Nonstandard micro-inertia terms in the relaxed micromorphic model: well-posedness for dynamics

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Abstract

We study the existence of the solution of some problems arising from the modelling of elastic materials using generalized theories of continua. In view of some evidence from physics of metamaterials we focus our effort on two nonstandard relaxed micromorphic models which consider some novel micro-inertia terms. These novel micro-inertia terms are needed to better capture the band-gap response. The existence proof is based on the Banach fixed point theorem.

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1 Introduction

1.1 Preliminaries and motivation

The micromorphic theory\textsuperscript{[15 37]} is a generalised theory of continua which, in order to describe both macro- and micro-deformation, considers that any point of the body is endowed with two fields functions, i.e. a vector field $u : \Omega \times [0, T] \to \mathbb{R}^3$ for the displacement of the macroscopic material points, and a tensor field $P : \Omega \times [0, T] \to \mathbb{R}^{3 \times 3}$ describing the micro-deformation (micro-distortion) of the substructure of the material. This theory was introduced 58 years before and an important part of its reason to be was to have a model which is capable to give a better agreement between the analytical and numerical results regarding wave propagation indicated by the model and those obtained in experiments. When the classical theories of linear elasticity are

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\end{footnotesize}
used, no dispersive effects can be predicted. However, the elastic wave propagation through heterogeneous media is generally dispersive, i.e. each wave number travels with a distinct wave speed. Some other possible applications of the micromorphic model are described in [16].

It is well known that the Cosserat theory (the case when the microdistortion $P$ is antisymmetric) is a particular case of the micromorphic theory. The Cosserat theory was introduced in 1909 [11] and it was not really taken into account until the micromorphic theory was introduced. However, the micromorphic theory was developed in order to be able to capture some other effects, beside those already given by the Cosserat model [40, 29, 45, 38, 39] and by another particular theory called the microstretch theory. The biggest shortcoming of the classical micromorphic theory is that it involves a very big number of constitutive coefficients, which have to be determined if one really want to use it in applications. In order to avoid this shortcoming, one solution is to adapt the micromorphic theory to the phenomena we want to model with its help. This is done for instance in the works [41, 21] in which $P$ is assumed to be symmetric.

In a recent paper [43], we have introduced a simplified micromorphic theory which we called the relaxed micromorphic model. This model completes the model which was proposed in [10] and which was critically discussed in [53]. Since Claus and Eringen have introduced such a model having in mind an application to the dislocation theory, the main criticism was about the non-symmetry of the Cauchy-stress tensor. At that time it was not clear that such a theory may have a chance to be well-posed from a mathematical point of view, even when the Cauchy-stress tensor is nonsymmetric. However, due to some new Poincaré-Korn type inequalities [4, 48], the relaxed micromorphic model is well posed also when the Cauchy-stress tensor is symmetric [23]. We have explained in [42] why the relaxed model is a particular case of the classical Mindlin-Eringen model and how the energy of the relaxed model may be obtained taking some suitable form of the constitutive coefficients in the classical theory. The relaxed micromorphic theory is still general enough to incorporate the most used particular theories of the classical micromorphic theory, see [43] and Figure 1.

![Figure 1: An illustration of the relations between different theories of materials with microstructure: the classical micromorphic model [18]; the linear microstretch model [16]; the linear isotropic Cosserat model [11]; the microstrain model [21]; the linear isotropic microvoids model [49]. Comparison with other models are discussed in [43].](image)

We have observed that considering a wave ansatz in the partial differential equations arising in the relaxed micromorphic model, some band gaps arise in the diagrams describing the dispersion curves. This means that there exists an interval of values of the wave frequencies for which no wave propagations may occur [35, 34, 36]. Such an interesting phenomena is intended to be obtained when metamaterials are designed and may not be captured by the classical micromorphic approach [13, 43]. These metamaterials are able to “absorb” or even “bend” elastic waves with no energetic cost (see e.g. [19, 20]). They are conceived arranging small components into periodic or quasi-periodic patterns in such a way that the resulting structure possesses new unimaginable properties with respect to the original material. This insight has opened a way to obtain new generalized continuum models allowing to describe the behavior of metastructures in the simplified framework.
of continuum mechanics with homogenized material behaviour. Moreover, due to the relation between macro
and micro parameters of the considered samples obtained in \[3\], the constitutive parameters of our model may
be identified on real metamaterials \[12\] even for anisotropic materials, opening the way to the efficient design
and realization of metastructures.

Besides the interesting property of the relaxed micromorphic theory to model certain band-gap metamateri-
als, the mathematical results which may be obtained are also surprising. We have to remark that all the results
which assume that the energy is positive definite in the relaxed model are in fact valid without assuming that
the corresponding energy is positive definite in the classical Mindlin-Eringen model. Therefore, for the relaxed
model we have obtained \[23\] extensions of the results established by Sóos \[12\], by Hlaváček \[26\], by Ieşan and
Nappa \[27\] and by Ieşan \[28\] which consider the positive definitness in the Mindlin-Eringen model. We mention
here also the results given by Picard et al. \[50\] which have proved an existence result in the linear theory
of micromorphic elastic solids using an interesting mother/descendant mechanism regarding various models.

Going back to the applications of the relaxed micromorphic model in the modelling of wave propagation
in metamaterials, we observed that for large wavenumbers the dispersion curves obtained from our model
have a different behaviour as those which are predicted by the plane strain Bloch-Floquet analysis. However,
by considering some nonstandard micro-inertia terms in the form of the kinetic energy the fitting between
the dispersion curves given by the Bloch-Floquet analysis and by the new model is considerably improved
\[32,31,12\]. The idea of considering some new terms in the form of the kinetic energy is not entirely new, it
may be in concordance to the linear theory of nonlocal elasticity introduced in \[15\] and derived from a more
general theory developed by Eringen and Edelen \[17\]. We would like to mention that the gradient elasticity can
also be used to capture the dispersive behaviour of waves that propagate through a heterogeneous material and
that the presence of micro-inertia terms was used, before the work of Eringen \[15\], also by Mindlin \[37\] page
68. The difference is that Mindlin used micro-inertia terms only in the context of gradients elasticity, while
Eringen used them without considering higher order derivatives of the displacement, i.e. the equations which
are also considered in \[2\]. Various formats of gradient elasticity with inertia gradients have been considered in
many works \[51,54,9,14,5,25\]. But there does not seem to exist a rigorous mathematical treatment of these
models.

The main aim of this paper is to present relaxed models including some nonstandard micro-inertia terms
and to prove that they are well-posed. Our analysis covers also the case when the stress tensor is symmetric
since, in our mathematical approach, the presence of the constitutive term which makes it non-symmetric is
not essential. We may say that for the mathematical analysis of the model the presence of the Cosserat couple
modulus is redundant. However, in the study of meta-materials this coefficient is essential when one wants to
model the band-gap phenomena \[35\]. The model considered in the present paper is a little different from that
studied in \[12\], but the mathematical results obtained here lead us to the expectation that this model will
further improve the results from \[12\].

