Testing Theories of Gravity with a Spherical Gravitational Wave Detector

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ABSTRACT

We consider the possibility of discriminating different theories of gravity using a recently proposed gravitational wave detector of spherical shape. We argue that the spin content of different theories can be extracted relating the measurements of the excited spheroidal vibrational eigenmodes to the Newman-Penrose parameters. The sphere toroidal modes cannot be excited by any metric GW and can be thus used as a veto.
The efforts aimed at the detection of gravitational waves (GW) started more than a quarter of century ago and have been, up to now, unsuccessful \[1, 2\]. Resonant bars have proved their reliability, being capable of continuous data gathering for long periods of time \[3, 4\]. Their energy sensitivity has improved of more than four orders of magnitude since Weber’s pioneering experiment. But a further improvement is still necessary to achieve successful detection. While further developments of bar detectors are under way, two new generations of earth based experiments have been proposed. While detectors based on large laser interferometers are already under construction \[5\], resonant detectors of spherical shape are under study \[2\]. In the present letter we report on a study about the physical information that can be obtained thanks to the spherical symmetry of the latter detectors. More in detail, we want to show that the measurements of the sphere vibrations can provide specific information on the field content of the gravitational theory predicting the observed features of the waves.

We would like to remind the reader of the very special position of Einstein’s general relativity (GR) among the possible gravitational theories. Theories of gravitation, in fact, can be divided into two families: metric and non-metric theories \[6\]. The former can be defined to be all theories obeying the following three postulates:

- spacetime is endowed with a metric;
- the world lines of test particles are geodesic of the above mentioned metric;
- in local free-falling frames, the non-gravitational laws of physics are those of special relativity.

It is an obvious consequence of these postulates that a metric theory obeys the principle of equivalence. More succintly a theory is said to be metric if
the action of gravitation on the matter sector is due exclusively to the metric tensor. GR is the most famous example of a metric theory. Kaluza-Klein type theories, also belong to this class along with the Brans-Dicke theory. Different representatives of this class differ for their equations of motion which in turn can be deduced from a lagrangian principle. Up to now non-Einsteinian and non-metric theories have been considered a curiosity by the majority of physicists since there seems to be no compelling experimental or theoretical reason to justify their introduction. This point should perhaps be reconsidered if we think of the unique role of string theories in this context.

String theory seems to lead to a consistent framework in which to quantize gravity and the other fundamental interactions. Gravity emerges from string theory in different forms depending on the chosen vacuum configuration. Allowed string vacua are solutions of the equations of motion resulting from conformal invariance of the sigma model which governs the string propagation. The string equations of motion can also be obtained \textit{a posteriori} via a variational principle. Indeed, the elementary excitations of the string with zero mass (light sector) can be described by an effective lagrangian which is obtained after integrating out the massive (heavy sector) modes of the theory in a procedure à la Wilson. A great variety of string solutions has appeared in literature: plane waves, solitons, instantons, rotating and non-rotating black holes. More recently a cosmological solution has appeared that seems to lead to a potentially detectable quantity of gravitational radiation in a wide spectrum of frequencies. These solutions can be both metric and non-metric but are certainly non-Einsteinian since all of these vacua contain a massless scalar called dilaton. When perturbations of these vacua are considered all fields fluctuate and the resulting theory is non-metric. But, as in string theory the expectation value of the dilaton plays the role of a coupling constant, it is generally believed that some non-perturbative effect
(the most serious candidate at the moment is supersymmetry breaking) gives a mass to the dilaton. If this is true, the dilaton interactions with the other fields may well be negligible \[10\] and the theory may revert to be metric.

These are our motivations for testing theories of gravity using the measurable toroidal and spheroidal vibrational eigenmodes of a

sphere. The signature of a non-metric theory could obviously also be detected in experiments on the equivalence principle.

Before discussing the interaction with an external GW field, let us consider the basic equations governing the free vibrations of a perfectly homogeneous, isotropic sphere of radius \( R \), made of a material having density \( \rho \) and Lamé coefficients \( \lambda \) and \( \mu \) \[11\].

