Clifford Algebras and Lorentz Group

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Abstract

Finite–dimensional representations of the proper orthochronous Lorentz group are studied in terms of spinor representations of the Clifford algebras. The Clifford algebras are understood as an ‘algebraic covering’ of a full system of the finite–dimensional representations of the Lorentz group. Space–time discrete symmetries $P$, $T$, and $PT$, represented by fundamental automorphisms of the Clifford algebras, are defined on all the representation spaces. Real, complex, quaternionic and octonionic representations of the Lorentz group are considered. Physical fields of the different types are formulated within such representations. The Atiyah–Bott–Shapiro periodicity is defined on the Lorentz group. It is shown that modulo 2 and modulo 8 periodicities of the Clifford algebras allow to take a new look at the de Broglie–Jordan neutrino theory of light and the Gell-Mann–Ne’emann eightfold way in particle physics. On the representation spaces the charge conjugation $C$ is represented by a pseudoautomorphism of the complex Clifford algebra. Quotient representations of the Lorentz group are introduced. It is shown that quotient representations are the most suitable for description of the massless physical fields. By way of example, neutrino field is described via the simplest quotient representation. Weyl–Hestenes equations for neutrino field are given.

Key words: Clifford algebras, Lorentz group, finite–dimensional representations, discrete symmetries, Atiyah–Bott–Shapiro periodicity, charge conjugation, quotient representations, neutrino field, Weyl–Hestenes equations.

1998 Physics and Astronomy Classification Scheme: 02.10.Tq, 11.30.Er, 11.30.Cp
2000 Mathematics Subject Classification: 15A66, 15A90, 20645

1 Introduction

Importance of discrete transformations is well–known, many textbooks on quantum theory began with description of the discrete symmetries, and famous Lüders–Pauli $CPT$–Theorem is a keystone of quantum field theory. However, usual practice of definition of the discrete symmetries from the analysis of relativistic wave equations does not give a full and consistent theory of the discrete transformations. In the standard approach, except a well studied case of the spin $j = 1/2$ (Dirac equation), a situation with the

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discrete symmetries remains vague for the fields of higher spin \( j > 1/2 \). It is obvious that a main reason of this is an absence of a fully adequate formalism for description of higher–spin fields (all widely accepted higher–spin formalisms such as Rarita–Schwinger approach \[51\], Bargmann–Wigner \[3\] and Gel’fand–Yaglom \[24\] multispinor theories, and also Joos–Weinberg \( 2(2j+1) \)–component formalism \[33, 64\] have many intrinsic contradictions and difficulties). The first attempt of going out from this situation was initiated by Gel’fand, Minlos and Shapiro in 1958 \[29\]. In the Gel’fand–Minlos–Shapiro approach the discrete symmetries are represented by outer involutory automorphisms of the Lorentz group (there are also other realizations of the discrete symmetries via the outer automorphisms \[45, 40, 58\]). At present the Gel’fand–Minlos–Shapiro ideas have been found further development in the works of Buchbinder, Gitman and Shelep \[14, 30\], where the discrete symmetries are represented by both outer and inner automorphisms of the Poincaré group.

In 1957, Shirokov pointed out \[57\] that an universal covering of the inhomogeneous Lorentz group has eight inequivalent realizations. Later on, in the eighties this idea was applied to a general orthogonal group \( O(p,q) \) by Dąbrowski \[18\]. As known, the orthogonal group \( O(p,q) \) of the real space \( \mathbb{R}^{p,q} \) is represented by the semidirect product of a connected component \( O_0(p,q) \) and a discrete subgroup \{1, P, T, PT\}. Further, a double covering of the orthogonal group \( O(p,q) \) is a Clifford–Lipschitz group \( \text{Pin}(p,q) \) which is completely constructed within a Clifford algebra \( \mathcal{C}_{p,q} \). In accordance with squares of elements of the discrete subgroup \( (a = P^2, b = T^2, c = (PT)^2) \) there exist eight double coverings (Dąbrowski groups \[18\]) of the orthogonal group defined by the signatures \( (a,b,c) \), where \( a, b, c \in \{-, +\} \). Such in brief is a standard description scheme of the discrete transformations. However, in this scheme there is one essential flaw. Namely, the Clifford–Lipschitz group is an intrinsic notion of the algebra \( \mathcal{O}_{p,q} \) (a set of the all invertible elements of \( \mathcal{O}_{p,q} \)), whereas the discrete subgroup is introduced into the standard scheme in an external way, and the choice of the signature \( (a,b,c) \) of the discrete subgroup is not determined by the signature of the space \( \mathbb{R}^{p,q} \). Moreover, it is suggest by default that for any signature \( (p,q) \) of the vector space there exist the all eight kinds of the discrete subgroups. It is obvious that a consistent description of the double coverings of \( O(p,q) \) in terms of the Clifford–Lipschitz groups \( \text{Pin}(p,q) \subset \mathcal{O}_{p,q} \) can be obtained only in the case when the discrete subgroup \{1, P, T, PT\} is also defined within the algebra \( \mathcal{O}_{p,q} \). Such a description has been given in the works \[60, 61, 62\], where the discrete symmetries are represented by fundamental automorphisms of the Clifford algebras. Moreover, this description allows to incorporate the Gel’fand–Minlos–Shapiro automorphism theory into Shirokov–Dąbrowski scheme and further to unite them on the basis of the Clifford algebras theory.

In the present paper such an unification is given. First of all, Clifford algebras are understood as ‘algebraic coverings’ of finite–dimensional representations of the proper Lorentz group \( \mathfrak{g}_+ \). In the section 2 Clifford algebras \( \mathbb{C}_n \) over the field \( \mathbb{F} = \mathbb{C} \) are associated with complex finite–dimensional representations \( \mathcal{E} \) of the group \( \mathfrak{g}_+ \). It allows to define a new class of the finite–dimensional representations of \( \mathfrak{g}_+ \) (quotient representations) corresponded to the type \( n \equiv 1 \) (mod 2) of the algebras \( \mathbb{C}_n \). In its turn, representation spaces of \( \mathcal{E} \) are the spinspaces \( \mathcal{S}_{2n/2} \) or the minimal left ideals of the algebras \( \mathbb{C}_n \). In virtue of this in the section 3 the discrete symmetries representing by spinor representations
of the fundamental automorphisms of $\mathbb{C}_n$ are defined for both complex and real finite-dimensional representations of the group $\mathfrak{G}_+$.  

A full system $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$ of the finite-dimensional representations of the group $\mathfrak{G}_+$ allows to define in the section 4 the Atiyah–Bott–Shapiro periodicity on the Lorentz group. In case of the field $\mathbb{F} = \mathbb{C}$ we have modulo 2 periodicity on the representations $\mathfrak{C}$, $\mathfrak{C} \cup \mathfrak{C}$, that allows to take a new look at the de Broglie–Jordan neutrino theory of light [11, 37]. In its turn, over the field $\mathbb{F} = \mathbb{R}$ we have on the system $\mathcal{M}$ the modulo 8 periodicity which relates with octonionic representations of the Lorentz group and the Günyadin–Gürsey construction of the quark structure in terms of an octonion algebra $\mathcal{O}$ [31]. In essence, the modulo 8 periodicity on the system $\mathcal{M}$ gives an another realization of the well-known Gell-Mann–Ne’emann eightfold way [26]. It should be noted here that a first attempt in this direction was initiated by Coquereaux in 1982 [17].

Other important discrete symmetry is the charge conjugation $C$. In contrast with the transformations $P$, $T$, $PT$ the operation $C$ is not space–time discrete symmetry. This transformation is firstly appeared on the representation spaces of the Lorentz group and its nature is strongly different from other discrete symmetries. By this reason in the section 5 the charge conjugation $C$ is represented by a pseudoautomorphism $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ which is not fundamental automorphism of the Clifford algebra. All spinor representations of the pseudoautomorphism $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ are given in Theorem 3.

Quotient representations of the group $\mathfrak{G}_+$ compose the second half $\mathcal{M}^-$ of the full system $\mathcal{M}$ and correspond to the types $n \equiv 1 \pmod{2}$ ($\mathbb{F} = \mathbb{C}$) and $p - q \equiv 1, 5 \pmod{8}$ ($\mathbb{F} = \mathbb{R}$). An explicit form of the quotient representations is given in the section 6 (Theorem 4). In the section 7 the first simplest physical field (neutrino field), corresponded to a fundamental representation $\mathfrak{C}^{1,0}$ of the group $\mathfrak{G}_+$, is studied within a quotient representation $\chi_{\mathfrak{C}^{1,0}} \cup \chi_{\mathfrak{C}^{0,-1}}$. Such a description of the neutrino was firstly given in the work [60], but in [60] this description looks like an exotic case, whereas in the present paper it is a direct consequence of all mathematical background developed in the previous sections. It is shown also that the neutrino field $(1/2, 0) \cup (0, 1/2)$ can be defined in terms of a Dirac–Hestenes spinor field [32, 33], and the wave function of this field satisfies the Weyl–Hestenes equations (massless Dirac–Hestenes equations).

2 Finite-dimensional representations of the Lorentz group and complex Clifford algebras

It is well-known [56, 23, 1] that representations of the Lorentz group play a fundamental role in the quantum field theory. Physical fields are defined in terms of finite–dimensional irreducible representations of the Lorentz group $O(1,3) \simeq O(3,1)$ (correspondingly, Poincaré group $O(1,3) \odot T(4)$, where $T(4)$ is a subgroup of four–dimensional translations). It should be noted that in accordance with [17] any finite–dimensional irreducible representation of the proper Lorentz group $\mathfrak{G}_+ = O_0(1,3) \simeq O_0(3,1) \simeq SL(2; \mathbb{C})/\mathbb{Z}_2$ is equivalent to some spinor representation. Moreover, spinor representations exhaust in essence all the finite–dimensional representations of the group $\mathfrak{G}_+$. This fact we will widely use below.

Let us consider in brief the basic facts concerning the theory of spinor representa-
tions of the Lorentz group. The initial point of this theory is a correspondence between transformations of the proper Lorentz group and complex matrices of the second order. Indeed, follows to [29] let us compare the Hermitian matrix of the second order

\[
X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \tag{1}
\]

to the vector \( v \) of the Minkowski space–time \( \mathbb{R}^{1,3} \) with coordinates \( x_0, x_1, x_2, x_3 \). At this point \( \det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = S^2(x) \). The correspondence between matrices \( X \) and vectors \( v \) is one–to–one and linear. Any linear transformation \( X' = aXa^* \) in a space of the matrices \( X \) may be considered as a linear transformation \( g_a \) in \( \mathbb{R}^{1,3} \), where \( a \) is a complex matrix of the second order with \( \det a = 1 \). The correspondence \( a \sim g_a \) possesses following properties: 1) \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim e \) (identity element); 2) \( g_a, g_{a_2} = g_{a_1a_2} \) (composition); 3) two different matrices \( a_1 \) and \( a_2 \) correspond to one and the same transformation \( g_{a_1} = g_{a_2} \) only in the case \( a_1 = -a_2 \). Since every complex matrix is defined by eight real numbers, then from the requirement \( \det a = 1 \) it follow two conditions \( \text{Re} \det a = 1 \) and \( \text{Im} \det a = 0 \). These conditions leave six independent parameters, that coincides with parameter number of the proper Lorentz group.

Further, a set of all complex matrices of the second order forms a full matrix algebra \( M_2(\mathbb{C}) \) that is isomorphic to a biquaternion algebra \( \mathbb{C}_2 \). In its turn, Pauli matrices

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}
\]

form one from a great number of isomorphic spinbasis of the algebra \( \mathbb{C}_2 \) (by this reason in physics the algebra \( \mathbb{C}_2 \simeq \mathbb{O}^+_1 \simeq \mathbb{O}_3 \) is called Pauli algebra). Using the basis (2) we can write the matrix (1) in the form

\[
X = x^\mu \sigma_\mu. \tag{3}
\]

The Hermitian matrix (3) is correspond to a spintensor \((1, 1)\) \( X^{\lambda\bar{\nu}} \) with following coordinates

\[
\begin{align*}
x^0 &= +(1/\sqrt{2})(\xi^1\xi^1 + \xi^2\xi^2), & x^1 &= +(1/\sqrt{2})(\xi^1\xi^2 + \xi^2\xi^1), \\
x^2 &= -(i/\sqrt{2})(\xi^1\xi^2 - \xi^2\xi^1), & x^3 &= +(1/\sqrt{2})(\xi^1\xi^1 - \xi^2\xi^2),
\end{align*} \tag{4}
\]

where \( \xi^\mu \) and \( \bar{\xi}^\mu \) are correspondingly coordinates of spinors and cospinors of spinspaces \( S_2 \) and \( \bar{S}_2 \). Linear transformations of ‘vectors’ (spinors and cospinors) of the spinspaces \( S_2 \) and \( \bar{S}_2 \) have the form

\[
\begin{align*}
\prime \xi^1 &= \alpha \xi^1 + \beta \xi^2, & \prime \xi^2 &= \alpha \xi^2 + \beta \xi^1, \\
\gamma \xi^1 + \delta \xi^2, & \gamma \xi^2 + \delta \xi^1, & \gamma \nu = \hat{\gamma} \xi^1 + \hat{\delta} \xi^2, \\
\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, & \hat{\sigma} = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}.
\end{align*} \tag{5}
\]

Transformations (5) form the group \( SL(2; \mathbb{C}) \), since \( \sigma \in M_2(\mathbb{C}) \) and

\[
SL(2; \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C}_2 : \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1 \right\} \simeq \text{Spin}_+(1, 3).
\]
The expressions (4) and (5) compose a base of the 2-spinor van der Waerden formalism \[64, 54\], in which the spaces \(S_2\) and \(S'_2\) are called correspondingly spaces of \textit{undotted} and \textit{dotted spinors}. The each of the spaces \(S_2\) and \(S'_2\) is homeomorphic to an extended complex plane \(\mathbb{C} \cup \infty\) representing an absolute (the set of infinitely distant points) of a Lobatchevskii space \(S^{1,2}\). At this point a group of fractional linear transformations of the plane \(\mathbb{C} \cup \infty\) is isomorphic to a motion group of \(S^{1,2}\) \[53\]. Besides, in accordance with \[39\] the Lobatchevskii space \(S^{1,2}\) is an absolute of the Minkowski world \(\mathbb{R}^{1,3}\) and, therefore, the group of fractional linear transformations of the plane \(\mathbb{C} \cup \infty\) (motion group of \(S^{1,2}\)) twice covers a ‘rotation group’ of the space–time \(\mathbb{R}^{1,3}\), that is the proper Lorentz group.

**Theorem 1.** Let \(C_2\) be a biquaternion algebra and let \(\sigma_i\) be a canonical spinor representations (Pauli matrices) of the units of \(C_2\), then \(2k\) tensor products of the \(k\) matrices \(\sigma_i\) form a basis of the full matrix algebra \(\mathbb{M}_{2k}(C)\), which is a spinor representation of a complex Clifford algebra \(C_{2k}\). The set containing \(2k+1\) tensor products of the \(k\) matrices \(\sigma_i\) is homomorphically mapped onto a set consisting of the same \(2k\) tensor products and forming a basis of the spinor representation of a quotient algebra \(\mathbb{C}_{2k}\).

**Proof.** As a basis of the spinor representation of the algebra \(C_2\) we take the Pauli matrices \(2\). This choice is explained by physical applications only (from mathematical viewpoint the choice of the spinbasis for \(C_2\) is not important). Let us compose now \(2k\) \(2^k\)-dimensional matrices:

\[
\begin{align*}
E_1 &= \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0, \\
E_2 &= \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0, \\
E_3 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\
&\cdots \cdots \\
E_k &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1, \\
E_{k+1} &= \sigma_2 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\
E_{k+2} &= \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\
&\cdots \cdots \\
E_{2k} &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2.
\end{align*}
\]

(6)

Since \(\sigma_i^2 = \sigma_0\), then for a square of any matrix from the set (6) we have

\[
E_i^2 = \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \quad i = 1, 2, \ldots, 2k,
\]

(7)

where the product \(\sigma_0 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0\) equals to \(2^k\)-dimensional unit matrix. Further,

\[
E_i E_j = -E_j E_i, \quad i < j; \quad i, j = 1, 2, \ldots 2k.
\]

(8)

Indeed, when \(i = 1\) and \(j = 3\) we obtain

\[
\begin{align*}
E_1 E_3 &= \sigma_1 \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\
E_3 E_1 &= \sigma_3 \sigma_1 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0.
\end{align*}
\]

Thus, the equalities (6) and (8) show that the matrices of the set (6) satisfy the multiplication law of the Clifford algebra. Moreover, let us show that a set of the matrices...
$E_{\alpha_1} E_{\alpha_2} \ldots E_{\alpha_{2k}}$, where each of the indices $\alpha_1, \alpha_2, \ldots, \alpha_{2k}$ takes either of the two values 0 or 1, consists of $2^{2k}$ matrices. At this point these matrices form a basis of the full $2^{2k}$-dimensional matrix algebra (spinor representation of $\mathbb{C}_{2k}$). Indeed, in virtue of $i\sigma_1\sigma_2 = \sigma_3$ from (3) it follows

$$N_j = E_j E_{k+j} = \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \cdots \otimes \sigma_0,$$

$$j = 1, 2, \ldots, k,$$

here the matrix $\sigma_3$ occurs in the $j$-th position. Further, since the tensor product $\sigma_0 \otimes \cdots \otimes \sigma_0$ of the unit matrices of the second order is also unit matrix $E_0$ of the $2^k$-order, then we can write

$$Z_{j^+} = \frac{1}{2}(\mathcal{L}_0 - N_j) = \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes Q^{++} \otimes \sigma_0 \otimes \cdots \otimes \sigma_0,$$

$$Z_{j^-} = \frac{1}{2}(\mathcal{L}_0 + N_j) = \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes Q^{--} \otimes \sigma_0 \otimes \cdots \otimes \sigma_0,$$

where the matrices $Q^{++}$ and $Q^{--}$ occur in the $j$-th position and have the following form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
has unit elements at the intersection of \((s_1, s_2, \ldots, s_k)-th\) row and \((r_1, r_2, \ldots, r_k)-th\) column, the other elements are equal to zero. In virtue of (7) and (8) each of the matrices \(L_j, L_{k+j}\) and, therefore, each of the matrices \(Z_{ji}^{s,r}\), is represented by a linear combination of the matrices \(E_{1}^{a_1}E_{2}^{a_2}\cdots E_{2k}^{a_{2k}}\). Hence it follows that the matrices \(\prod_{j=1}^{k}(Z_{ji}^{s,r})\) and, therefore, all the \(2^{k}\)-dimensional matrices, are represented by such linear combinations. Thus, \(2k\) matrices \(E_1,\ldots, E_{2k}\) generate a group consisting of the products \(\pm E_{1}^{a_1}E_{2}^{a_2}\cdots E_{2k}^{a_{2k}}\), and an enveloped algebra of this group is a full \(2^{k}\)-dimensional matrix algebra.

The following part of this Theorem tells that the full matrix algebra, forming by the tensor products (8), is a spinor representation of the algebra \(C_{2k}\). Let us prove this part on the several examples. First of all, in accordance with (6) tensor products of the following products (matrices of the eighth order):  

\[
\begin{align*}
E_1 &= \sigma_1 \otimes \sigma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
E_2 &= \sigma_3 \otimes \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
E_3 &= \sigma_2 \otimes \sigma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
E_4 &= \sigma_3 \otimes \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. 
\end{align*}
\]

It is easy to see that \(E_1^2 = E_0\) and \(E_1E_2 = -E_2E_1\) \((i, j = 1, \ldots, 4)\). The set of the matrices \(E_1^{a_1}E_2^{a_2}\cdots E_4^{a_4}\), where each of the indices \(a_1, a_2, a_3, a_4\) takes either of the two values \(0\) or \(1\), consists of \(2^4 = 16\) matrices. At this point these matrices form a basis of the full \(4\)-dimensional matrix algebra. Thus, we can define a one-to-one correspondence between sixteen matrices \(E_1^{a_1}E_2^{a_2}\cdots E_4^{a_4}\) and sixteen basis elements \(e_{i_1}e_{i_2}\cdots e_{i_4}\) of the Dirac algebra \(C_4\). Therefore, the matrices (11) form a basis of the spinor representation of \(C_4\). Moreover, from (11) it follows that \(C_4 \cong C_2 \otimes C_2\), that is, the Dirac algebra is a tensor product of the two Pauli algebras.

Analogously, when \(k = 3\) from (8) we obtain following products (matrices of the eighth order):

\[
\begin{align*}
E_1 &= \sigma_1 \otimes \sigma_0 \otimes \sigma_0, \quad E_2 = \sigma_3 \otimes \sigma_1 \otimes \sigma_0, \quad E_3 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1, \\
E_4 &= \sigma_2 \otimes \sigma_0 \otimes \sigma_0, \quad E_5 = \sigma_3 \otimes \sigma_2 \otimes \sigma_0, \quad E_6 = \sigma_3 \otimes \sigma_3 \otimes \sigma_2.
\end{align*}
\]

The set of the matrices \(E_1^{a_1}E_2^{a_2}\cdots E_6^{a_6}\), consisting of \(2^6 = 64\) matrices, forms a basis of the full \(8\)-dimensional matrix algebra, which is isomorphic to a spinor representation of
the algebra \( \mathbb{C}_6 \cong \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \). Generalizing we obtain that \( 2k \) tensor products of the \( k \) Pauli matrices form a basis of the spinor representation of the complex Clifford algebra

\[
\mathbb{C}_{2k} \cong \underbrace{\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2}_{k \text{ times}}.
\]  

Let us consider now a case of odd dimensions. When \( n = 2k + 1 \) we add to the set of the tensor products (8) a matrix

\[
\mathcal{E}_{2k+1} = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3,
\]

which satisfying the conditions

\[
\mathcal{E}_{2k+1}^2 = \mathcal{E}_0, \quad \mathcal{E}_{2k+1} \mathcal{E}_i = -\mathcal{E}_i \mathcal{E}_{2k+1}, \quad i = 1, 2, \ldots, k.
\]

It is obvious that a product \( \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_{2k} \mathcal{E}_{2k+1} \) commutes with all the products of the form \( \mathcal{E}_1^{\alpha_1} \mathcal{E}_2^{\alpha_2} \cdots \mathcal{E}_{2k}^{\alpha_{2k}} \). Further, let

\[
\varepsilon = \begin{cases} 
1 & \text{if } k \equiv 0 \pmod{2}, \\
i & \text{if } k \equiv 1 \pmod{2}.
\end{cases}
\]

Then a product

\[
U = \varepsilon \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_{2k} \mathcal{E}_{2k+1}
\]

satisfies the condition \( U^2 = \mathcal{E}_0 \). Let \( P \) be a set of all \( 2^{2k+1} \) matrices \( \mathcal{E}_1^{\alpha_1} \mathcal{E}_2^{\alpha_2} \cdots \mathcal{E}_{2k}^{\alpha_{2k}} \mathcal{E}_{2k+1}^{\alpha_{2k+1}} \), where \( \alpha_j \) equals to 0 or 1, \( j = 1, 2, \ldots, 2k + 1 \). Let us divide the set \( P \) into two subsets by the following manner:

\[
P = P^1 + P^0,
\]

where the subset \( P^0 \) contains products with the matrix \( \mathcal{E}_{2k+1} \), and \( P^1 \) contains products without the matrix \( \mathcal{E}_{2k+1} \). Therefore, products \( \mathcal{E}_1^{\alpha_1} \mathcal{E}_2^{\alpha_2} \cdots \mathcal{E}_{2k}^{\alpha_{2k}} \subset P^1 \) form a full \( 2^k \)-dimensional matrix algebra. Further, when we multiply the matrices from the subset \( P^0 \) by the matrix \( U \) the factors \( \mathcal{E}_{2k+1} \) are mutually annihilate. Thus, matrices of the set \( UP^0 \) are also belong to the \( 2^k \)-dimensional matrix algebra. Let us denote \( UP^0 \) via \( P^2 \). Taking into account that \( U^2 = \mathcal{E}_0 \) we obtain \( P^0 = UP^2 \) and

\[
P = P^1 + P^2,
\]

where \( P^1, P^2 \in \mathbb{M}_{2k} \). Let

\[
\chi : P^1 + UP^2 \to P^1 + P^2
\]

be an homomorphic mapping of the set (14) containing the matrix \( \mathcal{E}_{2k+1} \) onto the set \( P^1 + P^2 \) which does not contain this matrix. The mapping \( \chi \) preserves addition and multiplication operations. Indeed, let \( P = P^1 + UP^2 \) and \( Q = Q^1 + UQ^2 \), then

\[
P + Q = P^1 + UP^2 + Q^1 + UQ^2 \to P^1 + P^2 + Q^1 + Q^2,
\]
that is, an image of the products equals to the product of factor images in the same order. In particular, when \( P = U \) we obtain \( P^1 = 0, P^2 = \mathcal{E}_0 \) and

\[
U \longrightarrow \mathcal{E}_0.
\]  

(17)

In such a way, at the mapping \( \chi \) all the matrices of the form \( \mathcal{E}_1^{\alpha_1} \mathcal{E}_2^{\alpha_2} \cdots \mathcal{E}_{2k}^{\alpha_{2k}} - U \mathcal{E}_1^{\alpha_1} \mathcal{E}_2^{\alpha_2} \cdots \mathcal{E}_{2k}^{\alpha_{2k}} \) are mapped into zero. Therefore, a kernel of the homomorphism \( \chi \) is an expression \( \text{Ker} \chi = \{ P^1 - UP^1 \} \). We obtain the homomorphic mapping of the set of all the \( 2^{2k+1} \) matrices \( \mathcal{E}_1^{\alpha_1} \mathcal{E}_2^{\alpha_2} \cdots \mathcal{E}_{2k}^{\alpha_{2k}} \) onto the full matrix algebra \( M_{2k} \).

