THERMODYNAMICS OF TWO-DIMENSIONAL BLACK-HOLES

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ABSTRACT

We explore the thermodynamics of a general class of two dimensional dilatonic black-holes. A simple prescription is given that allows us to compute the mass, entropy and thermodynamic potentials, with results in agreement with those obtained by other methods, when available.
Introduction

The original paper [1] on two dimensional black-holes in string theory started a renewed interest in the longstanding problem of formulating a non trivial theory of two dimensional gravity. Progress has been made in finding solutions of WZW models, which are exactly conformal field theories, as well as in finding solutions of the low energy string effective actions. Here we will restrict our analysis to the latter kind of models, namely to effective actions of dilaton gravity in two dimensions. A vast class of these models is classically soluble, even in the presence of a potential for the dilaton field. For instance, for a dilaton potential of the type produced by string loops, the solutions have very interesting spacetime geometries, more complex than the Reissner-Nordstrøm one. They generically exhibit more than two horizons, and the black-hole tiles the entire plane in the Penrose diagram [2]. Some of these models have proved useful as toy models to investigate the evaporation and formation of black-holes [3], as well as for more formal pursuits such as the study of perturbative renormalizability [4].

These models also appear to be interesting laboratories for questions about black-hole thermodynamics. We will use them as such, and find a simple prescription to compute all thermodynamic functions. We will write the on-shell action as the sum of two boundary terms. One of them, related to the second fundamental form of the boundary, will give the (ADM) mass when computed at infinity and the entropy when computed on the horizon. The other boundary term vanishes at infinity and gives the charges and chemical potentials when evaluated at the horizon. The sum of these two terms will reproduce the free energy.

This prescription has similarities with methods advocated in the context of higher dimensional black-holes [5,6,7,8]. As analogous ones, it is rather heuristic and it is an interesting challenge to find a fully satisfying theoretical justification.

1. The model and its solutions

Our action is

\[
I = \int d^2 x \sqrt{-g} e^{-2\phi} \left[ R + \gamma g^{ab} \partial_a \phi \partial_b \phi - \frac{1}{4} e^{\epsilon \phi} F^2 + V(\phi) \right].
\]  (1.1)
Particular cases of eq. (1.1) reduce to a number of well known models of two dimensional gravity [9], as well as to the string bosonic effective action, and to the heterotic string effective action [2].

The model is also connected with four-dimensional spherically symmetric gravity. Indeed from four-dimensional pure gravity
\[ I^{(4)} = \int d^4x \sqrt{-g^{(4)}} R^{(4)} \] (1.2)
by using spherical coordinates
\[ (4) ds^2 = (2) ds^2 + e^{-2\phi} \Omega^2 \] (1.3)
one gets
\[ I^{(4)} = \int d^2x \sqrt{-g^{(2)}} \left[ e^{-2\phi} \left( R^{(2)} + 2 \nabla^a \phi \nabla_a \phi + 2 e^{2\phi} \right) + 4 \nabla^a (e^{-2\phi} \nabla_a \phi) \right] \] (1.4)
which, apart from a total derivative usually neglected, is of the form of eq. (1.1) with \( \gamma = 2 \) and an appropriate potential for the dilaton.

