Greatest lower bounds on Ricci curvature for toric Fano manifolds

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ABSTRACT: In this short note, based on the work of Wang-Zhu [7], we determine the greatest lower bounds on Ricci curvature for all toric Fano manifolds.

1 Introduction

On Fano manifolds \(X\), i.e. \(K^{-1}_X\) is ample, the Kähler-Einstein equation

\[
Ric(\omega) = \omega
\]

is equivalent to the complex Monge-Ampère equation:

\[
(\omega + \partial \bar{\partial} \phi)^n = e^{h_\omega - \phi} \omega^n
\]

where \(\omega\) is a fixed Kähler metric in \(c_1(X)\), and \(h_\omega\) is the normalized Ricci potential:

\[
Ric(\omega) - \omega = \partial \bar{\partial} h_\omega, \quad \int_X e^{h_\omega} \omega^n = \int_X \omega^n
\]

In order to solve this equation, the continuity method is used. So we consider a family of equations with parameter \(t\):

\[
(\omega + \partial \bar{\partial} \phi_t)^n = e^{h_\omega - t \phi} \omega^n
\]

Define \(S_t = \{t : (\ast)_t \text{ is solvable}\}\). It was known that the set \(S_t\) is open. To solve (\ast), the crucial thing is to obtain the closedness of this set. So we need some a priori estimates. By Yau’s \(C^2\) and Calabi’s higher order estimates (See [8], [4]), we only need uniform \(C^0\)-estimates for solutions \(\phi_t\) of (\ast)_t. In general one can not solve (\ast), and so can not get the \(C^0\)-estimates, due to the well known obstruction of Futaki invariant. So when \(t \to R(X)\), some blow-up happens.

It was first showed by Tian [5] that we may not be able to solve (\ast)_t on certain Fano manifold for \(t\) sufficiently close to 1. Equivalently, for such a Fano manifold, there is some \(t_0 < 1\), such that there is no Kähler metric \(\omega\) in \(c_1(X)\) which can have \(Ric(\omega) \geq t_0 \omega\). It is now made more precise.

Define

\[
R(X) = \sup\{t : (\ast)_t \text{ is solvable}\}
\]

It can be shown that \(R(X)\) is independent of \(\omega \in c_1(X)\). In fact, Székelyhidi [3] observed

**Fact:** \(R(X) = \sup\{t : Ric(\omega) > t \omega, \forall \text{ Kähler metric } \omega \in c_1(X)\}\)

He also showed \(R(Bl_p \mathbb{P}^2) = \frac{5}{6}\) and \(\frac{1}{2} \leq R(Bl_p,q \mathbb{P}^2) \leq \frac{11}{12}\).

Let \(\Lambda \cong \mathbb{Z}^n\) be a lattice in \(\mathbb{R}^n = \Lambda \otimes \mathbb{R}\). A toric Fano manifold \(X_\Delta\) is determined by a reflexive lattice polytope \(\Delta\) (For details on toric manifolds, see [2]). For example, the toric manifold \(Bl_p \mathbb{P}^2\) is determined by the following polytope.

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In this short note, we determine $R(X_\Delta)$ for every toric Fano manifold $X_\Delta$ in terms of the geometry of polytope $\Delta$.

Any such polytope $\Delta$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\Delta$ by $P_c$. If $P_c \neq O$, the ray $P_c + \mathbb{R}_{\geq 0} \cdot \bar{P}_c O$ intersects the boundary $\partial \Delta$ at point $Q$. Our main result is

**Theorem 1.** If $P_c \neq O$,

$$R(X_\Delta) = \frac{|OQ|}{|P_c Q|}.$$  

Here $|OQ|, |P_c Q|$ are lengths of line segments $OQ$ and $P_c Q$. If $P_c = O$, then there is Kähler-Einstein metric on $X_\Delta$ and $R(X_\Delta) = 1$.

**Remark 1.** Note for the toric Fano manifold, $P_c$ is just Futaki invariant. So the second statement follows from Wang-Zhu [7]. We will repeat the proof in next section.

