Engineering photonic Floquet Hamiltonians through Fabry–Pérot resonators

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Abstract

In this paper we analyze an optical Fabry–Pérot resonator as a time-periodic driving of the (2D) optical field repeatedly traversing the resonator, uncovering that resonator twist produces a synthetic magnetic field applied to the light within the resonator, while mirror aberrations produce relativistic dynamics, anharmonic trapping and spacetime curvature. We develop a Floquet formalism to compute the effective Hamiltonian for the 2D field, generalizing the idea that the intra-cavity optical field corresponds to an ensemble of non-interacting, massive, harmonically trapped particles. This work illuminates the extraordinary potential of optical resonators for exploring the physics of quantum fluids in gauge fields and exotic space–times.

Time-periodic modulation is under active development both theoretically and experimentally as a tool for Hamiltonian engineering in platforms ranging from cold atoms in optical lattices [1–3] to microwave photons in arrays of superconducting resonators [4, 5] and electrons in solids [6, 7]. By imposing external fields which couple states of different energies and symmetries, modulation enables time-reversal symmetry breaking and the introduction of synthetic gauge fields [8, 9], as well as manipulation of interactions [10].

In parallel, there is an aggressive effort to explore optical modes coupled to matter as a platform for quantum manybody phenomenology. Single- [11–14] and multi- [15, 16] mode optical resonators, as well as photonic crystal structures [17, 18] are under investigation to induce long-range interactions between atoms; near-planar resonator/exciton heterostructures [19–21] and dye filled cavities [22] have been employed to study interacting quantum fluids [19–21]; arrays of microwave resonators coupled to superconducting qubits [23] have been harnessed as model Bose–Hubbard systems; and Rydberg electromagnetically induced transparency in trapped atomic gases has recently been demonstrated as a platform for studying 1D quantum dynamics of strongly interacting photons [24–28].

Here we formalize a new approach to photonic quantum materials based upon exotic optical resonators; following up on our prior work describing Rydberg-dressed photons in a near-degenerate optical resonator as interacting, massive, harmonically trapped 2D particles in synthetic magnetic fields [29], we now provide detailed framework for designing the resonators and characterizing the resulting single-particle photonic Hamiltonian dynamics.

Our approach begins in the ray-optics picture where, assuming round-trip ray-trajectories are nearly closed, round-trip propagation may be coarse-grained using a Floquet formalism to provide an effective 2D time-continuous Hamiltonian for the photon (section 1 and figure 1). We quantize this Hamiltonian (section 2), leading to a wave-optics view of the resulting physics and the appearance of longitudinal modes due to the energy periodicity of the Floquet formalism. In section 3 we classify all of the terms in this Hamiltonian: an inertial mass tensor, a harmonic confinement tensor and a synthetic magnetic field, as well as non-physical gauge degrees of freedom. In section 4 we consider what happens when the coarse-graining breaks down because ray-trajectories are not nearly closed and explore a way to recover a simple Hamiltonian picture if the trajectories nearly close after multiple round-trips.

To illustrate the techniques developed in the preceding sections, we next consider several different resonator geometries (section 5), focusing in particular on the symmetric two-mirror resonator. We distinguish between...
mechanical- and canonical- ray momenta and show that while photons in near-planar cavities exhibit a positive mass, those in near-concentric cavities exhibit a negative mass. We then briefly analyze twisted resonators, which introduce synthetic magnetic fields for photons.

In section 6 we explore the impact of mirror aberrations and non-paraxial propagation on the photonic Hamiltonian. We show that these corrections provide a route to arbitrary potentials and dispersion relations for resonator photons, along with photonic dynamics on curved spatial manifolds.

Finally, in section 7 we suggest several spectroscopic and time-resolved techniques to experimentally characterize such photonic Hamiltonians in the single-particle sector, enabling precise tuning of the resonator to near-degeneracy and extraction of mirror-induced disorder.

1. Floquet formalism for rays in optical resonators

A paraxial optical resonator may be characterized by an ABCD matrix $M$, describing the round trip evolution of all light rays in a given transverse plane of the resonator. In particular, the ray described by $V = \begin{pmatrix} s \\ x \end{pmatrix}$, where $x$ is the transverse location of the ray and $s = \tan \theta \approx \theta$ its slope (see figure 2), becomes $MV$ under round-trip propagation. The matrix $M$ may be written as a product $M = \prod_j M_j$, where the matrices $M_j$ correspond to: (1) free-space propagation over a distance $Z$: $M_{\text{prop}} = \begin{pmatrix} I_z & ZI_2 \\ 0 & I_2 \end{pmatrix}$; and (2) reflection off of a mirror of radius of curvature $R$: $M_{\text{mirror}} = \begin{pmatrix} I_z & 0 \\ -\frac{1}{R^2}I_2 & I_2 \end{pmatrix}$. Thus, describes a discrete linear transformation in phase space and suggests that such stroboscopic dynamics (see figure 3(b)) are equivalent to periodically sampled continuous evolution under a linear differential equation, which in turn may be generated by a quadratic time-invariant Hamiltonian.

To develop a Hamiltonian formalism describing the continuous evolution of the ray within a particular transverse plane we must first convert the slope $s$ into a momentum $p$ which is canonically conjugate to $x$. This momentum is $p = \frac{\hbar k s}{c}$, with $k \equiv 2\pi/\lambda$ and $\lambda$ the optical wavelength. We may thus define a phase-space state-vector $\mu \equiv \begin{pmatrix} x \\ p \end{pmatrix}$ and a round-trip propagation matrix $B = \beta M \beta^{-1}$, with $\beta \equiv \begin{pmatrix} I_z & 0 \\ 0 & \frac{\hbar k}{c} \end{pmatrix}$ and $I_2$ the $2 \times 2$ identity matrix. The same round-trip propagation matrix applies in paraxial wave optics [31], where $x$ and $p$ are operators.

