Different asymptotic behaviors of thick branes in mimetic gravity

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In this paper, thick branes generated by mimetic scalar field with Lagrange multiplier formulation are investigated. We give three typical thick brane background solutions with different asymptotic behaviors and show that all the solutions are stable under tensor perturbations. The effective potentials of the tensor perturbations exhibit as volcano potential, Pöschl-Teller potential, and harmonic oscillator potential for the three background solutions, respectively. All the tensor zero modes (massless gravitons) of the three cases can be localized on the brane. We also calculate the corrections to the Newtonian potential. On a large scale, the corrections to the Newtonian potential can be ignored. While on a small scale, the correction from the volcano-like potential is more pronounced than the other two cases. Combining the latest results of short-range gravity experiments that the usual Newtonian potential $\propto 1/r$ holds down to a length scale at $52\mu m$, we get the constraint on the scale parameter as $k \gtrsim 10^{-4} eV$, and constraint on the corresponding five-dimensional fundamental scale as $bM_\star \gtrsim 10^5 TeV$.

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I. INTRODUCTION

The standard model of cosmology based on general relativity has made a lot of successful predictions which are in good agreement with observational data. In this standard model, dark energy contributes 68.3\% and dark matter contributes 26.8\% of the total energy content. So, the standard model cosmology indicates that dark matter constitutes 85\% of the total mass of matter. While, dark matter has not yet been observed in any laboratories directly and we do not know its nature, i.e., the mass and production of dark matter, and what forces they will interact besides gravity. So, the dark matter problem is one of the most important issues in recent cosmology and particle physics. As the study progressed, the theoretical physicists provide many candidates of dark matter, e.g., the weakly interacting massive particles, axion, sterile neutrinos, primordial black holes, and massive compact halo objects et al (see \[1-3\] more details).

Recently, Chamseddine and Mukhanov provided a new idea to explain dark matter \[4, 5\] with the mimetic gravity model. In this model, the physical metric $g_{\mu\nu}$ is determined by an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field $\phi$: $g_{\mu\nu} = \tilde{g}_{\mu\nu} \phi^{\alpha \beta} \partial_\alpha \partial_\beta \phi$. In such a setup, the conformal degree of freedom is separated in a covariant way, and this extra degree of freedom can be deemed to dynamic and mimic cold dark matter. The mimetic model could be transformed into a Lagrange multiplier formulation with a potential of the mimetic scalar field. With these methods, one can obtain a viable theory confronted with the cosmic evolution. It was pointed out that this model can also drive the late-time acceleration and early-time inflation \[6\]. For more recent works about mimetic gravity, see Refs. \[7-17\].

Besides, since the theories of extra dimensions and braneworlds can potentially address the hierarchy problem and cosmological constant problem \[18-20\], they have attracted more and more attention. Various extensions of Randall-Sundrum (RS) models are investigated, including thin branes \[21-26\] and thick branes \[27-42\]. In addition, the extra-dimensional theory can predict a series of massive particles beyond the standard model of particle physics, e.g., massive gravitons and massive vector particles. The corresponding massive propagators will correct the four-dimensional Newtonian potential and Coulomb potential at short distance.

It is known that a massive graviton will contribute a correction term to the Newtonian potential with a form of Yukawa potential, and the total contribution from all massive gravitons depends on the mass spectrum. Note that brane models with different background solutions could have different effective potentials along the extra dimension. Furthermore, the corresponding effective potentials will lead to different mass spectra of the Kaluza-Klein (KK) gravitons, and these differences will eventually be reflected in the correction behavior to the Newtonian potential.

Recently, the thick brane theory and mimetic gravity were combined to investigate the inner structures of thick branes with a volcano-like effective potential \[43, 44\]. As can be seen in these works, the action with the Lagrange multiplier formulation will give a new degree of freedom which allows one to construct more new types of thick branes with richer inner structures. The corresponding solutions have different effective potentials which can lead to different massive KK modes of the gravitons. In this paper, we will consider three typical solutions that correspond to three kinds of effective potentials for ten-

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should be a function of the extra dimension. Therefore, the authors set dimensional Poincare invariance.

The organization of this paper is as follows. In Sec. II, we briefly introduce the mimetic theory in the braneworld scenario and obtain the corresponding thick brane solutions. Then, we analyze the stability of the brane solutions under the tensor perturbations and check the localization of the massless graviton in Sec. III. After that, we derive the corresponding correction to the Newtonian potential, for which the final constraints about the mimetic thick brane are given in Sec. IV. Finally, the conclusion and discussion are given in Sec. V.

