$d = 11$ supergravity on almost flat $\mathbb{R}^4$ times a compact hyperbolic 7-manifold, and the dip and bump seen in ATLAS-CONF-2010-088

Chris Austin

33 Collins Terrace, Maryport, Cumbria CA15 8DL, England

Abstract

Rough estimates are presented to show that the bump at 1.7 to 1.9 TeV seen in ATLAS-CONF-2010-088 could arise from about $10^{30}$ approximately degenerate Kaluza-Klein states of the $d = 11$ supergravity multiplet in the $s$ channel, that could arise from compactification of $d = 11$ supergravity on a 7-manifold with a compact hyperbolic Cartesian factor of intrinsic volume around $10^{34}$ and curvature radius an inverse TeV. A first hypothesis that the modes in the bump arise from a large degeneracy that restores agreement between the spectral staircase and the Weyl asymptotic formula immediately above the spectral gap gives a number of modes that is too large by a factor of around 60000. An alternative hypothesis that the modes in the bump arise from harmonic forms on the compact 7-manifold that are classically massless and acquire approximately equal masses from the leading quantum corrections to the CJS action naturally explains the slight reduction on a logarithmic scale in the number of modes relative to the first hypothesis, and predicts that the bump is spin 0 if the compact hyperbolic factor of large intrinsic volume is 7-dimensional, and a mixture of spins 0 and 1 if it is 5-dimensional or 3-dimensional. Even dimensions probably give too many modes. A provisional solution of the quantum-corrected $d = 11$ Einstein equations on a compact hyperbolic 7-manifold times 4 almost flat extended dimensions whose de Sitter radius can easily be as large as the observed value is considered, and a Hořava-Witten boundary is introduced to accommodate the Standard Model fields.

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1Email: chris@chrisaustin.info
1 Introduction

The ATLAS collaboration has presented in ATLAS-CONF-2010-088 [1] a search for events with at least 3 Standard Model objects in the final state, invariant mass above 800 GeV, and total transverse momentum above 700 GeV, produced in 295 per nb of proton-proton collisions at centre-of-mass energy $\sqrt{s} = 7$ TeV. Although not commented on in the report, the graphs of the numbers of events against invariant mass $M_{\text{inv}}$, which are the right-hand graphs on pages 8 and 9 of the report, show that the measured QCD background decreases more rapidly than the Monte Carlo over the range $M_{\text{inv}} = 1.1$ to 1.6 TeV, and is just half the predicted background at 1.5 to 1.6 TeV, and there is then a bump at 1.7 to 1.9 TeV, in which the number of events per energy bin increases by a factor of two.

The data is 3 sigma below the Monte Carlo in the $M_{\text{inv}} = 1.5$ to 1.6 TeV bin, and although the data is only 1 sigma above the Monte Carlo in the 1.8 to 1.9 TeV bin, the data follow a very smooth curve, with almost no scatter, which suggests that the overall significance of the combined dip and bump is well above 3 sigma. If we assume that the curve followed by the data below and above the bump is the correct QCD background, then the total number of events in the bump above this measured background is 8, so the bump is just under 3 sigma evidence for some sort of object in the $s$ channel with a mass of 1.8 TeV, that produces the excess of events over the measured background.

No update to ATLAS-CONF-2010-088 has yet been published, but since ATLAS had collected 45 per pb of collisions by the end of 2010, the dip and bump will by now be around 36 sigma, if the trend in the graphs has continued.

A natural interpretation of the more rapid decrease of the QCD background than
predicted by the Monte Carlo is that it represents the onset of Dienes-Dudas-Gherghetta
or Arkani-Hamed-Cohen-Georgi accelerated unification of the Standard Model coupling
constants [2, 3, 4], which begins at about a third of the energy at which compact extra
dimensions accessible to the Standard Model gauge fields start to become visible.

In this article I shall present rough estimates to show that the bump could arise from
about $10^{30}$ approximately degenerate Kaluza-Klein states of the $d = 11$ supergravity
multiplet in the $s$ channel, which according to a conjecture by Brooks, reported in [5],
on the lowest nonzero eigenvalue of the Laplace-Beltrami operator on generic compact
hyperbolic manifolds of large intrinsic volume, might arise from compactification of
$d = 11$ supergravity on a manifold with a compact hyperbolic Cartesian factor of
intrinsic volume around $10^{34}$, and curvature radius around an inverse TeV.

In section [3] on page 28, I shall look for solutions of the quantum-corrected Einstein
equations of $d = 11$ supergravity [6] that have 4 flat extended dimensions and a compact
7-manifold with one or more compact hyperbolic factors, in the presence of magnetic
4-form fluxes whose bilinears are on average covariantly constant. The necessary flux-
dependent terms in the dimension 8 local term in the $d = 11$ quantum effective action
are not yet known, and I shall use a simple guess for them for the case when the
flux is covariantly constant, which on reduction to 10 dimensions with NS 3-form flux
agrees with the Kehagias-Partouche conjecture [7], supported by recent calculations
by Richards [8], up to correction terms that contain factors that occur in the classical
Einstein equation in 10 dimensions.

Using this action solutions are proved impossible for the $\tilde{H}^7$, $\tilde{H}^5 \times \tilde{H}^2$, and $\tilde{H}^4 \times \tilde{H}^3$
cases even when 5-branes are added whose world-volumes lie along the 4 extended di-
densions and wrap 2-cycles of the compact manifold $\mathcal{M}^7$, and various types of search
have found no solutions in the $\tilde{H}^5 \times S^2$, $\tilde{H}^4 \times S^3$, $\tilde{H}^3 \times S^4$, and $\tilde{H}^3 \times S^2 \times S^2$ cases. Here $\tilde{H}^n$
denotes a smooth compact quotient of $n$-dimensional hyperbolic space $H^n$
by a freely-acting discrete subgroup of SO($n, 1$), which is the isometry group of $H^n$.
However at this level of approximation, where only the leading quantum corrections
to the classical action of $d = 11$ supergravity are included, the existence of solutions
is affected by field redefinitions, which should have no effect when all orders of pertur-
bation theory are included. An admissible field redefinition results in the existence
of a solution in the $\tilde{H}^7$ case, and probably also in the other cases. A Hořava-Witten
boundary to accommodate the strong/electroweak Standard Model fields is introduced
in section [4] on page 33.
The motivation for considering such solutions of quantum-corrected $d = 11$ supergravity is that the space-time we live in is approximately flat up to distances larger, by a factor of around $10^{61}$, than the radius of curvature that would be expected from combining the Standard Model (SM) of the strong/electroweak interactions with Einstein gravity in $3 + 1$ dimensions. Rigid extended objects such as the jib of a crane, a wing or the hull of an aeroplane, or an oil tanker, which are much longer in one dimension than in their other two dimensions, have a structure that makes the fullest possible use of their extension in their two short dimensions, in order to stiffen them in their long dimension. These examples suggest that the large size and flatness of the space-time we live in are supported by structures that make use of the extension of the universe in small additional spatial dimensions.

Compact hyperbolic manifolds $\tilde{H}^n$ with $n \geq 3$ have an ideal structure for stiffening the extended dimensions in this way, because when the metric on them is locally symmetric, and their curvature is fixed, they are completely rigid \cite{9,10,11,12,13}. Their shape and size is fixed by their topology, and they can be arbitrarily large. In particular, the volume $\tilde{V}$ of a compact hyperbolic manifold $\tilde{H}^n$ when its sectional curvature is equal to $-1$, which I shall call its \textit{intrinsic volume}, is fixed by its topology, and can be arbitrarily large.

Supergravity in $10 + 1$ dimensions is the unique quantum field theory that includes gravity, and whose gauge invariances force the cosmological constant to vanish in its defining dimension \cite{6,14,15,16}, and $10 + 1$ is the largest number of dimensions in which supergravity can exist if there is just one time dimension and flat space-time is required to be a solution of the classical field equations \cite{17}. Thus a solution of the quantum corrected field equations of $d = 11$ supergravity on an almost flat $\mathbb{R}^4$ times a compact hyperbolic 7-manifold $\tilde{H}^7$ of large intrinsic volume, or failing that a compact 7-manifold $\mathcal{M}^7$ with at least one Cartesian factor $\tilde{H}^n$ with $n \geq 3$ and large intrinsic volume, might provide a reasonable candidate model for the space-time we live in.

$d = 11$ supergravity can only exist on manifolds whose topology allows them to have a spin structure. Such manifolds are called \textit{spin manifolds}. All orientable manifolds of dimensions $n \leq 3$ are spin manifolds, and Cartesian products of spin manifolds are spin manifolds \cite{18}. A manifold $\mathcal{M}$ is spin if and only if its second Stiefel-Whitney class $w_2(\mathcal{M}) \in H^2(\mathcal{M};\mathbb{Z}_2)$ vanishes \cite{19}, where $H^2(\mathcal{M};\mathbb{Z}_2)$ is the second cohomology group with $\mathbb{Z}_2$ coefficients \cite{20}. Thus in the absence of additional information, the chance that an $n$-dimensional orientable manifold $\mathcal{M}^n$ with $n \geq 4$ is spin is $2^{-B_2\mathbb{Z}_2}$,
where $B_{2,\mathbb{Z}_2}$, the second Betti number with $\mathbb{Z}_2$ coefficients, is the number of $\mathbb{Z}_2$ factors in $H^2(\mathcal{M}; \mathbb{Z}_2)$. The sum of the Betti numbers of an $n$-dimensional compact hyperbolic manifold $\bar{H}^n$ with intrinsic volume $\bar{V}$ is bounded above by $b\bar{V}$ for arbitrary coefficients, where $b$ is a constant that depends only on $n$ and the coefficient ring [21, 22], and the number of topologically distinct $\bar{H}^n$ with intrinsic volume $\bar{V}$ less than a fixed value $V$ is finite for fixed $n \geq 4$ [23], but grows with $V$ as $V^{cV}$ for sufficiently large $V$, where $c > 0$ is a constant that depends only on $n$ [24]. The Davis manifold, which is an $\bar{H}^4$, is spin [25, 26], so we may reasonably expect that for all $n \geq 2$, there are spin $\bar{H}^n$, and that for $n \geq 4$, their number will grow with $V$ as $V^{cV}$ for sufficiently large $V$, while for $n = 3$, their number is already infinite for $V \geq 2.03$ [27, 12].

Supergravity does not couple to any matter fields in $10 + 1$ smooth dimensions, but if the $10 + 1$ dimensional space-time has a $9 + 1$ dimensional smooth “boundary”, which could perhaps better be described as a flexible mirror, because all the fields on one side of the mirror are exactly copied, up to sign, on the other side of the mirror, then supergravity in the $10 + 1$ dimensional “bulk” couples to a supersymmetric Yang-Mills multiplet [28, 29], with gauge group $E_8$, on the $9 + 1$ dimensional “boundary”. The Yang-Mills multiplet is adjacent to the flexible mirror, but infinitesimally displaced from it, so that it has its own reflection infinitesimally on the other side of the mirror [30]. This is called Hořava-Witten (HW) theory [31, 32, 33, 34, 35, 36], and I shall assume that we live on an HW boundary, while the graviton is part of the supergravity multiplet in the $10 + 1$ dimensional bulk.

An $n$-dimensional compact hyperbolic space $\bar{H}^n$, $n \geq 2$, is a quotient of $n$-dimensional hyperbolic space, $H^n$, by a discrete subgroup $\Gamma$ of the symmetry group $\text{SO}(n, 1)$ of $H^n$, such that $\Gamma$ acts freely on $H^n$, which means that no element of $\Gamma$ other than the identity leaves any point of $H^n$ invariant. This is equivalent to the requirement that $\Gamma$ have no nontrivial finite subgroup, which in the language of group theory is expressed by saying that $\Gamma$ has no torsion. The construction of a large family of $\bar{H}^n$, called arithmetic manifolds [37], is reviewed in [38] and section 3 of [39]. Non-arithmetic manifolds were constructed from suitable pairs of arithmetic manifolds for all $n \geq 2$ in [40]. It is the rapid growth with intrinsic volume $\bar{V}$ of the number of topologically distinct finite coverings of the non-arithmetic manifolds that results in the $V^{cV}$ growth of the number of topologically distinct $\bar{H}^n$ with $\bar{V} \leq V$ [24].

For the special case of $n = 3$, an infinite number of topologically distinct $\bar{H}^3$ can be constructed from each finite-volume cusped hyperbolic 3-manifold $\hat{H}^3$ by applying
an operation called Dehn filling to each cusp, and each $\bar{H}^3$ constructed in this way has $\bar{V} < \hat{V}$, where $\hat{V}$ is the intrinsic volume of the initial $\hat{H}^3$ [12]. A large number of $\bar{H}^3$ of small $\bar{V}$ have been constructed and catalogued in this way, and their properties can be studied using computer programs [41, 42, 43]. The Cartesian product of two $\bar{H}^3$ of small $\bar{V}$ might be a possible topology for the compact Cartesian factor of the HW boundary that accommodates the Standard Model fields. A cusp of a finite-volume hyperbolic $n$-manifold $\hat{H}^n$ is a region of the manifold whose topology is the Cartesian product of an infinite half-line and a flat $(n-1)$-manifold, for example an $(n-1)$-torus, such that the area of the cross section decreases so rapidly along the half-line that the volume of the cusp is finite. The number of cusps is always finite.

The $n$-dimensional hyperbolic space $H^n$ can be realized as the hypersurface $t^2 - \vec{x}^2 = 1$, $t > 0$, in $n + 1$ dimensional Minkowski space. Using spherical polar coordinates for $\vec{x}$, the equation reduces to $t^2 - r^2 = 1$, $t > 0$, so we can choose $t = \cosh \rho$, $r = \sinh \rho$, and $\rho$ is then the geodesic distance of the point $(\rho, \theta_1, \ldots, \theta_{n-1})$ from the origin [44].

The Riemann tensor of $H^n$ has the form:

$$ R_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}, \quad (1) $$

where $g_{ij}$ is the metric, which means that $H^n$ has constant sectional curvature equal to $-1$. The Ricci tensor of $H^n$ is then $R_{ij} = -(n-1)g_{ij}$. The Riemann tensor of a $d$-dimensional space or space-time, with metric $G_{IJ}$, is defined in general by:

$$ R_{IJK}^L = \partial_I \Gamma_{J}^{K} L - \partial_J \Gamma_{I}^{K} L + \Gamma_{I}^{K} M \Gamma_{J}^{M} L - \Gamma_{J}^{K} M \Gamma_{I}^{M} L, \quad (2) $$

and the Ricci tensor and scalar are defined by $R_{IJK} = R_{KIJ}$, and $R = G^{IJ}R_{IJK}$. The standard Christoffel connection is $\Gamma_{I}^{J} K = \frac{1}{2}G^{JL}(\partial_I G_{LK} + \partial_K G_{LI} - \partial_L G_{IK})$.

I shall use units such that $\hbar = c = 1$. The gravitational coupling constant in 11 dimensions is $\kappa_{11}$. The metric is mostly +. Coordinate indices $I, J, K, \ldots$ run over all 11 dimensions, and coordinate indices $\mu, \nu, \sigma, \ldots$ are tangent to the four observed space-time dimensions. The bosonic fields of supergravity in 11 dimensions are the graviton $G_{IJ}$, and a 3-index antisymmetric tensor Abelian gauge field $C_{IJK}$, called a 3-form gauge field, with 4-form field strength $H_{IJKL} = \partial_I C_{JKL} - \partial_J C_{IKL} + \partial_K C_{ILJ} - \partial_L C_{IJK}$.

The bosonic part of the classical action of $d = 11$ supergravity is [6]:

$$ S^{(\text{bos})}_{\text{CJS}} = \frac{1}{2\kappa^2_{11}} \int_{\mathcal{B}} d^{11}x e \left( R - \frac{1}{48} H_{IJKL} H^{IJKL} - \frac{1}{144^2} \epsilon_{I_1 \ldots I_{11}} C_{I_1 I_2 I_3} H_{I_4 \ldots I_7} H_{I_8 \ldots I_{11}} \right), \quad (3) $$

6
where $B$ means the bulk. The sign of the Einstein term follows from the Riemann and Ricci tensor conventions stated above, and the normalization of the 4-form kinetic term, and the definition of $H_{IJKL}$ given above, are chosen to agree with [45]. $e = \sqrt{-G}$ is the determinant of the vielbein $e_{IJ}$, where $G$ is the determinant of the metric $G_{IJ}$, and the antisymmetric tensor $\epsilon^{I_1 \ldots I_{11}}_{J_1 \ldots J_{11}}$ is related to the SO $(10, 1)$ invariant tensor $\hat{\epsilon}^{I_1 \ldots I_{11}}_{J_1 \ldots J_{11}}$, with components $0, \pm 1$, by $\epsilon^{I_1 \ldots I_{11}}_{J_1 \ldots J_{11}} = e^{I_1}_{I_1} \ldots e^{I_{11}}_{I_{11}} \hat{\epsilon}^{J_1 \ldots J_{11}}_{J_1 \ldots J_{11}}$. Hatted indices are local Lorentz indices. $C_{IJK}$ here is related to $C_{IJK}$ of [32, 33, 34, 35, 36] by $C_{IJK}$ (here) = $6\sqrt{2}C_{IJK}$ (HW), and $H_{IJKL}$ is related to $G_{IJKL}$ of [32, 33, 34, 35, 36] by $H_{IJKL} = \sqrt{2}G_{IJKL}$.

I assume the unperturbed metric in 11 dimensions has the form:

$$ds^2_{11} = G_{IJ}dx^I dx^J = a(x^C)^2 \eta_{\mu\nu} dx^\mu dx^\nu + h_{AB} dx^A dx^B$$  \hspace{1cm} (4)

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the metric on $(3+1)$-dimensional Minkowski space, and $h_{AB}$ is the metric on the compact 7-dimensional manifold-with-boundary $\mathcal{M}^7$. Coordinate indices $A, B, C, \ldots$ are tangential to $\mathcal{M}^7$. The “warp factor” $a(x^C)$ is equal to a constant value $A$ everywhere on $\mathcal{M}^7$ except in the immediate vicinity of the HW boundary, and it is equal to 1 on the HW boundary. I shall show in section 4, starting on page 35, that a consistent semiclassical treatment of the boundary region is possible if $A$ is $< 1$ but comparable to 1, for example $A = 0.7$. I assume that $\mathcal{M}^7$ has an $n$-dimensional compact hyperbolic Cartesian factor $\tilde{H}^n$ of large intrinsic volume $\tilde{V}_n$, which has one boundary of small intrinsic area $\tilde{A}_{n-1}$, where $\tilde{A}_{n-1}$ is defined using the metric of sectional curvature $-1$ on $\tilde{H}^n$, and that the 10-dimensional HW boundary is the Cartesian product of that boundary, and the remaining Cartesian factor $\mathcal{M}^{7-n}$ of $\mathcal{M}^7$, and the 4 extended dimensions. $\tilde{H}^n$ is constructed by cutting a complete $n$-dimensional compact hyperbolic manifold $\tilde{H}^n$ of large intrinsic volume $\tilde{V}^n$ along a small $(n-1)$-cycle that separates it into two disconnected parts. The smaller of the two parts is discarded, and the other becomes $\tilde{H}^n$.

