QUANTUM MARKOV PROCESSES ON GRAPHS, MONITORING AND HITTING TIMES

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Abstract. We make use of matrix representations of completely positive maps in order to study open quantum dynamics on graphs, with emphasis on quantum walks and the associated trajectories obtained via a monitoring of the position. We discuss the discrete and continuous time setting and consider the statistics of particles located on a vertex with some internal degree of freedom. Other classes of semigroup generators, such as graph-induced generators are also considered. In the case of primitive quantum channels we present expressions of the mean hitting time for a particle to reach a vertex in terms of a quantum version of the fundamental matrix, thus extending recent results in the setting of open quantum random walks. An open quantum version of Kac’s Lemma for the expected return time is discussed. By considering appropriate block matrix expressions we are able to make computations in terms of matrix-valued transition rates, in analogy with the notion of rates from classical continuous-time Markov chains. Also inspired by the classical case a recurrence criterion for the continuous time open walk setting is presented.

Keywords: quantum walk; Markov chains; completely positive map; mean hitting times; recurrence; fundamental matrix; generating function; Kac’s Lemma.

1. Introduction

The problem of estimating the times of first visit and return in quantum settings has been addressed in recent years, this being in part motivated by the study of quantum walks [34] and related topics in quantum information theory. One notices that there are several quantum notions of a first visit, each with its own features [18, 26, 33], so establishing a consistent mathematical framework where problems on quantum dynamics can be posed properly is an important first step.

On one hand, it is certainly true that many questions on hitting times in quantum contexts are inspired by the corresponding problem in the setting of Markov chains [7, 26]. In the classical context there are well-known techniques which can be used to obtain concrete answers to particular problems and also general results in terms of matrix analysis, probability theory and orthogonal polynomials [17]. In attempting to obtain quantum versions of classical statistical notions we not only make use of an analogous set of tools but also take in consideration the so-called internal degrees of freedom of the particles considered (e.g. spin). As a consequence, the dynamical evolution will be described in terms of objects which are distinct from stochastic matrices. In our case we will consider completely positive maps acting on finite-dimensional Hilbert spaces, central objects in quantum information theory and such that the study of similarities and differences with the classical setting are a motivating factor [31, 32].

In this work we present open quantum versions of certain results on hitting times for Markov chains with the goal of proposing a basic setting for further developments in the field of open quantum dynamics, most particularly quantum random walks. We describe discrete and continuous time dissipative dynamics and there the model of Open Quantum Random Walks (OQWs) on graphs [3, 28], which we briefly review, will be our starting point. The OQW model, inspired by the quantum Markov chain description due to S. Gudder [15], is such that one keeps track of the position and some internal degree of the particle. We will also describe results which are not in this context, for instance, semigroup generators induced by the underlying graph so in such setting we keep track of the position of the particle, but not of internal degrees.

In particular, regarding the first visit of a particle moving along the vertices of some graph we will consider quantum trajectories [19] associated with a monitoring procedure of the position [6, 13, 33], so one has a formal setting where probabilities of various events can be obtained. It is also seen that such constructions can be adapted to both unitary and dissipative (open) evolutions [14, 22].

In this work we discuss the following: after a review of OQWs, quantum trajectories and the monitoring procedure in Section 2 we recall the mean hitting time formula (MHTF) from Markov chain theory [1] and then study an open quantum generalization for finite primitive OQWs, thus extending a recent result presented in [20]. This is done in Section 3, where we also prove a second hitting time formula regarding an initial vertex chosen randomly. In Section 4 we discuss a result on Kac’s Lemma for the expected return time in an open quantum context, also as a complement for a later discussion. In Section 5 we present a discussion on continuous time quantum dynamics. We recall basic facts and study
OQW versions of these constructions. In addition, motivated by [33] we obtain a formula for mean hitting time in terms of continuous time processes for which jumps are dictated by a fixed Poisson rate. In the mentioned work the authors consider unitary dynamics and here we discuss a simple adaptation for the open quantum setting. In Section 6 we discuss a connection between the approach in [33] and the one made in Section 5. In Section 7 we discuss a recurrence criterion in the continuous time OQW setting. We conclude with an outlook on future directions and an open question regarding the unitary case.

2. Preliminaries

2.1. Matrix notations and CP maps. We refer the reader to [5] for more on the matrix theory review here. Let $M_d = M_d(\mathbb{C})$ denote the algebra of order $d$ complex matrices. A hermitian matrix $A : \mathbb{C}^d \to \mathbb{C}^d$ is positive semidefinite, denoted by $A \geq 0$, iff $\langle Av, v \rangle \geq 0$, for all $v \in \mathbb{C}^d$. We say $\rho \in M_d(\mathbb{C})$ is a density matrix if $\rho \geq 0$ and $tr(\rho) = 1$. Let $\Phi : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ be linear. We say $\Phi$ is a positive operator whenever $A \geq 0$ implies $\Phi(A) \geq 0$. Define for each $k \geq 1$, $\Phi_k : M_k(M_n(\mathbb{C})) \to M_k(M_n(\mathbb{C}))$. $\Phi_k(A) = [\Phi(A_{ij})]$. $A \in M_k(M_n(\mathbb{C}))$, $A_{ij} \in M_n(\mathbb{C})$. We say $\Phi$ is $k$-positive if $\Phi_k$ is positive, and we say it is completely positive (CP) if $\Phi_k$ is positive for every $k = 1, 2, \ldots$. It is well-known that a map $\Phi$ is CP if and only if it can be written in the Kraus form

$$\Phi(\rho) = \sum_i V_i \rho V_i^*$$

where the $V_i$ are linear operators (Kraus matrices). We say $\Phi$ is trace-preserving if $Tr(\Phi(\rho)) = Tr(\rho)$ for all $\rho \in M_d(\mathbb{C})$, which is equivalent to $\sum_i V_i^* V_i = I$. We say $\Phi$ is unital if $\Phi(I) = I$, which is equivalent to $\sum_i V_i^* V_i = I$. In this work we are interested in completely positive trace-preserving (CPT) maps (also called quantum channels) acting on a finite dimensional space.

If $A \in M_n(\mathbb{C})$, the corresponding vector representation $vec(A)$ associated to it is given by stacking together the matrix rows. For instance, if $n = 2$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow vec(A) = [a_{11} \ a_{12} \ a_{21} \ a_{22}]^T$$

The vec mapping satisfies $vec(AXB^T) = (A \otimes B)vec(X)$ for any $A, B, X$ square matrices so, in particular, $vec(BXB^*) = vec(BX^T) = (B \otimes B^*)vec(X)$. Then we can obtain the matrix representation for the map $\Phi$ [16]:

$$[\Phi] := \sum_i V_i \otimes V_i \implies \Phi(X) = vec^{-1}(vec(A)vec(X))$$

We will also use the notation $[A] := A \otimes \overline{A}$ for any $A \in M_n(\mathbb{C})$ and let

$$A^v := vec(A), \quad Tr^v(A^v) := Tr(vec^{-1}(A^v))$$

for the vector form of a matrix. We will often omit the superscript $v$ when calculating certain traces, but it will always be clear from the context the calculation that should be performed: if the term inside the trace is a vector, this should be devectorized in order to calculate such trace.

2.2. Open quantum walks and its trajectories. Let $\{B_{ij}\}_{i,j=1,\ldots,k}$ belong to $M_n(\mathbb{C})$ such that

$$\sum_{i=1}^{k} B_{ij} = I, \quad j = 1, \ldots, k$$

where $I = I_n$ denotes the order $n$ identity matrix. We say that $B_{ij}$ is the effect matrix of transition from site $j$ to site $i$ (matrix indices are always read from right to left in this work), that $k$ is the number of sites (also called vertices in this work) and $n$ is the degree of freedom on each site. Let

$$\rho := \sum_{i=1}^{k} \rho_i \otimes |i\rangle\langle i|, \quad \rho_i \in M_n(\mathbb{C}), \quad \rho_i \geq 0, \quad \sum_{i=1}^{k} Tr(\rho_i) = 1$$

For a given initial density matrix of such form, the Open Quantum Random Walk (OQW) on $k$ sites induced by the $B_{ij}$, $i,j = 1, \ldots, k$ satisfying (2.5) is, by definition [3], the map

$$\Phi(\rho) := \sum_{i=1}^{k} \left( \sum_{j=1}^{k} B_{ij} \rho_j B_{ij}^* \right) \otimes |i\rangle\langle i|$$

We note that we can also write $\Phi(\rho) = \sum_{i,j} M_{ij} \rho M_{ij}^*$, where $M_{ij} = B_{ij} \otimes |i\rangle\langle j|$. In this way we have a description of the Hilbert space as the tensor product of two parts, one associated with the internal degree of the particle and another
with a so-called position on the graph. This interpretation is a convenient one and guides us in numerous analogies and comparisons with classical random walks on graphs. We recall that density matrices of the form (2.6) are preserved under the action of OQWs: given one such density we have that $\Phi(\rho)$ is also a summation of terms of the form $\eta_i \otimes |i⟩⟨i|$, $\eta_i \geq 0$, $\sum_i Tr(\eta_i) = 1$, see [3]. In particular, the sites $i$ (also denoted by $|i⟩$) serve as an index for the entries in a given vector and so we can identify the domain of the OQW with a direct sum space. If the state of the chain at time $n$ is $\rho(n) = \rho \otimes |j⟩⟨j|$ then at time $n+1$ it jumps to

$$\rho(n+1) = \frac{B_{ij}\rho B^*_{ij}}{p(j,i)} \otimes |i⟩⟨i|, \quad i \in \mathbb{Z}$$

with probability

$$p(j,i) = Tr(B_{ij}\rho B^*_{ij})$$

This is a well-known transition rule seen in quantum mechanics, which depends on a density matrix. In probability notation, we have a homogeneous Markov chain $(\rho_n, X_n)$ with values in $D_n \times \{1, \ldots, k\}$ (continuous state, discrete time), where $D_n$ consists of the set of density matrices on $M_n(\mathbb{C})$ (see [3, 21]), satisfying: from any position $(\rho, j)$ one jumps to

$$\left( \frac{B_{ij}\rho B^*_{ij}}{p(j,i)}, i \right)$$

with probability given by (2.9). This is the quantum trajectories formalism of OQWs. Informally, we have an open quantum process for which we perform measurements at each time step, that is, we have a monitored procedure. Also it is worth noting that the dynamics of OQWs are quite different from the usual (closed) quantum random walks, we refer the reader to [26] and [34] for more on this kind of walk.

Remark 2.1. It should be clear that the theory of OQWs is not simply a rewriting of Markov chain theory in terms of block matrices. Although $(\rho_n, X_n)_{n \in \mathbb{N}}$ is in fact a Markov chain, we are often interested in facts which do not follow immediately from such property and which do not have a classical correspondent. For instance, the sequence of positions $(X_n)$ alone is not a Markov chain (unless we consider order 1 density matrices). This has implications, for instance, on the problem of site recurrence for OQWs (see e.g. [4, 11]) and also on the problem of first visit to a vertex, which is studied in this work. Because of this, we have that, in general, a Markov chain-related proof is not automatically an OQW proof.

Note that the spectrum of the channel is given by the corresponding information extracted from $[\Phi]$ and this will be useful throughout this work. Moreover, we have a graph visualization and a block matrix representation of the OQW action. Consider for instance the example of 3 vertices, the general case being examined in a similar manner.

Define the block matrix representation $\hat{\Phi}$:

$$(2.11) \quad \hat{\Phi} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Note that $[\Phi]$ and $\hat{\Phi}$ are distinct representations, and the latter makes clear that we have the correspondence

$$(2.12) \quad \rho = \sum_i \rho_i \otimes |i⟩⟨i| \mapsto \Phi(\rho) \leftrightarrow \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} \mapsto \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} B_{11}\rho_1 B^*_{11} + B_{12}\rho_2 B^*_{12} + B_{13}\rho_3 B^*_{13} \\ B_{21}\rho_1 B^*_{21} + B_{22}\rho_2 B^*_{22} + B_{23}\rho_3 B^*_{23} \\ B_{31}\rho_1 B^*_{31} + B_{32}\rho_2 B^*_{32} + B_{33}\rho_3 B^*_{33} \end{bmatrix}$$
and the spectrum of the OQW is readily obtained from \( \hat{\Phi} \) as well. If we demand that \( \Phi \) is unital then, in addition to trace preservation \( \sum_{i=1}^{k} B_{ij}^* B_{ij} = I \), all \( j \), we also must have

\[
\sum_{j=1}^{3} B_{ij} B_{ij}^* = I, \quad i = 1, 2, 3
\]

(2.13)

We have the analogous facts for an arbitrary number of vertices and we can also construct the matrix in a row stochastic-like fashion, the adaptations being straightforward.

We refer the reader to [8,25,28] and references therein for more on discrete time OQWs. Regarding continuous time OQWs, see [8,25,28] the latter being the basic description in terms of master equations.

