A PROOF OF THE BUNKBED CONJECTURE ON
THE COMPLETE GRAPH FOR \( p \geq 1/2 \)

PAUL DE BUYER

Abstract. The bunkbed of a graph \( G \) is the graph \( G \times \{0, 1\} \).
It has been conjectured that in the independent bond percolation model, the probability for \((u, 0)\) to be connected with \((v, 0)\) is
greater than the probability for \((u, 0)\) to be connected with \((v, 1)\),
for any vertex \( u, v \) of \( G \). In this article, we prove this conjecture
for the complete graph in the case of the independent bond
percolation of parameter \( p \geq 1/2 \).

1. Introduction

Percolation theory has been widely studied over the last decades and
yet, several intuitive results are very hard to prove rigorously. This is
the case of the bunkbed conjecture formulated by Kasteleyn (published
as a remark in [9]) which investigates a notion of graph distance through
percolation theory.

Consider a graph \( G = (V, E) \) where \( E \) is the set of edges and delete
each edge independently with probability \( p \); we obtain a random graph
which law is noted \( \mathbb{P}_p \). The main question in percolation is to under-
stand and bound the probability of two set of vertices to be connected
(for a general introduction on the subject, see [2]). Furthermore, one
can properly build a distance \( d : V \times V \) between vertices using percola-
tion, saying that for three vertices \( u, v, w \in V \), \( v \) is further from \( u \) than
\( w \) from \( u \) if the probability for \( u \) and \( v \) to be connected is smaller than
the probability for \( u \) and \( w \) to be connected, and ask if this distance
coincide with the usual graph distance \( d_G \). However, the validity of
this property depends on the graph and the value of \( p \), see figure 1.

Before recalling the bunkbed conjecture, we give another problem
closely related. Consider the centered box of \( B_n = [-n; n]^d \cap \mathbb{Z}^d \) and
its border \( \partial B_n = \{ x \in B_n : \| x \|_{\infty} = n \} \), and consider the following
function \( f : x \mapsto \mathbb{P}_p(x \leftrightarrow \partial B_n) \). An open question of interest is to
ask if the minimum of the function \( f \) is achieved at the origin, \( i.e. \)
if \( \min_x f(x) = f(0) \). One can extend the question in the following way:

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consider $x$ and $y$ two vertices of $B_n$ such that $d_G(x, \partial B_n) < d_G(y, \partial B_n)$, is it true that $f(x) \geq f(y)$? To the best knowledge of the author, these questions remain open. In order to understand these problems, some structure has been added to the graph giving rise to the study of the bunkbed conjecture defined as follow.

A bunkbed graph $G = (V, E)$ of a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ is the graph given by the superposition of two identical copy of $\tilde{G}$ to which we add the edges connecting the symetrical vertices, see figure 2.1; informally, we note $G = \tilde{G} \times \{0;1\}$. Furthermore, we say that a vertex $u$ belongs to the bottom graph if $u = (x, 0)$, $u$ belongs to the top graph if $u = (x, 1)$, and $u'$ is the symmetrical of $u$ if $u = (x, i)$ and $u' = (x, 1 - i)$. The statement of the conjecture is the following:

**Conjecture.** Let $G$ be a bunkbed graph. Let $u$ and $v$ be two vertices of the bottom graph, and $v'$ the symmetrical of $v$, then for any $p \in [0;1]$ the following holds:

$$\mathbb{P}_p(u \leftrightarrow v) \geq \mathbb{P}_p(u \leftrightarrow v')$$

Litterature about the conjecture is fairly poor even if the problem aroused the interest of quite a number of researchers. The work of S. Linusson and M. Leander [5, 6] who proved that the conjecture holds for a subclass of the planar graphs, called outerplanar graphs and wheel graphs. The proof used in their paper, called the minimal counter-example, is combinatoric and relies on the structure of the graph making it difficult to extend to a more general class of graph.

O. Häggström proved in [4] a similar conjecture for the Ising model on a general graph. Recall that the Ising model assign a value 1 or $-1$
to each vertex according to a Gibbs measure, and it has been shown that the value of $u$ has more influence on the value of $v$ than $v'$ in the sense that $\mathbb{E}[\omega(u)\omega(v)] \geq \mathbb{E}[\omega(u)\omega(v')]$ where $\omega$ is a configuration in $\{-1;1\}^V$ and $\omega(u)$ is the value assigned to the vertex $u$. The proof relies on the link between FK-percolation and Ising model as well as some properties of the Gibbs measure which cannot extend to the independent case.

Related to the study of this conjecture, a type of Harris-FKG inequality conditioned by a decreasing event has been proven in [8, 9].

Furthermore, in the random walk field, an equivalent conjecture of the bunkbed conjecture formulated in [1] is that starting from a vertex $u$, the first time of reaching a vertex $v$ is shorter (in some sense) that the first time of reaching the vertex $v'$ which has been proved in [3].

