Chiral matter and transitions in heterotic string models

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In the framework of $N = 1$ supersymmetric string models given by the heterotic string on an elliptic Calabi-Yau $\pi : Z \to B$ together with a $SU(n)$ bundle we compute the chiral matter content of the massless spectrum. For this purpose the net generation number, i.e. half the third Chern class, is computed from data related to the heterotic vector bundle in the spectral cover description; a non-technical introduction to that method is supplied. This invariant is, in the class of bundles considered, shown to be related to a discrete modulus which is the heterotic analogue of the $F$-theory four-flux. We consider also the relevant matter which is supported along certain curves in the base $B$ and derive the net generation number again from the independent matter-related computation. We then illustrate these considerations with two applications. First we show that the construction leads to numerous 3 generation models of unbroken gauge group $SU(5), SO(10)$ or $E_6$. Secondly we discuss the closely related issue of the heterotic 5-brane/instanton transition resp. the F-theoretic 3-brane/instanton transition. The extra chiral matter in these transitions is related to the Hecke transform of the direct sum of the original bundle and the dissolved 5-brane along the intersection of their spectral covers. Finally we point to the corresponding $F$-theory interpretation of chiral matter from the intersection of 7-branes where the influence of four-flux on the twisting along the intersection curve plays a crucial role.

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1 Introduction

Perhaps the most extensively studied class of $N = 1$ supersymmetric superstring compactification models is the heterotic string on a Calabi-Yau $Z$ with a vector bundle turned on breaking $E_8 \times E_8$ to some GUT group (times a hidden $E_8$ which couples only gravitationally). The classical example of this construction uses the tangent bundle, where the embedding of the spin connection in the gauge connection leads to an unbroken $E_6$. This was soon generalised [1] to the case of embedding a $SU(n)$ bundle also for $n = 4$ or $5$, leading to unbroken $SO(10)$ resp. $SU(5)$ which are also interesting phenomenologically. For example $E_6$ is less favoured as its fundamental $27$, decomposing as $16 \oplus 10 \oplus 1$ under $SO(10)$, contains the $10 \oplus 1$ not observed experimentally. Similarly the $SO(10)$ is superior to the $SU(5)$ as $16 = \bar{5} \oplus 10 \oplus 1$ shows that all fermions of one generation occur in one representation (augmented by the right-handed neutrino). Another interesting feature of $SO(10)$ would be the absence of (renormalizable) baryon number violation as $SO(10)$ forbids a $16^3$ coupling. Here we will study the heterotic string on an elliptically fibered Calabi-Yau $Z \to B$ endowed with an $SU(n)$ bundle and compute the net number of the relevant massless chiral matter multiplets, thereby exhibiting easily numerous three-generation models. As it was quite difficult to construct three generation models using the tangent bundle ("embedding the spin connection in the gauge connection"), i.e Calabi-Yau three-folds of Euler number (of absolute value) 6, this demonstrates impressively the greater flexibility of the extended ansatz.

The reason we restrict to the class of elliptic $Z$ is twofold: first this allows one to use a fibrewise description of the bundle, which data are then pasted together globally along the base $B$; second one has a dual string model given by $F$-theory on a $K3$ fibered Calabi-Yau four-fold $X \to B$. Both points together led to a very satisfying description of the relevant moduli spaces and some non-perturbative brane-impurities to be turned on in a consistent vacuum [2],[3],[4],[5],[6],[7].

Here we will be interested in the net amount of chiral matter. For this purpose we compute the net generation number (half of the third Chern class) of a heterotic vector bundle given in the spectral cover description. As the third Chern class would vanish for $\tau$-invariant bundles ($\tau$ the fibre involution) the result we get is proportional to the deviation from $\tau$-invariance as measured by a certain discrete modulus: the $\gamma$ class, given as a half-integer number $\lambda$ times a certain characteristic topological class; this is the analogue of four-flux in $F$-theory [7]. This result should be compared with an $F$-theory computation related to chiral matter on the intersection of two 7-branes where, besides the twisting because of the curvedness of the compact part of the world-volume of the 7-branes, the influence of 4-flux on the twisting has to play a crucial role. Note that a relation that four-flux should imply chiral matter was anticipated in [9]. Actually we will also point to a second possibility to turn on $c_3(V)$: the spectral cover construction generalizes$^2$ naturally to include a second discrete modulus (this time an integer $l$ which multiplies a second universally given cohomology class, given by the exceptional divisor resolving a naturally occuring singularity).

$^2$In [2] this did not come up as there the focus concerning Chern classes was, because of the consideration of 5-branes, on $c_2$ for which essentially a consideration of a spectral cover on the level of surfaces $C/B$ is sufficient; here, as the focus is on chiral matter and so on $c_3$, we have to face the three-fold covering $Z \times_B C \to Z$, where then the issue of resolution of singularities of $Z \times_B C$ occurs.
We also discuss the closely related framework of a transition where a 5-brane disappears changing $V$ to a bundle $W$ of changed $f_B c_2$ to restore the anomaly cancellation \[\int_B c_2(Z) = n_5 + \sum_{i=1}^{2} \int_B c_2(V_i)\]

