Stability criteria and turbulence paradox problem for type II 3D shears

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Received 18 July 2011, in final form 13 March 2012
Published 11 April 2012
Online at stacks.iop.org/JPhysA/45/175501

Abstract
There are two types of 3D shears in channel flows: \((U(y, z), 0, 0)\) and \((U(y), 0, W(y))\). Both are important in organizing the phase-space structures of the channel flows. Stability criteria of type I 3D shears were studied in Li (2011 Q. Appl. Math. 69 379–87). Here we study the stability criteria of type II 3D shears. We also provide more support to the idea of resolution of a turbulence paradox, introduced in Li and Lin (2011 SIAM J. Math. Anal. 43 1923–54) by studying a sequence of type II 3D shears.

PACS numbers: 47.27.Cn, 47.27.nd, 47.10.ad, 47.10.Fg

1. Introduction

Turbulence is a notoriously difficult problem. There are two basic aspects of this problem: (1) discovering an effective description of turbulence [8] and (2) understanding the transition to turbulence and the structure of turbulence. The study on hydrodynamic stability is the initial step toward understanding the transition to turbulence. Only in the area of hydrodynamic stability, there has been impressive progress as elegantly summarized in [19]. Based upon hydrodynamic stability results, further exploration of transition to turbulence in phase space has long been an active area of research, and some progress has been made [16, 17, 14, 23–25].

Studying the phase-space structures of channel flows is an important and emerging area. Most of the existing works in this area are numerical. Like every other dynamical system study in a phase space, non-wandering objects such as fixed points, periodic orbits etc play a fundamental role in organizing the phase-space structure. When the Reynolds number is infinite (i.e. zero viscosity), the corresponding 3D Euler equations have two types of steady shears as fixed points in the phase space:

- Type I 3D shears \((U(y, z), 0, 0)\),
- Type II 3D shears \((U(y), 0, W(y))\),

where the boundaries of the channel are in the \(y\)-direction.

When the Reynolds number is not infinite but large, these shears turn into slowly drifting states under the corresponding 3D Navier–Stokes dynamics. The stability of such slowly
drifting states must take a new meaning. The best way to describe it is via Fenichel fibers in the following setting [3–5]. There is a locally invariant slow manifold \(W\). Local invariance means that orbits can enter or exit the manifold only through its boundary. Slow manifold means that orbits in it move slowly. Our slowly drifting states reside in the locally invariant slow manifold \(W\). The slow manifold \(W\) has a center-unstable manifold \(W^{cu}\) and/or a center-stable manifold \(W^{cs}\). Fenichel fibers exist inside \(W^{cu}\) and/or \(W^{cs}\). The bases of the fibers are points inside \(W\), like type I or II 3D shears (slowly drifting states frozen at any moment). The Fenichel fibers move from fiber to fiber, and are carried by the slow drifting of their bases. The Fenichel fibers inside \(W^{cu}/W^{cs}\) are called unstable/stable Fenichel fibers. The repelling/attracting rate of an unstable/stable fiber is given by the unstable/stable eigenvalue of its base, e.g. the unstable/stable eigenvalue of the Orr–Sommerfeld operator with the potential being a slowly drifting state frozen at any moment. It is based upon the above intuition that we focus our studies on type I and II 3D shears. We believe that the type of instability mentioned above (Fenichel-type instability) is crucial in the transition to turbulence [10, 7].

The 3D Navier–Stokes dynamics also has fixed points like the so-called lower and upper branches [14]. When the Reynolds number approaches infinity, the lower branch approaches one of the type I 3D shears. How to distinguish this particular one from the rest of the type I 3D shears is an interesting question (also posted in a list of problems by Yudovich [26]). A condition satisfied by this particular type I 3D shear was derived in [11]. Again like every other dynamical system study in a phase space, the stability of these 3D shears is crucial in understanding the phase-space structure. The stability criteria for type I 3D shears were studied in [9]. Here we shall study the stability criteria for type II 3D shears. It turns out that the linearized 3D Euler equations can be cast into formally the same form as the classical Rayleigh equation for 2D shears \(\mathcal{U}(y), 0\). But the nature of the stability criteria is fundamentally different from that of the 2D shears. One can ask the question: What is the ‘percentage’ among e.g. all type I 3D shears, that is unstable? The author’s conjecture is as follows.

