 > 0$ in $D$, we can have the following estimate from (4.9):

$$
|u_0(x,i)| \leq \mathbb{E}_{x,i} |\phi(Y(\tau_D))| + \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_s(Y(s))ds \right) |f(Y(t),i)|dt
$$

$$
\leq \sup_{x\in \partial D} \{|\phi(x,i)|\} + \sup_{x\in D} \{|f(x,i)|\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp(-\varepsilon_0 t)dt
$$

$$
\leq \sup_{x\in \partial D} \{|\phi(x,i)|\} + \sup_{x\in D} \{|f(x,i)|\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp(-\varepsilon_0 t)dt
$$

$$
\leq \sup_{x\in \partial D} \{|\phi(x,i)|\} + \varepsilon_0^{-1} \sup_{x\in D} \{|f(x,i)|\}. \quad (4.11)
$$

As a result of (4.6), (4.10), and (4.11),

$$
M_0 := \sup_{(x,i)\in D \times \mathbb{Z}^+} |u_0(x,i)| \leq \sup_{x\in \partial D, i\in \mathbb{Z}^+} \{|\phi(x,i)|\} + \varepsilon_0^{-1} \sup_{x\in D, i > n_0} \{|f(x,i)|\}
$$

$$
+ \sup_{x\in D, i \leq n_0} \{|f(x,i)|\} \sup_{x\in D, i \leq \tau_0} \{\mathbb{E}_{x,i}\tau_D\} < \infty. \quad (4.12)
$$
Since $u_0(x, i)$ is continuous in $\mathcal{D} \times \mathbb{Z}_+$ and $q_i(x) = \sum_{j \in \mathbb{Z}_+} q_{ij}(x)$ is continuous and bounded in $\mathcal{D} \times \mathbb{Z}_+$, it is easy to show that $\sum_{j \in \mathbb{Z}_+} q_{ij}(x)u_0(x, j)$ is continuous in $\mathcal{D} \times \mathbb{Z}_+$ and

$$
\sup_{(x, i) \in \mathcal{D} \times \mathbb{Z}_+} \left| \sum_{j \in \mathbb{Z}_+} q_{ij}u_0(x, j) \right| := \bar{M}_0 < \infty.
$$

(4.13)

Thus, for each $i \in \mathbb{Z}_+$, there exists a strong solution $u_1(x, i) \in W^{2,p}_{loc}(\mathcal{D}) \cap C(\mathcal{D})$ to

$$
\left\{ \begin{array}{l}
\mathcal{L}_i u_1(x, i) - q_i(x)u_1(x, i) = -\sum_{j \in \mathbb{Z}_+} q_{ij}(x)u_0(x, j) & \text{in } D \times \mathbb{Z}_+ \\
u_1(x, i)|_{\partial D} = \phi(x, i) & \text{on } \partial D \times \mathbb{Z}_+.
\end{array} \right.
$$

(4.14)

owing to [24, Theorem 9.1.5]. Similar to (4.12), we can use (4.13) to obtain that

$$
\sup_{(x, i) \in \mathcal{D} \times \mathbb{Z}_+} |u_1(x, i)| := M_1 < \infty.
$$

(4.15)

Continuing this way, we can define recursively $\{u_{m+1}(x, i)\} \in \mathbb{H}^{2,p}(D)$, the strong solution to

$$
\left\{ \begin{array}{l}
\mathcal{L}_i u_{m+1}(x, i) - q_i(x)u_{m+1}(x, i) = -\sum_{j \in \mathbb{Z}_+} q_{ij}(x)u_m(x, j) & \text{in } D \times \mathbb{Z}_+ \\
u_{m+1}(x, i)|_{\partial D} = \phi(x, i) & \text{on } \partial D \times \mathbb{Z}_+.
\end{array} \right.
$$

(4.16)

By the Feynman-Kac formula,

$$
u_{m+1}(x, i) = \mathbb{E}_{x,i} \left[ \phi(Y(\tau_D)) \exp \left( -\int_0^{\tau_D} q_i(Y(s))ds \right) \right] \\
+ \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( -\int_0^t q_i(Y(s))ds \right) \sum_{j \in \mathbb{Z}_+} q_{ij}(Y(t))u_m(Y(t), j)dt \\
- \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( -\int_0^t q_i(Y(s))ds \right) f(Y(t), i)dt.
$$

(4.17)

Let $\Delta_m(x, i) = u_{m+1}(x, i) - u_m(x, i)$ and

$$
\Delta_m^i = \sup\{|\Delta_{m+1}(x, i)| : x \in \mathcal{D}\}
$$

It follows from (4.17) that

$$
|\Delta_{m+1}(x, i)| = \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( -\int_0^t q_i(Y(s))ds \right) \sum_{j \in \mathbb{Z}_+} q_{ij}(Y(t))|\Delta_m(Y(t), j)|dt \\
\leq \sup_{i \in \mathbb{Z}_+} \{\Delta_m^i\} \int_0^{\tau_D} \exp \left( -\int_0^t q_i(Y(s))ds \right) q_i(Y(t))dt \\
= \sup_{i \in \mathbb{Z}_+} \{\Delta_m^i\} \mathbb{E}_{x,i} \left[ 1 - \exp \left( -\int_0^{\tau_D} q_i(Y(s))ds \right) \right].
$$

(4.18)

Let

$$
p := \max_{\{i \leq n_0\}} \mathbb{E}_{x,i} \left[ 1 - \exp \left( -\int_0^{\tau_D} q_i(Y(s))ds \right) \right] < 1.
$$

20
We have from (4.18) that
\[ \max_{\{i \leq n_0\}} \{\Delta_{m+1}^i\} \leq p \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_m^i\}. \] (4.19)

It also follows from (4.18) that
\[ \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_{m+1}^i\} \leq \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_m^i\}. \] (4.20)

For \( i > n_0 \), using (4.19) again and then using (4.19) and (4.20), we have
\[
|\Delta_{m+2}(x, i)| \leq \max_{\{i \leq n_0\}} \{\Delta_{m+1}^i\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) \sum_{j \leq n_0} q_{ij}(Y(t)) dt \\
+ \sup_{\{i > n_0\}} \{\Delta_{m+1}^i\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) \sum_{j \leq n_0} q_{ij}(Y(t)) dt \\
\leq p \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_{m+1}^i\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) \sum_{j \leq n_0} q_{ij}(Y(t)) dt \\
+ \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_{m+1}^i\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) \sum_{j \geq n_0} q_{ij}(Y(t)) dt \\
\leq p \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_{m+1}^i\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) \sum_{j \geq n_0} q_{ij}(Y(t)) dt \\
+ (1 - p) \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_{m+1}^i\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) \sum_{j > n_0} q_{ij}(Y(t)) dt. \tag{4.21}
\]

Let
\[ M_D = \sup_{(x,i) \in D \times N} q_i(x). \] (4.22)