In the result of this mapping we have a quotient algebra \( \mathcal{M}_{2k} \simeq P / \text{Ker} \chi \). As noted previously, \( 2k \) tensor products of the \( k \) Pauli matrices (or other \( k \) matrices defining spinor representation of the biquaternion algebra \( \mathbb{C}_2 \)) form a basis of the spinor representation of the algebra \( \mathbb{C}_{2k} \).

It is easy to see that there exists one–to–one correspondence between \( 2^{2k+1} \) matrices of the set \( P \) and basis elements of the odd–dimensional Clifford algebra \( \mathbb{C}_{2k+1} \). It is well–known \cite{12,19} that \( \mathbb{C}_{2k+1} \) is isomorphic to a direct sum of two even–dimensional subalgebras: \( \mathbb{C}_{2k+1} \simeq \mathbb{C}_{2k} \oplus \mathbb{C}_{2k} \). Moreover, there exists an homomorphic mapping \( \epsilon : \mathbb{C}_{2k+1} \rightarrow \mathbb{C}_{2k} \), in the result of which we have a quotient algebra \( \mathbb{C}_{2k} \simeq \mathbb{C}_{2k+1} / \text{Ker} \epsilon \), where Ker \( \epsilon = \{ A^1 - \epsilon N A^1 \} \) is a kernel of the homomorphism \( \epsilon \), \( A^1 \) is an arbitrary element of the subalgebra \( \mathbb{C}_2 \), \( \omega \) is a volume element of the algebra \( \mathbb{C}_{2k+1} \). It is easy to see that the homomorphisms \( \epsilon \) and \( \chi \) have a similar structure. Thus, hence it immediately follows an isomorphism \( \mathcal{C}_{2k} \simeq \mathcal{M}_{2k} \). Therefore, a basis of the matrix quotient algebra \( \mathcal{M}_{2k} \) is also a basis of the spinor representation of the Clifford quotient algebra \( \mathcal{C}_{2k} \), that proves the latter assertion of the theorem.

\( \square \)

Let us consider now spintensor representations of the proper Lorentz group \( O_0(1,3) \simeq SL(2; \mathbb{C})/Z_2 \simeq \text{Spin}_+(1,3)/Z_2 \) and their relations with the complex Clifford algebras.

From each complex Clifford algebra \( \mathbb{C}_n = \mathbb{C} \otimes \mathcal{O}_{p,q} \) \( (n = p + q) \) we obtain a spinspace \( S_{2n/2} \), which is a complexification of the minimal left ideal of the algebra \( \mathcal{O}_{p,q} \); \( S_{2n/2} = \mathbb{C} \otimes I_{p,q} = \mathbb{C} \otimes \mathcal{O}_{p,q} e_{pq} \), where \( e_{pq} \) is a primitive idempotent of the algebra \( \mathcal{O}_{p,q} \). Further, a spinspace corresponding the Pauli algebra \( \mathbb{C}_2 \) has a form \( S_2 = \mathbb{C} \otimes I_{2,0} = \mathbb{C} \otimes \mathcal{O}_{2,0} e_{20} \) or \( S_2 = \mathbb{C} \otimes I_{1,1} = \mathbb{C} \otimes \mathcal{O}_{1,1} e_{11} = (\mathbb{C} \otimes I_{0,2} = \mathbb{C} \otimes \mathcal{O}_{0,2} e_{02} \). Therefore, the tensor product \( \otimes \) of the \( k \) algebras \( \mathbb{C}_2 \) induces a tensor product of the \( k \) spinspaces \( S_2 \):

\[
S_2 \otimes S_2 \otimes \cdots \otimes S_2 = S_{2k}.
\]

Vectors of the spinspace \( S_{2k} \) (or elements of the minimal left ideal of \( \mathbb{C}_{2k} \)) are spintensors of the following form

\[
\zeta^{\alpha_1 \alpha_2 \cdots \alpha_k} = \sum \zeta^{\alpha_1} \otimes \zeta^{\alpha_2} \otimes \cdots \otimes \zeta^{\alpha_k},
\]  

(18)

where summation is produced on all the index collections \( (\alpha_1 \ldots \alpha_k) \), \( \alpha_i = 1, 2 \). In virtue of (12) for each spinor \( \zeta^{\alpha} \) from (18) we have a transformation rule \( \zeta^{\alpha}\bar{\zeta} = \sigma^{\alpha} \zeta^{\alpha} \). Therefore, in general case we obtain

\[
\zeta^{\alpha_1 \alpha_2 \cdots \alpha_k} = \sum \sigma^{\alpha_1}_{\alpha_1} \sigma^{\alpha_2}_{\alpha_2} \cdots \sigma^{\alpha_k}_{\alpha_k} \zeta^{\alpha_1 \alpha_2 \cdots \alpha_k}.
\]  

(19)
A representation \((\mathcal{A})\) is called undotted spintensor representation of the proper Lorentz group of the rank \(k\).

Further, let \(\mathbb{C}_2\) be a biquaternion algebra, the coefficients of which are complex conjugate. Let us show that the algebra \(\mathbb{C}_2\) is obtained from \(\mathbb{C}_2\) under action of the automorphism \(\mathcal{A} \to \mathcal{A}^\ast\) or antiautomorphism \(\mathcal{A} \to \tilde{\mathcal{A}}\). Indeed, in virtue of an isomorphism \(\mathbb{C}_2 \cong \mathcal{O}_{3,0}\) a general element

\[
\mathcal{A} = a^0 e_0 + \sum_{i=1}^{3} a^i e_i + \sum_{i=1}^{3} \sum_{j=1}^{3} a^{ij} e_{ij} + a^{123} e_{123}
\]

of the algebra \(\mathcal{O}_{3,0}\) can be written in the form

\[
\mathcal{A} = (a^0 + \omega a^{123}) e_0 + (a^1 + \omega a^{23}) e_1 + (a^2 + \omega a^{31}) e_2 + (a^3 + \omega a^{12}) e_3, \quad (20)
\]

where \(\omega = e_{123}\). Since \(\omega\) belongs to a center of the algebra \(\mathcal{O}_{3,0}\) (commutes with all the basis elements) and \(\omega^2 = -1\), then we can to suppose \(\omega \equiv i\). The action of the automorphism \(\ast\) on the homogeneous element \(\mathcal{A}\) of a degree \(k\) is defined by a formula \(\mathcal{A}^\ast = (-1)^k \tilde{\mathcal{A}}\). In accordance with this the action of the automorphism \(\mathcal{A} \to \mathcal{A}^\ast\), where \(\mathcal{A}\) is the element \((20)\), has a form

\[
\mathcal{A} \to \mathcal{A}^\ast = -(a^0 - \omega a^{123}) e_0 - (a^1 - \omega a^{23}) e_1 - (a^2 - \omega a^{31}) e_2 - (a^3 - \omega a^{12}) e_3. \quad (21)
\]

Therefore, \(\ast : \mathbb{C}_2 \to -\mathbb{C}_2\). Correspondingly, the action of the antiautomorphism \(\mathcal{A} \to \tilde{\mathcal{A}}\) on the homogeneous element \(\mathcal{A}\) of a degree \(k\) is defined by a formula \(\tilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}} \mathcal{A}\). Thus, for the element \((20)\) we obtain

\[
\mathcal{A} \to \tilde{\mathcal{A}} = (a^0 - \omega a^{123}) e_0 + (a^1 - \omega a^{23}) e_1 + (a^2 - \omega a^{31}) e_2 + (a^3 - \omega a^{12}) e_3, \quad (22)
\]

that is, \(\tilde{\ast} : \mathbb{C}_2 \to -\mathbb{C}_2\). This allows to define an algebraic analog of the Wigner’s representation doubling: \(\mathbb{C}_2 \oplus -\mathbb{C}_2\). Further, from \((20)\) it follows that \(\mathcal{A} = \mathcal{A}_1 + \omega \mathcal{A}_2 = (a^0 e_0 + a^1 e_1 + a^2 e_2 + a^3 e_3) + \omega(a^{123} e_0 + a^{23} e_1 + a^{31} e_2 + a^{12} e_3)\). In general case, by virtue of an isomorphism \(\mathbb{C}_{2k} \cong \mathcal{O}_{p,q}\), where \(\mathcal{O}_{p,q}\) is a real Clifford algebra with a division ring \(\mathbb{K} \simeq \mathbb{C}\), \(p - q \equiv 3, 7 \mod 8\), we have for a general element of \(\mathcal{O}_{p,q}\) an expression \(\mathcal{A} = \mathcal{A}_1 + \omega \mathcal{A}_2\), here \(\omega^2 = e_{12\ldots p+q}^{2} = -1\) and, therefore, \(\omega \equiv i\). Thus, from \(\mathbb{C}_{2k}\) under action of the automorphism \(\mathcal{A} \to \mathcal{A}^\ast\) we obtain a general algebraic doubling

\[
\mathbb{C}_{2k} \oplus -\mathbb{C}_{2k}. \quad (23)
\]

Correspondingly, a tensor product \(\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \simeq \mathbb{C}_{2r}\) of \(r\) algebras \(\mathbb{C}_2\) induces a tensor product of \(r\) spinspace \(\hat{S}_2\):

\[
\hat{S}_2 \otimes \hat{S}_2 \otimes \cdots \otimes \hat{S}_2 = \hat{S}_{2r}.
\]

Te vectors of the spinspace \(\hat{S}_{2r}\) have the form

\[
\xi^{\hat{a}_1 \hat{a}_2 \ldots \hat{a}_r} = \sum \xi^{\hat{a}_1} \otimes \xi^{\hat{a}_2} \otimes \cdots \otimes \xi^{\hat{a}_r}, \quad (24)
\]
where the each cospinor \( \xi^{\hat{\alpha}} \) from (24) in virtue of (3) is transformed by the rule \( \xi^{\hat{\alpha}'} = \sigma^{\hat{\alpha}'}_{\hat{\alpha}} \xi^{\hat{\alpha}} \). Therefore,

\[
\xi^{\hat{\alpha}'_1 \hat{\alpha}'_2 \ldots \hat{\alpha}'_r} = \sum \sigma^{\hat{\alpha}'_1}_{\hat{\alpha}_1} \sigma^{\hat{\alpha}'_2}_{\hat{\alpha}_2} \ldots \sigma^{\hat{\alpha}'_r}_{\hat{\alpha}_r} \xi^{\hat{\alpha}_1 \hat{\alpha}_2 \ldots \hat{\alpha}_r}.
\]

(25)

A representation (24) is called a dotted spintensor representation of the proper Lorentz group of the rank \( r \).

In general case we have a tensor product of \( k \) algebras \( \mathbb{C}_2 \) and \( r \) algebras \( \hat{\mathbb{C}}_2 \):

\[
\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \otimes \hat{\mathbb{C}}_2 \otimes \cdots \otimes \hat{\mathbb{C}}_2 \simeq \mathbb{C}_{2k} \otimes \hat{\mathbb{C}}_{2r},
\]

which induces a spinspace

\[
\mathbb{S}_2 \otimes \mathbb{S}_2 \otimes \cdots \otimes \mathbb{S}_2 \otimes \hat{\mathbb{S}}_2 \otimes \cdots \otimes \hat{\mathbb{S}}_2 = \mathbb{S}_{2k+r}
\]

with the vectors

\[
\xi^{\alpha_1 \alpha_2 \ldots \alpha_k \hat{\alpha}_1 \hat{\alpha}_2 \ldots \hat{\alpha}_r} = \sum \xi^{\alpha_1} \otimes \xi^{\alpha_2} \otimes \cdots \otimes \xi^{\alpha_k} \otimes \xi^{\hat{\alpha}_1} \otimes \xi^{\hat{\alpha}_2} \otimes \cdots \otimes \xi^{\hat{\alpha}_r}.
\]

(26)

In this case we have a natural unification of the representations (19) and (25):

\[
\xi^{\alpha'_1 \alpha'_2 \ldots \alpha'_k \hat{\alpha}'_1 \hat{\alpha}'_2 \ldots \hat{\alpha}'_r} = \sum \sigma^{\alpha'_1}_{\alpha_1} \sigma^{\alpha'_2}_{\alpha_2} \ldots \sigma^{\alpha'_k}_{\alpha_k} \sigma^{\hat{\alpha}'_1}_{\hat{\alpha}_1} \sigma^{\hat{\alpha}'_2}_{\hat{\alpha}_2} \ldots \sigma^{\hat{\alpha}'_r}_{\hat{\alpha}_r} \xi^{\alpha_1 \alpha_2 \ldots \alpha_k \hat{\alpha}_1 \hat{\alpha}_2 \ldots \hat{\alpha}_r}.
\]

(27)

So, a representation (27) is called a spintensor representation of the proper Lorentz group of the rank \((k, r)\).

In general case, the representations defining by the formulas (19), (25) and (27), are reducible, that is there exists possibility of decomposition of the initial spinspace \( \mathbb{S}_{2k+r} \) (correspondingly, spinspaces \( \mathbb{S}_{2k} \) and \( \mathbb{S}_{2r} \)) into a direct sum of invariant (with respect to transformations of the group \( \mathfrak{G}_+ \)) spinspace \( \mathbb{S}_{2\nu_1} \oplus \mathbb{S}_{2\nu_2} \oplus \cdots \oplus \mathbb{S}_{2\nu_s} \), where \( \nu_1 + \nu_2 + \ldots + \nu_s = k + r \).

Further, an important notion of the physical field is closely related with finite–dimensional representations of the proper Lorentz group \( \mathfrak{G}_+ \). In accordance with Wigner interpretation [27], an elementary particle is described by some irreducible finite–dimensional representation of the Poincaré group. The double covering of the proper Poincaré group is isomorphic to a semidirect product \( SL(2; \mathbb{C}) \circ T(4) \), or \( \text{Spin}_4(1, 3) \circ T(4) \), where \( T(4) \) is the subgroup of four–dimensional translations. Let \( \psi(x) \) be a physical field, then at the transformations \((a, \Lambda)\) of the proper Poincaré group the field \( \psi(x) \) is transformed by a following rule

\[
\psi'_m u(x) = \sum_{\nu} \mathfrak{C}_{\mu\nu}(\sigma) \psi_{\nu}(\Lambda^{-1}(x - a)),
\]

(28)

where \( a \in T(4) \), \( \sigma \in \mathfrak{G}_+ \), \( \Lambda \) is a Lorentz transformation, and \( \mathfrak{C}_{\mu\nu} \) is a representation of the group \( \mathfrak{G}_+ \) in the space \( \mathbb{S}_{2k+r} \). Since the group \( T(4) \) is Abelian, then all its representations are one–dimensional. Thus, all the finite–dimensional representations of the proper Poincaré group in essence are equivalent to the representations \( \mathfrak{C} \) of the group \( \mathfrak{G}_+ \). If
the representation \( \mathfrak{C} \) is reducible, then the space \( S_{2k+r} \) is decomposed into a direct sum of irreducible subspaces, that is, it is possible to choose in \( S_{2k+r} \) such a basis, in which all the matrices \( \mathfrak{C}_{\mu\nu} \) take a block-diagonal form. Then the field \( \psi(x) \) is reduced to some number of the fields corresponding to obtained irreducible representations of the group \( \mathfrak{G}_+ \), each of which is transformed independently from the other, and the field \( \psi(x) \) in this case is a collection of the fields with more simple structure. It is obvious that these more simple fields correspond to irreducible representations \( \mathfrak{C} \). As known \([17, 29, 60]\), a system of irreducible finite-dimensional representations of the group \( \mathfrak{G}_+ \) is realized in the space \( \text{Sym}_{(k, r)} \subset S_{2k+r} \) of symmetric spintensors. The dimensionality of \( \text{Sym}_{(k, r)} \) is equal to \((k + 1)(r + 1)\). A representation of the group \( \mathfrak{G}_+ \) by such spintensors is irreducible and denoted by the symbol \( \mathfrak{C}^{j,j'} \), where \( 2j = k \), \( 2j' = r \), numbers \( j \) and \( j' \) defining the spin are integer or half-integer. Then the field \( \psi(x) \) transforming by the formula \((23)\) is, in general case, a field of the type \((j, j')\). In such a way, all the physical fields are reduced to the fields of this type, the mathematical structure of which requires a knowledge of representation matrices \( \mathfrak{C}^{j,j'} \). As a rule, in physics there are two basic types of the fields:

1) The field of type \((j, 0)\). The structure of this field (or the field \((0, j)\)) is described by the representation \( \mathfrak{C}^{j,0} \) (\( \mathfrak{C}^{0,j} \)), which is realized in the space \( \text{Sym}_{(k, 0)} \subset S_{2k} \) (\( \text{Sym}_{(0, r)} \subset S_{2r} \)). At this point in accordance with Theorem \( \[ \] \) the algebra \( \mathbb{C}_{2k} \simeq \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \) (correspondingly, \( \mathbb{C}_2 \simeq \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \)) is associated with the field of type \((j, 0)\) (correspondingly, \((0, j')\)) The trivial case \( j = 0 \) corresponds to a Pauli–Weisskopf field describing the scalar particles. In other particular case, when \( j = j' = 1/2 \) we have a Weyl field describing the neutrino. At this point the antineutrino is described by a fundamental representation \( \mathfrak{C}^{1/2,0} = \sigma \) of the group \( \mathfrak{G}_+ \) and the algebra \( \mathbb{C}_2 \) related with this representation (Theorem \( \[ \] \)). Correspondingly, the neutrino is described by a conjugated representation \( \mathfrak{C}^{0,1/2} \) and the algebra \( \mathbb{C}_2^* \). In relation with this, it is hardly too much to say that the neutrino field is a more fundamental physical field, that is a kind of the basic building block, from which other physical fields built by means of direct sum or tensor product.

2) The field of type \((j, 0) \oplus (0, j)\). The structure of this field admits a space inversion and, therefore, in accordance with a Wigner’s doubling \([38]\) is described by a representation \( \mathfrak{C}^{j,0} \oplus \mathfrak{C}^{0,j} \) of the group \( \mathfrak{G}_+ \). This representation is realized in the space \( \text{Sym}_{(k, k)} \subset S_{2k} \). In accordance with \([23]\) the Clifford algebra related with this representation is a direct sum \( \mathbb{C}_{2k} \oplus \mathbb{C}_{2k}^* \simeq \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2 \oplus \mathbb{C}_2^* \otimes \mathbb{C}_2^* \otimes \cdots \otimes \mathbb{C}_2^* \). In the simplest case \( j = 1/2 \) we have bispinor (electron–positron) Dirac field \((1/2, 0) \oplus (0, 1/2)\) with the algebra \( \mathbb{C}_2 \oplus \mathbb{C}_2^* \). It should be noted that the Dirac algebra \( \mathbb{C}_4 \) considered as a tensor product \( \mathbb{C}_2 \otimes \mathbb{C}_2 \) (or \( \mathbb{C}_2 \otimes \mathbb{C}_2^* \)) in accordance with \([13]\) (or \([20]\)) gives rise to spintensors \( \xi^{\alpha\dagger\alpha} \) (or \( \xi^{\alpha\dagger\alpha} \)), but it contradicts with the usual definition of the Dirac bispinor as a pair \((\xi^{\alpha}, \xi^{\dagger\alpha})\). Therefore, the Clifford algebra associated with the Dirac field is \( \mathbb{C}_2 \oplus \mathbb{C}_2^* \), and a spinspace of this sum in virtue of unique decomposition \( S_2 \oplus S_2 = S_4 \) (\( S_4 \) is a spinspace of \( \mathbb{C}_4 \)) allows to define \( \gamma \)-matrices in the Weyl basis. The case \( j = 1 \) corresponds to Maxwell fields \((1, 0)\) and \((0, 1)\) with the algebras \( \mathbb{C}_2 \otimes \mathbb{C}_2 \) and \( \mathbb{C}_2 \otimes \mathbb{C}_2^* \). At this point the electromagnetic field
is defined by complex linear combinations $\mathbf{F} = \mathbf{E} - i\mathbf{H}$, $\mathbf{F}^* = \mathbf{E} + i\mathbf{H}$ (Helmholtz representation). Besides, the algebra related with Maxwell field is a tensor product of the two algebras $\mathbb{C}_2$ describing the neutrino fields. In this connection it is of interest to recall a neutrino theory of light was proposed by de Broglie and Jordan [11, 37]. In the de Broglie–Jordan neutrino theory of light electromagnetic field is constructed from the two neutrino fields (for more details and related papers see [22]). Traditionally, physicists attempt to describe electromagnetic field in the framework of $(1,0) \oplus (0,1)$ representation (see old works [11, 54, 56] and recent developments based on the Joos–Weinberg formalism [30, 66] and its relation with a Bargmann–Wightman–Wigner type quantum field theory [3, 20]). However, Weinberg’s equations (or Weinberg–like equations) for electromagnetic field obtained within the subspace $\text{Sym}_{k,r}$ with dimension $2(2j + 1)$ have acausal (tachionic) solutions [2]. Electromagnetic field in terms of a quotient representation $(1,0) \cup (0,1)$ in the full representation space $\mathbb{S}_{2k+r}$ will be considered in separate paper.

In this connection it should be noted two important circumstances related with irreducible representations of the group $\mathfrak{g}_+$ and complex Clifford algebras associated with these representations. The first circumstance relates with the Wigner interpretation of an elementary particle. Namely, a relation between finite–dimensional representations of the proper Lorentz group and complex Clifford algebras (Theorem 1) allows to essentially extend the Wigner interpretation by means of the use of an extraordinary rich and universal structure of the Clifford algebras at the study of space–time (and also intrinsic) symmetries of elementary particles. The second circumstance relates with the spin. Usually, the Clifford algebra is associated with a half–integer spin corresponding to fermionic fields, so–called ‘matter fields’, whilst the fields with an integer spin (bosonic fields) are eliminated from an algebraic description. However, such a non–symmetric situation is invalid, since the fields with integer spin have a natural description in terms of spinor representations of the proper Lorentz group with even rank and algebras $\mathbb{C}_{2k}$ and $\mathbb{C}^*_{2k}$ associated with these representations, where $k$ is even (for example, Maxwell field). In this connection it should be noted that generalized statistics in terms of Clifford algebras have been recently proposed by Finkelstein and collaborators [25, 8].

As known, complex Clifford algebras $\mathbb{C}_n$ are modulo 2 periodic [4] and, therefore, there exist two types of $\mathbb{C}_n$: $n \equiv 0 \pmod{2}$ and $n \equiv 1 \pmod{2}$. Let us consider these two types in the form of following series:

\[ \begin{array}{cccccc}
\mathbb{C}_2 & \mathbb{C}_4 & \cdots & \mathbb{C}_{2k} & \cdots \\
\mathbb{C}_3 & \mathbb{C}_5 & \cdots & \mathbb{C}_{2k+1} & \cdots 
\end{array} \]

Let us consider the decomposition $\mathbb{C}_{2k+1} \simeq \mathbb{C}_{2k} \oplus \mathbb{C}^*_{2k}$ in more details. This decomposition may be represented by a following scheme:
Therefore, the quotient algebra $\frac{1}{2}e_1e_2\cdots e_{2k+1}$, $\frac{1}{2}e_1e_2\cdots e_{2k+1}$, where

$$\varepsilon = \begin{cases} 1, & \text{if } k \equiv 0 \pmod{2}, \\ i, & \text{if } k \equiv 1 \pmod{2} \end{cases}$$

satisfy the relations $(\lambda^+)^2 = \lambda^+$, $(\lambda^-)^2 = \lambda^-$, $\lambda^+\lambda^- = 0$. Thus, we have a decomposition of the initial algebra $C_{2k+1}$ into a direct sum of two mutually annihilating simple ideals: $C_{2k+1} \cong \frac{1}{2}(1 + \varepsilon\omega)C_{2k+1} \oplus \frac{1}{2}(1 - \varepsilon\omega)C_{2k+1}$. Each of the ideals $\lambda^\pm C_{2k+1}$ is isomorphic to the subalgebra $C_{2k} \subset C_{2k+1}$. In accordance with Chisholm and Farwell [16] the idempotents $\lambda^\pm$ and $\lambda^-$ can be identified with helicity projection operators which distinguish left and right handed spinors. The Chisholm–Farwell notation for $\lambda^\pm$ we will widely use below.

Therefore, in virtue of the isomorphism $C_{2k+1} \cong C_{2k} \cup C_{2k}$ and the homomorphic mapping $\epsilon : C_{2k+1} \to C_{2k}$ the second series (type $n \equiv 1 \pmod{2}$) is replaced by a sequence of the quotient algebras $\epsilon C_{2k}$, that is,

$$\begin{array}{cccccc}
C_2 & C_4 & \cdots & C_{2k} & \cdots \\
\epsilon C_2 & \epsilon C_4 & \cdots & \epsilon C_{2k} & \cdots \\
\end{array}$$

Representations corresponded these two series of $C_n (n \equiv 0, 1 \pmod{2})$ form a full system $\mathcal{M} = \mathcal{M}^0 \oplus \mathcal{M}^1$ of finite–dimensional representations of the proper Lorentz group $\mathfrak{G}_+$. All the physical fields used in quantum field theory and related representations of the group $\mathfrak{G}_+$. $\mathfrak{C}^{j,0} (C_{2k})$, $\mathfrak{C}^{j,0} \oplus \mathfrak{C}^{0,j} (C_{2k} \otimes \mathfrak{C}_{2k})$ are constructed from the upper series (type $n \equiv 0 \pmod{2}$). Whilst the lower series (type $n \equiv 1 \pmod{2}$) is not considered in physics as yet. In accordance with Theorem [1] we have an isomorphism $\epsilon C_{2k} \cong \lambda^0 M_{2k}$.

