Homogeneous Cotton solitons

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Abstract

Left-invariant Cotton solitons on homogeneous manifolds are determined. Moreover, algebraic Cotton solitons are studied providing examples of non-invariant Cotton solitons, both in the Riemannian and Lorentzian homogeneous settings.

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1. Introduction

The objective of the different geometric evolution equations is to improve a given metric by considering a flow associated with the geometric object under consideration. The Ricci, Yamabe and mean curvature flows are examples which have been extensively studied in the literature. Under suitable conditions, the Ricci flow evolves an initial metric to an Einstein metric, while the Yamabe flow evolves an initial metric to a new one with constant scalar curvature within the same conformal class. Cotton, Yamabe and Ricci flows have many physical applications (we refer to [14, 15, 17] and references therein for more information). There are however certain metrics which, instead of evolving by the flow, remain invariant. Such is the case of those solitons associated with self-similar solutions of the flow.

In the study of the conformal geometry of dimensions greater than 3, the Weyl tensor plays a distinguished role, since its nullity characterizes local conformal flatness. The three-dimensional case must be studied in a different way due to the fact that the Weyl tensor vanishes identically. Moreover, the whole curvature is completely determined by the Ricci tensor, $\rho$. Local conformal flatness is characterized in dimension 3 by the fact that the Schouten tensor, defined by $S_{ij} = \rho_{ij} - \frac{\tau}{8} g_{ij}$, where $\tau$ denotes the scalar curvature, is a Codazzi tensor, or equivalently the Cotton tensor $C_{ijk} = (\nabla_i S)_{jk} - (\nabla_j S)_{ik}$, which is the unique conformal invariant in dimension 3, vanishes. We refer to [3, 16, 25] and references therein for more information on the usefulness of the Cotton tensor in describing the geometry of three-dimensional manifolds.
The Cotton tensor appears naturally in many physical contexts \[11, 12\], especially in Chern–Simons theory \[13\] or topologically massive gravity \[1, 9, 17\]. In particular, the field equation in topologically massive gravity implies a proportionality between the Einstein and Cotton tensors. The fact that the Einstein tensor consists of second-order derivatives on the metric whereas the Cotton tensor is of order 3 implies that an exact solution to this field equation is in general difficult to find. Indeed, most of the solutions for the field equation in the topological massive gravity are constructed on homogeneous spaces.

In \[2\], a new geometric flow based on the conformally invariant Cotton tensor was introduced. A Cotton flow is a one-parameter family \(g(t)\) of three-dimensional metrics satisfying

\[
\frac{\partial}{\partial t} g(t) = -\lambda C_{g(t)},
\]

where \(C_{g(t)}\) is the \((0, 2)\)-Cotton tensor corresponding to the metric \(g(t)\), obtained from the \((0, 3)\)-Cotton tensor by means of the \(\star\)-Hodge operator, and given by

\[
C_{ij} = \frac{1}{2\sqrt{g}} C_{nm} \epsilon^{nm\ell} g_{ij},
\]

where \(\epsilon^{ijk}\) denotes the Levi-Civita permutation symbol (\(\epsilon^{123} = 1\)).

While the Ricci flow is a second-order equation (a kind of nonlinear heat equation), the Cotton flow is a third-order equation. This makes a substantial difference with the Ricci and the Yamabe flows. The existence of the Cotton flow is not clear for a generic metric; in this sense, even the short-time existence is an open problem (see \[2\] for some information on the homogeneous background). Although the Yamabe and the Ricci flows are well posed in the Riemannian setting, they do not necessarily exist in the Lorentzian case, where even the existence of short-time solutions is not guaranteed in general due to the lack of parabolicity. Cotton solitons provide initial metrics for which the Cotton flow has a self-similar solution, at least on closed manifolds.

When comparing it with the Yamabe flow, the Cotton flow has in some sense an opposite behavior, since the Cotton flow changes the conformal class except in the conformally flat case. The genuine fixed points of the Cotton flow are the locally conformally flat metrics. However, there exist other geometric fixed points, which correspond to Cotton solitons. A pseudo-Riemannian manifold \((M, g)\) is a Cotton soliton if it admits a vector field \(X\) such that

\[
\mathcal{L}_X g + C = \lambda g,
\]

where \(\mathcal{L}_X\) denotes the Lie derivative in the direction of the vector field \(X\) and \(\lambda\) is a real number. A Cotton soliton is said to be shrinking, steady or expanding respectively, if \(\lambda > 0\), \(\lambda = 0\) or \(\lambda < 0\). Since there is no ambiguity, we call the Cotton soliton both to the pseudo-Riemannian manifold \((M, g)\) and to the vector field \(X\).

Cotton solitons are closely related to Ricci and Yamabe solitons, which are defined by \(\mathcal{L}_X g + \rho = \lambda g\) and \(\mathcal{L}_X g = (\tau - \lambda) g\), respectively. In particular, if \((M, g)\) is a locally conformally flat homogeneous manifold then the class of Cotton solitons coincides with the class of Yamabe solitons (see for example \[8\]). In such a case, the Cotton soliton is said to be trivial. On the other hand, if \((M, g)\) is a Lorentzian manifold which satisfies the field equation of a topologically massive gravity space, then \((M, g)\) is a Ricci soliton if and only if it is a Cotton soliton \[17\].

In the Riemannian signature, any compact Cotton soliton is locally conformally flat, while compact Lorentzian examples are available in the non-locally conformally flat setting \[7\]. Moreover, the fact that any left-invariant homogeneous Ricci or Yamabe soliton on a three-dimensional Riemannian Lie group is flat, while non-flat examples exist in the Lorentzian signature (see \[4, 8\]), motivates a study of homogeneous Cotton solitons in the Lorentzian setting.
Complete and simply connected three-dimensional Lorentzian homogeneous manifolds are either symmetric or a Lie group with a left-invariant Lorentzian metric [6]. Since three-dimensional locally symmetric Lorentzian manifolds are locally conformally flat, the purpose of this paper is twofold: firstly, to classify invariant homogeneous Cotton solitons on Lie groups, and secondly, to determine algebraic Cotton solitons on Lie groups and use them in order to obtain new examples of non-invariant Cotton solitons on homogeneous manifolds, both in the Riemannian and Lorentzian settings.

**Conventions and structure of the paper**

Throughout this paper, \((M, g)\) denotes a three-dimensional pseudo-Riemannian manifold and \((G, g)\) denotes a three-dimensional Lie group equipped with a left-invariant metric; as usual, \(R\) stands for the curvature tensor taken with the sign convention \(R(X, Y) = \nabla_X Y - [\nabla_X, \nabla_Y]\), where \(\nabla\) denotes the Levi-Civita connection. The Ricci tensor, \(\rho\), and the corresponding Ricci operator, \(\hat{\rho}\), are given by \(\rho(X, Y) = g(\hat{\rho}(X), Y) = \text{trace}[Z \mapsto R(X, Z)Y]\), and we denote the scalar curvature by \(\tau\). Finally, \((G, g)\) is always assumed to be connected and simply connected.

This paper is organized as follows. In section 2, we review the description of all three-dimensional Lorentzian Lie algebras. We analyze the existence of non-trivial left-invariant Cotton solitons on three-dimensional Lorentzian Lie groups in section 3. In section 4, we determine algebraic Cotton solitons on homogeneous manifolds, showing the existence of Cotton solitons which are non-invariant. Finally, in section 5, examples of non-invariant shrinking and expanding Lorentzian homogeneous Cotton solitons which are non-trivial are constructed on the Heisenberg group and on the \(E(1, 1)\) group.

2. Preliminaries

Let \(\times\) denote the Lorentzian vector product on \(\mathbb{R}^3\) induced by the product of the para-quaternions (i.e., \(e_1 \times e_2 = -e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2\), where \([e_1, e_2, e_3]\) is an orthonormal basis of signature \((++-))\). Then \([Z, Y] = L(Z \times Y)\) defines a Lie algebra, which is unimodular if and only if \(L\) is a self-adjoint endomorphism of \(g\) [23]. Considering the different Jordan normal forms of \(L\), we have the following four classes of unimodular three-dimensional Lorentzian Lie algebras.