The plan of the paper is now the following. In Section 2 we introduce the initial-boundary value problem
arising in the model when non-standard inertia terms are present and we prove the existence of the solution.
The used approach is not common in elastodynamics, it is based on the Banach-fixed point theorem and it was
suggested by an approach from thermo-visco-plasticity \[8\]. We would like to mention that the usual approaches
based on the semigroup theory of linear operators or on the Galerkin method seem to not be able to lead to
an existence result without assuming a priori some compatibilities between the domains of the two operators
involved in our partial differential equations. We avoid these assumptions, since they are satisfied only by some
operators, which from a mechanical point of view lead to a very particular model, which is not capable to
describe the behaviour of waves in meta-materials, i.e. it does not correspond to the main raison d’être of our
model. In the last section, in view of the remarks from the previous paragraph, we introduce a simplified model
and we prove that the corresponding initial-boundary value problem is still well-posed.
2 Relaxed micromorphic models including micro-inertia terms

In view of the motivations given in the Introduction, we introduce a new model which takes into account the influence of some nonstandard micro-inertia terms on the behaviour of the solution of the obtained initial-boundary value problem.

We consider a micromorphic continuum which occupies a bounded domain $\Omega$ and having a piecewise smooth surface $\partial \Omega$. The motion of the body is referred to a fixed system of rectangular Cartesian axes $Ox_i, \ (i = 1, 2, 3)$. Throughout this paper (if we do not specify otherwise) Latin subscripts take the values $1, 2, 3$.

Here, we have denoted by $\mathbb{R}^{3 \times 3}$ the set of real $3 \times 3$ matrices. For all $X \in \mathbb{R}^{3 \times 3}$ we set $\text{sym} X = \frac{1}{2}(X + X^T)$ and $\text{skew} X = \frac{1}{2}(X - X^T)$. Typical conventions for differential operations are implied such as comma followed by a subscript to denote the partial derivative with respect to the corresponding cartesian coordinate. For $a, b \in \mathbb{R}^3$ we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|_{\mathbb{R}^3} = \langle a, a \rangle_{\mathbb{R}^3}$. The standard Euclidean scalar product on $\mathbb{R}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3 \times 3}} = \text{tr}(XY^T)$, and thus the Frobenius tensor norm is $\|X\|_{\mathbb{R}^{3 \times 3}} = \langle X, X \rangle_{\mathbb{R}^{3 \times 3}}$. In the following we omit the index $\mathbb{R}^{3 \times 3}$. The identity tensor on $\mathbb{R}^{3 \times 3}$ will be denoted by $I$, so that $\text{tr}(X) = \langle X, I \rangle$. We adopt the usual abbreviations of Lie-algebra theory, i.e., $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3 \times 3} | X^T = -X \}$ is the Lie-algebra of skew symmetric tensors and $\text{Sym}(3)$ denote the set of symmetric tensors.

By $C_{0}^{\infty}(\Omega)$ we denote the set of smooth functions with compact support in $\Omega$. All the usual Lebesgue spaces of square integrable functions, vector or tensor fields on $\Omega$ with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3 \times 3}$, respectively will be generically denoted by $L^2(\Omega)$. Moreover, we use the standard Sobolev spaces $[1] [30] [24]$.

$$
H^1(\Omega) = \{u \in L^2(\Omega) | \text{grad } u \in L^2(\Omega)\}, \quad \|u\|^2_{H^1(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|\text{grad } u\|^2_{L^2(\Omega)},
$$
$$
H(\text{curl}; \Omega) = \{v \in L^2(\Omega) | \text{curl } v \in L^2(\Omega)\}, \quad \|v\|^2_{H(\text{curl}; \Omega)} := \|v\|^2_{L^2(\Omega)} + \|\text{curl } v\|^2_{L^2(\Omega)},
$$
of functions $u$ or vector fields $v$, respectively. Furthermore, we introduce their closed subspaces $H^1_0(\Omega)$, and $H_0(\text{curl}; \Omega)$ as the closure with respect to the associated graph norms of $C_{0}^{\infty}(\Omega)$. Roughly speaking, $H^1_0(\Omega)$ is the subspace of functions $u \in H^1(\Omega)$ which are zero on $\partial \Omega$, while $H_0(\text{curl}; \Omega)$ is the subspace of vectors $v \in H(\text{curl}; \Omega)$ which are normal at $\partial \Omega$ (see $[13] [16] [17]$). For vector fields $v$ with components in $H^1(\Omega)$ and tensor fields $P$ with rows in $H(\text{curl}; \Omega)$, we define

$$
v = (v_1, v_2, v_3)^T, \quad v_i \in H^1(\Omega), \quad P = (P_1^T, P_2^T, P_3^T)^T, \quad P_i \in H(\text{curl}; \Omega),
$$

$$
\nabla v := ((\text{grad } v_1)^T, (\text{grad } v_2)^T, (\text{grad } v_3)^T)^T, \quad \text{Curl } P := ((\text{curl } P_1)^T, (\text{curl } P_2)^T, (\text{curl } P_3)^T)^T.
$$

We note that $v$ is a vector field, whereas $P$, $\text{Curl } P$ and $\nabla v$ are second order tensor fields. The corresponding Sobolev spaces will be denoted by $H^1(\Omega)$ and $H(\text{curl}; \Omega)$, and $H^1_0(\Omega)$ and $H_0(\text{curl}; \Omega)$, respectively.

We recall that for a fourth order tensor $\mathbb{C}$ and $X \in \mathbb{R}^{3 \times 3}$, we have $\mathbb{C} X \in \mathbb{R}^{3 \times 3}$ with the components $(\mathbb{C} X)_{ij} = C_{ijkl} X_{kl}$, while for a sixth order tensor $\mathbb{L}$ we consider $\mathbb{L}_i Z \in \mathbb{R}^{3 \times 3 \times 3}$ for all $Z \in \mathbb{R}^{3 \times 3 \times 3}$, $(\mathbb{L}_i Z)_{ijk} = L_{ijklmn} Z_{mn}$, where Einstein’s summation rule is used.

