Extending binary linear codes to self-orthogonal codes

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Abstract
Kim et al. (2021) gave a method to embed a given binary $[n, k]$ code $C (k = 3, 4)$ into a self-orthogonal code of the shortest length which has the same dimension $k$ and minimum distance $d' \geq d(C)$. We extend this result by proposing a new method related to a special matrix, called the self-orthogonality matrix $SO_k$, obtained by shortening a Reed-Muller code $R(2,k)$. Using this approach, we can extend binary linear codes to many optimal self-orthogonal codes of dimensions 5 and 6. Furthermore, we partially disprove the conjecture (Kim et al. (2021)) by showing that if $31 \leq n \leq 256$ and $n \equiv 14, 22, 29 \pmod{31}$, then there exist optimal $[n, 5]$ codes which are self-orthogonal. We also construct optimal self-orthogonal $[n, 6]$ codes when $41 \leq n \leq 256$ satisfies $n \neq 46, 54, 61$ and $n \neq 7, 14, 22, 29, 38, 45, 53, 60 \pmod{63}$.

Keywords: binary linear code, optimal self-orthogonal code, Reed-Muller code, quantum code

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1 Introduction

Since the beginning of the coding theory, many researchers have studied self-orthogonal (abbr. SO) codes and their applications. It is well-known that they have connections to $t$-
designs [2] and lattices [9]. Self-orthogonal codes also have connections to quantum codes [11]-[13],[15], which are currently receiving much attention due to quantum computers.

One of the main topics in coding theory is to find the minimum distance optimal code among self-dual or self-orthogonal codes [17]. Bouklie et al. [3] investigated optimal \([n, k]\) SO codes of lengths for \(n \leq 40\) and \(k \leq 10\). There are some optimal linear codes in BKLC (best-known linear codes) database of MAGMA [5] which are also self-orthogonal. However, the highest minimum weight of some optimal \([n, k]\) SO codes are still unknown for some parameters \(n\) and \(k\).

Kim et al. [14] gave a novel algorithm for the construction of optimal SO codes by adding columns to the generator matrix of a linear code \(C\) of dimension \(k \leq 4\) and minimum distance \(d' \geq d(C)\). They investigated the characterization of self-orthogonality for a given binary linear code in terms of the number of column vectors in its generator matrix. However, the algorithm in [14] was not suitable to construct optimal SO codes of dimensions greater than or equal to 5.

In this paper, we extend this result for \(k = 5\) and 6 by proposing a new method related to a special matrix, called the self-orthogonality matrix \(SO_k\), obtained by shortening a Reed-Muller code \(R(2, k)\). Furthermore, we partially disprove the conjecture 25 in [14] by showing that if \(31 \leq n \leq 256\) and \(n \equiv 14, 22, 29\) (mod 31), then there exist optimal \([n, 5]\) codes that are self-orthogonal. We also show that if \(n \leq 256\) satisfies \(n \neq 46, 54, 61, 86\) and \(n \equiv 7, 14, 22, 29, 38, 45, 53, 60\) (mod 63), then there exist optimal \([n, 6]\) codes which are self-orthogonal.

The paper consists of 5 sections. Section 2 gives preliminaries. Section 3 defines the self-orthogonality matrix \(SO_k\) and describes our two main theorems, Theorems [11] and [13]. In Section 4, we propose a shortest SO embedding algorithm, partially disprove the conjecture 25 in [14], and construct optimal SO \([n, 5]\) and \([n, 6]\) codes. We also give an example of quantum codes based on self-orthogonal codes (see Corollary [23] and Example [24]). We give a conclusion in Section 5.

2 Preliminaries

Let \(\mathbb{F}\) be the finite field of order 2. A subspace \(C\) of \(\mathbb{F}^n\) is called a linear code of length \(n\). For \(n, k \in \mathbb{Z}^+\), a \(k\)-dimensional linear code \(C \subseteq \mathbb{F}^n\) is called an \([n, k]\) code. The elements of \(C\) are called codewords. A generator matrix for \(C\) is a \(k \times n\) matrix \(G\) whose rows form a basis for \(C\).

For \(x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n\), let \(x \cdot y := \sum_{i=1}^{n} x_i y_i\). For a linear code \(C\), the code
\[
C^\perp := \{ x \in \mathbb{F}^n \mid x \cdot y = 0 \text{ for all } y \in C \}
\]
is called the dual of \(C\). A linear code \(C\) satisfying \(C \subseteq C^\perp\) (resp. \(C = C^\perp\)) is called self-orthogonal (resp. self-dual).

