Quotients of complex algebraic supergroups

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To the memory of V. S. Varadarajan

Abstract

In this paper we prove that the etale sheafification of the functor arising from the quotient of an algebraic supergroup by a closed subsupergroup is representable by a smooth superscheme.

1 Introduction

The purpose of this paper is to provide a construction of the quotient of a complex algebraic supergroup by a closed subsupergroup. This construction is already available in a more general setting in the literature (see \cite{11}), however here we present a different and more geometric proof, that is closer to the original approach by Chevalley (see \cite{1} Ch. II).

We start by reviewing the ordinary construction. Suppose $G$ is a complex algebraic group and $H$ a closed subgroup. Then, $G/H$ admits a unique algebraic variety structure, compatible with the group multiplication. In fact, there exists a rational representation of $G$ in a finite dimensional vector space $V$ and a line $L$ in $V$ whose stabilizer is $H$. Hence, we have an action of $G$ on the projective space $\mathbb{P}(V)$ and $H$ is the stabilizer subgroup of the point $[L]$ in $\mathbb{P}(V)$. We can thus identify set-theoretically the quotient $G/H$ with the orbit $Y$ of the point $[L]$; $Y$ being an orbit is also an algebraic variety,
because of Chevalley’s theorem. The uniqueness of this structure is obtained by the universal property of the quotient (see [1] Ch. II).

We want to replicate this geometric construction in the super setting. There are two major obstructions: the $\mathbb{C}$-points of a supervariety do not carry enough information on its geometry, as it happens for the ordinary counterpart. Also, quotients of supergroups may not admit a projective embedding. We overcome the first difficulty by making use of the functor of points of superschemes and introducing etale coverings and etale sections, which mimic in some sense the differential approach to the construction of quotients (see [7, 3]). As for the latter problem, we replace the projective superspace with Grassmannian superschemes. In supergeometry the projective superspace appears somehow too rigid and it is necessary to allow for more general structures, as the Grassmannians. In this way we can realize an embedding of an orbit of a supergroup action into a suitable Grassmannian, hence identifying it with a smooth superscheme. In this sense, our proof will also provide a variation of the ordinary construction of quotients of complex algebraic groups and goes beyond a mere translation of the known recipe into the super context.

Our main result is the following.

**Theorem 1.1.** Let $G$ be a complex algebraic supergroup, $H$ a closed sub-supergroup. Then, the sheafification in the etale topology of the functor $T \mapsto G(T)/H(T)$, $T$ a superscheme, is representable in the category of superschemes, by a smooth superscheme.

We shall prove this result in several steps. In Sec. 2 we give some preliminaries and notation on algebraic supervarieties and superschemes, while in Sec. 3 we establish some results on smoothness. In Sec. 4 we prove the representability of the etale sheafification of the functor $T \mapsto G(T)/H(T)$, when $H$ is the stabilizer of a point for an action of $G$ on a superscheme. Finally, in Sec. 5 we give our main result, Thm 5.5 and a comparison with [11] and the definition by Brundan in [2].

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2 Supervarieties and Superschemes

In this section we collect some facts of supergeometry. For more details see [5, 10, 4, 14].

Let \( C \) be our ground field. Let \((\text{salg})\) be the category of commutative superalgebras and let \( A = A_0 \oplus A_1 \in (\text{salg}) \). Let us consider a non-zero \( f \in A_0 \) and \( A_f \) the localization of the \( A_0 \)-module \( A \) at \( f \). The assignment:

\[
U_f := \{ x \in |\text{Spec}(A_0)| \mid f(x) \neq 0 \} \longrightarrow A_f
\]

defines a \( B \)-sheaf on \(|\text{Spec}(A_0)|\). Hence, there exists a unique sheaf of superalgebras \( \mathcal{O}_A \) on \(|\text{Spec}(A_0)|\) such that \( \mathcal{O}_A|_{U_f} = A_f \).

