Black Hole Solutions in Braneworlds with Induced Gravity

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Abstract

We extend our previous study on spherically symmetric braneworld solutions with induced gravity, including non-local bulk effects. We find the most general static four-dimensional black hole solutions with \(g_{tt} = -g_{rr}^{-1}\). They satisfy a closed system of equations on the brane and represent the strong-gravity corrections to the Schwarzschild-\((A)dS_4\) spacetime. These new solutions have extra terms which give extra attraction relative to the Newtonian-\((A)dS_4\) force; however, the conventional limits are easily obtained. These terms, when defined asymptotically, behave like \(AdS_4\) in this regime, while when defined at infinitely short distances predict either an additional attractive Newtonian potential or an attractive potential which scales approximately as \(\sqrt{r}\). One of the solutions found gives extra deflection of light compared to Newtonian deflection.

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1. Introduction

Branes are solitonic solutions of ten-dimensional string theories. In the most simplified picture of the braneworld scenario, our physical world is realized as a four-dimensional hypersurface embedded in a five-dimensional space called bulk. All matter and gauge interactions live on the brane, while the gravitational interactions are effective in the whole five-dimensional space. This novel approach of visualizing our world, offers a new understanding of the four fundamental forces. While in the so-called symmetric picture all known interactions were tried to be unified under the same symmetry group, the braneworld scenario treats the weak, electromagnetic and strong interactions differently than gravitational interactions. This allowed to define a gravitational scale of the whole space, which if the extra dimensions are large, can be as low as the $TeV$ scale $[1, 2, 3]$, while the four-dimensional gravitational scale of our world is at the Planck scale. This happens because our four-dimensional world is confined on the brane and can “see” only the four-dimensional localized gravitational field.

Braneworld solutions can give us information about the structure and nature of the extra dimensions. Very recently, we have a plethora of observational data, both cosmological and astrophysical. Consistency of cosmological and local braneworld solutions with these data can give information about the parameters of the theory, like the energy scale, the size of the extra dimensions or the strength of the gravitational force of the extra dimensions. For example, spherically symmetric local braneworld solutions can give information about the crossover scale above which the extra dimensions appear, how the Newton’s constant changes with matter density, or what are the corrections of the gravitational potential at high energies.

Braneworld solutions can be obtained following two different approaches. In the first approach, the dynamics and the geometry of the whole space is primarily considered, and then, the dynamics on the brane is extracted using mainly consistency checks like the Israel matching conditions. The second approach is to specify the dynamics and the geometry on the brane first, and then try to extent the solution to the bulk. A disadvantage of this method is that finding the bulk geometry in which the brane consists its boundary may be a very difficult task. Another difficulty in this approach is, as we will discuss later, that it is not always possible to obtain a closed set of equations on the brane, so that only with data on the brane to be able to predict the behavior of the fields on the brane. However, this method is basically the only way we have for finding non-trivial braneworld solutions (i.e. solutions not arising from factorizable bulk geometries).
The effective brane equations have been obtained \[4\] when the effective low-energy theory in the bulk is higher-dimensional gravity. However, a more fundamental description of the physics that produces the brane could include \[5\] higher order terms in a derivative expansion of the effective action, such as a term for the scalar curvature of the brane, and higher powers of curvature tensors on the brane. If the dynamics is governed not only by the ordinary five-dimensional Einstein-Hilbert action, but also by the four-dimensional Ricci scalar term induced on the brane, new phenomena appear. In \[6\], it was observed that the localized matter fields on the brane (which couple to bulk gravitons) can generate via quantum loops a localized four-dimensional worldvolume kinetic term for gravitons (see also \[7, 8, 9, 10\]). That is to say, four-dimensional gravity is induced from the bulk gravity to the brane worldvolume by the matter fields confined to the brane. It was also shown that an observer on the brane will see correct Newtonian gravity at distances shorter than a certain crossover scale, despite the fact that gravity propagates in extra space which was assumed there to be flat with infinite extent; at larger distances, the force becomes higher-dimensional.

A realization of the induced gravity scenario in string theory was presented in \[11\]. Furthermore, new closed string couplings on Dp-branes for the bosonic string were found in \[12\]. These couplings are quadratic in derivatives and therefore take the form of induced kinetic terms on the brane. For the graviton in particular these are the induced Einstein-Hilbert term as well as terms quadratic in the second fundamental tensor. Considering the intrinsic curvature scalar in the bulk action, the effective brane equations have been obtained in \[13\]. Results concerning cosmology have been discussed in \[14, 15, 16, 17, 18, 19, 20\].

In our previous paper \[21\], we discussed the gravitational field of an uncharged, non-rotating spherically symmetric rigid object when in the dynamics there is a contribution from the brane intrinsic curvature invariant. We found all the possible exterior braneworld solutions. Some of these solutions are of the Schwarzschild-(\(A\))dS\(_4\) form. In two cases, we also solved the interior problem which reduces to a generalization of the Oppenheimer-Volkoff solution. It was shown that the gravitational constant get corrected for very small matter densities. The conventional solar system bounds of General Relativity have set the crossover scale below the \(TeV\) scale. All the above results were obtained by setting the electric part of the Weyl tensor, \(E_{\mu\nu}\), vanishing on the brane, as the boundary condition of the propagation equations in the bulk space.

