Some aspects of braided geometry:  

differential calculus, tangent space, gauge theory  

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March 30, 2022  

Abstract  
A new approach is suggested to quantum differential calculus on certain quantum varieties. It consists in replacing quantum de Rham complexes with differentials satisfying Leibniz rule by those which are in a sense close to Koszul complexes from [G1]. We also introduce the tangent space on a quantum hyperboloid equipped with an action on the quantum function space and define the notions of quantum (pseudo)metric and quantum connection (partially defined) on it. All objects are considered from the viewpoint of flatness of quantum deformations. A problem of constructing a flatly deformed quantum gauge theory is discussed as well.  

1 Introduction  
In this paper we consider some problems which can be gathered together under a general name "braided (quantum, twisted or q-deformed) geometry". This type of geometry has had a real gold rush since the creation of the quantum group (QG) theory. This phenomenon is motivated by a common desire to generalize methods of ordinary geometry for the needs of mathematical physics since, in accordance with a widespread opinion, the future of this discipline is connected with models which are covariant w.r.t. special Hopf algebras rather than to ordinary transformation groups.  

Nevertheless, it turned out that not all objects of the ordinary geometry have their consistent q-analogues. For example, all attempts initiated by L.Woronowicz [W1], [W2] to develop a bicovariant differential calculus on quantum function space Fun(SL\(_q\)(n)) with two properties: flatness of deformation of the differential algebra\(^1\) and Leibniz rule
for the corresponding differential have failed.

Moreover, such a differential calculus does not exist. It was shown in [AAM] by considering the corresponding quasiclassical object, namely the graded Poisson-Lie structure, which is an extension of the Sklyanin-Drinfeld bracket to the differential algebra (cf. also the last section of [Ar]).

The problem is that a consistent $q$-deformation of the differential calculus which is well defined on Lie group $GL(n)$ (more precisely, on the corresponding matrix algebra $\text{Mat}(n)$), cf. [I], is not compatible with the constrains resulting from the equation $\det_q = 1$ where $\det_q$ is the quantum determinant. However, some $q$-deformed differential algebra equipped with a differential without Leibniz rule exists in the $SL(n)$ case, cf. [Ar], [FP1]. The authors of [FP1] recall a claim of L.Faddeev that the Leibniz rule is not reasonable in the quantum case.

One of the main purposes of this paper is to suggest a regular way to construct de Rham type complexes without any Leibniz rule. The essence of such a complex is close to that of Koszul complex (of the first kind) introduced in [G1] (we recall the construction of such a complex in section 2). In order to introduce a differential we fix a base in a $q$-deformed differential algebra and define it only in this base. This saves us from checking the fact that the differential respects the relations which define the algebra in question. Moreover, we realize in the classical case a spectral analysis of de Rham complex, i.e. we study the behavior of the classical differential on irreducible components of the initial complex and define a quantum differential with similar properties but in the $q$-deformed category. This approach is realizable when the spectral structure of the complex in question is simple enough (it can be also applied to a non-quasiclassical case).

We apply this approach to a quantum hyperboloid. By construction, its cohomology is just the same as in the classical case. (In general, the following conjecture seems to be very plausible: once a quantum de Rham complex is constructed in a proper way it has for a generic $q$ the same cohomology as its classical counterpart, cf. for example [FP2], [HS] for an illustration of this conjecture.) The corresponding construction is described in section 3.

In this connection we also discuss the following problem: what is a proper definition of the tangent space on the quantum hyperboloid (in other words, what is the phase space corresponding to the quantum hyperboloid considered as a configuration space)?

We introduce such a tangent space $T(H_q)$ (which is treated as $A$-module where $A$ is the quantum function space in question) and equip it with an action

$$T(H_q) \otimes A \rightarrow A$$

converting elements of the tangent space into "braided vector fields". Let us remark that our construction of braided vector fields is realized without (once more!) any Leibniz rule (cf. [A]).

We also introduce (in section 4) the notions of a (pseudo)metric and a connection (partially defined) on the tangent space on the quantum hyperboloid. In all our constructions we impose only two properties on any $q$-deformed object in question: $U_q(sl(2))$-covariance and flatness of the deformation.

In section 5 we consider the problem of constructing a quantum gauge theory from this viewpoint. In spite of numerous attempts to generalize the classical gauge theory
from the above viewpoint up to now this has not been satisfactory. We are rather sceptic about a possibility to introduce a consistent q-deformation of the classical gauge theory. We discuss this in section 5.

Throughout the whole paper the basic field $k$ is $\mathbb{R}$ or $\mathbb{C}$ and the parameter $q \in k$ is assumed to be generic.

### 2 De Rham and Koszul complexes: comparative description

First, let us consider some complexes related to the QG $U_q(sl(n))$. The most popular complexes of such a type are de Rham complexes connected with the first fundamental modules of the QG $U_q(sl(n))$ [WZ] and those defined on the q-deformed matrix algebra $\text{Mat}(n)$ [T]. Whereas the former one is one-sided $U_q(sl(n))$-covariant, the latter one is bicovariant. (Such complexes exist for any Hecke symmetry, see footnote 1.)

A de Rham complex related to the first fundamental $U_q(SO(n))$-module was constructed in [CSW].

However, all the above complexes are, in a sense, objects of quantum (braided or q-) linear algebra than of quantum geometry. Quantum geometry deals rather with quantum varieties different from vector spaces. A typical example of such a variety is $SL_q(n)$ defined by the equation $\det_q = 1$ (by an abuse of the language we speak about a variety although in fact we deal with the corresponding ”quantum function space”). As we said above, the quantum differential calculus well defined on the vector space $\text{Mat}(n)$ cannot be restricted to the variety in question if we want it to be a flat deformation of its classical counterpart and its differential to obey the Leibniz rule. We refer the reader to the survey [I] where this problem is discussed.

Another interesting class of varieties connected with the QG $U_q(\mathfrak{g})$ are quantum homogeneous spaces which are one-sided $U_q(\mathfrak{g})$-modules (the products in the corresponding algebras are assumed to be $U_q(\mathfrak{g})$-covariant in the following sense

$$Z(a \cdot b) = Z_{(1)} a \cdot Z_{(2)} b, \ Z \in U_q(\mathfrak{g}), \ Z_{(1)} \otimes Z_{(2)} = \Delta(Z).$$

A quantum homogeneous space is usually introduced via a couple of QG in the spirit of a homogeneous space $G/H$. However, it is desirable to have its more explicit description by some system of equations.

