Multi-bubble nodal solutions to slightly subcritical elliptic problems with Hardy terms in symmetric domains

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Abstract We consider the slightly subcritical elliptic problem with Hardy term
\[
\begin{aligned}
-\Delta u - \mu \frac{|u|}{|x|^2} &= |u|^{2^*-2-\epsilon} u & \text{in } \Omega \subset \mathbb{R}^N, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]
where \(0 \in \Omega\) and \(\Omega\) is invariant under the subgroup \(SO(2) \times \{\pm E_{N-2}\} \subset O(N)\); here \(E_n\) denotes the \(n \times n\) identity matrix. If \(\mu = \mu_0 e^\alpha\) with \(\mu_0 > 0\) fixed and \(\alpha > \frac{N-4}{N-2}\) the existence of nodal solutions that blow up, as \(\epsilon \to 0^+\), positively at the origin and negatively at a different point in a general bounded domain has been proved in [5]. Solutions with more than two blow-up points have not been found so far. In the present paper we obtain the existence of nodal solutions with a positive blow-up point at the origin and \(k = 2\) or \(k = 3\) negative blow-up points placed symmetrically in \(\Omega \cap (\mathbb{R}^2 \times \{0\})\) around the origin provided a certain function \(f_k : \mathbb{R}^+ \times \mathbb{R}^+ \times I \to \mathbb{R}\) has stable critical points; here \(I = \{t > 0 : (t,0,\ldots,0) \in \Omega\}\). If \(\Omega = B(0,1) \subset \mathbb{R}^N\) is the unit ball centered at the origin we obtain two solutions for \(k = 2\) and \(N \geq 7\), or \(k = 3\) and \(N\) large. The result is optimal in the sense that for \(\Omega = B(0,1)\) there cannot exist solutions with a positive blow-up point at the origin and four negative blow-up points placed on the vertices of a square centered at the origin. Surprisingly there do exist solutions on \(\Omega = B(0,1)\) with a positive blow-up point at the origin and four blow-up points on the vertices of a square with alternating positive and negative signs. The results of our paper show that the structure of the set of blow-up solutions of the above problem offers fascinating features and is not well understood.

2010 Mathematics Subject Classification: 35B44, 35B33, 35J60.

Key words: Hardy term; Critical exponent; Slightly subcritical problems; Nodal solutions; Multi-bubble solutions.

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1 Introduction

The paper is concerned with the semilinear singular problem

\begin{equation}
\begin{aligned}
-\Delta u - \mu \frac{u}{|x|^2} &= |u|^{2^* - 2 - \varepsilon} u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\end{equation}

(1.1)

where \( \Omega \subset \mathbb{R}^N, N \geq 7 \), is a smooth bounded domain with \( 0 \in \Omega; 2^* := \frac{2N}{N-2} \) is the critical Sobolev exponent.

In [5], we obtained the existence of two-bubble nodal solutions to problem (1.1) that blow up positively at the origin and negatively at a different point in a general bounded domain, as \( \varepsilon \to 0^+ \) and \( \mu = \mu_0 \varepsilon^\alpha \) with \( \mu_0 > 0 \) and \( \alpha > \frac{N-4}{N-2} \). The location of the negative blow-up point is determined by the geometry of the domain.

The existence of nodal bubble tower solutions has been proved in [6]. These are superpositions of positive and negative bubbles with different scalings, all blowing up at the origin. It seems to be a difficult and open problem whether solutions with a blow-up point at the origin and more than one blow-up point outside the origin exist in a general domain. In the present paper we investigate this problem in symmetric domains, in particular for the model case of the ball \( \Omega = B(0,1) \).

In the case of a ball it is natural to place one blow-up point, say positive, at the origin and \( k \) blow-up points, say negative, at the vertices of a regular \( k \)-gon with center at the origin. We shall prove that solutions of this shape exist for \( 2 \leq k \leq 3 \) but, somewhat surprisingly, not for \( k = 4 \). On the other hand, we prove the existence of solutions with four blow-up points, two positive and two negative ones, at the vertices of a square, centered symmetrically around the positive blow-up point at the origin. Our results show that the existence of solutions of (1.1) with three or more blow-up points is interesting and far from being understood.

When \( \mu = 0 \) the blow-up phenomenon for positive and for nodal solutions to problem (1.1) has been studied extensively, see for instance [2–4, 7, 9, 14, 19, 22, 24, 26, 28–31] and the references therein. However for \( \mu \neq 0 \), few results are known about the existence of positive or nodal solutions with multiple bubbles to problem (1.1). Positive solutions have been obtained in [12]. Related results, though for different equations, can be found in [16, 17, 27]. We also want to mention the papers [10, 11, 15, 18, 21, 23, 25, 32, 33, 35] dealing with the critical exponent, i.e. \( \varepsilon = 0 \).

An important role will be played by the limit problem

\begin{equation}
\begin{aligned}
-\Delta u - \mu \frac{u}{|x|^2} &= |u|^{2^* - 2} u & \text{in } \mathbb{R}^N, \\
u &\to 0 & \text{as } |x| \to \infty
\end{aligned}
\end{equation}

(1.2)

which has been investigated in [13, 35]. Positive solutions of (1.2) in the range \( 0 \leq \mu < \overline{\mu} := \frac{(N-2)^2}{4} \) are given by

\[ V_{\mu,\sigma}(x) = C_{\mu} \left( \frac{\sigma}{\sigma^2|x|^{\beta_1} + |x|^{\beta_2}} \right)^{\frac{N-2}{2}} \]
with \( \sigma > 0, \beta_1 := (\sqrt{\mu} - \sqrt{\mu - \rho})/\sqrt{\mu}, \beta_2 := (\sqrt{\mu} + \sqrt{\mu - \rho})/\sqrt{\mu}, \) and \( C_\mu := \left( \frac{A(N(1-\rho))}{N-2} \right)^{\frac{N+2}{N-2}}. \) These solutions minimize
\[
S_\mu := \min_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 - \mu \frac{u^2}{|x|^2})\,dx}{(\int_{\mathbb{R}^N} |u|^2 \,dx)^{2/2}},
\]
and there holds
\[
\int_{\mathbb{R}^N} \left(|\nabla V_{\mu,\sigma}|^2 - \mu \frac{|V_{\mu,\sigma}|^2}{|x|^2} \right) \,dx = \int_{\mathbb{R}^N} |V_{\mu,\sigma}|^2 \,dx = S_\mu^\frac{N}{N-2}.
\]
In the range \( 0 < \mu < \| \) these are all positive solutions of (1.2). In the case \( \mu = 0 \) these are all solutions with maximum at \( x = 0. \) Of course, if \( \mu = 0 \) also translates of \( V_{\mu,\sigma} \) are solutions of
\[
\begin{aligned}
\left\{ \begin{array}{ll}
-\Delta u = |u|^{2-2}u & \text{in } \mathbb{R}^N, \\
u \to 0 & \text{as } |x| \to \infty.
\end{array} \right.
\end{aligned}
\]
We will write
\[
U_{k,\xi}(x) = C_0 \left( \frac{\delta}{\delta^2 + |x - \xi|^2} \right)^{\frac{N+2}{2}}
\]
for the solutions of (1.3) where \( \frac{\delta}{\xi} > 0, \xi \in \mathbb{R}^N \) and \( C_0 := (N(N - 2))^{\frac{N+2}{N-2}}. \) These are the well known Aubin-Talenti instantons (see [1][3][4]).

Now we state our main results. We consider domains satisfying the condition
\[
(A_1) \; \Omega \subset \mathbb{R}^N \text{ is a bounded domain with } 0 \in \Omega, \text{ and it is invariant under the subgroup } SO(2) \times \{ \pm E_{N-2} \} \subset \mathcal{O}(N).
\]
We use the notation \( x = (x', x'') \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^{N-2} \) and write \( A(x', x'') = (Ax', x'') \) for \( A \in SO(2). \) For \( k \in \mathbb{N} \) let \( R_k = \left( \begin{array}{ccc} \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} \\ \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} \end{array} \right) \in SO(2), \) and set \( I = \{ t > 0 : (t, 0, \ldots, 0) \in \Omega \} \subset \mathbb{R}. \)

