Testing the equality of error distributions from $k$ independent GARCH models

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Abstract. In this paper we study the problem of testing the null hypothesis that errors from $k$ independent parametrically specified generalized autoregressive conditional heteroskedasticity (GARCH) models have the same distribution versus a general alternative. First we establish the asymptotic validity of a class of linear test statistics derived from the $k$ residual-based empirical distribution functions. A distinctive feature is that the asymptotic distribution of the test statistics involves terms depending on the distributions of errors and the parameters of the models, and weight functions providing the flexibility to choose scores for investigating power performance. A Monte Carlo study assesses the asymptotic performance in terms of empirical size and power of the three-sample test based on the Wilcoxon and Van der Waerden score generating functions in finite samples. The results demonstrate that the two proposed tests have overall reasonable size and their power is particularly high when the assumption of Gaussian errors is violated. As an illustrative example, the tests are applied to daily individual stock returns of the New York Stock Exchange data.

Keywords: GARCH model; residuals; empirical process; linear test statistics; asymptotic normality; bootstrap; Wilcoxon test; Van der Waerden test; empirical size; power.

1 Introduction

Analysis of volatility in financial time series is certainly the subject of considerable attention with huge literature having been published. In the seminal papers by Engle (1982) and Bollerslev (1986), generalized autoregressive conditional heteroskedasticity (GARCH) models have been proposed to capture special features of financial volatilities. Since then, numerous variations and extensions of GARCH models have been proposed to possibly explain and model risk and uncertainty in pricing derivative securities, in stochastic modelling of the term structure of interest rates, in applications related to fixed-income portfolio management, in asset pricing studies, and in the riskiness of financial returns which provides a volatility measure that can be used in financial decisions concerning risk analysis. Several excellent surveys of the GARCH methodology in finance are available, such as Bollerslev et al. (1992), Engle (1995), Gouriéroux (1997), Mikosch (2003) and Bauwens et al. (2006).

For time series data, residuals must be taken into account as they typically depend on parameter estimates, and inference based on these residuals, especially various diagnostic checks, is a basic tool in the statistical analysis of linear time series models (see Brockwell and Davis (1994)). By contrast, asymptotic theory for the residuals of nonlinear time series models has been surveyed by Berkes and Horváth (2002). For a GARCH($p,q$) model, Berkes and Horváth (2003) derived the asymptotic distribution of the empirical
process of residuals and showed that, unlike the residuals of autoregressive moving average (ARMA) models, these residuals do not behave in this context like asymptotically independent random variables, and the asymptotic distribution involves a term depending on the parameters of the model.

The classical two-sample problem is one of the central themes of nonparametric testing theory. One of the problems most frequently encountered in statistics is to test the hypothesis of no difference between two independent populations primarily on the basis of samples drawn at random from these two populations. Some of the earliest and most classical tests of nonparametric nature for this problem are Wilcoxon’s test, the Mann and Whitney test, the Mood and Brown test, Lehmann’s test, the Cramér-von Mises test and Van der Waerden’s test. Moreover, the classical limit theorem of normalized two-sample linear test statistics which generated much interest in this context is the celebrated Chernoff–Savage (1958) theorem. It is well known that the theorem is widely used to study the asymptotic power and power efficiency of the above two-sample tests. Further refinements on their conditions of this theorem, extensions and related results, are due to Durbin (1973), Puri and Sen (1993) and references therein.

The natural extension of the two-sample problem is the \(k\)-sample problem, where observations are taken under a variety of different and independent conditions. The nonparametric test procedures which have been developed for this \(k\)-sample problem require no assumptions beyond continuous populations and therefore are applicable under any circumstances. The classical tests in this context are the Kruskal-Wallis \(H\) test, Terpestra’s \(k\)-sample test, the Mood and Brown \(k\)-sample test, Kiefer’s \(k\)-sample analogues of the Kolomogorov-Smirnov test and the Cramér-von Mises \(k\)-sample test. To this end, it is of interest to state that Puri (1964) generalized the situation covered by the Chernoff–Savage (1958) theorem to the \(k\)-sample problem.

If GARCH errors were observable, the problem that we consider here would be the classical \(k\)-sample problem studied by Puri (1964). In our context, we do not observe these errors, but assume that well-behaved estimators of the parameters of the model are available. Hence, our test procedure can be thought of as an extension of the \(k\)-sample problem. More specifically, we are concerned with testing the null hypothesis that errors from \(k\) independent parametrically specified GARCH models have the same distribution versus a general alternative in the spirit of Chernoff and Savage (1958), Puri (1964), and Berkes and Horváth (2003). In contrast with the independent, identically distributed or ARMA setting, this study highlights some interesting features of \(k\) GARCH residual-based test statistics.

Potential applications of the \(k\)-sample test are to be found especially in studies of the behavior of speculative prices, such as stock prices or exchange rates, usually in view of testing market efficiency. One important problem, for example, the stock return of a company is defined as the error from a GARCH model, and the researcher is often interested in comparing the distributions of stock return of companies from \(k\) independent groups. Another related problem in this context is that the researcher may be interested in comparing the distribution functions of standardized real variables like exports or output growth rates with data from \(k\) independent companies. In other areas of financial markets it is often of interest to test whether \(k\) observable variables belong to the same location-scale family, which is also a special case of the test that we study. In all these situations, the usual approach to test for the equality of the distribution functions is to test the equality of just some moments to propose parametric models for the errors and then test whether the parameters estimated are equal. Instead, we propose to compare the entire distribution functions without assuming any parametric form for them.
The objective of this paper is to study the asymptotic behavior of \( k \) GARCH residual-based linear test statistics. The rest of the paper is organized as follows. Section 2 introduces the construction of \( k \) GARCH residual-based empirical distribution functions and proposes linear test statistics pertaining to these residual-based empirical distribution functions. In Section 3, we establish the asymptotic validity of the test. Section 4 reports the results in terms of empirical size and power of a Monte Carlo study for validating the three-sample test based on the Wilcoxon and Van der Waerden score generating functions for finite sample sizes. As an example, the two tests are applied to daily individual stock returns of the New York Stock Exchange data. The proof of the result in Section 3 is provided in Section 5.

## 2 \( k \) GARCH residual-based linear test statistics

In this section, we propose a family of linear test statistics pertaining to empirical processes of residuals in order to test the null hypothesis that errors from \( k \) parametrically specified GARCH models have the same distribution against a general alternative. We shall formulate the \( k \)-sample problem as follows. Let us consider the \( k \) independent random samples generated from the GARCH\((p_j, q_j)\) models given by

\[
\begin{align*}
X_{j,t} &= \sigma_{j,t} \varepsilon_{j,t}, \\
\sigma_{j,t}^2 &= \omega_0 + \sum_{i=1}^{p_j} \alpha_{0j}^i X_{j,t-i}^2 + \sum_{i'=1}^{q_j} \beta_{0j}^{i'} \sigma_{j,t-i'}^2, \quad 1 \leq t \leq n_j, \quad 1 \leq j \leq k,
\end{align*}
\]

(1)

where the \( \varepsilon_{j,t} \) are independent and identically distributed random variables such that \( E(\varepsilon_{j,t}^2) = 1 \), \( \omega_0 > 0 \), \( \alpha_{0j}^i \geq 0 \), \( 1 \leq i \leq p_j \), \( \beta_{0j}^{i'} \geq 0 \), \( 1 \leq i' \leq q_j \), and the \( \varepsilon_{j,t} \) is independent of \( X_{j,s}, s < t \). Henceforth, it is tacitly assumed that \( \alpha_{0j}^{p_j} > 0 \) when \( p_j \geq 1 \), and \( \beta_{0j}^{q_j} > 0 \) when \( q_j \geq 1 \).

In this paper, we are primarily concerned with the \( k \)-sample problem of testing

\[
H_0 : F_1(x) = \cdots = F_k(x) \text{ for all } x
\]

against

\[
H_A : F_i(x) \neq F_j(x) \text{ for at least some } x, \text{ and } i \neq j,
\]

(2)

where \( F_j(\cdot) \) is the distribution function of \( \{\varepsilon_{j,t}\} \), which is assumed to be absolutely continuous with respect to the Lebesgue measure, but unspecified. Henceforth, we assume that \( f_j(x) = F_j'(x) \) exists and is defined over \((-\infty, \infty)\).

We first proceed to describe the quasi-maximum likelihood (QML) estimation of model (1). The vector of parameters is \( \theta_j = (\theta_{1,j}, \ldots, \theta_{j,p_j+q_j+1})^T = (\omega_j, \alpha_{0j}^1, \ldots, \alpha_{0j}^{p_j}, \beta_{0j}^1, \ldots, \beta_{0j}^{q_j})^T \) which belongs to a compact parameter space \( \Theta_j \subset (0, \infty)^j \times [0, \infty)^{p_j+q_j+1} \). The true vector of parameters is unknown and is denoted by \( \theta_{0j} = (\omega_{0j}, \alpha_{0j}^1, \ldots, \alpha_{0j}^{p_j}, \beta_{0j}^1, \ldots, \beta_{0j}^{q_j})^T \).

Suppose that an observed stretch \( X_{j,1}, \ldots, X_{j,n_j} \) from \( \{X_{j,t}\} \) is available. Note that if \( \{\varepsilon_{j,t}\} \) is Gaussian, the quasi-likelihood function with respect to initial values \( X_{j,0}, \ldots, X_{j,1-p_j}, \hat{\sigma}_{j,0}^2, \ldots, \hat{\sigma}_{j,1-q_j}^2 \), is given by

\[
\mathbb{L}_{n_j}(\theta_j) = \sum_{t=1}^{n_j} \frac{1}{\sqrt{2\pi \hat{\sigma}_{j,t}^2}} \exp\left(-\frac{X_{j,t}^2}{2\hat{\sigma}_{j,t}^2}\right),
\]

where the \( \hat{\sigma}_{j,t}^2, t \geq 1 \) are defined recursively by

\[
\hat{\sigma}_{j,t}^2 = \omega_j + \sum_{i=1}^{p_j} \alpha_{0j}^i X_{j,t-i}^2 + \sum_{i'=1}^{q_j} \beta_{0j}^{i'} \hat{\sigma}_{j,t-i'}^2, \quad 1 \leq j \leq k.
\]
As an example, one can choose the initial values as \( X_{j,0}^2 = \cdots = X_{j,1-p_j}^2 = \hat{\sigma}_{j,0}^2 = \cdots = \hat{\sigma}_{j,1-q_j}^2 \equiv \omega_j \) or \( X_{j,0}^2 = \cdots = X_{j,1-p_j}^2 = \hat{\sigma}_{j,0}^2 = \cdots = \hat{\sigma}_{j,1-q_j}^2 = X_{j,1}^2 \).

