Article

$q$-Generalized Linear Operator on Bounded Functions of Complex Order

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Abstract: This article presents a $q$-generalized linear operator in Geometric Function Theory (GFT) and investigates its application to classes of analytic bounded functions of complex order $S_q(c; M)$ and $C_q(c; M)$ where $0 < q < 1$, $0 \neq c \in \mathbb{C}$, and $M > \frac{1}{2}$. Integral inclusion of the classes related to the $q$-Bernardi operator is also proven.

Keywords: $q$-difference operator; subordinating factor sequence; bounded analytic functions of complex order; $q$-generalized linear operator

MSC: Primary 30C45; Secondary 30C50; 30H05

1. Introduction

Quantum calculus or $q$-calculus is attributed to the great mathematicians L. Euler and C. Jacobi, but it became popular when Albert Einstein used it in quantum mechanics in his paper [1] published in 1905. F.H. Jackson [2,3] introduced and studied the $q$-derivative and $q$-integral in a proper way. Later, quantum groups gave the geometrical aspects to $q$-calculus. It is pertinent to mention that $q$-calculus can be considered an extension of classical calculus discovered by I. Newton and G.W. Leibniz. In fact, the operators defined as:

$$d_h f(z) = \frac{f(z + h) - f(z)}{h}$$

and:

$$d_q f(z) = \frac{f(z) - f(qz)}{(1 - q)z}, \quad 0 < q < 1,$$

where $z \in \mathbb{C}$ and $h > 0$ are the $h$-derivative and $q$-derivative, respectively, where $h$ is Planck’s constant, are related as: $q = e^{i \hbar} = e^{2\pi i \tilde{\hbar}}$ where $\tilde{\hbar} = h/2\pi$. Srivastava [4] applied the concepts of $q$-calculus by using the basic (or $q$-) hypergeometric functions in Geometric Function Theory (GFT). Ismail [5] and Agarwal [6] introduced the class of $q$-starlike functions by using the $q$-derivative. The $q$-close-to-convex functions were defined in [7], and Sahoo and Sharma [8] obtained several interesting results for $q$-close-to-convex functions. Several convolution and fractional calculus $q$-operators were defined by the researchers, which were repositioned by Srivastava in [9]. Darus [10] defined a new differential operator called the $q$-generalized operator by using $q$-hypergeometric functions. Let $A$ be the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

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analytic in the open unit disc $E = \{ z : |z| < 1 \}$.

Let $f(z)$ be given by (1) and $g(z)$ defined as:

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of $f$ and $g$ is defined by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let $f, h$ be analytic functions. Then, $f$ is subordinate to $h$, written as $f \prec h$ or $f(z) \prec h(z)$, $z \in E$, if there exists a Schwartz function $\omega(z)$ analytic in $E$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$, such that $f(z) = h(\omega(z))$. If $h$ is univalent in $E$, then $f \prec h$, if and only if $f(0) = h(0)$ and $f(E) \subset h(E)$.

A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is a subordinating factor if, whenever $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $a_1 = 1$ is regular, univalent, and convex in $E$, we have $\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z)$, $z \in E$.

We recall some basic concepts from q-calculus that are used in our discussion and refer to [2,3,12] for more details.

A subset $B \subset \mathbb{C}$ is called q-geometric if $q z \in B$ whenever $z \in B$, and it contains all the geometric sequences $\{z^q\}_{0}^{\infty}$. In GFT, the q-derivative of $f(z)$ is defined as:

$$d_q f(z) = \frac{f(z) - f(qz)}{(1 - q)z}, \quad q \in (0,1), \quad (z \in B \setminus \{0\}),$$

and $d_q f(0) = f'(0)$. For a function $g(z) = z^k$, the q-derivative is:

$$d_q g(z) = [k] z^{k-1},$$

where $[k] = \frac{1 - q^k}{1 - q} = 1 + q + q^2 + \ldots + q^{k-1}$.

We note that as $q \to 1^-$, $d_q f(z) \to f'(z)$, which is the ordinary derivative. From (1), we deduce that:

$$d_q f(z) = 1 + \sum_{k=2}^{\infty} [k] a_k z^k.$$

Let $f(z)$ and $g(z)$ be defined on a q-geometric set $B$. Then, for complex numbers $a, b$, we have:

$$d_q(a f(z) \pm b g(z)) = a d_q f(z) \pm b d_q g(z).$$

$$d_q(f(z)g(z)) = f(qz)d_q g(z) + g(z)d_q f(z).$$

$$d_q \left( \frac{f(z)}{g(z)} \right) = \frac{g(z) d_q f(z) - f(z) d_q g(z)}{g(z)g(qz)}, \quad g(z)g(qz) \neq 0.$$

$$d_q (\log f(z)) = \frac{\ln q^{-1} d_q f(z)}{1 - q} f(z).$$

Jackson [2] introduced the q-integral of a function $f$, given by:

$$\int_{0}^{z} f(t)d_q t = z(1 - q) \sum_{k=0}^{\infty} q^k f(q^k z),$$

provided that the series converges.