The Einstein action in the 4 extended dimensions has the form:

$$S_{\text{Ein}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} (g) ,$$  \hspace{1cm} (5)

where $g_{\mu\nu}$ differs from $\eta_{\mu\nu}$ by a small perturbation, that depends on the coordinates $x^\sigma$ on the 4 extended dimensions, but not on the coordinates $x^A$ on $\mathcal{M}^7$. $G_N$ is Newton’s constant, with the value [46]:

$$G_N = 6.7087 \times 10^{-33} \text{ TeV}^{-2} ,$$  \hspace{1cm} (6)
so that $\sqrt{G_N} = 8.1907 \times 10^{-17}$ TeV$^{-1} = 1.6160 \times 10^{-35}$ metres. Comparing with (3), and noting that $R_{ij}^{K_L}$ and hence $R_{ij}$ are unaltered by rescaling the metric by a constant factor, so that $\sqrt{-G}G^{\mu\nu}R_{\mu\nu}(G) = A^4 \sqrt{-g}\sqrt{h}g^{\mu\nu}R_{\mu\nu}(g)$ everywhere on $\mathcal{M}^7$ except in the immediate vicinity of the HW boundary, where $G_{ij}$, here represents the metric obtained from (4) by replacing $\eta_{\mu\nu}$ by $g_{\mu\nu}$, we find, in the approximation of neglecting the volume of the region where $a(x^C)$ differs appreciably from $A$, that the volume $V_7 \equiv \int_{\mathcal{M}^7} d^7x^C \sqrt{h}$ of $\mathcal{M}^7$ is given by \[ A^2 V_7 \frac{1}{2\kappa_{11}^2} = \frac{1}{16\pi G_N} \quad (7) \]

The experimental limits on the gravitational coupling constant in $D$ dimensions are expressed in terms of a mass $M_D$, such that for $D = 11$, $M_{11} = (2\pi)^{7/9} \kappa_{11}^{-2/9} = 4.1764 \kappa_{11}^{-2/9}$. The latest limit on $M_{11}$ for 7 flat extra dimensions, from [51], including bounds from non-observation of effects of virtual graviton exchange and graviton emission at the LHC, is that $M_{11}$ is larger than about 1.5 TeV, so that $\kappa_{11}^{-2/9}$ is larger than about 0.36 TeV. The limit might be less stringent for hyperbolic large extra dimensions, since the light Kaluza-Klein modes will be far fewer or absent. The dimension $n$ of the large intrinsic volume hyperbolic Cartesian factor $\tilde{H}^n$ of $\mathcal{M}^7$ might be less than 7, but the lower limit on $M_D$ found in [51] is about 1.5 TeV for all $D \geq 6$. If $\kappa_{11}^{-2/9}$ was about 0.36 TeV and $A$ was 1, then from (6) and (7), $V_7$ would be about $5.8 \times 10^{34}$ TeV$^{-7}$.

The classical field equations of $d = 11$ supergravity do not have any solutions with 4 flat extended dimensions if $\mathcal{M}^7$ has a compact hyperbolic factor, for any arrangement of 4-form fluxes such that the flux bilinears are on average covariantly constant, but in section 3 I shall show that when the leading quantum corrections to the classical action are taken into account, together with the freedom to make redefinitions of the fields of the form $G_{ij}, C_{IJK} \rightarrow G_{ij} + X_{ij}, C_{IJK} + Y_{IJK}$, where $X_{ij}$ and $Y_{IJK}$ are polynomials in the fields and their derivatives, with sufficiently small coefficients, then there is a solution with an almost flat $\mathbb{R}^4$ times a compact hyperbolic 7-manifold $\tilde{H}^7$ of large intrinsic volume, with magnetic 4-form fluxes wrapping 4-cycles of the $\tilde{H}^7$. Field redefinitions of this type are like a change of coordinates in “field space”, so they do not change the physical content of the theory. They do not alter the $S$-matrix [52, 53, 54], and when the leading higher-order corrections to the classical CJS action are calculated by requiring anomaly cancellation on 5-branes [55, 56, 57, 58, 59] and in Hořava-Witten theory [32, 60, 61, 62, 30, 64, 58, 65, 35], and supersymmetry
the result is only determined up to the freedom to add arbitrary linear combinations of terms that vanish when the classical field equations are satisfied, because the coefficients of such linear combinations of terms can be adjusted arbitrarily by small amounts, by field redefinitions of this type.

The leading quantum corrections are local terms that depend on the curvature of $\mathcal{M}^7$ but not directly on its topology, so the curvature radius $B$ of the $\bar{H}^7$ is around $\kappa_{11}^{2/9}$. The value of $B$ depends on the field redefinition parameter, but using the “principle of minimal sensitivity” advocated for dealing with the renormalization scheme dependence of low order calculations in perturbative QCD [70], the best value of $B$ is around $0.43 \kappa_{11}^{2/9}$. Thus if $\kappa_{11}^{2/9}$ was about 0.36 TeV, $B$ would be around 1.2 TeV$^{-1}$, so if $A$ was 1, so that $V_7$ was about $5.8 \times 10^{34}$ TeV$^{-7}$, the intrinsic volume $\bar{V}$ of the $\bar{H}^7$ would be around $1.6 \times 10^{34}$. The $\mathbf{R}^4$ could be exactly flat but for the quantization of the 4-form fluxes [71, 72], and for an $\bar{H}^7$ of intrinsic volume $\bar{V}^7 \sim 10^{34}$, the de Sitter radius of the $\mathbf{R}^4$ can easily be as large as the observed de Sitter radius, which is 16.0 Gyr $= 1.51 \times 10^{26}$ metres $= 0.94 \times 10^{61}\sqrt{G_N}$ [73, 74, 75].

The diameter $L$ of a compact manifold is by definition the maximum over all pairs of points of the manifold of the shortest geodesic distance between them, and I shall call $\bar{L}$, the diameter of a compact hyperbolic manifold $\bar{H}^n$ when its sectional curvature is equal to $-1$, its intrinsic diameter. If $\bar{H}^n$ is reasonably isotropic, in the sense that it has a fundamental domain in $H^n$ that is approximately spherical, then using spherical polar coordinates on $H^n$ as above, the intrinsic volume $\bar{V}_n$ of $\bar{H}^n$ is approximately related to its intrinsic diameter $\bar{L}$ by

$$\bar{V}_n \simeq S_{n-1} \int_0^{L/2} \sinh^{n-1} \rho d\rho,$$

where $S_{n-1}$ is the area of the $(n-1)$-sphere of unit radius. Thus if $\bar{H}^n$ is reasonably isotropic and has large intrinsic volume $\bar{V}_n$, the relation between $\bar{V}_n$ and $\bar{L}$ is approximately

$$\bar{V}_n \simeq \frac{S_{n-1}}{2^{n-1}(n-1)} e^{(n-1)\bar{L}/2}.$$  

Thus for reasonably isotropic $\bar{H}^7$ with volume $V_7 \simeq 5.8 \times 10^{34}$ TeV$^{-7}$ and curvature radius $B$ around 1.2 TeV$^{-1}$, so that $\bar{V}_7 \simeq 1.6 \times 10^{34}$, we find from $S_6 = \frac{16}{15} \pi^3$ that the intrinsic diameter of $\bar{H}^7$ is $\bar{L} \simeq 27$, so its actual diameter is $L \simeq 32$ TeV$^{-1}$.

On a flat Riemannian manifold of diameter $L$, the lowest non-zero eigenvalue of the negative of the Laplace-Beltrami operator $\Delta \equiv \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j \right)$, which is the generally
covariant form of the Laplacian, is typically \((\frac{2\pi}{L})^2\). However there is some evidence that the lowest non-zero eigenvalue \(\lambda_1\) of \(-\Delta\) on a reasonably isotropic compact hyperbolic manifold \(\bar{H}^n\) with sectional curvature \(-1\) and large diameter \(\bar{L}\) will be \(\sim 1\) rather than \(\sim \frac{1}{\bar{L}^2} [5]\). Firstly, the spectrum of \(-\Delta\) on \(H^n\), which is completely continuous, starts at \(\lambda = \frac{(n-1)^2}{4}\), rather than 0 [76 77]. Secondly, for \(n = 2\), Brooks and Makover have constructed very large families of compact Riemann surfaces with sectional curvature \(-1\) and large genus, all of which have \(\lambda_1 > \frac{3}{16} [78 79 80 81]\).

And thirdly, we can use Cheeger’s inequality [82, 83], which states that for a compact Riemannian manifold \(\mathcal{M}\), the lowest non-zero eigenvalue \(\lambda_1\) of \(-\Delta\) on \(\mathcal{M}\) is bounded below by:

\[
\lambda_1 (\mathcal{M}) \geq \frac{h^2 (\mathcal{M})}{4}, \tag{10}
\]

where \(h (\mathcal{M})\), called the Cheeger isoperimetric constant of \(\mathcal{M}\), is defined by:

\[
h (\mathcal{M}) \equiv \min_{\mathcal{N}} \left( \frac{A (\mathcal{N})}{\min (V (\mathcal{M}_1), V (\mathcal{M}_2))} \right), \tag{11}
\]

where \(\mathcal{N}\) denotes a hypersurface in \(\mathcal{M}\) that divides \(\mathcal{M}\) into two disconnected parts \(\mathcal{M}_1\) and \(\mathcal{M}_2\), \(V (\mathcal{M}_i)\) is the volume of \(\mathcal{M}_i\), and \(A (\mathcal{N})\) is the area of \(\mathcal{N}\), or in other words, if \(\mathcal{M}\) is \(n\)-dimensional, the \((n - 1)\)-volume of \(\mathcal{N}\), calculated in the given metric on \(\mathcal{M}\).

To apply Cheeger’s inequality to a large volume compact hyperbolic manifold \(\bar{H}^n\) with sectional curvature \(-1\) that is reasonably isotropic, in the sense that it has an approximately spherical fundamental domain, we note that a totally geodesic hypersurface through the centre of the approximately spherical fundamental domain will not, in general, divide the manifold into two disconnected parts, because the surface of the fundamental domain consists of a large number of \((n - 1)\)-dimensional polyhedral faces, each of which is identified with another face typically roughly on the opposite side of the fundamental domain. To minimize the value of \(\frac{A (\mathcal{N})}{\min (V (\mathcal{M}_1), V (\mathcal{M}_2))}\), we first look for minimal area topologically non-trivial hypersurfaces \(\mathcal{N}\) that divide \(\bar{H}^n\) into two disconnected parts of approximately equal volume, and we see that a typical such \(\mathcal{N}\) will consist of one totally geodesic hypersurface through the centre of the fundamental domain, plus polyhedral pieces that cover exactly half the surface of the fundamental domain. Using spherical polar coordinates as before, we see that for \(\bar{H}^n\) of large intrinsic volume and diameter \(\bar{L}\), the area of the part of \(\mathcal{N}\) that covers half the surface of the fundamental domain is

\[
A (\mathcal{N}_{\text{half sphere}}) \approx \frac{S_{n-1}}{2n} e^{(n-1)\frac{\bar{L}}{2}} \approx \frac{(n-1)}{2} \bar{V}_n, \tag{12}
\]
and the area of the part of $\mathcal{N}$ that is a totally geodesic hypersurface through the origin is negligible in comparison to this, suggesting that the Cheeger constant of $\hat{H}^n$ is $h(\hat{H}^n) \simeq n - 1$. As a check on this, we consider spherical $\mathcal{N}$ of radius $\rho$ centred at the centre of the fundamental domain. For all $\rho \gg 1$ the area of such an $\mathcal{N}$ is $n - 1$ times the volume it encloses, so for all $\rho \gg 1$, $\rho \leq L - \frac{2n^2}{n-1}$, the value of $\frac{A(\mathcal{N})}{\min(V(M_1),V(M_2))}$ is again $n - 1$. Thus it looks likely that for all reasonably isotropic compact hyperbolic manifolds $\hat{H}^n$ with sectional curvature $-1$ and large intrinsic volume, the lowest nonzero eigenvalue $\lambda_1$ of $-\Delta$ on $\hat{H}^n$ will by Cheeger’s inequality be bounded below by:

$$\lambda_1 \geq \frac{(n-1)^2}{4}$$

(13)

This is the value of $\lambda$ at which the spectrum starts on $H^n$ \[84\], and the lower bound on $\lambda$ for $H^n$ could be obtained from Cheeger’s inequality using spheres of radius $\rho \gg 1$ as above, if Cheeger’s inequality could be applied to $H^n$.

Furthermore, considering now $\hat{H}^n$ with sectional curvature $-1$ and large intrinsic volume that depart somewhat from being reasonably isotropic, there is for $n = 3$ a family of manifolds called Bianchi manifolds of arbitrarily large but finite intrinsic volume, that depart to the greatest possible extent from being reasonably isotropic, namely they have a finite number of cusps as described just before (1) above, for which $\lambda_1$ has been bounded below by $21/25 = 0.84$ and is conjectured to be equal to 1 \[85, 86, 87, 88\]. The diameter of a cusped hyperbolic $n$-manifold $\hat{H}^n$ of finite volume and sectional curvature $-1$ is infinite, and each cusp contributes a continuous part to the spectrum of $-\Delta$ on $\hat{H}^n$. However just as for $H^n$, the continuous part of the spectrum starts at $\lambda = \frac{(n-1)^2}{4}$, rather than 0 \[89, 90\]. For $n = 3$ the Dehn filling construction produces from any cusped $\hat{H}^n$ of finite volume an infinite sequence of compact $\hat{H}^n_i$ that converge to $\hat{H}^n$ as $i \to \infty$ in the sense that the intrinsic volumes converge, and the $\hat{H}^n_i$ look almost identical to $\hat{H}^n$ except for regions that move further and further out along the cusps of $\hat{H}^n$ as $i \to \infty$ \[12\], and the eigenvalues of $-\Delta$ below the bottom of the continuous spectrum of $\hat{H}^n$ are limits of the eigenvalues of the $\hat{H}^n_i$ \[91, 92\]. Thus for $n = 3$ there are infinite sequences of compact $\hat{H}^n_i$ with sectional curvature $-1$ that converge to fixed cusped Bianchi manifolds, which have arbitrarily large but finite intrinsic volumes, and these $\hat{H}^n_i$ have $\lambda_1$ converging to a number that is $\geq 0.84$ and conjectured to be 1, while their diameters tend to $\infty$ and their intrinsic volumes tend to fixed finite values, that can, however, be arbitrarily large.

So it seems possible, at least for $n = 2$ and $n = 3$, that a compact hyperbolic
manifold $\bar{H}^n$ with sectional curvature $-1$ and arbitrarily large intrinsic volume will have $\lambda_1 \simeq \frac{(n-1)^2}{4}$ even if it departs substantially from being isotropic. This is in agreement with a conjecture by Brooks reported in [3]. I shall now assume that this is the case, which means that classically, the lightest massive spin 2 Kaluza-Klein graviton would have mass $\sim \kappa_{11}^{-2/9}$ and thus around a TeV in the geometry considered above.

In contrast with the spectrum of $-\Delta$, the spectrum of the Dirac operator on the $n$-dimensional hyperbolic space $H^n$ is the whole real line [93, 94]. Thus in view of the correspondence between the spectrum of $-\Delta$ on $H^n$ and on a compact hyperbolic manifold $\bar{H}^n$ of large intrinsic volume $\bar{V}$ and intrinsic diameter $\bar{L}$ as considered above, it seems likely that classically, the lightest gravitino modes would have masses $\sim \frac{2\pi}{L\bar{L}}$, and thus around 0.2 TeV in the geometry considered above, but just as in the case of large flat extra dimensions, their effects would not show up at the LHC below energies around $M_{11}$, which is about 1.5 TeV in this example.

The $\mathbb{R}^4_{\text{flat}} \times \bar{H}^7$ solution of the quantum-corrected field equations of $d = 11$ supergravity found in section 3 depends essentially on the leading quantum corrections to the classical action, so there will be corresponding corrections $\sim \kappa_{11}^{-2/9}$ to the masses of all the Kaluza-Klein modes. I shall assume provisionally that the effect of these corrections is to increase the masses of the Kaluza-Klein modes. This is particularly significant for the 3-form gauge field $C_{IJK}$, since a compact hyperbolic manifold $\bar{H}^n$ of large intrinsic volume $\bar{V}$ has Betti numbers $B_1, \ldots, B_{n-1}$ that in a particular family of examples are bounded below by constants times powers $> 0$ and $< 1$ of $\bar{V}$ [95], and for $n = 7$ these would lead classically to $B_1$ massless 2-form Abelian gauge fields, each equivalent in $3 + 1$ dimensions to a massless scalar [96, 97], $B_2$ massless Abelian vector gauge fields, and $B_3$ massless scalars. There would also be $B_1$ additional massless Abelian vector gauge fields arising from $G_{1J}$. I shall assume provisionally that all these classically massless modes acquire masses $\sim \kappa_{11}^{-2/9}$ from the leading quantum corrections.

If we write the eigenvalues of $-\Delta$ on an $n$-dimensional compact manifold $\mathcal{M}^n$, with volume $V_n$, as $k^2$, then for large $\bar{k}$, the number of modes with $k \leq \bar{k}$ is approximately given by the Weyl asymptotic formula as

$$\frac{V_n}{(2\pi)^n} S_{n-1} \bar{k}^n, \quad (14)$$

which corresponds to a density of states $\frac{V_n}{(2\pi)^n}$ in momentum space. In numerical studies of the spectrum of $-\Delta$ on compact hyperbolic 3-manifolds of small intrinsic volume,
Inoue found that the Weyl asymptotic formula is actually approximately valid down to $\tilde{k}^2 \simeq \lambda_1$, the lowest non-zero eigenvalue, so that if $\lambda_1$ occurs at a larger value of $\tilde{k}$ than would be expected from the Weyl formula, there is a degeneracy or approximate degeneracy of eigenvalues near $\lambda_1$, that restores agreement with the Weyl formula for $\tilde{k}^2$ above $\lambda_1$.

It seems reasonable to expect that a similar approximate degeneracy of eigenvalues near $\lambda_1$, that restores agreement with the Weyl asymptotic formula (14) for $\tilde{k}^2 > \lambda_1$, will also occur for compact hyperbolic manifolds of large intrinsic volume, since the number of modes up to $\tilde{k} = \sqrt{\lambda_1}$ given by (14) is already large, comparable to the intrinsic volume $\bar{V}_n$. I shall provisionally assume that this happens, which means that for the geometry described above, we expect in the region of $10^{34}$ approximately degenerate Kaluza-Klein modes of the gravitational multiplet, close to a mass $\simeq \frac{n-1}{2A} = \frac{3}{B} \simeq 2.5$ TeV. In partial support of this hypothesis, every $\bar{H}^n$ has pairs of finite covers of arbitrarily large volume ratio, whose sets of eigenvalues of $-\Delta$, ignoring multiplicities, are identical [99], so since the Weyl asymptotic formula (14) is certainly valid for sufficiently large $\tilde{k} = \sqrt{\lambda}$, every $\bar{H}^n$ has finite covers whose eigenvalues have arbitrarily large multiplicities, for sufficiently large $\tilde{k}$.

