Heterotic/type I duality, D-instantons and a $N = 2$ ADS/CFT correspondence

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Abstract

D-instanton effects are studied for the IIB orientifold $T^2/(-1)^F\Omega I$ of Sen using type I/heterotic duality. An exact one loop threshold calculation of $t_8\text{tr}F^4$ and $t_8(\text{tr}F^2)^2$ terms for the heterotic string on $T^2$ with Wilson lines breaking $SO(32)$ to $SO(8)^4$ is related to D-instanton induced terms in the worldvolume of D7 branes in the orientifold. Introducing D3 branes and using the AdS/CFT correspondence in this case, these terms are used to calculate Yang-Mills instanton contributions to four point functions of the large $N_c$ limit of $N = 2 USp(2N_c)$ SYM with four fundamental and one antisymmetric tensor hypermultiplets.

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1. Introduction

One of the simplest examples of F-theory on $K3$ was introduced by Sen [1]. This compactification is realized as a perturbative IIB orientifold $T^2/(-1)^F \Omega I$, where $\Omega$ is the worldsheet orientation reversal, and $I$ is the inversion $z \rightarrow -z$ on the torus. Orientifold seven planes are located at the four fixed point of the inversion $I$ on $T^2$. In addition four D7 branes are on top of each of the orientifold seven planes. The sources for the dilaton and the Ramond-Ramond scalar coming from the D7 branes and the orientifold seven planes cancel locally and hence the dilaton is constant over the base $T^2/I$. The D7 branes lead to an enhanced gauge symmetry $SO(8)^4$. Each $SO(8)$ factor can be associated with four D7 branes on top of one of the four orientifold planes.

When a probe of $N_c$ parallel D3 branes [2] is moving in the vicinity of one of the orientifold planes the low energy field theory on the probe is given by a $N = 2$, $USp(2N_c)$ gauge theory with four hypermultiplets transforming in the fundamental representation and one hypermultiplet transforming in the second rank antisymmetric tensor representation of $USp(2N_c)$ respectively.

Such a theory has an exactly vanishing beta function and defines a conformal field theory for any $N_c$ when the expectation values for the scalars in the hypermultiplets are zero. In [3][4] the large $N_c$ limit of the probe field theory was considered using the AdS/CFT correspondence of Maldacena [5]. The conformal field theory is related to IIB superstring theory compactified on an orientifold $AdS_5 \times S_5/Z_2$.

Sen’s IIB orientifold is related by T-duality to type I theory compactified on $T^2$ with Wilson lines turned on. This theory is in turn mapped by S-duality to the heterotic string theory on $T^2$ with Wilson lines breaking $SO(32)$ to $SO(8)^4$. In the following this chain of dualities is used to relate a one loop calculation of four derivative threshold correction on the heterotic side to D-instanton induced four derivative terms localized in the worldvolume of the seven branes.

In the Maldacena limit the seven branes wrap a $S_3$ in $S_5/Z_2$ and fill $AdS_5$. Therefore terms in the worldvolume action of the seven brane induce vertices in the $AdS_5$ which contribute to correlation function in the CFT using the prescription introduced in [3].
2. Two torus compactification with Wilson lines

We are interested in the $SO(32)$ heterotic string compactified on a two torus. The Kähler and complex structure modulus of $T^2$ are denoted $T$ and $U$ respectively. The $SO(32)$ gauge symmetry will be broken to $SO(8)^4$ by introducing Wilson lines on the two torus of the following form

$$Y_i^1 = (0^4, 0^4, \frac{1}{2}, \frac{1}{2}), \quad Y_i^2 = (0^4, \frac{1}{2}, 0^4, \frac{1}{2}).$$

This choice of Wilson lines on the heterotic side is dual to the IIB orientifold $T^2/(-1)^F \Omega I$. Each $SO(8)$ factor can be associated with four D7 branes on top of one of the four orientifold planes located at the fixed points of $T^2/I$.

The one loop heterotic thresholds discussed [7][8] are related by supersymmetry to anomaly canceling terms [9] and are presumably exact at one loop. The one loop integrands involved are almost holomorphic since only BPS-states propagate in the loop and are related to the 'elliptic genus' [9].

For simplicity a square torus with radii $R_1, R_2$ will be considered here. The Kähler and complex structure moduli are then given by

$$T = B_{12}^{NS} + i R_1 R_2, \quad U = i \frac{R_2}{R_1}.$$  \hspace{1cm} (2)

Under heterotic-type I duality the coupling constants, metric and AST field are related by

$$\lambda_{het} = 1/\lambda_I, \quad \lambda_I g^{het}_{\mu\nu} = g^I_{\mu\nu}, \quad B^{het}_{\mu\nu} = B^I_{\mu\nu}.$$  \hspace{1cm} (3)

Two T-dualities invert the radii $R_i \rightarrow 1/R_i, \ i = 1, 2$. Under this operation type I gets mapped to the type IIB orientifold of Sen [1], where the radii and other fields are related by

$$R^I_1 = 1/R_1, \quad R^I_2 = 1/R_2, \quad R^I_1 R^I_2 \lambda = \lambda^I, \quad B^{RR,I}_{12} = \chi.$$  \hspace{1cm} (4)

Here the superscript $I$ denotes type I fields and no superscripts denotes fields of the IIB orientifold. $\chi$ denotes the RR scalar. Hence the heterotic moduli $T$ and $U$ get mapped to the following fields in the IIB orientifold

$$T \rightarrow \tau = \chi + i \frac{1}{\lambda}, \quad U \rightarrow U = i \frac{R_1}{R_2}.$$  \hspace{1cm} (5)

It follows from (3) that the heterotic modulus $T$ can be interpreted as the D-instanton action on the orientifold side. Hence worldsheet instantons on the heterotic side (i.e.}
fundamental string worldsheets wrapping $T^2$) which are weighted by a factor $\exp(-2\pi T^2)$ will be identified on the orientifold side with D-instantons which are weighted by a factor $\exp(-2\pi/\lambda)$.

In the following we will be interested in the one loop thresholds of the form $t_8 \text{tr}(F^4)_1$ and $t_8 (\text{tr}(F^2)_1)^2$ in the presence of the Wilson line (I). The subscript $1$ on the field strength indicates that the trace is taken over the first $SO(8)$ factor and without loss of generality can be associated with the four D7 branes at one of the orientifold planes.

These calculations were first performed by Lerche and Stieberger in [10] using methods developed in [11]. In particular the thresholds in eq (6) can be read off from eq (2.22) in [10]. Other one loop calculations of thresholds in related contexts can be found in [12][13][14].

For the interested reader the details of the threshold calculation are presented in the appendix using a somewhat different method (also employed in [15] for $SO(16)^2$). Only the result given in eq. B.8 and B.16 will be important for the main arguments of the paper.

It turns out that the $U$ and $T$ dependence of the result is decoupled and the $U$ dependent terms come from the trivial and degenerate orbits, whereas the $T$ dependent terms are given by the non degenerate orbits. Since we are interested in D-instanton induced terms on the worldvolume of the seven branes only the contributions of the non degenerate orbits are considered here. The results from appendix B are (see eq. B.8 and B.16):

$$I_7 = \int d^8 x \ t_8 \text{tr}(F^4) \sum_N \left( \frac{1}{2} \sum_{N|m} \frac{1}{m} e^{2\pi i 2NT} - \frac{1}{2} \sum_{N|m} \frac{1}{m} e^{2\pi i 4NT} \right)$$

$$+ \int d^8 x \ t_8 (\text{tr}(F^2))^2 \sum_N \left( \frac{1}{4} \sum_{N|m} \frac{1}{m} e^{2\pi i 2NT} - \frac{1}{8} \sum_{N|m} \frac{1}{m} e^{2\pi i 4NT} \right) + \text{cc.} \quad (6)$$

Using (5) together with the fact that since it is a one loop amplitude, (6) is independent of the heterotic string coupling (which in turn is related to the volume of $T^2/I$). It is then easy to see that (6) corresponds to D-instanton induced terms localized on the worldvolume of the seven branes. In (6) only the parity even contribution was calculated, the parity odd contribution which contains an eight dimensional epsilon tensor $\epsilon_8$ instead of $t_8$ can be calculated in the same way and the only difference is a minus sign instead of a plus

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2 The parameter $U$ is related to the conformal cross-ratio of the position of the seven branes in $T^2/I$
sign between the holomorphic (instanton) and and antiholomorphic (anti-instanton) part in (3).