Therefore, the quotient algebra $\epsilon C_{2k}$ induces a spin space $\epsilon S_{2k}$ that is a space of a quotient representation $\chi^0 C_j$ of the group $\mathfrak{G}_+$. Analogously, a quotient representation $\chi^0, j$ is realised in the space $\epsilon S_{2k}$ which induced by the quotient algebra $\epsilon C_{2k}$. In general case, we have a quotient representation $\chi^0, j$ defined by a tensor product $\epsilon C_{2k} \otimes \epsilon C_{2k}$. Thus, the complex type $n \equiv 1 \pmod{2}$ corresponds to a full system of irreducible finite–dimensional quotient representations $\chi^0, j$ of the proper Lorentz group. Therefore, until now in physics only one half $(n \equiv 0 \pmod{2})$ of all possible finite–dimensional representations of the Lorentz group has been used.

Let us consider now a full system of physical fields with different types. First of all, the field

$$(j, 0) = (1/2, 0) \otimes (1/2, 0) \otimes \cdots \otimes (1/2, 0) \quad (29)$$

in accordance with Theorem [2] is a tensor product of the $k$ fields of type $(1/2, 0)$, each of which corresponds to the fundamental representation $\mathfrak{C}^{1/2,0} = \sigma$ of the group $\mathfrak{G}_+$ and the biquaternion algebra $C_2$ related with fundamental representation. In its turn, the field

$$(0, j') = (0, 1/2) \otimes (0, 1/2) \otimes \cdots \otimes (0, 1/2) \quad (30)$$
is a tensor product of the $r$ fundamental fields of the type $(0,1/2)$, each of which corresponds to a conjugated representation $\mathcal{C}^{0,1/2} = \tilde{\sigma}$ and the conjugated algebra $\mathbb{C}_2$ obtained in accordance with (21)–(22) under action of the automorphism $\mathcal{A} \to \mathcal{A}^*$ (space inversion), or under action of the anti-automorphism $\mathcal{A} \to \tilde{\mathcal{A}}$ (time reversal). The numbers $j$ and $j'$ are integer (bosonic fields) if in the products (29)–(30) there are $k, r \equiv 0 \pmod{2}$ (1/2, 0) (or (0, 1/2)) factors, and the numbers $j$ and $j'$ are half–integer (fermionic fields) if in the products (29)–(30) there are $k, r \equiv 1 \pmod{2}$ factors. Further, the field 

$$(j, j') = (1/2, 0) \otimes (1/2, 0) \otimes \cdots \otimes (1/2, 0) \otimes (0, 1/2) \otimes (0, 1/2) \otimes \cdots \otimes (0, 1/2)$$

(31)

is a tensor product of the fields (29) and (30). As consequence of the doubling (23) we have the field of type $(j, 0) \oplus (0, j)$:

$$(j, 0) \oplus (0, j) = (1/2, 0) \otimes (1/2, 0) \otimes \cdots \otimes (1/2, 0) \oplus (0, 1/2) \otimes (0, 1/2) \otimes \cdots \otimes (0, 1/2)$$

In general, all the fields (29)–(31) describe multiparticle states. The decompositions of these multiparticle states into single states provided in the full representation space $\mathcal{S}_{2k+r}$, where $\text{Sym}_{(k,r)}$ and $\text{Sym}_{(k,k)}$ with dimensions $(2j+1)(2j'+1)$ and $2(2j+1)$ are subspaces of $\mathcal{S}_{2k+r}$ (for example, the Clebsh–Gordan decomposition of two spin 1/2 particles into singlet and triplet: $(1/2, 1/2) = (1/2, 0) \otimes (0, 1/2) = (0, 0) \oplus (1, 0)$). In the papers [34, 19, 59] a multiparticle state is described in the framework of a tensor product $\mathcal{A}_{3,0} \otimes \cdots \otimes \mathcal{A}_{3,0}$. It is easy to see that in virtue of the isomorphism $\mathcal{A}_{3,0} \cong \mathbb{C}_2$ the tensor product of the algebras $\mathcal{A}_{3,0}$ is isomorphic to the product $[\mathbb{C}_2]$. Therefore, the Holland approach naturally incorporates into a more general scheme considered here. Finally, for the type $n \equiv 1 \pmod{2}$ we have quotient representations $\mathcal{C}_n$ of the group $\mathfrak{G}_+$. The physical fields corresponding to the quotient representations are constructed like the fields (29)–(31). Due to the decomposition $\mathcal{C}_n \simeq \mathbb{C}_{n-1} \cup \mathbb{C}_{n-1}$ ($n \equiv 1 \pmod{2}$) we have a field 

$$(j, 0) \cup (j, 0) = (1/2, 0) \otimes (1/2, 0) \otimes \cdots \otimes (1/2, 0) \cup (0, 1/2) \otimes (0, 1/2) \otimes \cdots \otimes (0, 1/2),$$

(32)

and also we have fields $(0, j) \cup (0, j)$ and $(j, 0) \cup (0, j)$ if the quotient algebras $\mathcal{C}_{n-1}$ admit space inversion or time reversal. The field

$$(j, 0) \cup (0, j) = (1/2, 0) \otimes (1/2, 0) \otimes \cdots \otimes (1/2, 0) \cup (0, 1/2) \otimes (0, 1/2) \otimes \cdots \otimes (0, 1/2)$$

(33)

is analogous to the field $(j, 0) \oplus (0, j)$, but, in general, the field $(j, 0) \cup (0, j)$ has a quantity of violated discrete symmetries. An explicit form of the quotient representations and their relations with discrete symmetries will be explored in the following sections.

3 Discrete symmetries on the representation spaces of the Lorentz group

Since all the physical fields are defined in terms of finite–dimensional representations of the group $\mathfrak{G}_+$, then a construction of the discrete symmetries (space inversion $P$, time
reversal $T$ and combination $PT$) on the representation spaces of the Lorentz group has a primary importance.

In the recent paper [60] it has been shown that the space inversion $P$, time reversal $T$ and full reflection $PT$ correspond to fundamental automorphisms $\mathcal{A} \rightarrow \mathcal{A}^\star$ (involution), $\mathcal{A} \rightarrow \tilde{A}$ (reversal) and $\mathcal{A} \rightarrow \mathcal{A}^*$ (conjugation) of the Clifford algebra $\mathcal{C}$. Moreover, there exists an isomorphism between a discrete subgroup $\{1, P, T, PT\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ ($P^2 = T^2 = (PT)^2 = 1$, $PT = TP$) of the orthogonal group $O(p, q)$ and an automorphism group $\text{Aut}(\mathcal{C}) = \{\text{Id}, \star, \sim, \tilde{\star}\}$:

|       | Id   | \(\star\) | \(\sim\) | \(\tilde{\star}\) |
|-------|------|------------|-----------|-------------------|
| Id    | Id   | \(\star\) | \(\sim\)  | \(\tilde{\star}\) |
| \(\star\) | \(\star\) | Id | \(\sim\)  | \(\tilde{\star}\) |
| \(\sim\) | \(\sim\) | \(\tilde{\star}\) | Id | \(\star\) |
| \(\tilde{\star}\) | \(\tilde{\star}\) | \(\star\) | Id | \(\tilde{\star}\) |

Further, in the case $P^2 = T^2 = (PT)^2 = \pm 1$ and $PT = -TP$ there is an isomorphism between the group $\{1, P, T, PT\}$ and automorphism group $\text{Aut}(\mathcal{C}) = \{1, W, E, C\}$. Spinor representations of the fundamental automorphisms of the algebras $\mathbb{C}_n$ was first obtained by Rashevskii in 1955 [52]:

1) Involution: $\mathcal{A}^* = WAW^{-1}$, where $W$ is a matrix of the automorphism $\star$ (matrix representation of the volume element $\omega$); 2) Reversion: $\tilde{A} = E\mathcal{A}^T\mathcal{E}_1^{-1}$, where $E$ is a matrix of the antiautomorphism $\sim$ satisfying the conditions $\mathcal{E}_1 E - E\mathcal{E}_1^T = 0$ and $E^T = (-1)^{m(m-1)}E$, here $\mathcal{E}_1 = \gamma(e_i)$ are matrix representations of the units of the algebra $\mathcal{C}$; 3) Conjugation: $\mathcal{A}^* = \mathcal{C}A^T\mathcal{C}^{-1}$, where $C = EW^T$ is a matrix of the antiautomorphism $\tilde{\star}$ satisfying the conditions $C\mathcal{E}_1^T + \mathcal{E}_1 C = 0$ and $C^T = (-1)^{m(m+1)/2}C$.

So, for the Dirac algebra $\mathcal{C}_4$ in the canonical $\gamma$–basis there exists a standard (Wigner) representation $P = \gamma_0$ and $T = \gamma_1\gamma_3$ [1], therefore, $\{1, P, T, PT\} = \{1, \gamma_0, \gamma_1\gamma_3, \gamma_0\gamma_1\gamma_3\}$. On the other hand, the automorphism group of the algebra $\mathcal{C}_4$ for $\gamma$–basis has a form $\text{Aut}(\mathcal{C}_4) = \{1, W, E, C\} = \{1, \gamma_0\gamma_1\gamma_2\gamma_3, \gamma_1\gamma_3, \gamma_0\gamma_2\}$. In [60] it is shown that $\{1, P, T, PT\} = \{1, \gamma_0, \gamma_1\gamma_3, \gamma_0\gamma_1\gamma_3\} \simeq \text{Aut}(\mathcal{C}_4) \simeq \mathbb{Z}_4$, where $\mathbb{Z}_4$ is a complex group with the signature $(+, -, -, -)$.

In general case, according to Theorem 1 a space of the finite–dimensional representation of the group $SL(2; \mathbb{C})$ is a spinspace $\mathcal{S}_{2k+r}$, or a minimal left ideal of the algebra $\mathcal{C}_{2k} \otimes \tilde{\mathcal{C}}_{2r}$. Therefore, in the spinor representation the fundamental automorphisms of the algebra $\mathcal{C}_{2k} \otimes \tilde{\mathcal{C}}_{2r}$ (all spinor representations of the fundamental automorphisms have been found in [51]) by virtue of the isomorphism (34) induce discrete transformations on the representation spaces (spinspaces) of the Lorentz group.

### Theorem 2
1) The field $\mathbb{F} = \mathbb{C}$. The tensor products $\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2$, $\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2$ of the Pauli algebra $\mathcal{C}_2$ correspond to finite–dimensional representations $\mathcal{C}^{l_0+l_1-1,0}$, $\mathcal{C}^{0, l_0-l_1-1}$, $\mathcal{C}^{l_0+l_1-1, l_0-l_1-1}$ of the proper Lorentz group $\mathcal{G}_+$, where $(l_0, l_1) = (\frac{k}{2}, \frac{k}{2} + 1)$, $(l_0, l_1) = (\frac{k}{2}, \frac{k}{2} + 1)$, $(l_0, l_1) = (\frac{k}{2}, \frac{k}{2} + 1)$, and spinspaces $\mathcal{S}_{2k}$, $\mathcal{S}_{2r}$, $\mathcal{S}_{2k+r}$ are representation spaces of the group $\mathcal{G}_+$. $\mathcal{C}_2 \leftrightarrow \mathcal{C}^{1,0}$ ($\mathcal{C}_2 \leftrightarrow \mathcal{C}^{0,-1}$) is a
fundamental representation of $\mathfrak{g}_+$. Then in a spinor representation of the fundamental automorphisms of the algebra $C_n$ for the matrix $W$ of the automorphism $A \to A^*$ (space inversion) and also for the matrices $E$ and $C$ of the antiautomorphisms $A \to \tilde{A}$ (time reversal) and $A \to \tilde{A}^*$ (full reflection) the following permutation relations with infinitesimal operators of the group $\mathfrak{g}_+$ take place:

\[
[W, A_{23}] = [W, A_{13}] = [W, A_{12}] = 0, \quad \{W, B_1\} = \{W, B_2\} = \{W, B_3\} = 0 \quad (35)
\]

\[
[W, H_+] = [W, H_-] = [W, H_3] = 0, \quad \{W, F_+\} = \{W, F_-\} = \{W, F_3\} = 0. \quad (36)
\]

\[
[E, A_{23}] = [E, A_{13}] = [E, A_{12}] = 0, \quad \{E, B_1\} = \{E, B_2\} = \{E, B_3\} = 0, \quad (37)
\]

\[
[C, A_{23}] = [C, A_{13}] = [C, A_{12}] = 0, \quad [C, B_1] = [C, B_2] = [C, B_3] = 0, \quad (38)
\]

\[
[E, H_+] = [E, H_-] = [E, H_3] = 0, \quad \{E, F_+\} = \{E, F_-\} = \{E, F_3\} = 0, \quad (39)
\]

\[
[C, H_+] = [C, H_-] = [C, H_3] = 0, \quad \{C, F_+\} = \{C, F_-\} = \{C, F_3\} = 0. \quad (40)
\]

\[
[E, A_{23}] = [E, A_{13}] = [E, A_{12}] = 0, \quad [E, B_1] = [E, B_2] = [E, B_3] = 0, \quad (41)
\]

\[
[C, A_{23}] = [C, A_{13}] = [C, A_{12}] = 0, \quad \{C, B_1\} = \{C, B_2\} = \{C, B_3\} = 0, \quad (42)
\]

\[
[E, H_+] = [E, H_-] = [E, H_3] = 0, \quad \{E, F_+\} = \{E, F_-\} = \{E, F_3\} = 0, \quad (43)
\]

\[
[C, H_+] = [C, H_-] = [C, H_3] = 0, \quad \{C, F_+\} = \{C, F_-\} = \{C, F_3\} = 0. \quad (44)
\]

\[
[E, A_{23}] = 0, \quad \{E, A_{13}\} = \{E, A_{12}\} = 0, \quad \{E, B_1\} = 0, \quad \{E, B_2\} = \{E, B_3\} = 0, \quad (45)
\]

\[
[C, A_{23}] = 0, \quad \{C, A_{13}\} = \{C, A_{12}\} = 0, \quad \{C, B_1\} = 0, \quad \{C, B_2\} = \{C, B_3\} = 0. \quad (46)
\]

\[
[E, A_{23}] = 0, \quad \{E, A_{13}\} = \{E, A_{12}\} = 0, \quad \{E, B_1\} = 0, \quad \{E, B_2\} = \{E, B_3\} = 0, \quad (47)
\]

\[
[C, A_{23}] = 0, \quad \{C, A_{13}\} = \{C, A_{12}\} = 0, \quad \{C, B_1\} = 0, \quad \{C, B_2\} = \{C, B_3\} = 0. \quad (48)
\]

\[
\{E, A_{23}\} = \{E, A_{13}\} = 0, \quad \{E, A_{12}\} = 0, \quad \{E, B_1\} = [E, B_2] = 0, \quad \{E, B_3\} = 0, \quad (49)
\]

\[
\{C, A_{23}\} = \{C, A_{13}\} = 0, \quad \{C, A_{12}\} = 0, \quad \{C, B_1\} = \{C, B_2\} = \{C, B_3\} = 0, \quad (50)
\]

\[
\{E, H_+\} = \{E, H_-\} = 0, \quad \{E, H_3\} = 0, \quad \{E, F_+\} = \{E, F_-\} = 0, \quad \{E, F_3\} = 0, \quad (51)
\]

\[
\{C, H_+\} = \{C, H_-\} = 0, \quad \{C, H_3\} = 0, \quad \{C, F_+\} = \{C, F_-\} = 0, \quad \{C, F_3\} = 0. \quad (52)
\]

\[
\{E, A_{23}\} = 0, \quad \{E, A_{13}\} = 0, \quad \{E, A_{12}\} = 0, \quad \{E, B_1\} = 0, \quad \{E, B_2\} = \{E, B_3\} = 0, \quad (53)
\]

\[
\{C, A_{23}\} = 0, \quad \{C, A_{13}\} = 0, \quad \{C, A_{12}\} = 0, \quad \{C, B_1\} = 0, \quad \{C, B_2\} = \{C, B_3\} = 0. \quad (54)
\]

\[
\{E, A_{23}\} = 0, \quad \{E, A_{13}\} = 0, \quad \{E, A_{12}\} = 0, \quad \{E, B_1\} = 0, \quad \{E, B_2\} = \{E, B_3\} = 0, \quad (55)
\]

\[
\{C, A_{23}\} = 0, \quad \{C, A_{13}\} = 0, \quad \{C, A_{12}\} = 0, \quad \{C, B_1\} = 0, \quad \{C, B_2\} = \{C, B_3\} = 0. \quad (56)
\]

\[
\{E, A_{23}\} = \{E, A_{13}\} = 0, \quad \{E, A_{12}\} = 0, \quad \{E, B_1\} = \{E, B_2\} = 0, \quad \{E, B_3\} = 0, \quad (57)
\]

\[
\{C, A_{23}\} = \{C, A_{13}\} = 0, \quad \{C, A_{12}\} = 0, \quad \{C, B_1\} = \{C, B_2\} = 0, \quad \{C, B_3\} = 0. \quad (58)
\]

\[
\{E, H_+\} = \{E, H_-\} = 0, \quad \{E, H_3\} = 0, \quad \{E, F_+\} = \{E, F_-\} = 0, \quad \{E, F_3\} = 0, \quad (59)
\]

\[
\{C, H_+\} = \{C, H_-\} = 0, \quad \{C, H_3\} = 0, \quad \{C, F_+\} = \{C, F_-\} = 0, \quad \{C, F_3\} = 0. \quad (60)
\]
where \( A_{23}, A_{13}, A_{12} \) are infinitesimal operators of a subgroup of three-dimensional rotations, \( B_1, B_2, B_3 \) are infinitesimal operators of hyperbolic rotations.

2) The field \( \mathbb{F} = \mathbb{R} \). The factorization \( \mathcal{O}_{s_i, t_j} \otimes \cdots \otimes \mathcal{O}_{s_i, t_j} \) of the real Clifford algebra \( \mathcal{O}_{p,q} \) corresponds to a real finite-dimensional representation of the group \( \mathfrak{g}_+ \), with a pair \((l_0, t_1) = \left( \frac{p+q}{4}, 0 \right)\), that is equivalent to a representation of the subgroup \( \text{SO}(3) \) of three-dimensional rotations \( (B_1 = B_2 = B_3 = 0) \). Then there exist two classes of real representations \( \mathfrak{r}_{0,2} \) of the group \( \mathfrak{g}_+ \) corresponding to the algebras \( \mathcal{O}_{p,q} \) with a division ring \( \mathbb{K} \simeq \mathbb{R} \), \( p - q \equiv 0, 2 \) (mod 8), and also there exist two classes of quaternionic representations \( \mathfrak{r}_{4,6} \) of \( \mathfrak{g}_+ \) corresponding to the algebras \( \mathcal{O}_{p,q} \) with a ring \( \mathbb{K} \simeq \mathbb{H} \), \( p - q \equiv 4, 6 \) (mod 8). For the real representations \( \mathfrak{r}_{0,2} \) operators of the discrete subgroup of \( \mathfrak{g}_+ \) defining by the matrices \( W, E, C \) of the fundamental automorphisms of \( \mathcal{O}_{p,q} \) with \( p - q \equiv 0, 2 \) (mod 8) are always commute with all the infinitesimal operators of the representation. In turn, for the quaternionic representations \( \mathfrak{r}_{4,6} \) following relations hold:

\[
[W, A_{23}] = [W, A_{13}] = [W, A_{12}] = 0, \quad [W, H_+] = [W, H_-] = [W, H_3] = 0. \tag{61}
\]

\[
[E, A_{23}] = [E, A_{13}] = [E, A_{12}] = 0, \quad [C, A_{23}] = [C, A_{13}] = [C, A_{12}] = 0, \tag{62}
\]

\[
[E, H_+] = [E, H_-] = [E, H_3] = 0, \quad [C, H_+] = [C, H_-] = [C, H_3] = 0. \tag{63}
\]

\[
[E, A_{23}] = 0, \quad [E, A_{13}] = [E, A_{12}] = 0, \quad [C, A_{23}] = 0, \quad [C, A_{13}] = [C, A_{12}] = 0. \tag{64}
\]

\[
[E, A_{23}] = [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [C, A_{23}] = [C, A_{13}] = 0, \quad [C, A_{12}] = 0, \tag{65}
\]

\[
[E, H_+] = [E, H_-] = 0, \quad [E, H_3] = 0, \quad [C, H_+] = [C, H_-] = 0, \quad [C, H_3] = 0. \tag{66}
\]

\[
[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [C, A_{23}] = 0, \quad [C, A_{13}] = 0, \quad [C, A_{12}] = 0. \tag{67}
\]

**Proof.** 1) Complex representations.

As noted previously, a full representation space of the finite-dimensional representation of the proper Lorentz group \( \mathfrak{g}_+ \) is defined in terms of the minimal left ideal of the algebra \( \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \mathbb{C}_2 \simeq \mathbb{C}_{2k} \). Indeed, in virtue of an isomorphism

\[
\mathbb{C}_{2k} \simeq \mathcal{O}_{p,q}, \tag{68}
\]

where \( \mathcal{O}_{p,q} \) is a Clifford algebra over the field \( \mathbb{F} = \mathbb{R} \) with a division ring \( \mathbb{K} \simeq \mathbb{C} \), \( p - q \equiv 3, 7 \) (mod 8), we have for the minimal left ideal of \( \mathbb{C}_{2k} \) an expression \( S = \mathcal{O}_{p,q} f \), here

\[
f = \frac{1}{2}(1 \pm e_{a_1}) \frac{1}{2}(1 \pm e_{a_2}) \cdots \frac{1}{2}(1 \pm e_{a_t})
\]

is a primitive idempotent of the algebra \( \mathcal{O}_{p,q} \) \cite{12}, and \( e_{a_1}, e_{a_2}, \ldots, e_{a_t} \) are commuting elements with square 1 of the canonical basis of \( \mathcal{O}_{p,q} \) generating a group of order \( 2^t \). The values of \( t \) are defined by a formula \( t = q - r_{q-p} \), where \( r_i \) are the Radon–Hurwitz numbers \cite{13, 14}, values of which form a cycle of the period 8: \( r_{i+8} = r_i + 4 \). The values of all \( r_i \) are
The dimension of the minimal left ideal $S$ is equal to $2^k = 2^{\frac{k+q-1}{2}}$. Therefore, for each finite–dimensional representation of the group $G$ we have $2^k$ copies of the spin space $S_{2^k}$ (full representation space). It should be noted that not all these copies are equivalent to each other, some of them give rise to different reflection groups (see [61]).

