The Exact Solution of the Falling Body Problem in Three-Dimensions: Comparative Study

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Abstract: Very recently, the system of differential equations governing the three-dimensional falling body problem (TDFBP) has been approximately solved. The previously obtained approximate solution was based on the fact that the Earth’s rotation (ER) is quite slow and hence all high order terms of \( \omega \) in addition to the magnitude \( \omega^2 R \) were neglected, where \( \omega \) is the angular velocity and \( R \) is the radius of Earth. However, it is shown in this paper that the ignorance of such magnitudes leads, in many cases, to significant errors in the estimated falling time and other physical quantities. The current results are based on obtaining the exact solutions of the full TDFBP-system and performing several comparisons with the approximate ones in the relevant literature. The obtained results are of great interest and importance, especially for other planets in the Solar System or exterior planets, in which \( \omega \) and/or \( \omega^2 R \) are of considerable amounts and hence cannot be ignored. Therefore, the present analysis is valid in analyzing the TDFBP near to the surface of any spherical celestial body.

Keywords: falling body problem; angular velocity; projectile motion; three dimensions; Earth’s rotation; Laplace transform

1. Introduction

In the past decades, some attention was given to the study of the falling body problem [1–3] and two-dimensional projectile motion [4–14]. In the literature [1–3], the falling body problem was modeled as the vertical motion of a particle near the Earth’s gravitational field, i.e., as a problem formulated in one dimension. Hence, the results obtained in [1–3] and in [4–14] were only valid in a fixed frame, that is, a non-rotating one. By this, the effect of Earth’s rotation (ER) on the falling body problem and also on the projectile motion was not considered. In addition, the effect of ER on the three dimensional-falling body problem (TDFBP) was analyzed very recently by El-Zahar et al. [15]. However, the analysis introduced by El-Zahar et al. [15] still has some restrictions where all high order terms of \( \omega \) in addition to the magnitude \( \omega^2 R \) were neglected. In this paper, we consider the effect of the ER on the TDFBP without any constraints/restrictions where all of the physical quantities are left arbitrary and therefore the present results generalize the previously published ones in [15]. The physical problem is described as follows. Let \( \vec{r} = (x(t), y(t), z(t)) \) be the position vector of a particle located at a point Q near the Earth’s surface, relative to P (a surface point) as in Figure 1.
The present problem may be of practical importance in many sciences due to the possibility of finding meteorites, and other space objects (such as the Mir Space Station and the International Space Station) orbit the Earth at different altitudes from the surface of the Earth. The Earth’s planet is also exposed, annually, to many crashes of some aircrafts and meteorites, which cause a potential danger to vital installations and regions of population density in some countries. Therefore, the accurate identification of the possible location of these falling objects clearly helps in the possibility of finding and removing them from the Earth’s surface.

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these falling objects clearly helps in the possibility of finding them successfully; hence, the importance of the current study. So, the objective of this paper is to solve the present system, taking into account the contributions of all terms without any constraints/restrictions on any of the physical quantities.

2. The Exact Solution

In this section, the exact solution of the full system (1)–(4) will be obtained without neglecting any of the higher orders of $\omega$ and also the magnitude $\omega^2 R$. The present approach implements the Laplace transform (LT) to exactly solving the full model (1)–(4). The Laplace transform and its properties are listed in Appendix A. Applying the LT on system (1)–(4) and in view of Equations (A4)–(A6), we have:

$$\left(s^2 - B\right)X(s) - CZ(s) - AY(s) + \left(Ay_0 - x_0s - \frac{CR}{s}\right) = 0,$$

$$\left(s^2 - \omega^2\right)Y(s) + AX(s) + DsZ(s) - \left(Ax_0 + y_0s + Dz_0\right) = 0,$$

$$\left(s^2 - E\right)Z(s) - CX(s) - DsY(s) + \left(\frac{g}{s} - \frac{ER}{s} - z_0s + Dy_0\right) = 0,$$

where:

$$A = 2\omega \cos \lambda, \quad B = \omega^2 \cos^2 \lambda, \quad C = \omega^2 \sin \lambda \cos \lambda,$$

$$D = 2\omega \sin \lambda, \quad E = \omega^2 \sin^2 \lambda,$$

and $X(s)$, $Y(s)$, and $Z(s)$ are the LT of $x(t)$, $y(t)$, and $z(t)$, respectively. Solving system (6)–(8) for $X(s)$, $Y(s)$, and $Z(s)$, and implementing (9)–(10), we obtain:

$$X(s) = \frac{2x_0s^6 + \left(\omega^2 k_1\right)s^4 + \left(4\omega^2 y_0 \cos \lambda\right)s^3 + \left(\omega^2 \left(\omega^2 k_2 + 3g \sin 2\lambda\right)s^2 + \left(\omega^4 g \sin 2\lambda\right)s + \left(\omega^4 g \sin 2\lambda\right)\right)}{2s^2(s^2 + \omega^2)^2},$$

$$Y(s) = \frac{y_0s^3 + \left(3\omega^2 y_0\right)s + \left(\omega k_3\right)}{s^2(s^2 + \omega^2)^2},$$

$$Z(s) = \frac{z_0s^6 + \omega x_0s^4 + \left(2\omega^2 y_0 \sin \lambda\right)s^3 + \left(\omega^2 k_3\right)s^2 - \left(\omega^4 g \cos^2 \lambda\right)}{s^3(s^2 + \omega^2)^2},$$

where:

$$k_1 = (R + z_0) \sin 2\lambda + (5 + \cos 2\lambda)x_0,$$

$$k_2 = \omega^2 \left(6x_0 - k_1\right),$$

$$k_3 = -2\omega^2 x_0 \cos \lambda - 2\left(-g + \omega^2 (z_0 + R)\right) \sin \lambda,$$

$$k_4 = -g + \omega^2 \left(R \sin^2 \lambda + \frac{1}{2} x_0 \sin 2\lambda + \frac{1}{2} z_0 (5 - \cos 2\lambda)\right),$$

$$k_5 = 3\omega^2 z_0 - 3g \cos^2 \lambda - k_4.$$
Applying the inverse LT on (11)–(13), we obtain the explicit forms of \( x(t) \), \( y(t) \), and \( z(t) \) as:

\[
x(t) = \frac{1}{4w^2} [2a^2k_2 + g(2 + a^2 t^2) \sin 2\lambda] - \frac{w}{2\omega} [4w y_0 \cos \lambda + \left(\omega^2(k_1 - k_2 - 2x_0) - 2g \sin 2\lambda\right)t] \sin \omega t - \frac{1}{2\omega^2} [\omega^2(k_2 - 2x_0) + g \sin 2\lambda + (2\omega^3 y_0 \cos \lambda)t] \cos \omega t,
\]

\[
y(t) = \left( y_0 - \frac{k_3 t}{2\omega} \right) \cos \omega t + \left( \omega y_0 t + \frac{k_3}{2\omega^2} \right) \sin \omega t,
\]

\[
z(t) = \frac{1}{4\omega^2} \left[ 4k_5 + 2g(4 - a^2 t^2) \cos^2 \lambda \right] - \frac{w}{\omega} \left[ k_5 + 2g \cos^2 \lambda - \omega^2 z_0 + \left( \omega^3 y_0 \sin \lambda \right)t \right] \cos \omega t - \frac{1}{4\omega^2} \left[ -4\omega y_0 \sin \lambda + 2(k_5 - k_4 + \omega^2 z_0 + g \cos^2 \lambda) \right] \sin \omega t.
\]

Equations (19)–(21) express the solution of the TDFBP model (1)–(4) without any assumptions or restrictions on \( \omega \) and \( \omega^2R \). Expanding \( x(t) \), \( y(t) \), and \( z(t) \) in (19)–(21) as power series in \( \omega \), yields:

\[
x(t) = x_0 + \frac{t^2}{8} \cos \lambda (4x_0 \cos \lambda + \left( gt^2 + 4z_0 + 4R \right) \sin \lambda) + a^2 + \left( \frac{t^3}{3} y_0 \cos \lambda \right) a^3 + \ldots,
\]

\[
y(t) = y_0 + \left( \frac{gt^3}{3} \sin \lambda \right) \omega + \left( \frac{y_0 t^2}{2} \right) + a^2 + \left( \frac{t^3}{3} y_0 \cos \lambda \right) a^3 + \ldots
\]

\[
z(t) = \left( z_0 - \frac{gt^2}{2} \right) + \frac{t^2}{8} \sin \lambda \left( 4x_0 \cos \lambda + \left( gt^2 + 4z_0 + 4R \right) \sin \lambda \right) a^2 + \left( \frac{y_0 t^2}{3} \sin \lambda \right) a^3 + \ldots
\]

As \( a^2 \to 0 \), the results (22)–(24) reduce to:

\[
x(t) = x_0, \quad y(t) = y_0 + \left( \frac{gt^2}{3} \sin \lambda \right) \omega, \quad z(t) = z_0 - \frac{gt^2}{2},
\]

which are the corresponding results obtained in [15] when higher powers of \( \omega \), i.e., \( \omega^n, n \geq 2 \), are neglected. Furthermore, as \( \omega^2 \to 0 \), the horizontal and vertical displacements \( x(t) \) and \( z(t) \), respectively, are independent of the angular velocity \( \omega \). Moreover, Equations (22)–(24) reduce to some useful results at several special cases as declared in the subsequent sections.

3. Coordinates and Characteristics of the Falling Point

3.1. Coordinates of the Falling Point

Let \( G \) be the point at which the falling body hits the \( xy \)-plane, see Figure 2. At such point, the vertical displacement \( z(t) \) vanishes. Suppose that \( T \) is the time required such that \( z(T) = 0 \); then, the Cartesian coordinates of the falling point \( G \) in the local frame \((x, y, z)\) can be expressed as \((x_G, y_G, z_G) = (x(T), y(T), 0)\). In addition, the Polar coordinates \((\rho, \delta)\) of \( G \) are:

\[
\rho = \sqrt{(x(T))^2 + (y(T))^2}, \quad \delta = \tan^{-1} \left( \frac{y(T)}{x(T)} \right).
\]

In Appendix C, the coordinates of the falling point at various special cases are introduced.
respectively, given from (27) and (28) by

\[
x(t) = \frac{\sin \lambda}{\omega^2} \left[ 2 \left( g - \omega^4 z_0 + R \right) \left( 1 - \cos \omega t \right) + \omega \left( -2g + \omega^2 \left( 1 + \omega^2 \right) z_0 + R \right) \right] T \sin \omega t +
\]

\[
y(t) = \frac{\sin \lambda}{\omega^2} \left( g - \omega^4 z_0 + R \right) \left( \sin \omega t - \omega t \cos \omega t \right),
\]

\[
z(t) = \frac{1}{\omega^2} \left[ g - \omega^2 \left( R - z_0 \right) - \omega^2 g^2 \cos^2 \lambda - \left( g - \omega^2 \left( z_0 + R \right) \right) \left( \cos 2\lambda + 2 \sin^2 \lambda \right) \left( \cos \omega t + \omega t \sin \omega t \right) \right].
\]

At such point G, the component \( z(t) \) vanishes at a particular value \( t = T \). Hence, \( x_G \) and \( y_G \) are, respectively, given from (27) and (28) by

\[
x_G = \frac{\sin 2\lambda}{\omega^2 g T^2} \left[ 2 \left( g - \omega^4 z_0 + R \right) \left( 1 - \cos \omega T \right) + \omega \left( -2g + \omega^2 \left( 1 + \omega^2 \right) z_0 + R \right) \right] T \sin \omega T +
\]

\[
y_G = \frac{\sin \lambda}{\omega^2} \left( g - \omega^4 z_0 + R \right) \left( \sin \omega T - \omega T \cos \omega T \right),
\]

where \( T \) is a non-trivial solution of the following equation \( (z(T) = 0): \)

\[
g - \omega^2 \left( R - z_0 \right) - \omega^2 g^2 \cos^2 \lambda - \left( g - \omega^2 \left( z_0 + R \right) \right) \left( \cos 2\lambda + 2 \sin^2 \lambda \left( \cos \omega T + \omega T \sin \omega T \right) \right).
\]

3.2. Characteristics of the Falling Point

In this section, we highlight some of the characteristics of the particle’s falling point G. In particular, G lies on the Earth’s surface when some conditions are satisfied. As \( \rho \to 0 \), G belongs to the Earth’s surface, as shown from Figure 2. However, \( \rho \to 0 \) implies \( x_G = y_G = 0 \). The following lemmas address these issues.

