Delayed Recursive State and Input Reconstruction

Roshan A Chavan† and Harish J. Palanthandalam-Madapusi‡

SysIDEA Lab
IIT Gandhinagar,
Ahmedabad - 382424, India.

Abstract

The unknown inputs in a dynamical system may represent unknown external drivers, input uncertainty, state uncertainty, or instrument faults and thus unknown-input reconstruction has several widespread applications. In this paper we consider delayed recursive reconstruction of states and unknown inputs for both square and non-square systems. That is, we develop filters that use current measurements to estimate past states and reconstruct past inputs. We further derive necessary and sufficient conditions for convergence of filter estimates and show that these convergence properties are related to multivariable zeros of the system. With the help of illustrative examples we highlight the key contributions of this paper in relation with the existing literature. Finally, we also show that existing unbiased minimum-variance filters are special cases of the proposed filters and as a consequence the convergence results in this paper also apply to existing unbiased minimum-variance filters.

1. INTRODUCTION

Unknown inputs in a dynamical system may represent unknown external drivers, input uncertainty, state uncertainty, or instrument faults. Thus both reconstruction of unknown-inputs and

* This work was supported by Department of Science and Technology through projects SR/S3/MERC/0064/2012 and SERC/ET-0150/2012, and in part by IIT Gandhinagar.

† Graduate Student, Mechanical Engineering, System Identification, Data based Estimation and Analysis Lab, IIT Gandhinagar, Ahmedabad - 382424, India, roshan_chavan@iitgn.ac.in.

‡ Assistant Professor, Mechanical Engineering, System Identification, Data based Estimation and Analysis Lab, IIT Gandhinagar, Ahmedabad - 382424, India, +91 79 3245 9899, harish@iitgn.ac.in.
estimation of states in the presence of unknown inputs, have numerous applications in all fields of engineering. These are fundamental problems that have been of interest for the last several decades with a range of papers relating to state estimation and unknown-input reconstruction\cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 9, 15, 16, 17}. While, both discrete-time and continuous-time versions of the problem received attention, in the discrete-time setting, the problem can be stated in its simplest form as the problem of estimating the state $x_k$ and/or the unknown inputs $e_k$ for linear systems of the form

$$x_{k+1} = Ax_k + He_k,$$  \hspace{1cm} (1.1)

$$y_k = Cx_k,$$  \hspace{1cm} (1.2)

using knowledge of the model equations and measurements of the outputs $y_k$ alone.

The early works in this area \cite{1, 2, 3} approached this as a system inversion problem and focussed on observability conditions under which estimation of unknown inputs are possible. Subsequently, a number of papers over the next several years focussed on construction of observers for state estimation in the presence of unknown inputs \cite{4, 5, 6, 7, 8, 9, 10, 11, 13, 14} with varying approaches.

More recently, interest has turned to reconstructing the unknown inputs in addition to estimation of the states \cite{12, 18, 19, 16, 20, 21, 22, 23}. Some work like \cite{9, 23} suggest that input reconstruction can be conceived as an added step after unbiased estimates of the states of the systems are obtained. However, both the state estimation literature and input reconstruction literature focussed on estimating the states or inputs at the immediate previous time step given output measurements until the current time step. Such an approach invariably led to an assumption that $CH$ has full column rank or a closely related assumption. This assumption ensured that all the unknown inputs at time step $k - 1$ directly affected the outputs at time step $k$ (as is obvious from a simple substitution in (1.1) and (1.2)), and therefore was a necessary condition for being able to estimate $x_k$ and/or $e_{k-1}$ from $y_k$. This becomes a fairly restrictive assumption as there are large classes of systems in which the effect of all the unknown inputs may not be seen in the output in the immediate next time step but may be seen in subsequent
time steps (when $CH$ does not have full column rank). Furthermore, convergence results for the filters developed are largely missing the literature.

Recent work on input and state observability [17] question the need for this assumption and in fact conclude that in the case that $CH$ is not full column rank but other conditions are satisfied, it may be possible to estimate inputs and states with a time shift (with a delay). That is, it may be possible to estimate $e_{k-r}$ or $x_{k-r}$ given measurements of $y_k$. However, [17] does not provide a robust, recursive way to estimate these states and inputs. [15], [24], [25], [16] take advantage of this idea to explore recursive filter-based methods to estimate past (delayed) states and inputs based on measurements of current outputs. [25] represent some initial preliminary efforts in this direction, while [24] develops a heuristic method for square systems (dimension of inputs are same as dimension of outputs). [16] incorporates a reconstruction delay with the purpose of negating the effect of non-minimum-phase zeros on the reconstruction error. Note that it has been established in [17] and other related works that non-minimum-phase invariant zeros in the system present a fundamental limitation in reconstruction of unknown inputs and states.

[15] develops a filter that uses a bank of measurements from current time ($k$) to a past time ($k-r$) to estimate the states at time instance $k-r$. While this paper is able to successfully relax the assumption that $CH$ must have full column rank and drawing connections with presence of invariant zeros in the system, it does not focus on input reconstruction and focuses on state estimation alone. Further, the results are not connected to observability results present in the literature.

In this paper, we develop a novel but relatively simple class of filters that incorporates a reconstruction delay in estimating the states and the unknown inputs of the system. This reconstruction delay allows us to relax the assumption of $CH$ having full column rank and thus are applicable to a larger class of systems. These filters use measurements up to time step $k$ to reconstruct states and inputs up to time step $k-r$, where $r$ is a non-negative integer (see Figure 1). This is referred to as reconstruction of the inputs with a delay of $r$ time steps and should not be confused with the system dynamics having a delay. We further show that the reconstruction delay $r$ is not the choice of the user but rather dependent on the system and can be characterised in terms of the rank of matrices containing markov parameters.
For these new filters, we then develop necessary and sufficient conditions for convergence and investigate their relationship with the multivariable invariant zeros of the system. We further establish conditions under which a stable filter exists and relate these conditions to system invertibility results.

Thus we make three important contributions. First, we develop a new class of filters that apply to a wide range of systems by incorporating an appropriate delay in input reconstruction, with no restrictions on the nature of input. Second, we develop sufficient conditions for convergence of the estimates that is largely absent in the literature. By further establishing that several existing filters are special cases of the current filter, we are also indirectly proving convergence results for several filters in the literature. Third, we establish a connection between invertibility results in the literature with conditions for existence and convergence of the proposed filters. Since invertibility and observability literature and filtering literature have been mostly disparate thus far, this is a contribution that helps connect two sets of results in the literature.

We first start by introducing and developing the new filter in the following section.

2. Unbiased Filter with General Delay

Consider state estimation and input reconstruction with a delay of \( r \) time steps, that is, measurements up to time step \( k \) are used to estimate states and inputs at time step \( k - r \). Consider a state-space system

\[
x_{k+1} = Ax_k + Bu_k + He_k + w_k,
\]

(2.1)

\[
y_k = Cx_k + Du_k + v_k,
\]

(2.2)
$x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^l$, $e_k \in \mathbb{R}^p$, are the state, known input, output measurement, unknown input vectors, respectively, $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^l$ are zero-mean, white process and measurement noise, respectively, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $H \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{l \times n}$, and $D \in \mathbb{R}^{l \times m}$. Note that $e_k$ is an arbitrary unknown input and can represent either deterministic or stochastic unknown signals. First we consider the simplifications $B = 0$ and $D = 0$. Note that the filter derivation is independent of $B$ and $D$ matrices, and thus the assumption on $B$ and $D$ matrices is for convenience alone. Without loss of generality, we assume $l \leq n$ also we assume $p < n$ and $\text{rank}(H) = p$. $l > n$ would imply the presence of redundant sensors.

