A REFINEMENT OF RASMUSSEN’S S-IN VARIANT

ROBERT LIPSHITZ AND SUCHARIT SARKAR

Abstract. In [LSa] we constructed a spectrum-level refinement of Khovanov homology. This refinement induces stable cohomology operations on Khovanov homology. In this paper we show that these cohomology operations commute with cobordism maps on Khovanov homology. As a consequence we obtain a refinement of Rasmussen’s slice genus bound \( s \) for each stable cohomology operation. We show that in the case of the Steenrod square \( Sq^2 \) our refinement is strictly stronger than \( s \).

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1. Introduction

In [LSa] we gave a space-level refinement of Khovanov homology. That is, given a link \( L \) we produced a family of suspension spectra \( \mathcal{X}^j_{Kh}(L) \) with the property that \( \tilde{H}^i(\mathcal{X}^j_{Kh}(L)) = \)
$Kh^{i,j}(L)$. In this paper, we use these Khovanov spectra to give a family of potential improvements of Rasmussen’s celebrated $s$ invariant [Ras10]; and we show that at least one of these is, in fact, an improvement on $s$.

These refinements are fairly easy to state. To wit, let $Kh^{i,j}(L; \mathbb{F})$ denote Khovanov homology with coefficients in a field $\mathbb{F}$. There is a spectral sequence $Kh^{i,j}(L; \mathbb{F}) \Rightarrow \mathbb{F}^{2(L)}$, coming from a filtered chain complex $(C_*, \mathcal{F}_*)$. (Here, $\mathcal{F}_*$ is a descending filtration, with $Kh^{i,j}(L; \mathbb{F}) = H_*(\mathcal{F}_i C/\mathcal{F}_{i+2} C)$.) Originally, this was defined by Lee [Lee05], for fields $\mathbb{F}$ of characteristic different from 2. A variant which works for all fields (in fact, all rings) was studied by Bar-Natan [BN05] and Turner [Tur06]. We will work with this variant, which is reviewed in Section 2.

The Rasmussen $s$ invariant for a knot $K$ is defined by

$$s^\mathbb{F}(K) = \max\{q \in 2\mathbb{Z} + 1 \mid i_*: H_*(\mathcal{F}_q C) \to H_*(C) \cong \mathbb{F}^2 \text{ surjective}\} + 1$$

and gives a lower bound for the slice genus: $|s^\mathbb{F}(K)| \leq 2g_4(K)$. It is shown in [MTV07] that if $\text{char}(\mathbb{F}) \neq 2$ then it makes no difference whether one uses the Bar-Natan deformation or the Lee deformation in the definition of $s$; see the discussion around Theorem 2.5, below.

To define the improvements, let $\alpha: \tilde{H}^*(\cdot; \mathbb{F}) \to \tilde{H}^{*+n}(\cdot; \mathbb{F})$ be a stable cohomology operation (for some $n > 0$).

**Definition 1.1.** Fix a knot $K$. Call an odd integer $q$ $\alpha$-half-full if there exist elements $\tilde{a} \in Kh^{-n,q}(K; \mathbb{F}), \tilde{a} \in Kh^{0,q}(K; \mathbb{F}), a \in H_0(\mathcal{F}_q; \mathbb{F})$ and $\overline{a} \in H_0(C; \mathbb{F})$ satisfying:

1. the map $\alpha: Kh^{-n,q}(K; \mathbb{F}) = \tilde{H}^{-n}(\mathcal{X}_{Kh}^q; \mathbb{F}) \to \tilde{H}^0(\mathcal{X}_{Kh}^q; \mathbb{F}) = Kh^{0,q}(K; \mathbb{F})$ sends $\tilde{a}$ to $\tilde{a}$;
2. the map $H_0(\mathcal{F}_q; \mathbb{F}) \to Kh^{0,q}(K; \mathbb{F}) = H_0(\mathcal{F}_q/\mathcal{F}_{q+2}; \mathbb{F})$ sends $a$ to $\tilde{a}$;
3. the map $H_0(\mathcal{F}_q; \mathbb{F}) \to H_0(C; \mathbb{F})$ sends $a$ to $\overline{a}$; and
4. $\overline{a} \in H_0(C; \mathbb{F}) = \mathbb{F} \oplus \mathbb{F}$ is a generator.

(Note that $\tilde{a}$ and $\tilde{a}$ are allowed to be zero.)

Call an odd integer $q$ $\alpha$-full if there exist elements $\tilde{a}, \tilde{b} \in Kh^{-n,q}(K; \mathbb{F}), \tilde{a}, \tilde{b} \in Kh^{0,q}(K; \mathbb{F}), a, b \in H_0(\mathcal{F}_q; \mathbb{F})$ and $\overline{a}, \overline{b} \in H_0(C; \mathbb{F})$ satisfying:

1. the map $\alpha: Kh^{-n,q}(K; \mathbb{F}) = \tilde{H}^{-n}(\mathcal{X}_{Kh}^q; \mathbb{F}) \to \tilde{H}^0(\mathcal{X}_{Kh}^q; \mathbb{F}) = Kh^{0,q}(K; \mathbb{F})$ sends $\tilde{a}, \tilde{b}$ to $\tilde{a}, \tilde{b}$;
2. the map $H_0(\mathcal{F}_q; \mathbb{F}) \to Kh^{0,q}(K; \mathbb{F}) = H_0(\mathcal{F}_q/\mathcal{F}_{q+2}; \mathbb{F})$ sends $a, b$ to $\tilde{a}, \tilde{b}$;
3. the map $H_0(\mathcal{F}_q; \mathbb{F}) \to H_0(C; \mathbb{F})$ sends $a, b$ to $\overline{a}, \overline{b}$; and
4. $\overline{a}, \overline{b} \in H_0(C; \mathbb{F}) = \mathbb{F} \oplus \mathbb{F}$ form a basis.

(1) For justification of this equality in the case of fields of characteristic 2, see Proposition 2.6, below. Note that, while it is claimed in [MTV07] that $s^\mathbb{F}$ is independent of $\mathbb{F}$, there is a gap in the proof of [MTV07, Proposition 3.2].
(Again, note that \( \hat{a}, \hat{b}, \bar{a} \) and \( \bar{b} \) are allowed to be zero.)

In other words, \( q \) is \( \alpha \)-half-full if the following configuration exists:

\[
\begin{array}{c}
\langle \bar{a} \rangle \\
\downarrow \\
Kh^{-n,q}(K; \mathbb{F}) \xrightarrow{\alpha} Kh^{0,q}(K; \mathbb{F}) \\
\downarrow \\
\langle a \rangle \\
\downarrow \\
H_0(\mathcal{F}_q; \mathbb{F}) \rightarrow H_0(C; \mathbb{F}).
\end{array}
\]

while \( q \) is \( \alpha \)-full if the following configuration exists:

\[
\begin{array}{c}
\langle \hat{a}, \hat{b} \rangle \\
\downarrow \\
Kh^{-n,q}(K; \mathbb{F}) \xrightarrow{\alpha} Kh^{0,q}(K; \mathbb{F}) \\
\downarrow \\
\langle \alpha \rangle \\
\downarrow \\
H_0(\mathcal{F}_q; \mathbb{F}) \rightarrow H_0(C; \mathbb{F}).
\end{array}
\]

Definition 1.2. For a knot \( K \), define \( r_\alpha^\pm(K) = \max\{q \in 2\mathbb{Z} + 1 \mid q \) is \( \alpha \)-half-full\}\} + 1 and \( s_\alpha^\pm(K) = \max\{q \in 2\mathbb{Z} + 1 \mid q \) is \( \alpha \)-full\}\} + 3. If \( \overline{K} \) denotes the mirror of \( K \), define \( r_\alpha^\pm(K) = -r_\alpha^\pm(\overline{K}) \) and \( s_\alpha^\pm(K) = -s_\alpha^\pm(\overline{K}) \).

It is immediate from their definitions that \( s_\alpha^\pm \) and \( r_\alpha^\pm \) are knot invariants. The reason they are of interest is the following:

Theorem 1. Let \( \alpha \) be a stable cohomology operation and \( S \) a connected, embedded cobordism in \([0, 1] \times S^3\) from \( K_1 \) to \( K_2 \). If \( S \) has genus \( g \) then

\[
|s_\alpha^\pm(K_1) - s_\alpha^\pm(K_2)| \leq 2g \\
|r_\alpha^\pm(K_1) - r_\alpha^\pm(K_2)| \leq 2g \\
|s_\alpha^\pm(K_1) - s_\alpha^\pm(K_2)| \leq 2g \\
|r_\alpha^\pm(K_1) - r_\alpha^\pm(K_2)| \leq 2g.
\]

So, each of the number \( |r_\alpha^\pm|, |r_\alpha^\pm|, |s_\alpha^\pm|, |s_\alpha^\pm| \) gives a slice genus bound:

\[
\max\{|r_\alpha^\pm(K)|, |r_\alpha^\pm(K)|, |s_\alpha^\pm(K)|, |s_\alpha^\pm(K)|\} \leq 2g_4(K).
\]

(Note that if \( s_\alpha^\pm \) (respectively \( r_\alpha^\pm \)) differs from \( s \), then \( s_\alpha^\pm \) is not a slice homomorphism, as two homomorphisms to \( \mathbb{Z} \) cannot have a bounded difference.)

Of course, the numbers \( r_\alpha^\pm \) and \( s_\alpha^\pm \) are only interesting if they sometimes give better information than \( s \). In Section 5 we show:

Theorem 2. Let \( Sq_2^2 : \tilde{H}^\ast(\cdot; \mathbb{F}_2) \rightarrow \tilde{H}^{*+2}(\cdot; \mathbb{F}_2) \) denote the second Steenrod square. Then there are knots \( K \) so that \( |s_\alpha^\pm(K)| > |s(K)| \).

Theorem 3. Let \( Sq_1^1 : \tilde{H}^\ast(\cdot; \mathbb{F}_2) \rightarrow \tilde{H}^{*+1}(\cdot; \mathbb{F}_2) \) denote the first Steenrod square. Then there are knots \( K \) so that \( s_\alpha^\pm(K) \neq s^{F_2}(K) \).

A key step in the proof of Theorem 1 is that the cobordism maps on Khovanov homology commute with cohomology operations:
**Theorem 4.** Let $S$ be a smooth cobordism in $[0,1] \times S^3$ from $L_1$ to $L_2$, and let $F_S: Kh^*(L_1) \to Kh^{*+\chi(S)}(L_2)$ be the map associated to $S$ in [Jac04] (see also [Kho02, BN05, Kho06]). Let $\alpha: \tilde{H}^*(\cdot; \mathbb{F}) \to \tilde{H}^{*+n}(\cdot; \mathbb{F})$ be a stable cohomology operation. Then the following diagram commutes up to sign:

$$
\begin{align*}
Kh^{i,j}(L_1; \mathbb{F}) &= \tilde{H}^i(X_{Kh}^j(L_1); \mathbb{F}) \xrightarrow{\alpha} \tilde{H}^{i+n}(X_{Kh}^j(L_1); \mathbb{F}) = Kh^{i+n,j}(L_1; \mathbb{F}) \\
Kh^{i,j+\chi(S)}(L_2; \mathbb{F}) &= \tilde{H}^i(X_{Kh}^{j+\chi(S)}(L_2); \mathbb{F}) \xrightarrow{\alpha} \tilde{H}^{i+n}(X_{Kh}^{j+\chi(S)}(L_2); \mathbb{F}) = Kh^{i+n,j+\chi(S)}(L_2; \mathbb{F}).
\end{align*}
$$

(1.1)

In particular:

**Corollary 5.** Let $A_p$ denote the modulo-$p$ Steenrod algebra. Then the cobordism map $F_S: Kh^*(L_1; \mathbb{F}_p) \to Kh^{*+\chi(S)}(L_2; \mathbb{F}_p)$ associated to a smooth cobordism $S$ from $L_1$ to $L_2$ is a homomorphism of $A_p$-modules; that is, $Kh(\cdot; \mathbb{F}_p)$ is a projective functor of $A_p$-modules.

