Polyhedra without cubic vertices are prism-hamiltonian

Simon Špacapan\textsuperscript{1,2}

\textsuperscript{1}University of Maribor, Faculty of Mechanical Engineering (FME), Department of Mathematics, Maribor, Slovenia
\textsuperscript{2}IMFM, Ljubljana, Slovenia

Correspondence
Simon Špacapan, University of Maribor, Faculty of Mechanical Engineering (FME), Department of Mathematics, Smetanova 17, 2000 Maribor, Slovenia. Email: simon.spacapan@um.si

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Abstract
The prism over a graph $G$ is the Cartesian product of $G$ with the complete graph on two vertices. A graph $G$ is prism-hamiltonian if the prism over $G$ is hamiltonian. We prove that every polyhedral graph (i.e., 3-connected planar graph) of minimum degree at least four is prism-hamiltonian.

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circuit graph, Hamiltonian cycle

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1 \hspace{1em} INTRODUCTION

The study of hamiltonicity of planar graphs is largely concerned with finding subclasses of 3-connected planar graphs for which each member of the subclass is hamiltonian or has some hamiltonian-type property. One such result was obtained in 1956 by Tutte who proved that all 4-connected planar graphs are hamiltonian [23]. Another result was obtained in [9] by Gao and Richter who proved that every 3-connected planar graph has a 2-walk (i.e., a closed spanning walk that visits every vertex at most twice). It is well known that every hamiltonian graph is prism-hamiltonian and that every prism-hamiltonian graph has a 2-walk. The converse is not true, moreover there exist 3-connected planar graphs that are not prism-hamiltonian [20].

There is an extensive list of non-hamiltonian 3-connected planar graphs with special properties, such as graphs with small order and size [2], plane triangulations [17], regular graphs [22, 24], $K_{2,6}$-minor-free graphs [7], and graphs with few 3-cuts [4]. However some classes of graphs mentioned...
above are prism-hamiltonian. For example every plane triangulation is prism-hamiltonian [3], and every cubic 3-connected graph is prism-hamiltonian [5, 18].

Rosenfeld and Barnette [19] conjectured that every 3-connected planar graph is prism-hamiltonian (see also [12]). This conjecture was recently refuted in [20] where vertex degrees play a central role in construction of counterexamples. In particular every counterexample to Rosenfeld-Barnette conjecture given in [20] has many cubic vertices and two vertices of “high” degree (linear in order of the graph). In [10] the authors show that there is an infinite family of 3-connected planar graphs, each of them not prism-hamiltonian, such that the ratio of cubic vertices tends to 1 when the order goes to infinity, and maximum degree stays bounded by 36.

Vertex degrees in relation to hamiltonicity properties were first discussed by Dirac in [6] who proved that any graph of minimum degree at least \( n/2 \) is hamiltonian (here \( n \) is the order of the graph). Later this result was generalized by Ore in [14], and some related results were obtained by Jackson and Wormald in [11]. Let \( \sigma_k(G) \) be the minimum sum of vertex degrees of an independent set of \( k \) vertices. Ore showed that \( \sigma_2(G) \geq n \) implies that \( G \) is hamiltonian, and Jackson and Wormald showed that \( \sigma_3(G) \geq n \) implies that \( G \) has a 2-walk (provided that \( G \) is connected). This was strengthened by Ozeki in [15] who showed that \( \sigma_5(G) \geq n \) implies that \( G \) is prism-hamiltonian.

In [16] Ozeki outlines a proof that the prism over any 3-connected planar graph of minimum degree at least 4 is hamiltonian. Equivalently, every 3-connected planar graph which is not prism-hamiltonian must have at least one cubic vertex. In particular this implies that all regular 3-connected planar graphs are prism-hamiltonian. Note however that not every 3-connected planar graph of minimum degree at least 4 is traceable (even when restricted to plane triangulations, or to regular graphs), see [17] and [24].

In this paper we give full details of a proof of the result given in [16]. In fact we even prove a slightly stronger statement which implies Ozeki’s result outlined in [16]. The proof we give in this article builds on results obtained in [9], where a method of decomposing graphs into plane chains is developed. In [9] the authors work with circuit graphs, which were originally defined in [1], and with cactuses (see Section 2 for the definition of a circuit graph and a cactus). The main result obtained in [9] is that any circuit graph, and hence also any 3-connected plane graph, has a spanning cactus as a subgraph. Here we improve this result by proving that any circuit graph with no internal cubic vertex has a spanning bipartite cactus as a subgraph. Every cactus has a 2-walk while every bipartite cactus is prism-hamiltonian. Our result thus implies that circuit graphs with all internal vertices of degree at least 4 are prism-hamiltonian.

It is interesting to note that it is possible to construct arbitrarily large 3-connected planar graphs that have small number of cubic vertices (say, less than 400), yet their prisms are not hamiltonian (see the construction given in [10]).

We mention that 3-connected planar graphs of minimum degree at least 4 also appear in [21] where the author proved that no graph in this class is hypohamiltonian.

2 \ | \ Preliminaries

We refer to [13] for terminology not defined here. Let \( G = (V(G), E(G)) \) be a graph, \( x \in V(G) \) and \( X \subseteq V(G) \). We say that \( x \) is adjacent to \( X \), if \( x \) is adjacent to some vertex of \( X \). If \( u \) and \( v \) are adjacent then \( e = uv \) denotes the edge with endvertices \( u \) and \( v \); the subgraph induced by \( u \) and \( v \) is a path denoted by \( u, v \). The union of graphs \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) is
the graph \( G \cup H = (V(G) \cup V(H), E(G) \cup E(H)) \) and the intersection of \( G \) and \( H \) is \( G \cap H = (V(G) \cap V(H), E(G) \cap E(H)) \). The graph \( G - X \) is obtained from \( G \) by deleting all vertices in \( X \) and edges incident to a vertex in \( X \). Similarly, for \( M \subseteq E(G) \), \( G - M \) is the graph obtained from \( G \) by deleting all edges in \( M \). If \( X = \{x\} \) we write \( G - x \) instead of \( G - \{x\} \).

Let \( G \) be a plane graph. Vertices and edges incident to the unbounded face of \( G \) are called external vertices and external edges, respectively. If a vertex (or an edge) is not an external vertex (or edge), then it is called an internal vertex (or an internal edge). A path \( P \) is an external or an internal path of \( G \) if all edges of \( P \) are external or internal edges, respectively.

We use \([n]\) to denote the set of positive integers not greater than \( n \). A path of odd/even length is called an odd/even path, respectively. Similarly we define odd and even faces, based on the parity of their degree. In particular, a bounded odd face of a 2-connected plane graph \( G \) is a face \( F \) such that the boundary of \( F \) is an odd cycle of \( G \) and \( F \) lies in the interior of this cycle. We say that a cycle \( C \) of a plane graph \( G \) bounds a face \( F \) if \( F \) lies in the interior of \( C \).

A prism over a graph \( G \) is the Cartesian product of \( G \) and the complete graph on two vertices \( K_2 \). A cactus is a connected graph \( G \) such that every block of \( G \) is either a \( K_2 \) or a cycle, and such that every vertex of \( G \) is contained in at most two blocks of \( G \) (the last condition is usually omitted, however for us it will be crucial, so we include it in the definition). The following proposition is given in [8] (theorem 2.3). For the sake of completeness we include the proof of it also here.

**Proposition 2.1.** Every bipartite cactus is prism-hamiltonian.

**Proof.** We denote \( V(K_2) = \{a, b\} \). We use induction to prove the following stronger statement. Every prism \( G \square K_2 \) over a bipartite cactus \( G \) has a hamiltonian cycle \( C \) such that for every vertex \( x \in V(G) \) which is not a cutvertex of \( G \), we have \( (x, a)(x, b) \in E(C) \). This is clearly true when \( G \) is an even cycle or \( K_2 \).

Let \( G \) be a bipartite cactus and assume that the statement is true for all bipartite cactuses with fewer vertices than \( V(G) \). If \( G \) has no cutvertex, then \( G \) is an even cycle or \( K_2 \). Otherwise there is a cutvertex \( u \) of \( G \), and by the definition of a cactus, \( u \) is contained in exactly two blocks of \( G \).

Let \( G_1 \) and \( G_2 \) be connected components of \( G - u \), and let \( G_1 = G - G_2 \) and \( G_2 = G - G_1 \). Both \( G_1 \) and \( G_2 \) are bipartite cactuses. Moreover, \( u \) is not a cutvertex in \( G_i \), for \( i = 1, 2 \). By induction hypothesis there is a hamiltonian cycle \( C_i \) in the prism over \( G_i \) such that \( C_i \) uses the edge \( (x, a)(x, b) \) for every vertex \( x \in V(G_i) \) which is not a cutvertex of \( G_i \) (note that this is in particular true when \( x = u \)). The desired hamiltonian cycle in \( G \square K_2 \) is \( (C_1 \cup C_2) - e \), where \( e = (u, a)(u, b) \). Observe that every vertex which is not a cutvertex of \( G \) is also not a cutvertex of \( G_1 \) or \( G_2 \). It follows that for every vertex \( x \in V(G) \) which is not a cutvertex of \( G \), we have \( (x, a)(x, b) \in E(C) \).

If \( G \) is plane graph and \( H \) is a subgraph of \( G \), then \( H \) is also a plane graph and the embedding of \( H \) in the plane is the one given by \( G \).

Let \( G \) be a plane graph and \( G^+ \) the graph obtained from \( G \) by adding a vertex to \( G \) and making it adjacent to all external vertices of \( G \). The graph \( G \) is a circuit graph if \( G^+ \) is 3-connected. It follows from this definition that a plane graph is a circuit graph if and only if it is obtained from a 3-connected plane graph \( H \) by deleting all vertices that lie in the exterior of a cycle of \( H \). Another characterization of circuit graphs is given in the following proposition (which is not difficult to prove).
Proposition 2.2. Let $G$ be a plane graph with outer cycle $C$. Then $G$ is a circuit graph if and only if it is 2-connected and for any separating set $S \subseteq V(G)$ of size 2 every connected component of $G - S$ intersects $C$.

Note also that any circuit graph is 2-connected and hence the boundary of every face of a circuit graph is a cycle.

A graph $G$ is a chain of blocks if the block-cutvertex graph of $G$ is a path. We denote the blocks and cutvertices of $G$, by

$$B_1, b_1, B_2, ..., b_{n-1}, B_n,$$

where $B_i$ are blocks for $i \in [n]$, and $b_i \in V(B_i) \cap V(B_{i+1})$ are cutvertices of $G$ for $i \in [n - 1]$. A plane graph $G$ is a plane chain of blocks if it is a chain of blocks

$$G = B_1, b_1, B_2, ..., b_{n-1}, B_n$$

such that every external vertex of $B_i, i \in [n]$ is also an external vertex of $G$. The following lemma is given in [9] (lemma 3).

Lemma 2.3. Let $G$ be a circuit graph with outer cycle $C$ and let $x \in V(C)$. Let $x'$ and $x''$ be the neighbors of $x$ in $C$. Then

(i) $G - x$ is a plane chain of blocks $B_1, b_1, B_2, ..., b_{n-1}, B_n$ and each nontrivial block of $G - x$ is a circuit graph, and

(ii) setting $x' = b_0$ and $x'' = b_n$, then $B_i \cap C$ is a path in $C$ with endvertices $b_{i-1}$ and $b_i$, for every $i \in [n]$.

