Renormalization group theory of generalized multi-vertex sine-Gordon model

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We investigate the renormalization group theory of generalized multi-vertex sine-Gordon model by employing the dimensional regularization method and also the Wilson renormalization group method. The vertex interaction is given by \( \cos(k_j \cdot \phi) \) where \( k_j \) (\( j = 1, 2, \ldots, M \)) are momentum vectors and \( \phi \) is an \( N \)-component scalar field. The beta functions are calculated for the sine-Gordon model with multi cosine interactions. The second-order correction in the renormalization procedure is given by the two-point scattering amplitude for tachyon scattering. We show that new vertex interaction with momentum vector \( k_\ell \) is generated from two vertex interactions with vectors \( k_i \) and \( k_j \) when \( k_i \) and \( k_j \) meet the condition \( k_\ell = k_i \pm k_j \) called the triangle condition. Further condition \( k_i \cdot k_j = \pm 1/2 \) is required within the dimensional regularization method. The renormalization group equations form a set of closed equations when \( \{ k_j \} \) form an equilateral triangle for \( N = 2 \) or a regular tetrahedron for \( N = 3 \). The Wilsonian renormalization group method gives qualitatively the same result for beta functions.

I. INTRODUCTION

The sine-Gordon model is an interesting universal model that appears as an effective model in various fields of physics\[1,14\]. The two-dimensional (2D) sine-Gordon model can be mapped to the Coulomb gas model that has logarithmic Coulomb interaction\[17,18\]. The 2D sine-Gordon model has been investigated by several methods, especially by using the renormalization group method. The physics of the sine-Gordon model is closely related to that of the Kosterlitz-Thouless transition of the 2D XY model\[19,20\].

The sine-Gordon model is the model of scalar field under the periodic potential. This model can be generalized in several ways. The massive chiral model is regarded as a generalization of the sine-Gordon model where the potential term \( \text{Tr}(g + g^{-1}) \) is considered for \( g \) in a gauge group (Lie group) \( G \) (\( g \in G \))\[21,23\]. The chiral model was generalized to include the Wess-Zumino term as the Wess-Zumino-Witten model\[24–27\]. The other way of generalization is to include the potential terms of high frequency modes\[28\]. A generalized potential term is given as

\[
V = \frac{1}{L} \sum_{n=1}^{L} \alpha_n \cos(n\phi), \tag{1}
\]

where \( \phi \) is a one-component scalar field and \( L \) is an integer. In the Wilson renormalization group method, the cosine potential \( \cos((n - m)\phi) \) is generated from \( \cos(n\phi) \) and \( \cos(m\phi) \) in the second order perturbation. Thus there will be the correction to the beta function of \( \alpha_n \) in the form \( \alpha_n \alpha_m \) with \( n = |\ell - m| \). For the hyperbolic sine-Gordon model, \( \alpha_n \) has a correction from \( \alpha_\ell \) and \( \alpha_m \) satisfying \( n = \ell + m \)[29].

This kind of model can be generalized to a multi-component scalar field. In this paper we investigate a generalized multi-component sine-Gordon model with multiple cosine potentials. The cosine vertex interaction is given by \( \cos(\sum_\ell k_\ell \phi_\ell) \) where \( \phi = (\phi_1, \ldots, \phi_N) \) is a scalar field and \( k_j = (k_{j1}, \ldots, k_{jN}) \) (\( j = 1, \ldots, M \)) are momentum vectors of real numbers. \( k_j \) represents the direction of oscillation of field \( \phi \). The model for \( M = 3 \) was considered in \[30\]. The condition to generate a new vertex interaction shown above is generalized to \( k_n = k_\ell \pm k_m \). This is called the triangle condition in this paper.

It was pointed out that there is a close relation between the sine-Gordon model and string propagation in a tachyon background\[31\]. In fact, two-vertex correction in the renormalization procedure is given by the two-point scattering amplitude for tachyon scattering in the second order perturbation theory. The multi-vertex correction will be given by the multi-point tachyon scattering amplitude.

This paper is organized as follows. In section III we present the generalized sine-Gordon model. We show the renormalization procedure based on the dimensional regularization method in section IV. We applied the Wilsonian renormalization group method to our model in section V. We consider the generalized multi-vertex sine-Gordon model and calculate the beta functions in section VI. We give summary in the last section.

II. MULTI-VERTEX SINE-GORDON MODEL

We consider a \( N \)-component real scalar field \( \phi = (\phi_1, \ldots, \phi_N) \). The model is a \( d \)-dimensional generalized multi-vertex sine-Gordon model given by

\[
\mathcal{L} = \frac{1}{2t_0} (\partial_\mu \phi)^2 + \frac{1}{t_0} \sum_{j=1}^{M} \alpha_{0j} \cos(k_j \cdot \phi), \tag{2}
\]

where \( t_0(>0) \) and \( \alpha_{0j} \) (\( j = 1, \ldots, M \)) are bare coupling constants and \( k_j \) (\( j = 1, \ldots, M \)) are \( N \)-component constant vectors. We use the notation \( (\partial_\mu \phi)^2 = \sum_\ell (\partial_\mu \phi_\ell)^2 \) and \( k_j \cdot \phi = \sum_\ell k_{j\ell} \phi_\ell \) for \( k_j = (k_{j1}, \ldots, k_{jN}) \). We use the Euclidean notation in this paper. The second term is the potential energy with multi cosine interactions. The
dimensions of $t_0$ and $\alpha_0$ are given as $[t_0] = \mu^{-d}$ and $[\alpha_0] = \mu^2$ for the energy scale parameter $\mu$. The analysis is performed near two dimensions $d = 2$. We introduce the renormalized coupling constants $t$ and $\alpha_j$ where the renormalization constants are defined as

$$t_0 = t \mu^{2-d} Z_t, \quad \alpha_j = \alpha_j \mu^2 Z_{\alpha_j},$$  

(3)

where we set that $t$ and $\alpha_j$ are dimensionless constants. The renormalized field $\phi_R$ is introduced with the renormalization constant $Z_\phi$ as follows

$$\phi = \sqrt{Z_\phi} \phi_R.$$  

(4)

In the following $\phi$ denotes the renormalized field $\phi_R$ for simplicity. Then the Lagrangian density is given as

$$\mathcal{L} = \frac{\mu^{d-2} Z_\phi}{2t Z_t} (\partial_\mu \phi)^2 + \frac{\mu^d}{t Z_t} \sum_j Z_{\alpha_j} \alpha_j \cos(k_j \cdot \phi).$$  

(5)

We examine the renormalization group procedure for this model in section III and section IV. We also investigate the component dependence of renormalization in section V by generalizing the model as follows.

$$\mathcal{L} = \sum_j \frac{\mu^{d-2} Z_\phi}{2t_j Z_{\phi_j}} (\partial_\mu \phi_j)^2 + \sum_j \frac{\mu^d \alpha_j Z_{\alpha_j}}{t_j Z_t} \cos\left(\sqrt{Z_\phi} k_j \cdot \phi\right).$$  

