Pseudoparticle-operator description of an interacting bosonic gas

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We write the Hamiltonian of the Bose gas with two-body repulsive δ-function potential in a pseudoparticle operator basis which diagonalizes the problem via the Bethe ansatz. In this operator basis the original bosonic interactions are represented by zero-momentum forward-scattering interactions between Landau-liquid pseudoparticles. We find that this pseudoparticle operator algebra is complete: all the Hamiltonian eigenstates are generated by acting pseudoparticle operators on the system vacuum. It is shown that one boson of vanishing momentum and energy is a composite of a one-pseudoparticle excitation and a collective pseudoparticle excitation. These excitations have finite opposite momenta and cannot be decomposed. Our formalism enables us to calculate the various quantities which characterize the static and dynamic behavior of the system at low energies.

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I. INTRODUCTION

The interest on exact solutions of many-body problems has increased in the last years due to the “non-conventional” behavior of new materials which cannot be described by the usual perturbative approaches. The dynamics of elementary excitations in these systems is constrained to low-dimensional manifolds, one (for instance, in quasi-one dimensional conducting polymers and synthetic metals or edge states of the fractional quantum Hall effect) or two (as in the cuprates) spatial dimensions. It is known that the usual perturbation theory does not give correct results since in low dimensions (and specially in one dimension) the characteristics of the elementary excitations, due to phase-space considerations, change abruptly when the interaction between the particles is turned on.

The lack of exact solutions in higher dimensions contributed to the interest on one-dimensional integrable systems. These have been used as a paradigm to understand what is called today strongly interacting problems. The low dimensionality of these systems implies that a weak interaction is dramatically enhanced, which can be naively understood from the fact that particles in one dimension cannot “go around” each other (as they can in higher dimensions).

On the one hand, exact solutions in one dimension (specifically, the Bethe-ansatz [BA] solutions) have been considered often too “hermetic” to reveal the full physical information contained in them. This misconception follows from the fact that elementary excitations of models solvable by the BA seem to have non-universal features. For instance, at a first glance, the bosonic model [1-3] (which will be treated in this paper), with its fermionic excitation spectrum described by real BA rapidities and the SO(4) Hubbard model [4,5] which has the “strings” (described by complex, non-real, rapidities) as its elementary eta spin and spin lowest-weight-state (LWS) and highest-weight-state (HWS) excitations [6], have many different features.

However, it was noticed recently [7,8] that the universality of the models solvable by the BA becomes clearer when we look at the sectors of parameter space of lowest symmetry.
In the case of Hamiltonians with a non-Abelian symmetry these sectors correspond to finite values of the generalized “magnetic fields” associated with the diagonal generators of the non-Abelian algebras. The presence of these “magnetic fields” lowers the symmetry of the quantum problem to $[U(1)]^\nu$, where $\nu$ is both the number of fields and of the associated $U(1)$ Cartan sub-algebras. (At finite eigenvalues of the diagonal generators the symmetry of the problem is a global phase, $U(1)$, in each of the $\nu$ channels of interaction [eta spin, spin, etc].) A central point is that the expressions of many physical quantities in the sectors of symmetry $[U(1)]^\nu$ provides the corresponding values of these quantities in the sectors of higher symmetry $[10,11]$. In addition, in these sectors the different solvable BA problems have common universal properties, all gapless low-energy branches of excitations being of pseudoparticle-pseudohole type $[1]$.

One of the first applications of the BA was the study of a continuum problem within this universality class, i.e. a problem with a $U(1)$ symmetry. This was the solution of the many-body problem of bosons interacting by a two-body $\delta$-function potential $[1,3]$. As a field theory this is the repulsive quantum nonlinear Schrödinger model $[12]$. This was also the first model to which the quantum inverse-scattering method (QISM) was applied $[13–16]$.

It was firstly noted by Girardeau $[17]$ that, although the original problem is given in terms of bosons, the excitation spectrum has a fermionic character when the interaction is infinitely strong (the bosons acquire a hard core). Afterwards, Lieb $[2]$ pointed out that the elementary excitations of this model were not quasiparticles in the sense of the Landau theory of the Fermi liquid $[18,19]$ because quasiparticles decay in time while the elementary excitations, related to the excitation spectrum and, therefore, to the exact diagonalization of the problem, never decay. However, at that time, no operational method was proposed to deal directly with the elementary excitations in alternative to the original bosons of the problem. (As we have pointed out, in the bosonic basis the problem is nonperturbative.) The thermodynamics of the model was studied by Yang and Yang and also revealed the fermionic character of the elementary excitations $[3]$. The asymptotic behavior of the correlation functions was studied by several authors $[15,16,21,22]$.  


In this paper we deal directly with the *elementary excitations* which in the $U(1)$ sectors of the systems solvable by the BA can be generated by applying on the particle vacuum pseudoparticle fermionic operators $[10][11][23]$. In the present model the pseudoparticle operators generate a complete orthonormal basis that spans the whole Hilbert space. (This is in contrast to models with non-Abelian symmetries where the pseudoparticle algebra refers to the low-energy Hilbert space only $[11]$.) One of the motivations of our study is the simplicity of the model which has a simple $U(1)$ Abelian symmetry.

In the above operator basis the elementary excitations are described by a Landau liquid of fermionic pseudoparticles. By a Landau liquid we mean here the generalization of the Landau’s Fermi liquids $[18][19]$ introduced in Refs. $[10][11][23]$. (These works refer mainly to the one-dimensional Hubbard model.) As in the case of the class of fermionic integrable quantum liquids of Refs. $[11]$, the Hamiltonian, charge current, and other operators of the present model can be written in the pseudoparticle-operator basis. At fixed number of bosons, $N$, *all* the excited states can be generated by products of one-pair “pseudoparticle-pseudohole” operators acting on the $N$ interacting ground state (exactly as found by Lieb $[2]$).

The study of the present quantum problem in the pseudoparticle basis gives some new insights and provides a deeper understanding of the previously known aspects of the physics of the model. In particular, we will discuss the relationship between Lieb’s excitation spectrum and the pseudoparticle basis. This reveals that the pseudoparticle description is the natural representation for the quantum system. Moreover, we show that the original bosons of the theory are formed by two pseudoparticle excitations of opposite momentum. As in the case of the electrons of the Hubbard chain $[24]$, these excitations cannot be decomposed and the pseudoparticles cannot be removed or added to the many-particle system. They exist only inside that system, their configuration occupancies describing the Hamiltonian eigenstates. However, at infinite bosonic interaction (Girardeau’s case $[17]$) the pseudoparticles become free entities.

In the pseudoparticle basis and at fixed number of bosons, $N$, the ground state has
a non-interacting form \[11\]. The normal ordered Hamiltonian has in that basis an infinite number of terms, each corresponding to a pseudoparticle forward-scattering order, as we will see in Sec. II. While the low-energy physics requires considering the first two terms only, a full study of the physics of the model at all energy scales requires the use of all infinite terms of the Hamiltonian. Elsewhere we will study another integrable model of simple \( U(1) \) Abelian symmetry. This is the Calogero-Sutherland model associated with quantum chaos in Ref. \[25\]. The use of our operator basis reveals that the normal-ordered pseudoparticle Hamiltonian has for this model a finite number of terms. This further simplifies the study of its physics.

The various low-energy physical quantities can be derived as in the Landau theory of Fermi liquids. In particular, we can obtain the compressibility and the specific heat of the system and confirm that the pseudoparticle transport mass is density independent and that the charge conductivity spectrum reduces to the Drude peak, as required by the translational invariance of the model. In the pseudoparticle basis and at all energy scales the present quantum problem has zero-momentum forward scattering only. While at high energy the physics is determined by multipseudoparticle forward scattering of all orders, the low-energy quantities are determined by two-pseudoparticle scattering only. This reflects the perturbative character of the pseudoparticle basis \[11\]. We also discuss the properties of the system at the critical point (scaling limit), including the study of the operator representation for the Virasoro algebra \[11\].

