We study a log-gas on a network (a finite, simple graph) confined in a bounded subset of a local field (i.e. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, the field of $p$-adic numbers). In this gas, a log-Coulomb interaction between two charged particles occurs only when the sites of the particles are connected by an edge of the network. The partition functions of such gases turn out to be a particular class of multivariate local zeta functions attached to the network and a positive test function which is determined by the confining potential. The methods and results of the theory of local zeta functions allow us to establish that the partition functions admit meromorphic continuations in the parameter $\beta$ (the inverse of the absolute temperature). We give conditions on the charge distributions and the confining potential such that the meromorphic continuations of the partition functions have a pole at a positive value $\beta_{UV}$, which implies the existence of phase transitions at finite temperature. In the case of $p$-adic fields the meromorphic continuations of the partition functions are rational functions in the variable $p^{-\beta}$. We give an algorithm for computing such rational functions. For this reason, we can consider the $p$-adic log-Coulomb gases as exact solvable models. We expect that all these models for different local fields share common properties, and that they can be described by a uniform theory.

1. Introduction

In this article we study log-Coulomb gases on finite simple graphs confined in bounded regions. The partition function of these gases are local zeta functions (in the sense of Gel’fand, Atiyah, Igusa, Denef, Loeser, among others). By using the theory of local zeta functions, we establish the existence of phase transitions at finite temperature. The coordinates of the sites having the charged particles can be taken from any local field $K$, for instance $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$.

An ultrametric space $(M,d)$ is a metric space $M$ with a distance satisfying the strong triangle inequality $d(A,B) \leq \max\{d(A,C),d(B,C)\}$ for any three points $A, B, C$ in $M$. The field of $p$-adic numbers $\mathbb{Q}_p$ constitutes a central example of an ultrametric space. The ultrametricity, which is the emergence of ultrametric spaces in physical models, was discovered in the middle 1980s by Parisi et al. in the context of the spin glass theory, see e.g. $[11], [14]$. Ultrametric spaces constitute the right framework to formulate models where hierarchy plays a central role. Ultrametric models have been applied in many areas, including, quantum physics, $p$-adic string theory, $p$-adic Feynman integrals, brain and mental states models, relaxation of complex systems, evolutionary dynamics, cryptography and geophysics, among other areas, see e.g. $[2], [3]-[4], [7], [10], [29]-[30], [32]-[40], [42]-[43], [52]-[54]$, and the references therein.

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The Ising models over ultrametric spaces have been studied intensively, see e.g. [17], [22], [28], [30], [34], [37], [38], [39], [40], [47] and the references therein. An important motivation comes from the hierarchical Ising model introduced in [17]. The hierarchical Hamiltonian introduced by Dyson in [17] can be naturally studied in $p$-adic spaces, see e.g. [34], [22]. In [43], see also [31], Parisi and Sourlas presented a $p$-adic formulation of replica symmetry breaking. In this approach ultrametricity is a natural consequence of the topology of the $p$-adic numbers. This work raises the problem of knowing if it is possible to have a rigorous $p$-adic formulation of the replication method. This requires, among other things, a rigorous mathematical understanding of objects such as partition functions in a $p$-adic framework. This is precisely the objective of the present work. This article continues the investigation on $p$-adic graphs confined in the $p$-adic balls of arbitrary dimension. A natural problem is to study these models in the Archimedean framework and to compare them with the non-Archimedean counterparts. To the best of our knowledge this type of systems has not been studied yet.

By a generalized Mehta integral, we mean an integral of the form

$$Z_{\phi}(s) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_N) \prod_{1 \leq i < j \leq N} |x_i - x_j|^{s_{ij}} \prod_{i=1}^{N} dx_i,$$

where $\varphi$ is a Schwartz function, and $s = (s_{ij})_{1 \leq i < j \leq N} \in \mathbb{C}^{N(N-1)/2}$ with $\text{Re} (s_{ij}) > 0$ for any $1 \leq i < j \leq N$. The original Mehta integral $F_N(\gamma)$ is exactly $F_N(\gamma) = \frac{1}{(2\pi)^N} Z_{\varphi}(s) \big|_{s_{ij} = 2\gamma}$, with $\varphi(x_1, \ldots, x_N) = e^{-\frac{1}{2} \sum_{i=1}^{N} x_i^2}$, and it is the partition function of a 1D log-Coulomb gas, see e.g. [19], [18]. The integral $Z_{\varphi}(s)$ is a particular case of a multivariate local zeta function. These functions admit meromorphic continuations to the whole $\mathbb{C}^{N(N-1)/2}$, see e.g. [36]. Nowadays, there exists a uniform theory of local zeta functions over local fields of characteristic zero, e.g. $(\mathbb{R}, |\cdot|)$, $(\mathbb{C}, |\cdot|)$, and the field of $p$-adic numbers $(\mathbb{Q}_p, |\cdot|_p)$, see [25], [26], see also [13], [14], [20], [36], [51] and the references therein. By using this theory, we can construct incarnations of the integral $Z_{\varphi}(s)$ over $\mathbb{C}$ and $\mathbb{Q}_p$, which admit meromorphic continuations to the whole $\mathbb{C}^{N(N-1)/2}$. In addition, the possible poles of all these functions can be described in a geometric way.

Given a local field $(\mathbb{K}, |\cdot|_\mathbb{K})$ and a finite, simple graph $G$, we attach to them a 1D log-Coulomb gas and a local zeta function. By a gas configuration we mean a triple $(x, e, G)$, with $x = (x_v)_{v \in V(G)}$, $e = (e_v)_{v \in V(G)}$, where $e_v \in \mathbb{R}$ is a charge located at the site $x_v \in \mathbb{K}$, and the interaction between the charges is determined by the graph $G$. Given a vertex $u$ of $G$ ($u \in V(G)$), the charged particle at the site $x_u$ can interact only with those particles located at sites $x_v$ for which there exists an edge between $u$ and $v$ (we denote this fact as $u \sim v$). The Hamiltonian is given
by

\begin{align}
H_{K}(x; e, \beta, \Phi, G) = - \sum_{u,v \in V(G)} \ln |x_u - x_v|_{K}^{e_u e_v} + \frac{1}{\beta} P(x),
\end{align}

where \( \beta = \frac{1}{k_B T} \) (with \( k_B \) the Boltzmann constant, \( T \) the absolute temperature), \( P : \mathbb{K}[V(G)] \to \mathbb{R} \) is a confining potential such that \( \Phi(x) = e^{-P(x)} \) is a test function, which means that \( P = +\infty \) outside of a compact subset.

The partition function attached to the Hamiltonian (1.1) is given by

\begin{align}
Z_{G,K,e}(\beta) = \int_{K[V(G)]} e^{\sum_{u,v \in V(G)} |x_u - x_v|_{K}^{e_u e_v} \prod_{v \in V(G)} dx_v}.
\end{align}

In order to study this integral, using geometric techniques, it is convenient to extend \( e_u e_v \beta \) to a complex variable \( s(u,v) \), in this way the partition function (1.2) becomes a local zeta function. Then the partition function is recovered from the local zeta function taking \( s(u,v) = e_u e_v \beta \).

The local zeta function attached to \( G \), \( \Phi \) is defined as

\begin{align}
Z_{\Phi}(s; G, \mathbb{K}) = \int_{\mathbb{K}[V(G)]} e^{\sum_{u,v \in V(G)} |x_u - x_v|_{K}^{s(u,v)} \prod_{v \in V(G)} dx_v},
\end{align}

where \( s = (s(u,v)) \) for \( u, v \in V(G) \) for \( u \sim v \), \( s(u,v) \) is a complex variable attached to the edge connecting the vertices \( u \) and \( v \), and \( \prod_{v \in V(G)} dx_v \) is a Haar measure of the locally compact group \( (\mathbb{K}[V(G)], +) \). The integral converges for \( \text{Re}(s(u,v)) > 0 \) for any \( (u,v) \). The partition function \( Z_{G,K,e}(\beta) \) of \( H_{K}(x; e, \beta, \Phi, G) \) is related to the local zeta function of the graph by

\begin{align}
Z_{G,K,e}(\beta) = Z_{\Phi}(s; G, \mathbb{K})|_{s(u,v) = e_u e_v \beta}.
\end{align}

The zeta function \( Z_{\Phi}(s; G, \mathbb{K}) \) admits a meromorphic continuation to the whole complex space \( \mathbb{C}[E(G)] \), see [36, Théorème 1.1.4].

For a charge configuration \( e = (e_v)_{v \in V(G)} \) satisfying that \( e_u e_v > 0 \) for any \( u \sim v \), the partition function \( Z_{G,K,e}(\beta) \) is analytic for \( \beta > 0 \). If the sign of \( e_u e_v \), for \( u \sim v \), changes along the graph, then the partition function becomes an integral of a ‘rational function’ on a compact subset, and in the general case, the analyticity for \( \beta > 0 \) does not hold anymore. The existence of a meromorphic continuation for \( Z_{G,K,e}(\beta) \) having positive poles, say at \( \beta = \beta_{UV} > 0 \), implies that the function \( \ln Z_{G,K,e}(\beta) \) has a pole at \( \beta = \beta_{UV} \), and thus any canonical free energy defined using \( \ln Z_{G,K,e}(\beta) \) has a pole at \( \beta = \beta_{UV} \). Notice that the existence of such a pole does not require to pass to the thermodynamic limit. Since the canonical energy is not analytic around \( \beta = \beta_{UV} \), this point is a phase-transition point. We will say that \( Z_{G,K,e}(\beta) \) has a phase transition at temperature \( k_B / \beta_{UV} \). The determination of the actual poles for \( Z_{\Phi}(s; G, \mathbb{K}) \) is a difficult open problem. If \( \mathbb{K} \) is a \( p \)-adic field then \( Z_{G,K,e}(\beta) \) admits a meromorphic continuation as a rational function in the variables \( p^{-e_u e_v \beta} \), \( u \sim v \). For this reason we can consider the \( p \)-adic log-Coulomb gases as exact solvable models.

We establish the existence of phase transitions by showing the existence of a convergence interval \( (0, \beta_{UV}) \) for the integral \( Z_{G,K,e}(\beta) \), such that the meromorphic continuation of \( Z_{G,K,e}(\beta) \) has a pole at \( \beta = \beta_{UV} \). We provide two different types of criteria for the existence of such intervals. The first type is specific for the \( p \)-adic
case and requires that \( \Phi \) be the characteristic function of the unit ball \( \mathbb{Z}_p^{V(G)} \), but this criterion works with arbitrary charge distributions. Second type of criteria works on any local field of characteristic zero, but it requires that the support of \( \Phi \) be sufficiently small, and that the charge distribution be such that in \( (1.1) \)
\[ e_u e_v = \pm 1 \] for any \( u, v \in V(G) \). In terms of phase transitions, the log-Coulomb gases studied here behave similarly to the classical Ising model.

The above mentioned results were established by using the techniques developed in \( (51) \). In the \( p \)-adic setting, in the case in which \( \Phi(x) \) is the characteristic function of the \( |V(G)|- \) dimensional unit ball, the corresponding partition functions (or local zeta functions) are rational functions that can be computed explicitly using combinatorial techniques.

