PARABOLICITY, BROWNIAN ESCAPE RATE AND PROPERNESS OF SELF-SIMILAR SOLUTIONS OF THE DIRECT AND INVERSE MEAN CURVATURE FLOW

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ABSTRACT. We study some potential theoretic properties of homothetic solitons $\Sigma^m$ of the MCF and the IMCF. Using the analysis of the extrinsic distance function defined on these submanifolds in $\mathbb{R}^{n+m}$, we observe similarities and differences in the geometry of solitons in both flows. In particular, we show that parabolic MCF-solitons $\Sigma^m$ with $n > 2$ are self-shrinkers and that parabolic IMCF-solitons of any dimension are self-expanders. We have studied too the geometric behavior of parabolic MCF and IMCF-solitons confined in a ball, the behavior of the Mean Exit Time function for the Brownian motion defined on $\Sigma$ as well as a classification of properly immersed MCF-self-shrinkers with bounded second fundamental form, following the lines of [3].

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1. Introduction

The potential theory on a complete manifold is mainly devoted to the study of harmonic (or subharmonic) functions defined on it, and, more generally, to the study of the relation among the geometry of the manifold and the properties of the solutions of some distinguished PDEs raised using the Laplace-Beltrami operator, such us Laplace and Poisson equations. The interplay between geometric information, (encoded in the form of bounds for the curvature, for example) and functional theoretic properties, (such as the existence of bounded harmonic or subharmonic functions) constitutes a rich arena at the crossroads of Functional Analysis, Differential Geometry and PDEs theory where the problems we are going to study are placed. To address these problems, we will add in this paper the point of view of submanifold theory, in relation with some distinguished submanifolds in the Euclidean space. In particular, we are going to focus in the study of the parabolicity of homothetic solitons for the Mean Curvature Flow and for the Inverse Mean Curvature Flow and the relation of this concept with the geometry of these submanifolds. We are going to apply the same technique, namely, the analysis of the extrinsic distance defined on the submanifold, on MCF and IMCF solitons, in order to highlight similarities and differences among them.

We recall that a non-compact, complete $n$-dimensional manifold $M^n$ is parabolic if and only if every subharmonic $(\Delta u \geq 0$ when $u \in C^2(M))$, and bounded $(\sup_M u = u^* < \infty)$ continuous function $u : M \rightarrow \mathbb{R}$ defined on it is constant. If such non-constant function exists, then $M$ is non-parabolic. This functional property holds in compact manifolds as a direct application of the strong Maximum Principle, so parabolicity can be viewed as a generalization of compactness.

In fact, and if we modify slightly our point of view, parabolicity can be viewed as a stronger version of the following weak Maximum Principle: given $M$ a (not necessarily complete) Riemannian manifold, it satisfies the weak Maximum Principle if an only if for any bounded function $u \in C^2(M)$ with $\sup_M u = u^* < \infty$, there exists a sequence of points $\{x_k\}_{k \in \mathbb{N}} \subseteq M$ such that $u(x_k) > u^* - \frac{k}{m}$ and $\Delta u(x_k) < \frac{1}{m}$ (see [1]).

Let us consider now an isometric immersion $X : \Sigma \rightarrow \mathbb{R}^{n+m}$ of the manifold $\Sigma^n$ in $\mathbb{R}^{n+m}$. A question that arises naturally when studying the parabolicity of $\Sigma$ consists in to obtain a geometric description of this potential theoretic property, relating it, for example, with the behavior of its mean curvature. In this sense, when the dimension of the submanifold is $n = 2$, minimality does not imply parabolicity nor non-parabolicity: some minimal surfaces in $\mathbb{R}^3$ are parabolic, (e.g. Costa’s surface, Helicoid, Catenoid), while some others (like P-Schwartz surface or Scherk doubly periodic surface,) are non-parabolic.

However, something can be said in this context. In particular, we have, by one hand, that complete and minimal isometric immersions $\varphi : \Sigma^2 \rightarrow \mathbb{R}^n$ included in a ball $\varphi(\Sigma) \subseteq B^n_R$ are non-parabolic. The proof of this theorem follows from the fact that coordinate functions $x_i : \Sigma \rightarrow \mathbb{R}$ are harmonic, bounded in $\varphi(\Sigma) \subseteq B^n_R$ and non-constant. Recall that in the paper [27], N. Nadirashvili constructed a complete (non-proper) immersion of a minimal disk into the unit ball in $\mathbb{R}^3$.

On the other hand, and when the dimension of the submanifold is bigger or equal than 3, we have that complete and minimal proper isometric immersions $\varphi : \Sigma^n \rightarrow \mathbb{R}^{n+m}$ with $n \geq 3$ are non-parabolic (see [25]). The proof in this case is based on obtaining bounds for the capacity at infinity of a suitable precompact set in the submanifold.

Since solitons for MCF and IMCF satisfy a geometric condition on its mean curvature, namely, equations (2.8) and (2.9) in Definitions (2.7) and (2.9) respectively, and inspired by the results above mentioned, it could be interesting to establish a geometric description of parabolicity of a complete and non-compact soliton for the MCF and IMCF, and to study the behavior of parabolic solitons confined in a ball. To do that, we have used the analysis of the Laplacian of radial functions depending on the extrinsic distance, and Theorem (2.3)
In what follows, we are going to give an account of our main results concerning these and other related questions.

In Theorem 5.1 we prove that parabolic solitons for the MCF with dimension \( n \geq 3 \) are self-shrinkers and in Corollary 5.3 we prove that self-expanders for the MCF are non-parabolic. In this line and using the techniques mentioned before, we have proved in Theorem 4.1 that parabolic solitons for the IMCF are self-expanders, and that self-shrinkers for the IMCF with \( n \geq 2 \), and self-expanders for the IMCF with \( n \geq 3 \) and velocity \( C > \frac{1}{n-2} \) are non-parabolic, (Corollary 4.2).

Another line of research that we mentioned above is the study of the behavior of solitons included in a ball or in a half-space containing the origin. We can find in the literature several works dealing with this question, for example the paper [13], where it is extended the Hoffman-Meeks Halfspace Theorem to properly immersed self-shrinkers for the MCF, or the work [32], where some classification results for self-shrinkers for MCF are presented, assuming some restrictions on the norm of its second fundamental form, and considering that the self-shrinker is confined in a ball or a generalized cylinder and it has bounded mean curvature.

Our results in this line of research are Theorem 6.1, where it is proved that complete and parabolic self-shrinkers for the MCF confined in the ball \( B^{n+m}(\sqrt{\lambda}) \) centered at \( \bar{0} \in \mathbb{R}^{n+m} \) must be compact minimal submanifolds of the sphere \( S^{n+m-1}(\sqrt{\lambda}) \) and, as a corollary, that the only complete and connected parabolic self-shrinkers for the MCF with codimension 1 confined in the ball \( B^{n+m}(\sqrt{\lambda}) \) are the spheres of radius \( \sqrt{\lambda} \). Moreover, we have proved that there are not complete and non-compact parabolic self-expanders for MCF confined in a ball of any radius, (Theorem 6.3). Concerning solitons for the IMCF we have proved in Theorem 6.4 that complete and non-compact parabolic solitons confined in a \( R \)-ball are compact minimal submanifolds of a sphere of radius less or equal than \( R \).

In regard to classification results using bounds for the norm of the second fundamental form, in the paper [13], the authors obtained a classification theorem for complete self-shrinkers of MCF without boundary and with polynomial volume growth satisfying that the squared norm of its second fundamental form is less or equal than 1, (\( \lambda \) in the case we consider \( \lambda \)-self-shrinkers). Using the Mean Exit Time function, (whose behavior is closely related with the notion of parabolicity) defined on the extrinsic balls of the solitons, we have obtained some classification results for them. In particular, in first place, (Theorem 7.2), we have established an isoperimetric inequality satisfied by properly immersed MCF-self-shrinkers \( X : \Sigma^n \to \mathbb{R}^{n+m} \) and, from this result we have shown: first, that the properly immersed self-shrinkers confined in the \( \sqrt{\lambda} \)-ball \( B^{n+m}(\sqrt{\lambda}) \) or included in the complementary set \( \mathbb{R}^{n+m} \setminus B^{n+m}(\sqrt{\lambda}) \) must be compact minimal submanifolds of the sphere \( S^{n+m-1}(\sqrt{\lambda}) \) (Theorem 7.5), and secondly, (Theorem 7.10), that, if in addition the squared norm of the second fundamental form of these \( \lambda \)-self-shrinkers is bounded by the quantity \( \frac{1}{\sqrt{\lambda}} \), then they must be the sphere \( S^{n+m-1}(\sqrt{\lambda}) \), or, alternatively, this sphere separates the soliton into two parts. We present finally a characterization of IMCF-solitons in terms of the Mean Exit Time function defined on its extrinsic balls, (Theorem 8.3).

1.1. Outline of the paper. The structure of the paper is as follows:

In the preliminaries, Section 2.1 we recall the preliminary concepts and properties of extrinsic distance function. In subsection 2.2 it is presented and studied the notion of parabolicity, together a result due to Alias, Mastrolia and Rigoli, which extends the maximum principle to complete and non-compact manifolds that shall be widely used along the paper. We finish the preliminaries defining the solitons for the MCF and IMCF, (subsection 2.3) and relating them with the minimal spherical immersions, (subsection 2.4).
We shall prove Theorem 3.1 and Corollary 3.3 in subsection 3.1 of Section 3 and Theorem 4.1 and Corollary 4.2 in subsection 4.2 of Section 4. In Section 5 we check some of the parabolicity and non-parabolicity criteria we have proved on some examples. In Section 6 we shall study solitons confined in a ball. We prove Theorem 6.1 in subsection 6.1 obtaining Corollary 6.2 and Theorem 6.3. In subsection 6.2 we have proved Theorem 6.4 and Corollaries 6.5 and 6.6.

Finally, in subsection 7.2 the isoperimetric inequality, Theorem 7.2 is proved. Then, in subsection 7.3 we have the classification theorem 7.10. The characterization Theorem 8.3 is given in subsection 8.1 of Section 8 and an isoperimetric inequality for IMCF solitons is presented in Theorem 8.4 in subsection 8.2.

2. Preliminaries

2.1. The extrinsic distance function. Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete isometric immersion into the Euclidean space $\mathbb{R}^{n+m}$. The extrinsic distance function of $X$ to the origin $\vec{0} \in \mathbb{R}^{n+m}$ is given by

$$r : \Sigma \to \mathbb{R}, \quad r(p) = \text{dist}_{\mathbb{R}^{n+m}}(\vec{0}, X(p)) = \|X(p)\|.$$ 

In the above equality, $\| \cdot \|$ denotes the norm of vectors in $\mathbb{R}^{n+m}$ induced by the usual metric $g_{\mathbb{R}^{n+m}}$. The gradients of $r(x) = \text{dist}_{\mathbb{R}^{n+m}}(\vec{0}, x)$ in $\mathbb{R}^{n+m}$ and in $\Sigma$ are denoted by $\nabla_{\mathbb{R}^{n+m}} r$ and $\nabla_{\Sigma} r$, respectively. Then we have the following basic relation,

$$(2.1) \quad \nabla_{\mathbb{R}^{n+m}} r = \nabla_{\Sigma} r + (\nabla_{\mathbb{R}^{n+m}} r) \perp \text{on } \Sigma$$

where $(\nabla_{\mathbb{R}^{n+m}} r) \perp (X(x)) = \nabla_{\Sigma} r(X(x))$ is perpendicular to $T_x \Sigma$ for all $x \in \Sigma$.

Definition 2.1. Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete isometric immersion into the Euclidean space $\mathbb{R}^{n+m}$. We denote the extrinsic metric balls of radius $R > 0$ and center $\vec{0} \in \mathbb{R}^{n+m}$ by $D_R$. They are defined as the subset of $\Sigma$:

$$D_R = \{x \in \Sigma : r(x) < R\} = \{x \in \Sigma : X(x) \in B^{n+m}_R(\vec{0})\} = X^{-1}(B^{n+m}_R(\vec{0}))$$

where $B^{n+m}_R(\vec{0})$ denotes the open geodesic ball of radius $R$ centered at the pole $\vec{0} \in \mathbb{R}^{n+m}$. Note that the set $X^{-1}(\vec{0})$ can be the empty set.

Remark a. When the immersion $X$ is proper, the extrinsic domains $D_R$ are precompact sets, with smooth boundaries $\partial D_R$. The assumption on the smoothness of $\partial D_R$ makes no restriction. Indeed, the distance function $r$ is smooth in $\mathbb{R}^{n+m} - \{\vec{0}\}$ since $\vec{0}$ is a pole of $\mathbb{R}^{n+m}$. Hence the composition $r|_{\Sigma}$ is smooth in $\Sigma$ and consequently the radii $R$ that produce non-smooth boundaries $\partial D_R$ have 0-Lebesgue measure in $\mathbb{R}$ by Sard’s theorem and the Regular Level Set Theorem.

Remark b. Along the paper, we shall denote as $S^{n+m-1}(R)$ and as $B^{n+m}(R)$ or $B^{n+m}_R(\vec{0})$ the spheres and the balls centered at $\vec{0}$ in $\mathbb{R}^{n+m}$. In the classification results, (as Corollaries 6.2 and 6.6 or Theorem 7.10)), we are also using this notation to denote the $n$-dimensional $R$-spheres $S^n(R)$ considered as Riemannian manifolds, where the center it is not relevant. Another place where the center of the balls and spheres is not relevant is in the Poisson problem (7.2). In all the cases we are using the same notation, and the relevance or not of the center and if we are considering the spheres immersed or not will be clear from the context.

A technical result which we will use is the following:
Lemma 2.2. Let \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a complete isometric immersion into the Euclidean space \( \mathbb{R}^{n+m} \). Let \( r : \Sigma \rightarrow \mathbb{R} \), \( r(p) = \text{dist}_{\mathbb{R}^{n+m}}(X(p), \bar{0}) = \|X(p)\| \) the extrinsic distance of the points in \( \Sigma \) to the origin \( \bar{0} \in \mathbb{R}^{n+m} \). Given any function \( F : \mathbb{R} \rightarrow \mathbb{R} \), we have that

\[
\Delta^\Sigma F(r(x)) = \left( \frac{F''(r(x))}{r''(x)} - \frac{F'(r(x))}{r'(x)} \right) \|X^T\|^2 + \frac{F'(r(x))}{r'(x)} \left( n + \langle X, \vec{H} \rangle \right)
\]

(2.2)

where \( X^T \) denotes the tangential component of \( X \) with respect to \( X(\Sigma) \) and \( \vec{H} \) denotes the mean curvature vector field of \( \Sigma \).

2.2. Parabolicity and capacity estimates. Parabolicity extends the maximum principle to complete and non-compact parabolic manifolds in the following way, (see [1]):

Theorem 2.3. Let \( M \) be a complete non compact and parabolic Riemannian manifold. Then for each \( u \in C^2(\Sigma) \), \( \sup u < \infty \), \( u \) nonconstant on \( \Sigma \), there exists a sequence \( \{x_k\} \subset \Sigma \) such that \( u(x_k) > \sup u - \frac{1}{k} \), \( \Delta u(x_k) < 0 \), \( \forall k \in \mathbb{N} \).

To relate this functional property with the geometry of the underlying manifold, we shall establish bounds for the capacity of \( M \). When \( \Omega \subset M \) is precompact, it can be proved, (see [15]), that the capacity of the compact \( K \) in \( \Omega \) is given as the following integral:

\[
cap(K, \Omega) = \int_{\Omega} \|\nabla \phi\|^2 \, dV_g = \int_{\partial K} \|\nabla \phi\| \, d\mu
\]

(2.3)

where \( \phi \) is the solution of the Laplace equation on \( \Omega - K \) with Dirichlet boundary values:

\[
\begin{cases}
\Delta u = 0 \\
u \big|_{\partial K} = 1 \\
u \big|_{\partial \Omega} = 0
\end{cases}
\]

Moreover, for any compact \( K \subset \Sigma \) and any open set \( G \subset \Sigma \) containing \( K \), we have

\[
cap(K, \Sigma) \leq \cap(K, G)
\]

(2.4)

The relation among capacity and parabolicity is given by the following result, (see [15]):

Theorem 2.4. Let \((M, g)\) be a Riemannian manifold. \( M \) is parabolic iff \( M \) has zero capacity, i.e., there exists a non-empty precompact \( D \subset M \) such that \( \cap(D, M) = 0 \).

On the other hand, it can be proved that given \( K \subset M \) a (pre)compact subset of \( M \), if we consider \( \{\Omega_i\}_{i=1}^{\infty} \) an exhaustion of \( M \) by nested and precompact sets, such that \( K \subset \Omega_i \) for some \( i \), then the capacity of \( K \) in all the manifold, \( \cap(K, M) = \cap(K) \) is given as the following limit:

\[
cap(K, M) = \lim_{i \rightarrow \infty} \cap(K, \Omega_i)
\]

This definition is independent of the exhaustion. Another result concerning bounds for the capacity of a manifold is following:

Theorem 2.5 ([15]). Let \( \Sigma \) be a complete and non-compact Riemannian manifold. Let \( G \subset \Sigma \) be a precompact open set and \( K \subset G \) be compact. Suppose that a Lipschitz function \( u \) is defined in \( \overline{G \setminus K} \) such that \( u = a \) on \( \partial K \) and \( u = b \) on \( \partial G \) where \( a < b \) are real constants. Then,

\[
cap(K, G) \leq \left( \int_a^b \left( \frac{dt}{\|\nabla u(x)\| \, dA(x)} \right) \right)^{-1}
\]

(2.5)

To obtain sufficient conditions for parabolicity, we shall apply the following criterion of Hasminskii.
Theorem 2.6 ([16]). Let $M$ be a Riemannian manifold. If there exists $v : M \to \mathbb{R}$ superharmonic outside a compact set, and $v(x) \to \infty$ when $x \to \infty$, then $M$ is parabolic.

