Nonsingular Cosmologies from Branes

Anindya Biswas, Sudipta Mukherji and Shesansu Sekhar Pal

Institute of Physics, Bhubaneswar-751 005, India

anindyab, mukherji, shesansu@iopb.res.in,

We analyse possible cosmological scenarios on a brane where the brane acts as a dynamical boundary of various black holes with anti-de Sitter or de Sitter asymptotic. In many cases, the brane is found to describe completely non-singular universe. In some cases, quantum gravity era of the brane-universe can also be avoided by properly tuning bulk parameters. We further discuss the creation of a brane-universe by studying its wave function. This is done by employing Wheeler-De Witt equation in the mini superspace formalism.
1 Introduction

Motivated by string theory, the AdS/CFT correspondence and the hierarchy problem in particle physics, brane-world models have been a focus of interest in recent years [1] - [6]. In these models, our universe is realized as a boundary of a higher-dimensional space-time. In this context, a well studied example is when the bulk is an AdS space. The gravitational interaction among matter on this brane is found to be described by standard laws when one considers distance scale much larger than the AdS length scale [7].

In the cosmological context, many authors have considered (see for example [8] - [28]) the embedding of a four dimensional Friedmann-Robertson-Walker (FRW) universe in five dimensional bulk geometry. The bulk is often described by AdS or AdS Schwarzschild black hole metric. In the later case, the mass of the black hole is found to act effectively as an invisible energy density on the brane with the same equation of state of radiation matter. This, however, is found to have a nice interpretation in terms of AdS/CFT correspondence. The correspondence allows one to reinterpret the AdS-Schwarzschild geometry as a source for a four dimensional CFT at finite temperature. As pointed out in [29] - [31], it is easy then to understand why the radiation-dominated FRW universe emerges. This is because all CFTs have the same equations of state up to numerical constants and hence the FRW equation takes the same form as in the case of radiation dominated universe. Replacing the bulk now by a five dimensional AdS-Reissner-Nordst"orm black hole may seem like a straight forward generalization. However, as it was found in [32], such geometry has far reaching consequences on the brane evolution. First, FRW equation now describes the brane as dominated by induced radiation and “stiff” matter. However, in the equation, the stiff matter contribution comes with an opposite sign*. As a result, the universe (flat, open and closed) is never found to shrink to zero size. It is important to note that the bulk

*This was first noticed in [13].
electric charge plays the role of a “regulator” on the brane. As we take the charge to zero, big bang singularities are found to reappear\(^1\).

Here, in this paper, we further analyse the \(D\) dimensional brane evolution in various \(D + 1\) dimensional AdS or dS black hole backgrounds of various topologies. We first study these backgrounds carefully and find critical values of mass and other parameters in order these geometries to correspond to black holes with non-degenerate horizons. We believe that some of these explicit results here are new. We then study the brane dynamics in these geometries. It is found to be described by FRW equation which has contributions from induced radiation matter, stiff matter and an effective cosmological constant. This constant depends on a certain combination of bulk cosmological constant and brane tension. We then analyse the classical dynamics of the brane by solving the FRW equation in case of vanishing and nonvanishing effective cosmological constant.

We find that all open, flat and closed universes have no big bang singularities when the bulk has a nonvanishing electric charge. For the closed universe, the size is bounded from above and from below. On the other hand, for open and flat universe, the brane starts with finite radius and expands for ever. We also find here that the minimum radius of the universe can be made sufficiently large by tuning the bulk mass and charge to large values. This, in turn, means that for such universes we can avoid reaching a quantum gravity era in the past. This scenario, in a more general set up, is analysed recently in \(34\). We then analyse the brane universe when there is an effective (negative or positive) cosmological constant on the brane. Though the FRW equations here can not be solved analytically in a closed form, we study the equation by numerical means. The non-singular nature of the solutions is also found to persist in this case. However, is all the above cases, universe is found to be singular when the charge is set to zero. We, therefore, turn our attention to the fate of these singularities from the perspective of “quantum cosmology”. By employing the Wheeler De Witt (WDW) equation in the mini superspace formalism, we find that the minimum size of the brane universe is of the order of the Planck length. We feel that this may indicate that the universe stabilizes against collapse due to quantum effects. However, we may add here that quantum cosmology at this present stage has certain inherent ambiguities. These are associated with the orderings of operators, choice of boundary conditions of the wave functions etc. That is why our results here will be more of speculative in nature.

This paper is organized as follows. In the next section, we critically analyse various topological AdS and dS black hole backgrounds. Considering brane as a dynamical boundary of these geometries, we set up brane equation of motion by studying the brane effective action. In section 3, we analyses classical evolution of the brane by explicitly solving brane equation of motion.\(^4\) We then critically analyse various physical properties of our solutions with a particular emphasize on their early and late time behavior. Section 4 of this paper deals with the brane dynamics from the WDW perspective. We solve relevant WDW equations, in mini superspace, for tensionless branes and analyse its behavior at short distance scales. For branes with tension, one needs to resort to some approximate methods. We study these branes in section 5 via WKB approximation. The paper ends with a discussion of our results.

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\(^1\)Some earlier work on brane cosmology can be found in \(32, 33\).

