Hyperplane arrangements, $M$-tame polynomials and twisted cohomology

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1 Introduction

Let $\mathcal{A} = \{H_1, ..., H_d\}$ be an affine essential hyperplane arrangement in $\mathbb{C}^{n+1}$, see [OT1], [OT2] for general facts on arrangements. We set as usual $M = M(\mathcal{A}) = \mathbb{C}^{n+1} \setminus X$, $X$ being the union of all the hyperplanes in $\mathcal{A}$. One of the main problems now in hyperplane arrangement theory is to study the cohomology of the complement $M$ with coefficients in some local system $L$ on $M$, see for instance the introduction and the references in [CDO] as well as [OT2]. A rank one local system $L$ on $M(\mathcal{A})$ corresponds to a homomorphism

$$\pi_1(M(\mathcal{A})) \sim \mathbb{Z}^d \to \mathbb{C}^*$$

i.e. such a local system $L$ is determined by a collection $\lambda(L) = (\lambda_1(L), ..., \lambda_d(L))$ of $d$ non-zero complex numbers. Here $\lambda_j(L)$ is the monodromy of the local system $L$ about the hyperplane $H_j$. We call the local system $L$ equimonodromical if all these monodromies $\lambda_j(L)$ are the same, i.e. there is $\lambda \in \mathbb{C}^*$ such that $\lambda_j(L) = \lambda$ for all $j = 1, ..., d$. In such a situation we denote the corresponding local system by $L_\lambda$. We assume in the sequel that $\lambda_j(L) \neq 1$ for all $j = 1, ..., d$, the remaining cases being essentially reduced to this one using [Q]. Then, there are unique integers $N > 1$ and $0 < e_j < N$ for $j = 1, ..., d$ such that

$$g.c.d(e_1, ..., e_d) = 1 \quad \text{and} \quad \lambda_j(L) = \exp(2\pi i e_j/N)$$

for all $j = 1, ..., d$. We set $e = (e_1, ..., e_d)$.

For any $i = 1, ..., d$, let $\ell_i = 0$ be an equation for the hyperplane $H_i$ and consider the product

$$f_e = \prod_{i=1,d} \ell_i^{e_i} \in \mathbb{C}[x_0, ..., x_n].$$
Let \( d_e = e_1 + \ldots + e_d \) be the degree of the polynomial \( f_e \). When \( e_j = 1 \) for all \( j \), then we simply write \( f \) for the corresponding product. Note that \( \text{deg}(f) = d \) and \( f = 0 \) is an equation for the union \( X \).

When the arrangement \( A \) is central, i.e. \( 0 \in H_i \) for all \( i = 1, \ldots, d \), the above polynomial \( f \) is homogeneous and there is a lot of interest in the associated Milnor fiber

\[
F = F(A) = f^{-1}(1)
\]

and the corresponding monodromy action \( h^q : H^q(F, \mathbb{C}) \to H^q(F, \mathbb{C}) \) coming from the obvious fibration

\[
F \to M \to \mathbb{C}^*
\]

see for instance [CS]. In particular, it is known that

\[
dim H^q(M, L_\lambda) = \dim \ker (h^q - \lambda Id) + \dim \ker (h^{q-1} - \lambda Id)
\]

see for instance [DN2]. If we denote by \( M^* = M^*(A) \) the quotient \( M(A)/\mathbb{C}^* \subset \mathbb{P}^n \) and if \( \lambda^d = 1 \), then there is an induced equimonodromical local system \( L_\lambda^* \) on \( M^*(A) \) and we have

\[
dim H^q(M^*, L_\lambda^*) = \dim \ker (h^q - \lambda Id)
\]

see [CS]. When the local system \( L \) is not equimonodromical, then one still has an equality

\[
dim H^q(M^*, L) = \dim \ker (h^q - a Id) + \dim \ker (h^{q-1} - a Id)
\]

where \( a = \exp(2\pi i/N) \), \( F_e = f^{-1}_e(1) \) and \( h_e : F_e \to F_e \) is the corresponding monodromy operator, see [DN2].