It follows from the above lemma that for every nontrivial block $B_i$ of $G - x$, with outer cycle $C_i$, $C_i$ is the union of two $b_{i-1}b_i$-paths $P_i$ and $P'_i$, where $P_i$ is an internal path in $G$ and $P'_i$ is an external path in $G$.

3 | AN OUTLINE OF THE PROOF

In this section, we give essential definitions and an outline of the proof of Theorem 3.1.

Theorem 3.1. Let $B$ be a circuit graph such that every internal vertex of $B$ is of degree at least 4. Then $B$ has a spanning bipartite cactus.

It follows from the above theorem and Proposition 2.1 that every circuit graph with all internal vertices of degree at least 4 (and hence also every polyhedral graph of minimum degree 4) is prism-hamiltonian.

If $G$ is a circuit graph then it will be very useful whenever the prism over $G$ has a hamiltonian cycle which uses vertical edges (edges between the two layers of $G \square K_2$) at two predetermined vertices $x$ and $y$ of $G$. Such a cycle does not always exist and the following definition describes situations in which it does not (see Figure 1).
Definition 3.2. Let $G$ be a circuit graph with outer cycle $C$ and let $x, y \in V(C), x \neq y$. We say that $G$ is bad with respect to $x$ and $y$ if

(i) $G$ has exactly one bounded odd face $F$;
(ii) $x$ and $y$ are incident to $F$, and
(iii) if $x$ and $y$ are adjacent, then $e = xy$ is an internal edge of $G$.

We say that $G$ is good with respect to $x$ and $y$ if it is not bad with respect to $x$ and $y$.

It turns out (and follows from the proof of Theorem 3.1) that for any circuit graph $G$ with all internal vertices of degree at least 4, and any external vertices $x$ and $y$ of $G$ such that $G$ is good with respect to $x$ and $y$, there is a hamiltonian cycle in $G \Box K_2$ that uses vertical edges at $x$ and $y$. Conversely, if $G$ is a circuit graph with all internal vertices of degree at least 4 and $G$ is bad with respect to $x$ and $y$, then a hamiltonian cycle in $G \Box K_2$ that uses vertical edges at $x$ and $y$ need not exist. For example, an odd cycle is bad with respect to any two nonadjacent vertices and the prism over an odd cycle has no hamiltonian cycle that uses vertical edges at two nonadjacent vertices of this cycle. In fact, we are not aware of an example of a circuit graph which is bad with respect to $x$ and $y$ and its prism has a hamiltonian cycle such that this cycle uses vertical edges at $x$ and $y$; moreover, we conjecture that such a circuit graph does not exist.

Note also that a bipartite circuit graph $B$ is good with respect to any two external vertices of $B$ (this fact we shall use frequently). To simplify the formulation of statements, we also say that the complete graph $K_2$ is good with respect to both of its vertices.

Definition 3.3 (given below) hints at the main idea of the proof of Theorem 3.1, which we can describe by the following construction. Given two circuit graphs $B_1$ and $B_2$ that share a common vertex $b_1$, and assuming that prisms over $B_1$ and $B_2$ both have a hamiltonian cycle that uses the vertical edge at $b_1$, we can form a hamiltonian cycle in the prism over $B_1 \cup B_2$ (take the union of both hamiltonian cycles in prisms over in $B_1$ and $B_2$ and remove the vertical edge at $b_1$). This motivates the following definition.

Definition 3.3. Let $G = B_1, b_1, B_2, ..., b_{n-1}, B_n$ be a plane chain of blocks such that each nontrivial block $B_i$ is a circuit graph. Let $b_0 \neq b_1$ be an external vertex of $B_1$, and $b_n \neq b_{n-1}$...
be an external vertex of \( B_n \). We say that \( G \) is a good chain with respect to \( b_0 \) and \( b_n \) if \( B_i \) is good with respect to \( b_{i-1} \) and \( b_i \) for every \( i \in [n] \). Furthermore, we say that \( G \) is good with respect to \( b_0 \) if \( B_i \) is good with respect to \( b_{i-1} \) and \( b_i \) for every \( i \in [n - 1] \).

Note that by Definition 3.3, if \( G = B_i \) has only one block then \( G \) is a good chain with respect to any external vertex of \( G \). In the sequel “chain” is an abbreviation for “plane chain of blocks.” Two chains are disjoint if they have no common vertex.

We use good chains in the following definition which plays a central role in the proof of Theorem 3.1 (see Figure 2).

**Definition 3.4.** Let \( B \) be a circuit graph with outer cycle \( C \), and let \( u_1, u_2, u_3 \in V(C) \). A (possibly empty) set of pairwise disjoint chains \( \mathcal{C} \) is an \([u_1, u_2, u_3]\)-set of chains in \( B \) if there exists an even cycle \( C' \) in \( B \) such that

1. \( V(B) \setminus V(C') \subseteq \bigcup \mathcal{C} \);
2. for every \( G \in \mathcal{C} \), \( G \) intersects \( C' \) in exactly one vertex \( x_G \), where \( x_G \) is an external vertex of an endblock of \( G \), and \( G \) is a good chain with respect to \( x_G \);
3. no chain of \( \mathcal{C} \) contains more than one vertex in \( \{u_1, u_2, u_3\} \), and
4. for \( j \in [3] \), either there exists a chain \( G \in \mathcal{C} \) which is good with respect to \( u_j \) and \( x_G \), or \( u_j \notin \bigcup \mathcal{C} \).

A cycle \( C' \) that fulfills (i)–(iv) is called a \( \mathcal{C} \)-cycle.

We supplement the above definition with the following. Let \( B, C \) and \( C' \) be as in Definition 3.4 and suppose that \( u_1, u_2 \in V(C) \) are given. If \( \mathcal{C} \) fulfills (i), (ii), and (iv) for \( j \in [2] \), and no chain of \( \mathcal{C} \) contains \( u_1 \) and \( u_2 \), then \( \mathcal{C} \) is called a \([u_1, u_2]\)-set of chains in \( B \). Note also that \( \mathcal{C} = \emptyset \) is an \([u_1, u_2, u_3]\)-set of chains in \( B \) if and only if there is a hamiltonian cycle of even order in \( B \).

**FIGURE 2** A \([u_1, u_2, u_3]\)-set of chains in \( B \). Each block of a chain is represented by a cycle (some edges of the graph are not drawn). All vertices of \( B \), which are not contained in \( C' \) (cycle denoted with bold lines), are contained in the union of chains \( G_i, i \in [6] \).
Lemmas 3.5 and 3.6 are the main tools in proving Theorem 3.1. Here we give the statements of these two results and postpone the proofs to the following section.

**Lemma 3.5.** Let $B$ be a bipartite circuit graph with outer cycle $C$, and let $u_1, u_2, u_3$ be any vertices of $C$. If all internal vertices of $B$ are of degree at least 4, then there exists a $[u_1, u_2, u_3]$-set of chains in $B$.

**Lemma 3.6.** Let $B$ be a nonbipartite circuit graph with outer cycle $C$. Suppose that $x, y \in V(C)$ and that $B$ is good with respect to $x$ and $y$. If every internal vertex of $B$ is of degree at least 4 and $B$ is not an odd cycle, then there exists an $[x, y]$-set of chains in $B$.

Now we can prove our main result.

**Proof of Theorem 3.1.** Let $C$ be the outer cycle of $B$. We prove a slightly stronger statement: if $B$ is a circuit graph such that every internal vertex of $B$ is of degree at least 4, and $x, y \in V(C)$ are vertices such that $B$ is good with respect to $x$ and $y$, then $B$ has a spanning bipartite cactus $T$ such that $x$ and $y$ are not cutvertices of $T$.

The proof is by induction on $|V(B)|$. The statement is clearly true if $B$ is an even cycle. If $B$ is an odd cycle and $B$ is good with respect to $x$ and $y$ then $x$ and $y$ are adjacent. A spanning $xy$-path in $B$ is a bipartite spanning cactus in $B$ such that $x$ and $y$ are not cutvertices of this cactus.

If $B$ is not an odd cycle, then by Lemma 3.6 (if $B$ is nonbipartite) and Lemma 3.5 (if $B$ is bipartite), there is an $xy$-set of chains $G$ in $B$. Let $C'$ be a $G$-cycle. If $B'$ is a block of a chain $G \in C$, and $B'$ is not an endblock of $G$, then $B'$ is good with respect to both cutvertices of $G$ contained in $B'$. If $B'$ is an endblock of $G$, then $B'$ is good with respect to a cutvertex of $G$ contained in $B'$ (if $B'$ contains a cutvertex of $G$) and with respect to the vertex of $C'$ contained in $B'$ (if $B'$ contains a vertex of $C'$). Moreover, either $x \notin \bigcup C$ or a chain of $C$ is good with respect to $x$ (a similar fact is true for $y$). We can use the induction hypothesis on $B'$, to obtain a spanning bipartite cactus $T(B')$ in $B'$ such that (both) cutvertices of $G$ contained in $B'$ are not cutvertices of $T(B')$. Moreover, the block $B_x$ that contains $x$ (if any) has a spanning bipartite cactus $T(B_x)$ such that $x$ is not a cutvertex of $T(B_x)$ (a similar fact is true for $y$). Let $B$ be the set of all blocks of chains $G \in C$ and define $T = C' \cup \bigcup_{B' \in B} T(B')$. This gives the required bipartite cactus in $B$. □

# 4 TECHNICAL DETAILS

In this section, we give a series of technical lemmas with the final objective to prove Lemma 3.5 and Lemma 3.6.

**Lemma 4.1.** Let $B$ be a bipartite circuit graph with outer cycle $C$ such that all internal vertices of $B$ are of degree at least 4. Then $B$ has at least 4 external vertices of degree 2.

**Proof.** Let $C$ be a $k$-cycle, $k \geq 4$. Let $F$ be the set of faces of $B$, and set $e = |E(B)|$, $v = |V(B)|$ and $f = |F|$. Since $B$ is bipartite

$$2e = \sum_{F \in F} \deg(F) \geq 4(f - 1) + k.$$
We use Euler's formula to obtain
\[ \sum_{x \in V(B)} \deg(x) = 2e \leq 4v - k - 4. \]

Since every internal vertex of \( B \) is of degree at least 4 we get
\[ \sum_{x \in V(C)} \deg(x) \leq 3k - 4. \]

Since all vertices of \( C \) are of degree at least 2, the claim of the lemma follows from the pigeonhole principle.

The following lemma is a well known fact.

**Lemma 4.2.** A plane graph \( G \) is bipartite if and only if all bounded faces of \( G \) are even.

The following lemma is a fundamental result which will be applied frequently in subsequent results, in particular in the proof of Lemma 4.4 and Lemma 4.5.

**Lemma 4.3.** Let \( B \) be a circuit graph with outer cycle \( C \) such that all internal vertices of \( B \) are of degree at least 4. Let \( x \) and \( y \) be any vertices of \( C \), and \( Q \) an \( xy \)-path in \( C \). Suppose that all vertices in \( V(C) \setminus V(Q) \) are of degree at least three in \( B \). Then \( B \) is good with respect to \( x \) and \( y \).

