On blow-up criteria for a coupled chemotaxis fluid model

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Abstract
We consider a coupled chemotaxis fluid model and prove some blow-up criteria of the local strong solution.

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1 Introduction
We consider the following coupled chemotaxis fluid model [1]:

\begin{align}
  u_t + (u \cdot \nabla) u + \nabla \pi - \Delta u + n \nabla \phi &= 0, \\
  \text{div} \, u &= 0, \\
  n_t + (u \cdot \nabla) n - \Delta n &= -\nabla \cdot (n \chi(p) \nabla p), \\
  p_t + (u \cdot \nabla) p &= -nf(p), \\
  (u, n, p)(x, 0) &= (u_0, n_0, p_0)(x) \quad \text{in } \mathbb{R}^3.
\end{align}

Here \( u \) denotes the velocity vector field of the fluid and \( \pi \) is the pressure scalar, \( p \) and \( n \) denote the concentration of oxygen and bacteria, respectively. \( \nabla \phi \) is the gravitation force. \( f(p) \geq f(0) = 0 \) and \( \chi(p) \geq 0 \) are two given smooth functions of \( p \).

When \( \phi = 0 \), (1.1) and (1.2) are the well-known Navier-Stokes system. Kozono et al. [2] and Kozono and Shimada [3] proved the following blow-up criteria:

\begin{align}
  u &\in L^2(0, T; \dot{B}^0_{\infty, \infty}), \\
  u &\in L^2(0, T; \dot{B}^\theta_{\infty, \infty}) \quad \text{with } 0 < \theta < 1, \\
  \omega &:= \text{curl} \, u \in L^3(0, T; \dot{B}^0_{\infty, \infty}).
\end{align}

Here \( \dot{B}^p_{\theta, q} \) denotes the homogeneous Besov space. Zhang et al. [4] showed the following blow-up criterion in terms of pressure:

\begin{align}
  \pi &\in L^\frac{2}{\theta_1}(0, T; \dot{B}^r_{\infty, \infty}) \quad \text{with } -1 \leq r \leq 1.
\end{align}

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When \( u = \nabla \phi = 0 \), (1.3) and (1.4) are the Keller-Segel model which was studied in [5, 6]. Very recently, Chae et al.\[7\] showed the local well-posedness of smooth solutions to problem (1.1)-(1.5) and the following blow-up criterion:

\[
\begin{align*}
  u &\in L^{\frac{2d}{d-1}}(0,T;L^d) \quad \text{and} \quad n \in L^2(0,T;L^\infty) \quad \text{with} \quad 3 < q \leq \infty. 
\end{align*}
\]  

(1.10)

The aim of this paper is to refine (1.10) further; we will prove the following.

**Theorem 1.1** Let the initial data \((u_0, n_0, p_0)\) be given in \(H^1 \times H^{-1} \times \mathbb{R} \), \( n_0 \geq 0 \) in \( \mathbb{R}^d \) and \( \int_{\mathbb{R}^d} n_0 \, dx < \infty \). Suppose that \( \phi \) is a smooth function. Let \((u, n, p)\) be a local smooth solution on \([0, \tilde{T})\) for some \( \tilde{T} \leq \infty \). If \( u \) satisfies (1.6) or (1.7) or (1.8) or \( \pi \) satisfies (1.9) \((r = -1)\) and \( n \) satisfies

\[
\begin{align*}
  n &\in L^2(0,T;L^\infty) 
\end{align*}
\]  

with \( \tilde{T} \leq T < \infty \), then the solution \((u, n, p)\) can be extended beyond \( T > 0 \).

**Corollary 1.1** If \( u \) satisfies (1.6) or (1.7) or (1.8) or \( \pi \) satisfies (1.9) and \( \nabla p \) satisfies

\[
\begin{align*}
  \nabla p &\in L^{\frac{2d}{d-1}}(0,T;L^d) \quad \text{with} \quad 3 < q \leq \infty, 
\end{align*}
\]  

(1.12)

with \( \tilde{T} \leq T < \infty \), then the solution \((u, n, p)\) can be extended beyond \( T > 0 \).