This paper is organized as follows. **Section 2** reviews the Bar-Natan complex and the $s$-invariant defined using it, collecting some results we will need later. **Section 3** starts by reviewing the construction of the Khovanov homotopy type. The rest of **Section 3** is devoted to proving **Theorem 4**. In **Section 4** we recall the definitions of $r_\alpha^\pm$ and $s_\alpha^\pm$ and prove **Theorem 1**. **Section 5** contains some computations of the invariants $r_\alpha^\pm(K)$ and $s_\alpha^\pm(K)$ for some particular $\alpha$’s and $K$’s, and in particular gives proofs of **Theorem 2** and **Theorem 3**. We conclude, in **Section 6**, with some remarks and questions.

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2. **The $s$ invariant from Bar-Natan’s complex**

In this section we review some results on Bar-Natan’s filtered Khovanov complex. For our purposes, it serves as an analogue of the Lee deformation but which works over any coefficient ring. Almost all of the ideas and many of the results in this section are drawn from [Lee05, Ras10, BN05, Tur06, MTV07], but a few of the results have not appeared in exactly the form we need them. We start by reviewing the filtered complex itself and its basic properties in **Subsection 2.1**, and then turn in **Subsection 2.2** to the properties of the $s$-invariants obtained from the filtered complex.

2.1. **Bar-Natan’s complex.** Like the Khovanov complex, the Bar-Natan complex is obtained by feeding the cube of resolutions for a knot diagram into a particular Frobenius algebra:
Definition 2.1. The Bar-Natan Frobenius algebra is the deformation of $H^*(S^2)$ with multiplication $m$ given by
\[
    x_+ \otimes x_+ \mapsto x_+ \quad x_+ \otimes x_- \mapsto x_- \quad x_- \otimes x_+ \mapsto x_- \quad x_- \otimes x_- \mapsto x_-,
\]
comultiplication $\Delta$ by
\[
    x_- \mapsto x_- \otimes x_- \quad x_+ \mapsto x_+ \otimes x_- + x_- \otimes x_+ - x_+ \otimes x_+,
\]
unit $\iota$ by
\[
    1 \mapsto x_+
\]
and counit $\eta$ by
\[
    x_+ \mapsto 0 \quad x_- \mapsto 1.
\]
(These maps are obtained from the usual Khovanov maps from [Kho00] by adding the terms in red. These maps make sense over any ring; but we will continue to assume that we are working over a field $\mathbb{F}$.)

Feeding the cube of resolutions for a knot diagram $K$ into this Frobenius algebra gives a chain complex $C(K; \mathbb{F})$, which we will sometimes call the Bar-Natan complex. As an $\mathbb{F}$-module, the underlying chain group is identified with the Khovanov complex $KC(K; \mathbb{F})$. The Bar-Natan differential increases homological grading on the Khovanov complex by 1 and does not decrease the quantum grading. Write $C = \bigoplus_{i,j} C^{i,j}$, where $i$ denotes the homological grading and $j$ denotes the quantum grading on the Khovanov complex. Consider the subcomplex $F_q = \bigoplus_{j \geq q} C^{i,j}$. This gives a (finite) filtration
\[
    \cdots \subset F_{q+2} \subset F_q \subset F_{q-2} \subset \cdots \subset C.
\]

Theorem 2.2. [Tur06] If $K$ is a knot then $H_*(C(K; \mathbb{F})) = \mathbb{F} \oplus \mathbb{F}$, with both copies in homological grading 0. More generally, for $L$ an $\ell$-component link, $H_*(C(L; \mathbb{F})) \cong \mathbb{F}^{2\ell}$; a basis for $H_*(C(L; \mathbb{F}))$ is canonically identified with the set of orientations for $L$.

Sketch of proof. This is a special case of [MTV07, Propositions 2.3 and 2.4]. Following [Tur06, MTV07], write $x_1 = x_+ - x_-$ and consider the new basis $\{x_1, x_-\}$ for the Bar-Natan Frobenius algebra. In this basis, the multiplication, the comultiplication, the unit and the counit become
\[
\begin{align*}
    x_1 \otimes x_1 \xrightarrow{m} x_1 & \quad x_1 \otimes x_- \xrightarrow{m} 0 \quad x_- \otimes x_1 \xrightarrow{m} 0 \quad x_- \otimes x_- \xrightarrow{m} x_- \\
    x_1 \xrightarrow{\Delta} -x_1 \otimes x_1 & \quad x_- \xrightarrow{\Delta} x_- \otimes x_- \\
    1 & \xrightarrow{\iota} x_1 + x_- \\
    x_1 & \xrightarrow{\eta} -1 \quad x_- \xrightarrow{\eta} 1.
\end{align*}
\]
(2.1)

Since this change of basis diagonalizes the Frobenius algebra, the rest of the argument from [Lee05] goes through essentially unchanged. \qed
The following is essentially due to Bar-Natan:

**Theorem 2.3. [BN05]** The filtered complex $C$ is projectively functorial with respect to link cobordisms, in the sense that given a link cobordism $S$ from $L_1$ to $L_2$ there is an associated chain map $F_S: C(L_1) \to C(L_2)$, well-defined up to multiplication by $\pm 1$. The chain map $F_S$ preserves the homological grading and increases the quantum grading by at least $\chi(S)$, and is well-defined up to filtered homotopy\(^{(ii)}\) (and sign). The map of associated graded complexes induced by $F_S$ agrees with the usual cobordism map on Khovanov homology (as defined in [Jac04]).

**Sketch of proof.** It suffices to show that $C(L)$ is obtained by composing Bar-Natan’s formal-complex-valued invariant \([BN05, Definition 6.4]\) and some functor $\text{Cob}^3_\mathcal{L} \to \text{Kom}_\mathbb{Z}$. With coefficients in $\mathbb{F}_2$ instead of $\mathbb{Z}$ this is essentially \([BN05, Exercise 9.5]\), and is discussed further in \([Tur06]\). To obtain the result over $\mathbb{Z}$, by \([BN05, Theorem 5]\) it suffices verify that the topological field theory corresponding to the Frobenius algebra in Definition 2.1 satisfies the $S$, $T$ and $4\text{Tu}$ relations.

The argument is essentially the same as \([BN05, Proposition 7.2]\). The $S$ relation—that spheres evaluate to 0—follows from the fact that $\eta \circ \iota = 0$. The $T$ relation—that tori evaluate to 2—corresponds to the composition

$$1 \xrightarrow{\iota} x_+ \xrightarrow{\Delta} x_+ \otimes x_- + x_- \otimes x_+ - x_+ \otimes x_+ \xrightarrow{m} x_- + x_- - x_+ \xrightarrow{\eta} 2.$$ 

For the $4\text{Tu}$ relation, with notation as in \([BN05, Proposition 7.2]\), it suffices to show that $L = R$. Computing,

$$L = (\Delta \circ \iota) \otimes \iota \otimes \iota + \iota \otimes \iota \otimes (\Delta \circ \iota)$$

$$= [x_+ \otimes x_- + x_- \otimes x_+] \otimes x_+ \otimes x_+ + [x_+ \otimes x_+ \otimes x_- + x_- \otimes x_+ \otimes x_+]$$

$$= x_+ \otimes x_- \otimes x_+ + x_- \otimes x_+ \otimes x_+ - x_+ \otimes x_+ \otimes x_+ + x_+ \otimes x_+ \otimes x_- + x_- \otimes x_+ \otimes x_+$$

$$= x_{+++} + x_{++-} + x_{+--} + x_{---} - 2x_{+++}.$$ 

Given a vector space $V$, let $s_{i,j}: V^{\otimes n} \to V^{\otimes n}$ be the map which exchanges the $i^{\text{th}}$ and $j^{\text{th}}$ factors. Then $R = s_{23} \circ L$; but $L$ is invariant under $s_{23}$.  

The following is well-known to experts:

**Proposition 2.4.** Let $S$ be a connected cobordism from a knot $K_1$ to a knot $K_2$. Then the induced map $F_S: C(K_1) \to C(K_2)$ is a quasi-isomorphism.

\(^{(ii)}\)i.e., homotopies which decrease the homological grading by 1 and increase the quantum grading by at least $\chi(S)$. 
Proof. The analogous statement using the Lee deformation was proved by Rasmussen [Ras10]. Using Turner’s change of basis (Equation (2.1)), Rasmussen’s argument applies without essential changes. □

2.2. The s invariants.

Theorem 2.5. [MTV07] Let \( \mathbb{F} \) be a field of characteristic different from 2, so Rasmussen’s \( s \) over \( \mathbb{F} \) is well-defined. Then the numbers

\[
\begin{align*}
  s_{\min}^\mathbb{F}(K) &= \max\{q \in 2\mathbb{Z} + 1 \mid i_* : H_*(\mathcal{F}_q C) \to H_*(C) \cong \mathbb{F}^2 \text{ surjective} \} \\
  s_{\max}^\mathbb{F}(K) &= \max\{q \in 2\mathbb{Z} + 1 \mid i_* : H_*(\mathcal{F}_q C) \to H_*(C) \cong \mathbb{F}^2 \text{ nonzero} \}
\end{align*}
\]

defined using Bar-Natan’s complex agree with Rasmussen’s \( s \) invariants defined in [Ras10, Definition 3.1] (but using the field \( \mathbb{F} \) instead of \( \mathbb{Q} \)).

Proof. This is immediate from [MTV07, Proposition 3.1]. □

Proposition 2.6. Let \( K \) be a knot and \( \mathbb{F} \) a field. Then \( s_{\max}^\mathbb{F}(K) = s_{\min}^\mathbb{F}(K) + 2 \).

Proof. In the case that \( \text{char}(\mathbb{F}) \neq 2 \) this follows from Theorem 2.5 and [Ras10, Proposition 3.3] (but with \( \mathbb{F} \) used in place of \( \mathbb{Q} \)). For the case that \( \text{char}(\mathbb{F}) = 2 \) we need a further argument.

First, we show that \( s_{\max}^\mathbb{F}(K) \neq s_{\min}^\mathbb{F}(K) \). There is an involution \( I \) of the modulo 2 Bar-Natan Frobenius algebra defined by

\[
I(x_+) = x_- \\
I(x_-) = x_+.
\]

or equivalently

\[
I(x_+) = x_- \\
I(x_-) = x_- + x_+.
\]

The map \( I \) induces an involution \( I_* : C(K) \to C(K) \). Since \( I \) exchanges \( x_- \) and \( x_+ \), \( I_* \) is the nontrivial involution of \( H_*(C(K)) = \mathbb{F} \oplus \mathbb{F} \).