The paper is organized as follows: In Sec. II we present the pseudoparticle basis and how it can be obtained directly from the BA equations. The elementary excitations and pseudoparticle description of the bosons are discussed in Sec. III. In Sec. IV we study the static and dynamical properties and consider the low-temperature thermodynamics. The operator description of the Virasoro algebras is introduced in Sec.V. In Sec. VI we study the charge instabilities of the system. Finally, Sec.VII contains the concluding remarks.
II. THE PSEUDOPARTICLE OPERATOR BASIS

Let us consider a set of particles with bosonic statistics which interact via Dirac’s delta function. We describe their dynamics by the following Hamiltonian \[1,2,5\]

\[
\hat{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{l>j} \delta(x_j - x_l) + \mu N , \tag{1}
\]

where \(x_j\) is the position of the \(j^{th}\) particle, \(c\) gives the strength of the interaction (here we are interested in the repulsive case, \(c > 0\)), and \(\mu\) is the chemical potential (we work in units such that \(\hbar = 1\) and the bare mass is \(m = 1/2\)).

Originally, the diagonalization of this problem was carried out by means of the BA \[1-3\]. Later, it was shown by Thacker \[12\] that the two-body scattering problem of the Hamiltonian (1) can be solved in perturbation theory by considering all infinite set of diagrams. Furthermore, all the eigenstates are in the continuum since the interaction is repulsive (no bound states) and is localized in space. In other words, the spectrum of the non-interacting problem and the interacting problem “look” the same, exactly as predicted by the formal theory of scattering \[26\]. However, the similarity of the two problems is only apparent because turning on the interaction changes the character of the quantum numbers which label the Hamiltonian eigenstates. In contrast to the non-interacting \(c = 0\) problem, the new quantum numbers of the interacting system show global shifts in their values upon changing the number of bosons \(N\) by odd integers. This implies that the above spectrum corresponds to new fermionic objects, the pseudoparticles, and not to the original bosons, as we discuss below.

It will be useful to consider the second quantized form of the Hamiltonian (1),

\[
\hat{H} = \int dx \left( \frac{\partial \phi^\dagger(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} + \mu \phi^\dagger(x) \phi(x) + c \phi^\dagger(x) \phi^\dagger(x) \phi(x) \phi(x) \right) , \tag{2}
\]

where the operators \(\phi\) and \(\phi^\dagger\) obey the usual bosonic algebra. Direct inspection of the Hamiltonian (2) reveals that it is invariant under a global \(U(1)\) transformation,

\[
\phi(x) \to e^{i\theta} \phi(x) \tag{3}
\]

\[
\phi^\dagger(x) \to e^{-i\theta} \phi^\dagger(x) ,
\]
where $\theta$ is an arbitrary phase. This Hamiltonian can be diagonalized ($\hat{H} \mid \Psi \rangle = E \mid \Psi \rangle$) by the following wave function (the BA wave function) \cite{12}

$$
\mid \Psi(k_1, k_2, \ldots, k_N) \rangle = \int \left[ \prod_{j=1}^{N} e^{i k_j x_j} dx_j \right] \prod_{j<n \leq N} \left[ 1 - \frac{i c}{k_j - k_n} \text{sgn}(x_n - x_j) \right] \phi^\dagger(x_j) \mid V \rangle, \quad (4)
$$

where \( \{k_j\} \) is a set of \( N \) rapidity numbers (they are canonically conjugated to \( \{x_j\} \)), \( \mid V \rangle \) is the vacuum of the problem, \( sgn(x) \) is 1 for \( x > 0 \) and \( -1 \) for \( x < 0 \), and \( \Delta(k) \) is the two-particle “phase shift” which is obtained from the the equation,

$$
e^{i \Delta(k)} = \frac{k - ic}{k + ic}. \quad (5)
$$

The wave function (4) is a superposition of plane waves which is modulated by step functions. This is a basic characteristic of the BA wave function since in one dimension it is possible to order the particles in a line as \( 0 \leq x_1 < x_2 < \ldots < x_N \leq L \), where \( L \) is the length of the system. The spectrum of the problem, as in any problem in quantum mechanics, is obtained from the boundary conditions. As usual, we assume periodic boundary conditions and from the wave function we find

$$
e^{ik_j L} = \prod_{n \neq j} e^{i \Delta(k_n - k_j)}, \quad j = 1, \ldots, N. \quad (6)
$$

Equation (6) can be rewritten in the following form,

$$
q_j = k_j + \frac{1}{L} \sum_{l=1}^{N} \Delta(k_j - k_l) \quad j = 1, \ldots, N, \quad (7)
$$

where we choose

$$
\Delta(k) = -2 \tan^{-1} \left( \frac{k}{c} \right), \quad (8)
$$

and due to the boundary conditions

$$
q_j = \frac{2\pi}{L} n_j, \quad n_j = 0, \pm 1, \pm 2\ldots, \quad N \text{ odd}; \quad n_j = \pm 1/2, \pm 3/2\ldots, \quad N \text{ even}, \quad (9)
$$

are the true quantum numbers of the problem.
The wave function (4) does not only describes the ground state of the system but all Hamiltonian eigenstates. Each eigenstate corresponds to a different occupancy configuration of the quantum numbers \( q_j \), as we discuss below. This is in contrast to Hamiltonians with non-Abelian symmetries where the BA equations refer to the LWS or HWS of the corresponding algebras only [8,9,11]. In the present case each Hamiltonian eigenstate is characterized by a distribution of \( N \) occupied quantum numbers \( \{q_j\} \) over the infinite \( q_j \) available values, as we discuss below. We call the \( N \) occupied values pseudoparticles. In the present model the number of bosons \( N \) equals the number of pseudoparticles. Eqs. (6) and (7) refer to the \( N \) occupied values only, whereas the available set of numbers (9) corresponds to all integers or half integers. For each set of \( N \) rapidity numbers \( \{k_j\} \) there will be one, and only one, set of \( N \) quantum numbers \( \{q_j\} \). This one to one relation is defined by Eq. (7). This equation defines the values of the rapidities

\[
k_j = k(q_j) .
\]

(10)

(It also defines, implicitly, the values of the rapidities corresponding to the unoccupied values of \( q_j \).) Moreover, the energy of an arbitrary Hamiltonian eigenstate is given by (substitute (4) in (1))

\[
E = \sum_{j=1}^{N} [k_j]^2 + \mu N .
\]

(11)

The total momentum is

\[
K = \sum_{j=1}^{N} k_j ,
\]

(12)

where the sums run over the \( N \) occupied \( \{q_j\} \) values only. While the energy (11) involves the rapidity function (10), the total momentum (12) can be expressed as a sum of the pseudomomenta only. Summing both sides of (7) over the particle index \( j \) leads to

\[
K = \sum_{j=1}^{N} q_j .
\]

(13)

This is a general feature of many models solvable by BA. It follows from the fact that the two-particle “phase shift” is antisymmetric with respect to interchange of momentum.
A very important property of the BA wave function is that it vanishes if two occupied pseudomomenta $q_j$ are equal \[6,11,12\]. Suppose, for simplicity, that we have two particles in the problem only. In this case $N = 2$ and there are two occupied $\{q_j\}$ values, which we call $q_1$ and $q_2$. From (7) it is easy to find that

$$q_1 - q_2 = k_1 - k_2 + 4 \tan^{-1} \left( \frac{k_1 - k_2}{c} \right), \quad (14)$$

and, therefore, if $q_1 = q_2$ we conclude from (14) that $k_1 = k_2$ is a unique solution. Thus, from direct inspection of (4) we find $\Psi = 0$. It clearly implies the fermionic character of the excitation spectrum, although the wave function (4) is symmetric with respect to its arguments as we would expect for a bosonic wave function, i.e. its symmetry has nothing to do with the nature of the spectrum. However, the bosonic character of the wave function (4) has effects on the instabilities of the one-dimensional quantum liquid as, for example, the absence of the fermionic Peierls instability \[27\]. Although the fermionic pseudoparticles lead to the existence of a pseudo-Fermi surface, the Peierls instability only shows up at infinite repulsion (or zero density) where the pseudoparticles become free fermionic entities, as we find in Sec. VI.