2.3. Monitoring of quantum evolutions. The idea behind monitoring is motivated by a classical reasoning. Let \( P = (p_{ij}) \) be a (column) stochastic matrix and let \( p_{ij}^{(n)} = (P^n)_{ij} \), the \( (i,j) \)-th entry of matrix \( P \). This number corresponds to the probability of reaching vertex \( i \) at the \( n \)-th step given that at time 0 the walk was at vertex \( j \). By a matrix computation we see that we are summing the probabilities of all possible paths from \( j \) to \( i \), noting that for some paths it may happen that one reaches \( i \) before the \( n \)-th step as well.

Now suppose one is interested in calculating \( f_{ij}^{(n)} \), the probability that beginning at vertex \( j \) at time 0, one reaches \( i \) for the first time at the \( n \)-th step. That is, we do not consider paths that reach \( i \) before the \( n \)-th step. This is calculated in the following manner: let \( P_i \) be the projection matrix on vertex \( i \), and let \( Q = I - P_i \) be its complement. Then it is clear that

\[
f_{ij}^{(n)} = (i,j)\text{-th entry of } P_i P(Q,P)^{n-1} P_j
\]

(2.14)

Read from right to left we see that the computation means: begin at vertex \( j \); evolve one step, discard paths reaching vertex \( i \) and repeat such process \( n - 1 \) times; evolve one more step and then project onto vertex \( i \). This reasoning in terms of projections onto a space of interest and its complement is called a monitoring procedure, and has been employed in [33] so that hitting times could be studied, in [6,13] regarding the recurrence of unitary iterations and in [21,22,11] where the authors consider monitoring of OQWs. Furthermore, in [14] the authors consider generating functions and extend this monitoring formalism in a way that allows them to study Schur functions, FR-functions and associated splitting rules in the setting of random walks, unitary walks, open quantum walks and, more generally, contractions on Banach spaces.

Regarding the general problem of monitoring the time of first visit of a quantum dynamics, we highlight two cases:

(1) **First visit/return to a vertex.** Given an initial density \( \rho \otimes |i⟩⟨i| \) on vertex \( |i⟩ \), what is the first time it reaches vertex \( |j⟩ \), regardless of its accompanying state? This is the simplest problem of first visit, for which the projections are the one associated to the vertices alone, so that any internal state is acceptable (see [21] and the references above).

(2) **First visit/return to a state.** Given an initial state \( \rho \otimes |j⟩⟨j| \) on site \( |i⟩ \), will it ever evolve to some other state \( \eta \otimes |j⟩⟨j| \) on site \( |j⟩ \)? Recently the recurrence problem of states has been studied in the context of unitary evolutions with the monitoring formalism presented here (see [13] and the references above).

In this work we will focus on the first visit/return to vertices in the OQW setting, but state visit is considered in the setting of generators over \( M_n(\mathbb{C}) \) as well. The problem of state visit/recurrence for OQWs will be studied in a future work.

In mathematical terms we see a distinction between the notions of performing a measurement on a quantum system and the one of monitoring (a position, for instance): the former is associated with the square modulus of a summation of amplitudes, while the latter is associated with orthogonal projections. Both notions can be combined in calculations as illustrated by the results just mentioned and below we exemplify some of these constructions.

**Example 2.2.** Consider the nonunital OQW \( \Phi \) on two sites determined by

\[
B_{11} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_{21} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_{12} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_{22} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\]

(2.15)

Then the matrix representations \( [B_{ij}] \) are order 4 matrices and the block matrix representation for \( \Phi \) becomes the order 8 block matrix

\[
\tilde{\Phi} = \begin{bmatrix} [B_{11}] & [B_{12}] \\ [B_{21}] & [B_{22}] \end{bmatrix}
\]

(2.16)

Let \( P_i \) denote the projection on (the space generated by) vertex \( |i⟩ \),

\[
P_1 = \begin{bmatrix} I_4 & 0_4 \\ 0_4 & 0_4 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0_4 & 0_4 \\ 0_4 & I_4 \end{bmatrix}, \quad Q_i = I - P_i, \quad i = 1, 2
\]

(2.17)
where $I_4$ and $0_4$ denote the order 4 identity and zero matrices, respectively. Then, for instance, by letting $\rho = \rho_1 \otimes |1\rangle\langle 1|$ we have that $Tr^v(\mathbb{P}_2 \tilde{\Phi} vec(\rho)) = Tr(B_{21} \rho_1 B_{21}^t)$ denotes the probability that the walk will be at vertex $|2\rangle$ after one step. Also, write the first visit functions

$$F_{21}(z) = \sum_{n=0}^{\infty} \mathbb{P}_2 \tilde{\Phi} (z\mathbb{Q}_2 \tilde{\Phi})^n \mathbb{P}_1 = \mathbb{P}_2 \tilde{\Phi} (I - z\mathbb{Q}_2 \tilde{\Phi})^{-1} \mathbb{P}_1,$$  \hspace{0.1cm} |z| < 1 \hspace{0.1cm} (2.18)

which is a generating function for the first visit to vertex $|2\rangle$, given an initial state concentrated at vertex $|1\rangle$. Then, the probability that the walk will ever reach vertex $|2\rangle$ (with any accompanying density), given an initial density $\rho = \rho_1 \otimes |1\rangle\langle 1|$, is calculated as

$$Pr(\rho \otimes |1\rangle\langle 1| \rightarrow |2\rangle\langle 2|) = Tr^v(F_{21}(1) vec(\rho))$$  \hspace{0.1cm} (2.19)

where we write $F_{ij}(1) := \lim_{z \to 1} F_{ij}(z)$. Calculations with (2.15) produces that

$$F_{21}(z) = \begin{bmatrix} 0_4 & 0_4 \\ f_{21}(z) & 0_4 \end{bmatrix}, \hspace{0.1cm} f_{21}(z) = \frac{1}{2-z} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 - z & 1 - z & 1 \end{bmatrix}$$  \hspace{0.1cm} (2.20)

and so $Tr^v(F_{21}(1) vec(\rho)) = 1$ for every density $\rho$. Similarly, the mean hitting time operator can be obtained from the generating function

$$G_{21}(z) = \mathbb{P}_2 \tilde{\Phi} (I - z\mathbb{Q}_2 \tilde{\Phi})^{-2} \mathbb{P}_1$$  \hspace{0.1cm} (2.21)

from which the mean hitting time from $|1\rangle$ to $|2\rangle$ is given by $E(\rho \otimes |1\rangle\langle 1| \rightarrow |2\rangle\langle 2|) = Tr^v(G_{21}(1) vec(\rho)) = 2(1 + Re(\rho_{12}))$, where $\rho = (\rho_{ij})_{i,j=1,2}$. This example will be further developed in Section 3.

3. Mean hitting time formulae, discrete case

The discussion on hitting probabilities from the previous section can be further formalized in the following equivalent way. In the context of OQWs this has been first described in \[22\].

**Definition 3.1.** The probability of first visit to site $i$ at time $r$, starting at $\rho_j \otimes |j\rangle\langle j|$ is denoted by $b_r(\rho_j; i)$. This is the sum of the traces of all paths starting at $\rho_j \otimes |j\rangle\langle j|$ and reaching $i$ for the first time at the $r$-th step. The probability starting from $\rho_j \otimes |j\rangle\langle j|$ that the walk ever hits site $i$ is

$$h_{ij}(\rho_j) := \sum_{r=0}^{\infty} b_r(\rho_j; i), \hspace{0.1cm} i \neq j$$  \hspace{0.1cm} (3.1)

and $h_{jj}(\rho_j) := 1$. This is the probability of visiting site $i$, given that the walk started at site $j$. For fixed initial state and final site, the mean hitting time is

$$k_{ij}(\rho_j) := \sum_{r=1}^{\infty} r \hspace{0.1cm} b_r(\rho_j; i)$$  \hspace{0.1cm} (3.2)

Let $\pi(i \leftarrow j; r)$ be the set of all products of matrices corresponding to the sequences of sites that a walk is allowed to perform with $\Phi$, beginning at site $|j\rangle$, first reaching site $|i\rangle$ in $r$ steps. For instance, reading matrices and indices from right to left, we have $B_{43}B_{32}B_{21}B_{12}B_{21} \in \pi(4 \leftarrow 1; 5)$ as this corresponds to moving right, moving left and then moving right 3 times. Let $\pi(i \leftarrow j) = \cup_{r=1}^{\infty} \pi(i \leftarrow j; r)$. Then

$$h_{ij}(\rho_j) = \sum_{r=0}^{\infty} b_r(\rho_j; i) = \sum_{r=0}^{\infty} \sum_{C \in \pi(i \leftarrow j; r)} Tr(C \rho_j C^*) = \sum_{C \in \pi(i \leftarrow j)} Tr(C \rho_j C^*)$$  \hspace{0.1cm} (3.3)

If $C \in M_n(\mathbb{C})$, define $M_C : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, $M_C(X) := CXC^*$, $X \in M_n(\mathbb{C})$. Then, for $k = 2$, let

$$\hat{H} = \begin{bmatrix} \hat{h}_{11} & \hat{h}_{12} \\ \hat{h}_{21} & \hat{h}_{22} \end{bmatrix} := \begin{bmatrix} \sum_{C \in \pi(1 \leftarrow 2)} [M_C] & \sum_{C \in \pi(1 \leftarrow 2)} [M_C] \\ \sum_{C \in \pi(2 \leftarrow 1)} [M_C] & \sum_{C \in \pi(2 \leftarrow 1)} [M_C] \end{bmatrix}$$  \hspace{0.1cm} (3.4)

The hat notation $\hat{h}_{ij}$ means that we are referring to the block matrix in position $(i,j)$ instead of the $(i,j)$-th numerical entry of $\hat{H}$ and an analogous convention is used throughout this work. We give the analogous definition for $k > 2$ sites.
In words, \( \hat{h}_{ij} \) denotes the map associated with summing all possible ways of going from \(|j]\) to \(|i]\). Then for every density \( \rho_j \) we have \( h_{ij}(\rho_j) = \Tr(\hat{h}_{ij}\rho_j) \). In a similar way as in the definition for \( \hat{H} \), define the mean hitting time operator

\[
\hat{K} = \begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} \\
\hat{k}_{21} & \hat{k}_{22} \end{bmatrix} := \sum_{r} \sum_{C \in \pi(1 \cdots r)} [M_C] \sum_{r} \sum_{C \in \pi(1 \cdots 2r)} [M_C] \sum_{r} \sum_{C \in \pi(1 \cdots 2r)} [M_C]
\]

from which \( k_{ij}(\rho_j) = \Tr(\hat{k}_{ij}\rho_j) \), \( i \neq j \), noting that \( \hat{k}_{ii} \) is distinct from \( k_{ii}(\rho) = 0 \) for every \( \rho \) density (the latter begins counting visits at time zero). We give the analogous definition for \( k > 2 \) sites.

Let \( G \) be some fixed, finite graph and let \( \mathcal{C} \) denote the set of primitive (sometimes called ergodic) channels acting on \( G \), that is, the set of \( \Phi \) such that its iterates converge to some channel. We write \( \Omega = \Omega_\Phi = \lim_{n \to \infty} \Phi^n \). As we are considering finite graphs the convergence can be specified, for instance, as entrywise convergence of the block matrix representation \( \hat{\Phi} \). For every \( \Phi \in \mathcal{C} \) acting on a finite collection of sites, define

\[
\hat{\mathcal{Z}} := \hat{I} + \sum_{r \geq 1} (\Phi^r - \hat{\Omega}) = (\hat{I} - \hat{\Phi} + \hat{\Omega})^{-1}
\]

See also [31] for a related discussion on this map in the context of quantum channels. We state the following result, which does not assume unitality, thus extending the result in [20]. The proof is a simple variation of the result seen in the mentioned work, and it is briefly outlined in the Appendix for convenience of the reader.

