ON RELATION BETWEEN ATTRACTORS FOR SINGLE AND MULTIVALUED SEMIFLOWS FOR A CERTAIN CLASS OF PDES

PIOTR KALITA
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Łojasiewicza 6, 30-348 Kraków, Poland

GRZEGORZ LUKASZEWICZ
Faculty of Mathematics, Informatics, and Mechanics
University of Warsaw
ul. Banacha 2, 02-097 Warszawa, Poland

JAKUB SIEMIANOWSKI
Faculty of Mathematics and Computer Sciences
Nicolaus Copernicus University
ul. Chopina 12/18, 87-100 Toruń, Poland

Abstract. Sometimes it is not possible to prove the uniqueness of the weak solutions for problems of mathematical physics, but it is possible to bootstrap their regularity to the regularity of strong solutions which are unique. In this paper we formulate an abstract setting for such class of problems and we provide the conditions under which the global attractors for both strong and weak solutions coincide and the fractal dimension of the common attractor is finite. We present two problems belonging to this class: planar Rayleigh–Bénard flow of thermomicropolar fluid and surface quasigeostrophic equation on torus.

1. Introduction. In recent years a lot of effort has been put to the study of global attractors for problems without uniqueness of solutions. Several theories have been developed: theory of generalized semiflows [4], multivalued semiflows [29, 30], trajectory attractors [7], and of evolutionary systems [9]. The comparison of the first three of these theories with each other was done in [6, 24]. Within all these frameworks a large number of results has been obtained on the existence of global attractors, and their upper semicontinuous dependence on the problem data. Fairly recent overview of key results and open problems has appeared in the review paper [3].

While the question of the global attractor finite dimensionality has been studied very thoroughly for the problems where the solutions are unique, there are very few results available for the situation without the solution uniqueness. In fact, Balibrea, Caraballo, Kloeden, and Valero in their review article [3, Section 4.3 B)] state that

2010 Mathematics Subject Classification. Primary: 35B41, 35B65, 76D03, 80M35.

Key words and phrases. m-semiflow, global attractor, fractal dimension, thermomicropolar fluid, quasigeostrophic equation.

Work of P.K. G.L., and J.S. was supported by National Science Center (NCN) of Poland under project No. DEC-2017/25/B/ST1/00302, work of P.K. and J.S. was partially supported by NCN of Poland under project No. UMO-2016/22/A/ST1/00077.
the question of global attractor finite dimensionality is an important open problem for the case of multivalued semiflows.

In this paper we define a class of problems where the weak solution can possibly be nonunique and hence those weak solutions must be described by means of a multivalued theory. Moreover, we assume that in the considered class of problems we can bootstrap the regularity of weak solutions to the regularity of strong solutions, and we have the weak-strong uniqueness. In such case the attractor for the strong solutions can be described using the single-valued theory and we prove that the global attractors for weak and strong solutions coincide.

We illustrate the abstract scheme by two problems: Rayleigh–Bénard problem in two dimensions for thermomicropolar fluids and surface quasigeostrophic equation on torus. To show that the dimension of the attractor—for both problems—is finite we use the uniqueness of the strong solutions and a standard semigroup approach.

We remark here that for the class of problems that we consider, the dynamics on the global attractor is in fact single-valued, and hence the general question of attractor dimension for multivalued systems remains unanswered (there is a strong evidence that the dimension can be in fact infinite [1]). In this sense our examples are similar to that of [39] and give only a partial answer to the still open and difficult question of the finiteness of the fractal dimension for problems with multivalued dynamics on the attractor.

2. Abstract setting.

2.1. Existence and relations between attractors for strong and weak solutions. We assume that $\mathcal{V}$ and $\mathcal{H}$ are two Banach spaces such that $\mathcal{V} \subset \mathcal{H}$ with a continuous embedding. By $\mathcal{P}(\mathcal{H})$ (respectively, $\mathcal{B}(\mathcal{H})$) we denote the family of nonempty (respectively, nonempty and bounded) subsets of $\mathcal{H}$. Similar notation is used for $\mathcal{V}$. The Hausdorff semidistance in the Banach space $\mathcal{X}$, where either $\mathcal{X} = \mathcal{H}$ or $\mathcal{X} = \mathcal{V}$ is denoted by

$$
\text{dist}_\mathcal{X}(A, B) = \sup_{a \in A} \inf_{b \in B} \|b - a\|_\mathcal{X}.
$$

We consider two dynamical systems:

- a **single-valued semigroup** $\{S_\mathcal{V}(t)\}_{t \geq 0}$, i.e. a family of mappings $S_\mathcal{V}(t) : \mathcal{V} \to \mathcal{V}$, such that $S_\mathcal{V}(0) = \text{Id}_\mathcal{V}$ and $S_\mathcal{V}(s + t) = S_\mathcal{V}(s)S_\mathcal{V}(t)$ for $s, t \geq 0$,

- a **multivalued semiflow** (m-semiflow) $\{S_\mathcal{H}(t)\}_{t \geq 0}$, i.e. a family of mappings $S_\mathcal{H}(t) : \mathcal{H} \to \mathcal{P}(\mathcal{H})$, such that $S_\mathcal{H}(0)x = \{x\}$ for $x \in \mathcal{H}$ and $S_\mathcal{H}(s + t)x \subset S_\mathcal{H}(s)S_\mathcal{H}(t)x$ for $s, t \geq 0$ and $x \in \mathcal{H}$.

If, in definition of a multivalued semiflow, we replace the inclusion with the equality, i.e. we require that $S_\mathcal{H}(s + t)x = S_\mathcal{H}(s)S_\mathcal{H}(t)x$, then we say that the m-semiflow is strict. Note, however, that in the present paper we never make any assumptions on the m-semiflow strictness. In applications the multivalued semiflow $S_\mathcal{H}$ is defined by weak solutions of a given problem for which we do not know the uniqueness, and the semigroup $S_\mathcal{V}$ is given by the strong solutions which are known to be unique. To establish the existence and relations between attractors for strong and weak solutions we make the following three assumptions.

(i) **The semigroup of weak solutions has a compact absorbing set in $\mathcal{V}$**, i.e. there exists a set $B_0 \subset \mathcal{V}$ compact in $\mathcal{V}$ such that for any $B \in \mathcal{B}(\mathcal{H})$ there exists
$t_0 = t_0(B)$ with the property

$$\bigcup_{t \geq t_0} S_H(t)B \subset B_0.$$ 

(ii) The semigroup of strong solutions is continuous on $B_0$, i.e. the mappings $S_V(t)|_{B_0} : B_0 \to V$ are continuous (where both the domain and the image are endowed with the norm topology of $V$).

(iii) Strong solutions are weak solutions and there holds the weak-strong uniqueness, i.e. for all $t \geq 0$

$$S_H(t)|_V = S_V(t).$$

The proof of the following result can be found for instance in [17] where a more general case of pullback attractors was considered.

**Lemma 2.1.** Assuming (i), there exists a set $A_H \in \mathcal{B}(V)$ (global attractor for weak solutions), which is the smallest compact set in $H$ that attracts in $H$ the sets from $\mathcal{B}(H)$, i.e.

$$\lim_{t \to \infty} \text{dist}_H(S_H(t)B, A_H) = 0 \quad \text{for every} \quad B \in \mathcal{B}(H).$$

Note that we do not assume continuity of $S_H(t)$ and thus, in particular, we cannot expect any invariance (neither positive nor negative semi-invariance) of its attractor $A_H$ with respect to $S_H(t)$. The key observation is that in our situation to get the invariance it is enough to have the continuity of $S_V(t)$, cf. Theorem 2.3 below.

The following result is standard in the theory of global attractors for single valued semigroups, see [33, 37].

**Lemma 2.2.** Let $\{S_V(t)\}_{t \geq 0}$ be a semigroup of mappings $S_V(t) : V \to V$. Assume that there exists a compact in $V$ absorbing set $B_0$, i.e.

for every $B \in \mathcal{B}(V)$ there exists $t_0 = t_0(B)$ such that $\bigcup_{t \geq t_0} S_V(t)B \subset B_0$.

Moreover assume that the mappings $S_V(t)|_{B_0} : B_0 \to V$ are continuous in $V$. Then there exists a set $A_V \in \mathcal{B}(V)$ (global attractor for strong solutions), which is the smallest compact set in $V$ that attracts in $V$ the sets from $\mathcal{B}(V)$, i.e.

$$\lim_{t \to \infty} \text{dist}_V(S_V(t)B, A_V) = 0 \quad \text{for every} \quad B \in \mathcal{B}(V).$$

Moreover, the set $A_V$ is invariant with respect to $S_V$, i.e. $S_V(t)A_V = A_V$ for any $t \geq 0$.

Note that under assumptions (i), (ii) and (iii), the hypotheses of the above lemma are satisfied, and hence these hypotheses guarantee the existence of the set $A_V$—the global attractor for $\{S_V(t)\}_{t \geq 0}$. The proof of the next result can be found in [23, Theorem 9.4] for the case of thermomicropolar fluids. Here, we provide the abstract version of this theorem.

**Theorem 2.3.** Assume (i), (ii), (iii). There holds $A_V = A_H := A$. Moreover $S_V(t)A = S_H(t)A = A$ for all $t \geq 0$, and

$$\lim_{t \to \infty} \text{dist}_V(S_H(t)B, A) = 0 \quad \text{for every} \quad B \in \mathcal{B}(H).$$
Proof. As $A_V \in \mathcal{B}(V)$ we have $A_V \in \mathcal{B}(H)$. Hence
\[ \lim_{t \to \infty} \text{dist}_H(S_H(t)A_V, A_H) = 0. \]
But $A_V \subset V$, so (iii) and Lemma 2.2 imply that
\[ S_H(t)A_V = S_V(t)A_V = A_V \quad \text{for every} \quad t \geq 0. \]
Hence
\[ \text{dist}_H(A_V, A_H) = 0, \]
and, as both sets are closed in $H$ it follows that $A_V \subset A_H$. We will show that $A_V$ is attracting with respect to $S_H(t)$. Then $A_H \subset A_V$, since $A_V$ is compact in $H$ and $A_H$ is the smallest compact attracting set. We take $B \in \mathcal{B}(H)$, $t \geq t_0(B)$ and compute
\[
\text{dist}_H(S_H(t)B, A_V) = \text{dist}_H(S_H(t - t_0(B) + t_0(B))B, A_V) \\
\leq \text{dist}_H(S_H(t - t_0(B))S_H(t_0(B))B, A_V) \leq \text{dist}_H(S_H(t - t_0(B))B_0, A_V) \\
= \text{dist}_H(S_V(t - t_0(B))B_0, A_V) \leq C\text{dist}_V(S_V(t - t_0(B))B_0, A_V),
\]
where we have used (i) and (iii). As $B_0$ is bounded in $V$, we can pass with $t$ to $\infty$ and the right-hand side of the last expression tends to zero. Hence
\[ \lim_{t \to \infty} \text{dist}_H(S_H(t)B, A_V) = 0, \]
and the assertion is proved.

Finally, for $B \in \mathcal{B}(H)$ and $t \geq t_0(B)$
\[ \text{dist}_V(S_H(t)B, A) \leq \text{dist}_V(S_H(t - t_0(B))S_H(t_0(B))B, A) \leq \text{dist}_V(S_V(t - t_0(B))B_0, A), \]
and the right-hand side converges to zero, which concludes the proof of the theorem.

In applications, instead of verifying the assumption (i), it will be more convenient to verify the following two assumptions.

(iv) We can bootstrap the regularity of weak solutions, i.e. for any $B \in \mathcal{B}(H)$ there is $\epsilon > 0$ such that
\[ S_H(\epsilon)B \in \mathcal{B}(V). \]

(v) There exists a compact absorbing set for strong solutions, i.e. there exists $B_0 \in \mathcal{B}(V)$, a compact set in $V$, such that for any $B \in \mathcal{B}(V)$ there exists $t_0 = t_0(B)$ such that
\[ \bigcup_{t \geq t_0} S_V(t)B \subset B_0. \]

Lemma 2.4. If the semiflows $S_H$ and $S_V$ satisfy (iii), (iv), and (v), then $S_H$ satisfies (i).

Proof. Take $B \in \mathcal{B}(H)$. Let $\epsilon = \epsilon(B)$ be as in (iv). We take $t > \epsilon$ and get
\[ S_H(t)B \subset S_H(t - \epsilon)S_H(\epsilon)B. \]
Denote $B_1 = S_H(\epsilon)B$ and take $t_0 = t_0(B_1)$ from (v). We have
\[
\bigcup_{t \geq t_0(B_1) + \epsilon} S_H(t)B \subset \bigcup_{t \geq t_0(B_1) + \epsilon} S_H(t - \epsilon)B_1 = \bigcup_{t \geq t_0(B_1) + \epsilon} S_V(t - \epsilon)B_1 \subset B_0.
\]
The assertion is proved. \qed
The following result is a straightforward consequence of Theorem 2.3 and Lemma 2.4.

**Theorem 2.5.** The assertion of Theorem 2.3 holds if we assume (ii), (iii), (iv), and (v).

2.2. **Comparison with the theory of bi-space attractors.** The idea of bi-space attractors originates from the work of Babin and Vishik [2] who define them in the following way.

