Abstract

It is developed a systematic approach to contact and Jacobi structures on graded supermanifolds. In this framework, contact structures are interpreted as symplectic principal $\mathbb{R}^\times$-bundles. Gradings compatible with the $\mathbb{R}^\times$-action lead to the concept of a graded contact manifold, in particular a linear contact structure. Linear contact structures are proven to be exactly the canonical contact structures on first jets of line bundles. They give rise to linear Kirillov (or Jacobi) brackets and the concept of a principal Lie algebroid, a contact analog of a Lie algebroid. The corresponding cohomology operator is represented not by a vector field (a de Rham derivative) but a first-order differential operator. It is shown that one can view Kirillov or Jacobi brackets as homological Hamiltonians on linear contact manifolds. Contact manifolds of degree 2 are also studied as well as contact analogs of Courant algebroids. Lifting procedures to tangent and cotangent bundles are described and provide constructions of canonical examples of the structures in question.

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1 Introduction

The role played in geometry by symplectic structures cannot be overestimated. The theory of symplectic and Poisson manifolds, together with the theory of (pseudo)Riemannian ones, are two legs of the contemporary differential geometry. It is not only because of the fact that each other really important geometric structure has its symplectic origins, but also because of the prominent position of symplectic forms and Poisson brackets in physics, especially in classical mechanics, quantum field theory, and quantization. Actually, supergeometric versions of symplectic structures appeared first in the physics literature before being recognized by mathematicians. In particular, graded symplectic supermanifolds have been known to physicists since the 1970s, providing framework for the so-called BRST formalism. Its main geometrical background consists of an odd symplectic structure such that the fields and their respective anti-fields are conjugate with respect to the corresponding odd Poisson bracket $(\cdot, \cdot)$ (the ‘antibracket’), and an action functional $S$ obeying what is called the classical master equation $\{S, S\} = 0$.

In the early 1980s Batalin and Vilkovisky [8] developed a generalization of the BRST procedure in terms of so called Batalin-Vikovisky algebras which allows us, in principle, to handle symmetries of arbitrary complexity, while a detailed mathematical study of the classical BRST algebra and its quantization was undertaken in the late 1980s by Kostant and Sternberg [39] who related the BRST cohomology to the Lie algebra cohomology.

In mid 1990s, in turn, Alexandrov, Kontsevich, Schwarz, and Zaboronsky [1] found a simple and elegant procedure (referred to as the AKSZ formalism now) for constructing solutions to the classical master equation. Their approach uses mapping spaces of supermanifolds equipped with a compatible integer grading and a differential (see also [56]).

On the other hand, in 1986 Courant and Weinstein [11, 10] developed the concept of a Dirac structure providing a geometric setting for Dirac’s theory of constrained mechanical systems which was...
then algebraically generalized and applied to integrable systems by Dorfman [14]. The integrability condition for a Dirac structure makes use of a natural bracket operation on sections of the Pontryagin bundle $\mathcal{T}M = \mathcal{T}M \oplus_M \mathcal{T}^*M$, the direct sum of the tangent and the cotangent bundles, known nowadays as the Courant bracket. The Courant bracket, originally skew-symmetric, does not satisfy the Leibniz rule with respect to the multiplication by functions nor the Jacobi identity. Its non-symmetric version we will use in the sequel, going back to Dorfman, fulfills the Jacobi identity for the price of skew-symmetry. The nature of the Courant bracket itself remained unclear until several years later when it was observed by Liu, Weinstein and Xu [43] that $\mathcal{T}M$ endowed with the Courant bracket plays the role of a ‘double object’ in the sense of Drinfeld [15] for a pair of Lie algebroids (see [47]) over $M$. Let us recall that, in complete analogy with Drinfeld’s Lie bialgebras, in the category of Lie algebroids there also exist ‘bi-objects’, Lie bialgebroids, introduced by Mackenzie and Xu [47] as linearizations of Poisson groupoids. It is well known that every Lie bialgebra has a double which is a Lie algebra that is not true for a general Lie bialgebroid. Instead, Liu, Weinstein and Xu showed that the double of a Lie bialgebroid is a more complicated structure they call a Courant algebroid, the Pontryagin bundle $\mathcal{T}M$ with the Courant bracket being a special case.

Only recently become clear (see [54, 55]) that, again, any Courant algebroid is actually a certain Hamiltonian system on a graded symplectic supermanifold $\mathcal{M}$. The standard Courant algebroid $\mathcal{T}M = \mathcal{T}M \oplus_M \mathcal{T}^*M$ corresponds in this way to the symplectic supermanifold $\mathcal{M} = \mathcal{T}^*\Pi\mathcal{M}$ with the Hamiltonian associated with the de Rham vector field $d$ on $\Pi\mathcal{M}$. Note that all the theory of Lie algebroids or, equivalently, linear Poisson structures can be put into the framework of Courant algebroids.

Our aim in this paper is to develop and study contact versions of these formalisms, therefore including contact structures on graded supermanifolds. This task is slightly more complicated than its purely symplectic variant. First, because the standard definition of a contact form as a 1-form $\alpha$ on an $(2n+1)$-dimensional manifold satisfying $\alpha \wedge (d\alpha)^n \neq 0$ clearly does not make sense in the context of supergeometry, so one has to find a proper and nice substitute which works in general.

Second, canonical contact structures playing the role of the cotangent bundle $T^{\ast}\mathcal{M}$ in the symplectic case turn out to be associated not with the trivial bundles $\mathbb{R} \times \mathcal{M}$ but with, generally nontrivial, line bundles $L \to \mathcal{M}$ that introduces an additional complexity. As a result, we must work not with derivations, vector fields, and Poisson brackets, but with first-order differential operators acting on sections of line bundles and Jacobi brackets [12, 13] or, better to say, Kirillov brackets, as Kirillov was the first who studied local Lie brackets on sections of line bundles [33].

Our understanding of a supermanifold will be standard, i.e., we will use an atlas of local coordinates, some of which are even and some of which are odd. For details we refer to [13, 53, 62, 65] and references therein. In this paper we will work also with graded manifolds which are supermanifolds with an additional (usually $\mathbb{Z}^n$ or $\mathbb{N}^n$) gradation in the structure sheaf, so that we can choose local coordinates which have certain weights besides the parity. In such cases there is often a confusion about the relation between the additional grading (weight) and the $\mathbb{Z}_2$-grading (parity) responsible for the sign rule. We will follow the point of view of Voronov [67] that the grading (weight) is not directly related to parity, unless explicitly assumed, and work mainly with $\mathbb{N}^n$-gradings. Some information about graded manifolds one can find in first chapters of the book [62] and, if the $\mathbb{Z}$-grading is concerned, in the thesis of Mehta [48]. Certain important particular cases can be also found in the works of Kontsevich [34], Roytenberg [54, 56], Ševera [60], Voronov [67, 68], and Grabowski and Rotkiewicz [24, 29]. Since we will not develop a general theory of graded manifolds, we will only point out main subtleties that differ graded manifolds from standard manifolds, on one hand, and from pure supermanifolds, on the other.

In a standard description, a contact structure on a manifold $M$ is viewed as certain ‘maximally non-integrable’ one-codimensional subbundle in the tangent bundle $TM$. This subbundle is locally the kernel of a 1-form on a manifold $M$ which induces a rank-one subbundle in $T^*M$. Two such forms are viewed equivalent if they have the same kernel, i.e., they differ by a factor which is an invertible function. On the other hand, with every contact or, more generally, Jacobi structure one associates its symplectification (resp., Poissonization) [43, 20]. In our approach we will not really distinguish between the symplectification which is a certain symplectic submanifold in the (canonically symplectic) cotangent bundle $T^*M$ and the contact structure itself.

Viewing contact structures as particular symplectic manifolds is very convenient in the context of the graded (super)geometry in which we will work. Actually, the symplectification which is an (even) contact structure on a (super)manifold $\mathcal{M}$ will be understood as a principal $\mathbb{R}^\times$-bundle $P$ over $\mathcal{M}$ equipped with an (even) symplectic form which is homogeneous with respect to the $\mathbb{R}^\times$-action.
Such an object we will call a symplectic principal $\mathbb{R}^x$-bundle. The corresponding Legendre bracket is a Kirillov bracket on sections of the line bundle $L^*$, dual to the line bundle $L$ associated with $P$. We simply interpret sections of $L^*$ as 1-homogeneous functions on $P$ and use the fact that they are closed with respect to the symplectic Poisson bracket. In the case of an even contact form $\alpha$ on a manifold $M$, this principal $\mathbb{R}^x$-bundle $P$ is just the one-dimensional subbundle $(\alpha)$ in $T^*M$ generated by $\alpha$ with the $0$-section removed. We consider as well odd contact structures but we will remain with the even case in the introduction not to produce an additional complexity. Of course, we can consider as well homogeneous Poisson structures on $P$ which give rise to principal Poisson $\mathbb{R}^x$-bundles. In any case, the corresponding Kirillov brackets are, like the Legendre brackets for contact structures, restrictions of principal Poisson brackets to 1-homogeneous functions.

The main observation in this context is that the canonical linear contact structure on the first jet bundle $J^1L$ of a line bundle $L \to M$ can be identified with the cotangent bundle $T^*(L^*)^x$ of the principal $\mathbb{R}^x$-bundle $(L^*)^x$ associated with the dual $L^*$. The cotangent bundle $T^*(L^*)^x$ has a canonical symplectic embedding into a line subbundle $C_1$, of $T^*J^1L$, thus the corresponding Legendre bracket is defined on sections of $C_1$. If $L = \mathbb{R} \times M$ is trivial, then $J^1L = \mathbb{R} \times T^*M$ with the canonical contact form $\alpha = dz - p_a dx^a$, and the Legendre bracket is the ‘standard Legendre bracket’ of functions on $\mathbb{R} \times T^*M$,

$$\{F, G\}_\alpha = \frac{\partial F}{\partial p_a} \frac{\partial G}{\partial x^a} - \frac{\partial F}{\partial x^a} \frac{\partial G}{\partial p_a} + \frac{\partial F}{\partial z} \left( G - p_a \frac{\partial G}{\partial p_a} \right) - \left( F - \frac{\partial F}{\partial p_a} p_a \right) \frac{\partial G}{\partial z}.$$  

The cotangent bundle $T^*(L^*)^x$ is canonically also a symplectic principal $\mathbb{R}^x$-bundle over the first jet bundle $J^1L$. Actually, it is a double bundle, as it is simultaneously a vector bundle, and both structures are compatible in the sense that the Euler vector field of the vector bundle structure is 1-homogeneous with respect to the $\mathbb{R}^x$-action; we will call such structure a linear principal $\mathbb{R}^x$-bundle and the whole contact structure, when a homogeneous symplectic form of the bi-degree $(1, 1)$ is given, a linear contact structure. One of important tools will be therefore the theory of $n$-graded (super)manifolds and $n$-graded principal $\mathbb{R}^x$-bundles, with double or, more generally, $n$-tuple vector bundles as basic examples. A natural definition refers to compatibility condition expressed in terms of the commutation of the corresponding homogeneity structures represented by the homotheties induced by the weight or Euler vector fields, as described in [24, 25].

More precisely, an $n$-graded (super)manifold is a (super)manifold $M$ equipped with $n$ homogeneity structures, i.e. smooth actions $h^t : \mathbb{R} \times M \to M$, $t = 1, \ldots, n$, of the monoid $(\mathbb{R}, \cdot)$ of multiplicative reals. The compatibility condition says that all the homotheties commute, $h^t \circ h^s = h^{ts} = h^s \circ h^t$, where $h^1 = h(t, \cdot)$. Each homogeneity structure induces an $N$-gradation on $M$ represented by a certain weight vector field, so we obtain an $\mathbb{N}^n$-gradation. For $n = 1$, these are just $N$-manifolds in the sense of Severa and Roytenberg, if the weight of a function represents its parity. One difference with a principal $\mathbb{R}^x$-action is that $h^0$ represents a projection onto a submanifold in $M$, the base of the corresponding fibration, that is not present in the case of an $\mathbb{R}^x$-bundle, but we can consider the same compatibility condition between $\mathbb{R}^x$-actions and homogeneity structures. This leads to the concept of an $n$-graded principal $\mathbb{R}^x$-bundle. Such a bundle, equipped with a 1-homogeneous (with respect to the $\mathbb{R}^x$-action) symplectic form of weight $k \in \mathbb{N}^n$, becomes an $n$-graded contact structure of degree $k$. If, additionally, the total weight represents the parity, we speak about a contact $k$-manifold. Since there are canonical procedures of lifting weight vector fields and $\mathbb{R}^x$-actions to the tangent and cotangent bundles, we can produce $n$-graded contact structures out of $(n-1)$-graded principal $\mathbb{R}^x$-bundles, in particular, linear contact structures out of pure principal $\mathbb{R}^x$-bundles. For example, starting with the trivial bundle $\mathbb{R}^x \times M$, we obtain the canonical contact structure on $T^*M \times \mathbb{R}$.

Analogously one can define $n$-graded, in particular, linear principal Poisson $\mathbb{R}^x$-bundles which, according to a general knowledge, correspond to certain ‘contact Lie algebroids’, called by us principal Lie algebroids. A principal Lie algebroid can be equivalently defined as an ‘invariant’ Lie algebroid structure on a vector bundle $E$ over a principal $\mathbb{R}^x$-bundle $P_0$. If $P_0$ is trivial, $P_0 = \mathbb{R}^x \times M$, then we actually deal with what was called a Jacobi algebroid in [22] (or a generalized Lie algebroid in [24]). The corresponding cohomology can be defined analogously to the Lie algebroid case by means of a homological ‘$Q$-operator’ and, in the spirit of Vaǐntrob [63], a principal Lie algebroid can be viewed as a ‘$Q$-line bundle’. This time, however, $Q$ is not a vector field but a first-order differential operator. Of course, we can recover in this way some cohomology known in the Jacobi setting, e.g. the Lichnerowicz-Jacobi cohomology, as particular cases of our construction.
Note that canonical linear principal Poisson $\mathbb{R}^\times$-bundles arise as the tangent lifts of Poisson principal $\mathbb{R}^\times$-bundles (Kirillov brackets). The corresponding principal Lie algebroid can be viewed then as associated with the Kirillov bracket. In particular, a Jacobi bracket on $\mathcal{M}$ induces in this way a Lie algebroid structure on $T^*\mathcal{M} \times \mathbb{R}$, as has been observed by Kerbrat and Souici-Benhammadi [51] (see also [53] [22]).

A natural contact analog of the theorem stating that any linear symplectic structure (an NQ-manifold of degree 1 in the terminology of Ševera [60]) is equivalent to a cotangent bundle $T^*\mathcal{M}$ equipped with its canonical symplectic form, is Theorem 10.1 which implies that any linear even (resp., odd) contact structure is isomorphic with the canonical contact structure on the first jet bundle of a line bundle or, equivalently, with the cotangent bundle $T^*\bar{P}$ (resp., $\text{II}T^*\bar{P}$) of a principal $\mathbb{R}^\times$-bundle $\bar{P}$. This is generalized to $\alpha$-linear contact structures in the spirit of [23] Theorem 8.1. The supergeometric setting is crucial in interpreting Kirillov brackets on a line bundle $L$ as bi-homogeneous Hamiltonians on $\text{II}T^*(L^\times)^\times$: 1-homogeneous with respect to the $\mathbb{R}^\times$-action and quadratic with respect to the linear structure, which are homological with respect to the Poisson (or Legendre) bracket.

A natural problem is then the description of contact 2-manifolds (contact N-manifolds of degree 2 in the terminology of Roytenberg and Ševera) which turn out to be symplectic principal $\mathbb{R}^\times$-bundles of degree 2. A contact analog of the Roytenberg’s result [55] Theorem 3.3] says that contact 2-manifolds are in one-to-one correspondence with pseudo-Euclidean principal $\mathbb{R}^\times$-bundles (Theorem 12.2). A canonical example is $T^*[2]F[1]$, where $F$ is a linear principal $\mathbb{R}^\times$-bundle, more particularly $T^*[2]T[1](\mathbb{R}^\times \times \mathcal{M})$.

Next, in view of the Roytenberg’s description of Courant algebroids [56], ‘contact Courant algebroids’ called by us principal Courant algebroids are defined as contact 2-manifold equipped with an $\mathbb{R}^\times$-homogeneous cubic homological Hamiltonian. The ‘classical’ description is proven to be the following: a principal Courant algebroid is a vector bundle $\mathcal{E}$ over a manifold $\mathcal{M}$ equipped with
- a Loday (Leibniz) bracket $\langle \cdot, \cdot \rangle^1$ on sections of $\mathcal{E}$,
- a pseudo-Euclidean product $\langle \cdot, \cdot \rangle^2$ with values in a line bundle $L$,
- a vector bundle morphism $\rho^1 : \mathcal{E} \to DO^1(L, L)$ associating with any section $X$ of $\mathcal{E}$ a first-order differential operator $\rho^1(X)$ acting on sections of $L$ such that

$$\langle \{X, Y\}^1, Y \rangle^1 = \langle X, \{Y, Y\}^1 \rangle^1 \quad \text{and} \quad \rho^1(X)\{Y, Y\}^1 = 2\langle \{X, Y\}^1, Y \rangle^1$$

for all sections $X, Y$ of $\mathcal{E}$. The canonical homogeneous Courant algebroid structure on $T(\mathbb{R}^\times \times \mathcal{M})$ gives rise to a canonical Courant-Jacobi algebroid on $(\mathbb{R} \times T\mathcal{M}) \oplus_{\mathcal{M}} (\mathbb{R}^\times \times T^*\mathcal{M})$ considered already in [23].

We should admit that, although our framework seems to be original as a whole, some of our observations concerning contact forms and contact structures, especially in the pure even case, can be found spread over the existing literature. The existence of a correspondence between Jacobi manifolds and degree-one contact NQ-manifolds was previously mentioned by Ševera [60]. More recently, Antunes and Laurent-Gengoux [2] studied Jacobi structures from the supergeometric point of view. Additionally, Jacobi and contact structures on supermanifolds were considered in a series of papers by Bruce [3] [6] [7] [8]. Their approaches, however, are more traditional than what we propose here. After writing these notes we realized also that some of these questions, in the simplest version concerning contact forms and Jacobi brackets, have been recently studied by Mehta [49]. He mentioned also that developing a general theory of contact NQ-manifolds in the degree 2 case should provide a natural generalization of Courant algebroids, together with a ‘Courantization’ process and that this approach may be useful in studying Jacobi-Dirac and generalized contact structures [30] [51]. This is exactly what we propose in this note.

Our paper is organized as follows. In section 2 we present rudiments of the graded differential geometry, define weight vector fields, their homogeneity completeness, and homogeneity structures, as well as even and odd Poisson and Jacobi brackets. In Section 3 we define lifting procedures for graded structures. Section 4 is devoted to introducing $n$-graded principal $\mathbb{R}^\times$-bundles which are fundamental objects in our framework. Contact forms on supermanifolds and contact structures are defined in Section 5, where also their equivalence with symplectic principal $\mathbb{R}^\times$ bundles is proven. Moreover, local forms of even and odd contact forms on supermanifolds are derived. Graded, especially linear contact structures are introduced in Section 6, together with a fundamental example associated with first jets of line bundles which is studied deeper in Section 7. In turn, Section 8 is devoted to Jacobi and, more generally, Kirillov brackets. Linear principal Poisson structures as corresponding to contact analogs of Lie algebroids, principal Lie algebroids, are investigated in Section 9. As a by-product, we
present in Section 10 a full description of contact $n$-vector bundles. In particular, all linear contact structures turn out to be just the canonical ones associated with first jet bundles. In Section 11 we provide basic concepts concerning the cohomology of principal Lie algebroids. In Section 12 and 13 we introduce and study contact 2-manifolds and and principal Courant algebroids, contact analogs of symplectic manifolds of degree 2 and Courant algebroids.

2 Rudiments of graded differential geometry

2.1 Graded manifolds and weight vector fields

In supergeometry, a vector bundle $E$ of dimension $(p, q)$ over a base $M$ is a (super)manifold which is locally like $U \times \mathbb{R}^{p|q}$ in a consistent way. More precisely, we have an atlas on $E$ with charts whose coordinate functions split into two classes: basic functions, i.e. coordinate functions in a chart $U$ of $M$, and $p+q$ ‘linear functions’, $p$ even coordinates which vary through all $\mathbb{R}$ and $q$ odd coordinates, spanning together a free module over $C^\infty(U)$. Such coordinates we will call affine coordinates. Moreover, changes of coordinates respect these splitting, producing isomorphisms of the corresponding free modules and giving rise to a globally defined and locally free module $\mathcal{A}^1(E)$ over $C^\infty(M)$ of linear functions on $E$. The linear functions from $\mathcal{A}^1(E)$ are usually identified with sections of the dual bundle $E^*$, where $E^*$ is defined as a vector bundle whose local linear functions form the free modules dual to that of local linear functions on $E$. Note that the identification $\mathcal{A}^1(E) \simeq \mathfrak{sec}(E^*)$ makes no problems, while viewing sections as actual smooth maps causes some difficulties, since if the section (linear function) is not even, it does not correspond to a smooth map $\alpha : M \to E$ of the corresponding (super)manifolds, because the parity is not respected. The base $M$ of the vector bundle $E$ is a submanifold in $E$ defined locally by vanishing of all linear coordinates. The canonical projection $\tau : E \to M$, in turn, corresponds to the identification of basic functions with functions on $M$.

From the above description it is clear that the vector bundle structure on $E$ determines certain $\mathbb{N}$-gradation in the structure sheaf in which basic functions on $E$ (identified with functions on $M$) are of degree 0, linear functions are of degree 1, and these classes generate the gradation. Note that to any $\mathbb{N}$-gradation corresponds a canonical derivation of the ring. The corresponding vector field, the weight vector field. In the case of a vector bundle $E$, the weight vector field we will denote $\Delta_E$ and call the Euler vector field of the vector bundle structure. An Euler vector field can be characterized [24] as a complete vector field of the local form $\Delta_E = \sum w_i x^a \partial \alpha$, $w_i \in \{0, 1\}$, induced by the action of the multiplicative monoid $\mathbb{R}$ on $E$, $h : \mathbb{R} \times E \to E$, by homotheties $h_t$, $t \in \mathbb{R}$. The relation between $h$ and $\Delta_E$ is given by

$$\Delta_E(x) = \frac{d}{dt} \bigg|_{t=1} h_t(x).$$

Conversely, having an $\mathbb{N}$-gradation in $E$ that comes from an action $h$ of the monoid $(\mathbb{R}, \cdot)$ of multiplicative reals such that we can find an atlas with charts whose coordinates have degrees $\leq 1$, we deal actually with a vector bundle structure in $E$. Given $n$ compatible vector bundle structures in the sense that the corresponding homotheties (the corresponding Euler vector fields) pairwise commute, we deal with $n$-vector bundles [24].

More generally, we can allow $K^n$-graded manifolds, $K = \mathbb{Z}, \mathbb{N}$. In simple words, a $K^n$-graded manifold will be understood as a supermanifold $M$ with an additional $\mathbb{Z}^n$ (resp., $\mathbb{N}^n$) grading in the structure sheaf. Besides the degree (parity) $g \in \mathbb{Z}_2 = \{0, 1\}$, we have defined also the weight $w \in K^n$ which should be compatible with the degree in the sense that the subsheaf of homogeneous functions $F$ of degree $i = (i_1, \ldots, i_n) \in K^n$, $w(F) = i$, splits into the subspaces of odd, $g(F) = 1$, and even, $g(F) = 0$, functions. One can also think that we actually have a $K^n \times \mathbb{Z}_2$-grading in the structure sheaf. Of course, the algebra generated by homogeneous functions should be sufficiently large. We will assume that there is an atlas of so called affine charts whose coordinate functions have defined weight and parity (i.e., they are $K^n \times \mathbb{Z}_2$-homogeneous) and the weight and parity are respected by the changes of coordinates. Thus the space $\mathcal{A}^i(M)$ of functions $f$ which are homogeneous of weight $i$ (we will write also $w(f) = i$) is well defined and the algebra of homogeneous functions (or a homogeneous algebra) $\mathcal{A}(M) = \bigoplus_{i \in K^n} \mathcal{A}^i(M)$ is dense in the algebra $C^\infty(M)$ of smooth functions with respect to any reasonable $C^\infty$-topology. In particular, homogeneous functions span all finite jets. Morphisms in the category of graded manifolds are smooth maps respecting the grading. The subalgebra in $\mathcal{A}(M)$ consisting of homogeneous functions with weights in $\mathbb{N}^n$ will be called the polynomial algebra of $M$. 
In the above example of a vector bundle, the degree is not directly related to the parity, but we will consider as well the case where the grading respects the parity, i.e. where homogeneous functions of weight $i = (i_1, \ldots, i_n) \in \mathbb{K}_n$ are even if the total weight $|i| = \sum_i |i_i|$ is even, while functions with odd total weight are odd. For such structures, called $\mathbb{K}^n$-manifolds, the $\mathbb{Z}_2$-grading is induced by the $\mathbb{K}^n$-grading. A $\mathbb{K}^n$-manifold $\mathcal{M}$ possesses a coordinate atlas of affine charts with local coordinates $(x^n)$ to which weights $w_a = w(x^n) = (w_1(x^n), \ldots, w_n(x^n)) \in \mathbb{K}^n$ are assigned (and respected by the changes of coordinates), and induce the commutation rules
\[ x^a x^b = (-1)^{|w_a| - |w_b|} x^b x^a. \]

In other words, $g(x^n) = (-1)^{|w_a|}$.

