RELATIVE HYPERBOLICITY AND RIGHT-ANGLED COXETER GROUPS

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Abstract. We show that right-angled Coxeter groups are relatively hyperbolic in the sense defined by Farb, relative to a natural collection of rank-2 parabolic subgroups.

1. Introduction

The theory of (word) hyperbolic groups, initiated in the 1980s by M. Gromov in [7], is a rich one. The desirable geometric structure of such groups often provides a great deal of insight into the combinatorial structure of the group itself. Gromov suggested a generalization of the notion of hyperbolicity, wherein a group would be said to be relatively hyperbolic with respect to a given collection of subgroups (known as parabolic subgroups).

In [6], B. Farb presented the first rigorous definition of relative hyperbolicity. Though his definition is the one adopted in this paper, as well as in a number of others, it is not the only definition. B. Bowditch ([4]) and A. Yaman ([17]) have presented alternative definitions of relative hyperbolicity. A. Szczepański (in [14]) studied the relationship of these definitions, showing that the latter two are equivalent, and imply the first. In [6], Bowditch shows that if a group is relatively hyperbolic in the sense of Farb and furthermore satisfies Farb’s “bounded coset penetration” (BCP) property, then it is relatively hyperbolic in the sense of Bowditch and Gromov-Yaman. If $G$ is relatively hyperbolic in the sense of Farb, it is often said to be relatively hyperbolic in the weak sense.

The objects of study in this paper are right-angled Coxeter groups. Recall that a Coxeter system $(W, S)$ is a pair in which $W$ is a group (called a Coxeter group) and $S = \{s_i\}_{i \in I}$ for which there is a presentation

$$\langle S \mid R \rangle$$

where

$$R = \{(s_is_j)^{m_{ij}} \mid m_{ij} \in \{1, 2, ..., \infty\}, m_{ij} = m_{ji}, \text{ and } m_{ij} = 1 \iff i = j\}.$$  

(In case $m_{ij} = \infty$, the element $s_is_j$ has infinite order.)

Recall that for each subset $T \subseteq S$, the subgroup $W_T$ of $W$ generated by the elements of $T$ is also a Coxeter group, with system $(W_T, T)$ (see [3]). Such a subgroup is called a (standard) parabolic subgroup.

2000 Mathematics Subject Classification. 20F28, 20F55.

Key words and phrases. Coxeter group, relatively hyperbolic.

The author was supported by an NSF VIGRE postdoctoral grant.
If \( m_{ij} \in \{1, 2, \infty\} \) for every \( i, j \in I \), we call \((W, S)\) (and \(W\)) right-angled. In his dissertation [18], D. Radcliffe showed that the Coxeter presentation corresponding to a given right-angled Coxeter group is essentially unique, in that for any two fundamental generating sets \( S \) and \( S' \), there exists an automorphism \( \alpha \in \text{Aut}(W) \) such that \( \alpha(S) = S' \).

The information contained in a Coxeter system can be captured graphically by means of a Coxeter diagram. The Coxeter diagram \( \mathcal{V} = \mathcal{V}(W, S) \) corresponding to the system \((W, S)\) is an edge-labeled graph with vertex set in one-to-one correspondence with the set \( S \), and for which there is an edge \([s_i, s_j]\) labeled \( m_{ij} \) between any two vertices \( s_i \neq s_j \) whenever \( m_{ij} < \infty \). (In this paper, we omit the comma in the notation for an edge of a graph.) It is easy to see that Radcliffe’s rigidity result is equivalent to saying that to a given right-angled Coxeter group, there is a unique diagram, every edge of which has label 2. For this reason, we often suppress mention of the generating set \( S \).

The following theorem follows from the work of G. Moussong [10]:

**Theorem 1.1. [Moussong]** Let \( W \) be a right-angled Coxeter group, with corresponding diagram \( \mathcal{V} \). Then \( W \) is word hyperbolic if and only if \( \mathcal{V} \) contains no achordal simple circuits \( \{[ab], [bc], [cd], [da]\} \) of length 4.

That is, \( W \) is hyperbolic provided we see no squares without diagonals in \( \mathcal{V} \). (It is not hard to see that such squares imply the existence of flats in the Cayley graph of \( W \), arising from the presence of the subgroup \( D_\infty \times D_\infty \).) A very natural way to attempt to overcome this obstacle to hyperbolicity would be to “quotient” by one of the diagonals of such a square, in effect collapsing one of the dimensions in the corresponding subgroup \( H \cong D_\infty \times D_\infty \). In fact, that is just what we will do. The result is this paper’s main theorem:

**Theorem 1.2.** Let \( W \) be a right-angled Coxeter group with diagram \( \mathcal{V} \). For each achordal simple circuit \( \{[a_i b_i], [b_i c_i], [c_i d_i], [d_i a_i]\} \) in \( \mathcal{V} \), select a diagonal \( \{a_i, c_i\} \). Then \( W \) is relatively hyperbolic (in the sense of Farb) relative to the collection of subgroups \( \{W_{\{a_i, c_i\}}\} \) for the above choice of diagonals.

2. RELATIVE HYPERBOLICITY

We now recall the definition of relative hyperbolicity due to Farb.