In general, all the finite–dimensional representations of group $G$ in the spin space $S_{2^k}$ are reducible. Therefore, there exists a decomposition of the spin space $S_{2^k} \cong S_2 \otimes S_2 \otimes \cdots S_2$ into a direct sum of invariant subspaces $\text{Sym}_{(k_1,0)} \oplus \text{Sym}_{(k_2,0)} \oplus \cdots \oplus \text{Sym}_{(k_s,0)}$,

where $k_1 + k_2 + \cdots + k_s = k$, $k_j \in \mathbb{Z}$. At this point there exists an orthonormal basis with matrices of the form

$$\begin{pmatrix}
A_i^0 & A_i^{1/2} & \cdots & A_i^s \\
A_i^{1/2} & B_i^0 & \cdots & B_i^s \\
& B_i^{1/2} & \cdots & B_i^s \\
& & & B_i^s
\end{pmatrix},$$

where for the matrices $A_i^1, A_i^2, A_i^3, B_i^1, B_i^2, B_i^3$ (matrices of the infinitesimal operators of $G$) in accordance with Gel’fand–Naimark formulas [29, 47]

$$A_{23} \xi_{l,m} = -\frac{i}{2} \sqrt{(l + m + 1)(l - m)} \xi_{l,m+1} - \frac{i}{2} \sqrt{(l + m)(l - m + 1)} \xi_{l,m-1},$$

(69)

$$A_{13} \xi_{l,m} = \frac{1}{2} \sqrt{(l + m)(l - m + 1)} \xi_{l,m-1} - \frac{1}{2} \sqrt{(l + m + 1)(l - m)} \xi_{l,m+1},$$

(70)

$$A_{12} \xi_{l,m} = -im \xi_{l,m},$$

(71)

$$B_1 \xi_{l,m} = -\frac{i}{2} C_l \sqrt{(l - m)(l - m - 1)} \xi_{l-1,m+1} + \frac{i}{2} A_l \sqrt{(l - m)(l + m + 1)} \xi_{l,m+1} - \frac{i}{2} C_{l+1} \sqrt{(l + m + 1)(l + m + 2)} \xi_{l+1,m+1} + \frac{i}{2} A_l \sqrt{(l + m)(l - m + 1)} \xi_{l,m-1} + \frac{i}{2} C_{l+1} \sqrt{(l - m + 1)(l - m + 2)} \xi_{l+1,m-1},$$

(72)
\[ B_2 \xi_{l,m} = -\frac{1}{2} C_l \sqrt{(l + m)(l + m - 1)} \xi_{l-1,m-1} - \frac{1}{2} A_l \sqrt{(l + m)(l - m + 1)} \xi_{l,m-1} - \frac{1}{2} C_{l+1} \sqrt{(l - m + 1)(l - m + 2)} \xi_{l+1,m-1} - \frac{1}{2} A_{l+1} \sqrt{(l - m)(l + m + 1)} \xi_{l,m+1} - \frac{1}{2} C_l \sqrt{(l - m)(l + m + 1)} \xi_{l,m+1} - \frac{1}{2} A_l \sqrt{(l - m)(l + m + 1)} \xi_{l+1,m+1}, \quad (73) \]

\[ B_3 \xi_{l,m} = -i C_l \sqrt{l^2 - m^2} \xi_{l-1,m} + i A_l m \xi_{l,m} + i C_{l+1} \sqrt{(l + 1)^2 - m^2} \xi_{l+1,m}, \quad (74) \]

\[ A_l = \frac{il_0 l_1}{l(l + 1)}, \quad C_l = \frac{i}{l} \sqrt{\frac{(l^2 - l_0^2)(l^2 - l_1^2)}{4l^2 - 1}} \quad (75) \]

\[ m = -l, -l + 1, \ldots, l - 1, l \]

\[ l = l_0, l_0 + 1, \ldots \]

we have

\[ A_{j}^{23} = -i \frac{1}{2} \left[ \begin{array}{cccccc}
0 & \alpha_{-l_j+1} & 0 & \cdots & 0 & 0 \\
\alpha_{-l_j+1} & 0 & \alpha_{-l_j+2} & \cdots & 0 & 0 \\
0 & \alpha_{-l_j+2} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & \alpha_{l_j} \\
0 & 0 & 0 & \cdots & 0 & \alpha_{l_j} \\
\end{array} \right] \quad (76) \]

\[ A_{j}^{13} = \frac{1}{2} \left[ \begin{array}{cccccc}
0 & \alpha_{-l_j+1} & 0 & \cdots & 0 & 0 \\
-\alpha_{-l_j+1} & 0 & \alpha_{-l_j+2} & \cdots & 0 & 0 \\
0 & -\alpha_{-l_j+2} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & \alpha_{l_j} \\
0 & 0 & 0 & \cdots & 0 & -\alpha_{l_j} \\
\end{array} \right] \quad (77) \]

\[ A_{j}^{12} = \left[ \begin{array}{cccccc}
il_j & 0 & 0 & \cdots & 0 & 0 \\
0 & il_j & 0 & \cdots & 0 & 0 \\
0 & 0 & il_j & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -i(l_j - 1) & 0 \\
0 & 0 & 0 & \cdots & 0 & -il_j \\
\end{array} \right] \quad (78) \]
B_j^i = \frac{i}{2}A_j \begin{bmatrix}
0 & \alpha^{-l_j+1} & 0 & \ldots & 0 & 0 \\
\alpha^{-l_j+1} & 0 & \alpha^{-l_j+2} & \ldots & 0 & 0 \\
0 & \alpha^{-l_j+2} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \alpha_{l_j} \\
0 & 0 & 0 & \ldots & \alpha_{l_j} & 0 
\end{bmatrix}
\tag{79}

B_j^2 = \frac{1}{2}A_j \begin{bmatrix}
0 & -\alpha^{-l_j+1} & 0 & \ldots & 0 & 0 \\
\alpha^{-l_j+1} & 0 & -\alpha^{-l_j+2} & \ldots & 0 & 0 \\
0 & \alpha^{-l_j+2} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -\alpha_{l_j} \\
0 & 0 & 0 & \ldots & \alpha_{l_j} & 0 
\end{bmatrix}
\tag{80}

B_j^3 = \frac{1}{2}A_j \begin{bmatrix}
il_j & 0 & 0 & \ldots & 0 & 0 \\
0 & il_j - 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & il_j - 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -i(l_j - 1) & 0 \\
0 & 0 & 0 & \ldots & 0 & -il_j 
\end{bmatrix}
\tag{81}

where \(\alpha_m = \sqrt{(l_j + m)(l_j - m + 1)}\). The formulas (69)–(75) define a finite-dimensional representation of the group \(G^+\) when \(l_j^2 = (l_0 + p)^2\), \(p\) is some natural number, \(l_0\) is an integer or half-integer number, \(l_1\) is an arbitrary complex number. In the case \(l_j^2 \neq (l_0 + p)^2\) we have an infinite-dimensional representation of \(G^+\). We will deal below only with the finite-dimensional representations, because these representations are most useful in physics.

The relation between the numbers \(l_0, l_1\) and the number \(k\) of the factors \(\mathbb{C}_2\) in the product \(\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2\) is given by a following formula

\[(l_0, l_1) = \left(\frac{k}{2}, \frac{k}{2} + 1\right),\]

whence it immediately follows that \(k = l_0 + l_1 - 1\). Thus, we have a complex representation \(\mathfrak{c}^{l_0 + l_1 - 1,0}\) of the proper Lorentz group \(G^+\) in the spin space \(S_{2k}\).

Let us calculate now infinitesimal operators of the fundamental representation \(\mathfrak{c}^{1,0}\) of \(G^+\). The representation \(\mathfrak{c}^{1,0}\) is defined by a pair \((l_0, l_1) = (\frac{1}{2}, \frac{3}{2})\). In accordance with
From (76)–(81) we obtain

\[ A_{23}^{1/2} = -\frac{i}{2} \begin{bmatrix} 0 & \alpha_{1/2} \\ \alpha_{1/2} & 0 \end{bmatrix} = -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{82} \]

\[ A_{13}^{1/2} = \frac{1}{2} \begin{bmatrix} 0 & \alpha_{1/2} \\ -\alpha_{1/2} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{83} \]

\[ A_{12}^{1/2} = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \tag{84} \]

\[ B_{1}^{1/2} = \frac{i}{2} A_{1/2}^{1} \begin{bmatrix} 0 & \alpha_{1/2} \\ \alpha_{1/2} & 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{85} \]

\[ B_{2}^{1/2} = \frac{1}{2} A_{1/2}^{1} \begin{bmatrix} 0 & -\alpha_{1/2} \\ \alpha_{1/2} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{86} \]

\[ B_{3}^{1/2} = \frac{i}{2} A_{1/2}^{1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{87} \]

The operators (82)–(87) satisfy the relations

\[ [A_{23}, A_{13}] = A_{12}, \quad [A_{13}, A_{23}] = A_{23}, \quad [A_{12}, A_{23}] = A_{13}, \]

\[ [B_{1}, B_{2}] = -A_{12}, \quad [B_{2}, B_{3}] = A_{23}, \quad [B_{3}, B_{1}] = A_{13}, \]

\[ [A_{23}, B_{1}] = 0, \quad [A_{13}, B_{2}] = 0, \quad [A_{12}, B_{3}] = 0, \]

\[ [A_{23}, B_{2}] = -B_{3}, \quad [A_{23}, B_{3}] = B_{2}, \]

\[ [A_{13}, B_{3}] = -B_{1}, \quad [A_{13}, B_{1}] = B_{3}, \]

\[ [A_{12}, B_{1}] = B_{2}, \quad [A_{12}, B_{2}] = -B_{1}. \tag{88} \]

From (85)–(87) it is easy to see that there is an equivalence between infinitesimal operators $B_i^{1/2}$ and Pauli matrices:

\[ B_{1}^{1/2} = -\frac{1}{2} \sigma_1, \quad B_{2}^{1/2} = \frac{1}{2} \sigma_2, \quad B_{3}^{1/2} = -\frac{1}{2} \sigma_3. \tag{89} \]

In its turn, from (82)–(84) it follows

\[ A_{23}^{1/2} = -\frac{1}{2} \sigma_2 \sigma_3, \quad A_{13}^{1/2} = -\frac{1}{2} \sigma_1 \sigma_2, \quad A_{12}^{1/2} = \frac{1}{2} \sigma_1 \sigma_2. \tag{90} \]

It is obvious that this equivalence takes place also for high dimensions, that is, there exists an equivalence between infinitesimal operators (76)–(81) and tensor products of the Pauli matrices. In such a way, let us suppose that

\[ A_{23}^j = -\frac{1}{2} \mathcal{E}_a \mathcal{E}_b, \quad A_{13}^j = -\frac{1}{2} \mathcal{E}_a \mathcal{E}_b, \quad A_{12}^j = \frac{1}{2} \mathcal{E}_c \mathcal{E}_a, \tag{91} \]

\[ B_{1}^j = -\frac{1}{2} \mathcal{E}_c, \quad B_{2}^j = \frac{1}{2} \mathcal{E}_a, \quad B_{3}^j = -\frac{1}{2} \mathcal{E}_b. \tag{92} \]
where $E_i$ $(i = a, b, c)$ are $k$-dimensional matrices (the tensor products (8)) and $c < a < b$. It is easy to verify that the operators (91)–(92) satisfy the relations (94). Indeed, for the commutator $[A_{23}, A_{13}]$ we obtain

$$[A_{23}, A_{13}] = A_{23}A_{13} - A_{13}A_{23} = \frac{1}{4}E_aE_bE_cE_b - \frac{1}{4}E_cE_bE_aE_b = -\frac{1}{4}E_aE_c + \frac{1}{4}E_cE_a = \frac{1}{2}E_cE_a = A_{12}$$

and so on. Therefore, the operator set (91)–(92) isomorphically defines the set of infinitesimal operators of the group $\mathfrak{g}_+$. In accordance with Gel’fand–Yaglom approach (27) (see also (29, 47)) an operation of space inversion $P$ commutes with all the operators $A_{ik}$ and anticommutes with all the operators $B_i$:

$$PA_{23}P^{-1} = A_{23}, \quad PA_{13}P^{-1} = A_{13}, \quad PA_{12}P^{-1} = A_{12},$$

$$PB_1P^{-1} = -B_1, \quad PB_2P^{-1} = -B_2, \quad PB_3P^{-1} = -B_3.$$  \tag{93}

Let us consider permutation conditions of the operators (91)–(92) with the matrix $W$ of the automorphism $A \rightarrow A^*$ (space inversion). Since $W = E_1E_2\cdots E_n$ is a volume element of $\mathbb{C}_n$, then $E_a, E_b, E_c \in W, E_i^2 = 1, W^2 = 1$ at $n \equiv 0 \pmod{4}$ and $W^2 = -1$ at $n \equiv 2 \pmod{4}$. Therefore, for the operator $A_{23} \sim -\frac{1}{2}E_aE_b$ we obtain

$$A_{23}W = -(1)^{a+b-2} \frac{1}{2}E_1E_2\cdots E_{a-1}E_{a+1}\cdots E_{b-1}E_{b+1}\cdots E_n,$$

$$WA_{23} = -(1)^{2n-a-b} \frac{1}{2}E_1E_2\cdots E_{a-1}E_{a+1}\cdots E_{b-1}E_{b+1}\cdots E_n,$$

whence it immediately follows a comparison $a + b - 2 \equiv 2n - a - b \pmod{2}$ or $2(a + b) \equiv 2(n+1) \pmod{2}$. Thus, $W$ and $A_{23}$ are always commute. It is easy to verify that analogous conditions take place for the operators $A_{13}, A_{12}$ (except the case $n = 2$). Further, for $B_1 \sim -\frac{1}{2}E_c$ we obtain

$$B_1W = -(1)^{c-1} \frac{1}{2}E_1E_2\cdots E_{c-1}E_{c+1}\cdots E_n,$$

$$WB_1 = -(1)^{n-c} \frac{1}{2}E_1E_2\cdots E_{c-1}E_{c+1}\cdots E_n,$$

that is, $c - 1 \equiv n - c \pmod{2}$ or $n \equiv 2c - 1 \pmod{2}$. Therefore, the matrix $W$ always anticommute with $B_1$ (correspondingly with $B_1, B_2$), since $n \equiv 0 \pmod{2}$. Thus, in full accordance with Gel’fand–Yaglom relations (53) we have\footnote{Except the case of the fundamental representation $\mathfrak{e}^{1,0}$ for which the automorphism group is $\text{Aut}_+(\mathbb{C}_2) = \{1, W, E, C\} = \{\sigma_0, -\sigma_3, \sigma_1, -i\sigma_2\} \approx D_4/\mathbb{Z}_2$. It is easy to verify that the matrix $W \sim \sigma_3$ does not satisfy the relations (53). Therefore, in case of $\mathfrak{e}^{1,0}$ we have an anomalous behaviour of the parity transformation $W \sim P$. This fact will be explained further within quotient representations.}

$$WA_{23}W^{-1} = A_{23}, \quad WA_{13}W^{-1} = A_{13}, \quad WA_{12}W^{-1} = A_{12},$$

$$WB_1W^{-1} = -B_1, \quad WB_2W^{-1} = -B_2, \quad WB_3W^{-1} = -B_3.$$  \tag{94}
where the matrix \( W \) of the automorphism \( \mathcal{A} \rightarrow \mathcal{A}^* \) is an element of an Abelian automorphism group \( \text{Aut}_-(\mathbb{C}_n) \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) with the signature \((+, +, +)\) at \( n \equiv 0 \) (mod 4) and also a non–Abelian automorphism group \( \text{Aut}_+(\mathbb{C}_n) \cong Q_4/\mathbb{Z}_2 \) with the signature \((- , - , -)\) at \( n \equiv 2 \) (mod 4), here \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) is a Gauss–Klein group and \( Q_4 \) is a quaternionic group (see Theorem 9 in [60]).

Let us consider now permutation conditions of the operators (91) – (92) with the matrix \( E \) of the antiautomorphism \( \mathcal{A} \rightarrow \tilde{\mathcal{A}} \) (time reversal). Over the field \( \mathbb{F} = \mathbb{C} \) the matrix \( E \) has two forms (Theorem 9 in [60]): 1) \( E = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_m \) at \( m \equiv 1 \) (mod 2), the group \( \text{Aut}_+(\mathbb{C}_n) \cong Q_4/\mathbb{Z}_2 \); 2) \( E = \mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_n \) at \( m \equiv 0 \) (mod 2), the group \( \text{Aut}_+(\mathbb{C}_n) \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \). Obviously, in both cases \( W = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_m \mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_n \), where the matrices \( \mathcal{E}_i \) are symmetric for \( 1 < i \leq m \) and skewsymmetric for \( m < i \leq n \).

So, let \( E = \mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_n \) be a matrix of \( \mathcal{A} \rightarrow \tilde{\mathcal{A}}, \ m \equiv 0 \) (mod 2). Let us assume that \( \mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c \in \mathcal{E} \), then for the operator \( A_{23} \sim -\frac{1}{2} \mathcal{E}_a \mathcal{E}_b \) we obtain

\[
A_{23}E = -(-1)^{b+a-2} \frac{1}{2} \mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_{a-1} \mathcal{E}_{a+1} \cdots \mathcal{E}_{b-1} \mathcal{E}_{b+1} \cdots \mathcal{E}_n,
\]

\[
EA_{23} = -(-1)^{2m-a-b} \frac{1}{2} \mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_{a-1} \mathcal{E}_{a+1} \cdots \mathcal{E}_{b-1} \mathcal{E}_{b+1} \cdots \mathcal{E}_n,
\]

(95)

that is, \( b + a - 2 \equiv 2m - a - b \) (mod 2) and in this case the matrix \( E \) commutes with \( A_{23} \) (correspondingly with \( A_{13}, A_{12} \)). For the operator \( B_1 \) we have (analogously for \( B_2 \) and \( B_3 \)):

\[
B_1E = -(-1)^{c-1} \frac{1}{2} \mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_{c-1} \mathcal{E}_{c+1} \cdots \mathcal{E}_n,
\]

\[
EB_1 = -(-1)^{m-c} \frac{1}{2} \mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_{c-1} \mathcal{E}_{c+1} \cdots \mathcal{E}_n,
\]

(96)

that is, \( m \equiv 2c - 1 \) (mod 2) and, therefore, the matrix \( E \) in this case always anticommutes with \( B_i \), since \( m \equiv 0 \) (mod 2). Thus,

\[
EA_{23}E^{-1} = A_{23}, \quad EA_{13}E^{-1} = A_{13}, \quad EA_{12}E^{-1} = A_{12},
\]

\[
EB_1E^{-1} = -B_1, \quad EB_2E^{-1} = -B_2, \quad EB_3E^{-1} = -B_3.
\]

(97)

Let us assume now that \( \mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c \not\in \mathcal{E} \), then

\[
A_{23}E = (-1)^{2m} EA_{23},
\]

(98)

that is, in this case \( E \) and \( A_{ik} \) are always commute. For the hyperbolic operators we obtain

\[
B_iE = (-1)^m EB_i \quad (i = 1, 2, 3)
\]

(99)

and since \( m \equiv 0 \) (mod 2), then \( E \) and \( B_i \) are also commute. Therefore,

\[
[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [E, B_1] = 0, \quad [E, B_2] = 0, \quad [E, B_3] = 0.
\]

(100)
Assume now that $E_a, E_b \in E$ and $E_c \not\in E$. Then in accordance with (93) and (99) the matrix $E$ commutes with $A_{23}$ and $B_1$, and according to (100) anticommutes with $B_2$ and $B_3$. For the operator $A_{13}$ we find

$$A_{13}E = -(-1)^{b-1} \frac{1}{2} E_a E_{m+1} E_{m+2} \cdots E_{b-1} E_{b+1} \cdots E_n,$$

$$EA_{23} = -(-1)^{2m-b} \frac{1}{2} E_a E_{m+1} E_{m+2} \cdots E_{b-1} E_{b+1} \cdots E_n,$$ (101)

that is, $2m \equiv 2b - 1 \pmod{2}$ and, therefore, in this case $E$ always anticommutes with $A_{13}$ and correspondingly with the operator $A_{12}$ which has the analogous structure. Thus,

$$[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [E, B_1] = 0, \quad [E, B_2] = 0, \quad [E, B_3] = 0.$$ (102)

If we take $E_a \in E, E_b, E_c \not\in E$, then for the operator $A_{23}$ it follows that

$$A_{23}E = -(-1)^{a-2} \frac{1}{2} E_a E_{m+1} E_{m+2} \cdots E_{a-1} E_{a+1} \cdots E_n,$$

$$EA_{23} = -(-1)^{2m-a} \frac{1}{2} E_a E_{m+1} E_{m+2} \cdots E_{a-1} E_{a+1} \cdots E_n.$$ (103)

that is, $a - 2 \equiv 2m - a - 1 \pmod{2}$ or $2m \equiv 2a - 1 \pmod{2}$. Therefore, $E$ anticommutes with $A_{23}$. Further, in virtue of (98) $E$ commutes with $A_{13}$ and anticommutes with $A_{12}$ in virtue of (101). Correspondingly, from (99) and (100) it follows that $E$ commutes with $B_1, B_3$ and anticommutes with $B_2$. Thus,

$$[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [E, B_1] = 0, \quad [E, B_2] = 0, \quad [E, B_3] = 0.$$ (104)

Cyclic permutations of the indices in $E_i, E_j \in E, E_k \not\in E$ and $E_i \in E, E_j, E_k \not\in E$, $i, j, k = \{a, b, c\}$, give the following relations

$$[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [E, B_1] = 0, \quad [E, B_2] = 0, \quad [E, B_3] = 0, \quad E_a, E_c \in E, E_b \not\in E.$$ (105)

$$[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [E, B_1] = 0, \quad [E, B_2] = 0, \quad [E, B_3] = 0, \quad E_b, E_c \in E, E_a \not\in E.$$ (106)

$$[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [E, B_1] = 0, \quad [E, B_2] = 0, \quad [E, B_3] = 0, \quad E_b \in E, E_a, E_c \not\in E.$$ (107)

$$[E, A_{23}] = 0, \quad [E, A_{13}] = 0, \quad [E, A_{12}] = 0, \quad [E, B_1] = 0, \quad [E, B_2] = 0, \quad [E, B_3] = 0, \quad E_c \in E, E_a, E_b \not\in E.$$ (108)

Let us consider now the matrix $E$ of the group $\text{Aut}_+(\mathbb{C}_n) \simeq Q_4/\mathbb{Z}_2$. In this case $E = E_1 E_2 \cdots E_m, m \equiv 1 \pmod{2}$. At $E_a, E_b, E_c \in E$ from (95) it follows that $2(a + b) \equiv
2(m + 1) (mod 2), therefore, in this case $E$ always commutes with $A_{ik}$. In its turn, from (90) it follows that $E$ always commutes with $B_i$, since $m \equiv 1$ (mod 2). Thus, we have the relations (100). At $E_a, E_b, E_c \not\in E$ (except the cases $n = 2$ and $n = 4$) from (78) it follows that $E$ always commutes with $A_{ik}$, and from (93) it follows that $E$ always anticommutes with $B_i$. Therefore, for the case $E_a, E_b, E_c \not\in E$ we have the relations (97). Analogously, at $E_a, E_b \in E, E_c \not\in E$ in accordance with (75) and (76) $E$ commutes with $A_{23}$ and $B_1, B_3$, and in accordance with (101) and (99) anticommutes with $A_{13}, A_{12}$ and $B_1$. Therefore, for this case we have the relations (108). At $E_a \in E, E_b, E_c \not\in E$ from (103) it follows that $E$ anticommutes with $A_{23}$. In virtue of (78) $E$ commutes with $A_{13}$ and anticommutes with $A_{12}$ in virtue of (101). Correspondingly, from (93) and (96) it follows that $E$ anticommutes with $B_1, B_3$ and commutes with $B_2$. Therefore, for the case $E_a \in E, E_b, E_c \not\in E$ we have the relations (100). Further, cyclic permutations of the indices in $E_i, E_j, E_k \in E$ and $E_i \in E, E_j, E_k \not\in E, i, j, k = \{a, b, c\}$, give for $E_a, E_c \in E, E_b \not\in E$ the relations (107), for $E_b, E_c \in E, E_a \not\in E$ the relations (104), for $E_b \in E, E_a, E_c \not\in E$ the relations (105) and for $E_c \in E, E_a, E_b \not\in E$ the relations (102).

Let us consider now the permutation conditions of the operators (91)–(92) with the matrix $C$ of the antiautomorphism $\mathcal{A} \to \tilde{\mathcal{A}}^*$ (full reflection). Over the field $F = C$ the matrix $C$ has two different forms (Theorem 9 in [90]): 1) $C = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m$ at $m \equiv 0$ (mod 2), the group $\text{Aut}_{-}(C_n) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$; 2) $C = \varepsilon_{m+1} \varepsilon_{m+2} \cdots \varepsilon_n$ at $m \equiv 1$ (mod 2), the group $\text{Aut}_{+}(C_n) \simeq Q_4/\mathbb{Z}_2$. So, let $C = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m$ be a matrix of $\mathcal{A} \to \tilde{\mathcal{A}}^*, m \equiv 1$ (mod 2), then by analogy with the matrix $E = \varepsilon_{m+1} \varepsilon_{m+2} \cdots \varepsilon_n$ of $\mathcal{A} \to \tilde{\mathcal{A}}^*, m \equiv 0$ (mod 2), we have for $C$ the relations of the form (77)–(108). In its turn, the matrix $C = \varepsilon_{m+1} \varepsilon_{m+2} \cdots \varepsilon_n$ is analogous to $E = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m, m \equiv 1$ (mod 2), therefore, in this case we have also the relations (77)–(108) for $C$.

Now we have all possible combinations of permutation relations between the matrices of infinitesimal operators (91)–(92) of the proper Lorentz group $\mathbf{G}_+$ and matrices of the fundamental automorphisms of the complex Clifford algebra $\mathbb{C}_n$ associated with the complex representation $\mathfrak{c}^{0+h-1,0}$ of $\mathbf{G}_+$. It is obvious that the relations (94) take place for any representation $\mathfrak{c}^{0+h-1,0}$ of the group $\mathbf{G}_+$. Further, if $\mathfrak{c}^{0+h-1,0}$ with $2(l_0 + l_1 - 1) \equiv 0$ (mod 4) and if $\text{Aut}_-(C_n) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with $E = \varepsilon_{m+1} \varepsilon_{m+2} \cdots \varepsilon_n, C = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m$, $m \equiv 0$ (mod 2), then at $E_a, E_b, E_c \in E$ ($E_a, E_b, E_c \not\in C$) we have the relations (77) for $E$ and the relations of the form (100) for $C$ and, therefore, we have the relations (77) and (108) of the present Theorem. Correspondingly, at $E_a, E_b, E_c \not\in E$ ($E_a, E_b, E_c \not\in C$) we have the relations (100) for $E$ and the relations of the form (77) for $C$, that is, the relations (11) and (12) of Theorem. At $E_a, E_b \in E, E_c \not\in E$ ($E_a, E_b \not\in C, E_c \not\in C$) we obtain the relations (102) for $E$ and the relations of the form (108) for $C$ ((15)–(16) in Theorem) and so on. Analogously, if $\mathfrak{c}^{0+h-1,0}$ with $2(l_0 + l_1 - 1) \equiv 2$ (mod 4) and if $\text{Aut}_+(C_n) \simeq Q_4/\mathbb{Z}_2$ with $E = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_m$, $C = \varepsilon_{m+1} \varepsilon_{m+2} \cdots \varepsilon_n, m \equiv 1$ (mod 2), then at $E_a, E_b, E_c \in E$ ($E_a, E_b, E_c \not\in C$) we have the relations (100) for $E$ and the relations of the form (77) for $C$ (relations (38) and (37) in Theorem) and so on.

