October 2, 2019

ABOUT THE BLOCH-CONNELLY-HENDERSON THEOREM ON THE SIMPLEXWISE LINEAR HOMEOMORPHISMS OF A CONVEX 2-DISK

JEAN CERF

ABSTRACT. This paper gives an improved version of the original proof of Bloch-Connelly-Henderson’s theorem about the space of SL homeomorphisms of a convex 2-disk. A major improvement is related to the main lemma of the original paper.

INTRODUCTION

A simplexwise linear disk, briefly an SL disk, of $\mathbb{R}^2$ is a submanifold with boundary homeomorphic to the 2-disk $D^2$ and triangulated by means of affine simplices. Its boundary is called an SL circle of $\mathbb{R}^2$. An SL circle of $\mathbb{R}^2$ is said to be convex if the disk it bounds is convex, and strictly convex if in addition it is never flat at a vertex. The 1984 paper by Bloch, Connelly and Henderson (that will be quoted as [BCH]) proves the following results.

**Theorem 1.** Let $K$ be an SL disk of $\mathbb{R}^2$ and $f$ an SL embedding of $\partial K$ in $\mathbb{R}^2$. If $f(\partial K)$ is strictly convex, or more generally if $K$ has no $f$-obstructive 1-simplex (see §4) then $f$ can be extended to an SL embedding of $K$ in $\mathbb{R}^2$.

**Theorem 2.** If $K$ is convex and has $m$ interior vertices the space $\mathcal{E}_{\partial K}(K)$ of SL homeomorphisms of $K$ keeping $\partial K$ fixed, endowed with the compact-open topology, is homeomorphic to $\mathbb{R}^{2m}$.

Theorem 2 was stated as the main result in the introduction of [BCH]. Although Theorem 1 was not stated, it is implicitly contained in the final statement ([BCH], Theorem 5). This paper aims to give a simplified and clarified version of the original proofs. It is self-contained, giving detailed proofs even of the parts whose treatment is similar to the original one (§1 and §2).

What makes the proof of these theorems difficult is the fact that $\mathcal{E}_{\partial K}(K)$ is not an invariant of the abstract model of $K$. In other words, two SL disks $K$ and $K'$ may be SL isomorphic while the corresponding spaces $\mathcal{E}_{\partial K}(K)$ and $\mathcal{E}_{\partial K'}(K')$ are not homeomorphic.

Clearly, $\mathcal{E}_{\partial K}(K)$ is a $2m$-dimensional manifold; indeed, ordering the set of the interior vertices of $K$ identifies $\mathcal{E}_{\partial K}(K)$ with an open subspace of $\mathbb{R}^{2m}$. The study of that manifold easily reduces

2010 Mathematics Subject Classification. 57Q15.
to the case where $K$ is strictly convex and simple (i.e. no 1-simplex separates $K$ in two parts). The proof proceeds by induction on the number $p$ of 2-simplices of $K$.

A projective transformation of the plane changes neither convexity nor simplicity of $K$ nor again the topology of $E_{\partial K}(K)$. So, one may suppose that $K$ has been put in reduced form. This means that some 1-simplex $\sigma$ of $\partial K$ is the $[0,1]$ segment of the $x$-axis, the remaining arc of $\partial K$ being the graph of a concave $SL$ function over this segment (see Lemma 1). The 1-simplex $\sigma$ is a facet of some 2-simplex $\tau$ whose third vertex $s$ is interior. Removing $\tau$ from $K$ leads therefore to an $SL$ disk $L$ with $(p-1)$ 2-simplices. The disk $L$ is not convex, but deforming $\partial L$ into a strictly convex $SL$ circle is easy; indeed, keeping all vertices except $s$ fixed, one moves $s$ vertically (i.e. along a parallel to the $y$-axis) to a point $s'$ outside of $K$ and close to $\sigma$. In order to extend this deformation to the whole $L$, the authors made use of what they called the “basic lemma” ([BCH], Lemma 3.1).

By introducing keys and twin-keys, Lemma 3 of the present paper leads to a much simpler version of the “basic lemma” (Lemma 4), which is stated in the strictly convex case as:

If the $SL$ disk $K$ is transverse to the verticals (i.e. any intersection with a parallel of the $y$-axis is connected), then any vertical $SL$ embedding $f : \partial K \to \mathbb{R}^2$ with strictly convex image can be extended to a vertical $SL$ embedding of $K$ (i.e. an embedding moving vertically each vertex).