II. THE MODEL

In this section, we consider the five-dimensional mimetic gravity with the following action

$$S = \int d^5x \sqrt{-g} \left( \frac{R}{2k^2} + \lambda \left[ \partial_M \phi \partial^M \phi - U(\phi) \right] - V(\phi) \right),$$

where $\kappa^2 = 1/M^3$ with $M$ being the five-dimensional fundamental scale and $\lambda$ is a Lagrange multiplier. For simplicity, we chose the natural unit with $\kappa^2 = 1$. In the original mimetic gravity, the authors set $U(\phi) = -1$ [4]. This assumption was extended to the case with $U(\phi) < 0$ by Astashenok et al. [45]. Note that the mimetic field $\phi$ is a function of time $t$ in these works, while in thick brane theory generated by a scalar field, the scalar field $\phi$ should be a function of the extra dimension. Therefore, we choose the condition $U(\phi) = g^{MN} \partial_M \phi \partial_N \phi > 0$ proposed in Ref. [43]. By varying the action (1) with respect to $g_{MN}$, $\phi$, and $\lambda$, respectively, we get the equations of motion as

$$G_{MN} + 2\lambda \partial_M \phi \partial_N \phi - L_\phi g_{MN} = 0,$$

(2)

$$2\lambda \Box (\phi) + 2g^{MN} \partial_M \lambda \partial_N \phi + \lambda \frac{\partial U}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0,$$

(3)

$$g^{MN} \partial_M \phi \partial_N \phi - U(\phi) = 0.$$

(4)

Here, $L_\phi = \lambda \left[ g^{MN} \partial_M \phi \partial_N \phi - U(\phi) \right] - V(\phi)$, and the five-dimensional d’Alembert operator is defined as $\Box = g^{MN} \nabla_M \nabla_N$. The Latin indices ($M, N = 0, 1, 2, 3, 5$) stand for the five-dimensional coordinate indices, and the Greek indices ($\mu, \nu = 0, 1, 2, 3$) represent the brane coordinate indices.

We consider the following braneworld metric with four-dimensional Poincare invariance

$$ds^2 = e^{2A(z)} \left( \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right),$$

(5)

where $e^{A(z)}$ or $A(z)$ is called as the warp factor. Then, Eqs. (2)-(4) can be rewritten as

$$e^{-2A} \left( 3A'^2 + 3A'' \right) + V(\phi) + \lambda \left( U(\phi) - e^{-2A} \phi^2 \right) = 0,$$

(6)

$$6A'^2 + e^{2A} V'(\phi) + e^{2A} \lambda \left( U(\phi) + e^{-2A} \phi^2 \right) = 0,$$

(7)

$$\lambda \left( 6A' \phi' + 2 \phi'' + e^{2A} \frac{\partial U}{\partial \phi} \right) + 2 \lambda \phi' + e^{2A} \frac{\partial V}{\partial \phi} = 0,$$

(8)

$$e^{-2A} \phi^2 - U(\phi) = 0.$$

(9)

where the prime denotes the derivative with respect to the extra-dimensional coordinate $z$. The above equations are not independent of each other. Combining them we get three largely simplified equations for $\lambda$, $U(\phi)$, and $V(\phi)$

$$\lambda = \frac{3(A'' - A'^2)}{2 \phi'^2},$$

(10)

$$U(\phi) = e^{-2A} \phi'^2,$$

(11)

$$V(\phi) = -3e^{-2A} (A'^2 + A'').$$

(12)

Note that all the expressions of $\lambda$, $U(\phi)$, and $V(\phi)$ depend on the warp factor $A(z)$ and the mimetic field $\phi(z)$. So, once $A(z)$ and $\phi(z)$ are given, the profiles of $\lambda$, $U(\phi)$, and $V(\phi)$ could be determined. On the one hand, there are no constraints on $A(z)$ and $\phi(z)$ from the equations of motion, so they can be chosen arbitrarily in principle. However, on the other hand, a viable thick braneworld model should satisfy the minimal requirement, namely, the localization of massless tensor mode (massless graviton). Therefore, not all the choices of $A(z)$ and $\phi(z)$ are achievable. Here, we will consider three typical solutions.

A. Volcano (VO) type thick brane

Firstly, we consider the case of the warp factor $e^{A(z)}$ as a power function of the extra-dimensional $z$, and the mimetic field $\phi(z)$ is a kink function. The solutions can be given as

$$e^{A(z)} = \frac{1}{\sqrt{k^2 z^2 + 1}}, \quad \phi(z) = v \left( \frac{k z}{\sqrt{1 + k^2 z^2}} \right)^\gamma,$$

(13)

where $k$ is the scale parameter which controls the thickness of the brane, $\gamma$ is a positive integer, and $v$ is a positive parameter determining the limit of the scalar field. The corresponding expressions of $\lambda$, $U(\phi)$, and $V(\phi)$ can be expressed as

$$\lambda = -\frac{3}{2 \gamma^2 v^2} \left( k^2 z^2 + 1 \right)^\gamma e^{-1},$$

(14)

$$U(\phi) = k^2 v^2 \phi^{2-1/\gamma} \left( \phi^{2/\gamma} - 1 \right)^2,$$

(15)

$$V(\phi) = -3k^2 (3\phi^{2/\gamma} - 1),$$

(16)

where $\Phi = \phi/v$. Such brane solution will give a volcano type effective potential of the tensor perturbations.
B. Pöschl-Teller (PT) type thick brane

