EXISTENCE OF INFINITELY MANY MINIMAL HYPERSURFACES IN HIGHER-DIMENSIONAL CLOSED MANIFOLDS WITH GENERIC METRICS

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Abstract. In this paper, we show that a closed manifold $M^{n+1}$ ($n \geq 7$) endowed with a $C^\infty$-generic (Baire sense) metric contains infinitely many singular minimal hypersurfaces with optimal regularity. Moreover, for $2 \leq n \leq 6$, our argument also implies the denseness of the minimal hypersurfaces realizing min-max widths for generic metrics. This partially supports equidistribution of the minimal hypersurfaces realizing min-max widths conjectured by F.C. Marques, A. Neves and A. Song in [MNS19].

1. Introduction

In Riemannian geometry, the existence and regularity of minimal hypersurfaces is one of the central problems. In 1982, motivated by the existence results in $(n+1)$-dimensional closed manifolds by G.D. Birkhoff ([Bir17], $n = 1$), J. Pitts ([Pit81], $2 \leq n \leq 5$) and R. Schoen and L. Simon ([SS81], $n \geq 6$), S.-T. Yau proposed the conjecture of existence of infinitely many minimal surfaces in closed 3-dimensional Riemannian manifolds.

Conjecture 1.1 (Yau’s conjecture, [Yau82]). Any closed three-dimensional manifold must contain an infinite number of immersed minimal surfaces.

In [IMN18], K. Irie, F.C. Marques and A. Neves, using the Weyl law [LMN18] for volume spectra by Y. Liokumovich and the last two named authors, proved a stronger version of Yau’s conjecture in the generic case.

Theorem 1.1 (Density of minimal hypersurfaces in the generic case, [IMN18]). Let $M^{n+1}$ be a closed manifold of dimension $n + 1$, with $3 \leq n + 1 \leq 7$. Then for a $C^\infty$-generic Riemannian metric $g$ on $M$, the union of all closed, smooth, embedded minimal hypersurfaces is dense.

Later, in [MNS19], F.C. Marques, A. Neves and A. Song gave a quantitative description of the density, i.e., the equidistribution of a sequence of minimal hypersurfaces under the same condition.

The Yau’s conjecture for $2 \leq n \leq 6$ for general $C^\infty$ metrics was finally resolved by A. Song [Son18] using the methods developed by F.C. Marques and A. Neves in [MN17].

Recently, X. Zhou [Zho20] confirmed Marques-Neves multiplicity one conjecture for bumpy metrics, which combined with work of Marques-Neves [MN21] on the Morse index leads to:

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Theorem 1.2 ([MN21], Theorem 8.4). Let $g$ be a $C^\infty$-generic (bumpy) metric on a closed manifold $M^{n+1}$, $3 \leq (n + 1) \leq 7$. For each $k \in \mathbb{N}$, there exists a smooth, closed, embedded, multiplicity one, two-sided, minimal hypersurface $\Sigma_k$ such that

$$\omega_k(M, g) = \text{area}_g(\Sigma_k), \quad \text{index}(\Sigma_k) = k,$$

and

$$\lim_{k \to \infty} \frac{\text{area}_g(\Sigma_k)}{k^{\frac{n-1}{n}}} = a(n)\text{vol}(M, g)^{\frac{1}{n}},$$

where $\omega_k(M, g)$ is the volume spectrum, more precisely, the min-max $k$-width, and $a(n) > 0$ is a dimensional constant in Weyl law.

Note that most of the results above were obtained in the Almgren-Pitts min-max setting (Zhou’s result on the multiplicity one conjecture was based on a new regularization of the area functional in Caccioppoli min-max setting developed by him and J. Zhu [ZZ20]). In the Allen-Cahn min-max setting, P. Gaspar and M.A.M. Guaraco [GG19] and O. Chodosh and C. Mantoulidis [CM20] $(n = 2)$ gave similar results. In particular, O. Chodosh and C. Mantoulidis proved the multiplicity one conjecture in 3-manifolds before Zhou’s result. Most of the results above rely on two important ingredients, the upper bounds for Morse index and the denseness of bumpy metrics ([Whi91]). However, they could not be easily generalized in higher dimension $(n \geq 7)$ directly from the literature above, especially in the Almgren-Pitts setting. Thus, Yau’s conjecture is still left open in higher-dimensional closed manifolds.

In this paper, we will confirm Yau’s conjecture in higher dimension $(n \geq 7)$ for a closed manifold $M$ with a $C^\infty$-generic metric $g$. Due to the existence of nontrivial singularities in area-minimizing currents in higher dimension, we can at best expect that the minimal hypersurfaces have optimal regularity, i.e., $\text{codim}(\text{Sing}) \geq 7$.

Theorem 1.3 (Main Theorem). Given a closed manifold $M^{n+1}(n \geq 7)$, there exists a (Baire sense) generic subset of $C^\infty$ metrics such that $M$ endowed with any one of those metrics contains infinitely many singular minimal hypersurfaces with optimal regularity.

Here is the outline of the proof of our main theorem.

First, we establish the compactness for the Almgren-Pitts Realization $\mathcal{APR}(\Pi)$ of the min-max width of an $m$-parameter homotopy family $\Pi$, where $\mathcal{APR}(\Pi)$ is a nonempty subset of minimal hypersurfaces with optimal regularity, volume $L(\Pi)$ and certain stability property.

Then, by showing that the $p$-width could be achieved by the min-max width of an $m$-parameter homotopy family $\mathcal{P}_{p,m}$, we also have the compactness for the Almgren-Pitts Realization $\mathcal{APR}_p$ of $p$-width.

