CONSTRAINED DYNAMICS OF AN ANOMALOUS \((g \neq 2)\) RELATIVISTIC SPINNING PARTICLE IN ELECTROMAGNETIC BACKGROUND

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Abstract

In this paper we have considered the dynamics of an anomalous \((g \neq 2)\) charged relativistic spinning particle in the presence of an external electromagnetic field. The constraint analysis is done and the complete set of Dirac brackets are provided that generate the canonical Lorentz algebra and dynamics through Hamiltonian equations of motion. The spin-induced effective curvature of spacetime and its possible connection with Analogue Gravity models are commented upon.

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The symplectic structure (which, in the special case of a point particle lagrangian in a first order formalism, is essentially the terms containing time derivatives) completely dictates the kinematics of the particle as it generates the phase space algebra. This in turn determines the correct structure of Lorentz generators (or angular momenta) that will satisfy the Lorentz algebra. The mass-energy dispersion relation is also induced from the symplectic structure. Through Hamilton’s equations the dynamics is determined once a potential energy term is prescribed. The canonical symplectic structure, \( \sim p^i \dot{q}_i \) (\( p^i, q_i \) being the momenta and coordinates respectively) yields the canonical set of Poisson Brackets (PB) \( \{q_i, q_j\} = \{p^i, p^j\} = 0, \{q_i, p_i\} = \delta^i_j \). But more complicated form of symplectic structures \( \sim p^i A^j_i(p, q) \dot{q}_j \), in general yields more complicated brackets depending up on the specific form of \( A^j_i(p, q) \). In general the latter is referred as Non-Commutative (NC) phase space since additional terms in the canonical PB structure appear. NC spacetime is a special case where \( \{q_i, q_j\} \neq 0 \). There are two related ways to derive the algebra from a given symplectic structure: the Dirac constraint analysis formalism [1] or Faddeev-Jackiw symplectic formalism [2], both obviously leading to the same results. In the present work we will exploit the Dirac formalism. The symplectic structure induces constraints that determine the generalization of PB, known as Dirac Brackets (DB), via a systematic procedure. In a particular problem it is possible that one can fix the constraints in an intuitive way from the physics of the problem. In that case one does not need the explicit form of the symplectic structure and can directly compute the DB and subsequently obtain the particle kinematics and dynamics. This is the case for a charged relativistic spinning particle in the presence of an external electromagnetic field, to be discussed below. As we will mention at the end of the paper the present work can have interesting consequences in the topically active areas of Analogue Gravity models [3].

Before proceeding to the main topic let us mention some earlier works to get the perspective of the present work. The NC phase space - relativistic spinning particle connection in 3 + 1-dimensions appeared explicitly first in [4]. Later similar types of spinning particle
models \[\text{[3, 4, 5]}\] became popular in the context of Anyons, excitations of arbitrary spin in 2 + 1-dimensions \[\text{[8]}\]. We will closely follow the work of Chaichian, Gonzalez Felipe and D.Louis Martinez \[\text{[9]}\]. We aim to construct the NC phase space for a 3 + 1-dimensional classical spinning particle of charge \(e\) and an anomalous gyromagnetic ratio \(g \neq 2\) in the presence of a constant and weak electromagnetic field. The resulting Hamiltonian model in our case turns out to be structurally different from that of \[\text{[9]}\] for reasons that will be discussed as we proceed. Furthermore, we will provide a detailed constraint analysis, compute DBs and subsequently apply them in revealing the dynamics of the degrees of freedom. Our results will be valid to first non-trivial order in the electromagnetic field.

