Fibre bundle formulation of nonrelativistic quantum mechanics

V. Theory’s interpretation, summary and discussion

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Short title: Bundle quantum mechanics: V

Basic ideas: → March 1996
Began: → May 19, 1996
Ended: → July 12, 1996
Revised: → December 1996 – January 1997,
Revised: → April 1997, September 1998
Last update: → November 16, 1998
Composing/Extracting part V: → September 27/October 4, 1997
Updating part V: → February 20, 1999
Produced: → November 20, 2021

LANL xxx archive server E-print No.: quant-ph/9902068

Subject Classes:
Quantum mechanics; Differential geometry

1991 MSC numbers: 81P05, 81P99, 81Q99, 81S99
1996 PACS numbers: 02.40.Ma, 04.60.-m, 03.65.Ca, 03.65.Bz

Key-Words:
Quantum mechanics; Geometrization of quantum mechanics;
Fibre bundles

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Abstract

We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one but it is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it a pure state of some quantum system is described by a state section (along paths) of a (Hilbert) fibre bundle. Its evolution is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and the density operator are described via bundle morphisms (along paths). The mentioned quantities are connected by a number of relations derived in this work.

This is the fifth closing part of our investigation. We briefly discuss the observer’s rôle in the theory and different realizations of the space-time model used as a base space in the bundle approach to quantum mechanics. We point the exact conditions for the equivalence of Hilbert bundle and Hilbert space formulations of the theory. A comparison table between the both description of nonrelativistic quantum mechanics is presented. We discuss some principal moments of the Hilbert bundle description and show that it is more general than the Hilbert space one. Different directions for further research are pointed too.
1. Introduction

The present paper is the fifth closing part of our series of works [1, 2, 3, 4] devoted to the fibre bundle formulation of nonrelativistic quantum mechanics. It is organized as follows.

In Sect. 2 is paid attention on the observer’s rôle in the theory and are considered and interpreted some modifications of the proposed approach to quantum mechanics. Possible fields for further research are sketched too.

In Sect. 3 are briefly summarized our results and it is presented a comparison table between the conventional, Hilbert space, and the new, Hilbert bundle, formulations of nonrelativistic quantum mechanics.

In Sect. 4 are discussed certain aspects of the bundle formulation of nonrelativistic quantum mechanics and are pointed some its possible generalizations and applications.

The notation of the present part of the series is identical with the one of the preceding ones for which the reader is referred to [1, 2, 3, 4].

The references to sections, equations, footnotes etc. from the previous four parts of the series, namely [1], [2], [3], and [4], are denoted by the corresponding sequential reference numbers in these parts preceded by the Roman number of the part in which it appears and a dot as a separator. For instance, Sect. I.5 and (IV.2.11) mean respectively section 5 of part I, i.e. of [1], and equation (2.11) (equation 11 in Sect. 2) of part IV, i.e. of [4].

2. On observer’s rôle and theory’s interpretation

The concept of an ‘observer’ is more physical than mathematical one as it is not very well mathematically rigorously defined; some times its meaning is more intuitive than strict one. Generally an ‘observer’ is a physical system whose state is assumed to be ‘completely’ known and which has a double rôle with respect to the other systems(s) which is (are) under consideration. From one hand, it provides certain reference point, generally a set of objects and their properties, with respect to which are determined (all of) the quantities characterizing the investigated system(s) in some problem. From the other hand, it is supposed the ‘observer’ can perform certain procedures, called ‘measurements’ or ‘observations’, by means of which ‘he’ finds (determines) the parameters, properties, quantities, etc. describing the studied system(s). This second rôle of the observers is out of the subject of the present investigation and will not be discussed here (for some its aspects see, e.g., [5, 6]).

In this work the observers are supposed to be local and point-like, i.e. they are material points that can perform measurements at the points at which they are situated. They are moving (along paths) in some differ-
entiable manifold $M$.\(^1\) Namely their trajectories (world lines in special or
general relativity interpretation (see below)) are the reference objects with
respect to which we study the behaviour of the quantum systems. The ‘observational’ properties of an observer are assumed fixed and such that: (i)
Allow the observer to determine the initial values of the quantities character-
izing the state of the studied system(s) at some instant of time $t_0$; (ii)
Give certain correspondence rules according to which to any dynamical vari-
able $\mathbf{A}$, connected to the investigated system(s), is assigned some observable
which is a Hermitian operator (resp. morphism along paths), say $\mathbf{A}$ (resp. $A$), in the Hilbert space (resp. bundle) description that has a complete set
of (maybe orthonormal) eigenvectors (resp. eigensections along paths).

