Diffractive Electron-Nucleus Scattering and Ancestry in Branching Random Walks

A. H. Mueller\textsuperscript{1} and S. Munier\textsuperscript{2}

\textsuperscript{1}Department of Physics, Columbia University, New York, NY 10027, USA
\textsuperscript{2}Centre de physique th\textsuperscript{e}orique, \'{E}cole polytechnique, CNRS, Universit\textsuperscript{e} Paris-Saclay, 1 route de Saclay, 91128 Palaiseau, France

(Dated: May 25, 2018)

We point out an analogy between diffractive electron-nucleus scattering events, and realizations of one-dimensional branching random walks selected according to the height of the genealogical tree of the particles near their boundaries. This correspondence is made transparent in an event-by-event picture of diffraction emphasizing the statistical properties of gluon evolution, from which new quantitative predictions straightforwardly follow: We are able to determine the distribution of the total invariant mass produced diffractively, which is an interesting observable that can potentially be measured at a future electron-ion collider.

Abstract

We introduce an analogy between diffractive electron-nucleus scattering events, and realizations of one-dimensional branching random walks selected according to the height of the genealogical tree of the particles near their boundaries. This correspondence is made transparent in an event-by-event picture of diffraction emphasizing the statistical properties of gluon evolution, from which new quantitative predictions straightforwardly follow: We are able to determine the distribution of the total invariant mass produced diffractively, which is an interesting observable that can potentially be measured at a future electron-ion collider.

Introduction. Diffraction is an elementary consequence of the particle-wave duality postulated by quantum mechanics. Therefore, diffractive patterns are expected to be observed in the scattering of elementary particles off extended objects such as hadrons or nuclei. However, the microscopic interpretation of diffraction turns out to be subtle. Indeed, it is well-known that nuclei are loose compounds of hadrons, which themselves appear as fragile bound states of quarks as soon as they are involved in collisions at center-of-mass energies much larger than typically the mass energy of a nucleon. Naively, an energetic electron colliding with a hadron or nucleus, a process known as “deep-inelastic scattering” (DIS), would knock out a quark in each scattering event; Then, as a consequence of confinement, the final state would almost systematically consist in many new hadrons distributed all over the detector.

But this is not at all what has been seen experimentally. Indeed, one of the outstanding results of the DESY-HERA electron-proton collider is the observation of a significant fraction of the events (about 10%) in which the scattered proton is left intact, and is surrounded by an angular region of variable size empty of particles. What has been observed in electron-proton collisions should also happen in electron-nucleus scattering. Testing whether this expectation is true can be achieved at a future electron-ion collider.

Diffraction in DIS on protons has been studied extensively, both experimentally (for a review, see \cite{1}) and theoretically (see \cite{2} and references therein). But its quantitative theoretical description in the framework of the established theory of the strong interaction, quantum chromodynamics (QCD), remains a challenge. While it is known that the total diffractive cross section can be explained economically and elegantly by saturation models \cite{3}, little analytical insight has been gained for more exclusive diffractive observables.

In this Letter, we focus on the diffractive events in deep-inelastic scattering off a large nucleus in which the nucleus is left intact, but a hadronic state of large invariant mass $M_X$ is nevertheless produced. We explain how to characterize them microscopically, and we show that these hadrons are generated from a similar mechanism as the common ancestor of a set of particles at the frontier of a one-dimensional branching random walk. We deduce from this very mechanism a simple analytical prediction, Eq. (4) below, which we test against the numerical integration of a previously known equation governing the energy-dependence of high-mass diffraction.

Picture of electron-nucleus scattering at high energy. The scattering of the electron off the nucleus necessarily proceeds through the exchange of a virtual photon $\gamma^*$. We shall denote its virtuality by $Q$, and the center-of-mass energy of the $\gamma^* -$ nucleus subprocess by $W$. These variables are enough to label the total cross section. In the case of diffractive scattering (see Fig. 1), the cross section also depends on the total invariant mass $M_X$ of the produced hadrons. It is convenient to use, instead of $M_X$, the dimensionless rapidity $y_0 \equiv \ln(M_X^2 + Q^2)/Q^2$. The gap can then be characterized by the rapidity $y_0 \equiv Y - \bar{y}_0$, where $Y \equiv \ln(Q^2 + W^2)/Q^2$ is the total relative rapidity of the photon with respect to the nucleus.