Finally, in the article [7] of Rudzinski and Smyth, an extensive list of equivalent reformulation of the Bunkbed Conjecture is presented.

In this article, in the context of independent bond percolation model, we prove the following theorem:

**Theorem 1.1.** The bunkbed conjecture is verified for the complete graph with $n \geq 1$ vertices when the parameter of percolation $p \geq 1/2$.

The paper is organized in the following way. In the second section, we introduce formally all the notations and the results. In the third section, we give the proof of the main theorem which uses two technical lemmas proven in the fourth section. In the fifth section, we prove the secondary results, give deeper explanations the consequences of the proof, and possible leads to solve the conjecture.

2. Notations et results

In this section, we introduce formally the model, the notations and the main result. We recall that the complete graph with $n$ vertices is noted $K_n$.

We call the bunkbed graph $G = (V, E)$ of a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ (called the original graph) is defined by:

$\begin{align*}
V &= \tilde{V} \times \{0, 1\} \\
E &= \{(x, 0), (y, 0)\} : \{x, y\} \in \tilde{E} \} \cup \{(x, 1), (y, 1)\} : \{x, y\} \in \tilde{E} \} \\
&\quad \cup \{(x, 0), (x, 1)\}
\end{align*}$

An example is given in the figure 2.1. As previously explained in the introduction, it is natural to distinguish the vertices in the bottom graph, the set of vertices which can be written $(x, 0)$, and the vertices
in the top graph, the set of \((x, 1)\). We call the symmetrical of a vertex 
\(u = (x, i)\), the vertex \(u' = (x, 1 - i)\). In the rest of the article, letters \(u\) and \(v\) will designate vertices of the bottom graph.

The percolation model is defined as follow. Consider a graph \(G = (V, E)\) where \(V\) is the set of vertices and \(E\) the set of unoriented edges. We open each edges of \(E\) independently with probability \(p\) and close them with probability \(1 - p\) and we write \(P_p\) the law associated to this percolation model. We call a configuration, an element \(\omega = (\omega_e)_{e \in E} \in \{0, 1\}^E\) corresponding to the bond percolation model where \(\omega_e = 0\) means that the edge \(e\) is closed and \(1\) means that the edge \(e\) is open. We call a path between vertices \(u\) and \(v\) the set \(\gamma = \{e_1, ..., e_n \in E; e_1 = (u, x_1), e_2 = (x_2, x_3), ..., e_n = (x_{n-1}, v)\}\). For a configuration \(\omega\), we call an open path a path of open edges of \(\omega\). For two vertices \(x, y \in V\), we write \(x \leftrightarrow y\) if there exists an open path between \(x\) and \(y\). When there is no ambiguity, we will omit the dependence of \(\omega\) and write \(x \leftrightarrow y\) instead of \(x \leftrightarrow y\). By convention, for any configuration, a vertex is always connected to itself, i.e. \(x \leftrightarrow x\). We recall our main theorem given in the introduction.

**Theorem.** Let \(G = (V, E)\) be the bunkbed graph of \(K_n\) with \(n \geq 1\), then for all vertices of the bottom graph \(u, v \in V\) and all \(p \geq 1/2\):

\[
P_p(u \leftrightarrow v) \geq P_p(u \leftrightarrow v')
\]

**Remark 2.1.** We chose to study the complete graph since calculations are easier. Furthermore, a way to solve the conjecture would be to prove that if the conjecture is verified for a graph \(G\), then it is verified for the graph \(G' \setminus \{e, e'\}\) where we removed an edge \(e\) and it symmetric \(e'\).
Remark 2.2. Among the trivialities around the conjecture, for any graph $G$, there exists a constant $p_G \in (0; 1]$ such that the conjecture is verified for all $p \leq p_G$. Indeed, when the percolation parameter is small enough, only the shortest paths can connect two vertices according to $P_p$, the other paths contributing negligibly. Since the shortest path between $u$ and $v$ has a shorter length of 1 compared to the shortest path between $u$ and $v'$, the conjecture is proved.

As an auxiliary result, we prove that the conjecture holds in mean in a more general setting, suggesting that the conjecture should be true. Instead of keeping each edge with probability $p$, we keep the edge $e$ with probability $p_e$, and we define a vector of percolation parameter $p = (p_e)_{e \in E}$. In the context of the bunkbed conjecture, we say that the vector of percolation parameter is constrained if an edge $e = \{x, y\}$ has the same percolation parameter as its symmetrical $e' = \{x', y'\}$, i.e. for all $e \in E$, $p_e = p_{e'}$.

