Modulation instabilities in birefringent two-core optical fibres

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Abstract

Previous studies of the modulation instability (MI) of continuous waves (CWs) in a two-core fibre (TCF) did not consider effects caused by co-propagation of the two polarized modes in a TCF that possesses birefringence, such as cross-phase modulation (XPM), polarization-mode dispersion (PMD) and polarization-dependent coupling (PDC) between the cores. This paper reports an analysis of these effects on the MI by considering a linear-birefringence TCF and a circular-birefringence TCF, which feature different XPM coefficients. The analysis focuses on the MI of the asymmetric CW states in the TCFs, which have no counterparts in single-core fibres. We find that the asymmetric CW state exists when its total power exceeds a threshold (minimum) value, which is sensitive to the value of the XPM coefficient. We consider, in particular, a class of asymmetric CW states that admit analytical solutions. In the anomalous dispersion regime, without taking the PMD and PDC into account, the MI gain spectra of the birefringent TCF, if scaled by the threshold power, are almost identical to those of the zero-birefringence TCF. However, in the normal dispersion regime, the power-scaled MI gain spectra of the birefringent TCFs are distinctly different from their zero-birefringence counterparts, and the difference is particularly significant for the circular-birefringence TCF, which takes a larger XPM coefficient. On the other hand, the PMD and PDC only exert weak effects on the MI gain spectra. We also simulate the nonlinear evolution of the MI of the CW inputs in the TCFs and obtain good agreement with the analytical solutions.

(Some figures may appear in colour only in the online journal)

1. Introduction

Modulation instability (MI) of plane or continuous waves (CWs) arises in many fields of physics, e.g. Bose–Einstein condensates [1, 2], fluid mechanics [3] and optics [4, 5]. This study focuses on optical fibres, where the dynamics is determined by the interplay of dispersive and nonlinear effects, as has been demonstrated in many diverse settings [4–9]. The MI is a physically important problem, which is closely related to the Fermi–Pasta–Ulam recurrence effect and the formation of solitons [5, 10].
This paper addresses the MI in a two-core fibre (TCF), i.e. a fibre that consists of two linearly coupled identical parallel cores. The phenomenon of periodic optical power transfer between the two cores along a TCF [19] is widely used in many practical fibre-optic devices. Various aspects of the MI in a TCF have been studied [20–22]. The MI characteristics of the symmetric and antisymmetric CW states in a TCF are qualitatively similar to those in a conventional single-core fibre [20]. On the other hand, the spontaneous symmetry breaking in linearly coupled systems gives rise to asymmetric CW states in a TCF [23], which makes its MI characteristics qualitatively different from those of a single-core fibre [21]. In particular, the dispersion (or wavelength dependence) of the coupling coefficient between the two cores can drastically modify the MI bands of the asymmetric states in both the anomalous and normal dispersion regimes [22].

All the previous studies of the MI in a TCF [20–22] ignored potentially significant effects caused by co-propagation of the two polarized modes in a TCF that possesses birefringence, such as cross-phase modulation (XPM), polarization-mode dispersion (PMD) and polarization-dependent coupling (PDC) between the cores. In reality, TCFs, especially the recent ones based on photonic-crystal structures [24], can exhibit strong linear birefringence. The objective of this work is to study the effects of birefringence on the MI characteristics of a TCF, by considering a linear-birefringence TCF and a circular-birefringence TCF. A linear-birefringence TCF, where each core supports two orthogonal linearly polarized modes, is just the ordinary TCF used nowadays (see, for example, [24]). A circular-birefringence TCF, where each core supports the right- and left-circularly polarized modes, is a special fibre that could be fabricated by making the two cores rotate around the central axis of the fibre rapidly (e.g., by rapidly spinning the fibre preform during the fibre drawing process or by strongly twisting an ordinary TCF) [25]. The circular-birefringence TCF features much stronger XPM than the linear-birefringence TCF and, therefore, a comparison of their MI characteristics can help to highlight the significance of the XPM effects.

In this paper, the asymmetric CW states are considered, rather than the symmetric/antisymmetric CW states, for which the situation is too similar to previously studied cases [20]. We find that the asymmetric CW state emerges when the total input power exceeds a minimum (threshold) value, which strongly depends on the XPM coefficient. While the most general asymmetric CW states are not tractable analytically, we consider a special class of asymmetric CW states that admit analytical solutions. In the anomalous dispersion regime, without considering the PMD and PDC effects, the MI gain spectra of both birefringent TCFs are almost identical to those of the zero-birefringence TCF, if scaled by the respective threshold powers. However, in the normal dispersion regime, the power-scaled MI gain spectra of the birefringent TCFs are notably different from those of the zero-birefringence TCF and the difference is much stronger for the circular-birefringence TCF. On the other hand, the PMD and PDC in these fibres only have a weak influence on the MI characteristics. In addition, we verify the predictions on the dominant unstable mode from the analytical MI solutions, by direct numerical simulations of the coupled NLS equations.

