ON TIGHT GENERALIZED FRAMES

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ABSTRACT. Frames can be thought of as collections of rank-one, positive semidefinite operators that, if tight, enable signal decompositions like orthogonal bases. One refers to generalized frames if the rank-one constraint is withdrawn. Given a generalized frame, we construct - in a linear fashion - a tight generalized frame whose elements have equal norm and are close to the original frame. For integers \(p\), we also introduce stochastic generalized \(p\)-frames and verify that common random matrices used in compressed sensing satisfy tightness conditions. We then suggest a refinement of the notion of frame redundancy and discuss a few supporting examples.

1. Introduction

Frames are basis-like systems that span a vector space but allow for linear dependency. The inherent redundancy of frames can yield features unavailable with orthonormal bases [18, 22, 23, 24]. However, signal reconstruction requires the inversion of the frame operator, often a numerically cumbersome task. Such drawbacks are void when dealing with tight frames as their frame operator is a multiple of the identity. If, in addition, the frame elements have unit norm, then frame coefficients measure the signal energy in a uniform fashion, often more suitable for signal processing tasks.

Given a (unit norm) frame, the Paulsen problem asks for the closest tight frame whose elements have unit norm and for an algorithm to find it. This problem has been addressed and partially solved in [6, 11, 15].

Generalized frames were introduced in [34] as a tool for signal decomposition using a set of positive semi-definite (PSD) operators. In [17], collections of orthogonal projectors were considered under the name fusion frames, and the concept of tight \(p\)-fusion frames was developed in [3]. In the present paper, we address the following points:

1. We first introduce the Paulsen problem in the context of generalized frames. Given a generalized frame \(\{T_j\}_{j=1}^n\) whose elements have unit Hilbert-Schmidt norm, we present an algorithm that yields a unit norm tight generalized frame close to \(\{T_j\}_{j=1}^n\). The latter is the main result of this paper.

2. We introduce tight stochastic generalized \(p\)-frames as subclasses of random matrices in \(\mathbb{K}^{r \times d}\), where \(\mathbb{K}\) is \(\mathbb{R}\) or \(\mathbb{C}\), and the case \(p = r = 1\).
yields the concept of tight continuous frames as considered in [11, 25]. We verify that a sample from this random matrix forms an almost-tight generalized frame with high probability. Note that the concept of almost-tightness was introduced in [19] and enables signal reconstruction up to any desired precision without inverting the frame operator.

(3) Many types of random matrices are used in compressed sensing [13, 21] to model linear measurements. We shall verify that Gaussian matrices, circulant matrices, structured time-frequency measurements, and Fourier measurements are stochastic tight generalized $p$-frames for certain values of $p$.

(4) The lower and upper redundancy of a frame was introduced in [7], see also [5]. According to the notion in [7], the two frames \{\(e_1, e_1, e_2\)\} and \{\(e_1, e_2, e_1 + e_2\)\} in \(\mathbb{R}^2\) have the same lower and upper redundancy 1 and 2, respectively. However, the local redundancy, for instance, appears quite different in those cases. We shall suggest a refinement of the notion of redundancy in Section 7 and introduce the frame signature as an extension of the frame redundancy.

(5) Switching gears we consider ordinary frames in the space of matrices. We verify that weighted orthogonal projectors cannot be tight frames for the space of symmetric matrices.

The outline is as follows: In Section 2 we introduce the concept of (stochastic) frames and present the Paulsen problem in frame theory. In Section 3 we develop the algorithm related to the Paulsen problem. We introduce (stochastic) generalized frames in Section 4 and verify our results on the random choice of generalized frames. In Section 5 we introduce the Paulsen problem in the context of generalized frames and present our algorithm that determines a unit norm tight generalized frame. Compressed sensing random matrices are addressed in Section 6 and the notion of frame redundancy is refined in Section 7. In section 8 we verify that a collection of weighted orthogonal projectors cannot be a tight frame for the space of symmetric matrices. Conclusion are given in Section 9.

2. Finite frames

2.1. Definitions and elementary properties. Let \(\mathbb{K}\) be \(\mathbb{R}\) or \(\mathbb{C}\). Following [18] a finite collection of points \(\{x_i\}_{i=1}^n \subset \mathbb{K}^d\) is called a finite frame for \(\mathbb{K}^d\) if there are two constants \(0 < A \leq B\) such that

\[
A\|x\|^2 \leq \sum_{j=1}^n |\langle x, x_j \rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{K}^d.
\]

The constants \(A\) and \(B\) are called lower and upper frame bounds, respectively. We have the following simple characterization of finite frames:

**Lemma 2.1** ([18]). The sequence \(\{x_j\}_{j=1}^n \subset \mathbb{K}^d\) is a finite frame for \(\mathbb{K}^d\) if and only if it spans \(\mathbb{K}^d\).
For any collection \( \{x_j\}_{j=1}^n \subset \mathbb{K}^d \), the frame operator
\[
S : \mathbb{K}^d \to \mathbb{K}^d, \quad x \mapsto \sum_{j=1}^n \langle x_j, x \rangle x_j
\]
is self-adjoint and positive semi-definite. If \( \{x_j\}_{j=1}^n \) is a finite frame, then
\( S \) is positive definite and, in particular, invertible, cf. [18]. In this case, the
following reconstruction formula holds,
\[
(2) \quad x = \sum_{j=1}^n \langle S^{-1}x_j, x \rangle x_j = \sum_{j=1}^n \langle x_j, x \rangle S^{-1}x_j, \quad \text{for all } x \in \mathbb{K}^d,
\]
and \( \{S^{-1}x_j\}_{j=1}^n \) is a finite frame too, called the canonical dual frame.

A collection of points \( \{x_j\}_{i=1}^n \subset \mathbb{K}^d \) is called a finite tight frame for
\( \mathbb{K}^d \) if there is a positive constant \( A \) such that
\[
(3) \quad A\|x\|^2 = \sum_{j=1}^n |\langle x, x_j \rangle|^2, \quad \text{for all } x \in \mathbb{K}^d.
\]
The constant \( A \) is called the tight frame bound. If, in addition, \( A = 1 \) holds,
then we call \( \{x_j\}_{i=1}^n \) a Parseval frame. Tight frames and Parseval frames
directly lead to the reconstruction formula
\[
(4) \quad x = \frac{1}{A} \sum_{j=1}^n \langle x, x_j \rangle x_j, \quad \text{for all } x \in \mathbb{K}^d,
\]
since their frame operator is \( A \) times the identity, cf. [18].

2.2. The Paulsen problem. Here, we shall find Parseval frames that are
close to a given frame. If we do not require any further features, then the
following Theorem yields the closest Parseval frame together with its explicit
construction:

**Theorem 2.2 ([16]).** If \( \{x_j\}_{j=1}^n \subset \mathbb{K}^d \) is a finite frame with frame operator
\( S \), then \( \{S^{-1/2}x_j\}_{j=1}^n \) is a Parseval frame and, for any other Parseval frame
\( \{y_j\}_{j=1}^n \), we have
\[
\sum_{j=1}^n \|x_j - S^{-1/2}x_j\|^2 \leq \sum_{j=1}^n \|x_j - y_j\|^2.
\]
Equality holds if and only if \( \{y_j\}_{j=1}^n = \{S^{-1/2}x_j\}_{j=1}^n \).

The Parseval frame \( \{S^{-1/2}x_j\}_{j=1}^n \) is also called the canonical tight frame
associated to \( \{x_j\}_{j=1}^n \). It becomes a much harder problem if the elements
of the Parseval frame are requested to have equal norm. In this case, a
characterization as in Theorem 2.2 has not yet been found and has become
known as the Paulsen problem, cf. [6, 11, 15].

Normalization of a Parseval frame yields a finite tight frame whose ele-
ments lie on the sphere \( S^{d-1} \) but, in general, it may not be tight anymore.
An algorithm to compute a finite tight frame \( \{y_i\}_{i=1}^n \subset S^{d-1} \) that is close to a given collection \( \{x_i\}_{i=1}^n \) has been developed in [15] and led to the following result. Here, \( \|M\|_{HS}^2 \) denotes the Hilbert-Schmidt norm of a matrix \( M \in \mathbb{K}^{r \times d} \), and the inner product of \( M_1, M_2 \in \mathbb{K}^{r \times d} \) is \( \langle M_1, M_2 \rangle := \text{trace}(M_1^* M_2) \), so that \( \|M\|_{HS}^2 = \langle M, M \rangle \).