Further, let us consider the following combinations of $A_{ik}$ and $B_i$ (rising and lowering operators):

\[
H_+ = iA_{23} - A_{13}, \quad H_- = iA_{23} + A_{13}, \quad H_3 = iA_{13},
\]
\[
F_+ = iB_1 - B_2, \quad F_- = iB_1 + B_2, \quad F_3 = iB_3.
\]

(109)
satisfying in virtue of (88) the relations

\[
\begin{align*}
[H_+, H_3] &= -H_+, & [H_-, H_3] &= H_-, & [H_+, H_-] &= 2H_3, \\
[H_+, F_+] &= [H_-, F_-] &= [H_3, F_3] &= 0, \\
[F_+, F_3] &= -H_+, & [F_-, F_3] &= H_-, & [F_+, F_-] &= -2H_3, \\
[H_+, F_3] &= F_+, & [H_-, F_3] &= -F_-, & [H_-, F_+] &= 2F_3, \\
[F_+, H_3] &= -F_+, & [F_-, H_3] &= F_.
\end{align*}
\] (110)

It is easy to see that for \( W \) from (94) and (103) we have always the relations (35). Further, for \( \text{Aut}_-(\mathbb{C}_n) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) at \( \mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c \in \mathbb{E} \) from (97) and (100) in virtue of (108) it follow the relations (38) and (40). Analogously, at \( \mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c \notin \mathbb{E} \) from (100) and (77) we obtain the relations (14) and (24). In contrast with this, at \( \mathcal{E}_a, \mathcal{E}_b \in \mathbb{E}, \mathcal{E}_c \notin \mathbb{E} \) the combinations (102) and (108) do not form permutation relations with operators \( H_{+,3} \) and \( F_{+,3} \), since \( \mathbb{E} \) and \( \mathbb{C} \) commute with \( A_{23} \) and anticommute with \( A_{13} \), and \( B_1 \) commutes with \( \mathbb{C} \) and anticommutes with \( \mathbb{E} \) (inverse relations take place for \( B_2 \)). Other two relations (57)–(60) and (19)–(22) for \( \text{Aut}_-(\mathbb{C}_n) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) correspond to \( \mathcal{E}_a, \mathcal{E}_c \in \mathbb{E}, \mathcal{E}_b \notin \mathbb{E} \) and \( \mathcal{E}_b \in \mathbb{E}, \mathcal{E}_a, \mathcal{E}_c \notin \mathbb{E} \). It is easy to see that in such a way for the group \( \text{Aut}_+(\mathbb{C}_n) \simeq Q_4/\mathbb{Z}_2 \) we obtain the relations (83)–(84), (89)–(91), (14)–(16) and (14)–(16) correspondingly for \( \mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c \in \mathbb{E}, \mathcal{E}_a, \mathcal{E}_b \notin \mathbb{E}, \mathcal{E}_a, \mathcal{E}_c \in \mathbb{E}, \mathcal{E}_b \notin \mathbb{E} \) and \( \mathcal{E}_b \in \mathbb{E}, \mathcal{E}_a, \mathcal{E}_c \notin \mathbb{E} \).

In accordance with (29) a representation conjugated to \( \mathbb{C}^{\mu_0+\delta_1-1,0} \) is defined by a pair

\[
(l_0, l_1) = \left( -\frac{r}{2}, 0 \right) + \left( 0, \frac{r}{2} \right) + 1,
\]

that is, this representation has a form \( \mathbb{C}^{\mu_0-\delta_1+1} \). In its turn, a representation conjugated to fundamental representation \( \mathbb{C}^{1,0} \) is \( \mathbb{C}^{0,-1} \). Let us find infinitesimal operators of the representation \( \mathbb{C}^{0,-1} \). At \( l_0 = -1/2 \) and \( l_1 = 3/2 \) from (69)–(75) and (76)–(81) we obtain

\[
\begin{align*}
A_{23}^{-1/2} &= -\frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & A_{13}^{-1/2} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & A_{12}^{-1/2} &= \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \\
B_1^{-1/2} &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & B_2^{-1/2} &= \frac{1}{2} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, & B_3^{-1/2} &= \frac{i}{2} \begin{bmatrix} 0 & 0 \\ -i & i \end{bmatrix}.
\end{align*}
\]

Or

\[
\begin{align*}
A_{23}^{-1/2} &= -\frac{1}{2} \sigma_2 \sigma_3, & A_{13}^{-1/2} &= -\frac{1}{2} \sigma_1 \sigma_3, & A_{12}^{-1/2} &= \frac{1}{2} \sigma_1 \sigma_2, \\
B_1^{-1/2} &= \frac{1}{2} \sigma_1, & B_2^{-1/2} &= \frac{1}{2} \sigma_2, & B_3^{-1/2} &= \frac{1}{2} \sigma_3.
\end{align*}
\] (111)

It is easy to see that operators (111) differ from (89)–(91) only in the sign at the operators \( B_i \) of the hyperbolic rotations. This result is a direct consequence of the well-known definition of the group \( SL(2; \mathbb{C}) \) as a complexification of the special unimodular group \( SU(2) \) (see (88)). Indeed, the group \( SL(2; \mathbb{C}) \) has six parameters \( a_1, a_2, a_3, ia_1, ia_2, ia_3 \), where \( a_1, a_2, a_3 \in SU(2) \). It is easy to verify that operators (111) satisfy the relations...
Therefore, as in case of the representation \( \mathcal{C}^{(0,l_0-l_1+1)} \), infinitesimal operators of the conjugated representation \( \mathcal{C}^{(0,l_0-l_1)} \) are defined as follows:

\[
A_{23}'' \sim \frac{1}{2} \mathcal{E}_a \mathcal{E}_b, \quad A_{13}'' \sim \frac{1}{2} \mathcal{E}_c \mathcal{E}_b, \quad A_{12}'' \sim \frac{1}{2} \mathcal{E}_c \mathcal{E}_a,
\]

\[
B_1'' \sim \frac{1}{2} \mathcal{E}_c, \quad B_2'' \sim \frac{1}{2} \mathcal{E}_a, \quad B_3'' \sim \frac{1}{2} \mathcal{E}_b,
\]

(112)

where \( \mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c \) are tensor products of the form \( \mathcal{O} \). It is not difficult to verify that operators (112) satisfy the relations \((88)\), and their linear combinations satisfy the relations \((110)\). Since the structure of the operators (112) is analogous to the structure of the operators \((21)-(22)\), then all the permutation conditions between the operators of discrete symmetries and operators \((21)-(22)\) of the representation \( \mathcal{C}^{(0,l_0-l_1+1)} \) are valid also for the conjugated representation \( \mathcal{C}^{(0,l_0-l_1+1)} \) and, obviously, for a representation \( \mathcal{C}^{(0,l_0-l_1+1)} \).

2) Real representations.

As known [29], if an irreducible representation of the proper Lorentz group \( \mathfrak{G}_+ \) is defined by the pair \((l_0, l_1)\), then a conjugated representation is also irreducible and defined by a pair \((l_0, -l_1)\). Hence it follows that the irreducible representation is equivalent to its conjugated representation only in case when this representation is defined by a pair \((0, l_1)\) or \((l_0, 0)\), that is, either of the two numbers \(l_0\) and \(l_1\) is equal to zero. We assume that \(l_1 = 0\). Then, for the complex Clifford algebra \( \mathbb{C}_n (\mathbb{C}_n) \) associated with the representation \( \mathcal{C}^{(0,l_0-l_1+1)} \) of the group \( \mathfrak{G}_+ \) the equivalence of the representation to its conjugated representation induces a relation \( \mathbb{C}_n = \mathbb{C}_n \), which, obviously, is fulfilled only in case when the algebra \( \mathbb{C}_n = \mathbb{C} \otimes \mathcal{O}_{p,q} \) is reduced into its real subalgebra \( \mathcal{O}_{p,q} \) \( (p + q = n) \). Thus, a restriction of the complex representation \( \mathcal{C}^{(0,l_0-l_1+1)} \) of the group \( \mathfrak{G}_+ \) onto a real representation, \((l_0, l_1) \rightarrow (l_0, 0)\), induces a restriction \( \mathbb{C}_n \rightarrow \mathcal{O}_{p,q} \). Further, over the field \( \mathbb{F} = \mathbb{R} \) at \( p + q \equiv 0 \pmod{2} \) there are four types of real algebras \( \mathcal{O}_{p,q} \): two types \( p - q \equiv 0, 2 \pmod{8} \) with a real division ring \( \mathbb{K} \cong \mathbb{R} \) and two types \( p - q \equiv 4, 6 \pmod{8} \) with a quaternionic division ring \( \mathbb{K} \cong \mathbb{H} \). Thus, we have four classes of the real representations of the group \( \mathfrak{G}_+ \):

\[
\mathfrak{R}^{(0)}_0 \leftrightarrow \mathcal{O}_{p,q}, \quad p - q \equiv 0 \pmod{8}, \quad \mathbb{K} \cong \mathbb{R},
\]

\[
\mathfrak{R}^{(0)}_2 \leftrightarrow \mathcal{O}_{p,q}, \quad p - q \equiv 2 \pmod{8}, \quad \mathbb{K} \cong \mathbb{R},
\]

\[
\mathfrak{H}^{(0)}_4 \leftrightarrow \mathcal{O}_{p,q}, \quad p - q \equiv 4 \pmod{8}, \quad \mathbb{K} \cong \mathbb{H},
\]

\[
\mathfrak{H}^{(0)}_6 \leftrightarrow \mathcal{O}_{p,q}, \quad p - q \equiv 6 \pmod{8}, \quad \mathbb{K} \cong \mathbb{H},
\]

(113)

We will call the representations \( \mathfrak{H}^{(0)}_4 \) and \( \mathfrak{H}^{(0)}_6 \) are quaternionic representations of the group \( \mathfrak{G}_+ \). It is not difficult to see that for the real representations with the pair \((l_0, 0)\) all the coefficients \( A_l = il_0 l_1 / l(l + 1) \) are equal to zero, since \( l_1 = 0 \). Therefore, all the infinitesimal operators \( B_1'', B_2'', B_3'' \) (see formulas \((79)-(81)\)) of hyperbolic rotations are also equal to zero. Hence it follows that the restriction \((l_0, l_1) \rightarrow (l_0, 0)\) induces a restriction of the group \( \mathfrak{G}_+ \) onto its subgroup \( SO(3) \) of three-dimensional rotations. Thus, real representations with the pair \((l_0, 0)\) are representations of the subgroup \( SO(3) \). This result directly follows from the complexification of \( SU(2) \) which is equivalent to \( SL(2; \mathbb{C}) \). Indeed, the
parameters $a_1, a_2, a_3$ compose a real part of $SL(2; \mathbb{C})$ which under complex conjugation remain unaltered, whereas the parameters $\pm ia_1, \pm ia_2, \pm ia_3$ of the complex part of $SL(2; \mathbb{C})$ under complex conjugation are mutually annihilate for the representations with the pairs $(l_0, l_1)$ and $\pm (l_0, -l_1)$.

Let us find a relation of the number $l_0$ with dimension of the real algebra $\mathcal{O}_{p,q}$. If $p + q \equiv 0 \pmod{2}$ and $\omega^2 = e_{12}^{p+q} = 1$, then $\mathcal{O}_{p,q}$ is called positive ($\mathcal{O}_{p,q} > 0$ at $p - q \equiv 0, 4 \pmod{8}$) and correspondingly negative if $\omega^2 = -1$ ($\mathcal{O}_{p,q} < 0$ at $p - q \equiv 2, 6 \pmod{8}$). Further, in accordance with Karoubi Theorem [38, Prop. 3.16] it follows that if $\mathcal{O}(V, Q) > 0$, and dim $V$ is even, then $\mathcal{O}(V \oplus V', Q \oplus Q') \simeq \mathcal{O}(V, Q) \otimes \mathcal{O}(V', Q')$, and also if $\mathcal{O}(V, Q) < 0$, and dim $V$ is even, then $\mathcal{O}(V \oplus V', Q \oplus Q') \simeq \mathcal{O}(V, Q) \otimes \mathcal{O}(V', -Q')$, where $V$ is a vector space associated with $\mathcal{O}_{p,q}$, $Q$ is a quadratic form of $V$. Using the Karoubi Theorem we obtain for the algebra $\mathcal{O}_{p,q}$ a following factorization

$$\mathcal{O}_{p,q} \simeq \mathcal{O}_{s_i t_j} \otimes \mathcal{O}_{s_i t_j} \otimes \cdots \otimes \mathcal{O}_{s_i t_j}$$

(114)

where $s_i, t_j \in \{0, 1, 2\}$. For example, there are two different factorizations $\mathcal{O}_{1,1} \otimes \mathcal{O}_{0,2}$ and $\mathcal{O}_{1,1} \otimes \mathcal{O}_{2,0}$ for the spacetime algebra $\mathcal{O}_{1,3}$ and Majorana algebra $\mathcal{O}_{3,1}$. It is obvious that $l_0 = r/2$ and $n = 2r = p + q = 4l_0$, therefore, $l_0 = (p + q)/4$.

So, we begin with the representation of the class $\mathfrak{H}_0$. In accordance with Theorem 4 in [11] for the algebra $\mathcal{O}_{p,q}$ of the type $p - q \equiv 0 \pmod{8}$ a matrix of the automorphism $\mathcal{A} \to \mathcal{A}^*$ has a form $W = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_{p+q}$ and $W^2 = 1$. It is obvious that for $\mathcal{E}_a^2 = \mathcal{E}_b^2 = \mathcal{E}_c^2 = 1$ permutation conditions of the matrix $W$ with $A_{ik}$ are analogous to (111), that is, $W$ always commutes with the operators $A_{ik}$ of the subgroup $SO(3)$. It is sufficient to consider permutation conditions of $W$ with the operators $A_{ik}$ of $SO(3)$ only, since $B_i = 0$ for the real representations. In this case the relations (88) take a form

$$[A_{23}, A_{13}] = A_{12}, \quad [A_{13}, A_{12}] = A_{23}, \quad [A_{12}, A_{23}] = A_{13}.$$  

(115)

Assume now that $\mathcal{E}_a^2 = \mathcal{E}_b^2 = \mathcal{E}_c^2 = -1$, then it is easy to verify that operators

$$A_{23} \sim \frac{1}{2} \mathcal{E}_a \mathcal{E}_b, \quad A_{13} \sim \frac{1}{2} \mathcal{E}_c \mathcal{E}_b, \quad A_{12} \sim -\frac{1}{2} \mathcal{E}_c \mathcal{E}_a$$

(116)

satisfy the relations (113) and commute with the matrix $W$ of $\mathcal{A} \to \mathcal{A}^*$. It is easy to see that at $\mathcal{E}_a^2 = -1, \mathcal{E}_b^2 = \mathcal{E}_c^2 = 1$ and $\mathcal{E}_a^2 = \mathcal{E}_b^2 = -1, \mathcal{E}_c^2 = 1$ (i, j, k = \{a, b, c\}) the operators $A_{ik}$ do not satisfy the relations (113). Therefore, there exist only two possibilities $\mathcal{E}_a^2 = \mathcal{E}_b^2 = \mathcal{E}_c^2 = 1$ and $\mathcal{E}_a^2 = \mathcal{E}_b^2 = \mathcal{E}_c^2 = -1$ corresponding to the operators (91) and (116), respectively.

Further, for the type $p - q \equiv 0 \pmod{8}$ ($p = q = m$) at $E = \mathcal{E}_{p+1} \mathcal{E}_{p+2} \cdots \mathcal{E}_{p+q}$ and $C = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_p$ there exist Abelian groups $\text{Aut}_-(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the signature $(+, +, +)$ and $\text{Aut}_-(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4$ with $(+, -, -)$ correspondingly at $p, q \equiv 0 \pmod{4}$ and $p, q \equiv 2 \pmod{4}$, and also at $E = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_p$ and $C = \mathcal{E}_{p+1} \mathcal{E}_{p+2} \cdots \mathcal{E}_{p+q}$ there exist non–Abelian groups $\text{Aut}_+(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(+, -, +)$ and $\text{Aut}_+(\mathcal{O}_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(+, +, -)$ correspondingly at $p, q \equiv 3 \pmod{4}$ and $p, q \equiv 1 \pmod{4}$ (Theorem 4 in [11]). Besides, for the algebras $\mathcal{O}_{8t,0}$ of the type $p - q \equiv 0 \pmod{8}, t = 1, 2, \ldots$, the matrices $E \sim 1$, $C \sim \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_p$ and $W$ form an Abelian group $\text{Aut}(-\mathcal{O}_{p,0}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Correspondingly,
for the algebras $C_{0,8t}$ of the type $p - q \equiv 0 \pmod{8}$ the matrices $E \sim E_1 E_2 \cdots E_q$, $C \sim I$ and $W$ also form the group $\text{Aut}_-(C_{0,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

So, let $E = E_{m+1} E_{m+2} \cdots E_{2m}$ and $C = E_1 E_2 \cdots E_m$ be matrices of $A \to \tilde{A}$ and $A \to \tilde{A}^*$, $p = q = m$, $m \equiv 0 \pmod{2}$. Then for the operators (117) we obtain

$$A_{ik} E = (-1)^{2m} E A_{ik}, \quad (117)$$

$$A_{ik} C = \pm (-1)^{i+j-1} \frac{1}{2} \sigma(i) \sigma(j) E_{i+1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_{2m},$$

$$C A_{ik} = \pm (-1)^{2m-i-j} \frac{1}{2} \sigma(i) \sigma(j) E_{i+1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_{2m}, \quad (118)$$

since $E_a, E_b, E_c \not\in E$ and a function $\sigma(n) = \sigma(m-n)$ has a form

$$\sigma(n) = \begin{cases} 
-1 & \text{if } n \leq 0 \\
+1 & \text{if } n > 0 
\end{cases}$$

Analogously, for the operators (116) we find

$$A_{ik} E = \pm (-1)^{i+j-1} \frac{1}{2} \sigma(i) \sigma(j) E_{m+1} E_{m+2} \cdots E_{i-1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_{2m},$$

$$E A_{ik} = \pm (-1)^{2m-i-j} \frac{1}{2} \sigma(i) \sigma(j) E_{m+1} E_{m+2} \cdots E_{i-1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_{2m}, \quad (119)$$

$$A_{ik} C = (-1)^{2m} C A_{ik}. \quad (120)$$

It is easy to see that in both cases the matrices $E$ and $C$ always commute with the operators $A_{ik}$. Therefore, for the type $p - q \equiv 0 \pmod{8}$ at $p, q \equiv 0 \pmod{4}$ and $p, q \equiv 2 \pmod{4}$ the elements of the Abelian groups $\text{Aut}_-(C_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ and $\text{Aut}_+(C_{p,q}) \simeq \mathbb{Z}_4$ with $(+, -, +)$ and $(-, +, -)$ are always commute with infinitesimal operators $A_{ik}$ of $SO(3)$. Correspondingly, for the algebra $C_{8t,0}$ the elements of $\text{Aut}_-(C_{8t,0}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ are also commute with $A_{ik}$. The analogous statement takes place for other degenerate case $\text{Aut}_-(C_{0,8t}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$. In the case of non–Abelian groups $\text{Aut}_+(C_{p,q}) \simeq D_4/\mathbb{Z}_2$ (signatures $(+, -, +)$ and $(-, +, -)$) we obtain the same permutation conditions as (117)–(120), that is, in this case the matrices of the fundamental automorphisms always commute with $A_{ik}$. Thus, for the real representation of the class $O^{\theta}_0$ operators of the discrete subgroup always commute with all the infinitesimal operators of $SO(3)$.

Further, for the real representation of the class $O^{\theta}_2$, type $p - q \equiv 2 \pmod{8}$, at $E = E_{p+1} E_{p+2} \cdots E_{p+q}$ and $C = E_1 E_2 \cdots E_p$ there exist Abelian groups $\text{Aut}_-(C_{p,q}) \simeq \mathbb{Z}_4$ with $(+, -, +)$ and $\text{Aut}_+(C_{p,q}) \simeq \mathbb{Z}_4$ with $(-, +, -)$ correspondingly at $p \equiv 0 \pmod{4}$, $q \equiv 2 \pmod{4}$ and $p \equiv 2 \pmod{4}$, $q \equiv 0 \pmod{4}$, and also at $E = E_1 E_2 \cdots E_p$ and $C = E_{p+1} E_{p+2} \cdots E_{p+q}$ there exist non–Abelian groups $\text{Aut}_+(C_{p,q}) \simeq Q_4/\mathbb{Z}_2$ with $(+, -, -)$ and $\text{Aut}_+(C_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(-, +, -)$ correspondingly at $p \equiv 3 \pmod{4}$, $q \equiv 1 \pmod{4}$ and $p \equiv 1 \pmod{4}$, $q \equiv 3 \pmod{4}$ (Theorem 4 in [61]). So, for the Abelian groups at $E_a, E_b, E_c \not\in E$ (operators (117)) we obtain

$$A_{ik} E = (-1)^{2q} E A_{ik}, \quad (121)$$
\[
A_{ik} C = \pm (-1)^{i+j-2} \frac{1}{2} \sigma(i)\sigma(j) E_1 E_2 \cdots E_{i-1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_p,
\]
\[
C A_{ik} = \pm (-1)^{2p-i-j} \frac{1}{2} \sigma(i)\sigma(j) E_1 E_2 \cdots E_{i-1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_p.
\] (122)

Correspondingly, for the operators (116) \((E_a, E_b, E_c \in E)\) we have
\[
A_{ik} E = \pm (-1)^{i+j-2} \frac{1}{2} \sigma(i)\sigma(j) E_{p+1} E_{p+2} \cdots E_{i-1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_{p+q},
\]
\[
E A_{ik} = \pm (-1)^{2p-i-j} \frac{1}{2} \sigma(i)\sigma(j) E_{p+1} E_{p+2} \cdots E_{i-1} E_{i+1} \cdots E_{j-1} E_{j+1} \cdots E_{p+q},
\] (123)
\[
A_{ik} C = (-1)^{2p} C A_{ik}.
\] (124)

The analogous relations take place for the non–Abelian groups. From (121)–(124) it is easy to see that the matrices \(E\) and \(C\) always commute with \(A_{ik}\). Therefore, for the real representation of the class \(R^0\) operators of the discrete subgroup always commute with all the infinitesimal operators of \(SO(3)\).

Let us consider now quaternionic representations. Quaternionic representations of the classes \(R^0\) and \(R^0\), types \(p - q \equiv 4, 6 \pmod{8}\), in virtue of the more wide ring \(K \simeq \mathbb{H}\) have a more complicated structure of the reflection groups than in the case of \(K \simeq \mathbb{R}\). Indeed, if \(E = E_{j_1} E_{j_2} \cdots E_{j_k}\) is a product of \(k\) skewsymmetric matrices (among which \(l\) matrices have ‘+’-square and \(t\) matrices have ‘−’-square) and \(C = E_{i_1} E_{i_2} \cdots E_{i_{p+q-k}}\) is a product of \(p + q - k\) skewsymmetric matrices (among which there are \(h\) ‘+’-squares and \(g\) ‘−’-squares), then at \(k \equiv 0 \pmod{2}\) for the type \(p - q \equiv 4 \pmod{8}\) there exist Abelian groups \(\text{Aut}_-(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2\) and \(\text{Aut}_-(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4\) with \((+, -, -)\) if correspondingly \(l - t, h - g \equiv 0, 1, 4, 5 \pmod{8}\) and \(l - t, h - g \equiv 2, 3, 6, 7 \pmod{8}\), and also at \(k \equiv 1 \pmod{2}\) for the type \(p - q \equiv 6 \pmod{8}\) there exist \(\text{Aut}_-(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4\) with \((-, +, -)\) and \(\text{Aut}_-(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_4\) with \((-, -, +)\) if correspondingly \(l - t \equiv 0, 1, 4, 5 \pmod{8}\), \(h - g \equiv 2, 3, 6, 7 \pmod{8}\) and \(l - t \equiv 2, 3, 6, 7 \pmod{8}\), \(h - g \equiv 0, 1, 4, 5 \pmod{8}\). Inversely, if \(E = E_{i_1} E_{i_2} \cdots E_{i_{p+q-k}}\) and \(C = E_{j_1} E_{j_2} \cdots E_{j_k}\), then at \(k \equiv 1 \pmod{2}\) for the type \(p - q \equiv 4 \pmod{8}\) there exist Abelian groups \(\text{Aut}_+(\mathcal{O}_{p,q}) \simeq D_4 / \mathbb{Z}_2\) with \((+, +, +)\) and \(\text{Aut}_+(\mathcal{O}_{p,q}) \simeq D_4 / \mathbb{Z}_2\) with \((+, +, -)\) if correspondingly \(h - g \equiv 2, 3, 6, 7 \pmod{8}\), \(l - t \equiv 0, 1, 4, 5 \pmod{8}\) and \(h - g \equiv 0, 1, 4, 5 \pmod{8}\), \(l - t \equiv 2, 3, 6, 7 \pmod{8}\), and also at \(k \equiv 1 \pmod{2}\) for the type \(p - q \equiv 6 \pmod{8}\) there exist \(\text{Aut}_+(\mathcal{O}_{p,q}) \simeq \mathbb{Z}_2 / \mathbb{Z}_2\) with \((-, +, -)\) and \(\text{Aut}_+(\mathcal{O}_{p,q}) \simeq D_4 / \mathbb{Z}_2\) with \((-, +, +)\) if correspondingly \(h - g, l - t \equiv 0, 1, 4, 5 \pmod{8}\) (see Theorem 4 in [111]).

