Bounds on Factors in $\mathbb{Z}[x]$

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Abstract

We gather together several bounds on the sizes of coefficients which can appear in factors of polynomials in $\mathbb{Z}[x]$; we include a new bound which was latent in a paper by Mignotte, and a few minor improvements to some existing bounds. We compare these bounds and show that none is universally better than the others. In the second part of the paper we give several concrete examples of factorizations where the factors have “unexpectedly” large coefficients. These examples help us understand why the bounds must be larger than you might expect, and greatly extend the collection published by Collins.

1 Introduction

How large can the coefficients of a factor of a polynomial be? Let’s try an example. The polynomial $f$ given below has an irreducible factorization of the form $f(x) = g(x) g(-x)$ in $\mathbb{Z}[x]$. The largest coefficient in $f$ is 2; but how big is the largest coefficient of $g$?

$$f = x^{80} - 2x^{78} + x^{76} + 2x^{74} + 2x^{70} + x^{68} + 2x^{66} + x^{64} + x^{62} + 2x^{60} + 2x^{58} - 2x^{54} + 2x^{52} + 2x^{50} + 2x^{48} - x^{44} - x^{42} + 2x^{32} + 2x^{30} + 2x^{28} - 2x^{26} + 2x^{22} + 2x^{20} + x^{18} + x^{16} + 2x^{14} + x^{12} + 2x^{10} + 2x^{8} + x^{6} + x^{4} - 2x^{2} + 1$$

The effective factorization of polynomials in $\mathbb{Z}[x]$ (and thus also in $\mathbb{Q}[x]$ by Gauss’s lemma) is one of computer algebra’s success stories. Modern implementations running on current hardware take only a few seconds to factorize even quite large polynomials with degrees in the hundreds — a feat which would have been utterly impossible before the advent of computer algebra. Being able to compute reasonable bounds on the sizes of factors is a crucial part of this success.

Let us see why these bounds are so important. About forty years ago the ideas and algorithms of Berlekamp [Ber67] (and later [Ber70]) and Zassenhaus [Zas69] finally made polynomial factorization feasible. Now their essence lies at the heart of every general implementation. This Berlekamp-Zassenhaus

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1The much older algorithm of Newton was feasible only for very small inputs.
scheme has maintained its ubiquity, despite the appearance of a number of relatively recent results of considerable theoretical importance (e.g. showing that polynomial factorization is polynomial time), simply because it works so well in practice. A good survey of the development of factorization algorithms can be found in [Kal82], [Kal90], and [Kal92].

We recall briefly the principal steps in the Berlekamp–Zassenhaus algorithm to factorize a polynomial $f \in \mathbb{Z}[x]$:

1. make $f$ primitive and square-free
2. pick a suitable prime $p$
3. determine the factorization in $\mathbb{F}_p[x]$
4. lift to a factorization modulo a large enough power $p^k$
5. recover the true factors in $\mathbb{Z}[x]$

In this article we look at the answer to the question: What value of $k$ should we use in step (4)? In fact, this is just the same as asking: How large can the coefficients of an irreducible factor of $f \in \mathbb{Z}[x]$ be? Intuitively we would expect the factors to be “smaller” than $f$: of course, the degree is smaller, but the coefficients need not be (as we shall in sections 6 and 7).

As we shall see below, the need for bounds on the factors is exemplified by the polynomial $x^4 + 1$. It is irreducible in $\mathbb{Z}[x]$ but factorizes in every $\mathbb{F}_p[x]$ into either four linears or two quadratics. This phenomenon is quite general: it is well known that a general polynomial in $\mathbb{Z}[x]$ often has a complete modular factorization in $\mathbb{F}_p[x]$ which is finer than that in $\mathbb{Z}[x]$. This knowledge is commonly exploited in a process called degree analysis: we compute factorizations modulo several different primes, and use the degrees of the modular factors to deduce possible degrees for the true factors in $\mathbb{Z}[x]$. In some cases degree analysis alone can prove that a polynomial is irreducible.

In the case of $x^4 + 1$ degree analysis by itself cannot prove irreducibility, as all modular factorizations are compatible with the existence of two quadratic factors in $\mathbb{Z}[x]$. In fact, we know that $x^4 + 1$ is irreducible, so regardless of the chosen prime $p$, and for every exponent $k$, the lifted factorization modulo $p^k$ contains “false” factors. Now, if we had no bound on the sizes of the coefficients in the factors, we would have to continue lifting indefinitely because we could never be sure whether a few more lifting steps might produce the true factors in $\mathbb{Z}[x]$. In other words, without a bound on the factors, the ideas of Berlekamp and Zassenhaus produce a method which would fail to terminate on some inputs.

Thus good computable bounds for the sizes of coefficients in factors are essential for making the Berlekamp-Zassenhaus method into a general algorithm. Interestingly, Newton’s method does not need any such bounds.

In section 2 we set out the notation used in this paper. In section 3 we present the classical “degree aware” bounds for the coefficients of factors, and make improvements to two of them. Using the often overlooked fact that each of these bounds gives limits for each individual coefficient of the factor, we show how they can be combined together to produce a tighter result than any
one in isolation. We compare the various bounds using concrete examples. In section 4 we present a new “single factor” bound (which was latent in an article of Mignotte [Mig74]) together with the bound from [BTW93]. Again we use concrete examples in the comparison. In section 5 we confront the two types of bound: those from section 3 and those from section 4. In light of the examples in our comparisons, we conclude that no bound is universally better or worse than the others.

One common feature of all these bounds is that they are almost always surprisingly large — one would be tempted to say unreasonably large. In section 6 we exhibit some examples which help us understand why the bounds in section 3 must be so large; and in section 7 there are further examples which help explain why the bounds in section 4 have to be large. These examples are the fruits of extensive computations using ad hoc C++ programs assisted by CoCoA or CoCoALib [CoCoA]. The examples extend considerably the collection published by Collins [Col04].

Note The answer to the question right at the start is: the largest coefficient of \( g \) is 36. This is the irreducible factorization with largest ratio that we have found.

## 2 Notation

We introduce the notation and terminology we shall use throughout the paper.

Let \( f \in \mathbb{Z}[x] \) be a polynomial. We say that \( f \) is **primitive** if there is no common factor greater than 1 dividing its coefficients. We shall write \( \text{lc}(f) \) to denote the **leading coefficient** of \( f \), and \( \text{tc}(f) \) for its **trailing coefficient**: i.e. \( \text{tc}(f) = f(0) \). Define the **reverse** of a polynomial: \( \overline{f}(x) = x^df(\frac{1}{x}) \) where \( d \) is the degree of \( f \). We generalize slightly the usual notion of palindromic: we say that \( f \) is **\( \pm \)-palindromic** if \( f(x) = \pm \overline{f}(x) \). We define a **\( \ast \)-symmetric factorization** to be one of the form \( f(x) = g(x) \cdot g^*(x) \) where \( g^*(x) = \overline{g}(-x) \); here is an example

\[
\begin{align*}
  f &= 12x^8 + 2x^7 + 4x^6 - 8x^5 + 5x^4 + 8x^3 + 4x^2 - 2x + 12 \\
  &= (3x^4 + 8x^3 + 12x^2 + 10x + 4)(4x^4 - 10x^3 + 12x^2 - 8x + 3)
\end{align*}
\]

We define the **height** of a polynomial to be the largest absolute value of a coefficient: i.e. if \( f = \sum a_i x^i \) then \( \text{ht}(f) = \max\{|a_i|\} \) where, as usual, \(|z|\) denotes the absolute value of the complex number \( z \in \mathbb{C} \). The height will be our main measure of the size of a polynomial. Define the \( l_p \)-norm of the coefficients of \( f = \sum a_i x^i \in \mathbb{C}[x] \) to be \( |f|_p = \left( \sum |a_i|^p \right)^{1/p} \). While we will occasionally use the \( l_2 \)-norm, we are most interested in the case \( p = \infty \) because \( \text{ht}(f) = |f|_\infty \).

We define the **ratio** of a factorization \( f = g_1 g_2 \cdots g_s \in \mathbb{C}[x] \) to be

\[
\frac{1}{\text{ht}(f)} \min\{\text{ht}(g_1), \text{ht}(g_2), \ldots, \text{ht}(g_s)\}
\]

The ratio measures how large the factors are compared to their product. We shall be particularly interested in factorizations in \( \mathbb{Z}[x] \) with a ratio greater
than 1. We concentrate primarily on the case $s = 2$, and will look at the case $s > 2$ in section 8.

We recall Mahler’s measure $M(f) = |c(f)| \prod \max(1, |\alpha_i|)$ the product being taken over all the complex roots of $f$. Observe that Mahler’s measure is invariant under reversal, i.e. $M(\bar{f}) = M(f)$. Actually computing the value of $M(f)$ seems to be difficult, but a good approximation can be calculated using the ideas in [CMP87] and [DM90]. In particular, Mignotte [Mig74] proved that we can always use $|f|^2$ as an upper bound for $M(f)$.

3 Degree Aware Factor Coefficient Bounds

In the Introduction we mentioned how important factor bounds are, so it comes as no surprise to learn that it is a topic which has already attracted some attention. We distinguish two types of bound: the degree aware bounds which make use of information (e.g. obtained from degree analysis) about the possible degrees of factors, and the newer single factor bounds which apply to at least one factor (not necessarily irreducible). It seems that the single factor bounds cannot exploit knowledge about possible degrees of factors. In contrast, all the degree aware bounds are increasing with the degree (at least up to $\frac{1}{2} \deg(f)$), so each of them can be used as a single factor bound just by computing the bounds for a factor of degree $\frac{1}{2} \deg(f)$. We shall look at the single factor bounds in the next section. In this section we recall and compare several of the degree aware bounds which have appeared in the literature. We also present a minor improvement to two of the bounds.

In this section we shall assume we are given $f \in \mathbb{Z}[x]$ and also $\delta \in \mathbb{N}$, and the aim is to bound the coefficients of any factor $g \in \mathbb{Z}[x]$ whose degree is at most $\delta$. The four methods we present actually produce bounds for the magnitudes of the coefficients of any factor $g \in \mathbb{C}[x]$ satisfying a natural scaling hypothesis (described below). Previous uses of these bounds generally used each method to produce just a single number, namely a height bound valid for the whole of $g$ — such an overall height bound is all that is needed to determine how far one must lift. In fact each of the bounding methods gives individual limits for the separate coefficients of $g$, and we shall exploit this to improve the binomial bound (sect. 3.1) and the Knuth–Cohen bound (sect. 3.3). We give examples to show that none of the bounds is universally superior, i.e. for each of the bounding methods there are cases where it gives a lower overall height bound than the others. In section 3.5.5 we shall combine the individual coefficient bounds to obtain a result better than any one of the bounds in isolation.

The Scaling Hypothesis

Ideally we want bounds valid only for irreducible factors in $\mathbb{Z}[x]$, but at the moment the only bounds known to us are valid for a much wider class of factors, namely factors in $\mathbb{C}[x]$ which are suitably scaled to avoid problems with scalar factors. So, even though we are primarily interested in factorizations in $\mathbb{Z}[x]$,
we shall be considering polynomials with complex coefficients in this section. So, let
\[ f = \sum_{i=0}^{d} a_i x^i \in \mathbb{C}[x] \]
be the polynomial of degree \( d \) whose factors we wish to bound. For convenience we shall assume that \( a_0 \neq 0 \), i.e. \( \text{tc}(f) \neq 0 \). The factor whose coefficients we wish to bound is
\[ g = \sum_{i=0}^{\delta} b_i x^i \in \mathbb{C}[x] \]
We assume the following scaling hypothesis: we require that the factor \( g \) satisfy both \( |\text{lc}(g)| \leq |\text{lc}(f)| \) and \( |\text{tc}(g)| \leq |\text{tc}(f)| \); note that this hypothesis is automatically satisfied if \( f \) and \( g \) are images in \( \mathbb{C}[x] \) of a polynomial and one of its factors in \( \mathbb{Z}[x] \).

The Reversal Trick
We now make a simple observation which allows us to improve two of the bounds below. Polynomial multiplication and reversal commute: i.e. if \( f = g_1 g_2 \) then \( \bar{f} = \bar{g}_1 \bar{g}_2 \). Thus a bound for the coefficient of \( x^{\delta-k} \) in a degree \( \delta \) factor of \( \bar{f} \) is also valid as a bound for the coefficient of \( x^k \) in a factor of \( f \). So the idea is simply to compute two sets of individual coefficient bounds: one for a degree \( \delta \) factor of \( f \), and the other for a degree \( \delta \) factor of \( \bar{f} \). Then we combine them to get potentially improved bounds for the coefficient of \( x^k \) in a degree \( \delta \) factor of \( f \). Curiously, this simple idea does not seem to have been described before even though it often produces usefully lower height bounds.