Similarly on the $F$-theory side one has the process where a 3-brane disappears by being dissolved into a finite size instanton, i.e. a background gauge bundle $T$ inside the 7-brane. This exchange is governed by \[\frac{1}{24} \int_X c_4(X) = n_3 + \sum_j k_j + \frac{1}{2} G^2\]

where the contribution from the four-flux $G = \frac{1}{2\pi} dC$ is the analogue [7] of the contribution $-\frac{1}{2\pi} \gamma^2$ inside $\int_B c_2(V)$ in the $n_5$ formula and $k_j = \int_{S_j} c_2(T_j)$ are the instanton numbers of possible background gauge bundles $T_j$ inside the 7-brane (of compact part of world-volume given by a surface $S_j \subset B_3$). This breaks partially the gauge group which would be given, without background gauge bundle, by the singularity type of the degeneration over the 7-brane (we will consider the compact part of the worldvolume of the 7-brane to be given by 'sections' like the 'visible' two-fold base $B$ lying embedded inside the three-fold type IIB base $B_3$ of $F$-theory; $B_3$ is a $P^1$ bundle ober $B$). Note that the unbroken gauge group, heterotically the commutator of $V$ in $E_8$, corresponds in $F$-theory to that piece of the gauge group (related to the singularity type of $S$) which is left over after the breaking by $T$. We will see that in the transition extra chiral matter occurs if the original bundle had some (cf. the examples of the interesting non-perturbative phenomenon of chirality changing transitions [10]). The extra chiral matter in the transition is related heterotically to the Hecke transform of the direct sum of the original bundle and the dissolved 5-brane along the intersection of their spectral covers, resp. in $F$-theory to the intersection of 7-branes.

In section 2 we give a non-technical description of the spectral cover method. In section 3 we discuss the net generation number (whose direct numerical evaluation from spectral cover data is given in the Appendix), give an independent derivation of this number from computing the net matter supported on a certain curve and give many 3 generation models. Section 4 treats the question of transition and gives some heuristic dual $F$-theory considerations. The Appendix provides the calculation of the third Chern class.

2 The spectral cover description of heterotic string models

This section provides a non-technical introduction to the spectral cover method intended for the non-expert.

Let $\pi : Z \rightarrow B$ be an elliptically fibered Calabi-Yau three-fold of section $\sigma$, $V$ a $SU(n)$ vector bundle of $c_1(V) = 0$ over $Z$. The idea of the spectral cover description of $V$ is to consider it first on an elliptic fibre and then paste together these descriptions, using global data in the base $B$. More precisely the data pertaining to $Z$ will be $c_1 := c_1(B)$
and $\sigma \in H^2(Z)$, $V$ will be codified by a class $\eta \in H^{1,1}(B)$ and a half-integral number $\lambda$. Actually, as we will describe below, the contruction generalizes easily to include a second discrete modulus, essentially given by an integer $l$.

**fibre description**

We start on an elliptic fibre $E$, given in Weierstrass representation

$$y^2 = 4x^3 - g_2x - g_3$$

with a distinguished reference point $p$ given, the 'origin', i.e. the zero in the group law, 'infinity', i.e. the point $x = y = \infty$ in the Weierstrass description. $V$, assumed to be fibrewise semistable, decomposes on $E$ as a direct sum of line bundles of degree zero. Such a line bundle on $E$ is associated with a unique point on $E$. The fact that the product of the line bundles is trivial (as we consider an $SU(n)$ bundle) translates to the fact that the points sum up to zero. For such an $n$-tuple of points there exists a unique (up to multiplication by a complex scalar) meromorphic function $w$ vanishing to first order at the points and having a pole (at most of $n$th order) only at $p$. The last condition means that $w$ is a polynomial in $x$ and $y$

$$w = a_0 + a_2x + a_3y + a_4x^2 + a_5xy + \ldots + a_nx^{n/2}$$

The double interpretation of $E$ as set of points resp. parameter space of degree zero line bundles on itself is formalized by introducing the Poincare bundle $P$, a line bundle on $E_1 \times E_2$ whose restriction to $Q \times E_2$ is the line bundle on $E_2$ associated to the point $Q \in E_1$ (actually one uses a symmetrized version of this). Note for later use that the naturally given involution $\tau$ on $E$ (i.e. $a \to -a$ in the group law) corresponds to the dualization of the bundle on $E$.

**global description**

We assume a global choice of reference point to be given by the existence of a section $\sigma$. The $x, y, g_2, g_3$ now become sections of the $2, 3, 4, 6$ power of a line bundle over $B$ of class $c_1$ (this is the Calabi-Yau condition). The variation over $B$ of the $n$ points in a fibre leads to a hypersurface $C \subset Z$, a ramified $n$-fold cover (the 'spectral cover') of $B$ by $\pi|_C$, given by an equation $s = a_0 + a_2x + a_3y + \ldots + a_nx^{n/2} = 0$. Here the pole order condition leads to $s$ being a section of $O(\sigma)^n$ but now one can still twist by an arbitrary line bundle $\mathcal{M}$ over $B$ of $c_1(\mathcal{M}) =: \eta$ so that $s$ actually will be supposed to be a section of $O(\sigma)^n \otimes \mathcal{M}$. The class $\eta \in H^{1,1}(B)$ is the single most important globalization datum (topological class) of the construction. The cohomology class of $C$ in $Z$ is now given by

$$C \simeq n\sigma + \eta$$

Now the idea in the spectral cover description of $V$ is to trade in the $SU(n)$ bundle $V$ over $Z$ for a line bundle $\mathcal{R}$ over the $n$-fold cover $p : Z \times_B C \to Z$: the fibre of $V$ over a point $z \in Z$ with the $n$ preimages $\tilde{z}_i$ (this describes the situation outside the ramification

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3The line bundles resp. the points are determined by $V$ up to permutation, i.e. up to the action of the Weyl group.

4Note that $x$ resp. $y$ has a double resp. triple pole at $p$; the displayed equation is for $n$ even, for $n$ odd the last term is $a_nx^{(n-3)/2}y$. 