**Remark 1.1** By the very same calculations as those in Zhou [8], we can prove the following blow-up criteria:

\[
\begin{align*}
  \pi &\in L^{\frac{2d}{d+1}}(0,T;L^d) \quad \text{with} \quad 3/2 < q \leq \infty, 
\end{align*}
\]  

(1.13)

or

\[
\begin{align*}
  \nabla \pi &\in L^{\frac{2d}{d-1}}(0,T;L^d) \quad \text{with} \quad 1 < q \leq \infty, 
\end{align*}
\]  

(1.14)

and \( n \) satisfies (1.11). We omit the details here.

**2 Preliminary**

Here we recall the definitions and some properties of spaces.

Let \( \mathcal{B} = \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\} \) and \( \mathcal{C} = \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\} \). Choose two nonnegative smooth radial functions \( \chi, \varphi \) supported, respectively, in \( \mathcal{B} \) and \( \mathcal{C} \) such that

\[
\begin{align*}
  \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbb{R}^d, \\
  \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. 
\end{align*}
\]
We denote \( \psi_j = \varphi(2^{-j}x) \), \( h = \mathcal{F}^{-1}\varphi \) and \( \tilde{h} = \mathcal{F}^{-1}x \), where \( \mathcal{F}^{-1} \) stands for the inverse Fourier transform. Then the dyadic blocks \( \Delta_j \) and \( S_j \) can be defined as follows:

\[
\Delta_j f = \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) f(x-y) \, dy,
\]

\[
S_j f = \sum_{k\leq j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^j y) f(x-y) \, dy.
\]

Formally, \( \Delta_j = S_j - S_{j-1} \) is a frequency projection to annulus \( \{\xi : C_1 2^j \leq |\xi| \leq C_2 2^j\} \), and \( S_j \) is a frequency projection to the ball \( \{\xi : |\xi| \leq C 2^j\} \). One can easily verify that, with our choice of \( \varphi \),

\[
\Delta_j \Delta_k f = 0 \quad \text{if} \quad |j-k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} \Delta_k f) = 0 \quad \text{if} \quad |j-k| \geq 5.
\]

With the introduction of \( \Delta_j \) and \( S_j \), let us recall the definition of the Besov space.

**Definition 2.1 ([9, 10])** Let \( s \in \mathbb{R} \), \( (p, q) \in [1, \infty]^2 \), the homogeneous space \( \dot{B}^s_{p,q} \) is defined by

\[
\dot{B}^s_{p,q} = \{ f \in \mathcal{S}^{'}; \| f \|_{\dot{B}^s_{p,q}} < \infty \},
\]

where

\[
\| f \|_{\dot{B}^s_{p,q}} = \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} 2^{jsq} \| \Delta_j f \|_{L^p}^q \right)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_{L^p}, & \text{for } q = \infty,
\end{cases}
\]

and \( \mathcal{S}^{'} \) denotes the dual space of \( \mathcal{S} = \{ f \in \mathcal{S}(\mathbb{R}^d); \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \text{ multi-index} \} \) and can be identified by the quotient space of \( \mathcal{S}'/\mathcal{P} \) with the polynomials space \( \mathcal{P} \).

**Lemma 2.1 ([4])]** Let a measurable function \( \pi \) satisfy

\[
\pi \in \dot{B}^{r}_{\infty,\infty}(\mathbb{R}^3)
\]

for some \( r \) with \(-1 \leq r \leq 1\), then there exists a decomposition \( \pi := \pi_\ell + \pi_h \) such that

\[
\nabla^2 \pi_\ell \in L^\infty(\mathbb{R}^3) \quad \text{and} \quad \pi_h \in \dot{W}^{r-1,\infty}(\mathbb{R}^3),
\]

and

\[
\| \nabla^2 \pi_\ell \|_{L^\infty} + \| \pi_h \|_{\dot{W}^{r-1,\infty}} \leq C(e + \| \pi \|_{\dot{B}^{r}_{\infty,\infty}}),
\]

\[
\| \pi_\ell \|_{L^2} \leq C \| \pi \|_{L^2}, \| \nabla \pi_h \|_{L^2} \leq C \| \nabla \pi \|_{L^2}.
\]

**3 Proof of Theorem 1.1**

This section is devoted to the proof of Theorem 1.1. Since local existence results have been proved in [7], we only need to prove \textit{a priori} estimates.
To begin with, it is easy to see that
\[ n \geq 0, \quad 0 \leq p \leq C, \quad \int_{\mathbb{R}^3} n \, dx = \int_{\mathbb{R}^3} n_0 \, dx < \infty. \quad (3.1) \]