Our interest will focus on static configurations. We will work in the gauge
\[ ds^2 = -g(x) dt^2 + \frac{1}{g(x)} dx^2. \] (1.5)
The equations of motion are then
\[
\begin{align*}
(g\phi')' - 2g\phi'^2 - \frac{1}{4} f^2 e^{\epsilon \phi} + \frac{1}{2} V(\phi) &= 0 \\
2\phi'' + (\gamma - 4) \phi'^2 &= 0 \\
(f e^{(\epsilon - 2)\phi})' &= 0
\end{align*}
\] (1.6)
where \( \cdot' = \partial/\partial x \) and \( F = f(x) dx \wedge dt \). Substituting eqs. (1.6) into the action eq. (1.1) (without using any explicit solution), we get
\[ I|_{\text{on shell}} = - \int dtdx \partial_x \left[ 4 e^{-2\phi} g \partial_x \phi + e^{-2\phi} \partial_x g \right] |_{\text{on shell}}. \] (1.7)
This relation, already mentioned for \( \gamma = 4 \) and no dilaton potential in ref. [10], is actually valid for all the models in eq. (1.1). It is the generalization of an analogous relation for 4D pure gravity. Since on shell \( R^{(4)} = 0 \), it follows from eq. (1.4) that
\[ \int d^2x \sqrt{-g^{(2)}} e^{-2\phi} \left( R^{(2)} + 2 \nabla^a \phi \nabla_a \phi + 2 e^{2\phi} \right) |_{\text{on shell}} \]
\[ = - \int d^2x \sqrt{-g^{(2)}} \nabla^a \left[ 4 e^{-2\phi} \nabla_a \phi \right] |_{\text{on shell}}. \] (1.8)
Indeed eq. (1.8) is equivalent to eq. (1.7) since for 4D pure gravity \( \int d^2x \partial_x (e^{-2\phi} \partial_x g) = 0 \). Eq. (1.7) can be written in fully covariant form in terms of the boundary term in eq. (1.8) and the second fundamental form of the boundary in the 2D metric.

Another way of understanding the result in eq. (1.7) can be achieved if one computes to the leading order the variation of eq. (1.1) under the dilaton transformation \( \phi \rightarrow \phi + a \), where \( a \) is space dependent and with compact support. The above transformation is not a symmetry of eq. (1.1), not even in the absence of potential; however, on shell, where the variation is zero anyway, one gets

\[
I\big|_{\text{on shell}} = \int d^2x \left[ -\gamma \nabla^a (e^{-2\phi} \nabla_a \phi) + \frac{1}{2} e^{-2\phi} \frac{\partial V}{\partial \phi} \right] \tag{1.9}
\]

By using the equation of motions in their covariant form [11], one can rewrite eq. (1.9) in the more illuminating form

\[
I\big|_{\text{on shell}} = \int d^2x \left[ -4 \nabla^a (e^{-2\phi} \nabla_a \phi) + e^{-2\phi} (R + 2 \nabla^2 \phi + (\gamma - 4)(\nabla \phi)^2) \right] \tag{1.10}
\]

Finally, using eqs. (1.6), one can show that also the second term in eq. (1.10) is a total derivative independent of \( \gamma \), and is exactly the second term in eq. (1.7).

The equation (1.7) will be the main input in our derivation of the thermodynamics relations, which will basically consist in evaluating eq. (1.7) on specific solutions of the equations of motion. To that purpose we list here the solutions of eqs. (1.6) we will consider.

**Solution 1** [2]

Setting

\[
\gamma = 4, \quad F = f(x) \, dx \wedge dt, \quad V(\phi) = \sum_{n \geq 0} a_n e^{2n\phi(x)}
\]

\[
\epsilon = 0, \quad a_0 = Q^2, \quad a_1 = 0 \tag{1.11}
\]

one gets

\[
\phi(x) = \phi_0 - \frac{1}{2} Q x, \quad g(x) = 1 - 2m e^{-Qx} + \sum_{n \geq 2} \frac{b_n}{(1-n)Q^2} e^{-nQx}
\]

\[
f(x) = f_0 e^{2\phi(x)} \tag{1.12}
\]

where \( b_n = e^{2n\phi_0} (a_n - \frac{1}{2} \delta_{n,2} f_0^2) \) and it is convenient to set \( q^2 = f_0^2 e^{4\phi_0} / 2Q^2 \).
A particular solution is given by \( b_2 = -q^2Q^2, b_n = 0 \ (n > 2), a_n = 0 \ (n \geq 1) \), this is the charged heterotic black-hole of ref. [2]. The above potential might be generated by string loops corrections. As discussed in ref. [2] a non-trivial potential of this kind generically gives rise to interesting space-time geometries with multiple horizons.