These hadrons are distributed all over the detector. This is nevertheless produced. We explain how to characterize them microscopically, and we show that these hadrons are generated from a similar mechanism as the common ancestor of a set of particles at the frontier of a one-dimensional branching random walk. We deduce from this very mechanism a simple analytical prediction, Eq. (4) below, which we test against the numerical integration of a previously known equation governing the energy-dependence of high-mass diffraction.

![Figure 1](image.png)

FIG. 1. Schematic representation of a diffraction event. The initial-state particles are incoming from the left, the final state is on the right. The interaction of the electron with the nucleus is mediated by a virtual photon. While the nucleus is transferred unaltered in its nature to the final state, the photon converts to a set hadrons of total invariant mass $M_X$. The rapidity gap is an angular region around the nucleus in which no particle is observed.
When the energy of the reaction is large, it is possible to choose a reference frame in which the photon is fast enough to almost always convert to a quark-antiquark pair (which we shall call “onium”) before interacting. For our purpose, the only relevant parameter to characterize the latter is the distance \( r \) between the trajectories of the quarks, which can be considered unchanged throughout a scattering at high relative rapidity. The distribution of \( r \) for a given photon virtuality follows from simple electrodynamics. Hence electron-nucleus scattering is tantamount to onium-nucleus scattering. A scattering event occurs as soon as at least one gluon is exchanged between the onium and the nucleus.

A nucleus is a priori a very complicated composite object. However, a large nucleus is made of many hadrons which can be considered uncorrelated. Considering furthermore the number \( N_c \) of colors to be a large parameter, the rapidity evolution of the forward elastic amplitude \( \mathcal{T}(r, y) \) for the scattering of the onium off the nucleus can be established within QCD in these limits. It is given by the Balitsky-Kovchegov (BK) equation [4]:

\[
\frac{\partial T(r, y)}{\partial y} = \tilde{\alpha} [\chi \cdot T(r, y) - T \otimes T(r, y)],
\]

(1)

where \( \tilde{\alpha} \) is proportional to the product of the strong coupling constant \( \alpha_s \) by the number of colors, \( \tilde{\alpha} = \alpha_s N_c/\pi \); \( \chi \) in the first term is the linear operator that acts on a function \( f \) of \( r \) as

\[
\chi \cdot f(r) = \int \frac{d^2 r'}{2\pi} \frac{r^2}{r'^2 (r - r')^2} \left[ f(r') + f(r - r') - f(r) \right];
\]

and finally, the second term in the r.h.s. of (1) is the convolution

\[
f \otimes f(r) = \int \frac{d^2 r'}{2\pi} \frac{r^2}{r'^2 (r - r')^2} f(r') f(r - r').
\]

The elastic onium-nucleus scattering cross section per unit surface is \( \sigma_{\text{el}} = T^2 \) (since \( T \) is essentially real at high energy) evaluated at rapidity \( y = Y \), and the total cross section is twice \( T \) as a consequence of the optical theorem: \( \sigma_{\text{tot}} = 2T \). (The total electron-nucleus cross section may then easily be calculated from \( \sigma_{\text{tot}} \).