We can now define the QML estimators of \( \theta_j \) by

\[
\hat{\theta}_{j,n_j} = \arg \max_{\theta_j \in \Theta_j} \Phi_{n_j}(\theta_j) = \arg \min_{\theta_j \in \Theta_j} \tilde{I}_{n_j}(\theta_j),
\]

where \( \tilde{I}_{n_j}(\theta_j) = \frac{1}{n_j} \sum_{t=1}^{n_j} \tilde{l}_t(\theta_j), \quad \tilde{l}_t(\theta_j) = \log \hat{\sigma}_{j,t}^2 + \frac{X_{j,t}^2}{\hat{\sigma}_{j,t}^2}, \quad 1 \leq j \leq k. \)

For \( \hat{\theta}_{j,n_j} \), it is assumed that

\[
||\hat{\theta}_{j,n_j} - \theta_{0j}|| = O_p(n_j^{-1/2}), \quad 1 \leq j \leq k, \tag{3}
\]

where \( || \cdot || \) denotes the Euclidean norm. The validity of (3) is established by Francq and Zakoian (2004) based on the conditions of Assumption 2 given below. Conditions (3) are also typically satisfied by the QML estimators of Straumann and Mikosch (2006).

Henceforth, the empirical residuals are given by

\[
\hat{\varepsilon}_{j,t} = X_{j,t}/\hat{\sigma}_t(\hat{\theta}_{j,n_j}), \quad 1 \leq j \leq k.
\]

For (2), we first collect some basic tools and then describe our approach in the spirit of Chernoff and Savage (1958), and Puri (1964). Write \( N = \sum_{j=1}^{k} n_j \) and \( \lambda_j = n_j/N, \quad 1 \leq j \leq k \). In the following, we assume that the inequalities \( 0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{kN} \leq 1 - \lambda_0 < 1 \) for some \( \lambda_0 \leq 1/k \). Define by

\[
H_N(x) = \sum_{j=1}^{k} \lambda_j F_j(x)
\]

the combined cumulative distribution function. Write \( F_{j,n_j}(x) = n_j^{-1} \sum_{t=1}^{n_j} [I(\varepsilon_{j,t} \leq x)] \) and \( \hat{F}_{j,n_j}(x) = n_j^{-1} \sum_{t=1}^{n_j} [I(\hat{\varepsilon}_{j,t} \leq x)] \), where \( I(\Omega) \) is the indicator function of the event \( \Omega \). Then the empirical distribution function is

\[
\hat{H}_{N}(x) = \sum_{j=1}^{k} \lambda_j F_{j,n_j}(x)
\]

and analogously,

\[
\hat{H}_{N}(x) = \sum_{j=1}^{k} \lambda_j \hat{F}_{j,n_j}(x). \tag{4}
\]

Set \( \hat{B}_{j,n_j}(x) = n_j^{1/2} (\hat{F}_{j,n_j}(x) - F_j(x)) \). Then by virtue of Berkes and Horváth (2003), it follows that

\[
\hat{B}_{j,n_j}(x) = \varepsilon_{j,n_j}(x) + A_j x f_j(x) + \xi_{j,n_j}(x), \tag{5}
\]

where \( \sup_x |\xi_{j,n_j}(x)| = o_p(1) \),

\[
\varepsilon_{j,n_j}(x) = n_j^{-1/2} \sum_{t=1}^{n_j} [I(\varepsilon_{j,t} \leq x) - F_j(x)], \quad A_j = \sum_{l=1}^{p_j+q_j+1} n_j^{1/2} (\hat{\theta}_{j,n_j}^l - \theta_{0j}^l) \tau_{j,l}
\]
with
\[ \tau_{j,1} = E[1/2\hat{\sigma}_i^2(\theta_0)], \quad \tau_{j,l} = E[X_{j,t-l}^2/2\hat{\sigma}_i^2(\theta_0)], \quad 2 \leq l \leq p_j + 1, \]
and \( \tau_{j,p_j+1+l'} = E[\hat{\sigma}_{l'}^2(\theta_0)/2\hat{\sigma}_i^2(\theta_0)] \), \( 1 \leq l' \leq q_j, \ 1 \leq j \leq k \). Hence, by analogy with (5), the asymptotic representation of (4) becomes
\[
\hat{\mathcal{H}}_N(x) = \mathcal{H}_N(x) + \sum_{j=1}^{k} n_j^{-1/2} \lambda_{jN} A_j x f_j(x) + o_p(N^{-1/2}).
\]

Decomposition (6) is basic and plays an important role in the sequel.

Define \( \hat{S}^{(j)}_{iN} = 1 \), if the \( i \)-th smallest of \( N = \sum_{j=1}^{k} n_j \) empirical residuals is from \( \{\hat{\varepsilon}_{j,i}\} \), and otherwise define \( \hat{S}^{(j)}_{iN} = 0, \ 1 \leq i \leq N, \ 1 \leq j \leq k \). Then, for (2), we shall consider a family of linear test statistics of the form
\[
\hat{T}_{jN} = \frac{1}{n_j} \sum_{i=1}^{N} E_{iN} \hat{S}^{(j)}_{iN}, \quad 1 \leq j \leq k,
\]
where the \( E_{iN} \) are given constants called weights or scores. The definition of \( \hat{T}_{jN} \) is the one traditionally used. We shall, however, use the representation given by
\[
\hat{T}_{jN} = \int J\left( \frac{N}{N+1} \hat{\mathcal{H}}_N(x) \right) d\hat{F}_{j,n_j}(x), \quad 1 \leq j \leq k,
\]
where \( J(u), \ 0 < u < 1, \) is a continuous score-generating function. Note that \( E_{iN} = J(i/(N+1)), \ 1 \leq i \leq N \) are functions of the ranks \( i (= 1, \ldots, N) \) and are explicity known. Some typical examples of \( J \) given in Puri and Sen (1993) are as follows:

(i) Wilcoxon’s \( k \)-sample test with \( J(u) = u, \ 0 < u < 1, \)

(ii) Van der Waerden’s \( k \)-sample test with \( J(u) = \Phi^{-1}(u), \ 0 < u < 1, \) where \( \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2} dt, \)

(iii) Mood’s \( k \)-sample test with \( J(u) = (u - \frac{1}{2})^2, \ 0 < u < 1, \)

(iv) Klotz’s normal \( k \)-sample test with \( J(u) = (\Phi^{-1}(u))^2, \ 0 < u < 1. \)

In the following, \( K \) will denote a generic constant taking many different values \( K > 0 \) which may depend on \( J \) but will not depend on \( F_j(\cdot), n_j \) and \( N \) for all \( 1 \leq j \leq k. \)

### 3 Asymptotic properties of \( \hat{T}_{jN} \)

In this section, our primary object is to show that (7) has an asymptotically normal distribution. For this purpose, let \( \{\Delta_{0,j,t}, 1 \leq j \leq k\} \) be the \((p_j + q_j) \times (p_j + q_j)\) matrices defined by
\[
\Delta_{0,j,t} = \begin{pmatrix}
\alpha_{0j} \varepsilon_{j,t}^{-2} & \cdots & \alpha_{0j} \varepsilon_{j,t}^{-2} & \beta_{0j} \varepsilon_{j,t}^{-2} & \cdots & \beta_{0j} \varepsilon_{j,t}^{-2} \\
\alpha_{0j} \varepsilon_{j,t}^{-2} & \cdots & \alpha_{0j} \varepsilon_{j,t}^{-2} & \cdots & \alpha_{0j} \varepsilon_{j,t}^{-2} \\
0_{(p_j-1) \times p_j} & \cdots & 0_{(p_j-1) \times p_j} & \cdots & 0_{(p_j-1) \times p_j} \\
0_{p_j \times (q_j-1)} & \cdots & 0_{p_j \times (q_j-1)} & \cdots & 0_{p_j \times (q_j-1)}
\end{pmatrix}.
\]

Assuming that
\[
E(\log^+ \|\Delta_{0,j,1}\|) \leq E\|\Delta_{0,j,1}\| < \infty,
\]
(8)
the top Lyapunov exponent is defined by $\gamma(\Delta_{0j}) \equiv \inf_{t \geq 1} t^{-1} E(\log \| \Delta_{0j,1} \Delta_{0j,2} \cdots \Delta_{0j,t} \|)$, where $\Delta_{0j} = \{\Delta_{0j,t}, 1 \leq j \leq k\}$. In particular, one can readily check that if $\{\varepsilon_{j,t}\}$ is Gaussian, (8) holds. Bougerol and Picard (1992a,b) showed that if (8) holds, a general GARCH($p_j$, $q_j$) process has a unique non-anticipative strictly stationary solution if and only if $\gamma(\Delta_{0j}) < 0$, $1 \leq j \leq k$.

To establish the asymptotic properties of (7), we impose the following regularity conditions.

**Assumption 1**

(A.1) $J(u)$ is not constant and has a continuous derivative $J'(u)$ on $(0,1)$.

(A.2) $|J(u)| \leq K[u(1-u)]^{-\frac{1}{2}+\delta}$ and $|J'(u)| \leq K[u(1-u)]^{-\frac{3}{2}+\delta}$ for some $\delta > 0$.

(A.3) $xf_j(x)$ and $xf_j'(x)$ are uniformly bounded continuous, and integrable functions on $(-\infty, \infty)$.

(A.4) There exist constants $c_j > 0$ such that $F_j(x) \geq c_j \{xf_j(x)\}$ for all $x > 0$.

A few remarks concerning the necessity of these conditions are in order. Assumptions (A.1) and (A.2) are basic conditions in our context. As noted by Chernoff and Savage (1958), typically (A.2) has two important functions: (i) it limits the growth of the function $J$ and (ii) it supplies certain smoothness properties. Both conditions can be easily verifiable in the preceding examples given by $J$. Assumption (A.3) is basic and necessary for studying residual empirical processes and establishing the convergence result of (7). This condition was also made for empirical processes pertaining to linear regression residuals by Bai (1996). Assumption (A.4) is virtually imposed in dealing with the convergence of higher order terms of (7). Finally, it is worth noting that conditions (A.1)–(A.4) are typically satisfied by several error distributions such as, normal, Student’s $t$, logistic, double exponential, gamma and Laplace.

To validate (3), we require the following additional regularity conditions, which can be found in Francq and Zakoïan (2004).

**Assumption 2**

(B.1) $\theta_{0j} \in \tilde{\Theta}_j$, where $\tilde{\Theta}_j$ denotes the interior of the compact parameter space $\Theta_j$.

(B.2) $\gamma(\Delta_{0j}) < 0$ and $\sum_{\nu=1}^{q_j} \beta_{0j}^{\nu} < 1$ for all $\theta_j \in \Theta_j$.

