Pair production of black holes in a $U(1) \otimes U(1)$ theory

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Abstract

Charged dilaton black hole solutions have recently been found for an action with two $U(1)$ gauge fields and a dilaton field. I investigate new exact solutions of this theory analogous to the C-metric and Ernst solutions of classical general relativity. The parameters in the latter solution may be restricted so that it has a smooth Euclidean section with topology $S^2 \times S^2 - \{pt\}$, which gives an instanton describing pair production of the charged dilaton black holes. These instantons generalize those found recently by Dowker et al.

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I. INTRODUCTION

The study of black hole evaporation explores some of the most important issues in quantum gravity. One of the most interesting issues raised by black hole evaporation is the information loss problem: in the semiclassical theory, the information describing the configuration of matter used to form the black hole doesn’t re-emerge in the radiation emitted during collapse, as this radiation is precisely thermal [1]. It is possible that higher-order quantum effects modify the radiation so that it is not precisely thermal, but consideration of toy theories in two dimensions suggest that such effects become important too late in the evaporation for this to resolve the information problem [2–4].

Another possible repository for the information lost to the black hole is some kind of long-lived remnant left behind by the black hole [3,5,6]. One of the major problems with this proposal is that there would have to be an infinite number of distinct remnants to account for all the information that could possibly have been dumped into the black hole, as we can form black holes from arbitrarily large amounts of matter. One would naïvely assume that each of these species would have a finite probability to be pair produced in a suitable background field, and there would therefore be problems with divergences in the total pair production of remnants [7,8]. Various suppression mechanisms have been suggested which could produce a finite answer despite this naïve argument, but they are the subject of extensive contention [6–8], so an explicit calculation of this rate is essential to the further consideration of the specifically, if black holes can form but never disappear, they will violate CPT [9], but I will not concern myself with this here.

For a neutral black hole (which the semiclassical theory predicts will evaporate completely), the endpoint of evaporation lies deep in the quantum regime, as the mass of the black hole becomes of order $M_{pl}$, and it is therefore inaccessible to semiclassical analysis. This problem has led to extensive interest in consideration of the quantum behavior of more complicated black holes [10–14]. The behavior of near-extreme charged black holes displays the same features and puzzles, and can be studied semiclassically with a fair degree of confidence. More complicated models have other advantages: in string theory, the singularity in the sigma-model metric disappears from the spacetime down an infinitely long tube as extremality is approached, and excitations living far down the throat are candidate remnants [3]. It is useful to consider a model with no charged particles, as we then have the considerable simplification that the charge of the black hole is constant, while the fundamental black hole physics remains unchanged.

The most common such model is the action

$$S = \int d^4x \sqrt{-g}(R - 2\partial^\mu \phi \partial_\mu \phi - e^{-2a\phi} F_{\mu\nu} F^{\mu\nu}).$$  \hspace{1cm} (1)

This action has been extensively considered, and in [15], instantons describing the pair production of black holes in this theory were developed. In the sum-over-histories approach to quantum mechanics, the action for these instantons gives a good approximation to the rate for pair production. Unfortunately, the instantons are regular only for $a < 1$ unless the black holes produced are extreme. As the action (1) for $a = 1$ is a part of the action for the low-energy limit of string theory, we would like to extend consideration to this case.

It has recently been suggested [13] that a particularly interesting generalization of (1) is an action with an additional gauge field,
\[ I_{SU(4)} = \int d^4x \sqrt{-g} (R - 2\partial^\mu \phi \partial_\mu \phi - e^{-2\phi} (F_{\mu\nu} F^{\mu\nu} + G_{\mu\nu} G^{\mu\nu})), \]  

where

\[
F_{\mu\nu} = \partial_{[\mu} A_{\nu]}, \quad G_{\mu\nu} = \partial_{[\mu} B_{\nu]}. 
\]

This is a special case of a more general theory with a rigid \( SU(4) \otimes SU(1,1) \) symmetry, vector fields transforming under a \( SU(2) \otimes SU(2) \) group and a complex scalar, which arises from dimensionally reduced superstring theory or \( N = 4 \) supergravity \[16\]. The action (2) is invariant under a duality transformation,

\[
F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} = \frac{1}{2} e^{-2\phi} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad G_{\mu\nu} \rightarrow \tilde{G}_{\mu\nu} = \frac{1}{2} e^{-2\phi} \varepsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}, \quad \phi \rightarrow -\phi, \]  

which is analogous to the ordinary electric-magnetic duality transformation of Einstein-Maxwell theory. If we set one of the gauge fields to zero, it becomes the action (1) with \( a = 1 \). Black hole solutions of (2) were found by Gibbons \[10\]. I seek to develop C-metric and Ernst solutions of (2) which will generalize the solutions of (1) discussed in \[10,15\].

