THE $p$-ADIC VALUATIONS OF SEQUENCES COUNTING
ALTERNATING SIGN MATRICES

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Abstract. The $p$-adic valuations of a sequence of integers counting
alternating sign symmetric matrices is examined for $p = 2$ and $3$. Sym-
metry properties of their graphs produce a new proof of the result that
characterizes the indices that yield an odd number of matrices.

1. Introduction

The magnificent book Proofs and Confirmations by David Bressoud [4]
tells the story of the Alternating Sign Matrix Conjecture (ASM) and its proof.
This remarkable result involves the counting functions

\[ T(n) = \prod_{j=0}^{n-1} \frac{(3j + 1)!}{(n+j)!} \]

and

\[ C(n) = \prod_{j=0}^{n-1} \frac{(3j + 1)!(6j)!(2j)!}{(3j)!(4j + 1)!(4j)!} \]

The survey by Bressoud and Propp [5] describes the mathematics underlying
this problem.

The fact that these numbers are integers is a direct consequence of their
appearance as counting sequences. Mills, Robbins and Rumsey [12] con-
jectured that the number of $n \times n$ matrices whose entries are $-1$, $0$, or $1$, whose
row and column sums are all 1, and such that in every row, and in every col-
umn the non-zero entries alternate in sign is given by $T(n)$. The first proof
of this ASM conjecture was provided by D. Zeilberger [13]. This proof had
the added feature of being pre-refereed. Its 76 pages were subdivided by the
author who provided a tree structure for the proof. An army of volunteers
provided checks for each node in the tree. The request for checkers can be
read in

http://www.math.rutgers.edu/~zeilberg/asm/CHECKING

The question of integrality of quotients of factorials, such as $T(n)$, has
been considered by D. Cartwright and J. Kupka in [6].
Theorem 1.1. Assume that for every integer $k \geq 2$ we have
\begin{equation}
\sum_{i=1}^{m} \left\lfloor \frac{a_i}{k} \right\rfloor \leq \sum_{j=1}^{n} \left\lfloor \frac{b_j}{k} \right\rfloor.
\end{equation}
Then the ratio of $\prod_{j=1}^{n} b_j!$ to $\prod_{i=1}^{m} a_i!$ is an integer.

The authors [6] use this result to prove that $T(n)$ is an integer.

Given an interesting sequence of integers, it is a natural question to explore the structure of their factorization into primes. This is measured by the $p$-adic valuation of the elements of the sequence.

Definition 1.2. Given a prime $p$ and a positive integer $x \neq 0$, write $x = p^m y$, with $y$ not divisible by $p$. The exponent $m$ is the $p$-adic valuation of $x$, denoted by $\nu_p(x)$. This definition is extended to $x = a/b \in \mathbb{Q}$ via $\nu_p(x) = \nu_p(a) - \nu_p(b)$. We leave the value $\nu_p(0)$ as undefined.

The reader will find in [1] an analysis of the sequence
\begin{equation}
A_{l,m} = \frac{l! m!}{2^{m-l}} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{l}
\end{equation}
for fixed $l \in \mathbb{N}$. The sequence of rational numbers
\begin{equation}
d_{l,m} = \frac{A_{l,m}}{l! m! 2^{m+l}}
\end{equation}
appeared in [3] in relation to the evaluation
\begin{equation}
\int_{0}^{\infty} \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{\sqrt{2m!(4(2a+1))^{m+1/2}}} \sum_{l=0}^{m} A_{l,m} \frac{a^l}{l!}.
\end{equation}
This is a remarkable sequence of integers and some of its properties are described in [11]. In [2] the reader will find similar studies for the Stirling numbers of the second kind.

In this paper we discuss the $p$-adic valuation of the sequence $T(n)$. The data seems erratic, as seen in the case of the first few primes
\begin{align*}
\nu_2(T(n)) &= \{0, 1, 0, 1, 0, 2, 2, 3, 2, 2, 0, 2, 4, 4, 5, 4, 4, 2, 2, \cdots \} \\
\nu_3(T(n)) &= \{0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 2, 3, 5, 5, 3, 2, 1, 0, 0, 0, \cdots \}.
\end{align*}
\begin{align*}
\nu_5(T(n)) &= \{0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 4, 3, 2, 1, 0, 0, 0, \cdots \}.
\end{align*}

The goal of this paper is to provide a complete description of the function $\nu_p(T(n))$ for the primes $p = 2$ and $p = 3$. The case $p \geq 5$ presents similar features and the techniques described here might be used to explain the graphs shown in Figure 5 and 6. A detailed study of the graph of $\nu_2 \circ T$ yields a new proof of a result of D. Frey and J. Sellers: the number $T(n)$ is odd if and only if $n$ is a Jacobstahl number $J_m$. These numbers are defined by
the recurrence $J_n = J_{n-1} + 2J_{n-2}$ with initial conditions $J_1 = 1$ and $J_2 = 3$.

The proof presented here is based on the fact that the graph of $\nu_2(T(j))$ is formed by blocks over the intervals $\{[J_n, J_{n+1}] : n \in \mathbb{N}\}$. Moreover, the part over $[J_{n+1}, J_n]$ contains, at the center, a vertical shift of the graph over $[J_{n-1}, J_n]$. This proves that the valuation $\nu \circ T$ can only vanish at the endpoints $J_n$.

Introduce a generalization of $T(n)$ as

$$T_p(n) := \prod_{j=0}^{n-1} \frac{(pj + 1)!}{(n+j)!}.$$  

We will establish that, for each $p$, the numbers $T_p(n)$ are integers and examine some of their divisibility properties. A combinatorial interpretation of $T_p(n)$ is left as an open question.

2. A recurrence

The integers $T(n)$ grow rapidly and a direct calculation using (1.1) is impractical. The number of digits of $T(10^k)$ is 12, 1136, 113622 and 11362189 for $1 \leq k \leq 4$. Naturally, the prime factorization of $T(n)$ is more promising, since every prime $p$ dividing $T(n)$ satisfies $p \leq 3n - 2$.

In this section we discuss a recurrence for the $p$-adic valuation of $T(n)$, that permits a fast computation of this function. The statement involves the function

$$f_p(j) := \nu_p(j!).$$

**Theorem 2.1.** Let $p$ be a prime. Then the $p$-adic valuation of $T(n)$ satisfies

$$\nu_p(T(n+1)) = \nu_p(T(n)) + f_p(3n+1) + f_p(n) - f_p(2n) - f_p(2n+1).$$

**Proof.** This follows directly from comparing the expression

$$\nu_p(T(n)) = \sum_{j=0}^{n-1} f_p(3j + 1) - \sum_{j=0}^{n-1} f_p(n + j)$$

with the corresponding one for $\nu_p(T(n+1))$ and the initial value $T(1) = 1$. \hfill \Box

Legendre [10] established the formula

$$f_p(j) = \nu_p(j!) = \frac{j - S_p(j)}{p - 1},$$

where $S_p(j)$ denotes the sum of the base-$p$ digits of $j$. The result of Theorem 2.1 is now expressed in terms of the function $S_p$. 
Corollary 2.2. The $p$-adic valuation of $T(n)$ is given by

\[(2.5) \quad \nu_p(T(n)) = \frac{1}{p-1} \left( \sum_{j=0}^{n-1} S_p(n+j) - \sum_{j=0}^{n-1} S_p(3j+1) \right). \]

Summing the recurrence (2.2) and using $T(1) = 1$ we obtain an alternative expression for the $p$-adic valuation of $T(n)$.