**Type Ia.** If \(L\) is diagonalizable with eigenvalues \([\alpha, \beta, \gamma]\) with respect to an orthonormal basis \([e_1, e_2, e_3]\) of signature \((++-))\), then the corresponding Lie algebra is given by

\[
(g_{Ia}) \quad [e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1.
\]

**Type Ib.** Assume \(L\) has a complex eigenvalue. Then, with respect to an orthonormal basis \([e_1, e_2, e_3]\) of signature \((++-))\), one has

\[
L = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \gamma & -\beta \\
0 & \beta & \gamma
\end{pmatrix}, \quad \beta \neq 0
\]

and thus the corresponding Lie algebra is given by

\[
(g_{Ib}) \quad [e_1, e_2] = \beta e_2 - \gamma e_3, \quad [e_1, e_3] = -\gamma e_2 - \beta e_1, \quad [e_2, e_3] = \alpha e_1.
\]

**Type II.** Assume \(L\) has a double root of its minimal polynomial. Then, with respect to an orthonormal basis \([e_1, e_2, e_3]\) of signature \((++-))\), one has

\[
L = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{2} + \beta & -\frac{1}{2} \\
0 & -\frac{1}{2} + \beta & -\frac{1}{2} + \beta
\end{pmatrix}
\]
and thus the corresponding Lie algebra is given by

$$(g_{III}) [e_1, e_2] = \frac{1}{2} e_2 - (\beta - \frac{1}{2}) e_3, \quad [e_1, e_3] = - (\beta + \frac{1}{2}) e_2 - \frac{1}{2} e_3, \quad [e_2, e_3] = \alpha e_1.$$  

**Type III.** Assume $L$ has a triple root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+++)$, one has

$$L = \begin{pmatrix} \alpha & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \alpha & 0 \\ \frac{1}{\sqrt{2}} & 0 & \alpha \end{pmatrix}$$

and thus the corresponding Lie algebra is given by

$$(g_{III}) [e_1, e_2] = -\frac{1}{\sqrt{2}} e_1 - \alpha e_3, \quad [e_1, e_3] = -\frac{1}{\sqrt{2}} e_1 - \alpha e_2, \quad [e_2, e_3] = \alpha e_1 + \frac{1}{\sqrt{2}} (e_2 - e_3).$$

Next, we treat the non-unimodular case. First of all, recall that a solvable Lie algebra $g$ belongs to the special class $\mathcal{S}$ if $[x, y]$ is a linear combination of $x$ and $y$ for any pair of elements in $g$. Any left-invariant metric on $\mathcal{S}$ is of constant sectional curvature [19, 20] and hence locally conformally flat. Now, consider the unimodular kernel, $u = \ker(\text{trace ad} : g \to \mathbb{R})$. It follows from [10] that non-unimodular Lorentzian Lie algebras not belonging to class $\mathcal{S}$ are given, with respect to a suitable basis $\{e_1, e_2, e_3\}$, by

$$(g_{IV}) [e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \quad \alpha + \delta \neq 0,$$

where one of the following holds.

- **IV.1** $\{e_1, e_2, e_3\}$ is orthonormal with $g(e_1, e_1) = -g(e_2, e_2) = -g(e_3, e_3) = -1$ and the structure constants satisfy $\alpha \gamma - \beta \delta = 0$.
- **IV.2** $\{e_1, e_2, e_3\}$ is orthonormal with $g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = 1$ and the structure constants satisfy $\alpha \gamma + \beta \delta = 0$.
- **IV.3** $\{e_1, e_2, e_3\}$ is pseudo-orthonormal with $g(e_1, e_1) = -g(e_2, e_3) = 1$ and the structure constants satisfy $\alpha \gamma = 0$.

As a matter of notation, henceforth, we will write

$$\nabla_{e_i} e_j = \sum_k \Phi^k_{ij} e_k$$

to represent the Levi-Civita connection corresponding to the left-invariant metric on the Lie group, where $\{e_1, e_2, e_3\}$ denotes the basis fixed in each case. Moreover, we will denote by $X = \sum x_i e_i = (x_1, x_2, x_3)$ a generic vector field expressed in the corresponding basis.

Three-dimensional locally conformally flat Lorentzian Lie groups have been studied by Calvaruso in [5]. We translate his classification to our context, in order to fit the notation used throughout this paper.

**Lemma 1.** A three-dimensional Lorentzian Lie group $(G, g)$ is locally conformally flat if and only if one of the following holds:

1. $(G, g)$ is locally symmetric and
   
   (a) of type Ia with $\alpha = \beta = \gamma$ or any cyclic permutation of $\alpha = \beta, \gamma = 0$ (in any of these cases $(G, g)$ is of constant sectional curvature), or
   
   (b) of type II with $\alpha = \beta = 0$, and hence flat, or
(c) of type IV.1 with constant sectional curvature, or otherwise $\alpha = \beta = \gamma = 0$ and $
abla \neq 0$, or $\beta = \gamma = \delta = 0$ and $\alpha \neq 0$, or

(d) of type IV.2 with constant sectional curvature, or otherwise $\alpha = \beta = \gamma = 0$ and $\delta \neq 0$, or $\beta = \gamma = \delta = 0$ and $\alpha \neq 0$, or

(e) of type IV.3 and flat, or otherwise $\gamma = \delta = 0$ and $\alpha \neq 0$, or

(f) of type $\mathcal{S}$ and therefore of constant sectional curvature.

(ii) $(G, g)$ is not locally symmetric and

(a) of type Ib with $\alpha = -2\gamma$ and $\beta = \pm \sqrt{3}\gamma$, or

(b) of type III with $\alpha = 0$, or

(c) of type IV.3 with $\gamma = 0$ and $\alpha \delta (\alpha - \delta) \neq 0$.

Finally, note that any two vector fields $X_1$ and $X_2$ satisfying equation (2) $\mathcal{L}_X g + C = \lambda_i g$, $i = 1, 2$) differ in a homothetic vector field since

$$\mathcal{L}_X g - \mathcal{L}_Y g - (\lambda_1 - \lambda_2) g = \mathcal{L}_X g - \lambda_1 g - \mathcal{L}_X g + \lambda_2 g = 0.$$ 

Conversely, adding a homothetic vector field to any Cotton soliton gives another Cotton soliton. As a consequence, if a homogeneous Lorentzian manifold admits two distinct Cotton solitons (i.e., for constants $\lambda_1 \neq \lambda_2$), then it is locally conformally flat and therefore trivial (see [8, 15] for more information on homogeneous manifolds admitting non-Killing homothetic vector fields).

3. Left-invariant Cotton solitons on Lorentzian Lie groups

Now, we consider the existence of left-invariant solutions of equation (2) on the Lie algebras discussed in section 2. We completely solve the corresponding equations, obtaining a complete description of all non-trivial left-invariant Cotton solitons.

Since the Cotton tensor is trace-free, if a Killing vector field $X$ satisfies equation (2), then $C = 0$ and $(M, g)$ is locally conformally flat. Conversely, if $(M, g)$ is locally conformally flat and homogeneous, then any Cotton soliton is a homothetic vector field and hence it is Killing, or otherwise the Ricci operator is two-step nilpotent and $(M, g, X)$ is also a Yamabe soliton [8]. In the homogeneous setting, a Cotton soliton is said to be trivial if $C = 0$. In what follows, we will focus on the non-trivial case and therefore we will repeatedly use lemma 1 to exclude the case of locally conformally flat Lie groups.

As a consequence of our analysis in this section, the following geometric characterization of Lorentzian Lie groups admitting invariant Cotton solitons will be obtained.

**Theorem 2.** A Lorentzian Lie group $(G, g)$ admits a non-trivial left-invariant Cotton soliton if and only if the Cotton operator is nilpotent. More precisely, the Lorentzian Lie group $(G, g)$ is

(i) of type II with $\alpha = \beta \neq 0$ and locally isometric to $O(1, 2)$ or $SL(2, \mathbb{R})$, or

(ii) of type II with $\alpha = 0 \neq \beta$ and locally isometric to $E(1, 1)$, or

(iii) of type III with $\alpha \neq 0$ and locally isometric to $O(1, 2)$ or $SL(2, \mathbb{R})$.

The previous result remarks on the difference between the Riemannian and Lorentzian settings, since any nilpotent self-adjoint operator vanishes identically in the Riemannian category.

3.1. Unimodular case

In this subsection, we consider the existence of left-invariant Cotton solitons on three-dimensional unimodular Lorentzian Lie groups whose corresponding Lie algebras were introduced in section 2.
In this first case, the Levi-Civita connection is determined by
\[ \Phi_{12}^1 = \Phi_{13}^1 = \frac{1}{2}(\alpha - \beta - \gamma), \]
\[ \Phi_{21}^2 = \Phi_{23}^2 = \frac{1}{2}(\alpha - \beta + \gamma), \]
\[ \Phi_{31}^3 = -\Phi_{32}^3 = \frac{1}{2}(\alpha + \beta - \gamma), \]
expressions which, for a left-invariant vector field \( X = \sum x_ie_i \), allow us to calculate the non-zero terms in the Lie derivative of the metric, \( \mathcal{L}_X g \), given by
\[ (\mathcal{L}_X g)_{12} = (\alpha - \beta)x_3, \quad (\mathcal{L}_X g)_{13} = (\gamma - \alpha)x_2, \quad (\mathcal{L}_X g)_{23} = (\beta - \gamma)x_1. \tag{3} \]
and also to determine the non-zero components of the Cotton tensor, \( C \), as
\[ C_{11} = \frac{1}{2}(2\alpha^2 - \alpha^2(\beta + \gamma) - (\beta - \gamma)^2(\beta + \gamma)), \]
\[ C_{22} = \frac{1}{2}(2\beta^2 - \beta^2(\alpha + \gamma) - (\alpha - \gamma)^2(\alpha + \gamma)), \]
\[ C_{33} = -\frac{1}{2}(2\gamma^2 - \gamma^2(\alpha + \beta) - (\alpha - \beta)^2(\alpha + \beta)). \tag{4} \]

In this first case, we show that left-invariant Cotton solitons are necessarily trivial.

**Theorem 3.** If a type Ia unimodular Lorentzian Lie group is a left-invariant Cotton soliton, then it is necessarily trivial.

