FOURIER MULTIPLIERS FOR TRIEBEL-LIZORKIN SPACES ON GRADED LIE GROUPS

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ABSTRACT. In this work we investigate the boundedness of Fourier multipliers on Triebel-Lizorkin spaces associated to positive Rockland operators on a graded Lie group. The found criterion is expressed in terms of the Hörmander-Mihlin condition on the global symbol of a Fourier multiplier.

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1. INTRODUCTION

1.1. Outline. This paper is devoted to the boundedness of Fourier multipliers of Hörmander-Mihlin type for Triebel-Lizorkin spaces on graded Lie groups. The boundedness of Fourier multipliers of Hörmander-Mihlin type has become an indispensable tool in harmonic analysis. The classical Mihlin multiplier theorem [23] states that if a function $\sigma \in C^\infty(\mathbb{R}^n \setminus \{0\})$, satisfies

$$|\partial_\xi^\alpha \sigma(\xi)| \lesssim |\xi|^{-|\alpha|}, \quad |\alpha| \leq [n/2] + 1,$$

(1.1)

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then the multiplier $A$ (of the Fourier transform\footnote{Defined for $f \in C_0^\infty(\mathbb{R}^n)$, by $\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi ix \cdot \xi} f(x)dx$.} on $\mathbb{R}^n$) defined by
\[
A f(x) \equiv T_\sigma f(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \sigma(\xi) \hat{f}(\xi) d\xi, \quad f \in C_0^\infty(\mathbb{R}^n),
\]
admits a bounded extension on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$. Hörmander’s generalisation of Mihlin’s result in [22] guarantees the $L^p$-boundedness of an extension of $A$ under the Sobolev condition
\[
\|\sigma\|_{l.u. L^2} := \sup_{r > 0} \|\sigma(r \cdot)\|_{L^2(\mathbb{R}^n)} < \infty,
\]
where $\eta \in \mathcal{D}(0, \infty)$, $\eta \neq 0$, and $s > n/2$. Later, Calderón and Torchinsky in [3] extended the Hörmander-Mihlin theorem to Hardy spaces $H^p(\mathbb{R}^n)$ by proving that $A : H^p(\mathbb{R}^n) \to H^p(\mathbb{R}^n)$ admits a bounded extension provided that (1.3) holds with $s > n(1/p - 1/2)$ and $0 < p \leq 1$. A different proof to the one by Calderón and Torchinsky was done by Taibleson and Weiss in [30]. The endpoint for the Hörmander-Mihlin condition in Hardy spaces was found by Baernstein and Sawyer in [2]. The existence of bounded extensions $A : H^1(\mathbb{R}^n) \to L^{1,2}(\mathbb{R}^n)$ was investigated by Seeger in [27] and [28] by considering the Besov condition on the symbol $\sigma \in B^{0}_{1,1}(\mathbb{R}^n)$. These estimates were extended to Triebel-Lizorkin spaces by Seeger in [29]. We also refer to the recent paper [25] of Park for the generalisation of Seeger’s results for Triebel-Lizorkin spaces $F^{r,q}_{p,q}(\mathbb{R}^n)$, related to the Hörmander-Mihlin condition.

Because of the numerous applications of the Hörmander-Mihlin condition on $\mathbb{R}^n$ to the Euclidean harmonic analysis, this condition is also of interest for the harmonic analysis of non-commutative structures, namely, Lie groups and other spaces of homogeneous type (see Coifman and De Guzmán [11]). In the context on Lie groups, the Hörmander-Mihlin condition was extended in [26] to arbitrary compact Lie groups\footnote{and also extended in [7, Section 5] to subelliptic Fourier multipliers on compact Lie groups.} generalising the same condition for SU(2) given by Coifman and Weiss in [12]. In the case of a graded Lie group $G$, it was proved in [15] that the Hörmander-Mihlin condition for a multiplier $A \equiv T_\sigma$ (of the Fourier transform\footnote{which, on a graded Lie group $G$, with unitary dual $\hat{G}$, is defined for $f \in C_0^\infty(G)$ by $\hat{f}(\pi) := \int_{\hat{G}} f(x) \pi(x)^* dx$, at $\pi \in \hat{G}$. The Fourier inversion formula is given by $f(x) = \int_{\hat{G}} \text{Tr}[\pi(x) \hat{f}(\pi)] d\pi$, where $d\pi$ is the Plancherel measure on $\hat{G}$. In this way, a Fourier multiplier $A$ is determined by the identity $Af(\pi) = \sigma(\pi) \hat{f}(\pi)$, for a.e. $\pi \in \hat{G}$.} on a graded Lie group $G$), which is defined by
\[
A f(x) \equiv T_\sigma f(x) := \int_{\hat{G}} \text{Tr}[\pi(x) \sigma(\pi) \hat{f}(\pi)] d\pi, \quad f \in C_0^\infty(G),
\]
implies the existence of a bounded extension of $A$ on $L^p(G)$, for $1 < p < \infty$. This estimate was extended for Hardy spaces in [21] generalising the theorem of Calderón and Torchinsky to the setting of graded Lie groups. We refer the reader to [8, Page 39] and to [9] where the Hörmander-Mihlin condition for right Besov spaces on graded Lie groups was discussed.

\[
\int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) dx.
\]
In this work we investigate the Hörmander-Mihlin condition for multipliers on Triebel-Lizorkin spaces $F_{r,q}^r(G)$ on a graded Lie group $G$, extending in Theorem 1.1 the estimate of Seeger [29] for multipliers in Triebel-Lizorkin spaces $F_{p,q}^r(\mathbb{R}^n)$ on $\mathbb{R}^n$.

1.2. Hörmander-Mihlin condition for Triebel-Lizorkin spaces. In order to present our main result, let us introduce the required preliminaries. First we present the Hörmander-Mihlin condition for graded Lie groups as introduced in [15]. By a graded Lie group $G$, we mean a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ may be decomposed as the sum of subspaces

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{g}_{i+j} = \{0\}$ if $i + j > s$. The homogeneous dimension of $Q$ is defined by

$$Q := \sum_{\ell=1}^s \ell \cdot \dim(\mathfrak{g}_\ell).$$

Graded Lie groups include the Euclidean space $\mathbb{R}^n$, the Heisenberg group $\mathbb{H}^n$ and any stratified group. Now, consider a positive Rockland operator $\mathcal{R}$ of homogeneous degree $\nu > 0$. To define the Triebel-Lizorkin spaces associated to $\mathcal{R}$, let us fix $\eta \in C_0^\infty(\mathbb{R}^+, [0, 1])$, $\eta \neq 0$, so that $\text{supp}(\eta) \subset [1/2, 2]$, and such that

$$\sum_{j \in \mathbb{Z}} \eta(2^{-j}\lambda) = 1, \quad \lambda > 0.$$  \hfill (1.4)

Fixing $\psi_0(\lambda) := \sum_{j=-\infty}^0 \eta_j(\lambda)$, and for $j \geq 1$, $\psi_j(\lambda) := \eta(2^{-j}\lambda)$, we have

$$\sum_{\ell=0}^\infty \psi_\ell(\lambda) = 1, \quad \lambda > 0,$$  \hfill (1.5)

and one can define the family of operators $\psi_j(\mathcal{R})$ using the functional calculus of $\mathcal{R}$. Then, for $0 < q < \infty$, and $1 < p < \infty$, the Triebel-Lizorkin space $F_{p,q}^r(G)$ consists of the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{F_{p,q}^r(G)} := \left\| \left( \sum_{\ell=0}^\infty 2^{\ell r\nu} |\psi_\ell(\mathcal{R})f| q \right)^{\frac{1}{q}} \right\|_{L^p(G)} < \infty.$$

The weak-$F_{p,q}^r(G)$ space is defined by the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{\text{weak-}F_{p,q}^r(G)} := \sup_{t > 0} \left\{ x \in G : \left( \sum_{\ell=0}^\infty 2^{\ell r\nu} |\psi_\ell(\mathcal{R})f(x)| q \right)^{\frac{1}{q}} > t \right\} < \infty.$$  \hfill (1.6)

In terms of the Sobolev spaces $L^2_s(\hat{G})$ on the unitary dual (see (2.5) for details), and of the family of dilations $\{\sigma(r \cdot \pi)\}_{\pi \in \hat{G}}$, $r > 0$, of the symbol $\sigma$ of a Fourier multiplier

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These are linear left invariant homogeneous hypoelliptic partial differential operators, in view of the Helffer and Nourrigat’s resolution of the Rockland conjecture in [20]. Such operators always exist on graded Lie groups and, in fact, the existence of such operators on nilpotent Lie groups does characterise the class of graded Lie groups (c.f. [16, Section 4.1]).
$A \equiv T_\sigma$, the Hörmander-Mihlin condition takes the form\(^5\) (see Subsection 2.4 for details)
\[
\|\sigma\|_{L^2_t\ell.u,R,\eta,\mathcal{R}} := \sup_{r>0} \|\{\sigma(r \cdot \pi)\eta(\pi(\mathcal{R}))\}\|_{L^2_t(G)} < \infty,
\] (1.7)
and
\[
\|\sigma\|_{L^2_t\ell.u,L,\eta,\mathcal{R}} := \sup_{r>0} \|\{\eta(\pi(\mathcal{R}))\sigma(r \cdot \pi)\}\|_{L^2_t(G)} < \infty.
\] (1.8)

The following theorem is the main result of this work.