Proposition 2.3. Let $G$ be a bunked graph. Let $X$ and $Y$ be two random variables independent and identically distributed on the vertices of the bottom graph of $G$ according to a law $P$. Then for any vector of percolation parameter $p$ constrained, the following holds:

$$E[P_p(X \leftrightarrow Y)] \geq E[P_p(X \leftrightarrow Y')]$$

Finally, we give a simple upper bound on the difference between $P_p(u \leftrightarrow v)$ and $P_p(u \leftrightarrow v')$.

Proposition 2.4. For any bunked graph $G$ and any vector of percolation parameter $p$,

$$|P_p(u \leftrightarrow v) - P_p(u \leftrightarrow v')| \leq P_p(u \not\leftrightarrow u' \cap v \not\leftrightarrow v')$$

In the case of the bunked graph of $K_n$ when the vector of percolation is constant, $\forall e, p_e = p$, note that (2.1) is bounded by $P_p(u \not\leftrightarrow u') \leq (1 - p)(1 - p^3)^{n-1}$. From now on, we fix $n \in \mathbb{N}$ and $G$ will denote the bunkbed graph of $K_n$.

3. Proof of Theorem 1.1

In this section, we prove the main theorem in the following manner. First we fix a vertex $u$ on the bottom graph. We define $G_{x,y,z}$ the set of the induced subgraphs of $G$ containing $u$ with $x$ vertices on the bottom graph, $y$ vertices on the top graph, and $z$ vertices of the bottom graph having their symmetrical in the subgraph. From this definition $x \geq 1$, since $u$ is always in the subgraphs, and $z \leq \min(x, y)$. An example of these graphs is given in the figure 2.1.
Secondly, we fix a vertex \( v \) on the bottom graph. We define \( G^1 \) the set of induced subgraph of \( G \) containing \( v \) and \( G^2 \) the set of induced subgraph containing \( v' \).

From these three definitions, we introduce the sets \( G^1_{x,y,z} = G^1 \cap G_{x,y,z} \) and \( G^2_{x,y,z} = G^2 \cap G_{x,y,z} \). Noting \( |X| \) the cardinality of the set \( X \), we introduce the functions \( C_1 : \mathbb{N}^3 \to \mathbb{N} \), \( C_2 : \mathbb{N}^3 \to \mathbb{N} \) and \( C_{\text{diff}} : \mathbb{N}^3 \to \mathbb{N} \) defined by:

\[
C_1(x, y, z) = |G^1_{x,y,z}|
\]
\[
C_2(x, y, z) = |G^2_{x,y,z}|
\]
\[
C_{\text{diff}}(x, y, z) = \begin{cases} 
C_1(x, y, z) - C_2(x, y, z) & \text{if } x \neq y \\
+ C_1(y, x, z) - C_2(y, x, z) & \text{if } x = y
\end{cases}
\]

Note that, since we considered the bunkbed of a complete graph, any graph of \( G_{x,y,z} \) is isomorph to any other graph of \( G_{x,y,z} \), so we introduce the function \( P : \mathbb{N}^3 \to \mathbb{N} \) which associates to a triplet \( (x, y, z) \) the probability for a graph of \( G_{x,y,z} \) to be connected.

We call the main component of a configuration \( \omega \), the set of vertices connected by an open path to \( u \). Since the main component is a spanning subgraph of \( G' \in G_{x,y,z} \), it has to be connected and isolated from the rest; thus we introduce the number edges that have to be closed on the boundary of the main component with the function \( B : \mathbb{N}^3 \to \mathbb{N} \).

Note that \( C_{\text{diff}}, P \) and \( B \) are symmetrical in their two first coordinates unlike the functions \( C_1 \) and \( C_2 \).

If the main component contains \( v \) (resp. \( v' \)), then there exist \( x, y, z \) such that it is a spanning subgraph of a \( G' \in G^1_{x,y,z} \) (resp. \( G' \in G^2_{x,y,z} \)). To quantify \( \mathbb{P}_p(u \leftrightarrow v) \), we will classify the configurations \( \omega \) according to which set \( G^1_{x,y,z} \) (resp. \( G^2_{x,y,z} \)) the main component is a spanning subgraph.

We illustrate these quantities with the figure 3.1 where, for the sake of simplicity, we draw the bunkbed graph of a square. Full edges represent open edges and dotted lines the closed ones. Green vertices and vertices belong to the main component. Red edges are the exterior edges that have to be closed. Blue vertices and edges are the remaining components of the graph.

