Pullback dynamics of a 3D Navier-Stokes equation with nonlinear viscosity

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Abstract. This paper is concerned with pullback dynamics of a 3D Navier-Stokes equations with variable viscosity and subject to perturbations of time-dependent external forces. Under suitable assumptions on the external force, which is possibly unbounded, we establish the existence of finite-dimensional minimal pullback attractor in a general setting involving tempered universe. We also present a sufficient condition on the viscosity coefficients in order for the attractors to be non-trivial. We conclude the paper by showing the upper semi-continuity of pullback attractors as the non-autonomous perturbation vanishes.

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1. Introduction

The well-known 3D incompressible Navier-Stokes equation describes the conservation law of momentum and mass of viscous fluid, which can be proposed by

\[
\begin{aligned}
    u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= g(t, x), \quad x \in \Omega, \ t > \tau, \\
    \nabla \cdot u &= 0.
\end{aligned}
\]  

(1.1)

For system (1.1) endowed with Dirichlet boundary \( u|_{\partial \Omega} = a \) and appropriate initial data \( u|_{t=0} = b(x) \), the local well-posedness was investigated by Leray [27, 28, 29] and Hopf [20], and also Ladyzhenskaya [25, Temam [41], Lions [33] and references therein.

One of the main motivations to study the Navier-Stokes equations is to understand fluid turbulence. In 1960s, Ladyzhenskaya asked the question that does system (1.1) determine the motion of viscous fluid flow completely? In
fact, the rigorous answer to the well-posedness problem of (1.1) with \( a, b(x) \) given and boundary \( S = \partial \Omega \) sufficiently smooth is still unknown to date. In this context of well-posedness of (1.1), other more specific problems are investigated by mathematicians. For example, Robinson asked in [37] what is a reasonable equation for fully developed homogeneous turbulence? Could the pressure uniquely determine the velocity field of the fluid?

Physics provides important insights of solving the well-posedness problem of (1.1). For example, Ladyzhenskaya further proposed in [25] to approximate the solution of (1.1) by using the solution of a class of regular Navier-Stokes equations

\[
\begin{cases}
  u_t - \nu_0 \text{div} \left[ (1 + \varepsilon \hat{u}^2) Du \right] + (u \cdot \nabla)u + \nabla p = h(t, x), \\
  \nabla \cdot u = 0, \\
  Du = \nabla u + \nabla u^T, \quad \hat{u}^2 = \|Du\|_{L^2}^2,
\end{cases}
\]  

(1.2)

and its special case

\[
\begin{cases}
  u_t - \text{div} \left[ (\nu_0 + \nu_1 \| \nabla u \|^2_{L^2(\Omega)} )Du \right] + (u \cdot \nabla)u + \nabla p = h(t, x), \\
  \nabla \cdot u = 0,
\end{cases}
\]  

(1.3)

which reflects the physical phenomena that \( \| \nabla u(x, t) \|_{L^2(\Omega)} \) should not be too large or infinite, see [38, 39]. In this line of work, Smagorinsky [39] in 1960s proposed a similar approximating equation, known as Ladyzhenskaya-Smagorinsky model,

\[
\begin{cases}
  u_t - \text{div} \left[ (\nu_0 + \nu_1 \| Du \|_{L^2(\Omega)}^{p-2} )Du \right] + (u \cdot \nabla)u + \nabla p = h(t, x), \\
  \nabla \cdot u = 0,
\end{cases}
\]  

(1.4)

the regularity of which is studied by da Veiga [2] in 2009.

The advantages of systems (1.2), (1.3) and (1.4) can be summarized as:

(a) They all have meaningful physical interpretations;

(b) These approximating systems with boundary condition and initial data given possess global weak solution in 3D and the weak solution is unique in some appropriate sub-critical cases. Therefore, they are all well posed in 3D, the desired property of the original Navier-Stokes equation in 3D;

(c) The Stokes principle holds for all these systems of equations.

However, even for these systems, the uniqueness and stability when Reynold number is large are still open questions. To overcome this difficulty and simplify Ladyzhenskaya models, J. Lions and Prodi [31, 32] replace \( Du \)
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by \(\nabla u\), and thus generate another two variants of (1.1)
\[
\begin{align*}
&\begin{cases}
  u_t - \nu_1 \Delta u - \sum_{i=1}^n \frac{\partial}{\partial x_i} (|\nabla u|^{p-1} \frac{\partial u}{\partial x_i}) + (u \cdot \nabla) u + \nabla p = h(t, x), \\
  \nabla \cdot u = 0
\end{cases} \\
\end{align*}
\]
(1.5)

and
\[
\begin{align*}
&\begin{cases}
  u_t - \sum_{i=1}^n \frac{\partial}{\partial x_i} (|\nabla u|^{p-1} \frac{\partial u}{\partial x_i}) + (u \cdot \nabla) u + \nabla p = h(t, x), \\
  \nabla \cdot u = 0
\end{cases} \\
\end{align*}
\]
(1.6)

They proved that if \(p \geq \frac{n+2}{2} \), \((n = \text{dim } \Omega)\), systems (1.5) and (1.6) with Dirichlet boundary condition possess unique global weak solution. However, for systems (1.5) and (1.6), the uniqueness of global weak solution remains open in 3D for \(p < 5/2\).

To summarize, rigorous mathematical analysis on 3D Navier-Stokes equation is highly non-trivial, yet important for many physical and engineering applications. This gives us the motivation to study the problem proposed in this work: the pullback dynamics of 3D Navier-Stokes equation, using a dynamical system approach. Inspired by Ladyzhenskaya, Smagorinsky, J. Lions, Prodi and many other mathematicians’ work, instead of attacking the problem of the pullback dynamics of 3D Navier-Stokes equation directly, we would instead investigate the pullback dynamics of the approximating NS equation (1.6) with variable viscosity in 3D, as proposed in [31]:
\[
\begin{align*}
&\begin{cases}
  u_t - (\nu + \nu_0 \|\nabla u\|^2) \Delta u + (u \cdot \nabla) u + \nabla p = f(t, x), \quad x \in \Omega, \quad t > \tau, \\
  \nabla \cdot u = 0, \\
  u|_{\partial \Omega} = 0, \\
  u(x, \tau) = u_0(x), \quad x \in \Omega,
\end{cases} \\
\end{align*}
\]
(1.7)

here \(\Omega\) is a bounded domain in \(\mathbb{R}^3\) with smooth boundary \(\partial \Omega\). The unknown variables \(u\) in (1.7) is the velocity field with components \(u^i = u^i(t, x) (i = 1, 2, 3)\) and \(p = p(t, x)\) is the scalar pressure of the fluid. The kinematic viscous coefficients \(\nu\) and \(\nu_0\) are positive constants. When the variable viscosity \(\nu + \nu_0 \|\nabla u\|^2 > 0\) is constant, (1.7) reduces to (1.1).

We recall some of the most significant results pertaining to (1.7). The global weak solution of problem (1.7) was proved for \(n \leq 4\) by Lions [31, Chap 2]. The uniqueness of problem (1.7) was studied in [32] for space dimension \(n \leq 3\), which is necessary for investigation of long-time behavior of (1.7). Araújo, Milla Miranda and Medeiros [1] studied the global existence of weak solution in non-cylindrical domain. The forward dynamics for (1.7) was investigated by [8]. If \(\nu\) and \(\nu_0\) depend on time \(t\), the pullback dynamics on unbounded domain was studied in [3]. However, as far as we know, there are no results of pullback dynamics and its related topics such as the structure of attractors for system (1.7) with time-dependent external force which is our
The main results of this paper are summarized as:

(I) The existence of a family of minimal and unique pullback attractors of (1.7) based on universes and its structure. The relation between families of pullback attractors in various universes is also obtained. See Theorems 3.6 and 3.14.

(II) Estimate on the finite fractal dimension for minimal family of pullback attractor in $H$. Our result differs from those for 2D and 3D classical NS equation, see Theorem 3.7.

(III) A sufficient (but not necessary) condition that ensures non-trivialness of the pullback attractors. (See Theorem 3.8).

(IV) The continuity of pullback attractors to global attractor as the perturbation from the external force to system (1.1) vanishes (see Theorem 3.15).

One of the main features of our results is that the variable viscosity $\nu_0$ plays an significant role. First of all, it would change the space where we seek the weak solution from. Second, the upper bound for the estimate on the finite fractal dimension for (1.7) depends on $\nu$ as shown in Theorem 3.7.

We should point out that the continuity between the original NS equation (1.1) and the approximating system (1.7) is still open. In particular, the convergence of the pullback attractors of (1.7) to trajectory attractors for 3D incompressible Navier-Stokes equation (1.1) is still unknown and should be a good problem to study in the future.

The rest of this paper will be arranged as follows. We present some preliminaries and notations in Section 2. This will be followed by the statement and discussion of main results in Section 3. In the last section, Section 4, we prove the main results.

2. Preliminaries

Throughout this paper, $C$ will denote for positive constants, which would depend on $\Omega$.

2.1. Notations and functional spaces

Let $X$ be a Banach space with norm $\| \cdot \|_X$. The Hausdorff metric $\text{dist}_X(B_1, B_2)$ in $X$ between $B_1 \subseteq X$ and $B_2 \subseteq X$ is defined by

$$\text{dist}_X(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} d_X(x, y)$$

for $B_1, B_2 \subset X$,

where $d_X(x, y)$ denotes the distance between two points $x$ and $y$.

Let the space $H_0^m(\Omega)$ and $H^m(\Omega)(m \in \mathbb{R})$ be the Sobolev space. Denote $E := \{ u | u \in (C_0^\infty(\Omega))^3, \nabla \cdot u = 0 \}$, $H$ is the closure of $E$ in $(L^2(\Omega))^3$ topology,
\[(u, v) = \sum_{j=1}^{3} \int_{\Omega} u_j(x)v_j(x)dx, \quad \forall \ u, v \in (L^2(\Omega))^3,\]
\[|u|^2 = (u, u), \quad \forall \ u \in (L^2(\Omega))^3.\]

\(V\) is the closure of \(E\) in \((H^1(\Omega))^3\) topology, \((\cdot, \cdot)\) and \(|\cdot|\) denote the inner product and norm of \(H\) respectively, i.e.,
\[(u, v) = \sum_{i,j=1}^{3} \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx, \quad \forall \ u, v \in (H^1_0(\Omega))^3,\]
\[|u|^2 = ((u, u)), \quad \forall \ u \in (H^1_0(\Omega))^3.\]

\(H'\) and \(V'\) are dual spaces of \(H\) and \(V\) respectively, the injections \(V \hookrightarrow H \equiv H' \hookrightarrow V'\) are dense and continuous. The notations \(|\cdot|_s\) and \((\cdot, \cdot)_s\) denote the inner product and norm of \(V\) respectively, i.e.,
\[(u, v)_s = \sum_{i,j=1}^{3} \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx, \quad \forall \ u, v \in (H^1_0(\Omega))^3,\]
\[|u|_s^2 = ((u, u)), \quad \forall \ u \in (H^1_0(\Omega))^3.\]

### 2.2. Abstract equivalent equation

Let \(P\) be the Helmholtz-Leray orthogonal projection operator from \((L^2(\Omega))^3\) onto \(H\). We define \(A := -P\Delta\) is the Stokes operator with domain \(D(A) = (H^2(\Omega))^3 \cap (H^1_0(\Omega))^3\), then the operator \(A : V \to V'\) has the property \((Au, v) = ((u, v))\) for all \(u, v \in V\) which is an isomorphism from \(V\) into \(V'\). \((\lambda_j)_{j=1}^{\infty} (0 < \lambda_1 \leq \lambda_2 \leq \cdots)\) are eigenvalues of operator \(A\) for the eigenvalue equation with Dirichlet boundary in \(L^2(\Omega), \{\omega_j\}_{j=1}^{\infty}\) is an orthonormal basis of \(A\) corresponding to \((\lambda_j)_{j=1}^{\infty}\), i.e., \(A\omega_j = \lambda_j \omega_j\).

We define the fractal operator \(A^s (s \in \mathbb{R})\) (see [4], [42]) as
\[A^s f = \sum_{j} \lambda_j^s (f, \omega_j) \omega_j, \quad s \in \mathbb{C}, \ j \in \mathbb{R},\]
\[V^s = D(A^s) = \left\{ g \in H : (A^s g) \in H, \sum_{i=1}^{\infty} \lambda_i^{2s} |(u, \omega_i)|^2 < +\infty \right\},\]
\[\|A^s u\| = \left( \sum_{i=1}^{\infty} \lambda_i^{2s} |(u, \omega_i)|^2 \right)^{1/2},\]
\(D(A^s)\) denotes the domain of \(A^s\) with the inner product and the norm \(|\cdot|_s\) as
\[(u, v)_s = (A^{\frac{s}{2}} u, A^{\frac{s}{2}} v), \quad \|u\|_s^2 = (u, u)_s.\]

Especially, \(V = V^1, \ V^2 = W := (H^2(\Omega))^3 \cap (H^1_0(\Omega))^3\). We denote \(V^{\frac{1}{s+1}} = D(A^{\frac{1}{s+1}})\) with the norm \(|\cdot|_{s+1} = |\cdot|_{1+s}\). Moreover, we have the continuous embeddings \(D(A^{\frac{1}{s+1}}) \hookrightarrow D(A^{\frac{1}{s+1}})\) for any \(s > r\) and
\[V^{\frac{1}{s}} \equiv D(A^{\frac{1}{s}}) \hookrightarrow (L^{\frac{6}{3-2s}}(\Omega))^3\]
for all \(s \in [0, \frac{3}{2})\).
The operator $A : V \to V'$ by $Au = -\nu_0 \|u\|^2 \Delta u$ and

$$\langle Au, v \rangle = \nu_0 \|u\|^2 \langle -\Delta u, v \rangle = \nu_0 \|u\|^2 (\langle u, v \rangle), \ \forall \ u, v \in V. \quad (2.6)$$

Noting that $A$ is a monotone operator from $V$ into $V'$, we have

$$\|Au\|_{L(V, V')} = \sup_{\|v\|=1, v \in V} |\langle Au, v \rangle| = \sup_{\|v\|=1, v \in V} \nu_0 \|u\|^2 |a(u, v)| \leq \nu_0 \|u\|^3.$$

The bilinear and trilinear operators are defined as (see [42])

$$B(u, v) := P((u \cdot \nabla)v), \ \forall \ u, v \in E, \quad (2.7)$$

$$b(u, v, w) = (B(u, v), w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x} w_j dx. \quad (2.8)$$

### 3. Main results and comments

#### 3.1. Main results I: Wellposedness and regularity of global solutions

**Definition 3.1.** We call the function $u(t, x) \in L^\infty(\tau, T; H) \cap L^4(\tau, T; V)$ is the weak solution of (1.7) if

$$\begin{cases}
\frac{d}{dt} (u(t), v) + (\nu + \nu_0 \|\nabla u(t)\|^2) (\langle u(t), v \rangle) + b(u, u, v) = \langle f(t, x), v \rangle, \\
u(t, x) = u_0, \ \tau \in \mathbb{R}.
\end{cases} \quad (3.1)$$

for $v \in V$ in the sense of distribution.

Based on the above notations, the non-autonomous system (1.7) can be rewritten equivalently as the following abstract functional equation

$$\begin{cases}
\frac{du}{dt} + \nu Au + Au + B(u, u) = f(t, x), \\
\nabla \cdot u = 0, \\
u(t, x) = u_0, \ \tau \in \mathbb{R}.
\end{cases} \quad (3.2)$$

The existence, uniqueness and regularity of global solution for (3.2) can be derived by the Galerkin approximation method and some energy estimates.

**Theorem 3.2 (Existence of global solution for non-autonomous equation).**

(a) Assume the external force $f(t, x) \in L^{4/3}_{\text{loc}}(\mathbb{R}, V')$ and $u_0 \in H$, then equation (3.2) has a unique weak solution

$$u(x, t) \in L^\infty(\tau, T; H) \cap L^4(\tau, T; V) \quad (3.3)$$

in dimension $n \leq 3$.