2.3. Solitons. Let $X_0 : \Sigma^n \to \mathbb{R}^{n+m}$ be an isometric immersion of an $n$-dimensional manifold $\Sigma$ into the Euclidean space $\mathbb{R}^{n+m}$. The evolution of $X_0$ by mean curvature flow (MCF) is a smooth one-parameter family of immersions satisfying

\[
\frac{\partial}{\partial t} X(p, t) = -\frac{\vec{H}(p, t)}{\|\vec{H}(p, t)\|^2} \quad \forall p \in \Sigma, \quad \forall t \geq 0
\]

(2.6)

where $\vec{H}$ is the mean curvature vector of the immersion $X$. Here, $\vec{H}_t = \vec{H}(\cdot, t)$ is the mean curvature vector of the immersion $X_t = X(\cdot, t)$ i.e., the trace of the second fundamental form $\alpha$, $(\vec{H}_t = tr_{\Sigma} \alpha_t = \triangle_{\Sigma} X_t)$. Likewise, the evolution of the initial immersion $X_0$ by the inverse of the mean curvature flow (IMCF) is a one-parameter family of immersions satisfying

\[
\frac{\partial}{\partial t} X(p, t) = \frac{\vec{H}(p, t)}{\|\vec{H}(p, t)\|^2} \quad \forall p \in \Sigma, \quad \forall t \geq 0
\]

(2.7)

\[
X(p, 0) = X_0(p), \quad \forall p \in \Sigma
\]

We are going to fix the notions we shall use along the paper, (see [?] and [22] for the definition of soliton).

Definition 2.7. A complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a $\lambda$-soliton for the MCF with respect $\bar{0} \in \mathbb{R}^{n+m}$, ($\lambda \in \mathbb{R}$), if and only if

\[
\vec{H} = -\lambda X^\perp
\]

where $X^\perp$ stands for the normal component of $X$ and $\vec{H}$ is the mean curvature vector of the immersion $X$.

Remark c. Note that, if we have a complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ satisfying the geometric condition (2.8), and we consider the family of immersions $X_t = \sqrt{1 - 2\lambda t} X$, it is straightforward to check that $\{X_t\}_{t=0}^\infty$ satisfies equation (2.6), so $X$ becomes the 0-slice of the family $\{X_t\}_{t=0}^\infty$ of solutions of equation (2.6).

Definition 2.8. A $\lambda$-soliton for the MCF with respect $\bar{0} \in \mathbb{R}^{n+m}$ is called a self-shrinker if and only if $\lambda > 0$. It is called a self-expander if and only if $\lambda < 0$.

Remark d. Note that a complete and minimal immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ can be considered as a “limit case” of $\lambda$-soliton for the MCF when $\lambda = 0$, because as $\vec{H}_\Sigma = 0$, then it satisfies equation (2.8).

For the inverse mean curvature flow we have the following definition:

Definition 2.9. The complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a $C$-soliton for the IMCF with respect $\bar{0} \in \mathbb{R}^{n+m}$, ($C \in \mathbb{R}$), if and only if

\[
\vec{H}(p) = -CX^\perp \quad \forall p \in \Sigma
\]

where $X^\perp$ stands for the normal component of $X$ and $\vec{H}$ is the mean curvature vector of the immersion $X$.

Remark e. Note that if we have a complete isometric immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ satisfying the geometric condition (2.9), and we consider the family of immersions $X_t = e^{Ct} X$, it is straightforward to check that $\{X_t\}_{t=0}^\infty$ satisfies equation (2.7), so $X$ becomes the 0-slice of the family $\{X_t\}_{t=0}^\infty$ of solutions of equation (2.7).

Definition 2.10. A $C$-soliton for the IMCF with respect $\bar{0} \in \mathbb{R}^{n+m}$ is called a self-shrinker if and only if $C < 0$. It is called a self-expander if and only if $C > 0$.

Remark f. A complete and minimal immersion $X : \Sigma^n \to \mathbb{R}^{n+m}$ cannot be considered as a $C$-soliton for the IMCF with respect $\bar{0} \in \mathbb{R}^{n+m}$ for any constant $C$ because $X$ cannot satisfy equation (2.9).
2.4. Solitons and spherical immersions. Let us consider now a spherical immersion, namely, an isometric immersion 
\[ X : 
\Sigma^n \rightarrow \mathbb{R}^{n+m} \] such that \( X(\Sigma) \subseteq S^{n+m-1}(R) \) for some radius \( R > 0 \). Then, we have the following characterization of self-shrinkers of MCF and self-expanders of IMCF. Assertion (3) concerning solitons for the IMCF was proved in [11], and it was proved in [4] that closed \( C \)-solitons for the IMCF are minimal spherical immersions with velocity \( C = \frac{1}{n} \).

Previous to the statement of the characterization, we recall Takahashi’s Theorem (see [56]), which will be used in our proof:

**Theorem 2.11.** If an isometric immersion \( \varphi : M^n \rightarrow \mathbb{R}^{n+m} \) of a Riemannian manifold satisfies \( \Delta^M \varphi + \lambda \varphi = 0 \) for some constant \( \lambda \neq 0 \), then \( \lambda > 0 \) and \( \varphi \) realizes a minimal immersion in a sphere \( S^{n+m-1}(R) \) with \( R = \sqrt{\lambda} \).

Now, the mentioned result:

**Proposition 2.12.** Let \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a complete spherical immersion. We have that:

1. If \( X \) is a \( \lambda \)-soliton for the MCF with respect \( \bar{0} \in \mathbb{R}^{n+m} \), then \( \lambda = \frac{1}{n} \) and \( X : \Sigma^n \rightarrow S^{n+m-1}(R) \) is a minimal immersion.
2. If \( X \) is a \( C \)-soliton for the IMCF with respect \( \bar{0} \in \mathbb{R}^{n+m} \), then \( C = \frac{1}{n} \) and \( X : \Sigma^n \rightarrow S^{n+m-1}(R) \) is a minimal immersion.
3. Conversely, if \( X : \Sigma^n \rightarrow S^{n+m-1}(R) \) is a minimal immersion, then \( X \) is, simultaneously, a \( \frac{1}{n} \)-soliton for the MCF with respect \( \bar{0} \in \mathbb{R}^{n+m} \) and a \( \frac{1}{n} \)-soliton for the IMCF with respect \( \bar{0} \in \mathbb{R}^{n+m} \).

**Proof.** First of all, note that, as \( \| X \| = R \) on \( \Sigma \) then \( X(q) \perp T_q \Sigma \) for all \( q \in \Sigma \). Hence
\[ \chi_1 \chi_1 = X \text{ and } X^T = 0. \]

To see (1), we have, as \( \Sigma \) is a \( \lambda \)-soliton for the MCF, that
\[ \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} = -\lambda X_{\perp} = -\lambda X. \]

On the other hand, \( \lambda \neq 0 \) because as \( r = R \) on \( \Sigma \), then, applying Lemma 2.2
\[ 0 = \Delta^\Sigma r^2 = 2n - 2\lambda R^2, \]
and hence \( \lambda = \frac{1}{n} \neq 0 \). Therefore, \( \Delta^\Sigma X = \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} = -\frac{1}{n} X \). We apply now Takahashi’s Theorem to conclude that \( X : \Sigma^n \rightarrow S^{n+m-1}(R) \) is a minimal immersion.

To see assertion (2), we have that, as \( \Sigma \) is a \( C \)-soliton for the IMCF, that
\[ \frac{\tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}}}{\| \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} \|^2} = -CX_{\perp} = -CX. \]

On the other hand, \( C \neq 0 \) because as \( r = R \) on \( \Sigma \), then, applying Lemma 2.2
\[ 0 = \Delta^\Sigma r^2 = 2(n - \frac{1}{C}) \]
and hence \( C = \frac{1}{n} \neq 0 \). Moreover,
\[ \frac{\| \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} \|}{\| \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} \|^2} = \frac{R}{n} \]
so \( \| \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} \| = \frac{R}{n} \), and therefore,
\[ \Delta^\Sigma X = \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} = -C\| \tilde{H}_{\Sigma \subset \mathbb{R}^{n+m}} \|^2 X = -\frac{n}{R^2} X. \]

Again we use Takahashi’s Theorem to conclude that \( X : \Sigma^n \rightarrow S^{n+m-1}(R) \) is a minimal immersion.
To prove assertion (3), let us suppose that \( X : \Sigma^n \rightarrow S^{n+m-1}(R) \) is a minimal immersion. Then use the equation, (see [6]):

\[
\vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}} = \vec{H}_{\Sigma \subseteq S^{n+m-1}(R)} - \frac{n}{R^2} X = -\frac{n}{R^2} X
\]

and we have that \( \Sigma \) is a \( \lambda \)-soliton for the MCF with \( \lambda = \frac{n}{R^2} \).

On the other hand, \( \| \vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}} \| = \frac{n}{R^2} \| X \| = \frac{n}{R^2} \), and hence

\[
\frac{\vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}}}{\| \vec{H}_{\Sigma \subseteq \mathbb{R}^{n+m}} \|^2} = -\frac{1}{n} X^\perp
\]

and we have that \( \Sigma \) is a \( C \)-soliton for the IMCF, independently of the radius \( R \).

\[ \square \]

3. A GEOMETRIC DESCRIPTION OF PARABOLICITY OF MCF-SOLITONS

3.1. Geometric necessary conditions for parabolicity. We start proving that parabolic solitons for MCF with dimension strictly greater than 2 are self-shrinkers.

**Theorem 3.1.** Let \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a complete and parabolic \( \lambda \)-soliton for the MCF with respect \( 0 \in \mathbb{R}^{n+m} \), with \( n > 2 \). Then \( X \) is a self-shrinker \((\lambda > 0)\) for the MCF.

**Proof.** To prove the theorem we are going to apply Theorem 2.3, with a family of bounded functions depending on \( \epsilon > 0 \) and constructed using the distance function. For any \( \epsilon > 0 \), let us consider the function \( f^1_\epsilon : \mathbb{R}^*_+ \rightarrow (-\infty, \frac{1}{\epsilon}) \) defined as

\[
f^1_\epsilon(s) = \frac{1}{\epsilon} \left(1 - \frac{1}{s^\epsilon}\right)
\]

The function \( f^1_\epsilon \) is smooth in \( \mathbb{R}^*_+ \) and strictly increasing in \( \mathbb{R}^*_+ \), so it is a bijection among \( \mathbb{R}^*_+ \) and its image \( \text{Im} f^1_\epsilon \). Moreover, as \( \lim_{r \to 0^+} f^1_\epsilon(r) = -\infty \) and \( \lim_{r \to \infty} f^1_\epsilon(r) = \frac{1}{\epsilon} \), then \( \sup_{\mathbb{R}^*_+} f^1_\epsilon \leq \frac{1}{\epsilon} < \infty \).

We are going to divide the rest of the proof in two cases. First, we shall consider a soliton \( \Sigma \) such that \( \emptyset \not\subseteq X(\Sigma) \). In this case, \( r^{-1}(0) = \emptyset \), and we define the functions,

\[
(3.1) \quad u^1_\epsilon : \Sigma \rightarrow \mathbb{R}, \quad x \rightarrow u^1_\epsilon(x) := f^1_\epsilon(r(x)).
\]

We have that \( \sup_{\Sigma} u^1_\epsilon = u^1_\epsilon^{-1} \leq \frac{1}{\epsilon} < \infty \), and, as \( \emptyset \not\subseteq \Sigma \), then \( r^{-1}(0) = \emptyset \), and these functions are smooth in \( \Sigma \). Then we can apply to them directly Theorem 2.3 in the following way:

If, for some \( \epsilon > 0 \), the function \( u^1_\epsilon \) is constant, then it is straightforward to check that all functions \( u^1_\epsilon \) are constant and, moreover, \( r|_{\Sigma} : \Sigma \rightarrow \mathbb{R} \), so \( X(\Sigma) \subseteq S^{n+m-1}(R) \), namely, \( X \) is a spherical immersion and hence, we apply Proposition 2.12 to get the conclusion (1), (for all \( n \geq 1 \)).

Alternatively, let us suppose that the test functions \( u^1_\epsilon \) are nonconstant on \( \Sigma^n \). Given \( \epsilon > 0 \), since \( \sup u < \infty \) and \( \Sigma \) is parabolic, we know by using Theorem 2.3 that there exists a sequence \( \{ x_k \} \subset \Sigma \), (depending on \( \epsilon \)), such that

\[
\Delta^\Sigma u^1_\epsilon(x_k) < 0
\]

Moreover, by equation (2.2)

\[
0 > \Delta u^1_\epsilon(x_k) = -\frac{2 + \epsilon}{r^{q+\epsilon}(x_k)} \| X^T(x_k) \|^2 + \frac{1}{r^{q+\epsilon}(x_k)} (n + \langle H, X \rangle)
\]

\[
\geq -\frac{2 + \epsilon}{r^{q+\epsilon}(x_k)} \| X(x_k) \|^2 + \frac{1}{r^{q+\epsilon}(x_k)} (n + \langle H, X \rangle)
\]

\[
= \frac{-2 - \epsilon + n - \lambda \| X^\perp(x_k) \|^2}{r^{2+\epsilon}(x_k)}
\]
where we have used that \( \langle H, X \rangle = -\lambda \|X^\perp\|^2 \) because \( X : \Sigma \to \mathbb{R}^{n+m} \) is a \( \lambda \)-soliton for the MCF with respect \( \bar{g} \in \mathbb{R}^{n+m} \). Therefore, for any \( \epsilon > 0 \), and for its associated sequence \( \{x_k\} \subset \Sigma \),

\[
\lambda \|X^\perp(x_k)\|^2 > n - 2 - \epsilon
\]

Then, if \( n > 2 \) there exists \( \epsilon_0 \) such that \( n - 2 - \epsilon_0 > 0 \) and we have that

\[
\lambda \|X^\perp(x_k)\|^2 > n - 2 - \epsilon_0 > 0
\]

so we conclude that \( \lambda > 0 \) and we have proved the theorem.

In the second case to consider, we assume that \( \bar{g} \in \Sigma \), namely, that \( X^{-1}(\bar{0}) \neq \emptyset \). Then, \( r^{-1}(0) \neq \emptyset \), so \( u'_1 \) is not smooth in \( r^{-1}(0) \subset \Sigma \). We are going to modify \( u'_1 \) to get \( u'_2 \in C^\infty(\Sigma) \) and we shall use the same argument than before on these modified functions with some care. This modification is given by the following

**Lemma 3.2.** Let \( X : \Sigma \to \mathbb{R}^{n+m} \) be an isometric immersion. Suppose that \( X^{-1}(\bar{0}) \neq \emptyset \).

Then, given \( \epsilon > 0 \) and the function \( u'_1 \) defined in equation (3.3), there exist a smooth function \( u'_2 : \Sigma \to \mathbb{R} \) and a positive real number \( x_0 > 0 \) such that

1. The function \( u'_2 \) satisfies that
   \[
   u'_2 = \begin{cases} 
   u'_1 & \text{on } \Sigma \setminus D_{\frac{\rho}{4}}, \\
   f'_1(\frac{s}{4}) & \text{on } D_{\frac{\rho}{4}}.
   \end{cases}
   \]

2. The function \( u_2 \) is not constant on \( \Sigma \), and \( \sup_{\Sigma} u'_2 > \sup_{\Sigma} u'_1 \).

Therefore,

\[
\sup_{\Sigma} u'_2 \leq u'_1 < \infty.
\]

**Proof.** To prove the Lemma, and given the function \( u'_1 \) defined in equation (3.1), let us consider an extrinsic ball \( D_{\rho}(\bar{0}) \subset \Sigma \) such that \( \Sigma \setminus D_{\rho}(\bar{0}) \neq \emptyset \). We have that \( u'_1 \in C^\infty(\Sigma \setminus D_{\rho}(\bar{0})) \). Let us fix \( 0 < x_0 < \rho \) and \( 0 < \delta_0 < f'_1(x_0) - f'_1(\frac{\rho}{2}) \), and let us define the function \( g' : (-\infty, \frac{\rho}{4}] \cup [\frac{\rho}{4}, \frac{\rho}{2}) \to (f'_1(\frac{\rho}{4}) - \delta_0, f'_1(\frac{\rho}{2}) + \delta_0) \) as

\[
g'(s) := \begin{cases} 
   f'_1(\frac{s}{4}) & \text{for } s \leq \frac{\rho}{4}, \\
   f'_1(s) & \text{for } s \geq \frac{\rho}{4}.
   \end{cases}
\]

The set \( A := (-\infty, \frac{\rho}{4}] \cup [\frac{\rho}{4}, \frac{\rho}{2}) \) is closed in \( N := (-\infty, \frac{\rho}{2}) \), and if we denote as \( M := (f'_1(\frac{\rho}{4}) - \delta_0, f'_1(\frac{\rho}{2}) + \delta_0) \), then \( g' \in C^\infty(A, M) \). Moreover, there is a continuous extension of \( g' \) to \( N \), given by

\[
h'(s) := \begin{cases} 
   f'_1(\frac{s}{4}) & \text{for } s \leq \frac{\rho}{4}, \\
   (s - \frac{\rho}{4})f'_1(\frac{\rho}{4}) + f'_1(\frac{\rho}{2}) & \text{for } \frac{\rho}{4} \leq s \leq \frac{\rho}{2}, \\
   f'_1(s) & \text{for } \frac{\rho}{2} \leq s \leq \frac{\rho}{4}.
   \end{cases}
\]

Then, applying the Extension Lemma for smooth maps, (see [19]), there exists an smooth extension \( \tilde{h}' : N \to M \) of \( g' \), i.e., \( \tilde{h}'|_A = g' \). This function \( \tilde{h}' \) can be trivially extended smoothly to all the real line defining \( f'_2 : (-\infty, \infty) \to (f'_1(\frac{\rho}{4}) - \delta_0, \frac{\rho}{4}) \) as

\[
f'_2(s) := \begin{cases} 
   \tilde{h}'(s) & \text{for } s < \frac{\rho}{4}, \\
   f'_1(s) & \text{for } s \geq \frac{\rho}{4}.
   \end{cases}
\]

because \( \tilde{h}'(s) = g'(s) = f'_1(s) \) for any \( s > \frac{\rho}{4} \), and hence, \( f'_2 = f'_1 = \tilde{h}' \) in the open set \((\frac{\rho}{4}, \frac{\rho}{2})\).

