Unimodular metagrawity vs. General Relativity with a scalar field

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Abstract

The unimodular metagrawity, with the graviscalar as a dark matter, is compared with General Relativity (GR) in the presence of a scalar field. The effect of the graviscalar on the static spherically symmetric metric is studied. An exact limit solution representing a new cosmic object, the (harmonic) graviscalar black hole, is given. The relation with the black hole in the environment of a scalar field in GR is discussed.

1 Introduction

The explicit violation of the general covariance may serve as the resource of the dark matter (DM) of the gravitational origin \(^1\). Under the residual unimodular covariance, when the local scale invariance alone is violated, the metric comprises precisely one extra degree of freedom, the (massive) scalar graviton, or the graviscalar, in addition to the massless tensor one. Such an extension of General Relativity (GR) may be termed the unimodular metagrawity \(^1\). In the present report, we consider the influence of the graviscalar on the spherically symmetric metric \(^2\). An exact limit solution to the unimodular metagrawity equations representing a new cosmic object, the (harmonic) graviscalar black hole, is given. The relation of the latter with the black hole in the presence of a scalar field in GR is discussed.

2 Metagrawity vs. GR

Metagrawity Lagrangian The Lagrangian of the unimodular metagrawity, with the graviscalar as DM, is superficially similar to the GR Lagrangian in the presence of a scalar field. More particularly,

\[
L = L_g + L_h + L_m + L_{gh} + L_{mh},
\]

(1)

\(^1\)To distinguish from the unimodular relativity \[^2\]_ which also possesses the residual unimodular covariance, but with one component of the metric explicitly removed.

\(^2\)Partly, this was studied in \[^3\].
with the gravity and graviscalar contributions being conventionally as follows:

\[ L_g = -\frac{1}{2}\kappa_g^2 R(g_{\mu\nu}) - \Lambda, \]
\[ L_h = \frac{1}{2} \partial\chi \cdot \partial\chi - V_h(\chi). \]  

(2)

In \( L_g \), \( R \) is the Ricci scalar, \( g_{\mu\nu} \) is the metric, \( \kappa_g \) is the gravity mass scale, \( \kappa_g^2 = 1/(8\pi G) \), with \( G \) standing for the Newton’s constant, and \( \Lambda \) is the cosmological constant. In \( L_h \), \( \chi \) is the graviscalar field, \( \partial\chi \cdot \partial\chi = g^{\mu\nu}\partial_{\mu}\chi\partial_{\nu}\chi \) and \( V_h \) is the graviscalar potential. Let \( \chi_0 \) be the position of a minimum of the potential. Normalize the latter by the condition \( V_h|_{\chi_0} = 0 \) attributing the rest to the \( \Lambda \)-term.

The peculiarity of the unimodular metagravity compared to GR with a scalar field is contained in the definition of the graviscalar field [1]:

\[ \chi = \frac{\kappa_h}{2} \ln \frac{g}{g_h}, \]  

(3)

with \( \kappa_h \leq \mathcal{O}(\kappa_g) \) standing for the unimodular metagravity mass scale and \( g = \det g_{\mu\nu} \). Of importance, \( g_h \) is a non-dynamical scalar density of the same weight as \( g \). Such a primordial density enters as a manifestation of the general covariance violation. Explicitly, \( g_h \) can be given by its dependence on the observer’s coordinates. Implicitly, it can be introduced by defining the (class of the) “canonical” coordinates, where \( g_{\mu\nu} = -1 \).

### Metagravity equations

Varying the action \( A = \int L \sqrt{-g} d^4x \) with respect to \( g_{\mu\nu} \), under fixed \( g_h \), we arrive at the equations of the unimodular metagravity:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{\kappa_g^2} (T_{m\mu\nu} + T_{h\mu\nu} + T_{\Lambda\mu\nu}), \]  

(4)

with \( R_{\mu\nu} \) being the Ricci curvature tensor. In the above, \( T_{m\mu\nu} \) is the energy-momentum tensor of the ordinary matter, \( T_{\Lambda\mu\nu} = \Lambda g_{\mu\nu} \) is the vacuum contribution and \( T_{h\mu\nu} \) is the graviscalar one. The latter has the form conventional for a scalar field:

\[ T_{h\mu\nu} = \partial_{\mu}\chi\partial_{\nu}\chi - \left( \frac{1}{2} \partial\chi \cdot \partial\chi - \ddot{V}_h \right) g_{\mu\nu}, \]  

(5)

except for \( \ddot{V}_h \) being the metapotential:

\[ \ddot{V}_h = V_h + \kappa_h \left( \frac{\partial V_h}{\partial\chi} + \nabla \cdot \nabla\chi \right), \]  

(6)

with \( \nabla_{\mu} \) standing for a covariant derivative.