Proposition 2.3. The $p$-adic valuation of $T(n)$ is given by

\[(2.6) \quad \nu_p(T(n)) = \frac{1}{p-1} \sum_{j=1}^{n-1} (S_p(2j) + S_p(2j+1) - S_p(3j+1) - S_p(j)). \]

In particular, for $p = 2$ we have

\[(2.7) \quad \nu_2(T(n)) = \sum_{j=0}^{n-1} (S_2(2j+1) - S_2(3j+1)) = \sum_{j=1}^{n} (S_2(2j-1) - S_2(3j-2)). \]

Corollary 2.4. For each $n \in \mathbb{N}$ we have

\[(2.8) \quad \sum_{j=1}^{n-1} S_2(2j+1) \geq \sum_{j=1}^{n-1} S_2(3j+1). \]

Note. The formula (2.6) can be used to compute $T(n)$ for large values of $n$. Recall that only primes $p \leq 3n - 2$ appear in the factorization of $T(n)$. For example, the number $T(100)$ has 1136 digits and its prime factorization is given by

\[T(100) = 2^{23} \cdot 3^{19} \cdot 13^{13} \cdot 17^{4} \cdot 29^{3} \cdot 41^{4} \cdot 61^{2} \cdot 67^{11} \cdot 71^{5} \cdot 73^{3} \cdot 151 \cdot 157^{5} \cdot 163^{9} \cdot 167^{11} \times 173^{5} \cdot 179^{19} \cdot 181^{21} \cdot 191^{27} \cdot 193^{29} \cdot 197^{31} \cdot 199^{33} \cdot 211^{30} \cdot 223^{26} \cdot 227^{24} \cdot 229^{24} \cdot 233^{22} \times 239^{20} \cdot 241^{40} \cdot 251^{16} \cdot 257^{14} \cdot 263^{12} \cdot 269^{10} \cdot 271^{10} \cdot 277^{8} \cdot 281^{6} \cdot 283^{6} \cdot 293^{2}.\]

The recurrence (2.2) could be employed to generate large amount of data related to number theoretical questions associated to $T(n)$. In this paper we address the simplest of all: characterize those indices $n$ for which $T(n)$ is odd.

3. When is $T(n)$ odd?

Figure 1 shows the 2-adic valuation of the sequence $T(n)$ for $1 \leq n \leq 10^5$. Observe that $\nu_2(T(n)) \geq 0$ in view of the fact that $T(n) \in \mathbb{N}$. Moreover, we see that $\nu_2(T(n)) = 0$ for a sequence of values starting with

\[(3.1) \quad 1, 3, 5, 11, 21, 43, 85, 171, 341, 683. \]
A search in *The On-Line Encyclopedia of Integer Sequences* identifies these numbers as terms in the *Jacobsthal sequence* (A001045), defined by the recurrence

\[(3.2) \quad J_n = J_{n-1} + 2J_{n-2}, \text{ with } J_0 = 1, J_1 = 1.\]

The empirical observation is that the sequence \(T(n)\) is odd if and only if \(n\) is a Jacobsthal number; i.e., \(n = J_m\) for some \(m\).

**Note.** The Jacobsthal numbers have many interpretations. Here is a small sample:

a) \(J_n\) is the numerator of the reduced fraction in the alternating sum

\[\sum_{j=1}^{n+1} \frac{(-1)^{j+1}}{2^j}.\]

b) Number of permutations with no fixed points avoiding 231 and 132.

c) The number of odd coefficients in the expansion of \((1 + x + x^2)^{2n-1-1}\).

Many other examples can be found at

http://www.research.att.com/~njas/sequences/A001045

In this section we present a new proof of the following result [7].

**Theorem 3.1.** The number \(T(n)\) is odd if and only if \(n\) is a Jacobsthal number.

The proof will employ several elementary properties of the Jacobsthal number \(J_n\), summarized here for the convenience of the reader.

\[(3.3) \quad J_n = J_{n-1} + 2J_{n-2}, \text{ with } J_0 = 1, J_1 = 1.\]
Lemma 3.2. For \( n \geq 2 \), the Jacobstahl numbers \( J_n \) satisfy

a) \( J_n = J_{n-1} + 2J_{n-2} \) with \( J_0 = 1 \) and \( J_1 = 1 \). (This is the definition of \( J_n \)).

b) \( J_n = \frac{1}{3}(2^{n+1} + (-1)^n) \).

c) \( 2^{n-1} + 1 \leq J_n < 2^n \).

d) \( J_n + J_{n-1} = 2^n \).

e) \( J_n - J_{n-2} = 2^{n-1} \).

Outline of the proof of Theorem 3.1. The argument is based on some observations from the graph of the function \( \nu_2 \circ T \) as seen in Figure 1. The proof is divided into a small number of steps, each one verified by an inductive procedure. The hypothesis assumes complete knowledge of the function \( \nu_2(T(j)) \) for \( 0 \leq j \leq J_n \). We now show how to describe the function \( \nu_2 \circ T \) in the interval \([J_n, J_{n+1}]\).

Step 1. The midpoint of the interval is \( j = 2^n \). The value there is \( \nu_2(T(2^n)) = J_{n-1} \). This is Theorem 3.4.

Step 2. The value \( T(J_n) \) is odd, that is, \( \nu_2(T(J_n)) = 0 \). This is the content of Theorem 3.5.

Step 3. Let \( 0 \leq i \leq 2J_{n-3} \). Then

\[
(3.4) \quad \nu_2(T(J_n + i)) = i + \nu_2(T(J_{n-2} + i)).
\]

This is Lemma 3.6. It describes the function \( \nu_2 \circ T \) in the interval \([J_n, 2^n - J_{n-2}]\). In particular, \( \nu_2(T(2^n - J_{n-2})) = 2J_{n-3} \) and \( \nu_2(T(j)) > 0 \) for \( J_n < j < 2^n - J_{n-2} \).

Step 4. Let \( 0 \leq i \leq 2J_{n-2} \). Then

\[
(3.5) \quad \nu_2(T(2^n - J_{n-2} + i)) = \nu_2(T(J_{n-1} + i)) + 2J_{n-3}.
\]

This is Proposition 3.7. It shows that the graph of \( \nu_2 \circ T \) on the interval \([2^n - J_{n-2}, 2^n + J_{n-2}]\) is a vertical shift, by \( 2J_{n-3} \), of the graph over the interval \([J_{n-1}, J_n]\).

Step 5. This is Proposition 3.8. Let \( 0 \leq i \leq J_{n-1} \). Then \( \nu_2(T(2^n - i)) = \nu_2(T(2^n + i)) \), explaining the symmetry of the graph about the point \( j = 2^n \) on the interval \([J_n, J_{n+1}]\).

This completes the proof of Theorem 3.1.
Note. As we vary \( m \in \mathbb{N} \), the graph of \( \nu_2(T(n)) \) in the interval \([J_m, J_{m+1}]\) resemble each other. These are depicted in Figure 2 that shows the value of \( \nu_2(T(n)) \) for \( J_{10} = 341 \leq n \leq 683 = J_{11} \). This suggests a possible scaling law for the graph of \( \nu_2 \circ T \). Figure 3 shows the first 15 such graphs, scaled to the unit square. The convergence to a limiting curve is apparent. The properties of this curve will be explored in the future.

