ON COBWEB ADMISSIBLE SEQUENCES
The Production Theorem

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Summary
In this note further clue decisive observations on cobweb admissible sequences are shared with the audience. In particular an announced proof of the Theorem 1 (by Dziemiańczuk) from [1] announced in India-Kolkata-December 2007 is delivered here. Namely here and there we claim that any cobweb admissible sequence F is at the point product of primary cobweb admissible sequences taking values one and/or certain power of an appropriate primary number p.

Here also an algorithm to produce the family of all cobweb-admissible sequences i.e. the Problem 1 from [1] i.e. one of several problems posed in source papers [2, 3] is solved using the idea and methods implicitly present already in [4].

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http://ii.uwb.edu.pl/akk/sem/sem_rota.htm

1 Preliminaries
The notation from [2, 3, 1] is being here taken for granted.

Definition 1 ([2, 3, 1]) A sequence F is called cobweb-admissible iff for any n, k ∈ N \{0\}
\[
\binom{n}{k}_F = \frac{n_F \cdot (n-1)_F \cdot \cdots \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot \cdots \cdot k_F} \in \mathbb{N}
\] (1)

Problem 1 ([2, 3, 1]) Find effective characterizations and/or an algorithm to produce the cobweb admissible sequences i.e. find all examples.
2 Primary cobweb admissible sequence

Throughout this paper we shall consequently use $p$ letter only for primary numbers.

**Definition 2** A cobweb admissible sequence $P(p) \equiv \{n_P\}_{n \geq 0}$ valued one and/or powers of one certain primary number $p$ i.e. $n_P \in \{1,p,p^2,p^3,...\}$ is called primary cobweb admissible sequence.

**Theorem 1** Any cobweb-admissible sequence $F$ is at the point product of primary cobweb-admissible sequences $P(p)$.

**Proof**

Given any cobweb admissible sequence $F = \{n_F\}_{n \geq 0}$, each of its elements can be represented as a product of primary numbers’ powers i.e. $n_F = \prod_{s \geq 1} p_s^{\alpha(n,s)}$. Therefore the sequence $F$ is at the point product of sequences $P(p_1), P(p_2), ...$ such that $P(p_s) \equiv \{n_{P_s}\}_{n \geq 0}$ and $n_{P_s} = p_s^{\alpha(n,s)}$. Each of primary sequences $P(p_s)$ where $s = 1, 2, 3, ...$ is cobweb admissible as following holds for any $n, k \in \mathbb{N} \cup \{0\}$

\[
\binom{n}{k}_{P(p_1)} \cdot \binom{n}{k}_{P(p_2)} \cdot \ldots = \binom{n}{k}_F \in \mathbb{N} \Rightarrow \\
\Rightarrow \binom{n}{k}_{P(p_s)} = n_F^{p_s^{n_s}} k_{P(p_s)}! = p_F^{p_N} \in \mathbb{N} \tag{2}
\]

where $N$ stands for the sum of index powers’ of primary numbers $p_s$ in $n_F, (n-1)_F, ..., (n-k+1)_F$ product expansion via primary numbers and correspondingly $K$ is the index powers’ sum for $k$ first elements of the sequence $F$.

3 Primary cobweb admissible sequences family

In this section we define a family $A(p)$ of all primary cobweb admissible sequences taking values one and/or certain power of an appropriate primary number $p$. In the next part of this section we present the family in the graph structure of a tree defined in algorithmic way in what follows.

For this aim let consider a primary cobweb admissible sequence $F \equiv P(p)$ and its corresponding family of sequences $B(F) \equiv \{n_{B(F)}\}_{n \geq 0}$ such that

\[
n_{B(F)} = m \leftrightarrow n_F = p^m
\]

In the sequel we shall consider sequences for arbitrary but fixed one primary number $p$, therefore we use abbreviation $P(p) \equiv P$. 

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Lemma 1  Natural number valued sequence \( F \equiv \{n_F\}_{n \geq 0} \) is primary cobweb admissible \( P(p) \) iff for any natural number \( n, n_F \in \{1, p, p^2, p^3, \ldots\} \) and
\[
\forall 1 \leq k \leq [n/2] \sum_{s=n-k+1}^{n} s_{B(F)} \geq \sum_{s=1}^{k} s_{B(F)}.
\]

PROOF
The first steep. Given any primary cobweb admissible sequence \( F \equiv \{n_F\}_{n \geq 0} \). From the Definition 2 we know that \( n_F \in \{1, p, p^2, \ldots\} \) for certain primary number \( p \). From the Definition 1 we readily infer that for any \( n, k \in \mathbb{N} \cup \{0\} \)
\[
\binom{n}{k}_F \left( = \frac{p^{n_{B(F)}} \cdots p^{(n-k+1)_{B(F)}}}{p^1_{B(F)} \cdot p^2_{B(F)} \cdots p^k_{B(F)}} \right) = \frac{p^N}{p^K} \in \mathbb{N} \Rightarrow N \geq K
\]
where \( N = \sum_{s=n-k+1}^{n} s_{B(F)} \) and \( K = \sum_{s=1}^{k} s_{B(F)} \).