All quantities in the present work are referred to an observer moving
along some path $\gamma: J \to M$ parameterized with $t \in J$, where $J \subseteq \mathbb{R}$ is a real
interval. Our intention is to interpret $t$ as a ‘time’. The possibility for such
an interpretation is connected with the specific choice of the manifold $M$.

If we assume $M$ to be the classical, coordinate, 3-dimensional Euclidean
space $\mathbb{E}^3$ of the classical and quantum mechanics, which we have done more
implicitly than explicitly throughout this investigation, there exists a global
\textit{time} $t_g \in (-\infty, +\infty) = \mathbb{R}$ (in the Newtonian sense). In this case the tra-
jectories, such as $\gamma$, of \textit{all} observers is natural and convenient to be param-
eterized with this global time which, in fact, is done in the conventional,\nHilbert space, quantum mechanics. (One can also parameterize each ob-
server’s trajectory with its own local time which is in one-to-one, usually $C^1$,
correspondence with $t_g$, but this is an inessential generalization as $t_g$ itself is
defined with some arbitrariness (usually a $C^1$ map $\mathbb{R} \to \mathbb{R}$).) Consequently
the assumptions $M = \mathbb{E}^3$, $J = \mathbb{R}$ and $t = t_g$ are necessary and sufficient for
the full equivalence of the Hilbert space and Hilbert bundle descriptions of
quantum mechanics. These assumptions lead to one inessential mathemati-
cal complication. Since the observer’s trajectory $\gamma$ can has self-intersections,
the sections along $\gamma$ and morphisms along $\gamma$ can be multiple-valued at the
points of self-intersection of $\gamma$, if any. But this does not have some serious
consequences.

Analogous to $M = \mathbb{E}^3$ is the case when the Hilbert bundle is identified
with the system’s configuration space. The only difference is that $\gamma: J \to M$
has to be interpreted as the trajectory of the system in this space rather then
the one of some observer. Practically the same is the case when $M$ is taken
as the system’s phase space but, since this situation has some peculiarities, we
shall comment on it in Sect. 4.

A similar, but slightly different, is the situation when $M$ is taken to
be the four dimensional Minkowski space-time $M^4$ of special relativity.\(^2\)

\(^1\)Mathematically the developed here theory is sensible also if $M$ is considered as a
‘more general’ object than a manifold, but, at present, there are not indications that such
a theory can be physically important.

\(^2\)A like construction, under the name ‘Schrödinger bundle’, is introduced in the para-
Nevertheless that now every observer has its own local preferred time, called proper or eigen-time, there is a global (e.g. coordinate) time \( t_g \) which is in 1:1 correspondence with these local times. In this case the observers are moving along paths, such as \( \gamma \), which are their world lines (paths or curves) and usually are parameterized with the global time \( t_g \). This is an important moment because the world lines of the real objects observed until now can not have self-intersections. Therefore, since the observers are supposed to be such, their world lines are without self-intersections. This implies the absence of the complication mentioned at the end of the previous paragraph, viz. now the sections along \( \gamma \) and morphisms along \( \gamma \) are single-valued and, in fact, are respectively sections and morphisms of the restriction \( (F, \pi, M)|_\gamma (J) \) of the Hilbert bundle \( (F, \pi, M) \) on the set \( \gamma (J) \). Otherwise the case \( M = M^4 \) is identical with the one with \( M = E^3 \). Therefore we can say that it represents the Hilbert bundle description of the nonrelativistic quantum mechanics over the special relativity space-time. This point is worth-mentioning as at it meet the relativistic and non-relativistic concepts whose unification leads to the relativistic quantum theory which will be considered elsewhere.

Another important possibility is \( M = V_4 \) where \( V_4 \) is the four dimensional pseudo-Riemannian space-time of general relativity.\(^3\) The crucial point here is the generic non-existence of some global time, so the world line of any particular observer, say \( \gamma \), with necessity has to be parameterize with its (specifically local) proper time \( t \). A consequence of this is that in the (conventional) Hilbert space description the parameter (‘time’) \( t \) also has to be considered as a local (proper) time for the observer which describes the quantum system under consideration. Hence the global sections and morphisms defined via (I.4.3) and (II.3.5) have no physical meaning now. As in special relativity case, \( M = M^4 \), now \( \gamma \) cannot have self-intersections; so the sections and morphisms along \( \gamma \) are single-valued. What concerns other aspects of the case \( M = V_4 \), it is identical with the one for \( M = E^3 \). Consequently, it represents the non-relativistic quantum mechanics over the general relativity space-time. Here we see again a meeting of (general) relativistic and nonrelativistic concepts whose unification will be investigated elsewhere.