The proof of Theorem 3.1 begins with an auxiliary lemma.
Lemma 3.3. Let \( n \in \mathbb{N} \). Introduce the notation \( S_{n,j}^+ := S_2(3 \cdot 2^n + 3j - 2) \) and \( S_{n,j}^- := S_2(3 \cdot 2^n - 3j + 1) \). Then

\[
S_{n,j}^+ = \begin{cases} 
S_2(3j - 2) + 2 & \text{if } 1 \leq j \leq J_{n-1}, \\
S_2(3j - 2) & \text{if } 1 + J_{n-1} \leq j \leq J_n, \\
S_2(3j - 2) + 1 & \text{if } 1 + J_n \leq j \leq 2^n;
\end{cases}
\]

and

\[
S_{n,j}^- = \begin{cases} 
n + 1 - S_2(3j - 2) & \text{if } 1 \leq j \leq J_{n-1}, \\
n + 2 - S_2(3j - 2) & \text{if } 1 + J_{n-1} \leq j \leq J_n, \\
n + 1 - S_2(3j - 2) & \text{if } 1 + J_n \leq j \leq 2^n.
\end{cases}
\]

Proof. Let \( 3j - 2 = a_0 + 2a_1 + \cdots + a_r 2^r \) be the binary expansion of \( 3j - 2 \). The corresponding one for \( 3 \cdot 2^{n-1} \) is simply \( 2^{n-1} + 2^n \). For \( 3j - 2 < 2^{n-1} \) these two expansions have no terms in common, therefore \( S_{n,j}^+ = S_2(3j - 2) + 2 \). On the other hand, if \( 2^{n-1} \leq 3j - 2 < 2^n \) then the index in the binary expansion of \( 3j - 2 \) is \( r = n - 1 \) with \( a_{n-1} = 1 \). The expansion of \( 3j - 2 + 3 \cdot 2^{n-1} \) is now

\[
a_0 + 2a_1 + \cdots + a_{n-2} 2^{n-2} + 2^{n-1} + 2^{n-1} + 2^n = a_0 + 2a_1 + \cdots + a_{n-2} 2^{n-2} + 2^{n+1},
\]

and this yields \( S_{n,j}^+ = a_0 + a_1 + \cdots + a_{n-2} + 1 = S_2(3j - 2) \). The remaining cases are treated in a similar form. \( \square \)

We now establish the 2-adic valuation at the center of the interval \( [J_{n-1}, J_n] \). This completes Step 1 in the outline.

Theorem 3.4. Let \( n \in \mathbb{N} \). Then

\[
\nu_2(T(2^n)) = J_{n-1}.
\]

Proof. We proceed by induction and split

\[
\nu_2(T(2^n)) = \sum_{j=1}^{2^{n-1}} [S_2(2j + 1) - S_2(3j + 1)]
\]

at \( j = 2^{n-1} - 1 \). The first part is identified as \( \nu_2(T(2^{n-1})) \) to produce

\[
\nu_2(T(2^n)) = \nu_2(T(2^{n-1})) + \sum_{j=0}^{2^{n-1}-1} S_2(2j + 1 + 2^n) - \sum_{j=1}^{2^{n-1}} S_2(3j - 2 + 3 \cdot 2^{n-1}).
\]

Now observe that \( 2j + 1 \leq 2^n - 1 < 2^n \) so that \( S_2(2j + 1 + 2^n) = S_2(2j + 1) + 1 \). Lemma 3.3 gives, for \( n \) even,

\[
\sum_{j=1}^{2^{n-1}} S_2(3j - 2 + 3 \cdot 2^{n-1}) = \sum_{j=1}^{(2^{n-1}+1)/3} [S_2(3j - 2) + 2] + \sum_{j=(2^{n-1}+1)/3}^{(2^{n-1}-1)/3} [S_2(3j - 2) + 1]
\]

\[
+ \sum_{j=(2^{n-1}+1)/3}^{(2^{n-1})/3} [S_2(3j - 2) + 2] + \sum_{j=(2^{n-1})/3}^{2^{n-1}} [S_2(3j - 2) + 1].
\]
and using (2.7) yields
\begin{equation}
\nu_2(T(2^n)) = 2\nu_2(T(2^n-1)) - 1 = 2J_{n-2} - 1.
\end{equation}
Elementary properties of Jacobsthal numbers show that $2J_{n-2} - 1 = J_{n-1}$ proving the result for $n$ even. The argument for $n$ odd is similar. □

The next theorem corresponds to Step 2 of the outline.

**Theorem 3.5.** Let $n \in \mathbb{N}$. Then $T(J_n)$ is odd.

**Proof.** Proposition 2.3 gives
\begin{equation}
\nu_2(T(J_n)) = \sum_{j=1}^{J_n-1} [S_2(2j + 1) - S_2(3j + 1)].
\end{equation}
Observe that $2^{n-1} \leq J_n - 1$, so
\begin{align*}
\nu_2(T(J_n)) &= \sum_{j=1}^{2^{n-1}-1} [S_2(2j + 1) - S_2(3j + 1)] + \\
&\quad + \sum_{j=2^{n-1}}^{J_n-1} [S_2(2j + 1) - S_2(3j + 1)] \\
&= \nu_2(T(2^{n-1})) + \sum_{j=2^{n-1}}^{J_n-1} [S_2(2j + 1) - S_2(3j + 1)].
\end{align*}
Therefore
\begin{equation}
\nu_2(T(J_n)) = \nu_2(T(2^{n-1})) + \sum_{j=0}^{J_n-1-2^{n-1}} [S_2(2j + 1 + 2^n) - S_2(3j + 1 + 3 \cdot 2^{n-1})].
\end{equation}
The elementary properties of Jacobsthal numbers give
\begin{equation}
J_n - 1 - 2^{n-1} = J_{n-2} - 1,
\end{equation}
so that
\begin{equation}
\nu_2(T(J_n)) = \nu_2(T(2^{n-1})) + \sum_{j=0}^{J_{n-2}-1} [S_2(2j + 1 + 2^n) - S_2(3j + 1 + 3 \cdot 2^{n-1})].
\end{equation}
Observe that
\begin{equation*}
2j + 1 \leq 2(J_{n-2} - 1) + 1 = 2J_{n-2} - 1 = J_n - J_{n-1} - 1 < 2^n,
\end{equation*}
resulting in
\begin{equation*}
S_2(2j + 1 + 2^n) = S_2(2j + 1) + 1.
\end{equation*}
Similarly $3j + 1 \leq 3J_{n-2} - 2 < 3(2^{n-1} + (-1)^n) - 2 \leq 2^{n-1} - 1$ and from $3 \cdot 2^{n-1} = 2^n + 2^{n-1}$ we obtain
\begin{equation*}
S_2(3j + 1 + 3 \cdot 2^{n-1}) = S_2(3j + 1) + 2,
\end{equation*}
for $0 \leq j \leq J_{n-2} - 1$. It follows that
\[
\nu_2(T(J_n)) = \nu_2(T(2^{n-1})) + \sum_{j=0}^{J_{n-2}-1} [S_2(2j + 1) - S_2(3j + 1)] - J_{n-2}.
\]

Theorem 3.4 shows that the first and third term on the line above cancel, leading to
\[
\nu_2(T(J_n)) = \nu_2(T(J_{n-2})).
\]
The result now follows by induction on $n$. □

We continue with the proof of Theorem 3.1. The next Lemma corresponds to Step 3 in the outline. It describes the values $\nu_2(T(j))$ for $J_n \leq j \leq J_n + 2J_{n-3} = 2^n - J_{n-2}$. The result of Lemma 3.6 shows that $\nu_2(T(j)) > 0$ for $J_n < j < 2^n - J_{n-2}$.