3.1 The Binomial Bound
We shall use the binomial expansion to bound the coefficients of \( g \). Here is the basic idea. Suppose we know a value \( \rho \) which is an upper bound for the magnitude of any complex root of \( f \). Then we see that the \( |b_i| \) are dominated by the corresponding coefficients of the polynomial \( |\text{lc}(g)| (x+\rho)^\delta \). We do not know what \( |\text{lc}(g)| \) is, but by the scaling hypothesis we have \( |\text{lc}(g)| \leq |\text{lc}(f)| \). Thus the \( |b_i| \) are surely dominated by the coefficients of \( |\text{lc}(f)| (x+\rho)^\delta \).

We can improve the binomial bound by using the reversal trick. In detail, let \( \bar{\rho} \) be a root bound for \( \bar{f} \) then by reasoning analogous that above we see that \( |b_i| \) is dominated by the smaller of the coefficients of \( x^i \) in \( |\text{lc}(f)| (x+\rho)^\delta \) and in \( |\text{tc}(f)| (\bar{\rho} x + 1)^\delta \).

3.1.1 Root Bounds for a Polynomial
We shall now investigate how to find a good root bound \( \rho \) for \( f \). Define the perfect root bound \( \text{RB}(f) = \max\{|\alpha| : f(\alpha) = 0\} \); so clearly we must have \( \rho \geq \text{RB}(f) \). Luckily, finding suitable values for \( \rho \) is not too hard. We can take
\( \rho = C(f) \), the unique positive root of \( \hat{f}(x) = |a_d|x^d - \sum_{i=0}^{d-1} |a_i|x^i \) because if we have \( f(\alpha) = 0 \) then \( |a_d|\alpha^d = |a_d\alpha^d| = |\sum_{i=0}^{d-1} a_i\alpha^i| \leq \sum_{i=0}^{d-1} |a_i||\alpha|^i \), hence \( \hat{f}(\alpha) \leq 0 \). It is reported in [DM90] that Cauchy knew of this bound. In general, \( C(f) > \text{RB}(f) \); indeed in [DM90] it is shown that \( \text{RB}(f) \leq C(f) \leq \text{RB}(f) \left(2^{1/d} - 1\right)^{-1} \approx \text{RB}(f) d/\log 2 \) with both limits being attainable.

Alternatively, if we prefer not to compute \( C(f) \), we can obtain a slightly looser bound using a formula given in [Zas69]:

\[
Z(f) = \frac{1}{2^{1/d} - 1} \max \left\{ \left( \frac{|a_{d-i}|}{|a_d|} \left( \frac{d}{i} \right)^{1/i} \right)^{1/i} \right\}
\]

Another formula was given as exercise 4.6.2–20 in [Knu69] (curiously, it does not appear in the second edition [Knu81]):

\[
K(f) = 2 \max \left\{ \left( \frac{|a_{d-i}|}{|a_d|} \right)^{1/i} : i = 1, 2, \ldots, d \right\}
\]

In fact, these formulas merely give upper bounds for \( C(f) \); nevertheless, as Knuth showed, we know that \( K(f) \) cannot exceed \( 2d\text{RB}(f) \). In practice, we can start from the smaller of \( Z(f) \) and \( K(f) \) then apply a few Newton iterations to obtain quickly a tighter upper bound for \( C(f) \). The topic of root bounds has been much studied; some more information can be found in, for example, [Wil61] or section 6.2 of [Yap2000].

Now, \( C(f) \) can be rather larger than \( \text{RB}(f) \), the best possible value for \( \rho \). A simple way to compute better approximations to \( \text{RB}(f) \) is given in [DM90]. They use Gräffe’s transformation repeatedly to obtain the polynomial \( f_\ast \) whose roots are the \( 2^* \)-th powers of the roots of \( f \), then they use the smaller of \( K(f_\ast) \) and \( Z(f_\ast) \) to bound \( C(f_\ast) \) and finally take the \( 2^* \)-th root of the result. Choosing \( s \approx 3 + \log \log d \) is enough to obtain an estimate for \( \text{RB}(f) \) within a small constant factor of optimal. As we shall be computing powers of \( \rho \) in the binomial expansion a larger value of \( s \) may be better for us, in [DM90] they suggest using the value \( s = \max\{3, \log d\} \).

**Note** Although applying Gräffe’s transformation many times will surely produce a better estimate, a single application of Gräffe’s transformation may lead to larger estimates for the largest root; e.g. let \( f = x^2 - 2 \), then we immediately see that \( 3/2 \) is an upper bound for \( C(f) \); however, \( g = x^2 - 4x + 4 \) is the Gräffe transform of \( f \) and \( C(g) = 2 + 2\sqrt{2} \approx 4.8 \) which is considerably larger than \((3/2)^2 = 2.25\).

### 3.1.2 A refinement of the binomial bound

It is possible to refine the binomial bound if we have more detailed information about the roots of \( f \). Specifically, if we know values \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_d \) where each satisfies \( \rho_i \geq |\alpha_i| \) for a suitable numbering of the complex roots of \( f \), then the magnitude of the coefficient of \( x^i \) in a factor of degree \( \delta \) is bounded by the coefficient of \( x^i \) in the product \( \prod_{j=1}^{\delta}(x + \rho_j) \). In the rest of this paper we shall use just the standard form of the binomial bound.
3.2 Mignotte’s Bound

Mignotte [Mig74] has refined an inequality he ascribed to Mahler to obtain the following bound on a factor \( g \) of degree \( \delta \):

\[
|g|_1 \leq 2^\delta M(g) \leq 2^\delta M(f) \leq 2^\delta |f|_2
\]

In fact, formula (2) in that paper gives an individual bound for each coefficient of \( g \): namely \( |b_i| \leq \binom{\delta}{i} M(f) \). Mignotte has published some other bounds but the one given here seems most relevant to our purposes. The bound is clearly invariant under reversal (since Mahler’s measure is invariant).

3.3 The Knuth–Cohen Bound

A slight refinement of Mignotte’s bound appears as exercise 4.6.2–20 in [Knu81]; it is also given as Theorem 3.5.1 in [Coh95]. This refined bound is sufficiently different from Mignotte’s to merit separate mention. The bound is:

\[
|b_i| \leq \left( \frac{\delta - 1}{i} \right) |f|_2 + \left( \frac{\delta - 1}{i - 1} \right) |\text{l.c.}(f)|
\]

and presumably for \( i = 0 \) we simply take \( |b_0| \leq |a_0| \).

Like the binomial bound, this bound is not invariant under reversal so we can improve it by using the reversal trick. We shall use this improved version for the comparisons below.

3.4 Beauzamy’s Bound

In 1992 Beauzamy derived a new bound [Bea92] from a result of Bombieri. Once again he gives individual bounds for each \( |b_i| \):

\[
|b_i| \leq \sqrt{\frac{1}{2} \binom{\delta}{i} \binom{d}{\delta} |f|_2}
\]

where \( |f|_2 = \sqrt{\sum_{j=0}^{d} |a_j|^2 / \binom{d}{j}} \) is a weighted norm derived from Bombieri’s norm. An interesting feature of this norm is that the central coefficients of \( f \) have particularly small weights, so polynomials whose central coefficients are larger than the peripheral ones (as often happens in practice) have a small norm. The bound is clearly invariant under reversal (since Bombieri’s norm is invariant).

3.5 Comparison of the Degree Aware Bounds

Here we give four example polynomials to show that each of the degree aware bounds can be the best one; then we give a fifth example where combining the bounds is best. We produced the examples by generating many random irreducible polynomials of degree 4 then we chose for each bound a product of
two of these polynomials where that bound was better than the others. Since
we are interested in factors in \( \mathbb{Z}[x] \) we have rounded down to integers the values
of the bounds in the tables. It is quite apparent that the bounds are often very
loose.

### 3.5.1 A case favourable for the binomial bound

Let us take

\[
   f = (x^4 + 4x^3 + 16x^2 + 9x - 1)(x^4 + 4x^3 + 15x^2 + 3x - 2) \\
   = x^8 + 8x^7 + 47x^6 + 136x^5 + 285x^4 + 171x^3 - 20x^2 - 21x + 2
\]

Factorizing \( f \) modulo 5 shows that the only possible degree for a true factor
is 4. For this polynomial we obtained a good root bound \( \rho \approx 3.84 \). Our
estimate for Mahler’s measure is about 197. And Bombieri’s norm is about 47.
The coefficient bounds are:

|           | \( x^4 \) | \( x^3 \) | \( x^2 \) | \( x^1 \) | Overall |
|-----------|-----------|-----------|-----------|-----------|---------|
| Binomial  | 1         | 15        | 88        | 84        | 2       | 88      |
| Mignotte  | 196       | 787       | 1181      | 787       | 196     | 1181    |
| Beauzamy  | 275       | 551       | 675       | 551       | 275     | 675     |
| Knuth–Cohen | 1     | 366       | 1093      | 369       | 2       | 1093    |

In the table above, and those of the subsections below, we see the values of
the individual coefficient bounds for each method, and also the resulting overall
height bound. In this instance we observe that the binomial method produces
a considerably smaller overall value than the other methods, and so is the best
in this case.

### 3.5.2 A case favourable for Mignotte’s bound

Let us take

\[
   f = (x^4 - 7x^3 + 7x^2 - 8x + 2)(2x^4 - 2x^3 - 2x^2 + 6x - 5) \\
   = 2x^8 - 16x^7 + 26x^6 - 10x^5 - 41x^4 + 89x^3 - 87x^2 + 52x - 10
\]

Factorizing \( f \) modulo 37 shows that the only possible degree for a true factor
is 4. We estimate the Mahler measure of \( f \) to be about 33.4. A good root bound
for \( f \) is \( \rho \approx 6.1 \), and for its reverse we get \( \bar{\rho} \approx 3.2 \). The Bombieri norm of \( f \) is
about 31. The resulting coefficient bounds are:

|           | \( x^4 \) | \( x^3 \) | \( x^2 \) | \( x^1 \) | Overall |
|-----------|-----------|-----------|-----------|-----------|---------|
| Binomial  | 2         | 48        | 439       | 129       | 10      | 439     |
| Mignotte  | 33        | 133       | 200       | 133       | 33      | 200     |
| Beauzamy  | 180       | 361       | 443       | 361       | 180     | 443     |
| Knuth–Cohen | 2     | 150       | 440       | 174       | 10      | 440     |
3.5.3 A case favourable for the Knuth–Cohen bound

Let us take
\[ f = (x^4 - 8x^3 - 7x^2 - 5x + 5)(2x^4 - 6x^3 - x^2 + 4x - 5) \]
\[ = 2x^8 - 22x^7 + 33x^6 + 44x^5 + 10x^4 - 13x^3 + 10x^2 + 45x - 25 \]

Factorizing \( f \) modulo 11 shows that the only possible degree for a true factor is 4. We estimate the Mahler measure of \( f \) to be about 75. A good root bound for \( f \) is \( \rho \approx 8.9 \), and for its reverse we get \( \bar{\rho} \approx 2.05 \). The Bombieri norm of \( f \) is about 32. The resulting coefficient bounds are:

|            | \( x^4 \) | \( x^3 \) | \( x^2 \) | \( x^1 \) | \( x^0 \) | Overall |
|------------|-----------|-----------|-----------|-----------|-----------|---------|
| Binomial   | 2         | 70        | 628       | 204       | 25        | 628     |
| Mignotte   | 74        | 298       | 447       | 298       | 74        | 447     |
| Beauzamy   | 189       | 378       | 463       | 378       | 189       | 463     |
| Knuth–Cohen| 2         | 86        | 248       | 155       | 25        | 248     |

3.5.4 A case favourable for Beauzamy’s bound

Let us take
\[ f = (x^4 + 5x^3 - 14x^2 - x - 3)(3x^4 + 8x^3 + 15x^2 - 5x - 1) \]
\[ = 3x^8 + 23x^7 + 13x^6 - 45x^5 - 253x^4 + 26x^3 - 26x^2 + 16x + 3 \]

Factorizing \( f \) modulo 37 shows that the only possible degree for a true factor is 4. A good root bound for \( f \) is \( \rho \approx 7.0 \); for the reverse we get \( \bar{\rho} \approx 7.0 \). The Mahler measure of \( f \) is about 259. The Bombieri norm of \( f \) is about 33. The resulting coefficient bounds are:

|            | \( x^4 \) | \( x^3 \) | \( x^2 \) | \( x^1 \) | \( x^0 \) | Overall |
|------------|-----------|-----------|-----------|-----------|-----------|---------|
| Binomial   | 3         | 83        | 880       | 83        | 3         | 880     |
| Mignotte   | 258       | 1035      | 1553      | 1035      | 258       | 1553    |
| Beauzamy   | 197       | 394       | 482       | 394       | 197       | 482     |
| Knuth–Cohen| 3         | 270       | 793       | 270       | 3         | 793     |

3.5.5 A case where the combined bound is best

Let us take
\[ f = (2x^4 + 8x^3 + 10x^2 + 9x - 1)(x^4 - 3x^3 + 5x^2 - 5) \]
\[ = 2x^8 + 2x^7 - 4x^6 + 19x^5 + 12x^4 + 8x^3 - 55x^2 - 45x + 5 \]

Factorizing \( f \) modulo 61 shows that the only possible degree for a true factor is 4. We obtain the following coefficient bounds:

|            | \( x^4 \) | \( x^3 \) | \( x^2 \) | \( x^1 \) | \( x^0 \) | Overall |
|------------|-----------|-----------|-----------|-----------|-----------|---------|
| Binomial   | 2         | 22        | 95        | 178       | 5         | 178     |
| Mignotte   | 63        | 255       | 382       | 255       | 63        | 382     |
| Beauzamy   | 118       | 236       | 290       | 236       | 118       | 290     |
| Knuth–Cohen| 2         | 81        | 231       | 90        | 5         | 231     |
If we consider only the overall height bounds produced by each method then the best we can conclude is that any factor of degree 4 can have height at most 178. However, taking together the bounds for each individual coefficient (i.e. the minimum of each column) we see that no coefficient can exceed 95.