Suppose that \( G \) is a finitely generated group with distinguished generating set \( S \). Let \( \Gamma = \Gamma(G, S) \) be the Cayley graph for \( G \) with respect to the set \( S \). We consider \( \Gamma \) as a metric graph, where each edge has length 1. We will denote the vertex of \( \Gamma \) corresponding to \( g \in G \) by \( v(g) \), or simply by \( g \) when this notation will not be confusing.

Let \( \{H_i\}_{i \in I} \) be a collection of subgroups of \( G \) (for our purposes \( |I| < \infty \)). We construct a new graph, \( \hat{\Gamma} = \hat{\Gamma}(\{H_i\}_{i \in I}) \) as follows. For each left coset \( gH_i \) \((g \in G, i \in I)\), we add a new vertex, denoted \( v(gH_i) \), to \( \bar{\Gamma} \); for each element \( g' \in G \) lying in the coset \( gH_i \), we add an edge from the new vertex \( v(gH_i) \) to the vertex \( v(g') \in \hat{\Gamma} \). Define the length of each of the new edges to be 1/2. The graph \( \hat{\Gamma} \) so obtained is called the coned-off Cayley graph of \( G \) with respect to the collection \( \{H_i\}_{i \in I} \).

The effect of adding the new vertices is to create “shortcuts” which pass through left cosets; in particular, the distance from \( v(1) \) to \( v(g) \) is 1 for any \( g \in H_i \), for some \( i \in I \).
Clearly the geometric structure of $\hat{\Gamma}$ may be very different from that of $\Gamma$. If $\hat{\Gamma}$, with the metric defined in the paragraphs above, is a hyperbolic metric space, we say that $G$ is \textit{relatively hyperbolic}, relative to the collection $\{H_i\}_{i \in I}$. For details on hyperbolic metric spaces and their desirable properties, we refer the reader to [5]. Here we merely remind that hyperbolic metric spaces can be roughly characterized by possessing “thin” geodesic triangles. (The precise condition that we will verify is contained below in Proposition 2.1.)

A number of results regarding relatively hyperbolic groups have been proven. A good source of examples is the paper [15] by A. Szczepański. Regarding the relative hyperbolicity of Coxeter groups and related groups, little has been shown. In [8], I. Kapovich and P. Schupp show that large-type Artin groups are relatively hyperbolic relative to a certain collection of rank-2 subgroups. As was done in the latter paper, here we will rely upon the following useful result due to P. Papasoglu (in [12]):

\textbf{Proposition 2.1.} Let $\Gamma$ be a connected graph with simplicial metric $d$. Then $\Gamma$ is hyperbolic if and only if there is a number $\delta > 0$ such that for any points $x, y \in \Gamma$ (not necessarily vertices), any two geodesic paths from $x$ to $y$ in $\Gamma$ are $\delta$-Hausdorff close.

(See the remark in [8] regarding applicability of the result as originally stated to our situation.)

A graph with the simplicial metric can be obtained obtained from $\hat{\Gamma}$ by subdividing each of the original edges of $\Gamma$. However, it is not difficult to see that if the condition in Proposition 2.1 can be established for the original metric on $\hat{\Gamma}$, then the condition holds also for the simplicial metric (perhaps with a different constant $\delta$, of course). In fact, it is clear that we need only verify that the condition holds for two vertices $x = v(g_1)$ and $y = v(g_2)$ in $\Gamma$.

3. Admissible Coxeter groups

We first focus our attention on a certain class of right-angled Coxeter groups which are (at least intuitively) easier to deal with than right-angled groups in general.

Let $W$ be right-angled, and let $\mathcal{V}$ be its diagram. A \textit{simple circuit of length $n$} in the diagram $\mathcal{V}$ is a closed path $C = \{[s_1s_2], ..., [s_ns_1]\}$ in $\mathcal{V}$ such that $s_i \neq s_j$ for all $1 \leq i < j \leq n$. We identify each circuit with its cyclic permutations. A circuit $C$ as above is called \textit{achordal} if for any $s_i, s_j$, $|j - i| > 1$, $s_i$ and $s_j$ are not adjacent in $\mathcal{V}$.

Any achordal simple circuit $\{ab, bc, cd, da\}$ in $\mathcal{V}$ of length 4 is called a \textit{square}. A \textit{diagonal} of the square above is either one of the pairs $\{a, c\}$ or $\{b, d\}$. We call $\mathcal{V}$ (and also $W$) \textit{admissible} if it is possible to choose a collection of diagonals, one for each square in $\mathcal{V}$, which are all disjoint from one another. We will also call such a choice of diagonals admissible. As examples, consider the Coxeter diagrams in Figure 1; $\mathcal{V}_1$ represents an admissible group, and $\mathcal{V}_2$, a group which is not admissible. (All edges are assumed to have label 2.)

Now suppose that $\mathcal{V}$ is admissible, and select an admissible set of diagonals, $\{a_i, c_i\}$. Let $H_i = W_{\{a_i, c_i\}} \cong D_{\infty}$. We now prove the main theorem in the restricted setting of admissible Coxeter groups:
Theorem 3.1. Let $W$ be an admissible Coxeter group. Then $W$ is relatively hyperbolic relative to any collection of parabolic subgroups $\{H_i\}$ chosen as above (that is, corresponding to an admissible choice of diagonals).