**Lemma 3.6.** For $0 < i \leq 2J_{n-3}$ we have
\[
(3.13) \quad \nu_2(T(J_n + i)) = i + \nu_2(T(J_{n-2} + i)).
\]

**Proof.** Assume that $n$ is even and consider
\[
\nu_2(T(J_n + i)) = \sum_{j=1}^{J_n+i} [S_2(2j + 1) - S_2(3j + 1)]
\]
\[
= \sum_{j=1}^{J_n-1} [S_2(2j + 1) - S_2(3j + 1)] + \sum_{j=J_n}^{J_n+i-1} [S_2(2j + 1) - S_2(3j + 1)].
\]
The first sum is $\nu_2(T(J_n)) = 0$, according to Theorem 3.5. Therefore, using Lemma 3.2 we have
\[
\nu_2(T(J_n + i)) = \sum_{j=J_n}^{J_n+i} [S_2(2j + 1) - S_2(3j + 1)]
\]
\[
= \sum_{j=J_n+i-2^n-1}^{J_n+i-1} [S_2(2^n + 2j - 1) - S_2(3 \cdot 2^{n-1} + 3j - 2)]
\]
\[
= \sum_{j=J_{n-2}+i}^{J_{n-2}+i+1} [S_2(2^n + 2j - 1) - S_2(3 \cdot 2^{n-1} + 3j - 2)].
\]
The index $j$ satisfies
\[
2j - 1 \leq 2(J_{n-2} + i) - 1 < 2(J_{n-2} + 2J_{n-3}) = 2J_{n-1} < 2^n,
\]
therefore $S_2(2^n + 2j - 1) = 1 + S_2(2j - 1)$. 

\[\]
The lower limit in the last sum is \( J_{n-2} + 1 = \frac{1}{3}(2^{n-1} + 1) + 1 \), and the upper bound is
\[
J_{n-2} + i \leq J_{n-2} + 2J_{n-3} = J_{n-1} = \frac{1}{3}(2^n - 1).
\]
Lemma 3.3 gives \( S_2(3 \cdot 2^{n-1} + 3j - 2) = S_2(3j - 2) \). Therefore
\[
\nu_2(T(J_n + i)) = \sum_{j=J_{n-2}+1}^{J_{n-2}+i} [S_2(2j - 1) + 1 - S_2(3j - 2)]
\]
\[
= i + \sum_{j=J_{n-2}+1}^{J_{n-2}+i} [S_2(2j - 1) - S_2(3j - 2)]
\]
\[
= i + \nu_2(T(J_{n-2} + i)).
\]
The result has been established for \( n \) even. The proof for \( n \) odd is similar. \( \square \)

The next result shows the graph of \( \nu_2 \circ T \) on the interval \([2^n - J_{n-2}, 2^n + J_{n-2}]\) is a vertical shift of the graph on \([J_{n-1}, J_n]\). This corresponds to Step 4 in the outline.

**Proposition 3.7.** For \( 0 \leq i \leq 2J_{n-2} \),
\[
\nu_2(T(2^n - J_{n-2} + i)) = \nu_2(T(J_{n-1} + i)) + \omega_n,
\]
where \( \omega_n = 2J_{n-3} \) is independent of \( i \).

**Proof.** We prove that the graph of \( \nu_2(T(J_{n-1} + i)) \) and \( \nu_2(T(2^n - J_{n-2} + i)) \) have the same discrete derivative. This amounts to checking the identity
\[
\nu_2(T(J_{n-1} + i)) - \nu_2(T(J_{n-1} + i - 1)) = \nu_2(T(2^n - J_{n-2} + i)) - \nu_2(T(2^n - J_{n-2} + i - 1))
\]
for \( 1 \leq i \leq 2J_{n-2} \). Observe that
\[
\nu_2(T(k)) - \nu_2(T(k-1)) = S_2(2k - 1) - S_2(3k - 2),
\]
and using \( 2^n - J_{n-2} = 2^{n-1} + J_{n-1} \), we conclude that the result is equivalent to the identity
\[
S_2(2^n + 2(J_{n-1} + i) - 1) - S_2(2(J_{n-1} + i) - 1) = S_2(3 \cdot 2^{n-1} + 3(J_{n-1} + i) - 2) - S_2(3(J_{n-1} + i) - 2),
\]
for \( 1 \leq i \leq 2J_{n-2} \). Define
\[
h_n(i) = \begin{cases} 
1 & \text{if } 1 \leq i \leq J_{n-2}; \\
0 & \text{if } J_{n-2} + 1 \leq i \leq 2J_{n-2}.
\end{cases}
\]
The assertion is that both sides in (3.18) agree with \( h_n(i) \). The analysis of the left hand side is easy: the condition \( 1 \leq i \leq J_{n-2} \) implies \( 2(J_{n-1} +
Proof. Start with

We conclude that the binary expansion of $x$ is of the form $n \alpha + a_1 \cdot 2 + \cdots + a_{n-1} \cdot 2^{n-1} + 1 \cdot 2^n$. It follows that $2^n + x$ and $x$ have the same number of 1's in their binary expansion. Thus $S_2(x) = S_2(x + 2^n)$ as claimed.

The analysis of the right hand side of (3.18) is slightly more difficult. Let $x := 3(J_{n-1} + i) - 2$ and it is required to compare $S_2(x)$ and $S_2(3 \cdot 2^{n-1} + x)$. Observe that

\begin{align}
3(J_{n-1} + 1) - 2 &= 2^n + (-1)^{n-1} + 1 \geq 2^n.
\end{align}

We conclude that the binary expansion of $x$ is of the form

\begin{align}
x &= a_0 + a_1 \cdot 2 + \cdots + a_{n-1} \cdot 2^{n-1} + 1 \cdot 2^n,
\end{align}

and the corresponding one for $3 \cdot 2^{n-1}$ is $2^n + 2^{n-1}$. An elementary calculation shows that $S_2(x + 3 \cdot 2^{n-1}) - S_2(x)$ is 1 if $a_{n-1} = 0$ and 0 if $a_{n-1} = 1$. In order to transform this inequality to a restriction on the index $i$, observe that $a_{n-1} = 1$ is equivalent to $x - 2^n \geq 2^{n-1}$. Using the value of $x$ this becomes $3(J_{n-1} + i) - 2 \geq 3 \cdot 2^{n-1}$. This is directly transformed to $i \geq J_{n-2} + 1$. This shows that the right hand side of (3.18) also agrees with $h_n$ and (3.18) has been established.

The final step in the proof of Theorem 3.1, outlined as Step 5, shows the symmetry of the graph of $\nu_2(T(j))$ about the point $j = 2^n$. The range covered in the next proposition is $2^n - J_{n-1} \leq j \leq 2^n + J_{n-1}$.