**Theorem 2.3 ([15])**. Suppose that \( n \) and \( d \) are relatively prime and \( n \geq d \). If the frame operator \( S \) of \( \{x_i\}_{i=1}^n \subset S^{d-1} \) satisfies \( \|S - \frac{n}{d} I_d\|_{HS}^2 \leq \frac{4}{d^2} \), then there is a tight frame \( \{y_i\}_{i=1}^n \subset S^{d-1} \) such that

\[
\sum_{i=1}^n \|x_i - y_i\|^2 \leq 64d^{10} n^{17} \|S - \frac{n}{d} I_d\|_{HS}^2.
\]

A lower bound on the distance between \( \{x_j\}_{j=1}^n \subset S^{d-1} \) and the closest tight frame on the sphere is given in the following proposition, in which the first estimate is already contained in [15], and the second estimate is a consequence of the first one:

**Proposition 2.4.** Suppose that \( \{x_i\}_{i=1}^n \subset S^{d-1} \) with frame operator \( S \). If \( \{y_i\}_{i=1}^n \subset S^{d-1} \) is a tight frame, then both of the following estimates hold,

\[
\sum_{i=1}^n \|x_i - y_i\|^2 \geq \|S^{1/2} - \sqrt{n/d} I_d\|_{HS}^2,
\]

\[
\sum_{i=1}^n \|x_i - y_i\|^2 \geq \frac{d}{n(d^{1/2} + 1)^2} \|S - \frac{n}{d} I_d\|_{HS}^2.
\]

Note that (7) is weaker than (6) but incorporates the frame operator directly instead of its square root.

3. **An algorithm related to the Paulsen problem**

Theorem 2.2 says that the Parseval frame closest to a given frame can simply be obtained by applying a linear map. In general, we cannot find a linear map if we additionally require the elements of the Parseval frame to have equal norm. Here, we shall linearly map the frame to another frame whose elements only require normalization in order to form a Parseval frame.

Given samples \( \{x_j\}_{j=1}^n \subset S^{d-1} \), the closest equal norm frame is simply derived by rescaling. We shall construct a unit norm tight frame that is close to \( \{x_j\}_{j=1}^n \), and rescaling by the factor \( \sqrt{d/n} \) yields an equal norm Parseval frame.

Let \( P \) be the collection of hermitian positive definite matrices in \( \mathbb{K}^{d \times d} \) with trace 1. For \( \Gamma_0 = \frac{d}{d} I_d \in P \), we define the iterative scheme

\[
\Gamma_{k+1} = \frac{\Gamma_k^{1/2} M_k^{-1} \Gamma_k^{1/2}}{\text{trace}(\Gamma_k M_k^{-1})},
\]

where \( M_k = \frac{1}{n_d} \sum_{i=1}^n (x_i y_i^*)(x_i y_i^*)^T \) and \( n_d = \frac{d}{n} \).
Figure 1. original frame \( \{ x_j \}_{j=1}^3 \) in blue, optimal frame \( \{ z_j \}_{j=1}^3 = \{ R \hat{z}_j \}_{j=1}^3 \) in green, and our proposed algorithm finds \( \{ y_j \}_{j=1}^3 \) in red. (Green and red lines are sometimes right on top of each other).

where

\[
M(\Gamma) = \frac{d}{n} \sum_{j=1}^n \frac{\Gamma^{1/2} x_j x_j^* \Gamma^{1/2}}{x_j^* \Gamma x_j} \quad \text{and} \quad M_k = M(\Gamma_k). 
\]

If \( \{ x_j \}_{j=1}^n \) spans \( \mathbb{K}^d \), then the scheme is well-defined and \( \Gamma_k \in \mathcal{P} \).

To ensure convergence, we assume that \( n > d(d-1) \) and that each subset of \( d \) vectors spans \( \mathbb{K}^d \). If the latter condition does not hold originally, then we may “wiggle” a bit, so that the new set of vectors satisfies the condition. Then Theorems 1 and 2 in [28] imply that the iterative scheme (8) converges towards \( \Gamma \in \mathcal{P} \). A direct computation yields that \( M(\Gamma) = I_d \) must hold. We now linearly map \( x_j \) into \( \tilde{y}_j \) by applying \( \Gamma^{1/2} \), i.e., \( \{ \tilde{y}_j \}_{j=1}^n = \{ \Gamma^{1/2} x_j \}_{j=1}^n \). Since \( M(\Gamma) = I_d \), we obtain that \( \{ y_j \}_{j=1}^n = \{ \frac{\tilde{y}_j}{\| \tilde{y}_j \|} \}_{j=1}^n \subset S^{d-1} \) is a finite tight frame.

There are some heuristics suggesting that the above scheme finds a tight frame on the sphere that is close to the original frame: The iterative scheme [8] is applied as an M-estimator in [28, 37] related to the population covariance of the samples \( \{ x_j \}_{j=1}^n \). Thus, we can expect that if the sample covariance (assuming zero mean, this is \( \frac{1}{n} \sum_{j=1}^n x_j x_j^* \)) is close to a multiple of the identity, then \( \Gamma \) is close to the identity, which would then imply that \( y_j \) is close to \( x_j \).
We illustrate the performance of our algorithm in the following example:

**Example 3.1.** Let \( d = 2, n = 3 \), and, for simplification, we only consider the situation when the given frame and the tight frame have real entries. We pick \( \alpha_1, \alpha_2, \alpha_3 \) from a uniform distribution on \([0, 2\pi]\) and define \( x_j = \left( \cos(\alpha_j), \sin(\alpha_j) \right) \), \( j = 1, 2, 3 \). By multiplication with \(-1\) and rotation of all 3 vectors, we can restrict the angles to lie between 0 and \( \frac{2}{3}\pi \). For each random choice \( \{x_j\}_{j=1}^3 \), we compute a tight frame \( \{y_j\}_{j=1}^3 \subset S^1 \) using our proposed algorithm. Up to rotations \( R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \) and multiplication by \(-1\), there is only one single tight frame having real unit norm vectors. We choose \( \{\tilde{z}_j\}_{j=1}^3 \), where \( \tilde{z}_j = \left( \cos(\beta_j), \sin(\beta_j) \right) \) and \( \beta_1 = 0, \beta_2 = 1/3\pi, \) and \( \beta_3 = 2/3\pi \). Therefore, to find the tight frame with unit norm that is closest to \( \{x_j\}_{j=1}^3 \), we minimize the distance to \( \{x_j\}_{j=1}^3 \) over all rotations, i.e.,

\[
\hat{\theta} := \arg \min_{\theta \in [-2/3\pi, 2/3\pi]} \sum_{j=1}^3 \|R_\theta \tilde{z}_j - x_j\|^2,
\]

and define the closest tight frame by \( \{z_j\}_{j=1}^3 := \{R_\hat{\theta} \tilde{z}_j\}_{j=1}^3 \). Note that we can suppress the multiplication by \(-1\) because the angles of \( \{x_j\}_{j=1}^3 \) only run in \([0, 2\pi]\). The average error of \( \sum_{j=1}^n \|z_j - y_j\|^2 \) over 1000 realizations is \( \approx 0.0016 = 1/625 \), see also Fig. 1 for a visualization of few examples. In our numerical experiments, we observed that our proposed algorithm finds a tight frame that is almost identical to the optimal tight frame if all pairs \( x_i \) and \( x_j \), for \( i \neq j \), are far enough from each other.

### 4. Generalized frames

Here, we introduce a more general concept of frames than in the previous sections and discuss their properties.

#### 4.1. Definitions and characterizations

Let \( \Omega \) be a locally compact Hausdorff space and \( \mathcal{B}(\Omega) \) be the Borel-sigma algebra on \( \Omega \) endowed with a probability measure \( \mu \). We first fuse the concepts of generalized frames and probabilistic frames as developed in [34] and [24], respectively:

**Definition 4.1.** Let \( p \geq 1 \) an integer. We say that a random matrix \( T : \Omega \to \mathbb{K}^{r \times d}, \ r \leq d, \) is a **stochastic generalized \( p \)-frame** if there are positive constants \( A_p, B_p > 0 \) such that

\[
A_p\|x\|^{2p} \leq \int_{\Omega} \|T(\omega)x\|^{2p} d\mu(\omega) \leq B\|x\|^{2p}, \text{ for all } x \in \mathbb{K}^d.
\]

A stochastic generalized \( p \)-frame is called **tight** if we can choose \( A_p = B_p \). If, in addition, \( \mu \) is a counting measure whose support is \( \{\omega_j\}_{j=1}^n \) and \( T(\omega_j) = T_j \), for \( j = 1, \ldots, n \), then we call \( \{T_j\}_{j=1}^n \) a **generalized \( p \)-frame**.
or a generalized tight $p$-frame, respectively. In case $p = 1$, we call \( \{ T_j \}_{j=1}^n \) Parseval if $A_1 = B_1 = 1$.