**Solution 2 [11]**

Setting

\[
\gamma = 2, \quad F = f(x) \, dx \wedge dt, \quad V(\phi) = \sum_{n \geq 0} a_n e^{2n\phi(x)}
\]

\[
\epsilon = 0, \quad a_0 = 0, \quad a_1 = 2
\]

one gets

\[
\phi(x) = -\log(x), \quad g(x) = 1 - \frac{2m}{x} + \sum_{n \geq 2} \frac{b_n}{6 - 4n} x^{2-2n}
\]

\[
f(x) = f_0 e^{2\phi(x)}
\]

where \( b_n = (a_n - \frac{1}{2}\delta_{n,2}f_0^2) \) and it is convenient to set \( q^2 = \frac{1}{4}f_0^2 \).

If all the \( b_n \) are zero except \( b_1 \), then of course one gets the Schwarzschild black-hole. If \( b_2 = -2q^2, b_n = 0 \ (n \geq 3), a_n = 0 \ (n \neq 1) \), one gets the charged Reissner-Nordstrøm black-hole.

**Solutions with \( 0 < \gamma < 4 \)**

This is a new class of solutions of the equations of motion eqs. (1.6) for \( 0 < \gamma < 4 \) of which Solution 2 is the one with \( \gamma = 2 \). Let

\[
0 < \gamma < 4, \quad F = f(x) \, dx \wedge dt, \quad V(\phi) = \sum_{n \geq 0} a_n e^{2n\rho\phi(x)}
\]

\[
a_0 = 0, \quad \epsilon \neq 0.
\]

Moreover, it is convenient to set

\[
\alpha = \frac{4}{\gamma - 4}, \quad \epsilon = 4 - 4\rho, \quad \rho = \frac{2}{\alpha}
\]

\[
b_n = e^{2n\rho\phi_0}(a_n - \frac{1}{2}\delta_{n,2}f_0^2), \quad b_1 = \alpha + \alpha^2.
\]

One then gets

\[
g(x) = 1 - 2m x^{\alpha+1} + \sum_{n \geq 2} \frac{b_n}{(2n-1)\alpha + \alpha^2} x^{2(1-n)}
\]

\[
\phi = \phi_0 + \frac{1}{2}\alpha \log(x), \quad f(x) = f_0 e^{(2-\epsilon)\phi(x)}
\]
where the following conditions must be satisfied: \( \alpha + 1 < 0 \) (which implies \( 0 < \gamma < 4 \)) and \( b_n = 0 \) if \( 2 \leq n < n_0 \) where \( n_0 \) is a fixed number greater than 2 which satisfies \( 2(1 - n_0) < \alpha + 1 \) (both these conditions come from the requirement that the metric is asymptotic Minkowskian, i.e. \( g(x) \to 1 \) for \( x \to +\infty \) and that the (ADM) mass does not diverge).

Solution 2 is obtained by setting \( \phi_0 = 0 \) and \( \gamma = 2 \) (which implies \( \rho = 1 \) and \( \alpha = -2 \)).

2. Thermodynamics

On general grounds it must be that \(-I = F/T_c = \beta F\) where \( F \) is the free energy and \( T_c \) the temperature [5]. From thermodynamics we also expect \( F = M - T_c S - \sum_i \mu_i Q_i \) where \( M \) is the (ADM) mass, \( S \) the entropy, \( \mu_i \) the chemical potentials and \( Q_i \) the associated charges. Therefore we must find

\[
-I \bigg|_{\text{on shell}} = \beta M - S - \beta \sum_i \mu_i Q_i .
\]  

(2.1)

We will need to evaluate eq. (1.7). A first important observation is in order. For the flat space solution \( g(x) = 1 \) one gets

\[
I \bigg|_{\text{on shell}} = -\int dt dx \partial_x \left[ 4 e^{-2\phi} \partial_x \phi \right] \bigg|_{\text{on shell}} .
\]  

(2.2)

For \( \gamma = 4 \) (Solution 1), one has

\[
I \bigg|_{\text{on shell}} = 2 \beta Q e^{-2\phi_0} \left[ e^{Qx} \right]_0^\infty ;
\]  

(2.3)

for \( \gamma = 2 \) (Solution 2), one has

\[
I \bigg|_{\text{on shell}} = 4 \beta \left[ x \right]_0^\infty .
\]  

(2.4)

In both cases the on-shell action diverges, while instead, since these solutions describe the flat (empty) space, one would like to have \( I \bigg|_{\text{on shell}} = 0 \).