For the state-space system (2.1), (2.2) (and with $B = 0$ and $D = 0$), we consider a filter of the form

$$\hat{x}_{k-r|k} = \hat{x}_{k-r|k-1} + L_k(y_k - C\hat{x}_{k|k-1}), \quad (2.3)$$

where

$$\hat{x}_{k-r|k-1} = A\hat{x}_{k-r-1|k-1}, \quad \hat{x}_{k|k-1} = A^{r+1}\hat{x}_{k-r-1|k-1}. \quad (2.4)$$

The unique feature of the above filter equations is that estimates are computed with a delay of $r$ time steps. That is, $\hat{x}_{k-r|k}$ is the state estimate at time step $k - r$ given output data (measurements) up to the current time step $k$. Note that $\hat{x}_{k|k-1}$ is a $r + 1$ step open loop prediction based on the previous state estimate $\hat{x}_{k-r-1|k-1}$.

Next, we define the state estimation error as

$$\varepsilon_{k-r} \triangleq x_{k-r} - \hat{x}_{k-r|k}, \quad (2.5)$$

and the error covariance matrix as

$$P_{k-r|k} \triangleq \mathbb{E}[\varepsilon_{k-r}\varepsilon_{k-r}^T]. \quad (2.6)$$

A. Necessary Conditions for Unbiasedness

**Definition 2.1.** The filter (2.3) - (2.4) is unbiased if $\hat{x}_{k-r|k}$ is an unbiased estimate of the state $x_{k-r}$.
Definition 2.1 implies that the filter (2.3)-(2.4) is unbiased if and only if

$$\mathbb{E}[\varepsilon_{k-r}] = \mathbb{E}[x_{k-r} - \hat{x}_{k-r|k}] = 0.$$  (2.7)

Next, we note that

$$\varepsilon_{k-r} = x_{k-r} - \hat{x}_{k-r|k}
= Ax_{k-r-1} + He_{k-r-1} + w_{k-r-1} - \hat{x}_{k-r|k-1} - L_k(Cx_k + v_k - \hat{x}_{k|k-1}).
$$

$$= (A - L_kCA^r)\varepsilon_{k-r-1} + (H - L_kCA^rH)e_{k-r-1} - L_kCA^{-1}He_{k-r} - L_kCA^{-2}He_{k-r+1} + ... +
L_kCAH e_{k-2} - L_kCH e_{k-1} + w_{k-r-1} - L_k(CA^r w_{k-r-1} + CA^{-1}w_{k-r} + ... + Cw_{k-1} + v_k).$$  (2.8)

**Theorem 2.1.** Let $L_k$ be such that the filter (2.3) - (2.4) is unbiased. Then

$$H - L_kCA^rH = L_kCA^{-1}H = L_kCA^{-2}H =
... = L_kCH = 0$$  (2.9)

**Proof:** Since by definition, filter (2.3) - (2.4) is unbiased if and only if $\mathbb{E}[\varepsilon_{k-r-1}] = 0$, it follows from (2.8) that

$$\mathbb{E}[\varepsilon_{k-r}] = \mathbb{E}[(A - L_kCA^r+1)\varepsilon_{k-r-1} + (H - L_kCA^rH)e_{k-r-1} - L_kCA^{-1}He_{k-r} - L_kCA^{-2}He_{k-r+1} + ... +
L_kCAH e_{k-2} - L_kCH e_{k-1} + w_{k-r-1} - L_k(CA^r w_{k-r-1} + CA^{-1}w_{k-r} + ... + Cw_{k-1} + v_k)].$$  (2.10)

Since (2.10) must hold for arbitrary input sequence $e_k$, it follows that (2.9) must hold for filter (2.3) - (2.4) to be unbiased.  \[\square\]
Corollary 2.1. Let $L_k$ be such that the filter $(2.3)$ - $(2.4)$ is unbiased. Then, the following conditions hold

i) $p \leq l$,

ii) $\text{rank}(CA^rH) = p$,

iii) $\text{rank}(L_k) \geq p$, for all $k$,

iv) $\text{rank}(CA^dH) \leq l - p$, for $d = 0, \ldots, r - 1$.

Proof. Since the filter $(2.3)$- $(2.4)$ is unbiased, it follows from Theorem 2.1 that $(2.9)$ holds and hence

$$L_k(CA^rH) = H. \quad (2.11)$$

Since $\text{rank}(H) = p$, it then follows from $(2.11)$ that iii) holds and

$$\text{rank}(CA^rH) \geq p \quad (2.12)$$

Since $CA^rH \in \mathbb{R}^{l \times p}$, it follows from $(2.12)$ that statement i) holds. Furthermore, it follows from $(2.12)$ and i) that statement ii) holds.

Finally to prove iv), since $(2.9)$ holds, it follows from [26, Proposition 2.5.9, p. 106] that $\text{rank}(L_kC^dH) = 0$ and therefore

$$\text{rank}(L_k) + \text{rank}(CA^dH) \leq \text{rank}(L_kCA^dH) + l$$

$$= l. \quad (2.13)$$

Furthermore, using iii), $(2.13)$ becomes

$$p + \text{rank}(CA^dH) \leq l, \quad (2.14)$$

that is, $\text{rank}(CA^dH) \leq l - p$. \hfill \Box$

Corollary 2.2. Let $l = p$ and let $L_k$ be such that the filter $(2.3)$ - $(2.4)$ is unbiased, and let $l = p$. Then, $CA^dH = 0$ for $d = 0, \ldots, r - 1$ and $\text{rank}(L_k) = p$ for all $k$ and $r$. 
Next, we define the Rosenbrock matrix $Z(s)$ as

$$Z(s) \doteq \begin{pmatrix} (sI - A) & H \\ C & 0 \end{pmatrix}.$$ 

For an unbiased filter, we know from Corollary 2.1 (i) that $p \leq l$, therefore the transfer function $G(s) \doteq C(sI - A)^{-1}H$ has normal rank equal to $p$.

**Definition 2.2.** $z \in \mathbb{C}$ is an invariant zero of (2.1), (2.2) if

$$\text{rank } Z(z) < \text{normal rank } Z(s).$$

**Definition 2.3.** The filter (2.3)-(2.4) is asymptotically unbiased if

$$\lim_{k \to \infty} E[\varepsilon_{k-r}] = 0$$

Definition 2.3 implies that the filter (2.3) - (2.4) is asymptotically unbiased if and only if $\tilde{x}_{k-r|k}$ converges to an unbiased estimate of the state $x_{k-r}$ as $k$ approaches infinity.

In the following subsection we first develop the filter and examine its convergence properties for a square system ($l = p$), and treat the non-square case later.

