A CHARACTERIZATION OF THE DELAUNAY SURFACES

EZEQUIEL BARBOSA AND L.C. SILVA

ABSTRACT. In this paper we use the Alexandrov Reflection Method to obtain a characterization to embedded CMC capillary annulus $\Sigma^2 \subset \mathbb{B}^3$. In especial, but using a new strategy, we present a new characterization to the critical catenoid. Precisely, we show that $\Sigma \subset \mathbb{B}^3$ being an embedded minimal free boundary annulus in $\mathbb{B}^3$ such that $\partial \Sigma$ is invariant under reflection through a coordinates planes, then $\Sigma$ is the critical catenoid.

1. INTRODUCTION

An important and well-know result in geometry, due to the Nitsche [2], state that the only minimal disks, free boundary in $\mathbb{B}^3$, are the equatorial flat discs. In the sense, in [3] we have the

**Conjecture 1.1** (Fraser and Li). The critical catenoid is the unique properly embedded free boundary minimal annulus in $\mathbb{B}^3$, up to rotations.

The question above is analogous to the question answered by Nitsche, the difference between the two questions is found in the topology of the surfaces - annular or disc. Philosophically, there exist a parallel between the conjecture (1.1) and

**Conjecture 1.2** (Lawson). The Clifford Torus is the only embedded minimal torus in $\mathbb{S}^3$, up to rotations.

The Lawson’s conjecture was resolved definitively by Brendle in [4]. However, there was previously a partial proof due to Ros, in [5]:

**Theorem 1.1** (Ros). Let $\Sigma \subset \mathbb{S}^3$ be an embedded minimal torus, symmetric with respect to the coordinate hyperplanes of $\mathbb{R}^4$. Then $\Sigma$ is the Clifford torus.

In the case of the conjecture (1.1), there is a result analogous to that obtained by Ros, due to McGrath [6]:

**Theorem 1.2** (McGrath). Let $\Sigma \subset \mathbb{B}^n$, $n \geq 3$, be an embedded free boundary minimal annulus. If $\Sigma$ is invariant under reflection through three hyperplanes, orthogonal to each other, $\Pi_i$, $i = 1, 2, 3$, then $\Sigma$ is the critical catenoid, up to rotation.

Is know well that, if $\Sigma$ is a minimal surface free boundary in $\mathbb{B}^3$, then it coordinate functions are solutions for the Steklov Problem

$$
\begin{cases}
\Delta u = 0, & \text{on } \Sigma, \\
\frac{\partial u}{\partial \eta} = u, & \text{along } \partial \Sigma.
\end{cases}
$$

In him proof, McGrath use the result below, that can be found in [7].
Theorem 1.3 (Fraser and Schoen). Suppose $\Sigma$ is a free boundary minimal annulus in $B^n$ such that the coordinate functions are first Steklov eigenfunctions. Then $n = 3$ and $\Sigma$ is congruent to the critical catenoid.

In this paper, we presented, in the case $n = 2$, an improvement for the McGrath Theorem, as consequence of the following result:

**Theorem 3.1** Let $\Sigma^2 \subset B^3$ be an embedded CMC capillary annulus, such that $\partial \Sigma$ is symmetrical with respect to the coordinated planes, then $\Sigma$ is a delanay surface.

This theorem makes significant contributions in comparison with the results found in the literature. In their hypotheses, we consider cmc capillary surfaces instead of free boundary minimal surfaces. Thus, with our work, we shed light on the path that leads to the answer to the following question:

**Question 1.1.** An embedded CMC capillary annulus $\Sigma^2 \subset B^3$ must be a delanay surface.

In the especial case, where $\Sigma^2 \subset B^3$ is an embedded free boundary minimal annulus, in Theorem 3.1, we have an improvement for McGrath’s Theorem:

**Corollary 3.1** Let $\Sigma^2 \subset B^3$ be an embedded free boundary minimal annulus. If $\partial \Sigma$ is symmetrical with respect to the coordinated planes, then $\Sigma$ is the critical catenoid.