Then, we come to the hyperbolic function form of the warp factor $e^{A(z)}$ and a different kink form of the mimetic field $\phi(z)$. The expressions of warp factor and mimetic field can be given as

$$e^{A(z)} = \text{sech}(kz), \quad \phi(z) = v\tanh^\gamma(kz),$$

for which the other functions can be solved as

$$\lambda = -\frac{3 \sinh^2(2kz) \tanh^{-2n}(kz)}{8k^2v^2},$$

$$U(\phi) = (k^2v^2)^2 \Phi^{2-2/\gamma} \left( \Phi^{2/\gamma} - 1 \right),$$

$$V(\phi) = \frac{3k^2}{\Phi^{2/\gamma} - 1}.$$

C. Harmonic oscillator (HO) type thick brane

Finally, we choose an exponential warp factor and a kink mimetic field:

$$e^{A(z)} = e^{-k^2z^2}, \quad \phi(z) = v \left( \frac{kz}{\sqrt{1 + k^2z^2}} \right)^\gamma.$$  

The specific expressions of $\lambda$, $U(\phi)$, and $V(\phi)$ are solved as follows

$$\lambda = -\frac{3 \left( 2k^2z^2 + 1 \right) \left( k^2z^2 + 1 \right)^{\gamma+2}}{\gamma^2k^2v^2 (k^2z^2)^{\gamma-2}},$$

$$U(\phi) = (k^2v^2)^2 \Phi^{2-2/\gamma} \left( 1 - \Phi^{2/\gamma} \right)^3 \frac{e^{3/2\gamma}}{e^{1/2\gamma - 1}},$$

$$V(\phi) = \frac{6k^2}{\Phi^{2/\gamma} - 1} \frac{e^{-3/2\gamma}}{e^{1/2\gamma - 1}}.$$  

The shapes of these three kinds of warp factors and the two kinds of mimetic scalar fields are shown in Fig. 1. Figures 1(a) and 1(b) show that if $\gamma$ is an odd integer, the mimetic scalar field would be a single-kink (the black dashed lines) for $\gamma = 1$, and it will become a double-kink (the red lines) with $\gamma \geq 3$. Besides, if $\gamma$ is an even integer, the scalar field will not be a kink configuration (the blue dashed line) anymore. For a general thick brane model, the background scalar field should be a kink configuration. While, in mimetic thick brane model, due to the Lagrange multiplier which can cause excess degrees of freedom, the non-kink scalar field can also generate a thick brane. Although these solutions have the same limit of $e^{A(z)}|_{z \to \infty} \to 0$, they differ with the asymptotic behaviors at infinity of the extra dimension, with the attenuation intensity $\text{HO} > \text{PT} > \text{VO}$ (see Fig. 1(c)). These different asymptotic behaviors will lead to different physical properties, including the potentials felt by the gravitons along the extra dimension and the corrections of the Newtonian potential caused by the massive gravitons.

III. LINEAR PERTURBATIONS AND LOCALIZATION

In this section, we consider the linear perturbations of the metric and their localization. It is well known that the linear perturbations of a background metric can be decomposed into three parts: the transverse-traceless tensor modes, the scalar modes, and the transverse vector modes (the so-called scalar-vector-tensor decomposition). By making a scalar-vector-tensor decomposition, the three kinds of modes decouple with each other. In this work, we will only consider the transverse-traceless tensor perturbations modes.

For the tensor perturbations, the perturbed metric is given by

$$\tilde{g}_{MN} = e^{2A(z)} (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu + dz^2.$$  

Here, $h_{\mu\nu} = h_{\mu\nu}(x^\mu, z)$ depends on all the coordinates. Combining the specific perturbed metric and the transverse-traceless (TT) condition, i.e., $\partial^\mu h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$, we can simplify the perturbed Ricci tensor as

$$\delta R_{\mu\nu} = -\frac{1}{2} \left( \Box^{(4)} + \partial_2^2 + 2A'' + 6A'^2 + 3A' \partial z \right) h_{\mu\nu},$$

$$\delta R_{55} = 0,$$

where the four-dimensional d’Alembertian is defined as $\Box^{(4)} \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$. Besides, the tensor perturbations of the equations of motion (2) can be expressed as the following form

$$\delta R_{MN} = \frac{2}{3} \delta g_{MN}(\lambda U + V).$$  

Then, by combining Eqs. (10), (11), (12), and the perturbed metric (25), the right hand side of Eq. (28) can be simplified as

$$\frac{2}{3} \delta g_{55}(\lambda U + V) = -A'' + 3A'^2) h_{\mu\nu},$$

$$\frac{2}{3} \delta g_{55}(\lambda U + V) = 0.$$

Therefore, the perturbed tensor equation can be obtained as

$$-\frac{1}{2} \Box^{(4)} h_{\mu\nu} - \frac{1}{2} h''_{\mu\nu} - \frac{3}{2} A' h'_{\mu\nu} = 0.$$  