**Definition 2.1.** We define **affine superscheme** \( X \) associated with \( A \) the pair \( X = (|X|, \mathcal{O}_A) \), where \(|X|\) is the spectrum \(|\text{Spec}(A_0)|\) of the ordinary algebra \( A_0 \), while \( \mathcal{O}_A \) is the sheaf described above. The **reduced superscheme** \( X_r \) underlying \( X \) is the ordinary scheme associated with \( A_r = A/J_A \), where \( J_A \) is the ideal of the odd elements in \( A \).

We shall also denote with \( \mathcal{O}_X \) the sheaf of the superscheme \( X \) and with \( \mathcal{O}(X) \) the superalgebra \( A \).

A **morphism** \( f : X \longrightarrow Y \) of affine superschemes is a pair \((|f|, f^*)\), where \(|f| : |X| \rightarrow |Y|\) is a continuous map and \( f^* : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X \) is a map of sheaves of superalgebras, such that \( f^*_p : \mathcal{O}_{Y, |f|^{-1}(p)} \rightarrow \mathcal{O}_{X, p} \) is a local morphism for all \( p \) in \(|X|\).

We define **superscheme** a pair \( X = (|X|, \mathcal{O}_X) \) consisting of a topological space \(|X|\) and a sheaf of superalgebras \( \mathcal{O}_X \), which is locally isomorphic to an affine superscheme.

**Definition 2.2.** Let \( X \) be an affine superscheme, \( \mathcal{O}(X) \) the corresponding superalgebra. We say that \( S \) is a **subscheme** of \( X \), if \( S \) is the affine subscheme corresponding to the superalgebra \( \mathcal{O}(X)/I \) for \( I \) ideal in \( \mathcal{O}(X) \). If \( X = (|X|, \mathcal{O}_X) \) is a superscheme, we say \( S = (|S|, \mathcal{O}_S) \) is a **subscheme** of \( X \), if \(|S|\) is a closed subspace of \(|X|\) and \( \mathcal{O}_S = \mathcal{O}_X/I \), where \( I \) is an ideal sheaf with the following property. For any affine cover \( \{U_i\} \) of \( X \), \( I(U_i) \) is an ideal in \( \mathcal{O}_X(U_i), \mathcal{O}_S(U_i) = \mathcal{O}_X(U_i)/I(U_i) \). and on such \( U_i \) the sheaf \( \mathcal{O}_S|_{U_i} \) is obtained starting from the superalgebra \( \mathcal{O}_S(U_i) \) as in (1).

We now come to the functor of points.
Definition 2.3. Let $S$ and $T$ be superschemes. A $T$-point of $S$ is a morphism $T \rightarrow S$. We denote the set of all $T$-points by $S(T)$. We define the functor of points of the superscheme $S$ as the functor:

$$S : (\text{sschemes})^o \rightarrow (\text{sets}), \quad T \mapsto S(T), \quad S(\phi)(f) = f \circ \phi,$$

where $(\text{sschemes})$ denotes the category of superschemes, $(\text{sets})$ the category of sets and the index $o$ as usual refers to the opposite category.

By a common abuse of notation the superscheme $S$ and the functor of points of $S$ are denoted with the same letter; whenever is necessary to make a distinction, we shall write $h_S$ for the functor of points of $S$.

Definition 2.4. Let $A$ be a commutative superalgebra, $J_A$ the ideal generated by the odd elements. We say that $A$ is an affine superalgebra, if $A_0$ is a finitely generated superalgebra, such that its reduced associated algebra $A_r = A/J_A$ is an affine algebra (i.e. finitely generated and with no nilpotents) and $A_1$ is a finitely generated $A_0$-module.

We say that $X$ is an affine supervariety, if $X = (\text{Spec} A, O_X)$ and $A_r$ is an integral domain, i.e. $X_r$ is an ordinary affine variety. A supervariety $X$ is a superscheme which is locally isomorphic to an affine supervariety.