(b) Moreover, since the solution is continuous dependent on the initial data and $\frac{du}{dt} \in L^2(\tau, T; V')$, from the Aubin-Lions lemma, we derive that $u(t, x) \in C(\tau, T; H)$ which generates a continuous process $S(t, \tau) : H \to H$.

**Proof.** See e.g., [3], [24], [31], [32], here we omit the details. \(\square\)
Since the problem (1.7) is parabolic-hyperbolic coupled, we can not use bootstrap procedure similar as parabolic equation to obtain more regularity by assuming $u_0 \in H$ only.

**Theorem 3.3 (Regular solutions for non-autonomous equation).**

(a) If $u_0 \in D(A^2_f)$ with $0 \leq \sigma \leq 1$, $f(t, x) \in L^2_{\text{loc}}(\mathbb{R}; H)$, then $u \in L^\infty(\tau, T; D(A^2_f)) \cap L^2(\tau, T; D(A^\infty_{\text{loc}}))$.

(b) Moreover, we can derive $u \in C(\tau, T; D(A^2_f))$ similarly as Theorem 3.2 which generates a continuous process $S(t, \tau) : D(A^2_f) \to D(A^2_f)$.

**Proof.** See e.g., Chen, Yang and Si [8].

### 3.2. Main results II: Forward dynamical systems

We denote $E_1 = L^2_{\text{loc}}(\mathbb{R}; (L^2(\Omega))^2)$, and define $\hat{E}_1 = L^2(\mathbb{R}; (L^2(\Omega))^2)$, $\tilde{E}_1 = L^2(\mathbb{R}; (L^2(\Omega))^2)$ as translation bounded, translation compact or normal functional spaces respectively. Choosing an arbitrary function $\sigma_0 \in \tilde{E}_1$, $\hat{E}_1$ or $\tilde{E}_1$ and fixed, then we can define the symbol space $\mathcal{H}(\sigma_0)$ which is called hull of $\sigma_0$ by

$$\Sigma = \mathcal{H}(\sigma_0) = [\sigma_0(t + h) = T(h)\sigma_0(t)|h \in \mathbb{R}^+]_{E_1},$$

where $[\cdot]_{E_1}$ denotes the closure in strong topology of $E_1$, $T(\cdot)$ denotes the translation semigroup.

**Theorem 3.4.** Let $f \in \Sigma$ be a symbol for $u_0 \in H$, then the global weak solution for problem (1.7) generates a family of processes $\{U_f(t, \tau)\}$ ($f \in \Sigma$) and skew product flow $S(t) = (U_f(t, \tau), T(t)) \in H \times \Sigma$. If the symbol space is chosen to be a translation compact, bounded or normal class functional space, the skew product flow possesses a compact global attractor $A$ in $H \times \Sigma$ as

$$A = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(\tau) \times \{\sigma\},$$

here $\mathcal{K}_\sigma(\tau)$ denotes all bounded complete trajectories under the process and translation semigroup.

Moreover, let $\Pi_1$ and $\Pi_2$ be two projections from $H \times \Sigma$ to $H$ and $\Sigma$ respectively, then

$$\Pi_1 A = A_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(\tau)$$

is the uniform attractor of process in $H$, $\Pi_1 A = \mathcal{U} = \Sigma$ is the global attractor of translation semigroup $T(\cdot)$.

**Proof.** See e.g., Chen, Yang and Si [8].

**Corollary 3.5.** (Global attractor for autonomous system) In the above Theorem, if the external force $f$ is time independent, then the well-posedness and regularity of global solutions also hold and equation (3.2) generates a continuous semigroup $S(t)$. The uniform attractor in Theorem 3.4 reduces to a global attractor $A \in H$ for $f = f(x) \in V'$, $u_0 \in H$.

Moreover, if $f(x) \in H$, $u_0 \in D(A^2_f)$, (1.7) has a regular global attractor $\hat{A} \in D(A^2_f)$. 
3.3. Main results III: pullback dynamics

For dissipative systems, the study of pullback attractors was originated in the 1990s \cite{6, 15, 16}. The forward invariance is then replaced by pullback invariance. In this manuscript, we shall investigate pullback dynamics and related topics of problem (1.7).

Assume that for all \( t \geq \tau \in \mathbb{R} \), the external force \( f(t, x) \) satisfies
\[
\int_{-\infty}^{t} e^{\nu \lambda s} \| f(s) \|^2 V' ds < +\infty, \quad f(t, x) \in L^2_{loc}(\mathbb{R}; V'),
\]
(3.7)
or
\[
\int_{-\infty}^{\tau} e^{\nu \lambda s} | f(s) |^2 ds < +\infty, \quad f(t, x) \in L^2_{loc}(\mathbb{R}; H).
\]
(3.8)

Firstly, we shall present the minimal family of pullback attractors for the problem (1.7) in \( H \) and \( D(A^\sigma_2) \).

**Theorem 3.6.** (1) Assume that \( u_0(x) \in H \), the external force \( f(t, x) \in L^2_{loc}(\mathbb{R}; V') \) and (3.7) holds, then the continuous process \( \{ S(t, \tau) \} \) generated by the unique global solution \( u(x, t) \) for the problem (1.7) possesses a minimal family of pullback attractors \( A_\mu(t) \) \( (\mu \in (0, \mu_0]) \) with \( \mu_0 = \nu \lambda_1 \) and \( A_{\mu_0}(t) \) in \( H \) with
\[
A^H_\mu = \bigcap_{T \leq t \leq T} \bigcup_{s \leq t \leq T} S(t, s)D_\mu(s)^H = \omega(D_\mu(\cdot), t),
\]
(3.9)
\[
A^H_{\mu_0} = \bigcap_{T \leq t \leq T} \bigcup_{s \leq t \leq T} S(t, s)D_{\mu_0}(s)^H = \omega(D_{\mu_0}(\cdot), t).
\]
(3.10)

From the pullback invariance property of pullback attractors, we can see that \( \omega \)-limit set \( \omega(D_{\mu_0}(\cdot), t) \) is invariant, which implies the structure of minimal family for pullback attractors as
\[
W^u(E(s))(t) \bigcup \mathcal{E} \subseteq A^H_{\mu_0}(t),
\]
(3.11)
here \( W^u(E(s))(t) \) is the unstable manifold defined by
\[
W^u(E(s))(t) = \{ x \in H | \text{there exists a global weak solution} \}
\]
\[
\phi : \mathbb{R} \rightarrow H \text{ in the sense of Theorem 3.2 such that } \phi(t) = x \text{ and } \lim_{s \rightarrow -\infty} \text{dist}_H(\phi(s), E(s)) = 0 \},
\]
(3.12)
\( \mathcal{E} \) be the set of all equilibriums.

(2) Assume that \( u_0(x) \in D(A^\sigma_2) \), the external force \( f(t, x) \in L^2_{loc}(\mathbb{R}; H) \) and (3.8) holds, then the continuous process \( \{ S(t, \tau) \} \) generated by the unique global solution \( u(x, t) \) for equation (1.7) possesses a minimal family of pullback

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Attractors $A^D_\mu(A^\mathbb{F}_\mathbb{F})(t)$ and $A^D_{\mu_0}(A^\mathbb{F}_\mathbb{F})(t)$ in $D(A^\mathbb{F}_\mathbb{F})$ defined by

$$A^D_\mu(A^\mathbb{F}_\mathbb{F}) = \bigcap_{T \leq t_s \leq T} \bigcup_{s \leq t} S(t, s)\hat{D}_\mu(s)^D(A^\mathbb{F}_\mathbb{F}) = \omega(\hat{D}_\mu(\cdot), t), \quad (3.13)$$

$$A^D_{\mu_0}(A^\mathbb{F}_\mathbb{F}) = \bigcap_{T \leq t_s \leq T} \bigcup_{s \leq t} S(t, s)\hat{D}_{\mu_0}(s)^D(A^\mathbb{F}_\mathbb{F}) = \omega(\hat{D}_{\mu_0}(\cdot), t). \quad (3.14)$$

Similarly, $\omega$-limit set $\omega(\hat{D}_{\mu_0}(\cdot), t)$ is invariant, which implies the structure of minimal family of pullback attractors as

$$\hat{W}^u(E(s))(t) \bigcup \hat{E} \subseteq A^D_{\mu_0}(A^\mathbb{F}_\mathbb{F})(t), \quad (3.15)$$

here $\hat{W}^u(E(s))(t)$ is the unstable manifold defined as

$$\hat{W}^u(E(s))(t) = \{ x \in D(A^\mathbb{F}_\mathbb{F}) \mid \text{there exists a global weak solution } \phi : \mathbb{R} \to D(A^\mathbb{F}_\mathbb{F}) \text{ in the sense of Theorem 3.3 such that } \phi(t) = x \text{ and } \lim_{s \to -\infty} \text{dist}_{D(A^\mathbb{F}_\mathbb{F})}(\phi(s), E(s)) = 0 \}, \quad (3.16)$$

$\hat{E}$ be the set of all equilibria.

**Proof.** See Section 4.4. □

Let $X$ be a separable real Hilbert space, with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Let $K \subset X$ be a non-empty compact subset and $\varepsilon > 0$, we denote $N_\varepsilon(K)$ to be the minimum number of open balls in $X$ with radius $\varepsilon$ which are necessary to cover $K$. The fractal dimension of $K$ is defined as

$$d_F(K) = \limsup_{\varepsilon \to 0^+} \frac{\log(N_\varepsilon(K))}{\log(\varepsilon)}. \quad (3.17)$$

We estimate the bound of fractal dimension for pullback attractors in $H$ via verifying the uniform differentiability of the process and using trace formula [1].

**Theorem 3.7.** (1) Assume $f(t) \in L^2_{\text{loc}}(\mathbb{R}; V')$ and satisfies [377], $u_0 \in H$, then the pullback $D$-family of attractors $A^H_\mu = \{ A^H_\mu(t) \}$ in $H$ for the system [17] has bounded fractal dimension and Hausdorff dimension.

(2) If $\frac{\pi \nu_0^2}{2 |\eta|^4} + \frac{\pi \nu_0^4}{|\eta|^2} > \frac{C}{\nu_0} M + \frac{2 \nu}{27}$, here $M =: \lim_{T \to +\infty} \frac{1}{T} \int_{t-T}^t \| f(r) \|_{V'}^2 \, dr$, then $\dim_F A^H_\mu(t) \leq n$.

(3) If $\frac{\pi \nu_0^2}{2 |\eta|^4} + \frac{\pi \nu_0^4}{|\eta|^2} < \frac{C}{\nu_0} M + \frac{2 \nu}{27}$ and $f(t) \in L^\infty(-\infty, T^*; H)$, then the fractal and Hausdorff dimension of pullback attractors has bounded as $\dim_F(A^H_\mu(t)) \leq \hat{C} G + \frac{2 \nu}{27}$, here $G = \frac{\| f(t) \|_{L^\infty(-\infty, T^*; H)}}{\nu_0^2 \lambda_1}$ is the Grashof number for the non-autonomous system.

(4) For the autonomous case, the global attractor also has the finite fractal dimension with $\hat{C}' G'$, $G' = \frac{\| \mu \|^2}{\nu_0^2 \lambda_1}$ is the Grashof number for the autonomous system.
**Proof.** See Section 4.5.

**Theorem 3.8.** The pullback attractor $A^H_\mu$ is nontrivial if

$$G(t) \geq \sqrt{\frac{\nu_0}{c\nu + 4\nu^2\nu\lambda_1}},$$

(3.18)

$G^2(t) = \frac{\langle |f|^2 \rangle}{\nu_0\lambda_1}$, i.e., the pullback attractors becomes a single trajectory if (3.18) does not hold.

**Proof.** See Section 4.6.

The estimate established in Theorem 3.8 is different from that for 2D incompressible Navier-Stokes equation where $G(t) < c_1^2$, $c_1$ is independence on constant viscosity $\nu$. However, (3.18) is a sufficient condition but not necessary.

### 3.4. Comments on the fractal dimension of attractors

Due to the presence of $\|\nabla u\|^2$ in the viscosity coefficient in (1.7), weak solutions for (1.1) and (1.7) belong to different topological spaces: for (1.1), its weak solution $u(t)$ belongs to the space $L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ which satisfy energy inequality in 3D and uniqueness in 2D; while for (1.7), it has a unique global weak solution $u(t, x) \in C(\tau, T; H) \cap L^4(\tau, T; V)$.

Define the Grashof number $G = \frac{|f|}{\nu^2\lambda_1}$ and Reynolds number $Re = \frac{1}{\nu\lambda_1^{1/2}}$, the fractal dimension of global attractor $A$ for (1.1) of 2D case in $H$ can be estimated as

$$\dim_F A \leq CG, \quad C \leq \left(\frac{2}{\pi}\right)^{1/2}(\lambda_1|\Omega|)^{1/2},$$

(3.19)

$$\dim_F A \leq CG^{2/3}(1 + \log G)^{1/3}$$

(3.20)

for Dirichlet boundary and periodic boundary conditions respectively, see Temam [42], Foias, Manley, Temam and Treve [17]. Moreover, if $f \in H$, then $\dim_F A \leq \frac{1}{\pi} \frac{|f||\Omega|}{\nu^{1/2}}$ for (1.1) with Dirichlet boundary, see Constantin, Foias and Temam [14].

In Chepyzhov and Vishik [10], if the external force is translation compact, the kernel section of uniform attractor $A_\Sigma = \bigcup_{s \in \mathbb{R}} \mathcal{K}(s)$ of (1.1) with Dirichlet boundary condition in 2D has finite dimension:

$$\dim_F \mathcal{K}(s) \leq \frac{|\Omega|}{\pi \nu^2} (M(|f|^2))^{1/2}$$

(3.21)

where $M(|f|^2) = \limsup_{T \to \infty} \frac{1}{T} \int_\tau^{\tau+T} |f(s)|^2 ds$. If the external force is weaker than translation compact, then this upper bound of finite dimension for kernel section also holds (see [34]).
In Carvalho, Langa and Robinson [7], the fibre of pullback attractors $\mathcal{A}(\cdot)$ for (1.1) has finite dimension in $H$ for Dirichlet boundary and periodic boundary conditions respectively,

$$\dim_F \mathcal{A}(\cdot) \leq CG(t),$$  \hspace{1cm} (3.22)

$$\dim_F \mathcal{A}(\cdot) \leq CG(t)^{2/3}(1 + \log G(t))^{1/3},$$  \hspace{1cm} (3.23)

where $G(t) = \|f\|_{L^2[\infty, t; V')}^{2/\lambda_1}$ is the non-autonomous Grashof number. But their union $\bigcup_{t \in \mathbb{R}} \mathcal{A}(t)$ can be infinite dimension. However, if $\Omega$ is unbounded and the Poincare inequality holds, the pullback attractors $\mathcal{A}(t)$ also have finite fractal dimension

$$\dim_F \mathcal{A}(t) \leq \max\{1, \frac{2\|f\|^2_{L^2[\infty, t; V')}}{\nu^4 \lambda_1}\}$$  \hspace{1cm} (3.24)

for all $t \in \mathbb{R}$ in $H$, see Langa, Lukaszewicz and Real [26].

For (1.1) in 3D, Chepyzhov and Ilyin [9] gave the estimate of invariant sets $X_A$ in $V$ as $\dim_H X_A \leq CG^3$ and $\dim_F X_A \leq 2CG^3$. However, since the uniqueness of global weak solution for 3D equation is still an open problem, estimates on the fractal dimension of trajectory attractor is unknown.

As we show in this article, for equation (1.7), the finite fractal dimension of the family of pullback attractors has an upper bound of $\hat{C}G^3 + \frac{2\nu}{27}$ (again $G$ here is the generalized Grashof number $G = \|f(t)\|_{L^\infty[\infty, t; H]}^{2/\nu^2 \lambda_1}$). This is again consistent with the fact that the viscosity coefficient in (1.7) is not a constant.