Note that
\[ \sum_{j > n_0} q_{ij}(x) \leq 1 - \sum_{j \leq n_0} q_{ij}(x) \leq 1 - \frac{\varepsilon_0}{M_D} =: \varepsilon_1 \text{ for } i > n_0, \]
which implies that
\[
\mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) \sum_{j > n_0} q_{ij}(Y(t)) dt \\
\leq \varepsilon_1 \mathbb{E}_{x,i} \int_0^{\tau_D} \exp \left( - \int_0^t q_i(Y(s)) ds \right) q_i(Y(t)) dt \\
\leq \varepsilon_1 \text{ for } i > n_0. \tag{4.23}
\]

In view of (4.21) and (4.23), we have
\[
\sup_{\{i > n_0\}} \Delta_{m+2}^i \leq [p + (1 - p)\varepsilon_1] \sup_{\{i \in \mathbb{Z}_+\}} \{\Delta_{m+1}^i\}. \tag{4.24}
\]

21
By (4.19) and (4.20),

$$\sup_{\{i \leq m_0\}} \{\Delta^i_{m+2}\} \leq p \max_{\{i \in \mathbb{Z}_+\}} \{\Delta^i_{m+1}\} \leq \max_{\{i \in \mathbb{Z}_+\}} \{\Delta^i_m\}. \tag{4.25}$$

By (4.25) and (4.24),

$$\sup_{\{i \in \mathbb{Z}_+\}} \{\Delta^i_{m+2}\} \leq p + (1-p)\varepsilon_1 \max_{\{i \in \mathbb{Z}_+\}} \{\Delta^i_m\}. \tag{4.26}$$

In view of (4.12) and (4.13), sup$_{i \in \mathbb{Z}_+} \{\Delta^i_1\} \leq M_0 + M_1 < \infty$. Since $p + (1-p)\varepsilon_1 < 1$, it follows from (4.20) and (4.26) that the series $\sum_{m=1}^{\infty} \sup_{i \in \mathbb{Z}_+} \{\Delta^i_{m+2}\}$ is convergent. Thus $u_m(x, i)$ converges uniformly in $(x, i)$ to a function $u(x, i)$. For each $l \in \mathbb{Z}_+$, since $q_l(x) = \sum_j q_{ij}(x)$ is continuous, the convergence $\lim_{m \to \infty} \sum_{j<k} q_{ij}(x) = q_l(x)$ is uniform. Thus, it is easy to show that as $m \to \infty$, $\sum_{j<k} q_{ij}(x)u_m(x, j)$ converges uniformly to $\sum_j q_{ij}(x)u(x, j)$, (which is also continuous in $x$). Using this uniform convergence, passing the limit in (4.17) we have

$$u(x, i) = \mathbb{E}_{x, i} \left[ \phi_s(Y(\tau_D)) \exp \left( -\int_0^{\tau_D} q_s(Y(s)) ds \right) \right]$$

$$+ \mathbb{E}_{x, i} \int_0^{\tau_D} \exp \left( -\int_0^t q_s(Y(s)) ds \right) \sum_{j \in \mathbb{Z}_+} q_{ij}u(Y(t), j) dt$$

$$- \mathbb{E}_{x, i} \int_0^{\tau_D} \exp \left( -\int_0^t q_s(Y(s)) ds \right) f(Y(t), i) dt. \tag{4.27}$$

Since $\sum_j q_{ij}(x)u(x, j)$ is continuous in $x$ for each $i \in \mathbb{Z}_+$, the representation (4.27) shows that $u(x, i)$ satisfies

$$\mathcal{L}_i u(x, i) - q_i(x)u(x, i) = f(x, i) - \sum_{j \in \mathbb{Z}_+} q_{ij}(x)u(x, j) \text{ in } D \times \mathbb{Z}_+$$

Since $u_m(x, i) = \phi(x, i)$ on $\partial D \times \mathbb{Z}_+$ for all $m \in \mathbb{Z}_+$, we have $u(x, i) = \phi(x, i)$ on $\partial D \times \mathbb{Z}_+$. The existence of solutions is therefore proved. To prove the uniqueness, it suffices to consider the uniqueness in $\mathbb{H}^p(D)$ of the system

$$\mathcal{L}_i v(x, i) - q_i(x)v(x, i) + \sum_{j=1}^{\infty} q_{ij}(x)v(x, j) = 0 \text{ in } D \times \mathbb{Z}_+$$

$$v(x, i) \mid_{\partial D} = 0 \text{ on } \partial D \times \mathbb{Z}_+. \tag{4.28}$$

Let $\{v(x, i)\} \in \mathbb{H}^p(D)$ be a solution of (4.28). Then we have

$$v(x, i) = -\mathbb{E}_{x, i} \int_0^{\tau_D} \exp \left( -\int_0^t q_s(Y(s)) ds \right) \sum_{j \in \mathbb{Z}_+} q_{ij}v(Y(s), j) ds. \tag{4.29}$$

Similar to (4.19), it follows from (4.29) that

$$\sup_{i \leq n_0, x \in D} \{|v(x, i)|\} \leq p \sup_{i \in \mathbb{Z}_+, x \in D} \{|v(x, i)|\}. \tag{4.28}$$

Similar to (4.24), the above inequality and (4.29) imply that

$$\sup_{i \in \mathbb{Z}_+, x \in D} \{|v(x, i)|\} \leq [p + (1-p)\varepsilon_1] \sup_{i \in \mathbb{Z}_+, x \in D} \{|v(x, i)|\}. \tag{4.29}$$

Thus $\sup_{i \in \mathbb{Z}_+, x \in D} \{|v(x, i)|\} = 0$, that is, (4.28) has a unique solution. \hfill \square
Lemma 4.2. Let \( D \) be an open and bounded set of \( \mathbb{R}^n \). Let \( \xi_0 = 0 \) and \( \xi_k = \inf\{t \geq 0 : \alpha(t) \neq \alpha(\xi_{k-1})\}, k \in \mathbb{Z}_+ \). Let \( f(x, i) \) and \( g(x, i) \) are bounded and measurable functions on \( D \times \mathbb{Z}_+ \) and \( \partial D \times \mathbb{Z}_+ \) respectively. Then

\[
\mathbb{E}_{x,i} 1_{\{\xi_1 \leq \tau_D\}} f(X(\xi_1), \alpha(\xi_1)) + \mathbb{E}_{x,i} 1_{\{\xi_1 > \tau_D\}} g(X(\tau_D), i)
= \mathbb{E}_{x,i} \int_0^{\tau_D} q_{ij}(Y(t)) f(Y(t), j) \exp \left( - \int_0^t q_i(Y(s)) ds \right)
+ \mathbb{E}_{x,i} g(Y(\tau_D), i) \exp \left( - \int_0^{\tau_D} q_i(Y(t)) dt \right).
\]

(4.30)

Proof. Define

\[
\beta^{x,i}(t) = i + \int_0^t \int_{\mathbb{R}} h(Y_t^{x,i}, \beta^{x,i}(t-), z) p(dt, dz).
\]