At fixed $N$ the ground state of the system is obtained by ordering the occupied set of pseudomomenta $\{q_j\}$ symmetrically around zero \[2,28\] from $-\pi [N - 1]/L$ to $\pi [N - 1]/L$. The boundary conditions of Eq. (9) assure that the two pseudo-Fermi points are symmetric both for $N$ odd and $N$ even. This observation, summed to the fact that the wave function vanishes for equal quantum numbers, is enough to define the excitation spectrum of the system in terms of a pseudo-Fermi sea, which is completely filled in the ground state. That spectrum can be generated from operators which produce excited states by “creating” new $q_j$’s out of the sea or “annihilating” $q_j$’s inside the sea. We associate with each pseudoparticle (occupied quantum number $q_j$) of pseudomomentum $q_j$ fermionic operators $b_{q_j}^\dagger$ and $b_{q_j}$ which create and annihilate, respectively, these pseudoparticles and obey the usual fermionic algebra \[11\],

$$\{b_{q_j}^\dagger, b_{q_j'}\} = \delta_{q_j,q_j'}, \quad (15)$$
and all other anticommutators vanish. This pseudoparticle operator algebra generates all
the Hamiltonian eigenstates from the vacuum $|V\rangle$ of Eq. (4), as we find below.

Since these states can, alternatively, be described in terms of the original bosonic operator
algebra, as in Eq. (4), basic properties of quantum mechanics imply that there is a canonical
transformation which connects the pseudoparticle and bosonic operators. Its evaluation is a
very involved problem, yet in this paper we are able to describe the original bosons in terms
of pseudoparticles in some limits.

The above pseudoparticle-algebra construction is absolutely consistent with the prop-
erties of the wave function and the excitation spectrum given by the BA. This is because
the BA solution refers to that operator algebra \cite{11}. In the presence of external fields the
pseudoparticles are the transport carriers of the system \cite{10,27}.

From now on we will be interested in the thermodynamical limit only, where the numbers
of particles, $N$, and the length of the system, $L$, goes to infinity but the density, $n = N/L$, is kept constant. In this limit we replace the sums by integrals over the pseudomomentum
(see the periodic boundary conditions \cite{9}). The total energy, the total momentum, and the
total number of particles are given by,

$$\frac{E}{L} = \int_{-\infty}^{+\infty} \frac{dq}{2\pi} N(q) (k(q)^2 + \mu),$$  \hspace{1cm} (16)

$$\frac{K}{L} = \int_{-\infty}^{+\infty} \frac{dq}{2\pi} N(q) q,$$  \hspace{1cm} (17)

and

$$\frac{N}{L} = n = \int_{-\infty}^{+\infty} \frac{dq}{2\pi} N(q),$$  \hspace{1cm} (18)

respectively, where $N(q)$ is the distribution function of the pseudomomenta for a given state.
In particular, for the ground state we have the filled sea up to the pseudo-Fermi momenta,
$\pm q_F$, and, therefore, the occupation number is given by

$$N^0(q) = \theta(q_F - |q|)$$  \hspace{1cm} (19)
where, in the thermodynamic limit,

$$q_F = \pi [n - \frac{1}{L}] \approx \pi n .$$  \hspace{1cm} (20)

The BA equation (7) is rewritten as

$$q = k(q) + \frac{1}{\pi} \int_{-\infty}^{+\infty} dq' N(q') \tan^{-1} \left( \frac{k(q) - k(q')}{c} \right).$$  \hspace{1cm} (21)

Since the number of pseudoparticles equals the number of bosons, $N$, we have constructed a scenario where we can act with our pseudoparticle operators on states which are generated from the vacuum $|V\rangle$ of Eq. (4). For a canonical ensemble with a fixed number of bosons $N$ the corresponding ground state is obtained by filling the pseudo-Fermi sea $|0; N\rangle = \prod_{q \in q_F} b_q^\dagger |V\rangle$, \hspace{1cm} (22)

where $q_F$ was defined in (20). The Hilbert space associated with each canonical ensemble is spanned by the set of Hamiltonian eigenstates $|ex; N\rangle$ which can be constructed from the ground state (22) as follows

$$|ex; N\rangle = \prod_{i,j = 1}^{N_{ph}} \left[ b_q^\dagger b_q \right] |0; N\rangle .$$  \hspace{1cm} (23)

This is a state with $N_{ph}$ excited pseudoparticles (and pseudoholes).

It is also natural to define the number operator $\hat{N}(q)$ which counts the number of pseudoparticles

$$\hat{N}(q) = b_q^\dagger b_q .$$  \hspace{1cm} (24)

When acting on any Hamiltonian eigenstate this operator gives

$$\hat{N}(q)|ex, N\rangle = N(q)|ex; N\rangle ,$$  \hspace{1cm} (25)

where the eigenvalue $N(q)$ is 1 for occupied pseudomomentum values and 0 otherwise. In the particular case of the ground state (22), $N(q) = N^0(q)$, where $N^0(q)$ is given in Eq. (19).
Equations (16), (17), and (18) can be seen as eigenvalue equations in the Hilbert space we have just constructed. Although the following pseudoparticle-operator representation is similar to that of multicomponent integrable quantum systems studied in Ref. [11], the simplicity of the present model makes it constructive and interesting for clarifying the physics to describe here that representation in some detail.

For instance, the total momentum operator and the total number operator are defined as [11],

\[ \hat{K} = \sum_q q \hat{N}(q), \]  
\[ \hat{N} = \sum_q \hat{N}(q), \]  

respectively. When acting on any state of the Hilbert space (with fixed number of pseudoparticles) these operators give

\[ \hat{K}|ex; N \rangle = K|ex; N \rangle, \]  
\[ \hat{N}|ex; N \rangle = N|ex; N \rangle, \]  

where \( K \) and \( N \) are given in (17) and (18), respectively.

Moreover, from (21) it is clear that \( k(q) \) is a functional of \( N(q) \) (that is, for each state labelled by the set of occupied \( q_j \) values, \( \{q_j\} \), we generate a different set \( \{k_j\} \)). Since all the states are eigenstates of the number operator, the above construction implies that they are also eigenstates of the operator \( \hat{k}(q) \) defined by the eigenvalue equation

\[ \hat{k}(q)|ex; N \rangle = k(q)|ex; N \rangle, \]  

and, in particular, for the ground state (22) we have that

\[ \hat{k}(q)|0; N \rangle = k_0(q)|0; N \rangle. \]
We conclude from Eq. (30) that the BA equation (20) can be seen as an operator identity, where the numbers are exchanged by the respective operators (since it will be valid for any of the Hamiltonian eigenstates which constitute a complete orthonormal basis in the Hilbert space). This relates the rapidity operator, $\hat{k}$, and the pseudoparticle operators defined above. However, this relationship is not trivial and in its full form can only be obtained by the explicit solution of (21). The Hamiltonian can be rewritten in the pseudoparticle operator basis as (see (11))

$$\hat{H} = \sum_q \hat{N}(q) \left( \hat{k}(q) \right)^2 + \mu \hat{N}.$$  \hspace{1cm} (32)

However, after expressing $\hat{k}$ in terms of the pseudoparticles operator (24) this expression is extremely complex. We can simplify the problem if we construct a normal-ordered expansion which gives us the operator $\hat{k}$ in terms of the operators (24), as in the case of multicomponent integrable quantum liquids [11].

