Entropy and the Link Action in the Causal Set Path-Sum

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Abstract

In causal set theory the gravitational path integral is replaced by a path-sum over a sample space $\Omega_n$ of $n$-element causal sets. The contribution from non-manifold-like orders dominates $\Omega_n$ for large $n$ and therefore must be tamed by a suitable action in the low energy limit of the theory. We extend the work of Loomis and Carlip on the contribution of sub-dominant bilayer orders to the causal set path-sum and show that the “link action” suppresses the dominant Kleitman-Rothschild orders for the same range of parameters.

1 Introduction

In any theory of quantum gravity the transition from the deep quantum regime to the semi-classical regime requires the suppression of non-classical “quantum spacetimes”. In the causal set approach to quantum gravity, a quantum spacetime corresponds to a causal set or locally finite order$^1$ and the path integral is replaced by a path-sum over a sample space of causal sets.

As shown in [1], if one were to randomly pick an order from the sample space $\Omega_n$ of finite $n$-element orders, it would overwhelmingly be a “Kleitman-Rothschild” (KR) order as $n$ becomes very large. A KR order has three levels with approximately $n/4$ elements in the top and bottom levels, $n/2$ elements in the middle level, and such that every element in the top level and the bottom level is linked to approximately half of the elements in the middle level. In causal set theory (CST) a causal set is said to be manifold-like only if it can be obtained from a (typical) Poisson sprinkling into a spacetime. Thus KR orders are far from manifold-like. This poses a challenge to CST, since continuum-like dynamics must arise from the fundamentally discrete dynamics in the semi-classical limit. The number of KR orders goes as $\sim 2^{\frac{3n^2}{8} + \frac{3n}{2} + o(n)}$, and is the dominant entropic contribution to the CST path-sum. This entropy therefore need to be suppressed in the semi-classical limit by an appropriate choice of action.

In addition to the KR orders, as shown by Dhar [2], there is a hierarchy of sub-dominant “$k$-level” orders which are also not manifold-like. Of these, the next dominant contribution to the entropy of the path-sum comes from the bilayer (BL) orders. In [3],

$^1$In this work we shall use the term “order” instead of “poset”.

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Loomis and Carlip (henceforth “LC”) showed that the discrete Einstein-Hilbert (EH) or Benincasa-Dowker (BD) action (which simplifies to the link action for BL orders) suppresses all BL orders in the CST path-sum. In this work we explicitly show how their analysis carries over to KR orders for the link action. While the BD action is a natural choice since it limits to the EH action as $n \to \infty$, the link action is not without interest. As in the continuum, we expect higher order corrections to the action in the semi-classical regime. The (linear) link action can be viewed as one such correction\(^2\).

In Section 2, we first define what is meant by a level. In the literature this is often used interchangeably with the term “layer”. More confusingly, the same terminology is used in subtly different ways, for example in [2] and [5] versus [1]. We distinguish these different usages by defining two new types of levels: the quasi-levels (QLs) and the pseudo-quasi-levels (PQLs). In Section 2.1 we give a brief review of both the BD and the link actions.

We present our main results in Section 3 after reviewing the work of LC in the language of QLs and PQLs in Section 3.1. In Section 3.2 we show that a subset $\mathcal{P}^{*}_{\bar{p},p,n} \subset \Omega_n$ which contains “typical” naturally labelled KR orders with fixed level sizes is suppressed by the link action to leading order. This is also true of any naturally labelled $k$-PQL order with fixed level size. Surprisingly, while the counting arguments are different from those used by LC, we find the leading order contribution to the path-sum to be the same, so that the results of LC carry over trivially. In Section 3.3, we expand the analysis to $k$-QL orders for any $k$ (and therefore all KR orders when $k > 2$) and again find that the leading order contribution is the same as that found in LC.

In order to emphasise the non-triviality of these results, we use a very different subset of $\Omega_n$ in Section 3.4. This “KR + dust” subset consists of “typical” $n'$-KR orders with fixed level sizes for $n' \leq n$, with the remaining $n - n'$ elements forming an antichain or dust. Unlike our earlier results, here we find that the leading order contribution to the CST path-sum is not suppressed by the link action. We summarise our results in Section 4 and discuss some of the open questions.

## 2 Preliminaries

We consider the sample space $\Omega_n$ of finite $n$-element posets or orders, which are labelled over the set of $n$ integers as in [1] [2] [5] [6]\(^3\). An order $c \in \Omega_n$ is said to be naturally labelled if $\forall e_r, e_s \in c$, $e_r < e_s \Rightarrow r < s$.

As in standard CST terminology: (i) a minimal element in $c$ is one with no preceding element (ii) a link $\prec^*$ is a relation not implied by transitivity, i.e., $e_r \prec^* e_s$ if $\nexists e_t \in c$ such that $e_r < e_t < e_s$ (iii) an order interval is the set $I[\{e_r, e_s\}] := \{ e_t \in c \mid e_r < e_t < e_s \}$ and is called a $j$-element order interval if $|I[\{e_r, e_s\}]| = j$.

\(^2\)See [4] for a discussion on link-based actions.