**Definition 2.6.** Let \( \mathcal{H} \) be a Banach space and let \( \{ S_{\mathcal{H}}(t) \}_{t \geq 0} \) be a semigroup of operators. A set \( \mathcal{A} \) is called a \( (\mathcal{H}, \mathcal{V}) \) global attractor of the semigroup \( \{ S_{\mathcal{H}}(t) \}_{t \geq 0} \) if the following properties hold:

- \( \mathcal{A} \) is invariant, i.e. \( S_{\mathcal{H}}(t) \mathcal{A} = \mathcal{A} \) for every \( t \geq 0 \),
- \( \mathcal{A} \) is compact in \( \mathcal{V} \) and bounded in \( \mathcal{H} \),
- \( \mathcal{A} \) is \( (\mathcal{H}, \mathcal{V}) \) attracting, i.e. there exists a time \( T_0 > 0 \) with \( S_{\mathcal{H}}(t) \mathcal{H} \subset \mathcal{V} \) for every \( t > T_0 \) and for every \( B \in \mathcal{B}(\mathcal{H}) \) there holds
  \[
  \lim_{t \to \infty} \text{dist}_{\mathcal{V}}(S_{\mathcal{H}}(t)B, \mathcal{A}) = 0.
  \]

Note that we always assume that \( \mathcal{V} \subset \mathcal{H} \), but in Babin and Vishik’s theory this embedding is not required. In fact, it may even hold that \( \mathcal{H} \subset \mathcal{V} \) or neither of these spaces needs to be included in another one [11, 10]. We relax the above definition by allowing the semigroup to be multivalued.

**Definition 2.7.** Let \( \mathcal{V}, \mathcal{H} \) be two Banach spaces. Let \( \{ S_{\mathcal{H}}(t) \}_{t \geq 0} \) be a multivalued semigroup of operators. A set \( \mathcal{A} \) is called a \( (\mathcal{H}, \mathcal{V}) \) global attractor of the multivalued semigroup \( \{ S_{\mathcal{H}}(t) \}_{t \geq 0} \) if the following properties hold:

- \( \mathcal{A} \) is invariant, i.e. \( S_{\mathcal{H}}(t) \mathcal{A} = \mathcal{A} \) for every \( t \geq 0 \),
- \( \mathcal{A} \) is compact in \( \mathcal{V} \) and bounded in \( \mathcal{H} \),
- \( \mathcal{A} \) is \( (\mathcal{H}, \mathcal{V}) \) attracting, i.e. there exists a time \( T_0 > 0 \) with \( S_{\mathcal{H}}(t) \mathcal{H} \subset \mathcal{V} \) for every \( t > T_0 \) and for every \( B \in \mathcal{B}(\mathcal{H}) \) there holds
  \[
  \lim_{t \to \infty} \text{dist}_{\mathcal{V}}(S_{\mathcal{H}}(t)B, \mathcal{A}) = 0.
  \]

Using the above definition we can reformulate Theorem 2.3 in the language of bi-space attractors.

**Theorem 2.8.** Let \( \mathcal{V}, \mathcal{H} \) be two Banach spaces such that \( \mathcal{V} \subset \mathcal{H} \) with a continuous embedding. Let \( \{ S_{\mathcal{H}}(t) \}_{t \geq 0} \) be a multivalued semigroup of operators on \( \mathcal{H} \). Define \( S_{\mathcal{V}}(t) = S_{\mathcal{H}}(t)|_{\mathcal{V}} \). If the following assumptions hold:

- for every \( t \geq 0 \) the operator \( S_{\mathcal{V}}(t) : \mathcal{V} \to \mathcal{V} \) is single valued,
- the multivalued semigroup \( \{ S_{\mathcal{H}}(t) \}_{t \geq 0} \) has an absorbing set in \( B_0 \) which is compact in \( \mathcal{V} \),
- the mappings \( S_{\mathcal{V}}(t) \) are continuous on \( B_0 \) for every \( t \geq 0 \),

then \( \{ S_{\mathcal{H}}(t) \}_{t \geq 0} \) has the \( (\mathcal{H}, \mathcal{V}) \) global attractor.

2.3. **Fractal dimension.** We will provide the sufficient condition which guarantees that the fractal dimension in \( \mathcal{V} \) (and thus also in \( \mathcal{H} \)) of the obtained global attractor \( \mathcal{A} \) is finite. To this end we reinforce the previous assumptions with the following one.

There exists \( T > 0 \) such that

- (vi) The mapping \( S_{\mathcal{V}}(T) \) is Lipschitz in the space \( \mathcal{V} \) on the attractor, i.e.
  \[
  \| S_{\mathcal{V}}(T)u - S_{\mathcal{V}}(T)v \|_{\mathcal{V}} \leq L \| u - v \|_{\mathcal{V}} \quad \text{for every} \quad u, v \in \mathcal{A} \quad \text{with a constant} \quad L > 0.
  \]
(vii) The mapping $S_V(T)$ satisfies the squeezing condition in the space $V$ on the attractor, i.e. $V$ is a Hilbert space and there exists the orthogonal projection $P$ leading from $V$ into its finite dimensional subspace and a constant $\delta \in (0, 1)$ such that for every $u, v \in A$ either

$$\| (I - P)(S_V(T)v - S_V(T)u) \|_V \leq \| P(S_V(T)v - S_V(T)u) \|_V,$$

or

$$\| S_V(T)v - S_V(T)u \|_V \leq \delta \| v - u \|_V.$$

Under the stated assumptions the attractor $A$ is a finite dimensional subset of the space $V$, see [19, Chapter 2] where a stronger result is obtained, namely it is proved that the squeezing condition guarantees the existence of a so called exponential attractor, or [12, Theorem 2.15], where the attractor finite dimensionality has been obtained with (vii) replaced by a weaker assumption valid not only in Hilbert spaces, but also in Banach spaces. The finite dimensionality result of [12, 19] is stated in the following lemma.

**Lemma 2.9.** If, in addition to the assumptions (i), (ii), and (iii) of Section 2.1 we impose assumptions (vi) and (vii), then the upper box counting dimension of the global attractor $A$ in the space $V$ is finite, i.e.

$$d_V^b(A) < \infty,$$

where

$$d_V^b(A) = \limsup_{\varepsilon \to 0^+} \frac{\ln N_V(A, \varepsilon)}{\ln \frac{1}{\varepsilon}},$$

with $N_V(A, \varepsilon)$ being the minimal cardinality of covering of the compact set $A$ by the closed balls in $V$ of radius $\varepsilon$.

Observe that, as $V \subset H$, then

$$d_H^b(A) \leq d_V^b(A),$$

and hence we deduce, that under assumptions of the above lemma also the fractal dimension in $H$ of the global attractor for the m-semiflow $\{S_H(t)\}_{t \geq 0}$ is finite. We will use Lemma 2.9 in Section 3 to demonstrate that the fractal dimension of the global attractor for the problem considered there is finite.

3. Rayleigh-Bénard problem for thermomicropolar fluids. Micropolar fluids are fluids that consist of particles which undergo the intrinsic spin independent of the motion of fluid itself. The constitutive equations of these fluids were introduced in [20]. For the mathematical investigation of micropolar fluid model, see [26, 27, 28].

Study of thermal processes in those fluids leads to the so-called thermomicropolar fluid model, introduced in [21]. A simplified version of this model was studied in [35, 36], and, recently, in [22], cf. also Straughan et al. [31, 34].

In this section we consider the Rayleigh–Bénard problem of a planar flow of thermomicropolar fluids. The existence of strong and weak solutions for the problem, the relation between these two classes of solutions as well as the existence of the global attractor have been recently proved in [23]. In particular, in [23] we have proved, using the single valued theory, that for the problem under consideration there exists the global attractor for the strong solutions, and, using the multivalued theory, that there exists the global attractor for the weak solutions. We have also shown in [23] that both attractors coincide. We recall these results in Subsection
3.2. The new results are contained in Subsection 3.3, where we show that the obtained global attractor has finite fractal dimension. We use the method based on the squeezing condition, cf. [19, 38], to show the finite dimensionality of global attractor for the strong solutions, and since both attractors coincide, we deduce the finite dimensionality of the global attractor for the weak solutions.

3.1. Problem formulation. Let \( \Omega_\infty = (-\infty, \infty) \times (0, 1) \). For a given \( l > 0 \) define \( \Omega = (0, l) \times (0, 1) \) and introduce the spaces

\[
\tilde{V}_S = \left\{ u|_{\tilde{\Omega}} : u \in C^\infty(\tilde{\Omega}_\infty)^2, \text{div} \ u = 0, \ u|_{x_2=0,1} = 0, \right. \\
\left. \text{and} \ u \text{ is} \ l \text{-periodic in} \ x_1 \text{-direction} \right\},
\]

\[
\tilde{V} = \{ \omega|_{\tilde{\Omega}} : \omega \in C^\infty(\tilde{\Omega}_\infty), \omega|_{x_2=0,1} = 0, \text{and} \ \omega \text{ is} \ l \text{-periodic in} \ x_1 \text{-direction} \},
\]

\[
H_S = \text{closure of} \ \tilde{V}_S \text{ in} \ L^2(\Omega)^2, \quad H = L^2(\Omega),
\]

\[
V_S = \text{closure of} \ \tilde{V}_S \text{ in} \ H^1(\Omega)^2, \quad \text{and} \ V = \text{closure of} \ \tilde{V} \text{ in} \ H^1(\Omega).
\]

Spaces \( H \) and \( H_S \) are Hilbert spaces with the inner product

\[
(u, v) = (u, v)_{L^2} = \int_{\Omega} u(x) \cdot v(x) \, dx.
\]

and the corresponding norms

\[
|v| = (v, v)^{1/2} \quad \text{for} \quad v \in H, H_S.
\]

Spaces \( V \) and \( V_S \) are Hilbert spaces with the norms

\[
\|v\| = (\nabla v, \nabla v)^{1/2} = ((v, v))^{1/2} \quad \text{for} \quad v \in V, V_S.
\]

We will formulate the problem in dimensionless variables. In its classical formulation the problem is governed by the following system of equations, cf. [23]

\[
\frac{1}{Pr} (u_t + (u \cdot \nabla) u + \nabla p) = \Delta u + K(2\text{rot} \omega + \Delta u) + R\theta e_2,
\]

\[
\text{div} u = 0,
\]

\[
\frac{M}{Pr} (\omega_t + u \cdot \nabla \omega) = L\Delta \omega + 2K(\text{rot} u - 2\omega),
\]

\[
\theta_t + u \cdot \nabla \theta = \Delta \theta + D\text{rot} \omega \cdot \nabla \theta + D\partial_{x_1} \omega + u_2,
\]

together with boundary conditions
\[
u|_{x_2=0,1} = 0, \quad \omega|_{x_2=0,1} = 0, \quad \theta|_{x_2=0,1} = 0,
\]

\[
u, \omega, \theta \text{ are restrictions of} \ l - \text{periodic in} \ x_1 \text{ functions defined on} \ (-\infty, \infty) \times (0, 1),
\]

and initial conditions
\[
u(0) = u_0, \quad \omega(0) = \omega_0, \quad \theta(0) = \theta_0.
\]

The physical properties of the fluid are described by six positive constants: the Rayleigh number \( R \), the Prandtl number \( Pr \), the micropolar constants \( K, L, M, \) and the thermomicro polar constant \( D \), see [23] for some discussion. We note here that the key difficulty in the study of the above system of equations comes from the term \( D\text{rot} \omega \cdot \nabla \theta \) in equation (4). Because of this term it appears impossible to get the weak solution uniqueness even in two space dimensions.
Throughout this section we will denote \( \mathcal{H} = H_S \times H \times H \) and \( \mathcal{V} = V_S \times V \times V \). Let us denote
\[
\kappa_1 = \Pr(1 + K) \quad \text{and} \quad \kappa_2 = \frac{L \Pr}{M}
\]
for short. We equip the Hilbert space \( \mathcal{H} \) with the following scalar product, for \((u_i, \omega_i, \theta_i) \in \mathcal{H}, i = 1, 2\)
\[
\left( (u_1, \omega_1, \theta_1), (u_2, \omega_2, \theta_2) \right) = \frac{1}{\kappa_1} (u_1, u_2) + \frac{1}{\kappa_2} (\omega_1, \omega_2) + (\theta_1, \theta_2).
\]
The associated norm will be denoted by \( | \cdot | \). We note here that the weights \( 1/\kappa_1 \) and \( 1/\kappa_2 \) appear in the definition of the norm \( | \cdot | \) will be of key importance in the proof of the squeezing condition in step 2 in Section 3.3 below.

The Hilbert space \( \mathcal{V} \) will be equipped with the following scalar product, for \((u_i, \omega_i, \theta_i) \in \mathcal{V}, i = 1, 2\)
\[
\left( (u_1, \omega_1, \theta_1), (u_2, \omega_2, \theta_2) \right) = \left( (u_1, u_2) + (\omega_1, \omega_2) + (\theta_1, \theta_2) \right).
\]
The associated norm will be denoted by \( \| \cdot \| \).

We define the standard trilinear forms (see [33] or [37])
\[
b_S(u, v, w) = \sum_{i,j=1}^{2} \int_{\Omega} u_i \partial_x v_j w_j \, dx \quad \text{and} \quad b(u, v, w) = \sum_{i=1}^{2} \int_{\Omega} u_i \partial_x v w \, dx.
\]
Let \( A \) be the minus Laplace operator associated with boundary conditions (5)–(6), \( A : D(A) \to H \), where
\[
D(A) = \{ v \in V \mid -\Delta v \in H \} \quad \text{and} \quad Av = -\Delta v, \quad v \in D(A).
\]
The eigenvectors \( \{ w_k \}_{k \geq 1} \subset D(A) \) of \( A \) form the orthonormal basis of \( H \) and \( Aw_k = \beta_k w_k \) and \( 0 < \beta_1 \leq \beta_2 \leq \ldots \leq \beta_k \to \infty \).