The extension from $\partial L$ to $L$ completes the inductive step in the proof of Theorem 1. Concerning Theorem 2, it remains to prove that $E_{\partial L}(L)$ is invariant under this extension. The present version of this part of the proof takes advantage of the fact that Lemma 4 works directly in the space of $SL$ embeddings whereas the “basic lemma” works in the space of $SL$ mappings with non-negative volumes. But the general line essentially remains the original one. Let us say that $f$ and $f'$ in $E_{\partial L}(L)$ are equivalent if their composition with the projection of $\mathbb{R}^2$ to the $x$-axis coincide (in other words, if $f' \circ f^{-1}$ is vertical). The proof relies on the fact that this equivalence relation leads to a product decomposition with fibers homeomorphic to $\mathbb{R}^m$.

I cannot close this introduction without expressing my gratitude to François Laudenbach. Without his constant and amical interest and help—including both mathematical and practical points of view—this paper would not exist.

Conventions and notations. The following conventions hold for the whole paper. The coordinates in $\mathbb{R}^2$ are denoted $(x,y)$. The projection to the $x$-axis (resp. $y$-axis) is denoted by $\pi_x$ (resp. $\pi_y$). The lines parallels to $Ox$ (resp. $Oy$) are called horizontals (resp. verticals).

The coordinates in $\mathbb{R}^n \times \mathbb{R}^{n'}$ are denoted by $((x_1, \ldots, x_n), (y_1, \ldots, y_{n'}))$ or by $(X,Y)$. The two projections are respectively denoted $\Pi_x$ and $\Pi_y$.

The vertices of an $SL$ disk being denoted by $s_1, \ldots, s_n$, the space of all its $SL$ mappings to $\mathbb{R}^2$ is identified with $\mathbb{R}^n \times \mathbb{R}^n$ by the homeomorphism

$$f \mapsto ((\pi_x f(s_1), \ldots, \pi_x f(s_n)), (\pi_y f(s_1), \ldots, \pi_y f(s_n))).$$
1. PROJECTIVE REDUCTION OF A CONVEX SL CIRCLE OF $\mathbb{R}^2$

Let $C$ be a convex SL circle of $\mathbb{R}^2$. A maximal line segment in $C$ will be called a natural edge. The SL circle whose 1-simplices are the natural edges of $C$ is called the strictly convex SL circle associated with $C$.

A projective transformation of $\mathbb{R}^2$ maps $C$ to an SL circle if and only if the inverse image of the infinity line does not meet $C$; convexity and strict convexity are preserved.

![Diagram](image)

**Lemma 1.** Let $C$ be a convex SL circle and $\mu$ be a natural edge of $C$. There exists a projective transformation $g$ of $\mathbb{R}^2$ such that $g(C)$ is an SL circle, $g(\mu)$ is $[0,1] \times \{0\}$ and $g(C \setminus \mu) \subset (0,1) \times \mathbb{R}^+$. 

One says that such a $g$ puts $C$ in reduced form. When $C$ is in reduced form and bounds an SL disk $K$, then $K$ is also said to be in reduced form.

**Proof.** The general case easily reduces to the strictly convex one: one replaces $C$ by the associated strictly convex SL circle.

Let $\sigma$ be any 1-simplex of the strictly convex $C$. Let $s_1$ and $s_2$ be the vertices of $\sigma$ and let $s_0$ (resp. $s_3$) be the vertex right before $s_1$ (resp. right after $s_2$). One denotes by $D$ the disk bounded by $C$. The line containing $[s_0,s_1]$ and the line containing $[s_2,s_3]$ are distinct since $D$ is salient at $s_1$ and $s_2$. So, they meet at a unique point $t$ (maybe at infinity) which is not on the line $\Delta$ carrying $\sigma$. The case of Figure 1 left ($t$ and $D$ are on opposite sides of $\Delta$) and the case where $t$ is at infinity, both cases reduce to the case of Figure 1 right (both $t$ and $D$ are at finite distance and lie on the same side of $\Delta$) by means of a projective transformation of $\mathbb{R}^2$ whose inverse image $\Delta'$ of the line at infinity intersects the open 1-simplices $(t,s_1)$ and $(t,s_2)$. In that case, $u$ being any interior point of $\sigma$, the disk $D$, except $s_1$ and $s_2$, is contained in the open band limited by the two parallels to the direction $\vec{ut}$ passing respectively through $s_1$ and $s_2$. The proof is achieved by means of a suitable affine isomorphism of $\mathbb{R}^2$. \hfill $\Box$