Comparing the well-posedness of solutions, fractal dimension of forward and pullback attractor between (1.1) for 2D or 3D and (1.7), we can conclude that the variable viscosity in (1.7) plays an important role in long time dynamics, which implies (1.7) has better property than (1.1) in 3D, but possesses less regularity and worse optimal upper bounded fractal dimension of attractors than (1.1) in 2D. To see more results of fractal dimension of attractors, we refer to [9], [11], [12], [13], [14], [22], [24], [26], [34], [42].

### 3.5. Main result IV: Relation of families of pullback attractors and comments

Let $\mathcal{P}(H)$ denote the family of all non-empty subsets of $H$, we want to show the relation among these pullback attractors from choosing different family of nonempty sets in $\mathcal{P}(H)$. Firstly, we give the definition of universes in $H$ and $D(A^T_\infty)$ which could be unbounded.

**Definition 3.9 (The universe in $H$).** Denote $\mathcal{P}(H)$ as the collection of all nonempty subsets in $H$, $\hat{D} = \{D(t)\} \subset \mathcal{P}(H)$, $B(0, \rho_\hat{D}(t))$ denotes a family of balls at center 0 with radius $\rho_\hat{D}(t)$ satisfying $e^{-\mu t} |\rho_\hat{D}(\tau)|^2 e^{\mu \tau} \to 0$ as $\tau \to -\infty$, i.e., there exists a pullback time $\tau_D \leq t$, such that for any $\tau < \tau_D$ and any fixed $t$,

$$|\rho_\hat{D}(\tau)|^2 e^{\mu \tau} \to 0.$$  \hspace{1cm} (3.25)
We define the universe $\mathcal{D}^H_\mu = \{ \hat{D} = \{ D(t) \} \}$ as

$$\mathcal{D}^H_\mu = \{ \hat{D} | D(t) \subset B(0, \rho_{\hat{D}}(t)) \text{ with } \rho_{\hat{D}} \text{ satisfying (3.25)} \} \tag{3.26}$$

and $\mathcal{D}^H_{\mu_0}$ as

$$\mathcal{D}^H_{\mu_0} = \{ D_{\mu_0}(t) \subset \mathcal{P}(H) | D_{\mu_0}(t) \subset B(0, \rho_{D_{\mu_0}}(t)), \lim_{\tau \to -\infty} (|\rho_{D_{\mu_0}}(\tau)|^2 e^{\mu \tau}) = 0 \}. \tag{3.27}$$

**Definition 3.10 (The universe in $D(A^\tau)$).** Denote $\mathcal{P}(D(A^\tau))$ as the collection of all nonempty subsets in $D(A^\tau)$, $\hat{D} = \{ D'(t) \} \subset \mathcal{P}(D(A^\tau))$, $B(0, \rho_{D'}(t))$ denotes a family of balls at center 0 with radius $\rho_{D'}(t)$ satisfying

$$e^{-\mu t} |\rho_{D'}(\tau)|^2 e^{\mu \tau} \to 0$$

as $\tau \to -\infty$, i.e., there exists a pullback time $\tau'_{D'} \leq t$, such that for any $\tau < \tau'_{D'}$, and any fixed $t$,

$$|\rho_{D'}(\tau)|^2 e^{\mu \tau} \to 0. \tag{3.28}$$

We define the universe $\mathcal{D}^{D(A^\tau)}_\mu = \{ \hat{D}' = \{ D'(t) \} \}$ as

$$\mathcal{D}^{D(A^\tau)}_\mu = \{ \hat{D}' | D'(t) \subset B(0, \rho_{D'}(t)) \text{ with } \rho_{D'} \text{ satisfying (3.28)} \} \tag{3.29}$$

and $\mathcal{D}^{D(A^\tau)}_{\mu_0}$ as

$$\mathcal{D}^{D(A^\tau)}_{\mu_0} = \{ \hat{D}_{\mu_0}(t) \subset \mathcal{P}(H) | D_{\mu_0}(t) \subset B(0, \rho_{D_{\mu_0}}(t)), \lim_{\tau \to -\infty} (|\rho_{D_{\mu_0}}(\tau)|^2 e^{\mu \tau}) = 0 \}. \tag{3.30}$$

Secondly, let us recall the theory of pullback attractors which is initiated from the research on random dynamic systems in Crauel, Debussche and Flandoli [15, 16].

**Theorem 3.11.** (See [15, 16]) Let $\phi$ be a cocycle on a non-empty complete metric space $Q$ induced by shift operator $\theta$, assume that (a) the pair $(\theta, \phi)$ is non-autonomous dynamic system defined on $Q \times X$, (b) $\phi$ is continuous, (c) $\mathcal{D} = \{ D_q \}_{q \in Q}$ is a family of compact pullback absorbing sets in $X$. Then there exists a pullback attractor $A_{CDF} = \{ A_q \}_{q \in Q}$ such that $A_q \subset D_q$ for all $q \in Q$.

This theorem assume that there exists a family of compact pullback absorbing sets the proof of which needs more regular estimates and compact embedding. Based on the property of bounded pullback asymptotic compactness, Wang, Zhong and Zhou in [45] derive the existence of bounded pullback attractor:

**Theorem 3.12.** (See [45]) Assume $(\theta, \phi)$ is non-autonomous dynamic system defined on $Q \times X$, and satisfies
Pullback dynamics of 3D Navier-Stokes equation with nonlinear viscosity

(a) (uniformly dissipative with respect to \( q \in Q \)) there exists bounded set \( D \subset X \) satisfies that for \( \forall B \in \mathcal{B}(X) \), \( \exists t_0(B) > 0 \) such that

\[
\bigcup_{t \geq t_0(B)} \bigcup_{q \in Q} \phi(t, q, B) \subset D,
\]  
(3.31)

here \( \mathcal{B}(X) \) is bounded sets family in \( X \).

(b) \( \phi \) is bounded pullback asymptotically compact.

Then \((\theta, \phi)\) possesses a bounded pullback attractor \( \mathcal{A}_b = \{A_q\}_{q \in Q} \).

Sun, Cao and Duan [40] subsequently improve the previous result by replacing (a-1) in Theorem 3.12 by a weaker condition:

(a-1) The set class \( D = \{D_q\}_{q \in Q} \) is a pullback absorbing set in \( X \) and satisfies

\[
D_{\theta-(q)} \subset D_q, \quad \forall q \in Q, \ t \geq 0.
\]  
(3.32)

In [40] and [45], the pullback absorbing sets should have class including property and \( \bigcup_{t \geq 0} A_{\theta-(q)} \) is bounded in \( X \). However, in some cases, the pullback absorbing set is uniformly dissipative or does not satisfy the class including property, such as the pullback absorbing set of 2D Navier-Stokes equation. Therefore, Caraballo, Łukaszewicz and Real [6] considered the \( D \)-pullback attractors in [7] where

\[
D_t = \left\{ u \in H : \|u\|_H^2 \leq C \int_{-\infty}^{t} e^{-\sigma(t-s)} \|f\|_{V^{'}}^2 ds \right\}, \quad \mathcal{D} = \{D_t\}_{t \in \mathbb{R}}.
\]  
(3.33)

Compared to these results available in the literature, the solution for system (1.7) considered in this paper also possesses the \( D \)-pullback absorbing set similar as in (3.33). However, the \( D \)-pullback absorbing sets for system (1.7) do not have an upper bound independent on time \( t \), which implies that the uniformly pullback dissipative property does not hold, a key difference from what is presented in [45]. This shortage, however, does not prevent us from getting an inclusion relation among the families of pullback attractors for (1.7) with respect to the universes we defined above.

Assume \( f(t, x) \in L^2_{loc}(\tau, \infty; V^{'}) \) satisfying (3.7) or (3.8), \( u_0 \in H \) or \( D(A_{\tau}^\sigma) \), \( \mu \in (0, \mu_0] \) with \( \mu_0 = \nu\lambda_1 \), the different universes can be stated as

**Definition 3.13.** (1) (Universe in \( H \) and \( D(A_{\tau}^\sigma) \)) We denote universes \( \mathcal{D}^H_\mu \) in (3.25) and \( \mathcal{D}^H_{\mu_0} \) as the class of nonempty subset, \( \mathcal{D}^{D(A_{\tau}^\sigma)}_\mu \) in (3.28) and \( \mathcal{D}^{D(A_{\tau}^\sigma)}_{\mu_0} \) as the class of nonempty subsets.

(2) (Fixed universe) We denote \( \mathcal{D}^F_\mu \) as the universe of fixed nonempty bounded subsets of \( H \) satisfying property specified in (3.25) and \( \mathcal{D}^{D(A_{\tau}^\sigma)}_F \) as the universe of fixed nonempty bounded subsets of \( D(A_{\tau}^\sigma) \) satisfying property specified in (3.28).

**Remark 3.1.** (a) From observing, we see that \( \mathcal{D}^H_F \subset \mathcal{D}^H_\mu \subset \mathcal{D}^H_{\mu_0} \) which are inclusion closed. From Theorem 3.6 the minimal families of pullback attractors
\(A^H_F(t), A^H_{\mu}(t)\) and \(A^H_{\mu_0}(t)\) corresponding to the above universe exists respectively. The existence of compact pullback attractors in the sense of \([16]\) as \(A_{cdf}(t)\) can also be derived.

(2) \(D^{D(A_{\mu}^F)}_F \subset D^{D(A_{\mu}^F)}_\mu \subset D^{D(A_{\mu_0}^F)}_\mu\) which are also inclusion closed.

From Theorem 3.14, the minimal families of pullback attractors \(A^D_F(A_{\mu}^F)(t), A^D_\mu(A_{\mu}^F)(t)\) and \(A^D_{\mu_0}(A_{\mu_0}^F)(t)\) corresponding to the above universe also exists.

In the sense of \([10]\), if \(f(t, x)\) is uniformly bounded \(||f(t)|| \leq \phi\) for all \(t \in \mathbb{R}\), then there exists a bounded pullback attractor (using similar technique as in Chapter 11 of \([7]\))

\[
A_B = \bigcap_{T \leq t} \bigcup_{s \leq T} S(t, s)B(s)^H. \tag{3.34}
\]

Moreover, the condition of \(f(t, x)\) can be relaxed to \(f \in L^2_{loc}(\mathbb{R}; H)\) and \(\int_{-\infty}^{\infty} e^{-\mu_0 s} |f(s)|^2 ds < +\infty\).

**Theorem 3.14.** We have the relation among these families of pullback attractors:

(a) \(A_{cdf} \subset A^H_F(t) \subset A^H_\mu(t) \subset A^H_{\mu_0}(t)\).

(b) \(A^D_F(A_{\mu}^F)(t) \subset A^D_\mu(A_{\mu}^F)(t) \subset A^D_{\mu_0}(A_{\mu_0}^F)(t)\).

(c) If \(\bigcup_t D_{\mu_0}(t)\) is bounded, then \(A_{cdf} = A^H_F(t) = A^H_\mu(t) = A^H_{\mu_0}(t)\).

(d) If \(\bigcup_t D^{D(A_{\mu_0}^F)}_\mu\) is bounded, then \(A^D_F(A_{\mu}^F)(t) = A^D_\mu(A_{\mu}^F)(t) = A^D_{\mu_0}(A_{\mu_0}^F)(t)\).

(e) For the bounded pullback attractors, if \(\bigcup_t D_{\mu_0}(t)\) and the union of bounded pullback absorbing set \(\bigcup_t B(t)\) are bounded, then \(A_B(t) = A^H_{\mu_0}(t)\). If the universe \(D^H_{\mu_0}\) contains all bounded sets in \(H\), then \(A_B(t) = A^H_{\mu_0}(t)\).

**Proof.** See, Section 4.7.

\[\square\]

### 3.6. Main result V: Continuity of pullback attractors

Considering the perturbed system (1.7) with \(f(t, x) = \varepsilon h(t, x)\). From Sections 3.2 and 3.3 we see that the continuous process \(\{U_\varepsilon(t, \tau)\}\) generated by equation (1.7) with perturbed external force \(f(t, x) = \varepsilon h(t, x)\) has a family of pullback attractors \(A_\varepsilon(t)\) for \(\varepsilon > 0\) in \(H\), and the semigroup of autonomous problem (1.7) possesses a global attractors \(A_0\) for the case \(\varepsilon = 0\) in \(H\). The upper semi-continuity of \(A_\varepsilon(t)\) to \(A_0\) as \(\varepsilon \to 0\) in \(H\) is stated in the following theorem.

**Theorem 3.15.** Assume \(u_0 \in H\), the external force \(h(t, x) \in L^2(\tau, T; H)\) satisfies

\[
\int_{-\infty}^{\tau} e^\eta s |h(s)|^2 ds < +\infty. \tag{3.35}
\]
Then the pullback attractors \( \mathcal{A}_\varepsilon = \{ \mathcal{A}_\varepsilon(t) \}_{t \in \mathbb{R}} \) and the global attractor \( \mathcal{A} \) for (1.7) with \( \varepsilon > 0 \) and \( \varepsilon = 0 \) respectively satisfy the property of upper semi-continuity in \( H \), i.e.,

\[
\lim_{\varepsilon \to 0^+} \text{dist}_H(\mathcal{A}_\varepsilon(t), \mathcal{A}) = 0 \quad \text{for any } t \in \mathbb{R}.
\]

\( \square \)

**Remark 3.2.** The upper semi-continuity of pullback attractors to trajectory attractor of (1.7) as \( \nu_0 \) goes to 0 is still open.

### 4. Proof of main results

In this section, we shall present the proof of main results based on the preliminary theory of pullback attractors and other related topics.

#### 4.1. The preliminary theory of pullback attractor

**Definition 4.1.** We call the family of subsets \( \mathcal{B} = \{ B(t) \}_{t \in \mathbb{R}} \) a pullback \( \mathcal{B} \)-absorbing set for the process \( U(\cdot, \cdot) \), if for every \( t \in \mathbb{R} \), any bounded subsets \( B \subset X \), there exists a time \( T(t, B) > 0 \), such that \( U(t, t - \tau)B \subset B(t) \) for all \( \tau \geq T(t, B) \).

**Definition 4.2.** Let \( \mathcal{B} = \{ B(t) \}_{t \in \mathbb{R}} \) be a family of subsets in \( X \). We say the evolutionary process \( U(\cdot, \cdot) \) is pullback \( \mathcal{B} \)-asymptotically compact in \( X \), if for any sequences \( \tau_n \to \infty \) and \( x_n \in B(t - \tau_n) \), the sequence \( \{ U(t, t - \tau_n)x_n \} \) is precompact in \( X \) for all \( t \in \mathbb{R} \).

**Theorem 4.3.** Let the family of sets \( \mathcal{B} = \{ B(t) \}_{t \in \mathbb{R}} \) be pullback absorbing set for the process \( U(\cdot, \cdot) \) which is pullback \( \mathcal{B} \)-asymptotically compact in \( X \). Then, the family \( \mathcal{A} = \{ A(t) \}_{t \in \mathbb{R}} \) is defined by \( A(t) = \Lambda(\mathcal{B}, t) \) is a pullback attractor for \( U(\cdot, \cdot) \) in \( X \) for the process \( \{ U(\cdot, \cdot) \} \), where \( \Lambda(\mathcal{B}, t) = \bigcap_{s \geq 0} \bigcup_{\tau \geq s} U(t, t - \tau)B(t - \tau) \) for each \( t \in \mathbb{R} \).

Let us denote by \( \mathcal{P}(X) \) the family of all nonempty subsets of \( X \), and consider a family of nonempty sets \( \hat{D}_0 = \{ D_0(t) | t \in \mathbb{R} \} \subset \mathcal{P}(X) \), which might not be bounded or compact.