**B. Square Systems**

1) **Sufficient Conditions for Unbiasedness:**

**Lemma 2.1.** Let $l = p$ and let $L_k$ be such that (2.9) holds. Then

$$L_k = H(CA^rH)^{-1}.$$  \hspace{1cm} (2.15)

**Proof.** The proof follows from (2.9) and (ii) of Corollary 2.1

**Lemma 2.2.** Let $l = p$ and let $L_k$ be such that (2.9) hold. Then all non-zero eigenvalues of $(A - L_kCA^{r+1})$ are invariant zeros of (2.1) and (2.2).
**Proof.** Let $\lambda \neq 0$ be an eigenvalue of $(A - L_k CA^{r+1})$. Then using Lemma 2.1 it follows that

$$\det(A - (CA^r H)^{-1}CA^{r+1} - \lambda I) = 0.$$ 

Next, let $\nu \neq 0$ be such that

$$(A - (CA^r H)^{-1}CA^{r+1} - \lambda I)\nu = 0,$$

and thus

$$(A - \lambda I)\nu - H(CA^r H)^{-1}CA^{r+1}\nu = 0.$$ 

Next, defining

$$\mu \triangleq -(CA^r H)^{-1}CA^{r+1}\nu,$$ (2.16)

it follows that

$$(A - \lambda I)\nu + H\mu = 0,$$ (2.17)

and thus yielding

$$A\nu = \lambda\nu - H\mu$$ (2.18)

and

$$A^{r+1}\nu = \lambda A^r\nu - A^r H\mu.$$ (2.19)

Repeatedly using (2.18) in (2.19), we get

$$A^{r+1}\nu = \lambda^{r+1}\nu - \lambda^r H\mu - \lambda^{r-1} AH\mu - \ldots - \lambda A^{r-1} H\mu - A^r H\mu.$$ 

Left multiplying by $C$ on both sides gives

$$CA^{r+1}\nu = \lambda^{r+1}C\nu - \lambda^r CH\mu - \lambda^{r-1} CAH\mu - \ldots - \lambda CA^{r-1} H\mu - CA^r H\mu.$$
Since from Corollary 2.2 it follows that $CA^dH = 0$ for $d = 0, \ldots, r - 1$ and since $CA^rH$ is invertible, we have

$$(CA^rH)^{-1}CA^{r+1}v = \lambda^{r+1}(CA^rH)^{-1}Cv - \mu,$$

and thus

$$\mu = -(CA^rH)^{-1}CA^{r+1}v + \lambda^{r+1}(CA^rH)^{-1}Cv.$$  \hspace{1cm} (2.20)

Comparing (2.20) with (2.16), it follows that

$$\lambda^{r+1}(CA^rH)^{-1}Cv = 0.$$  \hspace{1cm} (2.21)

Since $\lambda \neq 0$, and $(CA^rH)^{-1}$ is full rank, it follows that

$$Cv = 0.$$  \hspace{1cm} (2.21)

Combining (2.17) and (2.21), we have

$$\begin{bmatrix} (\lambda I - A) & -H \\ C & 0 \end{bmatrix} \begin{bmatrix} \nu \\ \mu \end{bmatrix} = 0.$$  \hspace{1cm} (2.22)

Since $\nu \neq 0$, it follows that $\begin{bmatrix} \nu \\ \mu \end{bmatrix} \neq 0$ and noting that normal rank $Z(s) = n + p$ from Corollary 12.10.6 in [26], it follows that

$$\text{rank} \begin{bmatrix} (\lambda I - A) & -H \\ C & 0 \end{bmatrix} < \text{normal rank} \begin{bmatrix} (z I - A) & -H \\ C & 0 \end{bmatrix} = n + p.$$  \hspace{1cm} (2.23)

Therefore $\lambda$ is an invariant zero of $(A, H, C)$. \hfill \Box

In Lemma 2.2, we established the relationship between the invariant zeros and the eigenvalues of $(A - L_kCA^{r+1})$. Next, we use this relationship to examine the convergence of the filter.

**Theorem 2.2.** Let $l = p$ and let $L_k$ be such that (2.9) hold. Then the filter (2.3) - (2.4) is unbiased if and only if (2.1), (2.2) have no invariant zeros, while the filter (2.3) - (2.4) is
asymptotically unbiased if and only if all of the invariant zeros lie within the unit circle.

**Proof.** First, taking the expected values of both sides of (2.8) and using (2.9), and noting that
\[ E[w_{k-r-1}] = E[v_k] = E[w_{k-r}] = E[w_{k-1}] = 0, \]
it follows that
\[ E[\varepsilon_{k-r}] = E[(A - LCA^{r+1})\varepsilon_{k-r-1}] = (A - LCA^{r+1})E[\varepsilon_{k-r-1}]. \] (2.24)

Therefore, filter (2.3) - (2.4) is unbiased if and only if \((A - LCA^{r+1})\) is zero or equivalently all eigenvalues of \((A - LCA^{r+1})\) are zero. Furthermore, (2.3) - (2.4) is asymptotically unbiased if and only if \((A - LCA^{r+1})\) is asymptotically stable. It follows from Lemma 2.2 that all non-zero eigenvalues of \((A - LCA^{r+1})\) are invariant zeros of (2.1), (2.2). Subsequently, the filter (2.3) - (2.4) is asymptotically unbiased if and only if no invariant zeros of (2.1) and (2.2) are non-minimum-phase and there are no zeros on the unit circle (all of the invariant zeros are within the unit circle). The filter gives rise to a persistent reconstruction error if the zeros lie on the unit circle. \(\square\)

C. Input Reconstruction

We discussed necessary and sufficient conditions to obtain the unbiased estimates of states. Next, we consider using these estimates to reconstruct the unknown inputs.

**Proposition 2.1.** Let \(l = p\) and let \(L_k\) be such that (2.9) hold, and let \(\hat{x}_{k-r|k}\) be an unbiased estimate of \(x_{k-r}\). Then
\[ \hat{e}_{k-r-1} \overset{\Delta}{=} (CA^r H)^{-1}(y_k - C\hat{x}_{k|k-1}) \] (2.25)
is an unbiased estimate of \(e_{k-r-1}\).

**Proof.** Since \(\text{rank}(CA^r H) = p\), we can define
\[ \hat{e}_{k-r-1} \overset{\Delta}{=} (CA^r H)^{-1}(y_k - C\hat{x}_{k|k-1}). \] (2.26)
Next, using (2.26), (2.3), (2.5), (2.15) it follows that

\[ \hat{e}_{k-r-1} = (CA^rH)^{-1}CA^rH(CA^rH)^{-1}(y_k - C\hat{x}_{k|k-1}) \]
\[ = (CA^rH)^{-1}CA^rL_k(y_k - C\hat{x}_{k|k-1}) \]
\[ = (CA^rH)^{-1}CA^r(\hat{x}_{k-r|k} - \hat{x}_{k-r|k-1}) \]
\[ = (CA^rH)^{-1}CA^r(x_{k-r} + \varepsilon_{k-r} - \hat{x}_{k-r|k-1}) \]
\[ = (CA^rH)^{-1}CA^r(Ax_{k-r-1} + He_{k-r-1} + w_{k-r-1} + \varepsilon_{k-r} - A\hat{x}_{k-r-1|k-1}) \]
\[ = (CA^rH)^{-1}CA^r(A\varepsilon_{k-r-1} + He_{k-r-1} + w_{k-r-1} + \varepsilon_{k-r}). \]  
(2.27)

Finally, taking the expected values of both sides and noting that

\[ \mathbb{E}[\varepsilon_{k-r}] = \mathbb{E}[\varepsilon_{k-r-1}] = \mathbb{E}[w_{k-r-1}] = 0, \]  
(2.28)

it follows that

\[ \mathbb{E}[\hat{e}_{k-r-1}] = (CA^rH)^{-1}CA^rH\mathbb{E}[e_{k-r-1}] = \mathbb{E}[e_{k-r-1}]. \]  
□

D. Non-square Systems

In the previous subsection we dealt with square systems. In this section we explore the possibility of input reconstruction with a delay for non-square systems \( l \neq p \). In the context of non square systems, first we show that the invariant-zeros of the non-square system are a subset of the eigenvalues of \( (A - L_kCA^{r+1}) \) as follows.