Finally, the proof of the main theorem follows the idea of Proposition 3.1 in [IMN18] with a tricky modification. Roughly speaking, to overcome the difficulty of the lack of bumpy metrics, we will apply the compactness results to obtain the openness of “good” metrics and use $\mathcal{M}_f$ (the set of metrics where Yau’s conjecture fails) to be the starting point of metric perturbation. If we divide the metric space into two parts $\mathcal{O} = \text{Int}(\mathcal{M}_f)$ and $\mathcal{O}^c$, then on the one hand, in $\mathcal{O}$, with two ingredients mentioned above, we can show that $\mathcal{M}_f \cap \mathcal{O}$ is meagre. On the other hand, by definition, it is clear that $\mathcal{M}_f \cap \mathcal{O}^c$ is nowhere dense. In summary, $\mathcal{M}_f$ is meagre.
In addition, together with White’s structure theorem on minimal submanifolds [Whi17], our proof above also implies the generic denseness of min-max minimal hypersurfaces.

**Theorem 1.4** (a.k.a. Corollary 4.1). For a closed manifold $M^{n+1}$ ($2 \leq n \leq 6$) with a $C^\infty$-generic metric $g$, min-max minimal hypersurfaces are dense.

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### 2. Preliminaries

#### 2.1. Basic Notations in the Almgren-Pitts Min-max Theory

In this paper, we will use $Z_n(M^{n+1}, Z_2)$ to denote the space of modulo two $n$-dimensional flat cycles endowed with the flat topology. In Almgren’s thesis [Alm62], he gave a natural isomorphism

$$
\pi_k(Z_n(M^{n+1}, Z_2), 0) = H_{n+k}(M^{n+1}, Z_2),
$$

and later, it was shown that $Z_n(M^{n+1}, Z_2)$ is weakly homotopic to $\mathbb{RP}^\infty$ ([MN16b], Section 4). We denote the generator of $H^1(Z_n(M^{n+1}, Z_2); Z_2) = Z_2$ by $\bar{\lambda}$.

In the cycles space, we can define $M$ to be the mass norm or mass functional (See [Fed96], Section 4.2). The nontrivial topology of $Z_n(M^{n+1}, Z_2)$ indicates that the min-max theory for the mass functional can be developed in it.

Let $X$ be a finite dimensional simplicial complex and we call a map $\Phi$ a $p$-sweepout if it is a continuous map $\Phi : X \rightarrow Z_n(M^{n+1}, Z_2)$ such that $\Phi^*(\bar{\lambda}^p) \neq 0$.

The $p$-admissible set $P_p = P_p(M, g)$ is the set of all $p$-sweepouts $\Phi$ that have no mass concentration (See [MN14] Section 4), i.e.

$$
\lim_{r \rightarrow 0} \sup \{ M(\Phi(x) \cap B_r(y)) : x \in X, y \in M \} = 0.
$$

The $p$-width $\omega_p(M, g)$ can be defined by $\inf_{\Phi \in P_p} \sup \{ M(\Phi(x)) : x \in dmn(\Phi) \}$ ([Gro03], [Gut09], [MN17], [LMN18]).

We also define a min-max sequence $S = \{ \Phi_i \}_{i \in \mathbb{N}}$ to be a sequence in $P_p$ satisfying

$$
\lim_{i \rightarrow \infty} \sup_{x \in X_i} M(\Phi_i(x)) = \omega_p.
$$

The critical set of $S$ is

$$
C(S) = \{ V ||V||(M) = \omega_p \text{ and } V = \lim_{j} |\Phi_{ij}(x_j)| \text{ for some sequence } \{ x_j \in X_{ij} \} \}.
$$

Note that the domains in $P_p$ could be different. In order to get the compactness, we need to make some restriction on the domains.

For convenience, we shall use some notions of cell complexes from [Pit81].

**Definition 2.1** ([Pit81], 4.1(1)).

- For $n \in \mathbb{N}^+$, let $I(1,n)$ denote the cell complex on the unit interval $I$ whose 1-cells are the intervals $[0, 1 \cdot 3^{-n}], [1 \cdot 3^{-n}, 2 \cdot 3^{-n}], \ldots, [1 - 3^{-n}, 1]$, and whose 0-cells are the endpoints $[0], [3^{-n}], [2 \cdot 3^{-n}], \ldots, [1]$. 


• For $n, m \in \mathbb{N}^+$, $I(m, n) = I(1, n)^m = I(1, n) \otimes I(1, n) \otimes \cdots \otimes I(1, n)$ is the cell complex on $I^m$. In addition, $I(m, n)_0$ denote the set of all 0-cells in $I(m, n)$.

Now we can give a definition of an $m$-parameter homotopy family in general, which need not be a subset of some $P_p$.

**Definition 2.2.** We call a subset $\Pi$ of continuous maps from finite dimensional simplicial complexes to $\mathbb{Z}_n(M^{n+1}, \mathbb{Z}_2)$ a (continuous) $m$-parameter homotopy family if the following properties hold.

- For any $\Phi \in \Pi$, $X = \text{dmn}(\Phi)$ is a subcomplex of $I(m, k)$ for some $k \in \mathbb{N}^+$ and $\Phi$ has no mass concentration.
- For any $\Phi \in \Pi$, every continuous $\Phi' : \text{dmn}(\Phi) \to \mathbb{Z}_n(M^{n+1}, \mathbb{Z}_2)$ homotopic to $\Phi$ in the flat topology also lies in $\Pi$, provided that $\Phi'$ has no mass concentration.

**Remark 2.1.** In Pitts’ original proof [Pit81], he considered discrete sweepouts in $[I^m, \mathbb{Z}_n(M; M; \mathbb{Z}_2)]^\#$. Fortunately, due to the remarkable interpolation results by F.C. Marques and A. Neves [MN14, Section 13 and 14], the discrete settings and the continuous settings are interchangeable in some sense. Thus, the $m$-parameter homotopy family defined here will preserve most of the properties that the discrete homotopy families have.

