1. Introduction

In this paper we construct the dotted cells and dotted lines in the next scheme:

[Diagram description]

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**QUATERNION LANDAU-GINSBURG MODELS AND NONCOMMUTATIVE FROBENIUS MANIFOLDS.**

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**Abstract.** We extend the classical topological Landau-Ginsburg model to a quaternion Landau-Ginsburg model, that satisfy the axioms of open-closed topological field theory. Later we prove, that a moduli space of quaternion Landau-Ginsburg models are non-commutative Frobenius manifold in means of [15].

**MSC:** 53C; 81T

**JGP SC:** Symplectic geometry; Quantum field theory; Strings and superstrings

**Keywords:** Landau-Ginsburg models; Frobenius manifolds
Explain this scheme in detail. The classical topological Landau-Ginsburg model was found by Vafa [17] (see also Dijkgraaf-Witten [5]). It generates an algebra over $\mathbb{C}$ on the tangent space to a polynomial $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ in the space $\text{Pol}(n)$ of all such polynomials. This algebra $A_p$ is associative, with the unity and a linear functional $l_p : A_p \to \mathbb{C}$, such that the bilinear form $(d_1, d_2) = l_p(d_1 d_2)$ is non degenerates. We call by Frobenius pairs all pairs $(A_p, l_p)$ with such algebraic properties. Algebra $A_p$ is commutative for classical topological Landau-Ginsburg model.

Commutative Frobenius pairs one-to-one correspond to topological field theories that appear from closed topological strings [3, 4, 7, 16]. These topological field theories naturally extend up to open-closed topological field theories, describing strings with boundary [10, 12], and even up to Kleinian topological field theories, describing strings with arbitrary world sheets [1]. In their turn, open-closed topological field theories one-to-one correspond to combinations from one commutative and one unrestricted Frobenius pairs, connected by Cardy condition [1]. We call such algebraic structure by Cardi-Frobenius algebra. Some classifications of Cardi-Frobenius algebras is contained in [1].

In the present paper we construct some extension of classical topological Landau-Ginsburg model to a Cardy-Frobenius algebra with a quaternion structure. Next we prove, that the set of such quaternion Landau-Ginsburg models over all polynomials $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ form a non-commutative Frobenius manifold in means of [15].

Let us explain this result in more detail. The moduli space of the classical topological Landau-Ginsburg models coincides with the space $\text{Pol}(n)$ of miniversale deformation for the singularity of type $A_n$ [2]. The metrics $(d_1, d_2) = l_p(d_1 d_2)$ on the algebras $A_p$ turn $\text{Pol}(n)$ into a Riemannian manifold with some additional properties [6, 7]. The differential-geometric structure, arising here, is an important example of Frobenius manifolds [7, 11]. The theory of Frobenius manifolds has a lot of applications in different areas of mathematics (integrable systems, singularity theory, topology of symplectic manifolds, geometry of moduli spaces of algebraic curves etc.).

The Dubrovin’s theory of Frobenius manifolds [7] is a theory of flat deformations of commutative Frobenius pairs. As we discussed, Frobenius pairs are extended up to Cardy-Frobenius algebras. This suggests on extension of Frobenius manifolds to Cardy-Frobenius manifolds. An approach to this problem, is contained in [15]. It is based on Kontsevich-Manin cohomological field theory [9].

In this paper we define Cardy-Frobenius bundles as spaces of some flat deformations of Cardy-Frobenius algebras and prove, that they are
Cardy-Frobenius (noncommutative) manifolds as regards to [15]. Moreover we prove that the family of all the quaternion Landau-Ginsburg models form a Cardy-Frobenius bundle with quaternion structure.

2. Cardy-Frobenius algebras.

Following [10,12] describe the algebraic structure (Cardy-Frobenius algebra), connected with open-closed topological field theory. It is follow from [1], that Cardy-Frobenius algebras one-to-one correspond to open-closed topological field theories.

2.1. Frobenius pairs.

Definition. By Frobenius pair over a field $\mathbb{K}$ we call a set $(D, l^D)$ where

1. $D$ is an associative algebra over $\mathbb{K}$ with the unity element $1^D$.
2. $l^D : D \rightarrow \mathbb{K}$ is a $\mathbb{K}$-linear functional such that the bilinear form $(d_1, d_2)^D = l^D(d_1d_2)$ is non-degenerated.

In this case $D$ is a Frobenius algebra [8].

Definition. By orthogonal sum of Frobenius pairs $(D_1, l^{D_1})$ and $(D_2, l^{D_2})$ we call the Frobenius pair $(D, l^D) = (D_1, l^{D_1}) \oplus (D_2, l^{D_2})$, where $D = D_1 \oplus D_2$ is the direct sum of algebras ($d_1d_2 = 0$ for $d_i \in D_i$) and $l^D = l^{D_1} \oplus l^{D_2}$.

Example 2.1. $\mathbb{K}$-numbers.

$\mathbb{K}(\lambda) = (\mathbb{K}, l_\lambda)$, where $\lambda \neq 0 \in \mathbb{K}$ and $l_\lambda(z) = \lambda z$ for $z \in \mathbb{K}$.

Example 2.2. Matrixes over $\mathbb{K}$.

$\mathbb{M}(n, \mathbb{K})/(\mu) = (\mathbb{M}(n, \mathbb{K}), l_\mu)$, where $\mathbb{M}(n, \mathbb{K})$ is the algebra of $n \times n$ $\mathbb{K}$-matrixes, $\mu \neq 0 \in \mathbb{K}$ and $l_\mu(z) = \mu \text{tr}(z)$ for $z \in \mathbb{M}(n, \mathbb{K})$.

Example 2.3. Quaternions over $\mathbb{K}$.

Let $\mathbb{R}_\mathbb{K} \in \mathbb{K}$ be a subfield, isomorphic to the field of real numbers. Let $\mathbb{H}_\mathbb{K}$ be the algebra of quaternions, that is the algebra over $\mathbb{R}$, generated by vectors $1^\mathbb{H}, I, J, K$, where $IJ = K, JK = I, KI = J$. Use the isomorphism $\mathbb{R}_\mathbb{K} \cong \mathbb{R}$ for define $\mathbb{H}_\mathbb{K} = \mathbb{H}_\mathbb{R} \otimes_{\mathbb{R}} \mathbb{K}$. We will consider $1^\mathbb{H}, I, J, K$ also as a basis of $\mathbb{H}_\mathbb{K}$ over $\mathbb{K}$. Put $\mathbb{H}_\mathbb{K}(\rho) = (\mathbb{H}_\mathbb{K}, l_\rho)$ where $\rho \neq 0 \in \mathbb{K}$ and $l_\rho : \mathbb{H}_\mathbb{K} \rightarrow \mathbb{K}$ be the $\mathbb{K}$-linear functional, defined by $l_\rho(1^\mathbb{H}) = 2\rho$, $l_\rho(I) = l^\mathbb{H}(J) = l^\mathbb{H}(K) = 0$.