Remark 2.5. We are also interested in the functor of points of algebraic supervarieties, which are a subcategory of the category of superschemes. The category of affine superschemes is equivalent to the category of commutative superalgebras (see [4] Ch. 10), moreover the functor of points of a superscheme is determined by its behaviour on affine superschemes. We can then regard the functor of points of an algebraic supervariety (or superscheme) $X$ as starting from the category of commutative superalgebras, that is $X : (\text{salg}) \rightarrow (\text{sets}), \ X(\phi)(f) = \phi \circ f$.

3 Smooth morphisms

We now introduce the notion of smooth morphism of relative dimension. For the ordinary setting see [13] Ch. 5.

Definition 3.1. We say that a morphism of superschemes $f : X \rightarrow Y$ is smooth at $x \in |X|$ of relative dimension $m|n$, if there exists two affine
neighbourhoods $U \subset X$ and $V \subset Y$ such that:

\[
U \xrightarrow{g} \text{Spec} R[x_1 \ldots x_{m+r}, \xi_1, \ldots, \xi_{n+s}] / (f_1, \ldots, f_r, \phi_1, \ldots, \phi_s)
\]

\[
V \xrightarrow{\pi} \text{Spec} R
\]

and the rank of the Jacobian is maximal, i.e.

\[
\operatorname{rk} \frac{\partial (f_i, \phi_j)}{\partial (x_k, \xi_l)}(x) = r|s
\]

(1 $\leq i \leq r$, 1 $\leq j \leq s$, 1 $\leq k \leq m + r$, 1 $\leq l \leq n + s$). $f$ is smooth of relative dimension $m|n$, if it is smooth of relative dimension $m|n$ at all $x \in |X|$.

We say that a morphism of superschemes is etale, if it is smooth of relative dimension $0|0$.

We say that $x \in |X|$ is a smooth point, if the corresponding morphism $X \rightarrow C$ is smooth ($|X|$ is identified with $X(C)$, see [4] Ch. 10, 10.6.4). The superscheme $X$ is smooth, if all $x \in |X|$ are smooth.

This notion of smoothness of a superscheme $X$ is equivalent to the one in [6] and [11] [12].

**Proposition 3.2.** A morphism of superschemes $f : X \rightarrow Y$ is smooth of relative dimension $m|n$ at $x \in |X|$ if and only if there exist an open $V \subset Y$, $U = f^{-1}(V) \subset X$ ($x \in |U|$) such that $f = \pi \circ g$,

\[
U \xrightarrow{g} V \times C^{m|n} \xrightarrow{\pi} V
\]

where $\pi$ is the projection and $g$ is etale.

**Proof.** One direction is clear, since the composition of smooth morphisms is smooth. Since the question is local, we can look at superalgebra maps, that is $f^* : R \rightarrow R[x_1, \ldots, x_m, \ldots x_{m+r}, \xi_1, \ldots, \xi_{n}, \ldots \xi_{n+s}] / (f_1, \ldots, f_r, \phi_1, \ldots, \phi_s)$.

We can write:

\[
R \xrightarrow{\pi^*} R[x_1, \ldots, x_m, \xi_1, \ldots, \xi_n] \xrightarrow{g^*} R[x_1, \ldots, x_m, \ldots x_{m+r}, \xi_1, \ldots, \xi_n, \ldots \xi_{n+s}] / (f_1, \ldots, f_r, \phi_1, \ldots, \phi_s)
\]

with

\[
\operatorname{rk} \frac{\partial (f_i, \phi_j)}{\partial (x_k, \xi_l)}(x) = r|s
\]

by the very definition of $g$ etale, the result follows immediately. \qed
Lemma 3.3. Let \( f : X \rightarrow Y \) be a smooth morphism of superschemes of relative dimension \( m|n \). Then, for any morphism \( Y' \rightarrow Y \) we have that \( \text{pr}_2 : X \times_Y Y' \rightarrow Y \) is smooth of relative dimension \( m|n \).

\[
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{\text{pr}_2} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\text{pr}_1} & Y
\end{array}
\]

In particular, if \( f : X \rightarrow Y \) is etale, also \( \text{pr}_2 : X \times_Y Y' \rightarrow Y \) is etale.