Due to the contracted Bianchi identity, the tensor of the total energy-momentum satisfies the covariant continuity law:

\[ \nabla_{\mu}(T_{m\mu} + T_{h\mu} + T_{\Lambda\mu}) = 0. \]  

(7)
By this token, the graviscalar can naturally be treated as DM of the gravitational origin. In the ordinary matter vacuum, $T_{m\mu\nu} = 0$, the continuity law results in the field equation of the graviscalar alone:

$$\nabla \cdot \nabla \chi + \frac{\partial \tilde{V}_h}{\partial \chi} = 0,$$

(8)

with the metapotential reducing to

$$\tilde{V}_h = V_h + \Lambda_h e^{-\chi/\kappa h},$$

(9)

where $\Lambda_h$ is an integration constant. In the limit $\Lambda_h = 0$, we recover the GR equations for gravity and a scalar field with the conventional $T_{h\mu\nu}$. At a finite $\Lambda_h$, the difference between the unimodular metagravity equations in the matter vacuum and the GR equations in the presence of a scalar field reduces to the indicated substitution $V_h \to \tilde{V}_h$.

**Spherical symmetry** Consider the spherically symmetric metric around a center. The line element for such a metric in the polar coordinates $x^0 = t, x^m = (r, \theta, \varphi), m = r, \theta, \varphi$, looks most generally like

$$ds^2 = adt^2 - bdr^2 - cr^2 d\Omega, \quad d\Omega = d\theta^2 + \sin^2 \theta d\varphi^2.$$

(10)

The field $\chi$ has the same form as in the quasi-Galilean coordinates:

$$\chi = \kappa_h \ln \sqrt{a b c / \sqrt{-g_h}}.$$

(11)

In the static case we consider, the metric variables $a, b$ and $c$ are the arbitrary functions of $r$ alone. The same is true for $g_h$. The graviscalar energy-momentum tensor is then

$$T_{h\mu\nu} = \left( \frac{1}{2b} \chi'' + \tilde{V}_h \right) \delta_\mu^\nu - \frac{1}{b} \chi' \delta_\mu^r \delta_\nu^r, \quad \mu, \nu = 0, r, \theta, \varphi,$$

(12)

with the prime designating the derivative with respect to $r$. In the matter vacuum, $T_{m\mu\nu} = 0$, the metapotential $\tilde{V}_h$ is given by eq. (9) but for a singular point $r = 0$.

Fix the choice of the radial coordinate by the condition $ab = 1$. Designating $A = a = 1/b$ and $C = cr^2$ we get the unimodular metagravity equations in the empty space as follows:

$$(\ln C)'' + \frac{1}{2} (\ln C)^2 = -\frac{1}{\kappa_g^2} \chi'^2,$$

$A'' - A \left( (\ln C)'' + (\ln C)^2 \right) + 2C^{-1} = 0,$

$A'' + A' (\ln C)' = -\frac{2}{\kappa_g^2} (V_h + \Lambda + \Lambda_h e^{-\chi/\kappa h}).$  

(13)
Harmonic solution  Consider the limit $V_h = \Lambda_h = 0$. It results in the harmonicity condition, $\nabla \cdot \nabla \chi = 0$. The respective solution of the metagravity equations may be called the harmonic one. Besides, we take $\Lambda = 0$. In this case, the unimodular metagravity equations in the empty space coincide with the GR equations in the presence of a free massless scalar field. The exact solution to the latter equations was first found in a different context by Buchdahl [4]. We straightforwardly recover the solution as follows:

$$A = \left(1 - \frac{r_h}{r}\right)^{\gamma_h},$$

$$C = r^2\left(1 - \frac{r_h}{r}\right)^{1-\gamma_h},$$

$$s = \frac{\sqrt{2}}{\kappa_g} \chi = \pm \sqrt{1 - \gamma_h^2} \ln \left(1 - \frac{r_h}{r}\right), \quad (14)$$

with $r_h$ and $\gamma_h$ being some integration constants. Thereof, we can find $g_h$ in the given coordinates. The graviscalar field is defined modulo an additive constant. Call the respective cosmic object the harmonic graviscalar black hole. For consistency, $|\gamma_h| \leq 1$. By taking $\gamma_h = 1$, we recover the Schwarzschild black hole: $a = 1/b = 1 - r_h/r$, $c = 1$ and $\chi = 0$. Incidentally, $g_h = -1$ in this case.