- The unstable percentage of type II 3D shears > the unstable percentage of type I 3D shears
  > the unstable percentage of 2D shears.

The paper is organized as follows. General necessary conditions for the linear instability of type II 3D shears are derived in section 2, and a general sufficient condition is derived in section 3. In section 4, we focus upon a specific sequence of type II 3D shears for which linear instability can be established. This sequence converges to the linear shear in the \(L^\infty\) norm of velocity. We shall provide more support to the idea of resolution of the turbulence paradox, introduced in [10] by studying this sequence of type II 3D shears.

The turbulence paradox is also called the Sommerfeld paradox [20] which roughly says that the linear shear in the plane Couette flow is linearly stable for all values of the Reynolds number, while in experiments, transition from the linear shear to turbulence occurs under arbitrarily ‘small’ perturbations when the Reynolds number is sufficiently large (specifically, the threshold in perturbation amplitude that must be exceeded to trigger transition scales as \(R^{-\mu}\) where \(1 \leq \mu < 21/4\) and \(R\) is the Reynolds number, depending on the type of perturbation [6]). Attempts on resolving the paradox started as early as the 1930s when Tollmien and Schlichting [21] were mainly focusing upon establishing linear instability. Instead of the Couette flow, they discovered the linear instability for the viscous plane Poiseuille flow. Finer calculation on this linear instability by Lin [13] revealed substantial discrepancy between the calculated and the experimental critical Reynolds number (the infimum Reynolds number that supports linear instability). More rigorously speaking, the turbulence paradox remained open for the plane Poiseuille flow even after the works of Tollmien, Schlichting and Lin, and our idea of the
resolution is still relevant! Historically, the turbulence paradox focused upon the initiator for the transition from the linear shear to turbulence. In terms of the bifurcation routes from the linear shear to turbulence, the problem is much more complicated. The current author believes that there is a general theory on the initiator(s) for transition, there is no general theory on transitional turbulence in-between the linear shear and the fully developed turbulence, and there may be a general theory on the fully developed turbulence. The status of other proposals for the initiator(s) of transition is still at a speculative (conjectural) stage, e.g. non-normal amplification and subcritical bifurcation. Non-normal amplification refers to the polynomial in the time factor of neutral or slowly stable higher order eigenfunctions of a non-normal linear operator (e.g. a Jordan form matrix). The speculation (conjecture) on the possible existence of non-normal amplification in the channel flow was first raised by Orr [15]. But until now, it is still elusive. Nevertheless, there has been constant pursuit along this direction using artificial models, see e.g. [22, 1]. There are also numerical pursuits, see e.g. [18, 19]. A confirmative result can only be obtained from hard analysis (at the very least heuristic analysis). One needs to prove that the corresponding linear Navier–Stokes operator (Orr–Sommerfeld operator) to be non-normal and its neutral or slowly stable higher order eigenfunctions indeed exist. Assuming such hard analysis is successful and the result is positive, it is still difficult to link such non-normal amplification to the development of the coherent structure at the initial stage of transition [2], since the neutral eigenfunctions tend to be streamwise independent! The speculation (conjecture) on the possible existence of subcritical bifurcation in the Couette flow stemmed from the subcritical bifurcation of a 1D map. Unlike the 1D map, a confirmative result on subcritical bifurcation in the Couette flow requires hard analysis (at least heuristic analysis) that is beyond our reach. Until such hard analytical result is achieved, it is hard to establish any affirmative result. Laboratory experiments are incapable of doing such a job. The reason is that the corresponding phase space is infinite dimensional, and the phase-space structure (even in the neighborhood of the linear shear) is extremely difficult to explore. On top of that, norms play a fundamental role [10]. Smallness in one norm does not mean the same in another norm. Numerical simulations are incapable of doing such a job either. The current computer ability is far from what is needed to scan an entire neighborhood of the linear shear in the infinite-dimensional phase space [24].