Note that only even number of instantons contribute to the four derivative terms in (3). The structure of these terms is simpler than the one found in related instanton induced threshold terms like $t^8 t^8 R^4$ terms in IIB [16]. The $t^8 \text{tr}(F)^4$ and the $t^8 (\text{tr}(F)^2)^2$ thresholds in (3) do not receive any corrections as a power series $T_2$, which on the IIB orientifold side are interpreted as perturbative corrections to the D-instanton (as in [14]). One possible interpretation is that there is an exact cancellation of bosonic and fermionic fluctuations in this case [17][18] in analogy to cancellations of bosonic and fermionic determinants in the background of supersymmetric Yang-Mills instantons [19].

3. Instanton induced interactions

A D-instanton is a D $p = -1$ brane, which means that the worldvolume is a point in space time and the bosonic coordinates of open strings ending on the D-instanton $X^\mu$ satisfy Dirichlet boundary conditions for all directions $\mu = 0, \cdots, 9$. The presence of an orientifold plane changes the representation of Chan-Paton factors labeling open strings stretched between several D-instantons. The worldvolume of the orientifold seven plane will be chosen to lie in the $\mu = 0, \cdots, 7$ directions. The Chan-Paton factors for $k$ D-instanton fall in representations of $SO(k)$. The vertex operator for the ’massless’ bosonic states is given by

$$V^i = M_{(IJ)} \partial_n X^i, \quad i = 0, \cdots, 7$$

$$V^a = M_{[IJ]} \partial_n X^a, \quad a = 8, 9. \tag{7}$$

The fact that the longitudinal vertex $V^i$ and transverse vertex $V^a$ transform as second rank symmetric and antisymmetric tensors of $SO(k)$ respectively is determined by the consistency of action of the orientifold $(-1)^F \Omega I$ on the open string vertices and the Chan-Paton factors [20].

In addition there are fermionic collective coordinates which like their bosonic partners in (7) come in two representations.

$$V^a = M_{(IJ)} e^{-\frac{1}{2} \phi} S^\alpha e^{-\frac{i}{2} H_5},$$

$$V^\dot{a} = M_{[IJ]} e^{-\frac{1}{2} \phi} S^{\dot{\alpha}} e^{+\frac{i}{2} H_5}. \tag{8}$$

Here $\phi$ is the bosonized superghost and the fermionic vertices are in the $-1/2$ picture. A $SO(10)$ spin field $\Sigma^a$ of definite chirality is decomposed as $S^{\alpha} e^{-\frac{i}{2} H_5}$ and $S^{\dot{\alpha}} e^{+\frac{i}{2} H_5}$. Here
\( S^\alpha, S^{\dot{\alpha}} \) are \( SO(8) \) spin fields of opposite chirality and a \( SO(2) \) spin field is bosonized \( e^{\pm i\frac{1}{2} H_5} \) using the free boson \( H_5 \) in (8). We identify the \( SO(8) \) directions with the worldvolume directions of the orientifold seven plane. The sign of \( e^{\pm i\frac{1}{2} H_5} \) determines the parity under the orientifold projection and hence the representation of the \( SO(k) \) Chan-Paton factors in (8).

The presence of D7 branes introduces further collective coordinates, corresponding to stretched strings between the D7 brane and the D-instanton. The analysis of such open string states is quite subtle, since there are eight ND directions, like in the D0-D8 system [21] [22]. An analysis of the normal ordering constants for the Virasoro constraint of this system shows that all states in the NS sector are massive and that there is a single physical state in the Ramond sector (after GSO projection) which can be interpreted as a fermionic stretched string.

The vertex operator for this state when inserted on the boundary of a worldsheet changes the boundary condition from Neumann (7-brane) to Dirichlet (D-instanton) for \( X^i, i = 0, \cdots, 7 \). Such boundary condition changing operators can be constructed using bosonic \( Z_2 \) twist fields, which are familiar from string compactifications on orbifolds [23]. For a single complex boson the operator product expansion of the twist field \( \sigma \) is

\[
\sigma(z)\partial X(z) = (z - w)^{-\frac{1}{2}} \tau(w) + \cdots
\]

Here \( \tau \) is an excited twist field. The conformal dimension of \( \sigma \) is \( h = 1/8 \). The vertex operator for the fermionic ground state of the string stretched between the seven brane and the D-instanton is then given by

\[
V_\chi = e^{-1/2\phi} \sigma_1 \sigma_2 \sigma_3 \sigma_4 e^{-\frac{1}{2} H_5}.
\]

Here \( \sigma_i, i = 1, \cdots, 4 \) are twist fields which change the boundary conditions for the eight coordinates in the seven brane directions \( X^{2i-2} + iX^{2i-1}, \ i = 1, \cdots, 4 \). The conformal dimension of the vertex (10) is \( h = 3/8 + 4 \times 1/8 + 1/8 = 1 \). Since a stretched open string has one end on the D7 branes and the other on the D-instanton, the Chan-Paton factors are transforming as \( (8, k) \) of \( SO(8) \times SO(k) \). The only nontrivial coupling of the vertices (7) (8) and (10) is given by the following three point function on the disk.

\[
\langle cV_\phi(x_1)cV_\chi(x_2)cV_\chi(x_3) \rangle = \langle ce^{-\phi(\psi^8 - i\psi^9)}(x_1) ce^{-\frac{1}{2}\phi} \prod_i \sigma_i e^{i\frac{1}{2} H_5(x_2)} ce^{-\frac{1}{2}\phi} \prod_i \sigma_i e^{i\frac{1}{2} H_5(x_3)} \rangle.
\]

\( (11) \)
Where $c$ denotes the reparameterization ghosts, and inserting three of them fixes the Moebius invariance of the disk amplitude. The vertex $V^a$ in (7) has been transformed into the $-1$ picture in order to saturate the superghost anomaly on the disk. To evaluate (11) note that the fermion in can be bosonized $\psi^8 - i\psi^9 = \exp(-iH_5)$. Using the free field correlators is is easy to see that (11) is equal to a constant. Similarly the calculations of three point functions reveals that two $\theta$ couple to $\phi_1 + i\phi_2$ and two $\lambda$ couple to $\phi_1 - i\phi_2$.