**Proof.** Suppose to the contrary, that \( B \) is bad with respect to \( x \) and \( y \). Then \( x \) and \( y \) are incident to odd face \( F \) of \( B \), and \( F \) is the only bounded odd face of \( B \). Moreover, \( x \) and \( y \) are not adjacent in \( C \). It follows that \( B - \{x, y\} \) has exactly two components.

Let \( H \) be the component of \( B - \{x, y\} \) that contains a vertex of \( Q \). If \( xy \in E(B) \) and \( F \) is contained in the exterior of the cycle \( E(Q) \cup \{xy\} \) define \( H' = (B - V(H)) - xy \). Otherwise define \( H' = B - V(H) \). The graph \( H' \) is a plane chain of blocks and each nontrivial block of \( H' \) is a bipartite circuit graph (by Lemma 4.2). Let
\[ H' = D_1, d_1, D_2, \ldots, d_m, D_m, \]
and denote \( d_0 = x \) and \( d_m = y \). If \( j \in [m] \) and \( u \in V(D_j) \setminus \{d_{j-1}, d_j\} \), then \( \deg_{D_j}(u) > 2 \). So if \( D_j \) is nontrivial, then it has at most two vertices of degree 2 in \( D_j \); since \( D_j \) is bipartite this contradicts Lemma 4.1. It follows that all blocks of \( H' \) are trivial. If \( H' \) is \( K_2 \) then \( x \) and \( y \) are adjacent in \( C \) (a contradiction), otherwise a vertex in \( V(C) \setminus V(Q) \) is of degree \( \leq 2 \) (this contradicts the assumption of the lemma).

**Lemma 4.4.** Let \( B \) be a circuit graph with outer cycle \( C \) such that all internal vertices of \( B \) are of degree at least 4. Let \( x \in V(C) \) be any vertex and
\[ B - x = B_1, b_1, B_2, \ldots, b_{n-1}, B_n. \]
Let $b_0 \in V(B_1)$ and $b_n \in V(B_n)$ be the neighbors of $x$ in $C$. Then for every $i \in [n]$, $B_i$ is good with respect to $b_{i-1}$ and $b_i$.

**Proof.** Let $B_i$ be a nontrivial block with outer cycle $C_i$, and define $Q = C \cap B_i$. The graph $Q$ is a path in $C_i$ with endvertices $b_{i-1}$ and $b_i$, and every vertex in $V(C_i) \setminus V(Q)$ is of degree more than 2 in $B_i$. By Lemma 4.3, $B_i$ is good with respect to $b_{i-1}$ and $b_i$. \qed

The following lemma is another technical result frequently applied later.

**Lemma 4.5.** Let $B$ be a circuit graph with outer cycle $C$ such that all internal vertices of $B$ are of degree at least 4. Let $x \in V(C)$ be any vertex, and let

$$B - x = B_1, b_1, B_2, ..., b_{n-1}, B_n.$$

Then for every $k \in [n - 1]$ the graph

$$G = B - \bigcup_{i=k+1}^{n} V(B_i)$$

is a good chain with respect to $x$.

**Proof.** Let $k \in [n - 1]$ be given, and let $b_0 \in V(B_1)$ be the neighbor of $x$ in $C$. Denote the path $x, b_0$ by $B_0$.

**Case 1.** Suppose that $B_k$ is trivial.

If $x$ is not adjacent to a vertex in $G - b_0$, then $G$ is a plane chain of blocks

$$B_0, b_0, B_1, ..., b_{k-2}, B_{k-1}$$

and, by Lemma 4.4, $B_i$ is good with respect to $b_{i-1}$ and $b_i$ for $i \in [k - 1]$. Assume therefore that $x$ is adjacent to a vertex in $G - b_0$. Let $\ell \in [k]$ be the maximum number such that $x$ is adjacent to $B_{\ell} - \{b_{\ell-1}, b_{\ell}\}$. Since $B_k$ is trivial, $\ell \neq k$. The graph $H$ induced by $\bigcup_{i=0}^{\ell} V(B_i)$ is a nontrivial block of $G$. Moreover, since $H$ is a subgraph of $B$ bounded by a cycle of $B$, $H$ is a circuit graph. Note also that if $x$ and $b_{\ell}$ are incident to a bounded face $F$ of $H$, then $x$ and $b_{\ell}$ are adjacent, moreover $xb_{\ell}$ is an external edge of $H$. It follows that $H$ is good with respect to $x$ and $b_{\ell}$, and therefore

$$G = H, b_{\ell}, B_{\ell+1}, ..., b_{k-2}, B_{k-1}$$

is a good chain with respect to $x$.

**Case 2.** Suppose that $B_k$ is nontrivial.

By Lemma 2.3, $B_k - b_k$ is a plane chain of blocks, so let

$$B_k - b_k = D_1, d_1, D_2, ..., d_{m-1}, D_m.$$
Let $C_k$ be the outer cycle of $B_k$, and $d_0 \in V(D_1)$, $d_m \in V(D_m)$ be the neighbors of $b_k$ in $C_k$. Without loss of generality assume that $b_k, d_0$ is an internal edge of $B$. Note that $D_1$ is nontrivial if $b_{k-1} \neq d_0$, for otherwise $\deg_B(d_0) \leq 3$ (this is a contradiction because $d_0$ is an internal vertex of $B$ if $b_{k-1} \neq d_0$).

Suppose that $x$ is adjacent to $D_1 - \{b_{k-1}, d_1\}$ (it is possible that $b_{k-1} = d_1$). Then $D_1$ is nontrivial. Let $j \in [m]$ be such that $b_{k-1} \in V(D_j) \setminus \{d_{j-1}\}$ and let $H'$ be the graph induced by

$$
\bigcup_{i=0}^{k-1} V(B_i) \cup \bigcup_{i=1}^j V(D_i).
$$

The graph $H'$ is bounded by a cycle of $B$, so it is a circuit graph. We shall prove that $H'$ is good with respect to $x$ and $d_j$. Suppose that $x$ and $d_j$ are incident to a face $F'$ of $H'$, and that $F'$ is the only bounded odd face of $H'$. Then $d_j = b_{k-1}$, and all bounded faces of $D_1$ are even. This contradicts Lemma 4.1, because $\deg_{D_1}(u) > 2$ for every $u \in V(D_1) \setminus \{d_0, d_1\}$. It follows that

$$
G = H', d_j, D_{j+1}, ..., d_{m-1}, D_m
$$

is a good chain with respect to $x$.

Assume therefore that $x$ is not adjacent to $D_1 - \{b_{k-1}, d_1\}$. We claim that $|V(D_1) \cap V(C)| \geq 2$. To prove the claim suppose the contrary, that $|V(D_1) \cap V(C)| < 2$. Then $\{b_k, d_1\}$ is a separating set in $B$, and $D_1 - d_1$ is a component of $B - \{b_k, d_1\}$ disjoint with $C$. It follows that $B$ is not a circuit graph (see Proposition 2.2), a contradiction. This proves the claim.

Define $\ell'$ and $H$ as in Case 1. We claim that

$$
G = H, b_{\ell'}, B_{\ell'+1}, ..., B_{k-1}, b_{k-1}, D_1, d_1, ..., d_{m-1}, D_m
$$

is a good chain with respect to $x$ (see Figure 3). We have already shown (in Case 1) that $H$ is good with respect to $x$ and $b_{\ell'}$. By Lemma 4.4, $B_i$ is good with respect to $b_{i-1}$ and $b_i$ for

![Figure 3](image-url)
The following definition (see Figure 4) is crucial for the rest of the paper—we use it to prove Lemma 3.5 and Lemma 3.6.

**Definition 4.6.** Let $B$ be a circuit graph with outer cycle $C$. Let $\{x, y\} \subseteq V(C)$ and $\{u_1, u_2\} \subseteq V(C)$ be any sets. A (possibly empty) set of pairwise disjoint chains $C$ is an $(x, y; u_1, u_2)$-set of chains in $B$ if there exists an $xy$-path $P$ in $B$ such that

(i) $V(B) \setminus V(P) \subseteq \bigcup C$;
(ii) for every $G \in C$, $G$ intersects $P$ in exactly one vertex $x_G$, and $G$ is a good chain with respect to $x_G$;
(iii) no chain of $C$ contains $u_1$ and $u_2$,
(iv) for $j \in [2]$, either there exists a chain $G \in C$ which is good with respect to $u_j$ and $x_G$, or $u_j \notin \bigcup C$.

A path $P$ that fulfills (i)–(iv) is called a $C$-path. The set $C$ is an odd or an even $(x, y; u_1, u_2)$-set of chains if there exists an odd or an even $C$-path, respectively.

We supplement the above definition with the following. Let $B$, $C$ and $P$ be as in Definition 4.6 and suppose that $u_1 \in V(C)$ is given. We say that a set of pairwise disjoint chains $C$ is an $(x, y; u_1)$-set of chains if it satisfies (i),(ii), and (iv) for $j = 1$. Moreover, $C$ is an $(x, y)$-set of chains if it satisfies (i) and (ii) of Definition 4.6.

We also use Definition 4.6 in slightly more general settings in which $B$ is a plane chain of blocks (and each block is a circuit graph). More precisely, if $B$ is a plane chain of blocks, $x$, $y$ are two external vertices of $B$, and $C$ is a set of pairwise disjoint plane chains that satisfy (i)–(iv), then $C$ is an $(x, y; u_1, u_2)$-set of chains in $B$. 

![Figure 4](image-url)

*Figure 4* An $(x, y; u_1, u_2)$-set of chains in $B$. All vertices of $B$, which are not on the $xy$-path $P$ (denoted with bold lines), are contained in the union of chains $G_i$, $i \in [8]$. 

\[ i \in [k - 1] \setminus [\ell], \text{ and } D_i \text{ is good with respect to } d_{i-1} \text{ and } d_i \text{ for } i \in [m], \ i \neq 1. \] It remains to prove that $D_1$ is good with respect to $b_{k-1}$ and $d_1$. Let $C'$ be the outer cycle of $D_1$ and let $Q = C \cap D_1$ (or equivalently $Q = C \cap C'$). Note that for every vertex $z \in V(C') \setminus V(Q)$, $\deg_{D_1}(z) > 2$. By Lemma 4.3, $D_1$ is good with respect to $b_{k-1}$ and $d_1$. \[ \square \]
Suppose that $G$ is a plane chain of blocks and $B$ is a block of $G$. Given an $(x, y; u, v)$-set of chains $C$ in $B$ it is sometimes possible to add a couple of chains to $C$ to obtain an $(x, y; u', v')$-set of chains in $G$. This has to be done in such a way that all newly added chains are attached at an endblock of a chain of $C$ or at a “free” vertex of the $C$-path. This is more precisely described in the following lemma.

**Lemma 4.7.** Let $G$ be a bipartite plane chain of blocks

$$G = B_1, b_1, B_2, ..., b_{n-1}, B_n$$

such that for $i \in [n]$ each nontrivial block $B_i$ of $G$ is a circuit graph with outer cycle $C_i$. Suppose that $u, x, y \in V(C_j), u \neq b_j$, and that $C$ is an $(x, y; u, b_j)$-set of chains in $B_j$ for some $j \in [n]$. Then for every $\ell > j$ and any vertex $v \in V(C_\ell) \setminus V(C_{\ell-1})$, there is an $(x, y; u, v)$-set of chains in $\bigcup_{i=j}^{\ell} B_i$. Moreover, if $u = b_{j-1}$, then for every $\ell' < j$ and any vertex $v' \in V(C_{\ell'}) \setminus V(C_{\ell'+1})$, there is an $(x, y; v', v)$-set of chains in $\bigcup_{i=\ell'}^{\ell} B_i$.