Next, we make a KK decomposition $\tilde{h}_{\mu\nu} = \sum e^{\mu}_{\mu\nu}(x) \tilde{\Psi}_n(z)$, where the polarization tensor $e^{\mu}_{\mu\nu}$ also satisfies the TT condition $\partial^\mu e^{\mu}_{\mu\nu} = e^{\mu}_{\mu\nu} = 0$. Bringing the KK decomposition into the perturbed tensor equation (31), we can get a four-dimensional massive Klein-Gordon equation for the polarization tensor $e^{\mu}_{\mu\nu}(x)$ and an equation for the extra-dimensional part $\tilde{\Psi}_n(z)$:

$$\left( \Box^{(4)} - m_n^2 \right) e^{(n)}_{\mu\nu}(x) = 0,$$

$$-\tilde{\Psi}_n''(z) - 3A' \tilde{\Psi}_n'(z) = m_n^2 \tilde{\Psi}_n(z).$$
Furthermore, by redefining the extra dimensional part as $\Psi_n(z) = e^{-\frac{2}{5}A} \Psi_n(z)$, we obtain a Schrödinger-like equation for the new function $\Psi_n(z)$ of the extra-dimensional part:

$$-\partial_z^2 \Psi_n(z) + V_I(z) \Psi_n(z) = m_n^2 \Psi_n(z),$$

with the effective potential $V_I(z)$ given by

$$V_I(z) = \frac{3}{2} A'' + \frac{9}{4} A^2.$$  \hspace{1cm} (35)

We should note that with this effective potential (35), the above Schrödinger-like equation can be rewritten as $\mathcal{H} \Psi_n(z) = m_n^2 \Psi_n(z)$, where the Hamiltonian operator is given as $\mathcal{H} = \mathcal{Q}^+ \mathcal{Q}$, with $\mathcal{Q} = -\partial_z + \frac{3}{2} \partial_z A$. Since the eigenvalue of the Hamiltonian operator $\mathcal{H}$ is positive definite, there do not exist negative $m_n^2$ modes, namely, there are no tachyonic tensor modes.

The abstract expression of the effective potential $V_I(z)$ shows that it only depends on the warp factor $A(z)$, namely, different asymptotic behaviors of the warp factor $A(z)$ can result in different properties of the effective potentials. We will discuss the properties of the effective potentials with the warp factors $A(z)$ given in Sec. II.

Shapes of three kinds of effective potentials $V_I(z)$ are shown in Fig. 2. From Fig. 2(a), we can see that the effective potential $V_I(z)$ of case VO is a volcano-like potential with a single potential well, and the potential well becomes narrower and deeper with the parameter $k$ increases. The asymptotic behavior of the volcano-like potential shows that there are no other bound states except the zero mode, and the mass spectrum of the massive excited states is continuous from $m_n > 0$. Figure 2(b) describes the shape of effective potential $V_I(z)$ for case PT. It shows that the second warp factor $A(z)$ leads to a PT potential behavior as $V_I(z)|z \to +\infty = 9k^2/4$. The scale parameter $k$ determines the width and the depth of the potential well. For this PT effective potential, there are two bound states, i.e., the zero mode with $m_0 = 0$ and the first excited state with $m_1 = \sqrt{2}k$, and the mass spectrum is also continuous from $m_n \geq \sqrt{2}k$. For case HO, Fig. 2(c) shows that the effective potential $V_I(z)$ has the behavior of a harmonic-oscillator potential with the parameter $k$ controls the the mass spectrum. For the harmonic-oscillator potential, all of the states are bound states with mass $m_n = \sqrt{6nk}$, the index $n$ means $n$–th eigenstate. These different potentials lead to different mass spectra of massive gravitons, which can result in different corrections to the Newtonian potential.

At the end of this section, we consider the zero mode of the tensor perturbations by setting $m_n^2 = 0$ in Eq. (34). It is easy to get the solution of the zero mode:

$$\Psi_0(z) \propto e^{\frac{3}{2}A(z)},$$

One can verify that the zero modes for the above three different types of warp factors are square-integrable and hence all the zero modes are localized around the brane. Thus, the four-dimensional Newtonian potential can be realized on the brane.

IV. THE CORRECTION TO NEWTONIAN POTENTIAL

In the above section, we have considered the tensor perturbations and obtained the Schrödinger-like equation (34), we also demonstrated that the zero mode can be localized on the brane for all the cases to recover the four-dimensional gravity. In this section, we consider the massive KK modes of the gravitons which can cause the correction to the Newtonian potential in four-dimensional theory.