**Proposition 3.8.** For $1 \leq i \leq J_{n-1}$,

\begin{align}
\nu_2(T(2^n - i)) = \nu_2(T(2^n + i)).
\end{align}

**Proof.** Start with

\begin{align}
\nu_2(T(2^n)) - \nu_2(T(2^n - i)) &= \sum_{j=2^n-i+1}^{2^n} [S_2(2j - 1) - S_2(3j - 2)] \\
&= \sum_{k=1}^{i} [S_2(2^{n+1} - (2k - 1)) - S_2(3 \cdot 2^n - (3k - 1))].
\end{align}
The first term in the sum satisfies
\[ S_2(2^{n+1} - (2k - 1)) = n + 2 - S_2(2k - 1). \] (3.25)

To check this, write \( 2k - 1 = a_0 + a_1 \cdot 2 + \cdots + a_r \cdot 2^r \) with \( a_0 = 1 \) because \( 2k - 1 \) is odd. Now, \( 2^{n+1} = (1 + 2 + 2^2 + \cdots + 2^n) + 1 \) and we conclude that
\[ 2^{n+1} - (2k - 1) = (2^n + 2^{n-1} + \cdots + 2^r + 1) \]
\[ + (1 - a_r) \cdot 2^r + (1 - a_{r+1}) \cdot 2^{r-1} + \cdots + (1 - a_1) \cdot 2 + 1 \]

Therefore
\[ S_2(2^{n+1} - (2k - 1)) = n + 1 - (a_r + a_{r-1} + \cdots + a_1) \]
\[ = n + 2 - S_2(2k - 1). \]

We conclude that
\[ \nu_2(T(2^n)) - \nu_2(T(2^n - i)) = (n + 2)i - \sum_{k=1}^{i} S_2(2k - 1) - \]
\[ \sum_{k=1}^{i} S_2(3 \cdot 2^n - (3k - 1)). \] (3.26)

Similarly
\[ \nu_2(T(2^n + i)) - \nu_2(T(2^n)) = \sum_{j=2^n+1}^{2^n+i} (S_2(2j - 1) - S_2(3j - 2)) \]
\[ = \sum_{k=1}^{i} (S_2(2^{n+1} + 2k - 1) - S_2(3 \cdot 2^n + 3k - 2)). \]

The inequality
\[ 2k - 1 \leq 2i - 1 \leq 2J_n - 1 \leq 2 \cdot 2^{n-1} - 1 \leq 2^n - 1 < 2^{n+1} \] (3.27)
shows that \( S_2(2^{n+1} + 2k - 1) = 1 + S_2(2k - 1). \) Lemma 3.3 yields the identity
\[ S_2(3 \cdot 2^n + 3k - 2) + S_2(3 \cdot 2^n - 3k + 1) = n + 3. \] (3.28)

Therefore
\[ \nu_2(T(2^n + i)) - \nu_2(T(2^n)) = \sum_{k=1}^{i} (S_2(2^{n+1} + 2k - 1) - S_2(3 \cdot 2^n + 3k - 2)) + i \]
\[ + \sum_{k=1}^{i} S_2(2k - 1) - (n + 3 - S_2(3 \cdot 2^n - 3k + 1)). \]

It follows that
\[ \nu_2(T(2^n)) - \nu_2(T(2^n - i)) = -[\nu_2(T(2^n - i)) - \nu_2(T(2^n))], \]
and symmetry has been established. \( \square \)
Note. The identity (3.28) can be given a direct proof by inducting on $k$. It is required to check that the left hand side is independent of $k$ and this follows from the identity

$$S_2(m + 3) - S_2(m) = \begin{cases} 2 - \omega_2 \left( \frac{m}{2} \right) & \text{if } m \equiv 0 \mod 2; \\ -\omega_2 \left( \left\lfloor \frac{m}{4} \right\rfloor \right) & \text{if } m \equiv 1 \mod 2. \end{cases}$$

(3.29)

Here $\omega_2(m)$ is the number of trailing 1’s in the binary expansion of $m$. For $m = 829$ we have $S_3(829) = 7$ and $S_3(832) = 3$. The binary expansion of $m = 207 = \lfloor 829/4 \rfloor$ is 11001111 and the number of trailing 1’s is 4. This observation is due to A. Straub.

The next result shows that every positive integer $k$ is attained as $\nu_2(T(n))$.

**Theorem 3.9.** Every nonnegative integer appears as $\nu_2(T(n))$ for some $n$, i.e.,

$$\mathbb{N} = \{\nu_2(T(n)) : n \in \mathbb{N}\}.$$ 

Furthermore, each positive integer $m$ appears only finitely many times, and the last appearance is when $n = J_{2m+1} - 1$.

**Proof.** From the results before, we know that

$$\nu_2(T(J_n + i)) > \nu_2(T(J_{n+1} - 1)),$$

for $1 < i < J_{n+1} - J_n - 2$ and $\nu_2(T(J_{n+2} - 1)) = \nu_2(T(J_n - 1)) + 1$. This shows that the minimum values of the graph of $\nu_2(T(n))$ around $2^n$ are attained exactly at $J_n + 1$ and $J_n + 1 - 1$. These values are also strictly increasing along the even and odd indices. Thus, $m < \nu_2(T(i))$ for any given $m$, provided $i$ is large enough.

To determine the last appearance of $m$, we only need to determine the last occurrence of $n$ such that $\nu_2(T(J_n - 1)) = m$. Since $\nu_2(T(J_2 - 1)) = \nu_2(T(J_3 - 1)) = 1$, we conclude that $\nu_2(T(J_{2n} - 1)) = \nu_2(T(J_{2n+1} - 1)) = n$. Therefore the last occurrence for $m$ is at $J_{2m+1} - 1$. \hfill \Box

Note. Define $\lambda(m)$ to be the number $m$ is attained by $\nu \circ T$. The values for $1 \leq m \leq 8$ are shown below.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $\lambda(m)$ | 2 | 8 | 5 | 12 | 5 | 14 | 8 | 14 |

**Table 1.** The first 8 values in the range of $\nu_2 \circ T$

For example, the values of $n$ for which $\nu(T(n)) = 5$ are 16, 342, 682, 684 and $J_{11} - 1 = 1364$ and the eight solutions to $\nu(T(n)) = 7$ are 26, 38, 46, 82, 5462, 10922, 10924 and $J_{15} - 1 = 21844$.

Note. In sharp contrast to the 2-adic valuation, D. Frey and J. Sellers [8, 9] show that if $p \geq 3$ is a prime, then for each nonnegative integer $m$ there exist infinitely many positive integers $n$ for which $\nu_p(T(n)) = m$. 


4. The 3-adic valuation of $T(n)$

The analysis of the 2-adic valuation of $T(n)$ is now extended to the prime $p = 3$. The discussion employs the expansion of $n \in \mathbb{N}$ in base 3, given by

$$n = a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + \cdots + a_r \cdot 3^r$$

and the function

$$S_3(n) := a_0 + a_1 + \cdots + a_r.$$

Figure 4 presents a well-defined symmetry for $\nu_3(T(n))$. This is explained in Theorem 4.1.

The first result characterizes the values $n$ for which $\nu_3(T(n)) = 0$.

**Theorem 4.1.** Let $n \in \mathbb{N}$ with \eqref{eq:base3} as its expansion in base 3. Then $\nu_3(T(n)) = 0$ if and only if there is an index $0 \leq i \leq r$ such that $a_0 = a_1 = \cdots = a_{i-1} = 0$ and $a_{i+1} = a_{i+2} = \cdots = a_r = 0$ or 2, with $a_i$ arbitrary.

