TIP: Typifying the Interpretability of Procedures

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Abstract

We provide a novel notion of what it means to be interpretable, looking past the usual association with human understanding. Our key insight is that interpretability is not an absolute concept and so we define it relative to a target model, which may or may not be a human. We define a framework that allows for comparing interpretable procedures by linking it to important practical aspects such as accuracy and robustness. We characterize many of the current state-of-the-art interpretable methods in our framework portraying its general applicability. Finally, principled interpretable strategies are proposed and empirically evaluated on synthetic data, as well as on the largest public olfaction dataset that was made recently available [12]. We also experiment on MNIST with a simple target model and different oracle models of varying complexity. This leads to the insight that the improvement in the target model is not only a function of the oracle models performance, but also its relative complexity with respect to the target model.

1 Introduction

What does it mean for a model to be interpretable? From our human perspective, interpretability typically means that the model can be explained, a quality which is imperative in almost all real applications where a human is responsible for consequences of the model. However good a model might have performed on historical data, in critical applications, interpretability is necessary to justify, improve, and sometimes simplify decision making.

A great example of this is a malware detection neural network [6] which was trained to distinguish regular code from malware. The neural network had excellent performance, presumably due to the deep architecture capturing some complex phenomenon opaque to humans, but it was later found that the primary distinguishing characteristic was the grammatical coherance of comments in the code, which were either missing written poorly in the malware as opposed to regular code. In hindsight, this seems obvious as you wouldn’t expect someone
writing malware to expend effort in making it readable. This example shows how the interpretation of a seemingly complex model can aid in creating a simple rule.

The above example defines interpretability as humans typically do: we require the model to be understandable. This thinking would lead us to believe that, in general, complex models such as random forests or even deep neural networks are not interpretable. However, just because we cannot always understand what the complex model is doing does not necessarily mean that the model is not interpretable in some other useful sense. It is in this spirit that we define the novel notion of $\delta$-interpretability that is more general than being just relative to a human.

We offer an example from the healthcare domain [3], where interpretability is a critical modeling aspect, as a running example in our paper. The task is predicting future costs based on demographics and past insurance claims (including doctor visit costs, justifications, and diagnoses) for members of the population. The data used in [3] represents diagnoses using ICD-9-CM (International Classification of Diseases) coding which had on the order of 15,000 distinct codes at the time of the study. The high dimensional nature of diagnoses led to the development of various abstractions such as the ACG (Adjusted Clinical Groups) case-mix system [20], which output various mappings of the ICD codes to lower dimensional categorical spaces, some even independent of disease. A particular mapping of IDC codes to 264 Expanded Diagnosis Clusters (EDCs) was used in [3] to create a complex model that performed quite well in the prediction task.
2 Defining $\delta$-Interpretability

Let us return to the opening question. Is interpretability simply sparsity, entropy, or something more general? An average person is said to remember no more than seven pieces of information at a time [14]. Should that inform our notion of interpretability? Taking inspiration from the theory of computation [19] where a language is classified as regular, context free, or something else based on the strength of the machine (i.e. program) required to recognize it, we look to define interpretability along analogous lines.

Based on this discussion we would like to define interpretability relative to a target model (TM), i.e. $\delta$-interpretability. *The target model in the most obvious setting would be a human, but it doesn’t have to be.* It could be a linear model, a decision tree or even an entity with superhuman capabilities. The TM in our running healthcare example [3] is a linear model where the features come from an ACG system mapping of IDC codes to only 32 Aggregated Diagnosis Groups (ADGs).

Our model/procedure would qualify as being $\delta$-interpretable if we can somehow convey information to the TM that will lead to improving its performance (e.g., accuracy or AUC or reward) for the task at hand. Hence, the $\delta$-interpretable model has to transmit information in a way that is consumable by the TM. For example, if the TM is a linear model our $\delta$-interpretable model can only tell it how to modify its feature weights or which features to consider. In our healthcare example, the authors in [3] need a procedure to convey information from the complex 264-dimensional model to the simple linear 32-dimensional model. Any pairwise or higher order interactions would not be of use to this model. Thus, if our "interpretable" model came up with some modifications to pairwise interaction terms, it would not be considered as an $\delta$-interpretable procedure for the linear TM.

Ideally, the performance of the TM should improve w.r.t. to some target distribution. The target distribution could just be the underlying distribution, or it could be some reweighted form of it in case we are interested in some localities of the feature space more than others. For instance, in a standard supervised setting this could just be generalization error (GE), but in situations where we want to focus on local behavior the error would be w.r.t. the new reweighted distribution that focuses on the specific region. *In other words, we allow for instance level interpretability as well as global interpretability and capturing of local behaviors that lie in between.* The healthcare example focuses on mean absolute prediction error (MAPE) expressed as a percentage of the mean of the actual expenditure (Table 3 in [3]). Formally, we define $\delta$-interpretability as follows:

**Definition 2.1.** $\delta$-interpretability: Given a target model $M_T$ belonging to a hypothesis class $\mathcal{H}$ and a target distribution $D_T$, a procedure $P_I$ is $\delta$-interpretable if the information $I$ it communicates to $M_T$ resulting in model $M_T(I) \in \mathcal{H}$ satisfies the following inequality: $e_{M_T(I)} \leq \delta \cdot e_{M_T}$, where $e_M$ is the expected error of $M$ relative to some loss function on $D_T$. 

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The above definition is a general notion of interpretability that does not require the interpretable procedure to have access to a complex model. It may use the complex model (CM) but it may very well act as an oracle conjuring up useful information that will improve the performance of the TM. The more intuitive but special case of Definition 2.1 is given below which defines $\delta$-interpretability for a CM relative to a TM as being able to transfer information from the CM to the TM using a procedure $P_I$ so as to improve the TM’s performance. These concepts are depicted in figure 1.

**Definition 2.2. CM-based $\delta$-interpretability:** Given a target model $M_T$ belonging to a hypothesis class $\mathcal{H}$, a complex model $M_C$, and a target distribution $D_T$, the model $M_C$ is $\delta$-interpretable relative to $M_T$, if there exists a procedure $P_I$ that derives information $I$ from $M_C$ and communicates it to $M_T$, resulting in model $M_T(I) \in \mathcal{H}$ satisfying the following inequality:

$$
e_{M_T(I)} \leq \delta \cdot e_{M_T},$$

where $e_M$ is the expected error of $M$ relative to some loss function on $D_T$.

One may consider the more intuitive definition of $\delta$-interpretability when there is a CM.