Recall that we introduced these functions to decompose the quantities \( \mathbb{P}_p(u \leftrightarrow v) \) and \( \mathbb{P}_p(u \leftrightarrow v') \). The configurations \( \omega \) have as their main components a connected spanning subgraph of \( G' \in G_{x,y,z} \), which are for fixed \( x, y, z \) isomorph to each other and whose probability to be connected is \( P(x, y, z) \), and a number \( B(x, y, z) \) of closed boundary
Figure 3.1. Decomposition of a configuration

edges. Writing $MC(\omega)$ the main component of $\omega$, we obtain:

$$P_p(u \leftrightarrow v) = \sum_{x,y,z} \sum_{G'=(V',E')\in G_{x,y,z}} \sum_{\omega} P(MC(\omega) = V')$$

$$= \sum_{x,y,z} \sum_{G'=(V',E')\in G_{x,y,z}} P(G' \text{ is connected}) \times (1-p)^{B(x,y,z)}$$

$$= \sum_{x,y,z} C_1(x,y,z) \times P(x,y,z) \times (1-p)^{B(x,y,z)}$$

Likewise, it holds:

$$P_p(u \leftrightarrow v') = \sum_{x,y,z} C_2(x,y,z) \times P(x,y,z) \times (1-p)^{B(x,y,z)}$$

Taking the difference between these two last quantities, and reindexing, we obtain:

$$P_p(u \leftrightarrow v) - P_p(u \leftrightarrow v')$$

$$= \sum_{x,y,z} \left( P \times (1-p)^B \times (C_1 - C_2) \right) (x,y,z)$$

$$= \sum_{x,y,z} \sum_{i \in \mathbb{N}} \left( P \times (1-p)^B \times C_{diff} \right) (x,y,z)$$

$$= \sum_{k>0} \sum_{i \in \mathbb{N}} \sum_{x \geq y} \left( P \times (1-p)^B \times C_{diff} \right) (k+i,k-i,z)$$

$$+ \sum_{k>0} \sum_{i \in \mathbb{N}} \sum_{z \geq 0} \left( P \times (1-p)^B \times C_{diff} \right) (k+1+i,k-i,z)$$

(3.1)
The proof of the theorem relies on the two key lemmas:

**Lemma 3.1.** Let \( p \geq 1/2 \). \( \forall \epsilon \in \{0; 1\} \) there exists \( i_0 := i_0(n) \) such that \( \forall i < i_0 \), we have:

\[
(1 - p)^B \times P (k + i + \epsilon, k - i, z) \leq ((1 - p) \times P) (k + i_0 + \epsilon, k - i_0, z)
\]

\[
C_{diff} (k + i + \epsilon, k - i, z) < 0
\]

and for all \( i \geq i_0 \):

\[
(1 - p)^B \times P (k + i + \epsilon, k - i, z) \geq ((1 - p)^B \times P) (k + i_0 + \epsilon, k - i_0, z)
\]

\[
C_{diff} (k + i + \epsilon, k - i, z) \geq 0
\]

**Remark 3.2.** One may ask if the lower bounds \(1/2\) is optimal. Coarse estimates on the function \( P \) have been used to prove this lemma to get a simplified proof. One can using better estimates and lower the bound. The author has obtained a bound of \(0.42\) using computer calculation.

**Lemma 3.3.** For all \( z > 0 \) and for all \( k \geq z \), we have:

\[
\sum_{i=0}^{k-z} C_{diff} (k + i, k - i, z) = \sum_{i=0}^{k-z} C_{diff} (k + i + 1, k - i, z) = 0
\]

Once these lemmas stated, we can end the proof of the theorem. We start by noting that using lemma 3.1 when \( i > i_0 \), the quantity \( C_{diff} \) is positive, and we can lower bound \( (1 - p)^B \times P (k + i, k - i, z) \) by \( (1 - p)^B \times P (k + i_0, k - i_0, z) \), whereas when \( i < i_0 \), the quantity \( C_{diff} \) is negative, we can upper bound \( (1 - p)^B \times P (k + i, k - i, z) \) by \( (1 - p)^B \times P (k + i_0, k - i_0, z) \). Finally, when \( z = 0 \), meaning that none of the vertices are connected to their symmetrical, no configuration connect \( u \) to \( v' \). In this way, we get:

\[
(3.1)
\]

\[
\geq \sum_{z>0} ((1 - p)^B \times P (k + i_0, k - i_0, z)) \sum_{i=0}^{k-z} C_{diff} (k + i, k - i, z)
\]

\[
+ \sum_{z>0} ((1 - p)^B \times P (k + i_0 + 1, k - i_0, z)) \sum_{i=0}^{k-z} C_{diff} (k + i + 1, k - i, z)
\]

\[
= 0
\]

Where the last inequality is obtained using lemma 3.3 concluding the proof of the main theorem.
BUNKBED CONJECTURE ON THE COMPLETE GRAPH FOR $p \geq 1/2$

4. PROOF OF THE TECHNICAL LEMMAS

We start by proving the two technical lemmas used in the proof of the main theorem. Recall that we considered the bunkbed of the complete graph with $n$ vertices. We start by proving lemma 4.1 using three intermediates lemmas.