Denoting the wave functions of the bosonic vertices $V^i$ and $V^a$ as $X^i$ and $\phi^a$, the fermionic vertices $V^\alpha$ and $V^{\dot{\alpha}}$ with $\theta^\alpha$ and $\lambda^{\dot{\alpha}}$ respectively, the matrix mechanics governing $k$ D-instantons near 4 D7 branes on top of an orientifold seven plane is given by the following action

$$S = \frac{1}{2} \text{tr} \left( \frac{1}{2} [X_i, X_j]^2 + [\phi_a, X_i]^2 + \frac{1}{2} [\phi_a, \phi_b]^2 + ig\theta[(\phi_1 + i\phi_2), \theta] + ig\lambda[(\phi_1 - i\phi_2), \lambda] + ig\Gamma_i[X^i, \lambda] + ig\chi_I(\phi_1 - i\phi_2)\chi_I \right).$$

This matrix action can also be deduced by applying a T-duality on action of the type I$'$ D0 brane quantum mechanics [21] [22]. The fields $X^i$ and $\theta^\alpha$ transform in the symmetric second rank tensor representation of $SO(k)$ have a trace part which decouples from (12) and are 'center of mass' degrees of freedom. In particular the trace part of the fermion $\theta^\alpha$ can be viewed as the fermionic collective coordinates associated with the broken supersymmetries of the D-instanton in the D7-O7 background. In the absence of a second quantized formulation of string theory the treatment of the D-instanton moduli space involves a certain amount of guesswork. Using ideas which have worked well for IIB instantons [24], we will define an integration over the collective coordinates as an integral over the matrix degrees of freedom associated with the D-instanton weighted with the Matrix action (12). Note that the trace part of $\theta$ does not appear in the matrix action (12) hence the integration over these fermionic variables has to be saturated by additional insertions which will produce instanton induced interaction vertices.

Another important feature is the coupling of the bosonic field $\phi_1 \pm i\phi_2$ to the fermionic degrees of freedom $\theta, \lambda$ and $\chi$. The fermionic integrals are saturated by pulling bilinears in the fermions from the action. This gives only a non vanishing result if there are an equal number of $\phi_1 + i\phi_2$ and $\phi_1 - i\phi_2$ appearing in this process, since otherwise the phase integrations of $\phi$ will kill the matrix integral. Since for $SO(2k)$ matrix mechanics the number of $\theta_\alpha$ is $8 \times 2k(2k+1)/2 - 8$ and the number of $\lambda$ is $4 \times 2k(2k-1)/2$ it follows that we need to pull down $8 \times (2k-1)$ $\chi$’s from the action (12). Note that there are $8 \times 2k$
\[\chi \] altogether. This implies that the integration over eight \(\chi\) cannot be saturated by the action and has to come from other insertions. Note that the \(\text{tr}(\theta \Gamma_i [X^i, \lambda])\) term in (12) saturates an equal number of \(\theta\) and \(\lambda\) integrations and does not change the counting of the deficit above.

In summary the integration over eight trace components of \(\theta\) and eight \(\chi\) have to be saturated by additional insertions. These insertion will be chosen here to produce the \(\text{tr}F^4\) and \((\text{tr}F^2)^2\) instanton induced interactions. Since the \(SO(8)\) gauge fields are associated with the seven brane and the fermionic zero modes with the D-instanton, the simplest string diagram which involves both is given by the disk with two boundary changing (twist) operators (10) inserted such that a part of the boundary of the world sheet ends on the seven brane and a part ends on the D-instanton. The simplest amplitude involves one gauge field vertex \(V_F = F_{ij}(X^i \partial X^j + i\psi^i \psi^j)\) and two fermionic zero modes together with two boundary condition changing operators (10) inserted. This amplitude has the form (only the \(F_{ij}\psi^i \psi^j\) part of the vertex contributes).

\[
A = F^{A}_{ij} T^{A}_{IJ} \theta^a \theta^b \int dx_2 dx_4 \langle e^{-1/2 \phi} S^a(x_1) e^{-1/2 \phi} V^I(x_2) c \psi^i \psi^j(x_3) e^{-1/2 \phi} V^I(x_4) e^{-1/2 \phi} S^b(x_5) \rangle \\
= \int_{x_1}^{x_3} dx_2 \int_{x_3}^{x_5} dx_4 \frac{x_5 - x_1}{(x_4 - x_2) \sqrt{(x_2 - x_1)(x_4 - x_1)(x_5 - x_2)(x_5 - x_4)}} F^{A}_{ij} \chi^I T^{A}_{IJ} \chi^J \theta \gamma_{ij} \theta.
\]

In evaluating (13) the Moebius invariance of the disk amplitude is fixed by inserting three reparameterization ghosts \(c\) which fix the positions of the vertices at \(x_1, x_3\) and \(x_5\).
correlators are evaluated using standard results for the superghosts, spin fields and twist fields \[25\][23]. Choosing the positions of the fixed vertex operators to be \(x_1 = 0, x_3 = 1\) and \(x_5 = \infty\) the remaining integrations over \(x_2\) and \(x_4\) in (13) are elementary and give

\[
\int_0^1 dx_2 \int_1^\infty dx_4 \frac{1}{(x_4 - x_2)\sqrt{x_2 x_4}} = \int_1^\infty dx_4 \frac{\ln \left( \frac{\sqrt{x_4} + 1}{\sqrt{x_4} - 1} \right)}{x_4} = \frac{\pi^2}{2}.
\]

(14)

The eight fermionic zero modes \(\theta^a\) associated with the supersymmetries broken by the presence of the D instanton are soaked up by four disk diagrams (13). Furthermore the four disk diagrams also saturate the integrations over the eight remaining \(\chi\) and induce a term with four \(F\) gauge fields. Note that the integral over \(\theta^a\) reproduces the well known kinematic tensor \(t_8\) since

\[
\int d^8 \theta \, \theta \gamma_{ij} \theta \gamma_{kl} \theta \gamma_{mn} \theta \theta_{\gamma pq} \theta = i^{ijklmnqp}.
\]

(15)

In the simplest case of \(SO(2)\) matrix mechanics it is straightforward to check that the integration over \(\chi\) produces the factors \(\text{tr}(F^4)\) and \((\text{tr}F^2)^2\) in the correct relative normalization as determined by (1). It is a natural generalization of the arguments involving type IIB thresholds and \(SU(N)\) matrix integrals [24] that the threshold amplitudes (6) are related to matrix integrals with action (12). Since the matrix mechanics for \(SO(2k)\) for general \(k\) is very complicated we do not attempt to check this correspondence here, although it would be very interesting to try to apply the methods of [28] to evaluate the matrix integrals for arbitrary \(k\). Note that because of the insertions described above, a certain correlation function involving eight \(\chi\) and not the partition function itself has to be compared to the results given in this paper.

4. D-Instanton in supergravity

The D-instanton solution of IIB supergravity [27] is a BPS solution which preserves half the supersymmetries,

\[
\delta \lambda = i P_\mu \epsilon^*, \quad \delta \psi_\mu = D_\mu \epsilon.
\]

(16)

Where the scalar field strength, covariant derivative and composite \(U(1)\) gauge field are defined as

\[
P_\mu = i \frac{\partial_\mu \tau}{2\tau_2}, \quad D_\mu \epsilon = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} - \frac{1}{2} i Q_\mu \right) \epsilon, \quad Q_\mu = - \frac{\partial_\mu \tau_1}{2\tau_2}.
\]

(17)
and the complex scalar given by a combination of RR scalar and dilaton $\tau = \chi + i e^{-\phi}$. Solutions to (14) which are BPS satisfy (in Euclidean space) $\partial_\mu \chi = \pm i \partial_\mu e^{-\phi}$. The D-instanton solution is then $g_{\mu\nu} = \eta_{\mu\nu}$ and the dilaton satisfies

$$\partial_\mu \partial^\mu e^\phi = 0 \tag{18}$$

i.e. the dilaton profile for a D-instanton centered at $y^\mu$ is given in terms of a harmonic function

$$e^\phi(x) = e^{\phi_0} + \frac{2K\alpha'^4}{\pi^4|x-y|^8} \tag{19}$$

When the three other singularities are far away the geometry in the vicinity of one of the orientifold singularities on $T^2/Z_2$ is locally $R^2/Z_2$. In the supergravity this implies that under the inversion $I$ $x_a \rightarrow -x_a$, $a = 8, 9$ transform according to the orientifold projection $(-1)^F \Omega I$. In particular this relates at $z$ to fields at $-z$. From the 32 supersymmetries of IIB only sixteen satisfying

$$\eta = (1 + i\Gamma^{89})\epsilon, \tag{20}$$

are unbroken by the orientifold projection.