We introduce the fractional power of the Laplacian, see [33, Chapters 3 and 6]. Let
\[
D(A^{3/2}) = \left\{ u \in H \mid u = \sum_{k \geq 1} (u, w_k) w_k \ (\text{in} \ H), \sum_{k \geq 1} (u, w_k)^2 \beta_k^3 < \infty \right\},
\]
and define
\[
A^{3/2} u = \sum_{k \geq 1} \beta_k^{3/2} (u, w_k) w_k.
\]
To make \( D(A^{3/2}) \) a Hilbert space we equip it with the inner product
\[
(u, v)_{D(A^{3/2})} = (A^{3/2} u, A^{3/2} v)
\]
which gives the corresponding norm equivalent to the norm of \( H^3(\Omega) \).
\[
\| u \|_{D(A^{3/2})} = |A^{3/2} u|.
\]
Now, we introduce the Stokes operator \( A_S \), see [33] or [37] for details. Let
\[
D(A_S) = \{ u \in V_S \mid \exists w \in H_S, \forall \varphi \in V_S \ (w, \varphi) = (\nabla u, \nabla \varphi) \}
\]
and define
\[
A_S(u) = w.
\]
It is known that $A_S = -\Pi \Delta$, where $\Pi$ stands for the Helmholtz–Leray projector from $L^2(\Omega)^2$ onto $H_S$. Similar as in the case of the Laplace operator, the eigenfunctions $\{v_k\}_{k=1}^\infty$ of the operator $A_S$ given by

$$A_S v_k = \alpha_k v_k \quad \text{and} \quad 0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k \to \infty$$

form the orthonormal basis of $H_S$ and they are smooth. Using these eigenfunctions it is possible to define, similar as for the Laplace operator,

$$D(A_S^{3/2}) = \left \{ u \in H_S \mid u = \sum_{k \geq 1} (u, v_k) v_k \ (\text{in } H_S), \sum_{k \geq 1} (u, v_k)^2 \alpha_k^3 < \infty \right \},$$

and

$$A_S^{3/2} u = \sum_{k \geq 1} \alpha_k^{3/2} (u, v_k) v_k.$$ 

The space $D(A_S^{3/2})$ is a Hilbert space equipped with the following inner product and norm, equivalent to that of $H^3(\Omega)^2$,

$$(u, v)_{D(A_S^{3/2})} = (A_S^{3/2} u, A_S^{3/2} v), \quad \|u\|_{D(A_S^{3/2})} = |A_S^{3/2} u|.$$ 

We are in position to define weak and strong solutions for the formulated problem. We also recall the results on their existence obtained in [23].

**Definition 3.1.** Let $T > 0$, $(u_0, \omega_0, \theta_0) \in \mathcal{H}$. By a weak solution of problem (1)–(7) we mean a triple of functions $(u, \omega, \theta)$,

$$u \in L^2(0, T; V_S) \cap C([0, T], H_S) \cap W^{1,2}(0, T; V_S'),$$

$$\omega \in L^2(0, T; V) \cap C([0, T], H) \cap W^{1,2}(0, T; V^*),$$

$$\theta \in L^2(0, T; V) \cap C_u([0, T]; H) \cap W^{1,2}(0, T; D(A^{3/2})^*)$$

such that $u(0) = u_0$, $\omega(0) = \omega_0$, $\theta(0) = \theta_0$ satisfying the following identities.

$$\frac{1}{Pr} \left( \frac{d}{dt} (u(t), \varphi) + b_S(u(t), u(t), \varphi) \right) + (\nabla u(t), \nabla \varphi)$$

$$+ K \left[ (\nabla u(t), \nabla \varphi) - 2(\nabla \omega(t), \varphi) \right] = Ra(\theta(t)) e_2, \varphi \quad (11)$$

for every $\varphi \in V_S$,

$$\frac{M}{Pr} \left( \frac{d}{dt} (\omega(t), \psi) + b(u(t), \omega(t), \psi) \right) + K \left[ A(\omega(t), \psi) - 2(\nabla u(t), \psi) \right] + L(\nabla \omega(t), \nabla \psi) = 0 \quad (12)$$

for every $\psi \in V$,

$$\frac{d}{dt} (\theta(t), \eta) + b(u(t), \theta(t), \eta) + (\nabla \theta(t), \nabla \eta)$$

$$= -D(\theta(t), \nabla \omega(t) \cdot \nabla \eta) + D(\theta_x(t), \omega(t), \eta) + (u_2(t), \eta) \quad (13)$$

for every $\eta \in D(A^{3/2})$, in the sense of scalar distributions on $(0, \tau)$. Additionally, we require that $|\theta(\cdot)|$ is upper-semicontinuous from the right at $t = 0$,

$$\limsup_{t \to a^+} |\theta(t)| \leq |\theta_0|. \quad (14)$$

Existence of the weak solution satisfying the above definition has been established in [23, Theorem 5.1].
Definition 3.2. Let $T > 0$, $(u_0, \omega_0, \theta_0) \in \mathcal{V}$. By a strong solution of problem (1)–(7) we mean a triple of functions $(u, \omega, \theta)$,

$$u \in L^2(0, T; D(A_S)) \cap C([0, T], V_S) \cap W^{1,2}(0, T; H_S),$$

$$\omega, \theta \in L^2(0, T; D(A)) \cap C([0, T], V) \cap W^{1,2}(0, T; H),$$

such that $u(0) = u_0$, $\omega(0) = \omega_0$, $\theta(0) = \theta_0$ and the following identities hold

$$\frac{1}{Pr} \left( \frac{d}{dt}(u(t), \varphi) + b_S(u(t), u(t), \varphi) \right) + (-\Delta u(t), \varphi) + K\left[(-\Delta u(t), \varphi) - 2(\text{rot } \omega(t), \varphi)\right] = \text{Ra}(\theta(t)e_2, \varphi), \quad (15)$$

for every $\varphi \in H_S$,

$$\frac{M}{Pr} \left( \frac{d}{dt}(\omega(t), \psi) + b(u(t), \omega(t), \psi) \right) + K\left[4(\omega(t), \psi) - 2(\text{rot } u(t), \psi)\right] + L(-\Delta \omega(t), \psi) = 0, \quad (16)$$

for every $\psi \in H$,

$$\frac{d}{dt}(\theta(t), \eta) + b(u(t), \theta(t), \eta) + (-\Delta \theta(t), \eta) = D(\text{rot } \omega(t) \cdot \nabla \theta(t), \eta) + D(\partial_{x_1} \omega(t), \eta) + (u_2(t), \eta), \quad (17)$$

for every $\eta \in H$, in the sense of scalar distributions on $(0, \tau)$.

Existence and uniqueness of strong solutions satisfying the above definition has been proved in [23, Theorem 7.2].

3.2. Attractors for single and multivalued semiflows and their equality. In [23] we proved that weak and strong solutions of problem (1)–(7) given by Definitions 3.1 and 3.2, respectively, exist and that their corresponding attractors coincide. Our results show, in particular, that abstract assumptions (i), (ii), (iii) from Section 2.1 hold in this case. We recall the results of [23] in Lemmas 3.3, 3.4, 3.5, 3.6, and 3.7.

First we have to define the corresponding semiflows.

The family of multivalued maps $\{S_H(t)\}_{t \geq 0}$ with $S_H(t) : \mathcal{H} \to \mathcal{P}(\mathcal{H})$ is given by

- $S_H(t)(u_0, \omega_0, \theta_0) := \{(u(t), \omega(t), \theta(t)) : \text{ where } (u, \omega, \theta) \text{ is (possibly nonunique)}$ weak solution of problem (1)–(7) with the initial data $(u_0, \omega_0, \theta_0)$ in $\mathcal{H}$, given by Definition 3.1\}

We also define the family of single valued operators $\{S_V(t)\}_{t \geq 0}$, $S_V(t) : \mathcal{V} \to \mathcal{V}$, by

- $S_V(t)(u_0, \omega_0, \theta_0) := \{(u(t), \omega(t), \theta(t)) : \text{ where } (u, \omega, \theta) \text{ is a unique strong solution}$ of problem (1)–(7) with the initial data $(u_0, \omega_0, \theta_0)$ in $\mathcal{V}$, given by Definition 3.2\}

We remind the following two Lemmas.

Lemma 3.3. [23, Lemma 8.2] The family $\{S_H(t)\}_{t \geq 0}$ is an $m$-semiflow on $\mathcal{H}$.

Lemma 3.4. [23, Lemma 9.1] The family $\{S_V(t)\}_{t \geq 0}$ is a semiflow on $\mathcal{V}$.

We shall use Theorem 2.3 to prove existence and equality of global attractors for weak and strong solutions. Thus, we have to show that $\{S_H(t)\}_{t \geq 0}$ and $\{S_V(t)\}_{t \geq 0}$ satisfy hypotheses (i), (ii), and (iii) from Section 2.1. Below we recall respective lemmas from [23].
Lemma 3.5. [23, Lemma 8.3] There exists a ball
\[ B_0 = \{(u, \omega, \theta) \in D(A_S) \times D(A) \times D(A) : \|u\|_{D(A_S)}^2 + \|\omega\|_{D(A)}^2 + \|\theta\|_{D(A)}^2 \leq R^2 \} \]
such that for every \( B \in \mathcal{B}(\mathcal{H}) \) there exists a time \( t_0 = t_0(B) \) such that
\[ \bigcup_{t \geq t_0} S_\mathcal{H}(t)B \subset B_0. \]
In consequence, the hypothesis (i) is satisfied.

Continuity of \( S_\mathcal{V}(t) \) on the absorbing ball \( B_0 \) also follows from the a priori estimates proved in [23]. In fact we also have the Lipschitz continuity on \( B_0 \) which we establish in the following Lemma.

Lemma 3.6. Let \((\bar{u}_0, \bar{\omega}_0, \bar{\theta}_0) \in B_0 \) and \((\bar{u}_0, \bar{\omega}_0, \hat{\theta}_0) \in B_0 \). Then there exists an increasing function \( L : [0, \infty) \to [1, \infty) \) such that
\[ \|S_\mathcal{V}(t)(\bar{u}_0, \bar{\omega}_0, \bar{\theta}_0) - S_\mathcal{V}(t)(\bar{u}_0, \bar{\omega}_0, \hat{\theta}_0)\| \leq L(t)\|(\bar{u}_0, \bar{\omega}_0, \bar{\theta}_0) - (\bar{u}_0, \bar{\omega}_0, \hat{\theta}_0)\|. \]
In consequence, assumptions (ii) and (vi) of Section 2.1 are satisfied.

Proof. A priori estimates which use the uniform Gronwall lemma proved in [23, Lemma 8.3] imply the existence of the absorbing ball \( B_0 \) in \( D(A_S) \times D(A) \times D(A) \). The same argument implies that if the initial data belongs to the ball \( B_0 \), then the following bounds hold
\[ \sup_{t \geq 0}\{\|u(t)\|_{D(A_S)}, \|\omega(t)\|_{D(A)}, \|\theta(t)\|_{D(A)}\} \leq C(R). \]
The assertion follows from these bounds by using the estimates on the difference of two strong solutions obtained in [23, Theorem 6.4, Theorem 7.2].

Finally, we have:

Lemma 3.7. [23, Theorem 7.2] Each strong solution given by Definition 3.2 is also a weak solution given by Definition 3.1. Moreover, if \((u_0, \omega_0, \theta_0) \in \mathcal{V} \) then the strong solution with the initial data \((u_0, \omega_0, \theta_0) \) is unique also in the class of weak solutions. In consequence, the hypothesis (iii) is satisfied.

Using Lemmas 3.5, 3.6, and 3.7 it follows that all assumptions of Theorem 2.3 hold, whence we get the following result.

Theorem 3.8. Both dynamical systems, the multivalued one on \( \mathcal{H} \) generated by weak solutions and the single valued one on \( \mathcal{V} \) generated by strong solutions have the global attractors, and both attractors coincide. Denoting the common attractor by \( \mathcal{A} \) we have,
\[ \lim_{t \to \infty} \text{dist}_\mathcal{V}(S_\mathcal{H}(t)B, \mathcal{A}) = 0 \quad \text{for every} \quad B \in \mathcal{B}(\mathcal{H}). \]
Moreover, \( \mathcal{A} \subset D(A_S) \times D(A) \times D(A) \) and the dynamics on this attractor is single valued and given by \( S_\mathcal{V} \). This attractor is compact in \( \mathcal{V} \), and invariant, i.e. \( S_\mathcal{H}(t)\mathcal{A} = S_\mathcal{V}(t)\mathcal{A} = \mathcal{A} \) for every \( t \geq 0 \). The attractor is minimal in the class of \( \mathcal{H} \)-closed sets attracting in \( \mathcal{H} \) with respect to \( S_\mathcal{H} \) and maximal in the class of \( \mathcal{H} \)-bounded sets which are invariant with respect to either \( S_\mathcal{H} \) or \( S_\mathcal{V} \).
3.3. Finite dimensionality of the global attractor. The attractor finite dimensionality will follow from Lemma 2.9. Since we have already established hypothesis (vi) in Lemma 3.6, it suffices to prove that the squeezing condition given in hypothesis (vii) holds. The proof of the latter consists of two steps. In the first step we establish an additional result on the attractor regularity. We need to show that the functions \( \omega(t) \) and \( \theta(t) \) are uniformly bounded in \( D(A^{3/2}) \) on the attractor. This result will be used in the second step of the proof to show the squeezing condition by the method of [19, 38]. To use this method we need to appropriately define both the linear operator \( A : D(A) \subset H \to H \) and the scalar product on the space \( H \). We have already defined the scalar product on \( H \) by the formula (8), and the operator \( A : D(A) \subset H \to H \) will be defined by the formula (34) below.

**Step 1.** We prove the following theorem

**Theorem 3.9.** There exists \( K_0 > 0 \) such that for every \((u_1, \omega_1, \theta_1) \in A\) there holds

\[
(u_1, \omega_1, \theta_1) \in D(A_2) \times D(A) \times D(A) \quad \text{and} \quad \|u_1\|_{D(A_2)} + \|\omega_1\|_{D(A)} + \|\theta_1\|_{D(A)} \leq K_0,
\]

\[
(\omega_1, \theta_1) \in D(A^{3/2}) \times D(A^{3/2}) \quad \text{and} \quad \|\omega_1\|_{D(A^{3/2})} + \|\theta_1\|_{D(A^{3/2})} \leq K_0.
\]

**Proof.** The regularity and estimate (18) follow from Lemma 3.5. To get (19) we need to bootstrap the regularity of \( \omega \) and \( \theta \) to \( D(A^{3/2}) \times D(A^{3/2}) \), a closed subspace of \( H^3(\Omega)^2 \). The desired estimates in (19) will be obtained by applying the Laplace operator to equations (3)–(4).