\(^3\)As discussed in [7], the labelling introduces a factor of at most $n!$ which is sub-dominant to the entropic factor of $2^{n^2/4}$. Hence much of our analysis carries over to the unlabelled case, unless stated otherwise, as in Sec. 3.2.
Definition 2.1. The level $L_j$ of an order $c$, with $j = 1, 2, 3, \ldots k$, is the set of minimal elements that remain after deleting all elements in levels $L_m$, $m < j$. In particular, $L_1$ contains all the minimal elements of $c$ [1].

Definition 2.2. A KR order has three levels $L_1, L_2, L_3$ satisfying the following properties:

1. $|L_1|, |L_3| = n/4 + o(n)$ and $|L_2| = n/2 + o(n)$.
2. $e_r \prec e_s$ and $e_r \in L_j$ implies $e_s \in L_{j+1}$.
3. Each element in a level $L_j$ is connected to asymptotically half of the elements in $L_{j-1}$ and half of the elements in $L_{j+1}$.
4. For all $e_r \in L_1$ and $e_s \in L_3$, $e_r \prec e_s$.

Def. 2.2 is not explicitly stated in the original paper of Kleitman and Rothschild [1]. However since the dominant contribution comes from those orders satisfying all the conditions in Def. 2.2, this additional condition was imposed in [2, 5, 6].

Definition 2.3. [2, 5, 6] In a $k$-layer order $c \in \Omega_n$ it is possible to assign a layer $\zeta(e_r) \in \{1, 2, \ldots, k\}$ to each element $e_r \in c$, such that:

1. $e_r < e_s \Rightarrow \zeta(e_r) < \zeta(e_s)$.
2. $\zeta(e_s) > \zeta(e_r) + 1 \Rightarrow e_r < e_s$.

Let $D_n^k$ denote set of these orders.

Definition 2.4. A subset $\hat{c} \subset c$ is causally disconnected if there exists no relation between elements of $\hat{c}$ and its complement $\hat{c}'$ in $c$. $\hat{c}$ is an irreducible causally disconnected subset of $c$ if further, it contains no non-trivial causally disconnected proper subsets.

Definition 2.5. In a $k$-quasi-level ($k$-QL) order $c \in \Omega_n$ it is possible to assign to each element $e_r \in c$ a quasi-level (QL), $\eta(e_r) \in \{1, 2, \ldots, k\}$, such that:

1. For $e_r, e_s \in c$, if $e_r \prec e_s$ then $\eta(e_s) = \eta(e_r) + 1$.
2. For every causally disconnected subset $\hat{c} \subset c$, $\exists e_\alpha \in \hat{c}$ such that $\eta(e_\alpha) = 1$.

Let $Q_n^k$ denote the set of $k$-QL orders.

Claim 1. There is a unique assignment of QLs for any $c \in Q_n^k$, i.e., every $k$-QL order is a unique labelled order in $\Omega_n$.

Proof. Let $c \in Q_n^k$ and let $\eta$ and $\eta'$ be two distinct QL assignments on $c$. Thus $\exists e_r \in c$ such that $\eta(e_r) \neq \eta'(e_r)$. Wlog let $\eta'(e_r) = \eta(e_r) + \eta_0$ with $\eta_0 > 0$. If $\hat{c} \supset e_r$ is the unique irreducible causally disconnected subset containing $e_r$, it follows from Def. 2.5-1 that for every $e_s \in \hat{c}$ which is linked to $e_r$, $\eta(e_s) = \eta(e_r) + 1$ and $\eta'(e_s) = \eta'(e_r) + 1$, i.e., $\eta'(e_s) = \eta(e_s) + \eta_0$ (where the choice “±” depends on whether $e_s$ is to the future or past of $e_r$). Since $\hat{c}$ is irreducible, every element in $\hat{c}$ is “connected” via a set of future and past relations to every other element in $\hat{c}$. Therefore $\forall e_r, e_s \in \hat{c}$, $\eta'(e_r) = \eta(e_r) + \eta_0$ and since $\eta(e_r) \geq 1 \Rightarrow \eta'(e_r) \geq 1 + \eta_0$. Since $\eta'$ is also a QL, $\exists e_\alpha \in \hat{c}$ such that $\eta'(e_\alpha) = 1$, which is true only if $\eta_0 = 0$ thus implying that $\eta = \eta'$. \qed
\( Q^k_n \) is therefore a proper subset of \( \Omega_n \). It is evident that \( D^k_n \cap Q^k_n \neq \emptyset \), but that one is not nested inside the other. This is because \( \exists c \in D^k_n \) in which there are links between non-consecutive layers, and hence Def. 2.5-1 is not satisfied. Conversely \( \exists c \in Q^k_n \) such that the elements in \( L_{i+2} \) are not all related to those in \( L_i \) and hence Def. 2.3-2 is not satisfied. Importantly, since the KR orders satisfy both Def. 2.5-1 and Def. 2.3-2, they lie in \( D^k_n \cap Q^k_n \). We give examples of these differences in Fig. 1. Note also that typical manifold-like orders do not lie in either \( Q^k_n \) or \( D^k_n \).

![Diagram](image)

**Figure 1:** The first two orders belong to both \( D^k_n \) and \( Q^k_n \), the third to \( D^k_n \) but not to \( Q^k_n \), and the fourth to \( Q^k_n \) but not to \( D^k_n \).

**Definition 2.6.** In a bilayer or BL order all relations are links.

Clearly, every \( c \in Q^2_n \) is a BL order. Conversely, to any BL order \( c \), we can assign the QL order \( \eta = 1 \) for all minimal elements and \( \eta = 2 \) otherwise. This satisfies Def. 2.5-1 and 2.5-2 and hence \( c \in Q^2_n \).