Since the dilaton and RR scalar sources of of four D7 branes on top of the orientifold cancel locally, the D-instanton solution of IIB (19) can be used to write down a solution of the orientifolded IIB supergravity.

We have to find a solution which is invariant under the orientifold projection acting on the fields of IIB supergravity. Since there is no nontrivial monodromy of $\tau$ under the orientifold projection it relating the fields at $z$ and $-z$. Therefore the simplest invariant dilaton profile

$$e^\phi(x^i, z) = e^{\phi_0} + \frac{2K\alpha'^4}{\pi^4((x-x_1)^2 + |z-z_1|^2)^4} + \frac{2K\alpha'^4}{\pi^4((x-x_1)^2 + |z+z_1|^2)^4} \tag{21}$$

Where we split the coordinate $x^\mu, \mu = 0, \cdots, 9$ into $x^i, i = 0, \cdots, 7$ and $z, \bar{z}$. It is obvious that (21) is invariant under $z \rightarrow -z$. Since the solution (19) satisfies the charge quantization condition minimally, i.e. corresponds to singly charged D-instanton, it is easy to see that the projected instanton solution (21) has instanton charge 2. A 'stuck' instanton can carry instanton charge one

$$e^\phi(x, z) = e^{\phi_0} + \frac{2K\alpha'^4}{\pi^4((x-x_1)^2 + |z|^2)^4}, \tag{22}$$

and is also invariant under $z \rightarrow -z$ but cannot move away from the orientifold singularity $z = 0$. 

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In [28], the D-instanton solution in $AdS_5 \times S^5$ was discussed. Since $AdS_5 \times S^5$ is conformally flat, the solution can easily be obtained from the flat space D-instanton solution. Here we shall only be interested in the appropriately rescaled D-instanton solution in the limit when the profile is evaluated at the boundary of $AdS_5$.

$$\lim_{\rho \to 0} \rho^{-4}(e^\phi - e^{\phi_\infty}) = \frac{2K\alpha'^4}{L^8\pi^4} \frac{\rho_0^4}{((x-x_0)^2 + \rho_0^4)^2}. \quad (23)$$

Here $\rho_0, x_0$ label the position of the D-instanton in the bulk and $x$ is a coordinate on the boundary of $AdS_5$. Note that in the limit $\rho \to 0$ the position on the $S^5$ has disappeared from (23). In particular this implies that even in the orientifolded $AdS_5 \times S^5$ the D-instanton solution will be of the form (23) (with twice the charge).

5. $N = 2 USp(N_c)$ theories and the AdS/CFT correspondence

As mentioned in the introduction a probe of $N_c$ D3 brane moving in Sen’s $Z_2$ IIB orientifold produces a $USp(2N_C) N = 2$ gauge theory on the worldvolume with four hypermultiplets transforming in the fundamental and one hypermultiplet transforming in the second rank antisymmetric tensor of $USp(2N_c)$ [2]. The $SO(8)$ gauge symmetry on the D7 branes corresponds to a global $SO(8)$ symmetry of the four fundamental hypermultiplets.

The $USp(2N_c)$ $N = 2$ with the matter content described above has vanishing beta function and for zero Higgs expectation values defines a conformal field theory for any integer value of $N_c$. Hence we can consider the large $N$ limit. In [3][4] the AdS/CFT correspondence was generalized to this conformal field theory and it was shown that the $USp(2N_c) N = 2$ theory is dual to IIB supergravity on $AdS_5 \times S_5/Z_2$. The metric on the $S_5/Z_2$ is given by

$$ds^2 = d\theta^2 + \sin^2(\theta)d\phi^2 + \cos^2(\theta)d\Omega_3^2, \quad (24)$$

here $\phi \in [0, \pi]$ is periodic with period $\pi$ instead of $2\pi$ for an ordinary $S_5$ and some of the fields of IIB supergravity have nontrivial monodromies as $\phi \to \phi + \pi$ [3]. The $S^3$ can be regarded as the fixed point of the orientifold and the original sevenbranes are filling $AdS_5$ and are wrapped on the $S^3$ at $\theta = 0$. The changed periodicity and monodromy of type IIB fields modifies the spectrum of the chiral primaries coming from the bulk of the supergravity as analyzed in [3][4]. In addition there are chiral primaries coming from fields localized on the seven branes [4]. Such states carry $SO(8)$ charges. All fields in the supergravity and field theory are characterized by the charges they carry with respect to

$$SU(2)_R \times SU(2)_L \times U(1)_R \times SO(8) \times USp(2N_c), \quad (25)$$
where $SU(2)_R \times U(1)_R$ is identified with the R-symmetry of the the $N = 2$ superconformal algebra and $SU(2)_L$ can be interpreted as a flavor symmetry of the AST hypermultiplet, $SO(8)$ is a gauge symmetry on the D7 branes which is a global flavor symmetry from the perspective of the D3 branes and $USp(2N)$ denotes the gauge symmetry of the D3 brane. Geometrically the $SU(2)_L \times SU(2)_R$ can be identified with the $SO(4)$ symmetry of the $S^3$.

On the AdS side we will mainly be interested in the fields related to the vector field $A_M$ of the heterotic string. In the $AdS$ limit this vector field decomposes into three different fields which are all transforming in the adjoint of $SO(8)$,

Firstly the $M = 0, \cdots, 4$ components of the vector state are related to a vector field in $AdS_5$ with $A_\mu = \sum_k A^k(\mathbf{x})Y^k(\mathbf{y})$ which transforms as $(k,k)_0$ under under $SU(2)_R \times SU(2)_L \times U(1)_R$. Where $Y^k(\mathbf{y})$ is the $k$-th scalar KK mode on the 3 sphere. The $M = 5, 6, 7$ components are related to vectors with polarizations taking values in the tangent space of $S^3$. These states lead to scalar fields in the $AdS_5$. The $A_a = \sum_k A^k(\mathbf{x})Y^k_a(\mathbf{y})$ transform as $(k,k + 2)_0 + (k+2,k)_0$. Where $Y^k_a$ is the $k$-th vector spherical harmonics on $S^3$. $M = 8, 9$ components are related to the scalar $z = \sum_k z_k(\mathbf{x})Y^k(\mathbf{y})$ which transforms as $(k,k+2)_0$.

The part of the $k = 1$ KK mode of the internal vector $A_a$ which transforms as $(3,1)_0$ represents a chiral primary with dimension $\Delta = 2$.

In the gauge theory side these fields are identified with composite operators which are chiral primaries and have the same dimension and quantum numbers. $A^{k=1}_a$ is associated with the bilinear made of $q^I$

$$z^{[IJ]} = q^{[I}_A q^{J]}_B J^{AB}. \tag{26}$$

Where $q^I$ are the scalars in the fundamental hypermultiplets, and $J^{AB}$ is the invariant second rank $USp(2N)$ tensor and we have suppressed the $SU(2)_R$ indices. Other members of this supermultiplet can be obtained by acting with supercharges on the chiral primary field $z^{[IJ]}$. Of particular interest the $k = 1$ vector field $A^{k=1}_\mu$ transforms in the $(1,1)_0$ and has conformal dimension $\Delta = 3$. The vector field $A^{k=1}_m$ is identified with the global $SO(8)$ current

$$J^{[IJ]}_\mu = q^{[I}_\mu q^{J]} + i\bar{\psi}^[I \gamma_\mu \psi^{[J]}, \tag{27}$$

\[\text{A WZ coupling of the RR four form potential and the gauge field in the seven brane leads to a split of the (3,1)_0 and (1,3)_0 and a shift of their masses, and only the (3,1)_0 component corresponds to a chiral primary with } \Delta = 2. \]
which is given by acting with $Q\bar{Q}$ on the chiral primary (26). Furthermore the $z^{k=1}$ scalar is
associated with fields $qXq$ and $\psi\psi$ and is given by acting with $Q^2$ (or $\bar{Q}^2$ for the conjugate) on (26).

