Yang-Baxter Equation for the R-matrix of 1-D SU(n) Hubbard Model

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Abstract

Based on the tetrahedral Zamolodchikov algebra, we prove the Yang-Baxter equation for the R-matrix of 1-D SU(n) Hubbard model. Furthermore, we present generalizations of the model.

Keyword: Hubbard model, R-matrix, tetrahedral Zamolodchikov algebra

1 Introduction

The Hubbard model is one of the significant models in the study of strongly correlated electronic systems which might reveal an enlightening role in understanding the mysteries of the high-$T_C$ superconductivity. The 1-D Hubbard model also favours a lot of properties of integrable models in non-perturbative quantum field theory and mathematical physics. Since Lieb and Wu\textsuperscript{[1]} solved the 1-D Hubbard model by Bethe ansatz method in 1968, based on their results (Lieb and Wu’s Bethe ansatz equations), many works\textsuperscript{[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]} have been done extensively to clarify the physical properties of this model. Although there were lots of works on the Hubbard model, the integrability was finished until 1986 by Shastry\textsuperscript{[15]}, Olmedilla and Wadati\textsuperscript{[16]} in both boson and fermion graded versions. However, the Yang-Baxter equation for the $R$-matrix of the model obtained by Shastry was proved until 1995 by Shiroishi and Wadati in Refs.\textsuperscript{[31, 32, 33]} and a generalization of the Shastry’s bilayer vertex model was also presented in\textsuperscript{[31]}. Moreover, the eigenvalue of the

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transfer matrix related to the Hubbard model was suggested in Ref. [23] and proved through
different methods in Refs. [17, 18].

Based on the knowledge of Lie algebra, Maassarani and Mathieu succeeded in constructing
the Hamiltonian of the \( \text{SU}(n) \) XX model and showed its integrability [22]. Considered two
coupled \( \text{SU}(n) \) XX models, by using Shastry’s method, Maassarani constructed the \( \text{SU}(n) \)
Hubbard model [23] and found the related \( R \)-matrix which ensures the integrability of the 1-D
\( \text{SU}(n) \) Hubbard model [24]. (It was also proved by Martins for \( n = 3, 4, 25 \), and by Yue and
Sasaki [26] for general \( n \) in terms of Lax pair formulism.) The exact Solution of the \( \text{SU}(3) \)
Hubbard model was given in Ref. [27]. But the Yang-Baxter equation for the given \( R \)-matrix
was not proved.

The main purpose of the present paper is to prove the Yang-Baxter equation for the
\( R \)-matrix of 1-D \( \text{SU}(n) \) Hubbard model following method suggested in Ref. [31]. In section
2 we review the model and its integrability. We present the \( L \)-operator and the \( R \)
\( \text{Zamolodchikov} \) algebra related to the \( \text{SU}(n) \) Hubbard model and formulate the Yang-Baxter relation. In section
3 we construct the tetrahedral Yamolodchikov algebra related to the \( \text{SU}(n) \) Hubbard model. The Yang-Baxter equation for the
corresponding \( R \)-matrix was proved in section 4 and we also present a generalization of
the model in this section. In section 5 we make some conclusion remarks.

\section{The 1-D \( \text{SU}(n) \) Hubbard model and its integrability}

The Hamiltonian of the 1-D \( \text{SU}(n) \) Hubbard model is:

\[
H = \sum_{k=1}^{L} \sum_{\alpha=1}^{n-1} (E_{\alpha,k} E_{\alpha,k+1}^{\text{nn}} + E_{\alpha,k}^{\text{nn}} E_{\alpha,k+1} + E_{\tau,k}^{\text{nn}} E_{\tau,k+1} + E_{\tau,k} E_{\tau,k+1}) + \frac{U n^2}{4} \sum_{k=1}^{L} C_k^{(\sigma)} C_k^{(\tau)},
\]

where \( U \) is the Coulomb coupling constant, and \( E_{\alpha,k} (a = \sigma, \tau) \) is a matrix with an one at
row \( \alpha \) and column \( \beta \) and zeros otherwise:

\[(E_{\alpha\beta})_{lm} = \delta_{l}^{\alpha} \delta_{m}^{\beta}.\]