Our method is based on Wang-Zhu’s [7] theory for proving the existence of Kähler-Ricci solitons on toric Fano manifolds. In view of the analysis in [7], if $R(X_\Delta) < 1$, then as $t \to R(X_\Delta)$, the blow-up happens exactly because the minimal points of a family of proper convex functions go to infinity, or, equivalently, the images of minimal points under the momentum map of a fixed metric tend to the boundary of the toric polytope. The key identity relation in [Section 2,(11)] and some uniform a priori estimates enable us to read out $R(X_\Delta)$ in terms of geometry of $\Delta$.

This note is partly inspired by the Székelyhidi’s paper [3] and Donaldson’s survey [1]. The author thanks Professor Gang Tian for constant encouragement.

## 2 Consequence of Wang-Zhu’s theory

First we recall the set up of Wang-Zhu [7]. For a reflexive lattice polytope $\Delta$ in $\mathbb{R}^n = \Lambda \otimes_\mathbb{Z} \mathbb{R}$, we have a Fano toric manifold $(\mathbb{C}^*)^n \subset X_\Delta$ with a $(\mathbb{C}^*)^n$ action. Let $\{z_i\}$ be the standard coordinates of the dense orbit $(\mathbb{C}^*)^n$, and $z_i = \log |z_i|^2$. Let $\{p_\alpha\}_{\alpha = 1, \ldots, N}$ be the lattice points contained in $\Delta$.

We take the fixed Kähler metric $\omega$ to be given by the potential (on $(\mathbb{C}^*)^n$)

$$\tilde{u}_0 = \log \left( \sum_{\alpha = 1}^N e^{\langle p_\alpha, x \rangle} \right) + C$$  \hfill (2)

$C$ is some constant determined by normalization condition:

$$\int_{\mathbb{R}^n} e^{-\tilde{u}_0} dx = Vol(\Delta) = \frac{1}{n!} \int_{X_\Delta} \omega^n = \frac{c_1(X_\Delta)^n}{n!}$$  \hfill (3)

By standard toric geometry, each lattice point $p_\alpha$ contained in $\Delta$ determines, up to a constant, a $(\mathbb{C}^*)^n$-equivariant section $s_\alpha$ in $H^0(X, K_X^{-1})$. We can embed $X_\Delta$ into $P(H^0(X, K_X^{-1})^*)$ using these sections. Let $s_0$ be the section corresponding to the origin $0 \in \Delta$, then its Fubini-Study norm is

$$|s_0|^2_{FS} = \sum_{\alpha = 1}^N |s_\alpha|^2 = \left( \sum_{\alpha = 1}^N \prod_{i=1}^n |z_i|^{2p_{\alpha,i}} \right)^{-1} = \left( \sum_{\alpha = 1}^N e^{\langle p_\alpha, x \rangle} \right)^{-1} = e^C e^{-\tilde{u}_0}$$
So the Kähler metric $\omega = \frac{\alpha}{2\pi} \partial \bar{\partial} u_0$ is the Fubini-Study metric.

On the other hand, $Ric(\omega)$ is the curvature of Hermitian line bundle $K_M^{-1}$ with Hermitian metric determined by the volume form $\omega^n$. Note that on the open dense orbit $(C^*)^n$, we can take $s_0 = z_1 \frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n \frac{\partial}{\partial z_n}$. Since $\frac{\partial}{\partial \log z_i} = \frac{1}{2} \left( \frac{\partial}{\partial \log |z_i|} - \sqrt{-1} \frac{\partial}{\partial \phi} \right)$, when acting on any $(S^1)^n$ invariant function on $(C^*)^n$, we have

$$|s_0|^2_{\omega^n} = \left| z_1 \frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n \frac{\partial}{\partial z_n} \right|^2_{\omega^n} = \det \left( \frac{\partial^2 \tilde{u}_0}{\partial \log z_i \partial \log z_j} \right) = \det(\tilde{u}_{0,ij})$$

It’s easy to see from definition of $h_\omega$ (1) and normalization condition (3) that

$$e^{h_\omega} = e^{-C \frac{|s_0|^2_{\omega^n}}{|s_0|^2_{\bar{\omega}^n}}} = e^{-\tilde{u}_0 \det(\tilde{u}_{0,ij})}^{-1}$$

Then using the torus symmetry, $(*)_t$ can be translated into real Monge-Ampère equation [7] on $\mathbb{R}^n$.

$$\det(u_{ij}) = e^{-(1-t)\tilde{u}_0 - tu} = e^{-w_t} \quad (**)_t$$

The solution $u_t$ of $(**)_t$ is related to Kähler potential $\phi_t$ in $(*)_t$ by the identity:

$$u = \tilde{u}_0 + \phi_t \quad (4)$$

where $\phi_t$ is viewed as a function of $x_i = \log |z_i|^2$ by torus symmetry.

