Navier–Stokes equations: an analysis of a possible gap to achieve the energy equality

Paolo Maremonti · Francesca Crispo · Carlo Romano Grisanti

Received: 14 February 2020 / Revised: 4 May 2020 / Published online: 5 June 2020
© Università degli Studi di Napoli "Federico II" 2020

Abstract
The paper is concerned with the IBVP of the Navier–Stokes equations. The goal is to evaluate the possible gap between the energy equality and the energy inequality deduced for a weak solution. This kind of analysis is new and the result is a natural continuation and improvement of a result obtained by the same authors in Crispo et al. (Some new properties of a suitable weak solution to the Navier–Stokes equations. arXiv:1904.07641).

Keywords
Navier–Stokes equations · Weak solutions · Energy equality

Mathematics Subject Classification 35Q30 · 35B65 · 76D05
1 Introduction

This note concerns the 3D-Navier–Stokes initial boundary value problem:

\[ v_t + v \cdot \nabla v + \nabla \pi v = \Delta v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \Omega, \]
\[ v = 0 \text{ on } (0, T) \times \partial \Omega, \quad v(0, x) = v_0(x) \text{ on } \{0\} \times \Omega. \]  

(1)

In system (1) \( \Omega \subseteq \mathbb{R}^3 \) is assumed to be bounded or exterior, and its boundary is smooth. The symbol \( v \) denotes the kinetic field, \( \pi v \) is the pressure field, \( v_t := \frac{\partial}{\partial t} v \) and \( v \cdot \nabla v := v_k \frac{\partial}{\partial x_k} v \). We investigate on the existence of weak solutions. Here the notion of weak solution (see Definition 1) is meant in the sense of Leray’s weak solution, but with no change in the arguments it is possible to consider a weak solution in the sense of Caffarelli–Kohn–Nirenberg [1]. In this connection we refer the interested reader to the paper [2] for details.

We set \( J^2(\Omega) \) and \( J^{1,2}(\Omega) \) as the completion of \( C_0^0(\Omega) \) in \( L^2(\Omega) \) and in \( W^{1,2}(\Omega) \), respectively. The symbol \( C_0^0(\Omega) \) denotes the set \( \{ \varphi : \varphi \in C^\infty_0(\Omega) \text{ and } \nabla \cdot \varphi = 0 \} \).

**Definition 1** Let \( v_0 \in J^2(\Omega) \). A field \( v : (0, \infty) \times \Omega \rightarrow \mathbb{R}^3 \) is said a weak solution if the field \( v \) verifies

i. for all \( T > 0, v \in L^\infty(0, T; J^2(\Omega)) \cap L^2(0, T; J^{1,2}(\Omega)) \),

ii. \( \lim_{t \to 0} \| v(t) - v_0 \|_2 = 0 \),

iii. for all \( t, s \in (0, T), v \) satisfies the equation:

\[
\int_s^t \left[ (v, \varphi_\tau) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) \right] d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),
\]

for all \( \varphi \in C^1_0((0, T) \times \Omega) \), with \( \nabla \cdot \varphi = 0 \).

It is well known that any Leray’s weak solution enjoys the energy inequality in strong form [3,5,7]:

\[
\| v(t) \|_2^2 + \int_s^t \| \nabla v(\tau) \|_2^2 d\tau \leq \| v(s) \|_2^2, \quad \text{for all } t > s, \text{ a.e. in } s > 0 \text{ and for } s = 0. \]  

(2)

In the recent paper [2] the energy inequality (2) becomes a particular case of a suitable energy relation. In [2], assuming only \( v_0 \in J^2(\Omega) \), the existence of a suitable approximating sequence \( \{ v^m \} \), solutions to problem (25) (see Sect. 2), that, for all \( p \in [1, 2] \) and \( T > 0 \), enjoys the strong convergence in \( L^p(0, T; W^{1,2}(\Omega)) \cap L^2(0, T; L^2(\Omega)) \) to the weak solution \( v \) of (1) is obtained. This is the first result of strong convergence of the gradient. Its interest is connected with the possibility of deducing an energy equality. This result surely holds if the convergence is strong in \( L^2(0, T; J^{1,2}(\Omega)) \). Of course this is not our case. The idea is to handle the strong convergence in \( L^p(0, T; J^{1,2}(\Omega)) \).
\[ p \in [1, 2), \text{ suitably considering also the energy relation that holds for the approximation } \{v^m\}: \]
\[
\frac{1}{2} \frac{d}{dt} \|v^m(t)\|_2^2 + \|\nabla v^m(t)\|_2^2 = 0, \quad \text{for all } n \in \mathbb{N} \text{ and } t > 0. \tag{3}
\]
Unfortunately this it is not enough for the energy equality. Actually, by our arguments for the field \(v\) only
\[
\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau - \|v(s)\|_2^2 = -H(t, s), \quad \text{a.e. in } t > s > 0 \text{ and for } s = 0, \tag{4}
\]
and
\[
2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau = F(t, s)(\|v(s)\|_2^2 - \|v(t)\|_2^2), \quad \text{a.e. in } t > s > 0 \text{ and for } s = 0. \tag{5}
\]
are fulfilled. The functions \(H(t, s)\) and \(F(t, s) \in (0, 1]\) have the expressions:
\[
H(t, s) := \lim_{\gamma \to 0} \lim_{m \to \infty} \gamma \int_s^t \frac{\|v^m(\tau)\|_2^2}{(K + \|\nabla v^m(\tau)\|_2^2)^{\gamma+1}} \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau, \quad \text{for } s > 0, \tag{6}
\]
\[
H(t, 0) := \lim_{s \to 0} H(t, s)
\]
for any arbitrary constant \(K > 0\), as well as \(v\) satisfies relation (5) with
\[
F(t, s) := \lim_{\gamma \to 0} \lim_{m \to \infty} \frac{1}{(K_1 + \|\nabla v^m(t_{\gamma, m})\|_2)^\gamma}, \tag{7}
\]
for any arbitrary constant \(K_1 \geq 0\). Finally, the following property of inclusion holds: \(G_1 := \{t, s \text{ such that } (4) \text{ is true} \} \subseteq G_2 := \{t, s \text{ such that } (5) \text{ is true} \}. \) Actually in paper [2] these sets of instants are the ones for which the strong convergence of the sequence \(\{v^m\}\) holds in \(J^{1,2}(\Omega)\) and in \(J^2(\Omega)\), respectively. In the proof of formulas (8) and (12) we also use the set \(G_1\) as set of instants for which the quoted strong convergence holds.

