ON SMOOTHLY SUPERSLICE KNOTS

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ABSTRACT. We find smoothly slice (in fact doubly slice) knots in the 3-sphere with trivial Alexander polynomial that are not superslice, answering a question posed by Livingston and Meier.

1. Introduction

A recent paper of Livingston and Meier raises an interesting question about superslice knots. Recall [3] that a knot $K$ in $S^3$ is said to be superslice if there is a slice disk $D$ for $K$ such that the double of $D$ along $K$ is the unknotted 2-sphere $S$ in $S^4$. We will refer to such a disk as a superslicing disk. In particular, a superslice knot is slice and also doubly slice, that is, a slice of an unknotted 2-sphere in $S^4$. Livingston and Meier ask about the converse in the smooth category.

Problem 4.6 (Livingston-Meier [10]). Find a smoothly slice knot $K$ with $\Delta_K(t) = 1$ that is not smoothly superslice.

The corresponding question in the topological (locally flat) category is completely understood [10, 12], for a knot $K$ with $\Delta_K(t) = 1$ is topologically superslice.

In this note we give a simple solution to problem 4.6, making use of Taubes’ proof [10] that Donaldson’s diagonalization theorem [5] holds for certain non-compact manifolds. For $K$ a knot in $S^3$, we write $\Sigma_k(K)$ for a $k$-fold cyclic branched cover of $S^3$ branched along $K$. The same notation will be used for the corresponding branched cover along an embedded disk in $B^4$ or sphere in $S^4$.

Theorem 1.1. Suppose that $J$ is a knot with Alexander polynomial 1, so that $\Sigma_k(J) = \partial W$, where $W$ is simply connected and the intersection form on $W$ is definite and not diagonalizable. Then the knot $K = J\# - J$ is smoothly doubly slice, but is not smoothly superslice.

An unpublished argument of Akbulut says that the positive Whitehead double of the trefoil is a knot $J$ satisfying the hypotheses of the theorem, with
\[ k = 2. \] The construction is given as \[1\] Exercise 11.4 and is also documented, along with some generalizations, in the paper \[4\]. Hence \( J \) gives an answer to Problem 4.6. We remark that for the purposes of the argument, it doesn’t matter if \( W \) is positive or negative definite, as one could replace \( J \) by \(-J\) and change all the signs.

We need a simple and presumably well-known algebraic lemma.

**Lemma 1.2.** Suppose that

\[
\begin{array}{c}
  A \\
  \downarrow i_1 \\
  B \\
  \downarrow i_2 \\
  \downarrow j_2 \\
  C \\
  \uparrow j_1 \\
  B \\
  \uparrow i_1
\end{array}
\]

is a pushout of groups, and that \( i_1 = i_2 \). Then \( C \) surjects onto \( B \).

*Proof.* This follows from the universal property of pushouts; the identity map \( \text{id}_B \) satisfies \( \text{id}_B \circ i_1 = \text{id}_B \circ i_2 \), and hence defines a homomorphism \( C \rightarrow B \) with the same image as \( \text{id}_B \). \[\square\]

Applying Lemma 1.2 to the decomposition of the complement of the unknot in \( S^4 \) into two disk complements, we obtain the following useful facts. (The first of these was presumably known to Kirby and Melvin; compare \[8\] Addendum, p. 58, and the second is due to Gordon and Sumners \[6\].)

**Corollary 1.3.** If \( K \) is superslice and \( D \) is a superslicing disk, then

\[
\pi_1(B^4 - D) \cong \mathbb{Z} \quad \text{and} \quad \Delta_K(t) = 1.
\]

*Proof.* The lemma says that there is a surjection \( \mathbb{Z} \cong \pi_1(S^4 - S) \rightarrow \pi_1(B^4 - D) \). Hence \( \pi_1(B^4 - D) \) is abelian and so must be isomorphic to \( \mathbb{Z} \). This condition implies, using Milnor duality \[13\] in the infinite cyclic covering, that the homology of the infinite cyclic covering of \( S^3 - K \) vanishes, which is equivalent to saying that \( \Delta_K(t) = 1 \). \[\square\]

*Proof of Theorem 1.1.* It is standard \[15\] that any knot of the form \( J \# -J \) is doubly slice. In fact, it is a slice of the 1-twist spin of \( J \), which was shown by Zeeman \[17\] to be unknotted.

Suppose that \( K \) is superslice and let \( D \) be a superslicing disk, so \( D \cup_K D = S \), an unknotted sphere. Then \( S^4 = \Sigma_k(S) = V \cup_Y V \), where we have written \( Y = \Sigma_k(K) \) and \( V = \Sigma_k(D) \). By Claim 1.3 the \( k \)-fold cover of \( B^4 - D \) has \( \pi_1 \cong \mathbb{Z} \), so the branched cover \( V \) is simply connected.

Note that \( \Sigma_k(K) = \Sigma_k(J) \# -\Sigma_k(J) \). Since \( \Delta_J(t) = 1 \), the same is true for \( \Delta_K(t) \), moreover this implies that both \( \Sigma_k(J) \) and \( \Sigma_k(K) \) are homology
spheres. An easy Mayer–Vietoris argument says that $V = \Sigma_k(D)$ is a homology ball; in fact Claim 1.3 implies that it is contractible. Adding a 3-handle to $V$, we obtain a simply-connected homology cobordism $V'$ from $\Sigma_k(J)$ to itself. By hypothesis, there is a manifold $W$ with boundary $\Sigma_k(J)$ and non-diagonalizable intersection form. Stack up infinitely many copies of $V'$, and glue them to $W$ to make a definite periodic-end manifold $M$, in the sense of Taubes [16]. Since $\pi_1(V)$ is trivial, $M$ is admissible (see [16, Definition 1.3]), and Taubes shows that its intersection form (which is the same as that of $W$) is diagonalizable. This contradiction proves the theorem.

□

The fact that $\pi_1(B^4 - D) \cong \mathbb{Z}$ for a superslicing disk leads to a second obstruction to supersliceness, based on the Ozsváth–Szabó $d$-invariant [14]. Recall from [11] (for degree 2 covers) and [7] in general that for a knot $K$ and prime $p$, that one denotes by $\delta_p(K)$ the $d$-invariant of a particular spin structure $s$ on $\Sigma_p^p$ pulled back from the 3-sphere. The fact that a $p^n$ fold branched cover of a slicing disk is a rational homology ball implies that if $K$ slice then $\delta_p(K) = 0$. For a non-prime-power degree $k$, the invariant $\delta_k(K)$ might not be defined, because $\Sigma_k(K)$ is not a rational homology sphere. (One might define such an invariant using Floer homology with twisted coefficients as in [2, 9], but there’s no good reason that it would be a concordance invariant.)

**Theorem 1.4.** If $K$ is superslice, then for any $k$, the $d$-invariant $d(\Sigma_k(K), s_0)$ is defined and vanishes.

**Proof.** Since by Claim 1.3 the Alexander polynomial is trivial, so $\Sigma_k(K)$ is a homology sphere, and hence $d(\Sigma_k(K), s_0)$ is defined. (There is only the one spin structure.) As in the proof of Theorem 1.1 the branched cover $\Sigma_k(D)$ is contractible, and hence [14, Theorem 1.12], $d(\Sigma_k(K), s_0) = 0$.

□

Sadly, we do not know any examples of a slice knot where Theorem 1.4 provides an obstruction to it being superslice. For such a knot would not be ribbon, so we would also have a counterexample to the slice-ribbon conjecture!

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