Noting that round-trip propagation requires a time $T_r = L_{\text{RT}}/c$, where $L_{\text{RT}}$ is the total round-trip length along the resonator axis and $c$ is the speed of light, we may now view $B$ as a stroboscopic time evolution operator: $\mu(t + T_r) = B \mu(t)$. We would like to determine a continuous time evolution which reproduces the stroboscopic evolution $B$ and parametrize this evolution with a Hamiltonian. The most general time-invariant quadratic Hamiltonian may be written as:

$$H \equiv \frac{1}{2} \mu^\top G \mu,$$

where $G = \begin{pmatrix} 0 & I_z \\ -I_z & 0 \end{pmatrix}$ and $Q$ encodes the Hamiltonian’s matrix elements and is what we would like to solve for. Hamilton’s equations imply $[32] \frac{d\mu}{dt} = Q\mu$, which we can integrate to yield: $\mu(t + \tau) = \exp(Q\tau)\mu(t)$. Using

![Figure 1. Schematic three-mirror Fabry–Pérot resonator. A single ray (thin red line) is followed over many roundtrips through the resonator. The intersection pattern of this ray (red spheres) in a chosen transverse plane (gray polygon) of the resonator traces out a stroboscopic evolution corresponding, in this case, to a massive, harmonically trapped particle and its image reflected across the $x$- and $y$-axes. These dynamics may be formally predicted using the Floquet formalism described in this work.](image-url)
\[ T = B^T \exp(\mathbb{R}) = \mathbb{I}. \]

Solving for \( Q \), we arrive at
\[ QB \mathbb{I} \log_2 \mathbb{I} (\mathbb{R}) = -1, \]
where \( l \) is the identity matrix. Substituting for \( Q \) in (1.1) (and identifying \( BM = -\mathbb{I} \)) yields the effective Floquet Hamiltonian:
\[ H = \frac{c}{L} \left[ \frac{1}{2} \mathbb{I} \mathbb{G} (\log \beta \mathbb{M} \beta^{-1}) \mu - i \pi l \cdot \sum_{i=1}^{2} \mathbb{I} \mathbb{I} \right]. \tag{1.2} \]

In appendix A, we discuss the mathematical properties of this mapping in more detail. We employ the standard definition of the matrix logarithm as the inverse of the matrix exponential, which is itself defined in terms of its Taylor series. The logarithm is defined only modulo \( 2\pi \), but so long as \( x \) and \( p \) commute (as they do in the ray-optics limit), the resulting term drops out of the Hamiltonian, leaving:
\[ H_{\text{classical/ray}} = \frac{c}{L} \left[ \frac{1}{2} \mathbb{I} \mathbb{G} (\log \beta \mathbb{M} \beta^{-1}) \mu \right]. \tag{1.3} \]

2. Quantum mechanical treatment

Quantizing the ray matrix dynamics turns \( x \) and \( p \) into canonically conjugate (non-commuting) operators. Noting that \( [x_i, p_j] = i\hbar \delta_{ij} \), the Hamiltonian (equation 1.2) becomes:
\[ H_{\text{quantum/wave}} = H_{\text{classical/ray}} + \frac{\hbar^2}{L^2} 2\pi \cdot l. \tag{2.1} \]

The additional \( \frac{\hbar^2}{L^2} 2\pi \cdot l \) in the energy reflects the fact that we are considering a Floquet Hamiltonian [8, 9]; the periodic influence of the mirrors on the optical field means that the eigen-frequencies are only defined up to the inverse round-trip time; this is analogous to the quasi-momentum being defined only up to the lattice spacing in a crystal (see figure 3(a)). It is interesting to note that this energy periodicity corresponds to the cavity free-spectral range and the \( l \) quantum number is actually the familiar longitudinal mode index.

3. Decomposing a general quadratic Hamiltonian

A general paraxial Fabry–Pérot may include non-commuting non-planar reflections and mirror astigmatism, corresponding to a near-arbitrary \( 4 \times 4 \) ABCD matrix \( M \) and Floquet Hamiltonian \( H_{\text{Floquet}} = \frac{c}{L} \left[ \frac{1}{2} \mathbb{G} (\log \beta \mathbb{M} \beta^{-1}) \mu \right] \). We would like to be able to ascertain, for such a resonator, what types of dynamics we can engineer and for a particular resonator design, what we have engineered.

Without worrying about the fact that the Hamiltonian arises from ray optics, we can say that the most general quadratic Hamiltonian for a single particle in 2D may be represented by a symmetric \( 4 \times 4 \) quadratic form in \( x, y, p_x \) and \( p_y \), with 10 independent parameters. This quadratic form may then be decomposed in the following physically illuminating way:
\[ H = \frac{1}{2} (p - \beta \sigma \cdot x)^T m_{\text{eff}} (p - \beta \sigma \cdot x) + \frac{1}{2} x^T \omega_{\text{trap}}^T m_{\text{eff}}^{-1} \omega_{\text{trap}} x \tag{3.1} \]
with: $m_{\text{eff}}^{-1} = \frac{1}{m} R_\theta \cdot \left( \begin{array}{cc} 1 + \epsilon_t & 0 \\ 0 & 1 - \epsilon_t \end{array} \right) \cdot R_{-\theta}, \quad \omega_{\text{trap}} = \omega \cdot R_\theta \cdot \left( \begin{array}{cc} 1 + \epsilon_t & 0 \\ 0 & 1 - \epsilon_t \end{array} \right) \cdot R_{-\theta}, \quad \sigma^k \equiv [I, \sigma^x, \sigma^y, \sigma^z].$

$\beta_k \equiv [\delta, \Delta_\alpha, -iB_z/2, \Delta_+], \quad R_\phi \equiv \left( \begin{array}{cc} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{array} \right)$ and the Pauli matrices operate on the real-space vector $x$.