2. TRV DISKS: EMBEDDINGS AND MAPPINGS WITH NON-NEGATIVE VOLUMES

Let $K$ be an SL disk of $\mathbb{R}^2$ whose vertices are denoted by $s_i$ ($i = 1, \ldots, n$) and the 2-simplices by $\tau_j$ ($j = 1, \ldots, p$). The space of all mappings from $K$ to $\mathbb{R}^2$ being identified to $\mathbb{R}^n \times \mathbb{R}^n$, the function which maps $f$ to the algebraic volume of $f(\tau_j)$, noted $\text{vol}(f(\tau_j))$ (see Section 7),
identifies to an alternate bilinear form $\ell_j$ on the vector space $\mathbb{R}^n$ taking a positive value at $f$ when $f : K \hookrightarrow \mathbb{R}^2$ is the natural injection. The space of all $SL$ mappings with non-negative volumes identifies with the convex polyhedral cone defined by

$$\ell_j(X, Y) \geq 0 \quad \text{for } j = 1, \ldots, p.$$  \hspace{1cm} (1)$$

The system obtained by adding to (1) the equation $X = X_0$, with $X_0 = \Pi_x(s_1, \ldots, s_n)$ defines the subspace of vertical $SL$ mappings. We will usually write

$$\ell_{j,X_0}(Y) \geq 0 \quad \text{for } j = 1, \ldots, p,$$  \hspace{1cm} (2)$$

where $\ell_{j,X_0}$ is a linear form on $\{X_0\} \times \mathbb{R}^n$ identified to $\mathbb{R}^n$. This linear form has a positive value at the point $\Pi_y(s_1, \ldots, s_n)$.

**Definitions 1.** Let $K$ be an $SL$ disk of $\mathbb{R}^2$. One says that $K$ is *transverse to the verticals* (we will say $TrV$) if any non-empty intersection of $K$ with a vertical line is connected.

In this setting, the *roof* of $K$ is the 1-disk of $\partial K$ made of the 1-simplices $\sigma$ such that no ascending vertical half-line issued from a point of $\sigma$ meets the interior of $K$.

Similarly, one defines a $TrH$ disk (transverse to the horizontals).

**Remark.** Some 1-simplices of the roof of a $TrV$ disk, including the first and the last ones, may be vertical.

**Notations.** Let $K$ be a $TrV$ disk. One denotes by $W(K)$ the space of vertical $SL$ mappings from $K$ to $\mathbb{R}^2$ with non-negative volumes (recall that vertical means that each vertex moves vertically). One denotes by $W_T(K)$ the subspace of mappings which fix the roof $T$ of $K$. One denotes by $V(K)$ (resp. $V_T(K)$) the subspace of $W(K)$ (resp. $W_T(K)$) consisting of the orientation preserving embeddings.

The vertices of $K$ being numbered from 1 to $n$ one assumes that the first $q$ ones belong to $T$.

**Lemma 2.** The space $W(K)$ identifies to the convex polyhedral cone of dimension $n$ of $\mathbb{R}^n$ defined by the system (2). The subspace $W_T(K)$ identifies to the convex polyhedron of dimension $n - q$ of $\mathbb{R}^{n-q}$ defined by (2) and

$$\tan \pi_i Y = \tan \pi_j Y \quad \text{for } i = 1, \ldots, q.$$ \hspace{1cm} (3)$$

The space $V(K)$ is the interior of $W(K)$ and $V_T(K)$ is the interior of $W_T(K)$.

**Proof.** Clearly $V$ (resp. $V_T$) is a non-empty open subset of $W$ (resp. $W_T$). It remains to prove that $V$ is the interior of $W$, in other words that any $f$ which is solution of the system

$$\ell_{j,X_0}(Y) > 0 \quad \text{for } j = 1, \ldots, p$$ \hspace{1cm} (2')$$

is an injective mapping. A sufficient condition for this is that $f|(K \cap V)$ is injective on every vertical line $V$. Now, $K \cap V$ is either one point or, $K$ being $TrV$, is an $SL$ 1-disk. By (2') $f$ is strictly increasing on any 1-simplex of this 1-disk. □
3. TrV disks continued: keys and twin-keys

Definitions 2. Let $K$ be a TrV disk. A 2-simplex $\tau$ of $K$ is said to be a key if one of its faces $\sigma$ is in the roof $T$ and if the link of $\sigma$ (i.e. the vertex of $\tau$ opposite to $\sigma$) is interior to $K$ and projects vertically to the interior of $\sigma$.

A pair $(\tau_\ell, \tau_r)$ of adjacent 2-simplices is said to be a twin-key if $\tau_\ell \cap T$ and $\tau_r \cap T$ are 1-simplices whose common link is interior to $K$ and if in addition $\tau_\ell \cap \tau_r$ is vertical.

Definitions 3. Let $K$ be an SL disk of $\mathbb{R}^2$.

An 1-simplex $\sigma$ of $K$ is said to be spanning if $\partial \sigma = \sigma \cap \partial K$.

If $K$ has no spanning 1-simplex, it is said to be simple.