**Lemma 2.3.** Let \( L_k \) be such that (2.9) is satisfied. Then the invariant zeros of system (2.1), (2.2) are a subset of the eigenvalues of \( (A - L_kCA^{r+1}) \).

**Proof.** Let \( z \) be an invariant zero of (2.1), (2.2), and let vector \( \begin{bmatrix} \nu \\ \mu \end{bmatrix} \neq 0 \) be such that

\[ \begin{bmatrix} (zI - A) & -H \\ C & 0 \end{bmatrix} \begin{bmatrix} \nu \\ \mu \end{bmatrix} = 0. \]  
(2.29)
Therefore

\[(zI - A)\nu - H\mu = 0, \quad (2.30)\]

\[C\nu = 0. \quad (2.31)\]

Note that if \(\nu = 0\), it can be seen from (2.30) that \(H\mu = 0\), but since \(\text{rank}(H) = p, \mu = 0\)
violating the assumption \(\begin{bmatrix} \nu \\ \mu \end{bmatrix} \neq 0\). Hence \(\nu \neq 0\).

Next, left multiplying (2.30) by \(L_kCA^r\) and rearranging,

\[zL_kCA^r\nu - L_kCA^{r+1}\nu = L_kCA^rH\mu. \quad (2.32)\]

Also it follows from (2.30) that,

\[A\nu = z\nu - H\mu. \quad (2.33)\]

Next using (2.33) in (2.32) and rearranging,

\[z^{r+1}L_kC\nu - z^rL_kCH\mu - z^{r-1}L_kCAH\mu - \ldots - zL_kCA^{r-1}H\mu - L_kCA^{r+1}\nu = L_kCA^rH\mu. \quad (2.34)\]

and using (2.31), (2.9),

\[-L_kCA^{r+1}\nu = H\mu. \quad (2.35)\]

Using (2.35) in (2.30),

\[(zI - A)\nu + L_kCA^{r+1}\nu = 0,\]

\[z\nu - A\nu + L_kCA^{r+1}\nu = 0,\]
and further rearranging, we have

$$(A - L_kCA^{r+1} - zI)\nu = 0.$$  \hfill (2.36)

Since $\nu \neq 0$, it follows that $z$ is an eigenvalue of $(A - L_kCA^{r+1})$.

For a non-square system, we note however that the converse of Lemma 2.3 does not hold as the following counter example demonstrates.

Consider a state space system characterized by the following $A, H, C$ matrices.

$$A = \begin{bmatrix}
-0.95 & -0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.025 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.97 & -0.06 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.05 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.95 & -0.05 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.98 & -0.04 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.05 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.95 & -0.08 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & -0.06 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 1 \\
\end{bmatrix}, \quad H = \begin{bmatrix}
0.4 & 0 \\
0 & 0 \\
0.2 & 0 \\
0 & 0 \\
0.2 & 0 \\
0.2 & 0 \\
0 & 0 \\
0 & 0 \\
0.2 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
0.25 & 2 & 0 & 0 & 0 & 0 & 0.5 & 2 & 0 & 0 \\
0 & 0 & 0.5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 1 & 0 & 0 & 0 & 0 & 0.5 \\
\end{bmatrix}. \hfill (2.37)

It is seen that this system has an invariant zero at $0.8$ of multiplicity two. The eigenvalues of $(A - L_kCA^2)$, a delay of one-time step, $(r = 1)$, are found to be $0.7528, 0.9782 \pm 0.0707i, 0.9782 - 0.0707i, 0.9750 + 0.0443i, 0.9750 - 0.0443i, 1.0076, 0.8, 0.8$. It can be seen that the invariant zeros of the non-square system are the eigenvalues of $(A - L_kCA^2)$ in accordance with lemma 2.3 but along with other spurious eigenvalue of which are not the invariant zeros of the system. This result is obtained using a value of $L_k$ that satisfies (2.9). The calculation of $L_k$ in the example (2.37) is based on the procedure which is discussed next.

We note that in the non-square case, an infinite number of solutions for $L_k$ that satisfy (2.9) are possible. The following results thus derive the $L_k$ that minimizes the trace of the error.
covariance and hence the minimum variance gain. $Q_k$ and $R_k$ are the process noise covariance and sensor noise covariance respectively.

**Fact 2.1.** Let $L_k$ be such that the filter (2.3) - (2.4) is unbiased. Then

$$P_{k-r|k} = (A - L_k CA^{r+1})P_{k-r-1|k-1}(A - L_k CA^{r+1})^T$$

$$+ (I - L_k CA^r)Q_{k-r-1}(I - L_k CA^r)^T + (L_k CA^{r-1})Q_{k-r}(L_k CA^{r-1})^T +$$

$$\ldots + (L_k C)Q_{k-1}(L_k C)^T + L_k R_k L_k^T$$

(2.38)

**Proof.** The proof follows by substituting (2.8) in (2.6) and using (2.9). □

Next, define the cost function $J$ as the trace of the error covariance matrix

$$J(L_k) = \text{tr}E[\varepsilon_{k-r}\varepsilon_{k-r}^T] = \text{tr}P_{k-r|k}.$$  

(2.39)

Therefore, it follows from (2.38) that

$$J(L_k) = \text{tr}[(A - L_k CA^{r+1})P_{k-r-1|k-1}(A - L_k CA^{r+1})^T$$

$$+ (I - L_k CA^r)Q_{k-r-1}(I - L_k CA^r)^T + (L_k CA^{r-1})Q_{k-r}(L_k CA^{r-1})^T +$$

$$\ldots + (L_k C)Q_{k-1}(L_k C)^T + L_k R_k L_k^T].$$

(2.40)

To derive the unbiased minimum-variance filter gain, we minimize the objective function (2.40) subject to the constraints (2.9) while noting that from Corollary 2.1, we have \(\text{rank}(CA^dH) \leq l - p\).

**Theorem 2.3.** Suppose there exists at least one $L_k$ that satisfies (2.9), then the unbiased minimum-variance gain $L_k$ is

$$L_k = \left[T_k A^r C^T + N_k Z_k^1 [CA^r H \ CA^{r-1} H \ \ldots \ CH]^T \right] S_k^{-1}.$$

(2.41)
\[ T_k \triangleq Q_{k-r-1} + AP_{k-r-1\mid k-1}A^T, \]
\[ S_k \triangleq CA^T_kA^T + CA^{-1}Q_{k-r}A^{r-1T}C^T + \ldots + CQ_{k-1}C^T + R_k, \]
\[
Z_k = \begin{bmatrix}
(CA^rH)^T S_{k-1}^{-1} (CA^rH) & (CA^rH)^T S_{k-1}^{-1} (CA^{r-1}H) & \ldots & (CA^rH)^T S_{k-1}^{-1} (CH) \\
(CA^{r-1}H)^T S_{k-1}^{-1} (CA^rH) & (CA^{r-1}H)^T S_{k-1}^{-1} (CA^{r-1}H) & \ldots & (CA^{r-1}H)^T S_{k-1}^{-1} (CH) \\
\vdots & \vdots & \ddots & \vdots \\
(CH)^T S_{k-1}^{-1} (CA^rH) & (CH)^T S_{k-1}^{-1} (CA^{r-1}H) & \ldots & (CH)^T S_{k-1}^{-1} (CH)
\end{bmatrix},
\]

and
\[ N_k = \begin{bmatrix}
H - T_kA^{rT}C^TS_{k-1}CA^rH & -T_kA^{rT}C^TS_{k-1}CA^{r-1}H & \ldots & -T_kA^{rT}C^TS_{k-1}CH
\end{bmatrix}. \]

