SASAKI-EINSTEIN METRICS ON A CLASS OF 7-MANIFOLDS

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Abstract. In this note we give an explicit construction of Sasaki-Einstein metrics on a class of simply connected 7-manifolds with the rational cohomology of the 2-fold connected sum of $S^2 \times S^5$. The homotopy types are distinguished by torsion in $H^4$.

Introduction

Sasaki-Einstein metrics on 7-manifolds continue to play an important role in M-theory as well as black hole physics [Spa11, GLPS17, MT18]. An important reason for this is that Sasaki-Einstein manifolds admit supersymmetry and are used in the AdS-CFT correspondence. For such reasons it seems important to have a large list of explicit examples of Sasaki-Einstein 7-manifolds that can be used as possible models. Since these SE metrics are toric, general existence of such metrics in well known [FOW09, Leg16]. What is new here is an explicit construction of such metrics, their relation with Bott manifolds (orbifolds), and the topological description of the 7-manifolds. We give an explicit construction of toric Sasaki-Einstein (SE) 7-manifolds which can be represented as $S^1$ orbibundles over 2-twist stage 3 Bott orbifolds. All of these are obtained by adding orbifold structures to certain stage 3 Bott manifolds which were studied in [BCTF18]. The 7-manifolds all have the rational cohomology of the 2-fold connected sum $2(S^2 \times S^5)$ and are generalizations of the SE 7-manifolds given by Theorem 1.2 of [BTF15].

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1. Stage Three Bott Towers and Orbifolds

Following [GK94] and [BCTF18] we consider Bott towers which in arbitrary dimension is represented by a lower triangular unipotent matrix $A$ over $\mathbb{Z}$. Here we deal only with stage 3 Bott towers, so the matrix $A$ in [GK94, BCTF18] takes the form

$$A = \begin{pmatrix}
1 & 0 & 0 \\
a & 1 & 0 \\
b & c & 1
\end{pmatrix}$$

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with $a, b, c \in \mathbb{Z}$. The Bott manifold can be realized as the quotient of $S^3 \times S^3 \times S^3$ by the $T^3$ action

$$ (z_j^0, z_j^\infty)_{j=1}^3 \mapsto (t_j z_j^0, \prod_{i=1}^3 t_i^A_j z_j^\infty). $$

The quotient $M_3$ which is called a Bott manifold can be represented as a sequence, called a Bott tower

$$ M_3 \xrightarrow{\pi_3} M_2 \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = pt, $$

where the $j$th $S^3$ is written as $|z_j^0|^2 + |z_j^\infty|^2 = 1$. Then $M_k$ is the compact complex manifold arising as the total space of the $\mathbb{CP}^1$ bundle $\pi_k: \mathbb{P}(\mathbb{I} \oplus \mathcal{L}_k) \to M_{k-1}$. At each stage we have zero and infinity sections $\sigma^0_k: M_{k-1} \to M_k$ and $\sigma^\infty_k: M_{k-1} \to M_k$ which respectively identify $M_{k-1}$ with $\mathbb{P}(\mathbb{I} \oplus 0)$ and $\mathbb{P}(0 \oplus \mathcal{L}_k)$. We consider these to be part of the structure of the Bott tower $(M_k, \pi_k, \sigma^0_k, \sigma^\infty_k)_{k=1}^n$. Here in our case $M_3 = M_3(a, b, c)$ is a stage 3 Bott manifold, $M_2(a)$ is a Hirzebruch surface $\mathcal{H}_a = \mathbb{P}(\mathbb{I} \oplus \mathcal{O}(a)) \to \mathbb{CP}^1$, and $M_1 = \mathbb{CP}^1$. $M_3(a, b, c)$ can be viewed as the total space of $\mathbb{CP}^1$ bundle over the Hirzebruch surface $\mathcal{H}_a$, and also a bundle of Hirzebruch surfaces over $\mathbb{CP}^1$ with fiber $\mathcal{H}_c$. Bott towers form the object set $\mathcal{BT}_0$ of a groupoid whose morphisms $\mathcal{BT}_1$ are biholomorphisms $[\text{BCTF18}]$, and elements of the quotient space $\mathcal{BT}_0/\mathcal{I}_1$ are identified with biholomorphism classes of Bott manifolds. Since Bott manifolds are toric, they are described by a fan, and it follows from the Bott tower description (3) that the fan of the Bott tower $M_3(a, b, c)$ is described by the primitive collections (cf. [CLS11])

$$ \{v_1, u_1\}, \{v_2, u_2\}, \{v_3, u_3\} $$

with normal vectors $u_1 = -v_1 - av_2 - bv_3, u_2 = -v_2 - cv_3, u_3 = -v_3$, and thus has the combinatorial type of a cube. The symmetry group of a cube is the Coxeter group $BC_3 \cong \text{Sym}_3 \times \mathbb{Z}_2^3$ where $\text{Sym}_3$ is the symmetric group on 3 letters. However, not all elements of $BC_3$ are induced by equivalences. We refer to [BCTF18] and references therein for details.

The structure of Bott towers implies the existence of 3 pairs of $T^3$ invariant divisors $\{D_{v_j}, D_{u_j}\}$ which are the zero and infinity sections of $M_j \xrightarrow{\pi_j} M_{j-1}$ with $j = 1, 2, 3$. Thus, elements of the subgroup $\mathbb{Z}_3^2$ are induced by the fiber inversion maps that interchange the zero and infinity sections, so these equivalences always exist. However, elements of $\text{Sym}_3$ are induced by equivalences only in special cases (see Lemma 1.11 and Example 1.6 of [BCTF18] for details).

Recall (Definition 1.6 of [BCTF18]) that the holomorphic twist of an $n$ dimensional Bott tower is the number $t \in \{0, \ldots, n-1\}$ of holomorphically nontrivial $\mathbb{CP}^1$ bundles in the tower. So for $n = 3$ there are only 3 possibilities $t = 0, 1, 2$. Of course, $t = 0$ is the well understood product $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, whereas $t = 1$ leads to the Koiso-Sakane case, so here we restrict our attention to $t = 2$. For the $t = 2$ case we have a nontrivial holomorphic bundle over a Hirzebruch surface $\mathcal{H}_a$ with $a \neq 0$ whose fiber is $\mathbb{CP}^1$. The case of interest to us can be obtained as an $S^3_w$-join with the $Y^{p,q}$ structures of [GMSW04] on $S^2 \times S^3$, that is, $(S^2 \times S^3) \ast_{t_1, t_2} S^3_w$. So the stage 3 Bott manifold has an additional orbifold structure which we now describe.
1.1. **Invariant Divisors.** Here we consider the $T^3$ invariant divisors $D_{v_i}, D_{u_i}$ defined by the normal vectors $v_i, u_i$ with equivalences

$$D_{v_1} \sim D_{u_1}, \quad D_{v_2} \sim aD_{u_1} + D_{u_2}, \quad D_{v_3} \sim bD_{u_1} + cD_{u_2} + D_{u_3}. $$

We have 4 sets of distinguished invariant bases of the Chow group $A_2(M_3)$ of invariant divisor classes

(5) $\{[D_{u_1}], [D_{u_2}], [D_{u_3}]\}$,

(6) $\{[D_{u_1}], [D_{u_2}], [D_{v_1}]\}$,

(7) $\{[D_{u_1}], [D_{v_2}], [D_{v_3}]\}$,

(8) $\{[D_{u_1}], [D_{v_2}], [D_{v_3}]\}$.

This gives rise to the 4 sets of dual bases of cohomology classes in $H^2(M_3, \mathbb{Z})$, viz

(9) $\{x_1, x_2, x_3\}$,

(10) $\{x_1, x_2, y_3\}$,

(11) $\{x_1, y_2, x_3\}$,

(12) $\{x_1, y_2, y_3\}$,

where $x_i$ is dual to $[D_{u_i}]$ and $y_i$ is dual to $[D_{v_i}]$. Both the ample and Kähler cones can be easily worked out, see Example 3.3 of [BCTF18].

1.2. **Bott Orbifolds and log pairs.** We are interested in these Bott manifolds, but with an additional special orbifold structure along the $T^3$-invariant divisors $D_{v_i}, D_{u_i}$.

