Spacetime Lagrangian formulation of Barbero–Immirzi gravity

L Fatibene\(^1\), M Francaviglia\(^1,2,3\) and C Rovelli\(^4\)

\(^1\) Department of Mathematics, University of Torino, Italy
\(^2\) INFN, Iniziativa Specifica Na12, Italy
\(^3\) ESG, University of Calabria, Italy
\(^4\) Centre de Physique Théorique de Luminy, Université de la Méditerranée, F-13288 Marseille, France

Received 14 June 2007
Published 31 July 2007
Online at stacks.iop.org/CQG/24/4207

Abstract
We shall discuss a new spacetime gauge-covariant Lagrangian formulation of general relativity by means of the Barbero–Immirzi SU(2)-connection on spacetime. To the best of our knowledge, the Lagrangian based on SU(2) spacetime fields seems to appear for the first time here.

PACS numbers: 04.20.Fy, 04.60.Pp

1. Introduction

In a previous paper of ours [1] we introduced new gauge-covariant spacetime variables that are suited to providing a spacetime interpretation of the Barbero–Immirzi SU(2)-connection (BI connection) that, in turn, enters the formulation of loop quantum gravity (LQG).

We shall discuss here the classical dynamics of general relativity (GR) in terms of these new spacetime variables introduced in [1, 2].

The main idea is very simple: we shall pull back Holst’s action in the new variables and then restrict it to a spacelike hypersurface \(S \subset M\) to obtain the constraint equations, see [3] for theoretical motivations.

The new Holst’s Lagrangian is a functional of the spacetime fundamental fields \((e^\mu_a, A^i_\mu, K^i_\mu)\). These fields have to be considered as being independent. We shall call this Lagrangian the \(\text{Barbero–Immirzi–Holst (BIH) Lagrangian}\).

Field equations of the BIH action will provide manifestly gauge-covariant equations that are in fact equivalent to the field equations of Holst action (though directly written in terms of the spacetime BI connection), which are in turn equivalent to standard GR equations.

By projecting onto a hypersurface \(S \subset M\) field equations split into some evolution equations, some constraint equations (that are in fact the starting point for LQG, see [3–5])
and some further (algebraic) constraint equations which determine the $K$ field as a function of the (densitized) triad $E_i^j$ and the BI connection $A^j_A$.

Although the general relation with the Hamiltonian multisymplectic framework will be investigated elsewhere it must be noted that the equations obtained by the projection of the Lagrange equations on a hypersurface $S \subset M$ actually coincide for GR with the Hamiltonian constraints (see [3]). This derivation is quite simple from the conceptual and computational viewpoint. Moreover, it is quite useful to discuss on the Lagrangian side the gauge properties of the model together with its relations to other equivalent frameworks, such as the Dirac–Bergman Hamiltonian reduction.

2. Notation

We shall briefly recall here the notation introduced [1] and adapt, to the present case, the results developed in [2] for the self-dual case.

Let $M$ be an orientable connected and paracompact manifold of dimension $m = 4$. Let us fix either the Euclidean (or the Lorentzian) signature $\eta = (4, 0)$ (or $\eta = (3, 1)$, respectively). For later convenience, we shall introduce a signature-dependent quantity $\sigma$, being $\sigma = 1$ in the Euclidean signature and $\sigma = i$ in the Lorentzian one.

The spacetime manifold is assumed to allow global $\eta$-metrics and spin structures of the relevant signature. This is equivalent to requiring the first and second Stiefel–Whitney classes of $M$ to vanish, which in turn implies that the third Stiefel–Whitney class vanishes as well, see [6, 7].

The group $\text{Spin}(4)$ is known to be canonically isomorphic to $SU(2) \times SU(2)$. The first $SU(2)$ factor is called the self-dual part of the spin group, while the second factor is the anti-self-dual part; the projection on the first factor is $p_{\pm} : \text{Spin}(4) \rightarrow SU(2)$.

We also introduce a group homomorphism $\iota : SU(2) \rightarrow \text{Spin}(\eta)$; in the Euclidean case the group homomorphism $\iota : SU(2) \rightarrow \text{Spin}(4)$ is defined as $\iota(S) = (S, S)$, where the isomorphism $\text{Spin}(4) \simeq SU(2) \times SU(2)$ has been understood. In the Lorentzian case, the spin group $\text{Spin}(3, 1)$ is canonically isomorphic to $SL(2, \mathbb{C})$, which is the ‘complexified’ version of $SU(2)$; in this case, the group homomorphism $\iota : SU(2) \rightarrow \text{Spin}(3, 1)$ exhibits $SU(2)$ as a real section of $SL(2, \mathbb{C})$.

Let us choose a $\text{Spin}(\eta)$-principal bundle $P$ over $M$ such that global spin frames exist, see [8]. We stress that this is considerably less than asking $M$ to be parallelizable (namely, to allow global sections of the general bundle $L(M)$ of frames or, equivalently, the tangent bundle $TM$ to be trivial). For example, one can define this structure on all spheres despite the even-dimensional spheres being not parallelizable.

