QUANTUM CORRECTIONS IN TWO-DIMENSIONAL
NON-SUPERSYMMETRIC HETEROаблиц STRINGS

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We study quantum corrections for a family of 24 non-supersymmetric heterotic strings in
two dimensions. We compute their genus two cosmological constant using the hyperelliptic
formalism and the genus one two-point functions for the massless states. From here we get
the mass corrections to the states in the massless sector and discuss the role of the infrared
divergences that appear in the computation. We also study some tree-level aspects of these
theories and find that they are classified not only by the corresponding Niemeier lattice but
also by their hidden right-moving gauge symmetry.

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1 Introduction

It is well known that string theory is not just equivalent to a collection of quantum fields. Atkin-Lehner symmetry is probably the best mathematical formulation of this physical statement as we have a net number of bosonic massless fields not balanced with fermionic ones and at the same time the one-loop vacuum energy vanishes \[^{[1]}\]. The vanishing of the cosmological constant due to Atkin-Lehner symmetry is known to be a one-loop effect and the common belief is that interactions should actually induce a non-zero vacuum energy. Our objective here will be to study these quantum corrections not only to the model with Atkin-Lehner symmetry but also to all its 23 relatives constructed on the Niemeier lattices.

Quantum corrections in heterotic string theories have been mainly studied in the context of supersymmetric models \[^{[2, 3, 4, 5, 6]}\]. In these cases non-renormalization theorems ensure that some loop correlation functions for less than four external massless particles have to vanish \[^{[7]}\] since they are related to couplings in the low-energy action that are fixed by supersymmetry. However in the case of non-supersymmetric heterotic strings one expects to have an induced cosmological constant and/or a mass renormalization for the massless states.

Non-supersymmetric heterotic strings are peculiar for a number of reasons. Among all the possible models that can be constructed only a relatively small subset gives rise to well-behaved string theories in which the breaking of space-time supersymmetry does not introduce tachyonic states. This is the case, for example, of the ten-dimensional heterotic string with gauge group \(SO(16) \times SO(16)\) \[^{[8]}\]: this model is tachyon free and has a finite non-vanishing one-loop cosmological constant. However the theory is non-finite due to divergences appearing in the computation of some loop amplitudes. More recently the concept of misaligned supersymmetry \[^{[9]}\] has been introduced in order to establish the minimum requirements a non-supersymmetric heterotic string has to fulfill in order to be tachyon free and then to have finite one-loop induced vacuum energy. Nevertheless, the computation of scattering amplitudes for these models may be afflicted with infrared divergences as we will discuss in more detail later.

In the context of the model with Atkin-Lehner symmetry there arises an obvious question which is how two-loops string physics modifies the delicate cancellation of the one-loop vacuum energy. The best way to answer this question is of course to compute its genus two cosmological constant. This is not an easy task as it can be seen from the physics literature during past years. The best way to accomplish such a computation seems to be using hyperelliptic formalism which has provided good results for supersymmetric heterotic strings \[^{[10, 11]}\]. However the expressions for the two-loops cosmological constant in a non-supersymmetric model may, and will, be quite unmanageable. This is a little bit disappointing but by no means the end of the story; as it was described in \[^{[12]}\] in the context of the bosonic string, if we look at the regions in the boundary of the genus two moduli space we can get some information about possible divergences. In particular, when a tubular neighborhood of a non-trivial homology cycle gets long and skinny we can rewrite the contribution to the genus-two vacuum energy in terms of the on-shell two-point function on
the torus for the states in the model [12]. In a sense we can partly satisfy our curiosity about
the physics that arise at two-loops by just looking at some one-loop amplitudes.

The paper is organized as follows. In Sec. 2 we describe the two-dimensional models
under study and make a summary of some well-known and not-so-well-known facts about
them. In Sec. 3 we study the underlying symmetries of the models and give an alternative
contraction based on free world-sheet fermions. We finish the section with the computation
of tree-level correlation functions and the construction of the low-energy effective action for
the massless states. In Sec. 4 the genus two cosmological constant for these heterotic strings
is computed. Secs. 5-6 are devoted to the computation of the two-point functions on the
torus for external massless states and in Sec. 7 we study the possibility of calculating mass
corrections from the gotten results. Finally in Sec. 8 we will summarize the conclusions.
For the sake of self-containment, some useful results about lattice theta functions, Riemann
surfaces in hyperelliptic formalism and the Weierstrass elliptic function are presented in the
Appendices.

2 Heterotic Strings in Two-Dimensions

We are going to focus ourselves in the study of a family of two-dimensional heterotic strings
[1, 13, 14, 15]. These models are constructed by directly compactifying the left-moving sector
of a 26-dimensional bosonic string into one of the 24 Niemeier lattices [16]. The right-movers
are those of a type II string compactified using the $\Gamma_8$ lattice\footnote{i.e., the root lattice of $E_8$, the only even, self-dual lattice in eight dimensions.}. However, as they stand, these
24 string models are supersymmetric. To break supersymmetry we mod out the right moving
sector by the operator [17]

$$\alpha = (-1)^F e^{2\pi i P_R \cdot \delta},$$

(2.1)

where $F$ is the target space fermion number, $P_R$ is the momentum in the $\Gamma_8$ lattice and $\delta$ is
a vector such that $2\delta \in \Gamma_8$. In our case we take

$$\delta = \left( \left( \frac{1}{2} \right)^4, 0^4 \right).$$

(2.2)

As it is usual in order to preserve modular invariance, in addition to the untwisted sector
whose states are $\alpha$-invariant we must add up the twisted states in which the string closes
modulo a transformation by $\alpha$ and then project again onto the states invariant under this
operator. At the end, we have to consider four subsectors, two of which belong to the
untwisted sector and correspond to the following pairing between the four conjugacy classes
of $SO(8)$ and certain set of vectors [17]

$$\left( \Gamma_8^+, v \right), \quad \left( \Gamma_8^-, s \right),$$

where $\Gamma_8^\pm$ are the subset of vectors in $\Gamma_8$ such that their scalar product with $\delta$ is respectively
an integer or a half-integer. In the twisted sector we have

$$\left( \Gamma_8^+ + \delta, o \right), \quad \left( \Gamma_8^- + \delta, c \right).$$
It is easy to see what the massless spectrum for each of these models is. Before the modding, the massless states are
\[
\begin{align*}
\hat{\alpha}^{-1}_{-1}|P_L^2 = 0\rangle \otimes |i\rangle, & \quad \hat{\alpha}^{-1}_{-1}|P_L^2 = 2\rangle \otimes |i\rangle, \\
& \quad |P_L^2 = 0\rangle \otimes |a\rangle, \\
& \quad |P_L^2 = 2\rangle \otimes |a\rangle,
\end{align*}
\]
where \{\{|i\rangle, |a\rangle\} span the \(8_v \oplus 8_s\) of \(Spin(8)\). The states in the first line correspond to neutral particles under the left-moving gauge group, while the particles in the second line are charged. All the states have \(P_R^2 = 0\).

When modding out by \(\alpha\) none of the fermionic states in the massless sector survive the projection, since they have \(\alpha = -1\) so we are left with one-half of the states (those in the first column). To construct the states in the twisted sector we go to the mass formula
\[
\alpha^{'2} m_R^2 = \frac{1}{2} P_R^2 + \sum_{n>0} \alpha^{-i}_n \alpha^{-i}_{n^i} + \sum_{r>1/2} r S^a_n S^a_r - \frac{1}{2},
\]
where \(P_R \in \Gamma_8^+ + \delta\) and \(r\) runs over positive half-integers\(^2\). The only way to get an \(\alpha\)-symmetric state with \(m^2 = 0\) is to have \(P_R^2 = 1\) and \(N_S = N_o = 0\), which means that it must belong to the \((\Gamma_8^+ + \delta, o)\) sector. In fact it can be seen that there are 16 such states corresponding to the 16 points in the \(\Gamma_8^+ + \delta\) lattice at \(P_R^2 = 1\). Then we finally find the following states in the massless sector
\[
\begin{align*}
\hat{\alpha}^{-1}_{-1}|P_L^2 = 0\rangle \otimes |i\rangle, & \quad |P_L^2 = 2\rangle \otimes |i\rangle, \\
& \quad |P_L^2 = 0\rangle \otimes |P_R^2 = 1; 0\rangle, \\
& \quad |P_L^2 = 2\rangle \otimes |P_R^2 = 1; 0\rangle;
\end{align*}
\]
0\(t\) in the right-moving part indicates the twisted vacuum defined by \(S^a_r|0\rangle = 0\) with \(r \geq 1/2\).

As a matter of fact we can divide these states into \(24 \times 24 = 576\) neutral bosonic particles plus \(24 \times r_\Gamma(1)\) charged bosons where \(r_\Gamma(1)\) is the number of sites at \(P_L^2 = 2\) in the corresponding Niemeier lattice. A quite remarkable property of this family of models is that the spectrum is Bose-Fermi degenerate in all mass levels except in the massless sector. In fact we can rewrite the massless states using the Neveu-Schwarz-Ramond (NSR) rather than Green-Schwarz (GS) formulation; this will be useful later when constructing vertex operators for the massless states. The states in the untwisted sector in NSR language can be easily read from the ones in GS formulation. In the case of the twisted states one only has to take into account that the ground state in the scalar conjugacy class of SO(8) is the standard NS vacuum \(|0_{NS}\rangle\), so we have
\[
\begin{align*}
\hat{\alpha}^{-1}_{-1}|P_L^2 = 0\rangle \otimes b^{-}_{\frac{1}{2}}, & \quad |P_L^2 = 2\rangle \otimes b^{-}_{\frac{1}{2}}|0_{NS}\rangle, \\
& \quad |P_L^2 = 0\rangle \otimes |P_R^2 = 1; 0\rangle, \\
& \quad |P_L^2 = 2\rangle \otimes |P_R^2 = 1; 0\rangle.
\end{align*}
\]

Two-dimensional heterotic strings can also be formulated using fermionic constructions\(^[18, 19, 20]\). In the fermionic model the right-moving sector is made out of a set of 24

\(^2\)The appearance of half-integers \(r\) and the normal ordering constant \(-1/2\) are due to the fact that in the twisted sector we have \(S^a(\sigma + \pi) = (-1)^F S^a(\sigma) = -S^a(\sigma)\).
free Majorana-Weyl fermions. In our case we will consider that all of them have the same boundary conditions on the world-sheet. In the path integral computation each sector of boundary conditions contributes with a definite sign that is fixed by the requirement that the resulting amplitude must be modular invariant. At one-loop level it can be seen that the correct choices for the signs are

\[ C(A, A) = -C(P, A) = -C(A, P) = 1, \]

where A and P stand for periodic or antiperiodic boundary conditions along each of the two homology cycles of the torus. The computation of the partition function and other observables can be simplified by bosonizing the fermions; then we are left with 12 free bosons living in the \( D_{12} \) root lattice. The choice of signs implies that the only conjugacy classes that contribute to the partition function are the vectorial and one of the spinorials. The massless spectrum can then be constructed. To this purpose one can use the familiar techniques used in the light-cone quantization of the superstring to find

\[ \bar{\alpha}^{A-1}|P_L^2 = 0\rangle \otimes b^{\frac{1}{2}}|NS\rangle, \]
\[ |P_L^2 = 2\rangle \otimes b^{\frac{1}{2}}|NS\rangle, \]

with \( A = 1, \ldots, 24 \) and \( |NS\rangle \) the Neveu-Schwarz vacuum (not to be confused with \( |0_{NS}\rangle \), the Neveu-Schwarz vacuum in the bosonic construction of the right-moving sector). It can be seen that the lowest state in the Ramond sector is in the first massive level \([13] \). Again, with \( m^2 = 0 \) we have \( 24 \times 24 = 576 \) neutral states and \( 24 \times r_\Gamma(1) \) charged ones. One can wonder now whether or not the bosonic and fermionic constructions give the same theory. In fact their partition functions are equal, they have the same massless spectrum and it can be seen that the number of states in the vectorial+scalar of \( SO(8) \) in the bosonic construction equals the number of states in the vectorial of \( SO(24) \) in the fermionic representation\(^3\). In the next section we will argue whether a fermionic construction exists for the 24 heterotic models described earlier in a bosonic fashion. To find such a construction will allow us to use either the bosonic or the fermionic formulation depending on what is the representation in which the computation is simpler.

Among the menagerie of 24 models, either in the fermionic or the bosonic realization, described above there is one whose properties deserve some attention. This is the model which is built up using the Leech lattice; the main characteristic of this lattice is the fact that it has no points at \((\text{length})^2 = 2\). This means that \( r_\Gamma(1) = 0 \) and then the massless spectrum is only made of neutral bosons. It most unexpected property is that it has no one-loop induced cosmological constant \([1, 13] \). The vanishing of the one loop vacuum energy is mathematically explained by the presence of a discrete symmetry, called Atkin-Lehner symmetry, which acts on the torus modular parameter \( \tau \). Although this cancellation mechanism was for some time regarded as a promising candidate to solve the cosmological constant problem, it was soon realized \([21] \) that this two-dimensional model was essentially the only consistent theory\(^4\). Since both theories are Bose-Fermi degenerate for \( m > 0 \) the number of fermionic states are also the same.

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\(^4\)
with Atkin-Lehner symmetry, since for any theory in higher dimensions the presence of this symmetry leads to the existence of fermionic tachyons which is forbidden by Lorentz invariance.

One can ask for the physical meaning of this phenomenon. It is well known that in field theory the only way to set the cosmological constant to zero without fine tuning is by considering theories in which there is a Bose-Fermi degeneration, i.e. supersymmetric theories.\(^4\) In this case however we have a theory that while having a net number of bosonic massless states has no one-loop vacuum energy. In spite of the stringy nature of this cancellation one would like to understand it in terms of field-theoretical degrees of freedom. The key to such an interpretation was given in [14]. There the toroidal compactification of these models into \(\mathbb{R} \times S^1\) was studied and a non-analytic behavior of the partition function as a function of the compactification scale was found at the self-dual size. It was also shown that the part of the partition function below the self-dual radius contains the contribution of \(24 \times 24\) bosonic states with the wrong sign, while above \(\sqrt{\alpha'}\) it contains a constant term which is equal to \(-24 \times 24\) times the vacuum energy of the \(c=1\) model. This exactly cancels the contribution to the vacuum energy of the net bosonic states in the Atkin-Lehner model as computed using the analog model.