The orbifold structure on $M_3(a, b, c)$ that we are interested in is given by the log pair $(M_3(a, b, c), \Delta_m)$ where $\Delta_m$ is the branch divisor

(13) $\left(1 - \frac{1}{m_1^0}\right)D_{v_1} + \left(1 - \frac{1}{m_1^\infty}\right)D_{u_1} + \left(1 - \frac{1}{m_2^0}\right)D_{v_2} + \left(1 - \frac{1}{m_2^\infty}\right)D_{u_2} + \left(1 - \frac{1}{m_3^0}\right)D_{v_3} + \left(1 - \frac{1}{m_3^\infty}\right)D_{u_3}$

where $m_j^0, m_j^\infty \in \mathbb{Z}^+$ are the ramification indices. We define $\Delta_{m_3} = \Delta_m$ and

$$\Delta_{m_2} = \left(1 - \frac{1}{m_1^0}\right)D_{v_1} + \left(1 - \frac{1}{m_1^\infty}\right)D_{u_1} + \left(1 - \frac{1}{m_2^0}\right)D_{v_2} + \left(1 - \frac{1}{m_2^\infty}\right)D_{u_2},$$

$$\Delta_{m_1} = \left(1 - \frac{1}{m_1^0}\right)D_{v_1} + \left(1 - \frac{1}{m_1^\infty}\right)D_{u_1}. $$

Clearly we have

$$\Delta_{m_1} \subset \Delta_{m_2} \subset \Delta_{m_3}. $$

From [GK94] or [BCTF18] one easily sees that

**Lemma 1.1.** We have the sequence of Bott towers of log pairs

(14) $(M_3(a, b, c), \Delta_{m_3}) \xrightarrow{\pi_3} (M_2(a), \Delta_{m_2}) \xrightarrow{\pi_2} (M_1, \Delta_{m_1}) \xrightarrow{\pi_1} (\{pt\}, \emptyset),$

where $M_2(a) = \mathcal{H}_a$ is a Hirzebruch surface, $M_1 = \mathbb{CP}^1$, and $\pi_i$ is the natural projection.

The invariant branch divisors are related to the section maps by

$$D_{v_3} = \sigma_2^0(\mathcal{H}_a), \quad D_{u_3} = \sigma_2^\infty(\mathcal{H}_a), \quad D_{v_2} = \pi_3^{-1}(\sigma_1^0(\mathbb{CP}^1)), \quad D_{u_2} = \pi_3^{-1}(\sigma_1^\infty(\mathbb{CP}^1)).$$
and

\[ D_{v_1} = (\pi_2 \circ \pi_3)^{-1}(\sigma_1^0(\{pt\})), \quad D_{u_1} = (\pi_2 \circ \pi_3)^{-1}(\sigma_1^\infty(\{pt\})). \]

We denote by \( \mathfrak{BO}_A \) the set of all such Bott orbifolds.

### 1.3. The Orbifold First Chern Class.

We can now compute the orbifold canonical divisor \( K_{\text{orb}} \) and dually the orbifold first Chern class. We compute the orbifold first Chern class \( c_1^{\text{orb}} \) in the \( \{x_j\} \) basis for any n-dimensional Bott manifold \( M_n(A) \);

\[
c_1^{\text{orb}}(M_n(A), \Delta_m) = c_1(M_n(A)) - \sum_{i=1}^n \left( (1 - \frac{1}{m_i^0}) y_i + (1 - \frac{1}{m_i^\infty}) x_i \right)
= \sum_{i=1}^n (x_i + y_i) - \sum_{i=1}^n \left( (1 - \frac{1}{m_i^0}) y_i + (1 - \frac{1}{m_i^\infty}) x_i \right)
= \sum_{j=1}^n \left( \frac{1}{m_j^0} y_j + \frac{1}{m_j^\infty} x_j \right) = \sum_{i=1}^n \left( \frac{1}{m_i^0} + \frac{1}{m_i^\infty} x_i + \sum_{j=1}^{i-1} \frac{A_j^i}{m_i^0} x_j \right)
= \sum_{j=1}^{n-1} \left( \frac{1}{m_j^0} + \frac{1}{m_j^\infty} + \sum_{i=j+1}^n \frac{A_j^i}{m_i^0} \right) x_j + \left( \frac{1}{m_n^0} + \frac{1}{m_n^\infty} \right) x_n.
\]

In dimension \( n \) there are \( 2^{n-1} \) invariant bases in which to compute \( c_1^{\text{orb}} \). This becomes much more manageable for \( n = 3 \). We have \( c_1^{\text{orb}}(M_3(a, b, c), \Delta_m) \) in the four bases \([9]-[12]\), respectively

\[
(16) \quad \left( \frac{1}{m_1^0} + \frac{1}{m_1^\infty} \right) x_1 + \left( \frac{1}{m_2^0} + \frac{1}{m_2^\infty} \right) x_2 + \left( \frac{1}{m_3^0} + \frac{1}{m_3^\infty} \right) x_3,
(17) \quad \left( \frac{1}{m_1^0} + \frac{1}{m_1^\infty} + \frac{a}{m_2^0} + \frac{b}{m_3^\infty} \right) x_1 + \left( \frac{1}{m_2^0} + \frac{1}{m_2^\infty} - \frac{c}{m_3^\infty} \right) x_2 + \left( \frac{1}{m_3^0} + \frac{1}{m_3^\infty} \right) y_3,
(18) \quad \left( \frac{1}{m_1^0} + \frac{1}{m_1^\infty} - \frac{a}{m_2^0} + \frac{b - ac}{m_3^\infty} \right) x_1 + \left( \frac{1}{m_2^0} + \frac{1}{m_2^\infty} + \frac{c}{m_3^\infty} \right) y_2 + \left( \frac{1}{m_3^0} + \frac{1}{m_3^\infty} \right) x_3,
(19) \quad \left( \frac{1}{m_1^0} + \frac{1}{m_1^\infty} - \frac{a}{m_2^0} - \frac{b - ac}{m_3^\infty} \right) x_1 + \left( \frac{1}{m_2^0} + \frac{1}{m_2^\infty} - \frac{c}{m_3^\infty} \right) y_2 + \left( \frac{1}{m_3^0} + \frac{1}{m_3^\infty} \right) y_3.
\]

Note also that as a cohomology class \( c_1^{\text{orb}}(M_3(a, b, c), \Delta_m) \) as an element of \( H^{1,1}(M_n(a, b, c), \mathbb{R}) \) makes perfect sense for all \( m_j^0, m_j^\infty \in \mathbb{R}^+ \). We shall make use of this fact shortly.

Equations \([16]-[19]\) implies

\textbf{Lemma 1.2.} Let \( (M_3(a, b, c), \Delta_m) \) be a Bott orbifold. Then the following are equivalent:

1. \( (M_3(a, b, c), \Delta_m) \) is log Fano,
2. \( c_1^{\text{orb}}(M_3(a, b, c), \Delta_m) \) lies in the Kähler cone \( \mathcal{K}(M_3(a, b, c)) \),
(3) the inequalities
\[
\begin{align*}
\frac{1}{m_1^0} + \frac{1}{m_1^\infty} + \frac{a}{m_2^0} + \frac{b}{m_2^\infty} &> 0, \\
\frac{1}{m_1^0} + \frac{1}{m_1^\infty} + \frac{a}{m_2^0} - \frac{b}{m_2^\infty} &> 0, \\
\frac{1}{m_1^0} + \frac{1}{m_1^\infty} - \frac{a}{m_2^0} + \frac{(b - ac)}{m_3^\infty} &> 0, \\
\frac{1}{m_1^0} + \frac{1}{m_1^\infty} - \frac{a}{m_2^0} - \frac{(b - ac)}{m_3^\infty} &> 0,
\end{align*}
\]
hold.

2. \(M^7 = \text{the join } Y^{p,q} \ast_{l_1,l_2} S^3_w\)

Not every \(S^1\) orbibundle over a Bott orbifold can be realized as a join; however, the 2 twist stage 3 Bott orbifolds that we study here can be realized as Kähler quotients of the join \(Y^{p,q} \ast_{l_1,l_2} S^3_w\) where \(Y^{p,q}\) are the well known SE structures on \(S^2 \times S^3\) discovered by the physicists \(\text{GMSW04}\). In fact, since \(Y^{p,q}\) is itself a join of two \(S^3\)'s, it is an iterated join of three \(S^3\)'s which in the terminology of \(\text{BHLTF18}\) is completely cone decomposable. Now from Example 6.8 of \(\text{BTF16}\) we have

\[
(20) \quad Y^{p,q} = S^3 \ast_{l,p} S^3 \ast_{l,\frac{p+q}{l}} \ast_{l,\frac{p-q}{l}},
\]

where \(l = \gcd(p + q, p - q)\) which equals 2 if \(p, q\) are both odd, and equals 1 if \(p, q\) have opposite parity. Note that we choose the standard SE structure on the lefthand \(S^3\) factor, whereas it is the weighted Sasakian structure with weights \((\frac{p+q}{l}, \frac{p-q}{l})\) on the righthand \(S^3\).