In this note we give further partial contributions for an energy equality that are of some interest. Firstly we provide a generalization of properties (4) and (5) respectively. Actually we are able to prove
\[
G(t, s) = \|v(s)\|_2^2 - \|v(t)\|_2^2 - 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau, \quad \text{a.e. in } t \geq s > 0 \text{ and for } s = 0, \tag{8}
\]
the functions \( G(t, s) \) has the expression:

\[
G(t, s) := \lim_{\beta \to 0} \lim_{m \to \infty} \beta \int_s^t \|v^m(\tau)\|_2^2 g^{\beta - 1}(\|\nabla v^m(\tau)\|_2^2) g'(\|\nabla v^m(\tau)\|_2^2) \frac{d}{d\tau} \|\nabla v^m(\tau)\|_2^2 d\tau,
\]

for \( s > 0 \), \( G(t, 0) := \lim_{s \to 0} G(t, s) \).

(9)

for any arbitrary function \( g(\rho) > 0 \) which is Lipschitz continuous on \([0, \infty)\) and such that \( \lim_{\rho \to \infty} g(\rho) = 0 \). Also \( v \) satisfies the relation

\[
2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau = P(t, s)(\|v(s)\|_2^2 - \|v(t)\|_2^2), \quad \text{a.e. in } t \geq s > 0 \text{ and for } s = 0.
\]

(10)

with the following alternative expressions for the function \( P(t, s) \):

\[
\lim_{R \to \infty} p_R^\alpha(h(\alpha, R)) \in (0, 1], \quad \lim_{\alpha \to 0^+} \lim_{R \to \infty} p_R^\alpha(h(\alpha, R)) \in (0, 1],
\]

\[
\lim_{\alpha \to \infty} \lim_{R \to \infty} p_R^\alpha(h(\alpha, R)) \in (0, 1),
\]

(11)

where the exponent \( \alpha \) is real, \( p_R : [0, \infty) \to [0, 1] \) is a positive continuous cut off function with \( p_R(\rho) = 1 \) for \( \rho \in [0, R] \), \( p_R(\rho) = 0 \) for \( \rho \geq 2R \), and \( p_R(\rho) \in (0, 1) \) for \( \rho \in (R, 2R) \), and finally \( h(x, R) \in [0, 2R) \). Of course by proving that \( G(t, s) \leq 0 \) or \( P(t, s) = 1 \) one deduces the energy equality. No contraindication is known for these claims. Moreover we give the following

\[
E(t, s) = \|v(s)\|_2^2 - \|v(t)\|_2^2 - 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau, \quad \text{a.e. in } t \geq s > 0 \text{ and for } s = 0,
\]

(12)

with

\[
E(t, s) := \lim_{R \to \infty} E(t, s, R)
\]

\[
E(t, s, R) := \lim_{m \to \infty} \left[ \sum_{h=1}^{p_{22}(m, R)} 2 \left[ \|v^m(t_h^{22})\|_2^2 - \|v^m(s_h^{22})\|_2^2 \right] \right.
\]

\[
+ \sum_{h=1}^{p_{21}(m, R)} \left[ \|v^m(t_h^{21})\|_2^2 - 2\|v^m(s_h^{21})\|_2^2 \right] \right. \left. + \sum_{h=1}^{p_{11}(m, R)} \left[ \|v^m(t_h^{11})\|_2^2 - \|v^m(s_h^{11})\|_2^2 \right] \right.
\]

\[
+ \sum_{h=1}^{p_{12}(m, R)} \left[ 2\|v^m(t_h^{12})\|_2^2 - \|v^m(s_h^{12})\|_2^2 \right] \right],
\]

\[
E(t, 0) := \lim_{s \to 0} E(t, s),
\]

(13)
where, for all $R > 0$, $i_{h}^{ij} > s_{h}^{ij}$ and

$$
p_{22}(m, R) \sum_{h=1}^{22} (t_{h}^{22} - s_{h}^{22}) + p_{12}(m, R) \sum_{h=1}^{12} (t_{h}^{12} - s_{h}^{12}) + p_{11}(m, R) \sum_{h=1}^{11} (t_{h}^{11} - s_{h}^{11}) + \sum_{h=1}^{p_{21}(m, R)} (t_{h}^{21} - s_{h}^{21}) < \frac{\|v_{0}\|_{2}^{2}}{2R}. \tag{14}
$$

Since

$$\|v^{m}(t_{h}^{ij})\|_{2} - \|v^{m}(s_{h}^{ij})\|_{2} < 0, \quad \text{for all } i, j, h,$$

we get that on the right-hand side of (13) only the last sum can be positive, the other sums surely are negative. In addition, by construction the equality

$$p_{21}(m, R) = p_{12}(m, R) \tag{15}$$

holds. Hence we get

$$E(t, s, R) = \lim_{m \to \infty} \left[ B(m, R) + \|v^{m}(t_{1}^{12})\|_{2}^{2} - \|v^{m}(s_{21}^{21})\|_{2}^{2} \right], \quad \text{with } B(m, R) < 0, \quad \text{for all } R > 0. \tag{16}$$

This is of some interest. Actually if $E(t, s) \leq 0$, then we trivially have that the energy equality holds.

In the linear case or for 2D-Navier–Stokes equations we get $G(t, s) = 0$, $P(t, s) = 1$ and $E(t, s) = 0$. The first two claims are already proved in [2]. In the case of $E(t, s)$, for $R$ sufficiently large, we are able to prove that $p_{ij}(m, R) = 0$ for all $i, j = 1, 2$.