(B.3) $\varepsilon_{j,t}^2$ has a non-degenerate distribution with $E(\varepsilon_{j,t}^2) = 1$.

(B.4) $\kappa_j \equiv E(\varepsilon_{j,t}^2) < \infty$.

(B.5) If $q_j > 0$, $A_{\theta_{0j}}(z)$ and $B_{\theta_{0j}}(z)$ have no common root, $A_{\theta_{0j}}(1) \neq 0$, and $\alpha_{0j}^{p_j} + \beta_{0j}^{q_j} \neq 0$, where $A_{\theta_{0j}}(z) = \sum_{i=1}^{p_j} \alpha_{i0j}^{\nu} z^i$ and $B_{\theta_{0j}}(z) = 1 - \sum_{\nu=1}^{q_j} \beta_{j}^{\nu} z^\nu$. Conventionally, $A_{\theta_{0j}}(z) = 0$ if $p_j = 0$ and $B_{\theta_{0j}}(z) = 1$ if $q_j = 0$.

We now justify that conditions (B.1)–(B.5) are necessary for the model under consideration. These conditions were essentially made by Francq and Zakoïan (2004) for the validity of (3). We first note that the compactness of $\Theta_j$ is always assumed.

Assumption (B.1) is typically necessary to obtain the asymptotic normality of the QML estimators $\hat{\theta}_{j,n_j}$, $1 \leq j \leq k$. In the case of $\alpha_{0j} \equiv \alpha_{0j}^1 = 0$, the limit distribution
of $\sqrt{n_j}(\hat{\alpha}_j - \alpha_{0j})$ is non-normal over $[0, \infty)$. Assumption (B.2) is a sufficient condition for the stationarity and ergodicity of model (1). This condition implies that the roots of $B_{\theta_j}(z)$ are outside the unit disc. Moreover, if $\gamma(\Delta_{0j}) < 0$, there exists $s > 0$ such that $E(\sigma_{j,t}^{2s}) < \infty$ and $E(X_{j,t}^{2s}) < \infty$. Assumption (B.3) is made for model identification is not restrictive provided $E(\varepsilon_{j,t}^{2}) < \infty$. This moment condition is clearly necessary to establish the asymptotic normality of the Gaussian QML estimator as in Berkes and Horváth (2003). The existence of a fourth-order moment given by (B.4) is a strengthening of (B.3) required for the finiteness of the variance of the score vector $\partial u_{t}(\theta_{0j})/\partial \theta_{j}$. Note also that this condition does not imply the existence of a second-order moment for the observed process $\{X_{j,t}\}$. It is often the case that the existence of the second-order moments is found to be inappropriate for financial applications.

Finally, the assumption that the polynomials whose common roots uniquely identify $\theta_{j}$ was also made by Berkes et. al (2003). This condition is typically satisfied when $p_{j} > 1$ and $q_{j} > 1$. If $p_{j} = 1$ and $\alpha_{0j} \neq 0$, the unique root of $A_{\theta_{0j}}(z) = 0$ and $B_{\theta_{0j}}(z) \neq 0$. If $q_{j} = 1$ and $\beta_{0j} \equiv \beta_{0j} \neq 0$, the unique root of $B_{\theta_{0j}}(z) = 1/\beta_{0j} > 0$, and because $\alpha_{0j} > 0$ produces $A_{\theta_{0j}}(1/\beta_{0j}) \neq 0$. Moreover, it can be noted that (B.5) implies that $\theta_{0j}$ does not necessarily have to belong to the interior of $\Theta_{j}$. This is essentially important when dealing with situations of over-specification. When a GARCH($p_{j}, q_{j}$) is fitted, one can show that an ARCH($p_{j}$) model can be estimated consistently. In a general sense, either $p_{j}$ or $q_{j}$ can be over-specified, but not both of them. Indeed, it is required that $\alpha_{0j} > 0$ for some $i$ when $p_{j} > 0$. If this assumption is dropped, the model solution would simply reduce to an i.i.d. white noise of the form $\sigma_{j,t}^{2} = \sigma_{j}^{2}(1 - \beta_{0j}) + \beta_{0j}\sigma_{j,t-1}^{2}$, where $\sigma_{j}^{2} = \omega_{0j}/(1 - \beta_{0j})$.

In order to state the main result, we shall introduce the following notation:

$$U(\theta_{0j}) = E \left[ \frac{1}{\sigma_{j}^{2}(\theta_{0j})} \frac{\partial \sigma_{j}^{2}(\theta_{0j})}{\partial \theta_{j}} \frac{\partial \sigma_{j}^{2}(\theta_{0j})}{\partial \theta_{j}} \right], \quad u_{t}(\theta_{j}) = \frac{1}{\sigma_{j}^{2}(\theta_{j})} \frac{\partial \sigma_{j}^{2}(\theta_{0j})}{\partial \theta_{j}}, \quad 1 \leq j \leq k.$$

By virtue of (B.4), it is seen that the $i$th element of each $\hat{\theta}_{j,n_{j}}, 1 \leq j \leq k$ admits the asymptotic representation,

$$\hat{\theta}_{j,n_{j}}^{i} - \theta_{0j}^{i} = \frac{1}{n_{j}} \sum_{t=1}^{n_{j}} Z_{t}^{i}(\theta_{j})(\varepsilon_{j,t}^{2} - 1) + o_{p}(n_{j}^{-1/2}),$$

where $Z_{t}^{i}(\theta_{j})$ is the $i$th element of $[U(\theta_{0j})]^{-1}u_{t}(\theta_{j}), 1 \leq i \leq p_{j} + q_{j} + 1$. As shown by Francq and Zakoian (2004), $U(\theta_{0j})$ is positive definite for all $1 \leq j \leq k$. These considerations motivate the following result, whose proof is relegated to Section 5.

**Theorem 1.** Suppose that Assumptions 1 and 2 hold and that, in addition, $\{\hat{\theta}_{j,n_{j}}\}$ is a sequence of QML estimators typically satisfying (3). Then, as $N \to \infty$,

$$N^{1/2}\Sigma_{N}^{-1/2}s_{N} \overset{d}{\rightarrow} \mathcal{N}(0, I_{k}),$$

where $I_{k}$ is the $k \times k$ identity matrix, $\Sigma_{N}$ is the $k \times k$ positive definite dispersion matrix whose entries are given by (15) and (16), and $s_{N} = (\hat{T}_{jN} - \mu_{jN})_{1 \leq j \leq k}$ with $\mu_{jN} = \int J(H_{N})dF_{j}(x)$.

**Remark 1.** If $J(\cdot)$ and $\Sigma_{N}$ were known, an immediate consequence of Theorem 1 is that the quadratic statistic $\mathcal{L}_{N} = N\Sigma_{N}^{-1}s_{N}$ has an approximate $\chi^{2}(k)$ distribution with $k$ degrees of freedom under $H_{0}$ (cf. Theorem 2.8 in Seber (1977)). Unfortunately, the covariance structure of $\Sigma_{N}$, in general, depends on the unspecified distribution function
$F_j(\cdot)$, the unknown parameter vector $\theta_{0j}$ and some expectations. Thus, it is not possible to perform a consistent test based on $\mathcal{L}_N$. Replacing $\Sigma_N$ by a consistent estimator $\hat{\Sigma}_N$ (for details, see Section 4), we can effectively estimate $\mathcal{L}_N$ by $\hat{\mathcal{L}}_N = Ns_N^{-1}\hat{\Sigma}_N^{-1}s_N$. Writing $\tilde{s}_N = N^{1/2}(\hat{T}_N - \mu)_{1 \leq j \leq k}$, we have $\hat{\mathcal{L}}_N / \mathcal{L}_N = \hat{s}_N^T \hat{\Sigma}_N^{-1} \tilde{s}_N / \tilde{s}_N^T \Sigma_N^{-1} \tilde{s}_N$, and using Lemma 1 given in Section 5, it follows that $\text{ch}_k(\hat{\Sigma}_N \Sigma_n^{-1}) \leq (\hat{\mathcal{L}}_N / \mathcal{L}_N) \leq \text{ch}_1(\Sigma_N \Sigma_n^{-1})$, where $\text{ch}_j(\Lambda)$ is the $j$th characteristic root of $\Lambda$. Moreover, by the ergodic theorem we have $\hat{\Sigma}_N \Sigma_n^{-1} \xrightarrow{p} I_k$, which implies $\text{ch}_1(\hat{\Sigma}_N \Sigma_n^{-1}) \xrightarrow{p} 1$ and $\text{ch}_k(\hat{\Sigma}_N \Sigma_n^{-1}) \xrightarrow{p} 1$. Observing that $\hat{\mathcal{L}}_N / \mathcal{L}_N \xrightarrow{d} 1$, and writing $\hat{\mathcal{L}}_N = \mathcal{L}_N \times (\hat{\mathcal{L}}_N / \mathcal{L}_N)$ we may conclude from Slutsky’s theorem that $\hat{\mathcal{L}}_N \xrightarrow{d} \chi^2(k)$ under $H_0$, as was to be proved.

## 4 Simulation and empirical studies

In this section we study the finite sample performance of the proposed test procedure by means of a simple numerical experiment and an empirical example. The ideal way to carry out the former case would be first to generate data from some specific GARCH model, and then estimate a GARCH model either correctly specified or not and check the asymptotic behavior of $\hat{\mathcal{L}}_N$ in terms of empirical size and power.

For simplicity and clarity, we shall consider three-independent random samples generated from the GARCH(1,1) model

$$X_{j,t} = \sigma_j(\theta_j) \varepsilon_{j,t}, \quad \sigma_j^2(\theta_j) = \omega_j + \alpha_j X_{j,t-1}^2 + \beta_j \sigma_{t-1}^2(\theta_j), \quad 1 \leq t \leq n_j, \quad 1 \leq j \leq 3,$$  (9)

where the $\varepsilon_{j,t}$ are independent and identically distributed random variables such that $E(\varepsilon_{j,t}^2) = 1$, $\theta_j = (\omega_j, \alpha_j, \beta_j)^T$, $\omega_j > 0$, $\alpha_j \geq 0$, $\beta_j \geq 0$ are unknown parameters, and the $\varepsilon_{j,t}$ are independent of $X_{j,s}, s < t$. Note that model (9) is the most commonly used in the literature, and enjoy substantial application in the finance setting.

In the following, we are concerned with the three-sample problem of testing

$$H_0 : F_1(\cdot) = F_2(\cdot) = F_3(\cdot) \quad \text{against} \quad H_A : F_1(\cdot) \neq F_2(\cdot) \neq F_3(\cdot),$$

where $F_j(\cdot)$ is an absolutely continuous distribution function of $\{\varepsilon_{j,t}\}$, but unspecified. For testing $H_0$, we propose to use the statistic $\hat{\mathcal{L}}_N = Ns_N^T \hat{\Sigma}_N^{-1} s_N$, which has an approximate $\chi^2(3)$ distribution with 3 degrees of freedom and $0 < r < 1$ is the preassigned level of significance.

