Coherence, squeezing and entanglement
– an example of peaceful coexistence

Katarzyna Górska, Andrzej Horzela and Franciszek Hugon Szafraniec

Abstract
After exhaustive inspection of bosonic coherent states appearing in physical literature two of us, Horzela and Szafraniec, came in 2012 to the reasonably general definition which relies exclusively on reproducing kernels. The basic feature of coherent states, which is the resolution of the identity, is preserved though it now achieves advantageous form of the Segal-Bargmann transform.

It turns out that the aforesaid definition is not only extremely economical but also puts under a common umbrella typical coherent states as well as those which are squeezed and entangled. We examine the case here on the groundwork of holomorphic Hermite polynomials in one and two variables. An interesting side of this story is how some limit procedure allows disentangling.

1 Coherent states - a smooth introduction

Coherent states (CSs in short) constitute a vivid topic in Quantum Optics besides being of interest from the mathematical point of view. This Section provides a short though solid introduction culminating in the fairly recent extension of the notion

Katarzyna Górska
H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, Division of Theoretical Physics, ul. Eliasza-Radzikowskiego 152, PL 31-342 Kraków, Poland, e-mail: katarzyna.gorska@ifj.edu.pl

Andrzej Horzela
H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, Division of Theoretical Physics, ul. Eliasza-Radzikowskiego 152, PL 31-342 Kraków, Poland, e-mail: andrzej.horzela@ifj.edu.pl

Franciszek Hugon Szafraniec
Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, 30 348 Kraków, Poland, e-mail: umszafr@cyf-kr.edu.pl
which has a novel and pretty interesting application (cf. Section 4). As a kind of shorthand to this Section the presentation [48] may serve.

1.1 Standard coherent states

What are coherent states? The standard harmonic oscillator coherent state, originated in [38], are simply

\[ c_z = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} h_n, \quad z \in \mathbb{C}, \quad (1) \]

with \( h_n \)'s being the Hermite functions

\[ h_n(x) = 2^{-n/2}(n!)^{-1/2} \pi^{-1/4} e^{-x^2/2} H_n(x), \quad H_n(x) \text{ is the } n\text{-th Hermite polynomial.} \]

As the Hermite functions \( h_n \) are residing in \( L^2(\mathbb{R}) \) the safest way to consider convergence in (1) is to require it in this space.

Immediately from the definition (1) one usually derives that such introduced \( c_z \) are:

(a) normalized;
(b) continuous functions in \( z \);
(c) never orthogonal, even more \( \langle c_z, c_w \rangle = e^{-|z-w|^2} \);
(d) temporally stable [16, p.32]

\[ e^{iHt} c_z = e^{-i\frac{\omega t}{2}} c_{e^{i\omega t} z}, \quad H \text{ is the harmonic oscillator Hamiltonian;} \]

and, last but not least, satisfy the celebrated relation [e]

(e) \( I = \int |z\rangle \langle z| \frac{dz}{2\pi}, \) where \( |z\rangle \) states for \( c_z \).

**Remark 1.** (e) is customarily called the resolution of the identity, sometimes referred to as (over)completeness.

In the literature one finds three ways of constructing CSs:

(A) as the (normalized) eigenvectors of the annihilation operator.

1 They bear different names like canonical, classical, orthodox, Glauber-Klauder-Sudarshan (GKS in short, [18, 26, 39]), etc. though the most explicative way would be to call them, as it becomes clearer later, Gaussian coherent states upon the Gaussian kernel involved in (1).

2 In this section standard Hilbert space notation is employed; there are two exceptions when Dirac’s notation is in use: here and on p. 8.

3 The basic monographs [1, 16, 28, 29, 36] of the subject can be completed with other articles like [10, 40] and [51].

4 They are in fact in the domain of the closure of annihilation rather than in its domain as usually seems to be thought of; see [47] for more on the operators.
(B) as the orbit of the vacuum under a square integrable representation of a unitary group;
(C) as states which minimize the Heisenberg uncertainty relation.

It turns out that for the GKS CSs these three lead to the same provided in (B) the group is that of the displacement operator.

**A CLOSER LOOK AT THE PROPERTIES (a)-(e)**

Property (a) is superfluous, the normalization it serves for can be achieved any time it is needed because CSs are vectors in a Hilbert space and as such they have “finite” norms. In general normalization may destroy holomorphicity of CSs in the case it may be present (*vide* the Segal-Bargmann space).

Properties (b) and (d) depend on circumstances or in other words on structure of the set which CSs are parametrised by; removing normalizability as suggested above makes the GKS CSs even holomorphic.

The angle between CSs calculated in most of the cases supports property (c).

The resolution of the identity property (e) is our main concern in this Section and different its aspects will be discussed.

### 1.2 After 1963

Since their rediscovery in 1963 CSs have begun spreading out and a plethora of their different versions rooted in various branches of physics have appeared (for a fairly account of most of those diversities see e.g. [10], [16] and [28]). In particular, MPs (this an apparent abbreviation for either Mathematical Physics itself or its admirers), still keeping in mind the postulates (a) – (e) and following any of the directives (A), (B) and (C), have been trying to find either generalizations of CSs or to provide any evidence of existence of CSs in various fields of physics.

Being joined to these efforts we adopt as our starting point to give a precise meaning to what is in Remark [1] and by the way to respond to the call formulated by J. R. Klauder as the Postulate 3 *Completeness and resolution* in his seminal paper [26]. Klauder’s approach, later on pushed forward in [27], has found its further development in [17] where J.-P. Gazeau and J. R. Klauder have proposed to make the following replacements in (1) (we refer to those as to KG CSs though pretty often they are referred to as nonlinear CSs [40]):

---

5 “A resolution of unity in the Hilbert space $\mathcal{H}$ exists as an integral over projection operators onto individual vectors in the (coherent states) set $S$.”

6 “Traditionally, coherent states rely heavily for their construction and analysis on properties of suitable Lie algebra generators appropriate to some specific group. Hence, most of the properties of the coherent states are inherited from the group itself...We entirely set aside any group ..., and proceed more generally. We are led to an extremely wide class of coherent states that includes group-defined coherent states as a small subset” (underlined by the authors of the present paper).
\[ h_n, \text{'s} \mapsto \text{arbitrary orthonormal basic vectors in some Hilbert space (in which the would-be CSs have to reside)}; \]
\[ n! \mapsto \rho_n = \epsilon_1 \ldots \epsilon_n > 0 \text{ in the way which ensures convergence; the sequence } (\epsilon_n)_n \text{ is usually assumed to be related to the spectrum of the Hamiltonian (describing the physical system under consideration) in a way which guarantees temporal stability and so-called "action identity");} \]
\[ \exp(-|z|^2/2) \mapsto \text{a suitable normalization factor if any;} \]
\[ d^2z \text{ in (e) on p. 2} \mapsto \text{a rotationally invariant measure on } \mathbb{C} \text{ with a radial factor coming from (and solving) the Stieltjes moment problem}; \]
\[ \text{one checks that this secures the property (e)}. \]

The final step in such realized generalization of the CSs concept is that proposed by J.-P. Gazeau in the Ch.5 of [16]: CSs are introduced as continuous in \( x \) and normalizable linear combinations

\[ c_{x} = \sum_{n \in \mathbb{N}} \phi_n(x)e_n, \quad x \in X, \]

where \((e_n)_n\) are normalized eigenvectors of a self-adjoint operator \( A \) and \((\phi_n(x))_n\) is an orthonormal system of functions in \( L^2(X, \nu) \) being in one-to-one correspondence with \((e_n)_n\) and satisfying \( \sum_n |\phi_n(x)|^2 < \infty \) for all \( x \in X \) (normalization condition). This allows to get the property (e)

\[ I = \int_X |x\rangle\langle x|\nu(dx) \]

where, as previously, \( |x\rangle \) states for \( c_x \).

Everything happens in the presence of a measure which makes the resolution of the identity possible; this is out of any discussion there. Even if a measure exists it may not be unique and if the latter happens a plenty of non-rotationally invariant measures always have to appear. More than, no measure may exist though suitably understood resolution of identity can be done which makes the new (generalized) CSs good sense. This may be painful and in this Section we propose a cure for that.

### 1.3 Reproducing kernel Hilbert space - instructional material

The tool is the reproducing kernel Hilbert space (RKHS in short) approach a gentle introduction to which follows.

\[ \text{A set } X \text{ given, call it basic or supporting.} \]

\[^7\text{This takes place for a vast majority of examples present in the literature (see e.g. [30, 33, 35]).}\]
Given a Hilbert space $\mathcal{H}$ of complex functions on $X$ and a function $K : X \times X \mapsto \mathbb{C}$; 
$(\mathcal{H}, K)$ is called a RKHS couple if 
\begin{itemize}
  \item $K_x \overset{\text{def}}{=} K(\cdot, x) \in \mathcal{H}$, $x \in X$;
  \item $f(x) = (f, K_x)$, $f \in \mathcal{H}$, $x \in X$.
\end{itemize}

The second fact is just referred to as the celebrated reproducing kernel property. Therefore, we call $K$ the reproducing kernel. There is a list of properties coming out of this definition and each of them may work for construction the couple, cf. [45]. Among them one finds positive definiteness of the kernel $K$ and boundedness of the evaluation functional on $\mathcal{H}$.

Fundamental for us, however, is Zaremba’s formula ([50]) and its consequences. Given a sequence $(\Phi_n)_{n=0}^{\infty}$ of complex functions on $X$ such that
\begin{equation}
\sum_{n=0}^{\infty} |\Phi_n(x)|^2 < +\infty, \quad x \in X.
\end{equation}

Then
\begin{equation}
K(x, y) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \Phi_n(x) \bar{\Phi}_n(y), \quad x, y \in X,
\end{equation}
is a positive definite kernel and, consequently, due to Moore-Aronszajn’s construction, see [5] or [45] for instance, it uniquely determines its partner, denoted by $\mathcal{H}_K$ further on, so that they both together constitute a reproducing kernel couple. This may serve as a very practical way of constructing RKHS.