We now clarify the use of the term Information $I$ in the definition. In a normal binary classification task, training label $y \in \{+1, -1\}$ can be considered to be a one bit information about the sample $x$, i.e. "Which label is more likely given $x$?", whereas the confidence score $p(y|x)$ holds richer information, i.e. "How likely is the label $y$ for the sample $x$?". From an information theoretic point of view, given $x$ and only its training label $y$, there is still uncertainty about $p(y|x)$ in the interval $[1/2, 1]$ prior to training. According to our definition, an interpretable method can provide useful information $I$ in the form of a sequence of bits or parameters about the training data that can potentially reduce this uncertainty of the confidence score of the TM prior to training. Moreover, the new $M_T(I)$ is better performing if it can effectively use this information.

Our definitions above are motivated by the fact that when people ask for an interpretation there is an implicit quality requirement in that the interpretation should be related to the task at hand. We capture this relatedness of the interpretation to the task by requiring that the interpretable procedure improve the performance of the TM. Note the TM does not have to be interpretable, rather it just is a benchmark used to measure the relevance of the provided interpretation. Without such a requirement anything that one can elucidate is then an explanation for everything else, making the concept of interpretation pointless. Consequently, the crux for any application in our setting is to come up with an interpretable procedure that can ideally improve the performance of the given TM.

The closer $\delta$ is to 0 the more interpretable the procedure. Note the error reduction is relative to the TM model itself, not relative to the complex model. An illustration of the above definition is seen in figure 1. Here we want to interpret a complex process relative to a given TM and target distribution. The interaction with the complex process could just be by observing inputs and
outputs or could be through delving into the inner workings of the complex process. In addition, it is imperative that $M_T(I) \in \mathcal{H}$ i.e. the information conveyed should be within the representational power of the TM.

The advantage of this definition is that the TM isn’t tied to any specific entity such as a human and thus neither is our definition. We can thus test the utility of our definition w.r.t. simpler models (viz. linear, decision lists, etc.) given that a humans complexity maybe hard to characterize. We see examples of this in the coming sections.

Moreover, a direct consequence of our definition is that it naturally creates a partial ordering of interpretable procedures relative to a TM and target distribution, which is in spirit similar to complexity classes for time or space of algorithms. For instance, if $\mathcal{R}^+$ denotes the non-negative real line $\delta_1$-interpretability $\Rightarrow$ $\delta_2$-interpretability, where $\delta_1 \leq \delta_2 \forall \delta_1, \delta_2 \in \mathcal{R}^+$, but not the other way around.

### 3 Understanding Definition 2.1

In order to better understand our definition of $\delta$-interpretability, we ask the following question: how can one (i.e. an observer) validate that information shared by one entity (or procedure) can be interpreted by another entity (or
target model? The only way to evaluate such a proposition is to test the target model, e.g., evaluate the target model before and after information is shared and observe whether or not there is a change in state/performance. One may argue that the target model may communicate this directly to the observer, however this then assumes that the two are able to communicate. The most general setting is where no such assumption is made in which case the observer can only detect tangible transfer of information from the procedure to the target model by change in performance of the target model. The case where one wants to understand/interpret some concept is just a special case of this setting, where the observer and the target model can be considered to communicate perfectly or in essence are one and the same. Even in this case, the only way to really know that you understood the concept is to test oneself on the task at hand.

We depict this concept in figure 2. An observer sits at the bottom and asks whether or not a given procedure is interpretable. A target model is selected and its initial state (viz., performance on a benchmark data set) is observed, depicted by a weeping individual. The procedure conveys information $I$ to the target model, resulting in an updated target model whose state, depicted by a jovial individual, is observed. The observer thus has his answer regarding interpretability of the procedure. Hence, interpretability measures the change in behavior or performance (here being a change from sadness to happiness), and thus we truly need to define $\delta$-interpretability rather than simply interpretability in order to capture the impact of this change.

We can also think of this framework as the procedure being a teacher and the target model being a student, and the observer (i.e., the student’s parent) wants to measure the quality of the teacher. The parent observes the student a priori on some task, and then measures the student on the same task after the teacher’s lesson. The student’s change in performance would dictate the teacher’s ability to effectively convey information understandable/interpretable by the student. Note that the student’s performance could become worse in which case $\delta > 1$ indicating that the teacher was interpretable but bad.

4 Practical Definition of Interpretability

We first extend our $\delta$-interpretability definition to the practical setting where we don’t have the target distribution, but rather just samples. We then show how this new definition reduces to our original definition in the ideal setting where we have access to the target distribution.

4.1 $$(\delta, \gamma)$$-Interpretability: Performance and Robustness

Our definition of $\delta$-interpretability just focuses on the performance of the TM. However, in most practical applications robustness is a key requirement. Imagine a doctor advising a treatment to a patient. He better have high confidence in the treatments effect before prescribing.
So what really is a robust model? Intuitively, it is a notion where one expects the same (or similar) performance from the model when applied to “nearby” inputs. In practice, this is many times done by perturbing the test set and then evaluating performance of the model \[2\]. If the accuracies are comparable to the original test set then the model is deemed robust. Hence, this procedure can be viewed as creating alternate test sets on which we test the model. Thus, the procedures to create adversarial examples or perturbations can be said to induce a distribution \(D_R\) from which we get these alternate test sets. The important underlying assumption here is that the newly created test samples are at least moderately likely w.r.t. target distribution. Of course, in case of non-uniform loss functions the test sets on whom the expected loss is low are uninteresting. This brings us to the question of when is it truly interesting to study robustness.

It seems that robustness is really only an issue when your test data on which you evaluate is incomplete i.e. it doesn’t include all examples in the domain. If you can test on all points in your domain, which could be finite, and are accurate on it then there is no need for robustness. That is why in a certain sense, low generalization error already captures robustness since the error is over the entire domain and it is impossible for your classifier to not be robust and have low GE if you could actually test on the entire domain. The problem is really only because of estimation on incomplete test sets \[23\]. Given this we extend our definition of \(\delta\)-interpretability for practical scenarios.