2.1 Description of the mechanical model

The mechanical model is formulated in the variational context. This means that we consider an action functional on an appropriate function-space. The space of configurations of the problem is

$$
\mathcal{Q} := \{(u, P) \in C^1([\Omega \times [0, T], \mathbb{R}^3]) \times C^1([\Omega \times [0, T], \mathbb{R}^{3 \times 3}) : (u, P) \text{ verifies conditions } (B_1) \text{ and } (B_2)\}
$$

where

- $(B_1)$ are the boundary conditions $u(x, t) = \varphi(x, t) \text{ and } P_i(x, t) \times n = \psi_i(x, t), \ i = 1, 2, 3, \ (x, t) \in \partial \Omega \times [0, T]$, where $n$ is the unit outward normal vector on $\partial \Omega \times [0, T], P_i, \ i = 1, 2, 3$ are the rows of $P$ and $\varphi, \psi_i$ are prescribed functions and $[0, T]$ is the time interval;

- $(B_2)$ are the initial conditions $u|_{t=0} = u_0, u_t|_{t=0} = u_0, P|_{t=0} = P_0, P_t|_{t=0} = P_0$ in $\Omega$, where $u_0(x), u_0(x), P_0(x), P_0(x)$ are prescribed functions.
The action functional $\mathcal{A} : Q \rightarrow \mathbb{R}$, is the sum of the internal and external action functionals $\mathcal{A}^{\text{int}}, \mathcal{A}^{\text{ext}} : Q \rightarrow \mathbb{R}$ defined as follows

$$\mathcal{A}^{\text{int}} [(u, P)] := \int_0^T \int_{\Omega} \mathcal{L} (u_t, P_t, \nabla u_t, \nabla u, P, \nabla P) \, dv \, dt,$$

$$\mathcal{A}^{\text{ext}} [(u, P)] := \int_0^T \int_{\Omega} ((f, u) + (M, P)) \, dv \, dt,$$

where $\mathcal{L}$ is the Lagrangian density of the system and $f, M$ are the body force and double body force. We recommend also the paper by Germain [22] for some explanations about the physical significance of the involved quantities. In order to find the stationary points of the action functional, we have to calculate its first variation:

$$\delta \mathcal{A} = \delta \mathcal{A}^{\text{int}} = \delta \int_0^T \int_{\Omega} \mathcal{L} (u_t, P_t, \nabla u_t, \nabla u, P, \nabla P) \, dv \, dt.$$

For the Lagrangian energy density we assume the standard split in kinetic minus potential energy density:

$$\mathcal{L} (u_t, P_t, \nabla u_t, \nabla u, P, \nabla P) = J (u_t, P_t, \nabla u_t, \nabla u, P, \nabla P) - W (\nabla u, P, \nabla P),$$

In general anisotropic linear elastic micromorphic homogeneous media, we consider that the kinetic energy and the potential energy density have the following expressions

$$J (u_t, P_t, \nabla u_t, \nabla u, P, \nabla P) = \frac{1}{2} \langle \rho u_t, u_t \rangle + \frac{1}{2} \langle J P_t, P_t \rangle$$

$$+ \frac{1}{2} \langle \tilde{C}_c \text{sym} (\nabla u_t - P_t), \text{sym} (\nabla u_t - P_t) \rangle + \frac{1}{2} \langle \tilde{C}_c \text{skew} (\nabla u_t - P_t), \text{skew} (\nabla u - P) \rangle$$

$$+ \frac{1}{2} \langle \tilde{C}_\text{micro} \text{sym} P_t, \text{sym} P_t \rangle + \mu \frac{L_c^2}{2} \langle \tilde{L}_\text{aniso} \text{Curl} P_t, \text{Curl} P_t \rangle,$$

$$W (\nabla u, P, \nabla P) = \frac{1}{2} \langle C_c \text{sym} (\nabla u - P), \text{sym} (\nabla u - P) \rangle$$

$$+ \frac{1}{2} \langle C_c \text{skew} (\nabla u - P), \text{skew} (\nabla u - P) \rangle$$

$$+ \frac{1}{2} \langle \tilde{C}_\text{micro} \text{sym} P, \text{sym} P \rangle + \frac{1}{2} \langle \tilde{C}_c \text{skew} (\nabla u - P), \text{skew} (\nabla u - P) \rangle$$

$$+ \frac{1}{2} \langle \tilde{L}_\text{aniso} \text{Curl} P, \text{Curl} P \rangle,$$

where

- $\tilde{C}_c, \tilde{C}_\text{micro} : \text{Sym}(3) \rightarrow \text{Sym}(3)$ are the dimensionless 4th order elasticity tensors with 21 independent components,
- $\tilde{C}_c, \tilde{C}_\text{micro} : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3)$ are dimensionless 4th order tensors, with 6 independent components,
- $\tilde{L}_\text{aniso} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ are dimensionless 4th order tensors, with almost 45 independent components,

the positive constants $\rho, J > 0$ are the macro-inertia and micro-inertia density, $L_c \geq 0$ is the characteristic length of the relaxed micromorphic model and $\mu \geq 0$ is a parameter used for dimension compatibility of the involved terms.

In the form of our energies we omit for simplicity the mixed terms for some reasons. One reason is that as long as there exist no clear mechanical interpretation of the influence of these terms on the process we want to model, we decided to keep the formulation as simple as possible. If, from some practical problems, the presence of these terms is requested, then they may be included. Another motivation, and the most important for the reason of being of the present paper, is that our mathematical analysis can be extended in a straightforward manner to the case when the mixed terms are also present in the total energy.

We remark that the curvature dependence is reduced to a dependence only on the micro-dislocation tensor $\alpha := - \text{Curl} P \in \mathbb{R}^{3 \times 3}$ instead of $\gamma = \nabla P \in \mathbb{R}^{27} = \mathbb{R}^{3 \times 3 \times 3}$. Doing so, the first main advantage of the relaxed
micromorphic model is that the number of constitutive coefficients is drastically reduced. A second strong point of the relaxed model is that it was possible to show that in the limit case $L_c \to 0$ (which corresponds to considering very large specimens of a microstructured metamaterial) the meso- and micro-coefficients of the relaxed model can be put in direct relation with the macroscopic stiffness of the medium via a fundamental homogenization formula, in contrast to the Eringen-Mindlin theory [13, 37], where it is not possible to obtain this kind of results, see [3, 44]. Another important aspect is the following: in the relaxed micromorphic theory it is possible to prove the existence and uniqueness of the solution assuming that $\hat{L}$ is only semi-positive definite. In order to explain this fact, we consider the linear operator $\hat{L} : \mathbb{R}^{3 \times 3 \times 3} \to \mathbb{R}^{3 \times 3}$, defined as follows

$$
\mu \frac{L_c^2}{2} \langle \ll_{\text{aniso}} \text{Curl } P, \text{Curl } P \rangle = \langle \hat{L} \nabla P, \nabla P \rangle,
$$

for all $P \in H_0(\text{Curl}; \Omega)$. In particular, we have that $\ll_{\text{aniso}}$ acting on linear subspaces of $\mathfrak{gl}(3) \cong \mathbb{R}^{3 \times 3}$, defines only a positive semi-definite energy in terms of $\nabla P$. In conclusion, an existence result based on the assumption that $\ll_{\text{aniso}}$ is positive definite covers a particular situation when $\hat{L}$ from the classical Eringen-Mindlin micromorphic theory is semi-positive definite. Moreover, the case $\mathsf{C}_c = 0$, i.e. the situation in which the Cauchy-stress tensor is symmetric, or the situation when $\mathsf{C}_c$ is only semi-positive definite are also covered by our analysis.