So, let \(E = E_{j_1} E_{j_2} \cdots E_{j_k}\) and \(C = E_{i_1} E_{i_2} \cdots E_{i_{p+q-k}}\) be the matrices of \(A \to \tilde{A}\) and \(A \to \tilde{A}^*\), \(k \equiv 0 \pmod{2}\). Assume that \(E_a, E_b, E_c \in E\), that is, all the matrices \(E_i\) in the operators (91) and (116) are skewsymmetric. Then for the operators (91) and (116) we obtain
\[
A_{23} E = -(-1)^{b+a-2} \frac{1}{2} \sigma(j_a)\sigma(j_b) E_{j_1} E_{j_2} \cdots E_{j_{a-1}} E_{j_{a+1}} \cdots E_{j_{b-1}} E_{j_{b+1}} \cdots E_{j_k},
\]
\[
E A_{23} = -(-1)^{2k-a-b} \frac{1}{2} \sigma(j_a)\sigma(j_b) E_{j_1} E_{j_2} \cdots E_{j_{a-1}} E_{j_{a+1}} \cdots E_{j_{b-1}} E_{j_{b+1}} \cdots E_{j_k}.
\] (125)
that is, $E$ and $C$ commute with $A_{23}$ (correspondingly with $A_{13}$, $A_{12}$). It is easy to see that relations (123) and (126) are analogous to the relations (13) and (98) for the field $F = \mathbb{C}$. Therefore, from (123) and (126) we obtain the relations (124) of Theorem. Further, assume that $E_a, E_b, E_c \notin E$, that is, all the matrices $E_i$ in the operators (91) and (116) are symmetric. Then

$$A_{23}E = (-1)^{2k}EA_{23}, \quad (127)$$

and analogous relations take place for $A_{13}, A_{12}$. It is easy to verify that from (127) and (128) we obtain the same relations (122), since (122) are relations (37) or (11) at $B_i = 0$. Therefore, over the ring $K \simeq \mathbb{H}$ (quasicomplex case) the elements of Abelian reflection groups of the quaternionic representations $S^0, S^6$ satisfy the relations (17)–(61) over the field $F = \mathbb{C}$ at $B_i = 0$. Indeed, at $E_a, E_b \in E$, $E_c \notin E$ and $E_c \in E$, $E_a, E_b \notin E$ we have relations (64) which are particular cases of (13) at $B_i = 0$ and so on. It is easy to verify that the same relations take place for non–Abelian reflection groups at $E = E_i, E_{i+1} \cdots E_{p+q-k}$ and $C = E_{j_1}E_{j_2} \cdots E_{j_k}, k \equiv 1 \pmod{2}$.

**Remark.** Theorem exhausts all possible permutation relations between transformations $P, T, PT$ and infinitesimal operators of the group $\mathfrak{G}_+$. The relations (37)–(60) take place always, that is, at any $n \equiv 0 \pmod{2}$ (except the case $n = 2$). In turn, the relations (37)–(60) are divided into two classes. The first class contains relations with operators $H_{+,-3}$ and $F_{+,-3}$ (the relations (37)–(40), (11)–(14), (49)–(52), (57)–(60)). The second class does not contain the relations with $H_{+,-3}, F_{+,-3}$ (the relations (13)–(16), (17)–(18), (53)–(55), (55)–(56)). Besides, in accordance with (91) for the transformation $T$ there are only two possibilities $T = P$ and $T = -P$ (both these cases correspond to relation (37)). However, from other relations it follows that $T \neq \pm P$, as it should be take place in general case. The exceptional case $n = 2$ corresponds to neutrino field and further it will be explored in the following sections within quotient representations of the group $\mathfrak{G}_+$. Permutations relations with respect to symmetric subspaces $\text{Sym}_{(k,r)}$ can be obtained by similar manner.

4 **Atiyah–Bott–Shapiro periodicity on the Lorentz group**

In accordance with the section 2 the finite–dimensional representations $\mathcal{C}, \mathcal{C}^\ast, \mathcal{C} \oplus \mathcal{C}^\ast$ related with the algebras $\mathbb{C}_{2k}, \mathbb{C}_{2r}, \mathbb{C}_{2k} \oplus \mathbb{C}_{2k}$ of the type $n \equiv 0 \pmod{2}$ and the quotient representations $\chi_\mathcal{C}, \chi_\mathcal{C}^\ast, \chi_\mathcal{C} \cup \chi_\mathcal{C}^\ast (\chi_\mathcal{C} \oplus \chi_\mathcal{C}^\ast)$ related with the quotient algebras $\mathbb{C}_{2k}, \mathbb{C}_{2r}$,
appears. Explicitly, showed on the Trautmann-like diagram (spinorial clock \(12, 13\), see also [61]):

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that for the Clifford algebra with odd dimensionality, the isomorphisms are as follows:

\[C^{\ell_0+\ell_1,0} \simeq C^{\ell_0+\ell_1-1,0} \otimes C^{1,0},\]

\[C^{0,\ell_0-\ell_1} \simeq C^{0,\ell_0-\ell_1+1} \otimes C^{0,-1},\]

and correspondingly for \(C \otimes C\)

\[C^{\ell_0+\ell_1,\ell_0-\ell_1} \simeq C^{\ell_0+\ell_1-1,\ell_0-\ell_1+1} \otimes C^{1,-1}.\]

Thus, the action of \(BW_C \simeq \mathbb{Z}_2\) form a cycle of the period 2 on the system \(M\), where the basic 2-period factor is the fundamental representation \(C^{1,0} (C^{0,-1})\) of the group \(G_+\). This cyclic structure is intimately related with de Broglie–Jordan neutrino theory of light and, moreover, this structure is a natural generalization of BJ–theory. Indeed, in the simplest case we obtain two relations \(C^{1,0} \simeq C^{1,0} \otimes C^{1,0}\) and \(C^{0,-2} \simeq C^{0,-1} \otimes C^{0,-1}\), which, as it is easy to see, correspond two helicity states of the photon (left– and right–handed polarization). In such a way, subsequent rotations of the representation 2-cycle give all other higher spin physical fields and follows to de Broglie and Jordan this structure should be called as ‘neutrino theory of everything’.

Further, when we consider restriction of the complex representation \(C\) onto real representations \(R\) and \(\mathcal{H}\), that corresponds to restriction of the group \(G_+\) onto its subgroup \(SO(3)\), we come to a more high–graded modulo 8 periodicity over the field \(F = \mathbb{R}\). Indeed, the Clifford algebra \(\mathcal{C}_{p,q}\) is central simple if \(p - q \neq 1, 5 \,(\text{mod} \, 8)\). It is known that for the Clifford algebra with odd dimensionality, the isomorphisms are as follows:

\[\mathcal{C}^+_{p,q+1} \simeq \mathcal{C}^+_{p,q} \text{ and } \mathcal{C}^+_{p+1,q} \simeq \mathcal{C}^+_{q,p} \text{ [52] [49]. Thus, } \mathcal{C}^+_{p,q+1} \text{ and } \mathcal{C}^+_{p+1,q} \text{ are central simple algebras. Further, in accordance with Chevalley Theorem [15] for the graded tensor product there is an isomorphism } \mathcal{C}^+_{p,q} \otimes \mathcal{C}^+_{p',q'} \simeq \mathcal{C}^+_{p+p',q+q'} \text{. Two algebras } \mathcal{C}^+_{p,q} \text{ and } \mathcal{C}^+_{p',q'} \text{ are}

\[2\text{Such a description corresponds to Helmholtz–Silberstein representation of the electromagnetic field as the complex linear combinations } F = E + i\mathcal{H}, \bar{F} = E - i\mathcal{H} \text{ that form a basis of the Majorana–Oppenheimer quantum electrodynamics [48] [53] [24] [21] (see also recent development on this subject based on the Joos–Weinberg and Bargmann–Wigner formalisms [24]).}

\[3\text{Physical feilds defined within such representations describe neutral particles, or particles at rest such as atomic nuclei.}
said to be of the same class if \( p + q \equiv p' + q \pmod{8} \). The graded central simple Clifford algebras over the field \( \mathbb{F} = \mathbb{R} \) form eight similarity classes, which, as it is easy to see, coincide with the eight types of the algebras \( \mathcal{A}_{p,q} \). The set of these 8 types (classes) forms a Brauer–Wall group \( BW_\mathbb{R} \) that is isomorphic to a cyclic group \( \mathbb{Z}_8 \). Therefore, in virtue of identifications (113) a group action of \( BW_\mathbb{R} \simeq \mathbb{Z}_8 \) can be transferred onto the system \( \mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^- \). In its turn, a cyclic structure of the group \( BW_\mathbb{R} \simeq \mathbb{Z}_8 \) is defined by a transition \( \mathcal{A}_{p,q}^+ \overset{h}{\rightarrow} r_h \mathcal{A}_{p,q} \), where the type of the algebra \( \mathcal{A}_{p,q} \) is defined by a formula \( q - p = 8r \), here \( h \in \{0, \ldots, 7\} \), \( r \in \mathbb{Z} \) \([12, 13] \). Thus, the action of \( BW_\mathbb{R} \simeq \mathbb{Z}_8 \) on \( \mathcal{M} \) is defined by a transition \( \mathcal{D}^0 \overset{h}{\rightarrow} r_h \mathcal{D} \), where \( \mathcal{D}^0 = \{ \mathcal{A}_{0,2}^0, \mathcal{A}_{1,6}^0, \mathcal{A}_{3,7}^0, \mathcal{A}_{0,2}^0 \cup \mathcal{A}_{0,2}^0, \mathcal{A}_{1,6}^0 \cup \mathcal{A}_{1,6}^0 \} \), and \( \mathcal{D}^{r/2} \simeq \mathcal{D}^{-r/2} \) when \( \mathcal{D} \in \mathcal{M}^+ \) and \( (\mathcal{D}^{r/2} \cup \mathcal{D}^{-r/2})^+ \simeq \chi \mathcal{D}^{r/2} \) when \( \mathcal{D} \in \mathcal{M}^-, r \) is a number of tensor products in \((114)\). Therefore, a cyclic structure of the group \( BW_\mathbb{R} \simeq \mathbb{Z}_8 \) induces on the system \( \mathcal{M} \) modulo 8 periodic relations which can be explicitly showed on the following diagram (the round on the diagram is realized by an hour–hand):

\[
\begin{array}{c}
\mathcal{A}_{0,2}^0 \cup \mathcal{A}_{0,2}^0 & \quad & \mathcal{A}_{0,2}^0 \\
7 & \quad & 1 \\
6 & \quad & 2 \\
5 & \quad & 3 \\
\mathcal{A}_{1,6}^0 \cup \mathcal{A}_{1,6}^0 & \quad & \mathcal{A}_{1,6}^0 \\
p - q \equiv 1 \pmod{8} & \quad & p - q \equiv 0 \pmod{8} \\
p - q \equiv 2 \pmod{8} & \quad & p - q \equiv 1 \pmod{8} \\
p - q \equiv 3 \pmod{8} & \quad & p - q \equiv 2 \pmod{8} \\
p - q \equiv 4 \pmod{8} & \quad & p - q \equiv 3 \pmod{8}
\end{array}
\]

Fig.2 The action of the Brauer–Wall group \( BW_\mathbb{R} \simeq \mathbb{Z}_8 \) on the full system \( \mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^- \) of real representations \( \mathcal{D} \) of the proper Lorentz group \( \mathfrak{g}_+ \), \( l_0 = \frac{p+q}{4} \).

Further, it is well–known that a group structure over \( \mathcal{A}_{p,q} \), defined by \( BW_\mathbb{R} \simeq \mathbb{Z}_8 \), immediately relates with the Atiyah–Bott–Shapiro periodicity \([4] \). In accordance with \([4] \), the Clifford algebra over the field \( \mathbb{F} = \mathbb{R} \) is modulo 8 periodic: \( \mathcal{A}_{p+8,q} \simeq \mathcal{A}_{p,q} \otimes \mathcal{A}_{8,0} \) (\( \mathcal{A}_{p,q} \otimes \mathcal{A}_{0,8} \)). Therefore, we have a following relation

\[
\mathcal{D}^{l+2} \simeq \mathcal{D}^l \otimes \mathcal{A}_{8,0}^2,
\]

since \( \mathcal{A}_{8,0}^2 \leftrightarrow \mathcal{A}_{8,0} \) (\( \mathcal{A}_{0,8} \)) and in virtue of Karoubi Theorem from \((14)\) it follows that \( \mathcal{A}_{8,0} \simeq \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \) (\( \mathcal{A}_{0,8} \simeq \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \otimes \mathcal{A}_{2,0} \)) \( \text{therefore, } r = 4, \)

\footnote{The minimal left ideal of \( \mathcal{A}_{8,0} \) is equal to \( S_{16} \) and in virtue of the real ring \( \mathbb{K} \simeq \mathbb{R} \) is defined within the full matrix algebra \( M_{16}(\mathbb{R}) \). At first glance, from the factorization of \( \mathcal{A}_{8,0} \) it follows that \( M_2(\mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{H} \otimes M_2(\mathbb{R}) \not\simeq M_{16}(\mathbb{R}) \), but it is wrong, since there is an isomorphism \( \mathbb{H} \otimes \mathbb{H} \simeq M_{4}(\mathbb{R}) \) (see Appendix B in \([4]\)).}\n
$l_0 = r/2 = 2$. On the other hand, in terms of minimal left ideal the modulo 8 periodicity looks like

$$S_{n+8} \simeq S_n \otimes S_{16}.$$  

In virtue of the mapping $\gamma_{8,0} : \mathcal{O}_{8,0} \to M_2(\mathbb{O})$ (see also excellent review [5]) the latter relation can be written in the form

$$S_{n+8} \simeq S_n \otimes \mathbb{O}^2,$$

where $\mathbb{O}$ is an octonion algebra. Since the algebra $\mathcal{O}_{8,0} \simeq \mathcal{O}_{0,8}$ admits an octonionic representation, then in virtue of the modulo 8 periodicity the octonionic representations can be defined for all high dimensions and, therefore, on the system $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$ we have a relation

$$\mathcal{D}^{l_0+2} \simeq \mathcal{D}^{l_0} \otimes \mathcal{O},$$

where $\mathcal{D}$ is an octonionic representation of the group $\mathfrak{G}_+ (\mathfrak{O} \simeq \mathfrak{R}_0^2)$. Thus, the action of $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ form a cycle of the period 8 on the system $\mathcal{M}$. This is intimately related with an octonionic structure. In 1973, Günaydin and Gürsey showed that an automorphism group of the algebra $\mathcal{O}$ is isomorphic to an exceptional Lie group $G_2$ that contains $SU(3)$ as a subgroup [3]. The Günaydin–Gürsey construction allows to incorporate the quark phenomenology into a general algebraic framework. Moreover, this construction allows to define the quark structure on the system $\mathcal{M}$ within octonionic representations of the proper Lorentz group $\mathfrak{G}_+$. It is obvious that within such a framework the quark structure cannot be considered as a fundamental physical structure underlying of the world (as it suggested by QCD). This is fully derivative structure firstly appeared in 8-dimension and further reproduced into high dimensions by the round of 8-cycle generated by the group $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ from 8 ad infinitum (growth of quark’s flavors with increase of energy). One can say that such a description, included very powerful algebraic tools, opens an another way of understanding of the Gell-Mann–Ne’emann eightfold way in particle physics.

5 Pseudoautomorphism $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ and charge conjugation

As noted previously, an extraction of the minimal left ideal of the complex algebra $\mathbb{C}_n \simeq \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2$ induces a space of the finite–dimensional spin–tensor representation of the group $\mathfrak{G}_+$. Besides, the algebra $\mathbb{C}_n$ is associated with a complex vector space $\mathbb{C}^n$. Let $n = p + q$, then an extraction operation of the real subspace $\mathbb{R}^{p \times q}$ in $\mathbb{C}^n$ forms the foundation of definition of the discrete transformation known in physics as a charge conjugation $C$. Indeed, let $\{e_1, \ldots, e_n\}$ be an orthobasis in the space $\mathbb{C}^n$, $e_i^* = 1$. Let us remain the first $p$ vectors of this basis unchanged, and other $q$ vectors multiply by the factor $i$. Then the basis

$$\{e_1, \ldots, e_p, ie_{p+1}, \ldots, ie_{p+q}\}$$  

(129)
allows to extract the subspace $\mathbb{R}^{p,q}$ in $\mathbb{C}^n$. Namely, for the vectors $\mathbb{R}^{p,q}$ we take the vectors of $\mathbb{C}^n$ which decompose on the basis (129) with real coefficients. In such a way we obtain a real vector space $\mathbb{R}^{p,q}$ endowed (in general case) with a non–degenerate quadratic form

$$Q(x) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2,$$

where $x_1, \ldots, x_{p+q}$ are coordinates of the vector $x$ in the basis (129). It is easy to see that the extraction of $\mathbb{R}^{p,q}$ in $\mathbb{C}^n$ induces an extraction of a real subalgebra $\mathcal{O}_{p,q}$ in $\mathbb{C}^n$. Therefore, any element $A \in \mathbb{C}^n$ can be unambiguously represented in the form

$$A = A_1 + iA_2,$$

where $A_1, A_2 \in \mathcal{O}_{p,q}$. The one–to–one mapping

$$A \rightarrow \overline{A} = A_1 - iA_2 \quad (130)$$

transforms the algebra $\mathbb{C}^n$ into itself with preservation of addition and multiplication operations for the elements $A$; the operation of multiplication of the element $A$ by the number transforms to an operation of multiplication by the complex conjugate number. Any mapping of $\mathbb{C}^n$ satisfying these conditions is called a pseudoautomorphism. Thus, the extraction of the subspace $\mathbb{R}^{p,q}$ in the space $\mathbb{C}^n$ induces in the algebra $\mathbb{C}^n$ a pseudoautomorphism $A \rightarrow \overline{A}$.

Let us consider a spinor representation of the pseudoautomorphism $A \rightarrow \overline{A}$ of the algebra $\mathbb{C}^n$ when $n \equiv 0 (\text{mod} \ 2)$. In the spinor representation the every element $A \in \mathbb{C}^n$ should be represented by some matrix $\hat{A}$, and the pseudoautomorphism (130) takes a form of the pseudoautomorphism of the full matrix algebra $M_{2n/2}$:

$$A \rightarrow \overline{A}.$$

On the other hand, a transformation replacing the matrix $\hat{A}$ by the complex conjugate matrix, $\hat{A} \rightarrow \hat{\overline{A}}$, is also some pseudoautomorphism of the algebra $M_{2n/2}$. The composition of the two pseudoautomorphisms $\hat{A} \rightarrow A$ and $A \rightarrow \overline{A}$, $\hat{\overline{A}} \rightarrow \hat{A} \rightarrow \overline{A}$, is an internal automorphism $\hat{\overline{A}} \rightarrow \overline{A}$ of the full matrix algebra $M_{2n/2}$:

$$\overline{A} = \Pi \hat{A} \Pi^{-1}, \quad (131)$$

where $\Pi$ is a matrix of the pseudoautomorphism $A \rightarrow \overline{A}$ in the spinor representation. The sufficient condition for definition of the pseudoautomorphism $A \rightarrow \overline{A}$ is a choice of the matrix $\Pi$ in such a way that the transformation $A \rightarrow \Pi \hat{A} \Pi^{-1}$ transfers into itself the matrices $\mathcal{E}_1, \ldots, \mathcal{E}_p, i\mathcal{E}_{p+1}, \ldots, i\mathcal{E}_{p+q}$ (the matrices of the spinbasis of $\mathcal{O}_{p,q}$), that is,

$$\mathcal{E}_i \rightarrow \mathcal{E}_i = \Pi \hat{\mathcal{E}}_i \Pi^{-1} \quad (i = 1, \ldots, p + q). \quad (132)$$

**Theorem 3.** Let $\mathbb{C}_n$ be a complex Clifford algebra when $n \equiv 0 (\text{mod} \ 2)$ and let $\mathcal{O}_{p,q} \subset \mathbb{C}_n$ be its subalgebra with a real division ring $\mathbb{K} \simeq \mathbb{R}$ when $p - q \equiv 0, 2 \ (\text{mod} \ 8)$ and quaternionic division ring $\mathbb{K} \simeq \mathbb{H}$ when $p - q \equiv 4, 6 \ (\text{mod} \ 8)$, $n = p + q$. Then in dependence
on the division ring structure of the real subalgebra $\mathcal{C}_{p,q}$ the matrix $\Pi$ of the pseudoautomorphism $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ has the following form:

1) $\mathbb{K} \simeq \mathbb{R}, \ p - q \equiv 0, 2 \ (\text{mod} \ 8)$. The matrix $\Pi$ for any spinor representation over the ring $\mathbb{K} \simeq \mathbb{R}$ is proportional to the unit matrix.

2) $\mathbb{K} \simeq \mathbb{H}, \ p - q \equiv 4, 6 \ (\text{mod} \ 8)$. $\Pi = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a}$ when $a \equiv 0 \ (\text{mod} \ 2)$ and $\Pi = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b}$ when $b \equiv 1 \ (\text{mod} \ 2)$, where a complex matrices $\mathcal{E}_{\alpha}$ and $b$ real matrices $\mathcal{E}_{\beta}$, form a basis of the spinor representation of the algebra $\mathcal{C}_{p,q}$ over the ring $\mathbb{K} \simeq \mathbb{H}, \ a + b = p + q, \ 0 < t \leq a, \ 0 < s \leq b$. At this point

\[
\Pi \Pi = \begin{cases} 1 & \text{if } a, b \equiv 0, 1 \ (\text{mod} \ 4), \\ -1 & \text{if } a, b \equiv 2, 3 \ (\text{mod} \ 4), \end{cases}
\]

where $I$ is the unit matrix.

Proof. The algebra $\mathbb{C}_n \ (n \equiv 0 \ (\text{mod} \ 2), \ n = p + q)$ in virtue of $\mathbb{C}_n = \mathbb{C} \otimes \mathcal{C}_{p,q}$ and definition of the division ring $\mathbb{K} \simeq f\mathcal{C}_{p,q}f$ ($f$ is a primitive idempotent of the algebra $\mathcal{C}_{p,q}$) has four different real subalgebras: $p - q \equiv 0, 2 \ (\text{mod} \ 8)$ for the real division ring $\mathbb{K} \simeq \mathbb{R}$ and $p - q \equiv 4, 6 \ (\text{mod} \ 8)$ for the quaternionic division ring $\mathbb{K} \simeq \mathbb{H}$.

1) $\mathbb{K} \simeq \mathbb{R}$.

Since for the types $p - q \equiv 0, 2 \ (\text{mod} \ 8)$ there is an isomorphism $\mathcal{C}_{p,q} \simeq M_{\frac{p+q}{2}}(\mathbb{R})$ (Wedderburn–Artin Theorem), then all the matrices $\mathcal{E}_i$ of the spinbasis of $\mathcal{C}_{p,q}$ are real and $\hat{\mathcal{E}}_i = \mathcal{E}_i$. Therefore, in this case the condition (132) can be written as follows

\[
\mathcal{E}_i \rightarrow \mathcal{E}_i = \Pi \mathcal{E}_i \Pi^{-1},
\]

whence $\mathcal{E}_i \Pi = \Pi \mathcal{E}_i$. Thus, for the algebras $\mathcal{C}_{p,q}$ of the types $p - q \equiv 0, 2 \ (\text{mod} \ 8)$ the matrix $\Pi$ of the pseudoautomorphism $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ commutes with all the matrices $\mathcal{E}_i$. It is easy to see that $\Pi \sim I$.

2) $\mathbb{K} \simeq \mathbb{H}$.

In turn, for the quaternionic types $p - q \equiv 4, 6 \ (\text{mod} \ 8)$ there is an isomorphism $\mathcal{C}_{p,q} \simeq M_{\frac{p+q}{2}}(\mathbb{H})$. Therefore, among the matrices of the spinbasis of the algebra $\mathcal{C}_{p,q}$ there are matrices $\mathcal{E}_{\alpha}$ satisfying the condition $\hat{\mathcal{E}}_{\alpha} = -\mathcal{E}_\alpha$. Let $a$ be a quantity of the complex matrices, then the spinbasis of $\mathcal{C}_{p,q}$ is divided into two subsets. The first subset $\{\hat{\mathcal{E}}_{\alpha} = -\mathcal{E}_\alpha\}$ contains complex matrices, $0 < t \leq a$, and the second subset $\{\hat{\mathcal{E}}_{\beta} = \mathcal{E}_\beta\}$ contains real matrices, $0 < s \leq p + q - a$. In accordance with a spinbasis structure of the algebra $\mathcal{C}_{p,q} \simeq M_{\frac{p+q}{2}}(\mathbb{H})$ the condition (132) can be written as follows

\[
\mathcal{E}_{\alpha_t} \rightarrow -\mathcal{E}_{\alpha_t} = \Pi \mathcal{E}_{\alpha_t} \Pi^{-1}, \quad \mathcal{E}_{\beta_s} \rightarrow \mathcal{E}_{\beta_s} = \Pi \mathcal{E}_{\beta_s} \Pi^{-1}.
\]

Whence

\[
\mathcal{E}_{\alpha_t} \Pi = -\Pi \mathcal{E}_{\alpha_t}, \quad \mathcal{E}_{\beta_s} \Pi = \Pi \mathcal{E}_{\beta_s}.
\] (133)
Thus, for the quaternionic types $p - q \equiv 4, 6 \pmod{8}$ the matrix $\Pi$ of the pseudoautomorphism $\mathcal{A} \to \overline{\mathcal{A}}$ anticommutes with a complex part of the spinbasis of $\mathcal{A}_{p,q}$ and commutes with a real part of the same spinbasis. From (133) it follows that a structure of the matrix $\Pi$ is analogous to the structure of the matrices $E$ and $C$ of the antiautomorphisms $\mathcal{A} \to \overline{\mathcal{A}}$ and $\mathcal{A} \to \mathcal{A}^*$, correspondingly (see Theorem 4 in [11]), that is, the matrix $\Pi$ of the pseudoautomorphism $\mathcal{A} \to \overline{\mathcal{A}}$ of the algebra $\mathbb{C}_n$ is a product of only complex matrices, or only real matrices of the spinbasis of the subalgebra $\mathcal{A}_{p,q}$.