The subscripts \( a \) and \( k \) stand for two different \( E \) operators at \( k \)-th site \((k = 1, \cdots, L)\). The
\( n \times n \) diagonal matrix \( C \) is defined by \( C = \sum_{\alpha<n} E_{\alpha\alpha} - E_{\text{nn}} \). We also assume the periodic
boundary condition: \( E_{k+L}^{\alpha\beta} = E_{k}^{\alpha\beta} \).

In this model, the system has two types of particles named \( \sigma \) and \( \tau \) respectively, and each
particle can occupy \((n - 1)\) possible states. The same type of particles cannot appear in
one site, but two different types of particles can occupy a same site. We denote \(|n\rangle_{j}\) as the
vacuum state of \( j \)-th site, \(|1\rangle_{j}, |2\rangle_{j}, \cdots, |n - 1\rangle_{j}\) as the \((n - 1)\) possible one particle states of
\( j \)-th site. Under the following basis:

\[
|1\rangle_{j} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{j},
|2\rangle_{j} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{j},
\cdots, |n - 1\rangle_{j} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}_{j},
|n\rangle_{j} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}_{j}.
\]
It could be easily proved that: \( E^\alpha_j |n\rangle_j = |\alpha\rangle_j, E^n_\beta |\alpha\rangle_j = 0, E^n_\beta |n\rangle_j = 0. \) This means that the operators \( E^\alpha \) and \( E^n_\beta \) can be interpreted as the particle creation and annihilation operators respectively. \( E^\alpha_j \) can create a \(|\alpha\rangle_j\) state particle over the vacuum state \(|n\rangle_j\) of \( j\)-th site, and \( E^n_\beta \) annihilate a \(|\alpha\rangle_j\) state particle to vacuum state of \( j\)-th site.

The \( SU(n) \) Hubbard model is constructed by considering two coupled \( SU(n) \) XX model, so the Hamiltonian \( \mathcal{H} \) consists of two \( SU(n) \) XX model with an interaction term between them. The Hamiltonian of the \( SU(n) \) XX model is:

\[
H_{XX} = \sum_{k=1}^{L} \sum_{\alpha=1}^{n-1} (E_k^\alpha E_{k+1}^\alpha + E_k^\alpha E_{k+1}^\alpha),
\]

and the corresponding \( R \)-matrix is:

\[
R(\lambda) = a(\lambda)[E^{nn} \otimes E^{nn} + \sum_{\alpha,\beta<n} E^{\alpha\beta} \otimes E^{\beta\alpha}] + b(\lambda) \sum_{\alpha<n} (x E^{nn} \otimes E^{\alpha\alpha} + x^{-1} E^{\alpha\alpha} \otimes E^{nn}) + c(\lambda) \sum_{\alpha<n} (E^{\alpha\alpha} E^{nn} + E^{nn} E^{\alpha\alpha}),
\]

where \( x = e^{i\delta} \) and \( a(\lambda) = \cos(\lambda), b(\lambda) = \sin(\lambda), c(\lambda) = 1. \) The functions \( a(\lambda), b(\lambda), c(\lambda) \) satisfy the free-fermion relation: \( a^2(\lambda) + b^2(\lambda) = c^2(\lambda). \)

