The role of noncomutativity in the evolution of the Universe

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Abstract

In the present work, we study the noncommutative version of a quantum cosmology model. The model has a Friedmann-Robertson-Walker geometry, the matter content is a radiative perfect fluid and the spatial sections have zero constant curvature. We work in the Schutz’s variational formalism. We quantize the model and obtain the appropriate Wheeler-DeWitt equation. In this model the universe is spatially bounded. Therefore, its quantum mechanical version has a discrete energy spectrum. We compute the discrete energy spectrum and the corresponding eigenfunctions. The energies depend on a noncommutative parameter $\theta$. We compute the scale factor expected value $\langle a \rangle$ for several values of $\theta$. For all of them, $\langle a \rangle$ oscillates between maximums and minimums values. We observe that, $\langle a \rangle$ grows with the decrease of $\theta$. We also observe that, the smaller the value of $\theta$, the greater is the interval where $\langle a \rangle$ takes values. This behavior proceeds until the commutative case, where $\theta$ vanishes. There, the universe is not spatially bounded anymore and $\langle a \rangle$ grows with time, without limit. In that picture, initially, the noncommutativity effect was very strong ($\theta >> 1$), then, with the passage of time, that effect was diminishing until it disappeared ($\theta = 0$). Therefore, the noncommutativity may furnishes a mechanism by which the Universe begins spatially bounded and evolves to a latter spatially unbounded one.

Keywords: Quantum cosmology, Noncommutative spacetime

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Noncommutative ideas were first introduced, a long time ago, by Snyder \cite{1,2}. There, the noncomutativity was imposed between the spacetime coordinates and his main motivation was to eliminate the divergences in quantum

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field theory. Recently, the interest in those ideas of noncommutativity between spacetime coordinates were renewed due to some important results obtained in superstring, membrane and $M$-theories \[3, 4, 5, 6, 7\]. For more information on those important results we refer to the reviews \[8, 9\]. Since then, noncommutativity has been applied to many other physical systems, such as: quantum harmonic oscillator \[10, 11, 12\], hydrogen atom \[13\], quantum Hall effect \[14, 15, 16\], Einstein’s gravity theory \[17, 18, 19\], cosmology \[20, 21, 22, 23, 24\], black hole physics \[32, 33, 35, 34, 36, 37, 38, 39, 40, 41\], quantum cosmology \[42, 43, 44\], to name only but a few. For a more complete list of references see \[45\].

One important arena where noncommutative (NC) ideas may play an important role is cosmology. In the early stages of its evolution, the Universe may have had very different properties than the ones it has today. Among those properties some physicists believe that the spacetime coordinates were subjected to a noncommutative algebra. Inspired by these ideas some researchers have considered such NC models in quantum cosmology \[42, 43, 44\]. It is also possible that some residual NC contribution may have survived in later stages of our Universe. Based on these ideas some researchers have proposed some NC models in classical cosmology in order to explain some intriguing results observed by WMAP. Such as a running spectral index of the scalar fluctuations and an anomalously low quadrupole of CMB angular power spectrum \[21, 22, 23, 24, 25\]. Another relevant application of the NC ideas in semi-classical and classical cosmology is the attempt to explain the present accelerated expansion of our Universe \[28, 29, 31\].

Superstring theory \[46, 47\] is one strong candidate for the unified description of all known physical interactions, and many physicists believe that it will correctly describe the quantum gravity effects that took place at the beginning of our Universe. There is an important conjecture \[48\] which states that Type IIB string theory on \((AdS_5 \times S_5)_N\) plus some appropriate boundary conditions is dual to the $U(N)$ super-Yang-Mills theory with $N = 4$ and $d = (3 + 1)$. It means that, possibly, for an appropriate description of the known physical interactions by superstring theories, strings have to exist in an Anti-DeSitter spacetime. As the latter has a negative cosmological constant, such possibility encourages the study of spacetimes with a negative cosmological constant, for the sake of understanding more about a possible initial state of the Universe. The presence of a negative cosmological constant in a cosmological model has, in general, the important effect of constraining the size of the Universe to take values in a bounded domain. With the discovery that, presently, our Universe is expanding in an accelerated way \[49, 50\], it is clear, in that picture, that some process must have happened so that the effect of a negative cosmological constant is no longer present. Instead, today, we must have the presence of an exotic form of matter accelerating the expansion of the Universe. In the present paper, we would like to investigate a mechanism that may explain such transition from a quantum cosmology model where initially the size of the Universe takes values in a bounded domain to a later model where it is unbounded.