**Theorem 1.1.** Let $G$ be a graded Lie group of homogeneous dimension $Q$. Let $\sigma \in L^2(\hat{G})$. If
\[
\|\sigma\|_{L^2_t\ell.u,L,\eta,\mathcal{R}}, \|\sigma\|_{L^2_t\ell.u,R,\eta,\mathcal{R}} < \infty,
\] (1.9)
with $s > Q/2$, then the corresponding multiplier $A \equiv T_\sigma$ extends to a bounded operator from $F^r_{p,q}(G)$ into $F^r_{p,q}(G)$ for all $1 < p < \infty$, and all $r \in \mathbb{R}$. Moreover
\[
\|T_\sigma\|_{L^2_r(F^r_{p,q}(G))} \leq C \max\{\|\sigma\|_{L^2_t\ell.u,L,\eta,\mathcal{R}}, \|\sigma\|_{L^2_t\ell.u,R,\eta,\mathcal{R}}\},
\] (1.10)
and for $p = 1$, $A \equiv T_\sigma$ admits a bounded extension from $F^r_{1,q}(G)$ into weak-$F^r_{1,q}(G)$, and
\[
\|T_\sigma\|_{L^2_r(F^r_{1,q}(G),\text{weak-}F^r_{1,q}(G))} \leq C \max\{\|\sigma\|_{L^2_t\ell.u,L,\eta,\mathcal{R}}, \|\sigma\|_{L^2_t\ell.u,R,\eta,\mathcal{R}}\},
\] (1.11)
for any $1 < q < \infty$.

Now, we discuss briefly our result.

**Remark 1.2.** In the case $G = \mathbb{R}^n$, and taking $\mathcal{R} = (-\Delta_x)^{\frac{1}{2}}$, where $\Delta_x$ is the negative Laplacian on $\mathbb{R}^n$, Theorem 1.1 recovers the Hörmander-Mihlin theorem in Seeger [29] for Triebel-Lizorkin spaces on $\mathbb{R}^n$. Also, in view of the Littlewood-Paley theorem in [8], $F^0_{p,2}(G) = L^p(G)$, for all $1 < p < \infty$, Theorem 1.1 recovers the $L^p(G)$-Hörmander Mihlin theorem in [16].

**Remark 1.3.** The Hörmander-Mihlin theorem has been extended by several authors to spectral multipliers of Laplacian and sub-Laplacians, and settings that go beyond the Euclidean case. The literature is so broad that it is impossible to provide complete list here. We refer the reader to [1, 5, 10] and to the extensive list of references therein.

**2. Preliminaries**

In this section, we recall some preliminaries on graded and homogeneous Lie groups $G$. The unitary dual of these groups will be denoted by $\hat{G}$. We also present the notion of Rockland operators and Sobolev spaces on $G$ and on the unitary dual $\hat{G}$ by following [15], to which we refer for further details on constructions presented in this section. For the general aspects of the harmonic analysis on nilpotent Lie groups we refer the reader to [16, 18].

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\(^5\)Here $\pi(\mathcal{R})$, a.e. $\pi \in \hat{G}$, is the symbol of $\mathcal{R}$, characterised by the condition $\hat{\mathcal{R}} f(\pi) = \pi(\mathcal{R}) \hat{f}$, a.e. $\pi \in \hat{G}$, and $\eta(\pi(\mathcal{R}))$ is defined by the spectral calculus of Rockland operators (see [16, Page 178]).
2.1. **Dilations on a graded Lie group.** Let $G$ be a graded Lie group. This means that $G$ is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ may be decomposed as the sum of subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and $\mathfrak{g}_{i+j} = \{0\}$ if $i + j > s$. This implies that the group $G$ is nilpotent because the sequence $\mathfrak{g}(1) := \mathfrak{g}, \mathfrak{g}(n) := [\mathfrak{g}, \mathfrak{g}(n-1)]$ defined inductively terminates at $\{0\}$ in a finite number of steps. Examples of such groups are the Heisenberg group $H^n$ and more generally any stratified groups where the Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1$. The exponential mapping from $\mathfrak{g}$ to $G$ is a diffeomorphism, then, we can identify $G$ with $\mathbb{R}^n$ or $\mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s$ as manifolds. Consequently, we denote by $S(G)$ the Schwartz space of functions on $G$, by considering the identification $G \equiv \mathbb{R}^n$. Here, $n$ is the topological dimension of $G$, $n = n_1 + \cdots + n_s$, where $n_k = \dim \mathfrak{g}_k$. A family of dilations $D_r$, $r > 0$, on a Lie algebra $\mathfrak{g}$ is a family of linear mappings from $\mathfrak{g}$ to itself satisfying the following two conditions:

- For every $r > 0$, $D_r$ is a map of the form $D_r = \text{Exp}(\ln(r)A)$ for some diagonalisable linear operator $A$ on $\mathfrak{g}$.
- $\forall X, Y \in \mathfrak{g}$, and $r > 0$, $[D_rX, D_rY] = D_r[X, Y]$.

We call the eigenvalues of $A$, $\nu_1, \nu_2, \ldots, \nu_n$, the dilations weights or weights of $G$. A homogeneous Lie group is a connected simply connected Lie group whose Lie algebra $\mathfrak{g}$ is equipped with a family of dilations $D_r$. In such case, and with the notation above, the homogeneous dimension of $G$ is given by

$$Q = \text{Tr}(A) = \sum_{l=1}^{s} l \cdot \dim \mathfrak{g}_l.$$

We can transport dilations $D_r$ of the Lie algebra $\mathfrak{g}$ to the group by considering the family of maps

$$\exp_G \circ D_r \circ \exp_G^{-1}, \ r > 0,$$

where $\exp_G : \mathfrak{g} \to G$ is the usual exponential function associated to the Lie group $G$. We denote this family of dilations also by $D_r$ and we refer to them as dilations on the group. If we write $r x = D_r(x)$, $x \in G$, $r > 0$, then a relation on the homogeneous structure of $G$ and the Haar measure $dx$ on $G$ is given by

$$\int_{G} (f \circ D_r)(x) dx = r^{-Q} \int_{G} f(x) dx.$$

2.2. **The unitary dual and the Plancherel theorem.** We will always equip a graded Lie group with the Haar measure $dx$. For simplicity, we will write $L^p(G)$ for $L^p(G, dx)$. We denote by $\hat{G}$ the unitary dual of $G$, that is the set of equivalence classes of unitary, irreducible, strongly continuous representations of $G$ acting in separable Hilbert spaces. The unitary dual can be equipped with the Plancherel measure $d\mu$. 
So, the Fourier transform of every function $\varphi \in \mathcal{S}(G)$ at $\pi \in \hat{G}$ is defined by

$$(F_G \varphi)(\pi) \equiv \hat{\varphi}(\pi) = \int_G \varphi(x) \pi(x)^* dx,$$

and the corresponding Fourier inversion formula is given by

$$\varphi(x) = \int_{\hat{G}} \text{Tr}(\pi(x) \hat{\varphi}(\pi)) d\mu(\pi).$$

In this case, we have the Plancherel identity

$$\|\varphi\|_{L^2(G)} = \left( \int_{\hat{G}} \text{Tr}(\hat{\varphi}(\pi) \hat{\varphi}(\pi)^*) d\mu(\pi) \right)^{1/2} = \|\hat{\varphi}\|_{L^2(\hat{G})}.$$ 

We also denote $\|\hat{\varphi}\|_{\text{HS}}^2 = \text{Tr}(\hat{\varphi}(\pi) \hat{\varphi}(\pi)^*)$ the Hilbert-Schmidt norm of operators. A Fourier multiplier is formally defined by

$$T_{\sigma} u(x) = \int_{\hat{G}} \text{Tr}(\pi(x) \sigma(\pi) \hat{f}(\pi)) d\mu(\pi),$$

where the symbol $\sigma(\pi)$ is defined on the unitary dual $\hat{G}$ of $G$. For a rather comprehensive treatment of this quantization we refer to [16].

2.3. Homogeneous linear operators and Rockland operators. A linear operator $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ is homogeneous of degree $\nu \in \mathbb{C}$ if for every $r > 0$

$$T(f \circ D_r) = r^\nu(Tf) \circ D_r$$

holds for every $f \in \mathcal{D}(G)$. If for every representation $\pi \in \hat{G}$, $\pi : G \to U(H_\pi)$, we denote by $H_\pi^\infty$ the set of smooth vectors, that is, the space of elements $v \in H_\pi$ such that the function $x \mapsto \pi(x)v$, $x \in \hat{G}$ is smooth, a Rockland operator is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree $\nu = \nu_{\mathcal{R}}$ and such that, for every unitary irreducible non-trivial representation $\pi \in \hat{G}$, $\pi(\mathcal{R})$ is injective on $H_\pi^\infty$; $\sigma(\mathcal{R}) = \pi(\mathcal{R})$ is the symbol associated to $\mathcal{R}$. It coincides with the infinitesimal representation of $\mathcal{R}$ as an element of the universal enveloping algebra. It can be shown that a Lie group $G$ is graded if and only if there exists a differential Rockland operator on $G$. If the Rockland operator is formally self-adjoint, then $\mathcal{R}$ and $\pi(\mathcal{R})$ admit self-adjoint extensions on $L^2(G)$ and $H_\pi$, respectively. Now if we preserve the same notation for their self-adjoint extensions and we denote by $E$ and $E_\pi$ their spectral measures, by functional calculus we have

$$\mathcal{R} = \int_{-\infty}^{\infty} \lambda dE(\lambda), \text{ and } \pi(\mathcal{R}) = \int_{-\infty}^{\infty} \lambda dE_\pi(\lambda).$$

We now recall a lemma on dilations on the unitary dual $\hat{G}$, which will be useful in our analysis of spectral multipliers. For the proof, see Lemma 4.3 of [15].