The \( R \)-matrix of the \( SU(n) \) XX model satisfies regularity property \( R(0) = P, \) unitarity condition \( R_{12}(\lambda) R_{21}(-\lambda) = \cos^2(\lambda) I \) and Yang-Baxter equation (YBE):

\[
R_{31}(\lambda_1) R_{32}(\lambda_2) R_{12}(\lambda_2 - \lambda_1) = R_{12}(\lambda_2 - \lambda_1) R_{32}(\lambda_2) R_{31}(\lambda_1),
\]

where \( P \) is a permutation operator on the tensor product of two \( n \)-dimensional spaces. It is easy to verify that it also satisfies a decorated Yang-Baxter equation (DYBE):

\[
R_{31}(\lambda_1) R_{32}(\lambda_2) C_2 R_{12}(\lambda_2 - \lambda_1) = R_{12}(\lambda_2 - \lambda_1) C_2 R_{32}(\lambda_2) R_{31}(\lambda_1).
\]

For the two \( SU(n) \) XX model, the \( R \)-matrix without interaction term is given by:

\[
\tilde{R}_{ij}(\lambda) = R_{ij}(\lambda),
\]

here \( R_{ij}(\lambda) \) and \( R_{ij}(\lambda) \) denote the \( R \)-matrices of two \( SU(n) \) XX models. Since both \( R_{ij}(\lambda) \) and \( R_{ij}(\lambda) \) satisfy the YBE and DYBE, the product \( \tilde{R}_{ij}(\lambda) \) also satisfy the YBE:

\[
\tilde{R}_{31}(\lambda_1) \tilde{R}_{32}(\lambda_2) \tilde{R}_{12}(\lambda_2 - \lambda_1) = \tilde{R}_{12}(\lambda_2 - \lambda_1) \tilde{R}_{32}(\lambda_2) \tilde{R}_{31}(\lambda_1)
\]

and DYBE:

\[
\tilde{R}_{31}(\lambda_1) \tilde{R}_{32}(\lambda_2) C_2^{(c)} C_2^{(c)} \tilde{R}_{12}(\lambda_2 - \lambda_1) = \tilde{R}_{12}(\lambda_2 - \lambda_1) C_2^{(c)} C_2^{(c)} \tilde{R}_{32}(\lambda_2) \tilde{R}_{31}(\lambda_1)
\]


A linear combination of (5) and (8) yields:

\[ R_{31}(\lambda_1)R_{32}(\lambda_2) \left\{ \alpha R_{12}(\lambda_2 - \lambda_1) + \beta C_2^{(\sigma)} C_2^{(\tau)} R_{12}(\lambda_2 + \lambda_1) \right\} = \left\{ \alpha R_{12}(\lambda_2 - \lambda_1) + \beta \bar{R}_{12}(\lambda_2 + \lambda_1) C_2^{(\sigma)} C_2^{(\tau)} \right\} \bar{R}_{32}(\lambda_2)R_{31}(\lambda_1), \]  

(9)

here \( \alpha \) and \( \beta \) are combination coefficients and arbitrary.

For the \( SU(n) \) Hubbard model, the two coupled \( SU(n) \) XX model, we look for a solution of the Yang-Baxter relation (YBR):

\[ L_{31}(\lambda_1)L_{32}(\lambda_2)R_{12}^h(\lambda_1, \lambda_2) = R_{12}^h(\lambda_1, \lambda_2)L_{32}(\lambda_2)L_{31}(\lambda_1) \]

(10)

in the form:

\[ R_{12}^h(\lambda_1, \lambda_2) = \alpha R_{12}(\lambda_2 - \lambda_1) + \beta \bar{R}_{12}(\lambda_2 + \lambda_1) C_2^{(\sigma)} C_2^{(\tau)}, \]

(11)

\[ L_{ij}(\lambda) = \bar{L}_{ij}(\lambda) \exp\{h(\lambda)C_j^{(\sigma)} C_j^{(\tau)}\}. \]

(12)

Comparing eq. (9) with the Yang-Baxter relation (10), we get a relation:

\[ I_1(\lambda_1)I_2(\lambda_2)R_{12}^h(\lambda_1, \lambda_2)I_1^{-1}(\lambda_1)I_2^{-1}(\lambda_2) = \alpha R_{12}(\lambda_2 - \lambda_1) + \beta C_2^{(\sigma)} C_2^{(\tau)} \bar{R}_{12}(\lambda_2 + \lambda_1), \]