With this brief introduction we now proceed to the main body of our work. The phase space degrees of freedom with the following set of coordinate and spin variables, \(x_\mu, p_\mu\) and \(n_\mu, p_\mu^{(n)}\) respectively with \(\mu = 0, 1, 2, 3\) and our metric is \(\eta_{\mu\nu}, \eta_{00} = -\eta_{ii} = 1\). The above canonically conjugate pairs are independent with the PB,

\[
\{x_\mu, p_\nu\} = -\eta_{\mu\nu}, \quad \{x_\mu, x_\nu\} = \{p_\mu, p_\nu\} = 0, \quad (1)
\]

\[
\{n_\mu, p_\mu^{(n)}\} = -\eta_{\mu\nu}, \quad \{n_\mu, n_\nu\} = \{p_\mu^{(n)} , p_\nu^{(n)}\} = 0. \quad (2)
\]

The \(n_\mu, p_\mu^{(n)}\) sector will describe the spin. All cross brackets between \(n_\mu, p_\mu^{(n)}\) and \(x_\mu, p_\mu\) vanish.

The particle moves in a constant electromagnetic field \(F_{\mu\nu}\) given in terms of the potential by \(A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu\). Sometimes it is convenient to use the variable \(\pi_\mu = p_\mu - eA_\mu\) with the canonical PB

\[
\{\pi_\mu, \pi_\nu\} = -eF_{\mu\nu} \quad (3)
\]

We also consider the total angular momentum operator \(J_{\mu\nu}\) consisting of a rotation part \(l_{\mu\nu}\) and a spin part \(s_{\mu\nu}\),

\[
J_{\mu\nu} = l_{\mu\nu} + s_{\mu\nu} = (x_\mu p_\nu - x_\nu p_\mu) + (n_\mu p_\nu^{(n)} - n_\nu p_\mu^{(n)}), \quad (4)
\]

obeying the conventional Lorentz algebra,

\[
\{J_{\mu\nu}, J_{\alpha\beta}\} = J_{\mu\beta}g_{\nu\alpha} + J_{\nu\alpha}g_{\mu\beta} + J_{\alpha\mu}g_{\nu\beta} + J_{\beta\nu}g_{\mu\alpha}. \quad (5)
\]
In a relativistic formulation of spinning particle one needs to introduce constraints such that the spin tensor reduces to a three dimensional vector in the particle rest frame \[1, 6, 9, 5, 7\]. This is the Hamiltonian analogue of the well known Frenkel condition

\[
(s_{\mu\nu}\dot{x}^\nu)/d\tau = 0.
\]

The constraints are the following \[6, 9\] (see also \[4, 7\]):

\[
\Phi_1 \equiv (\pi p^{(n)}) \approx 0; \quad \chi_2 \equiv (\pi n) \approx 0. \tag{6}
\]

We use a shorthand notation \(A^\mu B_\mu = (AB)\). This pair of Hamiltonian constraint is Second Class (in the sense of Dirac constraint analysis \[1\]) that is they do not commute at the PB level,

\[
\{\Phi_1, \Phi_2\} = \pi^2 + \frac{e}{2} F_{\mu\nu}\pi^{\mu\nu}. \tag{7}
\]

Hence we are obliged to replace the PB by DB as defined below for two generic dynamical variables \(A, B\),

\[
\{A, B\}_{DB} = \{A, B\}_{PB} - \{A, \Phi_i\}_{PB}(\{\Phi_i, \Phi_j\}_{PB})^{-1}\{\Phi_i, B\}_{PB}. \tag{8}
\]

From here on all the brackets are DB and hence we drop the subscript \(DB\) in the bracket. The constraints can also be treated as strong relations, \(\Phi_1 = \Phi_2 = 0\). The constraint matrix and its inverse are given by,

\[
\{\Phi_i, \Phi_j\}_{PB} = \begin{pmatrix}
0 & \pi^2 + \frac{e}{2} F_{\mu\nu}\pi^{\mu\nu} \\
-\pi^2 - \frac{e}{2} F_{\mu\nu}\pi^{\mu\nu} & 0
\end{pmatrix}
\]

\[
\{\Phi_i, \Phi_j\}_{PB}^{-1} = \begin{pmatrix}
0 & -\frac{1}{\pi^2 + \frac{1}{2} F_{\mu\nu}\pi^{\mu\nu}} \\
\frac{1}{\pi^2 + \frac{1}{2} F_{\mu\nu}\pi^{\mu\nu}} & 0
\end{pmatrix}
\]