For any non-negative integer $n$, the q-number shift factorial is defined as:

$$[n]! = \begin{cases} 1, & \text{if } n = 0, \\ [1][2] \ldots [n], & \text{if } n \neq 0. \end{cases}$$
Let $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$; the $q$-generalized Pochhammer symbol is defined as:

$$[\lambda]_n = [\lambda] [\lambda + 1] [\lambda + 2] \ldots [\lambda + n - 1].$$

The $q$-Gamma function is defined for $\lambda > 0$ as:

$$\Gamma_q(\lambda + 1) = [\lambda] \Gamma_q(\lambda) \quad \text{and} \quad \Gamma_q(1) = 1.$$

For complex parameters $a_i$ ($1 \leq i \leq l$), $b_j \neq 0$, $-1, -2, \ldots (1 \leq j \leq m)$ with $l \leq m + 1$, the basic $q$-hypergeometric function is defined as:

$$f_m(a_1, \ldots a_l; b_1, \ldots, b_m, z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_l)_k}{(q)_k (b_1)_k \ldots (b_m)_k} \left( (-1)^k q^{\binom{k}{2}} \right)^{1+m-l} z^k.$$  \hspace{1cm} (2)

with $\binom{n}{2} = \frac{n(n-1)}{2}$ and $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here, the $q$-shifted factorial is defined for $a \in \mathbb{C}$ as:

$$(a)_k = \begin{cases} (1 - a) (1 - aq) \ldots (1 - aq^{k-1}) & \text{if } k \in \mathbb{N}, \\ 1 & \text{if } k = 0. \end{cases}$$

Let $l = m + 1$, $a_1 = q^{l+1} (-1)$, $a_i = q \forall 2 \leq i \leq l$, and $b_j = q \forall 1 \leq j \leq m$, and by using the property $(q^a)_k = \Gamma_q(a + k) (1 - q)^k / \Gamma_q(a)$, from (2), we get the function,

$$F_{q,\lambda+1}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_q(\lambda + k)}{\Gamma_q(\lambda + 1)} \frac{[\lambda + 1]_k}{[k-1]!} z^k = z + \sum_{k=2}^{\infty} \frac{[\lambda + 1]_k}{[k-1]!} z^k, \ z \in E.$$ 

In [13], the $q$-Srivastava–Attiya convolution operator is defined as:

$$G_{q,a}^s(z) = z + \sum_{k=2}^{\infty} \left( \frac{[1 + a]}{[k + a]} \right)^s \frac{[\lambda + 1]_{k-1}}{[k-1]!} a_k z^k, \ z \in E,$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}^-; s \in \mathbb{C} \text{ when } |z| < 1; \text{ Re}(s) > 1 \text{ when } |z| = 1).$$

Using convolution, the operator $D_{q,a,\lambda}^s$ for $\lambda > -1$ is defined as:

$$D_{q,a,\lambda}^s f(z) = f_{q,a,\lambda}^s(z) \ast f(z) = z + \sum_{k=2}^{\infty} \left( \frac{[k + a]}{[1 + a]} \right)^s \frac{[\lambda + 1]_{k-1}}{[k-1]!} a_k z^k, \ z \in E,$$

where:

$$f_{q,a,\lambda}^s(z) = \left( G_{q,a}^s(z) \right)^{-1} \ast F_{q,\lambda+1}(z) = z + \sum_{k=2}^{\infty} \left( \frac{[k + a]}{[1 + a]} \right)^s \frac{[\lambda + 1]_{k-1}}{[k-1]!} z^k.$$ 

It is a convergent series with a radius of convergence of one. We observe that $D_{q,a,0,0}^0 f(z) = f(z)$ and $D_{q,a,0,0}^1 f(z) = zd_q f(z)$. The operator $D_{q,a,\lambda}^s$ reduces to known linear operators for different values of parameters $a, s$, and $\lambda$ as:

(i) If $q \to 1^-$, it reduces to the operator $D_{a,\lambda}^s$ discussed by Noor et al. in [14].
(ii) For $s = 0$, it is a $q$-Ruscheweyh differential operator [15].
(iii) If $s = -1$, $\lambda = 0$, and $q \to 1^-$, it is an Owa–Srivastava integral operator [16].
(iv) If $s \in \mathbb{N}_0$, $a = 1$, $\lambda = 0$, and $q \to 1^-$, it reduces to the generalized Srivastava–Attiya integral operator [17].
(v) If $s \in \mathbb{N}_0$, $a = 0$, $\lambda = 0$, it is a $q$-Salagean differential operator [18].
(vi) For $s, \lambda \in \mathbb{N}_0$, and $a = 0$, it is the operator defined in [19].
The following identities hold for the operator $D^{s}_{q,a,\lambda}f(z)$,

$$zd_q\left(D^{s}_{q,a,\lambda}f(z)\right) = \left(\frac{1+a}{q^a}\right)D^{s+1}_{q,a,\lambda}f(z) - \frac{a}{q^a}D^{s}_{q,a,\lambda}f(z)$$

$$zd_q(D^{s}_{q,a,\lambda}f(z)) = \left(\frac{1+\lambda}{q^\lambda}\right)D^{s}_{q,a,\lambda+1}f(z) - \frac{[\lambda]}{q^\lambda}D^{s}_{q,a,\lambda}f(z).$$