2. Coupled-mode equations and the analysis of MI

2.1. Coupled-mode equations

In the high-birefringence limit for the TCF, the propagation of slowly varying amplitudes of the electric fields along the $z$ coordinate is described by four coupled generalized NLS equations:

$$i \left( \frac{\partial a_{1x}}{\partial z} + \beta_{1x} \frac{\partial a_{1x}}{\partial t} \right) = -\frac{1}{2} \beta_{2x} \frac{\partial^2 a_{1x}}{\partial t^2} + \gamma \left( |a_{1x}|^2 + \sigma |a_{1y}|^2 \right) a_{1x} + C_{1x} a_{2x} + i C_{1x} \frac{\partial a_{1x}}{\partial t} = 0,$$

$$i \left( \frac{\partial a_{2x}}{\partial z} + \beta_{1x} \frac{\partial a_{2x}}{\partial t} \right) = -\frac{1}{2} \beta_{2x} \frac{\partial^2 a_{2x}}{\partial t^2} + \gamma \left( |a_{2x}|^2 + \sigma |a_{2y}|^2 \right) a_{2x} + C_{1x} a_{1x} + i C_{1x} \frac{\partial a_{2x}}{\partial t} = 0,$$

$$i \left( \frac{\partial a_{1y}}{\partial z} + \beta_{2y} \frac{\partial a_{1y}}{\partial t} \right) = -\frac{1}{2} \beta_{2y} \frac{\partial^2 a_{1y}}{\partial t^2} + \gamma \left( |a_{1y}|^2 + \sigma |a_{1x}|^2 \right) a_{1y} + C_{1y} a_{1y} + i C_{1y} \frac{\partial a_{1y}}{\partial t} = 0,$$

$$i \left( \frac{\partial a_{2y}}{\partial z} + \beta_{2y} \frac{\partial a_{2y}}{\partial t} \right) = -\frac{1}{2} \beta_{2y} \frac{\partial^2 a_{2y}}{\partial t^2} + \gamma \left( |a_{2y}|^2 + \sigma |a_{2x}|^2 \right) a_{2y} + C_{1y} a_{1y} + i C_{1y} \frac{\partial a_{2y}}{\partial t} = 0.$$

(1)

Here $a_{mj}$ ($m = 1, 2$ and $j = x, y$) are the amplitudes of the $j$ polarization in the $m$th core, $\beta_{1j}$ is the group delay of the respective polarization, $\beta_{2j}$ is its GVD coefficient at the carrier frequency ($\beta_{2j}$ are negative and positive for anomalous and normal dispersion, respectively), $\gamma_j$ is the nonlinearity coefficient of the $j$ polarization, $\sigma$ is the relative XPM coefficient, $C_{ij} = dC_{ij}/d\omega$, evaluated at the carrier frequency $\omega$, represents the dispersion of the coupling coefficient in the $j$ polarization. The latter effect is equivalent to the intermodal dispersion arising from the group-delay difference between the even and odd supermodes of the TCF [26, 27]. Nonlinear coupling between the two cores is ignored in equation (1), which is justified by the fact that, in most practical situations, the spatial overlap between the fields of the modes propagating in the two cores is negligibly small. Nonlinear coupling may need to be included, however, in unusual situations where the two cores are very close to each other and the modes are operated in the close-to-cutoff regime (i.e. when the two cores are exceptionally strongly coupled in the linear sense).

The subscripts $x$ and $y$, attached to the polarization components in equation (1), naturally refer to the orthogonal linearly polarized (in the $x$- and $y$-directions) modes of the individual cores of the linear-birefringence TCF, whose XPM coefficient is $\sigma = 2/3$ [28]. For the sake of the uniformity of the notation, we apply the same subscripts, $x$ and $y$, to the clockwise and counter-clockwise circularly polarized modes of the individual cores of the circular-birefringence TCF, whose XPM coefficient is $\sigma = 2$ [28]. As usual, rapidly
oscillating four-wave mixing terms are neglected in equation (1) [5].

Usually, the polarization dependences of the GVD, the nonlinearity coefficient and the coupling coefficient dispersion are weak. Consequently, we set \( \beta_{21} = \beta_{22} \equiv \beta_2 \), \( \gamma_1 = \gamma_2 \equiv \gamma \) and \( C_{12} = C_{13} \equiv C_1 \). The PMD and PDC coefficients in the TCF are defined as \( \Gamma \equiv \beta_1 - \beta_2 \) and \( \Delta C \equiv C_2 - C_\gamma \). The objective of the study is to understand how the polarization-dependent parameters \( \sigma, \Gamma \) and \( \Delta C \) affect the MI characteristics of the TCF, when CWs carried by both polarized modes of the fibre are launched into the fibre.

It is relevant to mention that a system of four coupled equations meant for a birefringent TCF was postulated in some earlier works in the literature [29], but the system there does not include any linear coupling between the two cores, and hence cannot be considered as a valid model for a TCF. As shown in the following sections (and is obvious anyway), the linear coupling coefficients, namely \( C_1 \) and \( C_\gamma \), which are absent in the earlier models [29], play a central role in the analysis of the MI characteristics of a birefringent TCF.