Remark. Definitions 3 depend only on the abstract model of $K$.

Lemma 3. Let $K$ be a simple TrV disk having more than one 2-simplex. Then $K$ possesses a key or a twin-key.

Remark. A simple SL disk having more than one 2-simplex has at least three 2-simplices.

Proof. One denotes by $s_i$ ($i = 1, \ldots, q$) from the left to the right the vertices of the roof, by $\sigma_i$ ($i = 1, \ldots, q - 1$) the 1-simplex starting at $s_i$, by $\tau_i$ the 2-simplex whose $\sigma_i$ is a facet, and by $s'_i$ the link of $\sigma_i$. Since $K$ is simple, $s'_i \in \partial K$ would imply $\partial \tau_i \subset \partial K$ and hence $K = \tau_i$ which is excluded. So, $s'_i$ is an interior point of $K$ for every $i$.

If $q = 2$, then $\tau_1$ is a key. So, we may suppose $q > 2$. If $\tau_1$ is not a key then $\pi_x(s'_1) \geq \pi_x(s_2)$.

Suppose that some $i \leq q - 2$ satisfies $\pi_x(s'_i) \geq \pi_x(s_{i+1})$; if neither $\sigma_{i+1}$ is a key nor $(\sigma_i, \sigma_{i+1})$ is a twin-key, then $\pi_x(s'_{i+1}) \geq \pi_x(s_{i+2})$. This is impossible if $\sigma_{i+1}$ is vertical descending or if $i = q - 2$. \qed

4. TrV disks completed: the main Lemma

Definition 4. Let $K$ be an SL disk and $f$ be an SL embedding of $\partial K$ in $\mathbb{R}^2$ with a convex image. A spanning 1-simplex $\sigma$ is said to be $f$-obstructive if the $f$-image of one of the arcs that $\partial \sigma$ bounds in $\partial K$ is flat.

Property 1. Let $\sigma$ be a spanning 1-simplex which divides $K$ into $K_1$ and $K_2$. If $\sigma$ is not $f$-obstructive then $f$ canonically defines an SL embedding $f_i: \partial K_i \to \mathbb{R}^2$. It has the property that if $K$ has no $f$-obstructive spanning 1-simplex, then $K_i$ is so for $f_i$ ($i = 1, 2$).

Proof. If $K_i$ has a spanning 1-simplex $\sigma_i$, then the arc of $\partial K_i$ containing $\sigma$ cannot be flattened by $f$ because $f(K)$ is convex. The same is true for the remaining arc because $\sigma_i$ is not $f$-obstructive. \qed
**Property 2.** One supposes that \( K \) is \( TrV \) with roof \( T \) and that \( f \) is vertical. If \( K \) has an \( f \)-obstructive spanning 1-simplex \( \sigma \) with one end point in \( T \backslash \partial T \) then the other end point lies in \( T \).

**Proof.** Suppose \( \sigma \) has one end point in \( \partial K \backslash \partial T \). Then both arcs of \( \partial K \backslash \partial \sigma \) would have non-injective projection on the \( x \)-axis. So, they could not be flattened by any vertical embedding. □

---

**Figure 2.** On the left, a *key*; on the right, a *twin-key*.

The notations concerning keys and twin-keys that we will use in the proof of the next lemma are respectively shown on Figures 2 left and right. On Figure 2 left, \( s \) is the vertical projection of \( s' \) to \( \sigma \).

**Lemma 4 (the main lemma).** Let \( K \) be a \( TrV \) disk and \( v \) be an \( SL \) vertical embedding of \( \partial K \) to \( \mathbb{R}^2 \). If \( v(\partial K) \) is convex and \( K \) has no \( v \)-obstructive 1-simplex then \( v \) extends to an \( SL \) vertical embedding of \( K \) to \( \mathbb{R}^2 \).

**Proof.** One may suppose that \( v \) has the positive orientation. The method is an induction on the number \( p \) of 2-simplices of \( K \). The statement is obvious for \( p = 1 \). So, one assumes that \( p > 1 \) and that the statement is proved for every \( TrV \) disk having a number of 2-simplices less than \( p \).

If \( K \) has a spanning 1-simplex \( \sigma \) which divides \( K \) into \( K_1 \) and \( K_2 \), then by Property 1 both \( K_1 \) and \( K_2 \) satisfy the induction assumption. The \( SL \) vertical embeddings defined by \( v \) on \( \partial K_1 \) and \( \partial K_2 \) coincide along \( \sigma \). Therefore, their vertical extensions to \( K_1 \) and \( K_2 \) define a vertical extension of \( v \) to \( K \). So, we may suppose that \( K \) is simple, and hence, has a key \( \tau \) or a twin-key \( \tau_l \cup \tau_r \) (see Lemma 3 and Figure 2). Recall that in both cases the vertex \( s' \) is interior to \( K \).