2. Necessary conditions for instability

The inviscid channel flow is governed by the 3D Euler equations

$$\frac{\partial}{\partial t} u_i + u_k u_{i,k} = -p_{,i}, \quad u_{i,j} = 0,$$  \hfill (2.1)

where \((u_1, u_2, u_3)\) are the three components of the fluid velocity along the \((x, y, z)\) directions and \(p\) is the pressure. The boundary condition is the so-called non-penetrating condition

$$u_2(x, a, z) = 0, \quad u_2(x, b, z) = 0,$$  \hfill (2.2)

where \(a < b\) are the boundary locations of the channel in the \(y\)-direction.

We start with type II 3D steady shear solutions of the 3D Euler equations:

$$u_1 = U(y), \quad u_2 = 0, \quad u_3 = W(y), \quad p = p_0 \text{ (a constant)}.$$

Of particular importance are those profiles which also satisfy the non-slip boundary condition

$$U(a) = \alpha, \quad U(b) = \beta, \quad W(a) = W(b) = 0,$$
where \( \alpha \) and \( \beta \) are the velocities of the two walls of the channel. Such profiles may be the viscous limiting profiles when the viscosity approaches zero. Linearizing the 3D Euler equations by setting

\[
u = e^{ik_1 x + ik_3 z - \sigma t} u(y) + \text{c.c.}, \quad \nu = e^{ik_1 x + ik_3 z - \sigma t} v(y) + \text{c.c.},
\]

\[
u = e^{ik_1 x + ik_3 z - \sigma t} w(y) + \text{c.c.}, \quad \rho \rightarrow \rho_0 + e^{ik_1 x + ik_3 z - \sigma t} \rho(y) + \text{c.c.};
\]

where \( k_1 \) and \( k_3 \) are the wave numbers and \( \sigma \) is a complex constant, we obtain the linearized 3D Euler equations

\[
i(k_1 U + k_3 W - \sigma) u + U' v = -ik_1 \rho,
\]

\[
i(k_1 U + k_3 W - \sigma) v - U' \sigma = -p',
\]

\[
i(k_1 U + k_3 W - \sigma) w + W' v = -ik_3 \rho,
\]

\[
i k_1 u + U' \sigma + i k_3 w = 0.
\]

Two forms of simplified systems can be derived:

\[
v'' - \frac{k_1 U'' + k_3 W''}{k_1 U + k_3 W - \sigma} v = (k_1^2 + k_3^2) v,
\]

with the boundary condition \( v(a) = v(b) = 0 \), and

\[
(k_1 U + k_3 W - \sigma)^2 (k_1 U + k_3 W - \sigma - \sigma') \rho' = (k_1^2 + k_3^2) \rho.
\]

with the boundary condition \( \rho'(a) = \rho'(b) = 0 \).

Note that formally equation (2.7) was in the same form as the classical Rayleigh equation where the 2D shear \( U(y) \) is now replaced by \( k_1 U + k_3 W \). We have the following extension of the classical Rayleigh’s inflection-point theorem.

**Theorem 2.1.** If \((k_1, k_3)\) is an unstable mode, then

\[
(k_1 U'' + k_3 W'')(y_*) = 0 \quad \text{for some } y_* \in (a, b).
\]

**Remark 2.2.** Even though the Rayleigh inflection-point theorem is a special case of the above theorem, the claim in the above theorem in general is fundamentally different from that of Rayleigh. Rayleigh’s claim basically says that if \((U(y), 0)\) is linearly unstable, then \(U(y)\) has an inflection point. Here on the other hand, condition (2.9) can be satisfied by most of type II 3D shears \((U(y), 0, W(y))\). In fact, one can choose \( k_1 = W''(y_*),\) \( k_3 = -U''(y_*) \) for any \( y_* \in (a, b) \), then (2.9) is satisfied. Therefore, in general, (2.9) is a very weak necessary condition for the linear instability of \((U(y), 0, W(y))\). This may indicate that type II 3D shears are more often to be unstable than 2D shears. The author’s conjecture is that type I 3D shears are between type II 3D shears and 2D shears in terms of the probability of being unstable.