**Proof.** From equations (3) and (4), a left-invariant vector field \( X = (x_1, x_2, x_3) \) is a Cotton soliton if and only if
\[
\begin{align*}
x_1(\beta - \gamma) &= x_2(\gamma - \alpha) = x_3(\alpha - \beta) = 0, \\
\alpha^2(\beta + \gamma) + (\beta - \gamma)^2(\beta + \gamma) - 2\alpha^2 + 2\lambda &= 0, \\
\beta^2(\alpha + \gamma) + (\alpha - \gamma)^2(\alpha + \gamma) - 2\beta^2 + 2\lambda &= 0, \\
\gamma^2(\alpha + \beta) + (\alpha - \beta)^2(\alpha + \beta) - 2\gamma^2 + 2\lambda &= 0. \tag{5}
\end{align*}
\]
Note that the first equation in (5) implies that the existence of non-zero Cotton solitons is possible only if \( \alpha = \beta, \alpha = \gamma, \) or \( \beta = \gamma \). Next assume that \( \alpha = \beta \) (the study of the other two cases is analogous). Under this condition, again the first equation in (5) implies that either \( \beta = \gamma \) or \( x_1 = x_2 = 0 \). In the first case, if \( \beta = \gamma \), the Lie group is of constant sectional curvature \( -\frac{\alpha^2}{\lambda} \). In the second case, if \( x_1 = x_2 = 0 \), equation (5) reduces to
\[ \gamma^2(\beta - \gamma) - 2\lambda = 0, \quad \gamma^2(\beta - \gamma) + \lambda = 0, \]
and it follows that if \( \beta \neq \gamma \), then necessarily \( \gamma = 0 \) and the Lie group is flat. We conclude that, in any case, the Cotton soliton is trivial. \( \square \)

### 3.1.2. Type Ib

The Levi-Civita connection is determined by
\[ \Phi_{12}^1 = \Phi_{13}^1 = \frac{1}{2}(\alpha - 2\gamma), \]
\[ \Phi_{21}^2 = -\Phi_{22}^2 = -\Phi_{31}^3 = -\Phi_{32}^3 = -\beta, \]
\[ \Phi_{23}^3 = \Phi_{33}^3 = \Phi_{32}^3 = -\frac{u}{2}, \]
and a straightforward calculation shows that the Lie derivative of the metric is determined by
\[ (\mathcal{L}_X g)_{12} = x_3\beta + x_3(\alpha - \gamma), \]
\[ (\mathcal{L}_X g)_{13} = x_3\beta + x_2(\gamma - \alpha), \]
\[ (\mathcal{L}_X g)_{23} = (\mathcal{L}_X g)_{33} = -2\beta x_1. \tag{6} \]
while the Cotton tensor is characterized by
\[ C_{11} = -2C_{22} = 2C_{33} = \alpha^2 - \alpha^2\gamma + 4\beta^2\gamma, \]
\[ C_{23} = \frac{1}{2}\beta(\alpha^2 + 4\beta^2 - 8\gamma^2 + 4\alpha\gamma). \tag{7} \]

As in the type Ia unimodular case, we do not obtain non-trivial left-invariant Cotton solitons in this second case.
Theorem 4. A type Ib unimodular Lorentzian Lie group does not admit any non-zero left-invariant Cotton solitons.

Proof. For a left-invariant vector field $X = (x_1, x_2, x_3)$, equations (6) and (7) imply that $X$ is a Cotton soliton if and only if

$$\begin{align*}
x_2 \beta + x_3 (\alpha - \gamma) &= 0, \\
\beta (\alpha^2 + 4 \alpha \gamma + 4 \beta^2 - 8 \gamma^2) &= 0, \\
\alpha^3 - \alpha^2 \gamma + 4 \beta^2 \gamma - \lambda &= 0, \\
\alpha^3 - \alpha^2 \gamma + 4 \beta^2 \gamma + 2 \lambda + 4 \beta x_1 &= 0, \\
\alpha^3 - \alpha^2 \gamma + 4 \beta^2 \gamma + 2 \lambda - 4 \beta x_1 &= 0.
\end{align*}$$

(i)

Note that $\beta \neq 0$. Hence, the first equation in (8) implies that $x_2 = \frac{x_3 (\alpha - \gamma)}{\beta}$ and, as a consequence, the second equation in (8) is equivalent to $x_3 (\beta^2 + (\gamma - \alpha)^2) = 0$. Therefore, it follows that $x_3 = 0$ and hence $x_2 = 0$. Finally, from the last two equations in (8) one obtains $x_1 = 0$, and this ends the proof.

3.1.3. Type II. In this case, the Levi-Civita connection is determined by

$$\begin{align*}
\Phi^1_{12} &= \Phi^1_{33} = \frac{1}{2} (\alpha - 2 \beta), & \Phi^2_{21} &= -\Phi^1_{22} = -\Phi^1_{31} = -\Phi^1_{33} = -\frac{1}{2}, \\
\Phi^3_{21} &= \Phi^2_{33} = \frac{1}{2} (\alpha - 1), & \Phi^3_{31} &= -\Phi^3_{32} = \frac{1}{2} (\alpha + 1),
\end{align*}$$

and thus the Lie derivative of the metric is characterized by

$$\begin{align*}
(L_X g)_{12} &= \frac{1}{2} (x_2 + (2 \alpha - 2 \beta - 1) x_3), \\
(L_X g)_{13} &= \frac{1}{2} (x_3 - (2 \alpha - 2 \beta + 1) x_2), \\
(L_X g)_{22} &= (L_X g)_{33} = -(L_X g)_{23} = -x_1.
\end{align*}$$

Moreover, the Cotton tensor is determined by

$$\begin{align*}
C_{11} &= \alpha^2 (\alpha - \beta), \\
C_{22} &= -\frac{1}{4} (2 \alpha^3 - 8 \beta^2 + 4 \alpha \beta - \alpha^2 (2 \beta - 1)), \\
C_{23} &= \frac{3}{4} (\alpha^2 - 8 \beta^2 + 4 \alpha \beta), \\
C_{33} &= \frac{1}{4} (2 \alpha^3 + 8 \beta^2 - 4 \alpha \beta - \alpha^2 (2 \beta + 1)).
\end{align*}$$

(10)

Next we determine the left-invariant Cotton solitons in the type II unimodular case.

Theorem 5. A unimodular Lorentzian Lie group $(G, g)$ of type II admits non-trivial left-invariant Cotton solitons if and only if one of the following conditions holds:

(i) $\alpha = \beta \neq 0$, and then $G$ is locally isometric to $O(1, 2)$ or $SL(2, \mathbb{R})$,

(ii) $\alpha = 0 \neq \beta$, and then $G$ is locally isometric to $E(1, 1)$.

Moreover, in these cases, the Cotton solitons are always steady and respectively given by:

(i) $X = \frac{1}{2} \beta^2 e_1 + \kappa (e_2 + e_3)$, where $\kappa \in \mathbb{R}$.

(ii) $X = 2 \beta^2 e_1$. 

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Proof. Considering equations (9) and (10), the Cotton soliton condition for a left-invariant vector field $X = (x_1, x_2, x_3)$ can be expressed as

$$
\begin{align*}
&x_2 + x_3 (2\alpha - 2\beta - 1) = 0, \\
&x_3 - x_2 (2\alpha - 2\beta + 1) = 0, \\
&\alpha^2 - 8\beta^2 + 4\alpha\beta + 4x_1 = 0, \\
&\alpha^2 (\alpha - \beta) - \lambda = 0, \\
&2\alpha^3 - 8\beta^2 + 4\alpha\beta - \alpha^2 (2\beta - 1) + 4x_1 + 4\lambda = 0, \\
&2\alpha^3 + 8\beta^2 - 4\alpha\beta - \alpha^2 (2\beta + 1) - 4x_1 + 4\lambda = 0.
\end{align*}
$$

Adding the first and the second equations in (11), we obtain $(\alpha - \beta)(x_2 - x_3) = 0$. Hence, either $\alpha = \beta$ or $x_2 = x_3$. Suppose first that $\alpha = \beta$; in this case, the fourth equation in (11) implies that $\lambda = 0$. Now, the first equation in (11) leads to $x_2 = x_3$ and finally the last equation in (11) implies that $x_1 = \frac{1}{2} \beta^2$. With this set of conditions equation (11) holds and the Cotton soliton is non-trivial whenever $\alpha = \beta \neq 0$, which shows (i).

Next, we analyze the case $x_2 = x_3$, with $\alpha \neq \beta$. In this case, the first equation in (11) implies that $x_1 (\alpha - \beta) = 0$ and hence $x_1 = 0$. On the other hand, from the fourth equation in (11) we have $\lambda = \alpha^2 (\alpha - \beta)$. Moreover, from the last two equations in (11) one easily obtains $\alpha^2 (\alpha - \beta) = 0$, and therefore $\alpha = 0$. Thus, equation (11) reduces to $x_1 = \frac{1}{2} \beta^2$, which shows (ii).

Finally, the classification of three-dimensional Lorentzian Lie groups obtained in [6, theorem 4.1] shows that $G$ is locally isometric to $O(1, 2)$ or $SL(2, \mathbb{R})$ in case (i), and to $SL(2, \mathbb{R})$ in case (ii). □

Remark 6. Left-invariant Cotton solitons on a unimodular Lorentzian Lie group of type II are gradient if and only if $\alpha = 0 \neq \beta$. To show this, we analyze the two different cases obtained in theorem 5. First, assume that $\alpha = \beta \neq 0$; in this case, left-invariant Cotton solitons are of the form $X = \frac{1}{2} \beta^2 e_1 + \kappa (e_2 + e_3), \kappa \in \mathbb{R}$, and hence the dual form $X^b$ is given by

$$
X^b = \frac{1}{2} \beta^2 e_1^l + \kappa (e_2^l - e_3^l).
$$

A straightforward calculation shows that

$$
\begin{align*}
\text{d} X^b &= -\beta \kappa (e_1^l \wedge e^l_2 - e^l_1 \wedge e^l_3) - \frac{1}{2} \beta^3 e_2^l \wedge e^l_3
\end{align*}
$$

and therefore the Cotton soliton is never gradient. Now suppose that $\alpha = 0 \neq \beta$; in this second case, the left-invariant Cotton soliton is given by $X = 2\beta^2 e_1$. The dual form is

$$
X^b = 2\beta^2 e_1^l
$$

and, as a consequence, $\text{d} X^b = 0$. Thus, we conclude that there exists a smooth function $f$ such that $X = \nabla f$.

