Good upper bounds for the
total rainbow connection of graphs*

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Abstract

A total-colored graph is a graph $G$ such that both all edges and all vertices of $G$ are colored. A path in a total-colored graph $G$ is a total rainbow path if its edges and internal vertices have distinct colors. A total-colored graph $G$ is total-rainbow connected if any two vertices of $G$ are connected by a total rainbow path of $G$. The total rainbow connection number of $G$, denoted by $trc(G)$, is defined as the smallest number of colors that are needed to make $G$ total-rainbow connected. These concepts were introduced by Liu et al. Notice that for a connected graph $G$, $2diam(G) - 1 \leq trc(G) \leq 2n - 3$, where $diam(G)$ denotes the diameter of $G$ and $n$ is the order of $G$. In this paper we show, for a connected graph $G$ of order $n$ with minimum degree $\delta$, that $trc(G) \leq 6n/(\delta + 1) + 28$ for $\delta \geq \sqrt{n-2} - 1$ and $n \geq 291$, while $trc(G) \leq 7n/(\delta + 1) + 32$ for $16 \leq \delta \leq \sqrt{n-2} - 2$ and $trc(G) \leq 7n/(\delta + 1) + 4C(\delta) + 12$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{\frac{21\log(\delta^3 + 2\delta^2 + 2\delta + 3) - 3(\log 3 - 1)}{\delta - 3}} - 2$.

This implies that when $\delta$ is in linear with $n$, then the total rainbow number $trc(G)$ is a constant. We also show that $trc(G) \leq 7n/4 - 3$ for $\delta = 3$, $trc(G) \leq 8n/5 - 13/5$ for $\delta = 4$ and $trc(G) \leq 3n/2 - 3$ for $\delta = 5$. Furthermore, an example shows that our bound can be seen tight up to additive factors when $\delta \geq \sqrt{n-2} - 1$.

Keywords: total-colored graph; total rainbow connection; minimum degree; 2-step dominating set.

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1 Introduction

In this paper, all graphs considered are simple, finite and undirected. We refer to book [2] for undefined notation and terminology in graph theory. Let $G$ be a connected graph on $n$ vertices with minimum degree $\delta$. A path in an edge-colored graph $G$ is a rainbow path if its edges have different colors. An edge-colored graph $G$ is rainbow connected if any two vertices of $G$ are connected by a rainbow path of $G$. The rainbow connection number, denoted by $rc(G)$, is defined as the smallest number of colors required to make $G$ rainbow connected. Chartrand et al. [6] introduced these concepts. Notice that $rc(G) = 1$ if and only if $G$ is a complete graph and that $rc(G) = n - 1$ if and only if $G$ is a tree. Moreover, $diam(G) \leq rc(G) \leq n - 1$. A lot of results on the rainbow connection have been obtained; see [13, 14].

From [4] we know that to compute the number $rc(G)$ of a connected graph $G$ is NP-hard. So, to find good upper bounds is an interesting problem. Krivelevich and Yuster [11] obtained that $rc(G) \leq 20n/\delta$. Caro et al. [3] obtained that $rc(G) \leq \ln n (1 + o(1))$. Finally, Chandran et al. [5] got the following benchmark result.

Theorem 1. [5] For every connected graph $G$ of order $n$ and minimum degree $\delta$, $rc(G) \leq 3n/(\delta + 1) + 3$.

The concept of rainbow vertex-connection was introduced by Krivelevich and Yuster in [11]. A path in a vertex-colored graph $G$ is a vertex-rainbow path if its internal vertices have different colors. A vertex-colored graph $G$ is rainbow vertex-connected if any two vertices of $G$ are connected by a vertex-rainbow path of $G$. The rainbow vertex-connection number, denoted by $rvc(G)$, is defined as the smallest number of colors required to make $G$ rainbow vertex-connected. Observe that $diam(G) - 1 \leq rvc(G) \leq n - 2$ and that $rvc(G) = 0$ if and only if $G$ is a complete graph. The problem of determining the number $rvc(G)$ of a connected graph $G$ is also NP-hard; see [7, 8]. There are a few results about the upper bounds of the rainbow vertex-connection number. Krivelevich and Yuster [11] proved that $rvc(G) \leq 11n/\delta$. Li and Shi [12] improved this bound and showed the following results.