Thanks to these notations, the structure of the BK equation (1) is quite clear. The first term, linear in \( T \), encodes the rise of the amplitude due to the multiplication of the gluons in the state of the onium as the rapidity is increased, i.e. as shorter-lived quantum fluctuations become relevant for the scattering. It is well-known that in the large-\( N_c \) limit and in a lightcone gauge, the Fock state of an onium can conveniently be represented by a set of dipoles of various sizes, and rapidity evolution can be thought of as a cascade of independent \( 1 \rightarrow 2 \) splittings of color dipoles [3]. A lightcone perturbation theory calculation in the framework of QCD leads to the expression of the splitting probability density of a dipole of size \( r \) into dipoles of sizes \( r' \), \( r - r' \) as its rapidity is increased by \( dy \); It reads

\[
\frac{dp(r \rightarrow r', r - r')}{dy} = \tilde{\alpha} dy \frac{d^2 r'}{2\pi} \frac{r^2}{r'^2 (r - r')^2}.
\]

The operator \( \chi \), which is constructed from the integral of \( (1/\tilde{\alpha}) dp/ dy \), is also the kernel of the evolution equation solved by the mean number density \( n(r, y) \) of dipoles at rapidity \( y \) in an onium of initial size \( r \): \( \partial_y n = \tilde{\alpha} \chi \cdot n \), which is nothing but the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [3]. The second term in the BK equation (1), significant only when \( T = O(1) \), keeps the amplitude unitary \((T \leq 1)\) throughout the evolution.

It is useful to note that the BK equation is in essence similar to the nonlinear diffusion equation, known as the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [7,8]: These two equations actually belong to the same universality class [9]. Starting from this correspondence, one can take advantage of the available mathematical knowledge on the FKPP equation (for a review, see Ref. [10]). One knows that for a vast class of initial conditions, its solution converges to a traveling wave at large \( y \), namely a front connecting \( T = 1 \) for \( r \) large to \( T = 0 \) for \( r \) small, the rapidity-evolution of which consists in a mere translation in \( r \). The transition region is located around a rapidity-dependent size \( r_s(y) \) related to the saturation momentum \( Q_s \) by \( r_s = 1/Q_s \). The analytical expression of \( Q_s(y) \) for \( \tilde{\alpha} y \gg 1 \) reads

\[
Q_s^2(y) = Q_{\text{MV}}^2 \left( \frac{\tilde{\alpha} y}{\gamma_0} \right)^{27/2},
\]

(2)

up to a multiplicative constant of order one depending on the very definition of \( Q_s \). The complex function \( \chi(\gamma) \) is the set of the eigenvalues of the \( \chi \) operator associated to its eigenfunctions \( r^{2\gamma} \), and \( \gamma_0 \) solves \( \chi(\gamma_0) = \gamma_0 \chi ' (\gamma_0) \). Explicitly, \( \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) \) where \( \psi \) is the digamma function, and \( \gamma_0 \approx 0.63 \). Equation (2) holds whenever the initial condition falls fast enough as \( r \) decreases. More precisely, if \( T(r, 0) \sim_{r \to 0} r^{2\gamma} \), then \( \beta \) must be larger than \( \gamma_0 \) [11].

An analytical expression for the asymptotic shape of the front is also known. It reads

\[
T(r, y) = c_T \ln \frac{1}{r^2 Q_s^2(y)} \left[ r^2 Q_s^2(y) \right]^{\gamma_0},
\]

(3)

where \( c_T \) is a numerical constant. This equation is valid when \( T \ll 1 \), and in the so-called “scaling region” [12]. These two conditions translate into the inequalities \( 1 \ll \ln r^2 Q_s^2(y) \ll \sqrt{\chi'(\gamma_0)\tilde{\alpha} y} \). Throughout, we will always assume that \( r \) is such that both these inequalities are fulfilled.

The initial condition for \( T \) describes the interaction amplitude of the onium with the nucleus at low energy. A nucleus in its restframe is a dense system of quarks. In the so-called McLerran-Venugopal (MV) model [13], it is characterized by a momentum scale \( Q_{\text{MV}} \) function of
the atomic number. (Its value is of order 1 GeV for heavy nuclei such as lead or gold). The scattering amplitude of an onium with a nucleus may be approximated by $T(r, y = 0) = 1 - e^{-r^2 Q^2_{\text{tot}}/4}$. Onia of size much larger than typically $r_{\text{MV}} \equiv 1/Q_{\text{MV}}$ are absorbed, while the nucleus appears transparent to onia of size much smaller than $r_{\text{MV}}$. We note that $T(r, y = 0) \sim r^2$ for small $r$: Hence the solution (2),(3) indeed applies.