(13)

where

\[ I_j(\lambda) = \exp\{h(\lambda)C_j^{(\sigma)} C_j^{(\tau)}\}. \]

(14)

From (13), we have:

\[ \frac{\beta a(\lambda_2 + \lambda_1)c(\lambda_2 + \lambda_1)}{\alpha a(\lambda_2 - \lambda_1)c(\lambda_2 - \lambda_1)} = \tanh(h(\lambda_2) - h(\lambda_1)), \]

\[ \frac{\beta b(\lambda_2 + \lambda_1)c(\lambda_2 + \lambda_1)}{\alpha b(\lambda_2 - \lambda_1)c(\lambda_2 - \lambda_1)} = \tanh(h(\lambda_2) + h(\lambda_1)), \]

(15)

which give the ratio of \( \alpha \) and \( \beta \) and constraints on \( h(\lambda_1) \) and \( h(\lambda_2) \). The constraints can be written in more explicit form \[24\]:

\[ \frac{\sinh(h(\lambda_1))}{\sin(2\lambda_1)} = \frac{\sinh(h(\lambda_2))}{\sin(2\lambda_2)} = \frac{n^2U}{4}. \]

(16)

Now we have obtained the \( R \)-matrix of the 1-D \( SU(n) \) Hubbard model \[24\]:

\[ R_{12}^h(\lambda_1, \lambda_2) = R_{12}^{(\sigma)}(\lambda_2 - \lambda_1)R_{12}^{(\tau)}(\lambda_2 - \lambda_1) + \frac{\cos(\lambda_2 - \lambda_1)}{\cos(\lambda_2 + \lambda_1)} \tanh(h(\lambda_2) - h(\lambda_1)) \times R_{12}^{(\sigma)}(\lambda_2 + \lambda_1)R_{12}^{(\tau)}(\lambda_2 + \lambda_1)C_2^{(\sigma)} C_2^{(\tau)}. \]

(17)

which satisfy the Yang-Baxter relation (10). This \( R \)-matrix depends not only on the difference of the spectral parameters \( \lambda_2 - \lambda_1 \), but also on the sum of the spectral parameters \( \lambda_2 + \lambda_1 \).
This non-additive property allows us to generalize the Hamiltonian of the 1-D \( SU(n) \) Hubbard model (see section 4).

The monodromy matrix of the model can be defined as:

\[
T_a(\lambda) = L_{La}(\lambda) L_{L-1a}(\lambda) \cdots L_{1a}(\lambda).
\]  

(18)

From the Yang-Baxter relation (10) we know the monodromy matrix satisfies the global Yang-Baxter relation:

\[
T_1(\lambda_1) T_2(\lambda_2) R_{12}^h(\lambda_1, \lambda_2) = R_{12}^h(\lambda_1, \lambda_2) T_2(\lambda_2) T_1(\lambda_1).
\]  

(19)

The corresponding transfer matrix is defined by:

\[
\tau(\lambda) = \text{tr}_a[T_a(\lambda)].
\]  

(20)

and from (19) it can be easily proved the existence of a commuting family of transfer matrices

\[
[\tau(\lambda_1), \tau(\lambda_2)] = 0.
\]  

(21)

Then the integrability of the model was proved.

