Relaxation regimes of the holographic electrons after a local quench of chemical potential.

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\textbf{Abstract:} In this work we study the relaxation of the system of strongly correlated electrons, when the chemical potential undergoes a local change. We use holographic duality to describe the system as a classical Reissner-Norström black hole in curved 4-dimensional AdS spacetime. Assuming the amplitude of the quench is small, we neglect the backreaction on the geometry. We numerically study the two relaxation regimes: the adiabatic relaxation when the quench is slow and the oscillatory relaxation governed by the quasinormal modes of the system, when the quench is fast. We confirm the expectation that the scale of separation between the slow and fast regimes is set by the characteristic frequency of the quasinormal modes.

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1 Introduction

The quantum features of the macroscopic systems always draw a lot of interest and lead to the new unexpected phenomena which can in many cases lead to the development of useful quantum devices. The few examples are superconductors, light-emitting diodes, quantum computers. The development of the latter ones addresses among others the interesting problem of quantum non-equilibrium dynamics, which can demonstrate quite unusual features.

The prominent example of the substantially quantum non-equilibrium behavior is the effect called orthogonality catastrophe (OC). Using the fundamental features of the weakly coupled quantum many-electron systems one can show that the response of such a system to the sudden introduction of the local impurity behave as a power law in time, which furthermore leads to the appearance of the power-law edge singularities in the X-ray absorption spectra [1]. This articular problem has recently gained more interest since the corresponding weakly interacting fermionic systems can be realized with the cold atom systems [2, 3].

The standard treatment of the orthogonality catastrophe relies on the assumption that the fermionic degrees of freedom of the system (quasiparticles) can be described as a nearly free Fermi gas. The natural question arises therefore whether the similar phenomenon can be seen in the strongly correlated electronic systems, where the quasiparticle description is unavailable. In order to address this question we invoke the AdS/CFT duality [4–6] (holography) which fundamentally relates the strongly coupled quantum field theory (which serves as an exact opposite to the nearly-free Fermi gas) to the classical gravity in a curved spacetime with one extra dimension. In holographic framework the system of strongly correlated electrons in 2+1 dimensions at finite chemical potential and temperature is
equivalent to the charged Reissner-Norström (RN) black hole the 3+1 dimensional Anti-de-Sitter spacetime. The charge of the black hole being controlled by the chemical potential and the horizon radius – by the temperature of the strongly correlated theory.

In this work we start the investigation of the orthogonality catastrophe in strongly correlated quantum systems by first addressing the basic problem of the non-equilibrium evolution of the system after a local quench in the chemical potential. At time $t = 0$ we local depression in the spacial profile of the chemical potential is created. This corresponds roughly to the X-ray absorption experiment, when the core hole is created and therefore the electrons in the conduction band feel the defect in the background potential.

We will focus on the simplest case when the perturbation of the chemical potential is small relative to its mean value. In this regime one can neglect the backreaction of the metric and the problem reduces to the study of non-equilibrium dynamics of the U(1) gauge field in the background of AdS-RN black hole. The convenient advantage of the holographic framework is that one can study the time evolution problems at finite temperature by directly solving the classical hyperbolic equations of motion.

As mentioned above the characteristic feature of the OC is the unusual power law approach to the equilibrium state at late times after the quench. We study the different cases, changing the timescale at which the defect in the chemical potential develops. We observe that the adiabatic evolution, when the system follows the development of the defect, which arises very slowly, gets substituted by the oscillatory relaxation, in case of the fast quench, when the quasinormal modes (QNMs) of the system get excited. The transition between the two regimes happens roughly at the scale of the lowest QNM frequency, which is controlled by the temperature. In the particular example under consideration we do not observe the power law relaxation characteristic to the OC, and we attribute this to the effective linear structure of the equations of motion. Which points out the necessity to include the full nonlinear gravitational backreaction in the treatment.

The paper is organized as follows. In Sec. 2 we outline the holographic setup, then we discuss the non-equilibrium evolution after a local quench in Sec. 3. In Sec. 4 we compute the spectrum of spherical quasinormal modes, which govern the late time evolution of the system in case of the fast quench and we conclude and discuss the future directions in Sec. 5.