All the above considerations are condensed in the following statement which is the chief result of the note:

**Theorem 1** For all $v_{0} \in J^{2}(\Omega)$ there exists a weak solution $v$ to problem (1) which is a weakly continuous function of $t \geq 0$. The field $v$ is the weak* limit in $L^{\infty}(0, T; J^{2}(\Omega))$ and weak limit in $L^{2}(0, T; J^{1,2}(\Omega))$ of a suitable sequence $\{v^{m}\}$ enjoying the energy equality (3). The sequence $\{v^{m}\}$ strongly converges to $v$ in $L^{2}(0, T; L^{2}(\Omega)) \cap L^{p}(0, T; W^{1,2}(\Omega))$ for all $p \in [1, 2)$. Moreover the limit $v$ satisfies the energy relations (8), (10) and (12) with functions $G(t, s)$, $P(t, s)$ and $E(t, s)$ given by (9), (11) and (13) respectively. In particular a.e. in $s > 0$ and in $s = 0$, $v$ is continuous on the right in the $L^{2}$-norm. Finally, in the case of equality (12) estimates (14)–(16) hold.

**Remark 1** By virtue of (8), (10) and (12) the energy inequality (2) is a particular case again. We stress that if the energy equality does not hold, then the gap with the equality is governed by the terms $G(t, s)$, $P(t, s)$ and $E(t, s)$. Since for the definition of $G$ we have arbitrary auxiliary Lipschitz continuous functions $g(\rho)$ and $o(1)$ for large $\rho$, for the definition of $P(t, s)$ arbitrary exponents $\alpha$ and its limits to 0 or $\infty$ respectively,
we have a wide set of expressions related to the same value of the gap. In the case of the term \( E(t, s) \), by means of a possible series, the gap is expressed by means of the \( L^2 \)-norm of the approximations.

We remark that in the case of a bounded domain \( \Omega \) the results also hold assuming the Galerkin approximation as sequence, as proposed in [8] and in [4].

Finally, if we consider the validity of the energy equality for the weak limit, for example by assuming, for all \( \varepsilon > 0 \), \( v \in L^4(\varepsilon, T; L^4(\Omega)) \), as made in [6], which is compatible with the assumption \( v_0 \in J^2(\Omega) \), then we get \( G(t, s) = E(t, s) = 0 \) and \( P(t, s) = 1 \). It is of some interest to understand if in turn these new relations can imply an information on the partial regularity or on the singularity of the solution \( v \), limit of the sequence \( \{v^m\} \) quoted in Theorem 1.

## 2 Notations and some preliminary result

The symbol \( \mathcal{C}_0(\Omega) \) indicates the set of function \( C_0^\infty(\Omega) \) and divergence free. In this paper we denote by \( J^2(\Omega) \) and \( J^{1,2}(\Omega) \) the completion spaces of \( \mathcal{C}_0(\Omega) \) with respect to the metric of \( L^2(\Omega) \) and \( W^{1,2}(\Omega) \) respectively. By \( L^p(\varepsilon, T; X) \) we mean the space of functions from \( (\varepsilon, T) \) into \( X \) such that \( \int_0^T \|u(\tau)\|^p_X d\tau < \infty \). The following lemma is not known to the authors, hence the proof is proposed.

**Lemma 1** Let \( \{h^m(t)\} \subset L^1((0, T)) \) be a \( L^1 \)-bounded sequence. Assume that \( h^m(t) \to h(t) \) a.e. in \( t \in (0, T) \) and \( h \in L^1((0, T)) \). Let \( p(s) \in C([0, \infty)) \) with \( p(s) > 0 \) and \( \lim_{s \to \infty} p(s) = 0 \). Then, there exists a function \( h_0(t) \equiv h(t) \) a.e. in \( t \in (0, T) \), such that, for all \( \alpha > 0 \), we get

\[
\lim_m \int_0^T \left| h^m(t) p^{\alpha}(|h^m(t)|) - h(t) p^{\alpha}(|h_0(t)|) \right| dt = 0. \tag{17}
\]

Moreover, the following limit property also holds:

\[
\lim_{\alpha \to 0} \lim_m \int_0^T h^m(t) p^{\alpha}(|h^m(t)|) dt = \int_0^T h(t) dt. \tag{18}
\]

**Proof** Since \( h \in L^1((0, T)) \), the estimate \( |h(t)| < \infty \) holds for all \( t \in (0, T) - A \) with \( \text{meas}(A) = 0 \). We set \( h_0(t) = h(t) \) for \( t \in (0, T) - A \) and \( h_0(t) = 0 \) for \( t \in A \).
We write the left-hand side of (17) as follows:

\[
\begin{align*}
    \int_0^T |h^m(t)p^\alpha(|h^m(t)|) - h(t)p^\alpha(|h_0(t)|)|dt \\
    = \int_0^T \left| [h^m(t) - h(t)]p^\alpha(|h^m(t)|) + h(t)\left[p^\alpha(|h^m(t)|) - p^\alpha(|h_0(t)|)\right] \right| dt \\
    \leq \int_0^T |h^m(t) - h(t)|p^\alpha(|h^m(t)|)dt + \int_0^T |h(t)||p^\alpha(|h^m(t)|) - p^\alpha(|h_0(t)|)|dt.
\end{align*}
\]

(19)

Since \( |p^\alpha(|h^m(t)|) - p^\alpha(|h_0(t)|)| \leq 2\|p\|_{\infty}^p \), for all \( t \in (0, T) \), and \( h^m(t) \to h(t) \) a.e. in \( t \in (0, T) \), applying the Lebesgue dominated convergence theorem, we arrive at

\[
\lim_m \int_0^T |h(t)||p^\alpha(|h^m(t)|) - p^\alpha(|h_0(t)|)|dt = 0.
\]

(20)