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locus) will be given by the sum of fibres of $\mathcal{R}$ at the $\tilde{z}_i$, written globally as $V = p_* \mathcal{R}$. If one takes for $\mathcal{R}$ the global version of $\mathcal{P}$ one gets indeed that fibrewise $V = p_* \mathcal{P}$ as $V$ was on an $E_b$ ($b \in B$) a sum of line bundles which corresponded to an $n$-tuple of points in $E_b$ which are collected in $C_b$ and then retransformed to line bundles via $\mathcal{P}$ and summed up by $p_*$. As the twist by a line bundle $L$ over $C$ leaves the fibrewise isomorphism class untouched ($L$ being locally trivial along $C$) the construction generalizes to

$$V = p_*(p^*_C L \otimes \mathcal{P})$$

The condition $c_1(V) = 0$ translates to a fixing\(^5\) of $c_1(L)$ in $H^{1,1}(C)$ up to a class in $\ker \pi_* : H^{1,1}(C) \rightarrow H^{1,1}(B)$; such a class is known to be of the form $\gamma = \lambda (n\sigma - (\eta - nc_1))$ with $\lambda$ half-integral. Note that the $\tau$-invariant bundles, i.e. bundles with $\tau^* V = V^*$ are characterised\(^2\) by $\gamma = 0$.

### 3 Chiral matter and the generation number for heterotic string models

Concerning the net generation number $\frac{1}{2}c_3(V) = -(h^1(Z, V) - h^1(Z, V^*))$ for the stable vector bundle $V$ note first that, to avoid $\tau$-invariance of $V$ which makes $c_3(V)$ trivially vanishing, $c_3(V)$ has to involve $\lambda$ which measures (cf. the remark above) the deviation from $\tau$-invariance of $V$. Actually one arrives from a Grothendieck-Riemann-Roch calculation for $V = p_*(p^*_C L \otimes \mathcal{P})$ given in the Appendix at the relation (between numbers)

$$\frac{1}{2}c_3(V) = \lambda\eta(\eta - nc_1)$$

**Remark** Actually the construction is naturally slightly generalised leading to a dependence on a further discrete parameter: $Z \times_B C$ will have a set $S$ of isolated singularities when in the base the discriminant referring to the $Z$-direction meets the branch locus referring to the $C$ direction. Their resolution $Y \rightarrow Z \times_B C$ leads to the possibility to formulate the whole construction on $Y$ and twist there by the line bundle corresponding to a multiple $l \in \mathbb{Z}$ of the exceptional divisor $E$. Including this twist we end up with the final formula (the number of singularities is $|S| = 12c_1 n(2\eta - (n - 1)c_1)$)

$$\frac{1}{2}c_3(V) = \lambda\eta(\eta - nc_1) + \frac{l(l - 1)(2l - 1)}{6}|S|$$

Unless stated otherwise we will restrict in the following to the sector $l = 0$.

**matter**

Now let us consider an understanding of this net generation number more directly in terms of matter, i.e. as $-(h^1(Z, V) - h^1(Z, V^*))$. Note\(^2\) that by the Leray spectral sequence $H^1(Z, V)$ is localised\(^6\) along the fibers $E_b$ of $h^i(E_b, V) \neq 0$ for $i = 0$ or 1, i.e. where one of the line bundle summands on $E_b$ is trivial. As this corresponds to the point at infinity this condition will not be fulfilled generically but only along the curve

\(^5\) $\pi_*(c_1(L)) = -\pi_* \frac{c_1(C) - c_b}{2}$

\(^6\) This generalizes some insights of [12],[13] for the case of one-dimensional $B$. 

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$A = C_V \cdot \sigma \subset \sigma(B)$ of cohomology class $\eta - nc_1$ in $B \cong \sigma(B)$. Now $V$ restricted to $\sigma(B)$ is given by $\pi_*L$ and $\chi_1(L|_A) = deg\frac{K_C-K_B}{2}|_A + deg \gamma|_A = \frac{1}{2}deg K_A - deg K_B|_A + \gamma \cdot \sigma$ with $\gamma \cdot \sigma = -\lambda \eta \sigma$ where the $\sigma$ becomes in $A \subset B \cong \sigma(B)$ the $\eta - nc_1$. Consideration of the exact sequence derived from the Leray spectral sequence $H^p(B, R^q\pi_*V) \Rightarrow H^{p+q}(Z,V)$

$$0 \to H^1(B, R^0\pi_*V) \to H^1(Z,V) \to H^0(B, R^1\pi_*V) \to H^2(B, R^0\pi_*V)$$

which leads because of the localization here to $H^1(Z,V) \cong H^0(B, R^1\pi_*V)$ leads by taking into account relative Serre duality $\pi_* (V^* \otimes \omega_{Z/B}) \cong (R^1\pi_*V)^*$ with the relative dualizing sheaf $\omega_{Z/B} = \pi^*K_B^{-1}$ to the reinterpretation of the fermions as $H^1(Z,V) = H^0(A, R^1\pi_*V|_A) = H^0(A, L|_A \otimes K_B|_A)$ and considering the appropriate $\bar{\partial}$-equation on $A$, the index theorem gives the net-chirality as

$$- \left( h^0(A, L|_A \otimes K_B|_A) - h^1(A, L|_A \otimes K_B|_A) \right) = - \left( \frac{\chi_A}{2} + deg L|_A + deg K_B|_A \right) = \lambda \eta(\eta - nc_1)$$

As we will see shortly a number of 3 generation models with unbroken gauge group $SU(5), SO(10)$ or $E_6$ (times some hidden group) is thereby easily constructed. For this let us first recall the connection between the breaking representation of $V$ and the representation of the matter charged under the unbroken gauge group.

$$248 = (3, 27) \oplus (3, \bar{27}) \oplus (1, 78) \oplus (8, 1)$$
$$= (4, 16) \oplus (4, \bar{16}) \oplus (6, 10) \oplus (1, 45) \oplus (15, 1)$$
$$= (5, 10) \oplus (5, \bar{10}) \oplus (10, 5) \oplus (10, \bar{5}) \oplus (1, 24) \oplus (24, 1)$$

So in the case of unbroken $SO(10)$ the consideration of the fundamental $V = 4$ still suffices (the relevant fermions are in the 16 only, one does not need the 10 which would be coupled with the 6 = $\Lambda^24$), whereas in the case of unbroken $SU(5)$ one has also to consider $\Lambda^2V = 10$ (to get the 5 part of the fermions 10 $\oplus$ 5; the 10 and the 5 will come in the same number of families by anomaly considerations). Note that instead of the $27^3$ coupling of the $E_6$ case one has in the $SO(10)$ case especially the $16 \cdot 16 \cdot 10$ with the massless Higgs bosons sitting in the 10 to give mass to the fermions.