The correspondance

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1^\mathbb{H}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \mapsto I, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto J, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto K$

defines an isomorphism between $\mathbb{M}(2, \mathbb{K})(\rho)$ and $\mathbb{H}_\mathbb{K}(\rho)$.

2.2. Cardy-Frobenius algebras.

Definition. By Cardy-Frobenius algebra over $\mathbb{K}$ we call a set $\{ (A, l^A), (B, l^B), \phi \}$, where

1. $(A, l^A)$ and $(B, l^B)$ are Frobenius pairs over $\mathbb{K}$ and $A$ is a commutative algebra;
2. \( \phi : A \to B \) is a homomorphism of algebras, such that \( \phi(A) \) belong to centre of \( B \);

3. For any \( x, y \in B \) is fulfilled \( (\phi^*(x), \phi^*(y))^A = tr(W_{x,y}) \), where \( \phi^* : B \to A \) is the linear operator, such that \( (a, \phi^*(b))^A = (\phi(a), b)^B \), for \( a \in A, b \in B \), and \( W_{x,y} : B \to B \) is the linear operator, defined by \( W_{x,y}(b) = xby \).

The condition of this type first appear in works of Cardy and usually has him name. Let us give a coordinate representation of Cardy condition, using bases \((\alpha_1, \ldots, \alpha_n) \subset A \) and \((\beta_1, \ldots, \beta_n) \subset B \). Let \( \{F^{\alpha_i, \alpha_j}\} \) and \( \{F^{\beta_i, \beta_j}\} \) be matrixes inverse to \( F_{\alpha_i, \alpha_j} = (\alpha_i, \alpha_j)^A \) and \( F_{\beta_i, \beta_j} = (\beta_i, \beta_j)^B \).

Then condition 3. is equivalent to

3'. For any \( x, y \in B \) is fulfilled \( F^{\alpha_i, \alpha_j} l^B(\phi(a_i) x) l^B(\phi(a_j) y) = F^{\beta_k, \beta_l} l^B(x_{\beta_k y, \beta_l}) \).

Proof of equivalent. The equality \( (\phi^*(x), \phi^*(y))^A = F^{\alpha_i, \alpha_j} l^B(\phi(a_i) x) l^B(\phi(a_j) y) \) is obviously. Consider a basis of \( B \) such that \( (\beta_k, \beta_l) = \delta_{kj} \). Then \( F^{\beta_k, \beta_l} l^B(x_{\beta_k y, \beta_l}) = \sum_{k=1}^n W_{x,y}(\beta_k, \beta_k)^B = tr(W_{x,y}) \).

**Example 2.4.** \( \mathbb{K} \) - numbers and matrixes.

\( \{\mathbb{K}(\mu^2), \mathbb{M}(\mu, \mathbb{K})(\mu), \phi_M\} \), where \( \phi_M : \mathbb{K} \to \mathbb{M}(\mu, \mathbb{K}) \) is the nature homomorphism of numbers to diagonal matrixes.

Check the axiom 3. Choice the elementary matrixes \( E_{ij} \) as a base of \( \mathbb{M}(\mu, \mathbb{K}) \). Left and right parts of axiom 3’ are equal to 0, if one of the matrixes \( x \) or \( y \) are not diagonal. Left and right parts of axiom 3’ are equal to 1 for \( x = E_{ii}, y = E_{jj} \).

**Definition.** By orthongonal sum of Cardy-Frobenius algebras \( \{(A_1, l^{A_1}), (B_1, l^{B_1}), \phi_1\} \) and \( \{(A_2, l^{A_2}), (B_2, l^{B_2}), \phi_2\} \) we call the Cardy-Frobenius algebra \( \{(A, l^A), (B, l^B), \phi\} = \{(A_1, l^{A_1}), (B_1, l^{B_1}), \phi_1\} \oplus \{(A_2, l^{A_2}), (B_2, l^{B_2}), \phi_2\} \), where \( (A, l^A) = (A_1, l^{A_1}) \oplus (A_2, l^{A_2}), (B, l^B) = (B_1, l^{B_1}) \oplus (B_2, l^{B_2}) \) and \( \phi = \phi_1 \oplus \phi_2 \).

A Cardy-Frobenius algebra is called semisimple, if \( A \) and \( B \) are semisimple algebras. It is follow from [1], that any semisimple Cardy-Frobenius algebra is isomorphic to orthogonal sum of some numbers of algebras of \( \{\mathbb{K}(\mu^2), \mathbb{M}(\mu, \mathbb{K})(\mu), \phi_M\} \) and some numbers of algebras \( \{\mathbb{K}(\lambda_i), 0 , 0\} \).

**Example 2.5.** \( \mathbb{K} \) - numbers and quaternions.

\( \{\mathbb{K}(\rho^2), \mathbb{H}(\rho), \phi_\mathbb{H}\} \), where homomorphism \( \phi_\mathbb{H} : \mathbb{K} \to \mathbb{H} \) is defined by \( \phi_\mathbb{H}(1) = 1^\mathbb{H} \).

The isomorphism between \( \mathbb{M}(2, \mathbb{K})(\rho) \) and \( \mathbb{H}(\rho) \) generate the isomorphism between \( \{\mathbb{K}(\rho^2), \mathbb{M}(2, \mathbb{K})(\rho), \phi_M\} \) and \( \{\mathbb{K}(\rho^2), \mathbb{H}(\rho), \phi_\mathbb{H}\} \).

3. QUATERNION LANDAU-GINSBURG MODELS.

The classical topological Landau-Ginsburg model [17] of degree \( n \) is generated by a complex polynomial \( p \) in the form \( p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n \), such that all roots \( \alpha_1, \ldots, \alpha_n \) of its derivative \( p'(z) \) are simple.
The set of all such polynomials form a complex manifold \( Pol(n) \) of complex dimension \( n \). Its tangent space \( A_p \) in a point \( p \) consists of all polynomials of degree \( n - 1 \). The Landau-Ginsburg model generates on \( A_p \) a structure of algebra, where the multiplication \( q = q_1 \ast_p q_2 \) for polynomials \( q_1 = q_1(z) \) and \( q_2 = q_2(z) \in A_p \) is defined by condition \( q(z) = q_1(z)q_2(z) \mod p'z) \).