The vanishing of the third Stiefel–Whitney class implies (see [6, 7]) the existence of a $SU(2)$-reduction, namely of a $SU(2)$-principal bundle $^+P$ together with a principal morphism relative to the group morphism $\iota : SU(2) \rightarrow \text{Spin}(\eta)$:

\[
\begin{array}{ccc}
+P & \rightarrow & P \\
\downarrow & & \downarrow \\
M & \rightarrow & M \\
\end{array}
\]

A local trivialization (also known as a local gauge) of $^+P$ amounts to fixing a local section $\sigma^{(a)} : U_\alpha \subset M \rightarrow ^+P$ on a chart domain $U_\alpha$. Using the reduction (2.1), a local trivialization of $^+P$ induces a local trivialization $\hat{\sigma}^{(a)}$ of $P$. By construction, two local trivializations on $^+P$
are mapped one into the other by a SU(2)-gauge transformation \( \sigma^{(\beta)} = \sigma^{(\alpha)} \cdot \varphi(\alpha \beta)(x) \) where \( \varphi(\alpha \beta)(x) \in SU(2) \). The same happens on \( P \), namely, \( \delta^{(\beta)} = \delta^{(\alpha)} \cdot \ell(\varphi(\alpha \beta)(x)) \). Hence \( P \) allows, by construction, a trivialization with transition functions with values in SU(2) \( \hookrightarrow \text{Spin}(\eta) \).

For future convenience, we shall use on \( P \) only this sort of reduced trivializations.

The standard Holst’s fields are a spin tetrad \( e^\mu_a \) and a spin connection \( \theta^a_{\mu} \). The spin frame \( e^\mu_a \) is a global section of the bundle \( P_\lambda \) associated with \( P \times L(M) \) by means of the following action of the group \( GL(4) \times \text{Spin}(\eta) \) on \( GL(4) \):\( \lambda : GL(4) \times \text{Spin}(\eta) \times GL(4) \rightarrow GL(4) : (J, S, e) \mapsto J \cdot e \cdot \ell(S^{-1}) \) (2.2)

where \( \ell : \text{Spin}(\eta) \rightarrow SO(\eta) \) is the covering map exhibiting the spin group as a double covering of the relevant orthogonal group. The spin frame bundle \( P_\lambda \) has local coordinates \( (x^\mu, e^\mu_a) \), and, by construction, it is assumed to allow global sections (also when \( M \) is non-parallelizable).

As discussed in [1], \( A^\mu_i \) is a SU(2)-connection on \( ^*P \) while \( K^\mu_i \) is a su(2)-valued 1-form on \( ^*P \). We shall call \( A^\mu_i \) the (spacetime) BI connection and \( K^\mu_i \) the extrinsic (spacetime) field. Both fields live on a bundle associated with \( ^*P \) and are hence SU(2)-objects.

3. BIH Lagrangian

The standard Holst’s Lagrangian (see [9, 10]) reads

\[
L_\gamma (j^1 \theta, e) = \frac{1}{4\kappa} R^{ab} \wedge e^c \wedge e^d e_{abcd} + \frac{1}{2\kappa} R^{ab} \wedge e_a \wedge e_b
\]

where \( R^{ab} \) denotes the curvature 2-form of the spin connection \( \theta_{ab}^{\mu} \). Let us denote by \( \nabla \) the covariant derivative with respect to the connection \( A^\mu_i \).

By using the transformation (2.3) it is easy to prove that

\[
\begin{align*}
R^b_i &= 2(\nabla K^i + \gamma \theta^i_j K^j ) + 2 \tilde{\nabla} K^i \\
R^i_j &= \epsilon^{ij}_k F^k - 2\gamma \epsilon^{ij}_k \nabla K^k - 2(\sigma^2 + \gamma^2) K^i \wedge K^j
\end{align*}
\]

(3.2)

where \( \tilde{\nabla} \) denotes the covariant derivative with respect to the connection \( A - \gamma K \).

We can now use (3.2) to pull back the Lagrangian (3.1) along the new variables (2.3) to obtain

\[
L_\gamma (e, j^1 A, j^1 K) = -\frac{1}{4\kappa} (8(\sigma^2 - \gamma^2) \nabla K^i \wedge e_i \wedge e^0 + 4\gamma(\sigma^2 - \gamma^2) \epsilon_{ijk} K^i \wedge K^j \wedge e^k \wedge e^0 \\
+ 4\gamma F^k \wedge e_k \wedge e^0 - 2\epsilon_{ijk} F^i \wedge e^j \wedge e^k - 4(\sigma^2 - \gamma^2) K^i \wedge e_i \wedge K^j \wedge e_j)
\]

(3.3)

