Weak-Coupling, Strong-Coupling and Large-Order Parametrization of the Hypergeometric-Meijer Approximants.

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Abstract

Without Borel or Padé techniques we show that for a divergent series with \( n! \) large-order growth factor, the Hypergeometric series \( \text{k+1}F_{\text{k-1}} \) is a suitable approximant. The point is that Hypergeometric series \( \text{k+1}F_{\text{k-1}} \) can be parametrized to give the full large-order behavior of the series. The divergent \( \text{k+1}F_{\text{k-1}} \) series are then resummed via their representation in terms of the Meijer-G function. The choice of \( \text{k+1}F_{\text{k-1}} \) accelerates the convergence even with only weak-coupling information as input. For more acceleration of the convergence, we employ the strong-coupling and large-order information. The expected acceleration of convergence of this algorithm is tested by obtaining the exact result for the zero-dimensional partition function of the scalar \( \phi^4 \) theory. The algorithm is also applied for the resummation of the ground state energies of \( \phi^4_{0+1} \) and \( i\phi^3_{0+1} \) scalar field theories. We get accurate results for the whole coupling space and the precision is improved systematically in using higher orders. A very precise calculation for the critical exponent \( \nu \) of the \( O(5) \)-symmetric model in three dimensions is also reported. As an example for non-Borel summable series, the exact partition function of the degenerate-vacua \( \phi^4 \) theory has been obtained by summing the corresponding resurgent transseries in using only strong-coupling and large order information.

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In quantum field theory, perturbative calculations are always producing divergent series of zero radius of convergence \[1–6\]. Being divergent, one cannot rely on their predictions because we ignored terms that might contribute more than the ones taken into account. To overcome such problems, resummation techniques are introduced. The most famous one is Borel \[1, 3\] resummation technique and its extension Borel-Padé \[4\]. Recently, a Hypergeometric resummation technique has been introduced and applied to various examples \[7–13\]. Although it results in precise predictions in resumming a divergent series, the algorithm has some limitations \[8, 14\]. As reported in Refs.\[8, 14\], one might not be able to get aimed precision for small coupling values because of the use of Hypergeometric function of finite radius of convergence \(2F1\) to resum a divergent series with zero radius of convergence. This issue has been solved (by the same authors) for \(2F1\) resummation in Ref.\[8\] by brute-force disposition of the branch-cut (make it run from 0 to \(\infty\)). Another resummation algorithm (Borel-Hypergeometric) has employed in Ref.\[7\] too and extended to Meijer-G approximant algorithm in Ref.\[14\]. In fact, the algorithm in Ref.\[14\] is shown to have precise predictions from relatively low orders of perturbation series. In Ref.\[15\], a closely related algorithm has been used where the authors match the Borel-transformed series by a linear combination of asymptotic series of confluent Hypergeometric functions. These algorithms can overcome the problem of precision at small coupling values. For instance, the series expansion of the used Meijer-G functions \[14, 16, 17\] has zero-radius of convergence while for the work in \[15\] they are matching a Borel-transformed series with confluent Hypergeometric functions which are in turn having finite radius of convergence.

Motivated by the work in both Refs.\[7, 14\], we aim in this work to introduce a resummation algorithm that has the same level of simplicity as the first algorithm in Ref.\[7\] (Hypergeometric resummation one) but on the other hand can give precise results for the whole coupling space as well as can employ the large-order and Strong-Coupling information. By simple we mean no usage of Borel or Padé techniques but rather using Hypergeometric functions having a zero-radius of convergence to approximate the given series and then resum them using a representation in terms of Mellin-Barnes integrals. The suggested algorithm will not only stress simplicity but also can guarantee faster convergence as it will be able to accommodate available information from asymptotic large-order and strong coupling data.