In the end of this short comparison, we point out another difference between these two approaches. The variational setting allows to prescribe tangential boundary conditions, i.e. $P_i(x,t) \times n(x) = 0$, $i = 1, 2, 3$, $(x,t) \in \partial \Omega \times [0,T]$. For the rest of the paper we assume that the constitutive coefficients $\mathsf{C}_c, \mathsf{C}_{\text{micro}}, \mathsf{C}_c, L_c$ are constant and they have the following symmetries

$$(\mathsf{C}_c)_{ijrs} = (\mathsf{C}_c)_{rsij}, \quad (\mathsf{C}_{\text{micro}})_{ijrs} = (\mathsf{C}_{\text{micro}})_{rsij}, \quad (\mathsf{C}_c)_{ijrs} = -(\mathsf{C}_c)_{jirs} = (\mathsf{C}_c)_{rsij}, \quad (\ll_{\text{aniso}})_{ijrs} = (\ll_{\text{aniso}})_{rsij}.$$  

(2.1)

We find, after considering the first variation of the action functional $\mathcal{A}$, that the general equations of the relaxed micromorphic model including the nonstandard inertia terms are

balance of forces :

$$\rho \dddot{u} - \text{Div} [\tilde{\mathsf{C}}_c, \text{sym} (\nabla \ddot{u} - P_{tt}) + \tilde{\mathsf{C}}_c, \text{skew} (\nabla \dot{u} - P_{tt})] = \text{Div} [\tilde{\mathsf{C}}_c, \text{sym} (\nabla u - P) + \mathsf{C}_c, \text{skew} (\nabla u - P)] + f,$$

balance of moment stresses :

$$J \dddot{P}_{tt} + \mu L_c^2 \text{Curl} \ll_{\text{aniso}} \text{Curl } P_{tt} - \tilde{\mathsf{C}}_c, \text{sym} (\nabla \ddot{u} - P_{tt}) - \tilde{\mathsf{C}}_c, \text{skew} (\nabla \dot{u} - P_{tt}) + \tilde{\mathsf{C}}_{\text{micro}}, \text{sym} P_{tt} = - \mu L_c^2 \text{Curl} \ll_{\text{aniso}} \text{Curl } P + \mathsf{C}_c, \text{sym} (\nabla u - P) + \mathsf{C}_c, \text{skew} (\nabla u - P) - \mathsf{C}_{\text{micro}}, \text{sym} P + M.$$  

To the above system of partial differential equations, we adjoin the boundary conditions

$$u(x,t) = 0,$$

and the tangential condition $P_i(x,t) \times n(x) = 0$, $i = 1, 2, 3$, $(x,t) \in \partial \Omega \times [0,T]$,  

(2.3)

where $P_i, i = 1, 2, 3$ are the rows of $P$ and the initial conditions

$$u(x,0) = u_0(x), \quad u_{tt}(x,0) = u_{tt_0}(x), \quad P(x,0) = P_0(x), \quad P_{tt}(x,0) = P_{tt_0}(x), \quad x \in \Omega,$$  

(2.4)

where $u_0, u_{tt_0}, P_0$ and $P_{tt_0}$ are prescribed functions.

We assume that the fourth order elasticity tensors $\mathsf{C}_c, \mathsf{C}_{\text{micro}}$ and $\ll_{\text{aniso}}$ are positive definite. Then, there are positive numbers $\mu_e^M, \mu_e^m > 0$ (the maximum and minimum elastic moduli for $\mathsf{C}_c$), $\mu_c^M, \mu_c^m > 0$ (the maximum and minimum elastic moduli for $\mathsf{C}_c$), $L_c^M, L_c^m > 0$ (the maximum and minimum moduli for $\ll_{\text{aniso}}$)

$$
\mu_e^m \|X\|^2 \leq \langle \mathsf{C}_c \cdot X, X \rangle \leq \mu_e^M \|X\|^2 \quad \text{for all } X \in \text{Sym}(3),
$$

$$
L_c^m \|X\|^2 \leq \langle \ll_{\text{aniso}} \cdot X, X \rangle \leq L_c^M \|X\|^2 \quad \text{for all } X \in \mathbb{R}^{3 \times 3},
$$

$$
\mu_{\text{micro}}^m \|X\|^2 \leq \langle \mathsf{C}_{\text{micro}} \cdot X, X \rangle \leq \mu_{\text{micro}}^M \|X\|^2 \quad \text{for all } X \in \text{Sym}(3).
$$

(2.5)
We introduce the bilinear forms in the model, while the final form of the equations is that which confirms the fact that mathematically the equations which satisfy
\[
\begin{align*}
\langle \mathbb{C}_c \cdot X, X \rangle \leq \mu_c^M \|X\|^2 & \quad \text{for all } X \in \mathfrak{so}(3). 
\end{align*}
\] (2.6)
Let us remark that, in our assumptions the constitutive tensor \( \mathbb{C}_c \) is either semi-positive definite, or it vanishes.

The same conditions are imposed on the constitutive coefficients \( \tilde{\mathbb{C}}_c, \mathbb{C}_c, \mathbb{C}_{micro}, \mathbb{C}_{aniso} \), respectively.

In the next subsection, we will prove that the above model is well-posed from a mathematical point of view.

As always in modelling, a first draft form of the equations is suggested by the effects which we intend to capture in the model, while the final form of the equations is that which confirms the fact that mathematically the model leads to a solution as suggested by practice, i.e. there exists a solution and it is unique.