I shall now provisionally assume that the bump at 1.7 to 1.9 TeV seen in the right-hand graphs on pages 8 and 9 of ATLAS-CONF-2010-088 arises from the production and decay of these approximately $10^{34}$ approximately degenerate gravitational Kaluza-Klein modes in the $s$ channel, and give rough estimates to test this interpretation. I shall show in section 4, starting on page 35, that a mode on $\bar{H}^7$ of intrinsic mass $k$ has mass $\frac{Ak}{B}$ as measured on the HW boundary, where $B \simeq 0.43\kappa_{11}^{2/9}$ is the curvature radius of the $\bar{H}^7$, and $A$ is the value of the “warp factor” $a(x^C)$ everywhere on the $\bar{H}^7$ except in the immediate vicinity of the HW boundary, as described after (14). Thus if the bump centred at 1.8 TeV corresponds to approximately degenerate modes of intrinsic mass $\frac{n-1}{2} = 3$ on the $\bar{H}^7$, and $\kappa_{11}^{-2/9}$ is at its current lower bound for 2 or more flat extra dimensions, $\kappa_{11}^{-2/9} \simeq 0.36$ TeV [51], as in the example above, then $A = \frac{1.8 \text{ TeV}}{2.5 \text{ TeV}} = 0.72$. This then revises $V_7$ from the value $5.8 \times 10^{34}$ TeV$^{-7}$ calculated from (17) assuming $A = 1$, to $1.1 \times 10^{35}$ TeV$^{-7}$, and the intrinsic volume $\bar{V}_7$ is revised to $3.1 \times 10^{34}$.

For an order of magnitude estimate, I shall treat each Kaluza-Klein mode of the graviton multiplet, with mass around 1.8 TeV, as an independent scalar particle $\varphi$. Then the total cross section for the process $gg \rightarrow \varphi \rightarrow uu$, for example, where $g$ represents a gluon, is given near 1.8 TeV by the Breit-Wigner cross-section [100], and
in order of magnitude is:

\[ \sigma (gg \to \varphi \to u\bar{u}) \sim \frac{\Gamma_{gg}\Gamma_{u\bar{u}}}{E^2} \frac{1}{(E - M_\varphi)^2 + \Gamma^2/4}, \] (15)

where \( \Gamma_{gg} \) and \( \Gamma_{u\bar{u}} \) are the partial widths for \( \varphi \) to decay to \( gg \) and \( u\bar{u} \) respectively, \( E \) is the invariant mass of the \( gg \) system, \( M_\varphi \simeq 1.8 \text{ TeV} \) is the mass of \( \varphi \), and \( \Gamma \) is the total width of \( \varphi \). The partial widths are estimated in order of magnitude by [101]:

\[ \Gamma_{gg} \sim \Gamma_{ggg} \sim \ldots \sim \Gamma_{u\bar{u}} \sim \Gamma_{gu\bar{u}} \sim \ldots \sim \Gamma_{\nu\bar{\nu}} \sim M_\varphi^3 G_N \simeq 3.9 \times 10^{-32} \text{TeV} \] (16)

The only factor in the cross-section (15) that varies significantly with energy over the energy range of interest is the final factor, and this arises from the squared magnitude of the energy-dependent factor:

\[ \frac{1}{E - M_\varphi + i\Gamma/2} \] (17)

in the corresponding amplitude. The Kaluza-Klein graviton is not present in the final state, so we have to sum the amplitude over the contributions of all the different Kaluza-Klein graviton modes, all of which give the same contribution to the amplitude, apart from slight differences in their masses.

The number of bosonic modes in the \( d = 11 \) supergravity multiplet is 128, so the number of approximately degenerate bosonic Kaluza-Klein states of mass \( \simeq 1.8 \text{ TeV} \), which is the number of states up to 1.8 TeV as given by the Weyl asymptotic formula (14), is around:

\[ 128 \frac{\tilde{V}_7}{7(2\pi)^7} S_6 3^7 \simeq 1.1 \times 10^{35} \] (18)

where I used \( \tilde{V}_7 \simeq 3.1 \times 10^{34} \) from just before (15), and \( S_6 = \frac{16}{15} \pi^3 \), and (13) with \( n = 7 \). I shall assume that these states are distributed over a smooth peak in the density of states from about 1.7 TeV to 1.9 TeV.

The total width \( \Gamma \) of \( \varphi \) is the product of the partial width (16) and the number of Standard Model states, which is 36, so \( \Gamma \) is extremely small. We can replace the sum over the individual Kaluza-Klein modes by an integral over their masses [51], and to a first approximation, when convoluted with a smooth density of states, the energy-dependent factor (17) is effectively \( \simeq -i\pi\delta (E - M) \), because the \( i\Gamma \) means that the integration path has to go around the singularity in the lower half of the complex \( M \) plane. Choosing the integration path to be along the real axis except for a small semicircle centred at \( M = E \), the contributions from the real axis approximately
cancel, and the semicircle gives $-i\pi$ times the density of states evaluated at $M = E$. Thus after summing the amplitude over the Kaluza-Klein modes, the energy-dependent factor in the amplitude is approximately replaced by:

$$-1.1 \times 10^{35} i\pi f(E),$$

where $f(E)$ has a smooth peak from about 1.7 TeV to 1.9 TeV, and $\int f(E)\,dE = 1$. Thus the cross-section is:

$$\sigma(gg \rightarrow \varphi \rightarrow u\bar{u}) \sim \frac{\Gamma_{gg}\Gamma_{u\bar{u}}}{E^2} \times 1.2 \times 10^{70} \pi^2 f(E)^2 \sim 5.6 \times 10^7 f(E)^2. \quad (19)$$

A monoenergetic high energy beam of protons with $N$ protons of energy $E$ per unit area per unit time is equivalent to a beam of partons, such that the number of $u$ quarks per unit area per unit time with energy between $xE$ and $(x + dx)E$ is $f_u (x) \, dx$, and similarly for the other types of parton, where $f_p (x), p = u, d, \bar{u}, \bar{d}, \ldots$ are the parton distribution functions (PDFs). The PDFs evolve logarithmically with $Q^2$, the square of the momentum transferred in a scattering process, and for a rough estimate I shall use the gluon distribution $f_g (x)$ from the plot [102] with $Q^2 = 10^4$ GeV$^2$. To produce $\varphi$ with mass 1.8 TeV at rest with 3.5 TeV per proton beam, each gluon needs $x = 0.26$. If the 4-momenta of the beams are $(P, 0, 0, P)$ and $(P, 0, 0, -P)$ and the momentum fractions of the gluons are $x_1$ and $x_2$, then their Mandelstam $s$ is $P^2 ((x_1 + x_2)^2 - (x_1 - x_2)^2) = 4P^2 x_1 x_2$. From the plot we find that $f_g (x) \approx 0.060 x^{-2.17}$ for $0.05 \leq x \leq 0.2$, but substantially smaller than this for $x \geq 0.3$. Then the total cross section to produce a $\varphi$ within the bump is roughly:

$$\int_0^{0.3} dx_1 \int_0^{0.3} dx_2 0.060^2 (x_1 x_2)^{-2.17} \times 5.6 \times 10^7 f (2P \sqrt{x_1 x_2})$$

$$\approx \frac{2.0 \times 10^5}{(0.2 \text{ TeV})^2} \times \int_0^{0.3} dx_1 \int_0^{0.27} dx_2 \frac{2x \, dx}{x^{4.34}} \approx 3.0 \times 10^6 \text{ nb}, \quad (21)$$

where I approximated $f(E)$ as a constant from 1.7 TeV to 1.9 TeV and 0 outside this interval. Multiplying by $4\pi$, and by 75 for the number of Standard Model helicity states, and by $\frac{1}{8}$ for the probability that the two initial gluons can form a colour singlet, and by $\frac{1}{4}$ for the average over the helicity states of the initial gluons, gives a final estimate of $8.8 \times 10^7 \text{ nb}$ for the total cross section to produce a $\varphi$ within the bump that decays to two Standard Model objects, if $\kappa_{11}^{-2/9} \approx 0.36$ TeV, and all the bosonic modes of the $d = 11$ supergravity multiplet that according to the Weyl asymptotic
formula (14) would have masses up to 1.8 TeV, are in fact approximately degenerate, with masses in the range 1.7 TeV to 1.9 TeV.

To reduce the background, the ATLAS measurement selected events with at least 3 Standard Model objects in the final state, and total transverse momentum above 700 GeV. However the partial widths for \( \varphi \) to decay to 2, 3, or 4 Standard Model states are all of the same order of magnitude (16). And for an order of magnitude estimate, the decay of \( \varphi \) is effectively isotropic, and the 700 GeV lower limit on the total transverse momentum will not result in missing a high percentage of the decays of the 1.8 TeV \( \varphi \) into 3 or 4 Standard Model states. The number of different types of 3 object final states is roughly 10 times the number of different types of 2 object final states, and omitting this factor could very roughly compensate for neglecting the effect of the 700 GeV lower limit on the total transverse momentum. Thus we find an estimate of \( 10^8 \) nb for the total cross section to produce a \( \varphi \) within the bump that is counted as an event in the ATLAS measurement, if \( \kappa_{11} = 2/9 \approx 0.36 \) TeV, and all the bosonic modes of the \( d = 11 \) supergravity multiplet that according to (14) would have masses up to 1.8 TeV, are in fact approximately degenerate, with masses in the range 1.7 TeV to 1.9 TeV. So in 295 per nb of proton-proton collisions there should be about \( 3 \times 10^{10} \) events above background in the bump. If we assume that the curve followed by the data below and above the bump is the correct QCD background, then the total number of events in the bump above this measured background is 8.

Thus the estimate of the number of modes in the bump, (18), is too large by a factor of around 60000, so the correct number is around \( 1.8 \times 10^{30} \). From (7) and (18) and the relation \( V_7 = B^2 \tilde{V}_7 \), where \( B \approx 0.43 \kappa_{11}^{2/9} \) is the curvature radius of the \( \tilde{H}^7 \), and the relation \( \frac{3A}{B} = 1.8 \) TeV, where \( A \) is the value of the “warp factor” \( a (x^C) \) everywhere on the \( \tilde{H}^7 \) except in the immediate vicinity of the HW boundary, as described after (4), we see that the intrinsic volume \( \tilde{V}_7 \) of \( H_7 \), which determines the number of modes in the bump if all the modes expected from (14) up to the expected intrinsic \( \lambda_1 = 3^2 \) are approximately degenerate at 1.8 TeV, is independent of \( \kappa_{11}^{2/9} \), so the discrepancy cannot be corrected by reducing \( \kappa_{11}^{2/9} \).

However the expected number of modes in the bump would be reduced by a factor of \( 3^{-7} = \frac{1}{2187} \) to around \( 5.0 \times 10^{31} \) if for some reason the intrinsic \( \lambda_1 \) was 1 rather than \( 3^2 \) as expected from Cheeger’s inequality. This might happen if the modes in the bump originate from modes of the 3-form \( C_{IJK} \) that are 2-forms on \( \tilde{H}^7 \) and vectors along the extended dimensions, because for \( p < \frac{n-1}{2} \), \( \lambda_1 \) for a \( p \)-form on \( H^n \) is \( \left( \frac{n-1-2p}{4} \right)^2 \).
In this case the relation $\frac{2A}{B} = 1.8$ TeV would be modified to $\frac{A}{B} = 1.8$ TeV, so since $A \leq 1$ by section 4 starting on page 35, we would find from $B \simeq 0.43\kappa_{11}^{2/9}$ that $\kappa_{11}^{2/9} \geq 0.78$ TeV, hence $M_{11} \geq 3.3$ TeV.

If this is the correct interpretation, then the number of modes would actually be reduced still further, possibly to around the correct value, because the factor 128 in (18) would be replaced by a smaller value, representing the fact that the modes in the bump represent only modes of $C_{IJK}$ that are 2-forms on $\bar{H}^7$ and vectors along the extended dimensions. In this case, there might be a second bump at around $2 \times 1.8$ TeV = 3.6 TeV from modes of $C_{IJK}$ that are 1-forms on $\bar{H}^7$ and 2-form Abelian gauge fields, equivalent in 4 dimensions to scalars $\kappa_2 \kappa_4^2$ along the extended dimensions, and from modes of the metric $G_{IJ}$ that are vectors on $\bar{H}^7$ and vectors along the extended dimensions, and a third bump at around $3 \times 1.8$ TeV = 5.4 TeV from the KK modes of the graviton that are scalars on $\bar{H}^7$ and traceless symmetric tensors along the extended dimensions. However the position is made less clear by the fact that, as discussed just before (14), on page 12, I had to assume that the classically massless modes of $C_{IJK}$ and $G_{IJ}$ resulting from the Hodge - de Rham harmonic forms on $\bar{H}^7$ all acquire masses $\sim \kappa_{11}^{2/9}$ from the leading quantum corrections to the classical CJS action (3).

If the intrinsic $\lambda_1$ is $3^2$, as assumed on the basis of Cheeger’s bound for scalars on $\bar{H}^7$ in the initial estimate above of the expected number of events above background in the bump, then we would have to conclude that just a fraction around $\frac{1}{60000}$ of the bosonic gravitational KK modes expected from (14) are approximately degenerate at 1.8 TeV.

An alternative hypothesis is that notwithstanding the rapid restoration of agreement between the spectral staircase and the Weyl asymptotic formula (14) found for small intrinsic volume $\bar{H}^3$’s in [98], and the possibility of arbitrarily large degeneracies of eigenvalues of $-\Delta$ for large intrinsic volume $\bar{H}^n$’s indicated by [99], the restoration of agreement with (14) for $\bar{H}^7$ occurs over a wider range of energies than 0.2 TeV, and the modes responsible for the bump from 1.7 to 1.9 TeV are instead the Hodge - de Rham harmonic 1-forms, 2-forms, and 3-forms on $\bar{H}^7$, which have all acquired approximately equal masses $\sim \kappa_{11}^{2/9}$ from the $(DH)^2R^2$ and $H^2R^3$ terms in the leading quantum corrections to the CJS action (3).

The number of linearly independent Hodge - de Rham harmonic $p$-forms, $0 \leq p \leq n$, on a compact orientable $n$-manifold, is equal to the $p$th Betti number with integer coefficients, $B_p$, which is called the $p$th Betti number. The sum of all the Betti numbers
of an \( \tilde{H}^n \) of intrinsic volume \( \tilde{V}_n \) is bounded above by \( c \tilde{V}_n \), where \( c \) is a constant that depends only on \( n \) \cite{21,22}. The \( p \)th Betti number \( B_p \) of finite coverings of an \( \tilde{H}^n \) grows linearly with the intrinsic volume of the covering if and only if there are square-integrable harmonic \( p \)-forms on the covering space \( H^n \) \cite{103,104}, which by \cite{90} means if and only if \( n \) is even and \( p = \frac{n}{2} \). In the other cases, the rate of growth of \( B_p \) with the intrinsic volume of the covering is bounded above by a rate of growth that is strictly less than linear, by an amount that depends on the rate of growth of the \emph{injectivity radius} of the covering. The injectivity radius \( \operatorname{Inj}(\mathcal{M},x) \) at a point \( x \) of a Riemannian manifold \( \mathcal{M} \) is the largest radius for which the map from the tangent space at \( x \), to \( \mathcal{M} \), defined by mapping each tangent vector \( v \) at \( x \) to the point reached by starting at \( x \) and going out a distance \( |v| \) along the geodesic starting in the direction \( v \) from \( x \), is a diffeomorphism, and \( \operatorname{Inj}(\mathcal{M}) \) is the minimum of \( \operatorname{Inj}(\mathcal{M},x) \) over all the points \( x \) of \( \mathcal{M} \) \cite{105}.

If \( \tilde{H}^n \) is a compact hyperbolic \( n \)-manifold, and \( X \) is a finite regular covering manifold of \( \tilde{H}^n \) with intrinsic volume \( \operatorname{Vol}(X) \) and intrinsic injectivity radius \( \operatorname{Inj}(X) \), then by Theorem 0.3 of \cite{104}, there are constants \( C > 0 \) and \( \beta_p > 0 \) such that:

- For \( n \) odd:
  - If \( p \neq \frac{n \pm 1}{2} \), then \( B_p(X) \leq C \frac{\operatorname{Vol}(X)}{e^{\beta_p \operatorname{Inj}(X)}} \).
  - For \( p = \frac{n \pm 1}{2} \), \( B_p(X) \leq C \frac{\operatorname{Vol}(X) \cdot \log \operatorname{Inj}(X)}{\operatorname{Inj}(X)} \).

- For \( n \) even:
  - If \( p \neq \frac{n}{2} \), then \( B_p(X) \leq C \frac{\operatorname{Vol}(X)}{e^{\beta_p \operatorname{Inj}(X)}} \).

By the generalized Gauss-Bonnet theorem \cite{106,107}, the Euler characteristic, or Euler number, \( \chi(\mathcal{M}^{2q}) \equiv \sum_{p=0}^{2q} (-)^p B_p \), of an arbitrary smooth \( 2q \)-manifold \( \mathcal{M}^{2q} \), is given by:

\[
\chi(\mathcal{M}^{2q}) = \frac{1}{(8\pi)^q q!} \int_{\mathcal{M}^{2q}} d^{2q}x \sqrt{g} e_{j_1 \ldots j_{2q}} e^{i_1 \ldots i_{2q}} R_{i_1 j_2} R_{i_2 j_3} \ldots R_{i_{2q-1} j_{2q}} R_{i_{2q}}. \quad (22)
\]

Thus for a \( 2q \)-dimensional compact hyperbolic manifold \( \tilde{H}^{2q} \) with sectional curvature \(-1\):

\[
\chi(\tilde{H}^{2q}) = (-)^q \frac{(2q)!}{(2\pi)^q 2^q q!} \tilde{V}_{2q} = (-)^q \frac{(2q-1)!!}{(2\pi)^q} \tilde{V}_{2q} = (-)^q \frac{2}{S_{2q}} \tilde{V}_{2q}. \quad (23)
\]
where \( S_{2q} = \frac{2(2\pi)^{q}}{(2q-1)!!} \) is the 2q-volume of the 2q-sphere of unit radius. In particular, \( \chi(\bar{H}^6) = -\frac{15}{8\pi^6} \bar{V}_6 \simeq -0.060 \bar{V}_6 \), \( \chi(\bar{H}^4) = \frac{3}{4\pi^4} \bar{V}_4 \simeq 0.076 \bar{V}_4 \), and \( \chi(\bar{H}^2) = -\frac{1}{2\pi^2} \bar{V}_2 \simeq -0.159 \bar{V}_2 \). Thus since by [103, 104] and [90], \( B_p(X) \) for a finite regular covering \( X \) of \( \bar{H}^{2q} \) can grow linearly with the intrinsic volume \( \text{Vol}(X) \) if and only if \( p = q \), \((-)^{q} \frac{B_p(X)}{\chi(X)} \rightarrow 1 \) as \( \text{Vol}(X) \rightarrow \infty \), hence \( \frac{B_p(X)}{\text{Vol}(X)} \rightarrow \frac{2}{S_{2q}} \) as \( \text{Vol}(X) \rightarrow \infty \).