Using the relation \( h(0) = 0 \) and \( h'(0) = \frac{n^2 U}{4} \) we can obtain the Hamiltonian of the 1-D \( SU(n) \) Hubbard model (1):  

\[
H = \frac{d}{d\lambda} \ln \tau(\lambda)|_{\lambda=0} = \tau^{-1}(0) \frac{d}{d\lambda} \tau(\lambda)|_{\lambda=0} \\
= \sum_{k=1}^L \sum_{a=1}^{n-1} (E_{\sigma,k}^{\text{an}} E_{\sigma,k+1}^{\text{an}} + E_{\sigma,k}^{\text{an}} E_{\sigma,k+1}^{\text{an}} + E_{\tau,k}^{\text{an}} E_{\tau,k+1}^{\text{an}} + E_{\tau,k}^{\text{an}} E_{\tau,k+1}^{\text{an}}) \\
+ \frac{U n^2}{4} \sum_{k=1}^L C_k^{(\sigma)} C_k^{(\tau)}. \tag{22}
\]

3. Tetrahedral Zamolodchikov algebra

In the above section we have shown the integrability of the 1-D \( SU(n) \) Hubbard model. It is natural to expect that the \( R \)-matrix (17) itself satisfy the Yang-Baxter equation (YBE):

\[
R_{31}^h(\lambda_3, \lambda_1) R_{32}^h(\lambda_3, \lambda_1) R_{12}^h(\lambda_1, \lambda_1) = R_{12}^h(\lambda_1, \lambda_2) R_{32}^h(\lambda_3, \lambda_2) R_{31}^h(\lambda_3, \lambda_1). \tag{23}
\]

In the \( SU(2) \) case, the YBE of the \( R \)-matrix was proved in ref. [31] by using the tetrahedral Zemolodchikov algebra (TZA) [34, 33]. In this section, we construct the TZA related to the \( SU(n) \) Hubbard model.

The TZA is defined by the following set of relations:

\[
L_{12}^a L_{32}^b L_{31}^c = \sum_{def} S_{def}^{abc} L_{31}^d L_{32}^e L_{12}^f. \tag{24}
\]
where $a, b, \ldots, f = 0, 1$ and $S_{abc}^{def}$ are some scalar coefficients.

We take $L^{0}_{jk}$ and $L^{1}_{jk}$ as follows:

\[
L^{0}_{0j} = R_{jk}(\lambda_k - \lambda_j), \\
L^{1}_{0j} = R_{jk}(\lambda_k + \lambda_j)C_k,
\]

where $R_{jk}(\lambda)$ is the $R$-matrix of the $SU(n)$ XX model as before. Then we could find the following relations which give the TZA (24):

\[
\begin{align*}
L^{0}_{12}L^{0}_{32}L^{0}_{31} &= L^{0}_{31}L^{0}_{32}L^{0}_{12}, \\
L^{1}_{12}L^{1}_{32}L^{0}_{31} &= L^{0}_{31}L^{1}_{32}L^{1}_{12}, \\
L^{1}_{12}L^{0}_{32}L^{1}_{31} &= L^{1}_{31}L^{0}_{32}L^{1}_{12}, \\
L^{1}_{12}L^{1}_{32}L^{1}_{31} &= S_{011}^{111}L^{1}_{31}L^{0}_{32}L^{0}_{12} + S_{010}^{111}L^{1}_{31}L^{0}_{32}L^{1}_{12} + S_{001}^{111}L^{0}_{31}L^{0}_{32}L^{1}_{12},
\end{align*}
\]

(26) (27) (28) (29) (30) (31)