The key point to achieve our goal is to consider the set of Hypergeometric series \(pFq\) which have zero-radius of convergence when \(p \geq q + 2\) \[18\]. Note that \(k+1Fk\) approximants
with finite radius of convergence is still suitable in resumming divergent series of finite radius of convergence like the strong coupling expansion series of the Yang-Lee model in Ref. [2]. However, when the series under consideration has a zero radius of convergence, it would be more suitable to use the \( _pF_q \) series with \( p \geq q + 2 \) to approximate the divergent series under investigation. For \( p \geq q + 2 \), the series expansion of \( _pF_q \) is divergent but it can be analytically continued via use of the Meijer-G function [18] where we have the representation:

\[
_pF_q(a_1, ... a_p; b_1, ... b_q; z) = \frac{\prod_{k=1}^{q} \Gamma (b_k)}{\prod_{k=1}^{p} \Gamma (a_k)} G_{p,q+1}^{1,0} \left( \begin{array}{c} 1-a_1, ..., 1-a_p \\ 0, 1-b_1, ..., 1-b_q \end{array} \right| z),
\]

(1)

where the Meijer-G function has the integral form [18]:

\[
G_{m,n}^{p,q}(c_1, ..., c_p; d_1, ..., d_q; z) = \frac{1}{2\pi i} \int_C \frac{\prod_{k=1}^{m} \Gamma (s-c_k+1) \prod_{k=1}^{n} \Gamma (d_k-s)}{\prod_{k=m+1}^{p} \Gamma (-s-c_k) \prod_{k=m+1}^{q} \Gamma (s-d_k+1)} z^s \, ds.
\]

(2)

A suitable choice of the contour \( C \) enables one to get an analytic continuation for \( _pF_q(a_1, ... a_p; b_1, ... b_q; z) \). For instance when \( C \) is taken from \(-i\infty \) to \(+i\infty [18]\), the integral above converges for \( p + q < 2(m + n) \). For reasons that will be clearer later, we are interested in the functions \( k+1F_{k-1}(a_1, ... a_{k+1}; b_1, ..., b_{k-1}; z) \) in our work. So in using Eq.(2) we have \( m = 1, n = k + 1 \) and thus we have \( p + q = 2k - 1 \) which is is smaller than \( 2(m + n) = 2k + 2 \). So the resummation of the series of \( k+1F_{k-1}(a_1, ... a_p; b_1, ..., b_q; z) \) is possible. Although here no Borel transform is used, but the Mellin-Barnes transform defining the G-function might suffer from Stokes phenomena [19], which is then equivalent to Non-Borel summability. There exits algorithms in literature [19] to smooth them out but it is out of the scope of this work. Instead when facing such problems, we will apply the Hypergeometric-Meijer resummation algorithm (introduced in this work) to resum the resurgent transseries [14, 19–22] associated with that problem. The example of the resummation of the non-Borel summable series representing the zero-dimensional partition function of degenerate-vcua \( \phi^4 \) scalar field theory will be given.

We mentioned above that toward the resummation of a divergent series with zero radius of convergence, the functions \( k+1F_{k-1}(a_1, ... a_p; b_1, ..., b_q; z) \) are suitable when the weak-coupling information are available up to some order. It is well known that employing strong-coupling as well as large-order data can accelerate the convergence of a resummation technique [4]. Now we need to show that the set of \( _pF_{p-2}(a_1, ... a_p; b_1, ..., b_{p-2}; z) \) functions are able to incorporate both strong-coupling as well as the large-order data of the perturbation series to be ressummed. To do that, consider the divergent series of the expansion of a physical quantity
\[ Q(g) \] such that:

\[ Q(g) = \sum \beta_n g^n. \quad (3) \]

In fact, for divergent series of renormalization group functions in quantum field theory, the large order behavior of the perturbation series takes the form:

\[ \beta_n \sim \gamma n!(-\sigma)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty. \quad (4) \]

For the resummation of a divergent series that has such kind of large order behavior, we suggest the use of a Hypergeometric function \( _pF_q \) with a constraint on the relation between the number \( p \) of numerator parameters and the number \( q \) of denominator parameters such that it can reproduce the above large order behavior. The Hypergeometric function \( _pF_q \) has the series expansion

\[ _pF_q \left( a_1, \ldots, a_p; b_1, \ldots, b_q; -\sigma g \right) = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n) \cdots \Gamma(a_p+n)}{n! \Gamma(b_1+n) \cdots \Gamma(b_q+n)} (-\sigma g)^n. \quad (5) \]

The Large order behavior in Eq.(4) can be reproduced from this expansion only when \( p = q + 2 \). In fact one can show that for \( p = q + 2 \):

\[ \frac{\Gamma(a_1+n) \cdots \Gamma(a_p+n)}{n! \Gamma(b_1+n) \cdots \Gamma(b_q+n)} (-\sigma)^n \sim \gamma n!(-\sigma)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \quad (6) \]

where

\[ \sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b, \quad (7) \]

and

\[ \gamma = \frac{\prod_{i=1}^{p-2} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(a_i)}. \]

