CONSECUTIVE CANCELLATIONS IN TOR MODULES OVER LOCAL RINGS

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Abstract. Let $M, N$ be finite modules over a Noetherian local ring $R$. We show that the bigraded Hilbert series of $\text{gr}(\text{Tor}^R(M, N))$ is obtained from that of $\text{Tor}^{\text{gr}(R)}(\text{gr}(M), \text{gr}(N))$ by negative consecutive cancellations, thus extending a theorem of Rossi and Sharifan.

1. Introduction

Given a module over a local or graded ring, a major problem in commutative algebra is to understand the behavior of its homological data. It is often convenient to deal with “approximations” of the module with nicer properties, and try to compare their invariants with those of the original module. A well-known phenomenon is the fact that the Betti numbers increase when passing to initial submodules with respect to term orders or weights (cf. [6, Theorem 8.29]), lex-segment submodules (cf. [1, 4, 5, 7]), or associated graded modules with respect to filtrations (cf. [3]). The idea of consecutive cancellations, introduced by Peeva [8] in the graded context, arises in the attempt of describing these inequalities more precisely. Peeva showed that the graded Betti numbers of a homogeneous ideal $I$ in a polynomial ring are obtained from those of the associated lex-segment ideal $\text{Lex}(I)$ by a sequence of zero consecutive cancellations, i.e. cancellations from two Betti numbers $\beta_{i+1,j}(\text{Lex}(I)), \beta_{i,j}(\text{Lex}(I))$. The theorem can be generalized to modules (cf. [8, Theorem 1.1] and the subsequent remark).

Now let $(R, m)$ be a local Noetherian ring and $\text{gr}_m(R)$ its associated graded ring. A common strategy used to study homological invariants over $R$ is to endow modules and complexes with filtrations and consider the associated graded objects, thus taking advantage of the rich literature available for graded rings. In this framework, Rossi and Sharifan [10, 11] investigate the relation between the minimal free

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resolution of an $R$-module $M$ and that of the associated graded module $\gr_{\mathcal{M}}(M)$ over $\gr_{\mathfrak{m}}(R)$, where $\mathcal{M}$ is the $\mathfrak{m}$-adic filtration or more generally an $\mathfrak{m}$-stable filtration of $M$ (see the next section for definitions). Inspired by Peeva’s result, they introduce the notion of negative consecutive cancellations, i.e. cancellations from two Betti numbers $\beta_{i+1,a}(\gr_{\mathcal{M}}(M)), \beta_{i,b}(\gr_{\mathcal{M}}(M))$ with $a < b$. They prove the following result for the case of a regular local ring:

**Theorem 1** ([11, Theorem 3.1]). Let $(R, \mathfrak{m})$ be a regular local ring and $\gr_{\mathfrak{m}}(R)$ its associated graded ring. If $M$ is a finite $R$-module with an $\mathfrak{m}$-stable filtration $\mathcal{M}$, the Betti numbers of $M$ are obtained from the graded Betti numbers of the $\gr_{\mathfrak{m}}(R)$-module $\gr_{\mathcal{M}}(M)$ by a sequence of negative consecutive cancellations.

Their argument relies on explicit manipulations on the matrices of the differential in a free $R$-resolution of $M$ obtained by lifting the minimal free $\gr_{\mathfrak{m}}(R)$-resolution of $\gr_{\mathcal{M}}(M)$. In this note we take an alternative approach, namely we employ a spectral sequence, introduced by Serre [12], arising from a suitable filtered complex. We provide the following generalization of Theorem 1 which holds for Tor modules over an arbitrary Noetherian local ring:

**Theorem 2.** Let $(R, \mathfrak{m})$ be a Noetherian local ring, $k = R/\mathfrak{m}$ its residue field, and $\gr_{\mathfrak{m}}(R)$ its associated graded ring. Given two finite $R$-modules $M, N$ with $\mathfrak{m}$-stable filtrations $\mathcal{M}, \mathcal{N}$, there exist $\mathfrak{m}$-stable filtrations on each $\text{Tor}_i^R(M, N)$ such that the Hilbert series

$$\sum_{i,j} \dim_k \text{gr}(\text{Tor}_i^R(M, N))_j z^i t^j$$

is obtained from the Hilbert series

$$\sum_{i,j} \dim_k \text{Tor}_i^{\gr_{\mathfrak{m}}(R)}(\gr_{\mathcal{M}}(M), \gr_{\mathcal{N}}(N))_j z^i t^j$$

by negative consecutive cancellations, i.e. subtracting terms of the form $z^{i+1}t^a + z^i t^b$ with $a < b$.