The organization of the rest of the paper is as follows. In section II I discuss the C-metric and black hole solutions of (2), and relate them to the solutions of (1) discussed in \[10,15\]. There are nodal singularities in the C-metric which cannot in general be removed by any choice of the period of the azimuthal coordinate. In section III I describe the appropriate Harrison transformations to eliminate the nodal singularities. These transformations give solutions analogous to the Ernst solutions, which reduce to a solution with two constant background fields when the size of the black holes is much less than the scale of the fields, and at spatial infinity. In section IV I discuss the Euclidean continuation of these solutions, and find that it is necessary to impose another condition on the parameters to obtain a regular solution. The regular instantons have the topology of \( S^2 \times S^2 - \{pt\} \), and thus describe pair production of the black holes in the presence of the background fields \[11\]. In the limit that one of the charges vanishes, the condition on the parameters becomes the condition for the black holes to be extreme \[15\]. Section V summarizes my results.

**II. C-METRIC SOLUTIONS**

The charged black hole solutions of (2) are \[10\]:

\[
\begin{align*}
    ds^2 &= -\lambda dt^2 + \lambda^{-1} dr^2 + R^2 d\Omega, \\
    e^{2\phi} &= e^{2\phi_0} \frac{r + \Sigma}{r - \Sigma}, \\
    F &= \frac{Q e^{\phi_0}}{(r - \Sigma)^2} dt \wedge dr, \quad G = P e^{\phi_0} \sin \theta d\theta \wedge d\varphi, \\
\end{align*}
\]  

\[ (5) \]
where

\[ \lambda = \frac{(r - r_+)(r - r_-)}{R^2}, \quad R^2 = r^2 - \Sigma^2, \] (6)

and [17]

\[ r_\pm = M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2}, \quad \Sigma = \frac{P^2 - Q^2}{2M}. \] (7)

There is a curvature singularity at \( r = |\Sigma| \). The physical degrees of freedom are \( P, Q, M \) and \( \phi_0 \); \( M \) is the mass of the black hole, \( e^{\phi_0}Q \) is its electric charge, and \( e^{\phi_0}P \) is its magnetic charge. I could keep the asymptotic value of the dilaton \( \phi_0 \) as a free parameter, but I will instead fix it by requiring that the dilaton match to an appropriate background value at infinity. The solution has a manifest dual symmetry, under

\[ Q \leftrightarrow P, \quad \Sigma \leftrightarrow -\Sigma, \quad \tilde{F} \leftrightarrow G, \quad \phi \leftrightarrow -\phi, \] (8)

corresponding to the general symmetry (4). The parameters are constrained to \( M \geq M_{\text{extr}} = (|P| + |Q|)/\sqrt{2} \) by positivity bounds arising from supersymmetry [13]. The black holes have one unbroken \( N = 1 \) supersymmetry in the extremal limit if both gauge charges are non-zero, and two if one of the gauge charges is zero. Unlike the dilaton black hole solutions of (4), where the temperature at infinity is ill-defined, the temperature of these black holes goes smoothly to zero as extremality is approached [10,13].

If we make the coordinate transformation \( r' = r + \Sigma \), this metric becomes

\[
\begin{align*}
\lambda &= \frac{(r' - r_+)(r' - r_-)}{R'^2}, \\
R^2 &= r'(r' - 2\Sigma),
\end{align*}
\]

and when \( Q = 0 \) (which implies \( r_- = \Sigma = P^2/2M \)), this reduces to the black hole solution of (4) with magnetic charge found in [10] (and independently in [17]) if we identify \( r'_\pm = r_\pm + \Sigma, \quad q = P \). Similarly, if we make a coordinate transformation \( r' = r - \Sigma \) in (3), we will see that it will reduce to the electrically charged dual solution [10] when \( P = 0 \) if we identify \( r'_\pm = r_\pm - \Sigma, \quad q = Q \).