Of course, some of the above concepts can be extended to gradings more general than $\mathbb{Z}^n$ or $\mathbb{N}^n$. In what follows the term manifold will mean supermanifold. In this sense an ‘ordinary’ manifold will be an even manifold.

**Remark 2.1.** Note that, in general, the algebra of homogeneous functions $\mathcal{A}(\mathcal{M})$ does not determine $\mathcal{M}$. For, consider an ordinary vector bundle $E$ over $\mathcal{M}$ viewed as a purely even manifold $E = \mathcal{M}$. The vector bundle structure induces a natural $\mathbb{N}$-grading in the subalgebra of $C^\infty(E)$ of functions which are polynomial along fibers: homogeneous functions of degree $k$ are $k$-homogeneous with respect to the corresponding Euler vector field $\Delta_E$, $\Delta_E(f) = kf$. In other words, basic functions have degree 0, and functions linear along fibers (i.e., functions associated with sections of the dual bundle $E^*$) have degree 1, and together generate the algebra of polynomial functions. These polynomial functions, when restricted to an open strip over the zero-section, generate the same polynomial algebra, but this open strip with restricted polynomials is not diffeomorphic with the vector bundle, as there is no smooth diffeomorphism of the open strip onto the whole vector bundle respecting degrees of homogeneous functions.

Let us observe that any $\mathbb{K}^n$-grading can be conveniently encoded by means of the collection of weight vector fields which are jointly diagonalizable. More precisely, there are globally defined vector fields $\Delta_\mathcal{M}$ written locally in homogeneous coordinates $(x^n)$ as
\[ \Delta_\mathcal{M}^a = \sum_a w_a^s x^a \partial_{x^a}, \quad s = 1, \ldots, n, \]
where $w_a = w(x^n) \in \mathbb{K}$, which completely determine the grading. Indeed, $f \in \mathcal{A}(\mathcal{M})$ if and only if $\Delta_\mathcal{M}(f) = i_s f$, since each coordinate $x^a$ is even or odd, the weight vector fields are even and each weight vector field $\Delta_\mathcal{M}$ defines a submanifold $\mathcal{M}_s$ of $\mathcal{M}$ on which it vanishes. The submanifold $\mathcal{M}_s$ is defined locally by constraints $x^a = 0$ for $w_a \neq 0$, and local coordinates on $\mathcal{M}_s$ are induced by those local coordinates on $\mathcal{M}$ which are of weight 0.

**Definition 2.1.** We say that an $\mathbb{N}^n$-graded manifold is of degree $k \in \mathbb{N}$ if local coordinates can be chosen such that their weights $w_s$ satisfy $w \leq k$, and of total degree $r \in \mathbb{N}$ if local coordinates can be chosen such that all the total weights $|w_a| = |w(x^n)| = |w_1^1| + \cdots + |w_n^1|$ do not exceed $r$. Here, the partial order $\preceq$ is defined by $t \preceq k \Leftrightarrow \forall j \{ t_j \leq k_j \}$.

The list of weight vector fields can be completed by the parity diffeomorphism
\[ Par(x^n) = g(x^n)x^n \]
preserving all weight vector fields.

The weight vector fields give rise to one-parameter groups of automorphisms $\Phi^s_t$, $s = 1, \ldots, n$, of the algebra of homogeneous functions, defined by $\Phi^s_t(f) = e^{at} f$ for $f \in \mathcal{A}(\mathcal{M})$. We can equivalently write down this action as an action $h^s$ of multiplicative group of positive reals by putting $h^s_t = \Phi^s_{\ln(t)}$. In the case of an $\mathbb{N}^n$-gradation, these automorphisms can be naturally extended to actions $h^s : \mathbb{R} \times \mathcal{A}(\mathcal{M}) \ni (t, f) \mapsto h^s_t(f) \in \mathcal{A}(\mathcal{M})$ of the monoid $(\mathbb{R}, \cdot)$ of multiplicative reals. All these actions can be collected into one action $h : \mathbb{R} \times \mathcal{A}(\mathcal{M}) \ni (t, f) \mapsto h_t(f) \in \mathcal{A}(\mathcal{M})$ of the monoid $(\mathbb{R}, \cdot)$, i.e., $\mathbb{R}$ with coordinate-wise multiplication) on the homogeneous algebra via the algebra homomorphisms
\[ h_t(f) = \prod_s t_{s}^s f, \quad f \in \mathcal{A}(\mathcal{M}). \]

In other words, $h_t$ is the composition of pairwise commuting maps $h^s_{t_s}$, $s = 1, \ldots, n$, where $h^s_{t_s}$ acts on $\mathcal{A}^i$, $i \in \mathbb{N}$, as the multiplication by $t_{s}^s$. 

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Definition 2.2. We say that the weight vector field $\Delta_M^s$ is homogeneitically complete (h-complete, for short), if the actions $h^s$ can be extended to a smooth action of the monoid $(\mathbb{R}, \cdot)$ on the supermanifold $\mathcal{M}$. This means that $h^s$ is actually a diffeomorphism of $\mathcal{M}$ if $t \neq 0$, and it is a smooth surjection onto a certain submanifold $\mathcal{M}_s$ if $t = 0$. Such actions we will call homogeneity structures (cf. [25]). Homogeneitically complete vector fields $\Delta = \sum w_a x^a \partial_a$ with $w_a = 0, 1$ we will call Euler vector fields. We say that an $N^n$-graded manifold $(\mathcal{M}, \Delta^{n_1}_{\mathcal{M}}, \ldots, \Delta^{n_r}_{\mathcal{M}})$ is homogeneitically complete (h-complete, for short), if all its weight vector fields (2.1) are h-complete, so define homogeneity structures $h^1, \ldots, h^n$. Homogeneitically complete $N^n$-manifolds $(\mathcal{M}, h^1, \ldots, h^n)$ of degree $k \in N^n$ (total degree $r$) we will call $n$-graded manifolds (or $n$-graded bundles) of degree $k$ (total degree $r$). If the grading respects the parity, an $n$-graded manifold of degree $k \in N^n$ we will call simply a manifold of degree $k$, a k-manifold, or a k-bundle.

Remark 2.2. The reason for calling h-complete graded manifolds ‘bundles’ is that the submanifold $\mathcal{M}_s$ of zeros of the weight vector field $\Delta^s$ is in this case a base of the locally trivial fibration $h^s_0 : \mathcal{M} \to \mathcal{M}_s$. As the homogeneity structures pairwise commute, we have the whole diagram of such fibrations,

\begin{equation}
(2.4) \quad h^{t_1}_0 \circ \cdots \circ h^{t_n}_0 : \mathcal{M} \to \mathcal{M}_{s_1, \ldots, s_n} = \mathcal{M}_{s_1} \cap \cdots \cap \mathcal{M}_{s_n},
\end{equation}

with fibers being homogeneity spaces (cf. [24, 25]).

Note that $N^n$-manifold is a particular case of an $\mathbb{Z}^n$-manifold and h-complete 1-manifolds of degree $k \in N$ are $N$-manifolds of degree $k$ as defined in Ševera [60] and Roytenberg [56]. Recall that an $N$-manifold is of degree $k \in N$ if there is an atlas with coordinates of weights $\leq k$ and that $N$-manifolds of degree $k \in N$ we will call just $k$-manifolds.

It is obvious that weight vector fields on an $N^n$-graded manifold pairwise commute. Adapting the methods of [24] to supermanifolds, one can show that for h-complete weight vector fields the converse is also true: if they commute they are jointly diagonalizable.

Theorem 2.1. Pairwise commuting actions $h^1, \ldots, h^n$ of the monoid $(\mathbb{R}, \cdot)$ on a manifold $\mathcal{M}$ induce an $N^n$-gradation on $\mathcal{M}$. In particular they are jointly diagonalizable.

As Euler vector fields correspond exactly to vector bundle structures on $\mathcal{M}$ (see [24]), an $n$-vector bundle can be equivalently characterized as an $n$-graded manifold of the degree $1^n \in N^n$ (or the total degree 1) [24]. If $\mathcal{M}$ is just an ordinary (purely even) manifold, then this concept of an $n$-vector bundle coincides with that of Pradines [52] and Mackenzie [15, 16] (see also [34, 38]). Moreover, $n$-manifolds are exactly superanalogous of $n$-tuple homogeneity structures (or $n$-tuple graded bundles) introduced in [25].

Remark 2.3. The lack of completeness is exactly that differs a strip from the whole vector bundle in Remark 2.1 the weight vector field is h-complete on vector bundle (see [24]) while the strip is not, since some homotheties move points of the strip outside the strip. On the other hand, it is clear that h-complete graded manifolds with isomorphic homogeneous algebras are diffeomorphic.

Note that the weight defined for functions can be canonically extended to tensor fields on $\mathcal{M}$: a tensor field $\Lambda$ is of weight $i = (i_1, \ldots, i_n)$ if $\mathcal{L}_{\Delta_M^s}\Lambda = i_s \cdot \Lambda$, $s = 1, 2, \ldots, n$, where $\mathcal{L}$ denotes the Lie derivative. In particular, we have $w(\Delta_M^s) = 0$ which means exactly that the weight vector fields commute. Similarly, the parity diffeomorphism can be extended to a map on tensor fields, mapping a tensor field $K$ to $\overline{K}$. If $K = K^0 + K^1$ is the decomposition of a tensor field $K$ into the even and the odd part, then $\overline{K} = K^0 - K^1$. Note that, for a vector field $X$ and a one-form $\alpha$, we have $\overline{i_X\alpha} = i_{\overline{X}}\alpha$.

On a $K^n$-graded manifold we can always switch the grading with the use of a new weight vector field $\Delta = \sum w_a x^a \partial_a$ which is jointly diagonalizable with the old weight vector fields by passing to a new collection of weight vector fields: $\Delta$ and $\Delta_M^s = \Delta_M + \Delta$, $s = 1, \ldots, n$. The corresponding graded supermanifold we will denote $\mathcal{M}(\Delta)$. If $\Delta$ is an Euler vector field, we can also change the parity according to the weights induced by $\Delta$. The new atlas is consistent and we get a new graded supermanifold in which the grading remains the same but the parity has been changed according to the rule $g'(x^a) = g(x^a) + w^a$ (mod 2). The corresponding graded supermanifold we will denote $\Pi_\Delta \mathcal{M}$. If the vector bundle structure represented by $\Delta$ is fixed, we will use the standard notation $\Pi \mathcal{M}$ instead of $\Pi_\Delta \mathcal{M}$. Of course, we can change both: the grading into the one given by the weight vector fields $\Delta_M^s$ and the parity according to the above rule. The corresponding graded supermanifold we will denote...
where $\Lambda$ is an even (resp., odd) bi-derivation which is (anti)symmetric in an appropriate way, $\Gamma$ is and

\[(2.8) \quad \Lambda + \Gamma\]

is a (super)Lie bracket. Property (2.8), we call super-anticommutativity and (2.7) super-Jacobi identity. They say together that the bracket is a (super)Lie bracket. Property (2.6), generalized super-Leibniz rule, just tells us that the adjoint map, $ad_\sigma = \{\cdot, \cdot\} : A \to A$, is a first-order differential operator on $A$ of parity $g(\varphi)+k \mod 2$.

It follows (cf. [23, theorem 12]) that the Jacobi bracket is represented by a first-order bidifferential operator

\[(2.9) \quad \Lambda + \Gamma \cdot I + f I \cdot I,\]

where $\Lambda$ is an even (resp., odd) bi-derivation which is (anti)symmetric in an appropriate way, $\Gamma$ is and even (resp., odd) derivation, $I$ is the identity understood as a zero-order differential operator, $f$ is 0 (resp., an odd) function, and ‘.$$ is an appropriate ‘exterior product’ of first-order linear differential operators. In the pure even case, $f = 0$ and the manifold equipped with the pair $(\Lambda, \Gamma)$ is traditionally called a Jacobi manifold [43].

In fact (see [10, Theorem 4.2]), if $A$ has no nilpotents, then any Lie bracket on $A$ given by a bidifferential operator must be of first order, so is a Jacobi bracket. This is an algebraic generalization of a geometric result due to Kirillov [33]: any local Lie bracket on sections of a line bundle over an even manifold is of first order with respect to each argument. This justifies our definition, this time for arbitrary (super)manifolds.

**Definition 2.4.** An even (resp., odd) Lie bracket, defined on sections of an even line bundle $L$ over a manifold $M$, which is a first-order differential operator with respect to each argument will be called an even (resp., odd) Kirillov bracket.

Obviously, in a local trivialization of the line bundle any Kirillov bracket is represented by a Jacobi bracket. The Kirillov bracket is even or odd if the local Jacobi bracket is even or odd, respectively.

**Remark 2.4.** By definition, if $[\cdot, \cdot]_L$ is a Kirillov bracket on sections of a line bundle $L$ over $M$ and $\sigma$ is a section of $L$, then the adjoint operator $ad_\sigma = [\sigma, \cdot]_L$ is a first-order differential operator from $L$ into $L$, thus a section of the corresponding vector bundle $DO^1(L, L)$ of ‘infinitesimal’ first-order differential operators on $L$. There is a canonical isomorphism $DO^1(L, L) = TL^\times / R^\times$, where $R^\times$ is the group of multiplicative reals and $L^\times$ is the principal $R^\times$-bundle associated with $L$ (cf. Section [I]). In other words, $DO^1(L, L)$ is the Atiyah algebroid of the principal $R^\times$-bundle $L^\times$, and sections of $DO^1(L, L)$ can be viewed as invariant vector fields on $L^\times$. This vector bundle is canonically a Lie algebroid with the (super)commutator bracket and the anchor given by passing to the principal part of a first-order differential operator (which is a vector field on $M$) or, equivalently, as the projection of the invariant vector field on $L^\times$ onto $M$. 

$\mathcal{M}([\Delta])$. The latter operation produces $K^\bullet$-manifolds from $K^\bullet$-manifolds. If $\Delta$ is the Euler vector field of a fixed vector bundle structure, we usually write $\mathcal{M}(r)$ and $\mathcal{M}[r]$ instead of $\mathcal{M}(r\Delta)$ and $\mathcal{M}[r\Delta]$, respectively.

We can also construct new gradings using the observation that the sum of (h-complete) weight vector fields is a (h-complete) weight vector field again. In particular, any $n$-graded manifold $(\mathcal{M}, h^1, \ldots, h^n)$ is canonically a 1-graded manifold $(\mathcal{M}, h)$: we just add all the weight vector fields and get a weight vector field corresponding to the composition of all homotheties, $h_t = h^1_t \circ \cdots \circ h^n_t$. In this way we can view double vector bundles (like $TTM$ or $T^*TM$) as 2-manifolds (N-manifolds of degree 2).
Remark 2.5. It is clear that Jacobi brackets are Kirillov brackets on trivial line bundles. In our opinion, there is no particular reason to study Jacobi brackets rather than Kirillov ones. First of all, we must stress that a Jacobi bracket itself is genuinely a module bracket and the product $\gamma \psi$ in (2.7) should be understood as the left-module product (where $\gamma$ is an element of the algebra and $\psi$ is an element of the left-regular $A$-module) rather than the algebra product. This is because of the role of this product under isomorphism of the Jacobi brackets which do respect the left-regular module structure and do not respect the algebra structure (see [17]). In particular, the unity 1 regarded as a module element can be replaced by another invertible element (nonvanishing section) that cannot be done by an algebra isomorphism. The fact that the product $\gamma \psi$ refers to the module and not to the algebra structure is, of course, hard to be seen in the algebra (trivial bundle) case and becomes obvious only when passing to general line bundles.

From this point of view, the concept of a Jacobi bracket is an auxiliary concept which serves well locally but has to be naturally extended to an arbitrary line bundle. The concept of a first-order differential operator works well for modules of sections of such bundles and the distinction of a particular section (constant) is not needed. This is what differs this situation from the case of a Poisson bracket, where we deal with derivations of an associative algebra which are operators vanishing on constants.

Given a supermanifold $\mathcal{M}$, we can consider the cotangent bundle $\mathbb{T}^* \mathcal{M}$ with local Darboux coordinates $(x^a, p_b)$, where the degree of $x^a$ and that of $p_b$ being the same as the degree of $x^a$ on $\mathcal{M}$. Note that both vector fields on $\mathcal{M}$ are represented by functions on $\mathbb{T}^* \mathcal{M}$ linear in the fibers, so linear in $p_b$’s. In this case, the canonical symplectic form $\omega_{\mathcal{M}} = dp_a dx^a$ on $\mathbb{T}^* \mathcal{M}$ is even and induces an even Poisson bracket $\{\cdot, \cdot\}_{\mathcal{M}}$ on $C^\infty(\mathbb{T}^* \mathcal{M})$, closed on the superalgebra $\mathcal{A}(\mathbb{T}^* \mathcal{M})$ of polynomial functions on $\mathbb{T}^* \mathcal{M}$. The latter algebra can be identified with the algebra of symmetric multivector fields on $\mathcal{M}$, the algebra of symbols of differential operators.

If we reverse the parity of the momenta, $g(p_b) = g(x^a) + 1 (\text{mod} 2)$, then we work on the manifold $\text{II} \mathbb{T}^* \mathcal{M}$ with the canonical symplectic form $\omega_{\mathcal{M}}^\Pi = dp_a dx^a$ which is now odd and induces an odd Poisson bracket $\{\cdot, \cdot\}^\Pi_{\mathcal{M}}$ on $C^\infty(\text{II} \mathbb{T}^* \mathcal{M})$, closed on the superalgebra $\mathcal{A}(\text{II} \mathbb{T}^* \mathcal{M})$ of polynomial functions on $\text{II} \mathbb{T}^* \mathcal{M}$. The latter algebra can be identified with the Grassmann algebra $\mathcal{X}(\mathcal{M}) = \bigoplus_{k=0}^\infty \mathcal{X}^k(\mathcal{M})$ of skew-symmetric multivector fields on $\mathcal{M}$ with an obvious N-grading. The odd Poisson bracket $\{\cdot, \cdot\}^\Pi_{\mathcal{M}}$ on $\mathcal{X}(\mathcal{M})$ we will call the Schouten bracket on $\mathcal{M}$, while the bracket $\{\cdot, \cdot\}_{\mathcal{M}}$ on $\mathcal{A}(\mathbb{T}^* \mathcal{M})$ we will call the symmetric Schouten bracket on $\mathcal{M}$. The following is well known.

Theorem 2.2. Any even (resp. odd) Poisson bracket on $C^\infty(M)$ is the derived bracket $\{\cdot, \cdot\}_J$ induced from the symplectic Poisson bracket $\{\cdot, \cdot\}^{\Pi}_{\mathcal{M}}$ (resp. $\{\cdot, \cdot\}_{\mathcal{M}}$) by a certain even (resp. odd) Poisson tensor $J$ viewed as a quadratic Hamiltonian in $\mathcal{A}(\text{II} \mathbb{T}^* \mathcal{M})$ (resp., $\mathcal{A}(\mathbb{T}^* \mathcal{M})$), so that

\begin{equation}
\{F, G\}_J = \{\{F, J\}^{\Pi}_{\mathcal{M}}, G\}^{\Pi}_{\mathcal{M}}
\end{equation}

in the even case and

\begin{equation}
\{F, G\}_J = \{\{F, J\}_{\mathcal{M}}, G\}_{\mathcal{M}}
\end{equation}

in the odd case.

Note that if $\mathcal{M}$ is a vector bundle over $\mathcal{M}$ with the Euler vector field $\Delta$ and the Poisson tensor $J$ on $\mathcal{M}$ is linear, i.e. $J$ is homogeneous of degree -1, $\mathcal{L}_\Delta J = -J$, then the Poisson bracket $\{\cdot, \cdot\}_J$ is closed on linear function on $\mathcal{M}$, thus defines a Lie bracket $[\cdot, \cdot]_{\mathcal{M}}$ on sections of the dual bundle $E = \mathcal{M}^*$ by

\begin{equation}
\iota_{[e, e']}_{\mathcal{M}} = \{\iota_e, \iota_{e'}\}_{\mathcal{J}}.
\end{equation}

Here, $\iota_e$ is the linear function on $\mathcal{M} = E^*$ corresponding to $e \in \text{Sec}(E)$. This Lie bracket admits an anchor, i.e., a vector bundle morphism $\rho : E \to \mathbb{T}\mathcal{M}$ covering the identity such that

\begin{equation}
[e, fe']_{\mathcal{J}} = f[e, e']_{\mathcal{J}} + \rho(e)(f)e'
\end{equation}

for any basic function $f \in C^\infty(M)$. In other words, the linear Poisson tensor $J$ makes the vector bundle $E$ into a Lie algebroid.
3 Tangent and phase lifts

In the standard approach, for any supermanifold \( \mathcal{M} \) with local coordinates \((x^a)\) the tangent bundle \( T\mathcal{M} \) with the adapted local coordinates \((x^a, \dot{x}^b)\), and the cotangent bundle \( T^*\mathcal{M} \) with the adapted local coordinates \((x^a, p_b)\) are viewed as supermanifolds with the parity of \( \dot{x}^a \) and \( p_b \) being the same as the parity of \( x^a \).

We will regard the graded superalgebra (Grassmann algebra) \( \Omega(\mathcal{M}) = \bigoplus_{k=0}^{\infty} \Omega^k(\mathcal{M}) \) of differential forms on \( \mathcal{M} \) as the superalgebra of polynomial functions on \( \Pi^H \mathcal{M} \), so that the degree of local coordinate \( x^a \in \Omega(\mathcal{M}) \) is the same as in \( \mathcal{M} \), and the degree of \( \dot{x}^a = dx^a \in \Omega(\mathcal{M}) \) is \( g(x^a) + 1 \) (mod 2). This will be the default superalgebra structure on \( \Omega(\mathcal{M}) \) we will use.

Note that both Grassmann algebras, \( \mathcal{X}(\mathcal{M}) \) and \( \Omega(\mathcal{M}) \), are canonically subalgebras of the algebra \( C^\infty(T^*\Pi^H \mathcal{M}) \cong C^\infty(T^*\Pi^H \mathcal{M}_s) \). The canonical symplectic form \( \omega_{\Pi^H \mathcal{M}} \) induces an even Poisson bracket \( \{\cdot, \cdot\}_p \) on \( C^\infty(T^*\Pi^H \mathcal{M}) \) called sometimes the big bracket \( [1] \).

If \( (\mathcal{M}, \Delta_1^n, \ldots, \Delta_n^\mathcal{M}) \) is a \( \mathbb{K}^n \)-graded supermanifold, then \( T\mathcal{M} \) is canonically \( \mathbb{N} \times \mathbb{K}^n \)-graded.

The corresponding weight vector fields on \( T\mathcal{M} \) are: \( \Delta_{\mathcal{M}} \), the Euler vector field of the vector bundle structure \( \tau_\mathcal{M} : \mathcal{T}\mathcal{M} \to \mathcal{M} \), and the so called tangent lifts, \( d_T \Delta_{\mathcal{M}} \), of the weight vector fields \( \Delta_{\mathcal{M}} \) on \( \mathcal{M} \). In local coordinates,

\[
\Delta_{T\mathcal{M}} = \dot{x}^a \partial_{x^a}, \quad d_T \Delta_{\mathcal{M}} = \sum_a w^a_a x^a \partial_\dot{x}^a + \sum_a w^a_a \dot{x}^a \partial_x^a.
\]

This graded manifold we will denote \( T(\mathcal{M}) \) (or \( T[1] \mathcal{M} \) if we reverse also the parity of momenta). It follows that if \( \mathcal{M} \) is an \( n \)-manifold of degree \( r \), then \( T(\mathcal{M}) \) is canonically an \((n + 1)\)-manifold of degree \((1, r)\) if we reverse also the parity of momenta. Indeed, if the weight vector fields \( \Delta_{\mathcal{M}} \) are complete and correspond to pair-wise commuting actions \( h^s \) of the monoid \((\mathbb{R}, \cdot)\) on \( \mathcal{M} \) (homogeneity structures), then the vector fields \( d_T \Delta_{\mathcal{M}} \) correspond to pair-wise commuting homogeneity structures \( T\mathcal{H}^s \), \((T\mathcal{H})^t \) on \( \mathcal{T}\mathcal{M} \), and the Euler vector field \( \Delta_{T\mathcal{M}} \) corresponds to the multiplication \( h : \mathbb{R} \times \mathcal{T}\mathcal{M} \to \mathcal{T}\mathcal{M} \) by scalars in the vector bundle \( T\mathcal{M} \), \( h_t(v) = t v \), that commutes with all the actions \( T\mathcal{H}^s \), \( s = 1, \ldots, n \). As the tangent lift of an Euler vector field is an Euler vector field, \( T[1] \mathcal{M} \) (as well as \( T\mathcal{M} \)) is canonically an \((n + 1)\)-vector bundle if \( \mathcal{M} \) is an \( n \)-vector bundle (see \([24]\)).