In the present work, we study the noncommutative version of a quantum
cosmology model. The model has a Friedmann-Robertson-Walker (FRW) geometry, the matter content is a radiative perfect fluid and the spatial sections have positive constant curvatures. We work in the Schutz’s variational formalism [51, 52]. The noncommutativity that we are about to propose is not the typical noncommutativity between usual spatial coordinates. We are describing a FRW model using the Hamiltonian formalism, therefore the phase space of the present model is given by the following canonical variables and conjugated momenta: \{a, P_a, \tau, P_\tau\}. Then, the noncommutativity, at the quantum level, we are about to propose will be between some of these phase space variables. Since these variables are functions of the time coordinate \(t\), this procedure is a generalization of the typical noncommutativity between usual spatial coordinates. The noncommutativity between those types of phase space variables have already been proposed in the literature. At the quantum level in Refs. [42, 43, 44] and at the semi-classical and classical levels in Refs. [28, 29, 30, 31]. We quantize the model and obtain the appropriate Wheeler-DeWitt equation. In this model the scale factor takes values in a bounded domain. Therefore, its quantum mechanical version has a discrete energy spectrum. We compute the discrete energy spectrum and the corresponding eigenfunctions. The energies grow with a noncommutative parameter \(\theta\). We compute the scale factor expected value (\(\langle a\rangle\)) for several values of \(\theta\). For all of them, \(\langle a\rangle\) oscillates between maximums and minimums values. We observe that, \(\langle a\rangle\) grows with the decrease of \(\theta\). We also observe that, the smaller the value of \(\theta\), the greater is the interval where \(\langle a\rangle\) takes values. This behavior proceeds until the commutative case, where \(\theta\) vanishes. There, the scale factor is not bounded anymore and \(\langle a\rangle\) grows with time, without limit. In that picture, initially, the noncommutativity effect was very strong (\(\theta >> 1\)), then, with the passage of time, that effect was diminishing until it disappeared (\(\theta = 0\)). Therefore, the noncommutativity may furnishes a mechanism by which the Universe begins spatially bounded and evolves to a later spatially unbounded one. Here, we shall consider only the noncommutativity effect in order to constrain the size of the Universe. We shall leave the inclusion of a negative cosmological constant to a future work.

The FRW cosmological models are characterized by the scale factor \(a(t)\) and have the following line element,

\[ ds^2 = -N^2(t)dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right), \tag{1} \]

where \(d\Omega^2\) is the line element of the two-dimensional sphere with unitary radius, \(N(t)\) is the lapse function and \(k\) gives the type of constant curvature of the spatial sections. Here, we are considering the case with zero curvature \(k = 0\) and we are using the natural unit system, where \(\hbar = c = G = 1\). The matter content of the model is represented by a perfect fluid with four-velocity \(U^\mu = \delta^\mu_0\) in the comoving coordinate system used. The total energy-momentum tensor is given by,

\[ T_{\mu, \nu} = (\rho + p)U_\mu U_\nu - pg_{\mu, \nu}, \tag{2} \]

where \(\rho\) and \(p\) are the energy density and pressure of the fluid, respectively.
Here, we assume that $p = \rho/3$, which is the equation of state for radiation. This choice may be considered as a first approximation to treat the matter content of the early Universe and it was made as a matter of simplicity. It is clear that a more complete treatment should describe the radiation, present in the primordial Universe, in terms of the electromagnetic field.

From the metric (1) and the energy momentum tensor (2), one may write the total Hamiltonian of the present model ($N\mathcal{H}$), where $N$ is the lapse function and $\mathcal{H}$ is the superhamiltonian constraint. It is given by [52],

$$N\mathcal{H} = -\frac{p_a^2}{12} + p_T,$$

(3)

where $p_a$ and $p_T$ are the momenta canonically conjugated to $a$ and $T$, the latter being the canonical variable associated to the fluid [52]. Here, we are working in the conformal gauge, where $N = a$. The commutative version of the present model was first treated in Ref. [53].

We wish to quantize the model following the Dirac formalism for quantizing constrained systems [54]. First we introduce a wave-function which is a function of the canonical variables $a$ and $T$,

$$\Psi = \Psi(a, T).$$

(4)

Then, we impose the appropriate commutators between the operators $a$ and $T$ and their conjugate momenta $p_a$ and $p_T$. Working in the Schrödinger picture, the operators $a$ and $T$ are simply multiplication operators, while their conjugate momenta are represented by the differential operators,

$$p_a \rightarrow -i \frac{\partial}{\partial a}, \quad p_T \rightarrow -i \frac{\partial}{\partial T}. \quad (5)$$

Finally, we demand that the operator corresponding to $N\mathcal{H}$ annihilate the wave-function $\Psi$, which leads to the Wheeler-DeWitt equation,

$$\frac{1}{12} \frac{\partial^2}{\partial a^2} \Psi(a, \tau) = -i \frac{\partial}{\partial \tau} \Psi(a, \tau),$$

(6)

where the new variable $\tau = -T$ has been introduced. This is the Schrödinger equation of an one dimensional free particle restricted to the positive domain of the variable.

