THE STABILITY RADIUS OF FREDHOLM LINEAR PENCILS

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Abstract. Let $T$ and $S$ be two bounded linear operators from Banach spaces $X$ into $Y$ and suppose that $T$ is Fredholm and $\dim N(T - \lambda S)$ is constant in a neighborhood of $\lambda = 0$. Let $d(T; S)$ be the supremum of all $r > 0$ such that $\dim N(T - \lambda S)$ and $\text{codim } R(T - \lambda S)$ are constant for all $\lambda$ with $|\lambda| < r$. It is a consequence of more general results due to H. Bart and D.C. Lay (1980) that $d(T; S) = \lim_{n \to \infty} \gamma_n(T; S)^{1/n}$, where $\gamma_n(T; S)$ are some non-negative (extended) real numbers. For $X = Y$ and $S = I$, the identity operator, we have $\gamma_n(T; S) = \gamma_n(T^n)$, where $\gamma$ is the reduced minimum modulus. A different representation of the stability radius $d(T; S)$ is obtained here in terms of the spectral radii of generalized inverses of $T$. The existence of generalized resolvents for Fredholm linear pencils is also considered.

1. Introduction

Let $T$ be a bounded linear operator on a Banach space $X$. Denote by $\gamma(T)$ the reduced minimum modulus of $T$. The limit

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} \quad (1.1)$$

was studied for Fredholm operators $T$ by K.H. Förster and M.A. Kaashoek [FK]. If $T$ is Fredholm, they proved that the limit (1.1) exists and is equal to the supremum of all $r > 0$ such that $\dim N(T - \lambda)$ and $\text{codim } R(T - \lambda)$ are constant on $\{\lambda : 0 < |\lambda| < r\}$. If $T$ is Fredholm and $0 \in \text{reg}(T)$, the generalized resolvent set of $T$, then the limit (1.1) coincides [M1] with $\text{dist}(0, \sigma_g(T))$, the distance of 0 to the generalized spectrum of $T$. We refer to [FK], [M1] and section 2 of this paper for the definition of unknown terms and to [A, BL, KM, P, Sc, Z1] for other papers related to the limit (1.1).

An extension of the limit (1.1) for operator pencils $\lambda \to T - \lambda S$ between two Banach spaces $X$ and $Y$ has been considered by H. Bart and D.C. Lay [BL]. They defined a sequence of non-negative (extended) real numbers $\gamma_n(T; S)$ and studied the limit

$$\lim_{n \to \infty} \gamma_n(T; S)^{1/n}. \quad (1.2)$$
It was proved in [BL] that if $T$ is Fredholm, then the limit (1.2) exists and coincides with the so-called stability radius $d(T;S)$ of $T$ and $S$. In the case when $T$ is Fredholm and $\dim N(T - \lambda S)$ is constant in a neighborhood of $\lambda = 0$, the stability radius $d(T;S)$ equals the supremum of all $r > 0$ such that $\dim N(T - \lambda S)$ and $\text{codim } R(T - \lambda S)$ are constant for $|\lambda| < r$.

A different representation formula was given recently [BM1] for the limit (1.1). It was proved there that if $T$ is Fredholm and $0 \in \text{reg}(T)$, the generalized resolvent set of $T$, then

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} = \sup \left\{ \frac{1}{r(L)} : TLT = T \right\},$$

(1.3)

where $r(L)$ is the spectral radius of $L$. The same result (1.3) is true without the condition of Fredholmness of $T$ for Hilbert space operators [BM2].

The aim of the present note is to extend to linear pencils the formula (1.3). Namely, we will prove that

$$d(T;S) = \sup \left\{ \frac{1}{r(SL)} : L \in \mathcal{L}(Y,X), TLT = T \right\}$$

$$= \sup \left\{ \frac{1}{r(SL)} : L \in \mathcal{L}(Y,X), TLT = T, LTL = L \right\},$$

if $T$ is Fredholm and $\dim N(T - \lambda S)$ is constant in a neighborhood of $\lambda = 0$. The above formula relates the limit (1.2) with the spectral radii of generalized inverses of $T$. It is an open problem if this representation holds without the condition of constancy of $\dim N(T - \lambda S)$.

One of the ingredients of the proof of the main result is the existence of generalized resolvents for some operator pencils. This is interesting in its own.

The paper is organized as follows. In the next section we recall some notation and known results. In section 3 we introduce and study generalized resolvents for operator pencils and their existence for some operator pencils is proved in section 4. This is used in section 5 to prove the main result concerning the stability radius.

2. NOTATION AND KNOWN RESULTS

We present in this section some notation, basic definitions and known results.

Recall that $X$ and $Y$ will denote complex Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of all continuous, linear operators from $X$ into $Y$; we abbreviate $\mathcal{L}(X,X)$ to $\mathcal{L}(X)$. We use $R(T) = TX$, $N(T) = \{x \in X : Tx = 0\}$, $\sigma(T)$ and $\rho(T) = \mathbb{C} \setminus \sigma(T)$ to denote the
range, the kernel, the spectrum and, respectively, the resolvent set of $T$. We denote by $r(T)$ the spectral radius of $T$. We write $X = E \oplus F$ to designate direct sums (i.e. $E \cap F = \{0\}$ and $E + F = X$). In this case we say that $F$ is the (direct) complement of $E$ in $X$. The distance from a point $x$ to a set $A$ is denoted by $\text{dist}(x, A)$ and $B(\lambda_0, r)$ is the open set $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$.