3.1.4. Type III. In this last unimodular case, the Levi-Civita connection is determined by

$$
\begin{align*}
\Phi^1_{11} &= -\Phi^1_{12} = -\Phi^1_{13} = \Phi^2_{22} = \Phi^2_{23} = \Phi^3_{32} = \frac{1}{\sqrt{2}}, \\
\Phi^1_{12} &= -\Phi^1_{21} = -\Phi^3_{21} = -\Phi^3_{23} = \Phi^3_{31} = \Phi^3_{32} = -\frac{\kappa}{2}, \\
&\text{and hence the Lie derivative of the metric is given by}
\end{align*}
$$

$$
\begin{align*}
(L_X g)_{11} &= -\sqrt{2} (x_2 + x_3), & (L_X g)_{12} &= (L_X g)_{13} = \frac{x_1}{\sqrt{2}}, \\
(L_X g)_{22} &= \sqrt{2} x_3, & (L_X g)_{23} &= \frac{x_1 - x_3}{\sqrt{2}}, & (L_X g)_{33} &= -\sqrt{2} x_2.
\end{align*}
$$

(12)
while the Cotton tensor is characterized by
\[ C_{12} = C_{13} = \frac{3\alpha^2}{2\sqrt{2}}, \quad C_{22} = C_{23} = C_{33} = 3\alpha. \] (13)

For this case, we obtain the following.

**Theorem 7.** A unimodular Lorentzian Lie group \((G, g)\) of type III admits non-trivial left-invariant Cotton solitons if and only if \(\alpha \neq 0\), and then \(G\) is locally isometric to \(O(1, 2)\) or \(SL(2, \mathbb{R})\). Moreover, in such a case, the Cotton soliton is steady and given by
\[ X = -\frac{3\alpha^2}{2} e_1 + \frac{3\alpha}{\sqrt{2}} (e_2 - e_3). \]

**Proof.** For a left-invariant vector field \(X = (x_1, x_2, x_3)\), equations (12) and (13) imply that \(X\) is a Cotton soliton if and only if
\[
\begin{align*}
2x_1 + 3\alpha^2 &= 0, \\
\sqrt{2}x_2 + \sqrt{2}x_3 + \lambda &= 0, \\
3\sqrt{2}\alpha - x_2 + x_3 &= 0, \\
3\alpha + \sqrt{2}x_3 - \lambda &= 0, \\
3\alpha - \sqrt{2}x_2 + \lambda &= 0.
\end{align*}
\] (14)

If \(\alpha = 0\), then it follows from equation (14) that the Lie group does not admit any non-zero left-invariant Cotton solitons. Next suppose that \(\alpha \neq 0\). The first equation in (14) implies that \(x_1 = -\frac{3\alpha^2}{2}\) and the second equation in (14) implies that \(x_2 = x_3 - \frac{\lambda}{\sqrt{2}}\). Now, from the last two equations in (14), we obtain \(\lambda = 0\) and \(x_3 = -\frac{3\alpha}{\sqrt{2}}\). Finally, note that the classification of three-dimensional Lorentzian Lie groups obtained in [6, theorem 4.1] shows that \(G\) is locally isometric to \(O(1, 2)\) or \(SL(2, \mathbb{R})\). \(\Box\)

**Remark 8.** A unimodular Lorentzian Lie group of type III does not admit any left-invariant gradient Cotton solitons. Indeed, considering the Cotton soliton determined in theorem 7, the dual form \(X^b\) is given by
\[ X^b = -\frac{3\alpha^2}{2} e^1 + \frac{3\alpha}{\sqrt{2}} (e^2 + e^3) \]
and a straightforward calculation shows that
\[ dX^b = \frac{3\alpha^2}{2\sqrt{2}} (e^1 \wedge e^2 + e^1 \wedge e^3 + \sqrt{2}\alpha e^2 \wedge e^3). \]
Hence, the Cotton soliton \(X\) is not gradient.

### 3.2. Non-unimodular case

In this subsection, the existence of left-invariant Cotton solitons on three-dimensional non-unimodular Lorentzian Lie groups is considered; the corresponding Lie algebras were introduced in section 2. We exclude the study of type \(S\), since any Lie group of that type is of constant sectional curvature and does not admit any non-zero Cotton solitons. Next, we show that in any other non-unimodular case the Cotton solitons are trivial.
3.2.1. Type IV.1. In this case, the Levi-Civita connection is determined by
\[ \Phi_{i1}^3 = \Phi_{i3}^1 = \alpha, \quad \Phi_{i2}^3 = -\Phi_{i3}^2 = \Phi_{i2}^3 = -\beta - \gamma, \]
\[ \Phi_{i2}^1 = -\Phi_{i3}^1 = -\delta, \quad \Phi_{i3}^2 = \Phi_{i2}^2 = -\beta + \gamma, \]
and thus the Lie derivative of the metric is characterized by
\[ (\mathcal{L}_X g)_{11} = -2\alpha x_3, \quad (\mathcal{L}_X g)_{12} = (\beta - \gamma)x_3, \quad (\mathcal{L}_X g)_{13} = \alpha x_1 + \gamma x_2, \]
\[ (\mathcal{L}_X g)_{22} = 2\delta x_3, \quad (\mathcal{L}_X g)_{23} = -\beta x_1 - \delta x_2. \]
Moreover, the Cotton tensor is determined by
\[ C_{11} = -\frac{1}{2}(\beta^3 - 2\gamma^3 - 2\alpha^2 \beta + \beta \gamma^2 + 2\gamma \delta^2 - \beta \delta^2), \]
\[ C_{12} = \delta(\alpha - \delta) - \beta(\beta - \gamma), \]
\[ C_{13} = -\frac{1}{2}(\beta + \gamma)(\alpha + \beta - \gamma - \delta)(\alpha - \beta + \gamma - \delta). \]
We show the non-existence of non-trivial left-invariant Cotton solitons in this case in the following result.

**Theorem 9.** If a type IV.1 non-unimodular Lorentzian Lie group is a left-invariant Cotton soliton then it is necessarily trivial.

**Proof.** Throughout the proof, recall that \( \alpha \gamma - \beta \delta = 0 \) and \( \alpha + \delta \neq 0 \). Equations (15) and (16) imply that a left-invariant vector field \( X = (x_1, x_2, x_3) \) is a Cotton soliton if and only if
\[
\begin{align*}
\alpha x_1 + \gamma x_2 &= 0, \\
\beta x_1 + \delta x_2 &= 0, \\
(\beta + \gamma)(\alpha + \beta - \gamma - \delta)(\alpha - \beta + \gamma - \delta) + 2\lambda &= 0, \\
\delta(\alpha - \delta) - \beta(\beta - \gamma) + (\beta - \gamma)x_3 &= 0, \\
\beta^3 - 2\gamma^3 - 2\alpha^2 \beta + \beta \gamma^2 + 2\gamma \delta^2 + \beta \delta^2 + 4\alpha x_3 - 2\lambda &= 0, \\
2\beta^3 - \gamma^3 - 2\alpha^2 \beta - \beta \gamma^2 + \gamma \delta^2 + \alpha \beta \delta - 4\delta x_3 + 2\lambda &= 0.
\end{align*}
\]
We analyze first the case \( \alpha = 0 \); in this case, \( \beta = 0 \) and \( \delta \neq 0 \). The second equation in (17) implies that \( x_3 = 0 \). On the other hand, the fourth equation in (17) reduces to \( \gamma x_3 = 0 \) and therefore either \( \gamma = 0 \) or \( x_3 = 0 \). If \( \gamma = 0 \), then the Lie group is locally conformally flat and hence the Cotton soliton is trivial. If \( x_3 = 0 \) and \( \gamma \neq 0 \), then the last two equations in (17) imply that \( (\gamma - \delta)(\gamma + \delta) = 0 \), which leads to the constancy of the sectional curvature equals to \(-\frac{\lambda}{\gamma} \); therefore the Cotton soliton must be trivial.

Assume now that \( \alpha \neq 0 \); in this case, \( \gamma = \frac{\beta \delta}{\alpha} \) and the first equation in (17) implies that \( x_1 = -\frac{\beta \delta x_3}{\alpha^2} \), while the second equation in (17) reduces to
\[ (\alpha^2 - \beta^2)\delta x_3 = 0. \]
If \( \alpha^2 - \beta^2 = 0 \) then \( \alpha = \beta \) and \( \gamma = \delta \), where \( s^2 = 1 \); under these conditions the Lie group is of constant sectional curvature \(-\frac{1}{2}(\beta + \delta)^2 \) and hence the Cotton soliton is trivial.

Next, we consider the case \( \delta = 0 \) in equation (18) and therefore \( \gamma = 0 \). The fourth equation in (17) is equivalent to \( \beta x_3 = 0 \). Now, for \( \beta = 0 \) the Lie group is locally conformally flat and hence the Cotton soliton is trivial. On the other hand, if \( x_3 = 0 \) and \( \beta \neq 0 \) then equation (17) reduces to
\[ \beta^3 - \alpha^2 \beta - 2\lambda = 0, \quad \beta^3 - \alpha^2 \beta + \lambda = 0. \]
Therefore, $\alpha^2 - \beta^2 = 0$, which shows that the Lie group is of constant sectional curvature and the Cotton soliton is again trivial.