**Theorem 3.2.** (First Mean Hitting Time Formula for OQWs). Let \( \Phi \in \mathcal{C} \) denote a finite ergodic OQW with \( k \geq 2 \) sites and let \( \hat{\mathcal{Z}} \) denote its fundamental matrix. Let \( \hat{D} \) be the diagonal matrix operator with diagonal entries \( \hat{k}_{ii} \) and let \( \hat{N} := \hat{K} - \hat{D} \). Then for every \( \rho \) density matrix, for all \( i, j = 1, \ldots, k \),

\[
\Tr(\hat{N}_{ij}\rho^n) = \Tr((\hat{D}\hat{\mathcal{Z}})_{ij} - (\hat{D}\hat{\mathcal{Z}})_{ij}\rho^n)
\]

Equation (3.7) is to be regarded as a quantum version of the classical result [1, 7]: if \( V_j \) is the time of first visit to vertex \( j \) then

\[
E_i(V_j) = \frac{Z_{jj} - Z_{ij}}{\pi_j}
\]

Now define

\[
F_{ji} := \sum_{i} \hat{h}_{ji} \pi_i^n, \quad N_{ji} := \sum_{i} \hat{k}_{ji} \pi_i^n
\]

so \( \Tr^n(F_{ji}) \) is the probability of ever reaching vertex \( j \) given some initial vertex chosen according to the density \( \pi \) and \( \Tr^n(N_{ji}) \) is the mean hitting time for vertex \( j \), given an initial density randomly chosen according to \( \pi \). These are naturally associated with the classical probabilistic notions given by

\[
E_\pi(X) := \sum_x xP_\pi(X = x), \quad P_\pi(X = x) := \sum_i \pi_i P_i(X = x), \quad P_i(X_{n+1} = j) := P(X_{n+1} = j | X_n = i)
\]

where \( X, X_i \) are any discrete random variables. Then we can state the following:

**Theorem 3.3.** (Second Mean Hitting Time Formula for OQWs). Let \( \Phi \in \mathcal{C} \) denote an ergodic OQW with \( k \geq 2 \) sites, let \( \hat{\mathcal{Z}} \) denote its fundamental matrix and \( \pi = (\pi_i) = \sum_i \pi_i \otimes |i\rangle\langle i| \) the unique stationary density for \( \Phi \). Let \( \hat{D} \) be the diagonal matrix operator with diagonal entries equal to \( \hat{k}_{ii} \). Then

\[
\Tr^n(N_{ji}) = \Tr^n((\hat{D}\hat{\mathcal{Z}})_{jj}F_{ji})
\]

Equation (3.11) is to be regarded as a quantum version of the classical result [1, 23]

\[
E_\pi(V_j) = \frac{Z_{jj}}{\pi_j}
\]

**Proof of Theorem 3.3.** Recall the notation \( A^v \equiv \text{vec}(A) \), for any matrix \( A \). Let, for any vertices \( x, y \),

\[
\hat{P}^n(x,y) := \sum_{C \in \pi(y-x:n)} [C]
\]

where \( [C] \) is the conjugation map associated to a path \( B_{i_{n+1}} \cdots B_{i_2}B_{i_1} \) allowed by the OQW \( \Phi \). Let

\[
\hat{f}_{ij}^k := \sum_i \sum_{C \in \pi_{j,i}(j \leftarrow i; k)} [C]\pi_i^v, \quad F(s) := \sum_{k=0}^\infty \hat{f}_{ij}^k s^k
\]
Thus, the constant sequence equal to \( \pi_j \) is the convolution of the sequence of matrices \( \{ u_k + \hat{\Omega}_{jj}\}_k \) with the sequence of vectors \( \{ \hat{f}_j \}_k \). Then, the generating function of \( \pi_j \) equals the product of the generating functions of \( \{ u_k + \hat{\Omega}_{jj}\}_k \), which is
\[
\sum_{m=0}^{\infty} \left[ u_m + \hat{\Omega}_{jj} \right] s^m = U(s) + \frac{\hat{\Omega}_{jj}}{1-s}
\]
and of \( \{ \hat{f}_j \}_k \) (which is \( F(s) \)). Therefore,
\[
\frac{\pi_j}{1-s} = \sum_{m=0}^{\infty} \pi_j s^m = \left[ U(s) + \frac{\hat{\Omega}_{jj}}{1-s} \right] F(s) = \pi_x = [(1-s)U(s) + \hat{\Omega}_{jj}] F(s)
\]
which also holds for \( s = 1 \). Differentiating with respect to \( s \) at \( s = 1 \) gives
\[
0 = -U(1)F(1) + \hat{\Omega}_{jj} F'(1)
\]
and so
\[
\hat{\Omega}_{jj} F'(1) = U(1)F(1) \Rightarrow \hat{\Omega}_{jj} F'(1) = \hat{Z}_{jj} F_{j\pi}
\]
Apply \( \hat{D}_{jj} \) on the left of both sides and, noting that \( Tr^v(\hat{D}_{jj}\hat{\Omega}_{jj}\rho^v) = Tr(\rho) \) and that \( (\hat{D}\hat{Z})_{jj} = \hat{D}_{jj}\hat{Z}_{jj} \) we obtain, after taking the trace,
\[
Tr^v(N_{j\pi}) = Tr^v((\hat{D}\hat{Z})_{jj} F_{j\pi})
\]
\( \square \)

**Example 3.4.** Consider once again Example 2.2. The fixed density and the limit channel are given by
\[
\pi = \frac{1}{9} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \otimes |1\rangle \langle 1| + \frac{1}{3} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes |2\rangle \langle 2|
\]
\[
\hat{\Omega} = \begin{bmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} \\ \hat{\Omega}_{21} & \hat{\Omega}_{22} \end{bmatrix}, \quad \hat{\Omega}_{11} = \frac{1}{9} \begin{bmatrix} 5 & 0 & 0 & 5 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{\Omega}_{21} = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}
\]
The mean return time functions are
\[
G_{11}(z) = P_1 \hat{\Phi}(I - zQ_1 \hat{\Phi})^{-2} P_1 = \frac{1}{2(z-3)^2} \begin{bmatrix} 9 & 2z^2 - 12z + 9 \\ -z(z-6) & z(z-6) \end{bmatrix} \begin{bmatrix} z(z-6) & z(z-6) \\ -z(z-6) & z(z-6) \end{bmatrix}
\]
\[
G_{22}(z) = P_2 \hat{\Phi}(I - zQ_2 \hat{\Phi})^{-2} P_2 = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{4}{(z-2)^2} & \frac{-2z^3 + 5z^2 + 4z - 4}{(z-2)^2} & \frac{-2z^3 + 5z^2 + 4z - 4}{(z-2)^2} & \frac{-4z^3 + 13z^2 - 4z + 4}{(z-2)^2} \end{bmatrix}
\]
and so
\[
k_{11} = \lim_{z \to 1} G_{11}(z) = \frac{1}{8} \begin{bmatrix} 9 & -1 & -1 \\ 5 & -5 & -5 \\ 5 & -5 & -5 \end{bmatrix}, \quad k_{22} = \lim_{z \to 1} G_{22}(z) = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 9 \end{bmatrix}
\]
Operators $k_{12}$ and $k_{21}$ are calculated in a similar way. Then, after calculating $Z$ we can verify the MHTF:

\begin{align}
E_1(T_2) & = Tr(N_{21}\rho^v) = Tr([\hat{D}\hat{Z}]_{22} - (\hat{D}\hat{Z})_{21}) = 2[1 + Re(\rho_{12})] \\
E_2(T_1) & = Tr(N_{12}\rho^v) = Tr([(\hat{D}\hat{Z})_{11} - (\hat{D}\hat{Z})_{12}]\rho^v) = 3[\rho_{11} + 1/2\rho_{22} - 1/2 Re(\rho_{12})]
\end{align}

As for the second MHTF, we need to calculate $E_\pi$. The operators $\hat{h}_{ij}$ are obtained by setting $z = 1$ on the corresponding generating functions, thus giving

\begin{align}
\hat{h}_{11} & = \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \\
\hat{h}_{12} & = \frac{1}{4} \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & -1 & 2 \end{bmatrix}, \\
\hat{h}_{21} & = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \\
\hat{h}_{22} & = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix}
\end{align}

and so

\begin{align}
E_\pi(T_1) & = Tr((\hat{D}\hat{Z})_{11}E_\pi(1)\rho) = Tr((\hat{D}\hat{Z})_{11}\hat{h}_{11}(\pi_1)) + Tr((\hat{D}\hat{Z})_{11}\hat{h}_{12}(\pi_2)) = \frac{3}{2} \\
E_\pi(T_2) & = Tr((\hat{D}\hat{Z})_{22}E_\pi(2)\rho) = Tr((\hat{D}\hat{Z})_{22}\hat{h}_{21}(\pi_1)) + Tr((\hat{D}\hat{Z})_{22}\hat{h}_{22}(\pi_2)) \approx 2.5555
\end{align}

\begin{example}
Consider the following unital OQW on 3 vertices, induced by matrices

\begin{align}
B_{11} = B_{22} = B_{33} = \frac{I}{2}, \\
B_{12} = B_{23} = B_{31} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \\
B_{13} = B_{32} = B_{21} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\end{align}

(indices follows the convention in expression (2.11)). Then $\pi = (\pi_i)_{i=1}^3$, $\pi_i = I/6$. The first MHTF can be verified as in the above example (noting since it is unital, this example is covered by the result in [20]). As for the second MHTF, matrices $\hat{h}_{ij}$ can be obtained explicitly after a long but routine calculation (we omit the generating functions leading to them):

\begin{align}
\hat{h}_{11} = \hat{h}_{22} = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{29}{172} & -\frac{27}{172} & \frac{1}{8} \\ -\frac{1}{8} & -\frac{27}{172} & \frac{29}{172} & \frac{1}{8} \\ \frac{1}{2} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{bmatrix},
\hat{h}_{12} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{11}{28} & \frac{23}{56} & -\frac{5}{56} & \frac{5}{28} \\ -\frac{11}{28} & -\frac{5}{56} & \frac{23}{56} & \frac{5}{28} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix},
\hat{h}_{21} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{5}{28} & \frac{23}{56} & -\frac{5}{56} & \frac{11}{28} \\ -\frac{5}{28} & -\frac{5}{56} & \frac{23}{56} & \frac{11}{28} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}
\end{align}

Then, for $i = 1, 2, 3$, we obtain

\begin{align}
Tr((DZ)_{ii}\rho^v) = E_\pi(T_i) = \frac{1}{3}k_{i1}(\rho) + \frac{1}{3}k_{i2}(\rho) + \frac{1}{3}k_{i3}(\rho) \approx 2.53846(\rho_{11} + \rho_{22}) = 2.53846
\end{align}

\end{example}

4. Kac’s Lemma for OQWs and the stationary measure

In this section we combine some recent results concerning the stationary measure of an irreducible OQW and an open version of Kac’s Lemma regarding the expected return time. We will revisit this point later in the continuous time setting.

We recall that we have a version of Kac’s Lemma for OQWs [4,11]:

\begin{align}
E_{\pi, \pi(z)}(V_x) = Tr\left[\hat{h}_{xx}\left(\frac{\rho_x}{Tr(\rho_x)}\right)\right] = \frac{1}{Tr(\rho_x)}
\end{align}

where it is assumed that the OQW is irreducible and that it admits a unique stationary measure $\rho = (\rho_i)$. We also refer the reader to [29] for a related result in the context of quantum channels. In this way we have a result that closely resembles the classical result, namely the expected return time to a vertex is essentially the (trace of the) inverse of the stationary measure. Let us give a characterization of such measure. Part of this discussion has been given in [21] and is in part motivated by [14]. Let $\rho_x$ be any fixed density at site $x$ and write, for any given OQW with transition matrices $B_i^j$,

\begin{align}
\mathcal{S}_{\rho_x}^n := \sum_{i_1, \ldots, i_{n-1} \neq x} B_{i_1i_{n-1}} \cdots B_{i_{n-1}x} B_{i_1x} \rho_x B_{i_1x} x^* B_{i_2x} \cdots B_{i_{n-1}x}^*, \quad n = 2, 3, \ldots
\end{align}

This is the matrix consisting of the sum of all possible transitions from site $x$ (with initial density $\rho_x$) to site $j$ in $n$ steps and such that no visit to $x$ occurs during the first $n$ steps (except the case where $x = j$, when the first return occurs at
the $n$-th step). If the degree of each vertex of the graph is finite, as it is usually assumed, this is a finite sum of products of $n$ matrices and is therefore well defined. By taking the trace, we obtain the probability of reaching site $j$ from $\rho_x \otimes |x\rangle$ in $n$ steps without intermediate visits to $x$.

Now we note the following: if $P$ denotes the orthogonal projection onto vertex $|x\rangle$ and $Q = I - P$, and if $P_x \rho = P \rho_P$, $Q_x \rho = Q \rho Q$, we can define

\begin{equation}
F(z) := \mathbb{P}T(I - zQT)^{-1}\mathbb{P}
\end{equation}

(recall the reasoning leading to eq. (2.18)) so we can define

\begin{equation}
\rho_{st,\rho_x}(x) := \sum_{n=1}^{\infty} S^n_{\rho_x,x} = F(1) \rho_x
\end{equation}

By [21], Thm. 5.7 it follows that $\rho_{st,\rho_x}(x)$ must be equal to $\rho_x$ in order to have that

\begin{equation}
\rho_{st,\rho_x} := \sum_j \rho_{st,\rho_x}(j) \otimes |j\rangle \langle j|, \quad \rho_{st,\rho_x}(j) := \sum_{n=1}^{\infty} S^n_{\rho_x,j}
\end{equation}

is a stationary operator. This amounts to find the fixed point of $F(1)$. Let us rewrite the above expression. We may write

\begin{equation}
\rho_{st,\rho_x}(j) = F_{j\leftarrow x}(1) \rho_x
\end{equation}

where $F_{j\leftarrow x}$ is the hitting function of vertex $j$, avoiding $x$ (after beginning at 0):

\begin{equation}
F_{j\leftarrow x} := \mathbb{P}_j T(I - zQ_x T)^{-1}\mathbb{P}_x
\end{equation}

We recall from [4, Remark 3.2] that, for any finite irreducible OQW the probability that the walk will ever return to any given vertex equals 1. In addition, the expected return time is finite, by [4, Thms. 4.3 and 4.5]. From [21], such walk is positive recurrent and by irreducibility there is a unique stationary state, which must be (4.5). From this we have:

**Proposition 4.1.** For an irreducible OQW on a finite graph with a stationary measure $\pi = \sum_i \pi(i) \otimes |i\rangle \langle i|$, we have that for every $j$, and for any choice of $x$, $\pi(j) = F_{j\leftarrow x}(\rho_x)$, where $\rho_x$ is the fixed point of $F_{x\leftarrow x} := \mathbb{P}_x T(I - zQ_x T)^{-1}\mathbb{P}_x$.