1. $m_{\text{eff}}$ and $\omega_{\text{trap}}$ are the transverse effective mass and trapping frequencies of the particle, respectively, arising from the interplay of cavity length and mirror curvature; they become anisotropic (with axes rotated by $\theta_{\text{in},t}$ and fractional difference $\epsilon_{\text{in},t}$) in the presence of mirror astigmatism, often caused by off-axis reflection [30] from spherical mirrors.

2. The $\beta_k$’s are the remaining four degrees of freedom and parameterize the gauge potential arising from common-mode defocus ($\delta$), rotated differential defocus ($\Delta_\alpha, \Delta_+$) and resonator twist ($B_z$). Defining the vector potential $A \equiv \beta_2 \sigma^k \cdot x$, we find $\nabla \times A = B_z \hat{z}$, indicating that the rest of the terms ($\delta, \Delta_+, \Delta_+$) may be gauged away via $A \to A - \nabla f$ for $f = \frac{1}{2} \delta (x^2 + y^2) + \Delta_+ x y + \frac{1}{2} \Delta_+ (x^2 - y^2)$. Thus the only physically significant term is $B_z$, the magnetic field induced by twist.

The recipe for going from an arbitrary resonator geometry to the physical parameters of the space in which the trapped photons live is to: (1) compute a round-trip $4 \times 4$ ABCD matrix for the resonator geometry under consideration, (2) from this compute a Floquet Hamiltonian and finally (3) decompose this Hamiltonian into the physically significant parameters.

4. Near-degeneracy after multiple round-trips

The stroboscopic time evolution under the round-trip ray matrix becomes indistinguishable from continuous time evolution in the limit where the transverse dynamics are slow relative to the longitudinal dynamics. In this limit, the frequency splittings between the quantized transverse modes described by the first term in equation (2.1) become small compared to the splitting between the longitudinal modes described by the second
term in (2.1) and the cavity is said to be nearly degenerate. In such a cavity, rays pass near their previous location after each round trip and the first term in (2.1) describes the slow precession of the rays over many round trips.

It is often the case that while the ray does not pass near its original phase-space location after a single round-trip, it may do so after several round trips (see figure 4(a)). In the wave-optics picture, such a resonator exhibits a Floquet spectrum where near-degeneracy arises from incrementing both longitudinal and transverse quantum numbers in the appropriate proportion (see figure 4(b)).

Formally, we can write $M_{\text{multi}} = M'$ and $L_{\text{multi}} = s \cdot L_{\text{fl}}$; replacing $M \rightarrow M_{\text{multi}}$, $L_{\text{fl}} \rightarrow L_{\text{multi}}$ in the equations from the preceding sections provides the resulting effective Hamiltonian, with the caveat that the ray appears in every plane with multiple images mirroring its dynamics.

Some typical examples of this phenomenon are (1) the near-confocal resonator, which provides the low energy dynamics of a massive particle in a harmonic trap after two round-trips ($s = 2$) and (2) the astigmatism-compensated twisted resonator, which provides the low energy dynamics of a massive particle in a magnetic field after a twist-controllable number of round trips. In the former case, the image rays are reflected across the resonator's longitudinal axis. In the latter case, the image rays are rotated about this axis.

5. Examples of simple resonators

5.1. Symmetric two-mirror Fabry–Pérot in the focal plane
Here we consider a two-mirror symmetric Fabry–Pérot resonator with length $L$ and mirror radii of curvature $R$. The round-trip ABCD matrix for a single transverse direction, in the central plane of the cavity (a distance $L/2$ from each mirror) is (for $g \equiv 1 - L/R$):

$$M_{\text{focal}} = \begin{pmatrix} -1 + 2g^2 & gR(1 - g^2) \\ -\frac{4gR}{R} & -1 + 2g^2 \end{pmatrix}. \tag{5.1}$$

The matrix logarithm may be obtained using a similarity transform to a rotation matrix, giving (for $-1 \leq g \leq 1$ as required for resonator stability):
with \( \alpha \equiv \sqrt{1 - g^2} \), \( \theta \equiv \cos^{-1}(-1 + 2g^2) \). Thus the Floquet Hamiltonian is given by:

\[
H = \frac{\hbar^2}{2m} + \frac{1}{2} m \omega^2 x^2, \quad \text{where } \omega \equiv \frac{c}{2L} \theta \text{ and } m \equiv \frac{2\hbar}{2\theta R_0}. 
\]

Quantizing and diagonalizing this Hamiltonian gives a quantum harmonic oscillator \( H = \hbar \omega (a^\dagger a + 1/2) \) with energy-level spacing \( \hbar \omega \), harmonic oscillator lengths \( x_0 = \sqrt{\frac{\hbar}{m \omega}} = \sqrt{\frac{R_0}{2 \pi \hbar^2 \sqrt{1 - g^2}}}, \)

\[ p_0 = \sqrt{\frac{\hbar}{m \omega}} = \sqrt{\frac{2\pi \hbar}{\pi R_0}}, \text{raising operator } a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{x}{x_0} + i \frac{p}{p_0} \right), \text{and Hermite–Gauss eigenstates } \psi_n = \frac{1}{\sqrt{\sqrt{2} n! x_0}} e^{-x^2} \frac{1}{\sqrt{\pi}} H_n(\frac{x}{x_0}) \text{.}
\]

These results are consistent with the standard expressions for the two-mirror Fabry–Pérot [30], where the transverse mode spacing is \( \hbar \omega \) and the \( 1/e^2 \) waist of the lowest mode is \( w_0 = \frac{\hbar \Lambda}{2 \pi \sqrt{1 - g^2}} = x_0 \sqrt{2} \); the factor of \( \sqrt{2} \) arises from the different conventions for optical mode waist and harmonic oscillator length.