Let \( \lambda^{x,i}_1 = \inf\{t \geq 0 : \beta^{x,i}(t) \neq i\} \). We have that

\[
(X^{x,i}(t), \alpha^{x,i}(t)) = (Y^{x,i}(t), \beta^{x,i}(t)) \text{ up to } \lambda^{x,i}_1 = \xi^{x,i}_1,
\]

where \( (X^{x,i}(t), \alpha^{x,i}(t)) \) is the solution to (2.3) with initial value \((x, i)\) and \( \xi^{x,i}_1 \) is the first moment of jump for \( \alpha^{x,i}(t) \). Thus,

\[
\mathbb{P}_{x,i} \{\xi_1 \wedge \tau_D < \infty\} = \mathbb{P}_{x,i} \{\lambda_1 \leq \tau_D\} \geq \mathbb{P}_{x,i} \{\tau_D < \infty\} = 1.
\]

(4.32)

In view of [16, Lemma 4.2],

\[
\mathbb{E}_{x,i} 1_{\{\lambda_1 \leq \tau_D\}} f(Y(\lambda_1), \beta(\lambda_1)) + \mathbb{E}_{x,i} 1_{\{\lambda_1 > \tau_D\}} g(Y(\tau_D), i)
= \mathbb{E}_{x,i} \int_0^{\tau_D} q_{ij}(Y(t)) f(Y(t), j) \exp \left( - \int_0^t q_i(Y(s)) ds \right)
+ \mathbb{E}_{x,i} g(Y(\tau_D), i) \exp \left( - \int_0^{\tau_D} q_i(Y(t)) dt \right).
\]

(4.33)

Combining (4.31) and (4.33), we obtain (4.30).

Lemma 4.3. Let \( D \) be an open bounded set in \( \mathbb{R}^n \). For any \( \varepsilon > 0 \), there is an \( n_2 = n_2(\varepsilon) > 0 \) such that

\[
\mathbb{P}_{x,i} \{\xi_{n_2} \leq \tau_D\} < \varepsilon
\]

for any \((x, i) \in \mathcal{B} \times \mathbb{Z}_+\). As a result,

\[
\mathbb{P}_{x,i} \{\tau_D < \infty\} = 1.
\]

Moreover, for any \( k > 0 \), there is a \( T > 0 \) such that

\[
\mathbb{P}_{x,i} \{\xi_k \wedge \tau_D > T\} < \varepsilon.
\]

Proof. For each \( i \in \mathbb{Z}_+ \), we have that

\[
p_{i,D} := \sup_{x \in D} \mathbb{E}_{x,i} \tau_D < \infty.
\]

23
By Lemma 4.2 with $M_D$ defined as in (4.22), we have
\[
\mathbb{P}_{x,i}\{\xi_1 > \tilde{\tau}_D\} = \mathbb{E}_{x,i} \exp \left(- \int_0^{\tau_B} q_i(Y(t))dt \right)
\geq \mathbb{E}_{x,i} \exp (-M_D\tau_D)
\geq \exp (-M_D\mathbb{E}_{x,i}\tau_D)
\geq \exp (-M_D p_{i,D}).
\] (4.34)

Let $\tilde{p} := \min_{i \leq n_0} \{\exp (-M_D p_{i,D})\}$. By (4.2), for $i > n_0$,
\[
\sum_{j \leq n_0} q_{ij}(x) \geq \varepsilon_0 M_D > 0, x \in D.
\]
Applying Lemma 4.2 again, we have
\[
\mathbb{P}_{x,i}\{\xi_1 \leq \tilde{\tau}_D, \alpha(\xi_1) \leq n_0\} = \mathbb{E}_{x,i} \int_0^{\tau_B} \sum_{j \leq n_0} q_{ij}(Y(t)) \exp \left(- \int_0^t q_i(Y(s))ds \right) dt
\geq \frac{\varepsilon_0}{M_D} \mathbb{E}_{x,i} \int_0^{\tau_B} q_i(Y(t)) \exp \left(- \int_0^t q_i(Y(s))ds \right) dt
\geq \frac{\varepsilon_0}{M_D} \mathbb{P}_{x,i}\{\xi_1 \leq \tilde{\tau}_D\}. \tag{4.35}
\]
By the strong Markov property, (4.34), and (4.35), we have for $i > n_0$ that
\[
\mathbb{P}_{x,i}\{\xi_1 \leq \tilde{\tau}_D, \xi_2 > \tilde{\tau}_D\} \geq \mathbb{P}_{x,i}\{\xi_2 < \tilde{\tau}_D, \xi_1 > \tilde{\tau}_D, \alpha(\xi_1) \leq n_0\}
\geq \mathbb{P}_{x,i}\{\xi_1 \leq \tilde{\tau}_D, \alpha(\xi_1) \leq n_0\} \left[ \inf_{y \in D, j \leq n_0} \mathbb{P}_{y,j}\{\xi_1 < \tilde{\tau}_D\} \right]
\geq \frac{\tilde{p}\varepsilon_0}{M_D} \mathbb{P}_{x,i}\{\xi_1 \leq \tilde{\tau}_D\}. \tag{4.36}
\]
Since $\tilde{p} < 1$ and $\frac{\varepsilon_0}{M_D} \geq 1$, we have from (4.35) that
\[
\mathbb{P}_{x,i}\{\xi_2 > \tilde{\tau}_D\} \geq \mathbb{P}\{\xi_1 > \tilde{\tau}_D\} + \mathbb{P}_{x,i}\{\xi_1 \leq \tilde{\tau}_D, \xi_2 > \tilde{\tau}_D\}
\geq \mathbb{P}\{\xi_1 > \tilde{\tau}_D\} + \frac{\tilde{p}\varepsilon_0}{M_D} \mathbb{P}_{x,i}\{\xi_1 \leq \tilde{\tau}_D\}
\geq \frac{\tilde{p}\varepsilon_0}{M_D} \text{ for } x \in D, i > n_0. \tag{4.37}
\]
In light of (4.34),
\[
\mathbb{P}_{x,i}\{\xi_2 > \tilde{\tau}_D\} \geq \mathbb{P}\{\xi_1 > \tilde{\tau}_D\} \geq \tilde{p} \text{ for } x \in D, i \leq n_0. \tag{4.38}
\]
Thus, for any $x \in D$ and $i \in \mathbb{Z}_+$, we have
\[
\mathbb{P}_{x,i}\{\xi_2 > \tilde{\tau}_D\} \geq \frac{\tilde{p}\varepsilon_0}{M_D} \tag{4.39}
\]
Using the strong Markov property, we have from (4.39) that
\[
P_{x,i}\{\xi_{2k} \leq \tilde{\tau}_D\} \leq \left(1 - \frac{\tilde{p}\varepsilon_0}{MD}\right)^k.
\] (4.40)

By letting \(n_2 = 2k_2 + 1\) with \(k_2\) being sufficiently large so that \(\left(1 - \frac{\tilde{p}\varepsilon_0}{MD}\right)^{k_2} < \varepsilon\), we complete the proof for the first part of this lemma.