2. **Proof of the main result**

We begin by providing the necessary background and definitions. Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field $k = R/\mathfrak{m}$. The ring $G = \gr_{\mathfrak{m}}(R) = \bigoplus_{j \geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}$ is known as the associated graded ring of $R$, and it is a standard graded $k$-algebra. Geometrically, if $R$ is the local ring of a variety $\mathcal{V}$ at a point $\mathfrak{p}$, then $G$ is the homogeneous coordinate ring of the tangent cone to $\mathcal{V}$ at $\mathfrak{p}$; for this reason $G$ is sometimes called the tangent cone of $R$. 


Let $M$ be a finitely generated $R$-module. A descending filtration $\mathcal{M} = \{M^j\}_{j \in \mathbb{N}}$ of $R$-submodules of $M$ is said to be $m$-stable if the following conditions are satisfied: $M^0 = M$, $mM^j \subseteq M^{j+1}$ for all $j \geq 0$, and $mM^j = M^{j+1}$ for sufficiently large $j$. If $\mathcal{M}$ is an $m$-stable filtration of $M$, we define the associated graded module of $M$ with respect to $\mathcal{M}$ as $\text{gr}_\mathcal{M}(M) = \bigoplus_{j \geq 0} M^j / M^{j+1}$; it is a finitely generated graded $G$-module.

The central example is that of the $m$-adic filtration $M = \{m^j M\}_{j \in \mathbb{N}}$, however it will be necessary to allow for more general ones. In order to simplify the notation, sometimes we will just write $\text{gr}(M)$ if the filtration is clear from the context.

Let $M, N$ be two $R$-modules with $m$-stable filtrations. An $R$-linear map $f : M \rightarrow N$ such that $f(M^j) \subseteq N^j$ for all $j$ induces a homogeneous $G$-linear map $\text{gr}(f) : \text{gr}(M) \rightarrow \text{gr}(N)$. In fact, $\text{gr}(\cdot)$ is a functor from the category of $R$-modules with $m$-stable filtrations to the category of graded $G$-modules, and in particular we can consider associated graded complexes of filtered complexes. An important result due to Robbiano [9] states that it is possible to “lift” a graded free $G$-resolution $G = \{G_j\}$ of $\text{gr}_\mathcal{M}(M)$ to a free $R$-resolution $F = \{F_i\}$ of $M$ together with $m$-stable filtrations $F_i = \{F^j_i\}_{j \in \mathbb{N}}$ on each free module $F_i$ such that $\text{gr}(F) = G$. The lifted resolution $F$ may not be minimal in general, even if we start from a minimal one, and the filtration on $F_i$ may not be the $m$-adic one. Compare also [10, Theorem 1.8].

We refer to [10], [2, Chapter 5] for further background on filtered modules and to [2, Appendix 3] for background on spectral sequences.

**Proof of Theorem 2** We analyze the convergence of the spectral sequence

$$\text{Tor}^G(\text{gr}_\mathcal{M}(M), \text{gr}_\mathcal{N}(N)) \Rightarrow \text{Tor}^R(M, N).$$

The spectral sequence is constructed as follows. Let $\mathcal{M} = \{M^j\}_{j \in \mathbb{N}}$ and $\mathcal{N} = \{N^j\}_{j \in \mathbb{N}}$ denote the given filtrations on $M, N$. Let $F = \{F_i\}$ be a free $R$-resolution of $M$ with $m$-stable filtrations $\mathcal{F}_i = \{F^j_i\}_{j \in \mathbb{Z}}$ on each $F_i$ such that the associated graded complex $\text{gr}(F)$ is a graded free $G$-resolution of $\text{gr}_\mathcal{M}(M)$. Consider the complex $L = F \otimes_R N$. We equip each $L_i = F_i \otimes_R N$ with filtrations $\mathcal{L}_i = \{L^j_i\}_{j \in \mathbb{N}}$ defined by