Let \((u_1, \omega_1, \theta_1) \in A\). Invariance of \( A \) implies that there exists \((u_0, \omega_0, \theta_0) \in A\) such that \((u_1, \omega_1, \theta_1) = S_T(1)(u_0, \omega_0, \theta_0)\). Consider a strong solution \((u, \omega, \theta)\) starting at \((u_0, \omega_0, \theta_0)\) such that \((u_1, \omega_1, \theta_1) = (u(1), \omega(1), \theta(1))\). To avoid technicalities we derive below only the formal estimates, actually they should be derived for the Galerkin approximation. In the proof we will denote by \( C \) a generic positive constant which changes from line to line, by \( \epsilon > 0 \) an appropriately chosen small constant, and by \( C(\epsilon) \) a positive constant which depends only on \( \epsilon \).

First, we take the Laplacian of each term of (3) to get

\[
\frac{M}{Pr} \Delta \omega_t + \frac{M}{Pr} \Delta (u \cdot \nabla \omega) = L \Delta \Delta \omega + K (\Delta \text{rot } u - 2 \Delta \omega).
\]  

Using the formula

\[
\Delta (u \cdot \nabla \omega) = (\Delta u) \cdot \nabla \omega + u \cdot \nabla (\Delta \omega) + 2 \sum_{i=1}^{2} \nabla u_i \cdot \nabla (\partial_{x_i} \omega),
\]

and substituting \( \eta = \Delta \omega \) in (20) we get the equation

\[
\frac{M}{Pr} \eta_t - L \eta = \frac{M}{Pr} \left( (\Delta u) \cdot \nabla \omega + u \cdot \nabla \eta + 2 \sum_{i=1}^{2} \nabla u_i \cdot \nabla (\partial_{x_i} \omega) \right) + K (\Delta \text{rot } u - 2 \eta).
\]  

We multiply (21) by \( \eta \) and integrate over \( \Omega \),

\[
\frac{M}{2Pr} \frac{d}{dt} |\eta|^2 + L |\eta|^2 = -\frac{M}{Pr} (b(\Delta u, \omega, \eta) + b(u, \eta, \eta))
\]

\[
- 2 \frac{M}{Pr} \sum_{i=1}^{2} \int_{\Omega} (\nabla u_i \cdot \nabla (\partial_{x_i} \omega)) \eta \, dx + K \int_{\Omega} (\text{rot } u) \eta \, dx - 2K |\eta|^2.
\]  

\[
\frac{d}{dt} |\eta|^2 + L |\eta|^2 = -\frac{M}{Pr} (b(\Delta u, \omega, \eta) + b(u, \eta, \eta)) - 2 \frac{M}{Pr} \sum_{i=1}^{2} \int_{\Omega} (\nabla u_i \cdot \nabla (\partial_{x_i} \omega)) \eta \, dx + K \int_{\Omega} (\text{rot } u) \eta \, dx - 2K |\eta|^2.
\]
Now, we estimate each term on the right-hand side by using (18),
\[
|b(\Delta u, \omega, \eta)| \leq \|u\|_{H^2} \|\nabla \omega\|_{L^4} \|\eta\|_{L^4} \leq C|A_S u| |A \omega| |\eta| \leq \epsilon \|\eta\|^2 + C(\epsilon),
\]
\[
b(u, \eta, \eta) = 0,
\]
\[
\left| \int_{\Omega} \nabla u_i \cdot \nabla (\partial_x \omega) \eta \, dx \right| \leq c \|\nabla u\|_{L^4} |A \omega| \|\eta\|_{L^4} \leq c |A_S u| |A \omega| |\eta| \leq \epsilon \|\eta\|^2 + C(\epsilon),
\]
\[
\left| \int_{\Omega} \Delta \text{rot} \, u \eta \, dx \right| \leq \|u\|_{H^2} \|\eta\| \leq c |A_S u| |\eta| \leq \epsilon \|\eta\|^2 + C(\epsilon).
\]
Hence, after choosing a suitable \( \epsilon > 0 \), we get
\[
\frac{d}{dt} \|\eta\|^2 + C \|\eta\|^2 \leq C.
\]
Integrating the above inequality from \( t \geq 0 \) to \( t + 1 \) and obtain
\[
\int_t^{t+1} \|\eta(\tau)\|^2 \, d\tau \leq C(1 + |\eta(t)|^2) \leq C,
\]
where we have used \( |\eta(t)| = |A \omega(t)| \) and (18).

Now, we take the Laplacian of each term of (1) and get
\[
\frac{1}{Pr} (\Delta u_t + \Delta [(u \cdot \nabla) u] + \Delta \nabla p) = (1 + K)\Delta \Delta u + 2K \Delta \text{rot} \omega + Ra \Delta \theta e_2. \tag{24}
\]
For \( i = 1, 2 \), we use the formula,
\[
\Delta((u \cdot \nabla) u_i) = \Delta u \cdot \nabla u_i + u \cdot \nabla (\Delta u_i) + 2 \sum_{j=1}^2 \nabla u_j \cdot \nabla (\partial_x u_i),
\]
and set \( \xi = -A_S u \) in (24) to get, after taking the scalar product with \( \xi \) and integrating over \( \Omega \)
\[
\frac{1}{2Pr} \frac{d}{dt} |\xi|^2 + (1 + K)|\xi|^2 = -\frac{1}{Pr} (b_S(\Delta u, u, \xi) + b_S(u, \Delta u, \xi))
- \frac{2}{Pr} \sum_{i,j=1}^2 \int_{\Omega} \nabla u_j \cdot \nabla (\partial_x u_i) \xi_i \, dx + 2K \int_{\Omega} \Delta \text{rot} \omega \xi \, dx + Ra \Delta \theta e_2. \tag{25}
\]
We proceed with the above equation in a similar way as with equation (22) and we end up with the estimate
\[
\int_t^{t+1} \|\xi(s)\|^2 \, ds \leq C(1 + |\xi(t)|^2) \leq C,
\]
valid for every \( t \geq 0 \).

Now, we multiply (21) by \( A \eta \) and integrate over \( \Omega \),
\[
\frac{d}{dt} \frac{M}{2Pr} \|\eta\|^2 + L|A \eta|^2 = -\frac{M}{Pr} (b(\Delta u, \omega, A \eta) + b(u, \eta, A \eta))
- 2\frac{M}{Pr} \sum_{i=1}^2 \int_{\Omega} (\nabla u_i \cdot \nabla (\partial_x \omega)) A \eta \, dx + K \int_{\Omega} \Delta (\text{rot} u) A \eta \, dx - 2K \|\eta\|^2. \tag{27}
\]
We estimate the terms on the right-hand side as follows,

\[ |b(\Delta u, \omega, \eta)| \leq \|\Delta u\|_{L^4}\|\nabla \omega\|_{L^4}|\eta| \leq C\|u\|_{H^3}|\omega\||\eta| \]

\[ \leq C\|\xi\| |\eta| \leq \epsilon|\eta|^2 + C(\epsilon)\|\xi\|^2, \]

\[ |b(u, \eta, \eta)| \leq \|u\|_{L^\infty}\|\eta\| |\eta| \leq \epsilon|\eta|^2 + C(\epsilon)|\eta|^2, \]

\[ \left| \int_{\Omega} (\nabla u_i \cdot \nabla (\partial_i \omega)) \eta \, dx \right| \leq \|\nabla u\|_{L^4}\|D^2 \omega\|_{L^4}|\eta| \leq C\|u_S\|\|\eta\| |\eta| \]

\[ \leq \epsilon|\eta|^2 + C(\epsilon)\|\eta\|^2, \]

\[ \left| \int_{\Omega} \Delta (\text{rot} u) \eta \, dx \right| \leq \|u\|_{H^3}|\eta| \leq C\|\xi\| |\eta| \leq \epsilon|\eta|^2 + C(\epsilon)\|\xi\|^2, \]

whence, and by (27), we obtain

\[ \frac{d}{dt} \|\eta\|^2 \leq C(\|\eta\|^2 + \|\xi\|^2). \]

Inequalities (26) and (23) enable us to use the uniform Gronwall lemma which yields

\[ \|\eta(t)\|^2 \leq C \quad \text{for every} \quad t \geq 1. \]  

(28)

We recall that \(\|\eta\| = |A^{3/2}\omega|\), hence

\[ \|\omega_1\|_{D(A^{3/2})} = \|\omega(1)\|_{D(A^{3/2})} \leq C. \]

In particular we have proved that \(\|\omega_1\|_{D(A^{3/2})} \leq C\) holds for every \((u_1, \omega_1, \theta_1) \in \mathcal{A}\).

As the trajectory \((u(t), \omega(t), \theta(t))\) lies in the attractor the above estimate implies that

\[ \|\omega(t)\|_{D(A^{3/2})} \leq C \quad \text{for every} \quad t \in \mathbb{R}. \]  

(29)

We still need to estimate \(\|\theta(t)\|_{D(A^{3/2})}\). To this end, we first take the Laplacian of each term of (4),

\[ \Delta \theta_t + \Delta (u \cdot \nabla \theta) = \Delta \Delta \theta + D \Delta (\text{rot} \omega \cdot \nabla \theta) + D \Delta \partial_x \omega + \Delta u_2. \]

Then, we use the formulas

\[ \Delta (u \cdot \nabla \theta) = \Delta u \cdot \nabla \theta + u \cdot \nabla \Delta \theta + 2 \sum_{i=1}^{2} \nabla u_i \cdot \nabla (\partial_i \theta), \]

\[ \Delta (\text{rot} \omega \cdot \nabla \theta) = (\text{rot} \Delta \omega) \cdot \nabla \theta + \text{rot} \omega \cdot \nabla (\Delta \theta) + 2 \sum_{i=1}^{2} \text{rot} (\partial_i \omega) \cdot \nabla (\partial_i \theta) \]

and set \(\zeta = \Delta \theta\) in the above evolution equation to obtain

\[ \zeta_t - \Delta \zeta = -\Delta u \cdot \nabla \theta - u \cdot \nabla \zeta - 2 \sum_{i=1}^{2} \nabla u_i \cdot \nabla (\partial_i \theta) \]

\[ + D \left( (\text{rot} \Delta \omega) \cdot \nabla \theta + \text{rot} \omega \cdot \nabla \zeta + 2 \sum_{i=1}^{2} \text{rot} (\partial_i \omega) \cdot \nabla (\partial_i \theta) \right) \]

\[ + D \partial_t \Delta \omega + \Delta u_2. \]  

(30)
We multiply the above equation by \( \zeta \) and integrate over \( \Omega \) to get

\[
\frac{1}{2} \frac{d}{dt}|\zeta|^2 + \|\zeta\|^2 = -b(\Delta u, \theta, \zeta) - b(u, \zeta, \zeta) - 2 \sum_{i=1}^{2} \int_{\Omega} \nabla u_i \cdot \nabla (\partial_x \theta) \zeta \, dx
\]

\[
+ D \left( \int_{\Omega} (\text{rot} \, \Delta \omega) \cdot \nabla \theta \, dx + \int_{\Omega} (\text{rot} \, \omega \cdot \nabla \zeta) \, dx \right)
\]

\[
+ 2D \sum_{i=1}^{2} \int_{\Omega} (\text{rot} \, \partial_x \omega) \cdot \nabla (\partial_x \theta) \zeta \, dx + D \int_{\Omega} \partial_1 \Delta \omega \zeta \, dx + \int_{\Omega} \Delta u_2 \zeta \, dx.
\]

We estimate the terms on the right-hand side of the above equation using (18) and the already established uniform estimate of \( \omega(t) \) in \( D(A^{3/2}) \), cf. (29). We get, respectively,

\[
|b(\Delta u, \theta, \zeta)| \leq C |A_S u||\nabla \theta||_{L^4} \|\zeta\|_{L^4} \leq C |A \theta||\zeta\| \leq \epsilon \|\zeta\|^2 + C(\epsilon),
\]

\[
b(u, \zeta, \zeta) = 0,
\]

\[
\left| \int_{\Omega} \nabla u_i \cdot \nabla (\partial_x \theta) \zeta \, dx \right| \leq \|\nabla u\|_{L^4} ||\theta||_{H^2} \|\zeta\|_{L^4} \leq C |A_S u||A \theta||\zeta\| \leq \epsilon \|\zeta\|^2 + C(\epsilon),
\]

\[
\left| \int_{\Omega} (\text{rot} \, \Delta \omega) \cdot \nabla \zeta \, dx \right| \leq C ||\omega||_{D(A^{3/2})} \|\nabla \theta\|_{L^4} \|\zeta\|_{L^4} \leq C |A \theta||\zeta\| \leq \epsilon \|\zeta\|^2 + C(\epsilon),
\]

\[
\int_{\Omega} (\text{rot} \, \omega \cdot \nabla \zeta) \, dx = \int_{\Omega} \text{rot} \, \omega \cdot \frac{1}{2} \nabla (\zeta)^2 \, dx = -\frac{1}{2} \int_{\Omega} \omega \text{rot} \, \nabla (\zeta)^2 \, dx = 0,
\]

\[
\left| \int_{\Omega} \text{rot} \, (\partial_x \omega) \cdot \nabla (\partial_x \theta) \zeta \, dx \right| \leq C ||\text{rot} \, (\partial_x \omega)||_{L^4} |A \theta||\zeta\|_{L^4} \leq C ||\omega||_{D(A^{3/2})} \|\zeta\| \leq \epsilon \|\zeta\|^2 + C(\epsilon),
\]

\[
\left| \int_{\Omega} \partial_1 \Delta \omega \zeta \, dx \right| \leq |A \omega||\zeta| \leq \epsilon \|\zeta\|^2 + C(\epsilon),
\]

\[
\left| \int_{\Omega} \Delta u_2 \zeta \, dx \right| \leq |A_S u||\zeta| \leq C.
\]