The moral of the story is that if we want to understand the zero of the cosmological constant in the model with Atkin-Lehner symmetry in terms of field theory we need to introduce some intruder states which contribute to the partition function with the wrong sign and that when compactifying one of the open dimensions only get excited below the Planck scale. In the decompactification limit their contribution is just given by

\[
\Lambda_{\text{intruder}} = 24 \times 24 \int_{\mathcal{F}} \frac{d^2 \tau}{\tau^2_2} \tag{2.6}
\]

and then it cancels exactly the regularized vacuum energy of the massless states in the Atkin-Lehner model\(^5\). Of course this by no means implies the real existence of these states. They are just the result of trying to explain a stringy phenomenon using field-theoretical words.

Let us look back to the partition function in the bosonic construction. The right moving part is

\[
Z_R = \frac{\theta_3^4 - \theta_4^4}{2\eta^{12}} \Theta_{\Gamma^+_s} - \frac{\theta_2^4}{2\eta^{12}} \Theta_{\Gamma^-_s} + \frac{\theta_3^4 + \theta_4^4}{2\eta^{12}} \Theta_{\Gamma^+_s + \delta} - \frac{\theta_2^4}{2\eta^{12}} \Theta_{\Gamma^-_s + \delta}. \tag{2.7}
\]

In fact, the theta functions associated with the four sets of vectors can be rewritten in terms of the theta function for \(\Gamma_s\) with characteristics

\[
\Theta_{\Gamma^\pm_s} = \frac{1}{2} \Theta_{\Gamma_s} \begin{bmatrix} 0 & \pm 1 \\ 0 & \delta \end{bmatrix},
\]

\(^4\)Witten has pointed out recently [22] that in (2+1)-dimensional supergravity, due to a conical singularity at infinity, there are no global supercharges and then the vanishing of the cosmological constant could be accomplished without having Bose-Fermi degeneracy.

\(^5\)We have defined the vacuum energy as minus the integral to the fundamental region of the partition function in such a way that bosonic states contribute with a minus sign.
Using this expression it is clear the orbifold-like structure of the partition function. Since its left-moving part is not affected by the operator \( \alpha \) we can write

\[
Z = \sum_{m,n=0}^{1} Z_L(\bar{\tau}) Z_R^{(m,n)}(\tau),
\] (2.10)

where \( Z_R^{(m,n)} \) is the right moving contribution for the string with boundary conditions for the bosons twisted by \( (\alpha^m, \alpha^n) \) along the two homology cycles of the torus. Let us notice that the first term in (2.9) is equal to zero because of Jacobi's aequatio. Individually, each term in (2.10) can be written as a sum over spin structures \( e \) with given phases \( C_e(m,n) \) which can be read from (2.9)

\[
Z_R^{(m,n)} = \frac{1}{4} \sum_e C_e(m,n) \frac{\theta^4_e}{\eta^4} \Theta_{(m,n)}(0|\tau);
\] (2.11)

from now on, in order to simplify the expressions, we will write

\[
\Theta_{(m,n)}(0|\tau) = \Theta \left[ \frac{m\delta}{n\delta} \right] (0|\tau).
\] (2.12)

There is no contribution coming from world-sheet fermions with space-time indices, since this is cancelled by the contribution of the conformal and superconformal ghosts. The theta functions can be computed with the result

\[
\Theta_{(0,0)} = \frac{1}{2}(\theta_3^8 + \theta_4^8 + \theta_2^8), \\
\Theta_{(1,0)} = \theta_3^4 \theta_4^4, \\
\Theta_{(0,1)} = \theta_3^8 \theta_2^8, \\
\Theta_{(1,1)} = \theta_2^8 \theta_4^8.
\] (2.13)

On the other hand the left-moving partition function, which is common to all the sectors, can be written in terms of the modular invariant function \( j(\tau) \) as

\[
Z_L(\bar{\tau}) = j(\tau) - 720 + r_{\Gamma}(1).
\] (2.14)
3 Fermionic Constructions, Gauge Symmetry and the Low Energy Field Theory.

In the previous section we have discussed the construction of two-dimensional heterotic string models without space-time supersymmetry. Now we are going to study more carefully the fermionic realization of the family of 24 heterotic strings. After doing this we will try to extract the effective low-energy field theory for the massless particles.

At first sight there is an obvious asymmetry between the massless sectors in the bosonic (B) and fermionic (F) construction. In the B models massless particles are of two very different types\(^6\); on one hand we have the 8 untwisted states \(|i; P_R = 0\rangle\) which are in the vector of \(SO(8)\) and on the other we find the 16 twisted states \(|P_R^2 = 1\rangle\) associated with the 16 vectors in \(\Gamma_8^+ + \delta\) with \(P_R^2 = 1\). On the contrary in the F model we are left with 24 states \(b_{1/2}^A|\text{NS}\rangle\) in the vector of \(SO(24)\). It is not very pleasant to have such an asymmetry when we would like to identify both models.

In order to solve the mystery, let us look more closely to the B model. The 16 possible vectors \(P_R\) have coordinates in the orthonormal basis

\[
\left(\left(\pm \frac{1}{2}\right)^4, 0^4\right), \quad \left(0^4, \left(\pm \frac{1}{2}\right)^4\right),
\]

with an even number of minus signs. In fact the 16 vectors \((3.1)\) can be ordered in 8 pairs \(\{P_R^{(i)}, -P_R^{(i)}\}\), \((i = 1, \ldots, 8)\) such that \(P_R^{(i)} \cdot P_R^{(j)} = 0\) when \(i \neq j\). One can easily see that the set \((3.1)\) is isomorphic to the root system of \(SU(2)^8 \cong SO(3)^8\) (modulo a rescaling of the roots). Moreover, the 8 states in the untwisted sector fill the states in the Cartan subalgebra of the same Lie algebra. With this result at hand the most we can say is that the states in the massless sector in the B construction fit in the adjoint representation of \(SU(2)^8\). However in order to show that this is realized as a gauge symmetry of the theory we have to give a step forward and prove that there is a realization of the current algebra of \(SU(2)^8\) in the algebra of vertex operators.

Using the NSR formulation, the right-moving parts of the vertex operators in the 0 picture for states in the massless sector are

\[
V_0^j(k, z) = [\partial_z X^j + i (k_{\mu} \psi^\mu) \psi^j] e^{ik_{\mu} X^\mu(z)},
\]
\[
V_0^{P_R}(k, z) = i P_R^i \psi^j e^{i P_R^i X^i(z) + ik_{\mu} X^\mu(z)}.
\]

The second expression can be easily obtained by taking into account that the oscillator part of the twisted state is just that of the NS vacuum\(^7\). If we compute the OPE of these vertex

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\(^6\) In what follows we will drop the left-moving parts of the states whenever they are not relevant for the discussion.

\(^7\) In the GS formulation, the construction of the vertex operator for the twisted states requires the introduction of a new field \(\sigma^i(z), (i = 1, \ldots, 8)\) which creates the \(|i\rangle\) vacuum out of the twisted vacuum, \(|i\rangle = \sigma^i(0)|0_t\rangle\).
operators at $k = 0$ we find

\[
V_0^{P_R}(z)V_0^{-P_R}(w) = \frac{1}{(z - w)^2} - \frac{1}{z - w} i P_R^i V_0^i(w),
\]

\[
V_0^i(z)V_0^{P_R}(w) = \frac{-i}{z - w} P_R^i V_0^{P_R}(w),
\]

\[
V_0^i(z)V_0^j(w) = \frac{-\delta^{ij}}{(z - w)^2},
\]

(3.3)

with $V^{P_R}(z)V^{P_R'}(w) = 0$ when $P_R' \neq -P_R$. At this point after rescaling $V^j \to i V^j$ one is tempted to identify (3.3) as the OPE corresponding to the $k = 1$ $SU(2)^8$ Kac-Moody algebra in the Cartan-Weyl basis. However this would not be correct; the reason is that $P_R^2 = 1$, contrary to the $SU(2)^8$ roots which are canonically normalized to $\alpha^2 = 2$. This is relevant, since the components of the roots correspond to the structure constants of the Lie algebra in the Cartan-Weyl basis. In order to recover the standard form of the $SU(2)^8$ affine algebra we have to renormalize the vertex operators as $V_0 \to \sqrt{2}V_0$. After this we get the canonical OPE of a level 2 $SU(2)^8$ Kac-Moody algebra. Then we see that the introduction of the twisted states enhances the right-moving gauge symmetry from $U(1)^8$ to $SU(2)^8$ and then the full symmetry of the string theory is $G_L \times SU(2)^8$ with $G_L$ the gauge group associated with the corresponding Niemeier lattice (or $G_L = U(1)^24$ for the Leech lattice [16]).

Let us move to the F models. Now the internal CFT is that of a system of 24 Majorana-Weyl fermions all of them with the same world-sheet boundary conditions. In such a system there is a $N = 1$ superconformal symmetry generated by non-linear transformations

\[
\delta_\epsilon \lambda^A = \frac{i \epsilon}{\sqrt{2C_2(G)}} f^{ABC} \lambda^B \lambda^C,
\]

(3.4)

where $f^{ABC}$ are the structure constants of a semisimple Lie algebra $G$ (dim $G = 24$), $C_2(G)$ is the quadratic Casimir of the adjoint representation of $G$ and $\epsilon$ is an anticommuting infinitesimal parameter. Combining the fermions $\lambda^a$ with the structure constants one can construct the following currents

\[
J^A(z) = \frac{i}{2} f^{ABC} \lambda^B \lambda^C,
\]

(3.5)

which generate an affine algebra $\hat{G}$ with level $k = C_2(G)/2$. In fact it can be shown that all fermionic models can be classified in terms of a pair of semi-simple Lie groups $G$, $H$ such that $H \subset G$ and $G/H$ is a symmetric space ($G = \text{Lie}(G)$). This last condition implies that the theory can be truncated without breaking $N = 1$ superconformal symmetry by projecting on the states with $(-1)^{F_{\text{pseudo}}} = 1$, where $F_{\text{pseudo}}$ is the fermion number for the $\lambda^i$ with indices in $G - H$. This modding breaks the actual gauge symmetry of the system from $G$ to $H$.

\[G/H\] is a symmetric space if there exists an involutive automorphism $\sigma$ in $G$ such that $G^{0}_\sigma \subset H \subset G_\sigma$, where $G_\sigma$ is the set of points in $G$ fixed by $\sigma$ and $G^{0}_\sigma$ is its identity component.
To connect with the B model we only have to take $G = H = SU(2)^8$ so we take the same GSO projection for all worldsheet fermions. The right moving part of the vertex operators for the massless states in the 0 picture are

$$V_0^A(k, z) = \left[ i \sqrt{2} f^{ABC} \lambda^B \lambda^C + i(k_\mu \psi^{\mu}) \lambda^A \right] e^{ik_\mu X^\mu(z)}. \quad (3.6)$$

These vertex operators create states that are in the adjoint representation of $SU(2)^8$ and that when taken at zero external momentum generate a $SU(2)^8_{k=2} \simeq SO(3)^8_{k=2}$ Kac-Moody algebra.

Then we have two constructions, bosonic and fermionic, with the same underlying symmetry, namely $SU(2)^8_{k=2}$. In fact the two constructions (B and F) give the same answers when computing scattering amplitudes as can be checked at tree level (genus zero). In forthcoming sections we will also see that this is true in one-loop calculations. This strongly suggests that the B and F constructions render string theories that are completely equivalent.

To finish this section we are going to get the low-energy effective theory for the on-shell massless states. As we show these states are 192 neutral untwisted bosons $\Phi^{(I)} (I = 1, \ldots, 24; i = 1, \ldots, 8)$, 384 neutral twisted ones $\Psi^{I,P_R}$ ($P_R$ runs over all $P_R \in \Gamma^+ + \delta$ with $P_R^2 = 1$) and the corresponding charged particles, $8 \times r_T(1) \Phi^{\alpha,i}$ and $16 \times r_T(1) \Psi^{\alpha,P_R}$ in the untwisted and twisted sectors respectively, with $\alpha$ running over the roots of the left-moving gauge group. However from our previous discussion we know that the massless states are in the $(adj, adj)$ representation of the gauge group $G_L \times SU(2)^8$. Then we can write them in shorthand as a single field $\Phi = \Phi^{AA} T^A_L \otimes T^A_R$ where $T^A_L$ and $T^A_R$ are the generators in the fundamental representation of $G_L$ and $SU(2)^8$ respectively.

To get the couplings between the low-energy fields we have to compute the scattering amplitudes for the corresponding vertex operators at tree level in the string loop expansion. Two point functions vanish, reflecting the fact that the string equations are satisfied at tree level. The coupling involving three fields can be easily computed using either the B or F model with the resulting term in the effective action

$$A_3 = \frac{1}{\sqrt{2}} f^{ABC}_L f^{ABG}_R \Phi^{AA} \Phi^{BB} \Phi^{CC} = \frac{1}{\sqrt{2}} \text{Tr} \{ \Phi, \Phi \} \Phi, \quad (3.7)$$

where we have introduced the left and right-moving structure constants and the commutator has to be understood as the tensor product of commutators for the left and right-moving generators. Because of the presence of $f^{ABC}_L$ we see that this coupling vanishes for the theory constructed in the Leech lattice, in which the left-moving group is abelian. In the general case in which $r_T(1) \neq 0$ the coupling exists but only between one untwisted and two twisted states with opposite values of $P_R$ (or in other words, between one neutral and two charged states with total $SU(2)^8$-charge equal to zero). This can be understood from the known results in lower-dimensional heterotic strings \[24\]: the right-moving part of the amplitude for three gauge bosons is given by the contraction of the polarization tensors with space-time momenta $\zeta^{\mu}_i k_{j\mu}$. For untwisted states, polarizations lie always in the internal space and thus
are orthogonal to all space-time momenta, forcing the amplitude to vanish. On the contrary when twisted states are present we have internal momenta \( P_R \) and then the right-moving part of the amplitude does not vanish but it is proportional to the \( P_R^i \) which are essentially the structure constants for \( SU(2)^8 \) in the Cartan-Weyl basis.