Let $P(q)$ be the class of functions of the form $p(z) = 1 + c_1z + c_2z^2 + ..., \text{ analytic in } E$, and satisfying:

$$|p(z) - \frac{1}{1-q}| \leq \frac{1}{1-q}, \quad (z \in E, q \in (0,1)).$$

It is known from [20] that $p \in P(q)$ implies $p(z) < \frac{1+z}{1-q}$. It follows immediately that $\text{Re} \, p(z) > 0$, $z \in E$.

The classes of bounded $q$-starlike functions $S_q(c,M)$ and bounded $q$-convex functions $C_q(c,M)$ of complex order $c$ were defined in [21], respectively, as:

$$S_q(c,M) = \left\{ f \in A : \left| \frac{c - 1 + zd_qf(z)}{zd_qf(z)} - M \right| < M \right\},$$

$$S_q(c,M) = \left\{ f \in A : \frac{zd_qf(z)}{f(z)} < \frac{1 + \{c(1+m) - m\}z}{1-mz} \right\},$$

$$\left( c \in \mathbb{C}^*; M > \frac{1}{2}, z \in E \right),$$

or equivalently,

$$S_q(c,M) = \left\{ f \in A : \frac{zd_qf(z)}{f(z)} < \frac{1 + \{c(1+m) - m\}z}{1-mz} \right\},$$

$$\left( c \in \mathbb{C}^*; m = 1 - \frac{1}{M}; M > \frac{1}{2} \right).$$

The class of bounded $q$-convex functions $C_q(c,M)$ of complex order $c$ is defined as:

$$C_q(c,M) = \left\{ f \in A : \left| \frac{c - 1 + \frac{d_q(zd_qf(z))}{d_qf(z)} - M \right| < M \right\},$$

$$\left( c \in \mathbb{C}^*; M > \frac{1}{2}, z \in E \right),$$

or equivalently,

$$C_q(c,M) = \left\{ f \in A : \frac{d_q(zd_qf(z))}{d_qf(z)} < \frac{1 + \{c(1+m) - m\}z}{1-mz} \right\},$$

$$\left( c \in \mathbb{C}^*; m = 1 - \frac{1}{M}; M > \frac{1}{2} \right).$$

Using the operator $D^{s}_{q,a,\lambda}f(z)$, we now define the following new classes $S_{q,a,s,\lambda}(c,M)$ and $C_{q,a,s,\lambda}(c,M)$ as:
Let \( A \) be complex numbers with \( \Re(A) \geq 0 \) and \( \Im(A) \geq 0 \). Let \( B \) be regular in \( D(0,1) \) and \( \beta \) be complex numbers with \( |\beta| < \frac{1}{4} \). We start the section with the necessary and sufficient condition for a function to be in the class \( A \). Let \( f \) be complex numbers with \( \Re(f(z)) > 0 \) and \( \Im(f(z)) > 0 \) in \( D(0,1) \). Let \( \eta, A, B \) with \( \eta = 0 \) discussed in [19].

A function \( f \in A \) is in the class \( S_{q,a,s,\lambda}(c, M) \) if and only if:

\[
A = \frac{zd_q(D_q^{s,a,\lambda}(f(z)))}{D_q^{s,a,\lambda}(f(z))} - 1 \quad \text{and} \quad B = -m.
\]

The class \( C_{q,a,s,\lambda}(c, M) \) is defined as:

\[
C_{q,a,s,\lambda}(c, M) = \left\{ f \in A : \frac{d_q(zd_q(D_q^{s,a,\lambda}(f(z))))}{d_q(D_q^{s,a,\lambda}(f(z)))} < \frac{1 + \{c(1 + m) - m\}z}{1 - mz}, z \in E \right\},
\]

\[
\left( 0 < q < 1, c \in \mathbb{C}^*; m = 1 - \frac{1}{M}; M > \frac{1}{2} \right).
\]

It is easy to see that \( f \in C_{q,a,s,\lambda}(c, M) \Leftrightarrow zd_q f \in S_{q,a,s,\lambda}(c, M) \). In order to develop results for the classes \( S_{q,a,s,\lambda}(c, M) \) and \( C_{q,a,s,\lambda}(c, M) \), we need the following:

**Lemma 1** ([27]). Let \( \beta \) and \( \gamma \) be complex numbers with \( \beta \neq 0 \), and let \( h(z) \) be regular in \( E \) with \( h(0) = 1 \) and \( \Re(\beta h(z) + \gamma) > 0 \). If \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) is analytic in \( E \), then \( p(z) + \frac{zd_q p(z)}{d_q p(z) + \gamma} < h(z) \Rightarrow p(z) < h(z) \).