\(^4\)Some cosmological scenarios on the brane have been reviewed in \(27\).
2 Brane effective action

Bulk with AdS asymptotics

The dynamics of a $D$ dimensional brane inside a bulk gravitational field is dictated by the following bulk-boundary action with a negative cosmological constant $\Lambda = \frac{-D(D-1)}{2l^2}$

$$S = \frac{1}{16\pi G_{D+1}} \int_M d^{D+1}x \sqrt{g} \left( R - F^2 + \frac{D(D-1)}{l^2} \right) - \frac{1}{8\pi G_{D+1}} \int_{\partial M} d^Dx \sqrt{\gamma} \left( T - \frac{8}{8\pi G_{D+1}} \int_{\partial M} d^Dx \sqrt{\gamma} \right),$$

where $F^2$ is the Maxwell kinetic energy term and $K$ is the trace of the extrinsic curvature $K_{\mu\nu}$ taken with respect to the induced metric on the brane $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. This term is required for a well-defined variational principle on the space time boundary [35]. Here $n_\mu$ is the unit normal vector to the brane. Furthermore, $T$ in (1) is the brane tension and this term is required in order to get a finite action and stress tensor [36, 37].

Variation of (1) with respect to the induced metric gives us

$$K_{\mu\nu} = \frac{T}{D-1} \gamma_{\mu\nu}. \quad (2)$$

A large class of solutions of bulk equations of motion can be represented by

$$ds^2 = -h(a)dt^2 + \frac{da^2}{h(a)} + a^2 \tilde{\gamma}_{ij} dx^i dx^j, \quad (3)$$

where

$$h(a) = k + \frac{a^2}{l^2} - \frac{\omega_D M}{a^{D-2}} + \frac{(D-1)\omega_D^2 Q^2}{8(D-2)a^{2D-4}}, \quad \omega_D = \frac{16\pi G_{D+1}}{(D-1)V_{D-1}}. \quad (4)$$

Here, $k = 0, \mp 1$, correspond to flat, hyperbolic and spherical geometries of $D$ dimensional subspace for a given $a$. $\tilde{\gamma}_{ij}$ is the metric for a constant curvature manifold $M^{D-1}$ with volume $V_{D-1} = \int d^{D-1}x \sqrt{\tilde{\gamma}}$. As can easily be seen from [33] and [11], for $M = Q = 0$, the bulk is a simple $D$ dimensional AdS space. For only $Q = 0$, the bulk is an AdS Schwarzschild black hole, while for all the parameters non-zero and within certain domain, the background corresponds to charged AdS black holes. The parameters $M$ and $Q$ can then be identified with Arnowitt-Deser-Misner mass and charge respectively. Some analysis of the causal structure of this metric has been performed in [38, 39].

We would like to comment on the horizon structure of the metric [33]. First, note that this metric asymptotically is anti-de Sitter for all values of parameters. However, for the metric to describe the exterior of black hole with non-degenerate horizon, we need to restrict the parameters. To have such a geometry, we would like $a^{(2D-4)}h(a)$ to have a simple root at $a = a_0$ such that $h(a)$ is positive for $a > a_0$. This gives a lower bound on the parameter $M$. This can be seen as follows. Let $h(a)$ be zero for $a = a_0$. Therefore, we can invert the relation to get mass $M$ as a

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8In general, one should add more derivative terms to (1), see [37].
function of \( a_0 \) and other parameters as

\[
M = \frac{a_0^{(D-2)}}{\omega_D} \left[ k + \frac{a_0^2}{l^2} + \frac{(D-1)\omega_D^2 Q^2 a_0^{2D+4}}{8(D-2)} \right].
\] (5)

Now, by taking first and second derivative of (5) with respect to \( a_0 \), we see that \( M \) has a minimum \( M_{\text{crit}} \) at \( a_{0_{\text{crit}}} \) when

\[
8D(a_0^{\text{crit}})^{2(D-1)} + 8(D-2)k l^2 (a_0^{\text{crit}})^{2D-4} - l^2 \omega_D^2 (D-1)Q^2 = 0.
\] (6)

\( M_{\text{crit}} \) can therefore be determined by first solving \( a_{0_{\text{crit}}} \) from (6) and then substituting it into (5). If we now define \( a_H(M, Q) \) as the larger of the positive solutions, we will have the following inequality:

\[
a_{0_{\text{crit}}}(Q) \leq a_H(M, Q) < \infty, \text{ as } M_{\text{crit}}(Q) \leq M < \infty.
\] (7)

Though, in higher dimension, expressions of \( M_{\text{crit}} \) and \( a_{0_{\text{crit}}} \) are either not illuminating or hard to obtain, but for \( D = 3 \), we have

\[
M_{\text{crit}} = \frac{\sqrt{2}(-k l^2 + 3\omega_3^2 Q^2 + kl \sqrt{k l^2 + 3\omega_3^2 Q^2})}{\sqrt{27\omega_3} \sqrt{(-kl + \sqrt{k l^2 + 3\omega_3^2 Q^2})^2}},
\]

\[
a_{0_{\text{crit}}} = \sqrt{-\frac{k l^2}{6} + \frac{l^2}{6} \sqrt{k^2 + \frac{3\omega_3^2 Q^2}{l^2}}},
\] (8)