**Definition 4.4 (Universe).** Let \( \mathcal{D} \) be a nonempty class of families with parameters in time \( \hat{D} = \{ D(t) | t \in \mathbb{R} \} \subset \mathcal{P}(X) \), the class \( \mathcal{D} \) is called a universe in \( \mathcal{P}(X) \).

**Definition 4.5 (Inclusion closed).** We say the family \( \mathcal{D} \) is inclusion closed, if for any \( \hat{D} \in \mathcal{D} \), and \( \hat{C} = \{ C(t) \}_{t \in \mathbb{R}} \subset 2^X \), such that \( C(t) \subset X \), \( C(t) \neq \Phi \), \( C(t) \subset D(t) \) for all \( t \in \mathbb{R} \), then one has \( \hat{C} \in \mathcal{D} \).

**Definition 4.6 (Pullback D-condition (MWZ)-see Wang and zhong [14]).** Let \( X \) be a Banach space or Hilbert space, we call the family of processes \( U(t, \tau) : X \to X \) satisfies pullback \( \mathcal{D} \)-condition (MWZ) if for any \( \mathcal{B} = \{ B(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \)
for any \( t \in \mathbb{R} \) and fixed, any \( \varepsilon > 0 \), there exists a pullback time \( \tau_{\varepsilon} = \tau(t, \varepsilon, B) \leq t \) and a finite dimensional subspace \( X_1 \subset X \) such that

(i) \( P(\bigcup_{s \leq \tau_{\varepsilon}} U(t, s)B(s)) \) is bounded,

(ii) \( \| (I - P)(\bigcup_{s \leq \tau_{\varepsilon}} U(t, s)B(s)) \|_X \leq \varepsilon, \)

where \( P \) is the bounded projection from \( X \) to \( X_1 \).

**Theorem 4.7.** (See García-Luengo, Marín-Rubio and Real [18]) Let \( X \) be a complete metric or Banach space, \( \{U(\cdot, \cdot)\} : \mathbb{R}^2 \times X \to X \) be a process, \( D = \{D(t)\}_{t \in \mathbb{R}} \subset \mathcal{P}(X) \) be the universe. Suppose the continuous process satisfies the following hypothesis: (1) \( \{U(\cdot, \cdot)\} \) yields pullback \( D \)-condition (MWZ); (2) \( \{U(\cdot, \cdot)\} \) admits a pullback \( D \)-absorbing family \( \hat{D}_0 = \{D_0(t)\}_{t \in \mathbb{R}} \) which are not necessary in the universe. Then, \( U(\cdot, \cdot) \) possesses a minimal pullback \( D \)-attractor \( A_D = \{A_D(\cdot)\} \).

The family \( A_D \) is minimal in the sense that if \( \hat{C} = \{C(\cdot)\}_{\cdot \in \mathbb{R}} \subset \mathcal{P}(X) \) is a family of closed sets such that for any \( \hat{D} = \{D(t)\}_{t \in \mathbb{R}} \subset D \),

\[
\lim_{\tau \to -\infty} \text{dist}_X(U(\cdot, \tau)D(\tau), C(\cdot)) = 0,
\]

then \( A_D(\cdot) \subset C(\cdot) \). Moreover, if \( \hat{D}_0 \subset D \) and \( D \) is inclusion closed, then \( A \) is unique and \( A \in D \).

### 4.2. Upper semi-continuity theory of pullback attractors

Consider the non-autonomous system with perturbed external force

\[
\frac{\partial u}{\partial t} = \hat{A}_\sigma u + \varepsilon \sigma(t, x)
\]  

(4.1)

our goal of this section is to show the relationship between pullback attractors \( A_\varepsilon = \{A_\varepsilon(t)\}_{t \in \mathbb{R}} \) and global attractor \( A \) for (4.1) with the cases \( \varepsilon > 0 \) and \( \varepsilon = 0 \) respectively. The upper semi-continuity of attractors was investigated firstly by Hale and Raugel [19] in 1988, then many mathematicians extended the theory to pullback attractor and random attractors for processes (cocycle), see Caraballo, Langa and Robinson [5], Carvalho, Langa and Robinson [7], Kloeden and Stonier [23], Wang and Qin [43] and references therein.

In what follows, we will show the upper semi-continuity of pullback attractors with respect to the parameter \( \varepsilon \in (0, \varepsilon_0) \) for the evolutionary process \( U_\varepsilon(\cdot, \cdot) \) of (4.1).

For each \( \tau \leq t \in \mathbb{R} \) and \( x \in X \), we assume

\[
(H_1) \quad \lim_{\varepsilon \to 0} \text{dist}_X(U_\varepsilon(t, t - \tau)x, S(t - \tau)x) = 0
\]  

(4.2)

holds uniformly on bounded sets of \( X \).

**Definition 4.8.** (See [7]) Let \( X \) be a Banach space, \( \Lambda \) be a metric space and \( A_\lambda(\lambda \in \Lambda) \) be a family of subsets of \( X \). We say that the family of pullback attractors \( A_\lambda \) is upper semi-continuous as \( \lambda \to \lambda_0 \) if

\[
\lim_{\lambda \to \lambda_0} \text{dist}_X(A_\lambda, A_{\lambda_0}) = 0.
\]  

(4.3)
Theorem 4.9. (See [5]) Assume that (H1) holds and there exist pullback attractors $A_{\varepsilon} = \{A_{\varepsilon}(t)\}_{t \in \mathbb{R}}$ for all $\varepsilon \in (0, \varepsilon_0]$. If there exists a compact set $K \subset X$, such that

$$\lim_{\varepsilon \to 0} \text{dist}_X(A_{\varepsilon}(t), K) = 0 \quad \text{for any } t \in \mathbb{R}. \quad (4.4)$$

Then $A_{\varepsilon}$ are upper semi-continuous to $A$, i.e.,

$$\lim_{\varepsilon \to 0} \text{dist}_X(A_{\varepsilon}(t), A) = 0 \quad \text{for any } t \in \mathbb{R}. \quad (4.5)$$

In the sequel, we shall present an approach to verify (H2) for the process, such that Theorem 4.9 can be applied to the upper semi-continuity between pullback attractors $A_{\varepsilon}(t)$ and global attractor $A$.

Theorem 4.10. (See [43]) Assume the family of sets $B = \{B(t)\}_{t \in \mathbb{R}}$ is pullback absorbing for the process $U(\cdot, \cdot)$, $K_{\varepsilon} = \{K_{\varepsilon}(t)\}_{t \in \mathbb{R}}$ is a family of compact sets in $X$ for each $\varepsilon \in (0, \varepsilon_0]$. Suppose the decomposition $U_{\varepsilon}(\cdot, \cdot) = U_{1,\varepsilon}(\cdot, \cdot) + U_{2,\varepsilon}(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \times X \to X$ satisfies

(i) for any $t \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0]$, 

$$\|U_{1,\varepsilon}(t, t - \tau) x_{t-\tau}\|_X \leq \Phi(t, \tau), \quad \forall \ x_{t-\tau} \in B(t - \tau), \quad \tau > 0, \quad (4.6)$$

where $\Phi(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ satisfies

$$\lim_{\tau \to +\infty} \Phi(t, \tau) = 0 \quad \text{for each } t \in \mathbb{R}$$

(ii) for any $t \in \mathbb{R}$ and $T \geq 0$,

$$\bigcup_{0 \leq \tau \leq T} U_{2,\varepsilon}(t, t - \tau) B(t - \tau) \text{ is bounded,}$$

and for any $t \in \mathbb{R}$, there exists a time $T_B(t) > 0$, which is independent of $\varepsilon$, such that

$$U_{2,\varepsilon}(t, t - \tau) B(t - \tau) \subset K_{\varepsilon}(t), \quad \forall \ \tau \geq T_B(t), \ \varepsilon \in (0, \varepsilon_0] \quad (4.7)$$

and there exists a compact set $K \subset X$, such that

$$\lim_{\varepsilon \to 0} \text{dist}_X(K_{\varepsilon}(t), K) = 0, \quad \text{for any } t \in \mathbb{R}. \quad (4.8)$$

Then (a) for each $\varepsilon \in (0, \varepsilon_0]$, the system (4.1) possesses a family of pullback attractors $A_{\varepsilon} = \{A_{\varepsilon}(t)\}_{t \in \mathbb{R}}$, (b) (H2) holds and hence $A_{\varepsilon}$ has the upper semi-continuity to $A$.

Remark 4.1. In order to obtain the upper semi-continuity of attractors of the system (4.1), the weak solution should have the same initial data, i.e., every trajectory should begin at the same point.

4.3. Proof of Corollary 3.5 Global attractor for autonomous case $f = f(x)$

In this subsection, we will prove the existence, regularity of global attractor for autonomous case of (1.7).

Lemma 4.11. (See [42]) The bilinear operator $B(u, v)$ and trilinear operator $b(u, v, w)$ in 3D has the properties

$$\begin{align*}
\|B(u, u)\|_{V'} &
\leq c_0 \|u\|_V, \quad \forall \ u \in V, \\
b(u, v, v) &= 0, \quad \forall \ u \in V, \ v \in (H^1_0(\Omega))^3, \\
b(u, v, w) &= -b(u, w, v), \quad \forall \ u, v, w \in V, \\
|b(u, v, w)| &\leq C |u|^\frac{1}{2} \|v\| \|w\|^\frac{1}{2}, \quad \forall \ u, v, w \in V.
\end{align*} \quad (4.9)$$
Lemma 4.12. (1) Assume \( f(x) \in V' \) and \( u_0 \in H \) in (1.7), then the semigroup \( \{ S(t) \} \) has a bounded absorbing ball \( B_0 = \{ u \in H : |u|_H \leq \rho \} \) in \( H \).

(2) If \( f(x) \in H \) and \( u_0 \in D(A^\frac{\sigma}{2}) \) in (1.7), then the semigroup \( \{ S(t) \} \) has a bounded absorbing ball \( B_0 = \{ u \in D(A^\frac{\sigma}{2}) : \| u \|_{D(A^\frac{\sigma}{2})} \leq \dot{\rho} \} \) in \( D(A^\frac{\sigma}{2}) \).

Proof. (1) Taking inner product of (1.7) with \( u \) and integrating by parts over \( \Omega \), we derive

\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \nu |u|^2 + \nu_0 |u|^4 = \langle f(x), u \rangle \leq \frac{8}{\nu} \| f(x) \|_{V'}^2 + \frac{\nu}{2} |u|^2 \\
\leq \frac{8}{\nu} \| f(x) \|_{V'}^2 + \frac{\nu}{2} |u|^2,
\]

since \( (B(u, u), u) = 0 \).

From the Poincaré inequality and Gronwall’s inequality, (4.10) yields

\[
|u(t, x)|^2 \leq e^{-\nu \lambda_1 (t-\tau)} |u_0|^2 + \frac{8}{\nu^2 \lambda_1} \| f(x) \|_{V'}^2.
\]

Choosing a time \( T_0 = \tau + \frac{1}{\nu \lambda_1} \ln \left( \frac{\nu^2 \lambda_1 |u_0|^2}{8 \| f(x) \|_{V'}^2} \right) + 1 \) such that \( e^{-\nu \lambda_1 (t-\tau)} |u_0|^2 \leq \frac{1}{\nu^2 \lambda_1} \| f(x) \|_{V'}^2 \), if \( t \geq T_0 \). Defining \( \rho^2 = \frac{16}{\nu \lambda_1} \| g(x) \|_{V'}^2 \), we conclude that \( B_0 = \{ u : |u| \leq \rho \} \) is the bounded absorbing ball in \( H \).

(2) Taking inner product of (1.7) with \( A^\sigma u \) and integrating by parts over \( \Omega \), we derive

\[
\frac{1}{2} \frac{d}{dt} |A^\frac{\sigma}{2} u|^2 + \nu |A^\frac{\sigma+1}{2} u|^2 + \nu_0 |u|^2 |A^\frac{\sigma+1}{2} u|^2 \leq b(u, u, A^\sigma u) + \langle f(x), A^\sigma u \rangle,
\]

From the Cauchy-Schwarz inequality, the generalized Poincaré inequality and the property of trilinear operator, we have

\[
\langle f(x), A^\sigma u \rangle \leq \| f(x) \| |A^\sigma u| \leq C \| f(x) \| |A^\frac{\sigma+1}{2} u| \\
\leq \frac{\nu}{4} |A^\frac{\sigma+1}{2} u|^2 + \frac{C}{\nu} \| f(x) \|^2.
\]

and

\[
|b(u, u, A^\sigma u)| \leq \int_\Omega |u||\nabla u||A^\sigma u|dx \\
\leq C \| u \|^{\frac{3}{2}} |A^\frac{\sigma+1}{2} u|^\frac{1}{2} \\
\leq \frac{C}{\nu} |u|^2 + \frac{\nu}{4} |A^\frac{\sigma+1}{2} u|^2.
\]

From the Poincaré inequality and Gronwall’s inequality, neglecting the third term in left side of (1.12), using (4.13) and (4.14), it yields

\[
|A^\frac{\sigma}{2} u(t, x)|^2 \leq e^{-\nu \lambda_1 (t-\tau)} |A^\frac{\sigma}{2} u_0|^2 + \frac{C}{\nu^2 \lambda_1} \| f(x) \|^2 + \frac{C}{\nu} \| u \|_{L^2(\tau, T; V)}^2.
\]

Choosing a time \( \hat{T}_0 = \tau + \frac{1}{\nu \lambda_1} \ln \left( \frac{\nu^2 \lambda_1 |u_0|^2}{\| f(x) \|^2 + \frac{\nu}{2} \| u \|_{L^4(\tau, T; V)}^4} \right) + 1 \) such that \( e^{-\nu \lambda_1 (t-\tau)} |A^\frac{\sigma}{2} u_0|^2 \leq \frac{C}{\nu^2 \lambda_1} \| f(x) \|^2 + \frac{C}{\nu} \| u \|_{L^4(\tau, T; V)}^4 \) if \( t \geq \hat{T}_0 \). Defining \( \hat{\rho}^2 = \frac{16}{\nu \lambda_1} \| g(x) \|_{V'}^2 \), we conclude that \( B_0 = \{ u : |u| \leq \hat{\rho} \} \) is the bounded absorbing ball in \( H \).
Lemma 4.13. Let $u_0 \in H$, then for the problem (4.16), there exists a constant $\varepsilon = \varepsilon(f, \delta) > 0$ such that the solution of (4.16) satisfies

$$|S_{\nu}(t)u_0|^2 = |v(x,t)|^2 \leq |u_0|^2 e^{-2\nu_0 \lambda_1 (t-\tau)} + \delta(t)$$

(4.18)

for any $t \geq \tau$, where $0 < \delta(t) = \frac{C \varepsilon (1 - e^{-\nu_1 \lambda_1 (t+\tau)})}{\nu^2 \lambda_1} < \frac{C \varepsilon}{\nu^2 \lambda_1}$.

Proof. Multiplying (4.16) with $v(t)$, noting that $b(v, v, v) = 0$ and $\nu \leq \nu_0$, integrating by parts and using the Young inequality, we derive

$$\frac{d}{dt} |v|^2 + 2\nu |v|^2 + 2\nu_0 |u|^2 |v|^2 \leq \frac{C}{\nu} \|f - f^\varepsilon\|^2_{L^2} + \nu |v|^2,$$

(4.19)

by applying the Gronwall inequality and Poincaré’s inequality, neglecting the third term in (4.19), we conclude the result. \qed

Lemma 4.14. For $f^\varepsilon \in H$ with arbitrary $\varepsilon > 0$ and any time $T > 0$, there exists a positive constant $M = M(T, f^\varepsilon, \|u\|_{L^2(\tau, T; V)}^2)$ such that the solution of the system (4.17) satisfies

$$\|w(t)\|_{D(A^{\frac{\sigma}{4}}}^2 \leq M$$

(4.20)

for all $0 < \sigma \leq 1$ and $M(T, f^\varepsilon, \|u\|_{L^2(\tau, T; V)}^2) = \frac{C\|u\|_{L^2(\tau, T; V)}^2}{\nu^2} e^{C T} + e^{C(T - \tau)} \|f^\varepsilon\|^2$.