### 2.2. MI analysis

CW solutions to equation (1) depend on two different propagation constants, \( k_1 \) and \( k_2 \), which pertain to the different polarization components, while a given polarization must be, obviously, carried by the same propagation constant in both cores:

\[
\begin{align*}
A_{1x} &= A_1 \exp(ik_1z), \\
A_{2x} &= A_2 \exp(ik_2z), \\
A_{1y} &= B_1 \exp(ik_2z), \\
A_{2y} &= B_2 \exp(ik_2z).
\end{align*}
\]  

Equations (1) and (2) admit one CW solution with the amplitudes obeying the following relations:

\[
\begin{align*}
A_1A_2 &= \frac{C_\gamma - \sigma C_\gamma}{\gamma(1 - \sigma^2)}, & B_1B_2 &= \frac{C_\gamma - \sigma C_\gamma}{\gamma(1 - \sigma^2)}, \\
\frac{B_1}{A_1} &= \frac{B_2}{A_2} = \sqrt{\frac{\sigma C_\gamma - C_\gamma}{\sigma C_\gamma - C_\gamma}}(3a)
\end{align*}
\]

and

\[
\begin{align*}
k_1 &= \gamma\left(1 + \frac{C_\gamma}{C_\chi + C_\gamma}\right)P, & k_2 &= \gamma\left(1 + \frac{C_\gamma}{C_\chi + C_\gamma}\right)P - \frac{1}{P}, (3b)
\end{align*}
\]

where \( P = A_1^2 + B_1^2 + A_2^2 + B_2^2 \) is the total CW power. The CW state given by equations (2) and (3) is symmetric or antisymmetric when \( A_1 = A_2 \) or \( A_1 = -A_2 \), being asymmetric otherwise. Note that equation (3a) yields physically relevant (real) solutions under the conditions \( \sigma C_\gamma < C_\gamma < C_\chi/\gamma \) for 0 \( \leq \sigma < 1 \) and \( \sigma C_\gamma > C_\gamma > C_\chi/\gamma \) for \( \sigma \geq 1 \).

In the CW solutions given by equations (2) and (3), one of the amplitudes, for instance, \( A_1 \), may be chosen arbitrarily, as only three relations out of four in equation (3a) are independent. The most general CW asymmetric solutions of equation (1) depend on two independent parameters, namely the two propagation constants, \( k_1 \) and \( k_2 \), in equation (2). It is, however, impossible to find them analytically and hence study their MI in an analytical form. Therefore, we focus on the special analytical solutions given by equations (2) and (3), which, as elaborated in section 3, are sufficient to demonstrate MI gain spectra that are notably different from their counterparts in a TCF without birefringence.

We also note that equations (2) and (3) produce asymmetric CW solutions provided that the total CW power, \( P \), exceeds a minimum (threshold) value, \( P_{\text{min}}(\sigma) \). With regard to equation (3a), this condition is cast into the following form:

\[
P \equiv A_1^2 + B_1^2 + A_2^2 + B_2^2 = (1 - \sigma) \frac{C_\chi + C_\gamma}{\gamma(1 + \sigma)} (A_1^2 + A_2^2)
\]  

\[
\geq P_{\text{min}}(\sigma) \equiv 2 \frac{C_\chi + C_\gamma}{\gamma(1 + \sigma)}. (4)
\]

At the threshold, \( P = P_{\text{min}}(\sigma) \), one has \( A_1 = A_2 = \sqrt{\frac{C_\chi - \sigma C_\gamma}{\gamma(1 - \sigma^2)}} \).

The minimum power \( P_{\text{min}}(\sigma) \) is sensitive to the value of the XPM coefficient \( \sigma \); the linear-birefringence TCF (\( \sigma = 2/3 \)) has a higher threshold power than the circular-birefringence TCF (\( \sigma = 2 \)). As shown in section 3, the minimum power plays a crucial role in the study of the MI gain spectra of the TCFS.

In the experiment, the transition to the asymmetric state can also be controlled, in an obvious way, by means of the total power of the beam coupled into the TCF. If the power exceeds \( P_{\text{min}}(\sigma) \), the asymmetric configuration will form by itself, through the instability of the symmetric state.

For a given total power, \( P > P_{\text{min}}(\sigma) \), the powers of the four components of the asymmetric CW solutions are

\[
A_1^2 = \frac{C_\chi - \sigma C_\gamma}{(1 - \sigma)(C_\chi + C_\gamma)} \left(P \pm \sqrt{P^2 - P_{\text{min}}^2(\sigma)}\right), (5)
\]

\[
B_1^2 = \frac{C_\chi - \sigma C_\gamma}{(1 - \sigma)(C_\chi + C_\gamma)} \left(P \pm \sqrt{P^2 - P_{\text{min}}^2(\sigma)}\right), (6)
\]

\[
A_2^2 = \frac{C_\chi - \sigma C_\gamma}{(1 - \sigma)(C_\chi + C_\gamma)} \left(P \mp \sqrt{P^2 - P_{\text{min}}^2(\sigma)}\right), (7)
\]

\[
B_2^2 = \frac{C_\chi - \sigma C_\gamma}{(1 - \sigma)(C_\chi + C_\gamma)} \left(P \mp \sqrt{P^2 - P_{\text{min}}^2(\sigma)}\right). (8)
\]

Hence, the power ratio between the two cores is

\[
\frac{A_1^2 + B_1^2}{A_2^2 + B_2^2} = \frac{P \mp \sqrt{P^2 - P_{\text{min}}^2(\sigma)}}{P \pm \sqrt{P^2 - P_{\text{min}}^2(\sigma)}} = R \mp \sqrt{R^2 - 1} (9)
\]

where \( R = P/P_{\text{min}}(\sigma) \) is the total power normalized to the minimum power.