where the coefficients $S_{abc}^{def}$ are given by

\[
S_{001}^{111} = \frac{\sin(\lambda_2 + \lambda_1) \cos(\lambda_2 + \lambda_3)}{\cos(\lambda_2 - \lambda_1) \sin(\lambda_2 - \lambda_3)}, \\
S_{100}^{111} = \frac{\sin(\lambda_1 + \lambda_3) \cos(\lambda_2 + \lambda_3)}{\cos(\lambda_1 - \lambda_3) \sin(\lambda_2 - \lambda_3)}, \\
S_{001}^{010} = \frac{\sin(\lambda_2 - \lambda_1) \sin(\lambda_1 + \lambda_3)}{\cos(\lambda_2 + \lambda_1) \cos(\lambda_1 + \lambda_3)}, \\
S_{010}^{010} = \frac{\sin(\lambda_2 - \lambda_1) \sin(\lambda_1 - \lambda_3)}{\cos(\lambda_2 + \lambda_1) \cos(\lambda_1 - \lambda_3)}, \\
S_{001}^{001} = \frac{\sin(\lambda_2 - \lambda_3) \cos(\lambda_2 + \lambda_3)}{\cos(\lambda_1 + \lambda_3) \sin(\lambda_2 - \lambda_3)}, \\
S_{010}^{001} = \frac{\sin(\lambda_2 - \lambda_3) \cos(\lambda_2 + \lambda_3)}{\cos(\lambda_1 + \lambda_3) \sin(\lambda_2 + \lambda_3)}, \\
S_{100}^{001} = \frac{\sin(\lambda_2 - \lambda_1) \sin(\lambda_1 - \lambda_3)}{\cos(\lambda_2 + \lambda_1) \cos(\lambda_1 - \lambda_3)}, \\
S_{111}^{001} = \frac{\sin(\lambda_2 - \lambda_1) \sin(\lambda_1 + \lambda_3)}{\cos(\lambda_2 + \lambda_1) \cos(\lambda_1 + \lambda_3)}.
\]

(32)

Eq. (28) and eq. (27) are equivalent to the YBE (3) and DYBE (8) respectively. In this sense, the TZA (24) can be regarded as a generalization of the YBE and DYBE.

It is important to notice that the products $L^{0}_{12}L^{0}_{32}L^{0}_{31}$ are not linearly independent as operators acting on $V_1 \otimes V_2 \otimes V_3$ and they satisfy the following relations:

\[
\begin{align*}
L^{0}_{12}L^{0}_{32}L^{0}_{31} &= x_0 L^{0}_{12}L^{1}_{32}L^{0}_{31} + y_0 L^{1}_{12}L^{0}_{32}L^{1}_{31} + z_0 L^{1}_{12}L^{1}_{32}L^{0}_{31}, \\
L^{1}_{12}L^{0}_{32}L^{1}_{31} &= x_1 L^{0}_{12}L^{0}_{32}L^{1}_{31} + y_1 L^{1}_{12}L^{0}_{32}L^{1}_{31} + z_1 L^{0}_{12}L^{1}_{32}L^{0}_{31},
\end{align*}
\]

(33) (34)

with

\[
\begin{align*}
x_0 &= -\frac{\cos(\lambda_1 - \lambda_3) \sin(\lambda_2 - \lambda_3)}{\cos(\lambda_1 + \lambda_3) \sin(\lambda_2 + \lambda_3)}, \\
y_0 &= \frac{\cos(\lambda_2 - \lambda_1) \cos(\lambda_1 - \lambda_3)}{\cos(\lambda_2 + \lambda_1) \cos(\lambda_1 + \lambda_3)}.
\end{align*}
\]

(35)
where $L$ of the 1-D then the conditions (39) are satisfied. This proves the Yang-Baxter equation for the (23) in the following form:

$$4 \quad \text{The Yang-Baxter equation for the } R\text{-matrix of the } SU(n) \text{ Hubbard model}$$

In this section we prove the Yang-Baxter equation for the $R$-matrix of the 1-D $SU(n)$ Hubbard model \((23)\). Taking into account the form of the $R$-matrix (17), we look for a solution of the YBE (23) in the following form:

$$R_{jk}^h(\lambda_j, \lambda_k) = R^{(\sigma)}_{jk}(\lambda_k - \lambda_j) + \alpha_{jk} R(\sigma)_{jk}(\lambda_k + \lambda_j) C^{(\sigma)}_{jk} R^{(\tau)}_{jk}(\lambda_k + \lambda_j) C^{(\tau)}_{jk}$$

$$L_{jk}^0(\sigma) L_{jk}^0(\tau) + \alpha_{jk} C_{jk}^{(\sigma)} C_{jk}^{(\tau)}$$