(6)

We need some conditions so that we have one fixed point for $t$. For this purpose we normalize $k$ vectors as

$$k_j^2 = \sum_{\ell=1}^N k_{j\ell}^2 = 1 \quad (j = 1, \cdots, M).$$  

(7)

From the two vertices with momentum vectors $k_i$ and $k_j$, new vertex is generated with momentum $k_m$ when the triangle condition is satisfied:

$$k_m = k_i \pm k_j.$$  

(8)

We assume that a set $\{\alpha_j\}$ includes all vertices that will be generated from multi-vertex interactions each other. For a triangle or a regular polyhedron which is composed of equilateral triangles, $M$ becomes finite since $\{k_j\}$ form a finite set. For example, we consider an equilateral triangle or a regular tetrahedron. For an equilateral triangle ($N = 2$, $M = 3$) or a regular tetrahedron ($N = 3$, $M = 6$), we have

$$\sum_{j=1}^M k_{j\ell}^2 = C(M) \quad \text{for } \ell = 1, \cdots, N,$$

(9)

where $C(M)$ is a constant depending upon $M$. These conditions will be explained in the following sections.

III. RENORMALIZATION BY DIMENSIONAL REGULARIZATION

We evaluate the beta functions for the multi-vertex sine-Gordon model by using the dimensional regularization method. The lowest order contributions to the renormalization of $\alpha_j$ are given by tadpole diagrams. Using the expansion

$$\cos \phi = 1 - \frac{1}{2} \phi^2 + \frac{1}{24} \phi^4 + \cdots,$$

the cosine potential is renormalized as

$$\cos(\sqrt{Z_\phi} k_j \cdot \phi) \rightarrow \left(1 - \frac{1}{2} Z_\phi (k_j \cdot \phi)^2 + \cdots\right) \cos(\sqrt{Z_\phi} k_j \cdot \phi).$$  

(10)

$\langle \phi^2 \rangle$ is regularized as

$$\langle \phi^2 \rangle = \frac{\langle \phi^2 \rangle}{\langle \phi^2 \rangle} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m_0^2} = \frac{\mu^{2-d} Z_t}{Z_\phi} \frac{\Omega_d}{(2\pi)^d}$$  

(11)

d for $d = 2 + \epsilon$ where a mass $m_0$ is introduced to avoid the infrared divergence. We set $Z_t = 1$ in the lowest order of $t$. We adopt that $\langle \phi_i \phi_j \rangle = \delta_{ij} \langle \phi_i^2 \rangle$ and $\langle \phi_i^2 \rangle$ is independent of $\ell$. Then the renormalization of the potential term is given as

$$\alpha_j Z_{\alpha_j} \left(1 - \frac{1}{2} Z_\phi k_j^2 \langle \phi^2 \rangle + \cdots\right) \cos(\sqrt{Z_\phi} k_j \cdot \phi) = \alpha_j Z_{\alpha_j} \left(1 + \frac{1}{2} k_j^2 \mu^{2-d} \Omega_d (2\pi)^d + \cdots\right) \cdot \cos(\sqrt{Z_\phi} k_j \cdot \phi).$$  

(12)

The renormalization constant $Z_{\alpha_j}$ is determined as

$$Z_{\alpha_j} = 1 - \frac{t}{4\pi} k_j^2.$$  

(13)

Since the bare coupling constant $\alpha_0 = \alpha_j \mu^2 Z_{\alpha_j}$ is independent of $\mu$, we have

$$\beta(\alpha_j) = \frac{\partial \alpha_j}{\partial \mu} = -2\alpha_j + \frac{1}{4\pi} k_j^2 t \alpha_j.$$  

(14)

The beta function of $\alpha_j$ has a zero at

$$t = t_{\alpha_j} = 8\pi/k_j^2 = 8\pi,$$

(15)

since $k_j^2 = 1$.

B. Vertex-vertex interaction

We investigate the corrections to $t$ and $\alpha_j$ from vertex-vertex interactions. The second-order contribution $I^{(2)}$
to the action is given by
\[
I^{(2)} = -\frac{1}{2} \left( \frac{\mu^d}{tZ_t} \right)^2 \int d^d x d^d x' \\
\times \sum_{ij} \alpha_i \alpha_j Z_{\alpha_i} Z_{\alpha_j} \cos \left( \sqrt{Z_t} k_i \cdot \phi(x) \right) \\
\times \cos \left( \sqrt{Z_t} k_j \cdot \phi(x') \right)
\]
\[
= -\frac{1}{4} \left( \frac{\mu^d}{tZ_t} \right)^2 \int d^d x d^d x' \\
\times \sum_{ij} \alpha_i \alpha_j Z_{\alpha_i} Z_{\alpha_j} \left[ \cos \left( \sqrt{Z_t} (k_i \cdot \phi(x) - k_j \cdot \phi(x')) \right) \\
+ \cos \left( \sqrt{Z_t} (k_i \cdot \phi(x) + k_j \cdot \phi(x')) \right) \right].
\]
(16)

We first examine the first term denoted as \( I_1^{(2)} \):
\[
I_1^{(2)} = -\frac{1}{4} \left( \frac{\mu^d}{tZ_t} \right)^2 \int d^d x d^d x' \\
\times \sum_{ij} \alpha_i \alpha_j Z_{\alpha_i} Z_{\alpha_j} \cos \left( \sqrt{Z_t} (k_i \cdot \phi(x) - k_j \cdot \phi(x')) \right).
\]
(17)

We evaluate the renormalization of cosine term by calculating \( \langle (k_i \cdot \phi - k_j \cdot \phi)^2 \rangle \). We adopt that \( \langle \phi_\ell(x) \phi_m(x') \rangle = \delta_\ell m, \phi(x)(\phi(x')) \rangle \), and \( \langle \phi_\ell(x) \phi_\ell(x') \rangle \) is independent of \( \ell \). Then
\[
\langle (k_i \cdot \phi - k_j \cdot \phi)^2 \rangle = \sum_\ell \left[ k_{i\ell}^2 \langle \phi_\ell(x)^2 \rangle + k_{j\ell}^2 \langle \phi_\ell(x')^2 \rangle - 2 k_{i\ell} k_{j\ell} \langle \phi_\ell(x) \phi_\ell(x') \rangle \right].
\]