Note that the $\Lambda^2V$-related matter is localised on a different curve [2]; as $\Lambda^2V$ decomposes along an $E_6$ into products of pairs of those line bundles, into which $V$ decomposes, the triviality condition this time means that two of the points of $C_6$ should be inverses of each other (in the additive group law, meaning inverse bundles), so the localization curve will be $C \cdot \tau C$, resp. its projection to $B$.

three-generation models

We now give some examples of rational bases $B$ and choices of the globalization datum $\eta$ describing some $SU(n)$ bundle leading to 3 generation models of unbroken gauge groups$^8$

$^7$note that $deg K_C|_A = C \cdot \sigma|C$ and $\sigma \cdot C|_\sigma = deg K_B|_A$ as the canonical divisors of the two surfaces $C$ resp. $B$, which cut out $A$, are also their divisors $n\sigma + \eta$ resp. $\sigma$ in the Calabi-Yau $Z$ (as they represent their normal bundles; this is for example for $B$ the relation $\sigma|_C = K_B|_A$ of the appendix); note that thereby $\frac{1}{2}(K_C - K_B)|_A = \frac{1}{2}K_A - K_B|_A$ as the normal directions of $\sigma(B)$ and $C$ add up on their intersection curve $A$, i.e. $K_A = K_B|_A + K_C|_A$.

$^8$times some hidden gauge group left over by a choice of a non-trivial bundle in the other $E_8$, necessary because of $n_1 + n_2 = 12c_1$ for anomaly cancellation; note also that only the generation number is considered here, leaving untouched the question which of these vacua can be ruled out by other requirements or the question of breaking the GUT-group.
2 SU(5), SO(10) or $E_6$. So in the following examples $n = 5, 4, 3$ is understood, as well as $l = 0$; also, unless stated otherwise\(^9\), $\lambda = -\frac{1}{2}$. The entries are $\eta$'s leading to 3 generations.

| base | $SU(5)$ | $SO(10)$ | $E_6$ |
|------|---------|----------|-----|
| $F_{2k}$ | $3(1, -2 + k)$ | $3(1, -3 + k)$ | $(0,1)$ |
| $F_{2k+1}$ | $-3(1, k) (\lambda = \frac{1}{2})$ | $3(1, -3 + k) (\lambda = -1)$ | $(0,1)$ |
| $dP_k$ | $l - E_1 - E_2$ | $l + E_1 - E_2 - E_3 - 2E_4$ | $l - E_1$ |
| | $2E_1 - E_2 - E_3$ | $2E_1 - E_2 - E_3$ | $2E_1 - E_2 - E_3$ |
| | $E_1$ | $-l + E_1 + E_2 (\lambda = 1)$ | $l + E_1 (\lambda = \frac{1}{2})$ |
| $dP_6$ | $-c_1 (\lambda = \frac{1}{2})$ | $c_1 (\lambda = -1)$ | $c_1$ |
| $dP_7$ | | | |
| $dP_8$ | $b$ | $b + f (\lambda = -1)$ | $2b + f$ |
| | $2b + 2f$ | $-b (\lambda = 1)$ | $-3b + f$ |
| | $-2b (\lambda = \frac{1}{2})$ | $b + 4f (\lambda = 1)$ | $-b - 2f (\lambda = \frac{1}{2})$ |

Here the Hirzebruch surfaces\(^10\) $F_m$ are $P^1_{(2)}$ fibrations over $P^1_{(1)}$ possessing a section of self-intersection $-m$ and have $c_1 = (2, 2 + m)$; above $k \geq 0$. The $dP_k$ ($k = 1, \ldots, 8$) denote the del Pezzo surfaces ($P^2$ blown up in $k$ points, leading to corresponding exceptional divisors $E_i$; they have $c_1 = 3I - \sum E_i$), and\(^11\) $dP_9 = \frac{1}{2}K3$ the plane blown up in the nine intersection points of two cubics leading to an elliptically fibered surface with\(^12\) embedded $P^1$ base $b$ and fibre $f = c_1$.

4 Chiral matter and transitions in heterotic and F-theory

We will consider now transitions where a heterotic 5-brane resp. a F-theory 3-brane is dissolved into an 'enlarging' of the heterotic bundle resp. into an instanton (background gauge bundle inside the 7-brane) in F-theory, leading in both descriptions to a breaking of the gauge group to a smaller one. We will see that if the original bundle had some chiral matter then in the transition a change of net-chirality occurs: namely extra chiral matter of amount proportional to the original net-chirality will occur. Mathematically the computation which matches the newly occurring chiral matter will be very similar to the discussion already given, the emphasis of the physical interpretation will be slightly different. First let us recall some general background pertaining to this question [2], [3].

the non-triviality of bundle degenerations

One sees already from the occurence of $n$ in the formulas for the Chern numbers that $V_n$ cannot degenerate (in the class of bundles under consideration) to $O \oplus V_{n-1}$.

\(^9\)Note that in many examples (of $|\lambda| = \frac{1}{2}$) one can get the small number of standard model generations because of the possible half-integrality of the $\gamma$ class (cf. for this issue also [7], [16] and the remark at the end of section 4).