An attempt to find such a system for certain q-deformed $SL(n)$-orbits in $sl(n)^*$ featured in [DGK] where a two parameter family of quantum algebras was constructed. The problem of which algebra of this family could be considered as a q-analogue of a commutative algebra was not so evident. In the following we consider a particular case of these q-deformed orbits, namely that related to $U_q(sl(2))$ and called quantum hyperboloid. Being equipped with a proper involution it becomes Podles’ quantum sphere [P1] (more precisely a particular case of Podles’ quantum sphere which is simply the ”$q$-commutative” case; note that in this low-dimensional case there is no problem with understanding ”$q$-commutativity”).

The first attempt to construct a q-deformed differential calculus on a quantum sphere was undertaken in [P2]. However, the corresponding differential algebra is not a flat
deformation of its classical counterpart. In section 3 we will present another approach to introducing a quantum de Rham complex with the flatness property.

Let us evoke now another type of complexes connected (in particular) with the QYBE, namely Koszul complexes (we are still working in the framework of quantum linear algebra). Let \( V \) be a vector space over \( k \) and \( I \subset V \otimes 2 \) be a subspace of \( V \otimes 2 \). Let us set

\[
I^{(0)} = k, \ I^{(1)} = V, \ I^{(n)} = I \otimes V \otimes (n-2) \cap V \otimes I \otimes V \otimes (n-3) \cap ... \cap V \otimes (n-2) \otimes I, \ n \geq 2
\]

and consider the quadratic algebra

\[
A = T(V)/\{I\}, \text{ where } \{I\} \text{ is the ideal generated by } I
\]

and \( T(V) \) stands for the free tensor algebra of the space \( V \).

Let \( A^{(n)} \) be its homogeneous component of degree \( n \). Note that \( A^{(0)} = k, \ A^{(1)} = V \) and \( A^{(n)}, \ n \geq 2 \) can be treated as the quotient

\[
V \otimes n / I^n \text{ where } I^n = I \otimes V \otimes (n-2) + V \otimes I \otimes V \otimes (n-3) + ... + V \otimes (n-2) \otimes I.
\]

Then the corresponding Koszul complex is defined by

\[
d : A \otimes I^{(n)} \rightarrow A \otimes I^{(n-1)}, \quad d(a \otimes x \otimes y) = ax \otimes y \quad \text{where} \quad a \in A, \ x \otimes y \in V \otimes V \otimes (n-1)
\]

and \( ax \) is the product in the algebra \( A \). In fact, this complex decomposes into a series of subcomplexes

\[
A^{(m)} \otimes I^{(n)} \rightarrow A^{(m+1)} \otimes I^{(n-1)}.
\]

**Definition 1** A quadratic algebra \( A \) is called Koszul if the cohomology of the complex (2.1) vanishes in all terms (except of course the trivial term \( A^{(0)} \otimes I^{(0)} \) consisting of constants, i.e. elements of \( k \)).

Let us suppose now that we have two nontrivial complementary subspaces \( I_+ \subset V \otimes 2 \) and \( I_- \subset V \otimes 2 \), i.e. such that \( I_+ \cap I_- = \emptyset \) and \( I_+ \oplus I_- = V \otimes 2 \) and associate to them two algebras

\[
A_+ = T(V)/\{I_-\} \quad \text{and} \quad A_- = T(V)/\{I_+\}
\]

(they are treated as "symmetric" and "skew symmetric" algebras whereas the elements of the subspaces \( I_\pm \subset V \otimes 2 \) are treated as "symmetric" and "skew symmetric" tensors). Then we can define two Koszul complexes

\[
d : I_+^{(n)} \otimes A_- \rightarrow I_+^{(n-1)} \otimes A_- \quad \text{and} \quad \delta : A_+ \otimes I_-^{(n)} \rightarrow A_+ \otimes I_-^{(n-1)}
\]

as it is described above.

If moreover, we can identify \( A_+^{(n)} \) with \( I_+^{(n)} \) and \( A_-^{(n)} \) with \( I_-^{(n)} \) (this means that the spaces \( I_+^{(n)} \) and \( I_-^{(n)} \) on the one hand and \( I_+^{(n)} \) and \( I_-^{(n)} \) on the other hand are complementary for \( n \geq 3 \), cf. [DS]) we can consider these two complexes as one (whose the terms are \( A_+^{(m)} \otimes A_-^{(n)} \)) but equipped with two differentials mapping in opposite directions.
This is just the case of the complexes constructed in [G1] (where \( V \) is a vector space equipped with a Hecke symmetry, cf. footnote 1). As shown in [G1], the algebras \( A_\pm \) are Koszul. In particular, this implies the classical relation

\[
P_+ (t) \ P_- (-t) = 1
\]

between the Poincaré series of the "symmetric" and "skew symmetric" algebras.

Let us remark that the above identification \( A_\pm^{(n)} \approx I_\pm^{(n)} \) was realized in [G1] by means of projectors

\[
P_\pm^n : \ V^\otimes n \to I_\pm^{(n)}
\]

whose kernels are just \( I_{\pm}^n \). This implies that the spaces \( I_\pm^{(n)} \) and \( I_{\mp}^{(n)} \) are complementary. Moreover, the differentials \( d \) and \( \delta \) are realized in [G1] directly in terms of these projectors.

We say that an element \( x \otimes y \in A_\pm^{(m)} \otimes A_\mp^{(n)} \) is given in a canonical (or base) form if \( x \otimes y \in I_\pm^{(m)} \otimes I_\mp^{(n)} \), i.e. it is realized as a sum of products of "symmetrized" and "skew symmetrized" elements. In virtue of [G1] any element of \( A_\pm^{(m)} \otimes A_\mp^{(n)} \) can be represented in a canonical form.

Let us now compare these complexes with the de Rham complex constructed in [WZ]. By applying the de Rham differential to the product \( x_{i_1} x_{i_2} \ldots x_{i_m} \) one obtains, by virtue of the Leibniz rule, a sum whose arbitrary summand is of the form

\[
x_{i_1} x_{i_2} \ldots x_{i_{p-1}} \ dx_{i_p} x_{i_{p+1}} \ldots x_{i_m}, \quad 1 \leq p \leq m.
\]

(The sign \( \otimes \) is systematically omitted.) Here \( \{ x_i \} \) is a base of the space \( V \) equipped with a Yang-Baxter operator of the Hecke type.

