Multi-algebras as tolerance quotients of algebras

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Abstract. If $A$ is an algebra and $\tau$ is a tolerance on $A$, then $A/\tau$ is a multi-algebra in a natural way. We give an example to show that not every multi-algebra arises in this manner. We slightly generalize the construction of $A/\tau$ and prove that every multi-algebra arises from this modified construction.

1. Tolerance quotients

Let $M$ be a nonempty set. Denote by $\text{Pow} M$ the set of all subsets of $M$ and by $\text{Pow}_+ M$ the set of all nonempty subsets of $M$. A multi-operation $f$ on $M$ (of arity $n$) is a function $f : M^n \to \text{Pow}_+ M$. A multi-algebra $(M; F)$ is a nonempty set $M$ with a set $F$ of multi-operations on $M$.

If $(A; F)$ is an algebra and $\alpha$ is a congruence on $A$, then the congruence classes of $\alpha$ form an algebra $(A/\alpha; F) = (A; F)/\alpha$. A congruence $\alpha$ is a reflexive, symmetric, and transitive binary relation on $A$ with the Substitution Property (so $\alpha$ is a subalgebra of $(A; F)^2$). If we drop the Substitution Property, $(A/\alpha; F) = (A; F)/\alpha$ becomes a multi-algebra. The converse was proved in G. Grätzer [3]: every multi-algebra can be obtained (up to isomorphism) in this fashion; for additional results in this direction, see G. Grätzer and G. H. Wenzel [5], H. Höft and P. E. Howard [6].

What happens if we drop the transitivity of $\tau$? Define a tolerance $\tau$ on an algebra $(A; F)$ as a reflexive and symmetric binary relation on $A$ with the Substitution Property. For an overview of tolerances in algebra, see I. Chajda [2].

Let $\tau$ be a binary relation on the set $A$. As in graph theory, we call a subset $B \subseteq A$ a clique if $B^2 \subseteq \tau$; we call $B$ a maximal clique if $B$ is maximal with respect to the property $B^2 \subseteq \tau$. By Zorn’s Lemma, every clique is contained in a maximal clique. A covering of a nonempty set $A$ is a collection $\mathcal{C}$ of pairwise incomparable subsets of $A$ whose union is $A$. It is easy to see that a reflexive and symmetric binary relation $\tau$ on $A$ is equivalent to a covering, $\mathcal{C}_\tau$, of $A$ where the sets in the covering are the maximal cliques of $\tau$. For a tolerance $\tau$ on an algebra $(A; F)$, we call a maximal clique a tolerance block.
Conversely, a covering $\mathcal{C}$ of $A$ *induces* a reflexive and symmetric binary relation $\tau$ on $A$ defined by $(a, b) \in \tau$ iff there is an $S \in \mathcal{C}$ such that $a, b \in S$. It is easy to see that each $S \in \mathcal{C}$ is a clique of $\tau$; however, $S \in \mathcal{C}$ need not be a maximal clique of $\tau$, and even if each $S \in \mathcal{C}$ is a maximal clique of $\tau$, there may be maximal cliques of $\tau$ that do not belong to $\mathcal{C}$.

**Example 1.1.** Let $|A| \geq 3$.

(1) Let $\mathcal{C}_2$ be the set of all 2-element subsets of $A$. Then $\mathcal{C}_2$ induces the full relation $\tau = A^2$ on $A$, and $A$ is its unique maximal clique.

(2) On $\text{Pow}_+ A$, define the *non-disjointness* relation $\nu$ by

$$
(X, Y) \in \nu \text{ iff } X \cap Y \neq \emptyset.
$$

Then every ultrafilter $U$ on $A$ is a maximal clique of $\nu$. But these are not the only maximal cliques of $\nu$. Let $\{a, b, c\}$ be a 3-element subset of $A$; then the set

$$\{\{a, b\}, \{a, c\}, \{b, c\}\}
$$

is a clique of $\nu$, but is contained in no ultrafilter of $A$. Hence, there is a maximal clique of $\nu$ which is not an ultrafilter. Notice that $\nu$ is induced by the set of all principal ultrafilters of $A$. The non-disjointness relation is well studied in the combinatorics of set systems, usually under the name *intersecting families* of sets; see [1].

**Construction 1.2.** Given a tolerance $\tau$ on the algebra $(A; F)$, we construct a multi-algebra $(M; F)$. Let $\mathcal{B}_\tau$ be the covering of $A$ consisting of all tolerance blocks of $\tau$. For $f \in F$ of arity $n$ and $B_1, \ldots, B_n \in \mathcal{B}_\tau$, define $f$ on $\mathcal{B}_\tau$ by

$$
f(B_1, \ldots, B_n) = \{ B \in \mathcal{B}_\tau \mid f(b_1, \ldots, b_n) \in B \text{ for all } b_i \in B_i \text{ and } 1 \leq i \leq n \}.
$$

Since $\tau$ is a tolerance, if $f(b_1, \ldots, b_n) \in B$ for some $b_i \in B_i$, then $f(b_1, \ldots, b_n) \in B$ for all $b_i \in B_i$. We denote by $(A; F)/\tau$ this multi-algebra defined on $\mathcal{B}_\tau$ and call it the *tolerance quotient* induced by the tolerance $\tau$ on the algebra $(A; F)$.