Finally, we analyze the case $x_2 = 0$ in equation (18); we assume $\alpha \neq 0$ and $\alpha^2 - \beta^2 \neq 0$ to avoid the previous cases. Note that, in this case, $x_1 = 0$ and the fourth equation in (17) transforms into

$$(\alpha - \beta)(\delta(\alpha^2 - \beta^2) + \beta x_3) = 0.$$  

If $\alpha = \delta$ then equation (17) reduces to $\lambda = 0$ and $\delta x_3 = 0$; hence $x_3 = 0$ and the Cotton soliton is zero. Therefore, we assume $\alpha \neq \delta$ and we have $\delta(\alpha^2 - \beta^2) + \beta x_3 = 0$. Note that $\beta$ must be non-zero since we are assuming $\delta(\alpha^2 - \beta^2) \neq 0$. Thus, $x_3 = -\frac{\delta(\alpha^2 - \beta^2)}{\beta}$ and from the third equation in (17) we have

$$\lambda = -\frac{\beta(\alpha^2 - \beta^2)(\alpha - \delta)^2(\alpha + \delta)}{2\alpha^3}$$

and equation (17) reduces to

$$\begin{align*}
\alpha^2\beta^2 - 3\beta^2\delta^2 + 2\alpha\beta^2\delta + 4\alpha^4 &= 0, \\
3\alpha^2\beta^2 - \beta^2\delta^2 - 2\alpha\beta^2\delta - 4\alpha^2\delta^2 &= 0.
\end{align*}$$

Hence, $(\alpha^2 + \beta^2)(\alpha^2 - \delta^2) = 0$, which is a contradiction, and this ends the proof. \(\square\)

3.2.2. Type IV.2. In this case, the Levi-Civita connection and the Lie derivative of the metric are determined by

$$\Phi^1_{11} = \Phi^1_{13} = \alpha, \quad \Phi^2_{12} = \Phi^3_{13} = \Phi^1_{23} = \frac{\beta + \gamma}{2},$$

$$\Phi^3_{22} = \Phi^2_{23} = \delta, \quad \Phi^1_{31} = -\Phi^1_{32} = -\frac{\beta - \gamma}{2},$$

and

$$\begin{align*}
(L_X g)_{11} &= 2\alpha x_3, \quad (L_X g)_{12} = (\beta + \gamma)x_3, \quad (L_X g)_{13} = -\alpha x_1 - \gamma x_2, \\
(L_X g)_{22} &= 2\delta x_3, \quad (L_X g)_{23} = -\beta x_1 - \delta x_2,
\end{align*}$$

(19)

respectively. Now, the Cotton tensor is given by

$$\begin{align*}
C_{11} &= \frac{1}{2}(\beta^3 + 2\gamma^3 + \alpha^2\beta + \beta\gamma^2 + 2\gamma\delta^2 + \beta\delta^2), \\
C_{12} &= \delta(\alpha - \delta) + \beta(\beta + \gamma), \\
C_{22} &= -\frac{1}{2}(2\beta^3 + \gamma^3 + 2\alpha^2\beta + \beta^2\gamma + \gamma\delta^2 - \alpha\beta\delta), \\
C_{33} &= -\frac{1}{2}(\beta - \gamma)((\alpha - \delta)^2 + (\beta + \gamma)^2).
\end{align*}$$

(20)

For this case, we have the following.

**Theorem 10.** If a type IV.2 non-unimodular Lorentzian Lie group is a left-invariant Cotton soliton then it is necessarily trivial.

**Proof.** Throughout the proof recall that $\alpha\gamma + \beta\delta = 0$ and $\alpha + \delta \neq 0$. For a left-invariant vector field $X = (x_1, x_2, x_3)$, equations (19) and (20) imply that $X$ is a Cotton soliton if and only if

$$\begin{align*}
\alpha x_1 + \gamma x_2 &= 0, \\
\beta x_1 + \delta x_2 &= 0, \\
(\beta - \gamma)((\alpha - \delta)^2 + (\beta + \gamma)^2) - 2\lambda &= 0, \\
\delta(\alpha(\alpha - \delta) + \beta(\beta + \gamma)) + (\beta + \gamma)x_3 &= 0, \\
\beta^3 + 2\gamma^3 + \alpha^2\beta + \beta\gamma^2 + 2\gamma\delta^2 + \beta\delta^2 + 4\alpha x_3 - 2\lambda &= 0, \\
2\beta^3 + \gamma^3 + 2\alpha^2\beta + \beta^2\gamma + \gamma\delta^2 - \alpha\beta\delta - 4\delta x_3 + 2\lambda &= 0.
\end{align*}$$

(21)
Assume first that \( \alpha = 0 \). In this case, \( \beta = 0 \) and \( \delta \neq 0 \), and from the third and fifth equations in (21) one obtains \( \gamma (y^2 + \delta^2) = 0 \). Therefore, \( \gamma = 0 \) and the Lie group is locally conformally flat; thus the left-invariant Cotton soliton is trivial.

Next suppose that \( \alpha \neq 0 \). Hence \( \gamma = -\frac{\delta \beta}{\alpha} \) and the first equation in (21) implies that \( x_1 = \frac{\delta x_3}{\alpha^2} \). Now, the second equation in (21) is equivalent to

\[
(\alpha^2 + \beta^2) \delta x_3 = 0,
\]

and therefore either \( \delta = 0 \) or \( x_3 = 0 \). If \( \delta = 0 \), then the fourth equation in (21) reduces to \( \beta x_3 = 0 \). Note that if \( x_3 = 0 \), then equation (21) reduces to

\[
\beta^3 + \alpha^2 \beta + \lambda = 0, \quad \beta^3 + \alpha^2 \beta - 2\lambda = 0,
\]

which implies that \( \beta = 0 \). Hence, in any case, necessarily \( \beta = 0 \), and it follows that the Lie group is locally conformally flat. Thus, the Cotton soliton is trivial.

Finally, we analyze the case \( x_3 = 0 \) in equation (22); we assume \( \delta \neq 0 \) to avoid the previous case. Note that, in this case, \( x_1 = 0 \) and the fourth equation in (21) is equivalent to

\[
(\alpha - \delta)(\delta(\alpha^2 + \beta^2) + \beta x_3) = 0.
\]

If \( \alpha = \delta \) then equation (21) reduces to \( \lambda = 0 \) and \( \delta x_3 = 0 \); hence \( x_3 = 0 \) and the Cotton soliton is zero. Now, if \( \alpha \neq \delta \) then \( \delta(\alpha^2 + \beta^2) + \beta x_3 = 0 \). Note that \( \beta \) must be non-zero since we are assuming that \( \delta \neq 0 \). Thus, \( x_3 = -\frac{\delta(\alpha^2 + \beta^2)}{\beta} \) and the third equation in (21) implies that

\[
\lambda = \frac{\beta(\alpha^2 + \beta^2)(\alpha - \delta)\delta(\alpha + \delta)}{2\alpha^3}.
\]

Now, equation (21) reduces to

\[
\begin{cases}
4\alpha^4 - \alpha^2 \beta^2 - 2\alpha \beta^2 \delta + 3\beta^2 \delta^2 = 0, \\
\alpha^2(3\beta^2 + 4\delta^2) - 2\alpha \beta^2 \delta - \beta^2 \delta^2 = 0,
\end{cases}
\]

and it follows that \( (\alpha - \beta)(\alpha + \beta)(\alpha - \delta)(\alpha + \delta) = 0 \). Since we are assuming that \( \alpha \neq \delta \) and, moreover, \( \alpha + \delta \neq 0 \), then we have \( \alpha = \pm \beta \) and in such a case the above system of equations has no solution (since \( \delta \neq 0 \)). This ends the proof.

3.2.3. Type IV.3. In this case, the Levi-Civita connection is determined by

\[
\begin{align*}
\Phi_{11}^1 &= \Phi_{13}^1 = \alpha, & \Phi_{12}^1 &= -\Phi_{13}^1 = \Phi_{21}^1 = \Phi_{23}^1 = -\Phi_{31}^1 = -\Phi_{32}^1 &= \frac{\gamma}{2}, \\
\Phi_{11}^3 &= \Phi_{13}^3 = -\beta, & \Phi_{12}^3 &= -\Phi_{13}^3 = -\delta.
\end{align*}
\]

Hence, the Lie derivative of the metric and the Cotton tensor are characterized by

\[
\begin{align*}
(L_xg)_{11} &= 2\alpha x_3, & (L_xg)_{12} &= \gamma x_3, & (L_xg)_{13} &= -\alpha x_1 - \gamma x_2 - \beta x_3, \\
(L_xg)_{23} &= -\delta x_3, & (L_xg)_{33} &= 2(\beta x_1 + \delta x_2),
\end{align*}
\]

and

\[
C_{11} = \gamma^3, \quad C_{23} = \frac{\gamma^3}{2}, \quad C_{33} = \frac{\beta \gamma^3}{2}.
\]

Next, we show that left-invariant Cotton solitons reduce to left-invariant Yamabe solitons in this case.