We now describe our goodness-of-fit test using a smoothed bootstrap procedure. To this end, note that the asymptotic distribution of $T_{j,N}$ depends crucially on the assumption of continuity and hence bootstrap samples must be generated from continuous distributions. The following steps provide an explicit description of the bootstrap test procedure based on $\hat{\mathcal{L}}_N$:

1. Having observed $X_{j,1}, \ldots, X_{j,n_j}$, obtain an estimate $\hat{\theta}_{j,n_j} = (\hat{\omega}_j, \hat{\alpha}_j, \hat{\beta}_j)^T$ of $\theta_j$ using the QML method described in Section 2.

2. Generate $B$ independent sequences of i.i.d. standard normal random variables with replacement, each of length $n_j + n_0$, where $n_0$ is the length of warm-up sequence to reduce the effect of initial conditions. Then define each of the $B$ sequences by $\varepsilon_{j,-n_0+1}^*, \ldots, \varepsilon_{j,0}^*, \varepsilon_{j,1}^*, \ldots, \varepsilon_{j,n_j}^*$.

3. Generate $B$ bootstrap GARCH(1,1) independent realizations $X_{j,1}^*, \ldots, X_{j,n_j}^*$ with replacement, where the $X_{j,t}^*$ by analogy with (9), satisfy

$$X_{j,t}^* = \sigma_t^*(\hat{\theta}_{j,n_j}) \varepsilon_{j,t}^*, \quad \sigma_t^2(\hat{\theta}_{j,n_j}) = \hat{\omega}_j + \hat{\alpha}_j X_{j,t-1}^2 + \hat{\beta}_j \sigma_{t-1}^2(\hat{\theta}_{j,n_j}).$$
Note that \( \{X_{j,t}^*\} \) is a smooth bootstrap version of the sample \( \{X_{j,t}\} \).

4. For each of the \( B \) samples \( X_{j,1}^*, \ldots, X_{j,n_j}^* \), obtain an estimate \( \hat{\theta}_{j,n_j}^* = (\hat{\omega}_{j}^*, \hat{\alpha}_j^*, \hat{\beta}_j^*)^T \) of \( \hat{\theta}_{j,n_j} \) and construct bootstrap empirical residuals

\[
\hat{\varepsilon}_{j,t}^* = X_{j,t}^*/\sqrt{\hat{\omega}_j^* + \hat{\alpha}_j^* X_{j,t-1}^* + \hat{\beta}_j^* \sigma_{t-1}^2(\hat{\theta}_{j,n_j}^*)}, \quad t = 2, \ldots, n_j, \quad 1 \leq j \leq 3.
\]

5. For the score generating functions \( J(u) = u \) (Wilcoxon) and \( J(u) = \Phi^{-1}(u) \) (Van der Waerden), evaluate the following integral by a rectangular numerical integration with \( m \) terms:

\[
\hat{T}_{jN}^* = \int J\left(\frac{N}{N+1} \hat{H}_{N}^*(x)\right) d\hat{F}_{j,n_j}^*(x), \quad 1 \leq j \leq 3,
\]

where \( \hat{F}_{j,n_j}^*(\cdot) \) denotes the empirical distribution function constructed from \( \{\hat{\varepsilon}_{j,t}^*\} \) and \( \hat{H}_{N}^*(\cdot) \) is the bootstrap version of (6). Then, for each of the \( B \) residuals \( \{\hat{\varepsilon}_{j,t}^*\} \), calculate \( \hat{L}_{N}^* = N s_{N}^2 \hat{\Sigma}_{N}^{-1} s_{N}^* \), where \( s_{N}^* = (\hat{T}_{jN}^* - \mu_{j,1})_{i \leq 3} \) and \( \hat{\Sigma}_{N}^* \) is a resampled version of \( \hat{\Sigma}_{N} \).

6. Finally, repeat step 5 \( B \) times and then reject \( H_0 \) with significance level \( r \) if the \( p\)-value \( \hat{r} = P(\hat{L}_{N}^* > L_{N}^*) < r \), where \( L_{N}^* \) is the \( 1-r \) sample quantile from \( \{\hat{L}_{N,b}^*\}_{b=1}^B \). Here \( B \) is chosen to be a sufficiently large integer.

In what follows we test the null hypothesis that the zero-mean unit-variance errors have the same distribution function at the 5\% significance level. For this purpose, we shall consider two data generating processes (DGPs):

\[
X_{j,t} = \sigma(\theta_j) \varepsilon_{j,t},
\]

DGP 1: \( \sigma^2_t(\theta_j) = 0.1 + 0.1 X_{j,t-1}^2 + 0.1 \sigma^2_{t-1}(\theta_j) \),

DGP 2: \( \sigma^2_t(\theta_j) = 0.5 + 0.4 X_{j,t-1}^2 + 0.4 \sigma^2_{t-1}(\theta_j) \), \quad 1 \leq j \leq 3,

where the \( \varepsilon_{1,t} \) are i.i.d. random variables with an \( \mathcal{N}(0, 1) \) distribution, the \( \varepsilon_{2,t} \) are i.i.d. random variables with mixture distribution \((1 - \varphi)\mathcal{N}(0, 1) + \varphi \mathcal{N}(2, 1), 0 \leq \varphi \leq 1 \) and the \( \varepsilon_{3,t} \) are i.i.d. random variables with Student’s \( t \) distribution having \( \varphi^{-1} \) degrees of freedom. The values of \( \varphi \) that we consider are \( \varphi \in \{0, 1/9, 1/5, 1/3\} \). Note that if \( \varphi = 0 \), the errors \( \varepsilon_{2,t} \) and \( \varepsilon_{3,t} \) are generated from a standard normal distribution. The choice of \( \varphi \) values, in principle, indicates that the last two error processes have a leptokurtic distribution whose tails are heavier than the ones of a normal distribution. Observe that \( H_0 \) holds true if and only if \( \varphi = 0 \). We also notice that the parameter \( \varphi \) represents the departure from \( \mathcal{N}(0, 1) \) in the sense that the larger the value of \( \varphi \), the larger the deviation from the null model. Here, the distributions of interest are re-scaled such that they have the required zero mean and unit variance.

We generate repeated trials of lengths \( n_1 = n_2 = n_3 \in \{100, 300, 500\} \) from DGP1 and DGP2, and compute the empirical size and power of the 3-sample bootstrap Wilcoxon (W) and Van der Waerden (VdW) tests at the 5\% nominal level based on the steps 1–6 for each trial. The number of Monte Carlo trials is 10000 with \( B = 1000 \) bootstrap replications each. Each configuration of parameters was estimated by the QML method.

Table 1 reports the empirical proportion of rejections of \( H_0 \) for the W and VdW tests based on the corresponding asymptotic \( \chi^2_{0.05}(3) \) distribution. For the sake of brevity, we do not include the results for Mood’s and Klotz’s normal tests, which are quite similar.
From Table 1, it can be seen that the values are stable with respect to the choice of sample sizes and parameters. We noted in our Theorem 1 that the empirical rate of convergence of the normalized random variable $\Sigma_{N}^{-1/2} s_N$ to the $k$-variate normal distribution $\mathcal{N}(0, I_k)$ depends on the parameters of the GARCH process. The smaller the parameters $\alpha_j$ and $\beta_j$, the faster the convergence. This is intuitively clear because larger values of $\alpha_j$ and $\beta_j$ imply not only more dependence, but also heavier tails of the error distributions (cf. Basrak et al. (2002)). More specifically, we observe that the power of the tests for the DGP 1 is generally higher than that for the DGP 2 with respect to the sample sizes.

Overall, the two bootstrap-based statistics perform reasonably well in terms of empirical size and power, and none of them provides an obvious answer to the question of what test statistic should be preferred. Therefore, in practice we cannot know in advance which of them would lead to a more powerful test. Moreover, as the sample sizes and $\varphi$ increase, the size of both the tests converge to the theoretical level and their powers generally increase. When the error distributions are sufficiently different, the power of the tests is adequate for three different choices of the sample size. It is worth noting that the highest power of such tests is attained at $\varphi = 1/3$.

Table 1: Proportion of rejections of $H_0$ for the bootstrap W and VdW tests at $r = 5$

| $\varphi$ | $n_1 = n_2 = n_3 = 100$ | $n_1 = n_2 = n_3 = 300$ | $n_1 = n_2 = n_3 = 500$ |
|-----------|--------------------------|--------------------------|--------------------------|
|           | W | VdW | W | VdW | W | VdW | W | VdW |
| 0         | 0.045 | 0.046 | 0.052 | 0.053 | 0.049 | 0.051 |
| 1/9       | 0.122 | 0.123 | 0.171 | 0.169 | 0.214 | 0.221 |
| 1/5       | 0.321 | 0.331 | 0.412 | 0.401 | 0.785 | 0.788 |
| 1/3       | 0.821 | 0.818 | 0.861 | 0.871 | 0.913 | 0.912 |

DGP 2

| $\varphi$ | $n_1 = n_2 = n_3 = 100$ | $n_1 = n_2 = n_3 = 300$ | $n_1 = n_2 = n_3 = 500$ |
|-----------|--------------------------|--------------------------|--------------------------|
|           | W | VdW | W | VdW | W | VdW | W | VdW |
| 0         | 0.041 | 0.042 | 0.056 | 0.055 | 0.045 | 0.046 |
| 1/9       | 0.102 | 0.104 | 0.151 | 0.148 | 0.193 | 0.195 |
| 1/5       | 0.313 | 0.314 | 0.393 | 0.401 | 0.712 | 0.717 |
| 1/3       | 0.801 | 0.796 | 0.815 | 0.817 | 0.897 | 0.894 |

We conclude this section with a simple empirical example based on daily data. For this purpose, we apply the bootstrap W and VdW tests to the series of residuals obtained from the estimation of a GARCH(1,1) on series of daily individual stock returns for the three companies (i) AMOCO, (ii) FORD and (iii) HP listed on New York Stock Exchange. Each series starts from July 3, 1962, to December 31, 1991 with 7420 observations. In our analysis, however, we consider the last 2000 data points from each series from February 2, 1984, to December 31, 1991.

