Phonon condensation and cooling via nonlinear feedback

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We show that in multimode mechanical systems, the amplification of the lowest mode and the damping of all the other modes can be realized simultaneously via nonlinear feedback. The feedback-induced dynamics of the multimode system is related to the formation of phonon condensation. The phonon statistics of the lowest mode are similar to those of a phonon laser. Finally, we show the coherence of the lowest mode can be improved by an order of magnitude.

\textbf{Introduction.} – Manipulation and control of both coherent and incoherent phonons and/or vibration energy is of great interest for both engineering applications and fundamental research. On the one hand, the coherent phonons and/or vibrations is of primary importance in applications ranging from conventional nondestructive testing \cite{1, 2}, high resolution imaging and sensing \cite{3}, to quantum information transfer and storage, etc \cite{4–6}. On the other hand, the control of incoherent phonons and/or vibrations is of primary importance in noise reduction \cite{7}, thermoelectric energy conversion \cite{8}, heat management such as heat dissipation and heat insulation \cite{9, 10}.

In applications like highly sensitive and ultra-fast sensing \cite{11–13}, acoustic actuation \cite{14–16}, information processing \cite{17–19}, biological imaging \cite{20–22}, amplifying vibration amplitude and narrowing phonon linewidth are critical for good performance. Active linear feedback control, namely, feedback force is proportional to the measured mechanical displacement or velocity but with a phase difference, is a well-known technique for achieving these two goals. The control consists of the real-time monitoring of mechanical motion and feedback loop. Depending on the phase difference, either positive or negative feedback can be realized. This technique works well when the resonator can be considered as a single mode system.

In general, however, mechanical resonators have a series of normal modes. A single linear feedback loop results in the simultaneous amplification or damping of multiple modes \cite{23, 24}. In the cases like energy transfer/harvesting and phonon lasing, where only one selected mode is interested, it is then quite straightforward to ask if amplifying only one mode while cooling all the other modes can be achieved via a single feedback loop.

In this Letter, we propose to use a single nonlinear feedback loop to realize the amplification of the lowest normal mode and the damping of all the other modes simultaneously in multimode mechanical systems. In the feedback-induced steady state, the phonon statistics indicate strong amplitude coherence in the lowest mode. The phase coherence of the lowest mode is largely improved as well.

The amplification of the lowest modes and damping of all the other modes is closely related to the well-known Fröhlich condensate \cite{25–34}, where the vibration energy of a collection of oscillators would condensate in the lowest mode once the external energy supply exceeds a threshold. In Fröhlich condensate, the essential process is the energy redistribution among vibration modes induced by nonlinear couplings, whereas in our case it is through the nonlinear feedback.

\textbf{Model.} – We consider a mechanical resonator with several modes of oscillations. The configuration is shown in Fig. (1). The reflected and reference optical field from probe laser are collected by an interferometer to measure the collective displacement of the mechanical resonator. The measured displacement is then fed through a feedback loop to determine the drive applied onto the resonator. The feedback force can be provided by another laser via the optomechanical or photothermal effect, or by an electrical signal via the electromechanical effect.