When the arrangement \( A \) is not central, the usual way to study the cohomology groups \( H^*(M(A), L) \) is to identify \( A \) to a projective arrangement \( A_p \) in \( \mathbb{P}^{n+1} \) by adding the hyperplane at infinity, hence \( |A_p| = |A| + 1 = d + 1 \), and then study the Milnor fibration of the central arrangement \( B = \text{Cone}(A_p) \) in \( \mathbb{C}^{n+2} \) since \( M^*(B) = M(A) \). This approach has at least two disadvantages:

(i) we have to increase dimensions by one, e.g. if we start with a line arrangement \( A \), the Milnor fiber \( F(B) \) is a surface;

(ii) if we are interested in the special class of equimonodromical local systems \( L_\lambda \) and if \( a^{d+1} \neq 1 \), then the local system on \( M^*(B) \) naturally associated to \( L_\lambda \) on \( M(A) \), is no longer equimonodromical.
The purpose of this note is to introduce a new approach to the study of the affine arrangement $\mathcal{A}$, generalizing the central arrangement case and avoiding the above two problems. This approach is based on the study of the topology of the function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and of its monodromy representation, using the tools developed over the years by many authors, see for instance [B], [NZ1], [NZ2], [PZ] and the new progress on Alexander invariants in [DN2].

2 Affine arrangements and $M$-tame polynomials

First we recall the notion of an $M$-tame polynomial introduced in [NZ1] and later studied in [NZ2], [NS]. For any polynomial $g \in \mathbb{C}[x_0, \ldots, x_n]$ consider the set

$$M(g) = \{x \in \mathbb{C}^{n+1} | \text{grad}(g)(x) = c\mathfrak{x} \text{ for some } c \in \mathbb{C}\}$$

where $\text{grad}(g)(x) = (g_0(x), \ldots, g_n(x))$, with $g_k$ the partial derivative of $g$ with respect to $x_k$ and $\mathfrak{x}$ is the complex conjugate of $x$.

**Definition 2.1** We say that the polynomial $g$ is $M$-tame if for any sequence $\{z_k\} \subset M(g)$ with $\lim|z_k| = +\infty$ we have $\lim|g(z_k)| = +\infty$.

It is clear that an $M$-tame polynomial can have only isolated singularities (see also the proof of Corollary 2.2 below). Therefore our polynomial $f$ associated to an affine arrangement cannot be $M$-tame as soon as $n > 1$ (except very special cases). Our first result says that this is not a major drawback.

**Theorem 2.1** Let $\mathcal{A} = \{H_1, \ldots, H_d\}$ be an affine essential hyperplane arrangement in $\mathbb{C}^{n+1}$ given by the equation $f = 0$. Then the following hold.

(i) For $n = 1$ the polynomial $f$ is $M$-tame.

(ii) For $n > 1$ as well as for $n = 1$ and $d_e > d$, the polynomial $f_e$ is $M_0$-tame in the following sense: for any sequence $\{z^k\} \subset M(f_e) \setminus X$ with $\lim|z^k| = +\infty$ we have $\lim|f_e(z^k)| = +\infty$.

**Proof.**

The proof of the first claim is easily reduced to the second and we leave it to the reader. The fact that for $n = 1$ the polynomial $f$ has a good behaviour at infinity also follows from our discussion in the next section.

The proof of the second claim above is an improved version of the proof of Lemma 4 in [B]. Assume that there is a sequence of points $\{z^k\} \subset M(f_e) \setminus X$ with $\lim|z^k| = +\infty$ and $\lim|f_e(z^k)| \neq +\infty$. Then, by passing to a subsequence, we can assume that $\lim f_e(z^k) = b \in \mathbb{C}$. 