Following the notation of \[12\], let \( x_i, i = 1, 2, 3 \) be the equilibrium position of the element of the elastic sphere and \( x'_i \) be the deformed position then \( u_i = x'_i - x_i \) is the displacement vector. Such vector is assumed small, so that the linear theory of elasticity is applicable. The strain tensor is defined as \( u_{ij} = (1/2)(u_{i,j} + u_{j,i}) \) and is related to the stress tensor by \( \sigma_{ij} = \delta_{ij}\lambda u_{ll} + 2\mu u_{ij} \). The equations of motion of the free vibrating sphere are thus

\[
\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x^j}(\delta_{ij}\lambda u_{ll} + 2\mu u_{ij})
\]

(1)

with the boundary condition:

\[
n_j \sigma_{ij} = 0
\]

(2)

at \( r = R \) where \( n_i \equiv x_i/r \) is the unit normal. These conditions simply state that the surface of the sphere is free to vibrate. The displacement \( u_i \) is a time-dependent vector, whose time dependence can be factorised as \( u_i(\vec{x}, t) = u_i(\vec{x}) \exp(i\omega t) \), where \( \omega \) is the frequency. The equations of motion then become:

\[
\mu \nabla^2 u_i + (\lambda + \mu)\nabla_i(\nabla_j u_j) = -\omega^2 \rho u_i
\]

(3)
Their solutions can be expressed as a sum of a longitudinal and two transverse vectors:

\[ \vec{u}(\vec{x}) = C_0 \vec{∇}\phi(\vec{x}) + C_1 \vec{L}\chi(\vec{x}) + C_2 \vec{∇} \times \vec{L}\chi(\vec{x}) \]  

(4)

where \( C_0, C_1, C_2 \) are constants and \( \vec{L} \equiv \vec{x} \times \vec{∇} \) is the angular momentum operator. Regularity at \( r = 0 \) restricts the scalar functions \( \phi \) and \( \chi \) to be expressed as \( \phi(r, \theta, \varphi) \equiv j_l(qr)Y_{lm}(\theta, \varphi) \) and \( \chi(r, \theta, \varphi) \equiv j_l(kr)Y_{lm}(\theta, \varphi) \). \( Y_{lm}(\theta, \varphi) \) are the spherical harmonics and \( j_l \) the spherical Bessel functions:

\[ j_l(x) = \left(\frac{1}{x} \frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right) \]  

(5)

\( q^2 \equiv \rho\omega^2/(\lambda + 2\mu) \) and \( k^2 \equiv \rho\omega^2/\mu \) are the longitudinal and transverse wave vectors respectively.

Imposing the boundary conditions (2) at \( r = R \) yields two families of solutions:

- **Toroidal modes**: these are obtained by setting \( C_0 = C_2 = 0 \), and \( C_1 \neq 0 \). The eigenfunctions have the form:

\[ \vec{u}^T_{nlm}(r, \theta, \varphi) = T_{nl}(r)\vec{L}Y_{lm}(\theta, \varphi) \]  

(6)

with \( T_{nl}(r) \) proportional to \( j_l(k_nr) \). The eigenfrequencies are determined by the boundary conditions (4) which read

\[ f_1(kR) = 0 \]  

(7)

where

\[ f_1(z) \equiv \frac{d}{dz}\left[\frac{j_l(z)}{z}\right]. \]  

(8)

- **Spheroidal modes**: these are obtained by setting \( C_1 = 0 \), \( C_0 \neq 0 \) and \( C_2 \neq 0 \). The eigenfunctions can be conveniently rewritten as

\[ \vec{u}^S_{nlm}(\vec{x}) = A_{nl}(r)Y_{lm}(\theta, \varphi)\vec{n} - B_{nl}(r)\vec{n} \times \vec{L}Y_{lm}(\theta, \varphi) \]  

(9)
where $A_{nl}(r)$ and $B_{nl}(r)$ are dimensionless radial eigenfunctions [12], which can be expressed in terms of the spherical Bessel functions and their derivatives. The eigenfrequencies are determined by the boundary conditions (2) which read

$$\det \left( \begin{array}{cc} f_2(qR) - \frac{k}{2} q^2 R^2 f_0(qR) & l(l+1)f_1(kR) \\ f_1(qR) & \frac{1}{2} f_2(kR) + \frac{l(l+1)}{2} - 1 f_0(kR) \end{array} \right) = 0$$

(10)

where

$$f_0(z) \equiv \frac{j_l(z)}{z^2} \quad f_2(z) \equiv \frac{d^2}{dz^2} j_l(z)$$

(11)

