How Wave - Wavelet Trading Wins and “Beat” the Market

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Abstract

The purpose of this paper is to showcase trading strategies that give solutions to three difficult and intriguing problems in business finance, economics and statistics. The paper discusses trading strategies for both commodities and stocks but the main focus is on stock market trading at the New York Stock Exchange.

Problem 1: Buy Low and Sell High

The buy low and sell high problem can be summarized like this: suppose the price of a commodity or stock fluctuates indefinitely, is there any explicit strategy for a trader to “ride the price waves” by buying low and selling high to eventually win even if price does not increase? Buy low and sell high is the basic tenet of most forms of trading. However, people who engage in this kind of trading usually lose since it is not clear what “high” and “low” mean and how much to sell when price is high and how much to buy when price is low. Part 1 of the paper shows how these problems can be dealt with by employing my webpage “Portfolio Checker” on my website AgateTrading.com. This strategy can be used as a guide for the trading of any stock and a trader using it always wins in the long run. The only condition assumed is that the standard deviation of the stock price traded stays bounded away from zero.

Problem 2: “Beat” the Market

The market here mainly refers to the S&P 500 index SPY which is an important benchmark of U.S. stock performance. The share price of SPY has increased close to 700 percent from January 1993 to December 2016. The following problem is of great interest: is there any trading system that can “beat” the market consistently? Part 2 of this paper addresses this question. While the trading system in Part 1 always wins eventually, it does not “beat” the market consistently since its return rises more slowly than the S&P 500 benchmark. In Part 2, the trading system presented in Part 1 is transformed into a strategy that always outperforms the market eventually. Using this modified strategy requires a computer program on my webpage Wavelet Trading at my website AgateTrading.com. The strategy also works well as a guide for trading
other stocks with similar traits as the SPY, for example, the Russell 2000 Index IWM or SPDR Dow Jones Industrial Index ETF DIA.

Problem 3: Can a Trader Outperform a geometric Brownian Motion?

The geometric Brownian motion (GBM) is employed in the Black-Scholes model and is a very popular model for stock market price. GBM is also known as exponential Brownian motion. GBM is always positive and tends to infinity with probability 1 if the drift parameter is positive. In addition, GBM is a Markov process, which means that all information on past historical prices is incorporated in the current price. The general belief is that it is impossible to “beat” a GBM since technical analysis of historical prices is useless in predicting future prices. The last part of the paper shows that the answer to Problem 3 is actually a “YES”, which is quite surprising.

The trading strategies presented are based mainly on information obtained from the movements of waves and wavelets created by large and small fluctuations of market prices. They do not involve any forecasting or prediction of future prices. Behavioral economics also plays a role in the decision making process of the Wavelet Trading program. My website AgateTrading.com is available to the public.

Key words and Phrases. Wavelets, efficient market, geometric Brownian motion.
1. Introduction

Many economists, financial analysts, researchers, traders, and people acquainted with modern business finance theories consider that trading commodities and stocks by any system of buying and selling to make a profit is an impossible task, as there is plenty of empirical evidence and statistics against the existence of such a strategy; traders who buy and sell frequently usually end up losing money. In addition, proponents of the random walk and efficient market hypotheses, considered to be cornerstones of modern finance theory and economics, assert that stocks take an unpredictable path and that it is impossible to outperform the overall market consistently because stocks always trade at their fair value on stock exchanges. Observing that most people lose money in stock market trading brings many to conclude that trading is purely a game of luck. This paper will show that trading commodities and stocks is, in reality, more a game of skill than of luck.

There is a large body of literature on the random walk and efficient market hypotheses. See Basu (1977), Chan, Gup and Pan (2003), Cootner (1964), Fama (1965a, 1965b, 1970), Guerrien and Gun (2011), Lo and MacKinlay (1999), Lo (2005), Malkiel (1973), Pearson (1905), Samuelson (1965), Schwert (2001), Sharpe (1964, 1975), Timmermann and Granger (2004), and the references therein for an account of this information. The random walk theory became well known since the year 1973 when Malkiel (1973) published his book “A Random Walk Down Wall Street.” The efficient market hypothesis evolved from the work of Fama (1965a, 1965b). Both the random walk and the efficient market hypotheses have a strong influence on economics, modern market finances, and stock market trading. There are daily articles in newspapers on trading and related topics. The list of references listed here is a tiny fraction of the actual amount available.

The market is implicitly assumed to be the Standard and Poor’s 500 SPY index, unless indicated otherwise. The SPY index represents a collection of 500 stocks of a diverse group of U.S. companies and is a popular benchmark to measure U.S. stock performance. The SPY data set (Histo SPY Data) displayed on the website plays an important role in the paper. It contains the dates and corresponding adjusted closing prices of SPY from January 29th, 1993 to December 30th, 2016. The adjusted closing prices represent a more accurate reflection of SPY’s performance. They take into account corporate actions such as stock splits and dividends. The data was downloaded on January 1st, 2017 from the Yahoo Finance website online. These prices may look different if they are downloaded on another day. Many examples given in the
paper use the data from the SPY data set. A date of the year will often be displayed in the same style of the SPY data set. For example, January 29th, 1993 is written as 1/29/93.

The tables posted on the website are integral parts of the paper but they are too long to be included here. Each table will be explained as we go along. Interactive Brokers (IB) is used as the brokerage in many examples. The commission charged by IB is $1.00 if the number of shares per trade is 200 or less; it is $0.005 per share if the trade involves more shares. All money referred to is in U.S. dollars. The paper is organized as follows:

**Part 1.** The buy low and sell high problem is discussed in this part which contains Sections 2 and 3. Section 2 presents the Portfolio Checker program (denoted by PC) on my website and gives instructions as to how to employ it as a guide for buying lows and selling highs. A lot of effort on my part has been devoted to make the computer program user friendly. Being able to operate the computer program is necessary to understand what comes later. Section 3 discusses the PC strategy in more detail.

**Part 2.** This part investigates the “Beat the Market” problem. It contains Sections 4-6. Section 4 presents the Wavelet Trading program (denoted by WT) and instructions for a user. To evaluate the performance of WT, I use the SPY data set displayed on the Website as a test case. Table 1 on the website shows the trades by WT on each trading day from 1/29/93 to 12/30/16. WT is programmed to always start the trading by buying 1,500 shares on the first trading day. A share of SPY cost $28.000838 on this day so WT has to pay $42,001.26 for the cost of the shares and another $7.50 in commission. Note that Table 1 contains some other vital statistics: commissions, cost, cumulative cost, and the profit made by the trader. There are many days with no trades. Table 2 is the same as Table 1 with the days of no trades deleted. Table 1 and Table 2 show that WT made a profit of $396,639.09 after her latest trade on 12/30/16.