Case 1. Let (1.6) and (1.11) hold true.
Testing (1.1) by \( u \) and using (1.2), we infer that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx = \int_{\mathbb{R}^3} n \nabla \phi \, u \, dx \\
\leq \|n\|_{L^\infty} \|\nabla \phi\|_{L^2} \|u\|_{L^2},
\]
which leads to
\[
\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C. \quad (3.2)
\]

In the following calculations, we will use the following elegant inequality [11, 12]:
\[
\|\nabla u\|_{L^2} \leq C \|u\|_{B_{\theta,\infty}^\theta} \|\Delta u\|_{L^2}.
\]

Testing (1.1) by \( \Delta u \), using (1.2) and the above inequality, we find that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\mathbb{R}^3} |\Delta u|^2 \, dx \\
= \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \, dx + \int_{\mathbb{R}^3} n \nabla \phi \Delta u \, dx \\
= \sum_{ij} \int_{\mathbb{R}^3} u_i \partial_i u \partial_j u \, dx + \int_{\mathbb{R}^3} n \nabla \phi \Delta u \, dx \\
= - \sum_{ij} \int_{\mathbb{R}^3} \partial_i u_i \partial_i u \partial_j u \, dx + \int_{\mathbb{R}^3} n \nabla \phi \Delta u \, dx \\
\leq C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^2} + \|n\|_{L^\infty} \|\nabla \phi\|_{L^2} \|\Delta u\|_{L^2} \\
\leq C \|u\|_{B_{\theta,\infty}^\theta} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2} + C \|n\|_{L^\infty} \|\Delta u\|_{L^2} \\
\leq \frac{1}{2} \|\Delta u\|_{L^2}^2 + C \|u\|_{B_{\theta,\infty}^\theta}^2 \|\nabla u\|_{L^2}^2 + C \|u\|_{L^\infty}^2,
\]
which gives
\[
\|u\|_{L^\infty(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)} \leq C. \quad (3.3)
\]

By (1.10), this completes the proof of Case 1.

Case 2. Let (1.7) and (1.11) hold true.
Testing (1.1) by \( -\Delta u \), using (1.2) and the following inequalities [3, 11]:
\[
\|u \cdot \nabla u\|_{L^2} \leq C \|u\|_{B_{\theta,\infty}^\theta} \|u\|_{B_{0,1}^{1/\theta}}, \quad 0 < \theta < 1, \quad (3.4)
\]
\[
\|u\|_{L^2}^\theta \leq C \|u\|_{L^2}^{1-\theta} \|\nabla u\|_{L^2}^\theta, \quad 0 < \theta < 1, \quad (3.5)
\]
we derive

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx \\
= \int_{\mathbb{R}^3} (u \cdot \nabla)u \cdot \Delta u dx + \int_{\mathbb{R}^3} n\nabla \Delta u dx \\
\leq \|u \cdot \nabla u\|_{L^2} \|\Delta u\|_{L^2} + \|n\|_{L^\infty} \|\nabla \phi\|_{L^2} \|\Delta u\|_{L^2} \\
\leq C\|u\|_{B^2_{\infty, \infty}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + C\|n\|_{L^\infty} \|\Delta u\|_{L^2} \\
\leq \frac{1}{2} \|\Delta u\|_{L^2}^2 + C\|u\|_{B^2_{\infty, \infty}} \|\nabla u\|_{L^2}^2 + C\|n\|_{L^\infty}^2 \|\Delta u\|_{L^2}^2,
\]

which yields (3.3); this completes the proof of Case 2 again by (1.10).

Case 3. Let (1.8) and (1.11) hold true.

Testing (1.1) by \(-\Delta u\), using (1.2), we deduce that

\[
1 \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\Delta u|^2 dx \\
= \sum_{i,j} \int_{\mathbb{R}^3} \partial_i u_i \partial_j u_j dx + \int_{\mathbb{R}^3} n\nabla \Delta u dx \\
=: I_1 + \int_{\mathbb{R}^3} n\nabla \Delta u dx. \tag{3.6}
\]

By the very same calculations as those in [13], we get

\[
I_1 \leq \frac{1}{8} \|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|\Delta u\|_{L^2} + C\|\nabla u\|_{B^2_{\infty, \infty}} \|\nabla u\|_{L^2}^2 \log(e + \|\nabla u\|_{L^2}^2). \tag{3.7}
\]

Inserting (3.7) into (3.6) and solving the resulting inequality, we arrive at (3.3). This completes the proof of Case 3.