We obtain the analogous result of Lemma 2.1 for (stochastic) generalized frames:

**Proposition 4.2.** Let \( \{ T_j \}_{j=1}^n \) be an arbitrary finite collection of matrices in $K^{r \times d}$, $r \leq d$, and let $T : \Omega \rightarrow K^{r \times d}$ be a random matrix with respect to a Borel probability measure $\mu$ on $\Omega$.

1) Then \( \{ T_j \}_{j=1}^n \) is a generalized $p$-frame if and only if $\bigcup_{j=1}^n \text{range}(T_j^*)$ spans $K^d$.

2) If $T$ satisfies the upper bound in (10), then $T$ is a stochastic generalized $p$-frame if and only if $\bigcup_{\omega \in \text{supp}(\mu)} \text{range}(T^*(\omega))$ spans $K^d$.

Note that Proposition 4.2 says that \( \{ T_j \}_{j=1}^n \) is a generalized $p$-frame for some $p$ if and only if it is a generalized $p$-frame for the entire range $p \geq 1$.

**Proof.** For the finite collection, the upper frame bound in (10) can be defined as $B := \sum_{j=1}^n \| T_j \|^{2p}$. Therefore, 1) follows from 2), and we only consider $T$. If $\bigcup_{\omega \in \text{supp}(\mu)} \text{range}(T^*(\omega))$ does not span $K^d$, then there exists a nonzero element $x \in \bigcap_{\omega \in \text{supp}(\mu)} \text{null}(T(\omega))$. Therefore, (10) cannot hold.

For the reverse implication, let us define

$$A := \inf_{x \in K^d} \left( \frac{\int_{\Omega} \| T(\omega)x \|^{2p} d\mu(\omega)}{\| x \|^{2p}} \right) = \inf_{x \in S^{d-1}} \left( \frac{\int_{\Omega} \| T(\omega)x \|^{2p} d\mu(\omega)}{\| x \|^{2p}} \right),$$

where $S^{d-1} = \{ z \in K^d : \| z \| = 1 \}$. The mapping $x \mapsto \int_{\Omega} \| T(\omega)x \|^{2p} d\mu(\omega)$ is continuous and the infimum is in fact a minimum since $S^{d-1}$ is compact. Let $x \in S^{d-1}$ such that $A = \int_{\Omega} \| T(\omega)x \|^{2p} d\mu(\omega)$. According to the assumption,

$$x \not\in \bigcap_{\omega \in \text{supp}(\mu)} \text{null}(T(\omega)).$$

Thus, there is $\omega_0 \in \text{supp}(\mu)$ such that $\| T(\omega_0)x \|^{2p} > 0$. Therefore, there is $\varepsilon > 0$ and an open subset $U_{\omega_0} \subset \Omega$ such that $\omega_0 \in U_{\omega_0}$ and $\| T(\omega)x \|^{2p} > \varepsilon$, for all $\omega \in U_{\omega_0}$. Since $\mu(U_{\omega_0}) > 0$, we obtain $A \geq \varepsilon \mu(U_{\omega_0}) > 0$, which concludes the proof.

Following the lines of the proof for rank one projectors considered in [22] yields that any stochastic generalized 1-frame $T$ w.r.t. $\mu$ satisfies

$$A_1 \leq \frac{1}{d} \int_{\Omega} \| T(\omega) \|^{2} d\mu(\omega) \leq B_1.$$
Similar to finite frames, if $T$ is a stochastic generalized $p$-frame, then the stochastic generalized frame operator

$$S : \mathbb{K}^d \to \mathbb{K}^d, \quad x \mapsto \int_\Omega T^*(\omega)T(\omega)xd\mu(\omega)$$

is positive, self-adjoint, and invertible, cf. [34] for counting measures $\mu$. Thus, we obtain the reconstruction formula

$$x = \int_\Omega S^{-1}T^*(\omega)T(\omega)xd\mu(\omega).$$

Moreover, $\mu$ is a tight stochastic generalized 1-frame if and only if $S = cI_d$, where $c = \frac{1}{d} \int_\Omega \|T(\omega)\|^2_{HS}d\mu(\omega)$. Next, we define and discuss a stochastic generalized frame potential whose minimizers characterize tight stochastic generalized 1-frames.

**Definition 4.3**. Given a Borel probability measure $\mu$ on $\Omega$ and an associated random matrix $T : \Omega \to \mathbb{K}^{r \times d}$ with stochastic generalized frame operator $S$, we call

$$\Phi(T, \mu) := \int_\Omega \int_\Omega \langle T^*(\omega)T(\omega), T^*(\omega')T(\omega') \rangle d\mu(\omega)d\mu(\omega')$$

its stochastic generalized frame potential.

Tight stochastic generalized frames can be characterized as minimizers of the above potential among all Borel probability measures on $\Omega$ and random matrices $T$ with fixed mean Hilbert Schmidt norm $m = \int_\Omega \|T(\omega)\|^2_{HS}d\mu(\omega)$:

**Proposition 4.4**. If $T$ is a random matrix with Borel probability measure $\mu$ on $\Omega$, then

$$\Phi(T, \mu) \geq \frac{1}{d} \left( \int_\Omega \|T(\omega)\|^2_{HS}d\mu(\omega) \right)^2,$$

and equality holds if and only if $T$ is a tight stochastic generalized 1-frame.

We observe that $\Phi(T, \mu) = \text{trace}(S^2)$ so that considering the eigenvalues of $S$ as in [14] [22] implies Proposition 4.4.

Tight stochastic generalized frames are closely related to the concept of positive operator valued measures (POVM) as widely used for quantum measurements:

**Definition 4.5** ([20]). We call $F : \mathcal{B}(\Omega) \to \mathbb{K}^{d \times d}$ a POVM on $\Omega$ if the following points hold:

(i) $F(A)$ is positive semi-definite, for all $A \in \mathcal{B}(\Omega)$,
(ii) $F(\Omega)$ is the identity matrix,
(iii) If $\{A_i\}_{i \in I}$ is a countable family of pairwise disjoint Borel sets in $\mathcal{B}(\Omega)$, then

$$F(\bigcup_{i \in I} A_i) = \sum_{i \in I} F(A_i).$$
If, in addition, \( \text{supp}(F) = \{\omega_j\}_{j=1}^n \) is finite, then we identify \( F \) with a finite collection of positive semi-definite matrices \( \{F_j\}_{j=1}^n \) that we also call a POVM.

Any stochastic tight generalized 1-frame induces a POVM by

\[
F(A) := \int \Omega ||T(\omega)||^2_{HS}d\mu(\omega) \int_A T^*(\omega)T(\omega)d\mu(\omega), \quad A \in B(\Omega).
\]

This means in the finite setting that any tight generalized 1-frame \( \{T_j\}_{j=1}^n \) gives rise to a POVM by normalizing \( \{T_j^*T_j\}_{j=1}^n \). Conversely, any POVM \( \{F_j\}_{j=1}^n \) induces a tight generalized 1-frame \( \{F_j^{1/2}\}_{j=1}^n \). Moreover, each \( F_j^{1/2} \) can be replaced with weighted projectors, see [12].

4.2. **Approximate dual generalized frames.** The inversion of the frame operator can be numerically cumbersome. To avoid it altogether, the concept of approximate dual frames was introduced in [19]. We shall extend this concept to generalized frames and use it with random samplings from a stochastic generalized 1-frame.

If \( \{T_j\}_{j=1}^n \) is a generalized \( p \)-frame with generalized frame operator \( S \), then \( \{T_jS^{-1}\}_{j=1}^n \) is called the canonical dual generalized \( p \)-frame. Indeed, \( \{T_jS^{-1}\}_{j=1}^n \) forms a generalized \( p \)-frame, and (13) implies the reconstruction formula

\[
x = \sum_{j=1}^n (T_jS^{-1})^*T_j(x), \quad \text{for all } x \in \mathbb{K}^d.
\]

Since \( \{T_j\}_{j=1}^n \) may inherit some redundancy, there can also exist alternative choices, and we call \( \{R_j\}_{j=1}^n \) a dual generalized \( p \)-frame if

\[
x = \sum_{j=1}^n R_j^*T_j(x), \quad \text{for all } x \in \mathbb{K}^d.
\]

The **analysis operator** of a generalized frame \( \{T_j\}_{j=1}^n \) is

\[
\mathcal{F} : \mathbb{K}^d \to \prod_{j=1}^n \mathbb{K}^r, \quad x \mapsto (T_j(x))_{j=1}^n.
\]

Its adjoint is the **synthesis operator**

\[
\mathcal{F}^* : \prod_{j=1}^n \mathbb{K}^r \to \mathbb{K}^d, \quad (c_j)_{j=1}^n \mapsto \sum_{j=1}^n T_j^*c_j,
\]

such that the generalized frame operator is \( S = \mathcal{F}^*\mathcal{F} \), which is consistent with [12]. The identity (15) means \( I_d = \mathcal{F}^*\mathcal{F}_T \mathcal{F}_R^* \), where \( \mathcal{F}_T \) and \( \mathcal{F}_R \) are the analysis operators of \( \{T_j\}_{j=1}^n \) and \( \{R_j\}_{j=1}^n \), respectively.