### 2.2 Existence of the solution

We introduce the bilinear forms \( \mathcal{W}_1, \mathcal{W}_2 : (H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)) \times (H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)) \rightarrow \mathbb{R} \)
\[
\begin{align*}
\mathcal{W}_1((u, P), (\varphi, \Phi)) &= \int_{\Omega} \left( \rho \langle u, \varphi \rangle + J \langle P, \Phi \rangle + \langle \mathbb{C}_c \cdot \text{sym}(\nabla u - P), \text{sym}(\nabla \varphi - \Phi) \rangle, \\
& \quad + \langle \tilde{\mathbb{C}}_c \cdot \text{skew}(\nabla u - P), \text{skew}(\nabla \varphi - \Phi) \rangle + \langle \mathbb{C}_{micro} \cdot \text{sym} P, \text{sym} \Phi \rangle \\
& \quad + \mu L_c^2 \langle \tilde{l}_{aniso}, \text{Curl} P, \text{Curl} \Phi \rangle \right) dv, \\
\mathcal{W}_2((u, P), (\varphi, \Phi)) &= \int_{\Omega} \left( \langle \mathbb{C}_c \cdot \text{sym}(\nabla u - P), \text{sym}(\nabla \varphi - \Phi) \rangle + \langle \tilde{\mathbb{C}}_c \cdot \text{skew}(\nabla u - P), \text{skew}(\nabla \varphi - \Phi) \rangle \\
& \quad + \langle \mathbb{C}_{micro} \cdot \text{sym} P, \text{sym} \Phi \rangle + \mu L_c^2 \langle \tilde{l}_{aniso}, \text{Curl} P, \text{Curl} \Phi \rangle \right) dv
\end{align*}
\] (2.7)
and for each \((f, M) \in H^{-1}(\Omega) \times (H_0(\text{Curl}; \Omega))^*\) we consider the linear operator \( l^{(f,M)} \in H^{-1}(\Omega) \times (H_0(\text{Curl}; \Omega))^* \) defined by
\[
l^{(f,M)} : H^1_0(\Omega) \times H_0(\text{Curl}; \Omega) \rightarrow \mathbb{R}, \quad l^{(f,M)}(\varphi, \Phi) = \langle f, \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + \langle M, \Phi \rangle_{(H_0(\text{Curl}; \Omega))^*, H_0(\text{Curl}; \Omega)},
\]
where \( H^{-1}(\Omega) \) and \((H_0(\text{Curl}; \Omega))^*\) are the dual spaces of \( H^1_0(\Omega) \) and \( H_0(\text{Curl}; \Omega) \), respectively. We equip the product space \( H^1_0(\Omega) \times H_0(\text{Curl}; \Omega) \) with the norm
\[
\|(u, P)\|_{H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)} = \left( \|u\|_{H^1_0(\Omega)}^2 + \|P\|_{H_0(\text{Curl}; \Omega)}^2 \right)^{1/2}.
\]
For every \((f, M) \in C([0,T]; H^{-1}(\Omega) \times (H_0(\text{Curl}; \Omega))^*)\), the pair \((u, P) \in C^2([0,T]; H^1_0(\Omega) \times H_0(\text{Curl}; \Omega))\) is a weak solution of the problem [2.2]–[2.4] provided
\[
\mathcal{W}_1((u_{tt}(t), P_{tt}(t)), (\varphi, \Phi)) = \mathcal{W}_2((u(t), P(t)), (\varphi, \Phi)) + l^{(f(t), M(t))}(\varphi, \Phi)
\] (2.8)
for each \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\) and for all \(t \in [0,T]\), and it satisfies
\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_0'(x), \quad P(x, 0) = P_0(x), \quad P_t(x, 0) = P_0'(x), \quad x \in \Omega.
\] (2.9)

The proof that any classical solution is a weak solution follows using analogue calculations as in [42].

In order to prove the existence and uniqueness of a weak solution of problem [2.2], we follow the strategy proposed in [8] and we first prove the following lemma.

**Lemma 2.1.** For every \((f, M) \in C([0,T]; H^{-1}(\Omega) \times (H_0(\text{Curl}; \Omega))^*)\), \((v, Q) \in C^2([0,T]; H^1_0(\Omega) \times H_0(\text{Curl}; \Omega))\), \((u_0, P_0), (u_0', P_0') \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\) and for all tensors \((\mathbb{C}_c, \mathbb{C}_{micro}, \tilde{\mathbb{C}}_c, \mathbb{C}_{aniso})\) and \((\tilde{\mathbb{C}}_c, \mathbb{C}_{micro}, \tilde{\mathbb{C}}_c, \mathbb{C}_{aniso})\) which satisfy [2.1], [2.5] and [2.6], there exists a unique function \((u, P) \in C^2([0,T]; H^1_0(\Omega) \times H_0(\text{Curl}; \Omega))\) such that
\[
\mathcal{W}_1((u_{tt}(t), P_{tt}(t)), (\varphi, \Phi)) = \mathcal{W}_2((v(t), Q(t)), (\varphi, \Phi)) + l^{(f(t), M(t))}(\varphi, \Phi),
\] (2.10)
for all \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\) and for all \(t \in [0,T]\), and it satisfies
\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_0'(x), \quad P(x, 0) = P_0(x), \quad P_t(x, 0) = P_0'(x), \quad x \in \Omega.
\] (2.11)
Proof. Let us consider a fixed time \( t \in [0, T] \). In all cases, \( C_c \) positive definite or semi-positive definite or zero, from the Cauchy-Schwarz inequality and Poincaré inequality, and since the constitutive coefficients satisfy (2.5) and (2.6) inequality, we find that for fixed \((w, R) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\) the map \( \mathcal{W}_2((w, R), \cdot) : H^1_0(\Omega) \times H_0(\text{Curl}; \Omega) \to \mathbb{R} \) is bounded, since

\[
\|\text{sym}(\nabla\psi - \Psi)\|^2 \leq 2\left(\|\text{sym}\nabla\psi\|^2 + \|\text{sym}\Psi\|^2\right), \quad \|\text{sym}\nabla\psi\|^2 \leq \|\nabla\psi\|^2, \quad \|\text{sym}\Psi\|^2 \leq \|\Psi\|^2, \tag{2.12}
\]

for all \( \psi \in H^1_0(\Omega) \) and for all \( \Psi \in L^2(\Omega) \). Similarly it follows that \( \mathcal{W}_1 \) is bounded. On the other hand

\[
\mathcal{W}_1((\psi, \Psi), (\psi, \Psi)) = \int_\Omega \left( \rho \|\psi\|^2 + J \|\Psi\|^2 + \langle \overline{C}_c \text{sym}(\nabla\psi - \Psi), \text{sym}(\nabla\psi - \Psi) \rangle \\
+ \langle \overline{C}_c \text{skew}(\nabla\psi - \Psi), \text{skew}(\nabla\psi - \Psi) \rangle + \langle \overline{C}_{\text{micro}} \text{sym}\Psi, \text{sym}\Psi \rangle \\
+ \mu L^2_c \langle \overline{L}_{\text{aniso}} \text{Curl} \Psi, \text{Curl} \Psi \rangle \right) \, dv
\]

for all \((\psi, \Psi) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\), since, if it is present, the tensor \( C_c \) is semi-positive definite. Using the coercivity of the quadratic form from the right hand side of the above inequality, which is a direct consequence of the classical Korn inequality.

Since the linear operator \( l^{(f(t), M(t))} \) is bounded for a fixed time \( t \) considered in the beginning of the proof, using the Lax-Milgram theorem we obtain the existence of a unique solution \((u^*(t), P^*(t)) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\) of the equation

\[
\mathcal{W}_1((u^*(t), P^*(t)), (\varphi, \Phi)) = \mathcal{W}_2((v(t), Q(t)), (\varphi, \Phi)) + l^{(f(t), M(t))}(\varphi, \Phi) \tag{2.14}
\]

for all \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\). From standard arguments it follows that if \( (f, M) \in C([0, T]; H^{-1}(\Omega) \times (H_0(\text{Curl}; \Omega))^*) \) and \((v, Q) \in C([0, T]; H^1_0(\Omega) \times H_0(\text{Curl}; \Omega))\), then the solution \((u^*, P^*)\) of (2.14) belongs to \( C([0, T]; H^1_0(\Omega) \times H_0(\text{Curl}; \Omega))\). Indeed, the coercivity of (2.14) which correspond to two times \( t_1, t_2 \in [0, T] \) satisfies