To study the stability of the CW state, we seek perturbed solutions in the following form:

\[
\begin{align*}
a_{1x} &= (A_1 + u_1) \exp(ik_1z), & a_{2x} &= (A_2 + u_2) \exp(ik_2z), & a_{1y} &= (B_1 + v_1) \exp(ik_2z), & a_{2y} &= (B_2 + v_2) \exp(ik_2z),
\end{align*}
\]

where \( u_1 \equiv u(z, t) \) and \( v_1 \equiv v(z, t) \) (\( i = 1, 2 \)) are small perturbations. With equation (10) inserted into equation (1),
the linearization with respect to $u_i$ and $v_i$ yields

$$\begin{align*}
\Gamma_i &\left(\frac{\partial u_i}{\partial x} + \beta_{1i} \frac{\partial u_i}{\partial t} \right) - \frac{1}{2} \beta_{1i}^2 \frac{\partial^2 u_i}{\partial t^2} + i c_1 \frac{\partial u_i}{\partial t} + c_{1i} u_i \\
+ y_i \left[ (A_i^2 - A_i^2 - \sigma B_i^2) u_i + A_i^2 v_i^2 + \sigma A_i B_i (u_i + v_i^2) \right] &= 0,
\end{align*}$$

with $c_{1i} = c_1 \exp(i \sigma k_i z)$ and $g_{1i} = g_1 \exp(-i \sigma k_i z)$.

Further, we seek solutions for the perturbation modes in the following natural form:

$$\begin{align*}
u_1 &= f_1 \exp(i k_1 z) + g_1 \exp(-i k_1 z), \\
u_2 &= f_2 \exp(i k_2 z) + g_2 \exp(-i k_2 z),
\end{align*}$$

(12)

where $f_i, g_i, f, g$ and $i = 1, 2$ are real amplitudes, and $K$ and $\Omega$ are the wave number and the frequency of the perturbations, respectively. After substituting equation (12) into equation (11), the existence of nontrivial solutions for $F_1, G_1, f_1$ and $g_1$ requires the vanishing of the determinant of the corresponding coefficient matrix $M_i$, i.e.

$$\det[M(K)] = 0 \quad (13)$$

with

$$M(K) = \left\{ \begin{array}{l}
(A_i^2 - A_i^2 - \sigma B_i^2) y_i - \beta_{1i} \Omega + \frac{1}{2} \beta_{1i}^2 \Omega^2 - K, \gamma_i, c_i, \zeta_i, 0, \sigma \lambda_i, \sigma \lambda_i, 0, 0 \\
\end{array} \right.$$}

(14)

The dispersion relation that determines the MI is given by equation (13) for $K$ as a function of real $\Omega$. MI will occur when there are complex solutions for $K$, and the MI gain is then given by

$$g(\Omega) = |\text{Im}(K)|. \quad (15)$$

In the general case, the dispersion relation is quite involved. In the following sections, we provide explicit MI results for several special cases.

2.3. Zero PMD and PDC: $\Gamma = 0$ and $\Delta C = 0$

In this special case, the asymmetric CW solution given by equations (2) and (3) degenerates into

$$a_{1i} = A_i \exp(ik_1 z), \quad a_{2i} = A_2 \exp(ik_2 z), \quad a_{1j} = A_1 \exp(ik_1 z), \quad a_{2j} = A_2 \exp(ik_2 z). \quad (16)$$

with the amplitudes, propagation constants and total power determined by the following relations:

$$A_1 A_2 = \frac{C}{\gamma \left(1 + \sigma\right)}, \quad k_1 = k_2 = \frac{1}{2} \gamma \left(1 + \sigma\right) P, \quad P = 2 \left(A_i^2 + A_i^2\right). \quad (17)$$

The dispersion relation (13) is simplified to

$$\begin{align*}
\left[ \left( K - \Omega \right) \left( \beta_{1i} + \frac{\sqrt{3}}{2} C_i \right) \right]^2 - r_1 \\
\times \left[ \left( K - \Omega \right) \left( \beta_{1i} - \frac{\sqrt{3}}{2} C_i \right) \right]^2 - r_2 \\
\times \left[ \left( K - \Omega \right) \left( \beta_{1i} + \frac{\sqrt{3}}{2} C_i \right) \right]^2 - r_4 \\
\times \left[ \left( K - \Omega \right) \left( \beta_{1i} - \frac{\sqrt{3}}{2} C_i \right) \right]^2 - r_5 \\
\times \left[ \left( K - \Omega \right) \left( \beta_{1i} + \frac{\sqrt{3}}{2} C_i \right) \right]^2 - r_6 \right) = 0, \quad (18)
\end{align*}$$