Concerning the first integral on the right-hand side of (19), we arrange the following artifice:

\[
\begin{align*}
    \int_0^T |h^m(t) - h(t)|p^\alpha(|h^m(t)|)dt &= \int_{I_M^m} |h^m(t) - h(t)|p^\alpha(|h^m(t)|)dt \\
    &\quad + \int_{(0,T) - I_M^m} |h^m(t) - h(t)|p^\alpha(|h^m(t)|)dt =: J^m_1 + J^m_2,
\end{align*}
\]

where, for some fixed \( M > 0 \), \( I_M^m := \{ t : |h^m(t)| \leq M \} \). By the definition of \( I_M^m \), we easily obtain

\[
J^m_2 \leq p_M^\alpha \int_0^T (|h^m(t)| + |h(t)|)dt \leq c p_M^\alpha, \quad \text{for all } m \in \mathbb{N},
\]

where \( c \) is independent of \( M \) and \( p_M := \sup_{(M, \infty)} p(s) \). Concerning \( J^m_1 \), we introduce the characteristic function of the set \( I_M^m \), so that

\[
H^m(t) := |h^m(t) - h(t)|p^\alpha(|h^m(t)|) \chi_{I_M^m}(t) \to 0 \text{ a.e. in } t \in (0, T),
\]

and, a.e. in \( t \in (0, T) \), \( H^m(t) \leq (M + |h(t)|)\|p\|_{\infty}^\alpha \) for all \( m \in \mathbb{N} \).
So that applying the Lebesgue dominated convergence theorem, we deduce that \( \lim_m J^m_1 = 0 \). The above limit property of \( J^m_1 \) and estimate on \( J^m_2 \), via (19)–(20), furnish

\[
\lim_m \int_0^T |h^m(t) p^\alpha(|h^m(t)|) - h(t) p^\alpha(|h_0(t)|)| dt \leq c p^\alpha_M, \quad \text{for all } M > 0.
\]

Since \( p \to 0 \) for \( s \to \infty \), letting \( M \to \infty \) we prove (17). By applying the Lebesgue dominated convergence theorem we immediately get (18).

\[\square\]

**Corollary 1** Let \( \{h^m(t)\} \subset L^1((0, T)) \) be a sequence with \( A := \sup_m \|h^m\|_1 < \infty \). Assume that \( h^m(t) \to h(t) \) a.e. in \( t \in (0, T) \) and \( h \in L^1((0, T)) \). Let \( p_R(s) \in C([0, \infty)) \) with \( p_R(s) = 1 \) for \( s \leq R \), \( p_R(s) = 0 \) for \( s \geq 2R \) and \( p_R(s) \in (0, 1) \) for \( s \in (R, 2R) \). Then, there exists a function \( h_0(t) \equiv h(t) \) a.e. in \( t \in (0, T) \), such that, for all \( R > (2T)^{-1} A \), we get

\[
\lim_m \int_0^T |h^m(t) p_R(|h^m(t)|) - h(t) p_R(|h_0(t)|)| dt = 0. \tag{21}
\]

Moreover, the following limit also holds:

\[
\lim_{R \to \infty} \lim_m \int_0^T h^m(t) p_R(|h^m(t)|) dt = \int_0^T h(t) dt. \tag{22}
\]

**Proof** The first claim of the corollary is a particular case of the previous lemma. Hence we get

\[
\int_0^T |h^m(t) p_R(|h^m(t)|) - h(t) p_R(|h_0(t)|)| dt
\]

\[
= \int_0^T \left[ |h^m(t) - h(t)| p_R(|h^m(t)|) + h(t) \left[ p_R(|h^m(t)|) - p_R(|h_0(t)|) \right] \right] dt \tag{23}
\]

\[
\leq \int_0^T |h^m(t) - h(t)| p_R(|h^m(t)|) dt + \int_0^T |h(t)||p_R(|h^m(t)|) - p_R(|h_0(t)|)| dt,
\]

and

\[
\lim_m \int_0^T |h(t)||p_R(|h^m(t)|) - p_R(|h_0(t)|)| dt = 0. \tag{24}
\]
Concerning the first integral on the right-hand side of (23), we get

$$
\int_0^T |h^m(t) - h(t)| p_R(|h^m(t)|) dt = \int_0^{T_R} |h^m(t) - h(t)| p_R(|h^m(t)|) dt =: J_R^m,
$$

where we set \( T_R^m := \{ t : 0 \leq h^m(t) < 2R \} \). Since

\[
H^m(t) := |h^m(t) - h(t)| p_R(|h^m(t)|) \to 0 \text{ a.e. in } t \in T_R^m, \quad H^m(t) \leq 2R, \quad \text{for all } m \in \mathbb{N} \text{ and } t \in T_R^m,
\]

applying the Lebesgue dominated convergence theorem, we deduce that, for all \( R > 0 \), letting \( m \to \infty \), \( J_R^m \to 0 \). The above limit property via (23)–(24) furnishes

$$
\lim_m \int_0^T |h^m(t) p_R(|h^m(t)|) - h(t) p_R(|h_0(t)|)| dt = 0,
$$

hence

$$
\lim_m \int_0^T h^m(t) p_R(|h^m(t)|) dt = \int_0^T h(t) p_R(|h_0(t)|) dt dt.
$$

The first limit proves (21). By applying Lebesgue’s theorem to the second identity, letting \( R \to \infty \), we immediately get (22). \( \square \)

We consider a mollified Navier–Stokes initial boundary value problem:

\[
v^m_i + J_m[v^m] \cdot \nabla v^m + \nabla \pi_v^m = \Delta v^m, \quad \nabla \cdot v^m = 0, \quad \text{in } (0, T) \times \Omega, \\
v^m = 0 \text{ on } (0, T) \times \partial \Omega, \quad v^m(0, x) = v_0^m(x) \text{ on } [0] \times \Omega,
\]

where \( J_m[\cdot] \) is a mollifier and \( \{ v_0^m \} \subset J^{1,2}(\Omega) \) converges to \( v_0 \) in \( J^2(\Omega) \). In [2] is proved