The relatively strong suppression of \( B_p(X) \) relative to \( \text{Vol}(X) \) for \( p \neq \frac{n+1}{2} \) when \( n \) is odd and for \( p \neq \frac{n}{2} \) when \( n \) is even arises from the fact that in these cases, the spectrum of the Hodge - de Rham \(-\Delta\) for \( p \)-forms on \( H^n \) has a gap [90]. For \( p = \frac{n+1}{2} \) when \( n \) is odd the spectrum of \(-\Delta\) on \( H^n \) has no gap, but there are no square-integrable harmonic \( p \)-forms on \( H^n \), while for \( p = \frac{n}{2} \) when \( n \) is even, there are an infinite number of linearly independent square-integrable harmonic \( p \)-forms on \( H^n \) [90].

For finite coverings \( X \) of \( \bar{H}^n \) that are reasonably isotropic, in the sense that they have a fundamental domain that is approximately spherical, we might expect that the intrinsic injectivity radius \( \text{Inj}(X) \), which is half the intrinsic length of the shortest closed geodesic on \( X \), is roughly a fixed fraction of the intrinsic diameter of \( X \), in which case for for \( p \neq \frac{n+1}{2} \) when \( n \) is odd and for \( p \neq \frac{n}{2} \) when \( n \) is even, the above bounds on \( B_p(X) \) become upper bounds by constant multiples of powers strictly less than 1 of \( \text{Vol}(X) \). In the special case of \( \bar{H}^n \) whose fundamental groups are arithmetic discrete subgroups \( \Gamma \) of \( \text{SO}(n,1) \) [37], and finite coverings \( X \) of \( \bar{H}^n \) whose fundamental groups are congruence subgroups of \( \Gamma \), which are roughly the subgroups obtained by Selberg’s Lemma [108], reviewed in subsection 3.1.2 of [39], there are lower bounds on \( B_p(X) \) of this form [93].

If we assume that the above results for finite coverings \( X \) of a fixed \( \bar{H}^n \) roughly describe the dependence of the Betti numbers \( B_p(\bar{H}^n) \) on the intrinsic volume \( \bar{V} \) of \( \bar{H}^n \) for typical \( \bar{H}^n \) of large \( \bar{V} \), at least for reasonably isotropic \( \bar{H}^n \), then for \( p \neq \frac{n+1}{2} \) when \( n \) is odd and for \( p \neq \frac{n}{2} \) when \( n \) is even, \( B_p(\bar{H}^n) \) is roughly a constant times a power \(< 1 \) of \( \bar{V} \), while for \( p = \frac{n+1}{2} \) when \( n \) is odd, \( B_p(\bar{H}^n) \) is roughly a constant times \( \frac{\bar{V}}{\ln \bar{V}} \), and for \( p = \frac{n}{2} \) when \( n \) is even, \( B_p(\bar{H}^n) \) is roughly \( \frac{2}{S_n} \bar{V} \).

Thus if the \( \bar{H}^n \) Cartesian factor of \( \mathcal{M}^7 \) of large \( \bar{V} \) had \( n \) even, the number of modes in the bump according to the alternative hypothesis would probably be too large by a factor of at least 1000. For the \( \bar{H}^7 \) case, the number of modes would be about right if \( B_3 \) and \( B_4 \) were around \( \frac{1}{1000} \frac{\bar{V}}{\ln \bar{V}} \).

If this is the correct hypothesis, then the bump will be spin 0 if the compact hyperbolic factor of large intrinsic volume is 7-dimensional, and a mixture of spins 0
and 1 if it is 5-dimensional or 3-dimensional. This could be tested by plotting the energies of the decay products in 3-body decays of candidate bump particles, in the reconstructed rest frame of the candidate bump particle, on a Dalitz plot [109,110,100].

Confirmation that the bump particles decay democratically to all Standard Model particles, and are thus gravity-like, but have only spin 0, would provide evidence for the existence of the 3-form gauge field $C_{IJK}$ of $d=11$ supergravity [6], because in the $\tilde{H}^7$ case, the bump particles arise only from $C_{IJK}$.

2 Hořava-Witten theory

I shall use Moss’s improved version of Hořava-Witten theory [33,34,35,36]. In the region of the HW boundary, the coordinates $x^I$ have the form $(\tilde{x}^U, y)$, where indices $U,V,W,...$ are tangential to a family of hypersurfaces foliating the $(10+1)$-dimensional manifold-with-boundary, one of these hypersurfaces coinciding with the boundary, and $y$ takes a constant value on each of these hypersurfaces, with the value of $y$ distinguishing the hypersurfaces. $y$ takes the value $y_1$ on the boundary, and $y > y_1$ in the bulk. The symbol $y$ is also used as the coordinate index for the $y$ coordinate. The bosonic part of the semiclassical action is:

$$S_{\text{HW}}^{(\text{bos})} = S_{\text{CJS}}^{(\text{bos})} + S_{\text{GH}}^{(\text{bos})} + S_{\text{YM}}^{(\text{bos})}$$ (24)

The supergravity term is [3], on page 6.

The Gibbons-Hawking term is:

$$S_{\text{GH}}^{(\text{bos})} = \frac{1}{\kappa_{11}^2} \int d^{10} \tilde{x} \tilde{e} K$$ (25)

Here $\beta$ denotes the $9+1$ dimensional boundary at $y = y_1$, and on the boundary, $\tilde{e} = \sqrt{-\tilde{G}}$ denotes the square root of minus the determinant of the induced metric $\tilde{G}_{UV}$, which is obtained from $G_{IJ}$ by dropping the row and column with an index $y$. $K = G^{IJ} K_{IJ}$ is the trace of the extrinsic curvature $K_{IJ} \equiv (\delta_I^K - n_I n^K) (\delta_J^L - n_J n^L) D_K n_L$, where $n_I$ is the outward unit normal, and $D_K$ is the covariant derivative. Thus since $n_I$ is a scalar factor times $\partial_I y$, the only nonvanishing component of $n_I$ is $n_y = -\frac{1}{\sqrt{G_{yy}}}$, so only the $D_U n_V = -\Gamma_U^{yV} n_y$ components of $D_K n_L$ contribute to $K_{IJ}$, so $K_{IJ}$ is symmetric, and $K = \frac{1}{\sqrt{G_{yy}}} \left( G^{UV} - \frac{G^{ly} G^{ly}}{G_{yy}} \right) \Gamma_U^{yV}$.
The Yang-Mills term is:

$$S^{(\text{bos})}_{\text{YM}} = -\frac{1}{16\pi\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \int d^{10}\tilde{x} \tilde{e} \left(\frac{1}{30}\text{tr}F_{UV}F^{UV} - \frac{1}{2}\tilde{R}_{UVW\hat{X}}\tilde{R}^{UV\hat{W}\hat{X}}\right)$$  \hspace{1cm} (26)

Here $F_{UV} = \partial_U A_V - \partial_V A_U + i[A_U, A_V]$ is the field strength of an $E_8$ Yang-Mills gauge field $A_U = T^A A^A_U$ localized on the boundary. $\text{tr}$ means the ordinary trace, not the modified trace used by HW. Indices $A, B, \ldots$ run over the 248 generators of $E_8$, and the hermitian generators $T^A$ in the fundamental/adjoint of $E_8$ satisfy $\text{tr} T^A T^B = 30 \delta^{AB}$. In the SO (16) basis for $E_8$, the $T^A$ are $-\frac{1}{2}i$ times the generators in Appendix 6.A of [111] or subsection 2.1 of [39], and in the SU (9) basis for $E_8$, the $T^A$ are the generators in subsection 5.2 of [39]. The coefficient of the first term in (26) is fixed by anomaly cancellation [32, 60, 61, 62, 30, 63, 64, 58, 65, 35] and has the value found by Conrad [61], which is slightly different from the original value found by HW. The $\tilde{R}_{UVW\hat{X}}\tilde{R}^{UV\hat{W}\hat{X}}$ term was derived by Moss [36], with:

$$\tilde{R}_{UV\hat{X}} = \partial_U \tilde{\omega}_{V\hat{X}} - \partial_V \tilde{\omega}_{U\hat{X}} + \tilde{\omega}_{U\hat{Y}} \tilde{\omega}_{V\hat{X}} - \tilde{\omega}_{V\hat{Y}} \tilde{\omega}_{U\hat{X}},$$  \hspace{1cm} (27)

where

$$\tilde{\omega}_{U\hat{V}} = \tilde{\omega}_{U\hat{V}} + \frac{1}{2}H_{gUV\hat{W}}$$  \hspace{1cm} (28)

and $\tilde{\omega}_{U\hat{V}} = e^X_{\hat{W}} \left(\tilde{\Gamma}_{U}^{Y\hat{X}} e_{Y\hat{V}} - \partial_U e_{X\hat{V}}\right)$ is the Levi-Civita connection for the vielbein $\tilde{e}_{U\hat{V}}$, that satisfies $\tilde{e}_{U\hat{W}} \tilde{e}_{V\hat{X}} \eta^{\hat{W}\hat{X}} = \tilde{G}_{UV}$, where $\eta^{\hat{U}\hat{V}}$ is the Minkowski metric on the 9 + 1 dimensional boundary. The sign choice in (28) is correlated with the chirality conditions on the gravitino, gaugino, and supersymmetry variation parameter on the boundary.

Variation of the metric in $S^{(\text{bos})}_{\text{HW}}$ leads to the Einstein equations:

$$R_{IJ} - \frac{1}{2}R G_{IJ} - \frac{1}{12}H_I^{KLM}H_{JKLM} + \frac{1}{96}H^{KLMN}H_{KLMN}G_{IJ} = 0$$  \hspace{1cm} (29)

and on the boundary to the Israel boundary conditions [112, 113, 114, 34]:

$$K^{UV} - K \tilde{G}^{UV} - \kappa_{11}^2 \tilde{T}^{(\text{bos})UV} = 0,$$  \hspace{1cm} (30)

where

$$\tilde{T}^{(\text{bos})UV} = \frac{2}{\tilde{e}} \frac{\delta S^{(\text{bos})}_{\text{YM}}}{\delta \tilde{G}_{UV}}.$$  \hspace{1cm} (31)

The boundary is equivalent to a double-sided mirror at $y = y_1$, because all the fields on one side of the mirror are exactly copied, up to sign, on the other side of the mirror.
The Yang-Mills multiplet is adjacent to the mirror, but infinitesimally displaced from it, so that it has its own reflection infinitesimally on the other side of the mirror. The bulk and its reflection on the other side of the mirror form two parts of a single closed manifold, and the formula for the classical action in the bulk applies on both sides of the mirror, with the same SO(10,1)-invariant tensor \( \epsilon_{11}^{I_1\ldots I_{11}} \) occurring in the definition of \( \epsilon_{11}^{I_1\ldots I_{11}} \) on both sides of the mirror. Thus \( \epsilon_{11}^{I_1\ldots I_{11}} \) transforms as a pseudo-tensor under reflection in the mirror, which means that its components are multiplied by an extra factor of \(-1\) under the reflection, relative to what they would have become if it had transformed as a tensor, because it would have been multiplied by \(-1\) under the reflection, if it had transformed as a tensor.

The metric \( G_{IJ} \) transforms as a tensor under the reflection, because if it transformed as a pseudo-tensor, the measure factor \( \sqrt{-G} \) would become imaginary under the reflection, and the Ricci scalar would change sign under the reflection. The Yang-Mills gauge field \( A_U^I \) also transforms as a tensor under the reflection, because it only has components tangential to the boundary, and if it transformed as a pseudo-tensor, its components would have to vanish identically, if they were continuous across the mirror. However to preserve the sign of the last term, called the Chern-Simons term, of the classical action in the bulk under reflection in the mirror, the 3-form \( C_{IJK} \) has to transform, like \( \epsilon_{11}^{I_1\ldots I_{11}} \), as a pseudo-tensor under the reflection.

Thus \( C_{UVW} \) and \( H_{UVWX} \) change sign on reflection in the mirror, so if \( H_{UVWX} \) is continuous across the mirror, \( H_{UVWX} = 0 \) at the mirror. Thus the boundary conditions for \( C_{IJK} \) are controlled not by the action, but by the pseudo-tensor transformation rule of \( C_{IJK} \) under reflection in the mirror, and continuity. Therefore variations of \( C_{IJK} \) in \( S_{HW} \) have \( \delta C_{UVW} = 0 \) at the boundary, and lead only to the 3-form field equations, whose bosonic terms are:

\[
D_L H^{LIJK} - \frac{1}{3456} \epsilon_{11}^{IJKLMNOPQRS} H_{LMNO} H_{PQRS} = 0
\]

At zeroth order in \( \kappa_{11} \), \( SYM \) is absent and \( H_{UVWX} \) is continuous across the mirror, so the boundary condition is \( H_{UVWX} = 0 \). At order \( \kappa_{11}^{2/3} \), the \( E_8 \) Yang-Mills multiplet is required to cancel the gravitational anomalies resulting from the chirality of the gravitino boundary condition, and \( H_{UVWX} \) is no longer continuous across the mirror. The boundary condition for \( H_{IJKL} \) at the surface of the mirror is determined by anomaly cancellation and supersymmetry, and the bosonic terms are:

\[
H_{UVWX} = \pm \frac{3}{2\pi} \left( \frac{\kappa_{11}}{4\pi} \right)^{2/3} \left( \frac{1}{30} \text{tr} F_{[UV} F_{WX]} - \frac{1}{2} \bar{R}_{[UV} \hat{\gamma}^Z \bar{R}_{WX]} \hat{\gamma}^Z - \frac{1}{3} \partial_U \Omega_{VW}^{(H)} \right)
\]
where

$$\Omega^{(H)}_{UVW} = 3H_{y|U}X_1X_2D_YH_{W|X_1X_2y} + 3H_{yX_1X_2[U}D^{X_1}H_{X_2}V_{W]}$$

$$- \frac{1}{36} \varepsilon_{UVW} x_1x_2x_3x_4x_5x_6x_7y_{x_1x_2x_3}D_XH_{X_4X_5X_6X_7y}$$

(34)

The sign choice in (33) is correlated with the chirality conditions on the gravitino, gaugino, and supersymmetry variation parameter on the boundary. Square brackets around two or more indices denote antisymmetrization with unit total weight.

The boundary condition (33) can be integrated to:

$$C_{UVW} = \pm \left( \frac{\kappa_{11}}{4\pi} \right)^{2/3} \left( \frac{1}{30} \Omega^{(Y)}_{UVW} - \frac{1}{2} \Omega^{(L)}_{UVW} - \frac{1}{2} \Omega^{(H)}_{UVW} \right) + \lambda_{UVW},$$

(35)

where

$$\Omega^{(Y)}_{UVW} = 6\text{tr} \left( A_{[U} \partial_{V}A_{W]} + \frac{2}{3} iA_{[U}A_{V}A_{W]} \right)$$

(36)

and

$$\Omega^{(L)}_{UVW} = -6 \left( \tilde{\omega}_{[U} \hat{x}_1 \hat{x}_2 \partial_{V} \tilde{\omega}_{W]} \hat{x}_2 \hat{x}_1 + \frac{2}{3} A_{[U,V,W]} \tilde{\omega}_{U} \hat{x}_1 \hat{x}_2 \tilde{\omega}_{V} \hat{x}_2 \hat{x}_3 \tilde{\omega}_{W} \hat{x}_3 \hat{x}_1 \right)$$

(37)

are Yang-Mills and Lorentz Chern-Simons 3-forms respectively, $\lambda_{UVW}$ is an arbitrary closed 3-form on the boundary, and the notation $A_{(U,V,W)}$ indicates that what follows it is to be antisymmetrized in the indices $U, V, W$, with unit total weight.

The classical field equations of $d = 11$ supergravity do not have any solutions with 4 flat extended dimensions if $\mathcal{M}^7$ has a compact hyperbolic factor, for any arrangement of 4-form fluxes such that the flux bilinears are on average covariantly constant, so it is necessary to include the leading quantum corrections to the classical action in the bulk.

The quantum-corrected field equations have the form $\delta \Gamma_{HW} = 0$, where $\Phi$ represents all the fields, and the bosonic part of the quantum effective action $\Gamma_{HW}$ has the form:

$$\Gamma_{HW}^{(bos)} = S_{HW}^{(bos)} + \Gamma_{SG}^{(8, bos)} + \Gamma_{HW}^{(h.o., bos)}.$$  

(38)

Here $\Gamma_{SG}^{(8, bos)}$ is a dimension 8 local polynomial in the fields and derivatives in the bulk, of order $\kappa_{11}^{4/3}$ relative to $S_{CJS}^{(bos)}$, and $\Gamma_{HW}^{(h.o., bos)}$, representing the higher order corrections, contains both local and non-local terms, and is of order $\kappa_{11}^2$ or higher relative to $S_{CJS}^{(bos)}$ in the bulk, and of order $\kappa_{11}^{4/3}$ or higher relative to $S_{GH}^{(bos)}$ on the boundary. The dimension of a local monomial in the fields and derivatives is defined by counting 0 for each metric or 3-form, $\frac{1}{2}$ for each gravitino, and 1 for each derivative, and on the boundary also counting 1 for each Yang-Mills gauge field and $\frac{3}{2}$ for each gaugino.
The leading quantum corrections to \(d = 11\) supergravity in the bulk have the form \[119\]:

\[
\Gamma_{\text{SG}}^{(8,\text{bos})} = \frac{1}{147456 \pi^2 \kappa_{11}^2} \left( \frac{\kappa_{11}}{4 \pi} \right)^{4/3} \int_B \frac{d^{11}x}{4!} \epsilon_{11} R^4 - \frac{1}{4} \epsilon_{11} \epsilon_{11} R^4 - \frac{1}{6} \epsilon_{11} t_8 C R^4 + \Xi^{(\text{flux})}
\]

(39)

Here for an arbitrary tensor \(X_{IJKL}\), that is antisymmetric in its first two indices and antisymmetric in its last two indices:

\[
t_{8} t_8 X^4 \equiv t_8^I t_8^J t_8^K t_8^L K_1 K_2 L_1 L_2 t_8^M t_8^N t_8^O t_8^P t_8^Q X_{I_1 I_2 M_1 M_2 N_1 N_2 O_1 O_2 P_1 P_2},
\]

(40)

where \(t_8^{IJKLMNOP}\) is a tensor built from \(G^{IJ}\) and antisymmetric in each successive pair of indices, such that for antisymmetric tensors \(A_{IJ}, B_{IJ}, C_{IJ}, D_{IJ}\):

\[
t_8^{IJKLMNOP} A_{IJ} B_{KL} C_{MN} D_{OP} =
\]

\[
= 8 \left( \text{tr} \left( ABCD \right) + \text{tr} \left( ACBD \right) + \text{tr} \left( ACDB \right) \right)
- 2 \left( \text{tr} \left( AB \right) \text{tr} \left( CD \right) + \text{tr} \left( AC \right) \text{tr} \left( BD \right) + \text{tr} \left( AD \right) \text{tr} \left( BC \right) \right) =
\]

\[
= 8 \left( A_{IJ} B_{JK} C_{KL} D_{LI} + A_{IJ} C_{JK} B_{KL} D_{LI} + A_{IJ} C_{JK} D_{KL} B_{LI} \right)
- 2 \left( A_{IJ} B_{JI} C_{KL} D_{LK} + A_{IJ} C_{JI} B_{KL} D_{LK} + A_{IJ} D_{JI} B_{KL} C_{LK} \right)
\]

(41)

Here and in the following, repeated lower coordinate indices are understood to be contracted with an inverse metric, for example \(A_{IJ} B_{JK} \equiv A_I^J B_{JK} = G^{IJ} A_{IL} B_{JK}\).