3.6 Preprocessing to get Better Bounds

From the examples above it is quite clear that often all the bounds are very loose. Now, any of the bounds above applied to some multiple of \( f \) will naturally be valid also for factors of \( f \) itself. So we are interested in finding small height multiples of \( f \). We find that Theorems 4 and 4’ in [Mig88] prove the existence of small height multiples of polynomials in \( \mathbb{Z}[x] \); unfortunately, they do not give a good way of actually finding them. One approach is to use LLL lattice reduction [LLL82] to look for a multiple of \( f \) with small \( l_2 \) norm.

Another approach is hinted at in Theorem B of [CMP87]: given a positive degree \( \hat{d} \) find the unique monic \( \hat{f}_d \in \mathbb{Q}[x] \) of degree \( \hat{d} \) which minimizes \( |\hat{f}_d f| \). The coefficients of \( \hat{f}_d \) are the solutions of a simple linear system, so we have an efficient and effective method for finding a small multiple of prescribed degree in \( \mathbb{Q}[x] \).

The principal drawback of this preprocessing idea is that only rarely does it produce better bounds: the reduction in height is not usually sufficient to offset the increase in degree — recall that all the degree aware bounds grow exponentially with degree.

4 Single Factor Bounds

We now turn our attention to the single factor bounds which we mentioned at the start of section 3. Even though our primary interest is in factors in \( \mathbb{Z}[x] \), once again we need to consider what happens in \( \mathbb{C}[x] \). Let \( f \in \mathbb{C}[x] \) have degree \( d \). Then a single factor bound for \( f \) is a value \( B \) such that for any non-trivial factorization \( f = g_1 g_2 \cdots g_s \in \mathbb{C}[x] \) at least one factor (wlog \( g_1 \)) satisfies \( \text{ht}(g_1) \leq B \). As already observed, any of the degree aware bounds from the previous section can be used as a single factor bound simply by computing the bounds for a factor of degree \( \lfloor d/2 \rfloor \). Here we see two other ways of obtaining such bounds. We start with a new bound which was latent in an article of Mignotte.

4.1 Mignotte’s Bound

It is immediate from Theorem 2 in [Mig74] that in any non-trivial factorization \( f \) must have at least one factor, \( g \), satisfying

\[
|g|_1 \leq \sqrt{2^d M(f)} \leq \sqrt{2^d |f|_2}
\]
though this fact is not exploited there. For some reason this was overlooked also in [BTW93]; they just took the simpler bound from Mignotte’s Theorem 2 and applied it to degree \([d/2]\).

In fact, tracing through Mignotte’s reasoning one can also obtain another, more convenient version which bounds directly the sizes of the coefficients of some factor. This alternative version is also slightly tighter. We now present this new bound.

Suppose that \(f = g_1g_2\) is a non-trivial factorization with \(d_1 = \deg(g_1)\) and \(d_2 = \deg(g_2)\). Clearly \(M(f) = M(g_1)M(g_2)\). Now \(|g_1|_\infty \leq \left(\frac{d_1}{d_1/2}\right)M(g_1)\), and similarly for \(g_2\). Since \(d = d_1 + d_2\) and both \(d_1\) and \(d_2\) are strictly positive, we have \(\left(\frac{d_1}{d_1/2}\right)\left(\frac{d_2}{d_2/2}\right) \leq 3\left(\frac{d}{d/2}\right)\). Putting it all together we find that

\[|g_1|_\infty |g_2|_\infty \leq \frac{2}{3}\left(\frac{d}{(d/2)}\right)M(f)\]

We may assume that \(|g_1|_\infty \leq |g_2|_\infty\); therefore \(|g_1|_\infty \leq \sqrt{2\left(\frac{d}{d/2}\right)}M(f)\). Using Stirling’s approximation we can estimate the binomial coefficient to be about \(2^{d+1/(2\pi d)^{1/2}}\). If we want, using Theorem 1 from [Mig74], we can replace \(M(f)\) by the bound \(|f|_2\) to obtain a closed form. Computationally it is better to use approximations from [CMP87] or [DM90] to estimate \(M(f)\).

4.2 The BTW Bound

Another single factor bound was described more recently in [BTW93]; it is based on Bombieri’s weighted norm. They show that at least one factor, \(g\), must satisfy

\[|g|_\infty \leq c\sqrt{2^{d-3/4}|f|_2}\]

for some constant \(c < 1.1\) (provided \(d = \deg(f) > 2\)). It is fairly evident that for large \(d\) Mignotte’s single factor bound is greater than BTW if we choose to use \(|f|_2\) in place of \(M(f)\) since \(|f|_2 \geq |f|_2\). However, if we compute a good approximation to \(M(f)\) then Mignotte’s bound can be substantially lower: e.g. consider \(f = (x + 1)^2\) for which \(M(f) = 1\) and \(|f|_2 = 2^d\), then Mignotte’s bound yields \(2^d(\pi d)^{-1/4}\) whereas the BTW bound is \(c2^{3d/2}(2d)^{-3/8}\) — almost 50% bigger than Mignotte’s bound, in logarithmic terms.

4.3 Comparison of the Single Factor Bounds

Here are two examples to show that neither bound is superior to the other. Obviously the value of Mignotte’s bound depends on the quality of the approximation to \(M(f)\) we use: for our computations we used the techniques in [CMP87] to obtain a fairly good approximation. The first example is

\[
\begin{align*}
f &= 6x^8 + 27x^7 + 65x^6 + 105x^5 + 123x^4 + 105x^3 + 65x^2 + 27x + 6 \\
&= (2x^4 + 5x^3 + 7x^2 + 6x + 3)(3x^4 + 6x^3 + 7x^2 + 5x + 2)
\end{align*}
\]
where we find that Mignotte’s bound is 21 whereas the BTW bound is 47. Conversely, if we look at the example

\[ f = x^8 - 6x^6 + 59x^4 - 6x^2 + 1 = (x^4 - 4x^3 + 5x^2 + 4x + 1)(x^4 + 4x^3 + 5x^2 - 4x + 1) \]

we find that the BTW bound is 21 whereas Mignotte’s is 52.

Once again, since neither bound is always better than the other, it is worth computing both, and taking the smaller result. In the next section we compare the single factor bounds with the degree aware ones.

5 Single Factor vs. Degree Aware Bounds

In this section we consider the question Which are better: single factor bounds or degree aware bounds? The answer is Neither. We shall see that the best answer is always to calculate both and pick whichever is the smaller. For simplicity we shall restrict attention to short factorizations: \( f = g_1g_2 \).

One situation apparently favourable for the degree aware bounds is when we know that one true factor must have particularly low degree (i.e. considerably smaller than \( d/2 \)) — recall that we might be able to learn this from degree analysis. Conversely, a situation apparently favourable for the single factor bounds is when both factors could have high degree (i.e. close to \( d/2 \)). We give two examples of both types: one where the single factor bounds are better and one where the degree aware bounds are better.

5.1 Factorizations with a Low Degree Factor

We present two example short factorizations both with factors of degree 2 and 18. In one case we see that the degree aware bounds are much smaller than the single factor bounds, in the other case we see the opposite. For convenience, these two examples have a special structure: the smaller factor \( g_1 \) is quadratic, the larger factor \( g_2 = g_1^9 + x^8 \), which can be viewed as a “small perturbation” of \( g_1^9 \).

Taking \( g_1 = x^2 + 5x + 9 \) we obtain a polynomial \( f \) whose factorization modulo 151 tells us that the only degree pattern for true factors is 2 + 18. A good root bound for \( f \) is about 3.9, and the binomial bound for a degree 2 factor tells us that the maximum coefficient cannot exceed 45; the other degree aware bounds are larger. In contrast, Mignotte’s single factor bound is larger than \( 2 \times 10^7 \); the BTW bound is larger still.

Taking instead \( g_1 = 8x^2 + 9x + 9 \) we obtain a polynomial \( f \) whose factorization modulo 61 tells us that the only degree pattern for true factors is 2 + 18. A good root bound for \( f \) is about 1.3, and the binomial bound for a degree 2 factor tells us that the maximum coefficient cannot exceed about \( 2.8 \times 10^9 \); the other degree aware bounds are even larger. In contrast, Mignotte’s single factor bound is less than \( 2.7 \times 10^7 \); the BTW bound is larger.
5.1.1 A Family having Large Single Factor Bounds

We can, with very high probability, construct examples for which the degree aware binomial bound is arbitrarily many times smaller than either single factor bound. The idea is simple: we construct a polynomial which has two irreducible factors, one of very low degree and one of high degree. The degree aware bounds have the advantage of being able to use the information that there is a very low degree factor.

Consider the polynomial

\[ f = (x+1)(x^{d-1} - 10^{d-1} + \epsilon) \]

where \( \epsilon \) is a polynomial of degree less than \( d-1 \) having small coefficients and such that \( f \) has only two irreducible factors over \( \mathbb{Z} \). It seems quite easy to find such \( \epsilon \) for which degree analysis proves that the only possible degrees for true factors are 1 and \( d-1 \).

By choosing \( d \) sufficiently large (and \( \epsilon \) sufficiently small), Knuth's root bound for \( f \) yields a value of about 20, so a good root bound will surely be no larger than this. Thus the binomial bound for the height of a factor of degree 1 is at most 20 whereas the BTW single factor bound is greater than \( (d-1)/2 \pi d - \frac{1}{4} \) (since \( M(f) \geq 10^{d-1} \)). In other words the binomial bound is about \( (d-1)/2 \) times smaller (in a logarithmic sense) than either of the single factor bounds.

5.2 Factorizations with High Degree Factors

If degree analysis does not exclude the existence of high degree true factors (i.e. of degree close to \( d/2 \)) then the single factor bounds are often better than the degree aware bounds. Nonetheless, it is not hard to find cases where the degree aware bounds are tighter than the single factor bounds even in these “unfavourable circumstances”. For instance, let us take the irreducible factors

\[
\begin{align*}
g_1 &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + 2x^3 + 4x^2 + 4x + 5 \\
g_2 &= x^{10} + x^9 + 2x^8 + 2x^7 + 3x^6 + 4x^5 + 4x^4 + 4x^3 + 5x^2 + 5x + 5
\end{align*}
\]

Then for the product \( f = g_1g_2 \) we see from the factorization modulo 463 that the only possible degree for a true factor is 10. A good root bound for \( f \) is 1.24, and the binomial bound for a degree 10 factor is thus 757. The single factor bounds are rather larger: Mignotte’s bound gives 1859, and the BTW bound is a little larger at 1920.

We now present a case where the single factor bounds are significantly smaller than the degree aware bounds. We take as irreducible factors

\[
\begin{align*}
g_1 &= x^{10} + 5x^9 - 5x^8 - 3x^7 + 5x^6 - 5x^5 - 2x^4 - 4x^3 - 5x^2 + x + 3 \\
g_2 &= x^{10} + x^9 - 3x^8 - x^7 + 5x^6 + x^5 + x^4 - 4x^3 + 5x^2 - 1
\end{align*}
\]

Then for the product \( f = g_1g_2 \) we see from the factorization modulo 587 that the only possible degree for a true factor is 10. The best degree aware bound is Knuth–Cohen which gives 16339. The single factor bounds are rather smaller: Mignotte’s bound gives 3071, and the BTW bound is smaller still at 713.
6 Polynomials in \( \mathbb{Z}[x] \) having a large factor

Looking at the examples in sections 3, 4 and 5, it is quite clear that the factor bounds are often very loose. Indeed, based on experience, it is natural to conjecture that the factors must have height no greater than that of the polynomial they divide. Remarkably, this is false: there are cyclotomic polynomials of arbitrarily great height \cite{Yau74}, yet they each divide a polynomial of the form \( x^d - 1 \) (for some \( d \) depending on the polynomial).

In this section we are interested in polynomials which have at least one large height factor. The existence of these polynomials having a particularly large factor helps explain why the degree aware bounds from section 3 have to be so generous: those bounds must be large enough to accommodate these unusually large factors. In contrast, these factorizations do not have similar implications for the single factor bounds of section 4 — in section 7 we will consider large ratio factorizations which do force also the single factor bounds to be large.