The operator $T \in \mathcal{L}(X, Y)$ is Fredholm if both the dimension of $N(T)$ and the codimension of $R(T)$ are finite. The range of a Fredholm operator is closed.

The reduced minimum modulus of $T \neq 0$, $T \in \mathcal{L}(X, Y)$, is defined by

$$\gamma(T) = \inf \{\|Tx\| : x \in X, \text{dist}(x, N(T)) = 1\}.$$ 

If $T = 0$ we set $\gamma(T) = \infty$.

Let $T \in \mathcal{L}(X, Y)$. An operator $L \in \mathcal{L}(Y, X)$ is called a generalized inverse of $T$ if $TLT = T$ and $LTL = L$. If $L$ satisfies only the first condition $TLT = T$, then $L_1 = LTL$ is a generalized inverse of $T$.

Let $T \in \mathcal{L}(X)$. Let $\text{reg}(T)$ denote \cite{M1, M2} the generalized resolvent set of $T$, that is the set of all complex numbers $\lambda$ for which there exists a neighborhood $V_\lambda$ of $\lambda$ and an analytic function $G$ on $V_\lambda$ such that $G(\mu)$ is a generalized inverse of $T - \mu I_X$ for each $\mu \in V_\lambda$. Then (cf. \cite{M1, M2}) $0 \in \text{reg}(T)$ if and only if $T$ has a generalized inverse and $N(T) \subseteq \text{R}(T^m)$ for every $m$. Let $\sigma_g(T) = \mathbb{C} \setminus \text{reg}(T)$ be the generalized spectrum of $T$.

Let $T, S \in \mathcal{L}(X, Y)$ and consider the linear operator pencil $\lambda \to T - \lambda S \in \mathcal{L}(X, Y)$. Define the subspaces $N_m = N_m(T; S)$ and $R_m = R_m(T; S)$ of $X$ by $N_0 = \{0\}, R_0 = X$ and

$$N_{m+1} = T^{-1}SN_m \quad , \quad R_{m+1} = S^{-1}TR_m,$$

for $m = 0, 1, \ldots$. Note that the sequence $(R_m)$ is decreasing. Define

$$X_\infty = \bigcap_m R_m \quad ; \quad Y_\infty = \bigcap_m TR_m.$$

The extended integer $k(T; S)$ defined by

$$k(T; S) = \dim N(T)/(N(T) \cap X_\infty)$$

is the stability number of $T$ and $S$. So $k(T; S) = 0$ means $N(T) \subset R_m$ for all $m$. For further reference we note that $k(T; S) = 0$ implies (cf. \cite{K}, Lemma 2.3) $TX_\infty = Y_\infty$ and $S^{-1}Y_\infty = X_\infty$. If $T$ is Fredholm, then $k(T; S) = 0$ if and only if $\dim N(T - \lambda S)$ is constant in a neighborhood of $\lambda = 0$, if and only if $\codim R(T - \lambda S)$ is constant in a neighborhood of $\lambda = 0$. We will use these equivalences several times.

Let $m \geq 1$. The $m$-tuple $(x_1, x_2, \ldots, x_m) \in X^m$ is a chain for $T$ and $S$ if $Tx_i = Sx_{i-1}$ for $i = 2, \ldots, m$. Let $\gamma_m = \gamma_m(T; S)$ denote the
supremum of all $c \geq 0$ with the property that $\|Tx_1\| \geq c \text{dist}(x_m, N_m)$ for all chains $(x_1, x_2, \cdots, x_m)$. Then $\gamma_1(T; S) = \gamma(T)$ is the reduced minimum modulus of $T$.

Let $T, S \in \mathcal{L}(X, Y)$. Suppose that $R(T)$ is closed and $k(T; S) < +\infty$. The stability radius $d(T; S)$ of $T$ and $S$ is defined \cite{BL} as the supremum of all $r > 0$ such that $R(T - \lambda S)$ is closed and $k(T - \lambda S) = 0$ for $0 < |\lambda| < r$. When $T$ is Fredholm, the stability radius $d(T; S)$ is equal to the supremum of all $r > 0$ such that $\dim N(T - \lambda S)$ and $\text{codim } R(T - \lambda S)$ are constant on $0 < |\lambda| < r$. If $T$ is Fredholm, then \cite{BL} $d(T; S) = \lim_{m \to \infty} \gamma_m(T; S)^{1/m}$.

3. Generalized resolvents for linear pencils

We start with the following definition.

**Definition 3.1.** Let $T$ and $S$ be two elements of $\mathcal{L}(X, Y)$. Let $U$ be an open set in the complex plane. The function $\quad U \ni \lambda \rightarrow G(\lambda) \in \mathcal{L}(Y, X)$

is called a generalized resolvent on $U$ of the linear pencil $\lambda \rightarrow T - \lambda S \in \mathcal{L}(X, Y)$ if

1. $(T - \lambda S)G(\lambda)(T - \lambda S) = T - \lambda S$, for all $\lambda \in U$;
2. $G(\lambda)(T - \lambda S)G(\lambda) = G(\lambda)$, for all $\lambda \in U$;
3. $G(\lambda) - G(\mu) = (\lambda - \mu)G(\lambda)SG(\mu)$ for all $\lambda$ and $\mu$ in the same connected component of $U$.