The above estimates and (31) yield the bound

\[
\frac{d}{dt}|\zeta|^2 + \|\zeta\|^2 \leq C,
\]

for a suitable constant \( C \). We integrate this bound from \( t \geq 0 \) to \( t + 1 \) to get

\[
\int_{t}^{t+1} |\zeta(s)|^2 \, ds \leq C.
\]

In the final step we multiply (30) by \( A \zeta \) and integrate over \( \Omega \), whence

\[
\frac{1}{2} \frac{d}{dt} \|\zeta\|^2 + |A \zeta|^2 = -b(\Delta u, \theta, A \zeta) - b(u, \zeta, A \zeta) - 2 \sum_{i=1}^{2} \int_{\Omega} \nabla u_i \cdot \nabla (\partial_x \theta) A \zeta', \, dx
\]

\[
+ D \left( \int_{\Omega} (\text{rot} \, \Delta \omega) \cdot \nabla \theta) A \zeta \, dx + \int_{\Omega} (\text{rot} \, \omega \cdot \nabla \zeta) A \zeta \, dx \right)
\]
We denote by $P^m_S : V_S \to V_S$ the orthogonal projection onto the subspace spanned by the first $m$ eigenvectors of the Stokes operator, i.e.

$$P^m_S(u) = \sum_{j=1}^{m} (u, v_j)v_j, \quad \text{for } u \in V_S.$$
and by $P^k : V \rightarrow V$ the orthogonal projection onto the subspace spanned by the first $k$ eigenvectors of the Laplace operator $A$, i.e.

$$P^k \omega = \sum_{j=1}^{k} (\omega, w_j)w_j, \quad \text{for } \omega \in V.$$  

Let $\mathcal{A}$ be the differential operator acting from $D(\mathcal{A}) = D(S_A) \times D(A) \times D(A)$ to $\mathcal{H}$ defined by

$$\mathcal{A}(u, \omega, \theta) = (\kappa_1 A_S u, \kappa_2 A \omega, A \theta), \quad \text{(34)}$$

for $(u, \omega, \theta) \in D(\mathcal{A})$. Clearly, $\mathcal{A}$ is positive self-adjoint operator. We can relabel the countable set $\{(\sqrt{\kappa_1} v_j, 0, 0)\}_{j \geq 1} \cup \{(0, \sqrt{\kappa_2} w_j, 0)\}_{j \geq 1} \cup \{(0, 0, w_j)\}_{j \geq 1} \subset \mathcal{H}$ so that

$$\{e_j\}_{j \geq 1} = \{(\sqrt{\kappa_1} v_j, 0, 0)\}_{j \geq 1} \cup \{(0, \sqrt{\kappa_2} w_j, 0)\}_{j \geq 1} \cup \{(0, 0, w_j)\}_{j \geq 1}$$

and $(e_j)$ is the orthonormal basis in $\mathcal{H}$ consisting of eigenvectors of $\mathcal{A}$ and satisfying

$$\mathcal{A}e_j = \lambda_j e_j,$$

where $(\lambda_j)$ is the increasing sequence of eigenvalues of $\mathcal{A}$. Then we have $\mathcal{V} = D(\mathcal{A}^{1/2})$ and, for $(u_1, \omega_1, \theta_1) \in \mathcal{V}$ we have

$$\left( \mathcal{A}^{1/2}(u_1, \omega_1, \theta_1), \mathcal{A}^{1/2}(u_2, \omega_2, \theta_2) \right) = \frac{1}{\kappa_1} \left( \sqrt{\kappa_1} A_S^{1/2} u_1, \sqrt{\kappa_1} A_S^{1/2} u_1 \right) + \frac{1}{\kappa_2} \left( \sqrt{\kappa_2} A^{1/2} \omega_1, \sqrt{\kappa_2} A^{1/2} \omega_2 \right)\right) = \left( (u_1, \omega_1, \theta_1), (u_2, \omega_2, \theta_2) \right)$$

Thus, we have

$$\|e_j\|^2 = \left( \mathcal{A}^{1/2} e_j, \mathcal{A}^{1/2} e_j \right) = (\mathcal{A} e_j, e_j) = \lambda_j |e_j|^2 = \lambda_j, \quad \text{(35)}$$

since each $e_j$ is normalized in $\mathcal{H}$.

We consider the orthogonal projection $P^n$ in $\mathcal{V}$ onto the first $n$ eigenvectors of $\mathcal{A}$, i.e.

$$P^n(u, \omega, \theta) = \sum_{j=1}^{n} ((u, \omega, \theta), e_j)e_j, \quad \text{for } (u, \omega, \theta) \in \mathcal{V}.$$  

Observe that, for every $(u, \omega, \theta) \in \mathcal{V}$ and $n \geq 1$, there are nonnegative integers $m$, $k$ and $l$ such that $n = m + k + l$ and

$$P^n(u, \omega, \theta) = (P^n_S u, P^k, P^l \theta).$$

In the sequel we will simply write $P = P^n$, $P_S = P^n_S$, $P = P^k$, and $P = P^l$, avoiding the superindexes for the sake of the ease of notation. Note that it is always clear from the context which projection $P$ is considered.

Let $(\overline{u}, \overline{\omega}, \overline{\theta})$ and $(\tilde{u}, \tilde{\omega}, \tilde{\theta})$ be two strong solutions to (1)–(4) starting at $(\overline{u}_0, \overline{\omega}_0, \overline{\theta}_0)$ and $(\tilde{u}_0, \tilde{\omega}_0, \tilde{\theta}_0)$, respectively. We put $(u, \omega, \theta) = (\overline{u} - \tilde{u}, \overline{\omega} - \tilde{\omega}, \overline{\theta} - \tilde{\theta})$. Then $u$ satisfies the equation

$$\frac{1}{P_T} \left( u_t + \nabla p \right) - (1 + K) \Delta u = \frac{1}{P_T} \left( (\tilde{u} \cdot \nabla) \tilde{u} - (\bar{u} \cdot \nabla) \bar{u} \right) + 2K \text{rot} \omega + R \theta e_2. \quad \text{(36)}$$

1Note that although $P^n_S$ and $P^k$ are projections into $V_S$ and $V$, respectively, we use the scalar product from $H_S$ and $H$.

2Note that although scaling changes the eigenvalues it does not affect eigenvectors.

3Here we have $p = \overline{p} - \bar{p}$.
We take the scalar product of the above equation and $A_S P_S^n u = A_S P_S u$, and integrate over $\Omega$

$$\frac{1}{2 \Pr} \frac{d}{dt} \|P_S u\|^2 + (1 + K)|A_S P_S u|^2 = \frac{1}{\Pr} \left( b_S(-u + \tilde{\nu}, A_S P_S u) - b_S(\tilde{\nu}, u + \tilde{\nu}, A_S P_S u) \right)$$

$$+ 2K \left( \text{rot} \omega, A_S P_S u \right) + \text{Ra}(\theta e_2, A_S P_S u).$$

We end with

$$\frac{1}{2 \Pr} \frac{d}{dt} \|P_S u\|^2 + (1 + K)|A_S P_S u|^2 = -\frac{1}{\Pr} \left( b_S(u, \tilde{\nu}, A_S P_S u) + b_S(\tilde{\nu}, u, A_S P_S u) \right)$$

$$+ 2K \left( \text{rot} \omega, A_S P_S u \right) + \text{Ra}(\theta e_2, A_S P_S u).$$

We estimate

$$|b_S(u, \tilde{\nu}, A_S P_S u)| \leq \|u\|_{L^4} \|\nabla \tilde{\nu}\|_{L^4} |A_S P_S u| \leq cK_0 \|u\| |A_S P_S u|,$$

$$|b_S(\tilde{\nu}, u, A_S P_S u)| \leq \|\tilde{\nu}\|_{L^\infty} \|u\| |A_S P_S u| \leq cK_0 \|u\| |A_S P_S u|,$$

$$(\text{rot} \omega, A_S P_S u) \leq \|\omega\| |A_S P_S u|,$$

$$(\theta e_2, A_S P_S u) \leq c\|\theta\| |A_S P_S u|.$$

We end with

$$\frac{1}{2} \frac{d}{dt} \|P_S u\|^2 \geq -\Pr(1 + K)|A_S P_S u|^2 - c|A_S P_S u| (\|u\| + \|\omega\| + \|\theta\|)$$

(37)

for a suitable constant $c > 0$.

If we take the inner product of (36) and $A_S(I - P_S)u$, and integrate over $\Omega$ then the similar estimates yield

$$\frac{1}{2} \frac{d}{dt} \|(I - P_S)u\|^2 \leq -\Pr(1 + K)|A_S(I - P_S)u|^2 + c|A_S(I - P_S)u| (\|u\| + \|\omega\| + \|\theta\|).$$

(38)

Now, we proceed as above with the equation for $\omega$

$$\frac{M}{\Pr} \omega_t - L \Delta \omega = \frac{M}{\Pr} \left( \tilde{\nu} \cdot \nabla \tilde{\omega} - \tilde{\pi} \cdot \nabla \overline{\omega} \right) + 2K \text{rot} u - 4K \omega.$$

We multiply the above equation by $AP^k \omega = AP \omega$ and integrate over $\Omega$

$$\frac{M}{2 \Pr} \frac{d}{dt} \|P \omega\|^2 + L |AP \omega|^2 = -\frac{M}{\Pr} \left( b(u, \tilde{\omega}, AP \omega) + b(\tilde{\pi}, \omega, AP \omega) \right)$$

$$+ 2K \left( \text{rot} u, AP \omega \right) - 4K (\omega, AP \omega).$$

We estimate as above

$$|b(u, \tilde{\omega}, AP \omega)| \leq \|u\|_{L^4} \|\nabla \tilde{\omega}\|_{L^4} |AP \omega| \leq cK_0 \|u\| |AP \omega|,$$

$$|b(\tilde{\pi}, \omega, AP \omega)| \leq \|\tilde{\pi}\|_{L^\infty} \|\omega\| |AP \omega| \leq cK_0 \|\omega\| |AP \omega|,$$

$$(\text{rot} u, AP \omega) \leq \|u\| |AP \omega|,$$

$$|\omega, AP \omega) \leq c\|\omega\| |AP \omega|.$$

We end with

$$\frac{1}{2} \frac{d}{dt} \|P \omega\|^2 \geq -\frac{L \Pr}{M} |AP \omega|^2 - c|AP \omega| (\|u\| + \|\omega\|),$$

(39)

for a suitable constant $c > 0$. If we replace $AP \omega$ by $A(I - P)\omega$ and repeat the calculations we get

$$\frac{1}{2} \frac{d}{dt} \|(I - P)\omega\|^2 \leq -\frac{L \Pr}{M} |A(I - P)\omega|^2 + c|A(I - P)\omega| (\|u\| + \|\omega\|).$$

(40)
Lastly, we test the equation for $\theta$

$$\frac{d}{dt} \theta - \Delta \theta = \tilde{u} \cdot \partial - \tilde{\pi} \cdot \nabla \tilde{\theta} + D \text{rot } \varpi \cdot \nabla \tilde{\theta} - D \text{rot } \tilde{\omega} \cdot \nabla \hat{\theta} + D \frac{\partial \omega}{\partial x_2} + u_2$$

with the function $AP\theta' = AP\theta$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|P\theta\|^2 + |AP\theta|^2 = -b(\pi, \theta, AP\theta) - b(u, \hat{\theta}, AP\theta)$$

$$+ \int_{\Omega} (\text{rot } \tilde{\omega} \cdot \nabla \theta) AP\theta \, dx + \int_{\Omega} (\text{rot } \omega \cdot \nabla \tilde{\theta}) AP\theta \, dx + D \left( \frac{\partial \omega}{\partial x_2}, AP\theta \right) + (u_2, AP\theta).$$

We estimate as follows

$$|b(\pi, \theta, AP\theta)| \leq \|\pi\|_{L^\infty} |\theta|| |AP\theta| \leq cK_0 |\theta|| |AP\theta|,$$

$$|b(u, \hat{\theta}, AP\theta)| \leq \|u\|_{L^4} |\vartheta|| |AP\theta| \leq cK_0 |\theta|| |AP\theta|,$$

$$\left| \int_{\Omega} (\text{rot } \tilde{\omega} \cdot \nabla \theta) AP\theta \, dx \right| \leq \|\text{rot } \tilde{\omega}\|_{L^\infty} |\theta|| |AP\theta| \leq cK_0 |\theta|| |AP\theta|,$$

$$\left| \int_{\Omega} (\text{rot } \omega \cdot \nabla \tilde{\theta}) AP\theta \, dx \right| \leq \|\omega\| |\nabla \tilde{\theta}|_{L^\infty} |AP\theta| \leq cK_0 \|\omega\|| |AP\theta|,$$

$$|((\partial_{x_2} \omega, AP\theta))| \leq \|\omega\|| |AP\theta|,$$

$$|(u_2, AP\theta)| \leq c\|u\|| |AP\theta|.$$ 

Note that the third and fourth of the above estimates were the reason why we needed to derive in Step 1 the estimates (19). This yields

$$\frac{1}{2} \frac{d}{dt} \|P\theta\|^2 \geq -|AP\theta|^2 - c|AP\theta| (\|u\| + \|\omega\| + \|\theta\|) \quad (41)$$

for a suitable constant $c > 0$. Testing the equation for $\theta$ with the function $A(I - P)\theta$ and repeating the calculations leads to

$$\frac{1}{2} \frac{d}{dt} \|(I - P)\theta\|^2 \leq -|A(I - P)\theta|^2 + c|A(I - P)\theta| (\|u\| + \|\omega\| + \|\theta\|). \quad (42)$$