Every strictly convex function $f$ appearing in $(**)_t$ ($f = \tilde{u}_0, u, w_t = (1-t)\tilde{u}_0 + tu$) must satisfy $Df(\mathbb{R}^n) = \Delta^0$ ($\Delta^0$ means the interior of $\Delta$). Since 0 is (the unique lattice point) contained in $\Delta^0 = Df(\mathbb{R}^n)$, the strictly convex function $f$ is properly.

Wang-Zhu’s [7] method for solving $(**)_t$ consists of two steps. The **first step** is to show some uniform a priori estimates for $w_t$. For $t < R(X_\Delta)$, the proper convex function $w_t$ obtains its minimum value at a unique point $x_t \in \mathbb{R}^n$. Let

$$m_t = \inf\{w_t(x) : x \in \mathbb{R}^n\} = w_t(x_t)$$

**Proposition 1** ([7], See also [1]).

1. There exists a constant $C$, independent of $t < R(X_\Delta)$, such that

$$|m_t| < C$$

2. There exists $\kappa > 0$ and a constant $C$, both independent of $t < R(X_\Delta)$, such that

$$w_t \geq \kappa |x - x_t| - C \quad (5)$$

For the reader’s convenience, we record the proof here.

**Proof.** Let $A = \{x \in \mathbb{R}^n : m_t \leq w(x) \leq m_t + 1\}$. $A$ is a convex set. By a well known lemma due to Fritz John, there is a unique ellipsoid $E$ of minimum volume among all the ellipsoids containing $A$, and a constant $\alpha_n$ depending only on dimension, such that

$$\alpha_n E \subset A \subset E$$

$\alpha_n E$ means the $\alpha_n$-dilation of $E$ with respect to its center. Let $T$ be an affine transformation with $\det(T) = 1$, which leaves $x' =$ the center of $E$ invariant, such that $T(E) = B(x', R)$, where $B(x', R)$ is the Euclidean ball of radius $R$. Then

$$B(x', \alpha_n R) \subset T(A) \subset B(x', R)$$
We first need to bound \( R \) in terms of \( m_t \). Since \( D^2w = tD^2u + (1-t)D^2\tilde{u}_0 \geq tD^2u \), by \((**)_t\), we see that
\[
\det(w_{ij}) \geq t^n e^{-w}
\]
Restrict to the subset \( A \), it’s easy to get
\[
\det(w_{ij}) \geq C_1 e^{-m_t}
\]
Let \( \tilde{w}(x) = w(T^{-1}x) \), since \( \det(T) = 1 \), \( \tilde{w} \) satisfies the same inequality
\[
\det(\tilde{w}_{ij}) \geq C_1 e^{-m_t}
\]
in \( T(A) \).