If we assume that \(p > q \geq 1\) are such that \(\sqrt{4p^2 - 3q^2} \in \mathbb{N}\), then the Sasaki-Einstein structure on \(Y^{p,q}\) is quasi-regular \(\text{GMSW04}\). Indeed, the \(\eta\)-Einstein structure corresponds to a ray in the so-called \(w\) subcone is determined by co-prime solutions \((v_2^0, v_2^\infty)\) of

\[
(21) \quad \int_{-1}^{1} \frac{((v_2^0 - v_2^\infty) - (v_2^0 + v_2^\infty))}{3}(((p+q)v_2^\infty + (p-q)v_2^0 + (p+q)v_2^\infty - (p-q)v_2^0)3) d3 = 0
\]

(from e.g. (68) in \(\text{BTF16}\)). Following Theorem 3.8 in \(\text{BTF16}\), for any choice of quasi-regular ray determined by the co-prime pair \((v_2^0, v_2^\infty)\), the quotient Hirzebruch orbifold is \((\mathcal{M}_a, \Delta_{m_2})\) where \(\Delta_{m_2} = (1 - \frac{1}{m_2^0})D_{v_2} + (1 - \frac{1}{m_2^\infty})D_{v_2^\infty}\) is the branch divisor with \(\mathcal{M}_2 = (m_2^0, m_2^\infty) = m_2(v_2^0, v_2^\infty)\),

\[
(22) \quad m_2 = \frac{p}{\gcd(p, \frac{p+q}{l}v_2^\infty + \frac{p-q}{l}v_2^0)},
\]

\[
a = \frac{(p+q)m_2^\infty - (p-q)m_2^0}{p}.
\]
The join $M_{i_1,i_2,w} = Y^{p,q} \ast_{i_1,i_2} S^3_w$, where $Y^{p,q}$ has a quasi-regular Sasaki structure as above, can then be obtained as the quotient of the following $\mathbb{T}^2$ action on $S^3 \times S^3 \times S^3$:
\begin{equation}
(x, u; u_1, u_2; z_1, z_2) \mapsto \left( x, e^{i \theta \theta} u; e^{i (l_2 + \theta \theta)} u_1, e^{i (l_1 + \theta \theta)} u_2; e^{-il_1 \phi z_1}, e^{-il_2 \phi z_2} \right).
\end{equation}

First we notice that without loss of generality we can assume that $\gcd(l_1, m_2) = 1$, for otherwise we can redefine $\phi$. So our gcd conditions are $\gcd(l_1, l_2 m_2) = 1$, $\gcd(w_1, w_2) = 1 = \gcd(v_0, v_0^{\infty})$, and $\gcd(p, q) = 1$. Note that $p + q$ and $p - q$ can have a common factor and only if both $p$ and $q$ are odd, in which case the common factor is 2. However, $M_{i_1,i_2,w}$ may not be a smooth manifold. We have

**Lemma 2.1.** The join $M_{i_1,i_2,w} = Y^{p,q} \ast_{i_1,i_2} S^3_w$, with Sasaki structure on $Y^{p,q}$ given by the Reeb field $\xi_\nu$, with $\nu = (v_0, v_0^{\infty})$, is a smooth manifold if and only if $\gcd(l_2 m_2, l_1 w_j) = 1$ for $i = 0, \infty$ and $j = 1, 2$, where $m_2 = \frac{p \gcd(p, |v_0^{\infty} - v_0|)}{\gcd(p, |v_0^{\infty} - v_0|)}$.

**Proof.** From Proposition 7.6.7 of [BG08] $M_{i_1,i_2,w}$ is smooth if and only if $\gcd(l_1 Y_2, l_2 Y_1) = 1$ where $Y_1$ is the order of $Y^{p,q}$ and $Y_2$ is the order of $S^3_w$. The latter is $Y_2 = w_1 w_2$ with $w_1, w_2$ coprime. The order $Y_1$ with quasi-regular Reeb field $\xi_{\nu_2}$ is $Y_1 = m_2 v_0 w_2^{\infty}$. \qed

The analysis in Section 3 of [BTF16] holds equally well when the manifold $M$ in the join $M \ast_{i_1,i_2} S^3_w$ has any quasi-regular Sasaki structure. The major difference is having more complicated computations. For example we need the Fano index of of $Y^{p,q}$. As described above, the quotient of any quasi-regular Sasaki structure in the $w$ subcone of $Y^{p,q}$ is a Hirzebruch orbifold of the form $(\mathcal{H}_a, \Delta_{m_2})$. We have

**Lemma 2.2.** Let $\xi_{\nu_2}$ be a quasiregular Reeb vector field with quotient Hirzebruch orbifold $(\mathcal{H}_a, \Delta_{m_2})$ with $a > 0$. Then its Fano index $\mathcal{J}_{\nu_2}$ is given by
\[
\mathcal{J}_{\nu_2} = \gcd(2m_2^0 v_0^2 + a v_0^{\infty}, v_0^2 + v_0^{\infty})
\]
where $m_2^0 = m_2(v_0^0, v_0^{\infty})$ and $v_0^0, v_0^{\infty}$ are coprime.

**Proof.** Recall (Definition 4.4.24 of [BG08]) that the Fano index $\mathcal{J}$ of an orbifold $Z$ is the largest positive integer $k$ such that $p^* c_1^{\text{orb}} / k$ is an element of $H^2(Z, \mathbb{Z}) = H^2(\mathbb{Z}, \mathbb{Z})$. Now the classifying map $p : BZ \longrightarrow Z$ is an $m_2 v_0^{\infty}$-fold cover, and $c_1^{\text{orb}}$ is
\begin{equation}
(24) \quad c_1^{\text{orb}} = (2 + \frac{a}{m_2 v_0^{\infty}}) x_1 + \frac{v_0 + v_0^{\infty}}{m_2 v_0^{\infty}} x_2 = \frac{1}{m_2 v_0^{\infty}} ((2m_2 v_0^0 + a) v_0^{\infty} x_1 + (v_0^2 + v_0^{\infty}) x_2).
\end{equation}
So $p^* c_1^{\text{orb}} = (2m_2 v_0^0 + a) v_0^{\infty} p^* x_1 + (v_0^2 + v_0^{\infty}) p^* x_2$ from which the result follows. \qed

From now on we assume that $p > q \geq 1$ are such that $\sqrt{4p^2 - 3q^2} \in \mathbb{N}$ and $Y^{p,q}$ has the quasi-regular Sasaki-Einstein structure. Thus co-prime $(v_0^0, v_0^{\infty})$ are chosen such that $[21]$ is satisfied. We then present a version of Theorem 3.8 of [BTF16] that allows the join $M_{i_1,i_2,w} = Y^{p,q} \ast_{i_1,i_2} S^3_w$

**Theorem 2.3.** Consider the join $M_{i_1,i_2,w} = Y^{p,q} \ast_{i_1,i_2} S^3_w$, where $Y^{p,q}$ has a quasi-regular Sasaki-Einstein structure determined by the co-prime pair $(v_0^0, v_0^{\infty})$ satisfying Equation $[21]$. Let $S_\nu = (\xi_{\nu}, \eta_{\nu}, \Phi_{\nu}, \gamma_{\nu})$ be a quasi-regular Sasaki structure that lies in the $w$ subcone of the Sasaki cone with $\nu = (v_0^0, v_0^{\infty})$ where $v_0^0, v_0^{\infty}$ are coprime. Then the
quotient of $M_{1,2,\mathbf{w}}$ by the $S^1$ action generated by $\xi_{\mathbf{w}}$, is the Bott orbifold given by the log pair $(M_3(a,b,c), \Delta_{\mathbf{m}})$, where $a$ is determined by (22),

$$b = n\hat{b} = \frac{n(2m_2v^0_2 + a)v^\infty_2}{j_{v_2}},$$

$$c = n\hat{c} = \frac{n(v^0_3 + v^\infty_3)}{j_{v_2}},$$

$$n = l_1 \frac{w_{1v_3}^\infty - w_{2v_3}^0}{\gcd(|w_1v_3^\infty - w_2v_3^0|, l_2)}, \ J_{v_2} \text{ is given by Lemma 2.2, and } \Delta_{\mathbf{m}} \text{ is the branch divisor}$$

$$(1 - \frac{1}{m_1^0})D_{v_1} + (1 - \frac{1}{m_1^\infty})D_{u_1} + (1 - \frac{1}{m_2^0})D_{v_2} + (1 - \frac{1}{m_2^\infty})D_{u_2} + (1 - \frac{1}{m_3^0})D_{v_3} + (1 - \frac{1}{m_3^\infty})D_{u_3}$$

with $(m_1^0, m_1^\infty) = (1, 1), (m_2^0, m_2^\infty)$ given by (22), and

$$(m_3^0, m_3^\infty) = m_3(v_3^0, v_3^\infty) = l_2 \frac{l_1}{\gcd(|w_1v_3^\infty - w_2v_3^0|, l_2)}(v_3^0, v_3^\infty).$$