The eigenfrequencies can be determined numerically for both toroidal and spheroidal vibrations. Each mode of order $l$ is $(2l+1)$-fold degenerate. In the table below we show the value of the $(kR)$ roots for the lowest toroidal and spheroidal modes of vibration.

| $l$ | $n$ | $(kR)_{\text{toroidal}}$ | $(kR)_{\text{spheroidal}}$ | $l$ | $n$ | $(kR)_{\text{toroidal}}$ | $(kR)_{\text{spheroidal}}$ |
|-----|-----|--------------------------|-----------------------------|-----|-----|--------------------------|-----------------------------|
| 0   | 1   | 5.4322                   | 3.9489                      | 3   | 1   | 3.8647                   | 3.9489                      |
|     | 2   | 12.138                   | 6.9695                      | 2   | 3   | 8.4449                   | 6.9695                      |
|     | 3   | 18.492                   | 9.9720                      | 3   | 3   | 11.882                   | 9.9720                      |
|     | 4   | 24.785                   | 12.900                      | 4   | 4   | 15.175                   | 12.900                      |
| 1   | 1   | 5.7635                   | 3.5895                      | 4   | 1   | 5.0946                   | 5.0662                      |
|     | 2   | 9.0950                   | 7.2306                      | 2   | 4   | 9.7125                   | 8.2994                      |
|     | 3   | 12.323                   | 8.4906                      | 3   | 3   | 13.211                   | 11.324                      |
|     | 4   | 15.515                   | 10.728                      | 4   | 4   | 16.544                   | 14.467                      |
| 2   | 1   | 2.5011                   | 2.6497                      | 5   | 1   | 6.2658                   | 6.1118                      |
|     | 2   | 7.1360                   | 5.0878                      | 2   | 2   | 10.951                   | 9.8529                      |
|     | 3   | 10.515                   | 8.6168                      | 3   | 3   | 14.511                   | 12.686                      |
|     | 4   | 13.772                   | 10.917                      | 4   | 4   | 17.886                   | 15.879                      |
The eigenfrequency values can be obtained from:

\[ \omega_{nl} = \sqrt{\frac{\mu(kR)_{nl}}{\rho R}} \]  

(12)

The detector is assumed to be non-relativistic (with sound velocity \( v_s \ll c \) and radius \( R \ll \lambda \) the GW wavelength) and endowed with a high quality factor \( (Q_{nl} = \omega_{nl}\tau_{nl} \gg 1, \text{where} \tau_{nl} \text{is the decay time of the mode} \, nl) \). The displacement \( \vec{u} \) of a point in the detector can be decomposed in normal modes as:

\[ \vec{u}(\vec{x}, t) = \sum_{N} A_N(t) \vec{u}_N(\vec{x}) \]  

(13)

where \( N \) collectively denotes the set of quantum numbers identifying the mode. The basic equation governing the response of the detector is

\[ \ddot{A}_N(t) + \tau_{-1}^{-1} \dot{A}_N(t) + \omega_N^2 A_N(t) = f_N(t) \]  

(14)

We assume that the gravitational interaction obeys the principle of equivalence which has been experimentally supported to high accuracy. In terms of the so-called electric components of the Riemann tensor \( E_{ij} \equiv R_{0i0j} \), the driving force \( f_N(t) \) is then given by

\[ f_N(t) = -M^{-1} E_{ij}(t) \int u_N^*(\vec{x})x^j \rho d^3x \]  

(15)

where \( M \) is the sphere mass and we consider the density \( \rho \) as a constant. In any metric theory of gravity \( E_{ij} \) is a \( 3 \times 3 \) symmetric tensor, which depends on time, but not on spatial components.

Let us now investigate which sphere eigenmodes can be excited by a metric GW, \textit{i.e.} which sets of quantum numbers \( N \) give a non-zero driving force.

a) \textit{Toroidal} modes
The displacement vector can be expressed as in eq. (6). Up to a normalisation constant $C$, the driving force is