To prove the second part, note that \(E_{x,i}\tau_D \leq p_{i,D} < \infty\), thus for any \(\varepsilon' > 0\), there is \(T_1 > 0\) such that
\[
P_{x,i}\{\xi_1 \wedge \tilde{\tau}_D \leq T_1\} = P_{x,i}\{\lambda_1 \wedge \tau_D \leq T_1\}
\geq P_{x,i}\{\tau_D \leq T_1\} > 1 - \varepsilon' \text{ for all } x \in D, i \leq n_0.\] (4.41)

For \(i > n_0\), we have
\[
P_{x,i}\{\xi_1 \wedge \tilde{\tau}_D > T\} = P_{x,i}\{\tau_D > T, \lambda_1 > T\}
\geq E_{x,i}\left[1_{\{\tau_D > T\}} \int_0^T q_i(Y(t)) \exp\left(-\int_0^t q_i(Y(s))ds\right)dt\right]
= E_{x,i}\left[1_{\{\tau_D > T\}} \exp\left(-\int_0^T q_i(Y(s))ds\right)\right]
\leq E_{x,i}\left[1_{\{\tau_D > T\}} \exp(-T\varepsilon_0)\right] \text{ (since } q_i(x) > \varepsilon \text{ if } x \in D, i > n_0).\] (4.42)

Let \(T_2 > T_1\) such that \(\exp(-T_2\varepsilon) < \varepsilon'\). We have from (4.42) that
\[
P_{x,i}\{\xi_1 \wedge \tilde{\tau}_D \leq T_2\} > 1 - \varepsilon' \text{ for } x \in D, i > n_0.\] (4.43)

Using (4.41) and (4.43),
\[
P_{x,i}\{\xi_1 \wedge \tilde{\tau}_D \leq T_2\} > 1 - \varepsilon' \text{ for } x \in D, i \in \mathbb{Z}_.\] (4.44)

Using the strong Markov property, it is easy to show that
\[
P_{x,i}\{\xi_k \wedge \tilde{\tau}_D \leq kT_2\} > (1 - \varepsilon')^k \text{ for } x \in D, i \in \mathbb{Z}_.\] (4.45)

By choosing \(\varepsilon'\) such that \((1 - \varepsilon')^k > 1 - \varepsilon\), we obtain the second part of this lemma. \(\square\)

**Lemma 4.4.** Let \(D \in \mathbb{R}^n\) be a bounded set. For \(i_0 \in \mathbb{Z}_+, T > 0, \varepsilon > 0\), there is a \(k_0 = k_0(i_0, T, \varepsilon) > 0\) such that
\[
P_{x,i_0}\{\zeta_{k_0} > T\} < \varepsilon, x \in D,
\]
where \(\zeta_k = \inf\{t > 0 : \alpha(t) \geq k\}\).

**Proof.** This lemma is a direct consequence of [16, Theorem 4.5] and the Heine-Borel covering theorem. \(\square\)

To proceed, we need the following lemma, which is a weak form of Harnack’s principle.
Lemma 4.5. Let $D$ be an open bounded set in $\mathbb{R}^n$ with $\partial D \in C^2$ and fix $(x_0, i_0) \in D \times \mathbb{Z}_+$. Let $B \subset \overline{B} \subset D$ be a ball centered at $x_0$. Then for any $\varepsilon > 0$, there is a $c_0 = c_0(B, i_0, \varepsilon) > 0$ satisfying

$$u(x, i_0) \leq c_0 u(x_0, i_0) + \varepsilon \sup_{\{y, i\} \in \partial D \times \mathbb{Z}_+} \{u(y, i)\}, x \in \overline{B},$$

where $\{u(x, i)\} \in \mathbb{H}^p(D)$ satisfies

$$L_i u(x, i) - q_i(x) u(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) u(x, j) = 0 \text{ in } D \times \mathbb{Z}_+.$$ 

Proof. Let $\phi(x, i) = u(x, i)|_{\partial D}$ and

$$\zeta_k = \inf\{t > 0 : \alpha(t) \geq k\}.$$

Let

$$u_k(x, i) = \begin{cases} u(x, i) & \text{if } i < k \\ 0 & \text{if } i \geq k. \end{cases}$$

By Itô’s formula,

$$\mathbb{E}_{x,i} u_k(X(\tau_D \wedge \zeta_k \wedge t), \alpha(\tau_D \wedge \zeta_k \wedge t))
= u_k(x, i) + \mathbb{E}_{x,i} \int_0^{\tau_D \wedge \zeta_k \wedge t} \mathcal{L} u_k(X(s), \alpha(s)) ds$$
$$= u_k(x, i) - \mathbb{E}_{x,i} \int_0^{\tau_D \wedge \zeta_k \wedge t} \sum_{j \geq k} q_{\alpha(s), j}(X(s)) u_k(X(s), j) ds.$$ 

Letting $k \to \infty$ and then $t \to \infty$, we obtain from the dominated convergence theorem that

$$u(x, i) = \mathbb{E}_{x,i} \phi(X(\tau_D), \alpha(\tau_D)).$$

As a result of Lemmas 4.3 and 4.4, there is a $k_1 = k_1(i_0, \varepsilon) \in \mathbb{Z}_+$ such that

$$\mathbb{P}_{x,i} \{\tau_D > \xi_{k_1} \} < \varepsilon.$$ 

In view of (4.47) and (4.48),

$$u(x, i_0) = \mathbb{E}_{x,i_0} 1_{(\tau_D < \xi_{k_1})} \phi(X(\tau_D), \alpha(\tau_D)) + \mathbb{E}_{x,i_0} 1_{(\tau_D > \xi_{k_1})} \phi(X(\tau_D), \alpha(\tau_D))$$
$$\leq \mathbb{E}_{x,i_0} 1_{(\tau_D < \xi_{k_1})} \phi(X(\tau_D), \alpha(\tau_D)) + \varepsilon \sup_{(y,j) \in \partial D \times \mathbb{Z}_+} \{\phi(y, j)\}. \tag{4.49}$$

Let

$$\tilde{u}(x, i) = \mathbb{E}_{x,i} 1_{(\tau_D < \xi_{k_1})} \phi(X(\tau_D), \alpha(\tau_D))$$

for $i < k$. The process $\{(X(t), \alpha(t)) \leq t < \xi_{k_1}\}$ can be considered as a switching diffusion process on $\mathbb{R}^n \times \{1, \ldots, k_1 - 1\}$ with lifetime $\xi_{k_1}$. Its generator is

$$\mathcal{L}_i f(x, i) = \mathcal{L}_i f(x, i) - q_i(x) u(x, i) + \sum_{j < k_1} q_{ij}(x) f(x, j),$$

26
for \( i = 1, \ldots, k - 1 \). Then [3, Theorem 3.6] reveals that \( \bar{u}(x, i) \) satisfying

\[
\begin{align*}
\mathcal{L}_i \bar{u}(x, i) &= 0 \text{ in } D \times \{1, \ldots, k-1\} \\
\bar{u}(x, i)|_{\partial D} &= \phi(x, i) \text{ on } \partial D \times \{1, \ldots, k-1\}.
\end{align*}
\]  

(4.50)

By the Harnack principle for weakly coupled elliptic systems (see e.g., [2]), there is a \( c_0 = c_0(k_1) \) such that

\[
\bar{u}(x, i_0) \leq c_0 \bar{u}(x, i_0) \leq c_0 u(x, i_0)
\]

(4.51)

The desired result follows from (4.49) and (4.51).