**Lemma 1.** If the particle is released from a point \( (0,0,z_0) \) in the system \( (x,y,z) \) such that \( \lambda \in [0,\pi] \), then G belongs to the Earth’s surface.

**Proof of Lemma 1.** Substituting \( x_0 = 0 \) and \( y_0 = 0 \) into (19)–(21) and implementing the corresponding \( k_i \) \( (i = 1,2,3,4,5) \) in (14)–(18), we have:

\[
x(t) = \frac{\sin \lambda}{\omega^2} \left[ \left( g - \omega^4 z_0 + R \right) \left( 1 - \cos \omega t \right) + \omega \left( -2g + \omega^2 \left( 1 + \omega^2 \right) z_0 + R \right) \right] T \sin \omega t +
\]

\[
y(t) = \frac{\sin \lambda}{\omega^2} \left( g - \omega^4 z_0 + R \right) \left( \sin \omega t - \omega t \cos \omega t \right),
\]

\[
z(t) = \frac{1}{\omega^2} \left[ g - \omega^2 \left( R - z_0 \right) - \omega^2 g^2 \cos^2 \lambda - \left( g - \omega^2 \left( z_0 + R \right) \right) \left( \cos 2\lambda + 2 \sin^2 \lambda \right) \left( \cos \omega t + \omega t \sin \omega t \right) \right].
\]

At such point G, the component \( z(t) \) vanishes at a particular value \( t = T \). Hence, \( x_G \) and \( y_G \) are, respectively, given from (27) and (28) by

\[
x_G = \frac{\sin 2\lambda}{\omega^2 g T^2} \left[ \left( g - \omega^4 z_0 + R \right) \left( 1 - \cos \omega T \right) + \omega \left( -2g + \omega^2 \left( 1 + \omega^2 \right) z_0 + R \right) \right] T \sin \omega T +
\]

\[
y_G = \frac{\sin \lambda}{\omega^2} \left( g - \omega^4 z_0 + R \right) \left( \sin \omega T - \omega T \cos \omega T \right),
\]

where \( T \) is a non-trivial solution of the following equation \( (z(T) = 0): \)

\[
g - \omega^2 \left( R - z_0 \right) - \omega^2 g^2 \cos^2 \lambda - \left( g - \omega^2 \left( z_0 + R \right) \right) \left( \cos 2\lambda + 2 \sin^2 \lambda \left( \cos \omega T + \omega T \sin \omega T \right) \right).
\]
It is clear from Equations (30) and (31) that $x_G = y_G = 0$ when $\sin 2\lambda = 0$ and $\sin \lambda = 0$. These two equations imply that $\lambda \in \{0, \pi\}$ which completes the proof. In addition, Equation (29) reduces to:

$$2z_0 - gT^2 = 0$$

which gives $T = \sqrt{\frac{2z_0}{g}}$. (33)

As a confirmation of this Lemma, we have from Appendix C (1 and 2) that $\rho = \sqrt{\left(1 + \omega^2T^2\right)(x_0^2 + y_0^2)}$ for $\lambda = 0$ or $\pi$. When $(x_0, y_0) \rightarrow (0, 0)$, we observe that $\rho \rightarrow 0$; hence, $G$ lies on the Earth’s surface. □

**Lemma 2.** If $x_0 = 0$ and $y_0 = 0$, then the falling point $G$ is equivalent to the North Pole and the South Pole of Earth when $\lambda = 0$ and $\lambda = \pi$, respectively.

**Proof of Lemma 2.** Let us first introduce the relation between the systems $(x, y, z)$ and $(X, Y, Z)$ described in Figure 1. It was shown in [16,17] that the following transformations hold after a time $t$:

$$X(t) = x(t) \cos \lambda \cos \omega t - y(t) \sin \omega t + (R + z(t)) \sin \lambda \cos \omega t,$$

$$Y(t) = x(t) \cos \lambda \sin \omega t + y(t) \cos \omega t + (R + z(t)) \sin \lambda \sin \omega t,$$

$$Z(t) = -x(t) \sin \lambda + (R + z(t)) \cos \lambda.$$ (36)

From Lemma 1, it was proved that $T = \sqrt{\frac{2z_0}{g}}$ when $x_0 = y_0 = 0$ and $\lambda \in \{0, \pi\}$, where $G = (x_G, y_G, z_G) = (0,0,0)$ in the local frame $(x, y, z)$. Accordingly, the Cartesian coordinates $(X_G, Y_G, Z_G)$ of $G$ in the inertial frame $(X, Y, Z)$ are:

$$X_G = x_G \cos \lambda \cos \omega \sqrt{\frac{2z_0}{g}} - y_G \sin \omega \sqrt{\frac{2z_0}{g}} + (R + z_G) \sin \lambda \cos \omega \sqrt{\frac{2z_0}{g}},$$

(37)

$$Y_G = x_G \cos \lambda \sin \omega \sqrt{\frac{2z_0}{g}} + y_G \cos \omega \sqrt{\frac{2z_0}{g}} + (R + z_G) \sin \lambda \sin \omega \sqrt{\frac{2z_0}{g}},$$

(38)

$$Z_G = -x_G \sin \lambda + (R + z_G) \cos \lambda.$$ (39)

Substituting $x_G = y_G = z_G = 0$ into Equations (37)–(39) yields:

$$X_G = R \sin \lambda \cos \omega \sqrt{\frac{2z_0}{g}},$$

(40)

$$Y_G = R \sin \lambda \sin \omega \sqrt{\frac{2z_0}{g}},$$

(41)

$$Z_G = R \cos \lambda.$$ (42)
At \( \lambda = 0 \), we obtain:
\[
X_G = 0, \quad Y_G = 0, \quad Z_G = R,
\]
which are the Cartesian coordinates of the North pole in the frame \((X, Y, Z)\). Moreover, we have from Equations (40)–(42) at \( \lambda = \pi \) that:
\[
X_G = 0, \quad Y_G = 0, \quad Z_G = -R,
\]
which are equivalent to the Cartesian coordinates of the South Pole. □