The generalization of the C-metric solution [18] to this theory is given by

\[
\begin{align*}
ds^2 &= \frac{1}{A^2(x - y)^2}[F(x)(G(y)dt^2 - G^{-1}(y)dy^2) \\
&\quad + F(y)(G^{-1}(x)dx^2 + G(x)d\varphi^2)],
\end{align*}
\]

\[
e^{-2\phi} = e^{-2\phi_0} \left( \frac{1 + \Sigma Ax}{1 - \Sigma Ax} \right) \left( \frac{1 - \Sigma Ay}{1 + \Sigma Ay} \right), \] (11)
\[ F_{yt} = \frac{\alpha e^{\phi_0}}{(1 + \Sigma Ay)^2}, \quad G_{x\phi} = \frac{\beta e^{\phi_0}}{(1 - \Sigma Ax)^2}, \]  
\[ (12) \]

where

\[ F(\xi) = 1 - \Sigma^2 A^2 \xi^2, \]  
\[ G(\xi) = \frac{(1 - \xi^2 - r_+ A \xi^3)(1 + r_- A \xi)}{(1 - \Sigma^2 A^2 \xi^2)}, \]  
\[ (13) \quad (14) \]

and

\[ \alpha^2 = \frac{1}{2}(r_+ - \Sigma)(r_- - \Sigma) + \frac{1}{2} A^2 \Sigma^3 (r_- - \Sigma), \]  
\[ = Q^2 + \frac{1}{2} A^2 \Sigma^3 (r_- - \Sigma), \]  
\[ (15) \]

\[ \beta^2 = \frac{1}{2}(r_+ + \Sigma)(r_- + \Sigma) - \frac{1}{2} A^2 \Sigma^3 (r_+ + \Sigma), \]  
\[ = P^2 - \frac{1}{2} A^2 \Sigma^3 (r_+ + \Sigma). \]  
\[ (16) \]

This metric has the same general form as the C-metric solutions in [15,18], and we choose coordinates so that the cubic factor in \( G \) still has the same form, but the functions \( G \) and \( F \) are more complicated. The parameters in this solution are still related by (7). The interpretation of the parameters is now essentially qualitative, as the mass and charges are given by \( M, Q \) and \( P \) only in the weak-field limit. The additional parameter \( A \) determines the strength of the acceleration. Note that this solution also has the manifest dual symmetry (8).

This C-metric solution tends to the charged black hole (5) as \( A \to 0 \). To see this, make the coordinate transformation

\[ r = -\frac{1}{A y}, \quad T = A^{-1} t, \]  
\[ (17) \]

which puts the metric in the form

\[ ds^2 = \frac{1}{(1 + A r x)^2} [F(x)(-H(r)dT^2 + H^{-1}(r)dr^2) + R^2(r)(G^{-1}(x)dx^2 + G(x)d\phi^2)], \]  
\[ (18) \]

where

\[ H(r) = \frac{(r - r_+ - A^2 r_3)(r - r_-)}{R^2(r)}, \]  
\[ (19) \]

\( F(x) \) is given by (13), and \( R(r) \) is given by (9). If we now set \( A = 0 \) and make the further coordinate transformation \( x = \cos \theta \), this metric reduces to the charged black hole metric (8). The other fields also reduce to the forms in (9). When \( \Sigma = 0 \) (i.e., \( P = Q \), the metric
reduces to the charged C-metric of Einstein-Maxwell theory [18], but with a different field content.

If either of the gauge charges vanish, this C-metric will reduce to the dilaton C-metric given in [15] with \( a = 1 \), although the parameters are not simply related to the parameters \( m, A, q \) of that solution. To demonstrate this in the case where \( Q \) is set to zero, we first perform a coordinate transformation

\[
y = \frac{y'}{1 + \Sigma Ay'}, \quad x = \frac{x'}{1 + \Sigma Ax'};
\]

the metric (10) becomes

\[
ds^2 = \frac{1}{A^2(x' - y')^2} \left[ F(x')(G(y')dt^2 - G^{-1}(y')dy'^2) + F(y')(G^{-1}(x')dx'^2 + G(x')d\varphi^2) \right],
\]

where

\[
F(\xi') = 1 + 2\Sigma A\xi',
\]

\[
G(\xi') = \frac{(1 + \Sigma A\xi')^3 - \xi'^2 - r'_+ A\xi'^3(1 + r'_- A\xi')}{F(\xi')},
\]

and \( r'_\pm = r_\pm + \Sigma \). If we now make a further coordinate transformation \([18]\)

\[
t = c_0 \hat{t}, \qquad \varphi = c_0 \hat{\varphi}, \quad x' = c_1 c_0 \hat{x} + c_2, \quad y' = c_1 c_0 \hat{y} + c_2,
\]

and restrict \( c_0, c_1, c_2 \) suitably, we can rewrite this as

\[
ds^2 = \frac{1}{A^2(\hat{x} - \hat{y})^2} \left[ \hat{F}(\hat{x})(\hat{G}(\hat{y})d\hat{t}^2 - \hat{G}^{-1}(\hat{y})d\hat{y}^2) + \hat{F}(\hat{y})(\hat{G}^{-1}(\hat{x})d\hat{x}^2 + \hat{G}(\hat{x})d\hat{\varphi}^2) \right],
\]

where now

\[
\hat{F}(\hat{\xi}) = 1 + 2\hat{\Sigma} \hat{A}\hat{\xi},
\]

and

\[
\hat{G}(\hat{\xi}) = \frac{(1 - \hat{\xi}^2 - \hat{r}_+ \hat{A}\hat{\xi}^3)(1 + \hat{r}_- \hat{A}\hat{\xi})}{\hat{F}(\hat{\xi})}.
\]