**Proof.** The Lagrangian for the constrained minimization problem is
\[
\mathcal{L}(L_k) \triangleq J(L_k) + 2\text{tr}\left([I - L_kCA^rH \ L_kCA^{r-1}H \ \ldots \ L_kCH] \Lambda_k^T\right), \tag{2.42}
\]
where \(\Lambda_k^T \in \mathbb{R}^{n \times (r+1)p}\) is the matrix of Lagrange multipliers. Next, differentiating (2.42) with respect to \(L_k\) and setting it equal to zero yields
\[
-2CA^rQ_{k-r-1} - 2CA^{r+1}P_{k-r-1\mid k-1}A^T + 2[CA^{r+1}P_{k-r-1\mid k-1}A^{r+1T}C^T
\]
\[+CA^rQ_{k-r-1}A^{rT}C^T + CA^{r-1}Q_{k-r}A^{r-1T}C^T + \ldots + CQ_{k-1}C^T + R_k]L_k^T
\]
\[-2[CA^rH \ CA^{r-1}H \ \ldots \ CH] \Lambda_k = 0. \tag{2.43}
\]
Next assuming \(S_k\) is invertible and solving (2.43) for \(L_k\), we get
\[
L_k = \left[T_kA^{rT}C^T + \Lambda_k^T [CA^rH \ CA^{r-1}H \ \ldots \ CH]^T\right] S_k^{-1}. \tag{2.44}
\]
Next, to solve for $\Lambda_k$, we substitute (2.44) in (2.9) to get

$$\begin{bmatrix} T_k A^T C^T + \Lambda_k^T [C A^r H \ C A^{r-1} H \ldots CH]^T \end{bmatrix} S_k^{-1} [C A^r H \ C A^{r-1} H \ldots CH] = \begin{bmatrix} H \ 0_{n \times p} \ldots \ 0_{n \times p} \end{bmatrix}. $$

Next we define $Z_k$ and $N_k$ as follows

$$Z_k = \begin{bmatrix} (C A^r H)^T S_k^{-1} (C A^r H) & (C A^r H)^T S_k^{-1} (C A^{r-1} H) & \ldots & (C A^r H)^T S_k^{-1} (C H) \\ (C A^{r-1} H)^T S_k^{-1} (C A^r H) & (C A^{r-1} H)^T S_k^{-1} (C A^{r-1} H) & \ldots & (C A^{r-1} H)^T S_k^{-1} (C H) \\ \vdots & \vdots & \ddots & \vdots \\ (C H)^T S_k^{-1} (C A^r H) & (C H)^T S_k^{-1} (C A^{r-1} H) & \ldots & (C H)^T S_k^{-1} (C H) \end{bmatrix}, $$

$$N_k = \begin{bmatrix} H - T_k A^T C^T S_k^{-1} C A^r H & -T_k A^T C^T S_k^{-1} C A^{r-1} H & \ldots & -T_k A^T C^T S_k^{-1} C H \end{bmatrix}. $$

Therefore (2.45) becomes

$$\Lambda_k^T Z_k = N_k. $$

Solving for $\Lambda_k^T$ we get

$$\Lambda_k^T = N_k Z_k^+. $$

Where $Z_k^+$ is a Moore-Penrose generalized inverse of $Z_k$. Substituting (2.49) in (2.44) we get

$$L_k = \begin{bmatrix} T_k A^T C^T + N_k Z_k^+[C A^r H \ C A^{r-1} H \ldots CH]^T \end{bmatrix} S_k^{-1}. $$

Note that the assumption of $S_k$ being invertible is ensured by demanding apriori that $R_k$ is positive definite for all $k$. This indicates the persistence of sensor noise and is a valid assumption on physical grounds. $Q_k$ is assumed to be nonnegative definite for all $k$. It should also be noted that since $Z_k$ is not full rank there are an infinitely many possible solutions for $\Lambda_k$. However at
any instant \( k \), any \( \Lambda_k \) that satisfies (2.49) will give the same minimum-variance gain \( (L_k) \).

**Corollary 2.3.** Suppose \( CA^dH = 0 \) for \( d = 0, 1, \ldots, r \) then the unbiased minimum-variance gain \( L_k \) is

\[
L_k = \left[ T_k A^{r_T} C^T + \Phi_k (CA^r H)^T \right] S_k^{-1},
\]

where

\[
T_k \triangleq Q_{k-r-1} + AP_{k-r-1[k-1]} A^T,
\]

\[
S_k \triangleq CA^T k A^{r_T} C^T + CA^{-1} Q_{k-r} A^{-1} C^T + \ldots + CQ_{k-1} C^T + R_k,
\]

\[
\Phi_k \triangleq \left[ H - T_k A^{r_T} C^T S_k^{-1} CA^r \right] \left( (CA^r H)^T S_k^{-1} CA^r H \right)^{-1}
\]

In the next section we present some numerical results using the previously developed filter.

### 3. Existence and Maximum Delay

According to (2.9), \( L_k \) should satisfy the condition

\[
L_k \begin{bmatrix} CA^r H & CA^{r-1} H & \ldots & CH \end{bmatrix} = \begin{bmatrix} H & 0 & \ldots & 0 \end{bmatrix}.
\]

**Lemma 3.1.** \( L_k \begin{bmatrix} CA^r H & CA^{r-1} H & \ldots & CH \end{bmatrix} \neq \begin{bmatrix} H & 0 & \ldots & 0 \end{bmatrix} \ \forall \ r \geq n
\]

**Proof.** Suppose on the contrary, assume \( L_k \begin{bmatrix} CA^r H & CA^{r-1} H & \ldots & CH \end{bmatrix} = \begin{bmatrix} H & 0 & \ldots & 0 \end{bmatrix} \ \forall \ r \geq n \). Let \( r = n \). The condition for unbiasedness becomes

\[
L_k \begin{bmatrix} CA^n H & CA^{n-1} H & \ldots & CH \end{bmatrix} = \begin{bmatrix} H & 0 & \ldots & 0 \end{bmatrix}
\]

Using Cayley-Hamilton theorem and after rearranging (3.2) becomes

\[
\begin{bmatrix} (d_1 L_k CA^{n-1} H + d_2 L_k CA^{n-2} H + \ldots + d_n L_k CH) & L_k CA^{n-1} H & \ldots & L_k CH \end{bmatrix} = \begin{bmatrix} H & 0 & \ldots & 0 \end{bmatrix},
\]

(3.3)
which results in a contradiction, which is also true for all \( r > n \). Hence our assumption is wrong and therefore

\[
L_k \begin{bmatrix} CA^r H & CA^{r-1} H & \ldots & CH \end{bmatrix} \neq \begin{bmatrix} H & 0 & \ldots & 0 \end{bmatrix} \quad \forall \ r > n \quad \square
\]

This shows that the choice of the delay, \( r \), cannot be arbitrary and has to be smaller than the system order. The lower bound on the delay is determined by the first Markov parameter which has full rank.

**Conjecture 3.1.** The filter will be unbiased for only one value of the delay \( r \), i.e. there exists only one value of \( r \) that satisfies condition (3.1).