Our first results are concerned with the existence of nodal solutions with \( k + 1 \) bubbles, one being positive and \( k \) being negative. Let \( G(x, y) = \frac{1}{|x-y|^{N-2}} - H(x, y) \) be the Green function (up to a coefficient involving the volume of the unit ball) for the Dirichlet Laplace operator in \( \Omega \) with regular part \( H. \) For \( k = 2, 3 \) we define the function \( f_k : \mathbb{R}^+ \times \mathbb{R}^+ \times I \to \mathbb{R} \) by
\[
f_k(\lambda_0, \lambda_1, t) := b_1 \left( H(0, 0) \lambda_0^{N-2} + k H(\xi(t), \xi(t)) \lambda_1^{N-2} + 2k G(\xi(t), 0) \lambda_0 \lambda_1^{N-2} \lambda_1^{N-2} \right)^{\lambda_1^{N-2} \lambda_1^{N-2}}
\]
where \( \xi(t) = (t, 0, \ldots, 0) \) and
\[
b_1 = \frac{1}{2} C_0 \int_{\mathbb{R}^N} U_{1,0}^{2-1} \quad \text{and} \quad b_2 = \frac{1}{2} \int_{\mathbb{R}^N} U_{1,0}^2.
\]
Finally we call a critical point of \( f_k \) stable if it is isolated and has nontrivial local degree. This is the case, for instance, if it is non-degenerate or an isolated local maximum or minimum.
Theorem 1.1. Suppose \( \Omega \) satisfies \((A_1)\), and suppose \((\lambda_0, \lambda_1, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times I\) is a stable critical point of \( f_k \), \( k = 2 \) or \( k = 3 \). Let \( \mu_0 > 0 \) and \( \alpha > \frac{N-2}{2} \) be fixed. Then there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \) problem \((1.1)\) with \( \mu = \mu_\varepsilon = \mu_0 \varepsilon^\alpha \) has a pair of solutions \( \pm u_\varepsilon \) satisfying

\[
\begin{align*}
\quad u_\varepsilon(x) &= V_{\mu_\varepsilon, \sigma_\varepsilon}(x) - \sum_{i=1}^k U_{\delta_\varepsilon, R_k^\varepsilon \xi(t_\varepsilon)}(x) + o(1) \\
&= C_{\mu_\varepsilon} \left( \frac{\sigma_\varepsilon}{(\sigma_\varepsilon)^2 + |x|^2} \right)^{\frac{N-2}{2}} - C_0 \sum_{i=1}^k \left( \frac{\delta_\varepsilon}{(\delta_\varepsilon)^2 + |x - R_k^\varepsilon \xi(t_\varepsilon)|^2} \right)^{\frac{N-2}{2}} + o(1),
\end{align*}
\]

where

\[
\sigma_\varepsilon = (\lambda_0 + o(1)) \varepsilon^{\frac{1}{2}}, \quad \delta_\varepsilon = (\lambda_1 + o(1)) \varepsilon^{\frac{1}{2}}, \quad \xi(t_\varepsilon) = (t_\varepsilon, 0, \ldots, 0) = (t + o(1), 0, \ldots, 0) \quad \text{as} \quad \varepsilon \to 0.
\]

These solutions satisfy the following symmetries:

\[
\begin{align*}
\quad u_\varepsilon(x', x'') &= u_\varepsilon(x', -x'') = u_\varepsilon(R_k x', x'') \quad \text{for} \quad (x', x'') \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}.
\end{align*}
\]

As Proposition \([1.3]\) below shows, for \( k = 4 \) a family \( u_\varepsilon \) as in Theorem \([1.1]\) need not exist even in the case of the ball. It is a challenging problem to find critical points of \( f_k \) for general domains satisfying \((A_1)\). We consider the special case where \( \Omega = B(0, 1) \subset \mathbb{R}^N \) is the unit ball.

Theorem 1.2. If \( \Omega = B(0, 1) \subset \mathbb{R}^N \), \( k = 2 \) and \( N \geq 7 \), or \( k = 3 \) and \( N \) is large enough, then \( f_k \) has two stable critical points, one is a local minimum, the other a mountain pass point with Morse index 1. As a consequence, problem \((1.1)\) has two families of solutions \( \pm u_1^k, \pm u_2^k \) as in Theorem \([1.1]\). They have the additional symmetry

\[
\begin{align*}
\quad u_\varepsilon(x', x'') &= u_\varepsilon(x', Ax'') \quad \text{for all} \ A \in \text{SO}(N-2).
\end{align*}
\]

Remark 1.3. a) In the proof of the case \( k = 3 \) we provide an explicit inequality for \( N \), so that the solutions as in Theorem \([1.2]\) exist if this inequality holds. Numerical computations show that this inequality is not satisfied for \( N = 7 \). We do not know the optimal value for \( N \) such that Theorem \([1.2]\) is true for \( k = 3 \); see also Remark \([1.2]\).

b) We conjecture that Theorem \([1.2]\) holds for other domains satisyin \((A_1)\), for instance for \( \Omega = B^2(0, 1) \times \Omega' \subset \mathbb{R}^2 \times \mathbb{R}^{N-2} \) with \( \Omega' = -\Omega' \subset \mathbb{R}^{N-2} \) a bounded symmetric neighborhood of 0. Our proof of Theorem \([1.2]\) uses the explicit knowledge of the Green function for the ball, hence it does not extend immediately to other domains.

The next result shows that Theorem \([1.2]\) is optimal.

Proposition 1.4. For \( \Omega = B(0, 1) \subset \mathbb{R}^N \), \( N \geq 7 \) and \( k = 4 \) there does not exist a family of solutions \( \pm u_\varepsilon \) as in Theorem \([1.4]\).

Remark 1.5. a) We conjecture that Proposition \([1.4]\) can be generalized to \( k \geq 4 \).
b) We cannot exclude the existence of solutions with a positive bubble at the origin and \( k = 4 \) negative bubbles placed somewhere in the ball and with possibly different blow-up parameters \( \delta \). However, we can show that there do not exist solutions with four negative bubbles at the vertices of a square centered at the origin even if one allows different blow-up speeds, i.e. if one replaces the \( \delta_i \) in (1.4) by \( \delta_{i,\varepsilon} \), \( i = 1, \ldots, 4 \). In fact, it is not difficult to prove that the blow-up parameters \( \delta_{i,\varepsilon} \) have to be independent of \( i \) if the vertices are at \( R_4^i \xi(t_\varepsilon) \), \( i = 1, \ldots, 4 \).

Considering Proposition [1.4] the following existence results of nodal solutions with five bubbles, three being positive and two being negative, is somewhat surprising.

**Theorem 1.6.** Let \( \Omega = B(0,1) \subset \mathbb{R}^N \), \( N \geq 7 \), \( \mu_0 > 0 \) and \( \alpha > \frac{N-4}{N-2} \) be fixed. Then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0,\varepsilon_0) \), there exist a pair of 5-bubble solutions \( \pm u_\varepsilon \) to problem (1.1) with \( \mu = \mu_\varepsilon = \mu_0 \varepsilon^\alpha \) of the shape

\[
    u_\varepsilon(x) = V_{\mu_\varepsilon, \sigma_\varepsilon}(x) + \frac{4}{\varepsilon} \sum_{i=1}^4 (-1)^i U_{\delta_{i,\varepsilon}, R_4^i \xi(t_\varepsilon)}(x) + o(1)
\]

where \( \sigma_\varepsilon = (\lambda_0 + o(1)) \varepsilon \frac{\Delta \xi}{\varepsilon^N} \), \( \delta_{1,\varepsilon} = \delta_{3,\varepsilon} = (\lambda_1 + o(1)) \varepsilon \frac{\Delta \xi}{\varepsilon^N} \), \( \delta_{2,\varepsilon} = \delta_{4,\varepsilon} = (\lambda_2 + o(1)) \varepsilon \frac{\Delta \xi}{\varepsilon^N} \), \( \xi(t_\varepsilon) = (t,0,\ldots,0) \), \( t \varepsilon \rightarrow 0 \) for some \( \lambda_0, \lambda_1, \lambda_2 > 0 \), \( t \in (0,1) \). These solutions satisfy the symmetries:

\[
    u_\varepsilon(x',x'') = u_\varepsilon(x', A x'') = -u_\varepsilon(R x', x'') \quad \text{for} \quad (x', x'') \in B(0,1) \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}, \ A \in SO(N-2).
\]

**Remark 1.7.** a) The parameters \( (\lambda_0, \lambda_1, \lambda_2, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times (0,1) \) in Theorem 1.6 are obtained as a critical point of a suitable limit function \( f_5 \). We conjecture that there exists a second solution in Theorem 1.6 but the computations for finding a second critical point of \( f_5 \) are intimidating.

b) It seems that for \( k = 2 \) in Theorem 1.2 it is still possible to obtain the information on the nodal sets of the solutions as in [3]. For \( k = 3 \) in Theorem 1.2 and for Theorem 1.6 it is an interesting problem to study the profile of the nodal sets of the solutions.

c) As stated in [3], the assumption \( \alpha > \frac{N-4}{N-2} \) is essential in our theorems.

The paper is organized as follows. In Section 2, we collect some notations and preliminary results. Section 3 is devoted to the method of finite dimensional reduction. Section 4 contains the proof of Theorems 1.1 and 1.2. Proposition 1.4 is proved in Section 5 and finally Theorem 1.6 is proved in Section 6.
2 Notations and preliminary results

Throughout this paper, positive constants will be denoted by $C, c$. By Hardy’s inequality the norm
\[
\|u\|_\mu := \left( \int_\Omega (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) \, dx \right)^{\frac{1}{2}}
\]
is equivalent to the norm $\|u\|_0 = \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}$ on $H^1_0(\Omega)$ provided $0 \leq \mu < \mu_\sigma$. This inequality is of course satisfied for $\mu = \mu_\sigma \epsilon^\alpha$ with $\alpha > 0$ and $\epsilon > 0$ small. We write $H_\mu(\Omega)$ for the Hilbert space consisting of the $H^1_0(\Omega)$ functions with the inner product
\[
(u, v)_\mu := \int_\Omega (\nabla u \nabla v - \mu \frac{uv}{|x|^2}) \, dx.
\]
As in [5, 16] let $\iota_\mu^* : L^{2N/(N+2)}(\Omega) \to H_\mu(\Omega)$ be the adjoint operator of the inclusion $\iota_\mu : H_\mu(\Omega) \to L^{2N/(N-2)}(\Omega)$, that is,
\[
\iota_\mu^*(u) = v \iff (v, \phi)_\mu = \int_\Omega u(x) \phi(x) \, dx, \quad \text{for all } \phi \in H_\mu(\Omega).
\]
There exists $c > 0$ such that
\[
\|\iota_\mu^*(u)\|_\mu \leq c \|u\|_{2N/(N+2)}.
\]
Then problem (1.1) is equivalent to the fixed point problem
\[
u = \iota_\mu^*(f_\epsilon(u)), \quad u \in H_\mu(\Omega),
\]
where $f_\epsilon(s) = |s|^{2^* - 2 - \epsilon} s$.