\[
\mathcal{W}_1((u^*(t_1) - u^*(t_2), P^*(t_1) - P^*(t_2)), (\varphi, \Phi)) = \mathcal{W}_2((v(t_1) - v(t_2), Q(t_1) - Q(t_2)), (\varphi, \Phi)) \\
+ \langle f(t_1) - f(t_2), \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + \langle M(t_1) - M(t_2), \Phi \rangle_{H^*_0(\text{Curl}; \Omega), H_0(\text{Curl}; \Omega)} 	ag{2.15}
\]

for all \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)\). Using the coercivity of \( \mathcal{W}_1 \) and the boundedness of \( \mathcal{W}_2 \) we obtain that

\[
\|u^*(t_1) - u^*(t_2)\|^2_{H^1_0(\Omega)} + \|P^*(t_1) - P^*(t_2)\|^2_{H_0(\text{Curl}; \Omega)} \\
\leq c(\|u^*(t_1) - u^*(t_2)\|^2_{H^1_0(\Omega)} + \|P^*(t_1) - P^*(t_2)\|^2_{H_0(\text{Curl}; \Omega)})^{1/2} \\
+ \left[\|v(t_1) - v(t_2)\|^2_{H^1_0(\Omega)} + \|Q(t_1) - Q(t_2)\|^2_{H_0(\text{Curl}; \Omega)}\right]^{1/2} + \|f(t_1) - f(t_2)\|_{H^{-1}(\Omega)} + \|M(t_1) - M(t_2)\|^2_{H^*_0(\text{Curl}; \Omega)}^{1/2},
\]

for all \((t_1, t_2) \in [0, T] \).
where \( c \) is a positive constant. Hence, there is a positive constant \( c \) such that
\[
\|u^*(t_1) - u^*(t_2)\|^2_{H^1_0(\Omega)} + \|P^*(t_1) - P^*(t_2)\|^2_{H_0(\text{Curl};\Omega)} \\
\leq c(\|v(t_1) - v(t_2)\|^2_{H^1_0(\Omega)} + \|Q(t_1) - Q(t_2)\|^2_{H_0(\text{Curl};\Omega)})^{1/2} + (\|f(t_1) - f(t_2)\|^2_{H^{-1}} + \|M(t_1) - M(t_2)\|^2_{H^1_0(\text{Curl};\Omega)})^{1/2},
\]
which prove the continuity.

Now, the unique solution \((u, P) \in C^2([0, T]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) of the problem defined by (2.10) and (2.11) will be
\[
u(t) = u_0 + \int_0^t \left( u_0 + \int_0^s u^*(\xi) d\xi \right) ds, \quad P(t) = P_0 + \int_0^t \left( P_0 + \int_0^s P^*(\xi) d\xi \right) ds \tag{2.16}
\]
for all \( t \in [0, T] \) and the proof is complete.

**Theorem 2.2.** For every \((f, M) \in C([0, T]; H^{-1}(\Omega) \times (H_0(\text{Curl};\Omega))^*)\), \((u_0, P_0) \in H^1_0(\Omega) \times H_0(\text{Curl};\Omega)\) and for all tensors \((C_{\varepsilon}, C_{\text{micro}}, C_{\varepsilon}, C_{\varepsilon}, L_{\text{aniso}})\) and \((C_{\varepsilon}, C_{\text{micro}}, C_{\varepsilon}, C_{\varepsilon}, L_{\text{aniso}})\) which satisfy (2.11), (2.15) and (2.16), there exists a unique solution \((u, P) \in C^2([0, T]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) of the problem defined by (2.8) and (2.9).

**Proof.** To prove this theorem, we will use the Banach fixed-point theorem for the mapping
\[
\mathcal{L} : C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega)) \rightarrow C^2([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega)) \subset C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))
\]
which, for fixed \((f, M) \in C([0, \delta]; H^{-1}(\Omega) \times (H_0(\text{Curl};\Omega))^*)\), \((u_0, P_0) \in H^1_0(\Omega) \times H_0(\text{Curl};\Omega)\), maps each \((v, Q) \in C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) to the solution of the corresponding problem defined by (2.10) and (2.11), where \( \delta > 0 \) will be suitably chosen.

Let us consider \((u^{(1)}, Q^{(1)}), (v^{(2)}, Q^{(2)}) \in C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) and their corresponding solutions \((u^{(1)}, P^{(1)}), (u^{(2)}, P^{(2)}) \in C^2([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) of the related problems defined by (2.10) and (2.11), i.e.
\[
u^{(\alpha)}(t) = u_0 + \int_0^t \left( u_0 + \int_0^s u^{(\alpha)*}(\xi) d\xi \right) ds, \quad P^{(\alpha)}(t) = P_0 + \int_0^t \left( P_0 + \int_0^s P^{(\alpha)*}(\xi) d\xi \right) ds, \quad \alpha = 1, 2,
\]
for all \( t \in [0, T] \), where \((u^{(\alpha)*}, P^{(\alpha)*}) \in C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) is the unique solution of the equation:
\[
\mathcal{W}_1((u^{(\alpha)*}(t), P^{(\alpha)*}(t)), (\varphi, \Phi)) = \mathcal{W}_2((v^{(\alpha)}(t), Q^{(\alpha)}(t)), (\varphi, \Phi)) + \int f(t), M(t))((\varphi, \Phi)
\]
for all \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl};\Omega)\) and for all \( t \in [0, T] \).

Then, because the solutions correspond to the same initial conditions and forces, we have
\[
\max_{t \in [0, \delta]} \left( \|u^{(1)}(t) - u^{(2)}(t)\|_{H^1_0(\Omega)} + \|P^{(1)}(t) - P^{(2)}(t)\|_{H_0(\text{Curl};\Omega)} \right) \\
\leq \max_{t \in [0, \delta]} \int_0^t \int_0^s \left( \|u^{(1)*}(\xi) - u^{(2)*}(\xi)\|_{H^1_0(\Omega)} + \|P^{(1)*}(\xi) - P^{(2)*}(\xi)\|_{H_0(\text{Curl};\Omega)} \right) d\xi ds.
\tag{2.17}
\]
Since the problem is linear, \((u^{(1)*} - u^{(2)*}, P^{(1)*} - P^{(2)*}) \in C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) satisfies
\[
\mathcal{W}_1((u^{(1)*}(t) - u^{(2)*}(t), P^{(1)*}(t) - P^{(2)*}(t)), (\varphi, \Phi)) = \mathcal{W}_2((v^{(1)}(t) - v^{(2)}(t), Q^{(1)}(t) - Q^{(2)}(t)), (\varphi, \Phi))
\tag{2.18}
\]
for all \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl};\Omega)\) and for all \( t \in [0, \delta] \). The coercivity of \( \mathcal{W}_1 \) and the boundedness of \( \mathcal{W}_2 \) lead us to
\[
\|u^{(1)*}(t) - u^{(2)*}(t)\|^2_{H^1_0(\Omega)} + \|P^{(1)*}(t) - P^{(2)*}(t)\|^2_{H_0(\text{Curl};\Omega)} \\
\leq c(\|v^{(1)}(t) - v^{(2)}(t)\|^2_{H^1_0(\Omega)} + \|Q^{(1)}(t) - Q^{(2)}(t)\|^2_{H_0(\text{Curl};\Omega)}),
\tag{2.19}
\]
where \( c \) is a positive constant. But
\[
\frac{1}{2}(\|u^{(1)*}(t) - u^{(2)*}(t)\|^2_{H^1_0(\Omega)} + \|P^{(1)*}(t) - P^{(2)*}(t)\|^2_{H_0(\text{Curl};\Omega)})^2 \\
\leq \|u^{(1)*}(t) - u^{(2)*}(t)\|^2_{H^1_0(\Omega)} + \|P^{(1)*}(t) - P^{(2)*}(t)\|^2_{H_0(\text{Curl};\Omega)},
\tag{2.20}
\]
and therefore we deduce that