Remark 4.2. In (4.9) and (1.17), we apply the Feynman-Kac formula for functions in the class \( W^{2,p}_{\text{loc}}(D) \cap C(D) \) rather than \( C^2(D) \). Feynman-Kac formula is proved using Itô’s formula, which is usually stated for \( C^2 \)-functions. However, Itô’s formula also holds for diffusion processes with functions in \( W^{2,p}_{\text{loc}}(D) \cap C(D) \) when \( p > n \). The proof for this claim can be found in [12, Theorem 2.10.2]. With a careful consideration, we can generalize the result for diffusion processes to switching diffusion processes in which the switching has a finite state space. Thus, (1.46) holds as long as \( u(\cdot, i) \in W^{2,p}_{\text{loc}}(D) \cap C(D) \).

Proof of Theorem 4.2. Let \( k_0 \in \mathbb{Z}_+ \) sufficiently large such that \( D_1 = \{ x \in \mathbb{R}^n : |x| < k_0 \} \). For \( k > k_0 \), define \( D_k = D \cap \{ x \in \mathbb{R}^n : |x| < k \} \). By (4.47), \( u_k(x, i) := \mathbb{E}_{x,i}\tilde{\tau}_{D_k} \) satisfies the equation

\[
\begin{align*}
\mathcal{L}_i u_k(x, i) - q_i(x)u_k(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x)u_k(x, j) &= -1 \text{ in } D_k \times \mathbb{Z}_+ \\
u_k(x, i)|_{\partial D_k} &= 1 \text{ on } \partial D_k \times \mathbb{Z}_+.
\end{align*}
\]  

(4.52)

Let \( B_1 \subset B_2 \) be two balls in \( D \) and fix \( (x_0, i_0) \in B_1 \times \mathbb{Z}_+ \) and let \( k_1 > k_0 \) be such that \( B_2 \subset D_{k_1} \). Suppose that \( \mathbb{E}_{x,i}\tilde{\tau}_D < M \) for any \( (x, i) \in B_2 \times \mathbb{Z}_+ \). Then \( u_k(x, i) < M \) for \( k > k_0 \) and \( (x, i) \in B_2 \times \mathbb{Z}_+ \). Let \( v_{k,m} = u_k(x, i) - u_m(x, i) \) for \( k > m > k_1 \), we have

\[
\begin{align*}
\mathcal{L}_i v_{k,m}(x, i) - q_i(x)v_{k,m}(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x)v_{k,m}(x, j) &= 0 \text{ in } B_2 \times \mathbb{Z}_+.
\end{align*}
\]  

(4.53)

By Lemma 4.5 for any \( \varepsilon > 0 \), there is a \( c_0 > 0 \) such that

\[
\begin{align*}
v_{k,m}(x, i_0) &\leq c_0 v_{k,m}(x_0, i_0) + \varepsilon \sup\{v_{k,m}(y, j) : (y, j) \in B_2 \times \mathbb{Z}_+\} \\
&\leq c_0 v_{k,m}(x_0, i_0) + M\varepsilon \text{ for any } x \in B_1.
\end{align*}
\]  

(4.54)

For any \( \varepsilon > 0 \), since \( u_k(x_0, i_0) = \mathbb{E}_{x_0,i_0}\tilde{\tau}_{D_k} \to \mathbb{E}_{x_0,i_0}\tilde{\tau}_D \) as \( k \to \infty \), there exists \( k_2 = k_2(\varepsilon) \) such that \( c_0 v_{k,m}(x_0, i_0) = c_0[u_k(x_0, i_0) - u_m(x_0, i_0)] < \varepsilon \) for any \( k > m > k_2 \). In view of (4.54),

\[
v_{k,m}(x, i_0) \leq (M + 1)\varepsilon \text{ for any } (x, i_0) \in B_1 \times \mathbb{Z}_+, k > m > k_2
\]  

(4.55)

Thus, \( u_k(x, i_0) \) converges uniformly in each compact subset of \( D \). The limit \( u(x, i_0) \) is therefore continuous for any \( i_0 \). Now, let \( \phi(x, i) = u(x, i)|_{\partial B_2} \). Since \( \phi(x, i) \) is continuous and uniformly bounded, by Lemma 4.11 for each \( i \in \mathbb{Z}_+ \), there exists \( \{\bar{u}(x, i)\} \in H^p(B_2) \) satisfying

\[
\begin{align*}
\mathcal{L}_i \bar{u}(x, i) - q_i(x)\bar{u}(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x)\bar{u}(x, j) &= -1 \text{ in } B_2 \times \mathbb{Z}_+ \\
\bar{u}(x, i)|_{\partial D} &= \phi(x, i) \text{ on } \partial B_2 \times \mathbb{Z}_+.
\end{align*}
\]  

(4.56)
Similar to (4.47), by applying Itô’s formula we have that
\[
\tilde{u}(x, i) = E_{x, i} \tilde{\tau}_{B_2} + E_{x, i} \phi(X(\tilde{\tau}_{B_2}), \alpha(\tilde{\tau}_{B_2})) \\
= E_{x, i} \tilde{\tau}_{B_2} + E_{x, i} E(X(\tilde{\tau}_{B_2}), \alpha(\tilde{\tau}_{B_2})) \vec{r}_D \\
= E_{x, i} \tilde{\tau}_D \text{ (due to the strong Markov property)} \\
= u(x, i).
\]

The proof is concluded.

Proof of Theorem 4.1. After having Lemma 4.5, we adapt the proof of [10, Theorem 3.10] to obtain the desired result. First, suppose that (4.3) has a unique solution in \(\mathbb{H}^p(D)\) for some \(p > 0\) given that \(f(x, i)\) is continuous and bounded on \(D \times \mathbb{Z}_+\). We define \(D_k\) as in the proof of Theorem 4.2 and \(v_k(x, i) := P_{x,i} \{X(\hat{\tau}_{D_k}) \in \partial D\}\). By (4.47), \(v_k(x, i)\) is the strong solution to
\[
\begin{aligned}
\mathcal{L}_i v_k(x, i) - q_i(x) v_k(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) v_k(x, j) &= 0 \text{ in } D_k \times \mathbb{Z}_+ \\
v_k(x, i) \big|_{\partial D} &= 1 \text{ on } \partial D \times \mathbb{Z}_+ \\
v_k(x, i) \big|_{\partial D} &= 0 \text{ on } \{y \in \mathbb{R}^n : |y| = k\} \times \mathbb{Z}_+.
\end{aligned}
\]

By the definition of \(v_k(x, i)\), we have that
\[
\lim_{k \to \infty} v_k(x, i) = v(x, i) := P_{x,i} \{\hat{\tau}_D < \infty\}
\]

On the other hand, owing to Lemma 4.5, we can use arguments in the proof of Theorem 4.2 to show that \(\{v(x, i)\} \in \mathbb{H}^p(D)\) is the solution to
\[
\begin{aligned}
\mathcal{L}_i v(x, i) - q_i(x) v(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) v(x, j) &= 0 \text{ in } D \times \mathbb{Z}_+ \\
v(x, i) \big|_{\partial D} &= 1 \text{ on } \partial D \times \mathbb{Z}_+.
\end{aligned}
\]