A \( p \)-adic number is a series of the form
\[ (1.3) \quad x = x_{-k} p^{-k} + x_{-k+1} p^{-k+1} + \ldots + x_0 + x_1 p + \ldots, \quad \text{with} \quad x_{-k} \neq 0, \]
where \( p \) denotes a fixed prime number, and the \( x_j \)'s are \( p \)-adic digits, i.e. numbers in the set \( \{0, 1, \ldots, p-1\} \). There are natural field operations, sum and multiplication, on series of the form \( (1.3) \). The set of all possible \( p \)-adic numbers constitutes the field of \( p \)-adic numbers \( \mathbb{Q}_p \). There is also a natural norm in \( \mathbb{Q}_p \) defined as \( |x|_p = p^k \), for a nonzero \( p \)-adic number of the form \( (1.3) \). We extend the \( p \)-adic norm to \( \mathbb{Q}_p^N \), by taking \( \|x_1, \ldots, x_N\|^p = \max_i |x_i|^p \).

The Hamiltonian of the \( N \)-dimensional \( p \)-adic Coulomb gas is
\[ H_N(x_1, \ldots, x_N; \beta) = \sum_{1 \leq i<j \leq N} e_i e_j E_\alpha \left( \|x_i - x_j\|_p \right) + \frac{1}{\beta} P(x_1, \ldots, x_N), \]
where \( e_j \) is the charge of a particle located at \( x_j \in \mathbb{Q}_p^N \), and \( P(x_1, \ldots, x_N) \) is a confining potential. We assume that \( P(x_1, \ldots, x_N) = +\infty \) outside of an open compact subset. The Coulomb kernel \( E_\alpha(\|x\|_p) \) is a fundamental solution of a ‘\( p \)-adic Poisson’s equation.’ More precisely, if
\[ E_\alpha(\|x\|_p) = \begin{cases} 1 - \frac{p^{-\alpha}}{p^N} \ln \|x\|_p, & \text{if } \alpha = N, \\ 1 - \frac{p^{-\alpha}}{p^N} \|x\|_p^{\alpha - N}, & \text{if } \alpha \neq N, \end{cases} \]
then \( D^\alpha E_\alpha = \delta \), where \( D^\alpha \), \( \alpha > 0 \), is the \( N \)-dimensional Taibleson operator which is a pseudodifferential operator defined as \( \mathcal{F}(D^\alpha \varphi) = \|\xi\|^\alpha \mathcal{F}\varphi \), where \( \mathcal{F} \) denotes the Fourier transform, see \( [15] \) Theorem 13 and \( [55] \) Chapter 5].

The study of \( p \)-adic Coulomb gases was initiated in \( [50] \), where some probabilistic aspects attached to Coulomb gases, involving the kernel \( \|x\|_p^{\alpha - N}, N > \alpha, \) were studied.

In this article we study 1D \( p \)-adic log-Coulomb gases, under the assumption that \( e^{\frac{1}{x}} \) is the characteristic function of the \( |V(G)| \)-dimensional unit ball \( \mathbb{Z}_p^{V(G)} \). In this case, the local zeta function attached to \( G \) is defined as
\[ Z(s; G) = \int_{\mathbb{Z}_p^{V(G)}} \prod_{u \neq v} \left| x_u - x_v \right|_p^{s(u,v)} \prod_{v \in V(G)} dx_v, \]
where \( s(u,v) \) is a complex variable attached to the edge connecting the vertices \( u \) and \( v \). The partition function \( Z_{G,p,\alpha}(\beta) \) of \( H_p(x; e, \beta, G) \) is related to the local zeta function of the graph by \( Z_{G,p,\alpha}(\beta) = Z(s; G)_{s(u,v) = e_u e_v \beta}. \)
Section 3 is dedicated to the study of the function $Z(s; G)$. This function admits a meromorphic continuation as a rational function in the variables $p^{-s(u,v)}$, see Proposition 1. We provide a recursive algorithm for computing $Z(s; G)$. The algorithm uses vertex colorings and chromatic polynomials, see Proposition 2. This algorithm allows us to describe the possible poles of $Z(s; G)$ in terms of the subgraphs of $G$, see Theorem 1 and Corollary 2.

In Section 4, we give conditions on the distribution of charges that guarantee the convergence of the integral $Z_{G,p,e}(\beta)$ in an interval $(\beta_{IR}, \beta_{UV})$, see Proposition 3. We also give conditions so that the meromorphic continuation of $Z_{G,p,e}(\beta)$ has a pole at $\beta = \beta_{UV}$, see Proposition 4. This result allows us to give criteria for the existence of phase transitions at finite temperature. In Section 5, we study the thermodynamic limit for a log-Coulomb gas attached to a star graph $S_M$, confined in the 1-dimensional ball $B_k$ of radius $p_k$, when $M \to \infty$, $k \to \infty$, and $\frac{M}{\rho}$ is constant. Assuming a neutral charge distribution satisfying $e_v = \pm 1$ for any $v \in V(G)$, we show that the dimensionless free energy per particle $\beta f$ has a singularity at $\beta = 1$, i.e. the gas has a phase transition at temperature $k_B$. We also compute the grand-canonical partition function for this gas.

There exists a large family of zeta functions attached to finite graphs, which can be considered as discrete analogues of the Riemann zeta function, see [50] and the references therein. There are also zeta functions attached to infinite graphs, see e.g. [9], [21], [23], and attached to hypergraphs [27]. From this perspective our graph zeta function is a 'new' mathematical object. On the other hand, our graph zeta functions are related to $p$-adic Feynman integrals. These integrals were studied by Lerner and Missarov in the context of quantum field theory, [34], [33], see also [15], [16], [42], and the references therein. In [33, Theorem 1], under a condition on all the connected subgraphs of $G$, it was established the convergence of $Z(s; G)$, and a recursive formula was given. Our Theorem 1 does not require these conditions.

The connections between zeta functions of number fields and statistical mechanics, especially phase transitions, have received great attention due to the influence of the work of Connes, see e.g. [10]-[12], see also [24]. To the best of our knowledge, the connection between phase transitions and local zeta functions is new. In [48] some aspects of the partition function for $p$-adic log-Coulomb gases attached to the complete graph were studied.

In Section 6 we review the basic aspects of the theory of local zeta functions for rational functions, on local fields of characteristic zero, developed in [51]. By using this theory, we give a criterion for the existence of phase transitions at finite temperature for a 1D log-Coulomb gas with Hamiltonian (1.1), under the supposition that the function $\Phi$ is supported on a sufficiently small neighborhood of a point, and that the charge distribution $e = \{e_v\}_{v \in V(G)}$ satisfies $\{e_v e_u; v, u \in V(G), u \sim v\} = \{+1, -1\}$, see Theorem 2.

2. Basic ideas on $p$-adic analysis

In this section we collect some basic results about $p$-adic analysis that will be used in the article. For an in-depth review of the $p$-adic analysis the reader may consult [1], [49], [52].

2.1. The field of $p$-adic numbers. Along this article $p$ will denote a prime number. The field of $p$-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of
rational numbers \( Q \) with respect to the \( p \)-adic norm \( | \cdot |_p \), which is defined as

\[
|x|_p = \begin{cases} 
0, & \text{if } x = 0 \\
p^{-\gamma}, & \text{if } x = p^\gamma a, 
\end{cases}
\]

where \( a \) and \( b \) are integers coprime with \( p \). The integer \( \gamma =: \text{ord}(x) \), with \( \text{ord}(0) := +\infty \), is called the \( p \)-adic order of \( x \).

Any \( p \)-adic number \( x \neq 0 \) has the form \( x = p^{\text{ord}(x)} \sum_{j=0}^\infty x_j p^j \), where \( x_j \in \{0, \ldots, p-1\} \) and \( x_0 \neq 0 \).

2.2. **Topology of \( \mathbb{Q}_p^N \)**. We extend the \( p \)-adic norm to \( \mathbb{Q}_p^N \) by taking

\[
||x||_p := \max_{1 \leq i \leq N} |x_i|_p, \quad \text{for } x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N.
\]

We define \( \text{ord}(x) = \min_{1 \leq i \leq N} \{\text{ord}(x_i)\} \), then \( ||x||_p = p^{-\text{ord}(x)} \). The metric space \((\mathbb{Q}_p^N, || \cdot ||_p)\) is a separable complete ultrametric space. Ultrametricity refers to the fact that the norm \( || \cdot ||_p \) satisfies \( ||x + y||_p \leq \max \{||x||_p, ||y||_p\} \). Furthermore, if \( ||x||_p \neq ||y||_p \), then \( ||x + y||_p = \max \{||x||_p, ||y||_p\} \).

For \( r \in \mathbb{Z} \), denote by \( B_r(a) = \{x \in \mathbb{Q}_p^N; ||x - a||_p \leq p^r\} \) the ball of radius \( p^r \) with center at \( a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N \), and take \( B_r := B_r(0) \). Note that \( B_r(a) = B_r(a_1) \times \cdots \times B_r(a_N) \), where \( B_r(a_i) := \{x_i \in \mathbb{Q}_p; ||x_i - a_i||_p \leq p^r\} \) is the one-dimensional ball of radius \( p^r \) with center at \( a_i \in \mathbb{Q}_p \). The ball \( B_r \) equals to the product of \( N \) copies of \( B_0 = \mathbb{Z}_p \), the ring of \( p \)-adic integers of \( \mathbb{Q}_p \).

2.3. **Test functions**. A complex-valued function \( \varphi \) defined on \( \mathbb{Q}_p^N \) is called locally constant if for any \( x \in \mathbb{Q}_p^N \) there exist an integer \( l(x) \in \mathbb{Z} \) such that \( \varphi(x + x') = \varphi(x) \) for \( x' \in B_{l(x)} \). A function \( \varphi : \mathbb{Q}_p^N \to \mathbb{C} \) is called a Bruhat-Schwartz function, or a test function, if it is locally constant with compact support. The \( \mathbb{C} \)-vector space of Bruhat-Schwartz functions is denoted by \( D := D(\mathbb{Q}_p^N) \).

2.4. **Integration and change of variables**. We denote by \( d^N x \) a Haar measure of the topological group \((\mathbb{Q}_p^N, +)\) normalized by the condition \( \int_{B_r} d^N x = 1 \).

A function \( h : U \to \mathbb{Q}_p \) is said to be analytic on an open subset \( U \subset \mathbb{Q}_p^N \), if there exists a convergent power series \( \sum a_i x^i \) for \( x \in \bar{U} \subset U \), with \( \bar{U} \) open, such that \( h(x) = \sum a_i x^i \) for \( x \in \bar{U} \), with \( x^i = x_1^i \cdots x_N^i \), \( i = (i_1, \ldots, i_N) \). In this case, \( \frac{\partial}{\partial x^i} h(x) = \sum a_i \frac{\partial}{\partial x^i} \) is a convergent power series. Let \( U, V \) be open subsets of \( \mathbb{Q}_p^N \). A mapping \( \sigma : U \to V, \sigma = (\sigma_1, \ldots, \sigma_N) \) is called analytic if each \( \sigma_i \) is analytic.