\(^10\)Note that the $E_6$-models over the $F_m$ are derived from 6D models which are adiabatically extended over a further $P^1_{(1)}$.

\(^11\)The corresponding $Z$ is the double elliptic $CY^{19,19}$ which is a $(K3 \times T^2)/\mathbb{Z}_2$, studied especially for example in [14]. The other Calabi-Yau three-folds over $dP_k$ resp. $F_m$ are studied in [4] resp. [15].

\(^12\)take $b = l - E_1 - E_2$, i.e. the proper transform of a line through two of the points
(thereby ’liberating’ a greater unbroken gauge group). The defining equation $s = a_0 + a_2 x + a_3 y + \ldots + a_n x^{n/2} = 0$ for the spectral cover $C_n$ related to $V_n$ (say $n$ even) develops, when one sends $a_n \to 0$ to reduce to a $V_{n-1}$ situation, besides a $C_{n-1}$ component also a second branch $\sigma$ (corresponding to $\mathcal{O}$) which intersects $C_{n-1}$. This means that one gets instead of the direct sum bundle $\mathcal{O} \oplus V_{n-1}$ a non-trivial extension (’elementary modification’/Hecke transform of $\mathcal{O} \oplus V_{n-1}$). By contrast $V_n$ could be possibly deformed to $\mathcal{O}' \oplus V_{n-1}$ with $\mathcal{O}'$ not the trivial line bundle but a rank one sheaf which is torsion-free, i.e. the ideal sheaf of a codim 2 subvariety: the fibers wrapped by the heterotic 5-branes ([2, cf. also [11]).

**The connection between the line bundle $L$ on $C$ and the gauge fields inside the $F$-theory 7-branes**

The guiding principle is that multiple components of the spectral cover correspond to 7-branes carrying non-abelian gauge groups inside them (breaking the gauge group which would otherwise correspond to the singularity type of the 7-brane)[3]. For this assume that $C'$ is a component of $C$ of multiplicity $m > 1$. The non-reduced surface $mC'$ will in general be equipped with a rank 1 sheaf $L'$, one possible version of which consists in a rank $m$ vector bundle $M$ on the reduced surface $C'$. For example, if one has a bundle $\pi^* M$ pulled back from a bundle $M$ of rank $m$ on the base $B$, then $\pi^* M$ is fibrewise trivial of rank $m$, i.e. the corresponding spectral surface is $m\sigma$ (with spectral bundle $M$). Now the bundle $M$ on the spectral surface should correspond to the gauge bundle inside the $F$-theory 7-brane. This can also be considered from the perspective of the symmetry breaking mechanism [3]: if one starts, say, from a bundle $V = \bigoplus_{i=1}^n U = U \otimes I_n$ (with $U$ irreducible and $I_n$ trivial of rank $n$) then $C_V = nC_U$ and $V$ has $SU(n)$ as an automorphism group leading to a corresponding unbroken gauge group; then deformations of the bundle $M$ over $nC_U$ break the $U \otimes I_n$ product structure, reducing thereby the automorphism group. Thus the bundle $M$ on a spectral surface provides the analogue to the symmetry breaking mechanism coming from a background gauge bundle inside the 7-brane in $F$-theory.

**The transition**

Now we want to follow the transition where a $F$-theory 3-brane disappears by being dissolved into a finite size instanton, i.e. a background gauge bundle $\tilde{M}$ inside the 7-brane [3]. Heterotically $V$ will be enhanced to a new bundle $W$, absorbing the instanton number of the dissolved 5-brane. As $W$ will be a deformation (Hecke transform) of $V \oplus \pi^* M$ and $c_2(M)$ counts the dissolved 5-branes just as $c_2(\tilde{M})$ the dissolved 3-branes the correspondence should be $M = \tilde{M}$ (both bundles are over $B$).

**Heterotic situation**

We have $C_W = C_V + m\sigma$ with $m = \text{rank}(M)$. To understand in more detail the part of the massless spectrum given by the deformations of $W$ look at the decomposition of the deformation space

$$H^1(Z, \text{End}(W)) = H^1(Z, \text{End}(V)) \oplus H^1(Z, \text{End}(\pi^* M))$$

$$\oplus H^1(Z, \text{Hom}(V, \pi^* M)) \oplus H^1(Z, \text{Hom}(\pi^* M, V))$$

in which the first two summands correspond to deformations of the individual summands.
whereas the latter two summands (of dimensions $N$ resp. $\bar{N}$, say) deform away from the direct sum $V \oplus \pi^*M$, the last one for example giving instead non-trivial extensions $0 \to V \to W \to \pi^*M \to 0$. These provide the moduli of all the non-trivial Hecke transforms of $V \oplus \pi^*M$ along the intersection curve $A$ (assumed to be smooth) of the spectral surfaces $C_V$ resp. $m\sigma$ of the two summands. Note that the cohomology class of $A = C_V \cdot \sigma$ in $B$ is $\eta - nc_1$ and that one gets for the net-chirality the result which depends on $M$ only through its rank [3]

$$N - \bar{N} = \frac{1}{2} \text{rank}(V)c_3(\pi^*M) - \frac{1}{2}mc_3(V) = -\frac{1}{2}mc_3(V)$$