**Definition 4.1.** \((\delta, \gamma)\)-interpretability: Given a target model \(M_T\) belonging to a hypothesis class \(\mathcal{H}\), a sample \(S_T\) from the target distribution \(D_T\), a sample \(S_R\) from the adversarial distribution \(D_R\), a model \(P_I\) is \(\delta, \gamma\)-interpretable relative to \((D_T \sim)S_T\) and \((D_R \sim)S_R\) if the information \(I\) it communicates to \(M_T\) resulting in model \(M_T(I) \in \mathcal{H}\) satisfies the following inequalities:

\[
\begin{align*}
&\hat{e}^{S_T}_{M_T(I)} \leq \delta \cdot \hat{e}^{S_T}_{M_T} \quad \text{(performance)} \\
&\hat{e}^{S_R}_{M_T(I)} - \hat{e}^{S_T}_{M_T(I)} \leq \gamma \cdot (\hat{e}^{S_R}_{M_T} - \hat{e}^{S_T}_{M_T}) \quad \text{(robustness)}
\end{align*}
\]

where \(\hat{e}^S_M\) is the empirical error of \(M\) relative to some loss function.

The first term above is analogous to the one in Definition \[21\]. The second term captures robustness and asks how representative is the test error of \(M_T(I)\) w.r.t. its error on other high probability samples when compared with the performance of \(M_T\) on the same test and robust sets. This can be viewed as an orthogonal metric to evaluate interpretable procedures in the practical setting. This definition could also be adapted to a more intuitive but restrictive definition analogous to Definition \[22\].

**4.2 Reduction to Definition \[21\]**

An (ideal) adversarial example is not just one which a model predicts incorrectly, but rather it must satisfy also the additional requirement of being a highly probable sample from the target distribution. Without the second condition even
Table 1: Above we see how our framework can be used to characterize interpretability of methods across applications.

| Interpretable Procedure | TM | δ | γ | Dataset (S_T) | Performance Metric |
|-------------------------|----|----|----|---------------|-------------------|
| EDC Selection           | OLS| 0.925| 0 | Identity      | Medical Claims    |
| Defensive Distillation  | DNN| 1.27| 0.8 | L_2 attack 2 | MNIST Classification error |
| MMD-critic              | NFC | 0.24 | 0.98 | Skewed MNIST | Classification error |
| LIME                    | SLR | 0.1 | 0 | Identity      | Books Feature Recall |
| Interpretable MDP       | Static | 0.579 | 0 | Identity      | TUI Travel Products Conversion Rate (Normalized) |
| Rule Lists (size \leq 4) | Human | 0.95 | 0 | Identity      | Manufacturing Yields |

highly unlikely outliers would be adversarial examples. But in practice this is not what people usually mean, when one talks about adversarial examples.

Given this, ideally, we should choose \( D_R = D_T \) so that we test the model mainly on important examples. If we could do this and test on the entire domain our Definition 4.1 would reduce to Definition 2.1 as seen in the following proposition.

**Proposition 1.** In the ideal setting, where we know \( D_T \), we could set \( D_R = D_T \) and compute the true errors, \((\delta, \gamma)\)-interpretability would reduce to \( \delta \)-interpretability.

**Proof Sketch.** Since \( D_R = D_T \), by taking expectations we get for the first condition: \( E[e_{M_T} - \delta e_{M_T}] \leq 0 \) and hence \( e_{M_T} - \delta e_{M_T} \leq 0 \). For the second equation we get: \( E[e_{M_T} - \gamma e_{M_T}] \leq 0 \) and hence \( e_{M_T} - \gamma e_{M_T} \leq 0 \), which implies \( 0 \leq 0 \).

The second condition vanishes and the first condition is just the definition of \( \delta \)-interpretability. Our extended definition is thus consistent with Definition 2.1.1 where we have access to the target distribution.

**Remark:** Model evaluation sometimes requires us to use multiple training and test sets, such as when doing cross-validation. In such cases, we have multiple target models \( M_T^i \) trained on independent data sets, and multiple independent test samples \( S_T^j \) (indexed by \( i = \{1, \ldots, K\} \)). The empirical error above can be defined as \( (\sum_{i=1}^K e_{M_T^i}^j) / K \). Since \( S_T^j \), as well as the training sets, are sampled from the same target distribution \( D_T \), the reduction to Definition 2.1.1 would still apply to this average error, since \( E[e_{M_T^h}^j] = E[e_{M_T^k}^j] \forall h, i, j, k. \)
5 Application to Existing Interpretable Methods

We now look at how some of the current methods and how they fit into our framework.

EDC Selection: The running healthcare example of [3] considers a complex model based on 264 EDC features and a simpler linear model based on 32 ACG features, and both models also include the same demographic variables. The complex model has a MAPE of 96.52% while the linear model has a MAPE of 103.64%. The authors in [3] attempt to improve the TM’s performance by including several EDC features. They develop a stepwise process for generating selected EDCs based on significance testing and broad applicability to various types of expenditures ([3], Additional File 1). This stepwise process can be viewed as a \((\delta, \gamma)\)-interpretable procedure that provides information in the form of 19 EDC variables which, when added to the TM, improve the performance from 103.64% to 95.83% and is thus \((0.925, 0)\)-interpretable. Note the significance since, given the large population sizes and high mean annual healthcare costs per individual, even small improvements in accuracy can have high monetary impact.

Distillation: DNNs are not considered interpretable. However, if you consider a DNN to be a TM then you can view defensive distillation as a \((\delta, \gamma)\)-interpretable procedure. We compute \(\delta\) and \(\gamma\) from results presented in [2] on the MNIST dataset, where the authors adversarially perturbs the test instances by a slight amount. We see here that defensive distillation makes the DNN slightly more robust at the expense of it being a little less accurate.

Prototype Selection: We implemented and ran MMD-critic [13] on randomly sampled MNIST training sets of size 1500 where the number of prototypes was set to 200. The test sets were 5420 in size which is the size of the least frequent digit. We had a representative test set and then 10 highly skewed test sets where each contained only a single digit. The representative test set was used to estimate \(\delta\) and the 10 skewed test sets were used to compute \(\gamma\). The TM was a nearest prototype classifier [13] that was initialized with random 200 prototypes which it used to create the initial classifications. We see from the table that mmd-critic has a low \(\delta\) and almost 1 \(\gamma\) value. This implies that it is significantly more accurate than random prototype selection while maintaining robustness. In other words, it should be strongly preferred over random selection.

LIME: We consider the experiment in [18] where they use sparse logistic regression (SLR) to classify a review as positive or negative on the Books dataset. Their main objective here is to see if their approach is superior to other approaches in terms of selecting the true important features. There is no robustness experiment here so \(D_R\) is identity which means same as \(D_T\) and hence \(\gamma\) is 0. Their accuracy however, in selecting the important features is significantly better than random feature selection which is signified by \(\delta = 0.1\). The other experiments can also be characterized in similar fashion. In cases where only explanations are provided with no explicit metric one can view as the experts
confidence in the method as a metric which good explanations will enhance.

