CORRIGENDUM

Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids

To cite this article: J C Marrero et al 2006 Nonlinearity 19 3003

View the article online for updates and enhancements.

Related content
- First integrals of the discrete nonconservative and nonholonomic systems
  Zhang Hong-Bin, Chen Li-Qun and Liu Rong-Wan
- The discrete variational principle in Hamiltonian formalism and first integrals
  Zhang Hong-Bin, Chen Li-Qun and Liu Rong-Wan
- Symmetries and variational calculation of discrete Hamiltonian systems
  Xia Li-Li, Chen Li-Qun, Fu Jing-Li et al.

Recent citations
- New developments on the geometric nonholonomic integrator
  Sebastián Ferraro et al
- Groupoids, Discrete Mechanics, and Discrete Variation
  Guo Jia-Feng et al
Corrigendum

Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids
J C Marrero, D Martín de Diego and E Martínez 2006 Nonlinearity 19 1313–1348

In our paper we developed a geometric description of discrete Lagrangian and Hamiltonian mechanics on Lie groupoids. There are some results in the paper which are not completely correct. In the following, we will give the correct version of these results.

First of all, suppose that \( G \) is a Lie groupoid over \( M \) with Lie algebroid \( \tau : AG \to M \) and that \( \{X_i\}_i \) is a local basis of \( \Gamma(\tau) \) on \( U \). Then, the function \( \overline{X}_i(L) \) (respectively, \( \overline{X}_i(L) \)) in remark 4.3 of our paper is defined in the open subset \( \beta^{-1}(U) \) (respectively, \( \alpha^{-1}(U) \)) of \( G \). Thus, the local expression of the Poincaré–Cartan 2-section \( \Theta_1 \) in this remark is only valid in the open subset \( \alpha^{-1}(U) \cap \beta^{-1}(U) \) of \( G \). In fact, in order to obtain the local expression of \( \Theta_1 \), \( \Theta_1^* \) and \( \Omega \), in an open neighbourhood of every point \( g \in G \), we must choose a local basis of \( \Gamma(\tau) \) on \( U \) and \( V \), where \( U \) and \( V \) are open subsets of \( M \) such that \( \alpha(g) \in U \) and \( \beta(g) \in V \). Therefore, the correct version of remark 4.3 in our paper is the following result.

**Remark 4.3.** Let \( g \) be an element of \( G \) such that \( \alpha(g) = x \) and \( \beta(g) = y \). Suppose that \( U \) and \( V \) are open subsets of \( M \), with \( x \in U \) and \( y \in V \), and that \( \{X_i\}_i \) and \( \{Y_j\}_j \) are local bases of \( \Gamma(\tau) \) on \( U \) and \( V \), respectively. Then, \( \{X_i^{(1,0)}, Y_j^{(0,1)}\} \) is a local basis of \( \Gamma(\pi^+) \) on the open subset \( \alpha^{-1}(U) \cap \beta^{-1}(V) \). Moreover, if we denote by \( \{(X^i)^{(1,0)}, (Y^j)^{(0,1)}\} \) the dual basis of \( \{X_i^{(1,0)}, Y_j^{(0,1)}\} \), we have that on the open subset \( \alpha^{-1}(U) \cap \beta^{-1}(V) \)

\[
\Theta_1 = -\overline{X}_i(L)(X^i)^{(1,0)}, \quad \Theta_1^* = \overline{Y}_j(L)(Y^j)^{(0,1)}, \\
\Omega = -\overline{X}_i(L)(X^i)^{(1,0)} \wedge (Y^j)^{(0,1)}.
\]

In the same vein, the correct version of remark 4.8 in our paper is the following result.

**Remark 4.8.** Note that \( (F^*L)(h) \in A^*_\alpha(h)G \). Furthermore, if \( U \) is an open subset of \( M \) such that \( \alpha(h) \in U \) and \( \{X_i\}_i \) is a local basis of \( \Gamma(\tau) \) on \( U \) then

\[
F^*L = \overline{X}_i(L)(X^i \circ \alpha),
\]

on \( \alpha^{-1}(U) \), where \( \{X^i\}_i \) is the dual basis of \( \{X_i\}_i \). In a similar way, if \( V \) is an open subset of \( M \) such that \( \beta(g) \in V \) and \( \{Y_j\}_j \) is a local basis of \( \Gamma(\tau) \) on \( V \) then

\[
F^*L = \overline{Y}_j(L)(Y^j \circ \beta),
\]

on \( \beta^{-1}(V) \).