Next, we introduce the following notations

\[
S_r \triangleq \begin{bmatrix} CA^r H & CA^{r-1} H & \ldots & CH \end{bmatrix},
\]

\[
S_{r-1} \triangleq \begin{bmatrix} CA^{r-1} H & CA^{r-2} H & \ldots & CH \end{bmatrix}.
\]

**Theorem 3.1.** There exists a matrix \( L_k \) that satisfies (3.1) if and only if

\[
\text{rank} \begin{bmatrix} S_r \\ H & 0 & \ldots & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} S_r \end{bmatrix} = p
\]

**Proof.** There exists a matrix \( L_k \) that satisfies (3.1) if and only if the matrix \( \begin{bmatrix} H & 0 & \ldots & 0 \end{bmatrix} \) is in the space spanned by the rows of \( S_r \). This is equivalent to the condition

\[
\text{rank} \begin{bmatrix} S_r \\ H & 0 & \ldots & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} S_r \end{bmatrix}
\]

\[
\text{rank} \begin{bmatrix} CA^r H & CA^{r-1} H & \ldots & CH \\ H & 0 & \ldots & 0 \end{bmatrix}
\]

September 22, 2015 DRAFT
Using the matrix identity

\[
\text{rank} \begin{bmatrix} A & B \end{bmatrix} = \text{rank} \begin{bmatrix} A \end{bmatrix} + \text{rank} \begin{bmatrix} B \end{bmatrix} - \dim \left( \mathcal{R} \begin{bmatrix} A \end{bmatrix} \cap \mathcal{R} \begin{bmatrix} B \end{bmatrix} \right),
\]

(3.6)

and noting that \( \text{rank}[H] = p \) we get

\[
\text{rank} \begin{bmatrix} S_r & \cdot & \cdot & \cdot \\ H & 0 & \ldots & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} S_{r-1} \end{bmatrix} + p.
\]

(3.7)

From (3.5) and (3.7) we get the condition

\[
\text{rank} \begin{bmatrix} S_r \end{bmatrix} = \text{rank} \begin{bmatrix} S_{r-1} \end{bmatrix} + p.
\]

Corollary 3.1. If \( \text{rank} \begin{bmatrix} S_r \end{bmatrix} - \text{rank} \begin{bmatrix} S_{r-1} \end{bmatrix} = p \) then \( \text{rank} \begin{bmatrix} M_r \end{bmatrix} - \text{rank} \begin{bmatrix} M_{r-1} \end{bmatrix} = p \), where

\[
M_r = \begin{bmatrix}
CH & 0 & 0 & \ldots & 0 \\
CAH & CH & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
CA^rH & CA^{r-1}H & CA^{r-2}H & \ldots & 0
\end{bmatrix}.
\]

Proof. Using (3.6) we have

\[
\text{rank} \begin{bmatrix} M_r \end{bmatrix} - \text{rank} \begin{bmatrix} M_{r-1} \end{bmatrix} = \text{rank} \begin{bmatrix}
CH \\
CAH \\
\vdots \\
CA^rH
\end{bmatrix} + \dim \left( \mathcal{R} \begin{bmatrix}
CH \\
CAH \\
\vdots \\
CA^rH
\end{bmatrix} \cap \mathcal{R} \begin{bmatrix} M_{r-1} \end{bmatrix} \right),
\]

(3.8)
where $M_{r-1}$ can also be represented as follows

$$
[M_{r-1}] = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
CH & 0 & \ldots & 0 \\
CAH & CH & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
S_{r-1}
\end{bmatrix}.
$$

(3.9)

Since, $\text{rank}[S_r] - \text{rank}[S_{r-1}] = p$ according the Theorem 3.1, there exists a matrix $L_k$ such that (3.1) and hence (2.9) is satisfied. Then from Corollary 2.1, (ii) we have $\text{rank}(CA^rH) = p$. Therefore

$$
\text{rank} \begin{bmatrix}
CH \\
CAH \\
\vdots \\
CA^rH
\end{bmatrix} = p.
$$

(3.10)

Next, Using (3.6) we have

$$
\text{rank}[S_r] - \text{rank}[S_{r-1}] = \text{rank}[CA^rH] + \text{dim} \left( \mathcal{R} \left[ CA^rH \right] \cap \mathcal{R} \left[ S_{r-1} \right] \right).
$$

(3.11)

Now since, $\text{rank}[S_r] - \text{rank}[S_{r-1}] = p$ and $\text{rank}(CA^rH) = p$, (3.11) leads to

$$
\text{dim} \left( \mathcal{R} \left[ CA^rH \right] \cap \mathcal{R} \left[ S_{r-1} \right] \right) = 0.
$$

(3.12)

From (3.8), (3.9) and (3.12) we have

$$
\text{dim} \left( \mathcal{R} \left[ CA^rH \right] \cap \mathcal{R} \left[ M_{r-1} \right] \right) = 0.
$$

(3.13)
Next, substituting (3.10) and (3.13) in (3.8) leads to

$$\text{rank} \left[ M_r \right] - \text{rank} \left[ M_{r-1} \right] = p.$$ 

This in turn implies that the system (2.1) and (2.2) is $r$-delay invertible [27]. However it should be noted that the $r$-delay invertibility of a system is not a sufficient condition for the existence of $L_k$ which satisfies the condition (3.1) as the following example shows. Consider the system

$$A = \begin{bmatrix}
0.5 & -0.6 & 0 & 0 \\
0.5 & 0 & 0 & 0 \\
0 & 0 & -0.5 & -0.6 \\
0 & 0 & 0.5 & 0
\end{bmatrix}, \quad H = \begin{bmatrix} 4 & 0 \end{bmatrix}, \quad C = \begin{bmatrix}
0.25 & 1.05 & 0.25 & 1.1 \\
0.25 & 1.15 & 0.25 & 1 \\
0.25 & 1.05 & 0.25 & 1.1
\end{bmatrix}.$$ 

For this system $\text{rank}(M_r) - \text{rank}(M_{r-1}) = 2 = p$, however $\text{rank}(S_r) - \text{rank}(S_{r-1}) < p$.

4. Numerical Results Using an Unbiased Filter with Delay of One Time Step

A. Numerical Results - Square Systems

![Fig. 2](image-url)

Fig. 2. This compartmental model comprises of six compartments that exchange mass or energy through mutual interaction. The inputs are provided to the first and the last compartment while the states from the second and the fifth compartment are measured and constitute the outputs.

To illustrate recursive input reconstruction, we consider a compartmental system comprised of $n$ compartments that exchange mass or energy through mutual interaction. This can repre-
sent physical models like collection of rooms which mutually exchange mass and energy. The conservation equations governing the compartmental model are

\begin{align*}
x_{1,k+1} &= x_{1,k} - \beta x_{1,k} + \alpha (x_{2,k} - x_{1,k}), \quad (4.1) \\
x_{i,k+1} &= x_{i,k} - \beta x_{i,k} + \alpha (x_{i+1,k} - x_{i,k}) - \alpha (x_{i,k} - x_{i-1,k}), \\
& \quad i = 2, \ldots, n-1, \quad (4.2) \\
x_{n,k+1} &= x_{n,k} - \beta x_{n,k} - \alpha (x_{n,k} - x_{n-1,k}), \quad (4.3)
\end{align*}

where $0 < \beta < 1$ is the loss coefficient and $0 < \alpha < 1$ is the flow coefficient. Figure 2 illustrates a schematic of the compartmental model wherein each block represents a compartment. The arrows labeled $e_1, e_2$ indicate the input to the system while the arrows labeled $y_1, y_2$ indicate the output. The system equations (4.1) - (4.3) can be written in the state space form (2.1), (2.2) with

\[
A = \begin{bmatrix}
1 - \beta - \alpha & \alpha & 0 & \cdots & 0 \\
\alpha & 1 - \beta - \alpha & \alpha & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \alpha & 1 - \beta - \alpha
\end{bmatrix}. \quad (4.4)
\]