**Definition 2.7.** \( c \in \Omega_n \) is said to admit a \( k \)-pseudo-quasi-level (k-PQL) assignment \( \vartheta(e_r) \in \{1, 2, \ldots, k\} \), \( \forall e_r \in c \) such that Def. 2.5-1 is satisfied but not necessarily Def. 2.5-2. We can build an order in \( \Omega_n \) by filling k-PQLs over the set of \( n \) integers while ensuring Def. 2.5-1. This gives us the set \( P^k_n \) of k-PQL orders.

Note that in LC the term “level” is the same as a PQL. Indeed for BL orders, since all relations are links, Def. 2.3-2 is trivially satisfied so that there is no distinction between layers and PQLs. Since we are interested in generalising the analysis to include 3-level orders, this distinction however does become important.

### 2.1 Causal Set Actions

The CST path-sum over \( \Omega_n \) is given by

\[
\mathcal{Z}_n = \sum_{c \in \Omega_n} \exp \left( \frac{i}{\hbar} S(c) \right),
\]

where \( S(c) \) denotes a choice of causal set action. The Kleitman-Rothschild result implies that if \( S(c) = 0 \), the KR orders dominate \( \mathcal{Z}_n \). The choice of \( S(c) \) is therefore crucial in taming the contribution of the KR orders. In analogy with the continuum, the natural choice for \( S(c) \) is the discrete EH action, or the \( d \) dimensional BD action

\[
\frac{1}{\hbar} S_{BD}^{(d)}(c,e) \equiv \mu(d,e) \left( n + \sum_{j=0}^{j_{\text{max}}(d)} \lambda_j(d,e) N_j \right),
\]

where

\[
\lambda_j(d,e) \equiv \frac{1}{\hbar} \int_{e-j}^{e} \frac{d^d \xi}{(2\pi \hbar)^d} \int_{e-j-1}^{e-j} \frac{d^d \eta}{(2\pi \hbar)^d} \vartheta(e_{r_1}) \vartheta(e_{r_2}) \ldots \vartheta(e_{r_d}) \vartheta(e_{r_{d+1}}) \vartheta(e_{r_{d+2}}) \ldots \vartheta(e_{r_{d+j}})
\]

\[
\mu(d,e) \equiv \frac{1}{(2\pi \hbar)^d} \int_{e-j}^{e} \frac{d^d \xi}{(2\pi \hbar)^d} \int_{e-j-1}^{e-j} \frac{d^d \eta}{(2\pi \hbar)^d} \vartheta(e_{r_1}) \vartheta(e_{r_2}) \ldots \vartheta(e_{r_d}) \vartheta(e_{r_{d+1}}) \vartheta(e_{r_{d+2}}) \ldots \vartheta(e_{r_{d+j}})
\]

\[
\vartheta(e_{r_1}) \vartheta(e_{r_2}) \ldots \vartheta(e_{r_d}) \vartheta(e_{r_{d+1}}) \vartheta(e_{r_{d+2}}) \ldots \vartheta(e_{r_{d+j}})
\]
where $N_j$ is the number of $j$-element order intervals in $c$, $\epsilon$ is a new “mesoscale” to suppress fluctuations and $\mu(d, \epsilon)$, $\lambda_j(d, \epsilon)$ and $j_{\text{max}}(d)$ are given in [8, 9, 10]. In the limit $n \to \infty$ the expectation value of $S(c)$ over different Poisson sprinkling gives the EH action, upto boundary terms [11, 12, 13].

The “link action” on the other hand depends only on the number of links $N_0$

$$\frac{1}{\hbar} S_L(c) \equiv \mu \left( n + \lambda_0 N_0 \right),$$

which can also be obtained from the BD action by putting $\lambda_j(d, \epsilon) = 0$, $\forall j > 0$.

The path-sum Eqn. [2.1] can be split into a sum over mutually disjoint subsets $\{\pi_1, \pi_2, \ldots\}$ of $\Omega_n$ with $\sqcup \pi_s = \Omega_n$, so that

$$Z_n = \sum_{\pi_s} Z_n \left|_{\pi_s} \right.,$$

where $Z_n \left|_{\pi_s} \right.$ denotes the restriction of $Z_n$ to $\pi_s$. While such a split is obviously non-unique, it allows us to isolate the contributions from specific classes of orders, like the KR orders. In LC, $Z_n$ was restricted to the subset of BL orders. Here we consider the path-sum using the link action $Z_n^{(L)}$ and its restriction to the QL and PQL orders.

# 3 KR Orders and the Link Action

In this section we compute the leading order contribution to the path-sum restricted to three different subsets of $\Omega_n$ each of which contains what we will loosely refer to as “KR-like” orders.

First we will need a few more definitions. The set $P^k_n$ is not in one to one correspondence with elements in $\Omega_n$ since the same order in $\Omega_n$ can be obtained from different fillings of a given set of $k$-PQLs. Therefore we cannot use $P^k_n$ to partition $Z_n^{(L)}$. Instead we look at a subset $P_{\vec{q},n}^* \subset P^k_n$ defined as follows.