Now, let us define, for each \( \epsilon > 0 \), the function \( u'_2 : \Sigma \to \mathbb{R} \) as \( u'_2(p) := f'_2(r(p)) \). Then, \( u'_2 \in C^\infty(\Sigma) \). Observe that this \( u'_2 \) satisfies the statement (1) of the lemma.

To prove statement (2) of the lemma note that since \( X^{-1}(\bar{0}) \neq \emptyset \) there exist at least one point \( p \in \Sigma \) such that \( p \in D_{\frac{\rho}{4}} \), (on the contrary, \( X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}_{\frac{\rho}{4}}(\bar{0}) \), so
\( X^{-1}(\emptyset) = \emptyset \). Then \( u_2(p) = f'_1(p) \). On the other hand, since \( \Sigma \setminus D_{x_0} \neq \emptyset \), then there exist at least one \( q \in \Sigma \setminus D_{x_0} \). Then, as \( f'_1 \) is strictly increasing, 
\[
u_2^*(q) = u_1^*(q) = f'_1(r(q)) \geq f'_1(x_0) > f'_1\left(\frac{x_0}{4}\right) = u_2^*(p)\.
\]
Hence, \( u_2^* \) is not constant on \( \Sigma \). Let us observe now that, as \( \delta_0 < f'_1(x_0) - f'_1\left(\frac{x_0}{4}\right) \), and \( f'_2(s) = h^2(s) \forall s < \frac{x_0}{4} \), then we have
\[
sup_{\Sigma} u_2^* \leq f'_1\left(\frac{x_0}{2}\right) + \delta_0 < f'_1(x_0)
\]
But again since \( \emptyset \neq \Sigma \setminus D_{x_0} \subseteq \Sigma \setminus D_{x_0} \), there exists \( q \in \Sigma \setminus D_{x_0} \) with \( u_2^*(q) > f'_1(x_0) \). Then,
\[
sup_{\Sigma} u_2^* > \sup_{\Sigma} u_2^*.
\]
Now, let us suppose that \( \sup_{\Sigma \setminus D_{x_0}} u_2^* > \sup_{\Sigma \setminus D_{x_0}} u_1^* \). Then, as \( \sup_{\Sigma} u_1^* \geq \sup_{\Sigma \setminus D_{x_0}} u_1^* = \sup_{\Sigma \setminus D_{x_0}} u_2^* \), we obtain \( \sup_{\Sigma \setminus D_{x_0}} u_2^* > \sup_{\Sigma \setminus D_{x_0}} u_2^* \) and therefore, \( \sup_{\Sigma \setminus D_{x_0}} u_2^* \geq \sup_{\Sigma} u_2^* \), which is a contradiction. Hence, \( \sup_{\Sigma \setminus D_{x_0}} u_2^* \leq \sup_{\Sigma} u_1^* \) and therefore, as we know that \( \sup_{\Sigma \setminus D_{x_0}} u_2^* = \sup_{\Sigma \setminus D_{x_0}} u_1^* \leq \sup_{\Sigma} u_1^* \), then \( \sup_{\Sigma} u_2^* \leq \sup_{\Sigma} u_1^* \).

\[\square\]

We can finish now the proof of the theorem by using as a test function in Theorem 2.3 the smooth function \( u_2^* \) given by Lemma 2.2. For any \( \epsilon > 0 \), since \( \sup_{\Sigma} u_2^* \leq \infty \) and \( \Sigma \) is parabolic, we know by using Theorem 2.3 that there exists a sequence \( \{x_k\} \subset \Sigma \) such that \( u_2^*(x_k) \geq u_2^* - \frac{1}{k} \) and
\[\Delta^\Sigma u_2^*(x_k) < 0\]
Then, as \( \sup_{\Sigma} u_2^* > \sup_{\Sigma \setminus D_{x_0}} u_2^* \), there exists \( \delta_1 > 0 \) such that \( \sup_{\Sigma} u_2^* - \delta_1 > \sup_{\Sigma \setminus D_{x_0}} u_2^* \). Given the sequence \( \{x_k\} \subset \Sigma \), let us consider the numbers \( k \) such that \( \frac{1}{k} < \delta_1 \). Then
\[u_2^*(x_k) > \sup_{\Sigma} u_2^* - \frac{1}{k} > \sup_{\Sigma} u_2^* - \delta_1 > \sup_{\Sigma \setminus D_{x_0}} u_2^* \]
so \( x_k \) belongs to \( \Sigma \setminus D_{x_0} \) for \( k \) large enough and we have
\[0 > \Delta u_2^*(x_k) = \Delta u_1^*(x_k) = -\frac{2 + \epsilon}{r^{4+\epsilon}(x_k)}\|X^\perp(x_k)\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}(n + \langle H, X \rangle)
\geq -\frac{2 + \epsilon}{r^{4+\epsilon}(x_k)}\|X(x_k)\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}(n + \langle H, X \rangle)
= -\frac{2 - \epsilon + n - \lambda\|X^\perp(x_k)\|^2}{r^{2+\epsilon}(x_k)}\]
and we follow the argument as in the first case.

\[\square\]

As a first Corollary of Theorem 3.1 we have the following result, which extends one of the results in [23], (namely, that complete, non-compact and minimal immersions \( X : \Sigma^n \to \mathbb{R}^{n+m} \) with \( n > 2 \) are non-parabolic), to self-expanders for the MCF, not necessarily proper.

**Corollary 3.3.** Let \( X : \Sigma^n \to \mathbb{R}^{n+m} \) be a complete \( \lambda \)-soliton for the MCF, with \( \lambda \leq 0 \) and \( n > 2 \). Then \( \Sigma \) is non-parabolic.

**Remark g.** We must note at this point that the converse in Theorem 3.1 is not true in general:

1. When \( n = 1 \), then complete \( \lambda \)-solitons for the MCF are parabolic for all \( \lambda \).
Moreover, if 
X
using the distance function.

To prove the first assertion, we are going to apply Theorem 2.3 as in Theorem 3.1

Proof. To prove the first assertion, we are going to apply Theorem 2.3 as in Theorem 3.1 with the same family of bounded functions \( u_1 \). Depending on \( \epsilon > 0 \) and constructed using the distance function.

Then, if we assume that \( \bar{0} \not\in X(\Sigma) \), we have, for all \( \epsilon > 0 \) and each function \( u_1 \in C^\infty(\Sigma) \), a sequence \( \{ x_k \} \subset \Sigma \), (depending on \( \epsilon \)), such that \( \Delta \Sigma u_1(x_k) < 0 \) and therefore

\[
\lambda \| X^\perp(x_k) \| ^2 > -\epsilon
\]

so

\[
\| X^\perp(x_k) \| ^2 < -\epsilon \lambda.
\]

Since \( \| \bar{H} \| ^2 = \lambda \| X^\perp \| ^2 \), we have, for each sequence \( \{ x_k \} \subset \Sigma \), depending on \( \epsilon \)

\[
\| \bar{H}(x_k) \| ^2 < -\epsilon \lambda,
\]

which implies that, for all \( \epsilon > 0 \),

\[
\inf_{\Sigma} \| \bar{H} \| ^2 \leq -\epsilon \lambda,
\]

and hence

\[
\inf_{\Sigma} \| \bar{H} \| ^2 = 0.
\]

On the other hand, if we assume that \( \bar{0} \in X(\Sigma) \), we argue as in the proof of Theorem 2.2 modifying \( u_1 \) to obtain a new function \( u_2 \in C^\infty(\Sigma) \) which satisfies Lemma 3.2. As we have seen before, these new functions cannot be constant, so we apply Lemma 2.2 and Theorem 3.1 again, obtaining, for each \( \epsilon > 0 \), and each function \( u_2 \in C^\infty(\Sigma) \), a sequence \( \{ x_k \} \subset \Sigma \), (depending on \( \epsilon \)),

\[
\lambda \| X^\perp(x_k) \| ^2 > -\epsilon
\]

Now the proof follows as above.

Finally, to prove second assertion, for any connected and unbounded component \( V \) of \( \Sigma \) we define the following function

\[
F^\perp(x) := \begin{cases}
   f_1(R) & \text{if } x \in D_R \\
   u_1(x) & \text{if } x \in (\Sigma \setminus D_{2R}) \cap V \\
   f_1(2R) & \text{if } x \in (\Sigma \setminus D_{2R}) \setminus ((\Sigma \setminus D_{2R}) \cap V)
\end{cases}
\]

Observe that \( F^\perp \) is a smooth function defined on \( D_R \cup (\Sigma \setminus D_{2R}) \) and has a continuous extension on \( D_{2R} \setminus D_R \). Then, by using similar arguments as the used in the proof of
there exists an smooth extension $F_V : \Sigma \to \mathbb{R}$. Since $F_V$ is bounded and is non-constant, by theorem there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ such that

$$F_V(x_k) > \sup_{\Sigma} F_V - \frac{1}{k}, \quad \Delta F_V(x_k) < 0.$$ 

This implies that $\{x_k\}$ belongs to $V$ for $k$ large enough, and hence $F_V(x_k) = u'_1(x_k)$. Furthermore,

$$\Delta F_V(x_k) = \Delta u'_1(x_k) < 0$$

Then,

$$\inf \|\tilde{H}\|^2 \leq \|\tilde{H}(x_k)\|^2 \leq -\epsilon \lambda.$$ 

Finally the corollary follows letting again $\epsilon$ tend to 0. \qed

**Remark h.** As a consequence of Corollary if $\Sigma^2$ is a proper self-expander for the MCF and $\|H_{\Sigma}\| > C$ out of a compact set in $\Sigma^2$, then $\Sigma^2$ is non parabolic

### 3.2. Geometric sufficient conditions for parabolicity

We are going to study now sufficient conditions for parabolicity of properly immersed solitons for the MCF. In the paper M. Rimoldi has shown the following theorem, which shows that proper self-shrinkers for the MCF with mean curvature bounded from below exhibits the opposite behavior than minimal immersions in the sphere even when they are not minimal immersions.

**Theorem 3.5.** Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and non-compact properly immersed $\lambda$-self-shrinker for the MCF with respect $0 \in \mathbb{R}^{n+m}$. If $\|H_{\Sigma}\| \geq \sqrt{n}\lambda$ outside a compact set, then $\Sigma$ is a parabolic manifold. In particular, if $\|H_{\Sigma}\| \to \infty$ when $x \to \infty$, then $\Sigma$ is parabolic.

**Proof.** Given $r^2 = \|X(p)\|^2$, we have that

$$\Delta r^2 = 2(n - \frac{1}{\lambda}) \|H_{\Sigma}\|^2 \leq 0$$

As $H_{\Sigma} \to \infty$ when $x \to \infty$ and $X$ is proper, then $\Delta r^2 \leq 0$ outside a compact set. Then, apply Theorem[2.6]to get the conclusion. \qed

**Remark i.** As a consequence of Corollary[3.4]and Theorem[3.5] we can conclude that, if they exists, all complete and non-compact non-parabolic $n$-dimensional self-shrinkers for the MCF, such that $\|H_{\Sigma}\| \geq \sqrt{n}\lambda$ outside a compact set, are not properly immersed.

Respectively, if they exists, all complete and non-compact parabolic $n$-dimensional self-expanders for the MCF ($n > 2$), such that $\|H_{\Sigma}\| \geq C$ outside a compact set, being $C$ any positive constant, are not properly immersed.

These affirmations come from the fact that, in case $X : \Sigma^n \to \mathbb{R}^{n+m}$ is a complete and non-compact properly immersed self-shrinker for the MCF, (resp. self-expander), satisfying $\|H_{\Sigma}\| \geq \sqrt{n}\lambda$ outside a compact set, (resp. $\|H_{\Sigma}\| \geq C$ outside a compact set, being $C$ any positive constant), then $\Sigma$ must be parabolic, (resp., non-parabolic).

To prove our last sufficient condition of parabolicity for properly immersed solitons for MCF, we shall prove first the following result, which shows that, in some sense, (see affirmation (3) in the statement of the Theorem), MCF-self-shrinkers behaves in a similar way than minimal immersions in the sphere even when they are not minimal immersions.

**Theorem 3.6.** Let $X : \Sigma \to \mathbb{R}^{n+m}$ be a complete properly immersed $\lambda$-self-shrinker for the MCF with respect $0 \in \mathbb{R}^{n+m}$, then

1. $\int_{\Sigma} e^{-\frac{\lambda}{2}r^2(p)}dV(p) < \infty$.
2. $\int_{\Sigma} r^2(p)e^{-\frac{\lambda}{2}r^2(p)}dV(p) < \infty$.  

\[\int_{\Sigma} e^{-\frac{\lambda}{2}r^2(p)}dV(p) < \infty.\]
(3) \( \lambda \int_{\Sigma} r^2 e^{-\frac{\lambda}{2} r^2} dV(p) = n \int_{\Sigma} e^{-\frac{\lambda}{2} r^2} dV(p) \)

where here, \( r(p) := \|X(p)\| \) and \( dV \) stands for the Riemannian volume density of \( \Sigma \).

**Proof.** If \( X \) is spherical, affirmations (1) and (2) are obvious. Moreover, in this case affirmation (3) follows from Proposition 2.12 because \( \lambda = \frac{\lambda}{r} \). On the other hand, if \( X \) is not a spherical immersion, the statement (1) of the theorem is proved in [8].

To prove (2) and (3), since the immersion is proper, the extrinsic ball \( D_R \), i.e., \( \{ x \in \Sigma : \|X(x)\| < R \} \) is a precompact set of \( \Sigma \) and its boundary

\( \partial D_R = \{ x \in \Sigma : \|X(x)\| = R \} \)

by the Sard’s theorem is a smooth submanifold of \( \Sigma \) for almost every \( R \) with unit normal vector field \( \frac{\nabla r}{\| \nabla r \|} \). Then by applying the divergence theorem on \( D_R \) to the vector field \( e^{-\frac{\lambda}{2} r^2} \nabla r^2 \), we obtain

\[
\int_{D_R} \text{div} \left( e^{-\frac{\lambda}{2} r^2} \nabla r^2 \right) dV = \int_{\partial D_R} e^{-\frac{\lambda}{2} r^2} \left( \nabla r^2, \frac{\nabla r}{\| \nabla r \|} \right) dA 
= 2Re^{-\frac{\lambda}{2} R^2} \int_{\partial D_R} \| \nabla r \| dA. \quad (3.5)
\]

But, taking into account that

\[
\text{div} \left( e^{-\frac{\lambda}{2} r^2} \nabla r^2 \right) = (\nabla e^{-\frac{\lambda}{2} r^2}, \nabla r^2) + e^{-\frac{\lambda}{2} r^2} \Delta r^2
= -2\lambda e^{-\frac{\lambda}{2} r^2} r^2 \| \nabla r \|^2 + e^{-\frac{\lambda}{2} r^2} \left( 2n - 2\lambda \| X \|^2 \right)
= -2\lambda e^{-\frac{\lambda}{2} r^2} \| X \|^2 + e^{-\frac{\lambda}{2} r^2} \left( 2n - 2\lambda \| X \|^2 \right)
= 2e^{-\frac{\lambda}{2} r^2} \left( n - \lambda r^2 \right), \quad (3.6)
\]

equation (3.5) can be written as

\[
\int_{D_R} e^{-\frac{\lambda}{2} r^2} \left( n - \lambda r^2 \right) dV = Re^{-\frac{\lambda}{2} R^2} \int_{\partial D_R} \| \nabla r \| dA \geq 0. \quad (3.7)
\]

Consequently,

\[
\lambda \int_{D_R} r^2 e^{-\frac{\lambda}{2} r^2} dV \leq n \int_{D_R} e^{-\frac{\lambda}{2} r^2} dV \leq n \int_{\Sigma} e^{-\frac{\lambda}{2} r^2} dV.
\]

But then,

\[
\lambda \int_{\Sigma} r^2 e^{-\frac{\lambda}{2} r^2} dV = \lim_{R \to \infty} \lambda \int_{D_R} r^2 e^{-\frac{\lambda}{2} r^2} dV \leq n \int_{\Sigma} e^{-\frac{\lambda}{2} r^2} dV
\]

and the statement (2) of the theorem is proved. To prove statement (3) of the theorem, observe that

\[
2n \text{vol}(D_R) \geq \int_{D_R} \Delta r^2 dV = 2R \int_{\partial D_R} \| \nabla r \| dA.
\]

Then by the equality (3.7),

\[
0 \leq \int_{D_R} e^{-\frac{\lambda}{2} r^2} \left( n - \lambda r^2 \right) dV \leq ne^{-\frac{\lambda}{2} R^2} \text{vol}(D_R) \leq nCe^{-\frac{\lambda}{2} R^2} R^n
\]

where we have applied that for [8] since \( X \) is proper \( \Sigma \) has at most Euclidean volume growth. Finally the theorem is proved by taking the limit \( R \to \infty \).