The four derivative threshold corrections in the heterotic string (5) are mapped to D-instanton induced $F^4$ terms living on the sevenbrane in the IIB orientifold. In the AdS/CFT correspondence the seven branes wrap the $S^3$ and fill $AdS_5$ and they lead to new bulk four point vertices in the $AdS_5$. Using the relation of the heterotic vector fields to the the vector $A_m$ and and the scalars $A_a$ and $z$, (5) will lead to four point correlation functions of these fields.

In the following the four point function involving four vector fields $A_m$ in $AdS_5$ are considered in detail, the four point vertices are given by the following expression,

$$I_4 = \int \frac{d^4zd\bar{z}}{\bar{z}_0^2}z_0^8s_{mnpqrst}D_mA^A_DpA^B_DqD^C_DsA^D_tA^A_D\left(F_{ABCD}(\tau) + G_{ABCD}(\tau)\right). \tag{28}$$

Here the indices $m,n,\cdots = 0,\cdots4$ denote coordinates in $AdS_5$ and the superscript in $A^A_A$ labels a basis of the adjoint representation of $SO(8)$. The tensor $t_8$ is constructed using $\delta_{mn}$ and the factor $z_0^8$ in (28) comes from four inverse metrics. The functions $F_{ABCD}$ and $G_{ABCD}$ in (28) contain the information about the D-instanton contribution and have the following structure

$$F_{ABCD}(\tau) = \text{tr}(t_At_Bt_Ct_D)\sum_N \left(\frac{1}{2}\sum_{N|m} \frac{1}{m}e^{i2\pi N\tau} - \frac{1}{2}\sum_{N|m} \frac{1}{m}e^{i4\pi N\tau} + \text{c.c.}\right)$$

$$G_{ABCD}(\tau) = \text{tr}(t_At_B)\text{tr}(t_Ct_D)\sum_N \left(\frac{1}{4}\sum_{N|m} \frac{1}{m}e^{i2\pi N\tau} - \frac{1}{8}\sum_{N|m} \frac{1}{m}e^{i4\pi N\tau} + \text{c.c.}\right). \tag{29}$$

Using the prescription developed in [6] the vertices (28) contribute to a correlation function in the CFT of four $SO(8)$ currents, $\langle J^{\mu A}(x_1)J^{\nu B}(x_2)J^{\rho C}(x_3)J^{\lambda D}(x_4) \rangle$. This calculation uses the bulk to boundary propagator for a vector field in $AdS_5$, given by $G_{m\mu}(z,x)$ which relates the bulk gauge field $A_m(z)$ to the boundary values $A_\mu(x)$. The bulk to boundary propagator defined in [29] has the following form

$$G_{m\mu}(z,x) = \frac{3}{\pi^2} \frac{z_0^2}{(z_0^2 + (z-x)^2)^3}J_{m\mu}(z-x). \tag{30}$$

Note that in (28) only the field strength $D_{[m}G_{n]\mu(z,x)$ appears and this expression is independent of the choice of gauge in (30). In addition the covariant derivative can be
replaced by an ordinary derivative and the result can be expressed in a simple manner (30).

\[ D_{[m} G_{n]\mu}(z, x) = \partial_{[m} G_{n]\mu}(z, x) = \frac{6}{\pi^2} \frac{z_0}{(z_0^2 + (z-x)^2)^3} J_{o[m}(z-x) J_{n]\mu(z-x). \] (31)

The conformal tensor \( J_{m\mu} \) is defined by

\[ J_{m\mu} = (z_0^2 + (z-x)^2) \frac{\partial}{\partial z_m} \left( \frac{(z-x)_\mu}{(z_0^2 + (z-x)^2)} \right), \] (32)

and \((z-x)_0 = z_0 \) is implied in (32). Plugging (31) into (28) gives the following contribution to the four point function of \( SO(8) \) currents

\[ \langle J_{\mu A}(x_1) J_{B}(x_2) J_{\rho}(x_3) J_{\lambda}(x_4) \rangle = t^m_{\ \nu pqrst} \left( \frac{6}{\pi^2} \right)^4 \int \frac{d^4 z d z_0}{z_0^5} \frac{z_{12} J_{o[m}(z-x_1) J_{n]\mu(z-x_1)}{(z_0^2 + (z-x_1)^2)^3} \times \frac{J_{o[p}(z-x_2) J_{q]\nu(z-x_2) J_{o[l}(z-x_3) J_{r]\rho(z-x_3) J_{o[s](z-x_4) J_{t]\lambda(z-x_4)}{(z_0^2 + (z-x_2)^2)^3}{(z_0^2 + (z-x_3)^2)^3}{(z_0^2 + (z-x_4)^2)^3} \times \left\{ F_{ABCD}(\tau) + G_{ABCD}(\tau) \right\}. \] (33)

This constitutes a prediction for the instanton contributions to the four point correlator of four \( J_{\mu A} \). It is easy to generalize this calculation to the four point functions involving the internal vector \( A_a \) and the scalar \( z \) using (6), as well as all Kaluza-Klein descendants.

6. D-instantons and Yang-Mills instantons

The map between the parameters of SYM and IIB string theory on \( AdS_5 \times S_5 \) is given by

\[ g_a = q_{YM}^2 / 4 \pi, \quad \chi = \theta_{YM} / 2 \pi, \quad R^2 / \alpha' = \sqrt{q_{YM}^2 N}. \] (34)

A first indication that D-instanton effects are related to YM-instanton effects is given by the observation that the action for a charge \( k \) D-instanton \( \exp(-2\pi k / g_s) \) is mapped to the charge \( k \) YM-instanton action \( \exp(-8\pi k / g_{YM}^2) \) [30]. In [28] more evidence for this correspondence was found by considering the D-instanton solution of IIB supergravity in the \( AdS_5 \times S_5 \) background. The gauge field \( \text{tr}(F^+)^2 \) of an \( SU(2) \) instanton of charge one is given by

\[ \text{tr}(F^-)^2 = \frac{4}{g_{YM}^2} \frac{\rho_0^4}{(\rho_0^2 + (x-x_0)^2)^4}. \] (35)
Remarkably agrees with the D-instanton solution in $AdS_5 \times S^5$ (23), where $x$ a point at the boundary and $(\rho_0, x_0)$ is the location of the D-instanton in the bulk of $AdS_5$. Note that the scale size $\rho$ of the YM instanton is the position of D-instanton in the radial direction of $AdS_5$. This relation is one example of the IR/UV relation in the gauge theory/supergravity correspondence.

In an impressive series of papers [31] the predictions for instanton contributions to correlators in the large $N_c$ limit of $N = 4$ $SU(N_c)$ SYM were confirmed using ADHM multi instanton calculus. One important fact which makes this correspondence possible is that the threshold corrections like $t_8 t_8 R^4$ in IIB only receive perturbative contribution at tree level and one loop. This implies that one can isolate the instanton contributions reliably even in the large $N_c$ limit.

For the $N = 2$ conformal field theory which arises in the IIB orientifold discussed in this paper a similar calculation of the instanton contribution in the gauge theory should be possible. Note that for $USp(2N_c)$ theories with hypermultiplets transforming in the fundamental representation there is a flavor parity symmetry [32] [33] which implies that only even instanton numbers contribute to these instanton induced interactions, in agreement with (6).

