Generalised Bent Criteria for Boolean Functions (I)

Constanza Riera, Matthew G. Parker

Abstract

Generalisations of the bent property of a boolean function are presented, by proposing spectral analysis with respect to a well-chosen set of local unitary transforms. Quadratic boolean functions are related to simple graphs and it is shown that the orbit generated by successive Local Completions on a graph can be found within the transform spectra under investigation. The flat spectra of a quadratic boolean function are related to modified versions of its associated adjacency matrix.

I. Introduction

It is often desirable that a boolean function, \( p \), used for cryptographic applications, is be highly \textit{nonlinear}, where nonlinearity is determined by examining the spectrum of \( p \) with respect to (w.r.t.) the \textit{Walsh Hadamard Transform} (WHT), and where the nonlinearity is maximised for those functions that minimise the magnitude of the spectral coefficients. To be precise, define the boolean function of \( n \) variables \( p : \text{GF}(2)^n \rightarrow \text{GF}(2) \), and the WHT by the \( 2^n \times 2^n \) unitary matrix \( U = H \otimes H \ldots \otimes H = \otimes_{i=0}^{n-1} H \), where the Walsh-Hadamard kernel \( H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). \( \otimes \) indicates the tensor product of matrices, and unitary means that \( U U^\dagger = I_n \), where \( ^\dagger \) means transpose-conjugate and \( I_n \) is the \( 2^n \times 2^n \) identity matrix. We further define a length \( 2^n \) vector, \( s = (s_{00}, s_{01}, s_{01}, \ldots, s_{11}) \) such that \( s_i = (-1)^{p(i)} \), where \( i \in \text{GF}(2)^n \). Then the Walsh-Hadamard spectrum of \( p \) is given by the matrix-vector product \( P = Us \), where \( P \) is a vector of \( 2^n \) real spectral coefficients, \( P_k \), where \( k \in \text{GF}(2)^n \).

The spectral coefficient, \( P_k \), with maximum magnitude tells us the minimum (Hamming) distance, \( d \), of \( p \) to the set of affine boolean functions, where \( d = 2^n-1 - 2^{2^n-2} |P_k| \). By Parseval’s Theorem, the extremal case occurs when all \( P_k \) have equal magnitude, in which case \( p \) is said to have a \textit{flat} WHT spectra, and is referred to as \textit{bent}. If \( p \) is bent, then it is as far away as it can be from the affine functions [31], which is a desirable cryptographic design goal. It is an open problem to classify all bent boolean functions, although many results are known [20], [30], [13], [21].

In this paper, we extend the concept of a bent boolean function to some \textit{Generalised Bent Criteria} for a boolean function, where we now require that \( p \) has flat spectra w.r.t. one or more transforms from a specified set of unitary transforms. The set of transforms we choose is not arbitrary but is motivated by cryptographic applications.
by the choice of unitary transforms that are typically used to action a local basis change for a pure $n$-qubit quantum state. We here apply such transforms to a $n$-variable boolean function, and examine the resultant spectra accordingly. In particular we apply all possible transforms formed from $n$-fold tensor products of the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the Walsh-Hadamard kernel, $H$, and the Negahadamard kernel $N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, where $i^2 = -1$. We refer to this set of transforms as the \{I, H, N\}$^n$ transform set, i.e. where all transforms are of the form \{I, H, N\}$^n = \bigotimes_{j \in R_I} I_j \bigotimes_{j \in R_H} H_j \bigotimes_{j \in R_N} N_j$, where the sets $R_I, R_H$ and $R_N$ partition $\{0, \ldots, n-1\}$, and $H_j$, say, is short for $I \otimes I \otimes \ldots \otimes I \otimes H \otimes I \otimes \ldots \otimes I$, with $H$ in the $j^{th}$ position. There are $3^n$ such transforms which act on a boolean function of $n$ variables to produce $3^n$ spectra, each spectrum of which comprises $2^n$ spectral elements (complex numbers). By contrast, the WHT can be described as \{H\}$^n$, which is a transform set of size one, where the single resultant output spectrum comprises just $2^n$ spectral elements.

A. The Quantum Context

The choice of $I$, $H$, and $N$, is motivated by their importance for the construction of Quantum Error-Correcting Codes (QECCs). This is because $I$, $H$, and $N$ are generators of the Local Clifford Group [11], [29] which is defined to be the set of matrices that stabilize the group of Pauli matrices \footnote{The Pauli matrices are $I$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\sigma_y = i \sigma_x \sigma_z$.} which, in turn, form a basis for the set of local errors that act on the quantum code. This implies that the set of locally-equivalent $n$-qubit quantum states, that occur as joint eigenspectra w.r.t. \{I, H, N\}$^n$, are equally robust to quantum errors from the Pauli error set. Stabilizer QECCs can also be interpreted as additive codes over GF(4) [11].

To evaluate the quantum entanglement of a pure $n$-qubit state one should really examine the spectra w.r.t. the infinite set of $n$-fold tensor products of all $2 \times 2$ unitary matrices [35]. Those states which minimise all spectral magnitudes w.r.t. this infinite transform set are as far away as possible from all generalised affine functions and can be considered to be highly entangled as the probability of observing (measuring) any specific qubit configuration is as small as possible, in any local measurement basis. However it is computationally intractable to evaluate, to any reasonable approximation, this continuous local unitary spectrum beyond about $n = 4$ qubits (although approximate results up to $n = 6$ are given in [35]). Therefore we choose, in this paper, a well-spaced subset of spectral points, as computed by the set of \{I, H, N\}$^n$ transforms, from which to ascertain approximate entanglement measures. Complete spectra for such a transform set can be computed up to about $n = 10$ qubits using a standard desk-top computer, although partial results for higher $n$ are possible if the $n$-qubit quantum state is represented by, say, a quadratic boolean function over $n$ variables.
B. The Graphical Context

The graphical description of certain pure quantum states was investigated by Parker and Rijmen [35]. They proposed partial entanglement measures for such states and made observations about a *Local Unitary (LU) Equivalence* between graphs describing the states w.r.t. the tensor product of $2 \times 2$ local unitary transforms. These graphs were interpreted as quadratic boolean functions and it was noted that bipartite quadratic functions are LU-equivalent to indicators for binary linear error-correcting codes. It was further observed that physical quantum graph arrays are relevant to the work of [35] and were already under investigation in the guise of *cluster states*, by Raussendorf and Briegel [38], [6]. These clusters form the 'substrate' for measurement-driven quantum computation.

Measurement-driven quantum computation on a *quantum factor graph* has been discussed by Parker [34]. Independent work by Schlingemann and Werner [11], Glynn [22], [23], and by Grassl, Klappenecker, and Rotteler [25] proposed to describe *stabilizer* Quantum Error-Correcting Codes (QECCs) using graphs and, for QECCs of dimension zero, the associated graphs can be referred to as *graph states*. The graph states are equivalent to the graphs described by [35] and therefore have a natural representation using quadratic boolean functions.

In [35] it was observed that the complete graph, star graph, and generalised GHZ (Greenberger-Horne-Zeilinger) states are all LU-equivalent. It turns out that LU-equivalence for graph states can be characterised, graphically, via the *Vertex-Neighbour-Complement* (VNC) transformation, which was defined by Glynn, in the context of QECCs, in [22] (definition 4.2) and also, independently, by Hein, Eisert and Briegel [27], and also by Van Den Nest and De Moor [11]. VNC is another name for *Local Complementation* (LC), as investigated by Bouchet [7], [8], [9] in the context of *isotropic systems*. By applying LC to a graph $G$ we obtain a graph $G'$, in which case we say that $G$ and $G'$ are *LC-equivalent*. Moreover, the set of all LC-equivalent graphs form an *LC-orbit*. LC-equivalence translates into the natural equivalence between GF(4) additive codes that keeps the weight distribution of the code invariant [11]. There has been recent renewed interest in Bouchet’s work motivated, in part, by the application of *interlace graphs* to the reconstruction of DNA strings [3], [2]. In particular, various *interlace polynomials* have been defined [2], [11], [4], [5] which mirror some of the quadratic results of part II of this paper [39]. We point out links to this work in part II but defer a thorough exposition of these links to the future.

C. The Boolean Context

Spectral analysis w.r.t. $\{I, H, N\}^n$ also has application to the cryptanalysis of classical cryptographic systems [17]. In particular, for a block cipher it models attack scenarios where one has full read/write access to a subset of plaintext bits and access to all ciphertext bits, (see [17] for more details). The analysis of spectra w.r.t. $\{I, H, N\}^n$ tells us more about $p$ than is provided by the spectrum w.r.t.
the WHT; for instance, identifying relatively high generalised linear biases for \( p \). In Part I of this paper our aim is to introduce these new generalised bent criteria. In Part II \(^{39}\) we enumerate the flat spectra w.r.t. \( \{I, H, N\}^n \) and its subsets. We are trying to answer the question: which boolean functions are as far away as possible from the set of generalised affine functions as defined by the rows of \( \{I, H, N\}^n \)? \(^2\)

The classification of bent quadratic (degree-two) boolean functions is well-known \(^{30}\), and is facilitated because the bent criteria is an invariant of affine transformation of the input variables. However, the classification of generalised bent criteria for a quadratic boolean function w.r.t. the \( \{I, H, N\}^n \) transform set is new, and the generalised bent criteria are not, in general, invariant to affine transformation of the inputs. This paper characterises these generalised bent criteria for both quadratic and more general boolean functions. We associate a quadratic boolean function with an undirected graph, which allows us to interpret spectral flatness with respect to \( \{I, H, N\}^n \) as a maximum rank property of suitably modified adjacency matrices. We interpret LC as an operation on quadratic boolean functions, and as an operation on the associated adjacency matrix, and we also identify the LC-orbit with a subset of the flat spectra w.r.t. \( \{I, H, N\}^n \). The spectra w.r.t. \( \{I, H, N\}^n \) motivates us to examine the properties of the WHT of all \( \mathbb{Z}_4 \)-linear offsets of boolean functions, the WHT of all subspaces of boolean functions that can be obtained by fixing a subset of the variables, the WHT of all \( \mathbb{Z}_4 \)-linear offsets of all of the above subspace boolean functions, the WHT of each member of the LC-orbit, and the distance of boolean functions to all \( \mathbb{Z}_4 \)-linear functions. This leads us to prove the following:

All quadratic boolean functions are \textit{bent}_4, \textit{I-bent} and \textit{I-bent}_4.