Thus for an arbitrary such tensor \(X_{IJKL}\):

\[
t_8 t_8 X^4 = 12 X_{IJKL} X_{JILK} X_{MNOP} X_{MNPO} + 24 X_{IJKL} X_{JMNP} X_{ONLK} X_{POMN}
- 96 X_{IJKL} X_{JILM} X_{MNPQ} X_{ONPK} - 48 X_{IJKL} X_{JIMN} X_{OPLM} X_{PONK}
- 96 X_{IJKL} X_{JMNL} X_{MNOP} X_{NIPQ} - 48 X_{IJKL} X_{JINP} X_{MPLK} X_{PION}
+ 192 X_{IJKL} X_{JMLN} X_{MONP} X_{OIPK} + 384 X_{IJKL} X_{JMLN} X_{MOPK} X_{OINP}
\]

(42)

And similarly, for an arbitrary such tensor \(X_{IJKL}\):

\[
\frac{1}{4!} \epsilon_{11} \epsilon_{11} X^4 \equiv \frac{1}{4!} \epsilon_{11} \epsilon_{11}^{IJKL} L_1 L_2 M_1 M_2 N_1 N_2 O_1 O_2 \left( \epsilon_{11} \right)^{IJKP} P_1 P_2 Q_1 Q_2 R_1 R_2 S_1 S_2
\]

\[
\times X_{L_1 L_2} P_1 P_2 X_{M_1 M_2} Q_1 Q_2 X_{N_1 N_2} R_1 R_2 X_{O_1 O_2} S_1 S_2
\]

\[
= - \frac{8!}{4} X_{L_1 L_2} \left[ L_1 L_2 \ X_{M_1 M_2} \ X_{N_1 N_2} \ X_{O_1 O_2} \right]
\]

(43)
And similarly:

\[
\frac{1}{6} \epsilon_{11ts} CR^4 \equiv 4 \epsilon^{I_1 \ldots I_{11}} C_{I_1 I_2 I_3} R_{I_4 I_5 J K} R_{I_6 I_7 K L} R_{I_8 I_9 L M} R_{I_{10} I_{11} M J} - \epsilon^{I_1 \ldots I_{11}} C_{I_1 I_2 I_3} R_{I_4 I_5 J K} R_{I_6 I_7 K J} R_{I_8 I_9 L M} R_{I_{10} I_{11} M L}
\]  \tag{44}

The relative coefficients of all terms in (39) are fixed by supersymmetry, up to the fact that arbitrary multiples of linear combinations of terms that vanish when the classical field equations (29) and (32) are satisfied can be added, because the overall coefficients of such linear combinations of terms can be adjusted arbitrarily by small amounts, by making redefinitions of the fields of the form

\[
G_{IJ}, C_{IJK} \rightarrow G_{IJ} + X_{IJ}, C_{IJK} + Y_{IJK},
\]

where \(X_{IJ}\) and \(Y_{IJK}\) are polynomials in the fields and their derivatives, with small coefficients \([45, 67, 69]\). The coefficient of the \(\epsilon_{11ts} CR^4\) term, which is known as the Green-Schwarz term because of its role in anomaly cancellation \([120]\), is fixed absolutely by anomaly cancellation on five-branes \([55, 56, 57, 58, 59]\), and confirmed by anomaly cancellation in Ho\(\check{r}\)ava-Witten theory \([60, 61, 62, 50, 63, 64, 58, 65, 35]\).

The flux term \(\Xi^{\text{flux}}\) in (39) is a linear combination of dimension 8 coordinate scalar monomials built from \(R_{IJKL}, G^{IJ}, \epsilon_{I_1 \ldots I_{11}}, H_{IJKL}\), and the covariant derivative \(D_I\), such that \(H_{IJKL}\) occurs at least once in each monomial, and the number of \(R_{IJKL}\) plus the number of \(H_{IJKL}\) is at least 4. \(\epsilon_{I_1 \ldots I_{11}}\) occurs once in each monomial with an odd number of \(D_I\), and is absent from monomials with an even number of \(D_I\). The only building blocks of odd dimension are \(H_{IJKL}\) and \(D_I\), so each occurs an odd number of times if the other does. If \(H_{IJKL}\) occurs exactly once then there are three \(R_{IJKL}\)'s and one \(D_I\), hence also an \(\epsilon_{I_1 \ldots I_{11}}\), and it is impossible to build a coordinate scalar that does not vanish identically by a Bianchi identity or the Ricci cyclic identity \([121, 122]\), so in fact there are at least two \(H_{IJKL}\)'s in each monomial.

The terms in \(\Xi^{\text{flux}}\) can be classified by the numbers of \(R_{IJKL}\) and \(H_{IJKL}\) in them. For general fluxes \(H_{IJKL}\) only the \(R^2 H^2, RH^3,\) and \(H^4\) terms are known at present \([121, 122, 123, 69]\). These terms simplify substantially in the special case where there is a particular coordinate, which for convenience of notation I shall call \(y\), although in general there is no HW boundary nearby, such that all nonvanishing components of \(H_{IJKL}\) have an index \(y\), all the fields are independent of \(y\), and, borrowing the notation used above for HW theory, the metric components \(G_{Uy}\) are 0. Then the \(RH^3\) terms vanish, because they have two \(H_{IJKL}\)'s contracted with an \(\epsilon_{I_1 \ldots I_{11}}\) \([121, 122]\), and by the correspondence with type IIA superstring theory \([124, 125]\), in the special case where
only the fields of the type I supergravity multiplet in 10 dimensions are nonzero, the 4-field terms in \( \Gamma_{SG}^{(8,\text{bos})} \) must reduce to the 4-field terms in:

\[
\left. \Gamma_{SG}^{(8,\text{bos})} \right|_{I,y} = \frac{1}{147456\pi^2\kappa_{11}^2} \left( \frac{\kappa_{11}}{4\pi} \right)^{4/3} \int_B d^{11}x e \left( t_8 t_8 \tilde{R}^4 - \frac{1}{4!} \epsilon_{11} \epsilon_{11} \tilde{R}^4 \right),
\]

where \( \tilde{R}_{UVWX} = e^w \hat{y} e^x \hat{z} \tilde{R}_{UVWZ} \), with \( \tilde{R}_{UVWX} \) defined by (27) and \( \tilde{\omega}_{UVW} \) defined by (28), \( \tilde{R}_{yUVW} = \tilde{R}_{WVYu} = R_{gUVW} = 0, \tilde{R}_{yUyV} = R_{yUyV}, \) and either sign in (28) may be used, since with \( \tilde{H}_{UVW} \equiv \tilde{H}_{gUVW}, \) the \( \tilde{R}_{UVWX} \) and

\[
-\frac{1}{4} \tilde{G}^{YZ} \left( \tilde{H}_{UYW} \tilde{H}_{VXZ} - \tilde{H}_{UYW} \tilde{H}_{UXZ} \right)
\]

terms in \( \tilde{R}_{UVWX} \) are unaltered by swapping \( UV \) with \( WX \), while the \( \pm \frac{1}{2} \left( \tilde{D}_U \tilde{H}_{VWXY} - \tilde{D}_V \tilde{H}_{UXWY} \right) \) terms change sign due to the Bianchi identity for \( H_{IJKL} \), so reversing the sign of \( \tilde{H}_{UVW} \) is equivalent to swapping the two \( t_8 \)’s or the two \( \epsilon_{11} \)’s in (45), so (45) only contains even powers of \( \tilde{H}_{UVW} \). The coefficient of \( \tilde{H}_{UVW} \) in \( \tilde{\omega}_{UVW} \) is fixed to \( \pm \frac{1}{2} \), in agreement with (28), by comparison with equations (2.8), (2.12), and (2.13) of [126], noting the unusual normalization of antisymmetrization brackets around indices used in that article, as shown by a statement after their equation (2.9), or equivalently by comparison with equations (5.10), (5.18), and (5.36) of [127], who use the standard normalization of antisymmetrization brackets, as defined after equation (34) above.

Richards [8] has found evidence that for the field configurations specified above, the formula (45) for \( \Gamma_{SG}^{(8,\text{bos})} \) might also be valid beyond the 4 field level, in agreement with an earlier suggestion by Kehagias and Partouche [7]. However the following example shows that (45) as it stands cannot be oxidized to a generally covariant formula in 11 dimensions.

We consider the Cartesian product of 7-dimensional Minkowski space and a 2-sphere of radius \( a \) and a 2-sphere of radius \( b \). We choose spherical polar coordinates \((\theta, \varphi)\) on the first 2-sphere, and spherical polar coordinates \((\eta, \xi)\) on the second 2-sphere, so the metric is:

\[
ds_{11}^2 = G_{IJ} dx^I dx^J = \eta_{\alpha\beta} dx^\alpha dx^\beta + a^2 d\theta^2 + a^2 \sin^2 \theta d\varphi^2 + b^2 d\eta^2 + b^2 \sin^2 \eta d\xi^2, \tag{46}
\]

where \( \eta_{\alpha\beta} = \text{diag} (-1, 1, \ldots, 1) \) is the metric on \((6 + 1)\)-dimensional Minkowski space. The non-vanishing components of \( H_{IJKL} \), up to index permutations, are:

\[
H_{\theta\eta\xi} = h \varepsilon_{\theta\varphi} \varepsilon_{\eta\xi} = h \sin \theta \sin \eta,
\]

where if indices \( i, j, \ldots \) are tangential to the first 2-sphere and indices \( p, q, \ldots \) are tangential to the second 2-sphere, \( \varepsilon_{ij} \) is a covariantly constant harmonic 2-form on
the first 2-sphere, with non-vanishing components $\varepsilon_{\theta\varphi} = -\varepsilon_{\varphi\theta} = \sin \theta$, and $\varepsilon_{pq}$ is a covariantly constant harmonic 2-form on the second 2-sphere, with non-vanishing components $\varepsilon_{\eta\xi} = -\varepsilon_{\xi\eta} = \sin \eta$. The flux (47) is a covariantly constant harmonic 4-form on the Cartesian product of the two 2-spheres.

For the metric (46) and the flux (47), either $\varphi$ or $\xi$ could be the special coordinate $y$ in (45). The $\varepsilon_{11} \varepsilon_{11} \bar{R}^4$ term gives 0 because $\bar{R}_{IJKL}$ is 0 unless all its indices are tangential to the Cartesian product of the two 2-spheres, and if we choose $\varphi$ as $y$ then using Cadabra [128, 129, 130, 131, 132, 133] we find:

$$t_8 t_8 \bar{R}^4 = \frac{576}{a^8} + \frac{384}{a^4 b^4} + \frac{576}{b^8} - \frac{192 h^2}{a^8 b^6} + \frac{576 h^4}{a^4 b^{10}} + \frac{72 h^4}{a^4 b^8} + \frac{264 h^4}{a^8 b^{12}} - \frac{60 h^6}{a^4 b^{14}} + \frac{45 h^8}{4a^{16} b^{16}}$$

(48)

Choosing $\xi$ as $y$ gives (48) with $a$ and $b$ swapped, so since (48) is not symmetrical in $a$ and $b$, (45) cannot be oxidized to a generally covariant expression in 11 dimensions, because any such expression would give a result symmetrical in $a$ and $b$.

Thus (45) must be subject to a correction if used beyond the 4-field level. I shall consider only the case where $H_{IJKL}$ is covariantly constant. For a metric $G_{IJ}$ as specified before (45), the non-vanishing Christoffel symbols are:

$$\Gamma_{U^V W} = \frac{1}{2} G^{VX} \left( \partial_U G_{XW} + \partial_W G_{UX} - \partial_X G_{UW} \right) = \tilde{\Gamma}_{U^V W}$$

$$\Gamma_{y^V W} = \frac{1}{2} G^{yy} \partial_W G_{yy}, \quad \Gamma_{y^V y} = -\frac{1}{2} G^{VX} \partial_X G_{yy}, \quad \Gamma_{U^V y} = \frac{1}{2} G^{yy} \partial_U G_{yy}$$

(49)

Thus with $H_{yVWX} = \frac{1}{\sqrt{G_{yy}}} H_{yVWX}$:

$$\tilde{D}_U H_{yVWX} = \frac{1}{\sqrt{G_{yy}}} D_U H_{yVWX}$$

(50)

So $H_{yVWX}$ is covariantly constant as a 3-form in the 10-dimensional sense, with the $\hat{y}$ index ignored, if and only if $H_{yVWX}$ is covariantly constant as a 4-form in the 11-dimensional sense.

If the 4-form is covariantly constant, then from (27) and (28):

$$R_{UVWX} = R_{UVWX} - \frac{1}{4} G^{yy} G^{YZ} H_{yVUX} H_{yZVX} + \frac{1}{4} G^{yy} G^{YZ} H_{yYVW} H_{yZUX}$$

(51)

Thus a simple guess for the oxidation of (45) to 11 dimensions when the 4-form is covariantly constant would be to replace $R_{IJKL}$ in (39) by:

$$X_{IJKL} = \varepsilon_{IJKL} - \frac{1}{8} H_{IKMN} H_{JL}^{MN} + \frac{1}{8} H_{JKMN} H_{IL}^{MN},$$

(52)
and discard $\Xi^{(\text{flux})}$, because terms with a covariant derivative acting on the 4-form now vanish, and terms without covariant derivatives are absorbed in the $X^4$ terms. This gives a correction to (45) when only the fields of the type I supergravity multiplet in 10 dimensions are nonzero, as required by the above example, because we then have:

$$X_{yUyV} = R_{yUyV} + \frac{1}{8} H_{yyM} H_{yV}^{MN}$$

(53)

The correction term $\frac{1}{8} H_{yyM} H_{yV}^{MN}$ occurs multiplied by $2G^{yy}$ as a source term in the classical Einstein equations (29) for $G_{UV}$, so terms containing it might arise from a modified treatment of the classical Einstein equation in extracting the $d = 10$ effective action from the superstring scattering amplitude.

3 The bulk

I shall now look for solutions of the quantum-corrected $d = 11$ Einstein equations that have 4 flat extended dimensions and a compact 7-manifold $\mathcal{M}^7$ with one or more compact hyperbolic factors, in the presence of magnetic 4-form fluxes whose bilinears are on average covariantly constant. The averages of terms containing covariant derivatives of the 4-form could in principle be non-zero, but I shall assume that these terms can be neglected for a first approximation. As a guess for the approximate oxidation of the Kehagias-Partouche form (45) when covariant derivatives of the 4-form can be neglected, I shall use (39) with $R_{IJKL}$ replaced by the $X_{IJKL}$ defined in (52), and $\Xi^{(\text{flux})}$ discarded. I shall consider $\mathcal{M}^7$ that are Cartesian products of compact hyperbolic and spherical factors, and work in the leading order of the Lukas-Ovrut-Waldram harmonic expansion of the energy-momentum tensor on $\mathcal{M}^7$ [134].

There is then one independent Einstein equation for each Cartesian factor of $\mathcal{M}^7$, plus one Einstein equation for the 4 flat extended dimensions, because no source terms for the off-diagonal blocks of the Ricci tensor can be built from the covariantly constant average flux bilinears, and the source term for each diagonal block of the Ricci tensor is proportional to the corresponding diagonal block of the metric. The Einstein equations are consistent with the symmetry of the metric ansatz, so by Palais’s Principle of Symmetric Criticality [135] [136] [137], the Einstein equations can be derived by varying the action with respect to an independent scale factor introduced for each Cartesian factor of $\mathcal{M}^7$, and a scale factor for the 4 flat extended dimensions.
Within these approximations we can also introduce 5-branes [138], which are magnetic sources such that the 5-form dual to the 6-dimensional world-volume of the 5-brane provides a source for the Bianchi identity of the 4-form, localized on the 5-brane world-volume. To preserve Lorentz invariance 4 dimensions of the 5-brane world-volume must lie along the 4 extended dimensions, and the remaining two dimensions of the 5-brane world-volume wrap a 2-cycle of $\mathcal{M}^7$. I do not consider 2-branes [139] since they are electric sources and their presence would violate Lorentz invariance, and I do not consider a flux $H_{\mu\nu\sigma\tau}$ proportional to $\epsilon_{\mu\nu\sigma\tau}$ with indices along the 4 extended dimensions, because by the HW boundary condition [33] it would require the presence either of Lorentz-violating Yang-Mills fields, or Lorentz-violating fluxes, or fluxes that violate translation-invariance along the extended dimensions, and it is not in general simultaneously measurable with the magnetic fluxes [140].

The simplest case is when $H^7$ is a compact hyperbolic 7-manifold $\bar{H}^7$. The metric ansatz (4) now takes the form:

$$ds^2_{11} = G_{IJ} dx^I dx^J = A^2 \eta_{\mu\nu} dx^\mu dx^\nu + B^2 g_{ij} dx^i dx^j,$$

(54)

where indices $i,j,\ldots$ are tangential to the $\bar{H}^7$, and $g_{ij}$ is a metric of sectional curvature $-1$ on the $\bar{H}^7$. We introduce fluxes $H_{ijkl}$ wrapping 4-cycles of $\bar{H}^7$ such that on average:

$$H_{ijkl} H_{klop} G^{mo} G^{np} = h^2 B^8 (G_{ik} G_{jl} - G_{jk} G_{il}),$$

(55)

where $h$ is a constant. The dependence on $B$ is fixed by the fact that $H_{ijkl}$ is independent of $B$, because the integral of $H_{ijkl} dx^i dx^j dx^k dx^l$ over a 4-cycle of $\bar{H}^7$ is quantized, and independent of $B$. From (55) and (52) we find:

$$X_{ijkl} = (G_{il} G_{jk} - G_{jl} G_{ik}) \left( \frac{1}{B^2} + \frac{h^2}{8B^8} \right)$$

(56)

The $\epsilon_{11}\epsilon_{11}X^4$ term (43) in the action vanishes because the indices $i,j,\ldots$ run over only 7 dimensions, and using Cadabra [128, 129, 130, 131, 132, 133] to evaluate the $t_5 t_5 X^4$ term (42), the action density is:

$$\frac{A^4B^7}{2\kappa_{11}^2} \left( -42 \frac{1}{B^2} - \frac{42}{48} \frac{h^2}{B^8} - \frac{k^2}{B^8} + \frac{254016}{73728\pi^2} \left( \frac{\kappa_{11}}{4\pi} \right)^{4/3} \left( \frac{1}{B^2} + \frac{h^2}{8B^8} \right)^4 \right)$$

(57)

The $-\frac{k^2}{B^8}$ term is the classical action of a distribution of 5-branes. There is evidence that in HW theory, a 5-brane can only have the fundamental tension $T_5 = \frac{1}{8\pi} \left( \frac{4\pi}{\kappa_{11}} \right)^{4/3}$.
The 5-brane extends with any cycle that it wraps, and the denominator factor is to cancel the measure factor for each cycle it does not wrap.