**Lemma 2.1.** For every $\pi \in \hat{G}$ let us define

$$D_r(\pi) \equiv r \cdot \pi := \pi^{(r)}$$
by \( D_r(\pi)(x) = \pi(r \cdot x) \) for every \( r > 0 \) and \( x \in G \). Then, if \( f \in L^\infty(\mathbb{R}) \) then \( f(\pi^{(r)}(\mathcal{R})) = f(r^\nu \pi(\mathcal{R})) \).

We refer to [16, Chapter 4] and references therein for an exposition of further properties of Rockland operators and their history, and to ter Elst and Robinson [13] for their spectral properties.

2.4. Hörmander-Mihlin multipliers on \( L^p(G) \). To define Sobolev spaces, we choose a positive left-invariant Rockland operator \( \mathcal{R} \) of homogeneous degree \( \nu > 0 \). With notations above one defines Sobolev spaces as follows (c.f [16]).

**Definition 2.2.** Let \( r \in \mathbb{R} \), the homogeneous Sobolev space \( \dot{L}^p_r(G) \) consists of those functions \( f \in \mathcal{D}'(G) \) satisfying

\[
\|f\|_{\dot{L}^p_r(G)} := \|\mathcal{R}^{\frac{r}{2}} f\|_{L^p(G)} < \infty.
\]  

(2.3)

Analogously, the inhomogeneous Sobolev space \( H^{r,p}(G) \) consists of those distributions \( f \in \mathcal{D}'(G) \) satisfying

\[
\|f\|_{L^p(G)} := \|(I + \mathcal{R})^{\frac{r}{2}} f\|_{L^p(G)} < \infty.
\]  

(2.4)

By using a quasi-norm \( | \cdot | \) on \( G \) we can introduce for every \( r \geq 0 \), the inhomogeneous Sobolev space of order \( r \) on \( \mathcal{G}, L^2_r(\mathcal{G}) \) which is defined by

\[
L^2_r(\mathcal{G}) = \mathcal{F}_G(L^2(G, (1 + | \cdot |^2)^{\frac{r}{2}} dx))
\]  

(2.5)

where \( \mathcal{F}_G \) is the Fourier transform on the group \( G \). In a similar way, for \( r \geq 0 \) the homogeneous Sobolev space \( \dot{L}^2_r(\mathcal{G}) \) is defined by

\[
\dot{L}^2_r(\mathcal{G}) = \mathcal{F}_G(L^2(G, | \cdot |^r dx)).
\]

As usual if \( r = 0 \) we denote \( L^2(\mathcal{G}) = \dot{H}^0(\mathcal{G}) = H^0(\mathcal{G}) \). Characterisations of Sobolev spaces on \( G \) and on the unitary dual \( \hat{G} \) in terms of homogeneous norms on \( G \) can be found in [15] and [16], respectively.

Finally we present the Hörmander-Mihlin theorem for graded nilpotent Lie groups. The formulation of such result requires a local notion of Sobolev space on the dual space \( \hat{G} \). We introduce this as follows. Let \( s \geq 0 \), we say that the field \( \sigma = \{\sigma(\pi) : \pi \in \hat{G}\} \) is locally uniformly in right-\( L^2_s(\hat{G}) \) (resp. left-\( L^2_s(\hat{G}) \)) if there exists a positive Rockland operator \( \mathcal{R} \) and a function \( \eta \in \mathcal{D}(G) \), \( \eta \neq 0 \), satisfying

\[
\|\sigma\|_{L^2_{s,t,u,R,\eta,\mathcal{R}}} := \sup_{r > 0} \|\{\sigma(r \cdot \pi) \eta(\pi(\mathcal{R}))\}\|_{L^2_s(\mathcal{G})} < \infty,
\]  

(2.6)

respectively,

\[
\|\sigma\|_{L^2_{s,t,u,L,\eta,\mathcal{R}}} := \sup_{r > 0} \|\{\eta(\pi(\mathcal{R})) \sigma(r \cdot \pi)\}\|_{L^2_s(\mathcal{G})} < \infty.
\]  

(2.7)

It important to mention that if \( \phi \neq 0 \), is another function in \( \mathcal{D}(0, \infty) \) then (see [15])

\[
\|\sigma\|_{L^2_{s,t,u,R,\eta,\mathcal{R}}} \asymp \|\sigma\|_{L^2_{s,t,u,L,\phi,\mathcal{R}}} \quad \text{and} \quad \|\sigma\|_{L^2_{s,t,u,L,\eta,\mathcal{R}}} \asymp \|\sigma\|_{L^2_{s,t,u,L,\phi,\mathcal{R}}}.
\]  

(2.8)

The following lemma shows how Sobolev spaces on the unitary dual interact with the family of dilations.
Lemma 2.3. Let $\sigma \in L^2(\hat{G})$. If $r > 0$ and $s \geq 0$ then
\[
\|\sigma \circ D_r\|_{L^2_s(\hat{G})} = r^{s - \frac{Q}{2}} \|\sigma\|_{L^2(\hat{G})}.
\]
This implies that $\sigma \in \dot{L}^2_s(\hat{G})$ if only if for every $r > 0$, $\sigma \circ D_r \in \dot{L}^2_s(\hat{G})$. Also, if $\mathcal{R}, \mathcal{S}$ are positive Rockland operators and $\eta, \zeta \in \mathcal{D}(0, \infty)$, $\eta, \zeta \neq 0$, then there exists $C > 0$ such that
\[
\|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)} \leq C \|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)}
\]
and
\[
\|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)} \leq C \|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)}.
\]

Proof. By Lemma 2.1 or Lemma 4.3 of [15] we have
\[
\|\sigma \circ D_r\|_{L^2(\hat{G})} = \|r \cdot \hat{\mathcal{F}}_G^{-1}(\sigma \circ D_r)\|_{L^2(\hat{G})} = \|r \cdot \hat{\mathcal{F}}_G^{-1}(\sigma)\|_{L^2(\hat{G})}
\]
\[
= r^{s - \frac{Q}{2}} \|\sigma\|_{L^2(\hat{G})}.
\]

With the equality above, it is clear that $\sigma \in \dot{L}^2_s(\hat{G})$ if only if for every $r > 0$, $\sigma \circ D_r \in \dot{L}^2_s(\hat{G})$. The second part of the Lemma has been shown in Proposition 4.6 of [15].

Now, we state the Hörmander-Mihlin theorem on the graded nilpotent Lie group $G$ (c.f. Theorem 4.11 of [15]):

Theorem 2.4. [15, $L^p$-Hörmander-Mihlin Theorem]. Let $G$ be a graded Lie group. Let $\sigma \in L^2(\hat{G})$. If
\[
\|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)} \leq C \|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)} < \infty,
\]
with $s > Q/2$, then the corresponding multiplier $A \equiv T_\sigma$ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$, and for $p = 1$, $T_\sigma$ is of weak $(1,1)$ type. Moreover
\[
\|T_\sigma\|_{L^1(G)} \leq C \|\sigma\|_{L^2(\hat{G})}, \quad \|T_\sigma\|_{L^1(G)} \leq C \max\{\|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)}, \|\sigma\|_{L^2_{\mathcal{I}, \mathcal{R}}(\eta, \zeta)}\}.
\]

In the proof of Theorem 1.1 we will use that every dyadic decomposition of a Fourier multiplier satisfying the Hörmander-Mihlin condition has a Calderón-Zygmund kernel and we will make use of the estimates proved in [15] for this family of kernels. So, in the following remark we record the Calderón-Zygmund estimates in [15].