In the present paper, we generalize our study of spherically symmetric braneworld solutions with induced gravity, by including the non-local bulk effects, as they are ex-
pressed by a non-vanishing electric part of the Weyl tensor on the brane. By choosing 
\( g_{tt} = -g_{rr}^{-1} \), the system of equations on the brane becomes closed and all the possible 
static black hole solutions are found for these metrics. These solutions have generic new 
terms which give extra attractive force compared to the Newtonian -\((A)dS_4\) force, and 
represent the strong-gravity corrections to the Schwarzschild-\((A)dS_4\) spacetime. However, 
the conventional limits are easily obtained. The new terms, when defined asymptotically, 
behave like \( AdS_4 \) in this regime, while when defined at infinitely short distances predict 
either an additional attractive Newtonian potential or an attractive potential which scales 
approximately as \( \sqrt{r} \). One of the solutions found gives extra deflection of light compared 
to Newtonian deflection.

The paper is organized as follows. In section 2, we give a general introduction to the 
induced gravity formalism and we review some of the solutions found in [21]. In section 
3, we include the electric part of the Weyl tensor in spherically symmetric braneworld 
configurations and we show how the modified Einstein equations supplemented by the 
Bianchi identities constitute a closed system of equations on the brane. In section 4, we 
present our solutions and discuss their physical implications. Finally, in the last section 
we summarize our results.

2. Induced Gravity Equations on the Brane

We consider a 3-dimensional brane \( \Sigma \) embedded in a 5-dimensional spacetime \( M \).
Capital Latin letters \( A, B, \ldots = 0, 1, \ldots, 4 \) will denote full spacetime, lower Greek \( \mu, \nu, \ldots = 
0, 1, \ldots, 3 \) run over brane worldvolume, while lower Latin ones span some 3-dimensional 
spacelike surfaces foliating the brane, i.e. \( i, j, \ldots = 1, \ldots, 3 \). For convenience, we can quite 
generally, choose a coordinate \( y \) such that the hypersurface \( y = 0 \) coincides with the 
brane. The total action for the system is taken to be:

\[
S = \frac{1}{2 \kappa_5^2} \int_M \sqrt{-g} \ (5) R - 2 \Lambda_5) d^5x + \frac{1}{2 \kappa_4^2} \int_\Sigma \sqrt{-g} \ (4) R - 2 \Lambda_4) d^4x \\
+ \int_M \sqrt{-g} L_{5, \text{mat}}^\text{mat} d^5x + \int_\Sigma \sqrt{-g} L_{4, \text{mat}}^\text{mat} d^4x. \tag{1}
\]

For clarity, we have separated the cosmological constants \( \Lambda_5, \Lambda_4 \) from the rest matter 
contents \( L_{5, \text{mat}}, L_{4, \text{mat}} \) of the bulk and the brane respectively. \( \Lambda_4/\kappa_4^2 \) can be interpreted as 
the brane tension of the standard Dirac-Nambu-Goto action and can include quantum 
contributions to the four-dimensional cosmological constant. We basically concern on the 
case with no fields in the bulk, i.e. \( (5) T_{AB} = 0 \).
From the dimensionful constants $\kappa_5^2$, $\kappa_4^2$ the Planck masses $M_5$, $M_4$ are defined as:

$$\kappa_5^2 = 8\pi G_5 = M_5^{-3}, \quad \kappa_4^2 = 8\pi G_4 = M_4^{-2},$$

with $M_5$, $M_4$ having dimensions of (length)$^{-1}$. Then, a distance scale $r_c$ is defined as:

$$r_c \equiv \frac{\kappa_5^2}{\kappa_4^2} = \frac{M_4^2}{M_5^3}. \quad (3)$$

Varying (1) with respect to the bulk metric $g_{AB}$, we obtain the equations

$$(5G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2 (5T_{AB} + \text{(loc)} T_{AB} \delta(y)), \quad (4)$$

where

$$\text{(loc)} T_{AB} \equiv -\frac{1}{\kappa_4^2} \sqrt{-\frac{-4}{(4)} g} \left( (4G_{AB} - \kappa_4^2 (4T_{AB} + \Lambda_4 h_{AB}) \right) \quad (5)$$

is the localized energy-momentum tensor of the brane. $^{(5)}G_{AB}$, $^{(4)}G_{AB}$ denote the Einstein tensors constructed from the bulk and the brane metrics respectively. Clearly, $^{(4)}G_{AB}$ acts as an additional source term for the brane through $\text{(loc)} T_{AB}$. The tensor $h_{AB} = g_{AB} - n_A n_B$ is the induced metric on the hypersurfaces $y = \text{constant}$, with $n^A$ the normal vector on these.

The way the $y$-coordinate has been defined, allows us to write, at least in the neighborhood of the brane, the 5-line element in the block diagonal form

$$ds^2_5 = -N^2 dt^2 + g_{ij} dx^i dx^j + dy^2, \quad (6)$$

where $N, g_{ij}$ are generally functions of $t, x^i, y$. The distributional character of the brane matter content makes necessary for the compatibility of the bulk equations (4) the following modified (due to $^{(4)}G_{\mu}^\nu$) Israel-Darmois-Lanczos-Sen conditions [22]

$$[K^\mu_\nu] = -\kappa_5^2 \left( \frac{\text{(loc)} T^\mu_\nu}{3} - \frac{\text{(loc)} T}{3} \delta^\mu_\nu \right), \quad (7)$$

where the bracket means discontinuity of the extrinsic curvature $K_{\mu\nu} = \partial_y g_{\mu\nu}/2$ across $y = 0$. A $\mathbb{Z}_2$ symmetry on reflection around the brane is considered throughout.