To get this, we modify the starting action subtracting the flat space contribution eq. (2.2). This is exactly what Gibbons and Hawking were forced to do as well in 4D gravity [5]. From eq. (1.8) it is indeed obvious that the same divergence occurs there (in our gauge
the l.h.s. of eq. (1.8) is exactly eq. (2.2)). Notice that eq. (2.2) is a pure boundary term, and thus does not contribute to the equations of motion. Therefore, instead of eq. (1.1), our starting point will be

\[ I = \int d^2x \sqrt{-g} e^{-2\phi} \left[ R + \gamma (\nabla \phi)^2 - \frac{1}{4} e^{\phi} F^2 + V(\phi) + \nabla^a (4e^{-2\phi} \nabla_a \phi) \right] \quad (2.5) \]

for which \( I(g=1) \) \( \big|_{\text{on shell}} \) = 0. Therefore, instead of eq. (1.7), we get

\[ I \big|_{\text{on shell}} = - \int dt dx \partial_x \left[ 4e^{-2\phi} (g-1) \partial_x \phi + e^{-2\phi} \partial_x g \right] \big|_{\text{on shell}} \quad (2.6) \]

which, for future convenience, we divide in two pieces

\[ I_1 = \int dt dx \left[ \partial_x (e^{-2\phi} \partial_x g) \right] \quad (2.7) \]

and

\[ I_2 = \int dt dx \left[ \partial_x (4e^{-2\phi} (g-1) \partial_x \phi) \right] . \quad (2.8) \]

An alternative way to deal with the above discussed infinity is to leave it there, and interpret it as the divergent contribution to the chemical potentials associated with a topologically conserved dilaton charge [10]. In two dimensions, the current

\[ J_a = \epsilon_a^b \nabla_b e^{-2\phi(x)} \quad (2.9) \]

is conserved and the associated charge is

\[ D = \int_{\Sigma} J_a d\Sigma^a \quad (2.10) \]

where \( \Sigma \) is a spacelike hypersurface. The charge \( D = [e^{-2\phi(x)}]_{+\infty} \) actually diverges, since the dilaton is a a long-range field, and its contribution to the free energy is explicitly given by eq. (2.2). Our prescription of subtracting the divergent term from the action can be interpreted therefore as eliminating from the free energy the contribution of the background dilaton field. This has the advantage that we will deal only with finite quantities and we will not need to put the theory in a finite volume [10].

To explicitly evaluate (2.6), we will work in the euclidean formalism. We will assume that the time is periodic of period \( \beta \) and the space coordinate \( x \) takes values between the outer horizon \( x = r_+ \) and infinity [5,6]. The horizons are defined as the roots of the
equation \( g(x) = 0 \) and \( r_+ \) is the largest one. For all models we consider here, it is possible to show in the gauge of eq. (1.5) that

\[
\frac{4\pi}{\beta} = \left. \frac{\partial g}{\partial x} \right|_{r+} .
\]  

(2.11)

Let’s look first to the contribution of the boundary term \( I_1 \). Without using any explicit solution but only the definition of \( r_+ \) and eq. (2.11), one finds

\[
I_1 = \beta \left[ e^{-2\phi} \partial_x g \right]_{+\infty} - 4\pi \left[ e^{-2\phi} \right]_{r+} .
\]  

(2.12)

We are then led to the following identifications

\[
M = \left[ e^{-2\phi} \partial_x g \right]_{+\infty} , \quad S = 4\pi \left[ e^{-2\phi} \right]_{r+} .
\]  

(2.13)

We will also show in all the specific cases we will study, that eq. (2.8) generates the thermodynamic potentials, i.e. \( I_2 = -\beta \sum \mu_i Q_i \), so that

\[
I_1 \bigg|_{\text{on shell}} = \beta M - S , \quad I_2 \bigg|_{\text{on shell}} = -\beta \sum \mu_i Q_i .
\]  

(2.14)