To analyse the field equations it is convenient to introduce rescaled parameters

\[
\tilde{B} = \frac{2^{29} \pi^5}{21^{\frac{1}{3}}} \frac{B}{\kappa_{11}^{2/3}}, \quad \tilde{h} = \frac{2^{29} \pi^5}{21} \frac{h}{\kappa_{11}^{2/3}}, \quad \tilde{k} = \frac{2^{29} \pi^5}{21^{\frac{1}{3}}} \frac{k}{\kappa_{11}^{1/3}}, \quad (58)
\]

in terms of which the action density becomes a constant multiple of:

\[
A^4 \tilde{B}^7 \left( -42 \frac{1}{B^2} - \frac{42 \tilde{h}^2}{48 B^6} - \frac{\tilde{k}^2}{B^5} + \left( \frac{1}{B^2} + \frac{\tilde{h}^2}{8B^8} \right)^4 \right) \quad (59)
\]

The field equations are then:

\[
4096 \tilde{B}^{27} \tilde{k}^2 - \tilde{h}^8 - 32 \tilde{B}^6 \tilde{h}^6 - 384 \tilde{B}^{12} \tilde{h}^4 + 3584 \tilde{B}^{24} \tilde{h}^2 - 2048 \tilde{B}^{18} \tilde{h}^2 + 172032 \tilde{B}^{30} - 4096 \tilde{B}^{24} = 0 \quad (60)
\]

\[
8192 \tilde{B}^{27} \tilde{k}^2 + 25 \tilde{h}^8 + 608 \tilde{B}^{26} \tilde{h}^6 + 4992 \tilde{B}^{12} \tilde{h}^4 - 3584 \tilde{B}^{24} \tilde{h}^2 \\
+ 14336 \tilde{B}^{18} \tilde{h}^2 + 860160 \tilde{B}^{30} + 4096 \tilde{B}^{24} = 0 \quad (61)
\]

Adding these two equations, we find:

\[
24 \left( 512 \tilde{B}^{27} \tilde{k}^2 + \tilde{h}^8 + 24 \tilde{B}^{26} \tilde{h}^6 + 192 \tilde{B}^{12} \tilde{h}^4 + 512 \tilde{B}^{18} \tilde{h}^2 + 43008 \tilde{B}^{30} \right) = 0 \quad (62)
\]

Thus there is no solution. Consideration of the \( \tilde{B}^{24} \tilde{h}^2 \) and \( \tilde{B}^{24} \) terms shows that this is the only linear combination, up to an overall factor, that gives a non-negative coefficient to every term.

I then studied the \( \tilde{H}^5 \times \tilde{H}^2 \) and \( \tilde{H}^4 \times \tilde{H}^3 \) cases, including all possible types of flux. In both cases, there was a unique linear combination of the 3 field equations, up to an overall factor, that gave a non-negative coefficient to every term, and thus proved there was no solution. I then studied the \( \tilde{H}^5 \times S^2, \tilde{H}^4 \times S^3 \), and \( \tilde{H}^3 \times S^4 \) cases, and also the \( \tilde{H}^3 \times S^2 \times S^2 \) case, with the radii of the two \( S^2 \)’s set equal after deriving the field equations. There was no longer any simple proof that no solution existed, but searches by various methods found no solution. TeXmacs files containing some of these calculations, together with Cadabra scripts and source code for a C++ program used in one of the searches, are available on the web page [141].

I therefore returned to the \( \tilde{H}^7 \) case, and studied whether a field redefinition could produce a solution, even though, in principle, field redefinitions should have no physical
effects when all orders of perturbation theory are included. Trying a redefinition that affects only the coefficient of the highest power of $\frac{1}{B}$ in the action, one finds that as soon as this coefficient becomes negative, a solution appears. One might suspect that a solution that disappears at the leading order it exists on performing a field redefinition is unphysical, and that this will show up through the presence of tachyonic modes or through large corrections at higher orders in $\kappa_{11}^{2/3}$.

However there is no absolutely preferred set of coordinates in “field space”, and I will now show that a solution also appears when we make a field redefinition that has the same form as the redefinition that transforms the action with the unexpanded $\epsilon_{11}\epsilon_{11}X^4$ term to the form that occurs naturally when supersymmetry is systematically implemented by the Noether method [66, 45, 67, 68], except that instead of making the redefinition with coefficient 1, we make it with a coefficient $< -0.34$. Thus a solution appears after a redefinition that is much closer to the identity than a redefinition that is certainly admissible.

To transform the action to the form that would naturally be obtained by the Noether method, we have to remove all the Ricci terms from the expansion of $\epsilon_{11}\epsilon_{11}X^4$ by using the classical Einstein equations (29). Let:

\[ V_{IK} \equiv X_{IJK}^J = R_{IK} - \frac{1}{8} H_I^{JMN} H_{KJMN} \]  

\[ U \equiv V_I^I = R - \frac{1}{8} H^{IJMN} H_{IJMN} \]

where I used (52). Let $\tilde{V}_{IK}$ and $\tilde{U}$ be the result of using the classical Einstein equations (29) to replace the Ricci terms in $V_{IK}$ and $U$ by $H^2$ terms. Then:

\[ \tilde{V}_{IK} = -\frac{1}{24} H_I^{JMN} H_{KJMN} - \frac{1}{144} H^{OLMN} H_{OLMN} G_{IK} \]  

\[ \tilde{U} = -\frac{17}{144} H^{IJMN} H_{IJMN} \]

So for the $\tilde{H}^7$ case we find from (55):

\[ \tilde{V}_{\mu\nu} = -\frac{7}{24} \frac{h^2}{B^8} G_{\mu\nu}, \quad \tilde{V}_{ij} = -\frac{13}{24} \frac{h^2}{B^8} G_{ij} \]  

\[ \tilde{U} = -\frac{119}{24} \frac{h^2}{B^8} \]
Now using Cadabra we find that:

\[
\begin{align*}
\frac{-1}{4!}\epsilon_{11}\epsilon_{11}X^4 &= 4U^4 - 96V_{1J}V_{1J}U^2 + 24X_{IJKL}X_{IJKL}U^2 \\
&\quad + 384X_{IJKL}V_{IK}V_{JL}U + 256V_{1J}V_{IK}V_{JK}U \\
&\quad - 384X_{IJKL}X_{IJKM}V_{LM}U + 32X_{IJKL}X_{IJMN}X_{KLMO}U \\
&\quad - 128X_{IJKL}X_{IMKN}X_{JNLM}U + 192V_{1J}V_{IK}V_{KL}U \\
&\quad - 96X_{IJKL}X_{IJKM}V_{MN} - 768X_{IJKL}X_{IMKN}V_{ILVM} \\
&\quad - 1536X_{IJKL}V_{IK}V_{JLM} + 768X_{IJKL}X_{IJKM}X_{LNMO}V_{NO} \\
&\quad - 384V_{1J}V_{IK}V_{JL}V_{KL} + 768X_{IJKL}X_{IJKM}V_{LNVM} \\
&\quad + 384X_{IJKL}X_{IJMN}V_{KM}V_{LN} - 384X_{IJKL}X_{IJMN}X_{KLMO}V_{NO} \\
&\quad + 768X_{IJKL}X_{IMKN}V_{JNLO}V_{MO} \\
&\quad + 12X_{IJKL}X_{IJKL}X_{MNOP}X_{MNOP} - 192X_{IJKL}X_{IJKM}X_{LNOP}X_{MNOP} \\
&\quad + 24X_{IJKL}X_{IJMN}X_{KLOP}X_{MNOP} - 384X_{IJKL}X_{IJMN}X_{KOMP}X_{LONP} \\
&\quad + 192X_{IJKL}X_{IMKN}X_{JLNP}X_{MONP} - 384X_{IJKL}X_{IMKN}X_{JLMP}X_{LONP}
\end{align*}
\]

(69)

This used the fact that \(X_{IJKL}\) in (52) has the same monoterm symmetries as the Riemann tensor, namely antisymmetry in the first two indices, antisymmetry in the last two indices, and symmetry under exchange of the first two indices with the last two indices, although it does not satisfy the Ricci cyclic identity, because it has a piece with [1111] Young tableau symmetry.

Let \(Z \equiv -\frac{1}{4!}\epsilon_{11}\epsilon_{11}X^4\), and let \(\tilde{Z}\) be the result of replacing \(V_{1J}\) by \(\tilde{V}_{1J}\) and \(U\) by \(\tilde{U}\) in the right-hand side of (69), so that \(Z = \tilde{Z}\) if the classical Einstein equations (29) are satisfied. Then \((\tilde{Z} - Z)\), with an arbitrary coefficient that is not too large, is an expression that can be added to the action at this order by means of a field redefinition. To reach the form of the action that would naturally be obtained by imposing supersymmetry by the Noether method, one has to add \((\tilde{Z} - Z)\) with coefficient 1 to the integrand of the dimension 8 local term (39), on page 24, which is here being taken with \(\Xi^{(\text{flux})}\) discarded, and \(R_{IJKL}\) replaced by the \(X_{IJKL}\) defined in (52), on page 27.

From (67), (68), and (69), we find using Cadabra and Maxima [142] that for the \(\bar{H}^7\) case, for which \(Z = 0\):

\[
\tilde{Z} - Z = 7 \left( -\frac{32832}{B^8} + \frac{17088h^2}{B^{14}} + \frac{161h^4}{2B^{26}} + \frac{20395h^6}{24B^{26}} + \frac{2225255h^8}{82944B^{32}} \right)
\]

(70)

Thus adding \((\tilde{Z} - Z)\) with a coefficient \(c\) to the integrand of (39) with the above replacements, we find that the action density in terms of the rescaled parameters (58),
with the 5-brane term omitted, becomes a constant positive multiple of:

\[
A^4 \tilde{B}^7 \left( -\frac{42}{B^2} - \frac{7 \tilde{h}^2}{8 B^8} + \left( \frac{1}{B^2} + \frac{\tilde{h}^2}{8 B^8} \right)^4 
+ c \left( -\frac{19}{21 B^8} + \frac{89 \tilde{h}^2}{189 B^{14}} + \frac{23 \tilde{h}^4}{10368 B^{20}} + \frac{20395 \tilde{h}^6}{870912 B^{26}} + \frac{2225255 \tilde{h}^8}{3009871872 B^{32}} \right) \right) \quad (71)
\]

Deriving the field equations, one finds that a flat \( \mathbb{R}^4 \) times \( \bar{H}^7 \) solution exists for \( c < c_{\text{max}} \equiv -\frac{734832}{2225255} \simeq -0.3302 \). Thus a solution exists for precisely the range of \( c \) for which the coefficient of the highest power of \( \frac{1}{B} \) in the action is negative. To study the field equations, it is convenient to define \( x \equiv \frac{\tilde{h}}{B^3} = \frac{h}{B^3} \). The equations then become two simultaneous linear equations for \( c \) and \( \tilde{B}^6 \). Solving the equations one finds a hyperbola-like relation between \( c \) and \( x \), with \( x \to +\infty \) as \( c \to c_{\text{max}} \) from below, and \( c \to -\infty \) as \( x \to x_{\text{min}} \simeq 0.9364 \) from above. The range of \( x \) perceived as the “corner” of the hyperbola depends on the range of \( x \) for which the curve is plotted, but when the lower limit of the plot is about \( x = 3 \), so that \( c \) does not drop below about \( 2c_{\text{max}} \), the “corner” of the hyperbola is from about \( x = 3 \) to about \( x = 8 \).

One possibility is that this solution is unphysical, and either has tachyonic modes or is subject to very large corrections at higher orders in \( \kappa_{11}^{2/3} \). Another possibility is that there might be a “field redefinition group” similar to the renormalization group, and that at low orders of perturbation theory, we should use “coordinates in field space” best suited to the geometry being studied. Doing that would minimize the size of the higher-order corrections. Sensitivity to field redefinitions should decrease as higher-order corrections are included, so we should use the \textit{principle of minimal sensitivity} [70] to choose the best field redefinition at low orders of perturbation theory, as in perturbative QCD. \( c \) is the coefficient of the difference from the identity of a perturbative field redefinition, that we are neglecting terms of quadratic and higher order in the Taylor expansion of the action with respect to, so the magnitude of \( c \) should also be as small as possible. Thus we should choose \( c \) at the “corner” of the hyperbola in the \((x, c)\) plane. Choosing \( x = 5.5 \), we find \( c \simeq -0.38 \), and \( \tilde{B} \simeq 0.91 \), which corresponds to

\[
B \simeq 0.43\kappa_{11}^{2/9} \quad (72)
\]

as the best value of the radius of curvature of the \( \bar{H}^7 \). The best value of \( h = B^3 x \) is then \( h \simeq 0.45\kappa_{11}^{2/3} \).
The first Pontryagin class of an $\tilde{H}^7$ is 0, so the fluxes $H_{ijkl}$ wrapping individual 4-cycles of the $\tilde{H}^7$ are constrained by flux quantization such that the integral of $\frac{1}{4!} H_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l$ over a 4-cycle is equal to $\frac{2\pi}{T_2}$ times an integer, where $T_2 = \frac{1}{2} \left( \frac{4\pi}{\kappa_{11}} \right)^{2/3}$ is the fundamental membrane tension \cite{71, 72, 143}. If the average flux $h$ is not exactly equal to the value required by the solution of the field equations with flat extended dimensions, then the extended dimensions will have to be slightly curved to compensate. With $\kappa_{11}^{2/9}$ around an inverse TeV, thus around $10^{-19}$ metres, the observed de Sitter radius of the extended dimensions, which is $16.0 \text{ Gyr} = 1.51 \times 10^{26}$ metres $= 0.94 \times 10^{61}\sqrt{G_N}$ \cite{75}, is around $10^{45}\kappa_{11}^{2/9}$, so the curvature of the extended dimensions is around $10^{-90}\kappa_{11}^{-4/9}$. For an $\tilde{H}^7$ with intrinsic volume $\tilde{V}_7$ around $10^{34}$, typical 4-cycles of the $\tilde{H}^7$ also have intrinsic 4-area around $10^{34}$, so individual flux intensities can be adjusted to a relative precision of around $10^{-34}$. By choosing incommensurate flux numbers, the average intensity of the fluxes wrapping any three 4-cycles can be adjusted to a relative precision of around $10^{-102}$. The number of independent fluxes is the number of linearly independent harmonic 4-forms, which is the 4th Betti number, which from the discussion on the pages before and after \cite{22}, near the end of section 1, is likely to be roughly in proportion to the intrinsic volume, up to a logarithmic correction factor \cite{21, 95, 104}. Thus flux quantization does not prevent the observed de Sitter radius from being attained.

The second derivative of (71) with respect to $\tilde{B}$, at the above solution, is $\simeq -19000 \tilde{A}^4$, so a deviation of $\tilde{B}$ from its value $\tilde{B} \simeq 0.91$ at the solution has positive energy, and this would normally correspond to positive mass\(^2\) for the mode where the deviation of $\tilde{B}$ from 0.91 depends on position in the 4 extended dimensions. This mode is a scalar on the extended dimensions, and its only position dependence is its position dependence along the extended dimensions. For a first approximation I shall take the derivative part of its kinetic term as given by the Einstein action plus gauge-fixing terms. The Einstein action density, up to a positive overall factor and a total derivative term which I shall temporarily drop, is:

$$\sqrt{-G}R = -\frac{1}{4}\sqrt{-G}G^{IJ}G^{KL}G^{MN}(\partial_I G_{KM}\partial_J G_{LN} - 2\partial_I G_{NK}\partial_M G_{JL}$$

$$+ 2\partial_I G_{KL}\partial_M G_{NJ} - \partial_I G_{KL}\partial_J G_{MN})$$

(73)

Extracting the part of this which has $\partial_{\mu}$ derivatives on $B$ in the metric ansatz (54), on page 29 we see that only the first and last terms in the right-hand side of (73)
contribute, giving
\[ -\frac{1}{4}\sqrt{-G} \frac{1}{B^2} g^{kl} g^{mn} (g_{km}g_{ln} - g_{kl}g_{mn}) \partial_\mu B \partial^\mu B \]
\[ = \frac{21}{2}\sqrt{-G} \frac{1}{B^2} \partial_\mu B \partial^\mu B. \]  \hfill (74)

Thus this is a mode for which the kinetic term coming from the Einstein action has the wrong sign, and its positive mass could potentially lead to exponential instead of oscillatory time dependence. However a gauge-fixing term such as
\[ \frac{1}{2\alpha} \sqrt{-G} G^{IJ} G^{KL} (\partial_K \varphi_{LI} - \beta \partial_I \varphi_{KL}) G^{MN} (\partial_M \varphi_{NJ} - \beta \partial_J \varphi_{MN}), \]  \hfill (75)

where \( \varphi_{IJ} \) is the perturbation of \( G_{IJ} \) from the solution found above, and any value of \( \beta \) except \( \beta = 1 \) is allowed, will generically alter the derivative part of the kinetic term of this mode, and can change its sign. \( \beta = 1 \) is not allowed because \( G^{KL} (\partial_K \varphi_{LI} - \partial_I \varphi_{KL}) \) has a residual invariance under \( \varphi_{IJ} \rightarrow \varphi_{IJ} + \partial_I \partial_J \lambda \), and can thus not be set equal to an arbitrary 11-vector by a gauge transformation \( \varphi_{IJ} \rightarrow \varphi_{IJ} + D_I \varepsilon_J + D_J \varepsilon_I. \) Thus no exponential time dependence of this mode is observable.

4 The HW boundary

Assuming provisionally that the solution found in the previous subsection is valid, I shall now show that it might be possible to introduce an HW boundary, with the region around the boundary treated in a consistent semiclassical approximation. This can at best be only semi-quantitative, because no higher derivative corrections to the boundary conditions, corresponding to the leading quantum corrections used in the bulk, are introduced. However higher derivative boundary conditions have multiple solutions, most of which will be unstable, with tachyonic excitations. We have to select the solution free of tachyons, which will be the smoothest one, for which the higher derivative corrections have the least possible significance. Thus it is useful to find a solution that satisfies the semiclassical boundary conditions (30) and (33), since it should provide a good starting point for finding the correct solution to the boundary conditions with the higher derivative corrections included. I shall assume that the boundary condition (33) for the fluxes can be satisfied in an average sense, and consider the boundary condition (31) for the metric.

For simplicity I shall consider a boundary whose topology is a smooth compact quotient \( \bar{H}^6 \) of \( H^6 \), although it is not known whether \( \bar{H}^6 \) of sufficiently small intrinsic volume \( \bar{V}_6 \) actually exist. Better choices for the boundary topology might be the
Cartesian product of two small intrinsic volume \( \bar{H}^3 \)'s, or an Anderson 6-manifold \( \bar{H}^7 \), but it does not appear to be known whether an \( \bar{H}^7 \) can have minimal-area 6-cycles of these topologies. I shall assume that the boundary is somewhere near a minimal-area 6-cycle of the closed \( \bar{H}^7 \) that is cut into two disconnected parts along the boundary, and use the notation of section 2 starting on page 20. The metric in this region has the form:

\[
ds_{11}^2 = G_{IJ} dx^I dx^J = a(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu + b(y)^2 h_{ab} dx^a dx^b + dy^2 \tag{76}
\]

Here \( a(y) \) is as in (4), on page 7, so \( a(y) \to A \) away from the boundary, and \( a(y) \to 1 \) on the boundary. Indices \( a, b, c, \ldots \) are tangential to the boundary, so that \( x^A \) in (4) is now \((x^a, y)\), and \( h_{ab} \) is a metric of sectional curvature \(-1\).

