On Multiflows in Random Unit-Disk Graphs, and the Capacity of Some Wireless Networks

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Abstract

We consider the capacity problem for wireless networks. Networks are modeled as random unit-disk graphs, and the capacity problem is formulated as one of finding the maximum value of a multicommodity flow. In this paper, we develop a proof technique based on which we are able to obtain a tight characterization of the solution to the linear program associated with the multiflow problem, to within constants independent of network size. We also use this proof method to analyze network capacity for a variety of transmitter/receiver architectures, for which we obtain some conclusive results. These results contain as a special case (and strengthen) those of Gupta and Kumar for random networks, for which a new derivation is provided using only elementary counting and discrete probability tools.

I. INTRODUCTION

A. The Capacity of Wireless Networks – Five Years Later

In March 2000 (exactly five years ago as of the writing of this paper), Gupta and Kumar published a landmark piece of work, where they presented a thorough study on the capacity of wireless networks [14]. For random networks, this problem was formulated as one of forming tessellations of a sphere, then defining routes in between cells, for which tight upper and lower bounds were obtained on their capacity. The main finding in [14] was actually a rather negative one: under a variety of very reasonable scenarios, in all cases the throughput available to each node in the network was found to be of the form $\Theta \left( \frac{1}{\sqrt{n}} \right)$ at most, for a network with $n$ nodes – that is, this throughput becomes vanishingly small for large networks.

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The results of [14] generated a flurry of activity in this area (surveyed below, in subsection I-D). However, five years later, although some progress has been made towards understanding the capacity of large networks in a regime in which the minimum distance among nodes remains fixed and the area covered grows unbound with the number of nodes, some questions related to the original setup in [14], dealing with high-density networks (meaning, networks with a growing number of nodes covering a fixed finite area) still remain, at best, only partially answered:

- The ability to generate directed beams of energy in a wireless network could potentially change its behavior rather drastically, making the network “look like” a wired one. What exactly is the impact of directional antennas then on network capacity?
- Despite some attempts, a pure information theoretic analysis on the capacity of high-density wireless networks still remains elusive.

In this paper, we revisit the problem of capacity for random networks considered in [14]. We consider an entirely different problem formulation: our formalization of the network capacity problem consists of finding the value of a multicommodity flow problem defined on a random graph, for which we are able to obtain a number of results that contain those of [14] as a special case, generalizing them in a number of interesting directions.

B. Problem Formulation

Consider the following network communication problem. \( n \) nodes are uniformly distributed on the closed set \([0,1] \times [0,1]\), forming a random graph \( G = (V, E) \). Each node \( s_i \) can only send messages to and receive messages from nodes within distance \( d_n \), where \( d_n \), in order for the graph to be connected with probability 1 (as \( n \to \infty \)), has to satisfy

\[
\pi d_n^2 = \frac{\log n + \xi_n}{n},
\]

for some \( \xi_n \to \infty \) [13]. Source-destination pairs are formed randomly: for each source node \( s_i \) one destination node \( t_i \) is chosen by sampling uniformly (without replacement) from the set of network nodes \((1 \leq i \leq n)\) – each node is both a source, a destination for some other node, and a relay for other nodes. All links have the same fixed finite capacity \( c \), independent of network size. This scenario is illustrated in Fig. 1.

Our goal in this paper is to determine the rate of growth of the maximum stable throughput (MST) for the network [30]—the rate at which all sources can inject packets, while maintaining stability for the system—and provided all sources inject data at the same rate.
Fig. 1. Problem setup. \( n \) randomly located transmitters send data to \( n \) randomly chosen receivers, all nodes act as sources/destinations/relays, and nodes can only exchange messages with nearby nodes (within range \( d_n \)).

The problem of determining MST under a fairness constraint is an instance of a multicommodity flow problem [5, Ch. 29]:

- There are \( n \) commodities: the packets available for transmission from transmitter \( s_i \) to receiver \( t_i \).
- The load on a single link contributed by all sources that use that link cannot exceed its capacity.
- Subject to these constraints, we want to find the largest number of packets per unit of time that can be injected simultaneously by all sources.

Representing our network by a graph \( G = (V, E) \), the capacity of an edge \( e = (u, v) \) by \( c(u, v) \), and letting our optimization variables be \( f_i(u, v) \) (the flow along edge \( (u, v) \) for the \( i \)-th commodity), then the maximum multiflow problem above can be formulated as a linear program, as shown in Table I.

### TABLE I

Linear programming formulation of the multicommodity flow problem with a fairness constraint.

| max \( \lambda_n \) |
|-------------------|
| subject to:       |
| \( \lambda_n = \sum_{(s_i, v) \in E} f_i(s_i, v) \), \( 1 \leq i \leq n \) |
| \( \sum_{i=1}^n f_i(u, v) \leq c(u, v) \), \( (u, v) \in E \) |
| \( f_i(u, v) = -f_i(v, u) \), \( (u, v) \in E, 1 \leq i \leq n \) |
| \( \sum_{v \in V} f_i(u, v) = 0 \), \( u \in V - \{s_i, t_i\}, 1 \leq i \leq n \) |

Our main task in this paper is to provide a characterization of the optimal value \( \lambda_n^* \). Note that since the graph is random, and the LP is a function of the random graph, \( \lambda_n^* \) is a random variable itself.

### C. Asymptotically Tight Bounds

Not much is known about the structure of optimal solutions to the maximum multiflow problem—the only technique we are aware of for deciding whether a particular amount of flow of each commodity
can be supported by the network consists of formulating this problem as a linear program, and then answering the non-emptyness question for its polytope of optimization using a standard LP solver (e.g., the Ellipsoid method [12]), or some of the efficient algorithms for maximum multiflow such as that of Karger and Plotkin [16]. Hence, we will not be able to use those formulations to do much more than obtain numerical results for our problem. We are thus motivated to search for an alternative formulation of the problem. And one such possible alternative is illustrated in Fig. 2.

Fig. 2. In this formulation, we only consider the traffic generated by sources on the left-half of the network, with destination on the right-half—the traffic generated by all other source/destination pairs is discarded.

Note that doing this amounts to introducing a restriction in the domain of optimization of the linear program from Table I: instead of considering all possible network flows, we only consider those which satisfy the constraints of Fig. 2. But what is crucial in this case is that, different from the problem of Fig. 1, this new problem involving flow going from the left to the right admits a regular single commodity flow formulation. The resulting linear program is shown in Table II.

**TABLE II**

Linear program for the single commodity flow problem.

| \[\text{max} \quad \sum_{u \in V} f(s, u), \quad u \in V\] |
| \[f(u, v) \leq c(u, v), \quad (u, v) \in E\] |
| \[f(u, v) = -f(v, u), \quad (u, v) \in E\] |
| \[\sum_{v \in V} f(u, v) = 0, \quad u \in V - \{s, t\}\] |

The interest in this new linear program is due to the fact that, since it corresponds to a classical single commodity problem, we can try to solve it analytically using the max-flow/min-cut theorem [7].
However, the relationship between the optimal value $\lambda^*_n$ for the “difficult” multicommodity problem, and $\nu^*_n$ for the “easier” single commodity problem, is not entirely straightforward. On one hand, the linear program in Table II is a \textit{restriction} of that in Table I, since in the former some flow variables are constrained to 0 (that is how we incorporate the constraint of flow going only from left to right). On the other hand, the linear program in Table II is a \textit{generalization} of that in Table I, since the latter removes the multicommodity constraints (all commodities are treated as a single commodity). Thus, an important question is that of giving a precise relationship between $\lambda^*_n$ and $\nu^*_n$.

\textbf{D. Related Work}

This work is primarily motivated by our struggle to understand the results of Gupta and Kumar on the capacity of wireless networks [14]. And the main idea behind our approach is simple: the transport capacity problem posed in [14], in the context of random networks, is essentially a throughput stability problem—the goal is to determine how much data can be injected by each node into the network while keeping the system stable—, and this throughput stability problem admits a very simple formulation in term of flow networks. Note also that because of the mechanism for generating source/destination pairs, all connections have the same average length (one half of one network diameter), and thus we do not need to deal with the bit-meters/sec metric considered in [14].

As mentioned before, [14] sparked significant interest in these problems. Follow up results from the same group were reported in [15], [33]. Some information theoretic bounds for large-area networks were obtained in [19]. When nodes are allowed to move, assuming transmission delays proportional to the mixing time of the network, the total network throughput is $O(n)$, and therefore the network can carry a non-vanishing rate per node [11]. Using a linear programming formulation, non-asymptotic versions of the results in [14] are given in [29]; an extended version of that work can be found in [28]. An alternative method for deriving transport capacity was presented in [18]. The capacity of large Gaussian relay networks was found in [8]. Preliminary versions of our work based on network flows have appeared in [23], [24]; and network flow techniques have been proposed to study network capacity problems (cf., e.g., [1], [6, Ch. 14.10]), and network coding problems [17]. From the network coding literature, of particular relevance to this work is the work on multiple unicast sessions [20].

\textbf{E. Main Contributions and Organization of the Paper}

Let $\lambda^*_n$ denote an optimal solution to the linear program in Table I, and let $\nu^*_n$ denote an optimal solution to the linear program in Table II. Our first result consists of finding the asymptotic value of $\lambda^*_n$: 

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with probability 1 as $n \to \infty$,

$$
\Theta(\lambda^* n) = \Theta(\nu^* n) = \Theta \left( \frac{\log^{3/2}(n)}{\sqrt{n}} \right).
$$

(2)

This result formally establishes the equivalence between the two linear programs, in a well defined sense: they both have solutions which differ by at most a constant factor, independent of network size.