Not all quadratic boolean functions are \textit{LC-bent}.

All boolean functions are \textit{I-bent}_4.

Not all boolean functions are \textit{bent}_4 or \textit{I-bent}.

There are no \textit{Z}_4\text{-bent} or \textit{Completely I-bent}_4 boolean functions.

where the above terms for generalised bent criteria will be made clear in the sequel. We are able to characterise and analyse the criteria for quadratic boolean functions by considering properties of the adjacency matrix for the associated graph state.

### D. Paper Overview

For the interested reader, Appendix A reviews the graph state and its interpretations in the literature. In Section \[\text{II}\] we review LC as an operation on an undirected graph \(^{22, 23}\), and provide an algorithm

\(^2\)A row of \( U_0 \otimes U_1 \otimes \ldots \otimes U_{n-1} \) for \( U \) a \( 2 \times 2 \) unitary matrix can always be written as \( u = (a_0, b_0) \otimes (a_1, b_1) \otimes \ldots \otimes (a_{n-1}, b_{n-1}) \), where \( a_i, b_i \) are complex numbers. For \( \alpha \) a \( r^\text{th} \) complex root of 1, and \( m \) an integer modulus, we can approximate an unnormalised version of \( u \) by \( u \simeq m(x) \alpha^{p(x)} \), for some appropriate choice of integers \( s \) and \( r \), where \( m : \text{GF}(2)^n \rightarrow \text{GF}(s) \), \( p : \text{GF}(2)^n \rightarrow \text{GF}(r) \), and \( x \in \text{GF}(2)^n \), such that the \( j^\text{th} \) element of \( u \), \( u_j = m(j) \alpha^{p(j)} \), where \( j \in \text{GF}(2)^n \) and \( u_j \) is interpreted as a complex number. When \( u \) is fully-factorised using the tensor product then \( m \) and \( p \) are affine functions and we say that \( u \) represents a generalised affine function (see \(^{35}\), Section 5, for more details).
for LC in terms of the adjacency matrix of the graph. In Section III, we show that the LC-orbit of a quadratic boolean function lies within the set of transform spectra w.r.t. tensor products of the $2 \times 2$ matrices, $I$, $\sqrt{-i\sigma_x}$, and $\sqrt{i\sigma_z}$, where $\sigma_x$ and $\sigma_z$ are Pauli matrices. We also show, equivalently, that the orbit lies within the spectra w.r.t. $\{I, H, N\}^n$. We show that doing LC to vertex $x_v$ can be realised by the application of the Negahadamard kernel, $N$, to position $v$ (and the identity matrix to all other positions) of the bipolar vector $(-1)^{p(x)}$, i.e.

$$\omega^a p'(x)+a(x)(-1)^{p'(x)} = U_v(-1)^{p(x)} = I \otimes \cdots \otimes I \otimes N \otimes I \otimes \cdots \otimes I (-1)^{p(x)},$$

where $p(x)$ and $p'(x)$ are quadratic, $p'(x)$ is obtained by applying LC to variable $x_v$, $\omega = \sqrt{i}$, and $a(x)$ is any offset over $\mathbb{Z}_8$. In Appendix B we identify spectral symmetries that hold for $p(x)$ of any degree w.r.t. $\{I, H, N\}^n$. In Section IV we introduce the concepts of bent, $\mathbb{Z}_4$-bent, (Completely) I-bent, LC-bent, and (Completely) I-bent boolean functions, and show how, for quadratic boolean functions, these properties can be evaluated by examining the ranks of suitably modified versions of the adjacency matrix.

II. Local Complementation (LC)

Given a graph $G$ with adjacency matrix $\Gamma$, define its complement to be the graph with adjacency matrix $\Gamma + I + 1 \pmod{2}$, where $I$ is the identity matrix and $1$ is the all-ones matrix. Let $\mathcal{N}(v)$ be the set of neighbours of vertex, $v$, in the graph, $G$, i.e. the set of vertices connected to $v$ in $G$.

Definition 1: Define the action of LC (or vertex-neighbour-complement (VNC)) on a graph $G$ at vertex $v$ as the graph transformation obtained by replacing the subgraph $G[\mathcal{N}(v)]$ by its complement.

By Glynn (see [22]), a self-dual quantum code $[[n, 0, d]]$ corresponds to a graph on $n$ vertices, which may be assumed to be connected if the code is indecomposable. It is shown there that two graphs $G$ and $H$ give equivalent self-dual quantum codes if and only if $H$ and $G$ are LC-equivalent.

For a study of the group of compositions of local complementations, see [7], [9], [8], [14], which describe the relation between local complementation and isotropic systems. Essentially, a suitably-specified isotropic system has graph presentations $G$ and $G'$ iff $G$ and $G'$ are locally equivalent w.r.t. local complementation.

A. LC in terms of the adjacency matrix

Let $p(x) : F_2^n \rightarrow F_2$ be a (homogeneous) quadratic boolean function, defined by,

$$p(x) = \sum_{0 \leq i < j \leq n-1} a_{ij} x^i x^j.$$

We can express $p(x)$ by the adjacency matrix of its associated graph, $\Gamma$, such that $\Gamma(i, j) = \Gamma(j, i) = a_{ij}$, $i < j$, $\Gamma(i, i) = 0$. The LC operation on the graph associated to $p(x)$ can be expressed in terms of
the adjacency matrix. Without loss of generality, we show how the matrix changes from \( \Gamma \) to \( \Gamma_0 \) after doing LC on vertex \( x_0 \):

\[
\Gamma_0 = \begin{pmatrix}
0 & a_{01} & a_{02} & a_{03} & \cdots & a_{0n} \\
a_{01} & 0 & a_{12} + a_{01}a_{02} & a_{13} + a_{01}a_{03} & \cdots & a_{1n} + a_{01}a_{0n-1} \\
a_{02} & a_{12} + a_{01}a_{02} & 0 & a_{23} + a_{02}a_{03} & \cdots & a_{2n} + a_{02}a_{0n-1} \\
a_{03} & a_{13} + a_{01}a_{03} & a_{23} + a_{02}a_{03} & 0 & \cdots & a_{3n} + a_{03}a_{0n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{0,n-1} & a_{1,n-1} + a_{01}a_{0,n-1} & a_{2,n-1} + a_{02}a_{0,n-1} & a_{3,n-1} + a_{03}a_{0,n-1} & \cdots & 0
\end{pmatrix}.
\]

The general algorithm, mod 2, is

\[
\begin{cases}
\Gamma_v(i,j) = \Gamma(i,j) + \Gamma(v,i) \ast \Gamma(v,j), & i < j, \ i, j = 1, \ldots, n \\
\Gamma_v(i,i) = 0 & \forall i \\
\Gamma_v(j,i) = \Gamma_v(i,j), & i > j
\end{cases}
\]

where \( \Gamma_v \) is the adjacency matrix of the function after doing LC to the vertex \( x_v \).

### III. Local Complementation (LC) and Local Unitary (LU) Equivalence

Hein et al.\(^{27}\) state that LC-Equivalence (and therefore Local Unitary (LU) Equivalence) of graph states is obtained via successive transformations of the form,

\[
U_v(G) = (-i\sigma_z^v)^{1/2} \prod_{b \in \mathcal{N}_v} (i\sigma_z^b)^{1/2}, \tag{1}
\]

where \( \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) are Pauli matrices, the superscript \( (v) \) indicates that the Pauli matrix acts on qubit \( v \) (with \( I \) acting on all other qubits)\(^3\), and \( \mathcal{N}_v \) comprises the neighbours of qubit \( v \) in the graphical representation. Define matrices \( x \) and \( z \) as follows:

\[
x = (-i\sigma_x)^{1/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i \\ i & -1 \end{pmatrix}
\]

and

\[
z = (i\sigma_z)^{1/2} = \begin{pmatrix} w & 0 \\ 0 & w^3 \end{pmatrix},
\]

where \( w = e^{2\pi i/8} \). Furthermore, let \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Define \( \mathbf{D} \) to be the set of \( 2 \times 2 \) diagonal or anti-diagonal local unitary matrices, i.e. of the form \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \) or \( \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \), for some \( a \) and \( b \) in \( \mathbb{C} \). We make extensive use of the fact that a final multiplication of a spectral vector by tensor products of members of \( \mathbf{D} \) does not change spectral coefficient magnitudes. In this sense a final multiplication by tensor products of members of \( \mathbf{D} \) has no effect on the final

\(^3\)For instance, \( \sigma_x^2 = I \otimes I \otimes \sigma_x \otimes I \otimes \ldots \otimes I \).
spectrum and does not alter the underlying graphical interpretation. For instance, applying $x$ twice to the same qubit is the same as applying $x^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, which is in $\mathcal{D}$. Therefore we can equate $x^2$ with the identity matrix, i.e. $x^2 \simeq I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Similarly, the action of any $2 \times 2$ matrix from $\mathcal{D}$ on a specific vertex is 'equivalent' the action of the identity on the same vertex. Note that $z \in \mathcal{D}$. The same equivalence holds over $n$ vertices, so we define an equivalence relation with respect to a tensor product of members of $\mathcal{D}$ by the symbol '$\simeq$'.

**Definition 2:** Let $u$ and $v$ be two $2 \times 2$ unitary matrices. Then,

$$u \simeq v \iff u = dv, \quad d \in \mathcal{D}.$$ 

This equivalence relation allows us to simplify the concatenation of actions of $x$ and $z$ on a specific qubit.

**Remark:** Note that $u \simeq v$ cannot be deduced from (and does not imply) $u = vd$ for some $d \in \mathcal{D}$.

We now show that the LC-orbit of an $n$-node graph is found as a subset of the transform spectra w.r.t. $\{I, x, xz\}^n$. Subsequently, it will be shown that we can alternatively find the LC-orbit as a subset of the transform set w.r.t. $\{I, H, N\}^n$. We then re-derive the single LC operation on a graph from the application of $x$ (or $N$) on a single vertex.