The dilaton and gauge fields in this coordinate system are

\[
e^{-2\phi} = e^{-2\phi_0} \frac{\hat{F}(\hat{y})}{\hat{F}(\hat{x})},
\]

\[
G_{\hat{x}\hat{\varphi}} = \beta e^{\phi_0}, \quad \text{and} \quad F_{\hat{y}\hat{t}} = \frac{\alpha e^{\phi_0}}{\hat{F}(\hat{y})^2},
\]

\[6\]
where we find

\begin{equation}
\alpha^2 = \frac{1}{2} (\hat{r}_+ - 2 \hat{\Sigma}) (\hat{r}_- - 2 \hat{\Sigma}) + \frac{1}{2} \hat{\Sigma}^3 \hat{A}^2 (\hat{r}_- - 2 \hat{\Sigma})
\end{equation}

(30)

and

\begin{equation}
\beta^2 = \frac{1}{2} \hat{r}_+ \hat{r}_-.
\end{equation}

(31)

The parameters are related by

\begin{equation}
\hat{A}^2 = \frac{A^2 c_1}{1 + 2 \Sigma A c_2},
\end{equation}

(32)

\begin{equation}
\hat{r}_- \hat{A} = \frac{r' A}{1 + 2 \Sigma A c_2}, \hat{\Sigma} \hat{A} = \frac{\Sigma A}{1 + 2 \Sigma A c_2},
\end{equation}

(33)

and

\begin{equation}
\hat{r}_+ \hat{A} = r' A c_3 c_1^2 - \Sigma^3 A^3 c_0 c_1^2.
\end{equation}

(34)

When \( Q = 0 \), \( r'_- = 2 \Sigma \), and (33) therefore implies \( \hat{r}_- = 2 \hat{\Sigma} \). Thus, (25) then reduces to the C-metric solution given in \([15]\) for \( a = 1 \). When \( P = 0 \), a similar transformation may be used to show that it reduces to the electric dual to the \( a = 1 \) solution in \([15]\).

For \( r_+ A < 2/(3 \sqrt{3}) \), the function \( G(\xi) \) has four real roots, which we denote in ascending order by \( \xi_1, \xi_2, \xi_3, \xi_4 \). We may restrict the parameters so that \( \xi_1 = -1/r_- A \) and \( \xi_1 < \xi_2 \leq \xi_3 < \xi_4 \). The surface \( y = \xi_0 \equiv -1/|\Sigma| A \) is singular; this surface is analogous to the singular surface at \( r = |\Sigma| \) in the black hole solutions \([3]\). As \( r_- \geq |\Sigma| \), \( \xi_1 \geq \xi_0 \). The surfaces \( y = \xi_1, y = \xi_2 \) are the black hole horizons, and \( y = \xi_3 \) is the acceleration horizon. If I allowed \( \xi_1 = \xi_2 \), the black hole would be extremal, although the horizons would still be regular. However, for this paper, I will restrict my attention to \( \xi_1 < \xi_2 \). The coordinates \((x, \varphi)\) are angular coordinates, and \( x \) is restricted to the range \( \xi_3 \leq x \leq \xi_4 \) in which \( G(x) \) is positive, so that the metric has the appropriate signature. At \( x = \{\xi_3, \xi_4\} \), the norm of \( \partial/\partial \varphi \) vanishes, so these points are interpreted as the poles of two-spheres around the black hole. There is a divergence in the metric at \( x = y \), which is interpreted as the point at infinity, so \( y \) is restricted to the range \( \xi_0 < y < x \). Spatial infinity is reached when \( y = x = \xi_3 \), and null or timelike infinity when \( y = x \neq \xi_3 \) \([19]\).