When it is sufficient to approximate \( x \) up to some precision, then we may not need an exact dual frame. We say that a generalized \( p \)-frame \( \{R_j\}_{j=1}^n \)
is an approximate dual to $\{T_j\}_{j=1}^n$ if
\begin{equation}
\|I_d - \mathcal{F}^*_R \mathcal{F}_T\|_\infty < 1.
\end{equation}

Any approximate dual gives rise to a sequence of approximate duals that allow reconstruction of $x$ as precise as needed:

**Proposition 4.6.** Let $\{R_j\}_{j=1}^n$ be an approximate dual generalized frame to the generalized frame $\{T_j\}_{j=1}^n$ and let $\varepsilon := \|I_d - \mathcal{F}^*_R \mathcal{F}_T\|_\infty < 1$. Then the analysis operator $\mathcal{F}_{Q^{(N)}}$ associated to the collection $\{Q^{(N)}_j\}_{j=1}^n$ defined by
\begin{equation}
Q^{(N)}_j := R_j + \sum_{i=1}^N R_j(I_d - \mathcal{F}^*_T \mathcal{F} R)^i
\end{equation}
satisfies $\|I_d - \mathcal{F}^*_Q \mathcal{F}_T\|_\infty \leq \varepsilon^{N+1} \to 0$ as $N \to \infty$.

The statement can be derived by minor modifications of the lines in the proof of Proposition 4.1 in [19], where standard frames are considered. Therefore, we omit it here.

If a generalized frame operator $S$ satisfies $\|I_d - S\|_\infty < 1$, then the underlying generalized frame is approximately dual to itself. In the following theorem, we shall verify that random samplings yield generalized frame operators that are sufficiently close to the identity. For results on random vectors, we refer to [39].

**Theorem 4.7.** Let $\{T_j\}_{j=1}^n$ be independent copies of a stochastic tight generalized 1-frame $T : \Omega \to \mathbb{K}^{r \times d}$, where $\|T\|_{HS}^2 = R$ with probability 1 for some positive constant $R$. For fixed $\varepsilon \in (0, 1)$, there are positive constants $\gamma_\varepsilon, c_\varepsilon > 0$ such that, for all $n \geq c_\varepsilon d \ln(d)$, the frame operator $S_n$ of the scaled collection $\{\sqrt{\frac{d}{n R^2}} T_j\}_{j=1}^n$ satisfies $\|I_d - S_n\|_\infty < \varepsilon$ with probability at least $1 - e^{-\gamma_\varepsilon n}$.

**Proof.** Let $\lambda_{\min}(S_n)$ and $\lambda_{\max}(S_n)$ denote the smallest and largest eigenvalue of $S_n$, respectively. The matrix Chernoff bounds as stated in [36] yield, for all $0 \leq \varepsilon \leq 1$,
\begin{align*}
\mathbb{P}(\lambda_{\min}(S_n) \leq 1 - \varepsilon) &\leq d \left(\frac{e^{-\varepsilon}}{(1 - \varepsilon)^{1-\varepsilon}}\right)^{n/d}, \\
\mathbb{P}(\lambda_{\min}(S_n) \geq 1 + \varepsilon) &\leq d \left(\frac{e^{\varepsilon}}{(1 + \varepsilon)^{1+\varepsilon}}\right)^{n/d}.
\end{align*}

Some calculus yields
\begin{equation}
(1 + \varepsilon)^{1+\varepsilon}(1 - \varepsilon)^{\varepsilon-1} \leq e^{2\varepsilon}, \quad \forall \varepsilon \in [0, 1],
\end{equation}
so that we derive
\begin{equation}
\mathbb{P}(\|I_d - \sum_{j=1}^n \frac{d}{n R} T_j^* T_j\|_\infty \geq \varepsilon) \leq 2d \left(\frac{e^{\varepsilon}}{(1 + \varepsilon)^{1+\varepsilon}}\right)^{n/d}.
\end{equation}
We can further compute
\[
2d \left( \frac{e^\varepsilon}{(1 + \varepsilon)^{1+\varepsilon}} \right)^{n/d} = 2d e^{-\frac{n}{d}(1+\varepsilon)\ln(1+\varepsilon)-\varepsilon} = e^{-\frac{n}{d}(1+\varepsilon)\ln(1+\varepsilon)-\varepsilon - \frac{d}{n} \ln(2) \ln(d)}.
\]
Since \((1 + \varepsilon)\ln(1 + \varepsilon) - \varepsilon > 0\), for all \(\varepsilon \in (0, 1]\), we can find a suitable constant \(\gamma_\varepsilon > 0\) if \(n\) is sufficiently large. □

**Remark 4.8.** The constants \(\gamma_\varepsilon\) and \(c_\varepsilon\) in Theorem 4.7 can be explicitly computed. By using \(a_\varepsilon := (1 + \varepsilon)\ln(1 + \varepsilon) - \varepsilon > 0\), we can choose \(c_\varepsilon > \frac{\ln(2)}{a_\varepsilon}\) and \(\gamma_\varepsilon = a_\varepsilon - \frac{\ln(2)}{c_\varepsilon}\).

5. **Parseval generalized frames close to a given generalized frame**

As with finite frames, it is natural to ask for the closest stochastic Parseval generalized frame to a given stochastic generalized frame. The following result generalizes Theorem 2.2.

**Theorem 5.1.** If \(T\) is a stochastic generalized frame with generalized frame operator \(S\), then \(TS^{-1/2}\) is a stochastic Parseval generalized frame and, for any other stochastic Parseval generalized frame \(R\), we have
\[
\int_\Omega \|T(\omega) - (T(\omega)S^{-1/2})\|_{HS}^2 d\mu(\omega) \leq \int_\Omega \|T(\omega) - R(\omega)\|_{HS}^2 d\mu(\omega).
\]
Equality holds if and only if \(R = TS^{-1/2}\) almost everywhere.

Most parts of the proof of Theorem 5.1 follow the lines in [16], where Theorem 2.2 is verified. Deviations due to the more general situation are quite straight-forward, so we omit the proof.

For \(r = 1\), we recover the finite frame \(\{x_j\}_{j=1}^n \subset \mathbb{K}^d\) as the rank one operators \(\{x_j^*\}_{j=1}^n\), and the corresponding generalized frame operator is
\[
\sum_{j=1}^n x_jx_j^* = \sum_{j=1}^n \langle x_j, \cdot \rangle x_j.
\]
The canonical tight frame \(\{S^{-1/2}x_j\}_{j=1}^n \subset \mathbb{K}^d\) is associated to the generalized frame \(\{x_j^*S^{-1/2}\}_{j=1}^n\), so that Theorem 5.1 matches Theorems 2.2 when the generalized frame is induced by a finite frame.

**Remark 5.2.** Similar to the standard Paulsen problem, cf. [6, 11, 15], the situation becomes more complicated if, given a stochastic generalized frame \(T\), we look for the closest stochastic Parseval generalized frame that inherits further structure, such as being formed by orthogonal projectors or similar conditions. Some structures are preserved from \(T\) itself, and, for instance, if \(T\) is self-adjoint a.e., then \(TS^{-1/2}\) is so too. Similarly, if \(T\) is positive (semi-)definite a.e., then so is \(TS^{-1/2}\). However, if \(T\) is an orthogonal projector
(or has unit Hilbert Schmidt norm) a.e., then $TS^{-1/2}$ may often not be an orthogonal projector (or does not have equal norm), and one needs an alternative construction. To our knowledge, these are still open problems.

Given a generalized frame, we shall adopt the algorithm proposed in Section 3 to construct a tight generalized 1-frame $\{R_j\}_{j=1}^n$ with unit norm $\|R_j\|_{HS} = 1$, for $j = 1, \ldots, n$. Rescaling by the factor $\sqrt{d/n}$ yields the desired equal norm Parseval generalized 1-frame.