**A. The LC-orbit Occurs Within the $\{I, x, xz\}^n$ Set of Transform Spectra**

We summarise the result of (1) as follows.

**Lemma 1:** Given graphs $G$ and $G'$ as represented by the quadratic boolean functions, $p(x)$ and $p'(x)$, then $G$ and $G'$ are in the same LC-orbit iff $(-1)^p'(x) \simeq U_{v_{t-1}}U_{v_{t-2}} \ldots U_{v_0}(-1)^p(x)$ for some series of $t$ local unitary transformations, $U_{v_i}$.

From Lemma 1, we see that, by applying $U_v(G)$ successively for various $v$ to an initial state, one can generate all LC-equivalent graphs within a finite number of steps. (It is evident from the action of LC on a graph that any LC-orbit must be of finite size). Instead of applying $U$ successively, it would be nice to identify a (smaller) transform set in which all LC-equivalent graphs exist as the spectra, to within a post-multiplication by the tensor product of matrices from $\mathcal{D}$. One can deduce from definition 2 that $zx \simeq x$, and it is easy to verify that

**Lemma 2:** $zxz \simeq I$, and $xzx \simeq zzx$

With these definitions and observations we can derive the following theorem.

**Theorem 1:** To within subsequent transformation by tensor products of matrices from $\mathcal{D}$, the LC-orbit of the graph, $G$, over $n$ qubits occurs within the spectra of all possible tensor product combinations of the $2 \times 2$ matrices, $I$, $x$, and $xz$. There are $3^n$ such transform spectra.

**Proof:** For each vertex in $G$, consider every possible product of the two matrices, $x$, and $z$. Using the equivalence relationship and lemma 2
Theorem 1 follows by recursive application of (1) with these rules, and by noting that the rules are unaffected by the tensor product expansion over \( n \) vertices.

For instance, for \( n = 2 \), the LC-orbit of the graph represented by the quadratic function \( p(x) \) is found as a subset of the \( 3^2 = 9 \) transform spectra of \( (-1)^{p(x)} \) w.r.t. the transforms \( I \otimes I, I \otimes x, I \otimes xz, x \otimes I, x \otimes x, x \otimes xz, xz \otimes I, xz \otimes x, \) and \( xz \otimes xz \). Theorem 1 gives a trivial and very loose upper bound on the maximum size of any LC-orbit over \( n \) qubits, this bound being \( 3^n \). It has been computed in [16] that the number of LC-orbits for connected graphs for \( n = 1 \) to \( n = 12 \) are \( 1 \), \( 1 \), \( 1 \), \( 2 \), \( 4 \), \( 11 \), \( 26 \), \( 101 \), \( 440 \), \( 3132 \), \( 40457 \), and \( 1274068 \), respectively (see also [27], [23], [28], [15], [42]).

B. The LC-orbit Occurs Within the \( \{I, H, N\}^n \) Set of Transform Spectra

One can verify that \( N \simeq x \) and \( H \simeq xz \). Therefore one can replace \( x \) and \( xz \) with \( N \) and \( H \), respectively, so the transform set, \( \{I, xz, x\} \) becomes \( \{I, H, N\} \). This is of theoretical interest because \( H \) defines a 2-point (periodic) Discrete Fourier Transform matrix, and \( N \) defines a 2-point negaperiodic Discrete Fourier Transform matrix. In other words a basis change from the rows of \( x \) and \( xz \) to the rows of \( N \) and \( H \) provides a more natural set of multidimensional axes in some contexts. For \( t \) a non-negative integer,

\[
N^{3t} \simeq I, \quad N^{3t+1} \simeq N, \quad N^{3t+2} \simeq H, \quad N^{24} = I,
\]

so \( N \) could be considered a 'generator' of \( \{I, H, N\} \). The \( \{I, H, N\}^n \) transform set over \( n \) binary variables has been used to analyse the resistance of certain S-boxes to a form of generalised linear approximation in [37]. It also defines the basis axes under which aperiodic autocorrelation of boolean functions is investigated in [17]. The Negahadamard Transform, \( \{N\}^n \), was introduced in [33]. Constructions for boolean functions with favourable spectral properties w.r.t. \( \{H, N\}^n \) (amongst others) have been proposed in [36], and [35] showed that boolean functions that are LU-equivalent to indicators for distance-optimal binary error-correcting codes yield favourable spectral properties w.r.t. \( \{I, H\}^n \).

C. A Spectral Derivation of LC

We now re-derive LC by examining the repetitive action of \( N \) on the vector form of the graph states, interspersed with the actions of certain matrices from \( D \). We will show that, as with Lemma 1, these repeated actions not only generate the LC-orbit of the graph, but also generate the \( \{I, H, N\}^n \) transform spectra. The LC-orbit can be identified with a subset of the flat transform spectra w.r.t.
\{I, H, N\}^n$. Let $s = (-1)^{p(x)}$, where $p(x)$ is quadratic and represents a graph $G$. Then the action of $N_v$ on $G$ is equivalent to $U_\nu s$, where:

$$U_v \simeq U'_v = I \otimes \cdots \otimes I \otimes N \otimes I \otimes \cdots \otimes I,$$

where $N$ occurs at position $v$ in the tensor product decomposition. Let us write $p(x)$, uniquely, as,

$$p(x) = x_v N_v(x) + q(x),$$

where $q(x)$ and $N_v(x)$ are independent of $x_v$ ($N_v(x)$ has nothing to do with the Negahadamard kernel, $N_v$). We shall state a theorem that holds for $p(x)$ of any degree, not just quadratic, and then show that its specialisation to quadratic $p(x)$ gives the required single LC operation. Express $N_v(x)$ as the sum of $r$ monomials, $m_i(x)$, as follows,

$$N_v(x) = \sum_{i=0}^{r-1} m_i(x).$$

For $p(x)$ of any degree, the $m_i(x)$ are of degree $\leq n - 1$. In the sequel we mix arithmetic, mod 2, and mod 4 so, to clarify the formulas for equations that mix moduli, anything in square brackets is computed (mod 2). The \{0,1\} result is then embedded in (mod 4) arithmetic for subsequent operations outside the square brackets. We must also define,

$$N'_v(x) = \sum_{i=0}^{r-1} [m_i(x)] \pmod{4}.$$

**Theorem 2:** Let $s' = U_v s$, where $s = (-1)^{p(x)}$ and $s' = i^{p'(x)}$. Then,

$$p'(x) = 2 \left[ p(x) + \sum_{j \neq k} m_j(x)m_k(x) \right] + 3N'_v(x) + 3[x_v] \pmod{4} \quad (3)$$

**Proof:** Assign to $A$ and $B$ the evaluation of $p(x)$ at $x_v = 0$ and $x_v = 1$, respectively. Thus,

$$A = p(x)_{x_v=0} = q(x).$$

Similarly,

$$B = p(x)_{x_v=1} = N_v(x) + q(x).$$

We need the following equality between mod 2 and mod 4 arithmetic.

**Lemma 3:**

$$\sum_{i=1}^{n} [A_i] \pmod{4} = [\sum_{i=1}^{n} A_i + 2\sum_{i \neq j} A_iA_j] \pmod{4} \quad \text{where } A_i \in \mathbb{Z}_2.$$
Observe the following action of $N$:

\[
\begin{bmatrix}
1 & i \\
1 & -i
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
= w
\begin{bmatrix}
1 & -i \\
1 & i
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & i \\
1 & -i
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
= w
\begin{bmatrix}
1 & -i \\
1 & i
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & i \\
1 & -i
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
= w
\begin{bmatrix}
1 & -i \\
1 & i
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & i \\
1 & -i
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix}
= w
\begin{bmatrix}
1 & -i \\
1 & i
\end{bmatrix}
\]

where $w = e^{2\pi i/8}$. We ignore the global constant, $w$, so that $N$ maps $(-1)^{00}$ to $i^{03}$, $(-1)^{10}$ to $i^{12}$, $(−1)^{01}$ to $i^{30}$ and $(−1)^{11}$ to $i^{21}$. In general, for $A,B \in \mathbb{Z}_2$, $\alpha, \beta \in \mathbb{Z}_4$, $(-1)^{AB}$ is mapped by $N_v$ to $i^{\alpha \beta}$, where

\[
\alpha = 2[AB] + [A] + 3[B] \pmod{4}
\]

\[
\beta = 2[AB] + 3[A] + [B] + 3 \pmod{4}
\]

Substituting the previous expressions for $A$ and $B$ into the above and making use of Lemma 3 gives,

\[
\alpha(x) = 2[q(x)] + 3[N_v(x)] \pmod{4}
\]

\[
\beta(x) = 2[q(x)] + [N_v(x)] + 3 \pmod{4}
\]

$p'(x)$ can now be written as,

\[
p'(x) = (3[x_v] + 1)\alpha(x) + [x_v]\beta(x) \pmod{4}
\]

Substituting for $\alpha$ and $\beta$ gives,

\[
p'(x) = 2[q(x)] + 2[x_vN_v(x)] + 3[N_v(x)] + 3[x_v] \pmod{4}
\]

Applying Lemma 3 to the term $3[N_v(x)]$,

\[
3[N_v(x)] = 2 \left[ \sum_{j \neq k} m_j(x)m_k(x) \right] + 3N'_v(x) \pmod{4}
\]

Furthermore, Lemma 3 implies that,

\[
2 \left[ \sum_{i=1}^{n} A_i \right] \pmod{4} = 2 \sum_{i=1}^{n} [A_i] \pmod{4} \quad \text{where } A_i \in \mathbb{Z}_2
\]

For $p(x)$ a quadratic function, $N_v(x)$ has degree one, so $N'_v(x)$ is a sum of degree-one terms over $\mathbb{Z}_4$. Therefore the $\mathbb{Z}_4$ degree-one terms, $N'_v(x)$ and $3[x_v]$, can be eliminated from (3) by appropriate
subsequent action by members of \( \{ D \}^n \) to \( s' \). As all monomials, \( m_i(x) \), are then of degree one, (3) reduces to,

\[
p'(x) \simeq p(x) + \sum_{j,k \in N, j \neq k} x_j x_k \quad (\text{mod } 2) .
\]

(4) precisely defines the action of a single LC operation at vertex \( v \) of \( G \), where we have used \( \simeq \) to mean that \( (-1)^{p'(x)} = BU (-1)^{p(x)} \), for some fully tensor-factorisable matrix, \( U \), and some \( B \in \{ D \}^n \). As \( p'(x) \) is also quadratic boolean, we can realise successive LC operations on chosen vertices in \( G \) via successive actions of \( N \) at these vertices, where each action of \( N \) must be interspersed with the action of a matrix from \( \{ D \}^n \) to eliminate \( \mathbb{Z}_4 \)-linear terms from (3). In particular, one needs to intersperse with tensor products of \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \).