It is not generally possible to choose the period \( \Delta \varphi \) of the azimuthal coordinate so that the nodal singularities at the two poles \( x = \xi_3 \) and \( x = \xi_4 \) are eliminated simultaneously. The deficit angles at the two poles are given by \([13]\)

\begin{equation}
\delta_3 = 2\pi - \frac{1}{2} \Delta \varphi |G'(\xi_3)|, \delta_4 = 2\pi - \frac{1}{2} \Delta \varphi |G'(\xi_4)|.
\end{equation}

(35)

By choosing \( \Delta \varphi = 4\pi/|G'(\xi_3)| \) we may eliminate the nodal singularity at \( x = \xi_3 \), but there will then in general be a negative deficit angle along the \( \xi_4 \) direction, which may be interpreted as a “line singularity” pushing the two black holes apart. In the next section, we will see how we can eliminate the nodal singularities by introducing external fields via a Harrison transformation \([20]\).
III. ERNST SOLUTIONS

In [13], Dowker et al showed that given an axisymmetric solution with \( A_i = g_i = 0 \), where \( x^i \) are the other three coordinates, a new axisymmetric solution may be obtained by the transformation

\[
g'_{ij} = \Lambda g_{ij}, \quad g'_{\phi\phi} = \Lambda^{-1} g_{\phi\phi},
\]

\[
e^{-2\phi'} = e^{-2\phi} \Lambda, \quad A'_{\phi} = -\frac{1}{BA}(1 + BA_{\phi}),
\]

\[
\Lambda = (1 + BA_{\phi})^2 + \frac{1}{2} B^2 g_{\phi\phi} e^{2\phi}.
\]

I will now construct an appropriate generalization of this transformation. In the string conformal gauge \( ds^2_T = e^{2\phi} ds^2 \), the action (2) is

\[
S = \int d^4x \sqrt{-g_T} e^{-2\phi} \left[ R_T + 4(\nabla\phi)^2 - F^2 - G^2 \right].
\]

If I write \( F_{\mu\nu} \) and \( G_{\mu\nu} \) in terms of vector potentials as

\[
G_{\mu\nu} = \partial_{[\mu} A_{\nu]}, \quad F_{\mu\nu} = \frac{1}{2} e^{2\phi} \epsilon_{\mu\nu\rho\sigma} \partial^\rho B^\sigma
\]

and introduce the definitions

\[
3g_{ij} = g_{Tij}, \quad V = g_{T\phi\phi}, \quad \tilde{\phi} = \phi - \frac{1}{4} \log V,
\]

then the action can be rewritten in the form

\[
S = \alpha \int d^3x \sqrt{-3g} e^{-2\tilde{\phi}} \left[ 3R + 4\partial_i \tilde{\phi} \partial^i \tilde{\phi} - \frac{1}{4} V^{-2} \partial_i V \partial^i V - 2V^{-1} \partial_i A_{\phi} \partial^i A_{\phi} - 2e^{\tilde{\phi}} \partial_i B_{\phi} \partial^i B_{\phi} \right].
\]

It is now a relatively easy exercise to show that this action is invariant under the transformations

\[
V' = \frac{1}{\Lambda^2} V, \quad A'_{\phi} = -\frac{1}{BA}(1 + BA_{\phi}),
\]

\[
3g'_{ij} = \Psi^2 3g_{ij}, \quad e^{-2\phi'} = \frac{1}{\Psi} e^{-2\tilde{\phi}},
\]

\[
B'_{\phi} = \frac{1}{E\Psi} (1 + EB_{\phi}),
\]
\[ \Lambda = (1 + BA_\varphi)^2 + \frac{1}{2}B^2V, \]  \hspace{1cm} (46)  

\[ \Psi = (1 + EB_\varphi)^2 + \frac{1}{2}E^2e^{-4\phi}. \]  \hspace{1cm} (47)

I may now construct new solutions by applying these transformations to axisymmetric solutions satisfying \( A_i = g_{i\varphi} = 0 \). The analogue of the Melvin solution \(^{21}\), obtained by applying these transformations to the vacuum, is\(^4\)

\[ ds^2 = \Lambda \Psi [-dt^2 + d\rho^2 + dz^2] + \frac{\rho^2 d\varphi^2}{\Lambda \Psi}, \]  \hspace{1cm} (48)  

\[ e^{-2\phi} = \frac{\Lambda}{\Psi}, \ A_\varphi = -\frac{B\rho^2}{2\Lambda}, \ B_\varphi = -\frac{E\rho^2}{2\Psi}, \]  \hspace{1cm} (49)  

\[ G_{\mu\nu} = \partial_{[\mu}A_{\nu]}, \ F^{\mu\nu} = \frac{1}{2}e^{2\phi}\epsilon_{\mu
u\rho\sigma} \partial_\rho B_\sigma, \]  \hspace{1cm} (50)

\[ \Lambda = 1 + \frac{1}{2}B^2\rho^2, \ \Psi = 1 + \frac{1}{2}E^2\rho^2. \]  \hspace{1cm} (51)