Let \( \varphi : V \to \mathbb{C} \) be a continuous function with compact support, and let \( \sigma : U \to V \) be an analytic mapping. Then

\[
(2.1) \quad \int_V \varphi(y) d^N y = \int_U \varphi(\sigma(x)) |\text{Jac}(\sigma(x))|_p d^N x,
\]

where \( \text{Jac}(\sigma(z)) = \det \left[ \frac{\partial \sigma_j}{\partial x_i} (z) \right]_{1 \leq i \leq N, 1 \leq j \leq N} \), see e.g. [S] Section 10.1.2].
3. Zeta functions for graphs

Along this article by a graph, we mean a finite, simple graph, i.e. a graph with no loops and no multiple edges, see e.g. [5, Definition 1.2.4].

Let $G$ be a graph. We denote by $V := V(G)$ its set of vertices and by $E := E(G)$ its set of edges. If $E(G) \neq \emptyset$, we denote by $i_G$ the incidence relation on $G$, i.e. a mapping from the set of edges to the set of pairs of vertices, where the corresponding two vertices are necessarily distinct. We use the notation $i_G(l) = \{u, v\}$ or the notation $u \sim v$. To each vertex $v \in V$ we attach a complex variable $x_v$, and to each edge $l \in E$ we attach a complex variable $s(l)$. We also use the notation $s(u, v)$ if $u \sim v$. We set $x := \{x_v\}_{v \in V}$, $s := \{s(l)\}_{l \in E}$.

Given $l \in E$, with $i_G(l) = \{u, v\}$, we set

$$F_l(x_u, x_v, s(l)) := |x_u - x_v|_p^{s(l)}$$

and

$$F_G(x, s) := \prod_{l \in E} F_l(x_u, x_v, s(l)) = \prod_{u,v \in V \atop u \sim v} |x_u - x_v|_p^{s(u,v)}.$$  

Remark 1. (i) If $V(G) \neq \emptyset$ and $E(G) = \emptyset$, then $G$ consists of a finite set of vertices without edges connecting them, thus incidence relation is not defined. In this case we set $F_G(x, s) := 1$. Due to technical reasons, we consider the empty set as a graph, in this case $F_G(x, s) := 1$.

Notation 1. (i) For a finite subset $A$, we denote by $|A|$ its cardinality.

(ii) We denote by $\mathcal{D}_{sym}(\mathbb{Q}_p^N)$ the $\mathbb{C}$-vector space of symmetric test functions, i.e. all the complex-valued test functions satisfying $\varphi(x_1, \ldots, x_N) = \varphi(x_{\pi(1)}, \ldots, x_{\pi(N)})$ for any permutation $\pi$ of $\{1, 2, \ldots, N\}$.

Let $G$ and $H$ be graphs. By a graph isomorphism $\sigma : G \to H$, we mean a pair of mappings $\{\sigma_E, \sigma_V\}$, where $\sigma_V : V(G) \to V(H)$, $\sigma_E : E(G) \to E(H)$ are bijections, with the property that $i_G(l) = \{u, v\}$ if and only if $i_H(\sigma_E(l)) = \{\sigma_V(u), \sigma_V(v)\}$. In the case of simple graphs, $\sigma_E$ is completely determined by $\sigma_V$. For the sake of simplicity, we will denote the pair $\{\sigma_E, \sigma_V\}$ as $\sigma$, see e.g. [5, Sections 1.2.9, 1.2.10].

We denote by Aut($G$) the automorphism group of $G$. Let $\sigma : G \to H$ be a graph isomorphism. Assume that the cardinality of $|V(G)| = |V(H)| = N$. Let $x_u$, $u \in V(G)$, be $p$-adic variables as before. Then the mapping

$$\sigma^* : \mathbb{Q}_p^N \to \mathbb{Q}_p^N$$

$$x_v \to x_{\sigma(v)}$$

is a $p$-adic analytic isomorphism that preserves the Haar measure of $\mathbb{Q}_p^N$, see [24.

Definition 1. Given $\varphi \in \mathcal{D}_{sym}(\mathbb{Q}_p^{|V(G)|})$, the $p$-adic zeta function attached to $(G, \varphi)$ is defined as

$$Z(\varphi; s; G) = \int_{\mathbb{Q}_p^{|V(G)|}} \varphi(x) F_G(x, s) \prod_{v \in V(G)} dx_v,$$

for $\Re(s(l)) > 0$ for every $l \in E$, where $\prod_{v \in V(G)} dx_v$ denotes the normalized Haar measure on $\left(\mathbb{Q}_p^{|V(G)|}, +\right)$. If $\varphi$ is the characteristic function of $\mathbb{Z}_p^{|V(G)|}$, we use the notation $Z(s; G)$. 

Lemma 1. Let $G$ and $H$ be graphs. If $\sigma : G \to H$ is a graph isomorphism, then
\[
Z_\varphi(\{s(l)\}_{l \in E(G)}; G) = Z_\varphi(\{s(l)\}_{l \in E(H)}; H).
\]
Furthermore, for any $\sigma = (\sigma_V, \sigma_E) \in \text{Aut}(G)$, it holds true that
\[
Z(\{s(l)\}_{l \in E(G)}; G) = Z(\{s(\sigma(l))\}_{l \in E(G)}; G),
\]
where the integrals exist.

Proof. By using that
\[
Z_\varphi(s; G) = \int_{\mathbb{Q}_p^{l(V(G))}} \varphi(\{x_v\}_{v \in V(G)}) \prod_{u,v \sim v} |x_u - x_v|^s(u,v) \prod_{v \in V(G)} dx_v,
\]
and changing variables as $\sigma^* : \mathbb{Q}_p^N \to \mathbb{Q}_p^N$, $x_v \mapsto x_{\sigma(v)}$, see (3.2), we have
\[
\varphi(\{x_v\}_{v \in V(G)}) = \varphi(\{x_{\sigma(v)}\}_{v \in V(G)}) = \varphi(\{x_{\sigma'(v)}\}_{v' \in V(H)}),
\]
because the list $\{x_v\}_{v' \in V(H)}$ is a permutation of the list $\{x_v\}_{v \in V(G)}$. In addition,
\[
\prod_{u,v \sim v} |x_u - x_v|^s(u,v) = \prod_{\sigma(u),\sigma(v) \in V(G)} \prod_{u,v \sim v} |x_{\sigma(u)} - x_{\sigma(v)}|^s(\sigma(u),\sigma(v))
\]
and by using that $\sigma^*$ preserves the Haar measure,
\[
\prod_{v \in V(G)} dx_v = \prod_{v' \in V(H)} dx_{\sigma(v)} = \prod_{v' \in V(H)} dx_{v'}.
\]
Consequently $Z_\varphi(\{s(l)\}_{l \in E(G)}; G) = Z_\varphi(\{s(l)\}_{l \in E(H)}; H)$.

Remark 2. We use the notation $G = G_1 \# \cdots \# G_k$ to mean that $G_1, \ldots, G_k$ are all the distinct connected components of $G$. Then $F_G(x,s) = \prod_{i=1}^k F_{G_i}(x,s)$ and
\[
Z(s; G) = \prod_{i=1}^k Z(s; G_i).
\]
Notice that $Z(s; G_i) = 1$, if $G_i$ consists of only one vertex.

The zeta functions $Z_\varphi(s; G)$ are a special type of multivariate Igusa zeta functions. These functions were studied in [36], in particular, the following result holds true:

Proposition 1 (F. Loeser [36 Théorème 1.1.4]). The zeta function $Z_\varphi(s; G)$ admits a meromorphic continuation to $\mathbb{C}^{l(E(G))}$ as a rational function in the variables $p^{-s(l)}$, $l \in E(G)$, more precisely,
\[
Z_\varphi(s; G) = \prod_{l \in T} \frac{P_\varphi(s)}{1 - p^{-N_0^l - \sum_{l \in E(G)} N_l^l s(l)}},
\]
where $T$ is a finite set, the $N_0^l$, $N_l^l$ are non-negative integers, and $P_\varphi(s)$ is a polynomial in the variables $\{p^{-s(l)}\}_{l \in E(G)}$. 
**Corollary 1.** The following functional equations hold true:

\[
\prod_{i \in T} \left( 1 - p^{-N_0 - \sum_{E \in (G)} N_i^{s(l)}} \right) = \prod_{i \in T} \left( 1 - p^{-N_0 - \sum_{E \in (G)} N_i^{s(l)} s(\sigma(l))} \right),
\]

for any \( \sigma = (\sigma_V, \sigma_E) \in \text{Aut}(G). \)

**Proof.** The results follows from by using the fact gives an equality between functions in an open set containing \( \{ \text{Re}(s(l)) > 0; l \in E(G) \}. \)

**Example 1.** Let \( K_2 \) be the complete graph with two vertices, \( v_0, v_1 \). We denote the corresponding edge as \( l \). Then \( F_{K_2} (x, s) = |x_{v_0} - x_{v_1}|^s \) and

\[
Z(s; K_2) = \int_{\mathbb{Z}_p^2} |x_{v_0} - x_{v_1}|^s \, dx_{v_0} \, dx_{v_1} = \int_{\mathbb{Z}_p} \left\{ \int_{\mathbb{Z}_p} |x_{v_0} - x_{v_1}|^s \, dx_{v_0} \right\} \, dx_{v_1}.
\]

By changing variables as \( y = x_{v_0} - x_{v_1}, z = x_{v_1} \), we have

\[
Z(s; K_2) = \int_{\mathbb{Z}_p} \left\{ \int_{\mathbb{Z}_p} |y|^s \, dy \right\} \, dz = \int_{\mathbb{Z}_p} |y|^s \, dy = \frac{1 - p^{-1}}{1 - p^{-1-s}}.
\]

**Example 2.** We denote by \( S_N \) the star graph with \( N \) vertices labeled as \( V(S_N) = \{1, \ldots, N\} \), where the vertex 1 is the center of the star, i.e.

\[
E(S_N) = \{(1, 2), \ldots, (1, l), \ldots, (1, N)\}.
\]

Then \( F_{S_N} (x, s) = \prod_{i=2}^N |x_1 - x_i|^s \) and

\[
Z(s; S_N) = \int_{\mathbb{Z}_p^N} \left\{ \int_{\mathbb{Z}_p} \prod_{i=2}^N |x_1 - x_i|^s \, dx_1 \right\} \, dx_2 \ldots \, dx_N.
\]

By changing variables as \( z_1 = x_1, z_i = x_1 - x_i \) for \( i = 2, \ldots, N \), we obtain that

\[
Z(s; S_N) = \int_{\mathbb{Z}_p^{N-1}} \prod_{i=2}^N |z_i|^s \, dz_1 \ldots \, dz_N = \frac{1 - p^{-1}}{1 - p^{-1-s_1}}.
\]

**Example 3.** Let \( T_N \) be a finite connected tree with \( N \) vertices. Then

\[
Z(s, T_N) = \frac{(1 - p^{-1})^{N-1}}{\prod_{(u,v) \in E(T_N)} 1 - p^{-1-s(u,v)}}.
\]