4. Results and Discussions

This section presents the main difference between the current exact results and those approximately obtained by El-Zahar et al. [15]. Besides, the two Lemmas presented in the previous section will be numerically and graphically validated in this discussion. First of all, the falling time \( T \) is approximately obtained in [15] by
\[
T_{\text{Approx.}} = \sqrt{\frac{2z_0}{g}} \quad \forall \lambda \in [0, \pi].
\]
At \( \lambda = 0, \pi \), it is noted from Appendix C (1 and 2) that the falling time \( T \) is obtained by
\[
T = \sqrt{\frac{2z_0}{g}}
\]
which agrees with the corresponding expression in [15]. In such special cases, there is no difference between the current exact formula of the falling time \( T \) and the corresponding approximate one in [15]. However, when \( \lambda \) takes other values, i.e., \( \lambda \in (0, \pi) \), there are differences in the calculations of \( T \). In these cases, the exact values of \( T \) \((T_{\text{Exact}})\) can be calculated by solving Equation (21) as \( z(T_{\text{Exact}}) = 0 \); hence, the error formula of \( T \) is defined as:
\[
\text{Error}(T) = T_{\text{Exact}} - \sqrt{\frac{2z_0}{g}}.
\]

Table 1 presents the comparisons between the current values of the falling time \( T \) and the corresponding results in [15] at twenty different cases of the colatitude \( \lambda \) and the release point Q. The results reveal that the obtained error increases as the colatitude \( \lambda \) increases, especially at the equator of the Earth \((\lambda = \pi/2)\). In [15], the approximate Cartesian coordinates \((x_G)_{\text{Approx.}}\) and \((y_G)_{\text{Approx.}}\) of the falling point G were given by:
\[
(x_G)_{\text{Approx.}} = x_0, \quad (y_G)_{\text{Approx.}} = y_0 + \left(\frac{\omega g}{3}\right) \left(\sqrt{\frac{2z_0}{g}}\right) \sin \lambda,
\]
and the approximate polar coordinates \((\rho)_{\text{Approx.}}\) and \((\delta)_{\text{Approx.}}\) of G were obtained as [15]:
\[
(\rho)_{\text{Approx.}} = \sqrt{x_0^2 + \left(y_0 + \left(\frac{\omega g}{3}\right) \left(\sqrt{\frac{2z_0}{g}}\right) \sin \lambda \right)^2},
\]
\[
(\delta)_{\text{Approx.}} = \tan^{-1}\left(\frac{3y_0 + \omega g \left(\sqrt{\frac{2z_0}{g}}\right) \sin \lambda}{3x_0}\right).
\]
Table 1. Comparisons between the present values of the falling time (T) (s) and the corresponding results in [15] at various values of the colatitude (λ) (radian) and the release point Q.

| Case | Colatitude (λ) (Radian) | Cartesian Coordinates of the Release Point Q | Falling Time (T) (S) |
|------|------------------------|---------------------------------------------|---------------------|
|      | x₀ (km) | y₀ (km) | z₀ (km) | Exact (Present) | Approximate [15] | Error |
| 1    | π/6     | 10      | 10      | 10            | 45.1949          | 45.1754 | 0.0195 |
| 2    | π/6     | 20      | 20      | 20            | 63.9154          | 63.8877 | 0.0277 |
| 3    | π/6     | 30      | 30      | 30            | 78.2803          | 78.2461 | 0.0342 |
| 4    | π/6     | 40      | 40      | 40            | 90.3905          | 90.3508 | 0.0397 |
| 5    | π/6     | 50      | 50      | 50            | 101.0598         | 101.0153 | 0.0445 |
| 6    | π/4     | 10      | 10      | 10            | 45.2145          | 45.1754 | 0.0391 |
| 7    | π/4     | 20      | 20      | 20            | 63.9431          | 63.8877 | 0.0534 |
| 8    | π/4     | 30      | 30      | 30            | 78.3143          | 78.2461 | 0.0682 |
| 9    | π/4     | 40      | 40      | 40            | 90.4299          | 90.3508 | 0.0791 |
| 10   | π/4     | 50      | 50      | 50            | 101.1040         | 101.0153 | 0.0887 |
| 11   | π/3     | 10      | 10      | 10            | 45.2340          | 45.1754 | 0.0586 |
| 12   | π/3     | 20      | 20      | 20            | 63.9708          | 63.8877 | 0.0831 |
| 13   | π/3     | 30      | 30      | 30            | 78.3483          | 78.2461 | 0.1022 |
| 14   | π/3     | 40      | 40      | 40            | 90.4692          | 90.3508 | 0.1184 |
| 15   | π/3     | 50      | 50      | 50            | 101.1480         | 101.0153 | 0.1327 |
| 16   | π/2     | 10      | 10      | 10            | 45.2535          | 45.1754 | 0.0781 |
| 17   | π/2     | 20      | 20      | 20            | 63.9984          | 63.8877 | 0.1107 |
| 18   | π/2     | 30      | 30      | 30            | 78.3820          | 78.2461 | 0.1359 |
| 19   | π/2     | 40      | 40      | 40            | 90.5082          | 90.3508 | 0.1574 |
| 20   | π/2     | 50      | 50      | 50            | 101.1916         | 101.0153 | 0.1763 |

Therefore, the errors in $x_G$, $y_G$, $\rho$, and $\delta$ at $\lambda = 0$ are given by:

$$\text{Error}(x_G) = \left(x_0 - \omega y_0 \sqrt{\frac{2 \omega_0}{\bar{g}}} \cos \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} + \left(y_0 + \omega x_0 \right) \sqrt{\frac{2 \omega_0}{\bar{g}}} \sin \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} \right) - x_0,$$  \hfill (49)

$$\text{Error}(y_G) = \left(y_0 + \omega x_0 \sqrt{\frac{2 \omega_0}{\bar{g}}} \cos \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} - \left(x_0 - \omega y_0 \right) \sqrt{\frac{2 \omega_0}{\bar{g}}} \sin \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} \right) - y_0,$$  \hfill (50)

$$\text{Error}(\rho) = \sqrt{(1 + \omega^2 T^2) (x_0^2 + y_0^2)} - \sqrt{x_0^2 + y_0^2},$$  \hfill (51)

$$\text{Error}(\delta) = \tan^{-1}\left(\frac{\left(x_0 + \omega x_0 \sqrt{\frac{2 \omega_0}{\bar{g}}} \cos \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} - \left(x_0 - \omega y_0 \right) \sqrt{\frac{2 \omega_0}{\bar{g}}} \sin \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} \right)}{\left(x_0 - \omega y_0 \right) \sqrt{\frac{2 \omega_0}{\bar{g}}} \cos \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} + \left(y_0 + \omega x_0 \right) \sqrt{\frac{2 \omega_0}{\bar{g}}} \sin \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} \right)} \right) - \tan^{-1}\left(\frac{y_0}{x_0}\right).$$  \hfill (52)