For \(x, y \in \mathbb{F}^n\), we define the (Hamming) distance \(d(x, y)\) between \(x\) and \(y\) by the number of coordinates in which \(x\) and \(y\) differ. The minimum distance of \(C\) is the smallest distance between any two distinct codewords. For \(n, k, d \in \mathbb{Z}^+\), an \([n, k, d]\) code \(C\) is an \([n, k]\) code whose minimum distance is \(d\). A linear \([n, k]\) code \(C\) is called optimal if its minimum distance \(d\) is the highest among all \([n, k]\) codes. For many values of \(n\) and
k, an optimal linear \([n,k]\) code is not self-orthogonal. So we say that a self-orthogonal \([n,k]\) code with the highest minimum weight among all self-orthogonal \([n,k]\) codes is an optimal SO code. We denote by \(d(n,k)\) and \(d_{so}(n,k)\) the minimum distance of an optimal \([n,k]\) code and optimal \([n,k]\) SO code, respectively.

We point out that \(d_{so}(n,k)\) is always even because of self-orthogonality and that the best possible minimum distance of a self-orthogonal code is \(2\lfloor d(n,k)/2 \rfloor\). In other words, if there exists a self-orthogonal code \(C\) with the minimum distance \(2\lfloor d(n,k)/2 \rfloor\), then \(C\) is an optimal SO code and therefore, \(d_{so}(n,k) = 2\lfloor d(n,k)/2 \rfloor\).

Let us collect some required notations. For any \([n,k]\) code \(C\) generated by \(G\), we denote by \(r_i(G)\) the \(i\)th row of \(G\) from the top for \(1 \leq i \leq k\) and \(c_j(G)\) the \(j\)th column of \(G\) from the left for \(1 \leq j \leq n\). If there is no danger of confusion to the matrix \(G\), then we will write \(r_i\) (resp. \(c_j\)) for \(r_i(G)\) (resp. \(c_j(G)\)). We denote by \(H_k\) the generator matrix of the binary simplex code \(S_k\). For example, \(S_3\) is the \([7,3]\) linear code generated by

\[
H_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}.
\]

For \(i = 1, 2, \cdots, 2^k - 1\), we let

\[
h_i := \text{the } i\text{th column vector of } H_k,
\]

and for a \(k \times n\) matrix \(G\) over \(\mathbb{F}\), we define

\[
\ell_{h_i}(G) := \text{the number columns of } G \text{ which is equal to } h_i.
\]

If there is no confusion we will simply write \(\ell_i\) for \(\ell_{h_i}\). We also define a vector \(\ell(G)\) over \(\mathbb{F}\) as

\[
\ell(G) := (\ell_1, \ell_2, \cdots, \ell_{2^k - 1}) \pmod{2}.
\]

**Example 1.** Let \(G_{8,3}\) be a \([8,3,3]\) code generated by

\[
G_{8,3} = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

Then \(c_1 = h_4, c_2 = h_4, c_3 = h_2, c_4 = h_6, c_5 = h_6, c_6 = h_5, c_7 = h_5, c_8 = h_3\), thus

\[
(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7) = (0, 1, 1, 2, 2, 2, 0),
\]

and we obtain the binary vector

\[
\ell(G_{8,3}) = (0, 1, 1, 0, 0, 0, 0).
\]

### 3 Binary self-orthogonality Matrix

**Definition 2.** For the matrix \(H_k\), let \(r_i\) be the \(i\)th row vector of \(H_k\). Following the notation of [10 Thm 1.4.3.(i)], let \(r_i \cap r_j\) be the vector in \(\mathbb{F}^n\), which has 1s precisely in those positions where both \(r_i\) and \(r_j\) have 1s. Let \(R(H_k)\) be a set of vectors as

\[
R(H_k) = \{ r_i \cap r_j \mid 1 \leq i \leq j \leq k \}.
\]

Then, we define the self-orthogonality matrix \(SO_k\) as a matrix with all vectors in \(R(H_k)\) as rows. Since \(R(H_k)\) is a set of \(\frac{k(k+1)}{2}\) vectors of length \(2^k - 1\), the size of \(SO_k\) is \(\frac{k(k+1)}{2} \times (2^k - 1)\).
We note that the vector \( r_i \cap r_j \) is equal to \( r_i \). Therefore, all the rows of \( H_k \) are also rows of the matrix \( SO_k \), thus we regard \( SO_k \) as a vertically concatenated matrix of \( H_k \) and the matrix consisting of the row vectors \( r_i \cap r_j \) for \( 1 \leq i < j \leq k \).