Finally, we will check that the thermodynamic relations are satisfied. In particular, when there is only one conserved charge, the electric charge, one has the following thermodynamical relations

\[
\beta = \left[ \frac{\partial S}{\partial M} \right]_{Q_{el}} , \quad \mu_{el} = T_c \left[ \frac{\partial S}{\partial Q_{el}} \right]_{M} .
\]  

(2.15)

2.1 Solution 1 (\( \gamma = 4 \))

Let us first apply the previous considerations to the Solution 1 and in particular to the case of the charged heterotic black-hole:

\[
g(x) = 1 - 2me^{-Qx} + q^2 e^{-2Qx} , \quad \phi(x) = \phi_0 - \frac{1}{2} Qx .
\]  

(2.16)

From the previous section one gets

\[
e^{Qr_+} = m + \sqrt{m^2 - q^2} , \quad \beta = \frac{2\pi}{Q} \cdot \frac{m + \sqrt{m^2 - q^2}}{\sqrt{m^2 - q^2}} .
\]  

(2.17)

\[
M = 2mQe^{-2\phi_0} , \quad S = 4\pi e^{-2\phi_0} \left( m + \sqrt{m^2 - q^2} \right) .
\]  

\[\text{This expression for } M \text{ has also been obtained in ref. [10].}\]
The electric charge and chemical potential are given by \[2,10\]

\[ Q_{el} = \sqrt{2} q e^{-2\phi_0} = \frac{q}{\sqrt{2m}} M, \quad \mu_{el} = -\frac{\sqrt{2q}}{m + \sqrt{m^2 - q^2}}. \] (2.18)

One can then check that \(^2\)

\[ I_2 = -\beta \mu_{el} Q_{el} = \beta \cdot 2Q e^{-2\phi_0} q^2 e^{-Qr_+}. \] (2.19)

To verify the thermodynamical relations eqs. (2.15), one needs to express \(S\) as a function of \(M\) and \(Q_{el}\) as follows

\[ S = 2\pi \frac{Q}{Q} \left( M + \sqrt{M^2 - 2Q^2_{el}} \right), \] (2.20)

then it is easy to see that eqs. (2.15) hold. The above results agree, after the appropriate change of variables, with those in ref. [10] \(^3\). Notice that in ref. [10] these results were obtained by exploiting thermodynamic relations rather than our prescription.

Let’s consider now the general Solution 1 with \(V(\phi) = \sum_{n \geq 0} a_n e^{2n\phi(x)}\). Obviously, it is not anymore possible to explicitly compute \(r_+\) so that the quantities depending on it will remain in an implicit form. Thus, one has

\[ M = 2mQ e^{-2\phi_0} \] (2.21)

\[ S = 4\pi \left[ e^{-2\phi} \right]_{r_+} = 4\pi \left[ e^{-2\phi_0} (2m - q^2 e^{-Qr_+}) + \sum_{n \geq 2} \frac{a_n e^{(n-1)(2\phi_0 - Qr_+)}}{(n-1)Q^2} \right]. \]

Notice that the (ADM) mass doesn’t change whereas the entropy does, as one would expect (for instance, we know that the entropy must change in the presence of an electromagnetic field, which is the \(a_2 \neq 0\) case). It is easy to check that for \(a_n = 0 (n \geq 2)\), one gets back eq. (2.17). Moreover

\[ I_2 = 2\beta Q e^{-2\phi_0} (2m - e^{-Qr_+}) = \beta \cdot 2Q e^{-2\phi_0} q^2 (1 + d(r_+)) e^{-Qr_+} \] (2.22)

where

\[ d(r_+) = \sum_{n \geq 2} \frac{a_n e^{n(2\phi_0 - Qr_+)}}{(1-n)q^2Q^2}. \] (2.23)

\(^2\) In deriving these results the following identity can be useful: \(m - \sqrt{m^2 - q^2} = q^2/(m + \sqrt{m^2 - q^2})\).