One denotes by \( K' \) (with roof \( T' \)) the \( TrV \) disk obtained from \( K \) by removing \( \tau \) (this case is represented on Figure 3) or by removing \( \tau_l \cup \tau_r \) in case of a twin-key.

Let \( v' \) be the \( SL \) vertical embedding \( \partial K' \rightarrow \mathbb{R}^2 \) which coincides with \( v \) on \( \partial K' \cap \partial K \) and verifies \( v'(s') = v(s) \). As \( K \) is simple \( s' \) belongs to \( \partial \sigma' \) for every spanning 1-simplex \( \sigma' \) of \( K' \). If \( \sigma' \) is \( v' \)-obstructive, Property 2 implies that the other vertex \( s'' \) of \( \sigma' \) lies in \( T' \); so, \( \sigma' \) cannot be vertical. Assume for instance that \( s'' \) lies to the left of \( s' \) then \( \sigma' \) is \( v' \)-obstructive if and only if \( v'(s'') \) is aligned with \( v(\sigma) \) in case of a key and with \( v(\sigma_l) \) in case of a twin-key. Denote by \( s'_l \) the leftmost point \( s'' \) when \( \sigma' \) runs over all \( v' \)-obstructive spanning 1-simplices of
are similar definitions of $K$ of the evaluation map at $\sigma$ joining the end points of Corollary. Let removing $K$ obvious. One assumes that the result is proved up to $f$.

Special case: $\pi$ the projection $\pi$ fixing $T$ is a vertical open half-line, infinite in the descending direction, whose upper bound lies above the line segment joining the end points of $T$.

Proof. We may suppose that the end points of $T$ are the points 0 and 1 of the $x$-axis. Denote the projection $\pi_x(u)$ by $u'$. One has $0 < u' < 1$. If $\pi_y(v(u)) < 0$, the image $v(\partial K)$ is strictly convex and hence, there exists no $v$-obstructive spanning 1-simplex. This property remains true if $v(u) = u'$, because a $v$-obstructive spanning 1-simplex should have one end point at $u'$, which is impossible. The fact that the evaluation mapping has an open image completes the proof.

5. Proof of Theorem 1

Recall the statement of Theorem 1: $K$ is an $SL$ disk of $\mathbb{R}^2$, $f : \partial K \to \mathbb{R}^2$ is an $SL$ embedding. If $K$ has no $f$-obstructive spanning 1-simplex then $f$ extends to an $SL$ embedding $K \to \mathbb{R}^2$.

Special case: $f(\partial K)$ is strictly convex.

The proof proceeds by induction of the number $p$ of 2-simplices of $K$. The case $p = 1$ is obvious. One assumes that the result is proved up to $p - 1$ with $p > 1$. If $K$ has a spanning
1-simplex dividing $K$ into $K_1$ and $K_2$, both are strictly convex and $f$ defines an embedding $f_i : \partial K_i \to \mathbb{R}^2$ ($i = 1, 2$). By induction assumption, $f_i$ extends to an $SL$ embedding $K_i \to \mathbb{R}^2$. Gluing these extensions together yields an extension of $f$ to $K$. So, one may suppose that $K$ is simple. Moreover by Lemma 1, one may assume that $f(\partial K)$ is in reduced form.

Let $\rho$ be the 1-simplex of $\partial K$ such that $f(\rho) = [0, 1] \times \{0\}$; one denotes by $\tau$ the 2-simplex of $K$ whose $\rho$ is a face and by $u$ the link of $\rho$. As $K$ is simple, $u$ is an interior vertex of $K$. One denotes by $L$ the $SL$ disk obtained by removing $\tau$ from $K$ (Figure 4).

**Figure 4.**

Let $u'$ be a point of $\mathbb{R}^2$ below $(0, 1) \times \{0\}$. Denote by $f'$ the $SL$ embedding $\partial L \to \mathbb{R}^2$ which coincides with $f$ on $\partial L \cap \partial K$ and maps $u$ to $u'$. The image $f'(L)$ is strictly convex and, by induction, there exists an $SL$ embedding extending $f'$ to $L$. One still denotes this extension by $f'$ and one denotes its image by $L'$ (see Figure 4); set $u'' = \pi_x(u')$ and choose a point $u'''$ on the vertical ascending from $u''$. Let $v$ be the $SL$ embedding $\partial L' \to \mathbb{R}^2$ fixing the roof and sending $u'$ to $u''$. Applying Lemma 4 to $(L', v)$ yields an extension $f''$ of $v$ to $L'$. Denote by $f'''$ the $SL$ mapping $L' \to \mathbb{R}^2$ which maps $u'$ to $u'''$ and fixes the other vertices. If $u'''$ is close enough to $u''$, then $f'''$ is an embedding. By construction, $f''' \circ f'$ has a canonical extension to $K$. $\square$