Let \( a \in \mathcal{F}_{s_{\min}^\mathbb{F}(K)} C \) be a cycle so that \( H_*(C(K)) = \mathbb{F}\langle a, I_*(a) \rangle \). In particular, \( a \) is chosen so that \( a + I_*(a) \) represents a nontrivial homology class. The map \( I \) respects the \( q \)-filtration; moreover, \( I \) induces the identity map on the associated graded complex. It follows that the lowest-order terms of \( a \) and \( I(a) \) are the same. Thus, \( a + I(a) \in \mathcal{F}_{q+2} C(K) \), so \( s_{\max}^\mathbb{F}(K) \geq s_{\min}^\mathbb{F}(K) + 2 \).

Next, we argue as in the proof of [Ras10, Proposition 3.3] that \( s_{\max}^\mathbb{F}(K) \leq s_{\min}^\mathbb{F}(K) + 2 \). Let \( U \) denote the unknot. Trivially, \( H_*(C(U)) = C(U) = \mathbb{F}\langle x_+, x_- \rangle \), with both \( x_+ \) and \( x_- \) lying in filtration level \( q = -1 \). Now, \( C(U) \) is a filtered ring, and choosing a basepoint \( p \) on \( K \) converts \( C(K) \) and \( H_*(C(K)) \) into filtered modules over \( C(U) \), where multiplication is filtered of degree \(-1\). The element \( a \), above, is homologous to one of the canonical generators \( a' \); without loss of generality, suppose that in the generator \( a' \), the component containing \( p \) is labeled by \( x_- \). Then \( x_-I(a) = 0 \) and \( x_-a = a \). Hence

\[
s_{\min}^\mathbb{F}(K) = s(a) = s(x_+(a + I_*(a))) \geq s(x_-) + s(a + I_*(a)) - 1 = s_{\max}^\mathbb{F}(K) - 2,
\]
as desired.

In view of Proposition 2.6 write \( s^F(K) = s^F_{\text{min}}(K) + 1 = s^F_{\text{max}}(K) - 1 \).

**Corollary 2.7.** Let \( K_1 \) and \( K_2 \) be knots and let \( S \) be a connected cobordism in \([0,1] \times S^3\) from \( K_1 \) to \( K_2 \). Then the invariant \( s^F \) defined using the Bar-Natan complex satisfies

\[
|s^F(K_1) - s^F(K_2)| \leq -\chi(S)
\]

In particular, for a knot \( K \),

\[
|s^F(K)| \leq 2g_4(K).
\]

**Proof.** In the case that \( \text{char}(F) \neq 2 \), this follows from Theorem 2.5 (i.e., \([\text{MTV07, Proposition 3.1}]\)) and Rasmussen’s result \([\text{Ras10, Theorem 1}]\) (except with \( F \) in place of \( \mathbb{Q} \)). But, using this as an opportunity to review Rasmussen’s argument, which will be adapted to a slightly more complicated setting below, we give a direct proof.

Let \( q = s^F_{\text{min}}(K_1) \). By Theorem 2.3, \( F_S : C(K_1) \to C(K_2) \) is a filtered map of filtration \( \chi(S) \), so we have the following commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{F}_q C(K_1) & \overset{F_S}{\longrightarrow} & \mathcal{F}_{q+\chi(S)} C(K_2) \\
\downarrow i & & \downarrow i \\
C(K_1) & \overset{F_S}{\longrightarrow} & C(K_2)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
H_0(\mathcal{F}_q C(K_1)) & \overset{F_S}{\longrightarrow} & H_0(\mathcal{F}_{q+\chi(S)} C(K_2)) \\
\downarrow i_* & & \downarrow i_* \\
H_0(C(K_1)) & \overset{\cong}{\longrightarrow} & H_0(C(K_2)).
\end{array}
\]

Choose \( a, b \in H_0(\mathcal{F}_q C(K_1)) \) so that \( i_*(a), i_*(b) \in H_0(C(K_1)) \) form a basis. By Proposition 2.4, \( F_S(i_*(a)) = i_*(F_S(a)) \) and \( F_S(i_*(b)) = i_*(F_S(b)) \) in \( H_0(C(K_2)) \) also form a basis. So, \( s^F_{\text{min}}(K_2) \geq q + \chi(S) = s^F_{\text{min}}(K_1) + \chi(S) \), or equivalently \( -\chi(S) \geq s^F_{\text{min}}(K_1) - s^F_{\text{min}}(K_2) \). Viewing \( S \) as a cobordism from \( K_2 \) to \( K_1 \) instead and applying the same argument gives

\[
-\chi(S) \geq |s^F_{\text{min}}(K_2) - s^F_{\text{min}}(K_1)|.
\]

Now, applying Proposition 2.6 gives the first half of the result. The second half follows from the trivial computation that \( s^F(U) = 0 \).

**Corollary 2.8.** The invariant \( s^F(K)/2 \) defines a homomorphism from the smooth concordance group to \( \mathbb{Z} \).

**Proof.** In the case that \( \text{char}(F) \neq 2 \) this follows from Theorem 2.5 (i.e., \([\text{MTV07, Theorem 4.2}]\)) and Rasmussen’s result \([\text{Ras10, Theorem 2}]\). For the general case, it follows from Corollary 2.7 that \( s^F \) descends to a function on the smooth concordance group. So, it only remains to prove that this function is a homomorphism, i.e., that

\[
s^F(K_1 \# K_2) = s^F(K_1) + s^F(K_2).
\]
This follows by the same argument as [Ras10, Proposition 3.12]. The main points are that there is a short exact sequence
\[ 0 \to H_*(C(K_1#K_2)) \to H_*(C(K_1)) \otimes H_*(C(K_2)) \to H_*(C(K_1#r(K_2))) \to 0 \]
where \( r \) denotes the reverse and both maps in the sequence are filtered of \( q \)-degree \(-1\) ([Ras10, Lemma 3.8]); and that for the mirror \( \overline{K} \) of a knot \( K \),
\[ s^F_{\min}(\overline{K}) = -s^F_{\max}(K) \]
([Ras10, Proposition 3.10]). The proofs of both statements carry over to the Bar-Natan complex without change. The first statement implies that \( s^F_{\min}(K_1#K_2) \leq s^F_{\min}(K_1) + s^F_{\min}(K_2) + 1 \). The second then implies (by considering the mirrors) that \( s^F_{\max}(K_1) + s^F_{\max}(K_2) \leq s^F_{\max}(K_1#K_2) + 1 \). Using Proposition 2.6 then gives \( s^F_{\min}(K_1)+s^F_{\min}(K_2)+4 \leq s^F_{\min}(K_1#K_2) + 3 \). Combining this with the first inequality gives \( s^F_{\min}(K_1#K_2) = s^F_{\min}(K_1) + s^F_{\min}(K_2) + 1 = s^F(K_1) + s^F(K_2) - 1 \) and \( s^F_{\max}(K_1#K_2) = s^F_{\max}(K_1) + s^F_{\max}(K_2) - 1 = s^F(K_1) + s^F(K_2) + 1 \), proving the result.

3. Cobordism maps commute with cohomology operations

We start this section with a brief introduction to the Khovanov homotopy type in Subsection 3.1. The rest of this section is devoted to showing that the cobordism maps on Khovanov homology commute with stable cohomology operations (like Steenrod squares). To prove this, we simply need to associate a map of Khovanov spectra to each cobordism and verify that the induced map on cohomology agrees with the homomorphism given in [Kho02, Jac04].

Like the cobordism maps on Khovanov homology, the cobordism maps on the Khovanov spectra are defined by composing maps associated to elementary cobordisms. We conjecture that up to (stable) homotopy the map of Khovanov spectra is independent of the decomposition into elementary cobordisms—i.e., is an isotopy invariant of the cobordism—but we will not show that here; see also Remark 3.1.

In the process of proving that the Khovanov homotopy type is a knot invariant, in [LSa] we associated maps to Reidemeister moves. These maps were homotopy equivalences, and in particular induce isomorphisms on Khovanov homology; but we did not verify in [LSa] that these isomorphisms agree with the standard ones. So, we give this verification in Subsection 3.2.

Given this, it remains to define maps associated to cups, caps, and saddles. Doing so is fairly straightforward; see Subsection 3.3. With these ingredients in hand, Theorem 4 follows readily; see Subsection 3.4.

3.1. A brief review of the Khovanov homotopy type. In this subsection, we summarize the construction of the Khovanov suspension spectrum from [LSa]. The construction is somewhat involved, and at places, fairly technical, so we only present the general outline,
and highlight some of the salient features. The reader may find it helpful to consult the examples in [LSa, Section 9] in conjunction with the discussion here.

Fix a link \( L \); the construction of the suspension spectrum \( \mathcal{X}_{Kh}(L) \) depends on several choices that are listed in Subsection 3.2. Of them, we treat choice (1), namely the choice of a ladybug matching, as a global choice. Choices (2)–(4) essentially correspond to the choice of a link diagram \( D \); those are the usual choices that one makes in defining the Khovanov chain complex; after making those choices, we can talk about a Khovanov chain complex \( KC(D) \), which is a chain complex equipped with distinguished set of generators, which we often refer to as the Khovanov generators.

The suspension spectrum \( \mathcal{X}_{Kh}(L) \) is constructed as (the formal desuspension of) the suspension spectrum of a CW complex \( |\mathcal{C}_K(D)| \). The cells in \( |\mathcal{C}_K(D)| \) (except the basepoint) canonically correspond to the Khovanov generators; furthermore, the correspondence induces an isomorphism between the reduced cellular cochain complex of \( |\mathcal{C}_K(D)| \) and \( KC(D) \) (after an overall grading shift).

The central idea is to construct an intermediate object \( \mathcal{C}_K(D) \), which is a framed flow category. We define a partial order on the Khovanov generators by declaring \( b \prec a \) if there is a sequence of differentials in \( KC(D) \) from \( b \) to \( a \). To such a pair \( b \prec a \), we associate a framed \((\text{gr}_h(a) - \text{gr}_h(b) - 1)\)-dimensional moduli space \( M(a, b) \) subject to the following conditions:

1. If \( a \) appears in the Khovanov differential applied to \( b \) with coefficient \( n_{ab} \), then \( M(a, b) \) consists of \( n_{ab} \) points, counted with sign. (Recall that a framed 0-manifold is a disjoint union of signed points.)
2. If \( c \prec b \prec a \), then \( M(a, b) \times M(b, c) \) is identified with a certain subset of \( \partial M(a, c) \), and the framings are coherent. (See, e.g., [LSa, Definition 3.12] for a precise version of this condition.)