Lemma 4.1. Let $x, x', y, y', z \in [0; n] \cap \mathbb{N}$ such that $x + y = x' + y'$ and $z \leq \min (x, x', y, y')$. If $|x - y| > |x' - y'|$ then $B(x, y, z) \geq B(x', y', z) + 2$

Proof. A simple calculation gives for all $x, y, z$

$$B(x, y, z) = (x + y) n - x^2 + x - y^2 + y - 2z$$

Then, to prove the result, it is enough to show the result for $x = x' + 1 \geq y = y' - 1$. Using the equality above, we obtain:

$$B(x + 1, y - 1, z) - B(x, y, z) = -2x + 2y - 2 \leq 0$$

□

Lemma 4.2. Let $p \geq 1/2$ and let $x, x', y, y', z \in [0; n] \cap \mathbb{N}$ such that $x + y = x' + y'$ and $z \leq \min (x, x', y, y')$. If $|x - y| > |x' - y'|$ then $((1 - p)^B P)(x, y, z) \geq ((1 - p)^B P)(x', y', z)$

Proof. Again, to prove this lemma and according to the previous one, it is enough to prove for $x > y \geq z$:

$$(4.1) \quad P(x + 1, y, z) \geq (1 - p)^2 P(x, y + 1, z)$$

As an upper bound for $P(x, y, z)$, we use the probability that there is at least one vertical edge is open, i.e. that $P(x, y, z) \leq 1 - (1 - p)^z$. Then we lower $P(x, y, z)$ by the event that the upper graph is connected as well as the lower graph and at least one vertical edge open:

$$(4.2) \quad P(x, y, z) \geq P(x, 0, 0) \times P(0, y, 0) \times (1 - (1 - p)^z)$$

Furthermore, one can use as a lower bound the probability for a complete graph to be disconnected is greater than $1$—the sum of the probability of any set of vertices of cardinality less that $n/2$ is connected
and isolated from the rest:

\[
P(n, 0, 0) \geq 1 - \sum_{k=1}^{n/2} \binom{n}{k} (1-p)^{k(n-k)} \\
\geq 2 - \sum_{k=0}^{n/2} \binom{n}{k} (1-p)^{kn/2} \\
= 2 - \left(1 + (1-p)^{n/2}\right)^n
\]

The function \( n \mapsto \left(1 + (1-p)^{n/2}\right)^n \) is a decreasing function as long as \( n \geq 2 \). Thus, for \( n \geq 10 \), we have that:

\[
P(n, 0, 0) \geq 0.6 \geq 1 - p
\]

For \( n \leq 10 \), one can lower bound the probability for the complete graph to be connected with \( p = 1/2 \) and use OEIS A001187\footnote{This sequence gives the number of connected graph with \( n \) vertices.} to see that this probability is greater than \( 1/2 \). Plugging these bounds into (4.1), one obtains:

\[
\frac{P(x + 1, y, z)}{P(x, y + 1, z)} \geq P(x, 0, 0) \times P(0, y, 0) \geq 1/4 \geq (1-p)^2
\]

which shows the desired result. \( \square \)

In the following, we will use the following equality valid for all \( k \in \mathbb{N} \):

\[
\frac{1}{(x-k)!} \mathbb{1}_{x \geq k} = \frac{1}{x!} \prod_{i=0}^{k} (x-i)
\]

Recall that we fixed three vertices \( u, v \) and \( v' \) (the symmetrical of \( v \)) of the bunkbed graph such that \( u \neq v \) in order to study the quantity \( \mathbb{P}(u \leftrightarrow v) - \mathbb{P}(u \leftrightarrow v') \).

**Proposition 4.3.** For all \( x, y \geq z \geq 1 \) such that \( x + y - z \leq n \)

\[
C_1(x, y, z) = \frac{(n-2)!x(x-1)}{(x-z)!z!(n-x-y+z)!(y-z)!}
\]

**Proof.** First of all, if \( z = 0 \) and \( y > 0 \), then a graph of \( G_{x,y,z} \) can’t be connected. Moreover, \( u \) and \( v \) must be in the set of vertices of the graph of \( G_{x,y,z} \), thus \( x \) has to be greater than 2 else \( G_{x,y,z} \) is an empty set. Then, we obtain a graph of \( G_{x,y,z}^1 \), we choose \( x - 2 \) vertices on the bottom graph among the \( n - 2 \) vertices left, \( z \) vertices above the \( x \)
vertices previously chosen, and finally \(y - z\) vertices among the \(n - x\) vertices left. We can write:

\[
C_1(x, y, z) = \binom{n - 2}{x - 2} \times \binom{x}{z} \times \binom{n - x}{y - z} \times 1_{x \geq 2}
\]