Further, for the numerical simulations, we choose $n = 6$, $\alpha = 0.1$ and $\beta = 0.1$, we assume we have no known inputs and therefore the $Bu_k$ and $Du_k$ terms disappear, and we assume two unknown inputs enter compartments 1 and 6, while the states in compartments 2 and 5 are measured as outputs. It then follows that

\[
H = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}^T, \quad C = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}. \quad (4.5)
\]

Note that since $CH = 0$, the filters in [9], [21], [11], [22] cannot be applied for input reconstruction (or state estimation). Further, since $CAH$ is full rank and there are no invariant zeros of the system, it follows from Theorem 2.2 that the filter (2.3) - (2.4), (2.15), (2.25) with $r = 1$ will provide an unbiased estimate of the unknown inputs. We choose the first input
to be a sawtooth and the second input to be a sinusoid. Figure 3 shows the actual unknown inputs and the estimated unknown inputs using the recursive filter developed previously. The compartmental model is a convenient example to illustrate delayed input reconstruction since by changing the compartments from which the states are measured, the product $CA^rH$ can be altered. For instance, if the inputs are given to the compartments 1 and 6 and the states in the compartments 3 and 4 are measured as outputs then it follows that

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \quad (4.6)$$

Note that in this case $CH = 0, CAH = 0$ and $CA^2H$ is full rank. This implies that input reconstruction is only possible using the filter $\eqref{eq:2.3} - \eqref{eq:2.4}, \eqref{eq:2.15}, \eqref{eq:2.25}$ with a minimum delay.
of two time-steps i.e, \( r = 2 \). In this case, results similar to those in Figure 3 are obtained, but not shown here due to space constraints. Note that the filters in [9], [21], [22] are no-delay filters and require \( CH \) to be full rank and hence cannot handle the case where \( CH = 0 \).

Next we present a numerical result illustrating delayed input reconstruction in presence of minimum phase zeros. Consider a state space system characterized by the following \( A, H, C \) matrices.

\[
A = \begin{bmatrix}
1.1 & -0.6 & 1 \\
0.5 & 0 & 1 \\
0 & 0.2 & 0.3
\end{bmatrix}, \quad H = \begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & 0.4 & 1
\end{bmatrix}.
\]

This system has a zero at \(-0.2\) (minimum phase) and the eigenvalues of \((A - L_kCA^2)\) are 0, 0, \(-0.2\) in accordance to Lemma 2.2. Figure 5 shows the actual unknown input and the estimated unknown input using the recursive delayed input reconstruction filter. Figure 6 shows

![Figure 5](image-url)

**Fig. 5.** Actual and estimated inputs for a system with minimum phase zero

the error in state estimation. The computed state estimation error represents the error computed using (2.24) and substituting \( r = 1 \) and assuming that the initial condition of the states is known. The computation was done using the eigenvalues of \((A - L_kCA^2)\) and the results were plotted against the actual state estimation error. The exact overlap between the actual and the computed state estimation error consolidates the finding that the invariant zeros of the system govern the dynamics of the state estimation error while the decaying nature of the error is in accordance
with the Theorem 2.2 with \( r = 1 \).

![Graphs showing error in state estimation for a system with minimum phase zero](image)

**Fig. 6.** Error in state estimation for a system with minimum phase zero

Now consider a state space system characterized by the following \( A, H, C \) matrices.

\[
A = \begin{bmatrix}
0.0725 & 1 & 0.2072 \\
-0.6158 & 0.0725 & 0.2339 \\
0 & 0 & -0.1449
\end{bmatrix}, \quad H = \begin{bmatrix}
0 \\
0 \\
4
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
5.005 \\
0 \\
0
\end{bmatrix}.
\]

This system has a zero at \(-1.0564\) (nonminimum phase) and the eigenvalues of \( A - L_k C A^2 \) are \(-1.0564, 0, 0\) in accordance with Theorem 2.2. Figure 7 shows the actual unknown input and the estimated unknown input using the recursive delayed input reconstruction filter. Figure 8 shows the actual and computed error in state estimation.

![Graphs showing actual and estimated inputs for a system with nonminimum phase zero](image)

**Fig. 7.** Actual and estimated inputs for a system with nonminimum phase zero

and computed error in state estimation.
B. Numerical Results - Non-square Systems

State estimation and input reconstruction with a delay of one time step is performed for the non-square state space system characterized by the following $A, H, C$ matrices.

$$
A = \begin{bmatrix}
0.0725 & 1 & 0.2072 \\
-0.6158 & 0.0725 & 0.2339 \\
0 & 0 & -0.1449
\end{bmatrix}, \quad H = \begin{bmatrix}
0 \\
0 \\
4
\end{bmatrix},
$$

$$
C = \begin{bmatrix}
5.005 & 0 & 0 \\
0 & 0.1 & 0
\end{bmatrix}.
$$

The minimum variance gain for the estimator is computed using Corollary 2.3. Figure 9 shows the actual unknown input and the estimated unknown input using the recursive delayed input reconstruction filter. It should be noted that the convergence of the filter in this case is asymptotic.

Fig. 8. Error in state estimation for a system with nonminimum phase zero

Fig. 9. Input reconstruction for a non-square system
In the next section we highlight the relationship between the filter developed in this paper and the filters developed in [9], [21], [22].

5. Relationship with Unbiased Minimum Variance Filters With No Delay

In this subsection, within the context of square systems ($l = p$), we show that the filters developed in [9], [21], [22] are special cases of the filter (2.3) - (2.4), (2.15), (2.25) with $r = 0$.

Proposition 5.1. Let $l = p$, then the unbiased minimum variance filters in [21, (3) - (6) along with (13) and (20)] and [9] are equivalent to the filter (2.3) - (2.4), (2.15), (2.25) with $r = 0$.

Proof. The proof follows by first noting that in the case $l = p$, $CH (F_k$ in [21]) is invertible, and then following straight-forward substitution and simplification.

Note that [21], [9] do not prove convergence of the respective filters, therefore the above result proves convergence of the filter through Theorem 2.2 with $r = 0$ by connecting the convergence with zeros. Furthermore, note that in the case $l = p$ with $r = 0$, from Theorem 2.1 the filter gain must satisfy the condition

$$LCH = H,$$  \hspace{1cm} (5.1)

and since rank $H = p$ and consequently rank $CH = p$, $L = H(CH)^{-1}$ is the only possible $L$ that satisfies (5.1) and hence there is a unique solution and no concept of a minimum variance solution exists.