The following proposition is from [5 Proposition 3.1].

Proposition 2.1. Let $0 < \mu < \mu_\sigma$, and let $\Lambda_i$, $i = 1, 2, \ldots$, be the eigenvalues of
\[
\begin{cases}
-\Delta u - \mu \frac{u}{|x|^2} = \Lambda |V_\sigma|^2 - 2 u & \text{in } \mathbb{R}^N, \\
|u| \to 0 & \text{as } |x| \to +\infty
\end{cases}
\]
in increasing order. Then $\Lambda_1 = 1$ with eigenfunction $V_\sigma$, $\Lambda_2 = 2^* - 1$ with eigenfunction $\frac{\partial V_\sigma}{\partial r}$.

Our main results will be proved using variational and singular limit methods applied to the energy functional
\[
J_\epsilon(u) := \frac{1}{2} \int_\Omega \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) \, dx - \frac{1}{2^* - \epsilon} \int_\Omega |u|^{2^* - \epsilon} \, dx
\]
defined on $H_\mu(\Omega)$.

Let us also recall that the Green’s function of the Dirichlet Laplacian $G(x, y) = \frac{1}{|x-y|^{N-2}} - H(x, y)$ and its regular part $H$ are symmetric: $G(x, y) = G(y, x)$ and $H(x, y) = H(y, x)$. If $\Omega$ is invariant under some $A \in O(N)$ then $G(Ax, Ay) = G(x, y)$, and the same holds for $H$. 

3 The finite dimensional reduction

First we recall some notation from [5]. Fix \( \mu_0 > 0, \alpha > \frac{N-4}{N-2} \) and an integer \( k \geq 0 \). For \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k) \in \mathbb{R}_{+}^{k+1} \) and \( \xi = (\xi_1, \xi_2, \ldots, \xi_k) \in \Omega^k \) we define

\[
W_{\varepsilon, \lambda, \xi} := \sum_{i=1}^{k} \text{Ker} \left( -\Delta - (2^* - 1)U_{\delta_i, \xi_i}^{2^* - 2} - (2^* - 1)V_{\mu, \sigma}^{2^* - 2} \right) \subset H^1(\mathbb{R}^N)
\]

where \( \delta_i = \lambda_i \varepsilon^{\frac{1}{N-2}}, \mu = \mu_0 \varepsilon^\alpha, \sigma = \lambda_0 \varepsilon^{\frac{1}{N-2}} \). By Proposition 2.1 and [8] we know that

\[
W_{\varepsilon, \lambda, \xi} = \text{span} \left\{ \Psi_j, \Psi_0, i = 1, 2, \ldots, k, j = 1, 2, \ldots, N \right\},
\]

where for \( i = 1, 2, \ldots, k \) and \( j = 1, 2, \ldots, N \):

\[
\Psi_j := \frac{\partial U_{\delta_i, \xi_i}}{\partial \xi_{i,j}}, \quad \Psi_0 := \frac{\partial U_{\delta_i, \xi_i}}{\partial \delta_i}, \quad \Psi_0 := \frac{\partial V_{\mu, \sigma}}{\partial \sigma}
\]

with \( \xi_{i,j} \) the \( j \)-th component of \( \xi_i \). For \( \eta \in (0,1) \) we define

\[
\mathcal{O}_\eta := \left\{ (\lambda, \xi) \in \mathbb{R}_{+}^{k+1} \times \Omega^k : \lambda_i \in (\eta, \eta^{-1}) \text{ for } i = 0, \ldots, k, \text{ dist}(\xi_i, \partial \Omega) > \eta, \right. \\
\left. |\xi_i| > \eta, \ |\xi_i - \xi_{i_0}| > \eta, \text{ for } i, i_1, i_2 = 1, \ldots, k, \ i_1 \neq i_2 \right\}.
\]

The projection \( P : H^1(\mathbb{R}^N) \to H^1_0(\Omega) \) is defined by \( \Delta Pu = \Delta u \) in \( \Omega \) and \( Pu = 0 \) on \( \partial \Omega \). We also need the spaces

\[
K_{\varepsilon, \lambda, \xi} := P W_{\varepsilon, \lambda, \xi}
\]

and

\[
K_{\varepsilon, \lambda, \xi}^\perp := \{ \phi \in H_{\mu}(\Omega) : (\phi, P\Psi)_{\mu_\varepsilon} = 0, \text{ for all } \Psi \in W_{\varepsilon, \lambda, \xi} \},
\]

as well as the \((\cdot, \cdot)_{\mu_\varepsilon}\)-orthogonal projections

\[
\Pi_{\varepsilon, \lambda, \xi} : H_{\mu_\varepsilon}(\Omega) \to K_{\varepsilon, \lambda, \xi}
\]

and

\[
\Pi_{\varepsilon, \lambda, \xi}^\perp := \text{Id} - \Pi_{\varepsilon, \lambda, \xi} : H_{\mu_\varepsilon}(\Omega) \to K_{\varepsilon, \lambda, \xi}^\perp.
\]

Then solving problem (1.1) is equivalent to finding \( \eta > 0, \varepsilon > 0, (\lambda, \xi) \in \mathcal{O}_\eta \) and \( \phi_{\varepsilon, \lambda, \xi} \in K_{\varepsilon, \lambda, \xi}^\perp \) such that:

\[
\Pi_{\varepsilon, \lambda, \xi} (V_{\varepsilon, \lambda, \xi} + \phi_{\varepsilon, \lambda, \xi} - \iota_\mu^*(f_\varepsilon(V_{\varepsilon, \lambda, \xi} + \phi_{\varepsilon, \lambda, \xi}))) = 0,
\]

and

\[
\Pi_{\varepsilon, \lambda, \xi} (V_{\varepsilon, \lambda, \xi} + \phi_{\varepsilon, \lambda, \xi} - \iota_\mu^*(f_\varepsilon(V_{\varepsilon, \lambda, \xi} + \phi_{\varepsilon, \lambda, \xi}))) = 0,
\]

where in the case of Theorem 1.2

\[
V_{\varepsilon, \lambda, \xi} = -\sum_{i=1}^{k} PU_{\delta_i, \xi_i} + PV_{\mu, \sigma}
\]
with $k = 2, 3$, and in the case of Theorem 1.6

\[ V_{\varepsilon,\lambda,\xi} = \sum_{i=1}^{k} (-1)^i P U_{\delta_i,\xi_i} + PV_{\mu,\sigma,\varepsilon} \]

with $k = 4$.

The following two propositions have been proved in [5].

**Proposition 3.1.** For every $\eta > 0$ there exist $\varepsilon_0 > 0$ and $c_0 > 0$ with the following property. For every $(\lambda, \xi) \in O_\eta$ and for every $\varepsilon \in (0, \varepsilon_0)$ there exists a unique solution $\phi_{\varepsilon,\lambda,\xi} \in K_{\varepsilon,\lambda,\xi} \perp$ of equation (3.1) satisfying

\[ \|\phi_{\varepsilon,\lambda,\xi}\|_{\mu,\varepsilon} \leq c_0 \left( \varepsilon^{N/2} + \varepsilon^{1+\alpha/4} \right). \]

The map $\Phi_{\varepsilon} : O_\eta \to K_{\varepsilon,\lambda,\xi}$ defined by $\Phi_{\varepsilon}(\lambda, \xi) := \phi_{\varepsilon,\lambda,\xi}$ is $C^1$.

Now we can define the reduced functional

\[ I_\varepsilon : O_\eta \to \mathbb{R}, \quad I_\varepsilon(\lambda, \xi) := J_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}). \]

**Proposition 3.2.** If $(\lambda, \xi) \in O_\eta$ is a critical point of $I_\varepsilon$ then $V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}$ is a solution to problem (1.1) for $\varepsilon > 0$ small.