Did WT outperform a buy-and-hold investor (denoted by BH) who bought 1,500 shares on 1/29/93 and left it alone in the brokerage until 12/30/16? Sections 5 and 6 give an affirmative “YES” answer to this question. Assume the following: WT deposited into the brokerage an initial amount of $42,001.26 and $7.50 in commission on 1/29/93. WT uses margins to buy shares when necessary according to the rules set up by Interactive Brokers (IB). Currently, IB allows a regular trader to use a 2:1 leverage using margin.
Under this scenario, the profit of $396,639.09 mentioned above is not WT’s actual return since it does not take into account the interests paid to (or earned from) the brokerage when she owed the brokerage money (or had surplus cash) in her account. Table 3 in Section 5 shows the computation of the returns of WT and BH from the starting date on 1/29/93 to the last day of trading on 12/30/16.

Section 6 compares the performance of WT and BH. Table 3 shows that the returns of WT and BH are, respectively, $393,169.57 and $293,286.24. By trading, WT has earned a return larger than BH’s return by $99,883.33 or 34.06 percent. Did WT earn a higher return by assuming additional risk? Computing the Sharpe ratios of WT and BH and comparing them is a way to answer this question. The yearly rate of return of WT has a standard deviation of $17.57 as compared with $17.24 for BH and the Sharpe Ratios of WT and BH are found to be 0.44 and 0.38. Therefore, WT has obtained much better risk adjusted returns than the market. The last part of this section shows that WT always “beats” the market eventually for any arbitrary starting trading date. Tables 4 and 5 are examples to persuade the reader that this is indeed the case. Section 7 compares PC versus WT and discusses some important points that were not mentioned in previous sections.

In Section 8, a geometric Brownian motion (GBM) is fitted to Histo Spy Data. Brownian motion can take on negative values but stock prices cannot. Thus, using Brownian motion to model stock prices is disputable. However, GBM only takes positive values and is a Markov process. The future values of GBM given the present value are independent of past values. GBM is often used to model stock prices in a non-arbitrage world. Using the fitted GBM, simulated data is generated to show that WT always “beats” BH eventually. It is quite surprising that a trader can “beat” a market having a perfect fair price by utilizing market fluctuations. At the end of the paper, a heuristic proof that the market is not efficient will be given.

There is a quick way to see that WT “beats” the market. From Table 1, WT makes $396,639.09 in profit while BH makes $223,529 × (223.529999 − 28.000838) − $7.50 = $293,296.2. Note that $7.50 is the commission paid to buy 1,500 shares on the first day of trading. WT’s return is only a few thousand dollars less if margin rates of interest are taken into account (see Table 3).

Throughout the paper, I will implicitly assume that the standard deviation of market prices stays bounded away from zero, that is, it stays above some positive number. When I use the words “tends to”, I always refer to a limit as time tends to infinity.
Part 1: Buy Low and Sell High

2. The Portfolio Checker Program

This section explains how to operate the portfolio checker. The user starts by entering the “Magic Number” on the bottom left side of the screen right below the “Generate Table” button. The Magic Number is currently set to be 8141946.

Remark 2.1. It is important to enter the Magic Number correctly on the screen.

She then needs to come up with three numbers before she makes a trade: the current market price of a share of stock being considered for selling or buying, the number of shares to buy (a positive integer) or sell (a negative integer) and the commission for the trade. She then asks PC if such a trade is feasible. This is done by entering the three numbers on the screen with a comma or a space as a separator and then activating the computer program by clicking on the button with the words “Generate Table”. On the screen, don’t separate digits in a number by using commas. For example: write 1250 instead of 1,250. PC analyzes these numbers and answers the question “Is my Portfolio OK?” with a simple “YES” or “NO”. The user needs to make sure that she gets a “YES” each time she makes a trade.

The following rules are set by PC:

Rule 1. The first trade must be a buy of at least 20 shares. In addition, the cost of the first trade must be at least $1,000.00.

Rule 2. Don’t sell if the current price of a share is less than the price at its last trade.

Rule 3. Don’t buy if the current price of a share is higher than the price at its last trade.

Rule 4. Don’t buy or sell an amount less than 50 times the commission.

Remark 2.2. If you buy a stock of $2.50 a share then to satisfy Rule 1, you need to start with at least 400 shares, while if you buy a stock of $250.00 a share, then you need to start with at least 20 shares, which cost $5,000.00. The number of shares you buy in your first transaction depends on the total amount of money you have allocated for that particular stock. Some funds should be available for additional purchases if necessary. Rules 2 and 3 are essential elements of the buy low and sell high scheme. Rule 4 is to ensure that you don’t buy or sell an amount that is too small to justify the commission that you pay.
PC does not recommend a number of shares to buy or sell. This gives more flexibility to the trader in deciding as to how much to buy or sell. The computer program is designed to assure the user that she will eventually make a profit if she gets a “YES” answer to the portfolio question before each trade that she makes. The trader controls her account balance. If her account balance gets too low, she should buy more often than sell to raise it up.

A “NO” answer occurs due to various reasons. The most common ones are:

(1) The price of the stock traded has not increased enough or decreased enough from the price at its last trade to warrant, respectively, a “sell” or a “buy”. A regular trader with 300 shares of SPY and a bank of about $100,000.00 should wait for the price to change about 2 percent or more from the price at its last trade before she attempts to trade.

(2) The number of shares entered is too high or too low.