Compared to McGrath’s results, we assume that $\partial \Sigma$ is invariant under reflection through three orthogonal hyperplanes, in contrast, he assumes such a propriety for $\Sigma$. Thus, when $n = 2$, the following corollary is an improvement that we give to McGrath’s theorem, in addition to using another strategy, namely, the Alexandrov Reflection Method (ARM), which we’ll talk more about in the next section. With this same methodology, we also present a new version, in the embedded case, of the proof from following result, that can be found in [9], due to Juncheo Pyo.

**Theorem 3.2 [Pyo]** Let $\Sigma^2$ be an embedded minimal surface in $\mathbb{R}^3$ with two boundary components and let $\Gamma$ be one component of $\partial \Sigma$. If $\Gamma$ is a circle and $\Sigma$ meets a plane along $\Gamma$ at a constant angle, then $\Sigma$ is part of the catenoid.

2. **Maximum Principles**

Let $A \subset \mathbb{R}^n$ an open set and

$$L(w) = \sum_{i,j} a_{ij}(x)w_{ij} + \sum_i b_i(x)w_i + c(x)w$$

where $w_i := \frac{\partial w}{\partial x_i}$, $w_{ij} := \frac{\partial^2 w}{\partial x_i \partial x_j}$ and the functions $a_{ij}$, $b_i$ and $c$ are continuous on $\bar{A}$, a differential elliptic operator on $A$, i.e., the matrix $[a_{ij}(x)]$ is positive definite for all $x \in A$, that is,

$$0 < \sum_{i,j=1}^n a_{ij} \xi_i \xi_j, \ \forall \ x \in A, \ \forall \ \xi \in \mathbb{R}^n \setminus \{0\}.$$
We called $L$ uniformly elliptic on $A$ if, there exist a constant $\kappa$ such that

\begin{equation}
\kappa|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j, \quad \forall \ x \in A, \forall \ \xi \in \mathbb{R}^n \setminus \{0\}.
\end{equation}

Now, we will present three maximum principles that we use during this work, especially in one step of the ARM, and can be found in [1]. The first of them, for points $x \in \text{int} A$:

**Lemma 2.1.** Let $L$ be an elliptic operator as in (2.2) and $w \in C^2(A)$ a function such that

\begin{equation}
L(w) \geq 0, \quad \text{on } A.
\end{equation}

If exist $x_0 \in A$ such that $w(x_0) = 0$ and $w \leq 0$ on $A$, then $w \equiv 0$ on $A$.

The second, for points $x \in \partial A$ such that $\partial A$ is of class $C^1$.

**Lemma 2.2.** Let $L$ be an uniformly elliptic operator as in (2.2), let $A$ be a region in $\mathbb{R}^2$ and suppose that in a neighborhood of $x_0 \in \partial A$, $\partial A$ is of class $C^1$. If

\begin{equation}
L(w) \geq 0, \quad \text{on } A,
\end{equation}

$w(x_0) = 0$, $w(x) \leq 0$, $\forall \ x \in \bar{A}$, and $\frac{\partial w}{\partial \nu} = 0$, where $\nu$ is the inward normal derivative, then $w \equiv 0$, on $A$.

Finally, the third, for points $x \in \partial A$, at a corner.

**Lemma 2.3** (Serrin’s Boundary Point Lemma at a Corner [10]). Let $A \subset \mathbb{R}^2$ be a bounded region which has a $C^2$ boundary in a neighborhood of $x_0 \in \partial A$. Consider $T$ be a normal plane to $\partial A$ at $x_0$ and $A^+$ be that component of $A$ lying on one side of $T$ which contains $x_0$ in its closure. Let $L$ be an uniformly elliptic operator on $A^+$. Suppose also that

\begin{equation}
|\sum_{i,j} a_{ij}(x) \xi_i \nu_j| \leq K \cdot ||(\xi \cdot \nu)|| + |\xi|d
\end{equation}

for some constant $K > 0$, all $x \in \bar{A}^+$, any $\xi = (\xi_1, \ldots, \xi_n)$, where $\nu = (\nu_1, \ldots, \nu_n)$ is an unit normal to $T$, and where $d$ is the distance from $x$ to $T$.

Let $w \in C^2(\bar{A}^+)$ satisfy $L(w) \geq 0$ on $\bar{A}^+$ and suppose that $w(x_0) = 0$, $w(x) \leq 0$, for all $x \in \bar{A}^+$, and that $\partial w/\partial s = \partial^2 w/\partial s^2 = 0$, in any direction which enters $A^+$ non-tangentially at $x_0$.