**What is the role played by the functions $\Phi_n$?**

1°. It follows from the Schwarz inequality applied to (B) that for any $\xi = (\xi_n)_{n=0}^{\infty} \in \ell^2$:
\begin{itemize}
  \item the series 
    \begin{equation}
    \sum_{n=0}^{\infty} \xi_n \Phi_n(x)
    \end{equation}
    is absolutely convergent for any $x$;
  \item the function 
    \begin{equation}
    f_\xi : x \mapsto \sum_{n=0}^{\infty} \xi_n \Phi_n(x)
    \end{equation}
    is in $\mathcal{H}_K$ with $\|f_\xi\| \leq \|\xi\|_{\ell^2}$;
  \item moreover, $\sum_n \xi_n \Phi_n$ is convergent in $\mathcal{H}_K$ to $f_\xi$.
\end{itemize}

---

8 This item as well as the other ([46]) contains excerpts from [43]; the proofs are contained in the latter.
In particular, \( \sum_n \Phi_n(x) \Phi_n \) is convergent in \( \mathcal{H}_K \) to \( K \), the functions \( \Phi_n \) are in \( \mathcal{H}_K \) and \( \| \Phi_n \| \leq 1 \).

\[ \text{2}^\circ. \] The sequence \( (\Phi_n)_{n=0}^\infty \) is always complete\(^9\) in \( \mathcal{H}_K \). Moreover, the following facts are equivalent

(i) \( \xi \in \ell^2 \) and \( \sum_n \xi_n \Phi_n(x) = 0 \) for every \( x \) yields \( \xi = 0 \);
(ii) the sequence \( (\Phi_n)_n \) is orthonormal in \( \mathcal{H}_K \).

It is recommended to notify that Zaremba’s construction guarantees always completeness of the sequence \( (\Phi_n)_{n=0}^\infty \) in \( \mathcal{H}_K \); it is an intrinsic feature of the approach. Orthonormality of \( (\Phi_n)_{n=0}^\infty \), on the other hand, requires additional effort as \( 2^\circ \) above shows. If the latter happens, \( (\Phi_n)_{n=0}^\infty \) must necessarily be a Hilbert space basis of \( \mathcal{H}_K \).

### 1.4 Horzela-Szafraniec’s CSs and the Segal-Bargmann transform

**Horzela-Szafraniec’s CSs**

The only data, which the Horzela-Szafraniec procedure\(^{23,24}\) requires besides the supporting set \( X \), are

- a sequence \( \Phi = (\Phi_n)_{n=0}^\infty \) of functions on \( X \) such that \( (2) \) holds;
- a separable Hilbert space \( \mathcal{H} \) (it can be thought of as a surrogate of the state space).

Now let \( K \) be the kernel on \( X \) got via Zaremba’s construction and \( \mathcal{H}_K \) its RKHS. Fix an orthonormal basis \( e = (e_n)_{n=0}^\infty \) in \( \mathcal{H} \). Introduce the family \( \{c_x \}_{x \in X} \)

\[ c_x = \sum_{n=0}^\infty \Phi_n(x) e_n \quad x \in X. \quad (3) \]

We do not suppose for a while that \( \Phi = (\Phi_n)_{n=0}^\infty \) are orthonormal.

---

\(^9\) Notice completeness of \( (\Phi_n)_n \) appears *a posteriori.*

\(^{10}\) Complete or total means the closed linear span \( \text{clolin} \{\Phi_n : n = 0, 1, \ldots \} \) is \( \mathcal{H}_K \). This is equivalent to saying that the only function in \( \mathcal{H}_K \) orthogonal to all the \( \Phi_n \)’s is 0.
Let us mention that positive definiteness of $K$, or rather some Schwarz type inequalities which follow, guarantees continuous or holomorphic dependence on $x$ of so introduced $c_x$ according to circumstances (cf. [25]); this refers to (b) on p. [2].

**THE SEGAL-BARGMANN TRANSFORM**

The transform

$$B_h \defeq \sum_{n=0}^{\infty} \Phi_n(h,e_n)_H, \quad h \in \mathcal{H},$$

(4)

is well defined and maps $\mathcal{H} \mapsto \mathcal{H}_K$ (notice $Be_n = \Phi_n$); convergence in (4) is that of $\mathcal{H}_K$. It is a contraction with a dense range.

Due to the reproducing property we have

$$(Bh)(x) = (Bh,K_x)_H = \sum_{n=0}^{\infty} \Phi_n(x)(h,e_n)_H = (c_x,h)_H, \quad h \in \mathcal{H}, \, x \in X,$$

(5)

with the convergence being uniform on those subsets of $X$ on which $K(x,x)$ is bounded.

Moreover if $(\Phi_n)_{n=0}^{\infty}$ is an orthonormal basis then (4) and the Parseval formula yields

$$\langle Bh,Bg\rangle_{\mathcal{H}_K} = \langle h,g\rangle_H, \quad g,h \in \mathcal{H};$$

(6)

hence $B$ is unitary.

**Theorem 1.** The following three facts are equivalent

- the transform $B$ is unitary;
- the family $\{c_x\}_{x \in X}$ is complete;
- the sequence $(\Phi_n)_{n=0}^{\infty}$ is orthonormal in $\mathcal{H}_K$.

In the GKS prototype, that is when $\Phi_n(z,w) = e^{wz}/\sqrt{n!}$ or $K(z,w) = e^{zw}$ and $e_n = h_n$ are Hermite functions, the transform $B$ becomes that of Segal-Bargmann [6, 21].

Now it is a right time to declare: call the vectors (states) $c_x, \, x \in X$ Horzela-Szafraniec coherent states $^a$ if they are given by (3) and the family $\{c_x\}_{x \in X}$ is complete in $\mathcal{H}$.

$^a$ Nicknamed HSz CSs
Horzela-Szafraniec coherent states back and forth

Universality of our definition of coherent states can be enhanced by the fact which follows

**Proposition 1.** Let $\mathcal{H}$ be a Hilbert space and $(e_n)_{n=0}^\infty$ be an orthonormal base in it (one can think of it as the Fock basis). Any family of vectors (states) $(c_x)_{x \in X}$ in $\mathcal{H}$ becomes a family of coherent states in a sense of Horzela-Szafraniec with respect of the uniquely determined reproducing kernel

$$K(x,y) = \sum_n \langle c_x, e_n \rangle \langle c_y, e_n \rangle, \quad x, y \in X.$$ 

This implies that all the coherent states already present in the literature (cases like A, B, C on page 2) fit in with the Horzela-Szafraniec class; the states mentioned at the end of Section 3 are within this class too.

In particular the Segal-Bargmann transform is valid and Theorem 1 holds.

Once more, notice that the resolution of the identity (e), p. 2 (which in our approach, as will be seen explicitly very soon, turns into the Segal-Bargmann transform) is an *a posteriori* fact coming out of the construction, not an *a priori* postulate.

Resolution of the identity for malcontents

**Definition 1.** If $X$ is a (subset of a) topological space and there is a positive measure $\mu$ on the completion $\overline{X}$ of $X$ such that $\mathcal{H}$ is embedded isometrically in “a natural way” in $\mathcal{L}^2(X, \mu)$ we say that $(\mathcal{H}, K)$ is integrable.

Let us emphasise that there are non-integrable RKHSs, look at p. 10.

If $\mu$ is any measure which makes integrability of RKHS possible then

$$\langle |h| \int_X |x| \langle x| \mu(dx) |g \rangle \rangle = \int_X \langle |h| |x| \mu(dx) |g \rangle \langle x| \mu(dx) \rangle$$

$$= \int_X \langle |h| |x| \mu(dx) \rangle \langle |x| |g| \mu(dx) \rangle$$

$$= \int_X (Bh)(x)(Bg)(x) \mu(dx)$$

$$= \langle Bh, Bg \rangle_{L^2(X, \mu)}$$

$$= \langle Bh, Bg \rangle_{\mathcal{H}^K}$$

$$= \langle h, g \rangle_\mathcal{H} = \langle |h|g \rangle.$$  

11 Notice Dirac’s notation is used for the second time in this section.
Resolution of the identity, the key feature of CSs, has been rescued in the full glory! Now it bears the name **Segal-Bargmann transform**.

All this justifies once more the use of term *coherent states* for the family \( \{ c(x) \}_{x \in X} \).

---

### 1.5 The measure – to be or not to be?

Three possibilities for the family \( \{ c(x) \}_{x \in X} \) of CSs may happen.

\( \mathcal{H}_K \) is integrable and the measure is unique.

Here is a list of assorted cases.

- **Standard CSs**
  \[ \Phi_n(z) = \frac{\sqrt{n!}}{\sqrt{n!}} z^n, \quad z \in \mathbb{C}, \quad e_n = \text{Hermite functions and } K(z,w) = e^{z\overline{w}}. \]

- van Eijndhoven–Meyers’ orthogonality, cf. p. 12.

- **CSs on the unit circle.** They come from the Szegö kernel; here
  \[ \Phi_n(z) = \sqrt{\frac{1}{2\pi}} z^n, \quad \text{with } K(z,w) = \frac{1}{2\pi} (1 - z\overline{w})^{-1}, \quad |z|, |w| < 1, \]

and \( \mathcal{H}_K \) is the space of holomorphic functions on the open unit disk \( \mathbb{D} \) which is customarily named after Hardy. The corresponding measure is supported on the unit circle \( T \subset \mathbb{D} \), cf. the definition of integrability on p. 8.