**Proof.** Since the question is local, we can assume to be in the affine case.

\[
\begin{array}{ccc}
R[x_i, \xi_j]/(f_k, \phi_l) \otimes_R S & \xrightarrow{\text{pr}_1} & S \\
\uparrow & & \uparrow \\
R[x_i, \xi_j]/(f_k, \phi_l) & \leftarrow R
\end{array}
\]

Since \( R[x_i, \xi_j]/(f_k, \phi_l) \otimes_R S \cong S[x_i, \xi_j]/(f_k, \phi_l) \) we obtain the result. \( \square \)

We now make some observations on Grothendieck topologies. For more details see \[15\] for the ordinary setting and \[8\] for the supergeometric one.

**Observation 3.4.** Let us consider the category \((	ext{sschemes})\) of superschemes and define coverings of a superscheme \( U \) to be collections of etale maps whose images cover \( U \). This is a Grothendieck topology, because of the existence and the properties of the fibered product in \((	ext{sschemes})\), together with Lemma 3.3. This topology defines the super Etale site. Similarly, we can define another Grothendieck topology by taking Zariski coverings, i.e. collections of open embeddings, and obtain the super Zariski site (see \[8\]). Notice that if \( U_i \rightarrow U \) is a Zariski covering of a superscheme, then it is also an etale covering. Hence the etale topology is finer than the Zariski one. By the previous observation, we immediately have that a sheaf on the etale topology is a sheaf in the Zariski one, but not vice-versa (see \[8\] Sec. 2, Prop. 2.5).

As in the ordinary setting, any etale morphism will admit an etale section; this fact is essential for our construction of quotients.

**Proposition 3.5.** Let \( f : X \rightarrow Y \) be a morphism of superschemes smooth of relative dimension \( m|n \) at \( p \in |X| \). Then there exist open \( V \subset Y, U \subset X \),
$p \in |U|$, an étale cover $\phi : W \to V$ and a morphism $W \to U$ making the following diagram commute:

$$
\begin{array}{ccc}
W & \to & V \\
\phi & \downarrow & \\
U & \to & Y \\
\end{array}
$$

**Proof.** By Prop. 3.2 we have that there exists $U$ open in $X$ and $V$ open in $Y$ such that $U \to V \times \mathbb{C}^{m|n} \to V$.

We can write immediately a section $s$ for the projection, $s : V \to V \times \mathbb{C}^{m|n}$.

$$
W = U \times_{V \times \mathbb{C}^{m|n}} V \\
\downarrow \\
U \to V \times \mathbb{C}^{m|n}
$$

By Lemma 3.3 since $g$ is étale, we have that $\text{pr}_2 : W = U \times_{V \times \mathbb{C}^{m|n}} V \to V$ is also étale.

The morphism $W \to U$ is called a local étale section of $f : X \to Y$.

**Proposition 3.6.** Let $f : X \to Y$ be a morphism of smooth superschemes of finite type, $X$ an algebraic variety. If $|f|$ is surjective, $(df)_x : T_x X \to T_{f(x)} Y$ is surjective and $\dim T_x X - \dim T_{f(x)} Y = m|n$ for all $x \in |X|$, then $f$ is smooth of relative dimension $m|n$.

**Proof.** The proof follows closely Cor. 5.4.6, Ch. V in [13]. We briefly recap here the main steps. The statement is local, so let $x \in |X|$. We can factor $f$ as:

$$
\begin{array}{ccc}
U & \to & Y \times \mathbb{C}^{m|n} \\
\downarrow & & \downarrow \\
Y & \to & V \\
\end{array}
$$

where $U \subset X$ is open, $x \in |U|$. In terms of superalgebra maps this diagram reads:

$$
\begin{array}{ccc}
R[x_1, \ldots, x_{m+r}, \xi_1, \ldots, \xi_{n+s}]/(f, \varphi_j) & \to & R[x_1, \ldots, x_{m+r}, \xi_1, \ldots, \xi_{n+s}] \\
\downarrow & & \downarrow \\
R & \to & R \\
\end{array}
$$

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with \( i = 1, \ldots, r, j = 1, \ldots, s \). Furthermore \((f_i, \varphi_j)\) can be chosen such that 
\[
\text{rk} \frac{\partial (f_i, \varphi_j)}{\partial (x_k, \xi_l)} = r|s
\]

This is because we can choose such \( f_i, \varphi_j \) so that their images \( f_i, \varphi_j \) in \( m_{X,x}/m_{X,x}^2 \) are linearly independent.