Proof. We can follow the proof of Theorem 3.8 of [BTF16] with the caveat that $N$ is a Hirzebruch orbifold $(\mathcal{H}_a, \Delta_{\mathbf{m}_3})$. As in Equation (3) of [BTF16] we have the commutative diagram

$$\begin{array}{ccc}
S^3 \times S^3 \times S^3 & \downarrow & Y^{p,q} \times S^3_w \\
\downarrow & \nearrow \pi_L & \\
Y^{p,q} \times S^3_w & \downarrow & Y^{p,q} \ast_{t_1, t_2, \mathbf{w}} S^3_w \\
\downarrow & \nearrow \pi_1 & \\
(\mathcal{H}_a, \Delta_{\mathbf{m}_2}) \times \mathbb{C}P^1[\mathbf{w}] & & \\
\end{array}$$

where the $\pi$s are the obvious projections, and the orbifold $(\mathcal{H}_a, \Delta_{\mathbf{m}_2})$ is the quotient by the locally free $S^1$ action on $Y^{p,q}$ generated by the quasi-regular Reeb vector field $\xi_{\mathbf{m}_2}$ where $\mathbf{m}_2 = m_2(v_2^0, v_2^\infty)$ and $a$ is given in Equations (22). The holomorphic line bundle $L_n = L^n$ now becomes the holomorphic line orbibundle with $L$ determined by the “primitive” Kähler class in $H^2((\mathcal{H}_a, \Delta_{\mathbf{m}}), \mathbb{Q})$, namely

$$(c_{t_{(\mathcal{H}_a, \Delta_{\mathbf{m}_2})}}^\text{orb})_{J_{v_2}} = \frac{1}{m_2v_2^0v_2^\infty} \frac{((2m_2v_2^0 + a)v_2^\infty x_1 + (v_2^0 + v_2^\infty x_2)}{J_{v_2}}. $$

We make note of the fact that $Y^{p,q} \rightarrow (\mathcal{H}_a, \Delta_{\mathbf{m}_2})$ is an $m_2v_2^0v_2^\infty$-fold covering map.

Now as in Section 3.5 of [BTF16] we want to describe the base orbifold $B_{1, v, \mathbf{w}}$ of the $S^1$ orbibundle generated by the locally free action of the quasi-regular Reeb vector field $\xi_{\mathbf{m}_3}$ of the SE structure on the join $Y^{p,q} \ast_t S^3_w$. We see that the analysis of Section 3.5 of [BTF16] goes through verbatim through Remark 3.7 with $M = Y^{p,q}$ and $N = (\mathcal{H}_a, \Delta_{\mathbf{m}_2})$. In particular, from Lemma 3.6 of [BTF16] we obtain the base orbifold
$B_{t,\nu, w} \approx (B_{t,1,w'}, \Delta)$ with
\[
\Delta = \left(1 - \frac{1}{m_3^0}\right)D_{v_3} + \left(1 - \frac{1}{m_3}\right)D_{u_3}, \quad w' = (v_3^\infty w_1, v_3^0 w_2),
\]
and from Theorem 3.8 of [BTF16] we have
\[
(27) \quad m_3 = m_3(v_3^0, v_3^\infty), \quad s = \gcd(|w_1 v_3^\infty - w_2 v_3^0|, l_2), \quad l_2 = sm_3, \quad n = \frac{l_1}{s}(w_1 v_3^\infty - w_2 v_3^0).
\]

Then from the proof of Theorem 3.8 we see that the quotient is the total space of the projective orbibundle $\mathbb{P}(\mathcal{I} \oplus L^n)$ over $(\mathcal{H}_a, \Delta_{m_2})$ whose invariant divisors are generally branch divisors of an orbifold. But then using Lemma 1.1 this is precisely a stage 3 Bott orbifold $(M_3(a,b,c), \Delta_m)$ for some $b, c$ and where $a$ is given in Equations (22). The fact that $Y^{p,q}$ as a join has the form of Equation (20) with the standard regular Sasakian structure on the first $S^3$ implies that the ramification indices $m_1 = (1, 1)$.

It remains to check the equations for $b$ and $c$. For this we make use of an orbifold version of Proposition 1.5 in [BCTF18]. Explicitly, we have

**Lemma 2.4.** The pullback of $c_1(L^n)$ of the orbifold line bundle $L^n$ to $M_3(a,b,c)$ is $b x_1 + c x_2$ where $n$ is given in Equations (27).

We know that $L^n$ is the $n$th tensor product of the line orbibundle $L$ which is determined by the Kähler class
\[
\frac{c_1^{orb}(\mathcal{H}_a, \Delta_{m_2})}{\mathcal{I}_{v_2}} = \frac{((2m_2 v_2^0 + a)v_2^\infty x_1 + (v_2^0 + v_2^\infty)x_2)}{m_2 v_2^0 v_2^\infty \mathcal{I}_{v_2}}.
\]

Now the projection $p : (M_3(a,b,c), \Delta) \to (\mathcal{H}_a, \Delta_{m_2})$ is a $m_2 v_2^0 v_2^\infty$-fold covering map. So pulling back we have $c_1(L^n) = nc_1(L) = np^*\left(\frac{c_1^{orb}(\mathcal{H}_a, \Delta_{m_2})}{\mathcal{I}_{v_2}}\right)$. The equations for $b$ and $c$ then follow by equating coefficients in this and in Lemma 2.4. \hfill \Box

**Remark 2.5.** The Poincaré dual to $c_1^{orb}(M_3(a,b,c), \Delta_{m})$ is a Q-divisor on $M_3(a,b,c)$ which is ample when $c_1^{orb}(M_3(a,b,c), \Delta_{m})$ is positive. Such a class gives a polarization to the orbifold $(M_3(a,b,c), \Delta_{m})$, and a $\mathbb{T}^3$ invariant orbifold 2-form representing $c_1^{orb}(M_3(a,b,c), \Delta_{m})$ gives an orbifold Kähler metric $g_{a,b,c,m}$ on $M_3(a,b,c)$.

**Remark 2.6.** Note that the real cohomology class $c_1^{orb}(M_3(a,b,c), \Delta_{m})$ makes perfect sense for $m_j^0, m_j^\infty \in \mathbb{R}^+$, and we denote the set of all such classes by $\mathfrak{BL}_A$. In this case $(M_3(a,b,c), \Delta_{m})$ can be understood as having Kähler metrics with conical singularities along the corresponding $\mathbb{R}$-divisors $D_{v_i}$ and $D_{u_i}$ with cone angle $\frac{2\pi}{m_i^0}$ and $\frac{2\pi}{m_i^\infty}$, respectively [Don12, CDS15]. By this we mean that there is a Kähler metric $\omega$ which is smooth on $M_3(a,b,c) \setminus \Delta_{m}$ which extends to $M_3(a,b,c)$ as a closed positive $(1, 1)$ current satisfying certain uniformity requirements. See Definition 1.3 of [CDS15] for the precise statement. For arbitrary $m$ we say that $c_1^{orb}(M_3(a,b,c), \Delta_{m})$ represents a cone singularity along the divisor $\Delta_{m}$. The rational entries in the interval $(0, 1)$ are related to the so-called ‘ramifolds’ [RT11]. As in this reference we shall also assume hereafter that $m_j^0, m_j^\infty \in (0, \infty)$. We denote by $\mathfrak{F}_A$ the subset of $\mathfrak{BL}_A$ whose Kähler metrics are log Fano and have cone singularities along the divisor $\Delta_{m}$. Then the natural map from $\mathfrak{F}_A$ to the
Kähler cone $\mathcal{K}(M_3(a, b, c))$ is surjective, and the conclusion of Lemma 2.2 holds for all $(M_3(a, b, c), \Delta_m) \in \mathcal{L}_3^A$.

**Remark 2.7.** We consider the action of the affine monoid $\mathcal{M}(\mathbb{R})$ on $(\mathbb{R}^+)^6$ defined by the affine linear map

$$ (m_j^0, m_j^\infty) \mapsto (\lambda_j^0 m_j^0 + a_j^0, \lambda_j^\infty m_j^\infty + a_j^\infty) = (\tilde{m}_j^0, \tilde{m}_j^\infty) $$

with $1 \leq \lambda_j < \infty$ and $0 \leq a_j < \infty$. Restricting to the positive integers gives an action of the submonoid $\mathcal{M}(\mathbb{Z})$ on $(\mathbb{Z}^+)^6$. One easily checks that this induces an action of $\mathcal{M}(\mathbb{R})$ on $\mathcal{B}L$ that leaves the subset $\mathcal{L}_3^A$ invariant for all $\lambda_j^0, \lambda_j^\infty \in [1, \infty)$ and $a_j^0, a_j^\infty \in [0, \infty)$ sending $c_1^{orb}(M_3(a, b, c), \Delta_m)$ to $c_1^{orb}(M_3(a, b, c), \Delta_m)$.