By following the ideas in (8) and (9), we define $\Gamma_0 := \frac{1}{d}I_d$ and, recursively,

$$
(18) \quad \Gamma_{k+1} = \frac{\Gamma_k^{1/2}M_k^{-1}\Gamma_k^{1/2}}{\text{trace}(\Gamma_k M_k^{-1})},
$$

where

$$
M(\Gamma) = \frac{d}{n} \sum_{j=1}^n \frac{\Gamma_j^{1/2}T_j^*T_j\Gamma_j^{1/2}}{\text{trace}(T_j T_j^*)} \quad \text{and} \quad M_k = M(\Gamma_k).
$$

Next, we use the scheme (18) to compute a unit norm tight generalized 1-frame. This generalizes results in [28, 37] for frames in $\mathbb{R}^d$. Our results here cover the more general setting of generalized frames in $\mathbb{R}^d$:

**Theorem 5.3.** Let $\{T_j\}_{j=1}^n$ be a generalized frame satisfying $\|T_j\|_{HS} = 1$, for $j = 1, \ldots, n$. Moreover, let $n > d(d - 1)$ and suppose that, for any subcollection of $q$ distinct operators $\{T_j\}_{j=1}^q$, $1 \leq q \leq d$, the set $\bigcup_{j=1}^q \text{range}(T_j^*)$ spans a subspace of $\mathbb{R}^d$ whose dimension is at least $q$. Then the recursive scheme (18) with $\Gamma_0 = \frac{1}{d}I_d$ converges towards a positive definite $\Gamma$ and $\{R_j\}_{j=1}^n$ defined by

$$
(19) \quad R_j := \frac{T_j \Gamma_n^{1/2}}{\|T_j \Gamma_n^{1/2}\|_{HS}}
$$

is a tight generalized 1-frame with $\|R_j\|_{HS} = 1$, for $j = 1, \ldots, n$.

**Proof.** Since $\{T_j\}_{j=1}^n$ is a generalized frame, the sequence $\{\Gamma_k\}_{k=1}^\infty$ is well-defined. It is also clear that $\Gamma_k$ is hermitian positive definite and $\text{trace}(\Gamma_k) = 1$, for all $k = 0, 1, 2, \ldots$. If we suppose that $\{\Gamma_k\}_{k=1}^\infty$ converges towards $\Gamma$, then $M(\Gamma) = I_d$ must hold, and a direct computation yields that $\{R_j\}_{j=1}^n$ is a tight generalized 1-frame with $\|R_j\|_{HS} = 1$, for $j = 1, \ldots, n$.

To verify convergence, we shall follow the ideas of the technical 3-step procedure used in [28, 37] for frames in $\mathbb{R}^d$.

**Step 1** (refers to Theorem 2.1 in [37]) We shall verify that, $M(\Gamma_a) = M(\Gamma_b)$ if and only if $\Gamma_b = c\Gamma_a$ for some positive factor $c$. Without loss of generality, we can assume that $\Gamma_b = I_d$. Otherwise, replace $\{T_j\}_{j=1}^n$ with $\{T_j \Gamma_b^{1/2}\}_{j=1}^n$. Let $\gamma_1$ be the largest eigenvalue of $\Gamma_a$ and $P_1$ be the associated eigenprojector. Moreover, let $\{P_i\}_{i=2}^r$ be one-dimensional eigenprojectors of $\Gamma_a$ associated to eigenvalues $\{\gamma_i\}_{i=2}^r$ such that $\gamma_1 > \gamma_i$, for $i = 2, \ldots, r$, and
We obtain
\[ \sum_{i=1}^{r} \gamma_i P_i = \Gamma_a. \] Note that \( \gamma_2, \ldots, \gamma_r \) do not need to be pairwise distinct. We obtain
\[
P_1 M(\Gamma_1) = \gamma_1 \frac{d}{n} \sum_{j=1}^{n} \frac{T_j^* P_j T_j}{\text{trace}(T_j^* \Gamma_1 T_j^*)} = \gamma_1 \frac{d}{n} \sum_{j=1}^{n} \sum_{i=1}^{r} \gamma_i \frac{\text{trace}(T_j^* P_j T_j)}{\gamma_i \text{trace}(T_j^* T_j^*)}
\]
\[ \leq \gamma_1 \frac{d}{n} \sum_{j=1}^{n} \frac{T_j^* P_j T_j}{\gamma_1 \text{trace}(T_j^* T_j^*)} = P_1 M(I_d). \]

Now, \( M(\Gamma_a) = M(I_d) \) implies that either \( \text{trace}(T_j^* P_j T_j^*) = 0 \), for all \( j = 1, \ldots, n \) and \( i = 2, \ldots, r \) or \( P_1 = I_d \) and, hence, \( \Gamma_a = \gamma_1 I_d \). Since \( \{T_j\}_{j=1}^{n} \) is a generalized frame, \( \text{trace}(T_j^* P_j T_j^*) \) cannot vanish simultaneously for all \( j = 1, \ldots, n \), so that, indeed, \( \Gamma_a \) is a nonzero multiple of the identity.

**Step 2** (refers to Lemma 2.1 in [37]) Let \( \lambda_{1,k} \) and \( \lambda_{d,k} \) be the largest and smallest eigenvalue of \( M_k \), respectively. Following the lines of the proof of Lemma 2.1 in [37] yields that \( \{\lambda_{1,k}\}_{k=1}^{\infty} \) is a decreasing sequence that converges towards \( \lambda_1 \geq 1 \). On the other hand, \( \{\lambda_{d,k}\}_{k=1}^{\infty} \) is an increasing sequence that converges towards \( \lambda_d \leq 1 \).

**Step 3** (refers to Theorem 2.2 and Corollary 2.2 in [37]) Here, we need \( n > d(d - 1) \) and the additional assumptions in the theorem. Then we can follow the lines of the proofs of Theorem 2.2 in [37] and Corollary 2.2 in [37], so that Steps 1 and 2 yield the convergence of \( \Gamma_k \) towards a positive definite \( \Gamma \).

\[ \square \]

Let us have a look at few pathological examples first:

**Example 5.4.**

1) If \( \{T_j\}_{j=1}^{n} \) is already a unit norm tight generalized 1-frame, then \( \Gamma_1 = \Gamma = \frac{1}{d} I_d \) and \( \{R_j\}_{j=1}^{n} = \{T_j\}_{j=1}^{n} \).

2) If the generalized frame consist of a single matrix \( T \in \mathbb{K}^{r \times d} \), then
\[
\Gamma_1 = \Gamma = T^* T \quad \text{and} \quad \{T_j\}_{j=1}^{n} \quad \text{yields} \quad R = \frac{1}{\sqrt{d}} T (T^* T)^{-1/2},
\]
which is a unit norm tight generalized 1-frame.

Next, we illustrate Theorem 5.3 with two numerical examples:

**Example 5.5.** For \( 0 \leq t \leq 1/2 \), define
\[
T_1(t) = \begin{pmatrix} \sqrt{1-t} & 0 \\ 0 & \sqrt{t} \end{pmatrix}, \quad T_2(t) = \begin{pmatrix} \sqrt{t} & \sqrt{t} \\ 0 & \sqrt{1-2t} \end{pmatrix}, \quad T_3(t) = \begin{pmatrix} \sqrt{\frac{1-t}{2}} & 0 \\ 0 & \sqrt{\frac{1+t}{2}} \end{pmatrix},
\]
and let \( S(t) \) denote the associated generalized frame operator. Note that \( \{T_j(t)\}_{j=1}^{3} \) satisfies the assumptions of Theorem 5.3 for all \( 0 \leq t \leq 1/2 \). If \( t = 0 \), then we have a tight generalized 1-frame with unit HS-norms. Our algorithm provides a Parseval generalized 1-frame \( \{\sqrt{2/3} R_1(t)\}_{j=1}^{3} \) with equal HS-norm, see Fig. 2 for the errors \( \sum_{j=1}^{3} \|T_j(t) - \sqrt{2/3} R_1(t)\|_{HS}^2 \) and \( \sum_{j=1}^{3} \|T_j(t) - T_j(t) S(t)\|_{HS}^{-1/2} \) for the errors.
Figure 2. $t$ versus the distance (blue) $\sum_{j=1}^{3} \|T_j(t) - \sqrt{2/3}R_j(t)\|_{HS}^2$ and (red) $\sum_{j=1}^{3} \|T_j(t) - T_j(t)S(t)^{-1/2}\|_{HS}^2$ in Example 5.5. Note that $\{T_j(0)\}_{j=1}^{3}$ is tight but needs rescaling to become the Parseval generalized frame $\{\sqrt{2/3}T_j(0)\}_{j=1}^{3}$. It is clear that $R_j(0) = T_j(0)$ and $\sqrt{2/3}T_j(0) = T_j(0)S(0)^{-1/2}$ holds, and we have $\sum_{j=1}^{3} \|T_j(0) - \sqrt{2/3}T_j(0)\|_{HS}^2 = 0.1010$. The latter explains why the distance plots do not start at 0.