Let $\mathcal{V}_n^k$ denote the set of all of possible assignments of PQLs over the set of $n$ integers. Since there are $k^n$ ways of making a PQL assignment, $|\mathcal{V}_n^k| = k^n$. Any element of $\mathcal{V}_n^k$ can be labelled by the filling fraction, $\vec{q} = (q_1, q_2, \ldots, q_k)$ where $q_n$ is the cardinality of the $i^{\text{th}}$ PQL, so that $\sum_{i=1}^k q_i = 1$. Since the elements are distinguishable, each choice of $\vec{q}$ comes with a multiplicity $m(\vec{q})$ given by

$$m(\vec{q}) = \frac{n!}{(q_1 n)! (q_2 n)! \ldots (q_k n)!} \Rightarrow \sum_{\vec{q}} m(\vec{q}) = k^n.$$  

**Definition 3.1.** A PQL assignment $\mathcal{V}_n^k \in \mathcal{V}_n^k$ is said to be **naturally labelled** if the first PQL is $\{e_1, \ldots, e_{q_1 n}\}$, the second PQL is $\{e_{q_1 n+1}, \ldots, e_{(q_1+q_2)n}\}$ and so on, and therefore is unique. We denote the set of naturally labelled $k$-PQL orders on $\mathcal{V}_n^k$ by $P_{\vec{q},n}^* \subset P^k_n$.

We further denote the set of all $k$-PQL orders on $\mathcal{V}_n^k$ with $pm^2$ links by $P_{\vec{q},p,n}^* \subset P_{\vec{q},n}^*$ and similarly the set of all $k$-QL orders with $pm^2$ links by $Q_{\vec{q},p,n}^* \subset Q_{\vec{q},n}^*$. 

5
Claim 2. There exists a one-to-one map from $\mathcal{P}_{q,p,n}^*$ to $\mathcal{Q}_{p,n}^k$, so that $\mathcal{P}_{q,p,n}^* \subseteq \mathcal{Q}_{p,n}^k \subseteq \Omega_n$ and therefore $|\mathcal{P}_{q,p,n}^*| \leq |\mathcal{Q}_{p,n}^k|$.

Proof. Let $\hat{c} = \{\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_k\}$ denote the set of all irreducible causally disconnected subsets of $c \in \mathcal{P}_{q,p,n}^*$. Let $t_i$ denote the smallest PQL in $\hat{c}_i$. Make the new PQL assignment

$$\eta(e_r) \equiv \vartheta(e_r) - t_i + 1, \quad \forall e_r \in \hat{c}_i. \quad (3.2)$$

Under $\eta$, consecutive levels are mapped to consecutive levels. Hence Def. 2.5-1 is still satisfied. Moreover, for each $\hat{c}_i$, there exists an $e_r$ such that $\vartheta(e_r) = t_i$ and hence, $\eta(e_r) = 1$. This “shuffling down” brings all the elements in the PQL $t_i$ associated with $\hat{c}_i$ to the PQL $\Omega = 1$, so that the number of PQLs either stays the same or decreases. Thus, $\eta$ satisfies Def. 2.5-2 and hence is also a QL. $\eta$ therefore defines a one-to-one map $\varphi: \mathcal{P}_{q,p,n}^* \rightarrow \mathcal{Q}_{p,n}^k$. Since for any pair of distinct $c, c' \in \mathcal{P}_{q,p,n}^*$, wlog, $\exists e_r, e_s \in c, c'$ such that $e_r \prec e_s$ in $c$ but not in $c'$. Since Def. 2.5-1 is still satisfied under $\eta$, $e_r \prec e_s$ in $\varphi(c)$ but not in $\varphi(c')$. Therefore $c \neq c' \Rightarrow \varphi(c) \neq \varphi(c')$, which ensures that the map $\varphi$ is one-to-one.

3.1 Review of the LC Result

We start with a brief review of the results of LC [3] on bilayer orders rephrased in the language of QLs and PQLs. LC noticed that for BL orders, the maximum number of possible links is $n^2/4$ and occurs only for the filling fraction $\bar{q}_b = (1/2, 1/2)$. A clever bounding argument (which we will also use in Sec. 3.3) shows that to leading order

$$\ln |\mathcal{Q}_{p,n}^2| = \ln |\mathcal{P}_{q,b,p,n}^*| + o(n^2), \quad |\mathcal{P}_{q,b,p,n}^*| = \left(\frac{n^2}{pn^2}\right)^2, \quad (3.3)$$

where $N_0 = pn^2 \equiv \bar{p}n^2/4$ and $p \leq 1/4$. Using Stirling’s approximation to leading order in $n$

$$\ln |\mathcal{P}_{q,b,p,n}^*| = \frac{n^2}{4} h(\bar{p}) + o(n^2), \quad (3.4)$$

where

$$h(\bar{p}) = -\bar{p} \ln \bar{p} - (1 - \bar{p}) \ln (1 - \bar{p}), \quad 0 < \bar{p} < 1, \quad (3.5)$$

is Dhar’s entropy function [2].