To such a framed flow category \( \mathcal{C}_K(D) \), one can associate an explicit CW complex \( |\mathcal{C}_K(D)| \). The construction was introduced by Cohen-Jones-Segal in [CJS95, pp. 309–312], and is described in more detail in [LSa, Subsection 3.3]. If one has a framed \( k \)-dimensional manifold in \( \mathbb{R}^n \), by the Pontryagin-Thom construction, one gets a (stable) map from \( S^n \) to \( S^{n-k} \), and thereby a CW complex with a 0-cell, an \((n-k)\)-cell, and an \((n+1)\)-cell. The Cohen-Jones-Segal construction is a refined version of the Pontryagin-Thom construction, allowing one to describe arbitrary CW complexes at the cost of working with manifolds with corners (organized into flow categories). One needs to choose a few parameters in order to pass from a framed flow category to a CW complex; these parameters are are listed as choice (7) in Subsection 3.2.

So all that remains is to construct the Khovanov flow category \( \mathcal{C}_K(D) \). The Khovanov chain complex \( KC(D) \) is modelled after the cube chain complex \( C^\ast_{\text{cube}}(N) \), which is obtained from the cube \( (\mathbb{Z} \to \mathbb{Z})^\otimes N \) after infusing the arrows with a sign assignment that makes every face anti-commute. One can define a framed flow category for the cube by defining the
moduli spaces to be permutahedra. This is choice (5) from Subsection 3.2. We define the Khovanov flow category by modelling it on the cube flow category $\mathcal{C}(N)$, via choice (6); in particular, we ensure that the each moduli space is a disjoint union of permutahedra.

More concretely, if $b \prec a$, and $\text{gr}_h(a) - \text{gr}_h(b) = 1$, we define $\mathcal{M}(a, b)$ to be a point (with the framing determined by the sign assignment). If $b \prec a$, and $\text{gr}_h(a) - \text{gr}_h(b) = 2$, there could be two or four broken flowlines from $b$ to $a$. In the former case, we define $\mathcal{M}(a, b)$ to be an interval. In the latter case, we define it to be a disjoint union of two intervals; there is a choice of matching that one needs to do in this case, and that is precisely the choice of the ladybug matching. The rest of the construction proceeds inductively: When we construct the $n$-dimensional moduli space $\mathcal{M}(a, b)$, all the lower dimensional moduli spaces in the Khovanov flow category $\mathcal{C}_K(D)$ have already been constructed, and the lower dimensional moduli spaces admit covering maps to the corresponding moduli spaces in the cube flow category $\mathcal{C}(N)$. Therefore, by (the precise version of) Condition (2) above, $\partial \mathcal{M}(a, b)$ has already been constructed, and it admits a covering map to the boundary of the $n$-dimensional permutahedron. When $n \geq 3$, by simple-connectedness, this forces $\partial \mathcal{M}(a, b)$ to be a disjoint union of boundaries of permutahedra, and therefore, we can define $\mathcal{M}(a, b)$ to be the corresponding disjoint union of permutahedra. When $n = 2$, there is a possible obstruction; however, an exhaustive case check annihilates that possibility. Thus we can define the Khovanov flow category, and via the Cohen-Jones-Segal construction, the Khovanov suspension spectrum $X_{Kh}(L)$.

3.2. The Reidemeister maps agree. Fix a link $L$. Recall from [LSa, Definition 5.5 and Section 6] that the construction of the suspension spectrum $X_{Kh}(L)$ depends on several choices:

1. A choice of ladybug matching (left or right).
2. An oriented link diagram $D$ for $L$, with $N$ crossings.
3. An ordering of the crossings of $D$.
4. A sign assignment $s$ for the cube $\mathcal{C}(N)$.
5. A neat embedding $\iota$ and a framing $\varphi$ for the cube flow category $\mathcal{C}_C(N)$ relative to $s$.
6. A framed neat embedding $\kappa$ of the Khovanov flow category $\mathcal{C}_K(L)$ relative to some $d$. This framed neat embedding is a perturbation of $(\iota, \varphi)$.
7. Integers $A, B$ and real numbers $\epsilon, R$.

It is proved in [LSa, Section 6] that, up to homotopy equivalence, $X_{Kh}(L)$ is independent of these auxiliary choices. We will view the ladybug matching as a global choice. The goal of this section is to prove that, on the level of homology, the rest of these homotopy equivalences agree with the isomorphisms on Khovanov homology.

Recall that $X_{Kh}(L)$ is a formal de-suspension of the realization $|\mathcal{C}_K(L)|$ in the sense of [LSa, Definition 3.23] of the Khovanov flow category $\mathcal{C}_K(L)$. As noted, this realization depends on the auxiliary choices above. For $D$ a link diagram, let $|\mathcal{C}_K(D)|_{o,s,\iota,\varphi,s,A,B,\epsilon,R}$ denote the Khovanov space defined using the ordering $o$ of the crossings of $D$, sign assignment...
For any set of auxiliary choices, there is a canonical identification of the cells of $|\mathcal{C}_K(D)|$ and generators of the Khovanov complex $KC(D)$, intertwining the cellular cochain differential on $|\mathcal{C}_K(D)|$ and the Khovanov differential on $KC(D)$. This gives a canonical identification between $\tilde{H}^*(|\mathcal{C}_K(D)|_{o,s,\iota,\varphi,\kappa,A,B,\epsilon,R})$ and $Kh(D)$. We need to show that the stable homotopy equivalences associated to changes of auxiliary data respect this identification. We start with choices (5)--(7):

**Proposition 3.1.** Let $D$ be a link diagram. Then for any two choices of auxiliary data $(\iota, \varphi, \kappa, A, B, \epsilon, R)$ and $(\iota', \varphi', \kappa', A', B', \epsilon', R')$ the stable homotopy equivalence

$$|\mathcal{C}_K(D)|_{o,s,\iota,\varphi,\kappa,A,B,\epsilon,R} \simeq |\mathcal{C}_K(D)|_{o,s,\iota',\varphi',\kappa',A',B',\epsilon',R'}$$

furnished by [LSa, Proposition 6.1] induces the identity map on Khovanov homology.

**Proof.** It is immediate from the proof of [LSa, Lemma 3.25] that the maps associated to changing $\epsilon$ and $R$ are isomorphisms of CW complexes respecting the identification of cells with generators of $KC(D)$; in particular, they induce the identity map on Khovanov homology. As in the proof of [LSa, Lemma 3.26], increasing $A'$, decreasing $B'$ or increasing $d$ has the effect of suspending the CW complex, and the map of realizations is the identity map

$$\Sigma^d|\mathcal{C}_K(D)|_{o,s,\iota,\varphi,\kappa,A,B,\epsilon,R} \xrightarrow{\cong} |\mathcal{C}_K(D)|_{o,s,\iota',\varphi',\kappa',A',B',\epsilon',R'}$$

In particular, the induced map on Khovanov homology is the identity.

By [LSa, Lemmas 3.22 and 4.13], any choices of $\kappa$, $\iota$ and $\varphi$ lead to isotopic framings of $\mathcal{C}_K(D)$. Again, from the proof of [LSa, Lemma 3.25], the map associated to isotoping the framing of $\mathcal{C}_K(D)$ is an isomorphism of CW complexes respecting the identification of cells with generators of $KC(D)$. So, again, these maps induce the identity map on Khovanov homology. \hfill \Box

The fact that the isomorphisms agree for changes in sign assignment and ordering of crossings is a bit more subtle. First, recall that a *sign assignment* is a choice of signs for the edges of the *cube chain complex*

$$C^*_{cube}(N) = (\mathbb{Z} \xrightarrow{\cong} \mathbb{Z})^\otimes N$$

so that $\partial^2 = 0$, i.e., so that each face anti-commutes. In [LSa], the sign assignment and ordering of crossings enter the construction of $|\mathcal{C}_K(D)|$ as follows. The ordering of crossings specifies a map from Khovanov generators to vertices of the hypercube $\{0,1\}^N$. Using this map, the sign assignment then specifies framings for the 0-dimensional moduli spaces in $\mathcal{C}_K(D)$. Thus, changing the ordering of the crossings is equivalent to changing the sign assignment, and we prefer to view the operation in the latter way (cf. [LSa, Proof of Proposition 6.1]).
Lemma 3.2. Given sign assignments \( s, s' \) for \( C^*_{\text{cube}}(N) \) there are exactly two gauge transformations \( t_1, t_2 \) from \( s \) to \( s' \). Moreover, \( t_2 = -t_1 \).

Proof. This is a straightforward induction argument, showing that the value of \( t \) on the vertex \((0, \ldots, 0)\), say, and the fact that \( v \mapsto t(v) \cdot v \) is a chain map uniquely determine \( t \). □

Proposition 3.3. Let \( \Phi_{s_0, s_1} \) denote the map of Khovanov homotopy types associated in [LSa, Proposition 6.1] to a change of sign assignments and \( F_{s_0, s_1} \) the map of Khovanov chain complexes induced by a change of sign assignments, as alluded to in [BN05, p. 1457]. Then the following diagram commutes up to sign:

\[
\begin{array}{ccc}
\tilde{H}^i(X^j_{Kh}(D, s_0)) & \cong & \tilde{H}^i(X^j_{Kh}(D, s_1)) \\
\downarrow & & \downarrow \\
Kh^{i,j}(D, s_0) & \xrightarrow{F_{s_0, s_1}} & Kh^{i,j}(D, s_1).
\end{array}
\]

Proof. By Lemma 3.2, it suffices to show that the map \( \Phi_{s_0, s_1} \) on cochain complexes takes each generator to \( \pm \) itself, where the signs are determined by a map of \( C^*_{\text{cube}}(N) \). We recall the definition of \( \Phi_{s_0, s_1} \). Consider the diagram \( D \sqcup U \) obtained by taking the disjoint union of \( D \) with a 1-crossing diagram for the unknot. Choose \( U \) so that the 0-resolution of \( U \) has two components and the 1-resolution of \( U \) has one component. Let \( \mathcal{C}_K(D \sqcup U)_+ \) denote the full subcategory generated by the objects in which the (one or two) circles corresponding to \( U \) are labeled by \( x_+ \). Write \( \mathcal{C}_K(D \sqcup U)_0 \) to be the subcategory of \( \mathcal{C}_K(D \sqcup U)_+ \) in which we take the 0-resolution of \( U \) and \( \mathcal{C}_K(D \sqcup U)_1 \) to be the subcategory of \( \mathcal{C}_K(D \sqcup U)_+ \) in which we take the 1-resolution of \( U \). Then each of \( \mathcal{C}_K(D \sqcup U)_0 \) and \( \mathcal{C}_K(D \sqcup U)_1 \) is isomorphic to \( \mathcal{C}_K(D) \). Choose a sign assignment \( s \) for \( D \sqcup U \) so that the induced sign assignment for \( \mathcal{C}_K(D \sqcup U)_i \) is \( s_i \). Thus, if we choose the embedding data compatibly, \( |\mathcal{C}_K(D \sqcup U)_0| = |\mathcal{C}_K(D)|_{s_0} \) and \( |\mathcal{C}_K(D \sqcup U)_1| = \Sigma |\mathcal{C}_K(D)|_{s_1} \). Moreover:

1. There is a cofibration sequence

\[
|\mathcal{C}_K(D)|_{s_0} \to |\mathcal{C}_K(D \sqcup U)_+|_s \to \Sigma |\mathcal{C}_K(D)|_{s_1}.
\]

2. The cohomology \( \tilde{H}^\ast(|\mathcal{C}_K(D \sqcup U)_+|_s) \) is trivial, so \( |\mathcal{C}_K(D \sqcup U)_+|_s \) is contractible.