Moreover, the Landau-Ginsburg model generates on \( A_p \) a non-degenerated bilinear form \( (q_1, q_2)_p = l_p(q_1q_2) = l_p(q_1 \ast_p q_2) \), where \( l_p(q) = \frac{1}{2\pi i} \oint \frac{a(z)dz}{p(z)} \). (Here and late the formula \( \frac{1}{2\pi i} \oint \) means ”minus residue in \( \infty \)). Thus \( A_p \) has a structure of Frobenius algebra.

The polynomials \( e_{p,\alpha}(z) = \prod_{j \neq i} \frac{z-\alpha_j}{\alpha_i-\alpha_j} \) form a basis of idempotents of \( A_p \), that is \( e_{p,\alpha}e_{p,\alpha} = \delta_{ij} e_{p,\alpha} \). Thus \( A_p \) is a semi-simple algebra. Put \( \mu_{p,\alpha} = l_p(e_{p,\alpha}) = \frac{1}{n+1} \prod_{j \neq i} \alpha_i-\alpha_j \). Let \( A_{p,\alpha} \) be the complex vector space generated by \( e_{p,\alpha} \) and \( t^A_{p,\alpha} = l_p|_{A_{p,\alpha}} \). Then \( (A_{p,\alpha}, t^A_{p,\alpha}) \) is a Frobenius pair, isomorphic to \( \mathbb{C}(\mu_{p,\alpha}) \).

Consider the Frobenius pair \( (B_{p,\alpha}, t^B_{p,\alpha}) \), where \( B_{p,\alpha} = A_{p,\alpha} \otimes \mathbb{C} \mathbb{H}_\mathbb{C} \cong \mathbb{H}_\mathbb{C} \) and \( t^B_{p,\alpha} \) is defined by \( t^B_{p,\alpha}(1) = 2\rho_{p,\alpha}, \quad t^B_{p,\alpha}(I) = t^B_{p,\alpha}(J) = 0 \). Let us define the homomorphism \( \phi_{p,\alpha} : A_{p,\alpha} \to B_{p,\alpha} \) by \( \phi_{p,\alpha}(e_{p,\alpha}) = e_{p,\alpha} \otimes \mathbb{C} \mathbb{H}_\mathbb{C} \). Then \( \{(A_{p,\alpha}, t^A_{p,\alpha}), (B_{p,\alpha}, t^B_{p,\alpha}), \phi_{p,\alpha}\} \) is a Cardy-Frobenius algebra, isomorphic to \( \{\mathbb{C}(\rho_{p,\alpha}), \mathbb{H}_\mathbb{C}, \phi_{\mathbb{H}_\mathbb{C}}\} \).

Put \( B_p = \bigoplus_{i=1}^n B_{p,\alpha} = A_p \otimes \mathbb{C} \mathbb{H}_\mathbb{C} \). Let \( \phi_p : A_p \to A_p \otimes \mathbb{C} \mathbb{H}_\mathbb{C} = B_p \) be the natural homomorphism. By quaternion Landau-Ginsburg model we call the Cardy-Frobenius algebra \( \{(A_p, t^A_p), (B_p, t^B_p), \phi_p\} = \bigoplus_{i=1}^n \{(A_{p,\alpha}, t^A_{p,\alpha}), (B_{p,\alpha}, t^B_{p,\alpha}), \phi_{p,\alpha}\} \).

4. Frobenius manifolds.

4.1. Frobenius manifolds and WDVV equations.

In middle of 90 years of the last century B.Dubrovin found and investigated a class of ”flat” deformation of commutative Frobenius algebras that appear in different domain of mathematics. He calls this structure Frobenius manifolds \([7]\). *Frobenius manifold* is a manifold with a Dubrovin connection. Present some equivalent definitions of the Dubrovin connection.

**Definition**. ([7]) Let \( M \) be a smooth (real or complex) manifold. By *Dubrovin connection* on \( M \) is called is a family of commutative Frobenius pairs \( (M_p, \theta_p) \) on the tangent spaces \( M_p = T_p M \) for any point \( p \in M \) such that

1. Tensors \( \theta = \{\theta_p|p \in M\} \), \( g(a, b) = \theta(ab) \), \( c(a, b, d) = \theta(abd) \) are smooth and \( d\theta = 0 \).
2. The Levi-Civita connection \( \nabla \) of the matric \( g \) is flat and such that \( \nabla e = 0 \), where \( e \) is the field of unity elements of the algebras \( M_p \).
3. Tensors $c(a, b, d)$ and $\nabla_f c(a, b, d)$ are symmetrical by variables $a, b, d, f$.

4. There exists a vector field $E$ (it is called *Euler field*), such that $\nabla(\nabla E) = 0$.

A Frobenius manifold $M$ is called semi-simple, if $M_p$ is a semi-simple algebra for any point $p \in M$.

If $\theta(e) = 0$, then Frobenius manifold has a special flat quasi-homogeneous coordinate system $t = (t^1, ..., t^n)$, such that $g = \sum_{ij} \delta_{i+j, n} dt^i \otimes dt^j$, $e = \partial/\partial t^1$; $d_n = 1$, $d_i r_i = 0$, $d_i + d_{n+1-i} = v + 2$ for all $i$; $E = \sum_{i=1}^n (d_i t^i + r_i)(\partial/\partial t^i)$ for some constants $d^i, r_i$. We call such Dubrovin connection *anti-diagonale*.

Recall, that any semi-simple commutative Frobenius algebra is a direct sum of one-dimensional one [8]. Thus it has a canonical basis $e_1, ..., e_n$. This is a basis with the properties $e_i e_j = \delta_{ij} e_i$, $l(e_i) \neq 0$. The canonical basis is defined uniquely up to enumeration of its elements.

**Definition.** ([13]) A semi-simple anti-diagonale Dubrovin connection on a smooth (real or complex) manifold $M$ is a family of commutative Frobenius pairs $(M_p, \theta_p)$ on the tangent spaces $M_p = T_p M$ for any point $p \in M$ such that

1. Tensors $\theta = \{\theta_p|p \in M\}$, $g(a, b) = \theta(ab)$, $c(a, b, d) = \theta(abd)$ are smooth and $d\theta = 0$.

2. There exists a covering $M = \bigcup \alpha U_\alpha$ by coordinate maps $(x^1_\alpha, ..., x^n_\alpha) : U_\alpha \rightarrow \mathbb{K}^n$, such that: a) the vectors $(\partial/\partial x^1_\alpha, ..., \partial/\partial x^n_\alpha)$ form a canonical basis in $M_p$ for any $p \in U_\alpha$; b) the field $E = \sum_{i=1}^n x^i_\alpha (\partial/\partial x^i_\alpha)$ don’t depend from $\alpha$; c) $L_E \theta = (v + 1)\theta$, where $L_E$ is the Li derivative by $E$. Such coordinate system is call canonical.