## 2 Holographic model

In this work we focus on the simplest setup with the strongly interacting electrons in 2+1 dimensions at finite chemical potential. In the holographic framework this system is described via the Reissner-Nordström black hole in 3+1 dimensional curved AdS spacetime. In what follows we will adopt the polar coordinates on the boundary therefore the metric
reads
\[ ds^2 = \frac{1}{z^2} \left( -f(z) dt^2 + \frac{dz^2}{f(z)} + dr^2 + r^2 d\phi^2 \right), \quad (2.1) \]
\[ f(z) = (1 - z)(z^2 + z + 1 - \mu^2 z^3), \quad (2.2) \]
\[ A_t(z) = \mu (1 - z), \quad (2.3) \]

where \( A_\mu \) is a gauge field dual to the \( U(1) \) conserved charge of the fermion particle number, the \( AdS \) boundary is located at \( z = 0 \) and we rescale the holographic coordinate in such a way that the horizon is located at \( z = 1 \). The above metric is a solution to the classical equations of motion following from the Einstein-Maxwell action
\[ S = \int \sqrt{g} \left[ \frac{1}{4} F^2 + R - 2\Lambda \right], \quad (2.4) \]

where \( F = dA \) is the \( U(1) \) gauge field strength tensor, \( R \) – the Ricci scalar and \( \Lambda = -3 \) – cosmological constant.

As usual [7], the chemical potential \( \mu \) and charge density \( \rho \) are related to the leading and subleading terms of the temporal gauge field component near the boundary of AdS:
\[ A_0(z) \bigg|_{z \rightarrow 0} = \mu + \rho z. \quad (2.5) \]

In the same fashion the radial current is related to the subleading term of the radial gauge field component
\[ A_r(z) \bigg|_{z \rightarrow 0} = J_r z. \quad (2.6) \]

In the end of the day, the temperature of the field theory is related to the horizon radius of the black RN hole [8],
\[ T = \frac{12 - \mu^2}{16\pi\mu}. \quad (2.7) \]

In what follows we will mostly focus on the low temperature case
\[ T = 0.01. \quad (2.8) \]

3 Evolution after a local quench

We are mostly interested in studying the non-equilibrium dynamics of the system (2.1) after the local quench in the profile of the chemical potential. We consider the local spherically-symmetric perturbation above the chemical potential \( \mu_0 \) with the characteristic size \( s \) which turns on at moment \( t = 0 \) and rises to the amplitude \( \delta \mu_0 \) at the characteristic time \( \tau \).
\[ \delta \mu(t,r) = -\delta \mu_0 \cdot e^{-2r/s^2} \cdot \tanh(t/\tau). \quad (3.1) \]

In order to be able to treat this perturbation as local, but at the same time keeping accessible to the numerical simulations, we chose the size to be reasonably small
\[ s = 0.1. \quad (3.2) \]
We will keep $\tau$ as the tunable parameter and explore the behavior of the system in the different regimes when the quench is “fast”, $\tau \ll 1$ or “slow”, $\tau \gg 1$.

Our task therefore is to study the time-evolution of the gravitational system (2.1) with the gauge field subject to the boundary condition (3.1) according to (2.5). In order to describe the causal time-evolution we have to keep the infalling boundary condition at the black hole horizon. This is most straightforwardly done by choosing the generalized Eddington-Finkelstein (EF) coordinates [9]: \{v, z, r, \phi\}, where

\[ v = t - z', \quad dz' = dz f(z)^{-1}. \]  

(3.3)

In these coordinates the geodesics with constant $v$ correspond to the infalling waves, therefore the necessary horizon boundary condition reduces to the regularity requirement of the solution in the EF coordinates (3.3). The metric takes the form

\[ ds^2 = \frac{1}{z^2} \left( -f(z) dv^2 + 2dvdz + dr^2 + r^2 d\phi^2 \right), \]  

(3.4)

Given that at $z \to 0$ it follows from (3.3) that $z' \sim z$, we see that the asymptotic expansion (2.5) does not change. Moreover, at the AdS boundary the new coordinate $v$ equals boundary time-coordinate

\[ v \big|_{z \to 0} = t, \]  

(3.5)

Therefore in the EF coordinates (3.3) one can directly use the boundary condition for $A_t$ (3.1) replacing $t$ by $v$.