This can be easily achieved using the limit of the \( \Gamma \) function:

\[ \lim_{n \to \infty} \frac{\Gamma(n + \alpha)}{\Gamma(n)} n^\alpha = 1, \]

and the asymptotic form of a ratio of \( \Gamma \) functions [27]:

\[ \frac{\Gamma(n + \alpha)}{\Gamma(n + \beta)} = n^{\alpha - \beta} \left(1 + \frac{(\alpha - \beta)(-1 + \alpha + \beta)}{n} + O\left(\frac{1}{n^2}\right)\right). \quad (8) \]
Accordingly, the suitable candidate to represent the perturbation series in Eq.(3) with the large order behavior in Eq.(4) is the function $pF_{p-2} (a_1, ......a_p; b_1, .......b_{p-2}; -\sigma g)$.

The strong coupling expansion of a physical quantity can also be obtained (for some cases) using methods in Refs.[24, 25]. The $a_i$ parameters in the function $pF_{p-2} (a_1, ......a_p; b_1, .......b_{p-2}; -\sigma g)$ are totally determined from powers in the strong coupling expansion. So one can get the whole set of parameters in $pF_{p-2} (a_1, ......a_p; b_1, .......b_{p-2}; -\sigma g)$ from the available orders of the perturbation series, large-order and strong-coupling information.

In view of the above explanations, let us then list the steps of the Hypergeometric-Meijer resummation algorithm in this work as follows:

1. **Matching the given perturbation series with the series expansion of $pF_{p-2} (a_1, ......a_p; b_1, .......b_{p-2}; \sigma g)$**: In case we have only weak coupling information, all the parameters in $pF_{p-2} (a_1, ......a_p; b_1, .......b_{p-2}; -\sigma g)$ are obtained by matching the expansion of $pF_{p-2}$ with that in Eq.(3). For example, the third order parametrization will lead to the $2F_0 (a_1, a_2; ; -\sigma g)$ where the matching will lead to the result:

$$a_1a_2\sigma = \beta_1$$

$$\frac{1}{2} a_1 (1 + a_1) a_2 (1 + a_2) \sigma^2 = \beta_2$$

$$\frac{1}{6} a_1 (1 + a_1) (2 + a_1) a_2 (1 + a_2) (2 + a_2) \sigma^3 = \beta_3$$

Solving these equations, the three parameters $a_1, a_2$ and $\sigma$ are fully determined. The fourth order Hypergeometric approximant is $3F_1 (a_1, a_2, a_3; b_1; -\sigma g)$ and so on.

2. **Hypergeometric to Meijer-G approximants**: We use the representation of the $pF_{p-2} (a_1, ......a_p; b_1, .......b_{p-2}; -\sigma g)$ in terms of the Meijer-G function in Eq.(1) to get a convergent result out of the divergent series $pF_{p-2}$. For instance, the fourth order Hypergeometric approximation is represented as:

$$3F_1 (a_1, a_2, a_3; b_1; \sigma z) = \frac{\Gamma (b_1)}{\prod_{k=1}^{3} \Gamma (a_k)} G_{3,2}^{1,3} (1-a_1,...,1-a_3 | 0,1-b_1 \sigma z).$$

One can accelerate the convergence of the algorithm by using the large-order information. To illustrate this, consider for simplicity the $2F_0 (a_1, a_2; ; -\sigma g)$ which needs three orders from
the perturbation series represented by the coefficients \( \beta_1, \beta_2 \) and \( \beta_3 \) above. In case we know the large-order information, one can match the parameters known from the large-order behavior of the given perturbation series with the large order form of \( 2F_0(a_1, a_2; -\sigma g) \). Then we get the parameter \( \sigma \) and also we should have the parameters constraint:

\[
\sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b. \tag{11}
\]

Accordingly, the large-order information lowers the third order parametrization of \( 2F_0(a_1, a_2; -\sigma g) \) into just first-order. This means that we need only the equation \( a_1 a_2 \sigma = \beta_1 \) from weak coupling data and the constraint in Eq.(11) to solve for \( a_1 \) and \( a_2 \).