The nonvanishing Riemann tensor components for the metric ansatz (76) are:

\[
R_{\mu\nu\sigma\tau} = \frac{\dot{a}^2}{a^2} \left( G_{\mu\sigma} G_{\nu\tau} - G_{\mu\tau} G_{\nu\sigma} \right),
\]

\[
R_{abcd} = \frac{b^2}{b^2} R_{abcd}(h) + \frac{\dot{b}^2}{b^2} \left( G_{ad} G_{bc} - G_{ac} G_{bd} \right),
\]

\[
R_{\mu\nu a b} = -\frac{\dot{a}}{a} G_{\mu\nu} G_{ab}, \quad R_{\mu\nu y y} = -\frac{\dot{a}}{a} G_{\mu\nu}, \quad R_{ayby} = -\frac{\ddot{b}}{b} G_{ab},
\tag{77}
\]

together with the components related to these by the antisymmetries of the Riemann tensor under the interchange of its first two indices, and under the interchange of its last two indices, where a dot denotes differentiation with respect to \( y \), and \( R_{abcd}(h) = h_{ad} h_{bc} - h_{ac} h_{bd} \) denotes the Riemann tensor calculated from the six-dimensional metric \( h_{ab} \).

Thus the nonvanishing Ricci tensor components for the metric ansatz (76) are:

\[
R_{\nu\tau} = -3 \frac{\dot{a}^2}{a^2} G_{\nu\tau} - 6 \frac{\dot{a} \dot{b}}{ab} G_{\nu\tau} - \frac{\ddot{a}}{a} G_{\nu\tau} = - \left( \frac{5 \dot{a}^2}{a^2} + 6 \frac{\dot{a} \dot{b}}{ab} + \frac{\ddot{a}}{a} \right) G_{\nu\tau}
\]

\[
R_{bd} = R_{bd}(h) - 5 \frac{\dot{b}^2}{b^2} G_{bd} - 4 \frac{\dot{a} \dot{b}}{ab} G_{bd} = - \left( \frac{5 \dot{a}^2}{a^2} + 6 \frac{\dot{a} \dot{b}}{ab} + \frac{\ddot{a}}{a} \right) G_{bd}
\]

\[
R_{yy} = -\frac{\dddot{a}}{a} - 6 \frac{\ddot{b}}{b}
\tag{78}
\]

where I used that \( R_{bd}(h) = -5 h_{bd} \).
I shall look for a solution where the metric in the region of the boundary is a small perturbation of what it would have been if the boundary was not there. In the absence of the boundary \( a \) is equal to the constant \( A \), and \( R_{ABCD} = \frac{1}{B^2} (G_{AD}G_{BC} - G_{AC}G_{BD}) \), the Riemann tensor on the closed \( \bar{H}^7 \), where \( B \) has the value \( B \approx 0.43 \kappa_{11}^{2/9} \) found in the previous section. Thus from the second line of (77), we require \( \frac{1}{b^2} + \frac{\kappa^2}{9} = \frac{1}{B^2} \), and assuming \( b(y) \) has its minimum value at \( y = 0 \), the solution of this is \( b = B \cosh \left( \frac{y}{B} \right) \). This then also gives the correct value \( R_{yby} = -\frac{1}{B^2} G_{ab} \), from the third line of (77). I shall now work out what the effective energy-momentum tensor components must be, in order for this form of \( b(y) \), with constant \( a(y) = A \), to solve the classical Einstein equations.

I assume the effective energy-momentum tensor \( T_{I,J} \) in the absence of the boundary has the form:

\[
T_{\mu\nu} = t^{(1)}(y) G_{\mu\nu}, \quad T_{ab} = t^{(2)}(y) G_{ab}, \quad T_{yy} = t^{(3)}(y)
\]  

(79)

The Einstein equations can be written:

\[
R_{I,J} + \kappa_{11}^2 \left( T_{I,J} - \frac{1}{9} G_{I,J} G^{K,L} T_{K,L} \right) = 0,
\]

(80)

and using the Ricci tensor components from (78), these are:

\[
\frac{\ddot{a}}{a} + 3 \frac{\dot{a}^2}{a^2} + 6 \frac{\dot{a} \dot{b}}{ab} + \frac{\kappa_{11}^2}{9} \left( 5 t^{(1)}(y) - 6 t^{(2)}(y) - t^{(3)}(y) \right) = 0
\]

(81)

\[
\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} + 4 \frac{\dot{a} \dot{b}}{ab} + \frac{5 \lambda}{b^2} + \frac{\kappa_{11}^2}{9} \left( -4 t^{(1)}(y) + 3 t^{(2)}(y) - t^{(3)}(y) \right) = 0
\]

(82)

\[
4 \frac{\ddot{a}}{a} + 6 \frac{\dot{b}}{b} + \frac{\kappa_{11}^2}{9} \left( -4 t^{(1)}(y) - 6 t^{(2)}(y) + 8 t^{(3)}(y) \right) = 0
\]

(83)

In (82) I have introduced a factor \( \lambda \) multiplying the \( \frac{\dot{b}}{b^2} \) term as a rough way of allowing for the possibility that the boundary might be, for example, the Cartesian product of two \( \bar{H}^3 \)'s, or an Anderson 6-manifold [144]. Setting \( \lambda = 1 \) gives the equation that follows from the Ricci tensor components (78).

Eliminating the double derivatives between the three Einstein equations, we find:

\[
\frac{\dot{a}^2}{a^2} + 4 \frac{\dot{a} \dot{b}}{ab} + \frac{5 \dot{b}^2}{2b^2} + \frac{5 \lambda}{2b^2} - \frac{1}{6} \kappa_{11}^2 t^{(3)} = 0,
\]

(84)

hence:

\[
\frac{\dot{a}}{a} = \frac{\dot{b}}{2b} - \frac{1}{2} \sqrt{\frac{\dot{b}^2}{b^2} - \frac{10 \lambda}{b^2} + \frac{2}{3} \kappa_{11}^2 t^{(3)}},
\]

(85)
where I assumed that \( \dot{b} \) does not change sign between the boundary and the main part of the bulk, so the sign of the square root follows from \( a = 0 \) and the convention stated at the start of section 2 on page 20 that \( y > y_1 \) in the bulk, where \( y_1 \) is the value of \( y \) at the boundary.

The second Einstein equation, (82), now becomes:

\[
\frac{\ddot{b}}{b} - 3 \frac{\dot{b}^2}{b^2} + 2 \frac{\dot{b}}{b} \sqrt{\frac{6 \dot{b}^2}{b^2} - \frac{10 \lambda}{b^2} + \frac{2}{3} \kappa_1^2 t^{(3)} + \frac{5 \lambda}{b^2} + \frac{\kappa_1^2}{9} (-4t^{(1)} + 3t^{(2)} - t^{(3)})} = 0 \tag{86}
\]

From subsection 2.4 of [39], a solution of (85) and (86) solves all three Einstein equations, provided the energy-momentum tensor is conserved, and the square root does not vanish identically for the solution. The square root does not vanish for the solution I will consider, so it is sufficient to consider (85) and (86).

Requiring that (85) and (86) are solved by \( a(y) = A, b(y) = B \cosh \frac{y}{B} \), we find the effective energy-momentum tensor components:

\[
\frac{2}{3} \kappa_1^2 t^{(3)} = \frac{10 \sinh^2 \frac{y}{B} + 10 \lambda}{B^2 \cosh^2 \frac{y}{B}} \tag{87}
\]

\[
\frac{\kappa_1^2}{9} (-4t^{(1)} + 3t^{(2)} - t^{(3)}) = - \left( \frac{1}{B^2} + 5 \frac{\sinh \frac{y}{B}}{B \cosh^2 \frac{y}{B}} + \frac{5 \lambda}{B^2 \cosh^2 \frac{y}{B}} \right) \tag{88}
\]

Substituting in these values of the \( t^{(i)} \), the Einstein equations (85) and (86) become:

\[
\frac{\dot{a}}{a} = -2 \frac{\dot{b}}{b} + \frac{1}{2} \sqrt{\frac{6 \dot{b}^2}{b^2} - \frac{10 \lambda}{b^2} + \frac{10 \sinh \frac{y}{B}}{B \cosh \frac{y}{B}} + \frac{10 \lambda}{B \cosh^2 \frac{y}{B}}} \tag{89}
\]

\[
\frac{\ddot{b}}{b} - 3 \frac{\dot{b}^2}{b^2} + 2 \frac{\dot{b}}{b} \sqrt{\frac{6 \dot{b}^2}{b^2} - \frac{10 \lambda}{b^2} + \frac{10 \sinh \frac{y}{B}}{B \cosh \frac{y}{B}} + \frac{10 \lambda}{B \cosh^2 \frac{y}{B}}} + \frac{5 \lambda}{b^2} - \left( \frac{1}{B^2} + 5 \frac{\sinh \frac{y}{B}}{B \cosh^2 \frac{y}{B}} + \frac{5 \lambda}{B^2 \cosh^2 \frac{y}{B}} \right) = 0 \tag{90}
\]

We now let \( a(y) = (1 + p(y)) A, b(y) = (1 + q(y)) B \cosh \frac{y}{B} \), and expand to first order in \( p \) and \( q \). Then (89) and (90) become:

\[
\dot{p} = -\frac{5}{4} \dot{q} + \frac{5 \lambda q}{4 B \sinh \frac{y}{B} \cosh \frac{y}{B}} \tag{91}
\]

\[
\ddot{q} + 7 \dot{q} \frac{\sinh \frac{y}{B}}{B \cosh \frac{y}{B}} - \frac{5 \lambda q}{B^2 \cosh^2 \frac{y}{B}} = 0 \tag{92}
\]
For a first estimate of the boundary conditions (30), on page 21, I shall neglect the flux terms in $\bar{R}_{UVWX}$, defined in (27), by assuming, if necessary, that $H_{UVWX}$ is smaller than its average value, near the boundary. The energy-momentum tensor on the boundary is then:

$$\tilde{T}_{UV} = \frac{1}{16\pi\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \left(\frac{4}{30} \text{tr} F_{UV} F_V W - \frac{1}{30} \tilde{G}_{UV} \text{tr} F_{WX} F_W X - 2 R_{UVWX} R_{WXY} + \frac{1}{2} \tilde{G}_{UV} R_{WXYZ} R_{WXYZ} \right)$$  (93)

I shall work in the leading order of the Lukas-Ovrut-Waldram [134] harmonic expansion of the energy-momentum tensor (93) on the boundary, and assume that $\text{tr} F_{ac} F^c_b$ is a multiple of $\tilde{G}_{ab}$, and also that $\hat{R}_{acde} \hat{R}_{b cde}$ is a multiple of $\tilde{G}_{ab}$, which it is if the boundary is an $\bar{H}^6$, but is not if the boundary is an $\bar{H}^3 \times \bar{H}^3$, for example. We then find:

$$\tilde{T}_{\mu\nu} = \frac{1}{16\pi\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \left(- \frac{1}{30} \text{tr} F_{cd} F^{cd} + \frac{1}{2} \hat{R}_{cdef} \hat{R}^{cdef} \right) \tilde{G}_{\mu\nu}$$  (94)

$$\tilde{T}_{ab} = \frac{1}{16\pi\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \left(- \frac{1}{90} \text{tr} F_{cd} F^{cd} + \frac{1}{6} \hat{R}_{cdef} \hat{R}^{cdef} \right) \tilde{G}_{ab}$$  (95)

We define $\tilde{t}^{(1)}$ and $\tilde{t}^{(2)}$ by:

$$\tilde{T}_{\mu\nu} = \tilde{t}^{(1)} \tilde{G}_{\mu\nu}, \quad \tilde{T}_{ab} = \tilde{t}^{(2)} \tilde{G}_{ab}$$  (96)

Then by subsection 2.3.9 of [39], the boundary conditions for $a(y)$ and $b(y)$ are:

$$\left. \frac{\dot{a}}{a} \right|_{y=y_1} = \frac{\kappa_{11}^2}{18} (-5\tilde{t}^{(1)} + 6\tilde{t}^{(2)}), \quad \left. \frac{\dot{b}}{b} \right|_{y=y_1} = \frac{\kappa_{11}^2}{18} (4\tilde{t}^{(1)} - 3\tilde{t}^{(2)})$$  (97)

Thus:

$$\left. \frac{\dot{a}}{a} \right|_{y=y_1} = -\frac{1}{96\pi} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \left(- \frac{1}{30} \text{tr} F_{cd} F^{cd} + \frac{1}{2} \hat{R}_{cdef} \hat{R}^{cdef} \right)$$  (98)

$$\left. \frac{\dot{b}}{b} \right|_{y=y_1} = \frac{1}{96\pi} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \left(- \frac{1}{30} \text{tr} F_{cd} F^{cd} + \frac{1}{2} \hat{R}_{cdef} \hat{R}^{cdef} \right)$$  (99)

The vacuum Yang-Mills fields on the boundary are quantized, for example by a Dirac quantization condition as in subsection 5.3 of [39] if they are in the Cartan subalgebra of $E_8$, so the right-hand sides of (98) and (99) are proportional to $b_1^{-s}$, where $b_1 \equiv b(y_1)$ is the value of $b$ on the boundary. Moss’s derivation of the $R_{UVWX} R^{UVWX}$ term in (26) used an expansion scheme in which Ricci tensor and scalar terms, if present,
would only show up at higher orders. If the $R_{UVWX}R^{UVWX}$ term was in fact the first term in a Lovelock-Gauss-Bonnet term of the form $R_{UVWX}R^{UVWX} - 4R_{UV}R^{UV} + R^2$, the size of the curvature terms in (98) and (99) would be increased by a factor of 6 if the boundary is an $\tilde{H}^6$, with the main contribution coming from the square of the Ricci scalar. This would make it easier to keep the right-hand side of (98) negative and the right-hand side of (99) positive, while introducing enough vacuum Yang-Mills fields to break $E_8$ to $SU(3) \times SU(2) \times U(1)_Y$ and produce the chiral fermions.

To find the solution of (91) and (92) such that $p$ and $q$ tend to 0 as $y \to \infty$, we define $\tanh \frac{y}{2} = x$, so that $x \to 1$ as $y \to \infty$. The equations then become:

$$\frac{dp}{dx} = -\frac{5}{4} \frac{dq}{dx} + \frac{5\lambda q}{4x}$$ (100)

$$\left(1 - x^2\right) \frac{d^2q}{dx^2} + 5x \frac{dq}{dx} - 5\lambda q = 0$$ (101)

For (101), we try $q \to (1 - x)^\alpha$ near $x = 1$, which gives $\alpha = \frac{7}{2}$. We then find the expansion:

$$q = (1 - x)^{\frac{7}{2}} + \frac{(20\lambda - 35) (1 - x)^{\frac{9}{2}}}{36} + \frac{(400\lambda^2 - 1240\lambda + 945) (1 - x)^{\frac{11}{2}}}{3168}$$

$$+ \frac{(8000\lambda^3 - 29200\lambda^2 + 32540\lambda - 10395) (1 - x)^{\frac{13}{2}}}{494208} + \ldots$$ (102)

For $\lambda = 1$, (102) looks qualitatively like the base of a parabola centred at $x = 1$, and is $\approx 0.6184$ for $x = 0$.

The solution of (100) is $p = -\frac{5}{4} q - \frac{5\lambda}{4} \int_x^1 \frac{q(x')}{x'} dx'$. Thus $p$ is $\leq 0$ for $0 < x \leq 1$, and has a logarithmic singularity as $x \to 0$.

We now try to satisfy the boundary conditions (98) and (99) with

$$a = (1 + kp) A, \quad b = (1 + kq) B \cosh \sqrt{6} y, \quad c = (1 + k\rho) C \sinh \sqrt{6} y,$$

where $p$ and $q$ are the solutions just found, and $k$ is a constant.

We write the boundary conditions (98) and (99) as:

$$\left. \frac{\dot{a}}{a} \right|_{y = y_1} = -\rho \frac{\kappa_{11}^{2/3}}{b_1^{4/3}}, \quad \left. \frac{\dot{b}}{b} \right|_{y = y_1} = \rho \frac{\kappa_{11}^{2/3}}{b_1^{4/3}}$$ (103)

where the number $\rho$ is:

$$\rho \equiv \frac{1}{V_6} \int_\beta d^6 \sqrt{h} \frac{1}{96\pi (4\pi)^{2/3}} h^{bc} h^{bd} \left( -\frac{1}{30} \text{Tr} F_{ab} F_{cd} + \frac{1}{2} \bar R_{ab}^e \bar R_{cde}^f \right)$$ (104)

where $V_6 \equiv \int_\beta d^6 \sqrt{\tilde{h}}$ is the intrinsic volume of the boundary $\beta$ in the metric $h_{ab}$, which if the boundary is an $\tilde{H}^6$ is defined to have sectional curvature $-1$. If the
boundary is an $\mathcal{H}^6$, with $\bar{R}^{c}_{ab}e_f = h_{af}\delta^c_b - h_{bf}\delta^c_a$, then the Riemann$^2$ term in $\rho$ is $\frac{60}{192\pi(4\pi)^{3/2}} \approx 0.01840$. If the spin connection is naturally embedded in the $E_8$ Yang-Mills gauge group [145, 146, 147, 148], which by the Bianchi identity for the Riemann tensor gives a solution of the classical Yang-Mills equations if $h_{ab}$ is an Einstein metric, then the Yang-Mills term in $\rho$ is $-2$ times the Riemann$^2$ term, so $\rho$ is negative.

I shall assume that for moderate values of $\bar{V}_6$ it is possible to introduce vacuum gauge fields in the $E_8$ Cartan subalgebra, topologically stabilized by a Dirac quantization condition as in subsection 5.3 of [39], with field strengths proportional to 4, 5, or 6 of a larger number of linearly independent harmonic 2-forms on the boundary, such that $E_8$ is broken to $SU(3) \times SU(2) \times U(1)_Y$ and the Standard Model chiral fermion zero modes are produced, and the Yang-Mills term in $\rho$ is sufficiently “diluted” that $\rho$ is positive. If the Riemann$^2$ term in (26) is the first term of the Lovelock-Gauss-Bonnet combination, then the Riemann and Ricci contribution to $\rho$ for $\mathcal{H}^6$ would be 6 times larger, so $\rho$ would be positive even if the spin connection was embedded in the gauge group.