So far everything works on a general bounded domain. Now we will use the invariance of $I_\varepsilon$ under certain symmetries for further reductions. For $A \in O(N)$, $\xi = (\xi_1, \ldots, \xi_k) \in (\mathbb{R}^N)^k$ and $u \in L^p(\mathbb{R}^N)$ we set $A\xi := (A\xi_1, \ldots, A\xi_k)$ and $A \ast u := u \circ A^{-1}$. This induces isometric actions of $O(N)$ on $(\mathbb{R}^N)^k$ as well as on $L^p(\mathbb{R}^N)$ and, if $A\Omega = \Omega$, on $L^p(\Omega)$ and on $H^1(\Omega)$ such that $\iota_\mu$ and $\iota_\mu^*$ are equivariant. Moreover we have

\[ U_{\delta, A\xi} = A \ast U_{\delta, \xi} \quad \text{and} \quad W_{\varepsilon, A\xi} = \{ A \ast u : u \in W_{\varepsilon, \xi} \}, \]

and analogously for $K_{\varepsilon,\lambda,\xi}$, $K_{\varepsilon,\lambda,\xi}$, $V_{\varepsilon,\lambda,\xi}$. As a consequence, the uniqueness statement in Proposition 3.1 implies

\[ \phi_{\varepsilon,\lambda,\xi} = A \ast \phi_{\varepsilon,\lambda,\xi}, \]

hence $I_\varepsilon$ is invariant with respect to the action $A \ast (\lambda, \xi) = (\lambda, A\xi)$ of $O(N)$ on $O_\eta$:

\[ I_\varepsilon(\lambda, A\xi) = I_\varepsilon(\lambda, \xi). \]

Now we apply the principle of symmetric criticality using the matrix $A_N := \begin{pmatrix} E_2 & 0 \\ 0 & -E_{N-2} \end{pmatrix} \in O(N)$. By assumption $A_N(\Omega) = \Omega$, hence $A_N$ acts on $O_\eta$ as above leaving $I_\varepsilon$ invariant. The principle of symmetric criticality implies that critical points of $I_\varepsilon$ constrained to the fixed point set

\[ O_\eta^{A_N} = \{ (\lambda, \xi) \in O_\eta : A_N\xi = \xi \} = \{ (\lambda, \xi) \in O_\eta : \xi_i = (\xi_i', 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, i = 1, \ldots, k \} \]
are critical points of $I_\varepsilon$. We also need the invariance of $I_\varepsilon$ with respect to permutations of the blow-up points. Here we need to distinguish between the cases where $V_{\varepsilon,\lambda,\xi}$ is of the form [5.2] or of the form [5.3]. Let $S_k$ denote the group of permutations of $\{1, \ldots, k\}$. For $\pi \in S_k$ and $(\lambda, \xi) \in \mathbb{R}^{k+1} \times (\mathbb{R}^N)^k$ we define

$$\pi \ast (\lambda_0, \lambda_1, \ldots, \lambda_k) := (\lambda_0, \lambda_{\pi(1)}, \ldots, \lambda_{\pi(k)}) \quad \text{and} \quad \pi \ast (\xi_1, \ldots, \xi_k) := (\xi_{\pi(1)}, \ldots, \xi_{\pi(k)}).$$

In the case when $V_{\varepsilon,\lambda,\xi}$ is of the form [5.2] it is obvious that

$$I_\varepsilon(\pi \ast \lambda, \pi \ast \xi) = I_\varepsilon(\lambda, \xi) \quad \text{for all } \pi \in S_k, (\lambda, \xi) \in O_\eta.$$

It follows that $I_\varepsilon$ is invariant under the map

$$\tau : O^{A_N}_\eta \to O^{A_N}_\eta, \quad \tau(\lambda_0, \lambda_1, \ldots, \lambda_k, \xi_1, \ldots, \xi_k) := (\lambda_0, \lambda_k, \lambda_1, \ldots, \lambda_{k-1}, R_k \xi_k, R_k \xi_1, \ldots, R_k \xi_{k-1}),$$

which induces an action of $\mathbb{Z}/k\mathbb{Z}$ on $O^{A_N}_\eta$; here $R_k(\xi', \xi'') := (R_k \xi', \xi'')$ where $R_k \in SO(2)$ is the rotation from Theorem 1.2. Therefore critical points of $I_\varepsilon$ constrained to the fixed point set of the above map, i.e. to

$$O^{A_N, \tau}_\eta := \{(\lambda, \xi) \in O^{A_N}_\eta : \lambda_i = \cdots = \lambda_1, \xi_i = R_k^{i-1} \xi_1, i = 2, \ldots, k\},$$

are critical points of $I_\varepsilon$.

In conclusion, for the proofs of Theorems 1.1 and 1.2 it remains to find critical points of $I_\varepsilon$ constrained to $O^{A_N, \tau}_\eta$ for $\varepsilon > 0$ small. The additional symmetry of the solutions stated in Theorem 1.2 and also in Theorem 1.6 is obtained as follows. Since the ball is invariant under the action of $A \in SO(N-2)$ defined by $A(x', x'') := (x', Ax'')$ and since $A \ast (\lambda, \xi) = (\lambda, A\xi) = (\lambda, \xi)$ for $(\lambda, \xi) \in O^{A_N}_\eta$ it follows from [3.4] that

$$A \ast \phi_{\varepsilon,\lambda,\xi} = \phi_{\varepsilon,\lambda,A\xi} = \phi_{\varepsilon,\lambda,\xi} \quad \text{for all } A \in SO(N-2),$$

hence $u_\varepsilon = V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}$ satisfies $A \ast u_\varepsilon = u_\varepsilon$, i.e. $u_\varepsilon(x', Ax'') = u_\varepsilon(x', x'')$, for all $A \in SO(N-2)$.

In Theorem 1.6 $V_{\varepsilon,\lambda,\xi}$ is of the form [5.3] with $k = 4$. Here $I_\varepsilon$ is invariant under the map

$$\tilde{\tau}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_0, \xi_1, \xi_2, \xi_3, \xi_4) := (\lambda_3, \lambda_4, \lambda_1, \lambda_2, R_4 \xi_1, R_4 \xi_2, R_4 \xi_3),$$

so, applying the principle of symmetric criticality once more, a critical point of $I_\varepsilon$ constrained to the set

$$O^{A_N, \tilde{\tau}}_\eta := \{(\lambda, \xi) \in O^{A_N}_\eta : \lambda_1 = \lambda_3, \lambda_2 = \lambda_4, \xi_i = R_k^{i-1} \xi_1, i = 2, 3, 4\}$$

is an unconstrained critical point of $I_\varepsilon$. This can of course be generalized to any even integer $k \geq 4$.

## 4 Proof of Theorems 1.1 and 1.2

In this section we consider $V_{\varepsilon,\lambda,\xi} = -\sum_{i=1}^{k} PU_{\delta_i, \xi_i} + PV_{\mu_i, \sigma_i}$ for $k = 2$ and $k = 3$. The reduced energy is expanded as follows; see [5 Proposition 5.1].
Lemma 4.1. For \( \varepsilon \to 0^+ \) there holds
\[
I_\varepsilon(\lambda, \xi) = a_1 + a_2 \varepsilon - a_3 \varepsilon^\alpha - a_4 \varepsilon \ln \varepsilon + \psi(\lambda, \xi) \varepsilon + o(\varepsilon)
\]

\( C^1 \)-uniformly with respect to (\( \lambda, \xi \)) in compact sets of \( O_\eta \). The constants are given by
\[
a_1 = \frac{1}{N}(k + 1)S_0^\varepsilon, \quad a_2 = \frac{(k + 1)}{2^*} \int_{\mathbb{R}^N} U_{1,0}^{2*} \ln U_{1,0} - \frac{k + 1}{(2^*)^2} S_0^\varepsilon, \quad a_3 = \frac{1}{2} S_0^\varepsilon \eta, \quad a_4 = \frac{k + 1}{2^*} \int_{\mathbb{R}^N} U_{1,0}^{2*},
\]

where \( S > 0 \) is defined by \( S_\mu = S_0 - \overline{S} \mu + O(\mu^2) \); see [3, Lemma A.10]. The function \( \psi : O_\eta \to \mathbb{R} \) is given by
\[
\psi(\lambda, \xi) = b_1 \left( H(0,0)\lambda_0^{N-2} + \sum_{i=1}^k H(\xi_i, \lambda_i)\lambda_i^{N-2} + 2 \sum_{i=1}^k G(\xi_i, 0)\lambda_i^{N-2} \right)
- 2 \sum_{i,j=1, i \neq j}^k G(\lambda_i, \lambda_j)\lambda_i^{N-2} \lambda_j^{N-2} - b_2 N - 2 \ln (\lambda_1 \lambda_2 \ldots \lambda_k \lambda_0),
\]

with
\[
b_1 = \frac{1}{2} C_0 \int_{\mathbb{R}^N} U_{1,0}^{2*} - 1, \quad b_2 = \frac{1}{2^*} \int_{\mathbb{R}^N} U_{1,0}^{2*}.
\]

It is well known that a stable critical point (\( \lambda, \xi \)) of \( I_\varepsilon \) for \( \varepsilon > 0 \) small, and that (\( \lambda_\varepsilon, \xi_\varepsilon \)) \( \to (\lambda, \xi) \) as \( \varepsilon \to 0 \). This applies in particular if (\( \lambda, \xi \)) is an isolated critical point of \( \psi \) with nontrivial local degree.