Construct an auxiliary function
\[
v(x) = C_1^+ e^{-\frac{m_t}{2}} \left( |x - x'|^2 - (\alpha_n R)^2 \right) + m_t + 1
\]
Then in \( B(x', \alpha_n R) \),
\[
\det(v_{ij}) = C_1 e^{-m_t} \leq \det(\tilde{w}_{ij})
\]
On the boundary \( \partial B(x', \alpha_n R) \), \( v(x) = m_t + 1 \geq \tilde{w} \). By the Bedford-Taylor comparison principle for Monge-Ampère operator, we have
\[
\tilde{w}(x) \leq v(x) \text{ in } B(x', \alpha_n R)
\]
In particular
\[
m_t \leq \tilde{w}(x') \leq v(x') = C_1^+ e^{-\frac{m_t}{2}} \left( \frac{R^2}{n^2} \right) + m_t + 1
\]
So we get the bound for \( R \):
\[
R \leq C_2 e^{\frac{m_t}{m}}
\]
So we get the upper bound for the volume of \( A \):
\[
Vol(A) = Vol(T(A)) \leq CR^n \leq Ce^{\frac{m_t}{m}}
\]
By the convexity of \( w \), it’s easy to see that \( \{x; w(x) \leq m_t + s\} \subset s \cdot \{x; w(x) \leq m_t + 1\} = s \cdot A \), where \( s \cdot A \) is the \( s \)-dilation of \( A \) with respect to point \( x_t \). So
\[
Vol(\{x; w(x) \leq m_t + s\}) \leq s^n Vol(A) \leq Cs^n e^{\frac{m_t}{m}}
\]
The lower bound for volume of sublevel sets is easier to get. Indeed, since \(|Dw(x)| \leq L \), where \( L = \max_{y \in \triangle} |y| \), we have \( B(x_t, s \cdot L^{-1}) \subset \{x; w(x) \leq m_t + s\} \). So
\[
Vol(\{x; w(x) \leq m_t + s\}) \geq Cs^n
\]
Now we can derive the estimate for \( m_t \). First note the identity:
\[
\int_{\mathbb{R}^n} e^{-w} dx = \int_{\mathbb{R}^n} \det(u_{ij}) dx = \int_{\triangle} d\sigma = Vol(\triangle)
\]
Second, we use the coarea formula
\[
\int_{\mathbb{R}^n} e^{-w} dx = \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} e^{-s} ds dx = \int_{-\infty}^{+\infty} e^{-s} ds \int_{\mathbb{R}^n} 1_{\{w \leq s\}} dx = \int_{m_t}^{+\infty} e^{-s} Vol(\{w \leq s\}) ds
\]
\[
= e^{-m_t} \int_0^{+\infty} e^{-s} Vol(\{w \leq m_t + s\}) ds
\]
Using the bound for the volume of sublevel sets \((6)\) and \((7)\) in \((9)\), and compare with \((8)\), it’s easy to get the bound for \(|m_\ell|\).

Now we prove the estimate \((5)\) following the argument of \([1]\). We have seen \(B(x_\ell, L^{-1}) \subset \{w \leq m_\ell + 1\}\) and \(\text{Vol}(\{w \leq m_\ell + 1\}) \leq C\) by \((6)\) and uniform bound for \(m_\ell\). Then we must have \(\{w \leq m_\ell + 1\} \subset B(x_\ell, R(C, L))\) for some uniformly bounded radius \(R(C, L)\). Otherwise, the convex set \(\{w \leq m_\ell + 1\}\) would contain a convex subset of arbitrarily large volume. By the convexity of \(w\), we have \(w(x) \geq \frac{1}{R(C, L)}|x - x_\ell| + m_\ell - 1\). Since \(m_\ell\) is uniformly bounded, the estimate \((5)\) follows.

The second step is trying to bound \(|x_\ell|\). In Wang-Zhu’s \([7]\) paper, they proved the existence of Kähler-Ricci soliton on toric Fano manifold by solving the real Monge-Ampère equation corresponding to Kähler-Ricci soliton equation. But now we only consider the Kähler-Einstein equation, which in general can’t be solved because there is the obstruction of Futaki invariant.

**Proposition 2** ([7]). The uniform bound of \(|x_\ell|\) for any \(0 \leq t \leq t_0\), is equivalent to that we can solve \((**)_t\), or equivalently solve \((*)_t\), for \(t\) up to \(t_0\). More precisely, (by the discussion in introduction,) this condition is equivalent to the uniform \(C^0\)-estimates for the solution \(\phi_t\) in \((*)_t\) for \(t \in [0, t_0]\).

Again we sketch the proof here.

**Proof.** If we can solve \((**)_t\) (or equivalently \((*)_t\)) for \(0 \leq t \leq t_0\). Then \(\{w(t) = (1 - t)\bar{u}_0 + tu; 0 \leq t \leq t_0\}\) is a smooth family of proper convex functions on \(\mathbb{R}^n\). So their minimal points are uniformly bounded in a compact set.