If necessary, a stationary density matrix is then obtained after normalization. The above result gives a more precise description of the stationary density and allows one to make comparisons with the structure of classical Markov chains.

In such context a stationary measure is obtained by studying the expected time spent on certain vertices [7].

**Example 4.2.** Let us consider the following 2-vertex nonunital, irreducible OQW given by

\begin{equation}
\Phi = \begin{bmatrix} U_2 & L \\ U_1 & R \end{bmatrix}, \quad U_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad L = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\end{equation}

The unique stationary state can be obtained by calculating an eigenvalue problem:

\begin{equation}
\pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \otimes |1\rangle \langle 1| + \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \otimes |2\rangle \langle 2|
\end{equation}

From the above proposition we have the following: if $P_1$ is the projection on site 1 and $Q_1 = I - P_1$, a calculation gives $F_1(z) = \mathbb{P}(I - zQ_1 \Phi)^{-1}\mathbb{P}_1$ and then

\begin{equation}
F_1(1) = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\end{equation}

which has as fixed point the matrix

\begin{equation}
\rho_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}
\end{equation}

Similarly, by [17], if

\begin{equation}
F_{2\leftarrow 1}(z) := \mathbb{P}_2 T(I - zQ_1 T)^{-1}\mathbb{P}_1
\end{equation}

then

\begin{equation}
F_{2\leftarrow 1}(1) = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
-\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4}
\end{bmatrix}
\end{equation}
and by (4.6),

\[ \rho_2 = F_{2t-1}(1)\rho_1 = \begin{bmatrix} \frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{8} \end{bmatrix} \]

Then a simple calculation shows that \( \pi = \frac{3}{8}(\rho_1 \otimes |1\rangle\langle 1| + \rho_2 \otimes |2\rangle\langle 2|) \), as expected. A similar calculation with (the also recurrent) vertex 2 produces the same result.

\[ \square \]

5. Continuous time processes

5.1. Basic constructions. Let \( \mathcal{L} \) denote the generator of a continuous time process on a finite number of vertices, described by the master equation \( \partial_t \rho = \mathcal{L}(\rho) \). A special case of valid generators which are of interest to us are the ones given by \( \mathcal{L} = \Phi - I \), for any quantum channel \( \Phi \) [33], this being analogous to the classical theory of Markov chains: given a stochastic matrix \( P \) we have that \( Q = P - I \) is a valid (classical) generator so \( e^{tQ} \) is stochastic for every \( t \geq 0 \). More generally, we would like to examine semigroups with a continuous time parameter associated to OQWs on some graph. In terms of calculations the (block) matrix representation of \( \mathcal{L} \) will prove to be quite useful and this is also our guiding tool in obtaining general expressions.

In the setting of stochastic matrices, the valid generators are called Q-matrices, that is, of the form

\[ Q = (q_{ij})_{i,j} \]

with

\[ 0 \leq -q_{ii} < \infty \quad \text{for all} \quad i, \quad q_{ij} \geq 0 \quad \text{for all} \quad i \neq j \quad \text{and} \quad \sum_{j} q_{ij} = 0 \quad \text{for all} \quad j. \]

The number \( q_i := -q_{ii} \) is called the rate of the process at vertex \( i \). We recall the following well-known results [2]:

1. Let \( X_t = e^{tQ}, \ t \geq 0 \), assume \( X(0) = i \) for some state \( i \) and define the holding time in state \( i \),

\[ T_i = \inf\{ t \geq 0 : X(t) \neq i \} \]

Then

\[ P(T_i > t | X(0) = i) = e^{-q_i t}, \quad t \geq 0 \implies E(T_i) = \int_0^\infty P(T_i > t)dt = \frac{1}{q_i} \]

In words: the process spends an average time equal to \( 1/q_i \) on each vertex before leaving it.

2. (Jump matrix) It holds that

\[ P(X(T_i) = j | X(0) = i) = q_{ji}/q_i, \quad j \neq i \]

With such results we are able to define the jump chain associated to the semigroup, given by the stochastic matrix \( \Pi = (\pi_{ij}) \) where if \( q_j \neq 0 \) then \( \pi_{ij} = q_{ij}/q_j \) if \( i \neq j \), and \( \pi_{jj} = 0 \). If \( q_i = 0 \) then \( \pi_{ij} = 0 \) for \( i \neq j \) and \( \pi_{ii} = 1 \). That is, \( \Pi \) captures only the jumps made by the given path.

Regarding the quantum setting, it is clear that a Lindblad generator \( \mathcal{L} \) will not be a Q-matrix in general. Nevertheless we may ask whether the block matrix \( \hat{\mathcal{L}} \) presents properties which are analogous to the ones listed above. We will see that this is in fact possible to some extent, and we will be able to compare such description with a related one seen in [33]. Let \( \mathcal{L} \) be a fixed generator, define \( \Lambda_t(\rho) = e^{t|\mathcal{L}|vec(\rho)}, \ t \geq 0 \), and

\[ p_{ij,\rho}(t) := Tr^v(\mathcal{P}_t \Lambda_t \mathcal{P}_j vec(\rho)) \quad q_{ii,\rho} := \lim_{t \downarrow 0} \frac{1 - p_{ii,\rho}(t)}{t} \quad q_{ij,\rho} := \lim_{t \downarrow 0} \frac{p_{ij,\rho}(t)}{t}, \quad i \neq j \]

As before, indices are read from right to left, so \( p_{ij,\rho}(t) \) is the probability of reaching vertex \( i \) at time \( t \), under the action of the semigroup \( \Lambda_t \) beginning at state \( \rho \otimes |j\rangle\langle j| \). Let us focus on the case of OQW semigroups, that is, \( \{\Lambda_t, t \geq 0\} \) dictates the evolution of density matrices of the form (2.6) with a generator describing the structure of the graph and the evolution of the internal degree.

Let \( \mathcal{L} \) be a generator for an OQW semigroup and let \( \hat{\mathcal{L}}_{ij} = \mathcal{P}_i \hat{\mathcal{L}} \mathcal{P}_j \) denote the \((i,j)\)th block matrix of \( \hat{\mathcal{L}} \). We say \( \mathcal{L} \) is complete if all of its block matrices \( \hat{\mathcal{L}}_{ij} \) are invertible. In this work we will, for simplicity, restrict our study to the case of complete generators.

**Proposition 5.1.** Let \( \mathcal{L} \) be a complete generator for an OQW semigroup and let \( T_i = \inf\{ t \geq 0 : X(t) \neq i \} \). Then a)

\[ q_{ii,\rho} = -\lim_{t \downarrow 0} \frac{\ln p_{ii,\rho}(t)}{t} = -Tr^v(\hat{\mathcal{L}}_{ii}\rho^v) \]

b)

\[ Pr(T_i > t | X(0) = (i; \rho)) = e^{-q_{ii} t} \]

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In particular, the expected time needed to leave state \((i; \rho)\) is \(1/q_{i; \rho}\). c) Starting from \((i; \rho)\), the probability that the first jump is to \(j\) is given by

\[
Pr(X(T_i) = j|X(0) = (i; \rho)) = q_{ij; \rho}/q_{i; \rho}, \quad j \neq i
\]

**Remark 5.2.** From Proposition 5.1 we see that the local characteristics \(q_{i; \rho}, q_{ij; \rho}\) depend not just on the particular vertex but also on the internal degree of freedom. In particular, from item c) we see that the use of the jump matrix needs to take in consideration the density matrix at each site, and not just the position as in jump chains induced by classical Markov chains.

In order to prove the proposition we recall the following well-known lemma [2].

**Lemma 5.3.** (Subadditive functions). Let \(\phi : (0, +\infty) \to [0, +\infty)\) be a function satisfying a) \(\phi(s + t) \leq \phi(s) + \phi(t)\) for all \(s, t \geq 0\), and b) \(\lim_{t \to 0} \phi(t)/t\) exists (may be infinite) and equals \(\sup_{t > 0} \phi(t)/t\).

**Proof of Proposition 5.1.** The reasoning below is inspired by [2, 7]. a) First recall that the semigroup satisfies

\[
\Lambda'(t) = \mathcal{L}\Lambda(t), \quad \Lambda'(0) = \mathcal{L}
\]

Then

\[
q_{i; \rho} = \lim_{t \to 0} \frac{1 - p_{ii; \rho}(t)}{t} = -\lim_{t \to 0} \frac{p_{ii; \rho} - 1}{t} = -\lim_{t \to 0} \frac{Tr^v(P_i\Lambda t P_i \rho^v) - 1}{t} = -\lim_{t \to 0} \frac{Tr^v((P_i\Lambda t P_i - \mathbb{1})\rho^v)}{t}
\]

(5.9)

Now, we need to show that

\[
q_{i; \rho} = -\lim_{t \to 0} \frac{\ln p_{ii; \rho}(t)}{t}
\]

In fact, let \(\phi(t) = -\ln p_{ii; \rho}(t)\), where \(p_{ii; \rho}(t) = Tr(P_i e^{t\mathcal{L} P_i} \rho^v)\). Since \(p_{ii; \rho}(t) \to 1\) as \(t \to 0\) we have that \(p_{ii; \rho}(t)\) can never vanish (see [2, Prop. 1.3]). Therefore \(\phi(t)\) is well defined and is subadditive. Therefore \(\lim_{t \to 0} \phi(t)/t\) exists and equals \(\sup_{t > 0} \phi(t)/t\). If such limit equals 0 then \(\phi(t)/t = 0\) for all \(t > 0\), so \(p_{ii; \rho}(t) = 1\) for all \(t \geq 0\) and so \(\lim_{t \to 0} [1 - p_{ii; \rho}(t)]/t = 0\), and we are done. Otherwise suppose such limit is greater than 0. Then, there is \(\delta > 0\) such that if \(0 < t < \delta\) then \(\phi(t) > 0\). With such \(t\), we have

\[
\lim_{t \to 0} \frac{1 - p_{ii; \rho}(t)}{t} = \lim_{t \to 0} \frac{1 - e^{-\phi(t)}}{\phi(t)} \cdot \frac{\phi(t)}{t} = \lim_{t \to 0} \frac{1 - e^{-\phi(t)}}{\phi(t)} \cdot \lim_{t \to 0} \frac{\phi(t)}{t} = \frac{\phi(t)}{t}
\]

(5.11)

Hence,

\[
q_{i; \rho} := \lim_{t \to 0} \frac{1 - p_{ii; \rho}(t)}{t} = -\lim_{t \to 0} \frac{\ln p_{ii; \rho}(t)}{t}
\]

(5.12)

b) Let \(P_{i; \rho}(\cdot) = Pr[\cdot|(X(0), \rho(0)) = (i, \rho)]\). Then

\[
P_{i; \rho}(T_i > t + s) = P_{i; \rho}(X(u) = i, 0 \leq u \leq t + s)
\]

\[
= P_{i; \rho}(X(u) = i, t \leq u \leq t + s|X(u) = i, 0 \leq u \leq t) \cdot P_{i; \rho}(X(u) = i, 0 \leq u \leq t)
\]

\[
= P_{i; \rho}(X(u) = i, t \leq u \leq t + s|X(t) = i) \cdot P_{i; \rho}(T_i > t)
\]

(5.13)

\[
P_{i; \rho}(X(u) = i, 0 \leq u \leq s|X(0) = i) \cdot P_{i; \rho}(T_i > t) = P_{i; \rho}(T_i > s)P_{i; \rho}(T_i > t)
\]

The only bounded and right continuous solution of this equation is \(P_{i; \rho}(T_i > t) = e^{-\alpha t}\) for some \(\alpha \geq 0\). In order to determine \(\alpha\) note that, since the sample paths are step functions,

\[
P_{i; \rho}(T_i > t) = Pr(X(u) = i, 0 \leq u \leq t|X(0) = i)
\]

(5.14)

\[
= \lim_{n \to \infty} Pr(X(u) = i, u = 0, t/n, 2t/n, \ldots, (n - 1)t/n, t|(X(0), \rho(0)) = (i, \rho)) = \lim_{n \to \infty} Tr^v\left(\left[P_i e^{t(n)|\mathcal{L}| P_i} \right]^n \rho^v\right)
\]