$$f_N^{(T)}(t) = -e^{-i\omega N t} \frac{3C}{4\pi R^3} \int_0^R drr^3 j_I(k_{nl}^{(T)} r) \int_0^\pi d\theta \int_0^{2\pi} d\phi \left\{ \frac{E_{yy} - E_{xx}}{2} \left( \sin \theta \sin 2\phi \frac{\partial Y_{lm}^*}{\partial \theta} + \cos \theta \cos 2\phi \frac{\partial Y_{lm}^*}{\partial \phi} \right) + E_{xy} \left( \sin \theta \cos 2\phi \frac{\partial Y_{lm}^*}{\partial \theta} - \cos \theta \sin 2\phi \frac{\partial Y_{lm}^*}{\partial \phi} \right) + E_{xz} \left[ -\sin \phi \cos \theta \frac{\partial Y_{lm}^*}{\partial \theta} + (\sin \theta \cos \phi - \frac{\cos^2 \theta}{\sin \theta} \cos \phi) \frac{\partial Y_{lm}^*}{\partial \phi} \right] + E_{yz} \left[ \cos \phi \cos \theta \frac{\partial Y_{lm}^*}{\partial \theta} + (\sin \theta \sin \phi - \frac{\cos^2 \theta}{\sin \theta} \sin \phi) \frac{\partial Y_{lm}^*}{\partial \phi} \right] + \left( E_{zz} - \frac{E_{xx} + E_{yy}}{2} \right) \cos \theta \frac{\partial Y_{lm}^*}{\partial \phi} \right\} \right)$$

(16)

Using the equations

$$\frac{\partial Y_{lm}^*}{\partial \theta} = (-)^m \left[ \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} \right] \frac{1}{2} P_l^m(\cos \theta) e^{-im\phi} \right)$$

(17)

and

$$\frac{\partial Y_{lm}^*}{\partial \phi} = -im(-)^m \left[ \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} \right] \frac{1}{2} P_l^m(\cos \theta) e^{-im\phi} \right)$$

(18)

the integration over $\phi$ can be performed. Eq. (16) then contains integrals over $\theta$ of the form:

$$\int_0^\pi \left[ (\sin^2 \theta - \cos^2 \theta) P_i^{\pm 1}(\cos \theta) - \sin \theta \cos \theta \frac{\partial P_i^{\pm 1}(\cos \theta)}{\partial \theta} \right] d\theta \right)$$

(19)

and

$$\int_0^\pi \left[ 2 \sin \theta \cos \theta P_i^{\pm 2}(\cos \theta) + \sin^2 \theta \frac{\partial P_i^{\pm 2}(\cos \theta)}{\partial \theta} \right] d\theta \right)$$

(20)

After integration by parts, the derivative terms in eqs. (19) and (20) exactly cancel the non-derivative ones. The remaining boundary terms vanish too, thanks to the periodicity of the trigonometric functions and to the regularity of the associated Legendre polynomials. The vanishing of the above integrals
has a profound physical consequence. It means that in any metric theory of gravity the toroidal modes of the sphere cannot be excited by GW and can thus be used as a veto in the detection. In this respect we disagree with the results of [14] that seems to find a zero result only for the even $l$ case.

b) Spheroidal modes

The forcing term is given by:

$$f^{(S)}_N(t) = -M^{-1} E_{ij}(t) \int x^j \left( \frac{x^i}{r} A_N(r) Y_{lm}^{(\theta, \varphi)} - B_N(r) \epsilon^{ijk} x_j L_k Y_{lm}^{(\theta, \varphi)} \right) \rho d^3 x$$

(21)

One is thus lead to compute integrals of the following types

$$\int x^j x^l Y_{lm}^{(\theta, \varphi)} d^3 x$$

(22)

and

$$\int x^j x^l L_k Y_{lm}^{(\theta, \varphi)} d^3 x$$

(23)

Since the product $x^j x^l$ can be expressed in terms of the spherical harmonics with $l = 0, 2$ and the angular momentum operator does not change the value of $l$, one immediately concludes that in any metric theory of gravity only the $l = 0, 2$ spheroidal modes of the sphere can be excited. At the lowest level there are a total of five plus one independent spheroidal modes that can be used for GW detection and study.

These results agree with the conclusion of [12] and generalize the results of [15, 16].

From the analysis of the spheroidal modes active for metric GW, we now want to infer the field content of the theory. For this purpose it is convenient to express the Riemann tensor in a null (Newman-Penrose) tetrad basis [6].