In order to prove this, we must verify that the condition in Proposition 2.1 holds for admissible Coxeter groups. We do this by explaining the structure of geodesic paths in $\hat{\Gamma} = \hat{\Gamma}(\{H_i\})$ between two vertices $v(w_1)$ and $v(w_2)$ in $\Gamma = \Gamma(W, S)$. Having described the structure of such paths, we will be able to show that any two such paths with the same endpoints are $\delta$-close, where $\delta$ is independent of the choice of endpoints.

Having proven Theorem 3.1, we will (in the final section) indicate the minor changes that need be made in the arguments of Section 4 in order to prove the main theorem in general.

4. Geodesics in $\hat{\Gamma}$

Throughout this section, both the group $W$ and the choice of diagonals made below are assumed to be admissible. We describe the collection of geodesics in $\hat{\Gamma}$ from 1 to a given vertex $w$ in $\Gamma$, in terms of one such geodesic. Our description will enable an easy proof of Proposition 2.1 in the presence of admissibility, and therefore of Theorem 3.1.

First of all, let us note the following fact, which can be proven very easily using van Kampen diagrams.

Lemma 4.1. Let $(W, S)$ be a right-angled Coxeter system, and let $w \in W$. Any two geodesic words (in the generators $S$) representing $w$ contain the same number of occurrences of each letter $s \in S$.

Lemma 4.1 actually follows from a more general result (which can also be proven using van Kampen diagrams):

Proposition 4.2. Let $(W, S)$ be an arbitrary Coxeter system, and let $w \in W$. Any two geodesic words (in the generators $S$) representing $w$ contain the same number of letter from each conjugacy class of generators from $S$.

Indeed, Lemma 4.1 follows from Proposition 4.2 because in an even system (one in which every relator is of the form $(st)^m$, $m$ even), no two distinct generators are conjugate. Lemma 4.1 will be used frequently without mention.

Now to the geodesics. We will show that there are essentially two operations which can be applied in order to obtain new geodesics from a fixed geodesic. Having
described these operations, we will show that the geodesics obtained from a single geodesic by means of these operations are uniformly close to the original geodesic.

Let \( \Gamma = \Gamma(W,S) \) and \( \hat{\Gamma} = \hat{\Gamma}\{\{H_i\}\} \) as in the preceding section. Let \( v(w_1) \) and \( v(w_2) \) be vertices in \( \Gamma \). By translation, we can assume that \( w_1 = 1 \), and we rewrite \( w_2 \) as \( w \). Through abuse of notation, we will often write the vertex \( v(w) \) as \( w \).

Suppose that \( \hat{\gamma} \) is a \( \hat{\Gamma} \)-geodesic from 1 to \( w \). The path \( \hat{\gamma} \) can be broken up into blocks which correspond to subpaths in \( \Gamma \) and subpaths in \( \Gamma \setminus \hat{\Gamma} \). That is, \( \hat{\gamma} = \alpha_1\beta_1 \cdots \alpha_k\beta_k \), where \( \alpha_i \) is a path

\[
\{[w_{i-1}, w_{i-1}s_1], [w_{i-1}s_1, w_{i-1}s_1s_2], \ldots [w_{i-1}s_1s_2 \cdots s_l, w_i]\}
\]

where every vertex and every edge lies in \( \Gamma \), and \( \beta_i \) is a path of combinatorial length 2:

\[
\{[w_{i-1}, v(w_{i-1}H_i)], [v(w_{i-1}H_i), w_i = w_{i-1}h_i]\},
\]

for some \( h_i \in H_i \). We call the paths \( \alpha_i \) the \( \Gamma \)-blocks of \( \hat{\gamma} \) and the paths \( \beta_i \) the \( \hat{\Gamma} \)-blocks of \( \hat{\gamma} \).

One fact is obvious (by the geodesity of \( \hat{\gamma} \)):

**Lemma 4.3.** Each \( \Gamma \)-block \( \alpha_i \) of \( \hat{\gamma} \) is a geodesic in \( \Gamma \).

When it will not cause confusion, we do not distinguish between the path \( \alpha_i \) and the word which labels this path.

We now construct a path, \( \gamma_i \), in \( \Gamma \), from \( \hat{\gamma} \). For each \( \hat{\Gamma} \)-block \( \beta_i \) in \( \hat{\gamma} \) as above, replace \( \beta_i \) with the path \( \hat{\beta}_i \) in \( \Gamma \) from \( w_{i-1} \) to \( w_i = w_{i-1}h_i \) defined by the (unique) geodesic word in \( W_{\{a_i, a_j\}} \) representing \( h_i \). If \( \hat{\beta}_i \) is a single letter, we call the corresponding \( \hat{\Gamma} \)-block trivial; otherwise, we call the block non-trivial. We will often abuse terminology and call \( \hat{\beta}_i \) (or even the label of this path) a \( \Gamma \)-block as well.

Performing all such replacements \( \beta_i \rightarrow \hat{\beta}_i \) yields the path \( \gamma \). In general, \( \gamma \) may not be a \( \Gamma \)-geodesic, but it is nearly so. Namely, we have the following

**Lemma 4.4.** Let \( \gamma \) be derived from \( \hat{\gamma} \) as above. Let \( h_i \) be the geodesic word in \( H_i \) corresponding to the \( \hat{\Gamma} \)-block \( \beta_i \). Then a \( \Gamma \)-geodesic \( \gamma \) from 1 to \( w \) can be obtained from \( \gamma \) by canceling at most 2 letters from each word \( h_i \) arising from a non-trivial \( \hat{\Gamma} \)-block.