In the thick brane scenario, the energy density of the brane has a distribution along the extra dimension. Therefore, for simplicity, Refs. [27, 46–49] considered the gravitational potential between two point-like sources of mass $M_1$ and $M_2$ located at the origin of the extra dimension, i.e., $z = 0$. We can express the gravitational potential between two masses on the brane as

$$V(r) = -\frac{M_1 M_2}{M_{pl}^2} \frac{1}{r} - \frac{M_1 M_2}{M_{st}^2} \sum_{n \neq 0} \frac{e^{-mr}}{r} |\Psi_n(0)|^2$$

$$= -\frac{M_1 M_2}{M_{pl}^2} \frac{1}{r} \left( 1 + \frac{M_{pl}^2}{M_{st}^2} \sum_{n \neq 0} \Delta u(r) \right),$$

\hspace{1cm} (37)
where $M_{pl}$ and $M_*$ are the effective four-dimensional Plank scale and the five-dimensional fundamental scale, respectively. $\sum_n$ stands for summation or integration (or both) with respect to $n$, depending on the respective discrete or continuous character of the massive KK modes. Besides, we set $\Delta u(r) = e^{-m_n r} |\Psi_n(0)|^2$.

We can focus on the curvature term of the action (1) from which we will derive the effective four-dimensional scale $M_{pl}$:

$$M_{pl}^2 \int d^5x \sqrt{-gR} \supset M_{pl}^2 \int d^4x \sqrt{-g^{(4)}(x^\mu)R^{(4)}(x^\mu)}. \quad (38)$$

Therefore, the relation between the effective Planck scale $M_{pl}$ and the fundamental scale $M_*$ is given by

$$M_{pl}^2 = M_*^3 \int_{-\infty}^{+\infty} dz e^{3A}. \quad (39)$$

So, the gravitational potential between two masses on the brane can be simplified as

$$V(r) = -\frac{M_1 M_2}{M_{pl}^2} \left( \frac{1}{r} + \Delta V(r) \right) = -\frac{M_1 M_2}{M_{pl}^2} \frac{1}{r} \left( 1 + \Delta U(r) \right), \quad (40)$$

$$\Delta U(r) = \left( \int_{-\infty}^{+\infty} dz e^{3A} \right) \sum_{m \neq 0} \Delta u(r), \quad (41)$$

where $\Delta V(r) = \frac{1}{r} \Delta U(r)$ is the correction term to the Newtonian potential, $\Delta U(r)$ the relative correction term and $\Delta u(r)$ the correction factor. Next, we calculate the corrections to the Newtonian potential for all cases.

**Case VO:** We substitute the warp factor (13) into the Schrödinger-like equation (34), and get the reduced Schrödinger-like equation

$$-\partial_z^2 \Psi_n + \frac{3k^2 (5k^2 z^2 - 2)}{4 (k^2 z^2 + 1)^2} \Psi_n = m_n^2 \Psi_n. \quad (42)$$

To solve the above equation, we consider the behavior of the effective potential at infinity of the extra dimension:

$$V(z) \sim \frac{15}{4z^2}. \quad (43)$$

The approximate solution is given by a linear combination of Bessel functions as

$$\Psi_n(z) = \sqrt{z} \left( C_1 J_2(m_n z) + C_2 Y_2(m_n z) \right), \quad (44)$$

where $C_1$ and $C_2$ are arbitrary constants, $J_2(m_n z)$ and $Y_2(m_n z)$ are the first and second Bessel functions, respectively. In Ref. [27], the authors calculated the expression of $\Psi_n(0)$:

$$\Psi_n(0) \sim \left( \frac{m_n}{k} \right)^{1/2}. \quad (45)$$

So, the correction factor $\Delta u(r)$ to the Newtonian potential with a massive graviton $m_n$ is

$$\Delta u(r) = e^{-m_n r} |\Psi_n(0)|^2 = \frac{m_n e^{-m_n r}}{k}. \quad (46)$$

From Fig. 2(a), we can see that the spectrum of the massive gravitons is continuous. Therefore, the relative correction term to the Newtonian potential resulted by all the massive gravitons is

$$\Delta U(r) = \left( \int_{-\infty}^{+\infty} dz e^{3A} \right) \int_0^\infty dm \Delta u(r) \quad = \frac{2}{k^2 z^2}. \quad (47)$$

The correction to the Newtonian potential for case VO is the same form as that of the Randall-Sundrum brane model $\Delta V(r) = \frac{1}{r} \Delta U(r) \sim 1/k^2 r^3$.

**Case PT:** For this case with the warp factor (17), the corresponding effective potential turns into

$$V_{\text{pt}}(z) = \frac{9}{4} k^2 - \frac{15}{4} k^2 \text{sech}^2(kz), \quad (48)$$

and the Schrödinger-like equation can be expressed as

$$\left( -\partial_z^2 - \frac{15}{4} k^2 \text{sech}^2(kz) \right) \Psi_n = E_n \Psi_n, \quad (49)$$

where $E_n = m_n^2 - \frac{9}{4} k^2$. It can be shown that there are two bound states in this potential. The first one is the ground state $\Psi_0(z)$ with $E_0 = -\frac{9}{4} k^2$, and it is in fact the zero.
mode since the mass is zero: \( m_0 = 0 \). The second one is the first excited state \( \Psi_1(z) \) with \( E_1 = -k^2/4 \), which represents a massive graviton with mass \( m_1 = \sqrt{2}k \). The two bound states wave functions are

\[
\Psi_0(z) = C_0 \text{sech}^{3/2}(kz), \\
\Psi_1(z) = C_1 \text{sinh}(kz) \text{sech}^{3/2}(kz).
\]

(50) and (51)

Here \( C_0 \) and \( C_1 \) are the normalization constants. We should note that \( \Psi_1(0) = 0 \) which means that the first excited state does not contribute to the correction of the Newtonian potential.