Similarly, the cotangent bundle \( T^* \mathcal{M} \) is canonically \( \mathbb{N} \times \mathbb{K}^n \)-graded. The corresponding weight vector fields on \( T^* \mathcal{M} \) are: \( \Delta_{T^* \mathcal{M}} \), and the so called cotangent lifts, \( d_T^* \Delta_{\mathcal{M}} \), of the weight vector fields \( \Delta_{\mathcal{M}} \) on \( \mathcal{M} \). In local Darboux coordinates \((x, p)\),

\[
\Delta_{T^* \mathcal{M}} = p_a \partial_{p_a}, \quad d_T^* \Delta_{\mathcal{M}} = \sum_a w^a_a x^a \partial_{\dot{x}^a} - \sum_a w^a_a p_a \partial_{p_a}.
\]

The cotangent lift of a h-complete weight vector field is not h-complete any more, since it includes negative weights. If we start with an \( \mathbb{N}^n \)-graded manifold of degree \( r = (r_1, \ldots, r_n) \in \mathbb{N}^n \), a solution depends on using the r-phase lifts \( T^*(r) \Delta_{\mathcal{M}} \) of the weight vector fields,

\[
(3.1) \quad T^*(r) \Delta_{\mathcal{M}} = d_T^* \Delta_{\mathcal{M}} + r_2 \Delta_{T^* \mathcal{M}} = \sum_a w^a_a x^a \partial_{\dot{x}^a} + \sum_a (r_s - w^a_s) p_a \partial_{p_a},
\]

instead of the cotangent lifts. Hence, we end up with an \( \mathbb{N}^{n + 1} \)-graded manifold which we will denote as \( T^*(r) \mathcal{M} \), or \( T^*[r] \mathcal{M} \) if we reverse the parity of momenta for all \( r_s \) being odd. If \( \mathcal{M} \) is an \( n \)-manifold of degree \( r \), these lifts, together with the Euler vector field \( \Delta_{T^* \mathcal{M}} \), make \( T^* \mathcal{M} \) into an \((n + 1)\)-manifold of degree \((1, r)\) \( \in \mathbb{N}^{n + 1} \). The homogeneity structures \( (T^*(r) \mathcal{H})^s \) associated with \( T^*(r) \Delta_{\mathcal{M}} \) are defined by

\[
(3.2) \quad (T^*(r) \mathcal{H})^t = T^*(r) \mathcal{H}^s_t^{r-1}
\]

for \( t \neq 0 \) and extend canonically to the whole \( \mathbb{R} \). Note that the canonical symplectic form \( \omega_{T^* \mathcal{M}} = dp_a dx^a \) on \( T^*(r) \mathcal{M} \) has the weight \((1, r) \in \mathbb{N}^{n + 1} \).

The 1-phase lifts, which we will call simply phase lifts, have the property that they produce Euler vector fields from Euler vector fields, so \( T^* \mathcal{M} \) is canonically an \((n + 1)\)-vector bundle if \( \mathcal{M} \) is an \( n \)-vector bundle (see \([24]\)). In fact, in this case \( T^* \mathcal{M} \) is a symplectic \((n + 1)\)-vector bundle, since the canonical symplectic form \( \omega_{\mathcal{M}} \) on \( T^* \mathcal{M} \) is homogeneous of degree 1 with respect to phase lifts and with respect to the Euler vector field \( \Delta_{T^* \mathcal{M}} \).
4 Graded principal $\mathbb{R}^\times$-bundles

Let $\mathbb{R}^\times = \mathbb{R}\setminus\{0\}$ be the multiplicative group on non-zero real numbers, viewed as an ordinary manifold (purely even supermanifold). With every principal $\mathbb{R}^\times$-bundle $P$ over $\mathcal{M}$

$$h : \mathbb{R}^\times \times P \to P, \quad (t, p) \mapsto h_t(p),$$

one can associate canonical line bundles over $\mathcal{M}$: the even line bundle $\mathcal{E}$ with the typical fibre $\mathbb{R} = \mathbb{R}^{1|0}$ and the odd line bundle $\mathcal{O}$ with the typical fibre $\mathbb{R}^{0|1}$. With $\mathcal{E}$ and $\mathcal{O}$ we will denote the dual line bundles, $\mathcal{E}^* = (\mathcal{E})^*$, $\mathcal{O}^* = (\mathcal{O})^*$.

On the other hand, using the transformation rules for local trivializations, with every line bundle $L$ over $\mathcal{M}$ one can associate canonically a principal $\mathbb{R}^\times$-bundle $P = L^\times$ over $\mathcal{M}$. These operations on bundles are mutual inverses: $L^\times = L$ and $\hat{L}^\times = \hat{L}$, for an even and odd line bundle $L$, respectively. If $L$ is even, one can obtain $L^\times$ just by removing the zero-section: $L^\times = L \setminus \{0\}$. Moreover, the fundamental vector field $\Delta_P$ of the $\mathbb{R}^\times$-action $\mathbb{R}^\times \times P \ni (t, p) \mapsto h_t(p) \in P$ on $P = L^\times$ is just the Euler vector field $\Delta_1$ of the vector bundle $L$, restricted to $L^\times$. With some abuse of terminology, we will call it the Euler vector field of the $\mathbb{R}^\times$-principal bundle and the $\mathbb{R}^\times$-action we will call the homogeneity structure of the $\mathbb{R}^\times$-principal bundle. For the standard coordinate $t$ on $\mathbb{R}$, thus $\mathbb{R}^\times$, used as coordinate in fibers for a fixed local trivialization and extended by coordinates in the base manifolds (such coordinates in $L^\times$ we will call homogeneous), we have $\Delta_L = t\partial_t$.

**Definition 4.1.** An n-graded principal $\mathbb{R}^\times$-bundle of degree $k$ over $\mathcal{M}$ is a principal $\mathbb{R}^\times$-bundle $P$ over $\mathcal{M}$ with a principal action $h^0$ of the group $\mathbb{R}^\times$ on $P$, equipped simultaneously with a structure of an n-graded manifold $(P, h^1, \ldots, h^n)$ of degree $k$ such that the action $h^0$ commutes with the homogeneity structures $h^i$,

$$h^0 \circ h^i_s = h^0 \circ h^0_s, \quad t, s \in \mathbb{R}^\times, \quad i = 1, \ldots, n.$$  

In particular, if $h^1, \ldots, h^n$ are homogeneity structures of an $n$-vector bundle, an n-graded principal $\mathbb{R}^\times$-bundle $(P, h^1, h^1, \ldots, h^n)$ of degree 1 we will call an n-linear principal $\mathbb{R}^\times$-bundle.

An analogous object, called simply a principal $\mathbb{R}^\times$-bundle of degree $k$, we obtain additionally that the parity is defined by the total weight.

**Remark 4.1.** Of course, $h^0 \circ h^i_s = h^1 \circ h^0_s$ for $t, s \in \mathbb{R}^\times$ implies that $h^0$ respects the bundle structures, $h^0 \circ h^0_s = h^0 \circ h^0_s$, and, in turn, that the diffeomorphisms $h^0_s$ preserve the weight vector fields $\Delta^1, \ldots, \Delta^n$ associated with the homogeneity structures $h^1, \ldots, h^n$, i.e. $(h^0)^\ast(\Delta^i) = \Delta^i$. Hence, the fundamental vector field $\Delta^0$ of the principal $\mathbb{R}^\times$-action commutes with $\Delta^i$, i.e. $[\Delta^0, \Delta^i] = 0$ for $i = 1, \ldots, n$, and it is easy to see that all these vector fields are jointly diagonalizable. On the other hand, the group $\mathbb{R}^\times$ is not connected and the fundamental vector field $\Delta^0$ determines only the action of its connected component $\mathbb{R}^\times_\ast$. To know the whole action, we have to know additionally the symmetry $h_{-1}$, so (4.2) means that $[\Delta^0, \Delta^i] = 0$ and $(h_{-1})^\ast(\Delta^i) = \Delta^i$. In any case, however, an n-graded principal $\mathbb{R}^\times$-bundle is canonically an $\mathbb{Z} \times \mathbb{N}^n$-graded manifold with the homogeneous algebra

$$\mathcal{A}(P) = \bigoplus_{j \in \mathbb{Z} \times \mathbb{N}^n} \mathcal{A}^j(P),$$

where $j_0$ of $(j_0, \ldots, j_n) \in \mathbb{Z} \times \mathbb{N}^n$ refers to the degree of homogeneity with respect to the $\mathbb{R}^\times$-action.

A local form of a graded principal $\mathbb{R}^\times$-bundle describes the following.

**Theorem 4.1.** Any n-graded principal $\mathbb{R}^\times$-bundle $(P, h^0, h^1, \ldots, h^n)$ over $\mathcal{M}$ induces a canonical n-graded manifold structure on $\mathcal{M}$ and admits an atlas of $\mathbb{R}^\times$-invariant charts on $P$ with homogeneous local coordinates $(t, x^1, \ldots, x^n)$, $t \in \mathbb{R}^\times$, such that $h^0_t(x^\alpha) = (st, x^\alpha)$ and the $h$-complete weight vector fields read as

$$\Delta^i = \sum_{a=1}^m w^i_a x^a \partial_{x^a},$$

for some $w^i_a \in \mathbb{N}$. In other words, $P$ is locally the product $\mathbb{R}^\times \times U$, where $U$ is an open n-graded submanifold of $\mathcal{M}$, with the obvious n-graded principal $\mathbb{R}^\times$-bundle structure.
Proof. It is easy to see that the commutation relations \((4.2)\) imply that \(h^1, \ldots, h^n\) induce on \(P/\mathbb{R}^x = \mathcal{M}\) a reduced \(n\)-manifold structure \(\tilde{h}^1, \ldots, \tilde{h}^n\). Consider a local trivialization \(\mathbb{R}^x \times U\) of \(P\), where \(U\) is an \(\tilde{h}^1, \ldots, \tilde{h}^n\)-invariant open subset of \(\mathcal{M}\), and corresponding coordinates \((t, x^a)\), \(a = 1, \ldots, m\). We can choose coordinates \((x^a)\) being homogeneous, so that

\[
\Delta^i = f^i(t, x)\partial_i + \sum_{a=1}^m w^i_a x^a \partial_{x^a}.
\]

As \((h^a_s)_*(\Delta^i) = \Delta^i\), we get that \(sf^i(t/s, x) = f^i(t, x)\) for all \(s, t \in \mathbb{R}^x\), so \(f^i(t, x) = F^i(t)\). If we had \(F^i(x) \neq 0\), we would get \(h^i_s(t, x) = (s^{f^i(x)} t, \tilde{h}^i(x))\) for \(s > 0\). Since \(h^i\) is a smooth map \(\mathbb{R} \times \mathbb{R}^x \times U \ni (s, t, x) \mapsto h^i_s(t, x) \in \mathbb{R}^x \times U\), considering the limit \(\lim_{s \to 0^+} h^i_s(t, x)\) the limit \((0, \tilde{h}^i(x)) \in \mathbb{R}^x \times U\), we would get \(0 \in \mathbb{R}^x\); a contradiction. Hence, \(F^i(x) \equiv 0\).

\[\square\]

Example 4.1. Like for vector bundles, the tangent \(TP\) and the cotangent bundle \(T^*P\) of an \(\mathbb{R}^x\)-principal bundle \(P\) are canonically \(\mathbb{R}^x\)-principal bundles, with the tangent \(Th\) and the phase \(T^*h\) lift of the \(\mathbb{R}^x\)-action, respectively:

\[
(Th)_t = T(h_t), \quad (T^*h)_t = (Th_{t^{-1}})^*, \quad t \neq 0.
\]

In the coordinates \((t, x^a)\) on \(P\) as above and the adopted coordinates \((t, x^a, p_0, p_a)\) in \(T^*P\), we have \(h^i_s(t, x^a) = (st, x^a)\), so \((T^*h)_s(t, x^a, p_0, p_a) = (st, x^a, sp_a, p_0)\) that commutes with the action \(h^i_s(t, x^a, p_0, p_a) = (t, x^a, up_a, wp_a)\) of \(\mathbb{R}\) by homotheties. We will study closer the case of the cotangent bundle in Section 7.

Formally the same formulae hold true when we are lifting a principal \(\mathbb{R}^x\)-bundle structure to the structures \(\Pi Th\) and \(\Pi T^*h\) on \(\Pi TP\) and \(\Pi T^*P\), respectively. As \(T\mathcal{M}\) and \(T^*\mathcal{M}\) are simultaneously vector bundles and the both structures are compatible, \(TP\) and \(T^*P\) (as well as \(\Pi TP\) and \(\Pi T^*P\)) are canonically linear principal \(R^x\)-bundles.

Remark 4.2. We will not especially indicate the lifting of vector bundle structures and the \(\mathbb{R}^x\)-action to the cotangent bundle and will write simply \(T^*P\) instead writing \(T^*(1)P\) or similar for an \(n\)-linear principal \(\mathbb{R}^x\)-bundle \(P\), just assuming implicitly that on \(T^*P\) the phase lifts are taken by default.

Suppose now that \((P, h^0, h^1)\) is a 1-graded principal \(\mathbb{R}^x\)-bundle with a principal bundle and a 1-graded manifold fibrations \(\tau_0 : P \to P_0\) and \(\tau_1 = h^0_1 : P \to P_1\), respectively, and with \(\text{the Euler vector fields } \Delta^0, \Delta^1\). According to Theorem 4.1, \(P_0 = P / \mathbb{R}^x\) is canonically a 1-graded manifold with the reduced homogeneity structure \(\tilde{h}^1\) and the bundle structure \(\tau_1 : P_0 \to \mathcal{M}\).

Similarly, the \(\mathbb{R}^x\)-action \(h^0\) is projectable onto the base \(P_1\) of the 1-graded bundle structure and induces there a principal \(\mathbb{R}^x\)-bundle structure over a base \(P_1 / \mathbb{R}^x\) which can be identified with \(\mathcal{M}\). Recall that \(P_1\) is canonically embedded in \(P\) (cf. Remark 2.2). Thus the picture is analogous to the case of a double vector bundle (see [24]),

\[
P \overset{\tau_0}{\underset{\tau_1}{\longrightarrow}} P_0, \quad P_1 \overset{\tau_0}{\underset{\tau_1}{\longrightarrow}} \mathcal{M}
\]

eXcept for the fact that one structure is not a vector bundle but a principal \(\mathbb{R}^x\)-bundle, thus \(P_0\) is not canonically embedded in \(P\). In this picture however, we have no core, so that the map

\[
\tau = (\tau_1, \tau_0) : P \to P_1 \times \mathcal{M} P_0
\]

is a fibration with one-point fibers, thus a diffeomorphism.

Indeed, using the local coordinates \((t, x^a)\) in \(P\) described in Theorem 4.1 we can write the map \(\tau\) as the identity, since coordinates on \(P_0\) are \((x^a)\), coordinates on \(P_1\) are \((t, x^A)\), and coordinates on \(\mathcal{M}\) are \((x^A)\), where with \((x^A)\) we denoted coordinates on \(P_0\) of degree 0. We can summarize these observations as follows.

\[
\tau = (\tau_1, \tau_0) : P \to P_1 \times \mathcal{M} P_0
\]
Theorem 4.2. Any 1-graded principal $\mathbb{R}^\times$-bundle $(P, h^0, h^1)$ induces the commutative diagram of morphisms of bundle structures and is therefore equivalent to the product $P_1 \times_M P_0$ of the principal $\mathbb{R}^\times$-bundle $P_1 = h_0^1(P)$ over $M = P_1/\mathbb{R}^\times$ and the 1-graded bundle $P_0 = P/\mathbb{R}^\times$ fibred over $M$, with the obvious 1-graded principal $\mathbb{R}^\times$-bundle structure on the product.

Remark 4.3. A particular example of a 1-graded principal $\mathbb{R}^\times$-bundle is a linear principal $\mathbb{R}^\times$-bundle for which $h^1$ represents homotheties in the vector bundle $\tau_1 : P \to P_1$. Here, $P_1$ represents just the 0-section in the vector bundle.

5 Contact forms and contact structures

5.1 Contact forms

Note that sections of a vector bundle over $M$ form canonically a $C^\infty(M)$-bimodule: we usually think about the left module structure, but the right module structure comes automatically according to the standard rules of super-commutation. Since the left- and the right-module structures are generally different, we encounter some problems when dealing with contractions. The standard (left) contraction $i_X \alpha$ of a vector field $X$ with a $k$-form $\alpha$ is bilinear with respect to the left-module structure on vector fields and the right module structure on forms. In particular, for a vector field $X$ and a one-form $\alpha$, the contractions $i_X \alpha$ and $i_{\alpha} X$ give, in general, different results. Of course, we can consider as well the right contraction $i_X' \beta$ which is bilinear with respect to the left-module structure on forms and the right module structure on vector fields. The left and the right module structures will be indicated by the subscripts ‘l’ and ‘r’. For example, $\Omega^1(M)$ is canonically the dual module of $\mathcal{X}^1(M)$.

Similarly, speaking about subbundles and quotient bundles as well as direct sums or products we will usually refer to certain locally free sub-bimodules or quotient bimodules. If, say, a left submodule is not generated by even or odd sections, it is, in general, not a bimodule of sections of a super-vector bundle, even if it is locally free.

Recall that the parity diffeomorphism of $M$ can be extended to arbitrary tensors: we will write $\overline{K}$ for the parity diffeomorphisms applied to a tensor field $K$, i.e. $\overline{K} = K^0 - K^1$, where $K^0, K^1$ are the even and the odd part of $K$, respectively.

Let us observe that if $\alpha$ is an arbitrary 1-form on $M$, not assumed to be even, so it cannot be viewed as a smooth map $\alpha : M \to T^*M$ between supermanifolds, the pull-back $\alpha^*(\beta)$ of a differential form $\beta$ on $T^*M$ which is basic or homogeneous of degree 1 is nevertheless well defined. In particular, the pull-back $\alpha^*(\sigma_M)$ of the canonical Liouville one-form $\sigma_M$ on $T^*M$ makes sense and $\alpha^*(\sigma_M) = \alpha$. Similarly, the pull-back of the canonical symplectic form $\omega_M = d\sigma_M$ on $T^*M$ gives $\alpha^*(\omega_M) = d\alpha$.

Indeed, in local coordinates $(x^a)$ in $M$ and the adapted Darboux coordinates $(x^a, p_i)$ in $T^*M$, we have $\alpha = \sum_a \alpha_a(x)dx^a$, so that

$$\alpha^*(\sigma_M) = \alpha^* \left( \sum_a p_a dx^a \right) = \sum_a \alpha_a(x)dx^a = \alpha,$$

as $\alpha^*$ is, by convention, the identity map on basic functions and intertwines the de Rham differential. In the sequel we will extensively use this observation.

A 1-form $\alpha \in \Omega^1(M)$ will call nowhere-vanishing if there is a vector field $Y \in \mathcal{X}^1(M)$ such that $i_Y \alpha = 1$. It is easy to see that, equivalently, the one-form $\alpha$ is nowhere-vanishing if and only if there is $Y' \in \mathcal{X}^1(M)$ such that $i_{Y'} \alpha = i_{\alpha} Y' = 1$. This implies that $\alpha$ and $Y'$ generate free 1-dimensional submodules $\langle \alpha \rangle$ and $\langle Y' \rangle$ in $\Omega^1(M)$ and $\mathcal{X}^1(M)$, respectively, and that we have the splittings

$$\mathcal{X}^1(M) = \langle Y \rangle \oplus \text{Ker}(\alpha), \quad \Omega^1(M) = \langle \alpha \rangle \oplus \text{Ker}(Y'),$$

where Ker$(\alpha) = \{ X \in \mathcal{X}^1(M) : i_X \alpha = 0 \}$ and Ker$(Y') = \{ \beta \in \Omega^1(M) : i_\beta Y' = 0 \}$ are the left kernels of $\alpha$ and $Y'$, respectively. Of course, the above splittings depend on the choice of $Y$ and $Y'$.

A two form $\omega$ is called non-degenerate if the contraction $\mathcal{X}^1 \ni X \mapsto i_X \omega \in \Omega^2(M)$ defines a module isomorphism, and symplectic if $\omega$ is non-degenerate and closed. Since on a supermanifold a 1-form need not to be a nilpotent element in the Grassmann algebra of differential forms and, in general, there is no module of ‘top forms’, it is clear that a contact form $\alpha$ has to be defined differently than via non-vanishing of $\alpha(\frac{d\alpha}{n})^n$, for a certain $n$, like in the standard differential geometry.
**Theorem 5.1.** Let $\alpha$ be a nowhere-vanishing 1-form on a supermanifold $\mathcal{M}$. The following are equivalent.

(a) $\langle \alpha \rangle$ and $\text{Im}(d\alpha) = \{ \beta \in \Omega^1(\mathcal{M}) : \exists X \in \mathcal{X}^1(\mathcal{M}) [\beta = i_Xd\alpha] \}$ are complementary submodules of $\Omega^1_t(\mathcal{M})$, 

\begin{equation}
\Omega^1_t(\mathcal{M}) = \langle \alpha \rangle \oplus \text{Im}(d\alpha),
\end{equation}

and the contraction $X \mapsto i_Xd\alpha$ constitutes a $C^\infty(\mathcal{M})$-module isomorphism between $\text{Ker}(\overline{\alpha})$ and $\text{Im}(d\alpha)$; 

(b) $\Omega^1_t(\mathcal{M}) = \langle \alpha \rangle \oplus \text{Im}(d\alpha)$ and 

\[ \#_\alpha : \mathcal{X}^1(\mathcal{M}) \rightarrow \Omega^1_t(\mathcal{M}), \quad \#_\alpha(X) = i_X\overline{\alpha} \cdot \alpha + i_Xd\alpha, \]

is a $C^\infty(\mathcal{M})$-module isomorphism;

(c) the two-form 

\begin{equation}
\omega = d(t \cdot \alpha) = dt \cdot \alpha + t \cdot d\alpha
\end{equation}

is symplectic on the trivial $\mathbb{R}^\times$-principal bundle $\mathbb{R}^\times \times \mathcal{M}$.

**Proof.** (a)$\Rightarrow$(b) Since $\alpha$ is nowhere-vanishing, we can find a vector field $Y$ with $i_Y\alpha = 1$. Then, $\overline{\alpha} = 1$, so $\overline{\alpha}$ is nowhere-vanishing as well. According to the splitting $\mathcal{X}^1(\mathcal{M}) = \langle Y \rangle \oplus \text{Ker}(\overline{\alpha})$ and the fact that the contraction $X \mapsto i_Xd\alpha$ constitutes a $C^\infty(\mathcal{M})$-module isomorphism between $\text{Ker}(\overline{\alpha})$ and $\text{Im}(d\alpha)$, the vector field $Y$ can be uniquely chosen from $\text{Ker}(d\alpha)$. We will call it the left Reeb vector field of $\alpha$. The map $\#_\alpha$ is clearly a $C^\infty(\mathcal{M})$-module isomorphism. This is because $\#_\alpha(X)$ acts on $\text{Ker}(\overline{\alpha})$ as an isomorphism onto $\text{Im}(d\alpha)$, and on $\langle Y \rangle$ as an isomorphism onto $\langle \alpha \rangle$.

(b)$\Rightarrow$(c) The two-form 

\[ \omega = d(t \cdot \alpha) = dt \cdot \alpha + t \cdot d\alpha, \quad t \neq 0 \]

is clearly closed, so we will show that it is non-degenerate. Any vector field on $\mathbb{R}^\times \times \mathcal{M}$ is of the form $\tilde{X} = f\partial_t + gX$, where $X$ is a vector field on $\mathcal{M}$ and $f, g$ are smooth function on $\mathbb{R}^\times \times \mathcal{M}$. We have 

\[ i_{\tilde{X}}\omega = f \cdot \alpha - g \cdot i_X(\overline{\alpha})dt + t \cdot g \cdot i_Xd\alpha, \]

so, in view of the decomposition (5.1), $i_{\tilde{X}}\omega = 0$ implies that $f = 0$, $g \cdot i_X(\alpha) = 0$, and $g \cdot i_Xd\alpha = 0$, thus $\tilde{X} = 0$. It remains to show that the contraction with $\omega$ is ‘onto’. Take $\tilde{\beta} \in \Omega^1(\mathbb{R}^\times \times \mathcal{M})$, $\tilde{\beta} = Fdt + G\beta$, where $\beta$ is a one-form on $\mathcal{M}$ and $F, G$ are smooth function on $\mathbb{R}^\times \times \mathcal{M}$. Take $X, \overline{Y} \in \mathcal{X}^1(\mathcal{M})$ such that $\#_\alpha(X) = \beta$ and $\#_\alpha(\overline{Y}) = \alpha$. Then, as easily seen, we get $\beta$ by contracting $\omega$ with $(t^{-1}G) \cdot X + (t^{-1}G - F) \cdot \overline{Y} + (G \cdot i_X(\overline{\alpha})) \partial_t$.