with

$$\begin{align*}
r_1 &= \frac{1}{4} \beta_{1i}^2 \Omega^4 + \left( \frac{1}{2} C_i^2 + \sqrt{2} \beta_{1i} C_i \right) \Omega^2 + 2 C_i^2 (R^2 - 1), \\
r_2 &= \frac{1}{4} \beta_{1i}^2 \Omega^4 + \left( \frac{1}{2} C_i^2 - \sqrt{2} \beta_{1i} C_i \right) \Omega^2 + 2 C_i^2 (R^2 - 1), \\
r_3 &= \frac{1}{2} \beta_{1i}^2 \Omega^4 C_i^2 \left[ 5 \beta_{1i}^2 C_i^2 + C_i^4 - 4 \beta_{1i}^2 R C_i^2 \right] \Omega^4 \\
&- 8 C_i^2 (R^2 - 1) (C_i \beta_{1i}^2 + C_i^2) \Omega^2 + 4 C_i^4 (R^2 - 1)^2, \\
r_4 &= \frac{1}{4} \beta_{1i}^2 \Omega^4 + \left( \frac{1}{2} C_i^2 + \sqrt{2} \beta_{1i} C_i - 2 R C_i \sigma \rho_{1} \right) \Omega^2 \\
&+ 2 C_i^2 (R^2 - 1 - \frac{1 - \sigma}{1 + \sigma}) - 4 \sqrt{2} C_i^2 R \frac{\sigma}{\sigma + 1}, \\
r_5 &= \frac{1}{4} \beta_{1i}^2 \Omega^4 + \left( \frac{1}{2} C_i^2 - \sqrt{2} \beta_{1i} C_i - 2 R C_i \sigma \rho_{1} \right) \Omega^2 \\
&+ 2 C_i^2 (R^2 - 1 - \frac{1 - \sigma}{1 + \sigma}) + 4 \sqrt{2} C_i^2 R \frac{\sigma}{\sigma + 1}, \\
r_6 &= \frac{1}{2} \beta_{1i}^2 \Omega^4 C_i^2 \left[ \rho_{1}^2 C_i^4 + (1 + \sigma)^2 \right] - \frac{4 \beta_{1i}^2 C_i^2 R^2}{(\sigma + 1)^2}, \\
&+ C_i^4 \left( \frac{1}{\sigma + 1} \right) \Omega^4 \\
&- 8 C_i^2 \beta_{1i}^2 \Omega^4 C_i^2 \left[ \rho_{1}^2 C_i^4 + (1 + \sigma)^2 \right] - \frac{4 \beta_{1i}^2 C_i^2 R^2}{(\sigma + 1)^2}, \\
&+ 4 C_i^2 \left( R^2 - 1 + \frac{1 - \sigma}{1 + \sigma} \right) + \left( \frac{1 - \sigma}{1 + \sigma} \right)^2, \\
R &= \frac{P}{P_{\text{min}}(\sigma)}, \quad P = 2 \left(A_i^2 + A_i^2\right). \quad (20)
\end{align*}$$

$$P_{\text{min}}(\sigma) = \frac{4 C}{\gamma \left(1 + \sigma\right)}. \quad (21)$$

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The above asymmetric CW solution, which is obtained for zero PMD and PDC, does not reduce to that for a zero-birefringence TCF [22]. The values of the XPM coefficients are different for different fibres, which generally lead to different MI characteristics.

2.4. Zero-birefringence TCF: \( \sigma = 0 \)

With \( \sigma = 0 \), the two polarized modes are uncoupled and equation (1) degenerates into two identical sets of coupled equations, with either set representing a zero-birefringence TCF. For either polarization, the corresponding dispersion relation from equation (13) reduces to

\[
\left( K - \Omega \left( \beta_1 + \frac{\sqrt{2}}{2} C_1 \right) \right)^2 - r_1 \times \left( K - \Omega \left( \beta_1 - \frac{\sqrt{2}}{2} C_1 \right) \right)^2 - r_2 - r_3 = 0, \tag{22}
\]

where \( \beta_1 \) can be either \( \beta_{1x} \) or \( \beta_{1y} \), and \( r_1, r_2 \) and \( r_3 \) are given by equation (19) with

\[
R = \frac{P}{P_{\text{min}}(\sigma = 0)}, \quad P = A_x^2 + A_y^2, \tag{23}
\]

\[
P_{\text{min}}(\sigma = 0) = \frac{2C_0}{\gamma}. \tag{24}
\]

Note that the minimum power given by equation (24) is smaller by a factor of 2 than that obtained by setting \( \sigma = 0 \) in equation (21). The reason is that the input power for the zero-birefringence fibre \((\sigma = 0)\), as defined by equation (23), is also smaller by a factor of 2 than that for the birefringent fibre, as defined by equation (20). Equations (22) and (24) have been derived elsewhere by directly solving the system of two coupled equations [22].

3. Numerical results

3.1. Linear-birefringence TCF \((\sigma = 2/3)\)

We first address the MI in the anomalous dispersion regime. The following physical parameters are taken here: \( \beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}, \gamma = 3 \text{ (kW m)}^{-1}, C_x = C_y = 200 \text{ m}^{-1} \text{ and } \Gamma = C_1 = 0 \). The range of the numerical values for the coupling coefficient in typical TCFs can be found in [26]. A coupling coefficient of 200 m\(^{-1}\) corresponds roughly to a TCF with a core-to-core separation of three to four times of the core radius, operating at the wavelength of 1.55 \(\mu\)m. To highlight the effects due to the XPM, both PMD and PDC are ignored.

Figure 1(a) shows the variation of the MI gain spectrum of the linear-birefringence TCF \((\sigma = 2/3)\) with the normalized total input power, \(P/P_{\text{min}}\). For comparison, figure 1(b) shows the same physical entities for the zero-birefringence TCF \((\sigma = 0)\). An examination of figures 1(a) and (b) shows that the results are almost identical, regardless of the fact that the two systems have different threshold powers, i.e. \( \sigma = 1/3 \) = 160 and \( \sigma = 0 \) = 133.3 kW. The level of similarity is shown more clearly in figure 1(c), where the MI gain spectra of the two systems are juxtaposed at several normalized input powers. The results suggest that the MI gain spectrum of the linear-birefringence TCF can be obtained from that of the zero-birefringence TCF by a straightforward rescaling of the input power, i.e. the XPM effects can simply be taken into account by means of power rescaling. This is a surprising result, considering the fact that the linear-birefringence TCF is described by four coupled equations, while the zero-birefringence TCF is described by two coupled equations.