Since the arrangement $\mathcal{A}$ is essential, the set of indices $j$ such that $\lim |\ell_j(z^k)| = +\infty$ is not empty. By renumbering the hyperplanes, we can assume that $\lim |\ell_m(z^k)| = 0$ exactly for $1 \leq m \leq q$ with $q \geq 1$ (this set is non-empty since $\lim f_m(z^k) = b$). We set

$$g = \prod_{m \leq q} \ell_m^{e_m} \quad \text{and} \quad h = \prod_{m > q} \ell_m^{e_m}.$$

By a linear unitary change of coordinates we can assume that $H_1 \cap \ldots \cap H_q : x_0 = \ldots = x_p = 0$ with $p \leq q - 1$. (The unitary requirement is essential, since the condition of $M_0$-tame is a condition of transversality of the fibers of $f_e$ with respect to large spheres centered at the origin, and such spheres being invariant by unitary transformations, it follows that the condition $M_0$-tame is also invariant.) Then $\ell_m$ for $1 \leq m \leq q$ is a linear combination of $x_0, \ldots, x_p$ and $g$ is a homogeneous polynomial of degree $e_i = e_1 + \ldots + e_q$ in $\mathbb{C}[x_0, \ldots, x_p]$. Now write $z^k = (z^k_0, \ldots, z^k_n)$ in the above fixed coordinate system and hence $z^k_m \to 0$ for $1 \leq m \leq p$. There is an integer $K > p$ such that $|z^k_K| \to +\infty$.

Consider the obvious equality

$$\frac{\text{grad}(f_e)}{f_e} = \frac{\text{grad}(g)}{g} + \frac{\text{grad}(h)}{h}.$$ 

By passing to a subsequence if necessary, we can assume that $\ell_j(z^k)$ is bounded away from 0 for $j > q$. It follows that $\frac{\text{grad}(h)}{h}$ is bounded on the sequence $z^k$. This implies that for $i > p$, $\frac{f_{e,i}}{f_e} = \frac{h_i}{h}$ is bounded on the sequence $z^k$.

Consider now the equality

$$\sum_{i=0,p} f_{e,i}(z^k) \frac{z_i^k}{f_e(z^k)} = \sum_{i=0,p} g_i(z^k) \frac{z_i^k}{g(z^k)} + \sum_{i=0,p} h_i(z^k) \frac{z_i^k}{h(z^k)}.$$

By Euler formula, the first term in the right hand side is equal to $q_e > 0$, while by the above discussion the second term tends to zero. It follows that there is an integer $L \leq p$ such that $\frac{f_{e,L}(z^k)}{f_e(z^k)} \to +\infty$. Since $z^k \in M(f) \setminus X$ we have

$$|z^k_K| \frac{f_{e,L}(z^k)}{f_e(z^k)} = |z^k_L| \frac{f_{e,K}(z^k)}{f_e(z^k)}.$$

This leads to a contradiction, as the left hand side goes to infinity while the right hand side goes to zero, by the definition of $K$ and $L$. 

This result has the following corollaries, saying that essentially \( f_e \) behaves like an \( M \)-tame polynomial. In fact, only the high connectivity of the general fiber \( F_e \) of \( f_e \) is lost. On the other hand, the defining condition on the multi-index \( e \) implies that this general fiber \( F_e \) is connected, see [DP1], Remark (I).

**Corollary 2.1** For any \( t \in \mathbb{C} \) the inclusion of the fiber \( F_t = f_e^{-1}(t) \) into the corresponding tube \( T_t = f_e^{-1}(D_t) \), with \( D_t \) a small disc in \( \mathbb{C} \) centered at \( t \), is a homotopy equivalence. In particular, both \( X = F_0 \) and \( T_0 \) have the homotopy type of a bouquet of \( n \)-dimensional spheres.

**Proof.** The condition of \( M \)-tame says that the fibers of \( f_e \) are transversal to large enough spheres in \( \mathbb{C}^{n+1} \) centered at the origin. The weaker condition \( M_0 \) says the same thing, if we interpret transversality to the special fiber \( X = F_0 \) in the stratified sense. So the retractions from \( T_t \) to \( F_t \) obtained in the \( M \)-tame case in [NZ1], [NZ2] by integrating vector fields exist in our case as well. The fact that \( X \) has the homotopy type of a bouquet of \( n \)-dimensional spheres is well known, see for instance [DP].

Let \( \mu(\mathcal{A}) \) denote the number of spheres in the above bouquet. This number is determined by the following result, see [Da] for a different approach.