**Theorem 3:** Given a graph, \( G \), as represented by \( s = (-1)^{p(x)} \), with \( p(x) \) quadratic, the LC-orbit of \( G \) comprises graphs which occur as a subset of the spectra w.r.t. \( \{ I, H, N \}^n \) acting on \( s \).

**Proof:** Define \( D_1 \subset D \) such that

\[ D_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a = 1, b = \pm 1 \right\} . \]

Similarly, define \( D_2 \subset D \) such that

\[ D_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a = 1, b = \pm i \right\}, \quad \text{where } i^2 = -1 . \]

Then it is straightforward to establish that, for any \( \Delta_1, \Delta'_1 \in D_1 \), any \( \Delta_2, \Delta'_2 \in D_2 \), and any \( c \in \{ 1, i, -1, -i \} \),

\[
N \Delta_1 = c \Delta'_1 N \quad \quad H \Delta_1 = c \Delta'_1 H \\
N \Delta_2 = c \Delta_1 H \quad \quad H \Delta_2 = c \Delta_1 N .
\]

(5)

Let \( \Delta_* \in D_1 \cup D_2 \). Then, for a vertex, successive applications of \( \Delta_* N \) can, using (5), be re-expressed as,

\[
\prod (\Delta_* N) = c \Delta_* \prod N \simeq \prod N .
\]

But, from (2), successive powers of \( N \) generate \( I, H, \) or \( N, \) to within a final multiplication by a member of \( D \). It follows that successive LC actions on arbitrary vertices can be described by the action on \( s \) of a member of the transform set, \( \{ I, H, N \}^n \), and therefore that the LC-orbit occurs within the \( \{ I, H, N \}^n \) transform spectra of \( s \). \( \blacksquare \)

**D. LC on Hypergraphs**

For \( p(x) \) of degree \( > 2 \), \( \mathcal{N}_v(x) \) will typically have degree higher than 1, and therefore the expansion of the sum will contribute higher degree terms. For such a scenario we can no longer eliminate the nonlinear and non-boolean term, \( \mathcal{N}'_v(x) \), from the right-hand side of (3) by subsequent actions from \( D \). Therefore, it is typically not possible to iterate LC graphically beyond one step. We would like
to identify hypergraph equivalence w.r.t. \textit{local} unitary transforms, in particular w.r.t. \(\{I, H, N\}^n\). Computations have shown that orbits of boolean functions of degree \(> 2\) and size greater than one do sometimes exist with respect to \(\{I, H, N\}^n\), although they appear to be significantly smaller in size compared to orbits for the quadratic case \[17\].

An interesting open problem is to characterise a 'LC-like' equivalence for hypergraphs.

Further spectral symmetries of boolean functions w.r.t. \(\{I, H, N\}^n\) are discussed in Appendix B.

IV. Generalised Bent Properties of Boolean Functions

A. Bent Boolean Functions

A bent boolean function can be defined by using the WHT. Let \(p(x)\) be our function over \(n\) binary variables. Define the WHT of \(p(x)\) by,

\[
P_k = 2^{-n/2} \sum_{x \in GF(2)^n} (-1)^{p(x) + k \cdot x},
\]

where \(x, k \in GF(2)^n\), and \(\cdot\) implies the scalar product of vectors.

The WHT of \(p(x)\) can alternatively be defined as a multiplication of the vector \((-1)^{p(x)}\) by \(H \otimes H \otimes \ldots \otimes H\). Thus,

\[
P = 2^{-n/2} (H \otimes H \otimes \ldots \otimes H)(-1)^{p(x)} = 2^{-n/2} \left( \bigotimes_{i=0}^{n-1} H \right)(-1)^{p(x)},
\]

where \(P = (P_{(0,\ldots,0)}, \ldots, P_{(1,\ldots,1)}) \in \mathbb{C}^{2^n}\).

\(p(x)\) is defined to be \textit{bent} if \(|P_k| = 1 \ \forall k\), in which case we say that \(p(x)\) has a \textit{flat spectra} w.r.t. the WHT. In other words, \(p(x)\) is bent if \(P\) is \textit{flat}.

Let \(\Gamma\) be the binary adjacency matrix associated to \(p(x)\) when \(p(x)\) is a quadratic.

\textit{Lemma 4: [30]}

\(p(x)\) is \textit{bent} \iff \(\Gamma\) has maximum rank, mod 2.

It is well-known \[30\] that all bent quadratics are equivalent under affine transformation to the boolean function \(\left( \sum_{i=0}^{n-1} x_{2i} \cdot x_{2i+1} \right) + c \cdot x + d\) for \(n\) even, where \(c \in GF(2)^n\), and \(d \in GF(2)\). More generally, bent boolean functions only exist for \(n\) even. It is interesting to investigate other bent symmetries where affine symmetry has been omitted. In particular, in the context of LC, we are interested in the existence and number of flat spectra of boolean functions with respect to the \(\{H, N\}^n\)-transform set (\textit{bent}_4), the \(\{I, H\}^n\)-transform set (\textit{I-bent}), and the \(\{I, H, N\}^n\)-transform set (\textit{I-bent}_4). In the following subsections we investigate the \textit{bent}_4, \(Z_4\)-bent, (Completely) \textit{I-bent}, \textit{LC-bent}, and (Completely) \textit{I-bent}_4 properties of connected quadratic boolean functions, where affine symmetry is omitted, and make some general statements about these properties for more general boolean functions.
B. Bent Properties with respect to \( \{H, N\}^n \)

We now investigate certain spectral properties of boolean functions w.r.t. \( \{H, N\}^n \), where \( \{H, N\}^n \) is the set of \( 2^n \) transforms of the form \( \bigotimes_{j \in R_H} H_j \bigotimes_{j \in R_N} N_j \), where the sets \( R_H \) and \( R_N \) partition \( \{0, \ldots, n-1\} \).

The following is trivial to verify:

\[
p(x) \text{ is bent } \iff p(x) + k \cdot x + d \text{ is bent},
\]

where \( k \in \text{GF}(2)^n \) and \( d \in \text{GF}(2) \). In other words, if \( p(x) \) is bent then so are all its affine offsets, mod 2. However the above does not follow if one considers every possible \( \mathbb{Z}_4 \)-linear offset of the boolean function. The WHT of \( p(x) \) with a \( \mathbb{Z}_4 \)-linear offset can be defined as follows.

\[
P_{k,c} = 2^{-n/2} \sum_{x \in \text{GF}(2)^n} (i)^{2[p(x)+k \cdot x]+[c \cdot x]} \quad k, c \in \text{GF}(2)^n.
\] (8)

**Definition 3:**

\( p(x) \) is bent_4 \( \iff \exists c \) such that \( |P_{k,c}| = 1 \forall k \in \text{GF}(2)^n \).

Let \( R_N \) and \( R_H \) partition \( \{0, 1, \ldots, n-1\} \). Let,

\[
U = \bigotimes_{j \in R_H} H_j \bigotimes_{j \in R_N} N_j.
\]

\( s' = U(-1)^{p(x)} \). (9)

**Lemma 5:** \( p(x) \) is bent_4 if there exists one or more partitions, \( R_N, R_H \) such that \( s' \) is flat.

**Proof:** The rows of \( U \) can be described by \( (i)^{f(x)} \), where \( x = (x_0, x_1, \ldots, x_{n-1}) \), where \( f \) is linear, \( f : \text{GF}(2)^n \to \text{GF}(4) \), and the coefficient of \( x_j \) in \( f \) for \( j \in R_H \) and \( f \in \{1, 3\} \) for \( j \in R_N \). Therefore \( s' \) can always, equivalently, be expressed as \( s' = (\bigotimes H)(i)^{2p(x)+[f'(x)]} \), where \( f' \) is linear, \( f' : \text{GF}(2)^n \to \text{GF}(2) \), and the coefficient of \( x_j \) in \( f' \) is 0 for \( j \in R_H \), and 1 for \( j \in R_N \).

An alternative way to define the bent_4 property for \( p(x) \) quadratic is via a modified form of the adjacency matrix.

**Lemma 6:** For quadratic \( p(x) \),

\[
p(x) \text{ is bent}_4 \iff \Gamma_v \text{ has maximum rank, mod 2, for some } v \in \text{GF}(2)^n.
\]

where \( \Gamma_v \) is a modified form of \( \Gamma \) with \( v_i \) in position \([i, i] \), where \( v = (v_0, v_1, \ldots, v_{n-1}) \).
Proof: We first show that the transform of \((-1)^{p(x)}\) by tensor products of \(H\) and \(N\) produces a flat spectra if and only if the associated periodic and negaperiodic autocorrelation spectra have zero out-of-phase values. We then show how these autocorrelation constraints lead directly to constraints on the associated adjacency matrix.

Consider a function, \(p\), of just one variable, \(x_0\), and let \(s = (-1)^{p(x_0)}\). Define the periodic autocorrelation function as follows,

\[
a_k = \sum_{x_0 \in GF(2)} (-1)^{p(x_0)+p(x_0+k)}, \quad k \in GF(2).
\]

Then it is well-known that \(s' = Hs\) is a flat spectrum if and only if \(a_k = 0\) for \(k \neq 0\).