The effects of the PMD on the MI gain spectrum in the anomalous dispersion regime are shown in figure 2. The PMD in a TCF should be similar to that in a single-core fibre, so the PMD values used in these examples are taken as typical ones for a single-core fibre [15]. It is seen from figure 2 that PMD leads only to a slight decrease of the MI gain. For a realistic TCF, the value of PDC is within ±0.1% of the coupling coefficient, which can hardly affect the MI gain spectrum.

Figure 1. Variations of the MI gain spectra with the normalized input power, \(P/P_{\text{min}}\), in the anomalous dispersion regime for (a) the linear-birefringence TCF, (b) the zero-birefringence TCF and (c) the comparison of the MI gain spectra in the linear-birefringence (dashed) and zero-birefringence (solid) TCFs at several normalized input powers. The results are obtained for the following parameters: \( \beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}, \gamma = 3 \text{ (kW m)}^{-1}, C_x = C_y = 200 \text{ m}^{-1} \text{ and } \Gamma = C_1 = 0 \).
Figure 2. (a) 3D and (b) 2D plots showing the variation of the MI gain spectrum with the PMD in the anomalous dispersion regime for the linear-birefringence TCF ($\sigma = 2/3$) with $\beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 3 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$ and $\Gamma = C_1 = 0$. The total input power is $P = (A_2^2 + B_2^2) = 200 \text{ kW}$. 

Next, we consider the MI in the normal dispersion regime, taking the following physical parameters: $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$ and $\Gamma = C_1 = 0$. Again, both PMD and PDC are set to zero to highlight the effects of XPM. 

Figures 3(a) and (b) show the evolution of the MI gain spectra with the normalized total input power for the linear-birefringence and zero-birefringence TCFs, respectively, and figure 3(c) compares the MI gain spectra of the two fibres at several normalized input powers. The results are obtained for the following parameters: $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$ and $\Gamma = C_1 = 0$. Note that the minimum power of the circular-birefringence TCF is $P_{\text{min}}(\sigma = 2) = 88.9 \text{ kW}$, which is significantly smaller than those of the linear-birefringence TCF $[P_{\text{min}}(\sigma = 2/3) = 160 \text{ kW}]$ and the zero-birefringence TCF $[P_{\text{min}}(\sigma = 0) = 133.3 \text{ kW}]$. 

3.2. Circular-birefringence TCF ($\sigma = 2$) 

In the anomalous dispersion regime, as in the case of the linear-birefringence TCF, the MI characteristics of the circular-birefringence TCF are almost identical to those of the zero-birefringence TCF, if the MI gain spectra of both fibres are expressed in terms of the respective normalized input powers, as shown in figure 5 for the same set of parameters as in figure 1: $\beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 3 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$ and $\Gamma = C_1 = 0$. Note that the minimum power of the circular-birefringence TCF is $P_{\text{min}}(\sigma = 2) = 88.9 \text{ kW}$, which is significantly smaller than those of the linear-birefringence TCF $[P_{\text{min}}(\sigma = 2/3) = 160 \text{ kW}]$ and the zero-birefringence TCF $[P_{\text{min}}(\sigma = 0) = 133.3 \text{ kW}]$. 

The situation in the normal dispersion regime is very different. The evolution of the MI gain spectra of the circular-birefringence TCF and that of the zero-birefringence TCF with the respective normalized input powers are shown in figure 6 for $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$, $\Gamma = C_1 = 0$, to allow a direct comparison with the results for the linear-birefringence TCF presented in figure 3. The minimum power of the circular-birefringence TCF is $P_{\text{min}}(\sigma = 2) = 44.4 \text{ kW}$. As shown in figure 6, for the circular-birefringence TCF, a
new MI band appears and quickly becomes dominant with an increase in the input power. This scenario is markedly different from the situation for the zero-birefringence TCF.

In both the anomalous and normal dispersion regimes, the PMD and PDC in the circular-birefringence TCF produce only weak effects on the MI characteristics, similar to the properties exhibited in a linear-birefringence TCF.

It is worthwhile to highlight that the XPM coefficient $\sigma = 2$ case also corresponds to a different, and meaningful, physical configuration, namely CWs of two different wavelengths being simultaneously launched into a zero-birefringence TCF. Consequently, the results obtained for the circular-birefringence TCF apply equally well to the two-wavelength case, where the PMD should be interpreted as the group-delay difference between the two wavelengths, and PDC as the wavelength-dependent coupling. A more general analysis of the MI in the two-wavelength case should also take into account the wavelength dependence of the GVD and the dispersion of the coupling coefficient.