- **Bergman kernels.** Here
  \[ \Phi_n(z) = \sqrt{\frac{1}{2\pi}} z^n, \quad \text{with } K(z,w) = \frac{1}{2\pi} (1 - z\overline{w})^{-2}, \quad |z|, |w| < 1, \]

and the corresponding space \( \mathcal{H}_K \) again is composed of holomorphic functions on the open unit disk \( \mathbb{D} \).

- **\( q \)-Gaussian CSs for \(-1 < q < 1\);** the corresponding \( q \) moment problem is determined and the operators appearing in the \( q \)-oscillator are bounded, see [44] and the references therein.

---

12 Notice there a bifurcation of names in this case.
\( \mathcal{H}_K \) is integrable and the measure is not unique

Two cases for the time being.

- Typical providers are indeterminate moment problems or rather orthonormal polynomials coming from them. If \( (\Phi_n)_n \) is such a sequence of polynomials then the well known consequence is that it satisfies (2). As already shown any of the orthogonality measures appearing in this problem works well for the resolution of the identity to be satisfied. It may create problems for further use of this property. Our construction of CSs and, in particular, of the Segal-Bargmann transform opens a way of overcoming obstacles which may appear.

- The case \( q > 1 \) is also considered in [44]. In [41] two different kinds of orthonormal bases and their RKHS’s are given explicitly: one measure is absolutely continuous with the Lebesque measure on \( \mathbb{C} \), and the other is supported on a countable family of circles tending both to zero and infinity. Needless to say, if \( q \to 1+ \) both RKHS converge do the GKS picture.

\( \mathcal{H}_K \) is not integrable

- The Sobolev space on [0, 1], which is a RKHS, cf. [7, p. 321], is recognized as an example of non-integrable RKHS; to see this perform an argument with logarithmic convexity like on p. 11.

- Consider now

\[ \Phi_n(z) \equiv \frac{n!}{z(z + 1) \cdots (z + n)}. \]

Then

\[ K(z, w) = \sum_{n=0}^{\infty} \frac{n!}{z(z + 1) \cdots (z + n) w(w + 1) \cdots (w + n)} = {}_3F_2(1, 1, 1; z + 1, w + 1; 1), \quad \Re z, \Re w > 1/2. \]

and the space \( \mathcal{H}_K \) is not integrable over \( X = (z, w): \Re z, \Re w > 1/2 \) though HSz CSs make sense. This is a kind of surprising, thought-provoking example, see [31].

- Notice that \( \mathcal{H}_K = \{ \sum_n \xi_n \Phi_n: (\xi_n)_n \in \ell^2 \} \) is the Segal-Bargmann type space of holomorphic functions on \( \{ (z, w): \Re z, \Re w > 1/2 \} \).

Another look at KG CSs

Suppose a sequence \( (k_n)_{n=0}^{\infty} \) of positive numbers (cf. the second item in the list of the KG postulates) is given such that

\[ X = \{ z \in \mathbb{C}: \sum_n k_n^2 |z|^{2n} < +\infty \} \neq \emptyset. \]
This set is rotationally invariant and so is the kernel

\[ K(z,w) = \sum_{n} k_n z^n \bar{w}^n, \quad z,w \in X, \]

which is well defined due to the Schwarz inequality. Because \( K \) is positive definite, we get RHKS \( \mathcal{H}_K \). Furthermore, the monomials \( \Phi_n = k_n^{1/2} z^n \) are orthonormal \(^{13} \) in \( \mathcal{H}_K \). Consequently,

\[ \|\Phi_n\|_{\mathcal{H}_K} = 1 = k_n^{1/2} \|z^n\|_{\mathcal{H}_K}. \tag{8} \]

Suppose for a while \( \mathcal{H}_K \) is integrable and using (8) write

\[ k_{m+n}^{-2} = \left( \int_{X} |z^{2m+n}|^2 \mu(dz) \right)^2 = \left( \int_{X} |z^{2m}|^2 \mu(dz) \right)^2 \leq \left( \int_{X} |z^{2n}|^2 \mu(dz) \right)^2 \int_{X} |z^{2m-n}|^2 \mu(dz) = k_{2m}^{-1} k_{2n}^{-1}. \]

What we have got from the above heuristic reasoning is

\[ k_{m+n}^{-2} \leq k_{2m}^{-1} k_{2n}^{-1}, \]

which is just logarithmic convexity of \( (k_n^{-1})_n \). Therefore logarithmic convexity is a necessary condition for integrability; it is important to know that.

Manipulating \( (k_n^{-1})_n \) to break down logarithmic convexity may provide at once examples of non-integrable \( \mathcal{H}_K \).

Start now from a measure \( \nu \) representing a Stieltjes moment sequence

\[ a_n = \int_{0}^{+\infty} x^n \nu(dx), \quad n = 0, 1, \ldots, \]

and define the rotationally invariant measure \( \mu \) on \( \mathbb{C} \)

\[ \mu(\sigma) = (2\pi)^{-1} \int_{0}^{2\pi} \int_{0}^{+\infty} \chi_{\sigma}(re^{i\varphi}) \nu(dr)d\varphi, \quad \sigma \text{ Borel subset of } \mathbb{C}, \]

where \( \chi_{\sigma} \) is the characteristic (indicator function) of \( \sigma \).

If \( k_n^{-1} \equiv 1_{2\pi} a_{2n} \) then because

\[ \int_{\mathbb{C}} |z|^{2n} \mu(dz) = \int_{0}^{+\infty} r^{2n} \nu(dr), \]

\( (\Phi_n)_n, n = 0, 1, \ldots, \) are orthonormal in \( L^2(\mathbb{C}, \mu) \) as well. Hence the inclusion is isometric and \( \mathcal{H}_K \) is integrable.

---

\(^{13} \) This is due to the fact that the sum appearing in 2o, (ii), p. 8 is holomorphic.
Warning: if the Stieltjes moment problem for \((a_n)_n\) is indeterminate, besides rotationally invariant \(\mu\)’s, non-rotationally invariant measures exist too - despite the fact that the kernel itself is rotationally invariant, cf. [32] and [44]. This never happens when \(\nu\) is determinate, in particular if it has a compact support. It is creditable to suggest here \(q\)-moments: determinate if \(0 < q \leq 1\) and indeterminate if \(q > 1\) which covers both cases [41, 44].

Remark 2. Introduce the sequence \(\sigma_0 \overset{\text{def}}{=} k_0^{-1/2}\), \(\sigma_n \overset{\text{def}}{=} k_n / k_{n+1}\) and define the weighted shift operators, cf. [16, p.146]
\[
a_+ \Phi_n \overset{\text{def}}{=} \sqrt{\sigma_n + 1} \Phi_n, \quad a_- \overset{\text{def}}{=} \sqrt{\sigma_n} \Phi_{n-1}, \quad a_- \Phi_0 \overset{\text{def}}{=} 0.
\]
They can viewed as generalized when compared with the standard definition of the creation and annihilation operators, p. 2 is put in an application. This holds independently of whether \(H_K\) is integrable or not.

2 Holomorphic Hermite polynomials

The holomorphic Hermite polynomials in one and two variables, as well as holomorphic Hermite functions determined by them, will be our main tool extensively used in the next Sections to construct coherent states. This Section serves as a kind of technical introduction and revokes the formulae derived and proved in [4] and [20].

2.1 Holomorphic Hermite polynomials in a single variable

The Hermite polynomials
\[
H_n(z) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2z)^{n-2m}}{m! (n-2m)!}, \quad z = x + iy,
\]
are treated here as functions of a single complex variable \(z\) and as such they become holomorphic.

van Eijndhoven-Meyers orthogonality

As is shown in [11] \(H_n(z)\) satisfy the orthogonality relations
\[
\int_{\mathbb{R}^2} H_m(x+iy)H_n(x+iy) e^{-(1-\alpha)x^2-(1-\alpha)y^2} \, dx dy = \frac{\pi \sqrt{\alpha}}{1-\alpha} \left( 2 \frac{1+\alpha}{1-\alpha} \right)^n n! \delta_{mn},
\]
where $0 < \alpha < 1$ is a parameter. The space $X^{\alpha}_{\text{hol},1}$ of entire functions $f$ such that

$$
\int_{\mathbb{R}^2} |f(z)|^2 e^{\alpha x^2 - \frac{1}{2}y^2} \, dx \, dy < \infty, \quad z = x + iy,
$$

is a Hilbert space with the inner product

$$
\langle f, h \rangle \overset{\text{def}}{=} \int_{\mathbb{R}^2} f(z) \overline{h(z)} e^{\alpha x^2 - \frac{1}{2}y^2} \, dx \, dy,
$$

in which $h^{(\alpha)}_n(z)$ defined by

$$
h^{(\alpha)}_n(z) \overset{\text{def}}{=} e^{-\frac{z^2}{2}} \left( \frac{\pi^{\frac{1}{4}}}{\alpha} \right)^{\frac{1}{2}} \frac{1}{\sqrt{n!}} \left( \frac{1 + \alpha}{1 - \alpha} \right)^n H_n(z), \quad z \in \mathbb{C},
$$

constitute, due to (10), an orthonormal basis. Moreover, because

$$
\sum_{n=0}^{\infty} |h^{(\alpha)}_n(z)|^2 < +\infty,
$$

the formula (2) allows us to initiate Zaremba’s procedure ensuring that the space $X^{\alpha}_{\text{hol},1}$ is RKHS with the kernel

$$
K^{(\alpha)}(z, w) \overset{\text{def}}{=} \sum_{n=0}^{\infty} h^{(\alpha)}_n(z) \overline{h^{(\alpha)}_n(w)}
= e^{-\frac{z^2 + w^2}{2}} \frac{1 - \alpha^2}{2\pi \alpha} \exp \left\{ -\frac{(1 - \alpha)^2}{4\alpha} (z^2 + w^2) + \frac{1 - \alpha^2}{2\alpha} z w \right\}, \quad z, w \in \mathbb{C}.
$$