**Interpretable MDP:** The authors used a constrained MDP formulation \[17\] to derive a product-to-product recommendation policy for the European tour operator TUI. The goal was to generate buyer conversions and to improve a simple product-to-product policy based on static pictures of the website and what products are currently looked at. The constrained MDP results in a policy that is just as simple but greatly improves the conversion rate (averaged over 10 simulations where customer behavior followed a mixed logit customer choice model with parameters fit to TUI data). The mean normalized conversion rate increased from 0.3377 to 0.6167.

**Rule Lists:** We built a rule list on a semi-conductor manufacturing dataset \[4\] of size 8926. In this data, a single datapoint is a wafer, which is a group of chips, and measurements, which correspond to 1000s of input features (temperatures, pressures, gas flows, etc.), are made on this wafer throughout its production. The goal was to provide the engineer some insight into, if anything, was plaguing his process so that he can improve performance. We built a rule list \[21\] of size at most 4 which we showed to the engineer. The engineer figured out that there was some issue with some gas flows, which he then fixed. This resulted in 1% more of his wafers ending up within specification. Or his yield increased from 80% to 81%. This is significant since, even small increase in yield corresponds to billions of dollars in savings.

6 Framework Generalizability

It seems that our definition of $\delta$-interpretability requires a predefined goal/task. While (semi-)supervised settings have a well-defined target, we discuss other applicable settings.

In unsupervised settings although we do not have an explicit target there are quantitative measures \[1\] such as Silhouette or mutual information that are used to evaluate clustering quality. Such measures which people use to evaluate quality would serve as the target loss that the $\delta$-interpretable procedure would aim to improve upon. The target distribution in this case would just be the instances in the dataset. If a generative process is assumed for the data, then that would dictate the target distribution.

In other settings such as reinforcement learning (RL) \[22\], the $\delta$-interpretable procedure would try to increase the expected discounted reward of the agent by directing it into more lucrative portions of the state space. In inverse RL on the other hand, it would assist in learning a more accurate reward function based on the observed behaviors of an intelligent entity. The methodology could also be used to test interpretable models on how well they convey the causal structure \[16\] to the TM by evaluating the TMs performance on counterfactuals before and after the information has been conveyed.

There maybe situations where a human TM may not want to take any explicit actions based on the insight conveyed by a $\delta$-interpretable procedure. However, an implicit evaluation metric, such as human satisfaction, probably
exists, which a good interpretable procedure will help to enhance. For example, in the setting where you want explanations for classifications of individual instances \[18\], the metric that you are enhancing is the human confidence in the complex model.

7 Candidate (Model Agnostic) $\delta$-Interpretable Procedures

In this section, we provide candidate theoretically grounded $\delta$-interpretable strategies that use the complex model as a black box and weight the training data that the TM trains on. Note although the procedures are described for binary classification they straightforwardly extend to multiclass settings. We then illustrate how our $\delta$-interpretable procedures can help improve the accuracy of a logistic model.

7.1 Derivation of $\delta$-interpretable Procedures

We discuss two popular frameworks for binary classifiers based on their training methods.

1. Expected Risk Minimization models (ERM): Suppose the target model is optimized according to empirical risk minimization on $m$ training samples using the risk function $r(y, x, \theta)$ given by: \[ \min_{\theta} \frac{1}{m} \sum_{i=1}^{m} r(y, x, \theta) \]. Let us assume that $0 \leq r(\cdot) \leq 1$. The classification rule for a new sample $x$ is \[ \arg\min \{ r(y, x, \theta) : y \in \{+1, -1\} \} \].

2. Maximum Likelihood Estimation Models (MLE): In this case, the binary classifier is specified directly by a conditional probability distribution $p_{TM}(y|x; \theta)$. Given $m$ training samples $(y_i, x_i)$, the following optimization is performed: \[ \min_{\theta} \frac{1}{m} \sum_{i=1}^{m} -\log p_{TM}(y_i|x_i; \theta) \]. The classification rule for a new sample $x$ in this case is: \[ \arg\max \{ p_{TM}(y|x) : y \in \{+1, -1\} \} \]

We next provide two $\delta$-interpretable methods. Two are well-suited for ERM models while the last is well-suited for a maximum likelihood estimator (MLE) model. The methods are motivated by the following theorem that offers bounds and reformulations of ERM and MLE generalization error. The function $f(\cdot)$ below is non-increasing on the domain $(0, \infty)$. We provide an explicit $f(\cdot)$ at the end of the next subsection; however, in practice $f(\cdot)$ can be any function that optimizes the margin of the given target model. Let the shorthand notation $r_1(x)$ denote $r(+1, x, \theta)$ while $r_2(x)$ denote $r(-1, x, \theta)$. Let $y'(x) = \arg\max_y p_{CM}(y|x)$, $r_1'(x) = r(y'(x), x, \theta)$, and $r_2'(x) = r(-y'(x), x, \theta)$.

Let us assume that the complex model CM is high performing or quite close to the true model. Hence, we assume that the data is generated according to the distribution $D_{CM} = p(x)p_{CM}(y|x)$. The error obtained by applying the TM on the data in the MLE case is given by: \[ E_{D_{CM}}[1_{p_{TM}(y|x; \theta) < 1/2}] \]. For the ERM case, it is given by: \[ E_{D_{CM}}[1_{r(y, x, \theta) > r(-y, x, \theta)}] \]. Theorem 2 below (proof in
obtaining the confidence scores two for the ERM case and one for the MLE case. Each procedure relies on three problems: ideas from [25]. The three procedures are to respectively solve the following data. If a classifier does not output confidence scores we can obtain them using to hard classifications. Thus, the theorem specifies a candidate in ERM (b) is the quantization error incurred from converting confidence scores to hard classifications. Thus, the theorem specifies a candidate $f(\cdot)$ for both the margin based $\delta$-interpretable methods.