(here \( j^1 \) refers to the first-order jet prolongations, simply meaning that the Lagrangian depends on the fields \( (A^\mu_i, K^\mu_i) \) together with their first derivatives). We stress that at this level the
fields \((e^\mu_a, A^\mu_i, K^\mu_j)\) have to be considered as independent fields. By varying the Lagrangian (3.3) we obtain field equations under the form

\[
\begin{align*}
\gamma (\nabla K^k) \otimes e \otimes e^0 &= \gamma (\nabla K^k) \otimes e \otimes e^0 \\
(\sigma^2 - \gamma^2) K^k \otimes e \otimes e^0 &= (\sigma^2 - \gamma^2) K^k \otimes e \otimes e^0 \\
F^k \otimes e \otimes e^0 &= (\sigma^2 - \gamma^2) K^k \otimes e \otimes e^0.
\end{align*}
\]

These field equations are obtained by varying the Lagrangian \(L_\gamma\) with respect to the fields \((K^\mu_j, A^\mu_i, e^\mu_0, e^\mu_k)\), respectively.

One can easily check that these field equations are equivalent to the field equations of the standard Holst’s Lagrangian (3.1), as expected, see appendix A.

4. Hamiltonian framework

Let us now fix a (spacelike) hypersurface \(S \subset M\). We stress that we are fixing a single hypersurface, not a foliation. Let us choose coordinates \(kA\) on \(S\) so that the canonical injection is locally expressed by \(i : S \rightarrow M : k \mapsto x(k)\).

The structure bundles (2.1) can be pulled back (i.e. restricted) to \(S\) obtaining

\[
\begin{align*}
+P &\quad \rightarrow \quad P \\
+\Sigma &\quad \rightarrow \quad \Sigma \\
M &\quad \rightarrow \quad M \\
S &\quad \rightarrow \quad S
\end{align*}
\]

The bundles \(\Sigma\) and \(\Sigma\) are \(SU(2)\) and \(Spin(\eta)\) bundles, respectively.

As shown in [2] by techniques adapted to the case of BI connection (see appendix B), the spin tetrad \(e^\mu_a\) canonically determines a spin triad \(e^A_i\) on \(S\) together with a vector \(u\) normal to \(S\). Of course, one has the identities

\[
\begin{align*}
u_\mu u^\mu &= \sigma^2 \\
u_\mu e^\mu_i &= 0 \\
u_\mu \partial_\mu x^\mu &= 0
\end{align*}
\]

that express the fact that \(u\) is orthogonal to \(S\) and it is unitary (and timelike in the Lorentzian case) with respect to the metric induced by the frame itself.

Note that the frame \(e^\mu_a\) is expressed by \(4 \times 4 = 16\) functions. The spin triad \(e^A_i\) is expressed by \(3 \times 3 = 9\) functions, \(u\) is expressed by 4 functions. As shown in [2], the reduction on \(S\) is achieved by canonically determining (out of the frame and the hypersurface \(S\)) an anti-self-dual transformation, which is expressed by other \(3 = \dim SU(2)\) functions. Thus, the new variables are again described by \(9 + 4 + 3 = 16\) functions (though 3 of them—namely, those connected to the anti-self-dual transformation—are canonically fixed as a function of the others once \(S\) is fixed; for this reason they will be systematically dropped below).

The BI connection \(A^\mu_i\) induces two fields on \(S\), namely

\[
\begin{align*}
A^i_A &= A^i_\mu \partial_\mu x^\mu \\
A^i &= A^i_\mu u^\mu.
\end{align*}
\]
Similarly, the extrinsic field induces
\[ \begin{align*}
K^\mu_A &= K^\mu_B \partial_A x^\mu \\
\tilde{K}^\mu &= A^\mu_A u^\mu.
\end{align*} \tag{4.4} \]

The field \( A^\mu_A \) is a \( SU(2) \)-connection on \( +\Sigma \) while \( K^\mu_B \) is a \( su(2) \)-valued 1-form on \( +\Sigma \). The fields \((\tilde{A}^\mu, \tilde{K}^\mu)\) are \( su(2) \)-valued scalar fields on \( \Sigma \).

Note that the original spin connection \( \partial^\mu_{\nu} \) is expressed by \( 6 \times 4 = 24 \) functions. Similarly, \( A^\mu_A \) and \( K^\mu_B \) are in fact \( 3 \times 4 = 12 \) functions each. Once projected on \( S \) we still have \( 3 \times 3 = 9 \) functions for \( A^\mu_A \), 3 functions for \( \tilde{A}^\mu \), 3 \( \times 3 = 9 \) functions for \( K^\mu_B \), thus again \( 9 + 3 + 9 + 3 = 24 \) functions. Preserving the number of independent quantities is of course necessary for any change of variables in the space of fields.

We are now going to project the field equations (3.4) onto \( S \), i.e. writing them in terms of the fields \((\epsilon^i_\mu, A^\mu_A, \tilde{A}^\mu, K^\mu_B, \tilde{K}^\mu)\), see appendix C for technicalities. We shall systematically split each equation into its component parallel to \( S \) (by multiplying it by \( \epsilon^{ABC} \partial_A x^\mu \partial_B x^\nu \partial_C x^\rho \)) and into its component orthogonal to \( S \) (by multiplying it by \( \epsilon^{ABC} u^\mu \partial_B x^\nu \partial_C x^\rho \)).

