1+3 Covariant Cosmic Microwave Background anisotropies I: Algebraic relations for mode and multipole representations.

Tim Gebbie † and George F.R. Ellis † ‡

† Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa
and
‡ Department of Mathematics, Queen Mary and Westfield College, Mile End Road, London E1 4NS, UK

October 17, 2018

Abstract

This is the first of a series of papers extending a 1+3 covariant and a gauge invariant treatment of kinetic theory in curved space-times to a treatment of Cosmic Background Radiation (CBR) temperature anisotropies arising from inhomogeneities in the early universe. This paper deals with algebraic issues, both generically and in the context of models linearised about Robertson-Walker geometries.

The approach represents radiation anisotropies by Projected Symmetric and Trace-Free tensors. The Angular correlation functions for the mode coefficients are found in terms of these quantities, following the Wilson-Silk approach, but derived and dealt with in 1+3 covariant and gauge invariant (CGI) form. The covariant multipole and mode-expanded angular correlation functions are related to the usual treatments in the literature. The CGI mode expansion is related to the coordinate approach by linking the Legendre functions to the Projected Symmetric Trace-free representation, using a covariant addition theorem for the tensors to generate the Legendre Polynomial recursion relation.

This paper lays the foundation for further papers in the series, which use this formalism in a CGI approach to developing solutions of the Boltzmann and Liouville equations for the CBR before and after decoupling, thus providing a unified CGI derivation of the variety of approaches to CBR anisotropies in the current literature.

Contents

1 Introduction 2

2 Temperature anisotropies 4

2.1 Covariant and gauge invariant angular correlation function 5

2.2 The Central-Limit Theorem 6

2.3 Gaussian perturbations 7

3 Multipole expansions 7

3.1 The PSTF part of $e^{A_{\ell}}$ 7

3.1.1 Relationship to Legendre polynomials 9

3.2 The mean square of PSTF coefficients : $\langle F_{A_{\ell}}F_{A_{\ell}}^\ast \rangle$ 9
1 Introduction

Ellis, Treciokas and Matravers (ETM) introduced a 1 + 3 covariant kinetic theory formalism in which an irreducible representation of the rotation group based on Projected Symmetric and Trace-Free (PSTF) tensors orthogonal to a physically defined 4-velocity $u^a$ gives a covariant
representation of the Cosmic Background Radiation (CBR) anisotropies, which is gauge-invariant when the geometry is an almost-Robertson Walker (RW) geometry. This 1 + 3 Covariant and Gauge-Invariant (CGI) formalism has been used in a previous series of papers [9, 10, 11, 13, 14, 15] to look at the local generation of CBR anisotropies by matter and spacetime inhomogeneities and anisotropies in an almost-Friedmann-Lemaître (FL) universe model [16]. By contrast, the present series of papers [17] uses this formalism to investigate CBR anisotropies in the non-local context of emission of radiation near the surface of last scattering in the early universe and its reception here and now (the Sachs-Wolfe (SW) effect and its further developments).

There is of course a vast literature investigating these anisotropies both from a photon viewpoint, developing further the methods of the original Sachs-Wolfe paper [29], and from a kinetic theory viewpoint, so it is useful to comment on why the CGI philosophy and programme [9] make the present series of papers worthwhile. Rather than beginning with a background described in particular coordinates and perturbing away from this background, this approach centres on 1 + 3 covariantly defined geometric quantities, and develops exact nonlinear equations for their evolution. These equations are then systematically linearised about a Friedmann-Lemaitre (FL) background universe with a Robertson-Walker (RW) geometry resulting in description by gauge-invariant variables and equations [16]. Because the definitions and equations used are coordinate-independent, one can adopt any suitable coordinate or tetrad system to specialise the tensor equations to specific circumstances when carrying out detailed calculations; a harmonic or mode analysis can be carried out at that stage, if desired.

This approach is geometrically transparent (see [30], [31]) because of the CGI variable definitions used. In contrast to the various gauge-dependent approaches to perturbations in cosmology, the differential equations used are of just the order that is needed to describe the true physical degrees of freedom, so no non-physical gauge modes occur. When a harmonic decomposition is introduced in the case of linear perturbations, the CGI variables used here provide a description that is equivalent to that obtained by approaches based Bardeen’s GI variables [33], see [34], but they do not imply linearisation of the equations from the outset, as occurs in that formalism.

Thus the benefit of the present formalism is precisely its 1 + 3 covariant and gauge invariant nature, together with the fact that we are able to write down the exact non-linear equations governing the growth of structure and the propagation of the radiation, and then linearise them in a transparent way in an almost-RW situation. This means it can be extended to non-linear analyses in a straightforward way [14], which will be essential in developing the theory of finer CBR anisotropy structure as reliable small-angle observations become available. Achievements of the CGI approach with respect to the CMB are the almost-EGS Theorem [1], related model-independent limits on inhomogeneity and isotropy [11, 12, 13, 14], and derivation of exact anisotropic solutions of the Liouville equation in a RW geometry ([36], see also [35]).

This paper, Part I, deals with algebraic issues, developing further the formalism of ETM: namely an irreducible representation of radiation anisotropies based on PSTF tensors [3, 1]. The paper considers this irreducible representation and its relation to observable quantities, both generically and in the context of models linearised about RW geometries [11]. In section 2 and 3, the underlying 1 + 3 decomposition is outlined and the basic CGI harmonic formalism for anisotropies developed.

In section 4, the angular correlation functions are constructed from CGI variables, assuming that the multipole coefficients are generated by superpositions of homogeneous and isotropic Gaussian random fields. The multipole expansion is discussed in detail, extending the results of ETM.

---

1 Here ‘Robertson-Walker’ refers to the geometry, whatever the field equations; ‘Friedman-Lemaître’ assumes that the Einstein gravitational field equations with a perfect fluid matter source are imposed on such a geometry.

2 And a series by Challinor and Lasenby [30], [31] which are similar in method and intent but different in detail and focus.
giving the construction of the multipole coefficient mean-square and developing its link to the angular correlation function. In section 5, the mode coefficients are found following the Wilson-Silk approach, but derived and dealt with in the CGI form; the covariant and gauge invariant multipole and mode expanded angular correlation functions are related to the usual treatments used in the literature \cite{17, 20, 22, 23, 21}. In this discussion, the CGI mode expansion is related to the coordinate approach by linking the Legendre Tensors to the PSTF representation, using a covariant addition theorem to generate the Legendre Polynomial recursion relation. The key result is the construction of the angular correlation functions in the CGI variables, and their link to the (non-local) GI Mode functions \cite{20}.

The following papers in the series look at the Boltzmann equation and multipole divergence relations, solution of the resulting mode equations, and relation of the kinetic theory approach to the photon based formalism of the original Sachs-Wolfe paper. Exact non-linear equations are obtained and then linearised, allowing a transparent linearisation process from the non-linear equations that is free from ambiguities and gauge modes.

2 Temperature anisotropies

A radiation temperature measurement is associated with an antenna temperature, \( T(x^i, e^a) \), measured by an observer moving with 4-velocity \( u^a \) at position \( x^i \) in a direction \( e^a \) on the unit sphere \( (e^a e_a = 1, e^a u_a = 0) \). We assume \( u^a \) can be uniquely define in the cosmological situation, corresponding to the motion of ‘fundamental observers’ in cosmology \cite{37}. The direction \( e^a \) can be given in terms of an orthonormal tetrad frame \cite{4}, for example by:

\[
e^a(\theta, \phi) = (0, \sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta).
\]  

The temperature \( T(x^i, e^a) \) can be unambiguously decomposed into the all-sky average bolometric temperature \( \bar{T}(x^i) \) at position \( x^i \), given by

\[
T(x^i) = \frac{1}{4\pi} \int_{4\pi} T(x^i, e^a) d\Omega.
\]  

where \( \Omega \) is the solid angle on the sky, and the anisotropic temperature perturbation \( \delta T(x^i, e^a) \) (the difference from the average over the unit sphere surrounding \( x^i \) \cite{10}), can be defined:

\[
T(x^i, e^a) = T(x^i) + \delta T(x^i, e^a).
\]  

From the Stefan-Boltzmann law it follows – if the radiation is almost black-body, which we assume – that the radiation energy density is given in terms of the average bolometric temperature by \( \rho_R(x^i) = rT^4(x^i) \). (\( r \) is Stefan-Boltzmann constant). Both the quantities \( T(x^i) \) and \( \delta T(x^i, e^a) \) are CGI, for \( T(x^i) \) is defined in a physically unique frame in the real universe (because \( u^a \) is assumed to be uniquely defined), and \( \delta T(x^i, e^a) \) vanishes in any background without temperature anisotropies.

We can define the fractional temperature variation \( \tau(x^i, e^a) \) by \cite{10}

\[
\tau(x^i, e^a) := \frac{\delta T(x^i, e^a)}{T(x^i)}.
\]  

\footnote{In the early universe, when matter and radiation average velocities differ, there may be several competing possibilities for covariant definition of \( u^a \); however once a choice has been made between these possibilities, this vector field is uniquely defined.}

\footnote{The description of such a tetrad frame is briefly discussed in last appendix}

\footnote{Note that this is not the same as the background temperature, for that quantity varies only with cosmic time \( t \), whereas the true isotropic component of the temperature varies with spatial position as well as time.}
and take a covariant (angular) harmonic expansion of this,

$$\tau(x^i, e^a) = \sum_{\ell=1}^{\infty} \tau_{a_1 a_2 a_3 \ldots a_{\ell}}(x^i) e^{a_1} e^{a_2} e^{a_3} \ldots e^{a_{\ell-1}} e^{a_\ell} \equiv \sum_{\ell \geq 1} \tau_{\ell} \hat{e}^{\ell}.$$  \hspace{1cm} (5)

We introduce the shorthand notation using the compound index $A_\ell = a_1 a_2 \ldots a_\ell$. Here $\tau_{a_1 a_2 a_3 \ldots a_{\ell}}(x^i)$ are trace-free symmetric tensors orthogonal to $u^a$:

$$\tau_{A_\ell} = \tau(A_\ell), \quad \tau_{A_\ell} h^{ab} = 0, \quad \tau_{A_\ell} u^a.$$  \hspace{1cm} (6)

Round brackets “(..)” denote the symmetric part of a set of indices, angle brackets “⟨..⟩” the (orthogonally-) Projected Symmetric Trace-Free (PSTF) part of the indices: $\tau_{A_\ell} = \tau(\langle A_\ell \rangle)$.

Because of (1), this expansion is entirely equivalent to a more usual expansion in terms of spherical harmonics:

$$\tau(x^i, e^a) = \sum_{\ell=1}^{\infty} A^n_{\ell}(x^i) Y^m_{\ell}(\theta, \phi)$$  \hspace{1cm} (7)

(see [5] for details), but is more closely related to a tensor description, and so results in more transparent relations to physical quantities.

We wish to measure the temperature in two different directions to find the temperature difference associated with the directions $e^a$ and $e'^a$ such that (using [3]):

$$\Delta T(x^i; e^a, e'^a) = T(x^i, e^a) - T(x^i, e'^a) \Rightarrow \Delta T(x^i; e^a, e'^a) = \delta T(x^i, e^a) - \delta T(x^i, e'^a).$$  \hspace{1cm} (8)

It follows from (8), (4) and (5) that $\Delta T(x^i; e^a, e'^a) = T(x^i) \sum_{\ell} \tau_{A_\ell} (e^{A_\ell} - e'^{A_\ell})$, where $\Delta T/T$ represents the real fractional temperature difference on the current sky. Due to the CGI nature of $T(x^i)$ we may relate this directly to the real temperature perturbations (no background model is involved in these definitions).

The relation between the two directions $e^a$ and $e'^a$ at $x^i$ is characterised by

$$e^a e'^a = \cos(\beta) =: X.$$  \hspace{1cm} (9)

i.e. they are an angular distance $\beta$ apart [4]. If analogous to (1) we write $e^a(\theta', \phi') \equiv e'^a = (0, \sin \theta' \sin \phi', \sin \theta' \cos \phi', \cos \theta')$, then it can be shown from (6) that

$$\cos \beta = \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'.$$  \hspace{1cm} (10)

In later applications, it is important to relate the different terms of the harmonic expansion to angular scales in the sky. A useful approximation is $l \approx \frac{1}{\beta}$, where $\theta$ is in radians.