One can derive from equations (4), (7) the induced brane gravitational dynamics [13], which consists of a four-dimensional Einstein gravity, coupled to a well-defined modified matter content. More explicitly, one gets

$$^{(4)}G^\mu_\nu = \kappa_4^2 (4T^\mu_\nu - \left( \Lambda_4 + \frac{3}{2} \alpha^2 \right) \delta^\mu_\nu + \alpha \left( L^\mu_\nu + \frac{L}{2} \delta^\mu_\nu \right), \quad (8)$$
where $\alpha \equiv 2/r_c$, while the quantities $L^\mu_\nu$ are related to the matter content of the theory through the equation

$$L^\mu_\nu L^\lambda_\nu - \frac{L^2}{4} \delta^\mu_\nu = T^\mu_\nu - \frac{1}{4}(3\alpha^2 + 2T^\lambda_\lambda) \delta^\mu_\nu,$$  \hspace{1cm} (9)

and $L \equiv L^\mu_\mu$. The quantities $T^\mu_\nu$ are given by the expression

$$T^\mu_\nu = \left(\Lambda_4 - \frac{1}{2}\Lambda_5\right) \delta^\mu_\nu - \kappa_4^2 \left((5)T^\mu_\nu + \frac{(5)T}{4} \delta^\mu_\nu\right) - \mathbb{E}^\mu_\nu,$$  \hspace{1cm} (10)

with $(5)T = (5)\bar{T}^A_A$, $(5)\bar{T}^A_B = g^{AC}(5)\bar{T}^C_B$. Bars over $(5)T^A_B$ and the electric part $\mathbb{E}^\mu_\nu = C^\mu_{AB}n^A n^B$ of the 5-dimensional Weyl tensor $C^A_{BCD}$ mean that the quantities are evaluated at $y = 0$. $\mathbb{E}^\mu_\nu$ carries the influence of non-local gravitational degrees of freedom in the bulk onto the brane [4] and makes the brane equations (8) not to be, in general, closed. This means that there are bulk degrees of freedom which cannot be predicted from data available on the brane. One needs to solve the field equations in the bulk in order to determine $\mathbb{E}^\mu_\nu$ on the brane [23].

Due to the contracted Bianchi identities, the following differential equations among $L^\mu_\nu$ arise from (8):

$$L^\mu_{\nu ; \mu} + \frac{L_{\nu ; \mu}}{2} = 0.$$  \hspace{1cm} (11)

The spherically symmetric braneworld line-element is

$$ds^2_{(4)} = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (12)

The matter content of the 3-universe is considered to be a localized spherically symmetric untilted perfect fluid (e.g. a star) $(4)T_{\mu_\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu_\nu}$ ($u^\mu$ stands for the four-velocity of the fluid) with $\rho = p = 0$ for $r > R$, plus the cosmological constant $\Lambda_4$. The matter content of the bulk is a cosmological constant $\Lambda_5$. In [21], we considered the case $\mathbb{E}^\mu_\nu = 0$ as the boundary condition of the propagation equations in the bulk space. This is somehow simplified from the viewpoint of geometric complexity, but it was the first step for investigating the characteristics carried by the brane curvature invariant on the local brane dynamics we are interested in. All the solutions outside a static localized matter distribution were found. One of these is the Schwarzschild-$(A)dS_4$ metric which is matched to a modified Oppenheimer-Volkoff interior. We will review this solution and in the next section we will discuss the strong gravity corrections resulting from the presence
of the electric part of the Weyl tensor on the brane. The other solutions have $AB \neq 1$ and will not be discussed here. The exterior solution was found to be

$$B_>(r) = \frac{1}{A_>(r)} = 1 - \frac{\gamma}{r} - \beta r^2 , \ r \geq R ,$$

(13)

where $\gamma$ is an integration constant and

$$\beta = \frac{1}{3} \Lambda_4 + \frac{1}{2} \alpha^2 - \frac{\alpha}{2 \sqrt{3}} \sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2} .$$

(14)

Considering a uniform distribution $\rho(r) = \rho_o = \frac{3M}{4\pi R^3}$ for the object, the interior solution was found, and its matching to the above exterior specified the integration constant $\gamma$. The result is

$$\frac{1}{A_<(r)} = 1 - (\beta + \frac{\gamma}{R^3}) r^2 , \ r \leq R ,$$

(15)

$$B_<(r) = \frac{1 - \frac{\gamma}{R} - \beta R^2}{\left(1 + \frac{4 \pi R^3}{3 M} p(r)\right)^2} , \ r \leq R ,$$

(16)

$$p(r) = -\rho_o \sqrt{1 - (\beta + \frac{\gamma}{R^3}) r^2} - \sqrt{1 - (\beta + \frac{\gamma}{R^3}) R^2} \sqrt{1 - (\beta + \frac{\gamma}{R^3}) r^2} - \omega \sqrt{1 - (\beta + \frac{\gamma}{R^3}) R^2} ,$$

