Field-dependent symmetries in Friedmann-Robertson-Walker models

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We consider effective actions of the cosmological Friedmann-Robertson-Walker (FRW) models and discuss their fermionic rigid BRST invariance. Further, we demonstrate the finite field-dependent BRST transformations as a limiting case of continuous field-dependent BRST transformations described in terms of continuous parameter $\kappa$. The Jacobian under such finite field-dependent BRST transformations is computed explicitly, which amounts an extra piece in the effective action within functional integral. We show that for a particular choice of the parameter the finite field-dependent BRST transformation maps the generating functional for FRW models from one gauge to another.

I. INTRODUCTION

The quantum cosmology is a branch of theoretical physics attempting to study the effect of quantum mechanics on the formation of the universe, or its early evolution [1, 2]. Despite many attempts, such as the Wheeler-DeWitt equation, and more recently loop quantum cosmology, the field remains a rather speculative branch of quantum gravity. The cosmological principle is an axiom that embodies the working assumption or premise that both the spatial homogeneity and the isotropy of universe is actually valid for the very large scale of the universe rather than the originally stated large scale. The homogeneous and isotropic spacetime symmetry was originally studied by Friedmann, Robertson, and Walker (FRW) [3–8] and therefore such universe models are known as the FRW models. In actual sense, the FRW models are the backbone of modern cosmology describing universe because most of the works on quantum cosmology are based on the FRW universe models. However, anisotropic models had also been studied some time (for instance see [9]). Even though almost all the models of dark energy meet some difficulties like cosmological constant problems, fine-tuning problems and so on but they get relevance in FRW spacetime. Therefore, for better realizations of modern cosmology a more careful investigation of FRW cosmology is quite demanded.

On the other hand, the realization of gauge symmetry in FRW models is well established. According to standard quantization procedure, the gauge invariant models can be quantized correctly by fixing the gauge which removes the redundant degrees of freedom in field variables. The well-known path integral procedure to employ gauge-fixing condition at quantum level is known as Faddeev-Popov trick which involves the so-called Faddeev-Popov ghosts too. The BRST supersymmetry was introduced in the mid-1970s [10, 11] and was quickly understood to justify the introduction of these Faddeev-Popov ghosts and their exclusion from “physical” asymptotic states when performing calculations. The BRST symmetry plays a prominent role in the standard paradigm of fundamental interactions [12].

Although the BRST symmetry has been found for FRW models in particular gauge [13, 14], the generalization of BRST symmetry by making the parameter field-dependent, so-called finite field-dependent BRST transformation, has not yet been studied. The finite field-dependent BRST formulation has many applications on gauge field theories [16, 25]. So, it is worth analysing such formulation for cosmological models describing universe at very large scale. However, a different kind of field-dependent symmetries in case of non-relativistic fluid model had already been studied [29]. This provides us sufficient motivation for present investigation. In this paper we demonstrate the nilpotent BRST symmetries of the FRW models in various gauges which secures the unitarity of the universe models. Further, we analyse the aspects of making the parameter of BRST symmetry field-dependent in rather different way than the finite field-dependent BRST formulation originally advocated in [16]. We found that such revised
formulation is simpler than the original one. Within the analysis we find that for a particular choice of field-dependent parameter the finite field-dependent BRST symmetry connects the generating functional of the model in two different gauges.

This paper is outlined as follows. In Sec. II, we present the FRW models in different gauges with their BRST invariance. Further, we analyse the finite field-dependent BRST symmetry in full generality in sec. III. Within Sec. III, we establish the connection between different gauges of FRW models using finite field-dependent BRST symmetry transformation.

II. BRST INVARIANT FRW MODELS

In this section, we discuss the preliminaries of cosmological FRW models describing homogeneous and isotropic universe having fermionic rigid BRST invariance. So, let us start with the FRW metric defined in spherical coordinates as follows,

$$ds^2 = N^2 dt^2 + a^2(t) \left( \frac{1}{1 - kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),$$

where $N$ is the lapse function and $a(t)$ is the unknown potential of the metric that encodes the size at large scales, more formally is the scale factor of the universe. Here both the lapse function and the scale factor depend on time only. However, the values of $k = 1, -1, 0$ correspond to a space of positive curvature (closed universe), negative curvature (open universe) and zero curvature (flat universe) respectively. Now, we define the classical Lagrangian density of the FRW models traditionally described in Arnowitt-Deser-Misner (ADM) variables as follows [14],

$$L_{inv} = -\frac{1}{2} \frac{a\dot{a}^2}{N} + \frac{k}{2} Na.$$ (2)