In case we know the strong-coupling information, then the parameters \( a_1, a_2 \) are known while \( \sigma \) is already known from large-order data. In other words, all the parameters in \( 2F_0(a_1, a_2; -\sigma g) \) function has been determined completely without the need of the weak coupling information. In other words, employing the strong coupling information beside the weak coupling accelerates the convergence to the extent that we are not in a need from weak coupling data for the lowest order approximant:

\[
2F_0(a_1, a_2; -\sigma g) = \frac{1}{\Gamma(a_1) \Gamma(a_2)} \mathcal{G}_{2,1}^{1,2} \left( \begin{array}{c} 1-a_1,1-a_2 \end{array} \mid \sigma z \right). \]

For a higher order Hypergeometric function \( _pF_{p-2}(a_1, ......a_p; b, .......b_{p-2}; -\sigma g) \) with their equivalent Meijer-G approximans, one need \( 2p-1 \) terms from the week-coupling information to solve for all unknown parameters \( a_i, b_i \) and \( \sigma \). If the strong coupling information are known, we need only \( p - 1 \) terms from week-coupling information. Knowing weak-coupling, strong-coupling and large-order information, then one needs \( p - 3 \) orders of the perturbation series to determine all the parameters in the \( _pF_{p-2}(a_1, ......a_p; b, .......b_{p-2}; -\sigma g) \) series.

An example of a divergent series with zero radius of convergence that is always used to test the success of a resummation algorithm is the partition function of zero-dimensional \( \phi^4 \) theory. Let us apply the algorithm here to resum the associated divergent perturbation series. We shall apply the algorithm three times for the same problem, one using weak coupling information only, another adding the large-order information and finally by adding strong-coupling information to monitor the effect of each step on the acceleration of convergence to the exact result. To do that, consider the partition function of that model given by:

\[
Z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi \exp \left( \frac{-1^2}{2} \phi^2 - \frac{g}{4!} \phi^4 \right), \tag{12}
\]
while the associated weak coupling perturbation series is of the form:

$$Z(g) = 1 - \frac{g}{8} + \frac{35}{384} g^2 - \frac{385}{3072} g^3 + O(g^4).$$  \hfill (13)

In fact, the lowest order $\, _pF_{p-2}(a_1, \ldots, a_p; b, \ldots, b_{p-1}; \sigma g)$ Hypergeometric approximant is 
$\, _2F_0(a_1, a_2; \sigma g)$ with only three unknown parameters. To determine the parameters $a_1$, $a_2$ and $\sigma$ we use Eq.(9) with the corresponding $\beta_i$ coefficients:

$$a_1 a_2 \sigma = -\frac{1}{8},$$

$$\frac{1}{2} a_1 (1 + a_1) a_2 (1 + a_2) \sigma^2 = \frac{35}{384},$$

$$\frac{1}{6} a_1 (1 + a_1) (2 + a_1) a_2 (1 + a_2) (2 + a_2) \sigma^3 = -\frac{385}{3072}.$$

The solution of these equations are given by: $a_1 = \frac{1}{4}, a_2 = \frac{3}{4}$ and $\sigma = -\frac{2}{3}$. Accordingly, the Hypergeometric-Meijer approximant of $Z(g)$ is

$$Z(g) = _2 F_0 \left( \frac{1}{4}, \frac{3}{4}; \left| -\frac{2}{3} g \right| \right) = \frac{1}{\Gamma \left( \frac{1}{4} \right) \Gamma \left( \frac{3}{4} \right)} G_{1,2}^{1,2} \left( \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ -\frac{2}{3} \end{array} \left| \frac{2}{3} \right| \right).$$

In using the identity

$$\, _2F_0(-n, n+1, x) = \frac{1}{\sqrt{n \pi}} \sqrt{-\frac{1}{x}} \exp \left( -\frac{1}{2x} \right) K_{n+\frac{1}{2}} \left( -\frac{1}{2x} \right),$$

we get the exact result reported in Ref.[14] but it has been obtained there at the fifth order while we obtained it from knowing only the first three terms of the weak coupling expansion.