Now, as an application of maximum principles above, we presented the steps of ARM.

1. Consider a subsidiary plane $P$ and an arbitrary family, $P_\lambda$, $\lambda \in \mathbb{R}$, of parallel planes with each other, and orthogonal to $P$.
2. Varying the parameter $\lambda$, a moving planes process is started by means of family $P_\lambda$. For some $\lambda \in \mathbb{R}$, $P_\lambda \cap \Sigma \neq \emptyset$ and can be considered the reflection, through $P_\lambda$, of the part of $\Sigma$ surpassed by $P_\lambda$.
3. For a critical parameter, $\lambda^*$, it is considered the reflection through $P_{\lambda^*}$ of the part of $\Sigma$ surpassed by $P_{\lambda^*}$, see Figure [1].
4. Considering an appropriate coordinated system, we use an appropriate maximum principle and it is concluded that the reflection, through $P_{\lambda^*}$, of the part of $\Sigma$ surpassed by $P_{\lambda^*}$ coincide, locally, with the part of $\Sigma$ non surpassed by $P_{\lambda^*}$.
(5) The single continuation principle is used and it is concluded that the reflection, through $P_{\lambda^*}$, of the part of $\Sigma$ surpassed by $P_{\lambda^*}$ coincide with the part of $\Sigma$ non surpassed by $P_{\lambda^*}$.

(6) Finally, from arbitrariness of $P_{\lambda}$, it is concluded that $\Sigma$ is symmetrical rotationally.

For more examples of this method see [1] and [8]. A natural question around the steps above:

**Question 2.1.** How to determine the critical parameter $\lambda^*$?

Consider

$$\Lambda \text{ the region bounded by } C_+ \cup \Sigma \cup C_- \subset \mathbb{B}^3,$$

(2.8)

where $C^+$ is the upper portion of $\mathbb{S}^2$ such that $\partial C^+ = \Gamma$; $C^-$ is the lower portion of $\mathbb{S}^2$ such that $\partial C^- = \Gamma'$. As $\Sigma$ is embedded, $\Lambda$ is connected (Figure 2).
Define
\[ \lambda^- = \min \{ \lambda \in \mathbb{R} : P_\lambda \cap \Sigma \neq \emptyset \} , \]
(2.9)
\[ \lambda^+ = \max \{ \lambda \in \mathbb{R} : P_\lambda \cap \Sigma \neq \emptyset \} , \]
(2.10)
and observe that, as \( \Sigma \subset \mathbb{B}^3 \), so \(-1 \leq \lambda^- < \lambda^+ \leq 1 \). To better organize the text, consider the following definition:
(i): \( \Sigma_\lambda \) being the part of \( \Sigma \) between \( P_{\lambda^-} \) and \( P_{\lambda} \), \( \lambda \in (\lambda^-, \lambda^+) \) is that, the part of \( \Sigma \) surpassed by \( P_\lambda \);
(ii): \( \tilde{\Sigma}_\lambda \) being the reflection of \( \Sigma_\lambda \) through \( P_\lambda \);
(iii): \( \Sigma \setminus \Sigma_\lambda \) being the part of \( \Sigma \) between \( P_{\lambda} \) and \( P_{\lambda^+} \), \( \lambda \in (\lambda^-, \lambda^+) \), is that, the part of \( \Sigma \) non surpassed by \( P_\lambda \).

For some value of parameter \( \lambda \), called \( \lambda^* \), we say that the reflected part, \( \tilde{\Sigma}_\lambda \), definitely extrapolates \( \Lambda \) if,
\[ \exists \, x^* \in \tilde{\Sigma}_\lambda : x^* + \mu \cdot N_{\lambda} \notin \Lambda, \forall \mu > 0 , \]
(2.11)
where \( N_{\lambda} \) is the unit normal vector to family \( P_\lambda \), pointing in the sense of increasing \( \lambda \). Thus, it is defined the critical parameter of moving planes process with respect to family \( P_\lambda \), see Figure 3. There are the following possibilities for this extrapolation:

\begin{itemize}
  \item [(P1)] At a point \( x^* \) on \( \text{int}(\tilde{\Sigma}_{\lambda^*}) \cap \text{int}(\Sigma \setminus \Sigma_{\lambda^*}) \).
  \item [(P2)] At a point \( x^* \in \partial \tilde{\Sigma}_{\lambda^*} \cap \partial (\Sigma \setminus \Sigma_{\lambda^*}) \).
  \item [(P3)] At a point \( x^* \) such that \( T_{x^*} \Sigma \perp P_{\lambda^*} \).
  \item [(P4)] At a point \( x^* \) such that \( T_{x^*} \partial \Sigma \perp P_{\lambda^*} \).
\end{itemize}

Note that, we should not worry with the possibility of \( \tilde{\Sigma}_{\theta,\lambda} \) definitely extrapolates \( \Lambda \) by \( C^+ \) or \( C^- \) and also in the possibility of a point along \( \partial \Sigma_{\theta,\lambda} \) definitively extrapolates \( \Lambda \), at a point \( p \in \text{int} \Sigma \setminus \Sigma_{\lambda^*} \) because \( \partial \Sigma \) is invariant under reflection through of the three coordinated hyperplanes and due to spherical geometry (this is another relevance of the \( \partial \Sigma \) symmetry).

\[ \text{Figure 3. The moving planes process in two moments, } \lambda \text{ and } \lambda^*. \]

**Observation 2.1.** We created and adopted the concept definitely extrapolates instead of touching, the latter already existing in the literature, to avoid the possibility of the touch occurring at the intersection of a boundary point of \( \Sigma_\lambda \) with an interior point of \( \Sigma \setminus \Sigma_\lambda \).

Once the Question 2.1 has been answered, we can present our results.
3. The Proof of the Theorem 3.1

Let \( \Sigma \subset \mathbb{B}^3 \) an embedded free boundary minimal annulus such that

\begin{align}
\text{(3.12)} & \quad \text{int}(\Sigma) \subset \text{int}(\mathbb{B}^3) \\
\text{(3.13)} & \quad \partial\Sigma = \Gamma \cup \Gamma' \subset \partial\mathbb{B}^3
\end{align}

where \( \Gamma \) and \( \Gamma' \) are the connected components of the boundary of \( \Sigma \). Consider a hyperplane \( \Pi \subset \mathbb{R}^3 \) and \( R_{\Pi} \) the map such that \( R_{\Pi}(x) \) is the orthogonal reflection of \( x \) through \( \Pi \). If \( R_{\Pi}(\Sigma) = \Sigma \), we say that \( \Sigma \) is \( \Pi \)-invariant. Note that, the map \( R_{\Pi} : \Sigma \to \Sigma \) is an isometry such that \( \partial\Sigma \mapsto \partial\Sigma \) and \( \text{int}(\Sigma) \mapsto \text{int}(\Sigma) \). From now on, consider

\begin{equation}
\Pi_i := \{(x_1, x_2, x_3) \mid x_i = 0\}, \tag{3.14}
\end{equation}

Let \( G = \{R_{\Pi_1}, R_{\Pi_2}, R_{\Pi_3}\} \) be the group of the reflection with respect to the coordinate planes. We say that \( \Sigma \) is \( G \)-invariant if, \( R_{\Pi_i}(\Sigma) = \Sigma \), for all \( i \in \{1, 2, 3\} \). In this paper, we consider \( \partial\Sigma = \Gamma \cup \Gamma' \) \( G \)-invariant.

In this moment, we prove that the \( G \)-invariant propriety of \( \partial\Sigma \) implies that it intersects the interior of each of the eight octants and there exist a plane such that \( \Gamma' \) is the reflection of \( \Gamma \) through it.