We can finally conclude using (4.3):

\[
C_1(x, y, z) = \frac{(n - 2)!}{(n - x)! (x - 2)!} \times \frac{x!}{(x - z)! z!} \times \frac{(n - x)!}{(n - x - y + z)! (y - z)!} 1_{x \geq 2}
\]

\[
= \frac{(n - 2)! (n - x)! (x - 2)!}{(n - x)! (x - 1)!} \times \frac{x!}{(x - z)! z!} \times \frac{(n - x)!}{(n - x - y + z)! (y - z)!}
\]

\[
\square
\]

**Proposition 4.4.** For all \(x, y \geq z \geq 1\) such that \(x + y - z \leq n\):

\[
C_2(x, y, z) = \frac{(n - 2)! (xy - z)}{(x - z)! (n - x - y + z)! (y - z)!}
\]

**Proof.** First of all, note that a graph of \(G_{x,y,z}^2\) has to belong to the set of vertices. Thus, to enumerate the number of graph \(G = (V, E)\) of \(G_{x,y,z}^2\), we distinguish 4 different cases: either \(v\) and \(u'\) belongs to \(V\); either \(v \in V\) and \(u' \notin V\); either \(v \notin V\) and \(u' \in V\); either \(u', v' \in V\). We can write:

\[
C_2(x, y, z)
\]

\[
= \binom{n - 2}{x - 2} \binom{x - 2}{z - 2} \binom{n - x}{y - z} \times 1_{x \geq 2, y \geq 2, z \geq 2}
\]

\[
+ \binom{n - 2}{x - 2} \binom{x - 2}{z - 1} \binom{n - x}{y - z} \times 1_{x > \max(1, z), y < n}
\]

\[
+ \binom{n - 2}{x - 1} \binom{x - 1}{y - 1} \binom{n - x - 1}{y - 1} \times 1_{x < n, y > \max(1, z)}
\]

\[
+ \binom{n - 2}{x - 1} \binom{x - 1}{z} \binom{n - x - 1}{y - 1} \times 1_{\max(1, z) < x < n, \max(1, z) < y < n}
\]
Using (4.3), we obtain:

\[
C_2(x, y, z) = \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!} \times z(z - 1) \\
+ \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!} \times z(x - z) \\
+ \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!} \times z(y - z) \\
+ \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!} \times (x - z)(y - z) \\
= \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!}(zy - z)
\]

Which is the desired result. $\square$

**Lemma 4.5.** For all $x, y, z \in \mathbb{N}\setminus\{0\}$, there exists an $x_0 := x_0(y, z)$ such that $y \leq x \leq x_0 \Rightarrow C_{diff}(x, y, z) \leq 0$.

**Proof.** Using propositions 4.3 and 4.4, we have that:

\[
C_1(x, y, z) - C_2(x, y, z) = \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!}(x^2 - xy + z)
\]

From the definition of $C_{diff}$, we have that $x \geq z \geq 1$:

\[
C_{diff}(x, x, z) = \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!}(z - x)
\]

and for all $x, y \geq z \geq 1$:

\[
(4.4) \quad C_{diff}(x, y, z) = \frac{(n - 2)!}{(x - z)!z!(n - x - y + z)!(y - z)!}(x^2 - 2xy + y^2 - x - y + 2z)
\]

Thus, we obtain:

\[
C_{diff}(x, y, z) \leq 0 \iff x \in \left[y + \frac{1 - \sqrt{8y - 8z + 1}}{2}; y + \frac{1 + \sqrt{8y - 8z + 1}}{2}\right]
\]

$\square$

Combining lemmas 4.1, 4.2 and 4.5, lemma 3.1 is shown.

Then we prove lemma 3.3 by proving two intermediate lemmas.

**Lemma 4.6.** For all $k \geq z \geq 1$, we have the following equality:

\[
\sum_{i=0}^{k-z} C_{diff}(k + i, k - i, z) = 0
\]
Proof. To prove this lemma, it is easier to show that:

$$\sum_{i=1}^{k-z} C_{diff}(k+i, k-i, z) = -C_{diff}(k, k, z)$$

Using the argument of the function $C_{diff}$, the 3-tuple $(k+i, k-i, z)$, some of the factors are independent of $i$. Indeed, we have:

$$C_{diff}(k+i, k-i, z) = \frac{4i^2 - 2k + 2z}{(k+i-z)! (k-i-z)!} \times \frac{(n-2)!}{z!(n-2k+z)!}$$

So to prove the lemma, it is enough to show that:

$$\sum_{i=1}^{k-z} \frac{4i^2 - 2k + 2z}{(k+i-z)! (k-i-z)!} = \frac{k-z}{(k-z)! (k-z)!}$$