\( I_1^{(2)} \) is renormalized as
\[
I_1^{(2)} = -\frac{1}{4} \left( \frac{\mu^d}{tZ_t} \right)^2 \int d^d x d^d x' \left[ \sum_i \alpha_i^2 Z_{\alpha_i}^2 \\
\times \exp \left( -Z_t k_i^2 \langle \phi_1(x)^2 \rangle + Z_t k_i^2 \langle \phi_1(x')^2 \rangle \right) \\
\times \cos \left( \sqrt{Z_t} (k_i \cdot \phi(x) - \phi(x')) \right) \right] \\
\times \sum_{ij} \alpha_i \alpha_j Z_{\alpha_i} Z_{\alpha_j} \left[ \cos \left( \sqrt{Z_t} (k_i \cdot \phi(x) - k_j \cdot \phi(x')) \right) \\
+ \cos \left( \sqrt{Z_t} (k_i \cdot \phi(x) + k_j \cdot \phi(x')) \right) \right].
\]
(19)

The two-point function is written as
\[
\langle \phi_\ell(x) \phi_\ell(y) \rangle = \frac{t \mu^{2-d} Z_t}{Z_\phi} \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip \cdot (x-y)}}{p^2 + m_0^2} \\
= \frac{t \mu^{2-d} Z_t}{Z_\phi} \sqrt{\frac{4\pi d}{d-2}} K_0(m_0|x-y|),
\]
(20)

where we introduced \( m_0 \) to avoid the infrared divergence and \( K_0 \) is the zero-th modified Bessel function.

C. Renormalization of \( t \)

The first term of \( I_1^{(2)} \) gives a contribution to the renormalization of the coupling constant \( t \). Since \( K_0(m_0 r) \) increases as \( r \rightarrow 0 \), we can expand in terms of \( r \). By using \( \cos(\sqrt{Z_t} k_i \cdot (\phi(x) - \phi(x + r))) \approx 1 - (1/2) Z_t (r \partial_\mu (k_i \cdot \phi(x)))^2 \) where \( \partial_\mu = \partial/\partial x_\mu \), the first term \( I_1^{(2)} \) of \( I_1^{(2)} \) is written as
\[
I_1^{(2)} \simeq \frac{1}{4} \left( \frac{\mu^d}{tZ_t} \right)^2 \int d^d x d^d r \sum_i \alpha_i^2 Z_{\alpha_i}^2 \frac{1}{4} Z_t (\partial_\mu \phi_i)^2 \\
\times r^2 \exp \left( -Z_t k_i^2 \langle \phi_i^2 \rangle + t Z_t k_i^2 \frac{\Omega_d}{(2\pi)^d} K_0(m_0 r) \right),
\]
(21)

where \( \phi_i = \sum \delta_{\ell m} \phi_\ell \). If \( \{k_\ell \} \in SO(N) \) (with \( M = N \)), we have \( \sum_i (\partial_\mu \phi_i)^2 = \sum_i (\partial_\mu \phi_i)^2 \). In general, we have
\[
\sum_i (\partial_\mu \phi_i)^2 = \sum_{i \neq j} k_{ij}^2 (\partial_\mu \phi_i)^2 + \sum_{i \neq m} k_{im} \partial_\mu \phi_i \partial_\mu \phi_m.
\]
(22)

As mentioned in section II, we consider the case where \( \{k_\ell \} \) form an equilateral triangle (\( M = 3 \)) or a regular tetrahedron (\( M = 6 \)), and we obtain \( \sum_k k_{ij}^2 = \text{const} \equiv C \) depending on \( M \) such as \( C = 3/2 \) for \( M = 3 \) and \( N = 2 \), and \( C = 2 \) for \( M = 6 \) and \( N = 3 \). In this case
\[
\sum_i (\partial_\mu \phi_i)^2 = C \sum_i (\partial_\mu \phi_i)^2 + \sum_{i \neq m} k_{im} \partial_\mu \phi_i \partial_\mu \phi_m.
\]
(23)

In order to recover the kinetic term in the original action, we use the approximation
\[
\sum_i \alpha_i^2 Z_{\alpha_i}^2 (\partial_\mu \phi_i)^2 \rightarrow \frac{1}{M} \sum_i \alpha_i^2 \cdot C (\partial_\mu \phi)^2
\]
\[
= \langle \phi_i^2 \rangle C (\partial_\mu \phi)^2.
\]
(24)

Otherwise the renormalization of the kinetic term becomes complicated since we must introduce \( \{t_\ell \} \) that depend on components of \( \phi \). This may not be essential for the renormalization group flow. We discuss this point later.

\( \langle \phi_i^2 \rangle \) is evaluated as
\[
\langle \phi_i^2 \rangle = \frac{t \mu^{2-d} Z_t}{Z_\phi} \sqrt{\frac{4\pi d}{d-2}} K_0(m_0 a),
\]
(25)

where \( a \) is a small cutoff. The \( r \)-integration is calculated as
\[
J_j := \int d^d r r^2 \exp \left( t k_j^2 \frac{\Omega_d}{(2\pi)^d} K_0(m_0 \sqrt{r^2 + a^2}) \right)
\]
\[
\approx \frac{\Omega_d}{c m_0^2(\sqrt{a^2 + r^2})},
\]
(26)
where \( c = (e^7/2)^2 \). We put
\[
d = 2 + \epsilon, \tag{27}
\]
and
\[
t = \frac{t}{8\pi} = 1 + v, \tag{28}
\]
since we normalize \( k_j^2 = 1 \). Then we have
\[
J_j = \Omega_d(c_m^2)^{-1/2} \int_0^\infty drr^{d+1} \frac{1}{(r^2 + \alpha^2)^{1+2\epsilon}}
\]
\[
= -\Omega_d(c_m^2)^{-1/2} \frac{1}{\epsilon} + O(v). \tag{29}
\]
This indicates
\[
I_{1a}^{(2)} = -\frac{C}{8} \left( \frac{\mu^d}{tZ_t} \right)^2 \left( \alpha_i^2 \right) \exp \left( -Z_\phi(\phi_i^2) \right)
\]
\[
\times \Omega_d(c_m^2)^{-1/2} \int d^d x \frac{1}{2} Z_k^2 (\partial_\mu \phi)^2 + O(v_t).
\]
\[
= -\frac{C}{8} \left( \frac{\mu^d}{tZ_t} \right)^2 \left( \alpha_i^2 \right)(cm^2 a^2)^{1/4} \Omega_d(c_m^2)^{-2/3}
\]
\[
\times \frac{1}{\epsilon} \int d^d x \frac{1}{2} Z_\phi (\partial_\mu \phi)^2 + O(v_t)
\]
\[
= -\frac{C}{8} \left( \alpha_i^2 \right) \frac{\Omega_d}{8\pi} \mu^{d+2} a^4 \frac{1}{\epsilon} \int d^d x \frac{d^d Z_\phi (\partial_\mu \phi)^2}{2tZ_t}
\]
\[
+ O(v_t). \tag{30}
\]
Then we choose
\[
Z_t = 1 - \frac{C}{32} (\alpha_i^2) \mu^{d+2} a^4 \frac{1}{\epsilon}, \tag{31}
\]
where we set \( Z_{\alpha_i} = 1 \) to the lowest order of \( \alpha_i \).