In the case of the four-fields coupling the computation is a little more involved since a Koba-Nielsen integral has to be performed. Taking the leading terms in the limit \( \alpha' \to 0 \) it can be seen that the corresponding contribution to the low-energy action is

\[
A_4 = \frac{1}{2} f_L^{ABE} f_L^{EDC} f_R^{ABE} f_R^{EDC} \Phi^{\bar{A}} \Phi^{\bar{B}} \Phi^{\bar{C}} \Phi^{\bar{D}} + \frac{\alpha'}{8} \left( f_L^{ABE} f_L^{ABE} \delta^{AC} \delta^{BD} + \frac{1}{2} \delta^{\bar{A}\bar{C}} \delta^{\bar{B}\bar{D}} f_R^{ABE} f_R^{CD} \partial_\mu \Phi^{\bar{A}} \Phi^{\bar{B}} \Phi^{\bar{C}} \Phi^{\bar{D}} \right).
\]

\[ (3.8) \]

Now we can construct the low-energy field theory for the massless fields \( \Phi \). Retaining only the leading terms in the \( \alpha' \) expansion the result is

\[
S = \frac{1}{2} \int d^2 x Tr \left\{ \partial_\mu \Phi \partial^\mu \Phi + \frac{g}{\alpha'} [\Phi, \Phi] \Phi + \frac{g^2}{\alpha'} [\Phi, \Phi]^2 \right\},
\]

\[ (3.9) \]

where \( g \) is the dimensionless gauge coupling constant which is proportional to the string coupling constant and inversely proportional to the square root of the product of the levels of the right and left-moving Kač-Moody algebras. The effective action get simpler if we particularize to the case of the Leech lattice since now all the commutators vanish

\[
S_{\text{Leech}} = \frac{1}{2} \int d^2 x \partial_\mu \Phi \partial^\mu \Phi,
\]

\[ (3.10) \]

i.e., we are left with a sigma-model defined on \( U(1)^{24} \otimes SU(2)^8 \).

### 4 Genus-Two Cosmological Constant for the Two-Dimensional Models

One way to study the physics that arises after turning on the interaction between strings is to compute the genus two vacuum energy. Higher genus computations in string theory have been source of discussion along the years. In the ten-dimensional \( E_8 \times E_8 \) or \( SO(32) \) heterotic string some expressions have been proposed which vanish, as it is expected from supersymmetry \[10, 25\]. However the main drawback of these computations is the fact that the vanishing expressions are not modular invariant. In ref. \[11\] a way of computing a two-loop (vanishing) modular invariant cosmological constant was finally designed. In the case of the supersymmetric heterotic string the two-loops cosmological constant can be written as an integral over the fundamental region of \( Sp(2, \mathbb{Z}) \) of an expression which is identically zero due to some combinations of standard Riemann identities. In our case, however, we do not expect this to be the case and therefore the usual argument in favor of the expressions
given in [10] (that zero is always modular invariant) cannot even be applied. In the following computation we will closely follow ref. [11], which we regard as the most clarifying approach, and we will be able to get a modular invariant expression for the integrand of the cosmological constant. We will use the fermionic construction of the model in which the computations notably simplify.

The starting point is a modification of the Knihznik formula [10] for the two-loops cosmological constant in hyperelliptic formalism (for definitions and notations see Appendix B)

\[ Z_{g=2} = \sum_{e} C(e) \int \prod_{i=1}^{6} d^{2}a_{i} \frac{1}{d^{2}v_{pr}} T^{-1} \prod_{k<l} a_{kl}^{-3} F(\Lambda_{24}) \prod_{k<l} a_{kl}^{-2}[P^{X} + P^{gh}] O_{e}^{3}; \]  

(4.1)

\( F(\Lambda_{24}) \) is the partition function for the left-moving bosonic sector and \( C(e) \) are the phases that generalize (2.4) for the right-moving world-sheet fermions at genus two. The correlation of the two PCOs now is a little bit different from the one for the ten-dimensional heterotic string since a PCO has a space-time and an internal part

\[ P_{+1}^{s-t} + P_{+1}^{int}. \]

The correlator then is written

\[ \langle P_{+1}(z)P_{+1}(w) \rangle = \langle P_{+1}^{s-t}(z)P_{+1}^{s-t}(w) \rangle + \langle P_{+1}^{int}(z)P_{+1}^{int}(w) \rangle. \]  

(4.2)

Nonetheless, as we take the limit \( z \rightarrow a_{1}, w \rightarrow a_{2} \) it can be seen that the internal part does not contribute to (4.1) so we have

\[ P^{X} = \frac{1}{5} P_{10}^{X}, \]  

(4.3)

where \( P_{10}^{X} \) is given by (B.6). For the ghost part we find just the same result since the ghost content of the two-dimensional models is the same than in the ten-dimensional case.

As it is argued in Appendix B, it is convenient to eliminate the \( SL(2, \mathbb{C}) \) redundancy by introducing the harmonic ratios (B.3). Then (4.1) can be rewritten in terms of \( \lambda_{i} \) as

\[ Z_{g=2} = \sum_{e} C(e) \int \prod_{i=1}^{3} d\lambda_{i} W_{2}^{-1}(\lambda_{i})F(\Lambda_{24})[U_{e}^{X} + U_{e}^{gh}], \]  

(4.4)

where

\[ F(\Lambda_{24}) = \prod_{i=1}^{3} d\lambda_{i} \frac{a_{12}a_{15}a_{25}a_{26}a_{36}a_{46}}{a_{45}a_{56}} \prod_{k<l} a_{kl}^{-3} \Theta(\Lambda_{24}), \]  

(4.5)

\( W_{2}(\lambda_{i}) = |a_{12}a_{45}a_{36}|^{2}T \) and

\[ U_{e}^{X} = \frac{1}{8} \prod_{k<l} a_{kl}^{-2} \frac{a_{12}a_{15}a_{25}a_{26}a_{36}a_{46}}{a_{12}a_{45}a_{56}} \left\{ a_{23}a_{24}a_{25} \left( \frac{a_{26}}{a_{16}} \right) P_{12} + (a_{1} \leftrightarrow a_{2}) \right\} O_{e}, \]  

(4.6)

\[ U_{e}^{gh} = \prod_{k<l} a_{kl}^{-2} \frac{a_{12}a_{15}a_{25}a_{36}a_{46}}{a_{45}a_{56}} P_{e}^{gh} O_{e} \]

and \( P_{12} \) is defined as

\[ P_{12} = \left( \frac{a_{26}}{a_{16}} \right)^{2} \frac{P_{12}}{T}. \]  

(4.7)
The strategy now is that of ref. [11]. \( Z_{g=2} \) as given by (4.4) is of the form

\[
Z_{g=2} = \sum_e I_e.
\]

(4.8)

However, in general, the contributions \( I_e \) are not even invariant under the subgroup of modular transformations \( \Gamma_e \) that leaves the spin structure \( e \) unchanged. Nevertheless not everything is lost since \( I_e \) is invariant under a subgroup of \( \Gamma' \subset \Gamma_e \). Then we can use the results of [26] and perform a coset extension from \( \Gamma' \) to \( \Gamma_e \). Once we have such a \( \Gamma_e \)-invariant extension \( I_e \) of \( I_e \) we can further extend it to the full modular group by the same procedure.

The final result of the coset extension to the full modular group will still depend on the spin structure we started with [11]. In fact we have two orbits of spin structure contributions \( I(e) \) which cannot be transformed into one another using modular transformations; these are the orbits that contain respectively the contributions of the spin structures \( (123) \) and \( (12) \). The way to decide between the two possible results is that the final expression has to have the good factorization properties. In the ten-dimensional heterotic string, and in our case also, the correct answer is gotten by starting with the \( (123||456) \) spin structure.

Then let us begin with \( I_{(123||456)} \). The permutations of the branch points that generate the subgroup \( \Gamma_{(123||456)} \subset \Gamma \) are \( (12), (23), (45), (56) \) and \( (14)(25)(36) \). \( I_{(123||456)} \) as read from (4.4) can be decomposed into a matter part and a ghost part \( I_{(123||456)} = I^X_1 + I^{gh}_1 \).

Furthermore, looking at the explicit expressions of \( I^X_1 \) and \( I^{gh}_1 \) we see that they can be written respectively as

\[
I^X_1 = I^X_{1,1} + (12)I^X_{1,1},
I^{gh}_1 = I^{gh}_{1,1} + (45)I^{gh}_{1,1} + (56)(45)I^{gh}_{1,1},
\]

(4.9)

where

\[
I^X_{1,1} = \frac{1}{8}(\wedge_{i=1}^3 d\lambda_i)W_2^{-1}F(\Lambda_{24}) \frac{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2 \lambda_3^2 (\lambda_2 - 1)^2 (\lambda_3 - 1)^2} P_{12},
\]

\[
I^{gh}_{1,1} = \frac{1}{4}(\wedge_{i=1}^3 d\lambda_i)W_2^{-1}F(\Lambda_{24}) \frac{(\lambda_2 - \lambda_1)^2(\lambda_3 - 1)^2}{\lambda_1^2 \lambda_2^2 \lambda_3^2 (\lambda_3 - 1)^2 (\lambda_2 - 1)^2 (\lambda_3 - 1)}. \]

(4.10)

Then, since \( I_{1,2} \) as well as \( I^{gh}_{1,2} \) and \( I^{gh}_{1,3} \) are obtained from \( I^X_{1,1} \) and \( I^{gh}_{1,1} \) by transformations that belong to \( \Gamma_{(123||456)} \) we can make the coset extension directly from (1.10).

Looking at \( I^{gh}_{1,1} \) and applying the generators of \( \Gamma_{(123||456)} \) we find

\[
(12)I^{gh}_{1,1} = -\frac{\lambda_2(\lambda_3 - \lambda_2)(\lambda_1 - 1)}{\lambda_1(\lambda_2 - 1)(\lambda_3 - \lambda_1)} I^{gh}_{1,1},
\]

\[
(23)I^{gh}_{1,1} = \frac{(\lambda_2 - \lambda_1)^2(\lambda_3 - 1)}{(\lambda_3 - \lambda_1)^2(\lambda_2 - 1)} I^{gh}_{1,1},
\]

\[
(45)I^{gh}_{1,1} = \frac{\lambda_2^2}{\lambda_1} I^{gh}_{1,1},
\]

12
(56) \( I_{1,1}^{gh} = I_{1,1}^{gh} \),

\[
(14)(25)(36) I_{1,1}^{gh} = -\frac{\lambda_2 - \lambda_1}{(\lambda_3 - \lambda_1)(\lambda_2 - 1)} I_{1,1}^{gh}.
\]

These are exactly the same transformations that one finds for the case of the ten-dimensional supersymmetric heterotic string. Following the same steps than in [11] we can see that after performing the coset extension to the full modular group we are going to have \( I_{(1)}^{gh} = 0 \). In fact the vanishing of the ghost contribution can be seen in a more general context. The ghost part of the correlation of the two PCOs, \( P_{gh} \), is a holomorphic function as can be easily seen from its expression (B.8) or (B.9). This means that the integrand of \( I_{(1)}^{gh} \) factorizes into a holomorphic and an antiholomorphic function of the period matrix

\[
\int I_{(1)}^{gh} = \int \prod_{i<j}^2 d^2 \tau_{ij} (\det \imath \tau)^{-1}(\Delta(2))^{-2} \Theta(\Lambda 24) Z_R^{gh}(\tau),
\]

where \( \Delta_2 = \prod e^\theta [e](0|\tau) \), the product being over the ten even spin structures. Since by construction \( I^{gh} \) is modular invariant, \( Z_R^{gh} \) must be a modular function of weight 2. Moreover if the theory has no right-moving tachyons (as it is the case for both the supersymmetric heterotic string and the two dimensional models under consideration) \( Z_R^{gh} \) must be not only a function but a weight 2 modular form under \( SP(2, \mathbb{Z}) \). However, as proved by Igusa [27], there is no modular functions of weight 2 at genus two, and then \( Z_R^{gh}(\tau) = 0 \) (cf. [28]).

Let us turn now to the matter part. Now the generators of \( \Gamma_{(123)|456} \) act on \( I_{1,1}^X \) as

\[
(12) I_{1,1}^X = \frac{\lambda_1(\lambda_3 - \lambda_1)(\lambda_2 - 1)^3}{\lambda_2(\lambda_3 - \lambda_2)(\lambda_1 - 1)^3} P_{12} I_{1,1}^X = M_{(1),21} I_{1,1}^X,
\]

\[
(23) I_{1,1}^X = -\frac{\lambda_3(\lambda_2 - \lambda_1)(\lambda_2 - 1)}{\lambda_2(\lambda_3 - \lambda_1)(\lambda_3 - 1)} P_{12} I_{1,1}^X = M_{(1),13} I_{1,1}^X,
\]

\[
(45) I_{1,1}^X = I_{1,1}^X,
\]

\[
(56) I_{1,1}^X = I_{1,1}^X,
\]

\[
(14)(25)(36) I_{1,1}^X = \frac{\lambda_2^2(\lambda_2 - \lambda_1)(\lambda_2 - 1)}{\lambda_2(\lambda_3 - \lambda_2)(\lambda_1 - 1)^2} P_{12} I_{1,1}^X = M_{(1),45} I_{1,1}^X,
\]

where \( P_{ij} \) are the obvious generalizations of \( P_{12} \). In general given a transformation \( g \in \Gamma_{(123)|456} \) which takes the pair \( \{12\} \) to \( \{ij\} \) we will have \( g \cdot I_{1,1}^X = M_{(1),ij} I_{1,1}^X \) with \( M_{(1),ij} \) proportional to \( P_{ij}/P_{12} (M_{(1)12} = 1) \). If we want to avoid overcounting we have to consider only one transformation \( g \) such that \( g(12) = (ij) \). This leaves only 12 transformations and the \( \Gamma_{(123)|456} \)-invariant extension of \( I_{1,1}^X \) is given by

\[
\tilde{I}_{1,1}^X = \sum_{i,j} M_{(1),ij} I_{1,1}^X.
\]

Now we have to perform the last coset extension from \( \Gamma_{(123)|456} \) to the full modular group \( \Gamma \). To do so we have to consider modular transformations that take the spin structure
(123|456) into any other of the remaining 9 even spin structures. Because \( I_{(1)} \) is the sum of 12 terms and we have ten even spin structures, the modular invariant result will be the sum of 120 terms of the type \( M_{(i),jk} I_{1,1}^X \)

\[
\sum_e I_e = \sum_{i=1}^{10} \sum_{j,k} M_{(i),jk} I_{1,1}^X.
\]  

(4.15)

In fact it is somewhat convenient to reorder the previous expression as a sum of 30 terms of the form

\[
(M_{(1),ij} + M_{(2),ij} + M_{(3),ij} + M_{(4),ij}) I_{1,1}^X,
\]

(4.16)

where \( M_{(k),ij} (k = 2, 3, 4) \) are obtained from \( M_{(1),ij} \) by 3 generators that leave invariant the pair \( \{ij\} \). The point is that all \( M_{(k),ij} \) are proportional to \( \frac{P_{ij}}{P_{12}} \). From (4.11) and (4.13) we find that for example

\[
4 \sum_{i=1} M_{(i),12} = 1 + \frac{\lambda_3^3 \lambda_2^3 (\lambda_3 - 1)^3 - \lambda_3^3 (\lambda_3 - 1)^3 - \lambda_2^3 (\lambda_1 - 1)^3 (\lambda_2 - 1)^3}{(\lambda_3 - \lambda_1)^3 (\lambda_3 - \lambda_2)^3}.
\]