We recall that a tree is an undirected graph in which any two vertices are connected by exactly one path. We fixed \( r \in V(T_N) \) and for \( r \in V(T_N) \) we denote by \( l_r(v) \) the length of path from \( r \) to \( v \). We now set \( l_r(T_N) := \max_{v \in V(T_N)} l_r(v) \). If \( l_r(T_N) = 1 \), then \( T_N \) is a star graph with \( N \) vertices. The announced formula is established by induction on \( l_r(T_N) \). The case \( l_r(T_N) = 1 \) was already established. Assume that \( l_r(T_N) \geq 2 \). Then there exists \( u' \in V(T_N) \) \( \{r\} \) with \( l_r(u') = l_r(T_N) \). We fix such \( u' \), then there exists a unique path from \( u' \) to \( r \), and consequently
unique $v' \in V(T_N)$ with $u' \sim v'$. We denote by $T_{N-1}'$ the tree obtained from $T_N$ by deleting the edge $u' \sim v'$. Notice that $T_{N-1}'$ has $N-1$ vertices. Then

$$Z(s, T_N) =$$

$$\int_{\mathbb{Z}_p^{\left|V(T_N)\right|}} \left( \prod_{u \in (V(T_{N-1}') \setminus \{u'\})} \left| x_u - x_{u'} \right|^p \right) \left| x_{u'} - x_{v'} \right|^p dx_{u'} \prod_{v \in (V(T_{N-1}) \setminus \{u'\})} \left| z_{u'} \right|^p dz_{u'}$$

We now change variables as $x_u \mapsto x_u$ if $u \neq u'$, and $x_u \mapsto z_u + x_{v'}$ if $u \neq u'$, in the above integral:

$$Z(s, T_N) =$$

$$\int_{\mathbb{Z}_p^{\left|V(T_{N-1}')\right|}} \int_{\mathbb{Z}_p^{\left|V(T_{N-1})\right|}} \left( \prod_{u \in (V(T_{N-1}') \setminus \{u'\})} \left| x_u - x_{u'} \right|^p \right) \left| z_{u'} \right|^p dz_{u'} \prod_{v \in (V(T_{N-1}) \setminus \{u'\})} \left| z_u \right|^p dz_u = Z(s, T_{N-1}') \left( \frac{1 - p^{-1}}{1 - p^{-1-s(u',v')}} \right).$$

Thus, by induction hypothesis,

$$Z(s, T_N) = \left( \frac{1 - p^{-1}}{1 - p^{-1-s(u',v')}} \right) \left( \frac{(1 - p^{-1})^{\left|V(T_{N-1})\right| - 1}}{\prod_{(u,v) \in E(T_{N-1})} 1 - p^{-1-s(u,v)}} \right) \left( \frac{(1 - p^{-1})^N}{\prod_{(u,v) \in E(T)} 1 - p^{-1-s(u,v)}} \right).$$

**Example 4.** Let $L_N$ denote the linear graph consisting of $N$ vertices labeled as $V(L_N) = \{1, \ldots, N\}$, and edges $E(L_N) = \{(1, 2), \ldots, (l-1, l), \ldots, (N-1, N)\}$. Then $F_{L_N}(x, s) = \prod_{i=2}^{N} |x_{i-1} - x_i|^s_i$ and

$$Z(s; L_N) = \int_{\mathbb{Z}_p^N} \prod_{i=2}^{N} |x_{i-1} - x_i|^s_i \prod_{i=2}^{N} dx_i.$$

By changing variables as $z_1 = x_1$, $z_i = x_{i-1} - x_i$ for $i = 2, \ldots, N$ and using the fact that this transformation preserves the normalized Haar measure of $\mathbb{Z}_p^N$, we obtain that

$$Z(s; L_N) = \prod_{i=2}^{N} \int_{\mathbb{Z}_p} |z_i|^s_i dz_i = \frac{(1 - p^{-1})^{N-1}}{\prod_{i=2}^{N} (1 - p^{-1-s_i})} = Z(s; S_N).$$

**Remark 3.** The assertion

if $Z(s; G) \neq Z(s; K)$, then $G$ is not isomorphic to $K$

is true, cf. Lemma 2 but Examples 2, 3 show that the assertion

if $Z(s; G) = Z(s; K)$, then $G$ is isomorphic to $K$

is false.
3.1. Vertex Colorings and Chromatic Functions. We recall that a graph \( H \) is called a subgraph of \( G \) if \( V(H) \subset V(G) \), \( E(H) \subset E(G) \). If \( E(H) \neq \emptyset \), \( i_H \) is the restriction of \( i_G \) to \( E(H) \). If \( E(H) = \emptyset \), \( H \) consists of a subset of vertices of \( G \) without edges, and thus \( i_H \) is not defined.

**Definition 2.** Let \( I \) be a non-empty subset of \( V(G) \). We denote by \( G_I \) (or \( G[I] \)) the subgraph induced by \( I \), which is the subgraph defined as \( V(G_I) = I \),

\[
E(G_I) = \{ l \in E(G) ; i_G(l) = \{ v, v' \} \text{ for some } v, v' \in I \},
\]

and \( i_{G_I} = i_G |_{E(G_I)} \). If \( I = \emptyset \), by definition \( G_I = \emptyset \).

Suppose that \( G_{I} = G^{(1)}_{I} \# \cdots \# G^{(m)}_{I} \). If \( G^{(j)}_{I} = \{ v \} \), we say that \( v \) is an isolated vertex of \( G_{I} \). We denote by \( G^{iso}_{I} \) the set of all the isolated vertices of \( G_{I} \).

Then

\[
G_{I} = G^{red}_{I} \bigcup G^{iso}_{I},
\]

where \( G^{red}_{I} := G^{(1)}_{I} \# \cdots \# G^{(m)}_{I} \) and \( |G^{(i)}_{I}| > 1 \) for \( k = 1, \ldots, l \). We call \( G^{(i)}_{I} \) the reduced subgraph of \( G_{I} \).

We adopt the convention that if \( I = \emptyset \), then \( G^{red}_{I} = G^{iso}_{I} = \emptyset \).

3.1.1. Colorings and Chromatic Functions. In this section we color graphs using \( p \) colors, more precisely, we attach to every element of \( \{0, 1, \ldots, p-1\} \) (which we identify with an element of \( \mathbb{F}_p \)) a color.

**Definition 3.** A vertex coloring of \( G \) is a mapping \( C : V(G) \rightarrow \mathbb{F}_p \). If \( v \) is a vertex of \( G \), then \( C(v) \) is its color. We denote by \( \text{Colors}(G) \), the set of all possible vertex-colorings of \( G \).

We call \( C \) an isolated vertex-coloring if \( C(v) = \emptyset \) for \( v \in V \). Notice that any coloring \( C \) is given by a vector \( a = (a_v)_{v \in V(G)} \in \mathbb{F}_p^{|V(G)|} \) with \( C(v) = a_v \) for \( v \in V \). We will identify \( C \) with \( a \). Our notion of vertex coloring is completely different from the classical one which requires that adjacent vertices of \( G \) receive distinct colors of \( \mathbb{F}_p \), see e.g. [5] Section 7.2.

**Definition 4.** Given a pair \( (G, C) \), we attach to it a colored graph \( G^C \) defined as follows: \( V(G^C) = V(G) \),

\[
E(G^C) = \{ l \in E(G) ; C(u) = C(v) \text{ where } i_G(l) = \{ u, v \} \}
\]

and \( i_{G^C} = i_G |_{E(G^C)} \).

We note that if \( G^1, \ldots, G^r \) are all the connected components of \( G^C \), then \( C \mid_{G^k} \) is constant for \( k = 1, \ldots, r \). If \( C \) is identified with \( a \) we use the notation \( G^a \). Definition 4 tells us how to color the edges of a graph if we have already assigned colors to the vertices of the graph. To an edge having its two vertices colored with the same color we assign the color of its vertices, in other case, we discard the edge.

**Definition 5.** We set \( \text{Colored}(G) := \{ G^C ; C \in \text{Colors}(G) \} \), and \( \text{Subgraphs}(G, |G|) \) to be the set of all graphs \( H \) such that \( V(H) = V(G) \), \( E(H) \subset E(G) \), and if \( E(H) \neq \emptyset \), \( i_H \) is the restriction of \( i_G \) to \( E(H) \). We define

\[
\Phi : \text{Colored}(G) \rightarrow \text{Subgraphs}(G, |G|)
\]

as follows: \( \Phi (G^C) = H \) if and only if \( V(H) = V(G^C) \), \( E(H) = E(G^C) \) and \( i_H = i_{G^C} \). We set \( \text{Subgraphs}(G, |G|) = \Phi (\text{Colored}(G)) \).
The family $\text{Colored}(G)$ is formed by all the possible colored versions of $G$, the operation ‘forgetting the coloring’ $\tilde{\mathcal{F}}$ assigns to an element of $\text{Colored}(G)$ a subgraph of $G$ having the same vertices as $G$. Any graph in $\text{Subgraphs}(G, |G|)$ is obtained from $G$ by deleting one or more edges, ‘but keeping’ the corresponding vertices.

**Definition 6.** We define $\text{Indgraphs}(G)$ to be the set of all connected graphs $H$ such that there exists a coloring $C$, with $G^C = G_1^C \# \cdots \# G_r^C$, and $H = G_i^C$ for exactly one index $i$.

By Definition 2 we have

$$\text{Indgraphs}(G) = \{G[I] : \emptyset \neq I \subset V(G) \text{ and } G[I] \text{ is connected}\},$$

where $G[I]$ denotes the subgraph induced by $I$.

### 3.1.2. The Chromatic Functions.

**Definition 7.** Given $H$ in $\text{Subgraphs}(G, |G|)$, we define its chromatic function as

$$\mathcal{C}(p; H) = \left| \left\{ C^C \in \text{Colored}(G) : \tilde{\mathcal{F}}(G^C) = H \right\} \right|.$$ 

Notice that if $G$ is connected, then $\mathcal{C}(p; G) = p$. Indeed, if we use at least two colors then $G^C$ has at least two connected components, and thus $\mathcal{F}(G^C) \neq G$. So we can use only constant colorings to have $\mathcal{F}(G^C) = G$.

Given $u, v \in V(G)$, we denote by $d(u, v)$ the length of the shortest path in $G$ joining $u$ and $v$. Given $H, W$ subgraphs of $G$, we set

$$d(H, W) = \min_{u \in V(H), v \in V(W)} d(u, v) \in \mathbb{N}.$$ 

**Remark 4.** Suppose that $H = H_1 \# \cdots \# H_r$. The condition $\tilde{\mathcal{F}}(G^C) = H$ implies that $C|_{H_i} = a_i \in \mathbb{F}_p$ for $i = 1, \ldots, r$. Now if $d(H_i, H_j) = 1$, then $a_i \neq a_j$, i.e., $a_i$ may be equal. We now define $D_1(H) := D_1 = \left\{ \{H_i, H_j\} : H_i, H_j \text{ are connected components of } H, d(H_i, H_j) = 1 \right\}$, and $D_2(H) := D_2 = \left\{ \{H_i, H_j\} : H_i, H_j \text{ are connected components of } H, d(H_i, H_j) \geq 2 \right\}$.