This solution has a manifest dual symmetry

\[ B \leftrightarrow E, \ \tilde{F} \leftrightarrow G, \ \phi \leftrightarrow -\phi. \]  \hspace{1cm} (52)

If we apply the transformations to (10), we will obtain an analogue of the Ernst solution \(^{22}\) which preserves the manifest dual symmetry of (10). The resulting solution is

\[ ds^2 = \frac{\Lambda \Psi}{A^2(x - y)^2}[F(x)(G(y)dt^2 - G^{-1}(y)dy^2) \]  \hspace{1cm} (53)

\[ \hspace{1cm} + F(y)G^{-1}(x)dx^2] + \frac{F(y)G(x)}{\Lambda \Psi A^2(x - y)^2}d\varphi^2, \]  

\[ e^{-2\phi} = e^{-2\phi_0} \frac{\Lambda}{\Psi} \left( \frac{1 + \Sigma Ay}{1 - \Sigma Ay} \right) \left( \frac{1 - \Sigma Ax}{1 + \Sigma Ax} \right), \]  \hspace{1cm} (54)  

\[ A_\varphi = -\frac{e^{\phi_0}}{B\Lambda} \left( 1 + \frac{B\beta x}{1 - \Sigma Ax} \right) + k, \]  \hspace{1cm} (55)

\(^1\)After we have made a further gauge transformation to make the vector potentials regular on the axis \( \rho = 0 \).
\[ B_\phi = -\frac{e^{-\phi_0}}{E\Psi} \left( 1 + \frac{E\alpha x}{1 + \Sigma A x} \right) + k', \quad (56) \]

\[ G_{\mu\nu} = \partial_{[\mu} A_{\nu]}, \quad F^{\mu\nu} = \frac{1}{2} e^{2\phi} \epsilon^{\mu\nu\rho\sigma} \partial_\rho B_\sigma, \quad (57) \]

\[ \Lambda = \left( 1 + \frac{B\beta x}{1 - \Sigma A x} \right)^2 + \frac{B^2(1 - x^2 - r_+ A x^3)(1 + r_+ A x)(1 - \Sigma A y)^2}{2A^2(x - y)^2(1 - \Sigma A x)^2}, \quad (58) \]

\[ \Psi = \left( 1 + \frac{E\alpha x}{1 + \Sigma A x} \right)^2 + \frac{E^2(1 - x^2 - r_+ A x^3)(1 + r_+ A x)(1 + \Sigma A y)^2}{2A^2(x - y)^2(1 + \Sigma A x)^2}, \quad (59) \]

where \( F(\xi) \) and \( G(\xi) \) are given by (13,14), and \( \alpha \) and \( \beta \) are given by (15,16). The constants \( \phi_0, k, \) and \( k' \) will be chosen so that the solution at infinity agrees with (48). This solution has the manifest dual symmetry

\[ Q \leftrightarrow P, \Sigma \leftrightarrow -\Sigma, B \leftrightarrow E, \tilde{F} \leftrightarrow G, \phi \leftrightarrow -\phi \quad (60) \]

(which implies \( k \leftrightarrow k' \) and \( \phi_0 \leftrightarrow -\phi_0 \) to preserve the agreement with (48) at infinity).

As in the original Ernst solution, the background fields provide the force necessary to accelerate the black holes. To eliminate the nodal singularities in this metric at \( x = \xi_3 \) and \( x = \xi_4 \) simultaneously, we must constrain \( B \) and \( E \) so that

\[ G'(\xi_3)\Lambda(\xi_4)\Psi(\xi_4) = -G'(\xi_4)\Lambda(\xi_3)\Psi(\xi_3) \quad (61) \]

and choose \( \Delta \phi = 4\pi|\Lambda\Psi/G'(x)|_{x=\xi}\). In the limit \( r_+ A \ll 1 \), this constraint reduces to Newton’s law,

\[ MA \approx BP + EQ. \quad (62) \]

This leads me to suppose that \( r_+ A \ll 1 \) is in some sense a point particle limit, which appears reasonable, as this is simply a statement that the black hole is small on the scale set by the background fields. In this limit, so long as \(|r_+ A y| \ll 1 \) as well, one finds that \( G(\xi) \approx 1 - \xi^2, F(\xi) \approx 1 \), and thus the metric (53) becomes

\[ ds^2 \approx \frac{\Lambda\Psi}{A^2(x - y)^2}[1 + (1 - y^2)dt^2 - (1 - y^2)^{-1}dy^2] \quad (63) \]

\[ + (1 - x^2)^{-1}dx^2 + \frac{1 - x^2}{\Lambda\Psi A^2(x - y)^2}d\phi^2, \]

\[ ^2 \text{Note that } \Lambda(\xi_i) \equiv \Lambda(x = \xi_i) \text{ and } \Psi(\xi_i) \equiv \Psi(x = \xi_i) \text{ are constants.} \]
\[ \Lambda \approx 1 + \frac{1}{2} B^2 \frac{1 - x^2}{A^2(x - y)^2}, \quad (64) \]

\[ \Psi \approx 1 + \frac{1}{2} E^2 \frac{1 - x^2}{A^2(x - y)^2}. \quad (65) \]