A second important contribution is to show an application of the proof methods developed to establish (2), to obtain the maximum stable throughput for various transmitter/receiver architectures:

- We consider first the case of omnidirectional antennas, where we show that the scaling laws obtained based on our proof method are identical to those of [14]: per-node throughput is $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$.

- Then we apply the same proof techniques to the determination of scaling laws for a new architecture, in which transmitter nodes can generate a single and arbitrarily narrow directed beam, and in which receivers can successfully decode multiple transmissions as long as the transmitters are not co-linear. And in this case we find that:
  
  - If only enough power to maintain the network connected is radiated at each node, the maximum stable throughput of this network is $\Theta \left( \frac{\log n}{\sqrt{n}} \right)$.
  
  - If now enough power is radiated to achieve MST linear in network size (certainly feasible with narrow beams), then the number of resolvable beams that each node must generate is $\Theta(n)$.

- Finally, we consider a node architecture in which each node is able to generate multiple and arbitrarily narrow directed beams, simultaneously to all nodes within its transmission range, and receivers operate as above. In this case we find that:
  
  - If only enough power to maintain the network connected is radiated at each node, the maximum stable throughput of this network is $\Theta \left( \frac{\log n / \sqrt{n}}{\sqrt{n}} \right)$.
  
  - If now enough power is radiated to achieve MST linear in network size (certainly feasible with narrow beams), then the number of resolvable beams that each node must generate is $\Theta(n^{3/2})$.

Essentially, our results show that both directional antennas, as well as the ability to communicate simultaneously with multiple nodes, can only achieve modest improvements in terms of achievable MST. While some performance gains are certainly feasible at reasonable complexities (in the order of a low-degree polynomial in $\log n$), the number of resolvable beams that need to be generated to increase the achievable MST by more than a polylog factor is polynomial in network size, and thus exponential in the minimum number of beams required to keep the network connected. How many beams need to be resolved is a reasonable measure of complexity, since the higher this number, the narrower these beams need to be made, and hence the higher the complexity of a practical implementation.
We also believe another original contribution is given by our proof techniques:

- Our results are obtained using only elementary network flow concepts, and the calculations involved require only basic probability theory, calculus and combinatorics. By formulating the problem of [14] as an elementary problem of flows in random graphs, we obtain what we believe is a number of interesting insights into the nature of this problem which were not obvious to us from their proof technique, as well as a set of meaningful generalizations to deal with the case of directional antennas.

- Except for some elements of the protocol model considered in [14], most of the work we are aware of on this subject (e.g., [8], [11], [15], [19], [29], [33]), has focused on the use of “continuous” tools, dealing with Gaussian signals, power constrained channels, etc. Our work instead takes a “discrete” approach to the network capacity problem, tackling it primarily using flow, counting and discrete probability tools. Thus, we believe our proof technique, while using only elementary tools, has some novelty in the context of the problem considered here.

An added benefit of our proof method is that we are able to prove strong convergence (meaning, convergence with probability 1) in all cases. In particular, when considering the specialization of our results to the setup of [14], our results are stronger, in that only weak convergence is established there.

The rest of this paper is organized as follows. In Section II we formulate upper and lower bounds on the value of $\lambda^*_n$: the upper bound is evaluated in Section III, and the lower bound is evaluated in Section IV. Then, applications of these results in the context of wireless networking problems follow: in Section V we present an alternative derivation for the results of Gupta and Kumar in the context of random networks [14], and in Section VI these results are extended to deal with two different cases involving directional antennas. The paper concludes with Section VII.

II. ASYMPTOTICALLY TIGHT BOUNDS ON THE VALUE OF THE LINEAR PROGRAM

In this section we start with some preliminaries presenting the tools used to carry out our analysis, to then go on to formulate upper and lower bounds on the value of $\lambda^*_n$.

A. Tools

To compute the maximum value of a single commodity flow in our network, we use a standard result in flow networks: the max-flow/min-cut theorem of Ford and Fulkerson [7]. We solve this problem by counting how many edges can be constructed so that they all simultaneously straddle a minimum cut.
1) The Max-Flow/Min-Cut Theorem: \( f \) is a flow of maximum value iff \( |f| = c(S, T) \) (for some cut \((S, T)\)). We focus our attention on one particular cut (shown also in Fig. 3):

\[
S = (x_i, y_i) \in V \cap [0, \frac{1}{2}) \times [0, 1],
\]

\[
T = (x_i, y_i) \in V \cap [\frac{1}{2}, 1] \times [0, 1].
\]

Fig. 3. To illustrate the choice of a cut to derive bounds. \( L \) and \( R \) are sections of the network on each side of the cut boundary, of width \( d_n \), the transmission range.

In this way, to compute the value of a maximum flow we need to determine how many edges straddle the \( x = \frac{1}{2} \) cut. To do that, we proceed in two steps. First, we compute the expected number of edges that straddle this cut, with this expectation taken as an ensemble average over all possible network realizations. Then, we derive a sharp concentration result: given an arbitrary network realization, with probability 1 as \( n \rightarrow \infty \), we show that in this network, the actual number of edges that straddle the cut has the exact same rate of growth (in the \( \Theta \) sense of [10]) as the ensemble mean does.

2) Mean Values: What is the average number of nodes in a subset \( A \subseteq [0, 1] \times [0, 1] \)? A simple calculation shows that

\[
E(\text{Number of nodes in } A) = nP(A) = n \int_A f_{XY} \, dA = n|A|,
\]

where \( |A| \) denotes the area of \( A \).

3) Chernoff Bounds: In addition, to prove sharp concentration results, we need to bound the probability of deviations from its mean by sums of independent random variables:

- Consider \( n \) points \( X_1 \ldots X_n \) iid and uniformly distributed on \([0, 1] \times [0, 1]\). We have a number of subsets \( A_j \subseteq [0, 1] \times [0, 1] \), for \( j = 1\ldots f(n) \) (the number of subsets may depend on the number of points \( n \)), and denote the area of any such subset by \( |A_j| \). Now we define some random variables:

\[
N_{ij} = \begin{cases} 
1, & X_i \in A_j \\
0, & \text{otherwise.}
\end{cases}
\]

Since the \( X_i \)'s are independent, the \( N_{ij} \)'s are also independent.
Now let $N_j$ be another random variable defining the number of points in $A_j$, i.e., $N_j = \sum_{i=1}^{n} N_{ij}$.

We see in this case that the $N_j$'s, $j = 1 \ldots f(n)$ are random variables where each is the sum of $n$ iid binary random variables (but not necessarily independent among the $N_j$'s themselves).

The expected number of points in $A_j$ is

$$E(N_j) = E\left(\sum_{i=1}^{n} N_{ij}\right) = \sum_{i=1}^{n} E(N_{ij}).$$

But, since $P(X_i \in A_j) = |A_j|$, we have that $E(N_{ij}) = 1|A_j| + 0(1 - |A_j|) = |A_j|$, and hence $E(N_j) = n|A_j|$.

For the family of variables $N_j$, we have the following standard results, known as the Chernoff bounds (see, e.g., [22, Ch. 4]):

1) For any $\delta > 0$:

$$P\left[N_j > (1 + \delta)n|A_j|\right] < \left(\frac{e^\delta}{(1 + \delta)^{1 + \delta}}\right)^{n|A_j|}.$$

2) For any $0 < \delta < 1$:

$$P\left[N_j < (1 - \delta)n|A_j|\right] < e^{-\frac{\delta}{2}n|A_j|\delta^2}.$$

With a few simple calculations we can rewrite the first bound as

$$P\left[(N_j - n|A_j|) > \delta n|A_j|\right] < \left(\frac{e^\delta}{e^{(1+\delta)\ln(1+\delta)}}\right)^{|A_j|} = \left(e^{\delta-(1+\delta)\ln(1+\delta)}\right)^{|A_j|} = e^{-\theta_1 n|A_j|},$$

where $-\theta_1 \triangleq \delta - (1 + \delta) \ln(1 + \delta)$. We can also rewrite the second bound as

$$P\left[(N_j - n|A_j|) < -\delta n|A_j|\right] < e^{(-\frac{\delta}{2}\delta^2)n|A_j|} = e^{-\theta_2 n|A_j|},$$

where $-\theta_2 \triangleq -\frac{1}{2}\delta^2$. Consider now the case of $0 < \delta < 1$: restricted to this range, we have that $\theta_1 > 0$; and $\theta_2$ is clearly positive as well. Thus, by defining $\theta(\delta) = \min(\theta_1, \theta_2)$, we have

$$P\left[|N_j - n|A_j| | > \delta n|A_j|\right] < e^{-\theta n|A_j|}.$$  (4)

Our interest in (4) is because, if we can prove probability bounds of that form, then we can claim that $\frac{N_j}{n} = \Theta(|A_j|)$ with probability 1, in the limit as $n \to \infty$. In other words, for the random variables $N_j$, as $n \to \infty$, there exist constants such that deviations from their mean by more than these constants occur with probability 0. Note that as $n \to \infty$, $e^{-\theta n|A_j|} \to 0$, so

$$\lim_{n \to \infty} P\left[|N_j - n|A_j| | > \delta n|A_j|\right] = 0,$$
or equivalently,

\[
1 = \lim_{n \to \infty} P\left[ |N_j - n|A_j| \leq \delta n|A_j| \right]
\]

\[
= \lim_{n \to \infty} P\left[ 0 \leq (1 - \delta)n|A_j| \leq N_j \leq (1 + \delta)n|A_j| \right]
\]

\[
= \lim_{n \to \infty} P\left[N_j = \Theta(n|A_j|) \right],
\]

where \( \Theta \) is defined in [10] as:

\[
\Theta(g(n)) = \{ f(n) : \exists c_1 > 0, c_2 > 0, n_0, \text{ for which } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0. \}
\]

**B. An Equivalent Linear Program**

As suggested in the Introduction, we will not work directly with the original linear program from Table I, but instead we will work with a new linear program, one in which flow is constrained to move from left to right. The new LP is formally stated in Table III.