**Lemma 3.1.** Let \( \Sigma^2 \subset \mathbb{B}^3 \) an embedded annulus such that \( \partial\Sigma \) is \( G \)-invariant. Then,

\begin{enumerate}
\item[(i)] there exist \( i, j \in \{1, 2, 3\}, i \neq j \), such that
\begin{align}
\Gamma &= R_{\Pi_i}(\Gamma) = R_{\Pi_i}(\Gamma), \\
\Gamma' &= R_{\Pi_j}(\Gamma') = R_{\Pi_j}(\Gamma'),
\end{align}
and
\item[(iii)] there exists \( k \in \{1, 2, 3\}, k \notin \{i, j\} \), such that
\begin{equation}
\Gamma' = R_{\Pi_k}(\Gamma) \quad \text{and} \quad \Gamma = R_{\Pi_k}(\Gamma') \tag{3.17}
\end{equation}
\end{enumerate}

**Proof of Lemma 3.1.** Let \( \partial\Sigma = \Gamma \cup \Gamma' \), where \( \Gamma \) and \( \Gamma' \) are the connected components of the boundary of \( \Sigma \), and

\begin{equation}
\Gamma \cap \mathcal{O} := \gamma : [0, 1] \to \partial\Sigma, \tag{3.18}
\end{equation}

where \( \mathcal{O} = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1, x_2, x_3 \geq 0\} \), the part of \( \Gamma \) contained in the first octant. As \( \partial\Sigma \) is \( G \)-invariant, we can find it by putting together the possible reflections of \( \gamma \), i.e., there are \( i, j, k \in \{1, 2, 3\} \), different from each other, such that

\begin{equation}
\partial\Sigma = \tilde{\gamma} \cup R_{\Pi_k}(\tilde{\gamma}) \tag{3.19}
\end{equation}

where

\begin{equation}
\tilde{\gamma} := \gamma \cup R_{\Pi_i}(\gamma) \cup R_{\Pi_j}(\gamma) \cup (R_{\Pi_i} \circ R_{\Pi_j})(\gamma) \subset \partial\Sigma \tag{3.20}
\end{equation}

As \( \Sigma \) is embedded, \( \gamma(0, 1) \) doesn’t intersect \( \Pi_i \), \( i \in \{1, 2, 3\} \), since \( \partial\Sigma \) is \( G \)-invariant, otherwise \( \Gamma \) would have self intersections. Once \( \Gamma \) and \( \Gamma' \) are closed curves, \( \gamma(0) \in \Pi_i \) and \( \gamma(1) \in \Pi_{j'}, i \neq j \), because \( \partial\Sigma \) is \( G \)-invariant. Indeed, if \( \gamma(1) \notin \Pi_{j'} \), then there exist \( p_j \in \Pi_j \) such that \( d(\gamma(1), p_j) = d > 0 \) and follows from \( (3.19) \) and \( (3.20) \) that, \( \partial\Sigma \) would not be the union of closed curves and this would be a contradiction, see Figure 4.
In the other hand, if $\gamma(0), \gamma(1) \in \Pi_i = \Pi_j$, see Figure 5, then the curve $\gamma \cup R_{\Pi_i}(\gamma) := \beta : [a, b] \rightarrow \partial \Sigma$ would be a closed curve contained in $\partial \Sigma$. Thus, the curves $\beta, R_{\Pi_i}(\beta), R_{\Pi_k}(\beta)$ and $(R_{\Pi_i} \circ R_{\Pi_k})(\beta)$ would be closed curves contained in $\partial \Sigma$, but this is also a contradiction, since $\Sigma$ is a topological annulus.

Then, if we define $\Gamma := \tilde{\gamma}$, we have (i) and (ii).

Let $T_\lambda, \lambda \in \mathbb{R}$, be a family of planes parallel to each other, where there is a relationship 1-1 between $\lambda \in \mathbb{R}$ and each plane $T \in T_\lambda$. We call moving planes, the process of varying the parameter $\lambda$, from the geometric viewpoint, we have a movement between this parallel planes.

**Theorem 3.1.** Let $\Sigma^2 \subset \mathbb{B}^3$ be an embedded CMC capillary annulus, such that $\partial \Sigma$ is symmetrical with respect to the coordinated planes, then $\Sigma$ is a delaunay surface.