Now, the canonically conjugate momenta corresponding to the lapse function $N$ and the scale factor $a$ are calculated by,

$$\pi_N = 0,$$ (3)
$$\pi_a = -\frac{a\dot{a}}{N}.$$ (4)

The momenta corresponding to $N$ reflect the primary constraint of the theory. It is now easy to the evaluate the canonical Hamiltonian density (exploiting Legendre transform) [14],

$$H_c = \pi_a \dot{a} - L_{inv} = -\frac{N\pi_a^2}{2a} - \frac{k}{2} Na.$$ (5)

Exploiting time conservation of the primary constraint, we calculate the secondary constraint of the theory as follows,

$$\frac{\pi_a^2}{2a} + \frac{k}{2} \dot{a} = 0.$$ (6)

Since both the constraints (3) and (6) are kind of first-class. This immediately confirms that the theory of universe embeds gauge invariance. The canonical variables transform under following gauge transformation [14]

$$\delta N = -N\dot{\eta} - \dot{N}\eta, \quad \delta a = -\dot{a}\eta,$$ (7)

where $\eta(t)$ is an infinitesimal parameter of transformation. Now, we follow the standard procedure to quantize the FRW models. Since before quantizing the theory, it is necessary to impose gauge-fixing condition to remove the redundancy in gauge degrees of freedom. However, the essential requirements
for gauge-fixing condition are as follows: (i) it must fix the gauge completely, i.e., there must not be any residual gauge freedom, and (ii) using the transformations it must be possible to bring any configuration, specified by \( N \) and \( a \) into one satisfying the gauge condition. Keeping the above conditions in mind we choose the following gauge condition \([14]\):

\[
\dot{N} = \frac{d}{dt} f(a),
\]

where \( f(a) \) is an arbitrary function of \( a \). This gauge condition \([8]\) can be employed in the theory at quantum level by adding following gauge-fixing term in the invariant Lagrangian density \([2]\) \([14]\):

\[
L_{gf} = \lambda \left( \dot{N} - \frac{d}{dt} f(a) \right),
\]

where \( \lambda \) is an auxiliary field.

Now, the determinant corresponding to the above gauge-fixing term can be compensated in the functional integral by further adding the following ghost term in the effective Lagrangian density:

\[
L_{gh} = \dot{\bar{c}} (\dot{N} - \frac{d}{dt} f(a)) c + \dot{c} \dot{N} \bar{c},
\]

where \( c \) and \( \bar{c} \) refer the Faddeev-Popov ghost and antighost fields respectively. Now, the complete extended Lagrangian density reads

\[
L_{\text{ext}} = L_{\text{inv}} + L_{gf} + L_{gh}.
\]

However, for the different choice of gauge-fixing condition \([15]\),

\[
\dot{N} = f(a),
\]

the gauge-fixing and ghost terms are demonstrated as follows,

\[
L_{gf}' = \lambda \left( \dot{N} - f(a) \right),
\]

\[
L_{gh}' = \dot{\bar{c}} \dot{N} c + \dot{c} \dot{N} \bar{c} + \dot{c} \frac{d}{dt} f(a) c.
\]

The complete extended Lagrangian density corresponding to the gauge condition \([12]\) is defined by,

\[
L_{\text{ext}}' = L_{\text{inv}} + L_{gf}' + L_{gh}'.
\]

The nilpotent BRST symmetry transformations are constructed by replacing the parameter \( \eta \) of \([7]\) by ghost field \( c \) as follows,

\[
\begin{align*}
\delta_b N &= (\dot{N} c + N \dot{c}), \\
\delta_b a &= \dot{\bar{c}}, \\
\delta_b c &= 0, \\
\delta_b \bar{c} &= -\lambda, \\
\delta_b \lambda &= 0,
\end{align*}
\]

under which both the extended Lagrangian densities \( L_{\text{ext}} \) and \( L_{\text{ext}}' \) are invariant up to total derivative. Since the combination of gauge-fixing and ghost terms for both gauges are BRST exact and, therefore, we can express these in terms of BRST variation of gauge-fixing fermion \( \Psi \) as follows,

\[
\begin{align*}
L_{gf} + L_{gh} &= s_b \Psi = -s_b \left[ \dot{c} \left( \dot{N} - \frac{d}{dt} f(a) \right) \right], \\
L_{gf}' + L_{gh}' &= s_b \Psi' = -s_b \left[ \dot{c} \left( \dot{N} - f(a) \right) \right].
\end{align*}
\]
where the gauge-fixing fermions have the following expressions \( \Psi = -\bar{c} \left( \dot{N} - \frac{d}{dt} f(a) \right) \) and \( \Psi' = -\bar{c} \left( \dot{N} - f(a) \right) \).