**Example 3.** Since \( H_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \), the set \( R(H_2) \) has three vectors \( r_1 \cap r_1, r_2 \cap r_2, \) and \( r_1 \cap r_2 \). Thus,

\[
SO_2 = \begin{pmatrix} r_1 \cap r_1 \\ r_2 \cap r_2 \\ r_1 \cap r_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]

**Example 4.** Since \( H_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \), the set \( R(H_3) \) has 6 vectors and

\[
SO_3 = \begin{pmatrix} r_1 \cap r_1 \\ r_2 \cap r_2 \\ r_3 \cap r_3 \\ r_1 \cap r_2 \\ r_1 \cap r_3 \\ r_2 \cap r_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\]

**Definition 5** ([14]). Let \( C \) be an \([n, k]\) code generated by \( G \).

1. An **SO embedding** of \( C \) is an SO code whose generator matrix \( \tilde{G} \) is obtained by adding a set \( S \) of column vectors to \( G \), i.e., \( \tilde{G} := [G \mid S] \).

2. An SO embedding of \( C \) is called a **shortest SO embedding** of \( C \) if its length is shortest among all SO embeddings of \( C \).

Kim et al. [14, p. 3705] remarked the following based on their complicated algorithms and left the other cases open, which will be solved in Theorem 13.

**Remark 6.**

1. A shortest SO embedding code of a binary \([n, 2]\) code and a binary \([n, 3]\) code can be obtained by adding exactly three or fewer columns.

2. A shortest SO embedding code of a binary \([n, 4]\) code can be obtained by adding exactly five or fewer columns.

For a \( k \times n \) matrix \( G \) and \( 0 < j \leq k \), let \( I(j) \) be a multiset

\[
I(j) := \left\{ I_j(G) \bigg| \begin{array}{ll}
(i) & 1 \leq i \leq n \\
(ii) & c_j(G) = h_t \text{ for } 1 \leq t \leq 2^k - 1 \\
 & \text{satisfying } [\frac{i}{2^t}] \equiv_2 1
\end{array} \right\}.
\]

In other words, \( I(j) \) is a multiset of the columns which has a 1 in the \( j \)-th row from the bottom of \( G \).

Then the next theorem is proved as a characterization for self-orthogonality in terms of \( I(j) \) in [14].

**Theorem 7** (Lemma 2 [14]). Let \( C \) be an \([n, k]\) code generated by \( G \). Then, \( C \) is self-orthogonal if and only if for all \( 1 \leq j \leq j', k \), \([I(j) \cap I(j')]\) is even.

The following main theorem characterizes self-orthogonality using the vector \( \ell(G) \) and the matrix \( SO_k \).
Theorem 11. Let $H$ belong to $R(n, k)$ and $u_1, u_2$ be the self-orthogonal if and only if

$$SO_k \ell(G)^T = 0.$$ 

Proof. By Theorem 8, we know that $H$ is self-orthogonal if and only if for all $1 \leq j \leq j', |I(j) \cap I(j')|$ is even. From definitions of $SO_k$, $\ell(G)$, and $I(j)$, it is easy to check that $|I(j) \cap I(j')| = (r_{k+1-j} \cap r_{k+1-j'}) \cdot \ell(G)$ for all $1 \leq j \leq j' \leq k$. Therefore, $C$ is self-orthogonal if and only if for all $1 \leq j \leq j' \leq k$, $(r_{k+1-j} \cap r_{k+1-j'}) \cdot \ell(G) = 0$, equivalently, $SO_k \ell(G)^T = 0$.

Corollary 9. Let $C$ be a binary $[n, k]$ code generated by $G$ and let $H$ be a binary code generated by the matrix $SO_k$. Then, $C$ is self-orthogonal if and only if the vector $\ell(G)$ belongs to $H \perp$.

Proof. By Theorem 8 we know that $C$ is self-orthogonal if and only if $SO_k \ell(G)^T = 0$. Since $SO_k$ is a generator matrix of $H$, the corollary follows.