The continuous spectrum starts at \( E_n = 0 \), corresponding to \( m_n^2 \geq 9k^2/4 \). These excited states asymptotically turn into plane waves, and represent delocalized KK massive gravitons. Their explicit expressions can be given in terms of the associated Legendre functions of the first kind:

\[
\Psi_n(z) = \sum_{\pm} C_{\pm}^n P^{3/2}_{3/2}(\sigma) \text{tanh}(kz),
\]

(52)

where \( C_{\pm}^n \) are \( m_n \)-dependent parameters and

\[
\sigma = \sqrt{\frac{9}{4}k^2 - m_n^2}.
\]

(53)

In order to calculate the correction from the continuous modes, we need to compute the normalization constants \( C_{\pm}^n \). According to Refs. [50-52], we can reduce the constants \( C_{\pm}^n \) as

\[
C_+^n = C_-^n = \frac{1}{\sqrt{2\pi}} \left[ \Gamma(1 + \sigma) \right].
\]

(54)

So, \( \Psi_n(0) \) can be expressed as the following form

\[
\Psi_n(0) = \frac{\Gamma(1 - \sigma)}{\Gamma(-\frac{1}{2} - \frac{\sigma}{2}) \Gamma(\frac{1}{2} - \frac{\sigma}{2})},
\]

(55)

and for a massive graviton \( \Delta u(r) \) is

\[
\Delta u(r) = e^{-m_n r} |\Psi_n(0)|^2 \\
= e^{-m_n r} \left[ \frac{\Gamma(1 - \sigma)}{\Gamma(-\frac{1}{2} - \frac{\sigma}{2}) \Gamma(\frac{1}{2} - \frac{\sigma}{2})} \right]^2.
\]

(56)

Although we have obtained the parsed expression of the Newtonian potential for a massive graviton, it is cumbersome for the final result to integrate (56) directly. So, our approach is to use \( |\Psi_n(0)|^2 \) as a fitting function with its approximate behavior, i.e.,

\[
|\Psi_n(0)|^2 \approx a_1 \text{tanh}(a_2(m - m_0)) + a_3,
\]

(57)

where \( a_1 = \frac{26003}{100000}, a_2 = \frac{13}{10}, a_3 = \frac{3977}{100000} \), and \( m_0 = 3k/2 \). The relative correction to the Newtonian potential for all the massive gravitons is

\[
\Delta U(r) = \int_{-\infty}^{+\infty} dz \int_{4k}^{\infty} dm \Delta u(r) \\
= b_1 e^{-\frac{3kr}{2}} \left[ -b_2 k r \psi^{(0)}(b_1k r) - b_3 + b_2 k r \psi^{(0)}(b_1k r + \frac{1}{2}) \right].
\]

(58)

where \( b_1 = \frac{10}{14}, b_2 = \frac{26003}{100000}, b_3 = \frac{3977}{100000} \), and \( \psi^{(0)}(x) \) is the logarithmic derivative of the Gamma function: \( \psi^{(0)}(x) \equiv \frac{d}{dx} \ln \Gamma(x) \). Note that the term in the square bracket of (58) is almost a constant, we can get an approximate expression of the relative correction term of the Newtonian potential:

\[
\Delta U(r) \propto e^{-\frac{3kr}{2}}.
\]

(59)

**Case HO**: The corresponding Schrödinger equation (34) for case HO can be expressed as the following form

\[
-\partial_z^2 \Psi_n + (9k^4z^2 - 3k^2) \Psi_n = m_n^2 \Psi_n.
\]

(60)

The normalized solution is given by

\[
\Psi_n = \sqrt{\frac{3}{\pi n!}} \sqrt{\frac{1}{n!}} e^{-\frac{3k^2z^2}{2}} H_n \left( \sqrt{3k}z \right),
\]

(61)

where \( n \) is a positive integer and \( H_n \) is Hermite polynomial. The corresponding mass spectrum is \( m_n = k\sqrt{6n} \), which means that the mass gap decreases with the mass. The expression of wave function \( \Psi_n \) at \( z = 0 \) can be rewritten as

\[
\Psi_n(0) = \frac{\sqrt{3}2^{n/2}}{\sqrt{n!} \Gamma \left( \frac{1}{2} - \frac{n}{2} \right)}.
\]

(62)

Then, \( \Delta u(r) \) for a massive graviton is

\[
\Delta u(r) = e^{-m_n r} |\Psi_n(0)|^2 \\
= e^{-k\sqrt{6n}r} \frac{\sqrt{3}2^n}{n! \Gamma \left( \frac{1}{2} - \frac{n}{2} \right)}.
\]