**Outlook on the F-theory side**

We close with a short heuristic remark on the treatment of the corresponding F-theory situation. Here the corresponding transition-related part of the spectrum is seen as follows. The first two summands correspond essentially to complex structure moduli of the F-theory CY$^4$ resp. moduli of the instanton background (in F-theory) whereas the latter two summands correspond to chiral matter multiplets $q$ and $\tilde{q}$, transforming in the (bi-)fundamental resp. anti-fundamental of $SU(m)$, supported on an intersection curve $A \subset B$ (assumed to be smooth) of the 7-branes (with compact part of world-volume) $B$ resp. $S$, say, corresponding to $m\sigma$ resp. $C_V$ on the heterotic side (we assume that inside $S$ no background gauge bundle is turned on). The index theorem gives their net-chirality as ($\mathcal{T}$ a twisting line bundle on $A$) [3]

$$H^0(A, M \otimes \mathcal{T}) - H^1(A, M \otimes \mathcal{T}) = m(\frac{X_A}{2} + \text{deg} \mathcal{T})$$

To outline some further possible steps, one would have to identify now $A$ and $\mathcal{T}$. Concerning the first question one should use something like the four-flux $G_c$ in $B$ whereas the latter two summands correspond to chiral matter multiplets $q$ and $\tilde{q}$, transforming in the (bi-)fundamental resp. anti-fundamental of $SU(m)$, supported on an intersection curve $A \subset B$ (assumed to be smooth) of the 7-branes (with compact part of world-volume) $B$ resp. $S$, say, corresponding to $m\sigma$ resp. $C_V$ on the heterotic side (we assume that inside $S$ no background gauge bundle is turned on). The index theorem gives their net-chirality as ($\mathcal{T}$ a twisting line bundle on $A$) [3]

$$H^0(A, M \otimes \mathcal{T}) - H^1(A, M \otimes \mathcal{T}) = m(\frac{X_A}{2} + \text{deg} \mathcal{T})$$

To outline some further possible steps, one would have to identify now $A$ and $\mathcal{T}$. Concerning the first question one should use something like the $\eta = 6c_1 + t$ connection for $E_8$ bundles [2] where $t$ describes the $P^1$ bundle $B^3$ over $B$. Concerning the second question one should expect a relation of the type $\text{deg} \mathcal{T} = \text{deg} \frac{K_A}{2} + G_\gamma \cdot \Sigma$ where the four-flux $G_\gamma$, corresponding to the heterotic $\gamma$, is part of the data like $S$ and $X$ which translate the heterotic bundle $V$; it 'couples' to the twisting line bundle $\mathcal{T}$ on $A$ by its contribution to $\text{deg} \mathcal{T}$ being $G_\gamma \cdot \Sigma$ with $\Sigma$ representing the section of the $K3$ fibration $X^4 \to B$. The idea would be then to use some relation like $G_\gamma \cdot \Sigma = \gamma \cdot \sigma$ (cf. also [7]).

**Remark** Concerning the critical question of $\lambda$-dependence we would like to point to a further (cf. the 3 generation models above) meaning of the generator value $\lambda = \frac{1}{2}$. For $A = \eta - nc_1$ one gets coincidence of the F-theory net-chirality with the heterotic one if $\text{deg} \mathcal{T} = -\frac{c_2(V)}{2} - \frac{\lambda(A)}{2} = -\lambda \eta A + \frac{K_A}{2} = (-\lambda \eta + \frac{A^2 + K_B}{2} + A) = \text{deg} \left( (\frac{1}{2} - \lambda) \eta + \frac{-nc_1 - c_1}{2} \right)_A$.

Now $\text{deg} \mathcal{T}$ is $\eta$-indep. for $\lambda = 1/2$ just as (cf. Appendix) the $n_5$-'relevant' part $\pi^*\omega$ of $\frac{c_2(V)}{2} = \eta \sigma + \pi^*\omega$, $\omega \in H^4(B)$.

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13In yet another direction we would like to add that everything in the F-theory side interpretation proposed here concerns so far the heterotic sector $l = 0$ only. This clearly deserves further study.

14This $P^1$ bundle is given as the projectivization of a vector bundle $O \oplus \Theta$ with a line bundle $\Theta$ over $B$ of $c_1 = t$, so if one uses homogeneous coordinates $a, b$, which are sections of $O(1)$ (of $c_1 = t$; it is fibrewise the usual $O(1)$ bundle) and $O(1) \otimes \Theta$, one has $r(r + t) = 0$. Note that then $B : B = r^2 = -rt$ in $B^3$ is $-t = \eta - 6c_1$ in $B$.

15We will always think of the four-flux in F-theory as a limit from M-theory and not go to an IIB interpretation of it; this allows more easily to keep contact with the four-fold $X^4$ and thereby with the heterotic side (cf. [7]).
Appendix

Computation of the generation number from spectral cover data

the set-up

The essential conceptual ideas of the spectral cover description of an SU(n) vector bundle over an elliptic Calabi-Yau three-fold are described in section 2. Here we will give the technical framework of cohomological formulas pertaining to the Chern class computation given later; for many details on the formulas in this subparagraph cf. [2].

Notation: Let $\pi : Z \to B$ be an elliptically fibered Calabi-Yau three-fold of section $\sigma$, $V$ a vector bundle of $c_1(V) = 0$ and rank $n$ over $Z$. $\sigma$ will also denote the class in $H^2(Z)$ represented by the divisor $\sigma(B)$. Cohomology classes pulled back from $B$, like $\pi^*\eta$, will be denoted by their expression on $B$, like $\eta$; similarly for $p^*_C$ and $r$ or $\gamma$. Unspecified Chern classes, like $c_1$, will always refer to $B$. $\pi|_C$ will often denoted by $\pi$ in the sequel and likewise $\sigma|_C$ by $\sigma$. $\Delta$ will denote the diagonal in $Z \times_B Z$ restricted to $Z \times_B C$, $\sigma_i$ ($i = 1, 2$) the classes pertaining to the embeddings of $B$ into the different fibre-factors; the class $r$ (cf. below) will always refer to the second factor, i.e. $r = n\sigma_2 + \eta + c_1$.