Our main interest is in irreducible factors in \( \mathbb{Z}[x] \): and in the context of polynomial factorization we can further restrict to irreducible factors of primitive square-free polynomials. However, the known bounds apply to any factor of any polynomial in \( \mathbb{C}[x] \) under the scaling hypothesis mentioned at the start of section 3. As we shall see, this wider applicability forces the bounds to be much larger than necessary for irreducible factors in \( \mathbb{Z}[x] \).

It appears to be very difficult to devise general bounds valid only for factors in \( \mathbb{Z}[x] \) — or ideally, irreducible factors in \( \mathbb{Z}[x] \). It is even unclear how to devise a bound valid only for factors of a primitive square-free polynomial in \( \mathbb{Z}[x] \). Yet the examples below indicate that such specialized bounds could be significantly tighter than the current ones we know.

6.1 Large Factors of \( x^d - 1 \)

In this subsection we look at how large factors of \( x^d - 1 \) can be. We chose to consider this family of polynomials for several reasons: the family depends on a single parameter (namely the degree \( d \)), there are already many interesting theoretical results, and the limited nature of the family permits computational experimentation up to moderately high degree.

6.1.1 Large height irreducible factors of \( x^d - 1 \in \mathbb{Z}[x] \)

The irreducible factors of \( x^d - 1 \) in \( \mathbb{Z}[x] \) are called cyclotomic polynomials. They enjoy numerous special properties; we recall just one of them: \( x^d - 1 = \prod_{n|d} \phi_n(x) \) where we write \( \phi_n(x) \) to denote the \( n \)-th cyclotomic polynomial. The first few cyclotomic polynomials all have height 1, but as we reach higher indices we find examples of greater height. Here is a table of heights of certain cyclotomic polynomials: these are successive maximums (up to index 100000). We used version 4.7.4 of the CoCoA system \cite{CoCoA} for the computations.
We notice that one needs to go to very high degrees to obtain large coefficients. These cyclotomic polynomials have much smaller heights than allowed by the factor bounds: e.g. even for the smallest case in the table, the best bound we get for a degree 48 factor of $x^{105} - 1$ is $(\frac{48}{24}) \approx 3 \times 10^{13}$ which is far larger than the actual height. In [Vau74] Vaughan showed that the asymptotic growth of $\text{ht}(\phi_d)$ for certain $d$ increases as $\exp(d \log 2 + o(1)) / \log \log d$.

### 6.1.2 Large height reducible factors of $x^d - 1$

Recalling that the bounds of section 3 do not distinguish between reducible and irreducible factors, we now look for large factors of $x^d - 1$ including reducible ones. Here is a table of successive maximums of the greatest height factor in $\mathbb{Z}[x]$ of $x^d - 1$ for $d$ up to 719. The values were computed directly: factorize $x^d - 1$ then try all possible products of the irreducible factors. We did not attempt $d = 720$ as $x^{720} - 1$ has 30 factors, and it would take a long time to generate and test all $2^{30}$ factors. These examples were computed by a dedicated C++ program which relied upon version 0.99 of CoCoALib [CoCoA].

| $\text{ht}(\phi_d)$ | $d$ | Factorization of $d$ |
|---------------------|-----|----------------------|
| 2                   | 105 | $3 \cdot 5 \cdot 7$ |
| 3                   | 385 | $5 \cdot 7 \cdot 11$ |
| 4                   | 1365| $3 \cdot 5 \cdot 7 \cdot 13$ |
| 5                   | 1785| $3 \cdot 5 \cdot 7 \cdot 17$ |
| 6                   | 2805| $3 \cdot 5 \cdot 11 \cdot 17$ |
| 7                   | 3135| $3 \cdot 5 \cdot 11 \cdot 19$ |
| 9                   | 6545| $5 \cdot 7 \cdot 11 \cdot 17$ |
| 14                  | 10465| $5 \cdot 7 \cdot 13 \cdot 23$ |
| 23                  | 11305| $5 \cdot 7 \cdot 17 \cdot 19$ |
| 25                  | 17255| $5 \cdot 7 \cdot 17 \cdot 29$ |
| 27                  | 20615| $5 \cdot 7 \cdot 19 \cdot 31$ |
| 59                  | 26565| $3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ |
| 359                 | 40755| $3 \cdot 5 \cdot 11 \cdot 13 \cdot 19$ |

| Factor Height | $d$ | Factor |
|---------------|-----|--------|
| 3             | 12  | $\phi_1 \phi_4 \phi_6$ |
| 4             | 20  | $\phi_3 \phi_5$ |
| 12            | 30  | $\phi_6 \phi_{10} \phi_{15}$ |
| 54            | 60  | $\phi_1 \phi_4 \phi_9 \phi_{10} \phi_{15} \phi_{60}$ |
| 55            | 84  | $\phi_1 \phi_4 \phi_9 \phi_{14} \phi_{21} \phi_{84}$ |
| 58            | 90  | $\phi_2 \phi_3 \phi_7 \phi_{15} \phi_{30} \phi_{45}$ |
| 72            | 105 | $\phi_3 \phi_5 \phi_7 \phi_{105}$ |
| 192           | 120 | $\phi_2 \phi_3 \phi_4 \phi_5 \phi_{24} \phi_{30} \phi_{40} \phi_{60}$ |
| 475           | 180 | $\phi_1 \phi_4 \phi_6 \phi_{10} \phi_{15} \phi_{18} \phi_{36} \phi_{45} \phi_{60} \phi_{90}$ |
| 10188         | 210 | $\phi_6 \phi_{10} \phi_{14} \phi_{15} \phi_{21} \phi_{35} \phi_{210}$ |
| 395796        | 420 | $\phi_1 \phi_4 \phi_6 \phi_{10} \phi_{14} \phi_{15} \phi_{21} \phi_{35} \phi_{60} \phi_{84} \phi_{140} \phi_{210}$ |
| 396660        | 630 | $\phi_2 \phi_3 \phi_5 \phi_7 \phi_{90} \phi_{30} \phi_{42} \phi_{45} \phi_{70} \phi_{90} \phi_{105} \phi_{126} \phi_{315}$ |
Note how the factor height increases with index $d$ much more rapidly than it did in the previous subsection where we measured the largest irreducible factor. Nevertheless these highest factors still lie well within the limits permitted by the factor bounds: for instance the best bound for a degree 57 factor of $x^{120} - 1$ is $\left(\frac{57}{25}\right) \approx 1.5 \times 10^{16}$ — far larger than the observed value of 192.

**Note:** In [PR07] Theorems 4.1 and 5.3 state that the asymptotic growth of the greatest height of a factor of $x^d - 1$ for certain exponents $d$ (with many factors) increases as $\exp\left(q^{(\log 3 + o(1))/\log \log d}\right)$. This impressive result is, however, relevant only for very large degrees well outside the realm of practical polynomial factorization.

### 6.1.3 Large factors of $x^d - 1$ in $\mathbb{R}[x]$ and $\mathbb{C}[x]$ 

Recalling that the bounds in section 3 on the heights do, in fact, apply to all factors in $\mathbb{C}[x]$ which satisfy the scaling hypothesis, we now consider such factors. We shall see that much greater heights can be achieved compared to what we found when looking at factors in $\mathbb{Z}[x]$. Our study is helped greatly by the particularly simple factorization: $x^d - 1 = \prod_{k=0}^{d-1}(x - \zeta^k) \in \mathbb{C}[x]$ where $\zeta$ is a primitive $d$-th root of unity.

Upon examining all possible factors in $\mathbb{C}[x]$ for small values of $d$, we quickly found a simple characterization of a factor of greatest height. There are three separate cases depending on the value of $d$ modulo 3: namely $d = 3t$, $d = 3t + 1$ and $d = 3t - 1$. In the cases $d = 3t$ and $d = 3t - 1$ a factor of greatest height is given by $g_d = \prod_{k=1}^{t-1}(x - \zeta^k)$; moreover, by symmetry we see that $g_d \in \mathbb{R}[x]$. In contrast, the case $n = 3t + 1$ is different: a best factor is $h_d = g_d(x) \cdot (x - \zeta^t) \notin \mathbb{R}[x]$ and there is no best factor which is real. Nevertheless also in the case $d = 3t + 1$ we see that $g_d$ continues to be a real factor of greatest height, and in fact it is not much smaller than $h_d$: empirically we observe that $\text{ht}(g_d)/\text{ht}(h_d) \geq 1/\sqrt{2}$ always, and that $\text{ht}(g_d)/\text{ht}(h_d) \to 1$ as $d$ increases.

Empirically, based on computed values up to $d = 100$ (or equivalently $t = 33$), we find that $\text{ht}(g_d) \approx 0.15 \times 1.37^d$ or equivalently $\log(\text{ht}(g_d)) \approx 0.316d - 1.9$. For a degree $\delta = \left\lceil \frac{(2d+1)/3}{\delta/2} \right\rceil$ which we can approximate using Stirling’s formula to obtain $2^\delta/\sqrt{\pi\delta/2} \approx 1.58^\delta/\sqrt{\delta}$ for large $d$. Here we see that the bounds are too large by roughly 42% in logarithmic terms.

### 6.2 Factors of Polynomials of Height 1 in $\mathbb{Z}[x]$ 

In the preceding section we looked at the family $x^d - 1$ which grows only slowly with degree: in each degree there is precisely one polynomial to consider. Here we consider all polynomials of height 1 and given degree. This family of polynomials increases exponentially with degree: there are $4 \times 3^{d-2}$ cases to consider in degree $d$ after excluding some obvious symmetries and multiples of $x$. We restricted attention to height 1 because allowing greater heights would result in
even faster exponential growth with degree. As a consequence our experimental results extend only to moderately high degrees.

The next two subsections below look at factors in \( \mathbb{Z}[x] \): first only the irreducible factors, then all the factors. We did not investigate factors in \( \mathbb{C}[x] \) since the CoCoALib library does not (yet) have the ability to compute approximate roots in \( \mathbb{C} \).

In the last subsection, we mention briefly the special case of height 1 multiples of powers of \( x + 1 \), a topic which has already attracted the attention of several authors.

### 6.2.1 Large height irreducible factors

In this subsection we are interested in polynomials of height 1 having a large irreducible factor in \( \mathbb{Z}[x] \). We note immediately that the greatest height in a given degree grows much faster than it did for cyclotomic polynomials.

The entries in this table were found using an *ad hoc* program written in C++ which relied upon CoCoALib for polynomial arithmetic and factorization. The approach was “brute force search”, i.e. we factorized each polynomial, and looked for the greatest height irreducible factor. For compactness, in the table we represent each polynomial by the list of its coefficients.
Here are a few observations on the values reported in the table. In every degree there was a largest factor having all positive coefficients. Except for degrees 5 and 9 the factors exhibit coefficients which increase weakly to a maximum value and then decrease weakly again. Several of these extremal examples exhibit ±-palindromic symmetry; in degrees 7 and 10 two palindromic maximal height factors were found, so we listed both cases in the table. In degrees 4, 6, 12, 15, 16, 19 and 21 the maximal height factor was essentially unique (up to reversal and mapping $x \mapsto -x$). In degrees 4 to 9 the cofactor is $x - 1$; in