Similar to $p$-width and the min-max sequence for $p$-width, we can also define a min-max invariant for the $m$-parameter homotopy family $\Pi$,

$$L(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \{M(\Phi(x))\},$$

and a min-max sequence $S = \{\Phi_i\} \subset \Pi$,

$$\limsup_{i \to \infty} \sup_{x \in X_i = \text{dmn}(\Phi_i)} M(\Phi_i(x)) = L(\Pi).$$

Now the critical set of $S$ is

$$C(S) = \{V \mid V\|(M) = L(\Pi) \text{ and } V = \lim_{j \to \infty} |\Phi_{ij}(x_j)| \text{ for some sequence } \{x_j \in X_{ij}\}\}.$$

2.2. Singular Minimal Hypersurfaces with Optimal Regularity and Compactness with Stability Condition.

**Definition 2.3.** We call a varifold $V$ in $M^{n+1}$ a singular minimal hypersurface with optimal regularity, if it is an $n$-dimensional stationary integral varifold and its singular part has dimension $\dim(\text{Sing}(V)) \leq n - 7$.

By Allard compactness (See [All72], [Sim84]), a sequence $\{V_j\}$ of singular minimal hypersurfaces with optimal regularity with bounded volume (up to a subsequence) will converge to an $n$-dimensional stationary integral varifold in the varifold sense. However, without extra information of the sequence, we could not obtain further regularity of the limit varifold.

If we further assume that $\{V_j\}$ is stable in some open subset $U$, then from the result of [SS81] (See also [Wic14]), we have the following compactness consequence.

**Proposition 2.1.** Given a sequence of singular minimal hypersurfaces with optimal regularity $\{V_j\}$ in a smooth closed manifold $(M, g)$ with uniformly bounded volume and converging to an $n$-dimensional stationary integral varifold $V$ in the varifold
sense, suppose that each $V_j$ is stable (See the definition in [SS81]) in an open subset $U$ of $M$, then $V$ is stable in $U$ and of optimal regularity in $U$.

**Remark 2.2.** This is also true when $V_j$ is defined on $(M, g_j)$ and $g_j$ converges to $g$ in $C^3$ ([SS81], Theorem 2).

Recently, A. Dey [Dev19] generalized this result with the assumption that the $p$-th eigenvalue is uniformly bounded from below.

3. Minimal Hypersurfaces from Almgren-Pitts Min-max Construction

In this section, we will prove some properties of singular minimal hypersurfaces obtained from Almgren-Pitts Min-max construction, especially realizations of $p$-widths and their compactness.

3.1. Almgren-Pitts Realizations of $m$-parameter Homotopy Families and Their Compactness. One of the novelties in Pitts’ monograph is the application of the combinatorial arguments, Proposition 4.9 and Theorem 4.10 in [Pit81], from which he could prove the regularity of varifolds in a critical set. Here, we shall show that these arguments could imply more properties of varifolds in a critical set.

First, we adapt Proposition 4.9 in [Pit81] to our current setup and improve the constant slightly.

In the following, $A(p, s, r)$ denotes an open annulus in $M$ centered at $p$ with inner radius $s$ and outer radius $r$, provided that $s < r$ and $r$ is no greater than the injective radius of $M$ at $p$. $\bar{A}(p, s, r)$ is simply its closure.

**Lemma 3.1.** Suppose that $X$ is a subcomplex of some $I(m, n)$. If every cell $\sigma$ of $X$ is associated with a point $p_\sigma \in M$ and a finite set $A(\sigma) = \{\bar{A}(p_\sigma, s_j, r_j)\}_{j=1,2,\ldots,5^m}$, where $r_j > s_j > 10r_{j+1} > 0$ $(r_1 < \text{inj}_M/2)$, then we can find a function defined on the set of cells of $X$ (denoted by $X(n)$),

$$\alpha : X(n) \to \bigcup_{\sigma \in X(n)} A(\sigma),$$

such that $\alpha(\sigma) \in A(\sigma)$ and $\alpha(\sigma) \cap \alpha(\tau) = \emptyset$ whenever $\sigma \neq \tau$ and $\sigma, \tau \subset \gamma$ for some cell $\gamma \in X(n)$.

**Proof.** We shall define $\alpha$ inductively.

Let $D \subset X(n)$ where $\alpha$ is already defined, and thus, at the beginning, $D = \emptyset$.

Whenever $D \neq X(n)$, we can take an annulus with the smallest outer radius in $\bigcup_{\sigma \in X(n) \setminus D} A(\sigma)$, say, $\bar{A}(p_{\sigma_0}, s, r) \subset A(\sigma_0)$ for some $\sigma_0 \in X(n) \setminus D$. Now, $\sigma_0$ can be added into $D$ as $\alpha(\sigma_0)$ is defined to be $\bar{A}(p_{\sigma_0}, s, r)$. Then, for each $\tau \in X(n) \setminus D$ which is a face of some cell $\gamma \in X(n)$ with $\sigma_0 \subset \gamma$, we remove $\bar{A}(p_\tau, s, r) \subset A(\tau)$ from the set $A(\tau)$ if $\bar{A}(p_\tau, s, r) \cap \bar{A}(p_{\sigma_0}, s, r) \neq \emptyset$. Since $\bar{A}(p_{\sigma_0}, s, r)$ is of the smallest outer radius, by the radius relations of the annuli in $A(\tau)$, it is easy to check that we at most remove one annulus from each $A(\tau)$.