We note here that \( a_{0_{\text{crit}}} \to 0 \) as \( Q \to 0 \). For \( D = 4 \), that is when the bulk is five dimensional, it is also possible to find \( M_{\text{crit}} \) and \( a_{0_{\text{crit}}} \) explicitly. However, the expressions are very big except for \( k = 0 \). In this case,

\[
M_{\text{crit}} = \frac{3\sqrt{2} \left(5\omega_4 Q^2 + \sqrt{l^4\omega_4^2 Q^4} \right)}{16 \left(l^2 \omega_4^2 Q^2 + \sqrt{l^4\omega_4^2 Q^4} \right)^{\frac{3}{2}}},
\]

\[
a_{0_{\text{crit}}} = \frac{3\sqrt{2}}{2} \left(l^2 \omega_4^2 Q^2 + \sqrt{l^4\omega_4^2 Q^4} \right)^{\frac{3}{2}}.
\] (9)

As we would like to find the dynamics of a \( D \) dimensional brane moving in (8), it is instructive to find out the effective action that controls the dynamics. For this purpose (and hence in the rest of the section), we do not need to explicitly specify the functional form of \( h(a) \). As we will see, effective action describing the boundary dynamics can be expressed in terms of \( h(a) \).

The extrinsic curvature, \( K_{\mu\nu} \), can be calculated following (10). Those are

\[
K_{\theta i} = \frac{\sqrt{h(a) + a^2}}{a},
\] (10)

and

\[
K^\tau = \frac{h'(a) + 2\ddot{a}}{2\sqrt{h(a) + a^2}},
\] (11)
where \( h'(a) = \frac{dh(a)}{da} \) and \( i = 1, \ldots, D - 1 \), \( \theta_i \)'s are the angular coordinates, \( \dot{a} = \frac{da}{d\tau} \) and \( \tau \) denotes the proper time as measured along the brane world volume. Details of these computations can be found in many earlier literature, see for example [40]. Inserting these values of extrinsic curvature in (2), we get

\[
\sqrt{h(a) + \dot{a}^2} = \frac{T}{D - 1}.
\]

(12)

The dynamics of the brane can be captured through an effective Lagrangian \( L \). Clearly, the only degree of freedom in \( L \) will be \( a \) - so the Lagrangian which we are about to write down is a construction in the mini-superspace. It is given by

\[
L = M_p \left[ \dot{a} \sinh^{-1} \left( \frac{\dot{a}}{\sqrt{h(a)}} \right) - \sqrt{h(a) + \dot{a}^2} + \frac{T}{D - 1} a \right],
\]

(13)

where \( M_p \) is Planck mass, a dimension full constant. The Euler-Lagrange equation for \( a \) then is given by

\[
\frac{h'(a) + 2\dot{a}}{2\sqrt{h(a)} + \dot{a}^2} = \frac{T}{D - 1}.
\]

(14)

\( L \) then reproduces the correct equation of motion of the brane (2) along with (12). We see from the Lagrangian, that the momentum conjugate to \( a \) is

\[
p = \frac{\partial L}{\partial \dot{a}} = M_p \sinh^{-1} \left[ \frac{\dot{a}}{\sqrt{h(a)}} \right].
\]

(15)

It is then straightforward to construct the hamiltonian,

\[
H = p\dot{a} - L = M_p \left[ \sqrt{h(a)} \cosh \left( \frac{p}{M_p} \right) - \frac{T}{D - 1} a \right].
\]

(16)

We will, in section 4, use this hamiltonian with suitable ordering to analyse a possible quantum scenario for the brane universe. However, before doing so, we will study the classical equation of motion of \( a \) in the next section.

**Bulk with dS asymptotics**

Brane evolution in de Sitter black hole background can simply be obtained from (4) by replacing \( l^2 \rightarrow -l^2 \) and setting \( k = 1 \). In this case, \( h(a) = 0 \) can have three positive roots. The largest one corresponds to the cosmological horizon and the rest two correspond to the inner and outer horizons of the black hole [41]. The brane equations remain same as in equations (12) and (14) with the hamiltonian as in (16).

### 3 Classical cosmological evolution of the brane

The classical cosmological evolution of the brane is determined from the solutions of (12). Substituting the form of \( h(a) \) from (4) in (12), we get

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{k}{a^2} + \Lambda_D + \frac{\omega_D M}{a^D} - \frac{(D - 1)\omega_D^2 Q^2}{8(D - 2)a^{2D - 2}},
\]

(17)
where $\Lambda = \frac{r^2}{(D-1)r^2 - \frac{1}{r^2}}$ is the effective cosmological constant on the brane. Note that for AdS bulk, the effective cosmological constant can be either zero, negative or positive. However, for dS bulk, $\Lambda$ is strictly positive. We can now find solutions of (17) in different situations, namely, for different values to $k$, for different choices of $\Lambda$ (positive, negative and zero). We thus have three distinct cases: (1) $Q = 0$, $M = 0$, (2) $Q = 0$, $M > 0$ and (3) $Q > 0$, $M > 0$ and for each of these three cases one have the following 9 situations

$$\begin{align*}
\Lambda &= 0, \quad k = 0, \pm 1 \\
\Lambda &> 0, \quad k = 0, \pm 1 \\
\Lambda &< 0, \quad k = 0, \pm 1.
\end{align*}$$

We will now discuss all these cases in the following.