(17)

where

$$\gamma = \frac{\kappa_4^2 M}{4 \pi} + \frac{\alpha R^3}{2 \sqrt{3}} \sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2} - \frac{\alpha R^3}{2 \sqrt{3}} \sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2} + \frac{3 \kappa_4^2 M}{\pi R^3} ,$$

(18)

$$\omega^{-1} = 1 - \frac{2}{\kappa_4^2 \rho_o} \left(\beta + \frac{\gamma}{R^3}\right) \left(1 - \frac{\sqrt{3} \alpha}{\sqrt{4 \Lambda_4 - 2 \Lambda_5 + 3 \alpha^2 + 4 \kappa_4^2 \rho_o}}\right)^{-1} .$$

(19)

The parameters $\gamma$ and $\beta$ of the Schwarzschild-$(A)dS_4$ exterior solution (13) can be constrained by solar system experiments. The bounds obtained fix the crossover scale below the $TeV$ range. The $(A)dS_4$ term $\beta r^2$ in this black hole solution gives at large distances additional force compared to the ordinary Newtonian force. Finally, the $\gamma$ parameter in the $1/r$ term modifies the Newton’s gravitational constant which, as it is seen from (18), for small matter densities deviates significantly from its conventional value.
3. Black Holes in Induced Gravity with Non-Local Bulk Effects

So far we considered local corrections to the Einstein equations on the brane. The presence of the electric part $E_{\mu\nu}$ of the Weyl tensor in (10) indicates the 5D gravitational stresses, which are known as massive KK modes of the graviton. For brane observers, these stresses are non-local. Local density inhomogeneities on the brane generate Weyl curvature in the bulk that “back-reacts” non-locally on the brane [23, 24, 25, 26, 27, 28, 29]. Therefore, in general, this term cannot be ignored.

With respect to the privileged direction $u^\mu$ defined by the perfect fluid, the symmetric and traceless tensor $E_{\mu\nu}$ is uniquely and irreducibly decomposed as follows

$$E_{\mu\nu} = U(u_\mu u_\nu + \frac{1}{3} h_{\mu\nu}) + P_{\mu\nu} + 2 Q_\mu u_\nu ,$$

(20)

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection operator normal to $u^\mu$, while $P_{\mu\nu} u^\nu = Q_\mu u^\mu = 0$. $U$ is the non-local energy density, $P_{\mu\nu}$ the non-local anisotropic stress, and $Q_\mu$ the non-local energy flux on the brane. Static spherical symmetry implies [23] that $Q_\mu = 0$, $P_{\mu\nu} = P(r) (r_\mu r_\nu - \frac{1}{3} h_{\mu\nu})$, (21)

where $r_\mu$ is the unit radial vector. Thus, the non-vanishing components of the electric part of the Weyl tensor are

$$E_0^0 = -U , \quad E_r^r = \frac{1}{3}(U + 2P) , \quad E_\theta^\theta = E_\phi^\phi = \frac{1}{3}(U - P) .$$

(22)

This means that for $^5T_B^A = 0$ in (10), the matrix $T_i^j$ has two distinct eigenvalues, namely

$$T_r^r = \Lambda_4 - \frac{1}{2} \Lambda_5 - \kappa_1^2 P - \frac{1}{3}(U + 2P) \equiv \tau_r ,$$

(23)

$$T_\theta^\theta = T_\phi^\phi = \Lambda_4 - \frac{1}{2} \Lambda_5 - \kappa_1^2 P - \frac{1}{3}(U - P) \equiv \tau_\theta \equiv \tau_\phi .$$

(24)

Consequently, due to the block-diagonal form of the metric, the solution of the algebraic system (13) is

$$L_i^0 = L_0^i = 0 ,$$

(25)

together with one of the following three distinct cases for $L_j^i, L_0^0$:

**Case 1**

$$L_j^i = \text{diag} \left( \sqrt{(L_0^0)^2 + \tau_r - T_0^0}, \sqrt{(L_0^0)^2 + \tau_\theta - T_0^0}, \sqrt{(L_0^0)^2 + \tau_\theta - T_0^0} \right)$$

(26)
Case 2

\[ L^i_j = \text{diag} \left( \sqrt{(L^0_0)^2 + \tau_r - T^0_0}, \sqrt{(L^0_0)^2 + \tau_\theta - T^0_0}, -\sqrt{(L^0_0)^2 + \tau_\theta - T^0_0} \right) \]  

Case 3

\[ L^i_j = \text{diag} \left( \sqrt{(L^0_0)^2 + \tau_r - T^0_0}, -\sqrt{(L^0_0)^2 + \tau_\theta - T^0_0}, -\sqrt{(L^0_0)^2 + \tau_\theta - T^0_0} \right) \]  

and the algebraic equation

\[ \sqrt{(L^0_0)^2 + \tau_r - T^0_0} + 2\epsilon \sqrt{(L^0_0)^2 + \tau_\theta - T^0_0} \pm L^0_0 = \sqrt{4(L^0_0)^2 + 2(\tau_r + 2\tau_\theta) + 3\alpha^2 - 2T^0_0} \]  

with \( \epsilon = +1, 0, -1 \) for the cases 1,2,3 respectively.