The second step of the procedure consists of moving the factor \( dx_{i_p} \) to the right side (for concreteness). So, the problem arises of finding a moving which would be compatible with the differential and would lead to a flat deformation of the initial differential algebra. If, moreover, one wants to restrict the differential to a quantum variety it is necessary to coordinate such a movement with constrains arising from the system of equations defining the variety in question.

Nevertheless, such a problem does not appear for the Koszul complex (2.1) since its differential \( d \) takes only one (namely, extreme) factor of the space \( I^{(n)} \) to the algebra \( A \). So, one should not transpose the elements from \( V \) and their differentials.

We can say that the Koszul complex from [G1] and the de Rham one from [WZ] are formed by the same terms. The difference is that all elements of the Koszul complex are represented in the canonical form. Moreover, it is easy to see that the differentials of these two complexes are proportional to each other on each term (and the coefficients are not trivial). This implies that their cohomologies are isomorphic (recall that \( q \) is generic).

Since the cohomology of the Koszul complex is trivial (apart from the \((0, 0)\) term) we obtain that it is also true for de Rham complex from [WZ] (a quantum version of the Poincaré lemma).

Let us remark that this scheme can be extended to other couples of subspaces \( I_\pm \) associated to the QYBE (including non-quasiclassical cases) but the crucial problem is to show that the associated spaces \( I_\pm^{(n)} \) and \( I_\mp^{(n)} \) (resp., \( \tilde{I}_\mp^{(n)} \) and \( \tilde{I}_\mp^{(n)} \)) are complementary (cf. [DS]).
3 De Rham type complex on quantum hyperboloid

Let us pass now to a quantum hyperboloid. Consider the QG $U_q(sl(2))$ generated by the generators $X$, $H$, $Y$ subject to the well known relations (cf. [CP]). Let us fix a coproduct and the corresponding antipode and consider the spin 1 $U_q(sl(2))$-module $V = V^q$.

In order to define a quantum hyperboloid we should fix a base in $V$ and write down the system of relations on the generators compatible with action of the QG in question. However, we want to represent this system in a symbolic way without referring to its specific coordinate form.

We need only the fact that the fusion ring for $U_q(sl(2))$-modules is exactly the same as in the classical case (we consider only the finite-dimensional $U_q(sl(2))$-modules which are deformations of the $sl(2)$-modules). Thus, if $V_i$ is the spin $i$ $U_q(sl(2))$-module then the classical formula

$$V_i \otimes V_j = \bigoplus_{k=|i-j|}^{i+j} V_k$$

is still valid although the Clebsch-Gordan coefficients (which depend on a base) are $q$-deformed.

In particular, we have

$$V^\otimes 2 = V_0 \oplus V_1 \oplus V_2.$$ 

We keep the notation $V$ for the initial space and $V_i$ for the component in $V^\otimes 2$ isomorphic to $V_i$. Let us fix in the spaces $V$, $V_0$, $V_1$ and $V_2$ some highest weight (h.w.) elements $v$, $v_0$, $v_1$ and $v_2$ respectively, and impose the relations (which are the most general relations compatible with action of the QG $U_q(sl(2))$)

$$v_0 = c, \quad v_1 = \hbar v.$$  \hspace{1cm} (3.2)

Here $c \in k$ and $\hbar \in k$ are some constants. One can now deduce the complete system of equations by applying to the second relation the decreasing operator $Y \in U_q(sl(2))$.

Let us denote $A_{c,\hbar}^q$ the algebra defined by (3.2) and derivative relations.

This algebra possesses the following property: it is multiplicity free. More precisely, any integer spin module occurs once in its decomposition into a direct sum of irreducible $U_q(sl(2))$-modules. Moreover, any element of $A_{c,\hbar}^q$ can be represented in a unique way as a sum of homogeneous elements belonging to the components $V_i \subset V^\otimes i$ (a proof of this fact can be deduced, for example from [GV]). This representation will be called canonical or base. Note that element $v^\otimes i$ is a h.w. one of the component $V_i$.

We treat a particular case of the algebra in question, namely $A_{0,\hbar}^q$, as a q-analogue of a commutative algebra and call it quantum hyperboloid if $c \neq 0$ and quantum cone if $c = 0$. Since the $sl(2)$-module $sl(2)^\otimes 2$ is multiplicity free we can introduce q-analogues $I_\pm$ of symmetric an skew symmetric subspaces of $sl(2)^\otimes 2$ by setting similarly to the classical case

$$I_+ = V_0 \oplus V_2 \quad \text{and} \quad I_- = V_1.$$

Let us emphasize that the corresponding algebras $A_\pm = T(sl(2))/\{I_\pm\}$ are flat deformations of their classical counterparts.

We will need also a q-deformed (braided) Lie bracket. It can be defined as a non-trivial map

$$[ \cdot, \cdot ]_q : V^\otimes 2 \rightarrow V.$$
\( V = \mathfrak{sl}(2) \) as linear spaces) being a \( \mathbb{U}_q(\mathfrak{sl}(2)) \)-morphism. By this request the bracket is defined in a unique way up to a factor.

**Remark 1** Let us remark that for the Lie algebras \( \mathfrak{g} = \mathfrak{sl}(n) \), \( n \geq 3 \) the \( \mathfrak{g} \)-module \( \mathfrak{g}^{\otimes 2} \) is not multiplicity free any more: it possesses two components isomorphic to \( \mathfrak{g} \) itself, one belongs to the symmetric part of \( \mathfrak{g}^{\otimes 2} \) and the other one to the skew symmetric part. This is reason why it is no so evident what are \( q \)-analogues of the symmetric and skew symmetric algebras of the space \( \mathfrak{g} \). However, there exists a subspace \( I_\mathfrak{g} \subset \mathfrak{g}^{\otimes 2} \) where \( \mathfrak{g}^q = \mathfrak{sl}(n) \) as vector spaces but equipped with a \( \mathbb{U}_q(\mathfrak{sl}(n)) \)-module structure such that the quadratic algebra \( \mathbb{T}(\mathfrak{g}^q)/\{I_\mathfrak{g}\} \) is a flat deformation of the symmetric algebra of \( \mathfrak{g} \) (cf. \([D]\)). A more explicit description of \( I_\mathfrak{g} \) can be given by means of the so-called reflection equation (RE)

\[
S L_1 S L_1 - L_1 S L_1 S = 0 \tag{3.3}
\]

where \( S \) is a solution of the QYBE (here of the Hecke type), \( L_1 = L \otimes \text{id} \) and \( L \) is a matrix with matrix elements \( (l^i_j) \). The quadratic algebra defined by the system (3.3) is usually called RE algebra.