**Lemma 2** ([11]). The sequence \( \{b_n\}_{n=1}^{\infty} \) is a subordinating factor sequence if and only if:

\[
\Re\left\{ 1 + 2 \sum_{k=1}^{\infty} b_k z^k \right\} > 0, \quad z \in E.
\]

2. Properties of Classes \( S_{q,a,s,\lambda}(c, M) \) and \( C_{q,a,s,\lambda}(c, M) \)

We start the section with the necessary and sufficient condition for a function to be in the class \( S_{q,a,s,\lambda}(c, M) \).
Theorem 1. Let \( f \in A \). Then, \( f \in S_{q,a,s,\lambda}(c,M) \) if and only if:

\[
\sum_{k=2}^{\infty} \left\{ [k] - 1 + |c(1+m) + m([k] - 1)| \right\} \left| \frac{[k+1]}{[k-1]} \right| \left| \frac{[k]}{[k+1]} \right| \left| a_k \right| < |c(1+m)|, \tag{6}
\]

where \( m = 1 - \frac{1}{M}, \) \( (M > \frac{1}{2}). \)

Proof. Let us assume first that Inequality (6) holds. To show \( f \in S_{q,a,s,\lambda}(c,M) \), we need to prove Inequality (5).

\[
\left| \frac{z(d(D_{q,a,s,\lambda}^z f(z)) - 1)}{A - B} \right| = \left| \frac{\sum_{k=2}^{\infty} \left( \frac{[k]}{[k+1]} \right)^{s} \frac{[k+1]}{[k-1]} (k - 1) a_k z^k}{A - B} \right| \leq \left| \frac{\sum_{k=2}^{\infty} \left( \frac{[k]}{[k+1]} \right)^{s} \frac{[k+1]}{[k-1]} (k - 1) a_k}{A - B} \right| \leq \left| \frac{\sum_{k=2}^{\infty} \left( \frac{[k+1]}{[k-1]} \right)^{s} \frac{[k+1]}{[k]} (k - 1) a_k}{A - B} \right| < 1.
\]

Hence, \( f \in S_{q,a,s,\lambda}(c,M) \) by using Inequality (6). Conversely, let \( f \in S_{q,a,s,\lambda}(c,M) \) be of the form (1), then:

\[
\left| \frac{z(d(D_{q,a,s,\lambda}^z f(z)) - 1)}{A - B} \right| = \left| \frac{\sum_{k=2}^{\infty} \left( \frac{[k]}{[k+1]} \right)^{s} \frac{[k+1]}{[k-1]} (k - 1) a_k z^k}{A - B} \right| \leq \left| \frac{\sum_{k=2}^{\infty} \left( \frac{[k+1]}{[k-1]} \right)^{s} \frac{[k+1]}{[k]} (k - 1) a_k}{A - B} \right| < 1.
\]

Since \( |\text{Re}| \leq |z| \), we have:

\[
\text{Re} \left[ \frac{\sum_{k=2}^{\infty} \left( \frac{[k]}{[k+1]} \right)^{s} \frac{[k+1]}{[k-1]} (k - 1) a_k z^k}{(A - B)z + \sum_{k=2}^{\infty} (A - B[k]) \left( \frac{[k]}{[k+1]} \right)^{s} \frac{[k+1]}{[k-1]} a_k z^k} \right] < 1.
\]

Now, we choose values of \( z \) on the real axis such that \( zd(D_{q,a,s,\lambda}^z f(z)) / D_{q,a,s,\lambda}^z f(z) \) is real. Letting \( z \to 1^{-} \) through real values, after some calculations, we obtain Inequality (6). \( \square \)

Remark 1. (i) If \( q \to 1^{-}, s \in \mathbb{N}_0, a = 0, \) and \( \lambda = 0, \) the above result reduces to the sufficient condition for \( f(z) \) to be in class \( H_q(c,M) \) \((c \in \mathbb{C}^{*}, M > \frac{1}{2}) \) discussed in [26]. (ii) If \( c = 1 - \alpha (\alpha \in [0,1]), m = 0, \lambda = 0, \) and \( q \to 1^{-}, \) the above result reduces to the sufficient condition for \( f(z) \) to be in class \( S_{q,a}^\alpha \) discussed in [28].