**The Segal-Bargmann transform**

By the classical Bargmann space $H^{1}_{\text{hol},1}$ [6] we mean the space of those functions in $L^2(\mathbb{C}, \pi^{-1} e^{-|z|^2} \, dz \, dw)$ which are entire, or equivalently, those which are the closure of all polynomials $\mathbb{C}[Z]$ in $L^2(\mathbb{C}, \pi^{-1} e^{-|z|^2} \, dz \, dw)$. Recall that the monomials

$$\Phi_n(z) \overset{\text{def}}{=} \frac{z^n}{\sqrt{n!}}, \quad z \in \mathbb{C}, \quad n = 0, 1, 2, \ldots,
$$

are an orthonormal basis in $H^{1}_{\text{hol},1}$. The unitary transform (namely the Segal-Bargmann one) between $H^{1}_{\text{hol},1}$ and the physical space $L^2(\mathbb{R})$, in which the functions

$$\psi_n(q) = \frac{\sqrt{\alpha}}{\sqrt{2^n n! \sqrt{\pi}}} e^{-aq^2/2} H_n(aq)$$

satisfy

$$
\int_{\mathbb{R}^2} |\psi_n(q)|^2 e^{\alpha x^2 - \frac{1}{2}y^2} \, dx \, dy < \infty, \quad z = x + iy,
$$

the Segal-Bargmann transform
are the orthonormal basis is given as an integral transform with the kernel
\[ A_1(q, z) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{\sqrt{2} a q z - \frac{1}{2} (z^2 + a^2 q^2)}. \] (11)

The similar Segal-Bargmann-like transform between \( H_{\text{hol},1} \) and \( X_{\text{hol},1}^\alpha \) was found in [4]. It is shown there that the mapping
\[ h_\alpha \mapsto \Phi_\alpha \]
\[ \Phi_\alpha(z) = \int_{\mathbb{R}^2} B_1(z, \bar{w}) h_\alpha(w) e^{\alpha z^2 - \frac{1}{2} \bar{w}^2} dw, \quad w = u + iv, \]
with
\[ B_1(z, \bar{w}) = \sum_{n=0}^{\infty} \Phi_n(z) h_\alpha(w) = \left( \frac{1 - \alpha}{\pi \sqrt{\alpha}} \right)^{1/2} e^{\sqrt{2} \alpha (z \bar{w} - \frac{1}{2} (z^2 + \bar{w}^2)), \quad z, w \in \mathbb{C},} \] (12)
is unitary; the notation \( \epsilon = (1 - \alpha)/(1 + \alpha) \) is adopted here and in what follows.

\textbf{Remark 3.} In constructing the transformation between the physical space \( L^2(\mathbb{R}) \) and \( X_{\text{hol},1}^\alpha \) we compose the above mappings and end up with the kernel \( C_1(q, \bar{w}) \)
\[ C_1(q, \bar{w}) = \int_{\mathbb{C}} A_1(q, \bar{w}) B_1(z, \bar{w}) e^{-|z|^2} \frac{dz}{\pi} \]
\[ = \frac{\sqrt{\alpha}}{\pi^{1/4}} \left( \frac{1 - \alpha^2}{2 \pi \alpha \sqrt{\alpha}} \right)^{1/2} \exp \left[ -\frac{1}{2\alpha} (a^2 q^2 + w^2) + \frac{\sqrt{1 - \alpha^2}}{\alpha} a q w \right], \]
which defines the unitary mapping via the integral transformation.

\textbf{Limits}

The van Eijndhoven-Meiers picture enjoys interesting limit properties: \( \alpha \to 0^+ \) and \( \alpha \to 1^- \). These two passages produce very different effects and must be treated separately. Having in mind our main purpose here we shall restrict ourselves to the case \( \alpha \to 1^- \) only. This limit preserves the crucial property needed for our construction of CSs: the existence of a suitable RKHS.

On the other hand the limit \( \alpha \to 0^+ \) leads to results which forbid to construct any kind of CSs being well-defined within our scheme. This is because performing this limit breaks down the fundamental condition [2] and, consequently, the normalizability of CSs [20, 49]. Nevertheless, the polynomials

---

\[ \text{The mass of one-dimensional harmonic oscillator is denoted by } M, \text{ its frequency by } \omega, \text{ and } a = \sqrt{M}\omega. \]
H_m,n(z, ¯z), which arise in the limit α → 0+ of the two variable generalization of the van Eijndhoven-Meiers picture (see the Section 2.2), have found plenty of interesting applications: to mention investigation of their relation with the entangled (in particular EPR) states begun more than 20 years ago [12], continued in [13], and still being the subject of extensive research (cf. references in footnote [15]).

Limit α → 1−
In order to make the limit procedure more efficient we redefine the holomorphic Hermite functions h(α)n(z) as

\[
k^{(α)}_n(z) \equiv \left( \frac{2}{1+α} \right)^{\frac{1}{2}} \left[ 1 - \alpha \right]^{\frac{1}{2}} \frac{1}{\sqrt{n!}} e^{\frac{1}{2} \sqrt{2} z} H_n \left( \sqrt{\frac{2α}{1-α}} z \right)
\]

(13)

Their orthogonality relation

\[
\int C_k^{(α)}(z) \overline{k^{(α)}_m(z)} e^{-|z|^2} \frac{dz}{π} = δ_{nm}
\]

can be immediately derived from (10). The reproducing kernel coincides with Bargmann’s

\[
K^{(α)}_1(z, w) \equiv \sum_{n=0}^{∞} k^{(α)}_n(z) \overline{k^{(α)}_n(w)} = e^{z\overline{w}}.
\]

The Segal-Bargmann transform ψ_n(q) → k^{(α)}_n(z) now is based on the kernel

\[
\hat{C}_1(q, z) = \sqrt{\frac{2πα}{1-α}} e^{\frac{1}{2} \sqrt{2} \alpha z^2} C_1(q, \sqrt{\frac{2α}{1-α}} z)
\]

\[
= \frac{\sqrt{πα}}{(πα)^{1/4}} \exp \left[ -\frac{1}{2α} (a^2 q^2 + α z^2) + \sqrt{2α} aqz \right]
\]

which tends to the kernel (11) when α → 1−.

The last unnumbered formula on p. 97 of [11] with \( t = \sqrt{(1-α^2)/2α} \) yields

\[
\lim_{α→1−} k^{(α)}_n(z) = \Phi_n(z),
\]

(14)

i.e. recovers the Bargmann basis. Details can be found in [42].
2.2 Holomorphic Hermite polynomials in two variables

Hermite polynomials in two complex variables are defined as

\[
H_{m,n}(z_1, z_2) \overset{\text{def}}{=} \sum_{k=0}^{\min(m,n)} \binom{n}{k} (-1)^k k! z_1^{m-k} z_2^n,
\]

where \( m, n = 0, 1, 2, \ldots \). The essence now is to think of them as holomorphic Hermite polynomials in \( z_1, z_2 \). The polynomials \( H_{m,n}(z_1, z_2) \) come from the generating function

\[
\exp(z_1 s + z_2 t - st) = \sum_{m,n} \frac{s^m t^n}{m!n!} H_{m,n}(z_1, z_2).
\]

It does not factorize as a product of two functions which may be generating functions of two other systems of orthogonal polynomials; the lack of factorization can be seen from the operational (raising and lowering) relations

\[
\begin{align*}
H_{m+1,n}(z_1, z_2) &= (z_1 - \partial_{z_1})H_{m,n}(z_1, z_2), \\
H_{m,n+1}(z_1, z_2) &= (z_2 - \partial_{z_2})H_{m,n}(z_1, z_2), \\
\partial_{z_2}H_{m,n}(z_1, z_2) &= nH_{m,n-1}(z_1, z_2), \\
\partial_{z_1}H_{m,n}(z_1, z_2) &= mH_{m-1,n}(z_1, z_2).
\end{align*}
\]

Using (16) the Hermite polynomials in two variables can be expressed, like it is shown in (20, Eq.(13)), in terms of the Hermite polynomials in a single variable (9)

\[
H_{m,n}(z_1, z_2) = 2^{-(m+n)} \sum_{k=0}^{m} \sum_{l=0}^{n} (-i)^{n-l} H_{k+l}(\frac{z_1 + z_2}{\sqrt{2}}) H_{m-n}(\frac{z_1 - z_2}{\sqrt{2}}),
\]

which does not undermine the just mentioned lack of factorizability.

Using (17) and formula (0.5) in (11) provides us with the orthogonality relations

\[
\int_{\mathbb{C}^2} H_{m,n}(z_1, z_2) \overline{H_{p,q}(z_1, z_2)} \exp\left(-\frac{1-\alpha}{4} |z_2 + z_1|^2 - \frac{1-\alpha}{4\alpha} |z_2 - z_1|^2\right) dz_1 dz_2
\]

\[
= \frac{\pi^{\alpha}}{(1-\alpha)^2} \frac{(1+\alpha)^{m+n}}{m!n!} \delta_{m,p} \delta_{n,q}
\]

---

15 Polynomials \( H_{m,n}(z_1, z_2) \) (mentioned previously as a special case of polynomials \( H_{m,n}(z_1, z_2) \)) for \( z_1 = z \) and \( z_2 = \bar{z} \), have been present in mathematical and physical literature for around 65 years and bear the names: Ito’s polynomials, incomplete or 2D Hermite polynomials, complex Hermite polynomials, Laguerre polynomials in two variables and possibly the other - for recent literature on the subject see [2, 3, 8, 9, 13, 19, 25, 33]. Here we want to emphasize that (15) defines polynomials in two complex variables with real coefficients while \( H_{m,n}(z, \bar{z}) \) are polynomials in two real variables \( x = \Re z \) and \( y = \Im z \) with complex coefficients. This may be somehow confusing when the term ”complex Hermite polynomials” appears for the latter, for more discussion see the introductory section in (20).
Coherence, squeezing and entanglement – an example of peaceful coexistence

valid for $0 < \alpha < 1$. Though algebraic properties of $H_{m,n}(z_1,z_2)$ have been widely considered in many papers (cf. references in [15], investigation of their analytic properties done in [20] is, according to our best knowledge, a novelty.