3. The Levi-Civita connection $\nabla$ of the matric $g$ is flat. It exists a coordinate system $t = (t^1, ..., t^n)$ on $M$, such that $g = \sum_{ij} \delta_{ij} dt^i \otimes dt^j$, $e = \partial/\partial t^1$ and $E = \sum_{i=1}^n (d_i t^i + r_i)(\partial/\partial t^i)$, where $d_i, r_i$ are constants.

By potential of Dubrovin connection $(M_p, \theta_p)$ on flat quasi-homogeneous coordinates $t = (t^1, ..., t^n)$, is called a function $F(t^1, ..., t^n)$ such that $\theta_p(\partial/\partial t^1, \partial/\partial t^2, \partial/\partial t^n) = \partial F/\partial t^1, \partial F/\partial t^2, \partial F/\partial t^n$.

**Definition.** Let $F(t^1, ..., t^n)$ be a function on a set $U \subset \mathbb{C}^n = (t^1, ..., t^n)$. Let $E = \sum_{i=1}^n (d_i t^i + r_i)(\partial/\partial t^i)$ be a vector field ,such that $d_n = 1$, $d_i r_i = 0$, $d_i + d_{n+1-i} = v + 2$ for all $i$. Then the pair $(F, E)$ is a solution of WDVV equations if:

1. $\sum_{q=1}^n \partial^3 F/\partial t^q \partial t^{q+1} \partial t^{q+2} = \sum_{q=1}^n \partial^3 F/\partial t^q \partial t^{q+1} \partial t^{q+2}$ (associativity equations);

2. $\partial^3 F/\partial t^i \partial t^j \partial t^k = \delta_{i+j, n+1}$ (normalization condition);

3. $L_E F = (v + 3)F + \sum_{ij} A_{ij} t^i t^j + \sum_i B_i t^i + C$, where $L_E$ is the Li derivative and $A_{ij}, B_i, C$ are constants.

(quasi-homogeneous conditions).
This equation was found in works of Witten [18] and Dijkgraaf, E. Verlinde, H. Verlinde [6] for a description of spaces of deformations of topological fields theories.

According to [7], any solution of WDVV equations is a potential of some anti-diagonale Dubrovin connection and moreover the correspondence (potential) \( \rightarrow \) (anti-diagonale Dubrovin connection) form one-to-one correspondence between solutions of WDVV equations and anti-diagonale Dubrovin connections.

If a solution \( F \) of WDVV equations has a representation in the form of Tailor series \( F(t) = \sum c(i_1, i_2, ..., i_k) t^{i_1} t^{i_2} ... t^{i_k} \), then associativity equations are equivalent to some relations between the coefficients \( c(i_1, i_2, ..., i_k) \). M.Kontsevich, Yu.Manin [9, 11] presented a family of coefficients with these relations by a special system of homomorphisms \( \mathbb{C}^\otimes l \rightarrow H^*(\mathcal{M}_{0,l}) \), where \( \mathcal{M}_{0,l} \) is the moduli space of spheres with \( l \) pictures. This gives some other method of description for Frobenius manifolds. It is called Cohomological Field Theory.

**4.2. Moduli spaces of classical topological Landau-Ginsburg models.**

The first and very important example of a complex Frobenius manifold is the moduli space \( Pol(n) \) of classical topological Landau-Ginsburg models. This space appear also in the theory of singularity, the theory of Coxeter groups, the theory of moduli spaces of Riemann surfaces, in matrix models of mathematical physics and in integrable systems. Let us describe this example more detailed, following on the whole [7].

**Theorem 4.1.** [6,7] The structure of Frobenius pairs \((\alpha_p, \lambda_p)\) for \( p \in Pol(n) \) generates a complex Dubrovin connection on the space \( Pol(n) \) of polynomials \( p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{-2} + ... + a_n \).

Proof. Axiom 1 directly follow from the definitions. Let \((\alpha_1, ..., \alpha_n)\) be the set of roots of \( p' \). Consider the functions \( x^i(p) = p(\alpha_i) \) as coordinates on \( Pol(n) \).

**Lemma 4.1.** The coordinates \((x^1, ..., x^n)\) are canonical, \( v = \frac{2}{n+1} - 1 \) and \( E = \sum_{i=k}^{n} \frac{k+1}{n+1} a_k \partial / \partial a_k \).

Proof. Prove, that \( \frac{\partial}{\partial x^j} = e_{p, \alpha_j} \). Really, \( \delta_{ij} = \frac{\partial x^i}{\partial x^j} = \frac{\partial (\alpha_{p})^{n+1} + a_1 (\alpha_{p})^{n-2} + ... + a_{n-1}}{\partial x^j} + \frac{\partial (\alpha_{p})^{n+1} + a_1 (\alpha_{p})^{n-2} + ... + a_{n-1}}{\partial \alpha_j} = p'(\alpha) \frac{\partial a_j}{\partial \alpha_j} + \frac{\partial p}{\partial \alpha_j} (\alpha) = \frac{\partial p}{\partial \alpha_j} (\alpha) \). Thus \( \frac{\partial p}{\partial \alpha_j} (\alpha) = \delta_{ij} = e_{p, \alpha_j} \). By definition, this means that the polynomial \( e_{p, \alpha_j} \) correspond to the tangent vector \( \frac{\partial}{\partial x^j} \). Thus \( (\partial / \partial x^1, ..., \partial / \partial x^n) \) form a canonical basis.

Prove now, that \( \lambda_p = \frac{a_0}{n+1} \). Really, considering the coefficients for \( z^{n-1} \) in \( \frac{\partial a_0}{\partial x^1} z^{n-1} + ... + \frac{\partial a_0}{\partial x^n} z^{n-1} = \frac{\partial p}{\partial x^i} (z) = e_{p, \alpha_j} (z) \) we find that \( \lambda_p (\partial / \partial x^i) = l_p (e_{p, \alpha_j}) = \frac{1}{n+1} \frac{\partial a_0}{\partial x^i} \). Thus \( \lambda_p = \sum_{i=1}^{n} \frac{1}{n+1} \frac{\partial a_0}{\partial x^i} = \frac{a_0}{n+1} \).
Thus, $E = \sum_{i=1}^{n} x_{\alpha}^{i} (\partial / \partial x_{\alpha}^{i})$ don't depend from $\alpha$ and $L_{E} l_{\rho} = (v + 1) p_{\rho}$. Prove at first, that $L_{E} p = p - \frac{1}{n+1} p'$. Really, the polynomials $L_{E} p$ and $p - \frac{1}{n+1} p'$ have the same degree $n - 1$ and they are the same values in the points $\alpha_{1}, ..., \alpha_{n}$, because $L_{E}(p)(\alpha_{k}) = (\sum_{i=1}^{n} x_{\alpha}^{i} (\partial / \partial x_{\alpha}^{i})(p))(\alpha_{k}) = x^{k} = p(\alpha_{k}) = p(\alpha_{k}) - \frac{1}{n+1} p'$. Consider now the vector field $F = \sum_{k=1}^{n} \frac{k+1}{n+1} a_{k} \partial / \partial a_{k}$. Then $L_{F}(p)(z) = \sum_{k=1}^{n} \frac{k+1}{n+1} a_{k} z^{n-k} = \sum_{k=1}^{n} (1 - \frac{n-k}{n+1}) a_{k} z^{n-k} = p(z) - \frac{1}{n+1} p'(z) = L_{E}(p)(z)$.