Table 2 displays the empirical proportion of rejections of $H_0$ for the W and VdW tests at the 5% significance level. The result shows that the tests have similar desirable size and power at the 5% level. To this end, the results provide enough evidence in support of the simulation results. For all the three considered series, the hypothesis of normality of the error distributions is rejected at the 5% level. The bootstrap tests we studied in this paper have reasonable size and can detect a misspecified probability distribution of the errors in a GARCH model with high probability.
| \( \varphi \) | 0  | 1/9  | 1/5  | 1/3  |
|---|---|---|---|---|
| W | 0.050 | 0.616 | 0.981 | 1.000 |
| VdW | 0.049 | 0.618 | 0.978 | 1.000 |

5 Proof and Auxiliary Lemma

In this section we provide Lemma 1 and the proof of Theorem 1. Lemma 1 is useful for ordering characteristic roots of a product of two matrices (see e.g., Sen and Singer (1993)).

**Lemma 1** (Courant). Let \( U \) and \( V \) be positive semi-definite matrices. Suppose that \( V \) is nonsingular and that \( x = (x_1, \ldots, x_k)^T \in (-\infty, \infty)^k \) is a characteristic vector. Then if the product \( UV^{-1} \) is well defined, and if \( \upsilon_i \) denotes the \( i \)th characteristic root of \( UV^{-1} \) for \( i = 1, \ldots, k \), we have

\[
ch_k(UV^{-1}) = \upsilon_k = \inf_{x} \frac{x^T Ux}{x^T Vx} \leq \sup_{x} \frac{x^T Ux}{x^T Vx} = \upsilon_1 = ch_1(UV^{-1}).
\]

Next we provide the proof of Theorem 1.

**Proof of Theorem 1.** Write \( d\hat{F}_{j,n_j} = d(\hat{F}_{j,n_j} - F_j + F_j) \) and

\[
J\left(\frac{N}{N+1}\hat{H}_N\right) = J(H_N) + (\hat{H}_N - H_N)J'(H_N) - \frac{\hat{H}_N}{N+1}J'(H_N)
+ \left[J\left(\frac{N}{N+1}\hat{H}_N\right) - J(H_N) - \left(\frac{N}{N+1}\hat{H}_N - H_N\right)J'(H_N)\right].
\]

Then the decomposition of (7) is given by

\[
\hat{T}_j = \mu_j + B_{1N,j} + B_{2N,j} + C_{1N,j} + C_{2N,j} + C_{3N,j},
\]

where

\[
B_{1N,j} = \int J(H_N)d(\hat{F}_{j,n_j} - F_j)(x),
\]

\[
B_{2N,j} = \int (\hat{H}_N - H_N)J'(H_N)dF_j(x),
\]

\[
C_{1N,j} = -\frac{1}{N+1}\int \hat{H}_N J'(H_N)d\hat{F}_{j,n_j}(x),
\]

\[
C_{2N,j} = \int (\hat{H}_N - H_N)J'(H_N)d(\hat{F}_{j,n_j} - F_j)(x),
\]

\[
C_{3N,j} = \int \left[J\left(\frac{N}{N+1}\hat{H}_N\right) - J(H_N)
- \left(\frac{N}{N+1}\hat{H}_N - H_N\right)J'(H_N)\right]d\hat{F}_{j,n_j}(x).
\]

To prove this theorem, it is necessary to show that (i) the vector \( N^{1/2}(B_{1N,j} + B_{2N,j})_{1 \leq j \leq k} \) when properly normalized has a limiting Gaussian distribution, and (ii) the \( C_\ast \) terms are uniformly of higher order. For (i), we observe that the difference
\[ N^{1/2}(\hat{T}_{jN} - \mu_{N,j})_{1 \leq j \leq k} - N^{1/2}(B_{1N,j} + B_{2N,j})_{1 \leq j \leq k} \text{ tends to zero in probability and so the vectors } N^{1/2}(\hat{T}_{jN} - \mu_{N,j})_{1 \leq j \leq k} \text{ and } N^{1/2}(B_{1N,j} + B_{2N,j})_{1 \leq j \leq k} \text{ possess the same limiting distribution.

Let us now proceed to show the statement (i). From (5), it is easily seen that

\[ B_{1N,j} = \int J(H_N)d(F_{j,n_j} - F_j)(x) + n_j^{-1/2}A_j \int J(H_N)d(xf_j(x)) + o_p(1). \]  

Integrating \( B_{2N,j} \) by parts, and using (6) and (10), it follows that

\[ N^{1/2}(B_{1N,j} + B_{2N,j}) = N^{1/2}\left(-\sum_{i=1 \atop i \neq j}^k \lambda_i N \int B_j(x)d(F_{i,n_i} - F_i)(x)\right. \]

\[ + \int (J(H_N) - \lambda_j B_j(x))d(F_{j,n_j} - F_j)(x) \]

\[ -n_j^{-1/2}A_j \sum_{i=1 \atop i \neq j}^k \lambda_i N \int xf_j(x)J'(H_N)dF_i(x) \]

\[ + \sum_{i=1 \atop i \neq j}^k \lambda_i N n_i^{-1/2}A_i \int xf_i(x)J'(H_N)dF_j(x) + o_p(1) \]

\[ = a_{jN} + b_{jN} + c_{jN} + d_{jN} + o_p(1) \text{ (say),} \]  

where \( B_j(x) = \int_{x_0}^x J'(H_N)dF_j(y) \) with \( x_0 \) determined somewhat arbitrarily, say by \( H_N(x_0) = 1/2. \)

In what follows, we shall first evaluate the asymptotic variance of (11) and then the asymptotic covariance to construct the dispersion matrix \( \Sigma_N. \) For this purpose, first consider \( a_{jN} \) and write it as

\[ -N^{1/2}\lambda_i N \int B_j(x)d(F_{i,n_i} - F_i)(x) \]

\[ = N^{1/2}\lambda_i N \int \int (F_{i,n_i} - F_i)J'(H_N)dF_j(x) \]

Then the mean is zero and the variance is

\[ E\left(N^{1/2}\lambda_i N \int (F_{i,n_i} - F_i)J'(H_N)dF_j(x)\right)^2 \]

\[ = E\left(N\lambda_i^2 N \int \int (F_{i,n_i}(x) - F_i(x))(F^{(i)}_{n_i}(y) - F_i(y))J'(H_N(x))J'(H_N(y))dF_j(x)dF_j(y)\right)^2 \]

\[ = 2\lambda_i N \int \int F_i(x)(1 - F_i(y))J'(H_N(x))J'(H_N(y))dF_j(x)dF_j(y). \]

Note that the application of Fubini’s theorem permits the interchange of integral and expectation.

By a similar argument, the variance of

\[ b_{jN} = -N^{1/2} \sum_{i=1 \atop i \neq j}^k \lambda_i N \int (F_{i,n_i} - F_i)J'(H_N)dF_j(x) \]
is given by
\[
\frac{2}{\lambda_{jN}} \sum_{i=1 \atop i \neq j}^{k} \int_{x<y} F_j(x)(1 - F_j(y))J'(H_N(x))J'(H_N(y))dF_i(x)dF_i(y)
\]
\[+ \frac{2}{\lambda_{jN}} \sum_{i,i'=1 \atop i \neq i',i \neq j,i' \neq j}^{k} \lambda_{iN} \lambda_{i'N} \int_{x<y} F_j(x)(1 - F_j(y))J'(H_N(x))J'(H_N(y))dF_i(x)dF_i'(y)
\]
\[+ \frac{2}{\lambda_{jN}} \sum_{i,i'=1 \atop i \neq i',i \neq j,i' \neq j}^{k} \lambda_{iN} \lambda_{i'N} \int_{y<x} F_j(y)(1 - F_j(x))J'(H_N(x))J'(H_N(y))dF_i(x)dF_i'(y).
\]
Therefore, by observing that \(a_{jN}\) and \(b_{jN}\) are mutually independent variables, it follows by the result of Puri (1964) that
\[
\sigma_{1N,jj} = Var(a_{jN} + b_{jN})
\]
\[= 2 \left\{ \sum_{i=1 \atop i \neq j}^{k} \lambda_{iN} \int_{x<y} \Gamma_{iN}(x,y)dF_j(x)dF_j(y)
\]
\[+ \frac{1}{\lambda_{jN}} \sum_{i=1 \atop i \neq j}^{k} \lambda_{iN}^{2} \int_{x<y} \Gamma_{jN}(x,y)dF_i(x)dF_i(y) \right\}
\]
\[+ \frac{1}{\lambda_{jN}} \sum_{i,i'=1 \atop i \neq i',i \neq j,i' \neq j}^{k} \lambda_{iN} \lambda_{i'N} \left\{ \int_{x<y} \Gamma_{jN}(x,y)dF_i(x)dF_i'(y)
\]
\[+ \int_{y<x} \Gamma_{jN}(y,x)dF_i(x)dF_i'(y) \right\}, \quad (12)
\]
where \(\Gamma_{jN}(u,v) = F_j(u)(1 - F_j(v))J'(H_N(u))J'(H_N(v))\). To evaluate the same for \(c_{jN}\) and \(d_{jN}\), recall the result of Francq and Zakoïan (2004) that
\[
Var\left(n_{j}^{1/2}(\hat{\theta}_{j,N} - \theta_{0j})\right) = (\kappa_{j} - 1)\left[U(\theta_{0j})\right]^{-1}, \quad 1 \leq j \leq k.
\]
In view of (5) and (11), it follows that
\[
\sigma_{2N,jj} = Var(c_{jN}) = (\kappa_{j} - 1)\omega_{jN}^{T}\left[U(\theta_{0j})\right]^{-1}\omega_{jN}, \quad (13)
\]
where \(\omega_{jN} = -\lambda_{jN}^{-1/2} \sum_{i=1 \atop i \neq j}^{k} \lambda_{iN} \int xf_j(x)J'(H_N)xF_i(x) \times \tau_j\) with \(\tau_j = (\tau_{j,1}, \ldots, \tau_{j,p_j+q_j+1})^T\), and analogously
\[
\sigma_{3N,jj} = Var(d_{jN}) = \sum_{i=1 \atop i \neq j}^{k} (\kappa_{i} - 1)\nu_{iN}^{T}\left[U(\theta_{0i})\right]^{-1}\nu_{iN}, \quad (14)
\]
where \(\nu_{iN} = \lambda_{iN}^{1/2} \int xf_j(x)J'(H_N)xF_j(x) \times \tau_i\). Moreover, by independence of \(X_{j,1}, \ldots, X_{j,n_j}\), it remains to evaluate
\[
K_{1N,j} = 2E(a_{jN}d_{jN}) \quad \text{and} \quad K_{2N,j} = 2E(b_{jN}c_{jN}).
\]
Using (11), we obtain
\[ K_{1N,j} = 2 \sum_{i=1 \atop i \neq j}^{k} \lambda_{iN} \int \int E[(n_i^{1/2}(F_{i,n_i}(x) - F_i(x)))A_i]\psi_iN(x,y)dF_j(x)dF_j(y), \]
where \( \psi_iN(u,v) = v f_i(v)J'(H_N(u))J'(H_N(v)) \). To obtain an explicit expression of \( K_{1N,j} \), it is necessary to evaluate \( E[i] \). From the result of Berkes and Horváth (2003) and (5), we find that
\[ E[n_i^{1/2}(F_{i,n_i}(x) - F_i(x))A_i] = \sum_{l=1}^{p_i+q_i+1} \tau_i h_i^l(x), \]
where \( h_i^l(v) = \delta_i^l \int_{u=}^{u}(u^2 - 1)f_i(u)du \) with \( \delta_i^l = E(Z_i^l(\theta_{0i})), 1 \leq i \leq k \). Then
\[ K_{1N,j} = 2 \sum_{i=1 \atop i \neq j}^{k} \sum_{l=1}^{p_i+q_i+1} \lambda_{iN} \tau_{i,l} \int \int h_i^l(x)\psi_iN(x,y)dF_j(x)dF_j(y), \]
and similarly
\[ K_{2N,j} = \frac{2}{\lambda_{jN}} \sum_{i=1 \atop i \neq j}^{k} \sum_{l=1}^{p_i+q_i+1} \lambda_{iN}^2 \tau_{j,l} \int \int h_j^l(x)\psi_jN(x,y)dF_i(x)dF_i(y). \]
Therefore, the variance terms when combined yield
\[ \sigma_{N,jj} = \sigma_{1N,jj} + \sigma_{2N,jj} + \sigma_{3N,jj} + \gamma_{N,jj}, \tag{15} \]
where \( \gamma_{N,jj} = K_{1N,j} + K_{2N,j} \).