Since \((df)_x : T_x X \to T_{f(x)}Y\) is surjective, we have an embedding \( m_{Y,f(x)}/m_{Y,f(x)}^2 \subset m_{X,x}/m_{X,x}^2 \). Using elementary facts of linear algebra, we have that \( \overline{f_i}, \overline{\varphi_j} \) are independent also in \( m_{Z,x}/(m_{Z,x}^2 + m_{X,x}O_{x,Z}) \) for \( Z = Y \times C^{m|n} \). This latter condition gives the independence of the differentials \( df_i, d\varphi_j \), hence the result. \( \square \)

## 4 Etale sections and quotients

In this section we examine supergroup actions and homogeneous superspaces.

**Definition 4.1.** A supergroup functor is a group valued functor from \((\text{salg})\) to \((\text{sets})\). An affine supergroup is a supervariety whose functor of points is group valued, that is to say, it associates a group to each superalgebra.

If \( G \) is an affine supergroup, then \( G \) is a closed subgroup of \( \text{GL}(m|n) \) and the superalgebra \( \mathcal{O}(G) \) has a natural Hopf superalgebra structure (see [4] Ch. 11). Furthermore, \( G \) is smooth (see [6]).

**Definition 4.2.** Let \( V \) be a super vector space. We define linear representation of \( G \) in \( V \) a morphism \( \rho : G \to \text{End}(V) \) where \( \text{End}(V) \) are the endomorphism of \( V \). We will also say that \( G \) acts on \( V \).

Let \( Y \) be a superscheme. We say that \( G \) acts on \( Y \) if we have a morphism of superschemes: \( a : G \times Y \to Y, \ g, x \mapsto a_T(g, x) := g \cdot x, \ x \in Y(T), \ g \in G(T) \), such that:

1. \( 1 \cdot x = x, \ \forall x \in Y(T) \)
2. \( (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x), \ \forall x \in Y(T), \ \forall g_1, g_2 \in G(T) \).

For \( p \in |Y| \) we define the orbit map \( a_p : G \to Y \) by \( a_p(g) = g \cdot p \ \forall g \in G(T) \).

Let \( Y \) be smooth. We say that the action \( a \) is transitive, if there exists a \( p \in |Y| \) such that \( |a_p| \) and \( (da_p)_T \) are surjective. In this case we call \( Y \) an homogeneous superspace.
Notice that according to Prop. 3.6, this is equivalent to ask that \( a_p \) is smooth of relative dimension \( m|n = \ker(da_p)_{1G} \).

**Proposition 4.3.** Let \( G \) be an affine supergroup acting transitively on a smooth superscheme \( Y \). Then there exists an étale cover \( \{W_i \to Y\} \) making the following diagram commute:

\[
\begin{array}{ccc}
  U_i & \subset & G \\
  \uparrow & & \downarrow \\
  W_i & \phi_i & \to & Y
\end{array}
\]

where the \( U_i \) are open and cover \( G \).

**Proof.** Immediate from Prop. 3.5. \( \square \)

**Lemma 4.4.** Let the notation be as above. Let \( \alpha \in Y(Z), Z \in \text{sschemes} \). Then there exists an étale covering \( \{\phi_i : Z_i \to Z\} \) and elements \( \beta_i \in G(Z_i) \) such that

\[
G(Z_i) \to Y(Z_i) \\
\beta_i \mapsto \alpha_i = \alpha \circ \phi_i
\]