We begin with some elementary results on the function $S_3$ which admit elementary proofs.

**Lemma 4.2.** Let $n \in \mathbb{N}$. Then

$$S_3(3n) = S_3(n), \quad S_3(3n + 1) = 1 + S_3(n) \quad \text{and} \quad S_3(3n + 2) = 2 + S_3(n).$$

**Lemma 4.3.** Let $n \in \mathbb{N}$. Then

$$S_3(4 \cdot 3^n + b) = 2 + S_3(b) \quad \text{for all} \quad 0 \leq b < 2 \cdot 3^n,$$

$$S_3(2 \cdot 3^n + b) = 2 + S_3(b) \quad \text{for all} \quad 0 \leq b < 3^n,$$

$$S_3(3^n + b - 1) = 1 + S_3(b - 1) \quad \text{for} \quad 1 \leq b < 3^n.$$

The next step in analyzing the function $\nu_3 \circ T$ is to produce a recurrence for this valuation. The symmetry observed in Figure 4 is a consequence of this result.
Proposition 4.4. Let $n \in \mathbb{N}$. Then $\nu_3(T(3n)) = 3\nu_3(T(n))$.

Proof. Legendre’s formula (2.2) shows that the result is equivalent to

\[
\sum_{j=0}^{3n-1} S_3(3n+j) - \sum_{j=0}^{3n-1} S_3(3j+1) - 3 \sum_{j=0}^{n-1} S_3(n+j) + 3 \sum_{j=0}^{n-1} S_3(3j+1) = 0. 
\]

Each term of (4.3) is now simplified. Lemma 4.2 shows that

\[
\sum_{j=0}^{3n-1} S_3(3n+j) = \sum_{j=0}^{n-1} S_3(3(n+j)) + \sum_{j=0}^{n-1} S_3(3(n+j)+1) + \sum_{j=0}^{n-1} S_3(3(n+j)+2) 
\]

\[
= 3n + 3 \sum_{j=0}^{n-1} S_3(n+j),
\]

and

\[
\sum_{j=0}^{3n-1} S_3(3j+1) = 3n + \sum_{j=0}^{3n-1} S_3(j) 
\]

\[
= 3n + \sum_{j=0}^{n-1} S_3(3j) + \sum_{j=0}^{n-1} S_3(3j+1) + \sum_{j=0}^{n-1} S_3(3j+2) 
\]

\[
= 6n + 3 \sum_{j=0}^{n-1} S_3(j),
\]

and, finally,

\[
\sum_{j=0}^{n-1} S_3(3j+1) = n + \sum_{j=0}^{n-1} S_3(j). 
\]

These identities show that the left-hand side of (4.3) vanishes. \qed

Corollary 4.5. For each $n \in \mathbb{N}$, we have $\nu_3(T(3^n)) = \nu_3(T(2 \cdot 3^n)) = 0$.

Proof. This follows directly from $T(1) = 1$ and $T(2) = 1$ and Proposition 4.4. \qed

For brevity, introduce the function

\[
\mu_3(j) := S_3(2j) + S_3(2j+1) - S_3(3j+1) - S_3(j).
\]

Thus Proposition 2.3 takes the form

\[
\nu_3(T(n)) = \frac{1}{2} \sum_{j=1}^{n-1} \mu_3(j).
\]

Observe that

\[
\mu_3(n-1) = 2(\nu_3(T(n)) - \nu_3(T(n-1))).
\]

Proposition 4.6. If $0 \leq a \leq 3^n$ then $\nu_3(T(a)) = \nu_3(T(2 \cdot 3^n + a))$. 
Proof. The limiting cases $a = 0$ and $a = 3^n$ follow from Corollary 4.5. The result follows from (4.5) and the identities $\mu_3(a) = \mu_3(2 \cdot 3^n + a)$ for $1 \leq a \leq 3^n$, that are direct consequence of Lemma 4.3. □

The proof of Theorem 4.1 is presented next.

Proof. Consider the representation of $n \in \mathbb{N}$ in base 3:

\begin{equation}
(4.7) \quad n = a_0 + 3a_1 + 3^2a_2 + \cdots + 3^ra_r.
\end{equation}

Corollary 4.5 and Proposition 4.6 show that the numbers $n$ with the form stated in the theorem satisfy $\nu_3(T(n)) = 0$. We need to prove that these are the only zeros of $\nu_3 \circ T$.

The proof is by induction and show that $\nu_3(T(a)) > 0$ for $3^n < a < 3^n+1$. Proposition 4.6 shows that, if $a_r = 2$, then $\nu_3(T(n)) > 0$. Proposition 4.7 treats the result for $a_r = 1$ and the first half of these numbers $0 \leq a - 3^r \leq 3^r$. Proposition 4.9 establishes a symmetry result that takes care of the second half. □

We now establish the symmetry of the function $\nu_3 \circ T$. The proof begin with some auxiliary steps.

**Proposition 4.7.** Let $n, a \in \mathbb{N}$ and assume $1 \leq a < 3^n$. Then

\[
\mu_3(3^n + a) = \begin{cases} 
\mu_3(a) + 2 & \text{if } 1 \leq a < \frac{1}{2}3^n; \\
\mu_3(a) & \text{if } a = \frac{1}{2}(3^n + 1); \\
\mu_3(a) - 2 & \text{if } \frac{1}{2}3^n + 1 < a \leq 3^n.
\end{cases}
\]

Proof. When $1 \leq b < \frac{1}{2}3^n$, the first part follows from Lemma 4.3. The other parts can be proved similarly, and thus omitted. □

**Lemma 4.8.** If $3 \nmid a$, $3 \nmid b$, $n < m$, and $b < 3^{m-n}$, then

\begin{equation}
(4.8) \quad \nu_3(T(3^m a - 3^n b)) = 2(m - n) + \nu_3(T(a)) - \nu_3(T(b)).
\end{equation}

**Proposition 4.9.** If $1 \leq i < \frac{3^n}{2}$, $\mu_3(3^n + i) = -\mu_3(2 \cdot 3^n - i + 1)$.

Proof. Let $A = 3^n + i$ and $B = 2 \cdot 3^n - i + 1$. We prove $\mu_3(A) = -\mu_3(B)$.

First we observe that

\[
\mu_3(A) = S_3(2 \cdot 3^n + 2i - 1) + S_3(2 \cdot 3^n + 2i - 2) - S_3(3^n + 2i - 2) - S_3(3^n + i - 1)
\]

\[
= (2 + S_3(2i - 1)) + (2 + S_3(2i - 2)) - (1 + S_3(3i - 2)) - (1 + S_3(i - 1))
\]

\[
= S_3(2i - 1) + S_3(2i - 2) - S_3(3i - 2) - S_3(i - 1) + 2.
\]