**4.1. Equation for \( \delta K^\mu_B \)**

Let us project the equation
\[ d\epsilon^i_0 \wedge e_i + K^j \wedge e_j \wedge e_i = \gamma \epsilon_{ijk} K^j \wedge e_k \wedge e^0 + \nabla e_i \wedge e^0. \tag{4.5} \]

The part parallel to \( S \) is
\[ \epsilon^{ABC} (\epsilon^i_\mu d_B u_C + K^j \epsilon_{ij\rho} \epsilon^\rho_\mu) = 0. \tag{4.6} \]

Now using equation (C.5) and multiplying by \( \epsilon_{EF\rho} \epsilon^\rho_\mu \) we obtain the equation
\[ K^\mu_{[AE]} = 0. \tag{4.7} \]

We remark that this is a consequence of spacetime field equations of the model under consideration and it is an algebraic relation between quantities defined on \( S \). We shall refer to this sort of equations as **constraints**, in opposition to **evolution equations** which will appear later. Hereafter, we shall be particularly interested in constraint equations, see [3].

The part orthogonal to \( S \) is
\[ \epsilon^i_{[A} d_B u_{|0]} + \tilde{K}^j \epsilon_{|i\rhoj]} = \sigma^2 \nabla \epsilon_{i\rhoj]}. \]

We stress that we used above the expression \( d_B u_{|0]} \) in place of \( \partial_B x^\nu \partial_C x^\rho \). As usual \([,\cdot,\cdot]\) denotes the antisymmetrization of homologous indices. Hereafter we shall set \( \Gamma = A - \gamma K \).

Equations (4.8) are nine equations. Note that the evolution part of these equations (namely, \( 2 \epsilon^{DBC} d_B u_{|0]} \)) enters through the antisymmetric part in \([CD]\). Let us thus split these nine equations into the symmetric (six equations) and antisymmetric (three equations) parts.

The symmetric part is
\[ 2 \epsilon^{ABC} d_A u_{|0]} - \epsilon^{D[} A u_{|0]} - \sigma^2 \epsilon^{ABC} \epsilon_i^{D]} \Gamma_{AE} e_i^j = 0. \tag{4.9} \]

The antisymmetric part is
\[ \epsilon^{ABC} \Gamma_{AE} e_i^j = 0. \tag{4.10} \]

Note that these are six further constraint equations, while equation (4.9) is not. We remark how the original 12 equations (4.5) have been split into 9 constraint equations (4.7) and (4.10) and 3 evolution equations (4.9).
4.2. Equation for $\delta A^k_i$

Let us project the equation
\[
\gamma^\alpha e^0 \wedge e_k + e_{ijk} \nabla e^i \wedge e^j = \gamma^\alpha \nabla e_k \wedge e^0 - (\sigma^2 - \gamma^2)e^j_{ik} K^i \wedge e_i \wedge e^0. \tag{4.11}
\]

The part parallel to $S$ is
\[
\epsilon^{ABC} \left( \gamma^\alpha d_{A[B} e^C_i + e^k_{ij} \nabla A^e_k e^C_i \right) = 0
\]
\[
\Rightarrow \epsilon^{ABC} e^A_{ij} \gamma^\alpha \nabla A^e k e^C_i = \epsilon^{ABC} e^D_{[A} \epsilon^D_{e]i} \nabla A^e k = 0
\]
\[
\Rightarrow \epsilon^{ABC} e^A_{ij} \gamma^\alpha \nabla A^e k e^C_i = 0
\]
\[
\Rightarrow \epsilon^{ABC} e^A_{ij} \gamma^\alpha \nabla A^e k e^C_i = 0
\]
\[
\Rightarrow \epsilon^{ABC} e^A_{ij} \gamma^\alpha \nabla A^e k e^C_i = 0
\]
where we set $\epsilon := \det e_A^i$ and $E^A_i := \epsilon e^A_i$.

This constraint condition can also be expressed in the following form:
\[
\epsilon^{ABC} \left( \gamma^\alpha d_{A[B} e^C_i \right) = 0 \Rightarrow \epsilon^{ABC} e^D_{[A} \epsilon^D_{e]i} \nabla A^e k = 0
\]
\[
\Rightarrow \epsilon^{ABC} e^D_{[A} \epsilon^D_{e]i} \nabla A^e k = 0
\]
\[
\Rightarrow \epsilon^{ABC} \left( \gamma^\alpha d_{A[B} e^C_i \right) = 0
\]
\[
\Rightarrow \epsilon^{ABC} \left( \gamma^\alpha d_{A[B} e^C_i \right) = 0
\]
where we used (4.7) in the last equation.

Note that (4.10) and (4.13) together imply
\[
\epsilon^{ABC} \epsilon^D_{[A} \epsilon^D_{e]i} \nabla A^e k = 0
\]
which in turn implies that $\Gamma = A - \gamma K$ is the connection $\Gamma(\epsilon)$ induced by the triad, see lemma 1.