**Corollary 2.2** The function \( f_e : M(\mathcal{A}) \to \mathbb{C} \) induced by the polynomial \( f_e \) has only isolated singularities and

\[
\sum_{x \in M(\mathcal{A})} \mu(f_e, x) = \mu(\mathcal{A}) = (-1)^{n+1} \chi(M(\mathcal{A})).
\]

**Proof.**

If \( f_e | M(\mathcal{A}) \) would have non-isolated singularities, then we can find \( t \in \mathbb{C}^* \) and an irreducible affine algebraic variety \( Y \subset F_t \cap \text{Sing}(f_e) \) with \( \text{dim } Y > 0 \). Any sequence of points in \( Y \) tending to infinity would then contradict the \( M_0 \)-tameness of \( f_e \).

To complete the proof, we can use the standard trick used already by Broughton in [B] and deduce that \( \mathbb{C}^{n+1} \) can be obtained from \( T_0 \) by adding \( (n+1) \)-cells in number equal to the above sum. Then we have just to use the obvious equalities \( \chi(X) = 1 + (-1)^n \mu(\mathcal{A}) \) and \( \chi(M(\mathcal{A})) = 1 - \chi(X) \).

The following result explains the interest of this point of view for the computation of the twisted cohomology of the complement \( M(\mathcal{A}) \) with values in a rank one local system \( L \). For basic facts on the monodromy at infinity of polynomials we refer to [DNT].
Theorem 2.2  (i) For any integer $k$ such that $0 < k < n$, the restriction of the constructible sheaf $R^k f_\ast \mathbb{Q}$ to $\mathbb{C}^*$ is a local system corresponding to the monodromy operator

$$M^k_e : H^k(F_e, \mathbb{Q}) \rightarrow H^k(F_e, \mathbb{Q}).$$

Here $F_e$ is the general fiber of the polynomial $f_e$ and $M^k_e$ can be taken to be either the monodromy about the fiber $F_0 = X$ or, equivalently, the monodromy at infinity of the polynomial $f_e$.

(ii) Let $F_e$ be the $\mathbb{Z}$-cyclic covering of $M(A)$ corresponding to the kernel of the morphism $f_{\ast} : \pi_1(M(A)) \rightarrow \pi_1(\mathbb{C}^*)$ and consider $H_\ast(F_e, \mathbb{Q})$ as a $\mathbb{Q}[t, t^{-1}]$-module in the usual way. Then there is an epimorphism of $\mathbb{Q}[t, t^{-1}]$-modules

$$H_\ast(F_e, \mathbb{Q}) \rightarrow H_\ast(F_e, \mathbb{Q})$$

where in the first module the multiplication by $t$ is either the monodromy about the fiber $F_0 = X$ or the monodromy at infinity of the polynomial $f$.

Proof.  The first claim follows from the fact that the isolated singularities of $f|_{M(A)}$ produce no changes in the topology of the fibers in dimensions $< n$. In particular, the two monodromy operators in the claim (i) above coincide.

Using the above construction of $M(A)$ starting from a punctured tube about $X = F_0$ (which can also be done starting from a punctured tube about the infinity, i.e. $f^{-1}(\mathbb{C}\setminus D_R)$, where $D_R$ is a disc in $\mathbb{C}$ of radius $R > 0$ centered at the origin), the proof is similar to the proofs in [DN2]. Easy examples in the case $n = 1$ (to be treated in detail in the next two sections) shows that the two monodromy operators in the claim (ii) above do not coincide in general.

Corollary 2.3  (i) For any integer $k$ such that $0 < k < n$, one has

$$\dim H^k(M(A), L) = N(k, a) + N(k - 1, a)$$

where $N(k, a) = \dim \ker (M^k_e - a\text{Id})$ and $a = \exp(2\pi i/N)$.

(ii) $\dim H^n(M(A), L) \leq N(n, a) + N(n - 1, a)$ and $\dim H^{n+1}(M(A), L) \leq N(n, a)$.

(iii) Both claims (i) and (ii) above hold for the trivial local system $\mathbb{C}_M$ by taking $a = 1$.