On the other hand, it should be noted that the implication (i) \( \Rightarrow \) (iii) in theorem 4.13 of our paper is not right. In fact, a counterexample is the discrete Lagrangian \( L \) on the pair groupoid \( G = \mathbb{R} \times \mathbb{R} \) defined by \( L(x, y) = (y - x)^4 \) (\( L \) admits a unique discrete Lagrangian evolution operator which is a global diffeomorphism and, however, the Poincaré–Cartan 2-form of \( L \) is not symplectic on \( G \)). The above problems are motivated by the definition of a regular discrete Lagrangian on a Lie groupoid (definition 4.12 in our paper). Thus, imitating the continuous case, we will replace definition 4.12 by the following notion which is more appropriate.
Definition 4.12. A Lagrangian $L : G \to \mathbb{R}$ on a Lie groupoid $G$ is said to be regular if the Poincaré–Cartan 2-section $\Omega_L$ is symplectic on the Lie algebroid $\mathcal{P}^*G \equiv V \beta \oplus_G V\alpha \to G$.

Remark. Using the correct version of remark 4.3, we deduce that the Lagrangian $L$ is regular if and only if for every $g \in G$ and every local basis $\{X_i\}$ (respectively, $\{Y_j\}$) of $\Gamma(\tau)$ on an open subset $U$ (respectively, $V$) of $M$ such that $\alpha(g) \in U$ (respectively, $\beta(g) \in V$) we have that the matrix $X_i(\gamma_j(L))$ is regular on $\alpha^{-1}(U) \cap \beta^{-1}(V)$.

Now, we have that the right version of theorem 4.13 in our paper is the following result.

Theorem 4.13. Let $L : G \to \mathbb{R}$ be a Lagrangian function. Then:

(a) The following conditions are equivalent:

(i) $L$ is regular.

(ii) The Legendre transformation $\mathcal{E}^L$ is a local diffeomorphism.

(iii) The Legendre transformation $\mathcal{E}^L$ is a local diffeomorphism.

(b) If $L : G \to \mathbb{R}$ is regular and $(g_0, h_0) \in G_2$ is a solution of the discrete Euler–Lagrange equations for $L$ then there exist two open subsets $U_0$ and $V_0$ of $G$, with $g_0 \in U_0$ and $h_0 \in V_0$, and there exists a (local) discrete Lagrangian evolution operator $\xi_L : U_0 \to V_0$ such that:

(i) $\xi_L(g_0) = h_0$,

(ii) $\xi_L$ is a diffeomorphism and

(iii) $\xi_L$ is unique, that is, if $U'_0$ is an open subset of $G$, with $g_0 \in U'_0$ and $\xi'_L : U'_0 \to G$ is a (local) discrete Lagrangian evolution operator then $\xi'_L|_{U_0 \cap U'_0} = \xi_L|_{U_0 \cap U'_0}$.

Proof.

(a) Proceeding as in the proof of the implication (iii) $\Rightarrow$ (iv) (respectively, (ii) $\Rightarrow$ (iii)) of theorem 4.13 in our paper we deduce that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) (respectively, (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i)).

(b) Using remark 4.11 in our paper, we have that

$$(\mathcal{F}^L)(g_0) = (\mathcal{F}^L)(h_0) = \mu_0 \in A^*G.$$ 

Thus, from the first part of this theorem, it follows that there exist two open subsets $U_0$ and $V_0$ of $G$, with $g_0 \in U_0$ and $h_0 \in V_0$, and an open subset $W_0$ of $A^*G$ such that $\mu_0 \in W_0$ and

$$\mathcal{F}^L : U_0 \to W_0, \quad \mathcal{F}^L : V_0 \to W_0$$

are diffeomorphisms. Therefore, using remark 4.11 in our paper, we deduce that

$$\xi_L = [(\mathcal{F}^L)^{-1} \circ (\mathcal{F}^L)]|_{U_0} : U_0 \to V_0$$

is a (local) discrete Lagrangian evolution operator. Moreover, it is clear that $\xi_L(g_0) = h_0$ and, from the first part of this theorem, we have that $\xi_L$ is a diffeomorphism.

Finally, if $U'_0$ is an open subset of $G$, with $g_0 \in U'_0$, and $\xi'_L : U'_0 \to G$ is another (local) discrete Lagrangian evolution operator then $\xi'_L|_{U_0 \cap U'_0} : U_0 \cap U'_0 \to G$ is also a (local) discrete Lagrangian evolution operator. Consequently, using remark 4.11 in our paper, we conclude that

$$\xi'_L|_{U_0 \cap U'_0} = [(\mathcal{F}^L)^{-1} \circ (\mathcal{F}^L)]|_{U_0 \cap U'_0} = \xi_L|_{U_0 \cap U'_0}.$$