(1) $M = Q = 0$

For $M = Q = 0$, the brane universe moves in AdS/dS bulk. Solving (17), we get the time dependence of the scale factor of the brane as

$$\begin{align*}
a &= \frac{1}{\sqrt{|\Lambda|}} e^{\sqrt{\Lambda} \tau} \quad \text{for} \quad k = 0, \\
&= \frac{1}{\sqrt{|\Lambda|}} \cosh(\sqrt{\Lambda} \tau) \quad \text{for} \quad k = +1, \\
&= \frac{1}{\sqrt{|\Lambda|}} \sinh(\sqrt{\Lambda} \tau) \quad \text{for} \quad k = -1.
\end{align*}$$

The above solutions are valid only for $\Lambda > 0$. For $k = 0$, we get a de-Sitter like expansion. For $k = 1$, the universe initially is infinitely large, then it shrinks at finite time to a finite size controlled by $\sqrt{\Lambda}$ and, subsequently, expands and reaches infinite size at late time $\tau$. For $k = -1$, the radius starts out from zero size and then grows to infinite size with time.

For the case of $\Lambda < 0$, there are no physically relevant solutions except for $k = -1$, where $a$ behaves as

$$a = \frac{1}{\sqrt{|\Lambda|}} \sin(\sqrt{|\Lambda|} \tau).$$

(2) $M > 0$, $Q = 0$

In this case, the brane moves in Schwarzschild black hole with AdS/dS asymptotics. The effective brane cosmological constant $\Lambda$ can be either zero or non-zero. We discuss both these cases for the $D + 1$ bulk in the following. Though in many cases exact solutions can be obtained, we resort to numerical analysis where exact results are either hard to obtain or less illuminating.

(i) $\Lambda = 0$
It turns out that, when the effective brane cosmological constant is zero, the exact solutions are most easily obtained in terms of conformal time $\eta$ defined as $a(\eta)d\eta = d\tau$. The solutions of (17) then gives,

\[
a(\eta) = \left[ \frac{D-2}{2} \sqrt{\Lambda_D} \eta \right]^{\frac{2}{D-2}} \text{ for } k = 0,
\]

\[
= (M\omega_D)^{\frac{1}{D-2}} \sin \left[ \frac{D-2}{2} \eta \right]^{\frac{2}{D-2}} \text{ for } k = +1,
\]

\[
= (M\omega_D)^{\frac{1}{D-2}} \sinh \left[ \frac{D-2}{2} \eta \right]^{\frac{2}{D-2}} \text{ for } k = -1.
\]

(21)

As can be seen from above, for $k = 0$, the brane starts out from singularity and expands to infinite size at late time. However, the rate of expansion and acceleration depend crucially on the dimension of the brane. For $D = 4$, it expands at a constant rate and for $D > 4$, the expansion rate decrease with time $\eta$ producing a deflationary scenario. In terms of cosmic time $\tau$, the brane expands as radiation dominated universe $a \sim \tau^{2/D}$. For $k = 1$, we get cyclic universe with maximum size of the brane is $a_{max} = (M\omega_D)^{1/(D-2)}$ where $M$ is the mass of the bulk black hole. For $k = -1$, on the other hand, we have an open universe as expected.

(ii) $\Lambda_D \neq 0$

When the brane has effective cosmological constant (negative or positive), the dynamics of the universe can be analysed in a quite simple manner. We discuss first the case of positive cosmological constant and then the case with $\Lambda_D < 0$.

(a) $\Lambda_D > 0$

In this case, (17) can be solved exactly in arbitrary $D$ for $k = 0$ with the result,

\[
a(\tau) = \left( \frac{M\omega_D}{\Lambda_D} \right)^{\frac{1}{D-2}} \sinh \left[ \frac{D\sqrt{\Lambda_D}}{2} \tau \right]^{\frac{D}{2}} \text{ for } k = 0.
\]

(22)

Therefore, the brane here emerges from the bulk singularity and then expands for ever. In particular, at late time, we have de-Sitter like inflation as can be seen from expanding (22) for large $\tau$.

However, for $k = \pm 1$, for only $D = 4$, exact analytical result can be found and is discussed for $k = 1$ in some detail in [42, 43]. For other dimensions we could get the behavior of the scale factor through numerical analysis. For $D = 4$, we get

\[
a(\tau) = \left[ \frac{e^{2\sqrt{\Lambda_4} \tau} + k}{2\Lambda_4} \right]^\frac{1}{4} \text{ for } 4\Lambda_4\omega_4M = 1,
\]

\[
= \frac{1}{\sqrt{2\Lambda_4}} \left[ \sqrt{1 - 4\omega_4\Lambda_4} \cosh(2\sqrt{\Lambda_4}\tau) + k \right]^\frac{1}{4} \text{ for } 4\omega_4\Lambda_4 < 1,
\]

8
\[
\frac{1}{\sqrt{2\Lambda_4}} \left[ \sqrt{4\omega_4 M \Lambda_4} - 1 \sinh(2\sqrt{\Lambda_4} \tau) + k \right]^{\frac{1}{2}} \text{ for } 4\omega_4 M \Lambda_4 > 1.
\]