(63)

Note that \( \Delta u(r) = 0 \) for an odd \( n \). Therefore, the odd modes of the massive gravitons do not contribute to the correction of the Newtonian potential. In order to calculate the correction to the Newtonian potential for all the massive gravitons, we should do some tedious but simple steps. The \( \Delta u(r) \) for this case can be rewritten as

\[
\Delta u(r) = e^{-m_n r} |\Psi_n(0)|^2 = e^{-2k\sqrt{3n}r} \frac{\sqrt{3}2^n}{(2n)! \Gamma \left( \frac{1}{2} - \frac{n}{2} \right)^2} \\
\approx e^{-2k\sqrt{3n}r} \frac{\sqrt{3}}{\pi \sqrt{a}} \quad (a = n/2 = \frac{m_n^2}{12k^2}).
\]

(64)

However, it is cumbersome to sum \( \Delta u \) directly. From Fig. 3, we can see that \( \sum_a \Delta u \) can be fitted by \( \int \Delta u \, da \). For simplicity, we replace sum with integration and obtain the following approximate result

\[
\Delta U(r) = \sum_{a=1}^{\infty} \Delta u(r) = \sum_{a=1}^{\infty} e^{-2k\sqrt{3a}r} \frac{\sqrt{3}}{\pi \sqrt{a}} \\
\approx \int_{1}^{\infty} e^{-2k\sqrt{3a}r} \frac{\sqrt{3}}{\pi \sqrt{a}} \, da = e^{-2\sqrt{3}kr}.
\]

(65)
Figure 4 shows the correction factor $\Delta u$ of the massive gravitons (plot as $\Delta u^{1/3}$) and the relative correction terms $\Delta U$ contributed by all the massive gravitons for the three cases, respectively. From Fig. 4(a), we can see that the correction factor $\Delta u$ is smaller for larger graviton mass, and the attenuation trends of the three cases are slightly different. On the one hand, the different forms of $|\Psi_n(0)|^2$, the square of the massive graviton mode on the brane, are given by

$$
|\Psi_n(0)|^2 \sim \begin{cases} 
\frac{m_n}{k}, & \text{case VO;} \\
\frac{a_1 \tanh(a_2(m_n - m_0)) + a_3}{n! t'\left(\frac{m_n}{m^*}\right)^2}, & \text{case PT;} \\
\frac{n^{m_n^2/6k^2}}{m^*}, & \text{case HO.}
\end{cases}
$$

(66)

On the other hand, for different models, the mass range of the massive gravitons which dominate the correction to the gravitational potential are different, i.e., $(0 \sim \infty)$, $[3k/2 \sim \infty)$, $[2\sqrt{3}k \sim \infty)$ for the three cases, respectively. Note that for case VO and case PT the mass spectra are continuous, while the mass spectrum of case HO is discrete. These differences lead to different forms of the relative correction terms $\Delta U$, which are shown in Fig. 4(b). From Fig. 4(b), we can see that on small scales $\Delta U > 1$, which means that the correction term dominates the Newtonian potential, and on large scales the relative correction term decays to zero rapidly. That is to say, the effect of all massive gravitons on the Newtonian potential can be ignored on large scales. We note that the correction term $\Delta U$ of case VO is more remarkable than the others.

So far, we have obtained the expressions of the three corrections to the Newtonian potential. Then, we can obtain the constraints on the parameters of our models by combining the latest tests of the gravitational inverse-square law [53–55]. In these experiments, the authors considered the following four-dimensional gravitational potential

$$
V(r) = V_N(r) \left[ 1 + \alpha \exp(-r/\lambda) \right],
$$

(67)

where the parameter $r$ is the separation between two masses, $\lambda$ and $\alpha$ are the length scale and strength of the Yukawa type correction. As shown in Refs. [54, 55], the corresponding values of $\alpha, \lambda, r$ are

$\{\alpha, r, \lambda\} = \{1, 210\mu m, 48\mu m\},$

(68)

$\{\alpha, r, \lambda\} = \{0.45, 52\mu m, 38.6\mu m\},$

(69)

and the magnitudes of the corresponding Yukawa correction term are

$$
\alpha e^{-r/\lambda} \sim \begin{cases} 
0.01, & r = 210\mu m, \\
0.1, & r = 52\mu m.
\end{cases}
$$

(70)

Note that, the magnitude of the correction term of the gravitational potential should be independent of its form. Therefore, it is natural to set the same magnitudes for the correction terms of our models when we choose the same separations as the separations in Refs. [54, 55]. Obviously, the Yukawa type correction $\alpha \exp(-r/\lambda)$ can be considered as a form of $\Delta U$ in our models. To get the constraints on the parameters, we can set that the upper limit of the correction term $\Delta U$ is the same with the magnitude of $\alpha e^{-r/\lambda}$, which means that $\Delta U < 0.01$ for $r = 210\mu m$ or $\Delta U \leq 0.1$ for $r = 52\mu m$.