**Proof:** Let $\ell > j$ and $v \in V(C_\ell) \setminus V(C_{\ell-1})$. Suppose that $C = \{G_1, ..., G_k\}$ is an $(x, y; u, b_j)$-set of chains in $B_j$ and that $P$ is a $C$-path. Then (a) or (b) occurs.

(a) There is a chain $G_r \in C$ such that $G_r$ is a good chain with respect to $x_r$ and $b_j$, where $\{x_r\} = V(G_r) \cap V(P)$;

(b) $b_j \not\in \bigcup_{i=1}^{\ell} V(G_i)$.

Since $G$ is bipartite every block of $G$ is good with respect to any two external vertices of this block. Therefore in case (a), $G' = G_r \cup \bigcup_{i=j+1}^\ell B_i$ is a good chain with respect to $x_r$ and $v$, and hence $C' = C \cup \{G'_r\} \setminus \{G_r\}$ is an $(x, y; u, v)$-set of chains in $\bigcup_{i=j}^{\ell} B_i$. In case (b), $G_0 = \bigcup_{i=j+1}^{\ell} B_i$ is a good chain with respect to $b_j$ and $v$, and therefore $C' = \{G_0, ..., G_k\}$ is an $(x, y; u, v)$-set of chains in $\bigcup_{i=j}^{\ell} B_i$. In both cases a $C'$-path is $P$. The last sentence of the lemma is proved analogously. 

**Remark 4.8.** A similar proof as above shows that an $(x, y; v)$-set of chains in $\bigcup_{i=j}^{\ell} B_i$ exists under the assumption that there is a $(x, y; b_j)$-set of chains in $B_j$ and $B_i$ is bipartite for $i > j$ (but $G$ may be nonbipartite), and that it has the same parity as the initial $(x, y; b_j)$-set of chains in $B_j$.

The following Theorem can be obtained directly from Theorem 5 in [9] by adding the assumption that $B$ is bipartite.

**Theorem 4.9.** Let $B$ be a bipartite circuit graph with outer cycle $C$. If $x, y \in V(C)$, then for any vertex $u \in V(C)$ (not necessarily distinct from $x$ and $y$) there exists an $(x, y; u)$-set of chains in $B$.

The following lemma is a technical detail which we need later in the proof of Lemma 4.11 and Lemma 4.14.
Lemma 4.10. Let $B$ be a circuit graph with outer cycle $C$ such that all internal vertices of $B$ are of degree at least 4. Let $x$ and $y$ be any vertices of $C$ and $Q$ an $xy$-path in $C$ such that all vertices in $V(C) \setminus V(Q)$ are of degree at least three in $B$. If $B - x$ is bipartite, then $|V(B_i \cap Q)| \geq 2$ for every block $B_i$ of $B - x$.

Proof. If $V(C) = V(Q)$, the lemma follows from Lemma 2.3. Assume $V(C) \neq V(Q)$, and let $u \in V(C) \setminus V(Q)$ be the neighbor of $x$ in $C$ (see Figure 5). The block $B_1$ of $B - x$ containing $u$ is nontrivial, for otherwise $\deg_B(u) = 2$. If $|V(B_1 \cap Q)| < 2$, then $B_1$ has at most two vertices of degree two in $B_1$ (vertices $u$ and $b_1$). Therefore, by Lemma 4.1, $B_1$ is nonbipartite and hence $B - x$ is nonbipartite, a contradiction. It follows that $b_1 \in V(Q)$ and $b_1 \neq y$ and therefore $|V(B_i \cap Q)| \geq 2$ for every block $B_i$ of $B - x$. □

We continue with a series of lemmas which are all proved in a similar way. The method of proving them can roughly be described as follows: suppose that we wish to prove the existence of a certain $(x, y; u, v)$-set of chains in a circuit graph $B$. We do this by deleting a vertex from $B$, say vertex $x$, and observe the obtained plane chain of blocks

$$B - x = B_1, b_1, B_2, ..., b_{n-1}, B_n.$$ 

Then for each nontrivial block $B_i$, $i \in I$ (where $I$ is a subset of $[n]$ determined later) we either use the induction hypothesis, if the proof is inductive, or we use one of the previously proved lemmas to obtain a suitable set of chains $C_i$ in $B_i$ with the corresponding path $P_i$ (which normally has endvertices $b_{i-1}$ and $b_i$, or possibly also $y$). If this is done appropriately, then $C = \cup_{i \in I} B_i$ is the desired $(x, y; u, v)$-set of chains in $B$ and $\cup_{i \in I} P_i$ is the corresponding $C$-path. The details are described in the sequel.

Lemma 4.11. Let $B$ be a bipartite circuit graph with outer cycle $C$. Suppose that $x, y \in V(C)$ and that $Q$ is an $xy$-path in $C$. If every internal vertex of $B$ is of degree at least 4 and every vertex in $V(C) \setminus V(Q)$ is of degree at least 3 in $B$, then there exists an $(x, y; x, y)$-set of chains in $B$.

![Figure 5](image-url) The blocks of $B - x$ are $B_1, ..., B_n$. The left $xy$-path in $C$ is $Q$. 
Proof. Note that an \((x, y; x, y)\)-set of chains in \(B\) is a set of chains \(\mathcal{C}\) such that \(x\) and \(y\) belong to no chain of \(\mathcal{C}\), and such that the chains of \(\mathcal{C}\) contain all vertices of \(B\) except possibly vertices of the corresponding \(\mathcal{C}\)-path. By Lemma 2.3, \(B - x\) is a plane chain of blocks

\[
B - x = B_1, b_1, B_2, ..., b_{n-1}, B_n.
\]

By Lemma 4.10, \(|V(B_i \cap Q)| \geq 2\) for \(i \in [n]\). We may assume, without loss of generality, that \(y \in V(B_1)\) and \(y \neq b_1\) (see Figure 6).

If \(B_1\) is nontrivial then \(B_1 - y\) is a plane chain of blocks

\[
B_1 - y = D_1, d_1, D_2, ..., d_{m-1}, D_m.
\]

Let \(d_0\) be the neighbor of \(y\) in \(Q\), and choose the notation so that \(d_0 \in V(D_i)\). Define \(k = \max\{i | D_i \cap Q \neq \emptyset\}\).

In the sequel we define an \(xy\)-path \(P\), and a set of good chains that form an \((x, y; x, y)\)-set of chains in \(B\) (with the corresponding path \(P\)). We use Theorem 4.9 to obtain an \((x, y; u)\)-set of chains in each of the blocks \(D_i\) and \(B_i\) (where \(u\) is a suitable vertex determined later). The paths that correspond to these \((x, y; u)\)-sets of chains are glued together to obtain the desired path \(P\). We distinguish two cases as follows.

Case 1. Suppose that \(D_k\) intersects \(Q\) in exactly one vertex (in this case \(d_{k-1}\)). Then \(G = \bigcup_{i=k}^{n} D_i\) is a good chain with respect to \(d_{k-1}\).

If \(D_i\) is trivial define \(P_i = D_i\) and \(C_i = \emptyset\), for \(i \in [k - 1]\). If \(D_i\) is nontrivial then, by Theorem 4.9, there is a \((d_{i-1}, d_i; d_{i-1})\)-set of chains \(C_i\) in \(D_i\), for \(i \in [k - 1]\). In this case let \(P_i\) be a \(C_i\)-path in \(D_i\).

If \(B_i\) is trivial define \(R_i = B_i\) and \(\mathcal{F}_i = \emptyset\), for \(i \in [n], i \neq 1\). If \(B_i\) is nontrivial then, by Theorem 4.9, there is a \((b_{i-1}, b_i; b_{i-1})\)-set of chains \(\mathcal{F}_i\) in \(B_i\), for \(i \in [n], i \neq 1\). In this case

![Figure 6](image-url) The blocks of \(B - x\) are \(B_1, ..., B_n\). In this figure the path \(Q\) goes from \(x\) to \(y\) in clockwise direction and \(d_{k-1} = b_1\).
let \( R_i \) be a \( \mathcal{F}_i \)-path in \( B_i \). Let \( b_n \) be the neighbor of \( x \) in \( Q \), and let \( R_{n+1} \) be the path \( x, b_n \). Additionally let \( P_0 \) be the path \( y, d_0 \). Define

\[
P = \bigcup_{i=0}^{k-1} P_i \cup \bigcup_{i=2}^{n+1} R_i.
\]

Note that \( P \) is an \( xy \)-path and that

\[
\mathcal{V} \setminus \mathcal{V}(P) \subseteq \bigcup_{i=0}^{k-1} \mathcal{C}_i \cup \mathcal{G} \cup \bigcup_{i=2}^{n+1} \mathcal{F}_i.
\]

Moreover, no chain \( \mathcal{C}_i \) or \( \mathcal{F}_i \) or \( \mathcal{G} \) contains \( x \) or \( y \). It follows that \( \mathcal{G} \) together with chains \( \mathcal{C}_i, \mathcal{F}_i, \mathcal{G} \) is an \( (x, y) \)-set of chains in \( B \). If we call this set of chains \( C \), then \( P \) is a \( C \)-path.

**Case 2.** Suppose that \( D_k \) intersects \( Q \) in more than one vertex. We assume in Case 2 the same notations as in Case 1. By Theorem 4.9, there is a \( (d_{k-1}, b_1; d_k) \)-set of chains \( \mathcal{H} \) in \( D_k \). By Lemma 4.7 (see also Remark 4.8) there is a \( (d_{k-1}, b_1) \)-set of chains in \( \bigcup_{i=2}^{m} D_i \). The rest of the proof is similar as in Case 1.

If \( B_1 \) is trivial, then \( x \) and \( y \) are adjacent in \( C \) (for otherwise \( u \in V(C) \setminus V(Q) \) and \( V(Q) = V(C) \). Define \( R_1 = B_1 \), and let \( R_i, \mathcal{F}_i \), for \( i \in [n], i \neq 1 \), and \( R_{n+1} \), be as defined in Case 1. Then \( \bigcup_{i=2}^{n} \mathcal{F}_i \) is an \( (x, y; x, y) \)-set of chains in \( B \); the corresponding path is \( \bigcup_{i=2}^{n+1} R_i \).

**Lemma 4.12.** Let \( B \) be a bipartite circuit graph with outer cycle \( C \). Let \( x, y, u_1, u_2 \in V(C) \) be such that \( \{x, y\} \neq \{u_1, u_2\} \). If every internal vertex of \( B \) is of degree at least 4, then there is an \( (x, y; u_1, u_2) \)-set of chains in \( B \).

**Proof.** Suppose that the claim of the lemma is not true; let \( B \) be a counterexample with minimum number of vertices. It is easy to verify the lemma when \( B \) is an (even) cycle. We may assume, without loss of generality, that \( y \notin \{u_1, u_2\} \) (otherwise \( x \notin \{u_1, u_2\} \) and we have an analogous proof).