Example 10.

1. Let $G_{8,3}$ be the generator matrix of $C_{8,3}$ in Example 1. We have

$$SO_3 = \begin{pmatrix} 001 & 111 \\ 110 & 011 \\ 101 & 101 \\ 000 & 011 \\ 000 & 011 \\ 001 & 001 \end{pmatrix}, \text{ and } \ell(G_{8,3}) = 0110000.$$

Thus,

$$SO_3 \ell(G_{8,3})^T = \begin{pmatrix} 001 & 111 \\ 110 & 011 \\ 101 & 101 \\ 000 & 011 \\ 000 & 011 \\ 001 & 001 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $C_{8,3}$ is not self-orthogonal by Theorem 8.

2. Let $G_{10,3}$ be a matrix

$$G_{10,3} = \begin{pmatrix} 110 & 111 & 100 & 001 \\ 001 & 110 & 011 & 111 \\ 000 & 110 & 111 & 110 \end{pmatrix}.$$ 

Then $\ell(G_{10,3}) = 0000000$, and we have

$$SO_3 \ell(G_{10,3})^T = 0.$$

Therefore, by Theorem 8 we know that $G_{10,3}$ generates a self-orthogonal code.

In the next theorem, we discuss the relationship between the self-orthogonality matrix $SO_k$ and Reed-Muller code $R(r, m)$ defined in [10, Chapter 1.10].

Theorem 11. Let $R(r, m)$ be the $r$th order binary Reed-Muller code of length $2^m$. Then for $k \geq 3$,
i) $SO_k$ is obtained by shortening on the first coordinate position from a binary Reed-Muller code $\mathcal{R}(2, k)$. Hence $\ell(G)$ is orthogonal to $SO_k$ if and only if $\ell(G)$ belongs to the puncture of a Reed-Muller code $\mathcal{R}(k - 3, k)$ on the first position.

ii) The covering radius $\rho$ of the puncture of a Reed-Muller code $\mathcal{R}(k - 3, k)$ is $k + 1$ if $k$ is even and $k$ if $k$ is odd.

**Proof.** i) From the definition of $SO_k$, it is easy to see that $SO_k$ is obtained by shortening on the first coordinate position from a binary Reed-Muller code $\mathcal{R}(2, k)$. The dual of this shortened RM code is the puncture of dual of $\mathcal{R}(2, k)$ on the first position by [10, Theorem 1.5.7], and so it is equal to the puncture of $\mathcal{R}(k - 3, k)$.

ii) By [16] the covering radius $\rho$ of a Reed-Muller code $\mathcal{R}(k - 3, k)$ is $k + 2$ if $k$ is even and $k + 1$ if $k$ is odd. It is easy to see that $\mathcal{R}(k - 3, k)$ is even since all-one vector is in its dual. Therefore by [10, Ex. 625], the covering radius $\rho$ of the puncture of a Reed-Muller code $\mathcal{R}(k - 3, k)$ is $k + 1$ if $k$ is even and $k$ if $k$ is odd. $\square$

**Corollary 12.** The rank of the matrix $SO_k$ is $\frac{k(k+1)}{2}$.

**Proof.** By [10] Theorem 1.5.7, we know that the dimension of Reed-Muller code $\mathcal{R}(2, k)$ equals

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} = 1 + k + \frac{k(k-1)}{2}.$$

Since $SO_k$ is obtained by shortening on the first coordinate position of $\mathcal{R}(2, k)$, the rank of the matrix $SO_k$ is one less than the dimension of $\mathcal{R}(2, k)$. Thus the rank of the matrix $SO_k$ is $\frac{k(k+1)}{2}$. $\square$

The following theorem generalizes Remark 6 for any $k \geq 2$.

**Theorem 13.** Given any binary $[n, k]$ code $C$ with $k \geq 2$, we can obtain a shortest SO embedding $\tilde{C}$ by adding exactly or less

i) $k + 1$ columns if $k$ is even or

ii) $k$ columns if $k$ is odd.