We take $n, m, k$ and $l$ such that $n = m + k + l$ and

$$\mathbb{P} = \mathbb{P}^n = P^m_\Sigma \oplus P^k \oplus P^l.$$

Let us denote

$$p = \|\mathbb{P}(u, \omega, \theta)\|$$

and

$$q = \|(I - \mathbb{P})(u, \omega, \theta)\|.$$ 

By the definition of scalar product we have

$$|\mathbb{A}(\mathbb{P}(u, \omega, \theta))|^2 = \frac{1}{\kappa_1} (\kappa_1 A_S P_S u, \kappa_1 A_S P_S u) + \frac{1}{\kappa_2} (\kappa_2 A_P \omega, \kappa_2 A_P \omega) + (AP\theta, AP\theta)$$

$$= \kappa_1 |A_S P_S u|^2 + \kappa_2 |AP\omega|^2 + |AP\theta|^2. \quad (43)$$

We add (37), (39) and (41), use (43) and choose a suitable constant $c > 0$ to get

$$\frac{1}{2} \frac{d}{dt} p^2 \geq -|\mathbb{A}(\mathbb{P}(u, \omega, \theta))|^2 - c|\mathbb{A}(\mathbb{P}(u, \omega, \theta))| (\|u, \omega, \theta\|).$$

Similarly, we add (38), (40) and (42) and, modifying $c > 0$ if necessary, we obtain

$$\frac{1}{2} \frac{d}{dt} q^2 \leq -|\mathbb{A}(I - \mathbb{P})(u, \omega, \theta)|^2 + c|\mathbb{A}(I - \mathbb{P})(u, \omega, \theta)| (\|u, \omega, \theta\|).$$
Since $\mathbb{AP}(u,\omega,\theta) = \sum_{j=1}^{n} ((u,\omega,\theta), e_j) \lambda_j e_j$ and by (35), we have
\[
|\mathbb{AP}(u,\omega,\theta)|^2 = \sum_{j=1}^{n} \lambda_j^2 ((u,\omega,\theta), e_j)^2 \leq \lambda_n \sum_{j=1}^{n} ((u,\omega,\theta), e_j)^2 \lambda_j
\]
\[
= \lambda_n \sum_{j=1}^{n} ((u,\omega,\theta), e_j)^2 \|e_j\|^2 = \lambda_n \|\mathbb{P}(u,\omega,\theta)\|^2.
\]

As a result, we have
\[
|\mathbb{AP}(u,\omega,\theta)| \leq \frac{\lambda_1}{2n} p
\]
and, similarly,
\[
|\mathbb{A}(I - \mathbb{P})(u,\omega,\theta)| \geq \frac{\lambda_1}{2n} q.
\]
The above inequalities yield
\[
\frac{1}{2} \frac{d}{dt} p^2 \geq -|\mathbb{AP}(u,\omega,\theta)| \left( \frac{\lambda_1}{2n} p + c \|u,\omega,\theta\| \right)
\]
and
\[
\frac{1}{2} \frac{d}{dt} q^2 \leq -|\mathbb{A}(I - \mathbb{P})(u,\omega,\theta)| \left( \frac{\lambda_1}{2n} q - c \|u,\omega,\theta\| \right).
\]
Since $\|u,\omega,\theta\| = \sqrt{p^2 + q^2} \leq p + q$, we have
\[
\frac{1}{2} \frac{d}{dt} p^2 \geq -\lambda_n^{1/2} p \left( \frac{\lambda_1}{2n} + c \right) p + cp
\]
and
\[
\frac{1}{2} \frac{d}{dt} q^2 \leq -|\mathbb{A}(I - \mathbb{P})(u,\omega,\theta)| \left( \frac{\lambda_1}{2n} - c \right) q - cp.
\]
If only
\[
(\lambda_n^{1/2} - c)q - cp \geq 0
\]
then
\[
\frac{1}{2} \frac{d}{dt} q^2 \leq -\lambda_n^{1/2} q \left( \frac{\lambda_1}{2n} - c \right) q - cp.
\]
We are about to show that the squeezing condition holds. If
\[
q(1) \leq p(1)
\]
then we are done. Thus, we may assume that
\[
q(1) > p(1)
\]
and that $n$ is sufficiently large so that
\[
\lambda_n^{1/2} - c > 2c.
\]
The above inequalities imply that
\[
\left( \lambda_n^{1/2} - c \right) q(t) > 2cp(t)
\]
in a certain neighbourhood of $t = 1$. There are two possibilities:
1. either (48) holds for every $t \in [0,1]$ or
2. (48) holds for $t \in (t_1,1]$, where $t_1 \geq 0$ and
\[
\left( \lambda_n^{1/2} - c \right) q(t_1) = 2cp(t_1).
\]
If (1) is satisfied then, from (45) and (46), we get

\[
\frac{d}{dt} q^2 \leq -\frac{\lambda_1}{2} \left( \lambda_{n}^{1/2} - c \right) q^2, \quad \text{for } t \in [0, 1].
\]

This yields

\[
q^2(1) \leq q^2(0) \exp \left( -\frac{\lambda_1}{2} \left( \lambda_{n}^{1/2} - c \right) \right).
\]

We have

\[
\| (u, \omega, \theta)(1) \| \leq \sqrt{2} q(1) \leq \sqrt{2} \exp \left( -\frac{1}{2} \frac{\lambda_1}{2} \left( \lambda_{n}^{1/2} - c \right) \right) \|(u_0, \omega_0, \theta_0)\|
\]

and appropriate choice of \( n \) ends the argument.

If (2) is satisfied then we transform (44) and (46) into

\[
\frac{d}{dt} p \geq -\frac{\lambda_1}{2} \left( (\lambda_{n}^{1/2} + c)p + cq \right),
\]

\[
\frac{d}{dt} q \leq -\frac{\lambda_1}{2} \left( (\lambda_{n}^{1/2} - c)q - cp \right),
\]

where we used the fact that \( \frac{1}{2} \frac{d}{dt} p^2 = p \frac{d}{dt} p \) and \( \frac{1}{2} q^2 = q \frac{d}{dt} q \). We consider the auxiliary function

\[
\Phi(p, q) = (p + q) \exp \left( \frac{\lambda_{n}^{1/2} q}{c(p + q)} \right).
\]

The inequalities (49) imply that, for every \( t \in [t_1, 1] \),

\[
\frac{d}{dt} \Phi(p(t), q(t)) \leq 0.
\]

Thus we obtain

\[
\Phi(p(1), q(1)) \leq \Phi(p(t_1), q(t_1)).
\]

We use the second part of condition (2) to get

\[
\Phi(p(t_1), q(t_1)) = \frac{\lambda_{n}^{1/2} + c}{2c} q(t_1) \exp \left( \frac{2\lambda_{n}^{1/2}}{\lambda_{n}^{1/2} + c} \right)
\]

and we use (47) to obtain

\[
\Phi(p(1), q(1)) \geq (p(1) + q(1)) \exp \left( \frac{\lambda_{n}^{1/2} q(1)}{c(p(1) + q(1))} \right) \geq q(1) \exp \left( \frac{\lambda_{n}^{1/2}}{2c} \right).
\]

Hence and from (47), we have

\[
\| (u, \omega, \theta)(1) \| \leq \sqrt{2} q(1) \leq \frac{\sqrt{2}}{2} \left( \lambda_{n}^{1/2} + c \right) \exp \left( \frac{2\lambda_{n}^{1/2}}{\lambda_{n}^{1/2} + c} - \frac{\lambda_{n}^{1/2}}{2c} \right) q(t_1)
\]

\[
\leq c \lambda_{n}^{1/2} \exp \left( -\frac{\lambda_{n}^{1/2}}{2c} \right) q(t_1)
\]

\[
\leq c \lambda_{n}^{1/2} \exp \left( -\frac{\lambda_{n}^{1/2}}{2c} \right) \|(u, \omega, \theta)(t_1)\|
\]

\[
\leq c \lambda_{n}^{1/2} \exp \left( -\frac{\lambda_{n}^{1/2}}{2c} \right) L(t_1) \|(u_0, \omega_0, \theta_0)\|.
\]
where \( \bar{c} > 0 \) does not depend on \( n \).\(^4\)

Summing up, for every \( \delta \in (0,1) \) there is \( n \geq 1 \) such that either

\[
\left\| (I - P^n) \left( S(1)(\bar{u}_0, \bar{\omega}_0, \bar{\theta}_0) - S(1)(\tilde{u}_0, \tilde{\omega}_0, \tilde{\theta}_0) \right) \right\| \\
\leq \left\| P^n \left( S(1)(\bar{u}_0, \bar{\omega}_0, \bar{\theta}_0) - S(1)(\tilde{u}_0, \tilde{\omega}_0, \tilde{\theta}_0) \right) \right\|
\]

or

\[
\left\| S(1)(\bar{u}_0, \bar{\omega}_0, \bar{\theta}_0) - S(1)(\tilde{u}_0, \tilde{\omega}_0, \tilde{\theta}_0) \right\| \leq \delta \left\| (\bar{u}_0, \bar{\omega}_0, \bar{\theta}_0) - (\tilde{u}_0, \tilde{\omega}_0, \tilde{\theta}_0) \right\|
\]

Thus we have proved the following result.

**Theorem 3.10.** The global attractor \( \mathcal{A} \) is a bounded set in \( D(A_S) \times D(A^{3/2}) \times D(A^{3/2}) \). Moreover, its fractal dimension in \( \mathcal{H} \) and \( \mathcal{V} \) is finite, i.e.

\[
d^f_{\mathcal{H}}(\mathcal{A}) \leq d^f_{\mathcal{V}}(\mathcal{A}) < \infty.
\]

4. **Surface quasigeostrophic equation on torus.** The second problem from mathematical physics which will serve as an example for our abstract framework will be the forced and critically damped surface quasigeostrophic (SQG) equation on two-dimensional torus \( \mathbb{T}^2 \). The equation models the temperature \( \theta \) on the boundary of a rapidly rotating half space with small Rosby and Eckmann numbers and constant potential vorticity, see [14]. We consider the vanishing viscosity weak solutions, namely, the class of the weak solutions being the limits of the vanishing viscosity approximations. For this class of weak solutions, although their uniqueness is unknown, Caffarelli and Vasseur [5] have proved that the multivalued semigroup has the regularizing effect (cf. also [25]) and that using the De Giorgi iteration method it is possible to bootstrap the regularity from \( L^2(\mathbb{T}^2) \) to \( L^\infty(\mathbb{T}^2) \). Existence of the global attractor for the multivalued semiflow governed by the viscosity weak solutions have been proved by Cheskidov and Dai [8] using the formalism of evolutionary systems developed by Cheskidov and Foiaş [9]. On the other hand, the method of nonlinear lower bounds developed by Constantin and Vicol [16] turned out useful to prove the existence of the global attractor for the unique strong solutions [15, 13]. In [18] it was proved that the global attractor for weak solutions obtained in [8] coincides with the global attractor of [15, 13] for the strong solutions but additional requirement of the strong convergence in \( L^2(\mathbb{T}^2) \) of the initial data needed for the multivalued semiflow to be strict was added in the definition of the weak solution. Using the developed formalism we show in this chapter that the requirement of the strong convergence of the initial data is unnecessary and actually the global attractor for the class of vanishing viscosity weak solutions considered in [5] (for the unforced case) and [8] coincides with the global attractor for the strong solutions considered in [15, 13].

4.1. **Problem formulation and definition of weak and strong solutions.** We briefly remind the setup of the problem. The two dimensional torus \( \mathbb{T}^2 \) is defined as \( (-\pi, \pi)^2 \) and \( C_p^\infty(\mathbb{T}^2) \) is the space of restrictions to \( \mathbb{T}^2 \) of smooth functions which are \( 2\pi \)-periodic in both variables and mean free. In the sequel we will always use the shorthand notation for various spaces of functions defines on \( \mathbb{T}^2 \), for example we will write \( H^s \) for the closure of \( C_p^\infty(\mathbb{T}^2) \), in \( H^s(\mathbb{T}^2) \) norm. All spaces consist of functions which are mean free and \( 2\pi \)-periodic with respect to both variables. The

\[^4\]Here, we use the Lipschitz condition given in Lemma 3.6.
scalar product on $L^2 = H^0$ will be denoted by $\langle \cdot, \cdot \rangle$ and the same notation will be used for the duality between various spaces and their duals.