Let us remark that the RE algebra is covariant w.r.t. \( \mathbb{U}_q(\mathfrak{sl}(n)) \) (cf. \([IP]\)) and it is a flat deformation of its classical counterpart \( \text{Sym}(W) \) where \( W = \text{span}(l^i_j) \) (cf. \([L]\)). It is not difficult to see that the space \( W \) is a sum of two irreducible \( \mathbb{U}_q(\mathfrak{g}) \)-modules: one-dimensional one with a generator \( l = \text{tr}_q L \) where \( \text{tr}_q \) is the \( q \)-trace and \( n^2 - 1 \)-dimensional one which can be identified with \( \mathfrak{g}^q \) above. By killing the component \( l \) (i.e. by passing to the quotient of the RE algebra over the ideal \( \{l\} \)) we get exactly the algebra mentioned above \( \mathbb{T}(\mathfrak{g}^q)/\{I_\mathfrak{g}\} \). In other words, the space \( I_\mathfrak{g} \) is defined by the relation (3.3) but with one component less.

Moreover, by means of the RE algebra one can get an algebra looking like the enveloping algebra of \( \mathbb{U}_q(\mathfrak{sl}(n)) \) introduced in \([LS]\). Before killing the component \( l \) let us realize a shift \( l^i_j \rightarrow l^i_j + \hbar \delta^i_j \). Then instead of a graded quadratic algebra we get a filtered algebra, defined by quadratic-linear relations. Now by killing \( l \) we get a quadratic-linear algebra with \( n^2 - 1 \) generators. This is just another realization of the enveloping algebra from \([L3]\) (if in the latter algebra we replace the Casimir element by a constant, cf. \([LS]\)) and a flat two parameter deformation of \( \text{Sym}(\mathfrak{g}) \) whose existence was stated in \([L]\). (However, to get a reasonable quasiclassical limit we should replace the parameter \( \hbar \) in this quadratic-linear algebra by \( \hbar/(q - 1) \).

If \( \mathfrak{g} \) is a simple Lie algebra different from \( \mathfrak{sl}(n) \) its tensor square is multiplicity free. This allows one to define a \( \mathbb{q} \)-deformed Lie bracket requiring it to be a non-trivial morphism in the category of \( \mathbb{U}_q(\mathfrak{g}) \)-modules (this defines the bracket uniquely up to a factor) and to introduce the corresponding enveloping algebra of the correspondent "braided Lie algebra" \( \mathfrak{g}_q \). Deformed analogues of the symmetric and skew symmetric algebras of the space \( \mathfrak{g} \) are also well defined. However, these algebras are not flat deformations of their classical counterparts (cf. \([GA]\)).

**Remark 2** Let us emphasize that the classical counterpart \( \mathcal{A}_{01} \) of the algebra \( \mathcal{A}_{0q} \) contains only polynomials restricted to the hyperboloid (or the cone). This is the reason why its properties and those of the function algebra on the sphere are similar: a passage from one algebra to the other one can be realized by a change of base. For example, this passage does not change the cohomology of the de Rham complex (see below).
Let us set $\Omega^0 = A^0_{0q}$. Our next step is to define the spaces of first- and second-order
differential forms over this algebra. First, consider the tensor products

$$\wedge^1 = A^0_{0q} \otimes V' \quad \text{and} \quad \wedge^2 = A^0_{0q} \otimes V_1''.$$ 

Their second factors are treated as pure differentials (say the element $x \otimes y \in \wedge^1$ is treated
as $x \, dy$ and in $x \otimes y \in \wedge^2$ the factor $y \in V_1''$ is a sum of products of two pure first order
differentials). The mark $'$ stands for a pure first-order differential term and that $''$ stands
for a pure second order differential term. Thus, the space $V'$ (resp., $V_1''$) is isomorphic to
the space $V_1$ itself; the isomorphism is defined by

$$dx_i \rightarrow x_i \ (\text{resp.,} \ dx_i \otimes dx_j \rightarrow x_i \otimes x_j).$$

Note that we treat the vector spaces $\wedge^i$ as left $A^0_{0q}$-modules. We do not endow their
sum $\bigoplus \wedge^i$ with any algebraic structure. So, we do not need any transposition rule for the
elements of $A^0_{0q}$ and their differentials.

Let us introduce now the first- and second-order differential forms on the quantum
hyperboloid by

$$\Omega^1 = \wedge^1 / \{(V \otimes V')_0\}, \quad \Omega^2 = \wedge^2 / \{(V \otimes V_1'')_1 + (v_0 - c) \otimes V_1''\}.$$ 

Here the terms in the denominators are not ideals but only left $A^0_{0q}$-submodules of $\wedge^1$ and
$\wedge^2$ respectively. The notation $(V \otimes V')_i$ means that in the product $V \otimes V'$ we take the
spin $i$ component (similarly for $(V \otimes V_1'')_i$). And $(v_0 - c) \otimes V_1''$ stands for the second-order
differential forms containing $v_0 - c$ as a factor.

To make this construction more explicit let us represent it in a base form (by restricting
ourselves to the classical case since it does not matter what case, classical or quantum we
deal with). Let $u, v, w$ be the usual base in $\text{Fun}(\text{sl}(2)^*) = \text{Sym}(\text{sl}(2))$. Then the above
denominators are generated respectively by

$$2udw + 2wdu + vdv \quad \text{and} \quad 2ue_2 + ve_1, \quad ue_3 - we_1, \quad 2we_2 + ve_3, \quad (2uw + 2wu + vv - c)e_i, \quad i = 1, 2, 3$$

with

$$e_1 = dudv - dvd, \quad e_2 = dudw - dwdu, \quad e_3 = dvdw - dwdv.$$ 

(we suppose here that $v_0 = 2uw + 2wu + vv$).