Theorem 2. [12] For a connected graph $G$ of order $n$ and minimum degree $\delta$, $rvc(G) \leq 3n/4 - 2$ for $\delta = 3$, $rvc(G) \leq 3n/5 - 8/5$ for $\delta = 4$ and $rvc(G) \leq n/2 - 2$ for $\delta = 5$. For sufficiently large $\delta$, $rvc(G) \leq (b \ln \delta)n/\delta$, where $b$ is any constant exceeding 2.5.

Theorem 3. [12] A connected graph $G$ of order $n$ with minimum degree $\delta$ has $rvc(G) \leq 3n/(\delta + 1) + 5$ for $\delta \geq \sqrt{n - 1} - 1$ and $n \geq 290$, while $rvc(G) \leq 4n/(\delta + 1) + 5$ for $16 \leq \delta \leq \sqrt{n - 1} - 2$ and $rvc(G) \leq 4n/(\delta + 1) + C(\delta)$ for $6 \leq \delta \leq 15$, where $C(\delta) = e^{3 \log(26^2 + 3) - 3 \log(3 - 1) - 2}$. 

2
Recently, Liu et al. [15] proposed the concept of total rainbow connection. A total-colored graph is a graph $G$ such that both all edges and all vertices of $G$ are colored. A path in a total-colored graph $G$ is a total rainbow path if its edges and internal vertices have distinct colors. A total-colored graph $G$ is total-rainbow connected if any two vertices of $G$ are connected by a total rainbow path of $G$. The total rainbow connection number, denoted by $trc(G)$, is defined as the smallest number of colors required to make $G$ total-rainbow connected. It is easy to observe that $trc(G) = 1$ if and only if $G$ is a complete graph. Moreover, $2diam(G) - 1 \leq trc(G) \leq 2n - 3$. The following proposition gives an upper bound of the total rainbow connection number.

**Proposition 1.** [15] Let $G$ be a connected graph on $n$ vertices and $q$ vertices having degree at least 2. Then, $trc(G) \leq n - 1 + q$, with equality if and only if $G$ is a tree.

From Theorem 1 and 3, one can see that $rc(G)$ and $rvc(G)$ are bounded by a function of the minimum degree $\delta$, and that when $\delta$ is in linear with $n$, then both $rc(G)$ and $rvc(G)$ are some constants. In this paper, we will use the same idea in [12] to obtain upper bounds for the number $trc(G)$, which are also functions of $\delta$ and imply that when $\delta$ is in linear with $n$, then $trc(G)$ is a constant.

## 2 Main results

Let $G$ be a connected graph on $n$ vertices with minimum degree $\delta$. Denote by $Leaf(G)$ the maximum number of leaves in any spanning tree of $G$. If $\delta = 3$, then $Leaf(G) \geq n/4 + 2$ which was proved by Linial and Sturtevant (unpublished). In [8, 10], it was proved that $Leaf(G) \geq 2n/5 + 8/5$ for $\delta = 4$. Moreover, Griggs and Wu [11] showed that if $\delta = 5$, then $Leaf(G) \geq n/2 + 2$. For sufficiently large $\delta$, $Leaf(G) \geq (1 - b \ln \delta/\delta)n$, where $b$ is any constant exceeding 2.5, which was proved in [10]. Thus, we can get the following results.

**Theorem 4.** For a connected graph $G$ of order $n$ with minimum degree $\delta$, $trc(G) \leq 7n/4 - 3$ for $\delta = 3$, $trc(G) \leq 8n/5 - 13/5$ for $\delta = 4$ and $trc(G) \leq 3n/2 - 3$ for $\delta = 5$. For sufficiently large $\delta$, $trc(G) \leq (1 + b \ln \delta/\delta)n - 1$, where $b$ is any constant exceeding 2.5.