**Theorem 11.** If a type IV.3 non-unimodular Lorentzian Lie group is a left-invariant Cotton soliton, then it is necessarily trivial.
Proof. Using equations (23) and (24), it follows that a left-invariant vector field \( X = (x_1, x_2, x_3) \) is a Cotton soliton if and only if

\[
\begin{align*}
\gamma x_3 &= 0, \\
\alpha x_1 + \gamma x_2 + \beta x_3 &= 0, \\
2\alpha x_1 + \gamma^3 - \lambda &= 0, \\
\gamma^3 - 2\delta x_3 + 2\lambda &= 0, \\
\beta \gamma^2 + 4\beta x_1 + 4\delta x_2 &= 0.
\end{align*}
\]  

(25)

Recall that \( \alpha \gamma = 0 \) and \( \alpha + \delta \neq 0 \). First, if \( \alpha = \gamma = 0 \), then the Lie group is flat and hence the left-invariant Cotton soliton is trivial. Now, if \( \alpha = 0 \) and \( \gamma \neq 0 \), then the first equation in (25) implies that \( x_3 = 0 \), and therefore from the third and fourth equations in (25), we easily obtain that \( \gamma = 0 \), which is a contradiction. Finally, if \( \gamma = 0 \) and \( \alpha \neq 0 \), then the Lie group is locally conformally flat and therefore the left-invariant Cotton soliton is trivial. \( \square \)

Remark 12. In the Riemannian setting, the unimodular Lie groups correspond to type Ia (just considering the usual cross product induced by the quaternions), while the non-unimodular case corresponds to type \( \mathcal{S} \) and type IV.2, as previously discussed [19].

With regard to types Ia and IV.2, the behavior is exactly the same in both the Riemannian and the Lorentzian cases, since theorems 3 and 10 remain true in the Riemannian setting. Thus, since locally symmetric spaces and Lie groups of type \( \mathcal{S} \) are locally conformally flat, one has that three-dimensional homogeneous Riemannian manifolds do not admit any non-trivial left-invariant Cotton soliton. Hence, any left-invariant Riemannian Cotton soliton is a Yamabe soliton, and in the Riemannian signature this implies the flatness of the manifold.

Let \( H_3 \) be the Heisenberg group and consider on \( H_3 \) the left-invariant metric given by \( g = dx^2 + dy^2 + (dz - xdy)^2 \). Proceeding as in section 5, it is obtained that \( (H_3, g) \) is a shrinking non-invariant Riemannian Cotton soliton which is not trivial.

Proof of theorem 2. A careful examination of the cases obtained in theorems 5 and 7 shows that the corresponding Cotton operator, \( \hat{C} \), is two-step nilpotent for unimodular Lorentzian Lie groups of type II with \( \alpha = \beta \neq 0 \) or \( \alpha = 0 \neq \beta \); indeed,

\[
\hat{C} = \Xi \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix},
\]

where \( \Xi = \frac{3\alpha^2}{\alpha^2 + 2\beta^2} \) if \( \alpha = \beta \neq 0 \), or \( \Xi = 2\beta^2 \) if \( \alpha = 0 \neq \beta \). Moreover, the degree of nilpotency is 3 for unimodular Lorentzian Lie groups of type III with \( \alpha \neq 0 \), since in this case we have

\[
\hat{C} = 3\alpha \begin{pmatrix} 0 & \alpha/2 & \alpha/2 \\ \alpha/2 & 1 & 1 \\ -\alpha/2 & -1 & -1 \end{pmatrix}.
\]

Conversely, a case by case examination of the Cotton operator in the different unimodular and non-unimodular Lorentzian Lie groups shows, after a long but straightforward calculation, that the cases with nilpotent Cotton operator are precisely those obtained in theorems 5 and 7. \( \square \)
Proposition 13. Let \((G, g)\) be a simply connected Lie group endowed with a left-invariant Lorentzian metric \(g\). If \((G, g)\) is an algebraic Cotton soliton, then it satisfies equation (26), then it is a Cotton soliton such that a vector field solving equation (2) is given by

\[
\hat{C} = \lambda \text{Id} + D,
\]

where \(\hat{C}\) stands for the Cotton operator \((g(\hat{C}(X), Y) = C(X, Y))\), \(\lambda\) is a real constant and \(D \in \text{Der}(g)\), i.e.,

\[
D[X, Y] = [DX, Y] + [X, DY]
\]

for all \(X, Y \in g\).

Next, we show that the algebraic Cotton soliton condition is stronger than the Cotton soliton one.

**Proposition 13.** Let \((G, g)\) be a simply connected Lie group equipped with a left-invariant Lorentzian metric \(g\). If \((G, g)\) is an algebraic Cotton soliton, i.e., it satisfies equation (26), then it is a Cotton soliton such that a vector field solving equation (2) is given by

\[
X = \frac{d}{dt} \bigg|_{t=0} \varphi_t(p), \quad \text{with} \quad d\varphi_t|_e = \exp \left( \frac{t}{2} D \right),
\]

where \(e\) denotes the identity element of \(G\).

**Proof.** Let \((G, g)\) be an algebraic Cotton soliton, i.e., \(\hat{C} = \lambda \text{Id} + D\), let \(\{e_i\}\) denote a pseudo-orthonormal basis and define a one-parameter family of automorphisms \(\varphi_t\) by setting \(d\varphi_t|_e = \exp \left( \frac{t}{2} D \right)\). Considering the vector field \(X\) given by \(X = \frac{d}{dt} \bigg|_{t=0} \varphi_t(p)\) a direct calculation shows that the Lie derivative in the direction of \(X\) is given by

\[
(L_X g)(e_i, e_j) = \frac{d}{dt} \bigg|_{t=0} g(e_i, e_j) = \frac{1}{2} \left( g(D e_i, e_j) + g(e_i, D e_j) \right),
\]

for \(i, j \in \{1, 2, 3\}\). Then,

\[
C(e_i, e_j) = \frac{1}{2} \left( g(\hat{C}(e_i), e_j) + g(e_i, \hat{C}(e_j)) \right)
\]

\[
= \frac{1}{2} \left( g((\lambda \text{Id} + D)(e_i), e_j) + g(e_i, (\lambda \text{Id} + D)(e_j)) \right)
\]

\[
= \lambda g(e_i, e_j) + (L_X g)(e_i, e_j),
\]

which shows that \((G, g, X)\) is a Cotton soliton.

In view of the proposition above, it seems natural to ask whether a Cotton soliton on a Lie group comes from an algebraic Cotton soliton or not. The rest of this section is devoted to clarifying this question. In this sense, the following theorem shows how algebraic Cotton solitons allow the construction of non-invariant Cotton solitons. The corresponding Lorentzian Lie groups are obtained from the classification in [6, theorem 4.1].

**Theorem 14.** A three-dimensional Lie group \((G, g)\) equipped with a left-invariant Lorentzian metric is an algebraic Cotton soliton if and only if one of the following conditions holds:

(i) \((G, g)\) is of type Ia with \(\alpha = \beta = 0\) and \(\gamma \neq 0\), or any cyclic permutation. In this case, \(\lambda = -2\gamma^3\) and \(G\) is locally isometric to the Heisenberg group \(H_3\).
\((ii)\) \((g, g)\) is of type \(Ib\) with \(\alpha = 0\) and \(\gamma = \frac{1+\sqrt{2}}{2}\). In this case, \(\lambda = 2\varepsilon\sqrt{2}\beta^3\) with \(\varepsilon^2 = 1\), and \(G\) is locally isometric to \(E(1, 1)\).

**Proof.** We analyze the existence of algebraic Cotton solitons on a three-dimensional Lorentzian Lie algebra. To do this, it is sufficient to study when an operator \(D\) of the form
\[
D = \tilde{C} - \lambda \text{Id}
\]
is a derivation on a Lie algebra \(g\).

We start with type \(Ia\). From equation (4) it is easy to obtain that \(D = \tilde{C} - \lambda \text{Id}\) is a derivation if and only if
\[
\begin{align*}
\gamma (\alpha^3 - \alpha^2 \beta - \alpha (\beta^2 - \gamma^2) + \beta^3 + \beta \gamma^2 - 2\gamma^3 - \lambda) &= 0, \\
\beta (\alpha^3 - \alpha^2 \gamma - \alpha (\gamma^2 - \beta^2) - 2\beta^3 + \beta^2 \gamma + \gamma^3 - \lambda) &= 0, \\
\alpha (2\alpha^3 - \alpha^2 (\beta + \gamma) - \beta^3 + \beta^2 \gamma + \beta \gamma^2 - \gamma^3 - \lambda) &= 0.
\end{align*}
\]
(27)

If all the structure constants vanish, then the Lie algebra is Abelian and hence the algebraic Cotton soliton is trivial. Hence, we assume that at least one of them is non-zero, for instance \(\gamma \neq 0\). Thus, from the first equation in (27), we obtain that \(\lambda = \gamma^2 (\alpha + \beta) + (\alpha - \beta)^2 (\alpha + \beta) - 2\gamma^3\).