**Lemma 2** For all \( v_0 \in J^2(\Omega) \) there exists a sequence of solutions \( \{(v^m_0, \pi_v^m)\} \) such that, for all \( m \in \mathbb{N} \) and \( T > 0 \), \( v^m \in C([0, T]; J^{1,2}(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega)) \).

\[1\] We stress that, for all \( m \in \mathbb{N}, \text{meas}(T_R^m) \neq 0 \). Actually, if for \( 2TR > A \) we guess that \( \text{meas}(T_R^m) = 0 \), then

\[
A < \int_0^T 2R \leq \int_0^T h^m(t) dt \leq A.
\]
Lemma 3 The sequence \( \{v^m\} \) of kinetic fields of Lemma 2 admits limit \( v \) which is weak* limit in \( L^\infty(0, T; J^2(\Omega)) \), weak limit in \( L^2(0, T; J^{1,2}(\Omega)) \) and, for all \( p \in [1, 2) \), strong limit in \( L^p(0, T; J^{1,2}) \cap L^2(0, T; J^2(\Omega)) \). Moreover, \( v \) is a weak solution to problem (1) with \( (v(t), \psi) \) continuous function of \( t \), for all \( \psi \in J^2(\Omega) \).

Proof Apart from the strong convergence properties, the result is well known. The properties of strong convergence are proved in [2]. \( \square \)

Lemma 4 Let \( \{(v^m, \pi v_m)\} \) be the sequence furnished by Lemma 2. Then, for all \( t \geq 0 \) and \( m \in \mathbb{N} \), \( \|v^m\|_{1,2} \neq 0 \) holds.

Proof See Lemma 10 in [2]. \( \square \)

Lemma 5 Let \( v \) be a weak solution. Assume that \( \|v(s)\|_2 \neq 0 \) and \( v \) is right continuous in the \( L^2 \)-norm. Then there exists \( \delta > 0 \) such that \( \|v(t)\|_2 \neq 0 \) for all \( t \in [s, s + \delta) \).

Proof The proof is given in [2]. However since it is very short, for the sake of the completeness we reproduce it. Assume that for all \( \delta > 0 \) we have a \( t \in (s, s + \delta) \) such that \( \|v(t)\|_2 = 0 \). Then we can select a sequence \( \{t_p\} \) converging to \( s \) such that \( \|v(t_p)\| = 0 \). Then, by virtue of the right-\( L^2 \)-continuity in \( s \), we get \( \lim_{t_p \to s^+} \|v(t_p) - v(s)\|_2 = 0 \), which is a contradiction with \( \|v(s)\|_2 \neq 0 \). \( \square \)

3 Proof of Theorem 1

Taking into account the results of Lemmas 2 and 3, our proof is reduced to prove the energy inequalities (8)–(10). In the sequel by \( G_1 \) we mean the set

\[
\{ t \geq 0: \|v^m(t) - v(t)\|_{1,2} \to 0 \text{ as } m \to \infty \}.
\]

Thanks to Lemma 3 we have \( \text{meas}(\mathbb{R}^+ - G_1) = 0 \). In the sequel by \( G_2 \) we mean the set

\[
\{ t \geq 0: \|v^m(t) - v(t)\|_2 \to 0 \text{ as } m \to \infty \}.
\]

We have \( \text{meas}(\mathbb{R}^+ - G_2) = 0 \), and a priori \( G_1 \subseteq G_2 \) holds.

Energy equality (8).
The starting point is the energy differential equation of \( (v^m, \pi v_m) \) related to the solution of the problem (25), that is

\[
\frac{d}{dt} \|v^m(t)\|_2^2 + 2\|\nabla v^m(t)\|_2^2 = 0, \quad t \geq 0.
\] (26)

We multiply Eq. (26) by \( g^\beta(\|\nabla v^m(t)\|_2^2) \). Here \( g(\rho) > 0 \) is a Lipschitz continuous function of \( \rho \geq 0 \) such that \( \lim_{\rho \to \infty} g(\rho) = 0 \) and \( \beta > 0 \) is an exponent. We consider
s, t ∈ G_1 with 0 < s < t. Integrating by parts on the interval (s, t), we get

\[-β \int_s^t \| v^m(τ) \|_2^2 s^β - 1 (\| ∇ v^m(τ) \|_2^2) G'(\| ∇ v^m(τ) \|_2^2) \frac{d}{dτ} \| ∇ v^m(τ) \|_2^2 dτ\]

\[= \| v^m(s) \|_2^2 s^β - 1 (\| ∇ v^m(s) \|_2^2) - \| v^m(t) \|_2^2 s^β - 1 (\| ∇ v^m(t) \|_2^2)\]

\[-2 \int_s^t g^β (\| ∇ v^m(τ) \|_2^2) \| ∇ v^m(τ) \|_2^2 dτ. \tag{27}\]

Recalling the definition of G_1, by virtue of Lemma 1, letting m → ∞ and subsequently letting β → 0, we arrive at (8) with G(t, s) given by (9).

\[\square\]

Energy equality (10).