**Proof.** We can choose a spanning tree $T$ with the most leaves. Denote $\ell$ the maximum number of leaves. Then color all non-leaf vertices and all edges of $T$ with $2n - \ell - 1$ colors, each receiving a distinct color. Hence, $trc(G) \leq 2n - \ell - 1$. \qed

**Theorem 5.** For a connected graph $G$ of order $n$ with minimum degree $\delta$, $trc(G) \leq 6n/(\delta + 1) + 28$ for $\delta \geq \sqrt{n - 2} - 1$ and $n \geq 291$, while $trc(G) \leq 7n/(\delta + 1) + 32$ for
16 ≤ δ ≤ \sqrt{n - 2} - 2 and trc(G) ≤ 7n/(δ + 1) + 4C(δ) + 12 for 6 ≤ δ ≤ 15, where
\[ C(δ) = e^{\frac{3 \log(n^4 + 2δ^2 + 3) - 31\log(3 - 1)}{4}} - 2. \]

Remark 1. The same example mentioned in [3] can show that our bound is tight up to additive factors when δ ≥ \sqrt{n - 2} - 1.

In order to prove Theorem 5, we need some lemmas.

**Lemma 1.** [11] If G is a connected graph of order n with minimum degree δ, then it has a connected spanning subgraph with minimum degree δ and with less than n(δ + 1/(δ + 1)) edges.

Given a graph G, a set D ⊆ V(G) is called a 2-step dominating set of G if every vertex of G which is not dominated by D has a neighbor that is dominated by D. A 2-step dominating set S is k-strong if every vertex which is not dominated by S has at least k neighbors that are dominated by S. If S induces a connected subgraph of G, then S is called a connected k-strong 2-step dominating set. These concepts can be found in [11].

**Lemma 2.** [12] If G is a connected graph of order n with minimum degree δ ≥ 2, then G has a connected δ/3强 2-step dominating set S whose size is at most 3n/(δ + 1) − 2.

**Lemma 3.** [1] (Lovász Local Lemma) Let A₁, A₂, ..., Aₙ be the events in an arbitrary probability space. Suppose that each event Aᵢ is mutually independent of a set of all the other events Aⱼ but at most d, and that P[Aᵢ] ≤ p for all 1 ≤ i ≤ n. If ep(d + 1) < 1, then Pr[\bigcap_{i=1}^{n} \bar{A}_i] > 0. □

Now we are arriving at the point to give a proof for Theorem 5.

**Proof of Theorem 5:** The proof goes similarly for the main result of [12]. We are given a connected graph G of order n with minimum degree δ. Suppose that G has less than n(δ + 1/(δ + 1)) edges by Lemma 1. Let S denote a connected δ/3-strong 2-step dominating set of G. Then, we have |S| ≤ 3n/(δ + 1) − 2 by Lemma 2. Let N¹(S) denote the set of all vertices at distance exactly k from S. We give a partition to N¹(S) as follows. First, let H be a new graph constructed on N¹(S) with edge set \[ E(H) = \{ uv : u, v \in N¹(S), uv \in E(G) or \exists w \in N²(S) such that uwv is a path of G \}. \]

Let Z be the set of all isolated vertices of H. Moreover, there exists a spanning forest F of V(H)\Z. Finally, choose a bipartition defined by this forest, denoted by X and Y. Partition N²(S) into three subsets: A = \{ u \in N²(S) : u \in N(X) \cap N(Y) \}, B = \{ u \in N²(S) : u \in N(X)\backslash N(Y) \} and C = \{ u \in N²(S) : u \in N(Y)\backslash N(X) \}; see Figure 1(a).