The first two conditions say that $G(\lambda)$ is a generalized inverse of $T - \lambda S$ for each $\lambda$, while the third one is an analogue of the classical resolvent identity. The assumption that the resolvent identity (3) holds only for $\lambda$ and $\mu$ in the same connected component is consistent with \cite{M2}. It is possible to have generalized resolvents with (3) not fulfilled for $\lambda$ and $\mu$ in distinct connected components \cite{M2, p. 376}. We refer to \cite{AC, M1, LM, M2, BM3} for properties of generalized resolvents in the classical case $X = Y, S = I_X$.

The following lemma shows that each generalized resolvent on $U$ of a linear pencil is analytic on $U$.

**Lemma 3.2.** Let $U$ be an open set in the complex plane and let $U \ni \lambda \rightarrow G(\lambda) \in \mathcal{L}(Y, X)$ be an operator function satisfying $G(\lambda) - G(\mu) = (\lambda - \mu)G(\lambda)SG(\mu)$ for all $\lambda$ and $\mu$ in the same connected component of $U$. Then $G$ is analytic on $U$. 
Proof. Let $\lambda_0$ be a fixed point in $U$ and let $r$ be a positive number such that $r < \|SG(\lambda_0)\|^{-1}$ and $B(\lambda_0, r)$ is included in the connected component of $U$ containing $\lambda_0$. Let $\lambda \in B(\lambda_0, r)$. The resolvent identity (3) implies that
\[ G(\lambda_0) = G(\lambda) \left[ I - (\lambda - \lambda_0)SG(\lambda_0) \right] \]
and thus
\[ G(\lambda) = G(\lambda_0) \left[ I - (\lambda - \lambda_0)SG(\lambda_0) \right]^{-1} \]
\[ = G(\lambda_0) \sum_{k \geq 0} (\lambda - \lambda_0)^k (SG(\lambda_0))^k \]
for each $\lambda \in B(\lambda_0, r)$. Thus $G$ is analytic on $U$.

Definition 3.3. Let $U$ be an open set in $\mathbb{C}$. The linear pencil $\lambda \to T - \lambda S \in \mathcal{L}(X, Y)$ is said to have fixed complements on $U$ if for each connected component $\Gamma$ of $U$ there exist two closed subspaces $E$ and $F$ of $X$ and $Y$ such that $X = N(T - \lambda S) \oplus E$ and $Y = R(T - \lambda S) \oplus F$ for all $\lambda \in \Gamma$.

Remark 3.4. Consider a linear pencil $\lambda \to T - \lambda S \in \mathcal{L}(X, Y)$ and let $w$ be a complex number such that $R(T - wS)$ is closed and there exists a bounded generalized inverse for $T - wS$. It follows from [1, Proposition I.2.2] that there exists a (connected) neighborhood $V_w$ of $w$ such that $\lambda \to T - \lambda S$ has fixed complements on $V_w$ if and only if the linear pencil $\lambda \to T - \lambda S$ is uniformly regular at $w$ [1], that is the function $\lambda \to \gamma(T - \lambda S)$ is continuous at $w$.

The following result gives a (global) characterization of the property of having fixed complements in terms of generalized resolvents.

Theorem 3.5. Let $T$ and $S$ be two elements of $\mathcal{L}(X, Y)$. Let $U \subset \mathbb{C}$ be an open set. There exists a generalized resolvent for $\lambda \to T - \lambda S$ on $U$ if and only if the linear pencil $\lambda \to T - \lambda S$ has fixed complements on $U$.

Proof. Via reduction to connected components, it is possible to assume that $U$ is connected.

Suppose that $\lambda \to T - \lambda S$ has fixed complements on $U$; that is, there exist two closed subspaces $E$ and $F$ of $X$ and $Y$ such that $X = N(T - \lambda S) \oplus E$ and $Y = R(T - \lambda S) \oplus F$ for all $\lambda \in U$. We will consider the projections $P(\lambda)$ and $Q(\lambda)$ onto $R(T - \lambda S)$ along $F$ and, respectively, onto $E$ along $N(T - \lambda S)$.

We will prove the existence of an operator valued function $G(\lambda) \in \mathcal{L}(Y, X)$, $\lambda \in U$, such that
(1) \((T - \lambda S)G(\lambda)(T - \lambda S) = T - \lambda S\), for all \(\lambda \in U\);
(2) \(G(\lambda)(T - \lambda S)G(\lambda) = G(\lambda)\), for all \(\lambda \in U\);
(3) \(G(\lambda) - G(\mu) = (\lambda - \mu)G(\lambda)SG(\mu)\) for all \(\lambda\) and \(\mu\) in \(U\).

The generalized resolvent \(G\) will be analytic on \(U\) by Lemma 3.2.

Let \(u \in Y\). Then \(P(\lambda)u \in R(T - \lambda S)\). Therefore, there exists \(v \in X\) such that \(P(\lambda)u = (T - \lambda S)v\). Set \(G(\lambda)u = Q(\lambda)v\).

This definition is correct. Indeed, if \(w \in X\) is such that \((T - \lambda S)w = P(\lambda)u = (T - \lambda S)v\), then \(v - w \in N(T - \lambda S) = N(Q(\lambda))\). Therefore \(Q(\lambda)v = Q(\lambda)w\). Hence \(G(\lambda)\) does not depend on the choice of \(v\).