For $p \neq p'$, $\mathcal{Q}_{p,n}^2$ and $\mathcal{Q}_{p',n}^2$ are disjoint, and one can therefore partition $\mathcal{Q}_{n}^2$ into disjoint subsets labelled by $p$. The path-sum for the BD action or equivalently the link action is then

$$Z_n^{(BD)}|_{\mathcal{Q}_{p,n}^2} = Z_n^{(BD)}|_{\mathcal{Q}_{p,n}^2} = \int_0^1 d\bar{p} \exp \left[\frac{n^2}{4} (i\mu \lambda_0 \bar{p} + h(\bar{p})) + o(n^2)\right]. \quad (3.6)$$

The integral in Eqn. (3.6) was evaluated to leading order by LC using the method of steepest descent. For the BD action, $\mu \lambda_0 < 0$ they showed that $Z_n^{(BD)}|_{\mathcal{Q}_{p,n}^2}$ is exponentially suppressed when

$$\tan \left(\frac{\mu \lambda_0}{2}\right) < -\sqrt{\frac{27}{4} e^{-1/2} - 1}. \quad (3.7)$$
However, for $Z_n^{(L)}|_{\mathcal{Q}_n^2}$ it is possible for $\mu \lambda_0 > 0$ and hence is exponentially suppressed when

$$\tan\left(\frac{\mu \lambda_0}{2}\right) > \sqrt{\frac{27}{4} e^{-1/2} - 1}.$$  

(3.8)

It is interesting to note the dimension dependence of the suppression in the BD action, where $\mu \lambda_0 = -\left(\frac{l}{l_p}\right)^{d-2} \beta_d C_1^{(d)}$ where $l$ is the discreteness scale [9, 10]. In $d = 2$, for example, $\mu \lambda_0 = -4$ and hence there is no suppression. For all $d \geq 3$, however, by adjusting $l/l_p$, one can find a suitable suppression regime. In $d = 4$, in particular, there is a suppression for all $l \geq 1.452 l_p$.

We note that the restricted path-sum in the LC calculation also includes BL orders which are not entropically dominant, for example the $(n - 1)$-element antichain with just one relation with the $n^{th}$ element. These nevertheless seem important in estimating the leading order contribution to the CST path-sum. This suggests that the restriction to appropriate subsets of $\Omega_n$ may be crucial in obtaining the right semi-classical approximation of the CST path-sum.

3.2 Naturally Labelled $k$-PQL Orders with a Fixed Filling Fraction

We now consider the restriction of $Z_n^{(L)}$ to $\mathcal{P}_{\vec{q},n}^* \subset \Omega_n$, which is the set of naturally labelled PQL orders with fixed filling fraction $\vec{q}$. For the choice $\vec{q}_{kr} = (1/4, 1/2, 1/4)$, $\mathcal{P}_{\vec{q}_{kr}, n}^*$ includes the set of naturally labelled “typical” KR orders. By this we mean Def. 2.2-1 and Def. 2.2-3 are satisfied exactly, i.e., without the $o(n)$ fluctuations. Fig. 2 is an example of such an order for $n = 8$.

![Figure 2: A naturally labelled $n = 8$ “typical” KR order with $\vec{q} = \vec{q}_{kr}$.](image)

The maximum number of links possible for any $c \in \mathcal{P}_{\vec{q},n}^*$ is

$$N_{max} = (q_1 q_2 + q_2 q_3 + \cdots + q_{k-1} q_k) n^2 =: \alpha(\vec{q}) n^2,$$

and therefore we can express the number of links $N_0$ for $c \in \mathcal{P}_{\vec{q},n}^*$ by

$$N_0 = \tilde{p} N_{max} = \tilde{p} n^2, \quad 0 \leq \tilde{p} \leq 1.$$  

(3.10)

We refer to $\tilde{p}$ as the linking fraction. As shown in Fig. 3 for a given $p$

$$|\mathcal{P}_{\vec{q},p,n}^*| = \left(\frac{N_{max}}{N_0}\right) = \left(\frac{\alpha(\vec{q}) n^2}{p n^2}\right),$$  

(3.11)
which to leading order in $n$ is

$$\ln |\mathcal{P}_{\tilde{q},p,n}^*| = \alpha(\tilde{q})n^2 h(\tilde{p}) + o(n^2).$$

(3.12)

This is identical to the LC expression Eqn. 3.4 with $n$ replaced by $2\sqrt{\alpha(\tilde{q})}n$, so that

$$Z_n|_{\tilde{q}^*,L} = \int_0^1 d\tilde{p} \exp \left[ \alpha(\tilde{q})n^2 \left( i\mu \lambda_0 \tilde{p} + h(\tilde{p}) \right) + o(n^2) \right],$$

(3.13)

which is the same as $Z_n|_{Q_{2,n}^k}$ up to leading order. Thus, from the analysis in [3], which is unaffected by any rescaling of $n$, we see that $Z_n|_{\tilde{q}^*,L}$ is exponentially suppressed for $\mu\lambda_0$ given by Eqn. (3.7) and Eqn. (3.8). The result is somewhat surprising, since it means that the path-sum in [3] captures a more general (leading order) feature of the full CST path-sum.

Figure 3: A naturally labelled 3-PQL order with $n = 8, \tilde{q} = \tilde{q}_{kr}, \tilde{p} = 0.25$. All the possible links are coloured black while those that are realised for this $\tilde{p}$ are coloured orange.

What we have calculated is the leading order contribution of $\mathcal{P}_{\tilde{q},p,n}^*$ to $Z_n$ for any $\tilde{q}$. Since the $\mathcal{P}_{\tilde{q},n}^*$ for different choices of $\tilde{q}$ overlap, one cannot however, further sum over $\tilde{q}$. Of special importance is the set $\mathcal{P}_{\tilde{q},k,n}^*$, which includes the naturally labelled "typical" KR-orders. Since the contribution from KR orders near typicality could also be entropically important, we now look for a larger class of orders which contain all of the KR orders.