Proof. Taking inner product of (4.17) with $A^\sigma w$, we obtain

$$\frac{1}{2} \frac{d}{dt} |A^\sigma w|^2 + \nu |A^{\frac{\sigma+1}{2}} w|^2 + \nu_0 |u|^2 |A^{\frac{\sigma+1}{2}} w|^2 \leq |b(w, w, A^\sigma w)| + |b(w, u, A^\sigma w)| + |b(u, w, A^\sigma w)| + \langle f^\varepsilon, A^\sigma w \rangle.$$  

(4.21)

Since $0 < \sigma \leq 1$, by the Cauchy inequality and property of trilinear operator, we have

$$\langle f^\varepsilon, A^\sigma w \rangle \leq \|f^\varepsilon\| |A^\sigma w| \leq \frac{C}{\nu} |f^\varepsilon|^2 + \frac{\nu}{4} |A^{\frac{\sigma+1}{2}} w|^2$$

(4.22)
and

\[ |b(w, w, A^\sigma w)| \leq \int_{\Omega} |w||\nabla w||A^\sigma w|dx \leq C|w||\nabla w|^\frac{1}{2}|A^\sigma w|^\frac{1}{2} \]
\[ \leq C|A^\frac{\sigma}{2}w||A^{\frac{1+\sigma}{2}}w|^\frac{1}{2} \]
\[ \leq \frac{C}{\nu}|A^\frac{\sigma}{2}w|^2 + \frac{\nu}{4}|A^{\frac{\sigma+1}{2}}w|^2 \]  \quad (4.23)

and

\[ |b(w, u, A^\sigma w)| \leq \int_{\Omega} |w||\nabla u||A^\sigma w|dx \leq |w|^\frac{1}{2}|\nabla u||A^\sigma w|^\frac{1}{2} \]
\[ \leq C|A^{\frac{1+\sigma}{2}}w|^\frac{1}{2}|u||A^{\frac{1+\sigma}{2}}w|^\frac{1}{2} \]
\[ \leq \frac{C}{\nu}|u|^2 + \frac{\nu}{4}|A^{\frac{\sigma+1}{2}}w|^2 \]  \quad (4.24)

and

\[ |b(u, w, A^\sigma w)| \leq \int_{\Omega} |u||\nabla w||A^\sigma w|dx \leq |\nabla w|^\frac{1}{2}|u||A^\sigma w|^\frac{1}{2} \]
\[ \leq C|A^{\frac{1+\sigma}{2}}w|^\frac{1}{2}|u||A^{\frac{1+\sigma}{2}}w|^\frac{1}{2} \]
\[ \leq \frac{C}{\nu}|u|^2 + \frac{\nu}{4}|A^{\frac{\sigma+1}{2}}w|^2. \]  \quad (4.25)

Substituting (4.22)-(4.25) into (4.21), then by Theorem 3.2, we derive

\[ \frac{d}{dt}|A^\frac{\sigma}{2}w|^2 \leq \frac{C}{\nu}\left(|u|^2 + |f^\varepsilon|^2\right) + \frac{C}{\nu}|A^\frac{\sigma}{2}w|^2. \]  \quad (4.26)

Neglecting the third term on the left-hand side of (4.21), using the Gronwall inequality and noting the initial data of \( w \), we conclude

\[ \|w(t)\|_{D(A^\frac{\sigma}{2})}^2 = |A^\frac{\sigma}{2}w|^2 \leq \frac{C}{\nu}\left(|u|^2 + |f^\varepsilon|^2\right)e^{\frac{CT}{\nu}} + e^{\frac{C}{\nu}(T-\tau)}|f^\varepsilon|^2 = M < +\infty. \]

which means the proof has completed. \( \square \)

**Lemma 4.15.** For any \( f(x) \in V' \) and \( u_0 \in H \), the semigroup \( \{S(t)\} \) generated by the system (3.2) is asymptotic smoothness in \( H \).

**Proof.** Using the Lemmas 4.13 and 4.14, since the embedding \( D(A^\frac{\sigma}{2}) \hookrightarrow H \) is compact, we can deduce the asymptotic smoothness for the semigroup. Combining with Lemma 4.12 we complete the proof. \( \square \)

**Lemma 4.16.** Let \( f(x) \in H, u_0 \in D(A^\frac{\sigma}{2}) \), the semigroup \( \{S(t)\} \) generated by the problem (3.2) satisfies the condition-(MWZ) which means the asymptotic smoothness in \( D(A^\frac{\sigma}{2}) \).

**Proof. Step 1:** From Lemma 4.12 we see that \( \hat{B}_0 \) be the bounded absorbing set, then there exists a forward time \( t_{\hat{B}_0} \) such that \( \|S(t)u_0\|_{D(A^\frac{\sigma}{2})} \leq \hat{\rho} \), i.e., the bounded of the semigroup \( S(t) : D(A^\frac{\sigma}{2}) \to D(A^\frac{\sigma}{2}) \).
Using the definition in (2.1) and Young’s inequality, we have

\[ \| \sigma u \|^2 \leq \| \sigma \|^2 + \nu \| u \|^2. \]

Taking inner product of (3.2) with \( \sigma u \) for \( m \geq 1 \), and hence the solution \( u \) has the decomposition

\[ u = Pu + (I - P)u := u_1 + u_2, \quad (4.27) \]

for \( u_1 \in V_1^\sigma \), \( u_2 \in V_2^\sigma \) with the initial data \( A^\sigma u_1(\tau) = PA^\sigma u_0 \) and \( A^\sigma u_2(\tau) = (I - P)A^\sigma u_0 \) which also are bounded in \( V^\sigma \).

**Step 2:** Since \( u_1 \) is the orthonormal projection of \( u \), from the existence of absorbing ball \( B_0 \) for the semigroup \( S(t) \), we derive that \( u \) is bounded in \( V^\sigma \), and hence \( u_1 \) is bounded in \( V^\sigma \), i.e., \( \|A^\sigma u_1\|_H^2 \leq \hat{\rho} \).

**Step 3:** The objective next is to obtain the \( V^\sigma \)-norm of \( u_2 \) is small enough as \( m \to +\infty \).

Taking inner product of \( 3.2 \) with \( A^\sigma u_2 \), noting \( (A^\sigma u_1, A^\sigma u_2) = 0 \) and \( (A^{\sigma+1} u_1, A^{\sigma+1} u_2) = 0 \) in \( V^\sigma \), it yields

\[
\frac{d}{dt} \|A^\sigma u_2\|^2 + \nu \|A^{\sigma+1} u_2\|^2 + \nu_0 \|u\|^2 \|A^{\sigma+1} u_2\|^2 \\
\leq |\langle (B(u, u), A^\sigma u_2) \rangle| + |\langle Pf(x), A^\sigma u_2 \rangle|.
\]

By the property of \( b(\cdot, \cdot, \cdot) \) in (4.9), using the \( \varepsilon \)-Young inequality, we have

\[
|\langle B(u, u), A^\sigma u_2 \rangle| = |b(u, u, A^\sigma u_2)| \leq \int_{\Omega} |u||\nabla u| |A^\sigma u_2| \, dx \\
\leq C |u|^\frac{1}{2} |A^\sigma u_2|^\frac{1}{2} |\nabla u| \\
\leq \frac{C}{\lambda_1^{\frac{1}{2}} \lambda_{m+1}} |u|^\frac{1}{2} |A^{\sigma+1} u_2|^\frac{1}{2} \\
\leq \frac{\nu}{4} |A^{\sigma+1} u_2|^2 + \frac{C}{\nu \lambda_1^{\frac{1}{2}} \lambda_{m+1}} |u|^2.
\]

Using the definition in (2.1) and Young’s inequality, we have

\[
|\langle f(x), A^\sigma u_2 \rangle| \leq \frac{C}{\nu \lambda_1^{\frac{1}{2}} \lambda_{m+1}} |f(x)|^2 + \frac{\nu}{4} |A^{\sigma+1} u_2|^2,
\]

here we only need \( \sigma \in [0, 1) \).

Combining (4.28)–(4.30), neglecting the second term in (4.28), we conclude

\[
\frac{d}{dt} |A^\sigma u_2|^2 + \nu |A^{\sigma+1} u_2|^2 \leq \frac{C}{\nu \lambda_1^{\frac{1}{2}} \lambda_{m+1}} |f(x)|^2 + \frac{C}{\nu \lambda_1^{\frac{1}{2}} \lambda_{m+1}} |u|^2.
\]
Using Poincaré’s inequality, noting the definition of $A^\beta$ ($\beta \geq 0$), since $\lambda_{m+1} \leq \lambda_{m+2} \leq \cdots$ and $A^\alpha u_2 = \sum_{j=m+1}^{\infty} \lambda_j \omega_j$ for $u_2 = \sum_{j=m+1}^{\infty} a_j \omega_j$, we derive that

$$|A^{\frac{\beta}{2}} u_2|^2 = \sum_{j=m+1}^{\infty} \lambda_j^{\frac{\beta}{2}} a_j \omega_j'^2 \geq \sum_{j=m+1}^{\infty} \lambda_j^{\frac{\beta}{2}} \lambda_{m+1} a_j \omega_j'^2 \geq \lambda_{m+1} |A^{\frac{\beta}{2}} u_2|^2,$$

(4.32)

thus, we conclude

$$\frac{d}{dt} |A^{\frac{\beta}{2}} u_2|^2 + \nu \lambda_{m+1} |A^{\frac{\beta}{2}} u_2|^2 \leq \frac{C}{\nu \lambda_{m+1}} |f(x)|^2 + \frac{C}{\nu \lambda_{m+1} \lambda_{m+1}} \|u\|^2,$$  

(4.33)

by applying the Gronwall inequality in $[\tau, t]$ to (4.33), we deduce that

$$|A^{\frac{\beta}{2}} u_2(t)|^2 \leq |A^{\frac{\beta}{2}} u_2(\tau)|^2 e^{-\nu \lambda_{m+1} (t-\tau)} + \frac{C}{\nu \lambda_{m+1} \lambda_{m+1}} \int_\tau^t e^{-\nu \lambda_{m+1} (t-s)} |f(x)|^2 ds$$

$$+ \frac{C}{\nu \lambda_{m+1} \lambda_{m+1}} \int_\tau^t \|u(s)\|^2 e^{-\nu \lambda_{m+1} (t-s)} ds$$

$$\leq e^{-\nu \lambda_{m+1} (t-\tau)} \rho^2 + \frac{C |f(x)|^2}{\nu^2 \lambda_{m+1}^2} (1 - e^{-\nu \lambda_{m+1} (t-\tau)})$$

$$+ \frac{C}{\nu \lambda_{m+1} \lambda_{m+1}^3} \|u\|^2_{L^2(\tau, T; V)},$$  

(4.34)

By the bounded of absorbing set, noting $\lim_{m \to \infty} \lambda_{m+1} = +\infty$ and the existence of global solution, then for $m$ large enough, it follows

$$|A^{\frac{\beta}{2}} u_2(\tau)|^2 e^{-\nu \lambda_{m+1} (t-\tau)} \leq \frac{\varepsilon}{3},$$

(4.35)

$$\frac{C |f(x)|^2}{\nu^2 \lambda_{m+1}^2} (1 - e^{-\nu \lambda_{m+1} (t-\tau)}) < \frac{\varepsilon}{3},$$

(4.36)

$$\frac{C}{\nu \lambda_{m+1} \lambda_{m+1}^3} \|u\|^2_{L^2(\tau, T; V)} < \frac{\varepsilon}{3},$$

(4.37)

combining (4.34)–(4.37), we conclude

$$\|(I - P) S(t) u_\tau\|^2_{V_\sigma} = |A^{\frac{\beta}{2}} u_2|^2 < \varepsilon$$

which implies the condition-(MWZ). The proof is completed by combining with Lemma 4.12.

4.4. Proof of Theorem 3.6 Pullback dynamics of non-autonomous systems

In this section, we shall proof the existence of pullback attractors in $H$ and $D(A^{\frac{\beta}{2}})$ for (1.7).
Lemma 4.17. Assume that $u_0(x) \in H$, the external force $f(t,x) \in L^2_{\text{loc}}(\mathbb{R}; V')$ and (3.7) holds, if we choose parameter $\mu \in (0, \mu_0]$ ($\mu_0 = \nu \lambda_1$) and fixed, then the solution $u$ to the problem (1.7) satisfies that for any $\tau \leq t$,

$$|u|^2 \leq |u_0|^2 e^{-\mu(t-\tau)} + \frac{Ce^{-\mu t}}{\nu} \int_{-\infty}^{t} e^{\mu s} \|f(s)\|_V^2 ds.$$ \hspace{1cm} (4.38)

Proof. Multiplying (1.7) with $u$ and integrating over $\Omega$, choosing an appropriate parameter $\mu$ such that $e^{-\mu(t-\tau)} \geq e^{-\mu_0(t-\tau)}$ and

$$\frac{e^{-\mu t}}{\nu} \int_{-\infty}^{t} e^{\mu s} \|f(s)\|_V^2 ds \geq \frac{e^{-\mu_0 t}}{\nu} \int_{-\infty}^{t} e^{\mu_0 s} \|f(s)\|_V^2 ds,$$

we can derive the result easily. \hfill $\square$

Lemma 4.18 (The pullback $D$-absorbing balls in $H$). Assume that $u_0(x) \in H$ and (3.7) holds, let $\hat{B}_0 = \{B_0(t)\}_{t \in \mathbb{R}}$ be a family of balls, where $B_0(t) = B(0, \rho_0(t))$ is a ball at center $0$ and radius $\rho_0(t)$ that satisfies

$$\rho_0^2(t) = 1 + \frac{4}{\nu \lambda_1} \int_{-\infty}^{t} e^{-\mu(t-s)} \|f(s)\|_V^2 ds.$$ \hspace{1cm} (4.39)

Then for any $0 < \varepsilon < \frac{1}{2}$ small enough, there exists a pullback time $\tau(t,\varepsilon)$, such that for any $\tau < \tau(t,\varepsilon) \leq t$, $\hat{B}_0(t)$ is a family of pullback $D$-absorbing sets for the continuous process $S(t,\tau)$.

Proof. Noting that

$$|S(t,\tau)u_0|^2 \leq |u_0|^2 e^{-\mu(t-\tau)} + \frac{Ce^{-\mu t}}{\nu} \int_{-\infty}^{t} e^{\mu s} \|f(s)\|_V^2 ds$$ \hspace{1cm} (4.40)

there exists a pullback time $\tau(t,\varepsilon)$, such that for any $\tau < \tau(t,\varepsilon)$, if follows $e^{-\mu t}\rho_0^2(\tau)|^2 e^{\mu \tau} \leq \varepsilon$. Hence, we have

$$|S(t,\tau)u_0|^2 \leq \varepsilon + \rho_0^2(t) - \frac{1}{2} \leq \rho_0^2(t),$$ \hspace{1cm} (4.41)

which implies that $S(t,\tau)D(\tau) \subset B_0(t)$, i.e., $\hat{B}_0(t)$ is a family of pullback $D$-absorbing balls. \hfill $\square$

The pullback $D$-asymptotically compact for the process in $H$ on unbounded domain was obtained by verify the pullback condition-(MWZ).

Lemma 4.19 (Pullback $D$-asymptotically compact in $H$). Assume that $u_0(x) \in H$, the external force $f(t,x) \in L^2_{\text{loc}}(\mathbb{R}; V')$ and (3.7) holds, then the processes $S(t,\tau)$ is pullback $D$-asymptotically compact in $H$ for the system (1.7).