(23)

Thus, for the case \(4\Lambda_4 \omega_4 M = 1\), we get for de-Sitter expansion for all values of \(k\) for large \(\tau\). However, for small \(\tau\), expansion is sensitive to \(k\). For \(k = 1\) the brane starts from a finite size \(\frac{1}{\sqrt{2\Lambda_4}}\), while for \(k = -1\), brane again emerges from bulk singularity. As can be seen from (23), when \(4\Lambda_4 \omega_4 M < 1\), the universe radius is bounded from below by \((\sqrt{1 - 4\Lambda_4 \omega_4 M} + k)^{\frac{1}{2}}/\sqrt{2\Lambda_4}\) for \(k = 1\).

As mentioned earlier, for \(D > 4\), analytical solutions in closed forms are hard to obtain. However, below are the plots of \(a\) with time for different \(k\).

![Figure 1: The universe for \(D = 6\) and \(k = 1\), \(\omega_6 M = 100\), \(\Lambda_6 = .001\). It begins at a finite size and expands for ever. Here and in other figures, all the quantities are measured in appropriate powers of length.](image-url)
Figure 2: The universe for $D = 6$ and $\omega_6 M = 100$, $\Lambda_6 = 1$. Here $k = -1$ corresponds to the solid line while dotted line is for $k = 0$. In both cases, the universe begins from singularity and expands for ever.

$(b) \Lambda_D < 0$

The case when effective cosmological constant is negative on the brane, can be exactly integrated for arbitrary $D$ for flat universe with the result

$$a(\tau) = \left( \frac{M \omega_D}{\Lambda_D} \right)^{\frac{1}{D}} \sin \left( \frac{D \Lambda_D}{2} \tau \right) \frac{D}{2}$$ for $k = 0$. 

(24)

We therefore have cyclic universe with minimum radius being zero while the maximum is determined by the ratio of black hole mass and cosmological constant.

As before, for $k = \pm 1$, exact solution can be found easily for $D = 4$. Those are given by

$$a(\tau) = \frac{1}{\sqrt{2\Lambda_4}} \left[ \sqrt{4M \omega_4 \Lambda_4 + 1} \sin \left( 2\sqrt{\Lambda_4} \tau \right) - k \right]^{\frac{1}{2}}$$ for $k = \pm 1$. 

(25)

For $D > 4$, the behavior of $a$ can be found by numerical means and is shown in the following figure.

$(3) M > 0$, $Q > 0$

In this subsection, we turn our attention to the case where the brane is moving in charged AdS background parametrized by mass $M$ and charge $Q$. In this case, as will be discussed below, we get a host of brane cosmologies which are completely non-singular during the brane evolution.
We start here with the case, where the effective cosmological constant $\Lambda_D$ is zero. In this case, (17) can be exactly solved for any $D$ with the result

$$a(\eta) = \left[ \frac{1}{M\omega_D} \left( \frac{(D-1)Q^2\omega_D^2}{8(D-2)} + \frac{M^2\omega_D^2}{4}(D-2)^2\eta^2 \right) \right]^{\frac{1}{D-2}} \text{ for } k = 0,$$

$$= \left[ \frac{M\omega_D}{2} \left( 1 + \sqrt{1 - \frac{D-1}{2(D-2)} \frac{Q^2}{M^2} \sin((D-2)\eta)} \right) \right]^{\frac{1}{D-2}} \text{ for } k = +1,$$

$$= \left[ \frac{M\omega_D}{2} \left( -1 + \sqrt{1 + \frac{D-1}{2(D-2)} \frac{Q^2}{M^2} \cosh((D-2)\eta)} \right) \right]^{\frac{1}{D-2}} \text{ for } k = -1.$$

(26)

It is important to note that $k = 1$ solution only makes sense when the charge satisfies the inequality

$$Q \leq \sqrt{2}M\sqrt{\frac{D-2}{D-1}}.$$  

(27)

For $D = 4$, these configurations are studied in some detail in [32]. As can be seen from the figure 4, all three solutions are non-singular. For $k = 0, -1$, universe starts from nonzero value of $a$ and expands forever. Thus for flat brane

$$\left[ \frac{1}{M\omega_D} \frac{(D-1)Q^2\omega_D^2}{8(D-2)} \right]^{\frac{1}{D-2}} \leq a \leq \infty,$$

(28)

and for $k = -1$

$$\left[ \frac{M\omega_D}{2} \left( -1 + \sqrt{1 + \frac{(D-1)Q^2}{2(D-2)M^2}} \right) \right]^{\frac{1}{D-2}} \leq a \leq \infty.$$  

(29)
On the other hand, we have bouncing non-singular universe for \( k = 1 \) where the universe oscillates between two non-zero values of \( a \) as

\[
\left[ \frac{M \omega_5 D}{2} \left( 1 - \sqrt{1 - \frac{(D-1)Q^2}{2(D-2)M^2}} \right) \right]^{\frac{1}{D-2}} \leq a \leq \left[ \frac{M \omega_5 D}{2} \left( 1 + \sqrt{1 - \frac{(D-1)Q^2}{2(D-2)M^2}} \right) \right]^{\frac{1}{D-2}}.
\]

(30)

Notice that as we take \( Q \) to zero, the minimum radius collapses to a singularity. So, in a sense, the charge parameter \( Q \) acts as a regulator. This behavior is also intuitively expected. As in this case, the brane moves in electrically charged background, the flux lines have to end on the brane. These fluxes, in turn, do not allow the brane to shrink to zero size. A typical behavior of the universe for \( k = 0, \pm 1 \) is shown in the following figure.