The equations of motion (EOM) of the resonator are

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Sketch of the multimode system considered. The reflected optical field provides information on the collective displacement of the resonator. Based on the detected signal, the feedback loop determines the drive applied onto the resonator. The feedback force can be realized using optomechanical, photothermal or electromechanical effect.}
\end{figure}
given by
\[ \dot{Q}_j = \omega_j P_j, \]
\[ \dot{P}_j = -\omega_j Q_j - \gamma_j P_j + \xi_j + H_{fb}^{(j)}, \]
where \( \omega_j \) is the frequency, \( Q_j = \sqrt{m/k_B T} \omega_j q_j \) and \( P_j = \sqrt{m/k_B T} \) are the dimensionless displacement and momentum of the \( j \)th normal mode so that the dimensionless vibration energy \( \frac{1}{2}(Q_j^2 + P_j^2) \) at thermal equilibrium is equal to one, \( \gamma_j \) is the damping rate, \( \xi_j \) is the thermal noise and \( H_{fb}^{(j)} \) is the feedback force acting on the \( j \)th mode. At high temperature limit \( k_BT \gg \hbar \omega_j \), the thermal noise satisfies \( \langle \xi_j(t)\xi_j(t') \rangle = 2\gamma_j \delta(t-t') \). The feedback force \( H_{fb}^{(j)} \) is determined by the measured collective displacement \( Q = \sum_j Q_j \) of the resonator. To realize the condensation of phonons at the lowest mode, we design the feedback loop as follows:
\[ F_I = \int_0^t Q(s) ds, \]
\[ F_D = Q, \]
\[ H_{fb}^{(j)} = -g_j \omega_j \tanh[\omega_j (F_I^2 F_D + 3Q^2 F_I)], \]
where \( g_j \) is the feedback gain term and \( \omega_{fb} \) is a reference frequency to make \( H_{fb}^{(j)} \) have correct unit. The terms \( F_I \) and \( F_D \) are like the integrator and differentiator circuits, respectively. A hyperbolic tangent function is introduced to \( H_{fb}^{(j)} \) to limit the strength of the feedback force between \( \pm g_j \omega_{fb} \). To understand how the feedback works, we first consider a simpler \( H_{fb,0}^{(j)} \) where the hyperbolic tangent function is replaced by the identity function, i.e., \( H_{fb,0}^{(j)} = -g_j \omega_j^2 (F_I^2 F_D + 3Q^2 F_I) \). After introducing the complex amplitude
\[ Q_j = \frac{1}{2} [a_j(t)e^{-i\omega_j t} + a_j^*(t)e^{i\omega_j t}] \]
with \( a_j(t) \) being slowly varying amplitudes (\( \dot{a}_j \ll \omega_j a_j \)), we can simplify the amplitude equations as
\[ \dot{a}_j = -\frac{\gamma_j}{2} a_j + \sum_i \frac{g_i \omega_i^2}{4 \omega_i^2 \omega_j} (\omega_i^2 - \omega_j^2) |a_i|^2 a_j + \Xi_j, \]
In the derivation we have assumed \( \omega_j \gg \gamma_j \), ignored off-resonant terms and averaged the thermal noise \( \xi_j(t) \) over the fast dynamics,
\[ \Xi_j(t) = \frac{\omega_j}{2\pi} \int_{t-\pi/\omega_j}^{t+\pi/\omega_j} ds \xi_j(s)e^{i\omega_j s}. \]
The slowly varying noise \( \Xi_j(t) \) satisfies
\[ \langle \Xi_j(t)\Xi_j^*(t') \rangle = 2\gamma_j \delta(t-t'). \]
FIG. 2. (color online). Phonon condensation in a system with $N = 4$ modes. (a) Vibration energy as a function of time. The dimensionless vibration energy at thermal equilibrium ($t = 0$) is one. The inset is zoom-in. The lowest mode is amplified by a factor of 10, while the other three modes are cooled to 0.57, 0.5 and 0.48, respectively. (b) The ratio of vibration energy in the lowest mode to the total vibration energy of the system as a function of time. In our simulation, the fre-
ed energy in the lowest mode to the total vibration energy of the system is one. The inset is zoom-in. The lowest mode is amplified by a factor of 10, while the other three modes are cooled to 0.57, 0.5 and 0.48, respectively.

The first four modes satisfy $\omega = 2.75$, $\omega_3/\omega_1 = 5.13$, $\omega_4/\omega_1 = 8.75$. The other parameters are $\gamma_j/\omega_j = 10^{-2}$, $g_j = g = 0.2$, $\omega_0/\omega_1 = 1$.

FIG. 3. (color online). Phonon statistics of the lowest mode at steady state. (a) Phase portrait of the lowest mode without feedback. (b) Phase portrait with feedback. (c) Phonon-number (energy) distribution without feedback. (d) Phonon-number (energy) distribution with feedback. The parameters used are the same as those in Fig. (2). The variance ($\sim 19$) in (d) is much smaller than that ($\sim 100$) in (c) of a thermal state with the same mean phonon number (energy).

By plotting the phase portrait, we show the amplitude coherence of the lowest mode in the feedback-induced steady state. We are also interested in the phase coherence, which can be determined by the linewidth of the noise power spectral density $S_{Q_1 Q_1}(\omega)$. The spectral density is obtained by the Fourier transform of autocorrelation function, i.e.,

$$S_{Q_1 Q_1}(\omega) = \int_{-\infty}^{+\infty} d\tau \langle Q_1(t) Q_1(t + \tau) \rangle e^{i\omega \tau},$$

where the overline denotes time average over $t$ and the angle brackets denote ensemble average. Fig. (4) shows the spectral density $S_{Q_1 Q_1}(\omega)$ without and with feedback. Without feedback, the intrinsic linewidth is $\gamma_1/\omega_1 = 1 \times 10^{-2}$, and the coherence time is given by $\omega_1 \tau_{coh} = 2 \omega_1 / \gamma_1 = 200$. With feedback, the linewidth is $\gamma_1/\omega_1 = 7 \times 10^{-4}$. The corresponding coherence time is $\omega_1 \tau_{coh} = 2857$, which is an order of magnitude longer than the intrinsic coherence time.

Discussions and conclusions. – The energy evolution in Fig. (2) is similar to the phonon number evolution in the formation of Fröhlich condensate [32, 34], illustrating the close connection between our model and Fröhlich condensate. This connection can be further seen from the similar formulas of the amplitude equations (4) and the rate equations of phonon numbers in Fröhlich’s model (see Appendix A). In Fröhlich’s model, there are third order terms in the Hamiltonian that couples the environment or auxiliary optical field with pairs of vibration modes [28–30, 34], inducing the energy redistribution among these modes. In our model, these interaction terms are replaced by the nonlinear feedback loop where a nonlinear functional of the collective motion $Q = \sum_j Q_j$ induces
the interactions between different vibration modes. From this perspective, we provide a method to realize Fröhlich-like phonon condensation even in linear systems.