This is just the Melvin solution \([48]\) in non-standard coordinates: the transformation

\[ \rho^2 = \frac{1 - x^2}{A^2(x - y)^2}, \quad \zeta^2 = \frac{y^2 - 1}{A^2(x - y)^2}, \quad (66) \]

\[ \hat{t} = \zeta \sinh t, \quad z = \zeta \cosh t, \quad (67) \]

puts it in the form \([48]\). The dilaton and gauge fields \((54)\) in this approximation are

\[ e^{-2\phi} \approx e^{-2\phi_0} \frac{\Lambda}{\Psi}, \quad A_\varphi \approx \frac{e^{\phi_0} B \rho^2}{2\Lambda}, \quad B_\varphi \approx \frac{e^{-\phi_0} E \rho^2}{2\Psi}, \quad (68) \]

where \(k\) and \(k'\) have been chosen so as to give regularity on the axis \(\rho = 0\), in agreement with \((49)\). This agrees with \((49)\) up to the arbitrary constant shift of the dilaton.

The Ernst solution \((53)\) also approaches \((48)\) at large spacelike distances. Spatial infinity corresponds to \(x, y \to \xi_3\), and in this limit it is convenient to use the change of coordinates given in \([15]\),

\[ x - \xi_3 = \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{\rho^2}{(\rho^2 + \zeta^2)^2}, \quad (69) \]

\[ \xi_3 - y = \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{\zeta^2}{(\rho^2 + \zeta^2)^2}, \quad (70) \]

\[ t = \frac{2\eta}{G'(\xi_3)}, \quad \varphi = \frac{2L^2 \hat{\varphi}}{G'(\xi_3)}, \quad (71) \]

where I have introduced \(L^2 = \Lambda(x = \xi_3)\Psi(x = \xi_3)\). Note that the choice of period of \(\varphi\) implies \(\hat{\varphi}\) has period \(2\pi\). For large \(\rho^2 + \zeta^2\), the Ernst solution in these coordinates reduces to

\[ ds^2 \to \tilde{\Lambda} \tilde{\Psi} (-\zeta^2 d\eta^2 + d\zeta^2 + d\rho^2) + \frac{\rho^2 d\hat{\varphi}^2}{\tilde{\Lambda} \tilde{\Psi}}, \quad (72) \]

where

\[ \tilde{\Lambda} = (1 + \frac{1}{2} \hat{B}^2 \rho^2) \text{ with } \hat{B}^2 = \frac{B^2 G'^2(\xi_3)}{4L^2 \Lambda(\xi_3)}, \quad (73) \]

and
\[ \tilde{\Psi} = (1 + \frac{1}{2} \hat{E}^2 \rho^2) \text{ with } \hat{E}^2 = \frac{E^2 G''(\xi_3)}{4 L^2 \Psi(\xi_3)}. \]  

(74)

If we now set \( \hat{t} = \zeta \sinh \eta, z = \zeta \cosh \eta \), we once again regain \( (18) \). For large \( \rho^2 + \zeta^2 \), the dilaton and gauge fields tend to

\[ e^{-2\phi} \to L^2 e^{-2\phi_0} \frac{\tilde{\Lambda}}{\Psi}, \]

(75)

\[ A_\tilde{\rho} \to L^{-1} e^{\phi_0} \frac{\hat{B} \rho^2}{2 \Lambda}, B_\tilde{\rho} \to L e^{-\phi_0} \frac{\hat{E} \rho^2}{2 \Psi}, \]

(76)

so if we set \( e^{\phi_0} = L \), we recover \( (19) \) in this limit. I will take this to define \( \phi_0 \) in general.

In summary, we recover the Melvin solution at large spacelike distances, with the physical background fields \( \hat{E} \) and \( \hat{B} \). In the limit \( r_+ A \ll 1, \hat{B} \approx B, \hat{E} \approx E \), as expected.