**TABLE III**

Linear programming formulation of the multicommodity flow problem, in which the supply of connections other than those going from left to right are set to 0.

| max \( \ell_n \) | subject to: |
|------------------|-------------|
| \( \ell_n = \sum_{(s_i,v) \in E} f_i(s_i,v) \), \( 1 \leq i \leq n \) and \( s_i \in S = \{ u \in V : u \in [0, \frac{1}{2}) \times [0, 1] \} \), \( t_i \in T = \{ u \in V : u \in [\frac{1}{2}, 1] \times [0, 1] \} \), | \( \sum_{i=1}^n f_i(u,v) \leq c(u,v) \), \( (u,v) \in E \) |
| \( f_i(u,v) = -f_i(v,u) \), \( (u,v) \in E, 1 \leq i \leq n \) | \( \sum_{v \in V} f_i(u,v) = 0 \), \( u \in V - \{s_i, t_i\}, 1 \leq i \leq n \) |
| \( f_i(s_i,v) = 0 \), \( \forall s_i \in T = \{ u \in V : u \in [\frac{1}{2}, 1] \times [0, 1] \} \), \( t_i \in S = \{ u \in V : u \in [0, \frac{1}{2}) \times [0, 1] \} \). | |

Considering sources on the left half of the network and destinations on the right, essentially says that in our linear programming formulation in Table I we must add the constraint of setting to 0 the demands of commodities such that either the source is located on the right or the sink is located on the left. But this constraint changes the result of the linear program only by a constant factor, and therefore asymptotically we get the same values from Table I and Table III. Intuitively, the reason is simple:
since nodes are uniformly distributed, we should have about \( n/2 \) nodes in \( S \) and about \( n/2 \) nodes in \( T \) with high probability; at the same time, since the source/destination pairs are uniformly distributed, about \( n/4 \) of the sources are placed on the left side of the network with destinations on the right side; therefore, by considering only traffic generated by sources in \( S \) for destinations only in \( T \) the value of the multicommodity problem should at most decrease by a factor of 4, and hence remains of the same order. To see this more formally, consider the following indicator variables:

\[
I_i^{(n)} = \begin{cases} 
1, & s_i \in S \land t_i \in T \\
0, & \text{otherwise}, 
\end{cases}
\]

where \( S \) and \( T \) are given in Table III. Then, \( I^{(n)} = \sum_{i=1}^{n} I_i^{(n)} \) is another random variable whose value is equal to the number of pairs with the source on the left half and the sink on the right half. We would like to compute how many are these pairs, to calculate the difference between the values of the two linear programs. To do this, we first compute the mean of \( I_i^{(n)} \), then we use the Chernoff bounds to prove a sharp concentration of this variable around its mean.

We start by computing \( E(I^{(n)}) \). We have:

\[
E(I^{(n)}) = E\left( \sum_{i=1}^{n} I_i^{(n)} \right) = \sum_{i=1}^{n} E(I_i^{(n)}),
\]

due to linearity of expectation. Now,

\[
E(I_i^{(n)}) = 1 \cdot P(s_i \in S \land t_i \in T) + 0 \cdot P(s_i \in T \lor t_i \in V) = P(s_i \in S \land t_i \in T).
\]

Since the nodes in our network are uniformly and independently distributed, we have that the events \( \{s_i \in S\} \) and \( \{t_i \in T\} \) are independent events, and therefore:

\[
E(I_i^{(n)}) = P(s_i \in S \land t_i \in T) = P(s_i \in S) \cdot P(t_i \in T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4},
\]

which finally gives us:

\[
E(I^{(n)}) = \sum_{i=1}^{n} E(I_i^{(n)}) = \frac{n}{4}.
\]

This expected number occurs with high probability as \( n \to \infty \) because, according to the Chernoff bound,

\[
P\left[ |I^{(n)} - E(I^{(n)})| > \delta E(I^{(n)}) \right] < e^{-\theta E(I^{(n)})} = e^{-\theta \frac{n}{4}} \to 0,
\]

as \( n \to \infty \). Thus, with high probability, there are about \( \frac{n}{4} \) sources in \( S \) with destinations in \( T \). As a result, from the fairness constraint we have that \( P(\ell_n^* = \lambda_n^*/4) \to 1 \), and herefore, \( \Theta(\lambda_n^*) = \Theta(\ell_n^*) \).
C. Formulation of the Bounds

With these tools, it is easy to describe asymptotic upper and lower bounds on \( \lambda^*_n \):

- To obtain an upper bound, we eliminate the multicommodity constraints from the linear program in Table III, and use the max-flow/min-cut theorem to compute the value of a maximum flow. Thus, we have that \( \Theta(\ell^*_n) \leq \Theta(\nu^*_n) \), for \( \nu_n \) as defined in Table II.
- To obtain a lower bound, we construct a feasible point for the linear program in Table III, to obtain a value \( \gamma_n \) for the LP for which, clearly, \( \gamma_n \leq \ell^*_n \).

In the next two sections we evaluate these bounds, to show that \( \Theta \left( \frac{\log \frac{3}{2} n}{\sqrt{n}} \right) = \gamma_n \leq \ell^*_n \leq \Theta \left( \frac{\log \frac{3}{2} n}{\sqrt{n}} \right) = \Theta \left( \frac{\log \frac{3}{2} n}{\sqrt{n}} \right) \), and thus conclude that \( \ell^*_n = \Theta \left( \frac{\log \frac{3}{2} n}{\sqrt{n}} \right) \) as well.

III. Evaluation of the Upper Bound

In this section, our goal is to show that \( \nu_n = \Theta \left( \frac{\log \frac{3}{2} n}{\sqrt{n}} \right) \), based on the methods outlined in the previous section.

A. Counting Edges Across a Minimum Cut

Fix a particular node on the left side of the minimum cut, \( L \). The number of edges that cross the cut for that one node is exactly the number of nodes in the right side of the cut, \( R \), that are within distance \( d_n \). Therefore, for an arbitrary point \( p = (x, y) \) in \( L = [\frac{1}{2} - d_n, \frac{1}{2}] \times [0, 1] \), we draw a circle of radius \( d_n \) and center \( (x, y) \). The points \( q = (u, v) \) in \( R = [\frac{1}{2}, \frac{1}{2} + d_n] \times [0, 1] \) that are inside the circle are equal to the number of edges we want to count. These points \( p \) and \( q \) for which an edge exists satisfy the following conditions: (1) \( \frac{1}{2} - d_n \leq x \leq \frac{1}{2} \); (2) either (a) \( 0 \leq y \leq 1 \), or (b) \( d_n \leq y \leq 1 - d_n \); (3) \( \frac{1}{2} < u \); and (4) \( (u - x)^2 + (v - y)^2 \leq d_n^2 \). The situation is illustrated in Fig. 4.

![Fig. 4. To illustrate constraints on edges.](image-url)
For each \( p = (x, y) \), we get the average number of points \( q = (u, v) \) within the shaded arc \( Q_p \) in Fig. 4 using eqn. (3): \( E(\text{Number of points in } Q_p) = n |Q_p| \).

To compute the area of \( Q_p \) (denoted \( |Q_p| \)), we let \( \vartheta \) denote the angle of the arc illustrated in Fig. 4. Then, it follows from elementary trigonometric identities that \( \sin \frac{x - \vartheta}{2} = \frac{x - \vartheta}{d_n} \), and so \( \cos \frac{\vartheta}{2} = \frac{x}{d_n} \). So, \( |Q_p| = \frac{\vartheta}{d_n} d_n^2 - \frac{\vartheta}{d_n} d_n \cos \frac{\vartheta}{2} d_n \sin \frac{\vartheta}{2} = \frac{\vartheta}{2} d_n^2 - \frac{\vartheta}{2} d_n^2 \sin \vartheta = \frac{\vartheta}{2} d_n^2 (\vartheta - \sin \vartheta) \). And plugging this expression into \( n |Q_p| \), we get \( n |Q_p| = n \frac{1}{2} d_n^2 (\vartheta - \sin \vartheta) \). But from the trigonometric identities above, we have that \( \vartheta = 2 \arccos \frac{\vartheta - x}{d_n} \) and hence, \( \sin \vartheta = 2 \sin \frac{\vartheta}{2} \cos \frac{\vartheta}{2} \), which implies \( \sin^2 \vartheta = 4 \sin^2 \frac{\vartheta}{2} \cos^2 \frac{\vartheta}{2} \), which again implies \( \sin^2 \vartheta = 4(1 - \cos^2 \frac{\vartheta}{2}) \cos^2 \frac{\vartheta}{2} \). Now, since \( 0 \leq \vartheta \leq \pi \), \( \sin \vartheta = 2 \cos \frac{\vartheta}{2} \sqrt{1 - \cos^2 \frac{\vartheta}{2}} \geq 0 \), and so, finally, we get an expression for \( n |Q_p| \) in terms of \( n \), \( d_n \), and the coordinates of the transmitter \( p = (x, y) \):

\[
\begin{align*}
n |Q_p| &= \frac{1}{2} n d_n^2 (\vartheta - \sin \vartheta) \\
&= \frac{1}{2} n d_n^2 \left( 2 \arccos \frac{\vartheta - x}{d_n} - 2 \cos \frac{\vartheta}{2} \sqrt{1 - \cos^2 \frac{\vartheta}{2}} \right) \\
&= n d_n^2 \left( \arccos \frac{\vartheta - x}{d_n} - \frac{\vartheta - x}{d_n} \sqrt{1 - \left( \frac{\vartheta - x}{d_n} \right)^2} \right).
\end{align*}
\]

The result above is the average number of edges that cross the cut, starting at a fixed point \( p = (x, y) \) in \( L \). To calculate the total number of edges \( S \) that cross the cut on average, we need to add up \( n |Q_p| \) over all nodes \( p \), (i.e., compute \( S = \sum_{p \in L} n |Q_p| \)). And our plan to do this is to approximate this sum by an integral.