Exploiting the definition of DB \(8\) we compute the latter,

\[
\{x_\mu, x_\nu\} = \frac{s_{\mu\nu}}{\pi^2 + \frac{1}{2} F_{\alpha\beta}s^{\alpha\beta}}, \quad \{x_\mu, \pi_\nu\} = -\eta_{\mu\nu} - \frac{es_\mu}{\pi^2 + \frac{1}{2} F_{\alpha\beta}s^{\alpha\beta}}, \quad \{x_\mu, n_\nu\} = \frac{\eta_{\mu \nu}}{\pi^2 + \frac{1}{2} F_{\alpha\beta}s^{\alpha\beta}}. \tag{9}
\]
\{x_\mu, p_\nu^{(n)}\} = \frac{p_\mu^{(n)} \pi_\nu}{\pi^2 + \frac{e}{2} F_{\alpha\beta} s_{\alpha\beta}}, \{\pi_\mu, \pi_\nu\} = -e F_{\mu\nu}, \{n_\mu, p_\nu^{(n)}\} = -g_{\mu\nu} + \frac{\pi_\mu \pi_\nu}{\pi^2 + \frac{e}{2} F_{\alpha\beta} s_{\alpha\beta}} \quad (10)

\{p_\mu^{(n)}, \pi_\nu\} = \frac{e \pi_\mu p_\nu^{(n)} F_{\alpha\nu}}{\pi^2 + \frac{e}{2} F_{\alpha\beta} s_{\alpha\beta}}, \{n_\mu, \pi_\nu\} = \frac{e \pi_\mu n_\nu F_{\alpha\nu}}{\pi^2 + \frac{e}{2} F_{\mu\nu} s_{\mu\nu}}

\{n_\mu, n_\nu\} = \{p_\mu, p_\nu\} = \{p_\mu^{(n)}, p_\nu^{(n)}\} = 0. \quad (11)

This shows that the constraints induce a NC spacetime. Notice the with respect to the DBs the spin and coordinate sectors have become mixed up. The above set of DBs are exact.

It is very important from our perspective to note that using the DBs (9,10,) the angular momentum $J_{\mu\nu}$ having the same canonical definition as given in (4) satisfies the undeformed Lorentz algebra (5) to $O(e)$. It is now straightforward to construct the Hamiltonian constraint (or equivalently the mass shell condition): we keep all terms that are Lorentz invariant, that is commutes in the sense of DB with $J_{\mu\nu}$, to $O(e)$. This is our guiding principle. Explicitly the Hamiltonian is,

\[ H = \frac{1}{m} \left[ \frac{\pi^2}{2} + \frac{eg}{2} F_{\alpha\beta} J^{\alpha\beta} + \frac{e(g - 2)}{m^2} F_{\alpha\beta} J_\gamma^\beta p_\alpha^\gamma \right] + \nu n_\alpha n^\alpha + \lambda p_\alpha^{(n)} p_\alpha^{(n)} + \sigma n^\alpha p_\alpha^{(n)}. \quad (12)\]

In the above $\nu, \lambda, \sigma$ are $c$-number parameters. Note that we have kept the same parametrization of constants $e, g$ as that of [9] but the terms are distinct from those of [9]. $g$ is identified as the gyromagnetic ratio and for $g = 2$ one recovers the conventional case with the last term in $H$ being absent. The last term is a manifestation of anomalous value of $g \neq 2$.