Proof.  This claim follows from the fact that $M(A)$ is obtained, exactly as in the proof above, from the punctured tube $T_0^* = T_0 \setminus X$ by attaching $(n + 1)$-cells, see also [DN2]. It follows that the inclusion $T_0^* \rightarrow M(A)$ induces an isomorphism

$$H^k(M(A), L) \simeq H^k(T_0^*, L)$$
for $0 < k < n$, and hence the result is obtained exactly as the corresponding result for central arrangements mentioned in the Introduction. For $k = n$ the inequality comes from the epimorphism in Theorem 2.2, (ii). The last claim is obvious from the previous discussion. □

Remark 2.1 (i) The $\mathbb{Q}[t, t^{-1}]$-modules $H_m(F, \mathbb{Q})$ are exactly the Alexander invariants of the hypersurface $X$ as discussed in [L], [D2], [DN2] and, in the case $n = 1$, in [K].

(ii) The $M_0$-tame polynomials have better topological properties than the semitame polynomials considered for instance in [PZ]. In particular, for an $M_0$-tame polynomial the monodromy at infinity can be realized as the monodromy à la Milnor, i.e. the total space can be chosen to be the complement of $X$ in a very large sphere in $\mathbb{C}^{n+1}$ centered at the origin as in the case of $M$-tame polynomials, see [NZ2].

(iii) It is not clear whether the monodromy operators $M^k_e : H^k(F_e) \to H^k(F)$ for $0 < k < n$ are semisimple. For $k = 1$, this is the case for the eigenvalue $\lambda = 1$, see [DS]. In the next section we also show that multiplication by $t$ on $H^1(F, \mathbb{C})$ is semisimple when $n = 1$.

The following result describes a way to compute the zeta-function

$$Z(f_e, 0)(t) = \prod_m (\det(Id - tM^m_e))^{-1}$$

of the monodromy operator $M_{e,0}$ of the polynomial $f_e$ about the fiber $X = F_0 = f_e^{-1}(0)$.

**Theorem 2.3** The direct image functor $Rf_*$ commutes on the constant sheaf $\mathbb{C}$ to the vanishing cycle functor $\varphi_f$. In particular

$$Z(f_e, 0) = \prod_{S \in S} Z(f_e, x_S)^{\chi(S)}$$

where $S$ is a constructible regular stratification of $X$ with connected strata such that all the cohomology sheaves $H^m(\varphi_f \mathbb{C})$ are locally constant along the strata of $S$, $x_S$ is an arbitrary point in the stratum $S$ and $Z(f_e, x_S)$ is the local zeta-function of the function germ $(f_e, x_S)$.

**Proof.**

Exactly as in the case of an $M$-tame polynomial treated in [NS], the direct image functor $Rf_*$ commutes on the constant sheaf $\mathbb{C}$ to the vanishing cycle functor $\varphi_f$. The formula for the zeta-function is similar to the one in the proper case obtained in [GLM] and is treated in detail for the case of tame polynomials in [D4]. □
Note that the above commutativity still holds when we replace the functor \( \varphi_f \) by the subfunctor \( \varphi_{f,\lambda} \) which takes only the vanishing cycles corresponding to a fixed eigenvalue \( \lambda \). In particular \( \varphi_{f,\lambda} \mathbb{C} = 0 \) implies \( N(k, \lambda) = 0 \) for all \( k \). This is an effective way to get vanishing (or upper bound) results for the cohomology groups \( H^*(M(\mathcal{A}), L_\varphi) \), compare to [CDO], Corollary 16. In particular, this remark combined with Corollary 2.3 yields the following.

**Corollary 2.4** If \( X \) is a normal crossing divisor and \( \lambda_j(L) \neq 1 \) for all \( j = 1, ..., d \), then \( H^q(M(\mathcal{A}), L) = 0 \) for all \( q < n \).