(4.17)

Using genus two Riemann theta functions this reads

\[
\left\{ \theta^{12} \left[ \begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \right] (0|\tau) \sum_{i=1}^4 M_{(i),12} \right\} = \theta^{12} \left[ \begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \right] (0|\tau) - \theta^{12} \left[ \begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array} \right] (0|\tau) + \theta^{12} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] (0|\tau) - \theta^{12} \left[ \begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{2} \end{array} \right] (0|\tau).
\]

(4.18)

The complete expression for the genus two cosmological constant will be the sum of 30 terms of the type (4.17) which can be obtained from it by modular transformations. As a concrete example, using the modular transformation (12) we find

\[
\sum_{i=1}^4 M_{(i),21} = - \left[ 1 + \frac{\lambda_3^3 \lambda_2^3 (\lambda_3 - 1)^3 - \lambda_3^3 (\lambda_3 - 1)^3 - \lambda_2^3 (\lambda_1 - 1)^3 (\lambda_2 - 1)^3}{(\lambda_3 - \lambda_1)^3 (\lambda_3 - \lambda_2)^3} \right] \times \frac{\lambda_1 (\lambda_3 - \lambda_1) (\lambda_2 - 1)^3 P_{21}}{\lambda_2 (\lambda_3 - \lambda_2) (\lambda_1 - 1) P_{12}}.
\]

(4.19)

Then we have arrived at a modular invariant expression for the genus two cosmological constant of the 24 two-dimensional heterotic models under study. As we expect from the fact that they are not supersymmetric, the integrand of the cosmological constant does not vanish identically contrary to the case of the ten-dimensional heterotic string \([11]\). However the expression gotten (of which (4.17) and (4.19) are just a piece) is rather difficult to work with. To check whether or not \( \Lambda_{2-loops} \) vanishes we should integrate this expression to the fundamental domain in the \( \lambda_i \)-space which seems a rather scary and maybe impossible task. We will follow a different path and will turn to the computation of the one-loop amplitude with two external massless states. This computation hopefully will serve us in a double way; from it we can get the mass corrections to the massless states in the theory and some indirect information about the genus two cosmological constant could be extracted along the lines of \([12]\).
The Two-Point Function for Massless Neutral Bosons at One Loop

For the computation of the two-point function for two massless states we will use the bosonic construction and then our task will be four-fold, since we will have to compute the amplitude for charged and neutral states in the untwisted and twisted sectors of the theory. In this section we will perform the computation for the states in the Cartan subalgebra of the left-moving gauge group for both twisted and untwisted states leaving for the next section the computation for charged states.

In the B formulation of the model, the world-sheet action is

\[ S[X^\mu, X^i, \psi^\mu, \psi^i, \phi^I] = S_{2d}[X^\mu, \psi^\mu] + S_{R,int}[X^i, \psi^i] + S_{L,int}[\phi^I], \quad (5.1) \]

where

\[ S_{2d} = -\frac{1}{8\pi} \int d^2z [\partial_z X^\mu \partial_z X_\mu + 2i\psi^\mu \partial_z \psi_\mu], \]
\[ S_{R,int} = -\frac{1}{8\pi} \int d^2z [\partial_\bar{z} X^i \partial_z X^i + 2i\psi^i \partial_\bar{z} \psi^i + \lambda_R \partial_z X^i \partial_\bar{z} X^i], \]
\[ S_{L,int} = -\frac{1}{8\pi} \int d^2z [\partial_\bar{z} \phi^I \partial_z \phi^I + \lambda_L \partial_z \phi^I \partial_\bar{z} \phi^I]. \quad (5.2) \]

and \( \lambda_{L,R} \) are Lagrange multipliers enforcing the chiral character of the bosons. In what follows we will use units in which \( \alpha' = 2 \).

Let us begin with neutral untwisted states. The vertex operators in the zero picture are

\[ V_0^{I,i}(k; z) = \frac{\kappa}{\pi} J^I(\bar{z}) [\partial_z X^i + i(k_\mu \psi^\mu)\psi^i](z)e^{ik_\mu X^\mu(z, \bar{z})}. \quad (5.3) \]

Here \( \mu = 0, 1 \) is a space-time index and \( i = 1, \ldots, 8 \) labels the eight internal dimensions in the right-moving sector. \( \kappa \) is the string coupling constant and the \( J^I \) are any of the 24 currents associated with the Cartan subalgebra of the left-handed gauge group \( G_L \)

\[ J^I(\bar{z}) = i\partial_\bar{z} \phi^I, \quad (5.4) \]

where \( \phi^I \) live in the 24-dimensional Niemeier lattice.

To compute the amplitude we have to evaluate the correlator of two vertex operators on the torus fixing simultaneously the spin structures \( e \) for the world-sheet fermions and the boundary conditions \( (\alpha^m, \alpha^n) \) for the bosons \( X^a \), and then sum over \( e, m \) and \( n \). We have

\[ A^{IJ,ij}_{(m,n)}(k) = \frac{\kappa^2}{\pi^2} \sum_e C_e(m, n) \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \int d^2z \eta(\tau) \frac{-24\theta^4[e]}{4\eta^{12}} \langle J^I(\bar{z})J^J(0) \rangle \times \langle [\partial_z X^i + i(k \cdot \psi)\psi^i](z) [\partial_z X^j - i(k \cdot \psi)\psi^j](0) \rangle_{(m,n)} \langle e^{ik \cdot X(z, \bar{z})}e^{-ik \cdot X(0)} \rangle. \quad (5.5) \]

The sub and superscripts in the second correlator indicate the boundary conditions on the torus and all the correlators are computed integrating over the matter fields in a fixed point.
of the moduli space. The first correlator can be computed by splitting \( \phi^I \) into a classical and a quantum piece \( \phi^I = \phi_{cl}^I + \phi_q^I \) and then summing over classical vacua and integrating the quantum fluctuations. Classical vacua are labeled by the vectors \( P_L \in \Lambda_{24} \):

\[
\phi_{cl}^I(z) = \phi_0^I + 2\pi P^I \int \bar{z} \omega,
\]

where \( \omega \) is the abelian 1-form on the torus. Using \( \langle \partial_z \phi_q^I \rangle = 0 \) we are left with

\[
\langle J^I(z)J^J(0) \rangle = -\langle \partial_z \phi_q^I(z) \partial_w \phi_q^J(0) \rangle \sum_{P \in \Lambda_{24}} e^{-i\pi P^2 L} \]  
- \( (2\pi)^2 \sum_{P_L \in \Lambda_{24}} P_L^I P_L^J e^{-i\pi P^2 L}. \]  

(5.7)

The quantum part is readily evaluated in terms of the prime form \( E(z,0) = \theta_1(z|\tau)/\theta_1'(0|\tau) \)

\[
\langle \partial_z \phi_q^I(z) \partial_w \phi_q^J(0) \rangle = -\delta^{IJ} \partial_z^2 \ln E(z,0) = \delta^{IJ} \partial_z^2 \ln E(z,0).
\]

(5.8)

At the same time, writing the second sum in (5.7) as the derivative of the theta function of \( \Lambda_{24} \) we finally find

\[
\langle J^I(z)J^J(0) \rangle = -\delta^{IJ} \partial_z^2 \ln E(z,0) \Theta_{\Lambda_{24}}(0|\tau) + \pi \delta^{IJ} \frac{\partial}{\partial \tau} \Theta_{\Lambda_{24}}(0|\tau).
\]

(5.9)

The third correlator in (5.5) is easily seen to be equal to

\[
\langle e^{ik \cdot X(z,w)} e^{-ik \cdot X(0)} \rangle = e^{k^2 (X(z,w)X(0))} = e^{k^2 G(z,0)},
\]

(5.10)

where the boson propagator is

\[
G(z,w) = -\ln |E(z,w)|^2 + \frac{2\pi}{\tau_2} (\text{Im} z)^2,
\]

(5.11)

and then

\[
\langle e^{ik \cdot X(z,w)} e^{-ik \cdot X(0)} \rangle = e^{\frac{2k^2}{\tau_2} (\text{Im} z)^2 |E(z,0)|^{-k^2}}.
\]

(5.12)

To finish, we are left with the computation of

\[
\langle [\partial_z X^i + i(k \cdot \psi)\psi^i](z)[\partial_w X^j - i(k \cdot \psi)\psi^j](0) \rangle_{(m,n)},
\]

(5.13)

which reduces to

\[
\langle \partial_z X^i(z) \partial_w X^j(0) \rangle_{(m,n)} = k^2 \delta^{ij} S_{\epsilon}(z,0)^2 \Theta_{(m,n)},
\]

(5.14)

where we have used the fermion propagator (Szegö kernel)

\[
\langle \psi(z) \psi(w) \rangle = S_{\epsilon}(z,w) = \frac{\theta[\epsilon](z-w|\tau)}{E(z,w)\theta[\epsilon](0|\tau)}.
\]

(5.15)
The first term in (5.14) has to be computed along the same lines as the \( \langle JJ \rangle \) correlator, but now taking into account that \( X^i \) has boundary conditions twisted by \((\alpha^m, \alpha^n)\) along the two homology cycles of the torus

\[
\langle \partial_z X^i(z) \partial_w X^j(m,n) \rangle = \delta^{ij} \delta^2 \ln E(z,0) \Theta_{(m,n)}(0|\tau) + \frac{\pi}{2i} \delta^{ij} \frac{\partial}{\partial \tau} \Theta_{(m,n)}(0|\tau). \tag{5.16}
\]

Putting all the ingredients together we get the amplitude for fixed boundary conditions \((m,n)\)

\[
A^{IJ,ij}_{(m,n)}(k) = \frac{k^2}{\pi^2} \delta^{IJ} \delta^{ij} \sum_e C_e(m,n) \int_{F} \frac{d^2 \tau}{\tau_2} \int d^2 z \bar{\eta}^{-24} \frac{\Theta^4[\epsilon]}{4\eta^{12}} F_L(\bar{\tau}) e^{k^2 G(z,0)} \times \left\{ \delta^2 z \ln E(z,0) \Theta_{(m,n)} + \frac{\pi}{2i} \frac{\partial}{\partial \tau} \Theta_{(m,n)} - k^2 S_e(z,0)^2 \Theta_{(m,n)} \right\}, \tag{5.17}
\]

where \( F_L(\bar{\tau}) \) is defined through \( \langle J^I(z) J^J(0) \rangle = \delta^{IJ} F_L(\bar{\tau}) \).

In all the calculation that have led us to (5.17) we have maintained \( k^2 \) without implementing the on-shell condition for the massless bosons \( k^2 = 0 \). At face value if we set \( k^2 = 0 \) in (5.17) we get rid of the term proportional \( S_e^2 \). However one has to be very careful, since after performing the integral in \( z \) we can have terms of the form \( 1/k^2 \) which might cancel the overall \( k^2 \) to give a finite result \[29\]. In the case of the \((0,0)\) sector this is not even needed, since we have

\[
\sum_e C_e(0,0) \Theta^4[\epsilon](0|\tau) S_e(z,0)^2 = 0, \tag{5.18}
\]
due to a Riemann identity \[30\]. In the other sectors, however, we do not have any Riemann identity so we have to study the limit \( k^2 \to 0 \). In principle the only source of divergence in the integral over \( z \) is the point \( z = 0 \) in which the two insertion points collide. In fact it can be checked that in the limit \( k^2 \to 0 \) there is no cancellation of the prefactor \( k^2 \) and then we find that these part of the regularized integral vanishes in that limit. Taking \( k^2 = 0 \) in (5.17) we have

\[
A^{IJ,ij}_{(m,n)}(k^2 = 0) = \frac{k^2}{\pi^2} \delta^{IJ} \delta^{ij} \sum_e C_e(m,n) \int_{F} \frac{d^2 \tau}{\tau_2} \bar{\eta}^{-12} \frac{\Theta^4[\epsilon]}{4\eta^{12}} \int d^2 z F_L(\bar{\tau}) \times \left\{ \delta^2 z \ln E(z,0) \Theta_{(m,n)}(0|\tau) + \frac{\pi}{2i} \frac{\partial}{\partial \tau} \Theta_{(m,n)}(0|\tau) \right\}. \tag{5.19}
\]

First of all, one has to check that this expression is modular invariant after summing over all boundary conditions \((m,n)\) and spin structures. This is easily done taking into account that if \( \Lambda \) is a \( d \)-dimensional self-dual lattice, then under \( S: \tau \to -1/\tau \), together with the transformation of \( \Theta_{\Lambda}(0|\tau) \) given in (A.3), we have

\[
\frac{\partial}{\partial \tau} \Theta_{\Lambda} \left[ \begin{array}{c} a \\ b \end{array} \right] (0|\tau) \to \tau^{\frac{d}{2}+1} \frac{\partial}{\partial \tau} \Theta_{\Lambda} \left[ \begin{array}{c} -b \\ a \end{array} \right] (0|\tau) + \frac{d}{2} \tau^{\frac{d}{2}+1} \Theta_{\Lambda} \left[ \begin{array}{c} -b \\ a \end{array} \right] (0|\tau). \tag{5.20}
\]
and the prime form transforms according to
\[ \partial_z^2 \ln E(z,0) \rightarrow \tau^2 \partial_z^2 \ln E(z,0) + 2\pi i \tau. \] (5.21)

The invariance of the amplitude under \( T : \tau \rightarrow \tau + 1 \) is also easy to show.

The next step is obviously to compute the integral over \( z \) in (5.19). The integral we are dealing with has the form
\[ I^{(m,n)} = \int d^2z F_L(\bar{z}|\bar{\tau}) F_R^{(m,n)}(z|\tau), \] (5.22)
where \( F_R^{(m,n)}(z|\tau) \) is the right-handed counterpart of the function \( F_L(\bar{z}|\bar{\tau}) \) defined above. A useful thing to notice is that it is possible to rewrite \( F_L \) as
\[ F_L = -\overline{\Theta}_{\Lambda_{24}} \partial_{\bar{z}} \rho(z,\bar{z}) + B_{14}(\tau,\bar{\tau}), \] (5.23)
where \( \rho(z,\bar{z}) \) is given by (cf. [31, 32])
\[ \rho(z,\bar{z}) = \partial_z \ln E(z,0) + \frac{\pi}{\tau_2} (z - \bar{z}) \] (5.24)
is a well defined function on the torus and
\[ B_{14}(\tau,\bar{\tau}) = \frac{\pi}{\tau_2} \Theta_{(m,n)} - \frac{\pi}{6i} \partial_\tau \Theta_{\Lambda_{24}} \] (5.25)
is independent of \( z \) and transforms as a modular function of weight 14.