Post-Newtonian approximation  To confront the harmonic solution with observations choose the isotropic coordinates, with a new radial coordinate $\hat{r}$ defined through $\hat{c} = \hat{b}$, so that

$$r = \hat{r}\left(1 + \frac{r_h}{4\hat{r}}\right)^2, \quad \hat{r} \geq r_h/4. \quad (15)$$

This results in

$$\hat{a} = \left(1 - \frac{r_h}{4\hat{r}}\right)^{2\gamma_h} / \left(1 + \frac{r_h}{4\hat{r}}\right)^{2\gamma_h},$$

$$\hat{b} = \left(1 - \frac{r_h}{4\hat{r}}\right)^{2(1-\gamma_h)} \left(1 + \frac{r_h}{4\hat{r}}\right)^{2(1+\gamma_h)},$$

$$\hat{s} = \pm 2\sqrt{1 - \gamma_h^2} \ln \left(\left(1 - \frac{r_h}{4\hat{r}}\right) / \left(1 + \frac{r_h}{4\hat{r}}\right)\right). \quad (16)$$

Decomposing the solution in $r_h/\hat{r}$ we get up to the terms $1/\hat{r}^2$:

$$\hat{a} = 1 - \frac{r_g}{\hat{r}} + \frac{1}{2} \frac{r_g^2}{\hat{r}^2},$$

$$\hat{b} = 1 + \frac{r_g}{\hat{r}} + \frac{3}{8} \frac{r_g^2 - r_s^2/3}{\hat{r}^2},$$

$$\hat{s} = \pm \frac{r_s}{\hat{r}}. \quad (17)$$

where we have replaced the two intrinsic parameters, $r_h$ and $\gamma_h$, by the two effective ones, $r_g$ and $r_s$, as follows

$$r_g = \gamma_h r_h, \quad r_s = \sqrt{1 - \gamma_h^2} r_h. \quad (18)$$

Identifying $r_g > 0$ with the gravitational radius, we see that the metric of the harmonic graviscalar black holes ($r_s \neq 0$) reproduce asymptotically the metric of the Schwarzschild black holes ($r_s = 0$) up to the first post-Newtonian approximation, i.e., $\hat{a}$ up to $1/\hat{r}^2$ and $\hat{b}$ up to $1/\hat{r}$. The only restriction is that $\gamma_h \gg (r_g/\hat{r})^{1/2}$ for $\hat{r}$ at hand.
Graviscalars vs. ordinary scalar

Clarify the physical content of the harmonic solution. Let $T_{\mu\nu} = T_{m\mu\nu} + T_{h\mu\nu}$ be the energy-momentum tensor of the ordinary matter and the graviscalars. Let $T_{g\mu\nu}$ be the Einstein gravitational pseudo-tensor in the quasi-Galilean coordinates $(x^0, x^n)$, $n = 1, 2, 3$. Neglect by the $\Lambda$-term. A result due to Tolman \[5\] states that the total energy (mass) $M$ of a static isolated distribution is given by

$$M = \int \left( T^0_0 + T^0_{g0} \right) \sqrt{-g} d^3x = \int \left( T^0_0 - T^0_n \right) \sqrt{-g} d^3x. \quad (19)$$

In view of the gravity equations, this results in

$$M = 2\kappa^2 \int R^0_0 \sqrt{-g} d^3x, \quad (20)$$

with the integral saturated by a $\delta$-function singularity. We have

$$R^0_0 = \left( CA' \right)' = -\frac{1}{2} g^{kl} \nabla_k \nabla_l \ln A, \quad (21)$$

where $\nabla_k$ is a spatial component of the covariant derivative and

$$g^{kl} = -An^k n^l - \frac{1}{c} (\delta^{kl} - n^k n^l), \quad (22)$$

with $n^k = x^k / r$, $r = |x| = (\delta_{kl} x^k x^l)^{1/2}$. The Gauss theorem reduces then the volume integral to the integral over a remote surface $\sigma$ as follows:

$$M = -\kappa^2 \int \sqrt{-g} g^{kl} \partial_l \ln A d\sigma_k, \quad (23)$$

with $\sqrt{-g} = c$. Returning back to the radial coordinate $r = |x|$, with $\partial_k = \delta_{kl} n^l \partial / \partial r$, $d\sigma_k = n_k r^2 d\Omega$, we get

$$M = 4\pi \kappa^2 \gamma h r_h = r_g / (2G). \quad (24)$$

Partite $M$ onto the contributions $M_m$ and $M_h$ of the ordinary matter and graviscalars, respectively, as follows:

$$M = M_m + M_h \equiv \int \left( \left( T^0_0 - T^0_m \right) + \left( T^0_0 - T^0_h \right) \right) \sqrt{-g} d^3x. \quad (25)$$