### 2.1 Covariant and gauge invariant angular correlation function

The two-point correlations are an indication of the fraction of temperature measurements, $T(x^i, e^a)$, that are the same for a given angular separation. This corresponds to the correlation between $\delta T(x^i, e^a)$ and $\delta T(x^i, e'^a)$ or equivalently between $\tau(x^i, e^a)$ and $\tau(x^i, e'^a)$, given by the angular position correlation function

$$C(e^a, e'^a) = \langle \tau(x^i, e^a), \tau(x^i, e'^a) \rangle,$$  \hspace{1cm} (11)

\footnote{It should be pointed out that $e'^a$ is distinct from $e^a$, the first denotes a direction vector different from $e^a$ in a given tetrad frame, while the second means the same direction vector in a different tetrad frame.}
where the angular brackets representing an angular average over the complete sky. Note this is a function in the sky. If we write \( \tau(x^i, e^a) \) and \( \tau(x^i, e'^a) \) in terms of the angular harmonic expansion, we can also define correlation functions \( C_\ell \) for the anisotropy coefficients \( \tau_A(\ell, e^a), \tau_A(\ell, e'^a) \) by

\[
C_\ell(e^a, e'^a) = (2\ell + 1)^{-1} \Delta_\ell \left\langle \tau_A(\ell, e^a) \tau_A(\ell, e'^a) \right\rangle.
\]  

(12)

Here the right-hand side term in brackets is the all-sky mean-square value of the \( \ell \)-th temperature coefficient \( \tau_A(\ell, e^a) \), and the coefficient \( \Delta_\ell \) is defined in (119). The numerical factor \( (2\ell + 1)^{-1} \Delta_\ell \) is included in order to agree with definitions normally used in the literature (see later). This can be thought of as the momentum space version of (11), as we have taken an angular fourier series of the quantities in that equation; it says, for each choice of \( e^a, e'^a \), how much power there is in that expression for that angular separation as contributed by a particular \( \ell \)-th valued multipole moment on average.

### 2.2 The Central-Limit Theorem

We consider an ensemble of temperature anisotropies, where a sequence of repeated trials is replaced by a complete ensemble of outcomes. The temperature anisotropy \( \tau_A \) found in a given member of the ensemble is a realization of the statistical process represented by the ensemble. The physically measured anisotropy is taken to be one such realization. The variance of the ensemble, for example \( \left\langle \tau_A(\ell, e^a) \tau_A(\ell, e'^a) \right\rangle \), is in principle found by averaging over a sufficiently large number of experiments, where we assume the results will approach the true ensemble variance – this is the assumption of ergodicity.

On Fourier transforming, we make the assumption that to a good approximation the phases of the various multipole moments are uncorrelated and random. This corresponds to treating the anisotropies as a form of random noise. The random phase assumption has a useful consequence: that the sum of a large number of independent random variables will tend to be normally distributed. By the central-limit theorem, this is true for all quantities that are derived from linear sums over waves. The end result is that one ends up with a Gaussian Random Field (GRF) which is fully characterized by a power spectrum. The central limit theorem holds as long as there exists a finite second moment, i.e. a finite variance.

We will assume that the angular variance \( \left\langle \tau_A(\ell, e^a) \tau_A(\ell, e'^a) \right\rangle \) is independent of position; this is the assumption of statistical homogeneity. Its plausibility lies in the underlying use of the weak Copernican assumption. Additionally it is convenient to assume that the power spectrum will have no directional dependence, thus it will be isotropic: \( P(k^a) = P(|k^a|) \). Together these imply the statistical distribution respects the symmetries of the RW background geometry.

One needs to be careful in using the central limit theory to motivate Gaussian random field, particularly in the presence of nonlinearity, which could result in the elements of the ensemble no longer being independent. While the assumptions of primordial homogeneous and isotropic GRF’s is plausible, because the perturbations are made up of a sufficiently large number of independent random variables, the key point is to realize that these are assumptions that should be tested if possible. The simplest test of weak non-Gaussianity is looking for a three-point angular or spatial correlation, for the Gaussian assumption ensures that all the odd higher moments are zero and that the even ones can be expressed in terms of the variance alone. If the primordial perturbations are made up of GRF’s, then non-Gaussianity of the CBR anisotropy spectrum should arise primarily from foreground contamination due to local physical processes. If the non-Gaussian effects due to these later physical processes or evolutionary effects are small enough, one can attempt to determine a cosmological primordial signature.
2.3 Gaussian perturbations

A general Gaussian perturbation [27], \( \tau(x^i, e^a) \), will be a superposition of functions, \( \tau_{A\ell} \), i.e. \([\mathbb{E}]\) is satisfied, where the probability, \( P \), of finding a particular valued temperature coefficient is given by \((\sigma_{A\ell}^2 = \langle \tau_{A\ell} \tau_{A\ell}^* \rangle ) \):

\[
P(\tau_{A\ell}) = \frac{1}{\sqrt{2\pi \sigma_{A\ell}^2}} \exp \left\{ \frac{-\tau_{A\ell} \tau_{A\ell}^*}{2\sigma_{A\ell}^2} \right\}.
\]

(13)

Note that \( \tau_{A\ell} \) is both the amplitude of the \( \ell \)-th component, and determines the probability of that amplitude. The probability of a temperature perturbation, \( \tau \), is given by the sum of the Gaussian probability distributions \([\mathbb{R}]\) weighting the various angular scales, given by \( \ell \), of the general perturbation \([\mathbb{F}]\).

Considering isotropic and homogeneous gaussian random fields, the angular position correlation function \( C(e^a, e'^a) \) is a function only of the angular separation \( \beta \) of the two temperature measurements. We then write \([\mathbb{I}]\) as

\[
\langle \tau \cdot \tau' \rangle = C(\beta) = W(X),
\]

(14)

where the expression on the left is shorthand for \( \langle \tau(x^i, e^a), \tau(x^i, e'^a) \rangle_\beta \), the 2-point angular correlation function for a given angular separation \( \beta \) between the on-sky temperature measurements, and \( X = e^a e'^a = \cos \beta \). This expression is now independent of position in the sky. Gaussian Fields are completely specified by the angular power spectrum coefficients \( C_\ell \) \([\mathbb{D}]\), which are now just constants, because \( \ell \) is uniquely related to \( \beta \), so the power spectrum is a function of the modulus of the wavenumber only. One thus expects the temperature perturbations in this case to be fully specified by the mean squares, \( \langle \tau_{A\ell} \tau_{A\ell}^* \rangle \), when \([\mathbb{E}]\) is substituted in \([\mathbb{F}]\). Equivalently they are uniquely determined by the angular Fourier transform of the 2-point angular correlation function.

3 Multipole expansions

In this section we examine the anisotropy properties of radiation described in terms of the covariant multipole formalism \([\mathbb{B}]\), which is equivalent to the usual angular harmonic formalism but much more directly related to space-time tensors. Note that the relations in this section hold at any point in the space-time, and in particular at the event \( R \) (‘here and now’) where observations take place. Here we consider the PSTF part of, \( e^{A\ell} \); some useful properties of \( e^{A\ell} \) are listed in appendix B.

3.1 The PSTF part of \( e^{A\ell} \)

Because the coefficients in \([\mathbb{B}]\) are symmetric and trace-free, the important directional quantities defined by directions \( e^a \) at a position \( x^i \) are the PSTF quantities

\[
O^{A\ell} = e^{(A\ell)} = e^{(a_1 e^{a_2} e^{a_3} ... e^{a_{\ell-1}} e^{a_{\ell})}},
\]

(15)

for clearly

\[
\tau(x^i, e^a) = \sum_{\ell=1}^{\infty} \tau_{A\ell}(x^i)e^{A\ell}(\theta, \phi) = \sum_{\ell=1}^{\infty} \tau_{A\ell}(x^i)O^{A\ell}(\theta, \phi).
\]

(16)

Indeed the standard spherical harmonic properties are contained in these quantities.

Now the symmetric trace-free (STF) part of a 3-tensor is given in general by Pirani \([\mathbb{I}]\),

\[
[F_{A\ell}]^{STF} = \sum_{n=0}^{[\ell/2]} B^{ln} h_{a_1 a_2 ... a_{2n-1} a_{2n}} F_{a_{2n+1} ... a_{\ell}} \quad \text{with} \quad B_{\ell n} = \frac{(-1)^n l!(2\ell - 2n - 1)!!}{(l - 2n)!(2\ell - 1)!!(2n)!!},
\]

(17)

7
Here \([\ell/2]\) means the largest integer part less than or equal to \(\ell/2\). The following definitions have also been used: \(\ell! = \ell(\ell - 1)(\ell - 2)(\ell - 3)...(1)\), and \(\ell!! = \ell(\ell - 2)(\ell - 4)(\ell - 6)...(2 \text{ or } 1)\).

The PSTF part of a tensor,
\[
T_{(ab)} = [T_{ab}]^{PSTF} = \left[ [T_{ab}]^{STF} \right]^P,
\]
(can be constructed recursively from a vector basis following EMT and a little algebra.

We take the PSTF part of \(e^{(A_\ell)}\) \([\ref{3}, \ref{4}, \ref{5}\]) to find
\[
O^{A_\ell} = e^{(A_\ell)} = \sum_{[\ell/2]} B_{lk} h^{(A_{2k} e^{A_{\ell - 2k}})} ,
\]
where \(h^{(A_{2k} e^{A_{\ell - 2k}})} \equiv h^{(a_1 a_2 ... a_{2k-1} a_{2k} e^{a_{2k+1}} e^{a_{2k+2}} ... e^{a_\ell})\) and \(B_{lk}\) are given by \([\ref{7}]\).

From \([\ref{3}]\) we can now construct recursion relations that play a key role later on. First,
\[
O^{(A_\ell) e^{A_{\ell + 1}}} = O^{(A_\ell) e^{a_{\ell + 1}}} - \frac{\ell}{2\ell + 1} e^{a_{\ell}} h_{ad} O^{(A_{\ell - 1}) h^{(a_{\ell} a_{\ell + 1})}}
\]
From \([\ref{20}], \ref{15}\), and using
\[
e_{a_{\ell}} O^{A_\ell} = \frac{\ell}{(2\ell - 1)} O^{A_{\ell - 1}},
\]
it can then be shown that
\[
O^{A_{\ell + 1}} = e^{(a_{\ell + 1} O^{A_\ell})} - \frac{\ell^2}{(2\ell + 1)(2\ell - 1)} h^{(a_{\ell + 1} a_{\ell} O^{A_{\ell - 1}})}
\]
relates the \((\ell + 1) - t\) th term to the \((\ell - t) - t\) th term and the \((\ell - 1) - t\) th term.

The orthogonality, addition theorem and double integral relations of \(O^{A_\ell}\) are listed in appendix B. Using the orthogonality relations we obtain the inversion of the harmonic expansion:
\[
\tau(x^i, e^a) = \sum_{\ell=0}^{\infty} \tau_{A_\ell}(x^i) O^{A_\ell} \iff \tau_{A_\ell}(x^i) = \Delta^{-1}_\ell \int d\Omega A_\ell \tau(x^i, e^a) .
\]

The polynomial \(L_\ell \equiv O^{A_\ell} O^{A'_\ell} = \sum_{m=0}^{[\ell/2]} B_{\ell m} X^{\ell - 2m}\), (see \([\ref{124}]\)) is the natural polynomial that arises in the PSTF tensor approach (equivalent to the Legendre polynomials, see below), where the coefficients \(B_{\ell m}\) are defined by \([\ref{17}]\). It follows from this that
\[
L_\ell(1) = O^{A_\ell} O^{A'_\ell} = \sum_{m=0}^{[\ell/2]} B_{\ell m} = : \beta_\ell \text{ with } \beta_\ell = \frac{([\ell])2\ell}{(2\ell)!} = \frac{\ell!}{(2\ell - 1)!!} .
\]
The \(\beta_\ell\)’s satisfy the recursive relations
\[
\beta_\ell = \frac{(2\ell + 1)}{(\ell + 1)} \beta_{\ell + 1}, \quad \beta_\ell = \frac{\ell}{(2\ell - 1)} \beta_{\ell - 1}, \quad \frac{\beta_{\ell + 1}}{\beta_{\ell - 1}} = \frac{\ell(\ell + 1)}{(2\ell + 1)(2\ell - 1)} .
\]

Any function \(W(X)\) can be expanded in terms of the polynomials \(L_\ell(X)\) \([\ref{124}]\) and then upon combining \([\ref{124}]\) and expansion in terms of \(L_\ell(X)\) to find the expansion in terms \(O^{A_\ell} :\)
\[
W(X) = \sum_{\ell=1}^{\infty} \hat{C}_{\ell} L_\ell(X) \quad \text{and} \quad W(X) = \sum_{n=1}^{\infty} \hat{C}_{n} O^{A_n} O^{A'_n} .
\]
When \(W(X)\) is the angular correlation function, the \(\hat{C}_{\ell}\) are the corresponding angular power spectrum coefficients (see below).