The above Theorem implies that proper self-shrinkers have finite weighted volume when we consider the density \( r^2 e^{-\frac{\lambda}{2} r^2} \), this property can be used to obtain a sufficient condition for parabolicity. We shall need the following
Definition 3.7. Let $X : \Sigma \to \mathbb{R}^{n+m}$ be a proper isometric immersion. Let us define the function $\Psi_{\Sigma} : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\Psi_{\Sigma}(R) := \int_{\{p \in \Sigma : \|X(p)\| > R\}} r^2(p) e^{-\frac{n}{2} r^2(p)} dV(p)$$

Because Theorem 3.6, if $X : \Sigma \to \mathbb{R}^{n+m}$ is a proper self-shrinker by MCF, then

$$\lim_{R \to \infty} \Psi_{\Sigma}(R) = 0.$$  

The rhythm of this decay implies in some cases consequences for the parabolicity of $\Sigma$ as the following theorem shows.

Theorem 3.8. Let $X : \Sigma \to \mathbb{R}^{n+m}$ be a complete properly immersed $\lambda$-self-shrinker for the MCF with respect $\bar{\Omega} \in \mathbb{R}^{n+m}$. Suppose that

$$\int_{-\infty}^{\infty} \frac{te^{-\frac{n}{2} r^2}}{\Psi_{\Sigma}(t)} dt = \infty.$$  

Then, $\Sigma$ is parabolic.

Proof. We are going to apply Theorem 2.5 to the function $u(x) = r(x) = \|X(x)\|$. By the equality in (3.7),

$$\int_{\{x : r(x) = t\}} \|\nabla r\| dA = \int_{\partial D_t} \|\nabla r\| dA = \frac{e^{-\frac{n}{2} r^2}}{t} \int_{D_t} e^{-\frac{n}{2} r^2} (n - \lambda r^2) dV$$

$$= \frac{e^{-\frac{n}{2} r^2}}{t} \left( \int_{\Sigma} e^{-\frac{n}{2} r^2} (n - \lambda r^2) dV - \int_{\Sigma \setminus D_t} e^{-\frac{n}{2} r^2} (n - \lambda r^2) dV \right)$$

$$\leq \frac{e^{-\frac{n}{2} r^2}}{t} \int_{\Sigma \setminus D_t} r^2 e^{-\frac{n}{2} r^2} dV = \frac{e^{-\frac{n}{2} r^2}}{t} \int_{\Sigma \setminus D_t} r^2 dV$$

By using inequality (2.4) and Theorem 2.5 with $K = D_\rho$ and $G = D_R$ with $R > \rho > 0$ we obtain,

$$\text{cap}(D_\rho, \Sigma) \leq \text{cap}(D_\rho, D_R) \leq \left( \frac{\rho}{\lambda \Psi_{\Sigma}(t)} \right)^{-1} \left( \frac{\rho}{\Psi_{\Sigma}(t)} \right)^{-1} \left( \frac{\rho}{\Psi_{\Sigma}(t)} \right)^{-1}$$

Finally the theorem is proved letting $R$ tend to $\infty$. \qed

4. A GEOMETRIC DESCRIPTION OF PARABOLICITY OF IMCF-SOLITONS

As in the previous section, we start with a necessary condition for parabolicity:

Theorem 4.1. Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete soliton for the IMCF, $(n \geq 1)$. Then, if $\Sigma$ is a parabolic manifold, then $X$ is a self-expander for the IMCF and

$$\frac{1}{n} \leq C \leq \frac{1}{n-2}.$$  

Moreover, if $C = \frac{1}{n-2}$, then $X : \Sigma^n \to S^{n+m-1}(R)$ is minimal for some radius $R > 0$.

Proof. Given $\epsilon > 0$, let us consider the test function $u_\epsilon(p) := \frac{1}{\epsilon} \left( 1 - \frac{1}{\epsilon r(p)} \right)$. We have that $\sup_{\Sigma} u_\epsilon < \infty$ and $u_\epsilon \in C^2(\Sigma)$ because $\bar{\Omega} \notin X(\Sigma)$. If any of these functions is constant for some $\epsilon > 0$, then all are constant and hence $r = R$ is constant on $\Sigma$. Then,
Hence, applying Proposition \(2.12\),
\[ C : \Sigma \subseteq S^{n+m-1}(R). \]

Applying Proposition \(2.12\),
\[ X : \Sigma \subseteq S^{n+m-1}(R). \]

Therefore, Corollary \(4.3\).

There are no complete, non-compact and smooth \(n > 2\) and hence,
\[ \sup \Sigma u_\epsilon < \infty \text{ and } \Sigma \text{ is parabolic}, \]
we know by using Theorem \(2.3\) that there exists a sequence \(\{x_k\} \subseteq \Sigma\) such that
\[ \Delta u_\epsilon(x_k) < 0 \]
Moreover, by equation \(2.2\)
\[ 0 > \Delta u_\epsilon(x_k) = -\frac{2 + \epsilon}{r^{4+\epsilon}(x_k)}\|X^T\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}(n + \langle H, X \rangle) \]
\[ \geq -\frac{2 + \epsilon}{r^{4+\epsilon}(x_k)}\|X\|^2 + \frac{1}{r^{2+\epsilon}(x_k)}(n + \langle H, X \rangle) \]
\[ = -2 - \epsilon + n - \frac{1}{r} \]
where we have used that \(\langle H, X \rangle = -\frac{1}{C}\) because \(X : \Sigma \rightarrow \mathbb{R}^{n+m}\) is a \(C\)-soliton of the IMCF. Therefore,
\[ \frac{1}{C} > n - 2 - \epsilon \]
for any \(\epsilon > 0\). Then \(\frac{1}{C} \geq n - 2\).

Now, let us consider the test function \(v : \Sigma \rightarrow \mathbb{R}\) defined as \(v(p) := -\|X(p)\|^2 = -r^2(p)\). If \(v\) is constant in \(\Sigma\), \(i.e., v(p) = -R^2\) for all \(p \in \Sigma\), then \(X : \Sigma \rightarrow \mathbb{R}^{n+m}\) is a complete \(C\)-soliton for the IMCF such that \(x(\Sigma) \subseteq S^{n+m-1}(R)\). Hence, applying Proposition \(2.12\) \(C = \frac{1}{n}\) and \(\Sigma\) is minimal in the sphere \(S^{n+m-1}(R)\).

On the other hand, if \(v\) is non constant on \(\Sigma\), as \(sup \Sigma u_\epsilon < \infty\), \(v \in C^\infty(\Sigma)\) and \(\Sigma\) is parabolic, we apply Theorem \(2.3\) to obtain a sequence \(\{x_k\} \subseteq \Sigma\) such that, using Lemma \(2.2\)
\[ \Delta v_\epsilon(x_k) = -2(n - \frac{1}{C}) < 0 \text{ } \forall k \in \mathbb{N} \]
and hence, \(n > \frac{1}{C}\), and the Theorem is proved.

Let us suppose now that \(X : \Sigma \rightarrow \mathbb{R}^{n+m}\) is a complete and non-compact, parabolic self-expander for the IMFC with \(C = \frac{1}{n}\). Then, using Lemma \(2.2\)
\[ \Delta^2 v(x) = -2(n - \frac{1}{C}) = 0 \]
As \(sup \Sigma v < \infty\), \(v \in C^\infty(\Sigma)\) and \(\Sigma\) is parabolic, then \(v\), and hence \(r\) are constant on \(\Sigma\).

Applying Proposition \(2.12\), \(X : \Sigma \rightarrow S^{n+m-1}(R)\) is minimal for some radius \(R > 0\).

Namely, parabolic self-expanders with velocity \(C = \frac{1}{n}\) always realizes as minimal submanifolds of a sphere of some radius. \(\square\)

As Corollaries of Theorem \(4.1\) we have that 2-dimensional self-shrinkers for IMCF are non-parabolic and that, when \(n \geq 3\), self-shrinkers and self-expanders with velocity \(C > \frac{1}{n-2}\) are non-parabolic.

**Corollary 4.2.** Let \(X : \Sigma \rightarrow \mathbb{R}^{n+m}\) be a complete and non-compact soliton for the IMCF. Then

1. If \(n = 2\) and \(C < 0\), \(\Sigma^n\) is non-parabolic.
2. If \(n \geq 3\) and \(C < 0\) or \(C > \frac{1}{n-2}\), \(\Sigma^n\) is non-parabolic.

**Corollary 4.3.** There are no complete, non-compact and smooth 1-dimensional solitons for the IMCF with velocity \(C \in (-1, 1)\).
Remark j. As a consequence of the proof of Corollary 4.4, if \( C \in (-\infty, -1] \cap [1, \infty) \). Hence, if \( C \in (-1, 1) \), then \( \Sigma^1 \) should be non-parabolic. But \( \Sigma^1 \), complete, non-compact, properly immersed non-parabolic soliton for the IMCF, then \( C < 1 \).

Finally, we shall follow the argument used by M. Rimoldi in [33] on solitons for the MCF, based in the application of Theorem 2.6 to obtain an extension of previous Corollary to solitons for the IMCF with dimension \( n > 1 \).

**Corollary 4.4.** There are no complete, connected and non-compact properly immersed solitons for the IMCF, \( X : \Sigma^n \to \mathbb{R}^{n+m} \), with velocity \( C \in [0, \frac{1}{n}] \).

**Proof.** If \( C \in [0, \frac{1}{n}] \), then \( \Sigma \) is parabolic because Theorem 2.6. In fact, given \( v(p) := v_0^2(p) = \|X(p)\|^2 \), as \( C \in [0, \frac{1}{n}] \), then

\[
\Delta v = 2(n - \frac{1}{C}) \leq 0
\]

Hence, \( v \) is superharmonic outside a compact and \( v(p) \to \infty \) when \( p \to \infty \) because \( \Sigma \) is properly immersed. Using Theorem 2.6, \( \Sigma \) is parabolic. Now, we apply Theorem 4.1 to conclude that \( C \in \left[\frac{1}{n+1}, \frac{1}{n-1}\right] \). Hence, \( C = \frac{1}{n} \), so \( X : \Sigma^n \to S^n(R) \) is a spherical and minimal isometric immersion for some radius \( R > 0 \). Therefore, \( \Sigma \) is compact, which is a contradiction. \( \square \)

**Remark j.** As a consequence of the proof of Corollary 4.4 if \( X : \Sigma^n \to \mathbb{R}^{n+m} \) is a complete and non-compact properly immersed non-parabolic soliton for the IMCF, then \( C < 0 \) or \( C > \frac{1}{n} \), namely, if they exists, all complete and non-compact non-parabolic solitons for the IMCF with velocity \( C \in (0, \frac{1}{n}) \) are not properly immersed.

5. Examples

Along this section we will analyze how to apply the geometric characterizations of parabolicity that appears in this paper for the case of generalized cylinders (example 5.1) and on the other hand, example 5.2 we will deduce geometric properties of the family of examples given in [5] where \( \mathbb{R}^2 \) is conformally immersed as a self-expander of the MCF in \( \mathbb{R}^4 \).

**Example 5.1** (Generalized cylinders). Given \( \rho > 0 \) and \( k \in \mathbb{N} \), the following hypersurface of \( \mathbb{R}^{n+1} \)

\[
C_k(\rho) := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{k+1}^2 = \rho^2\}
\]

is called generalized cylinder. The generalized cylinder \( C_k(\rho) \) is isometric to \( S^k(\rho) \times \mathbb{R}^{n-k} \), and the inclusion map

\[
X : C_k(\rho) \to \mathbb{R}^{n+1}, \quad x = (x_1, \ldots, x_{n+1}) \in C_k(\rho) \to X(x) = (x_1, \ldots, x_{n+1})
\]

is an immersion of \( C_k(\rho) \) in \( \mathbb{R}^{n+1} \). It is assumed that \( 0 \leq k \leq n \). In the extreme cases, \( k = 0 \) and \( k = n \), the generalized cylinders \( C_0(\rho) \) and \( C_n(\rho) \) are isometric to \( \mathbb{R}^n \) and \( S^n(\rho) \) respectively. Since \( C_k(\rho) \) is isometric to \( S^k(\rho) \times \mathbb{R}^{n-k} \) and \( S^k(\rho) \) is compact, it is known that \( C_k(\rho) \) is parabolic if and only if \( n - k \leq 2 \). In this example we will explain how deduce this behavior by using the geometric properties of \( C_k(\rho) \). First of all we must remark that since the mean curvature vector field of \( X : C_k(\rho) \to \mathbb{R}^n \) is given by

\[
H = -\frac{k}{\rho^2} X^\perp.
\]

Then \( X : C_k(\rho) \to \mathbb{R}^{n+1} \) can be considered as \( \lambda \)-self-shrinker for the MCF with \( \lambda = \frac{1}{\rho^2} \).

Likewise, since

\[
\frac{H}{\|H\|^2} = -\frac{1}{k} X^\perp.
\]
the immersion $X : C_k(\rho) \to \mathbb{R}^{n+1}$ is a C-self-expander for the IMCF with $C = \frac{1}{k}$. Hence, $X : C_k(\rho) \to \mathbb{R}^{n+1}$ is at the same time a self-shrinker for the MCF and a self-expander for the IMCF.

Looking at necessary conditions for parabolicity of solitons of the IMCF, by applying Theorem [4.7] to $X : C_k(\rho) \to \mathbb{R}^{n+1}$, we can deduce that if $C_k(\rho)$ is parabolic then

$$0 \leq n - k \leq 2$$

with $k = n$ if $C_k(\rho)$ is a minimal immersion of a sphere. This is obvious in this case because $C_k(\rho)$ is the $n$-sphere of radius $\rho$ in $\mathbb{R}^{n+1}$. Moreover, if $n - k > 2$, then $C_k(\rho)$ is non-parabolic.

Now, we can look for sufficient conditions for parabolicity of solitons of the MCF. The first geometric conclusion is that since $X$ is a proper immersion, by Theorem [3.6] we know the following relations for the weighted volumes

1. $\int_{C_k(\rho)} e^{-\frac{\lambda^2}{2}} dV < \infty$.  
2. $\int_{C_k(\rho)} r^2 e^{-\frac{\lambda^2}{2}} dV < \infty$.  
3. $\lambda \int_{C_k(\rho)} e^{-\frac{\lambda^2}{2}} dV = n \int_{C_k(\rho)} r^2 e^{-\frac{\lambda^2}{2}} dV$.

It is tedious but not difficult to check the above statements for $X : C_k(\rho) \to \mathbb{R}^{n+1}$. In fact if $n - k \geq 2$,

$$\int_{C_k(\rho)} e^{-\frac{\lambda^2}{2}} dV = e^{-k/2} \rho^{n-k} \Gamma \left[ \frac{n-k}{2} \right] \frac{n^{n-k-2}}{k^{n-k+1}} \text{vol}(S_1^k) \cdot \text{vol}(S_1^{n-k-1})$$

and

$$\int_{C_k(\rho)} r^2 e^{-\frac{\lambda^2}{2}} dV = e^{-k/2} n \rho^{n-k+2} \Gamma \left[ \frac{n-k}{2} \right] \frac{n^{n-k-2}}{k^{n-k+1}} \text{vol}(S_1^k) \cdot \text{vol}(S_1^{n-k-1})$$

Then

$$\frac{\int_{C_k(\rho)} e^{-\frac{\lambda^2}{2}} dV}{\int_{C_k(\rho)} r^2 e^{-\frac{\lambda^2}{2}} dV} = \frac{k}{n \rho^2} = \frac{\lambda}{n}$$

as it is predicted by statement (3) of Theorem [3.6].