3. **The Topology of $M^7 = Y_{l_1, l_2} S^3_w$**

It is important to remember that generally the topology of a join depends on the choice of Sasakian structure (through its Reeb vector field) of each factor. We assume that $(p, q)$ are relatively prime with $1 \leq q < p$ and that $l_1, l_2, w_1, w_2$ are chosen such that $M^7$ is smooth. We show first that our Sasaki 7-manifolds $M^7$ have the rational cohomology of the 2-fold connected sum $(S^2 \times S^5)\#(S^2 \times S^5)$. The integer cohomology groups are only distinguished by torsion in $H^4$. Moreover, the torsion depends on the choice of quasi-regular Sasakian structures on $Y_{p,q}$ and $S^3$. For the most generality we choose arbitrary quasi-regular Sasakian structure in the so-called $w$ subcone of the Sasaki cones for both $Y_{p,q}$ and $S^3$ (of course, the $w$ cone of $S^3$ is its entire Sasaki cone).

First we note that any quotient of a quasi-regular Reeb vector field $\xi_m$ in the $w$ cone of $Y_{p,q}$ has the form of a Hirzebruch orbifold $(H_\Delta, \Delta_m)$. Moreover, $Y_{p,q}$ is itself the join $S^3 \ast_{l_1, p} S^3_w$ with $w = (\frac{p+q}{l_1}, \frac{p-q}{l_1})$. Here $l_1 = 2$ if $p, q$ are both odd, and $l_1 = 1$ is $p, q$ have opposite parities. In any case the relation with the ramification indices is $m = m(v^0, v^\infty)$ with $v^0, v^\infty$ coprime and $m = p$.

The purpose of this section is to prove

**Theorem 3.1.** Let $Y_{p,q}$ have a quasi-regular Sasakian structure with Reeb vector field $\xi_m$. Then the 7-manifolds $M^7 = Y_{p,q} \ast_{l_1, l_2} S^3_w$ have the rational cohomology of the connected sum $(S^2 \times S^5)\#(S^2 \times S^5)$. Furthermore, the only torsion that occurs is $H^4(M^7, \mathbb{Z}) \approx \mathbb{Z}_{v^0v^\infty w_{l_1}^2} \oplus \mathbb{Z}_{w_1w_2l_2^2}$.

We begin with some lemmas.

**Lemma 3.2.** The 7-manifolds $M^7 = Y_{p,q} \ast_{l_1, l_2} S^3_w$ satisfy the following conditions:

1. $H_1(M^7, \mathbb{Z}) = \pi_1(M^7) = 0$,
2. $\pi_2(M^7) = \mathbb{Z}^2$,
3. $H^2(M^7, \mathbb{Z}) = H_2(M^7, \mathbb{Z}) = \mathbb{Z}^2$,
4. $H^3(M^7, \mathbb{Z})$ is torsion free.
5. $b_3(M^7) = b_4(M^7)$ is even.

**Proof.** From the long exact homotopy sequence for the fibration

$$ T^2 \longrightarrow S^3 \times S^3 \times S^3 \longrightarrow M^7 $$

we conclude that $M^7$ is simply connected and that $\pi_2(M^7) = \mathbb{Z}^2$. Thus, by Hurewicz $H_2(M^7, \mathbb{Z}) = \mathbb{Z}^2$ which implies (4) by universal coefficients, and then by item (1)
\(H^2(M^7, \mathbb{Z}) = \mathbb{Z}^2\). Item (5) follows from Poincaré duality and the fact that \(M^7\) admits a Sasakian structure.

\[\square\]

Actually we have

\[\textbf{Lemma 3.3.} \ H^3(M^7, \mathbb{Z}) = 0.\]

\[\textbf{Proof.}\] First by (4) of Lemma 3.2 \(H^3(M^7, \mathbb{Z})\) is torsion free, so it suffices to work with \(\mathbb{Q}\) coefficients. Since \(M^7\) is simply connected and is an \(S^1\) orbifold \((M_3(a, b, c), \Delta_m)\) we can apply the Leray-Serre Theorem with \(\mathbb{Q}\) coefficients. The differential \(d_2 : E_2^{0,1} \rightarrow E_2^{2,0} = H^2(M^7, \mathbb{Q})\) sends the class \(\alpha\) of the fiber \(S^1\) to the Kähler class \(c_1 x_1 + c_2 x_2 + c_3 x_3\) where \(c_i \in \mathbb{Q}^+.\) So by naturality we have \(d_2(\alpha \otimes x_1) = (c_1 x_1 + c_2 x_2 + c_3 x_3) x_1.\) Suppose there would exist a class \(w = w_1 x_1 + w_2 x_2 + w_3 x_3 \in E_2^{2,0}\) such that \(d_2(\alpha \otimes w) = 0.\) Then the 3-class \(\alpha \otimes w\) would survive to the limit giving a nonzero element in \(H^3(M^7, \mathbb{Q})\) by the Leray-Serre Theorem. Now the cohomology ring of \(M_3(a, b, c)\) is \([\text{CMS10}]\)

\[\mathbb{Z}[x_1, x_2, x_3]/(x_1^2, x_2(ax_1 + x_2), x_3(bx_1 + cx_2 + x_3)).\]

So computing the \(d_2\) differential we have

\[
0 = d_2(\alpha \otimes w) = d_2(\alpha) \otimes w = (c_1 x_1 + c_2 x_2 + c_3 x_3)(w_1 x_1 + w_2 x_2 + w_3 x_3) = (c_1 w_2 + c_2 w_1 - c_2 w_2 a)x_1 x_2 + (c_1 w_3 + c_3 w_1 - c_3 w_3 b)x_1 x_3 + (c_2 w_3 + c_3 w_2 - c_3 w_3 c)x_2 x_3
\]

which gives

\[\begin{pmatrix}
  c_2 & c_1 - c_2 a & 0 \\
  c_3 & 0 & c_1 - c_3 b \\
  0 & c_3 & c_2 - c_3 c
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  w_2 \\
  w_3
\end{pmatrix} = 0.
\]

Since the coefficients \(c_1, c_2, c_3\) are all positive, the rank of the matrix

\[C = \begin{pmatrix}
  c_2 & c_1 - c_2 a & 0 \\
  c_3 & 0 & c_1 - c_3 b \\
  0 & c_3 & c_2 - c_3 c
\end{pmatrix}\]

is either 3 or 2 which can be seen by putting \(C\) in Jordan canonical form. If the rank of \(C\) were 2 there would be exactly one solution to Equation \((31)\) which is an element of \(E_2^{1,2}\) and since \(E_2^{3,0} = H^3(M_3(a, b, c), \mathbb{Q}) = 0,\) there would be precisely one generator in \(H^3(M^7, \mathbb{Q})\) which contradicts the fact that \(b_3(M^7)\) is even. \(\square\)

\[\text{Proof of Theorem 3.1}\] It follows from Lemmas 3.2 and 3.3 and the fact that the homology (cohomology) of a connected sum is the direct sum of the homology (cohomology) of the two factors that \(M^7\) has the rational homology (cohomology) of the connected sum \((S^2 \times S^5) \# (S^2 \times S^5).\) By Poincaré duality and Lemma 3.2 it follows that \(H^5(M^7, \mathbb{Z})\) and \(H^6(M^7, \mathbb{Z})\) have no torsion. So it suffices to compute the torsion in \(H^4(M^7, \mathbb{Z}).\) First
Lemma 3.4. We have

\[ H^r_{\text{orb}}((\mathcal{H}_a, \Delta_{\mathbf{m}}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Z}^2 & \text{if } r = 2 \\ \mathbb{Z} \oplus \mathbb{Z}_{m^0} \oplus \mathbb{Z}_{m^\infty} & \text{if } r = 4 \\ 0 & \text{if } r \text{ is odd.} \end{cases} \]