\footnote{We include the dipole, \(\ell = 1\), however the monopole is dropped as we will only be considering the expansion of the CGI perturbations \([\ref{3}]\) where the isotropic part has been factored out according to \([\ref{4}]\). We must beware that this does not cause problems later by omitting the spatial gradients of the isotropic term.}
3.1.1 Relationship to Legendre polynomials

A Legendre polynomial \( P_\ell(X) \) is given by renormalising the polynomials \( L_\ell(X) \) defined in (124) so that \( P_\ell(1) = 1 \). By (24), this implies

\[
P_\ell(X) = (\beta_\ell)^{-1} L_\ell(X) \Rightarrow P_\ell(1) = 1 ;
\]

consequently from (124),

\[
O_A^\ell O_A'^\ell = \beta_\ell P_\ell(X),
\]

where \( \beta_\ell \) are given by (24). It follows from (28) that

\[
P_\ell(X) = \sum_{m=0}^{[\ell/2]} A_{\ell m} X^{\ell - 2m}, \quad \text{with} \quad A_{\ell k} = \frac{(-1)^k (2\ell - 2k)!}{2^k k!(\ell - k)!(\ell - 2k)!}.
\]

Any function \( W(X) \) can be expanded in terms of both sets of polynomials - see (26) and the corresponding expression

\[
W(X) = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell P_\ell(X).
\]

These two expansions can then be related as follows: equating (26) and (31), and using (29) and (19), gives

\[
\sum_{\ell=1}^{\infty} \sum_{m=0}^{[\ell/2]} \hat{C}_\ell B_{\ell m} X^{\ell - 2m} = \sum_{\ell=1}^{\infty} \sum_{m=0}^{[\ell/2]} \frac{(2\ell + 1)}{4\pi} C_\ell A_{\ell m} X^{\ell - 2m} \Rightarrow \hat{C}_\ell B_{\ell m} = \frac{(2\ell + 1)}{4\pi} C_\ell A_{\ell m},
\]

from (30) this gives the relation between the expansion coefficients :

\[
\hat{C}_\ell = \left[ \frac{(2\ell + 1)(2\ell)!}{4\pi 2^\ell (\ell!)^2} \right] C_\ell = \Delta_\ell^{-1} C_\ell.
\]

3.2 The mean square of PSTF coefficients : \( \langle F_A^\ell F_A^\ell \rangle \)

It is known as before, from evaluating \( \int d\Omega f^2 \) and constructing the orthogonality conditions on \( O_A^\ell \), that inversion

\[
F_A^\ell = \Delta_\ell^{-1} \int_{4\pi} d\Omega O_A^\ell f(x^i, e^a) \Leftrightarrow f(x^i, e^a) = \sum_{\ell=0}^{\infty} F_A^\ell O_A^\ell,
\]

can be constructed. From this we can build

\[
F_A^\ell F_B^m = \Delta_\ell^{-1} \Delta_m^{-1} \int_{4\pi} d\Omega O_A^\ell f(x^i, e^a) \int_{4\pi} d\Omega' O_B^m f(x^i, e'^a),
\]

to find

\[
F_A^\ell F_B^m = \Delta_\ell^{-1} \Delta_m^{-1} \int \int d\Omega d\Omega' O_A^\ell O_B^m f(x^i, e^a) f(x^i, e'^a).
\]
Taking the ensemble average \([27, 18]\) then gives

\[
\langle F_{A_l} F_{B_m} \rangle = \Delta^{-1} \Delta^{-1} \int \int d\Omega d\Omega' O_{A_l} O'_{B_m} \langle f \cdot f' \rangle
\]

(37)

where \(\langle ... \rangle\) indicates an ensemble average over sufficiently many realizations of the angular correlation function.

In order to evaluate this further we assume that the correlations between the function \(f(x^i, e^a)\), \(i.e., \langle f(x^i, e^a) f(x^i, e'^{a}) \rangle\) are a function of the angular separation between the two directions only,

\[
\langle f \cdot f' \rangle = W(e^a e'^a) = W(X).
\]

(38)

This is a consequence of the Gaussian assumption \([14]\), which allows one to evaluate the angular correlation functions \([11]\) in a straightforward way. With this assumption (37) becomes

\[
\langle F_{A_l} F_{B_m} \rangle = \Delta^{-1} \Delta^{-1} \int \int d\Omega d\Omega' O_{A_l} O'_{B_m} W(X).
\]

(39)

Substituting (26) into (39), we find

\[
\langle F_{A_l} F_{B_m} \rangle = \Delta^{-1} \Delta^{-1} \int \int d\Omega d\Omega' O_{A_l} O'_{B_m} \sum_{n=1}^{\infty} \hat{C}_n O_{A_n} O'_{A_n}.
\]

(40)

Rearranging terms,

\[
\langle F_{A_l} F_{B_m} \rangle = \Delta^{-1} \Delta^{-1} \sum_{n=1}^{\infty} \hat{C}_n \left\{ \int_{4\pi} d\Omega O_{A_n} O_{A_n} \right\} \left\{ \int_{4\pi} d\Omega' O'_{B_m} O'_{A_n} \right\},
\]

(41)

where the integrals can be evaluated using the orthogonality conditions on the \(O_{A_l}\)'s, \([19]\),

\[
\langle F_{A_l} F_{B_m} \rangle = \Delta^{-1} \Delta^{-1} \sum_{n=1}^{\infty} \hat{C}_n \left\{ \delta^{ln} \Delta_l h_{(A_l)}^{(A_n)} \right\} \left\{ \delta^{mn} \Delta_m \delta^{+1} h_{(B_m)}^{(A_n)} \right\}
\]

\[
= \sum_{n=1}^{\infty} \hat{C}_n \delta^{ln} \delta^{mn} h_{(A_l)}^{(A_n)} h_{(B_m)}^{(A_n)}.
\]

(42)

Thus

\[
\langle F_{A_l} F_{B_m} \rangle = \hat{C}_l \delta^{ln} h_{(A_l)}^{(B_m)}. \] to find \(\langle F_{A_l} F_{A_l} \rangle = \hat{C}_l h_{(A_l)}^{(A_l)}\),

(43)

so using \(h_{(A_l)}^{(A_l)} \) \([21]\) the mean-square is found to be

\[
\langle F_{A_l} F_{A_l} \rangle = \hat{C}_l (2l + 1)
\]

(44)

giving the angular power spectrum coefficients \(\hat{C}_l\) in terms of the ensemble-averages of the harmonic coefficients \([1]. If we use the Legendre expansion \([31]\) instead of the covariant expansion coefficients \([26]\), then from \([44]\) and \([33]\) the relation is

\[
\langle F_{A_l} F_{A_l} \rangle = (2l + 1) \Delta_l^{-1} C_l,
\]

(45)

where \(C_l\) are the usual Legendre angular power spectrum coefficients.\\

---

8Ideally one would prefer to evaluate the angular correlation function without using the Gaussian assumption, as this is one of the features one should test rather than assume.

9This corrects an error in \([3]\), removing a spurious factor of \(3^l = \hat{h}_{A_l}^{A_l}\) which follows from the orthogonality conditions in \([6]\) which are corrected here.
3.3 The CGI angular correlation function

We can now gather the results above in terms of the application we have in mind, namely anisotropy of the CBR. Consider on-sky perturbations made of Gaussian Random Fields: The angular correlation function \( C(\beta) = W(X) \) is given by (14); the angular power spectrum coefficients \( C_\ell \) are given by the mean-square of the \( \ell \)–th temperature coefficient through (45):

\[
\langle \tau_{A\ell} \tau^{A\ell} \rangle = (2\ell + 1)\Delta_\ell^{-1} C_\ell
\]

where the constants \( \Delta_\ell \) are given by (119). These quantities are related by (31):

\[
W(X) = \sum_{m=1}^{\infty} \frac{C_m (2m + 1)}{4\pi} P_m(X)
\]

where the \( P_m(X) \) are given by (27) from (31) and (24).

3.3.1 Cosmic variance

The observations are in fact of \( a_\ell^2 \) (which is \( \sum_{m=-\ell}^{+\ell} |a_{\ell m}|^2/4\pi \) in the usual notation). This is what is effectively found from experiments, such as the COBE-DMR experiment. This is a single realization of the angular power spectrum \( C_\ell \). The finite sampling of events generated by random processes (in this case Gaussian random fields) leads to an intrinsic uncertainty in the variance even in perfect experiments - this is sample variance, or in the cosmological setting, cosmic variance.

We are measuring a single realization of a process that is assumed to be random; there is an error associated with how we fit the single realization to the averaged angular power spectrum. The quantities \( \langle \tau_{A\ell} \tau^{A\ell} \rangle \) represent the averaged (over the entire ensemble of possible \( C_\ell \)'s) angular power spectrum, this is what one is in fact dealing with in the theory, as the reductions are done in terms of Gaussian Random Fields where the entire ensemble is considered rather than a single experimental realization. The \( a_\ell^2 \) are a sum of the \( 2\ell + 1 \) Gaussian Random Variables \( a_{\ell m} \), this is taken to be \( \chi^2 \) distributed with \( 2\ell + 1 \) degrees of freedom. Each multipole has \( 2\ell + 1 \) samples.

The key point here is that cosmic variance is proportional to \( \ell^{-1/2} \) and so is less significant for smaller angular scales than larger scales (as is popular wisdom), i.e. cosmic variance is not an issue on small scales. However on small (and perhaps intermediate scales) systematic errors could be underestimated.

Physical process deviations and instrument noise are expected to dominate the small scales rather than non-Gaussian effects in the primordial perturbations, but on large scales the uncertainty due to cosmic variance would swamp out a non-Gaussian signature. It then seems plausible that on both large and small scales the assumption of Gaussian Perturbations is acceptable; however on intermediate scales this is not the case, on these scales the effects of cosmic variance would be small enough to allow a non-Gaussian signature to be apparent.

4 Mode expansions

We now consider spatial harmonic analysis of the angular coefficients discussed in the previous section. Note that the relations in this section hold in space-like surfaces, namely the background space-like surfaces in an almost-FL model. The application in the following sections will be to the projection into these spacelike surfaces of null cone coordinates associated with the propagation of the CBR down the null cone.
Following the Wilson-Silk approach \cite{18, 20, 22} we consider the following CGI expansions. Eigenfunctions $Q(x^\nu)$ are chosen to satisfy the Helmholtz equation

$$D^a D_a Q = -k_{phys}^2 Q$$

in the (background) space sections of the given space-time of interest, where the $Q$'s are time-independent scalar functions with the physical wavenumber $k_{phys}(t) = k/a(t)$, the wave number $k$ being independent of time. These define tensors $Q_A(k^\nu, x^i)$ that are Projected, Symmetric, and Trace-Free, and in the case of scalar perturbations are chosen to be given by PSTF covariant derivatives of the eigenfunctions $Q$:

$$Q_{A_{\ell}} = (-k_{phys})^{-\ell} D_{(A_{\ell})} Q$$

Using these we define functions of direction and position:

$$G_{\ell}[Q](x^\nu, e^a) = O^A_{\ell} Q_{A_{\ell}}$$

with the $O^A_{\ell}$ defined by \cite{18}. Here the $G_{\ell}$ are call mode operators and the objects $G_{\ell}[Q]$ are called mode functions. It follows that

$$G_{\ell}[Q] = (-k_{phys})^{-\ell} O^A_{\ell} D_{(A_{\ell})} Q$$

and we can expand a given function $f(x^i, e^a)$ in terms of these functions. In our case this serves as a way of harmonically analysing the coefficients $\tau_{A_{\ell}}(x^i)$ in \cite{18} and \cite{18}: expanding the temperature anisotropy in terms of the mode functions,

$$\tau(x^i, e^a) = \sum_{\ell} \sum_k \tau_{\ell}(t, k) G_{\ell}[Q]$$

where the $\ell$-summation is the angular harmonic expansion and the $k$-summation the spatial harmonic expansion (in fact $k$ will be a 3-vector because space is 3-dimensional, see below). Using the expansion \cite{18} on the left and \cite{18} on the right,

$$\sum_{\ell} \tau_{A_{\ell}}(x^i) O^A_{\ell} = \sum_{\ell} \sum_k \tau_{\ell}(t, k) (-k_{phys})^{-\ell} O^A_{\ell} D_{(A_{\ell})} Q$$

and so

$$\tau_{A_{\ell}}(x^i) = \sum_k \tau_{\ell}(t, k) (-k_{phys})^{-\ell} D_{(A_{\ell})} Q(x^\nu)$$

which is the spatial harmonic expansion of the radiation anisotropy coefficients in terms of the symmetric, trace-free spatial derivatives of the harmonic function $Q$. The quantities $\tau_{\ell}(t, k)$ are the corresponding mode coefficients. Note that we have not as yet restricted the geometry of the $Q$'s: they could be either spherical or plane-wave harmonics, for example.