On the other hand, we can to apply Theorem [3.8] to see that $C_k(\rho)$ is parabolic if $n - k = 2$. To do it, we have to prove that

$$\int_0^\infty \frac{t e^{-t^2/4}}{\Psi_{C_k(\rho)}(t)} dt = \infty$$

where

$$\Psi_{C_k(\rho)}(R) = \text{vol}(S_1^k) \cdot \text{vol}(S_1^{n-k-1}) \cdot \int_0^\infty \frac{(t^2 + \rho^2)^{n-k-1}}{\sqrt{R^2 - \rho^2}} \left( t^2 + \rho^2 \right)^{n-k-1} e^{-\frac{\lambda^2}{2}}(t^2 + \rho^2) dt$$

$$= \text{vol}(S_1^k) \cdot \text{vol}(S_1^{n-k-1}) \cdot \int_{\sqrt{R^2 - \rho^2}}^\infty \frac{z^2 e^{-\frac{\lambda^2}{2}}}{\sqrt{R^2 - \rho^2}} (z^2 - \rho^2) \frac{n-k-2}{2} dz$$

In the case $n - k = 2$, we have that

$$\Psi_{C_k(\rho)}(R) = \frac{2 \rho^4 e^{-\frac{\lambda^2}{2}} R^2}{k^2} \text{vol}(S_1^k) \cdot \text{vol}(S_1^{n-k-1}) \left( \frac{k}{2 \rho^2} R^2 + 1 \right)$$
Hence finally,
\[ \int_{C_k(\rho)}^{\infty} t e^{-\frac{t^2}{2 \rho^2}} dt = \frac{1}{2 \rho^2 e^{-\frac{t^2}{2 \rho^2}}} \text{vol}(S^k_t) \cdot \text{vol}(S^{n-k-1}_t) \int_{C_k(\rho)}^{\infty} \frac{t}{2 \rho^2 t^2 + 1} dt \]
\[ = \frac{1}{2 \rho^2 e^{-\frac{t^2}{2 \rho^2}}} \lim_{t \to \infty} \log \left( \frac{k}{2 \rho^2 t^2 + 1} \right) \to \infty \]

and by using Theorem 3.8, \( C_k(\rho) \) is a parabolic manifold.

**Example 5.2** (Parabolic 2-dimensional self-expanders)

In the following example we will show how to deduce geometric properties from the conformal type of a soliton, applying our Corollary 3.4. I. Castro and A. Lerma have constructed in [5] the following \( 2 \)-dimensional \( \lambda \)-self-expander immersed in \( \mathbb{R}^4 = C^2 \) for any \( \delta > 0 \)

\[ X_\delta : \mathbb{R}^2 \to C^2, \quad X_\delta(s, t) := \frac{1}{\sqrt{-2 \lambda}} (i s_\delta \cosh(t) e^{-\frac{t^2}{2 \rho^2}}, t_\delta \sinh(t) e^{ic_\delta s}), \]

with \( s_\delta = \sinh(\delta), c_\delta = \cosh(\delta) \) and \( t_\delta = \tanh(\delta) \). This is a conformal immersion of \( \mathbb{R}^2 \) into \( C^2 \).

As the parabolicity is preserved on conformal changes of the metric, and the immersion \( X_\delta \) is conformal for any \( \delta > 0 \), \( (\mathbb{R}^2, X_\delta(g^{an}_2)) \) is parabolic for any \( \delta > 0 \), where \( X_\delta(g^{an}_2) \) is the pull-back of the canonical metric of \( C^2 \) given by \( X_\delta \). The conformal type of \( (\mathbb{R}^2, X_\delta(g^{an}_2)) \) implies certain behavior of the mean curvature vector field. More precisely, according to our Corollary 3.4 since \( (\mathbb{R}^2, X_\delta(g^{an}_2)) \) is parabolic, the infimum of the norm of the mean curvature vector field of \( X_\delta \) is therefore 0 with independence on \( \delta \). In fact, the mean curvature vector field can be explicitly computed as, (see proof of Proposition 2 of [5]),

\[ \vec{H}(s, t) = \frac{s_\delta^2 e^{-2u(t)}}{2c_\delta} \nabla \left( \frac{\partial}{\partial s} X_\delta(s, t) \right) \]

where \( J \) is the complex structure on \( C^2 \) and \( u(t) = \ln \left( \frac{1}{-2 \lambda} (t_\delta^2 \cosh^2(t) + s_\delta^2 \sinh^2(t)) \right) \).

Then

\[ \vec{H}(s, t) = \frac{s_\delta^2}{2c_\delta} \frac{4 \lambda^2}{(t_\delta^2 \cosh^2(t) + s_\delta^2 \sinh^2(t))^2} \left( i t_\delta \cosh(t) e^{-\frac{t^2}{2 \rho^2}}, -s_\delta \sinh(t) e^{ic_\delta s} \right) \]

and it is easy to check that,

\[ \lim_{t \to \infty} \vec{H}(s, t) = \vec{0}. \]

Hence

\[ \inf_{t \to \infty} ||\vec{H}|| = 0. \]

6. Solitons confined in a ball

6.1. Solitons for MCF confined in a ball. We are going to see, in the spirit of the results in [32], (see Proposition 5), that parabolic self-shrinkers for the MCF, \( X : \Sigma^n \to \mathbb{R}^{n+m} \), confined in a ball of radius \( \sqrt{\frac{n}{n+m}} \) realizes as minimal submanifolds of the sphere \( S^{n+m-1}(\sqrt{\frac{n}{n+m}}) \).

**Theorem 6.1.** Let \( X : \Sigma^n \to \mathbb{R}^{n+m} \) be a complete \( \lambda \)-self-shrinker for the MCF with respect \( \vec{0} \in \mathbb{R}^{n+m} \), ( \( \lambda > 0 \)). Let us suppose that \( \Sigma \) is parabolic. Then:

1. Either exists a point \( p \in \Sigma \) such that \( r(p) > \sqrt{\frac{n}{n+m}} \)
2. or \( X(\Sigma) \subseteq S^{n+m-1}(\sqrt{\frac{n}{n+m}}) \), and \( X : \Sigma^n \to S^{n+m-1}(\sqrt{\frac{n}{n+m}}) \) is minimal.
Remark k. Namely, there are no complete parabolic $\lambda$-self-shrinkers for the MFC inside the interior of a ball of radius $R \leq \sqrt{\frac{n}{\lambda}}$. If they are confined, i.e., $X(\Sigma) \subseteq B_{R}^{n+m}(\bar{0})$, with $R \leq \sqrt{\frac{n}{\lambda}}$, then $\Sigma$ realizes as minimal submanifolds of the sphere $\mathbb{S}^{n+m-1}(\sqrt{\frac{n}{\lambda}})$.

Proof. If there is no point $p \in \Sigma$ such that $r(p) > \sqrt{\frac{n}{\lambda}}$, then $r(p) \leq \sqrt{\frac{n}{\lambda}} \forall p \in \Sigma$, so we have that $X(\Sigma) \subseteq B_{R}^{n+m}(\bar{0})$ with $R \leq \sqrt{\frac{n}{\lambda}}$.

Let us consider the function $u : \Sigma \to \mathbb{R}$ defined as $u(p) := \|X(p)\|^2 = r^2(p)$. We assume by hypothesis, $(X(\Sigma) \subseteq B_{R}^{n+m}(\bar{0}))$, that

$$\sup_{\Sigma} u < \infty.$$ 

Moreover, since $X(\Sigma) \subseteq B_{R}^{n+m}(\bar{0})$, with $R \leq \sqrt{\frac{n}{\lambda}}$, we have that $\|X^\perp\|^2 \leq \|X\|^2 \leq \frac{n}{\lambda}$. Then, using Lemma $2.7$,

$$\Delta^\Sigma u(x) = 2(n - \lambda\|X^\perp\|^2) \geq 0$$

Then, as $\Sigma$ is parabolic, we conclude that $u$ is constant on $\Sigma$, so $r^2(x) = R^2 \forall x \in \Sigma$, for some $R \leq \sqrt{\frac{n}{\lambda}}$, (because $X(\Sigma) \subseteq B_{R}^{n+m}(\bar{0})$). Hence $X(\Sigma) \subseteq S^{n+m-1}(R)$.

On the other hand, as $X(x) \in T_x S^{n+m-1}(R) \subseteq T_x \Sigma \forall x \in \Sigma$, then $X = X^\perp$ and $X^T = 0$. But, as $u$ is constant on $\Sigma$ and $X = X^\perp$, then

$$\Delta^\Sigma u(x) = 2(n - \lambda\|X\|^2) = 0$$

and therefore, $R^2 = r^2(x) = \|X\|^2 = \frac{n}{\lambda}$. Hence $X(\Sigma) \subseteq S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ and, by Proposition $2.12$, $\Sigma$ is minimal in $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$.

Corollary 6.2. Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a complete and connected self-shrinker for the MCF, with $\lambda > 0$. Let us suppose that $\Sigma^n$ is parabolic and $X(\Sigma) \subseteq B_{R}^{n+1}(\bar{0})$, with $R \leq \sqrt{\frac{n}{\lambda}}$.

Then

$$\Sigma^n \equiv S^n\left(\sqrt{\frac{n}{\lambda}}\right)$$

Proof. In Theorem above, we have proved that $\|X\|^2 = \frac{n}{\lambda}$ on $\Sigma$. Hence $X : \Sigma^n \to S^n(\sqrt{\frac{n}{\lambda}})$ is a local isometry and therefore, as $\Sigma$ is connected and complete and $S^n(\sqrt{\frac{n}{\lambda}})$ is connected, then $X$ is a Riemannian covering, (see [34], p. 116). Moreover, as $S^n(\sqrt{\frac{n}{\lambda}})$ is simply connected, then $X$ is an isometry, (see [19], Corollary 11.24).

Remark l. If $n > 2$, it is enough to assume that $X : \Sigma^n \to \mathbb{R}^{n+1}$ is a complete and connected soliton for the MCF, by virtue of Theorem $3.1$.

Finally, we shall see that it is not possible to find complete and non-compact parabolic self-expanders confined in a ball

Theorem 6.3. There are no complete and non-compact parabolic self-expanders for MCF $X : \Sigma^n \to \mathbb{R}^{n+m}$ confined in a ball.

Proof. Let us consider $X : \Sigma^n \to \mathbb{R}^{n+m}$ a complete and non-compact parabolic self-expander. As $\Sigma$ is parabolic, then $n = 1$ or $n = 2$ by Corollary $3.3$. On the other hand, as $\lambda < 0$, we have, on $\Sigma$:

$$\Delta^\Sigma r^2 = 2n - 2\lambda\|X^\perp\|^2 \geq 0$$

Let us suppose that $X(\Sigma) \subseteq B_{R}^{n+m}(\bar{0})$ for some $R > 0$. Then, as $\sup_{\Sigma} r^2 \leq R < \infty$ and $\Sigma$ is parabolic, $r$ is constant on $\Sigma$, so $X(\Sigma) \subseteq S^n(R_0)$ with $R_0 \leq R$. By Proposition $2.12$, $\lambda = \frac{n}{R_0^2} > 0$, ($n = 1$ or $n = 2$), which is a contradiction. \qed
6.2. Solitons for IMCF confined in a ball.

Our aim in this subsection is the same than in subsection §6.1: we are going to see that parabolic self-expanders for the IMCF included in a $R$-ball, $X(\Sigma) \subseteq B_{R}^{n+m}(\vec{0})$, realizes as minimal submanifolds of a $r_0$-sphere with $r_0 \leq R$, and its velocity, (which do not depends on the radii $r$ and $R$), must be $C = \frac{1}{n}$ in this case.

**Theorem 6.4.** Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and non-compact soliton for the IMFC. Let us suppose that $\Sigma$ is parabolic. Then:

1. Either $\Sigma$ is unbounded
2. or $C = \frac{1}{n}$, $X(\Sigma) \subseteq S^{n+m-1}(r_0)$ with $r_0 \leq R$ and $X : \Sigma^n \to S^{n+m-1}(r_0)$ is minimal.

**Proof.** Let us suppose that $\Sigma$ is bounded, i.e., it is confined in a ball $X(\Sigma) \subseteq B_{R}^{n+m}(\vec{0})$. We are going to apply Theorem 2.3 to the function,

$$u(x) := r^2(x) = \|X(x)\|^2$$

Suppose that $u$ is nonconstant on $\Sigma$. Since $\sup_{\Sigma} u < \infty$, as $\Sigma$ is parabolic, by Theorem 2.3 there exists a sequence $\{x_k\}$ such that

$$\Delta u(x_k) < 0$$

but by Lemma 2.2,

(6.1) \hspace{1cm} 0 > \Delta u(x_k) = 2n - \frac{2}{C}$$

Hence, for any $x \in \Sigma^n$

$$\Delta u(x) = 2n - \frac{2}{C} < 0$$

On the other hand, let us consider the function,

$$v(x) := -r^2(x) = -\|X(x)\|^2$$

If we assume that $v$ is nonconstant on $\Sigma$ and since $\sup_{\Sigma} u \leq 0 < \infty$, we have that, applying again Theorem 2.3 there exists a sequence $\{x_k\}$ such that

$$\Delta v(x_k) = -2n + \frac{2}{C} < 0$$

Hence, $\frac{1}{n} < C < \frac{1}{n}$, so $u$ and $v$ must be constant functions. Therefore, applying Proposition 2.12 $C = \frac{1}{n}$, and $X : \Sigma^n \to S^{n+m-1}(r_0)$ is minimal. \hspace{1cm} \Box

As a corollary, and taking into account that every compact manifold is parabolic, we have the following result due to I. Castro and A. Lerma in [4].

**Corollary 6.5.** Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a complete soliton for the IMFC. Suppose that $\Sigma^n$ is compact. Then, $C = \frac{1}{n}$, and $X(\Sigma^n)$ is contained in a sphere $S^{n+m-1}(R) \subseteq \mathbb{R}^{n+m}$ of some radius $R$ centered at the origin of $\mathbb{R}^{n+m}$. Moreover, $X : \Sigma^n \to S^{n+m-1}(R) \subseteq \mathbb{R}^{n+m}$ is a minimal immersion into $S^{n+m-1}(R)$.

Another Corollary is the following analogous to Corollary 6.2 for parabolic and confined self-shrinkers for the MCF:

**Corollary 6.6.** Let $X : \Sigma^n \to \mathbb{R}^{n+1}$ be a connected and complete soliton for the IMFC. Let us suppose that $\Sigma^n$ is parabolic and $X(\Sigma) \subseteq B_{R}^{n+1}(\vec{0})$, for some $R > 0$. Then $\Sigma^n \equiv S^n(R)$

**Proof.** As $X(\Sigma) \subseteq B_{R}^{n+1}(\vec{0})$, for some $R > 0$, we have, applying Theorem 6.4, that $C = \frac{1}{n}$, $X(\Sigma) \subseteq S^n(r_0)$ with $r_0 \leq R$ and $X : \Sigma^n \to S^n(r_0)$ is minimal.

Hence $X : \Sigma^n \to S^n(\sqrt{R})$ is a local isometry and therefore, as $\Sigma$ is connected and complete and $S^n(\sqrt{R})$ is connected, then $X$ is a Riemannian covering. (see [34], p. 116).
Moreover, as $S^n(\sqrt{\lambda})$ is simply connected, then $X$ is an isometry, (see [19], Corollary 11.24).

\section{Mean Exit Time, and Volume of MCF-solitons}

The Mean Exit Time function for the Brownian motion defined on a precompact domain of the manifold satisfies a Poisson 2nd order PDE equation with Dirichlet boundary data, which, through the application of the divergence theorem, provides some information about the volume growth of the manifold. In the next sections and subsections we will explore these questions for MCF and IMCF solitons.

\subsection{Mean Exit time on Solitons for MCF.}

Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be an $n$-dimensional $\lambda$-soliton in $\mathbb{R}^{n+m}$ for the Mean Curvature Flow (MCF), with respect $\bar{0} \in \mathbb{R}^{n+m}$. Let us consider $r : \Sigma \to \mathbb{R}$ the extrinsic distance function from $\bar{0}$ in $\Sigma^n$. Given the extrinsic ball $D_R(\bar{0}) = X^{-1}(B_{R^{n+m}}^n(\bar{0}))$, let us consider the Poisson problem

\begin{equation}
\begin{cases}
\Delta^\Sigma E + 1 = 0 & \text{on } D_R, \\
E = 0 & \text{on } \partial D_R.
\end{cases}
\end{equation}

The solution of the Poisson problem on a geodesic $R$-ball $B^n(R)$ in $\mathbb{R}^n$

\begin{equation}
\begin{cases}
\Delta E + 1 = 0 & \text{on } B^0_R(R) \\
E = 0 & \text{on } S^{n-1}(R)
\end{cases}
\end{equation}

is given by the radial function $E_{R,n}^0(r) = \frac{R^2 - r^2}{2m}$.

Let us denote $E_R$ the solution of (7.1) in $D_R \subseteq \Sigma$. Transplanting the radial solution $E_{R,n}^0(r)$ to the extrinsic ball by mean the extrinsic distance function, we have $\tilde{E}_R : D_R \to \mathbb{R}$ defined as $\tilde{E}_R(p) := E_{R,n}^0(r(p))$.