We consider s, t ∈ G_2 with 0 ≤ s < t. We introduce a cutoff function p_R(ρ) which is Lipschitz continuous and such that p_R(ρ) = 1 for ρ ∈ [0, R], p_R(ρ) = 0 for ρ ≥ 2R and p_R(ρ) ∈ (0, 1) for ρ ∈ (R, 2R). We set

\[A = \int_0^t \| ∇ v(τ) \|_2^2 dτ, \text{ and } R_0 > (2t)^{-1} A\]

and such that \(\text{meas}([τ ∈ (0, t): \| ∇ v(τ) \|_2^2 < 2R_0]) \neq 0. \tag{28}\]

The claim (28) is consistent by virtue of Lemma 5. For α > 0, we get

\[p^α_R(\| ∇ v^m(τ) \|_2^2) \| ∇ v^m(t) \|_2^2 + p^α_R(\| ∇ v^m(τ) \|_2^2) \| ∇ v^m(t) \|_2^2 = 0. \tag{29}\]

Integrating on (0, t), considering R > R_0 and applying the mean value theorem, for some \(t^m_{α,R} \in (0, t)\), we get

\[p^α_R(\| ∇ v^m(t^m_{α,R}) \|_2^2) \| ∇ v^m(t) \|_2^2 - \| v^m(t) \|_2}{2} = − \int_0^t p^α_R(\| ∇ v^m(τ) \|_2^2) \| ∇ v^m(τ) \|_2^2 dτ. \]

By virtue of Lemma 4, for all m ∈ N the right-hand side is different from zero, we deduce that \(\| ∇ v^m(t^m_{α,R}) \|_2^2 \in (0, 2R)\) for all m ∈ N. Hence there exists a subsequence which admits a limit h(α, R) ∈ [0, 2R]. Since no confusion arises, we label again by m the subsequence. Letting m → ∞, firstly we obtain

\[\lim_m p^α_R(\| ∇ v^m(t^m_{α,R}) \|_2^2) = \frac{2 \int_0^t p^α_R(\| ∇ v(τ) \|_2^2) \| ∇ v(τ) \|_2^2 dτ}{\| v(0) \|_2^2 - \| v(t) \|_2^2} \neq 0, \tag{29}\]
as a consequence of (28). Then, since, for all \( \alpha > 0 \) and \( R > R_0 \), \( p^\alpha_R \) is continuous, we get \( \lim_m p^\alpha_R (\|\nabla v^m(t_{\alpha,R})\|_2^2) = p^\alpha_R (h(\alpha, R)) \). So that letting \( R \to \infty \), we get

\[
\lim_R p^\alpha_R (h(\alpha, R)) = \frac{2 \int_0^t \|\nabla v(\tau)\|_2^2 d\tau}{\|v(0)\|_2^2 - \|v(t)\|_2^2} \neq 0,
\]

which proves (10) with \( P(t, s) \) given by (11)\(_1\). The alternative expressions of \( P(t, s) \) are an immediate consequence of the fact that the right-hand side is independent of \( \alpha \).

\( \square \)

**Energy equality (12).**

We consider \( s, t \in G_1 \) with \( 0 < s < t \). We consider Eq. (29) with \( p_R(\rho) := \frac{2R - \rho}{R} \), where \( R > \max\{\|\nabla v(s)\|_2^2, \|\nabla v(t)\|_2^2\} \). Since \( s, t \in G_1 \) we have strong convergence in \( J^{1,2}(\Omega) \) of \( \{v^m(s, x)\} \) to \( v(s, x) \) and of \( \{v^m(t, x)\} \) to \( v(t, x) \), respectively. Hence, without losing the generality, we also claim that \( R > \max\{\|\nabla v^m(s)\|_2^2, \|\nabla v^m(t)\|_2^2\} \) for all \( m \in \mathbb{N} \). Integrating on \( (s, t) \), after an integration by parts we get

\[
- \int_s^t \|v^m(\tau)\|_2^2 \frac{d}{d\tau} p_R(\|\nabla v^m(\tau)\|_2^2) d\tau = \|v^m(s)\|_2^2 - \|v^m(t)\|_2^2 - 2 \int_s^t p_R(\|\nabla v^m(\tau)\|_2^2) \|\nabla v^m(\tau)\|_2^2 d\tau. \tag{30}
\]

We set

\[
I^m_R := \left\{ \tau \in (s, t) : R < \|\nabla v^m(\tau)\|_2^2 < 2R \right\}.
\]

The set \( I^m_R \), if not empty, is at most a countable union of disjoint open intervals. Actually, let \( J \) be the interior set of \( (s, t) - I^m_R \). Since \( \partial I^m_R \) is at most countable, we get

\[
\int_s^t \|v^m(\tau)\|_2^2 \frac{d}{d\tau} p_R(\|\nabla v^m(\tau)\|_2^2) d\tau = \int_{I^m_R} \|v^m(\tau)\|_2^2 \frac{d}{d\tau} p_R(\|\nabla v^m(\tau)\|_2^2) d\tau + \int_J \|v^m(\tau)\|_2^2 \frac{d}{d\tau} p_R(\|\nabla v^m(\tau)\|_2^2) d\tau.
\]

Let us fix \( \tau \in J \) and suppose that \( \|\nabla v^m(\tau)\|_2^2 \leq R \). Since \( J \) is open, it contains an interval centered at \( \tau \) where, by continuity, \( \|\nabla v^m(\tau)\|_2^2 \leq R \) holds. Hence we get \( \frac{d}{d\tau} p_R(\|v^m(\tau)\|_2^2) = 0 \). The same argument works in the case of \( \|\nabla v^m(\tau)\|_2^2 \geq 2R \).
Thus, we finally get
\[
\int_s^t \left| \nabla v^m(\tau) \right|^2 \frac{d}{d\tau} p_R(\left| \nabla v^m(\tau) \right|^2) d\tau = \int_s^t \left| \nabla v^m(\tau) \right|^2 \frac{d}{d\tau} p_R(\left| \nabla v^m(\tau) \right|^2) d\tau,
\]
which coupled with (30) furnish
\[
\int_s^t \frac{d}{d\tau} \left| \nabla v^m(\tau) \right|^2 d\tau = \left| v^m(s) \right|^2 - \left| v^m(t) \right|^2 - 2 \int_s^t p_R(\left| \nabla v^m(\tau) \right|^2) \left| \nabla v^m(\tau) \right|^2 d\tau.
\]

We denote by \( p_{ij}(m, R) \), \( i, j = 1, 2 \), a positive or null integer. If we have \( i \neq j \), then \( p_{ij}(m, R) \in \mathbb{N} \) (see item 3. below). If we have \( i = j \), then \( p_{ii}(m, R) \) can be \( \infty \).