**Proof of Theorem 1.1.** Since the symmetries of \( I_\varepsilon \) carry over to \( \psi \), for the existence of solutions \( u_\varepsilon \) as stated in Theorem 1.1 it is sufficient to find stable critical points of \( \psi \) constrained to
\[
O_{\eta}^{A, \varepsilon} = \{ (\lambda, \xi) \in O_{\eta}^A : \lambda_i = 1, \xi_i = R_{k_i}^{-1} \xi_1, i = 2, \ldots, k \},
\]
where \( k = 2, 3 \). Observe that \( I_\varepsilon \) and \( \psi \) are also invariant with respect to the action of \( A \in SO(2) \) given by \((x', 0) \mapsto (Ax', 0)\) acting on the \( \xi_i \). Therefore in the case \( k = 2 \), setting \( \xi_1 = (t, 0, \ldots, 0) \) for \( 0 < t < 1 \) and \( \xi_2 = R_2 \xi_1 = -\xi_1 \), it is sufficient to find stable critical points of the function \( f_2 : \mathbb{R}^+ \times \mathbb{R}^+ \times (0, 1) \to \mathbb{R} \) defined by
\[
f_2(\lambda_0, \lambda_1, t) = \psi(\lambda_0, \lambda_1, \lambda_1, 1, -\xi_1)
= b_1 \left( H(0,0)\lambda_0^{N-2} + 2H(\xi_1, 1)\lambda_1^{N-2} + 4G(\xi_1, 0)\lambda_1^{N-2} \right) - b_2 \frac{N - 2}{2} \ln (\lambda_1^2 \lambda_0).
\]

This proves Theorem 1.1 in the case \( k = 2 \). For \( k = 3 \) we set \( \xi_1 = (t, 0, \ldots, 0) \) for \( 0 < t < 1 \), \( \xi_2 = R_3 \xi_1 = \left(-\frac{t}{2}, \sqrt{3}t, 0, \ldots, 0\right) \), \( \xi_3 = R_3 \xi_2 = \left(-\frac{t}{2}, -\sqrt{3}t, 0, \ldots, 0\right) \). As above it is sufficient to find stable critical points of the function \( f_3 : \mathbb{R}^+ \times \mathbb{R}^+ \times (0, 1) \to \mathbb{R} \) defined by
\[
f_3(\lambda_0, \lambda_1, t) = \psi(\lambda_0, \lambda_1, \lambda_1, \lambda_1, \xi_2, \xi_3)
= b_1 \left( H(0,0)\lambda_0^{N-2} + 3H(\xi_1, 1)\lambda_1^{N-2} + 6G(\xi_1, 0)\lambda_0^{N-2} \right) - b_2 \ln (\lambda_1^2 \lambda_0).
Here we used that $G(\xi_1, 0) = G(\xi_2, 0) = G(\xi_3, 0)$ and $G(\xi_1, \xi_2) = G(\xi_1, \xi_3) = G(\xi_2, \xi_3)$, as well as $H(\xi_1, \xi_1) = H(\xi_2, \xi_2) = H(\xi_3, \xi_3)$. This proves Theorem 1.1 also in the case $k = 3$.

Proof of Theorem 1.2 Here we need to find stable critical points of $f_2$ and $f_3$ if $G$ is the Green function of the Dirichlet Laplace operator in the unit ball in $\mathbb{R}^N$. Our proof uses the explicit knowledge of $G$. In the case $k = 2$ the proof of [4, Lemma 3.1] applies almost verbatim and shows that $f_2$ has two isolated critical points: a local saddle point with Morse index 1, hence with local degree $-1$, and an isolated local minimum, hence with local degree 1. These are stable critical points, giving rise to critical points of $I_\varepsilon$ for $\varepsilon > 0$ small.

In the case $k = 3$ we set

$$
\gamma_1(t) := H(\xi_1, \xi_1) - 2G(\xi_1, \xi_2) = \frac{1}{(1 - t^2)^{N-2}} - \frac{2}{(\sqrt{3}t)^{N-2}} + \frac{2}{(t^3 + t^2 + 1)^{\frac{N-2}{2}}}
$$

and

$$
\tau_1(t) := G(\xi_1, 0) = \frac{1}{t^{N-2}} - 1
$$

so that

$$
f_3(\lambda_0, \lambda_1, t) = b_1 \left( H(0, 0)\lambda_0^{N-2} - 3\gamma_1(t)\lambda_1^{N-2} - 6\tau_1(t)\lambda_0^{\frac{N-2}{2}}\lambda_1^{\frac{N-2}{2}} \right) - b_2 \frac{N - 2}{2} \ln (\lambda_0^2 \lambda_0). \tag{4.1}
$$

One easily checks that $\gamma_1'(t) > 0$, $\gamma_1(t) \to -\infty$ as $t \to 0^+$, and $\gamma_1(\frac{1}{2}) > 0$. Thus there exists $t^* \in (0, \frac{1}{2})$ such that

$$
\gamma_1(t^*) = 0 \quad \text{and} \quad \gamma_1(t) > 0 \quad \text{for all} \quad t \in (t^*, 1).
$$

A direct computation shows that for $t \in (t^*, 1)$ there exist unique $\lambda_0(t)$, $\lambda_1(t)$ such that

$$
\frac{\partial f_3(\lambda_0(t), \lambda_1(t), t)}{\partial \lambda_0} = 0 \quad \text{and} \quad \frac{\partial f_3(\lambda_0(t), \lambda_1(t), t)}{\partial \lambda_1} = 0.
$$

In fact one obtains

$$
\lambda_0(t)^{\frac{N-2}{2}} = \alpha(\xi_1, \xi_2)\lambda_1(t)^{\frac{N-2}{2}} \quad \text{and} \quad \lambda_1(t)^{\frac{N-2}{2}} = \sqrt{\frac{1}{\beta(\xi_1, \xi_2)} \frac{b_2}{2b_1}}, \tag{4.2}
$$

where

$$
\alpha(x, y) = \frac{-2G(x, 0) + \sqrt{4G^2(x, 0) + 4H(0, 0)(H(x, x) - 2G(x, y))}}{2H(0, 0)}
$$

and

$$
\beta(x, y) = H(x, x) - 2G(x, y) + G(x, 0)\alpha(x, y).
$$
Moreover, continuing the computation one obtains

\[
\frac{\partial^2 f_3(\lambda_0(t), \lambda_1(t), t)}{\partial \lambda^2} = 3(N - 2)b_1 \left( (N - 3)\gamma_1(t)\lambda_1^{N-4} + \left( N - 4 \right) \frac{\lambda_0^{N-2}}{2} \frac{\lambda_1^{N-2}}{2} \right) + \frac{3(N - 2)b_2}{2\lambda_1^2}
\]

\[
= 3(N - 2)b_1 \left( (N - 2)\gamma_1(t)\lambda_1^{N-4} + \left( N - 4 \right) \frac{\lambda_0^{N-2}}{2} \frac{\lambda_1^{N-2}}{2} \right),
\]

\[
\frac{\partial^2 f_3(\lambda_0(t), \lambda_1(t), t)}{\partial \lambda^2} = (N - 2)b_1 \left( (N - 3)H(0, 0)\lambda_0^{N-4} + \left( N - 4 \right) \frac{\lambda_0^{N-2}}{2} \frac{\lambda_1^{N-2}}{2} \right)
\]

\[
+ \frac{(N - 2)b_2}{2\lambda_0^2}
\]

\[
= (N - 2)b_1 \left( (N - 2)H(0, 0)\lambda_0^{N-4} + \left( N - 2 \right) \frac{\lambda_0^{N-2}}{2} \frac{\lambda_1^{N-2}}{2} \right)
\]

\[
\frac{\partial^2 f_3(\lambda_0(t), \lambda_1(t), t)}{\partial \lambda_0 \partial \lambda_1} = \frac{3(N - 2)^2}{2} b_1 \gamma_1(t) \lambda_0^{\frac{N-4}{2}} \lambda_1^{\frac{N-4}{2}}.
\]

It follows that the Hessian matrix \(D^2 f_3(\lambda_0(t), \lambda_1(t), t)\) is positive definite, hence nondegenerate. Therefore it is sufficient to find stable critical points of the function

\[
\nu_1(t) := f_3(\lambda_0(t), \lambda_1(t), t) = 2b_2 - b_2 \frac{N - 2}{2} \ln \left( \lambda_1^2(t)\lambda_0(t) \right).
\]

As in \([1]\) (3.4) one sees that

\[
(4.3) \quad \lim_{t \to -1^+} \nu_1(t) = -\infty \quad \text{and} \quad \lim_{t \to 1^-} \nu_1(t) = +\infty.
\]

Now we prove \(\nu_1'(\frac{1}{2}) < 0\) for \(N\) large. Set

\[
\alpha(t) := \alpha(\xi_1, \xi_2) = -\tau_1(t) + \sqrt{\tau_1^2(t) + \gamma_1(t)},
\]

where we used \(H(0, 0) = 1\). We obtain

\[
\nu_1'(t) = \frac{\partial f_3(\lambda_0(t), \lambda_1(t), t)}{\partial t} = 3b_1 \left( \gamma_1'(t) + 2\alpha(t)\tau_1'(t) \right) \lambda_1^{N-2}.
\]

Then setting \(\epsilon_1(t) := \gamma_1'(t) + 2\alpha(t)\tau_1'(t)\), it is enough to show \(\epsilon_1(\frac{1}{2}) < 0\) for \(N\) large. In fact, since \(\frac{\gamma_1'(\frac{1}{2})}{\tau_1'(\frac{1}{2})} < 1\) for \(N\) large we see as in \([1]\) (3.9) that

\[
\epsilon_1(\frac{1}{2}) \leq \gamma_1'(\frac{1}{2}) + \frac{4\gamma_1(\frac{1}{2})}{5\tau_1(\frac{1}{2})} \tau_1'(\frac{1}{2}).
\]

A direct computation gives for \(N\) large the inequalities

\[
\gamma_1'(\frac{1}{2}) = (N - 2) \left( \left( \frac{4}{3} \right)^{N-1} + 4\left( \frac{2}{\sqrt{3}} \right)^{N-2} - \frac{3}{2} \left( \frac{1}{16} + \frac{1}{4} + 1 \right) \right) < \frac{11(N - 2)}{10} \left( \frac{4}{3} \right)^{N-1}
\]

and

\[
\tau_1'(\frac{1}{2}) = -(N - 2)2^{N-1}
\]

and

\[
\left( \frac{\gamma_1}{\tau_1} \right)(\frac{1}{2}) = \left( \frac{4}{3} \right)^{N-2} - 2\left( \frac{2}{\sqrt{3}} \right)^{N-2} + \frac{2}{(\frac{1}{16} + \frac{1}{4} + 1)^{\frac{1}{2}}} \geq \frac{11}{12} \left( \frac{4}{3} \right)^{N-2} \left( 2^{N-2} - 1 \right).
\]

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which yield \( t_1(\frac{1}{2}) < 0 \), hence \( \nu'_1(\frac{1}{2}) < 0 \) for \( N \) large enough. This together with (1.3) implies that \( \nu_1 \) has a local maximum \( t_1 \in (t^*, \frac{1}{2}) \) and a local minimum \( t_2 \in (\frac{1}{2}, \infty) \). These are nondegenerate because \( \nu_1 \) is analytic.