Our first result is a comparison for the Mean Exit Time function:

\begin{proposition}
Let $X : \Sigma^n \to \mathbb{R}^{n+m}$ be a properly immersed $\lambda$-soliton for the MCF, with respect $\bar{0} \in \mathbb{R}^{n+m}$. Let us suppose that $X(\Sigma) \not\subseteq S^{n+m-1}(R)$ for any radius $R > 0$. Given the extrinsic ball $D_R(\bar{0})$, we have

1. If $\lambda \geq 0$, $\tilde{E}_R(x) \leq E_R(x)$, $\forall x \in D_R$.

2. Or if $\lambda \leq 0$, $\tilde{E}_R(x) \geq E_R(x)$, $\forall x \in D_R$
\end{proposition}

\begin{proof}
We have, as $\tilde{E}_R(x) := E_{R,n}^0(r(x)) = \frac{R^2 - r(x)^2}{2m}$ and applying Lemma 2.2 that, on $D_R$

\begin{equation}
\Delta^\Sigma \tilde{E}_R = \left( E''_R(r) - E'_R(r) \frac{1}{r} \right) \|\nabla^\Sigma r\|^2 + E'_r(r) \left( \frac{n}{r} + \langle \nabla^{R^{n+m}} r, \tilde{H}_\Sigma \rangle \right) = -1 - \frac{1}{n} \langle r^{n+m} \nabla r, \tilde{H}_\Sigma \rangle
\end{equation}

On the other hand, $X(r) = r(p)\nabla^{R^{n+m}} r(p)$ for all $p \in \Sigma$, and, moreover, as we have that $\tilde{H}_\Sigma(p) = -\lambda X^{-1}(p)$ $\forall p \in \Sigma$, then

\begin{equation}
\langle r^{n+m} \nabla r, \tilde{H}_\Sigma \rangle = -\lambda \|X^{-1}\| = -\frac{\|\tilde{H}_\Sigma\|^2}{\lambda}
\end{equation}

Therefore, if $\lambda \geq 0$, we obtain

\begin{equation}
\Delta^\Sigma \tilde{E}_R = -1 + \frac{1}{n} \frac{\|\tilde{H}_\Sigma\|^2}{\lambda} \geq -1 = \Delta^\Sigma E_R
\end{equation}

\end{proof}
As \( \bar{E}_R = E_R \) on \( \partial D_R \), we apply now the Maximum Principle to obtain the inequality
\[
\bar{E}_R \leq E_R.
\]
Inequality (2) follows in the same way. \( \square \)

### 7.2. Volume of Self-shrinkers for MCF

As a consequence of the Proposition 7.1 and using the Divergence theorem we have the following isoperimetric inequality.

**Theorem 7.2.** Let \( X : \Sigma \to \mathbb{R}^{n+m} \) be a complete properly immersed \( \lambda \)-self-shrinker in \( \mathbb{R}^{n+m} \) for the MCF, with respect \( \vec{0} \in \mathbb{R}^{n+m} \). Let us suppose that \( X(\Sigma) \not\subseteq S^{n+m-1}(R) \) for any radius \( R > 0 \). Given the extrinsic ball \( D_R(\vec{0}) = \Sigma \cap B_R^{n+m}(\vec{0}) \), we have

\[
\frac{\text{Vol}(\partial D_R)}{\text{Vol}(D_R)} \geq \left( 1 - \frac{1}{n \lambda} \frac{\int_{D_R} H^2}{\text{Vol}(D_R)} \right) \frac{\text{Vol}(S^{n-1}_R)}{\text{Vol}(B_R^n)} \quad \text{for all } R > 0
\]

where

\[
1 - \frac{1}{n \lambda} \frac{\int_{D_R} H^2}{\text{Vol}(D_R)} \geq 0 \quad \forall R > 0
\]

**Proof.** We are going to prove first that

\[
1 - \frac{1}{n \lambda} \frac{\int_{D_R} H^2}{\text{Vol}(D_R)} \geq 0 \quad \forall R > 0
\]

To do that, let us consider the function \( r^2 : \Sigma \to \mathbb{R} \), defined as \( r(\Sigma) = \| X(\Sigma) \| = \| \bar{H}_\Sigma \| = \| \bar{H} \| \), where \( r \) is the extrinsic distance to \( \vec{0} \) in \( \Sigma \subseteq \mathbb{R}^{n+m} \). Then, applying Lemma 2.2 to the radial function \( F(r) = r^2 \)

\[
\Delta^\Sigma r^2 = 2n + 2 \langle r \nabla^R r, \bar{H}_\Sigma \rangle.
\]

Taking into account that \( \langle r \nabla^R r, \bar{H}_\Sigma \rangle = -\lambda \| X \| = -\frac{\| \bar{H} \|^2}{\lambda} \), we obtain

\[
\Delta^\Sigma r^2 = 2n - 2 \frac{\| \bar{H}_\Sigma \|^2}{\lambda}
\]

and hence

\[
\| \bar{H}_\Sigma \|^2 = n \lambda - \frac{\lambda}{2} \Delta^\Sigma r^2.
\]

Integrating on \( D_R \) equality above, and arranging terms, we have

\[
n \lambda \text{Vol}(D_R) - \int_{D_R} \| \bar{H}_\Sigma \|^2 d\sigma = \frac{\lambda}{2} \int_{D_R} \Delta^\Sigma r^2 d\sigma
\]

Now we apply Divergence theorem taking into account that the unitary normal to \( \partial D_R \) in \( \Sigma \), pointed outward is \( \mu = \frac{\nabla^\Sigma r}{\| \nabla^\Sigma r \|} \) and the fact that \( \nabla^\Sigma r = \nabla X \| X \| \),

\[
\int_{D_R} \Delta^\Sigma r^2 d\sigma = \int_{\partial D_R} \langle \nabla^\Sigma r^2, \frac{\nabla^\Sigma r}{\| \nabla^\Sigma r \|} \rangle d\mu = \int_{\partial D_R} 2r \| \nabla^\Sigma r \| d\mu = 2 \int_{\partial D_R} \| X \| d\mu
\]

so equation (7.11) becomes

\[
n \lambda \text{Vol}(D_R) - \int_{D_R} \| \bar{H}_\Sigma \|^2 d\sigma = \lambda \int_{\partial D_R} \| X \| d\mu
\]

and hence

\[
0 \leq \frac{\int_{D_R} \| \bar{H}_\Sigma \|^2 d\sigma}{\text{Vol}(D_R)} = n \lambda - \frac{\lambda \int_{\partial D_R} \| X \| d\mu}{\text{Vol}(D_R)} \leq n \lambda.
\]
which implies inequality \((7.7)\). On the other hand, integrating on \(D_R\) the first equality in \((7.21)\) we obtain

\[
- \int_{D_R} \Delta^\Sigma E_R d\sigma = \int_{D_R} \left( 1 - \frac{1}{n} \frac{\|\bar{H}_\Sigma\|^2}{\lambda} \right) d\sigma = \text{Vol}(D_R) - \frac{1}{n\lambda} \int_{D_R} \|\bar{H}_\Sigma\|^2 d\sigma.
\]

Now, applying Divergence Theorem, and taking into account, as before, that the unitary normal to \(\partial D_R\) in \(\Sigma\), pointed outward is \(\nabla\Sigma\), we have

\[
- \int_{D_R} \Delta^\Sigma E_R d\sigma = -\bar{E}'_R(R) \int_{\partial D_R} \|\nabla\Sigma\| d\sigma \leq \frac{\text{Vol}(B^0_R)}{\text{Vol}(S^{n-1}_R)} \text{Vol}(\partial D_R)
\]

Hence

\[
\text{Vol}(D_R) - \frac{1}{n\lambda} \int_{D_R} \|\bar{H}_\Sigma\|^2 d\sigma \leq \frac{\text{Vol}(B^0_R)}{\text{Vol}(S^{n-1}_R)} \text{Vol}(\partial D_R)
\]

so

\[
\frac{\text{Vol}(D_R)}{\text{Vol}(\partial D_R)} \leq \frac{\text{Vol}(B^0_R)}{\text{Vol}(S^{n-1}_R)} + \frac{1}{n\lambda} \int_{D_R} \|\bar{H}_\Sigma\|^2 d\sigma
\]

and therefore for all \(R > 0\),

\[
\frac{\text{Vol}(\partial D_R)}{\text{Vol}(D_R)} \geq \left( 1 - \frac{1}{n\lambda} \int_{D_R} \|\bar{H}_\Sigma\|^2 d\sigma \right) \frac{\text{Vol}(S^{n-1}_R)}{\text{Vol}(B^0_R)}.
\]

\(\square\)

### 7.3. Proper Self-shrinkers for MCF and their distance to the origin.

By using the above Proposition \(7.4\) and inequality \((7.6)\) we can state the following theorem which give us a dual description of the behavior of the self-shrinker when we change the hypothesis of parabolicity for the assumption that it is properly immersed.

**Theorem 7.3.** Let \(X : \Sigma^n \to \mathbb{R}^{n+m}\) be a complete properly immersed \(\lambda\)-self-shrinker in \(\mathbb{R}^{n+m}\) for the Mean Curvature Flow (MCF), with respect \(\bar{0} \in \mathbb{R}^{n+m}\). Then:

1. Either there exists a point \(p \in \Sigma\) such that \(r(p) < \sqrt{\frac{\lambda}{\lambda_n}}\).
2. Or \(\Sigma^n\) is compact and \(X : \Sigma \to S^{n+m-1}(\sqrt{\frac{\lambda}{\lambda_n}})\) is a minimal immersion.

**Proof.** Let us suppose that \(X(\Sigma) \not\subseteq S^{n+m-1}_R(\sqrt{\frac{\lambda}{\lambda_n}})\) for any radius \(R > 0\). If, in addition, there is no point \(p \in \Sigma\) such that \(r(p) < \sqrt{\frac{\lambda}{\lambda_n}}\), then \(r(p) \geq \sqrt{\frac{\lambda}{\lambda_n}} \forall p \in \Sigma\). Now, let us suppose that \(\inf_{\Sigma} r > \sqrt{\frac{\lambda}{\lambda_n}}\). Then, for any \(p \in \Sigma\), we have that \(1 - \frac{\lambda}{\lambda_n} r^2 < 0\). Hence

\[
\int_{D_R} \left( 1 - \frac{\lambda}{\lambda_n} r^2 \right) e^{\frac{\lambda}{\lambda_n} (r^2 - r^2)} d\sigma < 0
\]

Now, we need the following

**Lemma 7.4.** Let \(X : \Sigma^n \to \mathbb{R}^{n+m}\) be a complete properly immersed \(\lambda\)-self-shrinker in \(\mathbb{R}^{n+m}\) for the MCF, with respect \(\bar{0} \in \mathbb{R}^{n+m}\). Let us suppose that \(X(\Sigma) \not\subseteq S^{n+m-1}_R(\sqrt{\frac{\lambda}{\lambda_n}})\) for any radius \(R > 0\). Given the extrinsic ball \(D_R\), if \(\text{Vol}(D_R) > 0\), we have

\[
1 - \frac{\text{Vol}(D_R)}{\text{Vol}(D_R)} \int_{D_R} \frac{1}{n\lambda} \|\nabla\Sigma\|^2 d\sigma = \int_{D_R} \left( 1 - \frac{\lambda}{\lambda_n} r^2 \right) e^{\frac{\lambda}{\lambda_n} (r^2 - r^2)} d\sigma
\]

**Proof.** By applying the divergence theorem,

\[
\int_{D_R} \text{div} \left( e^{-\frac{\lambda}{\lambda_n} r^2} \nabla r^2 \right) d\sigma = \int_{\partial D_R} e^{-\frac{\lambda}{\lambda_n} r^2} \langle \nabla r, \frac{\nabla r}{\|\nabla r\|} \rangle d\mu = 2R^2 e^{-\frac{\lambda}{\lambda_n} R^2} \int_{\partial D_R} \|\nabla r\| d\mu
\]
By equation (7.13) we know that
\[ R \frac{n}{\text{Vol}(D_R)} \int_{D_R} \| \nabla r \| d\mu = 1 - \frac{\int_{D_R} \| H_\Sigma \|^2 d\sigma}{\lambda n \text{Vol}(D_R)} \]

Hence,
\[ 1 - \frac{\int_{D_R} \| H_\Sigma \|^2 d\sigma}{\lambda n \text{Vol}(D_R)} = \frac{e^{\frac{\lambda}{2} r^2}}{2n \text{Vol}(D_R)} \int_{D_R} \text{div} \left( e^{-\frac{\lambda}{2} r^2 \nabla r^2} \right) d\sigma \]

Finally, the proposition follows taking into account that, see equation (3.6),
\[ \text{div} \left( e^{-\frac{\lambda}{2} r^2 \nabla r^2} \right) = 2e^{-\frac{\lambda}{2} r^2 (n - \lambda r^2)} \]

□

Now, applying inequality (7.6) in Theorem 7.2 and Lemma 7.4 we have
\[ 0 \leq 1 - \frac{\int_{D_R} \| H_\Sigma \|^2 d\sigma}{n \lambda \text{Vol}(D_R)} = \frac{\int_{D_R} (1 - \frac{\lambda}{n} r^2) e^{\frac{\lambda}{2} (r^2 - r^2)} d\sigma}{\text{Vol}(D_R)} < 0 \]

which is a contradiction.

Hence, or \( X(\Sigma) \subseteq S^{n+m-1}(R) \) for some radius \( R_0 > 0 \), or \( \inf \Sigma r = \sqrt{\frac{1}{\lambda}} \).

In the first case, we have that \( X : \Sigma \to S^{n+m-1}(R_0) \) will be a spherical immersion and, by Proposition 7.12 as \( \Sigma \) is a \( \lambda \)-soliton for the MCF, then \( X \) is minimal and \( \lambda = \frac{1}{2R_0} \), namely, \( X : \Sigma \to S^{n+m-1}(\sqrt{\frac{1}{\lambda}}) \) is a minimal immersion.

In the second case we shall conclude the same: if \( \inf \Sigma r = \sqrt{\frac{1}{\lambda}} \), then \( \sqrt{\frac{1}{\lambda}} \leq r(p) \) for all \( p \in \Sigma \) and hence \( 1 - \frac{\lambda}{n} r^2 (p) \leq 0 \ ∀ p \in \Sigma \). Then by inequality (7.6) and equality (7.22) we have
\[ 0 \leq 1 - \frac{\int_{D_R} \| H_\Sigma \|^2 d\sigma}{n \lambda \text{Vol}(D_R)} = \frac{\int_{D_R} (1 - \frac{\lambda}{n} r^2) e^{\frac{\lambda}{2} (r^2 - r^2)} d\sigma}{\text{Vol}(D_R)} \leq 0 \]

Therefore, \( 1 - \frac{\lambda}{n} r^2 (p) \leq 0 \ ∀ p \in \Sigma \), so \( X(\Sigma) \subseteq S^{n+m-1}(\sqrt{\frac{1}{\lambda}}) \), and hence \( X : \Sigma \to S^{n+m-1}(\sqrt{\frac{1}{\lambda}}) \) is a complete spherical immersion, and as the radius \( R = \sqrt{\frac{1}{\lambda}} \), then by Proposition 7.12 \( \Sigma \) is minimal in the sphere \( S^{n+m-1}(\sqrt{\frac{1}{\lambda}}) \).

Finally, as \( X : \Sigma^n \to \mathbb{R}^{n+m} \) is proper, then \( \Sigma = X^{-1}(S^{n+m-1}(\sqrt{\frac{1}{\lambda}})) \) is compact. □

Another result which describes the position of properly immersed \( \lambda \)-self-shrinkers \( \Sigma^n \) with respect the critical ball \( B^{n+m}_{\sqrt{\frac{1}{\lambda}}} (\tilde{0}) \) is following. We must remark that the proof is based partially in the proof of Theorem 7.3 and that the same result has been proved in [17] as a corollary of the fact that properly immersed \( \lambda \)-self-shrinkers of MCF are \( h \)-parabolic submanifolds of the Euclidean space \( \mathbb{R}^{n+m} \) weighted with the Gaussian density \( e^{h(r)} \), \( h(r) = -\frac{\lambda}{2} r^2 \).

**Theorem 7.5.** Let \( X : \Sigma^n \to \mathbb{R}^{n+m} \) be a complete properly immersed \( \lambda \)-self-shrinker in \( \mathbb{R}^{n+m} \) for the Mean Curvature Flow (MCF), with respect \( \tilde{0} \in \mathbb{R}^{n+m} \). Let us suppose that:

(1) Either \( \Sigma \) is confined into the ball \( X(\Sigma) \subseteq B^{n+m}_{\sqrt{\frac{1}{\lambda}}} (\tilde{0}) \).

(2) or \( \Sigma \) yields entirely out this ball, \( X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}_{\sqrt{\frac{1}{\lambda}}} (\tilde{0}) \).

Then \( \Sigma^n \) is compact and \( X : \Sigma \to S^{n+m-1}(\sqrt{\frac{1}{\lambda}}) \) is a minimal immersion.

**Proof.** Let us suppose first that \( X(\Sigma) \subseteq B^{n+m}_{\sqrt{\frac{1}{\lambda}}} (\tilde{0}) \). Then \( \sqrt{\frac{1}{\lambda}} \geq r(p) \ ∀ p \in \Sigma \). Hence we have that \( \| X^+ \|^2 \leq \| X \|^2 \leq \frac{1}{\lambda} \). Then, using Lemma 7.28
\[ \Delta^2 r^2 (x) = 2(n - \lambda \| X^+ \|^2) \geq 0 \]
On the other hand, as $X$ is proper and $\Sigma = X^{-1}(B^{n+m}(\overline{0}))$, then $\Sigma$ is compact and hence, parabolic. In conclusion, $r^2(x) = R^2 \forall x \in \Sigma$, for some $R \leq \sqrt{\frac{n}{\lambda}}$. But as $\Sigma$ is a $\lambda$-soliton for the MCF, then $R = \sqrt{\frac{n}{\lambda}}$ by Proposition 2.12.