A detailed analysis of the multi instanton ADHM construction for $USp(2N_c)$ $N = 2$ SYM with the matter content described above is beyond the scope of the present paper. It would be nevertheless very interesting to see whether all or some of the features of the impressive $N = 4$ analysis [31] generalize in the case at hand. There are eight (instead of sixteen) zero modes associated with the broken superconformal symmetries which when soaked up by operator insertions should produce the four point correlators discussed above. All the other fermionic zero modes have to be lifted by quadrilinear terms in the multi instanton action. It would be interesting to see whether in the large $N_c$ saddle point approximation the dominant contribution comes from a single copy of $AdS_5$ and whether integrals in the $k$ instanton partition functions appear producing the $SO(2k)$ matrix integral (12).

7. Conclusions

In this paper D-instanton induced terms in the worldvolume of D7 branes in IIB orientifold were obtained by heterotic/type I duality from a one loop heterotic calculation. In the following we attempted to apply the methods which were applied successfully for thresholds in IIB and their relation to D-instantons as well as the relation of D-instantons
in $AdS_5 \times S_5$ and YM-instantons in $N = 4$ SYM. It is not clear whether this approach is justified since no independent check of the prediction coming from the heterotic loop calculation have been performed, unlike in the case of $N = 4$ SYM. It would therefore be very interesting to perform such (rather nontrivial) checks. It would also be interesting to analyze the type I formulation of the theory from the matrix string perspective \[18\] \[34\].

The arguments presented in this paper can be generalized in various ways. It is easy to calculate thresholds involving four gravitons, here the situation is more involved since there can also be contributions from the ten dimensional bulk as well as higher curvature terms living on the three branes \[35\]. The orientifold theory presented here is rather special it is an open question whether other heterotic loop calculations can be related to interesting quantities interesting conformal theories with $N = 2$ or $N = 1$ susy by similar arguments given in the paper.

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**Appendix A. Details of the heterotic loop amplitude**

In this appendix the calculation of one loop heterotic amplitudes with four gauge fields is outlined. The method is adapted from \[15\] for the Wilson line defined \[1\] and differs only slightly from the method used in \[10\]. The reader who is only interested in the result \[3\] may want to skip the details given in the appendix.

The integrals that will appear in this calculations are of the following form

$$I_Q = \int_P \frac{d^2 \tau}{\tau_2} \sum_A \frac{T_2}{\tau_2} \exp \left\{ 2\pi i T \det A - \frac{\pi T_2}{\tau_2 U_2} \right| (1U)A \left( \begin{array}{c} \tau \\ 1 \end{array} \right) \right|^2 \right\} QC(Y, A). \tag{A.1}$$

Here the matrix $A$ is given by $2 \times 2$ matrices with integer entries

$$A = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}, \quad m_1, m_2, n_1, n_2 \in \mathbb{Z}, \tag{A.2}$$

and $C(Y, A)$ is the partition function of the $SO(32)$ lattice which in general depends on the Wilson line $Y$ defined in \[1\] and the matrix $A$ \[A.2\] in the following way

$$C(Y, A) = \sum_{a,b=0,1} \prod_{k=1}^{16} e^{-i \pi (m^i n^j Y_i^k Y_j^k + b n^i Y_i^k)} \theta \left[ \begin{array}{c} a + 2m^i Y_i^k \\ b + 2n^i Y_i^k \end{array} \right] (0, \tau)$$

$$= \sum_{a,b} \theta^4 \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau) \theta^4 \left[ \begin{array}{c} a + m_2 \\ b + n_2 \end{array} \right] (0, \tau) \theta^4 \left[ \begin{array}{c} a + m_1 \\ b + n_1 \end{array} \right] (0, \tau) \theta^4 \left[ \begin{array}{c} a + m_1 + m_2 \\ b + n_1 + n_2 \end{array} \right] (0, \tau). \tag{A.3}$$
Where the standard notation for the theta functions is introduced
\[
\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \theta_1, \quad \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \theta_2, \quad \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \theta_3, \quad \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \theta_4, \quad .
\tag{A.4}
\]

The form of the operator \( Q \) in (A.1) depends on the threshold in question. The operator \( Q \) for \( \text{tr}(F^4) \) and \( (\text{tr}(F^2))^2 \) can be found by 'gauging' (A.3)[7]. The Wilson line (1) breaks the gauge group to \( SO(8)^4 \) and the thirty two free fermions of the \( SO(32) \) lattice are split into four sets of eight in (A.3). The result depends on the spin structures \([a, b]\) for the eight fermions which are associated with the first \( SO(8) \) in (A.3). For the \( \text{tr}(F^4) \) threshold the operators are given by
\[
Q_{\text{tr}(F^4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau) = -\frac{1}{2^{8/3}} \theta_3^4 \theta_4^4 (\tau),
\]
\[
Q_{\text{tr}(F^4)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) = \frac{1}{2^{8/3}} \theta_2^4 \theta_4^4 (\tau),
\]
\[
Q_{\text{tr}(F^4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau) = -\frac{1}{2^{8/3}} \theta_2^4 \theta_3^4 (\tau),
\tag{A.5}
\]

whereas for the \( (\text{tr}(F^2))^2 \) threshold the operator is given by
\[
Q_{(\text{tr}(F^2))^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau) = \frac{1}{2^{10/3}} (e_2 (\tau) + \hat{E}_2 (\tau))^2,
\]
\[
Q_{(\text{tr}(F^2))^2} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) = \frac{1}{2^{10/3}} (e_3 (\tau) + \hat{E}_2 (\tau))^2,
\]
\[
Q_{(\text{tr}(F^2))^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau) = \frac{1}{2^{10/3}} (e_4 (\tau) + \hat{E}_2 (\tau))^2.
\tag{A.6}
\]

Where the following notation has been introduced
\[
e_2 = \theta_3^4 + \theta_4^4, \quad e_3 = \theta_2^4 - \theta_4^4, \quad e_4 = -\theta_2^4 - \theta_3^4,
\tag{A.7}
\]

and \( \hat{E}_2 \) is the nonhomolomorphic (but modular) Eisenstein function of weight two defined as \( \hat{E}_2 = E_2 - 3/(\pi \tau^2) \). In the following it will be useful to expand (A.6) in powers of \( 1/\tau^2 \).
\[
Q_{(\text{tr}(F^2))^2} \begin{bmatrix} a \\ b \end{bmatrix} (\tau) = \sum_{r=0,1,2} \frac{1}{\tau^r} Q_{(\text{tr}(F^2))^2}^{(r)} \begin{bmatrix} a \\ b \end{bmatrix} (\tau)
\tag{A.8}
\]

For example it is easy to see that \( Q_{(\text{tr}(F^2))^2}^{(0)} \) is given by (A.6) with \( \hat{E}_2 \) replaced by \( E_2 \) and
\[
Q_{(\text{tr}(F^2))^2}^{(1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau) = -\frac{1}{2^{9/3} \pi} (e_2 (\tau) + E_2 (\tau)),
\]
\[
Q_{(\text{tr}(F^2))^2}^{(2)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau) = -\frac{1}{2^{10/3} \pi^2}
\tag{A.9}
\]

and similarly for the other charge insertions in (A.6).
Appendix B. Evaluation of integral

The integral (A.1) can be evaluated using the method of orbits \[36\]. In the present context this technique was discussed in \[17\] [37] and in \[10\], where type I thresholds with certain Wilson lines present were evaluated using results from \[11\]. Without Wilson lines it is straightforward to show that under the modular \(SL(2, \mathbb{Z})\) transformations \(\tilde{\tau} = \left(\frac{a\tau + b}{c\tau + d}\right)\) with \(a, b, c, d \in \mathbb{Z}, ad - bc = 1\)

\[
\frac{1}{\tau_2} |(1U)A \left(\frac{\tilde{\tau}}{1}\right)|^2 = \frac{1}{\tau_2} |(1U)A \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\frac{\tau}{1}\right)|^2.
\] (B.1)

The summation over all integer matrices matrices \(A\) can then replaced by the summation over all equivalence classes of \(SL(2, \mathbb{Z})\) orbits. There are three different cases, the trivial orbit \(A = 0\), the degenerate orbit \(\text{det}(A) = 0\) and the non degenerate orbit \(\text{det}(A) \neq 0\).