We set $\Pi_1 : A \times B \to A$, respectively $\Pi_2 : A \times B \to B$, for the canonical projections, and define $\bar{D} = \Pi_1 D_2 \cup \Pi_2 D_2$. Any coloring $C$ satisfying $\tilde{\mathcal{F}}(G^C) = H$ is determined by a set conditions of the following form. There exists a partition $\mathcal{P}(\bar{D}) = \{\bar{D}_1, \ldots, \bar{D}_k\}$, with $|\bar{D}_i| \geq 1$ for $i = 1, \ldots, k$, such that

$$\{C(H_i) \neq C(H_j) \text{ for } d(H_i, H_j) = 1; \}$$

(3.5)

$$\{ C(H_i) = C(H_j) = b_i \in \mathbb{F}_p, \text{ for any } \{H_i, H_j\} \in \bar{D}_i, \text{ with } b_i \neq b_m \text{ if } l \neq m, \text{ for } l, m \in \{1, \ldots, k\}. \}$$

(3.6)

The set of conditions (3.5)-(3.6) defines a relative closed subset of the affine space $\mathbb{F}_p^M$, for a suitable $M$, and the solution set of these conditions corresponds to the colorings defined by conditions (3.5)-(3.6).
Example 5. In this example, we compute the chromatic function $C(p; H)$, where $H$ is in $\text{Subgraphs}(G, |G|)$, with $G$ and $H$ as follows:

In this case $H = H_1 \# \cdots \# H_4$, where $H_i = \{x_i\}$ is the vertex $x_i$, for $i = 1, 2, 3, 4$. Set $C(H_i) = a_i$, for $i = 1, 2, 3, 4$. There are three different types of conditions (colorings) coming from $\mathfrak{S}(G^C) = H$:

\begin{align}
&\begin{cases}
  a_1 \neq a_2, a_1 \neq a_3, a_2 \neq a_3, a_3 \neq a_4; \\
  a_1 \neq a_4, a_2 \neq a_4;
\end{cases} \\
&\begin{cases}
  a_1 \neq a_2, a_1 \neq a_3, a_2 \neq a_3, a_3 \neq a_4; \\
  a_1 = a_4.
\end{cases} \\
&\begin{cases}
  a_1 \neq a_2, a_1 \neq a_3, a_2 \neq a_3, a_3 \neq a_4; \\
  a_2 = a_4.
\end{cases}
\end{align}

Consequently

$$C(p, H) = p(p - 1)(p - 2)(p - 3) + 2p(p - 1)(p - 2),$$

for any prime number $p$.

We now explain the connection between chromatic functions and the computation of certain $p$-adic integrals. Set

$$F_G(x, s) = |x_1 - x_2|^{s_{12}}_p |x_1 - x_3|^{s_{13}}_p |x_2 - x_3|^{s_{23}}_p |x_3 - x_4|^{s_{24}}_p,$$

and

$$I(s, a) = \int_{\mathbb{F}_p^4} F_G(x, s) \prod_{i=1}^4 dx_i,$$

where $a = (a_1, a_2, a_3, a_4) \in \mathbb{F}_p^4$. Assume that $a$ is a coloring of one the types \((3.7)-(3.9)\), i.e. $a$ is a solution of exactly one of the conditions systems \((3.7)-(3.9)\), then by using that

$$|a_1 - a_2 - p(x_1 - x_2)|^{s_{12}}_p |a_1 - a_3 - p(x_1 - x_3)|^{s_{13}}_p |a_2 - a_3 - p(x_2 - x_3)|^{s_{23}}_p \times$$

$$|a_3 - a_4 - p(x_3 - x_4)|^{s_{24}}_p = 1,$$

for any $x_1, x_2, x_3, x_4$,

we have $I(s, a) = p^{-4}$. Now notice that

$$|\{a \in \mathbb{F}_p^4; I(s, a) = p^{-4}\}| = C(p, H)$$

for any prime number $p$. 
Definition 8. Let \( G \) be a graph and let \( k \) be a positive integer. A proper \( k \)-coloring of the vertices of \( G \) is a function \( f : V(G) \to \{0, \ldots, k-1\} \) such that \( f^{-1}(j) \) is an independent set, i.e., for any \( u, v \in f^{-1}(j) \) there is no edge in \( E(G) \) joining them. Let \( \mathcal{P}(k;G) \) denotes the number of vertex \( k \)-colorings of \( G \). There exists a polynomial \( \mathcal{P}(x;G) \) (the chromatic polynomial of \( G \)), with integer coefficients, satisfying \( \mathcal{P}(x;G) \big|_{x=k} = \mathcal{P}(k;G) \) for any positive integer \( k \), see e.g. Proposition 9.2. The chromatic number \( \chi(G) \) of \( G \) is the positive integer defined as \( \chi(G) = \min \{ k \in \mathbb{N} \setminus \{0\} : \mathcal{P}(k;G) > 0 \} \).

Remark 5. We review the classical definitions of vertex colorings and chromatic polynomial. Let \( G \) be a graph and let \( k \) be a positive integer. A proper \( k \)-coloring of the vertices of \( G \) is a function \( f : V(G) \to \{0, \ldots, k-1\} \) such that \( f^{-1}(j) \) is an independent set, i.e., for any \( u, v \in f^{-1}(j) \) there is no edge in \( E(G) \) joining them. Let \( \mathcal{P}(k;G) \) denotes the number of vertex \( k \)-colorings of \( G \). There exists a polynomial \( \mathcal{P}(x;G) \) (the chromatic polynomial of \( G \)), with integer coefficients, satisfying \( \mathcal{P}(x;G) \big|_{x=k} = \mathcal{P}(k;G) \) for any positive integer \( k \), see e.g. Proposition 9.2. The chromatic number \( \chi(G) \) of \( G \) is the positive integer defined as \( \chi(G) = \min \{ k \in \mathbb{N} \setminus \{0\} : \mathcal{P}(k;G) > 0 \} \).

Proposition 2. For any graph \( G \) and any \( H \) in \( \text{Subgraphs}(G, |G|) \), \( \mathcal{C}(p;H) = \mathcal{P}(x;G) \big|_{x=p} \).

Proof. We assume that \( H = H_1 \# \cdots \# H_r \), as in Definition 8. The result follows by establishing a bijection between the following two sets:

\[
A(G^C, H) := \{ C \in \text{Colors}(G) : \mathfrak{F}(G^C) = H \},
B(G_H^*) := \{ p\text{-colorings of } G_H^* \}.
\]

Given a coloring \( C \in A(G^C, H) \), we define

\[
C^* : V(G_H^*) \to \{0, \ldots, p-1\}
\]

\[
H_i \to C(H_i).
\]

Now, if \( C_1, C_2 \in A(G^C, H) \) and \( C_1 \neq C_2 \), then there exists \( j \in \{1, \ldots, r\} \) such that \( C_{1|H_j} \neq C_{2|H_j} \), which implies that \( C_1^* \neq C_2^* \).

Given a \( p \)-coloring \( C^* \) of \( G_H^* \), we define

\[
C : V(G) \to \{0, \ldots, p-1\}
\]

\[
v \to C^*(H_i),
\]

for any \( v \in H_i \). Then \( C \in A(G^C, H) \). Indeed, by the definition of \( C \), \( G^C = H_1 \# \cdots \# H_r = H \), with \( C|_{H_i} = a_i \in \mathbb{F}_p \) for \( i = 1, \ldots, r \). Then \( V(G^C) = V(H) \).

Additionally, an edge \( l \in E(G^C) \) is and edge of \( G \), say \( i_G(l) = \{u, v\} \), satisfying \( C(u) = C(v) \). Then \( u, v \in V(H_i) \), and \( l \in E(H_i) \), i.e. \( E(G^C) \subset E(H) \). Conversely, given \( l \in E(H_i) \), with \( i_H(l) = \{u, v\} \), we have \( C(u) = C(v) = C^*(H_i) \), and thus \( l \in V(G^C) \).

3.2. Rationality and recursive formulas.

Theorem 1. Let \( G \) be a connected graph. Then, for any prime number \( p \), \( Z(s;G) \) satisfies:

\[
(i) \quad Z(s;G) = \frac{\sum_{H \in \text{Subgraphs}(G, |G|)} p^{-|V(G)|-\sum_{l \in E(H)} s(l)} \mathcal{C}(p; H) Z(s; H)}{1 - p^{-|V(G)|-\sum_{l \in E(G)} s(l)}}.
\]
(ii) $Z(s; G)$ admits a meromorphic continuation to $\mathbb{C}^{|E(G)|}$ as a rational function of $\{p^{-s(l)}; l \in E(G)\}$. More precisely,

$$Z(s; G) = \frac{M\left(\left\{ p^{-s(l)}; l \in E(G) \right\} \right)}{\prod_{H \in \text{Indgraphs}(G)} \left(1 - p^{1-|V(H)| - \sum_{l \in E(H)} s(l)}\right)},$$

where $M\left(\left\{ p^{-s(l)}; l \in E(G) \right\} \right)$ denotes a polynomial with rational coefficients in the variables $\{p^{-s(l)}\}_{l \in E(G)}$.

**Proof.** (i) We attach to $a = \{a_v\}_{v \in V(G)} \in \mathbb{F}_p^{V(G)}$ a color $C$ defined as $C(v) = a_v$, for $v \in V(G)$. We set

$$I(s; a) := \int_{\mathbb{F}_p^{V(G)}} F_G(x, s) \prod_{v \in V(G)} dx_v,$$

then

$$Z(s; G) = \sum_{a \in \mathbb{F}_p^{V(G)}} I(s; a).$$

Now

$$I(s; a) = p^{-|V(G)|} \int_{\mathbb{F}_p^{V(G)}} F_G(a + px, s) \prod_{v \in V(G)} dx_v,$$

where

$$F_G(a + px, s) = \prod_{l \in E(G)} \left| a_v - a_u + px_v - px_u \right|_{p}^{s(l)}$$

$$= \prod_{l \in E(G), (l) = \{v, u\}} \begin{cases} 1 & \text{if } C(v) \neq C(u) \\ p^{-s(l)} |x_v - x_u|_{p}^{s(l)} & \text{if } C(v) = C(u). \end{cases}$$

By attaching to $I(s; a)$ the colored graph $G^C = (G^C)^{\text{red}} \# (G^C)^{\text{iso}}$, and using $G^C = (G^C)^{\text{red}}$ by simplicity, we have

$$F_G(a + px, s) = p^{-\sum_{l \in E(G^C)} s(l)} \prod_{l \in E(G^C)} |x_v - x_u|_{p}^{s(l)},$$

and

$$I(s; a) = p^{-|V(G)| - \sum_{l \in E(G^C)} s(l)} Z\left(\{s(l)\}_{l \in E(G^C)}; \{x_v\}_{v \in V(G^C)}\right).$$