The second steep. Given any sequence \( F \equiv \{n_F\}_{n \geq 0} \) where \( n_F \in \{1, p, p^2, p^3, \ldots\} \) and \( \forall 1 \leq k \leq [n/2] \sum_{s=n-k+1}^{n} s_{B(F)} \geq \sum_{s=1}^{k} s_{B(F)} \) \((\ast)\). Then for any natural \( n, k \) below takes place
\[
\binom{n}{k}_F \left( = \frac{p^{n_{B(F)}} \cdots p^{(n-k+1)_{B(F)}}}{p^1_{B(F)} \cdot p^2_{B(F)} \cdots p^k_{B(F)}} \right) = C \land (\ast) \Rightarrow C \in \mathbb{N}
\]

Definition 3 (Primary cobweb admissible tree) Let \( G(p) \) to be a weighted tree \( G(p) = (V, E, \delta) \) where \( V \) stays for set of vertices, \( E \) denotes a set of nodes and function \( \delta \) which assigns weight for any vertex \( v \in V \) such that \( \delta(v) \in \{0, 1, 2, \ldots\} \) and \( p \) - primary number. We shall define the corresponding graph \( G(p) \) via the following recurrence:

1. \( v_0 \in V \) is called the root with weight \( \delta(v_0) = 0 \)
2. If \((v_0, v_1, \ldots, v_{n-1})\) is a path of graph \( G(p) \) then \((v_0, v_1, \ldots, v_{n-1}, v_n)\) is too if, and only if \( \forall 1 \leq k \leq [n/2]\) \( N_{n,k} \geq K_k \)

where \( N_{n,k} = \sum_{i=n-k+1}^{n} \delta(v_i) \) and \( K_k = \sum_{i=1}^{k} \delta(v_i) \).

Conclusion 1
Any path \((v_0, v_1, \ldots, v_n)\) from the root \( v_0 \) to vertex \( v_n \) encodes the first \( n \) terms with \( 0_F \) of primary cobweb admissible sequence \( F \equiv \{n_F\}_{n \geq 0} \), \( n_F \in \{1, p, p^2, \ldots\} \) with help of elements’ exponent powers’ sequence \( B(F) \) such that \( k_{B(F)} = \delta(v_k) \) i.e. the \( n + 1 \)-tuple
\[
(v_0, v_1, \ldots, v_n) \in V^{n+1} \leftrightarrow (\delta(v_0), \delta(v_1), \ldots, \delta(v_n))
\]
exactly encodes finite primary cobweb admissible sequence \( F \) valued by one and/or powers of primary number \( p \).
Observation 1  If any path \((v_0, v_1, ..., v_n)\) encodes \(n\) the first terms with 0\(_F\) of primary cobweb admissible sequence \(F\) then there exists infinite number of successors vertices \(v_{n+1}\) which encode primary cobweb admissible sequence \(F'\) specified by these \(n\) first terms with 0\(_F\) and the one additional \((n+1)F' = \delta(v_{n+1})\) term.

**Proof**

If any path \((v_0, v_1, ..., v_n)\) encodes \(n\) the first terms with 0\(_F\) of primary cobweb admissible sequence \(F\) then there exists infinite number of natural numbers \(M\) such that \(\delta(v_{n+1}) = M\) and \(N_{n,k-1} + M \geq K_k\).

Consequently, now we present an algorithm to generate primary cobweb admissible tree.

**Algorithm 1 (primary cobweb-admissible tree)** We shall begin with the root \(v_0\) of graph \(G(p)\) from Definition 2 and in the next steps, from any path \((v_0, v_1, ..., v_n)\) we obtain the very next one \((v_0, v_1, ..., v_n, v_{n+1})\).

**Input:** Any path \((v_0, v_1, ..., v_n)\) of \(G(p)\) which encodes \(n\) the first terms with 0\(_F\) of primary cobweb admissible sequence \(F\).

**Output:** Non-empty set \(\emptyset \neq \Delta_n \subseteq \{v_{n+1} : \delta(v_{n+1}) \in \{0,1,2,...\}\}\) with vertices’ successors for \(v_n\) vertex such that the paths \((v_0, v_1, ..., v_n, v_{n+1})\) where \(v_{n+1} \in \Delta_{n+1}\) encodes primary cobweb-admissible sequence, too.

