Frequency doubling in a pendulum

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Abstract

A smartphone is used as a pendulum bob to simultaneously measure the angular velocity and the acceleration of a pendulum. The data show a frequency doubling in the radial acceleration component compared with the angular velocity and the azimuthal acceleration component. This is mainly due to the centrifugal force and is an illustration of frequency doubling in a simple mechanical system. This experiment might be used in first year university physics courses to teach frequency analysis and frequency doubling in a home laboratory.

Keywords: pendulum, smartphone, frequency doubling

(Some figures may appear in colour only in the online journal)

1. Introduction

It is a general observation that nonlinearities of a physical system lead to the generation of higher harmonics in its frequency response. Examples are harmonic generation by the nonlinear current–voltage characteristics of pn-diodes [1] or frequency doubling from infrared to green light in Nd:YAG lasers by the use of nonlinear optical cavity materials such as lithium triborate [2]. In both cases the material response functions contain a quadratic term such that an input signal with harmonic time variation creates an output signal with harmonic time dependence at twice the input frequency by virtue of the trigonometric power formulas. Since this is a general principle and since the centrifugal acceleration is proportional to the square of the angular velocity, frequency doubling already occurs in pendulum motion. In traditional pendulum experiments, however, this is not revealed, since the focus is usually on measurements of the period or of the angular displacement as a function of time. The frequency doubling is then
overlooked, since it is only observed in the radial acceleration component, not in azimuthal acceleration or the angular velocity.

The aim of this paper is to provide an experimental setting and analysis of a pendulum experiment to demonstrate frequency doubling. The measurement device of choice is the smartphone. The target group are first year university students who already completed their mechanics course and should get some hands-on-experience in experimenting. Oscillations is a general physics concept permeating virtually all branches of physics, mechanical oscillations can be well detected by measuring the acceleration and angular velocity with the accelerometer and the gyroscope of a smartphone. Such experiments were described before for the physical pendulum [3–9] or for the spring pendulum [10, 11]. These usually deal with sinusoidal time variation of angular velocity, with the eigenfrequency and the damping constant. It is shown here that frequency doubling is clearly observed by comparison of the gyroscope and accelerometer readings. The smartphone sensors allow for a quick measurement of angular velocity and acceleration and enable modelling of the radial acceleration component. This experiment brings three new facets to the traditional pendulum experiment: (1) the introduction of the new physics concept of frequency doubling, (2) the experience that angular velocity and acceleration are vectors as well as (3) the creative task of making a smartphone pendulum.

2. Theory

2.1. Standard model of the physical pendulum in the laboratory system

The theory of the physical pendulum is well-known from numerous textbooks, see e.g. [5, 12–14], and is here only briefly reviewed as reference for the experimental section. Figure 1(a) shows a sketch of a rectangular body (smartphone) suspended on a string and deflected by an angle $\varphi_z$ from the vertical. The string is assumed to be inelastic and weightless, the moment of inertia of the pendulum bob with respect to the suspension point $P$ is given by $J_P$, its mass by $m$. Two coordinate systems are used to describe the motion, namely the inertial laboratory system (dashed lines in figure 1(a)) and the accelerated system $(x, y, z)$ rigidly attached to the smartphone, see figure 1(b). The oscillation occurs around the $z$-axis. Acceleration $\vec{a}$ and angular velocity $\vec{\omega}$ were measured by the in-built accelerometer and gyroscope, with respect to the accelerated system $(x, y, z)$ and are given by the vectors $\vec{a} = (a_x, a_y, a_z)$ and $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ with $\omega_z = \dot{\varphi}_z$. In the laboratory system, the angular momentum component $L_z$ with respect to $P$ is given by $L_z = J_P \omega_z$, the torque exerted by the force of gravity $\vec{G}$ is $\tau_z = -mgl \sin \varphi_z$, where $l$ denotes the length of the lever arm, i.e. the distance between suspension point $P$ and center of mass CM of the smartphone. Friction forces are neglected. The equation of motion follows from $\dot{L}_z = \tau_z$:

$$J_P \ddot{\varphi}_z = -mgl \sin \varphi_z.$$

In the small angle approximation $\sin \varphi_z \approx \varphi_z$ the equation of motion can be written in the following normal form:

$$\ddot{\varphi}_z + \frac{mgl}{J_P} \varphi_z = \dot{\varphi}_z + \omega_0^2 \varphi_z = 0,$$

with the solution

$$\varphi_z = \varphi_0 \cos(\omega_0 t + \alpha).$$
Figure 1. (a) Schematic of the pendulum. \( \vec{F}_T \) is the thread tension, CM the center of mass. (b) iPad Air as pendulum with axis conventions.

and the eigenfrequency

\[
    f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{mgl}{J_p}}
\]  

(4)

The amplitude \( \varphi_0 \) of the angular oscillations and the phase shift \( \alpha \) are determined by the initial conditions.