\[
\|u^{(1)}(t) - u^{(2)}(t)\|_{H^1_0(\Omega)} + \|P^{(1)}(t) - P^{(2)}(t)\|_{H_0(\text{Curl};\Omega)} \\
\leq \sqrt{2c} (\|v^{(1)}(t) - v^{(2)}(t)\|_{H^1_0(\Omega)}^2 + \|Q^{(1)}(t) - Q^{(2)}(t)\|_{H_0(\text{Curl};\Omega)}^2)^{1/2}. 
\]  

(2.21)

Hence, we obtain

\[
\max_{t \in [0, \delta]} (\|u^{(1)}(t) - u^{(2)}(t)\|_{H^1_0(\Omega)}^2 + \|P^{(1)}(t) - P^{(2)}(t)\|_{H_0(\text{Curl};\Omega)}^2)^{1/2} \\
\leq \max_{t \in [0, \delta]} (\|u^{(1)}(t) - u^{(2)}(t)\|_{H^1_0(\Omega)} + \|P^{(1)}(t) - P^{(2)}(t)\|_{H_0(\text{Curl};\Omega)}) \\
\leq \sqrt{2c} \max_{t \in [0, \delta]} \int_0^t \int_0^s (\|v^{(1)}(\xi) - v^{(2)}(\xi)\|_{H^1_0(\Omega)}^2 + \|Q^{(1)}(\xi) - Q^{(2)}(\xi)\|_{H_0(\text{Curl};\Omega)}^2)^{1/2} d\xi ds \\
\leq \delta^2 \sqrt{2c} \max_{t \in [0, \delta]} (\|v^{(1)}(t) - v^{(2)}(t)\|_{H^1_0(\Omega)}^2 + \|Q^{(1)}(t) - Q^{(2)}(t)\|_{H_0(\text{Curl};\Omega)}^2)^{1/2}.
\]

(2.22)

Therefore, we deduce

\[
\max_{t \in [0, \delta]} (\|u^{(1)}(t) - u^{(2)}(t)\|_{H^1_0(\Omega)}^2 + \|P^{(1)}(t) - P^{(2)}(t)\|_{H_0(\text{Curl};\Omega)}^2)^{1/2} \\
\leq \delta^2 c \max_{t \in [0, \delta]} (\|v^{(1)}(t) - v^{(2)}(t)\|_{H^1_0(\Omega)}^2 + \|Q^{(1)}(t) - Q^{(2)}(t)\|_{H_0(\text{Curl};\Omega)}^2)^{1/2}, 
\]  

(2.23)

where \(c\) is positive constant which is independent of time and also of the initial condition.

Hence, \(L\) is a contraction mapping on \(C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) when \(0 \leq \delta < \frac{1}{\sqrt{c}}\) with \(c\) derived from the above estimate. Hence, for \(0 \leq \delta < \frac{1}{\sqrt{c}}\), there exists a unique function \((u, P) \in C([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) so that

\[(u, P) = L(u, P) \in C^2([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega)). \]

(2.24)

Therefore, there exists a unique \((u, P) \in C^2([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) solution of the problem

\[W_1((u, P), t) = W_2((u, P), t) + l^{f(t), M(t)}(\varphi, \Phi) \]

for all \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl};\Omega)\) and for all \(t \in [0, \delta]\), which satisfies

\[u(x, 0) = u_0(x), \quad u_{\delta}(x, 0) = u_0(x), \quad P(x, 0) = P_0(x), \quad \varphi_{\delta}(x, 0) = P_0(x), \quad x \in \Omega\]

for all \(t \in [0, \delta]\).

By repeating the above analysis, since the positive constant \(c\) in (2.23) does not depend on the initial data, we may argue that there exists a unique solution \((\tilde{u}, \tilde{P}) \in C^2([0, \delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) of the problem

\[W_1((\tilde{u}, \tilde{P}) + l^{f(t), M(t)}(\varphi, \Phi) \]

for all \((\varphi, \Phi) \in H^1_0(\Omega) \times H_0(\text{Curl};\Omega)\) and for all \(t \in [0, \delta]\), which satisfies

\[\tilde{u}(x, \delta) = u(x, \delta), \quad \tilde{u}_{\delta}(x, \delta) = u_{\delta}(x, \delta), \quad \tilde{P}(x, \delta) = P(x, \delta), \quad \varphi_{\delta}(x, \delta) = P_0(x, \delta), \quad x \in \Omega.\]

Let us remark that putting together the solution on \([0, \delta]\) and on \([\delta, 2\delta]\) we obtain a solution \((u, P) \in C^2([0, 2\delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega))\) of the initial problem of the problem defined by (2.28) and (2.29) due to the fact that \((u, u_{\delta}, P, P_{\delta}) \in C([0, 2\delta]; H^1_0(\Omega) \times H_0(\text{Curl};\Omega)).\)

Hence, due to similar iterations, we may extend the already constructed solution to the interval \([0, T]\) by considering a big enough step \(n\), such that \(\frac{T}{n} < \frac{1}{c}\), where \(c\) is the constant from inequality (2.23). \(\Box\)

**Remark 2.1.** While in proving the existence of solution for the relaxed model without novel inertia terms (see [23, 42]) the request \(C_{\text{micro}} > 0\) was essential, in the relaxed model which include new inertia terms, the existence is still valid when \(C_{\text{micro}} = 0\) or \(\bar{C}_{\text{micro}} = 0\).
3 On the existence for a simplified model

The aim of this subsection is to investigate if the problem remains well-posed when we take into account the following simplified expression for the kinetic energy

\[ J(u_t, P_t, \nabla u_t, \text{Curl } P_t) = \frac{1}{2} \langle \rho u_t, u_t \rangle + \frac{1}{2} \left\langle \tilde{C} c \text{ sym } (\nabla u_t - P_t), \text{ sym } (\nabla u_t - P_t) \right\rangle + \left\langle \tilde{C}_{\text{micro sym }} P_t, \text{ sym } P_t \right\rangle + \mu \frac{L_c^2}{2} \left\langle \tilde{F}_{\text{aniso}} \text{ Curl } P_t, \text{ Curl } P_t \right\rangle \]

and to the same potential energy density as in the previous subsection. A justification of such a choice is that in the limit case \( P = \nabla u \), the variational formulation has to be related to the equations of the linear theory of nonlocal elasticity introduced by Eringen \[15\] to fit the acoustical branch of elastic waves within the Brillouin zone in periodic one dimensional lattices \[7\] (see also the work of Boutin \[6\] for an important comparison of the results using various models). We expect that the generalized model given in this section will improve the fitting of the dispersion curves, at least up to a value of the wave number compatible with the size of the microstructure of a considered microstructured material.