We next turn to evaluate the covariance terms. For this purpose, rewrite (11) as
\[ N^{1/2}(B_{1N,j} + B_{2N,j}) = N^{1/2} \sum_{i=1}^{k} \lambda_{iN}( - \int (F_{j,n_j}(x) - F_j(x))J'(H_N)dF_i(x) + \int (F_{n_i}^{(i)}(x) - F_i(x))J'(H_N)dF_j(x) - n_i^{-1/2} A_j \int x f_j(x)J'(H_N)dF_i(x) + n_i^{-1/2} A_i \int x f_i(x)J'(H_N)dF_j(x) ) + o_p(1) = a_{1N,j} + b_{1N,j} + c_{1N,j} + d_{1N,j} + o_p(1), \tag{say}. \]
By independence of \( X_{j,1}, \ldots, X_{j,n_j}, 1 \leq j \leq k \), we first compute
\[ \sigma_{1N,j'} = \text{Cov}(a_{1N,j} + b_{1N,j}, a_{1N,j'} + b_{2N,j'}) = E(a_{1N,j}b_{1N,j'}) + E(b_{1N,j}a_{1N,j'}) + E(b_{1N,j}b_{1N,j'}), \quad j \neq j' = 1, \ldots, k. \]
From
\[ a_{1N,j}b_{1N,j'} = -N \sum_{i=1}^{k} \sum_{l=1}^{k} \lambda_{iN} \lambda_{lN} \int \int (F_{j,n_j}(x) - F_j(x))(F_{i,n_i}(y) - F_i(y)) \times J'(H_N(x))J'(H_N(y))dF_i(x)dF_j(y), \]
it follows by using again the result of Puri (1964) that

\[
E(a_{1N,j}b_{1N,j'}) = - \sum_{i=1}^{k} \lambda_{iN} \int_{x<y} F_{j}(x)(1 - F_{j}(y))J'(H_{N}(x))J'(H_{N}(y))dF_{i}(x)dF'_{i}(y)
\]

\[- \sum_{i=1}^{k} \lambda_{iN} \int_{y<x} F_{j'}(y)(1 - F_{j'}(x))J'(H_{N}(x))J'(H_{N}(y))dF_{i}(x)dF'_{i}(y).
\]

In the same way, we have

\[
E(b_{1N,j}a_{1N,j'}) = - \sum_{i=1}^{k} \lambda_{iN} \int_{x<y} F_{j'}(x)(1 - F_{j'}(y))J'(H_{N}(x))J'(H_{N}(y))dF_{i}(x)dF'_{i}(y)
\]

\[- \sum_{i=1}^{k} \lambda_{iN} \int_{y<x} F_{j}(y)(1 - F_{j}(x))J'(H_{N}(x))J'(H_{N}(y))dF_{i}(x)dF'_{i}(y)
\]

and

\[
E(b_{1N,j}b_{1N,j'}) = \sum_{i=1}^{k} \lambda_{iN} \int_{x<y} F_{i}(x)(1 - F_{i}(y))J'(H_{N}(x))J'(H_{N}(y))dF_{j}(x)dF'_{j}(y)
\]

\[+ \sum_{i=1}^{k} \lambda_{iN} \int_{y<x} F_{i}(y)(1 - F_{i}(x))J'(H_{N}(x))J'(H_{N}(y))dF_{j}(x)dF'_{j}(y).
\]

Therefore,

\[
\sigma_{1N,jj'} = - \sum_{i=1}^{k} \lambda_{iN} \left( \int_{x<y} \Gamma_{jN}(x, y)dF_{i}(x)dF'_{j}(y) + \int_{y<x} \Gamma_{jN}(y, x)dF_{i}(x)dF'_{j}(y) \right)
\]

\[+ \sum_{i=1}^{k} \lambda_{iN} \left( \int_{x<y} \Gamma_{j'N}(x, y)dF_{i}(x)dF'_{j}(y) + \int_{y<x} \Gamma_{j'N}(x, y)dF_{i}(x)dF'_{j}(y) \right)
\]

\[+ \sum_{i=1}^{k} \lambda_{iN} \left( \int_{x<y} \Gamma_{iN}(x, y)dF_{j}(x)dF'_{j}(y) + \int_{y<x} \Gamma_{iN}(x, y)dF_{j}(x)dF'_{j}(y) \right).
\]

Now we turn to evaluate, for \( j \neq j' \),

\[
L_{1N,jj'} = E(a_{1N,j}d_{1N,j'}) + E(d_{1N,j}a_{1N,j'})
\] and

\[
L_{2N,jj'} = E(b_{1N,j}c_{1N,j'}) + E(c_{1N,j}b_{1N,j'}).
\]

In analogy with the preceding \( K \) terms, we have

\[
L_{1N,jj'} = - \sum_{i=1}^{k} \lambda_{iN} \left( \sum_{l=1}^{p_{j}+q_{j}+1} \tau_{j,l} \int_{x<y} h_{j}^{l}(x)\psi_{j}(x, y)dF_{i}(x)dF'_{j}(y) \right)
\]

\[+ \sum_{l=0}^{p_{j'}+q_{j'}+1} \tau_{j',l} \int_{x<y} h_{j'}^{l}(x)\psi_{j'}(x, y)dF_{i}(x)dF'_{j}(y) \]
and

\[
L_{2N,jj'} = -\sum_{i=1}^{k} \lambda_{iN} \left( \sum_{l=1}^{p_j+q_j+1} \tau_{j,l} \int \int h_j^l(y) \psi_j(y,x) dF_i(x) dF_j'(y) + \sum_{l=1}^{p_{j'}+q_{j'}+1} \tau_{j',l} \int \int h_{j'}^l(y) \psi_{j'}(y,x) dF_i(x) dF_{j'}(y) \right).
\]

Therefore, combining the covariance terms produces

\[
\sigma_{N,jj'} = \sigma_{1N,jj'} + \sigma_{2N,jj'},
\]

where \(\sigma_{2N,jj'} = L_{1N,jj'} + L_{2N,jj'}, j \neq j' = 1, \ldots, k\).

Hence, using (13)–(16) and the central limit theorems for martingale differences given by Berkes and Horváth (2003), and Francq and Zakoïan (2004), we may conclude that

\[
N^{1/2} \sum_{N}^{-1/2}(B_{1N,j} + B_{2N,j})_{1 \leq j \leq k} \xrightarrow{d} \mathcal{N}(0, I_k)
\]
as \(N \to \infty\).

Next, we turn to show statement (ii). For this purpose, we require the following elementary results (see Puri (1964)).

(i) \(H_N \geq \lambda_j N F_j \geq \lambda_0 F_j, \quad 1 \leq j \leq k\).

(ii) \((1 - F_j) \leq (1 - H_N)/\lambda_j \leq (1 - H_N)/\lambda_0, \quad 1 \leq j \leq k\).

(iii) \(F_j(1 - F_j) \leq H_N(1 - H_N)/\lambda_j^2 \leq H_N(1 - H_N)/\lambda_0^2, \quad 1 \leq j \leq k\).

(iv) \(dH_N \geq \lambda_j N dF_j \geq \lambda_0 dF_j, \quad 1 \leq j \leq k\).

(v) Let \((\vartheta_{1N}, \vartheta_{2N})\) be the interval \(S_{N,\epsilon}\) such that

\[
S_{N,\epsilon} = \{x : H_N(1 - H_N) > \eta_{\epsilon} \lambda_0/N\},
\]

where \(\epsilon > 0\) is arbitrarily small and \(\eta_{\epsilon}(>0)\) depends \(\epsilon\). Thus,

\[
\eta_{\epsilon} < N(H_N(\vartheta_{1N})), \quad (1 - H_N(\vartheta_{2N})) < \eta_{\epsilon}(1 + N^{-1} \eta_{\epsilon}).
\]

Hence, \(\eta_{\epsilon}\) can be chosen independently of \(F_j\) and \(\lambda_j N\) in such a way that

\[
N(H_N(\vartheta_{1N}) + (1 - H_N(\vartheta_{2N}))) \leq \epsilon.
\]

From (19), it follows that

\[
P(\varepsilon_{j,t} \in S_{N,\epsilon}, 1 \leq t \leq n_j, 1 \leq j \leq k) = P(\vartheta_{1N} \leq \varepsilon_{j,t} \leq \vartheta_{2N})
= \prod_{j=1}^{N} \left[ H_j(\vartheta_{2N}) - H_j(\vartheta_{1N}) \right]
= \prod_{j=1}^{N} \left[ 1 - [H_j(\vartheta_{1N}) + 1 - H_j(\vartheta_{2N})] \right]
\geq 1 - \sum_{j=1}^{N} [H_j(\vartheta_{1N}) + 1 - H_j(\vartheta_{2N})]
= 1 - N[H_N(\vartheta_{1N}) + (1 - H_N(\vartheta_{2N}))]
\geq 1 - \epsilon.
\]
Let us first evaluate \( C_{1N,j} \). By (6) and \( d \hat{F}_{j,n_i} = d(\hat{F}_{j,n_i} - F_j + F_j) \), we have

\[
C_{1N,j} = -\frac{1}{N + 1} \int \mathcal{H}_N x(\hat{H}_N) dF_{j,n_j}(x) \\
- \frac{1}{N(N + 1)} \sum_{i=1}^{k} n_i^{1/2} A_i \int x f_i(x) J'(H_N) dF_{j,n_j}(x) \\
- \frac{n_j^{-1/2}}{N + 1} A_j \int \mathcal{H}_N J'(H_N) d(x f_j(x)) \\
- \frac{n_j^{-1/2}}{N(N + 1)} \sum_{i=1}^{k} n_i^{1/2} A_i \int x f_i(x) J'(H_N) d(x f_j(x)) + o_p(N^{-1}) \\
= \sum_{i=1}^{4} C_{iN,j} + o_p(N^{-1}), \text{ (say)}.
\]