Note that $X$ is a subcomplex of $I(m, n)$, so for any cell $\tau \in X(n)$, we can find at most $5^m - 1$ other cells such that any one of them and $\tau$ are both faces of some cell $\gamma \in X(n)$. In other words, there will be at most $5^m - 1$ annuli being removed from each $A(\tau)$ and therefore, $A(\tau)$ will never be empty. This, together with the fact that there are only finitely many cells of $X$, guarantees that the procedure above could give the definition of $\alpha$ on the whole $X(n)$ as desired. 

}\hfill \square
Now, we can state our main lemma on the properties of varifolds in a critical set.

**Lemma 3.2.** Suppose $S = \{\Phi_i\}$ is a pulled-tight critical sequence for a $m$-parameter homotopy family $\Pi$ with $L(\Pi) > 0$. Then there exists a varifold $V \in C(S)$ such that for any $I_m = 5^m$ concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with $r_j > 0$, $s_j > 0$, $r_j - 2s_j > 10(r_{j+1} + 2s_{j+1})$, $r_{t_m} - 2s_{t_m} > 0$ and $r_1 < \text{inj}_M/2$, $V$ is almost minimizing (See Definitions 3.1 in [Φ81]) in at least one of the annuli. Therefore, by Theorem 3.3 in [Φ81], $V$ has property $(m)$:

For any $I_m$ concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with $r_j > 0$, $s_j > 0$, $r_j - 2s_j > 10(r_{j+1} + 2s_{j+1})$, $r_{t_m} - 2s_{t_m} > 0$ and $r_1 < \text{inj}_M/2$, $V$ is stable in at least one of the annuli.

**Proof.** Suppose not, and then for each $V$ in $C(S)$, there is $p^V \in M$ such that there exist $5^m$ concentric annuli $A(p^V, r_j - s_j', r_j' + s_j')$ with the conditions mentioned above and $V$ is not almost minimizing in any one of the annuli.

As explained in Remark 2.1, one could use a family of discrete sweepouts to approximate $S$. Roughly speaking, for each $\Phi_i \in S$ with $X_i = \text{d}mn(\Phi_i)$ where $X_i \subset I(m, n_i)$ is a subcomplex, we can take $N_i > n_i$ large enough and define

$$\psi_i : X_i \cap I(m, N_i) \rightarrow Z_\alpha(M; \mathbb{Z}_2),$$

which approximates $\Phi_i$. Note that the discrete sweepouts $S_\# = \{\psi_i\}$ satisfy that $C(S_\#) = C(S)$ and the Almgren extension $\Psi_i$ of $\psi_i$ is homotopic to $\Phi_i$ in the flat topology. Interested readers might refer to the proof for Theorem 3.8 in [MN16a] for more details. Although they require that each $\Phi_i$ is continuous with respect to the $F$ norm (See the definition in [Φ81] 2.1(20)), the same proof works in our setup as well.

For convenience, in the following, we may assume that $X_i$ itself is a subcomplex of $I(m, N_i)$, since one could always refine $X_i$ in a canonical way.

Now following the original proof of Theorem 4.10 in [Φ81], for $i$ large enough, we can assign to each face $\sigma \in X_i(N_i)$ a set $A(\sigma) = \{A(p^V, r_j - s_j', r_j' + s_j')\}_{j=1,...,5^m}$ for some $V_\sigma$ associated to $\sigma$, where $V_\sigma$ is one of the $\{V_1, ..., V_{5^m}\}$ therein. By Lemma 3.1, we could define $\alpha$ on $X_i(N_i)$. Therefore, the existence of a homotopic family of discrete sweepouts $S_\#^* = \{\psi_i^*\}$ such that $L(S_\#^*) < L(S_\#) = L(S)$ is just verbatim.

The Almgren extension $\Psi_i^*$ of $\psi_i^*$ is homotopic to $\Psi_i$ and thus to $\Phi_i$ which implies that $\{\Psi_i^*\} \subset \Pi$. As long as each $N_i$ is chosen large enough, $L(\{\Psi_i^*\}) = L(S_\#^*) < L(S)$, which gives a contradiction to the choice of $S$. Hence, the conclusion holds.

By applying Theorem 4 in [SS81], the varifold obtained in the lemma above is indeed a singular minimal hypersurface with optimal regularity.

**Definition 3.1.** We define **Almgren-Pitts Realization** of the $m$-parameter homotopy family $\Pi$, denoted by $\text{APR}(\Pi)$, to be the nonempty set of varifolds $V$ satisfying

- $\|V\|(M) = L(\Pi)$;
- $V$ is a singular minimal hypersurface with optimal regularity;
- $V$ has property $(m)$.

Now, we can show that $\text{APR}$ has following compactness property.
Proposition 3.1 (Compactness of APR). Given a sequence of m-parameter homotopy families $\Pi_i$ with $0 < \inf L(\Pi_i) \leq \sup L(\Pi_i) < \infty$ and $V_i \in APR(\Pi_i)$, there is a subsequence (still denoted using index $i$) $\Pi_i$ such that

\begin{equation}
\lim_{i \to \infty} L(\Pi_i) = L \in \mathbb{R}^+, \tag{12}
\end{equation}

and

\begin{equation}
V_i \rightharpoonup V, \tag{13}
\end{equation}

in the varifold sense. Moreover, $\operatorname{spt}(V)$ is also a singular minimal hypersurface with optimal regularity and property $(m)$.

Proof. By Allard compactness, we only need to check that $V$ has property $(m)$ and optimal regularity.

Firstly, we show that $V$ has property $(m)$.