3.3 $k$-QL Orders

We now consider the more general subset $Q_n^k$, which includes all $k'$-QL orders for $k' \leq k$. In particular for $k \geq 3$, this includes all KR orders.

Again we begin with the subset $Q_n^k$. For $p \neq p'$, $Q_{p,n}^k$ and $Q_{p',n}^k$ are disjoint, and one can therefore partition $Q_n^k$ into the disjoint subsets labelled by $p$ and use $p$ as an integration factor as before. Unlike our earlier calculation, however, $|Q_{p,n}^k|$ is harder to obtain directly. $|Q_{p,n}^k|$ was obtained by LC by saturating both a lower and an upper bound. We employ these same methods here.

Let $\mathcal{P}_{p,n}^k \subset \mathcal{P}_{p,n}^k$ be the set of all $k$-PQL orders with $pm^2$ links. Since $\mathcal{P}_{p,n}^k$ contains different PQL labellings of the same order, $Q_{p,n}^k \subset \mathcal{P}_{p,n}^k$. From Claim 2 for any filling
fraction $\vec{q}$,

$$|P_{q,p,n}^*| \leq |Q_p^k| \leq |P_p^k|. \tag{3.14}$$

In order to tighten these bounds, we vary over $\vec{q}$ and see that $\alpha(\vec{q})$ takes the maximum value $\alpha_m = 1/4$ for $\vec{q}_x = (1/4 - x, 1/2, 1/4 + x)$ where $-1/4 \leq x \leq 1/4$. This includes the two configurations: the symmetric $k = 2$ case, with $\vec{q}_0 = (1/2, 1/2)$ and the “typical KR” $k = 3$ case, with $\vec{q}_0 = (1/4, 1/2, 1/4)$. Since $|P_{q,p,n}^*|$ is a monotonically increasing function of $\alpha(\vec{q})$ for fixed $n$ and $p$, it achieves a maximum at $\alpha_m$. Let $\vec{q}_0$ denote one of these maximising configurations.

Thus

$$|P_{p,n}^k| = \sum_{\vec{q}} m(\vec{q})|P_{q,p,n}^*| \leq \sum_{\vec{q}} m(\vec{q})|P_{\vec{q}_0,p,n}^*| = k^n |P_{\vec{q}_0,p,n}^*|, \tag{3.15}$$

where $m(\vec{q})$ is defined in Eqn. (3.1) which implies that

$$|P_{\vec{q}_0,p,n}^*| \leq |Q_{p,n}^k| \leq k^n |P_{\vec{q}_0,p,n}^*|. \tag{3.16}$$

Note that this reduces to the calculation of LC when $k = 2$. The factor $k^n$ contributes only to the subleading order of $n$ and therefore

$$\ln |Q_{p,n}^k| = \ln |P_{\vec{q}_0,p,n}^*| + o(n^2). \tag{3.17}$$

Thus we see that the leading order contribution to $Z_n^{(L)}|_{Q_p^k}$ for any $k \geq 3$ comes from a subset of $\Omega_n$ which includes the KR orders as well as the symmetric BL orders. Moreover, it is again the same as $Z_n^{(L)}|_{Q_p^{\chi,n}}$ to leading order. Thus we see that the link action serves to suppress the contribution from all KR orders for $\mu \lambda_0$ satisfying Eqn. (3.7) and (3.8).

### 3.4 “KR + dust”

In order to illustrate the non-triviality of the previous calculations, we consider a wholly different class of orders, which contains “typical KR” orders of all cardinalities $n' < n$, along with the “dust” of an $(n-n')$-element antichain.

**Definition 3.2.** Let $c \in Q_n^3$ admit a partition $c = \hat{c}_1 \sqcup \hat{c}_2$ so that $\hat{c}_1$ is a $\chi n$ order and $\hat{c}_2$ is a $(1 - \chi)n$-element antichain, where $0 \leq \chi \leq 1$ so that:

1. $e_r \in \hat{c}_2 \Rightarrow \eta(e_r) = 1$.
2. The $\chi n$ elements of $\hat{c}_1$ are assigned QLs with filling fraction $\vec{q} = (1/4, 1/2, 1/4)$.
3. Each element in $\hat{c}_1$ with $\eta = 1$ is linked to exactly $\chi n/4$ elements with $\eta = 2$, and similarly each element with $\eta = 3$ is linked to exactly $\chi n/4$ elements with $\eta = 2$.
4. $e_r \in \hat{c}_1$ is minimal $\Leftrightarrow \eta(e_r) = 1$.

Let $T_{\chi,n}$ denote the set of these “KR + dust” orders, for a given $\chi$. Fig. 4 shows an example of such an order.