The proof of \( C_{11N,j} = o_p(N^{-1/2}) \) follows precisely the same arguments as in Puri (1964). Next we turn to \( C_{12N,j} \). By (A.2) and (A.3), we obtain

\[
|C_{12N,j}| \leq \frac{1}{N} \sum_{i=1}^{k} n_i^{1/2} |A_i| \frac{1}{N + 1} \int |x f_i(x)| J'(H_N) dF_{j,n_j}(x) \\
= \frac{1}{N} \sum_{i=1}^{k} n_i^{1/2} |A_i| \frac{1}{N} \int |J'(H_N)| dF_{j,n_j}(x).
\]

In a similar fashion as the proof for \( C_{11N,j} \), it follows that

\[
\frac{1}{N} \int |J'(H_N)| dF_{j,n_j}(x) = o_p(N^{-1/2}),
\]

which, combined with the fact

\[
\frac{1}{N} \sum_{i=1}^{k} n_i^{1/2} |A_i| = O_p \left( \frac{1}{N} \sum_{i=1}^{k} n_i^{1/2} \right), \tag{21}
\]

implies \( C_{12N,j} = o_p(N^{-1}) \). Next consider

\[
C_{13N,j} = -n_j^{-1/2} A_j (C_{13N,j}^* + C_{13N,j}^{**}),
\]

where

\[
C_{13N,j}^* = \frac{1}{N + 1} \int_{S_{N \xi}} \mathcal{H}_N J'(H_N) d(x f_j(x)),
\]

\[
C_{13N,j}^{**} = \frac{1}{N + 1} \int_{S_{N \xi}} \mathcal{H}_N J'(H_N) d(x f_j(x))
\]

and \( S_{N \xi}^c \) is the complementary event of \( S_{N \xi} \). Let us first deal with \( C_{13N,j}^* \). In view of (A.2), (A.3), (17) and (18), it follows that

\[
|C_{13N,j}^*| \leq \frac{K}{c_j N} \int_{S_{N \xi}} |J'(H_N)| dF_j(x)
\]
\[
\begin{align*}
&\leq \frac{K}{c_j N} \int_{S_{N\epsilon}} [H_N(1 - H_N)]^{-\frac{3}{2} + \delta} dH_N(x) \\
&\leq \frac{K}{c_j N} \int_{\mathcal{H}} H_N^{-\frac{3}{2} + \delta} dH_N(x) \\
&\leq \frac{K}{N^{\frac{3}{2} + \delta}}. 
\end{align*}
\tag{22}
\]

Now using the Markoff inequality, we obtain

\[
P(|C_{13N,j}^*| > mN^{-1/2}) \leq \frac{K}{N^{\frac{3}{2} + \delta} m} = \frac{K}{mN^{\delta}},
\]

where \( m > 0 \) and \( K \) may depend on \( \epsilon \). Next consider \( C_{13N,j}^{**} \). Write \( H_1 = H_N(\vartheta_1 N) \) and \( H_2 = H_N(\vartheta_2 N) \). Then from (17) and (18), we have \( H_1 = 1 - H_2 < K/N \). By (20), we are certain that \( \varepsilon_{j,t} \notin S_{N\epsilon}^c \) and

\[
|C_{13N,j}^{**}| \leq \frac{K}{c_j N} \left( \int_0^{H_1} [H_N(1 - H_N)]^{-\frac{3}{2} + \delta} dH_N(x) \\
+ \int_{H_2}^{1} [H_N(1 - H_N)]^{-\frac{3}{2} + \delta} dH_N(x) \right) \\
\leq \frac{K}{c_j N} \int_0^{H_1} H_N^{-\frac{3}{2} + \delta} dH_N(x) \\
\leq \frac{K}{N^{\frac{3}{2} + \delta}}. 
\tag{23}
\]

Therefore, by using (21), we have

\[
C_{13N,j} = o_p(N^{-1/2}). 
\tag{24}
\]

Similarly, it can be shown that \( C_{14N,j} = o_p(N^{-1}). \) Consequently, we have

\[
C_{1N,j} = o_p(N^{-1/2}).
\]

Next, we consider \( C_{2N,j} \). By analogy with the first \( C \) term, we have

\[
\begin{align*}
C_{2N,j} &= \int (H_N - H_N)J'(H_N)d(F_{j,n_j} - F_j)(x) \\
&\quad + \frac{1}{N} \sum_{i=1}^{k} n_i^{1/2} A_i \int x f_i(x) J'(H_N)d(F_{j,n_j} - F_j)(x) \\
&\quad + \frac{n_j^{-1/2}}{N} \sum_{i=1}^{k} n_i^{1/2} A_i \int x f_i(x) J'(H_N)d(x f_j(x)) \\
&\quad + n_j^{-1/2} A_j \int (H_N - H_N)J'(H_N)d(x f_j(x)) + o_p(N^{-1}) \\
&= \sum_{i=1}^{4} C_{2iN,j} + o_p(N^{-1}), \quad \text{(say)}.
\end{align*}
\]

The proof of \( C_{21N,j} = o_p(N^{-1/2}) \) is identical to that of Puri (1964). Next, we consider

\[
C_{22N,j} = \frac{1}{N} \sum_{i=1}^{k} n_j^{1/2} A_i (C_{22N,j}^* + C_{22N,j}^{**}),
\]
Therefore, it follows from (21) that
\[ C_{22N,j}^* = \int_{S_{N_c}} x f_j(x) J'(H_N) d(F_{j,n_j} - F_j)(x) = o_p(1), \quad (25) \]
\[ C_{22N,j}^{**} = \int_{S_{N_c}} x f_j(x) J'(H_N) d(F_{j,n_j} - F_j)(x) = o_p(1). \quad (26) \]

Note that from (A.2) and (A.3), we can find \( K > 0 \) such that \( |x f_j(x)| \leq KH_N(1 - H_N) \).
Then from (17), (18) and (22), it follows that (25) is dominated by
\[ |C_{22N,j}^*| \leq K \int_{S_{N_c}} \left| x f_j(x) \right| \left| J'(H_N) \right| \left| d(F_{j,n_j} - F_j)(x) \right| \]
\[ \leq K \int_{S_{N_c}} \left[ H_N(1 - H_N) \right]^{-\frac{1}{2} + \delta} \left| d(F_{j,n_j} - F_j)(x) \right| \]
\[ \leq n_j^{-\frac{1}{2}} \int_{K}^{1} O(N^{\frac{1}{2} - \delta}) \left| d[n_j^{1/2} (F_{j,n_j} - F_j)(x)] \right| = o_p(1). \]

Likewise, it is easy to show from (23) that (26) is dominated by
\[ |C_{22N,j}^{**}| \leq K \left( \int_{0}^{H_1} \left[ H_N(1 - H_N) \right]^{-\frac{1}{2} + \delta} \left| d(F_{j,n_j} - F_j)(x) \right| \right)
\[ + \int_{H_2}^{1} \left[ H_N(1 - H_N) \right]^{-\frac{1}{2} + \delta} \left| d(F_{j,n_j} - F_j)(x) \right| \]
\[ \leq n_j^{-\frac{1}{2}} \int_{0}^{H_1} O(N^{\frac{1}{2} - \delta}) \left| d[n_j^{1/2} (F_{j,n_j} - F_j)(x)] \right| = o_p(1). \]

Therefore, it follows from (21) that \( C_{22N,j} = o_p(N^{-1/2}) \). The proof for \( C_{23N,j} = o_p(N^{-1/2}) \)
is analogous to (24). To complete the assertion for \( C_{2N,j} \), it remains to evaluate \( C_{24N,j} = A_j(C_{24N,j}^* + C_{24N,j}^{**}) \), where
\[ C_{24N,j}^* = n_j^{-1/2} \int_{S_{N_c}} (H_N - H_N) J'(H_N) d(x f_j(x)), \]
\[ C_{24N,j}^{**} = n_j^{-1/2} \int_{S_{N_c}} (H_N - H_N) J'(H_N) d(x f_j(x)). \]

By virtue of Puri and Sen (1993, Theorem 2.11.10), write
\[ I_N(\delta') = \sup_{x} \left| \frac{N^{1/2} [H_N(x) - H_N(x)]}{H_N(x)(1 - H_N(x))} \right|^{\frac{1}{2} - \delta} \leq C^*, \quad \delta' > 0, \quad C^* > 0 \quad (27) \]
so that \( P(I_N(\delta')) \geq 1 - \epsilon \). Then, if we let \( \delta' < \delta \), it follows from (A.2)–(A.4), (22) and (27) that
\[ C_{24N,j}^* = n_j^{-1/2} \int_{S_{N_c}} |H_N - H_N| J'(H_N) dF_j(x) \]
\[ \leq K n_j^{-1/2} \int_{S_{N_c}} O(N^{-1/2})[H_N(1 - H_N)]^{\delta - \delta'} dH_N(x) \]
\[ \leq K n_j^{-1/2} O(N^{-1/2}) \int_{K}^{1} H_N^{\delta - \delta'} dH_N(x) \]
\[ = O(N^{\delta' - \delta - 1}) = o(N^{-1}) \]
and similarly from (23) that

$$C_{24N,j}^{**} \leq Kn^{-1/2} \mathcal{O}(N^{-1/2}) \int_0^{H_1} H_{N}^{\delta - \delta'} dH_{N}(x) = o(N^{-1}).$$