Remark 2.5 (On the proof of the $L^p$-Hörmander-Mihlin Theorem). Let us describe the fundamental steps of the proof of the $L^p$-Hörmander-Mihlin theorem (c.f. Theorem 4.11 of [15]) on graded Lie groups. For this, we follow [15, Page 19]. Let us fix
\[
\eta \in C^\infty_c(\mathbb{R}^+, [0, 1]), \eta \neq 0, \text{ so that supp}(\eta) \subset [1/2, 2], \text{ and such that}
\]
\[
\sum_{j \in \mathbb{Z}} \eta(2^{-j}\lambda) = 1, \quad \lambda > 0.
\]
By defining $\psi_0(\lambda) := \sum_{j=-\infty}^{0} \eta_j(\lambda)$, and for $j \geq 1$, $\psi_j(\lambda) := \eta(2^{-j}\lambda)$, we obviously have
\[
\sum_{\ell=0}^{\infty} \psi_\ell(\lambda) = 1, \quad \lambda > 0.
\]
Then, for a.e. $\pi \in \hat{G}$, $\sum_{\ell=0}^{\infty} \psi_\ell(\mathcal{R})$ converges towards the identity in the strong topology of the norm in $L^2(G)$. By decomposing

$$T_\sigma = \sum_{j \geq 0} T_j, \quad T_j := T_\sigma \psi_j(\mathcal{R}),$$

(2.16)

and using that the right-convolution kernels of the family $T_j$, $k_j$ summed on $j$, provide the distributional kernel of $T$, $k = \sum_j k_j$, which agrees with a locally integrable function on $G \setminus \{0\}$, such that, for every $c > 0$,

$$J_\ell := \sup_{x \in G} \int_{|x| > 4|z|} |2^{-\ell Q_k(2^{-\ell} \cdot z^{-1} x)} - 2^{-\ell Q_k(2^{-\ell} \cdot x)}| dx,$$  

(2.17)

satisfies, $J_\ell \lesssim 2^{-\ell \varepsilon_0} \max\{\|\sigma\|_{L^p_{\ell,u,L,R}}, \|\sigma\|_{L^p_{\ell,u,R,\eta}}\}$, for some $\varepsilon_0 > 0$, depending only of $c > 0$. The proof in [15, Page 19] shows that

$$\|T_j\|_{\mathcal{B}(L^p(G))} \leq J_\ell \max\{\|\sigma\|_{L^p_{\ell,u,L,R}}, \|\sigma\|_{L^p_{\ell,u,R,\eta}}\}$$

and consequently

$$\|T\|_{\mathcal{B}(L^p(G))} \lesssim \sum_j 2^{-j \varepsilon_0} \max\{\|\sigma\|_{L^p_{\ell,u,L,R}}, \|\sigma\|_{L^p_{\ell,u,R,\eta}}\},$$

proving the $L^p(G)$-boundedness of $T_\sigma$.

About, the Littlewood-Paley decomposition $\{\psi_\ell\}_{\ell=0}^{\infty}$, introduced in Remark 2.5, the following estimates were proved in [8]. The following result is the Littlewood-Paley theorem.

**Theorem 2.6.** [8, Littlewood Paley Theorem]. Let $1 < p < \infty$ and let $G$ be a graded Lie group. If $\mathcal{R}$ is a positive Rockland operator then there exist constants $0 < c_p, C_p < \infty$ depending only on $p$ and $\psi_0$ such that

$$c_p \|f\|_{L^p(G)} \leq \left\| \left( \sum_{\ell=0}^{\infty} |\psi_\ell(\mathcal{R}) f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \leq C_p \|f\|_{L^p(G)},$$

(2.18)

holds for every $f \in L^p(G)$. Moreover, for $p = 1$, there exists a constant $C > 0$ independent of $f \in L^1(G)$ and $t > 0$, such that

$$\left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |\psi_\ell(\mathcal{R}) f(x)|^2 \right)^{\frac{1}{2}} > t \right\} \leq \frac{C}{t} \|f\|_{L^1(G)}.$$

(2.19)

The action of dyadic decompositions on vector-valued functions is considered in the next theorem.

**Theorem 2.7.** [8, Vector-valued inequality for dyadic decompositions]. Let $1 < p, r < \infty$ and let $G$ be a graded Lie group. If $\mathcal{R}$ is a positive Rockland operator then there exist constants $C_p > 0$ depending only on $p$ and $\psi_0$, such that

$$\left\| \left( \sum_{\ell=0}^{\infty} |\psi_\ell(\mathcal{R}) f_\ell|^r \right)^{\frac{1}{r}} \right\|_{L^p(G)} \leq C_p \left\| \left( \sum_{\ell=0}^{\infty} |f_\ell(x)|^r \right)^{\frac{1}{r}} \right\|_{L^p(G)} := C_p \|f_\ell\|_{L^p(G, r^\phi(\eta))},$$

(2.20)
Moreover, for \( p = 1 \), there exists a constant \( C > 0 \) independent of \( \{f_{\ell}\} \in L^1(G, \ell^r(N_0)) \) and \( t > 0 \), such that
\[
\left\| x \in G : \left( \sum_{\ell=0}^{\infty} |\psi_{\ell}(\mathcal{R})f_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{C} \right\|_{L^1(G, \ell^r(N_0))} \leq \frac{C}{t} \left\| \{ f_{\ell} \} \right\|_{L^1(G, \ell^r(N_0))}. \tag{2.21}
\]

3. TRIEBEL-LIZORKIN SPACES ON GRADED LIE GROUPS

In this section, Triebel-Lizorkin spaces on graded Lie groups are introduced. They can be defined by using positive Rockland operators. As in the introduction, let us fix \( \eta \in C_0^\infty(\mathbb{R}^+, [0, 1]) \), \( \eta \neq 0 \), so that \( \text{supp}(\eta) \subset [1/2, 2] \), and such that
\[
\sum_{j \in \mathbb{Z}} \eta(2^{-j} \lambda) = 1, \ \lambda > 0. \tag{3.1}
\]
Fixing \( \psi_0(\lambda) := \sum_{j=-\infty}^{0} \eta_j(\lambda) \), and for \( j \geq 1 \), \( \psi_j(\lambda) := \eta(2^{-j} \lambda) \), we have
\[
\sum_{\ell=0}^{\infty} \psi_{\ell}(\lambda) = 1, \ \lambda > 0. \tag{3.2}
\]
Let \( \mathcal{R} \) be a positive Rockland operator and let us define the family of operators \( \psi_j(\mathcal{R}) \) using the functional calculus. Then, for \( 0 < q < \infty \), and \( 1 < p < \infty \), the Triebel-Lizorkin space \( F^{p,q}_{r}(G) \) consists of the distributions \( f \in \mathcal{D}'(G) \) such that
\[
\| f \|_{F^{p,q}_{r}(G)} := \left\| \left( \sum_{\ell=0}^{\infty} 2^{\frac{\ell}{r}} |\psi_{\ell}(\mathcal{R})f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} < \infty,
\]
and for \( p = 1 \), the weak-\( F^{q}_{1,q}(G) \) space is defined by the distributions \( f \in \mathcal{D}'(G) \) such that
\[
\| f \|_{\text{weak-} F^{q}_{1,q}(G)} := \sup_{t > 0} t \left\| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} 2^{\frac{\ell}{r}} |\psi_{\ell}(\mathcal{R})f(x)|^q \right)^{\frac{1}{q}} > t \right\} \right\| < \infty. \tag{3.3}
\]

Remark 3.1. In the formulation of the Triebel-Lizorkin spaces we use (smooth) dyadic decompositions instead of characteristic functions of intervals because, same as in \( \mathbb{R}^n \), characteristics functions applied to Rockland operators are in general unbounded operators on \( L^p(G) \), see [4, 14] for instance.

In the following theorem we study some embedding properties for Triebel-Lizorkin spaces and we show, that they are independent on the choice of the positive Rockland operator \( \mathcal{R} \). For a consistent investigation of Triebel-Lizorkin spaces on compact Lie groups, we refer the reader to [24] (and to [6, Chapter 6] for the Triebel-Lizorkin spaces associated to sub-Laplacians).

**Theorem 3.2.** Let \( G \) be a graded Lie group and let \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be two positive Rockland operators on \( G \) of homogeneous degrees \( \nu_1 > 0 \) and \( \nu_2 > 0 \), respectively. Then we have the following properties.
(1) For $1 < p, q < \infty$,
\[ F_{p,q}^r(G) := F_{p,q}^{r,R_1}(G) = F_{p,q}^{r,R_2}(G) \]
and for $p = 1$,
\[ \text{weak-}F_{1,q}^r(G) := \text{weak-}F_{1,q}^{r,R_1}(G) = \text{weak-}F_{1,q}^{r,R_2}(G), \]
where the coincidence of these spaces is understood in the sense that the topologies induced by their norms are equivalent.

(2) $F_{p,q1}^{r+,\ell}(G) \hookrightarrow F_{p,q2}^{r,\ell}(G) \hookrightarrow F_{p,\infty}^{r,\ell}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $0 \leq q_1 \leq q_2 \leq \infty$.

(3) $F_{p,q1}^{r+,\ell}(G) \hookrightarrow F_{p,q2}^{r,\ell}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.

(4) $F_{p,2}(G) = L^p_r(G)$ for all $r \in \mathbb{R}$, and all $1 < p < \infty$, where $L^p_r(G)$ are Sobolev spaces on $G$.