Case 4. Let (1.9) \((r = -1)\) and (1.11) hold true.

Testing (1.1) by \(|u|^2 u\) and using (1.2), we observe that

\[
1 \frac{d}{dt} \int_{\mathbb{R}^3} |u|^4 dx + \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla |u||^2 dx \\
= \int_{\mathbb{R}^3} (u \cdot \nabla)|u|^2 dx - \int_{\mathbb{R}^3} n\nabla |u|^2 u dx \\
=: I_2 + I_3. \tag{3.8}
\]

\(I_3\) can be bounded as follows:

\[
I_3 \leq \|n\|_{L^\infty} \|\nabla \phi\|_{L^2} \|u\|_{L^4}^3. \tag{3.9}
\]

We bounded \(I_2\) as follows:

\[
I_2 = \int_{\mathbb{R}^3} \pi u \cdot \nabla |u|^2 dx \\
\leq \|\nabla \phi\|_{L^2} \|u\|_{L^4} \|\nabla |u|^2\|_{L^2}
\]
\[
\leq \|\pi\|_{B^{1}_{\infty,\infty}}^{\frac{1}{2}} \|\nabla \pi\|_{L^2}^{\frac{1}{2}} \|u\|_{L^4} \|\nabla |u|^2\|_{L^2} \\
\leq \|\pi\|_{B^{1}_{\infty,\infty}}^{\frac{1}{2}} (\|u \cdot \nabla u\|_{L^2} + \|n \nabla \phi\|_{L^2})^{1/2} \|u\|_{L^4} \|\nabla |u|^2\|_{L^2} \\
\leq \|\pi\|_{B^{1}_{\infty,\infty}}^{\frac{1}{2}} (\|u \nabla u\|_{L^2} + \|n\|_{L^\infty})^{1/2} \|u\|_{L^4} \|\nabla |u|^2\|_{L^2} \\
\leq \frac{1}{8} \|u\|_{L^2}^2 + \frac{1}{8} \|\nabla u\|_{L^2}^2 + C \|\pi\|_{B^{1}_{\infty,\infty}} \|u\|_{L^4}^4 + C \|n\|_{L^\infty}^2,
\]
(3.10)

where we have used the elegant inequality [11, 12]
\[
\|\pi\|_{L^4}^2 \leq C \|\pi\|_{B^{1}_{\infty,\infty}} \|\nabla \pi\|_{L^2},
\]
(3.11)

and the pressure estimate
\[
\|\nabla \pi\|_{L^2} \leq C (\|u \cdot \nabla u\|_{L^2} + \|n \nabla \phi\|_{L^2}).
\]
(3.12)

Inserting (3.9) and (3.10) into (3.8) and using the Gronwall inequality, we conclude that
\[
\|u\|_{L^\infty([0,T;L^4])} \leq C.
\]
(3.13)

By (1.10), this completes the proof of Case 4.

4 Proof of Corollary 1.1

Testing (1.3) by \(n^{m-1}\) (\(m \geq 2\)), using (1.2) and (3.1) and denoting \(w := n^{\frac{m}{2}}\), we have
\[
\frac{1}{m} \frac{d}{dt} \int_{\mathbb{R}^3} w^2 \, dx + \frac{4(m-1)}{m^2} \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \\
\leq C \left| \int_{\mathbb{R}^3} \chi(p) \nabla p \cdot \nabla w \, dx \right| \\
\leq C \|\nabla p\|_{L^6} \|w\|_{L^{2\frac{m}{m-1}}} \|\nabla w\|_{L^2} \\
\leq C \|\nabla p\|_{L^6} \|w\|_{L^{2\frac{m}{m-1}}}^{1+\frac{3}{2}} \|\nabla w\|_{L^{2\frac{m}{m-1}}}^{1+\frac{3}{4}} \\
\leq \frac{m-1}{m^2} \|\nabla w\|_{L^2}^2 + C \|\nabla p\|_{L^{6\frac{m}{m-1}}}^{2\frac{m}{m-1}} \|w\|_{L^2}^2,
\]
which implies
\[
\|n\|_{L^\infty([0,T;L^m])} \leq C \quad \text{for} \quad m > 2.
\]
(4.1)