Now let $C \subset Z$ the spectral cover associated with $V$ which lies in $Z$ as a hypersurface given by an equation $s = a_0 + a_2x + a_3y + \ldots + a_{n-1}y^{n/2} = 0$. Here $s$ is a section of $\mathcal{O}(\sigma)^n \otimes \mathcal{M}$ with $\mathcal{M}$ a line bundle over $B$ of $c_1(\mathcal{M}) =: \eta$. So the cohomology class of $C$ in $Z$ is given by $n\sigma + \eta$; it is a ramified $n$-fold cover of $B$ by $\pi|_C$ of branch divisor $r = -(c_1(C) - c_1) = n\sigma + \eta + c_1$. Now the bundle $V$ is given in the spectral cover description as $V = p^*_s(p^*_CL \otimes \mathcal{P})$ with $L$ a line bundle over $C$. Here $\mathcal{P} = \mathcal{O}(\Delta - \sigma_1 - \sigma_2) \otimes \mathcal{Q}^{-1}$ is the suitably twisted Poincare line bundle for the family $Z \to B$, $\mathcal{Q} = \det TB$ of $c_1(\mathcal{Q}) = c_1$ so that $c_1(\mathcal{P}) = \Delta - \sigma_1 - \sigma_2 - c_1$. Before we move on let us collect a number of useful identities between the classes considered so far. One has $\sigma^2 = -\sigma_1$ so that $\pi^*\sigma|_C = \eta - n\sigma_1$; further for $\Delta$, the diagonal in $Z \times_B Z$ restricted to $Z \times_B C$, that $p^*\Delta = n\sigma + \eta$ and $\Delta^2 = -\Delta_c$; moreover from $\Delta_\sigma = \sigma_1\sigma_2$ one finds $\sigma_1(\mathcal{P}) = 0$ and $p^*_c(\mathcal{P}) = 0$.

$$\begin{array}{ccc}
Z \times_B C & \overset{p_C}{\longrightarrow} & C \\
\downarrow \pi & & \downarrow \pi|_C \\
Z & \overset{\pi}{\longrightarrow} & B
\end{array}$$

To understand the Chern classes of $V$ one uses Grothendieck-Riemann-Roch (GRR) $p^*_s(c_1(p^*_CL \otimes \mathcal{P})td(Z \times_B C)) = ch(V)td(Z)$, from the vanishing of whose first order term downstairs on $Z$, i.e. from the condition $c_1(V) = 0$ and the Calabi-Yau condition for $Z$, one gets on the $C/B$ level (i.e. restrict the consideration of GRR to $\sigma_1(B) \times_B C = C$ where $\mathcal{P}$ is trivial) the condition $\pi^*c_1(L) = -\pi^*\frac{c_1(C) - c_1}{2}$, i.e. $c_1(L) = \frac{r}{2} + \gamma$ with the universal class $\gamma = \lambda(n\sigma - \eta + nc_1)$ in ker $\pi^* : H^{1,1}(\mathcal{C}) \to H^{1,1}(B)$ with $\lambda$ half-integral. Concerning $\gamma$ note that although $\pi^*\gamma = 0$ one has nevertheless $\pi^*\gamma^2 = -\lambda^2\eta\eta(n - nc_1)$.
singularities

Actually this discussion has to be slightly modified as \( Z \times_B C \) will have a set \( \mathcal{S} \) of isolated singularities (generically ordinary double points) lying over points in the base \( B \) where the branch divisor \( r \) of \( \pi : C \to B \), resp. its image in \( B \) \( \pi_* r = n(2\eta - (n-1)c_1) \), meets the discriminant \( 12c_1 \) (cf. the case of an elliptic \( K3 \) over \( P^1 \)) of \( \pi : Z \to B \). Their number is (as detected in \( B \)) \( |\mathcal{S}| = 12c_1n(2\eta - (n-1)c_1) \). Let \( \nu : Y \to Z \times_B C \) be the resolution of the isolated singularity with \( E \) the exceptional divisor. For its discussion let us make a short digression.

\[
\begin{array}{c}
Y \\
\downarrow \nu \\
Z \times_B C \xrightarrow{p_C} C \\
\downarrow p \\
Z \xrightarrow{\pi} B
\end{array}
\]

For a surface it is well known that the blow-up of a point \( p \) leads to a divisor \( D = P^1 \) of \( D^2 \) being \(-1\) resp. \(-2\) for \( p \) smooth resp. an ordinary double point. In general on a \( n \)-fold \( X \) for a smooth point the local model of the exceptional divisor is just \( P^{n-1} \) with the tautological bundle \( O(-1) \) over it, so its self-intersection is minus a hyperplane \( P^{n-2} \) in the \( P^{n-1} \), so in the case of a 3-fold \( D^3 = +1 \) as \((-l)^2 = +1\) for \( l \) the line in the \( P^2 \). For a double point, described by a quadratic equation in an ambient \( (n+1) \)-fold \( W \), blow up \( W \) at the point to get a \( P^n \); in \( X \) this would lead, at a smooth point (described by a linear equation), to a \( P^{n-1} \), i.e. just the usual blow-up of \( X \). But for a double point one gets a degree 2 hypersurface in \( P^n \) (so in the surface case a conic, i.e. still a \( P^1 \)), again of normal bundle \( O(-1) \), which restricted to the degree 2 locus has degree \(-2\) (so in the surface case one gets \( O(-2) \) on \( P^1 \), i.e. self-intersection \(-2\)). For the 3-fold case \( D = P^1 \times P^1 \) where the \( P^1 \)'s are described linearly in \( P^3 \) and so \( D|_D = (-1, -1) \).