Suppose not, and then we can find a set of $I_m$ concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with the conditions mentioned in property $(m)$, such that $V$ is not stable in any annulus. By Proposition 2.1, we know that for each $j$, there is a positive integer $i_j > 0$ such that when $i > i_j$, $V_i$ is not stable in each annulus. If we take $i_0 = \max_{j=1, \ldots, l_m} \{i_j\}$, then $V_i$ is not stable in any one of $\{A(p, r_j - s_j, r_j + s_j)\}$ provided that $i > i_0$.

This contradicts to the definition of APR$(\Pi_i)$ that $V_i$ has property $(m)$.

Next, to show the optimal regularity of $V$, note that the same argument above together with Proposition 2.1 also implies that for any $p \in M$ and any $I_m$ concentric annuli $\{A(p, r_j - s_j, r_j + s_j)\}$ with the same properties mentioned above, $V$ is stable and of optimal regularity inside at least one of them. Thus, there exists a constant $r_p \in (0, \text{im} M/2)$ depending only on $p$ such that $V$ is stable and of optimal regularity in $A(p, s, r_p)$ for any $s \in (0, r_p)$ (See Theorem 4.10 in [Pit81]).

By Theorem 3.1 in [Wic14] and the Remark (3) before it, we know that $V$ is of optimal regularity in the open ball $B(p, r_p)$. Since $M$ is compact, taking a finite open cover, one can easily show that $V$ is of optimal regularity in $M$. \hfill $\square$

3.2. Almgren-Pitts Realizations of $(p, m)$-width and $p$-width and Their Compactness. In [Xu18], G. Xu defines $(p, m)$-width to be

\begin{equation}
\omega_{p,m}(M, g) = \inf_{\Phi \in \mathcal{P}_{p,m}} \max_{x \in \dimn(\Phi)} M(\Phi(x)), \tag{14}
\end{equation}

where $\mathcal{P}_{p,m}$ is the set of mass-concentration-free sweepouts from a subcomplex of some $I(m, k)$ into $\mathcal{Z}_n(M; \mathbb{Z}_2)$ detecting $\hat{\lambda}$. And he also proved that when $m \geq 2p + 1$, $\mathcal{P}_{p,m} \neq \emptyset$ and the $(p, m)$-width can be realized by a singular minimal hypersurface with optimal regularity.

Note that the only difference between $p$-width and $(p, m)$-width is the domain of the sweepouts. Since $\mathcal{P}_{p,m}$ is an $m$-parameter homotopy family, the realization of $(p, m)$-width is just a corollary of the compactness property (Proposition 3.1).

Corollary 3.1 ([Xu18], Theorem 1.12). For $m \geq 2p + 1$, there is a varifold $V$ such that $\|V\|(M) = \omega_{p,m}(M)$ and $\operatorname{spt}(V)$ is a singular minimal hypersurface with optimal regularity.

It is obvious that

\begin{equation}
\omega_{p,2p+1}(M) \geq \omega_{p,2p+2}(M) \geq \cdots \geq \omega_{p,m}(M) \geq \cdots \geq \omega_p(M). \tag{15}
\end{equation}
The question is whether for some $m \geq 2p + 1$, we can have the equality between $\omega_{p,m}(M)$ and $\omega_p(M)$. Here, we would like to confirm this by a simple argument.

**Proposition 3.2.** $\omega_{p,2p+1}(M, g) = \omega_p(M, g)$.

*Proof.* Given a min-max sequence $S = \{\Phi_i\}_{i \in \mathbb{N}} \subset \mathcal{P}_p$ for $\omega_p$, denote $X_i^{(p)}$ as the $p$-dimensional skeleton of $X_i = \text{dnn}(\Phi_i)$, and following the proof in Proposition 2.2 in [IMN18], we have

\[
H^l(X_i, X_i^{(p)}; \mathbb{Z}_2) = 0 \quad (l \leq p),
\]

and the exact sequence

\[
\cdots \rightarrow H^p(X_i, X_i^{(p)}; \mathbb{Z}_2) \rightarrow H^p(X_i; \mathbb{Z}_2) \rightarrow H^p(X_i^{(p)}; \mathbb{Z}_2) \rightarrow H^{p+1}(X_i, X_i^{(p)}; \mathbb{Z}_2) \rightarrow \cdots
\]

The pullback map from $H^p(X_i; \mathbb{Z}_2)$ to $H^p(X_i^{(p)}; \mathbb{Z}_2)$ is injective so $\Phi_i|_{X_i^{(p)}} \in \mathcal{P}_p$.

Since

\[
\omega_p(M, g) \leq \limsup_{i \to \infty} \text{sup} \{M(\Phi_i(x))| x \in X_i^{(p)}\} \leq \limsup_{i \to \infty} \text{sup} \{M(\Phi_i(x))| x \in X_i\} = \omega_p(M, g),
\]

we also have that

\[
\limsup_{i \to \infty} \text{sup} \{M(\Phi_i(x))| x \in X_i^{(p)}\} = \omega_p(M, g).
\]

As a consequence, we may assume that in the min-max sequence $S$, $\text{dnn}(\Phi_i) = X_i^{(p)}$.

Now that $X_i^{(p)}$ is a finite $p$-dimensional simplicial complex, and thus is homeomorphic to the support of some cubical subcomplex of some $N$-dimensional cube $I^N$ (Chapter 4 [BP02]). And we can take a canonical projection (a closed map) mapping $X_i^{(p)}$ into $I^{2p+1}$ within 1/2 of the interior. By the general position theorem for maps, Theorem 5.3 in [RS82], with $P_0 = \emptyset$ and $P = X_i^{(p)}$ therein, there exists a piecewise-linear embedding map from $X_i^{(p)}$ to some triangulation $T$ of $I^{2p+1}$ whose image has a distance, say, at least 1/6 to $\partial I^{2p+1}$.