In conclusion, \( f_1 \) has two critical points: \((\lambda_0(t_1), \lambda_1(t_1), t_1)\) with Morse index 1 and \((\lambda_0(t_2), \lambda_1(t_2), t_2)\) with Morse index 0. This concludes the proof of Theorem 1.2.

\[ \square \]

**Remark 4.2.** a) For \( k = 3, N = 7 \), numerical computations show that one cannot find \( t_0 \in (t^*, 1) \) such that \( \nu'_1(t_0) = 0 \). Therefore it is necessary to assume \( N \) large here.

b) For \( k = 4 \), the idea above cannot give the existence of nodal solutions with five bubbles, one positive at the origin and four negative as in Theorem 1.4. This is the content of Proposition 1.4.

## 5 Proof of Proposition 1.4

It follows from Lemma 4.1 that \( I_\varepsilon \) does not have critical points for \( \varepsilon > 0 \) small if \( \psi \) does not have critical points. This also holds if we constrain \( I_\varepsilon \) and \( \psi \) to \( O^{4N}_\varepsilon \). Setting \( \xi_1 = (t, 0, \ldots, 0) \) for \( 0 < t < 1 \), \( \xi_2 = R_4 \xi_1 = (0, t, 0, \ldots, 0) \), \( \xi_3 = R_4 \xi_2 = (-t, 0, \ldots, 0) \), and \( \xi_4 = R_4 \xi_3 = (0, -t, \ldots, 0) \) we need to consider the function

\[
f_4(\lambda_0, \lambda_1, t) := \psi(\lambda_0, \lambda_1, \lambda_1, \lambda_1, \xi_1, \xi_2, \xi_3, \xi_4)
= b_1 \left( H(0, 0) \lambda_0^{N-2} + 4H(\xi_1, \xi_1) \lambda_1^{N-2} + 8G(\xi_1, 0) \lambda_0^{N-2} \lambda_1^{N-2} \right.
\]

\[
- 8G(\xi_1, \xi_2) \lambda_1^{N-2} - 4G(\xi_1, \xi_3) \lambda_1^{N-2} \right) - b_2 \frac{N-2}{2} \ln(\lambda_1^4 \lambda_0),
\]

where we use

\[
H(\xi_1, \xi_1) = H(\xi_2, \xi_2) = H(\xi_3, \xi_3) = H(\xi_4, \xi_4) \quad \text{and} \quad G(\xi_1, 0) = G(\xi_2, 0) = G(\xi_3, 0) = G(\xi_4, 0)
\]

as well as

\[
G(\xi_1, \xi_2) = G(\xi_2, \xi_3) = G(\xi_3, \xi_4) = G(\xi_4, \xi_1), \quad G(\xi_1, \xi_3) = G(\xi_2, \xi_4).
\]

Proposition 1.4 follows if we can prove that \( f_4 \) does not have critical points. Let \( \tau_1(t) \) be as in (1.4) and define

\[
\gamma_2(t) := \frac{1}{(1 - t^2)^{N-2}} - \frac{2}{(\sqrt{2}t)^{N-2} + \frac{2}{(t^2 + 1)^{N-2}}} = \frac{1}{(2t)^{N-2}} + \frac{1}{(t^2 + 1)^{N-2}}
\]

so that

\[
f_4(\lambda_0, \lambda_1, t) = b_1 \left( H(0, 0) \lambda_0^{N-2} + 4\gamma_2(t) \lambda_1^{N-2} + 8 \tau_1(t) \lambda_0^{N-2} \lambda_1^{N-2} \right) - b_2 \frac{N-2}{2} \ln(\lambda_1^4 \lambda_0) .
\]

A direct computation shows that

\[
\gamma'_2(t) = (N-2) \left( \frac{2t}{(1 - t^2)^{N-1}} + \frac{2}{(\sqrt{2})^{N-2} t^{N-1}} - \frac{4t^3}{(t^4 + 1)^{N-2}} \right) + \frac{1}{2N-2t^{N-1}} - \frac{2t}{(t^2 + 1)^{N-1}} > 0.
\]
Clearly $\gamma_2(t) \to -\infty$ as $t \to 0^+$, and $\gamma_2\left(\frac{2}{\sqrt{2}}\right) > 0$. Then there exists $t^* \in (0, \frac{2}{\sqrt{2}})$ such that

\begin{equation}
\gamma_2(t^*) = 0 \quad \text{and} \quad \gamma_2(t) > 0 \quad \text{for all } t \in (t^*, 1).
\end{equation}

(5.1)

Notice that

$$\lambda_0^N = \alpha_1(\xi_1, \xi_2, \xi_3)\lambda_1^N,$$

$$\lambda_1 = \sqrt{\frac{1}{\beta_1(\xi_1, \xi_2, \xi_3)} \frac{b_2}{2b_1}},$$

where

$$H(\xi_1, \xi_2) = 2G(\xi_1, \xi_2) - G(\xi_1, \xi_2) > 0,$$

and

$$\beta_1(x, y, z) = H(x, y) - 2G(x, y) - G(x, z) + G(x, 0)\alpha_1(x, y, z).$$

Setting $\alpha_1(\xi_1, \xi_2, \xi_3) = \frac{3\tau_1(\xi_1^2 + \frac{\sqrt{3} - \sqrt{2}}{2})}{2}$ and $\alpha_2(\xi_1) = \gamma_2' + 2\alpha_1(\xi_2)$, a similar argument as above shows that problem (11) admits a solution with 5 bubbles, one positive at the origin and 4 negative as in Theorem 11, only if $\alpha_2(t)$ has a zero in $(t^*, 1)$. The following claim implies that this is not the case.

**Claim:** If $N \geq 7$ then $\alpha_2(t) > 0$ for any $t \in (t^*, 1)$.

We first show that $t^* > \frac{\sqrt{6} - \sqrt{2}}{2}$, where $t^*$ is from (5.1). In order to see this, it is enough to prove $\gamma_2\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right) < 0$. Since $2^{2/5} \cdot 2\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right)^2 < 1 < (\frac{\sqrt{6} - \sqrt{2}}{2})^4 + 1$, we have

$$\frac{1}{2} \cdot \frac{1}{\left(\sqrt{2} \cdot 2\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right)^2\right)^N} > \frac{1}{\left((\frac{\sqrt{6} - \sqrt{2}}{2})^4 + 1\right)^{N-2}},$$

for all $N \geq 7$.

On the other hand, it is easy to see that

$$\frac{1}{\left(1 - \left(\frac{\sqrt{6} - \sqrt{2}}{2}\right)^2\right)^N} = \frac{1}{\left(\sqrt{2} \cdot 2\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right)^2\right)^N} \quad \text{and} \quad \frac{1}{\left(2\left(\frac{\sqrt{6} - \sqrt{2}}{2}\right)^2\right)^N} > \frac{1}{\left((\frac{\sqrt{6} - \sqrt{2}}{2})^4 + 1\right)^{N-2}}.$$
which is equivalent to

\[(5.2)\quad (T^N + 3T) \cdot (1 - t^{N-2})^2 > 1.\]

It is obvious that if \( t \in [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}] \), then

\[(5.3)\quad 3T \cdot (1 - t^{N-2})^2 \geq 3(1 - t^5)^2 > 1,\]

and if \( t \in [\frac{3}{4}, 1] \), then

\[(5.4)\quad T^N \cdot (1 - t^{N-2})^2 \geq T^N \cdot (1 - t)^2 = \frac{t^4}{(1+t)^2} \cdot T^{N-2} > \left(\frac{4}{5}\right)^4 \left(\frac{4}{5}\right)^2 > 1.\]

Now we are left to prove (5.2) for \( t \in (t^*, \frac{1}{\sqrt{2}}) \). First of all, if \( t \in \left(\frac{\sqrt{N-1}}{2}, \frac{1}{\sqrt{2}}\right) \), then \( T \in \left(\frac{\sqrt{N-1}}{2}, 1\right) \). Setting

\[f(T) := 3T(1-t^{N-2})^2 = 3T \left(1 - \left(\frac{T}{1+T}\right)^{\frac{N-2}{2}}\right)^2,\]

a direct computation shows that

\[f'(T) = \left(1 - \left(\frac{T}{1+T}\right)^{\frac{N-2}{2}}\right) \left(3 - 3\left(\frac{T}{1+T}\right)^{\frac{N-2}{2}} - 3(N-2)\left(\frac{T}{1+T}\right)^{\frac{N-2}{2}} \left(\frac{1}{1+T}\right)\right)\]

\[\geq \left(1 - \left(\frac{1}{2}\right)^{\frac{N-2}{2}}\right) \left(3 - 3\left(\frac{1}{2}\right)^{\frac{N-2}{2}} - 3(N-2)\left(\frac{1}{2}\right)^{\frac{N-2}{2}} \left(\frac{1}{1 + \frac{\sqrt{N-1}}{2}}\right)\right)\]

where in the second inequality we use the fact that \(3 - 3\left(\frac{1}{2}\right)^{\frac{N-2}{2}} - 3(N-2)\left(\frac{1}{2}\right)^{\frac{N-2}{2}} \left(\frac{1}{1 + \frac{\sqrt{N-1}}{2}}\right)\) is increasing in \(N\). Now we conclude that

\[(5.5)\quad f(T) > \frac{\sqrt{3} - 1}{2} \left(1 - \left(\frac{\sqrt{N-1}}{2 + \sqrt{N-1}}\right)^{\frac{N-2}{2}}\right)^2 \quad \text{for all } N \geq 7.\]

The claim, hence Proposition 1.4 follows combining (5.2), (5.3), (5.4), and (5.5).