The graviscalar contribution is

$$M_h = -2 \int \tilde{V}_h \sqrt{-g} d^3x = -2\kappa h \int \nabla \cdot \nabla \chi \sqrt{-g} d^3x, \quad (26)$$

with the integral saturated by a singularity at $r = 0$. The term $\nabla \cdot \nabla \chi$ is produced ultimately by its self-consistent coupling with $R^0_0$ as a source. The Gauss theorem transforms the volume integral to the surface one with the result:

$$M_h = \pm 4\pi \kappa_g \kappa h \sqrt{2(1 - \gamma^2_h)} r_h = \pm \frac{\kappa h}{\kappa_g} \sqrt{2(1/\gamma^2_h - 1)} M. \quad (27)$$

The parameter $\gamma h$ characterizes thus the graviscalar contribution to the total mass. With $M > 0$, imposing $M_h \geq 0$ and $M_m \geq 0$ we get the restriction:

$$\frac{1}{1 + \kappa^2_g / (2\kappa^2_h)} \leq \gamma^2_h \leq 1, \quad (28)$$
The upper bound corresponds to the Schwarzschild black holes with $M_h = 0$, while the lower bound to a pure graviscalar cosmic objects, the graviballs, with $M_m = 0$. Finally, the physical meaning of the two effective parameters, $r_g$ and $r_s$, introduced previously is as follows: $r_g \sim M$ and $r_s/r_g \sim M_h/M$.

Let now $\chi$ be a scalar field in GR. Though the solution found is the same in GR and the unimodular metagravity, the physical interpretation of the solutions in the theories is quite different. First of all, in GR $\tilde{V}_h = V_h = 0$ everywhere, including the center, and hence $M_h = 0, M = M_m$. Further, consider the integral

$$\Delta = \int \nabla \cdot \nabla \chi \sqrt{-g} d^3 x.$$  \hspace{1cm} (29)

By means of the Gauss theorem we get

$$\Delta = \mp 4\pi \kappa g \sqrt{(1 - \gamma^2_h)/2 r_h} = \mp \sqrt{1/\gamma^2_h - 1} \sqrt{4\pi G M}.$$  \hspace{1cm} (30)

To saturate the volume integral, a central singularity is required. In distinction with the graviscalar, an ordinary scalar is not coupled with $R^0_0$. For consistency, we should thus admit that such a field is produced by the central matter as a source.

Let a black hole surrounded by an ordinary scalar field in GR, say a dilaton, model a typical cosmic object with $M = M_m \simeq m_N N$, where $m_N$ is the nucleon mass and $N$ is the number of nucleons in the object. Introduce the dilatonic “charge” per nucleon, $\delta_N = \Delta/N$. This fixes $\gamma_h$ as a universal parameter of the theory, not of a particular solution. The scalar mediated forces being attractive in nature, the interaction between two remote cosmic objects with the nucleon numbers $N$ and $n$ (masses $M$ and $m$, respectively) is given by

$$\delta U = -\frac{\delta^2_N N n}{r} \simeq (1 - 1/\gamma^2_h) \frac{4\pi G M m}{r}.$$  \hspace{1cm} (31)

This correction affects already the Newton’s limit and imposes the severe observational restriction on the model. Barring a fine tuning, we should require that $\gamma_h \simeq 1$, allowing in GR practically just the Schwarzschild black holes. In contrast, $\gamma_h$ in the metagravity may differ for the various objects. Thus, an analogous restriction for the graviscalar exchange can, in principle, be abandoned not excluding a priori the cosmic objects with $\gamma_h \neq 1$.

3 Conclusion

The unimodular metagravity is a viable extension of GR in the presence of a scalar field. The harmonic graviscalar black holes deserve further studying as an extension of the Schwarzschild black holes. Of particular interest are the pure graviscalar cosmic objects, the graviballs, with their counterpart in GR missing. Nevertheless, the peculiarity of the unimodular metagravity and graviscalar should expectedly manifest itself to the full extend in the nonharmonic black holes. Studying the latter ones is a crucial issue to ultimately treat the graviscalar as DM in the Universe.
References

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