If this Floquet analysis were performed starting in a different plane in the resonator, the Hamiltonian should remain unchanged in some physical sense. This story is explored more fully in appendix B where the conclusion is that while the particle mass and trapping potential are rescaled, the trap frequency (the 'physical' quantity) does not depend upon the chosen Floquet plane. Furthermore, the momentum undergoes a gauge transformation: outside of the focal plane of the resonator, the canonical momentum of the ray (corresponding to its slope as it propagates along the cavity axis) is no longer proportional to the mechanical momentum of the ray (the rate at which it moves in the 2D transverse plane under consideration). Instead, there is an additive correction which is linear in position, reflecting the wave-front curvature.

5.2. Near-concentric versus near-planar Fabry–Pérot

A separation of timescales between the cavity round-trip time and the harmonic oscillator period requires tuning the cavity geometry near a degeneracy point, where at least one of the transverse mode frequencies becomes much smaller than the cavity free spectral range. Both near-planar \( (L \ll R) \) and near-concentric \( (L \approx 2R) \) cavities exhibit such a near-degeneracy. One must be cautious in defining 'near-degenerate,' however, because while the ratio of the transverse—to longitudinal-mode spacing goes to zero in both cases, the transverse spacing itself only approaches zero when the appropriate parameter is tuned: the mirror radius of curvature in the near-planar case and the cavity-length in the near-concentric case.

The trap frequency and mass in the near-planar cavity are: \( \omega_{\text{trap}} \approx \frac{c}{\sqrt{LR/2}}, m \approx \frac{\hbar \Lambda}{c} \). In the near-concentric case they are: \( \omega_{\text{trap}} \approx \frac{2c}{T} \sqrt{1 - \frac{L}{2R}}, m \approx \frac{\hbar \Lambda}{1 - L/2R} \).

Note that:

- In the near-planar case the mass remains finite and the trap frequency approaches zero only if 'planarity' is approached by increasing mirror radius of curvature to infinity rather than by reducing cavity length to zero;
- In the near-concentric case the trap frequency goes to zero and the mass diverges, no matter how one approaches degeneracy (by adjusting resonator length or mirror curvature);
- The photon mass is negative in the near-concentric case. This reflects the fact that the direction of ray propagation out of the plane is opposite to the direction of motion of the particle within the plane (canonical and mechanical momenta are opposite), due to an inversion from the refocusing of the cavity mirrors.

5.3. Twisted resonators

The simplest resonators that exhibit synthetic magnetic fields for the photons traveling within them are (a) four-mirror resonators that do not reside in a plane and (b) three-mirror resonators with astigmatic mirrors that are twisted with respect to one another. What these resonator geometries have in common is a helicity to the round-trip manipulation of the photon trajectory, producing dynamics akin to a Floquet topological insulator [8]. Because (a) is easier to realize experimentally, it is the configuration that we will analyze here (see figure 5).

We consider a four-mirror resonator where the mirrors do not all reside in a plane, but where all mirrors are curvature-less (‘planar’), to keep the analysis simple. The resonator geometry consists of an opening angle \( \theta \) and a principal arm length \( L \). To analyze such a resonator requires \( 4 \times 4 \) ABCD matrices and careful transformation of coordinate bases at each reflection. The outcome is: \( M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes R(\phi) \), where \( R(\phi) \) is a 2D rotation.
through an angle $\phi$ given by: \[
\cos 3 \theta \cos 12 \cos 2 \theta \cos 3 \cos 4 \theta \approx 1 - \theta^4/2, \]
or $\phi \approx \theta^2$ for small $\theta$.

It is thus apparent that the resonator rotates the coordinate axes on each round-trip; the Floquet Hamiltonian is:

\[
H_{\text{Floquet}} = \frac{1}{\sqrt{1 + 2 \tan^2 \frac{\phi}{2}}} \left[ \frac{1}{2} \left( \frac{c}{\hbar k} \left( P_x^2 + P_y^2 \right) + \frac{c}{2l} \phi (y P_x - x P_y) \right) \right]
\]

\[
= \frac{(P - \frac{1}{2} q B_{\text{eff}} \times r)^2}{2m_{\text{eff}}} - \frac{1}{2} m_{\text{eff}} \omega_{\text{trap}}^2 r^2
\]

(5.3)

with an effective mass of $m_{\text{eff}} = \frac{\hbar k}{c} \sqrt{1 + 2 \tan^2 \frac{\phi}{2}}$ and rotation giving rise to an effective magnetic field $q B_{\text{eff}} = \frac{\hbar k}{c} \phi$ and corresponding centrifugal anti-trapping with frequency $\omega_{\text{trap}} = \frac{1}{\sqrt{1 + 2 \tan^2 \frac{\phi}{2}}}$. The magnetic length corresponding to the synthetic field is $l_B \equiv \frac{\hbar}{\sqrt{1 + 2 \tan^2 \frac{\phi}{2}}}$. Note that the effective photon charge $q$ and synthetic magnetic field $B_{\text{eff}}$ are individually meaningless; only the product $q B_{\text{eff}}$ is well defined.

In practice mirror curvature is essential to compensate the centrifugal anti-trapping; analysis of curved-mirror non-planar resonators in the presence of non-normal-reflection-induced astigmatism is beyond the scope of this work.

6. Higher-order perturbations to the resonator

Thus far we have analyzed the Hamiltonian for light trapped within a resonator composed entirely of quadratic optics, with paraxial (quadratic) propagation between these optics. It is well-known that resonator mirrors are measurably imperfect [33], both because they are spherical rather than parabolic and due to wavefront errors. Furthermore, the propagation of optical fields only approximately obeys the paraxial wave equation, with corrections arising at the same order as spherical aberration on the mirrors $\phi_{\text{S. aberration}} \propto \frac{1}{R_{\text{mirror}}}$. Such corrections become relevant even for low-order modes in moderate finesse $F \sim 10^4$ resonators (for 1 cm ROC mirrors) and become increasingly important for higher order modes. There are several calculations of resonator modes in the presence of such corrections [34, 35]; here take a different approach, instead computing the impact of such higher-order terms on the Floquet Hamiltonian.