Next, we can “thicken” $X_i^{(p)}$ in $I^{2p+1}$ to obtain a subcomplex $Y_i$ of some $I(2p + 1, k)$ such that $X_i^{(p)}$ is a retract of $Y_i$, which induces a map $\Psi_i : Y_i \rightarrow Z_n(M; \mathbb{Z}_2)$ by $\Psi_i(y) := \Phi_i(r(y))$. Moreover, $(\Psi_i)^*(\lambda^p) = r^* \circ (\Phi_i)^*(\lambda^p) \neq 0$, since $i^* \circ r^* = \text{id}^*$ implies that $r^*$ is an injection. It is also easy to see that $\max_{y \in Y_i}\{M(\Psi_i(y))\} = \max_{x \in X_i}\{M(\Phi_i(x))\}$.

Indeed, there exists a regular neighborhood $Z_i$ of $X_i^{(p)}$ in $T'$, a refinement of $T$ ([RS82] P.33). Moreover, by Corollary 3.30 in [RS82], we also know that $X_i^{(p)}$ is a deformation retract of $Z_i$, and thus, we have a retraction $r : Z_i \rightarrow X_i^{(p)}$. Now, since $d = \text{dist}(X_i^{(p)}, \partial Z_i) > 0$, we can take a large enough integer $k = k(p, d)$ such that the subcomplex $Y_i \subset I(2p + 1, k)$ with

\[
Y_i(k) = \{\alpha \in I(2p + 1, k)| \exists \beta \in I(2p + 1, k), \text{ s.t. } \alpha < \beta, \beta \cap X_i^{(p)} \neq \emptyset\},
\]

satisfies that $X_i^{(p)} \subset Y_i \subset Z_i$. Apparently, $r|_{Y_i}$ is also a retraction from $Y_i$ to $X_i^{(p)}$, which confirms our assertion.

In summary, $\omega_p(M, g)$ can be achieved by the sequence $\{\Psi_i\} \subset \mathcal{P}_{p,2p+1}$, so we have $\omega_p(M, g) \geq \omega_{p,2p+1}(M, g)$ and hence $\omega_p(M, g) = \omega_{p,2p+1}(M, g)$. □
Corollary 3.2 (Realization of $p$-width). Each $p$-width can be realized by a singular minimal hypersurface with optimal regularity and moreover, with property $(2p+1)$.

Remark 3.1. When $2 \leq n \leq 6$, this has been proved ([IMN18], Proposition 2.2), where they used the upper bound of Morse index of minimal hypersurfaces from min-max construction [MN16a]. However, when $n \geq 7$, without an adequate alternative of bumpy metrics defined in the singular setting, the technique in [MN16a] to make a critical sequence bypass all minimal hypersurface with large Morse index could not be applied directly. Thus, the compactness result using [Sha17] is still open.

Even worse, up to the author’s knowledge, it is still open whether the minimal hypersurfaces from min-max construction has finite Morse index.

Definition 3.2. We define the Almgren-Pitts realizations of $p$-width $APR_p(M,g)$ to be $APR(P_p,2p+1)$.

Now we have the following compactness for varying metrics.

Proposition 3.3 (Compactness of $APR_p$ for Varying Metrics). Given a smooth closed manifold $M$ and $C^\infty$ metrics $\{g_i\}$ and $g$ such that

$$\|V\|_i(M) = \omega_p(M,g),$$

and $V_i \in APR_p(M,g_i)$ for some $p > 0$, there is a subsequence of $\{V_i\}$ (still denoted by $\{V_i\}$) such that

$$V_i \rightharpoonup V,$$

in the varifold sense. Moreover, $V \in APR_p(M,g)$.

Proof. From the continuity of $p$-width ([IMN18], Lemma 2.1), we have that $\|V\|_i(M) = \omega_p(M,g)$. With Remark 2.2, the proof that $V$ has optimal regularity and property $(2p+1)$ is simply verbatim of the proof of Proposition 3.1. □

4. Proof of Main Theorem

For any open subset $U \subset M$, we define

$$\mathcal{M}_{U,p} := \{g \in \Gamma_{\infty}(M) | \forall V \in APR_p(M,g), \|V\|(U) > 0\},$$

where $\Gamma_{\infty}(M)$ is the set of all smooth Riemannian metrics on $M$ and let

$$\mathcal{M}_U := \bigcup_{p=1}^{\infty} \mathcal{M}_{U,p}.$$

Proposition 4.1. $\mathcal{M}_{U,p}$ is an open subset of $\Gamma_{\infty}(M)$ for any open subset $U$, and so is $\mathcal{M}_U$.

Proof. Given $g_0 \in \mathcal{M}_{U,p}$, we would like to show that there is an $\delta > 0$ such that $B_\delta(g_0,C^3) \cap \Gamma_{\infty}(M) \subset \mathcal{M}_{U,p}$.

Suppose not, there will be a sequence $g_i \in \Gamma_{\infty}(M)$ such that $g_i \rightharpoonup g_0$ but $g_i \notin \mathcal{M}_{U,p}$. Therefore, we can choose a sequence $\{V_i\}$ such that $V_i \in APR_p(M,g_i)$ but $V_i(U) = 0$. From Proposition 3.3, up to a subsequence,

$$V_i \rightharpoonup V,$$

where $V \in APR_p(M,g_0)$.

Since $U$ is open, $\|V\|(U) \leq \lim_{i \to \infty} \|V_i\|(U) = 0$ which gives a contradiction. □
Now we define $\mathcal{M}_f$ to be the set of metrics on $M$ where there are only finitely many singular minimal hypersurfaces of optimal regularity w.r.t. that metric.