Theorem 2. Let \( f_i \in S_{q,a,s,\lambda}(c,M) \) having the form:

\[
f_i(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad \text{for } i = 1, 2, 3, \ldots, l.
\]

Then, \( F \in S_{q,a,s,\lambda}(c,M) \), where \( F(z) = \sum_{i=1}^{l} c_i f_i(z) \) with \( \sum_{i=1}^{l} c_i = 1. \)

Proof. From Theorem 1, we can write:
where however due to (7), we have:

\[ \sum_{k=2}^{\infty} \left\{ \frac{\{k-1+|b(1+m)+m([k]-1)|\}}{|b(1+m)|} \frac{|k+a|}{[k-1]!} \right\} \sum_{i=1}^{\lambda+1 \frac{1}{[k-1]!}} \left( \frac{|k+a|}{[k-1]!} \right)^s a_{k,i} < 1. \]  \hspace{1cm} (7)

Therefore:

\[ F(z) = \sum_{i=1}^{l} c_i \left( z + \sum_{k=2}^{\infty} a_{k,i} z^k \right) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{l} c_i a_{k,i} \right) z^k; \]

where however due to (7), we have:

\[ \sum_{k=2}^{\infty} \left\{ \frac{\{k-1+|b(1+m)+m([k]-1)|\}}{|b(1+m)|} \frac{|k+a|}{[k-1]!} \right\} \sum_{i=1}^{\lambda+1 \frac{1}{[k-1]!}} \left( \frac{|k+a|}{[k-1]!} \right)^s c_i \leq 1; \]

Therefore, \( F \in S_{\nu,\alpha,\lambda}(c, M). \) \( \square \)

**Theorem 3.** Let \( f_i \) with \( i = 1, 2, ..., v \) belong to the class \( S_{\nu,\alpha,\lambda}(c, M) \). The arithmetic mean \( h \) of \( f_i \) is given by:

\[ h(z) = \frac{1}{v} \sum_{i=1}^{v} f_i(z) \]  \hspace{1cm} (8)

belonging to class \( S_{\nu,\alpha,\lambda}(c, M) \).

**Proof.** From (8), we can write:

\[ h(z) = \frac{1}{v} \sum_{i=1}^{v} \left( z + \sum_{k=2}^{\infty} a_{k,i} z^k \right) = z + \sum_{k=2}^{\infty} \left( \frac{1}{v} \sum_{i=1}^{v} a_{k,i} \right) z^k. \]  \hspace{1cm} (9)

Since \( f_i \in S_{\nu,\alpha,\lambda}(c, M) \) for every \( i = 1, 2, ..., v \), using (6) and (9), we have:

\[ \sum_{k=2}^{\infty} \left\{ [k-1+|b(1+m)+m([k]-1)|] \right\} \frac{|k+a|}{[k-1]!} \left( \frac{|k+a|}{[k-1]!} \right)^s \left( \frac{1}{v} \sum_{i=1}^{v} a_{k,i} \right) \]

\[ = \frac{1}{v} \sum_{i=1}^{v} \left( \sum_{k=2}^{\infty} \left\{ [k-1+|b(1+m)+m([k]-1)|] \right\} \frac{|k+a|}{[k-1]!} \left( \frac{|k+a|}{[k-1]!} \right)^s a_{k,i} \right) \]

\[ \leq \frac{1}{v} \sum_{i=1}^{v} \left\{ [b(1+m)] \right\} \leq [b(1+m)], \]

and this completes the proof. \( \square \)

Now, we give the subordination relation for the functions in class \( S_{\nu,\alpha,\lambda}(c, M) \) by using the subordination theorem.

**Theorem 4.** Let \( m = 1 - \frac{1}{M} \) \( (M > \frac{1}{2}) \). Furthermore, \( c \neq 0 \) with \( Re(c) > \frac{-m}{2(1+m)} \) when \( m > 0 \) and \( Re(c) < \frac{-m}{2(1+m)} \) when \( m < 0 \) and \( \lambda \geq 0 \). If \( f \in S_{\nu,\alpha,\lambda}(c, M) \), then:
The inequality (11) follows by taking $g(z) = \sum_{k=1}^{\infty} a_k z^k$ in (10).

where $g(z)$ is a convex function in $E$, $C_{\lambda,k} = \frac{\lambda + 1}{k-1} B_{s,a}(k) = \left| \left( \frac{k+1}{k-1} \right) \right|^s$, and:

\[
\text{Re } f(z) > 1 - \frac{\{q + c(1 + m) + mq\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + c(1 + m) + mq\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} (1 + m) |c| \tag{11}
\]

The constant \( \frac{\{q + c(1 + m) + mq\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + c(1 + m) + mq\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\} \) is the best estimate.