The value of \( |Q_p| \) is clearly dependent on the location of \( p \): for \( p \)'s in \( L \) near the boundary of the cut \( x \approx \frac{1}{2} \), \( \vartheta \approx \pi \) and hence the shaded area is large; for \( p \)'s still in \( L \) but far from the boundary of the cut \( x \approx \frac{1}{2} - d_n \), \( \vartheta \approx 0 \) and hence the shaded area is small. Furthermore, except near the top and bottom boundaries, the area of \( Q_p \) is independent of \( y \). Therefore, to obtain a simple expression for the sought sum, our first step consists of dividing \( L \) into \( \frac{d_n}{\Delta} \) thin strips of height 1 and width \( \Delta \) (for \( \Delta \ll d_n \)), and expanding \( \sum_{p \in L} n |Q_p| \) in two different ways:

\[
S_a = \sum_{k=1}^{d_n/\Delta} n |Q_{xy}| \cdot |\{ p = (x, y) \in L : 0 \leq y \leq 1 \}_{s_a}|
\]

\[
S_b = \sum_{k=1}^{d_n/\Delta} n |Q_{xy}| \cdot |\{ p = (x, y) \in L : d_n \leq y \leq 1 - d_n \}_{s_b}|
\]

(in both cases, we take \( \frac{1}{2} - d_n + (k - 1) \Delta \leq x \leq \frac{1}{2} - d_n + k \Delta \)). \( S_a \) is an upper bound on \( \sum_{p \in L} n |Q_p| \), since we may count edges that end up outside the network; \( S_b \) is a lower bound, since we may not count....
some valid edges close to the network boundary; but as long as \( d_n \to 0 \) as \( n \to \infty \), both bounds become tight and equal to \( \sum_{p \in L} n|Q_p| \).

The next step is to observe that once again we can approximate the size estimates \( s_a \) and \( s_b \) using eqn. (3): \( s_a = n \Delta \) and \( s_b = n(1 - 2d_n) \Delta \). Hence we get:

\[
S_a = \sum_{k=1}^{d_n/\Delta} n|Q_{xy}| \cdot n\Delta = n^2 \Delta \sum_{k=1}^{d_n/\Delta} |Q_{xy}|
\]

\[
\approx n^2 \int_{x=\frac{1}{2}-d_n}^{\frac{1}{2}} \int_{y=0}^{1} |Q_{xy}| \, dx \, dy;
\]

\[
S_b = \sum_{k=1}^{d_n/\Delta} n|Q_{xy}| \cdot n(1 - 2d_n) \Delta = n^2 (1 - 2d_n) \Delta \sum_{k=1}^{d_n/\Delta} |Q_{xy}|
\]

\[
\approx n^2 \int_{x=\frac{1}{2}-d_n}^{\frac{1}{2}} \int_{y=d_n}^{1} |Q_{xy}| \, dx \, dy,
\]

since \( \sum_{k=1}^{d_n/\Delta} |Q_{xy}| \) is a Riemann sum that, as we let \( \Delta \to 0 \), converges to the integral over an appropriate region of \( |Q_p| \).

And now we are almost done. Since \( S_b \leq \sum_{p \in L} n|Q_p| \leq S_a \), and we have that for \( n \) large, \( S_a \approx S_b \approx n^2 \int_L |Q_p| \) dp, we finally get:

\[
\sum_{p \in L} n|Q_p| \approx n^2 \int_L |Q_p| \, dp
\]

\[
= n^2 \int_{\frac{1}{2}-d_n}^{\frac{1}{2}} \int_{0}^{d_n} \left[ \arccos \frac{1}{d_n} - \frac{1}{d_n} \sqrt{1 - \left(\frac{1}{d_n} - x\right)^2} \right] \, dy \, dx
\]

\[
= n^2 d_n^2 \int_{\frac{1}{2}-d_n}^{\frac{1}{2}} \int_{0}^{1} \arccos \frac{1}{d_n} - \frac{1}{d_n} \sqrt{1 - \left(\frac{1}{d_n} - x\right)^2} \, dy \, dx - n^2 d_n^2 \int_{\frac{1}{2}-d_n}^{\frac{1}{2}} \int_{0}^{1} \frac{1}{d_n} - \frac{1}{d_n} \sqrt{1 - \left(\frac{1}{d_n} - x\right)^2} \, dy \, dx
\]

\[
= n^2 d_n^2 \int_{\frac{1}{2}-d_n}^{\frac{1}{2}} \arccos \frac{1}{d_n} - \frac{1}{d_n} \sqrt{1 - \left(\frac{1}{d_n} - x\right)^2} \, dx - n^2 d_n^2 \int_{\frac{1}{2}-d_n}^{\frac{1}{2}} \frac{1}{d_n} - \frac{1}{d_n} \sqrt{1 - \left(\frac{1}{d_n} - x\right)^2} \, dx
\]

\[
(a) = -n^2 d_n^3 \int_{1}^{0} \arccos u \, du + n^2 \int_{1}^{0} u \sqrt{1 - u^2} \, du
\]

\[
= n^2 d_n^3 \int_{0}^{1} \arccos u \, du - n^2 \int_{0}^{1} u \sqrt{1 - u^2} \, du
\]

\[
= n^2 d_n^3 - \frac{1}{2} n^2 d_n^3 = \frac{1}{4} n^2 d_n^3,
\]

where \( a \) follows from the change of variable \( \frac{1}{d_n} - x = u \).
B. Sharp Concentration Results

Our next goal is to show that the actual number of edges straddling the cut in any realization of the network is sharply concentrated around its mean. That is, in almost all networks, the number of edges across the cut is $\Theta(n^2 d_n^3) = \Theta(\sqrt{n \log^3(n)}).

Define a binary random variable $N_{ij}$, which takes the value 1 if the $i$-th node is within the transmission range of a node at coordinates $(x_j, y_j)$ on the other side of the cut, as illustrated in Fig. 4:

$$N_{ij} = \begin{cases} 
1, & X_i \in Q(x_j, y_j) \\
0, & \text{otherwise.} 
\end{cases}$$

Let $p$ denote the probability that $X_i$ is in $Q(x_j, y_j)$ (i.e., that $N_{ij} = 1$). Then, $p = |Q(x_j, y_j)| = \frac{1}{2} d_n^2 (\theta - \sin(\theta))$, with $0 \leq \theta \leq \pi$ is as in Fig. 4. Therefore, defining $\kappa_{\theta}$ as $\frac{1}{2} (\theta - \sin(\theta))$, we have $p = |Q(x_j, y_j)| = \kappa_{\theta} d_n^2 = \kappa_{\theta} \log n$.

Define $N_j = \sum_{i=1}^n N_{ij}$ as the number of points in $Q(x_j, y_j)$. In this case, we have $E(N_j) = E\left( \sum_{i=1}^n N_{ij} \right) = \sum_{i=1}^n p \cdot 1 + (1 - p) \cdot 0 = np = \kappa_{\theta} \log(n)$. Now, by eqn. (4), we have that

$$P\left( |N_j - \kappa_{\theta} \log(n)| > \delta \kappa_{\theta} \log(n) \right) < e^{-\theta \kappa_{\theta} \log(n)} = n^{-\theta \kappa_{\theta}},$$

As $n \to \infty$ this probability tends to zero, and therefore, in almost all network realizations, a node on the left side of the cut is connected to $\Theta(\log(n))$ nodes on the right side.\(^1\) By an analogous argument, we have $\Theta(nd_n) = \Theta(\sqrt{n \log n})$ nodes on the left half. Therefore, the actual number of edges across the cut is $\Theta(\log^2 n) \cdot \Theta(\sqrt{n \log n})$, so $n \nu_n^* = \Theta(\sqrt{n \log^2 n})$.

IV. Evaluation of the Lower Bound

To give a lower bound for $\ell_n^{*}$, we construct one feasible point: this is accomplished by giving a specific routing algorithm, and finding how much traffic this scheme can carry:

1) We start by proving that, with probability 1 as $n \to \infty$, there is a subgraph of the random graphs under consideration with a clear, regular structure.

2) We then develop a (very simple) routing technique that makes use of the links in the regular subgraph only.

3) Finally, we determine the throughput achieved in this way.