One can even accelerate the convergence to the exact result by using the large-order information. The large order behavior for the series $Z(g)$ can also be obtained as $n! n^{-1} \left( -\frac{2g}{3} \right)^n (1 + O \left( \frac{1}{n} \right))$ as $n \to \infty$. Accordingly, we have $\sigma = -\frac{2}{3}$. For the parameters $a_1$ and $a_2$, we use one equation from matching the weak coupling expansion with expansion of $\, _2F_0(a_1, a_2; \sigma g)$ to get:

$$\frac{-2}{3} a_1 a_2 = -\frac{1}{8},$$

while the other equation from matching the large-order behavior in Eq.(11):

$$a_1 + a_2 - 2 = -1.$$
Solving these equations one gets: \( a_1 = \frac{3}{4} \) and \( a_2 = \frac{1}{4} \). So in using the large-order data, the exact result has been obtained from first order in perturbation series. This means that the convergence to exact result in this example is now faster as the order in perturbation series needed has been lowered from third order obtained using week coupling information above to first order only.

One can also make the convergence even faster in case we know also the strong-coupling information. The strong coupling expansion of the integral in Eq. (12) can be obtained as:

\[
Z(g) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{3}{4}\right)} g^{-\frac{1}{4}} - \frac{\sqrt{3\pi}}{2\sqrt{\sqrt{3} \pi}} g^{-\frac{3}{4}} + O(g^{-\frac{5}{4}}).
\]  

(14)

When \( a_1 - a_2 \) is not an integer, the asymptotic behavior of \( _2F_0 (a_1, a_2; ; \sigma g) \) for large \( g \) values takes the form:

\[
_2F_0 (a_1, a_2; ; \sigma g) \sim c_1 g^{-a_1} + c_2 g^{-a_2}.
\]

Accordingly, we get \( a_1 = \frac{3}{4} \) and \( a_2 = \frac{1}{4} \). Thus we know \( \sigma \) from large order-data and \( a_1 \) and \( a_2 \) from strong-coupling data. This means that the exact partition function has been obtained from the knowledge of the large-order and strong-coupling information only (no week-coupling data needed). This means that the convergence now accelerated to zero order in perturbation series rather than first order using Large-order and week coupling data and third order in using week-coupling information only.

As another testing example, we apply the algorithm to resum the ground-state energy of the anharmonic oscillator where it is equivalent to the scalar \( \phi^4 \) theory in \( 0 + 1 \) space-time dimensions. The Hamiltonian density for this example is given by:

\[
H = \frac{\pi^2}{2} + \frac{1}{2} (\nabla \phi)^2 + \frac{m}{2} \phi^2 + \frac{g}{4} \phi^4.
\]

(15)

In \( 0 + 1 \) space-time dimensions and for \( m = 1 \), the perturbation series of the ground state energy has the form \( [26] \):

\[
E_0 = \frac{1}{2} + \frac{3}{4} g - \frac{21}{8} g^2 + \frac{333}{16} g^3 - \frac{30885}{128} g^4 + \frac{916731}{256} g^5 + O(g^6).
\]

(16)

The large order behavior is given also by \( -(-3)^n \sqrt{\frac{g}{\pi \pi \pi}} \Gamma \left(n + \frac{1}{2}\right) [26] \). It is clear here that the parameter \( \sigma \) is then given by \( \sigma = 3 \). A scaling operation can lead to the strong coupling expansion from which one can extract \( a_i \) as:
\[ a_1 = \frac{1}{3}, a_2 = \frac{1}{3}, a_3 = 1, a_4 = \frac{5}{3}, \ldots \]

Although the \( _3F_1, _4F_2, _5F_3 \) and \( _6F_4 \) approximants give good results with precision improvement from order to order, we will present here only the results from \( _8F_6 \) approximation (fifth order in using weak-coupling, large-order and strong coupling data). The \( b_i \) parameters in the \( _8F_6 \) functions can then be obtained from matching the coefficients of the series expansion of \( _8F_6 \) term by term with the coefficients in the perturbation series in Eq. (16). The results are shown in Table I. It is very clear that the algorithm gives accurate results from a relatively low order perturbation sires.

