Geometrical quantities of fuzzy sphere

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Abstract: In this paper, we consider the geometrical quantities on the fuzzy sphere from the spectral point of view, such as the area and the dimension. We find that, in contract to the standard sphere, the area and the dimension are the functions of the energy scale of the fuzzy sphere.

Keywords: Noncommutative geometry, fuzzy sphere, area function, fractal dimension.
1. Introduction

Fuzzy space [1] is a kind of noncommutative space, and it can provide an alternative way of regularizing field theories with finite degrees of freedom [2]. A good review of field theory on fuzzy sphere is [3]. Recently, the non-perturbative numerical studies of quantum fields on fuzzy sphere attracted many attention. It has been shown that there exist three distinct phases in the \( \phi^4 \) real scalar field theory [4, 5]. Also the phase structure of Yang-Mills-Chern-Simons field was given in [6].

In paper [7], we consider the behavior of the fuzzy sphere \( S_F^2 \) with the help of the spectral action, and find that in different energy scales, the fuzzy sphere behaves differently. In this paper, we consider two important geometrical quantities, that is the area \( A \) and the dimension \( D_s \), from the spectral point of view. We find that, unlike to the standard sphere, those two quantities are functions of the energy scale \( \Lambda \).

The paper is organized as follows. In section two we give the spectrum of the Dirac operator on the fuzzy sphere. In section three we calculate the area of the fuzzy sphere. In section four we calculate the dimension of the fuzzy sphere. In section five we present the conclusion and the discussion.
2. The spectrum of the Dirac operator

The algebra $A_N$ of fuzzy sphere $S^2_F$ is generated by the operators $x_i$ (i=1,2,3) satisfying
\[
[x_i, x_j] = i \frac{2l}{\sqrt{N(N+2)}} \varepsilon_{ijk} x_k, \quad (2.1)
\]
with constrain
\[
x_i x_i = l^2. \quad (2.2)
\]
Each operator of $A_N$ is represented by a matrix acting on the $N+1$-dimension Hilbert space $F_N$. Apparently, the matrices $x_i$ can identified with the generators of $su(2)$ Lie algebra and the generated algebra is equivalent to the algebra of $(N+1) \times (N+1)$ matrices, $M_C(N+1)$.

Dirac operator can be defined on the fuzzy sphere \[8\]. We cite the spectrum of the Dirac operator in \[8\]. The spectrum of the Dirac operator $D$ on the fuzzy sphere with $l = 1$ is as follows,
\[
D^2 \Psi_{jm} = \lambda_j^2 \Psi_{jm}, \quad (2.3)
\]
with
\[
\lambda_j^2 = (j + 1/2)^2 (1 + \frac{1 - (j + 1/2)^2}{N(N+2)}), \quad (2.4)
\]
where $j$ and $m$ are half integer and take the value
\[
1/2 \leq j \leq N + 1/2, \quad -j \leq m \leq j.
\]
From the expression \[2.4\] we can see that, unlike to the standard sphere, those eigenvalues are finite and not monotonically increasing.

3. The area function

To calculate the area of the fuzzy sphere, we have to define it though the spectrum of the Dirac operator. On standard sphere, we have the heat kernel expansion for the Dirac operator \[9\],
\[
Trace(e^{-tD^2}) = \sum_{n=1}^{\infty} 4ne^{-n^2 t} \simeq 1/(2\pi t) A - 1/3 + o(t), \quad (3.1)
\]
as \( t \to 0 \), where \( A \) is the area of the sphere. So we have, for the standard sphere,

\[
A \approx 2\pi t (\text{Trace}(\exp(-tD^2)) + 1/3),
\]

if \( t \) is not too big. Then we generalize this formula to the fuzzy sphere, that is, we define the area of the fuzzy sphere as follows,

\[
A \equiv 2\pi (\text{Trace}(\exp(-D^2/\Lambda^2)) + 1/3)/\Lambda^2,
\]

where \( t = 1/\Lambda^2 \), so \( \Lambda \) has the dimension of energy. Then with the spectrum given by (2.4), we can calculate the area of the fuzzy sphere for every integer \( N \).

\[
A(\Lambda) = 2\pi \left( \sum_{l=1}^{N} (4l) \exp(-l^2/\Lambda^2(1 + \frac{1 - l^2}{N(N + 2)}) + 1/3)/\Lambda^2. \right)
\]

Since we don’t have an analytic expansion for the area function, we just draw it out. See figure 1.

In the range \((0, 1)\), since the heat kernel expansion breaks down, we don’t rely on this figure. In this range, the first few eigenvalues play important role, so the fuzzy sphere behaves like the standard sphere. See [7] for the more discussion of this problem. From this figure we can see that for the standard sphere the area function is not constant. The reason is that the number of eigenvalue of the Dirac operator for the standard sphere is infinite, but we just choose the first 40 terms. The more terms we choose, the more closed the area function to constant 1. The area function for the fuzzy sphere is not constantly at all. The area function achieves a max value and then decrease to zero at high energy. With \( N \) increasing, the max value of the area function increases too.

4. The fractal dimension

In the paper [7], we find that, from spectral point of view, the fuzzy sphere can have different dimension for different energy scale. Since this result is very rough, we give more investigation to this issue. Fractal dimension for quantum sphere [10] and for loop quantum gravity [11] has been discussed, and we borrow the operational
Figure 1: The area function for standard sphere and fuzzy sphere with different $N$

definition from those two papers, that is, the dimension for the fuzzy sphere is,

$$D_s = -2 \frac{d \ln P(T)}{d \ln T}. \quad (4.1)$$

In our case, $P(T) = \text{Trace}(-D^2/\Lambda^2)$ and $T = 1/\Lambda^2$, so

$$D_s = - \frac{d \ln (\text{Trace}(e^{-D^2/\Lambda^2}))}{d \ln \Lambda}. \quad (4.2)$$

Let’s just figure it out for the fuzzy sphere. See figure 2.

This figure looks like the first figure. The dimension of the standard sphere is not constantly 2 because we just choose the first 40 terms. The fractal dimension for
Figure 2: The fractal dimension for standard sphere and fuzzy sphere with different $N$ the fuzzy sphere changes when the energy scale $\Lambda$ changes, and decay to zero at high energy. When $N$ becomes large, the max value of the fractal dimension becomes small and approach the standard dimension 2.

5. Conclusion

Dirac operator play an important role in noncommutative geometry [12]. In paper [13], the authors show that the eigenvalues of the Dirac operator can be treated as the dynamical variables for the Euclidean general relativity. From this point of view,
every geometrical quantities must be function of those eigenvalues. In this paper, we
calculate the area and the dimension of the fuzzy sphere from the spectrum of the
Dirac operator. We find that they are functions of the energy Λ. We hope that the
operational formula for the area and the dimension of a manifold will play some role
in quantum gravity.

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