Now, we define the source free generating functional for FRW models corresponding to gauge conditions (8) and (12) respectively as

\[
Z_1 = \int D\phi \, e^{i S_{\text{ext}}[\phi]}, \\
Z_2 = \int D\phi \, e^{i S'_{\text{ext}}[\phi]},
\]

(17)

where \( D\phi \) denotes the generic measure defined in terms of collective field \( \phi \) and the effective actions \( S_{\text{ext}} \) and \( S'_{\text{ext}} \) are defined, respectively, by

\[
S_{\text{ext}} = \int d^4x \, L_{\text{ext}}, \\
S'_{\text{ext}} = \int d^4x \, L'_{\text{ext}}.
\]

(18)

Here the Lagrangian densities \( L_{\text{ext}} \) and \( L'_{\text{ext}} \) are defined, respectively, in (11) and (14).

III. FINITE FIELD-DEPENDENT BRST TRANSFORMATION

In this section, we demonstrate the methodology of finite field-dependent BRST transformation, originally advocated in Ref. [16], in rather different and elegant way. Then, we discuss its application part.

A. Methodology

We start discussion by considering the fields \( \phi \) as a function of parameter \( \kappa : 0 \leq \kappa \leq 1 \) in such manner that the original fields and finitely transformed fields are described by its extremum values. For instance \( \phi(x, \kappa = 0) = \phi(x) \) defines the original fields, however, \( \phi(x, \kappa = 1) = \phi'(x) \) defines the field-dependent BRST transformed fields. Now, we define the infinitesimal field-dependent BRST transformation as in [16],

\[
\frac{d\phi(x, \kappa)}{d\kappa} = s_b \phi(x, \kappa) \Theta[\phi(x, \kappa)],
\]

(19)

Upon integration the above equation yields the following continuous field-dependent transformation,

\[
\phi(x, \kappa) = \phi(x, 0) + s_b \phi(x, 0) \Theta[\phi(x, \kappa)],
\]

(20)

which at boundary (\( \kappa = 1 \)) leads to the finite field-dependent BRST transformation [16],

\[
\phi'(x) = \phi(x) + s_b \phi(x) \Theta[\phi(x)].
\]

(21)

Furthermore, we compute the Jacobian of path integral measure under such finite field-dependent BRST transformation by start following the same procedure as discussed in [16] as follows,

\[
\mathcal{D}\phi(\kappa) = J(\kappa)\mathcal{D}\phi(\kappa) = J(\kappa + d\kappa)\mathcal{D}\phi(\kappa + d\kappa),
\]

(22)

which further reads

\[
\frac{J(\kappa)}{J(\kappa + d\kappa)} = \sum_{\phi} \pm \frac{\delta \phi(\kappa + d\kappa)}{\delta \phi(\kappa)},
\]

(23)
where ± signs are considered suggesting the nature of the fields \( \phi \) (+ for bosonic fields and − for fermionic ones). Utilizing the Taylor expansion, the relation (23) yields
\[
1 - \frac{1}{J} \frac{dJ}{d\kappa} = 1 + d\kappa \sum_{\phi} \pm s_b \phi(x, \kappa) \frac{\delta \Theta' \phi(x, \kappa)}{\delta \phi(x, \kappa)},
\]
(24)
Now it is easy to obtain the following expression from the above expression (24),
\[
\frac{d \ln J}{d\kappa} = - \int d^4 x \sum_{\phi} \pm s_b \phi(x) \frac{\delta \Theta' \phi(x)}{\delta \phi(x)},
\]
(25)
Performing further integration, we get the following expression:
\[
\ln J = - \int_0^1 d\kappa \int d^4 x \sum_{\phi} \pm s_b \phi(x, \kappa) \frac{\delta \Theta' \phi(x, \kappa)}{\delta \phi(x, \kappa)},
\]
\[
= - \left( \int d^4 x \sum_{\phi} \pm s_b \phi(x) \frac{\delta \Theta' \phi(x)}{\delta \phi(x)} \right)_{\kappa=1},
\]
(26)
and consequently we get the exact form of the Jacobian of functional measure as follows:
\[
J = \exp \left( - \int d^4 x \sum_{\phi} \pm s_b \phi(x) \frac{\delta \Theta' \phi(x)}{\delta \phi(x)} \right).
\]
(27)
Hence, with this expression of Jacobian, the generating functional for an effective theory described by an effective action \( S[\phi] \) changes under finite field-dependent BRST transformation as follows
\[
\int \mathcal{D} \phi' e^{iS[\phi']} = \int \mathcal{D} \phi e^{iS[\phi]} - \int d^4 x \left( \sum_{\phi} \pm s_b \phi \frac{\delta \Theta' \phi}{\delta \phi} \right),
\]
(28)
where \( \phi' \) refers the transformed fields collectively. Therefore, we are able now to draw following conclusion that under whole procedure the effective action of the theory gets modified from their original values by an extra piece. However, in the next subsection, we show that under such an analysis the theory does not change on physical ground but changes from one convention to another automatically which might be useful in computing the physical observable.