Using the fact that for every \(t \geq 0\) the term inside the trace is positive semidefinite, we apply the spectral theorem so we may write

\[
\alpha = -\frac{1}{t} \ln P_{i; \rho}(T_i > t) = -\frac{1}{t} \ln \left[ \lim_{n \to \infty} Tr^v\left(\left[P_i e^{t(n)|\mathcal{L}| P_i} \right]^n \rho^v\right) \right] = -\lim_{n \to \infty} \frac{Tr^v\left(\left[P_i e^{t(n)|\mathcal{L}| P_i} \right]^n \rho^v\right)}{t/n}
\]

(5.15)

\[
= -\lim_{n \to \infty} \frac{\ln \left[ p_{ii; \rho}(t/n) \right]}{t/n} = -\lim_{x \to 0} \frac{\ln \left[ p_{ii; \rho}(x) \right]}{x} = q_{i; \rho}
\]
c) With the assumption of right-continuity of the paths, we have that \(X(T_i)\) is the state the chain visits immediately upon leaving \(i\). Let
\[
R_{ij}(h) := \Pr(X(t + h) = j | \rho(t) = (\rho, i), X(t + h) \neq i)
\]
Note that this does not depend on \(t\) because of homogeneity, and that
\[
\Pr[X(T_i) = j | \rho(0) = (\rho, i)] = \lim_{h \to 0} R_{ij}(h)
\]
noting the LHS is the probability of a transition from \(i\) to \(j\), given that a transition out of \(i\) does occur. Then, by homogeneity,
\[
R_{ij}(h) = \Pr(X(h) = j | \rho(0) = (\rho, i), X(h) \neq i) = \frac{p_{ij;\rho}(h)}{h} \left(1 - \frac{p_{ii;\rho}(h)}{h}\right)
\]
from which we obtain \(\lim_{h \to 0} R_{ij}(h) = q_{ij;\rho}/q_{ii;\rho}\) if \(j \neq i\).

\[\square\]

Example 5.4. Consider the QW such that
\[
\hat{\Phi} = \begin{bmatrix} [L] & [R] \\ [R] & [L] \end{bmatrix}, \quad L = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\]
and let \(\mathcal{L} = \Phi - \mathbb{I}\), which is a complete generator, so we can define \(\Lambda_t(\rho) = e^{t[\mathcal{L}]} vec(\rho), \ t \geq 0\). For a generic density \(\rho = (\rho_{ij})\) we have, by letting,
\[
\beta = \beta(t) = \exp \left(\frac{\sqrt{3} - 3}{3} t\right), \quad \beta' = \beta'(t) = \exp \left(-\frac{\sqrt{3} - 3}{3} t\right)
\]
that
\[
p_{ii;\rho}(t) = \frac{1}{12} \left[6 + (3 + \sqrt{3})\beta + (3 - \sqrt{3})\beta' + 2\sqrt{3}(\beta - \beta') \left(2Re(\rho_{12}) - \rho_{11}\right)\right], \quad i = 1, 2
\]
\[
p_{ij;\rho}(t) = \frac{1}{12} \left[6 - (3 + \sqrt{3})\beta - (3 - \sqrt{3})\beta' - 2\sqrt{3}(\beta - \beta') \left(2Re(\rho_{12}) - \rho_{11}\right)\right], \quad i \neq j
\]
and so
\[
q_{ii;\rho} = \lim_{t \to 0} \frac{1 - p_{ii;\rho}(t)}{t} = \lim_{t \to 0} -\frac{\ln[p_{ii;\rho}(t)]}{t} = -Tr\left[v[L_{ii}\rho^v]\right] = \frac{1}{3} (1 + \rho_{11} - 2Re(\rho_{12})), \quad i = 1, 2
\]
\[
q_{ij;\rho} = \frac{1}{3} (1 + \rho_{11} - 2Re(\rho_{12})), \quad i, j = 1, 2
\]
this illustrating the equalities in the Proposition, item a). Note that such numbers are strictly greater than 0, since \(\rho\) is positive semidefinite. From item c) we see that the transition probabilities are all equal to 1, which is evident since the walk is on two vertices only.

\[\diamondsuit\]

5.2. Hitting times, continuous case dictated by a Poisson rate. The problem of hitting times for continuous time unitary evolutions on graphs has been addressed in [33]. There the authors obtain an explicit formula for the hitting probability and the mean hitting time for a particle to reach some target vertex, given a Hamiltonian \(H\), with vertex transition being dictated by a Poisson process with measurement rate \(\lambda\). In this section we make adaptations of such procedure so that under mild conditions a quite similar formula is available for open quantum evolutions (in a way that includes classical Markov chains as a particular case). In the next section we will relate this approach with another one which does not make use of a Poisson rate.

Let \(\omega = (t_1, t_2, \ldots)\) be a sequence of random times when the jumps of the Poisson process are observed, with \(t_n \in \mathbb{R}, \ t_0 = 0 < t_1 < t_2 < \ldots\). The sequences belong to a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which the Poisson process is defined: if \(\Omega\) is the collection of all sequences of random times then we write \(X_t : \mathbb{R} \times \Omega \to \{0, 1, 2, \ldots\}, X_t(\omega) = n, \ t \in [t_n, t_{n+1})\). For \(\omega \in \Omega\), let \(q_n(\omega)\) denote the probability of finding the particle at vertex \(j\), given that the system was not found in such state at times \(t_1, \ldots, t_{n-1}\). The mean hitting time for \(\omega \in \Omega\) is defined by
\[
\tau(\omega) := \sum_{n=1}^{\infty} t_n q_n(\omega)
\]
Note that at this point only times are being specified, but not the initial position, which is specified later. The mean hitting time for visiting vertex $|f\rangle$ is defined as the average over all possible sequences of random times:

$$\tau_h := \int_{\Omega} \tau(\omega) \, dP(\omega)$$

Now assume the $t_n$ are the events of a Poisson process, so the intervals $\tau_n := t_n - t_{n-1}$ are independent, identically distributed random variables, exponentially distributed with parameter $\lambda$ and probability density function $\lambda e^{-\lambda t}$. Therefore, we can reexpress $\tau_h$ as

$$\tau_h = \int_{\Omega} \tau(\omega) \, dP(\omega) = \prod_{i=1}^{\infty} \tau(\omega) e^{-\lambda \tau_i} d\tau_i$$

Fix an initial density $\rho \otimes |i\rangle\langle i|$, let $P_f$ be the projection on some target vertex $|f\rangle \neq |i\rangle$ and $Q_f := I - P_f$. Considering the semigroup with generator $\mathcal{L}$, we have

$$q_n(\omega) = Tr[P_f \prod_{m=1}^{n-1} (e^{(t_m+1-t_m)\mathcal{L}})Q_f] e^{t_1\mathcal{L}} \rho]$$

Now we are able to state the following result for OQWs, closely inspired by the unitary result seen in [33].

**Theorem 5.5.** Let $\mathcal{L}$ be a valid Lindblad generator for an OQW on a finite graph and $\Lambda_\lambda(\rho) := e^{t\mathcal{L}} vec(\rho)$. Let $P_f$ be the projection on some target vertex $|f\rangle$, $Q_f := I - P_f$ and define the dynamics determined by a Poisson rate via (5.27). Let

$$\mathcal{M}_\lambda := I - \frac{\mathcal{L}}{\lambda}, \quad \mathcal{N}_\lambda := \mathcal{M}_\lambda - Q_f$$

If a) $\mathcal{M}_\lambda$ is invertible and b) the eigenvalues of $Q_f \mathcal{M}_\lambda^{-1}$ are all strictly less than 1 in absolute value, then the following holds: the total probability of ever hitting the final vertex, given an initial density $\rho \otimes |i\rangle\langle i|, |i\rangle \neq |f\rangle$, is given by

$$p_h := \int_0^\infty \sum_{n=1}^{\infty} q_n(\omega) \, dP(\omega) = Tr[P_f \mathcal{N}_\lambda^{-1} \rho]$$

and the mean hitting time to reach vertex $|f\rangle$ equals

$$\tau_{h,f\rightarrow i}(\lambda; \rho) = \frac{1}{\lambda} Tr[P_f \mathcal{N}_\lambda^{-2} \rho]$$

Both expressions are open quantum versions of the formulae obtained in [33], noting that as a consequence the theorem also holds in the setting of classical Markov chains (stochastic matrices). We outline the proof of the theorem in the Appendix.

**Example 5.6.** (OQW example on $K_3$). We consider an OQW on the path with 2 vertices, so we assume an internal degree of freedom on each vertex. If we describe such degree by order 2 densities, then the OQW can be described in terms of an order 8 block matrix. As we are considering continuous-time dynamics, we need to specify a Lindblad generator. We will consider $\mathcal{L}_1 = \Phi_1 - I$ and $\mathcal{L}_2 = \Phi_2 - I$, where $\Phi_1$ and $\Phi_2$ are the discrete-time OQWs

$$\Phi_1 = \begin{bmatrix} 0 & [R_{\pi/4}] & [R_{\pi/2}] \\ \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} U_1 & U_2 \\ \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$U_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

We remark that in both examples below $\mathcal{M}_\lambda$ is not a normal operator.

a) Generator $\mathcal{L}_1 = \Phi_1 - I$. Consider the problem of mean hitting time for vertex $|2\rangle$, given that the walk begins at vertex $|1\rangle$. A routine calculation shows that, for every $\lambda > 0$, $\mathcal{M}_\lambda$ is invertible and the eigenvalues of $Q_f \mathcal{M}_\lambda^{-1}$ are strictly less than 1 in absolute value (the eigenvalues approach 1 as $\lambda \rightarrow \infty$, as it happens in an unitary example seen in [33]). In fact, we have

$$[\mathcal{M}_\lambda] = [I] - \frac{[\mathcal{L}_1]}{\lambda} = \frac{1}{\lambda} \begin{bmatrix} 1 + \lambda & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 + \lambda & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 + \lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 + \lambda & -1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 + \lambda & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 + \lambda & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 + \lambda & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 1 + \lambda \end{bmatrix}$$
and the nonzero eigenvalues of $Q_f M_\lambda^{-1}$ are

$$\alpha_{1,2} = \frac{\lambda^2 + 2\lambda + 1 \pm \sqrt{\lambda^2 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 2}}{\lambda^2 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 2}, \quad \alpha_{3,4} = \frac{\lambda + 1}{\lambda + 2}$$

and a simple inspection shows that $|\alpha_i| < 1$ for all $i$. Also,

$$N_\lambda = M_\lambda - Q_f = \frac{1}{\lambda} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 - \frac{\lambda}{2} & \frac{1 - \lambda}{2} & \frac{1 - \lambda}{2} & 1 + \lambda & 0 & 0 & 0 & 0 \\ \frac{1 - \lambda}{2} & \frac{1 - \lambda}{2} & \frac{1 - \lambda}{2} & 0 & 1 + \lambda & 0 & 0 & 0 \\ \frac{1 - \lambda}{2} & \frac{1 - \lambda}{2} & \frac{1 - \lambda}{2} & 0 & 0 & 1 + \lambda & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 1 + \lambda & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \end{bmatrix}$$

Then, for instance,

$$P_f N_\lambda^{-1} E_{11} = \frac{1}{2(\lambda^2 + 2\lambda + 2)} \begin{bmatrix} \lambda^2 + 3\lambda + 2 & -\lambda(\lambda + 1) \\ -\lambda(\lambda + 1) & \lambda^2 + 2\lambda + 2 \end{bmatrix} \otimes |2\rangle\langle 2| \implies p_h = Tr\left[ P_f N_\lambda^{-1} E_{11} \right] = 1$$

and a calculation shows that, for every $\rho$, $p_h = Tr\left[ P_f N_\lambda^{-1} \rho \right] = 1$. We also have that

$$P_f N_\lambda^{-2} E_{11} = \frac{1}{2(\lambda^2 + 2\lambda + 2)} \begin{bmatrix} \lambda^5 + 8\lambda^4 + 22\lambda^3 + 28\lambda^2 + 20\lambda + 8 & -\lambda^2(\lambda^2 + 4\lambda + 4) \\ -\lambda^2(\lambda^2 + 4\lambda + 4) & \lambda^5 + 4\lambda^4 + 10\lambda^3 + 20\lambda^2 + 20\lambda + 8 \end{bmatrix} \otimes |2\rangle\langle 2|$$

$$\implies \tau_h = Tr\left[ P_f N_\lambda^{-2} E_{11} \right] = 1 + \frac{2}{\lambda}$$

and a calculation shows that, for every $\rho$, $\tau_h = Tr\left[ P_f N_\lambda^{-1} \rho \right] = 1 + 2/\lambda$. Once again we have a simple interpretation in terms of the measurement rate: as the rate goes to infinity, the mean time equals 1 (so the quantum Zeno effect [33] does not occur), whereas if the rate goes to zero, the mean hitting time goes to infinity.