Proof. The Leray sheaf of the map

\[ p : B(\mathcal{H}_a, \Delta_{\mathbf{m}}) \longrightarrow \mathcal{H}_a \]

is the derived functor sheaf \( R^s p_* \mathbb{Z} \), that is, the sheaf associated to the presheaf \( U \mapsto H^s(\mathcal{H}_a, \Delta_{\mathbf{m}}) \). For \( s > 0 \) the stalks of \( R^s p_* \mathbb{Z} \) at points of \( U \) vanish if \( U \) lies in the regular locs of \((\mathcal{H}_a, \Delta_{\mathbf{m}})\), which is the complement of the union of the zero \( e_0 \) and infinity \( e_\infty \) sections of the natural projection \( \mathcal{H}_a \longrightarrow \mathbb{C}P^1 \). However, at points of \( e_0 \) and \( e_\infty \) the fibers of \( p \) are (up to homotopy) the Eilenberg-MacLane spaces \( K(\mathbb{Z}_{m^0}, 1) \) and \( K(\mathbb{Z}_{m^\infty}, 1) \), respectively. So at points of \( e_0(e_\infty) \) the stalks are the group cohomology \( H^s(\mathbb{Z}_{m^0}, \mathbb{Z})(H^s(\mathbb{Z}_{m^\infty}, \mathbb{Z})) \). This is \( \mathbb{Z} \) for \( s = 0 \) and \( \mathbb{Z}_{m^0}(\mathbb{Z}_{m^\infty}) \) at points of \( e_0(e_\infty) \) when \( s > 0 \) is even; it vanishes when \( s \) is odd. The \( E_2 \) term of the Leray spectral sequence of the map \( p \) is

\[ E_2^{r,s} = H^r(\mathcal{H}_a, R^s p_* \mathbb{Z}) \]

and by Leray’s theorem this converges to the orbifold cohomology \( H^r_{\text{orb}}((\mathcal{H}_a, \Delta_{\mathbf{m}}), \mathbb{Z}) \). Now \( E_2^{r,0} = H^r(\mathcal{H}_a, \mathbb{Z}) \) and \( E_2^{r,s} = 0 \) for \( r \) or \( s \) odd. For \( r = 0 \) since \( R^s p_* \mathbb{Z} \) has its support in the orbifold singular locus \( e_0 \cup e_\infty \), the only continuous section of \( R^s p_* \mathbb{Z} \) is the 0 section which implies that \( E_2^{0,s} = 0 \) for all \( s \). Now we have \( E_2^{2r,2s} = 0 \) for \( r > 1 \) and

\[ E_2^{2r,2s} = H^2(\mathcal{H}_a, R^{2s} p) = H^2(e_0, \mathbb{Z}_{m^0}) \oplus H^2(e_\infty, \mathbb{Z}_{m^\infty}) = \mathbb{Z}_{m^0} \oplus \mathbb{Z}_{m^\infty}. \]

One easily sees this spectral sequence collapses whose limit is the orbifold cohomology \( H^r_{\text{orb}}((\mathcal{H}_a, \Delta_{\mathbf{m}}), \mathbb{Z}) \) which implies the result. \( \square \)

To continue the proof of Theorem 3.1 as in [BTF16, BTF15], we use the commutative diagram of fibrations

\[
\begin{align*}
Y^{p,q} \times S^3_w & \longrightarrow M_{l_1,l_2,w} \longrightarrow BS^1 \\
\downarrow & \downarrow & \downarrow \psi \\
Y^{p,q} \times S^3_w & \longrightarrow B(\mathcal{H}_a, \Delta_{\mathbf{m}}) \times BCP^1[w] \longrightarrow BS^1 \times BS^1.
\end{align*}
\]

Here \( BG \) is the classifying space of a group \( G \) or Haefliger’s classifying space \( [\text{Hae84}] \) of an orbifold if \( G \) is an orbifold. The lower exact fibration is a product of fibrations. We denote the orientation classes of \( Y^{p,q} \times S^3 = S^2 \times S^3 \times S^3 \) by \( \alpha, \beta, \gamma \), respectively. As in 3.2.2 of [BTF15] we have \( d_4(\gamma) = w_1 w_2 s_2^2 \). For the fibration

\[ Y^{p,q} \longrightarrow B(\mathcal{H}_a, \Delta_{\mathbf{m}}) \longrightarrow BS^1 \]

we have \( d_4(\beta) = m^2 v^0 v^\infty s_2^2 \). The map \( \psi \) in Diagram (32) is induced by the map \( e^{i\theta} \mapsto (e^{il_2 \theta}, e^{-il_1 \theta}) \), so \( \psi^* s_1 = l_2 s \) and \( \psi^* s_2 = -l_1 s \) the result follows by the commutativity of Diagram (32). \( \square \)
Remark 3.5. Although the case \((p, q) = (1, 0)\) does not fit directly into the \(Y^{p,q}\) scheme of [GMSW04], it can, nevertheless, be identified with the homogeneous Sasaki-Einstein structure on \(S^2 \times S^3\). Then if we take \((v^0, v^\infty) = (w_1, w_2) = (l_1, l_2) = (1, 1)\) and \(m = 1\) we obtain the homogeneous Sasaki-Einstein structure on an \(S^1\) bundle over \(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1\) which is a 7-manifold with the integral cohomology of the 2-fold connected sum \(2(S^2 \times S^3)\). See Sections 11.1.1 and 11.4.2 of [BG08] for details. The general \(S^3_w\) join with \(Y^{1,0}\) was treated in Section 3.2.2 of [BTF15].

4. A Generalized Orbifold Calabi construction

We now discuss a family of explicit examples or orbifold Kähler-Einstein metrics that we may view as arising from a special case of the generalized Calabi construction as presented in [ACGTF04, Section 2.5] and further discussed in [ACGTF11, Section 2.3] and [ECTF18, Section 5.1] - here generalized to allowing certain mild orbifold singularities.

The base of the construction will in this case be a Kähler-Einstein Hirzebruch orbifold \((\mathcal{H}_a, \Delta_{m_2})\). This is the Kähler quotient of the quasi-regular Sasaki-Einstein \(Y^{p,q}\) examples produced in [GMSW04]. Now \(Y^{p,q}\) may be viewed as the total space of a \(S^1\) principal orbi-bundle, \(P\), over \((\mathcal{H}_a, \Delta_{m_2})\) defined by the class \(c^{orb}(\mathcal{H}_a, \Delta_{m_2}) = \frac{(2m_2v^0 + a)v^\infty x_1 + (v^0 + v^\infty)x_2}{m_2v^0v^\infty + \lambda x_2}\), where the notation is as in Lemma 2.2. In particular, the index of \((\mathcal{H}_a, \Delta_{m_2})\) is \(\mathcal{J}_{v^2} = \gcd(2m_2v^0 + a) \omega^\infty, v^0 + v^\infty)\), where \((m_0, m_2) = m_2(v^0, v^\infty)\) and \(v^0, v^\infty\) are coprime. Let \(g_{\text{base}}\) denote the Kähler-Einstein orbifold metric whose Kähler form, \(\omega_{\text{base}}\), satisfies that

\[
\left[ \frac{\omega_{\text{base}}}{2\pi} \right] = \frac{(2m_2v^0 + a)v^\infty x_1 + (v^0 + v^\infty)x_2}{m_2v^0v^\infty + \lambda x_2}. 
\]

As we saw in Example 6.8 of [BTF16], the metric \(g_{\text{base}}\) is explicit and "admissible" in the sense of [ACGTF08]. Note that the Ricci form of \(g_{\text{base}}\) is given by \(\rho_{\text{base}} = \mathcal{J}_{v^2} \omega_{\text{base}}\).

We consider the generalized Calabi construction of orbifold Kähler metrics on the bundle \(P \times_{S^1} \mathbb{C}P^1_{m, m^\infty} \rightarrow (\mathcal{H}_a, \Delta_{m_2})\). This may also be viewed as an admissible construction - extended to mild orbifold cases.

Definition 4.1. Generalized orbifold Calabi data for our purposes.

1. A log pair \((\mathcal{H}_a, \Delta_{m_2})\) with Kähler-Einstein structure \((\omega_{\text{base}}, g_{\text{base}})\) such that

\[
\left[ \frac{\omega_{\text{base}}}{2\pi} \right] = \frac{(2m_2v^0 + a)v^\infty x_1 + (v^0 + v^\infty)x_2}{m_2v^0v^\infty + \lambda x_2}.
\]

2. The weighted projective line \((\mathbb{C}P^1_{m_0, m^\infty} = \mathbb{C}P^1_{v^0,v^\infty}/\mathbb{Z}_{m_3}, g_{m_3}, \omega_{m_3})\) with rational Delzant polytope \([-1, 1] \subseteq \mathbb{R}^*\) and momentum map \(\mathfrak{z}: \mathbb{C}P^1_{m_0, m^\infty} \rightarrow [-1, 1]\). Here \((m_0, m^\infty) = m_3(v^0, v^\infty)\) and \(v^0, v^\infty\) are coprime.

3. A principal \(S^1\) orbi-bundle, \(P_n \rightarrow (\mathcal{H}_a, \Delta_{m_2})\), with a principal connection of curvature \(n \omega_{\text{base}} \in \Omega^{1,1}((\mathcal{H}_a, \Delta_{m_2}), \mathbb{R})\), where \(S^1\) acts on \(\mathbb{C}P^1_{m_0, m^\infty}\), \(n \in \mathbb{Z} \setminus \{0\}\), and \(\gcd(n, m_3) = 1\). Note that \(n \in \text{span}_\mathbb{Z}\{v^0, v^\infty\}\) (since \(v^0, v^\infty\) are coprime), so \(m_0n \in \text{span}_\mathbb{Z}\{m_0, m^\infty\}\).