We also remark that using this last result the evolution equation (4.9) simplifies to
\[
2 \epsilon^{ABC} d_{A[B} e_{C]} = -\epsilon^{ABC} K^i_{[A} e^i_{e]A}.
\]

The part orthogonal to $S$ is
\[
\epsilon^{ABC} \left( 2 \gamma^\alpha d_{A[B} e^C_i - \gamma \sigma^2 \nabla A^e k e^C_i + 2 e^k_{ij} \nabla (e^0) e^i_A + (\sigma^2 - \gamma^2) \sigma^2 e^k_{ij} K^i_{[A} e^j_{e]B} \right) = 0.
\]

By subtraction, equations (4.16) and (4.9) finally give
\[
\epsilon^{ABC} \left( 2 \gamma^\alpha d_{A[B} e^C_i + e^k_{ij} K^i_{[A} e^j_{e]B} - \gamma K^i_{[A} e^j_{e]B} \right) = 0
\]
which are nine evolutionary equations.

4.3. Equation for $\delta e^k_0$

Let us project the equation
\[
F^k \wedge e_k + (\sigma^2 - \gamma^2)e_{ijk} K^i \wedge K^j \wedge e^k + \frac{2 \sigma^2 - \gamma^2}{\gamma} \nabla K^i \wedge e_i = 0.
\]

The part parallel to $S$ is
\[
\epsilon^{ABC} \left( F^k_{[A} e^C_i + (\sigma^2 - \gamma^2)e_{ijk} K^i_{[A} e^j_{e]C} + 2 \frac{\sigma^2 - \gamma^2}{\gamma} \nabla K^i_{[A} e^j_{e]C} \right) = 0
\]
\[
\Rightarrow \epsilon^{ijk} F^k_{[A} e^A_i e^B_j + 2 (\sigma^2 - \gamma^2) K^i_{[A} K^j_{e]B} e^A_i e^B_j + 2 \frac{\sigma^2 - \gamma^2}{\gamma} \nabla K^i_{[A} e^j_{e]B} e^A_i e^B_j = 0.
\]

Then by lemma 2 we obtain
\[
\epsilon^{ijk} F^k_{[A} e^A_i e^B_j - 2 (\sigma^2 - \gamma^2) K^i_{[A} K^j_{e]B} e^A_i e^B_j = 0.
\]
The part orthogonal to $S$ is
\[ \epsilon^{ABC} \left( 2F^k_A e_{kB} + 2(\sigma^2 - \gamma^2)\epsilon_{iB} K^i_A e_B + 4\frac{\sigma^2 - \gamma^2}{\gamma} \nabla_{[B} K^i_{0]} \epsilon_{iA} \right) = 0 \] (4.21)
which are evolutionary equations.

### 4.4. Equation for $\delta e_i^A$

Let us project the equation
\[ 2(\sigma^2 - \gamma^2) K^i_A e_{kB} + \epsilon_{kij} F^j_{[A} e_{B]} + 2(\sigma^2 - \gamma^2) \nabla_{[A} e^i_{B]} + \gamma(\sigma^2 - \gamma^2) e_{kij} F^j_{[A} e_{B]} = 0. \] (4.22)

The part parallel to $S$ is
\[ \epsilon^{ABC} (2(\sigma^2 - \gamma^2) K^i_A e_{kB} + \epsilon_{kij} F^j_{[A} e_{B]} = 0 \] (4.23)
and using (4.7) we easily obtain
\[ F^i_{AB} e^A_i = 0. \] (4.24)

The part orthogonal to $S$ is
\[ \sigma^2 (\sigma^2 - \gamma^2) \nabla_{[A} K^i_{B]} - 2(\sigma^2 - \gamma^2) K^i_{[A} e_{B]} + 2\epsilon_{kij} F^j_{[A} e_{B]} + \gamma(\sigma^2 - \gamma^2) e_{kij} F^j_{[A} e_{B]} = 0 \] (4.25)
which are nine evolutionary equations ($F^i_A$ contains the evolutionary part for the field $A'_i$).

Hence, by finally collecting the constraint equations altogether one gets
\[ \begin{cases} \nabla_A E^A_i = 0 \\ F^i_{AB} E^A_i = 0 \\ \epsilon_{kij} F^k_{AB} E^A_i E^B_j - 2(\sigma^2 - \gamma^2) K^i_{[A} e_{B]} E^A_i E^B_j = 0 \end{cases} \] (4.26)

where $\gamma K'_i (A_i, j^i) = A_i - \Gamma^i_A$. These are the starting point of LQG, see [3].

### 5. Conclusions and perspectives

We produced a spacetime, manifestly covariant Lagrangian formulation of Ashtekar–Barbero–Immirzi gravity. The Lagrangian is simply the pull-back of the Holst’s Lagrangian along the field variables $(e, A, K)$.