Note that:

$$\frac{k-z}{(k-z)! (k-z)!} = \frac{4-2k+2z}{(k+1-z)! (k-1-z)!} + \frac{3(k+2-z)}{(k+2-z)! (k-2-z)!} 1_{k-z \geq 2}$$

and for all $k-z > i \geq 2$:

$$\frac{4i^2 - 2k + 2z}{(k+i-z)! (k-i-z)!} = \frac{4i^2 - 2k + 2z}{(k+i-z)! (k-i-z)!} + \frac{(2i+1)(k+i+1-z)}{(k+i+1-z)! (k-i-1-z)!}$$

And when $i = k-z$, then:

$$\frac{(2i-1)(k+i-z)}{(k+i-z)! (k-i-z)!} = \frac{4(k-z)^2 - 2k + 2z}{(2k-2z)!}$$

Which concludes the proof. \(\square\)

**Lemma 4.7.** For all $k \geq z \geq 1$, the following equality holds:

$$\sum_{i=0}^{k-z} C_{diff}(k+i+1, k-i, z) = 0$$

**Proof.** Following the proof of the lemma 4.6, it is enough to show:

$$\sum_{i=1}^{k-z} \frac{2i^2 + 2i - k + z}{(k+i+1-z)! (k-i-z)!} = \frac{k-z}{(k+1-z)! (k-z)!}$$
Since:
\[
\frac{k - z}{(k + 1 - z)! (k - z)!} = \frac{4 - k + z}{(k + 2 - z)! (k - 1 - z)!} + \frac{2 (k - z + 3)}{(k + 3 - z)! (k - 2 - z)!} \mathbb{1}_{k - z \geq 2}
\]
and for all \( k - z > i \geq 2 \), we have:
\[
\frac{i (k + i + 1 - z)}{(k + i - z)! (k - i - z)!} = \frac{2i^2 + 2i - k - z}{(k + i - z)! (k - i - z)!} + \frac{(i + 1)(k + i + 2 - z)}{(k + i + 1 - z)! (k - i - 1 - z)!}
\]
and when \( i = k - z \),
\[
\frac{i (k + i + 1 - z)}{(k + i + 1 - z)! (k - i - 1 - z)!} = \frac{2(k - z)^2 + 2(k - z) - k + z}{(2k - 2z + 1)!}
\]
Which concludes the proof of the lemma. \( \square \)

The proof of the lemma 3.3 is the combination of the lemmas 4.6 and 1.7.

5. Proof of Auxiliary Results and Remarks

5.1. Proof of the Proposition 2.3. Recall that the bunkbed conjecture can be reformulated with a set of parameter of percolation constrained as explained in the second section. In this context, we prove that the bunkbed conjecture is verified in mean by considering two independent random variables \( X \) and \( Y \) identically distributed on the vertices of the bottom graph.

Proof of Proposition 2.3. We show a slightly stronger result, for all configuration \( \omega \),
\[
E \left[ 1_{X \in \omega_Y} + 1_{X' \in \omega_Y} \right] \geq E \left[ 1_{X \in \omega_Y} + 1_{X' \in \omega_Y} \right]
\]
Given a configuration \( \omega \), we look at all of its clusters. For all \( x \), we note \( A (x) \) the cluster containing \( x \) intersected with the set of the vertices of the bottom and \( B (x) \) the cluster containing \( x \) intersected with the
set of the vertices of the top graph so that:

\[
E \left[ \mathbbm{1}_{X \leftrightarrow Y} + \mathbbm{1}_{X' \leftrightarrow Y'} \right] - E \left[ \mathbbm{1}_{X \leftrightarrow Y'} + \mathbbm{1}_{X' \leftrightarrow Y} \right] \\
= \sum_{x} P(X = x) \left[ \sum_{y \in A(x)} P(Y = y) - \sum_{y' \in B(x)} P(Y = y') \right] \\
+ \sum_{x'} P(X = x') \left[ \sum_{y \in B(x')} P(Y = y) - \sum_{y' \in A(x')} P(Y = y') \right] \\
= \sum_{x} P(X = x) [P(Y \in A(x)) - P(Y \in B(x))] \\
+ \sum_{x'} P(X = x') [P(Y \in B(x')) - P(Y \in A(x'))] \\
(5.1)
\]

Then, we sum over the different clusters \( C \in C(\omega) \) instead of the vertices and we note for each cluster a representative \( x_0 := x_0(C) \), so we get:

\[
(5.1) = \sum_{C \in C(\omega)} \sum_{x \in A(x_0)} P(X = x) [P(Y \in A(x)) - P(Y \in B(x))] \\
+ \sum_{x \in B(x_0)} P(X = x) [P(Y \in B(x')) - P(Y \in A(x'))] \\
= \sum_{C \in C(\omega)} \left[ P(X \in A(x_0)) - P(X \in B(x_0)) \right] \\
\times [P(Y \in A(x_0)) - P(Y \in B(x_0))] \\
= \sum_{C \in C(\omega)} [P(X \in A(x_0)) - P(X \in B(x_0))]^2 \geq 0
\]