In this work we assume the amplitude of the perturbation to be small as compared to the mean value of the chemical potential: $\delta \ll 1$. This allows us to linearize the equations of motion coming from (2.4) and neglect the backreaction of the gauge field to the metric. We study the spherically symmetric and time-dependent ansatz for the perturbed gauge field.

\[ A = A_t(v, r, z) dt + A_r(v, r, z) dr. \]  

(3.6)

The equations of motion on the background of the static metric (2.1) in EF coordinates read

\[ f(z) \partial^2_z A_t + 2\partial_v \partial_z A_t + f(z)^{-1} \partial^2_r A_t - \frac{f'(z)}{f(z)} \partial_v A_t + \partial^2_r A_t + \frac{1}{r} \partial_r A_t \]

\[ - \left( \frac{1}{r} + \partial_r \right) \partial_r A_r = 0 \]  

(3.7)

\[ f(z) \partial^2_z A_r + 2\partial_v \partial_z A_r + f'(z) \partial^2_r A_r + f(z)^{-1} \partial_v A_r = 0 \]  

(3.8)

\[ f(z)^{-1} \partial^2_r A_t + \partial_v \partial_z A_t - \left( \frac{1}{r} + \partial_r \right) \left( f(z) \partial_z A_r + \partial_v A_r \right) = 0 \]  

(3.9)

The latter equation is a constraint corresponding to our choice of the gauge $A_z = 0$. It can be used to eliminate the second time derivative $\partial^2_r A_t$ and reduce the problem to a system of two equations

\[ f(z) \partial^2_z A_t + \partial_v \partial_z A_t + \left( \partial_r + \frac{1}{r} \right) \partial_r A_t - \frac{f'(z)}{f(z)} \partial_v A_t + f(z) \left( \partial_r + \frac{1}{r} \right) \partial_z A_r = 0 \]  

(3.10)

\[ f(z) \partial^2_r A_r + f'(z) \partial_z A_r + 2\partial_v \partial_r A_r + f(z)^{-1} \partial_v A_r = 0. \]  

(3.11)
In what follows we solve the system (3.10) subject to the time-dependent boundary condition for $A_t$, set by the perturbation of the chemical potential (3.1):

$$z = 0 : \quad A_t(v, r, 0) = \mu + \delta \mu(v, r), \quad A_r(v, r, 0) = 0. \quad (3.12)$$

The boundary conditions at $r = 0$ are set by the generic structure of the polar coordinates

$$r = 0 : \quad \partial_r A_t(v, 0, z) = 0, \quad A_r(v, 0, z) = 0. \quad (3.13)$$

At radial infinity $r \to \infty$ the solution should approach the profile of the RN black hole (2.1)

$$r \to \infty : \quad A_t(v, r, z) = \mu(1 - z), \quad A_r(v, r, z) = 0. \quad (3.14)$$

And in the end of the day, as discussed above, we require the regularity of solution at the horizon

$$z = 1 : \quad A_t(v, r, 1) = \text{const}, \quad A_r(v, 1, z) = \text{const}. \quad (3.15)$$

The system (3.10) with the boundary conditions (3.12),(3.13),(3.14),(3.15) can be solved numerically by applying the “method of lines” (see Appendix A).

![Figure 1: The profile of the perturbation in the chemical potential (3.1). Left – radial profile given the characteristic size $s = 0.1$. Right – temporal profile of the fast quench $\tau = 0.1$.](image)

Once we obtain the solution at all times, we can extract the time-dependent values of the observables using the asymptotic formulae (2.5) and (2.6). In this way we study the response of the charge density and the radial current to the quench.

In case of the fast quench ($\tau = 0.1$), shown on Fig.2, one observes the oscillatory relaxation of the system, whose decay time is much longer then the timescale of the quench itself. These oscillatory behavior point out that the system is far from the equilibrium and the relaxation process is governed by the internal dynamics, independent of the precise profile of the quench.