6.1. Impact of non-quadratic optics on trapping potential, single-particle dispersion and spatial curvature

Consider an arbitrary lens providing a position dependent phase shift $\alpha(x)$ in a transverse plane that is a longitudinal distance $z_{\text{Lens}}$ from the plane where the Floquet Hamiltonian is defined (see figure 6). If the lens is weak enough that it couples only within Floquet bands, but not between them, its impact may be written as a perturbation to the Floquet Hamiltonian.

The simplest case is $z = 0$; a lens in the plane where the Floquet Hamiltonian is defined (henceforth the 'Floquet plane'). In this situation the phase shift $\alpha(x)$ per round trip corresponds to an energy (and thus effective potential) $\delta H_{\text{aberration}} = \frac{\hbar}{2m_{\text{eff}}} \alpha(\hat{x})$; the lens directly acts as a potential for the cavity photons. Note that $\alpha(\hat{x})$ is evaluated using the Taylor series expansion of $\alpha(x)$.
Figure 6. Impact of higher order perturbations out of the Floquet plane. An arbitrary lens produces a position-dependent phase-shift \( \alpha(x) \) on light propagating through the plane at \( z = z_{\text{cav}} \). Its impact on the field in the plane \( z = 0 \) may be understood in terms of a phase shift that reflects where the correspondingly propagated ray would hit the lens; that is, it produces a phase shift that is dependent upon both the position and momentum of the ray in the plane \( z = 0 \): 
\[
\alpha(x) = \alpha(x' + \frac{Z_{\text{eff}}}{\alpha}(p')).
\]

If the lens were placed within the cavity in a Fourier plane, the phase shift would be dependent upon the momentum in the Floquet plane and the corresponding Hamiltonian term would be 
\[
\delta H_{\text{aberration}} = \frac{\hbar}{L_{\text{en}}} \alpha \left( \hat{\mathbf{x}} + \frac{Z_{\text{eff}}}{\alpha} \hat{\mathbf{p}} \right),
\]
where \( f \) is effective focal length of the real-to-Fourier space imaging. Thus in this case, the lens acts to control the dispersion of the photons in the Floquet plane.

It is natural to ask what happens if the lens is placed between real- and Fourier- planes; in accordance with our ray-optics expectation, the correction is 
\[
\delta H_{\text{aberration}} = \frac{\hbar}{L_{\text{en}}} \alpha \left( \hat{\mathbf{x}} + \frac{Z_{\text{eff}}}{\alpha} \hat{\mathbf{p}} \right),
\]
as demonstrated in appendix C.

Note that an arbitrary perturbation in a single plane may always be understood (through a linear canonical transformation) as a pure real-space potential or a pure dispersion- two such perturbations in different planes are required to generate curvature that cannot be trivially removed through such a generalized coordinate transformation.

As an example, consider the case of quartic lenses \( \alpha(x) = \beta x^4 \) placed symmetrically around \( z = 0 \):
\[
\alpha(x - \mu p) + \alpha(x + \mu p).
\]
The resulting Hamiltonian contains terms quartic in \( x \), which we view as a quartic confining potential, terms quartic in \( p \), which we view as quartic single-particle dispersion and those quadratic in both \( x \) and \( p \) corresponding to manifold curvature. Noting that the non-relativistic Hamiltonian for a particle in curved space is given by: 
\[
H = \frac{\hbar}{2m} \mathbf{p}^2 + V(x, p),
\]
we can match terms to extract a metric tensor that induces this manifold curvature (assuming, for simplicity, the near-planar limit \( m \approx \hbar/c \)):
\[
dl^2 \equiv g_{ij} dx^i dx^j = dx^2 \left( 1 + \frac{3x^2 + y^2}{r_0^2} \right) + dy^2 \left( 1 + \frac{3y^2 + x^2}{r_0^2} \right) + 4dx dy \frac{\partial V}{\partial y} r_0^2
\]
and resulting scalar curvature \( R = \frac{-2y^2/r_0^4}{\left( 1 + 3y^2/r_0^2 \right)^2} \). Here \( r_0^2 \equiv \frac{\hbar^2}{8m \beta_{\text{planar}}} \) depends on the wavelength only because \( \alpha(x) \) is defined as a phase rather than a path-length. The manifold curvature is thus purely geometrical and persists in the ray optics limit.

6.2. Beyond paraxial optics

Taking the scalar wave equation \( c^2 \nabla^2 \psi = \partial_t^2 \psi \) and substituting \( \psi = \phi e^{ik(z-x)} \) yields:
\[
(\nabla_1^2 + 2ik \partial_x + \partial_t^2) \phi = 0. 
\]
At lowest order in \( \frac{k}{\alpha} \) this reduces to the paraxial wave equation: 
\[
(\nabla_1^2 + 2ik \partial_x) \phi = 0.
\]
Iterating this approximate solution for the second derivative yields a first Born correction:
\[
\left( 2ik \partial_x + \nabla_1^2 - \frac{\nabla_1^2 \phi}{4k^2} \right) \phi = 0. 
\]
This expression may be derived more directly via a Taylor expansion of \( k_x l \approx \sqrt{(\hbar k)^2 - p_x^2} \) to quartic order.

The above expression says that, up to a constant offset, the phase acquired from propagation along an arm of the cavity of length \( l \), is 
\[
k_x l = \frac{1}{2} k \left( \frac{p_x^2}{k^2} + \frac{1}{8} \frac{k^2}{\hbar^2} \right). 
\]
The first (quadratic) term gives rise to the kinetic dynamics we have been studying throughout this work and the second (quartic) term is a new correction.