**Proof of Theorem 3.1.** Let $\Sigma \subset \mathbb{B}^3$ an embedded CMC capillary annulus such that $\partial \Sigma = \Gamma \cup \Gamma'$ is $G$-invariant and

\begin{align}
(3.21) & \quad \text{int}(\Sigma) \subset \text{int}(\mathbb{B}^3) \\
(3.22) & \quad \partial \Sigma = \Gamma \cup \Gamma' \subset \partial \mathbb{B}^3
\end{align}
As $\partial \Sigma$ is $G$-invariant, follows from Lemma 3.1 that there exists a coordinated plane, without loss of generality, let’s say $\Pi_3$, such that

\begin{equation}
\Gamma' = R_{\Pi_3}(\Gamma) \quad \text{and} \quad \Gamma = R_{\Pi_3}(\Gamma')
\end{equation}

and

\begin{equation}
\Gamma = R_{\Pi_1}(\Gamma) = R_{\Pi_2}(\Gamma) \quad \text{and} \quad \Gamma' = R_{\Pi_1}(\Gamma') = R_{\Pi_2}(\Gamma')
\end{equation}

Consider the family $T_{\theta}$ of orthogonal planes to $\Pi_3$ and parallels to each other, such that

\begin{equation}
T_{\theta, \lambda} = \{(x_1, x_2, x_3) \in \mathbb{R}^3; \cos \theta \cdot x_1 + \sin \theta \cdot x_2 = \lambda\} \in T_{\theta},
\end{equation}

where $\theta \in [0, \pi)$ and $\lambda \in \mathbb{R}$. Let

\begin{equation}
\lambda^- = \min \{\lambda \in \mathbb{R} ; T_{\theta, \lambda} \cap \Sigma \neq \emptyset\},
\end{equation}

\begin{equation}
\lambda^+ = \max \{\lambda \in \mathbb{R} ; T_{\theta, \lambda} \cap \Sigma \neq \emptyset\},
\end{equation}

Define $\Sigma_{\theta, \lambda}$ as the part of $\Sigma$ between $T_{\theta, \lambda^-}$ and $T_{\theta, \lambda}$, $\lambda \in (\lambda^- \, , \lambda^+)$, precisely

\begin{equation}
\Sigma_{\theta, \lambda} := \{x \in \Sigma ; \lambda^- \leq \cos \theta \cdot x_1 + \sin \theta \cdot x_2 \leq \lambda\}.
\end{equation}

We will call $\bar{\Sigma}_{\theta, \lambda}$ the reflection of $\Sigma_{\theta, \lambda}$ through $T_{\theta, \lambda}$, i.e.,

\begin{equation}
\bar{\Sigma}_{\theta, \lambda} := \{x \in \Sigma ; \lambda \leq \cos \theta \cdot x_1 + \sin \theta \cdot x_2 \leq \lambda + \lambda^-\},
\end{equation}

and $\Sigma \setminus \Sigma_{\theta, \lambda}$ the part of $\Sigma$ between $T_{\theta, \lambda}$ and $T_{\theta, \lambda^+}$, $\lambda \in (\lambda^- \, , \lambda^+)$, see Figure 7.

![Figure 7. $\Sigma_{\theta, \lambda}$, $\bar{\Sigma}_{\theta, \lambda}$ and $\Sigma \setminus \Sigma_{\theta, \lambda}$](image)

For example, note that for $\theta = \lambda = 0$, $T_{0,0} = \Pi_1$ and for $\theta = \frac{\pi}{2}$ and $\lambda = 0$, $T_{\frac{\pi}{2},0} = \Pi_2$. In these cases, $\partial \Sigma_{\theta, \lambda} = \partial (\Sigma \setminus \Sigma_{\theta, \lambda})$, see Figure 8.

Let $x^*$ the extrapolation point, for some of the possibilities (P1) ~ (P4). Consider a coordinate system such that $x^* = (0, 0, 0)$ and smooth functions $u, v : \bar{A} \to \mathbb{R}$, where $A \subset \mathbb{R}^2$ is an open set and $(0, 0) \in \bar{A}$, such that

\begin{equation}
u(0, 0) = 0
\end{equation}
and $\Sigma \setminus \Sigma_{\theta,\lambda^*_0} = \text{graph}(u)$ and $\tilde{\Sigma}_{\theta,\lambda^*_0} = \text{graph}(v)$ in a neighborhood of $(0,0)$. Note that, as $\Sigma \setminus \Sigma_{\theta,\lambda}$ and $\tilde{\Sigma}_{\theta,\lambda^*_0}$ are CMC (for the same constant), $u$ and $v$ satisfy the same CMC equation. Hence, the function $w = v - u$ satisfy a homogeneous linear elliptic pde, see [1].