(c)$\Rightarrow$(a) According to (5.1), the contraction $i_X\omega$ is a one-form on $\mathcal{M}$ (a basic one-form) if and only if $X \in \text{Ker}(\overline{\alpha})$ and $f, tg$ are smooth function on $\mathcal{M}$. We can therefore write such $\tilde{X}$ uniquely as $\tilde{X} = f(x)\partial_t + t^{-1}X$, where $X \in \text{Ker}(\overline{\alpha})$. Since the form $\omega$ is non-degenerate, the map $f(x)\partial_t + t^{-1}X \mapsto f(x)\alpha + i_Xd\alpha$ is an isomorphism of the module $C^\infty(\mathcal{M}) \oplus \text{Ker}(\overline{\alpha})$ onto $\Omega^1_t(\mathcal{M})$. In particular, we have the splitting (5.1) and a module isomorphism $\text{Ker}(\overline{\alpha}) \ni X \mapsto i_X\alpha \in \text{Im}(d\alpha)$.

**Definition 5.1.** A nowhere-vanishing one-form $\alpha$ on a manifold $\mathcal{M}$ is called a contact form, if it satisfies one of the equivalent conditions of Theorem 5.1.

**Example 5.1.** On $\mathbb{R}^{1|1}$ with the even coordinate $x$ and the odd coordinate $\theta$, the one-form $\alpha = dx + \theta d\theta$ is an even contact form.

**Proposition 5.1.** Given an invertible smooth even function $\psi$ on a manifold $\mathcal{M}$, a one-form $\alpha$ on $\mathcal{M}$ is a contact form if and only if $\psi \cdot \alpha$ is a contact form.

**Proof.** According to Theorem 5.1(c), it suffices to show that if $d(t \cdot \alpha)$ is non-degenerate, then $d(t \cdot \psi \cdot \alpha)$ is non-degenerate. But $d(t \cdot \psi \cdot \alpha) = d(t \psi \cdot \alpha)$ is the pull-back $\Psi^*(d(t \cdot \alpha))$ of $d(t \cdot \alpha)$ with respect to the diffeomorphism $\Psi : \mathbb{R}^\times \times \mathcal{M} \rightarrow \mathbb{R}^\times \times \mathcal{M}$, $\Psi(t, x^\alpha) = (t \cdot \psi(x), x^\alpha)$, so $d(t \cdot \psi \cdot \alpha)$, exactly like $d(t \cdot \alpha)$, is non-degenerate.

We will say that the contact forms $\alpha$ and $\psi \cdot \alpha$ are equivalent.
Example 5.2. Note that in the above proposition we cannot take \( \psi \) being an arbitrary invertible function. Indeed, if in the example we put \( \psi = 1 + \theta \), then \( \psi \cdot \alpha = (1 + \theta)dx + \theta d\theta \) is no longer a contact form. The point is that \( \psi \) here is not even.

In what follows we will reduce ourselves to the case of contact forms of a determined parity. It is well known that on even manifolds contact forms can be written locally as \( \alpha = dz - p_a dx^a \). This can be easily extended to the super-case as follows (see [58, 59] for the complex setting). The proof refers to the results describing the local form of even and odd symplectic structures [32, 38, 58, 61].

Theorem 5.2. (a) Every even contact form \( \alpha \) on a supermanifold \( M \) can be locally written as

\[
\alpha = dz - p_a dx^a + \frac{\epsilon_j}{2} \theta^j d\theta^j, \quad \epsilon_j = \pm 1
\]

for certain local coordinates \((z, x^a, p_b, \theta^i)\) on \( M \) in which \((z, x^a, p_b)\) are even and \((\theta^i)\) are odd.

(b) Every odd contact form \( \alpha \) on a supermanifold \( M \) can be locally written as

\[
\alpha = d\xi - \theta^a dx^a
\]

for certain local coordinates \((x^a, \theta^b, \xi)\) on \( M \) in which \((x^a)\) are even and \((\theta^b, \xi)\) are odd.

Proof. It is easy to see that the above one forms are contact forms. Suppose now that a contact form \( \alpha \) is even (resp., odd) and let \( Y \) be the Reeb vector field for \( \alpha \), i.e. \( i_Y \alpha = 1 \) and \( i_Y d\alpha = 0 \). In particular, \( Y \) is even (resp., odd) nowhere vanishing and involutive \([Y, Y] = 0\) (which is a non-trivial condition in the odd case). This implies (cf. [39, 61]) that we can choose local coordinates \((y^i, y^j)\), \( i > 0 \), such that \( Y = \partial_0 = \frac{\partial}{\partial y^0} \) and \( \alpha = dy^0 - \alpha' \). Since \( i_{\partial_0} \alpha' = 0 \) and \( i_{\partial_0} d\alpha' = -i_{\partial_0} d\alpha = 0 \), we conclude that \( \alpha' \) depends only on coordinates \( y^i \), \( i > 0 \), and, due to properties of the contact form, \( d\alpha' \) is a symplectic form in coordinates \( y^i \). Now, we will use the local description of a symplectic form (Darboux Theorem). Since now the cases split, let us assume that \( \alpha \) is odd (the proof in the even case is similar). We can therefore assume that \((y^i) = (x^a, \theta^b), a, b = 1, \ldots, r\), where \( x^a \) are even and \( \theta^b \) are odd, and that \( d\alpha' = d\theta^a dx^a \). Since \( d(\theta^a dx^a) = d\alpha' \), the form \( \alpha'' = \alpha' - \theta^a dx^a \) is closed, so locally exact, \( \alpha'' = d f \) for a certain odd function \( f \) in coordinates \((x^a, \theta^b)\). Now we can introduce new coordinates \((x^a, \theta^b, \xi)\), where \( \xi = y_0 + f(x^a, \theta^b) \), in which \( \alpha \) takes the form \((5.5)\).

Note that the above formulae make sense only for contact forms with a determined parity. In general, there is much more freedom in choosing a contact form. For instance, if \( \alpha \) is an even contact form and \( F \) is any odd function, then \( \alpha + dF \) is again a contact form but we cannot build a new coordinate adding \( F \) to \( z \), as the function \( z + F \) has not a fixed parity. In what follows we will work only with even or odd contact forms.

If \( \alpha \) is a nowhere-vanishing even (resp., odd) one-form on a manifold \( M \), then it spans a trivial one-dimensional even vector subbundle \([\alpha]\) in \( T^* M \) (resp., in \( \Pi T^* M \)). With \( \alpha \) we associate a canonical smooth embedding \( I_\alpha : \mathbb{R} \times M \rightarrow T^* M \) (resp., \( I_\alpha : \mathbb{R} \times M \rightarrow \Pi T^* M \)) inducing an isomorphism of \( \mathbb{R} \times M \) with \([\alpha]\). If \((x^a)\) are local coordinates on \( M \) in which \( \alpha = f_b(x) dx^b \) and \((x^a, p_a)\) are the corresponding Darboux coordinates, then the canonical embedding \( I_\alpha \) reads

\[
I_\alpha (t, x^a) = (x^a, t \cdot f_b(x)).
\]

We can also consider the principal \( \mathbb{R}^\times \)-bundle \([\alpha]^\times\) which is an open submanifold in \([\alpha]\) described locally by the condition \( t \neq 0 \). It is easy to see that the restriction of the canonical symplectic form \( \omega_M \) (resp., \( \omega_M^H \)) to \([\alpha]^\times\) identified with \( \mathbb{R}^\times \times M \) is represented by \( I_\alpha^* (\omega_M) \) (resp., \( I_\alpha^H (\omega_M^H) \)) and coincides with \((5.2)\). Thus, in view of Theorem 4.1, we get the following.

Theorem 5.3. Let \( \alpha \) be a nowhere-vanishing even (resp., odd) 1-form on a manifold \( M \). Then, \( \alpha \) is a contact form if and only if the trivial principal \( \mathbb{R}^\times \)-bundle \([\alpha]^\times \simeq \mathbb{R}^\times \times M \) is, via the embedding \( I_\alpha \), a symplectic submanifold of \( T^* M \) (resp., \( \Pi T^* M \)).

In this case, the contact form \( \alpha \) can be reconstructed from the symplectic form \( \omega = I_\alpha^* (\omega_M) \) (resp., \( \omega = I_\alpha^H (\omega_M^H) \)) thanks to the formula

\[
i_\Delta \omega(t, x) = t \cdot \alpha(x),
\]
where $\Delta = \Delta[\alpha] = t \partial_t$ is the fundamental vector field of the $\mathbb{R}^\times$-principal bundle $[\alpha]^\times$. Moreover, this symplectic form is homogeneous with respect to the $\mathbb{R}^\times$-action, i.e.

$$
(h_t)^* (\omega) = t \omega, \quad \text{for } t \neq 0,
$$

so that

$$
L_\Delta \omega = d(t \cdot \alpha) = \omega.
$$

5.2 Contact structures

**Definition 5.2.** An $\mathbb{R}^\times$-principal bundle $(P,h)$ equipped with a 1-homogeneous symplectic form $\omega$, i.e., a symplectic form satisfying (5.8), we will call a symplectic principal $\mathbb{R}^\times$-bundle. More generally, a principal $\mathbb{R}^\times$-bundle $(P,h)$ equipped with a homogeneous Poisson structure $\mathcal{J}$ of degree -1, i.e., a Poisson structure satisfying

$$
(h_t)_* (\mathcal{J}) = -t \mathcal{J}, \quad \text{for } t \neq 0,
$$

we will call a principal Poisson $\mathbb{R}^\times$-bundle.

As a direct consequence of Theorem 5.3 we obtain the following.

**Theorem 5.4.** Let $\mathcal{C}$ be an even line subbundle (vector subbundle of rank 1) in $T^* \mathcal{M}$ (resp., $\Pi T^* \mathcal{M}$). The following are equivalent.

(a) $\mathcal{C}$ is locally generated by even contact (resp., odd contact) one-forms on $\mathcal{M}$.

(b) $\mathcal{C}^\times$ is a symplectic submanifold of $T^* \mathcal{M}$ (resp., $\Pi T^* \mathcal{M}$).

Moreover, any even (resp., odd) symplectic principal $\mathbb{R}^\times$ bundle over $\mathcal{M}$ has a canonical symplectic embedding into $T^* \mathcal{M}$ (resp., $\Pi T^* \mathcal{M}$) as a principal $\mathbb{R}^\times$-bundle of the form $\mathcal{C}^\times$. In other words the association $\mathcal{C} \mapsto \mathcal{C}^\times$ establishes a one-to-one correspondence between line subbundles in $T^* \mathcal{M}$ (resp., $\Pi T^* \mathcal{M}$) locally generated by even (resp., odd) contact forms and even symplectic (resp., odd symplectic) principal $\mathbb{R}^\times$-bundles over $\mathcal{M}$.

**Proof.** We will work with the even case; the proof in the odd case is parallel. In view of Theorem 5.3 the only nontrivial part is to show that any symplectic principal $\mathbb{R}^\times$-bundle $(P,\omega,h)$ over $\mathcal{M}$ is of the form $\mathcal{C}^\times$ for a $\mathcal{C}$ as in (a).

If $\Delta$ is the Euler vector field associated with $h$, then the 1-form $\tilde{\alpha} = i_\Delta \omega$ on $P$ is semi-basic and defines a map $\Psi : P \to T^* \mathcal{M}$ which is an embedding of the principal $\mathbb{R}^\times$-bundle $P$ onto $\mathcal{C}^\times \subset T^* \mathcal{M}$, where $\mathcal{C}$ is the line subbundle in $T^* \mathcal{M}$ spanned by the image of $\tilde{\alpha}$. Indeed, let us write in local bundle coordinates $(t,x)$ in $P$, with $\Delta = t \partial_t$, the symplectic form $\omega$ as $\omega = dt \wedge \alpha + \omega'$ for a certain semi-basic one-form $\alpha$ and a semi-basic two-form $\omega'$. Since $i_\Delta \omega' = 0$ and $i_\Delta \alpha = 0$, we get $\tilde{\alpha} = t \cdot \alpha$. As $\Delta$ is 0-homogeneous, $i_\Delta \omega$ is 1-homogeneous, so $\alpha$ is homogeneous of degree 0, thus $\mathbb{R}^\times$-invariant. Being simultaneously semi-basic it is actually basic, so it can be regarded as a 1-form on $\mathcal{M}$. Moreover, $d(t \cdot \alpha) = \omega$, so that $\alpha$ is a contact form. Hence, $\Psi(t,x) = t \cdot \alpha(x)$ is a principal bundle isomorphism which does not depend on the choice of homogeneous coordinates $(t,x)$, since a change in the choice of the local trivialization of $P$ results in multiplication of $\alpha$ by an invertible function on $\mathcal{M}$ and in multiplication of $t$ by the inverse of this function. \qed

**Definition 5.3.** A contact structure (resp., an odd contact structure) on a manifold $\mathcal{M}$ is an even line subbundle $\mathcal{C}$ in the cotangent bundle $T^* \mathcal{M}$ (resp., $\Pi T^* \mathcal{M}$) generated locally by contact forms. Equivalently, such a contact structure can be viewed also as an even (resp., odd) symplectic principal $\mathbb{R}^\times$-bundle.

**Remark 5.1.** The symplectic manifold $(\mathcal{C}^\times,\omega)$ is usually called the symplectization of the contact structure $\mathcal{C}$ (cf., $\mathcal{C}$). We prefer, finding it much more elegant and fruitful, to view the symplectization as the contact structure itself and to identify contact structures with symplectic principal $\mathbb{R}^\times$-bundles. We will use interchangeably both descriptions of a contact structure in the sequel.

Note also that the vector bundle $\mathcal{C}^\times$, dual to $\mathcal{C}$, is clearly the quotient $T\mathcal{M}/C^0$, where $C^0$ is the annihilator of $\mathcal{C}$, so locally $C^0$ is $\text{Ker}(\alpha)$ for any contact form $\alpha$ generating $\mathcal{C}$. Hence, local trivializations of $\mathcal{C}^\times$ are represented by cosets of Reeb vector fields of the corresponding contact forms. The subbundle $C^0$ in $T\mathcal{M}$ was another concept of a contact structure.
6 Graded contact structures

6.1 Contact structures of weight \( k \)

**Definition 6.1.** Let \( \mathcal{M} \) be an \( n \)-graded manifold. An even line subbundle \( C \) in \( T^*\mathcal{M} \) (resp., \( \Pi T^*\mathcal{M} \)) is a contact (resp., an odd contact) structure of weight \( k \in \mathbb{N^n} \) on \( \mathcal{M} \) if it is locally generated by contact one-forms of weight \( k \). If \( \mathcal{M} \) is an \( n \)-vector bundle, a contact structure of weight \( 1^n \) on \( \mathcal{M} \) we will call an \( n \)-linear contact structure or a contact \( n \)-vector bundle. A \( k \)-manifold \( \mathcal{M} \) equipped with a contact structure of weight \( k \) we will call a contact \( k \)-manifold.

**Remark 6.1.** The above definition is correct, i.e., the weight does not depend on the choice of the homogeneous contact form. Indeed, if \( \alpha' = f \cdot \alpha \) for certain homogeneous and invertible function, thus function of weight 0.

**Theorem 6.1.** An \( n \)-graded manifold \( (\mathcal{M}, h^1, \ldots, h^n) \) admits a contact structure of weight \( k \) only if \( M \) is of degree \( \leq k \). In particular, the \( k \)-phase lift \( \pi^* \mathcal{M} = (T^* \mathcal{M}, h^0, T^*(k_1)h^1, \ldots, T^*(k_n)h^n) \), where \( h^0 \) is the action of the multiplicative \( \mathbb{R} \) by homotheties in the vector bundle \( T^* \mathcal{M} \), is an \( \mathbb{N}^{n+1} \)-graded manifold.

**Proof.** Indeed, let \( \alpha \) be a local contact form generating the structure. Since \( \alpha \) and \( h^0 \alpha \) do generate the cotangent bundle, since \( w(\partial_{x^a}) \leq 0 \) and \( w(\alpha) = w(\alpha) = k \), all homogeneous local coordinates \( x^a \) have weight \( \leq k \).

**Definition 6.2.** An (even or odd) \( n \)-graded symplectic principal \( \mathbb{R}^\times \)-bundle of weight \( k \in \mathbb{N}^n \) on \( \mathcal{M} \) is an \( n \)-graded principal \( \mathbb{R}^\times \)-bundle \( (\mathcal{P}, h^0, h^1, \ldots, h^n) \) over \( \mathcal{M} = \mathbb{P} \mathbb{R}^\times \), equipped additionally with an (even or odd, respectively) \( \mathbb{R}^\times \)-homogeneous symplectic form \( \omega \) of weight \( k \).

\[
(h^i_\alpha)^* \omega = tw, \quad (h^i_\alpha)^* \omega = \lambda_i \omega \quad \text{for} \quad t \in \mathbb{R}, \quad i = 1, \ldots, n.
\]

An \( n \)-linear (even or odd) symplectic principal \( \mathbb{R}^\times \)-bundle \( (\mathcal{P}, h^0, \ldots, h^n) \) is an \( n \)-linear principal \( \mathbb{R}^\times \)-bundle \( (\mathcal{P}, h^0, \ldots, h^n) \) equipped with an (even or odd, respectively) symplectic form \( \omega \) that is 1-homogeneous, \( (h^0_i)^* \omega = tw_0 \) for \( t \neq 0 \), with respect to all actions \( h^0 \) (in particular, with respect to the Euler vector fields \( \Delta_0, \ldots, \Delta_0 \)).

More generally, a \( n \)-linear (even or odd) principal Poisson \( \mathbb{R}^\times \)-bundle \( (\mathcal{P}, \mathcal{J}, h^0, \ldots, h^n) \) is a similar structure with the role of the homogeneous symplectic form played by a Poisson tensor \( \mathcal{J} \) on \( \mathcal{P} \) which is homogenous of degree \(-1\) with respect to all homogeneity structures, i.e. \( (h^i_\alpha)^* \mathcal{J} = \mathcal{J} \) for \( t \neq 0 \) (in particular, \( \mathcal{E}_\Delta, \mathcal{J} = -\mathcal{J} \) for the Euler vector fields \( \Delta \), \( i = 0, \ldots, n \)).

**Example 6.1.** Let \( (\mathcal{P}, \mathcal{J}, h^0) \) be a principal Poisson \( \mathbb{R}^\times \)-bundle. It is easy to see that the complete tangent lift \( dT \mathcal{J} \) of \( \mathcal{J} \) (cf. \[26, 27\]) is homogeneous of bi-degree \((-1, -1)\) on \( TP \). In this way we obtain a linear principal Poisson \( \mathbb{R}^\times \)-bundle \( (\mathcal{P}, \mathcal{J}, h^0, h^1) \), the complete tangent lift of \( (\mathcal{P}, \mathcal{J}, h^0) \), where \( P = TP \), \( \mathcal{J} = dT \mathcal{J} \), \( h^0 = T \mathcal{J} \), and \( h^1 \) representing the canonical vector bundle structure \( TP \rightarrow P \).

**Theorem 6.2.** The correspondence \( C \mapsto C^\times \) establishes a one-to-one correspondence between even (resp., odd) contact structures of weight \( k \in \mathbb{N}^n \) on an \( n \)-graded manifold \( \mathcal{M} \) and even (resp., odd) \( n \)-graded symplectic principal \( \mathbb{R}^\times \)-bundles of weight \( k \in \mathbb{N}^n \) over \( \mathcal{M} \). In other words, contact structures of weight \( k \in \mathbb{N}^n \) can be viewed as \( n \)-graded symplectic principal \( \mathbb{R}^\times \)-bundles of weight \( k \).

**Proof.** As the proof is completely analogous in the odd case, to fix our attention let us assume that the case is even. Let \( C \subset T^*\mathcal{M} \) be a contact structure of weight \( k \in \mathbb{N}^n \) on an \( n \)-graded manifold \( (\mathcal{M}, h^1, \ldots, h^n) \). On the cotangent bundle \( T^*\mathcal{M} \) consider the phase lifts \( h^i = T^*(k_i)h^i \), \( i = 1, \ldots, n \), which induce an \( n \)-graded manifold structure on \( T^*\mathcal{M} \) (Theorem 6.1).

It is easy to see that if \( \alpha = \alpha(x) dx^a \) is a contact 1-form homogeneous of weight \( k \) on \( \mathcal{M} \), then the homogeneity structures \( h^i \), \( i = 1, \ldots, n \), are preserved the one-dimensional subbundle \( [\alpha] \) of \( T^*\mathcal{M} \) spanned by \( \alpha \). Indeed, if \( X_j = f^a_j(x)\partial x^a \), \( j = 1, \ldots, m-1 \), is a basis of local sections of \( \mathrm{Ker}(\alpha) \subset T\mathcal{M} \), then the local functions \( \iota_{X_j} \) define the submanifold \([\alpha]\) of \( T^*\mathcal{M} \) via the constraints \( f^a_j(x) = 0 \). Since \( \alpha \) is of weight \( k-\omega_a \), \( f^a_j(x)\alpha(x) = 0 \), the function \( f^a_j(x) \) is of weight \( w^a_j - k_j \). As \( h^i = e^{(k_i - w^a_j)\alpha} \), \( \alpha \neq 0 \) we have

\[
\iota_{X_j} \circ h^i = \left( \frac{w^a_j - k_j}{f^a_j(x)} \right) \left( e^{(k_i - w^a_j)\alpha} \right) = f^a_j(x) = \iota_{X_j}.
\]
therefore the constraints are preserved.

This implies that if $C \subset T^*M$ is a contact structure of weight $k$, then the homogeneous structures $h^i$ induce on $C^\times$ a structure of an $n$-graded principal $\mathbb{R}^\times$-bundle. Moreover, since the canonical symplectic form $\omega_{T^*M}$ is 1-homogeneous with respect to the canonical homogeneity structure $h^1$ in the vector bundle $T^*M$, and of weight $k$ with respect to $h^1, \ldots, h^n$, its restriction, $\omega$, to $C^\times$ is homogeneous of weight $k$.

Conversely, if $(P, h^0, h^1, \ldots, h^n)$ is an $n$-graded symplectic principal $\mathbb{R}^\times$-bundle over $M$ of weight $k$, then $h^i$, commuting with $h^0$, projects to a homogeneity structure $h^i$, $i = 1, \ldots, n$, on $M$ that turns $M$ into an $n$-graded manifold (Theorem 1.1). Moreover, if $A^0$ is the Euler vector field associated with the $\mathbb{R}^\times$-action $h^0$, then the 1-form $\alpha = i_{A^0}\omega$ on $P$ defines a map $\Psi : P \to T^*M$ which is a symplectic isomorphism of the principal $\mathbb{R}^\times$-bundle $P$ onto $C^\times \subset T^*M$, where $C$ is the line subbundle in $T^*M$ spanned by the image of $\alpha$ (see the proof of Theorem [5.4]). In the local coordinates $(t, x)$ described in Theorem [1.1], $\alpha(t, x) = t \cdot \alpha(x)$ for a contact form $\alpha$ and the image of $\Psi$ is locally generated by $\alpha$. As $\omega$ is of weight $k$, $t \cdot \alpha = i_{A^0}\omega$, and $t$ is of weight 0, the contact form is of weight $k$.

6.2 Canonical contact structures

To present some examples of canonical contact structures, let us start with an even line bundle $\tau : L \to M$ (for an odd line bundle the construction is similar). We can choose an open covering of $M$ by coordinate charts $U_i$ with coordinates $(x^a_i)$ and an open covering of $L$ by coordinate charts $V_i = \tau^{-1}(U_i)$, diffeomorphic to $\mathbb{R} \times U_i$, with coordinates $(z_i, x^a_i)$, where $z_i$ is even and varying through all $\mathbb{R}$. Let us fix $i$ and $j$. The change of coordinates takes the form

$$x^a_j = \varphi^a(x^a_i), \quad z_j = \psi(x^a_i) \cdot z_i,$$

for a certain diffeomorphism $\varphi : U_i \to U_j$ and an even smooth nowhere-vanishing function $\psi(x)$. The first jet bundle $J^1L$ is a vector bundle over $M$ of first jets of local sections of $L$ defined by local data as follows. In a given trivialization $V_i \simeq \mathbb{R} \times U_i$, any section $z_i = f(x^a_i)$ is represented by the function $f$ and $J^1L$ over $V_i$ is diffeomorphic to $\mathbb{R} \times T^*V_i$ with coordinates $(z_i, x^a_i, p^i_b)$. The coordinate change in $L$ results in the coordinate change in $J^1L$ of the form

$$x_j = \varphi(x^a_i), \quad z_j = \psi(x^a_i) \cdot z_i, \quad p^i_a = \left((\psi(x^a_i)) \cdot p^i_b + \frac{\partial \psi}{\partial x^a_i}(x^i \cdot z_i) \frac{\partial x^b_j}{\partial x^a_i}\right).$$

The first jet bundle $J^1L$ is a vector bundle with local linear coordinates $z_i$ and $p^i_b$. Every section $f$ of $L$ induces a section $j^1(f)$ of this bundle, its first jet prolongation, given locally by $z_i = f(x^a_i)$, $p^i_b = \frac{\partial f}{\partial x^a_i}(x^i)$. A similar construction of $J^1L$ can be done when starting with an odd line bundle $L$.