$$L^j_i = \sum_{j_1 + j_2 = j} \text{Image} \left( F^{j_1}_i \otimes_R N^{j_2} \rightarrow F_i \otimes_R N \right)$$

where the maps in parentheses are obtained by tensoring the natural maps $F^{j_1}_i \rightarrow F_i$ and $N^{j_2} \rightarrow N$. Since $\mathcal{F}_i$ and $\mathcal{M}$ are $m$-stable, $\mathcal{L}_i$ is $m$-stable as well. With this filtration we have the isomorphism of graded
free complexes over $G$

$$\text{gr}(L) \cong \text{gr}(F) \otimes_G \text{gr}_N(N) .$$

By the Artin-Rees lemma, $L_i$ in turn induces an $m$-stable filtration on $H_i(L) = \text{Tor}_R^1(M, N)$.

For the spectral sequence $\{\tau E\}_{r \geq 1}$ of the filtered complex $L$ we have

$$^1E_i^j = H_i(\text{gr}(L))_j = H_i(\text{gr}(F) \otimes_G \text{gr}_M(N))_j = \text{Tor}_R^1(\text{gr}_M(M), \text{gr}_N(N))_j ,$$

$$\infty E_i^j = \text{gr}(H_i(L))_j = \text{gr}(H_i(F \otimes_R N))_j = \text{gr}(\text{Tor}_R^1(M, N))_j .$$

The $k$-vector space $^r+1E_i^j$ is a subquotient of $^rE_i^j$ for each $r$, hence the cancellation from $^1E_i^j$ to $\infty E_i^j$ is the sum of the cancellations from $^rE_i^j$ to $^r+1E_i^j$ for all $r \geq 1$. In order to prove the theorem, it suffices to examine cancellations in two consecutive pages $^rE, ^r+1E$ of the spectral sequence, so we fix $r$ for the rest of the proof.

Let $d$ denote the differential of $L$. Cycles and boundaries in the spectral sequence are given by

$$r+1Z_i^j = \frac{\{z \in L_i^j : dz \in L_{i-1}^{j+r+1}\}}{L_i^{j+1} + dL_{i-1}^j} ,$$

$$r+1B_i^j = \frac{(L_i^j \cap dL_{i-1}^j + L_{i-1}^{j+1})}{L_i^{j+1} + dL_{i-1}^j} .$$

The inclusions $rB_i^j \subseteq r+1B_i^j \subseteq r+1Z_i^j \subseteq rZ_i^j$ induce maps of $k$-vector spaces

$$rE_i^j = \frac{rZ_i^j}{rB_i^j} \xrightarrow{\pi_{i,j}} \frac{rZ_i^j}{r+1B_i^j} \xrightarrow{\iota_{i,j}} \frac{r+1Z_i^j}{r+1B_i^j} = r+1E_i^j$$

with $\pi_{i,j}$ surjective and $\iota_{i,j}$ injective, so we have

$$\dim_k rE_i^j = \dim_k r+1E_i^j + \dim_k \ker(\pi_{i,j}) + \dim_k \text{coker}(\iota_{i,j}) .$$

Therefore a cancellation from page $r$ to page $r+1$ can occur either in $\ker(\pi_{i,j})$ or in $\text{coker}(\iota_{i,j})$. We write down explicitly

$$\text{coker}(\iota_{i,j}) = \frac{r+1B_i^{j+r}}{rB_i^{j+r-1}} = \frac{\{z \in L_i^j : dz \in L_{i-1}^{j+r+1}\}}{(L_i^{j+r} \cap dL_i^j) + L_{i-1}^{j+r+1}} .$$

$$\ker(\pi_{i-1,j+r}) = \frac{r+1B_i^{j+r}}{rB_i^{j+r-1}} = \frac{(L_i^{j+r} \cap dL_i^j + L_{i-1}^{j+r+1})}{(L_i^{j+r} \cap dL_i^j) + L_{i-1}^{j+r+1}} .$$

The differential $d$ induces a $k$-linear map $\delta : \text{coker}(\iota_{i,j}) \to \ker(\pi_{i-1,j+r})$ which sends $\overline{z}$ to $\overline{dz}$, and from the two expressions above we obtain immediately that $\delta$ is bijective. We conclude that cancellations in $\text{coker}(\iota)$ in homological degree $i$ and internal degree $j$ correspond bijectively to
cancellations in \( \ker(\pi) \) in homological degree \( i - 1 \) and internal degree \( j + r \), thus yielding the negative consecutive cancellations at each page.

\[ \square \]

**Remark 3.** We see from the proof that the degree \( r = b - a \) of a negative consecutive cancellation \( z^{i+1}t^a + z^jt^b \) corresponds to the page of the spectral sequence in which the cancellation occurs.