Clearly, \(v(x, i) \equiv 1\) is the solution to (4.60). By the uniqueness of solutions among the class \(\mathbb{H}^p(D)\), we have
\[
P_{x,i} \{\hat{\tau}_D < \infty\} = v(x, i) \equiv 1.
\]

Now, suppose that \(P_{x,i} \{\hat{\tau}_D < \infty\} \equiv 1\) and (4.3) has two solutions \(\{v^{(1)}(x, i)\}\) and \(\{v^{(2)}(x, i)\}\) for the same \(f(x, i)\) being continuous and bounded in \(\partial D \times \mathbb{Z}_+\). Let \(v^{(3)}(x, i) = v^{(1)}(x, i) - v^{(2)}(x, i)\). Then \(\{v^{(3)}(x, i)\} \in \mathbb{H}^p(D)\) and satisfies
\[
\begin{aligned}
\mathcal{L}_i v^{(3)}(x, i) - q_i(x) v^{(3)}(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) v^{(3)}(x, j) &= 0 \text{ in } D \times \mathbb{Z}_+ \\
v^{(3)}(x, i) \big|_{\partial D} &= 0 \text{ on } \partial D \times \mathbb{Z}_+.
\end{aligned}
\]

Let \(M^{(3)} = \sup_{x \in D \times \mathbb{Z}_+} \{v^{(3)}(x, i)\}\). In view of (4.46), for \(k > k_0 \lor |x|\), we have
\[
|v^{(3)}(x, i)| = |E_{x, i} 1_{\{X(\hat{\tau}_{D_k}) = k\}} v^{(3)}(X(\hat{\tau}_{D_k}), \alpha(\hat{\tau}_{D_k}))| \leq M^{(3)} |1 - P_{x,i} \{X(\hat{\tau}_{D_k}) \in \partial D\}|
\]

Letting \(k \to \infty\) and using \(P_{x,i} \{X(\hat{\tau}_{D_k}) \in \partial D\} \to P_{x,i} \{\hat{\tau}_D < \infty\} = 1\) as \(k \to \infty\), we obtain \(v^{(3)}(x, i) \equiv 0\).
A Appendix

Let \( Y^{x,i}(t) \) be the solution to
\[
dY(t) = b(Y(t), i) dt + \sigma(Y(t), i) dW(t), \quad t \geq 0
\] (A.1)
with initial condition \((x, i) \in \mathbb{R}^n \times \mathbb{Z}^+\). For \((\phi, i) \in C \times \mathbb{Z}^+_+\), we denote by \( Y^{\phi,i}(t), t \geq -r \)
be the process satisfying \( Y^{\phi,i}(t) = \phi \) if \( t \in [-r, 0] \) and \( Y^{\phi,i}(t) \) solves (A.1) for \( t > 0 \). Clearly
\( Y^{\phi,i}(t) = Y^{\phi(0),i}(t) \) for \( t \geq 0 \). Let \( \beta^{\phi,i} \) be the solution to
\[
\beta^{\phi,i}(t) = i + \int_0^t \int_{\mathbb{R}} h(Y^{\phi,i}_t, \beta^{\phi,i}(t-), z) p(dt, dz), \quad t \geq 0
\] (A.2)
satisfying \( Y^{\phi,i}(t) = \phi(t) \) in \([-r, 0]\) and \( \beta^{\phi,i}(0) = i \). Let \( \xi^{\phi,i}_1(t) \) and \( \lambda^{\phi,i}_1(t) \) be the first jump times of \( \alpha^{\phi,i}(t) \) and \( \beta^{\phi,i}(t) \), respectively. Clearly we have that
\[
X^{\phi,i}(t) = Y^{\phi,i}(t), \quad \alpha^{\phi,i}(t) = \beta^{\phi,i}(t) \quad \text{up to} \quad \xi^{\phi,i}_1(t) = \lambda^{\phi,i}_1(t).
\] (A.3)

Proof of Lemma 3.1 Since \( q_{ij} \cdot \) is continuous, there is an \( \varepsilon \in (0, 1) \) such that \( q_{ij} > 0 \)
given that \( \|\psi - \phi\| < \varepsilon \). Let \( M_{\phi} = \sup_{\psi \in C, \|\psi - \phi\| < 1} \{q_i(\psi)\} < \infty \). Let \( \delta_1 > 0 \) such that
\[
|\psi(s) - \phi(s')| < \frac{\varepsilon}{5} \quad \text{provided} \quad |s-s'| < \delta_1, s, s' \in [-r, 0].
\] (A.4)

Under either Assumption 2.1 or Assumption 2.2, standard arguments show that there exists a sufficiently small \( \delta_2 \in (0, \delta_1] \) satisfying
\[
\mathbb{P}_{\psi,i} \left\{ |Y(t) - \psi(0)| \leq \frac{\varepsilon}{5} \forall t \in [0, \delta_2] \right\} \geq \frac{1}{2}, \quad \forall \psi \in C, \|\psi - \phi\| < \varepsilon,
\] (A.5)
and
\[
\mathbb{P}_{\psi,j} \left\{ |Y(t) - \psi'(0)| \leq \frac{\varepsilon}{5} \forall t \in [0, \delta_2] \right\} \geq \frac{1}{2}, \quad \forall \psi' \in C, \|\psi' - \phi\| < \varepsilon.
\] (A.6)

In view of (A.4), it can be checked that
\[
\|Y^{\psi,i}_t - \phi\| \leq \frac{3\varepsilon}{5} \forall t \in [0, \delta_2] \text{ if } |Y^{\psi,i}(t) - \psi(0)| \leq \frac{\varepsilon}{5} \forall t \in [0, \delta_2] \text{ and } \|\psi - \phi\| < \frac{\varepsilon}{5}
\] (A.7)
and
\[
\|Y^{\psi,j}_t - \phi\| \leq \varepsilon \forall t \in [0, \delta_2] \text{ if } |Y^{\psi,j}(t) - \psi'(0)| \leq \frac{\varepsilon}{5} \forall t \in [0, \delta_2] \text{ and } \|\psi' - \phi\| < \frac{3\varepsilon}{5}.
\] (A.8)