**Lemma 4.1** (Key Lemma). For any open subset $O$ of $\Gamma_\infty$, if $\mathcal{M}_f$ is dense in $O$, then $\mathcal{M}_f \cap O$ is both open and dense in $O$ as well. Thus, $\mathcal{M}_f \cap O$ is a meagre set inside $O$.

**Remark 4.1.** As we will see in the proof, $\mathcal{M}_f$ plays the same role as bumpy metrics in the proof of Proposition 3.1 in [IMN18].

**Proof.** Fix $U$ as an open subset of $M$. For any $g$ in $O$, from the denseness of $\mathcal{M}_f$, there is a $g' \in \mathcal{M}_f$ such that $g'$ is arbitrarily close to $g$. Now, if $g' \in \mathcal{M}_U$ then we are done.

Suppose that $g' \notin \mathcal{M}_U$. We can follow the proof of Proposition 3.1 in [IMN18], since now the set

$$\mathcal{C} = \{ \sum_{j=1}^N m_j \text{vol}_{g'}(\Sigma_j) : N \in \mathbb{N}, \{m_j\}_{j=1}^N \subset \mathbb{N}, \{\Sigma_j\}_{j=1}^N \text{ are open, } \Sigma, \text{ singular minimal hypersurfaces with optimal regularity} \},$$

is countable and thus has empty interior.

Let $h$ be a smooth nonnegative function with spt($h$) $\subset U$ and $h(x) > 0$ for some $x \in U$. Let $g'(t) = (1 + th)g'$. Since $O$ is open, there is a $t_0 > 0$ such that $\{g'(t) : 0 \leq t \leq t_0\} \subset O$. Moreover, using the same argument in Proposition 3.1 in [IMN18], there exists a $t_1 \in (0, t_0]$ arbitrarily small and $p = p(t_0) \in \mathbb{N}$ such that $\omega_p(M, g'(t_1)) > \omega_p(M, g')$ and $\omega_p(M, g'(t_1)) \notin \mathcal{C}$. Now it suffices to show that $g'(t_1) \in \mathcal{M}_U$.

Suppose not, we can find $V \subset \mathcal{APR}_p(M, g'(t_1))$ such that $\|V\|(U) = 0$. Note that $g'(t_1) = g'$ outside $U$ so we have

$$\|V\|(M) = \sum_{j=1}^N m_j \text{vol}_{g'}(\Sigma_j) = \sum_{j=1}^N m_j \text{vol}_{g'}(\Sigma_j) \in \mathcal{C},$$

where $\{\Sigma_j\}$ is a finite set of singular minimal hypersurfaces with optimal regularity with respect to both $g'(t_1)$ and $g'$. This gives a contradiction.

Let $\{U_i\}$ be a countable basis of $M$, then $\mathcal{M} = \bigcap_{i} \mathcal{M}_{U_i}$ is of second Baire category in $O$ so $\mathcal{M} \cap O \subset \mathcal{M}^C \cap O$ is a meagre set. □

**Proof of Main Theorem.** Let $O = \text{Int}(\overline{\mathcal{M}_f})$ and it is easy to see that $\mathcal{M}_f \subset (\mathcal{M}_f \cap O) \cup \partial(\overline{\mathcal{M}_f})$. From Lemma 4.1, we know that $\mathcal{M}_f \cap O$ is meagre. Since $\partial(\overline{\mathcal{M}_f})$ is nowhere dense, $\mathcal{M}_f$ is also meagre. □

**Remark 4.2.** In the Key Lemma 4.1, we only use the fact that $\mathcal{C}$ is a set with empty interior. Thus, if $\mathcal{M}_{ei}$ is the set of metrics where $\mathcal{C}$ has empty interior and $\mathcal{O}_{ei} = \text{Int}(\overline{\mathcal{M}_{ei}})$, we also have that $\mathcal{M}$ is of second Baire category in $\mathcal{O}_{ei}$. As a consequence, we have the denseness of singular minimal hypersurfaces with optimal regularity in generic metrics inside $\mathcal{O}_{ei}$.

**Corollary 4.1.** For a closed manifold $M^{n+1}(2 \leq n \leq 6)$ with a $C^\infty$-generic metric $g$, the union of minimal hypersurfaces in $\mathcal{APR}(M, g) := \bigcup_{p=1}^\infty \mathcal{APR}_p(M, g)$, i.e., the minimal hypersurfaces realizing min-max widths, is a dense subset of $M$. 

**Proof.** By Theorem 2.7 in [Whi17] (an analogue could be referred to Theorem 9 in [ACS17]), the set of bumpy metrics is generic in $\Gamma_\infty(M)$. It follows from Sharp’s compactness theorem ([Sha17] Theorem 2.3 and Remark 2.4) that the bumpy metric belongs to $\mathcal{M}_{ei}$ and therefore, $\mathcal{M}_{ei}$ is also generic and thus dense in $\Gamma_\infty(M)$. From the remark above, we know that the Key Lemma leads to the conclusion. □

Parallel to the existence of bumpy metrics when $2 \leq n \leq 6$, we have the following conjecture.

**Conjecture 4.1.** $\mathcal{M}_{ei}$ is dense in $\Gamma_\infty(M)$.

**Corollary 4.2.** If Conjecture 4.1 above holds, singular minimal hypersurfaces with optimal regularity in $\mathcal{M}$ with generic metrics are dense.

In particular, let $\mathcal{M}_c$ be the set of the metrics where there are only countably many singular minimal hypersurfaces of optimal regularity w.r.t. that metric and then the following conjecture would imply Conjecture 4.1.

**Conjecture 4.2.** $\mathcal{M}_c$ is dense in $\Gamma_\infty(M)$.

Morally speaking, Conjecture 4.2 can even lead to upper bounds of Morse index following the techniques in [MN16a].