\(^1\)Observe that $\kappa_{\theta} = 0$ only over a set of measure zero (the set of network locations such that $x = \frac{1}{2} - d_n$), and thus the exponent can be assumed strictly positive.
A. Existence of a Regular Subgraph

Consider a partition of the network area (the closed set \([0, 1] \times [0, 1]\)) into square cells, each one of area \(c \frac{\log n}{n}\). To determine \(c\), we observe that the side of a cell is \(\sqrt{\frac{c \log n}{n}}\), and so \(c\) is chosen such that the inverse of this number, \(\sqrt{\frac{n}{c \log n}}\), is an integer – this is to guarantee that the cells form a partition of the whole network, as illustrated in Fig. 5.

![Diagram of network partition](image)

Fig. 5. To illustrate the presence of a structured subgraph for large \(n\). Consider the shaded center cell: all nodes within that cell are connected by an edge to every node in the cells above, below, left and right. To guarantee that all such edges can be formed, from Pithagoras, the transmission range \(d_n\) must be chosen as \(d_n = \sqrt{\frac{5c \log n}{n}}\). So, provided \(c > 0\), connectivity of the network is guaranteed [13].

With the construction of grid and choice of connectivity radius \(d_n\) shown in Fig. 5, each node within a cell will have an edge connecting it to all nodes in the four adjacent cells.

B. A Routing Algorithm

To describe the routing algorithm, we define first some notation:

- Given a cell \((i, j)\) \((1 \leq i, j \leq \sqrt{\frac{n}{c \log n}})\), \(v_{ij}\) denotes any arbitrary node \(v \in V\) contained in that cell.
- Given a node \(v \in V\), \((i(v), j(v))\) denotes the cell that contains \(v\).
- \(v_{\text{curr}}\): current node; \(v_{\text{dest}}\): destination node.

The algorithm executed at each node to decide the next hop of a message is as follows:

1) If \(j(v_{\text{curr}}) < j(v_{\text{dest}})\), send message to \((i(v_{\text{curr}}), j(v_{\text{curr}}) + 1)\).
2) Else, if \(i(v_{\text{curr}}) > i(v_{\text{dest}})\), send message to \((i(v_{\text{curr}}) - 1, j(v_{\text{curr}}))\).
3) Else, if \(i(v_{\text{curr}}) < i(v_{\text{dest}})\), send message to \((i(v_{\text{curr}}) + 1, j(v_{\text{curr}}))\).
4) Else, $v^{\text{curr}}$ and $v^{\text{dest}}$ are in the same cell (so $v^{\text{dest}}$ is reachable in one hop from $v^{\text{curr}}$), hence stop. These mechanics are illustrated in Fig. 6.

Fig. 6. To illustrate routing mechanics. A source in a cell left of the center cut sends messages to a destination on the right side by first forwarding data horizontally, then vertically. Under the assumption that all cells are non-empty, there is always a next hop.

For this algorithm to work properly, we must insure that no cells are empty: if this condition holds, then we can be sure that an L-shaped path as shown in Fig. 6 will always deliver packets to destination. But the fact that no cells are empty is not obvious, and requires proof. In fact, we will prove something stronger: the number of nodes contained in any arbitrary cell is $\Theta(\log n)$, with high probability as $n \to \infty$.

Define $X_{ij}$ as the number of nodes within a cell $(i, j)$ $(1 \leq i, j \leq \sqrt{\frac{1}{c \log n}})$. Then,

- **Mean value of $X_{ij}$**: 
  
  $$E(X_{ij}) = n \cdot \frac{c \log n}{n} = c \log n. \quad (5)$$

- **Chernoff bound on deviations from the mean for $X_{ij}$**: 
  
  $$P(|X_{ij} - c \log n| > \delta c \log n) \leq e^{-\theta c \log n} = \frac{1}{n^{\epsilon \theta}}.$$ 

- **Probability that the occupancy of none of the cells deviates significantly from its mean**: 
  
  $$P \left( \bigcap_{i,j} |X_{ij} - c \log n| < \delta c \log n \right) = 1 - P \left( \bigcup_{i,j} |X_{ij} - c \log n| > \delta c \log n \right)$$ 

  $$\geq 1 - \sum_{i,j} P (|X_{ij} - c \log n| > \delta c \log n).$$ 

Consider now any $\epsilon > 0$; there is a value $n_0(\epsilon)$ such that, for all $n > n_0(\epsilon)$,

$$\sum_{i,j} P (|X_{ij} - c \log n| > \delta c \log n) < \sum_{i,j} \frac{1}{n^{\epsilon \theta}} \stackrel{(a)}{=} \frac{1}{c \cdot n^{\epsilon \theta - 1} \log n} < \epsilon,$$
where \((a)\) follows from the fact that there are \(\frac{1}{c} \frac{n}{\log n}\) cells, and provided \(c > \frac{1}{\theta}\). Therefore,

\[
P \left( \bigcap_{i,j} |X_{ij} - c \log n| < \delta c \log n \right) \geq 1 - \epsilon,
\]

and thus all cells contain \(\Theta(\log n)\) nodes almost surely, as \(n \to \infty\).

With this, we see that all cells \(simultaneously\) will be non-empty in almost all networks. Thus, the routes defined by the proposed routing algorithm will always deliver data to destination. We still need to determine how much though.

C. Computation of the Achievable Throughput

The last step is to determine the throughput available to a source/destination pair constructed by the routing algorithm above.

We start by stating the relatively straightforward fact that, since the routes constructed by the algorithm above do not split the flow at any intermediate node, the throughput of a connection is determined by the capacity available to that connection at the link with highest load.\(^2\) Now, since links have a fixed finite capacity, and since in our problem we work under a \textit{fairness} constraint that forces all source/destination pairs to inject the same amount of data, we have that the capacity allocated to a commodity on any link is \(\frac{\text{raw link capacity}}{\# \text{ of commodities using that link}}\). Thus, our problem reduces to finding the maximum number of commodities sharing a link.

We claim that no link in the network is shared by more connections than the links which straddle the center cut:

- The number of commodities sharing a link across the center cut is exactly equal to the number of nodes within a horizontal strip, as illustrated in Fig. 7.

\[\text{Fig. 7. Each node left of the center cut generates traffic that must cross that cut.}\]

\(^2\)This intuitive fact can be formalized based on Robacker’s decomposition theorem for multiflows [24], [26].
Clearly, the number of commodities on horizontal links (meaning, links going from one cell to another cell either left or right) decreases as we move away from the center cut:

- Moving left, the number of sources decreases.
- Moving right, once a connection reaches the column on which the cell containing its destination lies, it starts moving along vertical links and never goes back to horizontal ones.

The number of commodities on vertical links is at most the same as the number on links in the center cut – but this requires proof.

To prove this last point, we need to count the number of commodities sharing a horizontal link crossing the center cut, and we have to give an upper bound on the number of commodities sharing an arbitrary vertical link.

In terms of the number of commodities sharing a horizontal link across the center cut:

- The average number of commodities across the center cut is just
  \[ n \cdot \frac{1}{2} \sqrt{\frac{c \log n}{n}} = \frac{\sqrt{c}}{2} \sqrt{n \log n}, \]
  and again from the Chernoff bounds, we have that this is not only the ensemble average, but that in almost all networks, this number is \( \Theta(\sqrt{n \log n}) \).

- By a similar argument, we have that the number of edges across the center cut in between two adjacent cells is \( \Theta(\log^2 n) \) – with high probability, \( \Theta(\log n) \) nodes in each cell, by construction there is a link between any two of those.

Thus, the number of commodities sharing a link across the center cut is
\[ \Theta(\frac{n \log n}{\Theta(\log^2 n)}) = \Theta\left(\frac{\sqrt{n}}{\log \frac{\pi}{2}(n)}\right). \]

In terms of the number of commodities sharing any vertical link, an upper bound on this number is given by \( \frac{\# \text{ of nodes in a vertical strip}}{\# \text{ of links between adjacent cells}} \). Why this is an upper bound is illustrated in Fig. 8. But then, since the number of nodes in a vertical strip is twice the number of nodes in \( \frac{1}{2} \) of a horizontal strip, from an argument entirely analogous to the count of commodities sharing a link across a center cut in the paragraph above, we have that the number of commodities sharing a vertical link is at most \( \Theta\left(\frac{\sqrt{n}}{\log \frac{\pi}{2}(n)}\right) \).

In summary, we have that the link sharing the largest number of commodities is shared by \( \Theta\left(\frac{\sqrt{n}}{\log \frac{\pi}{2}(n)}\right) \) of them. Therefore, by the fairness constraint, the capacity of this link is shared equally among all commodities, and thus this capacity is the sought \( \gamma_n \) value, i.e., \( \gamma_n = \Theta\left(\frac{\log \frac{\pi}{2}(n)}{\sqrt{n}}\right) \) is achievable for the linear program in Table III.
Fig. 8. Upper bound on the number of commodities sharing a vertical link. Clearly, not all commodities that use a vertical link share a link across the thick horizontal line: some will switch from horizontal to vertical above the line and reach their destination before crossing that line, and the same will happen below. By estimating the number of commodities sharing a vertical link by the number of nodes in a vertical strip, we effectively say that all commodities in that strip use *all* vertical edges. This is certainly an overestimate, based on which we obtain only an upper bound.

D. Remark

Note: the “spirit” of this proof is very similar to the proof in [14, Sec. IV]: in both cases, the goal is to give an explicit construction to show the achievability of certain throughput values. However, the methods employed to analyze the throughput achieved by the routing strategies proposed differ significantly – [14] relies heavily on VC theory [31], whereas we only use properties of flows and the Chernoff bound.

V. APPLICATIONS TO WIRELESS NETWORKING PROBLEMS I: THE GUPTA-KUMAR SETUP

Before considering more general node architectures in Section VI, we show in this section how, for the case of nodes equipped with omnidirectional antennas, using our proof techniques we obtain scaling laws identical to those reported in [14], but under strong convergence.