The ground state (22) is non interacting from the point of view of the pseudoparticles. This is not true “from the point of view” of the rapidities since as we change one or a few of the occupied pseudomomenta $\{q_j\}$ there is a “backflow” in “rapidity space” which can change all the corresponding $\{k_j\}$ configuration [10,12]. However, from the point of view of the pseudomomentum $q_j$ there is no backflow since these are the true quantum numbers which label the Hamiltonian eigenstates. At fixed values of $N$ the quantum problem is perturbative with respect to pseudoparticle scattering processes but it is not perturbative with respect to particle processes [11] (scattering of bosons). Therefore, a natural expansion is to create a small number (much smaller than the total number of pseudoparticles $N$) of pseudoparticle-pseudohole pairs close to the pseudo-Fermi surface. Any excited state can be described in terms of the occupation number, $N(q)$, which we rewrite as [10,23]

$$N(q) = N^0(q) + \delta N(q),$$ \hspace{1cm} (33)

where $N^0(q)$ is the ground state occupation number, (19), and $\delta N(q)$ is the small deviation from the ground state occupancy configuration. Equation (33) has a physically meaningful representation in terms of our operators. We define the operator $\hat{N}(q)$, such that
\[ : \hat{N}(q) := \hat{N}(q) - N^0(q) . \]  

This is \textit{normal ordered} relatively to the ground state \((\langle 0; N | : \hat{N}(q) : | 0; N \rangle = 0)\). By definition,

\[ N^0(q) = \langle 0; N | \hat{N}(q) | 0; N \rangle . \]  

The deviation \(\delta N(q)\) from the ground state to an excited state \(|ex; N\rangle\) is simply given by

\[ \delta N(q) = \langle ex; N | : \hat{N}(q) : | ex; N \rangle . \]

This deviation represents a change in the state of the system from the ground state \(|0; N\rangle\) (22) to an excited state \(|ex; N\rangle\) (23).

We can expand the BA equation (21) self-consistently in terms of the deviation operator, \( : \hat{N}(q) : \). The operator \(\hat{k}\) will be a functional of the deviation operator (see Eqs. (21) and (30)). Therefore, these two operators commute with each other. It is possible to show, after some straightforward but lengthy algebra, that the relation between these operators can be written as \([10,11]\)

\[ : \hat{k}(q) := k_0(: \hat{Q}(q) :) - k_0(q) , \]

where \(k_0(q)\) is the ground-state eigenvalue of Eq. (31) and the operator \( : \hat{Q}(q) : \) is given in terms of the expansion of the deviation operator in the form,

\[ : \hat{Q}(q) := \sum_{n=1}^{\infty} \hat{Q}_n(q) . \]

The operator \(\hat{Q}_n(q)\) corresponds to the \(n\)th pseudoparticle scattering order. Introducing (37) and (38) in the BA equation and expanding order by order provides the expression for all infinite terms of the expansion (38). For example, the first-order terms reads

\[ \hat{Q}_1(q) = \sum_{q'} : \hat{N}(q') : \Phi(q, q') . \]

Here \(\Phi(q, q')\) are the shifts in the phase of the pseudoparticle of pseudomomentum \(q'\) due to the zero-momentum forward-scattering collision with the pseudoparticle of pseudomomentum \(q\). This phase shift can be obtained self-consistently by direct substitution of the above
expansion in the BA equation. They are related to the usual BA phase shifts, $\Phi(k, k')$, which are expressed in terms of the ground-state rapidity numbers as

$$\Phi(q, q') = \Phi(k_0(q), k_0(q')).$$  \hfill (40)

In order to achieve consistency between our expansion and the BA equation we can show that the phase shifts must obey the following integral equation

$$\Phi(k, k') = -\frac{1}{\pi} \tan^{-1}\left(\frac{k - k'}{c}\right) + \int_{-Q}^{Q} dk'' R(k - k'')\Phi(k'', k'),$$ \hfill (41)

where the kernel is given by

$$R(k) = \frac{1}{\pi c} \left(\frac{1}{1 + (k/c)^2}\right),$$ \hfill (42)

and

$$Q = \pm k_0(\pm q_F).$$ \hfill (43)

It is worth noticing that higher order terms in the expansion (38) are exclusively given in terms of the phase shifts defined in (40) – (41). The physical reason for that is that higher-order scattering processes can be decomposed into two-pseudoparticle scattering processes. (38) is an expansion in the pseudoparticle scattering order [11]. The infinite expansion (38) is valid for the large class of integrable quantum system of Ref. [11] which refers to contact particle interactions. The normal-ordered pseudoparticle Hamiltonian has an infinite number of terms and is of the form

$$: \hat{H} := \sum_{i=1}^{\infty} \hat{H}_i,$$ \hfill (44)

where $\hat{H}_i$ is the Hamiltonian term of $i$ th pseudoparticle scattering order. The first-order term reads

$$\hat{H}_1 = \sum_q : \hat{N}(q) : \epsilon(q).$$ \hfill (45)

Here $\epsilon(q)$ defines the pseudoparticle band
\[ \epsilon(q) = \mu + [k_0(q)]^2 + 2 \int_{-Q}^{Q} dk k \Phi(k, k_0(q)) . \tag{46} \]

In Fig. 1 the pseudoparticle band (46) is plotted as a function of the density for \( c = 1 \). This band is a slightly distorted version of the non-interaction bosonic spectrum and it approaches the latter when \( n \to \infty \), which is the non-interacting bosonic limit.

The pseudoparticle group velocity is given by

\[ v(q) = \frac{d\epsilon(q)}{dq} , \tag{47} \]

and the velocity at the pseudo-Fermi points is defined as

\[ v_F = v(q_F) . \tag{48} \]

Up to first order in \( \delta N \) the pseudoparticles are free-fermionic entities which have a spectrum given by (46). Moreover, in the present thermodynamic limit the interacting terms are irrelevant for excitations of small momentum and low energy. This free behavior was also noted in previous studies of the model [2,12].

We can proceed in our calculation and go beyond the first order term [10,11]. This gives the spectrum for excitations involving a small but finite density of pseudoparticles, as we see in Sec. IV. Again, substituting the expansions in the BA equation and insuring consistency, we find the next order term

\[ \hat{H}_2 = \sum_{q, q'} : \hat{N}(q) :: \hat{N}(q') : \frac{1}{2} f(q, q') , \tag{49} \]

where

\[ f(q, q') = 2\pi v(q) \Phi(q, q') + 2\pi v(q') \Phi(q', q) + 2\pi v \sum_{j=\pm 1} \Phi(jq_F, q) \Phi(jq_F, q') . \tag{50} \]

Note the similarity between expression (49) and the usual Landau expansion of Fermi-liquid theory [19]. From (46) and (49), we can define the renormalized dispersion relation,

\[ \bar{\epsilon}(q) = \epsilon(q) + \frac{1}{2\pi} \int_{-\infty}^{\infty} dq' \delta N(q') f(q, q') . \tag{51} \]
Naturally, the functions $f(q, q')$ are interpreted as the $f$ functions of the theory and determine the two-pseudoparticle interactions of the quantum liquid. The form of expression (51) is an universal characteristic for the systems solvable by the BA in the sectors of parameter space of lowest $U(1)$ symmetry \[10,11\]. The main difference between these models is the form of the spectral parameters of the BA equations. While in the usual Fermi liquids the quasiparticles exist close to the Fermi surface only, and describe approximations to Hamiltonian eigenstates, in the present Landau liquid \[10,23\] pseudoparticle-pseudohole pairs are true Hamiltonian eigenstates for all energy scales and generate the whole spectrum of the system. In integrable systems with non-Abelian algebras this is true only for the Hilbert sub space spanned by the LWS or HWS of these algebras which are described by real rapidities \[10,11\]. (In the lowest-symmetry sectors and at low energy that sub space coincides with the full Hilbert space.)

Let us define the $f^{+1}$ function (related to the interaction of pseudoparticles at the same side of the pseudo-Fermi sea) and $f^{-1}$ function (related to the interaction at opposite sides of the pseudo-Fermi sea), as

$$f^{+1} = f(\pm q_F, \pm q_F), \quad f^{-1} = f(\pm q_F, \mp q_F).$$

(52)

As in a Fermi liquid, the symmetric and antisymmetric combinations of these functions define the Landau parameters

$$F^i = \frac{1}{2\pi} \sum_{j=\pm 1} (j)^i f^j, \quad i = 0, 1,$$

(53)

which play a relevant role in the low-energy physics.