**Proof.** We appeal to an algorithm of Tits in [16]. The results of this paper show that if a word \( w \) in the letters \( S \) is not geodesic, then a shorter representative for the same group element can be obtained from \( w \) by performing successive commutations to bring two occurrences of the same letter next to each other, which can then be canceled.

It is clear that there can be no such cancellation of the letter \( a \), where either \( a \) occurs in both \( \alpha_i \) and \( \alpha_j \) or \( a \) occurs in both \( \alpha_i \) and \( h_{ij} \). (Otherwise we would be able to replace the appropriate words with shorter words and readjust the path \( \gamma \) to obtain a shorter (relative to both both \( \Gamma \) and \( \hat{\Gamma} \) path from 1 to \( w \), contrary to the geodesity of \( \hat{\gamma} \).) The same holds for trivial \( \hat{\Gamma} \)-blocks \( \beta_i \).

Thus we need only consider the case where the letter \( a \) to be canceled occurs in two words \( h_{i1} \) and \( h_{i2} \) \((i_1 < i_2)\) coming from non-trivial \( \hat{\Gamma} \)-blocks \( \beta_{i1} \) and \( \beta_{i2} \). (By admissibility, the other letter, \( c \), in \( h_{i1} \) is the same as the other letter in \( h_{i2} \).) Suppose cancellation of \( a \) does occur. In this case, since \( ac \neq ca \), \( h_{i2} \) ends with \( a \) and \( h_{i2} \) begins with \( a \), and \( a \) commutes with every letter occurring in any word \( \alpha_j \).
or $h_j$ between $h_{i_1}$ and $h_{i_2}$. The letter $c$ cannot have such commutativity properties, as otherwise we would be able to commute the entire word $h_{i_1}$ past the intervening words and collapse the two $\Gamma$-blocks $\beta_{i_1}$ and $\beta_{i_2}$. We would thus obtain a $\Gamma$-path from 1 to $w$ with subpaths $\alpha_i$ identical to those of $\check{\gamma}$, but with one fewer $\Gamma$-block, contradicting $\check{\gamma}$'s geodesity. Therefore we may cancel only one letter from the end of $h_{i_1}$ and one from the beginning of $h_{i_2}$. It may be possible (if $h_{i}$ has length at least 3) to cancel both the first and the last letters of $h_{i}$ in this fashion. \hfill \Box

**Remark.** One can use van Kampen diagrams to prove the above result, instead of appealing to [16]. For applications of van Kampen diagrams to Coxeter groups and related groups, see [1], [2], and [8], for example. For details regarding van Kampen diagrams, the reader may consult [9] or [11].

The proof of Lemma 4.4 suggests a certain rigidity property that $\check{\Gamma}$-geodesics must satisfy. We say that two $\check{\Gamma}$-blocks $\beta_{i_1}$ and $\beta_{i_2}$ have the same type if they are both words in letters of the same diagonal, $\{a, c\}$. Arguments similar to the proof above can be used to verify Lemma 4.5.

Let $\check{\gamma}_1$ and $\check{\gamma}_2$ be two $\check{\Gamma}$-geodesics from 1 to $w$. Then the non-trivial $\check{\Gamma}$-blocks of $\check{\gamma}_1$ are the same type, up to multiplicity, as those of $\check{\gamma}_2$. (In fact, corresponding $\check{\Gamma}$-blocks $\beta$ and $\beta'$ give rise to words $h$ and $h'$ which differ by at most 2 letters, as in Lemma 4.4.)

As a consequence of these lemmata, there is only one fundamental operation by means of which we may obtain the $\check{\Gamma}$-blocks of one $\check{\Gamma}$-geodesic, $\check{\gamma}_1$, from the $\check{\Gamma}$-blocks of another, $\check{\gamma}_2$, with the same endpoints. Namely, suppose $\beta_{i_1}$ and $\beta_{i_2}$ are of the same type (corresponding to $H = W_{\{a, c\}}$). Suppose that the letter $a$ commutes with every letter in any block lying between $\beta_{i_1}$ and $\beta_{i_2}$. Our operation then consists of replacing $h_{i_1}$ with $h_{i_1}a$ and $h_{i_2}$ with $ah_{i_2}$. We call an application of this operation an insertion-deletion (ID). ID operations may be performed a number of times on a given geodesic.

Now note that it is possible that some $\check{\Gamma}$-blocks may commute with one another and so the relative order of the blocks in the geodesics $\check{\gamma}_1$ and $\check{\gamma}_2$ may not be the same. Likewise, given two $\check{\Gamma}$-geodesics, the letters which appear in the $\Gamma$-blocks of each geodesic may appear in a different order in each path as well. Furthermore, some such letters may commute with certain of the $\check{\Gamma}$-blocks, leading to greater variation. However, this variation is controllable: we now define a procedure by means of which every $\check{\Gamma}$-geodesic from 1 to $w$ can be obtained, once we have in hand one such geodesic, $\check{\gamma}$.