4. Comparison with numerical simulations

To verify the above MI analysis, equation (1) was solved numerically by launching asymmetric CW states, perturbed by small-amplitude white noise, into the fibre. The equations were solved by means of the pseudospectral method in the time domain, and the fourth-order Runge–Kutta scheme with an adaptive step-size control in the space domain. The power of the added white noise was typically about 0.01% of the input CW power, and the bandwidth of the noise covered the range of $[-1200 \text{ THz}, 1200 \text{ THz}]$, centred at the carrier optical frequency. In terms of the MI analysis, the maximum initial growth rate of the perturbations is expected to be in the vicinity of the dominant MI frequency, which corresponds to the maximum gain. As a result, the actual value of the dominant frequency at the onset of instability in the numerical simulations may be used to verify the MI analysis. The propagation distance is normalized by the coupling length, defined by $L_c = \pi / (2C)$, where $C_t = C_y \equiv C$ is assumed.

4.1. The anomalous dispersion regime

In the anomalous dispersion regime, the following fibre parameters were used, assuming a carrier wavelength of $1.5 \mu\text{m}$: $\beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 3 \text{ (kW m)}^{-1}$, $C_t = C_y = C = 200 \text{ m}^{-1}$ and $\Gamma = C_1 = 0$. The minimum powers for the corresponding zero-birefringence, linear-birefringence and
Variations of the MI gain spectra with the normalized input power $P/P_{\text{min}}$ in the normal dispersion regime for (a) the circular-birefringence TCF and (b) the zero-birefringence TCF and (c) the comparison of the MI gain spectra of the circular-birefringence (dashed) and zero-birefringence (solid) TCFs at several normalized input powers. The results are obtained for the following parameters: $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$ and $\Gamma = C_1 = 0$.

Evolution of the asymmetric CW input in a zero-birefringence TCF with $\beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 3 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$, $\Gamma = C_1 = 0$, $P_{\text{min}}(\sigma = 0) = 133.3$ kW and $P/P_{\text{min}}(\sigma = 0) = 1.2$.

Evolution of the asymmetric CW input in a linear-birefringence TCF with $\beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 3 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$, $\Gamma = C_1 = 0$, $P_{\text{min}}(\sigma = 2/3) = 160$ kW and $P/P_{\text{min}}(\sigma = 2/3) = 1.2$.

Evolution of the asymmetric CW input in a circular-birefringence TCF with $\beta_2 = -0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 3 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$, $\Gamma = C_1 = 0$, $P_{\text{min}}(\sigma = 2) = 88.9$ kW and $P/P_{\text{min}}(\sigma = 2) = 1.2$.

Figures 7–9 display the wave-propagation dynamics for the zero-birefringence ($\sigma = 0$), linear-birefringence ($\sigma = 2/3$) and circular-birefringence ($\sigma = 2$) TCFs, respectively. In each case, the total input power normalized by the respective threshold (minimum) power is fixed at 1.2, and the power ratio between the two cores is 3.47. The period of the modulated waves at the onset of the MI can be estimated from the propagation dynamics. From the results presented in figures 7–9, the periods for the zero-birefringence, linear-birefringence and circular-birefringence TCFs are found to be 33.5, 34.3 and 34.2 fs, respectively (all at $z = 4L_c$), which correspond to the modulation frequencies 29.9, 29.2 and 29.3 THz, in good agreement with the dominant MI frequencies from the MI analysis: 29.15, 29.15 and 29.24 THz. As predicted by the MI analysis, the three fibres have almost identical MI gain profiles and should start to produce the MI at similar propagation distances.

4.2. The normal dispersion regime

In the normal dispersion regime, the following fibre parameters are used: $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = C = 200 \text{ m}^{-1}$ and $\Gamma = C_1 = 0$. The minimum powers for the zero-birefringence, linear-birefringence and circular-birefringence TCFs are, respectively, $P_{\text{min}}(\sigma = 0) = 66.7$, $P_{\text{min}}(\sigma = 2/3) = 80$ and $P_{\text{min}}(\sigma = 2) = 44.4$ kW.

Figures 10–12 show the wave propagation dynamics for the single-polarization ($\sigma = 0$), linear-birefringence ($\sigma = 2/3$) and circular-birefringence ($\sigma = 2$) TCFs, respectively.
Figure 10. Evolution of the asymmetric CW state in a zero-birefringence TCF with $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$, $C_1 = 0$, $P_{\text{min}}(\sigma = 0) = 66.7 \text{ kW}$ and $P/P_{\text{min}}(\sigma = 0) = 2.5$.

Figure 11. Evolution of the asymmetric CW state for a linear-birefringence TCF with $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$, $\Gamma = C_1 = 0$, $P_{\text{min}}(\sigma = 2/3) = 80 \text{ kW}$ and $P/P_{\text{min}}(\sigma = 2/3) = 2.5$.

Figure 12. Evolution of the asymmetric CW state for a circular-birefringence TCF with $\beta_2 = 0.02 \text{ ps}^2 \text{ m}^{-1}$, $\gamma = 6 \text{ (kW m)}^{-1}$, $C_x = C_y = 200 \text{ m}^{-1}$, $\Gamma = C_1 = 0$, $P_{\text{min}}(\sigma = 2) = 44.4 \text{ kW}$ and $P/P_{\text{min}}(\sigma = 2) = 2.5$.