### 3 Line arrangements (equimonodromical case)

In this section we assume that \( \mathcal{A} \) is an essential line arrangement in the plane \( \mathbb{C}^2 \). Let \( n_k \) be the number of \( k \)-fold intersection points in \( X \). The following formulas are easy to deduce.

\[
\chi(M(\mathcal{A})) = \mu(\mathcal{A}) = 1 - d + \sum_{m \geq 2} n_m (m - 1).
\]

\[
b_1(F) = 1 - d + \sum_{m \geq 2} n_m (m - 1)m.
\]

Indeed, the first formula follows from Corollary 2.2 and the additivity of Euler characteristic with respect to constructible partitions. The second equality comes from the relation

\[
b_1(F) = \sum_{x \in \mathbb{C}^2} \mu(f, x) = \chi(M(\mathcal{A})) + \sum_{x \in X} \mu(f, x).
\]

Assume that the \( d \) lines in \( \mathcal{A} \) have \( p \) distinct directions and let \( k_j \) be the number of lines having the \( j \)-th direction. A standard computation shows that the genus (of a smooth projective model) of the general fiber \( F \) of the defining polynomial \( f \) is given by

\[
g = \text{genus}(F) = \frac{(d - 1)(d - 2)}{2} - \sum_{j=1,p} k_j (k_j - 1) \frac{2}{2}.
\]

One can determine the resolution graph of \( f \) as defined in [ACD] in a simple way. In fact \( X \) intersects the line at infinity \( L_\infty \) in exactly \( p \) points, say \( A_1, ..., A_p \) (corresponding to the \( p \) distinct directions of lines in \( X \)). Each of these points has to be blown-up, creating thus an exceptional curve \( E_j \). The proper transform of \( X \) cuts each \( E_j \) in exactly \( k_j \) points, and
each of them has to be blown-up several times to arrive at a dicritic of degree one. Hence
the total number of dicritics is
\[ \delta(f) = \sum_{j=1,p} k_j = d. \]

This gives the following.

**Corollary 3.1** Let \( n(F_t) \) denote the number of irreducible components of the fiber \( F_t \). Then Kaliman’s inequality
\[ \delta(f) - 1 \geq \sum_t (n(F_t) - 1) \]
is in our situation an equality. In particular, all the fibers \( F_t \) for \( t \neq 0 \) are irreducible.

It was known that this inequality is an equality when the general fiber \( F \) is a rational
curve (i.e. \( g = 0 \)), see [Ka], [ACD], but here we are not in this case in general, as can easily be verified using the above formula for the genus \( g \). One also has \( \dim \ker (M_{\infty}^1 - \text{Id}) = \delta(f) - 1 \)
for any polynomial \( f : \mathbb{C}^2 \rightarrow \mathbb{C} \), see [D3]. Therefore the equality \( \delta(f) = d \) implies that
\[ \dim \ker (M_{\infty}^1 - \text{Id}) = b_1(M(\mathcal{A})) - 1. \]

By Corollary 2.3 (iii), we get the same equality when \( n > 1 \).

The multiplicity of \( f \) along the line at infinity \( L_\infty \) is \( d \), along the exceptional curve \( E_j \) is \( d - k_j \) and then decreases to one for each exceptional curve just before a dicritic. Applying A’Campo’s formula for the zeta-function as in [ACD] gives the following formula for the characteristic polynomial of the monodromy at infinity acting on \( H^1(F, \mathbb{C}) \).
\[ \Delta_\infty(t) = (t-1)(t^d - 1)^{p-2} \prod_{j=1,p} (t^{d-k_j} - 1)^{k_j-1}. \]

Comparing the degree of this polynomial to the previous formula for \( b_1(F) \) we get the following relation among the numerical data associated to the line arrangement \( \mathcal{A} \).

**Corollary 3.2**
\[ 1 - d + \sum_{m \geq 2} n_m(m-1)m = (d - 1)^2 - \sum_{j=1,p} k_j(k_j - 1). \]
It is also easy to compute the characteristic polynomial of the monodromy at zero acting on $H^1(F)$. The result is the following.

$$\Delta_0(t) = (t - 1)^{\mu(A)} \prod_{m \geq 2} [(t - 1)(t^m - 1)^{m-2}]^{n_m}.$$ 

Moreover, in this case the multiplication by $t$ on $H_1(F, C)$ is semisimple. Indeed, using Theorem 2.2 we see that the multiplication by $t$ cannot have larger Jordan blocks for the eigenvalue $\lambda = 1$ since this is the case for the monodromy at infinity, see [D3] and, more generally, [DS]. But the multiplication by $t$ cannot have larger Jordan blocks for the eigenvalue $\lambda \neq 1$ since this is the case for the monodromy at zero, all the singularities on $X$ being weighted homogeneous. This proves the final claim in Remark 2.1 (iii).