Since the bare coupling constant \( t_0 = t\mu^{d-d} Z_t \) is independent of the energy scale \( \mu \), we have \( \partial_\mu t_0 / \partial \mu = 0 \). This results in
\[
\beta(t) := \frac{\partial t}{\partial \mu} = (d-2)t - t\mu \frac{\partial \ln Z_t}{\partial \mu}
\]
\[
= (d-2)t + \frac{C}{32}(\alpha_i^2), \tag{32}
\]
where we used \( \mu \partial_\mu t_0 / \partial \mu = -2(\alpha_i - t/8\pi) \), neglecting terms of the order of \( t^2 \alpha_i^2 \), and we put \( a = \mu^{-1} \).

### D. Vertex-vertex correction to \( \alpha_j \)

We consider the second term in \( I_{1b}^{(2)} \) that contains multi-vertex interaction:
\[
I_{1b}^{(2)} := -\frac{1}{4} \left( \frac{\mu^d}{tZ_t} \right)^2 \int d^d x d^d x' \sum_{i \neq j} \alpha_i \alpha_j Z_{\alpha_i} Z_{\alpha_j}
\]
\[
\times \exp \left[ -\frac{Z_\phi}{2} \left( k_i^2 \langle \phi_i(x)^2 \rangle + k_j^2 \langle \phi_j(x')^2 \rangle \right) + \frac{Z_\phi}{2} k_i \cdot k_j \langle \phi_i(x) \phi_j(x') \rangle \right]
\]
\[
\times \cos \left( \sqrt{Z_\phi} (k_i \cdot \phi(x) - k_j \cdot \phi(x')) \right). \tag{33}
\]
Let us examine the integral given by
\[
J_{ij} := \int d^d r \exp \left( Z_\phi k_i \cdot k_j \langle \phi(x) \phi(x) \rangle \right)
\]
\[
= \int d^d r \exp \left( k_i \cdot k_j t \mu^{d-2} Z_i \Omega_d(2\pi)^d K_0(m_0 r) \right)
\]
\[
\approx \Omega_d \int_0^\infty dr r^{d-1} \left( \frac{1}{c_m^2 (r^2 + a^2)} \right)
\]
\[
= \Omega_d \left( \frac{1}{c_m^2 a^2} \right) \frac{k_i \cdot k_j}{a^{d/2}} \Gamma \left( \frac{d}{2} \right) \times \Gamma \left( \frac{t}{4\pi} k_i \cdot k_j - \frac{d}{2} \right), \tag{34}
\]
where the cutoff \( a \) is introduced. We put \( t = 8\pi(1 + v) \), then we have a divergence near two dimensions when
\[
k_i \cdot k_j = 1/2. \tag{35}
\]
This means that two vectors \( k_i \) and \( k_j \) forms an equilateral triangle. When \( k_i \) and \( k_j \) satisfy this condition, we have
\[
J_{ij} = -\Omega_d(c_m^2)^{-1/2} \frac{1}{\epsilon} + O(v_t). \tag{36}
\]
Then we obtain
\[
I_{1b}^{(2)} \approx \frac{1}{4} \left( \frac{\mu^d}{tZ_t} \right)^2 \sum_{i \neq j} \alpha_i \alpha_j Z_{\alpha_i} Z_{\alpha_j} \left( cm^2 a^2 \right)^{d/4}
\]
\[
\times \frac{1}{\epsilon} \Omega_d(c_m^2)^{-1} \int d^d x \cos \left( \sqrt{Z_\phi} (k_i - k_j) \cdot \phi(x) \right)
\]
\[
\approx \frac{1}{\epsilon} \sum_{i \neq j} \alpha_i \alpha_j \frac{1}{tZ_t} a^{d/2} a^4 \int d^d x \times \cos \left( \sqrt{Z_\phi} (k_i - k_j) \cdot \phi(x) \right). \tag{37}
\]
Let \( k_\ell \) be a vector so that \( k_i, k_j, \) and \( k_\ell \) form an equilateral triangle where
\[
k_i - k_j = k_\ell. \tag{38}
\]
Then the potential term with coefficient \( \alpha_\ell \) has the correction as
\[
\frac{\mu^d}{tZ_t} \alpha_\ell Z_{\alpha_\ell} \left( 1 + \frac{t}{4\pi} + \frac{1}{16} \frac{\alpha_\ell \alpha_j}{\alpha_\ell} \mu^d c_m^2 a^4 \right) \cos \left( \sqrt{Z_\phi k_\ell} \cdot \phi \right). \tag{39}
\]
We choose the renormalization constant \( Z_{\alpha_\ell} \) as
\[
Z_{\alpha_\ell} = 1 - \frac{t}{4\pi} - \frac{1}{16} \frac{\alpha_\ell \alpha_j}{\alpha_\ell} \mu^d c_m^2 a^4. \tag{40}
\]
This leads to the beta function \( \beta(\alpha_\ell) \) with correction as
\[
\beta(\alpha_\ell) = -2\alpha_\ell \left( 1 - \frac{t}{8\pi} \right) + \frac{1}{8} c_m^2 a^2 \alpha_\ell \alpha_j. \tag{41}
\]
Since the coefficient of the correction term is dependent on cutoff parameters, we choose \( cm^2 \alpha^2 = 1 \) to have

\[
\beta(\alpha_t) = -2\alpha_t \left( 1 - \frac{t}{8\pi} \right) + \frac{1}{16} \alpha_i \alpha_j. \tag{42}
\]

There is also a contribution from the second term in \( I^{(2)} \) where \( k_j \) is replaced by \(-k_j\). In this case vertices with \( k_i \cdot k_j = -1/2 \) generate a new vertex with \( k_\ell \) satisfying

\[
k_i + k_j = k_\ell. \tag{43}
\]

In the dimensional regularization method, two vertices satisfying \( k_i \cdot k_j = \pm 1/2 \) generate a new vertex with \( k_\ell \) besides \( k_i \). As a result, the beta function for \( \alpha_t \) reads

\[
\beta(\alpha_t) = -2\alpha_t \left( 1 - \frac{t}{8\pi} \right) + \frac{1}{16} \sum_{ij} \alpha_i \alpha_j, \tag{44}
\]

where the summation should take for those satisfying \( k_\ell = k_i \pm k_j \) \((i,j,\ell = 1,\ldots,N)\).

When \( k_1, k_2 \) and \( k_3 \) form an equilateral triangle \((M = 3)\), the renormalization group equations for \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are closed within three equations. When \( k_1, k_2, \ldots, k_6 \) form a regular tetrahedron \((M = 6)\), we again have a closed set of equations for \( \alpha_j \) \((j = 1, 2, \ldots, 6)\). In the Wilsonian method, the same beta equation for \( \alpha_t \) is obtained which will be discussed in the next section. In the Wilson renormalization method, however, a new vertex marked by \( k_\ell \) is generated from any two vectors \( k_i \) and \( k_j \) except the case \( k_i \cdot k_j = 0 \).

\[
\begin{align*}
&\text{(a)} \quad \begin{array}{c}
\includegraphics[width=2cm]{triangle1.png}
\end{array} \quad \text{(b)} \quad \begin{array}{c}
\includegraphics[width=2cm]{triangle2.png}
\end{array}
\end{align*}
\]

**FIG. 1:** Triangles formed by wave vectors \( k_i, k_j \) and \( k_\ell \) for (a) \( k_i \cdot k_j = 1/2 \) and (b) \( k_i \cdot k_j = -1/2 \).