| Deg | Height | Largest Irreducible Factor & Height 1 Polynomial |
|-----|--------|--------------------------------------------------|
| 4   | 2      | $[1, 2, 1, 1]$ Unique                           |
|     |        | $[1, 1, -1, 0, -1]$                              |
| 5   | 2      | $[1, 2, 1, 2, 1]$                                |
|     |        | $[1, 1, -1, 1, -1, -1]$                          |
| 6   | 3      | $[1, 2, 3, 2, 2, 1]$ Unique                     |
|     |        | $[1, 1, 1, -1, 0, -1, -1]$                      |
| 7   | 3      | $[1, 2, 2, 3, 2, 2, 1]$                         |
|     |        | $[1, 1, 0, 1, -1, 0, -1, -1]$                    |
|     |        | $[1, 2, 3, 2, 3, 2, 1]$                         |
|     |        | $[1, 1, 1, -1, -1, -1, -1, -1]$                  |
| 8   | 4      | $[1, 1, 2, 3, 4, 3, 2, 1]$                       |
|     |        | $[1, 0, 1, 1, 1, -1, -1, -1]$                    |
| 9   | 4      | $[1, 2, 3, 4, 3, 4, 3, 2, 1]$                    |
|     |        | $[1, 1, 1, 1, -1, 1, -1, -1, -1, -1, -1, -1]$   |
| 10  | 5      | $[1, 3, 4, 5, 5, 5, 4, 3, 1]$                    |
|     |        | $[1, 1, -1, 0, -1, 0, -1, 0, -1, 1, 1]$         |
|     |        | $[1, 2, 4, 5, 5, 5, 4, 2, 1]$                    |
|     |        | $[1, 0, 1, -1, 0, -1, 1, 1, 0, 1]$               |
| 11  | 7      | $[1, 2, 4, 6, 7, 6, 4, 3, 1]$                    |
|     |        | $[1, 0, 1, 0, -1, -1, -1, -1, -1, -1, -1, -1]$  |
| 12  | 9      | $[1, 3, 5, 6, 8, 9, 8, 6, 3, 1]$ Unique          |
|     |        | $[1, 1, 1, 1, -1, 1, -1, -1, -1, -1, 1, 1, 1]$  |
| 13  | 9      | $[1, 2, 3, 5, 6, 8, 9, 8, 6, 3, 1]$              |
|     |        | $[1, 0, 0, 1, -1, 1, -1, -1, -1, -1, 1, 1, 1]$  |
| 14  | 11     | $[1, 3, 6, 8, 10, 11, 11, 11, 10, 8, 6, 3, 1]$  |
|     |        | $[1, 1, 1, -1, 0, -1, -1, -1, -1, 0, -1, 1, 1, 1]$ |
| 15  | 14     | $[1, 3, 6, 9, 11, 13, 14, 14, 13, 11, 8, 6, 3, 1]$ Unique |
|     |        | $[1, 1, 1, 0, -1, 0, -1, -1, -1, -1, 1, 1, 1, 1]$ |
| 16  | 16     | $[1, 3, 6, 8, 11, 13, 15, 16, 16, 15, 13, 10, 6, 3, 1]$ Unique |
|     |        | $[1, 1, 1, -1, 1, -1, 0, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1]$ |
| 17  | 17     | $[1, 3, 6, 9, 12, 14, 16, 17, 16, 14, 12, 9, 6, 3, 1]$ |
|     |        | $[1, 0, 0, -1, 0, -1, -1, -1, 1, 1, -1, 1, 1, 0, 1, 0, 0, -1]$ |
| 18  | 25     | $[1, 3, 6, 9, 13, 17, 21, 24, 25, 24, 22, 18, 13, 8, 4, 1]$ |
|     |        | $[1, 0, 0, -1, 1, -1, 0, -1, -1, 0, -1, 1, 1, 1, 0, 1, -1, -1]$ |
| 19  | 33     | $[1, 3, 7, 12, 18, 24, 29, 32, 33, 31, 27, 22, 17, 12, 8, 4, 1]$ Unique |
|     |        | $[1, 0, 1, -1, 0, -1, -1, 0, -1, 1, 1, 1, 0, 1, -1, 1, 1, -1, -1]$ |
| 20  | 39     | $[1, 4, 8, 14, 21, 28, 34, 38, 39, 38, 35, 31, 26, 21, 15, 9, 4, 1]$ |
|     |        | $[1, -1, 1, -1, -1, -1, -1, -1, -1, 1, 0, 1, 0, -1, 1, 1, -1, -1]$ |
| 21  | 43     | $[1, 4, 8, 14, 21, 28, 34, 39, 42, 43, 41, 37, 32, 27, 21, 15, 9, 4, 1]$ |
|     |        | $[1, 1, -1, 1, -1, -1, 1, 0, -1, 0, -1, 0, -1, 1, 1, 0, 1, 0, -1, -1]$ |
degrees 10 to 16 the cofactor is $(x - 1)^2$; and in degrees 17 to 21 the cofactor is $(x - 1)^3$.

From this small table it appears that the heights of the largest irreducible factor grow approximately as $0.7 \times 1.22^d$; however, since the table is rather small, extrapolation from this data is perhaps rather hazardous.

### 6.2.2 Large Height Reducible Factors

In this subsection we are interested in polynomials of height 1 having a large factor in $\mathbb{Z}[x]$. We note that the greatest height in a given degree appears to grow a little faster than it did for irreducible factors.

The table below summarises the results obtained. The entries in this table were found using an ad hoc program written in C++ which relied upon CoALib for polynomial arithmetic and factorization. The approach was “brute force search”: for each polynomial we computed all possible products of its irreducible factors, and measured their heights.

| Deg | Height | Largest Factor & Height 1 Polynomial |
|-----|--------|-------------------------------------|
| 3   | 2      | $[1, 2, 1]$                         |
|     |        | $[1, 1, -1, -1]$                    |
| 4   | 2      | $[1, 2, 1]$                         |
|     |        | $[1, 1, 0, 1, 1]$                   |
| 5   | 3      | $[1, 2, 3, 2, 1]$                   |
|     |        | $[1, 1, 1, -1, -1, -1]$             |
| 6   | 2      | $[1, 2, 1]$                         |
|     |        | $[1, 0, -1, 0, -1, 0, 1]$           |
| 7   | 4      | $[1, 2, 3, 4, 3, 2, 1]$             |
|     |        | $[1, 1, 1, 1, -1, -1, -1, -1]$      |
| 8   | 4      | $[1, 3, 4, 3, 1]$                   |
|     |        | $[1, 0, -1, 0, 0, -1, 0, 1]$        |
| 9   | 7      | $[1, 4, 7, 7, 4, 1]$                |
|     |        | $[1, 1, -1, -1, 0, 0, -1, -1, 1, 1]$|
| 10  | 10     | $[1, 4, 8, 10, 8, 4, 1]$            |
|     |        | $[1, 1, 0, -1, -1, 0, -1, -1, 0, 1, 1]$|
| 11  | 13     | $[1, 4, 9, 13, 13, 9, 4, 1]$        |
|     |        | $[1, 1, -1, -1, -1, -1, -1, -1, 1, 1, 1]$|
| 12  | 9      | $[1, 3, 6, 8, 9, 9, 8, 6, 3, 1]$    |
|     |        | $[1, 1, 1, -1, -1, 0, -1, -1, -1, 1, 1, 1]$|
| 13  | 22     | $[1, 5, 12, 19, 22, 19, 12, 5, 1]$  |
|     |        | $[1, 1, -1, -1, -1, 0, 0, 1, 1, 1, 1, 1, 1, -1, -1]$|
| 14  | 17     | $[1, 4, 9, 14, 17, 17, 14, 9, 4, 1]$|
|     |        | $[1, 0, 0, -1, 0, -1, 0, 1, 0, 1, 0, 0, -1]$|
| 15  | 30     | $[1, 5, 13, 23, 30, 30, 23, 13, 5, 1]$|
|     |        | $[1, 0, -1, -1, -1, 0, 1, 1, 0, 1, -1, -1, -1, 0, 1]$|
| 16  | 29     | $[1, 4, 9, 15, 21, 26, 29, 29, 26, 21, 15, 9, 4, 1]$|
|     |        | $[1, 1, 0, -1, -1, -1, -1, 0, 1, 1, 1, 1, 1, 0, -1, -1]$|

The $\pm$-palindromic symmetry present in these extreme examples is evident, as is the fact that the factors have positive coefficients which increase strictly
towards the maximal central values. There seems to be no particular pattern to the cofactors.

Beyond degree 16 it was impractical to conduct an exhaustive search. Instead, given that every extreme example up to degree 16 found by exhaustive search exhibited ±-palindromic symmetry, we considered only ±-palindromic polynomials in higher degrees — this restriction greatly reduced the number of cases to consider, and thus the computation time. It seems plausible to suppose that the examples found in this restricted search are nonetheless extremal amongst all height 1 polynomials of that same degree. To keep the table compact we give only about half the coefficients; the elided coefficients can easily be filled in because all the polynomials are ±-palindromic. The entries in this table were obtained using a slightly modified version of the program used for the table above.

| Deg | Height | Largest Factor & Height 1 Polynomial |
|-----|--------|-------------------------------------|
| 17  | 42     | [1, 5, 13, 24, 35, 42, 42, 35,...]   |
|     |        | [1, 0, −1, 0, −1, 1, 1, ... , 1]   |
| 18  | 42     | [1, 5, 13, 24, 35, 42, 42, 35,...]   |
|     |        | [1, 0, 0, 1, −1, −1, 1, 1, ... , 1] |
| 19  | 55     | [1, 4, 10, 19, 30, 41, 50, 55, 55, 50,...] |
|     |        | [1, 0, 0, −1, −1, 0, 1, 1, ... , 1] |
| 20  | 65     | [1, 6, 18, 36, 54, 65, 65, 54,...] |
|     |        | [1, 1, 0, −1, −1, 0, −1, 0, 1, 0,... , 1] |
| 21  | 110    | [1, 5, 14, 29, 49, 71, 91, 105, 110, 105,...] |
|     |        | [1, 0, −1, −1, −1, 0, 1, 1, 1, 1, 0,... , 1] |
| 22  | 110    | [1, 5, 14, 29, 49, 71, 91, 105, 110, 105,...] |
|     |        | [1, −1, −1, 0, 0, 1, 1, 0, 0, −1, 0,... , 1] |
| 23  | 161    | [1, 7, 24, 55, 96, 136, 161, 161, 136,...] |
|     |        | [1, 1, −1, −1, −1, 0, 0, 1, 1, 0, 0,... , 1] |
| 24  | 173    | [1, 6, 19, 42, 73, 107, 138, 161, 173, 173, 161,...] |
|     |        | [1, 0, −1, −1, 0, 0, −1, 1, 1, 1, 0, 1,... , 1] |
| 25  | 238    | [1, 7, 25, 61, 114, 173, 220, 238, 220,...] |
|     |        | [1, 0, −1, −1, −1, 1, 0, 1, 1, 0, 0,... , 1] |
| 26  | 233    | [1, 6, 19, 43, 78, 120, 162, 197, 221, 233, 223, 221,...] |
|     |        | [1, 0, −1, −1, −1, 0, 1, 1, 1, 1, 1, 0,... , 1] |
| 27  | 356    | [1, 7, 25, 62, 121, 197, 275, 334, 356, 334,...] |
|     |        | [1, 0, −1, −1, −1, 0, 1, 1, 1, 0, 0, −1,... , 1] |
| 28  | 371    | [1, 5, 15, 34, 64, 105, 155, 210, 265, 314, 351, 371, 371, 351,...] |
|     |        | [1, 0, 0, −1, −1, 0, 0, 1, 1, 1, 1, 0,... , 1] |
| 29  | 560    | [1, 6, 19, 44, 84, 140, 210, 289, 370, 445, 506, 546, 560, 546, 506,...] |
|     |        | [1, 1, −1, −1, −1, −1, 0, 1, 1, 1, 1, 1,... , 1] |

In both of the tables above every one of the largest factors is a product of powers of cyclotomic polynomials, with the exception of the height 560 polynomial which is a product of cyclotomic polynomials together with the irreducible polynomial $g_8 = x^8 + x^7 + x^5 + x^4 + x^3 + x + 1$.

From the values in this second table we see that the height of the largest factor increases roughly exponentially as $1.24^d$ where $d$ is the degree — though
it is also clear that the heights do not form an increasing sequence. We note that the rate of growth is only slightly greater than that for irreducible factors.

### 6.2.3 Height 1 multiples of \((x + 1)^n\) or \((x - 1)^n\)

Here we shall see a simple way of constructing height 1 polynomials having a large factor, namely \((x + 1)^n\) or \((x - 1)^n\). We recall that both \((x + 1)^n\) and \((x - 1)^n\) have height \(\lfloor n/2 \rfloor \approx 2^n/\sqrt{2\pi n}\). For our purposes \((x + 1)^n\) and \((x - 1)^n\) are virtually interchangeable since if \((x + 1)^n\) divides \(f(x)\) then \((x - 1)^n\) divides \(f(-x)\), and \textit{vice versa}.

Now, for any \(n\), we construct explicitly a polynomial of height 1 having \((x - 1)^n\) as a factor. Clearly \(x - 1\) divides \(x^d - 1\) for any positive integer \(d\). Consequently, \((x - 1)^n\) divides the product \(f = \prod_{k=1}^{n}(x^{e_k} - 1)\) for any choice of the exponents \(e_k\). In general the product may have large height, but choosing the exponents judiciously, for instance \(e_k = 2^{k-1}\), gives us a polynomial of height 1 (and degree \(2^n - 1\) in this instance). We note also that the cofactor \(f/(x - 1)^k\) is a product of polynomials of the form \((x^{e_k} - 1)/(x - 1)\) all of whose coefficients are equal to 1. We can easily find a lower bound for the height of the cofactor: by considering the degree of the cofactor, and the values at \(x = 1\) of each factor \((x^{e_k} - 1)/(x - 1)\) we see that its height is at least \(\prod e_j/(1 + \sum(e_j - 1))\).

A more general result about large factors of polynomials is Theorem 4 in [Mig88] which proves that we can find small height multiples of polynomials in \(\mathbb{Z}[x]\). In particular, from it we can deduce that for any \(n\) there exist height 1 multiples of \((x + 1)^n\) of degree less than \(n^2 \log n\) — a far lower degree than we get from the simple construction above. Alternatively, by writing \(d = n^2 \log n\), we can conclude that there are height 1 polynomials of degree at most \(d\) having a factor of height at least \(2^{d-1}/\sqrt{a}\) where \(a = \sqrt{2d}/\log d\).