![Figure 4: The universe for \( k = -1, 0, 1 \) are shown in solid, dotted and dashed lines respectively for \( D = 5, \omega_5 M = 500, \omega_5 Q = 1 \) and \( \Lambda_5 = 0 \). As can be seen, in all these cases, big bang like singularities are absent.](image-url)
We further would like to emphasize that for \( k = -1 \) and \( k = 1 \), the minimum size of the universe can be made sufficiently large by taking bulk mass and charge to large values. Thus, one would expect to have quantum gravity corrections to be small even when the brane is at its minimum size. In this sense, the universe avoids the usual quantum gravity era in the past.

(ii) \( \Lambda_D \neq 0 \)

It turns out that in this case, it is hard to integrate (17) to get a closed form expression for \( a \) as a function of \( \tau \). However, the equation can easily be integrated numerically. In what follows, we discuss the solutions for \( \Lambda_D > 0 \) as well as \( \Lambda_D < 0 \) for \( D = 4 \).

(a) \( \Lambda_D > 0 \)

As can easily be checked from (17) that for fixed \( \Lambda_D \), for certain range of parameters \( (M, Q) \), \( \dot{a} \) becomes zero at a positive value of \( a \) at some finite time \( \tau \) for all \( k \). The universe then begins at that radius. At late time, brane expands exponentially independent of the values of \( k \). The expansion rate of course depends on the effective cosmological constant \( \Lambda_D \). The complete evolution of the brane is shown in the figure below for certain choice of parameters.

![Figure 5](image)

Figure 5: This shows typical behavior of the universe for \( \Lambda_4 = 1 \). Here again \( k = -1, 0, 1 \) are shown in solid, dotted and dashed lines for \( D = 4 \), \( \omega_4 M = 50 \), \( \omega_4 Q = 0.1 \). Universe begins at a finite size and expands at late time.
In the case where the effective cosmological constant is negative, for fixed $\Lambda_D$, for certain choice of parameters $(M, Q)$, there are two real positive roots of the equation $\dot{a} = 0$ for all $k$. The universe then begins then at finite $a_{\text{min}}$ and ends at another value of $a_{\text{max}}$. Below $a_{\text{min}}$ and above $a_{\text{max}}$, the brane is unstable. The complete evolution of the brane for certain choice of parameters is shown in the figure below.

As we have now seen, in some instances, universe begins at the singularity of the bulk. This in turn imply big bang like singularity on the brane. In the next section, we study these singular universes from the perspective of “quantum cosmology”.

4 Quantum cosmologies arising from Wheeler-De Witt Equation

In this section, we turn our attention to a quantum version of the cosmological models of previous section. Due to lack of an existing formulation of quantum gravity, our task becomes really difficult. We, therefore, need to make drastic simplifications. The approximation that is most commonly used is the mini superspace truncation of the Wheeler-De Witt formalism. In what follows, we will make such approximation in order to discuss a suggestive quantum version of our previous models. Even with in the mini-superspace approximation, quantum cosmology suffers from many ambiguities. The operator ordering ambiguity in the hamiltonian operator, choice of boundary condition on the wave function are among many others. We may say that we have nothing to add to improve our understanding on those issues. We rather work with a particular operator ordering and chose a particular boundary condition on the wave function
which seems natural to us. As a result, our result in this section would at most be speculative in nature. We will closely follow the method developed in [44]. Creation of brane-universe in AdS space is discussed in [14].

First, the Wheeler-De Witt hamiltonian can be obtained from (16) by substituting $p \rightarrow -i \frac{\partial}{\partial a}$ and is

$$\hat{H} = M_p \left[ h^{\frac{1}{4}} \cos \left( \frac{1}{M_p} \frac{\partial}{\partial a} \right) h^{\frac{1}{4}} - \frac{T}{(D-1)} a \right],$$

(31)

and the evolution of the brane universe is then governed by

$$\hat{H} \psi(a) = 0.$$

(32)

Here, $\psi(a)$ is the wave function of the brane universe while $\hat{H}$ is a quantum hamiltonian of the brane. In writing down (31), one faces the ordering ambiguity. Hermiticity of the hamiltonian is used as guideline to set this operator ordering. We, however, may note that the ordering here is not unique and one can, in principle, work with a different ordering prescription. This, in turn, will change the nature of the quantum behavior.