Example 5.6. We choose each entry of each element in $\{T_j\}_{j=1}^{3} \subset \mathbb{K}^{2\times2}$ independently according to a uniform distribution on $[0,1]$ and normalize so that $\|T_j\|_{HS} = 1$. All numbers in the following are averaged over 10,000 realizations, and let $S$ denote the generalized frame operator of $\{T_j\}_{j=1}^{3}$. According to Theorem 5.1, $\{T_jS^{-1/2}\}_{j=1}^{3}$ is the Parseval generalized 1-frame that is closest to $\{T_j\}_{j=1}^{3}$, and we compute $\sum_{j=1}^{3} \|T_j - T_jS^{-1/2}\|_{HS}^2 \approx 0.61$. However, the elements of $\{T_jS^{-1/2}\}_{j=1}^{3}$ may not have equal HS-norm. Based on Theorem 5.3 the collection $\{\tilde{R}_j\}_{j=1}^{3} = \{\sqrt{2/3}R_j\}_{j=1}^{3}$ is a Parseval generalized 1-frame, its elements have equal HS-norm, and we compute $\sum_{j=1}^{3} \|T_j - \tilde{R}_j\|_{HS}^2 \approx 0.71$. Thus, the additional property of having equal HS-norm costs $\approx 0.10 = 0.71 - 0.61$. It remains open though if there are other Parseval generalized 1-frames whose elements have equal HS-norm and that are closer to $\{T_j\}_{j=1}^{3}$.

Let us also illustrate when the algorithm fails to converge:

Example 5.7. For $t = 0$, the collection $\{T_j(t)\}_{j=1}^{3}$, where

$$T_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2(t) = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad T_3(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

violates the conditions in Theorem 5.3 and indeed, the iterative scheme does not converge towards a positive definite matrix $\Gamma$. For $t > 0$, on the other hand, we observe convergence numerically, consistent with Theorem 5.3.
Remark 5.8. To the best of our knowledge, our proposed algorithm is the first attempt in the literature to compute a unit norm tight generalized 1-frame that is close to a given unit norm generalized 1-frame. Estimates on the distance to nearby unit norm tight generalized 1-frames do not seem to exist either.

6. Compressed sensing matrices

In this section we shall verify that the random matrices used in compressed sensing [13, 21] form tight stochastic generalized $p$-frames.

6.1. Gaussian matrices. Let $1 \leq k < d$ and consider the $k \times d$ random matrix $G$ whose entries are i.i.d Gaussian. Its joint element density is

$$M \mapsto \frac{1}{(2\pi)^{kd/2}} \exp\left(-\frac{1}{2} \|M\|_{HS}^2\right),$$

and the resulting self-adjoint matrix $G^*G \in \mathbb{R}^{d \times d}$ is a Wishart-matrix, whose distribution is singular since its rank is $k < d$, cf. [38]. According to (20) the distribution of $G$ is unitarily invariant, so that $G$ is a tight stochastic generalized $p$-frame for all integers $p$. By using the moments of the chi-squared distribution, we see that the bounds satisfy $A_p = B_p = k(k + 2) \cdots (k + 2p - 2)$.

6.2. Circulant matrices. Given a vector $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$, the corresponding circulant matrix is

$$C = \begin{pmatrix} x_1 & x_d & \ldots & x_2 \\ x_2 & x_1 & \ldots & x_3 \\ \vdots & \vdots & \ddots & \vdots \\ x_d & x_{d-1} & \ldots & x_1 \end{pmatrix}.$$

Each column of $C$ is a cyclic shift of the previous one, which can be performed by applying the matrix $T$, having ones in the lower secondary diagonal, another one in the upper right corner, and zeros anywhere else. The upper $k \times d$ block $D$ of the matrix $C$ was used as a compressed sensing measurement matrix in [31]. If the entries of $x$ are i.i.d. with zero mean and non-vanishing second moments, then $D$ is a stochastic tight generalized 1-frame with $A_1 = k\mathbb{E}(x_1^2)$. For instance, if $x$ is the Rademacher sequence, i.e., entries are independent and equal to $\pm 1$ with probability $1/2$, then $D$ is 1-tight but not 2-tight.

6.3. Fusion frames. Let us consider generalized frames that have additional structure. If the elements of a generalized frame $\{T_j\}_{j=1}^n \subset \mathbb{K}^{d \times d}$ are all multiples of orthogonal projectors of rank $k$, then we can identify them with subspaces $\{V_j\}_{j=1}^n \subset \mathcal{G}_{k,d}(\mathbb{K})$ and positive weights $\{v_j\}_{j=1}^n$ and call $\{(V_j, v_j)\}_{j=1}^n$ a fusion frame, cf. [17]. Hence, we have $T_j = \sqrt{v_j}P_{V_j}$, where $P_{V_j}$ denotes the orthogonal projector onto $V_j$. From a different perspective, a fusion frame can be identified with a weighted collection $\{(M_j, v_j)\}_{j=1}^n$ in
the Stiefel manifold $\mathcal{V}_{k,d} = \{ M \in \mathbb{K}^{k,d} : MM^* = I_k \}$, so that the rows of $M_j$ are an orthonormal basis of $V_j$ implying that $T_j^* T_j = v_j M_j^* M_j = v_j P_{V_j}$. Thus, both $\{ \sqrt{v_j} M_j \}_{j=1}^n \subset \mathbb{K}^{k \times d}$ and $\{ \sqrt{v_j} P_{V_j} \}_{j=1}^n \subset \mathbb{K}^{d \times d}$ “describe” the fusion frame $\{(V_j, v_j)\}_{j=1}^n$. Fusion frames were used for compressed sensing approaches in [10].

6.4. **Gabor.** Time-frequency structured measurement matrices were considered in [30] in relation to compressed sensing. The translation $T$ is performed by the cyclic shift as in Section 6.2, and the modulation operator on $\mathbb{C}^d$ is given by $M = \text{diag}(1, e^{2\pi i/d}, \ldots, e^{2\pi i(d-1)/d})$. For any nonzero $g \in \mathbb{C}^d$, the full Gabor system $\{ M^\ell T^k g : \ell, k = 0, \ldots, d-1 \}$ has cardinality $d^2$ and forms a tight frame for $\mathbb{C}^d$, cf. [29]. We shall use the $d \times d^2$ matrix $G$, whose columns are formed by the tight frame vectors. A short computation yields that, if $g$ is chosen at random as the Rademacher sequence, then $G$ is a stochastic tight generalized 1-frame. Moreover, each $G^* G$ is an orthogonal projector, so that $G^* G$ corresponds to a stochastic tight fusion frame. The same holds when $g$ is the Steinhaus sequence, i.e., each entry is uniformly distributed on the complex unit circle.

7. **Frame signature**

Given a generalized frame $\{T_j\}_{j=1}^n$, we call the sequence of lower and upper optimal generalized frame bounds $(A_p)_{p=1}^\infty$ and $(B_p)_{p=1}^\infty$ of the normalized system $\{T_j/\|T_j\|_{HS}\}_{j=1}^n$ the lower and upper generalized frame signature, respectively. We call $B_p - A_p$ the $p$-th gap of the generalized frame. For a stochastic generalized frame $T$, we define the signature consistently as the optimal lower and upper frame bounds of the random variable $\tilde{T}(\omega) := T(\omega)/\|T(\omega)\|$, for $T(\omega) \neq 0$ and $\tilde{T}(\omega) := 0$ otherwise.

For a frame $\{x_j\}_{j=1}^n \subset \mathbb{R}^d$, the first entry of the lower and upper frame signature coincides with the lower and upper frame redundancy as proposed in [7]. Note that the signature is invariant under the action of unitary operators, reweighting, and permutations, so that we are consistent with [7]. Moreover, the lower frame redundancy was aimed to yield the number of vectors that can be removed yet still leave a frame. The authors in [7] claim that $\lceil A_1 \rceil - 1$ many vectors can be deleted yet leave a frame. In fact, the number can be replaced with $\lceil A_1 \rceil - 1$, but, still, this is only a lower bound on this quantity. We claim that the frame signature allows a refinement and support our ideas with a collection of examples.