A. Transmitter/Receiver Model

In [14], transmissions were omnidirectional, and described based on a pure collision model: for a transmission to be successfully decoded, no other transmission has to be in progress within the range of the receiver under consideration. This setup is illustrated in Fig. 9.
B. Average Number of Edges Across the Cut

Our first task is to determine the average number of edges that can be simultaneously supported across the cut, average taken over all possible network realizations.

1) An Upper Bound: For a fixed receiver location \((x, y)\) in \(R\), there can only be one active transmitter within distance \(d_n\) of the receiver, for that transmission to be successfully received. Since to obtain an upper bound we only need worry about edges that cross the cut, we first consider all possible locations of one such transmitter in \(L\), by drawing a circle of radius \(d_n\) and center \((x, y)\). This region is illustrated in Fig. 10.

Fig. 10. For a receiver at location \((x, y)\), at most one transmitter in the shaded region \(T_{xy}\) can send a message (if this message is to be successfully decoded on the other side of the cut).

Denoting by \(|T_{xy}|\) the area of the shaded region \(T_{xy}\) in Fig. 10, we use eqn. (3) to estimate the number of transmitters located in \(T_{xy}\) as \(n|T_{xy}|\). However, since only one transmitter located within \(T_{xy}\) can transmit successfully to a receiver at \((x, y)\), the number of nodes that are able to transmit at the same time from \(L\) to \(R\) is upper bounded by

\[
\frac{E(\text{Number of nodes in } L)}{E(\text{Number of nodes in } T_{xy})} = \frac{nL}{nT_{xy}}
\]

This is an upper bound, because we are assuming that it is possible to find a set of locations \((x, y)\) in \(R\) such that no area in \(L\) is wasted—showing that this bound is indeed tight requires proof.
Now, the area of $L$ is $d_n$. To compute the area of $T_{xy}$, we have to determine the area of an arc of a circle with angle $\vartheta$, as shown in Fig. 10, and in a computation entirely analogous to that of the calculation of $|Q_p|$ in Section VI-C. In this case, we have that $\sin\left(\frac{1}{2}(\pi - \vartheta)\right) = \frac{x - \frac{1}{2}}{d_n} = \cos\left(\frac{1}{2}\vartheta\right)$, and since $\frac{1}{2} \leq x < \frac{1}{2} + d_n$ it is clear that we must have $0 < \vartheta \leq \pi$ and also $\sin \vartheta \geq 0$. Then, we get $|T_{xy}| = \frac{1}{2}\vartheta d_n^2 - \frac{1}{2}d_n \cos\left(\frac{1}{2}\vartheta\right)2d_n \sin\left(\frac{1}{2}\vartheta\right) = \frac{1}{2}\vartheta d_n^2 - \frac{1}{2}d_n^2 \sin \vartheta$, and therefore, $|T_{xy}| = \frac{1}{2}d_n^2(\vartheta - \sin \vartheta)$.

Hence, for each possible value of $\vartheta$, an upper bound on the number of nodes that are able to transmit at the same time from $L$ to $R$ is

$$\frac{nL}{nT_{xy}} = \frac{nd_n}{n\frac{1}{2}d_n^2(\vartheta - \sin \vartheta)} = \frac{1}{\frac{\vartheta}{d_n} - \frac{\sin \vartheta}{d_n}}.$$

Since this upper bound depends on the choice of receiver location (through the angle $\vartheta$), we will make this bound as small as possible by an appropriate choice of $\vartheta$. As noted above, $0 < \vartheta \leq \pi$, and $\sin \vartheta \geq 0$.

Hence, the number of transmitters in $L$ is smallest when $\vartheta = \pi$ and $\sin \vartheta = 0$, i.e., when the receivers are located close to the cut boundary (as it should be, since it is in this case when receivers “consume” the maximum amount of transmitter area). In this case, we get

$$\min_{0 < \vartheta \leq \pi} \left[ \frac{2}{\frac{\vartheta}{d_n} - \frac{\sin \vartheta}{d_n}} \right] = \frac{2}{\pi d_n}$$

as an upper bound on the number of edges across the cut. Furthermore, in this case we see immediately that to maximize capacity we must keep $d_n$ as small as possible—and we know from eqn. (1) that the smallest possible $d_n$ that will still maintain the network connected is $\Theta\left(\sqrt{n \log n} / n\right)$. Therefore, replacing for the optimal $d_n$, we finally get an upper bound of $\Theta\left(\sqrt{n \log n} / n\right)$.

2) **The Upper Bound is Asymptotically Tight:** To verify that the upper bound is tight, we give an explicit flow construction. Consider the placement of disks shown in Fig. 11.

Since the height of the square is 1, and we are placing nodes at distance $2d_n$ from each other, this guarantees that if there are nodes in each of the circles to create valid tx/rx pairs, then the number of successful simultaneous transmissions across the cut is $\frac{1}{2d_n} = \Theta\left(\sqrt{n \log n} / n\right)$. Whether all such pairs of nodes can be created simultaneously or not is the issue addressed next.
C. Uniform Convergence Issues

Next we prove that when \( n \) points are dropped uniformly over the square \([0,1] \times [0,1]\), we have that simultaneously (i.e., uniformly) over all \( \frac{1}{2dn} \) circles from Fig. 11, each one of the circles contains \( \Theta(\log(n)) \) points in almost all network realizations. From this, we conclude that the distribution of the number of edges across the cut is sharply concentrated around its mean, and hence that in a randomly chosen network, with probability approaching \( 1 \) as \( n \to \infty \), the actual number of straddling edges is indeed \( \Theta\left(\sqrt{n/\log(n)}\right) \).

1) Statement of the Result: Consider we have \( \frac{1}{2dn} \) circles centered along the \( x = \frac{1}{2} \) cut as shown in Fig. 11, with centers \( y_j = (2j - 1)d_n, j = 1 \ldots \frac{1}{2dn} \) and radius \( d_n \). Then, we have the following uniform convergence result:

**Proposition 1:** Define \( B_j := [\|N_j - \pi \log n\| < \delta \pi \log n] \). Then, as \( n \to \infty \), and for any \( \delta \in (x, 1) \) (\( x \approx 0.6 \)), we have that

\[
\lim_{n \to \infty} P \left[ \bigcap_{j=1}^{2dn} B_j \right] = 1.
\]

Essentially what this proposition says is that with very high probability and uniformly over \( j \), all \( A_j \)'s contain \( \Theta(\log n) \) nodes.

2) Proof: Note that the area of a circle in Fig. 11 is \( \pi d_n^2 = \pi \frac{\log n}{n} \). Then, from the Chernoff bound, we have that for any \( 0 < \delta < 1 \) we can find a \( \theta > 0 \) such that

\[
P[|N_j - \pi \log n| > \delta \pi \log n] < e^{-\theta \pi \log n} = n^{-\theta \pi}.
\]

Thus, we can conclude that the probability that the number of nodes in a circle deviates by more than a constant factor from the mean tends to zero as \( n \to \infty \). This is a key step in showing that all the events \( B_j := [\|N_j - \pi \log(n)\| < \delta \pi \log(n)] \) occur simultaneously. Now, from the union bound, we have that

\[
P \left[ \bigcup_{j=1}^{2dn} B_j \right] = 1 - \sum_{j=1}^{2dn} P[B_j^c] \geq 1 - \sum_{j=1}^{2dn} P[B_j^c].
\]

But, from eqn. (6), \( P[B_j^c] < n^{-\theta \pi} \), and therefore,

\[
\sum_{j=1}^{2dn} P[B_j^c] < \sum_{j=1}^{2dn} n^{-\theta \pi} = \frac{n^{-\theta \pi}}{2d_n} = \frac{n^{-\theta \pi}}{2\sqrt{\log n}} = \frac{\frac{1}{2} - \pi \theta}{2\sqrt{\log n}}.
\]

Putting everything together, and letting \( n \to \infty \), we have

\[
P \left[ \bigcap_{j=1}^{2dn} B_j \right] \geq 1 - \frac{n^{-\theta \pi}}{2\sqrt{\log n}} \to 1,
\]
if and only if $\pi \theta > \frac{1}{2}$. And this is true for $\delta \approx 0.6$ and above (this follows from the definition of $\theta$ and a simple numerical evaluation).

VI. APPLICATIONS TO WIRELESS NETWORKING PROBLEMS II: DIRECTIONAL ANTENNAS

A. On Directional Antennas and MST Issues

We consider now an application of the techniques that were used so far to analyze the network capacity problem in the context of directional antennas.

Why the interest in directional antennas? Because there is a question about wireless networks equipped with such antennas which we believe is very important, and for which we could not find a satisfactory answer in the literature. We discussed in Section V the vanishing throughput problem identified in [14]. But in a different segment of the research community, the use of directional antennas has also received a fair amount of attention in recent times. The rationale is that with omnidirectional antennas, existing MAC protocols require all nodes in the vicinity of a transmission to remain silent. With directional antennas however, it should be possible to achieve higher overall throughput, by means of a higher degree of spatial reuse of the shared medium, and a smaller number of hops visited by a packet on its way to destination (see, e.g., [4]). Furthermore, in the context of energy-efficient broadcast/multicast, it has been argued that the ability of a transmitter to reach multiple receivers is an important source of gains to take advantage of in the development of suitable protocols, such as BIP [32].