For the considered metric (12), one evaluates the Ricci tensor \( ^{(4)}R_{\mu\nu} \), and construct the field equations (8). The combination \( ^{(4)}R_{rr}/2A + ^{(4)}R_{\theta\theta}/r^2 + ^{(4)}R_{00}/2B \) provides the following differential equation for \( A(r) \):

\[ \frac{A'}{A} = \frac{1 - A}{r} + Ar \left[ \kappa_4^2 \rho + \Lambda_4 + \frac{3}{2} \alpha^2 - \frac{\alpha}{2} (3L^0_0 + L^r_r + 2L^\theta_\theta) \right] , \]  

\( A' \equiv \frac{dA}{dr} \). Eliminating \( \frac{A'}{A} \) in the \((\theta\theta)\) component of (8) using (30), we get an equation for \( \frac{B'}{B} \), from which we obtain

\[ \frac{(AB)'}{AB} = Ar \left[ \kappa_4^2 (\rho + p) - \alpha (L^0_0 - L^r_r) \right] . \]  

From the \((\theta\theta)\), \((\phi\phi)\) equations of (8) it arises that

\[ L^0_\theta = L^0_\phi . \]

Using (32), the Bianchi equations (11) take the form

\[ \frac{B'}{B}(L^r_r - L^0_0) + \frac{4}{r}(L^r_r - L^\theta_\theta) + (L^0_0 + 3L^r_r + 2L^\theta_\theta)' = 0 . \]  

The system of brane equations (8), due to the presence of non-local bulk effects is not closed. Equations (30), (31) and (33) consist a system of three equations with four unknown functions \( A, B, U, \mathcal{P} \). To make these equations closed we will look for solutions of the form \( AB = 1 \) and find the most general braneworld metrics for such configurations. Then, equation (31), outside the matter distribution is written equivalently as

\[ L^0_0 = L^r_r , \]
and equations (30), (33) get respectively the form

\[ \frac{A'}{A} = \frac{1 - A}{r} + Ar \left[ \Lambda_4 + \frac{3}{2} \alpha^2 - \alpha(2L_0^0 + L_\phi^0) \right], \quad (35) \]

\[ \frac{2}{r}(L_0^0 - L_\theta^0) + (2L_0^0 + L_\phi^0)' = 0. \quad (36) \]

Therefore, we have to solve the above two equations (35), (36) with \( L_0^0, L_r^0, L_\theta^0, L_\phi^0 \) satisfying (29), (32), (34) together with one of the relations (26) or (27) or (28).

To use relations (26), (27), (28) we have first to identify \( L_1, L_2, L_3 \) with \( L_r^0, L_\theta^0, L_\phi^0 \).

In doing so, we have to examine various cases depending on the choice of \( L \)'s. Demanding to have non-vanishing \( E_\mu^\nu \), only two cases remain, while all the others have \( E_\mu^\nu = 0 \) and have been thoroughly examined in [21]. These two cases are

\[ L_r^0 = \sqrt{(L_0^0)^2 + \tau_r - T_0^0}, \quad L_\theta^0 = L_\phi^0 = \epsilon \sqrt{(L_0^0)^2 + \tau_\theta - T_0^0}, \quad (37) \]

depending on the sign \( \epsilon = +1 \) or \( -1 \). In both cases, equation (34) is equivalent to the relation

\[ \tau_r = T_0^0 \Leftrightarrow 2U + \mathcal{P} = 0. \quad (38) \]

This equation, together with equation (29) which becomes

\[ L_0^0 \pm L_0^0 + 2\epsilon \sqrt{(L_0^0)^2 + \mathcal{P}} = \sqrt{4(L_0^0)^2 + 4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 + 2\mathcal{P}}, \quad (39) \]

relation \( L_\theta^0 = L_\phi^0 = \epsilon \sqrt{(L_0^0)^2 + \mathcal{P}} \), and equations (35), (36) are everything we have to satisfy. It is obvious that the two square roots appearing in (39) have to be well defined.

4. Black Hole Solutions including Non-Local Bulk Effects

a. First solution

We first consider the \( - \) of the \( \pm \) sign in the algebraic relation (33). The only solution with non-zero \( E_\mu^\nu \) corresponds to \( \epsilon = +1 \) with \( U, \mathcal{P} \) being constants

\[ \mathcal{P} = -2U = \frac{1}{2} \left( 4\Lambda_4 - 2\Lambda_5 + 3\alpha^2 \right). \quad (40) \]

Straightforward integration of (33) gives

\[ r = c \left( \sqrt{(L_0^0)^2 + \mathcal{P}} + L_0^0 \right)^{\frac{1}{4}} e^{\frac{3}{4} \sqrt{(L_0^0)^2 + \mathcal{P}} + L_0^0}, \quad (41) \]
with $c > 0$ an integration constant. It is convenient to define the positive dimensionless variable

$$z = \frac{9}{8|P|} \left( \sqrt{(L_0^0)^2 + P + L_0^0} \right)^2,$$

(42)

and then, integration of the remaining equation $(35)$ gives

$$B = \frac{1}{A} = 1 - \frac{\gamma}{r} - \beta r^2 + \frac{\delta}{r} \left[ \frac{128}{105} {}_1F_1 \left( \frac{15}{8}, \frac{23}{8}; sg(\zeta)z \right) z + \frac{9}{8} \left( \frac{1}{z} - sg(\zeta) \frac{8}{r} \right) e^{sg(\zeta)z} \right] z^{\frac{7}{8}},$$