**Proof.** Multiplying (2.7) by \( \bar{v} \),

\[
\int_a^b [v''^2 + (k_1^2 + k_3^2) |v|^2] \, dy + \int_a^b \frac{k_1 U'' + k_3 W''}{k_1 U + k_3 W - \sigma} |v|^2 \, dy = 0,
\]

the imaginary part of which is

\[
\sigma \int_a^b \frac{k_1 U'' + k_3 W''}{k_1 U + k_3 W - \sigma} |v|^2 \, dy = 0,
\]

where \( \sigma = \sigma_r + i\sigma_i \). Thus,

\[
(k_1 U'' + k_3 W'')(y_*) = 0, \quad \text{for some } y_* \in (a, b).
\]

We also have the following extension of Fjørtoft’s theorem.
Theorem 2.3. If \((k_1, k_3)\) is an unstable mode, then
\[ G''(y_0) [G(y_0) - G(y_*)] < 0 \] for some \(y_0 \in (a, b)\),
where \(G = k_1 U + k_3 W\) and \(y_*\) is the point at which \(G'' = 0\) is given by theorem 2.1.

Proof. The real part of equation (2.10) is
\[
\int_a^b \frac{G''(G - \sigma_r)}{|G - \sigma_r|^2} |v|^2 \, dy = -\int_a^b \left[ |v'|^2 + (k_1^2 + k_3^2) |v|^2 \right] \, dy.
\]
From (2.11), one has
\[
[\sigma_r - G(y_*)] \int_a^b \frac{G''}{|G - \sigma_r|^2} |v|^2 \, dy = 0,
\]
where \(y_*\) is chosen to be the point given by theorem 2.1, \(G''(y_*) = 0\). The above two equations imply that
\[
\int_a^b \frac{G''(y)[G(y) - G(y_*)]}{|G(y) - \sigma_r|^2} |v|^2 \, dy < 0,
\]
and the theorem is proved. □

Next we show the extension of Howard’s semi-circle theorem.

Theorem 2.4. If \((k_1, k_3)\) is an unstable mode, then its corresponding unstable eigenvalue lies inside the semi-circle in the complex plane
\[
\left( \sigma_r - \frac{M + m}{2} \right)^2 + \sigma_i^2 \leq \left( \frac{M - m}{2} \right)^2,
\]
where again \(\sigma = \sigma_r + i\sigma_i\), \(M = \max_{y \in [a, b]} (k_1 U + k_3 W)\) and \(m = \min_{y \in [a, b]} (k_1 U + k_3 W)\).

Proof. Multiply (2.8) with \(\overline{p}(G - \sigma)^{-2}\), integrate by parts, and split into real and imaginary parts, we obtain that
\[
\int_a^b GQ \, dy = \sigma_r \int_a^b Q \, dy,
\]
\[
\int_a^b G^2 Q \, dy = 2\sigma_r \int_a^b GQ \, dy + (\sigma_r^2 - \sigma_i^2) \int_a^b Q \, dy + (\sigma_i^2 + \sigma_r^2) \int_a^b Q \, dy
\]
by (2.12), where
\[
G = k_1 U + k_3 W,
Q = |G - \sigma|^{-4} |p'|^2 + (k_1^2 + k_3^2) |p|^2.
\]
Let
\[
M = \max_{y \in [a, b]} G, \quad m = \min_{y \in [a, b]} G,
\]
then
\[
\int_a^b (G - m)(M - G)Q \, dy \geq 0.
\]
Expanding this inequality and utilizing (2.12) and (2.13), we arrive at the semi-circle inequality in the theorem. □
3. A sufficient condition for instability

The zeroth mode \((k_1, k_3) = (0, 0)\) is trivially neutrally stable, so our interest is focused upon non-zero modes. Without loss of generality, we assume \(k_1 \neq 0\). Equation (2.7) can be re-written in the form

\[ v'' - \frac{U'' + \frac{k_3}{k_1} W''}{U + \frac{k_3}{k_1} W - \frac{\sigma}{k_1}} v = k_1^2 \left[ 1 + \left( \frac{k_3}{k_1} \right)^2 \right] v. \]

We will keep \( \kappa = k_3/k_1 \) fixed and vary \( k_1 \). Denoting

\[ H = U + \kappa W, \quad c = \frac{\sigma}{k_1}, \quad \alpha = k_1 \sqrt{1 + \kappa^2} \]

we obtain the following equivalent form of (2.7):

\[ v'' - \frac{H''}{H - c} v = \alpha^2 v, \quad (3.1) \]

with the boundary condition \( v(a) = v(b) = 0 \), which is in the same form as the classical Rayleigh equation [10]. Thus, we have the extension of Tollmien’s theorem on a sufficient condition for instability.