Example 2.1. Below is an example of 5 trades of the stock SPY, made respectively by a trader, on January 2nd, 6th, 8th, 14th and February 3rd in the year 2015. Before each trade, the trader used the Portfolio Checker to make sure that she got “YES” for an answer. The trader’s 5 transactions are summarized in the table below:

| Dates    | Price | Shares | Com | Cum Com | Cost  | Cum Cost | Cum Shares | MV       | Profit |
|----------|-------|--------|-----|---------|-------|----------|------------|----------|--------|
| 1/2/15   | 197.05| 300    | 1.50| 1.50    | 59115.00 | 59115.00 | 300        | 59115.00 | -1.50  |
| 1/6/15   | 191.66| 50     | 1.00| 2.50    | 9583.00  | 69698.00 | 350        | 67081.00 | -1620.50 |
| 1/8/15   | 197.50| -36    | 1.00| 3.50    | -7110.00 | 61588.00 | 314        | 62015.00 | 423.50 |
| 1/14/15  | 192.66| 36     | 1.00| 4.50    | 6935.76  | 68523.76 | 350        | 67431.00 | -1097.26 |
| 2/3/15   | 196.48| -28    | 1.00| 5.50    | -5501.44 | 63022.32 | 322        | 63266.56 | 238.74 |

The first four columns list respectively the trading dates, share prices at which trades are made, number of shares traded, and the corresponding commissions. A positive number for the shares means a “buy” and a negative one means a “sell”.

The fifth column (CumCom) lists cumulative commissions, which are total commissions paid up to the trading day. For example, on 1/8/15, after she sold 36 shares at the price of 197.50 a share, she has paid a commission of $1.50 + $1.00 + $1.00, which equals $3.50.

The sixth column (Cost) lists the cost of each trade which equals the number of shares traded multiplied by the price at which it is traded. You have to pay money if it is a positive number and you get back cash
if it is a negative number. A positive number for cost indicates a buy
and a negative number indicates a sell.

The seventh column (CumCost) lists the cumulative cost paid by
the trader. For example, after the fourth transaction is made on
1/14/15, the trader’s cumulative cost is obtained by adding $59,115.00,
$9,583.00, $7,110.00, and $6,935.76, which equals $68,523.76.

The eighth column (CumShare) lists the total number of shares held
by the trader after each trade. For example, after the fifth trade was
carried out on 2/3/15, the trader has a total of shares equal to 322,
which is the sum of all numbers in Column 3.

The ninth column lists the market value of the trader on each trading
day. The market value (MV) is computed by multiplying the cumu-
lative number of shares of the trader by the corresponding price. For
example, on 1/14/15, the market price of the trader is $192.66 × 350,
which equals $67,431.00.

Finally, the tenth column lists the profit after each trade. The profit
made by the trader after each trade is given by:

\[
\text{Profit} = \text{MV} - \text{CumCost} - \text{CumCom}.
\]

The profit after the last trade is $626.74, which is much more than $238.74
since the price of the stock fluctuates more.

It is a good idea for you to copy and paste the data into a text or
Excel file and keep it as your record. If you mistype and then activate
the program, the whole data that you have entered may disappear. In
case this happens, you still have a record of what you have typed, you
can copy and paste your data back onto the computer screen, correct
the mistype, and then reactivate the program. The website does not hold any data for you after you logout.

**Exercise 2.1.** As an exercise, you may try to enter the prices, shares, commissions in the table above on the screen as follows: 197.05, 300, 1.50, 189.66, 50, 1.00, 200.50, −36, 1.00, 187.66, 36, 1.00, 196.48, −28, 1.00. The numbers need to be separated by a comma or a space. The answer to the portfolio question is a “YES” after you activate the program.

**Remark 2.3.** Following Example 2.1, what should your next move be? There are 322 shares left. How about selling ten percent of the shares (32 shares) if your stock price goes up 2 percent or buying 32 additional shares if your stock price decreases 2 percent. The stock price becomes \(0.98 \times 196.48 = 192.55\) if it decreases 2 percent and \(1.02 \times 196.48 = 200.41\) if it increases 2 percent. You find out both of the trades get a “YES” answer by asking PC. Then you can make an advance order with your brokerage to perform whichever transaction that comes first.

There are many ways to enter the data on the “DATA TO CALCULATE” area. You may find it more to your liking to enter the data above as follows:

1. 197.05, 300, 1.50
2. 189.66, 50, 1.00
3. 200.50, −36, 1.00
4. 187.66, 36, 1.00
5. 196.48, −28, 1.00

**Exercise 2.2.** Assume that Scottrade is your brokerage and that your bank is $100,000.00. Note that Scottrade charges a commission fee of 7 dollars for each transaction no matter how many shares are involved. Using the computer program and the five prices of the stock SPY above, but starting with a buy of 350 shares on 1/2/2105, find 4 additional feasible trades on, respectively, the 6th, 8th, 14th and February 3rd of 2015. To get a feasible trade, you have to get a “YES” answer to the portfolio question before each trade is made. Do you make a profit after you finish with the last trade?

Example 2.1 shows that it is possible for you to make a profit even if the share price of the traded stock decreases if you trade properly. You can also lose if the share price of the traded stock increases if trading is not done right. An example is given below.
Example 2.2. Consider the following list of three transactions made by a trader using Interactive Brokers as the brokerage.

| Price | Shares | Com | Cum | Cost  | Cum | Cost  | Cum | Shares | MV   | Profit |
|-------|--------|-----|-----|-------|-----|-------|-----|--------|------|--------|
| 75.49 | 1000   | 5.00| 5.00| 75490.00 | 75490.00 | 1000 | 75490.00 | -5.00 |
| 86.03 | -100   | 1.00| 6.00| -8603.00 | 66887.00 | 900  | 77427.00 | 10534.00 |
| 82.97 | 200    | 1.00| 7.00| 16594.00 | 83481.00 | 1100 | 91267.00 | 7779.00 |

After the last transaction is made, she has a total of 1100 shares. If the share price drops down to $75.60, the market value of these shares would be $1100 \times 75.60 = 83,160.00$, which is less than the cumulative cost of \$83,481.00$. She ends up with a loss of \$83,481.00 - \$83,160.00 = \$328.00$, not counting the commission she has to pay if she cashes out. She loses money while the price of a share goes up from $75.49$ to $75.60$.

Let us ask the computer if these transactions are OK by entering $75.49$, $1000$, $5.00$ $86.03$, $-100$, $1.00$ $82.97$, $200$, $1.00$ on the screen and activate the program. The answer is a “NO”. The reason is because the amount of shares she bought on the last transaction is too high.