The orthogonality relations ([13]) allow to introduce normalized holomorphic Hermite functions

$$h_{m,n}^{(\alpha)}(z_1,z_2) = \frac{1 - \alpha}{\pi} \frac{1 - \alpha}{\alpha} \frac{\exp(-\frac{z_1^2 + z_2^2}{2})}{\sqrt{m!n!}} H_{m,n}(z_1,z_2),$$

where $z_1, z_2 \in \mathbb{C}$ and $0 < \alpha < 1$. These functions satisfy the relation

$$\sum_{m,n=0}^{\infty} |h_{m,n}^{(\alpha)}(z_1,z_2)|^2 = \frac{(1 - \alpha^2)^2}{4\pi^2 \alpha^2} e^{1 - \frac{1}{2} \frac{m^2}{\alpha^2} + \sum_{1}^{\infty} \frac{1}{m^2} (\alpha^2)} < +\infty,$$

which according to Zaremba’s procedure makes it possible to introduce $\mathcal{H}^{(\alpha)}$, being a RKHS with the kernel

$$K^{(\alpha)}(z_1,z_2,w_1,w_2) = \sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}(z_1,z_2) \overline{h_{m,n}^{(\alpha)}(w_1,w_2)}$$

$$= \frac{1 - \alpha^2}{4\pi^2 \alpha^2} \exp \left[ \frac{1 - \alpha^2}{4\alpha} (z_1 w_1 + z_2 w_2) - \frac{1 + \alpha^2}{4\alpha} (z_1 z_2 + w_1 w_2) \right],$$

calculated using either the formula (26) and Lemma 8 in [20] or [49] formula (5.2).

The Segal-Bargmann transform

The monomials

$$\Phi_{m,n}(z_1,z_2) = \frac{z_1^m z_2^n}{\sqrt{m!n!}}, \quad z_1, z_2 \in \mathbb{C}, \quad m, n = 0, 1, 2, \ldots,$$

form an orthonormal basis in the two variable Bargmann space $\mathcal{H}_{\text{hol,2}} = \text{Hol}(\mathbb{C}) \cap L^2(\mathbb{C}^2, \pi^{-3} \exp(-|z_1|^2 - |z_2|^2) dz_1 dz_2)$ and may be transformed into the square integrable functions describing the system of two independent harmonic oscillators

$$\psi_{m,n}(q_1,q_2) = \psi_m(q_1)\psi_n(q_2)$$

$$= \sqrt{ab} e^{-(a^2 q_1^2 + b^2 q_2^2)/2} H_m(aq_1)H_n(bq_2),$$

which are the basis in the physical Hilbert space $L^2(\mathbb{R}^2)$. The mapping $\mathcal{H}_{\text{hol,2}}$ into $L^2(\mathbb{R}^2)$ is unitary and has the kernel

---

16 We take that $h = 1$, $a = \sqrt{M\omega_x}$ and $b = \sqrt{M\omega_y}$, $M$ denotes the total mass of the system and frequencies of oscillations in $x$ and $y$ direction are $\omega_x$ and $\omega_y$, respectively.
\[ A_2(q_1, q_2, \bar{z}_1, \bar{z}_2) = A_1(q_1, \bar{z}_1)A_1(q_2, \bar{z}_2) \]
\[ = \sqrt{\frac{ab}{\pi}} e^{-\frac{\sqrt{\alpha}}{2}z_1\bar{z}_2 - \frac{\sqrt{\alpha}}{2}(a^2q_1^2 + b^2q_2^2)} e^{\sqrt{2\alpha}q_1\bar{z}_1 + bq_2\bar{z}_2}) . \]  

Thus, we have
\[ \Phi_{m,n}(z_1, z_2) = \int_{\mathbb{R}^2} \psi_{m,n}(q_1, q_2)A_2(q_1, q_2, \bar{z}_1, \bar{z}_2)dq_1dq_2. \]

Another Bargman-like transform acts between the spaces \( \mathcal{H}^{(\alpha)} \) and \( \mathcal{H}_{bil,2} \). It has been shown in [20, Section “Relating \( \mathcal{H}^{(\alpha)} \) to the Bargmann space”] that this transform is also unitary and possesses the kernel
\[ B_2(z_1, z_2, \bar{w}_1, \bar{w}_2) = \sum_{m,n=0}^{\infty} \Phi_{m,n}(z_1, z_2)h_{m,n}^{(\alpha)}(w_1, w_2) \]
\[ = 1 - \frac{\alpha}{\pi} \sqrt{\alpha} e^{-\frac{1}{\alpha}\bar{w}_1 + \sqrt{\alpha}(\bar{w}_1 + \bar{w}_2) + \epsilon z_1 z_2}, \]

where \( \epsilon \) was defined just after (12) and \( 0 < \alpha < 1 \). That leads to
\[ h_{m,n}^{(\alpha)}(w_1, w_2) = \int_{C^2} \Phi_{m,n}(z_1, z_2)B_2(z_1, z_2, \bar{w}_1, \bar{w}_2)e^{-|z_1|^2 - |z_2|^2} \frac{d\bar{z}_1d\bar{z}_2}{\pi^2}. \]

Extending Remark 3 to the 2D case we compose the transformations \( A_2 \) and \( B_2 \) and obtain the unitary mapping \( L^2 \to \mathcal{H}^{(\alpha)} \) with the kernel
\[ C_2(q_1, q_2, \bar{w}_1, \bar{w}_2) = \int_{C^2} A_2(q_1, q_2, \bar{z}_1, \bar{z}_2)B_2(z_1, z_2, \bar{w}_1, \bar{w}_2)e^{-|z_1|^2 - |z_2|^2} \frac{d\bar{z}_1d\bar{z}_2}{\pi^2} \]
\[ = \sqrt{\frac{ab}{\pi}} \frac{1 - \alpha^2}{2\pi\alpha} e^{-\frac{1}{\alpha}q_1^2(a^2q_1^2 + b^2q_2^2) - \frac{1}{\alpha}q_2^2(a^2q_1^2 + b^2q_2^2) - \frac{1}{\alpha}\sqrt{\alpha}q_2^2} \]
\[ \times e^{-\frac{1}{\alpha}(\sqrt{\alpha}q_1 \bar{w}_1 + \sqrt{\alpha}q_2 \bar{w}_2)} \sqrt{\frac{1 + \alpha}{\alpha}} \{1 + \alpha(aq_1 \sqrt{\alpha}q_1^2 + bq_2 \sqrt{\alpha}q_2^2) + (1 - \alpha)(aq_1 \bar{w}_1 + bq_2 \bar{w}_2)\}. \]

**Limit \( \alpha \to 1^- \)**

The limit case \( \alpha \to 1^- \) will be considered analogously to what was done for the Hermite polynomials in a single variable. We begin with redefining Hermite functions in two variables \( h_{m,n}^{(\alpha)}(z_1, z_2) \) as follows
\[ h_{m,n}^{(\alpha)}(z_1, z_2) \equiv \frac{2}{1 + \alpha} \left( \frac{1 - \alpha}{1 + \alpha} \right)^{m+n} e^{\frac{1}{1 + \alpha}z_1z_2} \frac{\alpha}{\sqrt{m!n!}} H_{m,n} \left( \frac{2\sqrt{\alpha}z_1}{\sqrt{1 - \alpha}}, \frac{2\sqrt{\alpha}z_2}{\sqrt{1 - \alpha}} \right) \]
\[ = \frac{2\alpha}{1 - \alpha^2} e^{\frac{1}{1 - \alpha^2}z_1z_2} h_{m,n}^{(\alpha)} \left( \frac{2\sqrt{\alpha}z_1}{\sqrt{1 - \alpha}}, \frac{2\sqrt{\alpha}z_2}{\sqrt{1 - \alpha}} \right). \]
They satisfy the orthogonality relation
\[
\int_{C^2} k_{m,n}(z_1, z_2) k_{m', n'}(z_1, z_2) \exp(-|z_1|^2 - |z_2|^2) \frac{dz_1 dz_2}{\pi^2} = \delta_{m,m'} \delta_{n,n'}
\]
and form RKHS \( \mathcal{K}^{(a)} \) with the kernel
\[
K^{(a)}(z_1, z_2, w_1, w_2) = \exp(z_1 w_1 + z_2 w_2),
\]
which again coincides with the two dimensional Bargmann one. The Segal-Bargmann transform connecting the spaces \( L^2(\mathbb{R}^2) \) and \( \mathcal{K}^{(a)} \) reads
\[
C_2(x, y, w_1, w_2) = \frac{2\pi \alpha}{1 - \alpha^2} e^{\frac{1+\alpha^2}{4\alpha} (a^2 q_1^2 + b^2 q_2^2)} e^{-\frac{1}{2} (q_1^2 + q_2^2)} e^{-\frac{1-\alpha^2}{2\alpha} a b q_1 q_2}
\]
\[
\times e^{\frac{1+\alpha^2}{2\alpha} (a^2 w_1^2 + b^2 w_2^2)} e^{-\frac{1}{2} (w_1^2 + w_2^2)} e^{-\frac{1-\alpha^2}{2\alpha} a b w_1 w_2},
\]
and in the limit \( \alpha \to 1- \) tends to (19). Analogously to (14) one gets
\[
\lim_{\alpha \to 1-} k_{m,n}(z_1, z_2) = \Phi_{m,n}(z_1, z_2),
\]
i.e. performing the limit procedure we end up on the 2D Bargmann basis.