(43)

where $\gamma$ is another integration constant (typically interpreted as $2G_NM$ with $M$ being the mass of the point particle, $G_N$ the Newton’s constant),

$$\beta = \frac{1}{3} \Lambda_4 + \frac{1}{2} \alpha^2, \quad \zeta = \frac{\alpha^2}{9} (4\Lambda_4 - 2\Lambda_5 + 3\alpha^2),$$

(44)

and $\delta = \frac{4}{9}(\frac{9}{8})^{\frac{1}{8}} |P|\alpha c^3 > 0$. Relation (41) becomes

$$r = \left( \frac{\delta}{\sqrt{|\zeta|}} \right)^{\frac{4}{9}} z^{\frac{1}{8}} e^{sg(\zeta)z^{3/8}}.$$  

(45)

The well-definiteness of the square roots in (39) and the fact that $L_0^0 \geq 0$ (eqs. (34), (37)) translate to $z \geq \frac{9}{8}$, which means that $r$ is larger (smaller) than $(9e^{3sg(\zeta)/8}^{1/8} (\delta/\sqrt{|\zeta|})^{1/3}$ for $\zeta > 0$ ($\zeta < 0$). Note that $r(z)$ is a monotonically increasing (decreasing) function for $\zeta > 0$ ($\zeta < 0$) in its range of validity.

Thus, the resulting solution is given in parametric form by equations (43), (44), containing two integration constants $\delta > 0$, $\gamma$ and two parameters $\beta$, $\zeta$ connected to $\alpha, \Lambda_4, \Lambda_5$ by relations (44). The electric components of the Weyl tensor are given by equation (40), $P = -2U = 9\zeta/(2\alpha^2)$.

Comparing the solution (43) with the solution (13) having $E_{\nu}^{\nu} = 0$, we note that there is a new term, besides the conventional Newtonian and $(A)dS_4$ terms, which carries the information of the gravitational field in the bulk. For $\zeta > 0$, the asymptotic behavior $r \to \infty$ of this new term in the solution (13) is seen to be $AdS_4$–like, i.e. $\sqrt{\zeta} r^2$. Thus, asymptotically, the effective cosmological constant is $\beta - \sqrt{\zeta}$. For $\zeta < 0$, the asymptotic behavior $r \to 0$ of the new term in the solution (13) is Newtonian, i.e. $-2\Gamma(7/8)\delta/r$. Thus, the effective Newton’s constant in this regime appears larger.

We know that for non-relativistic particles the effective potential is $2\Phi = B - 1$. As it can be seen, the new force corresponding to the above non-local term is always attractive. For $\zeta > 0$, its magnitude is monotonically increasing with distance, while
force
(a): $\zeta > 0$, $\beta = 0$
force
(b): $\zeta < 0$, $\beta = 0$
Figure 1: Dotted lines represent the Newtonian force. Continuous lines represent the total force, i.e. the sum of the Newtonian and the new force.

for $\zeta < 0$, this happens in decreasing distances (after a characteristic scale). In order for the new term not to disturb the well-measured Newtonian law at distances from the cm to the solar-distance scale, one has in both cases to adjust the quantity $\delta/\gamma$ as small as desired. For $\zeta > 0$ and for larger distances, the sum of the Newtonian and the new force decreases (in magnitude) slower than the Newtonian force, while for even larger distances, this sum grows to infinity (Figure 1a). For $\zeta < 0$, deviations between the total and the Newtonian force appear only at small distances (Figure 1b). The $(A)dS_4$ term $\beta r^2$ is generally considered to be of cosmological origin and is not considered here to be of importance at the local level.

Finishing with the above solution (43), we notice that this may have some interesting physical implications. For $\zeta > 0$, because the total gravitational force grows slower than the conventional Newtonian law, this force may serve as a possible qualitative explanation for the yet unresolved problem of galactic rotation curves. However, numerical fittings with real data remain to be done. On the other hand, the solution with $\zeta < 0$ could be considered if submillimeter deviations from the Newtonian law are discovered.

b. Second solution

Now, considering the + case of the $\pm$ sign in (39), we define the dimensionless variable

$$v = 2 + \epsilon \sqrt{1 + \frac{P}{(L_0^0)}}, \quad (46)$$
and equation (39) gets the form
\[(L_0^0)^2 = \frac{4\zeta}{9\alpha^2(v^2 - 3)}, \quad (47)\]
where
\[\zeta = \frac{9\alpha^2}{8}(4\Lambda_4 - 2\Lambda_5 + 3\alpha^2). \quad (48)\]
Thus, the electric components of the Weyl tensor are found in terms of \(\nu\) to be
\[\mathcal{P} = -2\mathcal{U} = \frac{4\zeta(v - 1)(v - 3)}{9\alpha^2(v^2 - 3)}. \quad (49)\]
The Bianchi equation (36) can be seen to become a simple separable differential equation for \(\nu(r)\):
\[\frac{d\nu}{dr} + \frac{2}{3r}(v - 3)(v^2 - 3) = 0. \quad (50)\]
Straightforward integration of (50) gives
\[r = \left(\frac{\delta}{\sqrt{|\zeta|}}\right)^\frac{1}{4} \frac{|\nu - \sqrt{3}|^{(\sqrt{3}+1)/8}}{|v - 3|^{1/4} |\nu + \sqrt{3}|^{(\sqrt{3}-1)/8} }, \quad (51)\]
with \(\delta > 0\) an integration constant.