By Lemma 2.3, \( B - x \) is a plane chain of blocks

\[
B - x = B_1, b_1, B_2, ..., B_{n-1}, B_n.
\]

Let \( Q \) and \( Q' \) be the \( xy \)-paths in \( C \). Let \( b_0 \in V(B_1) \) and \( b_n \in V(B_n) \) be the neighbors of \( x \) in \( Q \) and \( Q' \), respectively. We set \( B_0 = \emptyset \) (to avoid ambiguity in the following definitions). Let \( k \in [n] \) be such that \( y \in V(B_k) \setminus V(B_{k-1}) \), and let \( j \in [n] \) be such that \( u_j \in V(B_k) \setminus V(B_{k+1}) \) for \( j = 1, 2 \) (if \( x \in \{u_1, u_2\} \) this applies only to \( k_1 \) and we set \( u_2 = x \)). We may assume, without loss of generality, that \( y \neq b_0 \) (otherwise \( y \neq b_n \) and we have a similar proof) and that \( k_1 \leq k_2 \).

Since \( y \notin \{u_1, u_2\} \) we have the following possibilities: (1) \( u_1, u_2 \notin V(Q') \), (2) \( u_1 \notin V(Q'), u_2 \notin V(Q) \), (3) \( u_2 = x \) and \( u_1 \notin V(Q') \). All other possibilities are symmetric, and they can be obtained from one of the above cases by exchanging the roles of \( Q \) and \( Q' \); for
example, \( u_1, u_2 \notin V(Q') \) is symmetric to \( u_1, u_2 \notin V(Q) \). Therefore we can also assume that \( u_1 \notin V(Q') \).

In the sequel we define an \((x, y; u_1, u_2)\)-set of chains in \( B \) (with the corresponding path \( P \)). We use Lemma 4.11 and the minimality of \( B \) to obtain a set of chains \( C_i \) in \( B_i \) for \( i \in [n] \); the corresponding paths \( P_i \) of these sets of chains are glued together to obtain the desired path \( P \). We give details below.

For \( i \in [k] \), if \( B_i \) is trivial define \( C_i = \emptyset \) and \( P_i = B_i \). In the sequel we define \( C_i \) and \( P_i \) for nontrivial blocks \( B_i \).

By minimality of \( B \), Lemma 4.12 is true for every nontrivial block \( B_i \) of \( B - x \) and therefore, for every \( i \in [k] \) we can apply the statement of Lemma 4.12 to \( B_i \). Observe also that the statement of Lemma 4.12 assumes that \( \{x, y\} \neq \{u_1, u_2\} \) which becomes relevant in claims (ii) to (vii) below.

Denote the outer cycle of \( B_i \) by \( C_i \). Since \( Q_i = B_i \cap C \) is a \( b_{|i-1} \cdot b_i \)-path in \( C_i \) and every vertex of \( V(C_j) \setminus V(Q) \) is of degree at least 3 in \( B_i \), we can also apply Lemma 4.11 to \( B_i \). The following statements are obtained either by an application of Lemma 4.12 or Lemma 4.11 to \( B_i \). For \( i \in [k - 1] \) and \( j = 1, 2 \) there exists:

(i) a \((b_{i-1}, b_i; b_{i-1}, b_i)\)-set of chains in \( B_i \) (by Lemma 4.11);
(ii) a \((b_{k-1}, y; b_{k-1}, b_k)\)-set of chains in \( B_k \) (by minimality of \( B \) (i.e., by the statement of Lemma 4.12) if \( y \neq b_k \), and by Lemma 4.11 if \( y = b_k \), and)
(iii) a \((b_{k-1}, b_k; b_{k-1}, u_j)\)-set of chains in \( B_k \) (by minimality of \( B \) if \( u_j \neq b_k \), and by Lemma 4.11 if \( u_j = b_k \)).

We also have the following:

(iv) if \( k_j = k \) and \( u_j \neq y \), there is a \((b_{k-1}, y; b_{k-1}, u_j)\)-set of chains in \( B_k \) (by minimality of \( B \));
(v) if \( k_j = k \) and \( u_j \neq b_k \), there is a \((b_{k-1}, y; u_j, b_k)\)-set of chains in \( B_k \) (by minimality of \( B \));
(vi) if \( k_1 = k_2 \), there is a \((b_{k-1}, b_k; u_1, u_2)\)-set of chains in \( B_k \) (by Lemma 4.11 if \( k_1 = k_2 = 1 \) and \( \{b_{k-1}, b_k\} = \{u_1, u_2\} \), and by minimality of \( B \) otherwise), and
(vii) if \( k_1 = k_2 = k \), there is a \((b_{k-1}, y; u_1, u_2)\)-set of chains in \( B_k \) (by minimality of \( B \) and since \( y \notin \{u_1, u_2\} \) by an assumption).

With the assumption \( u_1 \notin V(Q') \), and recalling that one of the possibilities (1), (2), or (3) occurs, the following cases with regard to \( k, k_1 \) and \( k_2 \) may appear. Next to each particular case below we also write which of the above statements we use to prove the existence of an \((x, y; u_1, u_2)\)-set of chains in \( B \). Later we give detailed arguments.

(a) \( k_1 < k_2 < k \), we use (i) for \( i \in [k - 1] \setminus \{k_1, k_2\} \), (iii) for \( j \in [2] \), and (ii).
(b) \( k_1 < k_2 = k \), we use (i) for \( i \in [k - 1] \setminus \{k_1\} \), (iii) for \( j = 1 \), and (iv) for \( j = 2 \).
(c) \( k_1 < k < k_2 \), we use (i) for \( i \in [k - 1] \setminus \{k_1\} \), (iii) for \( j = 1 \), and (ii).
(d) \( k_1 = k_2 < k \), we use (i) for \( i \in [k - 1] \setminus \{k_1\} \), (vi) for \( j = 1 \), and (ii).
(e) \( k_1 = k_2 = k \), we use (i) for \( i \in [k - 1] \), and (vii).
(f) \( k_1 < k \) and \( u_2 = x \), we use (i) for \( i \in [k - 1] \setminus \{k_1\} \), (ii), and (iii) for \( j = 1 \).
(g) \( k_1 = k \) and \( u_2 = x \), we use (i) for \( i \in [k - 1] \), and (v) for \( j = 1 \).
We prove cases (a), (c), and (g) in detail. Cases (b), (d), and (e) are similar to case (a), and case (f) is similar to case (g), so here we skip the details.

Case (a). Suppose that \( k_1 < k_2 < k \). By (i), there is a \((b_{i-1}, b_i; b_{i-1}, b_i)\)-set of chains \( C_i \) in \( B_i \), for \( i \in \{k-1\} \setminus \{k_1, k_2\} \). By (iii), there is a \((b_{k-1}, b_{k_i}; b_{k-1}, u_j)\)-set of chains \( C_{k_i} \) in \( B_{k_i} \) for \( j = 1, 2 \). By (ii), there is a \((b_{k-1}, y; b_{k-1}, b_{k})\)-set of chains \( C_k \) in \( B_k \). Denote by \( P_i \) a \( C_i \)-path in \( B_i \), for \( i \in \{k\} \).

By Lemma 4.5, \( G_0 = B - \bigcup_{i=1}^{k} V(B_i) \) is a good chain with respect to \( x \). Let \( P_0 \) be the path \( x, b_0 \). Define \( P = \bigcup_{i=0}^{k} P_i \) (and recall that \( P_i = B_i \), if \( B_i \) is trivial). Then \( C = \{G_0\} \cup \bigcup_{i=1}^{k} C_i \) is an \((x, y; u_1, u_2)\)-set of chains in \( B \).

Case (c). Suppose that \( k_1 < k < k_2 \). By (i) there is a \((b_{i-1}, b_i; b_{i-1}, b_i)\)-set of chains \( C_i \) in \( B_i \), for \( i \in \{k-1\} \setminus \{k_1\} \). By (iii) there is a \((b_{k-1}, b_{k_i}; b_{k-1}, u_i)\)-set of chains \( C_{k_i} \) in \( B_{k_i} \). By (ii) there is a \((b_{k-1}, y; b_{k-1}, b_{k})\)-set of chains \( C_k \) in \( B_k \).

Since \( C_k \) is a \((b_{k-1}, y; b_{k-1}, b_{k})\)-set of chains in \( B_k \), we find by Lemma 4.7 that there is a \((b_{k-1}, y; b_{k-1}, u_2)\)-set of chains \( D_k \) in \( \bigcup_{i=1}^{k} B_i \).

By Lemma 4.5, \( G_1 = B - \bigcup_{i=1}^{k} V(B_i) \) is a good chain with respect to \( x \). Then \( G_1 \) together with chains in \( C_{i} \), \( i \in \{k-1\} \), and \( D_k \) forms an \((x, y; u_1, u_2)\)-set of chains in \( B \) (the corresponding path is \( P = \bigcup_{i=0}^{k} P_i \)).

Case (g). By (i) there is a \((b_{i-1}, b_i; b_{i-1}, b_i)\)-set of chains \( C_i \) in \( B_i \), for \( i \in \{k-1\} \). Since \( u_1 \notin V(Q') \), by an assumption, we have \( u_1 \neq b_k \). By (v), there is a \((b_{k-1}, y; u_1, b_{k})\)-set of chains \( C_k \) in \( B_k \). By Lemma 4.7 there is a \((b_{k-1}, y; u_1)\)-set of chains \( F_k \) in \( \bigcup_{i=k}^{n} B_i \).

Let \( P_i \) be a \( C_i \)-path in \( B_i \), for \( i \in \{k\} \), and define \( P = \bigcup_{i=0}^{k} P_i \). Then chains in \( C_{i} \), \( i \in \{k-1\} \), and \( F_k \) form an \((x, y; u_1, x)\)-set of chains in \( B \), with \( P \) being the corresponding path.

Now we are ready to prove the first of the two lemmas announced in Section 3.

**Proof of Lemma 3.5.** By Lemma 2.3, \( B - u_3 \) is a plane chain of blocks

\[ B - u_3 = B_1, b_1, B_2, ..., b_{n-1}, B_n. \]

Let \( b_0 \in V(B_i) \) and \( b_n \in V(B_n) \) be the neighbors of \( u_3 \) in \( C \), and let \( k_j \in [n] \) be such that \( u_j \in V(B_{k_j}) \setminus V(B_{k_{j-1}}) \) for \( j = 1, 2 \) (here we set \( B_0 = \emptyset \)). We shall now construct an \([u_1, u_2, u_3]\)-set of chains in \( B \) from sets of chains in blocks \( B_i \). For \( i \in [n] \), define \( P_i = B_i \) and \( C_i = \emptyset \), if \( B_i \) is trivial. In the sequel we define \( P_i \) and \( C_i \) for nontrivial blocks \( B_i \).

**Case 1.** \( k_1 \neq k_2 \). By Lemma 4.11 and 4.12 there is

(i) a \((b_{i-1}, b_i; b_{i-1})\)-set of chains \( C_i \) in \( B_i \), if \( i \in [n] \) and \( i \notin \{k_1, k_2\} \), and
(ii) a \((b_{i-1}, b_i; b_{i-1}, u_j)\)-set of chains \( C_i \) in \( B_i \), if \( i = k_j \) for \( j = 1, 2 \).
Let \( P_i \) be the corresponding \( C_i \)-path in \( B_i \), for \( i \in [n] \). Define \( P_0 = u_3, b_0 \) and \( P_{n+1} = b_n, u_3 \) and let \( C' = \bigcup_{i=0}^{n+1} P_i \) and

\[
C = \bigcup_{i=1}^{n} C_i.
\]

Note that, for \( j = 1, 2 \), either \( C_{kj} \) contains a chain which is good with respect to \( u_j \) or \( u_j \) is not contained in any chain of \( C_i \) if \( i \neq k_j \) (for \( j = 1, 2 \)). It follows that \( C \) is an \([u_1, u_2, u_3]\)-set of chains, and \( C' \) is the corresponding \( C \)-cycle.