In the possibility (P1), define a coordinate system such that $T_x \cdot \Sigma = \{z = 0\}$, where the axis $z$ pointing to $T_{\theta,\lambda^*_0}$ and use the Lemma 2.2 to conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\tilde{\Sigma}_{\theta,\lambda^*_0} = \Sigma \setminus \Sigma_{\theta,\lambda^*_0}$ in a neighborhood of $x^*$.

In (P2), define a coordinate system such that $T_x \cdot \partial \Sigma = \{x = z = 0\}$ and $T_{\nu} \cdot \Sigma = \{z = 0\}$, where the axis $z$ pointing to $T_{\theta,\lambda^*_0}$ and axis $x$ point to int$\Lambda$. So, use the Lemma 2.2 to conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\tilde{\Sigma}_{\theta,\lambda^*_0} = \Sigma \setminus \Sigma_{\theta,\lambda^*_0}$ in a neighborhood of $x^*$.

For the case (P3), define a coordinate system such that $T_x \cdot \Sigma = \{z = 0\}$, the plane $T_{\theta,\lambda^*_0}$ coincide with $\{x = 0\}$, where the axis $z$ pointing to $T_{\theta,\lambda^*_0}$ and axis $x$ point to int$\Lambda$. So, use the Lemma 2.2 to conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\tilde{\Sigma}_{\theta,\lambda^*_0} = \Sigma \setminus \Sigma_{\theta,\lambda^*_0}$ in a neighborhood of $x^*$.

In (P4), define a coordinate system such that $x^*$ is the origin of a coordinate system $(x, y, z)$, $T_x \cdot \partial \Sigma = \{x = z = 0\}$ and $T_{\nu} \cdot \Sigma = \{z = 0\}$, where the axis $z$ points into $\Sigma$ and axis $x$ pointing to $\Sigma_{\theta,\lambda^*_0}$. So, use the Lemma 2.3 and the capillarity of $\Sigma$ for conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\tilde{\Sigma}_{\theta,\lambda^*_0} = \Sigma \setminus \Sigma_{\theta,\lambda^*_0}$ in a neighborhood of $x^*$.

Using the unique continuation we concluded that $\tilde{\Sigma}_{\theta,\lambda^*_0} = \Sigma \setminus \Sigma_{\theta,\lambda^*_0}$.

**Affirmation:** $\lambda^*_0 = 0$, $\forall \theta \in [0, \pi)$.

Indeed, suppose absurdly, that $\lambda^*_0 < 0$.

Hereafter, as $\partial \Sigma_{\theta,\lambda^*_0} \subset S^2$, then $\Sigma$ is a surface whose boundary satisfy

\[(3.31) \quad \partial \Sigma = \partial \Sigma_{\theta,\lambda^*_0} \cup \partial \tilde{\Sigma}_{\theta,\lambda^*_0},\]

i.e., $\partial \Sigma$ does not contained in $S^2$. Contradiction, because $\phi$ is admissible! Then, the affirmation is true.

Finally, as $\lambda^*_0 = 0$ and $\tilde{\Sigma}_{\theta,0} = \Sigma \setminus \Sigma_{\theta,0}$, $\forall \theta \in [0, \pi)$, because $\theta$ was taken arbitrarily, if $\Pi$ is a plane parallel to $\Pi_0$, the straight line $r_0 := \Pi \cap T_{\theta,0}$ intersects orthogonally $\Sigma \cap \Pi$, for all $\theta \in [0, \pi)$. Besides that, as $\lambda^*_0 = 0$, $\forall \theta \in [0, \pi)$, all these
straight lines intersects each other at point \( p_0 \in \Pi \cap \text{axis } x_3 \), \( \forall \theta \in [0, \pi) \), i.e., \( \Sigma \cap \Pi \) is a circle.

Therefore, as \( \theta \) was taken arbitrarily, \( \Sigma \) is symmetrical rotationally.