### E. Relation to the tachyon scattering amplitude in a bosonic string theory

The two-vertex correction \( J_{ij} \) is related to the tachyon scattering amplitude in a bosonic string theory. The \( n \)-point scattering amplitude for tachyon scattering is given as

\[
A_n := \int d\mu \int DX \exp \left[ -\frac{1}{4\pi \alpha'} \int (\partial_\mu X_\nu \partial_\nu X^\mu) d^2 z \right.
\]

\[
+ i \sum_{i=1}^{n} k_\mu X^\mu \right]
\]

\[
\int d\mu \prod_{1 \leq i < j \leq n} |z_i - z_j|^{2\alpha' k_i \cdot k_j}, \tag{45}
\]

where the integration with the measure \( d\mu \) is an integral over the various \( z_i \). If we assume the correspondence

\[
2\pi \alpha' = t, \tag{46}
\]

the \( z_i \) dependence of the amplitude \( A_2 \) agrees with \( J_{ij} \). The vertex-vertex renormalization is given by the amplitude for tachyon scattering.

### F. Renormalization group flow

For an equilateral triangle configuration of \( \{k_i\} \) with \( M = 3 \) and \( N = 2 \), the equations read

\[
\mu \frac{\partial \alpha_1}{\partial \mu} = -2\alpha_1 \left( 1 - \frac{t}{8\pi} \right) + \frac{1}{16} \alpha_2 \alpha_3,
\]

\[
\mu \frac{\partial \alpha_2}{\partial \mu} = -2\alpha_2 \left( 1 - \frac{t}{8\pi} \right) + \frac{1}{16} \alpha_3 \alpha_1,
\]

\[
\mu \frac{\partial \alpha_3}{\partial \mu} = -2\alpha_3 \left( 1 - \frac{t}{8\pi} \right) + \frac{1}{16} \alpha_1 \alpha_2, \tag{47}
\]

and

\[
\mu \frac{\partial t}{\partial \mu} = (d-2)t + \frac{C}{32M} t \sum_{i=1}^{3} \alpha_i^2. \tag{48}
\]

We consider the simplified case where \( \alpha_i = \alpha \) \((i = 1, 2, 3)\). In this case the equations read

\[
\mu \frac{\partial \alpha}{\partial \mu} = -2\alpha \left( 1 - \frac{t}{8\pi} \right) + \frac{1}{16} \alpha^2,
\]

\[
\mu \frac{\partial t}{\partial \mu} = (d-2)t + \frac{C}{32} \alpha^2. \tag{49}
\]

In two dimensions \( d = 2 \), the equations become

\[
\mu \frac{\partial \alpha}{\partial \mu} = 2\alpha v + \frac{1}{16} \alpha^2,
\]

\[
\mu \frac{\partial v}{\partial \mu} = \frac{C}{32} \alpha^2, \tag{50}
\]

for \( t = 8\pi(1 + v) \). The renormalization group flow is shown in Fig. 2 for \( \alpha > 0 \). The dotted line indicates \( \alpha = -32v \) where \( \mu \partial \alpha / \partial \mu \) vanishes. The asymptotic line as \( \mu \to \infty \) is given by \( \alpha \sim b_+ v \) with

\[
b_+ = \frac{1}{C} \left( 1 + \sqrt{1 + 64C} \right), \tag{51}
\]
and $\alpha \sim b_\nu v$ with
\[
\frac{b_\nu}{C} = \frac{1}{1 - \sqrt{1 + 64C}}.
\]  

It is apparent from Fig. 2 that there is an asymmetry between positive $v$ and negative $v$. This is due to the two-vertex contribution. There is also an asymmetry between $\alpha > 0$ and $\alpha < 0$. The flow for $\alpha < 0$ is obtained just by extending straight lines into the negative $\alpha$ region.

![Graph](image)

**FIG. 2:** Renormalization group flow as $\mu \to \infty$ in the plane of $\alpha$ and $v$.

### IV. WILSONIAN RENORMALIZATION GROUP METHOD

We investigate the renormalization of the multi-vertex sine-Gordon model by using the Wilsonian renormalization group method. We obtain the same set of equations as that in the dimensional regularization method. The only difference is that two vertices satisfying $k_i \cdot k_j \neq 0$ generate a new vertex, while $k_i$ and $k_j$ should satisfy $k_i \cdot k_j = \pm 1/2$ in the dimensional regularization method.

#### A. Wilsonian renormalization procedure

We write the action in the following form.
\[
S = \int d^2x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \sum_j g_j \cos(\beta k_j \cdot \phi) \right],
\]

where $g_j = \alpha_j / t$ and $\beta = \sqrt{7}$. The field $\phi$ was scaled to $\beta \phi$. We reduce the cutoff $\Lambda$ in the following way:
\[
\Lambda \to \Lambda - d\Lambda = \Lambda - \Delta \ell = \Lambda e^{-d\ell}.
\]

The scalar field $\phi = (\phi_1, \cdots, \phi_N)$ is divided into two parts as $\phi(x) = \phi_1(x) + \phi_2(x)$ with $\phi_1(x) = (\phi_{\ell_1}, \cdots, \phi_{\ell_N}) (\ell = 1, 2)$ where
\[
\begin{align*}
\phi_1(x) &= \int_{\Lambda - d\Lambda}^{\Lambda} \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \phi_1(x), \\
\phi_2(x) &= \int_{-d\Lambda}^{\Lambda} \frac{d^2p}{(2\pi)^2} e^{ip \cdot x} \phi_2(x).
\end{align*}
\]

The action is written as
\[
S = \int d^2x \left[ \sum_{\ell=1}^{2} \frac{1}{2} (\partial_\mu \phi_{\ell})^2 + \sum_j g_j \cos(\beta k_j \cdot (\phi_1 + \phi_2)) \right]
\]

\[= S_0(\phi_1) + S_0(\phi_2) + S_1(\phi_1, \phi_2),
\]

where $S_1$ indicates the potential term. Then the partition function is given by
\[
Z = \int \mathcal{D}\phi e^{-S}
\]

\[= \int \mathcal{D}\phi \exp \left( -S_0(\phi_1) + \frac{1}{n!} \sum_n \Gamma_n(\phi_1) \right),
\]

where
\[
\sum_n \Gamma_n(\phi_1) = \langle \langle \sum n! (-1)^n S_1^n \rangle \rangle_{\text{conn}},
\]

with
\[
\langle \langle Q \rangle \rangle_{\text{conn}} = \frac{1}{Z_2} \int \mathcal{D}\phi_2 e^{-S_0(\phi_2)Q}.
\]

$\langle \langle \cdot \rangle \rangle_{\text{conn}}$ means keeping only connected diagrams in $\langle \langle \cdot \rangle \rangle$. $\Gamma_n (n = 1, 2, \cdots)$ represent contributions to the effective action.