Note that \(G(\lambda)\) is the generalized inverse of \(T - \lambda S\) with null space \(F\) and range \(E\). This gives the proofs of (1) and (2).

We also have

\[
(T - \lambda S)G(\lambda) = P(\lambda) \tag{3.1}
\]

and

\[
G(\lambda)(T - \lambda S) = Q(\lambda) \tag{3.2}
\]

Using these two equations and the fact that \(G(\lambda)\) is a generalized inverse of \(T - \lambda S\), we get

\[
Q(\lambda)G(\lambda) = G(\lambda) \tag{3.3}
\]

and

\[
G(\lambda)P(\lambda) = G(\lambda) \tag{3.4}
\]

The projection \(I - P(\lambda)\) is onto \(F\) and thus \(P(\mu)(I - P(\lambda)) = 0\). Similarly, \((I - Q(\lambda))Q(\mu) = 0\). We obtain

\[
P(\mu)P(\lambda) = P(\mu) \tag{3.5}
\]

and

\[
Q(\lambda)Q(\mu) = Q(\mu) \tag{3.6}
\]

Now we can write

\[
G(\lambda) - G(\mu) = G(\lambda)P(\lambda) - Q(\mu)G(\mu) \quad \text{(using (3.3) and (3.4))}
\]

\[
= G(\lambda)P(\lambda)P(\mu) - Q(\lambda)Q(\mu)G(\mu) \quad \text{(using (3.3) and (3.4))}
\]

\[
= G(\lambda)P(\mu) - Q(\lambda)G(\mu) \quad \text{(using (3.3) and (3.4))}
\]

\[
= G(\lambda)(T - \mu S)G(\mu) - G(\lambda)(T - \lambda S)G(\mu) \quad \text{(using (3.2) and (3.1))}
\]

\[
= (\lambda - \mu)G(\lambda)SG(\mu).
\]
This proves that \( G(\lambda) \) is a generalized resolvent of the linear pencil 
\( \lambda \to T - \lambda S \).

For the second part, suppose a generalized resolvent \( G \in \mathcal{L}(Y, X) \) 
exists on \( U \). We will show the existence of the fixed complements \( E \) 
and \( F \) of \( \lambda \to T - \lambda S \).

**STEP a The subspace \( E \)**

Using the resolvent identity, we have 
\[ G(\lambda) - G(\mu) = -[G(\mu) - G(\lambda)] = (\lambda - \mu)G(\mu)SG(\lambda). \]
We infer that 
\( G(\lambda) = G(\mu) + G(\mu)(\lambda - \mu)SG(\lambda), \)
yielding 
\( R(G(\lambda)) \subseteq R(G(\mu)). \)
Reversing the role of \( \lambda \) and \( \mu \) 
we get 
\( R(G(\mu)) = R(G(\lambda)). \)
Take \( E = R(G(\lambda)) \subset X \). Note that the 
range of \( G(\lambda) \), the generalized inverse of \( T - \lambda S \), is closed.

**STEP b The subspace \( F \)**

The resolvent identity (3) implies 
\( N(G(\mu)) \subseteq N(G(\lambda)) \) and thus 
\( N(G(\mu)) = N(G(\lambda)). \)
We denote \( F = N(G(\lambda)) \).

**STEP c \( Y = R(T - \lambda S) \oplus F \)**

We have 
\( u = [I - (T - \lambda S)G(\lambda)]u + (T - \lambda S)G(\lambda)u \) for each \( u \in Y \).
The first term of the sum is in \( N(G(\lambda)) = F \) since \( G(\lambda) \) is a generalized 
inverse of \( T - \lambda S \). The second term of the sum is clearly in \( R(T - \lambda S) \).

If \( y = (T - \lambda S)x \) and \( y \in F = N(G(\lambda)), \) then 
\( 0 = G(\lambda)y = G(\lambda)(T - \lambda S)x, \)
yielding 
\( 0 = (T - \lambda S)G(\lambda)(T - \lambda S)x = (T - \lambda S)x = y. \)

**STEP d \( X = N(T - \lambda S) \oplus E \)**

The proof of this equality is similar to that of Step c and will be 
omitted. \[ \blacksquare \]

4. Linear pencils with fixed complements

**Theorem 4.1.** Assume that for each \( \lambda \) in an open, connected set \( \Omega \subset \mathbb{C} \) 
the operator \( T - \lambda S \in \mathcal{L}(X,Y) \) is Fredholm, and that the dimension 
of the null space \( N(T - \lambda S) \) does not depend on \( \lambda \in \Omega \). Let \( U \) be a 
bounded, open and connected set with \( U \subset \overline{U} \subset \Omega \). Then there exists a 
generalized resolvent for \( \lambda \to T - \lambda S \) on \( U \).

Using Theorem 3.3, we have to show that \( \lambda \to T - \lambda S \) has fixed 
complements on \( U \), that is, there exist two closed subspaces \( E \) of \( X \) 
and \( F \) of \( Y \) such that 
\( X = N(T - \lambda S) \oplus E \) and \( Y = R(T - \lambda S) \oplus F \) 
for every \( \lambda \in U \). The existence of generalized resolvents (and thus of 
fixed complements) has been studied in \( [\text{AC, ML, M2, BM3}] \) in the case \( X = Y, \) 
\( S = I \). Proofs of Theorem 4.1 in the case \( X = Y, \) 
\( S = I \) can be found in \( [\text{S}] \) and \( [\text{M2}] \).