where $L_{jk}$ and $C_{jk}$ have been defined in (24). If $\alpha_{jk} = 0$, the $R$-matrix satisfies the YBE (23) in a trivial way. Now we look for a non-trivial solution. Substituting the expression (38) into the Yang-Baxter equation (23), by means of the tetrahedral Zamolodchikov algebra and relations (33) and (34), we could find that $\alpha_{jk}$ must satisfy the following conditions:

$$\alpha_{12} \sin 2(\lambda_1 + \lambda_2) + \alpha_{31} \sin 2(\lambda_1 + \lambda_3) = \alpha_{32} \sin 2(\lambda_2 + \lambda_3)$$

$$= \frac{1}{\alpha_{31}} \sin 2(\lambda_3 - \lambda_1) + \frac{1}{\alpha_{12}} \sin 2(\lambda_2 - \lambda_1).$$

If we take

$$\alpha_{jk} = \frac{\cos(\lambda_k - \lambda_j)}{\cos(\lambda_k + \lambda_j)} \tanh(h(\lambda_k) - h(\lambda_j)),$$

and impose the constrains

$$\frac{\sinh(h(\lambda_j))}{\sin(2\lambda_j)} = \frac{n^2 U}{4}, \quad (j = 1, 2, 3),$$

then the conditions (39) are satisfied. This proves the Yang-Baxter equaiton for the $R$-matrix of the 1-D $SU(n)$ Hubbard model.

Besides the YBE (24), the $R$-matrix (17) has the following properties:

$$R_{jk}^h(0, \lambda) = \frac{1}{\cosh(h(\lambda))} L_{jk}(\lambda),$$

$$R_{jk}^h(\lambda_0, \lambda_0) = \rho_{jk},$$

$$R_{jk}^h(\lambda_j, \lambda_k) R_{kj}^h(\lambda_k, \lambda_j) = \rho(\lambda_j, \lambda_k) \rho_{jk},$$
where
\[ \rho(\lambda_j, \lambda_k) = \cos^2(\lambda_k - \lambda_j) \left\{ \cos^2(\lambda_k - \lambda_j) - \tanh^2(h(\lambda_k) - h(\lambda_j)) \right\}, \] (45)
and the permutation operator is defined as
\[ \mathcal{P}_{jk} = \mathcal{P}^{(\sigma)}_{jk} \mathcal{P}^{(\tau)}_{jk}. \] (46)

The Yang-Baxter equation (23) implies a more general inhomogeneous model as:
\[ T_a(\lambda, \{\lambda_j\}) = R^h_{La}(\lambda, \lambda_N)R^h_{L_{-1}a}(\lambda, \lambda_{L-1}) \cdots R^h_{1a}(\lambda, \lambda_1), \] (47)
where \( \lambda_j \), \((j = 1, 2, \cdots, L)\) are the inhomogeneous parameters obeying the constraints
\[ \frac{\sinh(2h(\lambda_j))}{\sin(2\lambda_j)} = \frac{n^2U}{4}, \quad (j = 1, 2, \cdots, L). \] (48)

From the Yang-Baxter equation (23), we can obtain the global Yang-Baxter relation:
\[ T_1(\lambda, \{\lambda_j\})T_2(\mu, \{\lambda_j\})R^h_{12}(\lambda, \mu) = R^h_{12}(\lambda, \mu)T_2(\mu, \{\lambda_j\})T_1(\lambda, \{\lambda_j\}), \] (49)
which leads to the commutativity
\[ [\tau(\lambda, \{\lambda_j\}), \tau(\mu, \{\lambda_j\})] = 0, \] (50)
where \( \tau(\lambda, \{\lambda_j\}) \) is the transfer matrix of the model
\[ \tau(\lambda, \{\lambda_j\}) = tr_\sigma T_a(\lambda, \{\lambda_j\}). \] (51)