Observation 3.1. Follows from above affirmation that, for example, (P3) does not occur for \( \lambda < 0 \). So, the curve defined by intersection \( T^\perp_\theta \cap \Sigma \), where \( T^\perp_\theta \) is the plane containing the origin and orthogonal to \( \Pi_3 \) and \( T_\theta \), can be represented, globally, by the graph of a smooth function \( f(z) \), where \( z \in I \subset T^\perp_\theta \cap T_\theta \), see Figure 9. Thus, not exist the possibility of a boundary point of \( \bar{\Sigma}_\lambda \) intersects \( \Sigma \setminus \Sigma_\lambda \), i.e., we could considered the concept touching from the start.

![Figure 9](image_url)

Figure 9. As (P3) does not occur for \( \lambda < 0 \), the circled part doesn’t occur either.

In Theorem 1.2, McGrath assume that an embedded minimal free boundary annulus, \( \Sigma \subset \mathbb{B}^n \), is \( G \)-invariant, to prove that \( \Sigma \) is the critical catenoid. From Theorem 3.1, we improved the result of McGrath [6], to \( n = 2 \), because we assume only that \( \partial \Sigma \) is \( G \)-invariant.

Corollary 3.1. Let \( \Sigma^2 \subset \mathbb{B}^3 \) be an embedded minimal free boundary annulus. If \( \partial \Sigma \) is \( G \)-invariant, then \( \Sigma \) is the critical catenoid.

Proof of Corollary 3.1. Follows directly of proof from Theorem 3.1 and of fact that the critical catenoid is the only minimal surface rotationally symmetric free boundary in \( \mathbb{B}^3 \).

With this methodology, we also get a new demonstration for

Theorem 3.2 (Pyo). Let \( \Sigma^2 \) be an embedded minimal surface in \( \mathbb{R}^3 \) with two boundary components and let \( \Gamma \) be one component of \( \partial \Sigma \). If \( \Gamma \) is a circle and \( \Sigma \) meets a plane along \( \Gamma \) at a constant angle, then \( \Sigma \) is part of the catenoid.
Proof of Theorem 3.2. Let $\Pi$ the plane that contain $\Gamma$ and $\mathcal{T}_\theta$ be a family of parallel planes with each other and orthogonal to $\Pi$.

As $\Gamma$ is a circle, during the moving plane process, for some value of the parameter $\lambda$, $\tilde{\Sigma}_{\theta, \lambda}$ definitely extrapolates $\Sigma \setminus \tilde{\Sigma}_{\theta, \lambda}$, of some of the forms $(P1) \sim (P4)$. As we have no information about $\Gamma'$, another component connected of $\partial \Sigma$, we cannot say anything about the occurrence of the cases $(P2)$ and $(P4)$. If the cases $(P1)$ or $(P3)$ occur for some $\lambda^*_\theta \leq 0$, we have by ARM that

$$\tilde{\Sigma}_{\theta, \lambda^*_\theta} \text{ coincide to } \Sigma \setminus \tilde{\Sigma}_{\theta, \lambda^*_\theta}$$

and

$$\Gamma = \Gamma_{\theta, \lambda^*_\theta} \cup \tilde{\Gamma}_{\theta, \lambda^*_\theta},$$

But since $\Gamma$ is a circle, it follows of the circular geometry of $\Gamma$, which $\lambda^*_\theta = 0$.

Consider $r$ the orthogonal straight line to $\mathcal{T}_{\theta, 0}$ passing through the center of $\Gamma$ and let $p$ the point given by the intersection between $\Gamma$, $r$ and $\Sigma \setminus \tilde{\Sigma}_{\theta, 0}$. Once $\Gamma$ is a circle and $(P1)$ and $(P3)$ do not occur for $\lambda < 0$, we have $\tilde{\Sigma}_{\theta, 0}$ stays above $\Sigma \setminus \tilde{\Sigma}_{\theta, 0}$, relative to $\nu$, normal vector for $\tilde{\Sigma}_{\theta, 0}$ at the point $p$.

Consider a coordinated system such that $\{x_3 = 0\} = T_r \tilde{\Sigma}_{\theta, 0}$. Thus, using the ARM, the unique continuation principle, the arbitrariness in choosing $\theta$, as well as in the proof of Theorem 3.1, we conclude that $\Sigma$ is the critical catenoid.

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