**Proof.** By Remark 6(1), we may assume $k \geq 3$. Let $C$ be a given binary $[n, k]$ code with generator matrix $G$. By Theorem 11 we can check whether it is self-orthogonal. If not, by ii) of Theorem 11 any binary vector $\ell(G)$ can be corrected by changing at most $k + 1$ positions of $v$ if $k$ is even and $k$ positions of $v$ if $k$ is odd. Since we are embedding $C$ into a self-orthogonal codes, we need to add $k + 1$ columns (corresponding to those $k + 1$ positions) to $G$ if $k$ is even, and $k$ columns (corresponding to those $k$ positions) to $G$ if $k$ is odd. $\square$
4 Shortest SO embedding algorithm

In what follows, we describe a *shortest SO embedding algorithm* to construct an embedding of a given binary \([n, k]\) code. Normally, we consider a best-known linear \([n, k]\) code from *BKLC* database in Magma [5], hoping to obtain an optimal or best-known self-orthogonal code with the same dimension.

**Algorithm 14** (Shortest SO embedding algorithm).

- **Input:** A generator matrix \(G\) of the \([n, k]\) code \(C\).
- **Output:** A generator matrix \(G'\) for a shortest SO embedding.

1. Obtain the vector \(\ell(G)\) from \(G\). Go to (A2).
2. If \(SO_4 \ell(G)^T = 0\), then \(C\) is self-orthogonal. Go to (A6).
3. If \(SO_4 \ell(G)^T \neq 0\), then let \(s = SO_4 \ell(G)^T\) be the syndrome corresponding to the vector \(\ell(G)\), and go to (A4).
4. Let \(err_s\) be a coset leader correspond to the syndrome \(s\) and go to (A5).
5. For each index \(i\) of non-zero elements in \(err_s\), append \(G\) the column vector which is the binary representation of \(i\). Go to (A6).
6. Return \(G\) and terminate the algorithm.

We remark that a discussion of various decoding algorithms for Reed-Muller codes can be found in [1].

**Example 15.**

1. Let \(G_{8,3}\) be the generator matrix of \(C_{8,3}\) in Example 1. In Example 10, we obtained the syndrome \(s = 1001000\) of the vector \(\ell(G_{8,3}) = 0110000\). In this case, \(\ell(G_{8,3})\) itself is a coset leader \(err_s\) corresponding to the syndrome \(s\). Therefore, by adding two columns that are binary representations of 2 and 3, we obtain a matrix

\[
\tilde{G}_{8,3} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},
\]

which generates a self-orthogonal \([10, 3, 4]\) code.

2. Let \(C_{11,4}\) be the best-known linear \([11, 4, 5]\) code from *BKLC* database in Magma [5] with generator matrix

\[
G_{11,4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.
\]

Then \(\ell(G_{11,4}) = 110101110110111\). Since

\[
SO_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]
its syndrome is
\[ s = SO_4 \ell(G_{11,4})^T = 0011101011^T. \]
Since the coset leader corresponding to this syndrome is 000001000100001, we add three columns which are the binary representations of 6, 10, and 15, respectively. Consequently, we obtain a [14, 4, 6] self-orthogonal code with generator matrix
\[
\tilde{G}_{11,4} = \begin{pmatrix}
10010101110101110011011010110101100110111110011100
0000000111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
Table 1: New parameters of optimal $d_{so}(n, 5)$ for $45 \leq n \leq 256$ and $k = 5$

| $n$ | (mod 31) | $d_{so}(n, 5)$ |
|-----|----------|--------------|
|     |          | Known [ref]  | Our result |
| 45  | 14       | $\leq 22$ [8] | 22         |
| 53  | 22       | $\leq 26$ [8] | 26         |
| 60  | 29       | $\leq 30$ [8] | 30         |
| 76  | 14       | $\leq 38$ [8] | 38         |
| 84  | 22       | $\leq 42$ [8] | 42         |
| 91  | 29       | $\leq 46$ [8] | 46         |
| 107 | 14       | $\leq 54$ [8] | 54         |
| 115 | 22       | $\leq 58$ [8] | 58         |
| 122 | 29       | $\leq 62$ [8] | 62         |
| 138 | 14       | $\leq 70$ [8] | 70         |
| 146 | 22       | $\leq 74$ [8] | 74         |
| 153 | 29       | $\leq 78$ [8] | 78         |
| 169 | 14       | $\leq 86$ [8] | 86         |
| 177 | 22       | $\leq 90$ [8] | 90         |
| 184 | 29       | $\leq 94$ [8] | 94         |
| 200 | 14       | $\leq 102$ [8]| 102        |
| 208 | 22       | $\leq 106$ [8]| 106        |
| 215 | 29       | $\leq 110$ [8]| 110        |
| 231 | 14       | $\leq 118$ [8]| 118        |
| 239 | 22       | $\leq 122$ [8]| 122        |
| 246 | 29       | $\leq 126$ [8]| 126        |

Based on Theorem 17, we give modified conjectures as follows.