**General case.**

Let us name a *plateau* a natural edge with more than one 1-simplex. The previous case may be called the “zero plateau” case. From it, one argues by induction on the number of plateaus. One still supposes $f(\partial K)$ in reduced form. One denotes by $s_0, s_1, \ldots, s_r$ the vertices of $f(\partial K)$ on the $x$-axis. Let $A$ be the graph of a strictly convex function on $[0, 1]$ with end points $s_0$ and $s_r$, for instance the one yielded by a suitable circle with center $(1/2, y)$, $y > 0$. Let $s'_i$, $i = 1, \ldots, r - 1$, be the intersection of $A$ with the descending vertical of $s_i$ (see Figure 5). Denote by $v$ the $SL$ embedding $f(\partial K) \to \mathbb{R}^2$ sending $s_i$ to $s'_i$ for $i = 1, \ldots, r - 1$, and fixing the other vertices. Replacing $f$ by $v \circ f$ removes one plateau and does not introduce any obstructive spanning 1-simplex. Lemma 4 implies that $v^{-1}$ extends to an $SL$ embedding $f'' : f'(K) \to \mathbb{R}^2$. Then $f'' \circ f'$ is the desired extension. $\square$
6. Cairns parametrization of continuous families of convex polyhedral disks

Let \((\ell_{j,X}(Y))_{j=1,...,p}\) be a finite family of affine forms in \(\mathbb{R}^n\), where each of them depends continuously on the parameter \(X \in \mathbb{R}^m\). One denotes by \(P_X\) the part of \(\{X\} \times \mathbb{R}^n\) defined by
\[
(4) \quad \ell_{j,X}(Y) \geq 0 \quad \text{for } j = 1, \ldots, p.
\]
The subspace \(P_X\) is a convex closed polyhedron of \(\mathbb{R}^n\). When \(P_X\) is bounded and has a non-empty interior then \(P_X\) is homeomorphic to a disk \(D^n\) (this fact is a part of the above proof of Theorem 1). The following lemma is a parametrized version of this result.

**Lemma 5.** Let \(U\) be an open set in \(\mathbb{R}^m\) such that \(P_X\) is bounded and has a non-empty interior for every \(X \in U\). Then the projection to \(\mathbb{R}^m\) makes \(\bigcup_{X \in U} P_X\) a trivial fiber bundle homeomorphic to \(U \times D^n\).

**Proof.** The projection \(\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m\) has over \(U\) infinitely many sections \(X \mapsto c_X\) where \(c_X\) belongs to the interior of \(P_X\) for every \(X \in U\). For instance, one chooses \(c_X\) equal to the barycenter of \(P_X\). One parametrizes \(P_X\) radially, that is, the center \(O\) of \(D^n\) goes to \(c_X\) and for every \(\alpha \in S^{n-1}\) the radius \([O, \alpha]\) is mapped affinely to the parallel “radius” \([c_X, a_{X,\alpha}]\) of \(P_X\). The point \(a_{X,\alpha}\) is common to some hyperplanes
\[
(4') \quad \ell_{j,X}(Y) = 0
\]
where \(j\) belongs to some subset made of \(p' < p\) elements of \(\{1, \ldots, p\}\), say \(\{1, \ldots, p'\}\). For \((X', \alpha')\) close enough to \((X, \alpha)\) and \(j \leq p'\), let \(a_{X',\alpha',j}\) denote the point where the half-line starting at \(c_{X'}\) in the direction of \(\alpha'\) intersects the hyperplane \(\{\ell_{j,X'}(Y) = 0\}\). The length of the “radius” \([c_{X'}, a_{X',\alpha'}]\) is the smallest distance between \(c_{X'}\) and \(a_{X',\alpha',j}\) when \(j\) runs over \(\{1, \ldots, p'\}\), and hence, it varies continuously. \(\Box\)

7. Proof of Theorem 2

Recall the statement of Theorem 2: \(K\) is a convex \(SL\) disk with \(m\) interior vertices. Then the space \(\mathcal{E}_{\partial K}(K)\) of \(SL\) homeomorphisms of \(K\) keeping \(\partial K\) fixed is homeomorphic to \(\mathbb{R}^{2m}\).