Thus $L_{E}(l_{\rho}) = L_{E}(\frac{d_{a_{1}}}{n+1}) = \frac{1}{n+1} \frac{d_{a_{1}}}{n+1} = \frac{1}{n+1} l_{\rho}$. □

Construct now a flat coordinate systems on $Pol(n)$. Consider a function $\omega = \omega(p, z)$ on $Pol(n) \times \mathbb{C}$ such that $\omega^{n+1} = p(z)$. Put $z = \omega + \frac{\omega_{1}}{\omega} + \frac{\omega_{2}}{\omega^{2}} + ...$. The equality $\omega^{n+1} = p(\omega + \frac{\omega_{1}}{\omega} + \frac{\omega_{2}}{\omega^{2}} + ...)$ gives a possibility to find $\tilde{\omega}$ as a quasihomogenous polynomial of $\{a_{j}\}$. In particularity $\tilde{t}^{i} = -\frac{\alpha_{i}}{n+1} \tilde{t}^{2} = -\frac{\alpha_{i}}{n+1}$. Put $t^{1} = -(n+1) \tilde{t}^{n}$, $t^{n} = -\tilde{t}^{1}$ and $t^{i} = -\sqrt{n+1} \bar{t}^{i+1-n}$ for $i = 2, ..., n-1$.

**Lemma 4.2.** The coordinates $(t^{1}, ..., t^{n})$ are flat quasi-homogeneous and $d_{i} = n+2-i$.

Proof. Consider the set of polynomials $p(z, \tilde{t}) = z^{n+1} + a_{1}(\tilde{t}) z^{n-1} + a_{2}(\tilde{t}) z^{n-2} + ... + a_{n}(\tilde{t})$, where $\tilde{t} = (\tilde{t}^{1}, ..., \tilde{t}^{n})$. Consider the function $z(\omega, \tilde{t}) = \omega + \frac{\omega_{1}}{\omega} + \frac{\omega_{2}}{\omega^{2}} + ...$, such that $\omega^{n+1} = p(z(\omega, \tilde{t}))$. We will consider $\omega$ and $\{\tilde{t}^{1}, ..., \tilde{t}^{n}\}$ as independent variables. Then $0 = d_{\omega} n+1 = \frac{\partial p}{\partial \omega} + \frac{\partial p}{\partial \omega_{1}} = \frac{\partial p}{\partial \omega} + \frac{1}{\omega} \frac{\partial p}{\partial \omega^{2}}$. Thus $\frac{\partial p}{\partial \omega} = -\frac{1}{\omega^{2}}$.

Therefore $g(\partial / \partial \tilde{t}^{1}, \partial / \partial \tilde{t}^{n}) = \frac{p'(\partial / \partial \tilde{t}^{1}, \partial / \partial \tilde{t}^{i})}{\omega^{n}} = -\text{Res}_{\omega = \infty} \frac{\partial p}{\partial \omega} \frac{\partial p}{\partial \omega_{i+1}} dz = -\text{Res}_{\omega = \infty} \frac{p'(d_{a_{1}})}{\omega^{n+1}} = -\text{Res}_{\omega = \infty} \frac{d_{a_{1}}}{\omega^{n+1}} = (n+1) \delta_{i+1,n+1}$.

Thus, $g(\partial / \partial \tilde{t}^{i}, \partial / \partial \tilde{t}^{j}) = \delta_{i+1,n+1}$.

Let $e = \sum_{i=1}^{n} \rho_{\omega} \frac{\partial}{\partial \omega}$. Using this fact and lemma 4.1, we find that $\delta_{i+1,n+1} = d_{\tilde{t}^{1}} \theta(\partial / \partial \tilde{t}^{i}) = -\frac{n+1}{n+1} \rho_{\omega} \frac{\partial a_{1}}{\partial \omega} = -\rho_{\omega} \frac{\partial a_{1}}{\partial \omega} = \rho_{\omega} \frac{\partial a_{1}}{\partial \omega} = -\sum_{i=1}^{n} \rho_{i} \frac{\partial g}{\partial \omega} = -(n+1) \sum_{i=1}^{n} \rho_{i} \delta_{i+1,n+1} = -(n+1) \rho_{\omega} \delta_{i+1,n+1}$. Therefore, $e = -\frac{1}{n+1} \frac{\partial}{\partial \omega} = \frac{\partial}{\partial \omega}$.

It is follow from lemma 4.1. that $L_{E} a_{i} = \frac{i+1}{n+1} a_{i}$. Thus, by the definition of $\tilde{t}$, we find that $L_{E} \tilde{t}^{i} = \frac{n+2-i}{n+1} \tilde{t}^{i}$. Therefore $E(t^{i}) = \frac{n+2-i}{n+1} t^{i}$, and $E = \sum_{i=1}^{n} \frac{n+2-i}{n+1} t^{i}(\partial / \partial t^{i})$.

□

**Example 4.1.** Find the numbers $\mu_{p,a_{i}}$ as some functions of flat quasi-homogeneous coordinates for $n = 2$. If $p(z) = z^{3} + a_{1} z + a_{2}$, then $t^{2} = -\tilde{t}^{1} = \frac{a_{2}}{3}$. Moreover $p'(z) = 3 z^{2} + a_{1}$ and $\alpha_{i} = \pm \sqrt{-\frac{n-2}{3}} = \pm \sqrt{-t^{2}}$. Thus, $\mu_{p,a_{i}} = \pm \frac{1}{6+\sqrt{-t^{2}}}$.  

**Example 4.2.** The potential of the Frobenius manifold $Pol(n)$ is a polynomial $F_{n}$ [7]. Its coefficients was found in [14]. In particulary $F_{2} = \frac{1}{2} (t^{i})^{2} t^{2} + \frac{1}{24} (t^{2})^{4}$.
5. Non-commutative Frobenius manifolds.