By successively applying the background 3-space Ricci identity,

$$D_{abA_{\ell}} Q - D_{baA_{\ell}} Q = + \sum_{n=1}^{\ell} \frac{K}{a^2} \left( \delta_{b}^{a_n} h_{a_n a} - \delta_{a}^{b_n} h_{a_n b} \right) D_{A_{\ell}} Q,$$

\footnote{The function $Q$ will be associated with a direction vector $e_{(k)}^a$ and wave vector $k_{a} = ke_{(k)}^a$ normal to the surfaces $Q = \text{const}$, see the following subsection.}

\footnote{Note these are functions in phase space, not on $M$.}
where \( \tilde{A}_\ell = a_1...a_{n-1}b_na_{n+1}...a_\ell \), i.e., the sequence of \( \ell \) indices with the \( n \)-th one replaced with a contraction. First, the curvature-modified Helmholtz equation is found \( \text{using (133, 134, 132)} \):

\[
D^a D_a \bar{Q}_{\langle A_\ell \rangle} = -\tilde{k}_\ell^2 \bar{Q}_{\langle A_\ell \rangle} \quad \text{with} \quad -\tilde{k}_\ell^2 = \frac{1}{a^2} \left( K\ell(\ell + 2) - k^2 \right). \quad (56)
\]

Second, we are able to construct the mode recursion relation \( \text{using (130, 131 in 22)} \):

\[
e^aD_a \left[ G_{\ell}[Q] \right] = + k_{\text{phys}} \left[ \frac{\ell^2}{(2\ell + 1)(2\ell - 1)} \left( 1 - \frac{K}{k^2} (\ell^2 - 1) \right) G_{\ell-1}[Q] - G_{\ell+1}[Q] \right], \quad (57)
\]

The latter is the basis of the standard derivation of the linear-FRW mode hierarchy for scalar modes. The derivation of these are given in appendix C (we consider only scalar eigenfunctions). It will be seen, in paper II, that this relation can be used in place of the general divergence relations which allow the construction of generic multipole divergence equations \( \text{[5]} \) if one restricts oneself to constant curvature space-times.

Given the recursion relation one can immediately make the connection with the usual Legendre tensor treatment \( \text{[39]} \) this is shown in appendix C \( \text{[34, 41]} \).

### 4.1 \(|\tau_\ell|^2 \) in almost-FLRW universes

We now relate the multipole mean-squares \( \langle \tau_{A_\ell} \tau_{A_\ell} \rangle \) of the ensemble average over the multipole moments with that of the mode coefficient mean-squares \( |\tau(k)|^2 \).

In order to carry this out we relate two separate spatial harmonic expansions \( \text{[54]} \) for the same function: the first is one associated with plane wave harmonics \( (\bar{Q}^k) \), naturally used in describing structure existing at any time \( t \), and the second, one associated with radial and multipole harmonics \((O_{A_\chi} R_{A_\chi})\), i.e., a spherical expansion based at the point of observation, naturally arises when we project the null cone angular harmonics into a surface of constant time. These are both related to the Mode Function formulation which becomes useful in the non-flat constant curvature cases.

#### 4.1.1 Plane-waves and Mode functions

Considering flat FRW universes, each set of eigenfunctions satisfy \( \text{[48]} \). The temperature anisotropy \( \text{[4]} \) can be expressed in terms of its plane-wave spatial Fourier transform:

\[
\tau(x^i, e^a) = \sum_{k_\nu} \tau(k, t, e^{(k) a}, e^a) \bar{Q}_{flat}. \quad (58)
\]

(For a more detailed treatment see appendix D). It can be shown that for the flat case, \( K = 0 \), that is \( \text{[140]} \) hold in \( \text{[142]} \) and \( \text{[143]} \) to find from \( \text{[145]} \):

\[
D_{\langle A_\ell \rangle} \bar{Q}_{flat} = (-i k_{\text{phys}})^\ell \bar{Q}^{(k)}_{A_\ell} \bar{Q}_{flat} \quad (59)
\]

where \( \bar{Q}^{(k)}_{A_\ell} \) are the PSTF tensors associated with the direction \( e^{(k) a} \). Then from \( \text{[19]} \) we find that:

\[
Q_{flat A_\ell} = (-1)^\ell \bar{Q}^{(k)}_{A_\ell} Q_{flat}, \quad (60)
\]

along with \( \text{[54]} \) to get the temperature multipole:

\[
\tau_{A_\ell} = (-1)^\ell \sum_k \tau_{\ell}(t, k) Q_{flat} \bar{Q}^{(k)}_{A_\ell}, \quad (61)
\]

\(^{13}\)This will be necessary in order to switch from the almost-FLRW multipole divergence equations in part II to the mode representation where we use \( \text{[21]} \) to construct \( D^aD_{\langle a \rangle} Q_{\ell} \).
4.1.2 Radial expansions and Mode functions

Using the flat, $K = 0$, spherical eigenfunctions centred on a point $x_a^\mathbb{h}$, and with associated radial direction vector $e_a^{(\mathbb{h})}$. The latter is the same as the (spherically symmetric) projection into the constant time surfaces of the tangent vector $e^a$ of the radial null geodesics, so we need not distinguish it from that vector. In this case the $\ell$-th harmonic is

$$Q_\ell(x^\alpha, e_a^{(\mathbb{h})})|_X = R_{A_\ell}(r)O^{A_\ell}_X,$$

$D_a r = e_a$, $D^a e_a = \frac{2}{r}$ (62)

where $e^a = dx^a/dr$ is the unit radial vector. Cartesian coordinates in space are given by $r$ and $e^a$ through $x^i = re^i$. Defining the projection tensor

$$p_{ab} = h_{ab} - e_a e_b \Rightarrow p_{ab} u^a = 0 = p_{ab} e^a, \quad p_a^a = 2, \quad p^a_b p_b^b = p^a_c,$$

then ($e^a$ is shear and curl free)

$$D_a e_b = \frac{1}{r} p_{ab} \Rightarrow D_a p_{bc} = -\frac{1}{r} (p_{ab} e_c + p_{ac} e_b), \quad D^a p^b_a = -\frac{2}{r} e^b$$

(63)

so from (48)

$$D_a D^a (R_{A_\ell} O^{A_\ell}) = -h_{\mathbb{phys}}^2 (R_{A_\ell} O^{A_\ell})$$

(65)

To work out the l.h.s., we must first calculate $D_a Q_\ell$ (149) and $D_a D^a Q_\ell$ (150). These are then used to calculated : $D_a R_{A_\ell}$ (151), $D^a D_a R_{A_\ell}$ (152), $D_a R^{A_\ell}$ (153) and $D_a D^a O^{A_\ell}$ (154) respectively.

Putting these in (154) to find (156) from which we get :

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R_{A_\ell}}{\partial r} \right) + R_{A_\ell} \left( -\ell (\ell + 1) \frac{1}{r^2} \right) = -h^2_{\mathbb{phys}} R_{A_\ell},$$

(66)

the spherical Bessel equation. Thus providing the PSTF derivation of the spherical bessel equation in terms of the irreducible representation.

Now consider that we can choose any basis we like for the tensor basis here, independent of the spatial coordinates used. It is convenient to use the plane wave decomposition to get a parallel vector basis. We do this by writing

$$R_{A_\ell}(r) = \sum_{k^a} R^{(k)}_{A_\ell} = \sum_{k^a} R_{\ell}(k, r)O^{(k)}_{A_\ell},$$

(67)

this expresses the tensor eigenfunction in terms of the monopole eigenfunction. When this is substituted into (156) we obtain the radial equation which has solutions that are spherical Bessel functions in the flat case :

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial R_{\ell}}{\partial r} \right) + \left[ k^2_{\mathbb{phys}} r^2 - \ell (\ell + 1) \right] R_{\ell} = 0,$$

(68)

where $k^2_{\mathbb{phys}} = k^2/a^2$ and $\alpha_\ell$ are integration constants (the second set of constants for this second order equation vanish because we choose $R_\ell(0)$ to be finite; the Neumann functions are not finite at $r = 0$).

From (62), (67), and (68) we have found that the solutions to the Helmholtz equations give the eigenfunctions

$$Q_\ell(x^\nu) = \alpha_\ell \sum_{k^\nu} j_\ell(k_{\mathbb{phys}} r) O^{A_\ell}_{\nu}(x) = \alpha_\ell \sum_{k^\nu} j_\ell(k_{\mathbb{phys}} (k, r) r) L_\ell(X).$$

(69)
and we can set $\alpha_\ell = (\Delta_\ell)^{-1}$ so that $\int Q_\ell d\Omega = 1$.

It is important to notice that the functions $L_\ell(X)$ depend both on $k^a$ and on $e^a$, and so for each $k^a$ is a function of $(\theta, \phi)$, thus $Q$ is indeed a function of all spatial coordinates. We can pick any direction $k^a$ to find the particular eigenfunctions

$$Q^{(k)}_\ell = (\Delta_\ell)^{-1} j_\ell(k_{phys} r) O_A^{(k)}(x).$$

(70)

associated with that direction. The general $\ell$-th eigenfunction is a sum of such eigenfunctions over a basis of directions $k^a$.

Now we find $O^{A_\ell} D_{A_\ell} Q$ [161] in terms of $R_\ell$ (as show in the appendix using PSTF techniques, [157, 161]):

$$O_{(\chi)}^{A_\ell} D_{(A_\ell)} Q = (-k_{phys})^\ell O_{(\chi)}^{A_\ell} O^{(k)}_\ell R_\ell \Rightarrow O_{(\chi)}^{A_\ell} Q_{A_\ell} = (-k_{phys})^\ell Q_\ell.$$  

(72)

Putting this in the expansion (54)

$$\tau_{A_\ell}(x^i) = \sum_{k^n} \tau_{\ell}(t, k)(-k_{phys})^{-\ell} D_{(A_\ell)} Q(x^i) = \sum_{k^n} \sum_{\alpha} \tau_{\ell}(t, k)(-k_{phys})^{-\ell} D_{(A_\ell)} Q_n$$

(73)

gives the present version of (62)

$$\tau(x^i, e^a_{(\chi)}) = \sum_{\ell} \tau_{A_\ell} O^{A_\ell} = \sum_{\ell} \tau_{\ell}(k, t) G_{\ell}[Q];$$

(74)

directly analysing the coefficients $\tau_{A_\ell}$ in terms of these functions $Q$ - given that we have 3-dimensions worth of variability so as to represent arbitrary spatial functions - with purely time-dependent coefficients (parametrized by a vector $k^a$),

4.1.3 Radial expansions and plane-waves

Now consider the inversion

$$e^{(+ik_\alpha x^\alpha)} = \sum_{\ell} \tilde{R}_{A_\ell}(r) O^{A_\ell}_{(\chi)}, \quad \iff \quad \tilde{R}_{A_\ell} = (\Delta_\ell)^{-1} \int d\Omega(x) e^{+ik_\alpha x^\alpha} O^{(k)}_{A_\ell}.$$  

(75)

by taking a Taylor expansion using $k_\alpha x^\alpha = k_{phys} r e^b_k e^b_{(\chi)}$ we find

$$\tilde{R}_{A_\ell} = (\Delta_\ell)^{-1} \sum_{n=0}^\infty \frac{(-i)^n(k_{phys} r)^n}{n!} e^{(k)}_{B_n} \int d\Omega(x) e^{B_n}_{(\chi)}$$

$$= (\Delta_\ell)^{-1} \sum_{n=0}^\infty \frac{(-i)^n(k_{phys} r)^n}{n!} \frac{\delta^\ell+2n n!(n-\ell+1)!!}{(n-\ell+1)!(n+\ell+1)!!} 4\pi e^{(k)}_{B_n} h_{(A_\ell)}^{(B_n)}(B_\ell h_{B_n-\ell}).$$