By using the above assumptions, we get the critical points with $\Delta U(\tilde{r}) = 0.01$ with the separation $r = 210\mu m$ [54] and $\Delta U(\tilde{r}) = 0.1$ for $r = 52\mu m$ [55], where $\tilde{r} = kr$. After calculation, we get the critical values of $\tilde{r}$ for the three cases. Note that the relative correction term $\Delta U(\tilde{r})$ decreases monotonically with $\tilde{r}$, in other words, to make sure $\Delta U$ is less than the critical points for the test experiments, the scale parameter $k$ should satisfy the relation $k > \tilde{r}/r$. Besides, we can get the constraints on the five-dimensional fundamental scale $M_*$ based on Eq. (39).

We give the constraints on the parameter $k$ and the fundamental scale $M_*$ of our three models in Tables I and II. Comparing these results, we can see that the constraints of parameters $k$ and $M_*$ based on the experimental data in Ref. [55] are stronger than the constrains by Ref. [54]. The limits of $k$ and $M_*$ of case VO are stricter than other two cases for both the two experimental data. Therefore, we have a conclusion that the critical value of the scale parameter $k$ is at least $10^{-4}$eV, and the five-dimensional fundamental scale $M_*$ should be at least $10^5$TeV.
TABLE I: Constraints of the scale parameter $k$ and the fundamental $M_*$ for the three cases at $r = 210\mu$m.

| model | $\Delta U$ | $\tilde{r}$ | $k_{\text{min}}$(eV) | $M_*$ | $M^\text{min}_*$ (TeV) |
|-------|------------|-------------|----------------------|-------|----------------------|
| Case VO | $1/k^2r^2$ | 14.1 | $6.7 \times 10^{-3}$ | $\frac{k}{3}M^2_{pl}$ | $7.9 \times 10^5$ |
| Case PT | $\frac{\tilde{r}^2}{k^2r}$ | 1.6 | $7.4 \times 10^{-4}$ | $\frac{2k}{3}M^2_{pl}$ | $4.2 \times 10^5$ |
| Case HO | $\frac{e^{-2\sqrt{k}r}}{	ilde{r}}$ | 1.0 | $4.8 \times 10^{-4}$ | $\sqrt{\frac{3}{\pi k}}M^2_{pl}$ | $4.1 \times 10^5$ |

TABLE II: Constraints of the scale parameter $k$ and the fundamental $M_*$ for the three cases at $r = 52\mu$m.

| model | $\Delta U$ | $\tilde{r}$ | $k_{\text{min}}$(eV) | $M_*$ | $M^\text{min}_*$ (TeV) |
|-------|------------|-------------|----------------------|-------|----------------------|
| Case VO | $1/k^2r^2$ | 4.5 | $8.6 \times 10^{-3}$ | $\frac{k}{3}M^2_{pl}$ | $8.6 \times 10^5$ |
| Case PT | $\frac{\tilde{r}^2}{k^2r}$ | 0.7 | $1.4 \times 10^{-3}$ | $\frac{2k}{3}M^2_{pl}$ | $5.1 \times 10^5$ |
| Case HO | $\frac{e^{-2\sqrt{k}r}}{	ilde{r}}$ | 0.5 | $1.0 \times 10^{-3}$ | $\sqrt{\frac{3}{\pi k}}M^2_{pl}$ | $5.3 \times 10^5$ |

V. CONCLUSION

In this paper, we considered the thick brane model generated by a mimetic scalar field with the Lagrange multiplier formulation. With the existence of excess degrees of freedom, we constructed three background solutions. Although these three solutions have the same limit of $e^{A}|z| \to 0$, the asymptotic behaviors of these three cases are different with HO > PT > VO. These different asymptotic behaviors cause different effective potentials of tensor perturbations, which lead to different corrections to the Newtonian potential by the massive KK gravitons.

We got the specific expressions of the effective potentials for the three cases. They are volcano-like potential, PT potential, and harmonic oscillator potential. We showed that all the solutions are stable under the tensor perturbations and the zero modes of tensor perturbations can be localized on the branes. Therefore, the four-dimensional Newtonian potential can be recovered.

We also calculated the corrections to the Newtonian potential for the three cases. For case VO, the relative correction term $\Delta U \propto 1/(kr)^2$ is the same as the RS model [19]. Although, the relative correction terms $\Delta U$ of case PT and case HO have the same form with $\Delta U \approx \exp(-kr)/kr$, the specific values $\beta = -3/2$ and $\beta = -2\sqrt{3}$ will lead to big difference on a small scale. For the corrections to the Newtonian potential of these three cases, the results show that the four-dimensional Newtonian potential can be recovered on large scales. On a small scale, the three cases have different behaviors, the correction to the Newtonian potential of case VO is more pronounced than the other two cases. Combining the latest tests of the gravitational inverse-square law [54, 55], we obtained the constraints on the scale parameter $k$, i.e., $k$ is at least $10^{-4}$eV, and the corresponding five-dimensional fundamental scale $M_*$ should be at least $10^5$TeV.

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