To lowest non-trivial order in the perturbation the six independent "electric" components of the Riemann tensor may be expressed in terms of the
Newmann-Penrose (NP) parameters as

\[ E_{ij} = \begin{pmatrix}
-\text{Re}\Psi_4 - \Phi_{22} & \text{Im}\Psi_4 & -2\sqrt{2}\text{Re}\Psi_3 \\
\text{Im}\Psi_4 & \text{Re}\Psi_4 - \Phi_{22} & 2\sqrt{2}\text{Im}\Psi_3 \\
-2\sqrt{2}\text{Re}\Psi_3 & 2\sqrt{2}\text{Im}\Psi_3 & -6\Psi_2
\end{pmatrix} \tag{24} \]

The NP parameters allow the identification of the spin content of the metric theory responsible for the generation of the wave [6]. The classification can be summarized in order of increasing complexity as follows:

- **General Relativity** (spin 2): \( \Psi_4 \neq 0 \) while \( \Psi_2 = \Psi_3 = \Phi_{22} = 0 \).

- **Tensor-scalar theories** (spin 2 and 0): \( \Psi_4 \neq 0 \), \( \Psi_3 = 0 \), \( \Psi_2 \neq 0 \) and/or \( \Phi_{22} \neq 0 \) (e.g. Brans-Dicke theory with \( \Psi_4 \neq 0 \), \( \Psi_2 = 0 \), \( \Psi_3 = 0 \) and \( \Phi_{22} \neq 0 \)).

- **Tensor-vector theories** (spin 2 and 1): \( \Psi_4 \neq 0 \), \( \Psi_3 \neq 0 \), \( \Phi_{22} = \Psi_2 = 0 \).

- **Most General Metric Theory** (spin 2, 1 and 0): \( \Psi_4 \neq 0 \), \( \Psi_2 \neq 0 \), \( \Psi_3 \neq 0 \) and \( \Phi_{22} \neq 0 \) (e.g. Kaluza-Klein theories with \( \Psi_4 \neq 0 \), \( \Psi_3 \neq 0 \), \( \Phi_{22} \neq 0 \) while \( \Psi_2 = 0 \)).

In eq. (24), we have assumed that the wave comes from a localized source with wave vector \( \vec{k} \) parallel to the \( z \) axis of the detector frame. In this case the NP parameters (and thus the wave polarisation states) can be uniquely determined by monitoring the six lowest spheroidal modes. If the direction of the incoming wave is not known two more unknowns appear in the problem, i.e. the two angles of rotation of the detector frame needed to align \( \vec{k} \) along the \( z \) direction. In order to dispose of this problem one can envisage the possibility of combining the pieces of information from an array of detectors [17]. We restrict our attention to the simplest case in which the source direction is known.
The NP parameters of the incident wave can be easily obtained from the following argument. It is well known that any \( 3 \times 3 \) symmetric tensor such as (24) can be decomposed in the following way

\[
E_{ij}(t) = \sum_{l,m} c_{l,m}(t) S_{ij}^{(l,m)}
\]

where \( S_{ij}^{(0,0)} \equiv \delta_{ij}/\sqrt{4\pi} \) (with \( \delta_{ij} \) the Kronecker symbol) and \( S_{ij}^{(2,m)} \) \((m = -2, \ldots, 2)\) are five linearly independent symmetric and traceless matrices. The following explicit representation is particularly suited for exposing the spin content of the GW [12]

\[
\begin{align*}
S_{ij}^{(0,0)} &= \left( \frac{1}{4\pi} \right)^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} & S_{ij}^{(2,0)} &= \left( \frac{5}{16\pi} \right)^{\frac{1}{2}} \begin{pmatrix} -1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2 \end{pmatrix} \\
S_{ij}^{(2,2)} &= \left( \frac{15}{32\pi} \right)^{\frac{1}{2}} \begin{pmatrix} 1 & i & 0 \\
i & -1 & 0 \\
0 & 0 & 0 \end{pmatrix} & S_{ij}^{(2,-2)} &= \left( \frac{15}{32\pi} \right)^{\frac{1}{2}} \begin{pmatrix} 1 & -i & 0 \\
-i & -1 & 0 \\
0 & 0 & 0 \end{pmatrix} \\
S_{ij}^{(2,1)} &= -\left( \frac{15}{32\pi} \right)^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & i \\
i & 1 & 0 \end{pmatrix} & S_{ij}^{(2,-1)} &= \left( \frac{15}{32\pi} \right)^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & -i \\
i & -1 & 0 \end{pmatrix}
\end{align*}
\]

The matrices \( S^k \) are trace-orthogonal and are connected to the spherical harmonics by

\[
\begin{align*}
S_{ij}^{(0,0)} n_i n_j &= Y_{0,0}, & S_{ij}^{(2,0)} n_i n_j &= Y_{2,0} \\
S_{ij}^{(2,2)} n_i n_j &= Y_{2,2}, & S_{ij}^{(2,-2)} n_i n_j &= Y_{2,-2} \\
S_{ij}^{(2,1)} n_i n_j &= Y_{2,1}, & S_{ij}^{(2,-1)} n_i n_j &= Y_{2,-1}
\end{align*}
\]

The vector \( n_i \) in eqs. (27) has been defined after eq. (2).