The first jet bundle $J^1L$ is a tool for representing first-order differential operators on $L$ by vector bundle morphisms. Indeed, for any vector bundle $E$ over $M$, vector bundle morphisms $D \in \text{Hom}_M(J^1L, E)$ covering the identity on $M$ represent first-order differential operators $\tilde{D} \in \text{Sec}(\text{DO}^1(L, E))$ in the obvious way: $\tilde{D}(f) = D(j^1(f))$. We will often use the identification

$$\text{Hom}_M(J^1L, E) \simeq (J^1L)^* \otimes_M E \simeq \text{DO}^1(L, E).$$

In particular, $\text{DO}^1(L, L) = (J^1L)^* \otimes_M L$.

There is an even line subbundle $C_L$ in $T^*J^1L$ whose pull-back $J^1(f)^*(\beta)$ is 0 for all $f \in \text{Sec}(L)$. In local coordinates, these sections have the form $F(x^a_i) \cdot (dz_i - p^i_a dx^a_i)$, so the line bundle $C_L$ is generated locally by the contact form $dz_i - p^i_a dx^a_i$. This induces local coordinates $(t_i, z_i, x^a_i, p^a_i)$ on $C_L$. It follows that $C_L$ is an even contact structure, locally of the form $C^\times$ with no $\theta^0$ present.

Since $J^1L$ is a vector bundle over $M$, the cotangent bundle $T^*J^1L$ is a double vector bundle. The second non-obvious vector bundle structure is associated with a canonical fibration $\tau_1 : T^*J^1L \to (J^1L)^*$. The latter can be reduced to a vector bundle structure

$$\tau_1 : C^\times_L \to (L)^\times, \quad (t_i, z_i, x^a_i, p^a_i) \mapsto (t_i, x^a_i), \quad t \neq 0,$$

which makes $C^\times_L$ into a linear contact structure, as $\alpha_i = dz_i - p^a_i dx^a_i$ is linear. With $\Pi C^\times_L$ we will denote the parity shift associated with this vector bundle structure.
The local maps \((z_i, x_i^a, p_i^a) \mapsto (z_i, x_i^a)\) give rise to a vector bundle morphism \(J^1L \to L\) whose kernel has local coordinates \((x_i^a, p_i^a)\) with transformation rules
\[ x_j = \varphi(x_i), \quad p_j^i = \psi(x_i) \cdot p_i^j \frac{\partial x_i^j}{\partial x_j^i}, \]
so it can be identified with \(T^*M \otimes_M L\). Thus, we get the jet bundle exact sequences of vector bundle morphisms:
\[(6.5) \quad 0 \to T^*M \otimes_M L \to J^1L \to L \to 0,\]
if \(L\) is even, and
\[(6.6) \quad 0 \to \Pi T^*M \otimes_M L \to J^1L \to L \to 0,\]
if \(L\) is odd.

**Example 6.2.** (The case of the trivial line bundle)
Let \(L = \mathbb{R}^{1|0} \times M\) be the trivial even line bundle over \(M\). Then, \(J^1L = \mathbb{R}^{1|0} \times T^*M\) with local coordinates \((z, x^a, p_a)\), the jet bundle exact sequence of vector bundle morphisms reads as
\[ 0 \to T^*M \to \mathbb{R}^{1|0} \times T^*M \to \mathbb{R}^{1|0} \times M \to 0, \]
and the contact structure \(C_L\) is a trivial line subbundle in \(T^* (\mathbb{R}^{1|0} \times T^*M)\) generated by the contact form \(\alpha = dz - p_0dx^a\). Thus, \(C_L^\alpha = \mathbb{R}^\times \times \mathbb{R}^{1|0} \times T^*M\) with local coordinates \((t, z, x^a, p_a)\), \(t \neq 0\), and the symplectic form \(\omega = dt \wedge dz - p_a dt \wedge dx^a - tdp_a dx^a\) which is linear with respect to the vector bundle structure \(C_L^\alpha \ni (t, z, x^a, p_a) \mapsto (t, x^a) \in \mathbb{R}^\times \times M\).

If \(L = \mathbb{R}^{0|1} \times M\) is the trivial odd line bundle over \(M\), then \(J^1L = \mathbb{R}^{0|1} \times \Pi T^*M\) with local coordinates \((z, x^a, \bar{p}_a)\) (\(\bar{p}_a\) has the parity opposite to that of \(p_a\)), the jet bundle exact sequence of vector bundle morphisms reads as
\[ 0 \to \Pi T^*M \to \mathbb{R}^{0|1} \times \Pi T^*M \to \mathbb{R}^{0|1} \times M \to 0, \]
and the contact structure \(C_L\) is a trivial even line subbundle in \(\Pi T^* (\mathbb{R}^{0|1} \times \Pi T^*M)\) generated by the contact form \(\alpha = dz - \bar{p}_0dx^a\). Since in this case \(z\) is odd and \(x^a\) and \(\bar{p}_a\) have opposite parities, the contact form is odd. Thus, \(C_L^\alpha = \mathbb{R}^\times \times \mathbb{R}^{0|1} \times \Pi T^*M\) with local coordinates \((t, z, x^a, \bar{p}_a)\), \(t \neq 0\), and the symplectic form \(\omega = dt \wedge dz - \bar{p}_a dt \wedge dx^a - tdp_a dx^a\) which is odd and linear with respect to the vector bundle structure \(C_L^\alpha \ni (t, z, x^a, \bar{p}_a) \mapsto (t, x^a) \in \mathbb{R}^\times \times M\).

**Definition 6.3.** The even (resp., odd) contact structure \(C_L\) on \(J^1L\) associated with an even (resp., odd) line bundle \(L\) we will call canonical. The corresponding symplectic principal \(\mathbb{R}^\times\)-bundle \(C_L^\times\) will be called also, with some abuse of terminology, a canonical contact structure.

### 7 First jet bundles revisited

Let \(L\) be an even line bundle over \(M\). Note that the canonical contact structure \(C_L\) is a double vector bundle. Indeed, as \(J^1L\) is a vector bundle, \(T^*J^1L\) is a double vector bundle and \(C_L\) is a double vector subbundle of the latter. This is because the double vector bundle structure in \(T^*J^1L\) is determined by the commuting Euler vector fields: the Euler vector field \(\Delta_{T^*J^1L}\) of the vector bundle structure over \(J^1L\) and the Euler vector field \(\Delta_{J^1L}\), i.e. the phase lift of the Euler vector field \(\Delta_{J^1L}\) of the vector bundle structure on \(J^1L\), and the corresponding homotheties preserve \(C_L\). In local affine coordinates,
\[ \bar{\Delta}_0 = \Delta_{T^*J^1L|C_L} = t \partial t, \quad \bar{\Delta}_1 = \Delta_{J^1L|C_L} = p_a \partial p_a + z \partial z, \]
and the corresponding diagram of the double vector bundle structure reads

\[
\begin{array}{ccc}
C_L & \xrightarrow{\tau_0} & J^1L \\
\downarrow \tau_1 & & \downarrow \bar{\tau}_1 \\
L^* & \xrightarrow{\bar{\tau}_0} & M
\end{array}
\]
An important observation is that section of the line bundle $L$ can be identified with linear functions on $L^*$, i.e., in turn, with $(1,0)$-homogeneous functions with respect to the pair of the Euler vector fields $(\Delta_0, \Delta_1)$. The Euler vector fields $\Delta_i$, $i = 0, 1$, are tangent also to $C_L^\times$ which becomes a linear $\mathbb{R}^\times$-principal bundle $(C^\times_L, \widetilde{h^0}, \widetilde{h^1})$. The corresponding projections are $\tau_0 : C^\times_L \to J^1L$ and $\tau_1 : C^\times_L \to (L^*)^\times$, as the $\Delta_i$-invariant functions can be identified with functions on $(L^*)^\times$. To see the latter we will use another double bundle, namely $T^*(L^*)^\times$. One bundle structure is clearly the vector bundle structure over $(L^*)^\times$ associated with the Euler vector field $\Delta_0$ or the corresponding homogeneity structure $h^0$. The other is associated with the phase lift $h^0 = T^*h$ of the natural principal $\mathbb{R}^\times$-action $h$ on $(L^*)^\times$.

Let $\Delta_0$ be the Euler vector field associated with $h^0$. In local affine coordinates $(t, x^a)$, $t \neq 0$, on $(L^*)^\times$ and the corresponding Darboux coordinates $(t, x^a, z, p_a)$ on $T^*(L^*)^\times$,

$$\Delta_1 = z\partial_z + p_a\partial_{p_a}, \quad \Delta_0 = t\partial_t + p_a\partial_{p_a},$$

so that $h^0_a$ is the multiplication by $a$ in the coordinates $(z, p_a)$ and $h^1_a$ is the multiplication by $a$ in coordinates $(t, p_a)$. It is easy to see that the orbit space of the principal $\mathbb{R}^\times$ action $h^0$ on $T^*(L^*)^\times$ can be canonically identified with $J^1L$.

Of course, we can start with an arbitrary principal $\mathbb{R}^\times$-bundle $\tau : P \to M$ with the homogeneity structure $h$ instead of $(L^*)^\times \to M$ and considering the even and odd line bundles $P^{[0]}$ and $P^{[1]}$ associated with $P$, we get the following.

**Theorem 7.1.** The phase lift $h^0 = T^*h$ of the homogeneity structure of a principal $\mathbb{R}^\times$-bundle $\tau : P \to M$ defines canonical principal $\mathbb{R}^\times$-bundle structures on the fibrations $\tau : T^*P \to J^1(P^{[0]})^\times$ and $\tau^{[1]} : \Pi T^*P \to J^1(P^{[1]})^\times$.

The canonical symplectic structures: $\omega = d(t \cdot (dz - p_a dx^a))$ on $C^\times_L$ and $\omega_{(L^*)^\times} = dzdt + dp_a dx^a$ on $T^*(L^*)^\times$ are bi-homogeneous, $(\tilde{h}^0_a)\omega = uw_i, (\tilde{h}^1_a)\omega_{(L^*)^\times} = uw_i_{(L^*)^\times}$, $i = 0, 1$. Recall that we refer to such structures as to linear symplectic principal $\mathbb{R}^\times$-bundle structures or linear contact structures.

It is well known that linear symplectic structures are exactly those of cotangent bundles. Here, the map $\Psi : C_L^\times \to T^*(L^*)^\times$, written in homogeneous coordinates $(t, z, x^a, p_b)$, $t \neq 0$, in $C_L^\times$ and Darboux coordinates $(t, x^a, z, p_b)$ in $T^*(L^*)^\times$ as

$$(-t, z, x^a, p_b) \mapsto (t, x^a, z, tp_b),$$

is a symplectomorphism identifying the canonical symplectic structures and relating the double bundle structures: $(\Psi)_*\Delta_i = \Delta_i, i = 0, 1$. In particular, we have the commutative diagram of the corresponding fibrations

$$\begin{array}{ccc}
C^\times_L & \xrightarrow{\tau_0} & J^1L \\
\downarrow{\tau_1} & & \downarrow{\tau_1} \\
(L^*)^\times & \xrightarrow{\tau_0} & M
\end{array}$$

In the sequel we will usually identify the above linear symplectic principal $\mathbb{R}^\times$-bundles, where $\tau_0$ is a principal $\mathbb{R}^\times$-bundle and $\tau_1$ is a vector bundle fibration.

It is clear that we could start as well with an odd line bundle $L$, constructing similarly $J^1L$ and the odd contact structure $C_L$ on $J^1L$ as a subbundle in $\Pi T^*J^1L$. In this case, instead of the canonical Poisson bracket $\{\cdot, \cdot\}_\omega$ associated with the symplectic form $\omega = \omega_{(L^*)^\times}$ on $T^*(L^*)^\times \simeq C^\times_L$, we get the canonical Poisson bracket $\{\cdot, \cdot\}^{[1]}_\omega$ associated with the symplectic form $\omega^{[1]} = \omega^{[1]}_{(L^*)^\times}$ on $T^*(L^*)^\times \simeq C^\times_L$. If local coordinates are concerned, now $z$ is odd and the parity of $p_a$ is the opposite to the parity of $x^a$. Diagram (7.2) takes now the form

$$\begin{array}{ccc}
C^\times_L & \xrightarrow{\tau_0} & J^1L \\
\downarrow{\tau_1} & & \downarrow{\tau_1} \\
(L^*)^\times & \xrightarrow{\tau_0} & M
\end{array}$$

Thus, we get the following.
**Theorem 7.2.** Let $L$ be an even (resp., odd) line bundle over $\mathcal{M}$, let $L^\ast$ be its dual, and let $(L^\ast)^\times$ be the corresponding $\mathbb{R}^\times$-principal bundle with the $\mathbb{R}^\times$-action $\tau$. The canonical contact structure $(\mathcal{C}^\times_L, \omega, \tilde{h}^0, \tilde{h}^1)$ is canonically isomorphic to $(\mathcal{T}^\ast(L^\ast)^\times, \omega_L, h^0, h^1)$ (resp., to $(\Pi \mathcal{T}^\ast(L^\ast)^\times, \omega_L, h^0, h^1)$), where $h^0 = \mathcal{T}^\ast \tau$. In simple words, canonical even (resp., odd) contact structures are exactly (the parity shifts of) the cotangent bundles of principal $\mathbb{R}^\times$-bundles.

Note also that if we start with a principal $\mathbb{R}^\times$-bundle $P$ over $\mathcal{M}$, then diagrams (7.2) and (7.3) take the form

$$
\begin{array}{ccc}
\mathcal{T}^\ast P & \xrightarrow{\tau_0} & \Pi \mathcal{T}^\ast P \\
\downarrow \tau_1 & & \downarrow \tau_1 \\
\mathcal{M} & = & \mathcal{M}
\end{array}
$$

With the above notation we have $\Pi \mathcal{C}^\times_L = \Pi \mathcal{T}^\ast(L^\ast)^\times$ if the line bundle $L$ is even, and $\Pi \mathcal{C}^\times_L = \mathcal{T}^\ast(L^\ast)^\times$ if $L$ is odd. In other words, we can write $\Pi \mathcal{C}^\times_L = \Pi \mathcal{C}^\times_L$. Under this parity shift the canonical symplectic form $\omega$ on $\mathcal{C}^\times_L$ goes into the canonical symplectic form $\omega^\Pi$ on $\Pi \mathcal{C}^\times_L$ which is of the opposite parity. Note, however, that $(L^\ast)^\times = (\Pi L^\ast)^\times$, and so the principal $\mathbb{R}^\times$-bundle being the base of the vector fibration remains unchanged.

## 8 Brackets

Before considering arbitrary contact structures, let us consider the trivial case of a contact form which, however, can serve as a local model in general.

Let $\alpha$ be an even or odd contact one-form on $\mathcal{M}$ and let $\omega = d(t \cdot \alpha)$ be the corresponding symplectic form on $\widetilde{\mathcal{M}} = \mathbb{R}^\times \times \mathcal{M}$. Let $\{\cdot, \cdot\}_\omega$ be the Poisson bracket associated with the symplectic form $\omega$. Since $\omega$ is homogeneous of degree 1 with respect to the $\mathbb{R}^\times$-action on $\mathbb{R}^\times \times \mathcal{M}$ (thus with respect to the fundamental vector field $\Delta = t \partial_t$ of the action), the bracket is homogeneous of degree -1, so for any $F, G \in C^\infty(\mathcal{M})$ we have

$$
\{tF, tG\}_\omega = t\{F, G\}_\alpha
$$

for a certain function $\{F, G\}_\alpha \in C^\infty(\mathcal{M})$. Here, of course, $(tF)(t, x) = tF(x)$. The bilinear operation $(F, G) \mapsto \{F, G\}_\alpha$ on $C^\infty(\mathcal{M})$ we call the Legendre bracket associated with the contact form $\alpha$. The Legendre bracket is a particular example of a Jacobi bracket (cf. [23 43]).

It is easy to see that the equation (8.1) can be also taken as the definition of $\{\cdot, \cdot\}_\omega$ when $\{\cdot, \cdot\}_\alpha$ is given. This can be generalized to an arbitrary Jacobi bracket as follows (cf. [12 20]).

**Theorem 8.1.** There is a one-to-one correspondence between Jacobi brackets $\{\cdot, \cdot\}$ on $C^\infty(\mathcal{M})$ and homogeneous Poisson brackets $\{\cdot, \cdot\}_J$ on $\widetilde{\mathcal{M}} = \mathbb{R}^\times \times \mathcal{M}$ of degree -1, defined by the identity

$$
\{tF, tG\}_J = t\{F, G\}_J,
$$

valid for any $F, G \in C^\infty(\mathcal{M})$.

The bracket $\{\cdot, \cdot\}_J$ is usually called the **poissonization** of $\{\cdot, \cdot\}$. Under the poissonization, tensor fields on $\mathcal{M}$ go into the bivector field

$$
\mathcal{J} = \frac{1}{t} \Lambda + \Gamma \cdot \partial_t + tf \partial_t \cdot \partial_t
$$

on $\widetilde{\mathcal{M}}$ which is homogeneous of degree -1. This bivector field is an even (resp., odd) Poisson tensor in the sense that $\{\mathcal{J}, \mathcal{J}\}_\Pi = 0$ in the even case, and $\{\mathcal{J}, \mathcal{J}\}_\Pi = 0$ in the odd case. The even (resp., odd) Poisson tensor $\mathcal{J}$ represents a quadratics function on $\Pi \mathcal{T}^\ast \widetilde{\mathcal{M}}$ (resp., $\mathcal{T}^\ast \widetilde{\mathcal{M}}$). Note that in the even case $\partial_t \cdot \partial_t$ is automatically 0. The properties of even and odd Poisson bracket immediately imply the following.
Theorem 8.2. The tensor $\Lambda + \Gamma \cdot I$, where $\Lambda$ and $\Gamma$ are even elements from $X(\mathcal{M})$ of degrees, respectively, 2 and 1, defines an even Jacobi structure on $\mathcal{M}$ if and only if

$$\{\Lambda, \Lambda\}^\Pi_{\mathcal{M}} = 2\Gamma \cdot \Lambda, \quad \{\Gamma, \Lambda\}^\Pi_{\mathcal{M}} = 0.$$  

The tensor $\Lambda + \Gamma \cdot I + f I \cdot I$, where $\Lambda$, $\Gamma$, and $f$ are odd elements from $A(\mathcal{T}^*\mathcal{M})$ of degrees, respectively, 2, 1, and 0, defines an odd Jacobi structure on $\mathcal{M}$ if and only if

$$\{\Lambda, \Lambda\}_{\mathcal{M}} = 2\Gamma \cdot \Lambda, \quad \{\Gamma, \Lambda\}_{\mathcal{M}} = 2f \Lambda, \quad \{\Gamma, \Gamma\}_{\mathcal{M}} = 2(f \Gamma - \{f, \Lambda\}_{\mathcal{M}}), \quad \Gamma(f) = 0.$$  

Here $\{\cdot, \cdot\}^\Pi_{\mathcal{M}}$ (resp., $\{\cdot, \cdot\}_{\mathcal{M}}$) is the Schouten (resp., symmetric Schouten) bracket on $\mathcal{M}$.

Note that conditions (8.5) reduce to that considered in [5] for $f = 0$.

Example 8.1. On $\mathbb{R}^{1|1}$ with the even coordinate $x$ and the odd coordinate $\theta$, the triple

$$(\Lambda = \theta \cdot \partial_x \cdot \partial_x, \Gamma = \theta \cdot \partial_x, f = \theta)$$

represents an odd Jacobi structure.

We call a Jacobi bracket non-degenerate if its poissonization is symplectic, so that the tensor field $\mathcal{J}$ is invertible. As the poissonization is homogeneous of degree -1 with respect to $\Delta = t \partial_t$, the corresponding symplectic structure $\omega$ is 1-homogeneous and makes $\mathcal{M}$ into a symplectic $\mathbb{R}^x$-principal bundle. Hence, $\omega = d(t \cdot \alpha)$ for a contact form $\alpha$ and the Jacobi bracket is the Legendre bracket associated with this contact form. In other words, non-degenerate Jacobi brackets are Legendre brackets.

The Poisson bracket $\{\cdot, \cdot\}_\omega$ associated with the symplectic form $\omega = d(t \alpha)$, $t \neq 0$, where $\alpha$ is the even contact form [5,4], is determined by the following commutation rules of coordinate functions (lacking commutations rules not following from the antisymmetry are, by convention, trivially 0):

$$\{z, t\}_\omega = 1, \quad \{p_a, z\}_\omega = \frac{p_a}{t}, \quad \{\theta^i, z\}_\omega = \frac{\theta^i}{t}, \quad \{p_a, x^a\}_\omega = \frac{1}{t}, \quad \{\theta^i, \theta^j\}_\omega = \frac{\delta^i_j}{t},$$

so they are associated with the Poisson tensor $\mathcal{J}$,

$$\mathcal{J} = \partial_t \partial_z + \frac{1}{t} \left[ \partial_z \left( p_a \partial_{p_a} + \theta^i \partial_{\theta^i} \right) + \partial_{x^a} \partial_{p_a} - \frac{\epsilon^i_j}{2} \partial_{\theta^j} \partial_{\theta^i} \right],$$

viewed as a quadratic function on $\Pi \mathcal{T}^*\tilde{\mathcal{M}}$, as the derived brackets of the Schouten bracket $\{\cdot, \cdot\}^\Pi_{\mathcal{M}}$ on $\Pi \mathcal{T}^*\tilde{\mathcal{M}}$,

$$\{\varphi, \psi\}_\omega = \{\{\varphi, \mathcal{J}\}^\Pi_{\mathcal{M}}, \psi\}^\Pi_{\mathcal{M}}.$$  

Note that the derived bracket is even, and so for homogeneous $\varphi, \psi$ we have

$$\{\varphi, \psi\}_\omega = -(-1)^{\theta(\varphi)\theta(\psi)}\{\psi, \varphi\}_\omega,$$

and the corresponding Legendre bracket is an even (graded) Jacobi bracket and reads

$$\{F, G\}_\alpha = \frac{\partial F}{\partial p_a} \frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial z} \frac{\partial G}{\partial p_a} + \frac{\partial F}{\partial \theta^i} \frac{\partial G}{\partial \theta^i} + \frac{\partial F}{\partial \theta^i} \frac{\partial G}{\partial \theta^j} + \epsilon_j^i \frac{\partial F}{\partial \theta^j} \frac{\partial G}{\partial \theta^i} - \frac{\partial F}{\partial \theta^i} \frac{\partial G}{\partial \theta^j} - \frac{\partial F}{\partial \theta^j} \frac{\partial G}{\partial \theta^i} + (G - F) \frac{\partial F}{\partial \theta^i} \frac{\partial G}{\partial \theta^j} - \frac{\partial F}{\partial \theta^i} \frac{\partial G}{\partial \theta^j}.$$  

For odd contact form (8.10), the symplectic Poisson bracket is determined by

$$\{t, \xi\}_\omega = 1, \quad \{\theta^a, \xi\}_\omega = -\frac{\theta^a}{t}, \quad \{x^a, \theta^a\}_\omega = -1.$$  

It is associated with the Poisson tensor

$$\mathcal{J} = -\partial_t \partial_\xi + \frac{1}{t} \left[ \partial_{x^a} \partial_{\theta^a} - \partial_{\theta^a} \theta^a \partial_\xi \right],$$
viewed as a quadratic function on $T^*\tilde{M}$, as the derived bracket of the Poisson symplectic bracket \{·,·\}_{\tilde{M}} on $T^*\tilde{M}$,
\[
(8.12) \quad \{\varphi,\psi\}_\omega = \{(\varphi,\mathcal{J})_{\tilde{M}},\psi\}_{\tilde{M}}.
\]
Note that the bracket is odd, so for homogeneous $\varphi,\psi$ we have
\[
\{\varphi,\psi\}_\omega = -(-1)^{(g(\varphi)+1)(g(\psi)+1)}\{\psi,\varphi\}_\omega,
\]
and the corresponding Legendre bracket is an odd Jacobi bracket and reads
\[
(F,G)_\alpha = -\frac{\partial F}{\partial x^a}\frac{\partial G}{\partial \theta^a} + \frac{\partial F}{\partial \theta^a}\frac{\partial G}{\partial x^a} + F\cdot \frac{\partial G}{\partial \xi} - (-1)^{g(F)} \left(-\frac{\partial F}{\partial \theta^a}\frac{\partial G}{\partial x^a} + \frac{\partial F}{\partial x^a}\frac{\partial G}{\partial \theta^a} + \frac{\partial F}{\partial \xi}.G\right).
\]

If $P$ is a principal $\mathbb{R}^\times$-bundle over $\mathcal{M}$, then we have, in general, no trivialization $P = \mathbb{R}^\times \times \mathcal{M}$, so that formula (8.12) makes sense only locally. On the other hand, there is an obvious canonical identification of 1-homogeneous functions on $P$ with sections of the even line bundle $P$ (or the odd line bundle $P^o$). Conversely, this identification identifies sections $\sigma$ of an even (resp., odd) line bundle $L$ over $\mathcal{M}$ with 1-homogeneous functions $\iota_\sigma$ on $P = (L^*)^\times$. This observation yields a generalization of Theorem 8.1.