Now we want to call attention to a particular ‘degenerate’ case which falls out of our general interpretation of \( \gamma : J \rightarrow M \) as an observer’s world line (trajectory). Namely, it is possible to put \( M = J \), \( J \) being a real interval. For instance, as we pointed at the end of Sect. II.2, the considerations of [8] correspond to the choice \( M = \mathbb{R}_+ = \{ t : t \in \mathbb{R}, t \geq 0 \} \). Thus, if \( M = J \), graph containing equation (4) of [7]. This is a Hilbert bundle over \( M^4 \) having as a (standard) fibre \( \mathcal{F} \times \mathcal{F}^* \) instead of the conventional Hilbert space \( \mathcal{F} \) in our case.

\(^3\)Almost the same is the case when \( M \) represents the space-time model of other gravitational theories, like Einstein-Cartan and the metric-affine ones, in which \( M \) is a curved (non-flat) manifold with respect to some linear connection.
then \( \gamma : J \to J \) and, as we tend to interpret \( t \in J \) as time, it is natural now to assume that \( \gamma \) is one-to-one smooth \((C^1)\) map.\footnote{The case considered in reference [8] corresponds to \( \gamma = \text{id}_{\mathbb{R}^+} \).} If so, \( \gamma(t) \in J \) is also a ‘presentation’ of the time, but in other (re)parametrization. In this case \( \gamma \) is without self-intersections and, consequently, the sections and morphisms along \( \gamma \) are simply (single-valued) sections and morphisms of the bundle \((F, \pi, J)\) over the one-dimensional base \( J \). Therefore, in this situation, we can say that the evolution of a quantum system is described via linear transportation (of the state) sections of \((F, \pi, J)\) along the time, respectively the evolution of the observables is represented by linear transportation of (the observable) morphisms of \((F, \pi, J)\) along the time. Let us note that now the connection with the observers is not completely lost as the time does not exist by ‘itself’ in the theory; it is connected with (measured by) a concrete observer regardless of the fact that it can be global or local (see above).

Another degenerate case is when \( M \) consist of a single point, \( M = \{x\} \). Then \( F = \pi^{-1}(x) =: F_x \) and \( \gamma : J \to \{x\} \). Therefore \( \gamma(t) \equiv x \) for every \( t \in J \) and, if \( J \) is not compressed into a single real number, \( \gamma \) self-intersects at \( x \) infinitely many times. Also we have \( l_{\gamma(t)} = l_{\gamma(s)} = l_x =: l, \ s, t \in J \) with \( l : F \to \mathcal{F} \) being an isomorphism. So, in this way (see the paragraph after the one containing equation (I.4.2)), we obtain an isomorphic image in \( F \) of the quantum mechanics in \( \mathcal{F} \). Evidently, the conventional quantum mechanics is recovered by the choices \( \mathcal{F} = F \) and \( l = \text{id}_F \). Of course, now we can not interpreted \( \gamma \) as observer’s trajectory or world line but the interpretation of \( t \) as a ‘time’ can be preserved.

If the quantum system under consideration has a classical analogue, then the manifold \( M \) can be identified with the system’s configuration or phase space. In this case the path \( \gamma : J \to M \) can be taken to be the trajectory of the system’s classical analogue in the corresponding space. Thus we obtain an interesting situation: the (bundle) quantum evolution is described with respect to (is referred to) the corresponding classical evolution of the same system.

One can also take \( M \) to be the configuration or phase space of some observer. Then \( \gamma \) can naturally be defined as the observer’s trajectory in the corresponding space.

At this point we want to say a few words on the possibility to identify the Hilbert bundle’s base \( M \) with the phase space of certain system and to make some comments on [9], where this case is taken as a base for a bundle approach to quantum mechanics. Our generic opinion is that the phase space is not a ‘suitable’ candidate for a bundle’s base, the reason being the Heisenberg uncertainty principle by virtue of which the points of the phase space have no physical meaning [10, chapter IV]. This reason does not apply if as a base is taken the phase space of some observer as, by definition, the observers are treated as classical objects (systems). Therefore one can set
the base $M$ of the Hilbert bundle $(F, \pi, M)$ to be the phase space of some observer. Then the reference path $\gamma: J \rightarrow M$ can be interpreted as the observer’s phase-space trajectory which, generally, can have self-intersections. The further treatment of this case is the same as of $M = \mathbb{E}^3$. Regardless of the above-said, one can always identify $M$ with the system’s phase space, if it exists, as actually $M$ is a free parameter in the present work.