For a periodic and mean free distribution $\phi$ on $\mathbb{T}^2$ we can define its Fourier coefficients by

$$\hat{\phi}_k = \frac{1}{(2\pi)^2} \langle \phi, e^{-ik \cdot x} \rangle \quad \text{for} \quad k \in \mathbb{Z}_2^2,$$

where $\mathbb{Z}_2^2 = \mathbb{Z}^2 \setminus \{(0,0)\}$, and if the distribution $\phi$ is defined by a function, then its Fourier coefficients are given by the integrals

$$\hat{\phi}_k = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \phi(x) e^{-ik \cdot x} \, dx \quad \text{for} \quad k \in \mathbb{Z}_2^2.$$

Using the Fourier coefficients it is very easy to define the scale of Sobolev spaces $H^\sigma$ using the formula

$$H^\sigma = \left\{ \phi \in \mathcal{D}' : \|\phi\|_{H^\sigma}^2 = \sum_{k \in \mathbb{Z}_2^2} |k|^{2\sigma} |\hat{\phi}_k|^2 < \infty \right\}.$$

For $\sigma \in \mathbb{R}$ one can define the fractional Laplacian by means of the Fourier transform by the formula

$$(-\Delta)^{\sigma} \phi(x) = \sum_{k \in \mathbb{Z}_2^2} |k|^{2\sigma} \hat{\phi}_k e^{ik \cdot x} \quad \text{or} \quad (-\Delta)^{\sigma} \phi_k = |k|^{2\sigma} \hat{\phi}_k.$$

The fractional Laplacian is a linear and bounded operator from $H^\sigma$ to its dual $H^{-\sigma}$ and the norm on $H^\sigma$ can be equivalently defined by the formula $\|\phi\|_{H^\sigma} = \|(-\Delta)^{\sigma/2} \phi\|_{L^2}$. The Zygmund operator is defined as $\Lambda = (-\Delta)^{1/2}$, and, in general, $\Lambda^\sigma = (-\Delta)^{\sigma/2}$. Finally the $j$-th Riesz transform for $j = 1, 2$ is defined in terms of the Fourier coefficients as

$$\mathcal{R}_j \phi_k = \frac{ik_j}{|k|} \hat{\phi}_k \quad \text{for} \quad k \in \mathbb{Z}_2^2.$$

Moreover, there holds

$$\mathcal{R}_j \phi = \partial_{x_j} \Lambda^{-1} \phi.$$

We define $\mathcal{R}^\perp = (-\mathcal{R}_2, \mathcal{R}_1)$, whence $\mathcal{R} = (-\partial_{x_2} \Lambda^{-1}, \partial_{x_1} \Lambda^{-1})$. If $\phi \in H^{1/2}$, then $\mathcal{R}^\perp \phi \in H^{1/2} \times H^{1/2}$ is always divergence free.

The considered initial and boundary value problem has the form

$$\begin{cases}
\partial_t \theta + u \cdot \nabla \theta + (-\Delta)^{1/2} \theta = f, \\
u = \mathcal{R}^\perp \theta = \nabla \perp (-\Delta)^{-1/2} \theta, \\
\theta(0) = \theta_0, \quad \int_{\mathbb{T}^2} \theta_0(x) \, dx = 0.
\end{cases} \quad (50)$$

Note, that the function $f$ is always assumed to be mean free and time independent.

We are in position to define the weak solution for the problem (50)

**Definition 4.1.** A function $\theta \in L^2_{loc}(0, \infty; H^{1/2}) \cap C_w([0, \infty); L^2)$ is called a weak solution of (50) if for every $\varphi \in C_0^\infty((0, \infty); C_p^\infty)$ there holds for every $T > 0$

$$- \int_0^T \langle \theta(t), \partial_t \varphi(t) \rangle \, dt - \langle \theta_0, \varphi(0) \rangle + \langle \theta(T), \varphi(T) \rangle$$

$$- \int_0^T \langle \theta(t), u(t) \cdot \nabla \varphi(t) \rangle \, dt + \int_0^T \langle \Lambda^{1/2} \theta(t), \Lambda^{1/2} \varphi(t) \rangle \, dt = \int_0^T \langle f, \varphi(t) \rangle \, dt.$$
The existence of weak solutions (for $\theta_0 \in L^2$ and $f \in L^2$) has been proved in [32]. We will be interested in the subclass of all weak solutions, namely the vanishing viscosity weak solutions. They are defined as the limits of the following auxiliary problems parameterized by $\varepsilon > 0$

$$\partial_t \theta^\varepsilon - \varepsilon \Delta \theta^\varepsilon + R^\varepsilon \theta^\varepsilon \cdot \nabla \theta^\varepsilon + \Lambda \theta^\varepsilon = f.$$  
(51)

The class of vanishing viscosity weak solutions is defined in the following way.

**Definition 4.2.** A function $\theta \in L^2_{\text{loc}}([0,\infty); H^{1/2}) \cap C_w([0,\infty); L^2)$ is called a vanishing viscosity weak solution of (50) if it is a weak solution given by (4.1) and there exists a sequence $\varepsilon_n \to 0$ and corresponding solutions $\theta^{\varepsilon_n}$ to (51) with $\varepsilon = \varepsilon_n$ such that $\theta^{\varepsilon_n} \to \theta$ in $C_w([0,T]; L^2)$ for every $T > 0$.

**Remark 1.** In [18] the class of vanishing viscosity weak solutions with strongly converging initial data is considered. This restriction is needed to guarantee the strictness of the multivalued semiflow. Since in the abstract framework of the present article we do not need the strictness, we can work with the class of weak solutions given in Definition 4.2 which is possibly broader than that of [18]. Also note that Definition 4.2 is consistent with the corresponding definitions of [5, 8].

The proof of the existence of a vanishing viscosity weak solutions for every $\theta_0 \in L^2$ and $f \in L^\infty$ follows the lines of [5, Appendix C]. The following important property of vanishing viscosity weak solutions have been proved in [8].

**Theorem 4.3.** Let $\theta$ be a vanishing viscosity weak solution with $f \in L^\infty$ and $\theta_0 \in L^2$. Then for every $t > 0$ there holds $\theta(t) \in L^\infty$. Moreover, the following estimate holds for $t > 0$

$$\|\theta(t)\|_{L^\infty} \leq C \|\theta_0\|_{L^2} + C \|f\|_{L^\infty} \left(1 + \frac{1}{\sqrt{t}}\right),$$
and the next energy equation holds for every vanishing viscosity weak solution and every $0 \leq t_0 \leq t$

$$\frac{1}{2} \|\theta(t)\|^2_{L^2} + \int_{t_0}^t \|\Lambda^{1/2} \theta(s)\|^2_{L^2} ds = \frac{1}{2} \|\theta(t_0)\|^2_{L^2} + \int_{t_0}^t (f, \theta(s)) ds.$$  
(52)

**Remark 2.** It is clear that the above result implies that every vanishing viscosity weak solution belongs to $C([0,\infty); L^2)$.

4.2. **Attractors for single and multivalued semiflow and their equality.**

Define the (possibly multivalued) operator $S_{L^2}(t) : L^2 \to P(L^2)$ by the formula

$$S_{L^2}(t) \theta_0 = \{\theta(t) : \theta \text{ is a vanishing viscosity weak solution with } \theta(0) = \theta_0\}.$$  

The next result follows from the definition of vanishing viscosity weak solutions and their existence.

**Lemma 4.4.** Assume that $f \in L^\infty$. The mappings $\{S_{L^2}(t)\}_{t \geq 0}$ constitute a multivalued semiflow.

**Proof.** The fact that $S_{L^2}(t) \theta_0$ is nonempty follows from the existence of the vanishing viscosity weak solution for every $\theta_0 \in L^2$. On the other hand, the inclusion $S_{L^2}(s + t) \theta_0 \subseteq S_{L^2}(s) S_{L^2}(t) \theta_0$ follows from the so called translation property. Indeed, directly from Definitions 4.1 and 4.2 it follows that if $\theta$ is a vanishing viscosity weak solution with $\theta(0) = \theta_0$ then $\theta' = \theta(\cdot + t)$ is a vanishing viscosity weak solution with $\theta'(0) = \theta(t)$ and the proof is complete. $\square$
To use the framework of [13, 15] we will assume that $f \in L^\infty \cap H^1$, and under this regularity the class of strong solutions is defined on $H^1$. In fact we have the following result, cf. [15, Theorem 4.5], [13, Proposition 2.1].

**Theorem 4.5.** Let $f \in L^\infty \cap H^1$ and let $\theta_0 \in H^1$. There exists a unique strong solution $\theta \in C([0, \infty); H^1) \cap L^2_{loc}(0, \infty; H^{3/2})$ of the initial value problem (50).

Hence, we can define the single-valued operator $S_{H^1}(t) : H^1 \to H^1$ by means of the formula

$$S_{H^1}(t) \theta_0 = \{ \theta(t) : \theta \text{ is a strong solution with } \theta(0) = \theta_0 \}.$$ 

We will use the abstract Theorem 2.5 proved in Section 2.1 with $V = H^1$ and $H = L^2$ to get the equality of global attractors for strong and weak solutions. To use this theorem we need to show that its assumptions (ii), (iii), (iv), and (v) hold for $S_{L^2}$ and $S_{H^1}$. Hence, the next part of this chapter is devoted to the verification of these assumptions. Two of them follow directly from the results of [13]. The assertion (v) is expressed in the following

**Lemma 4.6.** [13, Theorem 6.1] There exists a constant $R = R(\|f\|_{L^\infty \cap H^1})$ such that for every $B \in \mathcal{B}(H^1)$ there exists $t_0 = t_0(B)$ such that the ball

$$B_0 = \{ \phi \in H^{3/2} : \|\phi\|_{H^{3/2}} \leq R \}$$

satisfies

$$\bigcup_{t \geq t_0} S_{H^1}(t)B \subset B_0.$$

On the other hand, the assertion (ii) directly follows from

**Lemma 4.7.** [13, Proposition 6.3] For every $t > 0$ the mapping $S_{H^1}(t)|_{B_0} : B_0 \to H^1$ is Lipschitz continuous in the topology of $H^1$.

In the next result we prove the assertion (iv).

**Lemma 4.8.** Let $B \in \mathcal{B}(L^2)$ and let $f \in L^\infty \cap H^1$. For every $\epsilon > 0$ there holds $S_{L^2}(\epsilon)B \in \mathcal{B}(H^1)$.

**Proof.** The result in fact follows from the consecutive a priori estimates obtained in [13, 15, 18] and uses the same bootstrapping argument as one which was used in [18]. Note that all estimates are obtained for the solutions of the problem with the added vanishing viscosity term, and, in consequence, they are preserved in the weak limit. The result is obtained by the bootstrap argument which consists of four steps.

**Step 1.** Bound in $L^\infty$. We have already stated the result which is the first step of the bootstrapping argument, namely the $L^\infty$ bound established in Theorem 4.3. So, after $t \geq \epsilon/4$ the trajectories starting from $B$ are absorbed by a bounded set in $L^\infty$.

**Step 2.** Bound in the Hölder seminorm $C^\alpha$. The second step of bootstrapping follows from [13, Lemma 4.7, Remark 4.8] where it has been established that if $\theta_0 \in L^\infty$, then for a certain $\alpha \in (0, 1/4]$, depending on $\|\theta_0\|_{L^\infty}$ and $\|f\|_{L^\infty}$, there holds

$$[\theta(t)]_{C^\alpha} \leq c(\|\theta_0\|_{L^\infty} + \|f\|_{L^\infty}) \text{ for } t \geq \epsilon/4.$$
Step 3. Bound in $H^{1/2}$. This bound follows from $[18$, Theorem B.1$]$, whence it has been proved that if for some $\beta \in (0, 1)$ we have the a priori bound

$$\|\theta(t)\|_{C^\beta} \leq K_\beta \text{ for every } t \geq 0,$$

then there holds a differential inequality

$$\frac{d}{dt}\|\theta(t)\|^2_{H^{1/2}} + \frac{1}{4}\|\theta(t)\|^2_{H^{1/2}} \leq cK^{\frac{4}{\beta}} + c\|f\|^2_{H^{1/2}}. \quad (53)$$

Integrating this inequality from $t - s$ to $t$ for $t \geq \epsilon/4$ and $s \in (0, \epsilon/4)$ yields

$$\|\theta(t)\|^2_{H^{1/2}} \leq \|\theta(t - s)\|^2_{H^{1/2}} + \frac{\epsilon}{4}cK_{\beta}^\frac{4}{\beta} + \frac{\epsilon}{4}c\|f\|^2_{H^{1/2}}.$$}

Integrating it once again over $s \in (0, \epsilon/4)$ yields

$$\|\theta(t)\|^2_{H^{1/2}} \leq \frac{4}{\epsilon} \int_{t-\epsilon}^t \|\theta(r)\|^2_{H^{1/2}} dr + \frac{\epsilon}{4}cK_{\beta}^\frac{4}{\beta} + \frac{\epsilon}{4}c\|f\|^2_{H^{1/2}}.$$}

The energy equation (52) implies that

$$\int_{t-\epsilon}^t \|\theta(r)\|^2_{H^{1/2}} dr \leq \|\theta(t - \epsilon)\|^2_{L^2} + \epsilon\|f\|_{L^2},$$

whence there follows the bound on $\|\theta(t)\|_{H^{1/2}}$ for $t = \epsilon/4$. The estimate on $\|\theta(t)\|_{H^{1/2}}$ for $t > \epsilon/4$ follows now directly from (53) by the Gronwall lemma.

Step 4. Bound in $H^1$. This bound also follows from $[18$, Theorem B.1$]$. Similar as in Step 3 if we have the a priori bound

$$\|\theta(t)\|_{C^\beta} \leq K_\beta \text{ for every } t \geq 0,$$

then there holds a differential inequality

$$\frac{d}{dt}\|\theta(t)\|^2_{H^1} + \frac{1}{4}\|\theta(t)\|^2_{H^1} \leq cK^{\frac{4}{\beta}} + c\|f\|^2_{H^1}. \quad (54)$$

Proceeding in a similar way as in Step 3 we obtain

$$\|\theta(t)\|^2_{H^1} \leq \frac{4}{\epsilon} \int_{t-\epsilon}^t \|\theta(r)\|^2_{H^1} dr + \frac{\epsilon}{4}cK_{\beta}^\frac{4}{\beta} + \frac{\epsilon}{4}c\|f\|^2_{H^1}.$$}

Since (53) yields

$$\int_{t-\epsilon}^t \|\theta(r)\|^2_{H^1} dr \leq 4\|\theta(t - 1)\|^2_{H^{1/2}} + \epsilon\|f\|^2_{H^{1/2}},$$

we obtain the needed assertion for $t = \epsilon/4$. The bound for $t > \epsilon/4$ follows from (54) and the Gronwall lemma.

Concluding, if $\theta_0 \in B \in \mathcal{B}(L^2)$, all trajectories starting from $\theta_0$ are absorbed by a bounded set in $H^1 \cap L^\infty \cap C^\beta$ after time $t = \epsilon$, and the proof is complete. Note that we have also proved that every vanishing viscosity weak solution belongs to $L^\infty(\epsilon, \infty); H^1 \cap L^\infty \cap C^\beta)$ for every $\epsilon > 0$. \hfill \square

It remains to obtain (iii). The proof of the following result follows the lines of $[15$, Theorem 4.4$]$.