In an earlier paper, Mignotte [Mig81] gave an (exponential) algorithm for finding polynomials with coefficients in \([-1, 0, 1]\) which are divisible by any specified power of \(x + 1\). We used the algorithm to find the lowest degree examples for each \(n \leq 8\); for the case \(n = 8\) we found [Abb89] the lowest degree height 1 multiple to be \(f_{41} = (x + 1)^8 \cdot g_{33}(-x)\) where

\[
\begin{align*}
f_{41} &= x^{41} - x^{40} - x^{39} + x^{36} + x^{35} - x^{33} + x^{32} - x^{30} - x^{27} + x^{23} + x^{22} \\
&\quad - x^{21} - x^{20} + x^{19} + x^{18} - x^{14} - x^{11} + x^{9} - x^{8} + x^{6} + x^{5} - x^2 - x + 1
\end{align*}
\]

and the cofactor is

\[
g_{33} = x^{33} + 7x^{32} + 27x^{31} + 76x^{30} + 174x^{29} + 343x^{28} + 603x^{27} + 968x^{26} + 1442x^{25} + 2016x^{24} + 2667x^{23} + 3359x^{22} + 4046x^{21} + 4677x^{20} + 5202x^{19} + 5578x^{18} + 5774x^{17} + 5774x^{16} + 5578x^{15} + 5202x^{14} + 4677x^{13} + 4046x^{12} + 3359x^{11} + 2667x^{10} + 2016x^9 + 1442x^8 + 968x^7 + 603x^6 + 343x^5 + 174x^4 + 76x^3 + 27x^2 + 7x + 1
\]

We observe that \(f_{41}(-x)\) is indeed of the form \(\prod_{k=1}^{n}(x^{e_k} - 1)\) with \(n = 8\) and exponents \(e_k : 1, 2, 3, 4, 5, 7, 8, 11\). A remarkable example of this form is
which has exponents $e_k : 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13$; it is a polynomial of
degree 69 having a factor of height $2930202$. These two polynomials along with
many other similar ones are mentioned in the article [BM00] where Borwein and
Mossinghof study specifically and in depth the question of the lowest degree
height 1 multiples of $(x + 1)^n$.

**Note** We observe that certain height 1 polynomials of the form $\prod_{k=1}^{n} (x^{e_k} - 1)$ have
the unusual property that some of the factors in the square-free decomposition have
large height; a concrete example is given by $e_k : 3, 4, 5$. We recall that one of the first
steps in polynomial factorization is to compute the square-free decomposition.

## 7 Large Ratio Factorizations in $\mathbb{Z}[x]$

In section 6 we looked at factorizations having (at least) one large factor, in this
section we shall look at factorizations where all the factors are large. Our main
interest is in “short” factorizations of the form $f = g_1g_2 \in \mathbb{Z}[x]$, *i.e.* with two
(non-trivial) factors. We concentrate our attention on this type of factorization
partly because that is conceptually the simplest situation, and partly because
that was the only case where it is feasible to conduct thorough searches on the
computer (beyond the lowest degrees). Later on in section 8 we will see how to
construct “longer” examples with more than two irreducible factors.

In the Conclusion of the paper [BTW93] the authors wondered whether a
polynomial $f \in \mathbb{Z}[x]$ must always have at least one small irreducible factor,
*i.e.* whose height is no greater than that of $f$. Collins gave a negative answer
in [Col04] where he published several quite simple examples, including two factorizations having ratio slightly greater than 2. In this section we present many
more counter-examples, most with much larger ratio, even exceeding 10 in a few
cases. On the basis of these examples it seems reasonable to conjecture that
there exist examples exhibiting arbitrarily great ratio.

The large ratio factorizations presented here also have a direct implication
for the single factor bounds of section 4 which bound the size of at least one
factor in any factorization. Thus the example factorizations in this section force
the single factor bounds to be “large”.

### 7.1 The Search is Theoretically Finite

Many of the examples we give here are probably extremal, *i.e.* there is no other
polynomial of the same degree having a factorization $f = g_1g_2 \in \mathbb{Z}[x]$ exhibiting
a greater ratio of $\min\{\text{ht}(g_1), \text{ht}(g_2)\}/\text{ht}(f)$. Unfortunately, proving extremality
seems to be rather difficult. In [Rum06] Rump studied the behaviour of the
two-norm under multiplication in $\mathbb{R}[x]$: in particular he looked at the minimal
value $\mu = \min\{|g_1g_2|_2 : |g_1|_2 = |g_2|_2 = 1\}$ where the degrees of $g_1$ and $g_2$
are fixed. The fact that $\mathbb{R}[x]$ contains no zero-divisors together with a simple
closedness and continuity argument show that the minimum must be strictly
positive. Rump reports that actually computing the minimum in general is hard.
Nevertheless, knowing that $\mu > 0$ lets us prove the following lemma — its
practical usefulness is severely limited by the difficulty in obtaining lower bounds for \(\mu\) and by the computational cost of a truly exhaustive search. However, the lemma does imply the existence of attainable rational upper bounds \(\beta(d_1, d_2)\) for the ratio of short factorizations \(f = g_1 g_2 \in \mathbb{Z}[x]\) with \(\text{deg}(g_1) = d_1\) and \(\text{deg}(g_2) = d_2\).

**Lemma 7.1** Let \(d_1, d_2 \in \mathbb{N}\) be positive. There exists a constant \(H\) depending on \(d_1\) and \(d_2\) such that if \(g_1, g_2 \in \mathbb{Z}[x]\) are of degree \(d_1\) and \(d_2\) respectively and \(\text{ht}(g_1) > H\) or \(\text{ht}(g_2) > H\) then \(\text{ht}(g_1 g_2) > \min\{\text{ht}(g_1), \text{ht}(g_2)\}\). In other words: polynomials in \(\mathbb{Z}[x]\) of fixed degree and having factorizations with ratio greater than 1 have bounded height.

**Proof** Suppose the contrary, i.e. there exists an infinite sequence of distinct pairs \((g_1^{(1)}, g_2^{(1)}), (g_1^{(2)}, g_2^{(2)}), \ldots\) of unlimited height exhibiting ratio greater than 1, with every \(\text{deg}(g_1^{(k)}) = d_1\) and \(\text{deg}(g_2^{(k)}) = d_2\). First note that the heights of both \(g_1^{(k)}\) and \(g_2^{(k)}\) are unlimited since otherwise there would be a finite limit on the height of \(g_1^{(k)} g_2^{(k)}\). But there are only finitely many polynomials of degree \(d_1 + d_2\) and bounded height, and unique factorization in \(\mathbb{Z}[x]\) would imply only finitely many possible pairs.

Thus for any \(B > 0\) there exists a pair \((g_1, g_2)\) with \(\text{ht}(g_1) > B\) and \(\text{ht}(g_2) > B\) and ratio \(R < 1\). Let \(l_1 = |g_1|^2\) and \(l_2 = |g_2|^2\), then we clearly also have \(l_1 \geq \text{ht}(g_1) > B\) and \(l_2 \geq \text{ht}(g_2) > B\). Put \(g_1 = g_1/l_1\) and \(g_2 = g_2/l_2\), so \(|g_1| = |g_2| = 1\). Considering their product, we see that

\[
|g_1 g_2|^2 \leq \sqrt{d_1 + d_2} \text{ht}(g_1 g_2) = \sqrt{d_1 + d_2} \text{ht}(g_1 g_2)/l_1 l_2 < \sqrt{d_1 + d_2} R \min\{\text{ht}(g_1), \text{ht}(g_2)\}/l_1 l_2 < \sqrt{d_1 + d_2} R/l_s < \sqrt{d_1 + d_2}/B
\]

where \(s\) is the index of whichever of \(g_1\) and \(g_2\) has greater height. This contradicts the strict positivity of \(\mu\). \(\square\)

**7.2 A Family of Irreducible Factorizations of Ratio > 1**

Here is a simple result showing that there are irreducible factorizations with ratio > 1 in every even degree (from 10 upwards) — the result depends on a conjecture (see below).

Let \(n > 1\) be an integer, and let \(g_1 = nx^2 - (2n - 1)x + n \in \mathbb{Z}[x]\) and \(g_2 = x^{n+1} + x^n + x^{n-1} + \cdots + 1)(x^n + x^{n+1} + x^n + \cdots + 1) \in \mathbb{Z}[x]\). Note that \(\text{ht}(g_2) = n + 2\). It is not hard to show that the product \(g_1 g_2\) has height \(n + 1\); similarly, it can be shown that both \(g_1(g_2 x + 1)\) and \(g_1(g_2 x^3 + x^2 + x + 1)\) also have height \(n + 1\). Now \(g_2\) is obviously reducible, but we make the following conjecture — verified for all \(n < 1000\).
Conjecture
(with the notation of the preceding paragraph)

• $g_2x + 1$ is irreducible whenever $n \not\equiv 2 \mod 3$,
• $g_2x^3 + x^2 + x + 1$ is irreducible whenever $n \equiv 2 \mod 3$.

Note In practice, for small values of $n$ it seems to be easy to find irreducible polynomials $\tilde{g}_2$ of height $2n - 1$ for which the product $g_1\tilde{g}_2$ has height $n$, though we have not managed to discern any obvious pattern in such polynomials. We do note that the minimal degree of suitable $\tilde{g}_2$ does rise quickly as $n$ increases: starting from $n = 2$ the first few minimal degrees are 7, 11, 14, 20, 22, 26.

7.3 A Family of Factorizations with Unlimited Ratio

Here we present an explicit family where the ratio increases exponentially with degree. Though we are primarily interested in irreducible factorizations, the bounds in section 4 apply to any factorization; so this family constrains those bounds to grow rapidly.

Let $f(x) = (x+1)(x^2+x+1) = x^3 + 2x^2 + 2x + 1$. Then $1 - x^6 = f(x) \cdot f(-x)$ is a $*$-symmetric factorization. We obtain our family simply by taking powers: namely, $(1 - x^6)^k = f^k(x) \cdot f^k(-x)$. We estimate the height of $(1 - x^6)^k$ using the binomial theorem and Stirling's approximation: $\text{ht}((1 - x^6)^k) \approx 2^{k+1}/\sqrt{2\pi k}$. We do not have an explicit formula for the asymptotic height of $f^k$ — but see Conjecture 1 in the Conclusion. Nevertheless, by considering the value of $f(x)$ at $x = 1$, we can deduce that $\text{ht}(f^k) \geq 6^k/(3k+1)$. Hence the ratio of the factorization is greater than $3^k/(3k+1)$. From this we obtain $1.2^d$ as an asymptotic lower bound for the greatest ratio among (reducible) factorizations of polynomials of degree $d$.

Note An exhaustive search in low degrees suggests that this family of factorizations is close to extremal.

7.4 Extremal Short Factorizations

In this section we present the results of some extensive computer searches for large ratio short factorizations in $\mathbb{Z}[x]$. The searches were conducted degree by degree, and in each degree we tried all possibilities for the degrees of the two factors. As we commented earlier it is hard to be absolutely certain that these examples are extremal; however, if there are other factorizations exhibiting a greater ratio then the factors must have considerably larger height than the example given in the table.

The examples in the table are not generally unique (i.e. in most cases there are other polynomials of the same degree whose factorizations exhibit the same ratio). Up to and including degree 14 the examples are almost certainly extremal; we think the examples in degrees 15, 16 and 17 are probably extremal,
but doubt that the example in degree 18 is extremal. The computations were done using an ad hoc program in C++.

| Deg | Ratio | Example factorization |
|-----|-------|-----------------------|
| 5   | 2     | $[1, 2, 1] \times [1, -2, 2, -1]$ |
| 6   | 2     | $[1, 2, 2, 1] \times [1, -2, 2, -1]$ |
| 8   | 4     | $[1, 3, 4, 3, 1] \times [1, -3, 4, -3, 1]$ |
| 9   | 4     | $[1, 3, 4, 3, 1] \times [1, -3, 4, -4, 3, -1]$ |
| 10  | 5     | $[1, 3, 5, 5, 3, 1] \times [1, -3, 5, -5, 3, -1]$ |
| 11  | 7     | $[1, 4, 7, 7, 4, 1] \times [1, -4, 8, -10, 8, -4, 1]$ |
| 12  | 7     | $[1, 4, 7, 7, 4, 1] \times [1, -4, 8, -11, 11, -8, 4, -1]$ |
| 13  | 11    | $[1, 5, 11, 14, 11, 5, 1] \times [1, -4, 8, -11, 11, -8, 4, -1]$ |
| 14  | 12    | $[2, 9, 19, 24, 19, 9, 2] \times [1, -5, 13, -22, 26, -22, 13, -5, 1]$ |
| 15  | 18    | $[1, 5, 12, 18, 12, 5, 1] \times [1, -5, 12, -19, 22, -19, 12, -5, 1]$ |
| 16  | 18    | $[1, 6, 17, 30, 36, 30, 17, 6, 1] \times [2, -10, 25, -41, 48, -41, 25, -10, 2]$ |
| 17  | 24    | $[1, 6, 18, 35, 48, 48, 35, 18, 6, 1] \times [2, -10, 25, -41, 48, -41, 25, -10, 2]$ |
| 18  | 23    | $[1, 5, 12, 19, 23, 23, 19, 12, 5, 1] \times [1, -5, 12, -19, 23, -23, 19, -12, 5, -1]$ |

At the higher degrees we limited the searches heuristically. Beyond degree 13 we considered only pairs whose degrees differ by at most 2, because all the best examples found in lower degrees had this property. Up to degree 14 it is very evident that there is always a best example whose factors are palindromic, thus beyond degree 14 we searched only for pairs of palindromic polynomials.