**TABLE I:** The fifth order Hypergeometric-Meijer resummation \( _8F_6 \) for the ground state Energy in Eq (16) compared to the exact results from Ref. [29].

| g  | \( _8F_6 \)  | Exact                  |
|----|--------------|------------------------|
| 0.5| 0.6961203131 | 0.6961758208           |
| 1  | 0.8037160010 | 0.8037706512           |
| 50 | 2.500620727  | 2.4997087726           |
| 1000| 6.702747381  | 6.694220850 5          |
| 20000| 18.16565096  | 18.137229073           |

Another example for a divergent series with zero radius of convergence is the ground state energy of the \( \mathcal{PT} \)-symmetric \( i\phi^3 \) theory with Hamiltonian density operator in the form:

\[
H = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 (x) + \frac{i \sqrt{g}}{6} \phi^3 (x).
\]  

(17)

In 0 + 1 space-time dimensions, the ground state energy of this theory has been resummed using different techniques in Ref. [2]. Our results are shown below compared to the 150th order of resummation method in Ref. [2] and also compared to exact results. Again it gives very accurate results using a relatively low order of the perturbation sires as an input.

For a more realistic example to test the Hypergeometric-Meijer resummation algorithm developed in this work, we consider the critical exponent \( \nu \) of \( O(5) \)-symmetric field theory where the \( \varepsilon \)-expansion of \( \nu^{-1} \) is given by [4]:

\[
\nu^{-1} (\varepsilon) = 2 - 0.5384616\varepsilon - 0.17365\varepsilon^2 + 0.09641\varepsilon^3 - 0.262298\varepsilon^4 + 0.622526\varepsilon^5 + O (\varepsilon^6) , \quad (18)
\]
TABLE II: The fifth order Hypergeometric-Meijer resummation $sF_6$ for the ground state energy corresponding to the Hamiltonian in Eq. (17) compared to the 150th order of resummation methods in Ref. [2] and also to exact results.

| $g$ | $sF_6$ | 150th Order in Ref. [2] | Exact |
|-----|--------|------------------------|-------|
| 0.5 | 0.5168918532764233 | 0.516891764253171978 |      |
| 1   | 0.5307847352189364 | 0.530781759304176671 | 0.5308175930417667 |
| 288/49 | 0.6130307602030971 | 0.612738106388984124 | 0.612738106388984125 |

where $\varepsilon = 4 - d$ and $d$ is the dimension of space-time. The large-order parameters for this divergent series are $\sigma = \frac{3}{13}$ and $b = \frac{13}{2}$. Note that (up to the best of our knowledge) the asymptotic behavior of this series for large $\varepsilon$ has not been obtained yet. In three diminutions ($\varepsilon = 1$), the third order (using small-$\varepsilon$ and large-order data) Hypergeometric-Meijer resummation $G_{3,2}^{1,3}(1-a_1; 1-b_1 | \sigma z)$ gives $\nu = 0.747609$ while the fifth order $G_{4,3}^{1,4}(1-a_1; 1-b_1 | \sigma z)$ gives $\nu = 0.766813$ which is in excellent agreement with the prediction of the variational resummation used for $\varepsilon$-expansion ($\nu = 0.766$) in Ref. [28]. We will not go far for such type of calculations as a full discussion of the critical exponents of the $O(N)$ model will appear in another work.

In some cases, the perturbation series is not Borel-summable and the Borel-summation of the series leads to complex ambiguities [20]. An example of such kind of perturbation series is the one associated with the integral representing the zero-dimensional partition function of the degenerate-vacua $\phi^4$ theory [14, 19, 21]:

$$
Z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi \exp \left( \frac{\phi^2}{2} \left( 1 - \sqrt{g}\phi \right)^2 \right),
$$

where up to third order we have:

$$
Z (g) \sim 1 + 6g + 210g^2 + 13860g^3 + O(g^4).
$$

It is clear that this series is not Borel-summable and the Borel sum will result in a complex ambiguity. The reason behind that is the existence of singular points on the contour used in the Borel transform and results in the Stokes phenomena. A similar situation can exist for the Meijer-G resummation since the Meijer-G function is defined through a Mellin-Barnes integrals where Stokes phenomena can exist too [21]. In such case a resurgent transseries can be obtained that can account for non-perturbative contributions for small coupling values associated with the expansion around the non-perurbative saddle point [19, 21].
transseries for the zero-dimensional partition function of the degenerate-vacua \( \phi^4 \) theory has been reported in Ref.[14] as:

\[
Z(g) = \pm i \sqrt{2} \exp \left( -\frac{1}{32g} \left( 1 - 6g + 210g^2 - 13860g^3 + O(g^4) \right) \right) + 2 \left( (1 + 6g + 210g^2 + 13860g^3 + O(g^4)) \right),
\]