Let us suppose now that $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}(\overline{0})$. This means that $\sqrt{\frac{n}{\lambda}} \leq r(p) \forall p \in \Sigma$ and hence, that there is not a point $p \in \Sigma$ such that $r(p) < \sqrt{\frac{n}{\lambda}}$. We apply Theorem 7.3 to conclude that $X: \Sigma \to S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ is a compact minimal immersion. □

Also as a corollary of Theorem 7.3 we have the following characterization of minimal spherical immersions

**Corollary 7.6.** Let $X: \Sigma^n \to \mathbb{R}^{n+m}$ be a complete and properly immersed $\lambda$-self-shrinker in $\mathbb{R}^{n+m}$ for the MCF, with respect $\overline{0} \in \mathbb{R}^{n+m}$.

Then, $X: \Sigma^n \to \mathbb{R}^{n+m}$ is a compact minimal immersion of a round sphere of radius $\sqrt{\frac{n}{\lambda}}$ centered at $\overline{0}$ if and only if $\inf_{\Sigma} r = \sqrt{\frac{n}{\lambda}}$.

**Remark.** Note that if either $\Sigma$ is confined into the ball $X(\Sigma) \subseteq B^{n+m}(\overline{0})$, or $\Sigma$ yields entirely out this ball, $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}(\overline{0})$, then by Theorem 7.5 we have that $\inf_{\Sigma} r = \sqrt{\frac{n}{\lambda}}$. Likewise, if either $\Sigma$ is confined into the ball $X(\Sigma) \subseteq B^{n+m}(\overline{0})$, or $\Sigma$ yields entirely out this ball, $X(\Sigma) \subseteq \mathbb{R}^{n+m} \setminus B^{n+m}(\overline{0})$, then by Theorem 7.5 we have that $\sup_{\Sigma} r = \sqrt{\frac{n}{\lambda}}$.

### 7.4. Comments on a classification of proper self-shrinkers for the MCF.

In [3] it was proved the following classification result for self-shrinkers with polynomial volume growth. We remark here that in [8] it was proved that properness of the immersion for self-shrinkers implies polynomial volume growth.

**Theorem 7.7.** Let $\Sigma^n \to \mathbb{R}^{n+m}$ be a complete $\lambda$-self-shrinker without boundary, polynomial volume growth and bounded norm of the second fundamental form by

$$\|A_{\Sigma}^{B^{n+m}}\|^2 \leq \lambda,$$

Then $\Sigma$ is one of the following:

1. $\Sigma$ is a round sphere $S^n(\sqrt{\frac{n}{\lambda}})$, (and hence $\|A_{\Sigma}^{B^{n+m}}\|^2 = \lambda$),
2. $\Sigma$ is a cylinder $S^k(\sqrt{\frac{k}{\lambda}}) \times \mathbb{R}^{n-k}$, (and hence $\|A_{\Sigma}^{B^{n+m}}\|^2 = \lambda$),
3. $\Sigma$ is an hyperplane, (and hence $\|A_{\Sigma}^{B^{n+m}}\|^2 = 0$).

We want to draw attention at this point on the following notion of separation of a submanifold:

**Definition 7.8.** We say that the sphere $S^{n+m-1}(\sqrt{\frac{n}{\lambda}})$ separates the $\lambda$-self-shrinker $X: \Sigma \to \mathbb{R}^{n+m}$ if

$$D_{\sqrt{\frac{n}{\lambda}}} = \left\{ p \in \Sigma : \|X(p)\| < \sqrt{\frac{n}{\lambda}} \right\} \neq \emptyset,$$

and

$$\Sigma \setminus D_{\sqrt{\frac{n}{\lambda}}} = \left\{ p \in \Sigma : \|X(p)\| > \sqrt{\frac{n}{\lambda}} \right\} \neq \emptyset,$$

**Remark.** When we consider any of the three proper and complete examples with $\|A_{\Sigma}^{B^{n+m}}\|^2 \leq \lambda$ in Theorem 7.7, the critical sphere of radius $\sqrt{\frac{n}{\lambda}}$ in $\mathbb{R}^{n+m}$ separates the self-shrinker $\Sigma$ unless $\Sigma$ is itself a round sphere $S^n(\sqrt{\frac{n}{\lambda}})$ and $\|A_{\Sigma}^{B^{n+m}}\|^2 = \lambda$. On the other hand,
Theorem 7.5 is telling us that non-separated $\lambda$-self-shrinkers by the critical sphere of radius $\sqrt{\frac{1}{2}}$ must be isometrically immersed in $S^n(\sqrt{\frac{1}{2}})$ as compact and minimal submanifolds.

In Theorem 7.10 of this section we will prove that the fact described in Remark 10 is still true when the squared norm of the second fundamental form of $\Sigma$ is bounded above by the greater constant $\frac{1}{2}\lambda$. More precisely, in Theorem 7.10 we will prove that the sphere of radius $\sqrt{\frac{1}{2}}$ separates any $\lambda$-self-shrinker properly immersed in $\mathbb{R}^{n+m}$ with $\|A_\Sigma\|^2 < \frac{3}{4}\lambda$ unless the self-shrinker is just the the $n$-sphere of radius $\sqrt{\frac{1}{2}}$.

To prove Theorem 7.10 we will make use of the classification provided by J. Simon, and S.S. Chern, M.P. Do Carmo and S. Kobayashi, for compact minimal immersions in the sphere, (see [35], [9], [2]), refined later by A.M. Li and J.M. Li, (see [20]). These results can be summarized in the following statement:

**Theorem 7.9** (Simon-Do Carmo-Chern-Kobayashi Classification after Li and Li).

Let $\varphi : (\Sigma^n, g) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$ be a compact and minimal isometric immersion.

1. If $m = 1$ or $m = 2$, let us suppose that $\|A_\Sigma^{S^{n+m-1}(1)}\|^2 \leq \frac{n}{2} = \frac{m-1}{2m-3}n$. Then,
   
   (a) either $A_\Sigma^{S^{n+m-1}(1)}$ is isometric to $S^n(1)$,
   
   (b) or either, (in case $m = 2$), $\|A_\Sigma^{S^{n+m-1}(1)}\|^2 = n$ and $(\Sigma^n, g)$ is isometric to a generalized Clifford torus $\Sigma^n = S^k(\sqrt{\frac{1}{2}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ immersed as an hypersurface in $S^{n+m}(1)$.

2. If $m \geq 3$, let us suppose that $\|A_\Sigma^{S^{n+m-1}(1)}\|^2 \leq \frac{an}{3}$, then,
   
   (a) either $(\Sigma^n, g)$ is isometric to $S^n(1)$, and $\|A_\Sigma^{S^{n+m-1}(1)}\|^2 = 0$
   
   (b) or when $n = 2$ and $m = 3$, then $(\Sigma^n, g)$ is isometric to the Veronese surface $S^2 = \mathbb{R}P^2(\sqrt{3})$ in $S^4(1)$, and $\|A_\Sigma^{S^{n+m-1}(1)}\|^2 = \frac{4}{3}$.

**Remark 6.** It is easy to check that the bound for the squared norm of the second fundamental form $\frac{an}{3}$, used in [20] and which do not depends on the codimension $m$, is bigger or equal than the bound $\frac{n-1}{2m-3}n$ used in [35], [9], [2], when $m \geq 3$. In fact, for all $n > 0$, the values are equal when $m = 3$ and $\frac{4n}{3} > \frac{n-1}{2m-3}n$ when $m > 3$.

Let us consider now $X : (\Sigma, g) \to (\mathbb{R}^{n+m}, g_{\text{can}})$ a complete and properly immersed $\lambda$-self-shrinker in $\mathbb{R}^{n+m}$. By Theorem 7.3 if the critical sphere of radius $\sqrt{\frac{1}{2}}$ does not separate $X(\Sigma)$, then $\Sigma$ is therefore compact and is minimally immersed in the round sphere $S^{n+m-1}(\sqrt{\frac{1}{2}})$ centered at $\bar{0}$.

We are going to present some computations to rescale the immersion $X$ in order to apply Theorem 7.9 to this situation. For that, we are interested in to know what is the relation between the squared norm $\|A_{\bar{X}}^{S^{n+m-1}(1)}\|^2$, (corresponding to the isometric immersion $\bar{X} : (\Sigma, \bar{g}) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$) and the squared norm $\|A_{\bar{X}}^{S^{n+m}}\|^2$, (which corresponds to the isometric immersion $X : (\Sigma, g) \to (S^{n+m-1}(\sqrt{\frac{1}{2}}), g_{S^{n+m-1}(\sqrt{\frac{1}{2}})})$).

The first thing to do is to relate the metrics on $\Sigma$, $g$ and $\bar{g}$. Note that, given the immersion $X : (\Sigma, g) \to (\mathbb{R}^{n+m}, g_{\text{can}})$, the rescaled map

$$\bar{X} : \Sigma \to \mathbb{R}^{n+m}, \ p \to \bar{X}(p) := \sqrt{\frac{n}{1}}X(p)$$

sends $\Sigma$ into $S^{n+m-1}(1)$, with codimension $m - 1$. Therefore,

$$\bar{X} : (\Sigma, \frac{\lambda}{n}g) \to (\mathbb{R}^{n+m}, g_{\text{can}})$$

is an isometric immersion, and in fact, $\bar{X} : (\Sigma, \frac{\lambda}{n}g) \to (S^{n+m-1}(1), g_{S^{n+m-1}(1)})$ realizes as a minimal immersion if $X$ is minimal. Hence $\bar{g} = \frac{\lambda}{n}g$. 


Moreover, it is straightforward to check from this that:

\[ \| A_{\Sigma}^{n+m-1}(1) \| = \frac{n}{\lambda} \| A_{\Sigma}^{n+m-1}(\sqrt{\lambda}) \| \]

and that

\[ \| A_{\Sigma}^{n+m} \| = \| A_{\Sigma}^{n+m-1}(\sqrt{\lambda}) \| + \lambda. \]

Then we conclude

(7.24) \[ \| \tilde{A}_{\Sigma}^{n+m-1}(1) \| = \frac{n}{\lambda} \| A_{\Sigma}^{n+m} \| - n. \]

With this last equation in hand, it is obvious that the bound for the squared norm of the second fundamental form

\[ \| A_{\Sigma}^{n+m-1}(1) \| \leq \frac{n}{2} - \frac{n}{m-1}. \]

is equivalent to the bound \( \| A_{\Sigma}^{n+m} \| \leq \frac{3m-4}{2m-\lambda} \).

Moreover, and in the same way, the bound for the squared norm of the second fundamental form given by \( \| A_{\Sigma}^{n+m-1}(1) \| \leq \frac{\lambda}{m} \) is equivalent to the bound \( \| A_{\Sigma}^{n+m} \| \leq \frac{\lambda}{2} \).

The previous comments allow us to state the following Theorem,

**Theorem 7.10.** Let \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a complete, connected and properly immersed \( \lambda \)-self-shrinker for the MCF with respect to \( \tilde{0} \in \mathbb{R}^{n+m} \). Let us suppose that

\[ \| A_{\Sigma}^{n+m} \| \leq \frac{5}{3} \lambda \]

Then, either

1. \( \Sigma^n \) is isometric to \( S^n(\sqrt{\lambda}) \) and \( \| A_{\Sigma}^{n+m} \| = \lambda \),
2. or, the sphere \( S^{n+m-1}(\tilde{0}) \) of radius \( \sqrt{\lambda} \) centered at \( \tilde{0} \in \mathbb{R}^{n+m} \) separates \( X(\Sigma) \).

**Remark p.** The bound \( \frac{\lambda}{3} \) is optimal in the following sense: the Veronese surface \( \Sigma^2 = \mathbb{R}P^2(\sqrt{3}) \) in \( \mathbb{R}^5 \) satisfies that \( \| A_{\Sigma}^{n+m} \| = \frac{\lambda}{3} \lambda \) and it is not separated by sphere \( S^{4}(\sqrt{\lambda}) \) of radius \( \sqrt{\lambda} \) centered at \( \tilde{0} \in \mathbb{R}^5 \).

**Proof.** We are going to see first that, if (1) is not satisfied, then it is satisfied (2). Namely, the fact that \( \Sigma^n \) is not isometric to \( S^n(\sqrt{\lambda}) \) or \( \| A_{\Sigma}^{n+m} \| \neq \lambda \), implies that the sphere \( S^{n+m-1}(\tilde{0}) \) of radius \( \sqrt{\lambda} \) centered at \( \tilde{0} \in \mathbb{R}^{n+m} \) separates \( X(\Sigma) \).

To see this, let us suppose that this sphere does not separates \( X(\Sigma) \). Then, by Theorem 7.9, \( X : (\Sigma, g) \rightarrow (S^{n+m-1}(\sqrt{\lambda}), g_{S^{n+m-1}(\sqrt{\lambda})}) \) is a compact and minimal immersion. Hence:

1. If \( m = 1 \), \( \Sigma^n \) is isometric to \( S^n(\sqrt{\lambda}) \) because \( X \) is a Riemannian covering and \( S^n(\sqrt{\lambda}) \) is simply connected, following the same arguments than in Corollaries 6.2 and 6.6. Hence \( \| A_{\Sigma}^{n+m} \| = \lambda \). But this is a contradiction with the assumption that \( \Sigma^n \) is not isometric to \( S^n(\sqrt{\lambda}) \) or \( \| A_{\Sigma}^{n+m} \| \neq \lambda \).
2. If \( m = 2 \), since \( \| A_{\Sigma}^{n+m} \| < \frac{\lambda}{3} \), \( \frac{\lambda}{3} \lambda \) then, applying Theorem 7.9 either (a) \( \Sigma \) is isometric to \( S^n(\sqrt{\lambda}) \) and \( \| A_{\Sigma}^{n+m} \| = \lambda \). But this is a contradiction with the assumption that \( \Sigma^n \) is not isometric to \( S^n(\sqrt{\lambda}) \) or \( \| A_{\Sigma}^{n+m} \| \neq \lambda \).

(b) or, \( \Sigma \) is isometric to the Clifford torus \( S^k(\sqrt{\frac{\lambda}{2\lambda}}) \times S^{n-k}(\sqrt{\frac{\lambda}{2\lambda}}) \) and \( \| A_{\Sigma}^{n+m} \| = 2\lambda \). But this is a contradiction with the hypothesis of norm of second fundamental form bounded from above by \( \| A_{\Sigma}^{n+m} \| < 2\lambda \).
(3) If \( m = 3 \), since \( \| A_{R}^{n+m} \|^2 < \frac{4}{3} \lambda \) then applying Theorem 7.9 either
(a) \( \Sigma \) is isometric to \( S^n\left( \frac{\sqrt{\lambda}}{R} \right) \) and \( \| A_{\Sigma}^{n+m} \|^2 = \lambda \). But this is a contradiction with the assumption that \( \Sigma^n \) is not isometric to \( S^n\left( \frac{\sqrt{\lambda}}{R} \right) \) or \( \| A_{\Sigma}^{n+m} \|^2 \neq \lambda \).
(b) or, \( \Sigma \) is isometric to the Veronese surface in \( S^4\left( \frac{\sqrt{\lambda}}{R} \right) \) and \( \| A_{\Sigma}^{n+m} \|^2 = \frac{2}{3} \lambda \).

But this is a contradiction with the hypothesis of \( \| A_{\Sigma}^{n+m} \|^2 < \frac{2}{3} \lambda \).

(4) If \( m > 3 \), then, applying Theorem 7.9, \( \Sigma \) should be isometric to \( S^n\left( \frac{\sqrt{\lambda}}{R} \right) \) and \( \| A_{\Sigma}^{n+m} \|^2 = \lambda \). But again this is a contradiction with the assumption that \( \Sigma^n \) is not isometric to \( S^n\left( \frac{\sqrt{\lambda}}{R} \right) \) or \( \| A_{\Sigma}^{n+m} \|^2 \neq \lambda \).

Conversely, if the sphere \( S^{n+m-1}(\vec{0}) \) of radius \( \sqrt{\lambda} \) centered at \( \vec{0} \in \mathbb{R}^{n+m} \) does not separate \( X(\Sigma) \), then, as we have argued before, by Theorem 7.5, \( X : (\Sigma, g) \rightarrow (S^{n+m-1}(\sqrt{\lambda}), g_{S^{n+m-1}(\sqrt{\lambda})}) \) is a compact and minimal immersion, and hence \( \tilde{X} : (\Sigma, \tilde{g}) \rightarrow (S^{n+m-1}(1), g_{S^{n+m-1}(1)}) \) realizes as a minimal immersion, with second fundamental form in the sphere satisfying \( \| \tilde{A}_{\Sigma}^{n+m-1}(1) \|^2 < \frac{2n}{3} \) because by hypothesis \( \| A_{\Sigma}^{n+m} \|^2 < \frac{4}{3} \lambda \). Therefore we apply Theorem 7.9 to conclude that

1. \( \Sigma^n \) should be isometric to \( S^n\left( \frac{\sqrt{\lambda}}{R} \right) \) and
2. \( \| A_{\Sigma}^{n+m} \|^2 = \lambda \)

\[ \square \]

8. Mean Exit Time, and Volume of IMCF-solitons

8.1. Mean Exit time on Solitons for IMCF.

We start studying the Mean Exit Time function on properly immersed solitons for IMCF \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \).