In the following we will consider only the non degenerate orbit, where the fundamental \(\mathcal{F}\) is unfolded into the double cover of the upper half plane \(\mathcal{H}\). The non degenerate \(SL(2, \mathbb{Z})\) orbits fall into the following equivalence classes

\[
A = \pm \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad k > 0, 0 \leq j < k, p \in \mathbb{Z}.
\] (B.2)

When Wilson lines are present, matters are more complicated but using the well known transformation properties of the theta functions under \(\tau \rightarrow \tau + 1, \tau \rightarrow -1/\tau\) is is easy to see that for both \(Q_{\text{tr}(F)^4} A\) and \(Q_{\text{tr}(F)^2} A\), \(QC(Y, A)\) defined in (A.3) behaves in the following way

\[
QC(Y, A)(\frac{a\tau + b}{c\tau + d}) = QC(Y, A \left(\begin{array}{cc} a & b \\ c & d \end{array}\right))(\tau).
\] (B.3)

Hence the method of orbits can be used to unfold the integral. For the non degenerate orbit we get

\[
I_{nd} = \int_{\mathcal{H}} \frac{d^2\tau}{\tau_2} \sum_{k > 0, 0 \leq j < k, p \in \mathbb{Z}} \frac{T_2}{\tau_2} \exp \left\{2\pi ikpT - \frac{\pi T_2}{\tau_2 U_2} |k\tau + j + pU|^2\right\} QC(Y, \left(\begin{array}{cc} k & j \\ 0 & p \end{array}\right))(\tau).
\] (B.4)

In order to evaluate (B.4) it is convenient to split the summation over equivalence classes \(A\) in (B.2) into four separate sectors \(A^{(i)}, i = 1, \cdots, 4,\)

\[
A^{(1)} = \begin{pmatrix} 2\tilde{k} & 2\tilde{j} \\ 0 & 2p \end{pmatrix}, \quad 0 \leq \tilde{j} < \tilde{k},
\]

\[
A^{(2)} = \begin{pmatrix} 2\tilde{k} + 1 & 2\tilde{j} \\ 0 & 2p \end{pmatrix}, \quad 0 \leq \tilde{j} \leq \tilde{k},
\]

\[
A^{(3)} = \begin{pmatrix} 2\tilde{k} + 1 & 2\tilde{j} + 1 \\ 0 & 2p \end{pmatrix}, \quad 0 \leq \tilde{j} < \tilde{k},
\]

\[
A^{(4)} = \begin{pmatrix} 2\tilde{k} & 2\tilde{j} \\ 0 & 2p + 1 \end{pmatrix}, \begin{pmatrix} 2\tilde{k} & 2\tilde{j} + 1 \\ 0 & 2p + 1 \end{pmatrix}, \begin{pmatrix} 2\tilde{k} & 2\tilde{j} + 1 \\ 0 & 2p \end{pmatrix} \quad 0 \leq \tilde{j} < \tilde{k}.
\] (B.5)
B1. The tr($F^4$) threshold

Using \( (A.3) \) and \( (A.5) \) we can express \( QC(A^{(i)}) \) for the \( \text{tr}(F_1)^4 \) threshold as

\[
QC_1 = QC(A^{(1)}) = \frac{1}{24 \eta^{24}} \left( -\theta_2^{16} \theta_3^4 \theta_4^4 + \theta_3^{16} \theta_2^4 \theta_4^4 - \theta_4^{16} \theta_2^4 \theta_3^4 \right) = 1, \\
QC_2 = QC(A^{(2)}) = \frac{1}{24 \eta^{24}} \theta_2^8 \theta_3^8 (-\theta_3^2 \theta_4^4 + \theta_2^4 \theta_4^4) = -\frac{1}{3}, \\
QC_3 = QC(A^{(3)}) = \frac{1}{24 \eta^{24}} \theta_2^8 \theta_3^8 (-\theta_3^2 \theta_4^4 - \theta_2^4 \theta_4^4) = -\frac{1}{3}, \\
QC_4 = QC(A^{(4)}) = \frac{1}{24 \eta^{24}} \theta_2^8 \theta_3^8 (2 \theta_2^4 \theta_4^4 - \theta_2^4 \theta_3^4) = -\frac{1}{3}.
\]

Where the following identities were used

\[
\theta_2^4 + \theta_4^4 - \theta_3^4 = 0, \quad \theta_2^4 \theta_3^4 \theta_4^4 = 16 \eta^{12}, \quad \theta_3^{12} - \theta_2^{12} - \theta_4^{12} = 48 \eta^{12}.
\]

Using the (trivial) fact that \( QC_1 = QC_2 + QC_3 + QC_4 + 2 \). The summation can be rearranged such that \( k, j \) and \( p \) run over the original summation range defined in \( (B.2) \).

\[
I_{nd}^{\text{tr}(F^4)} = 2 \int_{H} \frac{d^2 \tau}{\tau_2} \sum_{k>0,0 \leq j,k,p \in Z} \frac{T_2}{\tau_2} \exp \left\{ 2\pi i 4kpT - \frac{\pi 4T_2}{\tau_2 U_2} |k\tau + j + pU|^2 \right\} \\
- \int_{H} \frac{d^2 \tau}{\tau_2} \sum_{k>0,0 \leq j,k,p \in Z} \frac{T_2}{\tau_2} \exp \left\{ 2\pi i 2kpT - \frac{\pi 2T_2}{\tau_2 U_2} |k\tau + j + pU|^2 \right\} \\
= \sum_N \left( \frac{1}{2} \sum_{N|m} \frac{1}{m} e^{-2\pi N 4T} - \frac{1}{2} \sum_{N|m} \frac{1}{m} e^{-2\pi N 2T} \right) + cc
\]

where we used the formula for \( I_m \) derived in Appendix C for \( m = 4 \) and \( m = 2 \) to evaluate the terms in the first and second line of \( (B.8) \) respectively.