In summary, we have analyzed the prospects for using a feedback loop to realize the condensation of phonon or vibration energy in multimode mechanical systems. We have shown the proposed feedback decreases the effective damping rate of the lowest mode while increases the effective damping rate of other modes, which gives rise to the amplification of the lowest mode and the damping of all the other modes. For the phonon statistics and coherence of the lowest mode, the ring shape in the phase portrait, super-Poissonian phonon distribution and longer coherence time reveal intriguing similarities between the feedback-induced state and phonon laser. These features suggest the nonlinear feedback loop could be used in sensitive sensing and the design of novel monochromatic phonon laser where no two-level gain mediums are needed.

While we have used the continuum elasticity theory to model the mechanical resonator, the proposed feedback loop is applicable to general mechanical systems with incommensurable modes. The potential systems include nanoelectromechanical systems (NEMS), levitated nanoparticles in the optical tweezers, collective motions of cold atoms or ions in potential traps, etc. Further development from current model could replace the harmonic oscillators with more realistic nonlinear oscillators or self-sustained oscillators (e.g., Van der Pol oscillator). There are many interesting phenomena in coupled self-sustained oscillators, such as synchronization [38] and mode competition [39–41]. Our proposed feedback would be used in the control of these phenomena.

In our system, the essential part is the detailed form of the feedback loop, which determines the capability and efficiency of achieving phonon or energy condensation. While the proposed one works well, other feedback strategies might provide similar or even better results. Looking for better feedback strategies, especially with the help of the fast-developing machine learning models, is a promising direction [42]. Besides, the effect of a time delay and phase difference in the feedback loop is an interesting topic and deserves further study.

Appendix A: Comparison of our model and Fröhlich’s model

The rate equations of phonon numbers in Fröhlich’s model are given by

\[ \dot{n}_j = s - \gamma_j(n_j - \bar{n}_{j,\text{th}}) + \chi \sum_i [(n_j + 1)n_i - n_j(1 + n_i)e^{\hbar(\omega_j - \omega_i)/k_B T}], \]

(A1)

where \( s \) is the external pumping, \( \chi \) is the coupling strength of two-phonon process, \( \bar{n}_{j,\text{th}} \) is the thermal phonon number. In the limit of large phonon number \( n_j \gg 1 \), the equations are simplified as

\[ \dot{n}_j = s - \gamma_j(n_j - \bar{n}_{j,\text{th}}) + \chi \sum_i [1 - e^{\hbar(\omega_j - \omega_i)/k_B T}]n_in_j, \]

(A2)

Recently, there is a proposal to realize Fröhlich condensate in optomechanical systems [34]. The modified rate equations of phonon numbers are given by

\[ \dot{n}_j = -\gamma_j \left( n_j - \bar{n}_{j,\text{th}} \right) + \sum_{i \neq j} 4U_{ij}^2 \left[ \Gamma(\omega_i - \omega_j) - \Gamma(\omega_j - \omega_i) \right] n_in_j, \]

(A3)

where \( U_{ij} \) is coefficient and \( \Gamma(\omega) \) is a function of frequency. To compare the amplitude equations (4) with Eq. (A2) and (A3), we need to convert Eq. (4) to the rate equations of \( \langle |a_j(t)|^2 \rangle \). The formal solution of Eq. (4) is given by

\[ a_j(t) = \int_{-\infty}^{t} ds e^{-\gamma_j(t-s) + \sum_i \frac{\gamma_i^2}{2\hbar\omega_i} (\omega_i^2 - \omega_j^2) \int_t^{t'} dt' |a_i(t')|^2} \Xi_j(s). \]

(A4)

From Eq. (A4) we can get the formal solution of \( \langle |a_j(t)|^2 \rangle \),

\[ \langle |a_j(t)|^2 \rangle \approx 2\gamma_j \int_{-\infty}^{t} ds \left( e^{\gamma_j(t-s) + \sum_i \frac{\gamma_i^2}{2\hbar\omega_i} (\omega_i^2 - \omega_j^2) \int_t^{t'} dt' |a_i(t')|^2} \right), \]

(A5)
where we have used Eq. (6) and decorrelation approximation. Hence, the rate equations of $|a_j(t)|^2$ are given by

$$\frac{d(|a_j|^2)}{dt} \approx -\gamma_j(|a_j|^2) - 2 \sum_i g_{ji}^2 \frac{\omega_i^2}{\omega_j^2} \langle |a_j|^2 \rangle \langle |a_i|^2 \rangle$$

(A6)

under decorrelation approximation. In the high temperature limit, the phonon numbers are determined by

$$n_j = \langle |a_j|^2 \rangle k_B T / (2\hbar \omega_j).$$

From Eq. (A6), the rate equations of phonon numbers in our model are then given by

$$\dot{n}_j \approx -\gamma_j (n_j - \bar{n}_{j,th}) + \sum_i \frac{\hbar g_{ji} \omega_i^2}{2k_B T \omega_j} (\omega_i^2 - \omega_j^2) n_i n_j$$

(A7)

Eq. (A7) has the same form as Eq. (A2) and (A3) except the coupling function before $n_i n_j$ is different and there is no external pumping when compared with Eq. (A2).

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