**B. An application of finite field-dependent transformation**

The finite field-dependent BRST transformations for FRW models are constructed as
\[
f_b N = (\dot{N} c + N \dot{c}) \Theta[\phi],
\]
\[
f_b a = \dot{a} c \Theta[\phi],
\]
\[
f_b c = 0,
\]
\[
f_b \bar{c} = -\lambda \Theta[\phi],
\]
\[
f_b \lambda = 0,
\]
(29)
where the field-dependent BRST parameter is chosen as follows,
\[
\Theta[\phi] = \int_0^1 d\kappa \Theta'[\phi] = -i \int_0^1 d\kappa \int d^4 x \left[ \bar{c} \left( \frac{d}{dt} f(a) - f(a) \right) \right].
\]
(30)
Now, corresponding to this $\Theta'[\phi]$, we calculate the Jacobian for path integral measure with the help of formula given in (27) as follows,

$$J = \exp \left( i \int d^4 x \sum_\phi \pm s_b \phi(x) \frac{\delta}{\delta \phi(x)} \left[ \bar{c} \left( \frac{d}{dt} f(a) - f(a) \right) \right] \right),$$

$$= \exp \left( i \int d^4 x \left[ \lambda \left( \frac{d}{dt} f(a) - f(a) \right) + \dot{\bar{c}} \frac{d}{dt} f(a)c + \bar{c} \frac{d}{dt} f(a)c \right] \right),$$

(31)

Therefore, under finite field-dependent BRST transformation given in (29) the generating functional $Z_1$ changes as

$$\int D\phi' e^{iS_{ext}[\phi']} = \int J(\phi) D\phi e^{iS_{ext}[\phi]},$$

$$= \int D\phi \exp \left( i \int d^4 x \left[ L_{ext} + \lambda \left( \frac{d}{dt} f(a) - f(a) \right) + \dot{\bar{c}} \frac{d}{dt} f(a)c + \bar{c} \frac{d}{dt} f(a)c \right] \right),$$

$$= \int D\phi e^{iS'_{ext}[\phi]} = Z_2.$$  

(32)

Here, in intermediate steps, we have utilized the relation (31). So, this is nothing but the generating functional of FRW model corresponding to the gauge condition (12). This shows that the field-dependent BRST transformation changes generating functional from one gauge to another.

IV. CONCLUSION

The well-known models studying homogeneous and isotropic universe are known as the FRW models. In modern cosmology, these FRW models have extreme importance as these get relevance in most of the dark energy works. Recently, an interacting and non-interacting two-fluid scenario for dark energy models have studied in FRW universe [30, 31]. The models of mass condensation within the FRW universe lead to cosmological black holes [32]. These cosmological models assume zero cosmological constant that means the only force acting is gravity.

We have considered the BRST invariant FRW models describing flat, open and closed universe. The BRST symmetries of the models have been generalized by making the transformation parameter field-dependent. We have developed the formulation through continuous interpolation of a parameter $\kappa(0 \leq \kappa \leq 1)$ in fields such that fields at $\kappa = 0$ are the original ones, however, at $\kappa = 1$ these are the finite field-dependent BRST transformed ones. The Jacobian for finite field-dependent BRST transformation has been computed which depends explicitly on field-dependent parameter. We have found that under finite field-dependent BRST transformation with an appropriate choice of field-dependent parameter the generating functional for FRW models switches from one gauge to another. The present investigation will be useful in development of connection between the two different propagators and also may be helpful in renormalizing the universe models. It will be a step towards the development of full quantum theory of modern cosmology.

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