### 6 Proof of Theorem 1.6

In this section we consider solutions of the form \(V_{\varepsilon, \lambda, \xi} = \sum_{i=1}^{k} (-1)^i PU_{\delta, \xi, i} + PV_{\sigma}\). Then the reduced function in Lemma 1.1 becomes

\[\tilde{\psi}(\lambda, \xi) = b_1 \left(H(0, 0)\lambda_0^{N-2} + \sum_{i=1}^{k} H(\xi_i, \xi)\lambda_i^{N-2} + 2 \sum_{i=1}^{k} (-1)^{i-1} G(\xi_i, 0)\lambda_0^{N-2} \lambda_i^{N-2} \right.\]

\[+ \left. 2 \sum_{i,j=1, i<j}^{k} (-1)^{i+j} G(\xi, \xi_j)\lambda_i^{N-2} \lambda_j^{N-2} \right) - b_2 \frac{N-2}{2} \ln(\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_k),\]
where $b_1, b_2$ are as in Lemma 4.1.

**Proof of Theorem 1.6.** Let $k = 4$. Using the symmetry again we set $\xi_1 = (t, 0, \ldots, 0)$ for $0 < t < 1$, $\xi_2 = R_4 \xi_1 = (0, t, 0, \ldots, 0)$, $\xi_3 = R_4 \xi_2 = (-t, 0, \ldots, 0)$, and $\xi_4 = R_4 \xi_3 = (0, -t, \ldots, 0)$. As in the proof of Theorem 1.2 it is sufficient to find stable critical points of $\tilde{\psi}$ constrained to $\mathcal{O}^{N, \tilde{F}}$. Since

$$H(\xi_1, \xi_1) = H(\xi_2, \xi_2) = H(\xi_3, \xi_3) = H(\xi_4, \xi_4), \quad G(\xi_1, 0) = G(\xi_2, 0) = G(\xi_3, 0) = G(\xi_4, 0),$$

and

$$G(\xi_1, \xi_2) = G(\xi_2, \xi_3) = G(\xi_3, \xi_4) = G(\xi_4, \xi_1), \quad G(\xi_1, \xi_3) = G(\xi_2, \xi_4),$$

we need to find a critical point of the function

$$f_5(\lambda_0, \lambda_1, \lambda_2, t) := \tilde{\psi}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \xi_2, \xi_3, \xi_4)$$

and

$$= b_1 \left( H(0, 0) \lambda_0^{N-2} + 2H(\xi_1, \xi_1)(\lambda_1^{N-2} + \lambda_2^{N-2}) + 4G(\xi_1, 0)\lambda_0^{N-2} \left( \lambda_1^{N-2} - \lambda_2^{N-2} \right) \right)$$

$$+ 8G(\xi_1, \xi_2)\lambda_1^{N-2} \lambda_2^{N-2} - 2G(\xi_1, \xi_3)\lambda_1^{N-2} - 2G(\xi_1, \xi_4)\lambda_1^{N-2}$$

$$- b_2 \frac{N-2}{2} \ln \left( \lambda_0 \lambda_1^2 \lambda_2^2 \right).$$

**Claim 1:** There exist $t_1^* \in (0, \frac{1}{2})$ and $t_2^* \in (\frac{1}{2}, 1)$ such that for $t \in (0, t_1^*) \cup (t_2^*, 1)$ the equation

$$\nabla_{\lambda_0, \lambda_1, \lambda_2} f_5(\lambda_0, \lambda_1, \lambda_2, t) = 0$$

has a unique solution $(\lambda_0(t), \lambda_1(t), \lambda_2(t), t)$.

Observe that (6.1) is equivalent to the equations

$$H(0, 0) \lambda_0^{N-2} + 2G(\xi_1, 0)\lambda_0^{N-2} \left( \lambda_1^{N-2} - \lambda_2^{N-2} \right) = \frac{b_2}{2b_1}$$

and

$$(H(\xi_1, \xi_1) - G(\xi_1, \xi_1))\lambda_1^{N-2} + G(\xi_1, 0)\lambda_0^{N-2} \lambda_1^{N-2} + 2G(\xi_1, \xi_2)\lambda_1^{N-2} \lambda_2^{N-2} = \frac{b_2}{2b_1}$$

and

$$(H(\xi_1, \xi_1) - G(\xi_1, \xi_1))\lambda_2^{N-2} - G(\xi_1, 0)\lambda_0^{N-2} \lambda_2^{N-2} + 2G(\xi_1, \xi_2)\lambda_1^{N-2} \lambda_2^{N-2} = \frac{b_2}{2b_1}.$$

From (6.3) and (6.4) we deduce

$$\lambda_2^{N-2} - \lambda_1^{N-2} = \frac{G(\xi_1, 0)}{H(\xi_1, \xi_1) - G(\xi_1, \xi_1)}\lambda_0^{N-2},$$

which combined with (6.2) implies:

$$\lambda_0^{N-2} = \frac{H(\xi_1, \xi_1) - G(\xi_1, \xi_3)}{H(\xi_1, \xi_1) - G(\xi_1, \xi_3) - 2G^2(\xi_1, 0)} \cdot \frac{b_2}{2b_1}.$$
As a consequence of (6.3) we get
\[ \lambda_1^{N-2} + \lambda_2^{N-2} - 2\lambda_1^{\frac{N-2}{2}} \lambda_2^{\frac{N-2}{2}} = \lambda_0^{N-2}. \]
hence using (6.3) and (6.4) we deduce:
\[
(6.7) \quad \lambda_1^{\frac{N-2}{2}} \lambda_2^{\frac{N-2}{2}} = \frac{1}{H(\xi_1,\xi_1) - G(\xi_1,\xi_3) + 2G(\xi_1,\xi_2)} \frac{b_2}{2b_1},
\]
and
\[
(6.8) \quad \lambda_1^{N-2} + \lambda_2^{N-2} = \frac{1}{H(\xi_1,\xi_1) - G(\xi_1,\xi_3) + 2G(\xi_1,\xi_2)} b_2 + \lambda_0^{N-2}, \quad \left( \frac{G(\xi_1,0)}{H(\xi_1,\xi_1) - G(\xi_1,\xi_3)} \right)^2.
\]
Let \( \tau_1(t) = G(\xi_1,0) \) be as in (4.1) and set
\[
\gamma_3(t) := H(\xi_1,\xi_1) - G(\xi_1,\xi_3) = \frac{1}{(1 - t^2)^{N-2}} - \frac{1}{(2t)^{N-2}} + \frac{1}{(t^2 + 1)^{N-2}}
\]
and
\[
\gamma_4(t) := G(\xi_1,\xi_2) = \frac{1}{(\sqrt{2t})^{N-2}} - \frac{1}{(t^4 + 1)^{\frac{N-2}{2}}},
\]
so that
\[
f_5(\lambda_0, \lambda_1, \lambda_2, t) = b_1 \left( H(0,0)\lambda_0^{N-2} + 2\gamma_2(t) (\lambda_1^{N-2} + \lambda_2^{N-2}) + 8\gamma_4(t)\lambda_1^{\frac{N-2}{2}} \lambda_2^{\frac{N-2}{2}} + 4\tau_1(t)\lambda_0^{\frac{N-2}{2}} \left( \lambda_1^{\frac{N-2}{2}} - \lambda_2^{\frac{N-2}{2}} \right) \right) - b_2 \frac{N-2}{2} \ln (\lambda_1^4 \lambda_0).
\]
A direct computation shows that \( \gamma_3'(t) > 0, \gamma_3(t) \to -\infty \) as \( t \to 0^+ \), \( \gamma_3(t) \to +\infty \) as \( t \to 1^- \), and \( \gamma_3(\frac{1}{2}) > 0 \). Thus there exists \( t_1^* \in (0, \frac{1}{2}) \) such that
\[
\gamma_3(t_1^*) = 0 \quad \text{and} \quad \gamma_3(t) < 0 \quad \text{for all} \quad t \in (0, t_1^*).
\]
On the other hand, \( (\gamma_3(t) - 2\gamma_1(t))' \) \( > 0 \), \( \gamma_3(t) - 2\gamma_1(t) \to -\infty \) as \( t \to 0^+ \), \( \gamma_3(t) - 2\gamma_1(t) \to +\infty \) as \( t \to 1^- \), and \( \gamma_3(\frac{1}{2}) - 2\gamma_1(\frac{1}{2}) < 0 \). Thus there exists \( t_2^* \in (\frac{1}{2}, 1) \) such that
\[
\gamma_3(t_2^*) - 2\gamma_1(t_2^*) = 0 \quad \text{and} \quad \gamma_3(t) - 2\gamma_1(t) > 0 \quad \text{for all} \quad t \in (t_2^*, 1).
\]
It follows that for every \( t \in (0, t_1^*) \cup (t_2^*, 1) \) there exist unique \( \lambda_0(t), \lambda_1(t), \lambda_2(t) \) such that
\[
\nabla_{\lambda_0, \lambda_1, \lambda_2} f_5(\lambda_0(t), \lambda_1(t), \lambda_2(t), t) = 0,
\]
where \( \lambda_0(t), \lambda_1(t), \lambda_2(t) \) satisfy (6.3), (6.6), (6.7) and (6.8). This proves Claim 1.