An interesting bundle approach to quantum mechanics is contained in [9]. In it the evolution of a quantum system is described in a Hilbert bundle over the system’s phase space with the ordinary system’s Hilbert space as a (typical) fibre which is, some times, identified with the fibre over an arbitrary fixed phase-space point. The evolution itself is presented as a parallel transport in the bundle space generated via non-dynamical linear (and symplectic) connection which is closely related to the symplectic structure of the phase space. An important feature of [9] is that in it the bundle structure is derived from the physical content of the paper. In this sense [9] can be considered as a good motivation for the general constructions in this investigation.

Before comparing the mathematical results of [9] with the ones of this work, we have to say that the loc. cit. contains some incorrect ‘bundle’ expressions which, however, happily do not influence most of the conclusions made on their base. In [11] we point to and show possible ways for improving of a number of mathematically non-rigorous or wrong expressions, assertions, and definitions in [9]. We emphasize that all this concerns only the ‘bundle’ part of the mathematical structure of loc. cit. and does not deal with its physical contents. The general moral of the critical remarks of [11] is: most of the final results and conclusions of [9] are valid provided the pointed in [11] (and other minor) corrections are made. Below we shall suppose that this is carefully done. On this base we will compare [9] with the present work.

The main common point between [9] and this investigation is the consistent application of the fibre bundle theory to (nonrelativistic) quantum mechanics. But the implementation of this intention is quite different: in [9] we see a description of quantum mechanics in a new, but ‘frozen’, geometrical background based on a non-dynamical linear connection deduced from the symplectical structure of the system’s phase space, while the present work uses a ‘dynamical geometry’ (linear transport along paths, which may turn to be a parallel one generated by a linear connection) whose properties depend on the system’s Hamiltonian, i.e. on the physical system under consideration itself.

The fact that in [9] the system’s phase space is taken as a base of the used Hilbert bundle is not essential since nothing can prevent us from making the same choice as, actually, the base is not fixed here. In [9] is partially considered the dynamics of multispinor fields. This is an interesting problem, but, since it is not primary related to conventional quantum mechanics, we
think it is out of the scope of our work. The methods of its solution are
outlined in [9] and can easily be incorporated within our bundle quantum
mechanics.

The fields of (metaplectic) spinors used in [9] are simply sections of
the Hilbert bundle, while the “world-line spinors” in loc. cit. are sections
along paths in our terminology. The family of operators \( O_{\phi^a} \) or \( O_f(\phi) \) [9,
equations (4.8) and (4.9)] acting on \( F_{\phi} \) are actually bundle morphisms.

A central rôle in both works plays the ‘principle of invariance of the mean
values’: the mean values (mathematical expectations) of the morphisms cor-
responding to the observables (dynamical variables) are independent of the
way they are calculate. We have used this assumption many times (see,
e.g., Sections II.3, III.2, and IV.2, in particular, equations (II.3.3), (III.2.5),
(III.2.11), (III.2.28), and (IV.2.17)) without explicitly formulating it as a
‘principle’. But if one wants to build axiomatically the bundle quantum
mechanics, he will be forced to include this principle (or an equivalent to
it assertion) into the basic scheme of the theory. In [9] ‘the invariance of
the mean values’ is mentioned several times and it is used practically in the
form of the ‘background-quantum split symmetry’ principle, explained in [9,
sect. 4] (see, e.g., [9, equation (4.18)] and the comments after it). Its particu-
lar realizations are written as [9, equation (4.17) and second equation (4.42)]
which are equivalent to it in the corresponding context. A consequence of
the mean-value invariance is the ‘Abelian’ character of the compatible with
it connections, expressed by [9, equation (4.14)], which is a special case of
our result [12, equation (4.4)]. In [9] the mean values are independent of
the point at which they are determined. In our bundle quantum mechan-
ics this is not generally the case as different points correspond to different
time values (see, e.g., (II.3.3)). This difference clearly reflects the dynam-
ical character of our approach and the ‘frozen’ geometrical one of [9]. In
any case, the principle we are talking about is so important that without
it the equivalence between the bundle and conventional forms of quantum
mechanics can not be established.