Conversely, assume \(|x_\ell|\) is bounded. First note that \(\phi_t = u - \bar{u}_0 = \frac{1}{t}(w_t(x) - \bar{u}_0)\).

As in Wang-Zhu [7], we consider the enveloping function:

\[
v(x) = \max_{p_\in \Lambda \cap \Delta} \langle p_\alpha, x \rangle
\]

Then \(0 \leq \bar{u}_0(x) - v(x) \leq C\), and \(Dw(\xi) \cdot x \leq v(x)\) for all \(\xi, x \in \mathbb{R}^n\). We can assume \(t \geq \delta > 0\). Then using uniform boundedness of \(|x_\ell|\)

\[
\phi_t(x) = \frac{1}{t}(w_t(x) - \bar{u}_0) = \frac{1}{t}[(w_t(x) - w_t(x_\ell)) - v(x) + (v(x) - \bar{u}_0(x)) - w_t(x_\ell)]
\]

\[\leq \delta^{-1}(Dw_t(\xi) \cdot x - v(x) - Dw_t(\xi) \cdot x_\ell) - C \leq C'
\]

Thus we get the estimate for \(\sup \phi_t\). Then one can get the bound for \(\inf \phi_t\) using the Harnack inequality in the theory of Monge-Ampère equations. For details see ([7], Lemma 3.5) (see also [6]).

By the above proposition, we have

**Lemma 1.** If \(R(X_\Delta) < 1\), then there exists a subsequence \(\{x_{\ell_i}\}\) of \(\{x_{\ell}\}\), such that

\[
\lim_{\ell_i - R(X_\Delta)} |x_{\ell_i}| = +\infty
\]

The observation now is that

**Lemma 2.** If \(R(X_\Delta) < 1\), then there exists a subsequence of \(\{x_{\ell_i}\}\) which we still denote by \(\{x_{\ell_i}\}\), and \(y_\infty \in \partial \Delta\), such that

\[
\lim_{\ell_i - R(X_\Delta)} D\bar{u}_0(x_{\ell_i}) = y_\infty
\]

This follows easily from the properness of \(\bar{u}_0\) and compactness of \(\Delta\).

We now use the key relation (See [7], Lemma 3.3, and also [1] page 29)

\[
0 = \int_{\mathbb{R}^n} Dw(x)e^{-w}dx = \int_{\mathbb{R}^n} ((1 - t)D\bar{u}_0 + tDu)e^{-w}dx
\]
Since
\[ \int_{\mathbb{R}^n} Du e^{-w} dx = \int_{\mathbb{R}^n} Du \det(u_{ij}) dx = \int_{\triangle} yd\sigma = \text{Vol}(\triangle) P_c \]
where \( P_c \) is the barycenter of \( \triangle \), so
\[ \frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n} D\tilde{u}_0 e^{-w} dx = -\frac{t}{1-t} P_c \] (11)
We will show this vector tend to a point on \( \partial \triangle \) when \( t \) goes to \( R(X_\triangle) \). To prove this we use the defining function of \( \triangle \). Similar argument was given in the survey \([1]\), page 30.

3 Proof of Theorem 1

We now assume the reflexive polytope \( \triangle \) is defined by inequalities:
\[ \lambda_r(y) \geq -1, \ r = 1, \ldots, K \] (12)
\[ \lambda_r(y) = \langle v_r, y \rangle \] are fixed linear functions. We also identify the minimal face of \( \triangle \) where \( y_\infty \) lies:
\[ \lambda_r(y_\infty) = -1, \ r = 1, \ldots, K_0 \]
\[ \lambda_r(y_\infty) > -1, \ r = K_0 + 1, \ldots, K \] (13)
Clearly, Theorem 1 follows from

**Proposition 3.** If \( P_c \neq O \),
\[ -\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c \in \partial \triangle \]

Precisely,
\[ \lambda_r \left( -\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c \right) \geq -1 \] (14)
Equality holds if and only if \( r = 1, \ldots, K_0 \). So \( -\frac{R(X_\triangle)}{1-R(X_\triangle)} P_c \) and \( y_\infty \) lie on the same faces (13).