where the + sign for \( \text{Im}(g) > 0 \) and – sign for \( \text{Im}(g) < 0 \). This transseries has in fact incorporated the contributions from the Gaussian Saddle point and the instanton saddle point[22]. The two separate series in the transseries above can be resummed using the Hypergeometric-Meijer Resummation followed in this work and the exact result is obtained at the third order where we have:

\[
Z(g) = \frac{2}{\prod_{k=1}^{2} \Gamma(a_k)} G^{1,2}_{2,1} \left( \begin{array} {c} 1-a_1, 1-a_2 \end{array} \mid -32g \right) + \frac{\pm i \sqrt{2} e^{-\frac{\pi}{g}}}{\prod_{k=1}^{2} \Gamma(a_k)} G^{1,2}_{2,1} \left( \begin{array} {c} 1-a_1, 1-a_2 \end{array} \mid 32g \right),
\]

with \( a_1 = \frac{1}{4} \) and \( a_1 = \frac{3}{4} \). Note that this result is real and exact. In Ref.[30], the exact result is listed as (for \( \text{Re}(g) > 0 \)):

\[
Z(g) = e^{-\frac{1}{64} / g} D_{-\frac{1}{2}} \left( -\frac{1}{4\sqrt{g}} \right),
\]

where \( D_{\nu}(z) \) is the parabolic cylinder function. The Meijer G-function approximants above gives for \( Z(2) = 0.778225 \), \( Z(20) = 0.417229 \) and \( z(200) = 0.230612 \) which are the exact numerical values from the parabolic cylinder function above.

One can reduce the order to first order only in using the large-order information and to zero order in using strong-coupling information.

To conclude, we introduced what we can call the Hypergeometric-Meijer algorithm for a resummation of a divergent series with zero radius of convergence. The suggested algorithm is capable of accommodating the large-order and strong coupling information and thus is able to accelerate the convergence to the exact results. In Ref.[14], Héctor Mera et.al followed a Borel-Padé technique that led to a Meijer-G approximant algorithm which has been shown to produce precise results from weak coupling information as input. The algorithm we introduced however avoids Borel or Padé techniques used in Ref.[14] and instead starting from the parametrization of a Hypergeometric function that has the same \( n! \) growth factor characterizing the divergent series and then use the equivalent integral representation of Meijer G function as an approximant to the given perturbation series. In fact,
using weak coupling information in both the Hypergeometric-Meijer G approximant in our work and that in Ref. [14] leads to different parametrization. This can be seen for the exact partition function of zero-dimensional $\phi^4$ theory which has been obtained by a third order parametrization of Hypergeometric-Meijer G approximant in our work while in Ref. [14] the same result has been obtained at the fifth order.

Incorporation of the large-order information has been shown to accelerate the convergence and in adding the strong coupling data into the resummation technique, the convergence is even faster a fact that is traditionally known in resummation techniques [4]. In our work, we have tested this fact by obtaining the exact result of the zero-dimensional partition function of the $\phi^4$ theory from third order parametrization of the Meijer G approximant using only weak coupling information. By adding the large order information, the same exact result has been reached at the first order parametrization while in adding strong coupling data, the exact result is completely parametrized from large-order and strong-coupling data.

The algorithm is also applied to resum the ground state energies of the $\phi^4$ field theory as well as the $\mathcal{PT}$-symmetric $i\phi^3$ field theory in $0 + 1$ space-time dimensions (quantum mechanics. It shows precise predictions although few number of perturbative terms are employed.

Since the type of divergent series stressed in this work shares the same properties of the divergent series representing the renormalization group functions in quantum field theory [4], we applied it to calculate the critical exponent $\nu$ of the $O(5)$-symmetric model where we found that the fifth order parametrization of the Hypergeometric-Meijer approximants produces very precise result.

Since the Meijer-G function is represented by a Mellin-Barnes type of integrals, there is a possibility of Stokes phenomena existence [19]. So one can have Hypergeometric-Meijer non-summability like cases of Non-Borel summability. For such cases one resorts to resummation of resurgent transseries which then kills the complex ambiguity [20]. We applied this to resum the transseries of the partition function of degenerate vacu $\phi^4$ theory where we obtained exact result at the third order parametrization of Hypergeometric-Meijer approximant. After incorporating the large order data, the same result has been obtained using first order
parametrization.

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