Let

$$\check{\gamma} = \alpha_1 \beta_1 \cdots \alpha_k \tilde{\beta}_k$$

be obtained from $\check{\gamma}$ by means of the replacements $\beta_i \mapsto \tilde{\beta}_i$. As before, let $h_i$ be the unique geodesic word labeling $\beta_i$. Let $w = \alpha_1 h_1 \cdots \alpha_k h_k$; that is, $w$ is the word labeling $\check{\gamma}$.

We now describe all words $\check{\gamma}_1$ (with label $w_1 = W w$) which can be obtained from a $\check{\Gamma}$-geodesic $\check{\gamma}_1$ which in turn has the same endpoints as $\check{\gamma}$. Such a word $\check{\gamma}_1$ will be called nice; its label will also be called nice. In particular, each subword $h_i$ of $w$ appears undisturbed in the label of any nice word. (As an example, if $V$ consists...
of the square \( \{ [ab], [bc], [cd], [da] \} \), and if \( \{ a, c \} \) is the selected diagonal, then \( acab \)
and \( aca \) are both geodesics representing the same element as \( baca \), whereas only
the first is nice.) We assume (often without mention) throughout the remainder of
the section that every word \( \bar{\gamma} \) is nice.

We define a syllable of \( w \) to be either a letter in some \( \Gamma \)-block \( \alpha_i \) or a single \( \bar{\Gamma} \)-
block \( h_i \) in its entirety. We write \( \| w \| \) for the syllable length of \( w \) (that is, the number
of syllables occurring in \( w \)). Clearly every nice word \( w_1 \) satisfies \( \| w \| = \| w_1 \| \).

Certain of the syllables of the word \( w \) are forced to occur before certain others,
in any nice word \( w_1 \). That is, for instance, if \( st \neq ts \) and the first occurrence of \( s \)
precedes the first occurrence of \( t \) in \( w \), then in any nice word \( w_1 \), the first occurrence
of \( s \) must precede the first occurrence of \( t \). It is possible (although not necessary for
our purposes) to assemble a complete description of these precedence relationships
that arise in the word \( w \).

**Example.** Consider the group \( W \) with diagram shown in Figure 2.

![Figure 2](image)

We select the diagonals \( \{ a, c \} \) and \( \{ b, f \} \). Consider the word \( acabedbfbc \) (which
has 6 syllables). In any nice word \( w_1 \) such that \( w =_W w_1 \), \( aca \) must precede each
of \( e, d, c, \) and \( bfb \), but may follow the first \( b \). Also, \( d \) must follow \( aca, b, \) and \( e \),
and must precede \( bfb \), but may either precede or follow \( c \).

From similar considerations, it is possible to determine, for a given syllable \( x \)
of \( w \), which syllables of \( w \) may appear before \( x \) in some nice word \( w_1 \) such that
\( w =_W w_1 \). Of course, any syllable preceding \( x \) in \( w \) may precede \( x \) in some \( w_1 \).
Furthermore, any syllable \( x' \) which occurs after \( x \) in \( w \) and which commutes with
both every syllable lying between \( x \) and \( x' \), and with \( x \) itself can also precede \( x \) in
some nice word \( w_1 \).

**Example.** Returning to the previous example, \( d \) can be preceded by each of \( aca, b, e, \) and \( c \).

**Remark.** As the reader may have noticed, for the purposes of establishing rules
of precedence, as above, we may consider distinct occurrences of the same letter
\( s \) as different letters (or as indexed copies of the same letter). This fact is used
implicitly throughout the remainder of this section.

Consider any syllable \( x \) occurring in \( w \). We say that \( x \) is \( k \)-forced if in any nice
word \( w_1 =_W w \), \( x \) must appear in the first \( k \) syllables of \( w_1 \). For instance, every
syllable in \( w \) is \( \| w \| \)-forced, and in the above example, the syllable \( d \) is 5-forced.

As we will soon see, any prefix of a nice word \( w_1 \) of syllable length \( k \) will contain a
certain number of \( k \)-forced syllables, as well (perhaps) as a number of other syllables
are not \(k\)-forced. Syllables that occur in a prefix of \(w_1\) of syllable length \(k\) which are not \(k\)-forced will be called \(k\)-free.

**Lemma 4.6.** Let \(w_1\) be a nice word such that \(w =_W w_1\), and let \(p(k)\) be the prefix of \(w_1\) of syllable length \(k\). Then \(p(k) =_W p_1(k)p_2(k)\), where every syllable in \(p_1(k)\) is \(k\)-forced, and every syllable in \(p_2(k)\) is \(k\)-free.

**Proof.** Let \(x\) be any \(k\)-free syllable in \(p(k)\), and let \(y\) be a \(k\)-forced syllable occurring after \(x\) in \(p(k)\). It must be possible to commute \(x\) past \(y\) (and indeed past every intervening syllable), as otherwise \(x\) too would be \(k\)-forced. Therefore every \(k\)-free syllable can be commuted past every \(k\)-forced syllable, and \(p(k)\) can be rewritten as claimed. \(\Box\)

Being \(k\)-forced does not depend on the choice of nice word, so the word \(p_1(k)\) contains the same syllables, regardless of our choice of nice word \(w_1\). Moreover, although these syllables are bound by rules of precedence (as above), it is clear that the word \(p_1(k)\) obtained from a nice word \(w_1\) can be modified by commutations to obtain the word \(p_1(k)\) corresponding to any other nice word \(w_2\). The following fact results:

**Lemma 4.7.** We may choose the same word \(p_1(k)\) (as in Lemma 4.6) for every nice word \(w_1\) such that \(w =_W w_1\).