The BK equation (1) is also an equation for the $\tilde{y}$-evolution of the probability $P(r, \tilde{y}|R)$ that there be at least a dipole larger than some $R$ in an onium of initial size $r$ [14]. The initial condition in this case reads $P(r, \tilde{y} = 0|R) = \theta(ln r^2/R^2)$, which is of course “steep enough” for the asymptotic solution (3) to be valid. Then, thanks to the universality properties of the asymptotic solution to the BK equation, $P(r, \tilde{y}|R)$ has the same expression as $T(r, y)$ in Eq. (3) up to the substitutions $y \leftrightarrow \tilde{y}$ and $Q_{\text{MV}} \leftrightarrow 1/R$, and maybe up to the overall normalization constant. Thus one can write

$$\sigma_{\text{tot}} \times P(r, Y|1/Q_{\text{MV}}).$$

**Diffraction from rare fluctuations.** If $r$ is small compared to $1/Q_s(Y)$ – i.e. the onium is far from the saturation region of the nucleus –, then from Eq. (3), $T$ is small. Since the forward elastic amplitude $T$ is related to the total, elastic and inelastic cross sections through

$$\sigma_{\text{tot}} = 2T, \quad \sigma_{\text{el}} = T^2, \quad \sigma_{\text{in}} = \sigma_{\text{tot}} - \sigma_{\text{el}},$$

one sees that $\sigma_{\text{el}}$ is of second order in $T$ while $\sigma_{\text{in}}$ is of first order, and thus dominates $\sigma_{\text{tot}}$.

A diffractive event can occur with non-negligible probability only if a large dipole occurs in the Fock state of the onium at some point in the evolution. Indeed, only for such realizations of the evolution the scattering amplitude can be of order 1, and elastic scattering processes are thus probable. (Examples of Feynman diagrams contributing to the onium-nucleus diffractive vs total amplitudes are shown in Fig. 2.) Assume that such a dipole of size $r_0$ is produced at rapidity $\tilde{y}_0 = Y - y_0$. The condition that the whole partonic system scatters elastically with a significant probability is that $r_0$ be larger than the inverse saturation scale of the nucleus evaluated at the rapidity $Y - \tilde{y}_0 = y_0$: $r_0 \geq 1/Q_s(y_0)$. Such an event will exhibit a rapidity gap of size $y_0$.

From this picture, we can immediately infer that the diffractive cross section conditioned to a given rapidity gap $\tilde{y}_0$ is tantamount to the probability $P(r, \tilde{y}_0|1/Q_s(y_0))$. As discussed above, the latter is given by the solution to the BK equation (3) up to the appropriate substitution of the variables and parameters:

$$\frac{d\sigma_{\text{diff}}}{dy_0} = c_{\text{diff}} \ln \frac{1}{r^2 Q^2_s(y_0)} \left[ r^2 \tilde{Q}^2_s(y_0) \right]^{\gamma_0},$$

where $c_{\text{diff}}$ is a constant, and the momentum $\tilde{Q}_s$ reads

$$\tilde{Q}^2_s(y_0) = Q^2_s(y_0) e^{\tilde{y}_0 \Lambda(x)}(\tilde{y}_0)^{\gamma/2} \tilde{y}_0.$$
either (i) according to the Boltzmann weight \( e^{-\beta x} \) (i.e. the particle number \( j \) sitting at position \( x_j \) at time \( t \) is picked with probability \( e^{-\beta x_j} / \sum_k e^{-\beta x_k} \)), or (ii) to be exactly the two leftmost particles, and look for the first common ancestor splitting time \( t - t_0 \). Then, according to Ref. [15], \( t_0 \) is distributed as

\[
p(t_0) = c_p \left[ \frac{t}{t_0(t-t_0)} \right]^{3/2}, \text{ with } c_p = \frac{1}{\gamma} \frac{1}{\sqrt{2\pi \chi'(\gamma_0)}} \tag{5}
\]

where \( \gamma = \beta \) in case (i) and \( \gamma = \gamma_0 \) in case (ii). \( \gamma_0 \) solves \( \chi(\gamma_0) = \gamma_0 \chi'(\gamma_0) \).