By projecting on a spacelike hypersurface $S$ we reobtained in a reasonably simple way the Hamiltonian constraints which are the starting point for the LQG framework together with the expression of the extrinsic field $K$ in terms of the fields $(E, A)$. The Lagrangian written in the form (3.3) is, to the best of our knowledge, new in the literature.

We believe that this formulation might provide a better understanding of the gauge-covariant structure that LQG is based on. This structure is particularly important since it is the key fact which most of the results of LQG (namely, the discretization of space areas and volumes) are based on. It may also help a better understanding of spin foam models as it provides a spacetime formulation better adapted to the final LQG scheme when compared to the standard Holst’s formulation.

Further investigations will be devoted to the general behaviour of a gauge theory endowed with a reduction of the gauge group. It is particularly interesting to investigate the behaviour of conservation laws in such a case.
Acknowledgments

This work is partially supported by MIUR: PRIN 2005 on ‘Leggi di conservazione e termodinamica in meccanica dei continui e teorie di campo’. We also acknowledge the contribution of INFN (Iniziativa Specifica NA12) and the local research funds of Dipartimento di Matematica of Torino University.

Appendix A. Equivalence of field equations

We shall prove here the equivalence between field equations (3.4) ensuing from the BIH Lagrangian (3.3) and field equations of the usual Holst Lagrangian (3.1), namely

\[
\begin{align*}
\mathbf{\hat{w}}^\mu e^\mu &+ \frac{\gamma}{2} \varepsilon_{cd} e^a e^b \mathbf{\hat{w}}^c \wedge e^d = 0, \\
R^{ab} \wedge e_a + \frac{\gamma}{2} \varepsilon_{cd} e^a R^{cd} \wedge e_a = 0.
\end{align*}
\]  
(A.1)

Let us first note that

\[
\mathbf{\hat{w}}^0 e^0 = de^0 + \omega^0 \wedge e^0 = de^0 + K_i \wedge e^i
\]

and

\[
\mathbf{\hat{w}}^i e^i = de^i + \omega^i \wedge e^0 + \omega^0 \wedge e^i = \nabla e^i + \gamma e^i \sigma_j K^j \wedge e^k + \sigma^2 e^0 \wedge K^i
\]

where the inverse of transformation (2.3) has been used, namely

\[
\begin{align*}
\omega^{\mu} &\equiv K^\mu, \\
\omega^{ij} &\equiv \epsilon^{ij k}(A^k_{\mu} - \gamma K^k_{\mu}).
\end{align*}
\]  
(A.4)

By setting \([ab] = [0t]\) in the first equation of (A.1) we obtain

\[
0 = (de^0 + K_j \wedge e^j) \wedge e^i - (\nabla e^i + \gamma e^i j K^j \wedge e^k + \sigma^2 e^0 \wedge K^i) \wedge e^0 + \sigma^2 \gamma e^i j K^j \nabla e^i \wedge e^k + (\sigma^2 - \gamma^2) \sigma^2 K_k \wedge e^k \wedge e^i \equiv E^{ij}_0. 
\]  
(A.5)

By setting \([ab] = [ij]\) in the first equation of (A.1) we obtain

\[
0 = \nabla e^i \wedge e^j - \nabla e^j \wedge e^i + \gamma e_k^{ij} \nabla e^0 \wedge e^k - \gamma e_k^{ij} e^0 \wedge \nabla e^k + \nabla e^0 \wedge e^i \gamma e_k^{ij} \nabla e^k \wedge e^0 + 2(\sigma^2 - \gamma^2) e^0 \wedge K^0 \wedge e^i
\]

\[
+ 2 \gamma e_k^{ij} (e^0 \wedge e^i) \wedge e^j \wedge K^0 + \gamma e_k^{ij} e^0 K_i + e^i \equiv E^{ij}. 
\]  
(A.6)

One can easily check that \(e^i j k E^{jk} = 0\) coincides with the second field equation of (3.4); analogously the combination \(E^{0i} - \sigma^2 \gamma e^i j k E^{jk} = 0\) coincides with the first field equation of (3.4).

By setting \(b = 0\) in the second equation of (A.1) we obtain

\[
0 = R^{ik} \wedge e^i - \sigma^2 \gamma e^i j k R^{jk} \wedge e_i
\]

\[
= \sigma^2 (\gamma F^i \wedge e_i - 2(\sigma^2 - \gamma^2) \nabla K^i \wedge e_i - \gamma (\sigma^2 - \gamma^2) e^i j K^j \wedge K^k \wedge e_i)
\]  
(A.7)

where we used (3.2); the equation so obtained is equivalent to the third field equation of (3.4).