The operator $N\mathcal{H}$ is self-adjoint [53] with respect to the internal product,

$$(\Psi, \Phi) = \int_{0}^{\infty} da \, \Psi(a, \tau)^* \Phi(a, \tau),$$

(7)

if the wave functions are restricted to the set of those satisfying either $\Psi(0, \tau) = 0$ or $\Psi'(0, \tau) = 0$, where the prime $'$ means the partial derivative with respect to $a$. Here, we consider wave functions satisfying the former type of boundary condition and we also demand that they vanish when $a$ goes to $\infty$. For the boundary conditions mentioned above, the author of Ref. [53], showed that the scale factor expected value grows with $\tau$. It starts from a nonzero value and
grows without limit with a well defined function of $\tau$. For very large values of $\tau$, that function reduces to a first degree polynomial in $\tau$.

In order to introduce the noncommutativity in the present model, we shall follow the prescription used in Refs. [42, 43, 44]. In the present model, the noncommutativity will be between the two operators $p_a$ and $p_\tau$,

$$[\hat{p}_a, \hat{p}_\tau] = i\theta,$$

where $\hat{p}_a$ and $\hat{p}_\tau$ are the noncommutative version of the operators. This noncommutativity between those operators can be taken to functions that depend on the noncommutative version of those operators with the aid of the Moyal product [55, 56, 7, 8]. Consider two functions of $\hat{a}$ and $\hat{\tau}$, let’s say, $f$ and $g$. Then, the Moyal product between those two function is given by:

$$f(\hat{a}, \hat{\tau}) \star g(\hat{a}, \hat{\tau}) = f(\hat{a}, \hat{\tau}) \exp \left( \frac{i\theta}{2} \left( \partial \hat{a} \partial \hat{\tau} - \partial \hat{\tau} \partial \hat{a} \right) \right) g(\hat{a}, \hat{\tau}).$$

Using the Moyal product, we may adopt the following Wheeler-DeWitt equation for the noncommutative version of the present model,

$$\left[ \frac{\hat{p}_a^2}{12} - \hat{p}_\tau \right] \star \Psi(\hat{a}, \hat{\tau}) = 0. \quad (9)$$

It is possible to rewrite the Wheeler-DeWitt equation (9) in terms of the commutative version of the operators $p_a$ and $p_\tau$ and the ordinary product of functions. In order to do that, we must initially introduce the following transformation between the noncommutative and the commutative operators,

$$\begin{align*}
\hat{p}_a &= p_a + \frac{1}{2} \theta \tau, \\
\hat{p}_\tau &= p_\tau - \frac{1}{2} \theta a,
\end{align*} \quad (10)$$

and the transformations of the other noncommutative variables are trivial: $\hat{a} = a$ and $\hat{\tau} = \tau$. Then, we may write the commutative version of the Wheeler-DeWitt equation (9), to first order in the commutative parameter $\theta$, in the Schrödinger picture as,

$$\frac{1}{12} \frac{\partial^2 \Psi(a, \tau)}{\partial a^2} - \frac{i}{12} \theta a \frac{\partial \Psi(a, \tau)}{\partial a} - \frac{1}{2} \theta a \Psi(a, \tau) = -i \frac{\partial \Psi(a, \tau)}{\partial \tau}. \quad (11)$$

For a vanishing $\theta$ this equation reduces to the commutative Schrödinger equation described above.

In order to solve this equation, we start imposing that the wave function $\Psi(a, \tau)$ has the following form,

$$\Psi(a, \tau) = e^{i\theta a \tau/2} e^{-iE\tau} A(a). \quad (12)$$

Introducing this ansatz in Eq. (11), we obtain, to first order in $\theta$, the eigenvalue equation,
\[
\frac{d^2 A(a)}{da^2} - (12\theta a - 12E) A(a) = 0, \tag{13}
\]
where \( E \) is the eigenvalue and it is associated with the fluid energy.