b) Generator $L_1 = \Phi_2 - I$. We omit some of the calculation as these are routine and analogous to the ones of item a). Consider once again the problem of mean hitting time for vertex $|2\rangle$, given that the walk begins at vertex $|1\rangle$. Then,

$$det([M_\lambda]) = \lambda^{-7}(\lambda + 1)^4(\lambda^3 + 3\lambda^2 + 3\lambda + 1)$$

and the nonzero eigenvalues of $Q_f M_f^{-1}$ are

$$\alpha_1 = \frac{2\lambda + 1}{2\lambda + 2}, \quad \alpha_{2,3,4} = \frac{\lambda}{\lambda + 1}$$

and clearly $|\alpha_i| < 1$ for all $i$. Also, a calculation shows that, for every $\rho$, $p_h = Tr\left[ P_f N_\lambda^{-1} \rho \right] = 1$ and we see that in this example the mean hitting time depends on the initial density. A routine calculation gives that, for every $\rho$,

$$\tau_h = Tr\left[ P_f N_\lambda^{-2} \rho \right] = 2 + 2Re(\rho_{12}) + \frac{2}{\lambda}$$

We know that $Re(\rho_{12}) \in [-\frac{1}{2}, \frac{1}{2}]$ so that for a fixed $\lambda$ it holds that $\tau_h \in [1 + \frac{2}{3} + \frac{2}{3}, \frac{2}{3}]$. As the rate goes to infinity, the mean time equals a strictly positive number, whereas if the rate goes to zero, the mean hitting time goes to infinity.

\[\diamond\]

6. MHTF, CONTINUOUS CASE AND A RELATION WITH THEOREM 5.5

We claim that the continuous-time version of the MHTF is valid, the proof being a simple adaptation of the discrete

time case. The same happens in the classical setting, where the fundamental matrix is defined as

$$Z = (Z_{ij}), \quad Z_{ij} = \int_0^\infty (P_i(X_t = j) - \pi_j)dt$$

and analogously in the quantum case by making use of the entrywise computation of the integral of $\Lambda_t(\rho) - \hat{\Omega}$:

$$\hat{Z} = (Z_{ij}), \quad Z_{ij} = \int_0^\infty P_i(\Lambda_t - \Omega)P_j dt, \quad \Lambda_t(\rho) = e^{t[\hat{L}]}(vec(\rho))$$

Let us describe the continuous time notion of hitting probability.
Definition 6.1. Let $\Lambda_t$ be a semigroup with a continuous parameter and generator $L$ associated with a dynamics on some graph. The hitting time of vertex $i$ is the random variable defined by
\[
D^i(\omega) := \inf\{t \geq 0 : X_t(\omega) = i\}
\]
where $X_t(\omega)$ is the position of the walk at time $t$, and the first return time to vertex $i$ by
\[
D^{i+}(\omega) := \inf\{t > T_i : X_t(\omega) = i\}
\]
where $T_i$ is the holding time at vertex $i$, eq. (5.1).

Instead of making use of a Poisson rate to calculate mean hitting times, we would like to calculate $E_i(D^i)$ in terms of the local characteristics of the generator. In this case we are confronted with the problem of obtaining the hitting time operator $\hat{K} = (k_{ij})$ explicitly in the continuous time case. Before we discuss this we state the result, and the proof is given in the Appendix.

Theorem 6.2. (MHTF, continuous time OQWs). Let $\Phi \in C$ denote a finite ergodic OQW semigroup with $k \geq 2$ sites, let $\hat{Z}$ denote its fundamental matrix, given by eq. (6.2), and let $\hat{K} = (k_{ij})$ denote the associated hitting time operator. Let $\hat{D}$ be the diagonal matrix operator with diagonal entries $k_{ii}$ and let $\hat{N} := \hat{K} - \hat{D}$. Then for every $\rho$ density matrix, for all $i,j = 1, \ldots, k$,
\[
\text{Tr}(\hat{N}_{ij}\rho) = \text{Tr}((\hat{D}\hat{Z})_{ii} - (\hat{D}\hat{Z})_{ij})\rho)
\]
Although the formula is of theoretical interest, we note that its practical use is less straightforward than the discrete time case, since the RHS of eq. (6.5) has the operator $\hat{K}$ in mind, if we make that occur on the particular paths, and this information is contained in the generator of the semigroup only. With this in mind, in the quantum, discrete time case we have obtained an explicit matrix operator associated with $\hat{K}$. In the classical case this is often calculated by solving a linear system of equations. In the quantum, discrete time case we have obtained an explicit matrix operator associated with $\hat{D}$. But in the continuous time case, a simple expression is in general a nontrivial matter.

In order to address this, let us compare for a moment $\tau_h(\lambda)$ given by eq. (5.27), with $E(D^i)$. Expression $\tau_h(\lambda)$ averages the paths based on the holding times of the paths before jumps, these being weighted by a Poisson parameter. On the other hand, the mean $E(D^i)$ does not depend on a parameter $\lambda$, instead the mean is weighted on the jump probabilities that occur on the particular paths, and this information is contained in the generator of the semigroup only. With this in mind, in the expression for $\tau_h(\lambda)$, we will be equally averaging all possible holding times (note \[\lim_{\lambda \to \infty} \int_0^\infty \lambda e^{-\lambda t} \, dt = 1\]) and in this way we have recovered $E(D^i)$ from $\tau_h(\lambda)$. We have concluded the following:

Proposition 6.3. Under the assumptions of Theorem 5.5
\[
\lim_{\lambda \to \infty} \tau_{h:j\to i}(\lambda; \rho) = E_{i,\rho}(D^j)
\]
This result will be illustrated in the following section.

Remark 6.4. (On a continuous time OQW version of Kac’s Lemma). Suppose for a moment that we would like to make direct use of the continuous time MHTF above. In principle, we may proceed as follows: recall the classical result
\[
E_i(D^{i+}) = \frac{1}{q_i} \left( 1 + \sum_{j \neq i} q_{ji} E_j(D^i) \right) = \frac{1}{q_i \pi_i}
\]
where the first equality is just conditioning on the first step by making use of the jump matrix. The second equality is the continuous time version of Kac’s Lemma (recall Section 7). It is a natural question to ask whether a continuous time OQW version of Kac’s Lemma is available. At first sight an adaptation of the classical proof is not an obvious matter, the reason being that at each step we have a transition rate that depends in general not only on the vertex, but also on the density matrix degree of freedom, as explained in Proposition 5.7. Thus, up to our knowledge, a quantum version of Kac’s lemma in this setting remains an interesting open question. Nevertheless, for the case of generators on $M_n(\mathbb{C})$, we have a simpler problem, as in such case there is no internal degree of freedom to be considered. This is described shortly.

6.1. Examples, continuous time setting.

Example 6.5. (4-vertex continuous time classical walk). Let
\[
Q = \begin{bmatrix}
-2 & 2 & 3 & 0 \\
1 & -6 & 3 & 0 \\
1 & 2 & -9 & 1 \\
0 & 2 & 3 & -1
\end{bmatrix}
\]
Such matrix can be diagonalized so that the semigroup can be calculated efficiently. We have that $\lim_{t \to \infty} e^{tQ}$ equals the matrix with all columns equal to $\pi = [2637362 \ 0.09890109 \ 0.1098901 \ 0.5274725]^T$. Matrix $Z$ is calculated
accordingly to (6.1) with \( P_j(X_t = i) = (e^{tQ})_{ij} \). Suppose we would like to calculate the mean hitting time of vertex 1, given that the walk begins at vertex 3:

\[
\rho_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbb{P}_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbb{Q}_f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

Then, as expected the hitting probability equals 1 and the mean hitting time can be calculated by solving the linear system

\[
\begin{align*}
\begin{cases}
  k_2 = \frac{1}{6} + \frac{1}{3}k_3 + \frac{1}{2}k_4 \\
  k_3 = \frac{1}{9} + \frac{4}{3}k_4 + \frac{1}{2}k_2 \\
  k_4 = 1 + k_3 
\end{cases}
\end{align*}
\]

where \( k_i \) is the mean time to reach vertex 1. This has the solution \( k_3 = 2.375 \). By making use of (6.10) one obtains, as expected,

\[
\tau_h = \frac{3.79166 + 2.375\lambda}{\lambda} \to 2.375, \quad \lambda \to \infty
\]

\[\diamondsuit\]

Example 6.6. (3-vertex continuous time OQW). Let \( B_{13} = B_{22} = B_{31} = 0 \) and

\[
B_{11} = \frac{1}{\sqrt{3}}, \quad B_{12} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_{21} = \sqrt{\frac{2}{3}} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad B_{23} = B_{33} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad B_{32} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

and define the 3-vertex OQWs with block matrix \( \Phi = ([B_{ij}])_{i,j=1,3} \) and consider the generator \( \mathcal{L} = \Phi - I \). This is a nonunital ergodic channel. A calculation for the mean hitting time of vertex \( |1\rangle \), given an initial density \( \rho_2 \) concentrated on vertex \( |2\rangle \), produces

\[
\frac{1}{\lambda} \mathbb{P}_f \mathcal{N}_\lambda^{-2} \rho_2 = \begin{bmatrix} \lambda(12\lambda^2 - 4)\rho_{11} + 3\rho_{22} \\
-\frac{\lambda^2}{(1+\lambda)^2} \\
\lambda(12\lambda^2 - 4)\rho_{22} + 3\rho_{11} \end{bmatrix}
\]

\[
\Rightarrow \quad \frac{1}{\lambda} Tr(\mathbb{P}_f \mathcal{N}_\lambda^{-2} \rho_2) = \frac{12\lambda^2 - \lambda}{3(-1 + \lambda)^2}
\]

from which \( \lim_{\lambda \to \infty} \tau_h \) equals 4.

\[\diamondsuit\]

6.2. Generators on \( M_n(\mathbb{C}) \) and graph-induced generators. Besides generators associated with OQWs, which can be seen as acting on a direct sum space, one might also ask for generators acting on a space with less structure. Suppose one is interested in semigroups acting on \( M_n(\mathbb{C}) \), with no extra structure. Then we may assume the generator is acting on some graph by making the correspondence between its \( n \) vertices and the orthogonal projections \( P_1, \ldots, P_n \) on the diagonal matrices \( E_{ii} \) (entries all equal to 0 except the \( (i,i) \)-th entry, equal to 1). As discussed below in the case of graph-induced generators, this has the advantage of requiring less space than the representations used for OQWs in this work, but on the other side one does not carry an internal degree of freedom for the particle: only the position matters. This simpler construction may be quite useful in certain situations and allows for closer analogies with certain classical notions, such as the jump matrix discussed previously.

As examples of generators on \( M_n(\mathbb{C}) \), we recall that in \([24]\) the authors study a version of continuous-time open quantum walks for which the Lindblad generator is induced by the underlying graph \( G = (V,E) \). Define the Laplacian \( L = (L_{jk}) \in M_n(\mathbb{C}) \) and the canonical matrix of transition probabilities \( M = (M_{jk}) \in M_n(\mathbb{C}) \) by

\[
L_{jk} = \begin{cases}
\deg(j) & \text{if } j = k \\
-1 & \text{if } (j,k) \in E \\
0 & \text{if } (j,k) \notin E 
\end{cases} \quad M_{jk} = \begin{cases}
\frac{1}{\deg(j)} & \text{if } (j,k) \in E \\
0 & \text{otherwise}
\end{cases}
\]

Letting \( B_{jk} = \sqrt{M_{jk}} |j\rangle \langle k| \), one defines the graph-induced generator

\[
\mathcal{L}(\rho) := i[\rho, L] + \sum_{j,k} [B_{jk} \rho B_{jk}^* - \frac{1}{2} \{B_{jk}^*, B_{jk}, \rho\}]
\]

and the semigroup \( T_t(\rho) = e^{t\mathcal{L}}(\rho) \). It should be clear that this is one of several possible choices of generators which could also be induced by the graph in some form. Moreover, we note that in this model \( \dim(M) = \dim(L) = n \), which is the number of vertices in \( G \) so in particular the projections on the diagonal of \( \rho \) are associated with a location on the graph. As a consequence this model is not an OQW, since we are keeping track of the position of the particle on the graph, but the individual vertices do not carry an internal degree of freedom (like OQWs do). We also see that, unlike OQWs,
this construction admits a Kac’s Lemma in a simple way. In fact, equation \((6.7)\) holds in this setting, noting the proper positions of the entries \(q_{ij}\) in the matrix representation for the generator.