In each case, the total input power normalized by the respective minimum power is fixed at 2.5, and the power ratio between the two cores is 22.96. The periods of the modulated waves for the zero-birefringence, linear-birefringence and circular-birefringence TCFs are found to be 41.1 (at $z = 12L_c$), 43.5 (at $z = 12L_c$) and 39.9 fs (at $z = 5L_c$), which correspond, respectively, to the modulation frequencies 24.3, 23.0 and 25.1 THz, in good agreement with the results of the MI analysis: 24.45, 24.42 and 27.36 THz. As expected, the distances needed for the onset of the MI in the zero-birefringence and linear-birefringence TCFs are similar, which conforms to the finding from the MI analysis that the MI gain profiles are nearly identical in these two cases. On the other hand, the MI occurs at a much shorter distance in the circular-birefringence TCF, as predicted by the MI analysis, in view of the much larger growth rate.

5. Conclusions

In this work, we have analysed in detail the MI characteristics of linear-birefringence and circular-birefringence TCFs for a class of asymmetric CW inputs which admit analytical solutions. We have also verified the predictions of the MI analysis by means of direct simulations of wave propagation.
along different fibres. The asymmetric CW state exists when the total input power exceeds a minimum value, which depends on the fibre type (zero-birefringence, linear-birefringence or circular-birefringence). In the anomalous dispersion regime, the MI gain spectra of the three fibres are almost identical, if scaled with the respective minimum powers. The results suggest that the XPM interaction between the polarized modes of a birefringent TCF does not generate any new MI characteristics in the anomalous dispersion regime. In the normal dispersion regime, however, the power-scaled MI gain spectra corresponding to the three fibres are different and the difference is particularly significant for the circular-birefringence TCF, which has the largest XPM coefficient ($\sigma = 2$). In both regimes, the PMD and PDC show only weak effects on the MI characteristics of the TCF. Finally, it is relevant to highlight the significance of the MI in settings other than conventional optical fibres. In particular, linear couplings between parallel photonic nanowires [30–33] may display strong dispersion, which can consequently generate especially pronounced MI effects. The associated spatiotemporal solitary pulses and other related soliton phenomena [34] suggest fruitful directions of future research.

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References

[1] Kamchatnov A M and Salerno M 2009 J. Phys. B: At. Mol. Opt. Phys. 42 185303
[2] Das P, Vyas M and Panigrahi P K 2009 J. Phys. B: At. Mol. Opt. Phys. 42 245304
[3] Mei C C 1989 The Applied Dynamics of Ocean Surface Waves (Singapore: World Scientific)
[4] Taylor J R (ed) 1992 Optical Solitons: Theory and Experiments (London: Cambridge University Press)
[5] Kivshar Y S and Agrawal G P 2003 Optical Solitons: From Fibers to Photonic Crystals (San Diego, CA: Academic)
[6] Tai K, Hasegawa A and Tomita A 1986 Phys. Rev. Lett. 56 135
[7] Anderson D and Lisak M 1984 Opt. Lett. 9 468
[8] Shukla P K and Rasmussen J J 1986 Opt. Lett. 11 171
[9] Uzunov I M 1990 Opt. Quantum Electron. 22 529
[10] Beeckman J, Neyts K and Haelterman M 2008 J. Phys. B: At. Mol. Opt. Phys. 41 065402
[11] Agrawal G P 1987 Phys. Rev. Lett. 59 880
[12] Höök A and Karlsson M 1993 Opt. Lett. 18 1388
[13] Tanemura T, Ozeki Y and Kikuchi K 2004 Phys. Rev. Lett. 93 163902
[14] Menyuk C R 1987 IEEE. Quantum Electron. 23 174
[15] Wabnitz S 1988 Phys. Rev. A 38 2018
[16] Rothenberg J E 1990 Phys. Rev. A 42 682
[17] Millot G, Tchofo Dinda P, Seve E and Wabnitz S 2001 Opt. Fiber Technol. 7 170
[18] Chiu H S and Chow K W 2009 Phys. Rev. A 79 065803
[19] Snyder A W 1972 J. Opt. Soc. Am. 62 1267
[20] Trillo S, Wabnitz S, Stegeman G I and Wright E M 1989 J. Opt. Soc. Am. B 6 889
[21] Tascal R S and Malomed B A 1999 Phys. Scr. 60 418
[22] Li J H, Chiang K S and Chow K W 2011 J. Opt. Soc. Am. B 28 1693
[23] Snyder A W, Mitchell D J, Poladian L, Rowland D R and Chen Y 1991 J. Opt. Soc. Am. B 8 2102
[24] Betlej A, Sun'tsov S, Makris K G, Jankovic L, Christodoulides D N, Stegeman G I, Fini J, Bise R T and DiGiovanni D J 2006 Opt. Lett. 31 1480
[25] Tanemura T and Kikuchi K 2006 J. Light. Technol. 24 4108
[26] Chiang K S 1995 Opt. Lett. 20 997
[27] Chiang K S 1997 J. Opt. Soc. Am. B 14 1437
[28] Menyuk C R 1989 IEEE. J. Quantum Electron. 25 2674
[29] Li H and Huang D 2001 Chin. Phys. 10 626
[30] Panotu N C, Chen X and Osgood R M Jr 2006 Opt. Lett. 31 3609
[31] Benton C J, Gorbach A V and Skryabin D V 2008 Phys. Rev. A 78 033818
[32] Benton C J and Skryabin D V 2009 Opt. Express 17 5879
[33] Gorbach A V et al 2010 Phys. Rev. A 82 041802
[34] Roy U, Atre R, Sudheesh C, Kumar C N and Panigrahi P K 2010 J. Phys. B: At. Mol. Opt. Phys. 43 025003