Similarly, at $\lambda = \pi$, we have:

$$\text{Error}(x_G) = \left(x_0 + \omega y_0 \sqrt{\frac{2 \omega_0}{\bar{g}}} \cos \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} - \left(y_0 - \omega x_0 \right) \sqrt{\frac{2 \omega_0}{\bar{g}}} \sin \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} \right) - x_0,$$  \hfill (53)

$$\text{Error}(y_G) = \left(y_0 - \omega x_0 \sqrt{\frac{2 \omega_0}{\bar{g}}} \cos \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} + \left(x_0 + \omega y_0 \right) \sqrt{\frac{2 \omega_0}{\bar{g}}} \sin \omega \sqrt{\frac{2 \omega_0}{\bar{g}}} \right) - y_0,$$  \hfill (54)

$$\text{Error}(\rho) = \sqrt{(1 + \omega^2 T^2) (x_0^2 + y_0^2)} - \sqrt{x_0^2 + y_0^2},$$  \hfill (55)
Error(\(\delta\)) = \tan^{-1}\left( \frac{\sqrt{y_0-y_0 - \cos(\frac{\sqrt{x_0-y_0}}{2})} + \left(\sqrt{x_0+y_0 - \cos(\frac{\sqrt{x_0+y_0}}{2})}\right) \sin(\sqrt{\frac{x_0+y_0}{2}})}{\sqrt{x_0+y_0 - \cos(\frac{\sqrt{x_0+y_0}}{2})} - \left(\sqrt{x_0+y_0 - \cos(\frac{\sqrt{x_0+y_0}}{2})}\right) \sin(\sqrt{\frac{x_0+y_0}{2}})} \right) \tag{56} \\
\tan^{-1}\left( \frac{\sqrt{x_0-y_0}}{2} \right) \\

Table 2 presents the obtained errors in the Cartesian coordinates of the falling point \(G(x_G, y_G, 0)\) in the frame \((x, y, z)\) and the corresponding errors in the Polar coordinates \((\rho, \delta)\) at several points \(Q(x_0, y_0, z_0)\) when \(\lambda = 0, \pi\). The results of Table 2 indicate that the obtained errors in \(\rho\) and \(\delta\) are of the amount at \(\lambda = 0\) (North Pole) and \(\lambda = \pi\) (South Pole). In addition, the errors increase as the height \(z_0\) increases above the Earth’s surface. While the obtained errors in \(\rho\) and \(\delta\) in Table 2 seem very small at these particular values of the colatitude \(\lambda\), it will be declared that the errors in \(x_G\) and \(y_G\), \(\rho\), and \(\delta\) are significant at other values of \(\lambda \in (0, \pi)\). Repeating the analysis above at \(\lambda = \pi/6, \lambda = \pi/4, \lambda = \pi/3, \lambda = \pi/2\), we calculated the errors of \(x_G\) and \(y_G\), as introduced in Table 3.

Table 2. The obtained errors of the Cartesian coordinates of the falling point \(G(x_G, y_G, 0)\) in the frame \((x, y, z)\) and the corresponding errors in the polar coordinates \((\rho, \delta)\) at several points \(Q(x_0, y_0, z_0)\) when \(\lambda = 0, \pi\).

| Case | Colatitude \((\lambda)\) (Radian) | Cartesian Coordinates of the Release Point \(Q\) | Error of Cartesian Coordinates of \(G\) | Error of Polar Coordinates of \(G\) |
|------|----------------------------------|---------------------------------------------|----------------------------------|----------------------------------|
|      | \(0\)                            | \(10\) \(10\) \(10\)                       | \(0.0541\) \(0.0538\) \(0.0763\) | \(1.1819\)                       |
|      | \(0\)                            | \(20\) \(20\) \(20\)                       | \(0.2165\) \(0.2152\) \(0.3053\) | \(3.3429\)                       |
|      | \(0\)                            | \(30\) \(30\) \(30\)                       | \(0.4875\) \(0.4838\) \(0.6568\) | \(6.1413\)                       |
|      | \(0\)                            | \(40\) \(40\) \(40\)                       | \(0.8672\) \(0.8596\) \(1.2211\) | \(9.4550\)                       |
|      | \(5\)                            | \(50\) \(50\) \(50\)                       | \(1.3557\) \(1.3425\) \(1.9079\) | \(13.2137\)                      |
|      | \(\pi\)                          | \(10\) \(10\) \(10\)                       | \(0.2152\) \(0.0541\) \(0.0763\) | \(1.1819\)                       |
|      | \(\pi\)                          | \(20\) \(20\) \(20\)                       | \(0.4838\) \(0.2165\) \(0.3053\) | \(3.3429\)                       |
|      | \(\pi\)                          | \(30\) \(30\) \(30\)                       | \(0.8596\) \(0.4875\) \(0.6868\) | \(6.1413\)                       |
|      | \(\pi\)                          | \(40\) \(40\) \(40\)                       | \(1.3425\) \(0.8672\) \(1.2211\) | \(9.4550\)                       |
|      | \(\pi\)                          | \(50\) \(50\) \(50\)                       | \(1.3557\) \(1.3425\) \(1.9079\) | \(13.2137\)                      |

Table 3. The obtained errors of \(x_G\) and \(y_G\) at several release points \(Q(x_0, y_0, z_0)\) when \(\lambda = \pi/6, \pi/4, \pi/3, \pi/2\).