Which proves the result. \( \square \)

5.2. **Proof of Proposition 2.4.** The goal of the conjecture is to lower bound the quantity \( \mathbb{P}_p(u \leftrightarrow v) - \mathbb{P}_p(u \leftrightarrow v') \) by 0. We give a simple upper bound by proving Proposition 2.4.
Proof. Note that:

\[
P_p(u \leftrightarrow v) = P_p(u \leftrightarrow v \cap v \leftrightarrow v') + P_p(u \leftrightarrow v \cap v \not\leftrightarrow v' \cap u \leftrightarrow u') + P_p(u \leftrightarrow v')
\]

(5.2)

\[
P_p(u \leftrightarrow v' \cap v \leftrightarrow v') = P_p(u \leftrightarrow v' \cap v \not\leftrightarrow v' \cap u \leftrightarrow u') + P_p(u \leftrightarrow v' \cap v \not\leftrightarrow v' \cap u \not\leftrightarrow u')
\]

(5.3)

The first term of the right-hand of (5.2) and (5.3) are obviously equal as well as the second member by an argument of symmetry. Thus we get:

\[
P_p(u \leftrightarrow v) - P_p(u \leftrightarrow v') = P_p(u \leftrightarrow v | v \not\leftrightarrow v' \cap u \not\leftrightarrow u') - P_p(u \leftrightarrow v' | v \not\leftrightarrow v' \cap u \not\leftrightarrow u') \times P_p(v \not\leftrightarrow v' \cap u \not\leftrightarrow u')
\]

\[\geq -P_p(v \not\leftrightarrow v' \cap u \not\leftrightarrow u')
\]

We proceed in the same way to lower bound \(P_p(u \leftrightarrow v') - P_p(u \leftrightarrow v)\) and obtain the desired result.

\[\square\]

5.3. Trivial Case: the Line Segment. When we consider the bunkbed graph of the line segment, it is possible to quantify the difference of \(P_p(u \leftrightarrow v) - P_p(u \leftrightarrow v')\). Consider the line segment graph with \(n\) vertices and consider its bunkbed graph \(G\). Note \(u\) the point \((1,0)\) and \(v\) the point \((n,0)\). By symmetry, if the edge \{\((i,0),(i,1)\)\} is open then \(P((i,0) \leftrightarrow v) = P((i,1) \leftrightarrow v) = P((i,0) \leftrightarrow v')\). So, noting \(\tau\) the index of the first open vertical edge (starting from the left), and by convention, \(\tau = 0\) if none are opened, we get:

\[
P(u \leftrightarrow v) = \sum_{i=1}^{n} P(u \leftrightarrow (i,0) \cap (i,0) \leftrightarrow v \cap \tau = i) + P(u \leftrightarrow v \cap \tau = 0)
\]

Or, the part before the vertical edge \(e_{v_i}\) is independent of the part
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Figure 5.1. Case where $\tau = 4$ and $n = 6$

after, conditionally to $\tau = i$ (see figure 5.1), which gives:

$$\mathbb{P}(u \leftrightarrow v)$$

$$= \sum_{i=1}^{n} \mathbb{P}(u \leftrightarrow (i,0) | \tau = i) \mathbb{P}((i,0) \leftrightarrow v | \tau = i) \mathbb{P}(\tau = i)$$

$$+ \mathbb{P}(u \leftrightarrow v | \tau = 0)$$

$$= \sum_{i=1}^{n} \mathbb{P}(u \leftrightarrow (i,0) | \tau = i) \mathbb{P}((i,0) \leftrightarrow v' | \tau = i) \mathbb{P}(\tau = i)$$

$$+ \mathbb{P}(u \leftrightarrow v' | \tau = 0)$$

$$= \mathbb{P}(u \leftrightarrow v') + \mathbb{P}(u \leftrightarrow v | \tau = 0)$$

The difference of the probability is therefore given by:

$$\mathbb{P}(u \leftrightarrow v | \tau = 0) = \prod_{i=1}^{n-1} (1 - p_{v_i}) p_{\{(i,0),(i+1,0)\}}$$

We can remark that in this case, the difference of probabilities is the case where all the vertical edges are closed and $u$ is connected to $v$. In the proof of the theorem 1.1 we showed that the difference is strictly greater than this case (this can be seen in the case of the bunkbed graph of the triangle). In conclusion, when we consider $p = \frac{1}{2}$ in the theorem 1.1 it indicates that it would be a difficult task to build a surjection between the configurations connecting $u$ to $v$ and those connecting $u$ to $v'$. A second argument supporting this conclusion is the change of the sign intervening in the quantity $C_{\text{diff}}$.

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(P. de Buyer) Université Paris Nanterre - Modal’X, 200 avenue de la République 92000 Nanterre, France
E-mail address: debuyer@math.cnrs.fr