Theorem 2. The error bounds for the ERM and MLE cases are as follows:

**ERM case (a):**
$$
\mathbb{E}_{\mathcal{D}_{CM}}^2[1_{r(y, x, \theta) > r(-y, x, \theta)}] \leq \mathbb{E}_{p(x)} \left[ c \cdot p_{CM}(+1|x)r_1(x) + c \cdot p_{CM}(-1|x)r_2(x) \right] \\
\mathbb{E}_{p(x)} \left[ \log(1 + e^{-c|r_1(x) - r_2(x)|}) + 2e^{-2c|r_1(x) - r_2(x)|} \right]
$$

**ERM case (b):**
$$
\mathbb{E}_{\mathcal{D}_{CM}}^2[1_{r(y, x, \theta) > r(-y, x, \theta)}] = \mathbb{E}_{p(x)} \left[ 2 \left[ \frac{1}{2} - p_{CM}(y'(x)|x) \right] \cdot 1_{r_1'(x) > r_2'(x)} + \frac{1}{2} - \left[ \frac{1}{2} - p_{CM}(y'(x)|x) \right] \right]
$$

**MLE case:**
$$
\mathbb{E}_{\mathcal{D}_{CM}}^2[1_{p_{TM}(y|x; \theta) \leq 1/2}] \leq \mathbb{E}_{p(x)} \left[ -(p_{CM}(+1|x)) \log(p_{TM}(+1|x; \theta)) \\
- \left( p_{CM}(-1|x) \log(p_{TM}(-1|x; \theta)) + 2e^{-2\log p_{TM}(-1|x; \theta) - \log p_{TM}(+1|x; \theta)} \right) \right]
$$

The three results of this theorem motivate three $\delta$-interpretable procedures, two for the ERM case and one for the MLE case. Each procedure relies on obtaining the confidence scores $p_{CM}(y|x_i)$ for each sample $x_i$ of the training data. If a classifier does not output confidence scores we can obtain them using ideas from [25]. The three procedures are to respectively solve the following three problems:

**ERM case (a):**
$$
\min_{\theta} \frac{1}{m} \left[ \sum_{i=1}^m \sum_{y \in \{+1, -1\}} c \cdot p_{CM}(y|x_i)r(y, x_i, \theta) + f(c|r_1(x_i) - r_2(x_i)) \right]
$$

**ERM case (b):**
$$
\min_{\theta} \frac{1}{m} \left[ \sum_{i=1}^m \left[ \frac{1}{2} - p_{CM}(y'(x_i)|x_i) \right] r(y'(x_i), x_i, \theta) \right]
$$

**MLE case:**
$$
\min_{\theta} \frac{1}{m} \left[ \sum_{i=1}^m \sum_{y \in \{+1, -1\}} -p_{CM}(y|x_i) \log p_{TM}(y|x_i; \theta) \\
+ f(|| \log p_{TM}(+1|x_i; \theta) - \log p_{TM}(-1|x_i; \theta)||) \right]
$$

Note that there is a hyper-parameter $c > 0$ that must be tuned for ERM case (a). The three procedures are motivated directly by the respective three results of Theorem 2 above. The two ERM cases optimize the risk functionals while the MLE case directly optimizes the confidence of the TM.

Note that the ERM case (a) and MLE case suggest duplicating the training set for each label. However, the implementations of popular classification algorithms in standard packages are many times confused by this duplication. Hence, in practice, we suggest passing just one copy containing the predicted label with the corresponding weight when training the TM.
7.2 Evaluation on Simulated Data

We next illustrate how the above $\delta$-interpretable procedures can be used to improve a simple target model. Simulated data, a target model $M_T$ and improved target model $M_T(I)$ are shown in figure 3 (left). Two classes (green circles and red diamonds) are uniformly sampled from above and below the blue curve, so a linear model is clearly suboptimal. Label noise was added primarily in the upper left corner.

We show here that a k-nearest neighbors (knn) classifier is $(\delta, \gamma)$-interpretable relative to a linear model. The linear target model $M_T$ (dashed black line) is obtained by a logistic regression and achieves an accuracy of 0.766. A k-nearest neighbors classifier achieving accuracy 0.856 was used to generate confidence scores via [25], and solving the above problem for the MLE case results in the improved model $M_T(I)$ (solid red line) which achieves 0.782 accuracy. For the robustness test, 10% of the labels were flipped, and $M_T$ accuracy improved from 0.710 to 0.722 by $M_T(I)$, meaning we have $(0.931, 1.071)$-interpretability.

Figure 3 (middle) zooms in on a section of the left figure, exhibiting the benefit of the procedure: Several green circles misclassified by the target model $M_T$ are classified correctly by the reweighted model $M_T(I)$.

7.3 Evaluation on (Real) Olfaction Data

We evaluated our strategies on a recent publicly available olfaction dataset [12] which has hundreds of molecules and thousands of chemoinformatic features along with qualitative perceptions averaged across almost 50 individuals. We chose Pleasantness, which was one of the major percepts in this dataset as the target. The scale for this percept went from 0 to 100, where 0 meant that the odor was highly unpleasant while 100 meant that it was extremely pleasant. Hence, odors which were around 50 could be considered as neutral odors. For our binary classification setting we thus created two classes where class 1 was all odors with pleasantness $< 50$ while class 2 was all odors with pleasantness $\geq 50$.

We used a random forest (RF) model as our CM which had a test error of 0.24. Our TM was lasso which had a test error of 0.32. Using our MLE inter-
Figure 4: Above we see results on the MNIST dataset. In the above left figure we see that the complex model CM-4 which is of intermediate complexity and performance produces the greatest in improvement in our TM given our interpretable strategy. The right figure depicts the distribution of confidence scores for each CM used to weight the instances for training the TM.

interpretable procedure the error of lasso dropped to 0.26. While using our ERM (b) strategy the error dropped to 0.27. This is depicted in figure 3 (right). Hence, our MLE procedure was $(0.81, 0)$-interpretable, while our ERM (b) procedure was $(0.84, 0)$-interpretable. This illustrates a manner in which two interpretable procedure can be compared quantitatively, where in this example, the MLE procedure would be preferred.

7.4 Evaluation on MNIST

We now test the hypothesis if having a better complex model also implies that the $\delta$ will be lower for a given TM.

7.4.1 Setup

We build 5 complex models of decreasing complexity. The complexity could be characterized by the number of parameters used to train the models. The most complex model CM-1 we use is given as a candidate to use on MNIST
in keras [8] which has around 1.2 million parameters and is a 4 layer network. CM-2 [7] is of slightly lower complexity with around 670K parameters and is a 3 layer network. CM-3, CM-4 and CM-5 are 2 layer networks with 512, 64 and 32 rectified linear units respectively and a 10-way softmax layer. They have approximately 400K, 50K and 25K parameters. Our TM is a single layer network with just a softmax layer and has close to 8K parameters.