We have defined the spaces $\Omega^i, \ i = 1, 2$ as some quotients. Now, we want to define
the differentials in some bases of these spaces similarly to the Koszul complexes discussed
above. To define such bases we will realize a spectral analysis of the spaces $\Omega^i$, i.e.
decompose these spaces into a direct sum of irreducible $sl(2)$-modules. First, describe the
components in the products

$$V \otimes V', \ V_i \otimes V', \ V_i \subset A^0_{01}, \ i = 2, 3, ...$$

which are surviving in the quotient space $\Omega^1$. It is evident that in the product $V \otimes V'$ only
the components $(V \otimes V')_1$ and $(V \otimes V')_2$ survive since by construction the component
$(V \otimes V')_0$ is equal to 0 in the quotient.
By a similar reason in the product $V_2 \otimes V'$ the components $(V_2 \otimes V')_2$ and $(V_2 \otimes V')_3$ survive and that $(V_2 \otimes V')_1$ is equal to 0 modulo the terms of $k \otimes V' = V'$. This can be explained as follows. The elements of $\mu^{12}(V \otimes (V \otimes V'))_0$ are trivial in $\Omega^1$ by construction. Here $\mu$ stands for the product in $\mathcal{A}_{0q}$, the indexes 12 mean as usual that the operator $\mu$ is applied to the first two factors. By reducing any element of the product $V \otimes V$ to the canonical form we get a sum of an element from $V_2 \subset V^\otimes 2$ and another one from $k$. This completes the proof.

Similarly, in the product $V_i \otimes V'$ the component $(V_i \otimes V')_{i-1}$ is equal to 0 modulo the terms belonging to $V_j \otimes V'$, $j < i$. Thus, we have shown the following.

**Proposition 1** The base in the $\mathcal{A}_{0q}$-module $\Omega^1$ is formed by

1. $V'$, 2. $(V \otimes V')_{1,2}$, 3. $(V_2 \otimes V')_{2,3}$, 4. $(V_3 \otimes V')_{3,4}$ etc.

In a similar way one can perform a spectral analysis of the $\mathcal{A}_{0q}$-module $\Omega^2$ and describe its base.

**Proposition 2** The base in the $\mathcal{A}_{0q}$-module $\Omega^2$ is formed by

1. $V''_1$, 2. $(V \otimes V''_1)_{0,2}$, 3. $(V_2 \otimes V''_1)_3$, 4. $(V_3 \otimes V''_1)_4$ etc.

An evident difference between the modules $\Omega^1$ and $\Omega^2$ consists in the following. The module $\Omega^2$ is defined as a quotient of $\wedge^2$ over the sum of two submodules. Therefore, two components in the products $V_i \otimes V''_1$, $i \geq 2$ disappear and only one survives. The component $V_1 \otimes V'$ is exceptional because the relation $v_0 = c$ does not lead to any constraint for it. Let us consider now the de Rham complex in the classical case

$$0 \longrightarrow \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \longrightarrow 0.$$ \hspace{1cm} (3.4)

Since the differential commutes with the $sl(2)$ action it takes any irreducible $sl(2)$-module to either an isomorphic $sl(2)$-module or 0. Using propositions 1 and 2 it is not difficult to describe the irreducible $sl(2)$-modules of $\Omega^i$, $i = 0, 1, 2$ belonging to $\text{Ker } d$ and those belonging to $\text{Im } d$.

**Proposition 3** 1. In $\Omega^0$ the only trivial module, i.e. that consisting of the elements of $k$ belongs to $\text{Ker } d_0$.

2. $\text{Ker } d_1 = V' \oplus (V \otimes V')_2 \oplus (V_2 \otimes V')_3 \oplus (V_3 \otimes V')_4 \oplus ...$

and therefore the modules

$$(V \otimes V')_1, (V_2 \otimes V')_2, (V_3 \otimes V')_3, ...$$

go to isomorphic modules in $\Omega^2$.

**Corollary 1** The cohomology of the complex (3.4) is the following one

$$\dim H^0 = 1, \dim H^1 = 0, \dim H^2 = 1,$$

$H^0$ is generated by 1 and $H^2$ is generated by $(V \otimes V'')_0$. 9
Thus, it is just the cohomology of the sphere (see above, remark 2).

Let us extend now de Rham complex (3.4) to the quantum case. The terms of the quantum complex have just the same irreducible components as their classical counterparts (but these components become $U_q(sl(2))$-modules). Now we should define differentials. Define them on each term by requiring them to be $U_q(sl(2))$-morphisms and to be flat deformations of the classical differentials (by this demand the differentials are defined an each $U_q(sl(2))$-module in a unique way up to a factor). Finally, we have by construction just the same cohomology as in the classical case.

Comparing our construction with that from [P2] we repeat that the latter one is not any flat deformation of its classical counterpart meanwhile our deformation is flat by construction. On the other hand, we have lost the structure of an algebra in the $\mathcal{A}_{ho}^\varepsilon_q$-module $\Omega = \oplus \Omega^i$ and the Leibniz rule for the differentials.

4 Quantum tangent space and related structures

In the present section we introduce the tangent space on quantum hyperboloid and discuss some derived structures (metric, connection). We discuss also a way to realize the q-deformed tangent space by means of “braided vector fields”. Hopefully, this approach is valid for other quantum varieties like quantum orbits considered in [DGK]. Similarly to the previous section we avoid using any specific base form.

First, we consider a sphere $S^2$ given by $x^2 + y^2 + z^2 = c$. Let Fun $(S^2)$ be the space of the polynomials restricted to the sphere and Vect $(S^2)$ be the space of left vector fields. The latter space is generated as a left Fun $(S^2)$-module by three infinitesimal rotations

$$X = y \partial_z - z \partial_y, \quad Y = z \partial_x - x \partial_z, \quad Z = x \partial_y - y \partial_x.$$  

It is easy to check that the vector fields $X, Y, Z$ satisfy the following relation

$$x X + y Y + z Z = 0 \quad (4.5)$$

($x, y, z$ are treated here as operators via the product operator in the algebra Fun $(S^2)$).

So, as a Fun $(S^2)$-module Vect $(S^2)$ can be realized as the quotient $M/N$ where

$$M = \{a X + b Y + c Z, \ a, b, c \in \text{Fun} (S^2)\},$$

$$N = \{f (x X + y Y + z Z), \ f \in \text{Fun} (S^2)\}.$$  

In what follows we call this Fun $(S^2)$-module tangent space and denote it $T(S^2)$. In fact it is just the space Vect $(S^2)$ but we want to emphasize by this notation that we ignore the operator meaning of this space. As usual, the tangent space is introduced in local terms as a vector bundle. However, in the quantum case such a local description is not possible.

In a similar way there can be introduced the tangent space $T(H)$ on a hyperboloid $H$. Namely, it can be realized as the quotient of a free $\mathcal{A}_{ho}^\varepsilon_{1}$-module $M$ over its submodule

$$N = \{f (2u W + 2w U + v V)\}.$$

10
This is also motivated by the operator meaning of the generators: the generators $U, V, W$ are represented in the algebra $\text{Fun}(H) = \mathcal{A}_{01}^{c}$ by infinitesimal hyperbolic rotations.