In the Khovanov homology literature, it seems to be more typical to phrase results in terms of the standard sign assignment, but with different orderings of crossings; see, for instance, [BN05, p. 1457]. In our style, the change of ordering homomorphism of [BN05, p. 1457] is given as follows. Given sign assignments \( s, s' \) choose a map \( t \) from the vertices of \( C^*_{\text{cube}}(N) \) to \( \{\pm 1\} \) so that the map \( (C^*_{\text{cube}}(N), \partial_s) \to (C^*_{\text{cube}}(N), \partial_{s'}) \) given by \( v \mapsto t(v) \cdot v \) is a chain map. (We will call \( t \) a gauge transformation from \( s \) to \( s' \).) Then, the map of Khovanov complexes takes a generator \( x \) lying over a vertex \( v \) of the hypercube to \( t(v) \cdot x \).
The map $\Phi_{s_0,s_1} : \mathcal{X}^{ij}_{\mathcal{K}}(D, s_1) \to \mathcal{X}^{ij}_{\mathcal{K}}(D, s_0)$ is the Puppe map associated to the cofibration sequence in (1), which is an isomorphism by (2). (The suspension coming from the Puppe construction cancels with the suspension in $|\mathcal{C}_K(D)_{s_0}|$.)

The induced map $\Phi_{s_0,s_1}^*$ is the connecting homomorphism

$$\partial : Kh(D, s_0) = \tilde{H}^*|\mathcal{C}_K(D)|_{s_0}) \to \tilde{H}^*|\mathcal{C}_K(D)|_{s_1}) = Kh(D, s_1)$$

associated to the short exact sequence of chain complexes

$$0 \to KC(D, s_0) \xrightarrow{i} KC(D \amalg U, s)_+ \xrightarrow{p} KC(D, s_0) \to 0.$$ 

For $v \in \{0, 1\}^N$ let $t(v)$ be the sign assigned by $s$ to the edge from $(v, 0)$ to $(v, 1)$. Explicitly, the relevant maps are given by

$$i((v, x)) = (v, x) \otimes x_+ \quad p((v, x) \otimes x_+ \otimes x_+ = (v, x) \quad p((v, x) \otimes x_+ = 0$$

$$\delta((v, x) \otimes x_+ \otimes x_+) = t(v)(v, x) \otimes x_+ + \delta_{s_0}(v, x) \otimes x_+ \otimes x_+$$

$$\delta((v, x) \otimes x_+ = \delta_{s_1}(v, x) \otimes x_+.$$ (Here, the $x_+$'s are labels for the 0- or 1-resolution of $U$, and $\delta$ is the Khovanov differential for $KC(D \amalg U, s)_+$, and $(v, x)$ is a Khovanov generator for $KC(D)$ lying over the vertex $v \in \{0, 1\}^n.$) So, the connecting homomorphism $\partial$ is given by $\partial(v, x) = t(v)(v, x)$, which is as alluded to in [BN05, p. 1457].

In light of Propositions 3.1 and 3.3, it is safe to drop the auxiliary data $(o, s, \iota, \varphi, \kappa, A, B, \epsilon, R)$ from both the notation and the discussion, and we will do so for the rest of the paper.

**Proposition 3.4.** Let $D$ and $D'$ be link diagrams representing $L$, and choose a sequence of Reidemeister moves connecting $D$ and $D'$. Let $\Phi : \mathcal{X}^{ij}_{\mathcal{K}}(D') \to \mathcal{X}^{ij}_{\mathcal{K}}(D)$ denote the stable homotopy equivalence given by [LSa, Theorem 1] and let $F : Kh^{i,j}(D) \to Kh^{i,j}(D')$ denote the isomorphism given by [Kho02], [Jac04] or [BN05]. Then the following diagram commutes up to sign:

$$\begin{array}{ccc}
\tilde{H}^i(\mathcal{X}^{ij}_{\mathcal{K}}(D)) & \xrightarrow{\Phi^*} & \tilde{H}^i(\mathcal{X}^{ij}_{\mathcal{K}}(D')) \\
\cong & & \cong \\
Kh^{i,j}(D) & \xrightarrow{F} & Kh^{i,j}(D').
\end{array}$$

(Here, the vertical isomorphisms are also given by [LSa, Theorem 1].)

In [Kho02], Khovanov associated to any even integer $2m$ a ring $H^m$, and to any $(2m_1, 2m_2)$-tangle $T$ a bimodule $\mathcal{M}(T)$ over the rings $H^{m_1}$ and $H^{m_2}$. As he observed in [Kho06], these tangle invariants can be used to state and prove locality properties for the cobordism maps. **Proposition 3.4** will follow easily from the next lemma, which is along the same lines:
Lemma 3.5. The maps on Khovanov homology associated in [LSa, Propositions 6.2–6.4] to Reidemeister moves are induced by maps of tangles in the following sense. Let $T_k$ and $T'_k$ be the tangles shown in Figure 3.1, so $T_k$ differs from $T'_k$ by a Reidemeister move. Suppose that $D$ and $D'$ are knot diagrams so that $D'_k$ is obtained by replacing a copy of $T_k$ inside $D$ by a copy of $T'_k$. Let

$$
\Phi^*_k: \text{Kh}^{i,j}(D') \cong \widetilde{H}^i(\mathcal{X}^{i,j}_{\text{Kh}}(D')) \to \widetilde{H}^i(\mathcal{X}^{i,j}_{\text{Kh}}(D)) \cong \text{Kh}^{i,j}(D)
$$

be the map induced by the map of spaces $\Phi$ defined in [LSa, Proposition 6.2–6.4]. Then there are bimodule maps $F_k: \mathcal{M}(D'_k) \to \mathcal{M}(D_k)$ so that

$$
\Phi^*_k = \text{Id} \otimes F_k: H_*(\mathcal{M}(D \setminus T_k) \otimes \mathcal{M}(T'_k)) \to H_*(\mathcal{M}(D \setminus T_k) \otimes \mathcal{M}(T_k)).
$$

Proof. It is straightforward to verify that the maps (on homology) given in [LSa, Section 6] make sense on the level of tangles. The key point is that the maps are defined locally, by canceling acyclic subcomplexes and quotient complexes given by requiring certain circles in $T'_k$ to be decorated by one of $x_+$ or $x_-$. The same definitions define acyclic subcomplexes and quotient complexes of $\mathcal{M}(T'_k)$.

Proof of Proposition 3.4. It suffices to check the result when $D$ and $D'$ differ by a single Reidemeister move. In this case, by Lemma 3.5, the maps $\Phi^*$ are induced by isomorphisms of Khovanov’s tangle invariants. By definition, the maps $F$ are also induced by isomorphisms of tangle invariants. But the tangles $T_k, T'_k$ involved are invertible, so by [Kho06, Corollary 2] any two such isomorphisms agree up to sign.

3.3. Maps associated to cups, caps and saddles. Other than Reidemeister moves, there are three kinds of elementary cobordisms of knots: cups, saddles and caps, which correspond to index 0, 1 and 2 critical points of Morse functions, respectively.

3.3.1. Cups and caps. Let $L$ be a link diagram and $L' = L \amalg U$ the disjoint union of $L$ and an unknot; place $U$ so that it does not introduce new crossings. Passing from $L$ to
Each of $\text{Lemma 3.6}$. to be the projection and inclusion, respectively. There are maps
\[
F_\cup: Kh^{i,j}(L) \to Kh^{i,j+1}(L') \\
F_\cap: Kh^{i,j}(L') \to Kh^{i,j+1}(L)
\]
on Khovanov homology associated to a cup and a cap, respectively. We want to define maps
\[
\Phi_\cup: X_{Kh}^{j+1}(L') \to X_{Kh}^j(L) \\
\Phi_\cap: X_{Kh}^{j+1}(L) \to X_{Kh}^j(L')
\]
associated to a cup and a cap so that the induced maps on cohomology are $F_\cup$ and $F_\cap$.

In each resolution $L'_v$ of $L'$ there is a component $U_v$ corresponding to $U$. We can write $KC(L') = KC(L)_+ \oplus KC(L)_-$, where $KC(L)_+$ (resp. $KC(L)_-$) has basis those generators of $KC(L')$ in which $U_v$ is labeled by $x_+$ (resp. $x_-$). Each of $KC(L)_+$ and $KC(L)_-$ are canonically isomorphic to $KC(L)$. The cobordism maps on Khovanov homology associated to cups and caps are defined on the chain level by
\[
F_\cup: KC(L) \xrightarrow{\cong} KC(L)_+ \xhookrightarrow{} KC(L') \\
F_\cap: KC(L') \xrightarrow{} KC(L')_- \xrightarrow{\cong} KC(L),
\]
to be the inclusion and projection, respectively [Jac04, Figure 15]. Note that in the special case that $L$ is empty these maps restrict to the unit and counit on $H^*(S^2)$, respectively.

Similarly, the flow category $\mathcal{C}_K(L')$ is a disjoint union $\mathcal{C}_K(L') = \mathcal{C}_K(L)_+ \amalg \mathcal{C}_K(L)_-$, where $\mathcal{C}_K(L)_+$ (resp. $\mathcal{C}_K(L)_-$) is the full subcategory of $\mathcal{C}_K(L')$ whose objects are decorated resolutions $(v, x)$ with $U_v$ labeled by $x_+$ (resp. $x_-$). Thus, $|\mathcal{C}_K(L)| \cong |\mathcal{C}_K(L)_+| \lor |\mathcal{C}_K(L)_-|$. Each of $\mathcal{C}_K(L)_+$ and $\mathcal{C}_K(L)_-$ are canonically isomorphic to $\mathcal{C}_K(L)$. Define
\[
\Phi_\cup: |\mathcal{C}_K(L')| = |\mathcal{C}_K(L)_+| \lor |\mathcal{C}_K(L)_-| \to |\mathcal{C}_K(L)_+| \xrightarrow{\cong} |\mathcal{C}_K(L)| \\
\Phi_\cap: |\mathcal{C}_K(L)| \xrightarrow{\cong} |\mathcal{C}_K(L)_-| \xhookrightarrow{} |\mathcal{C}_K(L)_+| \lor |\mathcal{C}_K(L)_-| = |\mathcal{C}_K(L')|
\]
to be the projection and inclusion, respectively.

**Lemma 3.6.** The map on cohomology induced by $\Phi_\cup$ (resp. $\Phi_\cap$) is the cobordism maps $F_\cup$ (resp. $F_\cap$) associated in [Kho02, Jac04] to the cup (resp. cap) cobordism.