Lemma 4.9. Let $\theta_0 \in H^1$ and let $f \in H^1 \cap L^\infty$. If $\theta$ is a strong solution of (50) and $\overline{\theta}$ is its vanishing viscosity weak solution with the same initial data $\theta_0$, then $\theta(t) = \overline{\theta}(t)$ for every $t \geq 0$. 

Proof. Fix $0 < t_1 < t$. Lemma 4.8 implies that $\overline{\theta} \in L^\infty(t_1, t; L^\infty \cap H^1)$. Choose the test function $\varphi \in C^\infty_0([t_1, t]; C^\infty_p)$. Then
\[
\int_{t_1}^{t} \langle \partial_t \overline{\theta}(s), \varphi(s) \rangle \, ds - \int_{t_1}^{t} \langle \overline{\theta}(s), \nabla \varphi(s) \rangle \, ds + \int_{t_1}^{t} \langle \Lambda^{1/2} \overline{\theta}(s), \Lambda^{1/2} \varphi(s) \rangle \, ds
= \int_{t_1}^{t} \langle f, \varphi(s) \rangle \, ds.
\]
After integration by parts
\[
\int_{t_1}^{t} \langle \overline{\theta}(s), \varphi(s) \rangle \, ds + \int_{t_1}^{t} \langle \overline{\psi}(s) \cdot \overline{\nabla} \overline{\theta}(s), \varphi(s) \rangle \, ds + \int_{t_1}^{t} \langle \Lambda \overline{\theta}(s), \varphi(s) \rangle \, ds = \int_{t_1}^{t} \langle f, \varphi(s) \rangle \, ds.
\]
(55)

Hence, in the sense of distributions,
\[\overline{\theta}_t = f - \overline{\psi} \cdot \overline{\nabla} \overline{\theta} - \Lambda \overline{\theta}.
\]
Note that $\overline{\psi} \cdot \overline{\nabla} \overline{\theta} \in L^2(t_1, t; H^{-1/2})$. Indeed, using the fact that the Riesz transform is a linear and bounded operator from $L^4$ to $L^4$,
\[\left| \int_{t_1}^{t} \langle \overline{\psi}(s) \cdot \overline{\nabla} \overline{\theta}(s), \varphi(s) \rangle \, ds \right| \leq \int_{t_1}^{t} \| \overline{\psi}(s) \|_{L^4} \| \overline{\nabla} \overline{\theta}(s) \|_{L^2} \| \varphi(s) \|_{L^4} \, ds
\leq C \| \overline{\theta} \|_{L^\infty(t_1, t; H^1)} \| \varphi \|_{L^2(t_1, t; H^{-1/2})}.
\]
It follows that $\overline{\theta}_t \in L^2(t_1, t; H^{-1/2})$. Since at least the same regularity of $\partial_t \theta$ holds for the strong solutions we subtract (55) written for $\theta$ and $\overline{\theta}$ and test by $\varphi = \overline{\theta} - \theta$.

This yields
\[
\frac{1}{2} \| \overline{\theta}(t) - \theta(t) \|^2_{L^2} + \int_{t_1}^{t} \| \overline{\theta}(s) - \theta(s) \|^2_{H^{1/2}} \, ds
+ \int_{t_1}^{t} \langle (\overline{\psi}(s) - \psi(s)) \cdot \nabla \theta(s), \overline{\theta}(s) - \theta(s) \rangle \, ds = \frac{1}{2} \| \overline{\theta}(t_1) - \theta(t_1) \|^2_{L^2}.
\]

Now
\[
\left| \int_{t_1}^{t} \langle (\overline{\psi}(s) - \psi(s)) \cdot \nabla \theta(s), \overline{\theta}(s) - \theta(s) \rangle \, ds \right|
\leq C \int_{t_1}^{t} \| \overline{\theta}(s) - \theta(s) \|_{L^4} \| \nabla \theta(s) \|_{L^4} \| \overline{\theta}(s) - \theta(s) \|_{L^2} \, ds
\leq \int_{t_1}^{t} \| \overline{\theta}(s) - \theta(s) \|^2_{H^{1/2}} \, ds + C \int_{t_1}^{t} \| \theta(s) \|^2_{H^{3/2}} \| \overline{\theta}(s) - \theta(s) \|^2_{L^2} \, ds.
\]

It follows that
\[\| \overline{\theta}(t) - \theta(t) \|^2_{L^2} \leq \| \overline{\theta}(t_1) - \theta(t_1) \|^2_{L^2} + 2C \int_{t_1}^{t} \| \theta(s) \|^2_{H^{3/2}} \| \overline{\theta}(s) - \theta(s) \|^2_{L^2} \, ds.
\]

The Gronwall lemma implies that
\[\| \overline{\theta}(t) - \theta(t) \|^2_{L^2} \leq \| \overline{\theta}(t_1) - \theta(t_1) \|^2_{L^2} e^{2C \int_{t_1}^{t} \| \theta(s) \|^2_{H^{3/2}} \, ds}.\]
The assertion follows after passage to the limit as $t_1 \to 0^+$.

In Lemmas 4.6, 4.7, 4.9, and 4.8 we have established that the assertions (ii), (iii), (iv), and (v) of Section 2.1 hold. Hence, we are in position to use Theorem 2.5 to deduce the following result.
Theorem 4.10. Let \( f \in H^1 \cap L^\infty \). Both dynamical systems, the multivalued one on \( L^2 \) generated by vanishing viscosity weak solutions and the single valued one on \( H^1 \) generated by strong solutions have the global attractors and both attractors coincide. Denoting this attractor by \( \mathcal{A} \) there holds
\[
\lim_{t \to \infty} \text{dist}_{H^1}(S_{L^2}(t)B, \mathcal{A}) = 0 \quad \text{for every} \quad B \in \mathcal{B}(L^2).
\]
Moreover \( A \subset H^1 \) and the dynamics on this attractor is single valued and given by \( S_{H^1} \). This attractor is compact in \( H^1 \), and invariant, i.e. \( S_{L^2}(t)\mathcal{A} = S_{H^1}(t)\mathcal{A} = \mathcal{A} \) for every \( t \geq 0 \). The attractor is minimal in the class of \( L^2 \)-closed sets attracting in \( L^2 \) with respect to \( S_{L^2} \) and maximal in the class of \( L^2 \)-bounded sets which are invariant with respect to either \( S_{L^2} \) or \( S_{H^1} \).

Remark 3. Comparing the above results to [18, Theorem 1.1, Corollary 1.3] we have obtained attraction in \( H^1 \) and not only in \( L^2 \), and no strong convergence of initial datum is required in the definition of vanishing viscosity weak solutions.

The following result on further properties of the attractor follows directly from [15, Theorem 5.2, Theorem 6.4]

Theorem 4.11. Let \( f \in H^1 \cap L^\infty \). The global attractor \( \mathcal{A} \) is a bounded set in \( H^{3/2} \). Moreover, its fractal dimension in \( L^2 \) and \( H^1 \) is finite, i.e.
\[
d_{L^2}(\mathcal{A}) \leq d_{H^1}(\mathcal{A}) < \infty.
\]

REFERENCES

[1] J. M. Arrieta, A. Rodríguez–Bernal and J. Valero, Dynamics of a reaction diffusion equation with a discontinuous nonlinearity, Int. J. Bifurcat. Chaos, 16 (2006), 2965–2984.
[2] A. V. Babin and M. I. Vishik, Attractors of Evolution Equations, North Holland, Amsterdam, London, New York, Tokyo, 1992.
[3] F. Balibrea, T. Caraballo, P. E. Kloeden and J. Valero, Recent developments in dynamical systems: Three perspectives, Int. J. Bifurcat. Chaos, 20 (2010), 2591–2636.
[4] J. M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier–Stokes equations, Nonlinear Sci., 7 (1997), 475–502, Erratum, ibid 8 (1998), 233. Corrected version appears in Mechanics: from Theory to Computation, 447–474, Springer Verlag, 2000.
[5] L. A. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math., 171 (2010), 1903–1930.
[6] T. Caraballo, P. Marín-Rubio and J. C. Robinson, A comparison between two theories for multi-valued semiflows and their asymptotic behaviour, Set-Valued Anal., 11 (2003), 297–322.
[7] V. V. Chepyzhov and M. I. Vishik, Trajectory attractors for evolution equations, CR. Acad. Sci. I-Math., 321 (1995), 1309–1314.
[8] A. Cheskidov and M. Dai, The existence of a global attractor for the forced critical surface quasi-geostrophic equation in \( L^3 \) Journal of Mathematical Fluid Mechanics, 20 (2018), 213–225.
[9] A. Cheskidov and C. Foias, On global attractors of the 3D Navier Stokes equations, Journal of Differential Equations, 231 (2006), 714–754.
[10] J. W. Cholewa and T. Dlotko, Bi-spaces global attractors in abstract parabolic equations, Banach Center Publications, PWN, 60 2003, 13–26.
[11] J. W. Cholewa, R. Czaja and G. Mola, Remarks on the fractal dimension of bi-space global and exponential attractors, Bollettino dell’Unione Matematica Italiana, 1 (2008), 121–145.
[12] I. Chueshov and I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, American Mathematical Society, 195 2008, viii+183 pp.
[13] P. Constantin, M. Coti Zeliati and V. Vicol, Uniformly attracting limit sets for the critically dissipative SQG equation, Nonlinearity, 29 (2016), 298–318.
[14] P. Constantin, A. J. Majda and E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, Nonlinearity, 7 (1994), 1495–1533.
[15] P. Constantin, A. Tarfulea and V. Vicol, Long time dynamics of forced critical SQG, Communications in Mathematical Physics, 335 (2015), 93–141.
[16] P. Constantin and Y. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, *Geometric and Functional Analysis*, 22 (2012), 1289–1321.

[17] M. Coti Zelati and P. Kalita, Minimality properties of set-valued processes and their pullback attractors, *SIAM Journal on Mathematical Analysis*, 47 (2015), 1530–1561.

[18] M. Coti Zelati and P. Kalita, Smooth attractors for weak solutions of the SQG equation with critical dissipation, *Discrete and Continuous Dynamical Systems - Series B*, 22 (2017), 1857–1873.

[19] A. Eden, C. Foiaş, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, John Wiley & Sons/Masson, Chichester, New York, Brisbane, Toronto, Singapore/Paris, Milan, Barcelona, 1994.

[20] A. C. Eringen, Theory of micropolar fluids, *J. Math. Mech.*, 16 (1966), 1–18.

[21] M. Coti Zelati and P. Kalita, Minimality properties of set-valued processes and their pullback attractors, *SIAM Journal on Mathematical Analysis*, 47 (2015), 1530–1561.

[22] M. Coti Zelati and P. Kalita, Smooth attractors for weak solutions of the SQG equation with critical dissipation, *Discrete and Continuous Dynamical Systems - Series B*, 22 (2017), 1857–1873.

[23] P. Kalita, G. Łukaszewicz and J. Siemianowski, Rayleigh–Bénard problem for thermomicrofluids, *Topological Methods in Nonlinear Analysis*, accepted for publication.

[24] O. V. Kapustyan and J. Valero, Comparison between trajectory and global attractors for evolution systems without uniqueness of solutions, *Int. J. Bifurcat. Chaos*, 20 (2010), 2723–2734.

[25] A. Kiselev and F. Nazarov, A variation on a theme of Caffarelli and Vasseur, *Journal of Mathematical Sciences*, 166 (2010), 31–39.

[26] G. Łukaszewicz, *Micropolar Fluids. Theory and Applications*, Birkhäuser Boston, Inc., Boston, MA, 1999.

[27] G. Łukaszewicz, Long time behavior of 2D micropolar fluid flows, *Mathematical and Computer Modelling*, 34 (2001), 487–509.

[28] G. Łukaszewicz, Asymptotic behavior of micropolar fluid flows, *International Journal of Engineering Science*, 41 (2003), 259–269.

[29] V. S. Melnik and J. Valero, On attractors of multivalued semiflows and differential inclusions, *Set-Valued Anal.*, 6 (1998), 83–111.

[30] V. S. Melnik and J. Valero, Addendum to "On Attractors of Multivalued Semiflows and Differential Inclusions" [Set-Valued Anal. 6 (1998), 83–111], *Set-Valued Anal.*, 16 (2008), 507–509.

[31] L. E. Payne and B. Straughan, Critical Rayleigh numbers for oscillatory and nonlinear convection in an isotropic thermomicroscopic fluid, *International Journal of Engineering Science*, 27 (1989), 827–836.

[32] S. G. Resnick, *Dynamical Problems in Non-Linear Adveective Partial Differential Equations*, ProQuest LLC, Ann Arbor, MI, Ph.D. Thesis, 1995, The University of Chicago.

[33] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, Cambridge, UK, 2001.

[34] B. Straughan, *The Energy Method, Stability, and Nonlinear Convection*, Springer, 2004.

[35] A. Tarasińska, Global attractor for heat convection problem in a micropolar fluid, *Mathematical Methods in the Applied Sciences*, 29 (2000), 1215–1236.

[36] A. Tarasińska, Pullback attractor for heat convection problem in a micropolar fluid, *Nonlinear Analysis: Real World Applications*, 11 (2010), 1458–1471.

[37] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Second Edition, Springer–Verlag, New York, 1997.

[38] R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, Pennsylvania, 1983.

[39] J. Valero, Finite and infinite dimensional attractors of multi-valued reaction diffusion equations, *Acta Math. Hungary*, 88 (2000), 239–258.

Received January 2018; revised May 2018.

E-mail address: piotr.kalita@ii.uj.edu.pl
E-mail address: glukasz@mimuw.edu.pl
E-mail address: jsiem@mat.umk.pl