Under the convenient notation for vertices \(v(s) \equiv v_{n+1} \wedge \delta(v_{n+1}) = s\) note now the following.

**Steps:**

1. If \(n = 1\)
   \[\Delta_1 = \{v(0), v(1), v(2), ...\}\]
2. If \(n = 2\)
   \[\Delta_2 = \{v(m), v(m+1), v(m+2), ...\}, \text{ where } m = \delta(v_1)\]
   ...
3. For any natural \(n\)
   \[\Delta_n = \{v(m), v(m+1), v(m+2), ...\}, \text{ where } m = \max\{K_k - N_{n-1,k-1} : k = 1, 2, 3, ..., \lfloor n/2 \rfloor\}\]
   where \(K_k = \sum_{i=1}^{k} \delta(v_i)\) and \(N_{n,k} = \sum_{i=n-k+1}^{n} \delta(v_i)\).

**Definition 4** Denote with letter \(A(p)\) the family of all primary cobweb admissible sequences \(P(p)\).
Observation 2 The family $A(p)$ is labelled-designated by the set of infinite paths $(v_0, v_1, v_2, ...)$ of graph $G(p)$ from the root $v_0$ i.e.

$$F \in A(p) \iff (v_0, v_1, v_2, ...) \text{ is a path of graph } G(p)$$

where $F \equiv \{n_F\}_{n \geq 0}$ and $n_F = p^{\delta(v_n)}$.

**Proof**
This is a conclusion on graph $G(p)$ (Definition 3).

The first steep. If given any primary cobweb admissible sequence $F \equiv \{n_F\}_{n \geq 0}$, $n \in \{1, p, p^2, ..., \}$, then from the Definition 1 of admissibility and the Definition 3 of tree $G(p)$ for any natural numbers $n, k$ the following is true

$$\binom{n}{k}_F = \frac{p^{n_B(F)} \cdot \ldots \cdot p^{(n-k+1)_B(F)}}{p^{1_B(F)} \cdot p^{2_B(F)} \cdot \ldots \cdot p^{k_B(F)}} = \frac{p^N}{p^K} = \frac{p^{\delta(v_n)} \cdot \ldots \cdot p^{\delta(v_{n-k+1})}}{p^{\delta(v_1)} \cdot \ldots \cdot p^{\delta(v_k)}} \in \mathbb{N}$$

where $s_B(F) = \delta(v_s)$ from Conclusion 1. In view of the Definition 1 $N \geq K$ hence $(v_0, v_1, v_2, ...)$ is a path of graph $G(p)$.

The second steep. Take any given path $(v_0, v_1, v_2, ...)$ of the graph $G(p)$. Then by definition for any natural number $n, k$, $N_{n,k} \geq K_k$ where $N_{n,k} = \sum_{i=n-k+1}^n \delta(v_i)$ and $K_k = \sum_{i=1}^k \delta(v_i)$. Hence this path does encode the very primary cobweb admissible sequence $P(p)$.

**Theorem 2 (Cobweb Admissible Sequences Production Theorem)**
The family of all cobweb admissible sequences is a product of families $A(p_s)$ for $s = 1, 2, 3, ...$ i.e. for any cobweb admissible sequence $F$

$$F \in \times_{s=1} A(p_s)$$

**Proof**
This is the summarizing conclusion. Any cobweb admissible sequence $F$ is at the point product of primary cobweb admissible sequences $P(p)$ (Theorem 1) and the family of all primary cobweb admissible sequences $A(p)$ is defined by primary cobweb admissible tree $G(p)$ (Observation 2).

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References

[1] A. Krzysztof Kwaśniewski, M. Dziemiańczuk, *Cobweb posets - Recent Results*, ISRAMA 2007, December 1-17 2007 Kolcata, INDIA, [arXiv:0801.3985](http://arxiv.org/abs/0801.3985), 25 Jan 2008

[2] A. Krzysztof Kwaśniewski, *Cobweb posets as noncommutative prefabs*, Adv. Stud. Contemp. Math. vol. 14 (1) (2007) 37-47.

[3] A. Krzysztof Kwaśniewski, *On cobweb posets and their combinatorially admissible sequences*, [arXiv:math.CO/0512578](http://arxiv.org/abs/math.CO/0512578), 21 Oct 2007.

[4] M. Dziemiańczuk, *On cobweb posets tiling problem*, [arXiv:math.Co/0709.4263](http://arxiv.org/abs/math.Co/0709.4263), 4 Oct 2007