In \( F_R^{(m,n)}(z) \) we find the same structure than in \( F_L \) and therefore we can follow the same strategy and write
\[ F_R^{(m,n)} = \Theta_{(m,n)} \partial_z \rho(z,\bar{z}) - B_6^{(m,n)}(\tau,\bar{\tau}), \] (5.26)
where now
\[ B_6^{(m,n)}(\tau,\bar{\tau}) = \frac{\pi}{\tau_2} \Theta_{(m,n)} - \frac{\pi}{2i} \partial_\tau \Theta_{(m,n)}. \] (5.27)

Then \( I^{(m,n)} \) reads
\[ I^{(m,n)} = \Theta_{(m,n)} \int d^2z \partial_z \rho \partial_{\bar{z}} \rho + B_{14} \Theta_{(m,n)} \int d^2z \partial_z \rho + B_6^{(m,n)} \Theta_{\Lambda_{24}} \int d^2z \partial_{\bar{z}} \rho - B_{16} B_6^{(m,n)} \int d^2z. \] (5.28)

Now all the integrals can be explicitly calculated at the price of losing holomorphic factorization. A special care is needed in doing so, since the integrand in all the first three integrals is singular at \( z = 0 \) and the integrals are naively divergent. This divergence corresponds to the point in which the insertions of the two vertex operators come together. In order to regularize this divergence we are going to cut off a small circle \( |z| < \epsilon \) around \( z = 0 \). Then we have for the first integral
\[ I_1 = \int_{T^2} d^2z \partial_z \rho \partial_{\bar{z}} \rho = -\frac{1}{2i} \int_{T^2} \partial \rho \wedge \partial \rho \] (5.29)
where we have used the crucial fact that \( \partial \bar{\partial} \rho = 0 \). For the crossed terms we have

\[
I_2 = \int_{\mathcal{C}^2} d^2z \partial_\rho = \frac{1}{2i} \int_{|z| = \epsilon} d\bar{z} \rho. \tag{5.30}
\]

The last integral in (5.28) is simply equal to the area of the torus, minus the area of the removed circle namely \( \tau_2 - \pi \epsilon^2 \).

Since \( I_1 \) and \( I_2 \) are line integrals over \( |z| = \epsilon \) we only need to study the behavior of \( \rho(z, \bar{z}) \) near \( z = 0 \). \( E(z, 0) \) can be written

\[
E(z, 0) = z \exp \left[ - \sum_{k=1}^{\infty} \frac{z^{2k}}{2k} G_{2k}(\tau) \right], \tag{5.31}
\]

with \( G_{2k}(\tau) \) for \( k > 1 \) the \( k \)-th Eisenstein series

\[
G_{2k}(\tau) = \sum_{m,n \in \mathbb{Z}} (m\tau + n)^{-2k} \tag{5.32}
\]

and \( G_2(\tau) \) the holomorphic-regularized Eisenstein series of weight 2

\[
G_2(\tau) = -\frac{1}{3} \frac{\theta_1^{''''}}{\theta_1}. \tag{5.33}
\]

Then we find the following Laurent series for \( \rho(z, \bar{z}) \)

\[
\rho(z, \bar{z}) = \frac{1}{z} + \left[ \frac{\pi}{\tau_2} (z - \bar{z}) - G_2 \right] - \sum_{k=2}^{\infty} z^{2k-1} G_{2k}. \tag{5.34}
\]

Substituting in \( I_1 \) and \( I_2 \) and performing the phase integral we find

\[
I_1 = -\frac{\pi}{\epsilon^2} + \frac{\pi^2}{\tau_2} + \pi \left| \hat{G}_2 \right|^2 \epsilon^2 - \pi \sum_{k=2}^{\infty} \frac{\epsilon^{4k-2}}{2k} |G_{2k}|^2, \]

\[
I_2 = \pi \epsilon^2 \hat{G}_2, \tag{5.35}
\]

where now \( \hat{G}_2(\tau, \bar{\tau}) = G_2(\tau) - \frac{\pi}{\tau_2} \),

which is not holomorphic but transforms as a weight 2 modular function. Mixing all the ingredients we finally arrive at

\[
I^{(m,n)} = \frac{\pi}{\epsilon^2} \Theta_{\Lambda_{24}} \Theta_{(m,n)} - \frac{\pi^2}{\tau_2} \Theta_{\Lambda_{24}} \tau_2 - \frac{\pi}{\tau_2} B_{14} D_{6}^{(m,n)} + O(\epsilon^2). \tag{5.37}
\]

When computing the total amplitude we are going to have to sum over boundary conditions and spin structures, so we will need to evaluate the quantity \( \sum_{\epsilon} \sum_{m,n} C_{\epsilon}(m, n) \theta_4^4 |\epsilon| I^{(m,n)} \).
Using the definition of $B_6^{(m,n)}$ and the corresponding theta functions $\Theta_{(m,n)}$ as well as some well-known results about the ring of modular functions [33] we find

$$\sum_{e \sum_{m,n=0}} C_e(m,n) \theta^4[e] \Theta_{(m,n)} = 96\eta^{12},$$

$$\sum_{e \sum_{m,n=0}} C_e(m,n) \theta^4[e] B_6^{(m,n)} = -96\eta^{12}\hat{G}_2;$$

and

$$\mathcal{B}_{14} = -\bar{G}_2\Omega_{\Lambda_{24}} + \frac{\pi^2}{6\zeta(14)}\bar{G}_{14}.$$  \hfill (5.39)

The final result is

$$\sum_{e \sum_{m,n=0}} C_e(m,n) \theta^4[e] f^{(m,n)} = \frac{96\pi}{\epsilon^2} \eta^{12} \Omega_{\Lambda_{24}} - \frac{96\pi^2}{\tau_2} \eta^{12} \Omega_{\Lambda_{24}} - 96\tau_2 \eta^{12} \left[ \Omega_{\Lambda_{24}} |\hat{G}_2|^2 - \frac{\pi^2}{6\zeta(14)} \bar{G}_{14} \hat{G}_2 \right] + O(\epsilon^2).$$  \hfill (5.40)

Before going on any further, let us have a closer look at our cutoff $\epsilon$. We have regularized our integrals by removing a small circle with radius $\epsilon$ around $z = 0$. Let us assume that we perform a modular transformation on our torus. In that case we know that $z \rightarrow z/\tau$ and then we will have that after performing this transformation the boundary of our circle will also shrink according to $\epsilon \rightarrow \epsilon/|\tau|$. So in a sense we can say that $\epsilon$ is charged under the modular group, since maintaining $\epsilon$ invariant under a modular transformation would have the result of losing modular invariance in the expansion in powers of $\epsilon$. It would be much more convenient to have a neutral cutoff under modular transformations. Let us look at the problem in a more geometrical way; we want the radius $\epsilon$ of the circle we remove from the torus to be small in order to use the series expansion in powers of $z$ in the computation of the integrals. Nevertheless in the region in which $\tau_2 \rightarrow 0$ we are dealing with very small tori, and $\epsilon$ must go to zero in order the circle to be a well-defined neighborhood of $z = 0$; if the circle is too large it will intersect with itself, since now the size of the torus shrink to zero. However we are not interested in having a scaling of $\epsilon$ just as $\sqrt{\tau_2}$, since in that case in the region in which $\tau_2 \rightarrow \infty$ (large tori) we would have that the area of the circle would go to infinity although it can be small at the scale of the torus. What we want is the radius $\epsilon$ to be arbitrarily small, let us say of order $\tilde{\epsilon} \ll 1$, at all scales (i.e., all $\tau$) and to be at the same time small compared with the torus size which implies that $\epsilon$ must vanish when $\tau_2$ goes to zero. These conditions can be accomplished if we define our cutoff $\tilde{\epsilon}$ according to

$$\epsilon = \tilde{\epsilon} f(\tau, \bar{\tau}),$$  \hfill (5.41)

where $f(\tau, \bar{\tau})$ is of the order one at $\infty$, goes to zero when $\tau_2 \rightarrow \infty$, does not vanish anywhere else in the upper half plane and it is such that under $S$ we have $f(\tau, \bar{\tau}) \rightarrow f(\tau, \bar{\tau})/|\tau|$. $\tilde{\epsilon} \ll 1$
is now our new neutral cutoff. Let us notice that by expanding in powers of \( \bar{\epsilon} \) instead of \( \epsilon \) we do not modify the finite part but we can write all the coefficients in the expansion as integrals over \( \mathcal{F} \) of a modular invariant function.

A first question that arises about \( f(\tau, \bar{\tau}) \) is whether or not such a function exists. The easiest way to prove this existence theorem is just to construct a concrete example. Without much effort one can find, for example,

\[
f(\tau, \bar{\tau}) = 2 \left[ \sum_{i=2}^{4} |\theta_i(0|\tau)|^2 \right]^{-1}.
\]

Indeed \( f(\tau, \bar{\tau}) \) transforms in the right way under the modular group and does not vanish anywhere in the upper half plane, the theta series converging for every \( \tau \) such that \( \tau_2 > 0 \). Moreover, since the \( \theta_i \)’s \( (i = 2, 3, 4) \) do not vanish in the upper half plane, \( f(\tau, \bar{\tau}) \) is finite in the same region. Of course it is quite easy to provide different examples for \( f(\tau, \bar{\tau}) \). We will further discuss this ambiguity in Sec. 8.

Let us finally integrate over the fundamental region \( \mathcal{F} \). The resulting \( \tilde{\epsilon} \) expansion is

\[
A^{IJ,ij}(k^2 = 0) = \frac{24\kappa^2 F(-2)[f]}{\pi \tilde{\epsilon}^2} \delta^{IJ} \delta^{ij} \Lambda_{1\text{-}loop} - \frac{24\kappa^2}{\pi^2} F(0) \delta^{IJ} \delta^{ij} + O(\tilde{\epsilon}^2),
\]

where \( \Lambda_{1\text{-}loop} \) is the one-loop induced cosmological constant with bosonic states contributing with a minus sign; \( F(-2)[f] \) depends functionally on the regulating function \( f(\tau, \bar{\tau}) \)

\[
F(-2)[f] = \int_{\mathcal{F}} d^2\tau \frac{f(\tau, \bar{\tau}) - 2}{\tau_2} \left[ j(\tau) - 720 + r_\Gamma(1) \right]
\]

and \( F(0) \) is given by

\[
F(0) = \int_{\mathcal{F}} d^2\tau \tilde{G}_2 \left\{ \left[ \overline{j(\tau)} - 720 + r_\Gamma(1) \right] \overline{\tilde{G}_2} - \frac{\pi^2}{6\zeta(14)} \overline{G}_{14}^{24} \right\}.
\]

Before closing this section we will compute the amplitude for two external twisted states. The only change with respect to (5.17) appears in \( F_R^{m,n} \) since now the right-moving part of the vertex operator is

\[
V_0^R = iP_R^k \psi^i e^{i P_R X^i(z)}.
\]

If the two external states have internal momenta \( P_R \) and \( -P_R \) \( (P_R^2 = 1) \) we have

\[
F_R^{m,n}(z|\tau) = -S_e(z, 0) E(z, 0)^{-1} \Theta_{(m,n)}(z P_R^i |\tau),
\]

the left moving part \( F_L \) being just the one defined above. Summing over \((m, n)\) and the spin structure and using some theta function gymnastic one easily proves that

\[
\sum_{e} \sum_{m,n=0}^{1} C_e(m, n) \theta^4[e] F_{R, \text{twes}}^{m,n} = E(z, 0)^{-2} \sum_{i=2}^{3} C_i \theta_i^{12}(0|\tau) \frac{\theta_i^4(z|\tau)}{\theta_i^4(0|\tau)},
\]

(5.48)
$C_2 = C_4 = -C_3 = 1$. This expression seems a little unpleasant. It is worth noticing, however, that (5.48) is a holomorphic doubly periodic function with a double pole at $z = 0$ and its Laurent expansion around this point has no term in $z^0$. Using the results summarized in Appendix C we find

$$\sum_e \sum_{m,n=0}^1 C_e(m, n) \theta^4[e] F_{R,tws}^{m,n} = -96\eta^{12} P \left( \frac{1}{2}, \frac{\tau}{2} \right),$$  \hspace{1cm} (5.49)$$

where $P(z|\omega_1,\omega_2)$ is the Weierstrass elliptic function with semiperiods $\omega_1$ and $\omega_2$. This can be further related with the function $\rho(z, \bar{z})$ using the identity

$$P \left( \frac{1}{2}, \frac{\tau}{2} \right) = -\partial_z \rho(z, \bar{z}) - \hat{G}_2,$$  \hspace{1cm} (5.50)$$

which allows us to write finally

$$\sum_e \sum_{m,n=0}^1 C_e(m, n) \theta^4[e] F_{R,tws}^{m,n} = 96\eta^{12} \partial_z \rho(z, \bar{z}) + 96\eta^{12} \hat{G}_2.$$  \hspace{1cm} (5.51)$$

This is exactly the same result we got for the untwisted states (notice the overall minus sign in the definition of $B_6$).

Our final result is that the two-point amplitude on the torus for two neutral external states is given by (5.43) and it has the same expression for twisted and untwisted external states. This is no wonder, since we know that both kind of states in fact combine together in the adjoint representation of $SU(2)^8$. It is also easy to check that the result for the two-point amplitude can also be obtained using the fermionic construction. In fact it is clear that for example (5.48) can be rewritten in terms of fermion propagators and interpreted as the correlation function of the vertex operators in the F construction.