We illustrate Theorem 2 with a couple of examples.

**Example 4.** Let \( R = \mathbb{k}[[X, Y]] \) be the formal power series ring with \( m = (X, Y) \) and let \( M = R/(X^2 - Y^3), N = R/(X^2 - Y^a) \) for some integer \( a > 3 \). We compute

\[
\begin{align*}
\text{Tor}_0^R(M, N) & = M \otimes_R N = R/(X^2 - Y^3, X^2 - Y^a), \\
\text{Tor}_1^R(M, N) & = (X^2 - Y^3) \cap (X^2 - Y^a) = 0.
\end{align*}
\]

The associated graded ring of \( R \) is \( G = \mathbb{k}[x, y] \) where \( x, y \) are the initial forms of \( X, Y \), while the associated graded modules with respect to the \( m \)-adic filtrations are \( \text{gr}_m(M) = \text{gr}_m(N) = G/(x^2) \). We get

\[
\begin{align*}
\text{Tor}_0^G(\text{gr}_m(M), \text{gr}_m(N)) & = \text{gr}_m(M) \otimes_G \text{gr}_m(N) = G/(x^2), \\
\text{Tor}_1^G(\text{gr}_m(M), \text{gr}_m(N)) & = \frac{(x^2) \cap (x^2)}{(x^2)(x^2)} = \frac{(x^2)}{(x^4)},
\end{align*}
\]

hence their Hilbert series are respectively

\[
1 + 2 \sum_{j \geq 1} t^j \quad \text{and} \quad zt^2 + 2zt^3 \sum_{j \geq 3} t^j.
\]

Since the ideal \((X^2 - Y^3, X^2 - Y^a)\) is \( m \)-primary of colength 6 and the cancellations have negative degree by Theorem 2, we conclude that we have precisely the cancellations \( zt^2 + t^3 \) and \( 2zt^j + 2t^{j+1} \) for all \( j \geq 3 \).

Now we give an example of an infinite free resolution where no cancellation occurs.

**Example 5.** Let \((R, m)\) be a Noetherian local hypersurface ring with associated graded ring \( G \cong \mathbb{k}[x_1, \ldots, x_n]/(x_1^e) \) for some \( e \geq 3 \). Suppose \( M \) is a factor ring of \( R \) whose associated graded ring with respect to the \( m \)-adic filtration is of the form \( \text{gr}_m(M) \cong \mathbb{k}[x_1, \ldots, x_n]/I \) where

\[
I = (x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_m)
\]
for some $m \leq n$ and $2 \leq d < e$. Since $(x_i^t) \subseteq I$ are stable monomial ideals, we can use \cite{5} Theorem 3 to determine
\begin{equation}
\sum_{i,j} \dim_k \Tor_{i}^{\gr_m(R)}(\gr_m(M), k) z^i t^j = (1 + mzt^d + (m - 1)z^2t^{d+1}) \sum_{k \geq 0} z^{2k} t^{ke}.
\end{equation}
In particular, we obtain the degrees of the non-zero graded components of $\Tor_{i}^{\gr_m(R)}(\gr_m(M), k)$:
- if $i$ is odd then $\Tor_{i}^{\gr_m(R)}(\gr_m(M), k)$ is concentrated in the single degree \( \frac{i-1}{2}e + d \);
- if $i$ is even then $\Tor_{i}^{\gr_m(R)}(\gr_m(M), k)$ is concentrated in degrees $\frac{i-2}{2}e + d + 1$ and $\frac{i}{2}e$.

Since $d < e$, we conclude that the bigraded Hilbert series \( \mathbf{(1)} \) admits no negative consecutive cancellation. By Theorem 2 we get
\[
\gr(\Tor_{i}^{R}(M, k))_j = \Tor_{i}^{\gr_m(R)}(\gr_m(M), k)_j \quad \text{for all } i, j \geq 0.
\]

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