In the same way as in our diffraction calculation, the common ancestor of the boundary particles also corresponds to a fluctuation, in the form of a particle sent to the left of the expected position of the leftmost particle, occurring in the course of the evolution at time \( t - t_0 \). Hence the two problems are intimately related: Up to the overall normalization which is determined in the case of the genealogies but not in the case of diffraction, \( 1/\sigma_{tot} (d\sigma_{diff}/dy_0) \) corresponds to \( p(t_0) \), with the identifications \( \bar{a}Y \leftrightarrow t \), \( \bar{a}y_0 \leftrightarrow t_0 \).

**Numerical test.** An equation for the diffractive cross section with a rapidity gap \( y_0 \) was established some time ago in QCD by Kovchegov and Levin (KL) [16] (see also Ref. [2] [17]). It can be put in the form of two appropriately matched evolution equations in the total rapidity variable \( y \), which both turn out to be of the BK type. While this formulation has not led to much analytical insight, in particular for the gap distribution we are addressing here, it is very convenient for the numerical computation of the diffractive cross section, since the BK equation is easily discretized, implemented and solved using standard algorithms [13].

We have computed the rapidity-gap distribution for two values of the total rapidity, \( \bar{a}Y = 10 \) and 20. (These rapidities are of course too large to be realistic for phenomenology, but our goal here is to test our asymptotic prediction.) We have chosen \( r \) in such a way that \( |\ln r^2 Q_2^\gamma(Y)| \simeq 7.2 \), comfortably in the scaling region in both cases. The results are presented in Fig. 3 and compared to the analytical prediction [1], in appropriately rescaled variables chosen such that the expected asymptotic distribution be independent of \( \bar{a}Y \). The overall coefficient of the latter is not predicted in our approach. We could fit it to the numerical data, but interestingly enough, just setting it to be that predicted for BRW, namely \( c_p \) in Eq. [1] with \( \gamma = 1 \), leads to a remarkably good agreement between the numerical data and the prediction.

**Conclusion.** We have found that the distribution of the size \( y_0 \) of rapidity gaps in high-mass diffractive deep-inelastic scattering, or equivalently of the total mass of the diffracted system, can be calculated analytically for fixed large center-of-mass energies. Surprisingly enough, our quantitative prediction follows quite straightforwardly from simple considerations on the mechanism how the Fock state of a quark-antiquark pair evolves when one opens the phase-space for more short-lived fluctuations by increasing its rapidity. The essence of this evolution is that of a one-dimensional branching random walk, and viewed in such a picture, diffractive events are due to the existence of a large fluctuation in the evolution. The rapidity at which it occurs determines the size of the gap.

This large fluctuation can also be identified with the common ancestor of a few extreme objects generated by the BRW. The latter problem is of interest in the study of disordered system. It was known before that the BK and FKPP equations are in the same universality class [8], and also that the energy evolution of the scattering amplitude of ultra-high energy hadrons may be analogous to the time evolution of a reaction-diffusion process, the evolution of which is described by an equation belonging to the universality class of the stochastic FKPP equation [19]. But to our knowledge, this is the first time that the statistical properties of genealogical trees prove of direct relevance in the context of particle/nuclear physics. Hence beyond the new prediction we provide for a future electron-ion collider, our work contributes to bridge a priori unrelated fields of physics.

Further developments along with more numerical studies can be found in Ref. [20].