**Proof.** Let \( f_i : W_i \to V_i \) be the étale covering of \( Y \) described in Prop. 3.5 and \( \sigma_i : W_i \to U_i \subset G \) the corresponding étale sections. Let \( \alpha : Z \to Y \) be a \( Z \)-point of \( Y \). Define \( Z_i := W_i \times_Y Z \). We have the diagram:

\[
\begin{array}{ccc}
  Z_i = W_i \times_Y Z & \to & Z \\
  \pr_1 \downarrow & & \downarrow \\
  W_i & \to & Y
\end{array}
\]

Since the \( f_i \) are étale, we have that the \( g_i \) are étale. Take \( \beta_i := \sigma_i \circ \pr_1 : Z_i \to G \); by the very construction \( a_{p,Z_i}(\beta_i) = \alpha_i \). \( \square \)

Let \( H \) be the stabilizer functor of \( p \in |Y| \), that is \( H(Z) := \{g \in G(Z) | g \cdot p = p\} \). This is representable by a closed subgroup of \( G \) (see [4] Ch. 11). We can define the functor:

\[
G/H : \text{sschemes}^o \to \text{sets}, \quad (G/H)(Z) = G(Z)/H(Z)
\]

the definition on the arrows being clear.

The morphism \( a_p \) induces a natural transformation \( G/H \to Y \), with \( G(Z)/H(Z) \to Y(Z) \) injective for all \( Z \). In general, it will not be surjective, however we have the following (see [15] [8] for the notion of sheafification in this context).
Theorem 4.5. Let the notation be as above. The sheafification $\widetilde{G/H}$ in the \textit{etale} topology of the functor $Z \longrightarrow G(Z)/H(Z)$ is isomorphic to $Y$ and it is the functor of points of a superscheme.

\textit{Proof.} By our previous observation, we have a natural transformation: $\psi : G/H \longrightarrow Y$, that factors as:

$$G/H \longrightarrow \widetilde{G/H} \longrightarrow Y$$

We want to show that $\psi$ is an isomorphism. We only need to show it is surjective. Let $\alpha \in Y(Z)$. Then by Lemma \ref{lem:etale-cover} there exists an \textit{etale} cover $\phi_i : Z_i \longrightarrow Z$ and elements $\beta_i \in G(Z_i)$ such that

$$a_{p,Z_i} : G(Z_i) \longrightarrow Y(Z_i) \quad \beta_i \quad \mapsto \quad \alpha_i = \alpha \circ \phi_i$$

Let $\beta'_i$ be the projections of the $\beta_i$ onto $G(Z_i)/H(Z_i)$. We have the commutative diagram

$$
\begin{array}{ccc}
Z_i \times_{G/H} Z_j & \longrightarrow & Z_j \\
\downarrow & & \downarrow \beta'_j \\
Z_i & \longrightarrow & G/H
\end{array}
$$

Hence, the $\beta_i'$ correspond to a unique $\beta \in \widetilde{G/H}(Z)$, so this shows that $\widetilde{G/H}(Z) \cong Y(Z)$. \hfill $\Box$

5 Quotients

In this section we prove our main result.

Proposition 5.1. Let the $G$ be an affine algebraic supergroup and $H$ a closed subsupergroup. Then, there exists a finite dimensional representation $\rho$ of $G$ in $V$ and a subspace $W \subset V$, such that:

$$H(T) = \{ g \in G(T) | \rho(g)W = W \},$$

\textit{Proof.} See \cite{4}, 11.7.11. \hfill $\Box$

Once we fix suitable coordinates, the subsuperspace $W \subset V$ corresponds to a point $p \in |\text{Gr}|$, where $\text{Gr}$ is the Grassmannian of $r|s$ subsuperspaces of
\( \mathbb{C}^{m|n} \), where \( r|s = \dim W \) and \( m|n = \dim V \) (see [4] Ch. 10 for the definition of \( \text{Gr} \) as superscheme). So we have an action \( a = \rho|_G : G \times \text{Gr} \to \text{Gr} \), where \( H = \text{Stab} p \), and the corresponding orbit map \( a_p : G \to \text{Gr} \), \( a_p,T(g) = g \cdot p \), for all \( g \in G(T) \). Notice that both \( G \) and \( \text{Gr} \) are smooth algebraic varieties; \( a_p \) is of finite type. By Chevalley’s theorem \( \rho|_G \) is open in its closure, hence it defines a superscheme that we denote by \( G \cdot p \) and call the orbit of \( p \). We have then the following commutative diagram:

\[
\begin{array}{ccc}
\text{GL}(m|n) & \xrightarrow{\rho_p} & \text{Gr} \\
\uparrow & & \uparrow \\
G & \xrightarrow{a_p} & G \cdot p
\end{array}
\]