It is natural to wonder how general this construction is. As the next example shows, it is not completely general.

**Example 1.3.** Let $M = \{1, 2, 3\}$ and define the multi-groupoid $(M; +)$ by

$$
a + b = \{a, b\}
$$

for all $a, b \in \{1, 2, 3\}$. Let us assume that $(M; +)$ is isomorphic to the tolerance multi-groupoid induced by the tolerance $\tau$ on the groupoid $(A; +)$. Then $\tau$ has exactly 3 tolerance blocks: $B, C, D$. Further, $B + C = \{B, C\}$, $B + D = \{B, D\}$, and $C + D = \{C, D\}$. This means that there are $b_1, b_2 \in B$, $c_1, c_3 \in C$, and $d_2, d_3 \in D$ such that $b_1 + c_1 \in (B \cap C) - D$, $b_2 + d_2 \in (B \cap D) - C$, and $c_3 + d_3 \in (C \cap D) - B$. But then $\{b_1 + c_1, b_2 + d_2, c_3 + d_3\}$ is a 3-element clique in $\tau$ not contained in $B$, $C$, or $D$, so $\tau$ has at least 4 cliques. This is a contradiction, so there is no tolerance multi-groupoid isomorphic to the multi-groupoid $(M; +)$.
2. Full covering quotients

Let us re-examine the multi-groupoid $(M; +)$ of Example 1.3. We take $A = \text{Pow}_+ M$ and define $\nu$ to be the non-disjointness relation as in (1). Then $\nu$ has 4 maximal cliques:

- $S_1 = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$,
- $S_2 = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$,
- $S_3 = \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$,
- $S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Let $B, C \in A$. If $|B| = |C| = 1$, we define $B + C$ to be $B \cup C$; otherwise, define $B + C$ to be $M = \{1, 2, 3\}$. Then $\nu$ is readily seen to be a tolerance on $(A; +)$. Form $(A; +)/\nu$. We hope that $(M; +)$ is isomorphic to the subalgebra of $(A; +)/\nu$ on the subset $\{S_1, S_2, S_3\}$. But it is not: $S_1 + S_2 = \{S_1, S_2, S\}$, which is not a subset of $\{S_1, S_2, S_3\}$.

So we have to get rid of that troublesome tolerance block $S$.

**Construction 2.1.** Let $\tau$ be a tolerance on the algebra $(A; F)$. Let $\mathcal{C}$ be a covering of $A$ consisting of some tolerance blocks of $\tau$. If for $a, b \in A$, we have $(a, b) \in \tau$ iff there is some $C \in \mathcal{C}$ such that $a, b \in C$, then we call $\mathcal{C}$ a full covering. For $f \in F$ of arity $n$ and $C_1, \ldots, C_n \in \mathcal{C}$, define

$$f(C_1, \ldots, C_n) = \{C \in \mathcal{C} | f(c_1, \ldots, c_n) \in C \text{ for all } c_i \in C_i \text{ and } 1 \leq i \leq n \}.$$  

We denote this multi-algebra by $(A; F)/\mathcal{C}$ and call it the full covering quotient of the algebra $(A; F)$ induced by the full covering $\mathcal{C}$.

We shall now see that Construction 2.1 is completely general.

**Theorem 2.2.** Let $(M; F)$ be a multi-algebra. Then there is an algebra $(A; F)$ and a full covering $\mathcal{C}$ of $(A; F)$ that induces a tolerance $\tau$ on $(A; F)$ such that $(M; F)$ is isomorphic to $(A; F)/\mathcal{C}$.

**Proof.** Take $A = \text{Pow}_+ M$. For $n$-ary $f \in F$ and $m_1, \ldots, m_n \in M$, define $f$ on $\text{Pow}_+ M$ by:

$$f(\{m_1\}, \ldots, \{m_n\}) = \{f(m_1, \ldots, m_n)\},$$

and, otherwise,

$$f(B_1, \ldots, B_n) = M.$$  

For $m \in M$, let $M_m = \{S \subseteq M | m \in S\}$; then $\mathcal{C} = \{M_m | m \in M\}$ is a covering of $A$ by the principal filters of $M$. As noted in Example 1.1 (2), $\mathcal{C}$ induces $\nu$ on $A$. Each $M_m$ is a maximal clique of $\nu$.

To prove that $\nu$ is a tolerance, let $f \in F$ be $n$-ary and $(A_i, B_i) \in \nu$ for $1 \leq i \leq n$; we need to conclude that $(f(A_1, \ldots, A_n), f(B_1, \ldots, B_n)) \in \nu$. Note that $(M_N, N) \in \nu$ for all nonempty $N \subseteq M$; thus, our conclusion holds unless each $A_i$ and each $B_i$ is a $1$-element subset of $M$. But for $1$-element subsets $A, B$, we have $(A, B) \in \nu$ iff $A = B$. Because $\nu$ is reflexive, our conclusion
also holds in this case since then $A_i = B_i$ for all $i$. So $\nu$ is indeed a tolerance. Hence, $C$ is a full covering.

Now use Construction 2.1 to form $(A; F)/C$. It is obvious from the definitions that $(A; F)/C$ is isomorphic to $(M; F)$ via the map that sends $M_m$ to $m$. □

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