We split the MNIST training set randomly into two equal parts train1 and train2. We train our TM on train2. We then train the CMs on train1 and make predictions on train2. Using our MLE interpretable strategy we derive corresponding weights for instances in train2. We then train our TM using the corresponding weighted examples and obtain 5 corresponding versions of TM namely, TM-1 to TM-5. We then compute the error of all models i.e. CM-1, ..., CM-5 and TM, TM-1, ..., TM-5 on the MNIST test set of 10K examples. We use train2 to train the TM so as to get better estimates of confidence scores from the CMs as opposed to trusting an overfitted model. Moreover, this also gives us better resolution of weights from the CMs, as most of them have confidence scores close to 1 for almost all the train1 instances.

7.4.2 Observations

We see in figure 4 (left) that the most complex CM which is CM-1, has the best test performance. The performance drops monotonically as the CMs become less complex. From the TMs perspective we see that all the CMs help in reducing its test error. However, TM-4 has the lowest error amongst the TMs, which corresponds to CM-4. Thus, eventhough CM-4 is not the best performing CM it is the best teacher for the TM given our interpretable strategy. Consequently, in our framework, CM-4 is (0.95, 0)-interpretable, while CM-1 is (0.98, 0)-interpretable, with others lying in between.

To see why this happens we plot the distribution of the weights that are obtained by each CM which is observed in figure 4 (right). We see that the complicated CM is so good that for almost 98% of the instances it has a confidence score of $\sim 1$. The distribution starts to become more spread out as the complexity of the CMs reduces.

7.4.3 Insight

So what insight do the above observations convey. Given our interpretable strategies of weighting instances the improvement in the TM is a function of the performance of the CM and the diversity in its confidence scores. If the CM is so good that all its confidence scores are close to 1 then almost no new information is passed to the TM as the weighted training set is practically equivalent to the original unweighted one.

This leads to the following qualitative insight.
Having a teacher who is exceptional in an area may not be the best for the student as the teacher is not able to resolve what may be more difficult as opposed to less difficult and is thus unable to provide extra information that may give direction to help the student.

Of course, all of the above is contingent on the interpretable strategy and there may be better ways to extract information from complex models such as CM-1. Nonetheless, the above point we feel is thought provoking.

8 Related Work

There has been a great deal of interest in interpretable modeling recently and for good reason. In almost any practical application with a human decision maker, interpretability is imperative for the human to have confidence in the model. It has also become increasingly important in deep neural networks given their susceptibility to small perturbations that are humanly unrecognizable [2,10].

There have been multiple frameworks and algorithms proposed to perform interpretable modeling. These range from building rule/decision lists [21,24] to finding prototypes [13] to taking inspiration from psychometrics [11] and learning models that can be consumed by humans. There are also works [18] which focus on answering instance specific user queries by locally approximating a superior performing complex model with a simpler easy to understand one which could be used to gain confidence in the complex model.

The most relevant work to our current endeavor is possibly [5]. They provide an in depth discussion for why interpretability is needed, and an overall taxonomy for what should be considered when talking about interpretability. Their final TM is always a human even for the functionally grounded explanation as the TMs are proxies for human complexity. As we have seen, our definition of a TM is more general, as besides human, it could be any ML model or even something else that has superhuman cognitive skills. This generalization allows us to test our definition without the need to pin down human complexity. Moreover, we provide a formal strict definition for $\delta$-interpretability that accounts for key concepts such as performance and robustness and articulates how robustness is only an issue when talking about incomplete test sets. In addition we also propose a principled meta-interpretable strategy that works well in practice. Our meta strategy has relations to distillation and learning with privileged information [15] with the key difference being in the mechanics of how we use information which is by weighting the instances rather than modeling it as a target. This has the advantage of not having to change from a classification to regression setting. Moreover, weighting instances has an intuitive justification where if you view the complex model as a teacher and the TM as a student, the teacher is telling the student which aspects (e.g. instances) he should focus on and which he could ignore.
9 Discussion

In this paper we provided a formal framework to characterize interpretability. Using this framework we were able to quantify the performance many state-of-the-art interpretable procedures. We also proposed our own for the supervised setting that are based on strong theoretical grounding.

Our experiments led to the insight that having the best performing complex model is not necessarily the best in terms of improving a TM. In other words, it seems important to characterize the relative complexity of a (CM, TM) pair for useful information transfer. Trying to characterize this is part of future work. Of course, all of this is relative to the interpretable strategies that one can come up with. Hence, in the future we also want to design better interpretable strategies for more diverse settings.

From an information theoretic point of view, our work motivates the following two kinds of capacity notions: a) What is the least number of additional bits per training sample required for the TM to improve its performance by $\delta$? These additional bits would reduce the uncertainty in the confidence score about a target label than what is implied by the training data. b) What is the maximum number of additional bits per training sample that can be obtained from the CM towards reducing the uncertainty of the confidence scores of the TM? Based on these two questions, it may be possible to say when the second capacity exceeds the first capacity, then a $\delta$ improvement is possible. We intend to investigate this in the future.

We defined $\delta$ for a single distribution but it could be defined over multiple distributions where $\delta = \max(\delta_1, \ldots, \delta_k)$ for $k$ distributions and analogously $\gamma$ also could be defined over multiple adversarial distributions. We did not include these complexities in the definitions so as not to lose the main point, but extensions such as these are straightforward.

Another extension could be to have a sensitivity parameter $\alpha$ to define equivalence classes, where if two models are $\delta_1$- and $\delta_2$-interpretable, then they are in the same equivalence class if $|\delta_1 - \delta_2| \leq \alpha$. This can help group together models that can be considered to be equivalent for the application at hand. The $\alpha$ in essence quantifies operational significance. One can have even multiple $\alpha$ as a function of the $\delta$ values.

One can also extend the notion of interpretability where $\delta$ or and $\gamma$ are 1 but you can learn the same model with fewer samples given information from the interpretable procedure. In essence, have sample complexity also as a parameter in the definition.

Human subjects store approximately 7 pieces of information [14]. As such, we can explore highly interpretable models, which can be readily learned by humans, by considering models for TM that make simple use of no more than 7 pieces of information.

Feldman [9] finds that the subjective difficulty of a concept is directly proportional to its Boolean complexity, the length of the shortest logically equivalent propositional formula. We could explore interpretable models of this type.

Another model bounds the rademacher complexity of humans [26] as a func-
tion of complexity of the domain and sample size. Although the bounds are loose, they follow the empirical trend seen in their experiments on words and images.

Finally, all humans may not be equal relative to a task. Having expertise in a domain may increase the level of detail consumable by that human. So the above models which try to approximate human capability may be extended to account for the additional complexity consumable by the human depending on their experience.