So, let $0 < a < p + q$ and let $\Pi = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ be a matrix of $\mathcal{A} \to \overline{\mathcal{A}}$, then permutation conditions of the matrix $\Pi$ with the matrices $\mathcal{E}_{\beta_s}$ of the real part ($0 < s \leq p + q - a$) and with the matrices $\mathcal{E}_{\alpha_t}$ of the complex part ($0 < t < a$) have the form

$$\Pi \mathcal{E}_{\beta_s} = (-1)^a \mathcal{E}_{\beta_s} \Pi,$$  

(134)

$$\Pi \mathcal{E}_{\alpha_t} = (-1)^{a-t} \sigma(\alpha_t) \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_{t-1}}\mathcal{E}_{\alpha_{t+1}}\cdots\mathcal{E}_{\alpha_a},$$  

$$\mathcal{E}_{\alpha_t} \Pi = (-1)^{t-1} \sigma(\alpha_t) \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_{t-1}}\mathcal{E}_{\alpha_{t+1}}\cdots\mathcal{E}_{\alpha_a},$$  

(135)

that is, when $a \equiv 0 \pmod{2}$ the matrix $\Pi$ commutes with the real part and anticommutes with the complex part of the spinbasis of $\mathcal{A}_{p,q}$. Correspondingly, when $a \equiv 1 \pmod{2}$ the matrix $\Pi$ anticommutes with the real part and commutes with the complex part.

Further, let $\Pi = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{p+q-a}}$ be a product of the real matrices, then

$$\Pi \mathcal{E}_{\beta_s} = (-1)^{p+q-a-s} \sigma(\beta_s) \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{s-1}}\mathcal{E}_{\beta_{s+1}}\cdots\mathcal{E}_{\beta_{p+q-a}},$$  

$$\mathcal{E}_{\beta_s} \Pi = (-1)^{s-1} \sigma(\beta_s) \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{s-1}}\mathcal{E}_{\beta_{s+1}}\cdots\mathcal{E}_{\beta_{p+q-a}},$$  

(136)

$$\Pi \mathcal{E}_{\alpha_t} = (-1)^{p+q-a} \mathcal{E}_{\alpha_t} \Pi,$$  

(137)

that is, when $p + q - a \equiv 0 \pmod{2}$ the matrix $\Pi$ anticommutes with the real part and commutes with the complex part of the spinbasis of $\mathcal{A}_{p,q}$. Correspondingly, when $p + q - a \equiv 1 \pmod{2}$ the matrix $\Pi$ commutes with the real part and anticommutes with the complex part.

The comparison of the conditions (134)–(135) with the condition (133) shows that the matrix $\Pi = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ exists only at $a \equiv 0 \pmod{2}$, that is, $\Pi$ is a product of the complex matrices $\mathcal{E}_{\alpha_t}$ of the even number. In its turn, a comparison of (136)–(137) with (133) shows that the matrix $\Pi = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{p+q-a}}$ exists only at $p + q - a \equiv 1 \pmod{2}$, that is, $\Pi$ is a product of the real matrices $\mathcal{E}_{\beta_s}$ of the odd number.

Let us calculate now the product $\Pi \tilde{\Pi}$. Let $\Pi = \mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{p+q-a}}$ be a product of the $p + q - a$ real matrices. Since $\tilde{\mathcal{E}}_{\beta_s} = \mathcal{E}_{\beta_s}$, then $\tilde{\Pi} = \Pi$ and $\Pi \tilde{\Pi} = \Pi^2$. Therefore,

$$\Pi \tilde{\Pi} = (\mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_{p+q-a}})^2 = (-1)^{(p+q-a)(p+q-a-1)} \cdot 1.$$  

(138)

Further, let $\Pi = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a}$ be a product of the $a$ complex matrices. Then $\tilde{\mathcal{E}}_{\alpha_t} = -\mathcal{E}_{\alpha_t}$ and $\tilde{\Pi} = (-1)^a \Pi = \Pi$, since $a \equiv 0 \pmod{2}$. Therefore,

$$\Pi \tilde{\Pi} = (\mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_a})^2 = (-1)^{a(a-1)} \cdot 1.$$  

(139)
Let \( p + q - a = b \) be a quantity of the real matrices \( \varepsilon_\beta \) of the spinbasis of \( \mathcal{C}_{p,q} \), then \( p + q = a + b \). Since \( p + q \) is always even number for the quaternionic types \( p - q \equiv 4,6 \mod 8 \), then \( a \) and \( b \) are simultaneously even or odd numbers. Thus, from (138) and (139) it follows

\[
\Pi \tilde{\Pi} = \begin{cases} 
1, & \text{if } a, b \equiv 0, 1 \mod 4, \\
-1, & \text{if } a, b \equiv 2, 3 \mod 4,
\end{cases}
\]

which required to be proved.

In the present form of quantum field theory complex fields correspond to charged particles. Thus, the extraction of the subalgebra \( \mathcal{C}_{p,q} \) with the real ring \( \mathbb{K} \simeq \mathbb{R} \) in \( \mathbb{C}_n \), \( p - q \equiv 0, 2 \mod 8 \), corresponds to physical fields describing truly neutral particles such as photon and neutral mesons \( (\pi^0, \eta^0, \rho^0, \omega^0, \varphi^0, K^0) \). In turn, the subalgebras \( \mathcal{C}_{p,q} \) with the ring \( \mathbb{K} \simeq \mathbb{H} \), \( p - q \equiv 4, 6 \mod 8 \) correspond to charged or neutral fields.

As known [28], the charge conjugation \( C \) should be satisfied the following requirement

\[
CI^{ik} = I^{ik}C,
\]

(140)

where \( I^{ik} \) are infinitesimal operators of the group \( \mathfrak{g}_4 \). This requirement is necessary for the definition of the operation \( C \) on the representation spaces of \( \mathfrak{g} \). Let us find permutation conditions of the matrix \( \Pi \) with \( I^{ik} \) defined by the relations (11)–(12). It is obvious that in the case of \( \mathbb{K} \simeq \mathbb{R} \) the matrix \( \Pi \) commutes with all the operators \( I^{ik} \) and, therefore, the relations (140) hold. In the case of \( \mathbb{K} \simeq \mathbb{H} \) and \( \Pi = \varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_a} \) it is easy to verify that at \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \varepsilon_{\gamma_c} \in \Pi \) (all \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \varepsilon_{\gamma_c} \) are complex matrices) the matrix \( \Pi \) commutes with \( A_{ik} \) and anticommutes with \( B_i \) (permutation conditions in this case are analogous to (135) and (136)). In its turn, when \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \varepsilon_{\gamma_c} \notin \Pi \) (all \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \varepsilon_{\gamma_c} \) are real matrices) the matrix \( \Pi \) commutes with all the operators \( I^{ik} \). It is easy to see that all other cases given by the cyclic permutations \( \varepsilon_i, \varepsilon_j \in \Pi, \varepsilon_k \notin \Pi \) and \( \varepsilon_i \notin \Pi, \varepsilon_j, \varepsilon_k \in \Pi \) \((i, j, k \in \{a, b, c\})\) do not satisfy the relations (140). For example, at \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \varepsilon_{\gamma_c} \in \Pi \) and \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \varepsilon_{\gamma_c} \notin \Pi \) the matrix \( \Pi \) commutes with \( A_{23} \) and \( B_1 \) and anticommutes with \( A_{13}, A_{12}, B_2, B_3 \). Further, in the case of \( \Pi = \varepsilon_{\beta_1} \varepsilon_{\beta_2} \cdots \varepsilon_{\beta_b} \) it is not difficult to see that only at \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \in \Pi \) (all \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \) are real matrices) the matrix \( \Pi \) commutes with all \( I^{ik} \). Therefore, in both cases \( \Pi = \varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_a} \) and \( \Pi = \varepsilon_{\beta_1} \varepsilon_{\beta_2} \cdots \varepsilon_{\beta_b} \) the relations (140) hold when all the matrices \( \varepsilon_{\alpha_a} \varepsilon_{\beta_b} \varepsilon_{\gamma_c} \) belonging to (11)–(12) are real.

When we restrict the complex representation \( \mathfrak{g} \) (charged particles) of \( \mathfrak{g}_+ \) to real representation \( \mathfrak{h} \) (truly neutral particles) and \( \mathfrak{h} \) (neutral particles) we see that in this case the charge conjugation is reduced to an identical transformation \( I \) for \( \mathfrak{h} \) and to a particle–antiparticle conjugation \( C' \) for \( \mathfrak{h} \). Moreover, as follows from Theorem 3 for the real representations \( B_i = 0 \) and, therefore, the relations (140) take a form

\[
C' A_{ik} = A_{ik} C'.
\]

(141)

Over the ring \( \mathbb{K} \simeq \mathbb{R} \) the relations (141) hold identically. It is easy to verify that over the ring \( \mathbb{K} \simeq \mathbb{H} \) for the matrix \( \Pi = \varepsilon_{\alpha_1} \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_a} \) the relations (141) hold at \( \varepsilon_{\alpha_a}, \varepsilon_{\beta_b}, \varepsilon_{\gamma_c} \in \Pi \) and

\(^5\text{The requirement } CP = PC \text{ presented also in the Gel'fand–Yaglom work [28] is superfluous, since the inverse relation } CP = -PC \text{ is valid in BWW–type quantum field theories [3].} \)

\[ \]
\[ \xi^a = \Pi^a_\alpha \xi^\alpha, \]  
(142)

where \( \xi^\alpha = (\xi^a)^\ast \). In accordance with Theorem 3 for the matrix \( \Pi_\beta \) we have \( \dot{\Pi} = \Pi^{-1} \) or \( \dot{\Pi} = -\Pi^{-1} \), where \( \Pi^{-1} = \Pi_\beta^\ast \). Then a twice conjugated spinor looks like

\[ \xi^\alpha = \Pi^\alpha_\beta \xi^\beta. \]

Therefore, the twice conjugated spinor coincides with the initial spinor in the case of the real subalgebra of \( \mathbb{C}_2 \) with the ring \( \mathbb{K} \simeq \mathbb{R} \) (the algebras \( \mathcal{O}_{1,1} \) and \( \mathcal{O}_{2,0} \)), and also in the case of \( \mathbb{K} \simeq \mathbb{H} \) (the algebra \( \mathcal{O}_{0,2} \simeq \mathbb{H} \)) at \( a - b \equiv 0, 1 \) (mod 4). Since for the algebra \( \mathcal{O}_{0,2} \simeq \mathbb{H} \) we have always \( a - b \equiv 0 \) (mod 4), then a property of the reciprocal conjugacy of the spinors \( \xi^a \) \((a = 1, 2)\) is an invariant fact for the fundamental representation of the group \( \mathfrak{G}_+ \) (this property is very important in physics, since this is an algebraic expression of the requirement \( C^2 = 1 \)). Further, since the ‘vector’ (spintensor) of the finite-dimensional representation of the group \( \mathfrak{G}_+ \) is defined by the tensor product \( \xi^{a_1 a_2 \cdots a_k} = \sum \xi^{a_1} \otimes \xi^{a_2} \otimes \cdots \otimes \xi^{a_k} \), then its conjugated spintensor takes a form

\[ \xi^{a_1 a_2 \cdots a_k} = \sum \Pi^{a_1}_{\alpha_1} \Pi^{a_2}_{\alpha_2} \cdots \Pi^{a_k}_{\alpha_k} \xi^{\bar{a}_1 \bar{a}_2 \cdots \bar{a}_k}, \]  
(143)

It is obvious that the condition of reciprocal conjugacy \( \xi^{a_1 a_2 \cdots a_k} = \xi^{a_1 a_2 \cdots a_k} \) is also fulfilled for (143), since for each matrix \( \Pi^{a}_{\alpha} \) in (143) we have \( \dot{\Pi} = \Pi^{-1} \) (all the matrices \( \Pi^{a}_{\alpha} \) are defined for the algebra \( \mathbb{C}_2 \)).

Let us define now permutation conditions of the matrix \( \Pi \) of the pseudoautomorphism \( A \to \overline{A} \) (charge conjugation) with the matrix \( W \) of the automorphism \( A \to A^\ast \) (space inversion). First of all, in accordance with Theorem 3 in the case of \( \mathcal{O}_{p,q} \) with the real ring \( \mathbb{K} \simeq \mathbb{R} \) (types \( p - q \equiv 0, 2 \) (mod 8)) the matrix \( \Pi \) is proportional to the unit matrix and, therefore, commutes with the matrix \( W \). In the case of \( \mathbb{K} \simeq \mathbb{H} \) (types \( p - q \equiv 4, 6 \) (mod 8)) from Theorem 3 it follows two possibilities: \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \) is a product of \( a \) complex matrices at \( a \equiv 0 \) (mod 2) and \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \) is a product of \( b \) real matrices at \( b \equiv 1 \) (mod 2). Since \( a + b = p + q \), then the matrix \( W \) can be represented by the product \( \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \). Then for \( \Pi = \mathcal{E}_{\alpha_1} \mathcal{E}_{\alpha_2} \cdots \mathcal{E}_{\alpha_a} \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \), we have

\[ \Pi W = (-1)^{\frac{a(a-1)}{2}} \sigma(\alpha_{1})\sigma(\alpha_{2}) \cdots \sigma(\alpha_{a}) \mathcal{E}_{\beta_{1}} \mathcal{E}_{\beta_{2}} \cdots \mathcal{E}_{\beta_{b}}, \]

\[ W \Pi = (-1)^{\frac{b(b-1)}{2} + ab} \sigma(\beta_{1})\sigma(\beta_{2}) \cdots \sigma(\beta_{b}) \mathcal{E}_{\alpha_{1}} \mathcal{E}_{\alpha_{2}} \cdots \mathcal{E}_{\alpha_{a}}, \]

Hence it follows that at \( ab \equiv 0 \) (mod 2) the matrices \( \Pi \) and \( W \) always commute, since \( a \equiv 0 \) (mod 2). Taking \( \Pi = \mathcal{E}_{\beta_1} \mathcal{E}_{\beta_2} \cdots \mathcal{E}_{\beta_b} \), we obtain following conditions:

\[ \Pi W = (-1)^{\frac{b(b-1)}{2}} \sigma(\beta_{1})\sigma(\beta_{2}) \cdots \sigma(\beta_{b}) \mathcal{E}_{\alpha_{1}} \mathcal{E}_{\alpha_{2}} \cdots \mathcal{E}_{\alpha_{a}}, \]

\[ W \Pi = (-1)^{\frac{b(b-1)}{2}} \sigma(\beta_{1})\sigma(\beta_{2}) \cdots \sigma(\beta_{b}) \mathcal{E}_{\alpha_{1}} \mathcal{E}_{\alpha_{2}} \cdots \mathcal{E}_{\alpha_{a}}. \]
Hence it follows that \( ab \equiv 1 \pmod{2} \), since in this case \( b \equiv 1 \pmod{2} \), and \( p + q = a + b \) is even number, \( a \) is odd number. Therefore, at \( ab \equiv 1 \pmod{2} \) the matrices \( \Pi \) and \( W \) always anticommute.

It should be noted one important feature related with the anticommutation of the matrices \( \Pi \) and \( W \), \( \Pi W = -W \Pi \), that corresponds to relation \( CP = -PC \). The latter relation holds for Bargmann–Wightmann–Wigner type quantum field theories in which bosons and antibosons have mutually opposite intrinsic parities [3]. Thus, in this case the matrix of the operator \( C \) is a product of real matrices of odd number.

6 Quotient representations of the Lorentz group

**Theorem 4.** 1) \( F = \mathbb{C} \). Let \( A \to \overline{A} \), \( A \to A^* \), \( A \to \widetilde{A} \) be the automorphisms of the odd–dimensional complex Clifford algebra \( \mathbb{C}_{n+1} \) \((n + 1 \equiv 1, 3 \pmod{4})\) corresponding the discrete transformations \( C, P, T \) (charge conjugation, space inversion, time reversal) and let \( \epsilon \mathbb{C}_n \) be a quotient algebra obtained in the result of the homomorphic mapping \( \epsilon : \mathbb{C}_{n+1} \to \mathbb{C}_n \). Then over the field \( F = \mathbb{C} \) in dependence on the structure of \( \epsilon \mathbb{C}_n \) all the quotient representations of the Lorentz group are divided in the following six classes:

1) \( \chi_{\mathbb{C}_n}^{l_0 + l_1 - 1, 0} : \{ T, C \sim I \} \),
2) \( \chi_{\mathbb{C}_n}^{l_0 + l_1 - 1, 0} : \{ T, C \} \),
3) \( \chi_{\mathbb{C}_n}^{l_0 + l_1 - 1, 0} : \{ T, CP, CPT \} \),
4) \( \chi_{\mathbb{C}_n}^{l_0 + l_1 - 1, 0} : \{ PT, C, CPT \} \),
5) \( \chi_{\mathbb{C}_n}^{l_0 + l_1 - 1, 0} : \{ PT, CP \sim IP, CT \sim IT \} \),
6) \( \chi_{\mathbb{C}_n}^{l_0 + l_1 - 1, 0} : \{ PT, CP, CT \} \).

2) \( F = \mathbb{R} \). Real quotient representations are divided into four different classes:

7) \( \chi_{\mathbb{R}_n}^{l_0} : \{ T, C \sim I, CT \sim IT \} \),
8) \( \chi_{\mathbb{R}_n}^{l_0} : \{ T, CP \sim IP, CPT \sim IPT \} \),
9) \( \chi_{\mathbb{R}_n}^{l_0} : \{ T, C \sim C', CT \sim C'T \} \),
10) \( \chi_{\mathbb{R}_n}^{l_0} : \{ T, CP \sim C'P, CPT \sim C'PT \} \).

**Proof.** 1) Complex representations.
Before we proceed to find an explicit form of the quotient representations \( \chi \mathbb{C} \) it is necessary to consider in details a structure of the quotient algebras \( \epsilon \mathbb{C}_n \) obtaining in the result of the homomorphic mapping \( \epsilon : \mathbb{C}_{n+1} \to \mathbb{C}_n \). The structure of the quotient algebra \( \epsilon \mathbb{C}_n \) depends on the transfer of the automorphisms \( A \to A^* \), \( A \to \widetilde{A} \), \( A \to \widetilde{A}^* \), \( A \to A \) of the algebra \( \mathbb{C}_{n+1} \) under action of the homomorphism \( \epsilon \) onto its subalgebra \( \mathbb{C}_n \). As noted previously (see conclusion of Theorem [4]), the homomorphisms \( \epsilon \) and \( \chi \) have an analogous texture. The action of the homomorphism \( \epsilon \) is defined as follows

\[
\epsilon : A^1 + \epsilon \omega A^2 \longrightarrow A^1 + A^2,
\]
where \( A^1, A^2 \in C_n, \omega = e_{12...n+1} \), and

\[
\varepsilon = \begin{cases} 
1, & \text{if } n + 1 \equiv 1 \pmod{4}, \\
i, & \text{if } n + 1 \equiv 3 \pmod{4}; 
\end{cases}
\]

so that \((\varepsilon \omega)^2 = 1\). At this point \(\varepsilon \omega \to 1\) and the quotient algebra has a form

\[
C_n \simeq C_{n+1}/\text{Ker } \varepsilon,
\]

where \(\text{Ker } \varepsilon = \{A^1 - \varepsilon \omega A^1\}\) is a kernel of the homomorphism \(\varepsilon\).

For the transfer of the antiautomorphism \(A \to \tilde{A}\) from \(C_{n+1}\) into \(C_n\) it is necessary that

\[
\tilde{\varepsilon \omega} = \varepsilon \omega. \tag{144}
\]

Indeed, since under action of \(\varepsilon\) the elements 1 and \(\varepsilon \omega\) are equally mapped into the unit, then transformed elements \(\tilde{1}\) and \(\tilde{\varepsilon \omega}\) are also should be mapped into 1, but \(\tilde{1} = 1 \to 1\), and \(\tilde{\varepsilon \omega} = \pm \varepsilon \omega \to \pm 1\) in virtue of \(\tilde{\omega} = (-1)^{\frac{n(n-1)}{2}} \omega\), whence

\[
\tilde{\omega} = \begin{cases} 
\omega, & \text{if } n + 1 \equiv 1 \pmod{4}; \\
-\omega, & \text{if } n + 1 \equiv 3 \pmod{4}. 
\end{cases} \tag{145}
\]

Therefore, under action of the homomorphism \(\varepsilon\) the antiautomorphism \(A \to \tilde{A}\) is transferred from \(C_{n+1}\) into \(C_n\) only at \(n \equiv 0 \pmod{4}\).

In its turn, for the transfer of the automorphism \(A \to A^*\) it is necessary that \((\varepsilon \omega)^* = \varepsilon \omega\). However, since the element \(\omega\) is odd and \(\omega^* = (-1)^{n+1} \omega\), then we have always

\[
\omega^* = -\omega. \tag{146}
\]

Thus, the automorphism \(A \to A^*\) is never transferred from \(C_{n+1}\) into \(C_n\).

Further, for the transfer of the antiautomorphism \(A \to \tilde{A}^*\) from \(C_{n+1}\) into \(C_n\) it is necessary that

\[
(\tilde{\varepsilon \omega})^* = \varepsilon \omega. \tag{147}
\]

It is easy to see that the condition (147) is satisfied only at \(n + 1 \equiv 3 \pmod{4}\), since in this case from the second equality of (143) and (146) it follows

\[
(\tilde{\varepsilon \omega})^* = \tilde{\varepsilon \omega}^* = -\varepsilon \omega^* = \varepsilon \omega. \tag{148}
\]

Therefore, under action of the homomorphism \(\varepsilon\) the antiautomorphism \(A \to \tilde{A}^*\) is transferred from \(C_{n+1}\) into \(C_n\) only at \(n \equiv 2 \pmod{4}\).

Let \(n + 1 = p + q\). Defining in \(C_{n+1}\) the basis \(\{e_1, \ldots, e_p, ie_{p+1}, \ldots, ie_{p+q}\}\) we extract the real subalgebra \(C_{p,q}\), where at \(p - q \equiv 3, 7 \pmod{8}\) we have a complex division ring \(K \simeq \mathbb{C}\), and at \(p - q \equiv 1 \pmod{8}\) and \(p - q \equiv 5 \pmod{8}\) correspondingly a double real division ring \(K \simeq \mathbb{R} \oplus \mathbb{R}\) and a double quaternionic division ring \(K \simeq \mathbb{H} \oplus \mathbb{H}\). The product
\(e_1e_2 \cdots e_pe_{p+1} \cdots ie_{p+q} = i^q \omega \in \mathbb{C}_{n+1}\) sets a volume element of the real subalgebra \(\mathcal{A}_{p,q}\).

At this point we have a condition \((i^p \omega) = i^q \omega\), that is, \((-i)^q \omega = i^q \omega\), whence

\[
\omega = (-1)^q \omega.
\]  

When \(q\) is even, from (149) it follows \(\omega = \omega\) and, therefore, the pseudoautomorphism \(\mathcal{A} \to \mathcal{A}\) is transferred at \(q \equiv 0 \pmod{2}\), and since \(p + q\) is odd number, then we have always \(p \equiv 1 \pmod{2}\). In more detail, at \(n + 1 \equiv 3 \pmod{4}\) the pseudoautomorphism \(\mathcal{A} \to \mathcal{A}\) is transferred from \(\mathbb{C}_{n+1}\) into \(\mathbb{C}_n\) if the real subalgebra \(\mathcal{A}_{p,q}\) possesses the complex ring \(\mathbb{K} \simeq \mathbb{C}, p - q \equiv 3,7 \pmod{8}\), and is not transferred \((\omega = -\omega, q \equiv 1 \pmod{2}, p \equiv 0 \pmod{2})\) in the case of \(\mathcal{A}_{p,q}\) with double rings \(\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}\) and \(\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}\), \(p - q \equiv 1,5 \pmod{8}\). In its turn, at \(n + 1 \equiv 1 \pmod{4}\) the pseudoautomorphism \(\mathcal{A} \to \mathcal{A}\) is transferred from \(\mathbb{C}_{n+1}\) into \(\mathbb{C}_n\) if the subalgebra \(\mathcal{A}_{p,q}\) has the type \(p - q \equiv 1,5 \pmod{8}\) and is not transferred in the case of \(\mathcal{A}_{p,q}\) with \(p - q \equiv 3,7 \pmod{8}\). Besides, in virtue of (146) at \(n + 1 \equiv 3 \pmod{4}\) with \(p - q \equiv 1,5 \pmod{8}\) and at \(n + 1 \equiv 1 \pmod{4}\) with \(p - q \equiv 3,7 \pmod{8}\) a pseudoautomorphism \(\mathcal{A} \to \mathcal{A}^*\) (a composition of the pseudoautomorphism \(\mathcal{A} \to \mathcal{A}\) with the automorphism \(\mathcal{A} \to \mathcal{A}^*\) is transferred from \(\mathbb{C}_{n+1}\) into \(\mathbb{C}_n\), since

\[
\bar{\epsilon \omega}^* = \epsilon \omega.
\]

Further, in virtue of the second equality of (145) at \(n + 1 \equiv 3 \pmod{4}\) with \(p - q \equiv 1,5 \pmod{8}\) a pseudoantiautomorphism \(\mathcal{A} \to \mathcal{A}\) (a composition of the pseudoautomorphism \(\mathcal{A} \to \mathcal{A}\) with the antiautomorphism \(\mathcal{A} \to \mathcal{A}\)) is transferred from \(\mathbb{C}_{n+1}\) into \(\mathbb{C}_n\), since

\[
\bar{\epsilon \omega} = \epsilon \omega.
\]

Finally, a pseudoantiautomorphism \(\mathcal{A} \to \mathcal{A}^*\) (a composition of the pseudoautomorphism \(\mathcal{A} \to \mathcal{A}\) with the antiautomorphism \(\mathcal{A} \to \mathcal{A}\)), corresponded to CPT-transformation, is transferred from \(\mathbb{C}_{n+1}\) into \(\mathbb{C}_n\) at \(n + 1 \equiv 3 \pmod{4}\) and \(\mathcal{A}_{p,q}\) with \(p - q \equiv 3,7 \pmod{8}\), since in this case in virtue of (148) and (149) we have

\[
\bar{(\epsilon \omega^*)} = \epsilon \omega.
\]

Also at \(n + 1 \equiv 1 \pmod{4}\) and \(q \equiv 1 \pmod{2}\) we obtain

\[
\bar{(\epsilon \omega^*)} = -\bar{(\epsilon \omega^*)} = -\epsilon \omega^* = \epsilon \omega,
\]

therefore, the transformation \(\mathcal{A} \to \mathcal{A}^*\) is transferred at \(n + 1 \equiv 1 \pmod{4}\) and \(\mathcal{A}_{p,q}\) with \(p - q \equiv 3,7 \pmod{8}\).