It is now straightforward to solve (32) particularly for $T = 0$. The solution is

$$\psi_{mn}(a) = C_{mn}(\phi_m - \phi_n),$$

(33)

with

$$\phi_j = h^{-\frac{1}{4}} \exp \left[ -(j + \frac{1}{2}) a M_p \right].$$

(34)

$C_{mn}$ here are normalisation constants which can be explicitly evaluated. However, for our purpose, we would not require their explicit forms. In the above expression of the wave functions, $m, n$ are the integer valued quantum numbers describing the state of the brane. Negatives values of $m, n$ do not give normalizable wave function so we discard them. Same is the case for $m = n$. It is now quite easy to see that for both $M = 0$ and $Q = 0$, and $M \neq 0, Q \neq 0$, the solution, as written above, satisfies both the boundary condition (given later) and Wheeler-De Witt equation. However, for $M \neq 0, Q \neq 0$, we have another solution, which also satisfies both Wheeler-De Witt equation and the boundary condition, namely,

$$\psi_j(a) = C_j \phi_j,$$

(35)

where $\phi_j$ is as written above and $C_j$ is again a normalisation constant.

Note that our wave function is normalised as $\int_0^\infty |\psi|^2 da = 1$. Furthermore, we have chosen De-Witt boundary (see for example [45]) condition on the wave function.

$$\psi(a = 0) = 0.$$
two terms in $\psi_{mn}$ individually satisfy the differential equation $\hat{H}\psi = 0$, but do not individually satisfy the boundary condition for $M = 0$ and $Q = 0$. With the wave functions at hand, one can calculate the mean value of the “radius” of the brane universe, that is

$$\langle a \rangle = \frac{\int a |\psi_{mn}|^2 da}{\int |\psi_{mn}|^2 da}.$$  \hspace{1cm} (37)

By dimensional analysis this number can be expected to be the order of $\frac{1}{M_p}$. We can find the solution for $T = 0$, that is for $\Lambda_D = -\frac{1}{l^2}$. Using the form $h(a)$ as

$$h(a) = k - \Lambda_D a^2 - \frac{\omega DM}{a^{D-2}} + \frac{(D-1)\omega^2 DQ^2}{8(D-2)a^{2D-4}}. \hspace{1cm} (38)$$

Hence, the solution $\phi_j$ is

$$\phi_j = \frac{\exp[-(j + \frac{1}{2})\pi a M_p]}{(k - \Lambda_D a^2 - \frac{\omega DM}{a^{D-2}} + \frac{(D-1)\omega^2 DQ^2}{8(D-2)a^{2D-4}})^{\frac{1}{4}}}. \hspace{1cm} (39)$$

It is important to note that the wave function blows up when the denominator of $\phi_j$ vanishes for non-zero values of $a$. This happens at the horizon of the bulk black hole. The horizon singularity, in black hole physics, can be removed by proper choice of coordinates. Therefore, such divergences of the wave function may not be so worrying for us (see [44] for a discussion on this issue). On the other hand, the real singularity of the bulk is at $a = 0$. It is quite easy to check that the wave function approaches zero at this singularity satisfying the boundary condition.

Let us now proceed to analyse some consequences of the wave function we have just found for some specific cases. We first consider the case where $M = Q = 0, k = -1$ and $\Lambda_D < 0$. The classical behavior of the brane was analysed in the previous section. The universe then starts from singularity, expands till the maximum size $\frac{1}{\sqrt{|\Lambda_D|}}$ and then shrinks to singularity. From the quantum mechanical perspective, we find the wave function to be

$$\psi_{mn}(a) = \frac{C_{mn}}{\left(-1 + |\Lambda_D|a^2\right)^{\frac{1}{4}}} \left(e^{-(m+\frac{1}{2})\pi a M_p} - e^{-(n+\frac{1}{2})\pi a M_p}\right), \hspace{1cm} (40)$$

Consequently, we can evaluate the expectation value of the universe radius. This turns out to be

$$\langle a \rangle = \frac{\int_0^\infty da |\psi_{mn}|^2}{\int_0^\infty da |\psi_{mn}|^2}. \hspace{1cm} (41)$$

Further, expanding the denominator of $\psi_{mn}$ and keeping terms to linear in $\Lambda_D$ with $m = 1, n = 0$ gives

$$\langle a \rangle = \frac{1}{\pi M_p} \left(\frac{11}{6} + \frac{\Lambda_D}{(\pi M_p)^2} \frac{395}{108}\right) + O(\Lambda_D^2). \hspace{1cm} (42)$$

Similarly, for $M \neq 0, Q = 0, k = 1$ and $\Lambda_D = 0$ (with $T = 0, l \rightarrow \infty$), as written in the second line of eq. 21 and is

$$\psi_{mn}(a) = \frac{C_{mn}}{\left(1 - \frac{\omega DM}{a^{D-2}}\right)^{\frac{1}{4}}} \left(e^{-(m+\frac{1}{2})\pi a M_p} - e^{-(n+\frac{1}{2})\pi a M_p}\right), \hspace{1cm} (43)$$
the average value of \( a \) for small \( \omega_D M \) with \( m = 1 \) and \( n = 0 \) is

\[
\langle a \rangle = \frac{11}{6\pi M_p} + \frac{3\omega_D M}{2} \frac{\Gamma(4-D)}{(\pi M_p)^{3-D}} \left( \frac{1}{3^{4-D}} + 1 - \frac{1}{2^{2-D}} \right) - \frac{11\omega_D M}{4} \frac{\Gamma(3-D)}{(\pi M_p)^{3-D}} \left( \frac{1}{3^{3-D}} + 1 - \frac{1}{2^{2-D}} \right) + \mathcal{O}(\omega_D M)^2.
\]