**Case 2.** \( k_1 = k_2 \). By Lemma 4.11 and 4.12 there is

(i) a \([b_{i-1}, b_i; b_{i-1}, b_i]\)-set of chains \( C_i \) in \( B_i \), if \( i \neq k_i \), and

(ii) a \([b_{k_i-1}, b_{k_i}; u_1, u_2]\)-set of chains \( C_{k_i} \) in \( B_{k_i} \).

The rest of the proof is the same as above. □

The proof of the following lemma is similar to the proof of Lemma 4.7 (so we skip this proof).

**Lemma 4.13.** Let \( G \) be a bipartite plane chain of blocks

\[
G = B_1, b_1, B_2, ..., b_{n-1}, B_n.
\]

Let \( u \in V(B_j) \setminus \{b_{j-1}, b_j\} \) for some \( j \in \{2, ..., n - 1\} \), and suppose that there exists a \([b_{j-1}, b_j, u]\)-set of chains in \( B_j \). Then for any \( v' \in V(B_1) \setminus V(B_2) \) and \( v'' \in V(B_n) \setminus V(B_{n-1}) \), there exists a \([v', v'', u]\)-set of chains in \( G \).

**Lemma 4.14.** Let \( B \) be a nonbipartite circuit graph with outer cycle \( C \). Suppose that \( x, y \in V(C) \) and that \( Q \) is an \( xy \)-path in \( C \). If all internal vertices of \( B \) are of degree at least 4, every vertex in \( V(C) \setminus V(Q) \) is of degree at least 3 in \( B \), and \( B - x \) is bipartite, then for any vertex \( u \in V(C - x) \) there is an odd and an even \((x, y; u)\)-set of chains in \( B \).

**Proof.** Let \( u \) be a vertex of \( C - x \). We claim that for any neighbor \( z \) of \( x \), there is a \((x, y; u)\)-set of chains \( C \) in \( B \) such that a \( C \)-path contains the edge \( xz \). Before we prove the claim let us see how we prove the lemma using this claim. Since \( B \) is nonbipartite and 2-connected, there is an odd cycle \( C' \) containing \( x \). Let \( x_1 \) and \( x_2 \) be the neighbors of \( x \) in \( C' \). Let \( R \) be the \( x_i x_i \)-path in \( C' \) not containing \( x \). Suppose that \( R_i \) is an \( x_i y \)-path in \( B - x \) for \( i = 1, 2 \). Since \( B - x \) is bipartite, \( R_1 \cup R_2 \cup R \) is an even closed walk, and since \( R \) is odd, \( R_1 \) and \( R_2 \) have different parities. It follows that every \( x_i y \)-path in \( B - x \) is odd, and every \( x_2 y \)-path in \( B - x \) is even (or vice versa). Using the above claim and setting \( z = x_1 \) (resp. \( z = x_2 \)) we get an even (resp. odd) \((x, y; u)\)-set of chains in \( B \).

In the rest of the proof we prove the claim. Let

\[
B - x = B_1, b_1, B_2, ..., b_{n-1}, B_n
\]
and suppose that \( xz \in E(B) \). By Lemma 4.10, \( V(B_i \cap Q) \geq 2 \) for \( i \in [n] \), hence we may assume that \( y \in V(B_i) \setminus V(B_{n-1}) \). Let \( k_u, k_z \in [n] \) be such that \( u \in V(B_k) \setminus V(B_{k+1}) \) and \( z \in V(B_k) \setminus V(B_{k+1}) \) (here we set \( B_{n+1} = \emptyset \)). It follows from these definitions that \( u \neq b_k, z \neq b_k \).

We shall construct an \((x, y; u)\)-set of chains \( C \) in \( B \), and a \( C \)-path \( P \) in \( B \), so that \( P \) contains the edge \( xz \). We distinguish several cases with regard to \( k_u \) and \( k_z \). In each case we define \( P_i = B_i \), and \( C_i = \emptyset \), if \( B_i \) is trivial and \( i \in [n] \). Now we treat different cases and define \( P_i \) and \( C_i \), if \( B_i \) is nontrivial.

**Case 1.** Suppose that \( k_z < k_u < n \). By Lemma 4.11 and Lemma 4.12 there is

(i) a \((z, b_{k_z}; b_{k_z-1}, b_{k_z})\)-set of chains in \( B_k \), where \( k_z \neq k_u \);

(ii) a \((b_{i-1}, b_i; b_{i-1}, b_i)\)-set of chains in \( B_i \), for \( k_z < i < n \), \( i \neq k_u \);

(iii) a \((b_{n-1}, y; b_{n-1})\)-set of chains in \( B_n \), where \( n \neq k_u \), and

(iv) a \((b_{k_z-1}, b_{k_z}; u)\)-set of chains in \( B_u \).

Let \( C_i \) be the set of chains in \( B_i \) (as defined above), and let \( P_i \) be a \( C_i \)-path, where \( k_z \leq i \leq n \). By Lemma 4.5, \( G_0 = B - \bigcup_{i=k_z}^{n} V(B_i) \) is a good chain with respect to \( x \) in \( B \) (if \( k_z = 1 \) this is irrelevant). Let \( P_0 \) be the path \( x, z \), and define \( P = P_0 \cup \bigcup_{i=k_z}^{n} P_i \). Then

\[
C = \{G_0\} \cup \bigcup_{i=k_z}^{n} C_i
\]

is an \((x, y; u)\)-set of chains in \( B \) and \( P \) is a \( C \)-path. This proves the claim in case \( k_z < k_u < n \).

**Case 2.** Suppose that \( k_z = k_u < n \). In this case we use (ii) and (iii), and instead of (i) and (iv) we use

(v) there is a \((z, b_{k_z}; u)\)-set of chains in \( B_u \).

The rest of the proof is the same as above.

**Case 3.** Suppose that \( k_z < k_u = n \). In this case we use (i) and (ii), and instead of (iii) and (iv) we use

(vi) there is a \((b_{n-1}, y; u)\)-set of chains in \( B_n \)

and the rest of the proof is (again) the same as above (note that (v) and (vi) follow from Lemma 4.12).

**Case 4.** Suppose that \( k_u < k_z < n \). In this case we use (i), (ii) and (iii). By Lemma 4.7 and (i), there is a \((z, b_{k_z}; b_{k_z}, u)\)-set of chains \( F_{k_z} \) in \( \bigcup_{i=k_u}^{k_z} B_i \). By Lemma 4.5, \( G_1 = B - \bigcup_{i=k_u}^{n} V(B_i) \) is a good chain with respect to \( x \) in \( B \). It follows that

\[
C = \{G_1\} \cup F_{k_z} \cup \bigcup_{i=k_z+1}^{n} C_i
\]
is an \((x, y; u)-set\) of chains in \(B\). The path \(P\) (as defined above) is a \(C\)-path. This proves the claim when \(k_z \neq n\).

**Case 5.** Suppose that \(k_z = k_u = n\) and \(z \neq y\). By Lemma 4.12, there is

(vii) \((z, y; u)-set\) of chains \(\mathcal{H}_n\) in \(B_n\).

By Lemma 4.5, \(G_2 = B - V(B_n)\) is a good chain with respect to \(x\) in \(B\). It follows that \(C = \{G_2\} \cup \mathcal{H}_n\) is an \((x, y; u)-set\) of chains in \(B\), and \(P\) (as defined above) is a \(C\)-path.

**Case 6.** Suppose that \(k_z = k_u = n\) and \(z = y = u\). In this case \(xz\) is an edge of \(C\) (recall that \(y \neq b_{n-1}\) and that \(y\) is an external vertex of \(B\)). By Lemma 4.4, \(G_3 = B - y\) is a good chain with respect to \(x\). Therefore \(C = \{G_3\}\) is an \((x, y; u)-set\) of chains, where a \(C\)-path in \(B\) is the path \(x, y\).

**Case 7.** Suppose that \(k_z = k_u = n\) and \(z = y \neq u\). In this case \(B_n\) is a good chain with respect to \(y\) and \(u\), and \(G_2\) is a good chain with respect to \(x\). It follows that \(\{B_n, G_2\}\) is an \((x, y; u)-set\) of chains in \(B\); again a \(C\)-path in \(B\) is the path on two vertices \(x, y\).

**Case 8.** Suppose that \(k_u < k_z = n\). Suppose first that \(z \neq y\). By Theorem 4.9 there is

(viii) \((z, y; b_{n-1})-set\) of chains in \(B_n\).

By Lemma 4.7 and (viii), there is a \((z, y; u)-set\) of chains \(\mathcal{I}_n\) in \(\bigcup_{i=k_u}^{n} B_i\). In this case \(C = \{G_1\} \cup \mathcal{I}_n\) is an \((x, y; u)-set\) of chains in \(B\), where \(G_1 = B - \bigcup_{i=k_u}^{n} V(B_i)\).

If \(z = y\), then \(G_4 = \bigcup_{i=k_u}^{n} B_i\) is a good chain with respect to \(u\) and \(y\), hence \(C = \{G_1, G_4\}\) is an \((x, y; u)-set\) of chains in \(B\). This proves the claim and hence also the lemma. □

**Lemma 4.15.** Let \(B\) be a nonbipartite circuit graph with outer cycle \(C\), and let \(x, y \in V(C)\). If all internal vertices of \(B\) are of degree at least 4, then for any vertex \(u \in V(C)\) there is an \((x, y; u)-set\) of chains in \(B\).

**Proof:** Suppose the lemma is not true and let \(B\) be a counterexample of minimum order. We may assume that \(u \neq x\) (otherwise \(u \neq y\), and the proof is analogous). By Lemma 2.3, \(B - x\) is a plane chain of blocks

\[B - x = B_1, b_1, B_2, ..., b_{n-1}, B_n.\]

Let \(k_u, k_y \in [n]\) be such that \(u \in V(B_{k_u}) \setminus V(B_{k_u-1})\) and \(y \in V(B_{k_y}) \setminus V(B_{k_y-1})\) (here we set \(B_0 = \emptyset\)). Let \(b_0 \in V(B_1)\) and \(b_y \in V(B_n)\) be the neighbors of \(x\) in \(C\). We may assume, without loss of generality, that \(k_u \leq k_y\).

We give a construction of an \((x, y; u)-set\) of chains in \(B\), which we inductively construct from sets of chains in blocks \(B_i\). First we define \(P_1 = B_i\) and \(C_i = \emptyset\), if \(B_i\) is trivial. In the sequel we treat nontrivial blocks \(B_i\).
If \( k_u < k_y \) then, by minimality of \( B \) (if \( B_i \) is nonbipartite) and by Lemma 4.12 (if \( B_i \) is bipartite), there is

(i) a \((b_{i-1}, b_i; b_1)\)-set of chains in \( B_i \), for \( i \in [k_u - 1] \);
(ii) a \((b_{i-1}, b_i; u)\)-set of chains in \( B_i \), for \( i = k_u \);
(iii) a \((b_{i-1}, b_i; b_{i-1})\)-set of chains in \( B_i \), for \( i \in [k_y - 1] \setminus [k_u] \), and
(iv) a \((b_{i-1}, y; b_{i-1})\)-set of chains in \( B_i \), for \( i = k_y \).