If we take a step back, careful reading of these previous results raises an important question: how much exactly is there to gain from the use of directional antennas? Could directional antennas (in which the width of the beams tends to zero as $n$ gets large) be used to effectively overcome the vanishing maximum throughput of [14]? Although we have not been able to find answers to this question in the literature (and that motivated us to start working on this problem in the first place), we have found a couple of related results based on which we can say a-priori that the answer is probably no:

- In [14], the authors claim that their result holds irrespective of whether transmissions are omni-directional or directed, provided that in the case of directed antennas there is some lower bound (independent of network size) on how narrow the beams can be made.
- In [21], [27], for some regular networks, it is shown that enabling nodes with Multi-Packet Reception (MPR) capabilities [9] can only increase the total throughput of the network by a constant factor ($\approx 1.6$), independent of network size.

Given this state of affairs, it seems to us that deciding exactly how much there is to be gained by using directional antennas, and giving some measure of how complex the transmitters/receivers need to be made...
to achieve those gains, is indeed a topic worth being studied.

B. A Single Directed Beam

1) Transmitter/Receiver Model: In this section we consider the first model based on directional antennas: transmitters can generate a beam of arbitrarily narrow width aimed at any particular receiver, and receivers can accept any number of incoming messages, provided the transmitters are not in the same straight line. This results in a significant increase in the complexity of the signal processing algorithms required at each node, and in this section our goal is to determine if and how much it is possible to increase the achievable MST, compared to the omnidirectional case. This model is illustrated in Fig. 12.

Our goal in this subsection is to evaluate $\Theta(\nu_n^*)$ and $\Theta(\gamma_n)$, for this particular architecture.

2) Average Number of Edges Across the Cut: Since at most one edge per transmitter can be active at any point in time, the average number of edges going across the cut can be no larger than $nd_n$, the average number of transmitters on its left side. Since $L$ and $R$ have the same area, the average number of nodes on each side of the cut is the same (and equal to $nd_n$), and hence the maximum of $nd_n$ transmissions can actually be received, by “pairing up” every node from one side of the cut with every node on the other side. The pairing of nodes on each side of the cut is illustrated in Fig. 13.

Finally we note that, under the assumption of arbitrarily narrow and perfectly aligned beams, the only way in which we could have multiple receivers blocked out by a single transmission is by having them all
lying in a nearly straight line (i.e., a set of vanishing measure) under the beam of a single transmitter. But then, to have an actual edge count lower than $\Theta(nd_n)$, we would require an increasingly large number of nodes falling in a decreasingly small area: under our statistical model for node placement, this event occurs with vanishing probability, and therefore the average edge count is $\Theta(nd_n)$.

3) Sharp Concentration Results:

a) Number of Transmitters in $L$ and Receivers in $R$: Again, consider $n$ points $X_1...X_n$ uniformly distributed over the $[0,1] \times [0,1]$ plane, and consider the area $L$ on the left side of the cut, as shown in Fig. 13. We define variables

$$N_i = \begin{cases} 1, & X_i \in L \\ 0, & \text{otherwise.} \end{cases}$$

and $N = \sum_{i=1}^{n} N_i$. The probability $p$ of $X_i \in L$ is $p = |L| = 1 \cdot d_n$. Hence, $E(N_i) = 1 \cdot p + 0 \cdot (1 - p) = p = d_n$, and $E(N) = \sum_{i=1}^{n} E(N_i) = nd_n$. From the Chernoff bound, we know that

$$P(|N - nd_n| > \delta nd_n) < e^{-\theta nd_n}.$$ 

Since $\theta > 0$, we have that as $n \to \infty$, deviations of $N$ from its mean by a constant fraction (independent of $n$) occur with low probability, provided $d_n$ does not decay too fast.$^3$ Therefore, we conclude that in almost all realizations of the network, the number of transmitters in $L$ and the number of receivers in $R$ is $\Theta(nd_n)$.

b) Number of Edges Across the Cut: Knowing that we have $\Theta(nd_n)$ transmitters and receivers within range of each other on each side of the cut is not enough to claim that the number of edges that cross the cut is $\Theta(nd_n)$. This is because, in our model for directional antennas, a receiver can successfully decode two simultaneous incoming transmissions provided the angle formed by the receiver and the two transmitters is strictly positive: if all three are on the same straight line, collisions still occur, and those edges are destroyed. Therefore, we still need to show that the actual number of edges is $\Theta(nd_n)$. And to do this, we need to say something about the location of points that end up in $L$, and not just count how many. To proceed, we cut the area of $L$ into $nd_n$ rectangles of height $\frac{1}{nd_n}$ and width $d_n$, as illustrated in Fig. 14. Our goal then becomes to show that in “most” of these rectangles (meaning, in all but a constant fraction of them) we will have nodes capable of forming straddling edges.

$^3$Note that the fastest possible decay for $d_n$, according to eq. (1), is when $d_n \approx \sqrt{\frac{\log n}{n}}$. And in this case, $e^{-\theta nd_n} = e^{-\theta \sqrt{\frac{\log n}{n}}} \to 0$ as $n \to \infty$. If $d_n$ is any bigger, this probability goes to zero even faster. So the Chernoff bound applies for any connected network.
Counting how many of the $nd_n$ rectangles in Fig. 14 contain at least one of the $\Theta(nd_n)$ nodes that are dropped in $L$ is an instance of a classical occupancy problem, in which $k$ balls are thrown uniformly onto $m$ bins, in the case where $k = m = nd_n$ [22, Ch. 4]. Since $\frac{1}{m}$ is the probability that a ball falls in any particular bin, the probability $p$ of an empty bin after throwing all $m$ balls is $p = (1 - \frac{1}{m})^m$ which, for $m$ large, becomes approximately $\frac{1}{e}$. Therefore, the average number of empty bins is $mp \approx \frac{1}{e} \sqrt{n \log(n)}$.

And by the Chernoff bound, again we have that

$$P \left( Y - nd_n / e > \delta nd_n / e \right) < e^{-\theta nd_n / e},$$

where $Y$ is the number of empty bins. So, the probability that the number of empty bins is a constant factor away from its mean is small (again, provided $d_n$ does not decay too fast), and hence, for $n$ large, almost all network realizations will have $\Theta(nd_n)$ non-empty rectangles. But since transmitter/receiver pairs in different rectangles are not collinear, the number of edges across the cut is $\Theta(nd_n)$, qed.

4) Remarks:

a) **MST in a Minimally Connected Network:** In this section, we found that the MST achievable by the type of tx/rx pairs considered here depends on the connectivity radius $d_n$. If we replace $d_n$ with $\sqrt{\frac{e \log n}{n}}$ (the minimum radius required to maintain a connected network, from [13]), we get

$$nd_n \approx n \sqrt{\frac{e \log n}{n}} = \Theta \left( \sqrt{n \log n} \right).$$

Comparing this expression with its equivalent from Section V, we see that all we gain over the case of omnidirectional antennas is an increase in MST by a factor of $\Theta(\log n)$.

b) **Minimum Connectivity Radius Resulting in MST = $\Theta(n)$:** In this tx/rx architecture we are considering the use of arbitrarily narrow and perfectly aligned directed beams. Therefore, it does make sense to consider the use of a possibly larger transmission range than the minimum required to keep the network connected, since in this case a large range does not force other tx/rx pairs to remain silent while a given transmission is in progress. And since by increasing the transmission range now we can increase
throughput, our next goal is to determine the minimum range that would be required to achieve \( \text{MST} = \Theta(n) \).

Solving for \( d_n \) in \( \Theta(n) = \Theta(nd_n) \), we see that trivially, \( d_n = \Theta(1) \). That is, to achieve MST linear in the number of nodes using a single beam in each transmission, the radius of each transmission has to be a constant independent of \( n \).

c) Minimum Number of Simultaneous Beams: From a practical point of view, does it matter that to achieve linear MST we need to keep the transmission radius constant? In this section we argue that yes it does, very much. To see why this is so, next we count the minimum number \( \beta \) of narrow beams that a transmitter would have to generate simultaneously, if MST linear in the size of the network is to be achieved: this number gives a measure of the complexity of the beamforming transmitter, since \( 2\pi/\beta \) is an upper bound on the maximum angle of dispersion of the beam.

Since a node can generate a beam to any receiver within its transmission range (see Fig. 12), again using eqns. (3) and (4), we have that for \( n \) large, the number of points within a circle of radius \( d_n \) is \( \Theta(n \cdot \pi d_n^2) \). In the case of \( d_n \) only satisfying the requirement of keeping the network connected,

\[
\beta = n \cdot \pi d_n^2 = n \left( \frac{\pi c \log n}{n} \right) = \Theta(\log n).
\]

This fact was known already—see [34] for a more complete analysis (constants hidden by the \( \Theta \)-notation included), including also a number of interesting references on the history of this problem. But if now we consider a larger \( d_n \) satisfying the requirement of achieving linear MST, then

\[
\beta = n \cdot \Theta(1)^2 = \Theta(n).
\]

Therefore, we see \( \beta \) has an exponential increase relative to the number required to maintain minimum connectivity—it is on this fact that we base our claim about directional antennas not being able to provide an effective means of overcoming the issue with per-node vanishing throughputs.

C. Multiple Directed Beams

1) Transmitter/Receiver Model: In this section we consider another model based on directional antennas: transmitters can generate an arbitrary number of beams, of arbitrarily narrow width, aimed at any particular receiver; and receivers can accept any number of incoming messages, provided the transmitters are not in the same straight line. This is perhaps the most complex scheme that could be envisioned based on directed beams. Our goal is to determine if and how much it is possible to increase the achievable MST, compared to the previous two cases. This model is illustrated in Fig. 15.