#### B. Lowest order renormalization of $g_j$

The lowest order contribution $\Gamma_1 = -\langle \langle S_1 \rangle \rangle$ reads
\[
\Gamma_1 \simeq -\sum_j g_j \int d^2x \cos(\beta k_j \cdot \phi_1) \exp \left( -\frac{1}{2} \beta^2 \langle \langle (k_j \cdot \phi_2)^2 \rangle \rangle \right)
\]

\[= -\sum_j g_j \exp \left( -\frac{1}{2} \beta^2 G_{j\Delta}(0) \right) \int d^2x \cos(\beta k_j \cdot \phi_1),
\]

where the Green function $G_{j\Delta}$ is defined as
\[
G_{j\Delta}(x_1 - x_2) = \langle \langle \phi_{j_2}(x_1) \phi_{j_3}(x_2) \rangle \rangle
\]

\[= \frac{d\Delta}{2\pi} J_0(\Lambda|x_1 - x_2|),
\]

where $J_0$ is the zero-th Bessel function. Up to this order, the action is renormalized to
\[
S_{\Lambda - d\Lambda} = S_0(\phi_1) - \Gamma_1
\]

\[= \int d^2x \left[ \frac{1}{2} (\partial_\mu \phi_1)^2 + \sum_j g_j \left( 1 - \frac{\beta^2}{2} G_{j\Delta}(0) \right) \cos(\beta k_j \cdot \phi_1) \right].
\]
We perform the following scale transformation:
\[ x \rightarrow x' = e^{-dt}x, \]
\[ p \rightarrow p' = e^{dt}p, \]
\[ \phi_1(p) \rightarrow \tilde{\phi}_1(p') = \phi_1(p)\zeta^{-1}, \]
where \( \zeta \) is the scaling parameter for the field \( \phi_1 \). In the real space we have
\[ \phi_1(x) = \zeta e^{-2dt}\tilde{\phi}_1(x'). \] (64)
Then the effective action reads
\[ S_{A-dA} = \int d^2x' \left[ \zeta^2 e^{-4dt}\frac{1}{2}(\partial'_\mu \tilde{\phi}_1(x'))^2 + \sum_j g_j e^{2dt} \left( 1 - \frac{\beta^2 d\Lambda}{4\pi} \right) \times \cos \left( \beta \zeta e^{-2dt}k_j \cdot \tilde{\phi}_1(x') \right) \right]. \] (65)
Here we put
\[ \zeta^2 e^{-4dt} = 1, \] (66)
so that we obtain
\[ S_{A} = \int d^2x' \left[ \frac{1}{2}(\partial'_\mu \tilde{\phi}_1(x'))^2 + \sum_j g_j \left( 1 + 2 \frac{d\Lambda}{\Lambda} - \frac{\beta^2 d\Lambda}{4\pi} \right) \times \cos \left( \beta k_j \cdot \tilde{\phi}_1(x') \right) \right]. \] (67)
This leads to the renormalized \( g_{Rj} \) and \( \beta_R \) as
\[ g_{Rj} = g_j + \left( 2 - \frac{\beta^2}{4\pi} \right) g_j \frac{d\Lambda}{\Lambda}, \] (68)
\[ \beta_R = \beta. \] (69)
Then we have
\[ \Lambda \frac{dg_j}{d\Lambda} = \left( 2 - \frac{\beta^2}{4\pi} \right) g_j, \]
\[ \Lambda \frac{d\beta}{d\Lambda} = 0. \] (70)
Since \( \beta^2 = \tau \), these results agree with those obtained by the dimensional regularization method in two dimensions.

C. Multi-vertex contributions

The second-order contribution to the effective action is
\[ \Gamma_2 = \frac{1}{2} \langle \langle S^2 \rangle \rangle_{\text{conn}} \]
\[ = \frac{1}{2} \sum_{ij} g_i g_j \int d^2x d^2x' \left[ \langle \cos(\beta k_i \cdot \phi(x)) \cos(\beta k_j \cdot \phi(x')) \rangle \right. \]
\[ \left. - \langle \cos(\beta k_i \cdot \phi(x)) \rangle \langle \cos(\beta k_j \cdot \phi(x')) \rangle \right]. \] (71)
We integrate out contributions with respect to \( \phi_2 \) variable. For example, we use
\[ \langle \langle e^{i\beta(sk_i \cdot \phi_2(x) + s'k_j \cdot \phi_2(x'))} \rangle \rangle \]
\[ = \frac{1}{2} \beta^2 \left[ (k_i^2 + k_j^2)G_{dA}(0) + 2s s' k_i \cdot k_j G_{dA}(x-x') \right], \] (72)
where \( s \) and \( s' \) take \( \pm 1 \). The second-order effective action \( \Gamma_2 \) is given as
\[ \Gamma_2 = \frac{1}{4} \sum_{ij} g_i g_j \exp \left( -\beta^2 G_{dA}(0) \right) \int d^2x d^2x' \left[ \right. \]
\[ \left. \left( e^{-\beta^2 k_i \cdot k_j G_{dA}(x-x')} - 1 \right) \cos (\beta(k_i \cdot \phi_1(x) + k_j \cdot \phi_1(x')) \right) \]
\[ + \left( e^{\beta^2 k_i \cdot k_j G_{dA}(x-x')} - 1 \right) \cos (\beta(k_i \cdot \phi_1(x) - k_j \cdot \phi_1(x')) \right]. \] (73)
When \( k_i \cdot k_j > 0 \), the second term grows large for \( |x - x'| \to 0 \), while the first term becomes small. When \( k_i \cdot k_j < 0 \), the first term instead becomes large. Hence we have
\[ \Gamma_2 = \frac{1}{4} \sum_{ij} g_i g_j \exp \left( -\beta^2 G_{dA}(0) \right) \int d^2x d^2r \left[ \right. \]
\[ \left. \left( e^{\beta^2 k_i \cdot k_j G_{dA}(r)} - 1 \right) \cos (\beta(k_i \cdot \phi_1(x) \mp k_j \cdot \phi_1(x + r)) \right) \right], \] (74)
where \( \mp \) takes \(-\) when \( k_i \cdot k_j > 0 \) and \( + \) for \( k_i \cdot k_j < 0 \). Since the integrand is large when \( r \) is small, \( \Gamma_2 \) is written as
\[ \Gamma_2 = \frac{1}{4} \sum_{ij} g_i g_j \beta^2 |k_i \cdot k_j| \int d^2r G_{dA}(r) \]
\[ \cdot \left( \int d^2x \cos (\beta(k_i \mp k_j) \phi_1(x)) \right) \left( 1 - \frac{\beta^2}{2} (r \cdot \nabla (k_j \cdot \phi_1)^2) \right) \]
\[ \approx \frac{1}{4} \sum_{ij} g_i g_j \beta^2 |k_i \cdot k_j| \int d^2r G_{dA}(r) \]
\[ \cdot \left( \int d^2x \cos (\beta(k_i \mp k_j) \phi_1(x)) \right) \]
\[ - \frac{1}{8} \sum_j g_j^2 \beta^4 \int d^2x d^2r \frac{1}{2} G_{dA}(r) \int d^2x (\partial_\mu (k_j \cdot \phi_1))^2, \] (75)
where in the second term with derivative of \( \phi_1 \) we keep only \( k_i = k_j \) term since this term otherwise becomes small due to the oscillation of cosine function. As discussed before, we use the approximation \( \sum_j g_j^2 (\partial_\mu (k_j \cdot \phi_1))^2 \approx (g_j^2) C (\partial_\mu \phi_1)^2 \).
Then the effective action reads
\begin{align}
S_{\Lambda - d\Lambda} &= S_0(\phi_1) - \Gamma_1 - \Gamma_2 \\
&= \int d^2x \left[ \frac{1}{2} (\partial_{\mu} \phi_1)^2 \left( 1 + \frac{A}{8} \beta^4 (g_j^2) \frac{d\Lambda}{\Lambda^5} \right) d\Lambda \right] \\
&\quad + \sum_j g_j \left( 1 - \frac{\beta^2}{2} G_{d\Lambda}(0) \right) \cos(\beta k_j \cdot \phi_1) \\
&\quad - \frac{1}{4} B \sum_{ij} g_i g_j \beta^2 |k_i \cdot k_j| \frac{d\Lambda}{\Lambda^3} \cos(\beta (k_i \mp k_j) \phi_1(x)) \right], \\
&= \int d^2x \left[ \frac{1}{2} (\partial_{\mu} \phi_1)^2 \left( 1 + \frac{A}{8} \beta^4 (g_j^2) \frac{d\Lambda}{\Lambda^5} \right) \right] \\
&\quad + \sum_j g_j \left( 1 - \frac{\beta^2}{2} G_{d\Lambda}(0) \right) \cos(\beta k_j \cdot \phi_1) \\
&\quad - \frac{1}{4} B \sum_{ij} g_i g_j \beta^2 |k_i \cdot k_j| \frac{d\Lambda}{\Lambda^3} \cos(\beta (k_i \mp k_j) \phi_1(x)) \right],
\end{align}