For the proof we will need the following key result. Variants of this 
result can be found in \( [\text{Z2}, \text{M2}] \).

**Theorem 4.2.** Assume that for each \( \lambda \) in an open, connected set \( \Omega \subset \mathbb{C} \) 
the operator \( T - \lambda S \in \mathcal{L}(X,Y) \) is Fredholm, and that the dimension
of the null space \( N(T - \lambda S) \) does not depend on \( \lambda \in \Omega \). Let \( W \) be a closed subspace of \( Y \) such that
\[
\text{codim } [R(T - \lambda S) \oplus W] \geq 1, \quad \text{for all } \lambda \in \Omega. \tag{4.1}
\]
If \( U \) is a bounded, open, connected set such that \( U \subset \overline{U} \subset \Omega \), then there exists \( x \in Y \) such that \( x \notin R(T - \lambda S) \oplus W \) for all \( \lambda \in U \).

We start with the following Lemma.

**Lemma 4.3.** Assume that for each \( \lambda \) in an open, connected set \( \Omega \subset \mathbb{C} \) the operator \( T - \lambda S \in \mathcal{L}(X,Y) \) is Fredholm, and that the dimension of the null space \( N(T - \lambda S) \) does not depend on \( \lambda \in \Omega \). Let \( W \) be a closed subspace of \( Y \). Then there exists an analytic function \( \Omega \ni \lambda \rightarrow P(\lambda) \in \mathcal{L}(Y \oplus W) \) such that
\[
N(P(\lambda)) = R(T - \lambda S) \oplus W, \quad \lambda \in \Omega. \tag{4.2}
\]

**Proof.** Consider the Fredholm operator function
\[
F(\lambda) = (T - \lambda S) \oplus J : X \oplus W \to Y,
\]
where \( J : W \to Y \) is the inclusion operator. Then \( R(F(\lambda)) \) has constant finite codimension on \( \Omega \) and \( \text{N}(F(\lambda)) = N(T - \lambda S) \) has constant finite dimension. Then (cf. for instance [BKL]), the function \( F(\lambda) \) has a global analytic generalized inverse \( G(\lambda) \in \mathcal{L}(Y,X \oplus W) \) on \( \Omega \). Take
\[
P(\lambda) = I_{X \oplus W} - F(\lambda)G(\lambda) \in \mathcal{L}(X \oplus W).
\]
Then \( N(P(\lambda)) = R(F(\lambda)) = R(T - \lambda S) \oplus W \).

**Proof of Theorem 4.2.** Let \( W \) be a closed subspace of \( Y \) as in Theorem 4.2. Using the above Lemma, there exists an analytic function \( P(\lambda) \) such that (4.2) holds. Consider \( \lambda_0 \in U \) and let \( u \in Y \) be such that \( u \notin R(T - \lambda_0 S) \oplus W \). The existence of such an element \( u \) follows from condition (1.1). Then \( P(\lambda_0)u \neq 0 \).

Two situations can occur. If \( \lambda \rightarrow P(\lambda)u \) does not vanish in \( \overline{U} \), then we take \( x = u \). If not, the analytic function \( \lambda \rightarrow P(\lambda)u \) vanishes in the compact \( \overline{U} \) for finitely many points \( \lambda_1, \lambda_2, \cdots, \lambda_n \), each with finite order of multiplicity \( d_1, \cdots, d_n \).

**Claim.** There exists \( y \in Y \) such that the analytic function \( \lambda \rightarrow P(\lambda)y \) vanishes in \( \overline{U} \) only for \( \lambda_2, \cdots, \lambda_n \).

The proof of this claim is deferred to the next paragraph. Assuming this claim, we note that continuing this construction we are sure that in at most \( n \) steps we obtain an element \( z \in Y \) such that \( P(\lambda)z \neq 0 \). Then this \( z \) does not belong to \( R(T - \lambda S) \oplus W \) for any \( \lambda \in U \).
**Proof of the Claim.** The analytic function $\lambda \to P(\lambda)u$ vanishes in the compact $\overline{U}$ for $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since $P(\lambda_1)u = 0$, we infer that $u \in R(T - \lambda_1 S) \oplus W$. Thus there exist $v_1 \in X$ and $w_1 \in W$ such that $u = (T - \lambda_1 S)v_1 + w_1$. For any $\lambda \in \overline{U}$, $\lambda \neq \lambda_1$, we have

$$P(\lambda)u = P(\lambda)(T - \lambda_1 S)v_1 + P(\lambda)w_1 = P(\lambda)(T - \lambda S + \lambda S - \lambda_1 S)v_1 = (\lambda - \lambda_1)P(\lambda)Sv_1.$$

The equality $P(\lambda)u = (\lambda - \lambda_1)P(\lambda)Sv_1$ also holds for $\lambda = \lambda_1$ and thus it holds for all $\lambda \in \overline{U}$. In particular, zeros of $P(\lambda)Sv_1$ in $\overline{U}$ with $\lambda \neq \lambda_1$ are zeros of $P(\lambda)u$.

If $P(\lambda_1)Sv_1 \neq 0$, then the representation of $P(\lambda)u$ shows that $P(\lambda_1)Sv_1$ vanishes in $\overline{U}$ only for $\lambda_2, \ldots, \lambda_n$ and we take $y = Sv_1$.