Therefore

$$Z(s; G) = \sum_{G^C, C \in \text{Colors}(G)} p^{-|V(G)| - \sum_{l \in E(G^C)} s(l)} Z(s; G^C).$$
By fixing a graph $H$ in $\text{Subgraphs}_\mathcal{F}(G,|G|)$, we have

\begin{equation}
\sum_{\mathcal{F}(G^C) = H} p^{-|V(G)| - \sum_{l \in E(G)} s(l)} Z(s;G^C_{\text{red}}) = p^{-|V(G)| - \sum_{l \in E(H^C)} s(l)} C(p;H) Z(s;H),
\end{equation}

and consequently

\begin{equation}
Z(s;G) = \sum_{H \in \text{Indgraphs}(G,|G|)} p^{-|V(G)| - \sum_{l \in E(H)} s(l)} C(p;H) Z(s;H)
\end{equation}

By taking $H = G$, $C(p;H) = p$, in (3.11), we get

\[
\sum_{\mathcal{F}(G^C) = G} p^{-|V(G)| - \sum_{l \in E(G)} s(l)} Z(s;G^C) = p^{1-|V(G)| - \sum_{l \in E(G)} s(l)} Z(s;G)
\]

and thus from (3.12),

\begin{equation}
Z(s;G) = \sum_{H \in \text{Subgraphs}_\mathcal{F}(G,|G|): H \neq G} p^{-|V(G)| - \sum_{l \in E(H)} s(l)} C(p;H) Z(s;H) 1 - p^{1-|V(G)| - \sum_{l \in E(G)} s(l)}.
\end{equation}

Now, taking $H = H_1 \# \cdots \# H_{r(H)} \# H^\text{iso}$, where the $H_i$s are different graphs in $\text{Indgraphs}(H)$, we have

\begin{equation}
Z(s;H) = \prod_{j=1}^{r(H)} Z(s;H_j).
\end{equation}

By using recursively (3.13)-(3.14), and the formula for $Z(s;K_2)$, we obtain (3.10). Notice that at the beginning of any iteration of the formulas (3.13)-(3.14), with $|H_j| \geq 2$ for $j = 1, \ldots, r(H)$, we have

\[
\prod_{j=1}^{r(H)} Z(s;H_j) = \frac{A(s;H_1, \ldots, H_{r(H)})}{\prod_{j=1}^{r(H)} \left(1 - p^{1-|V(H_j)| - \sum_{l \in E(H_j)} s(l)}\right)}.
\]

where all the factors in the denominator are different since $H_j \cap H_i = \emptyset$ if $j \neq i$. \hfill \Box

**Corollary 2.** (i) Set $s(l) = \gamma \in \mathbb{C}$ for any $l \in E(G)$, and define $Z_{G,p}(\gamma) := Z(s(G_{\text{red}});s(l) = \gamma)$. Then the integral $Z_{G,p}(\gamma)$ converges for

\[
\text{Re}(\gamma) \geq \max_{H \in \text{Indgraphs}(G)} \frac{1 - |V(H)|}{|E(H)|} \geq \gamma_0.
\]

More generally, for $G$ and $p$ fixed, $Z_{G,p}(\gamma)$ is an analytic function in $\gamma$ for $\text{Re}(\gamma) \geq \gamma_0$.

(ii) Let $G = K_N$ be the complete graph with $N$ vertices. Then $Z_{G,p}(\gamma)$ is an analytic function in $\gamma$ for $\text{Re}(\gamma) \geq \frac{\gamma^2}{N}$.

(iii) Let $M \left( \left\{ p^{-s(l)}; l \in E(G) \right\} \right)$ be the polynomial defined in (3.11). Then the following functional equations hold true:

\[
M \left( \left\{ p^{-s(l)}; l \in E(G) \right\} \right) = M \left( \left\{ p^{-s(l)}; l \in E(G) \right\} \right)
\]

for any $\sigma = (\sigma_V, \sigma_E) \in \text{Aut}(G)$. 
Proof. (i) It follows directly from Theorem 1-(ii), by using the properties of the geometric series. (ii) It follows from the fact that any induced subgraph $H$ of $K_N$ is complete, say $H = K_l$, $|V(H)| = l$, $|E(H)| = \frac{l(l-1)}{2}$ for $l = 2, \ldots, N$. Then

$$\gamma_0 = \max_{2 \leq l \leq N} \frac{-2}{l} = \frac{-2}{N}.$$ 

(iii) It follows from Theorem 1-(ii) and Corollary 1 by using the fact that any isomorphism of $G$ induces a permutation on the set $\{H \in \text{Indgraphs}(G); |V(H)| \geq 2\}$. □

**Corollary 3.** (i) Let $G_1$ be an Indgraph of $G$ generated by $I \subset V(G)$. Then

$$Z(s; G_1) = Z(s; G) \big|_{s(t) = 0} .$$

(ii) If $\lim_{s(l) \to a_1} Z(s; G_1) = \infty$, then

$$\lim_{s(l) \to a_1} \lim_{l \in E(G_1) \setminus \{l_0\}} Z(s; G) = \infty.$$

(iii) Let $l_0 \in E(G)$ and let $K_2$ be the corresponding induced graph. Then

$$\lim_{s(l) \to 0} \lim_{l \in E(G_1) \setminus \{l_0\}} Z(s; G) = \infty.$$

Proof. (i) It follows from Theorem 1-(i). (ii) It follows from (i). (iii) It follows from (ii) by using the formula for $Z(s, K_2)$. □

4. Phase transitions at finite temperature I

4.1. Log-Coulomb gases on graphs. Let $G$ be a graph as before. Consider a log-Coulomb gas consisting of $|V(G)|$ charged, $e_v \in \mathbb{R}$ for $v \in V(G)$, which are located at $x_v \in \mathbb{Z}_p$ for $v \in V(G)$. We set as in the introduction $x = \{x_v\}_{v \in V(G)} \in \mathbb{Z}_p^{\left|V(G)\right|}$, $e_G = \{e_v\}_{v \in V(G)} \in \mathbb{R}^{\left|V(G)\right|}$. The Hamiltonian of the gas is

$$H_p(x; e, \beta, G) = -\sum_{u,v \in V(G)} \ln |x_u - x_v|_p^{e_u e_v} + \frac{1}{\beta} P(x),$$

where the confining potential is given by

$$P(x) = \begin{cases} 0 & \text{if } \{x_v\}_{v \in V(G)} \in \mathbb{Z}_p^{\left|V(G)\right|} \\ +\infty & \text{otherwise.} \end{cases}$$

The interaction between the two charged particles located at $x_u$ and $x_v$ is only possible when $u \sim v$. This condition can be naturally reformulated saying that the potential $V$ creates a potential well, supported in $\mathbb{Z}_p^{\left|V(G)\right|}$, whose geometry corresponds to the graph $G$.

The partition function of this gas is given by

$$Z_{G, p, e} (\beta) = \int_{\mathbb{Z}_p^{\left|V(G)\right|}} \prod_{u,v \in V(G)} |x_u - x_v|_p^{e_u e_v} \prod_{v \in V(G)} dx_v = Z_e(s; G)_{s(u,v) = e_u e_v \beta} .$$
where \( \varphi \) is the characteristic function of \( Z_p^{[V(G)]} \). The statistical mechanics of the gas is described by the corresponding Gibbs measure:

\[
dP_{G,p,e}(\mathbf{x}) = \frac{e^{-\beta H_p(\mathbf{x}, e, G)}}{Z_{G,p,e}(\beta)} \prod_{v \in V(G)} dx_v = \frac{\prod_{u,v \in V(G)} |x_u - x_v|^{\epsilon_i \epsilon_j \beta}}{Z_{G,p,e}(\beta)} 1_{p}^{V(G)}(\{x_v\}_{v \in V(G)}) \prod_{v \in V(G)} dx_v.
\]

The probability measure \( P_{G,p,e}(\mathbf{x}) \) gives the probability of finding the particles at \( \mathbf{x} \) at temperature \( \frac{1}{k_B \beta} \) given the charge distribution \( e \).

### 4.2. Phase transitions I.

For \( G, p, e \) fixed, the partition function \( Z_{G,p,e}(\beta) \) is a rational function in \( p^{-\beta} \) due to Theorem \( \Box \). The problem of determining the convergence region for \( Z_{G,p,e}(\beta) \) in terms of the poles of the meromorphic continuation of \( Z_{G,p,e}(\beta) \) is highly non-trivial. The integral \( Z_{G,p,e}(\beta) \) converges when the following conditions hold true:

\[
1 - |V(H)| - \sum_{u,v \in V(H)\atop u \sim v} e_u e_v \beta < 0 \text{ for } H \in \text{Indgraphs}(G), \quad |V(H)| \geq 2,
\]

see Theorem \( \Box \)(ii). For \( H \in \text{Indgraphs}(G), \quad |V(H)| \geq 2 \), we define

\[
\text{Char}_+(H) = \sum_{u,v \in V(H)\atop u \sim v; e_u e_v > 0} e_u e_v; \quad \text{Char}_-(H) = \sum_{u,v \in V(H)\atop u \sim v; e_u e_v < 0} e_u e_v.
\]

\( \text{Indgraphs}_-(G) := \{ \text{Indgraphs}(G); \text{Char}_+(H) + \text{Char}_-(H) < 0 \} \);

and \( \text{Indgraphs}_+(G) := \{ \text{Indgraphs}(G); \text{Char}_+(H) + \text{Char}_-(H) > 0 \} \).

With this notation, we rewrite (4.1) as

\[
1 - |V(H)| - \{ \text{Char}_-(H) + \text{Char}_+(H) \} \beta < 0 \text{ for } H \in \text{Indgraphs}(G), \quad |V(H)| \geq 2,
\]

and \( \text{Char}_+(H) + \text{Char}_-(H) \neq 0 \).

Then the integral \( Z_{G,p,e}(\beta) \) converges if

\[
\begin{align*}
\beta < \beta_+(H) := \frac{|V(H)| - 1}{|\text{Char}_-(H) + \text{Char}_+(H)|} & \quad \text{for } H \in \text{Indgraphs}_-(G), \quad |V(H)| \geq 2, \\
\beta > \beta_-(H) := \frac{1 - |V(H)|}{|\text{Char}_-(H) + \text{Char}_+(H)|} & \quad \text{for } H \in \text{Indgraphs}_+(G), \quad |V(H)| \geq 2.
\end{align*}
\]

If \( \text{Indgraphs}_-(G) \neq \emptyset \) and \( \text{Indgraphs}_+(G) \neq \emptyset \), we set

\[
\beta_{UV} := \min_{H \in \text{Indgraphs}_-(G)} \beta_+(H) \quad \text{and} \quad \beta_{IR} := \max_{H \in \text{Indgraphs}_+(G)} \beta_-(H).
\]

If \( \text{Indgraphs}_-(G) \neq \emptyset \) and \( \text{Indgraphs}_+(G) = \emptyset \), we set

\[
\beta_{UV} := \min_{H \in \text{Indgraphs}_-(G)} \beta_+(H) \quad \text{and} \quad \beta_{IR} := -\infty.
\]

If \( \text{Indgraphs}_-(G) = \emptyset \) and \( \text{Indgraphs}_+(G) \neq \emptyset \), we set

\[
\beta_{UV} := +\infty \quad \text{and} \quad \beta_{IR} := \max_{H \in \text{Indgraphs}_+(G)} \beta_-(H).
\]
In this way we obtain the following result:

**Proposition 3.** With the above notation, the integral \( Z_{G,p,e}(\beta) \) converges for \( \beta_{IR} < \beta < \beta_{UV} \).