The Landau parameters (53) appear in the low-energy expressions in the form of the “renormalized” velocities

$$v^i = v + F^i = v[\xi^i]^2, \quad i = 0, 1,$$

(54)

where $\xi^1$ is the dressed charge \[29\]. $\xi^0$ and $\xi^1$ can be written in terms of two-pseudoparticle phase shifts as
\[
\xi^i = 1 + \Phi(q_F, q_F) + (-1)^i\Phi(q_F, -q_F), \quad i = 0, 1.
\] (55)

In agreement with the results of Refs. [16,21,22], we find that \(\xi^0\) is the inverse of the dressed charge

\[
\xi^i = 1/\xi^{1-i}, \quad i = 0, 1,
\] (56)

and thus the following Luttinger-liquid relation [30] holds true

\[
v_0v_1 = (v)^2.
\] (57)

Equations (53)–(57) reveal that there is consistency between the Luttinger liquid of Haldane [30] and the Landau-liquid character of the BA integrable systems of lowest symmetry \(U(1)\). The Luttinger-liquid parameters (56) which define the three velocities of Eq. (57) arise here from the Landau parameters of Eqs. (53) and (54).

Equation (57) can be used to express the velocities (48) and (54) in terms of the dressed charge or its inverse as

\[
v = \frac{2\pi n}{[\xi^1]^2} = 2\pi n[\xi^0]^2,
\] (58)

and

\[
v_0 = \frac{|v|^2}{2\pi n}, \quad v_1 = 2\pi n,
\] (59)

in agreement with the results of Refs. [1,20–22,31]. The \(f^{\pm 1}\) functions (52) and Landau parameters (53) are plotted as a function of the density in Figs. (2), whereas the velocities (48), (54) and the dimensionless parameters (55) are plotted in Figs. 3 and 4, respectively. At \(c\) finite and \(n > 0\) we have \(v_1 > v > v_0\). (The density dependence of the functions plotted in Fig. 2 is commented in Sec. III.)

In contrast to other integrable quantum liquids [11], the simplicity of the present model can be seen in the fact that all observables depend on one quantity only, namely, \(\gamma = c/n\), as was pointed out in Ref. [2]. Also, the \(U(1)\) Abelian symmetry assures that the pseudoparticle operator algebra generates the whole Hilbert space from the vacuum of the theory.
This drastically simplifies the physical interpretation of the information contained in that operator basis which refers to the BA solution. In Sec. III we combine the data presented in this section with the study of the elementary excitations to extract new physical information about the quantum problem. Although the closed-form analytic form of the boson-pseudoparticle operator transformation remains an open question, we can obtain an interesting physical picture concerning the description of the bosons in terms of pseudoparticles.

III. BOSONS, PSEUDOPARTICLES, AND ELEMENTARY EXCITATIONS

The results of Sec. II confirm that the elementary excitations of the bosonic gas are fermionic and reveal that such gas can be described by a pseudoparticle Landau liquid. This has similar properties to the usual Fermi liquids with an important conceptual change, as it was already pointed out in Ref. [2]. The elementary excitations do not refer to quasiparticles. In our language this means that the pseudoparticles are not simple “dressed particles”. We find below that one boson is a collective excitation involving all pseudoparticles of the pseudo-Fermi sea. In addition, the pseudoparticle occupancy configurations define eigenstates of the quantum problem at all energy scales.

Otherwise, in what concerns two-pseudoparticle properties we use the well known machinery [18,19] of the conventional Fermi liquids to treat the physics of the pseudoparticles: in contrast to the one-particle problem, we find below that the boson pairs are not collective pseudoparticle excitations and involve two pseudoparticles only. This is because the collective pseudoparticle excitations of each boson “cancel” in this case. In particular, excitations which do not change the number $N$ of pseudoparticles and bosons can be treated as in a Fermi liquid.

The simplest case of the latter excitations are the one-pair pseudoparticles eigenstates. Equation (23) reveals that the low-energy excitations of the system with a fixed number of particles can be generated from the ground state (22) by pseudoparticle-pseudohole processes close to the pseudo-Fermi surface, as in a Fermi liquid. An one-pair excitation is written as
\begin{equation}
|q, k\rangle = b_{q+k}^\dagger b_q |0; N\rangle, \quad 0 < |q| < q_F, \quad |q + k| > q_F.
\end{equation}

In the thermodynamic limit the spectrum of this excitation does not involve the \( f \) function term of (44), which leads to corrections of order \( 1/L \) only. The spectrum reads

\[ \Delta E = \langle q, k | : \hat{H} : | q, k \rangle = \epsilon(q + k) - \epsilon(q). \]

This spectrum is shown in Fig. 5 for various densities and \( c = 1 \). Its boundaries are the continuum spectra I and II obtained by Lieb [4]. The spectra I and II correspond to removing and adding one boson from the system, respectively. The reason for this seems straightforward: since the number of pseudoparticles equals that of the bosons, if we remove or add one boson it is equivalent to diminish or increase the number of pseudoparticles by one and this creates an extra pseudohole or pseudoparticle relatively to the ground state (22), respectively. However, there is a \textit{fundamental} difference between the excitations which conserve the number of pseudoparticles and bosons, \( N \), as the one-pair states (60) and the general excitations (23), and these which change \( N \) by one (or any odd number).

The present picture has some basic similarities with the electrons of the Hubbard chain studied in Ref. [24]. One boson of vanishing momentum and energy includes two pseudoparticle excitations which cannot be decomposed: (a) one pseudoparticle of momentum \( \pm q_F \); and (b) a collective excitation of \textit{all} the remaining \( N - 1 \) pseudoparticles, each contributing with a small fraction \( \mp \pi/L \) to the momentum of the boson. Although in the present thermodynamic limit each fraction \( \mp \pi/L \) is vanishing small, if we multiply by the number of \( N - 1 \) pseudoparticles of the pseudo-Fermi sea this gives \( \mp \pi [n - 1/L] \) which, following Eq. (20), gives \textit{precisely} \( \mp q_F \). The fact that one boson decays in one pseudoparticle and a collective excitation involving the remaining pseudoparticles justifies the non-perturbative character of the bosonic basis [11].

Although the energy spectra I and II involve only the band \( \epsilon(q) \) of the corresponding “missing” or added “pseudoparticle”, this type of state \textit{is not} an one-pseudoparticle excitation and affects \textit{all} pseudoparticles of the pseudo-Fermi sea. Adding or removing one boson
includes both adding and removing one pseudoparticle and the above collective pseudoparticle excitation and these two pseudoparticle excitations cannot be decomposed. Therefore, the pseudoparticles are not asymptotic states of the many-body system. Although their occupancy configurations describe the Hamiltonian eigenstates and they are the transport carriers and couple to external fields [10,27], they are “confined” to the many-particle system. The only entities which we can add or remove are the bosons. These are the true asymptotic states of the problem. Since the number of bosons and pseudoparticles are equal, removing or adding one boson diminishes or increases the number of pseudoparticles by one. However, this “one-boson excitation” has in the pseudoparticle basis a collective character.

On the other hand, the pseudoparticle-pseudohole state (60) is a one-pair pseudoparticle excitation which involves the transfer from the pseudo-Fermi sea to the unoccupied pseudo-Brillouin zone of one pseudoparticle only. This excitation does not affect the remaining pseudoparticles of the pseudo-Fermi sea because it does not change $N$. Furthermore, removing two (or an even number $N$) of bosons can be seen, in the pseudoparticle basis, as a two-pseudoparticle (or $N$-pseudoparticle) excitation. This is because in this case there is no global pseudomomentum shifting, as confirmed by Eq. (9).

The spectrum I (or II) of Ref. [2] is degenerated with an excitation where we transfer one pseudoparticle from a pseudomomentum $q < q_F$ to $q_F^+$ (from $q = q_F$ to $q > q_F$). This is the lower (or upper) boundary of Fig. 5. Following the boundary conditions of Eq. (9), the $N$-particle and $N \pm 1$-particle ground states (22) have both zero momentum. This is because the pseudomomentum of the “removed” or “added” pseudoparticle is compensated by the global pseudomomentum shift $\pm \pi/L$, as we have discussed above. Since this collective pseudoparticle state involves $N \pm 1$ pseudoparticles, it has an excitation momentum of $\pm q_F$, as confirmed by Eq. (20). Therefore, the excitations I and II can be decoupled into two excitations: (i) a change from the $N$ ground state to the $N \pm 1$ ground state. In the thermodynamic limit this excitation has both vanishing energy and momentum; (ii) the above pseudoparticle-pseudohole excitation involving transfer of one pseudoparticle to or from the pseudo-Fermi surface of the $N \pm 1$ system.
If we remove two bosons of vanishing energy and momentum from the many-body system, the two collective pseudomomentum shifts have opposite momenta and cancel. Therefore, this can be seen as a two-pseudoparticle excitation. Since both the individual bosons and the pair of bosons have vanishing momentum, the pair of bosons is constituted by two pseudoparticles of pseudomomenta $q_F$ and $-q_F$. Therefore, we expect the pseudoparticle Hamiltonian to favour attraction between such pair and repulsion between two pseudoparticles with the same pseudomomentum $\pm q_F$.