If $P(\lambda_1)Sv_1 = 0$, then $Sv_1 \in R(T - \lambda_1 S) \oplus W$. We can write $Sv_1 = (T - \lambda_1 S)v_2 + w_2$, $v_2 \in X$. As above, we have $P(\lambda)Sv_1 = (\lambda - \lambda_1)P(\lambda)Sv_2$. Then $P(\lambda)u = (\lambda - \lambda_1)P(\lambda)Sv_1 = (\lambda - \lambda_1)P(\lambda)Sv_2$ for all $\lambda \in \overline{U}$.

If $P(\lambda_1)Sv_2 \neq 0$, then we can take $y = Sv_2$. If $P(\lambda_1)Sv_2 = 0$, then there exists $v_3 \in X$ such that $P(\lambda)u = (\lambda - \lambda_1)^2P(\lambda)Sv_3$.

This construction will lead eventually to an element $y = Sv_k$ satisfying the conditions of the Claim since the order $d_1$ of multiplicity of $\lambda_1$ is finite. This completes the proof of the claim and of the theorem.

**Proof of Theorem 4.1.** By considering the adjoint pencil $T^* - \lambda S^*$ acting between the dual spaces of $Y$ and $X$, it is sufficient to prove the existence of a fixed complement $F$ of $R(T - \lambda S)$ for $\lambda \in U$.

Without loss of any generality we can assume $0 \in U$. Let $n = \text{codim } R(T)$. If $n = 0$, take $F = \{0\}$. If $n \geq 1$, by Theorem 4.2 (with $W = \{0\}$), there exists $x \in Y$ such that $x \notin R(T - \lambda S)$ for all $\lambda \in U$. Then $R(T - \lambda S) \cap \text{lin}(x) = \{0\}$ and $\text{codim } (R(T - \lambda S) \oplus \text{lin}(x)) = n - 1$. Here $\text{lin}(x)$ is the subspace spanned by $x$. Note that $R(T - \lambda S) \oplus \text{lin}(x)$ is a closed subspace of $Y$. By a repeated application of Theorem 4.2 (with suitable $W$), we get a closed subspace $F$ such that $R(T - \lambda S) \cap F = \{0\}$ and $\text{codim } R(T - \lambda S) \oplus F = 0$. Therefore $R(T - \lambda S) \oplus F = Y$ for all $\lambda \in U$.

**Corollary 4.4.** Assume that $T \in \mathcal{L}(X,Y)$ is Fredholm and $\dim N(T - \lambda S)$ is constant in a neighborhood of $\lambda = 0$. Let $d = d(T; S)$ be the stability radius. Let $U \subset \overline{U} \subset B(0,d)$ be a bounded, open, connected subset of the open ball of radius $d$. Then there exists a generalized resolvent for $T - \lambda S$ on $U$.  


Proof. Recall that $k(T;S) = 0$. By [BL, Lemma 3.2], $T - \lambda S$ is Fredholm for each $\lambda \in \Omega := B(0,d)$ and [BL, p. 309] the dimension of the null space $N(T - \lambda S)$ does not depend on $\lambda \in \Omega$. The conclusion follows now from Theorems 4.1 and 3.5.  

5. The stability radius

The following result is the announced formula for the stability radius.

**Theorem 5.1.** Let $T$ and $S$ be two elements of $\mathcal{L}(X,Y)$ such that $T$ is Fredholm and $\dim N(T - \lambda S)$ is constant in a neighborhood of $\lambda = 0$. Let $d(T;S)$ be the stability radius of $T$ and $S$, which equals the supremum of all $r > 0$ such that $\dim N(T - \lambda S)$ and $\text{codim} R(T - \lambda S)$ are constant for all $\lambda$ with $|\lambda| < r$. Then we have

$$d(T;S) = \sup \{ \frac{1}{r(SL)} : L \in \mathcal{L}(Y,X), TLT = T \}$$

We start with the following auxiliary results.

**Lemma 5.2.** Suppose there exists $\delta > 0$ such that

$$(T - \lambda S)F(\lambda)(T - \lambda S) = T - \lambda S$$

for every $\lambda$ with $|\lambda| < \delta$ and $F(\lambda) = \sum_{n \geq 0} \lambda^n F_n$ for $|\lambda| < \delta$. Then

$$\gamma_m(T;S) \geq \frac{1}{\|F_{m-1}\|}$$

for every $m \geq 1$.

**Proof.** Let $(x_1, \ldots, x_m)$ be a chain for $T$ and $S$. Define

$$\psi(\lambda) = [I_X - F(\lambda)(T - \lambda S)] \left( \sum_{i=1}^{m} \lambda^{i-1} x_i \right).$$

We have $(T - \lambda S)\psi(\lambda) = 0$. Denote by $\psi_k, k \geq 0$, the Taylor coefficients of $\psi$ around zero. We then have $\psi_0 \in N(T) = N_1$ and $T\psi_k = S\psi_{k-1}$. We get recursively $\psi_{k-1} \in N_k$ for each $k \geq 0$. We have

$$\psi_{m-1} = x_m - F_{m-1} Tx_1$$

and therefore $x_m - F_{m-1} Tx_1 \in N_m$. This implies

$$\text{dist}(x_m, N_m) \leq \|F_{m-1} Tx_1\| \leq \|F_{m-1}\| \|Tx_1\|.$$ 

Since this holds for every chain $(x_1, \ldots, x_m)$, we obtain the desired estimate for $\gamma_m(T;S)$.  