Define the negaperiodic autocorrelation function as follows,

\[
b_k = \sum_{x_0 \in GF(2)} (-1)^{p(x_0)+p(x_0+k)+k(x_0+1)}, \quad k \in GF(2).
\]

Then \(s' = Ns\) is a flat spectrum if and only if \(b_k = 0\) for \(k \neq 0\). (For \(p\) a boolean function of just one variable, \(Hs\) is never flat and \(Ns\) is always flat, but this only holds for one variable).

We now elaborate on the above two claims. Define \(s(z) = s_0 + s_1z\), \(a(z) = a_0 + a_1z\), and \(b(z) = b_0 + b_1z\). Then the periodic and negaperiodic relationships between autocorrelation and fourier spectra, as claimed above, follow because periodic autocorrelation can be realised by the polynomial multiplication, \(a(z) = s(z)s(z^{-1}) \mod (z^2 - 1)\), with associated residue reduction, \(\mod (z - 1)\) and \(\mod (z + 1)\), realised by \(s' = Hs = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} s\) (with the Chinese Remainder Theorem realised by \(H_1s'\), where ‘\(\dagger\)’ means transpose conjugate). By Parseval, \(s'\) can only be flat if \(a_1 = 0\). Similarly, negaperiodic autocorrelation can be realised by the polynomial multiplication, \(b(z) = s(z)s(z^{-1}) \mod (z^2 + 1)\), with associated residue reduction, \(\mod (z - i)\) and \(\mod (z + i)\), realised by \(s' = Ns = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} s\) (with the Chinese Remainder Theorem realised by \(N_1s'\)). By Parseval, \(s'\) can only be flat if \(b_1 = 0\).

We extend this autocorrelation ↔ Fourier spectrum duality to \(n\) binary variables by defining multivariate forms of the above polynomial relationships. If we choose periodic autocorrelation for indices in \(R_H\) and negaperiodic autocorrelation for indices in \(R_N\), we obtain the autocorrelation spectra,

\[
A_{k,R_H,R_N} = \sum_{x \in GF(2)^n} (-1)^{p(x)+p(x+k)+\sum_{i=0}^{n-1} \chi_{R_N}(i)k_i(x_i+1)}, \quad (10)
\]

where \(k = (k_0, k_1, \ldots, k_{n-1}) \in GF(2)^n\), and \(\chi_{R_N}(i)\) is the characteristic function of \(R_N\), i.e,

\[
\chi_{R_N}(i) = \begin{cases} 
1, & i \in R_N \\
0, & i \notin R_N
\end{cases}
\]
In polynomial terms, with \( z \in \mathbb{GF}(2)^n \) and \( s(z) = \sum_{j \in \mathbb{GF}(2)^n} s_j \prod_{i=0}^{n-1} z_i^j \), we have,

\[
A_{R_hR_n}(z) = \sum_{k \in \mathbb{GF}(2)^n} A_{k,R_hR_n} \prod_{i=0}^{n-1} z_i^k
\]

\[
= s(z_0, z_1, \ldots, z_{n-1})s(z_0^{-1}, z_1^{-1}, \ldots, z_{n-1}^{-1}) \bmod \prod_{i=0}^{n-1}(z_i^2 - (-1)^{R_n(i)}) .
\]

Then, by appealing to a multivariate version of Parseval's Theorem, \( s' \) as defined in (9) is flat if and only if \( A_{k,R_hR_n} = 0, \forall k \neq 0 \).

These constraints on the autocorrelation coefficients of \( s \) translate to requiring a maximum rank property for a modified adjacency matrix, as follows. The condition \( A_{k,R_hR_n} = 0 \) for \( k \neq 0 \) is equivalent to requiring that, if we compare the function with its multidimensional periodic and negaperiodic rotations (but for the identity rotation), the remainder should be a balanced function. When dealing with quadratic boolean functions, the remainder is always linear or constant. This gives us a system of linear equations represented by the binary adjacency matrix, \( \Gamma \), of \( p(x) \), with a modified diagonal, that is with \( \Gamma_{i,i} = 1 \) for all \( i \in \mathbb{R}_n \), and \( \Gamma_{i,i} = 0 \) otherwise. Let

\[
p(x_0, x_1, \ldots, x_{n-1}) = a_{01}x_0x_1 + a_{02}x_0x_2 + \cdots + a_{ij}x_ix_j + \cdots + a_{n-2,n-1}x_{n-2}x_{n-1} .
\]

Therefore,

\[
p(x) + p(x + k) + \sum_{i=0}^{n-1} \chi_{R_n}(i)k_ix_i = k_0(\chi_{R_n}(0)x_0 + a_{01}x_1 + a_{02}x_2 + \cdots + a_{0,n-1}x_{n-1})
\]

\[
+ k_1(a_{01}x_0 + \chi_{R_n}(1)x_1 + a_{02}x_2 + \cdots + a_{0,n-1}x_{n-1}) + \cdots
\]

\[
+ k_{n-1}(a_{0,n-1}x_0 + \cdots + a_{n-2,n-1}x_{n-2} + \chi_{R_n}(n-1)x_{n-1}) .
\]

This is equal to:

\[
x_0(\chi_{R_n}(0)k_0 + a_{01}k_1 + \cdots + a_{0n}k_n) + x_1(a_{01}k_0 + \chi_{R_n}(1)k_1 + \cdots + a_{1,n-1}k_{n-1})
\]

\[
+ \cdots + x_{n-1}(a_{0,n-1}k_0 + a_{1,n-1}k_1 + \cdots + a_{n-2,n-1}k_{n-2} + \chi_{R_n}(n-1)k_{n-1}) ,
\]

which is balanced unless constant. The constant \( \sum_{i=0}^{n-1} \chi_{R_n}(i)k_i \) will not play any role in the equation \( A_k = 0 \), and can be ignored. We have the following system of equations:

\[
\chi_{R_n}(0)k_0 + a_{01}k_1 + a_{02}k_2 + \cdots + a_{0,n-1}k_{n-1} = 0
\]

\[
a_{01}k_0 + \chi_{R_n}(1)k_1 + a_{12}k_2 + \cdots + a_{1,n-1}k_{n-1} = 0
\]

\[
\vdots
\]

\[
a_{0,n-1}k_0 + a_{1,n-1}k_1 + \cdots + a_{n-2,n-1}k_{n-2} + \chi_{R_n}(n-1)k_{n-1} = 0 .
\]

Writing this system as a matrix, we have:

\[
\begin{pmatrix}
\chi_{R_n}(0) & a_{01} & a_{02} & \cdots & a_{0,n-1} \\
a_{01} & \chi_{R_n}(1) & a_{12} & \cdots & a_{1,n-1} \\
a_{02} & a_{12} & \chi_{R_n}(2) & \cdots & a_{2,n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{0,n-1} & a_{1,n-1} & a_{2,n-1} & \cdots & \chi_{R_n}(n-1)
\end{pmatrix} = 0 .
\]
This is a modification of $\Gamma$, with 1 or 0 in position $i$ of the diagonal depending on whether $i \in R_N$ or $i \in R_H$.

In general,

$$p(x) \text{ is bent } \Rightarrow p(x) \text{ is bent}_4 .$$

**Theorem 4**: All boolean functions of degree $\leq 2$ are bent$_4$.

**Proof**: Degree zero and degree one functions are trivial. Consider the adjacency matrix, $\Gamma$, associated with the quadratic boolean function, $p(x)$. We now prove that $\Gamma_v$ has maximum rank (mod 2) for at least one choice of $v$, where $\Gamma_v = \Gamma + \text{diag}(v)$ as before. Let $M$ be the minor associated with the first entry of $\Gamma$; in other words, let $\Gamma = \begin{pmatrix} 0 & \vdots \\ \vdots & M \end{pmatrix}$.

We prove by induction that there exists at least one choice of $v$ such that $\Gamma_v$ has maximum rank (mod 2). The theorem is true for $n = 2$: in this case, $\Gamma = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$.

Then, either $\det(\Gamma) = 1$, in which case we choose $v = (0, 0)$, or we have $a = 0$ (empty graph). In the last case we choose $v = (1, 1)$, so $\det(\Gamma_v) = 1 + a = 1$. Suppose the theorem is true for $n - 1$ variables. We will see that it is true for $n$ variables. If the determinant of $\Gamma$ is 1 we take $v = (0, \ldots, 0)$ and we are done. If $\det(\Gamma) = 0$, then we have two cases:

- $\det(M) = 1$: Take $v = (1, 0, \ldots, 0)$.
- $\det(M) = 0$: By the induction hypothesis there is at least one choice of $v(M) \in \mathbb{GF}(2)^{n-1}$, where $v(M) = (v_1, \ldots, v_{n-1})$ such that $M_v(M)$ has full rank. Let $v' = (0, v_1, \ldots, v_{n-1}) \in \mathbb{GF}(2)^n$. If $\det(\Gamma_{v'}) = 1$ we have finished. If $\det(\Gamma_{v'}) = 0$ we are in the first case again, so we take $v = (1, v_1, \ldots, v_{n-1})$, and we are done.

The theorem follows from lemma 6.

**Remark**: Theorem 4 is true even for boolean functions associated with non-connected or empty graphs.

**Lemma 7**: Not all boolean functions of degree $> 2$ are bent$_4$.

**Proof**: Counter-example - by computation there are no bent$_4$ cubics of three variables. Further computations show that there are no bent$_4$ boolean functions of four variables of degree $> 2$. Similarly, there are only 252336 bent$_4$ cubic boolean functions in five variables (out of a possible $2^{20} - 2^{10}$, not including affine offsets), and no bent$_4$ boolean functions of degree $\geq 4$ in five variables. Bent$_4$ cubics of six variables do exist. Lemma 7 identifies an open problem:

**What is the maximum algebraic degree of a bent$_4$ boolean function of $n$ variables?**

**Definition 4**: $p(x)$ is Z$_4$-bent $\iff |P_{k,c}| = 1 \quad \forall c, k \in \mathbb{GF}(2)^n$. 
The definition requires that all $\mathbb{Z}_4$-linear offsets of the boolean function, $p(x)$, are flat w.r.t. the WHT. We prove that no such boolean functions exist, first for all boolean functions of degree $\leq 2$, and then for all boolean functions.