If we consider the slow rise of the perturbation potential, ensuring that its characteristic timescale $\tau$ is long enough for the system to assume the local equilibrium at any given moment, we expect to observe the adiabatic evolution. Indeed, in case of the slow quench $\tau = 10$ we observe substantially different behavior as compared to the previous case, see Fig.3. All the observables change with the same rate as the quench itself with the timescale set by $\tau$. The system undergoes a sequence of quasi-equilibrium states and no oscillatory behavior is observed. The slow quench truly realizes the adiabatic process.
Figure 2: Fast quench $\tau = 0.1$. Left: the profile of the radial current depending on time $v$. The expanding wave of the current and the decaying oscillations at $v \ll \tau$ are seen. Right: the time-dependence of the local chemical potential and charge density at $r = 0$ and the current density at $r \approx 0.05$.

Figure 3: Slow quench $\tau = 10$. Left: the profile of the radial current depending on time $v$. No decaying oscillations or propagating waves are seen. Right: the time-dependence of the local chemical potential and charge density at $r = 0$ and the current density at $r \approx 0.05$.

4 Spherical quasinormal normal modes

The nature of the decaying oscillations in the fast quench solution, shown on Fig.2 is easily understood in the analysis of the quasinormal modes (QNM) of the RN black hole (2.1). The QNMs have finite negative imaginary part due to the absorption by the horizon, which is controlled by the temperature of the system. The spectrum of QNM can be obtained as a solution to the Sturm-Liouvile problem for system (3.10) with trivial boundary condition for the oscillatory modes of the form

$$\delta A(v, r, z) \sim e^{i\omega v}. \quad (4.1)$$

The values of $\omega$, for which the system has a nontrivial solution are the spectrum of QNMs, see Appendix B for more details. The modes with the lowest absolute value of the imaginary part control the relaxation time of the system. Importantly, according to (4), the relaxation
of these modes is always exponential. The quasinormal modes do not lead to the power law time tails, as expected for the orthogonality catastrophe like behavior. The modes with finite real part correspond to the decaying oscillations.

In the case under consideration $T = 0.01\mu$, the spectrum of QNMs is shown on Fig.4. One can discern a multitude of modes. The one which is closest to the real axis is the most long-lived one and controls the relaxation time $\langle \tau_r \rangle$ of the whole system,

$$\tau_r = 1/\text{Min} \left[ \text{Abs}[\text{Im}(\omega)] \right] \approx 5.$$  \hspace{1cm} (4.2)

Reconstructing the physical parameter one can express this relaxation time in terms of the chemical potential and the temperature (according to (2.7) we have $\mu = 3.22T$):

$$\tau_r \approx 16T/\mu$$  \hspace{1cm} (4.3)

The oscillatory modes with finite real part $\omega = \pm 180 - 0.75i$ are also present in the spectrum. These ones correspond to the fast oscillations, which are seen on Fig.2.

![Figure 4: Quasinormal modes of the Reissner-Nordström black hole (2.1). One can see a multitude of the diffusive modes $\text{Re}(\omega) = 0$ and a couple of the oscillatory ones $\text{Re}(\omega) \neq 0$](image)

5 Conclusion

In this work we consider the non-equilibrium evolution of the system of strongly correlated electrons after the local change in the chemical potential. This is a model problem for the X-ray spectrometry experiment, which in case of the weakly coupled electrons demonstrates the interesting quantum phenomenon of the X-ray edge anomaly. We use the holographic duality to describe the strongly correlated system and we work in the probe approximation, neglecting the backreaction of the gauge field profile on the geometry.

In the simple setup under consideration we observe that the dynamics of the system, even far from equilibrium, is described by the linear equations of motion (3.10). In this case the spectrum of quasinormal modes gives an exact representation of the possible non-equilibrium solutions. The internal time-scale of the QNM relaxation $\tau_r$, set by the
temperature and the mean chemical potential allows to distinguish between the two characteristic types of the evolution after a local quench. If the quench is slow and the defect develops at the time-scales longer then the equilibration time $\tau_r$, the dynamics is adiabatic. On the other hand, when the quench is fast, the higher order QNMs are excited and the relaxation dynamics has an oscillatory behavior.