Note that unlike the previous two cases, this fixes not only the filling fraction $\vec{q}$, but also the number of links $N_0 = (\chi n)^2/8$ in $T_{\chi,n}$. For $\chi \neq \chi'$, $T_{\chi,n}$ and $T_{\chi',n}$ are disjoint and therefore $T_n = \sqcup_{\chi} T_{\chi,n} \subset Q_n^3$. Thus

$$Z_n^{(L)}|_{T_n} \equiv \int_0^1 \mathrm{d} \chi |T_{\chi,n}| \exp (i S_L(\chi, n)). \tag{3.18}$$
In order to evaluate $|T_{\chi,n}|$ we consider the larger set of orders $T'_{\chi,n} \supset T_{\chi,n}$ which satisfy Def. 3.2-1-3 but not necessarily Def. 3.2-4. Therefore $T'_{\chi,n} \subset P^2_{n}$ since the elements of any $c \in T'_{\chi,n}$ can be assigned PQLs but not necessarily QLs. Let $G_{\chi,n} \subset T'_{\chi,n}$ such that for every $c \in G_{\chi,n}$, there is at least one element in the second PQL which is not linked to any element in the first PQL. From Def. 3.2 we see that $T_{\chi,n} = T'_{\chi,n} \setminus G_{\chi,n}$ and therefore

$$|T_{\chi,n}| = |T'_{\chi,n}| - |G_{\chi,n}|.$$  

(3.19)

Let us start with computing $|T'_{\chi,n}|$. The number of ways of separating out $(1 - \chi)n$ elements for $\hat{c}_2$ is $\binom{n}{\chi n}$ and the number of ways of distributing the remaining $\chi n$ elements so that it satisfies Def. 3.2-2 is $\binom{\chi n}{\chi n/4} \times \binom{3\chi n/4}{\chi n/4}$. Additionally, the number of ways of linking the elements so that Def. 3.2-3 is satisfied is $\binom{\chi n/2}{\chi n/4} \binom{\chi n}{\chi n/2} \binom{\chi n}{\chi n/2} \binom{\chi n}{\chi n/2}$. Hence, 

$$|T'_{\chi,n}| = \binom{n}{\chi n} \binom{\chi n}{\chi n/4} \binom{3\chi n/4}{\chi n/4} \binom{\chi n/2}{\chi n/4} \binom{\chi n}{\chi n/2} \binom{\chi n}{\chi n/2} \binom{\chi n}{\chi n/2} \binom{\chi n}{\chi n/2}.$$  

(3.20)

Now we compute $|G_{\chi,n}|$. The number of ways of separating out $(1 - \chi)n$ elements for $\hat{c}_2$ and the number of ways of distributing $\chi n$ elements so that it satisfies Def. 3.2-2 is the same as that for $|T'_{\chi,n}|$ and therefore 

$$|G_{\chi,n}| = \binom{n}{\chi n} \binom{\chi n}{\chi n/4} \binom{3\chi n/4}{\chi n/4} A_{\chi,n},$$  

(3.21)

where 

$$A_{\chi,n} = \frac{\chi n}{2} \left( \frac{\chi n/2 - 1}{\chi n/4} \right)^{\chi n/4} \left( \frac{\chi n/2}{\chi n/4} \right)^{\chi n/4} = \frac{\chi n}{2} \cdot 2^{-\chi n/4} \left( \frac{\chi n/2}{\chi n/4} \right)^{\chi n/2},$$  

(3.22)

denotes the number of ways in which the links can be assigned, in order to satisfy Def. 3.2-3, with at least one element in the second PQL not linked to any element in the first PQL. Thus 

$$|T_{\chi,n}| = \binom{n}{\chi n} \binom{\chi n}{\chi n/4} \binom{3\chi n/4}{\chi n/4} \binom{\chi n/2}{\chi n/4} \binom{\chi n}{\chi n/2} \left( 1 - \frac{\chi n}{2} \cdot 2^{-\chi n/4} \right).$$  

(3.23)

In the limit of large $n$, the second term is highly suppressed so that 

$$|T_{\chi,n}| \approx \binom{n}{\chi n} \binom{\chi n}{\chi n/4} \binom{3\chi n/4}{\chi n/4} \binom{\chi n/2}{\chi n/4} \binom{\chi n/2}{\chi n/4} \binom{\chi n}{\chi n/2} \left( 1 - \frac{\chi n}{2} \cdot 2^{-\chi n/4} \right).$$  

(3.24)
Using Stirling’s approximation,

\[
\ln |T_{\chi,n}| = \left(\frac{\chi n}{4}\right)^2 \ln 2 - \frac{\chi n}{4} \ln (\chi n) + o(n \ln n),
\]

and hence

\[
Z_n^{(L)} |_{T_n} = \int_0^1 d\chi |T_{\chi,n}| \exp \left( i\mu \left( n - \lambda_0 \left(\frac{\chi n}{8}\right)^2 \right) \right)
\]

\[
= \int_0^1 d\chi \exp \left( \frac{(\chi n)^2}{4} \left( \ln 2 - \frac{i\mu \lambda_0}{2} \right) + o(n^2) \right)
\]

\[
\approx -i \frac{\sqrt{\pi}}{n} \frac{\text{erf} \left( \frac{i\mu \sqrt{\ln 2 - \frac{i\mu \lambda_0}{2}}}{\sqrt{\ln 2 - \frac{i\mu \lambda_0}{2}}} \right)}{\sqrt{\ln 2 - \frac{i\mu \lambda_0}{2}}}. \tag{3.26}
\]

This is divergent in the limit of large \(n\) for \(any\) range of the parameters \(\mu\) and \(\lambda_0\) which means that \(T_n \subset \Omega_n\) is not suppressed in the CST path-sum, even though it contains “typical” KR orders with \(n' \leq n\). This underlines the importance of the choice of subset to which \(Z_n^{(L)}\) is restricted.