$\blacksquare$
Lemma 5.3. Let $T, S \in \mathcal{L}(X, Y)$ and suppose that $N(T) \subset R_m$ for all $m$. Let $L \in \mathcal{L}(Y, X)$ with $TLT = T$. Set $\alpha = \min(\|SL\|^{-1}; \|LS\|^{-1})$ and

$$F(\lambda) = L \left( I - \lambda SL \right)^{-1} = \sum_{k \geq 0} L(SL)^k \lambda^k$$

for $\lambda$ satisfying $|\lambda| < \alpha$. Then

$$(T - \lambda S)F(\lambda) = T - \lambda S$$

(5.1)

for each $\lambda$, $|\lambda| < \alpha$.

Proof. Recall that $N(T) \subset R_m$ for all $m$ is equivalent to $k(T; S) = 0$. We have $I - F(\lambda)(T - \lambda S) = \sum_{k \geq 0} (LS)^k (I - LT) \lambda^k$, the last series being convergent for $|\lambda| < \alpha$. This shows that

$$(T - \lambda S)F(\lambda)(T - \lambda S) = T - \lambda S - A(\lambda),$$

where $A(\lambda) = (T - \lambda S) [I - F(\lambda)(T - \lambda S)]$ is given by

$$A(\lambda) = \sum_{k \geq 1} (TL - I)(SL)^{k-1} S(I - LT) \lambda^k.$$  

(5.2)

The proof that all Taylor coefficients of $A(\lambda)$ are zero is obtained in several steps.

**Step $\alpha$** We have $S(I - LT)x \in Y_\infty$ for every $x \in X$.

Indeed, for every $x \in X$, $(I - LT)x \in N(T)$. Since $N(T) \subset R_m$ for all $m$, we have $(I - LT)x \in X_\infty$. Therefore [K Lemma 2.3]

$$S(I - LT)x \in SX_\infty \subseteq Y_\infty.$$  

**Step $\beta$** $L(Y_\infty) \subseteq X_\infty$.

We have [K Lemma 2.3] $TX_\infty = Y_\infty$. If $y \in Y_\infty$, $y = Tx$, $x \in X_\infty$, then $Ly = LTx = x + (LT - I)x \in X_\infty$. The latter follows from $x \in X_\infty$, $(LT - I)x \in N(T)$ and from $N(T) \subset X_\infty$.

**Step $\gamma$** For any $k \geq 1$, $(SL)^k(Y_\infty) \subseteq Y_\infty$.

Let $y \in Y_\infty$. Using Step $\beta$ we get $Ly \in X_\infty$ ; thus $SLy \in Y_\infty$. Applying this $k$ times, we get $(SL)^k(Y_\infty) \subseteq Y_\infty$.

**Step $\delta$** The restriction of $TL$ on $Y_\infty$ acts like the identity operator on $Y_\infty$.

Indeed, if $y \in Y_\infty$, then there exists $x \in X_\infty$ such that $y = Tx$. Then $TLy = TL(Tx) = Tx = y$.

By (5.2) we have $A(\lambda) = 0$ and thus (5.1) holds for each $\lambda$ satisfying $|\lambda| < \alpha$.

**$\blacksquare$**

Proof of Theorem 5.1. Note that the first sup is greater or equal than the second.

Let $\Omega = B(0, d)$ be the open disk of radius $d = d(T; S)$. Let $\varepsilon > 0$. Consider the disk $U = B(0, d/(1 + \varepsilon))$. Then $U \subset \overline{U} \subset \Omega$. 



THE STABILITY RADIUS OF FREDHOLM LINEAR PENCILS 11
It is a consequence of the hypothesis that $T$ is Fredholm and $k(T;S) = 0$. This gives the case $\lambda = 0$ of the following more general statement: The operator $T - \lambda S$ is Fredholm and $k(T - \lambda S;S) = 0$ for each $\lambda \in \Omega$. For $\lambda \neq 0$, this follows by combining [BL, Lemma 3.2] with [BL, Theorem 3.1].

By Corollary 4.4, there exists a generalized resolvent $G$ for the pencil $\lambda \rightarrow T - \lambda S$ on $U$. Set $G(\lambda) = \sum_{n=0}^{\infty} \lambda^n G_n$, $\lambda \in U$. Since $G$ satisfies the resolvent identity on $U$, we have $G(\lambda) - G_0 = G(\lambda) - G(0) = \lambda G(\lambda)SG(0)$. This implies $G_n = G_{n-1}SG_0$. Therefore $G_1 = G_0SG_0$ and, by recurrence, $G_n = G_0(SG_0)^n$ for all $n$.