The conditions for the transfer of the fundamental automorphisms of the algebra \(\mathbb{C}_{n+1}\) into its subalgebra \(\mathbb{C}_n\) under action of the homomorphism \(\epsilon\) allow to define in evident way an explicit form of the quotient algebras \(\mathcal{C}_n\).

1) The quotient algebra \(\mathcal{C}_n, n \equiv 0 \pmod{4}\).

As noted previously, in the case \(n + 1 \equiv 1 \pmod{4}\) the antiautomorphism \(\mathcal{A} \to \mathcal{A}\) and pseudoautomorphism \(\mathcal{A} \to \mathcal{A}\) are transferred from \(\mathbb{C}_{n+1}\) into \(\mathbb{C}_n\) if the subalgebra \(\mathcal{A}_{p,q}\) \(\subset \mathcal{C}_n\).
In the case $n$ divided into following two classes:

2) The quotient algebra $\mathcal{A} \rightarrow \mathcal{A}^\star$ and pseudoautormorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ are transferred if $\mathcal{A}_{p,q}$ has the complex ring $\mathbb{K} \simeq \mathbb{C}$ $(p - q \equiv 3, 7 \pmod{8})$. It is easy to see that in dependence on the type of $\mathcal{A}_{p,q}$ the structure of the quotient algebras $\mathfrak{C}_n$ of this type is divided into two different classes:

a) The class of quotient algebras $\mathfrak{C}_n$ containing the antiautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ and pseudoautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$. It is obvious that in dependence on a division ring structure of the subalgebra $\mathcal{A}_{p,q} \subset \mathbb{C}_{n+1}$ this class is divided into two subclasses:

$a_1$) $\mathfrak{C}_n$ with $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ at $\mathcal{A}_{p,q}$ with the ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$, $p - q \equiv 1 \pmod{8}$.

$a_2$) $\mathfrak{C}_n$ with $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ at $\mathcal{A}_{p,q}$ with the ring $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$, $p - q \equiv 5 \pmod{8}$.

b) The class of quotient algebras $\mathfrak{C}_n$ containing the transformations $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ if the subalgebra $\mathcal{A}_{p,q} \subset \mathbb{C}_{n+1}$ has the complex ring $\mathbb{K} \simeq \mathbb{C}$, $p - q \equiv 3, 7 \pmod{8}$.

2) The quotient algebra $\mathfrak{C}_n$, $n \equiv 2 \pmod{4}$.

In the case $n + 1 \equiv 3 \pmod{4}$ the antiautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$, pseudoautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ and pseudoantiautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ are transferred from $\mathbb{C}_{n+1}$ into $\mathbb{C}_n$ if the subalgebra $\mathcal{A}_{p,q} \subset \mathbb{C}_{n+1}$ possesses the complex ring $\mathbb{K} \simeq \mathbb{C}$ $(p - q \equiv 3, 7 \pmod{8})$, and also the pseudoautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ and pseudoantiautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ are transferred if $\mathcal{A}_{p,q}$ has the double rings $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$, $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ $(p - q \equiv 1, 5 \pmod{8})$. In dependence on the type of $\mathcal{A}_{p,q} \subset \mathbb{C}_{n+1}$ all the quotient algebras $\mathfrak{C}_n$ of this type are divided into following two classes:

c) The class of quotient algebras $\mathfrak{C}_n$ containing the transformations $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ if the subalgebra $\mathcal{A}_{p,q}$ has the ring $\mathbb{K} \simeq \mathbb{C}$, $p - q \equiv 3, 7 \pmod{8}$.

d) The class of quotient algebras $\mathfrak{C}_n$ containing the antiautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$, pseudoautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ and pseudoautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$. At this point, in dependence on the division ring structure of $\mathcal{A}_{p,q}$ we have two subclasses

$d_1$) $\mathfrak{C}_n$ with $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ and $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ at $\mathcal{A}_{p,q}$ with the ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$, $p - q \equiv 1 \pmod{8}$.

$d_2$) $\mathfrak{C}_n$ with $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$, $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ and $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ at $\mathcal{A}_{p,q}$ with the ring $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$, $p - q \equiv 5 \pmod{8}$.

Thus, we have 6 different classes of the quotient algebras $\mathfrak{C}_n$. Further, in accordance with the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ corresponds to space inversion $P$, the antiautomorphisms $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ and $\mathcal{A} \rightarrow \tilde{\mathcal{A}}^\star$ set correspondingly time reversal $T$ and full reflection $PT$, and the pseudoautomorphism $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$ corresponds to charge conjugation $C$. Taking into account this relation and Theorem 1 we come to classification presented in Theorem for complex quotient representations.

2) Real representations.
Let us define real quotient representations of the group $G_+$. First of all, in the case of types $p - q \equiv 3, 7 \pmod{8}$ we have the isomorphism (58) and, therefore, these representations are equivalent to complex representations considered in the section 3. Further, when $p - q \equiv 1, 5 \pmod{8}$ we have the real algebras $\mathcal{C}_{p,q}$ with the rings $K \simeq \mathbb{R} \oplus \mathbb{R}, K \simeq \mathbb{H} \oplus \mathbb{H}$ and, therefore, there exist homomorphic mappings $\epsilon : \mathcal{C}_{p,q} \to \mathcal{C}_{p,q-1}, \epsilon : \mathcal{C}_{p,q} \to \mathcal{C}_{q,p-1}$. In this case the quotient algebra has a form

$$\mathcal{C}_{p,q-1} \simeq \mathcal{C}_{p,q} / \text{Ker} \epsilon$$

or

$$\mathcal{C}_{q,p-1} \simeq \mathcal{C}_{p,q} / \text{Ker} \epsilon,$$

where $\text{Ker} \epsilon = \{A^1 - \omega A^1\}$ is a kernel of $\epsilon$, since in accordance with

$$\omega^2 = \begin{cases} -1 & \text{if } p - q \equiv 2, 3, 6, 7 \pmod{8}, \\ +1 & \text{if } p - q \equiv 0, 1, 4, 5 \pmod{8} \end{cases}$$

at $p - q \equiv 1, 5 \pmod{8}$ we have always $\omega^2 = 1$ and, therefore, $\epsilon = 1$. Thus, for the transfer of the antiautomorphism $\mathcal{A} \to \tilde{\mathcal{A}}$ from $\mathcal{C}_{p,q}$ into $\mathcal{C}_{p,q-1}$ ($\mathcal{C}_{q,p-1}$) it is necessary that

$$\tilde{\omega} = \omega$$

In virtue of the relation $	ilde{\omega} = (-1)^{(p+q)(p+q+1)} \omega$ we obtain

$$\tilde{\omega} = \begin{cases} +\omega & \text{if } p - q \equiv 1, 5 \pmod{8}, \\ -\omega & \text{if } p - q \equiv 3, 7 \pmod{8}. \end{cases} \quad (150)$$

Therefore, for the algebras over the field $\mathbb{F} = \mathbb{R}$ the antiautomorphism $\mathcal{A} \to \tilde{\mathcal{A}}$ is transferred at the mappings $\mathcal{C}_{p,q} \to \mathcal{C}_{p,q-1}, \mathcal{C}_{p,q} \to \mathcal{C}_{q,p-1}$, where $p - q \equiv 1, 5 \pmod{8}$.

In its turn, for the transfer of the automorphism $\mathcal{A} \to \mathcal{A}^*$ it is necessary that $\omega^* = \omega$. However, since the element $\omega$ is odd and $\omega^* = (-1)^{p+q} \omega$, then we have always

$$\omega^* = -\omega. \quad (151)$$

Thus, the automorphism $\mathcal{A} \to \mathcal{A}^*$ is never transferred from $\mathcal{C}_{p,q}$ into $\mathcal{C}_{p,q-1}$ ($\mathcal{C}_{q,p-1}$).

Further, for the transfer of the antiautomorphism $\mathcal{A} \to \tilde{\mathcal{A}}^*$ it is necessary that

$$\tilde{\omega}^* = \omega.$$

From (150) and (151) for the types $p - q \equiv 1, 5 \pmod{8}$ we obtain

$$\tilde{\omega}^* = \omega^* = -\omega. \quad (152)$$

Therefore, under action of the homomorphism $\epsilon$ the antiautomorphism $\mathcal{A} \to \tilde{\mathcal{A}}^*$ is never transferred from $\mathcal{C}_{p,q}$ into $\mathcal{C}_{p,q-1}$ ($\mathcal{C}_{q,p-1}$).
As noted previously, for the real representations of $\mathfrak{g}_+$ the pseudoautomorphism $A \to \overline{A}$ is reduced into identical transformation $I$ for $\mathfrak{g}_{0,2}$ and to particle–antiparticle conjugation $C'$ for $\mathfrak{g}_{0,6}$. The volume element $\omega$ of $C_{p,q}$ (types $p - q \equiv 1, 5 \pmod{8}$) can be represented by the product $e_1e_2\cdots e_pe_{p+1}'e_{p+2}'\cdots e_{p+q}'$, where $e_{p+j}' = ie_{p+j}$, $e_j' = 1$, $(e_{p+j})^2 = -1$. Therefore, for the transfer of $A \to \overline{A}$ from $C_{p,q}$ into $C_{p,q-1}$ ($C_{q,p-1}$) we have a condition

$$\overline{\omega} = \omega,$$

and in accordance with (149) it follows that the pseudoautomorphism $A \to \overline{A}$ is transferred at $q \equiv 0 \pmod{2}$. Further, in virtue of the relation (151) the pseudoautomorphism $A \to \overline{A}^*$ is transferred at $q \equiv 1 \pmod{2}$, since in this case we have

$$\overline{\omega} = \omega.$$

Also from (150) it follows that the pseudoantiautomorphism $A \to \overline{A}$ is transferred at $p - q \equiv 1, 5 \pmod{8}$ and $q \equiv 0 \pmod{2}$, since

$$\overline{\omega} = \omega.$$

Finally, the pseudoantiautomorphism $A \to \overline{A}^*$ ($CPT$–transformation) in virtue of (152) and (149) is transferred from $C_{p,q}$ into $C_{p,q-1}$ ($C_{q,p-1}$) at $p - q \equiv 1, 5 \pmod{8}$ and $q \equiv 1 \pmod{2}$.

Now we are in a position that allows to classify the real quotient algebras $\ast C_{p,q-1}$ ($\ast C_{q,p-1}$).

1) The quotient algebra $\ast C_{p,q-1}$ ($\ast C_{q,p-1}$, $p - q \equiv 1 \pmod{8}$).

In this case the initial algebra $C_{p,q}$ has the double real division ring $K \simeq \mathbb{R} \oplus \mathbb{R}$ and its subalgebras $C_{p,q-1}$ and $C_{q,p-1}$ are of the type $p - q \equiv 0 \pmod{8}$ or $p - q \equiv 2 \pmod{8}$ with the ring $K \simeq \mathbb{R}$. Therefore, in accordance with Theorem 3 for all such quotient algebras the pseudoautomorphism $A \to \overline{A}$ is equivalent to the identical transformation $I$. The antiautomorphism $A \to \overline{A}$ in this case is transferred into $C_{p,q-1}$ ($C_{q,p-1}$) at any $p - q \equiv 1 \pmod{8}$. Further, in dependence on the number $q$ we have two different classes of the quotient algebras of this type:

$$e_1) \ast C_{p,q-1}$ ($\ast C_{q,p-1}$) with $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $p - q \equiv 1 \pmod{8}$, $q \equiv 0 \pmod{2}$.

$$e_2) \ast C_{p,q-1}$ ($\ast C_{q,p-1}$) with $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $p - q \equiv 1 \pmod{8}$, $q \equiv 0 \pmod{2}$.

2) The quotient algebras $\ast C_{p,q-1}$ ($\ast C_{q,p-1}$), $p - q \equiv 5 \pmod{8}$.

In this case the initial algebra $C_{p,q}$ has the double quaternionic division ring $K \simeq \mathbb{H} \oplus \mathbb{H}$ and its subalgebras $C_{p,q-1}$ and $C_{q,p-1}$ are of the type $p - q \equiv 4 \pmod{8}$ or $p - q \equiv 6 \pmod{8}$ with the ring $K \simeq \mathbb{H}$. Therefore, in this case the pseudoautomorphism $A \to \overline{A}$ is equivalent to the particle–antiparticle conjugation $C'$. As in the previous case the antiautomorphism $A \to \overline{A}$ is transferred at any $p - q \equiv 5 \pmod{8}$. For this type in dependence on the number $q$ there are two different classes:

$$f_1) \ast C_{p,q-1}$ ($\ast C_{q,p-1}$) with $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $p - q \equiv 5 \pmod{8}$, $q \equiv 0 \pmod{2}$.

$$f_2) \ast C_{p,q-1}$ ($\ast C_{q,p-1}$) with $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $A \to \overline{A}$, $p - q \equiv 5 \pmod{8}$, $q \equiv 1 \pmod{2}$. 


Analysing the quotient representations of the group $G$, presented in Theorem 4, we see that only a representation of the class $c$ at $j = (l_0 + l_1 - 1)/2 = 1/2$ is adequate for description of the neutrino field. This representation admits full reflection $PT$, charge conjugation $C$ and $CPT$–transformation (space inversion $P$ is not defined). In contrast with this, the first three classes $a_1, a_2$ and $b$ are unsuitable for description of neutrino, since in this case $j$ is an integer number, $n \equiv 0 \mod 4$ (bosonic fields). In turn, the classes $d_1$ and $d_2$ admit $CT$–transformation that in accordance with $CPT$–Theorem is equivalent to space inversion $P$, which, as known, is a forbidden operation for the neutrino field.

So, we have an homomorphic mapping $\epsilon : \mathbb{C}_3 \rightarrow \mathbb{C}_2$, where $\mathbb{C}_3$ is a simplest Clifford algebra of the type $n + 1 \equiv 3 \mod 4$. In accordance with Theorem 3 under action of the homomorphism $\epsilon : \mathbb{C}_3 \rightarrow \mathbb{C}_2$ the transformations $A \rightarrow \overline{A}^\ast$, $A \rightarrow A$ and $A \rightarrow A^\ast$ are transferred from $\mathbb{C}_3$ into $\mathbb{C}_2$. At this point, the real subalgebra $\mathbb{C}_{3.0} \subset \mathbb{C}_3$ has the complex ring $\mathbb{K} \cong \mathbb{C}$, $p - q \equiv 3 \mod 8$, and, therefore, the matrix $\Pi$ of the pseudoautomorphism $A \rightarrow \overline{A}$ is not unit, that according to Theorem 4 under action of the homomorphism $\epsilon : \mathbb{C}_3 \rightarrow \mathbb{C}_2$ is a forbidden operation for the neutrino field.

Let $\varphi \in \mathbb{C}_3$ be an algebraic spinor of the form

$$\varphi = a^0 + a^1e_1 + a^2e_2 + a^3e_3 + a^{12}e_1e_2 + a^{13}e_1e_3 + a^{23}e_2e_3 + a^{123}e_1e_2e_3.$$ (154)

Then it is easy to verify that spinors

$$\varphi^+ = \lambda_+ \varphi = \frac{1}{2}(1 + ie_1e_2e_3)\varphi, \quad \varphi^- = \lambda_- \varphi = \frac{1}{2}(1 - ie_1e_2e_3)\varphi$$ (155)

are mutually orthogonal, $\varphi^+ \varphi^- = 0$, since $\lambda_+ \lambda_- = 0$, and also $\varphi^+ \in \mathbb{C}_2$, $\varphi^- \in \mathbb{C}_2^\ast$. Further, it is obvious that a spinospace of the algebra $\mathbb{C}_2 \cup \mathbb{C}_2^\ast$ is $S_2 \cup \hat{S}_2$. It should be
noted here that structures of the spinspaces $S_2 \cup \mathring{S}_2$ and $S_2 \oplus \mathring{S}_2$ are different. Indeed,

$$S_2 \cup \mathring{S}_2 = \begin{pmatrix} 00 & 00 \\ 10 & 10 \end{pmatrix}, \quad S_2 \oplus \mathring{S}_2 = \begin{pmatrix} 00 & 01 \\ 10 & 11 \end{pmatrix}.\$$

Since spinor representations of the quotient algebras $^*C_2$ and $^*\mathring{C}_2$ are defined in terms of Pauli matrices $\sigma_i$, then the algebraic spinors $\varphi^+ \in ^*C_2$ and $\varphi^- \in ^*\mathring{C}_2$ correspond to spinors $\xi^a \in S_2$ and $\xi^{\dot{a}} \in \mathring{S}_2$ ($i = 0, 1$). Hence it immediately follow Weyl equations

$$\left( \frac{\partial}{\partial x^0} - \sigma \frac{\partial}{\partial \mathbf{x}} \right) \xi^a = 0, \quad \left( \frac{\partial}{\partial x^0} + \sigma \frac{\partial}{\partial \mathbf{x}} \right) \xi^{\dot{a}} = 0. \quad (156)$$

Therefore, two–component Weyl theory can be naturally formulated within quotient representation $\chi C_{1,0} \cup \chi C_0^{-1}$ of the group $\mathfrak{g}_+. \quad $ Further, in virtue of an isomorphism $C_2 \simeq \mathcal{O}_{3,0} \simeq \mathcal{O}_{1,3}^+$ ($\mathcal{O}_{1,3}^+$ is the space–time algebra) the spinor field of the quotient representation $\chi C_0^{-1} \cup \chi C_{1,0}$ can be expressed via the Dirac–Hestenes spinor field $\phi(x) \in \mathcal{O}_{3,0}$.

Indeed, the Dirac–Hestenes spinor is represented by a following biquaternion number

$$\phi = a^0 + a^{01} \gamma_0 \gamma_1 + a^{02} \gamma_0 \gamma_2 + a^{03} \gamma_0 \gamma_3 + a^{12} \gamma_1 \gamma_2 + a^{13} \gamma_1 \gamma_3 + a^{23} \gamma_2 \gamma_3 + a^{0123} \gamma_0 \gamma_1 \gamma_2 \gamma_3,$$  

or using $\gamma$–basis

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}. \quad (158)$$

we can write (157) in the matrix form

$$\phi = \begin{pmatrix} \phi_1 & -\phi_2^* & \phi_3 & \phi_4^* \\ \phi_2 & \phi_1^* & \phi_4 & -\phi_3^* \\ \phi_3 & -\phi_4^* & \phi_1 & -\phi_2^* \\ \phi_4 & -\phi_3^* & \phi_2 & \phi_1^* \end{pmatrix}, \quad (159)$$

where

$$\phi_1 = a^0 - ia^{12}, \quad \phi_2 = a^{13} - ia^{23}, \quad \phi_3 = a^{03} - ia^{0123}, \quad \phi_4 = a^0 + ia^{02}.$$  

From (154)–(155) and (157) it is easy to see that spinors $\varphi^+$ and $\varphi^-$ are algebraically equivalent to the spinor $\phi \in C_2 \simeq \mathcal{O}_{3,0}$. Further, since $\phi \in \mathcal{O}_{1,3}^+$, then actions of the antiautomorphisms $A \to \tilde{A}$ and $A \to \mathring{A}^*$ on the field $\phi$ are equivalent. On the other hand, in accordance with Feynman–Stueckelberg interpretation, time reversal for the chiral field is equivalent to charge conjugation (particles reversed in time are antiparticles). Thus, for the field $\phi \in \chi C_0^{-1}$ we have $C \sim T$ and, therefore, this field is $CP$–invariant.  

\textsuperscript{6}See also [3].
The spinor (157) (or (159)) satisfies the Dirac–Hestenes equation
\[ \partial \phi \gamma_2 \gamma_1 - \frac{mc}{\hbar} \phi \gamma_0 = 0, \tag{160} \]
where \( \partial = \gamma^\mu \frac{\partial}{\partial x^\mu} \) is the Dirac operator. Let us show that a massless Dirac–Hestenes equation
\[ \partial \phi \gamma_2 \gamma_1 = 0 \tag{161} \]
describes the neutrino field. Indeed, the matrix \( \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_5 \) commutes with all the elements of the biquaternion (157) and, therefore, \( \gamma_5 \) is equivalent to the volume element \( \omega = e_1 e_2 e_3 \) of the biquaternion algebra \( \mathcal{O}_{3,0} \). In such a way, we see that idempotents
\[ P_+ = \frac{1 + \gamma_5}{2}, \quad P_- = \frac{1 - \gamma_5}{2} \]
cover the central idempotents (153). Further, from (161) we obtain
\[ P_\pm \gamma^\mu \frac{\partial}{\partial x^\mu} \phi \gamma_2 \gamma_1 = \gamma^\mu P_m P_\pm \frac{\partial}{\partial x^\mu} \phi \gamma_2 \gamma_1 = 0, \]
that is, there are two separated equations for \( \phi^\pm = P_\pm \phi \gamma_2 \gamma_1 \):
\[ \gamma^\mu \frac{\partial}{\partial x^\mu} \phi^\pm = 0, \tag{162} \]
where
\[ \phi^\pm = \frac{1}{2} (1 \pm \gamma_5) \phi \gamma_2 \gamma_1 = \frac{i}{2} \begin{pmatrix} \phi_1 \mp \phi_3 & \phi_2 \pm \phi_4^* & \phi_3 \mp \phi_1 & -\phi_4^* \mp \phi_2^* \\ \phi_2 \mp \phi_4 & -\phi_1 \pm \phi_3^* & \phi_4 \mp \phi_2 & \phi_3 \pm \phi_1^* \\ \mp \phi_1 \mp \phi_2 & \mp \phi_2^* - \phi_4^* & \mp \phi_3 \mp \phi_1 & \pm \phi_4^* + \phi_2^* \\ \mp \phi_2 \mp \phi_4 & \mp \phi_1^* + \phi_3 & \mp \phi_4 \mp \phi_2 & \mp \phi_3^* - \phi_1^* \end{pmatrix} \]
Therefore, each of the functions \( \phi^+ \) and \( \phi^- \) contains only four independent components and in the split form we have
\[ \phi^+ = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ -\psi_1 & -\psi_2 & -\psi_3 & -\psi_4 \end{pmatrix}, \quad \phi^- = \begin{pmatrix} \psi_5 & \psi_6 & \psi_7 & \psi_8 \\ \psi_5 & \psi_6 & \psi_7 & \psi_8 \end{pmatrix}, \]
where
\[
\begin{align*}
\psi_1 &= \frac{i}{2} \left( \phi_1 - \phi_3 \right), \\
\psi_2 &= \frac{i}{2} \left( \phi_2^* + \phi_4 \right), \\
\psi_3 &= \frac{i}{2} \left( \phi_3 - \phi_1 \right), \\
\psi_4 &= \frac{i}{2} \left( -\phi_1^* - \phi_2^* \right), \\
\psi_5 &= \frac{i}{2} \left( \phi_1 + \phi_3 \right), \\
\psi_6 &= \frac{i}{2} \left( \phi_2^* - \phi_4 \right), \\
\psi_7 &= \frac{i}{2} \left( \phi_3 + \phi_1 \right), \\
\psi_8 &= \frac{i}{2} \left( -\phi_1^* + \phi_2^* \right).
\end{align*}
\]
Thus, in the \( \gamma \)-basis we obtain from (162)
\[
\begin{align*}
\left( \frac{\partial}{\partial x^0} - \sigma \frac{\partial}{\partial x} \right) \psi_i &= 0, \\
\left( \frac{\partial}{\partial x^0} + \sigma \frac{\partial}{\partial x} \right) \psi_{i+4} &= 0, \quad (i = 1, 2, 3, 4)
\end{align*}
\]
These equations are equivalent to Weyl equations (158) and, therefore, should be called Weyl–Hestenes equations for neutrino field.
Acknowledgments

I am grateful to Prof. J.S.R. Chisholm for sending me his interesting works.

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