**References**

[ACS17] L. Ambrozio, A. Carlotto, and B. Sharp. Compactness analysis for free boundary minimal hypersurfaces. *Calculus of Variations and Partial Differential Equations*, 57(1):22, December 2017.

[All72] W. K. Allard. On the First Variation of a Varifold. *Annals of Mathematics*, 95(3):417–491, 1972.

[Alm62] F. J. Almgren. *The Homotopy Groups of the Integral Cycle Groups*. PhD thesis, 1962. OCLC: 22016723.

[Bir17] G. D. Birkhoff. Dynamical systems with two degrees of freedom. *Transactions of the American Mathematical Society*, 18(2):199–300, 1917.

[BP02] V. M. Buchstaber and T. E. Panov. *Torus Actions and Their Applications in Topology and Combinatorics*. Number v. 24 in University Lecture Series. American Mathematical Society, Providence, R.I, 2002.

[CM20] O. Chodosh and C. Mantoulidis. Minimal surfaces and the Allen–Cahn equation on 3-manifolds: Index, multiplicity, and curvature estimates. *Annals of Mathematics*, 191(1):213–328, 2020.

[Dey19] A. Dey. Compactness of certain class of singular minimal hypersurfaces. *arXiv:1901.05840 [math]*, January 2019.

[Fed96] H. Federer. *Geometric Measure Theory*. Classics in Mathematics. Springer, Berlin ; New York, 1996.

[GG19] P. Gaspar and M. A. M. Guaraco. The Weyl Law for the phase transition spectrum and density of limit interfaces. *Geometric and Functional Analysis*, 29(2):382–410, April 2019.

[Gro03] M. Gromov. Isoperimetry of waists and concentration of maps. *Geometric & Functional Analysis GAF A*, 13(1):178–215, February 2003.

[Gut09] L. Guth. Minimax Problems Related to Cup Powers and Steenrod Squares. *Geometric and Functional Analysis*, 18(6):1917–1987, March 2009.

[IMN18] K. Irie, F. C. Marques, and A. Neves. Density of minimal hypersurfaces for generic metrics. *Annals of Mathematics*, 187(3):963–972, May 2018.

[LMN18] Y. Liokumovich, F. C. Marques, and A. Neves. Weyl law for the volume spectrum. *Annals of Mathematics*, 187(3):933–961, May 2018.

[MN14] F. C. Marques and A. Neves. The Min-Max theory and the Willmore conjecture. *Annals of Mathematics*, 179(2):683–782, March 2014.
[MN16a] F. C. Marques and A. Neves. Morse index and multiplicity of min-max minimal hypersurfaces. *Cambridge Journal of Mathematics*, 4(4):463–511, 2016.

[MN16b] F. C. Marques and A. Neves. Topology of the space of cycles and existence of minimal varieties. *Surveys in Differential Geometry*, 21(1):165–177, 2016.

[MN17] F. C. Marques and A. Neves. Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. *Inventiones mathematicae*, 209(2):577–616, August 2017.

[MN21] F. C. Marques and A. Neves. Morse index of multiplicity one min-max minimal hypersurfaces. *Advances in Mathematics*, 378:107527, February 2021.

[MNS19] F. C. Marques, A. Neves, and A. Song. Equidistribution of minimal hypersurfaces for generic metrics. *Inventiones mathematicae*, 216(2):421–443, May 2019.

[Pit81] J. T. Pitts. *Existence and Regularity of Minimal Surfaces on Riemannian Manifolds*. Princeton University Press, 1981.

[RS82] C. P. Rourke and B. J. Sanderson. *Introduction to Piecewise-Linear Topology*. Springer Study Edition. Springer, Berlin, rev. printing edition, 1982. OCLC: 7948164.

[Sha17] B. Sharp. Compactness of minimal hypersurfaces with bounded index. *Journal of Differential Geometry*, 106(2):317–339, June 2017.

[Sim84] L. Simon. *Lectures on Geometric Measure Theory*. Number 3 in *Proceedings of the Centre for Mathematical Analysis / Australian National University*. Centre for Mathematical Analysis, Australian National University, Canberra, 1984. OCLC: 12264914.

[Son18] A. Song. Existence of infinitely many minimal hypersurfaces in closed manifolds. *arXiv:1806.08816 [math]*, June 2018.

[SS81] R. Schoen and L. Simon. Regularity of stable minimal hypersurfaces. *Communications on Pure and Applied Mathematics*, 34(6):741–797, November 1981.

[Whi91] B. White. The Space of Minimal Submanifolds for Varying Riemannian Metrics. *Indiana University Mathematics Journal*, 40(1):161–200, 1991.

[Whi17] B. White. On the bumpy metrics theorem for minimal submanifolds. *American Journal of Mathematics*, 139(4):1149–1155, 2017.

[Wic14] N. Wickramasekera. A general regularity theory for stable codimension 1 integral varifolds. *Annals of Mathematics*, 179(3):843–1007, May 2014.

[Xu18] G. Xu. The (p;m)-width of Riemannian manifolds and its realization. *Indiana University Mathematics Journal*, 67(3):999–1023, 2018.

[Yau82] S.-T. Yau. Problem section. In *Seminar on Differential Geometry. (AM-102)*, pages 669–706. Princeton University Press, 1982.

[Zho20] X. Zhou. On the Multiplicity One Conjecture in min-max theory. *Annals of Mathematics*, 192(3):767–820, 2020.

[ZZ20] X. Zhou and J. J. Zhu. Existence of hypersurfaces with prescribed mean curvature I – generic min-max. *Cambridge Journal of Mathematics*, 8(2):311–362, 2020.

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