In both works the quantum evolution is described via appropriate trans-
port along paths: In [9, see, e.g., equations (3.54) and (4.53)] this is an
‘Abelian’ parallel transport along curves, whose holonomy group is \( U(1) \) [9,
equation (4.38)], while in our investigation is employed a transport along
paths uniquely determined by the Hamiltonian (see Sect. I.5) which, gener-
ally, need not to be a parallel translation.

Now we turn our attention on the bundle equations of motion: in the
current work we have a single bundle Schrédinger equation (II.2.24) (see
also its matrix version (II.2.11)), while in [9, equation (5.54)] there is an
infinite number of such equations, one Schrödinger equation in each fibre
\( F_{\phi} \) for the system’s state vector \( |\psi(t)\rangle_{\phi} \) at every point \( \phi \in M \).
\(^5\) Analogous is

\(^5\)Note that the appearing in [9, equations (4.54)–(4.56)] operator \( O_H \) is an analogue of
the situation with the statistical operator (compare our equation (IV.2.17) or (IV.2.15) with [9, equation (4.56)]). This drastic difference is due to the different objects used to describe systems states: for the purpose we have used sections along paths (see Sect. I.4), while in [9] are utilized (global) sections of the bundle defined via (I.4.3) (cf. [9, equation (4.41)]). Hence, what actually is done in [9] is the construction of an isomorphic images of the quantum mechanics from the fibre $\mathcal{F}$ in every fibre $F_\phi, \phi \in M$ (see the comments after (I.4.3)).

To summarize the comments on part of the mathematical structures in [9]: It contains a fibre bundle description of quantum mechanics. The state vectors are replaced by (global) sections of a Hilbert bundle with the system’s phase space as a base and their (bundle) evolution is governed through Abelian parallel transport arising from the symplectical structure of the phase space. Locally, in any fibre of the bundle, the evolution is presented by a Schrödinger equation, specific for each fibre of the bundle. The work contains a number of incorrect mathematical constructions which, however, can be corrected so that the final conclusions remain valid. Some ideas of the paper are near to the ones of this investigation but their implementation and development is quite different in both works.

3. Summary

In this work we have proposed and developed a new invariant fibre bundle formulation of nonrelativistic quantum mechanics. It is fully equivalent to the usual formulation of the theory in the case when the underlying manifold is the (coordinate) three dimensional Euclidean space of classical (quantum) mechanics. In the new description a pure state of a quantum system is described by a state section along paths of a Hilbert fibre bundle. The time evolution of the state sections obeys the bundle Schrödinger equation (II.2.24). A mixed state of a quantum system is described via the density morphism along paths satisfying the Schrödinger (type) equation (IV.2.17). In the proposed bundle approach to any dynamical variable corresponds a unique observable which is a bundle morphism along paths of the Hilbert fibre bundle of the investigated system. The observed value of a dynamical variable is equal (by definition) to the mean value (the mathematical expectation) of the corresponding bundle morphism and it is calculated by means of the bundle state section or density morphism corresponding to the system’s state at the moment.

The correspondence between the conventional Hilbert space description and the new Hilbert bundle description of non-relativistic quantum mechanics is given in table 1 on page 10.
A feature of the bundle form of quantum mechanics is the inherent connection between physics and geometry: the system’s Hamiltonian (via equation (II.2.21)) completely determines the concrete properties of the system’s Hilbert bundle. In this sense, here the Hamiltonian plays the same rôle as the energy-momentum tensor in general relativity. Another viewpoint (also based on (II.2.21)) is to look on the Hamiltonian as a gauge field in the sense of Yang-Mills theories. In any case, we see in the bundle quantum mechanics a realization of the intriguing idea, going back to Albert Einstein and Bernhard Riemann, that the physical properties of the systems are responsible for the geometry of the spaces used for their description.