In the present study we develop several techniques and approaches which will be useful in the future developments: the method of lines for the integration of the gauge field evolution equations and the spherically symmetric setup suitable for treatment of the local defects.

We do not find a regime with the power law relaxation pattern, as would be similar to the pattern of the orthogonality catastrophe and this observation sets the possible future directions of the research program outlined in the Introduction. The obvious next step is to include the backreaction to the metric fields. This will make the evolution equations nonlinear and will allow for the solutions which are not representable by the linear combination of the QNMs. The other possible generalization is to include the features of the finite $N$ by making use of the Gauss-Bonet gravitational setups.

This work serves as a proof of principle that the holographic duality provides an adequate tools for treating the time-evolution problems in the system of strongly correlated electrons and has its advantages as compared to the adiabatic perturbation theory and the Schwinger-Keldysh approach.

Acknowledgments

I thank Oleg Lychkovskiy, Oleksandr Gamayun and Koenraad Schalm for proposing this problem. I benefited a lot from the useful discussions with Dmitry Ageev, Andrey Bagrov and Mikhail Katsnelson about the local quenches. I’m grateful to Aurelio Romero-Bermudez and Christian Ecker for discussing the time-evolution problems in holographic setups.

The work was supported by the Russian Science Foundation under the grant No. 17-71-20158.

A Method of lines

The numerical “method of lines” allows one to solve the hyperbolic differential equations with partial derivatives (PDEs), which describe the non-equilibrium dynamics of the system with a local defect (3.10). One introduces the calculation grid $\{r_i, z_j\}$ in the spacial part of the integration region and represents the derivatives as the finite differences of the functional values on this grid. After this discretization of the spacial directions the system of two differential equations (3.10) is represented by the set of $2*N_r \times N_z$ ordinary differential equations with only functional dependence on the “temporal” coordinate $v$ on the functions

$$A_{ij}^\mu(v) \equiv A_\mu(v, r_i, z_j)$$  \hspace{1cm} (A.1)
and their derivatives in $v$. The resulting system can be integrated with the standard methods suitable for the evolution of the ordinary first order differential equations, for instance Runge-Kutta of the 4-th order.

Before we turn to the numerical solution of the equations of motion (3.10), we introduce the compact spatial coordinate

$$x \equiv \frac{2}{\pi} \text{ArcTan}(\frac{r}{c}),$$

(A.2)

with the arbitrary parameter $c$, for which we will use the value $c = 0.5$ in what follows, Fig. 5. This coordinate transformation brings the infinite integration region in the radial coordinate into the $r \in [0, \infty)$ unit interval $x \in [0, 1)$. Therefore, in the new coordinates the full integration region for the numerical problem is a unit square: $[0, 1) \times (0, 1].$

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{figure5.png}
  \caption{The relation between the radial coordinate $r$ and the compactified one $x$, which we use in the numerical simulation (A.2), $c = 0.5$.}
\end{figure}

In the problem under consideration we use the pseudospectral collocation method to represent the partial derivatives on the 2-dimensional Chebyshev grid of the size $20_x \times 16_z$. We use the pseudospectral differentiation matrices discussed in i.e. [10, 11] and implement the boundary conditions in the linear differential operator itself, as discussed in [12]. We integrate the resulting linear system of time dependent ODEs using the standard evolution solver in Wolfram Mathematica [13].

B Quasinormal modes

In order to study the spectrum of quasinormal modes of the system we have to solve the Sturm-Liouville problem for the equations of motion (3.10) given the ansatz with constant frequency (4.1). We discretize the equations in the same manner as in App.A and end up with the system of the linear algebraic equations, since the time derivatives in the given ansatz get simply substituted by the $\omega$ factors. The matrix characterizing this system of equations depends on the parameter $\omega$ at most linearly. Therefore the solution to the
Sturm-Liouville problem involves finding the generalized eigenvalues of this matrix with respect to the parameter \( \omega \). We accomplish this task using the standard `EigenValue` routine in in `Wolfram Mathematica`.

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