The solutions to this equation are the Airy functions,
\[
A(a) = c_1 \text{Ai} \left( \frac{12\theta a - 12E}{(12\theta)^{2/3}} \right) + c_2 \text{Bi} \left( \frac{12\theta a - 12E}{(12\theta)^{2/3}} \right). \tag{14}
\]
The Airy functions \( \text{Bi} \) grow up exponentially when \( a \to \infty \). In order to eliminate this undesirable behavior, we put \( c_2 = 0 \). Then, the energy eigenfunctions for our model are,
\[
A(a) = c_1 \text{Ai} \left( \frac{12\theta a - 12E}{(12\theta)^{2/3}} \right). \tag{14}
\]
If we introduce the boundary condition that \( A(a = 0) = 0 \), we find from Eq. (14) the energy eigenvalues with the following expression,
\[
E_n = \frac{1}{12}(12\theta)^{2/3} \alpha_n \tag{15}
\]
where \( \alpha_n \) is positive and is the zero of order \( n \) of the Airy function \( \text{Ai} \). It is clear from this equation that the energy eigenvalues grow with \( \theta \).

The most general expression of \( \Psi(a, \tau) \) Eq. (12), which is a solution to Eq. (11), is a linear combination of the eigenfunctions \( A_n(a) \), Eq. (14), taking into account the energy eigenvalues Eq. (15), combined with the exponential factor present in Eq. (12), for a given \( \theta \) value.
\[
\Psi(a, \tau) = e^{i\theta a \tau / 2} \sum_{n=0}^{N} C_n \text{Ai} \left( \frac{12\theta a - 12E_n}{(12\theta)^{2/3}} \right) \exp \left( -iE_n \tau \right). \tag{16}
\]
The time evolution of the wave packets built from eq. (16) shows that they are null not only at the origin but they are asymptotically null at infinity as well. In the region near \( a = 0 \) these packets present strong oscillations, which decrease as \( a \) increases.

Now, we would like to verify that the wavefunction and quantities computed with it are well defined. In order to do that, we shall compute, the scale factor expected value, \( \langle a \rangle \), for different values of \( \theta \). First of all, we must define the wavepacket by choosing a finite number of eigenfunctions contributing in the linear combination Eq. (16) and the values of the \( C_n \)'s. Let us choose the first twenty one eigenfunctions, in other words \( N = 20 \), in Eq. (16) and for all \( n \), \( C_n = 1 \). Next, we compute the eigenvalues, \( E_n \) for those 21 eigenfunctions, with the aid of Eq. (15). In order to do that we must choose the values of \( \theta \) and the \( \alpha_n \). In the present situation, we shall choose several different values of \( \theta \). Finally, we must compute the scale factor expected value, using the following expression,
\[
\langle a \rangle (\tau) = \frac{\int_{0}^{\infty} a |\Psi(a, \tau)|^2 da}{\int_{0}^{\infty} |\Psi(a, \tau)|^2 da} \tag{17}
\]
After computing \( \langle a \rangle \) for several different values of \( \theta \) and various \( \tau \) intervals, we noticed that this quantity oscillates between a maximum and a minimum value and never vanishes. This result gives an indication that those models are free from singularities, at the quantum level. We also noticed that, \( \langle a \rangle \) grows with the decrease of \( \theta \) and the smaller the value of \( \theta \), the greater is the interval where \( \langle a \rangle \) takes values. This behavior proceeds until the commutative case, where \( \theta \) vanishes. There, \( \langle a \rangle \) is not spatially bounded anymore and it grows with time, without limit. In that picture, initially, the noncommutativity effect was very strong (\( \theta \gg 1 \)), then, with the passage of time, that effect was diminishing until it disappeared (\( \theta = 0 \)). Therefore, the noncommutativity may furnishes a mechanism by which the Universe begins spatially bounded and evolves to a later spatially unbounded one. It is important to mention that in the present paper this mechanism can only give a qualitative explanation of such transition. We hope, in the near future, to present a full quantitative explanation of such transition.

Figure 1 shows \( \langle a \rangle \) for \( \theta = 1 \) and the \( \tau \) interval \( 0 \leq \tau \leq 10000 \). It is clear from that figure that \( \langle a \rangle \) oscillates between a maximum and a minimum value and never vanishes. Figure 2, shows \( \langle a \rangle \) computed under the same conditions used to compute this quantity in Figure 1, except that in this new figure we considered \( \theta = 0.1 \). From Figure 2, it is clear that \( \langle a \rangle \) is larger than in the previous figure and also oscillates in a wider range of values. Those behaviors may be understood by the fact that the potential barrier, that confines the scale factor, grows with \( \theta \). Therefore, as \( \theta \) increases the scale factor is forced to oscillate in an ever decreasing region.

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Figure 2: ⟨a⟩ for θ = 0.1 and the time interval 0 ≤ τ ≤ 10000.

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