4. Discussion

Since the set of all sections of a vector bundle is a module over the ring of all \((C^0)\) functions on its base with values in the field with respect to which it has a vector structure [13, chapter 3, propositions 1.6], the set \(\text{Sec}(F, \pi, M)\) is a module over the ring of functions \(f: M \to \mathbb{C}\). Besides, this module is equipped with a scalar product. In fact, if \(\Phi, \Psi \in \text{Sec}(F, \pi, M)\), and \(f, g: M \to \mathbb{C}\), the vector structure of \(\text{Sec}(F, \pi, M)\) is given by

\[
(f\Phi + g\Psi): x \mapsto f(x)\Phi(x) + g(x)\Psi(x) \in F_x, \quad x \in M \tag{4.1}
\]

and its inner product is defined via \((\Phi, \Psi) \mapsto \langle \Phi|\Psi \rangle: M \to \mathbb{C}\) where

\[
\langle \Phi|\Psi \rangle: x \mapsto \langle \Phi|\Psi \rangle(x) := \langle \Phi(x)|\Psi(x) \rangle_x \in \mathbb{C}, \quad x \in M. \tag{4.2}
\]

Such a structure, module with an inner product, can naturally be called a Hilbert module.

Moreover, any bundle morphism \(A \in \text{Morf}(F, \pi, M)\) can be considered as an operator \(A: \text{Sec}(F, \pi, M) \to \text{Sec}(F, \pi, M)\) of the sections over \((F, \pi, M)\) whose action is defined by

\[
(A\Phi): x \mapsto A_x(\Phi(x)), \quad \Phi \in \text{Sec}(F, \pi, M), \quad x \in M \tag{4.3}
\]

and vice versa, to any operator \(B: \text{Sec}(F, \pi, M) \to \text{Sec}(F, \pi, M)\) there corresponds a unique morphism of \((F, \pi, M)\) whose restriction \(B_x\) on \(F_x\) is given via

\[
B_x(\Phi(x)) := (B\Phi)(x), \quad \Phi \in \text{Sec}(F, \pi, M), \quad x \in M. \tag{4.4}
\]

These mathematical results allow us, if needed, to reformulate (equivalently) the Hilbert bundle description of quantum mechanics in terms of vectors and operators, but now in the Hilbert module of sections of the Hilbert fiber bundle over the space-time.
Since any pure state of a quantum system can be described via a suitable density operator \([5, \text{ chapter VIII, \S 24}]\), the remark at the end of subsection IV.2.2 suggests that it is possible the nonrelativistic quantum mechanics to be formulated entirely in terms of morphisms along paths in the fibre bundle of the morphisms along paths of the Hilbert bundle of the considered system.

In the present investigation we have not fixed the base \(M\) of the Hilbert bundle \((F, \pi, M)\). We did not used even some concrete assumptions about \(M\), except the self-understanding ones, e.g. such as that it is an non-empty topological space. At the beginning of Sect. I.4 we assumed \(M\) to be the (coordinate) 3-dimensional Euclidean space \(E^3\) of quantum (or classical) mechanics. This is required for the physical interpretation of the developed here theory. This interpretation holds true also for any differentiable manifold \(M\) with \(\dim M \geq 3\). This is important in connection with further generalizations. For instance, such are the cases when \(M\) is chosen as the 4-dimensional Minkowski space-time \(M^4\) of special relativity, or the pseudo-Riemannian space-time \(V_4\) of general relativity, or the Riemann-Cartan space-time \(U_4\) of the \(U_4\)-gravitational theory. All this points to the great arbitrariness in the choice of the geometrical structure of \(M\). Generally it has to be determined by a theory different from quantum mechanics, such as the classical mechanics or the special or general relativity. As a consequence of this, there is a room for some kind of unified theories, for instance for a unification of quantum mechanics and general relativity. These problems will be investigated elsewhere.

The developed in the present investigation theory is global in a sense that in it we interpret \(M\) as being the whole space(-time) where the studied objects ‘live’. If by some reasons one wants or is forced to consider a system resided into a limited region of \(M\), then all the theory can be localized by replacing \(M\) with this region or simply via \textit{mutatis mutandis} restricting the already obtained results on it. At the same time, our theory is local in a sense that such are the used in it observers which are assumed to can make measurements only at the points they reside. The theory can slightly be generalized by admitting the observers can perform observations (measurements) at points different from their own residence. This puts the problem of defining the mean values of the observables at points different from the one at which the observer is situated in such a way as they to be independent of the possibly introduce for this purpose additional constrictions (cf. \([12, \text{ sect. 3}]\)). This problem will be solved elsewhere.