Actually, we distinguish the intervals of the union in the following way:

1. \( h \in \{1, \ldots, p_{11}(m, R)\} \), where a priori \( p_{11}(m, R) \in \mathbb{N} \cup \{\infty\} \), \((s_{h1}^{11}, t_{h1}^{11})\) is the interval such that \( \left| \nabla v^m(\tau) \right|^2 \in (R, 2R) \), for all \( \tau \in (s_{h1}^{11}, t_{h1}^{11}) \), with \( \left| \nabla v^m(s_{h1}^{11}) \right|^2 = \left| \nabla v^m(t_{h1}^{11}) \right|^2 = R \),
2. \( h \in \{1, \ldots, p_{12}(m, R)\} \), \((s_{h1}^{12}, t_{h1}^{12})\) is the interval such that \( \left| \nabla v^m(\tau) \right|^2 \in (R, 2R) \), for all \( \tau \in (s_{h1}^{12}, t_{h1}^{12}) \), with \( \left| \nabla v^m(s_{h1}^{12}) \right|^2 = \left| \nabla v^m(t_{h1}^{12}) \right|^2 = 2R \),
3. \( h \in \{1, \ldots, p_{21}(m, R)\} \), \((s_{h1}^{21}, t_{h1}^{21})\) is the interval such that \( \left| \nabla v^m(\tau) \right|^2 \in (R, 2R) \), for all \( \tau \in (s_{h1}^{21}, t_{h1}^{21}) \), with \( \left| \nabla v^m(s_{h1}^{21}) \right|^2 = \left| \nabla v^m(t_{h1}^{21}) \right|^2 = R \),
4. \( h \in \{1, \ldots, p_{22}(m, R)\} \), where a priori \( p_{22}(m, R) \in \mathbb{N} \cup \{\infty\} \), \((s_{h2}^{22}, t_{h2}^{22})\) is the interval such that \( \left| \nabla v^m(\tau) \right|^2 \in (R, 2R) \), for all \( \tau \in (s_{h2}^{22}, t_{h2}^{22}) \), with \( \left| \nabla v^m(s_{h2}^{22}) \right|^2 = \left| \nabla v^m(t_{h2}^{22}) \right|^2 = 2R \).

Since \( \left| \nabla v^m(s) \right|^2 < R \) and \( \left| \nabla v^m(t) \right|^2 < R \), we deduce that \( s < \min_{h}^{ij} \) and \( t > \max_{h}^{ij} \). Moreover

1. \( p_{ij}(m, R) = 0 \), for all \( i, j \in \{1, 2\} \), for all \( m \in \mathbb{N} \) if, and only if, \( \left| \nabla v^m(\tau) \right|^2 < R \) for all \( m \in \mathbb{N} \).
2. \( p_{12}(m, R) = p_{21}(m, R) = p_{22}(m, R) = 0 \), if, and only if, \( \left| \nabla v^m(\tau) \right|^2 < 2R \) for all \( \tau \in (s, t) \).
3. \( p_{12}(m, R) \neq 0 \) if, and only if, \( p_{21}(m, R) \neq 0 \), and in both cases \( p_{ij}(m, R) \in \mathbb{N} \).
   Actually, we have that \( \{(s_{h}^{12}, t_{h}^{12})\} \) is a sequence, the sequence collapses into a point \( \tilde{t} \) and as a consequence we get \( R = \lim_{s_{h}^{12} \to \tilde{t}} p_R(\left| \nabla v^m(s_{h}^{12}) \right|^2) = p_R(\left| \nabla v^m(\tilde{t}) \right|^2) = \lim_{t_{h}^{12} \to \tilde{t}} p_R(\left| \nabla v^m(t_{h}^{12}) \right|^2) = 2R \) which is a contradiction.
   Analogous argument works in the case of “21”.
4. We have \( p_{12}(m, R) = p_{21}(m, R) \). Actually, for fixed \( h \leq p_{12}(m, R) \), let us consider the set \( A := \{ k \in \mathbb{N} : s_{h}^{12} \leq s_{k}^{21} \} \). Since \( \left| \nabla v^m(t) \right|^2 < R \), by continuity, the set \( A \) is not empty. Let be \( \varphi(h) := \min(A) \). Then, by construction, \( \left| \nabla v^m(\tau) \right|^2 > R \) for all \( \tau \) such that \( \tau \in (t_{h}^{12}, t_{\varphi(h)}^{21}) \). It follows that
   \[ s_{\varphi(h)}^{21} < t_{\varphi(h)}^{21} < s_{h+1}^{12} < t_{h+1}^{12} < s_{\varphi(h+1)}^{21}. \]
Hence function \( \varphi \) is injective. Conversely, using the fact that \( \|\nabla v^m(s)\|_2^2 < R \), we can conclude that \( p_{12}(m, R) = p_{21}(m, R) \).

Since \( 2 \int_s^t \| \nabla v^m(t) \|^2_2 d\tau \leq \| v_0 \|^2_2 \), then, for all \( m \in \mathbb{N} \) and \( R \), we get \( \text{meas}(I_R^m) < \frac{\| v_0 \|^2_2}{2R} \) which proves (14). By the energy inequality the same property holds for the weak solution \( v \), that is

\[
I_R := \left\{ \tau \in (s, t): R \leq \| \nabla v(\tau) \|^2_2 \leq 2R \right\}, \quad \text{then } \text{meas}(I_R) < \frac{\| v_0 \|^2_2}{2R}. \tag{31}
\]