3 HSz CSs - holomorphic Hermite polynomials perspective

Let us recall that, according to our definition of coherent states evolved in Section 1.4 and starting from the formula (3), the basic requirement for some states to be called coherent is to be provided with (\( \Phi_{n,n} \)) satisfying (2). Now the Zaremba construction guarantees existence of the Segal-Bargmann transform, the property which is historically and not too rigorously identified with the overcompleteness and/or the resolution of the identity.

Our definition of coherent states allows to put traditional (like those in A, B and C on page 2) coherent and squeezed states on the same footing.

Considerations in the previous section show that introduced there complex Hermite functions fulfill the conditions imposed on RKHS. This opens a green light to engage it in construction of the coherent states. We shall do it in the next Section and show that so obtained states not only satisfy the resolution of the identity (which by the way is resulting from their construction) but they turn out to be also entangled.
To make the above statement more precise let us formulate the definition of what has to be understand under the notion of entanglement.

Suppose two separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are given. Let $\mathcal{H} \otimes \mathcal{K}$ be the state space. Call a state $c$ in $\mathcal{H} \otimes \mathcal{K}$ decomposable (or factorizable) if $c = c_{\mathcal{H}} \otimes c_{\mathcal{K}}$ with $c_{\mathcal{H}} \in \mathcal{H}$ and $c_{\mathcal{K}} \in \mathcal{K}$. A state which is not decomposable will be called entangled. Referring to CSs we can say that the family $\{c_x\}_{x \in X}$ is decomposable if

$$c_x = \sum_{m,n=0}^{\infty} \Phi_m(x) e_m \otimes \Psi_n(x) f_n, \quad x \in X,$$

$(\Phi_m)_{m=0}^{\infty}$ and $(\Psi_n)_{n=0}^{\infty}$ are orthonormal bases in suitable RKHSs.

If there are no such $(\Phi_m)_{m=0}^{\infty}$ and $(\Psi_n)_{n=0}^{\infty}$ making the above decomposition possible the family becomes entangled by definition. However, $\{c_x\}_{x \in X}$ as members of the state space $\mathcal{H} \otimes \mathcal{K}$ are HSz CSs anyway.

$^a$ In the literature there is no undoubtedly defined notion of the entanglement (cf. footnote 17).

In what follows we will provide the reader with a keystone example of new bosonic states, based on the holomorphic Hermite polynomials, which are coherent and entangled simultaneously. Even more, these new states appear to be squeezed. Surprisingly, the limit procedures which the Hermite polynomials enjoy allow to link entangled and decomposable states within the HSz coherent states framework. All this happens, due to the Proposition 1, under the guidance of HSz coherent states merging mathematical and physical aspects of the novel CSs.

4 CSs for holomorphic Hermite polynomials.

4.1 Single particle Hermite CSs - coherence and squeezing

Single particle CSs corresponding to the sequence $(\kappa_n^{(\alpha)})_n$ are defined as

$$c_z^{(\alpha)} = \sum_{n=0}^{\infty} \kappa_n^{(\alpha)} (z) e_n, \quad z \in \mathbb{C}.$$ 

Using the recurrence relation $H_{n+1} = 2zH_n(z) - 2nH_{n-1}(z)$ one gets

$^{17}$ States which are frequently appearing in the literature under the name coherent entangled states are bipartite Bell-like states constructed using tensor products of standard coherent states, usually $|z\rangle$ and $|-z\rangle$; they are obviously entangled but not coherent in any commonly acceptable sense.
Coherence, squeezing and entanglement – an example of peaceful coexistence

\[
k_{n+1}^{(a)}(z) = z \frac{2 \sqrt{\alpha}}{1 + \alpha} \frac{1}{\sqrt{n+1}} k_{n}^{(a)}(z) - \frac{1 - \alpha}{1 + \alpha} \sqrt{\frac{n}{n+1}} k_{n-1}^{(a)}(z)
\]

and shows that \( c_{z}^{(a)} \) appear to be eigenfunctions

\[
B_{-}c_{z}^{(a)} = zc_{z}^{(a)}
\]

of the operator

\[
B_{-} \equiv \frac{1 + \alpha}{2 \sqrt{\alpha}} b + \frac{1 - \alpha}{2 \sqrt{\alpha}} b^{\dagger},
\]

where \( b \) and \( b^{\dagger} \) denote the canonical annihilation and creation operators. \( B_{-} \) together with \( B_{+} \) given by

\[
B_{+} \equiv \frac{1 + \alpha}{2 \sqrt{\alpha}} b^{\dagger} + \frac{1 - \alpha}{2 \sqrt{\alpha}} b
\]

satisfy the commutation relations \([B_{-}, B_{+}] = 1 \) and \([B_{-}, B_{-}] = [B_{+}, B_{+}] = 0 \) which mean that (20) and (21) belong to the class of the Bogolubov transformations [15], the relation of which to the squeezed (coherent) states is well established [22]. So the states \( c_{z}^{(a)} \), primarily required only to satisfy the resolution of the identity, are also squeezed states in the sense of (A), p.2 and in the limit \( \alpha \to 1^{-} \) become exclusively coherent in the traditional meaning.

HERMITE CSs AND SINGLE MODE SQUEEZING OPERATION

The squeezed states \( \eta_{\xi}^{\epsilon} \) may also be introduced through squeezing operation acting on the standard coherent states

\[
\eta_{\xi}^{\epsilon} \equiv S(\xi)\eta_{\epsilon}, \quad S(\xi) = e^{\epsilon K_{-} - \epsilon^{-} K_{-}}, \quad \xi \in \mathbb{C},
\]

where \( K_{\pm} \) are the generators of \( su(1,1) \) algebra which, together with the third one \( K_{0} \), satisfy the commutation relations

\[
[K_{-}, K_{+}] = 2 K_{0}, \quad [K_{0}, K_{\pm}] = \pm K_{\pm}.
\]

Setting \( \zeta = \xi \tanh(|\xi|)/|\xi|, \ |\xi| < 1, \) the squeeze operator \( S(\xi) \) can be disentangled employing the well-known Zassenhaus formula

\[
S(\xi) = e^{\epsilon K_{+} - \epsilon^{-} K_{-}} = e^{\epsilon K_{+} e^{\ln(1+|\xi|^{2})} K_{0} e^{-\epsilon K_{-}}}.
\]

This may be used to obtain so-called squeezed basis \( \epsilon_{n}^{\xi} = S(\xi) \epsilon_{n}, \ n = 0, 1, 2, \ldots \) with which the squeezed states \( \eta_{\xi}^{\epsilon} \) are written as

\[
\eta_{\xi}^{\epsilon} = \sum_{n=0}^{\infty} \frac{z^{n}}{n!} \epsilon_{n}^{\xi}.
\]
Since the squeeze operator is unitary the squeezed basis is also orthonormal in the Bargmann space $\mathcal{H}_{\text{hol}}$. Squeezed states satisfy the same resolution of identity as $\eta_z$.

In the Bargmann representation the operators $K_+$ and $K_0$ have the form

$$K_+ = \frac{1}{2} z^2, \quad K_- = \frac{1}{2} \partial_z^2, \quad K_0 = \frac{1}{2} (1 + z \partial_z).$$

The squeezed RKHS basis $\Phi_n^\xi$ is determined by the action of $S(\xi)$ (given by (22) with (23) put in) on $\Phi_n(z)$. The calculation presented in [4] leads to

$$\Phi_n^\xi(z) = (1 - |\zeta|^2) \frac{1}{2} z^2 \zeta^m \sqrt{\frac{1}{2n!}} H_n \left( \sqrt{\frac{1 - |\zeta|^2}{2\zeta}} z \right).$$

From the algebraic relation $H_{n+1} = 2zH_n - H'_n$ we get

$$\sqrt{n+1} \Phi_{n+1}^\xi(z) = A_+ \Phi_n^\xi(z) \quad \text{with} \quad A_+ = (1 - |\zeta|^2)^{-\frac{1}{2}} (z - \zeta \partial_z),$$

while the twin relation $2nH_{n-1} = H'_n$ implies

$$\sqrt{n} \Phi_{n-1}^\xi(z) = A_- \Phi_n^\xi(z) \quad \text{where} \quad A_- = (1 - |\zeta|^2)^{-\frac{1}{2}} (\partial_z - \zeta).$$

Assuming that $\zeta = \epsilon$ (defined below (12)) we obtain, because of (24), that $\Phi_n^{\text{arctan}(\epsilon)}(z) = k_n^{\text{arctan}}(z)$ given by (13). That provides us the physical interpretation of the up-to-now mathematically contemplated parameter $\alpha$ [11, 42] - from now it is to be identified with the physical squeezing parameter which measures the ratio between coordinate and momentum uncertainties.

Comparing operators $B_-$ and $B_+$ with $A_-$ and $A_+$ for $\zeta = \epsilon$ we get the Segal-Bargmann representation of operators $b$ and $b^*$ [42]

$$b = \frac{1 + \alpha^2}{2\alpha} \partial_z - \frac{1 - \alpha^2}{2\alpha} z \quad \text{and} \quad b^* = \frac{1 + \alpha^2}{2\alpha} z - \frac{1 - \alpha^2}{2\alpha} \partial_z.$$

As it should be it goes to the standard Bargmann representation for $\alpha \to 1$.