**Proof.** Let $f(z) \in S_{q,a,\lambda}(c, M)$ and $g(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then:

\[
\begin{align*}
\frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} (f * g)(z) \\
= \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} \left( z + \sum_{k=2}^{\infty} a_k c_k z^k \right). 
\end{align*}
\]

Thus, (10) holds true if:

\[
\begin{align*}
\left\{ \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} a_k \left| z \right|^k \right\}_{k=1}^{\infty} 
\end{align*}
\]

is a subordinating factor sequence with $a_1 = 1$. From Lemma 2, it suffices to show:

\[
\text{Re } \left\{ 1 + \sum_{k=1}^{\infty} \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} a_k \left| z \right|^k \right\} > 0. \tag{14}
\]

Now, as \([k - 1 + |c(1 + m) + m(|k - 1|)]\) $C_{\lambda,k} B_{s,a}(k)$ is an increasing function of $k$ ($k \geq 2$), we have:

\[
\begin{align*}
\text{Re } \left\{ 1 + \sum_{k=1}^{\infty} \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} a_k \left| z \right|^k \right\} &= \text{Re } \left\{ 1 + \sum_{k=2}^{\infty} \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} a_k \left| z \right|^k \right\} \\
& \geq 1 - \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} r^k \\
\sum_{k=2}^{\infty} \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2)}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} a_k \left| z \right|^k \\
& \geq 1 - \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} r^k \\
& \geq 1 - \frac{\{q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|}{2\{(q + |c(1 + m) + mq|\} C_{\lambda,k} B_{s,a}(2) + |c(1 + m)|\}} r^k \\
& > 0. \quad (|z| = r < 1)
\end{align*}
\]

Hence, (14) holds true in $E$, and the subordination result (10) is affirmed by Theorem 4. The inequality (11) follows by taking $g(z) = \sum_{k=1}^{\infty} a_k z^k$ in (10).
Let us consider the function:

$$\phi(z) = z - \frac{|c(1 + m)|}{\{q + |c(1 + m) + mq|\} C_{q,a,s}(z) + |c(1 + m)|} z^2 \quad (z \in E)$$

which is a member of \(S_{q,a,s}(c, M)\). Then, by using (10), we have:

$$\frac{\{q + |c(1 + m) + mq|\} C_{q,a,s}(2)}{2 \{q + |c(1 + m) + mq|\} C_{q,a,s}(2) + |c(1 + m)|} \phi(z) \prec \frac{z}{1 - z}.$$ 

It is easily verified that:

$$\min \Re\left\{\frac{\{q + |c(1 + m) + mq|\} C_{q,a,s}(2)}{2 \{q + |c(1 + m) + mq|\} C_{q,a,s}(2) + |c(1 + m)|} \phi(z)\right\} = -\frac{1}{2} \quad (z \in E),$$

then the constant \(\frac{\{q + |c(1 + m) + mq|\} C_{q,a,s}(2)}{2 \{q + |c(1 + m) + mq|\} C_{q,a,s}(2) + |c(1 + m)|}\) cannot be replaced by a larger one.  

Remark 2. If \(s \in \mathbb{N}_0, a = 0, \lambda = 0, \) and \(q \to 1^-\), Theorem 4 reduces to the subordination result proven in [29].

Now, we discuss the inclusion results pertaining to classes \(S_{q,a,s}(c, M)\) and \(C_{q,a,s}(c, M)\) in reference to parameters \(s\) and \(\lambda\).

**Theorem 5.** For any complex number \(s\), \(S_{q,a,s+1}(c, M) \subset S_{q,a,s}(c, M)\) if \(\Re \left(\frac{1 + |c(1 + m) - m|}{1 - mz}\right) > \frac{1}{q^1(1-q)} \{1 - \cos(a_2 \ln q)\}\) where \(a = a_1 + ia_2\).

**Proof.** Let \(f \in S_{q,a,s+1}(c, M)\), then:

$$\frac{zd_q \left(D_{q,a,s}^{s+1} f(z)\right)}{D_{q,a,s}^{s+1} f(z)} \prec \frac{1 + \{c(1 + m) - m\}z}{1 - mz}, \quad (15)$$

Let:

$$h(z) = \frac{1 + \{c(1 + m) - m\}z}{1 - mz}$$

and:

$$r(z) = \frac{zd_q \left(D_{q,a,s}^{s} f(z)\right)}{D_{q,a,s}^{s} f(z)}.$$ 

We will show:

$$r(z) \prec h(z),$$

which would prove \(S_{q,a,s}(c, M) \subset S_{q,a,s+1}(c, M)\). From the identity relation (3), after a few calculations, we have:

$$\frac{zd_q \left(D_{q,a,s}^{s} f(z)\right)}{D_{q,a,s}^{s} f(z)} = \frac{[1 + a]}{q^a} D_{q,a,s}^{s+1} f(z) - \frac{[a]}{q^a}. \quad (16)$$