According to this model the pendulum swings harmonically with a fixed frequency \( f_0 \) that is entirely determined by the acceleration of gravity \( g \) and the geometry of the pendulum. If not treated within the small angle approximation, it is found that the eigenfrequency decreases with increasing amplitude \( \varphi_0 \) [15, 16] and that the time variation of the angle deviates from a harmonic dependence. In view of the accuracy of the accelerometer we consider a relative accuracy in the small angle approximation in (2) of 1% to be sufficient; this corresponds to a maximum angle of 14°.

2.2. Sensor acceleration in the accelerated smartphone frame of reference

The position of the accelerometer in the coordinate system attached to the smartphone is written as \( \vec{r}_S = (x_S, l_S, 0) \). The misalignment angle between the lines joining the points \( P \) either to the accelerometer or the center of mass is given by \( \tan \varphi_S = x_S/l_S \). In the laboratory frame the smartphone experiences the acceleration of gravity. In addition to this, in the accelerated smartphone frame, virtual accelerations appear, namely the centrifugal acceleration \( \vec{a}_c = -\vec{\omega} \times (\vec{\omega} \times \vec{r}_S) \) and the acceleration \( \vec{a}_\alpha = -\vec{\alpha} \times \vec{r}_S \) induced by the angular
acceleration $\ddot{a}$ [17]. The measured acceleration is then given by $\ddot{a} = \ddot{g} + \ddot{a}_c + \ddot{a}_s$. With $\ddot{\omega} = (0, 0, \ddot{\omega}_z)$ the components are

$$a_x = g \sin(\varphi_z + \varphi_s) + l_S \ddot{\varphi}_z + x_S \omega^2_z \quad (5)$$

$$a_y = g \cos(\varphi_z + \varphi_s) + l_S \omega^2_z - x_S \ddot{\varphi}_z. \quad (6)$$

Using equation (1) and writing the moment of inertia $J_P = m(l^2 + R_z^2)$ in terms of the radius of gyration $R_z$, $a_x$ can be written as (neglecting $\ddot{\varphi}_s$ and the $x_S$-term)

$$a_x = \frac{l^2 + R_z^2 - l_S}{l} \times g \sin \varphi_z = \frac{l_S - (l^2 + R_z^2)}{l} \ddot{\varphi}_z. \quad (7)$$

If the accelerometer was positioned at the center of mass ($l = l_S, x_S = 0$) and if the smartphone was pointlike ($R_z = 0$), the acceleration component $a_x$ would vanish. In any case, we expect that $a_x$ is considerably smaller than the respective acceleration of gravity component $g \sin \varphi_z$. The misalignment angle $\varphi_s$ is small, see below, therefore, in first approximation, in equations (5) and (6) terms proportional to $\ddot{\varphi}_s$ and $x_S$ can be neglected. The acceleration component $a_y$ is then a sum of $g \cos \varphi_z$, a term not larger than $g$, and the centrifugal force $l_S \ddot{\varphi}_z$ which is always positive. The quadratic dependence on the angular velocity leads to frequency doubling [18], as seen from the addition theorem

$$\cos^2(\omega_0 t) = \frac{1}{2} \left[ 1 + \cos(2\omega_0 t) \right]. \quad (8)$$

The first term in equation (6) is even more interesting, since it would in principle lead to the generation of even higher harmonics of arbitrary order. This can be easily seen by expanding the cos-function into a Taylor series and using higher order trigonometric power formulas for the even powers of $\varphi_z$. In the small angle approximation, the quadratic term of the Taylor series is dominant which also leads to a doubling of the frequency. The frequency doubling in $a_x$ is illustrated in figure 2 showing (a) a harmonic oscillation with frequency of 1 Hz and amplitude $10^\circ$ and (b) the main terms of equation (6) with $l_S = 0.5$ m. With this parameter choice the frequency doubling is mainly due to the centrifugal force.

Accordingly, we expect that $a_y$ shows a $2f_0$-oscillation, whereas $a_x$ oscillates just with the eigenfrequency $f_0$. The terms in equations (5) and (6) related to the sensor misalignment with respect to the $y$-axis generate spurious oscillations with frequency $f_0$ in $a_x$ and $2f_0$ in $a_x$.

If the oscillation occurs around the $x$-axis, the indices $x$ and $z$ in equations (5) and (6) have to be interchanged. Further, the terms proportional to $x_S$ are absent, since a displacement of the accelerometer along the oscillation axis does not effect the acceleration values in the $y$–$z$-plane. Accordingly, the spurious frequency components are expected to be absent in this configuration.