How much should she buy then? She should just try a smaller number that she is comfortable with and then ask the computer if buying the latter is ok before making such a trade. How about asking the portfolio checker if buying 60 at $82.97$ is ok? This is done by entering $75.49$, $1000$, $5.00$ $86.03$, $-100$, $1.00$ $82.97$, $60$, $1.00$ on the screen. The answer is “YES”. Suppose she buys 60 shares at $82.97$ per share, and then 40 shares if the price drops back to the starting price of $75.49$, she would make a profit of $597.20$ as shown in the table below.

| Price | Shares | Com | Cum | Cost  | Cum | Cost  | Cum | Shares | MV   | Profit |
|-------|--------|-----|-----|-------|-----|-------|-----|--------|------|--------|
| 75.49 | 1000   | 5.00| 5.00| 75490.00 | 75490.00 | 1000 | 75490.00 | -5.00 |
| 86.03 | -100   | 1.00| 6.00| -8603.00 | 66887.00 | 900  | 77427.00 | 10534.00 |
| 82.97 | 60     | 1.00| 7.00| 4978.20 | 71865.20 | 960  | 79651.20 | 7779.00 |
| 75.49 | 40     | 1.00| 8.00| 3019.60 | 74884.80 | 1000 | 75490.00 | 597.20 |

She again holds 1,000 shares which is the same number of shares she started with. The choice of buying 60 shares and then 40 is purely arbitrary. Buying 40 shares at $82.97$ a share and then 60 shares if
price falls to $75.49 would also get a “YES” answer from the Portfolio Checker. The answer to the question as to how many shares to buy or how many shares to sell depends on many variables: the commission you have to pay, cost of a share, among others. The best way to learn how to use the website is for you to play with it.

Exercise 2.3. In Example 2.2 above, buying 200 shares at $82.97 a share is not an acceptable trade. What is the maximum allowable number of shares that she can buy at $82.97 to get a “YES” answer?

Hint: Use portfolio checker and get the answer by trial and error. Decrease the number 200 slowly until you get “YES” for an answer.

It is fairly easy for you to get a “YES” answer from the portfolio checker after some practice. Before you quit using the website, make sure you save the historical data of your trades in some file. Click on the “Select Table” button on the upper right hand corner of the screen and use copy and paste to save your data.

You need to upload all the data back to the screen in case you want to ask the portfolio checker some time later if your “next” trade is ok or not. Whether your “next” trade is feasible or not depends not only on your “last” move but on the entire historical data of your trades.

3. Can PC “beat” the market?

The table below shows the trading of SPY from 4/6/98 to 9/8/03 when its share price moved from $79.60 on 4/6/98 to $79.03 on 9/8/03, which is a period of over 5 years. During this period, PC makes a profit of $14,721.15 while the market went down slightly. However, market price does not stay flat all the time. It can rise and fall dramatically for long periods of time in, respectively, a bull market and bear market. During these periods the market (or BH) can “beat” PC badly. Below are two main problems:

1. **Trader sells too many of her shares when market goes up.** Suppose a trader sells 10 percent of her shares whenever her stock price goes up 2 percent, she will have few shares left to sell if price goes straight up by 20 percent. And if price rises even more thereafter, the market ends up beating her badly. She would have won more if she had held to her shares without any selling.

2. **Trader buys too many of her shares when market goes down.** Suppose a trader buys to increase her shares by 10 percent whenever her stock price goes down 2 percent, she will have more than
twice as many shares as she started with if market price goes straight down 20 percent. She can lose big if price keeps falling thereafter.

As a remedy, use the following rule when you use PC: don’t let your cumulative cost vary by a large amount. This will prevent you from selling or buying too much when price is, respectively, too high or too low.

| Dates  | Price  | Shares | Com | Cum Com | Cost   | Cum Cost | Cum Shares | MV    | Profit |
|--------|--------|--------|-----|---------|--------|----------|------------|-------|--------|
| 4/6/98 | 79.60  | 1000   | 5.00| 5.00   | 79600.00| 79600.00 | 1000       | 79600.00| -5.00  |
| 4/27/98| 77.49  | 100    | 6.00| 7749.00 | 87349.00| 1100     | 85239.00  | -2116.00|
| 5/20/98| 80.11  | -100   | 7.00| -8011.00| 79338.00| 1000     | 80110.00  | 765.00  |
| 9/2/98 | 71.03  | 200    | 8.00| 14206.00| 93544.00| 1200     | 85236.00  | -8316.00|
| 11/4/98| 80.54  | -150   | 9.00| -12081.00| 81463.00| 1100     | 85239.00  | 3095.00 |
| 4/2/98 | 79.60  | 1000   | 5.00| 79600.00| 79600.00| 1000     | 79600.00  | -5.00   |
| 1/7/99 | 91.29  | -150   | 11.00| -13693.50| 50639.00| 700      | 63903.00  | 13252.50|
| 4/22/99| 98.26  | -100   | 12.00| -9826.00| 40813.50| 600      | 58956.00  | 18130.50|
| 6/2/99 | 93.72  | 150    | 13.00| 14058.00| 54871.50| 850      | 84567.00  | -8316.00|
| 9/2/99 | 87.80  | -100   | 14.00| -10016.00| 43906.00| 650      | 65280.00  | 20774.50|
| 11/4/99| 80.54  | -150   | 15.00| -15132.00| 39739.50| 600      | 65280.00  | 20774.50|
| 1/7/99 | 91.29  | -150   | 16.00| -13693.50| 50639.00| 700      | 63903.00  | 13252.50|

Exercise 3.1 Show that you always win eventually by using PC. Please be reminded that the standard deviation of SPY is assumed to always stay above a positive number. For your information, the standard deviation of SPY prices on Histo SPY Data is $47.39.