### 4.2 Bipartite CSs - coherence, squeezing and entanglement

**Coherent states** $c_{z_1, z_2}^{(a)}$

Our approach to CSs, based on the definition given in Section 1.4, is by no means restricted to the single particle case. It may be automatically extended to multipartite systems. Here we shall present an application to bipartite systems taking as a starting point holomorphic Hermite functions in two variables $k_{m,n}^{(a)}$ enabling to construct CSs. Taking as a state space $\mathcal{H} \otimes \mathcal{K}$, where each of $\mathcal{H}$ and $\mathcal{K}$ is a state space for itself, according to our scheme we can introduce the family of CSs.
Coherence, squeezing and entanglement – an example of peaceful coexistence

\[ c_{z_1, z_2}^{(a)} = \sum_{m,n} k_{m,n}^{(a)}(z_1, z_2)(\epsilon_m \otimes f_n), \quad z_1, z_2 \in \mathbb{C}^2, \quad (25) \]

which reside in the Hilbert space \( \mathcal{H} \otimes \mathcal{K} \).

The recurrence relations \([20], (12)\]

\[
H_{m+1,n}(z_1, z_2) = z_1 H_{m,n}(z_1, z_2) - nH_{m,n-1}(z_1, z_2), \\
H_{m,n+1}(z_1, z_2) = z_2 H_{m,n}(z_1, z_2) - mH_{m-1,n}(z_1, z_2)
\]

lead to

\[
k_{m+1,n}^{(a)}(z_1, z_2) = z_1 \frac{2 \sqrt{\alpha}}{\sqrt{1-\alpha^2}} \frac{1}{\sqrt{m+1}} k_{m,n}^{(a)}(z_1, z_2) - \frac{1-\alpha}{1+\alpha} \sqrt{n} \frac{1}{\sqrt{m+1}} k_{m,n-1}^{(a)}(z_1, z_2), \\
k_{m,n+1}^{(a)}(z_1, z_2) = z_2 \frac{2 \sqrt{\alpha}}{\sqrt{1-\alpha^2}} \frac{1}{\sqrt{n+1}} k_{m,n}^{(a)}(z_1, z_2) - \frac{1-\alpha}{1+\alpha} \sqrt{m} \frac{1}{\sqrt{n+1}} k_{m-1,n}^{(a)}(z_1, z_2),
\]

which enable one to show that the states \( c_{z_1, z_2}^{(a)} \) are common eigenvectors

\[
B_{1,-} c_{z_1, z_2}^{(a)} = z_1 c_{z_1, z_2}^{(a)} \quad B_{2,-} c_{z_1, z_2}^{(a)} = z_2 c_{z_1, z_2}^{(a)}, \quad z_1, z_2 \in \mathbb{C}^2,
\]

of the operators \( B_{1,-} \) and \( B_{2,-} \)

\[
B_{1,-} = \frac{1 + \alpha}{2 \sqrt{\alpha}} b_1 + \frac{1 - \alpha}{2 \sqrt{\alpha}} b_1^+, \quad B_{2,-} = \frac{1 - \alpha}{2 \sqrt{\alpha}} b_2 + \frac{1 + \alpha}{2 \sqrt{\alpha}} b_2^+
\]

where \( b_i^\dagger \) and \( b_i \) \( (i = 1, 2) \) denote the canonical creation and annihilation operators for the modes \( i = 1, 2 \). Operators \( B_{i,-} \) together with their adjoints \( B_{i,+} \), \( i = 1, 2 \) satisfy the standard canonical commutation relations \([B_{i,-}, B_{j,+}] = \delta_{ij}, \ [B_{i,-}, B_{j,-} = [B_{i,+}, B_{j,+}] = 0 \) for \( i, j = 1, 2 \). Proceeding further and using \([27]\) one shows that

\[
B_{1,-} \otimes B_{2,-} c_{z_1, z_2}^{(a)} = z_1 z_2 c_{z_1, z_2}^{(a)}, \quad z_1, z_2 \in \mathbb{C}^2.
\]

Taken together \([27]\) and \([29]\) mean that \( c_{z_1, z_2}^{(a)} \) fulfill the postulate (A) listed on the p. \([21]\) generalized here to the multimode case, i.e. to the set of mutually commuting operators playing the role of annihilators. Simultaneously, because of \([28]\), we see that this time we deal with the Bogolubov transformation which (unlike for the single particle case) mixes the modes. But, like previously, appearance of the Bogolubov transformation suggests that \( c_{z_1, z_2}^{(a)} \) may have something in common with squeezed states - this will be clarified in the next Section.

**HERMITE CSs AND TWO MODE SQUEEZING OPERATION**

Consider the two mode representation of the generators of \( su(1,1) \) algebra given by

\[
K_+ = z_1 z_2, \quad K_- = \partial z_1 \partial z_2, \quad K_0 = \frac{1}{2} (1 + z_1 \partial z_1 + z_2 \partial z_2),
\]

(30)
and extend the definition of the RKHS squeezed basis to the bipartite system

\[ \Phi_{m,n}(z_1, z_2) = S(\xi) \Phi_{m,n}(z_1, z_2), \quad \text{where} \quad S(\xi) = e^{\xi K_+ - \xi K_-}. \quad (31) \]

Then, using (31), (22), (30) and [9, (I.5.2d) on p. 24] we get

\[ \Phi_{m,n}(z_1, z_2) = \sqrt{1 - |\xi|^2} \sum_{n=0}^{\infty} \frac{\xi^{2n}}{n!} e^{\xi z_1 \bar{z}_2} H_{m,n}(z_1, z_2), \]

which span the appropriate RKHS being a subspace of \( L^2(\mathbb{C}^2, \pi^{-2} e^{-|z|^2} d^2 z_1 d^2 z_2) \).

In the Bargmann representation creation and annihilation operators acting on the functions \( f \in \text{lin}(\Phi_{m,n}^\xi) \) behave as

\[
(A_{1,+}^\xi f)(z_1, z_2) = \frac{z_1 - \bar{\xi} \partial_{z_2}}{\sqrt{1 - |\xi|^2}} f(z_1, z_2), \quad (A_{1,-}^\xi f)(z_1, z_2) = \frac{\partial_{z_1} - \xi z_2}{\sqrt{1 - |\xi|^2}} f(z_1, z_2), \\
(A_{2,+}^\xi f)(z_1, z_2) = \frac{z_2 - \bar{\xi} \partial_{z_1}}{\sqrt{1 - |\xi|^2}} f(z_1, z_2), \quad (A_{2,-}^\xi f)(z_1, z_2) = \frac{\partial_{z_2} - \xi z_1}{\sqrt{1 - |\xi|^2}} f(z_1, z_2)
\]

for \( z_1, z_2 \in \mathbb{C} \).

**Remark 4.** The justification of name annihilation and creation operators comes from the fact that operators \( A_{1,+/-} \) act on the first mode \( m \) as

\[ A_{1,+}^\xi \Phi_{m,n}^\xi = \sqrt{m+1} \Phi_{m+1,n}^\xi, \quad A_{1,-}^\xi \Phi_{m,n}^\xi = \sqrt{m} \Phi_{m-1,n}^\xi, \]

while \( A_{2,+/-} \) act on the second mode \( n \) as

\[ A_{2,+}^\xi \Phi_{m,n}^\xi = \sqrt{n+1} \Phi_{m,n+1}^\xi, \quad A_{2,-}^\xi \Phi_{m,n}^\xi = \sqrt{n} \Phi_{m,n-1}^\xi. \]

For \( \xi = \epsilon \) we have \( \Phi_{m,n}^{\text{arc tan}(\epsilon)}(z_1, z_2) = \Phi_{m,n}^{(\alpha)}(z_1, z_2) \). Comparing \( A_{1,+/-}^\epsilon \) with \( B_{i,+/-} \) we find the Bargmann representation of operators \( b_i^+ \) and \( b_i \), \( i = 1, 2 \)

\[
b_1^+ = \frac{1}{2\alpha} z_1 - \frac{1-\alpha^2}{2\alpha} \partial_{z_2}, \quad b_1 = \frac{1}{2\alpha} \partial_{z_1} - \frac{1-\alpha^2}{2\alpha} z_2, \\
b_2^+ = \frac{1}{2\alpha} z_2 - \frac{1-\alpha^2}{2\alpha} \partial_{z_1}, \quad b_2 = \frac{1}{2\alpha} \partial_{z_2} - \frac{1-\alpha^2}{2\alpha} z_1.
\]

We see that the parameter \( \alpha \) is responsible not only for squeezing but also for mixing the modes, one should also notice that both these effects disappear in the limit \( \alpha \to 1 \).
Coherence, squeezing and entanglement – an example of peaceful coexistence

ENTANGLED SQUEEZED COHERENT STATES

As said in the Section 3 the proper definition of the entanglement qualifies a state to be entangled if it is not factorizable. Because of (17) and operational rules satisfied by polynomials \( H_{m,n}(z_1,z_2) \) this is the case for the states \( c^{(\alpha)}_{z_1,z_2} \) which can not be represented as a product of factors depending separately on \( z_1 \) and \( z_2 \). But, as it has been demonstrated, \( c^{(\alpha)}_{z_1,z_2} \) are simultaneously coherent/squeezed which phenomenon at first glance may seem to be a little unexpected, nevertheless is shown to be a fact possible due to the generalization of coherence presented in our study.

Search for quantum states which are simultaneously coherent and entangled, or, more precisely, which satisfy some criteria allowing to call them coherent and entangled, is not new. Example of such states, called coherent-entangled, was provided in [13] where the authors found explicit form of bipartite states being common eigenvectors of the center of mass coordinate operator and the difference of canonical annihilators \( a_1 - a_2 \) and next linked superposition of these states to the standard example illustrating entanglement, namely to the EPR states, i.e. common eigenstates of the center of mass coordinate and relative momentum operators. Fan-Lu states, as may be seen from Eq.8 in [13], are nonfactorizable and satisfy the formal resolution of unity (e) but are not of a finite norm which means that they break one of requirements on which our construction is based. The problem becomes analogous to that which we have roughly mentioned in the Section 2.1 when have remarked on the limit case \( \alpha \to 0^+ \). The latter problem needs a very special and careful analysis which goes beyond the current research and this is why we have decided to exclude it from our considerations and restrict ourselves to the statement as follows:

As long as \( 0 < \alpha < 1 \) the states \( c^{\alpha}_{z_1,z_2} \) given by (25) exhibit the coherence/squeezing and entanglement peacefully coexisting and, moreover, somewhat interrelated. This is possible due to the HSz approach which proposes to look at the properties of coherent states through the reproducing kernel property and which enables us to see the resolution of the identity in much wider context, especially avoiding the restrictive assumption of rotational invariance of the measure in question. Linearity which is sitting in the heart of quantum physics and which enforces us to treat all linear combinations of elementary solutions on the same footing supports this kind of approach. A significant feature is that the limit \( \alpha \to 1^- \) switches off both entanglement and squeezing but does not loose anything of coherence.