2.3. Friction

In the experiment friction cannot be neglected. This might be modelled by viscous drag, i.e. by the introduction of an additional term $-2\delta J_P \omega_z$ with damping constant $\delta$ on the right-hand side of equation (1). This leads to an exponential decay of the amplitude, i.e. equation (3) acquires a factor $\exp(-\delta t)$ on the right-hand side. Moreover, the eigenfrequency is shifted to lower values

$$f_d = \frac{\sqrt{\omega_0^2 - \delta^2}}{2\pi}. \quad (9)$$
3. Experimental results and discussion

3.1. Experimental setup

The measurements were performed with an iPad Air, see figure 1(b). This was fixed in a 3D-printed frame to which the pendulum string was attached. Measurements were performed with the iPad oscillating around the z- and the x-axis. The damping constant was much larger for the x-axis oscillations. Accordingly, measurement times were shorter and the signals more noisy, since these were measured at smaller amplitudes. Therefore, we focus on the z-axis data. The measurements were cross-checked with a Huawei Mate 9 smartphone and analogous results were obtained. As seen in figure 1(b), in case of z-axis oscillations the string was simply slung over a steel rod such that the oscillation axis was the normal to the iPad screen (z-axis). This yielded an oscillation with a fairly constant rotation axis, although angular velocity components \( \omega_x \) and \( \omega_y \) were not exactly zero. However, \( \omega_z \) as well as \( a_x \) and \( a_y \) in case of z-axis oscillations and \( \omega_x \) as well as \( a_t \) and \( a_y \) in case of x-axis oscillations had significantly larger values than the other components and are reported here.

The data collection was facilitated with the app ‘phyphox’ [19]. Phyphox easily allows for the ‘simultaneous’ measurement of the gyroscope and accelerometer data by creating a new experiment (in the app press on ‘+’ > ‘new simple experiment’ and then select accelerometer and gyroscope). The measurement is not simultaneous, but gyroscope and accelerometer data are read alternatingly and are provided with the respective time stamps.

The location of the acceleration sensor was determined by placing the iPad centrally on a record player (set at 45 rpm) and measuring the angular velocity and acceleration, see also
Figure 3. (a) Angular velocity component $\omega_z$ and (b) acceleration component $a_y$ as a function of time. The inset to (a) shows $\omega_z$ on a longer time scale. The red lines are fits of an exponential function to the extremal points.

[20]. From the direction and magnitude of the centrifugal acceleration the position of the sensor was estimated to $x_S = -(10 \pm 5)$ mm and $y_S = -(56 \pm 3)$ mm with respect to the center of mass, i.e. the sensor is located above the center of mass (away from the home button) slightly to the right. In case of $z$-axis oscillations, the pendulum length (from suspension point to center of mass) was $l = (424 \pm 2)$ mm; along the $y$-axis the sensor had a distance $l_S = l + y_S = (368 \pm 5)$ mm from the suspension point $P$. The misalignment angle was $\varphi_S = (1.6 \pm 0.5)$ degrees.

3.2. Angular velocity and acceleration data

The time dependence of angular velocity component $\omega_z$ and acceleration component $a_y$ are shown in figures 3(a) and (b) for 20 s of the measurement. Whereas $\omega_z$ oscillates harmonically with one frequency $f_0$, $a_y$ oscillates clearly twice as fast and appears to contain two frequency components with frequencies $f_0$ and $2f_0$, since the amplitudes of subsequent oscillations are different. The insert to figure 3(a) shows the data over a time of 1400 s. The clear observation of the oscillation after such a long time shows that the damping was rather low. The minima and maxima were fitted with an exponential decay and yielded a damping constant $\delta = (0.0016 \pm 0.0002) \text{s}^{-1}$. One can see in the figure that the exponential decay is a good approximation, but it does not exactly match the time decay of the data. This means that there are other friction mechanisms active, especially sliding friction and air friction with quadratic
velocity dependence. The initial angular deflection was about 12°, so the small angle approximation should be valid. In case of oscillations around the $x$-axis the damping constant was about (0.015 ± 0.002) s$^{-1}$. Therefore, the measurements in this configuration could only be performed over about 200 s.

### 3.3. Frequency spectrum

The data in figure 3 were further analyzed by computing the fast Fourier transforms to obtain the corresponding frequency spectra. The latter are shown in figures 4(a) and (b) for $\omega_z$/t (black symbols), $a_x$ (blue symbols) and $a_y$/z (red symbols). The angular velocities from the gyroscope data show a single frequency peak at (0.7620 ± 0.0002) Hz (oscillation around $z$) or (0.848 ± 0.001) Hz (x), $a_t$ shows a frequency peak at twice that frequency: (1.5230 ± 0.0005) Hz (z) and (1.697 ± 0.001) Hz (x). This corresponds to the anticipated frequency doubling. $a_y$/z show a main frequency peak at the same value as the gyroscope. In case of the oscillation around the $z$-axis, $a_t$ shows a considerable spurious peak at $f_0$ and $a_x$ a spurious peak at $2f_0$. We attribute these peaks to the non-central position of the accelerometer as derived in the theory section. Note that the spurious peaks are much smaller in case of oscillations around the $x$-axis, as expected.