Then, equation (27) becomes
\[
\begin{align*}
\beta (\beta - \gamma) (\alpha^2 + 2\alpha (\beta + \gamma) - 3\beta^2 - 2\beta \gamma - 3\gamma^2) &= 0, \\
\alpha (\alpha - \gamma) (3\alpha^2 + 2\alpha (\gamma - \beta) - (\beta - \gamma) (\beta + 3\gamma)) &= 0.
\end{align*}
\]
(28)

A first non-trivial solution is obtained taking \(\alpha = \beta = 0\) and \(\lambda = -2\gamma^3\). For \(\alpha \neq 0\) and \(\beta = 0\), a straightforward calculation shows that \(\alpha = \gamma\), and in this case the manifold is locally conformally flat. The same conclusion is obtained for \(\alpha = 0\) and \(\beta \neq 0\). Finally, assume \(\alpha \neq 0 \neq \beta\). If \(\alpha = \gamma, \beta = \gamma\) or \(\alpha = \beta\), equation (28) implies that \(\alpha = \beta = \gamma\) and therefore the sectional curvature is constant; thus the manifold is locally conformally flat. If \(\alpha \neq \beta \neq \gamma \neq \alpha\), then equation (28) reduces to
\[
\begin{align*}
\alpha^2 + 2\alpha (\beta + \gamma) - 3\beta^2 - 2\beta \gamma - 3\gamma^2 &= 0, \\
3\alpha^2 + 2\alpha (\gamma - \beta) - (\beta - \gamma) (\beta + 3\gamma) &= 0.
\end{align*}
\]
(29)

From this last equation, we obtain \((\alpha - \beta)^2 + 3\gamma^2 = 0\), which is a contradiction. Proceeding in an analogous way for \(\alpha \neq 0\) or \(\beta \neq 0\) (i) is obtained.

We now consider the type \(Ib\). From equation (7), and taking into account that \(\beta \neq 0\), it follows that \(D = \tilde{C} - \lambda \text{Id}\) is a derivation if and only if
\[
\begin{align*}
\alpha^3 + 4\alpha \gamma^2 + 8\beta^2 \gamma - 8\gamma^3 - \lambda &= 0, \\
\alpha^2 + 4\alpha \gamma + 4\beta^2 - 8\gamma^2 &= 0, \\
\alpha \lambda &= 0.
\end{align*}
\]
(30)

The last equation in (30) implies that either \(\alpha = 0\) or \(\lambda = 0\). First, suppose that \(\alpha = 0\). Thus, equation (30) becomes
\[
\begin{align*}
8\beta^2 \gamma - 8\gamma^3 - \lambda &= 0, \\
\beta^2 - 2\gamma^2 &= 0.
\end{align*}
\]
(31)

from which (ii) is obtained. We now assume that \(\lambda = 0\) and \(\alpha \neq 0\). Then, equation (30) becomes
\[
\begin{align*}
8\gamma^3 - \alpha^3 - 4\alpha \gamma^2 - 8\beta^2 \gamma &= 0, \\
\alpha^2 + 4\alpha \gamma + 4\beta^2 - 8\gamma^2 &= 0.
\end{align*}
\]
(32)

A long but straightforward calculation shows that \(\alpha = -2\gamma\) and \(\beta = \pm \sqrt{3}\gamma\), and in this case the manifold is locally conformally flat.

The remaining types of three-dimensional Lorentzian Lie algebras do not provide non-trivial algebraic Cotton solitons. The proof is obtained by proceeding as in the previous cases. We omit the details for the sake of brevity since the calculations are standard. \(\blacksquare\)
Remark 15. Case (i) in theorem 14 is the Lie algebra associated with the Heisenberg group, while case (ii) corresponds to the Lie algebra associated with the solvable Lie group $E(1, 1)$. An analogous treatment can be performed in the Riemannian case to show that only the Riemannian Heisenberg group admits algebraic Cotton solitons. In this sense, in [7] the authors obtained the Cotton soliton associated with the Riemannian algebraic Cotton soliton on the Heisenberg group.

Remark 16. Theorem 14 highlights that left-invariant Cotton solitons do not come from algebraic Cotton solitons.

5. Non-invariant Lorentzian Cotton solitons

In this section, motivated by the existence of algebraic Cotton solitons, we show the existence of non-invariant Cotton solitons on three-dimensional homogeneous Lorentzian manifolds which are non-trivial. To do this, we consider the algebraic Cotton solitons obtained in theorem 14. Note that theorem 2 implies that any Lie group admitting a left-invariant Cotton soliton has necessarily the nilpotent Cotton operator and this condition does not hold for the Lie groups obtained in theorem 14. Thus, the Lie groups admitting algebraic Cotton solitons provide non-invariant examples of Cotton solitons.

5.1. Non-invariant Cotton solitons on the Heisenberg group

We start with the Lorentzian Heisenberg group. In [24], it is shown that this Lie group can be endowed with three different left-invariant Lorentzian metrics, up to isometry and scaling, given by

\begin{align*}
g_1 &= -dx^2 + dy^2 + (xy + dz)^2, \\
g_2 &= dx^2 + dy^2 - (xy + dz)^2, \\
g_3 &= dx^2 + (xy + dz)^2 - ((1 - x) dy - dz)^2.
\end{align*}

Metric $g_3$ is flat [20] and hence a trivial Cotton soliton, and metrics $g_1$ and $g_2$ are shrinking non-gradient Ricci solitons [21, 22]. Next, we analyze metrics $g_1$ and $g_2$ separately. Starting with the metric $g_1$, for a vector field $X = A(x, y, z) \partial_x + B(x, y, z) \partial_y + C(x, y, z) \partial_z$, the Lie derivative of the metric is given by

\begin{align*}
(L_X g_1)(\partial_x, \partial_x) &= -2A_x, \\
(L_X g_1)(\partial_y, \partial_y) &= -A_y + (x^2 + 1)B_y + xC_y, \\
(L_X g_1)(\partial_z, \partial_z) &= -A_z + xB_z + C_z, \\
(L_X g_1)(\partial_x, \partial_y) &= 2(xA + (x^2 + 1)B_y + xC_y), \\
(L_X g_1)(\partial_x, \partial_z) &= A + xB_y + (x^2 + 1)B_z + C_y + xC_z, \\
(L_X g_1)(\partial_y, \partial_z) &= 2(xB_z + C_z),
\end{align*}

and the $(0, 2)$-Cotton tensor is determined by

\begin{align*}
C(\partial_x, \partial_x) &= -\frac{1}{2}, \\
C(\partial_y, \partial_y) &= \frac{1}{2} - x^2, \\
C(\partial_z, \partial_z) &= -x, \\
C(\partial_x, \partial_y) &= -1.
\end{align*}
Now, equations (33) and (34) imply that $X$ is a Cotton soliton if and only if
\[
\begin{align*}
4A_x + 1 - 2\lambda &= 0, \\
2xB_x + 2C_x - 1 - \lambda &= 0, \\
A_x - xB_x - C_x &= 0, \\
A_y - (x^2 + 1)B_x - xC_x &= 0, \\
A + xB_y + (x^2 + 1)B_x + C_x - x - x\lambda &= 0, \\
4xA + 4(x^2 + 1)B_x + 4xC_y - 2x^2 + 1 - 2(x^2 + 1)\lambda &= 0.
\end{align*}
\tag{35}
\]

We start integrating first and second equations in (35) to obtain
\[
A(x, y, z) = A_1(y, z) + \frac{1}{2}(2\lambda - 1)x, \\
C(x, y, z) = C_1(x, y) - B(x, y, z)x + \frac{1}{2}(\lambda + 1)z.
\]

Now, the third equation in (35) transforms into
\[
B(x, y, z) = (C_1)_x(x, y) - (A_1)_z(y, z),
\]

and hence differentiating the fourth equation in (35) with respect to $x$ and $z$, we obtain $(A_1)_zz = 0$, and thus
\[
A_1(y, z) = A_2(y)z + A_3(y).
\]

Next, differentiating the fifth equation in (35) with respect to $z$, we obtain $A_2(y) = 0$ and the fourth equation in (35) transforms into $(C_1)_zz(x, y) - A'_1(y) = 0$, from which it follows that
\[
C_1(x, y) = \frac{1}{2}A'_1(y)x^2 + C_2(y)x + C_3(y).
\]

At this point, the fifth equation in (35) reduces to
\[
4A_1(y) + 4C'_1(y) + (2A'_1(y)x + 4C'_2(y) - 3)x = 0
\]

and from this equation a straightforward calculation shows that
\[
A_3(y) = \kappa_1 y + \kappa_2, \quad C_2(y) = \frac{3}{4}y + \kappa_3, \quad C_3(y) = -\frac{\kappa_1}{2}y^2 - \kappa_2 y + \kappa_4,
\]

where $\kappa_i \in \mathbb{R}$, $i = 1, \ldots, 4$. Thus, equation (35) finally reduces to $\lambda - 2 = 0$ and we conclude that $(H_3, g_1)$ is a Cotton soliton if and only if it is shrinking with $\lambda = 2$ and $X$ given by
\[
X = \left(\kappa_1 y + \frac{3}{4}y + \kappa_2\right)\partial_x + \left(\kappa_1 x + \frac{3}{4}y + \kappa_3\right)\partial_y - \left(\frac{\kappa_1}{2}(x^2 + y^2) + \kappa_2 y - \frac{3}{2}z - \kappa_4\right)\partial_z.
\]

Note that as a special case of the Cotton solitons obtained above, one has that $X = \frac{3}{2}x\partial_x + \frac{3}{2}y\partial_y + \frac{3}{2}z\partial_z$, defines a complete Cotton soliton on $(H_3, g_2)$.