After some calculations, we have:
\[
\frac{D^{s+1}_{q,a,\lambda} f(z)}{D^s_{q,a,\lambda} f(z)} = \frac{1}{[1 + a]} \left\{ q^{\alpha} zq (D^s_{q,a,\lambda} f(z)) + [a] \right\} \\
= \frac{1}{[1 + a]} (q^{\alpha} r(z) + [a]).
\]

Applying logarithmic \( q \)-differentiation, we have:

\[
\frac{zd_q(D^{s+1}_{q,a,\lambda} f(z))}{D^{s+1}_{q,a,\lambda} f(z)} = r(z) + \frac{zd_q r(z)}{r(z) + q^{-a} [a]}.
\]  \hspace{1cm} (16)

From (15) and (16), we have:

\[
r(z) + \frac{z[d_q r(z)]}{r(z) + q^{-a} [a]} < \frac{1 + \{c(1 + m) - m\} z}{1 - mz}.
\]

If \( \text{Re}(h(z)) > \frac{1}{q^{1(1-q)}} \{1 - \cos(a_2 \text{ln} q)\} \), then from Lemma 1, it implies:

\[
r(z) < h(z),
\]

which implies \( f(z) \in S_{q,a,s,\lambda}(c, M) \). Therefore, \( S_{q,a,s,\lambda}(c, M) \subset S_{q,a,s+1,\lambda}(c, M) \). \( \square \)

**Theorem 6.** For any complex number \( s \), \( C_{q,a,s+1,\lambda}(c, M) \subset C_{q,a,s,\lambda}(c, M) \) if \( \text{Re}(\frac{1 + [c(1 + m) - m] z}{1 - mz}) > \frac{1 - q^{-1}}{1 - q} \) where \( a = a_1 + ia_2 \).

**Proof.** It is obvious from the fact \( f \in C_{q,a,s,\lambda}(c, M) \leftrightarrow zd_q f \in S_{q,a,s,\lambda}(c, M) \). \( \square \)

**Theorem 7.** For any complex number \( s \), \( S_{q,a,s,\lambda+1}(c, M) \subset S_{q,a,s,\lambda}(c, M) \) if \( \text{Re}(\frac{1 + [c(1 + m) - m] z}{1 - mz}) > \frac{1 - q^{-1}}{1 - q} \), \( \lambda > -1 \).

**Proof.** Let \( f \in S_{q,a,s,\lambda+1}(c, M) \), then:

\[
\frac{zd_q(D^s_{q,a,\lambda+1} f(z))}{D^s_{q,a,\lambda} f(z)} < \frac{1 + \{c(1 + m) - m\} z}{1 - mz}.
\]  \hspace{1cm} (17)

Consider:

\[
h(z) = \frac{1 + \{c(1 + m) - m\} z}{1 - mz}
\]

and:

\[
q(z) = \frac{zd_q(D^s_{q,a,\lambda} f(z))}{D^s_{q,a,\lambda} f(z)}.
\]

We will show:

\[
q(z) < h(z),
\]

which would conveniently prove \( S_{q,a,s,\lambda+1}(c, M) \subset S_{q,a,s,\lambda}(c, M) \). From the identity relation (4), after a few calculations, we have:

\[
\frac{zd_q(D^s_{q,a,\lambda} f(z))}{D^s_{q,a,\lambda} f(z)} = \frac{[1 + \lambda]}{q^{\lambda}} \frac{D^s_{q,a,\lambda+1} f(z)}{D^s_{q,a,\lambda} f(z)} - \frac{[\lambda]}{q^{\lambda}}.
\]
After some calculations, we have:

\[
\frac{D^s_{q,a,\lambda+1}f(z)}{D^s_{q,a,\lambda}f(z)} = \frac{1}{1+\lambda} \left\{ q^s z d_q (D^s_{q,a,\lambda}f(z)) + [\lambda] \right\} = \frac{1}{1+\lambda} \left\{ q^s q(z) + [\lambda] \right\}.
\]

Applying logarithmic \(q\)-differentiation, we have:

\[
\frac{zd_q(D^s_{q,a,\lambda+1}f(z))}{D^s_{q,a,\lambda+1}f(z)} = q(z) + \frac{zd_q q(z)}{q(z) + q^{-\lambda} [\lambda]}.
\]

From (17) and (18), we have:

\[
q(z) + \frac{z[d_q q(z)]}{q(z) + q^{-\lambda} [\lambda]} < \frac{1 + \{c(1+m) - m\}z}{1 - mz}.
\]

If \(\text{Re}(h(z)) > \frac{1 - q^{-\lambda}}{1-q}\) for any value of \(\lambda > -1\), so by Lemma 1, we have \(q(z) < h(z)\), which implies \(f(z) \in S_{q,a,\lambda,1}(c, M)\). Therefore, \(S_{q,a,\lambda,1}(c, M) \subset S_{q,a,\lambda}(c, M)\).

