Abstract. We consider uniform approximations by trigonometric polynomials. The aim of the paper is to obtain good estimates of the Jackson–Stechkin constants $J_m$. We prove that $J_m \leq C2^{-m+5/2\log_2 m}$. Our proof is based on the difference analogue of the Bohr–Favard inequality.

1. Introduction

Let $C(T)$ be the space of continuous function on the unit circle $T = [-\pi, \pi]/2\pi Z$. Denote by $T_n$ the space of trigonometric polynomials $\sum_{j=-n}^{n} c_j e^{ijt}$ of degree $\leq n$. Let $C(T_n^\perp)$ be the class of functions from $C(T)$ which have no spectrum in $[-n, n]$.

We are interested in the uniform approximations of $f \in C(T)$ by trigonometric polynomials. Set

$$\|f\| := \max_{x \in T} |f(x)|, \quad \|f\|_1 := \int_T |f(u)| \, du.$$ 

We shall denote by

$$E_n(f) := \inf_{\tau \in T_n} \|f - \tau\|$$

the best approximation of $f \in C(T)$ by $\tau \in T_n$. Define $m$–difference of $f$ at a point $x$ with step $h$ as follows

$$\Delta_h^m f(x) := \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x - (m/2)h + ih).$$

The $m$–th modulus of smoothness of $f$ is defined by

$$\omega_m(f, \delta) := \sup_{x, |h| < \delta} |\Delta_h^m f(x)|.$$ 

The following theorem is one of the central results in approximation theory.

**Theorem J.** If $f \in C(T)$, then

$$E_n(f) \leq J_m \omega_m(f, \frac{2\pi}{n}), \quad n \geq m.$$ 

Theorem J was proved by Jackson [8] ($m = 1$), Akhiezer [1] ($m = 2$), Stechkin [10] ($m > 2$). Stechkin’s proof [4, p.205] does not allow us to obtain the estimate
$J_m < m^m$ for $m > m_0$. We shall show that constant $J_m$ tends to zero exponentially as $m \to \infty$. The main result of the paper is the following.

**Theorem 1.** For $f \in C(T)$, $\alpha := 2^{-5/2}3^{1/2}\pi < 1$,

$$E_n(f) \leq \frac{2(k+1)^2}{(2k)} \omega_{2k}(f, \frac{2\pi}{n} \alpha(1+1/k)), \quad n \geq 2k. \quad (1)$$

**Corollary.**

$$J_m \leq C \ 2^{-m+(5/2)\log_2 m}. \quad (2)$$

There is a connection between (1) and the following theorem for functions with uniform oscillation on $[a, b]$. We write $f \in O_n(a, b)$ if $f \in C(a, b)$ and

$$\int_a^{a+(b-a)/n} f(u) \, du = 0, \quad i = 1, \ldots, n.$$ 

**Theorem W.** For $f \in O_n(a, b)$,

$$\|f\| \leq W_n \omega_n(f, \frac{b-a}{n}), \quad n \geq m.$$ 

It is known that $W_n < 2 + 1/e^2$ and $W_n = 1$ for $n < 8$ [6,12]. Note that our conjecture $W_n = 1$ implies Sendov’s conjecture [9].

The condition $f \in O_n(a, b)$ is similar to the condition $f \in C(T_n^\perp)$. The Bohr–Favard [2,5] inequality reads

**Theorem F.** If $f, f^{(m)} \in C(T_n^\perp)$, then

$$\|f\| \leq F_m(n+1)^{-m}\|f^{(m)}\|, \quad F_m := \frac{4}{\pi} \sum_{i=0}^{\infty} (-1)^i (m+1)(2i+1)^{-m-1}.$$ 

The following difference analogue of Theorem F is the key result of the paper.