**Conjecture 19.** For $n \geq 32$, if $n \equiv 14, 22, 29 \ (\text{mod} \ 31)$, then $d_{so}(n, 5) = d(n, 5)$, i.e., there exist $[n, 5, d(n, 5)]$ SO codes.

**Conjecture 20.** If $n = 14, 21, 22, 28, 29$ or if $n \geq 32$ and $n \equiv 6, 13, 21, 28 \ (\text{mod} \ 31)$, then $d_{so}(n, 5) = d(n, 5) - 2$, i.e., there are no $[n, 5, d(n, 5)]$ SO codes.

### 4.2 optimal self-orthogonal codes of dimension 6

In this section, we introduce new parameters of optimal $[n, 6]$ SO codes for lengths $n \leq 256$. These SO codes are constructed using Algorithm 14. There are examples of $[n, 6]$ SO codes in [3]. In [3], optimal $[n, 6]$ SO codes of lengths only up to 40 are investigated and optimal $[n, 6]$ SO codes of lengths $n \leq 29$ and $n = 32, 33, 34, 35, 36, 40$ are obtained as results.

Besides optimal $[n, 6]$ SO codes from [3] and best-known linear codes (BKLC) database of MAGMA [5], we successfully constructed many new optimal $[n, 6]$ SO codes of lengths up to 256 starting from BKLC database of MAGMA [5] and their punctured
codes on at most six columns. The new parameters of SO codes of lengths up to 100 are listed in Table 2 and all the examples of lengths up to 256 are presented in the web [6]. In Table 2 we denote the $d_{so}(n, 6)$ by superscript * if $d_{so}(n, 6) = 2 \lfloor d(n, 6)/2 \rfloor$ is confirmed by our construction result.

Our computational result gives the following Theorem.

**Theorem 21.** For lengths $41 \leq n \leq 256$, if $n \neq 7, 14, 22, 29, 38, 45, 53, 60 \pmod{63}$ and $n \neq 46, 54, 61$, then $d_{so}(n, 6) = 2 \lfloor d(n, 6)/2 \rfloor$.

In [4], a construction method of quantum codes from binary linear codes was introduced as following Theorem.

**Theorem 22** (Theorem 9 in [4]). Let $C_1 \subseteq C_2$ be binary linear codes. By taking $C = \omega C_1 + \tilde{\omega} C_2$, we obtain an $[[n, k_2 - k_1, d]]$ code, where

$$d = \min\{\text{dist}(C_2 \setminus C_1), \text{dist}(C_1^{-} \setminus C_2^{-})\}.$$ 

If we focus on binary self-orthogonal codes, we have the following corollary.

**Corollary 23.** Let $C_1$ be a binary self-orthogonal $[n, k_1]$ code. If $d(C_1^{-}) \geq d(C_1)$, then by taking $C = \omega C_1 + \tilde{\omega} C_1$, we obtain an $[[n, n - 2k_1, d]]$ code, where $d = d(C_1^{-})$.

**Proof.** Since $C_1$ is self-orthogonal, $C_1 \subseteq C_1^{-}$. Thus, letting $C_2 = C_1^{-}$ in Theorem 22 we have $C_2^{-} = C_1$ and $k_2 = n - k_1$. Therefore, $k_2 - k_1 = n - k_1$, and $C_2 \setminus C_1 = C_1^{-} \setminus C_1 = C_1^{-} \setminus C_2^{-}$, thus $d = \text{dist}(C_1^{-} \setminus C_1)$. By the assumption that $d(C_1^{-}) \geq d(C_1)$, $C_1^{-}$ has no non-zero codeword having weight smaller than $d(C_1)$. Thus, $\text{dist}(C_1^{-} \setminus C_1) = d(C_1^{-})$, and the corollary follows.