7.1. **Examples of the frame signature.** The first entry of the lower and upper frame signature for both collections $\{e_1, e_1, e_2\}$ and $\{e_1, e_2, e_1 + e_2\}$ is $(A_1, B_1) = (1,1)$. However, the type of redundancy appears to be different in both sets. From the first set, we can remove one element and the remaining set has lower frame redundancy zero. Removal of an arbitrary element of
the second collection leaves us with a lower frame redundancy bigger than 0.29. It may be possible to “see” such differences in the frame signature.

By using results in [3], we observe that the signature of a frame \( \{x_j\}_{j=1}^n \) for \( \mathbb{R}^d \) satisfies \( A_p \leq n \frac{(1/2)^p}{(d/2)^p} \leq B_p \), and in the complex case we have \( A_p \leq n \frac{(1)^p}{(d)^p} \leq B_p \), where \( (a)p = a(a + 1) \cdots (a + p - 1) \).

**Example 7.1.**  1) The frame \( \{e_1, \ldots, e_1, e_2, e_3, \ldots, e_d\} \) for \( \mathbb{R}^d \), where \( e_1 \) occurs \( k \) times, satisfies \( B_p = k \), for all \( p \geq 1 \), and \( A_1 = 1 \). Some calculus yields \( A_p = (d - 1 + k^{-1/(p - 1)})^{1-p} \) for \( p \geq 2 \). Thus, the gap \( B_p - A_p \) converges towards \( k \).

2) The frame signature of \( \{e_1, e_2, e_3, e_2\} \subset \mathbb{R}^2 \) satisfies \( A_1 = 1 \) and \( B_1 = 2 \), and \( A_2 = 1/2 \) and \( B_2 = 3/2 \). We can compute using calculus that \( A_p = 2^{1-p} \) and \( B_p = 1 + 2^{1-p} \). Thus, the gap is \( B_p - A_p = 1 \) for all \( p \geq 1 \).

For \( d = k = 2 \), the lower bound \( A_p \) in Example 7.1 1) is bigger than in 2), but the gap is smaller in 2).

**Example 7.2.**  1) The frame signature of \( k \) copies of an orthonormal basis for \( \mathbb{R}^d \) is \( A_p = kd^{1-p} \) and \( B_p = k \), for all \( p \geq 1 \). The gap is then \( B_p - A_p = k(1 - \frac{1}{d^{p-1}}) \) and, therefore, increases in \( p \) and converges towards \( k \).

2) The frame \( \{e_1, e_2, \frac{1}{\sqrt{2}} (e_1 + e_2), \frac{1}{\sqrt{2}} (e_1 - e_2)\} \subset \mathbb{R}^2 \) is \( p \)-tight for \( p = 1, 2, 3 \), so that \( A_1 = B_1 = 2 \) and \( A_2 = B_2 = 3/2 \) and \( A_3 = B_3 = 5/4 \). It is not 4-tight since \( A_4 = 34/32 \) and \( B_4 = 36/32 \) while the tight 4-frame bound would be 35/32.

By some calculus and using the points \((1, 0)\) and \((\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{8}}, \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}}) = (\cos(\pi/8), \sin(\pi/8))\), we can derive

\[
A_p = 2^{1-p}((1 + \sqrt{\frac{1}{2}})^p + (1 - \sqrt{\frac{1}{2}})^p) \quad \text{and} \quad B_p = 1 + 2^{1-p}.
\]

For instance, we see that \( B_{10} - A_{10} \leq 0.6 \), and the gap converges from below towards 1.

For \( d = k = 2 \), the Examples 7.2 1) and 2) are both the union of two orthonormal bases, but 2) appears more uniformly distributed than 1). The following example is a modification of [7] Example 2.8:

**Example 7.3.** Let \( \varepsilon^2 \leq \frac{d-1}{d} \). The frame

\[
\{e_1, \sqrt{1 - \varepsilon^2} e_1 \pm \varepsilon e_2, \ldots, \sqrt{1 - \varepsilon^2} e_1 \pm \varepsilon e_d\}
\]

has \( 2d - 1 \) elements and satisfies

\[
A_p = \frac{2\varepsilon^{2p}}{(d-1)^{p-1}} \quad \text{and} \quad B_p = 1 + 2(d-1)(1 - \varepsilon^2)^p.
\]
Note that $B_p - A_p > 1$ for all $p$ and if $p$ tends to infinity, then the gap converges towards 1. If $p$ is fixed and $\varepsilon$ tends to zero, then the gap converges towards $1 + 2(d - 1)$. 

For a fusion frame $\{(V_j, v_j)\}_{j=1}^n$ with $\{V_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$, the fusion frame signature is consistently defined as the lower and upper generalized frame bounds of the system $\{k^{-1/2}P_{V_j}\}_{j=1}^n$. By using results in [3], we see that the fusion frame signature satisfies $A_p \leq nk^{-1/2} \frac{(k/2)^p}{(d/2)p} \leq B_p$ in the real case and $A_p \leq nk^{-1/2} \frac{(k_p)^p}{(d_p)^p} \leq B_p$ in the complex setting.

7.2. Theoretical result for $d = 2$. Let us formulate a result showing that the higher orders of frame gaps carry useful information:

**Proposition 7.4.** If the first $p$ gaps of the frame signature of $\{x_j\}_{j=1}^n \subset \mathbb{C}^d$ vanish, then any removal of $p$ elements leaves us with a frame.

To prove the above proposition, we need the following lemma:

**Lemma 7.5.** If the first $p$ gaps of the frame signature of $\{x_j\}_{j=1}^n \subset \mathbb{C}^d$ vanish and $Z = \left( \bigoplus_{j=p+1}^n x_j \mathbb{C} \right)^\perp$, then, for all $x \in S^{d-1} \setminus Z$,

$$\# \{ y \in S^{d-1} : (P_{x_{p+1}} c(y), \ldots, P_{x_n} c(y)) = (P_{x_{p+1}} c(x), \ldots, P_{x_n} c(x)) \} < \infty.$$

**Proof of Lemma 7.5.** If $p < \frac{n}{d}$, then $\{x_j\}_{j=p+1}^n$ is a frame and, therefore, spans $\mathbb{C}^d$, so that the statement holds. To address $p \geq \frac{n}{d}$, let $x \in S^{d-1} \setminus Z$ and denote $V_j := x_j \mathbb{C}$ with $a_j = 1/n$. Since the tight $p$-fusion frame $\{(V_j, a_j)\}_{j=1}^n$ is also a tight $\ell$-fusion frame for $1 \leq \ell \leq p$, the vector $(\|P_{V_1}(x)^2, \ldots, \|P_{V_\ell}(x)^2)$ is among the solutions of the system of algebraic equation

$$\sum_{j=1}^p a_j T_j^\ell \frac{(k)\ell}{(d)\ell} - \sum_{j=p+1}^n a_j \|P_{V_j}(x)^2, 1 \leq \ell \leq p. \tag{21}$$

According to [3] Proposition 7.1, the cardinality of the set of solutions $\mathcal{S}$ is finite (in fact, at most $p!$). Moreover, Proposition 7.2 in [3] not only holds in the real case but also for complex tight 2-frames, so that

$$P_{z \mathbb{C}} = \frac{1}{\alpha} \sum_{j=1}^n a_j \|P_{V_j}(z)^2P_{V_j} - I_d, \text{ for all } z \in S^{d-1}, \tag{22}$$

where $\alpha = \frac{1}{d(d+1)}$, see also [4]. We apply (22) by choosing $z = x$ and replacing $\{P_{V_j}(z)^2\}_{j=1}^p$ with $\{t_j\}_{j=1}^p$, where $(t_1, \ldots, t_p) \in \mathcal{S}$. If the right-hand side indeed defines a one-dimensional projector, we choose a vector in its range with norm one. Let $\{\xi_i\}_{i=1}^s \subset S^{d-1}$ denote this set of vectors. Since $x \notin Z$, the vector $(P_{V_{p+1}}(x), \ldots, P_{V_n}(x))$ must have at least one nonzero entry $P_{V_{p+j_0}}(x)$. We define

$$L := \{ \lambda \xi : \xi \in \{\xi_1\}_{i=1}^s, \exists \lambda \in \mathbb{C}, |\lambda| = 1, P_{V_{p+j_0}}(\lambda \xi) = P_{V_{p+j_0}}(x) \}$$
that is apparently a finite set. Finally, any \( \xi \in S^{d-1} \) satisfying \( P_{V_{p+1}}(\xi) = P_{V_{p+1}}(x), \ldots, P_{V_n}(\xi) = P_{V_n}(x) \) must lie in \( L \), which concludes the proof. \( \square \)

**Proof of Proposition 7.4.** Without loss of generality, we shall verify that the last \( n - p \) subspaces span \( C_d \). Aiming at a contradiction, we assume that they span a subspace of dimension less than \( d \) (it is impossible that the dimension is zero). Let \( Z \) be as in Lemma 7.5, and let \( x = x_a + x_b \in S^{d-1} \setminus Z \) such that \( \|x_b\| < 1 \). Since \( x \notin Z \), we see that \( x_a \neq 0 \neq x_b \).