Note that the symmetric algebra of the tangent space $T(S^2)$ can be treated as the function algebra on the underlying 4-dimensional algebraic variety embedded in the 6-dimensional space

$$(\text{span}(x, y, z, X, Y, Z))^*;$$

this variety is defined by the equation of the sphere and that (1.5) (if $k = \mathbb{R}$ it is true for $c > 0$). A similar description is also valid for the symmetric algebra of $T(H)$.

Unfortunately, there does not exist any quantum analogue of this algebra being its flat deformation (see below). Nevertheless, a reasonable q-deformation of tangent space equipped with an appropriated module structure exists. The aim of this section is to describe this deformation, i.e., to introduce the tangent space on the quantum hyperboloid as an $\mathcal{A}_{0q}^{c}$-module and to realize its elements as operators looking like vector fields on the classical object.

In order to do it we represent the defining relation of the tangent space $T(H)$ in a symbolic way:

$$(V \otimes V')_0 = 0 \quad (4.6)$$

(hereafter the mark $'$ designs the space $\text{span}(U, V, W)$). We treat the tangent space on the hyperboloid as a left $\mathcal{A}_{01}^{c}$-module (as a right $\mathcal{A}_{01}^{c}$-module the tangent space can be given by $(V' \otimes V)_0 = 0$).

It is evident that if we want to define the tangent space on the quantum hyperboloid as a flat deformation of its classical counterpart we should use the same formula (4.6) but in the category of $U_q(sl(2))$-modules. Let us make a precise. First, we introduce the left $\mathcal{A}_{0q}^{c}$-module $\wedge^1$ as in the previous section but with another signification of the space $V'$. This means that the generators $du, dv, dw$ are replaced by $U, V, W$, while the $U_q(sl(2))$-module structure of $V'$ is unchanged. Second, we define the tangent space on the quantum hyperboloid as its quotient like $\Omega^1$ above (fortunately, both the tangent and cotangent spaces as $\mathcal{A}_{0q}^{c}$-modules are defined by the same equation (1.6)). Let us denote the quotient object by $T(H_q)$ reserving the notation $H_q$ for the quantum hyperboloid.

**Proposition 4** The $\mathcal{A}_{0q}^{c}$-module $T(H_q)$ is a flat deformation of its classical counterpart.

Proof follows immediately from the explicit construction of the base of this quotient given in the previous section.

Let us assign now an operator meaning to the elements of the space $T(H_q)$.

**Proposition 5** There exists a map

$$\beta : T(H_q) \otimes \mathcal{A}_{0q}^{c} \rightarrow \mathcal{A}_{0q}^{c}. \quad (4.7)$$

such that the diagram

$$\begin{array}{ccc}
\mathcal{A}_{0q}^{c} \otimes T(H_q) \otimes \mathcal{A}_{0q}^{c} & \rightarrow & T(H_q) \otimes \mathcal{A}_{0q}^{c} \\
\downarrow & & \downarrow \\
\mathcal{A}_{0q}^{c} \otimes \mathcal{A}_{0q}^{c} & \rightarrow & \mathcal{A}_{0q}^{c}
\end{array}$$

11
is associative. Here the elements of $A_{0q}^c$ act on $A_{0q}^c$ (the low arrow) by the usual product. The vertical arrows are defined by means of $\beta$ and the top one makes use of the $A_{0q}^c$-module structure of $T(H_q)$. (Thus, the map $\beta$ realizes an action of the space $T(H_q)$ on the algebra $A_{0q}^c$.)

This proposition allows us to realize the tangent space as an operator algebra where the elements of the algebra $A_{0q}^c$ act via the product operator. We call the elements of the space $T(H_q)$ (left) braided vector fields if the operators $\beta(V')$ satisfy the relations

$$(\beta \otimes \beta)(V' \otimes V')_1 - \sigma \beta [\ , ]_q(V' \otimes V')_1 = 0$$

where $[\ , ]_q$ is the q-deformed Lie bracket introduced in section 3 and $\sigma \in k$ is a non-trivial factor. This means that $\beta$ realizes a representation (in the sense of [LS]) of the braided Lie algebra defined by the bracket $\nu [\ , ]_q$ with a proper factor $\nu$.

**Proposition 6** There exists a map $\beta$ from the previous proposition such that the elements of $T(H_q)$ being represented via $\beta$ becomes braided vector fields.

We refer the reader to [A] for proofs of these statements (the main idea of the construction has been suggested in [DC2]). Here we only want to say that the problem is to find good candidates for the role of q-analogues of the infinitesimal hyperbolic rotations $U, V, W$. They arise from the adjoint action of the q-Lie algebra $sl(2)_q$ onto itself (note that the operators $X, H, Y$ coming from the QG $U_q(sl(2))$ do not satisfy the relation (4.3)).

Let us remark that similar statements are valid for the tangent space treated as a right $A_{0q}^c$-module.

Thus, we have an embedding

$$sl(2)_q \hookrightarrow T(H_q) \quad (4.8)$$

where the tangent space is realized as braided vector fields space. This embedding is a deformation of its classical counterpart which is the simplest example of a so-called anchor (recall that an anchor consists of an variety $M$, a Lie algebra $g$ and an embedding of $g$ into the vector field space on $M$). This is reason why we call the embedding (4.8) *quantum anchor* in spite of the fact that the whole of the space $T(H_q)$ is not equipped with any q-deformed Lie bracket. We consider also the data $(T(H_q), A_{0q}^c)$ as a partial q-analogue of Lie-Rinehart algebras [R] ("partial" means here that the space $T(H_q)$ is not equipped with any "q-Lie algebra" structure properly coordinated with the product operator in the algebra $A_{0q}^c$).

After having represented the space $T(H_q)$ by braided vector fields it is natural to introduce the space of braided differential operators as that generated by the braided vector fields and the elements of $A_{0q}^c$ treated as 0-order operators (see above). In the classical case this space is spanned by the subspaces

$$A_{0q}^c \otimes V'^\otimes n.$$ 

The fact that this algebra is closed w.r.t. the operator product is assured by the Leibniz rule: by means of this rule it is possible to represent a product of two elements of this form as a linear combination of such elements.
Unfortunately, in the quantum case any form of the Leibniz rule does not exist (this fact can be checked by direct calculations). Roughly speaking, this means that there does not exist any reasonable way to transpose the elements of the algebra \( A_{0,q}^c \) and those of the space \( V' \). This is also the reason why there does not exist any “q-symmetric algebra” of the quantum tangent space \( T(H_q) \) being a flat deformation of its classical counterpart (see above). Without going into detail we say only that the Yang-Baxter operator (arising from the universal R-matrix) being at first glance a good candidate for the role of such a transposition leads to a non-flat deformation of the classical symmetric algebra. (See also below, remark 3).