IV. INSTANTONS

The solution \( (53) \) describes two black holes accelerating away from each other, propelled by the constant background fields. I now consider the Euclidean section obtained by taking \( \tau = it \) in \( (53) \). The Euclidean section gives an exact instanton describing pair production of the accelerating black holes \( (23) \). I will only consider the non-extreme or wormhole instantons, \textit{i.e.}, \( \xi_1 < \xi_2 \). I then find that it is necessary to impose another condition on the parameters to eliminate the possible conical singularities at the black hole horizon \( y = \xi_2 \) and the acceleration horizon \( y = \xi_3 \) simultaneously. Namely, we must take the period of \( \tau \) to be \( \Delta \tau = 4\pi / G'(\xi_2) \) and set

\[ |G''(\xi_2)| = |G'(\xi_3)|, \]

(77)

where \( G(\xi) \) is given by \( (14) \). This condition may be satisfied in either of two ways. Firstly, we may set \( \xi_3 = \xi_2 \), which gives a regular instanton with topology \( S^2 \times R^2 \), whose physical interpretation is unclear \( (15) \). Alternatively, we may set

\[ \left( \frac{\xi_2^2 - \xi_0^2}{\xi_2^2 - \xi_0^2} \right) \left( \frac{\xi_3 - \xi_1}{\xi_2 - \xi_1} \right) = \frac{\xi_4 - \xi_2}{\xi_4 - \xi_3}. \]

(78)

This condition provides a further restriction on the four parameters \( Q, P, M \) and \( A \), which it is useful to think of as determining \( r_+ A \) in terms of \( r_- A \) and \( \Sigma A \). It is difficult to give the explicit solution, as the roots \( \xi_2, \xi_3, \) and \( \xi_4 \) are complicated functions of \( r_+ A \). We can however make some interesting general remarks about the solution.

The left-hand side of \( (78) \) is greater than one, so the right-hand side must be greater than one as well. Since the first factor on the right-hand side is less than one, this requires that the second factor be significantly greater than one. Therefore, the condition will only be satisfied if \( \xi_2 - \xi_1 \) is sufficiently small (\textit{i.e.}, for black holes sufficiently close to extremality).

In the ‘point-particle’ limit \( r_+ A \ll 1 \), \( (78) \) reduces to \( r_+ \approx r_- \). Similarly, in the limit \( Q \to 0 \) (or \( P \to 0 \), but not both), it reduces to \( \xi_2 \approx \xi_1 \). Thus, we can only construct a regular
instanton for the production of extreme black holes in these two limits. This asymptotic behavior is comparable to that of the condition on the wormhole instantons in [15].

The topology of the instanton is $S^2 \times S^2 - \{pt\}$, where the removed point is $x = y = \xi_3$. These instantons are therefore a suitable generalization of the wormhole instantons of [15]. To see that they can be interpreted as a bounce, note that the $\tau = 0, \tau = \pi/2$ section has topology $S^2 \times S^1 - \{pt\}$, which is the topology of a wormhole attached to a spatial slice of the Melvin universe (48). It describes the production of a pair of oppositely charged black holes (5) connected by a wormhole throat, which subsequently accelerate away from each other.

V. CONCLUSIONS

We have seen that it is possible to extend the construction of analogues of the C-metric and Ernst solutions in [15] to the theory with two $U(1)$ gauge fields [13]. The resulting solutions share the property of dual symmetry with the black hole solution of this theory, (1), and the C-metric solution reduces to (5) when the acceleration parameter $A \to 0$. The C-metric solution also reduces to the dilaton C-metric solutions of [15] when one of the gauge charges vanish, although the parameters in (10) are not simply related to the parameters in the dilaton C-metric solution of [14].

The instantons discussed here extend the conclusions of [15] to the case $a = 1$. That is, they describe pair creation of black holes in a pair of background fields. The fact that this was possible in the action with two $U(1)$ gauge fields (2) and not in the previously considered action (1) is related to the thermodynamic properties of the black holes (5). In [13], it was pointed out that the puzzling thermodynamic behavior of dilaton black holes in the extremal limit may be resolved by considering a more general class of black holes, with a dilaton and two $U(1)$ gauge fields. The temperature of these black holes goes smoothly to zero in the extremal limit, so long as both charges are non-vanishing.

Well-behaved instantons do exist for $a = 1$. This leads me to believe that it should be possible to study the question of information loss and related issues in low-energy string theory semiclassically, at least where the temperature is well-defined. In particular, remnants provide a potentially viable solution to the information loss paradox even for $a = 1$. The instantons presented here also seem to avoid the problem of infinite pair production, although it will be necessary to calculate their action explicitly, and give a more careful consideration to quantum perturbations, before this problem can be said to be resolved. These calculations and the relation of these instantons to the extremal instantons of [15] will be the subject of a future paper.

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