where \( A \) and \( B \) are constants defined by
\begin{align}
A &= C \int_0^1 d\tau r^3 J_0(r), \\
B &= \int_0^1 d\tau r J_0(r).
\end{align}

We perform the scale transformation in eqs. \ref{eq:4} and \ref{eq:5}, where the parameter \( \zeta \) is chosen as
\begin{align}
\zeta^2 e^{-4d\tau} \left( 1 + \frac{A}{8} \beta^4 (g_j^2) \frac{d\Lambda}{\Lambda^5} \right) = 1.
\end{align}

Then the renormalized action is given by
\begin{align}
S_{\Lambda} &= \int d^2x \left[ \frac{1}{2} (\partial_{\mu} \phi_1)^2 \\
&\quad + \sum_j g_j \left( 1 + 2 \frac{d\Lambda}{\Lambda} - \frac{\beta^2}{4\pi} \frac{d\Lambda}{\Lambda} \right) \cos(\beta \zeta e^{-2d\tau} k_j \cdot \phi_1(x)) \\
&\quad - \frac{B}{4} \sum_{ij} g_i g_j |k_i \cdot k_j| \frac{d\Lambda}{\Lambda^3} \cos(\beta \zeta e^{-2d\tau} (k_i \mp k_j) \cdot \phi_1(x)) \right].
\end{align}

This results in the renormalization group equations as follows.
\begin{align}
\frac{d\beta}{d\Lambda} &= -\frac{A}{16\Lambda^4} \beta^5 (g_j^2), \\
\frac{d\beta}{d\Lambda} &= \left( 2 - \frac{\beta^2}{4\pi} \right) g_j - \frac{B}{4\Lambda^2} \beta^2 \sum_{j\ell} g_j g_j |k_i \cdot k_{\ell}|,
\end{align}

where the summation is taken for \( k_i \) and \( k_{\ell} \) satisfying \( k_j = k_i \mp k_{\ell} \).

The resulting equations are consistent with those obtained using the dimensional regularization. Note that the sign is different because the derivative is calculated in the descending direction \( \Lambda \to \Lambda - d\Lambda \) in the Wilsonian method. In the dimensional regularization method, the summation for \( g_i \) and \( g_\ell \) is restricted to \( k_i \) and \( k_{\ell} \) that satisfy \( k_i \cdot k_{\ell} = \pm 1/2 \). In the Wilsonian method, this condition is relaxed and new vertex is generated unils \( k_i \) and \( k_{\ell} \) are orthogonal.

V. GENERALIZED MULTI-VERTEX MODEL

A. Renormalization of \( \alpha_j \)

As shown in the evaluation of \( \beta(t) \), the corrections to \( t \) are dependent on momentum parameter \( \{k_i\} \). We examine this in this section. We consider the generalized Lagrangian given as
\begin{align}
\mathcal{L} &= \sum_j \frac{Z_\phi}{2t_j} \mu^{2-d} Z_{\ell_j} (\partial_\mu \phi_j)^2 + \sum_j \frac{\mu^d \alpha_j Z_{\alpha_j}}{t_j Z_{\ell_j}} \cos \left( \sqrt{Z_\phi} k_j \cdot \phi \right).
\end{align}

The potential term is renormalized to
\begin{align}
\alpha_j Z_{\alpha_j} \exp \left( -\frac{1}{2} Z_\phi \sum_\ell k^2_\ell (\phi^2_\ell) \right) \cos \left( \sqrt{Z_\phi} k_j \cdot \phi \right),
\end{align}

where
\begin{equation}
\langle \phi^2_\ell \rangle = \frac{t_\ell \mu^{2-d} Z_{\ell_\ell}}{Z_\phi} \int d^d p \frac{1}{(2\pi)^d p^2 + m_0^2} = -\frac{t_\ell \mu^{2-d} Z_{\ell_\ell}}{Z_\phi} \frac{1}{(2\pi)^d} \Omega_d.
\end{equation}

Then the correction is written as
\begin{align}
\alpha_j Z_{\alpha_j} \left( 1 + \frac{1}{2e} \sum_\ell k^2_\ell t_\ell \mu^{2-d} Z_{\ell_\ell} \right) \frac{\Omega_d}{(2\pi)^d} \cos \left( \sqrt{Z_\phi} k_j \cdot \phi \right).
\end{align}