With $\tanh \frac{x}{B} = x$ as before, and $x_1$ denoting the value of $x$ at the boundary, the sum of the boundary conditions (103) gives:

$$k \left(1 - x_1^2\right) \left(-\frac{1}{4} \frac{dq}{dx} \bigg|_{x=x_1} + \frac{5q(x_1)}{4x_1}\right) + x_1 = 0,$$

(105)

where (100) has been used with $\lambda = 1$. The logarithmic singularity of $p$ as $x \to 0_+$ means that we require $x_1 > 0$ for the assumption that $|kp| \ll 1$ to be valid, so since $\frac{dq}{dx} \leq 0$ and $q \geq 0$ for $0 \leq x \leq 1$, (105) implies that $k < 0$. Using $B \simeq 0.43\kappa_{11}^{2/9}$, from (72), on page 33, the second equation of (103) becomes:

$$k \left(1 - x_1^2\right) \frac{dq}{dx} \bigg|_{x=x_1} + x_1 \simeq \frac{201\rho(1 - 4kq(x_1))}{\left(\frac{1+x_1}{1-x_1} + \frac{1-x_1}{1+x_1}\right)^4},$$

(106)

Using (105) to express $k$ in terms of $x_1$, we find that $kq_1 \equiv kq(x_1)$, as a function of $x_1$, looks qualitatively like an upside-down parabola with a peak value of 0 at $x_1 = 0$, and $\simeq -0.25$ at $x_1 \simeq 0.63$. Substituting for $k$ from (105), we find from (106) that for $\rho = 0.01840$, $x_1 \simeq 0.165$, for which we have $kp_1 \equiv kp(x_1) \simeq 0.043$, hence $A \simeq 0.96$, and $kq_1 \simeq -0.020$, hence $b_1 \simeq 0.99B \simeq 0.43\kappa_{11}^{2/9}$. Thus working to first order in $kp$ and $kq$ has been justified. For $\rho \simeq 0.11$, corresponding to an $\mathcal{H}^6$ boundary with no vacuum Yang-Mills fields, if the Riemann$^2$ term in (26) is the start of the Lovelock-
Gauss-Bonnet combination, we find $x_1 \simeq 0.498$, for which $kp_1 \simeq 0.24$, hence $A \simeq 0.81$, and $kq_1 \simeq -0.16$, hence $b_1 \simeq 0.97B \simeq 0.42\kappa^{2/9}_{11}$.

From (105) we find that $\frac{b_1}{B}$, as a function of $x_1$, decreases smoothly from a peak value of 1 at $x_1 = 0$, to a minimum value $\simeq 0.967$ at $x_1 \simeq 0.56$, which is around the largest value of $x_1$ for which it is reasonable to work to first order in $kp$ and $kq$.

The Giudice-Rattazzi-Wells (GRW) estimate of the expansion parameter for graviton loop corrections in 11 dimensions, in the sense that perturbation theory must fail when the expansion parameter is $> 1$, is:

$$\frac{\kappa^{2/9}_{11}}{945 (2\pi)^4} \left(\frac{E}{2\pi}\right)^{9/2} \simeq \left(0.03\kappa^{2/9}_{11} E\right)^{9/2},$$

(107)

where $E$ is the relevant energy of the process. For a wave of wavelength $2\pi b_1$, $E$ is $\frac{1}{b_1}$, so the GRW estimate suggests that $b_1$ must be larger than around $0.03\kappa^{2/9}_{11}$. Thus for $b_1 \simeq 0.42\kappa^{2/9}_{11}$, this requirement is satisfied by a substantial margin.

From just before (15), on page 14, the first hypothesis considered in section 1 leads to the expectation that $0 < A \leq 0.72$ if $\frac{A}{B} = 1.8$ TeV, in order to satisfy the experimental constraint [51] that $\kappa^{2/9}_{11} \geq 0.36$ TeV, with $B \simeq 0.43\kappa^{2/9}_{11}$ from section 3, while from just before the start of section 2 if $\frac{A}{B} = 1.8$ TeV, then the requirement that $A \leq 1$, which follows from $k < 0$, leads to $\kappa^{2/9}_{11} \geq 0.78$ TeV, hence $M_{11} \geq 3.3$ TeV.

To study the Kaluza-Klein modes of the supergravity multiplet we have to expand $\Gamma_{\text{HW}}^{\text{(bos)}}$, in (38), including $\Gamma_{\text{SG}}^{(8,\text{bos})}$, in (39), to quadratic order in small fluctuations about the solution found above. For a first estimate I shall instead consider a massless scalar field $\Phi$ in the bulk, which is intended to represent a small fluctuation of a component of any of the supergravity fields, and retain only its classical action. Dropping initially also $R\Phi$ and $H^2\Phi$ terms, the equation for the small fluctuation $\Phi$ is then:

$$- \frac{1}{\sqrt{-G}} \partial_I \left( \sqrt{-G} G^{IJ} \partial_J \Phi \right) = 0$$

(108)

Trying an ansatz $\Phi(x^\mu, x^a, y) = \varphi(x^\mu) \psi(x^a, y)$, we find from (108) that:

$$- \frac{a^2}{b^2 \psi(x^c, y) \sqrt{h}} \partial_a \left( \sqrt{h} h^{ab} \partial_b \psi(x^c, y) \right) - \frac{a^2}{a^4 b^6 \psi(x^c, y)} \partial_y \left( a^4 b^6 \partial_y \psi(x^c, y) \right)$$

$$= \frac{1}{\varphi(x^\mu) \sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \varphi(x^\mu) \right)$$

(109)

The left-hand side of (109) is independent of $x^\mu$ and the right-hand side is independent of $x^a$ and $y$, hence each side must be a constant. The left-hand side is a positive
operator on a compact manifold so must be a non-negative constant $\frac{A^2}{B^2} m^2 \geq 0$. Away from the neighbourhood of the small boundary, the bulk is a compact hyperbolic 7-manifold $\tilde{H}^7$ with intrinsic volume $\tilde{V}_7 \sim 10^{34}$ and curvature radius $B \simeq 0.43 \kappa_{11}^{2/9}$, and $a = A$, so for most of the low-lying KK modes of the bulk, the KK masses will to a first approximation not be affected by the presence of the small boundary. So for most of the low-lying KK modes of the $d = 11$ supergravity multiplet, $m^2$ will to a first approximation be an intrinsic eigenvalue of the appropriate quantum-corrected differential operator, for example the negative of the quantum-corrected Laplace-Beltrami operator, on the corresponding closed $\tilde{H}^7$ with curvature radius 1, with the leading quantum corrections following from (39).

For the negative of the Laplace-Beltrami operator $\Delta$, with quantum corrections neglected, the left-hand side for the lightest massive modes is estimated from (13), on page 11, as $(\frac{n-1}{2^2} \frac{A^2}{B^2}) = 9 \frac{A^2}{B^2}$. So we find:

$$- \frac{1}{\sqrt{-g}} \partial_u \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \varphi (x^u) \right) + 9 \frac{A^2}{B^2} \varphi (x^u) = 0 \quad (110)$$

Thus for the first hypothesis in section 1, the mass of the lightest massive gravitational KK modes, as seen on the boundary, is $3 \frac{A}{B}$. Most of the modes in the bulk couple only with gravitational strength to the fields on the boundary, because the values on the boundary of their normalized wavefunctions are suppressed by the reciprocal of the square root of the large volume of the bulk. However if, as suggested in section 1, there are around $10^{30}$ approximately degenerate KK modes of the bulk with masses around $1.8$ TeV as seen on the boundary, the modes in the bump could equivalently be a relatively small number of linear combinations of these modes, whose normalized wavefunctions are large near the boundary and small away from the boundary, and these modes would have correspondingly large couplings to the fields on the boundary.

To keep the number $\rho$ defined in (104) positive, while introducing enough vacuum Yang-Mills fields to break $E_8$ to $SU(3) \times SU(2) \times U(1)_Y$ and produce the chiral fermions, it might be helpful if the boundary had a large enough intrinsic volume for the $E_8$ breaking and chiral fermions to arise from quantized Yang-Mills fluxes that are “dilute”, in the sense that only a small proportion of the possible quantized Yang-Mills fluxes are present, and these have flux numbers $\pm 1$. However if the boundary is an $\tilde{H}^6$, its intrinsic volume $\tilde{V}_6$ is approximately fixed by the value $\alpha_u$ of the QCD fine structure constant $\alpha_s = \frac{g^2_s}{4\pi}$ at unification, together with the fact that from the discussion after
Figure 1: Coxeter diagram for the $d = 6$ hyperbolic Coxeter polytope $T_3$

$$b_1 = b(y_1) \simeq B \simeq 0.43 \kappa_{11}^{2/9}.$$ 

I shall assume that the Yang-Mills $E_8$ at $y_1$ is broken to the Standard Model $SU(3) \times SU(2) \times U(1)_Y$ by topologically stabilized vacuum gauge fields in the Cartan subalgebra of the $E_8$. These were shown in subsection 5.3 of [39] to be restricted by Dirac quantization conditions to lie on a lattice of isolated points in the Cartan subalgebra. I shall assume that the unbroken $SU(3) \times SU(2) \times U(1)_Y$ is naturally embedded within an $SU(9)$ subgroup of $E_8$, and from subsection 5.2 of [39], the $E_8$ generators $T^A$ in (26) are normalized so that the structure constants of the $SU(3)$ subgroup have the same normalization as in the definition of $g_s$ in subsection 9.1 of [46], corresponding to $SU(3)$ generators $t^a$ normalized to $tr(t^a t^b) = \frac{1}{2} \delta^{ab}$, so performing the trace over the $E_8$ generators using $tr T^A T^B = 30 \delta^{AB}$, we find that

$$\alpha_U = \kappa_{11}^2 b_1^6 V_6 \left( \frac{4\pi}{\kappa_{11}} \right)^{2/3} = \frac{5.405}{V_6} \left( \frac{\kappa_{11}^{2/9}}{b_1} \right)^6 \simeq \frac{860}{V_6},$$ (111)

where $V_6$ is the volume of the boundary in the metric $h_{ab}$, which is the intrinsic volume of the boundary, if the boundary is an $\bar{H}^6$.

In the Dienes-Dudas-Gherghetta version of accelerated unification $\alpha_U \simeq \frac{1}{152}$ [2, 3], and in the Arkani-Hamed-Cohen-Georgi version, $\alpha_U \simeq \frac{1}{24}$ [4]. If the boundary is an $\bar{H}^6$, then by (111) these imply respectively $\bar{V}_6 \simeq 45000$ and $\bar{V}_6 \simeq 21000$, which by the generalized Gauss-Bonnet theorem (22), on page 18 correspond respectively to Euler numbers $\chi(\bar{H}^6) \simeq -2700$ and $\chi(\bar{H}^6) \simeq -1300$.

In 4 dimensions the Davis manifold [25] with $\chi = 26$, which is an orientable spin manifold [26], can be constructed from a hyperbolic Coxeter simplex by a variant of Selberg’s Lemma [108, 149]. There are no hyperbolic Coxeter simplexes in 5 or more dimensions [150], and no hyperbolic Coxeter polytopes with $n + 2 (n - 1)$-dimensional faces in 6 or more dimensions [151, 152]. There are exactly 3 hyperbolic Coxeter
polytopes with 9 5-dimensional faces in 6 dimensions \[153\], and the Coxeter diagram for one of these, which I will call \(T_3\), is shown in Figure 1. Twice the Gram matrix for \(T_3\) is:

\[
G \equiv \begin{pmatrix}
2 & -k & 0 & 0 & 0 & 0 & 0 \\
-k & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -2k & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 2 & -k \\
\end{pmatrix}, \quad (112)
\]

where \(k \equiv \frac{1 + \sqrt{5}}{2} = 2 \cos \frac{\pi}{5} \approx 1.6180\) is an algebraic integer that satisfies the equation:

\[
k^2 - k - 1 = 0. \quad (113)
\]

\(G\) has two linearly independent eigenvectors with eigenvalue 0, which may be chosen as:

\[
n_{(1)} \equiv \left( -3k - 2, \ -4k - 2, \ -3k - 1, \ -2k, \ -k, \ k + 1, \ k + 1, \ 1, \ 0 \right) \\
n_{(2)} \equiv \left( -5k - 3, \ -6k - 4, \ -4k - 3, \ -2k - 2, \ -1, \ k + 1, \ k + 1, \ 0, \ 1 \right) \quad (114)
\]

If we regard \(G_{ij}\) as a “metric” in 9 dimensions, then for any contravariant 9-vector \(v^i\) such that \(v^i v_i \neq 0\), where \(v_i \equiv v^j G_{ji}\), the matrix \(r(v)^i_j \equiv \delta^i_j - 2 \frac{v^i v^j}{v^k v_k}\) satisfies \(r(v)^i_k r(v)^k_j = \delta^i_j\) and \(G_{ik} r(v)^k_j = G_{ij}\), and is thus a reflection matrix with respect to the “metric” \(G_{ij}\). In particular, if we define \(u^i_j\), \(1 \leq i \leq 9\), to be the vector whose \(i\) th component is 1 and whose other components are 0, then since \(u^i_j u_{(i)j} = 2\) for all \(1 \leq i \leq 9\), and all the matrix elements of \(G\) are in the ring of algebraic integers \(\mathbb{Z}[k]\), the matrices \(r(u_{(i)})^j_k\), \(1 \leq i \leq 9\), are 9 reflection matrices whose matrix elements are in \(\mathbb{Z}[k]\). We now define \(\bar{u}^i_j \equiv u^i_j\), \(1 \leq i \leq 7\), \(\bar{u}^i_8 \equiv u^i_8 - n_{(1)}^i\), and \(\bar{u}^i_9 \equiv u^i_9 - n_{(2)}^i\). Then \(\bar{u}^i_j \bar{u}_{(i)j} = 2\) for all \(1 \leq i \leq 9\), and the last two components of \(\bar{u}^i_{(i)}\) are 0 for all \(1 \leq i \leq 9\). Let indices \(a, b, c, \ldots\) run from 1 to 7, \(H_{ab}\) be the \(7 \times 7\) symmetric matrix \(G_{ab}\), and \(\bar{u}^a_i\), \(1 \leq i \leq 9\), be the 7-vectors \(\bar{u}^a_{(i)}\). Then for all \(1 \leq i \leq 9\), the matrices \(r(\bar{u}_{(i)})^a_b\) are \(7 \times 7\) reflection matrices with respect to the “metric” \(H_{ab}\), and
their matrix elements are in \( \mathbb{Z}[k] \). Let:

\[
S \equiv \begin{pmatrix}
1 & k & \frac{3k+1}{2} & \frac{2k+1}{2} & 5k + 3 & 0 & 0 \\
0 & 1 & \frac{2k+4}{5} & k + 1 & 6k + 4 & 0 & 0 \\
0 & 0 & 1 & \frac{k+4}{2} & 4k + 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 2k + 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{k}{2} & \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\] (115)

Then \( S^T \mathcal{H} S = \text{diag} \left( 2, \frac{3-k}{2}, \frac{6-2k}{5}, \frac{2-k}{2}, -2k, 2, \frac{3-k}{2} \right) \), so \( \mathcal{H} \) has signature \((6,1)\), and \( \mathcal{H}_{ab} \) is the metric on 7-dimensional Minkowski space in a non-standard coordinate system. The dot products \( \bar{u}^a_{(i)} \bar{u}^b_{(j)} = \mathcal{H}_{ab} \bar{u}^a_{(i)} \bar{u}^b_{(j)} \) are equal to the corresponding elements of \( \mathcal{G}_{ij} \) for all \( 1 \leq i, j \leq 9 \), so the matrices \( r \left( \bar{u}^a_{(i)} \right)^b_a \) provide a representation of the Coxeter group \( \Gamma (T_3) \) corresponding to the Coxeter diagram \( T_3 \). Let superscript \( g \) denote the replacement of \( k \) by its Galois conjugate \( k^g = \frac{1 - \sqrt{5}}{2} \approx -0.6180 \), which is the other solution of (113). Then the diagonal matrix elements of \( S^g T \mathcal{H}^g S^g \) are \( > 0 \), so \( \mathcal{H}^g \) has signature \((7,0)\), so \( \Gamma (T_3) \) is of the arithmetic type \([37]\) reviewed in \([38]\) and subsection 3.1 of \([39]\).

A finite index torsionless normal subgroup \( \Lambda \) of \( \Gamma (T_3) \) can be obtained by the same variant of Selberg’s lemma \([108]\) as used in \([149]\) to obtain the 4-dimensional Davis manifold from a compact hyperbolic Coxeter 4-simplex. \( \Lambda \) consists of the elements of \( \Gamma (T_3) \) whose matrices in the representation just constructed are equal mod \( \sqrt{5} \) to the unit matrix. To calculate the index of \( \Lambda \) in \( \Gamma \), which gives the ratio of the intrinsic volume of the corresponding compact hyperbolic 6-manifold to the intrinsic volume of the compact hyperbolic Coxeter polytope \( \xi (T_3) \), we note that \( \frac{1 + \sqrt{5}}{2} + \sqrt{5} \frac{\sqrt{5} - 1}{2} = 3 \), hence \( k = 3 \) mod \( \sqrt{5} \). Thus the factor group \( \Gamma / \Lambda \), which is the group of all the distinct elements of \( \Gamma \) mod \( \sqrt{5} \), is obtained by replacing \( k \) by 3 in the \( 7 \times 7 \) matrix representations of the 9 generators of \( \Gamma \) constructed above, then reducing the resulting integer matrix elements mod 5 and doing the matrix multiplications with mod 5 arithmetic.

Using GAP \([154]\) we find Size \((\Gamma / \Lambda) = 460687500000000 = 2^83^45^{12}7^{13} \approx 4.6 \times 10^{14} \). Using the same procedure for the 6-dimensional compact hyperbolic Coxeter polytope with 9 5-dimensional faces whose Coxeter diagram is shown in Figure 4.3 of \([153]\), whose Gram matrix times 2 has matrix elements in \( \mathbb{Z}[k] \) after dividing 2 rows and 2 columns by \( \sqrt{2} \), and which is also of arithmetic type, we find that in that case,
Size \( \frac{\Gamma}{\Lambda} = 914004000000000 = 2^{11}3^45^97^{13}31 \simeq 9.1 \times 10^{14} \). The third 6-dimensional compact hyperbolic Coxeter polytope with 9 5-dimensional faces is obtained from the Coxeter polytope \( \xi(T_3) \) by cutting it in half along its hyperplane of reflection symmetry. By [155] and Table 2 of [156] the magnitudes of the fractional Euler numbers of these three 6-dimensional compact hyperbolic Coxeter polytopes are not less than \( \frac{67}{115200} \simeq 5.8 \times 10^{-5} \), so the intrinsic volumes of the compact hyperbolic 6-manifolds constructed in this way are too large by a factor of not less than around \( 10^7 \). However there are other methods of constructing torsionless subgroups of hyperbolic Coxeter groups [149, 157, 158, 159] that might provide examples of smaller intrinsic volume.

While version 3 of this article was being prepared, the XENON100 collaboration published the results of the first 100.9 days of data taking in the XENON100 direct search for dark matter, finding no evidence for dark matter interacting with the ultra-low background innermost 48 kg of their 62 kg liquid xenon target, and excluding spin-independent elastic WIMP-nucleon scattering cross-sections above \( 7.0 \times 10^{-45} \text{cm}^2 \) for a WIMP mass of 50 GeV at 90\% confidence level [160]. The model considered in this article could provide a variety of dark matter candidates with masses up to around a TeV that interact only gravitationally with ordinary matter, and are thus consistent with these limits, if there were from 1 to around 10 other HW boundaries distributed around the compact hyperbolic 7-manifold \( \bar{H}^7 \), possibly with different \( E_8 \) breakings on each of them.

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