By setting \(b = i\) in the second equation of (A.1) we obtain

\[
0 = R^{0i} \wedge e_0 - R^{ij} \wedge e_j - \sigma^2 \gamma e^i j k R^{jk} \wedge e_0 - \gamma e^i j k R^{0k} \wedge e_j
\]

\[
= 2(\sigma^2 - \gamma^2) \nabla K^i \wedge e^0 + 2(\sigma^2 - \gamma^2) K^i \wedge K^j \wedge e_j
\]

\[
- \gamma (\sigma^2 - \gamma^2) e^i j K^j \wedge K^k \wedge e^0 + \gamma F^i \wedge e^0 + e^i j F^j \wedge e^k
\]

which is equivalent to the fourth field equation of (3.4).
Appendix B. Projection of the frame

In [2] it was shown how one can project (with no gauge fixing) the spacetime tetrads to the triads on the hypersurface \( S \subset M \).

The construction was adapted to the self-dual formulation based on the projection on the self-dual bundle

\[
P \longrightarrow {}^+P \\
\downarrow \downarrow \\
M \quad M
\]

while we now need to deal with the Barbero–Immirzi formulation that is based on the reduction

\[
{}^+P \longrightarrow P \\
\downarrow \downarrow \\
M \quad M
\]

In the discussion about the self-dual formulation we basically resorted to the splitting \( \text{Spin}(4) \simeq SU(2) \times SU(2) \) and the fact that nothing depends on the anti-self-dual part.

In the Barbero–Immirzi case we have a different principal bundle diagram, namely the reduction

\[
\begin{array}{ccc}
{}^+P & \longrightarrow & P \\
\downarrow \downarrow & & \downarrow \downarrow \\
{}^\Lambda P & \longrightarrow & L(M) \\
i^\dagger & \longrightarrow & i^\dagger \\
{^+\Sigma} & \longrightarrow & {\Sigma} \\
M & \longrightarrow & M \\
i & \longrightarrow & i \\
S & \longrightarrow & S \\
\end{array}
\]

Moreover, in the Lorentzian case it is non-trivial to find a group of spin transformations fixing the BI connection. We hence revert to another argument, which remarkably leads to the same result.

Let us start with a point \((k, {}^+S) \in {}^+\Sigma\); it induces a point \((i(k), {}^+S) \in {}^+P\) and a point \((i(k), ^+S, {}^+S) \in P\). Still there exists a unique element \((\vec{I}, -S) \in \text{Spin}(4)\) such that \(\ell(\vec{I}, -S)(e_0) = u\); hence the frame \(\hat{e}_a = e_a \cdot \ell((\vec{I}, -S))\) is adapted to the submanifold \(S \subset M\). If the fields \((e^\mu_a, \theta^{ab}_\mu)\) provide a solution for the tetrad-affine formalism, then the corresponding \((\hat{e}^\mu_a, A'_\mu, K'_\mu)\) provide a solution of the Barbero–Immirzi formulation.

The Lorentzian case goes as the Euclidean one, except that the reduction is obtained by the group morphism \(i : \text{Spin}(3) \simeq SU(2) \rightarrow \text{Spin}(3, 1) \simeq SL(2, \mathbb{C}) : u \mapsto u\). Hence, one proves that there exists a unique element in the form \(S = \alpha + Ev \in \text{Spin}(3, 1)\) with \(\alpha \in \mathbb{R}\) such that the frame \(\hat{e}_a = e_a \cdot \ell((\vec{I}, -S))\) is adapted to the submanifold \(S \subset M\). If the fields \((e^\mu_a, \theta^{ab}_\mu)\) provide a solution for the tetrad-affine formalism, then the corresponding \((\hat{e}^\mu_a, A'_\mu, K'_\mu)\) provide a solution of the Barbero–Immirzi formulation.

In both cases the triad so obtained transforms as expected for a spin frame on \(^+\Sigma\).
Appendix C. Projection onto a hypersurface

We shall collect here a few tricks used in the above derivation of projected equations. Let us consider a (possibly local) 1-form \( \alpha = \alpha_\mu \, dx^\mu \); we can define two fields on \( S \):

\[
\begin{align*}
\alpha_A(k) &:= \alpha_\mu(x(k)) \partial_A x^\mu(k), \\
\tilde{\alpha}(k) &:= \alpha_\mu(x(k)) u^\mu(k).
\end{align*}
\] (C.1)

These two fields are a 1-form and a scalar over \( S \), respectively.

One can do the same with a 2-form \( \beta = \beta_{\mu\nu} \, dx^\mu \wedge dx^\nu \); we can define two fields on \( S \):

\[
\begin{align*}
\beta_{AB}(k) &:= \beta_{\mu\nu}(x(k)) \partial_A x^\mu(k) \partial_B x^\nu(k), \\
\tilde{\beta}_A(k) &:= \beta_{\mu\nu}(x(k)) u^\mu(k) \partial_A x^\nu(k).
\end{align*}
\] (C.2)

These two fields are a 2-form and a 1-form over \( S \), respectively.