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A Analysis: Proof of Theorem 2

We assume that any model for a given binary classification task is essentially a conditional probability distribution function \( p(y|x), \ y \in \{ +1, -1 \} \). All classifiers are assumed to follow the following classification rule: \( \text{argmax}(p(y|x) : y \in \{ +1, -1 \}) \). Let us denote the conditional probability distribution function for the complex model as \( p_{CM}(y|x) \). Let us denote the conditional probability distribution function for the target model with parameter \( \theta \) as \( p_{TM}(y|x; \theta) \). We unify the treatment through the lens of conditional probability scores. So one must define an explicit conditional probability score for the risk minimization models. We do so in the following subsection.

A.1 Risk Minimization models and pseudo-confidence scores

Suppose the target model is optimized according to empirical risk minimization on \( m \) training samples using the risk function \( r(y,x,\theta) \), i.e.

\[
\min_{\theta} \frac{1}{m} \sum_{i=1}^{m} r(y_i, x_i, \theta)
\]

Let us assume that \( 0 \leq r(\cdot) \leq 1 \). Let the shorthand notation \( r_1(x) \) denote \( r(+1,x,\theta) \) while \( r_2(x) \) denote \( r(-1,x,\theta) \). Define the conditional probability distribution on the target model based on the risk function as follows:

\[
p_{TM}(+1|x; \theta) = \frac{e^{-cr_1(x)}}{e^{-cr_1(x)} + e^{-cr_2(x)}}
\]

for some constant \( c > 0 \). Please note that, no matter what \( c \) is, the behavior of the classifier on actual data depends on whether \( r_1(x) > r_2(x) \) or not. Therefore for all \( c > 0 \), this is equivalent to checking if \( p_{TM}(+1|x; \theta) > 1/2 \) or not. In
fact, the behavior of the error term at the LHS of (3) depends only on whether \( r_1(x) > r_2(x) \) or not and this is independent of the choice of \( c \). So here, we don’t attach any real notion of confidence score to \( p_{TM}(\cdot) \) defined as above. They can be considered to be pseudo-confidence scores implied by the risk function \( r(\cdot) \) for the sake of analysis. So any risk function on the target model endows it a pseudo-confidence score.

A.2 Error Term

We will always treat an ERM case as an MLE case endowed with pseudo-confidence scores. We first note the following: The best GE one can obtain if the data is distributed according to \( D_{CM} \) is exactly \( E_{D_{CM}}[1_{p_{CM}(y|x) < 1/2}] \). This is because, even if the classifier knows the correct distribution, given that the output is either +1, −1, there will be some error due to this quantization.

We wish to find the optimum \( \theta \) that minimizes the classification error of the target model assuming that the samples arise from \( D_{CM} \). Based on the above observation, we split this error into two terms:

\[
E_{D_{CM}}[1_{p_{TM}(y|x;\theta) < 1/2}] = E_{D_{CM}}[1_{p_{TM}(y|x;\theta) < 1/2}] - E_{D_{TM,\theta}}[1_{p_{TM}(y|x;\theta) < 1/2}]
+ E_{D_{TM,\theta}}[1_{p_{TM}(y|x;\theta) < 1/2}]
\]  

(3)

The second term is the residual error of the perfect classifier on samples drawn according to the distribution defined by the target model, i.e. \( D_{TM,\theta} = p(x)p_{TM}(y|x;\theta) \).

A.3 Bounding the first difference term

Now, we bound the first difference term in (3) as follows:

**Theorem 3.**

\[
E_{D_{CM}}[1_{p_{TM}(y|x;\theta) < 1/2}] - E_{D_{TM,\theta}}[1_{p_{TM}(y|x;\theta) < 1/2}] \leq \sqrt{1/2 \cdot KL(p_{CM}(y|x)||p_{TM}(y|x;\theta))}
\]

(4)

**Proof.** Let \( d_{TV}(p, q) \) be the total variation distance between two distributions \( p \) and \( q \). We have the following simple chain of inequalities:

\[
E_{D_{CM}}[1_{p_{TM}(y|x;\theta) < 1/2}] - E_{D_{TM,\theta}}[1_{p_{TM}(y|x;\theta) < 1/2}] \leq d_{TV}(D_{TM,\theta}, D_{CM}) \leq \sqrt{1/2 \cdot KL(D_{CM}||D_{TM,\theta})}
\]

(5)

(a)- follows from the definition of total variation distance. (b) follows from Pinsker’s inequality connecting KL-divergence and total variation distance. This completes the proof. \( \square \)
Theorem 4. MLE case:

\[
\text{KL}(p_{CM}(y|x)\|p_{TM}(y|x;\theta)) \leq E_{p(x)}[-(p_{CM}(+1|x)) \log(p_{TM}(+1|x;\theta))] + \\
E_{p(x)}[-p_{CM}(-1|x) \log(p_{TM}(-1|x;\theta))] \tag{6}
\]

ERM case (a):

\[
\text{KL}(p_{CM}(y|x)\|p_{TM}(y|x;\theta)) \leq E_{p(x)}[(p_{CM}(+1|x)\left[cr_1(x) + (p_{CM}(-1|x))cr_2(x)\right) \\
+ E_{p(x)}[\log(1 + e^{-cr_1(x) - cr_2(x)})]] \tag{7}
\]

Proof. We have the following chain of inequalities:

\[
\text{KL}(p_{CM}(y|x)\|p_{TM}(y|x;\theta)) = E_{p_{CM}}[\log p_{CM}(y|x)] + E_{p(x)}[-p_{CM}(+1|x) \log(p_{TM}(+1|x;\theta)) \\
- p_{CM}(-1|x) \log(p_{TM}(-1|x;\theta))] \\
\leq E_{p(x)}[-p_{CM}(+1|x) \log(p_{TM}(+1|x;\theta)) \\
- p_{CM}(-1|x) \log(p_{TM}(-1|x;\theta))] \tag{8}
\]

(a)- This is because \(\log(p_{CM}(\cdot)) \leq 0\). This proves the result for the MLE case. For the ERM case, we further bound using risk functions.