Taking the scalar product we find

\[
c_{0,0}(t) = \frac{4\pi}{3} S_{ij}^{0,0} E_{ij}(t)
\]

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\[ c_{2,m}(t) = \frac{8\pi}{15} S_{ij}^{2,m} E_{ij}(t) \] (28)

For the NP parameters we find

\[ \Phi_{22} = \sqrt{\frac{5}{16\pi}} c_{2,0}(t) - \sqrt{\frac{1}{4\pi}} c_{0,0}(t) \quad \Psi_2 = -\frac{1}{12} \sqrt{\frac{5}{\pi}} c_{2,0}(t) - \frac{1}{12} \sqrt{\frac{1}{\pi}} c_{0,0}(t) \]

\[ \text{Re}\Psi_4 = -\frac{1}{16} \sqrt{\frac{15}{32\pi}} [c_{2,2} + c_{2,-2}] \quad \text{Im}\Psi_4 = -i \frac{1}{16} \sqrt{\frac{15}{32\pi}} [c_{2,2} + c_{2,-2}] \]

\[ \text{Re}\Psi_3 = \frac{1}{16} \sqrt{\frac{15}{\pi}} [c_{2,1} - c_{2,-1}] \quad \text{Im}\Psi_3 = \frac{i}{16} \sqrt{\frac{15}{\pi}} [c_{2,1} + c_{2,-1}] \] (29)

Eqs. (29) relate the quantities \( c_{i,m} \) with the GW polarization states, described by the NP parameters. A pictorial representation of the six polarization states connected with the NP parameters is given in Figure 1 where the wave vector is assumed parallel to the \( z \) axis as in (24). Starting from the top left corner of the figure we draw the polarization states \( \Psi_2, \Phi_{22}, \text{Re}\Psi_4, \text{Im}\Psi_4, \text{Re}\Psi_3, \text{Im}\Psi_3 \).

\[ \text{Figure 1} \]
Eq. (29) can be put in correspondence with the output of experimental measurements if the $c_{l,m}$ are substituted with their Fourier components at the quadrupole and monopole resonant frequencies which, for the sake of simplicity, we collectively denote by $\omega_0$. The $c_{l,m}(\omega_0)$ can be determined in the following way: once the Fourier amplitudes $A_N(\omega_0)$ are measured, by Fourier transforming (14) and (15) we get the Riemann amplitudes $E_{ij}(\omega_0)$ which, using (28), yield the desired result.

In order to determine the $A_N(\omega_0)$ amplitudes from a given GW signal two conditions must be fulfilled:

- the vibrational states of the five-fold degenerate quadrupole and monopole modes must be determined. The quadrupole modes can be studied by properly combining the outputs of a set of at least five motion sensors placed in independent positions on the sphere surface. Explicit formulas for practical and elegant configurations of the motion sensors have been reported by various authors [14, 18, 19]. The vibrational state of the monopole mode is provided directly by the output of any of the above mentioned motion sensors. If resonant motion sensors are used, since the quadrupole and monopole states resonate at different frequencies, a sixth sensor is needed.

- The spectrum of the GW signal must be sufficiently broadband to overlap with the antenna quadrupole and monopole frequencies.

From table 1 we see that the second order quadrupole spheroidal mode is close to the lowest order monopole and also to the toroidal mode $n = 1$, $l = 4$. Since it has been demonstrated [20] that the second order quadrupole spheroidal mode has high cross section for general relativistic GW (only a factor 2.6 lower than the cross section of the first order quadrupole spheroidal mode), the full analysis of the GW signal for the most general metric GW,
including the toroidal mode veto, can be hopefully performed in a small frequency range. The above reported analysis can then be applied to a large class of GW sources, including gravitational collapses, inspiralling and coalescing binary systems and stochastic sources.

The sensitivity of a multiton spherical detector, making use of ultracryogenic and superconducting techniques for noise reduction could be such to detect these events with a reasonable rate [14].

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