Proof. The proof of (2) and (3) are only an adaptation of the arguments presented in Triebel [31]. Because $F_{p,2}^p(G) = L^p_r(G)$, in view of the Littlewood-Paley theorem (Theorem 2.6) and the fact that $(1 + \mathcal{R}_1)^{\frac{1}{2}} : F_{p,2}^p(G) \rightarrow F_{p,2}^0(G)$, and $(1 + \mathcal{R}_1)^{\frac{1}{2}} : L^p_r(G) \rightarrow L^p_r(G)$ are isomorphisms for any $r \in \mathbb{R}$, we conclude that $F_{p,2}^p(G) = L^p_r(G)$ proving (4). So, to conclude the proof of the theorem, we will prove (1). Let us define the positive Rockland operator $\mathcal{R} := \mathcal{R}_1^{\nu_2} + \mathcal{R}_2^{\nu_1}$. Let us define for any $\ell \geq 0$, the operator $\psi_\ell(\mathcal{R})$ by using the functional calculus. So, the relation
\[ \sum_{\ell=0}^{\infty} \psi_\ell(\lambda) = 1, \lambda > 0, \]
implies that $\sum_{\ell=0}^{\infty} \psi_\ell(\mathcal{R}) = I$ converges to the identity operator $I$ on $L^2(G)$ in the strong topology induced by the norm of $L^2(G)$. Observe that from the properties of the supports of the dyadic decomposition $\psi_\ell$, $\ell \in \mathbb{N}$, for $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, and the fact that $\text{supp}(\psi_\ell) \subset [2^{\ell-1}, 2^{\ell+1}]$, $1 \leq \ell \leq 3$, we have
\[ \psi_{\ell_1}(\mathcal{R}_1)\psi_{\ell_2}(\mathcal{R})\psi_{\ell_3}(\mathcal{R}_2) \equiv 0, \quad |\ell_2 - \ell_1|, |\ell_3 - \ell_1| \geq M_0, \quad \text{for some } M_0 \in \mathbb{N}, \text{ independent of } \ell_1, \ell_2, \ell_3 \in \mathbb{N}_0. \]
These can be deduced from the properties of the spectral measures $dE_i$ associated to $\mathcal{R}_i$ and $dE$ associated to $\mathcal{R}$ respectively. Indeed, by following [15, Page 20], for all $a, b > 0$,
\[ E(-\infty, a^{\nu_1}]E[a, \infty) \equiv 0, \quad \text{and } E_1(b^{\nu_2}, \infty)E[b, \infty) \equiv 0. \]
Because
\[ \psi_{\ell_1}(\mathcal{R}_i) = E_i[2^{\ell_1-1}, 2^{\ell_1}]\psi_{\ell_1}(\mathcal{R}_i)E_i[2^{\ell_1-1}, 2^{\ell_1}], \]
and
\[ \psi_{\ell_3}(\mathcal{R}) = E[2^{\ell_3-1}, 2^{\ell_3}]\psi_{\ell_3}(\mathcal{R})E[2^{\ell_3-1}, 2^{\ell_3}], \]
the existence of $M_0 \in \mathbb{N}$ in (3.5) follows. Now, let us use this analysis to prove that $F_{p,q}^{0,R_1}(G) = F_{p,q}^{0,R_2}(G)$ for $1 < p < \infty$, and that weak-$F_{1,q}^{0,R_1}(G) = \text{weak-}F_{1,q}^{0,R_2}(G)$. From this case, the coincidence of the spaces $F_{p,q}^{r,R_1}(G) = F_{p,q}^{r,R_2}(G)$ and weak-$F_{1,q}^{r,R_1}(G) = \text{weak-}F_{1,q}^{r,R_2}(G)$ follows from the fact that $(1 + \mathcal{R}_1)^{\frac{1}{2}} : F_{p,q}^{r,R_1}(G) \rightarrow F_{p,q}^{0,R_2}(G)$ is an isomorphism, for any $r \in \mathbb{R}$. Now in view of (3.5), let us observe that for $1 < q < \infty$,
we can estimate
\[
\left( \sum_{\ell_1=0}^{\infty} \left| \psi_{\ell_1}(\mathcal{R}_1)f \right|^q \right)^{\frac{1}{q}} \leq \left( \sum_{\ell_1=0}^{\infty} \left( \sum_{\ell_2, \ell_3=0}^{\infty} \left| \psi_{\ell_1}(\mathcal{R}_1) \psi_{\ell_2}(\mathcal{R}) \psi_{\ell_3}(\mathcal{R}) f \right|^q \right) \right)^{\frac{1}{q}} \lesssim \sum_{\ell_1=0}^{\infty} \left( \sum_{|\ell_2-\ell_1|, |\ell_3-\ell_1| \leq M_0} \left| \psi_{\ell_1}(\mathcal{R}_1) \psi_{\ell_2}(\mathcal{R}) \psi_{\ell_3}(\mathcal{R}) f \right|^q \right)^{\frac{1}{q}}.
\]

So, we have
\[
\left( \sum_{\ell_1=0}^{\infty} \left| \psi_{\ell_1}(\mathcal{R}_1)f \right|^q \right)^{\frac{1}{q}} \leq \left( \sum_{\ell_1=0}^{\infty} \left( \sum_{|\ell_2-\ell_1|, |\ell_3-\ell_1| \leq M_0} \left| \psi_{\ell_1}(\mathcal{R}_1) \psi_{\ell_2}(\mathcal{R}) \psi_{\ell_3}(\mathcal{R}) f \right|^q \right) \right)^{\frac{1}{q}}. \tag{3.7}
\]

Now, taking the $L^p$-norm (or the $L^{1,\infty}$-norm) in both sides of (3.7), and by applying the vector-valued inequality (2.20), two times, we obtain
\[
\left\| f \right\|_{\mathcal{F}^{0, \mathcal{R}_1}} = \left\| \left( \sum_{\ell_1=0}^{\infty} \left| \psi_{\ell_1}(\mathcal{R}_1)f \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} \leq \left\| \left( \sum_{\ell_1=0}^{\infty} \left( \sum_{|\ell_2-\ell_1|, |\ell_3-\ell_1| \leq M_0} \left| \psi_{\ell_1}(\mathcal{R}_1) \psi_{\ell_2}(\mathcal{R}) \psi_{\ell_3}(\mathcal{R}) f \right|^q \right) \right)^{\frac{1}{q}} \right\|_{L^p(G)} \lesssim_{p, M_0} \left\| \left( \sum_{\ell_1=0}^{\infty} \left( \sum_{|\ell_2-\ell_1| \leq M_0} \left| \psi_{\ell_1}(\mathcal{R}_1) \psi_{\ell_2}(\mathcal{R}) f \right|^q \right) \right)^{\frac{1}{q}} \right\|_{L^p(G)} \lesssim_{p, M_0} \left\| f \right\|_{\mathcal{F}^{0, \mathcal{R}_2}}.
\]

This proves that $\left\| f \right\|_{\mathcal{F}^{0, \mathcal{R}_1}} \lesssim_{p, M_0} \left\| f \right\|_{\mathcal{F}^{0, \mathcal{R}_2}}$. In a similar way we can prove that $\left\| f \right\|_{\mathcal{F}^{0, \mathcal{R}_2}} \lesssim_{p, M_0} \left\| f \right\|_{\mathcal{F}^{0, \mathcal{R}_1}}$. The same analysis applying the $L^{1,\infty}$-norm in both sides of (3.7), and using the weak vector-valued inequality (2.21) imply that
\[
\left\| f \right\|_{\text{weak-}F^{0, \mathcal{R}_1}_{1,q}(G)} \lesssim_{p, M_0} \left\| f \right\|_{\text{weak-}F^{0, \mathcal{R}_2}_{1,q}(G)} \lesssim_{p, M_0} \left\| f \right\|_{\text{weak-}F^{0, \mathcal{R}_1}_{1,q}(G)}.
\]

So, we have proved (1). The proof of Theorem 3.2 is complete. \qed
4. Hörmander-Mihlin multipliers on Triebel-Lizorkin spaces: Proof of Theorem 1.1

Let us fix \( \eta \in C_0^\infty (\mathbb{R}, [0, 1]), \eta \neq 0 \), so that \( \text{supp}(\eta) \subset [1/2, 2] \), and such that

\[
\sum_{j \in \mathbb{Z}} \eta(2^{-j} \lambda) = 1, \; \lambda > 0.
\] (4.1)

By defining \( \psi(\lambda) := \sum_{j=\infty}^0 \eta_j(\lambda) \), and for \( j \geq 1 \), \( \psi_j(\lambda) := \eta(2^{-j} \lambda) \), we obviously have

\[
\sum_{\ell=0}^\infty \psi(\lambda) = 1, \; \lambda > 0.
\] (4.2)

Proof of Theorem 1.1. By observing that \( (1 + \mathcal{R})^\frac{r}{2} : F^r_{p,q}(G) \to F^0_{p,q}(G) \) and \( (1 + \mathcal{R})^{-\frac{r}{2}} : F^0_{p,q}(G) \to F^r_{p,q}(G) \) are isomorphism, it is suffices to prove that \( A \) admits a bounded extension from \( F^0_{p,q}(G) \) into \( F^0_{p,q}(G) \). For this, let us define the vector-valued operator \( W : L^2(G, \ell^2(\mathbb{N}_0)) \to L^2(G, \ell^2(\mathbb{N}_0)) \) by

\[
W(\{g_\ell\}_{\ell=0}^\infty) := (\{W_\ell g_\ell\}_{\ell=0}^\infty), \; W_\ell := A\psi_\ell(\mathcal{R}).
\] (4.3)

Observe that \( W \) is well-defined (bounded from \( L^2(G, \ell^2(\mathbb{N}_0)) \) into \( L^2(G, \ell^2(\mathbb{N}_0)) \)), because \( A \) admits a bounded extension on \( L^2(G) \) and also, in view of the following estimate

\[
\|W(\{g_\ell\}_{\ell=0}^\infty)\|_{L^2(G, \ell^2(\mathbb{N}_0))}^2 := \int G \sum_{\ell=0}^\infty |A\psi_\ell(\mathcal{R})g_\ell(x)|^2 \, dx
\]

\[
= \sum_{\ell=0}^\infty \int G |A\psi_\ell(\mathcal{R})g_\ell(x)|^2 \, dx
\]

\[
\leq \|A\|_{L^2(G)}^2 \sup_{\ell} \|\psi_\ell(\mathcal{R})\|_{L^2(G)}^2 \sum_{\ell=0}^\infty \int G |g_\ell(x)|^2 \, dx
\]

\[
\leq \|A\|_{L^2(G)}^2 \sup_{\ell} \|\psi_\ell\|_{L^2(\mathbb{R})}^2 \sum_{\ell=0}^\infty \int G |g_\ell(x)|^2 \, dx
\]

\[
= \|A\|_{L^2(G)}^2 \|\psi\|_{L^2(\mathbb{R})}^2 \sum_{\ell=0}^\infty \int G |g_\ell(x)|^2 \, dx
\]

\[
\lesssim \|\{g_\ell\}_{\ell=0}^\infty\|_{L^2(G, \ell^2(\mathbb{N}_0))}^2.
\]

So, observe that, in order to prove Theorem 1.1 it is enough to prove the following two lemmas.