Let us pass now to the problem of constructing a q-deformed metric on the quantum tangent space. To distinguish the quantum tangent spaces equipped with the left and right \( A_{0,q}^c \)-module structures we will use for the first (second) one the notation \( T(H_q)_l \) (\( T(H_q)_r \)).

**Definition 2** We say that an operator
\[
< , >: T(H_q)_l \otimes_k T(H_q)_r \to A_{0,q}^c
\]
is quantum (pseudo-)metric if it commutes with left and right multiplication by the elements from \( A_{0,q}^c \) in the following sense
\[
< f P, Q > = f < P, Q >, \quad < P, Q f > = < P, Q > f, \quad \forall f \in A_{0,q}^c, \quad P \in T(H_q)_l, \quad Q \in T(H_q)_r
\]
(in particular, \( P, Q \in V' \)) and if it is compatible with the action of \( U_q(sl(2)) \). The latter property means, as usual that
\[
Z < , > = < , > \Delta(Z), \quad \forall Z \in U_q(sl(2))
\]
(this relation is treated as operator one in \( T(H_q)_l \otimes_k T(H_q)_r \)). A metric is called symmetric if
\[
< , > (V' \otimes V')_1 = 0.
\]

**Proposition 7** There exists the unique (up to a factor) symmetric quantum metric on the quantum hyperboloid.

A proof of this fact is given in [A]. We do not reproduce it here. Let us indicate only the crucial idea of the proof. First, it is necessary to describe all pairings
\[
< , >: V' \otimes V' \to A_{0,q}^c
\]
compatible with the \( U_q(sl(2)) \) action. In order to do it we should decompose \( V' \otimes V' \) into a sum of the irreducible \( U_q(sl(2)) \)-modules. This gives rise to the following two parameter family of \( U_q(sl(2)) \)-covariant pairing
\[
< , > (V' \otimes V')_2 = aV_2, \quad < , > (V' \otimes V')_0 = b
\]
completed by relations \((4.10)\) (as usual, the relations are given in a symbolic way). On
the second step we should impose the condition
\[
<,>^{23} (V \otimes V')_0 \otimes V' = 0
\]
which results in a relation between the parameters \(a\) and \(b\). It remains to verify that this
relation is compatible with the following one
\[
<,>^{12} V' \otimes (V' \otimes V)_0 = 0
\]
and then to extend the metric to the whole \(T(H_q)_l \otimes_k T(H_q)_r\) by using the relations \((4.9)\).

Let us emphasize that although we call the above pairing metric ("pseudo" means
only that its classical analogue is not positive definite) it is well defined on the product
of a left and a right \(A^c_{0q}\)-modules. If we want now to define a similar pairing between two
left (or right) \(A^c_{0q}\)-modules we should proceed in the following way. Let us identify the
left tangent space \(T(H_q)_l\) and the right one \(T(H_q)_r\), i.e. define a map
\[
\alpha : T(H_q)_l \rightarrow T(H_q)_r
\]
being a \(U_q(sl(2))-\)isomorphism.

Then on setting by definition
\[
< X, Y > = < X, \alpha(Y) >, \ X, Y \in T(H_q)_l
\]
we get a pairing between two left \(U_q(sl(2))-\)modules. At first glance the map \(\alpha\) can be defined by means of the YB operator arising from the QG \(U_q(sl(2))\). However, this
operator which establishes a bijection between free \(A^c_{0q}\)-modules \(A^c_{0q} \otimes V'\) and \(V' \otimes \ A^c_{0q}\) is not any bijectivity on their factors \(T(H_q)_l\) and \(T(H_q)_r\) since it does not take
the denominator corresponding to \(T(H_q)_l\) to that corresponding to \(T(H_q)_r\). We suggest
another way to define such a \(U_q(sl(2))-\)isomorphism \(\alpha\).

Let us represent the both objects as sums of \(U_q(sl(2))-\)modules in the spirit of propo-
sition 1. Then the \(U_q(sl(2))-\)morphisms
\[
\alpha : (V \otimes V')_1 \rightarrow (V' \otimes V)_1, \ (V \otimes V')_2 \rightarrow (V' \otimes V)_2, \ (V_2 \otimes V')_2 \rightarrow (V' \otimes V_2)_2, ...
\]
are defined uniquely up to a factor on each couple of components (for the generating space
\(V'\) we put \(\alpha = \text{id} \)).

However, we can reduce this freedom by identifying the elements from
\[
(V \otimes V')_2 \quad \text{and} \quad (V' \otimes V)_2, \quad (V_i \otimes V')_{i+1} \quad \text{and} \quad (V' \otimes V_i)_{i+1}, \ i = 2, 3, ...
\]
which coincide if we replace \(V'\) by \(V\). As for the components
\[
(V \otimes V')_1 \quad \text{and} \quad (V' \otimes V)_1, \quad (V_i \otimes V')_i \quad \text{and} \quad (V' \otimes V_i)_i, \ i = 2, 3, ...
\]
their elements are identified if this operation leads to opposite images. It is not difficult
to see that in the classical case this identification and that defined by the flip coincide (it
is the motivation of our method).
Remark 3 Let us remark that for algebras looking like that $\mathcal{A}_{0,q}^c$ but connected to an involutory YB operator an identification of their left and right modules can be realized by means of this operator. Non-involutivity of the YB operator arising from the QG $U_q(sl(2))$ which leads to the above defect prevents us also from a reasonable definition of a tensor product $M_1 \otimes \mathcal{A}_{0,q}^c M_2$ of two (say) left $\mathcal{A}_{0,q}^c$-modules. The problem is that there do not exist any reasonable way to transpose the factor $f \in \mathcal{A}_{0,q}^c$ in the product

$$m_1 \otimes f m_2, \ m_1 \in M_1, \ m_2 \in M_2$$

on the left side so that the tensor product $\otimes \mathcal{A}_{0,q}^c$ is still associative and the module\mathcal{A}_{0,q}^c M_1$ and $M_2$ to be flat deformations of their classical counterparts. For an involutory YB operators this problem does not appear.