Here we used the Gagliardo-Nirenberg inequality
\[
\|w\|_{L^{\frac{2m}{m-1}}} \leq C \|w\|_{L^2}^{1+\frac{3}{2}} \|\nabla w\|_{L^2}^{\frac{3}{2}} \quad \text{with} \quad 3 < q \leq \infty.
\]
(4.2)

Now, since the proofs of other cases are very similar to those in Case 1, Case 2, Case 3 and Case 4, we only prove the following case: Let (1.9) \((-1 < r \leq 1)\) and (1.12) hold true.

We still have (3.8) and (3.9).
Using Lemma 2.1, (3.11), (3.12) and the pressure estimate
\[ \|\pi\|_{L^2} \leq C\left(\|u\|_{L^4}^2 + \left\|(-\Delta)^{-\frac{1}{2}} (n\nabla \phi)\right\|_{L^2}^2\right) \]
\[ \leq C\left(\|u\|_{L^4}^2 + \|n\nabla \phi\|_{L^2}^2\right) \]
\[ \leq C\left(\|u\|_{L^4}^2 + 1\right), \quad (4.3) \]
we bound \( I_2 \) as follows:
\[ I_2 = -\int_{\mathbb{R}^3} u\nabla \pi |u|^2 \, dx - \int_{\mathbb{R}^3} u\nabla \pi_h |u|^2 \, dx \]
\[ \leq \|\nabla \pi\|_{L^4\mathbb{R}^3} \|u\|_{L^4}^2 + \int_{\mathbb{R}^3} u\nabla \pi_h |u|^2 \, dx \]
\[ \leq \|\nabla \pi\|_{L^4\mathbb{R}^3} \frac{1}{2} \|\nabla^2 \pi\|_{\frac{n}{2} L^\infty\mathbb{R}^3} \|u\|_{L^4}^2 + \|u\|_{L^4\mathbb{R}^3} \|\nabla \pi_h\|_{L^2\mathbb{R}^3} \|u\|_{L^2}^2 \]
\[ \leq \|\nabla^2 \pi\|_{L^\infty\mathbb{R}^3} \left(\|u\|_{L^4}^4 + 1\right) + C \|\nabla \pi_h\|_{L^{\frac{n}{2}-1,\infty}\mathbb{R}^3} \left(\|u\|_{L^4}^2 \|\nabla u\|_{L^2}^2 + 1\right)^{\frac{1}{2}} \|u\|_{L^4\mathbb{R}^3} \|\nabla \pi_h\|_{L^2\mathbb{R}^3} \|u\|_{L^2}^2 \]
\[ \leq \frac{1}{8} \|\nabla u\|_{L^2}^2 + C \epsilon + \|\pi\|_{B^{\frac{\nu}{2},\infty}_{\infty}} \|u\|_{L^4}^4 + 1 + C. \quad (4.4) \]

Inserting (3.9) and (4.4) into (3.8), we obtain (3.13).

By the classical regularity theory of parabolic equations [14], it follows from (1.2), (1.3), (3.13) and (4.4) that
\[ \|\nabla u\|_{L^2(0,T;L^4)} \leq C \left(1 + \|u\|_{L^2(0,T;L^4)} + \|n\chi(p)\nabla p\|_{L^2(0,T;L^4)}\right) \]
\[ \leq C \left(1 + \|u\|_{L^\infty(0,T;L^4)} \|n\|_{L^\infty(0,T;\frac{\nu}{2} L^\infty)} + \|n\|_{L^\infty(0,T;\frac{\nu}{2} L^\infty)} \|\nabla p\|_{L^2(0,T;L^4)}\right) \]
\[ \leq C \quad (4.5) \]
for some \( 3 < \tilde{r} < 4 \) and \( \tilde{r} < q \).

Therefore,
\[ \|u\|_{L^2(0,T;L^\infty)} \leq C. \quad (4.6) \]

This completes the proof.

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
All authors of the manuscript have read and agreed to its content and are accountable for all the aspects of accuracy and integrity of the manuscript.

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