**Proof.** This is immediate from the definitions. \qed

### 3.3.2. Saddles

The remaining elementary cobordism is a saddle, which corresponds to making the local change shown Figure 3.2. The map $F_\ast$ on Khovanov homology associated to a saddle is defined as follows. Let $L_0$ and $L_1$ be the link diagrams before and after the saddle. Let $L$ be the link diagram obtained by replacing the region in which the saddle move is occurring by a crossing $c$, as in Figure 3.2. Then $KC(L_1)$ is a subcomplex of $KC(L)$.
Figure 3.2. A saddle. Parts (a) and (b) are related by an oriented saddle move. (There is another valid oriented saddle, obtained by reflecting the pictures vertically.) Notice that (a) is the 0-resolution of the crossing shown in (c), while (b) is the 1-resolution of the crossing shown in (c). There is no way to orient the crossing in (c) coherently with (a) or (b).

with corresponding quotient complex $KC(L_0)$. The map $F_s$ is defined to be the connecting homomorphism in the long exact sequence

$$
\cdots \to KC^{i-1,j-1}(L_1) \to KC^{i+a,j+b}(L) \to KC^{i,j}(L_0) \xrightarrow{F_s} KC^{i,j-1}(L_1) \to \cdots ,
$$

where $a$ and $b$ are integers which depend on how the orientation for $L$ is chosen. Equivalently, $F_s$ is the map occurring in the skein exact sequence associated to the crossing $c$; see, for instance, [BN05, p. 1472]. (Here, the gradings on $KC(L_0)$ and $KC(L_1)$ relate nicely because the saddle is oriented; and the grading on $KC(L)$ does not relate as nicely to these, because the orientations of $L_0$ and $L_1$ do not agree with an orientation of $L$.)

Essentially the same construction carries through on the space level. Briefly, a space-level version of the skein sequence is given in [LSa, Section 7], and we define $\Phi_s$ to be the Puppe map associated to this sequence. In more detail, the flow category $\mathcal{C}_K(L_0)$ is a downward-closed subcategory of $\mathcal{C}_K(L)$ (in the sense of [LSa, Definition 3.29]), with corresponding upward-closed subcategory $\mathcal{C}_K(L_1)$. So, there is a cofibration sequence

$$
|\mathcal{C}_K(L_0)| \to |\mathcal{C}_K(L)| \to |\mathcal{C}_K(L_1)|.
$$

The Puppe construction gives a map

$$
\Phi_s : |\mathcal{C}_K(L_1)| \to \Sigma |\mathcal{C}_K(L_0)|,
$$

and we define this to be the map of spaces associated to a saddle cobordism. Putting in the gradings, the map $\Phi_s$ has the form

$$
\Phi_s : \mathcal{X}^i_{Kh}(L_1) \to \mathcal{X}^{i+1}_{Kh}(L_0).
$$

Lemma 3.7. The map on cohomology induced by $\Phi_s$ is the cobordism maps $F_s$ associated in [Kho02, Jac04] to the saddle cobordism.

Proof. Again, this is immediate from the definitions. □
Remark 3.1. The reader might wonder where the sign ambiguity in the maps on Khovanov homology appears. Of course, we have not shown that the map of Khovanov spectra associated to a link cobordism is independent of the decomposition into elementary cobordisms, so the question is somewhat premature. But one possibility is that the ambiguity comes from the ambiguity in identifying \( \ast(X_{Kh}(L)) \) with \( KC(L) \): to make this identification one must orient the cells in \( \ast(X_{Kh}(L)) \), and that choice of orientation may be unnatural. If so, the map of Khovanov homotopy types could be completely well-defined, not just up to sign.

3.4. Completion of the proof of Theorem 4.

Proof of Theorem 4. Fix a diagram \( D_i \) for \( L_i \). We can view \( S \) as given by a sequence \( S_n \circ S_{n-1} \circ \cdots \circ S_1 \) of Reidemeister and Morse moves connecting \( D_1 \) to \( D_2 \). The map \( F_S \) is the composition \( F_S_n \circ \cdots \circ F_{S_1} \), so it suffices to prove that Diagram (1.1) commutes (up to sign) for a single \( S_i \). If \( S_i \) is a Reidemeister move then by Propositions 3.1, 3.3 and 3.4 the map \( \Phi_S \) of Khovanov homotopy types given by \([LSa, Theorem 1]\) induces the map \( \pm F_S \) on homology. Thus, by naturality of \( \alpha \), Diagram (1.1) commutes up to sign. If \( S_i \) is a Morse cobordism then by Lemma 3.6 or Lemma 3.7 there is a map \( \Phi_S \) of Khovanov homotopy types inducing the map \( F_S \) on cohomology. So, again, naturality of \( \alpha \) implies that Diagram (1.1) commutes up to sign. This completes the proof. □

Proof of Corollary 5. This is immediate from Theorem 4. □

4. New s-invariants

Fix some field \( \mathbb{F} \) and let \( \alpha: \tilde{H}^\ast(\cdot; \mathbb{F}) \to \tilde{H}^{\ast+n}(\cdot; \mathbb{F}) \) be a stable cohomology operation for some \( n > 0 \). Recall from Definition 1.1 that we defined an odd integer \( q \) to be \( \alpha \)-half-full if there is a configuration of the form

\[
\langle \tilde{a} \rangle \rightarrow \langle \tilde{a} \rangle \quad \langle a \rangle \rightarrow \langle \tilde{\pi} \rangle \neq 0
\]

\[
Kh^{-n,q}(K; \mathbb{F}) \xrightarrow{\alpha} Kh^{0,q}(K; \mathbb{F}) \xleftarrow{\hspace{2cm}} H_0(\mathbb{F}_q; \mathbb{F}) \xrightarrow{\hspace{2cm}} H_0(C; \mathbb{F}).
\]

and \( \alpha \)-full if there is a configuration of the form:

\[
\langle \tilde{a}, \tilde{b} \rangle \rightarrow \langle \tilde{a}, \tilde{b} \rangle \quad \langle a, b \rangle \rightarrow \langle \tilde{\pi}, \tilde{b} \rangle
\]

\[
Kh^{-n,q}(K; \mathbb{F}) \xrightarrow{\alpha} Kh^{0,q}(K; \mathbb{F}) \xleftarrow{\hspace{2cm}} H_0(\mathbb{F}_q; \mathbb{F}) \xrightarrow{\hspace{2cm}} H_0(C; \mathbb{F}).
\]

Clearly, if \( q \) is \( \alpha \)-full, then it is \( \alpha \)-half-full. Furthermore, it is easy to see that if \( q \) is \( \alpha \)-full (resp. \( \alpha \)-half-full), then \( q - 2 \) is \( \alpha \)-full (resp. \( \alpha \)-half-full). So, we defined the following knot
invariants in Definition 1.2:

\[ r^\alpha_+(K) = \max\{q \in \mathbb{Z} + 1 \mid q \text{ is } \alpha\text{-half-full}\} + 1 \]
\[ r^\alpha_-(K) = -r^\alpha_+(\overline{K}) \]
\[ s^\alpha_+(K) = \max\{q \in \mathbb{Z} + 1 \mid q \text{ is } \alpha\text{-full}\} + 3 \]
\[ s^\alpha_-(K) = -s^\alpha_+(\overline{K}), \]

where \( \overline{K} \) denotes the mirror of \( K \).

Let us study a few properties of these invariants before we delve into the proof of Theorem 1.

**Lemma 4.1.** If \( U \) denotes the unknot, then \( s^\alpha_\pm(U) = r^\alpha_\pm(U) = 0. \)

*Proof.* This is immediate if one considers the zero-crossing diagram of the unknot. \( \square \)

**Lemma 4.2.** For any knot \( K, r^\alpha_+(K), s^\alpha_+(K) \in \{s^\mathbb{F}(K), s^\mathbb{F}(K) + 2\}; \) therefore, \( r^\alpha_+(K) = s^\mathbb{F}(K) \) if \( s^\mathbb{F}(K) + 1 \) is not \( \alpha\text{-half-full} \) and \( r^\alpha_+(K) = s^\mathbb{F}(K) + 2 \) otherwise; and \( s^\alpha_+(K) = s^\mathbb{F}(K) \) if \( s^\mathbb{F}(K) - 1 \) is not \( \alpha\text{-full} \) and \( s^\alpha_+(K) = s^\mathbb{F}(K) + 2 \) otherwise.

*Proof.* If \( q \) is \( \alpha\text{-full} \), then there is a configuration of the form

\[ \langle \bar{a}, \bar{b} \rangle \xrightarrow{\alpha} \langle \bar{a}, \bar{b} \rangle \quad \text{and hence the map } H_0(\mathcal{F}_q; \mathbb{F}) \rightarrow H_0(\mathcal{F}_q; \mathbb{F}) \text{ is surjective. Therefore, } q \leq s^\mathbb{F}_{\min}(K) = s^\mathbb{F}(K) - 1. \]

This implies

\[ s^\alpha_+(K) = \max\{q \in \mathbb{Z} + 1 \mid q \text{ is } \alpha\text{-full}\} + 3 \leq s^\mathbb{F}(K) + 2. \]

For the other direction, we need to show that \( s^\mathbb{F}(K) - 3 \) is \( \alpha\text{-full} \). Let \( q = s^\mathbb{F} - 3 \). Choose \( a', b' \in H_0(\mathcal{F}_{q+2}; \mathbb{F}) \) so that \( a', b' \) maps to some basis \( \bar{a}, \bar{b} \in H_0(C; \mathbb{F}) \). Let \( a, b \in H_0(\mathcal{F}_q; \mathbb{F}) \) be the images of \( a', b' \) under the map \( H_0(\mathcal{F}_{q+2}; \mathbb{F}) \rightarrow H_0(\mathcal{F}_q; \mathbb{F}) \). The exact sequence

\[ H_0(\mathcal{F}_{q+2}; \mathbb{F}) \rightarrow H_0(\mathcal{F}_q; \mathbb{F}) \rightarrow Kh^{0,q}(K; \mathbb{F}) \]

implies that \( a, b \) maps to 0 in \( Kh^{0,q}(K; \mathbb{F}) \). Therefore, there is the following configuration

\[ \langle 0 \rangle \xrightarrow{\alpha} \langle 0 \rangle \quad \text{and hence } q \text{ is } \alpha\text{-full. Therefore, } s^\alpha_+(K) \geq q + 3 = s^\mathbb{F}(K). \]

The argument for \( r^\alpha_+(K) \) is similar. \( \square \)

**Corollary 4.3.** For any knot \( K, \) if \( \alpha(Kh^{-n,s^\mathbb{F}(K)+1}(K; \mathbb{F})) = 0, \) then \( r^\alpha_+(K) = s^\mathbb{F}(K); \) and if \( \alpha(Kh^{-n,s^\mathbb{F}(K)-1}(K; \mathbb{F})) = 0, \) then \( s^\alpha_+(K) = s^\mathbb{F}(K). \)
**Proof.** Assume $\alpha (Kh^{-n,s^F(K) - 1}(K; \mathbb{F})) = 0$ and $s^F_\alpha (K) \neq s^F (K)$. Therefore by Lemma 4.2, $s^F (K) - 1$ is $\alpha$-full. Let $q = s^F (K) - 1$. Since the image of $\alpha$ is zero, there exists the following configuration

$$\begin{align*}
\langle \hat{a}, \hat{b} \rangle & \quad \longrightarrow \quad \langle 0 \rangle & \quad \longrightarrow \quad \langle a, b \rangle & \quad \longrightarrow \quad \langle \overline{a}, \overline{b} \rangle \\
Kh^{-n,q}(K; \mathbb{F}) & \quad \xrightarrow{\alpha} \quad Kh^{0,q}(K; \mathbb{F}) & \quad \longleftarrow \quad H_0(\mathcal{F}_q; \mathbb{F}) & \quad \longrightarrow \quad H_0(C; \mathbb{F}).
\end{align*}$$

The exact sequence

$$H_0(\mathcal{F}_{q+2}; \mathbb{F}) \to H_0(\mathcal{F}_q; \mathbb{F}) \to Kh^{0,q}(K; \mathbb{F})$$

implies that $a, b$ are the images of some elements, say $a', b' \in H_0(\mathcal{F}_{q+2}; \mathbb{F})$. Therefore, the map $H_0(\mathcal{F}_{q+2}; \mathbb{F}) \to H_0(C; \mathbb{F})$ is surjective, which contradicts with the statement that $s^F_{\min}(K) = q$.