(2)

the vertical arrows being injections.

Without loss of generality, choose \( p \in \text{Gr} \) as the subsuperspace \( \langle e_1, \ldots, e_r, \epsilon_{n-s}, \ldots \epsilon_n \rangle \). So its stabilizer in \( \text{GL}(m|n) \) is:

\[
P(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix} \right\} \subset \text{GL}(m|n)(R)
\]

where \( a_{11}, a_{44} \) are \( r \times r \), \( s \times s \) matrices with entries in \( R_0 \), while \( a_{14} \) is \( r \times s \), \( a_{41} \) is \( s \times r \) matrix with entries in \( R_1 \) (similarly for the others).

**Observation 5.2.** Let

\[
g = \begin{pmatrix} g_{11} & g_{12} & \gamma_{13} & \gamma_{14} \\ g_{21} & g_{22} & \gamma_{23} & \gamma_{34} \\ \gamma_{31} & \gamma_{32} & g_{33} & g_{34} \\ \gamma_{41} & \gamma_{42} & g_{43} & g_{44} \end{pmatrix} \in \text{GL}(m|n)(R)
\]

with \( g_{11} \) and \( g_{44} \) invertible.

In the equivalence class \( gP(R) \in G(R)/H(R) \), we can choose a unique representative of the form:

\[
\begin{pmatrix} I_r & 0 & 0 & 0 \\ u & I_{m-r} & 0 & \eta \\ \xi & 0 & I_{n-s} & v \\ 0 & 0 & 0 & I_s \end{pmatrix}
\]

where \( I_t \) denotes the identity matrix of rank \( t \).
This is a straightforward calculation coming from the fact that the system:
\[
\begin{pmatrix}
g_{11} & g_{12} & \gamma_{13} & \gamma_{14} \\
g_{21} & g_{22} & \gamma_{23} & \gamma_{34} \\
\gamma_{31} & \gamma_{32} & g_{33} & g_{34} \\
\gamma_{41} & \gamma_{42} & g_{34} & g_{44}
\end{pmatrix} = \begin{pmatrix} I_r & 0 & 0 & 0 \\ u & I_{m-r} & 0 & \eta \\ \xi & 0 & I_{n-s} & v \\ 0 & 0 & 0 & I_s \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \alpha_{13} & \alpha_{14} \\ a_{22} & a_{22} & \alpha_{23} & 0 \\ a_{33} & a_{32} & a_{33} & 0 \\ \alpha_{41} & 0 & 0 & a_{44} \end{pmatrix}
\]
has a unique solution. It is given by:
\[
a_{11} = g_{11}, \quad a_{12} = g_{12}, \quad a_{13} = \gamma_{13}, \quad a_{14} = \gamma_{14}, \quad a_{41} = \gamma_{41}, \quad a_{44} = g_{44}
\]
\[
a_{22} = g_{22} - ug_{12}, \quad a_{23} = \gamma_{23} - u\gamma_{13}, \quad a_{32} = \gamma_{32} - \xi g_{12}, \quad a_{33} = g_{33} - \xi \gamma_{13}
\]
\[
\eta = (\gamma_{24} - u\gamma_{14})g_{44}^{-1}, \quad \eta = (\gamma_{31} - u\gamma_{41})g_{11}^{-1},
\]
\[
u = (g_{21} - \gamma_{24}g_{44}^{-1}\gamma_{41})(g_{11} - \gamma_{14}g_{44}^{-1}\gamma_{41})^{-1},
\]
\[
v = (g_{34} - \gamma_{31}g_{11}^{-1}\gamma_{14})(g_{44} - \gamma_{41}g_{11}^{-1}\gamma_{14})^{-1}
\]