The observers we have been dealing until now in this work can be called scalar and point-like as it is supposed that they have no internal structure and the only their characteristics in the theory are their positions (generally) in the space-time \(M\). Notice that as a set the manifold \(M\) coincides with the variety of all possible positions of all possible observers. This observation suggests that in the general case \(M\) has to be replaced with a set \(\widetilde{M}\) consis-
Table 1: Comparison between Hilbert space and Hilbert bundle descriptions.

| Hilbert space description | Hilbert bundle description | Remark(s) |
|---------------------------|---------------------------|-----------|
| Vector $\varphi \in F$   | Section $\Phi \in \text{Sec}(F, \pi, M)$ | $I_\varphi : F_\varphi \rightarrow F$ |
| Operator $A : F \rightarrow F$ | Bundle morphism $\Phi \in \text{Morf}(F, \pi, M)$ | $\Phi : \varphi \mapsto I_\varphi(\varphi)$ |
| State vector $\psi \in F$ | State section $\Psi$ along paths of $(F, \pi, M)$ | $\Psi(t) = (I_\varphi(t))$ |
| Observable $A : F \rightarrow F$ | Bundle morphism $A$ along paths of $(F, \pi, M)$ | $A_\varphi(t) = (I_\varphi(t)) \circ A(t) \circ I_\varphi(t)$ |
| Hermitian scalar product $\langle \varphi | \psi \rangle$ | Hermitian bundle scalar product $\langle \Phi_x | \Psi_x \rangle_x$ | $\langle \cdot | \cdot \rangle = \langle \cdot \rangle$ |

Hermitian conjugate operator $A^\dagger$ to an operator $A$:

1) Hermitian conjugate map $A_\varphi^\dagger : F_\varphi \rightarrow F_\varphi$ to a bundle map $A_{\varphi \rightarrow x} : F_{\varphi \rightarrow x} \rightarrow F_{\varphi \rightarrow x}$:

$\langle A^\dagger_\varphi \varphi | \Psi_y \rangle_y = \langle \Phi_x | A_{\varphi \rightarrow x} \Psi_x \rangle_x$

2) Hermitian conjugate morphism $A^\dagger$ to a bundle morphism $A$ along paths:

$\langle A^\dagger_\varphi \varphi | \Psi_x \rangle_x = \langle \Phi_x | A_x \Psi_x \rangle_x$

Unitary operator: $A^\dagger = A^{-1}$

1) Unitary bundle map:

$A_{\varphi \rightarrow y} = A_{\varphi \rightarrow x}^{-1} = (A_{\varphi \rightarrow x})^{-1}$

2) Unitary bundle morphism: $A^\dagger = A^{-1}$

1) Mean value of an operator $A$:

$\langle A(t) \rangle^\dagger = \frac{\langle \psi(t) | A(t) \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle}$

2) Mean value of a bundle morphism $A$:

$\langle A(t) \rangle^\dagger_{\Psi_x} = \frac{\langle \Phi_x(t) | A(t) \Phi_x(t) \rangle_{\Psi_x(t)}}{\langle \Phi_x(t) | \Phi_x(t) \rangle_{\Psi_x(t)}}$ |

Evolution operator $U$

Evolution transport $U$ along paths

1) Hamiltonian $\mathcal{H}$

2) Matrix Hamiltonian $\mathcal{H}^m$

1) Bundle Hamiltonian $H$ |

2) Matrix-bundle Hamiltonian $\mathcal{H}^m$

3) Matrix $\Gamma$ of the coefficients of the evolution transport

1) Schrödinger equation:

$i\hbar \frac{d\psi(t)}{dt} = \mathcal{H}(t)\psi(t)$

1) Bundle Schrödinger equation:

$D^*_\varphi \Psi_x = 0$

2) Matrix Schrödinger equation:

$i\hbar \frac{d\psi(t)}{dt} = \mathcal{H}^m(t)\psi(t)$

2) Matrix-bundle Schrödinger equation:

$\Gamma(t) = -\mathcal{H}^m(t)/i\hbar$

Equivalent equations

Density operator $\rho$

Density operator evolution:

$i\hbar \frac{d\rho(t)}{dt} = [\mathcal{H}(t), \rho(t)]_{\psi}$

Density morphism evolution:

$D^*_\rho (\Psi_x) = 0$ or $D^*_\rho (\Psi_x) = 0$

Equivalent equations

Schrödinger picture of motion:

$\psi(t)$, $A(t)$; $\rho(t)$, $A(t)$

Bundle Schrödinger picture of motion:

$\Psi_x(t)$, $A_x(t)$; $P_x(t)$, $A_x(t)$

See: (I.2.6), (II.2.24); (IV.2.5), (IV.2.15), (IV.2.17)

Heisenberg picture of motion:

$\psi^H(t_o)$, $A^H(t_o)$; $\rho^H(t_o)$, $A^H(t_o)$

Bundle Heisenberg picture of motion:

$\Psi^H_x(t_o)$, $A^H_x(t_o)$; $P^H_x(t_o)$, $A^H_x(t_o)$

See: (III.2.12), (III.2.28), (III.2.14), (III.2.15); (IV.2.18), (IV.2.19)

‘General’ picture of motion:

$\psi^G(t_o)$, $A^G(t_o)$; $\rho^G(t_o)$, $A^G(t_o)$

Bundle ‘general’ picture of motion:

$\Psi^G_x(t_o)$, $A^G_x(t_o)$; $P^G_x(t_o)$, $A^G_x(t_o)$

See: (III.2.36), (III.2.39)–(III.2.41), (III.2.31); (IV.2.26)–(IV.2.29)

Integral of motion $A$:

$\frac{d}{dt} A(t) + [A(t), H(t)]_{\psi} = 0$

Integral of motion $A$:

$D^*_\varphi (A_x) = 0$ or $D^*_\varphi (A_x) = 0$ (IV.2.17)

Equivalent concepts. See: (III.3.10), (III.3.20); (III.3.11), (III.3.19)
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...ting of all of the values of all parameters describing completely the states of any possible observer. Naturally, the set $\tilde{M}$ has to be endowed with some topological or/and smooth geometrical structure. For instance, consider an ‘anisotropic’ point-like observers characterized by their position $x$ in $M$ and some vector $V_x$ at it, i.e. by pairs like $(x, V_x)$, $x \in M$, $V_x \in L_x$ with $L_x$ being some vector space. In this case $\tilde{M}$ is naturally identified with the (total) bundle space $L$ of the arising over $M$ vector bundle $(L, \pi_L, M)$ with $L := \bigcup_{x \in M} \bigcup_{V_x \in L_x} (x, V_x)$ and $\pi_L(x, V_x) := x \in M$, i.e. we can put $\tilde{M} = L$. In particular, if $V_x$ is the observer’s velocity at $x$, we have $\tilde{M} = T(M)$ with $(T(M), \pi_{T(M)}, M)$ being the bundle tangent to $M$. Another sensible example is when $V_x$ is interpreted as observer’s spin etc. In connection with some recent investigations (see, e.g., [14, 9]) it is worth to be studied the special case when $\tilde{M}$ is taken to be the (classical) phase space of the observer. Returning to the general situation, we see that our theory can easily be modified to cover such generalizations. For this purpose we have simply to replace the Hilbert bundle $(F, \pi, M)$ over $M$ with the Hilbert bundle $(F, \pi, \tilde{M})$ over $\tilde{M}$. This, together with other evident corresponding changes, such as $(x \in M) \mapsto (x \in \tilde{M})$ and $(\gamma: J \to M) \mapsto (\gamma: J \to \tilde{M})$, allows us to apply the developed in the present investigation Hilbert bundle description to far more general situations than the one we were speaking about until now. In principle, the afore-described procedure is applicable to non-local observers too, but this is out of the subject of the present work.

We want also to mention that since any fibre of the Hilbert bundle $(F, \pi, M)$ is an isomorphic image of the Hilbert space $F$, the conventional probabilistic interpretation of the nonrelativistic quantum mechanics [5, 15] remains mutatis mutandis completely valid in the Hilbert bundle description too. For this purpose one has to replace state vectors and operators acting on $F$ with the corresponding state sections along paths and morphisms along paths of $(F, \pi, M)$ and then to follow the general rules outlined in this work and, for instance, in [5].

In connection with further applications of the bundle approach to the quantum field theory, we notice the following. Since in this theory the matter fields are represented by operators acting on (wave) functions from some space, the matter fields in their bundle modification should be described via morphisms (along paths) of a suitable fibre bundle whose sections (along paths) will represent the wave functions. We can also, equivalently, say that in this way the matter fields would be sections of the fibre bundle of bundle morphisms of the mentioned suitable bundle. An important point here is that the matter fields are primary related to the bundle arising over the space-time (or other space which includes it) and not to the space-time itself to which other structures are directly related, such as connections and the principle bundle over it.

The bundle approach to nonrelativistic quantum mechanics, developed
in the present investigation, seems also applicable to classical mechanics, statistical mechanics, relativistic quantum mechanics, and field theory. We hope that such a novel treatment of these theories will reveal new perspectives for different generalizations, in particular for their unification with the theory of gravitation.

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