Taking into account the previous considerations, integrating by parts on the intervals defining \( I_R^m \), we get

\[
E^m = -\frac{2}{R} \int_{I_R^m} \| \nabla v^m(\tau) \|^4_2 d\tau + \| v^m(s) \|^2_2 - \| v^m(t) \|^2_2
- 2 \int_s^t p_R(\| \nabla v^m(t) \|^2_2)\| \nabla v^m(\tau) \|^2_2 d\tau,
\]

where we considered that \( \frac{d}{dt} \| v^m(t) \|^2_2 = -2\| \nabla v^m(t) \|^2_2 \), and we set

\[
E^m := \sum_{h=1}^{p_{22}(m, R)} \left[ \| v^m(t_h^{22}) \|^2_2 - \| v^m(s_h^{22}) \|^2_2 \right] + \sum_{h=1}^{p_{21}(m, R)} \left[ \| v^m(t_h^{21}) \|^2_2 - \| v^m(s_h^{21}) \|^2_2 \right]
+ \sum_{h=1}^{p_{11}(m, R)} \left[ \| v^m(t_h^{11}) \|^2_2 - \| v^m(s_h^{11}) \|^2_2 \right] + \sum_{h=1}^{p_{12}(m, R)} \left[ 2\| v^m(t_h^{12}) \|^2_2 - \| v^m(s_h^{12}) \|^2_2 \right]. \tag{33}
\]

In the expression of \( E^m \) we have taken into account that the value of \( \| \nabla v^m(\tau) \|^2_2 \) in the end points of the intervals is alternatively \( R \) or \( 2R \) and for all the integral term there is the multiplicative factor \( \frac{1}{R} \). We discuss the first integral on the right-hand side of (32). By the definition of \( I_R^m \) we deduce that \( \frac{\| v^m(t) \|^2_2}{R^2} \leq 2 \) for all \( \tau \in I_R^m \) and for all \( m \in \mathbb{N} \). Hence, employing the energy relation for \( v^m \) and the Lebesgue theorem, we deduce that for all \( R \)

\[
\lim_{m} \frac{1}{R} \int_{I_R^m} \| \nabla v^m(\tau) \|^4_2 d\tau \leq 2 \lim_{m} \frac{1}{R} \int_{I_R^m} \| \nabla v^m(\tau) \|^2_2 d\tau \leq 2 \int_{I_R} \| \nabla v(\tau) \|^2_2 d\tau.
\]

Since for all \( T > 0 \) we have \( \| \nabla v(\tau) \|^2_2 \in L^1(0, T) \), then, via (31), we get

\[
\lim_{m} \frac{1}{R} \lim_{m} \frac{1}{R} \int_{I_R^m} \| \nabla v^m(\tau) \|^4_2 d\tau \leq 2 \lim_{m} \frac{1}{R} \int_{I_R} \| \nabla v(t) \|^2_2 d\tau = 0.
\]
Letting \( m \to \infty \) and then letting \( R \to \infty \) in formula (32), we arrive at (12) with \( E(t, s) \) given by (13). Finally, we justify (16). First of all we stress that by virtue of the energy inequality \( \| v^m(t) \|_2 < \| v^m(s) \|_2 \) true for all \( t > s \) and \( m \in \mathbb{N} \), we get that in (33) only the last sum can be positive. Concerning (16) it is enough to restrict ourselves to the finite sum with indexes \( h \in \{1, \ldots, p_{12}(m, R)\} \cup \{1, \ldots, p_{21}(m, R)\} \). Taking into account that \( p_{12}(m, R) = p_{21}(m, R) \), for all \( h \in \{1, \ldots, p_{12}(m, R)\} \) we have

\[
\| v^m(t_{12h}^1) \|_2^2 - \| v^m(s_{12h}^1) \|_2^2 < 0,
\]

\[
\| v^m(t_{21h}^2) \|_2^2 - \| v^m(s_{21h}^2) \|_2^2 < 0.
\]

Finally, if \( h \in \{2, \ldots, p_{12}(m, R)\} \), then the following holds:

\[
\| v^m(t_{12h}^1) \|_2^2 - \| v^m(s_{21h-1}^2) \|_2^2 < 0.
\]

Hence we obtain

\[
\sum_{h=1}^{p_{21}(m, R)} \left[ \| v^m(t_{21h}^2) \|_2^2 - 2\| v^m(s_{21h}^2) \|_2^2 \right] + \sum_{h=1}^{p_{12}(m, R)} \left[ 2\| v^m(t_{12h}^1) \|_2^2 - \| v^m(s_{12h}^2) \|_2^2 \right]
= B(m, R) + \| v^m(t_{121}^1) \|_2^2 - \| v^m(s_{21p_{21}}^2) \|_2^2,
\]

with \( B(m, R) < 0 \) and \( t_{121}^1 < s_{21p_{21}}^2 \).

References

1. Caffarelli, L., Kohn, R., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Commun. Pure Appl. Math. 35(6), 771–831 (1982)
2. Crispo, F., Grisanti, C.R., Maremonti, P.: Some new properties of a suitable weak solution to the Navier–Stokes equations. arXiv:1904.07641, to appear in Waves in Flows: The 2018 Prague-Sum Workshop Lectures, series: Lecture Notes in Mathematical Fluids Mechanics, editors: G.P.Galdi, T. Bodnar, S. Necasova, Birkhauser
3. Galdi, G.P., Maremonti, P.: Monotonic decreasing and asymptotic behavior of the kinetic energy for weak solutions of the Navier–Stokes equations in exterior domains. Arch. Ration. Mech. Anal. 94(3), 253–266 (1986)
4. Heywood, J.G.: The Navier–Stokes equations: on the existence, regularity and decay of solutions. Indiana Univ. Math. J. 29(5), 639–681 (1980)
5. Leray, J.: Sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math. 63(1), 193–248 (1934)
6. Maremonti, P.: A note on Prodi–Serrin conditions for the regularity of a weak solution to the Navier–Stokes equations. J. Math. Fluid Mech. 20(2), 379–392 (2018)
7. Miyakawa, T., Sohr, H.: On energy inequality, smoothness and large time behavior in \( L^2 \) for weak solutions of the Navier–Stokes equations in exterior domains. Math. Z. 199(4), 455–478 (1988)
8. Prodi, G.: Teoremi di tipo locale per il sistema di Navier–Stokes e stabilità delle soluzioni stazionarie. Rend. del Sem. Matem della Univ. di Padova 32, 173–182 (1962)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.