**Theorem 3.1.** If \( H'(y) \neq 0 \) for all \( y \in (a, b) \), \( H''(y_*) = 0 \) for some \( y_* \in (a, b) \) and the Sturm–Liouville operator

\[ L v = -v'' + \frac{H''}{H - H(y_*)} v \]

has a negative eigenvalue under the Dirichlet boundary condition \( v(a) = v(b) = 0 \), then equation (3.1) has an unstable eigenvalue (in fact, an unstable curve \( c = c(\alpha) \) for \( \alpha \) in some interval).

4. Turbulence paradox

Turbulence paradox is also called the Sommerfeld paradox. It originated from Sommerfeld’s analysis which concluded that the linear shear in the plane Couette flow is linearly stable for all values of the Reynolds number, whereas in fluid experiments, perturbations of the linear shear often lead to transition to turbulence. Such a paradox is universal among fluid flows, e.g. the pipe Poiseuille flow, plane Poiseuille flow etc. A resolution of this paradox is given in [10]. The main idea of the resolution is to show that even though the linear shear is linearly stable, states arbitrarily close to the linear shear can still be linearly unstable. Here different norms are crucial. It is shown in [10] that the sequence of 2D shears \((U_n(y), 0)\) is linearly unstable for all \( n \) and large enough Reynolds number (including infinity), where

\[ U_n(y) = y + \frac{A}{n} \sin(4n\pi y), \quad \left( \frac{1}{2} < A < \frac{1}{2\pi} \right). \quad (4.1) \]

Here, \( U_n(y) \) approaches the linear shear \( U = y \) in the \( L^\infty \) norm of velocity, but not in the \( L^\infty \) norm of vorticity. Note also that when the Reynolds number is not infinite but large, the shears \((U_n(y), 0)\) are not steady states rather slowly drifting states. The linear instability mentioned above is predicted by the Orr–Sommerfeld operator (linearized 2D Navier–Stokes operator) when the shears \((U_n(y), 0)\) are viewed frozen. Such an instability is in the spirit of Fenichel unstable fibers as mentioned in the introduction [3–5]. Such an instability is also observed numerically [7].

In this paper, we would like to add more support to the idea of resolution mentioned above by considering type II 3D shears \((U(y), 0, W(y))\).
Theorem 4.1. For any $\Lambda > 0$ and any integer $n \geq 1$, type II 3D shears (4.9) are linearly unstable under the 3D Euler dynamics. Specifically, there exists a 2D unstable eigenmode surface $(k_1, k_3; \sigma(k_1, k_3))$ with $\text{Im}(\sigma(k_1, k_3)) > 0$ for the equation (2.7), stemming from neutral modes of the form $(k_1(n), k_3(n); \frac{1}{2}k_1(n))$, where $|k_1(n)|^2 + |k_3(n)|^2 \geq Cn^2$ ($C > 0$ is independent of $n$). The corresponding eigenfunctions are in $C^\infty(0, 1)$. 

The viscous channel flow is governed by the Navier–Stokes equations

\[
\partial_t u_i + u_j u_{i,j} = -p_i + \epsilon u_{i,j j}, \quad u_{i,i} = 0; \tag{4.2}
\]

where again $(u_1, u_2, u_3)$ are the three components of the fluid velocity along the $(x, y, z)$ directions, $p$ is the pressure and $\epsilon = 1/R$ is the inverse of the Reynolds number $R$. The boundary condition is

\[
u_1(x, a, z) = \alpha, \quad u_1(x, b, z) = \beta, \quad u_j(x, a, z) = u_j(x, b, z) = 0, \quad (j = 2, 3), \tag{4.3}
\]

where $a < b$, $\alpha < \beta$. For the viscous channel flow, type II 3D shears mentioned above are no longer fixed points, instead they drift slowly in time (sometimes called quasi-steady solutions): \( (e^{i\epsilon t} U(y), 0, e^{i\epsilon t} W(y)) \).

By ignoring the slow drift and pretending they are still fixed points (or by using artificial body forces to stop the drifting), their unstable eigenvalues will lead to transient nonlinear growths as shown numerically [7]. A better explanation here is to use the theory of Fenichel fibers as discussed in the introduction [3–5, 12]. The slowly drifting 3D shears altogether form a locally invariant slow (center) manifold. The growth rate (or decay rate) of Fenichel fibers based upon this slow manifold can be estimated by ignoring the slow drift.