5.1. Extended WDVV equations.

A theory of deformations of closed strings is one of the sources for the theory of Frobenius manifolds. Its mathematical equivalent is flat deformations of commutative Frobenius pairs. But the theory of closed strings is only part of more general open-closed topological field theory. It is follow from [1,10,12], that a mathematica equivalent of an open-closed topological field theory is a Cardy-Frobenius algebra. A theory of flat deformation for Cardy-Frobenius algebras was suggested in [15]. It continue the Kontsevich and Manin approach [9,11] and it gives some extension of WDVV equations to differential equations on series of non-commutative variables.

Describe more detail these equations. Let \( t = (t^1, ..., t^n) \) (respectively \( s = (s^1, ..., s^m) \) be the standard coordinates on \( A \cong \mathbb{C}^n \) (respectively on \( B \cong \mathbb{C}^m \)). Consider the algebras of formal tensor series \( F = \sum c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) t^{i_1} \otimes t^{i_2} \otimes ... \otimes t^{i_k} \otimes s^{j_1} \otimes s^{j_2} ... s^{j_l} \), \( c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) \in \mathbb{C} \). Let \( F_A \) be the part of the series \( F \) that consists of all monomial without \( s^i \).

Partial derivatives of \( F \) are defined by partial derivatives of monomials. We consider that \( \partial(t^1 \otimes t^2 \otimes ... \otimes t^k \otimes s^1 \otimes s^2 \otimes ... \otimes s^k) \) is the sum of monomials \( t_i^1 \otimes t_i^2 \otimes ... \otimes t_i^{p-1} \otimes t_{i+1}^p \otimes ... \otimes t_i^k \otimes s^1 \otimes s^2 \otimes ... \otimes s^k \), such that \( i_p = i \).

Reciprocally \( \frac{\partial}{\partial s^j} \partial(t^1 \otimes t^2 \otimes ... \otimes t^k \otimes s^j \otimes s^{j_2} \otimes ... \otimes s^k) \) is the sum of monomials \( t_i^1 \otimes t_i^2 \otimes ... \otimes t_i^k \otimes s^{j_1} \otimes s^{j_2} \otimes ... \otimes s^{j_p-1} \otimes s^{j_p+1} \otimes ... \otimes s^k \), such that \( j_p = j \).

Put \( \frac{\partial^2}{\partial t^i \partial s^j} = \frac{\partial}{\partial t^i} \frac{\partial}{\partial s^j} \), \( \frac{\partial^2}{\partial t^i \partial s^j} = \frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j} \), \( \frac{\partial^2}{\partial s^i \partial s^j} = \frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j} \), \( \frac{\partial^2}{\partial s^i \partial s^j} = \frac{\partial}{\partial s^j} \frac{\partial}{\partial s^i} \).

The definition of the partial derivatives \( \frac{\partial^2}{\partial s^i \partial s^j} \) is more complicated. These partial derivatives are the sum of monomials \( t_i^1 \otimes t_i^2 \otimes ... \otimes t_i^k \otimes s^{k_2} \otimes ... \otimes s^{k_p-1} \otimes s^{k_p+1} \otimes ... \otimes s^{k_q-1} \otimes s^{k_q+1} \otimes ... \otimes s^{k_t} \) such that the sequences \( s_i^{k_2}, s_i^{k_2}, ..., s_i^{k_p-1}, s_i^{k_p+1}, ..., s_i^{k_q-1}, s_i^{k_q+1}, ..., s_i^{k_t} \) and \( (s_j^{k_2}, ..., s_j^{k_t}) \) are the same after an cyclic transposition.

We consider that a monomials \( t_i^1 \otimes ... \otimes t_i^k \otimes s_j^1 \otimes ... \otimes s_j^t \) and \( t_{i+1}^1 \otimes ... \otimes t_{i+1}^k \otimes s_{j+1}^1 \otimes ... \otimes s_{j+1}^t \) are equivalent, if \( \cup_{r=1}^{i_r} = \cup_{r=1}^{j_r} \) and \( \cup_{r=1}^{i_r} = \cup_{r=1}^{j_r} \).

Let \( \cup_{i \in \mathbb{Z}} t_i^1 \otimes ... \otimes s_i^j \) be the equivalent class of \( t_i^1 \otimes ... \otimes s_i^j \). The tensor series \( F = \sum c[i_1, ..., i_k|j_1, ..., j_l] a_{i_1} \otimes ... \otimes b_{j_l} \) generate the tensor series \( [F] = \sum c[i_1, ..., i_k|j_1, ..., j_l] [a_{i_1} \otimes ... \otimes b_{j_l}] \), where the sum is the sum by all equivalent classes of monomials and \( c[i_1, ..., i_k|j_1, ..., j_l] \) is the sum of all coefficients \( c[i_1, ..., i_k|j_1, ..., j_l] \), corresponding monomials from equivalent class \( [a_{i_1} \otimes ... \otimes b_{j_l}] \).

We say that a tensor series \( F = \sum c[i_1 \cdots i_k|j_1 \cdots j_l] t_i^1 \otimes ... \otimes t^k \otimes s_j^1 \otimes ... \otimes s_j^t \) satisfy extended WDVV equations on a space \( H = A \otimes B \), if the following conditions hold.
1. The coefficients \(c(i_1, \cdots, i_k | j_1, \cdots, j_\ell)\) are invariant under all permutation of \(\{i\}\).

2. The coefficients \(c(i, j)|\ c(|i, j)\) generate nondegenerate matrices. By \(F_{a}^{t_{ij}} \ F_{b}^{s_{s}j}\) denote the inverse matrices of \(c(i, j)|\ c(|i, j)\) respectively.

3. 
   \[
   \left[ \sum_{p,q=1}^{n} \frac{\partial^3 F_{A}}{\partial t^p \partial t^q \partial t^r} \right] \otimes F_{a}^{t_{ij}} = \left[ \sum_{p,q=1}^{n} \frac{\partial^3 F_{A}}{\partial t^p \partial t^q \partial t^k \partial t^l} \right] \otimes F_{a}^{t_{ij}} \ .
   \]

4. 
   \[
   \left[ \sum_{p,q=1}^{m} \frac{\partial^3 F}{\partial s^p \partial s^q \partial s^r} \right] \otimes F_{b}^{s_{s}j} = \left[ \sum_{p,q=1}^{m} \frac{\partial^3 F}{\partial s^p \partial s^q \partial s^k} \right] \otimes F_{b}^{s_{s}j} \ .
   \]

5. 
   \[
   \left[ \sum_{p,q=1}^{n} \frac{\partial^2 F}{\partial t^p \partial s^q} \right] \otimes F_{a}^{t_{ij}} = \left[ \sum_{p,q=1}^{n} \frac{\partial^2 F}{\partial s^p \partial s^q} \right] \otimes F_{b}^{s_{s}j} \ .
   \]