Proof. In fact, we can also prove the Pullback $D$-asymptotically compact via verifying the pullback condition-(MWZ) as in Lemma 4.16 here we omit the detail. \hfill $\square$
Lemma 4.20. Assume that $u_0(x) \in D(A^{\frac{\sigma}{2}})$, the external force $f(t, x) \in L^2_{loc}(\mathbb{R}; H)$ and (3.8) holds, if we choose fixed parameter $\mu \in (0, \mu_0]$ ($\mu_0 = \nu \lambda_1$) then the solution $u$ to the problem (1.14) satisfies that for any $\tau \leq t$,

$$
|A^{\frac{\sigma}{2}}u(t)|^2 \leq |A^{\frac{\sigma}{2}}u_0|^2 e^{-\mu(t-\tau)} + \frac{C e^{-\mu t}}{\nu} \int_{-\infty}^{t} e^{\mu s} |f(s)|^2 ds
$$

$$
+ \frac{C|u(s)|_{L^{\infty}(\tau, +\infty; H)}^\frac{\nu}{2}}{\nu^2 \lambda_1^\frac{\nu}{2}} \left[ 1 - e^{-\mu(t-\tau)} \right].
$$

(4.42)

Proof. Multiplying (1.7) with $A^\sigma u$ and integrating over $\Omega$, we derive that

$$
\frac{1}{2} \frac{d}{dt} |A^{\frac{\sigma}{2}}u|^2 + \nu |A^{\frac{\sigma+1}{2}}u|^2 + \nu_0 ||u||^2 |A^{\frac{\sigma+1}{2}}u|^2
\leq |b(u, u, A^\sigma u)| + | < Pf(t, x), A^\sigma u > |.
$$

(4.43)

By the property of $b(\cdot, \cdot, \cdot)$, using the $\varepsilon$-Young inequality and Hölder’s inequality, we have

$$
|b(u, u, A^\sigma u)| \leq \int_{\Omega} |u| ||\nabla u|| |A^\sigma u| dx
\leq C |u| |A^\sigma u|^{\frac{\nu}{2}} |\nabla u|^{\frac{\nu}{2}}
\leq C
\frac{\nu}{\lambda_1}\frac{\nu}{2} |u| |A^{\frac{\sigma+1}{2}}u|^{\frac{\nu}{2}} ||u||^{\frac{\nu}{2}}
\leq \nu_0 ||u||^2 |A^{\frac{\sigma+1}{2}}u|^2 + \frac{C}{\nu \lambda_1^\frac{\nu}{2}} |u|^\frac{\nu}{2}.
$$

(4.44)

Using the definition in (2.1) and Young’s inequality, we have

$$
| < f(t, x), A^\sigma u > | \leq \frac{C}{\nu} |f(t)|^2 + \frac{\nu}{2} |A^{\frac{\sigma+1}{2}}u|^2, \sigma \in [0, 1).
$$

(4.45)

Combining (4.43), (4.43) and (4.45), using the Poincaré inequality and Gronwall’s inequality, we can conclude

$$
|A^{\frac{\sigma}{2}}u(t)|^2 \leq |A^{\frac{\sigma}{2}}u_0|^2 e^{-\nu \lambda_1(t-\tau)} + \frac{C e^{-\nu \lambda_1 t}}{\nu} \int_{-\infty}^{t} e^{\nu \lambda_1 s} |f(s)|^2 ds
$$

$$
+ \frac{C|u(s)|_{L^{\infty}(\tau, +\infty; H)}^\frac{\nu}{2}}{\nu^2 \lambda_1^\frac{\nu}{2}} \left[ 1 - e^{-\nu \lambda_1(t-\tau)} \right].
$$

(4.46)

Choosing an appropriate parameter $\mu$ such that $e^{-\mu(t-\tau)} \geq e^{-\mu_0(t-\tau)}$, $1 - e^{-\nu \lambda_1(t-\tau)} \leq 1 - e^{-\nu \lambda_1(t-\tau)}$ and

$$
eq\frac{e^{-\mu t}}{\nu} \int_{-\infty}^{t} e^{\mu s} |f(s)|^2 ds \geq \frac{e^{-\mu_0 t}}{\nu} \int_{-\infty}^{t} e^{\mu_0 s} |f(s)|^2 ds,
$$

which means (4.42) holds. The proof is completed. \hfill \Box

Lemma 4.21 (The pullback $D$-absorbing balls in $D(A^{\frac{\sigma}{2}})$). Assume that $u_0(x) \in D(A^{\frac{\sigma}{2}})$, the external force $f(t, x) \in L^2_{loc}(\mathbb{R}; H)$ and (3.8) holds, let $\mathcal{B} =$
Pullback dynamics of 3D Navier-Stokes equation with nonlinear viscosity

\{B'_0(t)\}_{t \in \mathbb{R}} be a family of balls, where \(B'_0(t) = B'(0, \rho'_0(t))\) is a ball at center 0 and radius \(\rho'_0(t)\) that satisfies

\[
(\rho'_0(t))^2 = 1 + \frac{C|u(s)|^\frac{4}{3}\|L^\infty(\tau, +\infty; H)\|}{\nu^2 \lambda_1^{\frac{2}{3} - \frac{2\nu}{3}}} + \frac{4}{\nu \lambda_0} \int^t_\infty e^{-\mu(t-s)}\|f(s)\|^2 \nu ds. \tag{4.47}
\]

Then for any \(0 < \varepsilon < \frac{1}{2}\) small enough, there exists a pullback time \(\tau'(t, \varepsilon)\), such that for any \(\tau < \tau'(t, \varepsilon) \leq t\), \(\hat{B}'_0(t)\) is a family of pullback \(\mathcal{D}\)-absorbing sets for the continuous process \(S(t, \sigma)\).

**Proof.** Noting that

\[
\|S(t, \tau)u_0\|^2_{D(A^{\frac{2}{3}})} \leq \|u_0\|^2_{D(A^{\frac{2}{3}})} e^{-\mu(t-\tau)} + \frac{Ce^{-\mu t}}{\nu} \int^t_\infty e^{\mu s}|f(s)|^2 ds + \frac{C|u(s)|^\frac{4}{3}\|L^\infty(\tau, +\infty; H)\|}{\nu^2 \lambda_1^{\frac{2}{3} - \frac{2\nu}{3}}}, \tag{4.48}
\]

there exists a pullback time \(\tau'(t, \varepsilon)\), such that for any \(\tau < \tau'(t, \varepsilon)\), if follows \(e^{-\mu t}|\rho'_{D'}(\tau)|^2 e^{\mu \tau} \leq \varepsilon\). Hence, we have

\[
\|S(t, \tau)u_0\|^2_{D(A^{\frac{2}{3}})} \leq \varepsilon + (\rho'_0(t))^2 - \frac{1}{2} \leq (\rho'_0(t))^2, \tag{4.49}
\]

which implies that \(S(t, \tau)D'(\tau) \subset B'_0(t)\), i.e., \(\hat{B}'_0(t)\) is a family of pullback \(\mathcal{D}\)-absorbing balls in \(D(A^{\frac{2}{3}})\).

**Lemma 4.22.** (Pullback \(\mathcal{D}\)-asymptotic compactness in \(D(A^{\frac{2}{3}})\)) Assume that \(u_0(x) \in D(A^{\frac{2}{3}})\), the external force \(f(t, x) \in L^2_{loc}(\mathbb{R}; H)\) and (3.8) holds, then the processes \(S(t, \tau)\) satisfies pullback \(\mathcal{D}\)-condition (MWZ) which implies pullback \(\mathcal{D}\)-asymptotically compact in \(D(A^{\frac{2}{3}})\) for problem (1.7).

**Proof.** Step 1: Denote \(B' = \{B'_0(t)\}_{t \in \mathbb{R}}\) be the pullback \(\mathcal{D}\)-absorbing family given in Lemma 4.21, then there exists a pullback time \(\tau_{t, \varepsilon}\) such that \(|u|^2 = |S(t, \tau)u_0|^2 \leq \rho'_0(t)\).

The decomposition is similar to Lemma 4.16 hence the solution \(u\) has the decomposition

\[
u = Pu + (I - P)u := u_1 + u_2, \tag{4.50}
\]

for \(u_1 \in V_1^{\frac{2}{3}}, u_2 \in V_2^{\frac{2}{3}}\) with the initial data \(A^{\frac{2}{3}}u_1(\tau) = PA^{\frac{2}{3}}u_0\) and \(A^{\frac{2}{3}}u_2(\tau) = (I - P)A^{\frac{2}{3}}u_0\) which also are bounded in \(V^{\frac{2}{3}}\). From the existence of global solution and pullback \(\mathcal{D}\)-absorbing family of set, we know \(|u_1|^2 \leq \rho'_0(t)\), and next we only need to prove the \(V^{\frac{2}{3}}\)-norm of \(u_2\) is small enough.

**Step 2:** Taking inner product of (1.7) with \(A^{\sigma} u_2\), using the same technique in Lemma 4.16 we conclude

\[
\frac{d}{dt}|A^{\frac{2}{3}}u_2|^2 + \nu \lambda_{m+1}|A^{\frac{2}{3}}u_2|^2 \leq \frac{C}{\nu}|f(t)|^2 + \frac{C}{\nu \lambda_1^\frac{2(1-\sigma)}{3}}|u|^2, \tag{4.51}
\]
by applying the Gronwall inequality in \([\tau, t]\) to (4.51), we deduce that

\[
|A^\frac{2}{\sigma} u_2(t)|^2 \leq |A^\frac{2}{\sigma} u_2(\tau)|^2 e^{-\nu \lambda_{m+1}(t-\tau)} + \frac{C}{\nu \lambda_{m+1}} \int_{\tau}^{t} |f(s)|^2 ds
\]

\[
+ \frac{C}{\nu \lambda_{m+1}^{2(1-\sigma)}} \int_{\tau}^{t} \|u(s)\|^2 e^{-\nu \lambda_{m+1}(t-s)} ds. \tag{4.52}
\]

By the existence of bounded family of pullback \(D\)-absorbing sets, noting \(\lim_{m \to \infty} \lambda_{m+1}^{1-\sigma} = +\infty, \tau \to -\infty, f(s) \in L^2_{\text{loc}}(\mathbb{R}; H)\) and the existence of global solution, then for \(m\) large enough, there exists a pullback time \(\tau_0\), such that for all \(\tau \leq \tau_0\), it follows

\[
|A^\frac{2}{\sigma} u_2(\tau)|^2 e^{-\nu \lambda_{m+1}(t-\tau)} \leq (\rho_0'(t))^2 e^{\nu \lambda_{m+1} \tau} < \frac{\varepsilon}{3}, \tag{4.53}
\]

\[
\frac{C}{\nu \lambda_{m+1}} \int_{\tau}^{t} |f(s)|^2 ds < \frac{\varepsilon}{3}, \tag{4.54}
\]

\[
\frac{C}{\nu \lambda_{m+1}^{2(1-\sigma)}} \|u\|_2^2 L^2(\tau, T; V) < \frac{\varepsilon}{3}, \tag{4.55}
\]

combining (4.52)–(4.55), we conclude

\[
\|(I - P) S(t, \tau) u_\tau\|^2_{V^\sigma} = |A^\frac{2}{\sigma} u_2|^2 < \varepsilon \tag{4.56}
\]

which implies that the process satisfies pullback \(D\)-condition-(MWZ), hence \(\{S(t, \tau)\}\) is pullback \(D\)-asymptotically compact. The proof is completed. \(\square\)

**Proof of Theorem 3.6** From the pullback attractors theory, we prove the existence of the families of pullback \(D\)-absorbing sets in \(H\) and \(D(A^\frac{2}{\sigma})\) by Lemmas 4.18 and 4.21, then the pullback asymptotic compactness for the continuous process \(S(t, \tau)\) in \(H\) and \(D(A^\frac{2}{\sigma})\) are represented by Lemmas 4.19 and 4.22 respectively. Combining the continuity of process, the existence of pullback \(D\)-absorbing sets and the pullback \(D\)-asymptotic compactness for the process \(\{S(t, \tau)\}\), we finished the proof. \(\square\)

**4.5. Proof of Theorem 3.7** Finite fractal dimension of pullback attractors

Considering the Cauchy problem for first variation equation of (3.2) as

\[
\begin{cases}
\frac{dU}{dt} + \nu AU + A U + B(u, U) + B(U, u) = 0, \\
U(\tau) = \xi \in H.
\end{cases} \tag{4.57}
\]

Taking inner product of (4.57) by \(U\), we can prove the existence of unique solution \(U \in L^2(\tau, T; V) \cap L^4(\tau, T; V) \cap C(\tau, +\infty; H)\) for any \(T > \tau\) and \(\tau \in \mathbb{R}\).
Lemma 4.23. The solution of \((4.57)\) generates a bounded linear compact operator \(\Lambda(t, s; u_0)\xi = U(t) : H \to H\) satisfies
\[
\sup_{\tilde{u}_0, u_0 \in A(s)} \sup_{\|\tilde{u}_0 - u_0\| \leq \varepsilon} \frac{\|S(t, s)\tilde{u}_0 - S(t, s)u_0 - \Lambda(t, s; \tilde{u}_0)(\tilde{u}_0 - u_0)\|}{\|\tilde{u}_0 - u_0\|} \to 0 \quad (4.58)
\]
as \(\varepsilon \to 0\) which means the process is uniformly differentiable for \(t \geq s\).

Proof. (1) Assume \(w = u - \dot{u}\), \(u(t)\) and \(\dot{u}(t)\) be two solutions of
\[
\frac{du}{dt} + \nu Au + \mathbb{A}u + B(u, u) = Pf(t) \quad (4.59)
\]
with different initial data \(u(s) = u_0\) and \(\dot{u}(s) = \dot{u}_0\). Denoting \(U(t)\) be a solution of problem \((4.57)\) with initial data \(U(s) = u_0 - \dot{u}_0\), we can verify that \(\theta = u - \dot{u} - U\) satisfies the Cauchy problem
\[
\begin{cases}
\frac{d\theta}{dt} + \nu A\theta + \mathbb{A}\theta + B(\theta, u) + B(\theta, u) - B(w, w) = 0, \\
\theta(s) = 0. 
\end{cases} \quad (4.60)
\]
Taking inner product of \((4.60)\) by \(\theta\), using the property of operator \(\mathbb{A}\), we can derive \((4.58)\) easily.

(2) Taking inner product of \((4.57)\) with \(U\) and \(A^\sigma U\), integrating over \(\Omega\), by the same technique in the proof of regular pullback absorbing set, we can derive the uniform estimate in more regular space \(D(A^{\bar{\nu}})\). Since \(D(A^{\bar{\nu}})\) is compact in \(H\), then we can prove that the operator \(\Lambda(t, s; u_0)\) is compact. \(\square\)

Next, we shall use trace formula (Lemmas 4.19, 4.20 in [7]) to prove the bounded of fractal and Hausdorff dimension of pullback attractors in \(H\).

Proof of Theorem 3.7. From the definition of the family of pullback attractors, then for a fixed \(\tau^*\), \(\bigcup_{\tau \leq \tau^*} A(t)\) is precompact in \(H\). For each \(t \geq \tau\), \(\tau \leq \tau^*\) and \(u_0 \in H\), the linear operator is described as \(\Lambda(t, s; u_0) \cdot \xi = U(t)\), where \(U(t)\) is the solution of \((4.57)\). Denoting \(F(S(t, \tau)u_0, t) = -\nu A - \mathbb{A} - B(\cdot, \cdot) - B(\cdot, u)\), then from Lemma 4.23, we see that \(F(\cdot, t)\) is Gateaux differential in \(V\) at \(S(t, \tau)u_0\) which satisfies
\[
F'(S(t, \tau)u_0)U = -\nu AU - \mathbb{A}U - B(S(t, \tau)u_0, U) - B(U, S(t, \tau)u_0), \quad (4.61)
\]
this implies \(F'(S(t, \tau)u_0, t) \in \mathcal{L}(V, V')\) is a continuous linear operator satisfying the problem
\[
\begin{cases}
\frac{dU}{dt} = F'(S(t, \tau)u_0, t)U, & u_0 \in H, \\
U(\tau, x) = \xi 
\end{cases} \quad (4.62)
\]
which possesses a unique solution \(U(t) = U(t, \tau; u_0, \xi) \in L^2(\tau, T; V) \cap C(\tau, T; H)\).