Hint. In the beginning you can lose if share price drops. However, the assumption on the standard deviation ensures that share price has to stop dropping after some time. Thereafter, your gain by buy low and sell high will eventually surpass this initial loss.
Part 2: Wavelet Trading

4. The Wavelet Trading Program

Share prices are written with up to 6 decimals in this section. As an example, suppose that a trader started trading SPY on January 3rd, 2000. She entered the date and price of SPY (see Hist SPY Data on the website) on this date on the screen:

1/29/93  28.000838

She then asks WT what action to take by clicking on the button with the words “Generate Table”. WT analyzes these numbers and gives the following reply:

| Dates  | Price     | Shares | Com |
|--------|-----------|--------|-----|
| 1/29/93| 28.000838 | 1500   | 7.50|

Now she moved to the second day of trading and entered the dates and prices of SPY of both days:

1/29/93  28.000838
2/1/93   28.199990

and then clicked on Generate Table. WT replied with the following table:

| Dates  | Price     | Shares | Com |
|--------|-----------|--------|-----|
| 1/29/93| 28.000838 | 1500   | 7.50|
| 2/1/93 | 28.199990 | 0      | 0.00|

WT’s advice on 2/1/93 is not to buy any shares and WT computes the commission, cumulative cost and other statistics for the trader. She kept on repeating this process and her record for the first 15 days is:

| Dates  | Price     | Shares | Com | Cum | Cum |
|--------|-----------|--------|-----|-----|-----|
| 1/29/93| 28.000838 | 1500   | 7.50| 7.50| 7.50|
| 2/1/93 | 28.199990 | 0      | 0.00| 0.00| 0.00|
| 2/2/93 | 28.259704 | 0      | 0.00| 0.00| 0.00|
| 2/3/93 | 28.558465 | 0      | 0.00| 0.00| 0.00|
| 2/4/93 | 28.677956 | 0      | 0.00| 0.00| 0.00|
| 2/5/93 | 28.658009 | 0      | 0.00| 0.00| 0.00|
| 2/8/93 | 28.658009 | 0      | 0.00| 0.00| 0.00|
| 2/9/93 | 28.458857 | 0      | 0.00| 0.00| 0.00|
| 2/10/93| 28.498687| 0      | 0.00| 0.00| 0.00|
| 2/11/93| 28.638126| 0      | 0.00| 0.00| 0.00|
| 2/12/93| 28.419026| 200    | 1.00| 8.50| 5683.81|
| 2/13/93| 27.702077| 300    | 1.50| 10.00| 8310.62|
| 2/14/93| 27.682194| 0      | 0.00| 10.00| 0.00|
| 2/15/93| 27.662247| 0      | 0.00| 10.00| 0.00|
| 2/16/93| 27.761855| -300   | 1.50| 11.50| -8328.56|

MV: 47667.13  Profit: -483.48
Exercise 4.1. Table 1 on the website shows the number of shares traded by WT on each trading day from 1/29/93 to 12/30/16. Table 2 is the same as Table 1 with the days of no trades deleted. Go to the Wavelet Trading page of the website AgateTrading.com and paste the SPY data set onto the screen and activate the program. Download the files resultTable.csv and resultTableNoZeros.csv to your computer and verify that they are identical to Tables 1 and 2.

5. How Table 3 is established.

Assume that both WT and BH have a brokerage account with Interactive Brokers. An owner of a memorandum account with a minimum balance of $25,000 can use a 2:1 margin to buy additional shares. A trader with more than $100,000 can open a special memorandum account, which allows a 2.25:1 margin. The strategies of WT and BH are as follows: WT bought 1,500 shares of SPY at the price of $28.000838 on 1/29/93. The total cost is $42,008.76 which covers $42,001.26 for the cost of the shares bought and $7.50 in commission. WT may use margin to buy additional shares according to the rules set up by Interactive Brokers. BH took the same action as WT on 1/29/93 but BH made no other trades from there on. Note that the actions of WT and BH are identical on the first day of trading.

The performances of WT and BH will be compared in the next section. Table 3 plays an important role in this comparison. It is important to understand how the numbers in the table are computed and what the variables in each column in the table mean. The table contains all WT’s trades from 1/29/93 to 12/30/16. The numbers on Table 2 are used to make Table 3. The reader should know by now how the first ten columns A-I are. The rest of this section is to explain how the numbers in other columns are computed.

IRMB and IRMA. IRMB denotes the rate of interest that the trader pays the brokerage when she borrows money and IRMA is the rate of interest that she collects from the brokerage when she has surplus cash in her account. The rates of interest earned vary with different brokerages but currently Interactive Brokers charges about the lowest interest. If the trader has a lot of cash in the account, then she can move it into a sweep account and possibly collect higher interests than leaving it in the brokerage account. To make things simple, I just assume that when the trader has cash in the account, she collects interest from her brokerage at the rate equal to half the interest rate she pays when she borrows. The rate of interest on money borrowed at Interactive Brokers for 2016 is:
Interactive Brokers was established in 1993 and is the largest U.S. electronic brokerage firm by number of daily average revenue trades. I do not have the exact data on interest rates charged by Interactive Brokers for the years before 2016. The rates of interest listed below for the year 1993-2015 are set to be 1.05 times the interest rates set by the Federal Reserve board. These interest rates will be employed to calculate the interests the trader has to pay on margin when she borrows money from Interactive Brokers. Changing them by a few percentage points does not affect much the conclusion of my findings. From here on, when I write an interest rate of, say 0.013, I mean 1.3%.

| Year | IRMB | IRMA |
|------|------|------|
| 1993 | 0.060 | 0.030 |
| 1994 | 0.050 | 0.025 |
| 1995 | 0.060 | 0.030 |
| 1996 | 0.065 | 0.033 |
| 1997 | 0.065 | 0.033 |
| 1998 | 0.065 | 0.033 |
| 1999 | 0.065 | 0.033 |
| 2000 | 0.065 | 0.033 |
| 2001 | 0.065 | 0.033 |
| 2002 | 0.070 | 0.035 |
| 2003 | 0.040 | 0.020 |
| 2004 | 0.035 | 0.018 |
| 2005 | 0.050 | 0.025 |
| 2006 | 0.060 | 0.030 |
| 2007 | 0.060 | 0.030 |
| 2008 | 0.050 | 0.025 |
| 2009 | 0.040 | 0.020 |
| 2010 | 0.020 | 0.010 |
| 2011 | 0.020 | 0.010 |
| 2012 | 0.020 | 0.010 |
| 2013 | 0.020 | 0.010 |
| 2014 | 0.020 | 0.010 |
| 2015 | 0.020 | 0.010 |
| 2016 | 0.013 | 0.006 |

**Interest.** Interest (denoted by Int) on Table 3 lists the interest that trader has to pay or earns depending, respectively, on whether her “cash” is positive or negative. Next are some examples:

On 3/25/93, trader owed an interest of $14.05 on a debt of $14,243.35 held for 6 days. The rate of interest used to calculate interest is 0.06.

On 7/20/99, trader collects $38.58 in interest for the amount of $20,632.61 in cash that she held in her account for 21 days. The rate of interest used to calculate interest is now only 0.033.
I will later explain more on how interest is calculated when I discuss “Cash”.