6. 
   \[
   \left[ \sum_{p,q=1}^{m} \frac{\partial^2 F}{\partial s^p \partial t^q} \right] \otimes F_{b}^{s_{s}j} = \left[ \sum_{p,q=1}^{m} \frac{\partial F}{\partial s^p} \right] \otimes F_{b}^{s_{s}j} \ .
   \]

7. 
   \[
   \left[ \sum_{p,q=1}^{m} \frac{\partial^2 F}{\partial s^p \partial t^q} \right] \otimes F_{a}^{t_{ij}} = \left[ \sum_{p,q=1}^{m} \frac{\partial F}{\partial s^p} \right] \otimes F_{b}^{s_{s}j} \ .
   \]

In [15] are demonstrated that solutions of extended WDVV equations one-to-one correspond to potentials of some extension of Cohomological Field Theory and moreover they describe some class of deformations of Cardy-Frobenius algebras. Thus it is natural to consider solutions of extended WDVV equations as (non commutative) extension of Frobenius manifolds, that we call **Cardy-Frobenius manifolds**. Later we prove that quaternion Landau-Ginsburg models generate a Cardy-Frobenius manifold.

5.2. Cardy-Frobenius bundles.

**Definition.** Let \(M\) be a , were \(\mathbb{K}\) is the real or the complex field. By **Cardy-Frobenius bundle** on smooth (real or complex) manifold \(M\) we call a pair of bundles \(\varphi_A : A \rightarrow M\) and \(\varphi_B : B \rightarrow M\) with a flat connection \(\nabla_B\), and a set of Cardy-Frobenius algebras \(\{(A_p, l^A_p), (B_p, l^B_p), \phi_p\}\), where \(A_p = \varphi_A^{-1}(p), B_p = \varphi_B^{-1}(p)\), such that:

1. The algebras \(\{(A_p, l^A_p)\}\) form a Dubrovin connection.

2. The connection \(\nabla_B\) conserve the family of bilinear forms \(\{(b_1, b_2)p = l^B_p(b_1b_2)p | p \in M\}\).

Call by a flat system of coordinates on \(B\) a family of linear coordinates systems \(s = \{s_p = (s^1_p, ..., s^n_p) | p \in M\}\) on bands \(B_p\), that is invariant by \(\nabla_B\). It generates a basis \((\frac{\partial}{\partial s^1_p}, ..., \frac{\partial}{\partial s^n_p})\) on any vector space
B_p. Axiom 2. say that values \( \left( \frac{\partial}{\partial x_p}, \frac{\partial}{\partial y_p} \right) \) are constants on \( M \).

3. Let \( s = (s^1, ..., s^n) \) be a flat system of coordinates on \( B \). Then the tensor fields \( c^p(B) = l_p(B) \frac{\partial}{\partial x_p} \frac{\partial}{\partial y_p} \), are smooth as functions on \( M \). We call \( B \)-structures tensors.

4. The natural map \( \phi = \{ \cup \phi_p | p \in M \} : A \rightarrow B \) is smooth. It define smooth transition tensors field \( c^{AB}(a, b) = l_p^B(\phi(a)b) \).

**Theorem 5.1.** Let \( M \) be a semi-simple Frobenius manifold with a Dubrovin connection \( \{(A_p, l_p^A)| p \in M \} \). Then there exist Cardy-Frobenius bundles \( \{(A_p, l_p^A), (B_p, l_p^B), \phi \} \).

Proof. Let \( (x^1_\alpha, ..., x^n_\alpha) \) be canonical coordinates on \( M \). Put \( \lambda_{p,i} = l_p^A(\partial/\partial x^i_\alpha(p)) \). Let \( \mu_{p,i} \) be a smooth function on \( M \), such that \( \mu^2_{p,i} = \lambda_{p,i} \).

Consider the family of Frobenius pair \( (B_p, l_p^B) = \mathcal{M}(m, \mathbb{C})(\mu_{p,i}) \) from example 2.2. Put \( (B_p, l_p^B) = \bigoplus_i (B_{p,i}, l_{p,i}^B) \) and \( B = \bigoplus_{p \in M} B_p \).

Describe a connection \( \nabla_B \). The standard basis \( \{ E^{kr} \} \) of \( \mathcal{M}(m, \mathbb{C}) \) generate the basis the \( \{ E_{p,i}^{kr} \} \) of \( B_{p,i} \). We will be consider, that the connection \( \nabla_B \) generate the transfer \( E_{p,i}^{kr} \) to \( l_{p,i}^AE_{q,i}^{kr} \) for \( q \) from a neighborhood of \( p \).

Define a structure of smooth manifold on \( B \) considering that the projection \( \varphi_B(B_p) = p \) is smooth. Define the homomorphism \( \phi_p : A_p \rightarrow B_p \), considering that \( \phi_p(\partial/\partial x^i_\alpha) \) is the unit element of the algebra \( B_{p,i} \). According to example 2.4. the structure, that we constructed, is Cardy-Frobenius bundles.

\[ \square \]

**Definition.** Let \( \varphi_A : A \rightarrow M \), \( \varphi_B : B \rightarrow M \), \( \nabla_B \), \( \{(A_p, l_p^A), (B_p, l_p^B), \phi_p | p \in M \} \) be a Cardy-Frobenius bundle on \( M \). Let \( t = (t^1, ..., t^n) \) be a system of flat quasi-homogenies coordinates of the Dubrovin connection \( (A_p, l_p^A) \) and let \( s = (s^1, ..., s^n) \) be a system of flat coordinates on \( B \). By potential of this Cardy-Frobenius bundle is called the formal tensor series \( F(t|s) = \sum c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) t_1 \otimes t_2 \otimes ... \otimes t_k \otimes s_1 \otimes s_2 \otimes ... \otimes s_l \), where \( c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) \in \mathbb{C} \), such that

1. The matrixes \( c(i, j) = l_p^A(\frac{\partial}{\partial t^i \partial s^j}) \quad c(i, j) = l_p^B(\frac{\partial}{\partial x^i_\alpha \partial x^j_\alpha}) \) are non-degenerated. Let \( F^a(t|s) \) and \( F^b_{s^a|s^j} \) be the matrixes inverted to \( c(i, j) \) and \( c(i, j) \) respectively.

2. If \( F_A \) is the part of \( F \) that don’t depend from \( s \), that it pass to the potential of Dubrovin connection \( (M_p, l_p^A) \) after changing the tensor multiplication to the ordinary multiplication.

3. The formal tensor series \( \frac{\partial^2 F}{\partial s^a \partial s^a} \) are not depend from \( s \) and, after changing the tensor multiplication to the ordinary multiplication, coincide with the \( B \)-structure tensors of the bundle.