Let $\Delta_f$ be the greatest common divisor of the polynomials $\Delta_0$ and $\Delta_{\infty}$. Let $N_f(\lambda)$, $N_0(\lambda)$ and respectively $N_{\infty}(\lambda)$ be the multiplicity of $\lambda$ as a root of the polynomial $\Delta_f$, $\Delta_0$ and respectively $\Delta_{\infty}$. The following result can be proved exactly as Corollary 2.3.

**Corollary 3.3** For any $\lambda \in \mathbb{C}^*$, $\lambda \neq 1$, we have

$$\dim H^1(M(A), L_\lambda) \leq N_f(\lambda) = \min(N_0(\lambda), N_{\infty}(\lambda)).$$

It is interesting to compare this upper-bound to the upper-bound obtained in [CDO], Theorem 13. Since this latter result applies to equimonodromical rank one local systems on complements of projective line arrangements in $\mathbb{P}^2$, we have to assume that $\lambda^{d+1} = 1$ such that the local system $L_\lambda$ is a equimonodromical local system on the arrangement complement $M(A_p)$ as explained in the Introduction. Under this assumption, it follows that

$$N_{\infty}(\lambda) = \sum_j (k_j - 1)$$

where the sum is over all $j$ such that $\lambda^{k_j+1} = 1$. Since $k_j + 1$ is exactly the multiplicity of the corresponding projective arrangement $A_p$ at the point $A_j$, it follows that $N_{\infty}(\lambda)$ is exactly the upper-bound obtained in [CDO], Theorem 13 for the arrangement $A_p$ and the line at infinity $L_{\infty}$ as a chosen hyperplane.

On the other hand, it is easy to see that

$$N_0(\lambda) = \sum_m n_m (m - 2)$$

where the sum is over all $m \geq 2$ such that $\lambda^m = 1$. The interested reader will have no problem to find explicit examples of line arrangements showing that both inequalities $N_{\infty}(\lambda) > N_0(\lambda)$
and \( N_0(\lambda) > N_\infty(\lambda) \) are possible. Hence in some cases, the last corollary above gives better upper-bounds than Theorem 13 in [CDO] (for any choice of the line at infinity!). One such example (not very interesting) is \( f = xy(x+1)(y+1)(x+y+10)(x+y+11)(x-y+100)(x-y+101) \) and \( \lambda \) a cubic root of unity. Here any line in the associated projective arrangement contains at least a triple point (and hence \( N_\infty(\lambda) \geq 1 \) for any choice of the line at infinity), but \( X \) has only normal crossings and hence \( N_0(\lambda) = 0 \).

4 Line arrangements (general case)

In this section we continue to use the notation from the previous section, in particular \( X \cap L_\infty = \{A_1, ..., A_p\} \). These \( p \) line directions induce a partition \((I_1, ..., I_p)\) of the set of indices \( \{1, ..., d\} \) such that \( i \in I_j \) if and only if \( H_i \cap L_\infty = A_j \). Let \( C_t = \overline{F_t} \) be the closure in \( \mathbb{P}^2 \) of the fiber \( F_t = f_{-t}(t) \). Then \( C_t \) has exactly \( p \) singularities along the line at infinity (namely at the points \( \{A_1, ..., A_p\} \)), and an easy computation using the additivity of Milnor numbers under a blow-up, see [D1], Proposition (10.27) shows that

\[
\mu(C_t, A_j) = d_e(d_j - k_j) + d_j(k_j - 2) + 1.
\]

Here \( d_j = \sum_{i \in I_j} e_i \) and \( k_j = |I_j| \). This formula implies in the usual way the following equality

\[
b_1(F_e) = 1 + d_e(d - 1) - \sum_{j=1,p} d_j k_j.
\]

One surprising consequence of this formula when compared to Corrolary 2.2 is that for a fixed arrangement \( A \) we have \( \sup_e b_1(F_e) = \infty \), i.e. the topology of the general fiber \( F_e \) becomes more and more complicated as the multiplicities \( e \) increase.