If \( k_u = k_y \) and \( y \neq b_0 \) there is

(v) a \((b_{k-1}, y; u)\)-set of chains in \( B_k \).

Denote by \( C_i \) the set of chains in \( B_i \) defined by (i)–(iv) if \( k_u < k_y \); and defined by (i) and (v) if \( k_u = k_y \) and \( y \neq b_0 \). The corresponding \( C_i \)-path is denoted by \( P_i \), for \( i \in [k_2] \). Let \( P_0 \) be the path \( x, b_0 \). Define \( P = \bigcup_{i=0}^{k_2} P_i \). By Lemma 4.5, \( G_1 = B - \bigcup_{i=1}^{k_2} B_i \) is a good chain with respect to \( x \) in \( B \). Hence

\[ C = \{G_1\} \cup \bigcup_{i=1}^{k_2} C_i \]

is an \((x, y; u)\)-set of chains in \( B \). The path \( P \) is a \( C \)-path in \( B \).

If \( y = b_0 \) then \( u = y = b_0 \) (by our assumptions \( k_u \leq k_y \) and \( u \neq x \)). In this case \( G_2 = B - y \) is a good chain with respect to \( x \), by Lemma 4.4. Hence, \( C = \{G_2\} \) is an \((x, y; u)\)-set of chains, and \( P = x, y \) is the corresponding \( C \)-path. \( \square \)

Under more restrictive assumptions (than those of the above lemma) there is an odd and an even \((x, y; u)\)-set of chains in \( B \), as asserted in the following theorem.

**Theorem 4.16.** Let \( B \) be a nonbipartite circuit graph with outer cycle \( C \), and let \( x, y \in V(C) \). Suppose that all internal vertices of \( B \) are of degree at least 4 and that \( Q \) is an \( xy \)-path in \( C \) such that every vertex in \( V(C) \setminus V(Q) \) is of degree at least 3 in \( B \). Then for any vertex \( u \in V(Q) \) there is an odd and an even \((x, y; u)\)-set of chains in \( B \).

**Proof:** Suppose the theorem is not true and let \( B \) be a counterexample of minimum order. We may assume that \( u \neq x \) (otherwise \( u \neq y \), and the proof is analogous). If \( B - x \) is bipartite the theorem follows from Lemma 4.14, so assume that \( B - x \) is nonbipartite. By Lemma 2.3, \( B - x \) is a plane chain of blocks

\[ B - x = B_1, b_1, B_2, ..., b_{n-1}, B_n. \]

Let \( k_u, k_y \in [n] \) be such that \( u \in V(B_{k_u}) \setminus V(B_{k_u-1}) \) and \( y \in V(B_{k_y}) \setminus V(B_{k_y-1}) \) (here we set \( B_0 = \emptyset \)). Let \( b_0 \in V(B_1) \) and \( b_n \in V(B_n) \) be the neighbors of \( x \) in \( C \). We may assume that \( xb_0 \) is an edge of \( Q \) and \( xb_n \) is not an edge of \( Q \). By assumption \( u \in V(Q) \) and therefore \( k_u \leq k_y \).

We give a construction of an odd and an even \((x, y; u)\)-set of chains in \( B \), which we inductively construct from sets of chains in blocks \( B_i \). First we define \( P_i = B_i \) and \( C_i = \emptyset \), for \( i \in [n] \), if \( B_i \) is trivial. In the sequel we treat nontrivial blocks \( B_i \).
Case 1. $B_i$ is nonbipartite for some $i < k_y$ and $k_y \neq 1$, or $B_{k_y}$ is nonbipartite and $y = b_{k_y}$.

If $k_u < k_y$ then, by Lemma 4.15 (if $B_i$ is nonbipartite), and by Lemma 4.12 (if $B_i$ is bipartite), there is

(i) a $(b_{i-1}, b_i; b_i)$-set of chains in $B_i$, for $i \in [k_u - 1]$;
(ii) a $(b_{i-1}, b_i; u)$-set of chains in $B_i$, for $i = k_u$;
(iii) a $(b_{i-1}, b_i; b_{i-1})$-set of chains in $B_i$, for $i \in [k_y - 1] \setminus [k_u]$, and
(iv) a $(b_{i-1}, b_i; b_{i-1})$-set of chains in $B_i$, for $i = k_y$.

If $k_u = k_y$ and $y \neq b_0$ (note that in Case 1 we have $y \neq b_0$ by assumptions) there is

(v) a $(b_{k_y-1}, y; u)$-set of chains in $B_{k_y}$.

Note that for $i \in [k_y - 1]$ every vertex in $V(C) \setminus V(Q)$ is of degree at least 3 in $B_i$, where $C_i$ is the outer cycle of $B_i$ and $Q_i = Q \cap B_i$ (see Figure 7). By minimality of $B$ we may apply the statement of the theorem to $B_i$, if $B_i$ is nonbipartite. Hence, if $B_i$ is nonbipartite for some $i \in [k_y - 1]$, there is an odd and an even set of chains for (i), (ii), and (iii). Additionally, if $B_{k_y}$ is nonbipartite and $y = b_{k_y}$, then there is also an odd and an even set of chains $C_{k_y}$ for (iv) and (v) (by minimality of $B$). Thus, by assumptions of Case 1, we can choose $C_i$ to be odd or even for some $i \in [k_y]$.

Denote by $C_i$ the set of chains in $B_i$ defined by (i)-(iv) if $k_u < k_y$; and defined by (i) and (v) if $k_u = k_y$. The corresponding $C_i$-path is denoted by $P_i$, for $i \in [k_y]$. Let $P_0$ be the path $x, b_0$ and define $P = \bigcup_{i=0}^{k_y} P_i$. By Lemma 4.5, $G_1 = B - \bigcup_{i=1}^{k_y} B_i$ is a good chain with respect to $x$ in $B$. Hence

$$C = \{G_1\} \cup \bigcup_{i=1}^{k_y} C_i$$

is an $(x, y; u)$-set of chains in $B$, and $P$ is the corresponding $C$-path. Moreover, $C$ is an odd or an even set of chains subject to the (odd/even) choice of $C_i$ for an $i \in [k_y]$. This proves

\[\text{FIGURE 7} \quad \text{The blocks of } B - x \text{ are } B_1, ..., B_n. \text{ The "right" } xy\text{-path in } C \text{ is } Q.\]
the theorem if \( B_i \) is nonbipartite for some \( i \in [k_y - 1] \), or if \( B_{k_y} \) is nonbipartite and \( y = b_{k_y} \).

**Case 2.** assumptions of Case 1 are not fulfilled.

Note that assumptions of Case 1 are not fulfilled if and only if \( B_i \) is bipartite for \( i \leq k_y \), or \( B_i \) is bipartite for \( i < k_y \) and \( y \neq b_{k_y} \), or \( k_y = 1 \) and \( B_1 \) is bipartite, or \( k_y = 1 \) and \( y \neq b_1 \). Recalling that \( B - x \) is nonbipartite and therefore (in Case 2) \( B_i \) is nonbipartite for some \( i \in [n] \setminus \{k_y - 1\} \).

If \( k_u < k_y \) and \( y \neq b_{k_y} \) then, by Lemma 4.15 (if \( B_i \) is nonbipartite), and by Lemma 4.12 (if \( B_i \) is bipartite), there is

(vi) a \( (b_{i-1}, b_i; b_{i-1}) \)-set of chains in \( B_i \), for \( i \in [n] \setminus \{k_y\} \), and

(vii) a \( (b_i, y; b_{i-1}) \)-set of chains in \( B_i \), for \( i = k_y \).

If \( k_u = k_y \) and \( y \neq b_{k_y} \) there is

(viii) a \( (b_{k_y}, y; u) \)-set of chains in \( B_{k_y} \).

Denote by \( C_i \) the set of chains in \( B_i \) defined by (vi) and (vii) if \( k_u < k_y \) and \( y \neq b_{k_y} \); and defined by (vi) and (viii) if \( k_u = k_y \) and \( y \neq b_{k_y} \). The corresponding \( C_i \)-paths are denoted by \( P_i \), for \( i \in [n] \setminus \{k_y - 1\} \).

Let \( R_1 \) and \( R_2 \) be the \( yb_{k_y} \)-paths in \( C_{k_y} \) (where \( C_{k_y} \) is the outer cycle of \( B_{k_y} \)), and choose the notation so that \( b_{k_y - 1} \in V(R_1) \). Since every vertex in \( V(C) \setminus V(Q) \) is of degree at least 3 in \( B \), we find that every vertex in \( V(C_{k_y}) \setminus V(R_2) \) is of degree at least 3 in \( B_{k_y} \). Therefore, by minimality of \( B \), if \( B_{k_y} \) is nonbipartite, there is an odd and an even set of chains \( C_{k_y} \) for (vii) and (viii). Moreover, if \( B_i \) is nonbipartite for some \( i \in [n] \setminus \{k_y\} \), there is an odd and an even set of chains \( C_i \) for (vi). Recall that \( B_i \) is nonbipartite for some \( i \in [n] \setminus \{k_y - 1\} \) and therefore \( C_i \) can be chosen odd or even for some \( i \in [n] \setminus \{k_y - 1\} \). Let \( P_{n+1} \) be the path \( x, b_n \).

**Subcase 1.** Suppose that \( k_u < k_y \) and \( y \neq b_{k_y} \).

We distinguish two possibilities, \( u \neq b_{k_u} \) and \( u = b_{k_u} \).

If \( u \neq b_{k_u} \), then by (vii) and Remark 4.8 there is a \( (b_{k_y}, y; u) \)-set of chains \( D_{k_y} \) in \( \bigcup_{i=k_u}^{k_y} B_i \) (note that we can use Remark 4.8 because \( B_i \) is bipartite for \( i \in [k_y - 1] \)). By Lemma 4.5, \( G_2 = B - \bigcup_{i=k_u}^{n} B_i \) is a good chain with respect to \( x \). If we define \( P = \bigcup_{i=k_u}^{n+1} P_i \), then

\[
C = \{G_2\} \cup D_{k_y} \cup \bigcup_{i=k_y+1}^{n} C_i
\]

is an \( (x, y; u) \)-set of chains in \( B \) with the corresponding \( C \)-path \( P \). Since we can choose the parity of \( C_i \) for some \( i \in [n] \setminus \{k_y - 1\} \) (and by Remark 4.8 the parity of \( D_{k_y} \) is equal to the parity of \( C_{k_y} \)), we find that subject to this choice \( C \) is odd or even.
If \( u = b_{k_u} \), then by (vii) and Remark 4.8 there is a \((b_{k_y}, y; u)\)-set of chains \( E_{k_y} \) in \( \bigcup_{i=k_y+1}^{k_y} B_i \). By Lemma 4.5, \( G_3 = B - \bigcup_{i=k_y+1}^{n} B_i \) is a good chain with respect to \( x \). Therefore

\[
C = \{G_3\} \cup E_{k_y} \cup \bigcup_{i=k_y+1}^{n} C_i
\]

is an \((x, y; u)\)-set of chains in \( B \). The corresponding \( C \)-path is \( P = \bigcup_{i=k_y+1}^{n+1} P_i \). Since we can choose the parity of \( C_i \) for some \( i \in [n] \setminus [k_y - 1] \) we find that subject to this choice \( C \) is odd or even. This proves the theorem in subcase \( k_u < k_y \) and \( y \neq b_{k_y} \).