**Example 6.7.** (3 vertices, continuous time open walk, graph-induced generator). a) If we consider the 3-cycle \([24]\)

\[
(6.16)\quad L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad M = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}
\]

then the block matrix representation for \(L\) is easily calculated:

\[
(6.17)\quad [L] = \begin{bmatrix} -1 & -i & -i & i & \frac{1}{2} & 0 & i & 0 & \frac{1}{2} \\ -i & -1 & -i & 0 & i & 0 & 0 & i & 0 \\ -i & -i & -1 & 0 & 0 & i & 0 & 0 & i \\ i & 0 & 0 & -1 & -i & -i & i & 0 & 0 \\ \frac{1}{2} & i & 0 & -i & -1 & -i & 0 & i & \frac{1}{2} \\ 0 & 0 & i & -i & -1 & 0 & 0 & i & 0 \\ i & 0 & 0 & i & 0 & 0 & -1 & -i & -i \\ 0 & i & 0 & 0 & i & 0 & -i & -1 & -i \\ \frac{1}{2} & 0 & i & 0 & \frac{1}{2} & i & -i & -i & -1 \end{bmatrix}
\]

In this setting we calculate with the usual prescription \(p_{ij}(t) = Tr^\nu(P^t_iA_1P^\nu_jvcc(\rho))\), noting that the projections are over the diagonal entries of the order 3 density \(\rho\) on which the semigroup acts upon. A calculation shows that the mean hitting times from vertices 2 to 1, and 3 to 1 are equal:

\[
(6.18)\quad \lim_{\lambda \to \infty} \tau_{h:2\to 1}(\lambda) = \lim_{\lambda \to \infty} \tau_{h:3\to 1}(\lambda) = 2
\]

It is worth noting that the numbers \(q_{1} \) and \(q_{12}\) form a Q-matrix in the classical sense. Now we observe that since we are considering quantum trajectories, the density is normalized at each step (in this example to one of the projection densities \(\rho = \rho_{ii}, i=1,2,3\)) and so we may examine the mean hitting time in terms of a linear system, that is, in terms of the associated jump matrix. In this case

\[
(6.19)\quad \begin{cases} E_1(D^2) = \frac{1}{q_1} + q_{13}E_3(D^2) \\ E_3(D^2) = \frac{1}{q_3} + q_{31}E_1(D^2) \end{cases}
\]

which has as solution \(E_1(D^2) = E_3(D^2) = 2\), as expected.

b) If we consider the 3-path

\[
(6.20)\quad L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

then the block matrix representation for \(L\) is easily calculated:

\[
(6.21)\quad [L] = \begin{bmatrix} -1 & -i & 0 & i & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -i & -1+i & -i & 0 & i & 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 & 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & -1 & -i & -i & 0 & i & 0 \\ 1 & i & 0 & -i & -1 & -i & 0 & i & 1 \\ 0 & 0 & i & 0 & -i & -1 & 0 & 0 & i \\ 0 & 0 & 0 & i & 0 & 0 & -1 & -i & 0 \\ 0 & 0 & 0 & 0 & i & 0 & -i & -1+i & -i \\ 0 & 0 & 0 & 0 & \frac{1}{2} & i & 0 & -i & -1 \end{bmatrix}
\]

and a calculation shows that

\[
(6.22)\quad \lim_{\lambda \to \infty} \tau_{h:2\to 1}(\lambda) = 1, \quad \lim_{\lambda \to \infty} \tau_{h:3\to 1}(\lambda) = 4
\]

where the projections are given by, respectively,

\[
(6.23)\quad \mathbb{P}_2 = diag(0,0,0,0,1,0,0,0), \quad \mathbb{Q}_2 = diag(1,0,0,0,0,0,0,0,0)
\]

\[
(6.24)\quad \mathbb{P}_3 = diag(0,0,0,0,0,0,0,1), \quad \mathbb{Q}_3 = diag(1,0,0,0,1,0,0,0,0)
\]

In an analogous way as in item a), we have \(\tau_{2\to 1} = 1/q_{1} = 1\) and

\[
(6.25)\quad \tau_{3\to 1} = \frac{1}{q_{1}} + q_{12}\tau_{3\to 2} = \frac{1}{q_{1}} + q_{12}(\frac{1}{q_{2}} + q_{21}\tau_{3\to 1}) \implies \tau_{3\to 1} = \frac{1 + q_{12}}{1 - q_{12}q_{21}}
\]
Since $q_{12} = |\mathcal{L}|_{51} = 1$, $q_{21} = |\mathcal{L}|_{15} = 1/2$, we obtain $\tau_{3\rightarrow 4} = 4$, as expected.

Remark 6.8. It is natural to guess that some version of the MHTF is true for more general Lindblad generators like the ones obtained in the above example. However, it is not known whether the proof for the corresponding OQW result can be adapted to the general setting, due to the block matrix structure that appears in important ways in such reasoning. The latter is more readily observed by making dimension considerations and by writing the expression for $\Omega$ obtained in the above examples. In the OQW case one has a block matrix of the form (in the case of 3 vertices)

\begin{equation}
\hat{\Omega} = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} \\
\Omega_{31} & \Omega_{32} & \Omega_{33}
\end{bmatrix}
\end{equation}

something which does not occur in the graph-induced examples. Because of this, the validity of some MHTF formula for dissipative semigroups remains an interesting open question which might require a different approach than the one employed here.

7. OQW SITE RECURRENCE, CONTINUOUS TIME

In this section we give a brief description of site recurrence in the continuous time OQW setting, recalling that the problem in discrete time has been examined in \cite{4,11,21}. As in the case of Markov chain theory we are able to relate discrete and continuous time recurrence. In order to do this we will need to consider irreducible OQWs. We also remark that the construction given here holds for graphs with countably infinite vertices as well.

7.1. Recurrence and irreducibility. We recall that a CP map $\Phi$ is irreducible if for any $\rho \geq 0$, $\rho \neq 0$ trace class operator, there exists $t$ such that $e^{t\Phi}(\rho) > 0$. We refer the reader to \cite{9} for other equivalent definitions. Given $i,j$ vertices of a given graph, we say that $j$ is accessible from $i$, written $i \rightarrow j$ if $p_{ij}(t) > 0$ for some (and therefore all) $t \geq 0$ and for every $\rho$ density matrix. We say that $i$ and $j$ communicate, written $i \leftrightarrow j$ if $i$ and $j$ are accessible from each other. In the setting of discrete time OQWs, communicating classes have been discussed in \cite{9} and it follows that the irreducibility of an OQW corresponds to the existence of only one equivalence class with respect to accessibility.

Now we recall two notions of site recurrence in the discrete time case.

Definition 7.1. Let $\Phi$ be a discrete time OQW on some graph. a) We say vertex $|i\rangle$ is monitored recurrent with respect to $\Phi$ if the probability of ever return to $|i\rangle$, given any initial density $\rho \otimes |i\rangle \langle i|$, equals $1$. In symbols,

\begin{equation}
\sum_{n=0}^{\infty} Tr(P_i \Phi(Q_i \Phi)^n P_i \rho) = 1, \quad \forall \rho
\end{equation}

where $P_i$ is the projection onto vertex $|i\rangle$ and $Q_i = I - P_i$. We say $\Phi$ is monitored recurrent if every vertex is monitored recurrent with respect to $\Phi$. Otherwise we say vertex $|i\rangle$ is transient (with respect to monitoring). b) Let $p_{\text{vis}}(j)$ denote the probability of returning to site $|i\rangle$ in the $j$-th step, beginning with density $\rho_i \otimes |i\rangle \langle i|$. We say vertex $|i\rangle$ is SJK recurrent with respect to $\Phi$ if $\sum_{j=1}^{\infty} p_{\text{vis}}(j) = \infty$ for every density $\rho_i$ located at $|i\rangle$. We say $\Phi$ is SJK-recurrent if every vertex is SJK-recurrent with respect to $\Phi$. Otherwise we say vertex $|i\rangle$ is SJK-transient.

Remark 7.2. From the above we conclude that monitored recurrence for OQWs is a class property just as in the case of classical Markov chains. In particular, if some vertex is monitored recurrent with respect to an irreducible OQW then every vertex is monitored recurrent.

For the notions of monitored and SJK-recurrence in the discrete time unitary setting we refer the reader to \cite{13} and \cite{30}, respectively (SJK-recurrence is named after the initials of the authors of the latter work). In the case of discrete time Markov chains on a countable state space the equivalence between these two notions is well known. In the case of unitary walks, such notions are not equivalent \cite{13}. In the case of OQWs one has to take in consideration the degree of freedom given by the choice of density matrices and below we explain that such equivalence is valid in the case of irreducible OQWs.

Proposition 7.3. Let $\Phi$ be a discrete time irreducible OQW $\Phi$. If some (and consequently every) vertex is monitored recurrent then every vertex is SJK-recurrent. If for some faithful state $\rho_i \otimes |i\rangle \langle i|$ we have SJK-recurrence, then site $|i\rangle$ is monitored-recurrent with respect to every density $\rho$.

Proof. The first part has been proved in \cite{11}, Theorem 6.1. Let $t_i$ denote the time of first return to vertex $i$. Suppose $\sum_{j=1}^{\infty} p_i(j) = \infty$ for some faithful density $\rho_i \otimes |i\rangle \langle i|$. By \cite{11}, Thm. 6.2, site $|i\rangle$ is monitored-recurrent with respect to some density $\rho'$ accessible from $\rho_i$. By \cite{10} Remark 3.3 such $\rho'$ is also faithful. Then, by \cite{4} Corollary 3.5 Item 2, since
\(\rho'\) is faithful and \(P_{i,\rho'}(t_i < \infty) = 1\) then we must have that the expected number of visits to \(i\) is infinite for all \(\rho\). Then, by [2, Prop. 3.12], we conclude \(P_{i,\rho}(t_i < \infty) = 1\), for every \(\rho\).

\[\]

7.2. Recurrence criterion, continuous time case. With the discussion in the previous section we are able to examine the continuous time OQW case. As usual, let \(\{\Lambda_t, t \geq 0\}\) denote a continuous-time OQW on some graph having \(p_{ij,\rho}(t) = Tr^\rho(\Pi_j \Lambda_t \Pi_i \text{vec}(\rho))\) as its transition functions.

Definition 7.4. Given \(\delta > 0\), the discrete-time OQW \(\{\Lambda_{n\delta}, n = 0,1,2,\ldots\}\) having one-step transition probabilities \(p_{ij,\rho}(\delta)\) (and therefore \(n\)-step transition probabilities \(p_{ij,\rho}(n\delta)\)) is called the \(\delta\)-skeleton of \(\{\Lambda_t, t \geq 0\}\).

Definition 7.5. A vertex \(i\) is recurrent with respect to a continuous time OQW semigroup \(\Lambda_t\) if, for every \(\rho\) density, \(\int_0^\infty p_{ii,\rho}(t)dt = +\infty\), and transient otherwise.

At this point we must ask whether a relation between the discrete and continuous time OQWs holds with respect to recurrence, as in the classical case. The following is in part motivated by the classical result [2] and further justifies the definition of recurrence given above.

Proposition 7.6. Let \(\delta > 0\), \(|i|\) a vertex of the graph on which a continuous time OQW semigroup \(\Lambda_t\) is defined. Then \(|i|\) is recurrent with respect to \(\Lambda_t\) if, and only if, \(|i|\) is SJK-recurrent in the \(\delta\)-skeleton.

Proof. \(\sum_{n=0}^\infty \int_{n\delta}^{(n+1)\delta} p_{ii,\rho}(t)dt\) we can write
\[\delta \sum_{n=0}^\infty \min_{0 \leq s \leq \delta} p_{ii,\rho}(n\delta + s) \leq \int_0^\infty p_{ii,\rho}(t)dt \leq \delta \sum_{n=0}^\infty \max_{0 \leq s \leq \delta} p_{ii,\rho}(n\delta + s)\]

Now note that
\[p_{ii,\rho}(n\delta + s) = Tr(\Pi_i \Lambda_{n\delta+s} \Pi_i \rho) = \sum_k Tr(\Pi_i \Lambda_{n\delta+s} \Pi_k \Pi_i \rho)\]
\[= \sum_k Tr(\Pi_i \Lambda_{n\delta} \rho'_{ki}(s))Tr(\Pi_k \Lambda_{s} \Pi_i \rho) = \sum_k Tr(\Pi_i \Lambda_{n\delta} \rho'_{ki}(s))Tr(\Pi_k \Lambda_{s} \Pi_i \rho) = \sum_k p_{ik,\rho'_{ki}(s)}(n\delta)p_{ki,\rho}(s)\]
where we define, for every \(i,k\) vertices, \(s \geq 0\), \(\rho\) fixed density,
\[\rho'_{ki}(s) := \frac{\Pi_k \Lambda_{s} \Pi_i \rho}{Tr(\Pi_k \Lambda_{s} \Pi_i \rho)}\]
In particular \(p_{ii,\rho}(n\delta + s) \geq p_{ii,\rho'_{ki}(s)}(n\delta)p_{ii,\rho}(s)\), so that for any fixed \(\rho\),
\[\min_{0 \leq s \leq \delta} p_{ii,\rho}(n\delta + s) \geq \min_{0 \leq s \leq \delta} p_{ii,\rho'_{ki}(s)}(n\delta)\gamma, \quad \gamma := \min_{0 \leq s \leq \delta} p_{ii,\rho}(s) > 0\]
Next, \(p_{ii,\rho}((n+1)\delta) = p_{ii,\rho}(n\delta + s + \delta - s) \geq p_{ii,\rho'_{ki}(s)}(s)\)\(p_{ii,\rho}(\delta - s)\), so that
\[\max_{0 \leq s \leq \delta} p_{ii,\rho'_{ki}(s)}(n\delta + s) \leq p_{ii,\rho}(s) \frac{p_{ii,\rho}(n+1)\delta)}{\min_{0 \leq s \leq \delta} p_{ii,\rho}(\delta - s) = p_{ii,\rho}(n+1)\delta)}\]
Inserting (7.4) and (7.3) into (7.2) gives
\[\gamma \delta \sum_{n=0}^\infty \min_{0 \leq s \leq \delta} p_{ii,\rho'_{ki}(s)}(n\delta) \leq \int_0^\infty p_{ii,\rho}(t)dt \leq \frac{\delta}{\gamma} \sum_{n=0}^\infty p_{ii,\rho}(n+1)\delta)\]
Therefore, for every \(\rho\), the divergence of the integral in (7.5) implies the divergence of the series on the right. Also, if we suppose \(\sum_{n=0}^\infty p_{ii,\rho}(n\delta)\) diverges for every \(\rho\), then the series in the LHS of (7.5) diverges so the same happens to the integral.