There are three cases to consider according to the value of $i$ modulo 3. Assume first that $i \equiv 0 \mod 3$ and write $i = 3^x$, where $a > 0$ and $3 \nmid x$. Then
\[ \mu_3(A) = S_3(2i - 1) + S_3(2i - 2) - S_3(3i - 2) - S_3(i - 1) + 2 \]
\[ = S_3(2 \cdot 3^a x - 1) + S_3(2 \cdot 3^a x - 2) - S_3(3 \cdot 3^a x - 2) - S_3(3^a x - 1) + 2 \]
\[ = (S_3(2x) - 1 + 2a) + (S_3(2x) - 2 + 2a) - \\
(S_3(x) - 2 + 2(a + 1)) - (S_3(x) - 1 + 2a) + 2 \]
\[ = 2S_3(2x) - 2S_3(x) \]

\[ \mu_3(B) = S_3(4 \cdot 3^n - 2i + 1) + S_3(4 \cdot 3^n - 2i) - S_3(2 \cdot 3^{n+1} - 3i + 1) - S_3(2 \cdot 3^n - i) \]
\[ = S_3(4 \cdot 3^n - 2 \cdot 3^a x + 1) + S_3(4 \cdot 3^n - 2 \cdot 3^a x) \\
- S_3(2 \cdot 3^{n+1} - 2 \cdot 3^a x + 1) - S_3(2 \cdot 3^n - 3^a x) \]
\[ = (2n + 2 - S_3(2 \cdot 3^a x - 1)) + (2(n - a) + 2 - S_3(2x)) \\
- (2n + 4 - S_3(2 \cdot 3^{a+1} x + 1)) - (2(n - a) + 2 - S_3(x)) \]
\[ = (-S_3(2x) + 1) + (-S_3(2x)) - (-S_3(2x) - 1) - (-S_3(x)) - 2 \]
\[ = -2S_3(2x) + 2S_3(x) = -\mu_3(A), \]

as claimed. The cases \( i \equiv 1, 2 \mod 3 \) are analyzed by similar techniques. \( \square \)

**Note.** The techniques outlined in this paper can be used to present a complete description of the function \( \nu_p(T(n)) \) for \( p \geq 5 \) prime. We limit ourselves to showing the graphs for \( p = 5 \) and 7 in the range \( n \leq 5000. \)

![Figure 5. The 5-adic valuation of T(n)](image-url)

The rest of the section is devoted to develop an efficient procedure to compute \( \nu_3(T(n)) \). We begin with the ternary expansion of \( n \)

\[ n = \sum_{i=0}^{k} a_i 3^i \]
Figure 6. The 7-adic valuation of $T(n)$

and now define two sequence of integers:

\begin{equation}
    n_k = n'_k = n,
\end{equation}

and, for $0 \leq j < k$ and assume having

\begin{equation}
    n'_{j+1} = \sum_{i=0}^{j+1} b_{j+1,i} 3^i,
\end{equation}

then define recursively

\[
    n_j = \sum_{i=0}^{j} b_{j+1,i} 3^i, \\
    n'_j = \begin{cases} 
    n_j & \text{if } b_{j+1,j+1} = 0, 2; \\
    \min(n_j, 3^{j+1} - n_j) & \text{if } b_{j+1,j+1} = 1.
    \end{cases}
\]

**Theorem 4.10.** The 3-adic valuation of $T(n)$ satisfies

\begin{equation}
    \nu_3(T(n_j)) = \begin{cases} 
    \nu_3(T(n'_{j-1})) & \text{if } a_j = 0, 2; \\
    \nu_3(T(n'_{j-1})) + 2n'_{j-1} & \text{if } a_j = 1.
    \end{cases}
\end{equation}

**Note.** Observe that the time required to calculate $\nu_3(T(n))$ is $O(n^2 \ln n)$ using the definition of $T(n)$. Using Proposition 2.3 the computational time reduces to $O(n)$. The method described in Theorem 4.10 further reduces this time to $O(\ln n)$. A similar algorithm can be developed for $p = 2$.

**Example.** Let $n = 1280$, whose representation with base 3 is 1202102. Then $k = 6$ and we have
It follows that
\[ \nu_3(T(1280)) = 2n'_5 + \nu_3(T(n'_5)) \]
\[ = 2n'_5 + \nu_3(T(n_2)) \]
\[ = 2n'_5 + 2\nu_3(T(n'_1)) + \nu_3(T(n_1)) \]
\[ = 360. \]

5. A generalization

The sequence
\[ T_p(n) := \prod_{j=0}^{n-1} \frac{(pj+1)!}{(n+j)!}, \]
contains \( T(n) \) of (1.1) as the special case \( T(n) = T_3(n) \). In this section we present some elementary properties of this generalization.

**Theorem 5.1.** For a fixed prime \( p \geq 3 \), the numbers \( T_p(n) \) are integers.

**Proof.** Observe that
\[ T_p(n+1) = T_p(n) \times \frac{(pn+1)!n!}{(2n+1)! (2n)!}. \]

Define
\[ x_p(n) := \frac{(pn+1)!}{((p-1)n+1)!n!} = \binom{pn+1}{n}, \]
and observe that
\[ \frac{(pn+1)!n!}{(2n+1)! (2n)!} = x_p(n) \times \frac{((p-1)n+1)!}{(2n+1)! (2n)!} n!^2. \]

Iterating this argument yields
\[ \frac{(pn+1)!n!}{(2n+1)! (2n)!} = \prod_{r=0}^{k-1} x_{p-r}(n) \times \frac{((p-k)n+1)!}{(2n+1)! (2n)!} n!^{k+1}. \]
The choice $k = p - 4$ confirms that
\[
\frac{(pn + 1)! n!}{(2n + 1)! (2n)!} = \left(\frac{4n + 1}{2n}\right)^p \prod_{r=0}^{p-5} \binom{(p - r)n + 1}{n}
\]
is an integer. The recurrence (5.2) and the initial condition $T_p(1) = 1$ now show that $T_p(n)$ is also an integer. The explicit formula
\[
T_p(n) = \prod_{j=1}^{n-1} \left(\frac{4j + 1}{2j}\right)^j \prod_{r=0}^{p-5} \binom{(p - r)j + 1}{j}
\]
follows from the recurrence.

\[\square\]

Proof. An alternative proof of the fact that $\frac{(pn + 1)! n!}{(2n + 1)! (2n)!}$ is an integer was shown to us by Valerio de Angelis. Observe that, for $p \geq 4$, we have
\[
(pn + 1)! = N \times (4n + 1)! \text{ for the integer } N = (4n + 2)_{(p-4)n}.
\]
Therefore
\[
\frac{(pn + 1)! n!}{(2n + 1)! (2n)!} = (4n + 2)_{(p-4)n} \times \left(\frac{4n + 2}{2n}\right)n!.
\]
This leads to the explicit formula
\[
T_p(n) = \prod_{j=1}^{n-1} (4j + 2)_{(p-4)n} \left(\frac{4j + 1}{2j}\right)j!.
\]

\[\square\]

Proof. A third proof using Theorem 1.1 was shown to us by T. Amdeberhan. The required inequality states: if $n, k, p \in \mathbb{N}$ and $p \geq 3$, then
\[
\psi_k(n; p) := \sum_{j=0}^{n-1} \left\lfloor \frac{pj + 1}{k} \right\rfloor - \sum_{j=0}^{n-1} \left\lfloor \frac{n + j}{k} \right\rfloor \geq 0.
\]
It suffices to prove the special case $p = 3$, i.e. $\psi_k(n; 3) \geq 0$ which we denote by $\psi_k(n)$ for $k \geq 3, n \geq 1$. Write $n = ck + r$ where $0 \leq r \leq k - 1$.