Now, it is easy to see that the associated Cotton operator, $\hat{\mathcal{C}}$, is characterized by
\[
\hat{\mathcal{C}}(\partial_x) = \frac{1}{2}\partial_x, \quad \hat{\mathcal{C}}(\partial_y) = \frac{1}{2}\partial_y - \frac{3}{2}\partial_z, \quad \hat{\mathcal{C}}(\partial_z) = -\partial_z.
\]

As a consequence, it diagonalizes with eigenvalues $\{-1, \frac{1}{2}, \frac{1}{2}\}$ and hence the Cotton soliton cannot be left-invariant (see theorem 2). Moreover, the Ricci operator, $\hat{\rho}$, is given by
\[
\hat{\rho}(\partial_x) = \frac{1}{2}\partial_x, \quad \hat{\rho}(\partial_y) = \frac{1}{2}\partial_y - x\partial_z, \quad \hat{\rho}(\partial_z) = -\frac{1}{4}\partial_z,
\]

and thus it does not have any zero eigenvalues, which implies that the soliton is not gradient [7].

**Remark 17.** A completely analogous study can be developed with the metric $g_2$ to obtain that $(H_3, g_2)$ is a Cotton soliton if and only if it is expanding with $\lambda = -2$ and the soliton given by
\[
X = \left(\kappa_1 y - \frac{3}{4}y + \kappa_2\right)\partial_x + \left(\kappa_1 x - \frac{3}{4}y - \kappa_3\right)\partial_y + \left(\frac{\kappa_1}{2}(x^2 - y^2) - \kappa_2 y + \frac{3}{2}z + \kappa_4\right)\partial_z,
\]

where $\kappa_i \in \mathbb{R}$, $i = 1, \ldots, 4$. Again the Cotton soliton is not left-invariant and it is not gradient.
5.2. Non-invariant Cotton solitons on $E(1, 1)$

The Lie group $E(1, 1)$ admits the two kinds of algebraic Cotton solitons (up to some sign $\varepsilon = \pm 1$, see theorem 14) and it can therefore be equipped with a left-invariant metric with a non-nilpotent Cotton operator admitting some Cotton soliton. Moreover, these Cotton solitons will be necessarily non-invariant (see theorem 2).

For our purpose, we consider the frame [22]

$$E_1 = \partial_x, \quad E_2 = \frac{1}{2}(e^\varepsilon \partial_y + e^{-\varepsilon} \partial_z), \quad E_3 = \frac{1}{2}(e^\varepsilon \partial_y - e^{-\varepsilon} \partial_z),$$

and the associated coframe given by

$$E^1 = dx, \quad E^2 = e^{-\varepsilon}dy + e^\varepsilon dz, \quad E^3 = e^{-\varepsilon}dy - e^\varepsilon dz.$$

Henceforth, we consider left-invariant metrics corresponding to theorem 14(ii). First, we take the metric $g_1$ given by

$$g_1 = \frac{1}{2}(E^1)^2 - (E^2)^2 + 2\sqrt{6}(E^2 E^3) - 3(E^3)^2,$$

or in local coordinates

$$g_1 = \frac{1}{2}dx^2 + (2\sqrt{6} - 4)e^{-2\varepsilon}dy^2 - (2\sqrt{6} + 4)e^{2\varepsilon}dz^2 + 4dydz.$$

Now, for a vector field $X = A(x, y, z)\partial_x + B(x, y, z)\partial_y + C(x, y, z)\partial_z$, the Lie derivative of the metric is given by

$$(\mathcal{L}_X g_1)(\partial_y, \partial_z) = \frac{4}{2}A_x,$$

$$(\mathcal{L}_X g_1)(\partial_x, \partial_y) = \frac{1}{2}(A_y + 3(\sqrt{6} - 2)e^{-2\varepsilon}B_x + 3C_x),$$

$$(\mathcal{L}_X g_1)(\partial_y, \partial_z) = \frac{1}{2}A_z + 3B_x - 3(\sqrt{6} + 2)e^{2\varepsilon}C_x,$$

$$(\mathcal{L}_X g_1)(\partial_x, \partial_z) = 4C_x - 4(\sqrt{6} - 2)e^{-2\varepsilon}(A - B_y),$$

$$(\mathcal{L}_X g_1)(\partial_y, \partial_z) = 2((\sqrt{6} - 2)e^{-2\varepsilon}B_z + B_y + C_z - (\sqrt{6} + 2)e^{2\varepsilon}C_y),$$

$$(\mathcal{L}_X g_1)(\partial_z, \partial_x) = 4B_z - 4(\sqrt{6} + 2)e^{2\varepsilon}(A + C_x),$$

and a straightforward calculation shows that $X = \frac{3\varepsilon}{\sqrt{6}}\partial_y + \frac{3\varepsilon}{\sqrt{6}}\partial_z$ is a shrinking Cotton soliton with $\lambda = 2\sqrt{2}$. Moreover, due to the non-existence of homothetic vector fields on $E(1, 1)$ it follows that this is the unique shrinking Cotton soliton up to Killing vector fields [8].

Next, we consider the Lorentzian metric $g_2$ on $E(1, 1)$ given by

$$g_2 = \frac{1}{2}(E^1)^2 - 3(E^2)^2 + 2\sqrt{6}(E^2 E^3) - (E^3)^2,$$

or in local coordinates

$$g_2 = \frac{1}{2}dx^2 + (2\sqrt{6} - 4)e^{-2\varepsilon}dy^2 - (2\sqrt{6} + 4)e^{2\varepsilon}dz^2 - 4dydz.$$

Proceeding as in the previous case we obtain that $X = \frac{3\varepsilon}{\sqrt{6}}\partial_y - \frac{3\varepsilon}{\sqrt{6}}\partial_z$ is a expanding Cotton soliton for $\lambda = -2\sqrt{2}$. Furthermore, it is the unique expanding Cotton soliton up to Killing vector fields due to the results in [8].

**Remark 18.** As a final remark, it is worth pointing out once again that the non-invariant Cotton solitons constructed in this section come from algebraic Cotton solitons obtained in theorem 14. Moreover, let us emphasize that algebraic Cotton solitons have an opposite behavior with respect to invariant Cotton solitons on Lie groups since no left-invariant Cotton soliton can be obtained from an algebraic one.

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References

[1] Aliev A N and Nutku Y 1996 A theorem on topologically massive gravity *Class. Quantum Grav.* **13** L29–L32
[2] Kissel A U O, Sarıoğlu O and Tekin B 2008 Cotton flow *Class. Quantum Grav.* **25** 165019
[3] Bini D, Jantzen R T and Minniti G 2001 The Cotton, Simon–Mars and Cotton–York tensors in stationary spacetimes *Class. Quantum Grav.* **18** 4969–81
[4] Brozos-Vázquez M, Calvaruso G, García-Río E and Gavino-Fernández S 2012 Three-dimensional Lorentzian homogeneous Ricci solitons *Isr. J. Math.* **188** 385–403
[5] Calvaruso G 2007 Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds *Geom. Dedicata* **127** 99–119
[6] Calvaruso G 2007 Homogeneous structures on three-dimensional Lorentzian manifolds *J. Geom. Phys.* **57** 1279–91
[7] Calviño-Louzao E, García-Río E and Vázquez-Lorenzo R 2012 A note on compact Cotton solitons *Class. Quantum Grav.* **29** 205014
[8] Calviño-Louzao E, Seoane-Bascoy J, Vázquez-Abal M E and Vázquez-Lorenzo R 2012 Three-dimensional homogeneous Lorentzian Yamabe solitons *Abh. Math. Semin. Univ. Hamburg* **82** 193–203
[9] Chow D D K, Pope C N and Sezgin E 2010 Classification of solutions in topologically massive gravity *Class. Quantum Grav.* **27** 105001
[10] Cordero L A and Parker P E 1997 Left-invariant Lorentzian metrics on 3-dimensional Lie groups *Rend. Mat. Appl.* **17** 129–55
[11] Ferreiro-Pérez R 2010 Conserved current for the Cotton tensor, black hole entropy and equivariant Pontryagin forms *Class. Quantum Grav.* **27** 135015
[12] García A A, Hehl F W, Hemitke C and Macias A 2004 The Cotton tensor in Riemannian spacetimes *Class. Quantum Grav.* **21** 1099–118
[13] Guven J 2007 Chru–Simons theory and three-dimensional surfaces *Class. Quantum Grav.* **24** 1833–40
[14] Hall G S 1989 The global extension of local symmetries in general relativity *Class. Quantum Grav.* **6** 157–61
[15] Hall G S and Capocci M S 1999 Classification and conformal symmetry in three-dimensional space-times *J. Math. Phys.* **40** 1466–78
[16] Kroon J A Valiente 2004 Asymptotic expansions of the Cotton–York tensor on slices of stationary spacetimes *Class. Quantum Grav.* **21** 3237–49
[17] Lashkari N and Maloney A 2011 Topologically massive gravity and Ricci–Cotton flow *Class. Quantum Grav.* **28** 105007
[18] Lauret J 2001 Ricci soliton homogeneous nilmanifolds *Math. Ann.* **319** 715–33
[19] Milnor J 1976 Curvature of left invariant metrics on Lie groups *Adv. Math.* **21** 293–329
[20] Nomizu K 1979 Left-invariant Lorentz metrics on Lie groups *Osaka J. Math.* **16** 143–50
[21] Onda K 2010 Lorentz Ricci solitons on 3-dimensional Lie groups *Geom. Dedicata* **147** 313–22
[22] Onda K 2011 Examples of algebraic Ricci solitons in the pseudo-Riemannian case arXiv:1112.0424v3
[23] Rahmani S 1992 Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension trois *J. Geom. Phys.* **9** 295–302
[24] Rahmani N and Rahmani S 2006 Lorentzian geometry of Heisenberg group *Geom. Dedicata* **118** 133–40
[25] Sousa F C, Fonseca J B and Romero C 2008 Equivalence of three-dimensional spacetimes *Class. Quantum Grav.* **25** 035007