This is confirmed by the signs of the two-pseudoparticle $f^{\pm 1}$ functions (52) plotted in Fig. 2. When $\gamma \rightarrow 0$ the system is non-interacting from the point of view of the bosons, whereas when $\gamma \rightarrow \infty$ the bosons are strongly interacting and we obtain the case studied by Girardeau \cite{17}. By solving numerically the BA equations for a fixed $c$ and varying the density $n$ we obtain the behavior for the $f^{\pm 1}$ functions (52). Figure 2 confirms that when the density increases (or the interaction between the bosons decreases) the two-pseudoparticle repulsion increases in one channel ($f^1$) and the two-pseudoparticle attraction increases in the other channel ($f^{-1}$). Moreover, the pseudoparticle attraction always dominates the dynamics of the system at large densities. The interpretation of this behavior is straightforward from the point of view of pseudoparticles: when the bosons interact strongly (Girardeau’s case, $n \rightarrow 0$) the pseudoparticles tend to become free and their excitation spectrum is the one of free fermions. For finite values of the bosonic interaction the two pseudoparticles with opposite momentum which constitute a bosonic pair attract each other. This gives rise to bound states (analogous to Cooper pairs of the BCS theory of superconductivity). This attraction becomes more pronounced when the bosons interact weakly ($n \rightarrow \infty$). This confirms that the above bosons pairs of vanishing energy and momentum are nothing but pairs of pseudoparticles of opposite momentum.

Moreover, the attraction between the two pseudoparticles of pseudomomenta $q_F$ and $-q_F$ and the repulsion between the pairs of pseudoparticles with same pseudomomentum $\pm q_F$ is associated with the two collective pseudoparticle excitations of the individual bosons which constitute the pairs removed or added to the system. These collective pseudoparticle
excitations are due to the shift between the two boundary conditions of (9). Note that only when $\gamma$ is infinite the function (8) vanishes and Eq. (6) does not lead, necessarily, to the two boundary conditions of (9): in this case the numbers $n_j$ can be integers both for $N$ even and odd, as in a non-interacting system. However, any finite value of $\gamma$ implies the occurrence of the two boundary conditions (9) which are controlled by the parity of $N$. Therefore, when $1/\gamma > 0$ the bosons always include the collective pseudoparticle excitation. At finite $c$ the equality $1/\gamma = 0$ implies zero density, $n = 0$, and there are no bosons or pseudoparticles left. When $1/c = 0$ the infinite interaction between the bosons allows the pseudoparticles to be completely decoupled and they form a free-fermionic gas, as it was pointed out by Girardeau.

Our study justifies the crossover between the elementary excitations of this system: non-interacting bosons in one limit, and non-interacting fermionic pseudoparticles in the opposite limit.

IV. STATIC AND TRANSPORT QUANTITIES

Apart from the appealing physical interpretation of the bosons in terms of pseudoparticle excitations, which by itself is enough reason for the introduction of the formalism presented here, we can use the same procedures as in Landau’s Fermi liquid theory to calculate measurable quantities related to the response of the system to external fields.

While the study of the dynamic correlation functions at all energy scales requires considering all pseudoparticle scattering orders of the Hamiltonian (44), the perturbative character of the pseudoparticle basis determines that the low-energy physics is solely controlled by the first two terms of that Hamiltonian, as we have discussed in previous sections.

In what concerns excitations conserving the pseudoparticle number $N$ the present pseudoparticle liquid is very similar to the usual Fermi liquids of quasiparticles. As a simple example of the analogy with Fermi liquids we calculate the compressibility of the interacting bosonic gas. The compressibility is obtained as the change in the chemical potential due
to the change in the number of bosons. This refers to the energy changes of the ground states
with $N - 1$ and $N + 1$ bosons. Since $N - 1$ and $N + 1$ have the same parity, Eq. (9) assures
that the corresponding elementary excitation is of both two-pseudoparticle and two-boson
character, and does not involve pseudoparticle collective excitations of the type discussed in
Sec. III. Therefore, this excitation is equivalent to the change in the pseudo-Fermi momentum
$q_F$ by a small quantity $\delta q_F = \pi \delta n$ without changing the integer or half-integer character
of the numbers of Eq. (9).

In terms of the distribution functions (33) and (36) we can write,

$$N(q) = \Theta (q_F + \delta q_F - |q|),$$  \hspace{1cm} (62)

and, up to first order in $\delta q_F$ we obtain

$$\delta N(q) = \delta (q_F - |q|) \delta q_F.$$ \hspace{1cm} (63)

Following a calculation analogous to the one performed in Refs. \cite{23}, we easily obtain the
change of the chemical potential with the density as

$$\frac{\partial \mu(n)}{\partial n} = -v_0 \pi,$$ \hspace{1cm} (64)

where $v_0$ is defined in Eq. (54). The compressibility is easily obtained and reads

$$\chi = -\frac{1}{n^2} \frac{1}{\partial \mu(n)/\partial n} = \frac{1}{\pi n^2} \frac{1}{v_0}.$$ \hspace{1cm} (65)

Using the asymptotic values for the parameter $v_0$ given in Table 1, we find that at fixed
value of the interaction the compressibility diverges as $n^{-3}$ in the low density limit when
$n \rightarrow 0$ (strongly interaction of bosons). It goes to zero as $n^{-2}$ in the large density limit
$n \rightarrow \infty$ (weakly interaction of bosons). This behavior is clearly expected since there is a
crossover from fermionic to bosonic excitations. The inverse of the compressibility is plotted
in Fig. 6 as a function of the density $n$.

Also the low-temperature thermodynamics can be studied as in a Fermi liquid. This leads
to the same results as Yang in Ref. \cite{3}. The entropy is given in terms of the distribution
function $N(q)$ as
\[
S = -\frac{L}{\pi} \int_{-\infty}^{\infty} dq \{ N(q) \ln[N(q)] + (1 - N(q)) \ln[1 - N(q)] \}. \tag{66}
\]

It follows that the distribution function at finite small temperature \( T \) is given by

\[
N(q) = \frac{1}{1 + e^{\frac{\epsilon(q)}{k_B T}}}. \tag{67}
\]

(Note that \( \epsilon(q_F) = 0 \).) From (66) and (67) it is straightforward to obtain the low-temperature specific heat, which reads

\[
c_V/L = \left( \frac{k_B^2 \pi}{3} \right) \frac{1}{v} T = \left( \frac{k_B^2}{3n} \right) m^* T, \tag{68}
\]

where we have defined the effective mass to be

\[
m^* = \frac{q_F}{v}, \tag{69}
\]

and \( v \) is given in Eq. (48). At fixed temperatures the specific heat diverges as \( n^{-1} \) at low densities and goes to zero as \( n^{-1/2} \) at high densities (see Table 1).

These results are consistent with the non-interacting limit. At finite \( c \) the non-interacting bosonic system corresponds to the infinite-density limit. Since the total energy is always finite, the chemical potential must vanish in that limit. On the other hand, the dispersion relation for the free bosons with finite chemical potential is given by

\[
E_k = \sqrt{k^2 + \mu}, \tag{70}
\]

which reproduces exactly the result (68) for zero chemical potential with mass \( m = 1/2 \), as expected.

Following the same route as in the calculation of the static properties we can also use the transport equations of the usual Fermi liquids to study the transport properties of the present Landau liquid. In particular, we are interested here in the conductivity and corresponding transport mass.