**Margin.** Buying on margin means borrowing money from Interactive Brokers to buy stocks. The collateral for the borrowed funds is the stocks and cash in the investor’s account. Before buying on margin, an investor needs to open a margin account with the brokerage. Maintenance margin refers to the minimum amount of money that must exist in the account before the broker forces the investor to deposit more money.

In the United States, the Federal Reserve Board regulates the amount of margin that an investor must pay for a security. An investor is required to fund at least 50% of a security’s purchase price with cash. She may borrow the remaining 50% from a broker or a dealer.

**Cash.** The amount of cash in BH’s account is always zero since all of BH’s money is in stocks. The amount of cash in WT’s account varies and is rather complicated to figure out. If she buys on margin and owes the brokerage money, the cash amount is a negative number. She has to pay interest on this cash that she owes. The “Cash” variable of WT is zero between the day 1/29/93 until 2/12/93 when she bought on margin 200 shares at the price of $28.419026 a share. The cost is $5,683.81, which equals $5,683.81. Her cash after she bought 200 shares is −$5,684.81 which covers the $1.00 commission and the cost of the 200 shares. The negative sign indicates that she owed money after making this trade and she had to pay interest on this amount while she kept it.

Her next trade was a “buy” of 300 shares made on 2/16/93 at the price of $27.702077 a share. She has to pay $1.50 in commission and $8,310.62 for the cost of the shares. She now owes the interest on the $5,684.81 that she borrowed on 2/12/93. This period is four days long and the interest is

\[
\frac{5684.81 \times 4 \times 0.060}{365} = 3.74
\]

After the third transaction, her cash is

\[-5684.81 - 1.50 - 8310.62 - 3.74,\]

which equals -$14,000.67.

Let us compute her cash after one more transaction. On 2/19/93, she sold 300 shares at the price of $27.761855 a share, which amounts to $8,328.56. Her cash is equal to

\[-14000.67 - 1.50 + 8328.56 - 6.90 = -5680.51,\]
This amount consists of $14,000.67 in debt (carry over), an additional commission of $1.50, an amount $8,328.56 from the sale of 300 shares and $6.90 in interest on the debt of $14,000.67 which she held for 3 days.

Exercise 5.1. As an exercise, you may try to verify (using a calculator) that the trader’s cash is $35.84 after she made the next trade, which is a sell of 200 shares on 3/2/93 at the price of $28.638126 a share.

Buying Power. This is the money currently available for WT to buy additional shares. This amount is set to be at most twice the net liquidation value (NLV) of her account. Her NLV is her account balance which is the sum of her market value (MV) and cash. Hence, her buying power is

\[ 2NLV - MV = 2(MV + \text{Cash}) - MV = MV + 2\text{Cash}. \]

The Difference between Dates. DD on Table 3 denotes the difference between two trading dates. They are used in the computation of interests that WT pays if she borrows money from the brokerage to buy additional stocks. They are also used to compute the interest that the trader collects if she has surplus cash in her account.

6. Comparing the performances of The Trader versus the Buy-and-Hold Investor.

Table 3 shows that by 12/30/16, the returns of WT and BH are, respectively, $393,169.57 and $293,286.24. Thus WT’s return exceeds BH’s return by $99,883.33 or 34.06 percent.

Has WT earned much higher risk-adjusted returns than BH? This question can be answered by finding their Sharpe ratios. The Sharpe Ratio (denoted by SR) is widely used for risk-adjusted returns and can be computed using the formula:

\[ \text{SR} = \frac{(\text{Average Rate of Returns} - \text{Risk-free Rate})}{\sigma(\text{Rate of Returns})}, \]

where \( \sigma \) denotes the standard deviation. The numbers on the table below are derived from Table 3. They are used to compute Sharpe ratios of WT and BH.
I now explain how the numbers on the table are computed. WT's account balance is equal to the sum of her stock market value and available cash. BH’s account balance is equal to her stock market value (MV).

The starting account balance of WT in a year is her account balance at the first trading day of that year. During the period 1993-2015, the ending account balance of each year is set to be the starting account balance of the following year. For 2016, the ending account balance is set equal to her account balance on the last trading day of the year. Using Table 3, the ending account balance as of 12/30/2016 is found to be $447,060.00-$11,881.67 or $435,178.33. The return of a certain year is obtained by subtracting her beginning account balance from her ending account balance.

The starting and ending account balances of BH are defined similarly. Note that from Table 3, the ending account balance of BH for 2016 is $1500 × $223.529999 or $335,295.00. The account balances of WT and BH at the start of a year are abbreviated, respectively, by WTStartBal and BHStartBal. The yearly returns of WT and BH are denoted, respectively, by WTReturn and BHReturn. WT’s and BH’s rates of return are denoted, respectively, by WTRateRet and BHRateRet.

As an example, I calculate the returns and rates of returns for WT and BH for the year 1998. The return of WT for this year is equal
to $148,631.93 − $115,357.15 = $33,274.78. Note that $148,631.93 is the ending account balance of WT for 1998, which equals the beginning balance of 1999. Her beginning account balance for 1998 is $115,357.15. Thus WT’s rate of return (WTRateRet) for 1998 equals

\[
100 \times \frac{33274.78}{115357.15} = 28.85
\]

BH’s rates of returns are simpler to calculate. BH’s return for 1998 is equal to $132,853.32 − $104,235.36 = $28,617.96, and hence BH’s rate of return for this year equals

\[
100 \times \frac{28617.96}{104235.36} = 27.46
\]

I use 4% for the risk-free rate. The Sharpe ratios of the rates of returns for WT and BH are found, respectively, to be 0.44 and 0.38. They show that WT produces much better risk-adjusted returns than BH.

Next, I show that WT eventually outperforms the market using any day as her first day of trading. Table 4 calculates the returns of WT and BH using 7/16/07 as the starting date. This date is about the worst one for WT to start her trading since the market crashed soon thereafter. However, by 12/30/16, her return is $196,410.94 as compared with $145,112.45 for BH. The return of WT exceeds that of BH by 35.35 percent. Table 5 calculates the returns of WT and BH using 1/4/10 as the starting date of trading. This date was chosen rather arbitrarily. WT’s return is $244,508.53 as compared with $187,965.94 for BH. WT’s return exceeds BH’s return by 30.08 percent.