**Proof.** By (11) and defining function of \( \triangle \), we have
\[ \lambda_r \left( -\frac{t}{1-t} P_c \right) + 1 = \frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n} \lambda_r(D\tilde{u}_0) e^{-w} dx + 1 = \frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n} (\lambda_r(D\tilde{u}_0) + 1) e^{-w} dx \] (15)
The inequality (14) follows from (15) by letting \( t \to R(X_\triangle) \). To prove the second statement, by (15) we need to show
\[ \lim_{t \to R(X_\triangle)} \frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n} \lambda_r(D\tilde{u}_0) e^{-w_{r_1}} dx + 1 \]
\[ \begin{cases} = 0 & \text{for } r = 1, \ldots, K_0 \\ > 0 & \text{for } r = K_0 + 1, \ldots, N \end{cases} \] (16)
By the uniform estimate (5) and fixed volume (8), and since \( D\tilde{u}_0(\mathbb{R}^n) = \triangle^\circ \) is a bounded set, there exists \( R_\epsilon \), independent of \( t \in [0, R(X_\triangle)] \), such that
\[ \frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n \setminus B_{R_\epsilon}(x_i)} \lambda_r(D\tilde{u}_0) e^{-w_{r_1}} dx < \epsilon \]
and
\[ \frac{1}{\text{Vol}(\triangle)} \int_{B_{R_\epsilon}(x_i)} e^{-w_{r_1}} dx < \epsilon \] (17)
Now (16) follows from the following claim.

**Claim 1.** Let \( R > 0 \), there exists a constant \( C > 0 \), which only depends on the polytope \( \triangle \), such that for all \( \delta x \in B_R(0) \subset \mathbb{R}^n \),
\[ e^{-CR}(\lambda_r(D\tilde{u}_0(x_\ell)) + 1) \leq \lambda_r(D\tilde{u}_0(x_\ell + \delta x)) + 1 \leq e^{CR}(\lambda_r(D\tilde{u}_0(x_\ell)) + 1) \] (18)
Assuming the claim, we can prove two cases of (16). First by (10) and (13), we have
\[
\lim_{t_i \to R(X_{\triangle})} \lambda_r(D\tilde{u}_0(x_{t_i})) + 1 = \lambda_r(y_{\infty}) + 1 = \begin{cases} 
0 & : r = 1, \ldots, K_0 \\
\alpha_r > 0 & : r = K_0 + 1, \ldots, N
\end{cases}
\]  
(19)

1. \( r = 1, \ldots, K_0 \). \( \forall \epsilon > 0 \), first choose \( R_\epsilon \) as in (17). By (18) and (19), there exists \( \rho_\epsilon > 0 \), such that if \( |t_i - R(X_{\triangle})| < \rho_\epsilon \), then for all \( \delta x \in B_{R_\epsilon}(0) \subset \mathbb{R}^n \),
\[
0 \leq \lambda_r(D\tilde{u}_0(x_{t_i} + \delta x)) + 1 < e^{CR_\epsilon} (\lambda_r(D\tilde{u}_0)(x_{t_i})) + 1 < \epsilon
\]
in other words, \( \lambda_r(D\tilde{u}_0(x_{t_i} + \delta x)) + 1 \to 0 \) uniformly for \( \delta x \in B_{R_\epsilon}(0) \), as \( t_i \to R(X_{\triangle}) \). So when \( |t_i - R(X_{\triangle})| < \rho_\epsilon \),
\[
\frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n} \lambda_r(D\tilde{u}_0)e^{-w} \, dx + 1 = \frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n \setminus B_{R_\epsilon}(x_{t_i})} \lambda_r(D\tilde{u}_0)e^{-w} \, dx + \frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n \setminus B_{R_\epsilon}(x_{t_i})} e^{-w} \, dx
\leq 2\epsilon + \epsilon \frac{1}{\text{Vol}(\triangle)} \int_{B_{R_\epsilon}(x_{t_i})} e^{-w} \, dx \leq 3\epsilon
\]
The first case in (16) follows by letting \( \epsilon \to 0 \).