It is also clear that the factorizations are of the form $g_1(x) \cdot g_2(-x)$ where $g_1$ and $g_2$ have positive coefficients which increase strictly towards the central terms — the same phenomenon we observed in section 6.2.2. We note the curious fact that the examples in degrees 16 and 17 contain the same non-monic factor.

### 7.5 Extremal Irreducible Short Factorizations

Here we look at factorizations $f = g_1g_2 \in \mathbb{Z}[x]$ with the additional requirement that the two factors be irreducible. We will see that this extra requirement limits severely the growth of the ratio compared to what we observed in the previous subsection. Ideally, for the purposes of polynomial factorization, we would like to have a single factor bound which is valid only for irreducible factors in $\mathbb{Z}[x]$. The examples here show that a single factor bound specific to irreducible factors could be far smaller than the ones we currently know: from this table we see that the log of the ratio grows roughly as $0.07d - 0.3$ where $d$ is the degree.

The computations were done using an ad hoc program in C++ which conducted a “weak irreducibility” test (i.e. which simply tested for divisibility by a few commonly occurring polynomials, primarily low index cyclotomics). The final verification of irreducibility was effected using CoCoA.
### 7.6 Extremal Palindromic $*$-Symmetric Factorizations

In the previous subsection the exponential increase in candidate factorizations made it impractical to conduct full searches at higher degrees. Inspired by the appearance of palindromicity and $*$-symmetry in earlier examples, and realizing that these symmetries greatly reduce the number of candidates to consider, we decided to restrict attention to palindromic $*$-symmetric factorizations. While this will not let us find the probable extreme ratio in each degree it does at least establish a lower bound for that value.

Since any odd degree palindromic polynomial is always divisible by $x + 1$, and since we are seeking palindromic $*$-symmetric irreducible short factorizations, we consider only degrees which are multiples of 4. The search was effected using an *ad hoc* C++ program which conducted a brute force search with some

| Deg | Ratio | Example factorization |
|-----|-------|-----------------------|
| 7   | 1.25  | $[3,5,3] \times [-1,3,-5,4,-3,1]$ |
| 8   | 1.50  | $[2,4,6,5,2] \times [2,-5,6,-4,2]$ |
| 9   | 1.50  | $[2,3,2] \times [1,-1,1,0,-2,3,-2,1]$ |
| 10  | 1.50  | $[1,2,3,3,2] \times [1,-2,3,-1,-1,1]$ |
| 11  | 1.67  | $[2,4,5,4,2] \times [1,-1,0,2,-4,5,-3,1]$ |
| 12  | 2.00  | $[1,1,1,0,-1,-2,-1] \times [-1,2,-1,0,1,-1,1]$ |
| 14  | 2.00  | $[1,1,0,-1,0,1,2,1]$ |
| 16  | 2.16  | $[2,6,8,3,-6,-13,-12,-6,-1]$ |
| 18  | 2.71  | $[4,12,19,18,9,-2,-7,-6,-3,-1]$ |
| 20  | 3.00  | $[1,1,0,-1,-1,-1,0,2,3,2,1]$ |
| 22  | 3.50  | $[2,6,11,14,13,7,0,-5,-8,-9,-6,-2]$ |
| 24  | 4.25  | $[2,6,10,11,8,0,-10,-17,-16,-8,1,4,2]$ |

The examples given in the table are not generally unique (i.e. in most cases there are other pairs of polynomials exhibiting the same ratio). Up to and including degree 12 the examples are very probably extremal, i.e. any example exhibiting a greater ratio must have factors of considerably larger height. It is quite possible that in degrees 22 and higher there are examples of slightly greater height exhibiting greater ratios; the necessary computations were not attempted because they would take unreasonably long.

It is very evident that in the even degrees (up to and including degree 12) there is always a $*$-symmetric best example (i.e. where $g_1(x) = \bar{g}_2(-x)$). To save time in the searches beyond degree 12, we restricted ourselves to even degrees and to $*$-symmetric factorizations.

Note the example in degree 16 exhibits a greater ratio than Collins’s rather larger “winning” polynomial (on page 1518 in [Col04]): on the following page he claims an example of the same degree with ratio about 2.20, but we suspect that there is a misprint, and that that polynomial is the same as our example in degree 20 given in the next subsection. Our example in degree 20 is much smaller than Collins’s examples and exhibits a significantly larger ratio, namely 3.00.
simple pruning criteria; irreducibility of the factors was subsequently verified using CoCoA [CoCoA]. Here is the resulting table:

| Deg | Ratio | Corresponding $g_1$ |
|-----|-------|---------------------|
| 8   | 1.25  | [2, 4, 5, 4, 2]     |
| 12  | 1.48  | [5, 17, 31, 37, 31, 17, 5] |
| 16  | 1.69  | [13, 63, 157, 256, 299, 256, 157, 63, 13] |
| 20  | 2.20  | [29, 175, 543, 1119, 1683, 1921, 1683, 1119, 543, 175, 29] |
| 24  | 3.28  | [4, 14, 22, 13, −17, −53, −69, −53, −17, 13, 22, 14, 4] |
| 28  | 4.25  | [2, 6, 10, 10, 4, −6, −14, −17, −14, −6, 4, 10, 10, 6, 2] |
| 32  | 7.22  | [3, 13, 29, 41, 37, 12, −24, −54, −65, . . . , 3] |
| 36  | 11.37 | [6, 34, 98, 182, 234, 194, 41, −181, −377, −455, . . . , 6] |
| 40  | 13.75 | [6, 33, 93, 175, 243, 249, 158, −26, −248, −427, −495, . . . , 6] |

As in the other subsections here, it is difficult to be completely certain of extremality; nevertheless we believe the examples up to and including degree 32 to be extremal among palindromic $*$-symmetric factorizations since any example exhibiting greater ratio must have factors of considerably larger height. The example in degree 36 is probably extremal too, but the search was deliberately limited to where we expected to find a good example. The example in degree 40 is unlikely to be extremal: several *ad hoc* tricks were used to make the search faster (*i.e.* less slow).

We note that from degree 24 the best example in each degree contains negative coefficients, and has much smaller height compared to the best example in degree 20. Collins in [Col04] chose to consider only positive coefficients and reported finding no example with ratio greater than 2.20. We suspect that the factorization in degree 20 is the example hinted at on page 1519 in [Col04], though he reported (apparently mistakenly) that it was in degree 16. We point out that our degree 20 example in the previous subsection has a larger ratio.

### 7.7 Height 1 Polynomials with 2 Large Irreducible Factors

After looking at the examples in the preceding subsection one might think it is necessary to consider polynomials with ever larger coefficients in order to find large ratio (short) factorizations. Here we see that apparently it is enough to consider polynomials of height 1. While the few examples here and in the next subsection certainly do not prove that we can go on indefinitely, they seem to be enough to let us conjecture there are polynomials of height 1 whose irreducible factorizations exhibit arbitrarily great ratios.

These examples (and those in the following subsection) are particularly interesting because, together with the result in section S, they show that there are polynomials of height 1 with arbitrarily many irreducible factors of large height (at least up to 11).

Here is a small table of factorizations of polynomials of height 1 which exhibit increasing ratios. The first three examples are surely of minimal degree; we are not sure whether the others are of lowest possible degree exhibiting that ratio. The $*$-symmetric examples with ratios 4 and 5 are of minimal degree amongst
-symmetric factorizations with those ratios. The search was effected using an ad hoc C++ program which conducted a brute force search with some simple pruning criteria; irreducibility of the factors was subsequently verified using CoCoA [27].

| Ratio | Deg | Factorization |
|-------|-----|---------------|
| 2     | 12  | [1, 2, 1, 0, −1, −1, −1, −1, −1, 0, 1, 1, 1, 1] *-symmetric |
| 3     | 20  | [1, 2, 3, 2, 0, −1, −1, −1, 0, 1, 1, 1, 1] *-symmetric |
| 4     | 27  | [1, 1, 0, −2, −4, −4, −2, 0, 2, 3, 3, 2, 1] × [1, −2, 2, 0, −1, 0, 1, 0, −1, 1, 0, 0, −2, 4, −3, 1] |
| 5     | 30  | [1, 1, −1, −4, −5, −4, −1, 1, 1, 1] × [1, −2, 2, −1, −1, 3, −4, −2, −1, 4, −5, −4, −1, −2, 4, −4, 3, −1, −1, −2, −2, 1] |
| 6     | 32  | [1, 2, 2, 2, 2, 0, −3, −5, −5, −4, −2, 0, 2, 3, 3, 2, 1] *-symmetric |
| 7     | 40  | [1, 2, 2, 0, −2, −3, −1, 2, 4, 4, 3, 1, −2, −5, −6, −4, −1, 1, 2, 2, 1] *-symmetric |

7.8 Height 1 Polynomials with Large Ratio Factorizations: palindromic *-symmetric factors

Extending the searches of the previous subsection to ratios greater than 6 would have been prohibitively expensive. Somewhat arbitrarily, we decided to restrict consideration to palindromic *-symmetric factorizations in an attempt to find examples with greater ratio.

The examples here suggest that for any height \( H \) there exist polynomials of height 1 having irreducible *-symmetric palindromic factorizations of ratio at least \( H \). The table below exhibits for each height one of the factors; for compactness, we have used ellipsis for the higher degree factors. Up to and including height 6, the polynomials are surely of minimal degree; for heights 7 and greater, we restricted the search space, and so could conceivably have missed some lower degree example. We note that the degrees are not always increasing. There appears to be no obvious pattern.

| Ht | Deg | Palindromic factor |
|----|-----|-------------------|
| 2  | 32  | [1, 1, 1, 1, 1, 0, −1, −2, −2, −2, −1, 0, 1, 1, 1, 1] |
| 3  | 36  | [1, 1, 1, 1, 1, 1, 0, −1, −2, −3, −2, −1, 0, 1, 1, 1, 1] |
| 4  | 68  | [1, 3, 4, 3, 1, −1, −3, −4, −3, −1, 1, 2, 2, 1, 0, −1, −1, −1] |
| 5  | 76  | [1, 2, 2, 1, 0, 1, 2, 2, 1, 0, −1, −1, 0, 1, 1, 0, −2, −4, −5] |
| 6  | 72  | [1, 2, 2, 1, 0, −1, −2, −3, −3, −3, −3, −2, −1, 0, 2, 5, 6, 6] |
| 7  | 100 | [1, 3, 5, 6, 5, 2, −1, −2, −1, 1, 3, 3, 1, −2, −5, −7, −7, −5, −5, −2, 0, −1, −1, 0, 2, 4, 5, 5] |
| 8  | 84  | [1, 3, 5, 5, 2, −3, −7, −8, −6, −2, 2, 3, 1, −2, −4, −4, −1, 3, 5, 4, 2, 1, 1] |
| 9  | 88  | [1, 3, 5, 6, 6, 5, 3, 0, −3, −5, −6, −7, −8, −9, −9, −9, −7, −3, 1, 4, 6, 7, 7] |
| 10 | 120 | [1, 3, 4, 3, 1, −1, −3, −5, −7, −8, −7, −5, −3, −1, 1, 3, 5, 7, 9, 10, 9, 6, 3, 0, −3, −5, −5, −4, −3, −3, −3] |
| 11 | 100 | [1, 3, 5, 5, 2, −3, −7, −8, −6, −2, 3, 7, 8, 5, 0, −4, −5, −3, 1, 5, 7, 6, 2, −4, −9, −11] |
7.9 Large Ratio Factorizations of \( x^d - 1 \) in \( \mathbb{R}[x] \) and \( \mathbb{C}[x] \)

In this subsection we extend briefly our horizon to encompass factorizations in \( \mathbb{R}[x] \) and \( \mathbb{C}[x] \). The main reason for doing so is that the single factor bounds are valid for any factorization in \( \mathbb{C}[x] \) satisfying the scaling hypothesis. We shall see that this wide applicability of the bounds forces them to much larger than necessary for bounding factors in \( \mathbb{Z}[x] \).

In section 7 we concentrated on short factorizations (i.e., with just two factors), and it is natural to ask whether there are polynomials having more than two irreducible factors each of which has greater height than the original polynomial.

In this subsection we extend briefly our horizon to encompass factorizations in \( \mathbb{R}[x] \) and \( \mathbb{C}[x] \). We do not have a proof of correctness of the empirical formula above, but we estimate using Stirling’s formula to obtain \( 2^{\frac{d}{2}} \approx 0.97 \). Thus we see that the bounds are too large by only about 20% in logarithmic terms.