This fact is promising, as it indicates how we may prove that any two \(\hat{\Gamma}\)-geodesics are uniformly close to one another. Indeed, suppose that for every \(k, k - \|p_1(k)\| \leq M\) for some \(M\), and let \(\gamma(p_1)\) be the \(\hat{\Gamma}\)-geodesic from \(1\) to \(v = v(p_1)\) in \(\hat{\Gamma}\) which is determined by \(p_1\) in the obvious manner. Let \(w_1\) and \(w_2\) be nice words such that \(w =_W w_1, i = 1, 2\), and such that no ID operations need be performed in transforming \(w_1\) to \(w_2\). Let \(v_i\) be the point on the path in \(\hat{\Gamma}\) with label \(w_i\) of \(\hat{\Gamma}\)-distance \(k\) from \(1\). Then

\[
d_{\hat{\Gamma}}(v_1, v_2) \leq d_{\hat{\Gamma}}(v_1, v) + d_{\hat{\Gamma}}(v, v_2) \leq 2M.
\]

If we can uniformly bound \(M\) (that is, show that it is independent of the choice of \(w\)), we will be nearly done. (Because in this case, corresponding points on geodesic paths obtained from one another without the use of ID operations will be uniformly close.)

For a fixed system \((W, S)\) and admissible choice of diagonals, let \(M\) be one less than the maximal number of syllables which mutually commute with one another in any possible word \(w\). (For instance, in the above examples, \(M = 1\).) The value of \(M\) can be determined easily from the diagram \(V\) and from the choice of diagonals.

**Proposition 4.8.** Let \(w\) label a nice \(\Gamma\)-geodesic \(\gamma\). For each \(k, 1 \leq k \leq \|w\|\), define \(p_1(k)\) as above. Then for every \(k, 1 \leq k \leq \|w\|, k - \|p_1(k)\| \leq M\).

**Proof.** For any word \(w\) and any \(k, 1 \leq k \leq \|w\|\), denote by \(n(k, w)\) the number of syllables of \(w\) which are \(k\)-forced in \(w\). We need to show that \(k - n(k, w) \leq M\), for every \(k, 1 \leq k \leq r\).

We prove this fact by induction on the syllable length of \(w\). The result is clearly true if \(\|w\| = 1\). Assume it to be proven in case \(\|w\| \leq r - 1\), and let \(\|w\| = r\).

From \(w\) we create a shorter word, \(w'\), by removing a single syllable. Namely, let \(x\) be any syllable of \(w\) which is \(k\)-forced, for \(k\) minimal among all syllables of \(x\). We claim that \(x\) may be commuted to the front of \(w\). If this were not the case, some
syllable, $y$, which precedes $x$, satisfies $xy \neq yx$. In this case, strictly fewer syllables of $w$ can precede $y$ as can precede $x$, so that $y$ is $k$-forced for a smaller value of $k$ than is $x$, contrary to our choice of $x$.

Therefore $w = w' xw'$ for some $w'$, $\|w'\| = r - 1$. It is clear that $w'$ is a nice word (that is, that it labels a $\Gamma$-geodesic $\gamma'$ which arises from a $\tilde{\Gamma}$-geodesic $\tilde{\gamma}'$ by replacements $\beta_i \mapsto \tilde{\beta}_i$, as before).

Consider any syllable $y$, $y \neq x$ in $w$ (the same syllable appears in $w'$). If $y$ is $k$-forced in $w'$, then it is clearly $(k+1)$-forced in $w$ (x may precede it). Moreover, $x$ is $(k+1)$-forced in $w$ as well. Thus $n(k+1, w) \geq n(k, w') + 1$.

Fix $k \geq 2$ such that $n(k, w) \geq 2$. By the inductive hypothesis,

$$(k - 1) - n(k - 1, w') \leq M.$$ 

Thus

$$k - n(k, w) \leq k - (n(k - 1, w') + 1) = (k - 1) - n(k - 1, w') \leq M.$$ 

Therefore we need only show that $k - n(k, w) \leq M$ when $n(k, w) = 1$; that is, when $x$ is the only syllable which is $k$-forced. It suffices to show that for any choice of $w$, there is some syllable which is $(M + 1)$-forced.

As above, let $x$ denote any syllable which is $k$-forced, for $k$ minimal among all $x$ in $w$.

We construct a set $S_1$ of syllables of $w$, all of which commute with one another, in the following fashion. Initially, $S_1$ consists of the first syllable of $w$. We add to this set the second syllable of $w$ if it commutes with the first. Thereafter, we consider each syllable $y$ in turn, adding it to $S_1$ if and only if

1. $y$ commutes with each syllable already included in $S_1$, and
2. $y$ can be commuted to the front of the word $w$.

It is easy to show that $x \in S_1$.

Let $m = |S_1|$. By the definition of $M$, $m \leq M + 1$. We now claim that $x$ is $m$-forced (this will finish our proof).