Once again, the argument for $r^\alpha_+(K)$ is similar. □

**Corollary 4.4.** For any knot $K$, $r^\alpha_+(K), r^\alpha_-(K) \in \{ s^F(K), s^F(K) - 2 \}$; therefore, we have $\max\{|r^\alpha_+(K)|\}, \max\{|s^\alpha_+(K)|\} \in \{|s^F(K)|, |s^F(K)| + 2\}$.

**Proof.** This is immediate from Lemma 4.2, the definition of $r^\alpha_+$ and $s^\alpha_+$ (Definition 1.2) and the fact that $s^F(\overline{K}) = -s^F(K)$ (Corollary 2.8). □

**Proof of Theorem 1.** The proof essentially follows the proof of Corollary 2.7. Let $q = s^\alpha_+(K_1) - 3$. Choose elements $\tilde{a}, \tilde{b} \in Kh^{-n,q}(K_1; \mathbb{F})$, $\hat{a}, \hat{b} \in Kh^{0,q}(K_1; \mathbb{F})$, $a, b \in H_0(\mathcal{F}_q C(K_1); \mathbb{F})$ and $\overline{a}, \overline{b} \in H_0(C(K_1); \mathbb{F})$ satisfying:

$$\begin{align*}
\langle \tilde{a}, \tilde{b} \rangle & \quad \longrightarrow \quad \langle \hat{a}, \hat{b} \rangle & \quad \longrightarrow \quad \langle a, b \rangle & \quad \longrightarrow \quad \langle \overline{a}, \overline{b} \rangle \\
Kh^{-n,q}(K_1; \mathbb{F}) & \quad \xrightarrow{\alpha} \quad Kh^{0,q}(K_1; \mathbb{F}) & \quad \longleftarrow \quad H_0(\mathcal{F}_q C(K_1); \mathbb{F}) & \quad \longrightarrow \quad H_0(C(K_1); \mathbb{F}).
\end{align*}$$

By Theorem 2.3, the cobordism map $F_S: C(K_1) \to C(K_2)$ is a filtered map of filtration $\chi(S) = -2g$. By an abuse of notation, let $F_S$ also denote each of induced maps $H_0(C(K_1)) \to H_0(C(K_2))$, $H_0(\mathcal{F}_q C(K_1)) \to H_0(\mathcal{F}_{q-2g} C(K_2))$ and $Kh^{i,q}(K_1) \to Kh^{i,q-2g}(K_2)$.

Since the induced map on the associated graded complex commutes with the cohomology operation $\alpha$ up to a sign (by Theorem 4), we can pushforward the above configuration to get the following configuration:

$$\begin{align*}
\langle F_S(\tilde{a}), F_S(\tilde{b}) \rangle & \quad \longrightarrow \quad \langle F_S(\hat{a}), F_S(\hat{b}) \rangle & \quad \longrightarrow \quad \langle F_S(a), F_S(b) \rangle & \quad \longrightarrow \quad \langle F_S(\overline{a}), F_S(\overline{b}) \rangle \\
Kh^{-n,q-2g}(K_2; \mathbb{F}) & \quad \xrightarrow{\alpha} \quad Kh^{0,q-2g}(K_2; \mathbb{F}) & \quad \longleftarrow \quad H_0(\mathcal{F}_{q-2g} C(K_2); \mathbb{F}) & \quad \longrightarrow \quad H_0(C(K_2); \mathbb{F}).
\end{align*}$$
Finally, recall that since $a, b$ is a basis for $H_0(C(K_1))$, by Proposition 2.4, $F_S(\pi), F_S(\overline{b})$ is a basis for $H_0(C(K_2))$. Therefore, $q - 2g$ is $\alpha$-full for $K_2$ and hence,

$$s_+^\alpha(K_2) \geq q - 2g + 3 = s_+^\alpha(K_1) - 2g,$$

or

$$2g \geq s_+^\alpha(K_1) - s_+^\alpha(K_2).$$

By treating $S$ as a cobordism from $K_2$ to $K_1$, we get $2g \geq s_+^\alpha(K_2) - s_+^\alpha(K_1)$, and by combining, we get our desired inequality

$$|s_+^\alpha(K_1) - s_+^\alpha(K_2)| \leq 2g.$$

One can take mirrors to get a connected genus $g$ cobordism from $K_1$ to $K_2$. Therefore, we get

$$|s_+^\alpha(K_1) - s_+^\alpha(K_2)| = |s_+^\alpha(K_1) - s_+^\alpha(K_2)| \leq 2g.$$

Finally, since for the unknot $U$, $s_+^\alpha(U) = 0$ (from Lemma 4.1), we get the desired slice genus bounds. The story for $r_+^\alpha$ is similar.  

\section{5. Computations}

The first cohomology operation that comes to mind reduces to Rasmussen’s $s$ invariant:

\textbf{Lemma 5.1.} Suppose $\alpha$ is the zero map $\tilde{H}^*(\cdot; \mathbb{F}) \to \tilde{H}^{*+n}(\cdot; \mathbb{F})$ for some $n > 0$. Then $r_+^\alpha = s_+^\alpha = s^F$.

\textit{Proof.} This is immediate from Corollary 4.3. \hfill \Box

For most of the rest of the section we restrict our attention to the cohomology operations $\text{Sq}^2 : \tilde{H}^*(\cdot; \mathbb{F}_2) \to \tilde{H}^{*+2}(\cdot; \mathbb{F}_2)$ and $\text{Sq}^1 : \tilde{H}^*(\cdot; \mathbb{F}_2) \to \tilde{H}^{*+1}(\cdot; \mathbb{F}_2)$. We start with $\text{Sq}^2$.

\textbf{Lemma 5.2.} Let $K$ be a knot.

(1) If $Kh(K; \mathbb{F}_2)$ is supported on two adjacent diagonals, then

$$r_+^{\text{Sq}^2}(K) = s_+^{\text{Sq}^2}(K) = s^{\text{F}_2}(K).$$

(2) If $Kh(K; \mathbb{F}_2)$ is supported on three adjacent diagonals, then

$$r_+^{\text{Sq}^2}(K) = s^{\text{F}_2}(K).$$

\textit{Proof.} The first statement is obvious from Corollary 4.3 since the operation $\text{Sq}^2 : Kh^{i,j} \to Kh^{i+2,j}$ vanishes identically for these knots. For the second statement, observe that $Kh(K; \mathbb{F}_2)$ is non-zero on the bigradings $(0, s^{\text{F}_2} \pm 1)$ and hence has to be zero on the bigrading $(-2, s^{\text{F}_2} + 1)$. Therefore, once again via Corollary 4.3, we are done. \hfill \Box

\textit{Proof of Theorem 2.} Consider the knot $K = 9_{42}$. From direct computation, we get that $s^{\text{F}_2}(K) = 0$; and by direct computation or by comparing with the knot tables [KAT], we
learn that the ranks of $\text{Kh}(K; \mathbb{F}_2)$ are given by

\[
\begin{array}{ccccccc}
  & -4 & -3 & -2 & -1 & 0 & 1 & 2 \\
7 & . & . & . & . & . & . & 1 \\
5 & . & . & . & . & 1 & 1 & . \\
3 & . & . & . & 1 & 1 & . & . \\
1 & . & . & 2 & 2 & . & . & . \\
-1 & . & 1 & 2 & 1 & . & . & . \\
-3 & . & 1 & 1 & . & . & . & . \\
-5 & 1 & 1 & . & . & . & . & . \\
-7 & 1 & . & . & . & . & . & . \\
\end{array}
\]

From [LSb, Table 1], we see that $\text{Sq}^2: \text{Kh}^{-2, -1}(K; \mathbb{F}_2) \to \text{Kh}^{0, -1}(K; \mathbb{F}_2)$ is surjective.\(^{(iii)}\) Furthermore, since $\text{Sq}^2(K) = 0$, the map $H_0(\mathcal{F}_{-1}; \mathbb{F}_2) \to H_0(C; \mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2$ is also surjective. Therefore, $-1$ is $\text{Sq}^2$-full for $K$, and hence by Lemma 4.2, $s_{\pm}^{\text{Sq}^2}(K) = \text{Sq}^2(K) + 2 = 2$. \(\square\)

**Remark 5.1.** Recall that $\sigma(9_{42}) = 2$, so $9_{42}$ is not topologically slice. (The knot $9_{42}$ has $g_4 = 1$, both smoothly and topologically.) By contrast, the Ozsváth-Szabó concordance invariant $\tau$ does vanish for $9_{42}$.

In Table 1, we present some computations of these new invariants. Since the Khovanov homologies for all prime knots up to 11 crossings are supported on three adjacent diagonals, by Lemma 5.2, $s_{\pm}^{\text{Sq}^2} = \text{Sq}^2$; we have only listed the prime knots up to 11 crossings for which one of $s_{\pm}^{\text{Sq}^2}$ differs from $\text{Sq}^2$.

We use a Sage program (http://www.sagemath.org/) to compute $s_{\pm}^{\text{Sq}^2}$. The main program is main.sage; we extract data from the Knot Atlas ([KAT]) into the file extracted.sage; finally we use the program wrapper.sage to run the main program on these knots. All the programs are available at any of the following locations:

http://math.columbia.edu/~sucharit/programs/newSinvariants/

https://github.com/sucharit/newSinvariants

**Table 1**

| $K$ | $\text{sq}^2$ | $\text{sq}^2$ | $\text{sq}^2$ | $\text{sq}^2$ | $\text{sq}^2$ | $\text{sq}^2$ | $\text{sq}^2$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $9_{42}$ | 0 | 2 | 0 | 10_{132} | -2 | 0 | -2 | 10_{136} | 0 | 2 | 0 |
| $K_{11n19}$ | -2 | -2 | -4 | $K_{11n20}$ | 0 | 0 | -2 | $K_{11n24}$ | 0 | 2 | 0 |
| $K_{11n79}$ | 0 | 2 | 0 | $K_{11n92}$ | 0 | 0 | -2 | $K_{11n96}$ | 0 | 2 | 0 |
| $K_{11n12}$ | 2 | 2 | 0 | $K_{11n70}$ | 2 | 4 | 2 |
| $K_{11n138}$ | 0 | 2 | 0 |

\(^{(iii)}\) The action of $\text{Sq}^2$ on the Khovanov homology of knots has been computed independently by C. Seed (for knots up to 14 crossings) [See].
Examples are harder to find for $Sq^1$. Using computations of C. Seed’s (see also Remark 6.1), we can show that $K14n19265$ is one example.