Lemma 4.1. \( W : L^q(G, \ell^q(\mathbb{N}_0)) \to L^q(G, \ell^q(\mathbb{N}_0)) \) admits a bounded extension for all \( 1 < q < \infty \).

Lemma 4.2. \( W : L^1(G, \ell^q(\mathbb{N}_0)) \to L^1(\infty, \ell^q(\mathbb{N}_0)) \) admits a bounded extension for all \( 1 < q < \infty \).
Remark 4.3. Indeed, by the Marcinkiewicz interpolation, these two lemmas are enough to show that $W : L^p(G, \ell^q(N_0)) \to L^p(G, \ell^q(N_0))$ admits a bounded extension for all $1 < p \leq q < \infty$. The case $1 < q \leq p < \infty$ follows from the fact that $L^p(G, \ell^q(N_0))$ is the dual of $L^q(G, \ell^p(N_0))$ and also that Lemma 4.1 and Lemma 4.2 hold if we change $A$ by its standard $L^2$-adjoint. Now, note that by defining $\psi_{-1} = \psi_0$, we have

$$\|Af\|_{F^{p,q}_{\psi,q}(G)} = \left\| \left( \sum_{l=0}^{\infty} |A\psi_l(\mathcal{R})f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} \leq \left\| \left( \sum_{l=0}^{\infty} |A\psi_l(\mathcal{R})[\psi_{l-1}(\mathcal{R}) + \psi_{l}(\mathcal{R}) + \psi_{l+1}(\mathcal{R})]f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} = \|W\{[\psi_{l-1}(\mathcal{R}) + \psi_{l}(\mathcal{R}) + \psi_{l+1}(\mathcal{R})]f\}_{l=0}^{\infty}\|_{L^p(\ell^q)} \lesssim \|f\|_{F^{p,q}_{\psi,q}(G)}.$$  

Also note that from Lemma 4.2, we have

$$\|Af\|_{\text{weak-}F^{p,q}_{1,q}(G)} = \left\| \left( \sum_{l=0}^{\infty} |A\psi_l(\mathcal{R})f|^q \right)^{\frac{1}{q}} \right\|_{L^{1,\infty}(G)} \leq \left\| \left( \sum_{l=0}^{\infty} |A\psi_l(\mathcal{R})[\psi_{l-1}(\mathcal{R}) + \psi_{l}(\mathcal{R}) + \psi_{l+1}(\mathcal{R})]f|^q \right)^{\frac{1}{q}} \right\|_{L^{1,\infty}(G)} = \|W\{[\psi_{l-1}(\mathcal{R}) + \psi_{l}(\mathcal{R}) + \psi_{l+1}(\mathcal{R})]f\}_{l=0}^{\infty}\|_{L^{1,\infty}(\ell^q)} \lesssim \|f\|_{F^{p,q}_{1,q}(G)}.$$  

So, knowing that Lemmas 4.1 and 4.2 are enough for proving Theorem 1.1, we will proceed with their proofs.

Proof of Lemma 4.1. It suffices to prove that the operators $W_\ell$ are uniformly bounded on $L^q(G)$. This is trivial for $q = 2$, so it is suffices (by the duality argument) that the operators $W_\ell$ are uniformly bounded from $L^1(G)$ into $L^{1,\infty}(G)$. So, we are going to prove that there exists a constant $C > 0$, independent of $f \in L^1(G)$, and $\ell \in \mathbb{N}_0$, such that

$$|\{x \in G : |W_\ell f(x)| > t\}| \leq C \frac{\|f\|_{L^1(G)}}{t}. \quad (4.4)$$

We start the proof by applying the Calderón-Zygmund decomposition Lemma to the non-negative function $f \in L^p(G) \cap L^1(G) \subset L^1(G)$, under the identification $G \simeq \mathbb{R}^n$, (see, e.g. Hebisch [19]) in order to obtain a suitable family of disjoint open sets $\{I_j\}_{j=0}^{\infty}$ such that

- $f(x) \leq t$, for a.e. $x \in G \setminus \cup_{j \geq 0} I_j$,
• \[ \sum_{j \geq 0} |I_j| \leq \frac{C}{\ell} \|f\|_{L^1(G)}, \] and

• \[ |t|I_j| \leq \int_{I_j} f(x) dx \leq 2|I_j|t, \text{ for all } j. \]

Now, for every \( j \in \mathbb{N}_0 \), let us define \( R_j \) by

\[ R_j := \sup \{ R > 0 : B(z_j, R) \subset I_j, \text{ for some } z_j \in I_j \}, \tag{4.5} \]

where \( B(z_j, R) = \{ x \in I_j : |z_j^{-1}x| < R \} \). Then, we can assume that every \( I_j \) is bounded, and that \( I_j \subset B(z_j, 2R_j) \), where \( z_j \in I_j \) (see Hebish [19]).

**Remark 4.4.** Before continuing with the proof note that by assuming \( f(e_G) > t \), (this is just re-defining \( f \in L^p(G) \cap L^1(G) \) at the identity element) we should have that

\[ e_G \in \bigcup_j I_j, \tag{4.6} \]

because \( f(x) \leq t \), for a.e. \( x \in G \setminus \bigcup_{j \geq 0} I_j \).

Let us define, for every \( x \in I_j \),

\[ g(x) := \frac{1}{|I_j|} \int_{I_j} f(y) dy, \quad b(x) = f(x) - g(x), \tag{4.7} \]

and for \( x \in G \setminus \bigcup_{j \geq 0} I_j \),

\[ g(x) = f(x), \quad b(x) = 0. \tag{4.8} \]

Observe that for every \( x \in I_j \),

\[ |g(x)| = \left| \frac{1}{|I_j|} \int_{I_j} f(y) dy \right| \leq 2t. \]

By applying the Minkowski inequality, we have

\[ |\{ x \in G : |W_\ell f(x)| > t \}| \leq \left| \left\{ x \in G : |W_\ell g(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right|. \]

By the Chebyshev inequality, we have

\[
\begin{align*}
|\{ x \in G : |W_\ell f(x)| > t \}| & \leq \left| \left\{ x \in G : |W_\ell g(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \\
& = \left| \left\{ x \in G : |W_\ell g(x)|^2 > \frac{t^2}{2} \right\} \right| + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \\
& \leq \frac{2^2}{t^2} \int_G |W_\ell g(x)|^2 dx + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \\
& \leq \frac{2^2}{t^2} \sup_\ell \|W_\ell\|_{L^2(L^2(G))} \int_G |g(x)|^2 dx + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right|.
\end{align*}
\]
Taking into account that in view of the \(L_1\)-boundedness of \(A\) and the fact that the operators \(\psi_\ell(R)\) are \(L^2(G)\)-bounded uniformly in \(\ell\). Also, note that the estimate

\[
\|g\|_{L^2(G)}^2 = \int_G |g(x)|^2 dx = \sum_j \int_{I_j} |g(x)|^2 dx + \int_{G \setminus \cup_j I_j} |g(x)|^2 dx
\]

implies that,

\[
|\{ x \in G : |W_\ell f(x)| > t \}| \leq \frac{4}{t} \| f \|_{L^1(G)} + \left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right|.
\]

Taking into account that \(b \equiv 0\) on \(G \setminus \cup_j I_j\), we have that

\[
b = \sum_k b_k, \quad b_k(x) = b(x) \cdot 1_{I_k}(x).
\]  

(4.9)

Let us assume that \(I_j^*\) is an open set, such that \(I_j \subset I_j^*\), and \(|I_j^*| = K|I_j|\) for some \(K > 0\), and \(\text{dist}(\partial I_j^*, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G)\), where \(c > 0\) and \(e_G\) is the identity element of \(G\). So, by the Minkowski inequality we have,

\[
\left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| = \left| \left\{ x \in \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right|
\]

\[
\leq \left| \left\{ x \in G : x \in \cup_j I_j^* \right\} \right| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right|.
\]