8. Summary: Unitary evolutions and open questions

In this work we have studied a hitting time formalism for discrete and continuous time quantum dynamics on graphs dictated by completely positive maps, mostly in terms of open quantum random walks, but we have also seen that certain results are valid in other contexts as well. Hitting time formulae have been stated and proved, making clear certain similarities and differences with the classical case:

(1) We have established a MHTF for discrete time primitive OQWs on finite graphs. In the continuous case the MHTF formula also holds, and we have discussed the issue regarding its practical applicability (e.g., a simple Kac’s Lemma is not known, up to our knowledge). Nevertheless, we are still able to calculate mean hitting times by considering a Poisson rate \(\lambda\) and examining the limit \(\lambda \to \infty\).
(2) Although MHT formulae are not known for dynamics induced by generators on $M_n(C)$, we are still able to calculate mean hitting times by relying on the general theory associated with the local characteristics of the process.

We believe these results will serve as motivation for further studies in the context of quantum statistics on graphs. Besides the problem presented in the previous section (concerning a more general collection of MHT formulae), another question is to ask whether a MHTF exists for unitary evolutions on graphs. To some extent the proof for the dissipative case can be adapted, but there seem to be nontrivial difficulties in writing a formal proof (if one such natural formula exists at all). A first issue to be tackled is the iterative behaviour of the unitary map, which typically does not converge to a stationary operator. Based on certain examples one is motivated to consider von Neumann’s ergodic theorem together with averages of the iterated map. A second issue is that, unlike the case for OQWs, the block matrix representation is not available for unitary operators in general (but note that certain particular examples can be examined in such a way). In physical terms this is associated with the superpositions of the wave function and this suggests that a distinct approach to the hitting time problem is needed.

Another point which was not addressed in this work is the matter of obtaining criteria which allow us to determine whether infinite hitting times may occur. As it is known in the unitary setting, the symmetry of the graph can lead to infinite hitting times even if the Hamiltonian is not degenerate \([33]\). Up to our knowledge this problem has not been systematically examined in the OQW setting so far.

Finally, it is worth noting that the collection of hitting time formulae and results presented here consist of a subset of the results available for the time of first visit/return to states (not just vertices). This is in part illustrated via a generating function approach, such as the Schur function and first return functions (FR-functions) seen in \([14]\), (noting the emphasis of such work is on the recurrence problem) and also in part by the graph-induced quantum walks discussed previously. We expect to describe more results of this kind in the future.

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9. Appendix: proofs

9.1. Proof of Theorems 3.2 and 6.2 We are motivated by the unital, discrete time result seen in \([20]\). The proofs for nonunital discrete time and for continuous time OQWs follow from adapting these results. This is possible because such settings can be written in terms of the same block matrix computations, with the operators defined properly for each situation. We outline the details below. We recall that in the continuous time case the evolution is denoted by the semigroup $\Lambda_t$, $t \geq 0$ and $Z$ is defined by eq. (6.2).

Lemma 1. Let $\Phi \in C$. Then a) $\hat{Z}\hat{\Omega} = \hat{\Omega}\hat{Z} = \hat{\Omega}$. b) $\hat{Z}(I - \Phi) = I - \hat{Z}\hat{\Omega} = I - \hat{\Omega}$. c) $(I - \Phi)\hat{Z} = I - \hat{\Omega}\hat{Z} = I - \hat{\Omega}.$

Proof. The proof for the nonunital discrete time, and continuous time OQWs is identical to the one in \([20]\), as the algebraic relation between $Z$ and $\Omega$ is the same in all cases.

The second step is a technical lemma involving the mean hitting time operator $\hat{K}$. In both discrete and continuous time this is a linear map (given by \((3.5)\) in the discrete time case).

Lemma 2. Let $\Phi$ denote a finite ergodic OQW and let $\hat{Z}$ denote its fundamental matrix. Let $\hat{K}$ denote the mean hitting time operator. Let $\hat{D} = \text{diag}(k_{11}, \ldots, k_{nn})$ and let $\hat{L} := \hat{K} - (\hat{K} - \hat{D})\hat{\Phi}$ in the discrete time case and let $\hat{L}(t) := \hat{K} - (\hat{K} - \hat{D})\Lambda_t$ in the continuous time case. Then if $\rho_j$ is a density matrix concentrated on site $j$ then for all $i$ and all $c \in R$, $\text{Tr}(\hat{L}_{ij}c\rho_j) = \text{Tr}(c\rho_j) = c$.

Proof. From \([20], Lemma 2]\), which holds for the discrete time, unital case, we obtain the discrete time nonunital case, as the reasoning does not depend on the asymptotic limit $\hat{\Omega}$. Such proof is based on conditioning on the first step: if $i \neq j$,

\[
(9.1) \quad k_{ij}^{\rho_j} = 1 + \sum_{l \neq j} \text{Tr}(B_{ij}\rho_jB_{ij}^\dagger)k_{il}\left(\frac{B_{ij}\rho_jB_{ij}^\dagger}{\text{Tr}(B_{ij}\rho_jB_{ij}^\dagger)}\right),
\]

which is the OQW version of the classical conditioning $k_{ij} = 1 + \sum_{l \neq j} p_{ij}k_{il}$. As for the continuous time version of this, we replace \((9.1)\) with

\[
(9.2) \quad k_{ij}^{\rho_j} = \frac{1}{q_j} + \sum_{l \neq j} \frac{q_l}{q_j} k_{il}\left(\frac{p_{il}A(q_j)(\rho_j)}{\text{Tr}(p_{il}A(q_j)(\rho_j))}\right) = \frac{1}{q_j} \left[1 + \sum_{l \neq j} q_{ij}k_{il}\left(\frac{p_{il}A(q_j)(\rho_j)}{\text{Tr}(p_{il}A(q_j)(\rho_j))}\right)\right], \quad q_j = q_{j\rho_j}
\]

The remaining steps of the proof are identical to the ones in \([20]\), by appending $1/q_j$ together with the term $p_{il}A(q_j)$ and noting that the conclusion is invariant by scalar multiplication.

\[\text{Page 20}\]
Lemma 3. Let $\Phi$ denote a finite ergodic OQW and let $\mathcal{Z}$ denote its fundamental matrix. Let $\hat{K}$ be given by (3.5), Let $\hat{D}$ and $\hat{L}$ as in Lemma 2 and $\hat{N} := \hat{K} - \hat{D}$. Then

$$\hat{N}_{ij} = (\hat{D}\hat{Z})_{ii} - (\hat{D}\hat{Z})_{ij} + [(\hat{L}\hat{Z})_{ij} - (\hat{L}\hat{Z})_{ii}]$$

Proof. In the discrete time unital case $\Omega$ is a block matrix with all blocks equal, whereas in the general ergodic case (discrete and continuous time) we have a block matrix with all block columns equal (eq. (6.26) being the case of $n = 3$ vertices). The proof then follows the same steps as in [20], Lemma 3, and the same holds in the continuous time case. □

Then, Theorems 3.2 and 6.2 follow from the above preparation and [20], Theorem 1.

9.2. Proof of Theorem 5.5 We will obtain an explicit expression for $\tau_h$, by adapting a reasoning seen in [33]. Let $P_f = |f\rangle\langle f|$ be the orthogonal projection on the final state, and $Q_f = I - P_f$. In the setting of OQWs, we may take for instance the projection on some specific site, i.e., $P_f = I \otimes |f\rangle\langle f|$. We will employ the notation $P_f$ for simplicity, and have in mind hitting times for visiting a site (regardless of the accompanying state), but the general case is treated in the same way. The interval between jumps is written as $\tau_j = t_j - t_{j-1}$ and note that

$$q_n(\omega) = Tr\left[ P_f \prod_{m=1}^{n-1} e^{(t_{m+1} - t_{m})|\mathcal{C}|Q_f} e^{t_1|\mathcal{C}|} \rho \right] = Tr\left[ P_f \Lambda_{\tau_{n}}Q_f \Lambda_{\tau_{n-1}}Q_f \cdots \Lambda_{\tau_{1}}\rho \right]$$

where $P_f$ and $Q_f$ corresponds to the maps $X \mapsto P_fXP_f$ and $X \mapsto Q_fXQ_f$, respectively. Under the assumption of absolute convergence, we can rearrange terms and obtain

$$\tau(\omega) = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \sum_{m=1}^{n} t_m \sum_{\omega} q_n$$

The $t_n$ are the events of a Poisson process, so the intervals $\tau_m$ are i.i.d. random variables, exponentially distributed with parameter $\lambda$ and probability density function $\lambda e^{-\lambda t}$. Therefore, we can reexpress $\tau_h$ as

$$\tau_h = \int_{\Omega} \tau(\omega) \ d\mathbb{P}(\omega) = \prod_{l=1}^{\infty} \int_{0}^{\infty} \tau(\omega) \lambda e^{-\lambda t} dt = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \prod_{l=1}^{n} \int_{0}^{\infty} \sum_{m=1}^{n} t_m \sum_{\omega} q_n \lambda e^{-\lambda t} dt$$

From here, we obtain two kinds of integrals

$$A(X) = \int_{0}^{\infty} \Lambda_t(X) \lambda e^{-\lambda t} dt, \quad B(X) = \int_{0}^{\infty} t \Lambda_t(X) \lambda e^{-\lambda t} dt$$

Recalling that $\Lambda'_t(X) = |\mathcal{C}|\Lambda_t(X)$, we can obtain the following relation for $A(X)$, by integrating by parts:

$$A(X) = -\Lambda_t(X) e^{-\lambda t}\bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-\lambda t} \Lambda'_t(X) dt = X + \frac{|\mathcal{C}|}{\lambda} \int_{0}^{\infty} \Lambda_t(X) \lambda e^{-\lambda t} dt = X + \frac{|\mathcal{C}|}{\lambda} A(X)$$

(9.9)

$$\Rightarrow A(X) = \frac{|\mathcal{C}|}{\lambda} A(X) = X$$

Similarly, we obtain the following relation for $B(X)$:

$$B(X) = -\lambda t e^{-\lambda t} \Lambda_t(X)\bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-\lambda t} (\Lambda_t(X) + t|\mathcal{C}|\Lambda_t(X)) dt = \frac{1}{\lambda} A(X) + \frac{|\mathcal{C}|}{\lambda} B(X)$$

(9.10)

$$\Rightarrow B(X) - \frac{|\mathcal{C}|}{\lambda} B(X) = \frac{1}{\lambda} A(X)$$

(9.11)

and if we set

$$\mathcal{M}_\lambda(Y) := \left( I - \frac{|\mathcal{C}|}{\lambda} \right) Y$$

we obtain from (9.9) and (9.11) the system

$$\begin{cases}
\mathcal{M}_\lambda(A) = X \\
\mathcal{M}_\lambda(B) = \frac{1}{\lambda} A(X)
\end{cases}$$

(9.12)

Now two issues must be addressed: first, we need to verify whether $\mathcal{M}_\lambda$ is invertible. If this is the case, then $A = \mathcal{M}_\lambda^{-1}(X)$ and $B = \lambda^{-1}\mathcal{M}_\lambda^{-2}(X)$. Second, we need to verify whether the eigenvalues of $Q_f\mathcal{M}_\lambda^{-1}$ are all strictly less than 1 in absolute
value. If this is the case, by setting \( \mathcal{N}_\lambda := \mathcal{M}_\lambda - Q_f \) we obtain, via a reasoning which is identical to what is presented in [33] that

\[
\tau_h = \lambda^{-1} \text{Tr}\left[ P_f \mathcal{N}_\lambda^{-2} \rho \right]
\]

and we also have a formula for the mean probability to ever hit the final vertex, given by

\[
p_h := \int_\Omega \frac{\sum_{n=1}^\infty q_n(\omega)}{\sum_{n=1}^\infty q_n(\omega)} d\mathbb{P}(\omega) = \text{Tr}\left[ P_f \mathcal{N}_\lambda^{-1} \rho \right]
\]

\[\square\]

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