Hence, $C_{24N,j} = o_p(N^{-1/2})$. Consequently, we have

$$C_{2N,j} = o_p(N^{-1/2}).$$

Finally, we evaluate $C_{3N,j}$. Following the preceding $C_*$ term, and using

$$J\left(\frac{N}{N+1} \hat{H}_{N}\right) = J(H_N) + \left(\frac{N}{N+1} \hat{H}_{N} - H_N\right) \times J'\left(\frac{\varrho H_N + (1 - \varrho) N}{N+1} \hat{H}_{N}\right), \quad 0 < \varrho < 1,$$

we obtain

$$C_{3N,j} = \int \left(\frac{N}{N+1} \hat{H}_{N} - H_N\right) \times \left[ J'\left(\frac{\varrho H_N + (1 - \varrho) N}{N+1} \hat{H}_{N}\right) - J'(H_N) \right] dF_{j,n_j}(x)$$

$$+ \frac{1}{N+1} \sum_{i=1}^{k} n_i^{1/2} A_i \int x f_i(x)$$

$$\times \left[ J'\left(\frac{\varrho H_N + (1 - \varrho) N}{N+1} \hat{H}_{N}\right) - J'(H_N) \right] dF_{j,n_j}(x)$$

$$+ n_j^{-1/2} A_j \int \left(\frac{N}{N+1} \hat{H}_{N} - H_N\right) \times \left[ J'\left(\frac{\varrho H_N + (1 - \varrho) N}{N+1} \hat{H}_{N}\right) - J'(H_N) \right] d(x f_j(x))$$

$$+ \frac{n_j^{-1/2} A_j}{N+1} \sum_{i=1}^{k} n_i^{1/2} A_i \int x f_i(x)$$

$$\times \left[ J'\left(\frac{\varrho H_N + (1 - \varrho) N}{N+1} \hat{H}_{N}\right) - J'(H_N) \right] d(x f_j(x)) + o_p(N^{-1})$$

$$= \sum_{i=1}^{4} C_{3iN,j} + o_p(N^{-1}), \quad \text{(say)}.$$

First consider $C_{31N,j} = C_{31N,j}^* + C_{31N,j}^{**}$, where

$$C_{31N,j}^* = \int_{S_N} \left(\frac{N}{N+1} \hat{H}_{N} - H_N\right) \times \left[ J'\left(\frac{\varrho H_N + (1 - \varrho) N}{N+1} \hat{H}_{N}\right) - J'(H_N) \right] dF_{j,n_j}(x),$$

$$C_{31N,j}^{**} = \int_{S_N} \left(\frac{N}{N+1} \hat{H}_{N} - H_N\right) \times \left[ J'\left(\frac{\varrho H_N + (1 - \varrho) N}{N+1} \hat{H}_{N}\right) - J'(H_N) \right] dF_{j,n_j}(x).$$
To evaluate $C_{31N,j}^*$, first note from (6), (A.2), (A.3) and (21) that

$$H_N - \left( \varrho H_N + (1 - \varrho) \frac{N}{N + 1} \hat{H}_N \right)$$

$$= (1 - \varrho) \left( H_N - \frac{N}{N + 1} \hat{H}_N \right)$$

$$= (1 - \varrho) \left[ \left( H_N - \frac{N}{N + 1} \hat{H}_N \right) - \frac{N}{N + 1} \sum_{j=1}^{k} n_j^{-1/2} \lambda_{jN} A_j f_j(x) \right] + o_p(N^{-1/2})$$

$$= N^{-1/2} O(1) \{ 1 + [H_N(1 - H_N)]^{1/2 - \delta'} \}, \quad (28)$$

where $O(1)$ is uniform in $x$. Then from (18) and (28), it follows that

$$1 - \left( \varrho H_N + (1 - \varrho) \frac{N}{N + 1} \hat{H}_N \right) \times (1 - H_N)^{-1} = 1 + O(N^{-1/2}).$$

Thus, for sufficiently large $N > 0$, we can find $\zeta > 0$ such that

$$\inf_x \left( \varrho H_N(x) + (1 - \varrho) \frac{N}{N + 1} \hat{H}_N(x) \right) \left( \frac{1 - (\varrho H_N(x) + (1 - \varrho) \frac{N}{N + 1} \hat{H}_N(x))}{H_N(x)(1 - H_N(x))} \right) > \zeta \quad (29)$$

with probability $\geq 1 - \epsilon$. Now write

$$|C_{31N,j}^*| \leq \int \left| \frac{N}{N + 1} \hat{H}_N - H_N \right|$$

$$\times \left| J' \left[ \varrho H_N + (1 - \varrho) \frac{N}{N + 1} \hat{H}_N \right] - J'(H_N) \right| dF_{j,n_j}(x)$$

$$= \int Q_N dF_{j,n_j}(x), \quad (say). \quad (30)$$

Then it is easy to show from (A.2), (22), (27), (29) and (30) that

$$E \int_{S_{N_k}} Q_N dF_{j,n_j}(x) \leq K(1 + \zeta^{\delta - \frac{3}{2}}) O(N^{-1/2})$$

$$\times \int_{S_{N_k}} [H_N(1 - H_N)]^{\delta - \delta'} dH_N(x)$$

$$\leq K(1 + \zeta^{\delta - \frac{3}{2}}) O(N^{-1/2}) \int_{K}^1 H_N^{\delta - \delta'} dH_N(x). \quad (31)$$
Thus, $Q_N(x)$ is integrable and converges to 0 in probability. Hence, by virtue of the dominated convergence theorem and (31), it is seen that $C^*_{31N,j} = o_p(N^{-1/2})$. Similarly, we can show $C^{**}_{31N,j} = o_p(N^{-1/2})$ by using the arguments of (23) and (31). Next consider

$$C_{32N,j} = \frac{1}{N+1} \sum_{i=1}^{k} n_i^{1/2} A_i(C^*_{32N,j} + C^{**}_{32N,j}),$$

where

$$C^*_{32N,j} = \int_{S_{N_c}} x f_i(x) \left\{ J' \left[ q H_N + (1 - \varrho) \frac{N}{N+1} \hat{H}_N \right] \right\} dF_{j,n_j}(x)$$

and

$$C^{**}_{32N,j} = \int_{S_{N_c}} x f_i(x) \left\{ J' \left[ q H_N + (1 - \varrho) \frac{N}{N+1} \hat{H}_N \right] \right\} dF_{j,n_j}(x).$$

Let us first evaluate $C^*_{32N,j}$. Recalling $|x f_j(x)| \leq K H_N(1 - H_N)$, and using the arguments of $C^*_{31N,j}$ and (A.2), we obtain

$$E(|C^*_{32N,j}|) \leq \int_{S_{N_c}} |x f_i(x)| \left\{ J' \left[ q H_N + (1 - \varrho) \frac{N}{N+1} \hat{H}_N \right] \right\} dF_{j,n}(x) \leq K (1 + \zeta^{\delta - \frac{3}{2}}) \int_{S_{N_c}} |H_N(1 - H_N)|^{\delta - \frac{1}{2}} dH_N(x) \leq K (1 + \zeta^{\delta - \frac{3}{2}}) \int_{\frac{1}{N}} H_N^{\delta - \frac{1}{2}} dH_N(x).$$

(32)

In analogy with (23) and (32), we can show $C^{**}_{32N,j} = o_p(N^{-1/2})$. Hence, from (21), we have $C_{32N,j} = o_p(N^{-1/2})$. Next, we evaluate $C_{33N,j} = C^*_{33N,j} + C^{**}_{33N,j}$, where

$$C^*_{33N,j} = n_j^{-1/2} A_j \int_{S_{N_c}} \left( \frac{N}{N+1} \hat{H}_N - H_N \right) \times \left\{ J' \left[ q H_N + (1 - \varrho) \frac{N}{N+1} \hat{H}_N \right] \right\} d(x f_j(x))$$

and

$$C^{**}_{33N,j} = n_j^{-1/2} A_j \int_{S_{N_c}} \left( \frac{N}{N+1} \hat{H}_N - H_N \right) \times \left\{ J' \left[ q H_N + (1 - \varrho) \frac{N}{N+1} \hat{H}_N \right] \right\} d(x f_j(x)).$$

Following the arguments of $C^*_{31N,j}$, and using (A.2)–(A.4), we obtain

$$|C^*_{33N,j}| \leq \frac{n_j^{-1/2}}{c_j} |A_j| \int_{S_{N_c}} \left| \frac{N}{N+1} \hat{H}_N - H_N \right| \times \left\{ J' \left[ q H_N + (1 - \varrho) \frac{N}{N+1} \hat{H}_N \right] \right\} dF_{j}(x) \leq \frac{K n_j^{-1/2}}{c_j} (1 + \zeta^{\delta - \frac{3}{2}}) O(N^{-1/2}) \int_{\frac{1}{N}} H_N^{\delta - \delta^* - 1} dH_N(x).$$

Therefore, $C^*_{33N,j} = o_p(N^{-1/2})$. Similarly, in view of (22), we can show $C^{**}_{33N,j} = o_p(N^{-1/2})$. Hence, by (21), we have $C_{33N,j} = o_p(N^{-1/2})$. To complete the evaluation of $C_{3N,j}$, we
consider $C_{34N,j} = (C_{34N,j}^* + C_{34N,j}^{**})$, where

$$C_{34N,j}^* = \frac{n_j^{-1/2} \mathcal{A}_j}{N+1} \sum_{i=1}^{k} n_i^{1/2} \int_{S_{N_k}} x f_i(x)$$

$$\times \left\{ J' \left[ \phi H_N + (1 - \phi) \frac{N}{N+1} \hat{H}_N \right] - J'(H_N) \right\} d(x f_j(x)),$$

$$C_{34N,j}^{**} = \frac{n_j^{-1/2} \mathcal{A}_j}{N+1} \sum_{i=1}^{k} n_i^{1/2} \int_{S_{N_k}} x f_i(x)$$

$$\times \left\{ J' \left[ \phi H_N + (1 - \phi) \frac{N}{N+1} \hat{H}_N \right] - J'(H_N) \right\} d(x f_j(x)).$$

We first turn to evaluate $C_{34N,j}^*$. From (A.2)–(A.4), (21) and (32), it follows that

$$|C_{34N,j}^*| \leq K \frac{n_j^{-1/2} |\mathcal{A}_j|}{c_j N} \sum_{i=1}^{k} n_i^{1/2} |\mathcal{A}_i| \int_{S_{N_k}} H_N(1 - H_N)$$

$$\times \left\{ J' \left[ \phi H_N + (1 - \phi) \frac{N}{N+1} \hat{H}_N \right] - J'(H_N) \right\} dF_j(x)$$

$$\leq O_p \left( n_j^{-1/2} N^{-1} \sum_{i=1}^{k} n_i^{1/2} \right) \int_{\mathbb{R}} H_N^{k-\frac{1}{2}} dH_N(x). \quad (33)$$

Thus, $C_{34N,j}^* = o_p(N^{-1/2})$. By analogy with (23) and (33), we can show $C_{34N,j}^{**} = o_p(N^{-1/2})$. Consequently, we have

$$C_{34N,j} = o_p(N^{-1/2}).$$

This completes the proof of the theorem.

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