Empirically, based on computed values up to \( d = 100 \), we observe that \( \log \text{ht}(u_d) \approx 0.97 \times 1.34^n \), or equivalently \( \log \text{ht}(u_d) \approx 0.29d - 2.33 \). For a degree \( \lfloor d/2 \rfloor \) factor the best bounds are the binomial bound (with \( \rho = 1 \)) or Mignotte’s bound (with \( M(f) = 1 \)): they allow height up to \( \left( \begin{array}{c} d/2 \\ d/4 \end{array} \right) \) which we can approximate using Stirling’s formula to obtain \( 2^{1+d/2} / \sqrt{\pi d} \approx 1.41^{d/2} / \sqrt{d} \). Thus we see that the bounds are too large by only about 20% in logarithmic terms.

We do not have a proof of correctness of the empirical formula above, but we can prove a lower bound on the height of \( u_d \). We use the factorization \( u_d(x) = (x - 1) \prod_{k=1}^{t} (x^2 - 2 \cos(k \theta_d)x + 1) \). Since \( u_d \) has degree \( 2d + 1 \) it must have height at least \( \log |u_d(-1)| / (2d + 2) \). We estimate \( |u_d(-1)| \) using the factorization \( u_d(-1) = -2 \prod_{k=1}^{t} (2 + 2 \cos(k \theta_d)) \). Observing that for each index \( k \) we have \( \cos(k \theta_d) + \cos((t + 1 - k) \theta_d) > 1 \), it is easy to show that \( \prod_{k=1}^{t} (1 + \cos(k \theta_d)) > 2^{t/2} \). Hence \( |u_d(-1)| \geq 2^{t/2} = 2^{(3d+2)/8} \). And we can thus conclude that asymptotically \( \log \text{ht}(u_d) \geq 0.2599d \).

For the case \( d = 4t + 4 \), we obtain a largest ratio \( * \)-symmetric factorization \( x^d - 1 = v_d(x) \cdot v_d(-x) \in \mathbb{C}[x] \) where \( v_d(x) = u_d(x) \cdot (x + i) \). Empirically, we observe that \( \log \text{ht}(v_d) \approx 0.29d - 2.33 \) which is the same growth rate seen above for the factor \( u_d \). The asymptotic lower bound continues to be valid in this case too.

8 Longer Large Ratio Irreducible Factorizations

In section 7 we concentrated on short factorizations (i.e., with just two factors), and it is natural to ask whether there are polynomials having more than two irreducible factors each of which has greater height than the original polynomial.

Direct search is limited by its exponential nature. We did not succeed in finding in a reasonable time any small examples (i.e., with three irreducible
factors each of low degree and low height). Extending the search looks to be hopeless. Nevertheless we believe that fairly small examples exist, but we just do not know a good way to find them. Below we present a construction which allows us to create high degree examples with an even number of irreducible factors; currently we know of no similar method for constructing examples with an odd number of irreducible factors.

8.1 A Handy Lemma

Lemma 8.1 Let \( f \in \mathbb{Z}[x] \) be irreducible and not cyclotomic. Then we have the following:

1. \( f(x^p) \) is reducible in \( \mathbb{Z}[x] \) for only finitely many primes \( p \)

2. if \( p \) is an odd prime, and both \( f(x) \) and \( f(x^p) \) are irreducible then \( f(x^{p^k}) \) is irreducible for all \( k \in \mathbb{N} \).

3. if \( p \) is an odd prime then \( f(x^p) \) is irreducible if \( f(0)/\text{lcf}(f) \) is not a \( p \)-th power in \( \mathbb{Q} \).

Proof Let \( K : \mathbb{Q} \) be a splitting field extension for \( f \). So we have a factorization \( f(x) = \prod(x - \alpha_i) \in K \) where the \( \alpha_i \) are the roots of \( f \) in \( K \). Let \( p \) be a prime; then obviously we have a factorization \( f(x^p) = \prod(x^p - \alpha_i) \) in \( \mathbb{Z}[x] \). Now suppose that \( f(x^p) \) is reducible in \( \mathbb{Z}[x] \) and let \( g \in \mathbb{Z}[x] \) be one of its irreducible factors. Then each polynomial \( x^p - \alpha_i \) has a non-trivial factor \( \gcd(g(x), x^p - \alpha_i) \in K[x] \). But for any odd prime \( p \), by Capelli’s theorem, \( x^p - \alpha_i \) is reducible if and only if \( \alpha_i \) is a \( p \)-th power (or of the form \( -4\beta^4 \) if \( p = 2 \)). Since \( f \) is not cyclotomic, its roots are not roots of unity, and thus \( f(x^p) \) is reducible only for finitely many primes \( p \).

For the second claim, when both \( f(x) \) and \( f(x^p) \) are irreducible we know from Capelli’s Theorem that the \( \alpha_i \) are not \( p \)-th powers. The irreducibility of \( x^{p^k} - \alpha_i \) immediately follows, and hence \( f(x^{p^k}) \) is irreducible too.

For the third claim, the norm of each root \( \alpha_i \) is just \( f(0)/\text{lcf}(f) \). If \( \alpha_i \) is a \( p \)-th power then so must its norm be; conversely, if the norm is not a \( p \)-th power then \( \alpha_i \) cannot be, and \( f(x^p) \) is irreducible. I am indebted to Barry Trager for the outline of this proof.

\[ \Box \]

8.2 The Construction of Longer Factorizations

Starting from some of the high ratio short factorizations in section \( 7 \) we now show how to construct polynomials having at least 4 irreducible factors each of height greater than the original polynomial. We suppose we already have an irreducible factorization \( f = g_1g_2 \) where \( \text{ht}(f)^2 < \min(\text{ht}(g_1), \text{ht}(g_2)) \) — note that the height of \( f \) is squared. Any example from section \( 7.2 \) is a suitable candidate; for instance, the first example gives us a height 1 polynomial with irreducible factors \( g_1 = x^6 - x^5 + x^4 - x^2 + 2x - 1 \) and \( g_2 = x^6 + 2x^5 + x^4 - x^2 - x - 1 \).
Now we look for a power $k$ such that $ht(f(x) \cdot f(x^k)) = ht(f)^2$ and both $g_1(x^k)$ and $g_2(x^k)$ are irreducible. Clearly, choosing $k > \deg(f)$ automatically satisfies the first condition — some smaller values may also happen to work. Lemma [8.1] tells us that there are infinitely many values of $k$ which satisfy the irreducibility conditions. Thus we have infinitely many choices for $k$ which satisfy all the conditions. In our specific example we can take $k = 13$. This leads us to $f(x) \cdot f(x^{13})$ a polynomial of degree 168 and height 1 having four irreducible factors each of height 2. Here we see plainly that this construction produces polynomials of rather large degree.

Now, since $ht(f) = 1$ we can repeat the above process to produce a polynomial of height 1 having any specified even number of irreducible factors each of height 2. For instance, $f(x) \cdot f(x^{13}) \cdot f(x^{169})$ is a polynomial of degree 28560 and height 1 having six irreducible factors each of height 2.

Here is another example of the same construction. We take the palindromic $*$-symmetric example in degree 28 from section [7.6]: specifically we take

$$g_1 = 2x^{13} + 6x^{12} + 10x^{11} + 4x^{10} - 6x^9 - 14x^8 - 17x^7 - 14x^6 - 6x^5 + 4x^4 + 10x^3 + 10x^2 + 6x + 2$$

and $g_2(x) = g_1(-x)$, thus we have that $ht(g_1g_2)^2 < ht(g_1)$. By experimentation we find that the exponent $k = 15$ satisfies all the conditions. So the product $g_1(x)g_2(-x)g_1(x^{15})g_2(-x^{15})$ is a polynomial of degree 448 and height 16 having four irreducible factors each of height 17. However, this time we cannot repeat the process because $ht(g_1g_2)^3 > ht(g_1)$.

9 Conclusion

In this paper we have compared various factor coefficient bounds, and shown with concrete examples that no one bound is universally better or worse than the others — so it is always a good idea to compute them all and pick whichever comes out smallest. We have refined some existing bounds, and have made explicit a bound latent in an article by Mignotte. We have shown how the degree aware bounds can be combined to produce a better bound in some instances.

In the second part of the paper we have exhibited several examples of factorizations with unusually large factors which explain partly why the factor bounds have to be rather larger than one might expect. Nevertheless, the known bounds are almost always "far too big", most especially if we are concerned with the sizes of irreducible factors in $\mathbb{Z}[x]$. Indeed, all the degree aware bounds remain valid for (suitably scaled, reducible) factors in $\mathbb{C}[x]$, and this seems to be the main reason why the bounds are so loose. If good bounds can be found for irreducible factors in $\mathbb{Z}[x]$ then we expect a consequent significant improvement in speed when factorizing polynomials in $\mathbb{Z}[x]$.

Currently, in the context of practical polynomial factorization, the problem of overly large bounds is mitigated by using an "engineering" technique known as *early termination* during the lifting phase; we feel that better bounds would furnish a more "mathematical" response to the same problem.
Collins [Col04] had already published some examples (including some very small ones) answering the question in the Conclusion of [BTW93]. Here we have extended considerably the set of known examples, and have given examples exhibiting far larger ratio than those given by Collins. We have also improved the extremal ratio in degree 16 found by Collins. Our examples in section 7.6 boost credence in the conjecture that there exist factorizations with arbitrarily large ratios (even if we restrict to palindromic \(*\)-symmetric factorizations of height 1 polynomials), though they fall short of providing a proof or a concrete family.

In section 7.3 we exhibited a family of reducible \(*\)-symmetric factorizations with unbounded ratio, where the ratio grows exponentially with degree. This family thus forces any single factor bound to grow at least as fast.

The examples in section 7.6 compel any ideal “irreducible single factor bound” to grow with degree, though the rate of growth appears to be much slower than for single factor bounds valid for any (suitably scaled) factorization in \(\mathbb{C}[x]\). This suggests that such an ideal single factor bound could be very much smaller than the currently known ones.

We have also shown how to construct polynomials with any even number of irreducible factors and whose irreducible factorization has ratio greater than 1. Using the examples from section 7.8 we can create explicitly examples with ratio up to 11. So far we have not encountered any irreducible factorization with an odd number of factors and ratio greater than 1, though we think it very likely that such factorizations exist.

Questions and Conjectures

There are a number of unanswered questions related to the results contained in this article. Here are a few of them:

- Do two factor irreducible factorizations of arbitrarily great ratio exist?
- How does the maximal ratio of two factor factorizations increase with degree?
- Is there an easy way to construct high ratio two factor factorizations?
- Do high ratio factorizations with an odd number of factors exist?

We conclude with two more conjectures which arose during our studies. The first cropped up while conducting the studies presented in section 7.6. The second one is the consequence of some fruitless searches for polynomials whose square has lower height than the original — this search was inspired by the existence of high ratio palindromic \(*\)-symmetric factorizations which “look vaguely like the square of a polynomial”. The second conjecture has a natural extension to higher powers, but this extended form is backed up by far less computational evidence.

**Conjecture 1**

Let \(f \in \mathbb{C}[x]\) then asymptotically \(\text{ht}(f^n) \approx K_f |f|_o^n / \sqrt{n}\) where \(K_f\) is a constant depending on \(f\), and \(|f|_o = \max\{f(e^{2\pi i \theta}) : 0 \leq \theta < 1\}\) is the maximum modulus of \(f\) on the unit circle in \(\mathbb{C}\).
Note In the particular case of $f = x + 1$ we can use the binomial theorem and Stirling’s approximation to deduce that $K_f = \sqrt{2/\pi}$. For the general case, F. Amoroso kindly pointed out that it is not hard to show that

$$|f^n| (1 + n \deg f)^{-1} \leq \text{ht}(f^n) \leq |f^n|.$$

Conjecture 2

Let $f \in \mathbb{Z}[x]$ be non-zero and not of the form $\pm x^d$. Then $\text{ht}(f^2) \geq 2\text{ht}(f)$.

Note The factor 2 in the conjecture cannot be replaced by a larger value since we have equality for $(x + 1)^2 = x^2 + 2x + 1$. Indeed there are many other polynomials which achieve equality. For instance, taking $f = x^{14} - x^{12} - x^{10} - x^8 - 4x^7 + x^6 + x^4 + x^2 - 1$ we obtain a “non-trivial” example satisfying $\text{ht}(f^2) = 2\text{ht}(f)$. The largest height example with this property which we have found is a polynomial of height 30 and degree 1680. It seems likely that there are examples of arbitrarily great height.

Conjecture 2 (extended)

Let $f \in \mathbb{Z}[x]$ be non-zero and not of the form $\pm x^d$. Then for any integer $k > 0$ we have $\text{ht}(f^k) \geq R_k \text{ht}(f)$ where the factor $R_k = \binom{k}{\lfloor k/2 \rfloor}$.

Note Choosing $f = x + 1$ shows that the factor $R_k$ cannot be made larger. Equality can also be achieved for other polynomials: e.g. with $k = 3$ we can take $f = x^3 + x^4 - x + 1$. Indeed the only polynomials we have found which achieve equality are (obviously) all binomials of height 1, and certain quadrinomials of height 1.

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