Because the ray momentum is transformed after each mirror reflection, the total non-paraxial correction arising from the term in each arm of the cavity is:
\[
H_{\text{non-paraxial}} = \frac{\hbar c}{L_{\text{en}}} \int dx \frac{\nabla_1^2 \phi^2}{8k^3} = \frac{c}{8(\hbar k)^3} \sum_j \epsilon_j (D_j \phi + \hbar k \tilde{C}_j \phi)^4. 
\]
Here we have employed ABCD matrices that move from the Floquet plane to the region between the $j$th and $(j+1)$st mirrors and $\epsilon_j$, the fraction of the path length between those two mirrors.

We thus see that the lowest order correction to paraxial optics introduces a quartic potential, quartic dispersion and cross terms (including manifold curvature), akin to an out-of-focus quartic lens.

7. Hamiltonian characterization

Experimentally tuning the a resonator to near-degeneracy will require precise control of the resonator length and geometry. We now suggest several relatively straightforward techniques to characterize the single-particle Hamiltonian (see figure 7). It is apparent that photonic systems are ideal for Hamiltonian characterization, as the tools required are well-developed for standard optical-imaging and communication applications and there is virtually no limit to the amount of optical power that may be applied to the system in the detection process, providing high signal-to-noise.

**Transmission spectroscopy:** Ignoring perturbations from mirror aberrations above second order and non-paraxial corrections, the resonator spectrum is quadratic. Consequently its energy spectrum is fully characterized by one longitudinal mode spacing and two transverse mode spacings. These numbers may be extracted by observing the resonator transmission on a photodiode as a probe laser which is intentionally misaligned from the resonator axis (to ensure coupling to many modes) is swept in frequency. By performing this spectroscopy as the resonator length is varied, it will be possible to (a) tune to degeneracy and (b) observe residual avoided crossings resulting from corrections to the Hamiltonian: mirror imperfections, aberration and non-paraxial optics. This approach has been demonstrated by Kollar et al [36].

**Transmitted mode imaging:** An arbitrary quadratic Hamiltonian is not entirely characterized by its energy spectrum. For example, an electron which is harmonically trapped along $x$ but free along $y$ has the same energy spectrum as an electron in a magnetic field with a cyclotron frequency given by the aforementioned harmonic trap. To distinguish the two scenarios, one must observe the eigenmodes, in addition to their energies. This may be achieved with a CCD camera observing the resonator transmission near, but not at, degeneracy. Qualitatively speaking, the astigmatically harmonically trapped electron mentioned previously would exhibit Hermite–Gauss eigenmodes, while the electron in a magnetic field (plus a weak circular harmonic trap to slightly break the degeneracy of the Landau level) would exhibit Laguerre–Gausse eigenmodes; these are easily distinguished on a camera.

**Shack–Hartmann interferometry of transmitted modes:** Full reconstruction of the Hamiltonian requires both the intensity and the phase of the eigenmodes. Employing a high-resolution Shack–Hartmann interferometer...
(or interfering the transmitted light with a plane-wave of the same frequency) will enable direct extraction of the complex mode-functions.

- **Time dynamics:** Together, the complex eigenmodes and their energies provide complete information about the photonic Hamiltonian. Nonetheless, the spectroscopic approach may not provide the highest signal-to-noise ratio for extracting disorder potentials; temporal dynamics of the photon within the resonator-induced Hamiltonian would contribute dramatically to this objective. To this end, we propose to locally excite the resonator with a focused, temporally short probe pulse. Here ‘short’ means fast compared with the transverse dynamics, but slow compared with the longitudinal mode spacing. The temporal dynamics of the transmitted field may be observed with a small, high-speed photodiode whose location is scanned over many repetitions of the experiment to extract both spatial and temporal dynamics.

8. Conclusion

In this paper, we have harnessed the fact that an optical resonator may be viewed as a periodic drive applied to a 2D optical field to develop a Hamiltonian formalism for understanding photonic dynamics in such resonators. This approach applies both within the paraxial, quadratic approximation, where it results in arbitrary quadratic Hamiltonians tunable through resonator geometry and to perturbations which extend beyond the paraxial limit and produce exotic photon traps, dispersions and manifold curvatures. This work points to fascinating studies of wave dynamics on curved manifolds and in conjunction with Rydberg EIT to induce interactions between photons, an exciting route to strongly correlated photonic quantum materials, including those in the presence of synthetic gauge fields and manifold curvature.

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Appendix A. Mapping Hamiltonians to ray matrices

Here we show that every stable resonator can be described with a quadratic Hamiltonian.

We begin with optical resonator characterized by a round trip ray matrix \( M \); the ray vector after \( n \) round trips is \( V_n \) and \( V_{n+1} = MV_n \), where \( V \equiv \begin{pmatrix} x \\ z \end{pmatrix} \) as defined in the main text. If instead of working with the ray-slope \( s \), we would like to consider its transverse momentum \( p = \hbar ks \), we need employ a new round-trip matrix \( B = \beta M \beta^{-1} \) and \( \mu_{n+1} = \beta V_n \), where \( \beta \) is given in the main text and:

\[
\mu_{n+1} = B \mu_n. \tag{A.1}
\]

We want to find a differential equation that interpolates (A.1) to continuous times,

\[
\frac{d\mu}{dr} = Q\mu. \tag{A.2}
\]

and furthermore, we would like to find a Hamiltonian that generates (A.2). For this to be possible, \( Q \) must be a hamiltonian matrix, i.e. \( GQ \) must be symmetric [37]. We show below that such a \( Q \) always exists if \( B \) describes a stable optical resonator.

To show that a \( Q \) exists we observe that the solution to (A.2) is \( \mu(t + T) = \exp(Qt)\mu(t) \). Therefore it suffices to find a hamiltonian matrix \( Q \) such that \( \exp(QT_n) = B \).

We use the properties of \( B \) (derived by Habraken and Nienhuis [31]) to show that such a \( Q \) exists:

1. \( B \) is symplectic, \( B^T G B = G \), because it is a product of symplectic matrices. Therefore \( \det(B) = 1 \) (property of symplectic matrices).