6. Remarks

While the results in this paper indicate that if the system has nonminimum phase zeros or zeros on the unit-circle, the filter is not convergent, it is worthwhile to note however that there may be other approaches that may work in such cases as is being explored in [24], [28]. We also note here that for square systems having no/minimum phase zeros is a sufficient condition for unbiasedness/asymptotic unbiasedness. In non-square this is not the case as the example in section 2-D illustrates. Note that no-delay filters in [9], [21], [22] would only be applicable if $CH$ is full rank. So in the context of the numerical example of a compartmental model in
section [4-A] they would be applicable only if the output measurements were the states of the compartments that the unknown inputs were entering (in this case compartments 1 and 6). Finally, in the presence of additional known inputs $u_k$, note that the same filter equations and theory apply with the modified equations

$$\hat{x}_{k-r|k} = \hat{x}_{k-r|k-1} + L_k(y_k - C\hat{x}_{k|k-1} - Du_k),$$  \hfill (6.1)

$$\hat{x}_{k-r|k-1} = A\hat{x}_{k-r-1|k-1} + Bu_{k-r-1},$$  \hfill (6.2)

$$\hat{x}_{k|k-1} = A^{r+1}\hat{x}_{k-r-1|k-1} + A^{r}Bu_{k-r-1} + A^{r-1}Bu_{k-r} + \cdots + Bu_{k-1}$$  \hfill (6.3)

$$\hat{e}_{k-r-1} \triangleq (CH)^{-1}(y_k - C\hat{x}_{k|k-1} - Du_k).$$  \hfill (6.4)

instead of (2.3) - (2.4). It is to be noted that one drawback of the input reconstruction method is that it cannot differentiate between unknown noise and unknown input.

7. Conclusions

In this paper, we developed a technique that recursively use current measurements to estimate past states and reconstruct past inputs. Furthermore, we derived convergence results for the filters developed and established its relationship with invariant zeros of the system. Thus we developed a broader class of filters (than traditional unbiased minimum-variance filters), provided necessary and sufficient conditions for the filter to provide unbiased estimates. we also established that the unbiased-minimum variance filters in \[9, 21, 22, 23] are special cases of the filter developed in this note and provided numerical examples illustrating the key difference between the proposed filter and existing methods. The key results are listed below

1) Necessary conditions for unbiasedness of the filter.

2) Sufficient conditions for convergence of the filter for square systems.

3) Showing that the sufficient conditions for convergence of the filter for square systems do not hold for non-square systems.

4) Establishing an upper bound for the filter delay and deriving existence conditions for the filter.

Future work will focus on an in depth analysis of convergence in non-square systems and its relationship with the invariant parameters of the system.
REFERENCES

[1] M. K. Sain and J. L. Massey, “Invertibility of linear time-invariant dynamical systems,” IEEE Trans. on Automatic Contr., vol. AC-14, pp. 141–149, April 1969.

[2] L. M. Silverman, “Inversion of multivariable linear systems,” IEEE Trans. on Automatic Contr., vol. 14, pp. 270–276, June 1969.

[3] P. Dorato, “On the inverse of linear dynamical systems,” IEEE Trans. on System Science And Cybernetics., vol. SSC-5, pp. 43–48, January 1969.

[4] P. J. Moylan, “Stable inversion of linear systems,” IEEE Trans. on Automatic Contr., pp. 74–78, February 1977.

[5] S. P. Bhattacharyya, “Observer design for linear systems with unknown inputs,” IEEE Trans. on Automatic Contr., vol. AC-23, pp. 483–484, 1978.

[6] P. Kudva, N. Viswanadham, and A. Ramakrishna, “Observers for linear systems with unknown inputs,” IEEE Trans. on Automatic Contr., vol. 25, pp. 113–115, February 1980.

[7] R. J. Miller and R. Mukundan, “On designing reduced-order observers for linear time-invariant systems subject to unknown inputs,” Int. J. Contr., vol. 35, pp. 183–188, January 1982.

[8] F. W. Fairman, S. S. Mahil, and L. Luk, “Disturbance decoupled observer design via singular value decomposition,” IEEE Trans. on Automatic Contr., vol. AC-29, pp. 84–86, January 1984.

[9] P. K. Kitanidis, “Unbiased Minimum-variance Linear State Estimation,” Automatica, vol. 23, no. 6, pp. 775–578, 1987.

[10] F. Yang and R. W. Wilde, “Observers for linear systems with unknown inputs,” IEEE Trans. on Automatic Contr., vol. 33, pp. 677–681, July 1988.

[11] M. Darouach, M. Zasadzinski, and S. J. Xu, “Full-order observers for linear systems with unknown inputs,” IEEE Trans. on Automatic Contr., vol. 39, pp. 606–609, March 1994.

[12] M. Corless and J. Tu, “State and input estimation for a class of uncertain systems,” Automatica, vol. 34, no. 6, pp. 757–764, 1998.

[13] M. Hou and R. J. Patton, “Optimal filtering for systems with unknown inputs,” IEEE Trans. on Automatic Contr., vol. 43, pp. 445–449, March 1998.

[14] H. E. Emara-Shabaik, “Filtering of linear systems with unknown inputs,” Trans. of the ASME, J. of Dyn. Sys., Meas., and Contr., vol. 125, pp. 482–485, September 2003.

[15] S. Sundaram and C. Hadjicostis, “Delayed observers for linear systems with unknown inputs,” Automatic Control, IEEE Transactions on, vol. 52, pp. 334–339, Feb 2007.

[16] E. Z. G. Marro, “Unknown-state, unknown-input reconstruction in discrete-time nonminimum phase systems: Geometric methods,” Automatica, February 2010.

[17] S. Kirtikar, H. Palanthandalam-Madapusi, E. Zattoni, and D. Bernstein, “I-delay input and initial-state reconstruction for discrete-time linear systems,” Circuits, Systems, and Signal Processing, vol. 30, no. 1, pp. 233–262, 2011.

[18] Y. Xiong and M. Saif, “Unknown disturbance inputs estimation based on a state functional observer design,” Automatica, vol. 39, pp. 1389–1398, 2003.

[19] T. Floquet and J. P. Barbot, “An Observability Form for Linear Systems With Unknown Inputs,” Int. J. Contr., vol. 79, no. 2, pp. 132–139, 2006.
[20] G. Marro, D. Bernstein, and E. Zattoni, “Geometric methods for unknown-state, unknown input reconstruction in discrete time nonminimum-phase systems with feedthrough terms,” in Proc. of IEEE, Conf. on Decision and Control, (Atlanta, GA), December 2010.

[21] S. Gillijns and B. D. Moor, “Unbiased minimum-variance input and state estimation for linear discrete-time stochastic systems,” Internal Report ESAT-SISTA/TR 05-228, Katholieke Universiteit Leuven, Leuven, Belgium, November 2005.

[22] H. Palanthandalam-Madapusi, S. Gillijns, B. D. Moore, and D. S. Bernstein, “System identification for nonlinear model updating,” in Proc. of Amer. Contr. Conf., (Minneapolis, MN), June 2006.

[23] H. J. Palanthandalam-Madapusi and D. S. Bernstein, “Unbiased minimum-variance filtering for input reconstruction,” in Proc. of Amer. Contr. Conf., (New York, NY), pp. 5712 – 5717, July 2007.

[24] A. M. D’Amato, “Minimum-norm input reconstruction for nonminimum-phase systems,” in Proc. of Amer. Contr. Conf., (Washington, DC), pp. 3111 – 3116, June 2013.

[25] K. Fitch and H. Palanthandalam-Madapusi, “Unbiased minimum-variance filtering for delayed input reconstruction,” in American Control Conference (ACC), 2011, pp. 4859–4860, June 2011.

[26] D. S. Bernstein, Matrix Mathematics: Theory, Facts, and Formulas. Princeton University Press, second ed., 2009.

[27] A. S. Willsky, “On the invertibility of linear systems,” IEEE Trans. on Automatic Contr., vol. 19, pp. 272–274, June 1974.

[28] G. Singh, R. Chavan, and H. Palanthandalam-Madapusi, “Backward in time input reconstruction,” Proc. of European Contr. Conf. (submitted), 2014.