**Lemma 5.3.** The superscheme $G \cdot p$ is smooth.

**Proof.** It is enough to prove smoothness at $p$. Let $N$ be the closed subsupergroup of $GL(m|n)$ defined via functor of points by:

\[
N(R) = \left\{ \begin{pmatrix}
I_r & 0 & 0 & 0 \\ x & I_{m-r} & 0 & \nu \\ \mu & 0 & I_{n-s} & y \\ 0 & 0 & 0 & I_s
\end{pmatrix} \right\}
\]

Let $|U|$ be the open subset in $|GL(m|n)|$ defined by the open condition $g_{11}$ and $g_{44}$ invertible (see Obs 5.2 and $\pi(|U|)$ its projection on $|Gr|$, $\pi : |GL(m|n)| \to |Gr| = |GL(m|n)||P|$. Since $|U|$ and $\pi(|U|)$ are open respectively in $|GL(m|n)|$ and $|Gr|$, they define superschemes, that we denote with $U$ and $\pi(U)$.

Then, we have a functorial bijection:

\[
\rho_{p,R} : N(R) \to \pi(U)(R) \subset \text{Gr}(R)
\]

so $N$ and $\pi(U)$ are isomorphic supervarieties.

Let $N_G$ be the closed subsupergroup of $G$ defined as $N_G(R) = N(R) \cap G(R)$. We have $N_G(R) = \pi(U)(R) \cap (G \cdot p)(R)$, hence $N_G$ is isomorphic to the open subscheme $\pi(U) \cap G \cdot p$. \[\square\]
Proposition 5.4. The action \( a : G \times G \cdot p \to G \cdot p \) is transitive.

Proof. By the previous lemma, we know that \( G \cdot p \) is smooth and since \( G \) is an algebraic supergroup, by [6] it is smooth. It is enough to show that \( |a_p| \) is surjective (obvious) and \((da_p)_L\) is surjective. By the previous lemma this is clear.

Now we prove our main result.

Theorem 5.5. Let the \( G \) be an affine algebraic supergroup and \( H \) a closed subsupergroup. The etale sheafification of the functor

\[ T \to G(T)/H(T), \quad T \in (\text{sschemes}) \tag{3} \]

is representable in the category of superschemes, by a smooth superscheme.

Proof. By Prop. 5.4 \( G \) acts transitively on the smooth superscheme \( G \cdot p \). Hence, by Prop. 3.2 \( a_p \) is a smooth morphism of relative dimension \( m|n \) (for suitable \( m,n \). By Prop. 5.1 we have that \( H \) is the stabilizer of a point, so that we can apply Thm. 4.5 and obtain the result.

We conclude with a comparison with the results in [11] and the definition in [2].

Observation 5.6. 1. The functor of points of a superscheme is a sheaf in the following Grothendieck topologies: Zariski, etale and fppf (see [8, 11]). Thm 5.5 asserts that the etale sheafification of the functor (3) is representable by a superscheme \( G/H \), hence also its fppf sheafification has the same property. This is because \( G/H \) is already a sheaf in the fppf topology and the sheafification construction is unique up to isomorphism (see [15]).

2. Our realization of quotients satisfies the properties (Q1)-(Q3) in [2] Sec. 2. Properties (Q1), (Q2) are clear from our construction. As for property (Q3), notice that, in diagram (2), \( \rho_p \) is an affine morphism, and the embedding of \( G \) into \( \text{GL}(m|n) \) is also an affine morphism. The Grassmannian \( \text{Gr} \) is covered by affine open subsets \( V_i = \pi(U_i), \) \( V_i \cap G \cdot p \) is a closed subscheme of \( V_i \) hence affine and open in \( G \cdot p \). By the commutativity of (2), \( a_p^{-1}(V_i \cap G \cdot p) \) is affine.
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