As in Subsection 7.1, let us consider the Poisson problem defined on extrinsic \( R \)-balls \( D_R \subseteq \Sigma \)

\[ \Delta_{\Sigma} E + 1 = 0 \text{ on } D_R, \]
\[ E_{|\partial D_R} = 0. \]

(8.1)

We saw that the solution of the Poisson problem (7.2) on a geodesic \( R \)-ball \( B^n(R) \) in \( \mathbb{R}^n \) is given by the radial function \( E_{\Sigma}^{n,n}(r) = \frac{\rho^{2n}}{2n} \).

As in Subsection 8.2, we shall consider the transplanted radial solution of (7.1) \( \bar{E}_R(r) \) to the extrinsic ball by mean the extrinsic distance function, so we have \( \bar{E}_R : D_R \rightarrow \mathbb{R} \) defined as \( \bar{E}_R(p) := \bar{E}(r(p)) \forall p \in D_R \). Our first result here is again a comparison for the Mean Exit Time function:

Proposition 8.1. Let \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a complete properly immersed soliton in \( \mathbb{R}^{n+m} \) for the IMCF, with constant velocity \( C \neq 0 \) and with respect \( 0 \in \mathbb{R}^{n+m} \). Let us suppose that \( X(\Sigma) \not\subseteq S^{n+m-1}(R) \) for any radius \( R > 0 \). Given the extrinsic ball \( D_R(\vec{0}) = \Sigma \cap B_{R}^{n+m}(\vec{0}) \), we have that the mean exit time function on \( D_R, E_R \), satisfies

\[ E_R(x) = \frac{Cn}{Cn - 1} \bar{E}_R(x) \forall x \in D_R \]

(8.2)
We have, as \( E_R(x) := E^{0,n}_R(r(x)) = \frac{R^2 - r(x)^2}{2n} \) and applying Lemma 2.2 that, on \( D_R \)

\[
\Delta^\Sigma E_R = \left( \hat{E}''_R(r) - \frac{1}{r} \hat{E}'_R(r) \right) \| \nabla^\Sigma r \|^2
\]

(8.3)

\[
+ \hat{E}'_R(r)\left( \frac{n+1}{r} + \langle \nabla R^{n+1} r, \hat{H}_\Sigma \rangle \right) = -1 - \frac{1}{n} \langle r \nabla R^{n+1} r, \hat{H}_\Sigma \rangle
\]

On the other hand, \( X(p) = r(p)\nabla R^{n+1}(p) \) for all \( p \in \Sigma \), being \( X(p) \) the position vector of \( p \) in \( R^{n+m} \). And, moreover, as we have that \( \| \nabla H_\Sigma(p) \|^2 = -C X \perp(p) \), then

\[
\langle r \nabla R^{n+1} r, \hat{H}_\Sigma \rangle = \langle X, \hat{H}_\Sigma \rangle = \langle X, -C \| \hat{H}_\Sigma \|^2 X \perp \rangle
\]

(8.4)

\[
= -C \| \hat{H}_\Sigma \|^2 \| X \| ^2 = -\frac{1}{C}.
\]

Equation (8.3) becomes

\[
\Delta^\Sigma E_R = -1 + \frac{1}{Cn} = \frac{1 - Cn}{Cn}
\]

(8.5)

Therefore,

\[
\Delta^\Sigma \frac{Cn}{Cn - 1} E_R = \frac{Cn}{Cn - 1} \Delta^\Sigma E_R = \frac{Cn - 1}{Cn - 1} \frac{1 - Cn}{Cn}
\]

(8.6)

\[
= -1 = \Delta^\Sigma E_R \text{ on } D_R
\]

and, applying the Maximum Principle,

\[
\frac{Cn}{Cn - 1} E_R = E_R \text{ on } D_R
\]

\[\square\]

As a consequence, we obtain again Corollary 4.4.

**Corollary 8.2.** Let \( X: \Sigma^n \rightarrow R^{n+m} \) be a complete and non-compact, properly immersed soliton in \( R^{n+m} \) for the IMCF, with constant velocity \( C \neq 0 \) and with respect \( \bar{0} \in R^{n+m} \). Then

\[
C \notin \{0, \frac{1}{n}\}
\]

We finalize this subsection with a characterization of solitons for the IMCF in terms of the mean exit time function.

**Theorem 8.3.** Let \( X: \Sigma^n \rightarrow R^{n+1} \) be a proper immersion. Let us suppose that \( X(\Sigma) \nsubseteq S^n(R) \) for any radius \( R > 0 \). Then, if for all extrinsic R-balls \( D_R(\bar{0}) \), we have that \( E_R = \alpha \hat{E}_R \), with \( \alpha \neq 1 \) and \( \alpha \neq 0 \), then \( X \) is a soliton for the IMCF with respect \( \bar{0} \in R^{n+1} \), with velocity \( C = -\frac{\alpha}{\alpha - 1} \). Hence, if \( \alpha \in (1, \infty) \), then \( X \) is a self-shrinker and if \( \alpha \in (0, 1) \), then \( X \) is a self-expander.

**Proof.** We have, as \( \hat{E}_R(x) := E^{0,n}_R(r(x)) = \frac{R^2 - r(x)^2}{2n} \) and applying Lemma 2.2 that, on \( D_R \), for all \( R > 0 \),

\[
\Delta^\Sigma E_R = -1 - \frac{1}{n} \langle X, \hat{H}_\Sigma \rangle
\]

(8.7)

Hence, as we are assuming that \( E_R = \alpha \hat{E}_R \) for all \( R > 0 \), we have

\[
\Delta^\Sigma \alpha E_R = -\alpha - \frac{\alpha}{n} \langle X, \hat{H}_\Sigma \rangle = -1
\]

(8.8)

Therefore, on \( \Sigma \),

\[
\langle X, \hat{H}_\Sigma \rangle = \langle X \perp, \hat{H}_\Sigma \rangle = \frac{1 - \alpha}{\alpha n}
\]

(8.9)
so \( \|\vec{H}\| \neq 0 \).

But \( \vec{H}_\Sigma = h\nu \) where \( \nu \) is the unit normal vector field pointed outward to \( \Sigma \), so

\[
\langle X, \vec{H}_\Sigma \rangle = \langle X, \nu \rangle h = \frac{1 - \alpha}{\alpha} n
\]

and therefore

\[
\text{(8.10)} \quad \frac{1}{n} \frac{\alpha}{1 - \alpha} \langle X, \nu \rangle \nu = \frac{1}{h} \nu
\]

Hence, as \( X^\perp = \langle X^+, \nu \rangle \nu = \langle X, \nu \rangle \nu \)

\[
\text{(8.11)} \quad \frac{\vec{H}}{\|\vec{H}\|^2} = \frac{1}{h} \nu = \frac{\alpha}{1 - \alpha} \frac{1}{n} \langle X, \nu \rangle \nu = \frac{\alpha}{1 - \alpha} X^\perp
\]

and \( X \) is a soliton with \( C = -\frac{\alpha}{1 - \alpha} n \). \( \square \)

**Remark q.** Note that \( \alpha \neq 0, 1 \). If \( \alpha = 0 \), then \( E_R = 0 \) for all radius \( R > 0 \), so \( \Sigma \) reduces to a point. On the other hand, if \( \alpha = 1 \), then \( \Sigma \) is minimal in \( \mathbb{R}^{n+1} \), (see [21]), and hence \( X \) cannot be a soliton for the IMCF, (see Remark [1]).

### 8.2. Volume of Solitons for IMCF

As a consequence or the proof above, and using the Divergence theorem we have the following result:

**Theorem 8.4.** Let \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a complete properly immersed soliton in \( \mathbb{R}^{n+m} \) for the IMCF, with constant velocity \( C \neq 0 \) and with respect \( \vec{0} \in \mathbb{R}^{n+m} \). Let us suppose that \( X(\Sigma) \nsubseteq S^{n+m-1}(R) \) for any radius \( R > 0 \). Given the extrinsic ball \( D_R(\vec{0}) \), we have

\[
\text{(8.12)} \quad \frac{\text{Vol}(\partial D_R)}{\text{Vol}(D_R)} \geq \frac{Cn - 1}{Cn} \frac{\text{Vol}(S^{n-1}(R))}{\text{Vol}(B^n(R))} \text{ for all } R > 0
\]

**Proof.** Integrating on the extrinsic ball \( D_R \) the equality \( \Delta^\Sigma \frac{C_n}{C_n - 1} \vec{E}_R = -1 \) and applying Divergence theorem as in Theorem 7.2 we obtain, as \( C \in \mathbb{R} \sim [0, \frac{1}{n}] \):

\[
\text{(8.13)} \quad \text{Vol}(D_R) = \int_{D_R} -\Delta^\Sigma \frac{C_n}{C_n - 1} \vec{E}_R = -\frac{C_n}{C_n - 1} \vec{E}_R'(R) \int_{\partial D_R} \|\nabla^\Sigma r\| d\sigma
\]

\[
\leq \frac{Cn}{Cn - 1} \frac{\text{Vol}(B^n(R))}{\text{Vol}(S^{n-1}(R))} \frac{\text{Vol}(\partial D_R)}{\text{Vol}(D_R)}
\]

\( \square \)

**Remark r.** Equality in inequality (8.12) for all radius \( R \leq R_0 \) implies that the inequality \( \int_{\partial D_R} \|\nabla^\Sigma r\| d\sigma \leq \text{Vol}(\partial D_R) \) becomes an equality for all \( R \leq R_0 \). This implies that \( \|\nabla^\Sigma r\| = 1 = \|\nabla^\Sigma r_0\| \) in the extrinsic ball \( D_{R_0} \), so \( \nabla^\Sigma r = \nabla^\Sigma r_0 = \nabla r \) in \( D_{R_0} \), and \( \Sigma \) is totally geodesic in \( D_{R_0} \). Hence, \( \vec{H}_\Sigma = \vec{0} \) in \( D_{R_0} \), which is not compatible with the fact that \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a properly immersed soliton in \( \mathbb{R}^{n+m} \) for the IMCF. Therefore, if \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) is a properly immersed soliton in \( \mathbb{R}^{n+m} \) for the IMCF, then inequality (8.12) must be strict.

**Corollary 8.5.** Let \( X : \Sigma^n \rightarrow \mathbb{R}^{n+m} \) be a properly immersed soliton in \( \mathbb{R}^{n+m} \) for the IMCF, with constant velocity \( C \neq 0 \) and with respect \( \vec{0} \in \mathbb{R}^{n+m} \). Let us suppose that \( X(\Sigma) \nsubseteq S^{n+m-1}(R) \) for any radius \( R > 0 \). Let us define the volume growth function

\[
f(t) := \frac{\text{Vol}(D_t)}{\text{Vol}(B^n(t))^{\frac{n}{n+m}}}
\]

Then, given \( r_1 > 0 \), \( f(t) \) is non decreasing for all \( t \geq r_1 > 0 \).
Proof. As \( \frac{d}{dt} \text{Vol}(D_t) \geq \text{Vol}(\partial D_t) \) by the co-area formula, we have, applying Theorem 6.4,
\[
\frac{d}{dt} \ln f(t) \geq \frac{\text{Vol}(\partial D_t)}{\text{Vol}(D_t)} - \frac{Cn - 1}{Cn} \frac{\text{Vol}(\partial S^{n-1}(t))}{\text{Vol}(B^n(t))} \geq 0
\]
\( \square \)

REFERENCES

[1] L. Alias, P. Mastrolia, M. Rigoli, Maximum Principles and Geometric Applications Springer Monographs in Maths, Springer Verlag, Berlin, Heidelberg, New York, 2016

[2] B. Y. Cheng Riemannian submanifolds: a survey [arXiv:1307.1875v1 [math.DG], 7 Jul 2013.

[3] H.-D. Cao, H. Li A Gap theorem for Self-shrinkers of the Mean Curvature Flow in Arbitrary Codimension Calc. Var., 46, (2013), 879–889.

[4] I. Castro, A. Lerma Lagrangian Homothetic Solitons for The Inverse Mean Curvature Flow Results Math., 71, (2017), 1109–1125

[5] I. Castro, A. Lerma Hamiltonian stationary self-similar solutions for Lagrangian Mean Curvature Flow in the Complex Euclidean Plane Proc. Amer. Math. Soc., 138, n. 5, (2010), 1821–1832

[6] I. Chavel. Eigenvalues in Riemannian geometry. Including a chapter by Burton Randol. With an appendix by Jozef Dodziuk. Pure and Applied Mathematics, 115. Academic Press Inc., Orlando, FL, 1984. xiv+362 pp. ISBN: 0-12-170640-0

[7] I. Chavel. Riemannian geometry. A modern introduction. Second edition. Cambridge Studies in Advanced Mathematics, 98. Cambridge University Press, Cambridge, 2006. xvi+471 pp. ISBN: 978-0-521-61954-7.

[8] Xu Cheng and Detang Zhou Volume estimate about shrinkers, Proc. Amer. Math. Soc., 141, n. 2, (2013), 687–696.

[9] S. S. Chern, M. Do Carmo and S. Kobayashi Minimal Submanifolds of a Sphere with Second Fundamental Form of Constant Length. Shing-Shen Chern Selected Papers, 393–409, Berlin-Heidelberg-New York 1978.

[10] M. P. do Carmo. Riemannian geometry. Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory & Applications. Birkhäuser Boston Inc., MA, 1992. xiv+300 pp. ISBN: 0-8176-3490-8.

[11] G. Drugan, H. Lee and G. Wheeler Solitons for the Inverse Mean Curvature Flow Pacific. Jour. Math., 284, n. 2, (2016), 309–316

[12] E. B. Dynkin Markov processes Springer Verlag, Berlin, Heidelberg, New York, 1965.

[13] M. P. Cavalcante and J.M. Espinar Halfspace type theorems for self-shrinkers Bull. London Math. Soc., 48, (2016), 242-250

[14] R. Greene and H. Wu, Function theory on manifolds which possess a pole Lecture Notes in Math., vol. 699, Springer-Verlag, Berlin and New York, 1979.

[15] A. Grigor’yan. Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Amer. Math. Soc. (N.S.) 36 (1999), no. 2, 135–249.

[16] R. Z. Has’minskii, Probabilistic representation of the solution of some differential equations Proc. 6th All Union Conf. on Theor. Probability and Math. Statist. (Vilnius 1960), (1960)

[17] A. Hurtado, V. Palmer and C. Rosales Parabolicity criteria and characterization results for submanifolds of bounded mean curvature in model manifolds with weights, Preprint, 2018

[18] J. Jost. Riemannian geometry and geometric analysis. Third edition. Universitext. Springer-Verlag, Berlin, 2002. xiv+532 pp. ISBN: 3-540-42627-2 53-02

[19] J. Lee Introduction to smooth manifolds. Universitext. Springer-Verlag, Berlin,

[20] A.M. Li and J.M. Li An intrinsic rigidity theorem for minimal submanifolds in a sphere Archiv. Math., 58, (1992), 582–594.

[21] S. Markvorsen. On the mean exit time form a minimal submanifold Journal of Differential Geometry, 2, (1989), 1–8.

[22] C. Mantegazza Lecture Notes on Mean Curvature Flow Progress in Mathematics, 290. Birkhauser, Springer Basel AG, 2011.

[23] S. Markvorsen and M. Min-Oo, Global Riemannian Geometry: Curvature and Topology, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, Berlin (2003).

[24] Patrick McDonald. Exit times, moment problems and comparison theorems. Potential Anal., 38, (2013)1365–1372.

[25] S. Markvorsen and V. Palmer Transience and Capacity of minimal submanifolds Geometric and Functional Analysis, 13, (2003), 915–933

McM Patrick McDonald and Robert Meyers. Dirichlet spectrum and heat content. J. Funct. Anal., 200(1):150–159, 2003.
[27] N. Nadirashvili, *Hadamard and Calabi-Yau’s conjectures on negatively curved and minimal surfaces*, Inventiones Mathematicae, 126, n. 3, (1995), 457–465.

[28] B. O’Neill, *Semi-Riemannian Geometry; With Applications to Relativity* Pure and Applied Mathematics Series, Academic Press, San Diego, 1983.

[29] V. Palmer, *On deciding whether a submanifold is parabolic of hyperbolic using its mean curvature* Haesen S., Verstraelen L. (eds) Topics in Modern Differential Geometry. Atlantis Transactions in Geometry, vol 1. Atlantis, 2017.

[30] V. Palmer, *Mean exit time from convex hypersurfaces*, Proc. Amer. Math. Soc. 126 (1998), 2089–2094.

[31] V. Palmer, *Isoperimetric inequalities for extrinsic balls in minimal submanifolds and their applications*, Jour. London Math. Soc. 60 (2) (1999), 607–616.

[32] S. Pigola and M. Rimoldi, *Complete self-shrinkers confined into some regions of the space*, Ann. Glob. Anal. Geom. 45 (2014), 47–65.

[33] M. Rimoldi, *On a classification theorem for self-shrinkers*, Proc. Amer. Math. Soc. 124 (2014), 3605–3613.

[34] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs Volume 149, American Mathematical Society, 1996.

[35] J. Simon, *Minimal varieties in Riemannian manifolds*, Ann. Math., 88, (1968), 62–105.

[36] T. Takahasi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan, 18, n. 4, (1966), 380–385.

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