In consequence, we have the estimates,

\[
\left| \left\{ x \in G : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \leq \sum_j |I_j| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right|
\]

\[
= K \sum_j |I_j| + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right|
\]

\[
\leq \frac{CK}{t} \| f \|_{L^1(G)} + \left| \left\{ x \in G \setminus \cup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right|.
\]
The Chebyshev inequality allows to estimate the right hand side above as follows,

\[ \left| \left\{ x \in G \setminus \bigcup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \leq \frac{2}{t} \int_{G \setminus \bigcup_j I_j^*} |W_\ell b(x)| \, dx \]

\[ \leq \frac{2}{t} \sum_k \int_{G \setminus \bigcup_j I_j^*} |W_\ell b_k(x)| \, dx. \]

From now, let us denote by \( \kappa_\ell \) the right convolution kernel of \( W_\ell := A\psi_\ell(R) \). Observe that

\[ \left| \left\{ x \in G \setminus \bigcup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \leq \frac{2}{t} \sum_k \int_{G \setminus \bigcup_j I_j^*} |W_\ell b_k(x)| \, dx \]

\[ = \frac{2}{t} \sum_k \int_{G \setminus \bigcup_j I_j^*} |b_k * \kappa_\ell(x)| \, dx \]

\[ = \frac{2}{t} \sum_k \int_{G \setminus \bigcup_j I_j^*} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) \, dz \right| \, dx. \]

By using that the average of \( b_k \) on \( I_k \) is zero, \( \int_{I_k} b_k(z) \, dz = 0 \), we have

\[ \frac{2}{t} \sum_k \int_{G \setminus \bigcup_j I_j^*} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) \, dz \right| \, dx \]

\[ = \frac{2}{t} \sum_k \int_{G \setminus \bigcup_j I_j^*} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) \, dz - \kappa_\ell(x) \int_{I_k} b_k(z) \, dz \right| \, dx \]

\[ = \frac{2}{t} \sum_k \int_{G \setminus \bigcup_j I_j^*} \left| \int_{I_k} (\kappa_\ell(z^{-1}x) - \kappa_\ell(x)) b_k(z) \, dz \right| \, dx. \]

If we assume for a moment that

\[ M = \sup_k \sup_{x \in I_k} \sum_{t=0}^\infty \int_{G \setminus \bigcup_j I_j^*} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| \, dx < \infty, \quad (4.10) \]

then we have

\[ \left| \left\{ x \in G \setminus \bigcup_j I_j^* : |W_\ell b(x)| > \frac{t}{2} \right\} \right| \leq \frac{2M}{t} \sum_k \int_{I_k} |b_k(z)| \, dz \]

\[ = \frac{2M}{t} \|b\|_{L^1(G)} \leq \frac{6M}{t} \|f\|_{L^1(G)}. \]

So, if we prove the estimate (4.10) we obtain the weak (1,1) inequality (2.19) for \( f \in L^p(G) \cap L^1(G), \ f \geq 0 \). For the proof of (4.10) let us use the estimates of the
Calderón-Zygmund kernel of every operator $W_{\ell}$. Let us point out that (in view of (4.6) and from [8, Page 17]) for $x \in G \setminus \bigcup I_j$, and $z \in I_k$, $4c|z| = 4c \times \text{dist}(z, e_G) \lesssim \text{dist}(\partial I_k^*, \partial I_k) \leq |x|$. So, by a suitable variable change of variables and by using (2.17), we have

$$M_k := \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \bigcup I_j} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| \, dx$$

$$= \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \bigcup I_j} |2^{-\ell Q} \kappa_\ell(2^{-\ell} \cdot z^{-1}x) - 2^{-\ell Q} \kappa_\ell(2^{-\ell} \cdot x)| \, dx$$

$$\leq \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{|x| > 4c|z|} |2^{-\ell Q} \kappa_\ell(2^{-\ell} \cdot z^{-1}x) - 2^{-\ell Q} \kappa_\ell(2^{-\ell} \cdot x)| \, dx$$

$$= \sum_{\ell=0}^{\infty} \mathcal{J}_\ell \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon_0} = O(1).$$

Because

$$M_k := \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \bigcup I_j} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| \, dx \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon_0},$$

with the right hand side of the inequality being independent of $k$, we conclude that $M$ in (4.10) is finite. So, we have prove (4.4) for $f \in L^p(G) \cap L^1(G)$ with $f \geq 0$. Note that if $f \in L^p(G) \cap L^1(G)$ is real-valued, one can decompose $f = f^+ - f^-$, as the difference of two non-negative functions, where $f^+, f^- \in L^p(G) \cap L^1(G)$, and $|f| = f^+ + f^-$. Because $f^+, f^- \leq |f|$, we have

$$|[x \in G : |W_{\ell f}(x)| > t]|$$

$$\leq \left| \left\{ x \in G : |W_{\ell f^+}(x)| > \frac{t}{2} \right\} \right| + \left| \left\{ x \in G : |W_{\ell f^-}(x)| > \frac{t}{2} \right\} \right|$$

$$\leq \frac{C}{t} \|f^+\|_{L^1(G)} + \frac{C}{t} \|f^-\|_{L^1(G)}$$

$$\leq \frac{2C}{t} \|f\|_{L^1(G)}.$$

A similar analysis, by splitting a complex function $f \in L^p(G) \cap L^1(G)$ into its real and imaginary parts allows to conclude the weak (1,1) inequality (4.4) to complex functions. Thus, the proof of Lemma 4.1 is complete. □

**Proof of Lemma 4.2.** Now, we claim that

$$W : L^1(G, \ell^r(N_0)) \to L^{1,\infty}(G, \ell^r(N_0)), \quad 1 < r < \infty. \quad (4.11)$$
extends to a bounded operator. For the proof of (4.11), we need to show that there exists a constant $C > 0$ independent of $\{f_\ell\} \in L^1(G, \ell^r(\mathbb{N}_0))$ and $t > 0$, such that
\[
\left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell f_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \leq \frac{C}{t} \| \{f_\ell\} \|_{L^1(G, \ell^r(\mathbb{N}_0))}.
\] (4.12)

So, fix $\{f_\ell\} \in L^1(G, \ell^r(\mathbb{N}_0))$ and $t > 0$, and let $h(x) := (\sum_{\ell=0}^{\infty} |f_\ell(x)|^r)^{\frac{1}{r}}$, apply the Calderón-Zygmund decomposition Lemma to $h \in L^1(G)$, under the identification $G \simeq \mathbb{R}^n$, (see e.g. Hebish [19]) in order to obtain a disjoint collection $\{I_j\}_{j=0}^{\infty}$ of disjoint open sets such that

- $h(x) \leq t$, for a.e. $x \in G \setminus \bigcup_{j \geq 0} I_j$,
- $\sum_{j \geq 0} |I_j| \leq \frac{C}{t} \| h \|_{L^1(G)}$, and
- $t \leq \frac{1}{|I_j|} \int_{I_j} h(x) dx \leq 2t$, for all $j$.

Now, we will define a suitable decomposition of $f_\ell$, for every $\ell \geq 0$. Recall that every $I_j$ is diffeomorphic to an open cube on $\mathbb{R}^n$, that it is bounded, and that $I_j \subset B(z_j, 2R_j)$, where $z_j \in I_j$ (see Hebish [19]). Let us define, for every $\ell$, and $x \in I_j$,
\[
g_\ell(x) := \frac{1}{|I_j|} \int_{I_j} f_\ell(y) dy, \quad b_\ell(x) = f_\ell(x) - g_\ell(x).
\] (4.13)

and for $x \in G \setminus \bigcup_{j \geq 0} I_j$,
\[
g_\ell(x) = f_\ell(x), \quad b_\ell(x) = 0.
\] (4.14)

So, for a.e. $x \in G$, $f_\ell(x) = g_\ell(x) + b_\ell(x)$. Note that for every $1 < r < \infty$, $\| \{g_\ell\}\|_{L^r(\ell^r)} \leq t^{r-1} \| \{f_\ell\}\|_{L^1(\ell^r)}$. Indeed, for $x \in I_j$, Minkowski integral inequality gives,
\[
\left( \sum_{\ell=0}^{\infty} |g_\ell(x)|^r \right)^{\frac{1}{r}} \leq \left( \sum_{\ell=0}^{\infty} \left( \frac{1}{|I_j|} \int_{I_j} |f_\ell(y)| dy \right)^r \right)^{\frac{1}{r}} \leq \frac{1}{|I_j|} \int_{I_j} \left( \sum_{\ell=0}^{\infty} |f_\ell(y)|^r \right)^{\frac{1}{r}} dy
\]
\[
= \frac{1}{|I_j|} \int_{I_j} h(y) dy
\]
\[
\leq 2t.
\]

Consequently,
\[
\sum_{\ell=0}^{\infty} |g_\ell(x)|^r \leq (2t)^r,
\]
and from the fact that $h(x) \leq t$, for a.e. $x \in G \setminus \bigcup_{j \geq 0} I_j$, we have
\[
\| \{g_\ell\}\|_{L^r(\ell^r)} = \int_G \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx = \sum_j \int_{I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx + \int_{G \setminus \bigcup_{j \geq 0} I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx
\]
Now, by using the Minkowski and the Chebyshev inequality, we obtain

\[
= \sum_j \int_{I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r \, dx + \int_{G \setminus \bigcup_j I_j} \sum_{\ell=0}^{\infty} |f_\ell(x)|^r \, dx \\
\leq \sum_j \int_{I_j} (2t)^r \, dx + \int_{G \setminus \bigcup_j I_j} h(x)^r \, dx \\
\lesssim t^r \sum_j |I_j| + \int_{G \setminus \bigcup_j I_j} h(x)^{r-1} h(x) \, dx \\
\leq t^r \times \frac{C}{t} \|h\|_{L^1(G)} + t^{r-1} \int_{G \setminus \bigcup_j I_j} h(x) \, dx \lesssim t^{r-1} \|h\|_{L^1(G)} \\
= t^{r-1} \|\{f_\ell\}\|_{L^1(\mu)}.
\]