\[
\text{KL}(p_{CM}(y|x)\|p_{TM}(y|x;\theta)) = E_{p(x)}[-p_{CM}(+1|x) \log(p_{TM}(+1|x;\theta)) \\
- p_{CM}(-1|x) \log(p_{TM}(-1|x;\theta))] \\
= E_{p(x)}[p_{CM}(+1|x)cr_1(x)] + E_{p(x)}[p_{CM}(-1|x)cr_2(x)] \\
+ E_{p(x)}[\log(e^{-cr_1(x)} + e^{-cr_2(x)})] \\
\leq E_{p(x)}[(p_{CM}(+1|x))cr_1(x)] + E_{p(x)}[(p_{CM}(-1|x))cr_2(x) \\
- c \min(r_1(x), r_2(x)) + \log(1 + e^{-cr_1(x) - cr_2(x)})] \\
\leq E_{p(x)}[(p_{CM}(+1|x))cr_1(x)] + E_{p(x)}[(p_{CM}(-1|x))cr_2(x) \\
+ \log(1 + e^{-cr_1(x) - cr_2(x)})] \tag{9}
\]

The last inequality proves the ERM part of the theorem. \(\Box\)

A.4 Bounding the second term

ERM case (a): The second term in (3) can be expressed as follows:

\[
E_{D_{TM,\theta}}[1_{p_{TM}(y|x;\theta)\leq 1/2}] = E_{p(x)} \left[ \frac{e^{cr_1(x)}}{e^{cr_1(x)} + e^{cr_2(x)}} \right] \leq E_{p(x)}[e^{-cr_1(x) - cr_2(x)}] \tag{10}
\]
**MLE case:** We bound the second term in (3) as follows:

\[
\mathbb{E}_{D_{TM}, \theta}[1_{p_{TM}(y|x; \theta) \leq 1/2}] = \mathbb{E}_{p(x)} \left[ \min(p_{TM}(+1|x; \theta), p_{TM}(-1|x; \theta)) \right] \\
\leq \mathbb{E}_{p(x)} \left[ \frac{\min(p_{TM}(+1|x; \theta), p_{TM}(-1|x; \theta))}{\max(p_{TM}(+1|x; \theta), p_{TM}(-1|x; \theta))} \right] \\
\leq \mathbb{E}_{p(x)} \left[ e^{-|\log p_{TM}(-1|x; \theta) - \log p_{TM}(+1|x; \theta)|} \right] \quad (11)
\]

**A.5 Bounding the error term: Putting it together**

Therefore, we put everything together minimize the following upper bound on the square of the TM error with respect to the CM model as a function of \( \theta \).

**Proof of Theorem 2** ERM case (a): From (3), we have:

\[
\begin{align*}
(\mathbb{E}_{D_{CM}[1_{p_{TM}(y|x; \theta) \leq 1/2}]}^2 & \leq 2(\mathbb{E}_{D_{CM}[1_{p_{TM}(y|x; \theta) \leq 1/2}]} - \mathbb{E}_{D_{TM, \theta}[1_{p_{TM}(y|x; \theta) \leq 1/2}]}^2) \\
& + 2(\mathbb{E}_{D_{TM, \theta}[1_{p_{TM}(y|x; \theta) \leq 1/2}]}^2) \\
& \leq E_{p(x)} \left[ (p_{CM}(+1|x))c_1(x) + (p_{CM}(-1|x))c_2(x) \right] \\
& + E_{p(x)}[\log(1 + e^{-|c_1(x) - c_2(x)|})] + 2(E_{p(x)}[e^{-|c_1(x) - c_2(x)|}])^2 \\
& \leq E_{p(x)} \left[ (p_{CM}(+1|x))c_1(x) + (p_{CM}(-1|x))c_2(x) ight. \\
& + \log(1 + e^{-|c_1(x) - c_2(x)|}) + 2e^{-|c_1(x) - c_2(x)|}] \quad (12)
\end{align*}
\]

(a) - Jensen’s inequality on the convex function \( x^2 \). Similarly, one can show the result for the MLE case. \( \square \)

For ERM case (b), we provide the following analysis of the error of the target model assuming that the data is drawn according to the distribution \( D_{CM} = p(x)p_{CM}(y|x) \). Let \( y'(x) = \arg\max_y p_{CM}(y|x) \). Consider the the error of the target model in the ERM case:

\[
\begin{align*}
\mathbb{E}_{D_{CM}[1_{r(y,x, \theta) > r(-y,x, \theta)}}] &= \mathbb{E}_{x} \left[ \left( \frac{1}{2} + \frac{1}{2} - p_{CM}(y'(x)|x) \right) \cdot 1_{r(y'(x),x, \theta) > r(-y'(x),x, \theta)} + \\
\frac{1}{2} - \frac{1}{2} - p_{CM}(y'(x)|x) \right] \cdot 1_{r(-y'(x),x, \theta) > r(y'(x),x, \theta)} \\
& = \mathbb{E}_{x} \left[ \left( \frac{1}{2} + \frac{1}{2} - p_{CM}(y'(x)|x) \right) \cdot 1_{r(y'(x),x, \theta) > r(-y'(x),x, \theta)} + \\
\frac{1}{2} - \frac{1}{2} - p_{CM}(y'(x)|x) \right] \cdot 1_{r(-y'(x),x, \theta) > r(y'(x),x, \theta)} + \\
\frac{1}{2} - \frac{1}{2} - p_{CM}(y'(x)|x) \right] \\
& = \mathbb{E}_{x} \left[ \frac{1}{2} - \frac{1}{2} - p_{CM}(y'(x)|x) \right] \cdot 1_{r(y'(x),x, \theta) > r(-y'(x),x, \theta)} + \\
\mathbb{E}_{x} \left[ \frac{1}{2} - \frac{1}{2} - p_{CM}(y'(x)|x) \right] + \\
\mathbb{E}_{x} \left[ \frac{1}{2} - \frac{1}{2} - p_{CM}(y'(x)|x) \right] \quad (13)
\end{align*}
\]
During normal training, only the sequence of $y'(x)$ is given as a training label for the sample $x$ to the target model. However, a complex model can inform the target model of more information, i.e. $p_{CM}(y'(x)|x)$ (confidence of the complex model over the training labels). The second term in the right hand side of (13) is independent of the choice of $\theta$. This motivates an algorithm to minimize the first term in (13). This motivates the following heuristic:

Solve:

$$\min_{\theta} \frac{1}{m} \left[ \sum_{i=1}^{m} \left| 1 - p_{CM}(y'(x_i)|x_i) \right| r(y'(x_i), x_i, \theta) \right]$$  \hfill (14)

The above heuristic is motivated by the fact that normal training of a target model (through expected risk minimization) amounts to optimizing $\mathbb{E}_x \left[ 1_{r(y'(x), x, \theta) > r(-y'(x), x, \theta)} \right]$.

For the MLE model, replace $r(\cdot)$ by the negative log-likelihood to get an equivalent of the above heuristic.

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