(76)

on using (27). Putting (20) into (76) we find

$$\tilde{R}_{A_\ell} = (+i)^\ell (\Delta_\ell)^{-1} 4\pi \sum_{m=0}^\infty \frac{(-1)^m(k_{phys} r)^{\ell+2m}(2m+1)!!}{(2m+1)!(2m+2\ell+1)!!} O^{(k)}_{A_\ell}$$

$$= 4\pi (+i)^\ell (\Delta_\ell)^{-1} \sum_{m=0}^\infty \frac{(-1)^m(k_{phys} r)^{\ell+2m}}{2m m! (2m+2\ell+1)!!} O^{(k)}_{A_\ell}.$$  

(77)

14 Many treatments choose a particular direction for $k$: $k^a = k_3^a$ or similar and omit the summation.
15 Contrast with (52): there the l.h.s is the spherical eigenfunction; here it is the plane one, expressed in terms of spherical ones.
This can be re-expressed as

\[ \hat{R}_{\ell} \equiv 4\pi (i\ell) \Delta^{-1} j\ell(k_{\text{phys}} r)O_{\ell}^{(k)} = (i\ell) (2\ell + 1) \delta^{-1} j\ell(k_{\text{phys}} r)O_{\ell}^{(k)}. \] (78)

Hence

\[ Q_{_{\ell\ell}}(x, e^{(k)}) = e^{ikn_x} = 4\pi \sum_{n,k} (i\ell)^n j_n(k_{\text{phys}} r)O_{\ell}^{A_n}O_{\ell}^{(k)} \Delta^{-1} \] (79)

links the plane-waves to the spherical expansion. This recovers, from \( L_{\ell}(e^a e'_a) = \beta_{\ell}P_{\ell}(e^a e'_a) \) the more usual

\[ e^{ik_{\text{phys}} r} = \sum_{\ell} \hat{R}_{\ell} O_{\ell}^{(k)} = \sum_{\ell} (i\ell)^{(2\ell + 1)} j\ell(k_{\text{phys}} r)P_{\ell}(X) \quad (80) \]

4.1.4 \( |\tau_{\ell}|^2 \) for \( K = 0 \) (flat) almost-FLRW models

We now return to the relationship between the \( \tau_{\ell} \) and \( \tau_{\ell} \). Now from (143), (148), (74) and (73):

\[ \tau_{\ell} = (i\ell) \int \frac{k^2 dk}{(2\pi)^3} d\Omega_k \tau_{\ell}(k, t)O_{\ell}^{(k)} \sum_{n=0}^{\infty} (i\ell)^n j_n(\lambda r)O_{\ell}^{B_n}O_{\ell}^{(k)} \Delta^{-1}(2n + 1). \] (81)

This can be reduced further (162), where upon using (163) one finds that

\[ \tau_{\ell} = 4\pi O_{\ell}^{(k)} \int \frac{k^2 dk}{(2\pi)^3} \tau_{\ell}(k', t)R_{\ell}(k', \chi), \] (82)

where \( \chi = r/a \) and by using \( R_{\ell}(k, \chi) = j\ell(\lambda r) \) we can identify this with (162), the equivalent of the multipole moments found using the explicit form of the plane-waves.

In order to proceed further, from (82) we construct the ensemble average:

\[ \langle \tau_{\ell} \rangle = (4\pi)^2 O_{\ell}^{(k)} \int \frac{k^2 dk}{(2\pi)^3} \langle \tau_{\ell}(k', t) \rangle R_{\ell}(k', \chi), \] (83)

What has happened here is that we imagine an ensemble of universes and we use an ensemble average rather than the space average; we have to do this because \( \tau_{\ell} \) is not square-integrable, i.e., we cannot use the r.m.s value as we cannot integrate the square over the space in general; this operation is not well defined. In order to deal with the ensemble average over the mode coefficients, i.e. \( \langle \tau_{\ell} \rangle \), we assume the perturbations to be fairly homogeneously spread throughout the space and not confined in a particular region, and assume that there are no correlations between perturbations with different wavenumbers. Here the Gaussian assumption is useful, we have that

\[ \langle \tau_{\ell}(k, t) \rangle = (2\pi)^3 \delta(k' - k) \ell(k, t) \] (84)

\[ \langle j\ell(k')j\ell(k, r) \rangle_{_{x^n}} = 4\pi \left( \frac{3}{2} \right) \frac{1}{k^2} \delta(k' - k). \] (85)

---

\[ \text{Remembering that} \]

\[ \langle j\ell(k')j\ell(k, r) \rangle_{_{x^n}} = 4\pi \left( \frac{3}{2} \right) \frac{1}{k^2} \delta(k' - k). \] (86)
4.1.5 $|\tau_\ell|^2$ for almost-FLRW models

It is useful to notice that an alternative although equivalent avenue of approach is also possible proceeding directly from the mode expansion in $G_\ell[Q] = O^{A_\ell}_{(\chi)} Q_{A_\ell}$; this can be used for any $K$.

Notice that as before at the observer at $x^i_0$,

$$\tau(x^i_0, e^a) = \sum_\ell \tau_\ell O^{A_\ell}_\ell, \quad \iff \quad \tau_\ell(x^i_0) = \Delta^{-1}_\ell \int d\Omega_{A_\ell} \tau(x^i_0, e^a). \quad (84)$$

Now in the spatial section in general we can write

$$\tau(x^i, e^a_{(k)}) = \sum_\ell \tau_\ell O^{A_\ell}_{(k)}, \quad \iff \quad \tau_\ell(x^i) = \Delta^{-1}_\ell \int d\Omega^{(k)}_{A_\ell} \tau(x^i, e^a_{(k)}). \quad (85)$$

Now the point $x^i_0$ is chosen at some earlier time in a spatial section, with radial direction vector $e^a_{(\chi)}$, for FRW models we can consider $e^a_{(\chi)}$ and $e^a$ to be equivalent and write

$$\tau(x^i_0, e^a) = \tau(x^i_0, e^a_{(\chi)}) = \sum_\ell \tau_\ell(x^i_0) O^{A_\ell}_{(\chi)} \equiv \sum_\ell \tau_\ell(x^i_0) O^{A_\ell}$$

$$\iff \tau_\ell(x^i_0) = \Delta^{-1}_\ell \int d\Omega^{(\chi)}_{A_\ell} \tau(x^i_0, e^a_{(\chi)}). \quad (86)$$

Remember that in some spatial section (where the integral relations \([168, 169, 170]\) are useful)

$$\tau(x^i, e^a) = \sum_{\ell, k} \tau_\ell(k, t) O^{A_\ell}_{(\chi)} Q_{A_\ell}. \quad (87)$$

Now from

$$\tau_{A_\ell}(x^i_0) = \Delta^{-1}_\ell \int d\Omega \sum_{m, k^\alpha} \tau_{m} O_{A_\ell} G_{\ell}[Q] \quad (88)$$

we identify $O^{A_\ell}_{(\chi)}$ with $O^{A_\ell}$, to find

$$\tau_{A_\ell} = \sum_{k^\alpha} \tau_\ell(k, t) Q_{A_\ell}, \quad (89)$$

which means that for $K = 0$ ($R_\ell \propto j_\ell$).

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = \sum_{k^\nu} |\tau_\ell|^2 (Q_{A_\ell} Q^{A_\ell}) = (4\pi)b_\ell \sum_{k} |\tau_\ell|^2. \quad (90)$$

What is particularly useful about the last method of calculation is that for constant $K$ surfaces it can be shown on using relation \((174)\) recursively that

$$\int d\Omega_{k} \langle |G_\ell[Q]G_m[Q]| \rangle = 4\pi \Xi^2_\ell \delta^m, \text{ with } \Xi^2_\ell = \prod_{n=1}^{\ell} (\alpha_n)^2 \quad (91)$$

and $\langle \ldots \rangle$ denotes the ensemble average of the eigenfunctions. What is useful here is to notice that (using the normalisation for $K = 0$),

$$\Xi^2_{\ell}|_{K=0} = \prod_{n=1}^{\ell} \frac{n^2}{(2n+1)(2n-1)} = \frac{\beta^2_\ell}{2\ell + 1} = (4\pi)^{-2}(2\ell + 1)\Delta^2_\ell. \quad (92)$$
This means that for $K = 0$

$$\int d\Omega(k) \langle |G_{k}^{\ell}G_{m}^{\ell'}| \rangle = \delta_{\ell}^{m} \frac{2\ell + 1}{4\pi} \Delta_{g}^{2}. \quad (93)$$

Now we can extend the above to FL models with $K \neq 0$. The mean-square for constant $K$ is found by modifying the normalization after identifying the $K = 0$ normalizations in (170) and (93): 

$$\langle \tau_{A_{\ell}}\tau^{A_{\ell}} \rangle = \sum_{k, k'} |\tau_{\ell}|^{2} \langle Q_{A_{\ell}}Q^{A_{\ell}} \rangle = (4\pi)\beta_{\ell} \sum_{k} |\tau_{\ell}|^{2} \Xi_{\ell}^{2}, \quad (94)$$

or alternatively keeping the form of the mean-square in (90) by redefining the mode function expansion [24] by a wavelength-dependent coefficient; then

$$\tau(x, e^{\alpha}) = \sum_{\ell,k} \tilde{\tau}_{\ell}(k, t)M_{\ell}[Q], \quad \text{and} \quad M_{\ell}[Q] = \Xi_{\ell}(k, K)G_{\ell}[Q], \quad (95)$$

defines the new coefficients $\tilde{\tau}_{\ell}(k, t)$ so that

$$|\tilde{\tau}_{\ell}|^{2} = |\tau_{\ell}|^{2} \Xi_{\ell}^{2}. \quad (96)$$

Because we have redefined the mode functions in (95), the form of the equations for $K \neq 0$ is the same as in the case $K = 0$. However the coefficients are different because they are from a different expansion. Using the results from the $K = 0$ case, either (82) or (162), we have that

$$\langle \tau_{A_{\ell}}\tau^{A_{\ell}} \rangle = \frac{1}{2\pi^{2}} \beta_{\ell} \int \frac{dk}{k} k^{3} |\tilde{\tau}_{\ell}(k, t)|^{2}. \quad (97)$$

Hence, on using $\hat{C}_{\ell} = \Delta^{-1}C_{\ell}$ and $\langle \tau_{A_{\ell}}\tau^{A_{\ell}} \rangle = \hat{C}_{\ell}(2\ell + 1)$, we now have

$$\langle \tau(x, e^{\alpha})\tau(x, e'^{\alpha}) \rangle = \sum_{\ell=0}^{\infty} (2\ell + 1)^{-1} \langle \tau_{A_{\ell}}\tau^{A_{\ell}} \rangle O_{B_{\ell}}O'^{B_{\ell}} \quad (98)$$

which can now be written from (97) and (96) as

$$\langle \tau(x, e^{\alpha})\tau(x, e'^{\alpha}) \rangle = \left(\frac{1}{2\pi^{2}}\right) \sum_{\ell=0}^{\infty} (2\ell + 1)^{-1} \beta_{\ell} \int_{0}^{\infty} \frac{dk}{k} k^{3} |\tilde{\tau}_{\ell}(k, t)|^{2} O_{A_{\ell}}O'^{A_{\ell}}$$

$$= \left(\frac{1}{2\pi^{2}}\right) \sum_{\ell=0}^{\infty} \beta_{\ell}^{2} (2\ell + 1)^{-1} \int \frac{dk}{k} k^{3} |\tilde{\tau}_{\ell}(k, t)|^{2} P_{\ell}(e^{a}e'^{a}) \quad (99)$$

thus reobtaining the results of White and Wilson:

$$C_{\ell} = \frac{2}{\pi} \frac{\beta_{\ell}^{2}}{(2\ell + 1)^{2}} \int_{0}^{\infty} \frac{dk}{k} k^{3} |\tilde{\tau}_{\ell}(k, t)|^{2}. \quad (100)$$

One needs to be careful here with the factors of $(2\ell + 1)$. Equation (99) relates the amount of power in a given wavenumber, $|\tau_{\ell}(k, \eta_{0})|^{2}$, given the intersection of the null-cone fixed at the observer $x_{0}$, on a angular scale $\ell$ given that the angular correlations are found on the scale $X = e^{a}e'^{a}$, i.e., $X$ is the separation between measurements.
5 Conclusions

We have given here a comprehensive survey of the CGI representation of CBR anisotropies in almost-FLRW universes, and related this formalism to the other major formalisms in use for this purpose at the present time.