The work of AHM is supported in part by the U.S. Department of Energy Grant # DE-FG02-92ER40699. The work of SM is supported in part by the Agence Nationale de la Recherche under the project # ANR-16-CE31-0019. We thank Bernard Pire for urging us to try and make the material of the present Letter accessible to a wider audience, and for useful suggestions on the manuscript.
[1] L. Schoeffel, Prog. Part. Nucl. Phys. 65, 9 (2010), arXiv:0908.3287 [hep-ph].
[2] Y. V. Kovchegov and E. Levin, Quantum chromodynamics at high energy, Vol. 33 (Cambridge University Press, 2012).
[3] K. J. Golec-Biernat and M. Wusthoff, Phys. Rev. D59, 014017 (1998), arXiv:hep-ph/9807513 [hep-ph].
[4] I. Balitsky, Nucl. Phys. B463, 99 (1996), arXiv:hep-ph/9509348 [hep-ph] Y. V. Kovchegov, Phys. Rev. D61, 074018 (2000), arXiv:hep-ph/9905214 [hep-ph].
[5] A. H. Mueller, Nucl. Phys. B415, 373 (1994).
[6] L. N. Lipatov, Sov. J. Nucl. Phys. 23, 338 (1976), [Yad. Fiz.23,642(1976)]; E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, Sov. Phys. JETP 45, 199 (1977), [Zh. Eksp. Teor. Fiz.72,377(1977)]; I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978), [Yad. Fiz.28,1597(1978)].
[7] R. A. Fisher, Annals of Eugenics 7, 355 (1937) A. Kolmogorov, I. Petrovsky, and N. Piscounov, Bull. Univ. Etat Moscon A 1 (1937).
[8] S. Munier and R. B. Peschanski, Phys. Rev. Lett. 91, 232001 (2003), arXiv:hep-ph/0309177 [hep-ph].
[9] An exact mapping can be exhibited for the Fourier transform of $T$ defined as $\tilde{T}(k, y) = \int d^2 r / (2\pi r^2) e^{ikr} T(r, y)$ and in the so-called diffusive approximation; See Ref. [8].
[10] W. van Saarloos, Physics Reports 386, 29 (2003).
[11] M. Bramson, Mem. Amer. Math. Soc. 44, iv+190 (1983).
[12] A. M. Stasto, K. J. Golec-Biernat, and J. Kwiecinski, Phys. Rev. Lett. 86, 596 (2001), arXiv:hep-ph/0007192 [hep-ph], E. Iancu, K. Itakura, and L. McLerran, Nucl. Phys. A708, 327 (2002), arXiv:hep-ph/0203137 [hep-ph].
[13] L. D. McLerran and R. Venugopalan, Phys. Rev. D49, 2233 (1994), arXiv:hep-ph/9309289 [hep-ph].
[14] A. H. Mueller and S. Munier, Phys. Lett. B737, 303 (2014), arXiv:1405.3131 [hep-ph].
[15] B. Derrida and P. Mottishaw, EPL (Europhysics Letters) 115, 40005 (2016).
[16] Y. V. Kovchegov and E. Levin, Nucl. Phys. B577, 221 (2000), arXiv:hep-ph/9911523 [hep-ph].
[17] A. Kovner and U. A. Wiedemann, Phys. Rev. D64, 114002 (2001), arXiv:hep-ph/0106240 [hep-ph].
[18] E. Levin and M. Lublinsky, Phys. Lett. B521, 233 (2001), arXiv:hep-ph/0108265 [hep-ph], Eur. Phys. J. C22, 647 (2002), arXiv:hep-ph/0108239 [hep-ph], Nucl. Phys. A712, 95 (2002), arXiv:hep-ph/0207374 [hep-ph].
[19] E. Iancu, A. H. Mueller, and S. Munier, Phys. Lett. B606, 342 (2005), arXiv:hep-ph/0411008 [hep-ph], E. Iancu and D. N. Triantafyllopoulos, Nucl. Phys. A756, 419 (2005), arXiv:hep-ph/0411405 [hep-ph].
[20] A. H. Mueller and S. Munier, submitted to Phys. Rev. D (2018), arXiv:1805.02847 [hep-ph].