### 4 Conclusions

In this work we have shown that the link action can suppress the entropy of KR orders in the CST path-sum \(Z_n^{(L)}\) using techniques very similar to those used by LC to show the suppression of BL orders. The leading order contributions to \(Z_n^{(L)}\) found in Sec. 3.2 and Sec. 3.3 is the same as that found in LC. In the calculation of Sec. 3.3 this can be traced to the fact that the \(N_{\max}\) is maximised when the filling fraction is \(\vec{q} = (1/4 - x, 1/2, 1/4 + x), \ -1/4 \leq x \leq 1/4,\) which includes the symmetric BL orders. Thus the calculations of LC for BL orders already captures the essence of the contribution from the KR orders.

We have also examined the contribution to \(Z_n^{(L)}\) of naturally labelled \(k\)-PQL orders for any \(k\) with fixed \(\vec{q}\). As shown in Sec. 3.2 to leading order this too reduces to \(Z_n^{(L)} |_{Q_2^n}\) and is suppressed for the same range of parameters.

In order to emphasise the non-triviality of these results, we consider the restriction of \(Z_n^{(L)}\) to the “KR + dust” subset of \(\Omega_n\). We find \(no\) parameter range in which this contribution is suppressed. This example illustrates the importance of the choice of subset to which the CST path-sum is restricted and suggests that even to leading order in \(n\), subtle cancellations of the phases are important.

It would be of interest to extend these results to the other actions, like the relational action, or better still the BD action. In order to do this, one would have to be able to count the class of iso-action \(k\)-QL orders rather than those with fixed \(N_j\). This is beyond the scope of the present work and even a leading order estimation would be of great value.

We now present some numerical evidence that supports the idea suggested in LC that our link action result may be relevant to leading order even for the BD action.
In Fig. 5 and 6, we show the behaviour of the ratio of the number of order intervals \( N_j \) for \( j > 0 \) to the number of links \( N_0 \), with the linking fraction \( \tilde{p} \). We consider three different filling fractions \( \vec{q} \) with \( n \approx 400 \) in Fig. 5. The orders for each \( \vec{q} \) are generated by randomly choosing the links for a given linking fraction \( \tilde{p} \). The choice \( \vec{q}_{kr} \) (in blue)
includes the KR orders for which $\tilde{p} \sim 1/2$. The ratios $\langle N_j / N_0 \rangle$ for $j = 1, 2, 3$ can be seen to go to zero rapidly with $\tilde{p}$. In Fig. 6 we show the same effect enhanced for a much larger $n$ of 4000 with $\vec{q}_{kr}$.

We conclude by asking whether a similar type of suppression is possible for manifold-like orders with $d \neq 4 + D$, where $D$ corresponds to some fixed internal Kaluza-Klein type dimension. Crucial to such a calculation is the identification of the appropriate subsets of $\Omega_n$. It is possible that numerical investigations might provide useful clues and are a concrete way forward.

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References

[1] D. J. Kleitman and B. L. Rothschild, “Asymptotic enumeration of partial orders on a finite set,” Trans. Amer. Math. Soc., vol. 205, pp. 205–220, 1975.

[2] D. Dhar, “Entropy and phase transitions in partially ordered sets,” Journal of Mathematical Physics, vol. 19, pp. 1711–1713, 1978.

[3] S. Loomis and S. Carlip, “Suppression of non-manifold-like sets in the causal set path integral,” Class. Quant. Grav., vol. 35, no. 2, p. 024002, 2018.

[4] A. Eichhorn, “Towards coarse graining of discrete Lorentzian quantum gravity,” Class. Quant. Grav., vol. 35, no. 4, p. 044001, 2018.

[5] H. J. Prömel, A. Steger, and A. Taraz, “Phase Transitions in the Evolution of Partial Orders,” Journal of Combinatorial Theory, Series A, vol. 94, pp. 230–275, 2001.

[6] G. Brightwell, H. J. Prömel, and A. Steger, “The Average Number of Linear Extensions of a Partial Order,” Journal of Combinatorial Theory, Series A, vol. 73, pp. 193–206, 1996.

[7] J. Henson, D. P. Rideout, R. D. Sorkin, and S. Surya, “Onset of the Asymptotic Regime for Finite Orders.” arXiv:1504.05902, 4 2015.

[8] D. M. Benincasa and F. Dowker, “The Scalar Curvature of a Causal Set,” Phys. Rev. Lett., vol. 104, p. 181301, 2010.

[9] F. Dowker and L. Glaser, “Causal set d’Alembertians for various dimensions,” Class. Quant. Grav., vol. 30, p. 195016, 2013.

[10] L. Glaser, “A closed form expression for the causal set d’Alembertian,” Class. Quant. Grav., vol. 31, p. 095007, 2014.

[11] M. Buck, F. Dowker, I. Jubb, and S. Surya, “Boundary Terms for Causal Sets,” Class. Quant. Grav., vol. 32, no. 20, p. 205004, 2015.
[12] F. Dowker, “Boundary contributions in the causal set action.” arXiv:2007.13206, 7 2020.

[13] L. Machet and J. Wang, “On the continuum limit of Benincasa-Dowker-Glaser causal set action.” arXiv:2007.13192, 7 2020.