\(^3\) This change of variables is singular in the limit \(m \to \pm q\). This is the reason why this limit is unattainable in ref. [10].
This formula is very similar to eq. (2.19) which suggests us to define a renormalized electric charge as follows:

\[ q_{\text{ren}}^2 = q^2 (1 + d(r_+)) \quad (2.24) \]

\[ Q_{\text{el}}^\text{ren} = \sqrt{2} q_{\text{ren}} Q e^{-2\phi_0}, \quad \mu_{el}^\text{ren} = -\sqrt{2} q_{\text{ren}} e^{-Q r_+} \]

so that, again

\[ I_2 = -\beta \mu_{el}^\text{ren} Q_{el}^\text{ren} . \quad (2.25) \]

Finally, one can easily check that

\[ S = \frac{2\pi}{Q} \left( M + \sqrt{M^2 - 2(Q_{el}^\text{ren})^2} \right) \quad (2.26) \]

although it is not anymore possible to check eqs. (2.15) since we don’t know the explicit expression of \( r_+ \). The formula (2.26) suggests that the loop corrections can be interpreted as charge renormalization. This is the only reasonable interpretation, since, aside from the electromagnetic one, there is no other conserved charge in the game that could give rise to new chemical potentials.

2.2 Solution 2 (\( \gamma = 2 \))

The discussion just done for the Solution 1 can be repeated for the solutions with \( 0 < \gamma < 4 \) and in particular for those with \( \gamma = 2 \). Although it is possible to give the explicit formulæ for the general case \( 0 < \gamma < 4 \) (and the interested reader can easily work them out), for simplicity here we will just quote those for the Solution 2 (\( \gamma = 2 \)). One has

\[ M = 2 m e^{-2\phi_0} \quad (2.27) \]

\[ S = 4\pi [e^{-2\phi}]_{r_+} = 4\pi \left[ e^{-2\phi_0} (2mr_+ - q^2) - \sum_{n \geq 2} \frac{e^{2(n-1)\phi_0} a_n}{6 - 4n} r_+^{-2n} \right] \]

where \( q^2 = \frac{1}{4} e^{4\phi_0} f_0^2 \). Moreover

\[ I_2 = \beta \cdot 4 e^{-2\phi_0} (2m - r_+) \]

\[ = \beta \cdot \frac{4q^2 e^{-2\phi_0}}{r_+} \left( 1 + \frac{1}{q^2} \sum_{n \geq 2} \frac{a_n e^{2n\phi_0}}{6 - 4n} r_+^{-2n} \right) \quad (2.28) \]
which suggests to define

\[ q^2_{\text{ren}} = q^2 \left( 1 + \frac{1}{q^2} \sum_{n \geq 2} a_n e^{2n\phi_0} \frac{r^4 - 2n}{6 - 4n} \right) \]

\[ Q^\text{ren}_{el} = 2q_{\text{ren}} e^{-2\phi_0}, \quad \mu^\text{ren}_{el} = -\frac{2q_{\text{ren}}}{r_+} \] (2.29)

so that \( I_2 = -\beta \mu^\text{ren}_{el} Q^\text{ren}_{el} \). It is also simple to check that

\[ S = 4\pi e^{-2\phi_0} (r_+)^2 = 4\pi e^{-2\phi_0} \frac{e^{-2\phi_0}}{4} \left( M + \sqrt{M^2 - (Q^\text{ren}_{el})^2} \right)^2. \] (2.30)

Setting \( a_n = 0 \) for \( n \geq 2 \) and \( \phi_0 = 0 \), one gets

\[ g(x) = 1 - \frac{2m}{x} + \frac{q^2}{x^2}, \quad r_+ = m + \sqrt{m^2 - q^2} \] (2.31)

\[ \beta = 2\pi \left( 2\sqrt{m^2 - q^2} + 2m + \frac{q^2}{\sqrt{m^2 - q^2}} \right) \]

\[ M = 2m, \quad Q_{el} = 2q, \quad \mu_{el} = -\frac{2q}{r_+} \]

\[ S = 4\pi (r_+)^2 = \pi \left( M + \sqrt{M^2 - (Q_{el})^2} \right)^2. \]

These are the well known results for a 4D charged, spherically symmetric black-hole, i.e. the Reissner-Nordstrøm black-hole, obtained now from a two dimensional analysis.

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