| Case | Colatitude \((\lambda)\) (Radian) | Cartesian Coordinates of the Release Point \(Q\) | Error of Cartesian Coordinates of \(G\) |
|------|----------------------------------|---------------------------------------------|----------------------------------|
|      | \(\pi/6\)                        | \(10\) \(10\) \(10\)                       | \(14.9922\) \(0.0303\)           |
|      | \(\pi/6\)                        | \(20\) \(20\) \(20\)                       | \(30.1361\) \(0.1487\)           |
|      | \(\pi/6\)                        | \(30\) \(30\) \(30\)                       | \(45.4317\) \(0.3619\)           |
|      | \(\pi/6\)                        | \(40\) \(40\) \(40\)                       | \(60.8793\) \(0.6722\)           |
|      | \(\pi/6\)                        | \(50\) \(50\) \(50\)                       | \(76.4789\) \(1.0808\)           |
|      | \(\pi/4\)                        | \(10\) \(10\) \(10\)                       | \(17.3067\) \(0.0407\)           |
|      | \(\pi/4\)                        | \(20\) \(20\) \(20\)                       | \(34.7489\) \(0.1782\)           |
|      | \(\pi/4\)                        | \(30\) \(30\) \(30\)                       | \(52.3271\) \(0.4166\)           |
|      | \(\pi/4\)                        | \(40\) \(40\) \(40\)                       | \(70.0412\) \(0.7570\)           |
|      | \(\pi/4\)                        | \(50\) \(50\) \(50\)                       | \(87.8912\) \(1.2004\)           |
|      | \(\pi/3\)                        | \(10\) \(10\) \(10\)                       | \(14.9911\) \(0.0623\)           |
|      | \(\pi/3\)                        | \(20\) \(20\) \(20\)                       | \(30.0799\) \(0.2398\)           |
|      | \(\pi/3\)                        | \(30\) \(30\) \(30\)                       | \(45.2667\) \(0.5302\)           |
|      | \(\pi/3\)                        | \(40\) \(40\) \(40\)                       | \(60.5515\) \(0.9327\)           |
|      | \(\pi/3\)                        | \(50\) \(50\) \(50\)                       | \(75.9943\) \(1.4469\)           |
|      | \(\pi/2\)                        | \(10\) \(10\) \(10\)                       | \(4.8873 \times 10^{-8}\) \(2.1993\) |
|      | \(\pi/2\)                        | \(20\) \(20\) \(20\)                       | \(3.9099 \times 10^{-7}\) \(62.2715\) |
|      | \(\pi/2\)                        | \(30\) \(30\) \(30\)                       | \(1.3196 \times 10^{-6}\) \(114.4901\) |
|      | \(\pi/2\)                        | \(40\) \(40\) \(40\)                       | \(3.1280 \times 10^{-6}\) \(176.3861\) |
|      | \(\pi/2\)                        | \(50\) \(50\) \(50\)                       | \(6.1094 \times 10^{-6}\) \(246.6514\) |
The results show that the errors in $x_G$ and $y_G$ increases as $\lambda$, $x_0$, $y_0$, and $z_0$ increase. For example, when the release point $Q(x_0, y_0, z_0)$ takes the values $Q(10,10,10)$, the errors in $x_G$ are found as 14.9922 m at $\lambda = \pi/6$, 17.3067 m at $\lambda = \pi/4$, 75.9343 m at $\lambda = \pi/3$. However, the obtained maximum errors in $y_G$ are 1.4469 m when $\lambda = \pi/3$. Moreover, at $\lambda = \pi/2$ (the equator of Earth), the obtained errors in $x_G$ are small for the five cases of $Q(x_0, y_0, z_0)$, while the obtained error in $y_G$ becomes significant and considerable, equaling 246.6514 m for $Q(50,50,50)$. Of course, the significant errors in the Cartesian coordinates $x_G$ and in $y_G$ imply significant errors in the polar coordinates $\rho$ and in $\delta$. The above calculations implemented the real data of the Earth’s planet and can also be extended to cover other planets in the Solar System. Finally, the advantage of the present analysis over the previously published one is observable, especially when applied to specific planets of greater radius $R$ and greater angular velocity $\omega$ than those of the Earth.

In order to confirm such conclusion, the current exact solutions are applied for all planets in our Solar System, Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune. Astronomers usually divide the planets into two groups, which are known as the inner planets (Mercury, Venus, Earth, and Mars) and the outer planets (Jupiter, Saturn, Uranus, and Neptune) [18]. It will be indicated below that the obtained errors for the falling time $T$ and the landing positions $(x_G, y_G, 0)$ are bigger for the outer planets than those of the inner planets.

In Table 4, the physical data for all planets are introduced. Before launching to the main results, it can be seen from Table 4 that the value of angular velocity $\omega$ of any outer planet (Jupiter, Saturn, Uranus, and Neptune) is greater than the value of any inner planet. This means that the outer planets rotate more speedily about their axes than the inner planets. In another word, the Coriolis effect is expected to affect the calculated errors for outer planets more than it does for the inner planets. In addition, the radii of the outer planets are greater than the radii of the inner planets. So, it is expected that the differences between the present exact solutions and the approximate ones [15] will greatly appear in the cases of outer planets.

| Planet   | Ratio of Acceleration Due to Gravity [Earth = 1] | Acceleration Due to Gravity [m/s²] | Radius (km) | Rotation Period (in Earth Days) | Angular Velocity [s⁻¹] |
|----------|-----------------------------------------------|-----------------------------------|-------------|--------------------------------|-----------------------|
| Mercury  | 0.38                                          | 3.7278                            | 2439        | 58.65                          | 1.24 × 10⁻⁶           |
| Venus    | 0.9                                           | 8.829                             | 6052        | 243                            | 2.99 × 10⁻⁷           |
| Earth    | 1                                             | 9.80                              | 6378        | 1.00                           | 7.27 × 10⁻⁵           |
| Mars     | 0.38                                          | 3.7278                            | 3394        | 1.03                           | 7.10 × 10⁻⁵           |
| Jupiter  | 2.64                                          | 25.8984                           | 71,400      | 0.41                           | 1.80 × 10⁻⁴           |
| Saturn   | 0.93                                          | 9.1233                            | 60,000      | 0.44                           | 1.70 × 10⁻⁴           |
| Uranus   | 0.89                                          | 8.7309                            | 25,559      | 0.72                           | 1.0 × 10⁻⁴            |
| Neptune | 1.12                                          | 10.9872                           | 24,764      | 0.72                           | 1.0 × 10⁻⁴            |