The main part of the proof deals with the strictly convex case; the general case will easily follow. As for Theorem 1, the method is by induction on the number of 2-simplices of \(K\). Assume the result is proved if the number of 2-simplices is less that \(p\). One may suppose that \(K\) is in reduced form and simple (the proof of this claim is left to the reader).
One keeps the notations $\tau$, $L$, $T$ of §5. One denotes by $s_1, \ldots, s_m$ the interior vertices of $K$ so that $s_m$ is the vertex of $\tau$ interior to $K$. One will use the following notations:

- $\mathcal{E}_T(L)$ is the space of $SL$ embeddings with positive orientation, $L \hookrightarrow \mathbb{R}^2$, fixing $T$ and such that $\pi_x f(s_m) \in (0, 1)$;
- $\mathcal{F}_T(L)$ is the space of $SL$ mappings with non-negative volume, $L \to \mathbb{R}^2$, fixing $T$ and such that $\pi_x f(s_m) \in [0, 1]$;
- $\mathcal{E}^y_T(L)$, $\mathcal{E}^{\geq y}_T(L)$, $\mathcal{F}^y_T(L)$ etc. for which the exponent means the part of $\mathbb{R}$ assigned to $\pi_y f(s_m)$.

These spaces are all identified to subspaces of $\mathbb{R}^n \times \mathbb{R}^n$.

**Lemma 6.** 1) For every $y \leq 0$ the following holds:

$$
\Pi_x \mathcal{E}_T(L) = \Pi_x \mathcal{E}^y_T(L) = \Pi_x \mathcal{E}^{\geq y}_T(L) = \Pi_x \mathcal{E}^{\leq y}_T(L).
$$

2) For every $X \in \Pi_x \mathcal{E}_T(L)$ the intersections of $\Pi_x^{-1}(X)$ with $\mathcal{F}_T(L)$, $\mathcal{F}^{\leq y}_T(L)$, $\mathcal{F}^y_T(L)$ are convex polyhedra with respective dimension $m$, $m$, $m - 1$. They are bounded except the first one. Their respective interiors are $\mathcal{E}_T(L)$, $\mathcal{E}^{\leq y}_T(L)$, $\mathcal{E}^y_T(L)$.

**Proof.** The first item easily follows from the corollary of Lemma 4.

For the second item, the result concerning the pair $(\mathcal{F}_T(L), \mathcal{E}_T(L))$ implies the other ones by intersection with half-planes or hyperplanes. Let $X \in \Pi_x \mathcal{E}_T(L)$ and $f \in \Pi_x^{-1}(X) \cap \mathcal{E}_T(L)$. The mapping $f' \mapsto f' \circ f^{-1}$ is an homeomorphism of the pair $(\Pi_x^{-1}(X), \Pi_x^{-1}(X) \cap \mathcal{E}_T(L))$ onto the pair $(\mathcal{W}_T(f(L)), \mathcal{V}_T(f(L)))$ (see §2 for notations). Applying Lemma 4 to the disk $f(L)$ completes the proof of Lemma 6.

We come back to the proof of Theorem 2. For every $y < 0$ and every $f \in \mathcal{E}^y_T(L)$ the disk $f(L)$ is strictly convex and its number of 2-simplices is $p - 1$. On the other hand $\mathcal{E}_{\partial K}(K')$ is obviously homeomorphic to $\mathcal{E}^{\geq 0}_T(L)$. Theorem 2 (in the strictly convex case) is therefore consequence of the following two homeomorphisms:

$$
\begin{align*}
\mathcal{E}^{> 0}_T(L) &\cong \mathcal{E}^{< \frac{1}{2}}_T(L) \times (0, 1), \\
\mathcal{E}^{-\frac{1}{2}}_T(L) &\cong \mathcal{E}_{\partial f(L)}(f(L)) \times (0, 1) \quad \text{for every } f \in \mathcal{E}^{-\frac{1}{2}}_T(L).
\end{align*}
$$

**Proof of (6).** Consider the space of all $SL$ mappings $L \to \mathbb{R}^2$ identified with $\mathbb{R}^n \times \mathbb{R}^n$. (Notice that $L$ has the same vertices as $K$). Let $\tau_j$ be any 2-simplex of $L$ equipped with a numbering of its vertices which orients $\tau_j$ positively. It defines the alternate bilinear form $\text{vol}(f(\tau_j))$ by composing the projection $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^3 \times \mathbb{R}^3$ with the alternate bilinear form on $\mathbb{R}^3$:

$$
\begin{vmatrix}
1 & x_1 & y_1 \\
1 & x_2 & y_2 \\
1 & x_3 & y_3
\end{vmatrix}.
$$