This paper has been concerned with algebraic relations: specifically the Multipole, (e.g. for $\tau_{A\ell}$), and Mode, (e.g. for $\tau_\ell$), formalisms and the relationship between these. Where possible the multipole moments have been treated for a spacetime with generic inhomogeneity and anisotropy but small temperature anisotropies. The mode moments however are only meaningful in the restricted class of almost-FLRW universes.

The subsequent papers in the series consider the differential relations satisfied by the quantities mentioned here [4], and will show how both timelike and null integrations are used to lead to the standard results in the literature. Taken together, this will be is an ab initio demonstration of the way the different formalisms in use, and their major results, can be obtained from a single CGI approach, as well as providing the natural extension of the usual results into the non-linear (exact) theory.

Acknowledgements

We are grateful to Bill Stoeger, Bruce Bassett, Roy Maartens, and Peter Dunsby for useful comments and suggestions. This work was supported by the South African Foundation for Research and Development.
A Spherical Harmonics

A.1 Basic relations

A Spherical Harmonic (SH) \( Y_{\ell,m}(\theta, \phi) \) is related to an Associated Legendre Polynomial (ALP),

\[
Y_{\ell m} = C_{\ell m} e^{im\phi} P_{\ell m}(\cos \theta),
\]

\[
= C_{\ell m} (e^{i\phi} \sin \theta)^{m} \sum_{j=0}^{(\ell-m)/2} A_{\ellmj} (\cos \theta)^{\ell-2j} \quad \forall \ m \geq 0.
\]

Here,

\[
C_{\ell m} = (-1)^m \frac{(2\ell + 1)(\ell - m)!}{4\pi (\ell + m)!}, \quad \text{and} \quad A_{\ellmj} = \frac{(-1)^j (2\ell - 2j)!}{2^j j!(\ell - j)! (\ell - m - 2j)!},
\]

along with

\[
Y_{\ell m} = (-1)^m Y_{\ell|m*} \quad \forall \ m \leq 0.
\]

Now we can relate the SH, \( Y_{\ell m} \), to the direction vector product \( e^{A_{\ell}} \),

\[
Y_{\ell m} = Y_{\ell m} e^{A_{\ell}}(\theta, \phi),
\]

where following \[\PageIndex{1}\] (from making the substitution \( e^1 + ie^2 = e^{i\phi} \sin \theta \) and \( e^3 = \cos \theta \) into the above relation)

\[
Y_{\ell m} = C_{\ell m} \sum_{j=0}^{(\ell-m)/2} A_{\ellmj} \prod_{k=0}^{m} \left( h_{(a_k)}^1 + ih_{(a_k)}^2 \right) \prod_{p=m+1}^{\ell-2j} h_{a_p}^3
\]

\[
\times \prod_{q=1}^{j} \left( h_{a_{2q-1+\ell-2j}}^\alpha, h_{a_{2q+\ell-2j}}^\alpha \right).
\]

Furthermore, it can then be shown that from

\[
f = \sum_{\ell} F_{A_{\ell}} e^{A_{\ell}}, \quad \text{and} \quad F_{A_{\ell}} = \sum_{m=-\ell}^{m=+\ell} a_{\ell m} Y_{\ell m} e^{A_{\ell}}.
\]

This is not unexpected.

A.2 Consequences

A.2.1 Closure

\[
\sum_{\ell=0}^{\infty} \sum_{\ell=-\ell}^{\ell=m} Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega) = \delta(\Omega - \Omega').
\]

A.2.2 Addition

\[
\sum_{\ell=-m}^{\ell=m} Y_{\ell m}(\Omega') Y_{\ell m}(\Omega) = \frac{(2\ell + 1)}{4\pi} P_{\ell} = \Delta_{\ell}^{-1} O_{\ell} O'^{A_{\ell}}.
\]
A.2.3 Orthonormality

\[
\int_{4\pi} d\Omega Y_{\ell m}(\Omega') Y_{\ell m}(\Omega) = \delta_{\ell\ell'} \delta_{mm'}.
\] (110)

A.2.4 Matching plane-waves to spherical harmonics

\[
e^{(ik\chi)} = 4\pi \sum_\ell i^\ell j_\ell (k\chi) Y_{\ell m} Y'_{\ell,-m} = 4\pi \sum_\ell i^\ell j_\ell \Delta_{\ell}^{-1} O^{A\ell} O'^{A\ell},
\] (111)

\[
e^{(ik\chi)} = \sum_\ell i^\ell (2\ell + 1) j_\ell P_\ell = \sum_\ell i^\ell (2\ell + 1) j_\ell \beta_\ell^{-1} O^{A\ell} O'^{A\ell}.
\] (112)

B Multipole relations

B.1 Properties of \(e^{A\ell}\)

B.1.1 Normalization

The normalization for \(e^{A\ell}\) is found from [5], for odd and even \(\ell\) respectively:

\[
\frac{1}{4\pi} \int_{4\pi} d\Omega e^{A_{2\ell+1}} = 0, \quad \text{and} \quad \frac{1}{4\pi} \int_{4\pi} d\Omega e^{A_{2\ell}} = \frac{1}{2\ell + 1} h^{(A_{2\ell})}.
\] (113)

From which (contracting with \(h^{(A_{2\ell})}\)) it can be shown that

\[
h^{(A_{2\ell})} h^{(A_{2\ell})} = (2\ell + 1)
\] (114)

this can also be shown algebraically [7].

B.1.2 Orthogonality

From [5] we also have that

\[
\int_{4\pi} e^{A\ell} e^{Bm} d\Omega = \frac{4\pi}{\ell + m + 1} h^{(A\ell Bm)}.
\] (115)

if \(\ell + m\) is even, and is zero otherwise (this follows from the above because \(e^{A\ell} e^{Bm} = e^{A\ell+m}\) on relabeling indices: \(b_1, b_m \rightarrow a_{\ell+1}, a_{\ell+m}\)).

B.1.3 Addition Theorem

From [5] it follows that

\[
e^{A\ell} e'_{A\ell} = (X)^\ell = \sum_{\ell=0}^{\infty} e^{A\ell} e'_{A\ell} = \sum_{\ell} (X)^\ell = \frac{\cos \beta}{1 - \cos \beta},
\] (116)

where \(X = \cos \beta\). It may be useful to compare these to the relations for standard spherical harmonics, which are given in the appendix A. Note that

\[
\int_{4\pi} d\Omega e^{An} e'_{An} = \int_{4\pi} d\Omega (e^{a} e'^{a})^{n} = 2\pi \int_{-1}^{1} dXX^n = \begin{cases}
0 & \forall \ n \ \text{odd} \\
\frac{n+1}{4\pi} & \forall \ n \ \text{even}
\end{cases},
\] (117)

where the integral is taken over \(e^{a}\) with \(e'^{a}\) fixed.
B.1.4 Orthogonality of $O^A_{\ell}$

The orthogonality conditions can be found from

$$\sum_{m,\ell} \int d\Omega (F_{A\ell} O^{A\ell})(F_{Bm} O^{Bm}),$$  \hspace{1cm} (118)

see [5]. Here $F_{A\ell}$ are arbitrary PSTF harmonic components of some $f(e^a, x^i)$. Using (115), (19), and (17) we find

$$\int d\Omega O^{A\ell} O^{Bm} = \delta^\ell_m \Delta_{\ell} h^{(A\ell)} (B\ell) \text{ with } \Delta_{\ell} := \frac{4\pi}{(2\ell + 1)} \frac{2\ell(\ell)^2}{(2\ell)!}.$$  \hspace{1cm} (119)

where $h^{(A\ell)} (B\ell) = h^{(a_1 b_1 \cdots a_{\ell} b_{\ell})}$ and From this it follows that

$$e^{Bn} h^{(A\ell)} (B\ell h^{Bn-\ell}) = e^{(A\ell)} (+1)^{n-\ell} = O^{A\ell}.$$  \hspace{1cm} (120)

It should also be noticed that from (119),

$$h^{(A\ell)} (A\ell) = (2\ell + 1),$$  \hspace{1cm} (121)

can be also shown algebraically [7].

Using these relations we obtain the inversion of the harmonic expansion (5):

$$\tau(x^i, e^a) = \sum_{\ell=0}^{\infty} \tau_{A\ell}(x^i) O^{A\ell} \iff \tau_{A\ell}(x^i) = \Delta_{\ell}^{-1} \int d\Omega O^{A\ell} \tau(x^i, e^a).$$  \hspace{1cm} (122)

B.1.5 Addition of $O^A_{\ell}$

The addition theorem for $O^A_{\ell}$ can be found from

$$O^{A\ell} O^{A'\ell} = \sum_{k=0}^{[\ell/2]} \sum_{k'=0}^{[\ell'/2]} B_{\ell k} B_{\ell k'} h^{(A_{2k} h^{(A_{2k} e^a) e^{A'\ell})}.$$  \hspace{1cm} (123)

The resulting polynomial

$$L_\ell(X) \equiv O^{A\ell} O^{A'\ell} = \sum_{m=0}^{[\ell/2]} B_{\ell m} X^{\ell-2m},$$  \hspace{1cm} (124)

is the natural polynomial that arises in the PSTF tensor approach.

B.1.6 Double Integrals

First, note that

$$O^{A\ell} e^{Bn} \int d\Omega O^{A\ell} e^{Bn} = \int d\Omega O^{A\ell} O^{A\ell} e^{Bn} e^{Bn}$$

$$= \beta_{\ell} \frac{4\pi}{n + \ell + 1} \frac{n!(n-\ell+1)!}{(n-\ell+1)!(n+\ell-1)!}$$  \hspace{1cm} (125)

Note the contrast with (114).
on integrating a Legendre polynomial, where \( n \geq l \) and \((n - \ell)\) is even, so we can write \( n - \ell = 2m \) for \( m \) an integer\(^\text{18}\). Consequently, on remembering

\[
h^{\langle A \ell \rangle} (B_n h_{B_n \ldots}) O'_{A \ell} e' B_n = O'_{A \ell} e' A \ell = \beta_\ell, \quad (e'^a e'_a)^{(n - \ell)} = +1, \quad (126)
\]
we find

\[
\int d\Omega O^A e B_n = 4\pi \delta^{\ell+2m}_n \frac{n!(n - \ell + 1)!}{(n + \ell + 1)!} h^{\langle A \ell \rangle} (B_n h_{B_n \ldots}). \quad (127)
\]

Here \( m \) are positive integers. Also we will need

\[
\int d\Omega \epsilon B_k O^{(k)} O^{B_m} = \int d\Omega_k \int d\Omega_k' O^{(k')} O^{B_m} = 4\pi \int d\Omega_k O^{(k)} O^{B_m} = 4\pi \delta^m_\ell \Delta (h^{\langle A \ell \rangle})^{\langle B \ell \rangle} \delta (e^a_k - e^a_{k'}). \quad (128)
\]

\section{Mode relations}

**C.1 The curvature modified Helmholtz equation and the mode recursion relation**

By successively applying the background 3-space Ricci identity,

\[
D_{ab\ell \ell} Q - D_{b\ell a\ell} Q = + \sum_{n=1}^\ell \frac{K}{a^2} \left( \delta^b_n h_{an} - \delta^a_n h_{an} \right) D_{\ell \ell} Q, \quad (129)
\]

we find

\[
h^{(a_1 a_2) O^{A \ell} D_{A \ell+1}} Q = e^{a_1 O^{A \ell} D_{A \ell} Q} Q - \frac{1}{3} \frac{K}{a^2} \frac{\ell^2(\ell - 1)}{(2\ell - 1)} O^{A \ell+1} D_{A \ell+1} Q, \quad (130)
\]

\[
h^{(a_1 a_2) O^{A \ell-1} D_{A \ell+1}} Q = \left( -k^2_{\text{phys}} + \frac{K}{3} \frac{\ell(\ell + 2)}{a^2} \right) O^{A \ell-1} D_{A \ell-1} Q, \quad (131)
\]

\[
O^{A \ell} D_{c A \ell} Q = \left( -k^2_{\text{phys}} + \frac{K}{2} \frac{\ell(\ell + 3)}{a^2} \right) O^{A \ell} D_{A \ell} Q \quad (132)
\]

\[
O^{A \ell} D_{c A \ell} Q = O^{A \ell} D_{A c} Q + \frac{1}{2} \frac{K}{a^2} \ell(\ell - 1) O^{A \ell-1} D_{A \ell-1} Q, \quad (133)
\]