**Case 1.** δ ≥ \sqrt{n - 2} - 1.
Next we give a coloring to the edges and vertices of $G$. Let $k = 2|S| - 1$ and $T$ be a spanning tree of $G[S]$. Color the edges and vertices of $T$ with $k$ distinct colors such that $G[S]$ is total rainbow connected. Assign every $[X, S]$ edge with color $k + 1$, every $[Y, S]$ edge with color $k + 2$ and every edge in $N^1(S)$ with color $k + 3$. Since the minimum degree $\delta \geq 2$, every vertex in $Z$ has at least two neighbors in $S$. Color one edge with $k + 1$ and all others with $k + 2$. Assign every $[A, X]$ edge with color $k + 3$, every $[A, Y]$ edge with color $k + 4$ and every vertex of $A$ with color $k + 5$. We assign seven new colors from $\{i_1, i_2, ..., i_7\}$ to the vertices of $X$ such that each vertex of $X$ chooses its color randomly and independently from all other vertices of $X$. Similarly, we assign another seven colors to the vertices of $Y$. Assign seven colors from $\{j_1, j_2, ..., j_7\}$ to the edges between $B$ and $X$ as follows: for every vertex $u \in B$, let $N_X(u)$ denote the set of all neighbors of $u$ in $X$; for every vertex $u' \in N_X(u)$, if we color $u'$ with $i_t$ ($t \in \{1, 2, ..., 7\}$), then color $uu'$ with $j_t$. In a similar way, we assign seven new colors to the edges between $C$ and $Y$. All other edges and vertices of $G$ are uncolored. Thus, the number of all colors we used is

$$k + 33 = 2|S| - 1 + 33 \leq 2\left(\frac{3n}{\delta + 1} - 2\right) - 1 + 33 = \frac{6n}{\delta + 1} + 28.$$ 

We have the following claim for any $u \in B (C)$.

**Claim 1.** For any $u \in B (C)$, we have a coloring for the vertices in $X (Y)$ with seven colors such that there exist two neighbors $u_1$ and $u_2$ in $N_X(u)$ ($N_Y(u)$) that receive different colors. Hence, the edges $uu_1$ and $uu_2$ are also colored differently.
Notice that for every vertex $v \in X$, $v$ has two neighbors in $S \cup A \cup Y$. Moreover, $(\delta + 1)^2 \geq n - 2$. Thus, $v$ has less than $(\delta + 1)^2$ neighbors in $B$. For every vertex $u \in B$, $u$ has at least $\delta/3$ neighbors in $X$ since $S$ is a connected $\delta/3$-strong 2-step dominating set of $G$. Let $A_u$ denote the event that $N_X(u)$ receives at least two distinct colors. Fix a set $X(u) \subset N_X(u)$ with $|X(u)| = \lceil \delta/3 \rceil$. Let $B_u$ denote the event that all vertices of $X(u)$ are colored the same. Hence, $Pr[B_u] \leq 7^{-\lceil \delta/3 \rceil + 1}$. Moreover, the event $B_u$ is independent of all other events $B_v$ for $v \neq u$ but at most $((\delta + 1)^2 - 1)\lceil \delta/3 \rceil$ of them. Since $e \cdot 7^{-\lceil \delta/3 \rceil + 1}(((\delta + 1)^2 - 1)\lceil \delta/3 \rceil + 1) < 1$, for all $\delta \geq \sqrt{n - 2} - 1$ and $n \geq 291$, we have $Pr[\bigwedge_{u \in B} \bar{B}_u] > 0$ by Lemma 5. Therefore, $Pr[A_u] > 0$.