Similar considerations as in the previous section shows that \( \delta(f_e) = d \), hence the Kaliman’s inequality is an equality in this case as well and all the fibers \( F_t = f_{-t}(t) \) are irreducible for \( t \neq 0 \). Moreover, we get the following formula for the characteristic polynomial of the monodromy operator \( M_{1,e,\infty}^1 \) at infinity of the polynomial \( f_e \).

\[
\Delta_{e,\infty}(t) = (t - 1)(t^{de} - 1)^{p-2} \prod_{j=1,p} (t^{de-d_j} - 1)^{k_j-1}.
\]

Moreover, Theorem 2.3 can be applied in this situation and yields the following formula for the characteristic polynomial of the monodromy operator \( M_{1,e,0}^1 \) about the fiber \( F_0 = X \) of the polynomial \( f_e \).

\[
\Delta_{e,0}(t) = (t - 1) \prod_{\text{lines}} (t^{e_j} - 1)^{-\chi(H_j^0)} \prod_{\text{vertices}} (t^{d(I_v)} - 1)^{|I_v|^2}
\]
where the first product is over all the lines $H_j$ and $H_j^0 = H_j \setminus \cup_{i \neq j} H_i$, and the second product is over all the vertices $v$, $I_v$ denotes the set of $m$ such that $v \in H_m$ and $d(J) = \sum_{m \in J} e_m$.

Let us investigate the multiplicity of a root $a = \exp(2\pi i / N)$ in these two polynomials $\Delta_{e,\infty}$ and $\Delta_{e,0}$ under the assumption that $\lambda_j(L) = a^{e_j} \neq 1$ for any $j$. Using the above formula for $\Delta_{e,0}$ it is easy to see that this multiplicity is

$$N_0(a) = \text{mult}(a, \Delta_{e,0}) = \sum_{\text{vertices}} (|I_v| - 2)$$

where the sum is over all vertices $v$ in $\mathbb{C}^2$ such that $\prod_{j \in I_v} \lambda_j(L) = 1$. In a similar way

$$N_{\infty}(a) = \text{mult}(a, \Delta_{e,\infty}) = \sum_{\text{vertices}} (|I_v| - 2)$$

where the sum is over all vertices $v \in L_{\infty}$ of the corresponding projective arrangement $A_p$ in $\mathbb{P}^2$ such that $\prod_{j \in I_v} \lambda_j(L_p) = 1$, $L_p$ being the local system $L$ regarded as a local system on $M(A_p)$. Then we have the following result.

**Corollary 4.1** With the above notation, for any rank one local system $L$ on $M(A)$ such that $\lambda_j(L) \neq 1$ for all $j$ one has

$$\dim H^1(M(A), L) \leq \min(N_0(a), N_{\infty}(a)).$$

The upper-bound on $\dim H^1(M(A), L)$ obtained from $N_{\infty}(a)$ can be considered as a generalization of Theorem 13 in [CDO], which applies only to equimonodromical local systems.

On the other hand, it is easy to give a sheaf theoretic proof of the above Corollary. Indeed, the setting in the proof of Theorem 13 in [CDO] gives by a slight modification the upper-bound obtained from $N_{\infty}(a)$. To get the upper-bound $N_0(a)$, it is enough to play the same game of comparing the direct image $Rj_*L$ with the direct image with compact supports $Rj_!L$ as in [CDO], but replacing the affine space $\mathbb{C}^2$ by a large closed ball $B$ centered at the origin of $\mathbb{C}^2$ and taking $j$ to be the inclusion $M(A) \cap B \to B$. Indeed, it is known that the inclusion $M(A) \cap B \to M(A)$ is a homotopy equivalence, see for instance [D2] p. 26 and hence $H^1(M(A), L) \simeq H^1(M(A) \cap B, L)$. Further details will be given elsewhere.

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