The expression of the normal-ordered Hamiltonian (44), whose first two terms are given in Eqs. (45) and (49), is a sum of integrals over products of the pseudomomentum distribution operator (34). We find that the charge-current operator is also of that form. It
contains zero-momentum pseudoparticle forward-scattering terms only and commutes with that Hamiltonian. It follows that the conductivity spectrum has no incoherent part and is constituted by the Drude $\delta$ peak only. This is consistent with the translational invariance of the system which implies that the conductivity sum rule (32) involves the bare mass $m = 1/2$ and reads

$$\int_0^\infty d\omega \text{Re} \sigma(\omega) = \frac{\pi n}{m} = 2\pi n.$$ \hspace{1cm} (71)

Here we have used units of charge equal to 1. Since, following the translational invariance of the system, the sum rule (71) has to be exhausted by the Drude peak, the conductivity spectrum reads

$$\text{Re} \sigma(\omega) = 2\pi D\delta(\omega),$$ \hspace{1cm} (72)

where, combining Eqs. (71) and (72), we have

$$D = n = \frac{qF}{2\pi m},$$ \hspace{1cm} (73)

and $m = 1/2$.

To confirm that the pseudoparticles are the transport carriers, we rederive the spectrum (72) via kinetic equations, as in Ref. [10] for the case of the Hubbard chain. This confirms the expected relation between the velocity $v_1$ (54) and the stiffness. As is shown in Ref. [10], that parameter defines the elementary pseudoparticle current. The same method as in that reference leads to the conductivity spectrum (72) with

$$2\pi D = v^1.$$ \hspace{1cm} (74)

Since, following Eq. (59), $v^1 = 2\pi n$, we thus conclude that the pseudoparticle transport mass [10] coincides with the bare bosonic mass $m = 1/2$, as required by translation invariance.

V. CONFORMAL INVARIANCE

The properties of the system at low energies can be investigated from the point of view of conformal-field theory. Here we follow the operator analysis of Ref. [11]. The suitable
low-energy critical-point Hamiltonian is constructed in the pseudoparticle basis. Using the
same procedures as in Ref. [11], we linearize the bands of (45) and replace the $f$ functions
of (49) by the values (52) to obtain

$$
\hat{H} = \sum_{\kappa,\iota=\pm 1} \nu_{\iota} \hat{N}_\iota(\kappa) : + \frac{1}{2L} \sum_{\kappa,\kappa'\iota=\pm 1} \left[ f^1 : \hat{N}_\iota(\kappa) :: \hat{N}_{\iota}(\kappa') : \right] + f^{-1} : \hat{N}_\iota(\kappa) :: \hat{N}_{-\iota}(\kappa') : ,
$$

(75)

where the pseudo-wavevectors are written relative to the pseudo-Fermi momentum as $\kappa = q - \iota q_F$. Here $\iota = \text{sgn}(q)1$ defines the $\iota = 1$ right and $\iota = -1$ left movers. The operator $b_{\kappa,+1}$ (or $b_{\kappa,-1}$) refers to the annihilation of a pseudoparticle moving to the right (or left) with pseudomomentum $\kappa$ relative to the corresponding pseudo-Fermi point $q_F$ (or $-q_F$). In
(75) the $f^\pm$ functions (52) can be expressed in terms of the velocity (48) and dimensionless
parameters (56) as follows

$$
f^\iota = 2\pi v \left( -\delta_{\iota,1} + \frac{1}{2} \left[ (\xi^0)^2 + \iota(\xi^1)^2 \right] \right). 
$$

(76)

(These functions are plotted in Fig. 2.) The only scale in the quantum problem (75) is
the pseudo-Fermi velocity, $v$, which, following (76), is an overall multiplication factor in the
Hamiltonian (75) (all other parameters are dimensionless).

The Hamiltonian (75) can be rewritten as a pseudoparticle Luttinger liquid with right and
left potentials given by $\delta_{\kappa,0} f^{\pm 1}$ [24]. Haldane [31] has shown that the low-energy spectrum
of some of the systems solvable by BA can be mapped in that of the Luttinger model. Here
we have shown that the pseudoparticle basis provides an explicit operator construction for
that fact.

At the critical point the HWS of the Virasoro algebra are generated by the following
type of states: (A) Hamiltonian eigenstates which have a small density of pseudoparticles
or pseudoholes added to the ground state,

$$
| (A) \rangle = \prod_{\iota} \prod_{\kappa=0}^{\delta q_F} b^\dagger_{\kappa,\iota} | 0; N \rangle \quad \delta q_F > 0 ,
$$

$$
| (A) \rangle = \prod_{i\iota a} \prod_{\kappa=0}^{\iota \delta q_F} b_{\kappa,\iota} | 0; N \rangle \quad \delta q_F < 0 ,
$$

(77)
where \( \delta q_F = \pi \delta n \) is the change in the pseudo-Fermi momentum due to the corresponding change in the bosonic density; and (B) Hamiltonian eigenstates associated with a density of pseudoholes in one side of the pseudo-Fermi surface and the same density of pseudoparticles in the opposite side of that surface,

\[
| (B) \rangle = \prod_{\kappa=0}^{\tilde{\delta}q_F} b^{\dagger}_{\kappa,\iota} b_{\kappa,-\iota} |0\rangle \quad \iota q_F > 0. \tag{78}
\]

Here we have defined \( \tilde{\delta}q_F = \frac{2\pi \mathcal{D}}{L} \), where \( 2\mathcal{D} = \delta N_{\iota=1} - \delta N_{\iota=-1} \) is related to the number of pseudoparticles transferred across the pseudo-Fermi sea. \( N_{\iota} \) is the number of right (\( \iota = 1 \)) and left (\( \iota = -1 \)) carriers. The Hamiltonian eigenstates (B) have large momentum \( K = \mathcal{D} 2q_F \).

The exclusive zero-momentum forward-scattering character of the Hamiltonians (44) and (75) implies separated conservation laws for the numbers \( N_1 \) and \( N_{-1} \). Both the excitations (A) and (B) change these numbers. Note, however, that excitations (A) conserve the number \( N_1 - N_{-1} \), whereas excitations (B) conserve the number \( N = N_1 + N_{-1} \).

The tower of excitations, which conserve both numbers \( N_{\iota} \), can also be written in terms of the pseudoparticles operators. We denote these excitations by states (C) which involve a density of pseudoparticle-pseudohole pairs around the same pseudo-Fermi point. These are small-momentum and low-energy excitations given by

\[
| (C) \rangle = \mathcal{L}^\iota_{-N_{\iota ph}} |0; N\rangle, \tag{79}
\]

where \( (j < 0) \)

\[
\mathcal{L}^\iota_j = \prod_{\kappa_{p,\iota}} b^{\dagger}_{\kappa_{p,\iota}} b_{\kappa_{h,\iota}}, \tag{80}
\]

are the generators of the Virasoro algebra. In Eq. (79)

\[
N_{\iota ph}^\iota = \iota \frac{L}{2\pi} \left( \sum_p \kappa_{p,\iota} - \sum_h \kappa_{h,\iota} \right), \tag{81}
\]

is the total number of pseudoparticle-pseudohole pairs on the side \( \iota \) of the pseudo-Fermi sea. This quantity is given in terms of the net momentum involved by the creation of one
pseudoparticle with momentum $\kappa_{p,\iota}$ and the creation of a pseudohole with momentum $\kappa_{h,\iota}$ across the pseudo-Fermi surface. If we consider states with a fixed number of particles and small momentum, only the states (C) contribute.