It is worthwhile to compare and contrast our formalism and results with other earlier works in the same area: in particular that of van Holten [12] and Chaichian et.al. [9]. At the outset we point out that our formalism is more mathematical or algebraic in nature since we base our construction of the cherished Hamiltonian operator, (that generates the dynamics via Dirac Brackets computed here), purely by demanding that it be Lorentz invariant, (that is commutes with the Lorentz generators again in the sense of Dirac Brackets). On the other hand the abovementioned works are more intuitive and are principally based on
known (or expected) physical behaviour of a classical charged spinning particle in external field. Hence our setup is more general and but can be reconciled with the physically ori-
ented systems discussed in [12, 9]. In explicit terms one important difference between our
Hamiltonian and the previous ones in [12, 9] for \( g = 2 \) is that in our case the coupling term
\( eJ_{\mu\nu}F^{\mu\nu} = e(l_{\mu\nu} + s_{\mu\nu})F^{\mu\nu} \) whereas in the other it is \( es_{\mu\nu}F^{\mu\nu} \). In our case we have ensured
Lorentz invariance. Furthermore in [12] there is a prediction of a novel form of relativistic
time dilation effect even for a particle at rest. Similar effect will appear here as well but
with some additional complication coming from the extra term \( el_{\mu\nu}F^{\mu\nu} \) in our Hamiltonian.
Again in [12] the author has constructed a parallel setup in terms of anticommuting de-
grees of freedom to simulate spin dynamics of the corresponding quantum system where an
interacting Klein-Gordon form of equation for the spinning particle is advocated with the
canonical identification of
\( p_j = -i(\partial/\partial x_j) \). In principle this can be carried through in our
formalism but more work is involved since in our basic Dirac Bracket framework \( p_i, x_j \) and
\( x_i, x_j \) do not obey canonical algebra. It will be necessary in our framework to find a Darboux
map to a set of canonical variable with which one can proceed.

The \( g \neq 2 \) was studied in detail in [9]. One can now understand the difference in the
expressions of the Hamiltonian between our result [12] and [9]. In [9] the authors have
concentrated on keeping the Hamiltonian constraint First Class (in the Dirac sense [11]) that
is commuting with all other constraints whereas we have ensured that the Hamiltonian is
Lorentz invariant that is it commutes with the Lorentz generators \( J_{\mu\nu} \) in the DB sense. The
dynamics given in [9] (see equations 25-26), reduced to \( O(e) \), essentially matches similar
equations (15-22) in our model with the obvious differences coming from the extra terms
present in our Hamiltonian.

Exploiting the DBs (9, 10,) once again we can evaluate the dynamics by taking commu-
tators between each dynamical variable and \( H \) in (12). We give the result for \( \pi_\mu \) in detail
(keeping terms of \( O(e) \) throughout the rest of the paper):

\[
\{\pi_\mu, \frac{\pi^2}{2m}\} = -\frac{e}{m} F_{\mu\nu}p^\nu
\]  

(13)
\[
\{\pi_\mu, \frac{eg}{2m} F_{\alpha\beta} J^{\alpha\beta}\} = \frac{eg}{m} F_{\mu\nu} p^\nu
\] (14)

\[
\{\pi_\mu, \frac{e(g-2)}{m^3} F_{\alpha\beta} J^{\beta}_{\gamma\pi} \pi_\pi\} = -\frac{e(g-2)}{m} F_{\mu\nu} p^\nu
\] (15)

Combining the above we find,

\[
\dot{\pi}_\mu = \{\pi_\mu, H\} = \frac{e}{m} F_{\mu\nu} p^\nu.
\] (16)

Thus from (16) it is clear that there is no deviation from the Lorentz force to \(O(e)\) and the result is independent of \(g\).

The time derivatives of rest of the degrees of freedom are,

\[
\dot{x}_\mu = -\frac{1}{m} (\pi_\mu + e \frac{1}{m^2} s_{\mu\nu} F^\nu_p p_p + (g-2) \frac{1}{m^2} (F_{\mu\nu} J^\nu_p + J_{\mu\nu} F^\nu_p) p_p)
\] (17)

\[
\dot{n}_\mu = \frac{eg}{m} F_{\mu\nu} n^\nu + \frac{e(g-1)}{m^2} F_{\nu\rho} n^\rho p^\mu - \sigma n_\mu,
\] (18)

\[
\dot{p}_\mu^{(n)} = \frac{eg}{m} F_{\mu\nu} p^{(n)\nu} + \frac{e(g-1)}{m^3} F_{\nu\rho} p^{(n)\rho} p^\mu + \sigma p^{(n)}_\mu,
\] (19)