**Subcase 2.** Suppose that \( k_u = k_y \) and \( y \neq b_{k_y} \).

In this case, we use (vi) and (viii), and we recall that \( G_2 = B - \bigcup_{i=k_u}^{n} B_i \) is a good chain with respect to \( x \), and therefore

\[
C = \{G_2\} \cup \bigcup_{i=k_y}^{n} C_i
\]

is an \((x, y; u)\)-set of chains in \( B \). The corresponding \( C \)-path is \( P = \bigcup_{i=k_y+1}^{n+1} P_i \). As before, the parity of \( C_i \) can be chosen for some \( i \in [n] \setminus [k_y - 1] \), and subject to this choice \( C \) is odd or even. This proves Case 2 of the theorem if \( y \neq b_{k_y} \).

It remains to prove Case 2 when \( y = b_{k_y} \). If \( y = b_{k_y} \) then \( B_i \) is bipartite for \( i \leq k_y \) (by assumptions of Case 2) and hence \( k_y \neq n \) (for otherwise \( B - x \) is bipartite, which contradicts our assumption). In the remainder of this proof let \( C_i \) be the set of chains in \( B_i \) defined by (vi), for \( i \in [n] \setminus [k_y] \).

**Subcase 3.** Suppose that \( y = b_{k_y}, k_y \neq n \) and \( u \neq b_{k_u} \).

Since \( C_{k_y+1} \) is a \((b_{k_y}, b_{k_y+1}; u)\)-set of chains in \( B_{k_y+1} \), we find by Remark 4.8 that there is a \((b_{k_y}, b_{k_y+1}; u)\)-set of chains \( F_{k_y+1} \) in \( \bigcup_{i=k_y}^{k_y+1} B_i \) (note that we can use Remark 4.8 because in this subcase \( B_i \) is bipartite for \( i \leq k_y \)). Recall that \( G_2 = B - \bigcup_{i=k_u}^{n} B_i \) is a good chain with respect to \( x \) and hence

\[
C = \{G_2\} \cup F_{k_y+1} \cup \bigcup_{i=k_y+2}^{n} C_i
\]

is an \((x, y; u)\)-set of chains in \( B \). The corresponding \( C \)-path is \( \bigcup_{i=k_y+1}^{n+1} P_i \). Since \( B_i \) is bipartite for \( i \leq k_y \) we find that \( B_i \) is nonbipartite for some \( i \in [n] \setminus [k_y] \). Hence we can choose the parity of \( C_i \) for some \( i \in [n] \setminus [k_y] \) and so \( C \) is odd or even subject to this choice.

**Subcase 4.** Suppose that \( y = b_{k_y}, k_y \neq n \) and \( u = b_{k_u} \).

Let \( H_{k_y+1} \) be a \((b_{k_y}, b_{k_y+1}; u)\)-set of chains in \( \bigcup_{i=k_y+1}^{k_y+1} B_i \) (it exists; we use (vi) for \( i = k_y+1 \) and Remark 4.8) and recall that \( G_3 = B - \bigcup_{i=k_y+1}^{n} B_i \). In this case
\[ C = \{G_3\} \cup \mathcal{H}_{k+1} \cup \bigcup_{i=k+2}^n C_i \] is an \((x, y; u)\)-set of chains in \(B\). The corresponding \(C\)-path is \(\bigcup_{i=k+1}^{n+1} P_i\). With same arguments as in the previous subcase we can choose the parity of \(C_i\) for some \(i \in [n]\setminus [k,y]\) so that \(C\) is odd or even. \(\square\)

We now prove our second lemma announced in Section 3. Lemma 3.6 is a nonbipartite version of Lemma 3.5.

**Proof of Lemma 3.6.** By Lemma 2.3, \(B - x\) is a plane chain of blocks

\[ B - x = B_1, b_1, B_2, ..., b_{n-1}, b_n. \]

Let \(b_0 \in V(B_1)\) and \(b_n \in V(B_n)\) be the neighbors of \(x\) in \(C\), and define \(P_0 = x, b_0\) and \(P_{n+1} = b_n, x\). Let \(k \in [n]\) be such that \(y \in V(B_k)\setminus V(B_{k-1})\) (here we set \(B_0 = \emptyset\)).

If \(B_i\) is trivial, define \(C_i = \emptyset\) and \(P_i = B_i\), for \(i \in [n]\). Otherwise, if \(B_i\) is nontrivial, by Lemma 4.15 and Lemma 4.12 there is a nonbipartite \(k\)-path \(P_{ki}\)

\[ (i) \text{ a } (b_{i-1}, b_i; b_j)\text{-set of chains in } B_i, \text{ for } i \in [k-1]; \]

\[ (ii) \text{ a } (b_{i-1}, b_i; y)\text{-set of chains in } B_i, \text{ for } i = k, \text{ and} \]

\[ (iii) \text{ a } (b_{i-1}, b_i; b_{i-1})\text{-set of chains in } B_i \text{ for } i \in [n]\setminus [k]. \]

Denote by \(C_i\) the set of chains in \(B_i\) defined by (i), (ii) and (iii). The corresponding \(C_i\)-paths are denoted by \(P_i\) for \(i \in [n]\). Define \(C' = \bigcup_{i=0}^{n+1} P_i\).

Suppose that \(B_i\) is nonbipartite for some \(i \in [n]\). Then there is an odd and an even set of chains \(C_i\) for (i), (ii) or (iii). Hence, we can choose \(C_i\) and \(P_i\) so that \(C'\) is even, and therefore \(C = \bigcup_{i=1}^{n+1} C_i\) is an \([x, y]\)-set of chains in \(B\).

Suppose that all blocks \(B_i, i \in [n]\) are bipartite. Then all odd faces of \(B\) are incident to \(x\). Define \(B_{n+1} = P_{n+1}\).

If \(y \notin \{b_{k-1}, b_k\}\), then by Lemma 3.5 there exists a \([b_{k-1}, b_k, y]\)-set of chains \(D_k\) in \(B_k\).

Let \(G = B_1, b_1, B_2, ..., b_{n-1}, b_n, B_{n+1}\). By Lemma 4.13 there is an \([x, y]\)-set of chains in \(G\), which is also an \([x, y]\)-set of chains in \(B\) (because \(G\) is a spanning subgraph of \(B\)).

Assume now that \(y \in \{b_{k-1}, b_k\}\) (note that \(y = b_{k-1}\) is possible only when \(k = 1\)). Suppose that \(y \in \{b_0, b_n\}\). We may assume \(y = b_0\). If a block \(B_i\) of \(G\) is nontrivial, then by Lemma 3.5, there is a \([b_{i-1}, b_i]\)-set of chains \(F_i\) in \(B_i\). Therefore, by Lemma 4.13 there is an \([x, y]\)-set of chains in \(G\), which is also an \([x, y]\)-set of chains in \(B\). Otherwise all blocks \(B_i, i \in [n+1]\) are trivial, and therefore \(V(B) = V(C)\). If \(C\) is an even cycle, then \(C = \emptyset\) is an \([x, y]\)-set of chains in \(B\). Otherwise \(C\) is odd, and since \(B\) is not an odd cycle, \(C\) has a chord. Hence, \(B\) has an even cycle \(C_0\) (which goes through \(x\)). Clearly, \(C_0\) together with blocks \(B_i, i \in [n]\), such that \(V(B_i) \cap V(C_0) \leq 1\) forms an \([x, y]\)-set of chains in \(B\).

Hence we may assume that \(y = b_k\) where \(k \notin \{0, n\}\). Suppose that all bounded odd faces of \(B\) are incident to \(y\) (and recall that all bounded odd faces are incident to \(x\)). Then there are exactly one or two such faces (for otherwise, if there are more than two such faces, \(B - \{x, y\}\) has a component disjoint with \(C\), and hence \(B\) is not a circuit graph).

However, if there is exactly one bounded odd face in \(B\), and this odd face is incident to \(x\) and \(y\), then \(xy \in E(C)\) (follows from the fact that \(B\) is good with respect to \(x\) and \(y\)) and so \(y \in \{b_0, b_n\}\).
Therefore there are exactly two bounded odd faces in \( B \) (both adjacent to \( x \) and \( y \)). In this case the cycle \( C' = \bigcup_{i=0}^{n+1} P_i \) bounds exactly two odd faces of \( B \) and therefore \( C' \) is even. Hence, \( C = \bigcup_{i=1}^{n} C_i \) is an \([x, y]\)-set of chains in \( B \).

We may therefore assume that there is a bounded odd face \( F \) of \( B \), which is not incident to \( y \), and that \( y \not\in \{b_0, b_2\} \). Let \( x_0 \) and \( x_2 \) be edges incident to \( F \). Since \( F \) is not incident to \( y = b_k \) we may assume, without loss of generality, that \( x_1, x_2 \in \bigcup_{i=1}^{k} V(B_i) \). Since \( F \) is an odd face and \( G \) is bipartite, every \( x_1x \)-path in \( G \) is odd and every \( x_2x \)-path in \( G \) is even (or vice versa). Let \( k' \in [k] \) be such that \( x_i \in V(B_{k'}) \setminus V(B_{k'+1}) \). If \( B_{k'} \) is nontrivial then by Lemma 4.11 (if \( x_i = b_{k'-1} \)) and by Lemma 4.12 (if \( x_i \neq b_{k'-1} \)) there is

(iv) an \((x_i, b_{k'}; b_{k'-1}, b_{k'})\)-set of chains \( \mathcal{C}_{k'} \) in \( B_{k'} \).

If \( B_i \) is nontrivial, then by Lemma 4.12 there is

(v) a \((b_{i-1}, b_i; b_{i-1}, b_i)\)-set of chains \( \mathcal{C}_i \) in \( B_i \), for \( i \in [n] \setminus \{k'\} \).

Let \( P_i \) be the \( \mathcal{G}_i \)-path in \( B_i \) for \( i \in [n] \setminus \{k' \} \), if \( B_i \) is nontrivial (if \( B_i \) is trivial, define \( P_i = B_i \) and \( \mathcal{G}_i = \emptyset \)), and define \( C'' = \bigcup_{i=1}^{n+1} P_i \cup \{x_0\} \) (recall that \( P_{n+1} = b_n \)). Since every \( x_1x \)-path in \( G \) is odd, \( C'' \) is even. By (iv) and Lemma 4.7, there is a \((x_1, b_{k'}; b_{k'})\)-set of chains \( \mathcal{C}_{k'} \) in \( \bigcup_{i=k'}^{n} B_i \). Then \( \mathcal{G} = \bigcup_{i=k'+1}^{n} \mathcal{G}_i \cup \mathcal{H}_{k'} \) is an \([x, y]\)-set of chains in \( B \), and \( C'' \) is a \( \mathcal{G} \)-cycle in \( B \).

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Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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