Our point of view can be confirmed through the calculations in Tables 5 and 6 for $\lambda = \pi/6$ and $\lambda = \pi/4$, respectively. Such calculations are based on assuming that a projectile, at rest, is released from the point $Q(50,50,50)$. The results in Tables 5 and 6 reveal that the obtained errors of $T$ for outer planets are greater than those of inner ones. In addition, for the obtained errors of $y_G$, although small for all planets, we find that the corresponding errors of $x_G$ are significant, especially, for those outer planets. Therefore, the current exact analysis may be more helpful than the approximate one, particularly when studying the falling body problem near to the surface of a planet with greater values of $\omega$ and $R$ than those of Earth, as it already appeared in cases of Jupiter, Saturn, Uranus, and Neptune. Finally, we are in a period of new planets, new space bodies. For this reason, the current results may also be valid for new planets, which may be discovered later.
Table 5. The obtained errors of the falling time $T$, $x_G$, and $y_G$ for all planets in the Solar System at the release point Q(50,50,50) (km) when $\lambda = \pi/6$.

| Planet      | Errors of Falling Time ($T$) [s] | Errors of $x_G$ and $y_G$ |
|-------------|----------------------------------|---------------------------|
|             |                                   | Error of $x_G$ (m)        | Error of $y_G$ (m)        |
| Mercury     | 0.000022                          | 0.02334                   | 0.00103                   |
| Venus       | 0.000020                          | 0.00714                   | 0.00003                   |
| Earth       | 0.044593                          | 76.48                     | 1.08084                   |
| Mars        | 0.098546                          | 104.25                    | 2.80507                   |
| Jupiter     | 0.707271                          | 1982.68                   | 7.62458                   |
| Saturn      | 2.586710                          | 4332.41                   | 29.9594                   |
| Uranus      | 0.396289                          | 642.56                    | 0.43672                   |
| Neptune     | 0.271695                          | 493.98                    | 0.01481                   |

Table 6. The obtained errors of the falling time $T$, $x_G$, and $y_G$ for all planets in the Solar System at the release point Q(50,50,50) (km) when $\lambda = \pi/4$.

| Planet      | Errors of Falling Time ($T$) [s] | Errors of $x_G$ and $y_G$ |
|-------------|----------------------------------|---------------------------|
|             |                                   | Error of $x_G$ (m)        | Error of $y_G$ (m)        |
| Mercury     | 0.00004                           | 0.02637                   | 0.00103                   |
| Venus       | 0.00001                           | 0.00153                   | 0.00003                   |
| Earth       | 0.08874                           | 87.8912                   | 1.20038                   |
| Mars        | 0.19522                           | 119.268                   | 3.06653                   |
| Jupiter     | 1.43850                           | 2341.8                    | 3.25094                   |
| Saturn      | 5.37170                           | 5262.61                   | 16.1197                   |
| Uranus      | 0.79587                           | 746.424                   | 0.99274                   |
| Neptune     | 0.54491                           | 572.814                   | 0.99814                   |

5. Conclusions

In this paper, the three dimensional model of the falling body problem near the Earth’s surface was analyzed, taking into account the contribution of all physical quantities. Moreover, unlike the previous published paper [15], the values of the involved parameters were left arbitrary without any restrictions/constraints. Thus, the exact solutions of the full-three coupled equations of motion were obtained in terms of the release point $Q(x_0, y_0, z_0)$, the colatitude $\lambda$, the angular velocity $\omega$, and the radius $R$ of Earth. The present analysis was based on the Laplace transform as a method of solution. Furthermore, the properties of the falling point in the rotating frame and the original inertial frame were given by means of two lemmas. In addition, the conditions at which the falling point belongs to the Earth’s surface were theoretically proven. The effectiveness and validity of the present analysis over the published one in the literature [15] arises when applied to other planets of greater radius $R$ and angular velocity $\omega$ than those of the Earth. Hence, the present analysis is applicable to analyzing the TDFBP near enough to the surface of any spherical celestial body.

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Appendix A

Laplace Transform and Its Properties

The LT of a function \( u(t) \) is defined as \([19]\)
\[
L\{u(t)\} = \int_0^\infty e^{-st}u(t)dt = U(s),
\]
where the LT of \( u(t) \) is said to exist if the integral (A1) converges. The sufficient conditions for existence of the LT are detailed in [8], see pages (1–2). Moreover, the LT of the \( n \)-derivative of \( u(t) \) is given by \([19]\).
\[
L\{u^{(n)}(t)\} = s^nU(s) - \sum_{m=0}^{n-1} s^{n-m-1}u^{(m)}(0).
\]
For \( n = 2 \), we have
\[
L\{\ddot{u}(t)\} = s^2U(s) - su(0) - \ddot{u}(0).
\]
Applying the last formula on \( \ddot{x}(t) \), \( \ddot{y}(t) \), and \( \ddot{z}(t) \), yields
\[
\begin{align*}
L\{\ddot{x}(t)\} &= s^2X(s) - sx(0) - \dot{x}(0), \\
L\{\ddot{y}(t)\} &= s^2Y(s) - sy(0) - \dot{y}(0), \\
L\{\ddot{z}(t)\} &= s^2Z(s) - sz(0) - \dot{z}(0),
\end{align*}
\]

Appendix B

Appendix B.1. Special Cases

Appendix B.1.1. \( \lambda = 0 \) (North Pole)

At \( \lambda = 0 \), we have from (14)–(18) the following values for \( k_i, i = 1,2,3,4,5 \):
\[
k_1 = 6x_0, k_2 = 0, k_3 = -2a^2x_0, k_4 = -g + 2a^2z_0, k_5 = -2g + a^2z_0.
\]
Substituting (26) into (22)–(24), we obtain
\[
\begin{align*}
x(t) &= (x_0 - \omega y_0)\cos \omega t + (y_0 + \omega x_0)\sin \omega t, \\
y(t) &= (y_0 + \omega x_0)\cos \omega t - (x_0 - \omega y_0)\sin \omega t, \\
z(t) &= z_0 - \frac{gt^2}{2}.
\end{align*}
\]
It is noted from these equations that only \( z(t) \) is independent of the angular velocity \( \omega \).
Appendix B.1.2. \( \lambda = \pi \) (South Pole)

In this case, we have

\[
k_1 = 6x_0, \quad k_2 = 0, \quad k_3 = 2a^2x_0, \quad k_4 = -g + 2a^2z_0, \quad k_5 = -2g + a^2z_0.
\] (A11)

Accordingly, Equations (22)–(24) become

\[
x(t) = (x_0 + \omega y_0 t) \cos \omega t - (y_0 - \omega x_0 t) \sin \omega t,
\] (A12)

\[
y(t) = (y_0 - \omega x_0 t) \cos \omega t + (x_0 + \omega y_0 t) \sin \omega t,
\] (A13)

\[
z(t) = z_0 - \frac{gt^2}{2}.
\] (A14)

Equation (A14) indicates that \( z(t) \) is not influenced by \( \omega \) as in the previous special case.