4. A constant \(0 < |r_3| < 1\) of same sign as \(n\) [ensuring that the \((1, 1)\)-form \((1/r_3 + \mathfrak{z})n \omega_{\text{base}}\) is positive for \(\mathfrak{z} \in [-1, 1]\)].
From this data we may define the orbifold
\[ M_3 = P_n \times S^1 \mathbb{CP}^1_{m_3, m_3} = M_3 \times_{\mathcal{C}} \mathbb{CP}^1_{m_3, m_3} \rightarrow (\mathcal{H}_a, \Delta_{m_2}), \]
where \( M_3 = P_n \times S^1 (\mathbb{Z}^{-1}(-1, 1)) \). Since the curvature 2-form of \( P_n \) has type \((1, 1)\), \( M_3 \) is a holomorphic principal \( \mathbb{C}^* \) bundle with connection \( \theta \in \Omega^1(M_3, \mathbb{R}) \) and \( M_3 \) is a complex orbifold.

On \( M_3 \) we define Kähler structures of the form
\[ g = \frac{1}{r_3} \frac{n}{\Theta(3) d\bar{\Theta}(3)} + \Theta \theta^2 \]
where \( \frac{1}{\Theta(3)} = \frac{dU}{d\bar{z}} \) and \( U \) is the symplectic potential \([\text{Gui}94]\) of the chosen toric Kähler structure \( g_{m_3} \) on \( \mathbb{CP}^1_{m_3, m_3} \).

The generalized Calabi construction arises from seeing (33) as a blueprint for the construction of various orbifold Kähler metrics on \( M_3 \) by choosing various smooth functions \( \Theta(z) \) on \((-1, 1)\) satisfying that
- [boundary values] the following endpoint conditions are satisfied
  \[ \Theta(\pm 1) = 0 \quad \text{and} \quad \Theta'(\pm 1) = \begin{cases} 2/m_\infty^3 & \text{for } z = 1 \\ -2/m_0^3 & \text{for } z = -1. \end{cases} \]
- [positivity] the function \( \Theta(z) \) is positive for \( z \in (-1, 1) \).

Then (33) extends to an orbifold Kähler metric on \((M(a, b, c), \Delta_m)\), where \( b = n(2v_2^0 + a)\mathbb{Z}^\infty \), \( c = n^\frac{v_2^0 + v_2^\infty}{v_2^0} \), and \( m = (1, 1, m_2^0, m_2^\infty, m_3^0, m_3^\infty) \). Metrics constructed this way are called compatible Kähler metrics with compatible Kähler classes parametrized by \( r_3 \). From \([\text{ACG}06]\) (and specifically to our notation from \([\text{BTF}16, \text{Proposition 5.4}]\)) we have that the compatible metric defined by \( \Theta(z) \) is Kähler-Einstein exactly when
\[ 2r_3J_2/n = (1 + r_3)/m_3^\infty + (1 - r_3)/m_3^0 \]
and
\[ \int_{-1}^{1} ((1 - 3)/m_3^\infty - (1 + 3)/m_3^0)/(1 + 3) \, d\bar{z} = 0. \]

5. Explicit Sasaki-Einstein metrics

We now look more closely for explicit Sasaki-Einstein examples arising from the join from Theorem 2.3. The arguments in Sections 6.1 and 6.2 of \([\text{BTF}16]\) (see specifically page 1053) carry through so that we have an adapted version of Theorem 1.4 of \([\text{BTF}16]\):

**Theorem 5.1.** Consider the join \( M^7 = Y^{r,a} \ast_{l_1, l_2} S^3_w \) where \( Y^{r,a} \) has a quasi-regular Sasaki-Einstein structure with quotient Hirzebruch orbifold \((\mathcal{H}_a, \Delta_{v_2})\) with \( a > 0 \) and \( l_1, l_2 \) are given by
\[ l_1 = \frac{J_{v_2}}{\gcd(|w|, J_{v_2})}, \quad l_2 = \frac{|w|}{\gcd(|w|, J_{v_2})}. \]
Then for each vector \( \mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \) with relatively prime components satisfying \( w_1 > w_2 \) there exist a Reeb vector field \( \xi_{v_3} \) in the 2-dimensional \( \mathbf{w} \)-Sasaki cone on \( M \) such that the corresponding Sasaki structure is Sasaki-Einstein.

Specifically, using equation (59) from [BTF16], we know that if the ray defined by co-prime \( (v_3^0, v_3^\infty) \) is quasi-regular, then we ought to look at the Kähler class determined by \( r_3 = \frac{w_1v_3^\infty - w_2v_3^0}{w_1v_3^\infty + w_2v_3^0} \). Now with this \( r_3 \), and the above choice of \((l_1, l_2)\), (35) is automatically solved and \( k \) becomes (similarly to (68) in [BTF16])

\[
\int_{-1}^{1} \left( (v_3^0 - v_3^\infty) - (v_3^0 + v_3^\infty) \right) \left( (w_1v_3^\infty + w_2v_3^0) + (w_1v_3^\infty - w_2v_3^0) \right)^2 = 0.
\]

This equation defines a priori a quasi-regular Sasaki \( \eta \)-Einstein ray (and thus, up to transverse homothety, a Sasaki-Einstein structure), but, by the same arguments as in Section 6.1 of [BTF16], any solution \((v_3^0, v_3^\infty) \in \mathbb{R}^+ \times \mathbb{R}^+ \) of (37) defines a Sasaki \( \eta \)-Einstein ray in the \( \mathbf{w} \)-cone, which is irregular unless \( v_3^\infty / v_3^0 \in \mathbb{Q} \). This gives

**Corollary 5.2.** For any pair of relatively prime positive integers \((w_1, w_2)\) satisfying \( w_1 > w_2 \), we obtain an SE metric by solving the cubic equation

\[
3w_2k^3 + (2w_2 - w_1)k^2 - (2w_1 - w_2)k - 3w_1 = 0.
\]

The ray of the Reeb vector field of the SE metric is then given by

\[
v_3^\infty = \frac{3 + 2k + k^2}{(1 + 2k + 3k^2)} = k \frac{w_2}{w_1}.
\]

The SE metric is quasi-regular when \( k \) is a rational root of (38) and irregular when it is an irrational root.

It follows from the analysis in Section 6.2 of [BTF16] that the positive real root \( k \) lies in the open interval \((1, \infty)\).

5.1. **Quasi-regular examples.** Changing our point of view we see that for any \( k \in (1, +\infty) \cap \mathbb{Q} \) we can choose \((w_1, w_2)\) such that

\[
w_2 / w_1 = \frac{3 + 2k + k^2}{k(1 + 2k + 3k^2)}
\]

and then the Sasaki-Einstein metric from Theorem 5.1 is quasi-regular with ray \( v_3 \) defined by Equation (39). We now give some examples.

**Example 5.3.** Here we give an example that builds on a bouquet of Sasaki cones from Example 6.8 of [BTF16]. Corollary 5.5 of [BP14] describes the well known \( Y^{p,q} \) structures [GMSW04] on \( S^2 \times S^3 \) as a \(|\phi(p)|\)-bouquet of Sasaki cones where \(|\phi(p)|\) denotes the order of the Euler phi function \( \phi(p) \). Let us consider the example when \( p = 13 \). Since 13 is prime the bouquet consists of \( p - 1 = 12 \) Sasaki cones labeled by the 12 positive integers \( 1 \leq q < 13 \) and as such contains 12 Sasaki-Einstein metrics. However, in order to construct SE metrics on our 7-manifolds, we need the SE structure on \( Y^{p,q} \) to be quasi-regular. It is easy to check that for the bouquet \( \bigcup \{Y^{13,q}\}_{q=1}^{12} \) the only values of \( q \) where the SE metric is quasi-regular is for \( q = 7, 8 \), all the other SE metrics in the bouquet are irregular. Let us look at these two cases a bit closer.