7. Discussions and Comparisons Between PC and WT

1. Parts 1 and 2 present two different strategies. Since a share of SPY currently costs over 200 dollars, using WT requires a large initial amount of money. WT is designed mainly to trade the SPY index or other index stocks that represent a broad section of companies, for example, the Russell 2000 Index IWM or SPDR Dow Jones Industrial Index ETF DIA. Using PC would be a much better strategy for trading volatile stocks that do not exhibit any trends, for example, the Silver Standard Resources Inc. stock SSRI, the SPDR Gold Shares GLD, or VelocityShares Daily Inverse XIV. Seasonal stocks often exhibit this type of behavior.

2. PC requires that a trader waits until the price of the stock traded gets sufficiently low to buy or sufficiently high to sell. The waiting time to make the next trade may be long. A user needs some skill and practice to familiarize with the strategy before doing real life trading.
3. WT uses the movements of daily prices of SPY to make trading decisions. A trader using WT must enter the date and price of SPY of every trading day. She is required to buy, sell or hold according to the advice of WT. No practice or skill is needed to operate WT.

4. Suppose WT advises a trader to buy 200 shares at some price. The trader may not be able to buy 200 shares at exactly the price recommended since the price of the stock might have changed by a small amount by the time she actually buys. Buying at prices approximately equal to the prices recommended by WT would not affect the trader’s overall return much.

5. What if the user of WT trades more frequently, for example, every hour of the day? Trading here means she enters the date, share price and take the action recommended. This will probably increase her return. It would be interesting to see how WT performs with continuous data. I do not have a definite answer to the question at this time.

6. It is possible to create a sophisticated trading system which can monitor stock prices and trade automatically. This could even lead to a higher return.

7. WT can be adapted to use for any brokerage. A user can also start with an amount of shares different than 1,500. Table 6 shows the trades of WT starting with 750 shares and using Scottrade as the brokerage. Note that the number of shares of each of WT’s trade has been cut down by a half. Scottrade charges much higher margin rates of interest than Interactive Brokers. The numbers on the IRMB and IRMA columns of Table 6 are only rough estimates of Scottrade’s actual rates since I do not have historical data on its rates of interest. Her return exceeds the BH’s return by 17.15 percent. Scottrade charges $7.00 in commission per trade and high interest rates so WT’s return has been reduced substantially.

8. A hedge fund manager using a 5:1 margin leverage can increase her return substantially by using WT. Her buying power is now

\[ 5NLV - MV = 5(MV + \text{Cash}) - MV = 4MV + 5\text{Cash}. \]

Table 7 lists her trades from 1/29/93 to 12/30/16. She starts with 1,500 shares but the number of shares of each of her trade equals to twice the number of shares recommended by WT. Her return is $493,753.61 as compared with $293,286.24 for BH. By trading, her return exceeds the BH’s return by 68.35 percent. Table 8 shows that she never got a
margin call even if she started her trading on 9/28/07 before the crash of 2008-2009. Note that, with a 5:1 leverage, she never got a margin call since her buying power is always positive.

9. A trader gets a margin call if the buying power of her account becomes negative. The probability that WT gets a margin call is set very low. A stock market crash much worse than the crash in 2008-09 must occur for WT to get a margin call. Note that the buying power of WT is always a positive number in Tables 3-9.

10. A regular trader can buy and sell using a more aggressive strategy than WT (for example, by following the strategy of the hedge fund manager mentioned on point 8 above) to increase her return. The probability that she ever gets a margin call is still very small. However, rare events can happen after a long period of time. For example, if she started trading on 7/17/07 using the hedge fund manager’s strategy mentioned above, she would likely get a margin call by September 2009 if the brokerage only allowed her a 2:1 margin leverage.

11. WT is established under the assumption that SPY has an upward trend tending to infinity. WT generally outperforms the market during periods when market price is increasing. As a tradeoff, it might underperform the market during periods when market price is decreasing.

12. How long does it take for WT to “beat” the market? A bear market can last for years. Thus it can take years for WT to outperform the market. In Table 4, WT started trading on 7/17/07 at the beginning of a bear market. It took 5 years of trading for her return to exceed that of BH.

13. The important matters affecting WT’s trades are: “How high are the margin interest rates?”, “How much commissions does the brokerage charge?”, “How much margin leverage is WT allowed to use?”

14. Behavioral economics plays an important role in WT’s decision making process. WT often buys high when there is a big drop in share price. For example, WT bought 500 shares on 9/9/16 at share price $210.964611 but did not buy any shares on 7/7/16 at share price $207.255322 which is a lot less (see Table 1 on the website). Note that the drop in price on 9/9/16 was much bigger than the drop in price on 7/7/16.

Below are some exercises for the reader. Assume reasonable commissions and rates of margin interest.
Exercise 7.1. Download the historical data of the Russell index 2000. Use Table 3, show that WT always “beats” the Russell index in the long run.

Exercise 7.2. Suppose that an efficient market “in the sense that no trader can outperform it” exists. Show that its price tends to infinity.

Hint: It is not hard to see from Exercise 3.1 that the return of PC actually tends to infinity. Since no trader can “beat” an efficient market in the long run, BH’s return has to tend to infinity at least a bit faster than PC’s return. Note that BH’s return can only tend to infinity if the market price tends to infinity.

This exercise has an interesting consequence. It shows that stocks whose prices fluctuate widely without a clear upward or downward trend are inefficient.

Exercise 7.3. Persuade yourself that if the market price tends to infinity, then WT “beats” the market. You can do this by taking long sequences of historical prices of index stocks with increasing trends and use the Wavelet Trading page together with Table 3 to show that PC eventually earns a higher return than BH.