Theorem 5: There are no $\mathbb{Z}_4$-bent boolean functions of degree $\leq 2$.

Proof: This is trivial for degree zero and degree one functions. Consider the adjacency matrix, $\Gamma$, associated with the quadratic boolean function, $p(x)$. The theorem is equivalent to proving that there is a $v$ such that $\Gamma v$ has rank less than maximal. Then:

1. if $p(x)$ is not bent, then we take $v = (0, \ldots, 0)$ and we are done.
2. if $p(x)$ is bent, we take $M$ as in the proof for Theorem 4. If $\det(M) = 1$, we take $v = (1, 0, \ldots, 0)$ and we are done; if $\det(M) = 0$, modify the diagonal as in the proof for Theorem 4. If the determinant of the new matrix is equal to 0, we are done; if not, we are in case 1.

Theorem 6: There are no $\mathbb{Z}_4$-bent boolean functions.

Proof: Consider the proof of Lemma 6. We have established that, for a fixed choice of $R_H$ and $R_N$, $s'$, as defined in (9), is flat if and only if $A_{k:R_H,R_N} = 0$, $\forall k$, $k \neq 0$. Therefore $p(x)$ is $\mathbb{Z}_4$-bent iff $A_{k:R_H,R_N} = 0$, $\forall k$, $k \neq 0$, for all partitions $\{R_H, R_N\}$. In particular, if $p(x)$ is $\mathbb{Z}_4$-bent, then the polynomials, $A_{R_H,R_N}(z)$, as defined in (11), satisfy $A_{R_H,R_N}(z) = 2^n$ for all choices of $R_H$ and $R_N$ (i.e. their out-of-phase coefficients are all zero). By the Chinese Remainder Theorem (CRT) we can combine these polynomials for each choice of $R_H$ and $R_N$ to construct the polynomial,

$$r(z) \equiv r(z) \mod \prod_{j=0}^{n}(z_j^4 - 1)$$

where $r(z) = s(z_0, z_1, \ldots, z_{n-1})s(z_0^{-1}, z_1^{-1}, \ldots, z_{n-1}^{-1})$.

But as $r(z)$ comprises monomials containing only $z_i^{-1}, z_i^0, z_i^1$, the modular restriction in (12) has no effect on coefficient magnitudes, and

$$r(z) \equiv r(z) \mod \prod_{j=0}^{n}(z_j^4 - 1) .$$

to within a multiplication of the coefficients by $\pm 1$. It follows, by application of the CRT to (12) that, if $A_{R_H,R_N}(z) = 2^n$, $\forall R_H, R_N$, then $r(z) = 2^n$ also, i.e. $r(z)$ is integer. But this is impossible as the coefficients of the maximum degree terms, $\prod_j z_j^{-1} u_j$, $u_j \in \mathbb{Z}_2$, in $r(z)$ can never be zero, but are always $\pm 1$. Therefore $p(x)$ can never be $\mathbb{Z}_4$-bent.

Remark: Although we proved for boolean functions, it is possible to generalise the proof so as to state that no function from $\text{GF}(2)^n \rightarrow \text{GF}(q)$ can be $\mathbb{Z}_4$-bent, for any even integer $q$. 
C. Bent Properties with respect to \( \{I, H\}^n \)

We now investigate certain spectral properties of boolean functions w.r.t. \( \{I, H\}^n \), where \( \{I, H\}^n \) is the set of \( 2^n \) transforms of the form \( \bigotimes_{j \in R_I} I_j \bigotimes_{j \in R_H} H_j \), where the sets \( R_I \) and \( R_H \) partition \( \{0, \ldots, n-1\} \). [35] has investigated other spectral properties w.r.t. \( \{I, H\}^n \), such as weight hierarchy if the graph is bipartite.

The WHT of the subspace of a function from GF(2)\(^n\) to GF(2), obtained by fixing a subset, \( R_I \), of the input variables, can be defined as follows. Let \( \theta \in GF(2)^n \) be such that \( \theta_j = 1 \) iff \( j \in R_I \). Let \( r \preceq \theta \), where \( \preceq \) means that \( \theta \) 'covers' \( r \), i.e. \( r_i \leq \theta_i, \forall i \). Then,

\[
P_{k,r,\theta} = 2^{-(n-\text{wt}(\theta))/2} \sum_{x=r+y, y \leq \theta} (-1)^{p(x)+k \cdot x} k \preceq \overline{\theta}, r \preceq \theta .
\]

(13)

**Definition 5:**

\( p(x) \) is I-bent \( \iff \exists \theta \) such that \( |P_{k,r,\theta}| = 1 \) \( \forall k \preceq \overline{\theta}, \forall r \preceq \theta \),

where \( \text{wt}(\theta) < n \).

Let

\[
U = \bigotimes_{j \in R_I} I_j \bigotimes_{j \in R_H} H_j .
\]

(14)

\[
s' = U(-1)^{p(x)} .
\]

(15)

**Definition 6:** \( p(x) \) is I-bent if there exist one or more partitions, \( R_I, R_H \) such that \( s' \) is flat, where \( |R_I| < n \).

An alternative way to define the I-bent property of \( p(x) \) is via its associated adjacency matrix, \( \Gamma \). Let \( \Gamma_I \) be the adjacency matrix obtained from \( \Gamma \) by deleting all rows and columns of \( \Gamma \) with indices in \( R_I \).

**Lemma 8:** For quadratic \( p(x) \),

\( p(x) \) is I-bent \( \iff \Gamma_I \) has maximum rank, mod 2

for one or more choices of \( R_I \) where \( |R_I| < n \).

In general,

\( p(x) \) is bent \( \implies p(x) \) is I-bent .

**Theorem 7:** All boolean functions in two or more variables of degree \( \leq 2 \) are I-bent.

**Proof:** Degree zero and degree one functions are trivial. It is easy to show that all quadratic boolean functions of 2 variables are I-bent. The theorem follows by observing that all adjacency
matrices, $\Gamma$, representing quadratic functions of $n > 2$ variables contain $2 \times 2$ submatrices, obtained from $\Gamma$ by deleting all rows and columns of $\Gamma$ with indices $R_I$, for $|R_I| = n - 2$.

Lemma 9: Not all boolean functions of degree $> 2$ are I-bent.

Proof: Counter-example - by computation there are no I-bent cubics of three variables. Further computations show that there are only 416 I-bent cubics in four variables, and no I-bent quartics in four variables. There are only 442640 I-bent cubics, only 1756160 I-bent quartics in five variables, and no I-bent quintics in five variables. I-bent cubics in six variables do exist. Lemma 9 indicates an open problem:

What is the maximum algebraic degree of an I-bent boolean function of $n$ variables?

Definition 7:

$p(x)$ is Completely I-bent $\iff |P_{k,r,\theta}| = 1 \quad \forall \theta, k, r, \quad k \leq \bar{\theta}, r \leq \theta$.

Theorem 8: There are no Completely I-bent boolean functions.

Proof: Let $s = (-1)^{p(x)}$. Let $|R_I| = n - 1$. Then for $U$ as defined in (14), $s'$ cannot be flat.

D. Bent Properties with respect to $\{I, H, N\}$

The $\{H, N\}^{n-|R_I|}$ set of transforms of the subspace of a function from $GF(2)^n$ to $GF(2)$, obtained by fixing a subset, $R_I$, of the input variables, is defined as follows. Let $\theta \in GF(2)^n$ be such that $\theta_j = 1$ iff $j \in R_I$. Let $r \leq \theta$. Then,

$$P_{k,c,r,\theta} = 2^{-\left(n - \text{wt}((\theta))/2\right)} \sum_{x=r+y | y \leq \bar{\theta}} (i)^{2[p(x) \oplus k \cdot x] + [c \cdot x]} \quad k, c \leq \bar{\theta}, r \leq \theta \quad . \tag{16}$$

Definition 8:

$p(x)$ is I-bent$_4$ $\iff \exists c, \theta$ such that $|P_{k,c,r,\theta}| = 1 \quad \forall k \leq \bar{\theta}, \forall r \leq \theta$ ,

where $\text{wt}(\theta) < n$.

Let $R_I, R_H$ and $R_N$ partition $\{0, 1, \ldots, n - 1\}$. Let,

$$U = \bigotimes_{j \in R_I} I_j \bigotimes_{j \in R_H} H_j \bigotimes_{j \in R_N} N_j \quad . \tag{17}$$

$$s' = U(-1)^{p(x)} \quad . \tag{18}$$

Lemma 10: $p(x)$ is I-bent$_4$ if there exists one or more partitions, $R_I, R_H, R_N$ such that $s'$ is flat, where $|R_I| < n$. 
As a generalization of (10), we get flat spectra for one or more partitions $R_I, R_H, R_N$ iff

$$A_{k,R_I,R_H,R_N} = \sum_{x=r+y|y \leq \theta} (-1)^{p(x)+p(x+k)+\sum_{i=0}^{n-1} x_{RN}(i)k_i(x_i+1)} = 0, \quad \forall k \neq 0,$$

where $\theta_j = 1$ iff $j \in R_I$, $r \leq \theta$, and $r_j = k_j$ if $j \in R_I$.

An alternative way to define the I-bent$_4$ property when $p(x)$ is quadratic is via its associated adjacency matrix, $\Gamma$. Let $\Gamma_{I,v}$ be the matrix obtained from $\Gamma_v$ when we erase the $i^{th}$ row and column if $i \in R_I$.

Lemma 11: For quadratic $p(x)$,

$$p(x) \text{ is I-bent}_4 \iff \Gamma_{I,v} \text{ has maximum rank, mod } 2, \quad \text{where } v \leq \bar{\theta}$$

for one or more choices of $v$ and $\theta$ where $\text{wt}(\theta) < n$.

In general,

$$p(x) \text{ is bent } \Rightarrow \quad p(x) \text{ is bent}_4 \Rightarrow \quad p(x) \text{ is I-bent}_4.$$

Theorem 9: All boolean functions of degree $\leq 2$ are I-bent$_4$.

Proof: Follows from Theorems 14 and 7.