**Example 24.** Using Algorithm [4] with an optimal non-SO $[15, 5, 7]$ code, we construct a Reed-Muller $[16, 5, 8]$ SO code with generator matrix

$$\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

whose dual code is a $[16, 11, 4]$ code. Thus, by Corollary 23 we obtain an optimal $[[16, 6, 4]]$ quantum code by [7]. In the same manner, we obtain $[[11, 3, 3]]$ and $[[15, 7, 3]]$ quantum codes which are optimal by [7].

## 5 Concluding Remarks

We have made three major contributions. First, we obtain a new method for checking self-orthogonality using the self-orthogonality matrix $SO_k$ and a vector $\ell(G)$. Second, we solve the problem of finding additional columns needed to make the shortest SO embedding code from a given binary $[n, k]$ code for any $k \geq 2$. Finally, we give the shortest SO embedding algorithm for the construction of optimal self-orthogonal codes. Using this algorithm, we succeed in obtaining many new optimal self-orthogonal codes of dimensions 5 and 6 for $n \leq 256$.

As future work, it will be interesting to find new optimal SO codes with $n \geq 30$ and dimension $k \geq 7$. 

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| $n$ | $d_{50}(n, 6)$ | Known$[\text{ref}]$ | Our result |
|-----|---------------|---------------------|------------|
| 30  | $\leq 12$     | 12                  | $141$      |
| 31  | $\leq 12$     | $\leq 12$           | $142$      |
| 37  | $\leq 16$     | $16^*$              | $147$      |
| 38  | $\leq 16$     | 16                  | $148$      |
| 39  | $\leq 16$     | $16^*$              | $149$      |
| 41  | $\leq 18$     | $18^*$              | $150$      |
| 44  | $\leq 20$     | $20^*$              | $154$      |
| 45  | $\leq 22$     | $20^*$              | $155$      |
| 46  | $\leq 22$     | $20^*$              | $156$      |
| 47  | $\leq 22$     | $22^*$              | $157$      |
| 52  | $\leq 24$     | $24^*$              | $163$      |
| 53  | $\leq 26$     | 24                  | $164$      |
| 54  | $\leq 26$     | 24                  | $165$      |
| 55  | $\leq 26$     | $26^*$              | $166$      |
| 59  | $\leq 28$     | $28^*$              | $170$      |
| 60  | $\leq 30$     | 28                  | $171$      |
| 61  | $\leq 30$     | 28                  | $172$      |
| 62  | $\leq 30$     | $30^*$              | $173$      |
| 69  | $\leq 32$     | $32^*$              | $178$      |
| 70  | $\leq 34$     | 32                  | $179$      |
| 71  | $\leq 34$     | $34^*$              | $180$      |
| 72  | $\leq 34$     | $34^*$              | $181$      |
| 76  | $\leq 36$     | $36^*$              | $185$      |
| 77  | $\leq 38$     | 36                  | $186$      |
| 78  | $\leq 38$     | $38^*$              | $187$      |
| 79  | $\leq 38$     | $38^*$              | $188$      |
| 84  | $\leq 40$     | $40^*$              | $195$      |
| 85  | $\leq 42$     | 40                  | $196$      |
| 86  | $\leq 42$     | $42^*$              | $197$      |
| 87  | $\leq 42$     | $42^*$              | $198$      |
| 89  | $\leq 44$     | $44^*$              | $202$      |
| 90  | $\leq 44$     | $44^*$              | $203$      |
| 91  | $\leq 44$     | $44^*$              | $204$      |
| 92  | $\leq 46$     | 44                  | $205$      |
| 93  | $\leq 46$     | $46^*$              | $209$      |
| 94  | $\leq 46$     | $46^*$              | $210$      |
| 100 | $\leq 48$     | $48^*$              | $211$      |
| 101 | $\leq 50$     | 48                  | $212$      |
| 102 | $\leq 50$     | $50^*$              | $213$      |
| 103 | $\leq 50$     | $50^*$              | $217$      |
| 107 | $\leq 52$     | $52^*$              | $218$      |
| 108 | $\leq 54$     | 52                  | $219$      |
| 109 | $\leq 54$     | $54^*$              | $220$      |
| 110 | $\leq 54$     | $54^*$              | $226$      |
| 115 | $\leq 56$     | $56^*$              | $227$      |
| 116 | $\leq 58$     | 56                  | $228$      |
| 117 | $\leq 58$     | $58^*$              | $229$      |
| 118 | $\leq 58$     | $58^*$              | $233$      |
| 122 | $\leq 60$     | $60^*$              | $234$      |
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