So each of its \( |\mathcal{S}| \) components is a divisor \( D = P^1 \times P^1 \) of triple self-intersection \( D^3 = (-1, -1)^2 = +2 \) as \( D|_D = (-1, -1) \). Furthermore \( c_2(Y)|_D = c_2(D) + c_1(D) \cdot D|_D = 4 + (2, 2) \cdot (-1, -1) = 0 \), so \( c_2 \) remains unchanged as in the smooth case (cf. [17]) whereas the componentwise correction in \( c_1(Y) \) deviates from its value \(-2D \) in the smooth case (cf. [17]) as now \( c_1(Y)|_D = c_1(D) + D|_D = (2, 2) + (-1, -1) = -D|_D \), so here the correction is \(-D \). So one has

\[
c_1(Y) = \nu^* c_1(Z \times_B C) - E \\
c_2(Y) = \nu^* c_2(Z \times_B C)
\]

Now, more precisely, one formulates the spectral cover description on the resolved threefold \( Y \) which offers at the same time the possibility of a further twisting by (a multiple \( l \in Z \) of) the canonically given line bundle corresponding to the exceptional divisor. So one arrives at the description \( V = p_\ast \nu_* \mathcal{L} \) where \( \mathcal{L} = \nu^* p_C^* \mathcal{L} \otimes \nu^* \mathcal{P} \otimes \mathcal{O}_Y(lE) \).

In the main body of the paper we actually will always focus on the case \( l = 0 \); then the relation between \( \tau \)-invariance of \( V \) and vanishing of \( \gamma \) (cf. [2]) holds, and \( c_3(V) \) will be given by the \( \gamma \)-related term alone (cf. below). If the \( l \)-twist is turned on the \( E \)-contribution in \( c_1(Y) \) becomes important.
the computation

Now by Grothendieck-Riemann-Roch (GRR)

\[
p_*\nu_*(e^{c_1(L)}td(Y)) = ch(V)td(Z) = n + n\frac{c_2(Z)}{12} - c_2(V) + \frac{c_3(V)}{2}
\]

From the vanishing of the first order term of GRR downstairs on \(Z\) one gets the condition \((p\nu)_*c_1(L) = -\frac{1}{2}(p\nu)_*c_1(Y)\), which we met above on the \(C/B\) level, giving here \(c_1(L) = \frac{r}{2} + \gamma + c_1(P)\). Note further that from the Weierstrass representation one gets \(c_2(Z) = c_2 + 12\sigma_1 + 11c_1^2 = c_2 + 12(\sigma + c_1)c_1 - c_1^2\) [2]. Similarly one can derive the expressions \(c_1(Z \times_B Z) = -c_1\) and \(c_2(Z \times_B Z) = c_2 + 12[(\sigma_1 + c_1) + (\sigma_2 + c_1)]c_1 - c_1^3\) and from adjunction \(c(Z \times_B C) = \frac{c(Z \times_B Z)}{1 + n\sigma_2 + \eta}\) one finally finds

\[
c_1(Z \times_B C) = -(n\sigma_2 + \eta + c_1) = -r
\]

\[
c_2(Z \times_B C) = (n\sigma_2 + \eta + c_1)(n\sigma_2 + \eta) + c_2 + 12(\sigma_1 + \sigma_2)c_1 + 23c_1^2
\]

From the collected results one computes

\[
c_2(V) = \eta\sigma - \frac{n^3 - n}{24}c_1^2 - \frac{n}{8}\eta(\eta - nc_1) - \frac{1}{2}\pi_\gamma^2 = \eta\sigma - \frac{n^3 - n}{24}c_1^2 + (\lambda^2 - \frac{1}{4})\frac{n}{2}\eta(\eta - nc_1)
\]

and (for \(l = 0\) where only the \(\lambda\)-related term occurs)

\[
\frac{1}{2}c_3(V) = p_*\nu_*\left(c_3^2(L)\left(\frac{c_1(L)}{6} + \frac{c_1(Y)}{4}\right) + \frac{c_1(L)c_1^2(Y)}{12} + (c_1(L) + \frac{c_1(Y)}{2})c_2(Y)\right)
\]

\[
= p_*\left(-\frac{r^2}{24}(\gamma + c_1(P)) + \frac{(\gamma + c_1(P))^2}{6}(\gamma + c_1(P)) + \frac{c_2(Z \times_B C)}{12}(\gamma + c_1(P))\right)
\]

\[
= \frac{1}{6}p_*\left(3\gamma c_1^2(P)\right) = \lambda\sigma\eta(\eta - nc_1)
\]

After integration over the fiber one arrives at the relation between numbers\(^{16}\)

\[
\frac{1}{2}c_3(V) = \lambda\eta(\eta - nc_1)
\]

If one includes also the contribution coming from \(c_1(\mathcal{O}_Y(-E))\) one gets an additional term \(^{17}\) \(\frac{1}{12}(l(l-1)(2l-1)}2|S|\) in \(\frac{1}{2}c_3(V)\).

\(^{16}\) Concerning the treatment of \(c_3(V)\) computation given in the recent paper [18], which uses the parabolic bundle construction, note that the discussion given there is very restricted: for \(n\) even it is restricted to \(\tau\)-invariant bundles of \(\lambda = 0\), for which actually \(c_3 = 0\), resp. for \(n\) odd to bundles of \(\eta \equiv 0(n)\); and, especially important, by being restricted by his ansatz to a special point in moduli space (\(\lambda \sim \frac{1}{24}\)) the author there gets as coefficient for the main term a numerical factor \((1/n)\) whose extrapolation (as \(\sim \lambda^2 n\) instead of the actual \(\sim \lambda\)) leads to a misleading conceptual interpretation \((\pi_*^\gamma)^2\) of the term.

\(^{17}\) This contribution is even componentwise integral as for example \(\frac{1}{6}(l(l-1)(2l-1)} = \sum_{i=1}^{l-1} i^2\).
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