Acknowledgements The work of the third author is supported by the grant of NCN (National Science Center, Poland), decision No. DEC-2013/11/B/ST1/03613.
References

1. Ali S. T., Antoine J.-P., and Gazeau J.-P.: Coherent States, Wavelets and Their Generalizations (Springer, New York, 1999).
2. Ali S. T., Bagarello F., and Gazeau J.-P.: D-pseudo-bosons, complex Hermite polynomials, and integral quantization, SIGMA Symmetry Integrability Geom. Methods Appl. 11 (2015) 078 (23 pp).
3. Ali S. T., Ismail M. E. H., and Shah N. M.: Deformed complex Hermite polynomials, arXiv:1410.3908.
4. Ali S. T., Górka K., Horzela A., and Szafraniec F. H.: Squeezed states and Hermite polynomials in a complex variable, J. Math. Phys. 55 (2014) 012107 (11pp).
5. Aronszajn N.: Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950) 337–404.
6. Bargmann V.: On a Hilbert space of analytic functions and an associated integral transform, Commun. Pure Appl. Math. 14 (1961) 187–214.
7. Berlinet A., and Thomas-Agnan Ch.: Reproducing Kernel Hilbert Spaces in Probability and Statistics (Kluwer, Berlin, 2004).
8. Cotfas N., Gazeau J.-P., and Górka K.: Complex and real Hermite polynomials and related quantizations, J. Phys. A: Math. Theor. 43 (2010) 305304 (14 pp).
9. Dattoli G., Ottaviani P. L., Torre A., and Vázquez L.: Evolution operator equations: integration with algebraic and finite-difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory, La Rivista del Nuovo Cimento 20(4) (1997) 1–133.
10. Dodonov V. V.: Nonclassical’ states in quantum optics: a ‘squeezed’ review of the first 75 years, J. Opt. B: Quantum Semiclass. Opt. 4 (2002) R1–R33.
11. van Eijndhoven S. J. L., and Meyers J. L. H.: New orthogonality relations for the Hermite polynomials and related Hilbert spaces, J. Math. Anal. Appl. 146 (1990) 89–98.
12. Fan Hong-Yi, and Klauder J. R.: Eigenvectors of two particles’ relative position and total momentum, Phys. Rev. A. 49 (1994) 704–707.
13. Fan Hong-Yi, and Lu Hai-Liang: New two-mode coherent-entangled state and its application, J. Phys. A: Math. Theor. 37 (2004) 10993–11001.
14. Fan Hong-Yi, Wang Zhi-Long, Wu Ze, and Zhang Peng-Fei: A new kind of physical special function and its application, Chin. Phys. B 24 (2015) 100302 (4 pp).
15. Fetter A., and Walecka J.: Quantum theory of many body systems (Dover, 2003).
16. Gazeau J.-P.: Coherent states in quantum physics (Wiley-VCH, Weinheim, 2009).
17. Gazeau J.-P., and Klauder J. R.: Coherent states for systems with discrete and continuous spectrum, J. Phys. A: Math. Gen. 32 (1999) 123–132.
18. Glauber R. L.: Coherent and incoherent states of the radiation field, Phys. Rev. 131 (1963) 2766–2788.
19. Ghanmi A.: Operational formulae for the complex Hermite polynomials $H_{p,q}(z,\overline{z})$, Int. Trans. and Special Functions 24 (2013) 884–895.
20. Górka K., Horzela A., and Szafraniec F. H.: Holomorphic Hermite polynomials in two variables, arXiv:1706.04491.
21. Hall B. C.: Holomorphic methods in analysis and mathematical physics, Contemp. Math. 260 (2000) 1–59.
22. Henry R. W., and Glotzer S. C.: A squeezed state primer, Am. J. Phys. 56 (1988) 318–328.
23. Horzela A., and Szafraniec F. H.: A measure free approach to coherent states, J. Phys. A: Math. Theor. 45 (2012) 244018 (9 pp).
24. Horzela A., and Szafraniec F. H.: A measure free approach to coherent states refined, in Proc. of the XXIX Int. Colloquium on Group-Theoretical Methods in Physics 2012 Tianjin, China, Nankai Series in Pure, Applied Mathematics and Theoretical Physics 11, 277–282.
25. Ismail M. H. E., and Zhang R.: A review of multivariate orthogonal polynomials, J. Egyptian Math. Soc. 25 (2017) 91–110.
26. Klauder J. R.: Continuous-Representation Theory. I. Postulates of Continuous-Representation Theory J. Math. Phys. 4 (1963) 1055–1058.
27. Klauder J. R.: Coherent states without groups: Quantization on nonhomogeneous manifolds, *Mod. Phys. Lett.* **8** (1993) 1735–1738.
28. Klauder J. R., and Skagerstam B. S., (eds): *Coherent States: Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
29. Klauder J. R., and Sudarshan E.C.G.: *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).
30. Klauder J. R., Penson K. A., and Sixdeniers J.-M.: Constructing coherent states through solutions of Stieltjes and Hausdorff moment problems, *Phys. Rev. A* **64** (2001) 013817 (18pp).
31. Klopfenstein K. F.: A note on Hilbert spaces of factorial functions, *Indiana Univ. Math. J.* **25** (1976) 1073–1081.
32. Królak I.: Measures connected with Bargmann’s representation of the $q$-commutation relation for $q > 1$, *Banach Center Publ.* **43** (1998) 253–257.
33. Lv Cui-Hong, and Fan Hong-Yi: New complex function space related to both entangled state representation and spin coherent state, *J. Math. Phys.* **56** (2015) 082102 (7 pp).
34. Penson K. A., and Solomon A. I.: New generalized coherent states, *J. Math. Phys.* **40** (1999) 2354–2363.
35. Penson K. A., Blasiak P., Duchamp G. H. E., Horzela A., and Solomon A. I.: On certain non-unique solutions of the Stieltjes moment problem, *Discrete Math. and Theor. Comp. Sci.* **12** (2010) 295–306.
36. Perelomov A. M.: *Generalized Coherent States and Their Applications*, Sections 2.1-2.3, (Springer, Berlin, 1986).
37. Sanders B. C.: Review of entangled coherent states, *J. Phys. A; Math. Theor.* **45** (2012) 244002 (22 pp).
38. Schrödinger E.: Der stetige Übergang von der Mikro- zur Makromechanik, *Naturwiss.* **14** (1926) 664–666.
39. Sudarshan E. C. G.: Equivalence of semiclassical and quantum mechanical description of statistical light beams, *Phys. Rev. Lett.* **10** (1963) 277-279.
40. Sivakumar S.: Studies on nonlinear coherent states, *J. Opt. B; Quantum Semiclass. Opt.* **2** (2000) R61–R75.
41. Szafrański F. H.: A RKHS of entire functions and its multiplication operator. An explicit example, in *Linear Operators in Function Spaces*, Proceedings, Timișoara (Romania), June 6-16, 1988, Nelson H., B., Nagy B. Sz., and Vasilescu F.-H., (eds.) *Operator Theory: Advances and Applications*, vol. 43, pp. 309–312, (Birkhäuser, Basel, 1990).
42. Szafrański F. H.: Analytic models of the quantum harmonic oscillator, *Contemp. Math.*, **212** (1998) 269–276.
43. Szafrański F. H.: *Przestrzenie Hilberta z jądrem reprodukującym* (Reproducing kernel Hilbert spaces), in Polish (Wydawnictwo Uniwersytetu Jagiellońskiego, Kraków, 2004).
44. Szafrański F. H.: Operators of the $q$-oscillator, *Banach Center Publ.* **78** (2007) 293–307.
45. Szafrański F. H.: The reproducing kernel property and its space: the basics, in *Operator Theory vol. 1*, Alpay D. (ed.), 3–30, (Springer Reference, Berlin, 2015).
46. Szafrański F. H.: The reproducing kernel property and its space: more or less standard examples of applications, in *Operator Theory vol. 1*, Alpay D. (ed.), 31–58, (SpringerReference, Berlin, 2015).
47. Szafrański F. H.: Operators of the quantum harmonic oscillator and its relatives, in *Non-selfadjoint operators in quantum physics: mathematical aspects*, Bagarello F., Gazeau J.-P., Szafrański F. H., and Znojil M., (eds.), 59–120, (John Wiley & Sons, 2015).
48. Szafrański F. H.: Anatomy of coherent states, present. at *Coherent States and their Applications: A Contemporary Panorama*, Nov.14-18 2016, CIRM, Luminy, France, https://www.dropbox.com/sh/baic4lnzwmhcsu9/AABuH-0dHltCZz31B9fXi9a?dl=0
49. Wünsche A.: Generating functions for products of special Laguerre 2D and Hermite 2D polynomials, *Appl. Math.*, **6** (2015) 2142–2168.
50. Zaremba S.: *L’équation biharmonique et une classe remarquable de fonctions fondamentales harmoniques*, *Bulletin International de l’Académie des Sciences de Cracovie* (1907), 147–196.
51. Zhang Wei-Min, Feng Da Hsuan, and Gilmore R.: Coherent states: theory and some applications, *Rev. Mod. Phys* **62** (1990) 867–927.