Denote $M = \max\{\|G(\lambda)\| : |\lambda| \leq d/(1 + 2\varepsilon)\}$. Using Cauchy’s estimates, we obtain

$$\|G_0(SG_0)^n\| = \|G_n\| \leq M \left(\frac{1 + 2\varepsilon}{d}\right)^n,$$

for every $n \geq 0$. Therefore

$$\|(SG_0)^{n+1}\| = \|SG_0(SG_0)^n\| \leq \|S\| M \left(\frac{1 + 2\varepsilon}{d}\right)^n.$$

We obtain $r(SG_0) \leq (1 + 2\varepsilon)/d$. The fact that $G(\lambda)$ is a generalized inverse of $T - \lambda S$ implies $TG_0T = T$ and $G_0TG_0 = G_0$. Hence

$$\sup\{\frac{1}{r(SL)} : L \in \mathcal{L}(Y, X), TLT = T, LTL = L\} \geq \frac{1}{r(SG_0)} \geq \frac{d}{1 + 2\varepsilon}.$$

Since this holds for every $\varepsilon > 0$, we get that both suprema are not smaller than $d$. For the second inequality, let $L \in \mathcal{L}(Y, X)$ with $TLT = T$. Set

$$F(\lambda) = L \left(I - \lambda SL\right)^{-1} = \sum_{k=0}^{\infty} L(SL)^k \lambda^k$$

which is defined and analytic for $\lambda$ satisfying $|\lambda| < \alpha = \min(\|SL\|^{-1}; \|LS\|^{-1})$.

By Lemma 5.3 we have

$$(T - \lambda S)F(\lambda)(T - \lambda S) = T - \lambda S$$

for each $\lambda$ satisfying $|\lambda| < \alpha$. Using Lemma 5.2, this implies

$$\gamma_m(T;S) \geq \frac{1}{\|L(SL)^{m-1}\|} \geq \frac{1}{\|L\| \|(SL)^{m-1}\|}.$$
Using [BL], we have
\[ d = \lim_{m \to \infty} \gamma_m(T; S)^{1/m} \geq 1/r(SL), \]
for every \( L \in \mathcal{L}(Y, X) \) satisfying \( TLT = T \). This gives the desired inequality
\[ d \geq \sup \{ \frac{1}{r(SL)} : L \in \mathcal{L}(Y, X), TLT = T \}. \]
The proof is now complete.

**Remark 5.4.** Using the Kato [Ka] decomposition, the formula \( d(T; S) = \lim_{n \to \infty} \gamma_n(T; S)^{1/n} \) was proved in [BL] for \( T \) Fredholm and \( S \) arbitrary, without the condition that \( \dim N(T - \lambda S) \) is constant in a neighborhood of \( \lambda = 0 \). We do not know if Theorem 5.1 holds without this condition, equivalent to \( k(T; S) = 0 \).

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**References**

[A] C. Apostol : *The reduced minimum modulus*, Michigan Math. J. 32(1985), 279-294.

[AC] C. Apostol, K. Clancey : *On generalized resolvents*, Proc. Amer. Math. Soc. 58 (1976), 163-168.

[BM1] C. Badea ; M. Mbekhta : *Generalized inverses and the maximal radius of regularity of a Fredholm operator*, Integral Equations Operator Th. 28(1997), 133-146.

[BM2] C. Badea ; M. Mbekhta : *Compressions of resolvents and maximal radius of regularity*, Trans. Amer. Math. Soc. 351(1999), 2949-2960.

[BM3] C. Badea ; M. Mbekhta : *On existence of one-sided and generalized resolvents of bounded linear operators*, Rend. Circ. Mat. Palermo Ser.II-Supplement (Proc. Workshop on Operator Theory, Cefalu, 1997) 56(1998), 139-148.

[BKL] H. Bart, M. A. Kaashoek and D. C. Lay : *Relative inverses of meromorphic operator functions and associated holomorphic projection functions*, Math. Ann. 218 (1975), 199–210.

[BL] H. Bart ; D.C. Lay : *The stability radius of a bundle of closed linear operators*, Studia Math. 66(1980), 307-320.

[FK] K.H. Förster ; M.A. Kaashoek : *The asymptotic behaviour of the reduced minimum modulus of a Fredholm operator*, Proc. Amer. Math. Soc. 49(1975), 123-131.

[K] M.A. Kaashoek : *Stability theorems for closed linear operators*, Proc. Acad. Sci. Amsterdam A 68(1965), 452-466.

[Ka] T. Kato : *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. 6(1958), 261-322.

[KM] V. Kordula ; V. Müller : *The distance from the Apostol spectrum*, Proc. Amer. Math. Soc. 124(1996), 3055-3061.
[LM] J.-Ph. Labrousse, M. Mbekhta: Résolvant généralisé et séparation des points singuliers quasi-Fredholm, Trans. Amer. Math. Soc. 333 (1992), 299-313.

[M1] M. Mbekhta: Résolvant généralisé et théorie spectrale, J. Operator Th. 21 (1989), 69-105.

[M2] M. Mbekhta: On the generalized resolvent in Banach spaces, J. Math. Anal. Appl. 189 (1995), 362-377.

[P] P. W. Poon: The stability radius of a quasi-Fredholm operator, Proc. Amer. Math. Soc. 126 (1998), 1071-1080.

[S] P. Saphar: Sur les applications linéaires dans un espace de Banach II, Ann. Sci. Ecole Norm. Sup. Ser. 3, 82 (1965), 205-240.

[Sc] C. Schmoeger: The stability radius of an operator of Saphar type, Studia Math 113 (1995), 169-175.

[T] G.Ph.A. Thijse: Decomposition theorems for finite-meromophic operator functions, Thesis, Free University Amsterdam, 1978.

[Z1] J. Zemánek: The stability radius of a semi-Fredholm operator, Integral Equations Operator Th. 8 (1985), 137-144.

[Z2] J. Zemánek: An analytic Laffey-West decomposition, Proc. Roy. Irish Acad. Sect. A 92 (1992), 101-106.

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