8. CAN WAVELET TRADING “BEAT” A MARKET THAT IS A GEOMETRIC BROWNIAN MOTION?

The equation for a geometric Brownian motion (GBM) is given by:

\[ S_t = S_0 \exp \left( (\mu - \frac{\sigma^2}{2})t + \sigma W_t \right), \]

where \( W_t \) is standard Brownian motion. Here, \( S_t \) is the value of GBM at time \( t \) and \( S_0 \) is the initial value. GBM serves as an important example of a stochastic process satisfying a stochastic differential equation. The parameters \(-\infty < \mu < \infty\) and \( \sigma > 0 \) are constants. The mean and variance of \( S_t \) are:

\begin{align*}
\mathbb{E}(S_t) &= S_0 \exp(\mu t), \\
\text{Var}(S_t) &= (S_0)^2 \exp(2\mu t) \left( \exp(\sigma^2 t) - 1 \right),
\end{align*}

The question raised at the beginning of this section is answered in 3 steps:

Step 1. Fit a GBM to Histo SPY prices.
Let $X(t) = \log S_t - \log S_{t-1}$. Then

$$X(t) = \mu - (\sigma^2/2) + \sigma(W_t - W_{t-1}).$$

Then $X(t)$ is normally distributed with mean $\mu - (\sigma^2/2)$ and standard deviation $\sigma$. A total of 6,025 values of $X_t$ is obtained from Histo SPY Data. Using these values, estimates of the drift parameter $\mu - (\sigma^2/2)$ and standard deviation $\sigma$ are, respectively, 0.000345 and 0.011772. These estimates are, respectively, the sample mean and sample standard deviation of 6,025 observations of $X_t$'s. The parameter $\mu$ is estimated by 0.000414.

Since the drift parameter $\mu - (\sigma^2/2)$ is positive, $S_t$ tends to infinity with probability 1. This fits well with SPY which has been rising fairly steadily with time. The reader is referred to Oksendal, B. K. (2002) and Ross (2014) for more information on GBM.

Step 2. Generate Simulated Data Using the Fitted GBM.

I now show you how to employ the page GBM Sim Data (an Excel file) displayed on the website to generate simulated prices. The day 12/30/2016 is set to be $t_0$ and the initial price $S_0$ is set equal to the price of SPY on this day. The equation for the geometric Brownian motion obtained is

$$S_t = 223.529999 \exp \left( 0.000345 t + 0.011772 W_t \right).$$

Column 1 on GBM Sim Data lists 10,000 dates with 12/30/2016 as the first date. Column 2 lists the 10,000 time points from 0 to 9,999. The first number on column 3 is the value of standard Brownian motion at time 0, which is zero. The second number on column 3 is obtained by generating a standard normal random variable using the function NORMINV(RAND(),0,1) in Excel. The third number on column 3 is obtained by adding the second number to another randomly generated standard normal random variable, and so on until the last number, which is the sum of 9,999 independent standard normal random variables. The 3rd column contains 10,000 observations of simulated Brownian motion. Column 4 contains the values of simulated SPY prices computed using (8.3).
A simple computation shows that the expectation and variance of $S_{9999}$ are, respectively,

$$223.529999 \exp(0.000414 \times 9999) = 14032,$$

and

$$(223.529999)^2 \exp(8.279172) \times (0.385661) = 75940958.$$

\[\square\]

Step 3. Evaluate the Performance of WT Using Simulated Data.

After you have a list of simulated data, paste your data with dates and prices onto the Wavelet Trading page of the website and click on the Generate Table button. Wavelet Trading does the necessary computations and produces the output files resultTable.csv and resultTableNoZeros.csv. Download both the files to your computer. File resultTable.csv lists all of WT’s trades from day 0 to day 9999. This file includes many days with no trading and is for your inspection only. File resultTableNoZeros.csv contains only days with trades.

Table 9 on the website displays the returns of WT and BH using a set of simulated data that I obtained. I assume WT is a regular trader having a 2:1 margin leverage. I use the current margin of interest 0.013 as estimates for future years though interest rate will probably get higher in the future. I also use the current commissions set by Interactive Brokers as estimates of future commissions. After 10,000 days of trading, WT’s return is $21,181,912.67 as compared with $15,867,769.14 for BH. Thus, WT’s return exceeds BH’s return by 33.49 percent. WT still outperforms BH by a substantial amount if the rate of margin interest 0.013 is changed to 0.06, for example.

How many shares should a trader start with? The default number is set to be 1,500. If a trader can easily borrow money to deposit in the brokerage in case she gets a margin call, then she can start with a number smaller than 1,500, say, 1,000 shares. Decreasing the initial number of shares bought results in an increase in the probability of a margin call.

Using Sim Spy Data, the wavelet page, and Table 9, you can give a proof by simulation to show that WT always eventually outperforms BH. This is done by repeatedly generating samples of simulated observations and comparing the performances of WT and BH for each generated sample. More detail is provided in the exercise below.

**Exercise 8.1.** Show that WT generally outperforms BH after 10,000 days by going through the following steps:
(i) Download the Excel file “GBM Sim Data” posted on my website AgateTrading.com to your computer and generate 10,000 simulated values of SPY.

(ii) Paste your data containing only dates and prices of GBM on the Wavelet Trading page and click on the Generate Table button. Download the output files resultTable.csv and resultTableNoZeros.csv to your computer. Before you paste your data on the Wavelet Trading page, make sure that the trading dates are written with two numbers for the month, two numbers for the day and four numbers for the year. Do the trades on the two output files match?

(iii) Find the returns of WT and BH for your set of simulated data assuming that the margin interest rate is 0.013 and the margin leverage is 2:1. Does WT outperform BH? The page “Computation of Returns” on the website is an Excel file to help you compute the returns of WT and BH.

(iv) Repeat (i), (ii), (iii) over and over to show that WT outperforms BH.

□

A HEURISTIC PROOF THAT SPY IS NOT EFFICIENT

Actually, the proof follows from Exercises 7.2 and 7.3. Recall that an efficient market is as defined in Exercise 7.2. I will show that either PC or WT will beat the market consistently. Therefore the market cannot be efficient.

By the assumption made in the introduction, the standard deviation of the market price stays above a constant, say 10, for example. If the market price does not tend to infinity, then there must be some price that it will visit infinitely often. In this case, PC will “beat” the market again and again, making the market inefficient.

Thus market price has to tend to infinity to stay efficient. However, in this scenario WT will “beat” it. In this section I fit SPY with a GBM. The result that WT’s return increases at a faster rate than BH’s return would not change if SPY is fitted with another stochastic process.

In the proof, I assume that the commissions stay reasonable and a trader should be able to get a margin loan at leverage 2:1. The Black-Scholes model assumes that it is possible to borrow any amount of money at the risk free rate, and trades do not incur any fees. If this is the case, WT can increase her return considerably by employing a more aggressive strategy, for example, the strategy of the hedge fund manager mentioned in Point 8 of Section 7.
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