Lemma 12: All boolean functions are I-bent$_4$.

Proof: From Theorem 2, the action of a single $U_v$ on a boolean function, $p(x)$, of any degree, always gives a flat output spectra, for any value of $v$. This gives (at least) $n$ flat spectra for any boolean function.

Definition 9:

$$p(x) \text{ is Completely I-bent}_4 \iff |P_{k,c,r,\theta}| = 1 \quad \forall \theta, c, k, r, \quad k, c \leq \bar{\theta}, r \leq \theta.$$

Theorem 10: There are no completely I-bent$_4$ boolean functions.

Proof: Follows from theorems 10 or 11.

It is natural to ask whether, for a given quadratic, $p(x)$, there exists at least one member of its LC-orbit which is bent. If so, then we state that the graph state, $p(x)$, and its associated LC-orbit, is LC-bent. More formally,

Definition 10: The graph state, $p(x)$ (a quadratic boolean function), and its associated LC-orbit is LC-bent if $\exists p'(x)$ such that $p'(x) \in \text{LC-orbit}(p(x))$, and such that $p'(x)$ is bent.

For example, the bent function $x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$ is in the same LC-orbit as $x_0x_1 + x_0x_2 + x_0x_3$ so, although $x_0x_1 + x_0x_2 + x_0x_3$ is not bent, it is LC-bent.

In general, for $p(x)$ quadratic,

$$p(x) \text{ is bent } \Rightarrow \quad p(x) \text{ is LC-bent }.$$
Theorem 11: Not all quadratic boolean functions are LC-bent.

Proof: By computation, the LC-orbit associated with the \( n = 6 \)-variable boolean function, 
\[ x_0x_4 + x_1x_5 + x_2x_5 + x_3x_4 + x_4x_5 \] is not LC-bent.

By computation it was found that all quadratic boolean functions of \( n \leq 5 \) variables are LC-bent. Table I lists orbit representatives for those orbits which are not LC-bent, for \( n = 2 \) to 9, and provides a summary for \( n = 10 \), where the boolean functions are abbreviated so that, say, \( ab, de, fg \) is short for \( x_a x_b + x_d x_e + x_f x_g \). For those orbits which are not LC-bent we provide the maximum rank satisfied by a graph within the orbit.

| \( n \) | ANF for the orbit representative | Max. Rank within Orbit |
|--------|---------------------------------|------------------------|
| 2-5    | -                               | -                      |
| 6      | 04,15,25,34,45                  | 4                      |
| 7      | -                               | -                      |
| 8      | 07,17,27,37,46,56,67            | 6                      |
|        | 06,17,27,37,46,56,67            | 6                      |
|        | 07,17,25,36,46,57,67            | 6                      |
|        | 06,17,27,36,45,46,56,57,67      | 6                      |
| 9      | 08,18,28,38,47,57,67,78         | 6                      |
| 10     | 08,19,29,39,49,58,68,78,89      | 6                      |
|        | 51 other orbits                 | 8                      |

**TABLE I**
Representatives for all LC-Orbits which are not LC-bent for \( n = 2 \) to 10

V. Conclusion

This paper has examined the spectral properties of boolean functions with respect to the transform set formed by tensor products of the identity, \( I \), the Walsh-Hadamard kernel, \( H \), and the Negahadamard kernel, \( N \) (the \( \{I, H, N\}^n \) transform set). In particular, the idea of a bent boolean function was generalised in a number of ways to \( \{I, H, N\}^n \). Various theorems about the generalised bent properties of boolean functions were established. It was shown how a quadratic boolean function maps to a graph and it was shown how the local unitary equivalence of these graphs can be realised by successive application of the LC operation - Local Complementation - or, alternatively, by identifying a subset of the flat spectra with respect to \( \{I, H, N\}^n \). For quadratic boolean functions it was further shown how the \( \{I, H, N\}^n \) set of transform spectra could be characterised by looking at the ranks of suitably modified versions of the adjacency matrix. In the second part of the paper, we will apply this method to enumerate the flat spectra w.r.t. \( \{I, H\}^n \), \( \{H, N\}^n \) and \( \{I, H, N\}^n \) for certain concrete functions.
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VI. Appendix A - Various Interpretations of the Graph States

In this section we briefly characterise graph states.

A. Interpretation as a Quantum Error Correcting Code

Let $E$ be a $2n$-dimensional binary vector space, whose elements are written as $(a|b)$, where $a, b \in \text{GF}(2)^n$, and $E$ is equipped with the (symplectic) inner product $((a|b), (a'|b')) = a \cdot b' + a' \cdot b$. Define the weight of $(a|b) = (a_1, \ldots, a_n|b_1, \ldots, b_n)$ as the number of coordinates $i$ such that at least one of the $a_i$ or $b_i$ is 1. The distance between two elements $(a|b)$ and $(a'|b')$ is defined to be the weight of their difference.

Theorem 12: Let $S$ be a $(n-k)$-dimensional linear subspace of $E$, contained in its dual $S^\perp$ (with respect to the inner product), such that there are no vectors of weight $< d$ in $S \setminus S^\perp$. By taking an eigenspace of $S$ (for any chosen linear character) we obtain a quantum error-correcting code mapping $k$ qubits to $n$ qubits that corrects $[(d-1)/2]$ errors. Such a code is called an additive quantum error-correcting code (QECC), and is described by its parameters, $[[n, k, d]]$, where $d$ is the minimal distance of the code.

We show, later, that a $[[n, 0, d]]$ QECC can be represented by a graph. First we re-express the QECC as a GF(4) additive code.

B. Interpretation as a GF(4) Additive Code

From [11] we see how to interpret the binary space $E$ as the space GF(4)$^n$ and thereby how to derive a QECC from an additive (classical) code over GF(4)$^n$. Let $GF(4) = \{0, 1, \omega, \bar{\omega}\}$, with $\omega^2 = \omega + 1$, $\omega^3 = 1$; and conjugation defined by $\bar{\omega} = \omega^2 = \omega + 1$. The Hamming weight of a vector in GF(4)$^n$, written $wt(u)$, is the number of non-zero components, and the Hamming distance between $u, u' \in GF(4)^n$ is $\text{dist}(u, u') = wt(u + u')$. Define the trace function as: $tr(x) : GF(4) \rightarrow GF(2)$, $tr(x) = x + \bar{x}$. To each vector $v = (a|b) \in E$ we associate the vector $\phi(v) = a\omega + b\bar{\omega}$. The weight of $v$ is the Hamming weight of $\phi(v)$, and the distance between two vectors in $E$ is the Hamming distance of their images. If $S$ is a subspace of $E$ then $C = \phi(S)$ is a subset of GF(4)$^n$ that is closed under addition (defining thus an additive code). The trace inner product of $u, v \in GF(4)^n$ is

$$u \star v = Tr(u \cdot \bar{v}) = \sum_{i=1}^{n} (u_i \bar{v}_i + \bar{u}_i v_i) ,$$

Define the dual code $C^\perp$ as

$$C^\perp = \{ u \in GF(4)^n : u \star v = 0 \ \forall v \in C \} .$$

Now one can reformulate Theorem [12]
Theorem 13: Let $C$ be an additive self-orthogonal subcode of $\text{GF}(4)^n$, containing $2^{n-k}$ vectors, such that there are no vectors of weight $< d$ in $C \setminus C^\perp$. Then any eigenspace of $\phi^{-1}(C)$ is a QECC with parameters $[[n, k, d]]$.

By Glynn (see [22], [23]), we have: Let $S$ be a stabilizer matrix, that is $(n-k) \times n$ over $\text{GF}(4)$ and such that its rows are $\text{GF}(2)$-linearly independent. Then we define a QECC with parameters $[[n, k, d]]$ as the set of all $\text{GF}(2)$-linear combinations of the rows of $S$. The code is self-dual when $k = 0$.

C. The QECC as a Graph

Assume that each column of $S$ contains at least two non-zero values, for the columns that do not have this property may be deleted to obtain a better code. Following [22], a self-dual quantum code $[[n, 0, d]]$ corresponds to a graph on $n$ vertices, which may be assumed to be connected if the code is indecomposable. Let $\text{PG}(m, q)$ be the finite projective space defined from the vector space of rank $m + 1$ over the field $\text{GF}(q)$. Then, the Grassmannian of lines of $\text{PG}(n-1, 2)$, $G_1(\text{PG}(n-1, 2))$, regarded as a variety immersed in $\text{PG}(\binom{n}{2}, 2)$ is as follows: each line $l_i$ is defined by two points, $a_i$ and $b_i$. We associate to the set of lines all products $a_i b_j + a_j b_i, i \neq j \pmod{2}$. Define a mapping from a column of an $n \times n$ stabilizer matrix $S$ over $\text{GF}(4)$ to a vector of length $\binom{n}{2}$ with coefficients in $\text{GF}(2)$: We write each column over $\text{GF}(4)$ as $a + b \omega$, where $a, b \in \text{GF}(2)^n$.

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n 
\end{pmatrix} = \begin{pmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_n 
\end{pmatrix} + \omega \begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n 
\end{pmatrix}
\]

Taking all the $2 \times 2$ subdeterminants found when we put the two vectors into a matrix, we get the points of the Grassmannian. A point in $G_1(\text{PG}(n-1, 2)) \equiv$ a line in $\text{PG}(n-1, 2)$ $\equiv$ a column of length $n$ over $\text{GF}(4)$ (with at least two different non-zero components). A quantum self-dual code $[[n, 0, d]]$ corresponds to some set of $n$ lines that generate $\text{PG}(n-1, 2)$. As each line of $\text{PG}(n-1, 2)$ corresponds to a (star) kind of graph, the set corresponds to a graph in $n$ vertices.