It is easy to show that the Virasoro generator of zero order, which is related to spatial and temporal translations, is given by (we set $v = 1$)

$$L_0 = \iota \sum_\kappa \kappa : \hat{N}_\iota (\kappa) : + \frac{1}{L} \sum_\iota \sum_{\kappa,\kappa'} \left[ \left( \frac{\iota^1}{2\pi} + \frac{\iota^{\prime 1}}{4} \right) : \hat{N}_{\iota'} (\kappa) : + \hat{N}_{\iota'} (\kappa') : + \left( \frac{\iota^{-1}}{2\pi} + \frac{\iota^{\prime 1}}{4} \right) : \hat{N}_{\iota'} (\kappa) : + \hat{N}_{\iota'} (\kappa') : \right] .$$

(82)

The excitation energy $\Delta E$ and momentum $K$ corresponding to Hamiltonian eigenstates involving all three types of processes as in $| (A) \rangle$, $| (B) \rangle$, and $| (C) \rangle$ is written as

$$\Delta E = \langle : \hat{H} : \rangle = \frac{2\pi}{L} \sum_\iota \iota \left( h^t + N^t_{ph} \right) , \quad K = \frac{2\pi}{L} \sum_\iota \iota \left( h^t + N^t_{ph} \right) + D2q_F ,$$

(83)

where $h^t$ are the dimensions of the fields which are given by

$$h^t = \frac{1}{2} \left( \xi^1 D + \iota \xi^0 \delta N \right) .$$

(84)

Our pseudoparticle operator basis has allowed the study of the Virasoro algebra. Both the HWS and the towers states (C) are generated by acting the pseudoparticle operators on the ground state. This has the non-interacting form given in Eq. (22). We have written the generators of the algebra and other operators. Only the present pseudoparticle basis allows a simple operator representation for the generators of Virasoro algebras of BA integrable models [11].

The Virasoro algebra can also be used to evaluate the asymptotic of the correlation functions. We omit here that study. These functions have been obtained by other authors [15,16,20,22].

VI. CHARGE INSTABILITIES

The formalism we have presented in the previous sections also allows us finding and classifying the possible divergence in the response functions [27]. This provides the instabilities
of the many-boson system. For instance, the real part of the charge-charge response function at zero frequency can be written in terms of the dynamic-form factor $S(k, \omega)$ as

$$Re \chi(k, 0) = -2 \int_0^\infty d\omega \frac{S(k, \omega)}{\omega}. \quad (85)$$

Using the methods of Ref. [27] we can show that in the limit of small frequency and $k = \mathcal{D}2q_F$ the dynamical form factor is given by

$$S(\mathcal{D}2q_F, \omega) = S_0(\mathcal{D}2q_F)\omega^{\zeta(\mathcal{D}2q_F)} + h.o.t., \quad (86)$$

where the higher-order terms vanish as $\omega \to 0$ and $k = \mathcal{D}2q_F$ are the only values of momenta at which $S(k, \omega)$ can be non-vanishing at small $\omega$ [27]. In Eq. (86) $\zeta$ are the exponents which classify the divergence of the function (85). If $\zeta > 0$ there is no divergence in (85) and $S(\mathcal{D}2q_F, \omega)$ vanishes at $\omega = 0$, if $\zeta = 0$ there is a logarithmic divergence, and if $\zeta < 0$ there is a power-law divergence. The exponent $\zeta$ is related to the dressed charge $\xi^1 (55)$ by

$$\zeta(\mathcal{D}2q_F) = 2[ (\xi^1 \mathcal{D})^2 - 1]. \quad (87)$$

From the use of the solution of the BA equation for the phase shifts (41) we find that $\zeta$ is always greater than zero except for $\mathcal{D} = \pm 1$ and (a) for vanishing density at finite $c$; and (b) for infinite bosonic repulsion at finite $n$. In these limits (87) gives $\zeta(2q_F) = 0$. Therefore, in these particular limits and at the momenta $k = \pm 2q_F$ we have a logarithmic singularity in the response function (85). This singularity is expected for a fermionic gas. It is known as the Peierls instability [27]. Again, the free-fermionic character of a gas of bosons with hard cores appears in the response function and, therefore, it can be measured experimentally. For all other values of the interaction and density the bosonic character of the Hamiltonian eigenstates (4) “kills” the Peierls instability.

**VII. CONCLUDING REMARKS**

In this article we have shown that a gas of bosons interacting repulsively via hard core potentials can be seen as a liquid of interacting fermionic pseudoparticles. The bosons are at
vanishing energy and momentum an object constituted by two excitations which cannot be decomposed: one pseudoparticle of pseudomomentum $\pm q_F$ and a collective pseudoparticle excitation of momentum $\mp q_F$. In addition, one pair of bosons of vanishing momentum and energy is constituted by a pair of pseudoparticles with opposite momentum $q_F$ and $-q_F$, which form a bound state in the non-interacting bosonic limit. At infinite bosonic interaction the pairs break due to the scattering between the bosons. This gives rise to a gas of free fermionic pseudoparticles. In spite of the fermionic character of the excitations, the bosonic nature of the Hamiltonian eigenstates implies that only in this limit the system becomes really fermionic and develops a Peierls instability at the momenta $\pm 2q_F$.

Although some of the features of the model were already studied by many authors, we have presented a new operator representation for the problem. This uses a consistent framework valid in the $U(1)$ sectors of parameter space of all systems solvable by the BA. (In the present model this refers to all the parameter space.) Analogously to what happens in the sectors of lowest symmetry of other models as the Hubbard chain, the elementary excitations are described by a liquid of pseudoparticles with zero-momentum forward-scattering interactions only.

Our operator pseudoparticle algebra generates all Hamiltonian eigenstates from the bosonic vacuum. The pseudoparticle algebra refers to the BA solution whose equation was solved to provide a perturbative normal-ordered pseudoparticle Hamiltonian expansion. This uses the exact ground state of the problem as a reference state $|\Psi\rangle$. A central point is that in the pseudoparticle basis this is a non-interacting ground state. The above expansion is in the scattering order of the pseudoparticle processes and corresponds to a pseudoparticle perturbation theory. The perturbative character of the pseudoparticle basis assures its convergence. At high energies the physics is determined by pseudoparticle forward-scattering interactions of all orders. On the other hand, while in the case of small-momentum and low-energy excitations only the non-interacting Hamiltonian term (45) is relevant, in the case of low-energy excitations changing the values of conserving numbers $N_i$ both that term and the two-pseudoparticle interaction term (49) contribute. The two-pseudoparticle inter-
actions fully control the low-energy physics, as in a Fermi liquid.

Therefore, the use of the same kind of approaches as in the theory of Fermi liquids leads to the correct description of the static and dynamic properties of the system in a straightforward manner. We have also studied the conformal spectrum which is nothing but the low-energy spectrum of the above normal-ordered pseudoparticle Hamiltonian. The pseudoparticle basis allows an operator description for the generators of the Virasoro algebra.

The properties of the present bosonic model are a simple and clear manifestation of the Landau-liquid character of its fermionic elementary excitations. However, our study reveals that one boson of vanishing energy and momentum decays in all the pseudoparticles of the pseudo-Fermi sea. This is a non-Fermi liquid property which justifies the non-perturbative character of the bosonic basis.

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| $\xi^1$ | $v$      | $v_0$      | $\chi$            |
|---------|----------|------------|-------------------|
| $n \to 0$ | 1        | $2\pi n$  | $2\pi n$          | $1/(2\pi^2 n^3)$   |
| $n \to \infty$ | $\pi^{1/2} n^{1/4}$ | $2n^{1/2}$ | $2/\pi$          | $1/(2n^2)$         |

Table 1 - Limiting values of the parameter (55), velocities (58) – (59), and compressibility (65) for $c = 1$. 
FIGURES

FIG. 1. The pseudoparticle band $\epsilon(q)$ (46) for different values of the density and $c = 1$. Note that the zero-energy level corresponds to the pseudo-Fermi momenta $q = \pm q_F$ and that the pseudomomentum axis extends from $q = -\infty$ to $q = \infty$.

FIG. 2. The $f$ functions (a) $f^1$ and (b) $f^{-1}$ of Eq. (52), and (c) the symmetric and (d) antisymmetric Landau parameters (53) as a function of density and $c = 1$.

FIG. 3. The velocities $v$ (58), $v_0$, and $v_1$ (59) as a function of the density and $c = 1$.

FIG. 4. The dressed charge $\xi^1 = 1/\xi^0$ (56) as a function of the density and $c = 1$.

FIG. 5. The excitation spectrum (61) of the one-pair pseudoparticle-pseudohole eigenstates for $K > 0$, various values of the density, and $c = 1$. For $K < 0$ we note that $E(K) = E(-K)$.

FIG. 6. The inverse of the compressibility (65) as a function of the density and $c = 1$. 

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