Suppose that $S_1 = \{x_1, \ldots, x_{m-1}, x\}$. Without loss of generality, we can write $w$ as $x_1 \cdots x_{m-1} x w''$ for some word $w''$.

Assume that $x$ is not $m$-forced, and assume first that $x$ is a single letter. Thus there is some syllable $y \not\in S_1$ such that $yx = xy$, but $yx_i \neq x_i y$, for some $i$, $1 \leq i \leq m - 1$. We may assume that $y$ is the first such syllable to occur after $x$ in $w$, and by commuting, we may assume that $i = m - 1$. Assume for now that both $x_{m-1}$ and $y$ are also single letters. By the minimality in our choice of $x$, there must be a syllable $y'$ following $x$ in $w$ such that $x_{m-1}$ and $y$ both commute with $y'$, but $x$ does not. (In fact, from our choice of $y$, this must be true of the first syllable following $x$ which does not commute with $x$.) Assume for now that $y'$ too is a single letter.

We have found a square in the diagram $\mathcal{V}$, consisting of the letters $\{x, x_{m-1}, y', y\}$. Therefore either $\{x, y'\}$ or $\{x_{m-1}, y\}$ must have been selected as a diagonal. However, either choice violates the assumption that each of the letters $x$, $x_{m-1}$, $y$, and $y'$ is a syllable in its own right. (This is so because in either case we realize a non-trivial $\tilde{\Gamma}$-block consisting of two of these letters.)

There are a number (15, to be exact) of other cases to consider, depending on which of the syllables $x$, $x_{m-1}$, $y$, and $y'$ are single letters, and which stem from
non-trivial $\hat{\Gamma}$-blocks. In each case, we may argue much as above, obtaining either a similar contradiction, or a contradiction to the very admissibility of the group $W$.
(The latter contradiction arises, for instance, when $x$ and $y$ are single letters and $x_{m-1}$ and $y'$ come from non-trivial $\hat{\Gamma}$-blocks.)

Therefore $x$ is $m$-forced, and is thus $(M+1)$-forced. This concludes the proof. □

What have we now shown? Let $\hat{\gamma}_1$ and $\hat{\gamma}_2$ both be $\hat{\Gamma}$-geodesics between the vertices 1 and $w$ in $\hat{\Gamma}$. Then if $\hat{\gamma}_2$ can be obtained from $\hat{\gamma}_1$ without applying ID operations (i.e., by commutations of syllables only), $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are $2M$-Hausdorff close, where $M$ is defined as above.

In order to prove Proposition 2.1 (and therefore Theorem 3.1), it suffices to show that if $\hat{\gamma}_2$ and can be obtained from $\hat{\gamma}_1$ by applying only ID operations, then the paths are $\delta$-close, for some $\delta > 0$. (We can first apply ID operations to make sure that all $\hat{\Gamma}$-blocks which occur in two given nice geodesics are identical up to commutation, and then perform commutations to obtain one geodesic from the other.)

Indeed, this is so. The $\hat{\Gamma}$-distance between corresponding points on two $\hat{\Gamma}$-geodesics which differ only by application of ID operations is uniformly bounded by a constant which (like $M$) is related to the structure of $V$ and the choice of diagonals.

Each ID operation, like a balanced pair of parentheses, has an “opening” and a “closing”. In the notation used before to define ID operations, the “opening” occurs when the word $h_{i_1}$ is replaced by the word $h_{i_1}a$, and the closing when $h_{i_2}$ is replaced by $a h_{i_2}$. The greatest distance between corresponding points on two geodesics differing only by ID operations is obtained when the number of “unclosed” pairs is maximal. (This distance at most the number of unclosed pairs at that point). A necessary (though not in general sufficient) condition for each of the ID operations inserting and deleting the letters $a_1, a_2, ..., a_k$ to be simultaneously in an open state is that each $a_i$ commutes with $a_j$, $j > i$. Thus the maximal number of unclosed pairs at any time is bounded above by the maximal number of generators which commute with a given generator of $S$.

Therefore, Proposition 2.1 and Theorem 3.1 are proven. Moreover, an explicit “constant of relative hyperbolicity” can be computed, merely by examining the diagram $V$ and the admissible choice of diagonals.

5. The main theorem in general

We now indicate the changes that must be made in the above argument in order to prove Theorem 1.2 in general. The chief difficulty stems from the fact that if $W$ (or the choice of diagonals) is not admissible, then a single letter $s \in S$ may appear in more than one diagonal.

However, the first four results of Section 4 remain true in the general case. In particular, we need only cancel at most 2 letters from each non-trivial $\hat{\Gamma}$-block in order to obtain a $\Gamma$-geodesic from a $\hat{\Gamma}$-geodesic. Moreover, ID operations are still valid, although now an ID operation can effect the exchange of a letter between two $\hat{\Gamma}$-blocks of different type (which share a single letter).