In order to decide whether or not \( Z_{G,p,e}(\beta) \) converges for \( \beta = \beta_{UV} \), we require an additional condition. If the meromorphic continuation of \( Z_{G,p,e}(\beta) \) has a pole at \( \beta = \beta_{UV} \), then the integral \( Z_{G,p,e}(\beta) \) does not converge for \( \beta \geq \beta_{UV} \).

**Remark 6.** Notice that Corollary 3 is not useful to determine phase transition points of \( Z_{G,p,e}(\beta) \).

If there exists \( H \in \text{Indgraphs}(G) \) such that \( \beta_+(H) > 0 \) and \( Z_{G,p,e}(\beta) \) has a pole at \( \beta = \beta_+(H) \), then by Proposition 3 \( Z_{G,p,e}(\beta) \) has a pole at the temperature \( \beta_{UV} \). Notice that \( \beta_{UV} \) is not necessarily equal to \( \beta_+(H) \), since \( Z_{G,p,e}(\beta) \) may have other positive poles. In conclusion we have the following criteria:

**Proposition 4.** With the above notation, and fixing \( G, p, e \). If there exists \( H \in \text{Indgraphs}(G) \) such that \( \beta_+(H) > 0 \) and \( Z_{G,p,e}(\beta) \) has a pole at \( \beta = \beta_+(H) \), then \( Z_{G,p,e}(\beta) \) has a phase transition at the temperature \( \frac{1}{\kappa_b \beta_{UV}} \).

5. THE THERMODYNAMIC LIMIT IN STAR GRAPHS

The study of the thermodynamic limit in general graphs is a difficult matter since it requires explicit formulas for the partition functions. In this section we study the thermodynamic limit in star graphs, see Example 2. We consider a neutral gas of \( M = |V(S_M)| \) particles contained in a ball \( B_k = p^{-k}Z_p \), for \( k \in \mathbb{N} \). We assume that \( M \) is even, and label the vertices of \( S_M \) as \( V(S_M) = \{1, 2, \ldots, \frac{M}{2}, \frac{M}{2} + 1, \ldots, M\} \), where the vertex 1 is the center of the star. We assume a charge distribution \( e = \{e_i\}_{1 \leq i \leq M} \) of the form

\[
e_i = +1 \text{ for } i = 1, \ldots, \frac{M}{2} \text{ and } e_i = -1 \text{ for } i = \frac{M}{2} + 1, \ldots, M.
\]

We label the edges as \( V(E_M) = \{1, 2\}, \ldots, \{1, \frac{M}{2}\}, \{1, \frac{M}{2} + 1\}, \ldots, \{1, M\} \}, \) and attach the complex variable \( s_i = s(\{1, i\}) \) of the edge \( \{1, i\} \). Now we take \( s_i = e_i \beta \) for \( i = 1, \ldots, M \). We denote the partition function attached to \( (M, \beta, p, Z_p, e) \) as \( Z_{M,0}(\beta) \), and by \( Z_{M,k}(\beta) \) the partition function attached to \( (M, \beta, p, p^{-k}Z_p, e) \). Then by using Example 2 we get that

\[
Z_{M,0}(\beta) = Z(s; S_M) \big|_{s_i = e_i \beta} = \int_{Z_p} \prod_{i=2}^{\frac{M}{2}} \frac{|x_1 - x_i|^\beta}{p} \prod_{i=1+\frac{M}{2}}^{M} dx_i
\]

\[
= \frac{(1 - p^{-1})^{M-1}}{(1 - p^{-1-\beta})^{\frac{M}{2}-1} (1 - p^{-1+\beta})^{\frac{M}{2}}}
\]

Notice that the integral in (5.1) converges for \( \beta \in (-1, 1) \). By Proposition 3 there is a phase transition at \( \beta = 1 \). In order to determine \( Z_{M,k}(\beta) \) we use the Boltzmann
factor \( \prod_{i=2}^{M} |x_1 - x_i|_{p}^{\beta} \prod_{i=1+M}^{M} |x_1 - x_i|_{p}^{\beta} \), then

\[
Z_{M,k}(\beta) = \int_{p^{-\frac{k}{2}}}^{p^{\frac{k}{2}}} \frac{1}{M!} \prod_{i=1+M}^{M} |x_1 - x_i|_{p}^{\beta} \prod_{i=2}^{M} |x_1 - x_i|_{p}^{\beta} \, dx_i = p^{k(-\beta+M)} Z_{M,0}(\beta), \text{ for } k \in \mathbb{N}.
\]

The calculation of a thermodynamic limit requires to consider \( M = |V(S_M)| \to \infty \), \( \text{vol}(B_k) = p^k \to \infty \), with \( \frac{M}{p^k} = \rho \) fixed.

5.1. **The dimensionless free energy per particle.** Following the canonical formalism of statistical mechanics, we define the total dimensionless free energy \( \beta \mathcal{F} \) as

\[
\beta \mathcal{F} = -\ln \left( \frac{1}{((\frac{M}{2})!)^2} Z_{M,k}(\beta) \right).
\]

Notice that we use \( \{(\frac{M}{2})!\}^2 \) instead of \( M! \). This is the cardinality of the elements of \( \text{Aut}(S_N) \) preserving the charge distribution on \( S_N \). The dimensionless free energy per particle \( \beta f \) is defined as

\[
\beta f = \lim_{M, \text{vol}(B_k) \to \infty} \frac{1}{M} \beta \mathcal{F}.
\]

Then by using the Stirling formula we have \( \frac{2\ln(\frac{M}{2})}{M} \sim -1 + \ln \left( \frac{M}{2} \right) \), and

\[
\beta f = \lim_{M, \text{vol}(B_k) \to \infty} \frac{-1}{M} \ln \left( \frac{1}{(M-1)!} p^{k(-\beta+M)} \frac{(1-p^{-1})^{M-1}}{(1-p^{-1-\beta})^{\frac{M+1}{2}}(1-p^{-1+\beta})^{\frac{M-1}{2}}} \right)
\]

\[
= \lim_{M, \text{vol}(B_k) \to \infty} \left\{ -\ln p^k - \ln (1-p^{-1}) + \frac{1}{2} \ln (1-p^{-1-\beta}) + \frac{1}{2} \ln (1-p^{-1+\beta}) + \ln \left( \frac{M}{2} \right) - 1 \right\}
\]

\[
= \ln \rho - \ln (1-p^{-1}) + \frac{1}{2} \ln (1-p^{-1-\beta}) + \frac{1}{2} \ln (1-p^{-1+\beta}) - 1 - \ln 2.
\]

Now, in the high temperature limit \( \beta \to 0 \) and the \( \beta f \) tends to \( \ln \rho - 1 \). There is phase transition at \( \beta = 1 \), since \( \lim_{\beta \to 1} \beta f = -\infty \).

The mean energy per particle

\[
\mathcal{E} = \frac{\partial}{\partial \beta} \beta f = \lim_{M, \text{vol}(B_k) \to \infty} \left( \frac{1}{M} \langle H_{S_M} \rangle \right)
\]

\[
= \frac{\ln p}{2} \left( \frac{p^{-1-\beta}}{1-p^{-1-\beta}} - \frac{p^{-1+\beta}}{1-p^{-1+\beta}} \right).
\]
5.2. The grand-canonical potential. The grand-canonical distribution is the generating function for \( Z_{M,k}(\beta) \), for \( k \in \mathbb{N} \), i.e.

\[
Z_{\beta,k}(X) := \sum_{L=0}^{\infty} Z_{2L,\beta,k} X^{2L} := 1 + \sum_{L=1}^{\infty} Z_{2L,\beta,k} X^{2L}
\]

\[
= 1 + \left( \frac{1 - p^{-1-\beta}}{1 - p^{-1}} \right) p^{-k\beta} \sum_{L=1}^{\infty} \left\{ \frac{p^k (1 - p^{-1}) X}{\sqrt{(1 - p^{-1-\beta})(1 - p^{-1+\beta})}} \right\}^{2L},
\]

for \( 0 \leq \beta < 1 \). Notice that \( Z_{\beta,k}(X) \) has a pole at \( \beta = 1 \), i.e. there is phase transition at \( \frac{1}{k_B} \). Assuming that \( \frac{p^k (1 - p^{-1}) |X|}{\sqrt{(1 - p^{-1-\beta})(1 - p^{-1+\beta})}} < 1 \), for \( \beta \in [0, 1) \),

\[
(5.2) \quad Z_{\beta,k}(X) = 1 + \frac{p^{-k(\beta-2)} (1 - p^{-1}) X^2}{(1 - p^{-1+\beta})} \left( \frac{1}{1 - \left\{ \frac{p^k (1 - p^{-1})}{\sqrt{(1 - p^{-1-\beta})(1 - p^{-1+\beta})}} \right\}^2 X^2} \right)
\]

for \( \beta \in [0, 1) \). We now use the meromorphic continuation given (5.2) to interpolate the values of \( Z_{\beta,k}(X) \) for any \( X, k \), with \( \beta \in [0, 1) \). Now, we compute the grand-canonical potential, by using that \( \ln(1 + z) \sim z \) as \( z \to 0 \),

\[
\mathcal{X}_{\beta}(X) = \lim_{k \to \infty} \frac{1}{p^k} \ln Z_{\beta,k}(X) = 0 \quad \text{for} \quad \beta \neq 1.
\]

6. Local zeta functions for rational functions

In the 70s Igusa developed a uniform theory for local zeta functions and oscillatory integrals attached to polynomials with coefficients in a local field of characteristic zero, \([26], [25]\). In \([51]\) this theory is extended to the case of rational functions. We review some results of this article that are require here.

6.1. Local fields of characteristic zero. We take \( \mathbb{K} \) to be a non-discrete locally compact field of characteristic zero. Then \( \mathbb{K} \) is \( \mathbb{R} \), \( \mathbb{C} \), or a finite extension of \( \mathbb{Q}_p \), the field of \( p \)-adic numbers. If \( \mathbb{K} \) is \( \mathbb{R} \) or \( \mathbb{C} \), we say that \( \mathbb{K} \) is an \( \mathbb{R} \)-field, otherwise we say that \( \mathbb{K} \) is a \( \mathbb{p} \)-field.