D. Interpretation as a Modified Adjacency Generator Matrix over $\text{GF}(2)$ and $\text{GF}(4)$

From any connected graph we obtain an indecomposable code. Let $\Gamma$ be the adjacency matrix of a graph $G$ in $n$ variables. Then, $G_T = (I \mid \Gamma)$ (where $I$ is the $n \times n$ identity matrix) is the generator matrix of a binary linear code [13]. In other words,
generates a code over $\text{GF}(2)^n$. We can further interpret $G_T$ as a generating matrix of a code over $\text{GF}(4)^n$, as follows [11]:

$$G = \Gamma + \omega I = \begin{pmatrix}
\omega & a_{01} & \ldots & a_{0n} \\
a_{01} & \omega & \ldots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{0n} & a_{1n} & \ldots & \omega
\end{pmatrix}$$

is the generating matrix of an additive code over $\text{GF}(4)^n$. Different graphs may define the same code, but this relation is 1-1 with respect to LC-equivalence between graphs, as defined in section [11].

E. Interpretation as a Modified Adjacency Matrix over $\mathbb{Z}_4$

Define from a graph with adjacency matrix, $\Gamma$, the generating matrix of an additive code over $\mathbb{Z}_4^n$ as $2\Gamma + I$. This code has the same weight distribution over $\mathbb{Z}_4^n$ as $\Gamma + \omega I$ over $\text{GF}(4)^n$. Once again, LC-equivalent graphs define equivalent $\mathbb{Z}_4$ codes.

F. Interpretation as an Isotropic System

The graph state can also be viewed as an isotropic system (see [7], [9], [8], [14], [32]). Let $A$ be a 2-dimensional vector space over $\text{GF}(2)$. For $x, y \in A$, define a bilinear form, $<,>$, by

$$< x, y > = \begin{cases} 
1 & \text{if } x \neq y, x \neq 0 \text{ and } y \neq 0 \\
0, & \text{otherwise}
\end{cases}$$

Let $V$ be a finite set. Define the space of $\text{GF}(2)$-homomorphisms $A^V : V \rightarrow A$. Define in this $\text{GF}(2)$-vector space a bilinear form as:

$$\text{for } \phi, \psi \in A^V, \ < \phi, \psi > = \sum_{v \in V} < \phi(v), \psi(v) > \pmod{2} \ .$$

**Definition 11:** Let $L$ be a subspace of $A^V$. Then, $I = (V, L)$ is an isotropic system if $\dim(L) = |V|$ and $< \phi, \psi >= 0 \ \forall \ \phi, \psi \in L$.

For a graph $G$, $V(G)$ denotes the set of vertices of $G$. If $v \in V(G)$, $N(v)$ denotes the *neighbourhood* of vertex $v$, that is, the set of all its neighbours. For $P \subseteq V$, we set $N(P) = \sum_{v \in P} N(v)$. Let $K = \{0, x, y, z\}$ be the Klein group, which is a 2-dimensional vector space, and set $K' = K \setminus \{0\}$. Note that $x + y + z = 0$.
Lemma 13: ([39]) Let $G$ be a simple graph with vertex set $V$. Let $\phi, \psi \in K^n$ such that $\phi(v) \neq \psi(v)$ $\forall v \in V$, and set $L = \{\phi(P) + \psi(N(P)) : P \subseteq V\}$. Then $S = (V, L)$ is an isotropic system.

The triple $\Pi = (G, \phi, \psi)$ is called a graphic presentation of $S$.

For $\phi \in K^V$, we set $\hat{\phi} = \{\phi(P) : P \subseteq V\}$. $\hat{\phi}$ is a vector subspace of $K^V$.

Definition 12: For $\psi \in K^W$, the restricted Tutte-Martin polynomial $m(S, \psi; x)$ is defined by

$$m(I, \psi; x) = \sum (x - 1)^{\dim(L \cup \hat{\phi})},$$

where the sum is over $\phi \in K^W$ such that $\phi(v) \neq \psi(v), v \in V$.

Theorem 14: ([39]) If $G$ is a simple graph and $I$ is the isotropic system defined by a graphic presentation $(G, \phi, \psi)$, then

$$q(G; x) = m(I, \phi + \psi; x),$$

where $q(G; x)$ is the interlace polynomial of $G$.

We mention the interlace polynomial and its relation to our work in Part II of this paper [39].

G. Interpretation as a Quadratic Boolean Function

Let $p(x) : GF(2)^n \to GF(2)$ be a quadratic boolean function, defined by its Algebraic Normal Form (ANF), $p(x) = \sum_{0 \leq i < j \leq n-1} a_{ij}x^i x^j + \sum_{i=0}^{n-1} b_i x_i + \sum_{i=0}^{n-1} c_i$. We associate to $p(x)$ the non-directed graph that has variables as vertices (and vice-versa) [35]. The adjacency matrix, $\Gamma$, associated to $p(x)$, satisfies $\Gamma(i, j) = \Gamma(j, i) = a_{ij}, i < j, \Gamma(i, i) = 0$.

H. Interpretation of a Bipartite Quadratic Boolean Function as a Binary Linear Code

Quadratic ANFs, as represented by bipartite graphs, have an interpretation as binary linear codes [35]. Let $T_C$, $T_{C^\perp}$ be a bipartite splitting of $\{0, \ldots, n - 1\}$, and let us partition the variable set $x$ as $x = x_C \cup x_{C^\perp}$, where $x_C = \{x_i : i \in T_C\}$, and $x_{C^\perp} = \{x_i : i \in T_{C^\perp}\}$. Let $p(x) = \sum_k q_k(x_C)r_k(x_{C^\perp})$, where $\deg(q_k(x_C)) = \deg(r_k(x_{C^\perp})) = 1 \forall k$ (clearly, such a function corresponds to a bipartite graph), and let $s(x) = (-1)^{p(x)}$. Then the action of the transform $\bigotimes_{i \in T} H_i$, with $T = T_C$ or $T_{C^\perp}$, on $s(x)$ gives $s'(x) = m(x)$, with $m$ the ANF of a Boolean function. $s'$ is the binary indicator for a binary linear $[n, n - |T|, d]$ error correcting code.  

VII. Appendix B - Further Spectral Symmetries of Boolean Functions with respect to $\{I, H, N\}^n$  

The power spectrum of the WHT of a boolean function is invariant to within a re-ordering of the spectral elements after an invertible affine transformation of the variables of the boolean function.

[^4]: There is also an equivalent interpretation of bipartite graphs as binary matroids (e.g. [12]).
This implies that bent boolean functions remain bent after affine transform (see Section IV for a discussion of bent properties). However, the set of \(\{I, H, N\}^n\) power spectra are not an invariant of affine transformation. In this section we ascertain for which binary transformations (other than LC) the power spectra of the \(\{I, H, N\}^n\) transform remains invariant to within a re-ordering of the spectral elements within each spectrum. We refer to the complete set of \(3^n \times 2^n\) power spectral values w.r.t. \(\{I, H, N\}^n\) as \(S_{IHN}\). Moreover, 'invariance' is to within any re-ordering of the \(3^n \times 2^n\) spectral elements. From the discussion of sections III-A and III-B it is evident that \(S_{IHN}\) of a quadratic boolean function is LC-invariant. However the LC-orbit is not the only spectral symmetry exhibited with respect to \(S_{IHN}\). We identify the following symmetries.

**Lemma 14:** Let \(p(x)\) be a boolean function of any degree. Then \(S_{IHN}\) of \(p(x)\) and \(S_{IHN}\) of \(p(x) + l(x)\) are equivalent, where \(l\) is any affine function of its arguments.

**Lemma 15:** Let \(p(x)\) be a boolean function of any degree over \(n\) variables. Then \(S_{IHN}\) of \(p(x)\) and \(S_{IHN}\) of \(p(x + a)\) are equivalent, where \(a \in GF(2)^n\).

**Proof:** Replacing \(x_j\) with \(x_j + 1\) within any \(p(x)\) is equivalent to the action of the 'bit-flip' operator, \(\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), at position \(j\) of the transform on \((-1)^{p(x)}\), applying \(I\) in the rest of the positions.

We can rewrite \(H \sigma_x\) as follows,

\[
H \sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H = \sigma_z H .
\]

In other words, a bit-flip (or periodic shift) followed by the action of \(H\) is identical to the action of \(H\) followed by a 'phase-flip'. (This is well-known to quantum code theorists). The final phase-flip is a member of the set \(D\) (see Section III for a definition of \(D\)) so does not change the magnitude of the spectral values produced by \(H\). Therefore the power spectra produced by \(H\) is invariant to prior periodic shift.

We can rewrite \(N \sigma_x\) as follows,

\[
N \sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} N = -\sigma_y N ,
\]

where \(\sigma_y\) is one of the four Pauli matrices. In other words, a bit-flip (or periodic shift) followed by the action of \(N\) is identical to the action of \(N\) followed by a member of the set \(D\). Therefore the power spectra produced by \(N\) is invariant to a prior periodic shift.

The above argument is trivial with respect to \(I\). The argument extends naturally to any \(n\)-dimensional tensor product of \(I, H, \) and \(N\).

Let \(p(x)\) be a boolean function of any degree over \(n\) variables. We perform a combination of affine offset and periodic shift on \(p(x)\) by the following operation:

\[
p(x) \Rightarrow p(x + a) + c \cdot x + d ,
\]

The power of the \(k^{th}\) spectral element, \(P_k\), is given by \(|P_k|^2\), where \(P_k\) is defined in [6].
where \( a, c \in \text{GF}(2)^n \), \( d \in \text{GF}(2) \), and \( \cdot \) is the scalar product.

The symmetries generated by affine offset and periodic shift include all symmetries generated by any combination of periodic and negaperiodic shift, because we perform periodic and negaperiodic shifts on \( p(x) \) by the following operation:

\[
p(x) \Rightarrow p(x + a) + c \cdot x + \text{wt}(c), \quad c \preceq a,
\]

where \( a, c \in \text{GF}(2)^n \), \( c \preceq a \) means that \( c_i \leq a_i, \forall i \) (i.e. \( a \) covers \( c \)), and \( \text{wt}(c) \) is the binary weight of \( c \). The one positions in \( a \) identify variables \( x_i \) which are to undergo periodic or negaperiodic shift, and the one positions in \( c \) identify the variables \( x_i \) which are to undergo negaperiodic shift. The combined periodic and negaperiodic symmetry induced by \( \{I, H, N\}^n \) implies an aperiodic symmetry, as discussed further in [17].