### 6 The Case of the Charged Bosons

We now turn to the computation of the two point function for the $24 \times r_T(1)$ charged states both twisted and untwisted. Since the calculation will be very similar to the one made in the previous section we will skip here the details. We can make use of the formula (5.3) but now we have to use a different expression for the world-sheet currents $J(\bar{z})$. Charged bosons are related with the simple roots of the corresponding gauge group. These roots are precisely the vectors $\alpha^I$ of the left-moving lattice with $\alpha^2 = 2$. The current associated with the root $\alpha$ is

$$J^\alpha(\bar{z}) = c_\alpha e^{i\alpha \cdot \phi(\bar{z})},$$  \hspace{1cm} (6.1)$$

where $c_\alpha$ is a cocycle satisfying

$$c_\alpha c_{\alpha'} = e(\alpha, \alpha') c_{\alpha + \alpha'},$$  \hspace{1cm} (6.2)$$

and $e(\alpha, \alpha') = 0$ unless $\alpha + \alpha'$ is a root.
We are going to proceed as with the neutral bosons by writing \( \phi^I = \phi^I_0 + \phi^I_q \). Using (6.4) we get

\[
\langle J^\alpha(z)J^\beta(0) \rangle = e(\alpha, \beta)c_{\alpha+\beta} \langle e^{i\alpha\phi_q(z)} e^{i\beta\phi_q(0)} \rangle \sum_{PL \in \Lambda_{24}} e^{-i\pi \tau - 2i\tau \alpha \cdot P_L}. \tag{6.3}
\]

The integration over the zero mode \( \phi^I_0 \) gives rise to a delta function that enforces \( \alpha + \beta = 0 \) and that we will drop in the following. We can write the contribution to the total amplitude in the sector \((m, n)\) as

\[
A^{\alpha, ij}_{(m,n)}(k) = \frac{k^2}{\pi^2} \delta_{ij} \sum_{e} C_e(m, n) \int F \int d^2z \tilde{\eta}^{-24} \frac{\theta^4[e]}{4\eta^{12}} F_L^\alpha(\bar{z}|\tau)e^kG(z,0) \times \left\{ \left( \frac{\partial^2}{\partial z^2} \ln E(z,0)\Theta_{(m,n)} + \frac{\pi}{2i} \frac{\partial}{\partial \tau} \Theta_{(m,n)} - k^2 S_e(z,0)^2 \right) \right\}, \tag{6.4}
\]

where

\[
F_L^\alpha(\bar{z}|\tau) = \Theta_{\Lambda_{24}}(z \alpha^I|\tau) E(z,0)^{-2} \tag{6.5}
\]

and we have applied \( e(\alpha, -\alpha) = 1 \).

The trick to deal with this integral is somewhat similar to the one we used for the case of the twisted bosons. \( F_L^\alpha(\bar{z}|\tau) \) is a holomorphic doubly periodic function on the torus and then can be expressed in terms of the elliptic function, which in turn we know how to write in terms of \( \rho(z, \bar{z}) \) (in this discussion we will work with complex conjugate expressions in order to simplify the expression). Let us make use of some general properties of \( \Theta_{\Lambda_{24}}(z \alpha^I|\tau) \). Any of the 23 (in this discussion the Leech lattice is excluded) Niemeier lattices is a Lie algebra lattice which, in general, is composed of several factors \( L_1 \times \ldots \) where \( L_i \neq D_1 \). If we take a base of orthonormal vectors we can label the basis vectors in such a way that \( \alpha^I \) lies in the \( i \)-th factor and has coordinates \( \alpha = (1,1,0,\ldots,0) \). Then it is easy to see that \( \Theta_{\Lambda_{24}}(z \alpha^I|\tau) \) can be written (we drop the arguments when theta functions are evaluated in \( z = 0 \))

\[
\Theta_{\Lambda_{24}}(z \alpha^I|\tau) = C_1(\tau) \left[ \frac{\theta_1(z|\tau)}{\theta_1(0|\tau)} \right]^2 + \sum_{i=2}^4 C_i(\tau) \left[ \frac{\theta_i(z|\tau)}{\theta_i(0|\tau)} \right]^2. \tag{6.6}
\]

The transformation properties of \( C_i(\tau) \) can be gotten from the ones for \( \Theta_{\Lambda_{24}}(z \alpha^I|\tau) \), and by evaluating the expression at \( z = 0 \) we see that \( \Theta_{\Lambda_{24}} = \sum_{i=2}^4 C_i(\tau) \). Multiplying by \( E(z,0)^{-2} \) we find

\[
\Theta_{\Lambda_{24}}(z \alpha^I|\tau)E(z,0)^{-2} = C_1(\tau) + \sum_{i=2}^4 C_i(\tau) \left[ \frac{\theta_i(z|\tau)}{\theta_i(0|\tau)} \right]^2. \tag{6.7}
\]

We can now introduce the Weierstrass elliptic function by using (6.7)

\[
\Theta_{\Lambda_{24}}(z \alpha^I|\tau)E(z,0)^{-2} = \Theta_{\Lambda_{24}}P \left( z \left| \frac{1}{2}, \frac{\tau}{2} \right. \right) + C_1(\tau) - \sum_{i=2}^4 C_i(\tau) e_i. \tag{6.8}
\]

In fact it can be checked that the affine term in the last expression is a modular form of weight 14, and then

\[
F_L^\alpha(\bar{z}|\tau) = \Theta_{\Lambda_{24}}P \left( z \left| \frac{1}{2}, \frac{\tau}{2} \right. \right) + \frac{\pi^2}{6\zeta(14)} G_{14}. \tag{6.9}
\]

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where the coefficient of $G_{14}$ is fixed by comparing the series expansions. Taking into account (5.50) we can write

$$F_L^{(\alpha)}(z) = -\overline{\Theta}_{\Lambda_{24}} \partial \overline{\theta} - \overline{\Theta}_{\Lambda_{24}} \overline{G}_2 + \frac{\pi^2}{6\zeta(14)} G_{14}.$$

(6.10)

With this expression for $F_L^{(\alpha)}$ and the results of the previous section we have

$$\sum_e \sum_{m,n=0}^1 C_e(m,n) \theta^4[e] F_L^{(\alpha)} F_R^{(m,n)} = \frac{96\pi}{\epsilon^2} \eta_{12}^{12} \overline{\Theta}_{\Lambda_2} - \frac{96\pi^2}{\tau_2} \eta_{12}^{12} \overline{\Theta}_{\Lambda_{24}}$$

$$-96\tau_2 \eta_{12}^{12} \left[ \overline{\Theta}_{\Lambda_{24}} |\hat{G}_2|^2 - \frac{\pi^2}{6\zeta(14)} \eta^{12} G_{14} \hat{G}_2 \right] + O(\epsilon^2).$$

(6.11)

Multiplying by all the prefactors in (5.5) and integrating over the modular parameter we finally find

$$A^{\alpha,ij}(k^2 = 0) = \frac{24\kappa^2 F^{(-2)}[f]}{\pi \bar{\epsilon}^2} \delta^{ij} + \kappa^2 \Lambda_{1-loop} \delta^{ij}$$

$$- \frac{24\kappa^2}{\pi^2} F^{(0)} \delta^{ij} + O(\bar{\epsilon}^2).$$

(6.12)

In the case of charged twisted states no computation is necessary, since we have shown in Sec. 5 that the result has to be the one for untwisted states. Then (6.12) is valid for twisted and untwisted charged states.

### 7 The Infrared Behavior and Mass Corrections

In the preceding two sections we have computed the two-point function on the torus for the states in the massless sector of the 24 two-dimensional heterotic strings discussed in sec. 3. We have checked that the one loop correlator $\langle V_0 V_0 \rangle$ gives the same result for all the massless states (twisted or untwisted). This is not so surprising if we take into account that twisted and untwisted states in the B model add up to fill the adjoint representation of $SU(2)^8$ or that all states are on the same footing in the F construction.

However, in computing the correlator of the two vertex operators on the torus we are faced with the existence of divergences associated with the coincidence of the two insertion points on the worldsheet (the $z \to 0$ limit). This divergence looks like the ones arising in the computation of the one loop two-point graviton amplitude in the bosonic string [31]. The standard interpretation in the literature of these kind of divergences is that they are due to the propagation of an off-shell tachyon at zero momentum along the very long tube in fig. 1, which shows the factorization of the residue of the $1/\epsilon^2$ pole (see, for example, section 8.2.4 in ref. 2 and 31). Although this interpretation, in spite of involving off-shell guys, could be satisfactory for the bosonic string, in our case it is very unpleasant to link the divergence with the propagation of a tachyon since we are dealing with a tachyon free theory. Moreover,
Figure 1: Factorization of the double-pole singularity

if we try to relate the residue of the $\epsilon^{-2}$ pole to the factorization limit shown in fig. 1 we find that the vertex operator in the 0 picture must be

$$V_0^{tach}(k) \sim e^{i k \cdot X(\bar{z})},$$

(7.1)

where $X^\mu(\bar{z})$ is the antiholomorphic part of the $X^\mu(z, \bar{z})$ field. This vertex represents a rather weird state, being purely left-moving. Then the only conclusion to be extracted is that the interpretation of the divergence as caused by the propagation of off-shell tachyons is extremely unsatisfactory.

Using the naive regularization of the amplitude in which a $\tau$-independent cutoff $\epsilon$ is introduced \[31\], the final expansion in powers of $\epsilon$ is not modular invariant, in the sense that the coefficients of $\epsilon^n$ cannot be written as integrals over the fundamental domain of a modular invariant function, except for the finite part with $n = 0$. This seems to be a problem, since modular invariance is a necessary requisite for any sensible expression in string theory. It is precisely this symmetry which allows us to interpret any possible divergence appearing in any string amplitude as having an infrared origin ($\tau_2 \to \infty$) by excluding the ultraviolet region. The breaking of modular invariance in the $\epsilon$-expansion then makes difficult to see the divergence as due to an infrared instability of the theory.

In our analysis we have shown that a modular invariant cutoff $\epsilon[f] = \tilde{\epsilon} f(\tau, \bar{\tau})$ can be introduced to provide a modular invariant expansion in powers of $\tilde{\epsilon}$. Now, however, the residue of the pole in $\tilde{\epsilon}^{-2}$ cannot be interpreted in terms of the propagation of an off-shell tachyon along the tube in fig. 1, since this residue now depends functionally on the regulating function $f(\tau, \bar{\tau})$. In a sense this is satisfactory, since in a modular invariant description one expects to project out any off-shell tachyons propagating in long tubes. The divergence must then be interpreted in a different way; our theory, although tachyon free, is not finite and the arising divergence has an infrared origin, the only kind of divergences that any consistent string theory can contain. The problem left is then to look for a way in which one can get rid of this divergence. A first idea would be just to look for an analogue of the Fischler-Susskind mechanism \[34\] which removes the logarithmic divergences due to dilaton tadpoles by shifting the zero tree-level cosmological constant to the value induced at one loop. In
our case it is hard to find a similar mechanism since now there is no obvious parameter in
the sigma-model action whose analytic continuation could absorb the one-loop divergences.
In absence of a more elegant way to eliminate the divergence we will follow the procedure
of ref. \[31\] and just substract the pole. Then we find that, for both neutral and charged
bosons,
\[
A_2 = \kappa^2 \Lambda_{1-loop} - \frac{24 \kappa^2}{\pi^2} F^{(0)}. \tag{7.2}
\]
It is worth stressing that the presence of this kind of divergences associated with the
coincidence of the two insertions is ubiquitous in all the heterotic string models without
space-time supersymmetry, since the only way in which one can get rid of them is when
the integrand of the one-loop cosmological constant vanishes before integrating over the
fundamental region. This means that finitude seems to be a very difficult thing to get
whenever we deal with non-supersymmetric heterotic string models.

Going back to \(F^0\) as defined in (5.45) we can see that the term in the integrand propor-
tional to \(\tau^{-2}\) is modular invariant by itself and proportional to the one-loop cosmological
constant. Then we can separate this term to get
\[
A_2 = 2 \kappa^2 \Lambda_{1-loop} - \frac{24 \kappa^2}{\pi^2} \int \frac{d^2 \tau}{\tau_2} \left\{ -2\pi (\text{Re} G_2) |j(\tau) - 720 + r_\Gamma(1)| \right. \\
+ \left. \frac{\pi^3}{6 \zeta(14) \pi^{14}} + \tau_2 |G_2|^2 |j(\tau) - 720 + r_\Gamma(1)| - \frac{\pi^2 \tau_2}{6 \zeta(14) \pi^{14} |G_2|^2} \right\}. \tag{7.3}
\]
Let us now turn to the modular integral in (7.3) and analyze the infrared region (\(\tau_2 \to \infty\)).
We must remember that in the neighborhood of \(\tau = i\infty\) one must perform first the integral
over \(\tau_1\), which enforces the level-matching condition, and then integrate \(\tau_2\) all the way to
infinity. Doing so we find that all unphysical tachyons cancel; however the integral is infrared
divergent. In fact we have a logarithmic divergence and a lineal one in the proper time when
\(\tau_2 \to \infty\). These divergences are due to the fact that we are dealing with a two-dimensional
system (cf. \[35\]). The term \(\Lambda_{1-loop}\) can be known exactly due to the remarkable properties of
the modular invariant function. Introducing an infrared cutoff in proper time \(L^2\) to compute
the integrals we find
\[
A_2 = 16 \pi \kappa^2 [12 + r_\Gamma(1)] \ln L^2 - \frac{8 \pi^2 \kappa^2}{3} r_\Gamma(1) L^2 \\
- 16 \pi \kappa^2 r_\Gamma(1) + \kappa^2 A_{finite} + O(L^{-2}). \tag{7.4}
\]
A numerical analysis of \(A_{finite}\) yields
\[
A_{finite} = -519.865 + 27.436 \times r_\Gamma(1), \tag{7.5}
\]
where the numerical errors are in the third decimal place.

In fact, in order to understand this and the general structure of the two-point function,
one can try to construct an analog model for (7.3). The term proportional to the vacuum
energy can be interpreted in a standard way as just the contribution of the vacuum energies of the different fields with the subleties mentioned in sec. 2; we will discuss this term later on when trying to compute the mass corrections to massless states. Then let us center ourselves in the truncated amplitude $\tilde{A}$ without this term. Let us go to the region of large $\tau_2$ which corresponds to very long tori. In such a situation we can consider that only on-shell string states circulate in the loop since in that region we can impose the left-right level matching condition by integrating over $\tau_1$. Then we can write (trading $\tau$ by $\tau_1 + is$)

$$
\tilde{A} = \int_{\mu - 2}^{\infty} ds \int_{\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \left\{ 2(\text{Re } G_2) \left[ j - 720 + r_\Gamma(1) \right] - \frac{\pi^2}{6\zeta(14)} \frac{G_{14}}{\eta^{24}} \right\}
$$

Each integrand is a power series of the type $\sum_{m,n} a_{mn} e^{2\pi im\tau_1} e^{-2\pi ns}$ and the integration over $\tau_1$ restricts this sum to the $m = 0$ terms in such a way that we can write

$$
\tilde{A} = \sum_k \tilde{V}_{i,i,k,k} \int_{\mu - 2}^{\infty} ds \frac{e^{-m_k^2 s}}{s} + \sum_k \tilde{V}_{i,k,l} \tilde{V}_{k,l,i} \int_{\mu - 2}^{\infty} ds \frac{e^{-m_k^2 s}}{s}.
$$

Here $V_{i,i,k,k}$ and $V_{i,k,l}$ are effective couplings which in principle could be read from (7.6) and the sum is over all the states running in the loop. In fact, such a general structure for $\tilde{A}$ can be obtained from the Feynman diagrams in fig. 2. Now we can make an effective field theory interpretation of our result. In the large proper time limit the truncated two point function $\tilde{A}$ is the sum of two contributions. One of them comes from the degeneration of the torus into a four point function on the sphere with two of the states joined by a long tube (first diagram in fig. 2). The second one has its origin in a degeneration of the torus in which we have two three-point functions on the sphere joined by two long tubes. As a matter of fact the effective coupling $V_{i,j,k,l}$ must only include the $\alpha'$ leading contribution to the four-point tree level function; this diagram gives

$$
\frac{V_{i,i,k,k}}{4\pi} \int_{0}^{\infty} ds \frac{s}{e^{-m_k^2 s}}.
$$

Figure 2: Feynman diagrams for the mass shift in the analog model
with \(m\) the mass of the state running in the loop. The contribution coming from the second
diagram depends on whether or not the masses of the two internal states are equal. If they
are not we have

\[
\frac{V_{i,k,l}V_{k,l,i}}{4\pi(m_l^2 - m_k^2)} \int_0^\infty \frac{ds}{s} e^{-m_l^2 s} + \frac{V_{i,k,l}V_{k,l,i}}{4\pi(m_k^2 - m_l^2)} \int_0^\infty \frac{ds}{s} e^{-m_k^2 s}.
\]

But when the masses of the two states coincide \((m_k = m_l)\) we have

\[
\frac{V_{i,k,l}V_{k,l,i}}{4\pi} \int_0^\infty ds e^{-m_k^2 s}.
\]

From this and the Feynman diagrams in fig. 4 it is quite obvious why the effective vertex
\(V_{i,j,k,l}\) does not include subleading corrections in \(\alpha'\). These corrections are included in the
contribution of the second diagram when \(m_k\) and \(m_l\) are not equal.