Again, our goal in this subsection is to evaluate \( \Theta(\nu_n^*) \) and \( \Theta(\gamma_n) \), for this particular architecture.
2) Average Number of Edges Across the Cut: With minor variations, this calculation is essentially identical to that presented in Section III-A, and the final result is the same: the ensemble average number of edges straddling the center cut is $\Theta(n^2d_n^3)$. See Section III-A for details.

3) Sharp Concentration Results: Our next goal is to show that the actual number of edges straddling the cut in any realization of the network is sharply concentrated around its mean. That is, in almost all networks, the number of edges across the cut is $\Theta(n^2d_n^3)$, a) Number of Receivers per Transmitter: Define a binary random variable $N_{ij}$, which takes the value 1 if the $i$-th node is within the transmission range of a node at coordinates $(x_j, y_j)$ on the other side of the cut, as illustrated in Fig. 4:

$$
N_{ij} = \begin{cases} 
1, & X_i \in Q(x_j, y_j) \\
0, & \text{otherwise.}
\end{cases}
$$

Let $p$ denote the probability that $X_i$ is in $Q(x_j, y_j)$ (i.e., that $N_{ij} = 1$). Then, $p = |Q(x_j, y_j)| = \frac{1}{2}d_n^2(\vartheta - \sin(\vartheta))$, with $0 \leq \vartheta \leq \pi$ is as in Fig. 4. Therefore, defining $\kappa_\vartheta$ as $\frac{1}{2}(\vartheta - \sin(\vartheta))$, we have $p = |Q(x_j, y_j)| = \kappa_\vartheta d_n^2 = \kappa_\vartheta \log(n)\frac{n}{n}$.

Define $N_j = \sum_{i=1}^{n} N_{ij}$ as the number of points in $Q(x_j, y_j)$. In this case, we have $E(N_j) = \sum_{i=1}^{n} N_{ij} = \sum_{i=1}^{n} p \cdot 1 + (1 - p) \cdot 0 = np = \kappa_\vartheta \log(n)$. Now, again from the Chernoff bound, we have that

$$P(|N_j - \kappa_\vartheta \log(n)| > \delta \kappa_\vartheta \log(n)) < e^{-\theta \kappa_\vartheta \log(n)} = n^{-\theta \kappa_\vartheta},$$

for $\theta$ defined as in previous applications. As $n \to \infty$ this probability tends to zero, and therefore, in almost all network realizations, a transmitter on the left side of the cut will be able to reach $\Theta(\log(n))$ receivers on the right side.

b) Total Number of Edges: In a manner analogous to the situation discussed in Section VI-B, knowing that there are $\Theta(nd_n)$ transmitters on the left side of the cut, and that each transmitter can reach
\( \Theta(nd_n^2) \) receivers on the other side, is not enough to conclude that the total number of edges going across the cut must be \( \Theta(n^2d_n^2) \). This is because of our requirement that multiple transmitters not be perfectly aligned with a receiver for this receiver to decode all these messages simultaneously. Therefore, we still need to show that the actual number of edges is \( \Theta(n^2d_n^2) \). And to do this, we need to say something about the location of points in \( R \) that can be reached from \( L \), and not just count how many. To proceed then, we cut the area of \( Q_p \) into \( \kappa_\vartheta \log(n) \) slices, each slice of area \( \frac{|Q_p|}{\kappa_\vartheta \log(n)} = \frac{1}{n} \), as illustrated in Fig. 16.

![Fig. 16. Cutting the shaded arc \( Q_{xy} \) into regions of area \( \frac{1}{n} \), to formulate this as an occupancy problem analogous to that of Fig. 14.](image)

As in the occupancy problem considered in Section VI-B, our goal is to show that in “most” of these arc slices (most meaning, in all but a constant fraction of them) we will have nodes capable of forming straddling edges. This is again a problem of throwing \( k \) balls uniformly into \( m \) bins, where \( k = m = \kappa_\vartheta \log(n) \). And again, we have that with probability that tends to 1 as \( n \to \infty \), the number of empty bins is \( \kappa_\vartheta \log(n)/e \), and hence the number of occupied bins is \( \Theta(\log(n)) \).

Consider now a fixed transmitter located at some coordinates \( (x, y) \). Any other transmitter located at coordinates \( (x', y') \neq (x, y) \) defines a unique straight line that goes through \( (x, y) \) and \( (x', y') \). If there is a receiver on the other side of the cut along this line, within reach of both transmitters, then those two edges will be lost—and those will be the only lost edges, from among the \( \kappa_\vartheta \log(n) \) that each transmitter has. This situation is illustrated in Fig. 17.

![Fig. 17. To illustrate how we could end up losing edges: if the two black transmitters attempt simultaneously to communicate with the gray receiver, a collision will occur, and none of the edges will be created.](image)
And then we are done. We have established that in almost all network realizations, there are $\Theta(nd_n)$ transmitters within each side of the cut, that each transmitter can reach $\Theta(d_n^2)$ receivers on the other side of the cut, and that integrating out $\kappa, \theta$ we obtain exactly $\Theta(n^2d_n^3)$ edges going across the cut. Therefore, the actual number of edges across the cut is sharply concentrated around its mean, qed.

4) Remarks:

a) MST in a Minimally Connected Network: Substituting for $d_n = \sqrt{\frac{c \log n}{n}}$ in $\frac{2}{3}n^2d_n^3$, we get

$$\frac{2}{3}n^2 \left(\frac{\log n}{\pi n}\right)^\frac{\frac{2}{3}}{2} = \frac{2}{3} \sqrt{n} \log \frac{2}{\pi} n = \Theta\left(\sqrt{n} \log^2(n)\right)$$

Comparing this expression to the ones obtained in Sections V and VI-B, we see that the MST gain due to the use of multiple simultaneous, arbitrarily narrow beam is, at most, $\Theta\left(\log^2(n)\right)$.

b) Minimum Connectivity Radius Resulting in MST = $\Theta(n)$: The minimum $d_n$ resulting in linear MST is obtained by solving for $d_n$ in $\Theta(n^2d_n^3) = \Theta(n)$. Now, for $n$ large enough, there exist constants $c_1 < c_2 \in \mathbb{R}$ ($c_1 > 0$ and $c_2 < \infty$), such that $c_1 n < \frac{2}{3}n^2d_n^3 < c_2 n$, or equivalently, $c_1 3^{\frac{3}{4}n} < d_n < c_2 3^{\frac{3}{4}n}$. Therefore,

$$d_n = \Theta(n^{-\frac{1}{3}}).$$

c) Minimum Number of Simultaneous Beams: In Section VI-B, we said that keeping the transmission range constant resulted in an impractically large number of beams that the receiver needed to generate, if linear MST was to be achieved by increasing the complexity of the signal processing algorithms. But if we generate multiple beams, we have just shown that this minimum radius now is no longer a constant, but instead tends to zero as $\Theta(n^{-\frac{1}{3}})$. However, the situation is not much better compared to the single beam case, and to see this again we compute the minimum number of simultaneous beams that a transmitter would have to generate.

If now we consider the larger $d_n$ satisfying the requirement of achieving maximum stable throughput linear in network size, then

$$\beta = n \cdot \pi d_n^2 = n \Theta(n^{-\frac{2}{3}}) = \Theta(n^{\frac{1}{3}}).$$

Therefore, we see that while $\beta$ is smaller than in the case of the single beam, we still have an exponential increase relative to the number required to maintain minimum connectivity—so again, we claim that directional antennas are not able to provide an effective means of overcoming the issue with per-node vanishing throughputs.
VII. Conclusions

A. Summary of Contributions

In this paper, we have showed how network flow methods can be used to determine (to within constants) the maximum stable throughput achievable in a wireless network. This was done by formulating MST as a maximum multicommodity flow problem, for which tight upper and lower bounds were found. In the process, the difficult multicommodity problem was proved equivalent to a simpler single commodity problem, solvable using standard arguments based on flows and cuts.

As mentioned in the Introduction, this work grows out of our desire to cast what we deem to be the most useful insights in [14] (basically, that the constriction in capacity results from the need to share constant capacity links by a growing number of nodes), in a form that makes more intuitive sense to us. And we feel we have accomplished that:

- By reducing the problem to counting the average number of edges that cross a cut and then proving a sharp concentration result around this mean, the computational task becomes very simple, involving only elementary tools from combinatorics and discrete probability. In [14], similar results had been obtained based essentially on generalizations of the Glivenko-Cantelli theorem (that add uniformity to convergence in the law of large numbers), due to Vapnik and Chervonenkis [25, Ch. 2], [31].
- In our formulation, it is straightforward to see that capacity limitations arise essentially from the geometry of the problem: edges have a constant capacity, and only about $\sqrt{n}$ of them are available at a minimum cut to transport the traffic generated by $n$ sources.

B. Future Work

In terms of future work, there are a number of interesting questions opened up by this work. One deals with the generalization of these results to nodes distributed on arbitrary manifolds (instead of the square $[0,1] \times [0,1] \subset \mathbb{R}^2$). Another deals with exploring other combinatorial structures (such as hypergraphs [3]), to develop better collision models, especially in the omnidirectional case. Of particular interest to us however is the development of a purely information-theoretic formulation for the results in this paper, by exploiting the connections between Shannon information and network flow theory discovered in [2].

However, we certainly would have not been able to obtain our simpler and more general proofs without the insights provided by cultivating an appreciation for the line of reasoning employed by Gupta and Kumar in [14].
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