By virtue of (A.5), (A.7), and [16, Lemma 4.2], for \( \psi \in C, \|\psi - \phi\| < \frac{\varepsilon}{5} \) we have
\[
\mathbb{P}_{\psi,i} \left\{ \|Y_{\lambda_1} - \phi\| \leq \frac{3\varepsilon}{5} \text{ and } \lambda_1 < \delta_2, \beta(\lambda_1) = j \right\} = \mathbb{E}_{\psi,i} \left[ \int_0^{\delta_2} 1_{\|Y_t - \phi\| \leq \frac{\varepsilon}{5}} q_{i,j}(Y_t) \exp \left( -\int_0^t q_i(Y_s) ds \right) dt \right]
\geq \mathbb{E}_{\psi,i} \left[ \int_0^{\delta_2} 1_{\|Y_t - \phi(0)\| \leq \frac{\varepsilon}{5}, t \in [0, \delta_2]} q_{i,j}(Y_t) \exp \left( -\int_0^t q_i(Y_s) ds \right) dt \right]
\geq \frac{\delta_2}{2} \inf_{\phi' \in C, \|\phi' - \phi\| \leq \frac{\varepsilon}{5}} \left\{ q_{i,j}(\phi') \right\} \times \inf_{\phi' \in C, \|\phi' - \phi\| \leq \frac{\varepsilon}{5}} \left\{ \exp \left( -\int_0^{\delta_2} q_i(\phi'(s)) ds \right) \right\} := p_1 > 0.
\] (A.9)
Now, we have from the Markov property that
\[
\mathbb{P}_{\psi,i}\{\|X_{\delta_2} - \phi\| < \varepsilon, \alpha(\delta_2) = j\} \\
\geq \mathbb{P}_{\psi,i}\{\xi_1 < \delta_2, \alpha_1 = j, \|X_{\xi_1} - \phi\| < \frac{3\varepsilon}{5}\} \\
\times \mathbb{P}_{\psi,i}\{\|X_{\delta_2} - \phi\| < \varepsilon, \xi_2 > \delta_2 | \xi_1 < \delta_2, \alpha_1 = j, \|X_{\xi_1} - \psi\| < \varepsilon\}.
\]
(A.10)

By (A.5) and (A.8), if \(\|\psi' - \phi\| \leq \frac{3\varepsilon}{5}\), then
\[
\mathbb{P}_{\psi,j}\{\|X_t - \phi\| < \varepsilon \forall t \in [0, \delta_2], \xi_1 > \delta_2\} \\
= \mathbb{P}_{\psi,j}\{\|Y_t - \phi\| < \varepsilon \forall t \in [0, \delta_2], \lambda_1 > \delta_2\} \\
\geq \mathbb{E}_{\psi,j}\left[1_{\{\|Y_t - \phi\| < \varepsilon \forall t \in [0, \delta_2]\}} \exp\left(-\int_0^{\delta_2} q_i(Y_s)ds\right)\right] \\
\geq \mathbb{P}_{\psi,j}\left\{|Y(t) - \psi'(0)| \leq \frac{\varepsilon}{5} \forall t \in [0, \delta_2]\right\} \times \inf_{\phi' \in C: \|\phi' - \phi\| \leq \varepsilon} \left\{\exp\left(-\int_0^{\delta_2} q_i(\phi'(s))ds\right)\right\} \\
:= p_2 > 0.
\]
(A.11)

By the strong Markov property of \((X_t, \alpha(t))\), applying estimates (A.9) and (A.11) to (A.10), we obtain
\[
\sup_{\psi \in C: \|\psi - \phi\| \leq \frac{3\varepsilon}{5}} \mathbb{P}_{\psi,i}\{\|X_{\delta_2} - \phi\| < \varepsilon, \alpha(\delta_2) = j\} > p_1 p_2 > 0.
\]

Proof of Lemma 3.2. Using the Kolmogorov-Centsov theorem [9, Theorem 2.8], for each \(i \in \mathbb{Z}_+\) and \(R > 0\), there exists an \(h_{i,R} > 0\) such that
\[
\mathbb{P}_{\phi,i}\left\{\sup_{t,s \in [0,r], 0 < t - s < h_{i,R}} \frac{|Y(t) - Y(s)|}{(s-t)^{0.25}} \leq 4\right\} > \frac{1}{2}, \forall \|\phi\| \leq R.
\]
(A.12)

The detailed justification was given in the proof of [16, Lemma 4.5]. Let
\[
\mathcal{A} = \left\{\phi \in C: |\phi(-r)| \leq R, \sup_{t,s \in [-r,0], 0 < t - s < h_{i,R}} \frac{|\phi(t) - \phi(s)|}{(s-t)^{0.25}} \leq 4\right\}.
\]

Let \(R' > R\) such that \(\|\phi\| < R'\) for any \(\phi \in \mathcal{A}\) and \(M_{R'} = \sup_{\|\phi\| < R'} \{q_i(\phi)\} < \infty\). For \(\|\phi\| \leq R\), we have that
\[
\mathbb{P}_{\psi,i}\{X_r \in \mathcal{A}, \xi_1 > r\} = \mathbb{P}_{\psi,i}\{Y_r \in \mathcal{A}, \lambda_1 > r\} \\
= \mathbb{E}_{\psi,i}\left[1_{\{Y_r \in \mathcal{A}\}} \int_0^r \exp(-q_i(Y_s))ds\right] \\
\geq \mathbb{E}_{\psi,i}\left[1_{\{Y_r \in \mathcal{A}\}} \exp(-r M_{R'})\right] \\
\geq 0.5 \exp(-r M_{R'}).
which implies (3.2).

To prove (3.3), note that \( \sup_{(x,i) \in \mathbb{R}^n \times \mathbb{Z}_+} \mathbb{E}_{x,i} \tau_k < \infty \) where \( \tau_k = \inf \{ t \geq 0 : |Y(t)| \geq k \} \), (see e.g., [10, Corollary 3.3] or [26, Theorem 3.1]). Thus, there is a \( T > 0 \) such that

\[
\mathbb{P}_{x,i} \left\{ \tau_k < T \right\} > \frac{1}{2}, \quad \forall x \in \mathbb{R}^n.
\]

Denote \( \tilde{\tau}_k = \inf \{ t \geq 0 : \|X_t\| \geq k \} \). For \( \phi \in \mathcal{C} \) with \( \|\phi\| \leq R < k \) we have from (A.3) and [16, Lemma 4.2] that

\[
\mathbb{P}_{\phi,i} \left\{ \tilde{\tau}_k < T \right\} \geq \mathbb{P}_{\phi,i} \left\{ \tilde{\tau}_k < T, \alpha(t) = i \text{ for } t \in [0, \tilde{\tau}_k) \right\} = \mathbb{P}_{\phi,i} \left\{ \tilde{\tau}_k < T, \beta(t) = i \text{ for } t \in [0, \tau_k) \right\} = \mathbb{E}_{\phi,i} \left[ 1_{\{\tau_k < T\}} \exp \left( -\int_0^{\tau_k} q_i(Y_s)ds \right) \right] \geq \exp (-M_k T) \mathbb{E}_{\phi,i} 1_{\{\tau_k < T\}} \geq 0.5 \exp (-M_k T),
\]

where \( M_k = \sup_{\|\phi\| < k} \{ q_i(\phi) \} < \infty \). The proof is therefore complete. \( \square \)

We need an auxiliary lemma to obtain Lemma 3.3.

**Lemma A.1.** Fix \( i \in \mathbb{Z}_+ \) and suppose \( A(x,i) \) is elliptic uniformly in each compact subset of \( \mathbb{R}^n \). For \( D \) be a bounded open set in \( \mathbb{R}^n \) and \( K_1, K_2 \) be open sets whose closures are contained in \( D \). Then

\[
\inf_{\{\phi \in \mathcal{C} : \phi(0) \in K_1\}} \mathbb{P}_{\phi,i} \left\{ \{Y(T) \in K_2\} \cap \{Y(t) \in D\forall t \in [0,T]\} \right\} > 0 \tag{A.13}
\]

and there is a measure \( \nu \) on \( \mathcal{B}(\mathcal{C}) \) such that

\[
\mathbb{P}_{\phi,i} \{ Y_{T+r} \in \mathcal{B}, Y(t) \in D, t \in [0,T+r] \} \geq \nu(\mathcal{B}).
\]

Moreover, if \( \mathcal{B} \subset \{ \phi \in \mathcal{C} : \phi(t) \in D, t \in [-r,0] \} \) is an open set of \( \mathcal{C} \), then \( \nu(\mathcal{B}) > 0 \).