Now the origin of the divergences in (7.3) is clear. Due to the low value of the dimension
we will have divergences associated with large \(s\) which are logarithmic for the first diagram
and logarithmic and linear for the second one; the divergent parts as \(s \to \infty\) are

\[
16\pi\kappa^2[12 + r_T(1)] \ln s - \frac{8\pi^2\kappa^2}{3} r_T(1) s.
\]

This kind of infrared divergences appear in massless field theory whenever \(d \leq 2\) since in that
case the measure in the Feynman integrals cannot cancel the divergence in the propagator
as the internal momentum in the loop goes to zero.

In fact we can connect the general structure of the singular part with what we know from
the low energy field theory. As an example let us consider that we have neutral external
particles. In this case according to the computations of sec. 8 the leading contribution in
\(\alpha'\) comes from both diagrams with a charged particle (with left and right-moving charge)
running in the loop. Since the number of such states is proportional to \(r_T(1)\) we expect to
have both a linearly and a logarithmic divergent term proportional to \(r_T(1)\) as we indeed
have in (7.11). In addition, we also have a contribution coming from the second diagram
with one massive and one massless particles running in the loop, both charged only with
respect to \(SU(2)^8\), which corresponds to \(O(\alpha')\) terms in (3.13). This gives a contribution to
the logarithmic singularity which is independent of \(r_T(1)\). Of course, in order to reproduce
the concrete numbers in (7.11) one should sum over all the subleading contributios. In
any case we see that the structure of the result agrees qualitatively with the analysis done
in previous sections. At any rate it must be clear that this field theoretical interpretation of
the stringy result is by no means complete in the sense that it cannot reproduce the exact
result (7.3). In fact the lesson we learned from the study of the partition function is that no
field theoretical description of a string amplitude can reproduce the string theory calculation
unless \textit{intruder} (i.e., ghost-like) states are introduced in the game \([14, 15]\). The analysis of
the previous paragraph is simply intended to give a more physical insight of the stringy
result in terms of quantum fields.
One of the most interesting informations that can be extracted from our computation of the genus one two-point function is the existence of mass renormalization of the massless states. The point, however, is a little bit subtle for the 23 models without Atkin-Lehner symmetry. The interpretation of the on-shell two point function as quantum corrections to the tree-level mass for the massless states of the string can only be direct in the case of vanishing cosmological constant, since only in this case the perturbative expansion is consistent in the sense that the tree-level vacuum is also a good vacuum at one-loop level. In the case of models with one-loop induced vacuum energy \(r_T(1) \neq 0\) in our case the tree level vacuum is flat, but after the inclusion of the one-loop effects this vacuum no longer satisfies the equations of motion of the string, since now the string is propagating in a (Anti-)de-Sitter space-time. It is for this reason that only when \(\Lambda_{1-loop} = 0\) we can write

\[
\delta m^2_i = \left. -\langle V^0_i(k)V^0_i(-k)\rangle \right|_{k^2=0} - \text{massless tadpoles.} \tag{7.12}
\]

In the case of the model with Atkin-Lehner symmetry it is easy to see that all possible massless tadpoles vanish and then the mass shift for the states in the massless sector of this model is \((\alpha' = 2)\)

\[
\delta m^2_{Leech} \approx 519.865\kappa^2. \tag{7.13}
\]

Then we see that the massless sector does not survive the quantum corrections in the string coupling constant. For the remaining 23 models, things are not so easy as we have explained. In fact, when quantizing a scalar field \(\phi^2\) theory in curved space-time one must allow for a term in the action of the form \(\xi \phi^2(x)R(x)\) where \(R(x)\) is the scalar curvature and \(\xi\) is a coupling constant \([37]\). If we have our field propagating in a (Anti-)De-Sitter space-time with constant curvature \(R\), the two point function contributes to the renormalization of the wave function, the mass and \(\xi\) according to

\[
A_2^{1-loop}(p^2) = \delta Z p^2 - \delta m^2 - \delta \xi R. \tag{7.14}
\]

In our case this translates into a one-loop induced term in the effective action of the form

\[
S_{1-loop} = \int d^2x \left[ \frac{1}{2} \delta m^2 \mathrm{Tr} \Phi^2 + \frac{1}{2} \delta \xi R \mathrm{Tr} \Phi^2 \right]. \tag{7.15}
\]

Now we have to relate the scalar curvature with the parameters in our models. At tree-level we are perturbing around a vacuum in which all low-energy fields and the cosmological constant vanish (flat space-time). At one loop, however, our new vacuum has \(\Lambda \neq 0\) but none of the \(\Phi\) fields gets a vacuum expectation value, so from the dilaton beta function we must have \(R \sim 2\Lambda\). In fact since the one loop cosmological constant is proportional to \(r_T(1)\) we have that \(R \sim r_T(1)\). This means that all the finite terms in (7.4) which are proportional to \(r_T(1)\) may be readorsed in a renormalization of \(\xi\). In this way we find for the 23 models with non-vanishing cosmological constant (here we do not have massless tadpoles either)

\[
\delta m^2 \approx 519.865\kappa^2, \quad \delta \xi \approx 0.454\kappa^2. \tag{7.16}
\]
So the mass renormalization for the massless states would be the same for all the 24 two-dimensional heterotic models.

One can wonder about the possibility of having any breakdown of gauge symmetry because of these non-vanishing mass corrections. To clarify this point the best thing to do is to go to the analogous situation in field theory, that is, a theory with $N$ scalar fields in the adjoint representation of the gauge group. This theory can be viewed as the result of dimensional reduction of Yang-Mills theory in $d + N$ to $d$ dimensions where the scalars appear as the $N$ internal components of the gauge bosons. It is easy to see that the masses of such scalars are not protected by any Ward identity, since after dimensional reduction the gauge parameter loses any dependence in the internal coordinates and then the internal components of the gauge field are invariant under gauge transformations. A different problem is how non-vanishing and infrared divergent two-point functions for propagating gauge fields affect gauge invariance in non-supersymmetric string theories such as $SO(16) \times SO(16)$. This can only be addressed by studying the string Ward identities for such amplitudes and the possible anomalies that could arise in regularizing the amplitudes [38].

8 Conclusions

We have tried to clarify how quantum corrections in the string coupling constant modify the tree level structure of two-dimensional heterotic strings without space-time supersymmetry. We have found that the 24 models constructed from the left-moving bosonic string compactified on a Niemeier lattice and the right moving heterotic string on $\Gamma_8$ modded out by the operator $\alpha$ defined in Sec. 2 they all have a right-moving level 2, $SU(2)^8$ gauge symmetry. Using this fact we have been able to relate this bosonic construction of the right moving sector with a new one in terms of free worldsheet fermions. In ref. [13] a theorem was proved stating that for any two-dimensional heterotic string the partition function has to be of the form

$$Z = Z_R \int_F \frac{d^2 \tau}{\tau_2} [j(\tau) - 720 + r_\Gamma(1)], \quad (8.1)$$

where $Z_R$ must be a constant. This means that any two-dimensional heterotic string either is supersymmetric or at most supersymmetry is broken only at the massless level. We will see how this result constraint the possible fermionic constructions.

Let us consider the right-moving sector of the two-dimensional heterotic string as formed by a set of 24 free world-sheet Majorana-Weyl fermions. In Sec. 3 we said how such theories are classified by a pair of semi-simple Lie groups $G$ and $H$, $H \subset G$ [13]. The fact that our models live in two dimensions forces dim $G = 24$ and the dimension of $H$ will determine the mass of the lowest lying fermion in the model. Since from modular invariance we know that supersymmetry can at most be broken only for the massless states we can only have dim $H = 8, 24$. In the first case we single out one group of 8 world-sheet fermions transforming in the adjoint of $H$ and project down to $(-1)^{F_{\text{pseudo}}} = 1$. It is not difficult to realize that the lowest lying Ramond state is massless and we have the supersymmetric model ($Z_R = 0$).
If \( \dim H = 24 \) we take the same GSO projection over all the fermions and it can be easily seen that all massive levels have the same number of fermionic and bosonic degrees of freedom except for the massless sector in which there are only 24 bosons in the adjoint representation of \( H (Z_R = 24) \). Then we have seen that the only freedom we are left when constructing two-dimensional heterotic models is (besides the choice of the 24-dimensional lattice) the election of the right-moving gauge group. Different choices will differ in the actual couplings between the low-energy fields although (3.9) retains its general form. However other aspects such as the one-loop two point function or to some extent the two-loops cosmological constant appears to be quite independent of the model chosen. In the present paper we have centered ourselves in the study of one of these possibilities, namely the case \( G = H = SU(2)^8 \) since this is the model that results from the usual constructions in the previous literature [1, 14].

We have studied the genus two cosmological constant for the \( SU(2)^8 \) model and found, using the technique developed in [11], a modular invariant expression that does not vanish before integration on the fundamental region of \( Sp(2, Z) \). The question of the vanishing of this expression after integration over the harmonic ratios \( \lambda_i \) for the model with Atkin-Lehner symmetry \( (r_\Gamma(1) = 0) \) seems difficult to answer due to the unmanageable form of the integrand. However it is possible to give some indirect evidences that in fact this is not to be expected. The contribution to the genus two cosmological constant of Riemann surfaces in which a non-trivial homology cycle is pinched off could be written as [12]

\[
\Lambda_{2-loop} \sim \sum_i d_i \int_\mu^\infty \frac{ds}{s} e^{-m_i^2 s} A_{1-loop}^{i,i},
\]

(8.2)

where the sum is over all states in the string, \( A_{1-loop}^{i,i} \) is the one-loop two-point function for the \( i \)-th state with mass \( m_i \) and \( d_i \) is a degeneration factor that takes into account the number of physical degrees of freedom for each state. The boundary of the moduli space of genus two Riemann surfaces has two branches. One of them \( (B_1) \) is parametrized by the period matrix \( \tau_{ij} \) when one of its diagonal entries goes to \( i\infty \) (for example \( \tau_{11} \)). Geometrically this corresponds to the degeneration of a non-trivial homology cycle. The second branch \( (B_2) \) contains Riemann surfaces for which \( \tau_{12} \to 0 \), i.e., the trivial homology cycle is degenerated. Over \( B_1 \) the genus two partition function takes the form (8.2) where \( s \sim \tau_{11}, \tau_{12} \) is the relative coordinate of the two-insertions and \( \tau_{22} \) is the modular parameter of the remaining torus [12]. From our study of the genus one two-loop point function for massless states we know that they are divergent not only in the limit of coincidence of the two insertions \( \tilde{\epsilon} = 0 \) but also when \( \tilde{\epsilon} \neq 0 \) because of the low number of open space-time dimensions. Then we see that \( \Lambda_{2-loop} \) will have a divergent contribution not only from \( B_1 \cap B_2 \) but also from \( B_1 \). \( \tilde{\epsilon} \) can be seen as a coordinate over \( B_1 \) in a neighborhood of \( B_1 \cap B_2 \). This would suggest that the integrated genus two cosmological constant is divergent due to the same kind of infrared divergences that appear in the computation of one-loop scattering amplitudes.

In the study of the one-loop two point functions for these models we have found that in general they do not vanish. In fact during the computation we have been faced with divergences associated with the coincidence of the two insertions. We have studied the
origin of this divergence and argued that it cannot be explained in terms of off-shell tachyons propagating along degenerated Riemann surfaces. Using a modular invariant regulator we have identified the origin of this divergence as an infrared instability of the theory. Even after subtracting the pole $\tilde{\epsilon}^{-2}$ we have found that the finite part of the regulated amplitude is further afflicted from infrared divergences due to the fact that the string lives in a two-dimensional target space-time. Moreover, since these infinities are caused by massless states living in two dimensions, they are also present in the low-energy field theory described by the action (3.9). This is contrary to the infrared divergence associated with the pole $\tilde{\epsilon}^{-2}$. In this latter case, the divergence is due to the coincidence of two composite operators in the two-dimensional field theory on the world-sheet and then from a world-sheet point of view is of ultraviolet origin. However, looking at it from the two-dimensional target space the divergence is infrared and involves the full string theory. Then it cannot have any counterpart in the low-energy effective theory for the massless modes, since here we are integrating out all the massive states; it is a purely stringy infrared divergence.

In the case of the model based on the Leech lattice we have computed the one loop mass-shift and found it to be positive. For the other 23 models the interpretation of the one-loop two-point function as the first quantum correction to the mass of the state is rather problematic, since for them there is a one-loop induced cosmological constant and then the one loop vacuum does not satisfy the string equations of motion [31, 6]. We argue that the two-point function then contributes not only to the mass renormalization but also to the renormalization of the coupling $\xi$ between the massless scalar fields and the scalar curvature in the low-energy field theory. Identifying the terms in the two-point function proportional to $R \sim r_T(1)$ with the renormalization of $\xi$ we find that the renormalization of the mass for the massless states would be the same for all the 24 models.