**Proof of Theorem 3.** Consider the knot $K = K14n19265$. KnotTheory [KTP] provides us with $Kh(K; \mathbb{Z})$.

\[
\begin{array}{cccccccccc}
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\hline
Z & Z & Z & Z & Z & Z & Z & F_2 & Z & F_2 \\
Z & Z & Z & Z & Z & Z & F_2 & Z & F_2 & Z \\
Z & Z & Z & Z & Z & F_2 & Z & F_2 & Z & Z \\
Z & Z & Z & Z & Z & F_2 & Z & F_2 & Z & Z \\
Z & Z & Z & Z & Z & Z & F_2 & Z & F_2 & Z \\
Z & Z & Z & Z & Z & Z & Z & Z & F_2 & Z \\
Z & Z & Z & Z & Z & Z & Z & Z & Z & Z \\
Z & Z & Z & Z & Z & Z & Z & Z & Z & Z \\
Z & Z & Z & Z & Z & Z & Z & Z & Z & Z \\
\end{array}
\]

Direct computations done by C. Seed (see Remark 6.1) tell us that $s^{F_2}(K) = -2$, while $s^Q(K) = 0$ is forced by the form of $Kh(K; \mathbb{Q})$. We want to show that $s^{Sq^1}(K) = 0$ as well; using Lemma 4.2, we only need to show that $-3$ is $Sq^1$-full.

The quantum filtration on $C(K)/\mathcal{F}_-C(K)$ leads to a spectral sequence which starts at $\bigoplus_{q < -3} Kh^Kq (K; \mathbb{Z})$, and converges to $H_* (C(K)/\mathcal{F}_-C(K); \mathbb{Z})$, and whose differentials increase the homological grading by 1 and increase the quantum grading by at least 2. Therefore, from the form of $Kh(K; \mathbb{Z})$, we can conclude that $H_0 (C(K)/\mathcal{F}_-C(K); \mathbb{Z}) = \mathbb{Z}$ and $H_i (C(K)/\mathcal{F}_-C(K); \mathbb{Z}) = 0$ for all $i > 0$.

The short exact sequence

$$0 \to \mathcal{F}_-C(K) \xrightarrow{i} C(K) \xrightarrow{\pi} C(K)/\mathcal{F}_-C(K) \to 0$$

furnishes us with a long exact sequence

$$\cdots \to H_0 (\mathcal{F}_-C(K); R) \xrightarrow{\iota_0^R} H_0 (C(K); R) \cong R^2 \xrightarrow{\pi_0^R} H_0 (C(K)/\mathcal{F}_-C(K); R) \cong R \to \cdots$$

over any ring $R$.

Since $s_{\min}^Q (K) = -1 \geq -3$, the map $\iota_0^Q : \mathbb{Q}^2 \to \mathbb{Q}$ is surjective, and therefore the map $\pi_0^Q : \mathbb{Q}^2 \to \mathbb{Q}$ is zero. Lack of torsion implies that the map $\pi_0^Z : \mathbb{Z}^2 \to \mathbb{Z}$ is zero as well, and hence the map $\iota_0^Z$ is surjective.

Observe that since $H_* (C(K); \mathbb{Z})$ is torsion-free, the map $H_0 (C(K); \mathbb{Z}) \to H_0 (C(K); \mathbb{F}_2)$ is surjective. Choose a basis $\overline{\alpha}, \overline{\beta}$ of $H_0 (C(K); \mathbb{F}_2)$, and choose elements $\overline{\alpha}, \overline{\beta} \in H_0 (C(K); \mathbb{Z})$ that map to $\overline{\alpha}, \overline{\beta}$. Then choose $\alpha, \beta \in H_0 (\mathcal{F}_-C(K); \mathbb{Z})$ which map $\overline{\alpha}, \overline{\beta}$, and consider the
following induced configuration:

\[ \begin{array}{c}
\langle \hat{a}, \hat{b} \rangle \\
\downarrow \downarrow \\
Kh^{0,-3}(K; \mathbb{F}_2) \\
\downarrow \downarrow \\
H_0(F_{-3}; \mathbb{F}_2) \\
\downarrow \downarrow \\
H_0(C; \mathbb{F}_2)
\end{array} \]

Since \( \hat{a}, \hat{b} \in Kh^{0,-3}(K; \mathbb{F}_2) \) admit integral lifts, \( Sq^1(\hat{a}) = Sq^1(\hat{b}) = 0 \). From the form of \( Kh(K; \mathbb{Z}) \), we know that the following is exact

\[ Kh^{-1,-3}(K; \mathbb{F}_2) \xrightarrow{Sq^1} Kh^{0,-3}(K; \mathbb{F}_2) \xrightarrow{Sq^1} Kh^{1,3}(K; \mathbb{F}_2) \]

(the rank of the first map is 2, and the rank of the second map is 1, and \( Kh^{0,-3}(K; \mathbb{F}_2) \cong \mathbb{F}_2^3 \)). Therefore, \( \hat{a} \) and \( \hat{b} \) must lie in the image of \( Sq^1 \) as well. Thus we have a configuration

\[ \begin{array}{c}
\langle \hat{a}, \hat{b} \rangle \\
\downarrow \\
Kh^{0,-3}(K; \mathbb{F}_2) \\
\downarrow \\
H_0(F_{-3}; \mathbb{F}_2) \\
\downarrow \\
H_0(C; \mathbb{F}_2)
\end{array} \]

thereby establishing \(-3\) is \( Sq^1 \)-full, and hence \( s^{Sq^1}(K) = 0 \).

\[ \square \]

6. Further remarks

For convenience, throughout this paper we have used Khovanov homology with coefficients in a field. It is natural to wonder how the \( s \) invariants depend on the field. In a previous version of this paper, we asked:

**Question 6.1.** Let \( \mathbb{F} \) and \( \mathbb{F}' \) be fields. Is there a knot \( K \) so that \( s^\mathbb{F}(K) \neq s^{\mathbb{F}'}(K) \)?

**Remark 6.1.** It was claimed in [MTV07, Theorem 4.2] that \( s^\mathbb{F} \) is independent of \( \mathbb{F} \), but, as noted earlier, there is a gap in the proof of [MTV07, Proposition 3.2]. Since the first draft of this paper, using his package knotkit [See], Cotton Seed has found examples of knots, \( K14n19265 \) being one of them, where \( s^{\mathbb{F}_2} \neq s^\mathbb{Q} \).

A more quantitative version of **Question 6.1** is the following:
Question 6.2. Let $\mathcal{C}$ denote the smooth concordance group and let $\mathcal{I} \subset \mathcal{C}$ denote the subgroup generated by the topologically slice knots. Consider the homomorphism $s := (s^2, s^3, s^5, \ldots): \mathcal{C} \to \mathbb{Z}^\omega$. What are the images of $\frac{s}{2}$ and $\frac{s}{2}$?

If one considered Khovanov homology with coefficients in $\mathbb{Z}$, there are many possible variants of $s$. Specifically, for $m \in \mathbb{Z}$, we can consider the invariants

$$s^{Z_m}_{\text{min}}(K) = \max\{q \in 2\mathbb{Z} + 1 \mid \mathbb{Z}/m \text{ surjects onto } H_*(C(K; \mathbb{Z}))/i_*H_*(\mathcal{F}C(K; \mathbb{Z}))\}$$

$$s^{Z_m}_{\text{max}}(K) = \max\{q \in 2\mathbb{Z} + 1 \mid \mathbb{Z} \oplus \mathbb{Z}/m \text{ surjects onto } H_*(C(K; \mathbb{Z}))/i_*H_*(\mathcal{F}C(K; \mathbb{Z}))\}.$$ 

It is straightforward to verify that $s^{Z_m}_{\text{max}} - 1$ and $s^{Z_m}_{\text{min}} + 1$ give concordance invariants leading to slice genus bounds. Along the lines of Questions 6.1 and 6.2:

Question 6.3. For different $m \in \mathbb{Z}$, how are the invariants $s^{Z_m}_{\text{max}} - 1$ (resp. $s^{Z_m}_{\text{min}} + 1$) related?

In this context, one can use cohomology operations over $\mathbb{Z}$, similarly to Definition 1.1, to obtain other possibly new concordance invariants. One could go further and define more invariants by counting more complicated configurations using cohomology operations. For example, one could define $\bar{a}$ to be $S\bar{q}^{-2}S\bar{q}^{-1}$-half-full if there are elements $\bar{a} \in H_0(C(K; \mathbb{F}_2)$, $\bar{a} \in H_0(\mathcal{F}_q; \mathbb{F}_2)$, $\bar{a}_1 \in Kh^{0,q}(K; \mathbb{F}_2)$, $\bar{a}_2 \in Kh^{-2,q}(K; \mathbb{F}_2)$, $\bar{a}_3 \in Kh^{-3,q}(K; \mathbb{F}_2)$, and $\bar{a}_4 \in Kh^{-3,q}(K; \mathbb{F}_2)$ such that

$$S\bar{q}^{-2}(\bar{a}_1) = \bar{a}$$

$$S\bar{q}^{-1}(\bar{a}_1) = \bar{a}_2$$

$$S\bar{q}^{2}(\bar{a}_3) = \bar{a}_2,$$

and use this notion to define an invariant $r_{S\bar{q}^{-2}S\bar{q}^{-1}}$. (Here, $p: \mathcal{F}_q \to \mathcal{F}_q/\mathcal{F}_q^{+2}$ denotes the quotient map.) That is, $r_{S\bar{q}^{-2}S\bar{q}^{-1}}$ is defined by looking for configurations of the form

$$\xymatrix{ (\bar{a}_3) \ar[r] & (\bar{a}_2) & (\bar{a}_1) \ar[r] & (\bar{a}) \ar[r] & (a) \ar[r] & (\bar{a}) \neq 0 }$$

$$\xymatrix{ Kh^{-3,q}(K; \mathbb{F}_2) \ar[r]^{S\bar{q}^{-2}} & Kh^{-2,q}(K; \mathbb{F}_2) \ar[r]^{S\bar{q}^{-1}} & Kh^{-1,q}(K; \mathbb{F}_2) \ar[r]^{S\bar{q}^{-2}} & Kh^{0,q}(K; \mathbb{F}_2) \ar[r]^{p^*} & H_0(\mathcal{F}_q; \mathbb{F}_2) \ar[r]^{i_*} & H_0(C(K; \mathbb{F}_2).}$$

We have shown that $s^{S\bar{q}^{-2}}_{\pm}$ is distinct from $s^{S\bar{q}}$. It is natural to ask how many more of these invariants are new:

Question 6.4. For which $\alpha$'s are the invariants $r^\alpha_\pm$ (resp. $s^\alpha_\pm$) distinct? More generally, for which configurations are the resulting concordance invariants different?

Finally, in light of [FGMW10] and [KM], it is natural to ask:

Question 6.5. Does $s^\alpha_\pm$ or $r^\alpha_\pm$ give genus bounds for bounding surfaces in homotopy 4-balls?
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E-mail address: lipshitz@math.columbia.edu

E-mail address: sucharit@math.columbia.edu

Department of Mathematics, Columbia University, New York, NY 10027