2. \( r = K_0 + 1, \ldots, N \). We fix \( \epsilon = \frac{\epsilon}{2} \) and \( R_\epsilon \) in (17). By (18) and (19), there exists \( \rho > 0 \), such that if \( |t_i - R(X_{\triangle})| < \rho \), then for all \( \delta x \in B_{R_\epsilon}(0) \subset \mathbb{R}^n \),
\[
\lambda_r(D\tilde{u}_0(x_{t_i} + \delta x)) + 1 > e^{-CR_\epsilon} (\lambda_r(D\tilde{u}_0)(x_{t_i})) + 1 > e^{-CR_\epsilon} \frac{a_r}{2} > 0
\]
\[
\frac{1}{\text{Vol}(\triangle)} \int_{\mathbb{R}^n} \lambda_r(D\tilde{u}_0)e^{-w} \, dx + 1 \geq \frac{1}{\text{Vol}(\triangle)} \int_{B_{R_\epsilon}(x_{t_i})} (\lambda_r(D\tilde{u}_0) + 1)e^{-w} \, dx 
\geq e^{-CR_\epsilon} \frac{a_r}{2} \frac{1}{\text{Vol}(\triangle)} \int_{B_{R_\epsilon}(x_{t_i})} e^{-w} \, dx 
\geq e^{-CR_\epsilon} \frac{a_r}{2} \frac{1}{2} > 0
\]

Now we prove the claim. We can rewrite (18) using the special form of \( \tilde{u}_0 \) (2).
\[
D\tilde{u}_0(x) = \sum_{\alpha} \sum_{\beta} e^{<p_{\alpha}, x>} e^{<p_{\beta}, x>} p_{\alpha} = \sum_{\alpha} c_{\alpha}(x)p_{\alpha}
\]
Here the coefficients
\[
0 \leq c_{\alpha}(x) = \sum_{\beta} e^{<p_{\alpha}, x>} e^{<p_{\beta}, x>} \sum_{\beta} e^{<p_{\beta}, x>} \, \sum_{\alpha} c_{\alpha}(x) = 1
\]
So
\[
\lambda_r(D\tilde{u}_0(x)) + 1 = \sum_{\alpha} c_{\alpha}(x)(\lambda_r(p_{\alpha}) + 1) = \sum_{\{\alpha : \lambda_r(p_{\alpha}) + 1 > 0\}} c_{\alpha}(x)(\lambda_r(p_{\alpha}) + 1)
\]
Since \( \lambda_r(p_{\alpha}) + 1 \geq 0 \) is a fixed value, to prove the claim, we only need to show the same estimate for \( c_{\alpha}(x) \). But now
\[
c_{\alpha}(x_{t_i} + \delta x) = \frac{e^{<p_{\alpha}, x_{t_i} + \delta x>} e^{<p_{\alpha}, \delta x>}}{\sum_{\beta} e^{<p_{\alpha}, x_{t_i} + \delta x>} e^{<p_{\beta}, \delta x>}} \leq e^{<p_{\alpha}, \delta x>} e^{\max_{\beta} |p_{\beta}| |R|} \frac{e^{<p_{\alpha}, x_{t_i} + \delta x>}}{\sum_{\beta} e^{<p_{\beta}, x_{t_i} + \delta x>}} \leq e^{CR_\epsilon} e^{<p_{\alpha}, x_{t_i>}} = e^{CR_\epsilon} c_{\alpha}(x_{t_i})
\]
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And similarly
\[ c_\alpha(x, t_i + \delta x) \geq e^{-CR} c_\alpha(x, t_i) \]
So the claim holds and the proof is completed. \qed

4 Example

Example 1. \( X_\Delta = Bl_{p\mathbb{P}^2} \). See the figure in Introduction. \( P_c = \frac{1}{4}(\frac{1}{2}, -\frac{2}{3}), -6P_c \in \partial \Delta \), so \( R(X_\Delta) = \frac{4}{7} \).

Example 2. \( X_\Delta = Bl_{p,q \mathbb{P}^2} \), \( P_c = \frac{2}{9}(-\frac{1}{3}, -\frac{1}{3}), -\frac{21}{4}P_c \in \partial \Delta \), so \( R(X_\Delta) = \frac{21}{25} \).

\[ \begin{array}{c}
\text{P}_c \\
-\frac{21}{4} \cdot P_c
\end{array} \]

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