By projecting the differential \( \beta = d\alpha \), one obtains

\[
\begin{align*}
\beta_{AB} &:= d\left[\alpha_{A}^{\mu i} \partial_A x^\mu \partial_B x^i\right], \\
\tilde{\beta}_A &:= d\left[\tilde{\alpha}^{\mu} \partial_A x^\mu\right].
\end{align*}
\] (C.3)

By differentiating the first equation of (C.1) with respect to \( k^B \), we easily get

\[
d\left[\alpha_{A}^{\mu i} \partial_A x^\mu \partial_B x^i\right] = \beta_{AB}.
\] (C.4)

which means that the part of the differential \( d\alpha \) which is parallel to \( S \) is in fact the differential (on \( S \)) of the parallel part to \( S \) of \( \alpha \) itself.

We stress that the same trick does not hold for the orthogonal part; the orthogonal part of the differential cannot be computed on \( S \) alone. This means that, as expected, the differential in the orthogonal direction does somehow encode evolution of fields on \( S \) (or better it would encode evolution if we considered a foliation). Nevertheless, we shall call evolutionary the equations involving the orthogonal part of the differential.

Let us thus consider the special case in which \( \alpha_\mu = e_0^\mu \). In this case, we obtain \( \alpha_A = 0 \) and \( \tilde{\alpha} = \sigma^2 \); by applying (C.4) we directly obtain

\[
d\left[\beta_{A}^{\mu i} \partial_A x^\mu \partial_B x^i\right] = 0.
\] (C.5)

Let us now prove the following lemmas.

**Lemma 1.** There exists a unique connection \( \Gamma_A^i \), namely the connection \( \Gamma_A^i(\epsilon) \) induced by the triad, such that \( \nabla_A e_B^i = 0 \).

**Proof.** Let us introduce the quantity \( \Delta_{km}^i = (\Gamma_A^i - \Gamma_A^i(\epsilon)) e_m^A =: \frac{1}{2} \epsilon^{ijk} \Delta_{km}^{jk} \). The hypothesis can be written in terms of this quantity as

\[
\Delta_{km}^i = 0.
\] (C.6)

Hence, we obtain a quantity \( \Delta_{ijk} \) which is antisymmetric in \([ij]\) and symmetric in \((jk)\). Then one trivially proves

\[
\Delta_{ijk} = -\Delta_{ijk} = -\Delta_{jki} = \Delta_{kij} = -\Delta_{kij} = -\Delta_{ijk} \quad \Rightarrow \quad \Delta_{ijk} = 0
\] (C.7)

which in turn proves that \( \Gamma_A^i = \Gamma_A^i(\epsilon) \). \( \square \)

**Lemma 2.** \( \nabla_A K_B^k E_{iC}^A e^{ABC} = -2\gamma K_A^i K_B^j E_i^A E_j^B \)
Proof. We have

\[ \nabla_A K^i_B \epsilon^i_C e^{ABC} = d_A K^i_B \epsilon^i_C e^{ABC} - \epsilon^i_m K^i_B \epsilon^m_C e^{ABC} - \epsilon^i_m K^i_B \epsilon^m_C e^{ABC} = -K^i_B d_A \epsilon^i_C e^{ABC} - \epsilon^i_m K^i_B \epsilon^m_C e^{ABC} \]

\[ \overset{(4.14)}{=} -K^i_B \epsilon^i_l \Gamma^j_l e^{ABC} - \epsilon^i_m K^i_B \epsilon^m_C e^{ABC} \]

\[ = -K^i_B \epsilon^i_l (\gamma K^j_A - A^j_A) \epsilon^m_C e^{ABC} - \epsilon^i_m K^i_B \epsilon^m_C e^{ABC} \]

\[ = -2\gamma K^j_A \epsilon^j_B \epsilon^i_C e^{ABC} \]

(C.8)

from which the lemma readily follows. \( \Box \)

References

[1] Fatibene L, Francaviglia M and Rovelli C 2007 On a covariant formulation of the Barbero–Immirzi connection Class. Quantum Grav. 24 3055–66
[2] Fatibene L and Francaviglia M 2005 Spin structures on manifolds and Ashtekar variables Int. J. Geom. Methods Mod. Phys. 2 147–57
[3] Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[4] Perez A 2004 Introduction to loop quantum gravity and spin foams Proc. 2nd Int. Conf. on Fundamental Interactions (Pedra Azul, Brazil) (Preprint gr-qc/0409061)
[5] Thiemann T 2001 Introduction to modern canonical quantum general relativity Preprint gr-qc/0110034
[6] Antonsen F and Flagg M S N 2002 Spacetime topology (I)—chirality and the third Stiefel–Whitney class Int. J. Theor. Phys. 41
[7] Milnor J W and Stasheff J 1974 Characteristic Classes (Princeton, NJ: Princeton University Press)
[8] Fatibene L and Francaviglia M 2003 Natural and Gauge Natural Formalism for Classical Field Theories. A Geometric Perspective Including Spinors and Gauge Theories (Dordrecht: Kluwer)
[9] Holst S 1996 Barbero’s Hamiltonian derived from a generalized Hilbert–Palatini action Phys. Rev. D 53 5966
[10] Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report Class. Quantum Grav. 15 R53–152 (Preprint gr-qc/0404018)