The space $\mathcal{F}_T(L)$ (for which only $m$ vertices are movable) is defined in $\mathbb{R}^m \times \mathbb{R}^m$ by a system of $p - 1$ inequations of the type (4) with the next inequation added:

$$
0 \leq x_m \leq 1.
$$

10
In order to define $F_T^{>0}(L)$ one adds to the system (4) (8)

$$y_m \geq 0. \tag{9}$$

By 2) of Lemma 6, for every $X \in \Pi_T^{-1}(E_T^{>0}(L))$ the set $\Pi_T^{-1}(X) \cap F_T^{>0}(L)$ is a convex polyhedron whose interior $\Pi_T^{-1}(X) \cap E_T^{>0}(L)$ is non-empty and bounded. This implies that $\Pi_T^{-1}(X) \cap F_T^{>0}(L)$ is bounded. Therefore, Lemma 5 yields an homeomorphism

$$\Pi_T^{-1}(X) \cap F_T^{>0}(L) \cong (\Pi_T E_T^{>0}(L)) \times \mathbb{R}^m. \tag{10}$$

The space $F_T^{-\frac{1}{2}}(L)$ is defined by the system (4) (8) (9) modified by replacing $Y$ with $(y_1, \ldots, y_{m-1}, -\frac{1}{2})$ and $y_m > 0$ with $y_m = -\frac{1}{2}$. Applying 2) of Lemma 6 and Lemma 5 (with $n$ replaced by $m$ and $U$ replaced by $\Pi_T E_T^{-\frac{1}{2}}(L)$), one gets

$$E_T^{-\frac{1}{2}}(L) \cong \left(\Pi_T E_T^{-\frac{1}{2}}(L)\right) \times \mathbb{R}^{m-1}. \tag{11}$$

Applying 1) of Lemma 6 for $y = 0$ and $y = -1$ yields

$$\Pi_T E_T^{>0}(L) \cong \Pi_T E_T^{-\frac{1}{2}}(L). \tag{12}$$

Clearly (10), (11), (12) together imply (6). □

**Proof of (7).** Let $f \in E_T^{-\frac{1}{2}}(L)$ with for instance $f(s_m) = (\frac{1}{2}, -\frac{1}{2})$. One has an homeomorphism

$$E_T^{-\frac{1}{2}}(L) \cong E_T^{-\frac{1}{2}}(f(L)). \tag{13}$$

By the chosen representation of the $SL$ mappings (see the convention at the end of the introduction), the system which defines $E_T^{-\frac{1}{2}}(L)$ works as well for $E_T^{-\frac{1}{2}}(f(L))$. On the other hand, $f(L)$ is strictly convex. Therefore, it is transverse to the horizontals. Applying the horizontal version of Lemma 4 yields

$$\Pi_y E_T^{-\frac{1}{2}}(f(L)) \cong \Pi_y E_{\partial f(L)}(f(L)). \tag{14}$$

Henceforth, one will see the system defining $E_T^{-\frac{1}{2}}(f(L))$ as a system with $(m - 1)$ parameters, namely $(y_1, \ldots, y_{m-1})$ on the $\mathbb{R}^m$ space of the $X$ variable. Applying Lemma 5 in this setting, that is, $\Pi_y E_T^{-\frac{1}{2}}(f(L))$ as set of parameters, yields

$$E_T^{-\frac{1}{2}}(f(L)) \cong \left(\Pi_y E_T^{-\frac{1}{2}}(f(L))\right) \times \mathbb{R}^m. \tag{15}$$

By adding the equation $x_m = \frac{1}{2}$ in order to define $E_{\partial f(L)}(f(L))$ one similarly obtains

$$E_{\partial f(L)}(f(L)) \cong \left(\Pi_y E_{\partial f(L)}(f(L))\right) \times \mathbb{R}^{m-1}. \tag{16}$$

Clearly (13), (14), (15), (16) together imply (7). □

**Extension of Theorem 2 to the non-strictly convex case.** As for theorem 1, one argues by induction on the number of plateaus. One assumes $K$ is in reduced form. One again adopts the notations $s_i, s'_i, v$ (see Figure 5). As the extension $f'$ of $v$ to $K$ given by Lemma 4
is vertical the trivial fibration defined by the projection $\Pi_x$ on $E_{\partial K}(K)$ and on $E_{\partial f'(K)}(f'(K))$ have the same basis. On the other hand, both fibers are homeomorphic to $\mathbb{R}^m$. This completes the proof.

**Reference**

[BCH] Ethan D. Bloch, Robert Connelly, David W. Henderson, *The space of simplexwise linear homeomorphisms of a convex 2-disk*. Topology, vol. 23 (2) (1984), 161-175.

15, rue Sarrette, 75014 Paris, France

E-mail address: Jean Cerf <jean.cerf@math.u-psud.fr>