**Theorem 8.3.** Let $L$ be an even line bundle. There is a one-to-one correspondence between even (resp., odd) Kirillov brackets $[·,·]_L$ on $L$ and even (resp., odd) Poisson tensors $\mathcal{J}$ of degree -1 on $(L^*)^\times$, given by the formula
\[
(8.14) \quad \{\iota_\sigma,\iota_{\sigma'}\}_\mathcal{J} = \iota_{[\sigma,\sigma']}_L.
\]

In particular, any even or odd contact structure $C$ defines a Kirillov bracket $[·,·]_C^*$ on sections of the dual line bundle $C^*$.

**Definition 8.1.** Let $C$ be a contact structure. The Kirillov bracket $[·,·]_C^*$ on sections of the dual line bundle $C^*$ will be called the Legendre bracket associated with the contact structure $C$.

The Legendre bracket is nondegenerate in the sense that it is represented by a symplectic Poisson bracket on $C^*$. Conversely, if we have a nondegenerate Kirillov bracket on a line bundle $L$, then the corresponding Poisson structure $\mathcal{J}$ on $(L^*)^\times$ is symplectic and homogeneous, thus represents the contact structure $C^*_L$. This gives an identification of nondegenerate Kirillov brackets with Legendre brackets.

As usual, we can interpret an even Poisson tensors $\mathcal{J}$ on the principal $\mathbb{R}^\times$-bundle $P = (L^*)^\times$ as quadratic Hamiltonians on $\Pi T^*P = \Pi C^\times_L$ satisfying the homological condition
\[
(8.15) \quad \{\mathcal{J},\mathcal{J}\}_P^1 = 0,
\]
where $\{·,·\}_P^1$ is the Schouten bracket on $P$. As the Schouten bracket is of degree -1, it is closed on homogeneous functions of degree 1, i.e., functions from $\mathcal{A}^{(1,\ast)}(\Pi T^*P)$ if we consider the standard bi-gradation on $\Pi T^*P$. In the introduced terminology, the Schouten bracket $\{·,·\}_P^1$ reduced to $\mathcal{A}^{(1,\ast)}(\Pi T^*P)$ is the Legendre bracket $[·,·]_{\Pi T^*P^\circ}$ on sections of the line bundle $(\Pi T^*P)^\circ$. In the case of an odd Poisson tensor, the situation is analogous and we replace the Schouten bracket with the symmetric Schouten bracket. If $P = (L^*)^\times$, then, according to our conventions, we can write the Legendre bracket as $[·,·]_{\Pi C^\times_L}$ in the even as well as in the odd case.

Using our standard local coordinates $(t,z,x^a,p_a)$ on $\Pi C^\times_L$, with the bi-degrees $(1,0),(0,1),(0,0)$, and $(0,1)$, respectively, we can write locally any Poisson bracket of degree -1 as being derived from a quadratic Hamiltonian of degree 1,
\[
\mathcal{J} = t(f_{ab}(x)p_ap_b + h_a(x)p_a z + u(x)z^2).
\]
Note that we reversed the parity, so $z$ is even if and only if $L$ is odd, and $p_a$ and $x^a$ have different parities. Therefore, the term $u(x)z^2$ is non-zero only for even $L$. In Darboux coordinates $(t,z,x^a,p_a)$, $p_a = tp_a$, we get
\[
\mathcal{J} = \frac{1}{t} f_{ab}(x)p_ap_b + h_a(x)p_a z + tf(x)z^2.
\]
Since the Hamiltonian vector field of $z$ is $\partial_t$, this coincides with formula (8.3) in which the bivector field $\Lambda$ is represented by $f_{ab}(x)p_ap_b$ and the vector field $\Gamma$ is represented by $h_a(x)p_a$. This coincides also with (2.9) if we take into account that $t\partial_t$ acts on 1-homogeneous functions as the identity. Completely analogously to the classical result, describing Poisson brackets as derived brackets for Poisson tensors viewed as quadratic Hamiltonians with respect to the Schouten bracket, and to a similar result for Jacobi brackets [22, 23], we can now formulate the following description of Kirillov brackets.

**Theorem 8.4.** There is a one-to-one correspondence between Kirillov brackets $[\cdot, \cdot]_L$ on a line bundle $L$ and 1-homogeneous quadratic Hamiltonians $J \in A^{(1,2)}(\Pi C^*_L)$ satisfying the homological condition $[J, J]_{\Pi C^*_L} = 0$ with respect to the Legendre bracket on sections of $\Pi C^*_L$. In this case, the Kirillov bracket is given as the derived bracket

$$[\sigma, \sigma']_L = [[\sigma, J]_{\Pi C^*_L}, \sigma']_{\Pi C^*_L}$$

of the Legendre bracket $[\cdot, \cdot]_{\Pi C^*_L}$ with respect to $J$.

## 9 Principal Lie algebroids

It is well known that linear Poisson structures are nothing but Lie algebroids. This general philosophy suggests to consider linear principal Poisson $\mathbb{R}^\times$-bundles and the corresponding ‘algebroids’.

**Definition 9.1.** A principal Lie algebroid is a linear principal Poisson $\mathbb{R}^\times$-bundle. If this principal $\mathbb{R}^\times$-bundle is trivial, we will speak about a Jacobi algebroid.

To look closer on such structures, consider a linear principal Poisson $\mathbb{R}^\times$-bundle $P$ over $P_0$ and the corresponding fibrations $\tau_i : P \to P_i$, $i = 0, 1$. Let us denote with $A^{(k, l)}(P)$, $k \in \mathbb{Z}$, $l \in \mathbb{N}$, the space of those smooth functions on $P$ which are homogeneous of degree $k$ with respect to $h^0$ and homogeneous of degree $l$ with respect to $h^1$ (or $\Delta_1$). We will say simply that they are $(k, l)$-homogeneous. Recall that we have the decomposition $P = P_1 \times_{\mathcal{M}} P_0$, where $\mathcal{M} = h^0(P_0) = h^0(P)/\mathbb{R}^\times$ (Theorem 4.2). It is clear that $A^{(k, l)}(P) = A^k(P_1) \otimes_{C^\infty(M)} A^l(P_0)$ and that $A^{(1, \bullet)}(P) = \bigoplus_{l=0}^{\infty} A^{(1, l)}(P)$

is a bi-module over the algebra $A^{(0, \bullet)}(P) = \bigoplus_{l=0}^{\infty} A^{(0, l)}(P)$ of polynomial functions on the vector bundle $P_0$. Note that functions from $A^{(0, 1)}(P)$ can be identified with sections of the vector bundle $P_0^*$. It is also easy to see that elements of $A^{(1, 1)}(P)$ represent linear sections of the even line bundle $P^e$ associated with the principal $\mathbb{R}^\times$-bundle $P$, and elements of $A^{(1, 0)}(P)$ represent sections of the line bundle $P^*_L$. This line bundle can be described also as the restriction of $P^e$ to the zero-section $0_{P_0} \simeq \mathcal{M}$ of the vector bundle $P_0$. Note that, as $P \simeq P_1 \times_{\mathcal{M}} P_0$, the $A^{(0, 0)}(P)$-module $A^{(1, 1)}(P)$ can be identified with the module of sections of the vector bundle $P^*_L \otimes_{\mathcal{M}} P^*_0$ (or the vector bundle $P_0^* \otimes_{\mathcal{M}} P_0^*$). Indeed, any such section $\varepsilon = \sigma \otimes \varepsilon_0$ defines by duality a function $\varepsilon_{(1, 1)}$ of degree $(1, 1)$ on $P$ by $\tau_{(1, 1)}(x)$, where $\tau_{(1, 1)}$ is 1-homogeneous function on $P_1 = (L^*)^\times$ represented by $\sigma \in \text{Sec}(L)$, and $\varepsilon_0$ is the linear function on $P_0$ represented by the section $\varepsilon_0 \in \text{Sec}(P^*_0)$.

The Poisson bracket $\{\cdot, \cdot\} = \{\cdot, \cdot\}_J$ associated with the Poisson tensor $J$ on $P$ is homogeneous of bi-degree $(-1, -1)$, so

$$\{A^{(1,1)}(P), A^{(k,l)}(P)\} \subset A^{(k,l)}(P).$$

In particular, the modules $A^{(1, \bullet)}(P)$ over $A^{(0, \bullet)}(P)$ and $A^{(1, 1)}(P)$ over $A^{(0, 0)}(P)$ are closed with respect to the Poisson bracket.
Definition 9.2. The restriction of the Poisson bracket to the module \( A^{(1,\bullet)}(P) \) we will call the \textit{graded Kirillov bracket} associated with \((P, J, h^0, h^1)\) and denote as \([\cdot, \cdot]_J\), or simply \([\cdot, \cdot]\) if \(J\) is fixed. The restriction of the Poisson bracket to the module \( A^{(1,1)}(P) \) we will call the \textit{principal Lie algebroid bracket} associated with \((P, J, h^0, h^1)\) and denote as \([\cdot, \cdot]_J\), or simply \([\cdot, \cdot]\) if \(J\) is fixed.

Let us denote \(L = L(P) = P^*_J\) if \(J\) is even, and \(L = L(P) = P^*_J\) if \(J\) is odd. If we put \(E(P) = L \otimes_M P^*_0\), then this bracket, interpreted as a bracket on sections of \(E(P)\), is even and defines a Lie algebroid structure on \(E(P)\) with the anchor map

\[
\rho(\varepsilon)(f) = \{\iota_\varepsilon, f\}.
\]

This Lie algebroid we will call the \textit{Lie algebroid of the linear principal Poisson} \(\mathbb{R}^\times\)-bundle \((P, J, h^0, h^1)\).

We have also

\[
(9.2) \quad \{A^{(0,1)}(P), A^{(1,0)}(P)\} \subset A^{(0,0)}(P)
\]

that allows us to represent any \(\sigma \in A^{(0,1)}(P) \simeq \text{Sec}(P^*_0)\) as a first-order differential operator \(\Upsilon(\sigma) : \text{Sec}(L) \to C^\infty(M)\). Since

\[
(9.3) \quad \Upsilon : P^*_0 \to (J^1L)^*,
\]

covering the identity on \(M\). We have an obvious extension of this map to a vector bundle morphism

\[
(9.4) \quad \tilde{\rho} : E(P) = L \otimes_M P^*_0 \to (J^1L)^* \otimes_M L \simeq \text{DO}^1(L, L),
\]

that follows also directly from the inclusion

\[
\{A^{(1,1)}(P), A^{(1,0)}(P)\} \subset A^{(1,0)}(P).
\]

In other words,

\[
(9.5) \quad \iota_{\tilde{\rho}(\varepsilon)(\sigma)} = \{\iota_\varepsilon, \iota_\sigma\}.
\]

It follows immediately from the Jacobi identity that the map \(\tilde{\rho}\) induces a homomorphism of the Poisson bracket on \(A^{(1,1)}(P)\), viewed as the space of sections \(\sigma \in \text{Sec}(E(P))\), into the (super)commutator in \(\text{DO}^1(L, L)\), i.e., for \(\varepsilon, \varepsilon' \in \text{Sec}(E(P))\) of parities \(\delta, \delta'\), respectively, we have

\[
(9.6) \quad \tilde{\rho}^1(\varepsilon, \varepsilon') = [\tilde{\rho}^1(\varepsilon), \tilde{\rho}^1(\varepsilon')] = \tilde{\rho}^1(\varepsilon) \circ \tilde{\rho}^1(\varepsilon') - (-1)^{\delta \delta'} \tilde{\rho}^1(\varepsilon') \circ \tilde{\rho}^1(\varepsilon).
\]

The map \(\tilde{\rho}\) is actually a Lie algebroid morphism.

The dual of the map \(\Upsilon\), i.e., the map \(\Upsilon^* : J^1L \to P_0\), can be extended to

\[
(9.7) \quad \tilde{\Upsilon} : C_L \simeq (L^*)^\times \times_M J^1L \to (L^*)^\times \times_M P_0 \simeq P.
\]

It is easy to see that, for any section \(\varepsilon\) of \(E(P)\),

\[
\iota_\varepsilon \circ \tilde{\Upsilon} = \iota_{\tilde{\rho}(\varepsilon)}.
\]

Since the restriction of any \((-1, -1)\)-homogeneous Poisson bracket to \((1, 1)\)-homogeneous functions completely determines the bracket, we infer, in view of (9.6), that \(\tilde{\Upsilon}\) is a Poisson map. This is the contact analog of the map \(\rho^* : T^*M \to E^*\), dual to the anchor map \(\rho : E \to TM\), defined for a Lie algebroid on a vector bundle \(E\).

Note that we can change the parity in fibers of the vector bundle \(\tau_0 : P \to P_1\). In this way we get a new linear principal Poisson \(\mathbb{R}^\times\)-bundle \((\prod P, J^{\Pi}, h^0, h^1)\) over \(M\) with the reversed parity of the Poisson tensor and with the double bundle structure described by the diagram

\[
(9.8) \quad \begin{array}{ccc}
\Pi P & \xrightarrow{\tau_0} & \Pi P_0 \\
\downarrow \tau_1 & & \downarrow \rho_1 \\
P_1 & \xrightarrow{\tau_0} & M
\end{array}
\]
To be more precise, let us observe that, formally, $A^{(k,l)}(P) = A^{(k,l)}(IP)$ for $0 \leq k, l \leq 1$. We do not change the Poisson bracket on these spaces, i.e. $\{ \cdot , \cdot \}_J$ and $\{ \cdot , \cdot \}^H = \{ \cdot , \cdot \}_J^n$ coincide on $A^{(1,1)}(P) = A^{(1,1)}(IP)$. In other words, the corresponding Poisson tensors are formally the same, but they define Poisson brackets of the reversed parity. The situation is analogous to the one we encounter viewing the canonical symplectic form on $T^*M$ as a symplectic form also on $HT^*M$. Of course, the algebras $A^{(0,\bullet)}(P)$ and $A^{(1,\bullet)}(IP)$ differ, since elements from $A^{(0,1)}(P)$ and $A^{(0,1)}(IP)$ commute differently. The bracket $\{ \cdot , \cdot \}^H_J$ restricted to $A^{(1,\bullet)}(IP)$ we will call the graded Schouten-Kirillov bracket associated with $(P, J)$ and denote $[\cdot , \cdot ]^H_J$. In this way we get the following.

**Theorem 9.1.** Let $(P, J, h^0, h^1)$ be a linear principal Poisson $\mathbb{R}^X$-bundle over $M$, with the double bundle structure $(\{ \cdot , \cdot \}_J^H, L)$. and let $L = L(P)$ be the line bundle $P^\ast_1$ in the case when $J$ is even, and the line bundle $P^\ast_1$ in the case when $J$ is odd, with sections represented by functions from $A^{(1,0)}(P)$. Then, the corresponding Poisson bracket satisfies $(L, L)$ and induces:

(a) a Lie algebroid bracket $[,]^1$ on the vector bundle $E(P) = L \otimes_M P^\ast_1$ over $M$ who sections are represented by functions in the module $A^{(1,1)}(P)$ over $A^{(0,0)}(P) \simeq C^\infty(M)$;

(b) a morphism $\tilde{Y} : C^\infty_L \to P$ of linear principal Poisson $\mathbb{R}^X$-bundles covering the identity on $(L^\ast)^X$.

(c) a Lie algebroid morphism

$$\rho^1 : E(P) \to DO^1(L, L)$$

from $E(P)$ into the Lie algebroid $DO^1(L, L)$ of first-order differential operators from $L$ into $L$ with the (super)commutator bracket.

(d) a linear principal Poisson $\mathbb{R}^V$-bundle $(IP, J^H, h^0, h^1)$ over $M$ with the reversed parity of the Poisson tensor and a Kirillov bracket on the $A^{(0,0)}(IP)$-module $A^{(1,\bullet)}(IP)$, the Schouten bracket of $(P, J, h^0, h^1)$.

**Example 9.1.** Let $L$ be a line bundle over $M$ and $\{\cdot , \cdot \}_L$ be a Kirillov bracket on sections of $L$. According to Theorem 3.3 this Kirillov structure is associated with a principal Poisson $\mathbb{R}^X$-bundle structure $(P, J, h^0)$ on $\tilde{P} = (L^\ast)^X$. Passing to the tangent lifts (cf. Example 6.1), we get a linear principal Poisson $\mathbb{R}^X$-bundle $P = (T(L^\ast)^X, \rho, J, \tilde{T}^0, h^1)$. In this case, the Lie algebroid $E(P)$ is exactly the Lie algebroid associated with a Kirillov bracket (a local Lie algebra structure), discovered in [22]. In particular, if the line bundle $L$ is trivial and we deal with a Jacobi structure on $M$, then $P = T(\mathbb{R}^X \times M)$, so $P^\ast_0 = T^*M \times \mathbb{R}$ and the corresponding Lie algebroid structure lives on $E(P) = L \otimes_M P^\ast_0 = (T^*M \times \mathbb{R})^* = T^*M \times \mathbb{R}$. The latter is exactly the Lie algebroid structure on $T^*M \times \mathbb{R}$ associated with a Jacobi structure on $M$, as described in [31] (cf. also [64]).

Let us stress that the structure we have obtained on $E(P)$ is not only a Lie algebroid structure with a Lie bracket $[,]^1$, but also a Lie algebroid morphism $\rho^1 : E(P) \to DO^1(L, L)$. A part of this morphism defines the anchor map of the Lie algebroid structure by passing to the principal symbol of a first-order differential operator, but there is another part which is not directly seen by the Lie algebroid structure on $E(P)$. Let us recall that there is a natural identification of first-order differential operators $D$ between sections of $L$ with invariant vector fields $\tilde{D}$ on the principal $\mathbb{R}^X$-bundle $(L^\ast)^X$, i.e., with sections of the Atiyah algebroid $T(L^\ast)^X/\mathbb{R}^X$ of $(L^\ast)^X$. The vector field $\tilde{D}$ is uniquely defined by the formula

$$\tilde{D}(\iota_\sigma) = \iota_D(\sigma),$$

where $\sigma$ is any section of $L$ and $\iota_\sigma$ is the corresponding homogeneous function on $(L^\ast)^X$.

On the other hand, the linear principal $\mathbb{R}^X$-bundle $P$ can be reconstructed from $E(P)$. Indeed, denote the vector bundle dual to $P$ with $P^\ast$, so that $P^\ast = (L^\ast)^X \times_M P^\ast_0$. We have a canonical isomorphism of vector bundles over $(L^\ast)^X$,

$$S : (L^\ast)^X \times_M E(P) = (L^\ast)^X \times_M (L \otimes_M P^\ast_0) \to (L^\ast)^X \times_M P^\ast_0 = P^\ast,$$

given by

$$\sigma^*, \sigma \otimes \nu) \mapsto (\sigma^*, \sigma^*(\sigma)\nu).$$
This isomorphism identifies sections $\varepsilon$ of $\mathcal{E}(P)$, viewed as $\mathbb{R}^\times$-invariant (we will say also constant) sections $\tilde{\varepsilon}$ of $E(P) = (L^*)^\times \times \mathcal{M} \mathcal{E}(P)$, with 1-homogeneous sections $\varepsilon$ of the vector bundle $P^*$, thus homogeneous functions of bi-degree $(1, 1)$ on $P$. Note that the Lie algebroid structure on $\mathcal{E}(P)$, together with the Lie algebroid morphism \([9.9]\), defines a Lie algebroid structure on $E = (L^*)^\times \times \mathcal{M} \mathcal{E}(P)$, thus $P^*$, with a bracket $\{\cdot, \cdot\}$ and an anchor $\rho : E(P) \to T(L^*)^\times$ by

\begin{align}
[\tilde{\varepsilon}_1, \tilde{\varepsilon}_2] = [\varepsilon_1, \varepsilon_2], \\
\rho(\tilde{\varepsilon}) = \tilde{\rho}(\varepsilon). 
\end{align}

This Lie algebroid structure on $E$ over $(L^*)^\times$ is invariant with respect to the $\mathbb{R}^\times$-action in the sense that the bracket of invariant sections is invariant and the anchor of an invariant section is an invariant vector field. The Lie algebroid on $P^*$ completely determines the Poisson structure we started with. We can go back to $\mathcal{E}(P)$ by considering constant sections only. In this way we get the following.

**Theorem 9.2.** A principal Lie algebroid can be equivalently defined as an invariant Lie algebroid structure on a vector bundle $E$ over a principal $\mathbb{R}^\times$-bundle $P_0$. If $P_0$ is trivial, $P_0 = \mathbb{R}^\times \times \mathcal{M}$, then we deal with a Jacobi algebroid.

**Remark 9.1.** Isomorphism \([9.11]\) is a geometrical counterpart of a bi-degree shift in the linear principal bundle $(P^*, h^0, h^1)$, associated with a change of the $\mathbb{R}^\times$-action. Since the group $\mathbb{R}^\times$ acts canonically in each vector bundle by non-zero homotheties, we can consider a new principal $\mathbb{R}^\times$-action $h^0$ on $P^*$ by putting $h^0_t = h^0_0 \circ h^1_t$. This produces the shift $A^{(k,1)}(P^*) \mapsto A^{(k-1,1)}(P^*)$ in bi-degrees. In particular, linear 1-homogeneous tensors, i.e., tensors of the bi-degree $(1, 1)$, become linear invariant tensors, i.e., tensors of bi-degree $(0, 1)$. This explains how 1-homogeneous Lie algebroid structures have turned into invariant Lie algebroid structures.

**Example 9.2.** In the case of a Jacobi algebroid, $P_0 = \mathbb{R}^\times \times \mathcal{M}$, we can reduce the Lie algebroid structure to $\mathcal{E} = E/\mathbb{R}^\times$ over $\mathcal{M}$, interpreting sections of $\mathcal{E}$ as invariant sections of $E$. As the anchor map associates with any invariant section $\tilde{\varepsilon}$ of $E$ an invariant vector field $\rho(\tilde{\varepsilon})$ on $\mathbb{R}^\times \times \mathcal{M}$, its projection to $\mathcal{M}$ defines a map $\varepsilon \mapsto f$ represented by a nowhere-vanishing 1-form $\mu$ on $E$ (a section of $\mathcal{E}^*$), $\mu(\varepsilon) = f \in C^\infty(\mathcal{M})$. Since $\rho : E \to T(\mathbb{R}^\times \times \mathcal{M})$ is a Lie algebroid morphism, the 1-form $\mu$ is $\mathcal{E}$-closed, $d_{\mathcal{E}} \mu = 0$, and we arrive at the standard definition of a Jacobi algebroid (called also a generalized Lie algebroid) \([22, 29]\).

## 10 Contact $n$-vector bundles

In the case of a canonical contact structure $C_L$ and the corresponding symplectic Poisson tensor $\mathcal{J}$ on $P = C_L^*$, the maps $\Upsilon$, $\hat{\rho}$, and $\hat{\Upsilon}$, defined in the previous section, are all isomorphisms. Indeed, in this case $P_0 = J^1L$ and $P_1 = (L^*)^\times$, so $L(P) = L$, $\mathcal{E}(P) = DO^1(L, L)$, and the map $\Upsilon$, thus $\hat{\rho}$ and $\hat{\Upsilon}$, are all the identity maps. The space of linear sections of $C_L^*$ can be then identified with the space of first-order linear differential operators acting on sections of $L$. Fixing a local trivialization of $L$ and the corresponding affine coordinates $(t, x^a, z, p_0)$ in $C_L$, the linear section $\varepsilon$ of $C_L^*$ corresponding to the $(1, 1)$-homogeneous function $\epsilon_a = \epsilon(t(x_a) p_0 + h(x))$ on $C_L$ represents the differential operator $D_\varepsilon = f_a(x) \partial_{x^a} + h(x)$. These first-order differential operators are sections of the bundle $DO^1(L, L)$ over $\mathcal{M}$ with the basis of local sections represented by $p_0 = t \partial_a$ and $t z$. Moreover, the Legendre bracket $\{\varepsilon, \varepsilon\}_{C_L}$ of linear sections of given parity corresponds, via the identification $\varepsilon \mapsto D_\varepsilon$, to the (graded) commutator $[D_\varepsilon, D_\varphi]$ of differential operators.

Conversely, if $\mathcal{J}$ is invertible, i.e. $\omega = \mathcal{J}^{-1}$ is a symplectic form, then the map $\Psi_{\mathcal{J}}$ is an isomorphism. For, let us observe that the map $\Psi_{\mathcal{J}}$ is derived from a bilinear form $\langle \cdot, \cdot\rangle_{\mathcal{J}}$ on $P^*_0 \times \mathcal{M} J^1L$ defined by means of \([12]\). If the Poisson bracket is symplectic, then this map is a pairing, so $\Psi_{\mathcal{J}}$ is an isomorphism. Indeed, being of bi-degree $(-1, -1)$, the Poisson bracket $\{\cdot, \cdot\}_{\mathcal{J}}$ is completely determined by its restriction to $A^{(1,1)}(P) = A^{(1,0)}(P) \cdot A^{(0,1)}(P)$ and it is nondegenerate if and only if $\{\cdot, \cdot\}_{\mathcal{J}}$ is nondegenerate. Summarizing our observations, we get the following.

**Theorem 10.1.** For a linear principal Poisson $\mathbb{R}^\times$-bundle $(P, \mathcal{J}, h^0, h^1)$, the following are equivalent:

(i) The Poisson tensor $\mathcal{J}$ is invertible (symplectic);
(ii) the map
\[ \Upsilon : P_0^* \rightarrow J^1 L^* \]
is a vector bundle isomorphism;

(iii) the map
\[ \rho^* : \mathcal{E}(P) \rightarrow DO^1(L, L) , \]
described by \( [\mathcal{J}, \mathcal{A}] \), is an isomorphism of Lie algebroids.