For \(\epsilon = +1\), it is \(\nu > 2\) (with \(\nu \neq 3\)) and from (17) we have \(\zeta > 0\). More specifically, for \(2 < \nu < 3\) it is \(r > (2 - \sqrt{3})^\sqrt{3/4} (\delta/\sqrt{|\zeta|})^{1/3}\) and \(r(\nu)\) is monotonically increasing, while for \(\nu > 3\) it is \(r > (\delta/\sqrt{|\zeta|})^{1/3}\) with \(r(\nu)\) monotonically decreasing. For \(\epsilon = -1\), from (39) it is \(\mathcal{P} < 0\), thus (16) gives \(1 < \nu < 2\). Therefore, from (17) we have \(1 < \nu < \sqrt{3}\) for \(\zeta < 0\), and \(\sqrt{3} < \nu < 2\) for \(\zeta > 0\). More specifically, for \(1 < \nu < \sqrt{3}\) it is \(r < (\sqrt{3} - 1)^\sqrt{3/4} (\delta/\sqrt{|\zeta|})^{1/3}/2^{(\sqrt{3}+1)/8}\) with \(r(\nu)\) being monotonically decreasing, while for \(\sqrt{3} < \nu < 2\) it is \(r < (2 - \sqrt{3})^\sqrt{3/4} (\delta/\sqrt{|\zeta|})^{1/3}\) with \(r(\nu)\) monotonically increasing.

The remaining Einstein equation (33) gives
\[B = \frac{1}{A} = 1 - \frac{\gamma}{r} - \beta r^2 \oplus \frac{\delta}{r} \int |\nu - \sqrt{3}|^{-3(3 - \sqrt{3})/8} (\nu + \sqrt{3})^{-3(3 + \sqrt{3})/8} \frac{v}{|v - 3|^{7/4}} dv, \quad (52)\]
where \(\gamma\) is another integration constant (typically interpreted as \(2G_N M\) with \(M\) being the mass of the point particle, \(G_N\) the Newton’s constant) and
\[\beta = \frac{1}{3}\Lambda_4 + \frac{1}{2}\alpha^2. \quad (53)\]
The symbol \(\oplus\) means – for \(\epsilon = +1, \nu > 3\) or for \(\epsilon = -1, 1 < \nu < \sqrt{3}\); for \(\epsilon = +1, 2 < \nu < 3\) or for \(\epsilon = -1, \sqrt{3} < \nu < 2\) it means +. The above integral cannot be computed in
terms of known functions. However, this can be done in the asymptotic regimes $r \to \infty$ and $r \to 0$. For $r \to \infty (\epsilon = +1)$, the new term in the solution (52) becomes $AdS_4$-like, i.e. $(\sqrt{2\zeta}/3\sqrt{3}) r^2$ and thus, asymptotically, the effective cosmological constant is $\beta - (\sqrt{2\zeta}/3\sqrt{3})$. For $r \to 0 (\epsilon = -1)$, approximating the integral in (52) around $v = \sqrt{3}$, we find that the new term scales as $r^{2(2-\sqrt{3})}$, giving therefore extra attractive force $1/r^{2\sqrt{3}-3}$. Numerical evaluation of the integral in (52) leads for $\epsilon = +1$ qualitatively to the same picture as that of Figure 1a, where by adjusting the quantity $\delta/\gamma$ as small as desired, deviations from Newtonian law appear only at large distances. Similarly, for $\epsilon = -1$, the picture for the solutions resembles qualitatively to that of Figure 1b, where deviations from Newtonian law appear only at very small distances.

Thus, the resulting solution is given in parametric form by equations (51), (52), containing two integration constants $\delta > 0$, $\gamma$ and two parameters $\beta$, $\zeta$ connected to $\alpha$, $\Lambda_4$, $\Lambda_5$ by relations (48), (53). The electric components of the Weyl tensor are given by equation (49).

c. Deflection of light

We have considered so far the motion of non-relativistic particles. However, the motion of a freely falling photon in a static isotropic gravitational field (12) is described by the equation

$$\left(\frac{d\phi}{dr}\right)^2 = A r^4 \left(\frac{1}{J^2 B} - \frac{1}{r^2}\right)^{-1},$$

(54)

where $J$ is an integration constant. In the cases where the solutions (13), (52) deviate from Newton’s law at large distances, it is seen from (54) that $d\phi/dr \to 0$ as $r \to \infty$, and thus, the photon moves in a “straight” line of the background geometry in that region (even when a second horizon exists, we consider it of cosmological size compared to the local distances of interest). More specifically, at large distances, it arises from (54) that $\phi(r) - \phi(\infty) \simeq (\frac{1}{J^2} + \beta - \sqrt{\zeta})^{-1/2} \frac{1}{r}$ for the solution (13), while for the solution (52), $\sqrt{\zeta}$ is replaced by $\sqrt{2\zeta}/3\sqrt{3}$ in the last expression. This means that the “impact parameter” $b$ is $b = (\frac{1}{J^2} + \beta - \sqrt{\zeta})^{-1/2}$ for (13) (and respectively for (52) with the change of $\sqrt{\zeta}$). For our solutions, the total deflection angle in (54) cannot be computed explicitly. However, we can understand the influence of the new term on the motion of a photon and compare to the Newtonian deflection. For doing so, we have to refer to two photons with the same “initial conditions”, i.e. the same impact parameter $b$, one moving in a Schwarzschild-$(A)dS_4$ background (denoted by the subscript 1) and the other in the background defined
by the solutions (43), (52) (denoted by the subscript 2). The following equations are easily obtained for the solutions (43), (52) respectively:

\[
\frac{1}{\sqrt{\zeta} r^4} \frac{(dr_1)^2 - (dr_2)^2}{(d\phi)^2} = \frac{\sqrt{z}}{e^z} \left[ \frac{128}{105} \text{F}_1 \left( \frac{15}{8}, \frac{23}{8}; z \right) z + \frac{9}{8} \left( \frac{1}{z} - \frac{8}{7} \right) e^z \right] - 1, \tag{55}
\]

\[
\frac{3\sqrt{3}}{\sqrt{2}\zeta r^4} \frac{(dr_1)^2 - (dr_2)^2}{(d\phi)^2} = \frac{\sqrt{z}}{e^z} \left[ \frac{128}{105} \text{F}_1 \left( \frac{15}{8}, \frac{23}{8}; z \right) z + \frac{9}{8} \left( \frac{1}{z} - \frac{8}{7} \right) e^z \right] - 1. \tag{56}
\]

It is obvious that for the branch \( \epsilon = +1 \), \( v > 3 \) which extends to infinity, the right-hand side of equation (56) is negative, giving \( (dr_2)^2 > (dr_1)^2 \). Therefore, extra deflection of light compared to the Newtonian deflection arises. This situation of increased deflection (compared to that caused from the luminous matter) has been well observed in galaxies or clusters of galaxies and the above solution might serve as a possible way for providing an explanation. On the other hand, it is easily checked that equation (55) provides less deflection compared to Newtonian deflection at the distances of interest.

5. Conclusions

In this paper, we presented a new class of brane black hole solutions with induced gravity. It is known that the non-local bulk effects, as they are expressed via the projection of the Weyl tensor on the brane, do not make the brane dynamics closed. We need to know the geometry of the bulk space in order to be able to deal with the dynamics on the brane. In the case where \( g_{tt} = -g_{rr}^{-1} \), the system of equations consisting of the modified Einstein equations and the Bianchi identities is closed and we found all the possible black hole solutions. If we had to look for more general spherically symmetric solutions, some extra information would be needed for the non-local energy density \( U \) or the non-local anisotropic stress \( P \).

There has been argued \[30, 31\] on kinematical grounds, irrespectively from the gravitational dynamics, that the only spherically symmetric geometries which may be candidates for explaining from one side the extra deflection of light observed in galaxies and clusters of galaxies and from the other side the galactic rotation curves are of the form \( g_{tt} = -g_{rr}^{-1} \). However, severe criticism has appeared on this \[32\]. In the present paper, we use this interesting and reasonable condition to make the brane dynamics autonomous.
The black hole solutions we found, are in a sense generalizations of the spherically symmetric solutions we presented in [21]. They are representing strong-gravity corrections to the spherically symmetric Schwarzschild-(A)dS_4 braneworld. Their characteristic is that they predict a new attractive force. There are classes of solutions with increasing r, where this attractive force combined with the Newtonian one, results to a net force which decreases slower than the Newton’s force. This might have interesting physical implications for the explanation of galactic rotation curves. Within this class, a solution giving extra deflection of light compared to General Relativity predictions at galactic scales was found. It is interesting to observe that this solution has non-trivial (i.e. not constant) non-local energy $U$ and anisotropic stress $P$. In another class of solutions with decreasing $r$, the new force starts to deviate from the Newton’s force at small distances, indicating that at submillimeter scale we could have testable deviations from the Newtonian law.

In our previous work we also had deviations from the Newton’s law at large distances. This deviation was caused by the presence of the $(A)dS_4$ term $\beta r^2$, which for $\beta < 0$ can also give extra attraction. In our new solutions, the extra attractive force appears because of the presence of a new term which only asymptotically (when defined in this regime) behaves like $AdS_4$. This new term arises because of the presence of the electric part of the Weyl tensor and for an observer on the brane is a pure non-local effect. We had also found in [21] modifications to Newton’s law as a result of a change of the Newton’s constant due to the finite interior of the rigid object. This effect must have an analogous contribution here if one solves the interior problem.

We have followed a braneworld viewpoint for obtaining braneworld solutions, ignoring the exact bulk space. We have not provided a description of the gravitational field in the bulk space, but confined our interest to effects that can be measured by brane-observers. By making assumptions for obtaining a closed brane dynamics, there is no guarantee that the brane is embeddable in a regular bulk. This is the case for a Friedmann brane [32], whose symmetries imply that the bulk is Schwarzschild-AdS_5 [33, 34]. A Schwarzschild brane can be embedded in a “black string” bulk metric, but this has singularities [37, 38, 39, 40, 41]. The investigation of bulk backgrounds which reduce to Schwarzschild-(A)dS_4 or more general black holes is in progress.

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