*Proof.* For a bounded continuous function \( f(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) vanishing outside \( K_2 \), let \( u_f(t,x) \) be the solution to

\[
\begin{align*}
\frac{\partial u}{\partial t} + \mathcal{L}_i u &= 0 \text{ in } D \times [0,T) \\
u(T,x) &= f(x) \text{ on } D, \\
u(t,x) &= 0 \text{ on } \partial D \times [0,T].
\end{align*}
\]

(A.14)

It is well known (see, e.g., [13, Theorem 2.8.2]) that

\[
u_f(t,x) = \mathbb{E}_{x,i} \left[ f(Y(T-t)) 1_{\{Y(s) \in D \\forall s \in [0,T-t]\}} \right].
\]

Let \( g \) be a continuous function in \( D \) such that \( 0 \leq g(x) \leq 1 \ \forall x \in D \), \( g(x) = 0 \) outside \( K_2 \) and \( g(x) > 0 \) for some \( x \in K_2 \). By the strong maximum principle for parabolic equations (see [8, Theorem 7.12]), \( u_g(0,x) > 0 \) for all \( x \in D \), which implies that

\[
u_C := \inf \{ u_g(0,x) : x \in K_1 \} > 0. \tag{A.15}
\]
By the definition of $g(\cdot)$, we can obtain that
\[ \mathbb{P}_{x,i} \{ Y(T) \in K_2, Y(s) \in D \forall s \in [0, T] \} \geq u_g(0, x) \forall x \in D. \]  
(A.16)

The first desired result follows from (A.15) and (A.16). Moreover, in view of Harnack’s inequality (see [8, Theorem 7.10]), there is $\tilde{\rho}_i > 0$ such that $u_f(y, T) \geq \tilde{\rho}_i u_f(x_0, \frac{T}{2})$ for all $y \in K_2$ and $f$ being bounded continuous. Thus
\[ \mathbb{E}_{x,i} \left[ f(Y(T)) 1_{\{Y(s) \in D \forall s \in [0, T] \}} \right] \geq \tilde{\rho}_i \mathbb{E}_{x_0,i} \left[ f(Y(0.5T)) 1_{\{Y(s) \in D \forall s \in [0, T] \}} \right] \]
for any bounded and continuous function $f$. Thus, we obtain that
\[ \mathbb{P}_{x,i} \{ Y(T) \in B \text{ and } Y(s) \in D \forall s \in [0, T] \} \]
\[ \geq \tilde{\rho}_i \mathbb{P}_{x_0,i} \{ Y(0.5T) \in B, \text{ and } Y(s) \in D \forall s \in [0, 0.5] \} \]
\[ \geq \rho_i \tilde{\nu}(B) \]
for any Borel set $B$, where
\[ \nu(\cdot) = \mathbb{P}_{x_0,i} \{ Y(0.5T) \in \cdot, \text{ and } Y(s) \in D \forall s \in [0, 0.5T] \} \]
and $\rho_i = \tilde{\rho}_i \mathbb{P}_{x_0,i} \{ Y(s) \in D \forall s \in [0, 0.5T] \}$, which is positive due to (A.13). Denote $\widehat{D} = \{ \phi \in \mathcal{C} : \phi(t) \in D, t \in [-r, 0] \}$. For any Borel set $\mathcal{B} \subset \mathcal{C}$, we have from the Markov property of $Y^i(t)$ that
\[ \mathbb{P}_{x_0,i} \{ Y_{T+r} \in \mathcal{B} \text{ and } Y(s) \in D \forall s \in [0, T + r] \} = \mathbb{P} \left\{ Y_{T+r} \in \mathcal{B} \cap \widehat{D} \big| Y(T) = y \right\} \mathbb{P}_{x_0,i} \{ Y(T) \in dy \text{ and } Y(s) \in D \forall s \in [0, T + r] \} \]
\[ \geq \rho_i \int_{y \in D} \mathbb{P} \left\{ Y_{T+r} \in \mathcal{B} \cap \widehat{D} \big| Y(T) = y \right\} \tilde{\nu}(dy) \]
\[ = \nu(\mathcal{B} \cap \widehat{D}). \]

Now, let $\mathcal{B}$ be an open subset of $\widehat{D}$. Denote $\mathcal{B} = \{ \phi(-r) : \phi \in \mathcal{B} \}$. Then $\mathcal{B}$ is an open subset of $D$. By the support theorem (see [22, Theom 3.1]),
\[ \mathbb{P}_{y,i} \{ Y_{T+r} \in \mathcal{B} \} > 0 \text{ for any } y \in \mathcal{B}. \]  
(A.17)

In light of (A.13),
\[ \tilde{\nu}(\mathcal{B}) = \mathbb{P}_{\phi_0,i} \{ Y(0.5T) \in \mathcal{B}, \text{ and } Y(s) \in D \forall s \in [0, 0.5T] \} > 0. \]  
(A.18)

In view of (A.17) and (A.18),
\[ \nu(\mathcal{B}) \geq \int_{y \in \mathcal{B}} \mathbb{P}_{y,i} \{ Y_{T+r} \in \mathcal{B} \} \tilde{\nu}(dy) > 0 \]
if $\mathcal{B}$ is an open subset of $\widehat{D}$. 
\[ \square \]
Proof of Lemma 3.3. Let \( D = \{ x \in \mathbb{R}^n : |x| < R + 1 \} \), \( K_1 = K_2 = \{ x \in \mathbb{R}^n : |x| < R \} \) and \( M_{R,i} = \sup \{ q(\phi, t) : \| \phi \| \leq R + 1 \} < \infty \). In view of Lemma A.1,

\[
\mathbb{P}_{\phi,i^*} \{ Y_{T+r} \in \mathcal{B}, Y(t) < R + 1, t \in [0, T+r] \} \geq \nu(\mathcal{B})
\]

where \( \nu(\cdot) \) is defined as in Lemma A.1 with \( i \) replaced by \( i^* \). Thus,

\[
\mathbb{P}_{\phi,i^*} \{ X_{T+r} \in \mathcal{B} \text{ and } \alpha(T + r) = i^* \}
\geq \mathbb{P}_{\phi,i^*} \{ X_{T+r} \in \mathcal{B} \text{ and } \alpha(t) = i^*, \| X_t \| < R + 1 \forall t \in [0, T + r] \}
\geq \mathbb{E}_{\phi,i^*} \left[ 1_{\{ Y_{T+r} \in \mathcal{B}, \| Y_t \| < R \forall t \in [0, T+r] \}} \exp \left( - \int_0^{T+r} q_{i^*}(Y_t) dt \right) \right]
\geq \exp(-M_{R,i^*(T + r)}) \nu(\mathcal{B}).
\]

The proof is complete. \( \square \)

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34