For each \(\xi_1, \xi_2, \ldots, \xi_n \in H\), we denote \(U_i(t) = \Lambda(t, s; u_0) \cdot \xi_i\), which implies \(U_1(s) = U_1(s, \tau; u_0, \xi_1), U_2(s) = U_2(s, \tau; u_0, \xi_2), \ldots, U_n(s) = U_n(s, \tau; u_0, \xi_n)\) be the solution of problem \((4.62)\) with different initial data \(U_i(\tau) = \xi_i\).
(i = 1, 2, · · · , n) respectively, \( Q_n(s) \) denote the projection from \( H \) to the space \( \text{span}\{U_1(s), U_2(s), · · · , U_n(s)\} \), then by Lemma 4.19 in [7], it yields

\[
\|U_1(t) \wedge U_2(t) \wedge · · · \wedge U_n(t)\|_{\Lambda^v(H)} = \|\xi_1 \wedge \xi_2 \wedge · · · \wedge \xi_n\|_{\Lambda^v(H)} \exp \left( \int_t^s \text{Tr}_n(F'(\tau)u_0, s) \circ Q_n(s)ds \right).
\]

(4.63)

Let \( \{e_1(s), e_2(s), · · · , e_n(s)\} \) be an orthonormal basis for \( \text{span}\{U_1(s), U_2(s), · · · , U_n(s)\} \), then

\[
\text{Tr}_n(F'(\tau)u_0, s) = \sup_{\xi_i \in H, |\xi_i| \leq 1, i \leq n} \left( \sum_{i=1}^n \langle F'(\tau)u_0, s \rangle e_i, e_i \rangle \right).
\]

(4.64)

Since \( U_i(s) \in L^2(\tau, T; V) \), then \( U_i(s) \in V \) for a.e. \( s \geq \tau \), hence \( e_i(s) \in V \) for a.e. \( s \geq \tau \) and \( i = 1, 2, · · · , n \).

Noting that \( b(S(t, \tau)u_0, e_i(s), e_i(s)) = 0 \), we derive

\[
\text{Tr}_n(F'(\tau)u_0, s) \circ Q_n(s)
= \sum_{i=1}^n \langle F'(\tau)u_0, s \rangle e_i(s), e_i(s) \rangle
= \sum_{i=1}^n \left( -\nu \|e_i(s)\|^2 - b(S(\tau)u_0, e_i(s), e_i(s)) - b(e_i(s), S(\tau)u_0, e_i(s)) \right)
\leq -\nu \sum_{i=1}^n \|e_i(s)\|^2 - \nu_0 \sum_{i=1}^n \|e_i(s)\|^4 + \sum_{i=1}^n \|b(e_i(s), S(\tau)u_0, e_i(s))\|.
\]

(4.65)

For the second term in (4.65), by the Lied-Thirring inequality in 3D case \((p = 2, n = 3)\):

\[
\left( \int_\Omega \left( \sum_{i=1}^n |e_i(s)|^2 \right)^{\frac{p-1}{n}} dx \right)^{\frac{n}{p-1}} \leq C_1 \sum_{i=1}^n \int_\Omega |\nabla e_i(s)|^2 ds, \tag{4.66}
\]

where \( \frac{p}{2} < p \leq 1 + \frac{4}{7} \), we could proceed using the bounded

\[
\sum_{i=1}^n |b(e_i(s), S(\tau)u_0, e_i(s))| \leq C \int_\Omega \left( \sum_{i=1}^n |e_i(s)||\nabla S(\tau)u_0||e_i(s)| \right) dx
\leq \|S(\tau)u_0\| \left[ \int_\Omega \left( \sum_{i=1}^n |e_i(s,x)|^2 \right)^2 ds \right]^{\frac{3}{4} \times \frac{3}{4}}
\leq \frac{C}{\nu} \|S(\tau)u_0\|^2 + \frac{\nu}{2} \left[ \sum_{i=1}^n \|e_i(s)\|^2 \right]^{\frac{3}{4}}. \tag{4.67}
\]
Defining we derive where setting \( s \) and \( \tau \), we obtain

\[
Tr_n(F'(S(s, \tau)u_0, s)) \leq -\nu \sum_{i=1}^{n} \| e_i(s) \|^2 - \nu_0 \sum_{i=1}^{n} \| e_i(s) \|^4 + \frac{\nu}{2} \left[ \sum_{i=1}^{n} \| e_i(s) \|^2 \right] + \frac{C}{\nu} \| S(s, \tau)u_0 \|^2 
\]

Using the variational principle and we get

\[
\leq -\frac{2\nu}{27} \sum_{i=1}^{n} \lambda_i - \nu_0 \sum_{i=1}^{n} \lambda_i^2 + \frac{C}{\nu} \| S(s, \tau)u_0 \|^2
\]

\[
\leq -\frac{2\nu}{27} \frac{\pi \nu n^2}{|\Omega|} - \frac{\pi^2 \nu_0 n^4}{|\Omega|^2} + \frac{C}{\nu} \| S(s, \tau)u_0 \|^2.
\]

(4.68)

Defining

\[
q_n = \sup_{t \in \mathbb{R}} \sup_{u_0 \in A(t)} \left( \frac{1}{T} \int_{t-T}^{t} Tr_n(F'(S(s, \tau)u_0, s)) ds \right),
\]

(4.69)

\[
\bar{q}_n = \limsup_{T \to +\infty} q_n,
\]

(4.70)

we derive

\[
q_n \leq -\frac{2\nu}{27} \frac{\pi \nu n^2}{|\Omega|} - \frac{\pi^2 \nu_0 n^4}{|\Omega|^2} + \frac{C}{\nu} \sup_{t \in \mathbb{R}} \sup_{u_0 \in A(t)} \left( \frac{1}{T} \int_{t-T}^{t} \| S(s, \tau)u_0 \|^2 ds \right)
\]

(4.71)

and

\[
\bar{q}_n \leq -\frac{2\nu}{27} \frac{\pi \nu n^2}{|\Omega|} - \frac{\pi^2 \nu_0 n^4}{|\Omega|^2} + \frac{C}{\nu} q.
\]

(4.72)

where \( q = \lim \sup_{T \to +\infty} \sup_{t \in \mathbb{R}} \sup_{u_0 \in A(t)} \frac{1}{T} \int_{t-T}^{t} \| S(s, \tau)u_0 \|^2 ds \).

From the estimate of equation, we have

\[
\frac{\nu}{2} \int_s^t \| u(r) \|^2 dr + 2\nu_0 \int_s^t \| u(r) \|^4 dr \leq |u_0(s)|^2 + C\| f(r) \|_{L^2(s,t;V')}.
\]

(4.73)

Setting \( s = t - T \) in (4.73), using the bounded of solution in \( H \), it follows

\[
q \leq \frac{C}{\nu} \lim_{T \to +\infty} \frac{|u_0(t - T)|^2}{T} + \frac{C}{\nu} \lim_{T \to +\infty} \frac{1}{T} \int_{t-T}^{t} \| f(r) \|^2_{V'} dr
\]

\[
\leq \frac{C}{\nu} \lim_{T \to +\infty} \frac{1}{T} \int_{t-T}^{t} \| f(r) \|^2_{V'} dr.
\]

(4.74)

Defining \( M = \lim_{T \to +\infty} \frac{1}{T} \int_{t-T}^{t} \| f(r) \|^2_{V'} dr \), then

\[
\bar{q}_n \leq -\frac{2\nu}{27} \frac{\pi \nu n^2}{|\Omega|} - \frac{\pi^2 \nu_0 n^4}{|\Omega|^2} + \frac{C}{\nu_0^2} M.
\]

(4.75)

**Case 1:** If \( \frac{\pi \nu n^2}{|\Omega|} + \frac{\pi^2 \nu_0 n^4}{|\Omega|^2} > \frac{2
\nu}{27} + \frac{C}{\nu_0^2} M \), then by Lemma 4.19 in [7], we have \( \dim_B(A(t)) \leq n \).
From the procedure of pullback absorbing set, we see that, hence, multiplying (4.77) with \(w\), we obtain

\[
\dim(A(t)) \leq \frac{C|\Omega|^\frac{4}{3}}{\nu_0^2 \lambda_1} \|f(t)\|_{L^\infty(-\infty,T^*;H)}^2 + \frac{2\nu}{27} = \hat{G} + \frac{2\nu}{27}, \tag{4.76}
\]

here \(G = \frac{\|f(t)\|_{L^\infty(-\infty,T^*;H)}}{\nu_0^2 \lambda_1}\). This means the proof has been finished. \(\square\)

### 4.6. Proof of Theorem 3.8

When the pullback attractor becomes a single trajectory

Denoting \(\langle h \rangle \leq t = \limsup_{s \to -\infty} \frac{1}{t - s} \int_s^t h(r)dr\), the above pullback attractors \(A\) in \(H\) becomes a single trajectory for some special viscosity \(\nu, \nu_0\).

Let \(u(t), v(t)\) be two solutions of problem (1.7) with initial data \(u(0) = u_0\) and \(v(0) = v_0\) respectively, denote \(w = u(t) - v(t)\) and assume \(\|u(t)\| \geq \|v(t)\|\) (or else denote \(w = v - u\), then we see that \(w\) satisfies

\[
\begin{align*}
\nu \cdot w & = 0, \\
w|_{\partial \Omega} & = 0, \\
w(x, \tau) & = u_0 - v_0,
\end{align*}
\]

multiplying (4.77) with \(w\), using Poincaré’s inequality and the property of \(b(\cdot, \cdot, \cdot)\), it follows

\[
\frac{1}{2} \frac{d|w|^2}{dt} + [\nu + \nu_0(\|u\|^2 - \|v\|^2)]|w|^2 \leq |w|^2 \|v\|^2 \leq \frac{c}{\nu} \|w\|^2 \|v\|^4 + \frac{\nu}{2} \|w\|^2,
\]

hence,

\[
\frac{d|w|^2}{dt} \leq \left[ \frac{c}{\nu} \|v\|^4 + 2\nu_0 \lambda_1 \|v\|^2 - \nu \lambda_1 - 2\nu_0 \lambda_1 \|u\|^2 \right] \|w\|^2. \tag{4.78}
\]

If \(u_0\) and \(v_0\) fixed, we let \(\tau\) goes to \(-\infty\), it follows \(u(t) = v(t)\), which means the pullback attractors is a point provided that

\[
\frac{c}{\nu} \|v\|^4 + 2\nu_0 \lambda_1 \|v\|^2 - \nu \lambda_1 - 2\nu_0 \lambda_1 \|u\|^2 < 0. \tag{4.79}
\]

A sufficient but may be not optimal condition is

\[
\frac{c}{\nu} \|v\|^4 + 2\nu_0 \lambda_1 \|v\|^2 < \nu \lambda_1. \tag{4.80}
\]

From the procedure of pullback absorbing set, we see that

\[
\frac{d}{dt}|v|^2 + 2(\nu + \nu_0 \|v\|^2)\|v\|^2 \leq \frac{2|f|^2}{\nu \lambda_1} + \nu \|v\|^2 \tag{4.81}
\]
Pullback dynamics of 3D Navier-Stokes equation with nonlinear viscosity

\[ \nu \int_s^t \|v\|^2 \, dr + 2\nu_0 \int_s^t \|v\|^4 \, dr \leq (|v(t)|^2 - |v(s)|^2) + \frac{2}{\nu \lambda_1} \int_s^t |f(r)|^2 \, dr, \quad (4.82) \]

which implies

\[ \langle \|v\|^2 \rangle_{|t| \leq t} \leq \frac{\langle |f|^2 \rangle_{|t| \leq t}}{\nu^2 \lambda_1}, \quad \langle \|v\|^4 \rangle_{|t| \leq t} \leq \frac{\langle |f|^2 \rangle_{|t| \leq t}}{\nu_0 \lambda_1}. \quad (4.83) \]

Combining (4.80) and (4.83), it yields

\[ \frac{c \langle |f|^2 \rangle_{|t| \leq t}}{\nu^2 \lambda_1} + \frac{4\nu_0 \lambda_1 \langle |f|^2 \rangle_{|t| \leq t}}{\nu^2 \lambda_1} < \nu \lambda_1. \quad (4.84) \]

If we define the generalized Grashof number as \( G^2(t) = \frac{\langle |f|^2 \rangle_{|t| \leq t}}{\nu_0 \lambda_1^2} \), then we can derive a sufficient condition for pullback attractors to be single trajectory as

\[ G(t) < \sqrt{\frac{\nu_0}{c
\nu + 4\nu_0^2 \nu \lambda_1}}. \quad (4.85) \]

4.7. Proof of Theorem 3.14: The relation of pullback attractors

From [13] and [16], noting that

\[ \mathcal{A}_{CDF}(t) = \bigcup_{B \text{ bounded in } H} \Lambda(B, t), \quad (4.86) \]

where \( \Lambda(B, t) = \bigcap_{s \leq t} \bigcup_{t \leq s} U(t, \tau) B(\tau) \), and since the universes \( \mathcal{D}_F, \mathcal{D}_\mu \) and \( \mathcal{D}_{\mu_0} \) in \( H \) is no need to be bounded, \( \mathcal{D}_\mu \) is arbitrary, it follows that

\[ B \subset \mathcal{D}_F \subset \mathcal{D}_\mu \subset \mathcal{D}_{\mu_0}. \quad (4.87) \]

Using the structure of pullback attractors \( \mathcal{A}^H_\mu \) in Theorem 3.6 i.e., the property of pullback-\( \omega \) limit set

\[ \mathcal{A}^H_\mu = \bigcap_{T \leq t} \bigcup_{s \leq T} S(t, s) D_\mu(s), \quad (4.88) \]

we conclude that \( \mathcal{A}_{CDF}(t) \) is included in other pullback attractors and

\[ \mathcal{A}_F(t) \subset \mathcal{A}^H_\mu \subset \mathcal{A}^H_{\mu_0}. \quad (4.89) \]

The similar result also holds in \( D(A^F F) \), Which implies (a) and (b).

From the theory in [13, 36], if the union of universes or pullback absorbing sets in uniformly bounded, then (c)-(f) is true. The proof has been completed.
4.8. Proof: Upper semi-continuity of pullback attractors in $H$ for perturbed problem $g(t, x) = \varepsilon h(t, x)$ of $(1.7)$

Using the theory in Section 4.2 we shall use the decomposition of process to estimate the linear equation with non-homogeneous initial data and nonlinear equation with homogenous initial data, i.e., the solution $u_\varepsilon(t) = U_\varepsilon(t, \tau)u_\tau$ of perturbed problem $(3.2)$ with and $g(t, x) = \varepsilon h(t, x)$ and initial data $u_\tau \in H$ can be decomposed as

$$u_\varepsilon = S_\varepsilon(t, \tau)u_\tau = S_{1, \varepsilon}(t, \tau)u_\tau + S_{2, \varepsilon}(t, \tau)u_\tau,$$

(4.90)

here $S_{1, \varepsilon}(t, \tau)u_\tau = v(t)$ and $S_{2, \varepsilon}(t, \tau)u_\tau = w(t)$ solve the problems

$$\begin{cases}
  v_t + \nu A v + A v + B(v, v) = 0, \\
  v(x, t)|_{\partial \Omega} = 0, \\
  v(\tau, x) = u_0(x)
\end{cases}$$

(4.91)

and

$$\begin{cases}
  w_t + \nu A w + A w = B(w, w) - B(w, v) - B(v, w) + \varepsilon h(x, t), \\
  w(x, t)|_{\partial \Omega} = 0, \\
  w(\tau, x) = 0
\end{cases}$$

(4.92)

respectively.