4. The formal tensor series \( \frac{\partial^2 F}{\partial \delta t^i \partial \delta t^j} \) are not depend from \( s \) and, after changing the tensor multiplication to the ordinary multiplication, coincide with the transition function of the bundle.

**Theorem 5.2.** Let a Dubrovin connection \( (A_p, l_p^A) \) on \( M \) has a potential in form of Taylor series. Then any Cardy-Frobenius bundle
\( \varphi_A : A \to M, \varphi_B : B \to M, \nabla_B, \{(A_p, l^A_p), (B_p, l^B_p), \phi_p | p \in M\} \) also has a potential.

Proof. Let \( t = (t^1, ..., t^n) \) be a flat quasi-homogeneous coordinates of the Dubrovin connection \( \{(A_p, l^A_p)|p \in M\} \). Let \( s = (s^1, ..., s^m) \) be a flat coordinates system on \( B \). Consider the potential of the Dubrovin connection \( \{(A_p, l^A_p)|p \in M\} \). Changing the ordinary multiplication to the tensor multiplication we obtain a tensor series \( F_A \). Let \( F^i_A \) be the tensor series, such that \( \frac{\partial F_A^i}{\partial t^i} = F_A \). Then the tensor series \( F = F_A + \frac{1}{2} \sum l^A_i \left( \frac{\partial}{\partial t^j} \cdot \frac{\partial}{\partial s^l} \right) t^j \otimes t^l + \sum l^B_i \left( \frac{\partial}{\partial s^j} \cdot \frac{\partial}{\partial s^l} \right) s^j \otimes s^l + \frac{1}{3} \sum l^B_i \left( \frac{\partial}{\partial s^j} \cdot \frac{\partial}{\partial s^k} \right) s^j \otimes s^k \) is the potential of the Cardy-Frobenius bundle.

\[ \square \]

**Theorem 5.3.** The potential of any Cardy-Frobenius bundle satisfy the extended WDVV equations.

Proof. All relation follow from properties of Cardy-Frobenius algebras \( \{(A_p, l^A_p), (B_p, l^B_p), \phi_p \} \). In particulary relation 1 follow from the commutativity of \( A \). Relation 2 follow from the non-degeneracy the bilinear forms on \( A \) and \( B \). Relations 3 and 4 follow from the associativity of algebras \( A \) and \( B \). Relation 5. is true because \( \phi_p(A_p) \) belong to center of algebra \( B_p \). Relation 6 is true because the map \( \phi_p \) is a homomorphism. Relation 7 follow from Cardy axiom.

\[ \square \]

### 5.3. Moduli space of quaternion Landau-Ginsburg models.

Let us use the construction from theorem 5.1. for quaternion Landau-Ginsburg models. Then we have

**Corollary 5.1.** Family of quaternion Landau-Ginsburg models \( \{(A_p, l^A_p), (B_p, l^B_p), \phi_p | p \in \text{Pol}(n)\} \) form a Cardy-Frobenius bundles over \( \text{Pol}(n) \). The space \( B \) has a natural quaternion structure that is invariant by the connection \( \nabla_B \).

A flat system of coordinates \( t = (t^1, ..., t^n) \) for \( A \) is , described in section 4. A basis on \( B_p \) is \( \{l^\mathbb{H} e_{p,\alpha_i}, I e_{p,\alpha_i}, J e_{p,\alpha_i}, K e_{p,\alpha_i} | i = 1, ..., n\} \) from section 3. In a neighborhood of \( p \) it generate a flat coordinate system \( s_{q,i} = (s_{q,i}^i, s_{q,i}^l, s_{q,i}^j, s_{q,i}^K) = \frac{p_{\rho,\alpha}}{\rho_{\rho,\alpha}} (l^\mathbb{H} e_{p,\alpha_i}, I e_{p,\alpha_i}, J e_{p,\alpha_i}), \) where \( p_{\rho,\alpha_i} = \mu_{p,\alpha_i} \).

**Example 5.1.** For any \( V, W \in \{l^\mathbb{H}, I, J, K\} \), \( \frac{\partial}{\partial s_{q,i}^h} \frac{\partial}{\partial s_{q,i}^j} = \delta_{h,j} \delta_{i,j} \)

Thus we can to find the bilinear form \( \left( \frac{\partial}{\partial s_{q,i}^h}, \frac{\partial}{\partial s_{q,i}^j}\right)^B = l^B \left( \frac{\partial}{\partial s_{q,i}^h}, \frac{\partial}{\partial s_{q,i}^j}\right) \) and the structure tensor \( c^B \left( \frac{\partial}{\partial s_{q,i}^h}, \frac{\partial}{\partial s_{q,i}^j}, \frac{\partial}{\partial s_{q,i}^k}\right) = l^B \left( \frac{\partial}{\partial s_{q,i}^h}, \frac{\partial}{\partial s_{q,i}^j}, \frac{\partial}{\partial s_{q,i}^k}\right) \), by values \( p_{\rho,\alpha_i} \). Example 4.1. contain an algorithm for these calculations for \( n = 2 \). In this case \( p_{\rho,\alpha_i} = \pm \frac{1}{\sqrt{1-t^2}} \).
Example 5.2. Coupling between canonical $x = (x^1, ..., x^n)$ and flat quasi-homogeneous coordinates $t = (t^1, ..., t^n)$ generate the transition tensors. Demonstrate this for $n = 2$.

According to our definitions, $\frac{\partial}{\partial t^1} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial t^2} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$. Thus $\frac{\partial}{\partial t^2} = t^2 \frac{\partial}{\partial t^1} = t^2 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$.

On the other hand, according to example 4.2., $\frac{\partial}{\partial t^2} = \frac{1}{t}(t^1)^2 t^2 + \frac{1}{24} (t^2)^4$. Therefore $F_A^1 = \frac{1}{3} (t^1)^3 t^2 + \frac{1}{24} t^1 (t^2)^4$ and $F_A^2 = \frac{1}{4} (t^1)^2 (t^2)^2 + \frac{1}{120} (t^2)^5$.

Thus, we can find the potential of the Cardy-Frobenius bundle for $n = 2$. According to theorem 5.2 it is the tensor series

$$F = F_A + \frac{1}{3} \sum I^A \left( \frac{\partial}{\partial t^i} \frac{\partial}{\partial t^j} \right) t^i \otimes t^j + \sum l^B \left( \frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j} \right) s^i \otimes s^j + \sum F_A^i t^i \otimes s^i + \frac{1}{3} \sum I^B \left( \frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j} \frac{\partial}{\partial s^k} \right) s^i \otimes s^j \otimes s^k.$$ 

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