We thus notice that in both the above cases, the universe radius gets an average value which, in leading order, is proportional to \( 1/M_p \) and is independent of the other dimensionful constants that appear in \( \psi_{mn} \). One can further evaluate \( \langle a \rangle \) for small but non-zero \( \Lambda \). We have explicitly checked that the leading order term is again of the order \( 1/M_p \) and free of other dimensionful parameters. This may indicate that the universe, in stead of shrinking to zero size, may stabilize at the scale of \( 1/M_p \) due to quantum effects. In the next section, we would study the solution for nonzero \( T \) using WKB approximation.

## 5 WKB Approximation

It turns out to be too difficult to find the solution of the Wheeler-De Witt equation for \( T \neq 0 \). However, by employing WKB technique, it is possible to get a suggestive picture of the quantum universe for branes with non-zero tension.

Let us recapitulate the recipe to find the wave function of a system using WKB approximation. First, we have to solve the the equation \( \hat{H}(\hat{p}, \hat{a}) \psi = E \psi \), with \( H(p,a) \) as the Hamiltonian associated to classical system. Setting this classical Hamiltonian, \( H(p,a) = E \), gives us a relation between \( p = f(E,a) \). Using this approximation, we can find solution to the Hamiltonian equation for two regimes: for \( E > V \) and \( E < V \), the former corresponds to the classically allowed region and the latter to classically forbidden region. The solutions for these cases are

\[
\psi_A(a) = \frac{1}{\sqrt{|\frac{\partial H}{\partial p}|}} \times e^{\pm i \int_a^p dx} \quad \text{for classically allowed region},
\]

\[
\psi_F(a) = \frac{1}{\sqrt{|\frac{\partial H}{\partial p}|}} \times e^{-\int_a^p |p| dx} \quad \text{for classically forbidden region}. \tag{45}
\]

Following the prescription as written above, it is quite easy to find the momentum \( p \). It is given by

\[
p = M_p \cosh^{-1} \left[ \frac{Ta}{(D-1)\sqrt{h(a)}} \right] = M_p \left( \cosh^{-1} \left[ \frac{Ta}{(D-1)\sqrt{h(a)}} \right] + 2\pi n \right), \tag{46}
\]

where \( n \) is an integer. The conjugate momentum, as written above, has to satisfy the Bohr-Summerfield quantization condition

\[
\int p da = (m + \delta), \tag{47}
\]
where $m$ is an integer and associated to “radial” quantum number, and $\delta$ is typically a fraction. We will not require the explicit value of $\delta$. The wave function for the classically allowed region is then

$$
\psi_A(a) = \frac{1}{\sqrt{a}} \times e^{\int^a pdx} \times e^{-2\pi naM_p} e^{-\frac{(D-1)\omega^2Q^2}{8(D-2)a^{2D-4}}} \times e^{i\Theta(a)},
$$

(48)

where $\Theta(a)$ is

$$
\Theta(a) = M_p \int^a \cosh^{-1} \left[ \frac{Ta}{(D-1)\sqrt{h(a)}} \right].
$$

(49)

It is easy to see from (48) that the solution in the classically allowed region is oscillatory in nature and simultaneously exponentially damped for $n > 0$. The solution in the classically forbidden region is

$$
\psi_F(a) = \frac{1}{\sqrt{|a|}} e^{-2\pi naM_p} e^{-|\Theta(a)|},
$$

(50)

where $\Theta(a)$ is given in eq. (49).

6 Discussion

In this paper, we have initiated a systematic study of possible cosmological scenarios on a brane where the brane acts as a dynamical boundary of various asymptotically AdS or dS spaces. Our emphasize, thought out, was to isolate the cases where big bang singularity was absent on the brane. This happens, in particular, when the bulk represents $D + 1$ dimensional charged AdS or dS black hole. Early time singularities for $k = 0, \pm 1$ are found to be absent on the brane. This was found by explicitly solving the FRW equation when the effective cosmological constant on the brane was fine tuned to zero. Furthermore, by studying FRW equation through numerical means, we found that the non-singular feature continues even when the effective cosmological constant is non-zero. We have also found that, by taking mass and the charge of the bulk black hole to large values (but keeping their ratio fixed), the universe can avoid quantum gravity era at early time for the models with $k = \pm 1$. In section 4 and 5, we analysed the brane universe from the quantum mechanical perspective. This was done by employing Wheeler-De Witt equation in the mini superspace formalism and also via WKB methods. Within these approximation schemes, we studied the wave function of the brane-universe with a special attention to the scenarios where classical big bang singularities are present. We then found that the minimum size of the brane is of the order of Planck length.

As it is clear that many interesting issues of cosmological importance were not considered in this paper. No matter fields (except those that were induced from bulk) were introduced on the brane. These fields would certainly modify the scenarios that we have discussed so far. It would be nice to see, however, if the scenario of bounce followed by radiation still persists when...
such matter fields are included. Furthermore, fluctuation of such matter fields also remain an interesting arena to explore. We hope to address these issues in the future.

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