For \( a \in \mathbb{K} \), we define the modulus \( |a|_{\mathbb{K}} \) of \( a \) by

\[
|a|_{\mathbb{K}} = \begin{cases} 
\text{the rate of change of the Haar measure in } (\mathbb{K},+) \text{ under } x \to ax & \text{for } a \neq 0, \\
0 & \text{for } a = 0.
\end{cases}
\]

It is well-known that, if \( \mathbb{K} \) is an \( \mathbb{R} \)-field, then \( |a|_{\mathbb{R}} = |a| \) and \( |a|_{\mathbb{C}} = |a|^2 \), where \(|\cdot|\) denotes the usual absolute value in \( \mathbb{R} \) or \( \mathbb{C} \), and, if \( \mathbb{K} \) is a \( \mathbb{p} \)-field, then \( |\cdot|_{\mathbb{K}} \) is the normalized absolute value in \( \mathbb{K} \).
6.1.1. Structure of the $p$-fields. A non-Archimedean local field $K$ (or $p$-field) is a locally compact topological field with respect to a non-discrete topology, which comes from a norm $|\cdot|_K$ satisfying

$$|x + y|_K \leq \max\{|x|_K, |y|_K\},$$

for $x, y \in K$. Any such norm is called an ultranorm or non-Archimedean. Any non-Archimedean local field $K$ of characteristic zero is isomorphic (as a topological field) to a finite extension of $\mathbb{Q}_p$, and it is called a $p$-adic field. The field $\mathbb{Q}_p$ is the basic example of non-Archimedean local field of characteristic zero.

The ring of integers of $K$ is defined as

$$R_K = \{x \in K; |x|_K \leq 1\}.$$

Geometrically $R_K$ is the unit ball of the normed space $(K, |\cdot|_K)$. This ring is a domain of principal ideals having a unique maximal ideal, which is given by

$$P_K = \{x \in K; |x|_K < 1\}.$$

We fix a generator $p$ of $P_K$ i.e. $P_K = pR_K$. A such generator is also called a local uniformizing parameter of $K$, and it plays the same role as $p$ in $\mathbb{Q}_p$.

The group of units of $R_K$ is defined as

$$R_K^* = \{x \in R_K; |x|_K = 1\}.$$

The natural map $R_K \rightarrow R_K/P_K \cong \mathbb{F}_q$ is called the reduction mod $P_K$. The quotient $R_K/P_K \cong \mathbb{F}_q$, is a finite field with $q = p^d$ elements, and it is called the residue field of $K$. Every non-zero element $x$ of $K$ can be written uniquely as $x = p^{\text{ord}(x)}u$, $u \in R_K^*$. We set $\text{ord}(0) = \infty$. The normalized valuation of $K$ is the mapping

$$\begin{align*}
K & \rightarrow \mathbb{Z} \cup \{\infty\} \\
x & \rightarrow \text{ord}(x).
\end{align*}$$

Then $|x|_K = q^{-\text{ord}(x)}$ and $|p|_K = q^{-1}$.

We fix $\mathcal{S} \subset R_K$ a set which is mapped bijectively onto $\mathbb{F}_q$ by the reduction mod $P_K$. We assume that $0 \in \mathcal{S}$. Any non-zero element $x$ of $K$ can be written as

$$x = p^{\text{ord}(x)} \sum_{i=0}^{\infty} x_i p^i,$$

where $x_i \in \mathcal{S}$ and $x_0 \neq 0$. This series converges in the norm $|\cdot|_K$.

6.2. Local zeta functions for rational functions. If $K$ is a $p$-field, resp. an $\mathbb{R}$-field, we denote by $\mathcal{D}(K^N)$ the $\mathbb{C}$-vector space consisting of all $\mathbb{C}$-valued locally constant functions, resp. all smooth functions, on $K^N$, with compact support. An element of $\mathcal{D}(K^N)$ is called a test function. To simplify terminology, we will call a non-zero test function that takes only real and non-negative values a positive test function.

Let $f, g \in K[x_1, \ldots, x_N] \setminus K$ be polynomial functions such that $f/g$ is not constant. Let $\Phi : K^N \rightarrow \mathbb{C}$ be a test function. Then the local zeta function attached to $(f/g, \Phi)$ is defined as

$$Z_{\Phi}(s; f/g) = \int_{\mathbb{K}^N \setminus D_K} \Phi(x) \left| \frac{f(x)}{g(x)} \right|^s |x|^N \, dx,$$

for $s \in \mathbb{C}$ and $x \in \mathbb{K}^N$.
where \( s \in \mathbb{C} \), \( D_K = f^{-1}(0) \cup g^{-1}(0) \) and \( d^N x \) is the normalized Haar measure on \((\mathbb{K}^N,+)\). The convergence of the integral in (6.1) is not a straightforward matter; in particular the convergence does not follow from the fact that \( \Phi \) has compact support.

6.2.1. Numerical data. For any local field \( \mathbb{K} \) of characteristic zero, there exits a finite set of pair of integers depending on \((f/g, \Phi)\) of the form

\[
\{(n_i, v_i) ; i \in T_+, n_i > 0\} \cup \{(n_i, v_i) ; i \in T_-, n_i < 0\},
\]

where \( T_+ \) and \( T_- \) are finite sets. We now define

\[
\alpha := \alpha_\Phi = \begin{cases} 
\min_{i \in T_-} \left\{ \frac{v_i}{|n_i|} \right\} & \text{if } T_- \neq \emptyset \\
+\infty & \text{if } T_- = \emptyset
\end{cases}
\]

and

\[
\gamma := \gamma_\Phi = \begin{cases} 
\max_{i \in T_+} \left\{ -\frac{v_i}{n_i} \right\} & \text{if } T_+ \neq \emptyset \\
-\infty & \text{if } T_+ = \emptyset
\end{cases}
\]

6.2.2. Meromorphic continuation: \( p \)-field case. Let \( \mathbb{K} \) be a \( p \)-field. Then the following assertions hold: (1) \( Z_{\Phi}(s; f/g) \) converges for \( \gamma < \text{Re}(s) < \alpha \); (2) \( Z_{\Phi}(s; f/g) \) has a meromorphic continuation to \( \mathbb{C} \) as a rational function of \( q^{-s} \), and its poles are of the form

\[
s = \frac{v_i}{n_i} + \frac{2\pi \sqrt{-1}}{n_i \ln q} k, \quad k \in \mathbb{Z},
\]

for \( i \in T_+ \cup T_- \). In addition, the order of any pole is at most \( N \), cf. [51] Theorem 3.2.

6.2.3. Meromorphic continuation: \( \mathbb{R} \)-field case. Let \( \mathbb{K} \) be an \( \mathbb{R} \)-field. Then the following assertions hold: (1) \( Z_{\Phi}(s; f/g) \) converges for \( \gamma < \text{Re}(s) < \alpha \); (2) \( Z_{\Phi}(s; f/g) \) has a meromorphic continuation to \( \mathbb{C} \), and its poles are of the form

\[
s = -\frac{v_i}{n_i} - \frac{k}{[\mathbb{K} : \mathbb{R}] n_i}, \quad k \in \mathbb{Z}_{\geq 0},
\]

for \( i \in T_+ \cup T_- \), where \( [\mathbb{K} : \mathbb{R}] = 1 \) if \( \mathbb{K} = \mathbb{R} \) and \( [\mathbb{K} : \mathbb{R}] = 2 \) if \( \mathbb{K} = \mathbb{C} \). In addition, the order of any pole is at most \( N \), cf. [51] Theorem 3.5).

6.3. Existence of poles, largest and smallest poles. The theorems [51] Theorems 3.5, 3.2] above mentioned provide a list of the possible poles for the local zeta \( Z_{\Phi}(s, f/g) \) in terms of a list (the numerical data of resolution of singularities), which is not unique neither intrinsic. Now, if \( \gamma_\Phi \neq -\infty \), say it is equal to \(-\frac{v_i}{n_i}\) precisely for \( i \in T_+(\subset T_+) \), then by choosing a suitable positive \( \Phi, \beta_\Phi \) is a pole of \( Z_{\Phi}(s, f/g) \). And, if we assume that \( \alpha_\Phi \neq +\infty \), say it is equal to \( -\frac{v_i}{n_i} \) precisely for \( i \in T_-(\subset T_-) \), then by choosing a suitable positive \( \Phi, \alpha_\Phi \) is a pole of \( Z_{\Phi}(s, f/g) \), cf. [51] Theorem 3.9. This implies that \( \alpha_\Phi \) and \( \gamma_\Phi \) do not depend on the numerical data used to compute them, if we choose the test function \( \Phi \) conveniently.

In [51] Theorem 3.9] some criteria for the existence of positive and negative poles were developed. We review those criteria for the existence of positive poles, since we use them later on. Let \( U \) be an open subset of \( \mathbb{K}^N \). We assume that \( \Phi \) is a test function with support contained in \( U \). If there exists a point \( x_0 \in U \) such that
The Hamiltonian of this log-Coulomb gas has the form:

\[ H_1 \]

Then the charge distribution \( e \) subset of \( K \) for \( \beta = 1 \). Assume that Theorem 2.

7. Phase transitions at finite temperature II

In this section we consider a gas with \( |V(G)| \) particles confined in a compact subset of \( \mathbb{R}^{|V(G)|} \), which is the support of a positive test function \( \Phi \). We assume that the charge distribution \( e = \{e_v\}_{v \in V(G)} \) satisfies the following hypothesis:

\[
\text{(H1)} \quad \begin{cases} 
  e_v \in \{+1, -1\} & \text{for any } v \in V(G); \\
  \{e_v e_u; v, u \in V(G), u \sim v\} = \{+1, -1\}. 
\end{cases}
\]

The Hamiltonian of this log-Coulomb gas has the form:

\[
H_K(x; e) = - \sum_{u,v \in V(G)} \ln |x_u - x_v|_K^e - \frac{1}{\beta} \ln \Phi(x).
\]

The Boltzmann factor is \( \exp (-\beta H_K(x; e)) \) and the partition function is

\[
Z_{G,K,e} (\beta) = \int_{\mathbb{R}^{|V(G)|}} \Phi \left( \{x_v\}_{v \in V(G)} \right) \prod_{u,v \in V(G)} \frac{|x_u - x_v|^\beta_{K}}{\prod_{u,v \in V(G)} |x_u - x_v|^\beta_{K}} \prod_{u \sim v; e_u e_v = +1} dx_u.
\]

We now set

\[
f_{G,e}(x) := \prod_{u,v \in V(G)} (x_u - x_v) \quad \text{and} \quad g_{G,e}(x) := \prod_{u,v \in V(G)} (x_u - x_v).
\]

Assume that

\[
\text{(H2)} \quad \text{there exists } x_0 \in \mathbb{R}^{|V(G)|} \text{ such that } f_{G,e}(x_0) \neq 0 \text{ and } g_{G,e}(x_0) = 0.
\]

We pick a positive test function \( \Phi \) supported in a small enough neighborhood of \( x_0 \). Then there exits \( \beta_{UV} = \beta (\Phi, G) > 0 \) such that the integral \( Z_{G,K,e}(\beta) \) converges for \( \beta \in (0, \beta_{UV}) \), and the meromorphic continuation of \( Z_{G,K,e}(\beta) \) has a pole at \( \beta = \beta_{UV} \).

Theorem 2. Assume that \( G, K, \Phi, e \) are given, and the Hypotheses H1, H2 hold. Then \( Z_{G,K,e}(\beta) \) has a phase transition at the temperature \( \frac{1}{k_B \beta_{UV}} \).

Data Availability. No new data were created or analyzed in this study. Data sharing is not applicable to this article.

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