This results in
\begin{align}
Z_{\alpha_j} = 1 - \frac{1}{4\pi e} \sum_\ell k^2_\ell t_\ell.
\end{align}

Then we obtain
\begin{align}
\mu \left( \frac{\partial \alpha_j}{\partial \mu} \right) = -2\alpha_j \left( 1 - \frac{1}{8\pi e} \sum_\ell k^2_\ell t_\ell \right).
\end{align}

The fixed point of \( \{t_\ell\} \) is obtained as a zero of this equation. For an equilateral triangle where \( N = 2 \) and \( M = 3 \), we can choose \( \{k_j\} \) as
\begin{align}
k_1 = (1, 0), \quad k_2 = (1/2, \sqrt{3}/2), \quad k_3 = (-1/2, \sqrt{3}/2).
\end{align}

The critical value of \( t_\ell \) is obtained as
\begin{align}
t_{1c} = t_{2c} = t_{3c} = 8\pi.
\end{align}

For \( N = 3 \), we can consider a regular tetrahedron with \( M = 6, \{k_j\} \) are set as
\begin{align}
k_1 = (1, 0, 0), \quad k_2 = (1/2, \sqrt{3}/2, 0), \quad k_3 = (-1/2, \sqrt{3}/2, 0), \\
k_4 = (1/2, 1/2 \sqrt{3}, \sqrt{2}/3), \quad k_5 = (1/2, -1/2 \sqrt{3}, \sqrt{2}/3), \\
k_6 = (0, -1/\sqrt{3}, \sqrt{2}/3).
\end{align}

In this case, the fixed point of \( \{t_\ell\} \) is also given by \( t_{1c} = t_{2c} = \cdots = t_{6c} = 8\pi \).
From the second-order perturbation, there appears the term that renormalizes the kinetic term as shown in section III. We use the following approximation here:

\[
\cos \left( \sqrt{Z_\phi} k_j \cdot (\phi(x) - \phi(x + r)) \right) \\
= \cos \left( \sqrt{Z_\phi} r_\mu \partial_\mu (k_j \cdot \phi(x)) - \cdots \right) \\
= 1 - \frac{1}{2} Z_\phi (r_\mu \partial_\mu (k_j \cdot \phi(x)))^2 + \cdots \\
= 1 - \frac{1}{2} Z_\phi r_\mu r_\nu \sum_{\ell \mu} k_j k_{jm} \partial_\mu \phi_\ell \partial_\nu \phi_m + \cdots. \quad (90)
\]

We keep the diagonal terms \((\partial_\mu \phi)^2\), and then \(I_{1a}^{(2)}\) in section III becomes

\[
I_{1a}^{(2)} \simeq -\frac{1}{32} \sum_{\mu \neq \ell} \mu \partial^{2-2} Z_\phi \mu \partial^{2+2} \alpha_2 j \ell \cdot (\partial_\mu \phi_\ell)^2, \quad (91)
\]

where we put \(t_\ell = 8\pi (1 + \nu_\ell)\) and neglect the term of the order of \(\nu_\ell\). Then the kinetic term is renormalized into

\[
\sum_{\ell} \frac{Z_\phi}{2t_\ell \mu^{2-2} Z_\ell} \left[ 1 - \frac{1}{32} \mu \partial^{2+2} \alpha_2 j \ell \cdot (\partial_\mu \phi_\ell)^2 \right]. \quad (92)
\]

This leads to

\[
Z_{\ell \ell} = 1 - \frac{1}{32} \mu \partial^{2+2} \alpha_2 j \ell \cdot. \quad (93)
\]

Then we obtain

\[
\mu \frac{\partial t_\ell}{\partial \mu} = (d-2) t_\ell + \frac{1}{32} t_\ell \sum_j \alpha_2 j \ell \cdot. \quad (94)
\]

For \(N = 2\) and \(M = 3\), we use \(\{k_j\}\) for an equilateral triangle, the equations for \(t_1\) and \(t_2\) read

\[
\mu \frac{dt_1}{\partial \mu} = (d-2) t_1 + \frac{1}{32} t_1 \left( \alpha_2^2 + \frac{1}{4} \alpha_2^2 + \frac{1}{4} \alpha_3^2 \right), \quad (95)
\]

\[
\mu \frac{dt_2}{\partial \mu} = (d-2) t_2 + \frac{1}{32} t_2 \left( 3 \alpha_2^2 + \frac{3}{4} \alpha_2^2 + \frac{3}{4} \alpha_3^2 \right). \quad (96)
\]

This is the result for the generalized multi-vertex sine-Gordon model. The qualitative property is the same as that obtained for that in section III. When \(\alpha \equiv \alpha_1 \sim \alpha_2 \sim \alpha_3\), we have

\[
\mu \frac{dt_1}{\partial \mu} = (d-2) t_1 + \frac{C}{32} t_1 \alpha_2^2, \quad (97)
\]

with \(C = 3/2\). This agrees with the previous result.
VI. SUMMARY

We investigated the multi-vertex sine-Gordon model on the basis of the renormalization group theory. We employ the dimensional regularization method and the Wilsonian renormalization group method. Two results are consistent each other. The generalized sine-Gordon model contains multiple cosine (vertex) potentials labelled by momentum parameters \(\{k_j\}_{j=1,\ldots,M}\). The vertex-vertex scattering amplitude is given by tachyon model.

For two-component scalar field \((N = 2)\), \(\{k_j\}\) should form a triangle (Wilson method) or an equilateral triangle (dimensional regularization) for \(M = 3\). For three-component scalar field \((N = 3)\), a regular tetrahedron form a closed system for \(M = 6\). For these structures, the fixed point of \(\{t_j\}\) is given by \(t_1 = t_2 = \cdots = t_M\). A regular octahedron is also possible where there are six independent \(k_j\)s and thus \(M = 6\). For an equilateral triangle, regular tetrahedron and regular octahedron, we have \(\sum_j k_{jt}^2 = C(M)\) for \(j = 1,\ldots,M\) where we impose the normalization \(\sum_j k_{jt}^2 = 1\). We expect that there exist crystal structures in higher dimensions \(N \geq 3\) satisfying \(\sum_j k_{jt}^2 = \text{const. for any } j\).

The beta function of \(\alpha_t\) is generalized to include the product \(\alpha_t \alpha_j\) for which \(k_i \pm k_j = k_\ell\) is satisfied. This term is a non-trivial contribution compared to the conventional sine-Gordon model. The beta function of \(t_\ell\) has also contributions proportional to \(\alpha_t^2\). These terms are positive and thus do not change the renormalization group flow of \(t_\ell\). The additional terms to \(\beta(\alpha_t)\) change the flow of \((\alpha_t, t_j)\) qualitatively.

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