\(^1\)Example 6.8 in this reference has a small typo, namely \( v_3^0 \) and \( v_3^\infty \) got switched.
In the case $Y^{13.8}$ we have $a = 70$, $v_2^0 = 7$, $v_2^\infty = 5$, $m_2 = 13$, so $m_2^0 = 91$ and $m_2^\infty = 65$. This gives us that $I_y = 12$. If we put $k = 2$ in the quasi-regular SE prescription above given by Equation (38), we see that $(w_1, w_2) = (34, 11)$, so $l_1 = 4$ and $l_2 = 15$, and $(v_3^0, v_3^\infty) = (17, 11)$. Now we calculate that $s = \text{gcd}(|w_1 v_3^\infty - w_2 v_3^0|, l_2) = 1$ so $n = l_1 (w_1 v_3^\infty - w_2 v_3^0) = 748$, and $(m_3, m_3^\infty) = 15(17, 11)$. Then the quotient is the Bott orbifold given by the log pair $(M(a, b, c), \Delta_m)$, where

$$a = 70,$$
$$b = n \hat{b} = n\frac{(2m_3^0+a)v_3^\infty}{\hat{v}_2} = 78540,$$
$$c = n \hat{c} = n\frac{v_3^0+v_3^\infty}{\hat{v}_2} = 748,$$
$$m = (1, 1, 91, 65, 255, 165).$$

In the case where $p = 7$ in $Y^{p,q}$ we have $a = 36$, $v_2^0 = 4$, $v_2^\infty = 3$, $m_2 = 13$, so $m_2^0 = 52$ and $m_2^\infty = 39$. This gives us that $I_y = 7$. Again we put $k = 2$ in Equation (38) which gives $(w_1, w_2) = (34, 11)$ and $(v_3^0, v_3^\infty) = (17, 11)$ respectively. Now $l_1 = 7$ and $l_2 = 45$, and $s = \text{gcd}(|w_1 v_3^\infty - w_2 v_3^0|, l_2) = 1$ so $n = l_1 (w_1 v_3^\infty - w_2 v_3^0) = 1309$, and $(m_3, m_3^\infty) = 45(17, 11)$. Then the quotient is the Bott orbifold given by the log pair $(M(a, b, c), \Delta_m)$, where

$$a = 36,$$
$$b = n \hat{b} = n\frac{(2m_3^0+a)v_3^\infty}{\hat{v}_2} = 78540,$$
$$c = n \hat{c} = n\frac{v_3^0+v_3^\infty}{\hat{v}_2} = 1309,$$
$$m = (1, 1, 52, 39, 765, 495).$$

One can easily check from the torsion in Theorem 3.1 that the two SE 7-manifolds $Y^{13.8} \ast_{4,15} S^3_{(34,11)}$ and $Y^{13.7} \ast_{7,45} S^3_{(34,11)}$ are not homotopy equivalent. For both of these 7-manifolds the Reeb field that gives the SE metric is quasi-regular. Moreover, they are both induced from the same Reeb ray, namely $\{\xi_{a(17,11)}\}_{a>0}$ of the same $S^3_{(34,11)}$, and $b$ is the same for the quotient orbifolds.

For each choice of the rational number $k > 1$ we obtain a pair of quasi-regular SE 7-manifolds induced by the $S^3_w$ join and its Reeb field $\xi_{v_3}$, where $w$ and $v_3$ are determined by Equations (38) and (39).

**Example 5.4.** Here we give a 1-parameter family of smooth quasi-regular examples. First, let $k_2 \in \mathbb{Z}_{\geq 0}$ be given (using the subscript ”2” to indicate that this is a choice at the second stage). Then from [GMSW04] we get a quasi-regular Sasaki-Einstein $Y^{p,q}$ example by choosing

$$p = 12k_2^2 + 18k_2 + 7 \quad \text{and} \quad q = 12k_2^2 + 16k_2 + 5.$$

It is not hard to check that $\text{gcd}(p, q) = 1$ and $\text{gcd}(p + q, p - q) = 2$. Accordingly we recognize from (20) that

$$Y^{p,q} = S^3 \ast_{1,p} S^3_{2+4q, 2+q} = S^3 \ast_{2,(12k_2^2+18k_2+7)} S^3_{(2+3k_2)(3+4k_2),(1+k_2)}.$$
and that the quotient Hirzebruch orbifold of the quasi-regular Sasaki-Einstein metric is $(H_a, \Delta_{m_2})$ where, using (21) and (22),

$$m_2 = (m_2^0, m_2^\infty) = (12k_2^2 + 18k_2 + 7)((3 + 4k_2), 2(1 + k_2)),$$

$$m_2 = 12k_2^2 + 18k_2 + 7,$$

$$a = 6(1 + k_2)(1 + 2k_2)(3 + 4k_2).$$

We can calculate from Lemma 2.2 that $I_{v_2} = 5 + 6k_2$.

Now choosing $k = 3$ in (38) and (39), we have, from Theorem 5.1, a quasi-regular Sasaki-Einstein structure on $M^7 = Y_{p,q} \star_{l_1,l_2} S^3_w$ with quotient log pair $(M_3(a, b, c), \Delta_m)$ given by Theorem 2.3. Indeed, we have

$$a = 6(1 + k_2)(1 + 2k_2)(3 + 4k_2)$$

$$b = 4(1 + k_2)(2 + 3k_2)(3 + 4k_2)n$$

$$c = n$$

$$m = (m_1^0, m_1^\infty, m_2^0, m_2^\infty, m_3^0, m_3^\infty)$$

$$m_1^0, m_1^\infty = (1, 1)$$

$$m_2^0, m_2^\infty = (12k_2^2 + 18k_2 + 7)((3 + 4k_2), 2(1 + k_2))$$

$$m_3^0, m_3^\infty = m_3(v_3^0, v_3^\infty),$$

where

$$n = l_1 \frac{102}{\gcd(102, l_2)}$$

$$m_3 = l_2 \frac{l_2}{\gcd(102, l_2)}$$

$$l_1 = \frac{3l_2}{\gcd(20, 5 + 6k_2)}$$

$$l_2 = \frac{20}{\gcd(20, 5 + 6k_2)}$$

$$(w_1, w_2) = (17, 3)$$

$$(v_3^0, v_3^\infty) = (17, 9).$$

Using Lemma 2.1, we know that the corresponding Sasaki structure is a smooth manifold if and only if

$$\gcd(l_2(12k_2^2 + 18k_2 + 7)(3 + 4k_2), 17l_1) = 1$$

$$\gcd(l_2((12k_2^2 + 18k_2 + 7)(3 + 4k_2), 3l_1) = 1$$

$$\gcd(l_2(12k_2^2 + 18k_2 + 7)2((1 + k_2)), 17l_1) = 1$$

$$\gcd(l_2(12k_2^2 + 18k_2 + 7)2(1 + k_2), 3l_1) = 1$$
In order to get smooth examples, let us now assume \( k_2 = 255t + 10 \) with \( t \in \mathbb{Z}^+ \). Then we have
\[
I_{v_2} = 5(306t + 13) \\
l_1 = 306t + 13 \\
l_2 = 4 \\
n = 51(306t + 13) \\
m_3 = 2
\]
and then the corresponding Sasaki structure is a smooth manifold if and only if
\[
\gcd(4(1387 + 65790t + 780300t^2)(1020t + 43), 17(306t + 13)) = 1 \\
\gcd(4(1387 + 65790t + 780300t^2)(1020t + 43), 3(306t + 13)) = 1 \\
\gcd(8(1387 + 65790t + 780300t^2))(255t + 11), 17(306t + 13)) = 1 \\
\gcd(8(1387 + 65790t + 780300t^2))(255t + 11), 3(306t + 13)) = 1.
\]
if and only if
\[
\gcd((1387 + 65790t + 780300t^2)(1020t + 43), (306t + 13)) = 1 \\
\gcd((1387 + 65790t + 780300t^2))(255t + 11), (306t + 13)) = 1.
\]
Since \( \forall t \in \mathbb{Z}^+ \),
\[
6(255t + 11) - 5(306t + 13) = 1 \\
10(306t + 13) - 3(1020t + 43) = 1 \\
3(780300t^2 + 65790t + 1387) - (7650t + 320)(306t + 13) = 1,
\]
we have that this is always satisfied. Note that with \( k_2 = 255t + 10 \) we get
\[
a = 6(255t + 11)(510t + 21)(1020t + 43) \\
b = 204(255t + 11)(765t + 32)(1020t + 43)(306t + 13) \\
c = 51(306t + 13) \\
\mathbf{m} = (m_1^0, m_1^\infty, m_2^0, m_2^\infty, m_3^0, m_3^\infty) \\
(m_1^0, m_1^\infty) = (1, 1) \\
(m_2^0, m_2^\infty) = (1387 + 65790t + 780300t^2)(1020t + 43, 2(255t + 11)) \\
(m_3^0, m_3^\infty) = 2(17, 9).
\]
Finally, the \( p \) and \( q \) in \( Y^{p,q} \) are here given by
\[
p = 780300t^2 + 65790t + 1387
and
\[ q = 15(170t + 7)(306t + 13), \]
so the smooth Sasaki Einstein structures live on
\[ Y^{780300t^2 + 65790t + 1387, 15(170t + 7)(306t + 13)} \star^{306t + 13} S^3_{17,3}. \]

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