The equations of this model are

\[
\begin{align*}
\rho u_{tt} - \text{Div}[\tilde{C} c \text{ sym } (\nabla u_{tt} - P_{tt}) + \tilde{C} c \text{ skew } (\nabla u_{tt} - P_{tt})] &= \\
\text{Div}[\tilde{C} c \text{ sym } (\nabla u - P) + \tilde{C} c \text{ skew } (\nabla u - P)] &+ f, \\
\mu L_c^2 \text{ Curl}[\tilde{F}_{\text{aniso}} \text{ Curl } P_{tt}] - \tilde{C} c \text{ sym } (\nabla u_{tt} - P_{tt}) - \tilde{C} c \text{ skew } (\nabla u_{tt} - P_{tt}) + \tilde{C}_{\text{micro sym }} P_{tt} &= \\
- \mu L_c^2 \text{ Curl}[\tilde{F}_{\text{aniso}} \text{ Curl } P] + \tilde{C} c \text{ sym } (\nabla u - P) + \tilde{C} c \text{ skew } (\nabla u - P) - \tilde{C}_{\text{micro sym }} P + M.
\end{align*}
\]

For this model the bilinear form defined by the right hand side, i.e. \( W_1 : (H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)) \times (H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)) \rightarrow \mathbb{R} \)

\[
W_1((u, P), (\varphi, \Phi)) = \int_\Omega \left( \rho \langle u, \varphi \rangle + \langle \tilde{C} c \text{ sym } (\nabla u - P), \text{ sym } (\nabla \varphi - \Phi) \rangle + \langle \tilde{C} c \text{ skew } (\nabla u - P), \text{ skew } (\nabla \varphi - \Phi) \rangle + \langle \tilde{C}_{\text{micro sym }} P, \text{ sym } \Phi \rangle + \mu L_c^2 \langle \tilde{F}_{\text{aniso}} \text{ Curl } P, \text{ Curl } \Phi \rangle \right) dv
\]

remains bounded and also coercive. The boundedness follows without additional difficult. Regarding the coercivity, using the properties of the constitutive coefficients, we obtain that there is \( c > 0 \) such that

\[
W_1(w, w) \geq c \int_\Omega \left( \| u \|^2 + \| \text{sym } \nabla u \|^2 + \| \text{sym } P \|^2 + \| \text{Curl } P \|^2 \right) dv
\]

\[
\geq c \int_\Omega \left( \| \text{sym } \nabla u \|^2 + \| \text{sym } P \|^2 + \| \text{Curl } P \|^2 \right) dv
\]

for all \( (u, P) \in H^1_0(\Omega) \times H_0(\text{Curl}; \Omega) \).

Let us recall the following result \[18, 46, 47, 4\]:

Theorem 3.1. There exists a positive constant \( C \), only depending on \( \Omega \), such that for all \( P \in H_0(\text{Curl}; \Omega) \) the following estimates hold:

\[
\| P \|_{H^1(\text{Curl}; \Omega)} := \| P \|^2_{L^2(\Omega)} + \| \text{Curl } P \|^2_{L^2(\Omega)} \leq C (\| \text{sym } P \|^2_{L^2(\Omega)} + \| \text{Curl } P \|^2_{L^2(\Omega)}).
\]

While in the model introduced in the previous section, we have used only the Poincaré inequality and Korn inequality to show the coercivity of the bilinear form from the left hand side, in the model proposed in this section the above estimate is essential. Indeed, using also the Korn inequality, we obtain that \( W_1 \) is coercive for this model, too. Therefore, a similar analysis as in the previous section shows that the model presented in this section is well-posed and the solution \((u, P)\) belongs to \( C^2([0, T]; H^1_0(\Omega) \times H_0(\text{Curl}; \Omega)) \).

Let us remark that the coercivity is valid only when \( \tilde{C}_{\text{micro}} > 0 \) and also when \( \tilde{C} c \) either vanishes or it is only semi-positive definite. The presence of \( \tilde{C} c \) and \( \tilde{C}_{\text{micro}} \) is also not mandatory in order to have a well-posed model, as well as the presence of \( \| P_t \| \) or \( \| u_t \| \) in the expression of the kinetic energy.
4 Some final remarks

At the end of this article we mention that an approach based on the semigroup of linear operators may not lead to an existence result, even of a weak solution, without an a priori assumption on the compatibility of the domains of the operators defined by the left and right hand side, respectively, of the system of partial differential equations (which are Banach spaces endowed with the corresponding graph-norms, since the operators generate $C_0$-contractive semigroups in $L^2(\Omega) \times L^2(\Omega)$). We expect to have the same difficulties when the Galerkin method is used.

Another remark is that the first model considered in this paper remains well-posed also when the characteristic length scale $L_c$ vanishes, i.e. $\text{Curl} P_{\ell} \text{ and Curl} P$ are not present in the kinetic energy and in the potential energy density, respectively. The solution will belong to $C(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ as long as the forces are in $H^1_0(\Omega) \times L^2(\Omega)$ and the initial conditions are assumed to be in $H^1_0(\Omega) \times L^2(\Omega)$. Note also that for $L_c = 0$ we may assume only strong anchoring condition for $P$, i.e. we must prescribe $P$ on the boundary but not necessary zero.

We are not able to say the same for the second model, in the case $L_c = 0$, since the bilinear form $W_1$ may not be coercive, in general. However, when $C_c = 0$ and $C_c = 0$, $L_c = 0$ and $\|P_{\ell}\|$ is not taken into account in the form of the kinetic energy, the problem involves actually only the functions $u$ and sym $P$. Therefore, in this situation and when $C_{\text{micro}} > 0$, if sym $P(0), \text{sym} P_{\ell}(0) \in L^2(\Omega)$, $u(0), u_{\ell}(0) \in H^1_0(\Omega)$, $u = 0$ on the boundary, $P$ is also prescribed on the boundary, and the forces are in $C(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$, there exists a unique solution $(u, \text{sym} P) \in C^2([0,T]; H^1_0(\Omega) \times L^2(\Omega))$ of the problem in $u$ and sym $P$. We do not have information about skew $P$, since it is not involved in the equations and clearly it may not be uniquely determined, in order to satisfy also the initial and the boundary conditions for the full $P$.

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