Appendix B.1.3. \( \lambda = \pi/2 \) (Equator)

At \( \lambda = \pi/2 \) and proceeding as above, we have

\[
k_1 = 4x_0, \quad k_2 = 2a^2x_0, \quad k_3 = 2g - 2a^2(z_0 + R),
\] (A15)

\[
k_4 = -g + a^2(3z_0 + R), \quad k_5 = g - a^2R,
\] (A16)

and

\[
x(t) = a^2x_0 + (1 - a^2)x_0 \cos \omega t + \frac{1}{2} \omega(1 - a^2)x_0 t \sin \omega t,
\] (A17)

\[
y(t) = \frac{1}{\omega} \left[ \omega y_0 - \left( g - a^2(z_0 + R) \right) t \right] \cos \omega t + \frac{1}{\omega^2} \left[ a^3y_0 t + \left( g - a^2(z_0 + R) \right) t \right] \sin \omega t,
\] (A18)

\[
z(t) = \frac{g - a^2R}{\omega^2} + \frac{1}{\omega^2} \left[ -g + a^2(z_0 + R) - a^3y_0 t \right] \cos \omega t + \frac{1}{\omega} \left[ \omega y_0 + \left( -g + a^2(z_0 + R) \right) t \right] \sin \omega t.
\] (A19)

Here, it can be seen from Equations (A17)–(A19) that \( \omega \) is involved in all the components \( x(t) \), \( y(t) \), and \( z(t) \).

Appendix C

Appendix C.1. Coordinates of the Falling Point at Special Cases

Appendix C.1.1. \( \lambda = 0 \) (North Pole)

From (A10), by solving the equation \( z(T) = 0 \) for \( T \), we obtain

\[
T = \sqrt{\frac{2z_0}{g}}.
\] (A20)

Hence, \( x_G \) and \( y_G \) are given from (B.2,3) by

\[
x_G = (x_0 - \omega y_0 T) \cos \omega T + (y_0 + \omega x_0 T) \sin \omega T,
\] (A21)
\[ y_G = (y_0 + \omega x_0 T) \cos \omega T - (x_0 - \omega y_0 T) \sin \omega T. \] (A22)

From Equations (26), we have
\[ \rho = \sqrt{(1 + \omega^2 T^2)(x_0^2 + y_0^2)}, \] (A23)
\[ \delta = \tan^{-1}\left(\frac{(y_0 + \omega x_0 T) \cos \omega T - (x_0 - \omega y_0 T) \sin \omega T}{(x_0 + \omega y_0 T) \cos \omega T + (y_0 + \omega x_0 T) \sin \omega T}\right), x_0 \neq 0, y_0 \neq 0. \] (A24)

Appendix C.1.2. \( \lambda = \pi \) (South Pole)

In this case, we have
\[ x_G = (x_0 + \omega y_0 T) \cos \omega T - (y_0 - \omega x_0 T) \sin \omega T, \] (A25)
\[ y_G = (y_0 - \omega x_0 T) \cos \omega T + (x_0 + \omega y_0 T) \sin \omega T, \] (A26)
where \( T \) is defined by (C.1) and consequently
\[ \rho = \sqrt{(1 + \omega^2 T^2)(x_0^2 + y_0^2)}, \] (A27)
\[ \delta = \tan^{-1}\left(\frac{(y_0 - \omega x_0 T) \cos \omega T + (x_0 + \omega y_0 T) \sin \omega T}{(x_0 + \omega y_0 T) \cos \omega T - (y_0 - \omega x_0 T) \sin \omega T}\right), x_0 \neq 0, y_0 \neq 0. \] (A28)

Appendix C.1.3. \( \lambda = \frac{\pi}{2} \) (Equator)

In the previous cases, the time \( T \) was explicitly obtained as seen from (C.1). However, the present case is different and requires solving the following implicit equation in \( T \):
\[ \frac{g - \omega^2 R}{\omega^2} + \frac{1}{\omega^2}\left[-g + \omega^2(z_0 + R) - \omega^3 y_0 T\right] \cos \omega T + \frac{1}{\omega^4}\left[\omega y_0 + \left(-g + \omega^2(z_0 + R)\right)T\right] \sin \omega T = 0, \] (A29)
which can be simplified as
\[ (g - \omega^2 R) + \left[-g + \omega^2(z_0 + R) - \omega^3 y_0 T\right] \cos \omega T + \omega\left[y_0 + \left(-g + \omega^2(z_0 + R)\right)T\right] \sin \omega T = 0. \] (A30)

The corresponding \( x_G \) and \( y_G \) are given as and
\[ x_G = \omega^2 x_0 + \left(1 - \omega^2\right)x_0 \cos \omega T + \frac{1}{2}\omega(1 - \omega^2)x_0 T \sin \omega T, \] (A31)
\[ y_G = \frac{1}{\omega^4}\left[\omega y_0 - \left(g - \omega^2(z_0 + R)\right)T\right] \cos \omega T + \frac{1}{\omega^2}\left[\omega^3 y_0 T + \left(g - \omega^2(z_0 + R)\right)\right] \sin \omega T. \] (A32)

Hence
\[ \rho = \sqrt{x_G^2 + y_G^2}. \] (A33)
\[
\delta = \tan^{-1} \left( \frac{\alpha (\omega y_0 - \left( g - \omega^2 (z_0 + R) \right) T \cos \omega T + \left( \frac{g - \omega^2 (z_0 + R)}{\omega^4 x_0 + \alpha^2 (1 - \alpha^2) x_0 \cos \omega T + \frac{1}{2} \alpha^3 (1 - \alpha^2) x_0 \sin \omega T} \right) \sin \omega T}{\omega^4 x_0 + \alpha^2 (1 - \alpha^2) x_0 \cos \omega T + \frac{1}{2} \alpha^3 (1 - \alpha^2) x_0 \sin \omega T} \right) \right)^\frac{1}{2}, \quad (A34)
\]

where \( x \neq 0 \).

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