Lemma 4.5 is no longer valid. As before, given two $\hat{\Gamma}$-geodesics with the same endpoints in $\hat{\Gamma}$, $\hat{\gamma}_1$ and $\hat{\gamma}_2$, we obtain $\Gamma$-geodesics $\gamma_1$ and $\gamma_2$ by first replacing each $\beta_i$ with $\hat{\beta}_i$, and then by applying ID operations. In order to obtain $\gamma_2$ from $\gamma_1$, we
must modify the $\hat{\Gamma}$-blocks that occur in $\gamma_1$ without changing the syllable length of its label. This can be done step-by-step, where at each step we either leave fixed the number of $\hat{\Gamma}$-blocks (merely shifting letters from one block to another), or decrease the number of $\hat{\Gamma}$-blocks by one, adding 1 to the total length of the $\Gamma$-blocks $\alpha_i$.

An operation which accomplishes this first goal is called a shift operation. Let \{a, b\}, \{b, c\}, \{c, d\}, and \{d, e\} all be selected diagonals. Suppose that the word $w$ contains the subword $(ab)^n w_1 cdw_2 (cd)^n$, and that both $bw_1 = w_1 b$ and $dw_2 = w_2 d$ hold in $W$. The most general form of shift involves replacing the subword above with the subword $(ab)^{m-1} aw_1 bc w_2 (de)^n d$ (both of which have the same number of syllables), or vice versa. Of course, the words $w_i$ may be trivial. More than one shift can be performed in an overlapping sequence, each syllable passing on a single letter to the next, in the obvious fashion. In this case only the first and the last syllables involved in the shift can have length greater than 2.

A second way in which the first goal can be accomplished is by applying an exchange operation. Suppose that \{a, b\}, \{a, c\}, \{b, d\}, and \{c, d\} are all selected diagonals, and that $bc = cb$. Furthermore, assume that both $b$ and $c$ commute with the word $w_1$. In its most general form, an exchange consists of replacing the subword $abw_1 cd$ of $w$ with the subword $acw_1 bd$. As with shifts, there can be sequences of overlapping exchanges.

How do we effect a transformation in which a $\hat{\Gamma}$-block disappears and the total length of the $\Gamma$-blocks $\alpha_i$ is increased by 1? The disappearing $\hat{\Gamma}$-block must have length 2, and one of its letters must be “absorbed” into another $\hat{\Gamma}$-block. We’ve seen this behavior already, in ID operations which replace $(ab)^n w_1 (ac)$ with $(ab)^m aw_1 c$, for example. Another way this can be accomplished is by a variant sort of exchange, in which the word $(ab)w_1 (cd)w_2 (ed)^n$ is replaced by the word $(ac)bw_1 w_2 (de)^n d$ (or vice versa), where $\hat{\Gamma}$-blocks are indicated by parentheses.

These new operations differ from ID operations in that the type of syllables present is (in general) modified. However, each single new operation can be seen as a composition of two ID operations, where the word resulting from the application of the first ID operation is not a $\hat{\Gamma}$-geodesic. (In an exchange, there is a commutation of letters in between the two ID operations.) For this reason, we may treat shifts and exchanges much as we would ID operations. Indeed, keeping track of “openings” and “closings” of sequences of shifts and exchanges as we did with ID operations, it is not difficult to verify the following fact.

**Proposition 5.1.** Let $\gamma_1$ and $\gamma_2$ be $\hat{\Gamma}$-geodesics which can be obtained from one another through application of ID operations, shifts, and exchanges only. Then corresponding points on these two geodesics are uniformly $\hat{\Gamma}$-close.

We are almost done now. We first note the following analogue of Lemma 4.6.

**Proposition 5.2.** Let $\gamma_1$ and $\gamma_2$ be $\hat{\Gamma}$-geodesics with the same endpoints, 1 and $w$, in $\hat{\Gamma}$. Then the $\Gamma$-blocks of $\gamma_2$ can be obtained from the $\hat{\Gamma}$-blocks of $\gamma_1$ by application of ID operations, shifts, and exchanges.

**Proof.** Let $\bar{\gamma}_1$ be the $\Gamma$-path resulting from $\gamma_1$ be the replacements $\beta_i \mapsto \bar{\beta}_i$, as before. Both $\gamma_1$ and $\bar{\gamma}_2$ are nice words representing the same group element. By applying ID operations, we can transform each of these paths into a $\Gamma$-geodesic. The label of one of these paths can be obtained from the label of the other by commutations whose net effect must not increase the syllable length of the word in question. We
have defined the operations above precisely in order to describe all ways in which this can be done.

Now suppose we are given two $\hat{\Gamma}$ geodesics, $\hat{\gamma}_1$ and $\hat{\gamma}_2$, with the same endpoints, $1$ and $w$, in $\Gamma$. By applying ID operations, shifts, and exchanges, we may obtain from $\hat{\gamma}_1$ a new $\hat{\Gamma}$-geodesic, $\hat{\gamma}_1'$, which has the same $\hat{\Gamma}$-blocks as $\hat{\gamma}_2$, although in a different order. By Proposition 5.1, corresponding points on $\hat{\gamma}_1$ and $\hat{\gamma}_1'$ are uniformly close to one another in $\hat{\Gamma}$. In order to prove Theorem 1.2, we need only show that geodesics which differ only by the order of their syllables are uniformly close.

However, the proof of Proposition 4.8 in the more general setting goes through almost exactly as before. The only difference is that now we can no longer derive a contradiction to admissibility (such admissibility assumptions are not made!). We only contradict the syllabification of the geodesic, as was done in the proof of Proposition 4.8 above.

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