Now, by using the Minkowski and the Chebyshev inequality, we obtain

\[
\left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell f_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \\
\leq \left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell g_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| + \left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \\
\lesssim \frac{2^r}{t^r} \int_{G} \sum_{\ell=0}^{\infty} |W_\ell g_\ell(x)|^r \, dx + \left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right|.
\]

In view of Lemma 4.1, \( W : L^r(G, \ell^r(\mathbb{N}_0)) \to L^r(G, \ell^r(\mathbb{N}_0)) \), extends to a bounded operator and

\[
\int_{G} \sum_{\ell=0}^{\infty} |W_\ell g_\ell(x)|^r \, dx = \|W\{g_\ell\}\|_{L^r(\mu)} \lesssim \|\{g_\ell\}\|_{L^r(\mu)} \leq t^{r-1} \|\{f_\ell\}\|_{L^1(\mu)}.
\]

Consequently,

\[
\left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell f_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \\
\lesssim \frac{1}{t} \|\{f_\ell\}\|_{L^1(\mu)} + \left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right|.
\]

Now, we only need to prove that

\[
\left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \lesssim \frac{1}{t} \|\{f_\ell\}\|_{L^1(\mu)}.
\]
Taking into account that \( b_t \equiv 0 \) on \( G \setminus \bigcup_I I_j \), we have that
\[
b_t = \sum_k b_{t,k}, \quad b_{t,k}(x) = b_t(x) \cdot 1_{I_k}(x). \tag{4.17}
\]

Let us assume that \( I^*_j \) is a open set, such that \( |I^*_j| = K |I_j| \) for some \( K > 0 \), and \( \text{dist}(\partial I^*_j, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G) \), where \( c \) is defined in (2.17) and \( e_G \) is the identity element of \( G \). So, by the Minkowski inequality we have,
\[
\left| \left\{ x \in G : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|
= \left| \left\{ x \in \bigcup_j I^*_j : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|
+ \left| \left\{ x \in G \setminus \bigcup_j I^*_j : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|
\leq \left| \left\{ x \in G : x \in \bigcup_j I^*_j \right\} \right|
+ \left| \left\{ x \in G \setminus \bigcup_j I^*_j : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|
\leq \sum_j |I^*_j|.
\]

Since
\[
\left| \left\{ x \in G : x \in \bigcup_j I^*_j \right\} \right| \leq \sum_j |I^*_j|,
\]
we have
\[
\left| \left\{ x \in G : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|
\leq \sum_j |I^*_j| + \left| \left\{ x \in G \setminus \bigcup_j I^*_j : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|
= K \sum_j |I_j| + \left| \left\{ x \in G \setminus \bigcup_j I^*_j : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|
\leq \frac{CK}{t} \|f\|_{L^1(G, e^r)} + \left| \left\{ x \in G \setminus \bigcup_j I^*_j : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|
\leq \frac{CK}{t} \|f\|_{L^1(G, e^r)} + \frac{2}{t} \int_{G \setminus \bigcup_j I^*_j} \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^r \right)^{\frac{1}{r}} dx.
\]

Observe that the Chebyshev inequality implies
\[
\left| \left\{ x \in G \setminus \bigcup_j I^*_j : \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|
\leq \frac{2}{t} \int_{G \setminus \bigcup_j I^*_j} \left( \sum_{\ell = 0}^\infty |W_{\ell}b_{\ell}(x)|^r \right)^{\frac{1}{r}} dx.
\]
Now, we will proceed as follows. By using that
\[
\kappa \text{ and by using that } \sum_{\ell=0}^{\infty} \left| \left( W_{\ell} \left( \sum_{k} b_{\ell,k} \right) \right) (x) \right|^r dx
\]
\[
\leq \frac{2}{t} \sum_{k} \int_{G \setminus \bigcup_{j} I_{j}^{*}} \left( \sum_{\ell=0}^{\infty} \left| (W_{\ell} b_{\ell,k}) (x) \right|^r \right) dx.
\]

Now, if \( \kappa_{\ell} \) is the right convolution Calderón-Zygmund kernel of \( W_{\ell} \), (see Remark 2.5), and by using that \( \int_{I_{k}} b_{k,\ell}(y) dy = 0 \), we have that
\[
\left( \sum_{\ell=0}^{\infty} \left| (W_{\ell} b_{\ell,k}) (x) \right|^r \right)^{\frac{1}{r}} = \left( \sum_{\ell=0}^{\infty} \left| b_{\ell,k} \ast \kappa_{\ell} (x) \right|^r \right)^{\frac{1}{r}}
\]
\[
= \left( \sum_{\ell=0}^{\infty} \left| \int_{I_{k}} \kappa_{\ell}(y^{-1}x) b_{\ell,k}(y) dy - \kappa_{\ell}(x) \int_{I_{k}} b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}}
\]
\[
= \left( \sum_{\ell=0}^{\infty} \left| \int_{I_{k}} \left( \kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x) \right) b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}}.
\]

Now, we will proceed as follows. By using that \( |b_{\ell,k}(y)|^r \leq \sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \), we have, by an application of the Minkowski integral inequality,
\[
\left( \sum_{\ell=0}^{\infty} \left| (W_{\ell} b_{\ell,k}) (x) \right|^r \right)^{\frac{1}{r}} \leq \left( \sum_{\ell=0}^{\infty} \left| \int_{I_{k}} \left( \kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x) \right) b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}}
\]
\[
\leq \int_{I_{k}} \left( \sum_{\ell=0}^{\infty} \left| \kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x) \right|^r \right)^{\frac{1}{r}} \left( \sum_{\ell=0}^{\infty} \left| b_{\ell,k}(y) \right|^r \right)^{\frac{1}{r}} dy
\]
\[
\leq \int_{I_{k}} \left( \sum_{\ell=0}^{\infty} \left| b_{\ell,k}(y) \right|^r \right)^{\frac{1}{r}} \left( \sum_{\ell=0}^{\infty} \left| \kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x) \right|^r \right)^{\frac{1}{r}} dy.
\]

Consequently, we deduce,
\[
\frac{2}{t} \sum_{k} \int_{G \setminus \bigcup_{j} I_{j}^{*}} \left( \sum_{\ell=0}^{\infty} \left| (W_{\ell} b_{\ell,k}) (x) \right|^r \right)^{\frac{1}{r}} dx.
\]
By following [8, Page 17], for \( x \in G \setminus \cup_j I_j^* \), and \( y \in I_k \), we have \( 4c|y| = 4c \times \text{dist}(y, e_G) \leq \text{dist}(\partial I_k^*, \partial I_k) \leq |x| \). So,
\[
\{ x \in G : x \in G \setminus \cup_j I_j^* \} \subset \{ x \in G : \text{ for all } z \in I_k, \ 4c|z| \leq |x| \}.
\]

Now, from the estimate (2.17) in Remark 2.5, we deduce
\[
\int_{G \setminus \cup_j I_j^*} \left( \sum_{\ell=0}^{\infty} |\kappa_\ell(y^{-1}x) - \kappa_\ell(x)|^r \right)^{\frac{1}{r}} \, dx \leq \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} |\kappa_\ell(y^{-1}x) - \kappa_\ell(x)| \, dx
\]
\[
\leq \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_\ell(y^{-1}x) - \kappa_\ell(x)| \, dx
\]
\[
\leq \sum_{\ell=0}^{\infty} \int_{|x| > 4c|y|} |2^{-\ell Q} \kappa_\ell(2^{-\ell}y^{-1}x) - 2^{-\ell Q} \kappa_\ell(2^{-\ell}x)| \, dx
\]
\[
\lesssim \sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon_0} = O(1).
\]

Thus, we have proved that
\[
\left| \left\{ x \in G : \left( \sum_{\ell=0}^{\infty} |W_\ell b_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \lesssim \frac{2}{t} \sum_k \int_{I_k} \left( \sum_{\ell=0}^{\infty} |b_{\ell,k}(y)|^r \right)^{\frac{1}{r}} \, dy
\]
\[
= \frac{2}{t} \int_{\cup_k I_k} \left( \sum_{\ell=0}^{\infty} |b_{\ell}(y)|^r \right)^{\frac{1}{r}} \, dy
\]
\[
\lesssim \frac{1}{t} \| \{ f_\ell \} \|_{L^1(r)}.
\]

Thus, the proof of the weak (1,1) inequality is complete and we have that
\[
W : L^1(G, \ell^r(N_0)) \to L^{1,\infty}(G, \ell^r(N_0)), \quad 1 < r < \infty,
\]
(4.18)

admits a bounded extension. The proof of Lemma 4.2 is complete.

Now, in view of Lemmas 4.1 and 4.2, and the duality argument in Remark 4.3, we have proved Theorem 1.1.
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