**Theorem 2.** If $f \in C(T_n^\perp)$, then

$$\|f\| \leq \frac{k+1}{(2k)} \omega_{2k}(f, \frac{2\pi}{n} \alpha), \quad n \geq 2k.$$ 

The third theorem in this paper gives a link between Theorem 2 and Theorem 1. It is devoted to the approximation by Vallée Poussin means. Denote by $s_i$ the operator of $i$–th partial Fourier sum. Let

$$v_{k,m} := \frac{1}{m} \sum_{i=km}^{(k+1)m-1} s_i.$$
Theorem 3. If \( f \in C(T) \), then
\[
\| f - v_{k,m}f \| \leq \frac{2(k + 1)^2}{\binom{2k}{k}} \omega_{2k}(f, \frac{2\pi}{km} \alpha), \quad m \geq 2.
\]

We shall use special pointwise difference operators \( w_{x,h}^s \). The operators \( w_{x,h}^1 \) were introduced by Ivanov and Takev [7] under the influence of Beurling’s proof of Whitney’s theorem in \( R \) and \( R^+ \) [11, p.83]. We should also mention Brudnyi’s paper [3]. It contains the construction of smoothing operators, which is widely used in approximation theory [4, p.177]. The operators \( w_{x,h}^s \) are similar to Brudnyi’s operators, but provide more delicate estimates.

2. Smoothing operators

Let \( I_x \) be the identity operator at the point \( x \in T \).

\[
I_xf := f * \delta_x = f(x).
\]

Averaging operator on \([x - h, x + h]\), \(0 < h < \pi\), will be denoted by \( I_{x,h} \).

\[
I_{x,h}f := \frac{1}{2h} \int_{x-h}^{x+h} f(u) \, du.
\]

It is clear that
\[
I_{x,h}f = \int_T f(u) B^1_h(x-u) \, du = (f * B^1_h)(x),
\]

where
\[
B^1_h(x) := \begin{cases} 
\frac{1}{2h}, & x \in [-h, h] \\
0, & x \in [-h, h]^c.
\end{cases}
\]

Set \( I^0_{x,h} := I_x \), \( I^1_{x,h} := I_{x,h} \). Define \( I^s_h \), \( s = 2, \ldots \), by
\[
I^s_{x,h}f := (I_{x,h})^s f = (f * B^s_h)(x), \quad B^s_h(x) := (B^1_h * B^{s-1})^s(x).
\]

Note that the nonnegative function \( B^s_h \) has the following properties:
\[
\int_T B^s_h(u) \, du = 1,
\]
\[
\text{supp } B^s_h(u) = [-sh, sh], \quad 0 < sh < \pi.
\]

Operator of differentiation \( D \) acts on \( I_{x,h} \) in the following way:
\[
DI^s_{x,h} = (2h)^{-1} (I^{s-1}_{x+h,h} - I^{s-1}_{x-h,h}).
\]

This implies
\[
D^m I^s_{x,h} = (2h)^{-m} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} I^s_{x-mh+2ih}, \quad s \geq m.
\]
The operators $w^s_{x,h,2k}$ measuring the local $2k$–th smoothness of a function are defined by

$$w^s_{x,h,2k} := \sum_{i=0}^{2k} (-1)^{k-i} \binom{2k}{i} I^{s}_{2k}(x,(i-k)h) = I_x - 2 \sum_{i=1}^{k} (-1)^{i+1} \binom{2k}{k+i} I^s_{x,ih}, \quad I^s_{x,-ih} := I^s_{x,ih}. \quad (3)$$

We can write $w^s_{x,h,2k} f$ as

$$w^s_{x,h,2k} f = (-1)^k \binom{2k}{k}^{-1} \int_T \Delta^{2k} u f(x) B^s_h(u) du, \quad 0 < skh < \pi.$$

Note that

$$\|w^s_{x,h,2k} f\| \leq \binom{2k}{k}^{-1} \omega_{2k}(f, sh).$$

3. Proof of Theorem 2

Rewrite (3) in the following form

$$I_x = w^s_{x,h,2k} + 2 \sum_{i=1}^{k} (-1)^{i+1} \binom{2k}{k+i} I^s_{x,ih}. \quad (4)$$

We shall show that for $f \in C(T_n^\perp)$ one can choose $s, h$ such that

$$\|I^s_{x,ih} f\| \leq c(h, s, i) \|f\|,$$

and

$$2 \sum_{i=1}^{k} \binom{2k}{k+i} c(h, s, i) \leq b(k) < 1.$$

From

$$I^s_{x,ih} f = (B^s_{ih} \ast f)(x),$$

we have for $f \in C(T_n^\perp), \tau \in T_n$,

$$I^s_{x,ih} f = ((B^s_{ih} - \tau) \ast f)(x).$$

This gives

$$\|I^s_{x,ih} f\| \leq \inf_{\tau \in T_n} \|B^s_{ih} - \tau\|_1 \|f\|.$$

By Favard–Nikolskii theorem [4] p.215

$$\inf_{\tau \in T_n} \|B^s_{ih} - \tau\|_1 \leq F_{s-1} n^{-s+1} \|D^{s-1} B^s_{ih}\|_1.$$