B2. The \( (\text{tr}(F^2))^2 \) threshold

The operator \( Q \) for the \( (\text{tr}(F^2)_1)^2 \) threshold depending on the spin structures was defined in \( (A.9) \). Together with \( (A.3) \) \( QC(A^{(i)}) \) become

\[
QC_1 = QC(A^{(1)}) = \frac{1}{210 \eta^{24}} \left\{ \theta_2^{16} (e_2 + \hat{E}_2)^2 + \theta_3^{16} (e_3 + \hat{E}_2)^2 + \theta_4^{16} (e_4 + \hat{E}_2)^2 \right\}, \\
QC_2 = QC(A^{(2)}) = \frac{1}{210 \eta^{24}} \theta_2^s \theta_3^s \left\{ (e_2 + \hat{E}_2)^2 + (e_3 + \hat{E}_2)^2 \right\}, \\
QC_3 = QC(A^{(3)}) = \frac{1}{210 \eta^{24}} \theta_2^s \theta_4^s \left\{ (e_2 + \hat{E}_2)^2 + (e_4 + \hat{E}_2)^2 \right\}, \\
QC_4 = QC(A^{(4)}) = \frac{1}{210 \eta^{24}} \theta_3^s \theta_4^s \left\{ (e_3 + \hat{E}_2)^2 + (e_4 + \hat{E}_2)^2 \right\}.
\]

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Using the identity (see also [10])

\[ QC_1 = QC_2 + QC_3 + QC_4 - \frac{1}{2} \]  

we can redistribute the summation over \( A^{(1)} \) over the contributions of the \( A^{(i)}, i = 2, 3, 4 \). The integral involving the constant term on the right hand side of (B.10) is then easily evaluated.

\[
I_{nd}^1 = -\frac{1}{2} \int_H \frac{d^2 \tau}{\tau_2} \sum_{k>0, 0 \leq j < k, p \in Z} \frac{T_2}{\tau_2} \exp \left\{ 2\pi i4kpT - \frac{\pi 4T_2}{\tau_2U_2} |kT + j + pU|^2 \right\}
\]

\[
= -\frac{1}{8} \sum_{N} \sum_{N|m} \frac{1}{m} e^{-2\pi N4T}
\]

(B.11)

Where \( I_m \) for \( m = 4 \) form Appendix C was used.

To calculate the contribution to the integral of the terms containing \( QC_i, i = 2, 3, 4 \) we first use \( QC_2(\tau - 1) = QC_3(\tau) \) to eliminate \( QC_3 \) in favour of \( QC_2 \) with an enlarged summation range. Keeping carefully track of the range of summations \( A \) involved gives

\[
I_{nd}^{2+3+4} = \int_H \frac{d^2 \tau}{\tau_2} \sum_A \frac{T_2}{\tau_2} \exp(2\pi i2kT) \left\{ \exp \left( -\frac{\pi T_2}{\tau_2U_2} |(1U) \begin{pmatrix} k & 2j \\ 0 & 2p \end{pmatrix} |2\right)QC_2(\tau) + \exp \left( -\frac{\pi T_2}{\tau_2U_2} |(1U) \begin{pmatrix} 2k & j \\ 0 & p \end{pmatrix} |2\right)QC_4(\tau) + \exp \left( -\frac{\pi T_2}{\tau_2U_2} |(1U) \begin{pmatrix} 2k & j+k \\ 0 & p \end{pmatrix} |2\right)QC_4(\tau) \right\}
\]

(B.12)

Note that the summation range of \( k, j, p \) is now taken over the original nondegenerate orbit (B.2). We can define a new integration variables \( \tau \to \tau/2 \) for the first summand, \( \tau \to 2\tau \) for the second and \( \tau \to 2\tau + 1 \) for the third. Since the integration region is the entire upper half plane and the measure is invariant under this change the integrals will remain unchanged under this redefinition

\[
I_{nd} = \int_H \frac{d^2 \tau}{\tau_2} \sum_A \frac{T_2}{\tau_2} \exp \left( 2\pi i2kT - \frac{\pi 2T_2}{\tau_2U_2} |(1U) \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} |2\right) \times \left\{ QC_2(2\tau) + QC_4(\frac{\tau}{2}) + QC_4(\frac{\tau}{2} - \frac{1}{2}) \right\}
\]

(B.13)

we now have the remarkable identity [10]

\[ QC_2(2\tau) + QC_4(\frac{\tau}{2}) + QC_4(\frac{\tau}{2} - \frac{1}{2}) = \frac{1}{2} \]  

(B.14)

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from which it follows that all the nontrivial dependence on powers of \( e^{2\pi i\tau} \) cancels out of (B.12) and it can be written as

\[
I_{nd}^{(2)+(3)+(4)} = 2 \int_{H} \frac{d^2 \tau}{\tau_2} \sum_{k>0,0\leq j<k,p\in\mathbb{Z}} \frac{T_2}{\tau_2} \exp \left\{ 2\pi i 2kpT - \frac{\pi 2T_2}{\tau_2 U_2} |k\tau + j + pU|^2 \right\}
\]

\[
= \sum_{N} \frac{1}{2} \sum_{N|m} e^{-2\pi N^2 T} + \text{cc.}
\]

(B.15)

Putting (B.11) and (B.15) together the result for \((\text{tr} F^2)^2\) threshold is

\[
I_m^{(\text{tr} F^2)^2} = \sum_{N} \left( \frac{1}{4} \sum_{N|m} \frac{1}{m} e^{2\pi i 2NT} - \frac{1}{8} \sum_{N|m} \frac{1}{m} e^{2\pi i 4NT} \right) + \text{cc.}
\]

(B.16)

Note that the identities (B.10) and (B.14) imply that the nonholomorphic part of the integrals which is produced by the presence of the modular but not holomorphic \( \tilde{E}_2 \) in (B.9) does not contribute to the threshold integrals. This is rather remarkable and can be checked in detail, by expanding (B.9) in powers of \(1/\tau^2\). Using the same notation \( QC_k^{(i)} \), \( i = 1, 2 \) for terms proportional to \(1/\tau^2\) as in (A.9), the vanishing of these contributions is then a consequence of the following identities.

\[
QC_1^{(i)} - QC_2^{(i)} + QC_3^{(i)} + QC_4^{(i)} = 0, \quad i = 1, 2
\]

\[
\frac{1}{2} QC_2^{(1)}(2\tau) + 2 \left( QC_4^{(1)}(\frac{1}{2}\tau) + QC_4^{(1)}(\frac{1}{2}\tau - \frac{1}{2}) \right) = 0
\]

\[
\frac{1}{4} QC_2^{(2)}(2\tau) + 4 \left( QC_4^{(2)}(\frac{1}{2}\tau) + QC_4^{(2)}(\frac{1}{2}\tau - \frac{1}{2}) \right) = 0
\]

(B.17)

With the same rearrangement of the summation sectors as above.

**Appendix C. Evaluation of the integrals**

In this appendix we review the evaluation of integrals appearing in the heterotic threshold calculation. The basic technique was developed in [36] for more details in this context see [17][37].

\[
I_m = T_2 \sum_{k>0,0\leq j<k,p\neq 0} e^{2\pi i mkpT} \int \frac{d^2 \tau}{\tau_2^2} \exp \left( m \frac{\pi T_2}{\tau_2 U_2} |k\tau + j + pU|^2 \right)
\]

(C.1)

Where the parameter \( m \) takes the values \( m = 2, 4 \) for the integrals considered in Appendix B. Integrating over \( \tau_1 \) gives

\[
I = \sum_{k>0,0\leq j<k,p\neq 0} \sqrt{\frac{T_2 U_2}{k\sqrt{m}}} e^{2\pi i mkpT} \int \frac{d\tau_2}{\tau_2^{3/2}} e^{-\pi T_2 \frac{mk^2 \tau_2}{\tau_2^2}} e^{-\pi mp^2 T_2 U_2 / \tau_2}.
\]

(C.2)
The integral over \( \tau_2 \) can be done using the formula
\[
\int_0^\infty \frac{dx}{x^{3/2}} e^{-ax - b/x} = \sqrt{\frac{\pi}{b}} e^{-2\sqrt{ab}},
\]
where
\[
a = \frac{\pi T_2}{U_2} mk^2, \quad b = \pi mp^2 T_2 U_2. \tag{C.4}
\]

The final result is given by
\[
I_m = \frac{1}{m} \sum_{0 \leq j < k > 0, p > 0} \frac{1}{k|p|} e^{2\pi imkpT} + \text{cc.}
\]
\[
= \frac{1}{m} \sum_N \sum_{N|n} \frac{1}{n} e^{2\pi imNT} + \text{cc.} \tag{C.5}
\]
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