2. \( B \) describes a stable resonator so the eigenvalues have unit modulus.

3. \( B \) is real. Therefore, if \( \lambda \) is an eigenvalue, so is \( \lambda^* \).

These conditions imply that \( B \) has a logarithm \( A \) and that, moreover, \( A \) can be chosen to be real and hamiltonian (proof below). Now we have a solution: \( Q = A/T_n = \log(B)/T_n \).

The Hamiltonian is then \( H = \frac{1}{2} \mu^T G^T Q \mu \), in the sense that Hamilton’s equations give (A.2).
Proof that $A$ is real and hamiltonian: Meyer, Hall and Offin, p 88 [37], proves that that a symplectic matrix has a real hamiltonian logarithm if its negative eigenvalues occur in pairs [38]. The only allowed negative eigenvalue is $-1$ because the resonator is stable. If none of the eigenvalues is $-1$, then we have completed our proof, so we assume at least one is $-1$. The eigenvalues are then $-1, a, b, c$, so $abc = -1$. If $a, b, c$ are all real then they are all $\pm 1$ and it’s clear that $-1$ occurs in pairs. So we assume $a$ is not real. $B$ is real, so $a^2 = a$ is also an eigenvalue, which we call $b$. Then $-1 = |a|^2c = c$. Therefore, in every allowed case, negative eigenvalues occur in pairs and the aforementioned theorem applies, so the logarithm exists and $A$ is hamiltonian and real.

A.1. Deriving a ray matrix from a Hamiltonian

Given an arbitrary quadratic Hamiltonian $H = \frac{1}{2} \mu^T G^T Q \mu$, is there an optical resonator that implements $H$? Yes, but it is not always a stable resonator.

To prove this, we assume without loss of generality that $Q$ is hamiltonian. The exponential of a hamiltonian matrix is symplectic, so $\exp(\epsilon H)$ is always symplectic and therefore it is a valid ray matrix.

A.2. Stability

When does $\exp(\epsilon H)$ describe a stable optical resonator? The Hamiltonian function $H$ must be positive definite or negative definite in order to map to a stable resonator [37]. Physically, if the particle has positive (negative) mass, it also has to be in a harmonic trap with a local minimum (maximum).

Note: positive (negative) definiteness of the Hamiltonian is equivalent to the eigenvalues of $G^T Q$ being all positive (negative). This translates into a condition on $Q$: the eigenvalues of $Q$ must be purely imaginary and non-zero and $Q$ must be diagonalizable.

Appendix B. Two mirror resonator out of the focal plane

We now fully analyze the behavior of a symmetric two-mirror resonator in a plane other than its central (focal) plane. We will work this out backwards first, using knowledge of the resonator eigenmodes and scalar diffraction theory and then applying the full machinery of the Floquet formalism.

Clearly the eigen-energies of the resonator cannot change (since the eigenstates of the paraxial wave equation are solutions over the full 3D resonator). Furthermore, we know from scalar diffraction theory [30] that the impact of diffraction on the mode-functions is (1) a radial rescaling according to $w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_l}\right)^2}$; (2) a quadratic wavefront curvature of the form $e^{-\frac{1}{2} \frac{(z-w)^2}{w^2}}$, for $R(z) \equiv z \left[1 + \left(\frac{z}{z_l}\right)^2\right]$; and (3) a mode-dependent Gouy phase shift $\zeta_n(z) \equiv n \cdot \tan^{-1} \frac{z}{z_l}$. Here the Rayleigh range is defined by $z_l \equiv \frac{2w_0^2}{\lambda}$.

The Gouy phase may be gauged away through a trivial pre-factor on the wavefunction and we are left with a Hamiltonian system with uniformly spaced eigenvalues and Hermite–Gauss eigenfunctions $\psi_n = \frac{1}{\sqrt{\sqrt{2\pi} \sigma_{w(z)} w(z)}} e^{-\frac{(z-w)^2}{w^2}} H_n \left(\frac{\sqrt{2} z}{w(z)}\right)$.

The question then, is what Hamiltonian has these mode-functions? We would know a (quantum-harmonic oscillator Hamiltonian $H_{QHO} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$, were it not for the wave-front curvature term. We can remove this term by a unitary transformation $U = e^{-\frac{1}{2} w^2}$. The resulting Hamiltonian is $H = U H_{QHO} U^\dagger = \frac{(p + i k s / R(z))^2}{2m} + \frac{1}{2} m \omega^2 q^2$.

We now work forwards, arriving at this Hamiltonian via Floquet techniques. The round-trip ray-matrix for the same two-mirror symmetric Fabry–Pérot resonator considered previously, but for a plane located at $z = \frac{z_L}{2}$ from the resonator focal plane, is:

$$M_{\text{non focal}} = \begin{pmatrix} -1 + 2g(0 - (1 - g)) & gR(1 - g^2 + (1 - g)^2 e^{2\epsilon}) \\ -\frac{4g}{R} & -1 + 2g^2 + 2ge(1 - g) \end{pmatrix}. \quad (B.1)$$

A bit of arithmetic yields:

$$Q_{\text{non focal}} = Q_{\text{focal}} + \frac{c}{2L} \epsilon \frac{e^{2\epsilon}}{1 + e^{2\epsilon}} \begin{pmatrix} 1 - \frac{g}{1 + g} & (1 - g) e^{2\epsilon} \frac{R}{2\hbar} \\ 1 & -1 \end{pmatrix}. \quad (B.2)$$

The off-diagonal correction term in $Q_{\text{non focal}}$ modifies the photon mass, as it impacts only $\frac{\partial^2 H}{\partial \phi \partial \phi}$; it corresponds to a change in the mode-waist due to diffraction. More interesting are the diagonal corrections to $Q_{\text{non focal}}$: $Q_{\text{focal}}$ lacks any such terms, which correspond to a $\frac{\partial^2 H}{\partial \phi \partial \phi}$ term and reflect a term in the Hamiltonian...
proportional to $x_p$. Following the calculation through, we arrive at: $H = \frac{(p + bx)^2}{2m} + \frac{1}{2}m\omega^2x^2$, where $\omega = \frac{c}{2l}$, 

$m = \frac{\hbar k}{2(\hbar R_0)} \int \frac{1}{1 + \frac{1}{\hbar^2}k^2} b = \frac{2\hbar k}{(1 + g)k + (1 - g)k^2} = \frac{\hbar k}{2R(z)}$ for $R(z)$ defined as above. This expression coincides with our expectation from the paraxial wave equation and indeed, the trap frequency does not depend upon defocus.