However, any element in \( \mathcal{L} \cap S^{d-1} \), where
\[
\mathcal{L} := \{ \lambda x_a + x_b : \lambda \in \mathbb{C} \},
\]
would lead to the same set \( \{ P_{V_i}(x) \}_{i=p+1}^n \). Since the cardinality of \( \mathcal{L} \cap S^{d-1} \) is infinite, we have a contradiction to Lemma 7.5. \( \square \)

The maximal number of equiangular lines in \( \mathbb{C}^d \) is \( d^2 \) and the existence is settled for small \( d \) at least, cf. \([32, 8, 33, 35, 9]\). Equivalently, we may say that the maximal number of distinct points \( \{ x_j \}_{j=1}^n \) on the unit complex sphere is \( d^2 \) if we suppose that \( |\langle x_i, x_j \rangle| = c \) for all \( i \neq j \) and some constant \( c \).

**Example 7.6.** Let \( \{ f_j \}_{j=1}^4 \) be unit norm vectors in \( \mathbb{C}^2 \), so that each of them lies on exactly one line of a collection of 4 equiangular lines. The lower frame redundancy tells us that we can remove one arbitrary element and are still left with a frame since \( A_1 = 2 \). On the other hand, \( \{ f_j \}_{j=1}^4 \) is 2-tight, cf. \([2, 3]\), so that Proposition 7.4 implies that we can even remove 2 elements.

**7.3. Equivalence of frames.** We call two stochastic frames \( X \) and \( Y \) for \( \mathbb{K}^d \) equivalent if there is a unitary matrix \( U \) such that \( \pi(Y) = \pi(UX) \) almost everywhere on \( X \neq 0 \), where \( \pi \) is the projection from \( \mathbb{K}^d \) into projective space. Consistently, we say that two frames are equivalent if one can be derived from the other by a unitary transform, permuting its elements, and multiplication of each frame vector with some nonzero number. Clearly, two equivalent frames share the same frame signature. In the following we investigate on the converse relation: If the frame signatures coincide, are the two frames equivalent? We shall provide some examples supporting an affirmative answer.

The frame gaps of the uniform distribution on the unit sphere are all zero. The uniform distribution is unitarily invariant and its projection into projective space yields the uniform distribution there. The moments in projective space of any other distribution on the sphere whose frame gaps all vanish must coincide with those of the uniform distribution. Due to the uniqueness of moments in projective space, the distribution coincide there and, hence, must be equivalent on the sphere. For instance, a random vector of i.i.d Gaussian entries is equivalent to the uniform distribution.

Let us revisit Example 7.1 1):
Example 7.7. Let \( \{x_j\}_{j=1}^{d+k} \subset \mathbb{R}^d \). If its signature satisfies \( B_p = k \), for all \( p \geq 1 \), and \( A_1 = 1 \), then it is equivalent to an orthogonal basis, in which one element is repeated \( k \) times. Indeed, the upper signature yields that there is one element repeated \( k \) times and the other elements are orthogonal to them. \( A_1 = 1 \) then implies that the remaining elements are pairwise orthogonal.

For certain numbers of frame vectors, the first or second frame gap being zero can determine the frame up to equivalence:

Example 7.8.  
1) Let the signature of \( \{x_j\}_{j=1}^{d+1} \subset \mathbb{K}^d \) satisfy \( A_1 = B_1 \). Then the vectors are equiangular \([27, 26]\) and the \( \{x_j\}_{j=1}^{d+1} \) are determined up to equivalence.

2) Let the signature of \( \{x_j\}_{j=1}^{n} \subset \mathbb{K}^d \) satisfy \( A_2 = B_2 \). If \( n = 1/2d(d+1) \) in the real case (or \( n = d^2 \) in the complex setting), then \( \{x_j\}_{j=1}^{n} \) are equivalent to an equiangular collection on the sphere \([3, 2]\). The \( n \)-point homogeneity of the unit sphere yields that the \( \{x_j\}_{j=1}^{n} \) are determined up to equivalence.

8. Fusion frames cannot be tight frames for \( \mathbb{R}^{d \times d} \)

So far, we considered elements \( \{T_j\}_{j=1}^{n} \subset \mathbb{R}^{d \times d} \) that form certain types of frames for \( \mathbb{R}^d \). We now shift perspectives and explore their frame properties in the space \( \mathbb{R}^{d \times d} \) itself. The main result of the present section is the following negative finding:

**Theorem 8.1.** Let \( \{V_j\}_{j=1}^{n} \subset \mathbb{R}^d \) be linear subspaces of dimension \( 1 \leq k_j < d \), \( j = 1, \ldots, n \), respectively, and \( \{P_{V_j}\}_{j=1}^{n} \subset \mathbb{R}^{d \times d} \) the associated orthogonal projectors. Then there are no numbers \( \{c_j\}_{j=1}^{n} \) such that \( \{c_j P_{V_j}\}_{j=1}^{n} \) is a tight frame for the space of symmetric matrices in \( \mathbb{R}^{d \times d} \).

Note that Theorem 8.1 also implies that \( \{c_j P_{V_j}\}_{j=1}^{n} \) cannot be a tight frames for \( \mathbb{R}^{d \times d} \).

**Proof.** Let \( \mathcal{H} \) denote the space of symmetric matrices in \( \mathbb{R}^{d \times d} \) and suppose that \( \{c_j P_{V_j}\}_{j=1}^{n} \) is a tight frame for \( \mathcal{H} \). Then there is a positive constant \( A > 0 \) such that, for all \( X \in \mathcal{H} \),

\[
\sum_{j=1}^{n} c_j^2 \langle X, P_{V_j} \rangle P_{V_j} = AX, \quad \sum_{j=1}^{n} c_j^2 |\langle X, P_{V_j} \rangle|^2 = A \|X\|^2.
\]

Using the left-hand side of (23), choosing \( X = e_i^* e_i \), and summing up over \( i \) yield

\[
\sum_{j=1}^{n} c_j^2 k_j P_{V_j} = AI.
\]
Results in [3] then imply $A = \frac{1}{d} \sum_{j=1}^{n} c_j^2 k_j^2$. If we make use of the right-hand side of (23) and take $X = xx^*$, then we obtain, for all $x \in \mathbb{R}^d$,
\[
\sum_{j=1}^{n} c_j^2 \|P_{V_j}(x)\|^4 = A \|x\|^4.
\]
As in [3] Proof of Theorem 4.2, we can use the Laplacian on both sides and derive
\[
\sum_{j=1}^{n} c_j^2 (1 + k_j/2) \|P_{V_j}(x)\|^2 = A (1 + d/2) \|x\|^2,
\]
which yields $A (1 + d/2) = \frac{1}{d} \sum_{j=1}^{n} c_j^2 (1 + k_j/2) k_j$, cf. [3]. We now have two equations for $A$ that contradict each other. \hfill \Box

9. Conclusions

As the main result of the present paper, we derived an algorithm to construct unit norm tight generalized frames that are close to a given unit norm generalized frame. We also derived almost-tight generalized frames as samples from a suitable probability distribution that we called stochastic tight generalized $p$-frames. Such almost-tight generalized frames yield signal reconstruction up to a certain precision without the cumbersome inversion of the frame operator. Examples of stochastic tight generalized $p$-frames were derived from compressed sensing, and we introduced the frame signature as a measure of redundancy and discussed its relation to erasures.

The above results give rise to the following questions:

1) Given a unit norm almost-tight generalized frame (or fusion frame), can we find sharp estimates on its distance to the nearest unit norm tight generalized frame (or fusion frame)?

2) Does the frame signature inherit sufficient information to derive sharp statements about how many arbitrary elements of a frame can be removed, yet still have a frame? For fixed $d$ and $n$, does the signature even determine the frame up to equivalence?

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