Now consider \( O^{A \ell} D_{c A \ell} Q \) from (133) we find

\[
O^{A \ell} D_{c A \ell} Q = O^{A \ell} D_{A c} Q + \frac{1}{2} \frac{K}{a^2} \ell(\ell - 1) O^{A \ell-1} D_{c A \ell-1} Q \quad (134)
\]

where the first term on the left of the equality above, (134), can be reduced to one in terms of \( O^{A \ell} D_{A \ell} Q \) using (132) while the last term can also be rewritten in terms of \( O^{A \ell} D_{A \ell} Q \). Now on dropping the \( O^{A \ell} \) and making the identification of \( Q_{A \ell} = (-k^2_{\text{phys}})^{-\ell} D_{\langle A \ell \rangle} Q \) it is then shown that the \( Q_{A \ell} \) satisfy the curvature-modified Helmholtz equation

\[
D_0^a Q_{\langle A \ell \rangle} = -k^2_{\text{phys}} + \frac{K}{a^2} \ell(\ell + 2)) Q_{\langle A \ell \rangle}, \quad (135)
\]

\(^{18}\)This can also be seen from using \( F_{A \ell} h^{\langle A \ell B_n \rangle} = (n!/(n - \ell)!!(n + \ell - 1)!!) F_{A \ell} h^{\langle B_n \rangle} h_{B_n \ldots} \), the definition of \( O^{A \ell} \) in terms of \( e^{A \ell} \) and evaluating \( \int d\Omega F_{A \ell} O^{A \ell} e B_n \) for \( F_{A \ell} \) PSTF.
i.e., the Helmholtz equation with modified wavelength using as before \( k_{\text{phys}} = k/a \)

\[
-k_\ell^2 = \frac{1}{a^2}(K\ell(\ell + 2) - k^2).
\]  

(136)

On using (135) and taking the PSTF part of the lower indices, \( D^aD_{(aQ_{A\ell})} \), and using the PSTF tensor relation (21), to find:

\[
D^aD_{(aQ_{A\ell})} = \frac{(\ell + 1)}{(2\ell + 1)} \left( -\frac{k^2}{a^2}\right) \left[ 1 - \frac{K}{k^2}(\ell^2(\ell + 2)) \right] Q_{(A\ell)}.
\]  

(137)

On substituting the first two relations (130, 131) into the recursion relation for the PSTF tensors (22), we find

\[
e^aD_a[G_\ell(Q)] = +k_{\text{phys}} \left[ \frac{\ell^2}{(2\ell + 1)(2\ell - 1)} \left( 1 - \frac{K}{k^2}(\ell^2 - 1) \right) G_{\ell - 1}[Q] - G_{\ell + 1}[Q] \right],
\]  

(138)

C.2 Legendre tensors and the PSTF tensors

One can immediately make the connection between this formulation and the one usually used in terms of Legendre tensors, and see that the Legendre tensors used in Wilson (13) in the coordinate basis (indicated by late romans) can be related to irreducible representation \( O^{A\ell} \) in terms of its associated tetrad frame \( E^a_{\ell} = \{u^a, e^a_{\mu} \} \). The direction vectors \( e^a \) in the triad with components \( e^a_{\mu} \) are related to \( \gamma^i \), the direction cosines used in the Wilson-Silk coordinate basis treatment:

\[
P_{i_1i_2...i_\ell}^{(\ell)}(\gamma^i) = (\beta_\ell)^{-1}e_{a_1}^{i_1}e_{a_2}^{i_2}...e_{a_\ell}^{i_\ell}O^{A\ell}(e^a).
\]  

(139)

This connection to the Legendre polynomials can be seen by using the relation between spherical harmonics and the PSTF tensor along with the addition theorem for the PSTF tensors:

\[
O^{A\ell} = \gamma^A_{\ell m}Y^{\ell m}(\Omega), \quad \Leftrightarrow \beta_\ell P_{\ell}(e^a e^\ell_a) = O^{A\ell} O^{A\ell}.
\]  

(140)

Multiply (22) by \( \beta_{\ell-1} \) and use (133) and (25) to find the recursion relation for the Wilson-Silk Legendre polynomials

\[
P_{i_1...i_\ell}^{(\ell)} = \frac{(2\ell + 1)}{(\ell + 1)}\gamma(\ell)P_{i_1...i_{\ell+1}}^{(\ell)} - \frac{\ell}{(\ell + 1)}\gamma(\ell)P_{i_1...i_{\ell+1}}^{(\ell-1)}.
\]  

(141)

It is seen that the recursion relations (22) for the irreducible representation \( O^{A\ell} \) can be reduced to that of the Legendre polynomial. This links the CGI-PSTF approach to the usual GI-Legendre tensor approach.

D Plane-waves, spherical-waves and mode functions

D.1 Plane-wave and mode function relations

We consider only flat, \( K = 0 \), universes at present. Each set of harmonic functions \( Q_{(k)}(x^\alpha) \) satisfying (18) which has associated with it \( k_{\text{phys}} = k/a \), the physical wavenumber, a variation vector field, \( q^a \), and a direction \( e^a \ (e^a e_\alpha = 1, e^a u_\alpha = 0) \) determined by (23)

\[
D_aQ = Qq_a, \quad q_a = qe_a, \quad q^2 = q^aq_a
\]  

(142)

\(^{19}\)The vector \( e^a \) defined here is in general different from that associated with the angular harmonic expansion in (3). When ambiguity can arise, we explicitly put in the \( k \)-dependence : \( q_{(k)}^a \), to signify both this dependence and the definition of \( e^a \) from (142) : thus strictly we should write, for example, \( D_aQ_{(k)} = Q_{(k)}q^{(k)}_{a} = Q_{(k)}q^{(k)}e^{(k)}_a \). We will suppress the \( k \) when this causes no ambiguity.
the first equality defining \( q_a(x^i) \) (but not necessarily so as to factor out \( Q \)) and the second splitting it into its magnitude and direction. It follows that

\[
D^aD_aQ = Qq^2 + QD^aq_a = Q(q^2 + e^aD_aq + qD^ae_a)
\]  
(143)

so that (148) becomes

\[
q^2 + e^aD_aq + qD^ae_a = -k_{phys}^2, \quad \iff \quad D^aq_a = -q^2 - k_{phys}^2.
\]  
(144)

Using the \( K = 0 \) plane-wave eigenfunctions with associated direction vector \( e^{(k)}_a \):

\[
Q(x^i, e^{(k)}_a)|_{flat} = \exp\left\{ -ik_{phys}e^{(k)}_a x^a \right\},
\]  
(145)

where \( k_{phys}(k, t) = k/a \), expresses the temperature anisotropy (4) in terms of its plane-wave spatial Fourier Transform (58). In this case

\[
q_a = -ik_{phys}e^{(k)}_a, \quad D_aq_b = 0 = D_a e^{(k)}_b, \quad q^2 = \frac{k^2}{a^2} = -k_{phys}^2
\]  
(146)

holds in equation (142) and (143) respectively we find (from (145))

\[
D_{\langle A \ell \rangle}Q(x^i, e^{(k)}_a)|_{flat} = (-i k_{phys})^{\ell} O^{(k)}_{A \ell} Q(x^i, e^{(k)}_a)|_{flat}
\]  
(147)

where the \( O^{(k)}_{A \ell} \) are the PSTF tensors associated with the direction \( e^{(k)}_a \) in the tangent spaces on the spatial section. Thus from (143)

\[
(Q(x^i, e^{(k)}_a)|_{flat})_{\langle A \ell \rangle} = (-1)^{\ell} O^{(k)}_{A \ell} Q(x^i, e^{(k)}_a)|_{flat}
\]  
(148)

and (54) to find (61).

### D.2 Radial expansion and mode function relations

\[
D_aQ_\ell = D_a(R_{A \ell} O^{A \ell}) = O^{A \ell} D_aR_{A \ell} + R_{A \ell} D_a O^{A \ell},
\]  
(149)

which implies that

\[
D_aD^aQ_\ell = (D^aD_aR_{A \ell}) O^{A \ell} + 2(D_aR_{A \ell} D^a O^{A \ell}) + R_{A \ell} (D_aD_a O^{A \ell}).
\]  
(150)

Now we need to work out \( D_aR_{A \ell}, D^aD_aR_{A \ell}, D_a O^{A \ell}, \) and \( D_aD^a O^{A \ell} \), say (a), (b), (c), (d) respectively. Calculating (a):

\[
D_aR_{A \ell}(r) = \frac{\partial R_{A \ell}}{\partial r} D_ar = \frac{\partial R_{A \ell}}{\partial r} e_a,
\]  
(151)

hence (b) follows:

\[
D^aD_aR_{A \ell}(r) = \frac{\partial^2 R_{A \ell}}{\partial r^2} + 2 \frac{\partial R_{A \ell}}{r \partial r}.
\]  
(152)

Next, (c) is:

\[
D_a O^{A \ell} = \frac{\ell}{r} p^{(a \ell)}_a O^{A \ell-1} \Rightarrow e^a D_a O^{A \ell} = 0
\]  
(153)
which gives (d):

$$D^aD_a O^{A \ell} = \frac{\ell}{r} D^a p^{(a \ell)} O^{A_{\ell-1}} + \frac{\ell}{r} p^{(a \ell)} (D_a O^{A_{\ell-1}}) - \frac{\ell}{r^2} D^a (r) p^{(a \ell)} O^{A_{\ell-1}},$$

$$= -\frac{2\ell}{r^2} e^{(a \ell)} O^{A_{\ell-1}} + \frac{\ell(\ell - 1)}{r^2} p^{(a \ell - 1)} O^{A_{\ell-2}} - \frac{\ell}{r^2} e^{a \ell} p^{(a \ell)} O^{A_{\ell-1}},$$

$$= -\frac{\ell(\ell + 1)}{r^2} O^{A \ell}. \quad (154)$$

Now put these in (154) to find,

$$D_a D^a Q_\ell = \left[ \frac{\partial^2 R_{A \ell}}{\partial r^2} + \frac{2}{r} \frac{\partial R_{A \ell}}{\partial r} \right] O^{A \ell} + 2 \left[ \frac{\partial R_{A \ell a}}{\partial r} \right] \left[ \frac{\ell}{r} p^{(a \ell)} O^{A_{\ell-1}} \right]$$

$$+ R_{A \ell} \left[ -\frac{\ell(\ell + 1)}{r^2} O^{A \ell} \right] = -k^2 r_{\text{phys}} R_{A \ell} O^{A \ell}, \quad (155)$$

which simplifies to

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R_{A \ell}}{\partial r} \right) O^{A \ell} + R_{A \ell} \left[ -\frac{\ell(\ell + 1)}{r^2} O^{A \ell} \right] = -k^2 r_{\text{phys}} R_{A \ell} O^{A \ell}. \quad (156)$$

Now we need to find $O^{A \ell}_{(\ell)} D_{(A \ell)} Q$ in terms of $R_0$, then relate the resulting Rodrigues formulae of $R_\ell$ to get $R_\ell$ in terms of $R_0$ and hence $Q^{A \ell}$ in terms of $R_\ell$. In this regard, now in (157), $D_a e^{(k)}_b = 0$ (see (144)) which implies $D_a O^{(k)}_{A \ell} = 0$. Thus

$$D_a R_{A \ell} (r) = \sum_k \frac{\partial R_{\ell}(k, r)}{\partial r} e_a O^{(k)}_{A \ell} = \frac{\partial R_0(k, r)}{\partial r} e_a. \quad (157)$$