In both cases the damping constant does not affect the measured frequency within experimental uncertainties, see equation (9) and the values for $\delta$. According to equation (4) the
Figure 5. Measured acceleration $a_z$ (black lines) and acceleration calculated from equation (6) (red lines) shown for (a) 1400 s and (b) 20 s. (c) Measured acceleration $a_x$ (black line) and acceleration calculated from equation (7) (red line).

Eigenfrequencies $f_0$ are given by the acceleration of gravity $g = 9.81 \text{ m s}^{-2}$ (taken exactly) as well as pendulum length and radius of gyration. Let $J_{CM/z}$ denote the moment of inertia with respect to the $x/z$-axis through the center of mass, then $J_P$ is given by the parallel axis theorem as $J_P = mR^2 + J_{CM/z} = m(l^2 + R^2_z)$. As an estimate for the present pendulum, the printed frame is neglected, the iPad Air is assumed to be a cuboid with homogeneous mass density and dimensions $a = 174.1 \text{ mm}$, $b = 250.6 \text{ mm}$ and $c = 6.1 \text{ mm}$. Then $R^2_z = (b^2 + c^2)/12 = 5236 \text{ mm}^2$ and $R^2_y = (a^2 + b^2)/12 = 7760 \text{ mm}^2$. We estimate an uncertainty of 20% in the radii of gyration due to inhomogeneities in the mass distribution and the disregard of the printed frame. Then, from equation (4), frequencies $f_0 = (0.750 \pm 0.005) \text{ Hz (z)}$ and $f_0 = (0.855 \pm 0.006) \text{ Hz (x)}$ are obtained. The measured values fall in a range of three (z) or one (x) standard errors of the calculated values.

### 3.4. Modelling of the acceleration components

In case of the oscillation around the $z$-axis, the measured acceleration components $a_x$ and $a_y$ can be modelled with the gyroscope data using equations (5) and (6); the displacement of the sensor by $x_S$ was neglected. The centrifugal acceleration $l_S \omega^2_z$ depends on the distance $l_S$ between accelerometer and suspension point as well as the angular velocity $\omega_z$ which were both measured. The angle $\varphi_z$ needed for the decomposition of the acceleration of gravity along the coordinate axes of the iPad was obtained from numerical integration of $\omega_z$. Figure 5 shows the
measured $a_y$ values (black line) and the values calculated with equation (6) for (a) long times and (b) 20 s. The value of the acceleration of gravity was fixed at $9.84 \, \text{m s}^{-2}$. By comparing the readings of various smartphone sensors, we found that this was in the typical range of deviation from the standard value. Note that the oscillation does not occur symmetrically to the value for the acceleration of gravity. The oscillations above this value are due to the centrifugal acceleration, the oscillations below due to the $\cos(\varphi_z)$-term in equation (6), see also figure 2. Figure 5(c) shows measured acceleration $a_x$ compared to values calculated with equation (7). Overall agreement is very good, albeit not exact. The discrepancies are not explained by taking the $x_S$-displacement into account, but these might be related to sensor uncertainties, misalignment of the oscillation plane, or other factors. However, the overall agreement proves that there is frequency doubling in the radial acceleration component due to the centrifugal acceleration and the cosine-component of the vector decomposition.

4. Conclusions

The aim of the present paper was to introduce a new facet into experimenting with a pendulum. A smartphone enables measurement of not only the angular velocity, but also of the acceleration. This gives experimental access to the centrifugal acceleration and beautifully shows frequency doubling due to its quadratic dependence on the angular velocity.

Traditional undergraduate physics laboratories usually contain fundamental experiments which students are expected to perform during their educational careers to deepen their understanding of central concepts. Recent research has shown that this aim is often not achieved, but that the final exam success rate is vastly independent of whether an accompanying laboratory course was taken or not [21]. This can be attributed—at least partially—to the design of traditional experiments which often let students complete these experiments mechanically, following a given recipe with a pre-designed apparatus. Compared to conventional pendulum experiments this home lab might stimulate students’ creativity both through the challenge to plan and build the setup themselves as well as through the interplay of rather extended measurement capabilities and data analysis with simulation of theoretical relationships that have a certain complexity at that particular stage of their learning, i.e. in the first year of university studies. The digital transformation of teaching enables the creation of de-centralized digital laboratory spaces which are or can be equipped with inexpensive measurement devices that allow for a wide range of creative experiments performed at home. Examples are the use of smartphones as well as electronic equipment based on the Arduino or raspberry pi platforms. We have the hope that experiments like this would set students on a learning trajectory outside the rigid traditional lab structure within virtual or smart labs.

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