From the above relations it is straightforward to compute

\[
\dot{s}_{\mu\nu} = \frac{2e}{m^3} (s_{\mu\nu} n^\rho - s_{\nu\rho} n_\mu) p_\lambda F^{\kappa\lambda} - \frac{eg}{m} (s_{\mu\nu} g_{\nu\beta} - s_{\nu\beta} g_{\mu\nu}) F^{\alpha\beta}
\]  
\[+ \frac{e(g-2)}{m^3} (s_{\beta\nu} p_\nu - s_{\beta\mu} p_\mu) p_\alpha F^{\alpha\beta}.
\] (20)

We define the Pauli-Lubanski vector as

\[
s_\mu = \epsilon_{\mu\alpha\beta} s^{\nu\alpha} n^\beta
\] (21)

and compute

\[
\dot{s}_\mu = \frac{e}{m} F_{\mu\nu} s^\nu - \frac{e}{2m^3} (g-2) p_\mu F_{\alpha\beta} p^\alpha s^\beta.
\] (22)
It is very important to check that the Frenkel condition is maintained as this ensures that the relativistic spin tensor $s_{\mu\nu}$ reduces to the spin vector in the particle rest frame. In our generalized system we find

$$d(s_{\mu\nu}\dot{x}^\nu)/d\tau = \frac{e\sigma(g-2)}{m^2}s_{\mu\nu}(F^{\nu\alpha}J_{\alpha\beta}p^\beta + J^{\nu\alpha}F_{\alpha\beta}p^\beta).$$ (23)

This relation demands that we restrict our model to $\sigma = 0$ in (12) so that the Frenkel condition is time invariant. This ensures further that

$$\ddot{s}_\mu = 0.$$ (24)

In the present work we have studied the spinning particle model in a Hamiltonian framework. Hamiltonian analogues of the Frenkel condition appear as constraints and yields a non-commutative spacetime through the Dirac Brackets. The latter are essential for a proper quantum treatment of the model. This is beyond the scope of the present paper as the operatorial nature of the Dirac Brackets make the quantization non-trivial. However there is another area where the present work can have interesting consequences: Analogue Gravity Models.

In a particular framework, in Analogue Gravity approach, Rivelles [3] has shown that NC $U(1)$ gauge theory, obtained by Seiberg-Witten map [10] can induce an effective curved spacetime. (For general reviews on NC quantum field theories and applications of Seiberg-Witten map see for example [11].) The effective metric is composed of the gauge fields and the anti-symmetric NC parameter $\theta_{\mu\nu}$. In a similar vein it is very tempting to suggest that the Hamiltonian (12) can be expressed as $H \sim G^{\mu\nu}(F,J)\pi_\mu\pi_\nu$ with $G^{\mu\nu} \sim \eta^{\mu\nu} + e(g-2)/m^2F^{\mu\beta}J^\beta_\nu$ as a simple possibility. This identification was suggested earlier by [12] and by some of us [13]. Then it is very revealing to compare the works of [3] and ours. We find the spin tensor $s_{\mu\nu}$ in the spinning particle model can be identified with the NC parameter $\theta_{\mu\nu}$ in [3]. In this way noncommutativity and the resulting Analogue Gravity is not put by hand from outside as is done conventionally (eg. [3]) but appears as an effect of the particle spin. But for this to materialise truly there is still a lot more work to do primarily for two
reasons:
(a) The NC parameter $\theta_{\mu\nu}$ is a $c$-number parameter in [3] whereas $s_{\mu\nu}$ in the present case is a dynamical variable. Hence some approximations are needed.
(b) Interpreting $G^{\mu\nu}$ along the lines of Rivelles [3] has to be done with care since the spacetime (or phase space) is noncommutative in nature. It would have been more convenient if there was a Darboux-like map that expresses the NC variables in terms of canonical variables (see for example [14] for application of this approach in a different model). Then the NC or spin induced effects will appear as additional interaction terms but everything expressed in terms of canonical variables. In fact this is not unlike the Seiberg-Witten map [10], at least in spirit. Work is in progress along these lines.

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