Let us discuss now the problem of defining a (torsion free) $U_q(sl(2))$-covariant connection on the tangent space $T(H_q)$. Such a partially defined connection was introduced in [A]. "Partially defined" means here that the operators of covariant derivatives are defined only on a subspace of $T(H_q)$, namely on $V'$. More precisely, there exists a $U_q(sl(2))$-morphism

$$\nabla : T(H_q)_l \otimes V' \rightarrow T(H_q)_l,$$

such that

$$\nabla_{fX}Y = f\nabla_XY, \ X \in T(H_q), \ Y \in V', \ f \in \mathcal{A}_{0,q}^c$$

and

$$a^{ij}\nabla_{X_i}X_j = [X_i, X_j]_q, \ X_i, X_j \in V'$$

(4.11)

where $a^{ij}X_i \otimes X_j \in V_1$ and $[\ , \ ]_q$ is the above mentioned q-deformed Lie bracket.

We would be able to extend this (partially defined) connection to the whole $T(H_q) \otimes T(H_q)$ if we could extend the bracket $[\ , \ ]_q$ to the whole $T(H_q)$ and to understand what is the enveloping algebra of this extended q-deformed Lie algebra (we need this in order to write suitable expressions in the l.h.s. of (4.11)). Unfortunately, we do not know any way to do it.

Let us remark that all naive extensions of the bracket $[\ , \ ]_q$ is not compatible with the equation (1.13).

There exists a number of papers introducing the notions of quantum metric and connection in another way (cf. [HM] and the references therein). Our approach is motivated by our desire to control the flatness of deformation of classical objects (see also the next section).

Completing this section we want to mention a very important property of the $\mathcal{A}_{0,q}^c$-module $T(H_q)$ (if $c \neq 0$): it is projective in the category of $U_q(sl(2))$-modules. This means that it is a direct component in a free $\mathcal{A}_{0,q}^c$-module and there exists a projector of the latter one onto the module in question being a $U_q(sl(2))$-morphism (cf. [A]).

Some other projective modules over quantum sphere have been considered in [HM]. We plan to devote a subsequent paper to quantum projective modules in a more general context.
5 On quantum gauge theory

There exist two approaches to q-deformed gauge theory. One of them deals with the usual manifolds (varieties) and deforms only a structure of fibers. The second approach deals with quantum varieties looking like the quantum hyperboloid above.

First, let us evoke the paper [S] as the most advanced contribution to the first kind approach. The gauge potential $A_\mu$ introduced in this paper is a vector field

$$A_\mu(x) = A_i^\mu(x) X_i$$

with values in the quantum Lie algebra $g_q$. Here $g_q$ is the q-deformed Lie algebra $g = sl(n)$ (or $su(n)$) as defined in [LS] (note that the case $n = 2$ was previously considered in [DG1]). Thus, the factors $X_i$ are elements of this quantum Lie algebra and those $A_i^\mu(x)$ are usual functions depending on a "space-time point" $x$ (or more generally, on a point of a usual variety).

In virtue of [LS] the quantum Lie algebra $g_q$ is realized as a subspace in $U_q(g)$ so that it is stable w.r.t. the adjoint action of the QG $U_q(g)$ on itself and

$$\Delta(X_i) = X_i \otimes C + u_i^j \otimes X_j$$

where $\Delta$ is the coproduct in $U_q(g)$, $C$ is a central (Casimir) element of $U_q(g)$ and $u_i^j$ are some elements of $U_q(g)$.

The crucial point of any gauge theory is a transformation law of $A_\mu$ under an action of a gauge group or a gauge Hopf algebra. In [S] it is supposed to be

$$A_\mu \mapsto A'_\mu(x) = h(x)(1) A_\mu s(h(x)(2)) - \alpha^{-1} s(C)^{-1} \partial_\mu(h(x)(1)) s(h(x)(2))$$  \hspace{1cm} (5.12)

where $\alpha$ is a coupling constant, $s$ is the antipode, $h(x)$ is a function of $x$ with values in the QG $U_q(g)$ and

$$h(x)(1) \otimes h(x)(2) = \Delta h(x).$$

However, a problem arises to "distribute the $x$-dependence of the coproduct $\Delta(h(x))$ between two factors" in the second term of (5.12) so that it becomes an element of the space $g_q$ for a fixed $x$ (let us emphasize that it is not a trivial task). Such a distribution has been found in [S].

Nevertheless, it was indicated in [S] that if we consider a (say) bosonic field $\psi(x)$ then its defining relations cannot be introduced in a way compatible with quantum gauge transformations. We will try to explain it as follows. Let us consider the "quantum covariant derivative" of the field $\psi$

$$D_\mu \psi = \partial_\mu \psi + \beta \rho(A_\mu) \psi$$

where $\rho$ is the representation of the QG $U_q(g)$ corresponding to $\psi$ and $\beta = \alpha \rho(s(C))$ is a constant. In the r.h.s. of this formula the operator $\partial_\mu$ commutes with the $U_q(g)$ action but the operator $\rho(A_\mu)$ does not. This implies that such a covariant derivative cannot preserve the relations valid for $\psi$. (We would have a similar effect in a supertheory if we allowed the summand $A_\mu$ to be an odd operator.)
Let us discuss now the approach of the second kind, i.e. we suppose that the base
variety is quantum as well. An axiomatic way to suggest such an approach was considered
in numerous papers. We do not give here an exhausting list of these papers and only refer
the reader to the papers [S] and [BM] where such a list is given.

We will point out the common features of all of them. First, a quantum variety in
question is given in a way which does not allow us to control the flatness of deformation
(as a rule this problem is not even evoked). Another crucial defect of this approach is that
a connection is introduced habitually via a Leibniz rule similarly to the classical case but
as we have seen this implied the non-flatness of the deformation. Another reason of the
non-flatness of the deformation is that in the formulae analogueical to (5.12) the second
summand does not belong usually to the fiber.

This explains our scepticism about a possibility of constructing a quantum gauge the-
ory related to the QG $U_q(g)$ which would be a flat deformation of its classical counterpart.

Anyway, it would be desirable to precede any attempt to construct such a theory by
a quasiclassical study in the spirit of [Ar] confirming or refuting a possibility to do it.

Acknowledgment The authors are very indebted to P.Pyatov and P.Saponov for
discussions and valuable suggestions. One of the authors (D.G.) was supported by the
grant CNRS PICS-608.

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