We will show that $G$ is total-rainbow connected. Take any two vertices $u$ and $w$ in $V(G)$. If they are all in $S$, there is a total rainbow path connecting them in $G[S]$. If one of them is in $N^1(S)$, say $u$, then $u$ has a neighbor $u'$ in $S$. Thus, $uu'Pu$ is a required path, where $P$ is a total rainbow path in $G[S]$ connecting $u'$ and $w$. If one of them is in $X \cup Z$, say $u$, and the other is in $Y \cup Z$, say $w$, then $u$ has a neighbor $u'$ in $S$ and $w$ has a neighbor $w'$ in $S$. Hence, $uu'Pu_w$ is a required path, where $P$ is a total rainbow path connecting $u'$ and $w'$ in $G[S]$. If they are all in $X$, then there exists a $u' \in Y$ such that $u$ and $u'$ are connected by a single edge or a total rainbow path of length two. We know that $u'$ and $w$ are total-rainbow connected. Therefore, $u$ and $w$ are connected by a total rainbow path. If one of them is in $A \cup B$, say $u$, and the other is in $A \cup C$, say $w$, then $u$ has a neighbor $u'$ in $X$, and $w$ has a neighbor $w'$ in $Y$. Thus, they are total-rainbow connected. If they are all in $B$, by Claim 1 $u$ has two neighbors $u_1$ and $u_2$ in $X$ such that $u_1$, $u_2$, $uu_1$ and $uu_2$ are colored differently. Similarly, we also have that $w$ has two neighbors $w_1$ and $w_2$ in $X$ such that $w_1$, $w_2$, $ww_1$ and $ww_2$ are colored differently. Hence, $u$ and $w$ are total-rainbow connected. We can check that $u$ and $w$ are total-rainbow connected in all other cases.

**Case 2.** $6 \leq \delta \leq \sqrt{n - 2} - 2$.

We partition $X$ into two subsets $X_1$ and $X_2$. For any $u \in X$, if $u$ has at least $(\delta + 1)^2$ neighbors in $B$, then $u \in X_1$; otherwise, $u \in X_2$. Similarly, we partition $Y$ onto two subsets $Y_1$ and $Y_2$. Note that $|X_1 \cup Y_1| \leq n/(\delta + 1)$ since $G$ has less than $n(1 + 1/(\delta + 1))$ edges. Partition $B$ into two subsets $B_1$ and $B_2$. For any $u \in B$, if $u$ has at least one neighbor in $X_1$, then $u \in B_1$; otherwise, $u \in B_2$. In a similar way, we partition $C$ into two subsets $C_1$ and $C_2$; see Figure 1(b).

For $16 \leq \delta \leq \sqrt{n - 2} - 2$, assume that $C(\delta) = 5$; for $6 \leq \delta \leq 15$, assume that $C(\delta) = e^{(3 \log(\delta^3 + 2\delta^2 + 3) - 3(\log \delta - 1))} - 2$. Now we give a coloring to the edges and vertices of $G$. Let $k = 2|S| - 1$ and $T$ be a spanning tree of $G[S]$. Color the edges and vertices of $T$ with $k$ distinct colors. Assign every $[X, S]$ edge with color $k + 1$, every $[Y, S]$ edge with color $k + 2$ and every edge in $N^1(S)$ with color $k + 3$. Since every vertex in $Z$ has at least two neighbors in $S$, color one edge with $k + 1$ and all others with $k + 2$. Assign every
For every vertex \( u \in C \), similarly, we assign another \( X \) set. Thus, color the edge incident with \( X \) new colors to the vertices of \( Y \). Randomly and independently from all other vertices of \( C \), we have \( P \geq 2 \) -strong \( 2 \)-step dominating set of \( G \). Moreover, the event that all vertices of \( X \) are colored the same. Therefore, \( Pr[|X| \leq 1] \leq (C(\delta) + 2)^{-[\delta/3]+1} \). Moreover, the event \( B_u \) is independent of all other events \( B_v \) for \( v \neq u \) but at most \( ((\delta + 1)^2 - 1)[\delta/3] \) of them. Since \( e \cdot (C(\delta) + 2)^{-[\delta/3]+1}(((\delta + 1)^2 - 1)[\delta/3] + 1) < 1 \), we have \( Pr[\bigcup_{u \in B_2} B_u] > 0 \) by Lemma 3. Hence, we have \( Pr[A_u] > 0 \).

Similarly, we can check that \( G \) is also total-rainbow connected. 

The proof is now complete.

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