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Appendix A: Theta Functions for Lattices

In this Appendix we will summarize some results about theta functions for lattices [4, 39]. The theta series associated with a lattice $\Lambda$ is defined to be [39]

$$\Theta_\Lambda(0|\tau) = \sum_{P \in \Lambda} e^{i\pi \tau P \cdot P}. \quad (A.1)$$

This definition can be easily generalized to include non-vanishing first argument and characteristics. Let be $a^l$ and $b^l$ two vectors not in $\Lambda$ but such that $2a, 2b \in \Lambda$. Then we
In the case of null characteristics we will simply denote the corresponding theta function by \( \Theta_\Lambda(v^I|\tau) \). Notice that since the sum is extended to all vectors in \( \Lambda \) the characteristic vectors \( a^I \) and \( b^I \) are defined modulo shifts by vectors in the lattice, i.e., they can be taken to live in \((\Lambda/2)/\Lambda\).

Let us consider from now on that \( \Lambda \) is an even, self-dual lattice. Then under the modular group generators \( T \) and \( S \) the theta functions behave as

\[
\Theta_\Lambda \left[ \begin{array}{c} a \\ b \end{array} \right] (v^I|\tau + 1) = e^{-i\pi a^2} \Theta_\Lambda \left[ \begin{array}{c} a+b \\ a \end{array} \right] (v^I|\tau),
\]

\[
\Theta_\Lambda \left[ \begin{array}{c} a \\ b \end{array} \right] \left( \frac{v^I}{\tau} - \frac{1}{\tau} \right) = (-i\tau)^{d^2} e^{i\pi \tau^2/\tau} \Theta_\Lambda \left[ \begin{array}{c} -b \\ a \end{array} \right] (v^I|\tau). \tag{A.3}
\]

It is also useful to derive the quasiperiodicity properties; given \( p^I, q^I \in \Lambda \)

\[
\Theta_\Lambda \left[ \begin{array}{c} a \\ b \end{array} \right] (v^I + \tau p^I + q^I|\tau) = e^{-i\pi \tau q^2 - 2\pi i p^I(v+b) + 2\pi i a \cdot q} \Theta_\Lambda \left[ \begin{array}{c} a \\ b \end{array} \right] (v^I|\tau). \tag{A.4}
\]

Lattice theta functions with characteristics \( \{a^I, b^I\} \) will be even as functions of \( v^I \) if and only if \( 4a \cdot b \) is an even integer. If this is not the case then the theta function will be odd and in particular will vanish at \( v^I = 0 \). The zeroes of the \( \Theta_\Lambda(v^I|\tau) \) are actually related to the existence of odd characteristics; given two vectors \( \delta^I_1, \delta^I_2 \) such that \( 2\delta_1, 2\delta_2 \in \Lambda \) and \( 4\delta_1 \cdot \delta_2 \in 2\mathbb{Z} + 1 \), we have

\[
\Theta_\Lambda(\tau \delta^I_1 + \delta^I_2|\tau) = 0. \tag{A.5}
\]

This formula can be checked by writing the theta function with characteristics \( \{\delta^I_1, \delta^I_2\} \) in terms of \( \Theta_\Lambda(v^I + \tau \delta^I_1 + \delta^I_2|\tau) \) and taking into account that, being an odd function of \( v^I \), it has to vanish at \( v^I = 0 \).

### Appendix B: Riemann Surfaces in Hyperelliptic Formalism and the Knizhnik Formula

A genus \( g \) hyperelliptic surface is defined as a two-dimensional surface that uniformizes

\[
y(z)^2 = \prod_{i=1}^{2g+2} (z - a_i), \tag{B.1}
\]

where \( a_i = z(P_i) \) with \( z \) are holomorphic coordinates in \( \mathbb{C}P^1 \). Every Riemann surface with \( g \leq 2 \) is hyperelliptic. Using a \( SL(2,\mathbb{C}) \) transformation we can fix the locations of three branching points, the canonical choice being \( a_{2g} = 0, a_{2g+1} = 1 \) and \( a_{2g+2} = \infty \). The
remaining $2g - 1$ points on $\mathbb{CP}^1$ provide us with good coordinates in the moduli space $M_g$ of genus $g$ hyperelliptic Riemann surfaces and then we have $\dim \mathbb{C} M_g = 2g - 1$.

Let us focus on the $g = 2$ case. We have 6 branch points and the complex dimension of the moduli space is 3 which is the number of complex parameters of the genus-two period matrix. We represent this surface schematically in fig. 3 with the basis for the homology cycles. Modular transformations in the hyperelliptic language amounts to permutations of the branching points $a_i$ and then the five generators of the genus-two modular group are in one-to-one correspondence with the generators of the braid group on the sphere $B_5$. On the other hand the ten even spin structures on a genus two Riemann surface are in one-to-one correspondence with partitions of the set of the six branch points $(P_{i1}, P_{i2}, P_{i3}|P_{j1}, P_{j2}, P_{j3})$. The exact correspondence can be found in the Appendix A of [11]. For $e = (P_{i1}, P_{i2}, P_{i3}|P_{j1}, P_{j2}, P_{j3})$ the corresponding theta function can be written in terms of the $a_i$’s using Thomae’s formula

$$\theta^8[e] = (\det \sigma)^{-4} \prod_{k,l} a_{ik}^2 a_{jk}^2,$$

where $a_{ij} = a_i - a_j$ and $\sigma_{ij}$ is the matrix which relates the $g$ abelian differentials $v_i = z^{i-1}y(z)^{-1}dz$ with the canonical homology basis $\omega_i = \sum_j \sigma_{ij}v_j$. In order to eliminate explicitly the $SL(2, \mathbb{C})$ freedom when choosing the branch points on the sphere it is convenient to define the following harmonic ratios

$$\lambda_i = \frac{a_{i4}a_{56}}{a_{i5}a_{46}} \quad i = 1, 2, 3.$$

Modular transformations now act on $\lambda_i$; for example under a Dehn twist along $A_2$ we have $T_2 : \lambda_i \rightarrow \lambda_i/(\lambda_i - 1)$.

The computation of the higher genus cosmological constant for the heterotic string has been a rather controversial issue. In what follows we will briefly review the main problems found in such computations and the main features of the expression found by Knizhnik in [10] for the genus-two cosmological constant of heterotic strings.
While evaluating the functional integral for a heterotic string over a genus-two Riemann surface the main problem comes from the integration over the fermionic part of the supermoduli, i.e., the zero modes of the worldsheet gravitino. The two-loop cosmological constant in general can be written as an integral over the supermoduli $m^I$ of an integrand which factorizes into a holomorphic and antiholomorphic part (with respect the $m^I$). However after integration over the fermionic moduli this factorization property is in general destroyed; using bosonization \cite{41} is is argued that the integration over the gravitino zero modes is equivalent to the insertion of $2g - 2 \ (g \geq 2)$ Picture Changing Operators (PCOs) whose correlation function destroys the holomorphic factorization of the original expression. Using this, Knizhnik proposed the following expression for the genus-two cosmological constant of the ten-dimensional heterotic string

\begin{equation}
Z_{g=2} = \sum_{e, f, g} C(e, f, g) \int \prod_{i=1}^{6} \frac{1}{d\nu_{pr}} T^{-5} \prod_{k<l}^{6} a_{kl}^{-3} \hat{a}_{kl}^{-2} \hat{O}_{g}^{2} [\mathcal{P}^X_{e} + \mathcal{P}^{gh}_{e}] O_e,
\end{equation}

where $(e, f, g)$ are even spin structures, $C(e, f, g)$ are the phases dictated by the GSO projection in the different heterotic string models; $O_e = (\det \sigma)^{2} \theta^{4}[e]$ are the partition functions for each set of eight world-sheet fermions, matter and gauge, $T$ is given by

\begin{equation}
T = \int d^{2}z_{1}d^{2}z_{2}|(z_{1} - z_{2})y^{-1}(z_{1})y^{-1}(z_{2})|^{2}
\end{equation}

and $\mathcal{P}^X, \mathcal{P}^{gh}_e$ are respectively the matter and ghost part of the correlator of two PCOs. Their explicit expressions are

\begin{equation}
\mathcal{P}^X = \frac{5}{8} a_{12}^{-1} \left[ a_{23}a_{24}a_{25}a_{26} \frac{P_{12}}{T} + a_1 \leftrightarrow a_2 \right],
\end{equation}

with

\begin{equation}
P_{12} = \int d^{2}z_{1}d^{2}z_{2} \frac{(a_1 - z_1)(a_1 - z_2)}{(a_2 - z_1)(a_2 - z_2)} \left| \frac{z_1 - z_2}{y(z_1)y(z_2)} \right|^{2}
\end{equation}

and

\begin{equation}
\mathcal{P}^{gh}_e = \frac{1}{4} a_{12}^{-1} \sum_{i=1}^{3} (a_1 - A^e_i)(a_1 - B^e_i)(a_1 - B^e_{i+1})(a_2 - B^e_{i+2}),
\end{equation}

when $e = (12A^e_2 || B^e_1 B^e_2 B^e_3)$ or

\begin{equation}
\mathcal{P}^{gh}_e = \frac{1}{4} a_{12}^{-1} (a_1 - A^e_2)(a_1 - A^e_3)(a_2 - B^e_2)(a_2 - B^e_3)
\end{equation}

if $e = (1A^e_2 A^e_1 || 2B^e_2 B^e_3)$. $d\nu_{pr}$ is just the volume of the $SL(2, \mathbb{C})$ projective group

\begin{equation}
d\nu^2_{pr} = \frac{d^2a_4d^2a_5d^2a_6}{|a_{45}a_{46}a_{56}|^2}.
\end{equation}

To get (B.4) it has been assumed that the two PCOs have been located respectively at $a_1$ and $a_2$, as can be seen from the fact that their correlation function diverges when
This fact is actually behind the lack of modular invariance of \( a_{12} \to 0 \) since modular transformations interchange the branch points and then do not preserve the insertion points of the PCOs.

This expression can be easily applied not only to the ten-dimensional supersymmetric heterotic string but also to other models without supersymmetry and/or compactified dimensions. This is done simply by taking different choices for the \( C(e, f, g) \) phases and/or adding new internal fermionic sectors.

\[ \text{Appendix C: Weierstrass Elliptic Function} \]

In this third appendix we will collect some useful results about the Weierstrass elliptic function \( \mathcal{P}(z|\omega_1, \omega_2) \) \[42\].

A meromorphic function \( f(z) \) is said to be an elliptic function if it is doubly periodic with semiperiods \( \omega_1, \omega_2 \in \mathbb{C} \)

\[ f(z) = f(z + 2m\omega_1 + 2n\omega_2), \quad (C.1) \]

with \( m, n \in \mathbb{Z} \). Given \( \text{(C.1)} \) we see that \( f(z) \) is determined on the whole complex plane by its value in the fundamental parallelogram \( OABC \) (fig. \[4\]). Then as a corollary we see that any elliptic function without singularities in the fundamental parallelogram must be a constant. In the same way it can be proven that the sum of the residues at the poles in \( OABC \) vanishes. The Weierstrass elliptic function \( \mathcal{P}(z|\omega_1, \omega_2) \) is uniquely determined from the following three properties

- \( \mathcal{P}(z) \) is an elliptic function with a single pole located in \( z = 0 \).
- Its principal part in $z = 0$ is $1/z^2$.
- $\mathcal{P}(z) - z^{-2}$ tends to zero as $z \to 0$.

It is defined by

$$
\mathcal{P}(z|\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{m,n} \left[ \frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right]. \tag{C.2}
$$

Expanding in power series around $z = 0$ we find

$$
\mathcal{P}(z|\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{k=2}^{\infty} (2k - 1)G_{2k}z^{2k-2}, \tag{C.3}
$$

with $G_{2k} = \sum' (2m\omega_1 + 2n\omega_2)^{-2k}$.

An important result concerning elliptic functions is that any elliptic function $f(z)$ can be written in terms of $\mathcal{P}(z)$ and its derivative $\mathcal{P}'(z)$ which is itself an elliptic function also.

Suppose $f(z)$ is even, has a pole of order $2s$ at $z = 0$ (or a zero for $s < 0$) and its remaining poles and zeroes in the fundamental parallelogram are located respectively at $\{\beta_1,\ldots,\beta_k\}$ and $\{\alpha_1,\ldots,\alpha_k\}$ then

$$
f(z) = C\mathcal{P}(z)^s \prod_{i=1}^k \frac{\mathcal{P}(z) - \mathcal{P}(\alpha_i)}{\mathcal{P}(z) - \mathcal{P}(\beta_i)}. \tag{C.4}
$$

with $C$ a complex constant. If $f(z)$ is an odd elliptic function then $f(z)/\mathcal{P}'(z)$ is even and we can apply (C.4) and in the case of a general $f(z)$ one can always write it as the sum of an even and an odd piece.

Since $\mathcal{P}'(z)^2$ is an even elliptic function we know from what we said in the last paragraph that it can be expressed in terms of $\mathcal{P}(z)$. Locating the zeroes and the poles of $\mathcal{P}'(z)$ we find

$$
\mathcal{P}'(z)^2 = [\mathcal{P}(z) - e_1][\mathcal{P}(z) - e_2][\mathcal{P}(z) - e_3], \tag{C.5}
$$

where $e_i$ can be written in terms of Jacobi’s theta functions

$$
e_1 = \frac{\pi^2}{12\omega_1^2} [\theta_3^4(0|\tau) + \theta_4^4(0|\tau)],
\quad e_2 = \frac{\pi^2}{12\omega_1^2} [\theta_3^4(0|\tau) - \theta_4^4(0|\tau)],
\quad e_3 = -\frac{\pi^2}{12\omega_1^2} [\theta_2^4(0|\tau) + \theta_3^4(0|\tau)]. \tag{C.6}
$$

After some algebra it can be proven that

$$
\mathcal{P}(z|\omega_1,\omega_2) - e_k = \left[ \frac{\theta_1(0|\tau)}{2\omega_1 \theta_{k+1}(0|\tau)} \frac{\theta_{k+1}(v|\tau)}{\theta_1(v|\tau)} \right]^2, \tag{C.7}
$$

where $k = 1, 2, 3$ and $v = z/(2\omega_1)$. 37
To finish, let us define the Weierstrass $\zeta$-function

$$\int_0^z dz \left[ P(z) - \frac{1}{z^2} \right] = -\zeta(z) + \frac{1}{z}. \quad (C.8)$$

Taking derivatives with respect to $z$ in both sides of the last formula we have

$$P(z) = -\zeta'(z); \quad (C.9)$$

$\zeta(z)$ is an odd function of $z$ but it is not an elliptic function, since it can be easily checked that

$$\zeta(z + 2m\omega_1 + 2n\omega_2) = \zeta(z) + 2m\eta_1 + 2n\eta_2; \quad (C.10)$$

where $\eta_1$ and $\eta_2$ are related to the semiperiods by

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{i\pi}{2}. \quad (C.11)$$
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