**Remark 3.** If we consider \(q \to 1^-\) with \(\text{Re} a \geq 0\), \(c = 1\), \(m = 1\) in Theorem 5 and \(\lambda \geq 0\), \(c = 1\), \(m = 1\) in Theorem 7, we obtain the special cases of the inclusion results, Theorems 2.4 and 2.5 in [19].

In [30], the \(q\)-Bernardi integral operator \(L_b f(z)\) is defined as:

\[
L_b f(z) = \frac{[1+b]}{z^b} \int_0^z t^{b-1} f(t) d_q t = z + \sum_{k=2}^{\infty} \left( \frac{[1+b]}{k+b} \right) a_k z^k, \quad b = 1, 2, 3, \ldots.
\]

Now, we apply the generalized operator \(D^s_{q,a,\lambda}\) on \(L_b f(z)\) as:

\[
D^s_{q,a,\lambda}(L_b f(z)) = z + \sum_{k=2}^{\infty} \left( \frac{[k+a]}{[1+a]} \right)^s \frac{[\lambda+1][k-1]}{[k-1]!} \left( \frac{[1+b]}{k+b} \right) a_k z^k.
\]

The identity relation of \(D^s_{q,a,\lambda}(L_b f(z))\) is given as:

\[
zd_q \left[ D^s_{q,a,\lambda} \{ L_b f(z) \} \right] = \left( \frac{[1+b]}{q^b} \right) D^s_{q,a,\lambda} f(z) - \frac{[b]}{q^b} D^s_{q,a,\lambda} \{ L_b f(z) \}.
\]

The following theorems are the integral inclusions of the classes \(S_{q,a,s,\lambda}(c, M)\) and \(C_{q,a,s,\lambda}(c, M)\) with respect to the \(q\)-Bernardi integral operator.

**Theorem 8.** If \(f(z) \in S_{q,a,s,\lambda}(c, M)\) then \(L_b f(z) \in S_{q,a,s,\lambda}(c, M)\) if \(\text{Re} \left( \frac{1 + [c(1+m) - m]z}{1 - mz} \right) > \frac{1 - q^{-\lambda}}{1-q}\) for any complex number \(s\).

**Proof.** Let \(g(z) \in S_{q,a,s,\lambda}(c, M)\), then:

\[
zd_q \left( D^s_{q,a,\lambda} g(z) \right) < \frac{1 + \{c(1+m) - m\}z}{1 - mz}.
\]

Consider:
\[ h(z) = \frac{1 + \{c(1 + m) - m\}z}{1 - mz} \]

and:

\[ u(z) = \frac{zd_{q}(D_{q,a,\lambda}^{s}L_{b}g(z))}{D_{q,a,\lambda}^{s}L_{b}g(z)}. \]

We will show:

\[ u(z) \prec h(z), \]

which would prove \( L_{b}g(z) \in S_{q,a,s,\lambda}(c, M). \) From the identity relation (19), after some calculations, we have:

\[ \frac{zd_{q}(D_{q,a,\lambda}^{s}L_{b}g(z))}{D_{q,a,\lambda}^{s}L_{b}g(z)} = \left( \frac{[1 + b]}{q^{b}} \right) \frac{D_{q,a,\lambda}^{s}g(z)}{(D_{q,a,\lambda}^{s}L_{b}g(z))} - \frac{[b]}{q^{b}}. \]

After some calculations, we have:

\[ \frac{D_{q,a,\lambda}^{s}g(z)}{D_{q,a,\lambda}^{s}L_{b}g(z)} = \frac{1}{[1 + b]} \left[ q^{b}zd_{q}(D_{q,a,\lambda}^{s}L_{b}g(z)) \right] + [b] \]

Applying logarithmic \( q \)-differentiation, we have:

\[ \frac{zd_{q}(D_{q,a,\lambda}^{s}g(z))}{D_{q,a,\lambda}^{s}g(z)} = u(z) + \frac{z[d_{q}u(z)]}{u(z) + q^{-b}[b]} \quad (21) \]

From (20) and (21), we have:

\[ u(z) + \frac{z[d_{q}u(z)]}{u(z) + q^{-b}[b]} < \frac{1 + \{c(1 + m) - m\}z}{1 - mz} \]

If \( \text{Re}(h(z)) > \frac{1-q^{-b}}{1-q} \), so by Lemma 1, we have \( u(z) \prec h(z) \), which implies \( L_{b}g(z) \in S_{q,a,s,\lambda}(c, M). \) \( \square \)

**Theorem 9.** If \( f(z) \in C_{q,a,s,\lambda}(c, M) \), then \( L_{b}f(z) \in C_{q,a,s,\lambda}(c, M) \) for any complex number \( s \).

**Proof.** It is an immediate consequence of the fact \( C_{q,a,s,\lambda}(c, M) \Leftrightarrow zd_{q}f \in S_{q,a,s,\lambda}(c, M). \) \( \square \)

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