If this is the case, then \((P, J^{-1}, h^0, h^1)\) is a linear symplectic principal \( \mathbb{R}^\times \)-bundle and the map \([9, 7]\) constitutes a canonical isomorphism between \((P, J^{-1})\) and the linear symplectic principal \( \mathbb{R}^\times \)-bundle \((C_L^\times, \omega)\), associated with the canonical contact structure on the first jet bundle \( J^1 L \).

**Corollary 10.1.** Any linear even (resp., odd) contact structure is isomorphic with a canonical contact structure on the first jet bundle of an even (resp., odd) line bundle, i.e., with the cotangent bundle \( T^* \bar{P} \) (resp., \( \Pi T^* \bar{P} \)) of a principal \( \mathbb{R}^\times \)-bundle \( \bar{P} \).

**Remark 10.1.** The above result is a contact analog of the well-known fact that any linear symplectic manifold can be identified with the cotangent bundle of a manifold with its canonical symplectic structure.

**Remark 10.2.** It is easy to see that the space \( \mathcal{A}^{(1, \bullet)}(L) \) represents the space of principal symbols of differential operators in \( L \) and that the Poisson bracket \{ , \} is closed on principal symbols representing the principal part of the (super)commutator of differential operators. We will not go into details, just referring to [23] and the literature cited there. Note only that the observation that the canonical Poisson structure on \( T^* M \), as the bracket in principal symbols viewed as polynomial functions in \( T^* M \), can be obtained from the commutator of differential operators is, up to our knowledge, due to Vinogradov [66].

**Example 10.1.** Consider the contact structure of the trivial even line bundle \( L = \mathbb{R}^\times \times M \) over \( M \), i.e., the canonical linear principal symplectic \( \mathbb{R}^\times \)-bundle \( P = T^*(\mathbb{R}^\times \times M) = \mathbb{R}^\times \times (\mathbb{R} \times T^* M) \). In the standard Darboux coordinates \((t, z, x^a, p_a)\) of the bi-degrees, respectively, \((1, 0), (0, 1), (0, 0), (1, 1)\), the Poisson tensor takes the form \( J = \partial_t \partial_z + \partial_a \partial_{p_a} \). On the vector bundle \( T^*(\mathbb{R}^\times \times M)/\mathbb{R}^\times = J^1 (\mathbb{R} \times T^* M) \) we have coordinates \((z, x^a, p_a)\), where \( p_a = p_a \). The vector bundle \( \mathcal{E}(P) \) is in this case \( \mathcal{E}(P) = (\mathbb{R} \times T^* M)^* = \mathbb{R} \times TM = DO^1(M) \). Any its section \( \epsilon = g(x) + f^a(x)p_a \) is a first-order differential operator on \( M \) and corresponds to the function \( \epsilon_x = g(x)z + f^a(x)p_a = t(g(x)z + f^a(x)p_a) \) of the bi-degree \((1, 1)\) on \( P \). It is easy to see that the Poisson bracket of such functions corresponds to the commutator of the associated first-order differential operators. The corresponding principal Lie algebroid is \( E(P) = \mathbb{R}^\times \times \mathbb{R} \times TM \), with local coordinates \((t, \dot{t}, x^a, \dot{x}^a)\), and a vector bundle isomorphism \( \mathcal{J} : E(P) \rightarrow T(\mathbb{R}^\times \times M) = P^* \) is given in the adapted coordinates \((t, x^a, \dot{t}, \dot{x}^a)\) in \( T(\mathbb{R}^\times \times M) \) of the bi-degrees \((1, 0), (0, 0), (1, 1), (0, 1)\), respectively, by
\[ (t, x^a, \dot{t}, \dot{x}^a) = (t, x^a, \dot{t}, \dot{x}^a). \]

The principal Lie algebroid is therefore \( T(\mathbb{R}^\times \times M) \) with the bracket of vector fields, for which \( \mathbb{R}^\times \)-invariant vector fields \( g(x)\partial_t + f^a(x)p_a \) form a Lie subalgebra.

Recall that an \( n \)-vector bundle \( M \) with an even (odd) contact structure \( C \subset T^* M \) of degree \( 1^n \) we will call an even (odd) contact \( n \)-vector bundle. It can be also described as an \( n \)-linear symplectic principal \( \mathbb{R}^\times \)-bundle. In this language, Corollary 10.1 tells us that even (odd) contact vector bundles are exactly the canonical contact structures, i.e., they are represented by the cotangent bundles of principal \( \mathbb{R}^\times \)-bundles. We have the following straightforward generalization.

**Theorem 10.2.** The cotangent bundle \( T^* \bar{P} \) (resp., \( \Pi T^* \bar{P} \)) of an \((n-1)\)-linear principal \( \mathbb{R}^\times \)-bundle \((P, h^0, h^1, \ldots, h^{n-1})\) is canonically an even (resp., odd) contact \( n \)-vector bundle with the homogeneity structures \( h^i = T^* h^i \), \( i = 0, \ldots, n-1 \), and the homogeneity structure \( h^n \) associated with homotheties in the vector bundle \( \pi_{\bar{P}} := T^* \bar{P} \rightarrow \bar{P} \).

In addition, any even (resp., odd) contact \( n \)-vector bundle \((P, \omega, h^0, \ldots, h^n)\) is of this form, where as \( \bar{P} \) can be taken any of the side vector bundles \( h^i_0 : P \rightarrow \bar{P}_i, i = 1, \ldots, n \). In particular, all contact \( n \)-vector bundles \( T^* \bar{P}_i \) (resp., \( \Pi T^* \bar{P}_i \)) are canonically isomorphic as well as all the first jet bundles \( J^1 L_i \), where \( L_i = P_i^c \) (resp., \( L_i = P_i^o \)), \( i = 1, \ldots, n \).
Proof. As the proof is completely analogous in the odd case, let us assume that our case is even. If \((P, \tilde{h}, \tilde{h}^0, \ldots, \tilde{h}^{n-1})\) is an \((n-1)\)-linear principal \(\mathbb{R}^\times\)-bundle, with the principal \(\mathbb{R}^\times\)-bundle structure \(\tilde{\tau}_0 : \tilde{P} \to P_0\) and the vector bundle structures \(\tilde{\tau}_i : \tilde{P} \to \tilde{P}_i\), \(i = 1, \ldots, n-1\), then \(P = T^*\tilde{P}\) is canonically an \(n\)-linear symplectic principal \(\mathbb{R}^\times\)-bundle with the canonical symplectic structure, with the principal \(\mathbb{R}^\times\)-bundle structure associated with \(h^0 = T^*\tilde{h}^0\) (see theorem \(11.1\)) and with the vector bundle structures \(\tilde{h}^1, \ldots, \tilde{h}^n\), where \(h^i = T^*\tilde{h}^i\), \(i = 1, \ldots, n-1\), and \(h^n\) is associated with the vector bundle structure \(\pi_P := T^*\tilde{P} \to \tilde{P}\).

Conversely, if \((P, \omega, h^0, \ldots, h^n)\) is an \(n\)-linear symplectic \(\mathbb{R}^\times\)-principal bundle, \(n \geq 1\), then, as linear symplectic structures are exactly those of the cotangent bundle (cf. \([24]\)), the linear symplectic structure \((P, \omega, h^n)\) is canonically isomorphic with \((T^*\tilde{P}, \omega_{\tilde{P}}, h_{\tilde{P}})\), where \(\tilde{P} = \tilde{P}_n = \tau_n(P)\) is the base of the \(n\)th vector bundle structure and \(h_{\tilde{P}} = \text{homogeneity structure of the vector bundle}\) \(\pi_{\tilde{P}} : T^*\tilde{P} \to \tilde{P}\). Since \(h^i\) commute with \(h^n\), the corresponding Euler vector fields \(\Delta_i\) are linear on \(T^*\tilde{P}\) and project to Euler vector fields \(\Delta_i\) on \(\tilde{P}\), \(i = 1, \ldots, n-1\). As the linear vector fields on \(T^*\tilde{P}\), making the canonical symplectic form \(\omega_{\tilde{P}}\) homogeneous of degree 1, are completely determined by their values on \(\tilde{P}\) (cf. \([23, Proposition 5.1]\)), we have \(\Delta_i = T^*\Delta_i, i = 1, \ldots, n-1\). Hence, \(h^i = T^*\tilde{h}^i\), \(i = 1, \ldots, n-1\).

Similarly, \(h^0\) projects to an \(\mathbb{R}^\times\)-action \(\tilde{h}^0\) on \(\tilde{P}\) that makes \((\tilde{P}, \tilde{h}^0, \ldots, \tilde{h}^{n-1})\) into an \((n-1)\)-linear principal \(\mathbb{R}^\times\)-bundle. Since \(\omega_{\tilde{P}}\) is 1-homogeneous with respect to \(h^0\), it is necessarily the phase lift of \(\tilde{h}^0\). Of course, we can as well take one of \(\tilde{P}_1, \ldots, \tilde{P}_{n-1}\) instead of \(\tilde{P}_n\) with the same effect. Writing diagram \((7.2)\) for these realizations, we get

\[
\begin{array}{ccc}
P \simeq T^*\tilde{P} & \xrightarrow{\tau_0} & P_0 \simeq J^1L_i \\
\downarrow \tau_i & & \downarrow \tau_i \\
\tilde{P}_i & \xrightarrow{\tau_0} & \tilde{M}_i \simeq \tilde{P}_i/\mathbb{R}^\times
\end{array}
\]

and the theorem follows. \(\square\)

Remark 10.3. The above theorem is a contact analog of a theorem about symplectic \(n\)-vector bundles \((24, Theorem 6.1)\) stating that any such bundle, \((F, \omega)\), is canonically isomorphic with the cotangent bundle \(T^*F_i\) for each of its side bundles \(F_i\), \(i = 1, \ldots, n\). In particular, all \(T^*F_i\), \(i = 1, \ldots, n\), are canonically symplectically isomorphic. The latter is, in turn, a generalization of the well-known fact that, for a vector bundle \(E\), the double vector bundles \(T^*E\) and \(T^*E^*\) are canonically symplectically isomorphic.

11 Principal Lie algebroid cohomology

Consider again a linear principal Poisson \(\mathbb{R}^\times\)-bundle \((P, \mathcal{J}, \tilde{h}^0, \tilde{h}^1)\). To fix our attention, let us assume for a moment that \(\mathcal{J}\) is even. The principal Lie algebroid bracket on \(P^*\) is associated with the Poisson bracket \(\{\cdot, \cdot\}_{\mathcal{J}}\) which, according to Theorem \([8,3]\), is the derived bracket generated from the Schouten bracket \(\{\cdot, \cdot\}_{\mathbb{P}}\), by the linear 1-homogeneous Poisson tensor \(\mathcal{J}\) viewed as a homological 1-homogeneous quadratic Hamiltonian on \(\Pi C^\times_{P} = C^\times_{\Pi P} = \Pi T^*\Pi P^* \simeq \Pi T^*\Pi P^*\).

This Schouten bracket is of the tri-degree \((-1, -1, -1)\) on \(\mathcal{A}(\Pi T^* P)\), so that the quadratic Hamiltonian \(\mathcal{J}\), being of the tri-degree \((1, 1, 2)\) and homological, induces a cochain complex

\[
\begin{equation}
(A^{1,0,\cdot}(\Pi T^* P), d_{\mathcal{J}}), \quad d_{\mathcal{J}} = \{\mathcal{J}, \cdot\}_{\mathbb{P}},
\end{equation}
\]

of \(\mathcal{A}^{0,0,0}(\Pi T^* P)\)-modules. Note that \(\mathcal{A}^{0,0,0}(\Pi T^* P) = C^\infty(M)\), where \(M\) is the base of the vector bundle \(P_0\), and since we can identify elements of \(\mathcal{A}^{1,0,\cdot}(\Pi T^* P)\) with 1-homogeneous basic functions of degree \(l\) on \(\Pi T^* P^*\), thus functions from \(\mathcal{A}^{(1,0)}(\Pi P^*)\), our cochain complex is isomorphic with \(\left(\mathcal{A}^{1,\cdot}(\Pi P^*), d_{\mathcal{J}}\right)\). Observe that \(\mathcal{A}^{(1,\cdot)}(\Pi P^*)\) is canonically an \(\mathcal{A}^{0,\cdot}(\Pi P^*)\)-module and \(d_{\mathcal{J}}\) is a first-order differential operator on this module. A more vivid description would say that \(d_{\mathcal{J}}\) is a homological first-order differential operator acting on polynomial sections of the line bundle \(\Pi P \to \Pi P_0\). If \(\mathcal{J}\) is odd, we get an analogous complex \(\left(\mathcal{A}^{1,\cdot}(P^*), d_{\mathcal{J}}\right)\).
**Definition 11.1.** Consider the principal Lie algebroid $E(P) \simeq P^*$ associated with a linear principal $\mathbb{R}^\times$-bundle $(P, J, h^0, h^1)$. The cohomology of the complex $(A^{(1,0)}(P^*), d_J)$ if $J$ is even, and of the complex $(A^{(1,\bullet)}(P^*), d_J)$ if $J$ is odd, will be called the cohomology of the principal Lie algebroid $E(P)$.

**Example 11.1.** For a purely even $\mathcal{M}$, consider $P = T^*(\mathbb{R}^\times \times M) = \mathbb{R}^\times \times (\mathbb{R} \times T^*M)$, with the canonical Poisson tensor $J = \partial\partial_x + \partial_z\partial_p_a$ as in Example 11.1. We have $P^* = T(\mathbb{R}^\times \times M)$ with the adapted coordinates $(t, x^a, \dot{t}, \dot{x}^a)$ and $\Pi^*P^* = \Pi^*\Pi(T(\mathbb{R}^\times \times M))$ with the adapted Darboux coordinates $(t, x^a, \dot{t}, \dot{x}^a, z, \dot{p}_a, \dot{z}, \dot{p}_a)$. If, according to

$$\Pi^*\Pi(T(\mathbb{R}^\times \times M)) \simeq \Pi^*T^*(\mathbb{R}^\times \times M) = \Pi^*P,$$

we will regard $(t, x^a, z, \dot{p}_a, \dot{z}, \dot{p}_a, \dot{t}, \dot{x}^a)$ as coordinates in $\Pi^*P$ of the tri-degrees, respectively,

$$(1, 0, 0), (0, 0, 0), (0, 1, 0), (1, 1, 0), (0, 1, 1), (1, 1, 1), (1, 0, 1), (0, 0, 1),$$

and the parity indicated by the last term in the bi-degree, the canonical Poisson tensor on $T^*P$ takes the form

$$\partial_t\partial_z + \partial_t\partial_x + \partial_{x^a}\partial_{p_a} + \partial_{z\dot{p}_a},$$

so that $J$ is represented by the quadratic and odd Hamiltonian of the tri-degree $(1, 1, 2)$,

$$\iota_J = \dot{z}\partial_z + \dot{p}_a\partial_{p_a} + \dot{x}^a\partial_{x^a}.$$

The corresponding Hamiltonian vector field is therefore

$$\dot{z}\partial_z + \dot{p}_a\partial_{p_a} + \dot{x}^a\partial_{x^a},$$

which, restricted to functions from

$$A^{(0,0)}(\Pi^*P) = A(\Pi P^*),$$

i.e., functions in variables $t, x, \dot{t}, \dot{x}$, gives

$$d_J = \dot{t}\partial_t + \dot{x}^a\partial_{x^a}.$$
12 Contact 2-manifolds and principal Courant algebroids

Following the ideas of Roytenberg [54] in the symplectic case, let us consider contact 2-manifolds, i.e., 2-manifolds equipped with a contact structure of weight 2. According to our general philosophy, we will prefer an alternative definition which follows from the following variant of Theorem 6.2.

**Theorem 12.1.** A contact 2-manifold is canonically associated with a symplectic principal $\mathbb{R}^{\times}$-bundle $(P, \omega, h^0, h^1)$ of degree 2.

In the above, $h^0$ represents the free $\mathbb{R}^{\times}$-action on the principal bundle $\tau_0 : P \to P_0$ and $h^1$ is the homogeneity structure associated with an $h$-complete weight vector field $\Delta$. The symplectic form $\omega$ is 1-homogeneous with respect to $h^0$ and is of weight 2 with respect to $\Delta$, thus even, as in this case the parity is determined by the weight.

It is well known [54] that with a 2-manifold $M$ we have associated a tower of fibrations,

$$M = M_{(2)} \to M_{(1)} \to M_{(0)},$$

corresponding to the filtration $A_{(0)}(M) \subset A_{(1)}(M) \subset A_{(2)} = A(M)$ of the polynomial algebra on $M$, where $A_{(i)}$ is the subalgebra in $A$ generated by polynomial functions of weight $\leq i$, $i = 0, 1, 2$. Hence, $M_{(i)}$ is an $i$-manifold. In particular, $M_{(0)}$ is an even manifold and $M_{(1)} = F[1]$ is a vector bundle over $M_{(0)}$ with odd fibers. In the case of an $\mathbb{R}^{\times}$-principal 2-manifold, the fibrations intertwine the $\mathbb{R}^{\times}$-action, so $M_{(0)}$ and $M_{(1)}$ are also principal $\mathbb{R}^{\times}$-bundles and $M \to M_{(1)} \to M_{(0)}$ is a morphism of principal $\mathbb{R}^{\times}$-bundles. If, additionally, $(M, \omega)$ is a symplectic $\mathbb{R}^{\times}$-principal 2-manifold, the bundle $F[1]$ is a pseudo-Euclidean with a pseudo-scalar product $\langle \cdot, \cdot \rangle$ and $(M, \omega)$ is the pullback bundle of the affine fibration

$$T^*[2]F[1] \to (F \oplus_{M_{(0)}} F^*)[1],$$

equipped with the pulled-back canonical symplectic form, with respect to the isometric bundle embedding $F[1] \to (F \oplus_{M_{(0)}} F^*)[1]$ associated with $\langle \cdot, \cdot \rangle$. The pseudo-Euclidean product induces a Poisson structure on the principal $\mathbb{R}^{\times}$-bundle $M_{(1)} = F[1]$ that makes it into a linear odd principal Poisson $\mathbb{R}^{\times}$-bundle, as the pseudo-Euclidean product is an $\mathbb{R}^{\times}$-homogeneous linear odd Poisson bracket. Such linear odd principal Poisson $\mathbb{R}^{\times}$-bundles we will call **pseudo-Euclidean principal $\mathbb{R}^{\times}$-bundles**.

**Example 12.1.** Let $(\bar{P}, \bar{h}^0, \bar{h}^1)$ be a linear principal $\mathbb{R}^{\times}$-bundle with the vector bundle fibration $\bar{\tau}_1 : \bar{P} \to \bar{P}_1$. According to Theorem 10.2, the cotangent bundle $P = T^*\bar{P}$, equipped with the cotangent lifts of the homogeneity structures $h^i = T^*h^i$, $i = 0, 1$, and the canonical homogeneity structure $h^2$ of the vector bundle structure $\pi_P : T^*\bar{P} \to \bar{P}$, is a contact 2-vector bundle with respect to the canonical symplectic form $\omega = \omega_P$. Since $\omega$ is 1-homogeneous with respect to $h^1$, $i = 0, 1, 2$, we can produce a 1-graded contact manifold of weight 2, $(\bar{P}, \omega, \bar{h}^0, \bar{h}^1)$, out of it by composing the two vector bundle homogeneity structures, $h = h^1 \circ h^2$. Since the parity should be determined by the weight, $\bar{P}_1$ must be a purely even principal $\mathbb{R}^{\times}$-bundle over a purely even manifold $M$. The linear principal $\mathbb{R}^{\times}$-bundle $\bar{P}$ can be written as the product $\bar{P} = \bar{P}_1 \times_M \bar{P}_0$, where $\bar{P}_0 \to M$ is a vector bundle with odd linear fibers, so that $\bar{P}_0 = F_0[1]$ is the weight shift of a purely even vector bundle $\zeta : F_0 \to M$. Therefore,

$$\bar{P} = \bar{P}_1 \times_M F_0[1]$$

and finally

$$P = T^*[2] (\bar{P}_1 \times_M F_0[1]) = T^*[2] \bar{P}_1 \times_{T^*[2]M} T^*[2] F_0[1],$$

doing so $P_{(0)} = \bar{P}_1$ and $P_{(1)} = F[1] = \bar{P}_1 \times_M (F_0 \oplus_{M} F_0^*)[1]$ is a vector bundle over $\bar{P}_1$.

For trivial bundles, $\bar{P}_1 = \mathbb{R}^{\times} \times M$ and $F_0 = V \times M$, with coordinates $(u, x^a)$ and $(\xi^t, \eta^i)$, thus locally in general, we can write

$$P = T^*[2] \mathbb{R}^{\times} \times T^*[2]M \times T^*[2] V.$$

Denoting the corresponding coordinates as $(u, z, x^a, p_a, \xi^t, \eta^i)$, $t \neq 0$, where $u, x^a$ are of weight 0, $\xi^t$, $\eta^i$ are of weight 1, and $z, p_a$ are of weight 2, we can write the symplectic form (which is of weight 2) as

$$\omega = dz du + dp_a dx^a + d\eta_i d\xi^i,$$

and the corresponding Poisson tensor as

$$\mathcal{J} = \partial_u \partial_u + \partial_{p_a} \partial_{x^a} + \partial_{\eta_i} \partial_{\xi^i}.$$
The pseudo-Euclidean structure on the bundle $E[1]$ is represented by the form $d\eta d\xi^i$. The $\mathbb{R}^\times$-action is the phase lift of the action on $\mathbb{R}^\times$ by translations, so that

$$h^0_t(u, z, x^a, p_a, \xi^i, \eta) = (t' u, z, x^a, t' p_a, \xi^i, t' \eta).$$

In the coordinates $(t, z, x^a, p_a, \xi^i, \eta)$, where $t = -u$, $u p_a = p_a$ and $w\eta = \eta$, the action looks simpler, namely

$$h^0_t(t, z, x^a, p_a, \xi^i, \eta) = (t' t, z, x^a, p_a, \xi^i, \eta),$$

but the form of $\omega$ is more sophisticated,

$$\omega = dt dz - t (dp_a dx^a + d\eta d\xi^i) - dt (p_a dx^a + \eta d\xi^i) = d (dz - p_a dx^a - \eta d\xi^i).$$

The $\mathbb{R}^\times$-homogeneous pseudo-Euclidean form on $E[1] = \mathbb{R}^\times \times_M V[1] \times V^*[1]$ is represented by the Poisson tensor $-\frac{1}{t} \partial \eta \partial t$, being of weight -1 with respect to the vector field $t \partial_t$. On the other hand, $\omega$ is manifestly associated with the contact form $\alpha = dz - p_a dx^a - \eta d\xi^i$ of weight 2 on $J^1\left( P^*_1 \times_M F_0[1] \right) = \mathbb{R}[2] \times M \times V[1] \times V^*[1]$.

The above example shows that, in general, if $F$ is a purely even linear principal $\mathbb{R}^\times$-bundle over $\mathcal{M}$ with the linear $\mathbb{R}^\times$-action $h^0$ then $F \oplus_{\mathcal{M}} F^*$ is canonically a pseudo-Euclidean principal $\mathbb{R}^\times$-bundle with respect to action $h^0_\xi = h^0 \oplus t_\ast (h^0_{t-1} \xi)$ and the pseudo-Euclidean product induced from the canonical pairing between $F$ and $F^*$,

$$\langle X + \mu, Y + \nu \rangle = \frac{1}{2} (i_X \nu + i_Y \mu).$$

This structure can be viewed as a reduction of the symplectic principal $\mathbb{R}^\times$-bundle $T^*[2] F[1]$ of degree 2. If $F$ is a pseudo-Euclidean principal $\mathbb{R}^\times$-bundle, then the isometric bundle embedding

$$F[1] \to (F \oplus_{\mathcal{M}} F^*)[1], \quad X \mapsto X + \langle X, \cdot \rangle,$$

is simultaneously a morphism of linear principal $\mathbb{R}^\times$-bundles. We then let $P$ be the symplectic principal bundle of degree 2 being the pull-back of $T^*[2] F[1]$, i.e. completing the commutative diagram of morphisms of linear principal Poisson $\mathbb{R}^\times$-bundles

(12.4) \[
\begin{array}{ccc}
P & \longrightarrow & T^*[2] F[1] \\
\downarrow & & \downarrow \\
F[1] & \longrightarrow & (F \oplus_{\mathcal{M}} F^*)[1]
\end{array}
\]

This leads to the following ‘contact variant’ of the Roytenberg’s result [55, Theorem 3.3].

**Theorem 12.2.** Contact 2-manifolds are in one-to-one correspondence with pseudo-Euclidean principal $\mathbb{R}^\times$-bundles. The correspondence is given by the above construction.