The corresponding linear Navier–Stokes operator at $(U(y), 0, W(y))$ is given by the following counterpart of (2.3)–(2.6): \( i(k_1 U + k_3 W - \sigma) u + U' v = -i k_3 p + \epsilon [u'' - (k_1^2 + k_3^2) u], \)

\[
i(k_1 U + k_3 W - \sigma) v = -p' + \epsilon [v'' - (k_1^2 + k_3^2) v], \tag{4.5}
\]

\[
i(k_1 U + k_3 W - \sigma) w + W' v = -i k_3 p + \epsilon [w'' - (k_1^2 + k_3^2) w], \tag{4.6}
\]

\[
i k_1 u + u' + i k_3 w = 0. \tag{4.7}
\]

The simplified system as the counterpart of (2.7) is

\[
\frac{\dot{\xi}}{i \alpha} \left[ \frac{d^2}{dy^2} - \alpha^2 \right]^2 v + H' v - (H - c) \left[ \frac{d^2}{dy^2} - \alpha^2 \right] v = 0, \tag{4.8}
\]

with the boundary condition $v = v' = 0$ at $y = a, b$, where \( \dot{\xi} = \epsilon \sqrt{1 + \kappa^2}, \) as before $\kappa = k_3/k_1$ (fixed), $\alpha = k_1 \sqrt{1 + \kappa^2}, c = \sigma/k_1$ and $H = U + \kappa W$. Now both equations (3.1) and (4.8) are formally in the same form as those in [10]. We can specify

\[
a = 0, \quad b = 1, \quad \alpha = 0, \quad \beta = 1;
\]

and consider the following sequence of type II 3D shears:

\[
(U_n(y), 0, W_n(y)) = \left( y, 0, \frac{A}{n} \sin(4n\pi y) \right). \tag{4.9}
\]

The problem of linear instability of the type II 3D shears (4.9) is then cast into the same problem as that of the 2D shears (4.1). By the results of [10], we have the following.

**Theorem 4.1.** For any $\Lambda > 0$ and any integer $n \geq 1$, type II 3D shears (4.9) are linearly unstable under the 3D Euler dynamics. Specifically, there exists a 2D unstable eigenmode surface $(k_1, k_3; \sigma(k_1, k_3))$ with $\text{Im}(\sigma(k_1, k_3)) > 0$ for the equation (2.7), stemming from neutral modes of the form $(k_1(n), k_3(n); \frac{1}{2}k_1(n))$, where $|k_1(n)|^2 + |k_3(n)|^2 \geq Cn^2$ ($C > 0$ is independent of $n$). The corresponding eigenfunctions are in $C^\infty(0, 1)$. 


Proof. Fix $\kappa = k_3 / k_1$ such that
\[ \frac{1}{2} \frac{1}{4\pi} < \kappa A < \frac{1}{4\pi}, \]
then the problem is reduced to that of theorem 3.2 in [10]. □

**Theorem 4.2.** Let $(k_1^0, k_2^0, \sigma^0)$ be a point on the 2D unstable eigenmode surface given by theorem 4.1. Then when $\epsilon$ is sufficiently small, there exists an unstable eigenmode $(k_1^*, k_2^0; \sigma^*)$ with $\Im \{\sigma^*\} > 0$ for equation (4.8) with $H$ given by (4.9). When $\epsilon \to 0^+$, $\sigma^* \to \sigma^0$.

Proof. For the fixed $(k_1^0, k_2^0)$, $\kappa$ has a fixed value; then $\hat{\epsilon}$ and $\epsilon$ are equivalent. The problem is reduced to that of theorem 4.1 in [10]. □

**Remark 4.3.** Note that the sequence of type II 3D shears (4.9) is linearly unstable for all $A > 0$, while the sequence of 2D shears (4.1) is proved linearly unstable for $\frac{1}{2} \frac{1}{4\pi} < A < \frac{1}{4\pi}$. The sequence of type II 3D shears (4.9) also approaches the linear shear $U = y$ in the $L^\infty$ norm of velocity, but not in the $L^\infty$ norm of vorticity.

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