We approach a reduction process by breaking down the respective sums as follows.
\[
\sum_{j=0}^{n-1} \left\lfloor \frac{3j + 1}{k} \right\rfloor = \sum_{j=0}^{ck-1} \left\lfloor \frac{3j + 1}{k} \right\rfloor + \sum_{j=0}^{r-1} \left\lfloor \frac{3(ck + j) + 1}{k} \right\rfloor
\]
\[
= \sum_{j=0}^{ck-1} \left\lfloor \frac{3j + 1}{k} \right\rfloor + 3cr + \sum_{j=0}^{r-1} \left\lfloor \frac{3j + 1}{k} \right\rfloor,
\]
Theorem 5.2. Let 

\[ \psi(5.11) \]

and using Legendre’s formula we obtain

\[ \psi(p) \]

T(5.10)

Proof. Observe that

\[ \nu_p(T_p(pn)) = pn \nu_p(T_p(n)) + \frac{1}{2} p(p - 3)n^2. \]

and using Legendre’s formula we obtain

\[ (p - 1) \nu_p(T_p(pn)) = \sum_{j=0}^{pn-1} pj + 1 - S_p(pj + 1) - \sum_{j=pn}^{2pn-1} j - S_p(j). \]

Combining these expressions, we find that \( \psi_k(ck + r) = \psi_k(ck) + \psi_k(r) \). A similar argument with \( r \) replaced by \( k \) produces \( \psi_k(ck + k) = \psi_k(ck) + \psi_k(k) \).

We conclude \( \psi_k \) is \( k \)-Euclidean, i.e.

\[ \psi_k(ck + r) = c\psi_k(k) + \psi_k(r). \]

Therefore, we just need to verify the assertion \( \psi_k(r) \geq 0 \). In fact, we will strengthen it by giving an explicit formula in vectorial form

\[ [\psi_k(0), \ldots, \psi_k(k - 1)] = [0, 0^{k'}, 1, 2, \ldots, [k''/2], [k''/2], \ldots, 2, 1, 0^{k'}]; \]

where \( k' = \lfloor \frac{k+1}{3} \rfloor, k'' = k - 1 - 2k' \) and \( 0^{k'} \) means \( k' \) consecutive zeros. This admits an elementary proof. Note that \( \psi_k(ck) = 0 \), hence \( \psi_k \) is \( k \)-periodic and it satisfies \( \psi_k(ck + r) = \psi_k(r) \). \( \Box \)

We now discuss a recurrence for the valuation of the sequence \( T_p(n) \). The special role of the prime \( p = 3 \) becomes apparent.

Theorem 5.2. Let \( p \) be prime. Then the sequence \( T_p(n) \) satisfies

\[ \nu_p(T_p(pn)) = pn \nu_p(T_p(n)) + \frac{1}{2} p(p - 3)n^2. \]

Proof. Observe that

\[ T_p(pn) = \prod_{j=0}^{pn-1} (pj + 1)! / \prod_{j=pn}^{2pn-1} j! \]

and using Legendre’s formula we obtain

\[ (p - 1) \nu_p(T_p(pn)) = \sum_{j=0}^{pn-1} pj + 1 - S_p(pj + 1) - \sum_{j=pn}^{2pn-1} j - S_p(j). \]
The terms independent of the function $S_p$ add up to $n^2p(p - 3)/2$ and we obtain

$$
\nu_p(T_p(pm)) - p\nu_p(T_p(n)) = \frac{1}{2}n^2p(p - 3) + \frac{1}{p - 1}W_{p,n},
$$

where

$$
W_{p,n} = - \sum_{j=0}^{pn-1} S_p(pj + 1) + \sum_{j=pm}^{2pn-1} S_p(j) + p\sum_{j=0}^{n-1} S_p(pj + 1) - p\sum_{j=0}^{n-1} S_p(n + j).
$$

We now show that $W_{p,n} = 0$, this established the result.

Use $S_p(pj + 1) = 1 + S_p(j)$ to get that

$$
W_{p,n} = - \sum_{j=0}^{pn-1} S_p(j) + \sum_{j=pm}^{2pn-1} S_p(j) + \sum_{j=0}^{n-1} S_p(j) - \sum_{j=0}^{2n-1} S_p(j).
$$

In the second sum, write $j = pr + k$ with $0 \leq k \leq p - 1$ and $n \leq r \leq 2n - 1$, to obtain

$$
\sum_{j=pm}^{2pn-1} S_p(j) = \sum_{r=n}^{2n-1} \sum_{k=0}^{p-1} S_p(pr + k) = \sum_{r=n}^{2n-1} \sum_{k=0}^{p-1} (k + S_p(r)) = \frac{n}{2}p(p - 1) + p \sum_{r=n}^{2n-1} S_p(r).
$$

This term is now combined with the fourth one to simplify the sum. A similar calculation on the first term gives the result. Indeed,

$$
\sum_{j=0}^{pn-1} S_p(j) = \sum_{r=0}^{n-1} \sum_{k=0}^{p-1} S_p(pr + k) = \sum_{r=0}^{n-1} \sum_{k=0}^{p-1} (k + S_p(r)) = \frac{n}{2}p(p - 1) + p \sum_{r=0}^{n-1} S_p(r).
$$

\[\square\]

**Corollary 5.3.** For $p$ a prime, we have

$$
\nu_p(T_p(p^n)) = \frac{p^n(p - 3)(p^n - 1)}{2(p - 1)}.
$$
Proof. Replace $n$ by $p^n$ in the Theorem to obtain
\begin{equation}
\nu_p(T_p(p^{n+1})) = p\nu_p(T_p(p^n)) + \frac{1}{2}(p - 3)p^{2n+1}.
\end{equation}
Iterating this identity yields the result. \qed

Problem. The sequence $T_p(n)$ comes as a formal generalization of the original sequence $T_3(n)$ that appeared in counting alternating symmetric matrices. This begs the question: what do $T_p(n)$ count?

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References

[1] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of a sequence arising from a rational integral. Jour. Comb. A, 115:1474–1486, 2008.
[2] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of Stirling numbers. Experimental Mathematics, 17:69–82, 2008.
[3] G. Boros and V. Moll. An integral hidden in Gradshteyn and Ryzhik. Jour. Comp. Applied Math., 106:361–368, 1999.
[4] D. Bressoud. Proofs and Confirmations: the story of the Alternating Sign Matrix Conjecture. Cambridge University Press, 1999.
[5] D. Bressoud and J. Propp. How the Alternating Sign Matrix Conjecture was solved. Notices Amer. Math. Soc., 46:637–646, 1999.
[6] D. Cartwright and J. Kupka. When factorial quotients are integers. Austral. Math. Soc. Gaz., 29:19–26, 2002.
[7] D. Frey and J. Sellers. Jacobsthal numbers and Alternating Sign Matrices. Journal of Integer Sequences, 3:1–15, 2000.
[8] D. Frey and J. Sellers. On powers of 2 dividing the values of certain plane partitions. Journal of Integer Sequences, 4:1–10, 2001.
[9] D. Frey and J. Sellers. Prime power divisors of the number of $n \times n$ Alternating Sign Matrices. Ars Combinatorica, 71:139–147, 2004.
[10] A. M. Legendre. Theorie des Nombres. Firmin Didot Freres, Paris, 1830.
[11] D. Manna and V. Moll. A remarkable sequence of integers. Preprint, 2009.
[12] W. H. Mills, D. P. Robbins, and H. Rumsey. Proof of the MacDonald conjecture. Inv. Math., 66:73–87, 1982.
[13] D. Zeilberger. Proof of the Alternating Sign Matrix conjecture. Elec. Jour. Comb., 3:1–78, 1996.