**Example 12.2.** A canonical example of a contact 2-vector bundle is $P = T^*[2] T[1](L^*)^\times$ for a line bundle $L$ over $M$, with the diagram of vector and principal $\mathbb{R}^\times$-bundle morphisms

(12.5) \[
\begin{array}{ccc}
T^*[1](L^*)^\times & \longrightarrow & T^*[2] T[1](L^*)^\times \\
\downarrow & & \downarrow \\
J^1 [1] C_L^* & \longrightarrow & (L^*)^\times \\
\downarrow & & \downarrow \\
J^1 L[1] & \longrightarrow & T[1](L^*)^\times \\
\downarrow & & \downarrow \\
M & \longrightarrow & DO^1[1](L, L)
\end{array}
\]

Here, $J^1 C_L^*$ is the first jet bundle of the dual line bundle of $C_L \to J^1 L$, where $C_L \subset T^* J^1 L$ is the canonical contact bundle associated with $L$.
13 Principal Courant algebroids

Courant algebroids, originally defined in terms the standard differential geometry [44], have been recognized by Roytenberg [51, 55] as certain Hamiltonian systems on symplectic 2-manifolds. To be more precise, let us start with a symplectic 2-manifold $(\mathcal{M}, \omega, \hbar)$ with $\mathcal{M}_{(1)} = F[1]$, so that elements of $\mathcal{A}^1(\mathcal{M})$ (i.e., of weight 1) represent linear functions on $F$, and any section $e$ of $F$ represents a linear function $\iota_e$ on $F^*$.

Let $\{\cdot, \cdot \}_\mathcal{M}$ be the Poisson bracket associated with the symplectic form $\omega$ of weight 2. The symplectic Poisson bracket induces a nondegenerate pairing

\begin{equation}
\{\cdot, \cdot \}_\mathcal{M} : \mathcal{A}^1(\mathcal{M}) \otimes \mathcal{A}^0(\mathcal{M}) \to \mathcal{A}^0(\mathcal{M}),
\end{equation}

inducing an identification

$$F \simeq F^*,$$

i.e., an identification of the module $\text{Sec}(F)$ of sections of $F$ and the $\mathcal{A}^0(\mathcal{M}) = C^\infty(\mathcal{M}_{(0)})$-module $\mathcal{A}^1(\mathcal{M})$ of homogeneous functions of degree 1 by $e = \{\iota(e), \cdot \}_\mathcal{M}$, and a pseudo-Euclidean product on $F$ defined by

\begin{equation}
\langle e, e' \rangle = \{\iota(e), \iota(e') \}_\mathcal{M}.
\end{equation}

If $H \in \mathcal{A}^3(\mathcal{M})$ is a homological cubic Hamiltonian, $\{H, H\}_\mathcal{M} = 0$, then the derived bracket,

\begin{equation}
\{\varphi_1, \varphi_2 \} = \{\{\varphi_1, H\}_\mathcal{M}, \varphi_2 \}_\mathcal{M},
\end{equation}

is even and satisfies the Jacobi identity (2.7) with $k = 0$. Bilinear brackets (not necessarily skew-symmetric) satisfying the Jacobi identity are called Loday (or Leibniz) brackets. Being of degree -2, the bracket is closed on $\mathcal{A}^1(\mathcal{M})$, so that it can be viewed as a Loday bracket, called the Courant-Dorfman bracket, on sections of the vector bundle $F$. This bracket will be denoted, with some abuse of notation, also $\{\cdot, \cdot \}$, i.e., $\{e, e'\} = \{\iota(e), \iota(e')\}$. The bracket $\{\iota(e), f\}$ between functions $\iota(e) \in \mathcal{A}^1(\mathcal{M})$ and basic functions $f \in \mathcal{A}^0(\mathcal{M}) = C^\infty(\mathcal{M}_{(0)})$ is a derivative with respect to $f$ and corresponds to a vector bundle morphism $\rho : F \to T\mathcal{M}_{(0)}$ (the anchor map). The structures $\{\cdot, \cdot \}$ and $\rho$, enriched with the pseudo-Euclidean product (13.2), form a so called Courant algebroid structure on $F$. Moreover, maps $\{\cdot, \cdot \}$, $\rho$, and $\langle \cdot, \cdot \rangle$ arise in this way if and only if the following conditions hold true (cf. [23]):

\begin{align}
\langle \{e, e'\}, e' \rangle &= \langle e, \{e', e'\} \rangle, \label{eq:13.4} \\
\rho(e)\{e', e'\} &= 2\langle e, e' \rangle, \label{eq:13.5}
\end{align}

for all section $e, e'$ of $F$. Note that (13.5) implies that $\rho$ is the (left) anchor of the Courant-Dorfman bracket $\{\cdot, \cdot \}$ in the usual sense, i.e.,

\begin{equation}
\{e, f e'\} = f \{e, e'\} + \rho(e)(f)e',
\end{equation}

for all sections $e, e'$ of $F$ and all $f \in C^\infty(\mathcal{M}_{(0)})$. Indeed, the polarization of (13.1) gives

\begin{equation}
\rho(e)\{e', e''\} = \{\langle e, e' \rangle, e'' \} + \{\langle e, e'' \rangle, e' \}.
\end{equation}

Replacing $e'$ in the above equality with $fe'$, we conclude that

$$\rho(e)(f)e' = \{\langle e, f e' \rangle, e'' \},$$

thus (13.5), as the product $\langle \cdot, \cdot \rangle$ is nondegenerate. The anchor property (13.5) implies, in turn, that the anchor map is a bracket homomorphism (cf. [18]),

\begin{equation}
\rho(\{e, e'\}) = [\rho(e), \rho(e')],
\end{equation}

where the bracket on the right-hand side is clearly the bracket of vector fields. The latter follows also directly from the graded Jacobi identity applied to the bracket $\{\cdot, \cdot \}_\mathcal{M}$. To write a formula similar to (13.4) with respect to the multiplication by function in the first argument, let us first polarize (13.3) to get

$$\{\{e, e'\}, e''\} + \{\{e, e''\}, e'\} = \langle e, \{e', e''\} + \{e'', e'\} \rangle.$$
that, combined with (13.7), yields
\[(13.9)\quad \rho(e)(e', e'') = \langle e, \{e', e''\} + \{e'', e'\} \rangle.
\]
Defining a derivation \(D : C^\infty(M) \to \text{Sec}(F)\) by means of the pseudo-Euclidean product by
\[(13.10)\quad \langle D(f), e \rangle = \rho(e)(f),
\]
we can rewrite (13.9) in the form
\[(13.11)\quad \{e', e''\} + \{e'', e'\} = D(e', e'').
\]
Replacing \(e'\) with \(fe'\) in the latter and using (13.6), we get
\[(13.12)\quad \{fe', e''\} = f\{e', e''\} - \rho(e'')(f)e' + \langle e', e''\rangle D(f).
\]

If we start not with a bare symplectic 2-manifold but with a contact 2-manifold, i.e., with a symplectic principal \(\mathbb{R}^\times\)-bundle \((P, \omega, h^0, h)\) of degree 2, we can repeat the above construction. Note that in this case, \(P, \pi_1 = F[1]\), and \(P_{(0)}\) are additionally principal \(\mathbb{R}^\times\)-bundles. As our structure is richer than in the case of a Courant algebroid, e.g. the bracket \(\{\cdot, \cdot\}\) is homogeneous of degree -1 with respect to the \(\mathbb{R}^\times\) action \(h^0\), we can consider the space \(A^{(1,0)}(P) = \oplus_{n=0}^\infty A^{(1,0)}(P)\) which is closed with respect to this Poisson bracket, where \(A^{(1,0)}(P)\) is the subspace in the space of those functions of weight \(i\) on \(P\) which are simultaneously 1-homogeneous with respect to \(h^0\). In this notation, elements of \(A^{(1,0)}(P)\) represent homogeneous basic functions, i.e., functions on \(P_{(0)}\) which are 1-homogeneous with respect to the reduced \(\mathbb{R}^\times\)-action on \(P_{(0)}\), and elements of \(A^{(1,1)}(P)\) represent homogeneous sections of \(F\).

If we now choose an \(\mathbb{R}^\times\)-homogeneous cubic homological Hamiltonian \(H \in A^{(1,3)}(P)\), \(\{H, H\} = 0\), the derived bracket (13.3) is of degree \((-1, -1)\). In particular,
\[\{A^{(1,1)}(P), A^{(1,1)}(P)\} \subset A^{(1,1)}(P),\]
so it is closed on homogeneous sections of \(F\). This suggests the following definition a la Roytenberg.

**Definition 13.1.** A principal Courant algebroid is a contact 2-manifold \(P\) equipped with an \(\mathbb{R}^\times\)-homogeneous cubic homological Hamiltonian \(H \in A^{(1,3)}(P)\), \(\{H, H\} = 0\).

In this sense, a principal Courant algebroid is an \(\mathbb{R}^\times\)-homogeneous Courant algebroid. To find a more ‘classical’ description of the principal Courant algebroid, note that the space \(A^{(1,1)}(P)\), as well as the space \(A^{(1,0)}(P)\) of homogeneous functions on \(P_{(0)}\), is an \(A^{(0,0)}(P)\)-module, where elements of \(A^{(0,0)}(P)\) represent functions on the base \(M = P_{(0)}/\mathbb{R}^\times\) of the principal \(\mathbb{R}^\times\)-bundle \(P_{(0)} \to M\). Homogeneous functions on \(P_{(0)}\) represent, in turn, sections of the line bundle \(L = P_{(0)}^e\). Since
\[\{A^{(1,1)}(P), A^{(1,0)}(P)\} \subset A^{(1,0)}(P),\]
with any homogeneous linear section \(e\) of \(E\) we can associate a first-order differential operator \(\hat{\rho}^e\) from \(L\) into \(L\) represented by
\[\{e, \cdot\} : A^{(1,0)}(P) \to A^{(1,0)}(P).
\]
Similarly to decomposition (12.11), one can prove that \(F = P_{(0)} \times_M F_1\) for a vector bundle \(F_1\) over \(M\), so that elements of \(A^{(1,1)}(P)\) represent, via the identification \(e \mapsto \iota(e)\), sections of the vector bundle \(E(F) = L \otimes_M F_1\), and \(\hat{\rho}^e\) can be viewed as a vector bundle morphism
\[(13.13)\quad \hat{\rho} : E(F) \to DO^1(L, L).
\]
As
\[\{A^{(1,1)}(P), A^{(1,1)}(P)\} \subset A^{(1,0)}(P),\]
the symplectic Poisson bracket induces an \(L\)-valued pseudo-Euclidean product on \(E(F)\) defined by
\[(13.14)\quad \langle X, Y \rangle^1 = \{\iota(X), \iota(Y)\}.
\]
The Loday bracket induced on sections of $\mathcal{E}(F)$ by $\{\cdot,\cdot\}$, the Courant-Jacobi bracket, we will denote with $\{\cdot,\cdot\}_1$. The graded Jacobi identity for $\{\cdot,\cdot\}$ immediately implies the identities

$$
\begin{align}
\langle\{X,Y\}^1,Y\rangle^1 &= \langle X,\{Y,Y\}\rangle^1, \\
\hat{\rho}^1(X,Y)^1 &= 2\{\{X,Y\}^1,Y\},
\end{align}
$$

this time valid for sections $X,Y$ of $\mathcal{E}(F)$ and the pseudo-Euclidean product $\langle \cdot,\cdot\rangle$ with values in the line bundle $L$. Property (13.16) implies also that

$$\rho^1(X,Y)^1 = [\rho^1(X),\rho^1(Y)].$$

Here, of course, the right-hand side bracket in (13.17) is the commutator bracket of first-order differential operators. To prove (13.17), consider the polarization of (13.16),

$$\rho^1(X,Y)^1 = \langle\{X,Y\}^1,Y\rangle^1 + \langle\{X,Y\}^1,\{Y,Y\}^1\rangle,$$

This implies

$$\begin{align}
\rho^1(X')\rho^1(X,Y)^1 &= \langle\{X',\{X,Y\}^1\},\{Y\}\rangle^1 + \langle\{X',\{X,Y\}^1\},\{Y,Y\}\rangle^1 \\
&+ \langle\{X',\{X,Y\}^1\},\{Y,Y\}\rangle^1 + \langle\{X,Y\}^1,\{Y,Y\}\rangle^1,
\end{align}$$

so

$$\begin{align}
[\rho^1(X'),\rho^1(X)](Y)^1 &= \langle\{\{X',X\}^1,Y\},\{Y\}\rangle^1 + \langle\{\{X',X\}^1,Y\},\{Y,Y\}\rangle^1 \\
&= \hat{\rho}^1(\{X',X\}^1)\langle Y,Z\rangle^1
\end{align}$$

and (13.17) follows, since $\langle\cdot,\cdot\rangle^1$ is nondegenerate and $\langle Y,Z\rangle^1$ can be arbitrary.

Like in the case of a linear principal Poisson $\mathbb{R}^\times$-bundle, we can consider the linear principal $\mathbb{R}^\times$-bundle $E = E(F) = (L^*)^\times \times_M \mathcal{E}(F)$ and the canonical isomorphism $\tilde{\mathcal{E}}: E(F) \to F^*$ given by (9.12).

In particular, any section $\varepsilon$ of $\mathcal{E}(F)$ defines an invariant section $\tilde{\varepsilon}$ of $E$ and, via the isomorphism $\tilde{\mathcal{E}}$, a $1$-homogeneous section of $F^*$, thus again a function $\iota_\varepsilon \in A^{(1,1)}(F)$.

The pseudo-Euclidean product on the vector bundle $F \simeq F^*$ can be reconstructed from the $L$-valued pseudo-Euclidean product on $\mathcal{E}(F)$ as the unique one satisfying the formula

$$
\langle\tilde{\varepsilon},\tilde{\varepsilon}'\rangle((*)^*) = l^* \left(\langle\varepsilon,\varepsilon'\rangle^1\right).
$$

We can obtain the anchor $\rho: E \to TP(0) = (L^*)^\times$, using a natural identification of first-order differential operators $D$ from $L$ to $L$ with invariant vector fields $\hat{D}$ in the principal $\mathbb{R}^\times$-bundle $(L^*)^\times$,

$$\rho(\tilde{\varepsilon}) = \rho(\varepsilon),$$

and hence the derivation $D: C^\infty(P(0)) \to \text{Sec}(F)$ which is defined in terms of $\rho$ and $\langle\cdot,\cdot\rangle$. We can also reconstruct the bracket $\{\cdot,\cdot\}$ on sections of $F$. First of all,

$$\{\tilde{\varepsilon},\tilde{\varepsilon}'\} = \{\varepsilon,\varepsilon'\}^1.$$

Sections of the form $\tilde{\varepsilon}$ generate the vector bundle $E$, so the above expression uniquely defines $\{\cdot,\cdot\}$, in view of identities (13.6) and (13.12), that will give

$$\{f\tilde{\varepsilon},g\tilde{\varepsilon}'\} = fg\{\varepsilon,\varepsilon\} + f\rho(\varepsilon)g(\varepsilon')\varepsilon' - g\rho(\varepsilon')\varepsilon + g\langle\varepsilon,\varepsilon'\rangle\text{D}(f).$$

Conversely, suppose that we have a structure $(\mathcal{E},L,\{\cdot,\cdot\}_1,\{\cdot,\cdot\}_1,\rho^1)$ such that

(a) $\mathcal{E}$ is a vector bundle over $M$ and $L$ is a line bundle over $M$,

(b) $\{\cdot,\cdot\}_1$ is a Loday bracket on sections of $\mathcal{E}$,

(c) $\langle\cdot,\cdot\rangle^1$ is a pseudo-Euclidean product with values in $L$,

(d) $\rho^1: \mathcal{E} \to \text{DO}^1(L,L)$ is a vector bundle morphism, associating with any section $X$ of $\mathcal{E}$ a first-order differential operator $\rho^1(X)$, acting on sections of $L$,
(e) identities (13.15) and (13.16) are satisfied for all sections $X, Y$ of $E$.

Let us define a linear principal $\mathbb{R}^\times$-bundle $E = (L^*)^\times \times_M E$, an $\mathbb{R}^\times$-homogeneous pseudo-Euclidean product $\langle \cdot, \cdot \rangle$ on $E$ by (13.21), an anchor map $\rho : E \to TM$ by (13.14), and a Loday bracket $\{\cdot, \cdot\}$ on sections of $E$, satisfying (13.9) and (13.12), by (13.20). It can be checked directly in local coordinates that the constructions are correct and give us a Courant algebroid structure on $E$. As in this case $E$ is simultaneously a principal $\mathbb{R}^\times$-bundle, it is easy to see that all these structures are $\mathbb{R}^\times$-homogeneous, so the cubic Hamiltonian on the minimal symplectic realization, defining the derived bracket, is homogeneous as well. Summing up, we can propose the following.

**Theorem 13.1.** There is a one-to-one correspondence between principal Courant algebroids and structures $\langle E, L, \{\cdot, \cdot\}, \langle \cdot, \cdot\rangle, \rho \rangle$ described by items (a)-(d) above.

**Remark 13.1.** If the line bundle is trivial, $L = \mathbb{R} \times M$, the above description of Courant-Jacobi algebroids coincides with the one in [23, Definition 1] if we take into account that identity (13.17) follows from (13.16), so is superfluous.

**Example 13.1.** Consider a principal Courant algebroid $\langle E, L, \{\cdot, \cdot\}, \langle \cdot, \cdot\rangle, \rho \rangle$ defined as above. We can always find a local trivializations of $L$ and $E$ with associated local affine coordinates $(x^a, z)$ and $(x^a, \theta^i)$, respectively. In the local trivialization of $L$, the pseudo-Euclidean product with values in $L$ is a standard product with values in $\mathbb{R}$. Moreover, a basis $\{e_i\}$ of local sections of $E$ can always be chosen such that $\langle e_i, e_j \rangle = g_{ij}$ and the basic functions $g_{ij}$ are constants. Since by construction $\theta^i(e_j) = \delta^i_j$, our identification of sections with linear functions via the pseudo-Euclidean product yields $\varepsilon_i = g_{ij} \theta^j$.

Let us write in local coordinates

$$\rho(e_i) = r_i^a(x) \partial x^a + r_i(x)$$

and

$$\Theta(e_i, e_j, \varepsilon_k) = \langle \{e_i, e_j\}, \varepsilon_k \rangle = A_{ijk}(x).$$

Since the product $\langle \cdot, \cdot \rangle$ is nondegenerate, the above formula defines the brackets $\{e_i, e_j\}$ uniquely. Note that $A_{ijk}(x)$ is totally skew-symmetric in $(i, j, k)$. Indeed, combining (13.15) and (13.16), we get

$$\rho(\varepsilon_k)(e_i, e_j) = \langle \varepsilon_k, \{e_i, e_j\} \rangle + \langle \{e_i, \varepsilon_k\}, e_j \rangle = A_{ijk}(x) + A_{ikj}(x).$$

As $\langle e_i, e_j \rangle$ is constant, the left-hand side is 0, thus $A_{ijk}(x)$ is skew-symmetric with respect to the first two arguments. Similarly, (13.18) gives

$$\rho(\varepsilon_i)(e_j, \varepsilon_k) = \langle \{e_i, e_j\}, \varepsilon_k \rangle + \langle \{e_i, \varepsilon_k\}, e_j \rangle = A_{ijk}(x) + A_{ikj}(x) = 0,$$

so $A_{ijk}(x)$ is skew-symmetric also with respect to the last two arguments, hence totally skew-symmetric.

The linear principal $\mathbb{R}^\times$-bundle $E = (L^*)^\times \times_M E$ has local coordinates $(t, x^a, \theta^i)$, $t \neq 0$, and the $\mathbb{R}^\times$-action reduces to the regular action on the first argument, $h_s(t, x^a, \theta^i) = (st, x^a, \theta^i)$. Homogeneous sections $e_i = \tilde{e}_i$ of $E$ read $e_i(t, x) = t e_i(x)$ and correspond to linear homogeneous functions $tg_{ij} \theta^j$. The pseudo-Euclidean product is determined by

$$\langle e_i, e_j \rangle = tg_{ij}$$

and corresponds to the odd Poisson structure

$$\Lambda = -\frac{1}{2t} \sum_{i,j} g^{ij} \partial \theta_i \partial \theta_j$$

on $E$. Here, of course $(g^{ij}) = (g_{ij})^{-1}$. The minimal symplectic realization $P$ is a symplectic principal $\mathbb{R}^\times$ bundle of degree 2. In local Darboux coordinates $(t, x^a, \theta^i, z, p_a)$, the symplectic form reads as

$$\omega = dz dt + dp_a dx^a + \frac{t}{2} g_{ij} d\theta^i d\theta^j.$$

Note that, among local coordinates, $t, x^a$ are of weight 0 thus even, $\theta^i$ are of weight 1 thus odd, and $z, p_a$ are of weight 2 thus even, so that $\omega$ is of weight 2 and even. Moreover, with respect to the
\[ \mathbb{R}^x \text{-action, } x^a, \theta^i, z \text{ are invariant, and } t, p_a \text{ are homogeneous of degree 1, so that } \omega \text{ is homogeneous of degree 1. The corresponding Poisson tensor } \mathcal{J} = \omega^{-1} \text{ of weight 2 takes the form (cf. (8.7))} \]

\[ \mathcal{J} = \partial_t \partial_z + \partial_x \partial p_a - \frac{1}{2t} g^{ij} \partial \theta_i \partial \theta_j. \] 

If we use invariant coordinates \( p_a \) instead of \( p_a = tp_a \), the Poisson tensor reads (cf. (8.7))

\[ \mathcal{J} = \partial_t \partial_z + \frac{1}{t} \left[ \partial_z \left( p_a \partial p_a + \theta^j \partial \theta^j \right) + \partial_x \partial p_a - \frac{1}{2} g^{ij} \partial \theta_i \partial \theta_j \right]. \] 

The homogeneous Courant-Dorfman bracket on sections of \( E \) satisfies, clearly,

\[ \{ \{ e_i, e_j \}, e_k \} (t, x) = tA_{ijk}(x) \] 

and its anchor map, \( \rho : E \to (L^\ast)^{x} \), reads

\[ \rho(e_i) = r^a_i(x) \partial_x^a + r_i(x) t \partial_t. \]

All these data imply that the corresponding homogeneous cubic Hamiltonian must be of the form

\[ H = \theta^i (r^a_i(x)p_a + r_i(x)tz) - \frac{t}{6} A_{ijk}(x) \theta^i \theta^j \theta^k. \]

Indeed, since \( \{ e_i, \theta^j \} = \delta^i_j \), we have

\[ \{ e_i, H \} = r^a_i(x)p_a + r_i(x)tz - \frac{t}{2} A_{ijk}(x) \theta^i \theta^j \theta^k, \]

so that

\[ \{ \{ e_i, e_j \}, e_k \} = \{ \{ e_i, H \}, e_j \}, e_k \} = tA_{ijk}(x) \]

and

\[ \rho(e_i)(f) = \{ \{ e_i, H \}, f \} = r^a_i(x) \frac{\partial f}{\partial x^a} + r_i(x) \frac{\partial f}{\partial t}, \]

for any basic function \( f = f(t, x) \). Of course, the Jacobi identity for the derived bracket \( \{ \cdot, \cdot \} \), together with (13.4) and (13.5), is equivalent with the homological condition \( \{ H, H \} = 0 \).

**Example 13.2.** Consider, like in Example 11.1, the contact 2-manifold \( T^*[2]T^*[1](\mathbb{R}^x \times M) \) for a purely even manifold \( M \). As the cubic Hamiltonian \( H \), associated with the canonical vector field on \( T^*[1](\mathbb{R}^x \times M) \) being the de Rham derivative, is 1-homogeneous, we obtain a homogeneous Courant bracket on the linear principal \( \mathbb{R}^x \)-bundle \( F = T(\mathbb{R}^x \times M) \oplus_{\mathbb{R}^x \times M} T^*(\mathbb{R}^x \times M) \). Since we can interpret \( \mathcal{E}(F) \) as the bundle \( \mathcal{E} = (\mathbb{R} \times TM) \oplus_M (\mathbb{R}^* \times T^*M) \) whose sections are \( (X, f) + (\alpha, g) \), where \( f, g \in C^\infty(M) \), \( X \) is a vector field, and \( \alpha \) is a one-form on \( M \), we conclude that the principal Courant algebroid structure on \( E \) consists of

(a) a Loday bracket on sections of \( \mathcal{E} \) of the form

\[ \{ (X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2) \}^1 = \{ [X_1, X_2], X_1(f_2) - X_2(f_1) \} + (\mathcal{L}_{X_1} \alpha_2 - i_{X_2} d \alpha_1 + f_1 \alpha_2 - f_2 \alpha_1 + f_2 dg_1 + g_2 df_1, X_1(g_2) - X_2(g_1) + i_{X_2} \alpha_1 + f_1 g_2), \]

(b) a pseudo-Euclidean product of the form

\[ \langle (X, f) + (\alpha, g), (X, f) + (\alpha, g) \rangle^1 = \langle X, \alpha \rangle + fg, \]

(c) a vector bundle morphism \( \rho^1 : \mathcal{E} \to DO^1(M) \) of the form

\[ \rho^1 ((X, f) + (\alpha, g)) = X + f. \]

The skew-symmetrization of bracket (13.31) gives exactly the bracket introduced by Wade to define so called \( \mathcal{E}^1(M) \)-Dirac structures. In other words, (13.31) is the Wade’s bracket in the Dorfman form.
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