The equality

$$D^m B^s_h(x) = (2h)^{-m} \Delta^s_{2h} B^s_{ih}(x), \quad s \geq m,$$

implies

$$\inf_{\tau \in T_n} \|B^s_{ih} - \tau\|_1 \leq F_{s-1} (2ihn)^{-s+1} \|\Delta^s_{2h} B^1_{ih}\|_1.$$
Choose \( s = 3 \), \( \beta := \frac{\pi}{\sqrt{6}} \), \( h = \beta \pi (2n)^{-1} \). We thus get
\[
c(h, 3, i) \leq \inf_{\tau \in T} \| B_{i h}^s - \tau \|_1 \leq 4F_2(\beta \pi i)^{-2} = \frac{1}{2\beta^2 i^2},
\]
and
\[
2 \sum_{i=1}^{k} \left( \frac{2k}{k+i} \right) c(h, 3, i) \leq \frac{1}{\beta^2} \left( \frac{2k}{k+1} \right) \left( \sum_{i=1}^{k} i^{-2} \right) \leq \frac{2k}{(2k)^2} < 1.
\]
The identity (4) at the extremal point \( x_0 \), such that \( |f(x_0)| = \|f\| \), gives
\[
\|f\| (1 - \frac{(2k+1)}{(2k)}) \leq \| w_{3,h,2k}^3 f \|.
\]
The inequality
\[
\| w_{3,h,2k}^3 f \| \leq \frac{1}{(2k)} \omega_{2k}(f, 3h),
\]
implies
\[
\|f\| \leq \frac{1}{(2k)} \omega_{2k}(f, \frac{3\beta \pi}{2n}) = \frac{k+1}{(2k)} \omega_{2k}(f, \frac{2\pi}{n}).
\]
This completes the proof of Theorem 2.

4. **Proofs of Theorem 3 and Theorem 1**

Note that Vallée Poussin means \( v_{k,m} \) are the simple combination of Fejer’s means.

\[
v_{k,m} = 1_m \sum_{i=km}^{(k+1)m-1} s_i = (k+1)\sigma_{(k+1)m-1} - k\sigma_{km-1},
\]

where
\[
\sigma_j := \frac{1}{j+1} \sum_{i=0}^{j} s_i.
\]
For Fejer’s means \( \sigma_j \) we have
\[
\| \sigma_j f \| \leq \| f \|, \quad \Delta^{2k}_h \sigma_j f = \sigma_j \Delta^{2k}_h f.
\]
Therefore
\[
\omega_{2k}(v_{k,m} f, h) \leq (2k+1) \omega_{2k}(f, h).
\]
Let \( g := f - v_{k,m} f \in C(T_{km}^\perp) \). Theorem 2 implies
\[
\|g\| \leq \frac{k+1}{(2k)} \omega_{2k}(g, \frac{2\pi}{km}) \leq \frac{2(k+1)^2}{(2k)} \omega_{2k}(f, \frac{2\pi}{km}).
\]
Theorem 3 is proved.
For the best approximation by trigonometric polynomials of degree $n = km + i$, $i \in [0, m - 1]$, $m \geq 2$, we get

$$E_n(f) \leq E_{km}(f) \leq \|f - v_{m,k}f\| \leq \frac{2(k + 1)^2}{(2k)} \omega_{2k}(f, \frac{2\pi}{km} \alpha).$$

From

$$\frac{n}{km} \leq \frac{(km + m - 1)/(km) < 1 + 1/k},$$

we have

$$E_n(f) \leq \frac{2(k + 1)^2}{(2k)} \omega_{2k}(f, \frac{2\pi}{n} \alpha(1 + 1/k)).$$

This proves Theorem 1.

The inequalities

$$\omega_{2k}(f, \delta) \leq 2\omega_{2k-1}(f, \delta), \quad \frac{4^k}{\sqrt{\pi(k + 1/2)}} < \left(\frac{2^k}{k}\right) < \frac{4^k}{\sqrt{\pi k}},$$

lead to (2).

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