We can find $Q_{(A \ell)}$ from (159) obtaining

$$D_{(A \ell)} Q = \sum_{m, k^\nu} D_{(A \ell)} R_{B \ell} O^{B \ell}_{(\ell)} = \sum_{m, k^\nu} D_{(A \ell)} R_{m}(r, k) O^{(k)}_{B \ell} O^{(B \ell)}_{(\ell)} = D_{(A \ell)} R_0(k, r),$$

$$= D_{(A \ell - 1)} \frac{\partial R_0}{\partial r} e_{a \ell} = D_{(A \ell - 2)} \left( \frac{\partial^2 R_0}{\partial r^2} e_{a \ell - 1} e_{a \ell} + \frac{1}{r} p_{a \ell - 1 a \ell} \frac{\partial R_0}{\partial r} \right),$$

$$= D_{(A \ell - 3)} \left( \frac{\partial^3 R_0}{\partial r^3} e_{a \ell - 2} e_{a \ell - 1} e_{a \ell} + \frac{3}{r} \frac{\partial^2 R_0}{\partial r^2} p_{a \ell - 2 a \ell - 1} e_{a \ell} ight)$$

$$- \frac{2}{r} \frac{\partial R_0}{\partial r} p_{a \ell - 2 a \ell - 1} e_{a \ell} - \frac{1}{r} \frac{\partial R_0}{\partial r} (D_{a \ell - 2} r) p_{a \ell - 1 a \ell} \right). \quad (158)$$

Now we note that

$$O^{A \ell}_{(\ell)} D_{(A \ell)} Q = O^{A \ell}_{(\ell)} D_{(A \ell)} R_0(k, r) = O^{A \ell}_{(\ell)} D_{(A \ell - 2)} \left( \frac{\partial^2 R_0}{\partial r^2} - \frac{1}{r} \frac{\partial R_0}{\partial r} \right) e_{a \ell - 1} e_{a \ell},$$

$$= O^{A \ell}_{(\ell)} (-r)^\ell \left( -\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell R_0 e_{(A \ell)} \quad (159)$$
which follows from $p_{ab}e^i(a e^b) = h_{ab}O^{ab} - e_a e_b O^{ab} = -O_{ab}O^{ab}$ and that
\[ R_\ell = \frac{r^\ell}{k^\ell_{\text{phys}}} \left( -\frac{1}{r} \frac{\partial}{\partial r} \right) R_\ell . \]  
(160)

Used in (159) this gives that
\[ O^{A_\ell}_{(\chi)} D_{(A_\ell)} Q = (-k_{\text{phys}})^\ell O^{A_\ell}_{(k)} R_\ell . \]  
(161)

### E Mode mean square relations

#### E.1 Flat relations

\[ \tau_{A_\ell} = (i)^\ell \int \frac{k^2 dk}{(2\pi)^3} \int d\Omega_k \tau_k(k, t) O^{(k)}_{A_\ell} \sum_{n=0}^\infty (i)^n j_n(\lambda r) O^{B_n}_{(k)} O^{(\chi)}_{(k)} \beta_n^{-1}(2n + 1), \]  
(162)

Equivalently from
\[ \tau(x^i, e^a) = \sum_m \tau_m Q_m, \quad Q_m = \sum_{k^a} R_{C_m D_m} O^{D_m}_{(k)} O^{C_m}_{(\chi)}, \quad \text{and} \quad R_{C_m D_m} = R_{\ell(k, r)} h_{C_m D_m}. \]  
(163)

Invert the multipole expansion
\[ \tau_{A_\ell} = \Delta_\ell^{-1} \int d\Omega_k O^{(k)}_{A_\ell} \left\{ \sum_{m, k^a} \tau_m R_{C_m B_m} O^{B_m}_{(k)} O^{C_m}_{(\chi)} \right\}, \]  
(164)

to find, on using (128), that
\[ \tau_{A_\ell} = \frac{1}{2\pi^2} \Delta_\ell^{-1} \sum_m \int k'^2 dk' \tau_m(k', t) R_{C_m B_m} O^{C_m}_{(\chi)} \left[ \delta^m_\ell \Delta_\ell^{(B_\ell)} \right], \]  
(165)

and hence that
\[ \tau_{A_\ell} = \frac{1}{2\pi^2} \int k'^2 dk' \tau_k(k', t) R_{A_\ell C_\ell} O^{C_\ell}_{(\chi)}. \]  
(166)

Using (163) this becomes
\[ \tau_{A_\ell} = \frac{1}{2\pi^2} O^{(\chi)}_{A_\ell} \int k'^2 dk' \tau_k(k', t) R_{\ell(k', \chi)}, \]  
(167)

#### E.2 Constant curvature relations

We now have that
\[ \int d\Omega_{(\chi)} O^{A_\ell}_{(\chi)} G_\ell[Q] = \delta^\ell_m \Delta_\ell Q^{A_\ell}, \]  
(168)

\[ \int d\Omega_{(\chi)} G_\ell[Q] G_m[Q] = \delta^m_\ell Q^{A_\ell} A_\ell \Delta_\ell, \]  
(169)

\[ \int d\Omega_{(k)} G_\ell[k] G_m[k] = \int d\Omega_{(k)} O^{A_\ell}_{(k)} R_{\ell A_\ell} O^{B_m}_{(k)} R_{m B_m} O^{(k)}_{(\chi)} = \delta^m_\ell \Delta_\ell R_{\ell}^2 O^{(\chi)}_{A_\ell} O^{A_\ell}_{(\chi)}, \]  
\[ \Rightarrow \int d\Omega_{(k)} G_\ell G_m = \delta^\ell_m \Delta_\ell^2 \frac{2\ell + 1}{4\pi} R_{\ell}^2. \]  
(170)
Furthermore, we have that from the recursion relations (57)
\[ e_{(\chi)}^a D_a [G_\ell] = +k_{\text{phys}}[\alpha_\ell^2 G_{\ell-1} - G_{\ell+1}] \] (171)
where
\[ (\alpha_\ell)^2 = \frac{\ell^2}{(2\ell + 1)(2\ell - 1)} \left( 1 - \frac{K}{k^2}(\ell^2 - 1) \right) \] (172)
and using [17] [20],
\[ \int d\Omega_k \int dx^\nu e_{(\chi)}^a D_a [G_\ell G_{\ell-1}] = 0 \] (173)
to find
\[ \int d\Omega^{(k)} \langle |G_n|^2 \rangle = (\alpha_n)^2 \int d\Omega^{(k)} \langle |G_{n-1}|^2 \rangle . \] (174)
Here we have defined the mean square to pick out the power spectrum which is a function only of the absolute value of the wavelength for a Gaussian distribution (there is no directional dependence, the modulus is only dependent on the wave number).

F 1+3 Ortho-Normal Tetrad relations

We use an orthonormal tetrad approach (cf. 3). Consider an orthonormal tetrad basis \( E_a \) with components \( E^i_a \) relative to a coordinate basis; here indices a,b,c,..., that is early letters, are used for the tetrad basis, while late letters i,j,k,... are used for the coordinate basis. The differential operators \( \partial_a = E^i_a \partial_i \) are defined by the inverse basis components
\[ E^a_i E^b_j = \delta^i_j \iff E^a_i E^i_b = \delta^a_b . \] (175)
The tetrad components of a vector \( X^i \) are \( X^a = E^a_i X^i \), and similarly for any tensor. Tetrad indices are raised and lowered using the tetrad components of the metric
\[ g_{ab} = g_{ij} E^i_a E^j_b = \text{diag}(-1, +1, +1, +1), \quad g^{ab} g_{bc} = \delta^a_c \] (176)
the form of these components being the necessary and sufficient condition that the tetrad basis vectors used are orthonormal, which we will always assume.

For an observer with 4-velocity \( u^a \), there is a preferred family of orthonormal tetrads associated with \( u^a \). For a frame for which the time-like tetrad basis \( E^0 \) is parallel to the velocity \( u^a \). In such a tetrad basis
\[ u^a = \delta^a_0 \] (177)
\[ h_{ab} = \text{diag}(0, +1, +1, +1) \] (178)
All our work is based on such a tetrad, which leads to a preferred set of associated rotation coefficients. In paper 1, the form of these rotation coefficients is unimportant, so we defer their consideration to Paper 2. The issue for the present is that we have a preferred family of local orthonormal frames at each point (usually matter flow aligned), and carry out our algebraic analysis of observational quantities relative to that orthonormal frame.

\[ \text{The idea is to use this to fix the normalization of } \int d\Omega_{(k)} G_\ell [Q] = D_\ell \int d\Omega_{(k)} Q (x^i, e^a_{(k)}) O^{A_i}_{(k)} O^{(k)}_A . \]
References

[1] F. A. E. Pirani, Introduction to gravitational radiation Theory. Lectures in GR, Brandeis Lectures, Ed. Deser and Ford, (Prentice Hall, 1964) (sec 2.3).

[2] R. J. Adler, The Geometry of Random Fields, (Wiley, 1981).

[3] K. S. Thorne, Rev. Mod. Phys. 52 299 (1980).

[4] K. S. Thorne, Mon. Not. R. Ast. Soc. 194 439-473 (1981).

[5] G. F. R. Ellis, D. R. Matravers, R. Treciokas, Ann. Phys 150 455 (1983a).

[6] G. F. R. Ellis, R. Treciokas, D. R. Matravers, Ann. Phys 150 487 (1983b).

[7] S.R de Groot, W.A van Leeuwen, Ch. G. van Weert Relativistic Kinetic Theory (North-Holland, 1980)

[8] J Ehlers, R Geren, R K Sachs, J. Math. Phys. 9 1344 (1968).

[9] W. R. Stoeger, R Maartens, G. F. R. Ellis, Astrophys. J. 443 (1995).

[10] R. Maartens, G. F. R. Ellis, W. R. Stoeger, Phys. Rev. D 51 1525 (1995a).

[11] R. Maartens, G. F. R. Ellis, W. R. Stoeger, Phys. Rev. D 51 5942 (1995b).

[12] R. Maartens, G. F. R. Ellis, W. R. Stoeger, Astron. Astrophys 309 L7 (1996).

[13] W. R. Stoeger, M. Araujo, T. Gebbie, Astrophys. J. 476 435 (1997).

[14] R. Maartens, T. Gebbie, G. F. R. Ellis, Phys. Rev D 59 083506, astro-ph/9808163.

[15] P. J. E. Peebles, J. T. Yu : Astrophys. J. 162 815 (1970).

[16] G. F. R. Ellis, M. Bruni, Phys. Rev. D 40 6 1804 (1989).

[17] M. L. Wilson, J. Silk, Astrophys. J. 243 14 (1981).

[18] M. L. Wilson, Astrophys. J. 273 2 (1983).

[19] L. F. Abbott, R. K. Schaeffer, Astrophys. J. 308 546 (1986).

[20] N. Gouda, N. Sugiyama, M. Sasaki, Prog. Th. Phys. 85 1023 (1991).

[21] G. Efstathiou, J.R. Bond, S. D. M. White, Mon. Not. R. Astr. Soc. 258 5 (1992).

[22] W. Hu, N. Sugiyama, Astrophys. J. 444 489 (1995).

[23] W. Hu, N. Sugiyama, Phys. Rev. D 51 2599 (1995).

[24] M. White, D. Scott, Astrophys. J. 459 415 (1996).

[25] A. R. Liddle, D. H. Lyth, Phys. Rep. 231 57 (1993).

[26] P. G. Ferreira, J. C. R. Magueijo, Phys. Rev D 55 3358 (1997).

[27] D Lyth, A Woszczyna, Phys. Rev. D 52 6 3338 (1995).
[28] J. C. R. Magueijo, Phys. Rev. D 47 R353 (1993).
[29] R. K. Sachs, A. M. Wolfe, Astrophys. J. 147 73 (1967).
[30] A. D. Challinor, A. N. Lasenby, Phys. Rev. D. 58 (1998); astro-ph/9804150
[31] A. D. Challinor, A. N. Lasenby, Ap. J. 513, 1 (1999); astro-ph/9804301
[32] P. K. S. Dunsby, Class. Quant. Grav. 14 3391 (1997).
[33] J. M. Bardeen, Phys. Rev. D 22 1882 (1980).
[34] M Bruni, P. K. S. Dunsby and G. F. R. Ellis, Astrophys. J. 395 34 (1992).
[35] G. F. R. Ellis, D. R. Matravers, R. Treciokas, Gen. Rel. Grav. 15 931 (1983).
[36] D. R. Matravers and G. F. R. Ellis, Class. Quant. Grav. 6 369 (1989); 7, 1869 (1990); S. D. Maharaj, R. Maartens Class. Quant. Grav 19 499 (1987); 1217; 1223; R. Maartens, F. P. Wolvaardt Class. Quant. Grav. 14 535 (1997).
[37] G. F. R. Ellis, “Relativistic Cosmology”. In General Relativity and Cosmology, Proc Int School of Physics “Enrico Fermi” (Varenna), Course XLVII. Ed. R K Sachs (Academic Press, 1971), 104.