**Claim 2:** The Hessian matrix \( D^2_{\lambda_0, \lambda_1, \lambda_2} f_5(\lambda_0(t), \lambda_1(t), \lambda_2(t), t) \) is nondegenerate for any \( t \in (0, t_1^*) \cup (t_2^*, 1) \).
A direct computation using \(6.2\), \(6.3\), and \(6.4\) shows that, writing \(\lambda_i\) instead of \(\lambda_i(t)\),

\[
\frac{\partial^2 f_5(\lambda_0, \lambda_1, \lambda_2, t)}{\partial \lambda_0^2} = (N - 2)b_1 \left( (N - 3)H(0, 0)\lambda_0^{N-4} + (N - 4)\tau_1(t)\lambda_0^{N-6} \left( \lambda_1^{N-2} - \lambda_2^{N-2} \right) \right) + \frac{(N - 2)b_2}{2\lambda_0^2},
\]

\[
= (N - 2)^2b_1 \left( H(0, 0)\lambda_0^{N-4} + \tau_1(t)\lambda_0^{N-6} \left( \lambda_1^{N-2} - \lambda_2^{N-2} \right) \right),
\]

\[
\frac{\partial^2 f_5(\lambda_0, \lambda_1, \lambda_2, t)}{\partial \lambda_1^2} = (N - 2)b_1 \left( 2(N - 3)\gamma_3(t)\lambda_1^{N-4} + (N - 4)\tau_1(t)\lambda_0^{N-6} \lambda_1^{N-2} - \lambda_2^{N-2} \right) + \frac{(N - 2)b_2}{\lambda_1^2},
\]

\[
= (N - 2)^2b_1 \left( 2\gamma_3(t)\lambda_1^{N-4} + \tau_1(t)\lambda_0^{N-6} \lambda_1^{N-2} - \lambda_2^{N-2} \right) + 2\gamma_4(t)\lambda_1^{N-6} \lambda_2^{N-2} + 2\gamma_4(t)\lambda_1^{N-6} \lambda_2^{N-2},
\]

\[
\frac{\partial^2 f_5(\lambda_0, \lambda_1, \lambda_2, t)}{\partial \lambda_2^2} = (N - 2)b_1 \left( 2(N - 3)\gamma_3(t)\lambda_2^{N-4} - (N - 4)\tau_1(t)\lambda_0^{N-6} \lambda_2^{N-2} - \lambda_1^{N-2} \right) + \frac{(N - 2)b_2}{\lambda_2^2},
\]

\[
= (N - 2)^2b_1 \left( 2\gamma_3(t)\lambda_2^{N-4} - \tau_1(t)\lambda_0^{N-6} \lambda_2^{N-2} - \lambda_1^{N-2} \right) + 2\gamma_4(t)\lambda_1^{N-6} \lambda_2^{N-2} + 2\gamma_4(t)\lambda_1^{N-6} \lambda_2^{N-2},
\]

\[
\frac{\partial^2 f_4(\lambda_0, \lambda_1, \lambda_2, t)}{\partial \lambda_0 \partial \lambda_1} = (N - 2)b_1 \tau_1(t)\lambda_0^{N-4} - \lambda_1^{N-4},
\]

\[
\frac{\partial^2 f_4(\lambda_1, \lambda_2, t)}{\partial \lambda_0 \partial \lambda_2} = -(N - 2)b_1 \tau_1(t)\lambda_0^{N-4} \lambda_2^{N-4},
\]

\[
\frac{\partial^2 f_4(\lambda_1, \lambda_2, t)}{\partial \lambda_1 \partial \lambda_2} = 2(N - 2)b_1 \gamma_4(t)\lambda_1^{N-4} - \lambda_2^{N-4}.
\]

For simplicity, we introduce the notation

\[
X := \lambda_0^{N-2}, \quad Y := \lambda_1^{N-2}, \quad Z := \lambda_2^{N-2}.
\]

In order to prove that \(D^2_{\lambda_0\lambda_1\lambda_2}f_5(\lambda_0(t), \lambda_1(t), \lambda_2(t), t)\) is nondegenerate for any \(t \in (0, t_1') \cup (t_2', 1)\), it suffices to show that the matrix

\[
\begin{pmatrix}
X + \tau_1(t)(Y - Z) & \tau_1(t)\frac{X^{N-4}}{Y^{N-4}}Y & \tau_1(t)\frac{X^{N-4}}{Z^{N-4}}Z & -\tau_1(t)\frac{X^{N-4}}{Z^{N-4}}Z \\
\tau_1(t)\frac{X^{N-4}}{Y^{N-4}}Y & 2\gamma_3(t)Y + \tau_1(t)X + 2\gamma_4(t)Z & 2\gamma_4(t)Y & 2\gamma_3(t)Z - \tau_1(t)X + 2\gamma_4(t)Y \\
-\tau_1(t)\frac{X^{N-4}}{Z^{N-4}}Z & 2\gamma_4(t)Y & 2\gamma_3(t)Z & 2\gamma_3(t)Z - \tau_1(t)X + 2\gamma_4(t)Y
\end{pmatrix}
\]

is nondegenerate. Using \(6.2\), \(6.3\) and \(6.4\) this is equivalent to showing that the matrix

\[
\begin{pmatrix}
\frac{X}{Y} + \frac{X}{Z} & \gamma_3(t)Y + \frac{\lambda_0^{N-4}}{Y} \cdot \frac{1}{Y} & \gamma_4(t)Y & \gamma_3(t)Z + \frac{\lambda_0^{N-4}}{Y} \cdot \frac{1}{Y}
\end{pmatrix}
\]

is nondegenerate. A direct computation, using \(6.7\), shows that the determinant of the above matrix has the same sign as \(\gamma_3(t)\), hence is nontrivial, proving Claim 2.

Theorem \ref{thm:main} now follows from
Claim 3: The function \( \nu_2(t) := f_5(\lambda_0(t), \lambda_1(t), \lambda_2(t), t) \) has a critical point \( t_1 \in (0, t^*_1) \).

Observe that, writing again \( \lambda_i \) instead of \( \lambda_i(t) \),

\[
\nu_2'(t) = \frac{\partial f_4(\lambda_0(t), \lambda_1(t), \lambda_2(t), t)}{\partial t} = 2b_1 \left( \gamma_3'(t) \left( \lambda_1^{N-2} + \lambda_2^{N-2} \right) + 2\tau_1'(t) \lambda_0^{N-1} \left( \lambda_1 \lambda_2 \lambda_3 - \lambda_2 \lambda_3 \lambda_3 \right) + \frac{4}{3} \gamma_4'(t) \lambda_1^{N-2} \lambda_2^{N-2} \right),
\]

where \( \lambda_0, \lambda_1, \lambda_2 \) satisfy (6.5), (6.6), (6.7) and (6.8). Therefore, \( \nu_2'(t) = 0 \) for \( t \in (0, t^*_1) \) is equivalent to

\[
i_3(t) := \gamma_3'(t) \left( 2\gamma_3(t)(\gamma_3(t) - 2\tau_1^2(t)) + \tau_1^2(t)(\gamma_3(t) + 2\gamma_4(t)) \right) - 2\tau_1'(t) \tau_1(t) \gamma_3(t)(\gamma_3(t) + 2\gamma_4(t))
+ 4\gamma_4'(t) \gamma_3(t)(\gamma_3(t) - 2\tau_1^2(t))
= 0.
\]

It is easy to check that \( i_3(t) \to -\infty \) as \( t \to 0^+ \) and \( i_3(t^*_1) > 0 \) because \( \gamma_3'(t^*_1) > 0 \), \( \gamma_4(t^*_1) > 0 \) and \( \gamma_3(t^*_1) = 0 \). Hence there exists \( t_1 \in (0, t^*_1) \) such that \( i_3(t_1) = 0 \). Claim 3 follows, finishing the proof of Theorem 1.6. \( \square \)

Remark 6.1. We conjecture that there should also exist \( t_2 \in (t^*_2, 1) \) such that \( i_3(t_2) = 0 \). This is not considered here because the computations get enormous.

Acknowledgements: The authors would like to thank Professor Daomin Cao for many helpful discussions during the preparation of this paper. This work was carried out while Qianqiao Guo was visiting Justus-Liebig-Universität Gießen, to which he would like to express his gratitude for their warm hospitality.

Funding: Qianqiao Guo was supported by the National Natural Science Foundation of China (Grant No. 11971385) and the Natural Science Basic Research Plan in Shaanxi Province of China (Grant No. 2019JM275).

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