In short: outside of the focal plane of the resonator, the canonical momentum of the ray (corresponding to its slope as it propagates along the cavity axis) is no longer proportional to the mechanical momentum of the ray (the rate at which it moves in the 2D transverse plane under consideration). Instead, there is an additive correction which is linear in the position, reflecting the wave-front curvature.

### Appendix C. Formal treatment of aberrations

Here, we perform the simplest version of the calculation of the impact of aberration on a the cavity Floquet Hamiltonian: an arbitrary lens a distance $z$ from our Floquet plane, in one transverse dimension. More sophisticated calculations in two transverse dimensions with an arbitrary ABCD matrix in-between, are simply extensions of this technique.

Consider the Hamiltonian for an arbitrary lens in the plane $z$, which produces a round-trip phase-shift of $\alpha(x)$. We can compute its expansion in the plane at $z = 0$ by inserting identity operators:

$$H_{\text{lens}} = \frac{\hbar c}{L_{\text{rt}}} \alpha(\mathbf{x}; z) = \frac{\hbar c}{L_{\text{rt}}} \int \alpha(x)|x; z\rangle \langle x; z| dx$$

$$= \frac{\hbar c}{L_{\text{rt}}} \int \alpha(x)|x; z = 0\rangle \langle x; z = 0| dx \; dx_i \; dx_j.$$ (C.1)

We now relate localized excitations in the different planes via the free-space Green-function in the paraxial (Fresnel) approximation [39, 40]: $(\langle x; z = 0|x; z \rangle = \frac{e^{i\mathbf{k} \cdot \mathbf{z}}}{\sqrt{4\pi z}} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})}$. 

$$H_{\text{lens}} = \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\lambda z} \int \alpha(x) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} e^{-i\mathbf{k} \cdot \mathbf{z}} |x; z = 0\rangle \langle x; z = 0| dx \; dx_i \; dx_j.$$ (C.2)

redefining $x_j \rightarrow x_j + x$ we have:

$$H_{\text{lens}} = \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\lambda z} \int dx_j \; dx_i \; dx e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} \int dx \alpha(x)|x + x_j; z = 0\rangle \langle x + x_j; z = 0|$$ (C.3)

and identifying:

$$\int dx \; |x + x_j; z = 0\rangle \langle x + x_j; z = 0| = e^{\mathbf{q}_x \cdot \mathbf{x}} e^{-i\mathbf{q}_x \cdot \mathbf{x}} = e^{i\mathbf{q}_x \cdot \mathbf{x}} = e^{i\mathbf{q}_x \cdot \mathbf{x}} = e^{i\mathbf{q}_x \cdot \mathbf{x}}$$ (C.4)

yields:

$$H_{\text{lens}} = \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\lambda z} \int dx_j \; dx_i \; dx e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} \alpha(x) = \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\lambda z} \int dx_j \; dx_i \; dx e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} \alpha(x)$$ (C.5)

Performing the $x_j$ integral yields:

$$H_{\text{lens}} = \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\lambda z} \int dx_j \; dx_i \; dx e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} \alpha(x + x_j)$$

$$= \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\sqrt{-i\lambda z}} \int dx_j \; dx_i \; dx e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} \alpha(x + x_j) e^{i\mathbf{q}_x \cdot \mathbf{x}}$$

$$= \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\sqrt{-i\lambda z}} \int dx_j \; dx_i \; dx e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})} \alpha(x + x_j) e^{i\mathbf{q}_x \cdot \mathbf{x}}$$

$$= \frac{\hbar c}{L_{\text{rt}}} \frac{1}{\sqrt{-i\lambda z}} \alpha(\mathbf{x} + \frac{\mathbf{q}_x}{\hbar k}) \int dx_j \; dx_i \; dx e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{z})}$$

$$= \frac{\hbar c}{L_{\text{rt}}} \alpha(\mathbf{x} + \frac{\mathbf{q}_x}{\hbar k}).$$ (C.6)

Where we have (1) inserted an identity operator; (2) required that $\alpha(x)$ be analytic; (3) used the Baker–Campbell–Hausdorf formula; and (4) performed the remaining Gaussian integration. In two transverse dimensions it may be shown that an arbitrary lens produces a correction to the Hamiltonian: $\frac{\hbar c}{L_{\text{rt}}} \alpha(\mathbf{x} + \frac{\mathbf{q}_x}{\hbar k})$.

One might be inclined to attempt to draw a parallel to a Trotterized cold atom implementation, where atoms are allowed to evolve in a harmonic trap and then a quarter of a trap period later, when they are in momentum space, an optical potential is briefly applied to them with the hope that it would provide a effective momentum-dependent force (when viewed another quarter trap cycle later). A simple calculation reveals that this does not
work, because the atoms evolve back to real space in a way that depends upon the optical potential applied to
them. The key to this idea working for photons where it fails for atoms is that the potential may be weak enough
that it does not appreciably impact the photons within a single round-trip (it does not mix different Floquet/
longitudinal manifolds), but is still strong enough to substantially change the transverse dynamics within a single
near-degenerate Floquet manifold; in short, the cavity photons live simultaneously in real space, momentum
space and everywhere in-between.

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