The corresponding Hamiltonian is defined as the logarithmic derivative of the transfer matrix under all inhomogeneous parameters \( \lambda_j = \lambda_0, \(j = 1, 2, \cdots, L)\):
\[ H_{\lambda_0} = \frac{d}{d\lambda} \ln \tau(\lambda, \{\lambda_j = \lambda_0\})|_{\lambda = \lambda_0} = \tau^{-1}(\lambda_0, \{\lambda_j = \lambda_0\}) \frac{d}{d\lambda} \tau(\lambda, \{\lambda_j = \lambda_0\}) \]
\[ = \sum_{j=1}^{L} \sum_{\alpha<n} (E^{(\alpha)}_{\sigma j} E^{(\alpha)}_{\sigma j+1} + E^{(\alpha)}_{\tau j} E^{(\alpha)}_{\tau j+1}) \]
\[ + \frac{n^2U}{4 \cosh(2h(\lambda_0))} \sum_{j=1}^{L} B_{jj+1}^{(\sigma)} B_{jj+1}^{(\tau)}, \] (52)
where
\[ B_{jj+1} = \cos(2\lambda_0) (E^{(\alpha)}_{jj+1} + \sum_{\alpha, \beta<n} E^{(\alpha)}_{j} E^{(\beta)}_{j+1}) + \sin(2\lambda_0) \sum_{\alpha<n} (E^{(\alpha)}_{j} E^{(\alpha)}_{j+1} + E^{(\alpha)}_{j} E^{(\alpha)}_{j+1}) \]
\[ + \sum_{\alpha<n} (E^{(\alpha)}_{j} E^{(\alpha)}_{j+1} + E^{(\alpha)}_{j} E^{(\alpha)}_{j+1}). \] (53)

The arbitrariness of the parameter \( \lambda_0 \) comes from the non-additive property of the spectral parameters. If we take \( \lambda_0 = 0 \), this new Hamiltonian reduces to (4).

Thus, we have obtained a new 1-D \( SU(n) \) Hubbard by the Yang-Baxter equation of the \( R \)-matrix (23).
5 Conclusions

In this paper we have proved the $R$-matrix of the 1-D $SU(n)$ Hubbard model satisfying the Yang-Baxter equation. We notice that the tetrahedral Zamolodchikov algebra play an essential role in the proof.

In most lattice systems, the existence of $R$-matrix ensures the integrability and the $R$-matrix is isomorphic to the $L$-operator. Thus, the Yang-Baxter equation is a consequence of the Yang-Baxter relation $R_{12}L_1L_2 = L_2L_1R_{12}$. But, for Hubbard model, the situation is quite different. The $R$-matrix can not be obtained from $L$-operator, even if we limit to $SU(2)$ Hubbard model $^{15,16}$. The $R$-matrix satisfying the Yang-Baxter equation together with $L$-operator constitute the complete proof of the integrability.

For $SU(n)$ Hubbard model, the $R$-matrix does not isomorphic to the $L$-operator. This provides a method to construct a new kind of integrable system by considering the $R$-matrix as a $L$-operator (fundamental representation of same algebra). The general representation can be obtained by fusing the multi fundamental rep. ($R$-matrix). Therefore, one can get the full representation of the algebra in principle.

In the present paper we have derived out a new Hamiltonian $^{(52)}$ from the $R$-matrix. The last term introduces a new kind of interaction. This Hamiltonian is quite different from the original one $^{(1)}$. It also renders a question how to find the eigenvalue of this Hamiltonian $^{(52)}$. We will consider it late.

In the derivation of eq. (52), we have assumed all the parameters $\lambda_j$ are same $\lambda_0$. This is not necessary. The different choice will give out different Hamiltonian. On the other hand, one can consider the $R$-matrix as a Boltzmann weight in statistical mechanics. This provides an inhomegeneous lattice statistical medel. The partition function could be derived out in a similar way.

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