**Lemma 4.24.** Let $R_\eta = \{r : \mathbb{R} \to (0, +\infty) | \lim_{\xi \to -\infty} e^{\eta t}r^2(\xi) = 0\}$ and denote by $D_\eta$ the class of families $\tilde{D} = \{D(t) : t \in \mathbb{R}\} \subset D(H)$ as universe such that $D(t) \subset B(0, r_{\tilde{D}}(t))$, where $B(0, r_{\tilde{D}}(t))$ is the closed ball in $H$ centered at zero with radius $r_{\tilde{D}}(t)$.

Suppose that $u_0 \in H$, the external force $h(t, x) \in L^2(\tau, T; H)$ satisfies (3.35). Then for any bounded set $B \subset H$ and any fixed $t \in \mathbb{R}$, there exists a time $T(B, t) > 0$, such that

$$\| S_\varepsilon(t, t - \tau)u_{t-\tau} \|^2_{H} \leq R_\varepsilon^2(t) \quad \forall \ \tau \geq T(B, t), \ u_{t-\tau} \in B,$$

(4.93)

where $R_\varepsilon^2(t) = \frac{2C_\varepsilon}{\nu} e^{-\eta t} \int_{-\infty}^{t} e^{\eta s}|h(s)|^2 \, ds$. Moreover, setting $B_\varepsilon(t) = \{u_\varepsilon \in H : |u_\varepsilon|^2 \leq R_\varepsilon^2(t)\}$, then $B_\varepsilon = \{B_\varepsilon(t)\}_{t \in \mathbb{R} \in D_\eta}$ is the family of pullback absorbing sets in $H$, i.e.,

$$\lim_{t \to -\infty} e^{\eta t}R_\varepsilon(t) = 0 \quad \forall \ \varepsilon > 0.$$

(4.94)

**Proof.** Let $t \in \mathbb{R}$ be fixed, then for any $\tau \in \mathbb{R}$ and $u_0 \in H$, we denote

$$u_\varepsilon(r) = u(r; t - \tau, u_0) = u_\varepsilon(r - t + \tau, t - \tau, u_0) \text{ for } r \geq t - \tau.$$

(4.95)

Multiplying perturbed problem $(3.2)$ $(f(t) = \varepsilon h(t, x))$ with $e^{\eta t}u_\varepsilon (\eta)$ will be determined later), noting that $(B(u_\varepsilon, u_\varepsilon, u_\varepsilon) = 0$, we derive that

$$\frac{d}{dt} \left( e^{\eta t}|u_\varepsilon(t)|^2 \right) + 2\nu e^{\eta t}||u_\varepsilon(t)||^2 + 2r_0 e^{\eta t}||u_\varepsilon(t)||^4$$

$$\begin{align*}
  &= \eta e^{\eta t}|u_\varepsilon(t)|^2 + 2e^{\eta t}(\varepsilon h(t), u_\varepsilon(t)) \\
  &\leq \eta e^{\eta t}|u_\varepsilon(t)|^2 + \nu e^{\eta t}||u_\varepsilon(t)||^2 + \frac{C_\varepsilon}{\nu} e^{\eta t}|h(t)|^2,
\end{align*}$$

(4.96)
Pullback dynamics of 3D Navier-Stokes equation with nonlinear viscosity

holds for all \( u_\varepsilon \in H \), then using the Poincaré inequality, choosing \( \eta = \frac{\nu \lambda_1}{2} \) and neglecting the third term in (4.96), we have
\[
\frac{d}{dt} \left( e^{\eta t} |u_\varepsilon(t)|^2 \right) + \frac{\nu \lambda_1}{2} e^{\eta t} |u_\varepsilon(t)|^2 \leq \frac{C \varepsilon}{\nu} e^{\eta t} |h(t)|^2,
\]
which implies
\[
|u_\varepsilon(t)|^2 \leq e^{-\eta(t-\tau)} \|u_0\|^2 + \frac{C \varepsilon}{\nu} \int_{\tau}^{t} e^{-\eta(t-\xi)} |h(\xi)|^2 d\xi,
\]
for all \( \tau \in \mathbb{R} \).

Let \( \hat{D} \in D_\eta \) be given above, then for any \( u_0 \in D(\tau) \) and \( t \geq \tau \), it yields
\[
|S_\varepsilon(t, t-\tau) u_{t-\tau}|^2 \leq e^{-\eta(t-\tau)} \frac{\nu^2}{\varepsilon} + \frac{C \varepsilon}{\nu} \int_{-\infty}^{t} e^{-\eta(t-\xi)} |h(\xi)|^2 d\xi,
\]
Setting \( e^{-\eta(t-\tau)} \frac{\nu^2}{\varepsilon} \leq \frac{C \varepsilon}{\nu} \int_{-\infty}^{t} e^{-\eta(t-\xi)} |h(\xi)|^2 d\xi \), then for each fixed \( t \in \mathbb{R} \), we denote \( R_\varepsilon(t) > 0 \) as
\[
(R_\varepsilon(t))^2 = \frac{2C \varepsilon}{\nu} \int_{-\infty}^{t} e^{-\eta(t-\xi)} |h(\xi)|^2 d\xi.
\]
Considering the family of closed balls \( \hat{B}_\varepsilon \) for any fixed \( t \geq \tau \) in \( H \) defined by
\[
B_\varepsilon(t) = \{ u_\varepsilon \in H \| u_\varepsilon \|^2 \leq 2R_\varepsilon^2(t) \},
\]
it is easily to check that \( B_\varepsilon(t) \in D_\eta \) and hence \( B_\eta(t) \) is the family of \( D_\eta \)-pullback absorbing sets for the process \( \{ S_\varepsilon(t, t-\tau) \} \). \( \square \)

**Lemma 4.25.** Let \( R_\varepsilon(t) \), \( B_\varepsilon(t) \) are defined in Lemma 4.24, then for any \( t \geq \tau \in \mathbb{R} \), the solution \( v(t) = S_{1,\varepsilon}(t, t-\tau) u(t-\tau) \) of (4.91) satisfies
\[
|S_{1,\varepsilon}(t, t-\tau) u_{t-\tau}|^2 \leq e^{-2\nu \lambda_1 \tau} R_\varepsilon^2(t-\tau),
\]
\[
\int_{t-\tau}^{t} \|v(s)\|^2 ds \leq J_\varepsilon(t)
\]
for all \( \tau \in \mathbb{R} \) and \( u_{t-\tau} \in B_\varepsilon(t-\tau) \), where \( J_\varepsilon(t) \) is dependent on \( \tau \), \( R_\varepsilon^2(t-\tau) \), \( \nu \) and \( \lambda_1 \).

**Proof.** Multiplying (4.91) with \( v \) and integrating by part over \( \Omega \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \nu \|v(t)\|^2 + \nu_0 \|v\|^4 \leq 0,
\]
here we use \( \langle B(v, v), v \rangle = 0 \). By using the Poincaré inequality, neglecting the third term in (4.103), it yields
\[
\frac{d}{dt} \|v(t)\|^2 + 2\nu \lambda_1 \|v(t)\|^2 \leq 0.
\]
Applying Gronwall’s inequality to (4.103) from \( t-\tau \) to \( t \), we get
\[
|S_{1,\varepsilon}(t, t-\tau) u_{t-\tau}|^2 \leq |u_{t-\tau}|^2 \leq e^{-2\nu \lambda_1 \tau} \leq e^{-2\nu \lambda_1 \tau} R_\varepsilon^2(t-\tau), \quad \forall \ t \geq \tau,
\]
(4.103) is the direct result of (4.106), this completes the proof. \( \square \)
Lemma 4.26. Let the family of pullback absorbing sets \( B_\varepsilon(t) = \{ B_\varepsilon(t) \}_{t \in \mathbb{R}} \) be given by Lemma 4.24 and (4.94) holds, then for any fixed \( t \geq \tau \in \mathbb{R} \), there exist a time \( T_\varepsilon(t, B) > 0 \) and a function \( I_\varepsilon(t) > 0 \), such that the solution \( S_{2,\varepsilon}(t, \tau)u_\tau = w(t) \) of (4.92) satisfies
\[
\| S_{2,\varepsilon}(t, \tau)u_{t-\tau} \|_{D(A_{\varepsilon}^\frac{3}{2})}^2 \leq I_\varepsilon(t),
\]
for all \( \tau \geq T_\varepsilon(t, B) \) and any \( u_{t-\tau} \in B_\varepsilon(t-\tau) \).

Proof. Taking inner product of (4.92) with \( A^\sigma w(t) \) in \( H \), integrating by parts over \( \Omega \), we derive
\[
\frac{1}{2} \frac{d}{dt} |A_{\varepsilon}^\frac{3}{2} w(t)|^2 + \nu |A^{\sigma+1} w(t)|^2 + \nu_0 \| w(t) \|^2 |A^{\sigma+1} w(t)|^2
= b(w, w, A^\sigma w) - b(w, v, A^\sigma w) - b(v, w, A^\sigma w) + \varepsilon \langle h(t), w \rangle. \tag{4.108}
\]
By the property of trilinear operator \( b(\cdot, \cdot, \cdot) \) and Young’s inequality, we obtain
\[
|b(w, w, A^\sigma w)| \leq \frac{C}{\nu} |A_{\varepsilon}^\frac{3}{2} w|^2 + \frac{\nu}{3} |A^{\sigma+1} w|^2 \tag{4.109}
\]
and
\[
|b(v, w, A^\sigma w)| \leq \frac{C}{\nu_0 \lambda^{2(1-\alpha)}} |v|^\frac{3}{2} + \nu_0 \| w \|^2 |A^{\sigma+1} w|^2 \tag{4.110}
\]
and
\[
|b(w, v, A^\sigma w)| \leq \frac{C}{\nu} \| v \|^2 + \frac{\nu}{3} |A^{\sigma+1} w|^2 \tag{4.111}
\]
and
\[
\langle \varepsilon h(t), A^\sigma w \rangle \leq \frac{\nu}{3} |A^{\sigma+1} w(t)|^2 + \frac{C \varepsilon^2}{\nu} |h(t)|^2. \tag{4.112}
\]
Hence combining (4.108)–(4.112), we derive
\[
\frac{d}{dt} |A_{\varepsilon}^\frac{3}{2} w(t)|^2 \leq \frac{C}{\nu} |A_{\varepsilon}^\frac{3}{2} w(t)|^2 + \frac{C}{\nu_0 \lambda^{2(1-\alpha)}} |v|^\frac{3}{2} + \frac{C}{\nu} \| v \|^2 + \frac{C \varepsilon^2}{\nu} |h(t)|^2. \tag{4.113}
\]
Applying the Gronwall inequality to (4.113) from \( t-\tau \) to \( t \), using Lemma 4.25, we conclude that
\[
A_{\varepsilon}^\frac{3}{2} w(t)|^2 \leq I_\varepsilon(t) = I_\varepsilon(t, \tau, R_\varepsilon(t-\tau), J_\varepsilon, \nu, \nu_0, \int_{-\infty}^{t} e^{\beta s} \| h(s) \|^2 ds \tag{4.114}
\]
for all \( t \geq \tau \). This achieve the proof of desiring lemma. \( \square \)

Lemma 4.27. For any fixed \( t \geq \tau \in \mathbb{R} \), if \( u_0 \) takes its value in some bounded set, then the solution \( u_\varepsilon(t) = S_\varepsilon(t, t-\tau)u_0 \) of perturbed non-autonomous problem \( (f(t) = \varepsilon h(t, x) \) of (1.7) \) converges to the solution \( u(t) = S(t)u_0 \) of the autonomous case \( (f(t) = 0 \) of (1.7) \) with \( \varepsilon = 0 \) uniformly in \( H \) as \( \varepsilon \to 0^+ \), which means
\[
\lim_{\varepsilon \to 0^+} \sup_{u_0 \in B} \| u_\varepsilon(t) - u(t) \|_H = 0, \tag{4.115}
\]
where \( B \) is a bounded subset in \( H \).
Proof. Denote
\[ y^\varepsilon(t) = u^\varepsilon(t) - u(t), \]
then we can verify that \( y^\varepsilon(t) \) satisfies the problem
\[
\begin{cases}
\frac{dy^\varepsilon}{dt} + \nu Ay^\varepsilon + Ay^\varepsilon = -B(u^\varepsilon, u^\varepsilon) + B(u, u) + \varepsilon h(t, x), \\
y^\varepsilon|_{\partial\Omega} = 0, \\
y^\varepsilon|_{t=\tau} = (u^\varepsilon)_{\tau} - u_{\tau} = 0.
\end{cases}
\]
Multiplying (4.117) by \( y^\varepsilon(t) \), using the property of \( b(\cdot, \cdot, \cdot) \), we have
\[
\frac{1}{2} \frac{d}{dt} |y^\varepsilon|^2 + \nu \|y^\varepsilon\|^2 + \nu_0 \|y^\varepsilon\|^4 \\
= \langle B(u, u) - B(u^\varepsilon, u^\varepsilon), y^\varepsilon \rangle + \langle \varepsilon h(t), y^\varepsilon \rangle \\
\leq \langle B(u, u) - B(u^\varepsilon, u^\varepsilon), y^\varepsilon \rangle + \frac{\nu}{2} \|y^\varepsilon(t)\|^2 + \frac{C\varepsilon^2}{\nu} |h(t)|^2.
\]
By Young’s inequality, noting that \( b(u^\varepsilon, y^\varepsilon, y^\varepsilon) = 0 \), we get
\[
|\langle B(u, u) - B(u^\varepsilon, u^\varepsilon), y^\varepsilon \rangle| = |b(y^\varepsilon, u, y^\varepsilon)| \leq C \|y^\varepsilon\|^{\frac{3}{2}} \|y^\varepsilon\|^{\frac{1}{2}} \|A^\frac{1}{3} u\|^{\frac{1}{2}} \\
\leq \frac{\nu}{2} \|y^\varepsilon\|^2 + \frac{C}{\nu} \|y^\varepsilon\|^2 \|A^\frac{1}{3} u\|^2.
\]
Hence, neglecting the third term in (4.118), it follows
\[
\frac{d}{dt} |y^\varepsilon|^2 \leq \frac{C}{\nu} \|y^\varepsilon\|^2 \|A^\frac{1}{3} u\|^2 + \frac{C\varepsilon^2}{\nu} |h(t)|^2.
\]
Using Theorem 3.3, Lemmas 4.24-4.27, and (3.35), noting that \( h(t, x) \in L^2_{loc}(\mathbb{R}, H) \), using the Gronwall inequality to (4.120), we conclude
\[
|y^\varepsilon|^2 \leq \frac{C\varepsilon^2}{\nu} e^{\int_{t-\tau}^t \frac{\nu}{\sqrt{2}} \|u\|^2_{L^2(\mathbb{R},T,D(A^\frac{1}{3}))} ds} \int_{t-\tau}^t \|h(s)\|^2_H ds \\
\leq \frac{C\varepsilon^2}{\nu} e^{\int_{t-\tau}^t \frac{\nu}{\sqrt{2}} \|u\|^2_{L^2(\mathbb{R},T,D(A^\frac{1}{3}))} ds} \|h(s)\|^2_{L^2_{loc}(\mathbb{R},H)} \leq C' \varepsilon \to 0
\]
as \( \varepsilon \to 0^+ \), which implies (4.115). The proof has finished.

\[ \square \]

Proof of Theorem 3.15: By the continuous theory of pullback attractors, we have prove the upper semi-continuity of pullback attractors in \( H \) via Lemma 4.27.

\[ \square \]

5. Conclusion and further research

From the discussion in this paper, we can see that the 3D Navier-Stokes equation with nonlinear viscosity (1.7) has better dissipative property than 3D classical model (1.1). One shortage is that (1.7) does not satisfy the Stokes principle. On the other side, since the well-posedness of 3D Navier-stokes equation is still open, we would like to study the long-time dynamics of a class of physically justified Ladyzhenskaya models (1.4) that satisfy the Stokes principle and are also well-posed.
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