On the Convergence of the Variational Iteration Method

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Abstract

Convergence results are stated for the variational iteration method applied to solve an initial value problem for a system of ordinary differential equations.

Keywords: variational iteration method, ordinary differential equations, convergence

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1 Introduction

The Ji-Huan He’s Variational Iteration Method (VIM) was applied to a large range of problems for ordinary and partial differential equations. The purpose of this paper is to prove convergence theorem when the method is applied to solve an initial value problem for a system of ordinary differential equations. In advance there is presented a convergence result in the case of an initial value problem for an ordinary differential equation. The result as well the proof may be found in Salkuyeh D. K., Tavakoli A., [5], too. In the case of a system of linear differential equations a convergence result is given by Salkuyeh D. K., [4]. Variants of the convergence of the VIM are studied in [7], [3], [6].

In the last section there are presented some results of our computational experiences.

2 The case of an ordinary differential equation

Let $f : [t_0, t_f] \times \mathbb{R} \to \mathbb{R}$ ($t_0 < t_f < \infty$), be a continuous function such that it has the first and second order partial derivatives in the second variable.

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Denoting $\frac{\partial f(t, x)}{\partial x} = f_x(t, x)$, we suppose that there exists $L > 0$ such that

$$|f_x(t, x)| \leq L \quad \forall (t, x) \in [t_0, t_f] \times \mathbb{R}.$$ 

As a consequence the function $f$ satisfies the Lipschitz property

$$|f(t, x) - f(t, y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}.$$ 

To solve the initial value problem

\begin{align*}
x'(t) &= f(t, x(t)) \quad t \in [t_0, t_f] \\
x(t_0) &= x^0
\end{align*}

the VIM is applied, [2]. There is considered the sequence

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda(s)(x'(s) + \delta u_n'(s) - f(s, x(s) + \delta u_n(s)))ds, \quad n \in \mathbb{N}. \quad (3)$$

where $\lambda$ is the so called the Lagrange multiplier [1]. There is supposed that $u_n(t_0) = x^0$ and that $u_n$ is a continuous derivable function.

Denoting $x(t)$ the solution of the initial value problem (1)-(2), if $u_n(t) = x(t) + \delta u_n(t)$ and $u_{n+1}(t) = x(t) + \delta u_{n+1}(t)$ then (3) implies

$$\delta u_{n+1}(t) = \delta u_n(t) + \int_{t_0}^{t} \lambda(s)(x'(s) + \delta u_n'(s) - f(s, x(s) + \delta u_n(s)))ds =$$

$$= \delta u_n(t) + \int_{t_0}^{t} \lambda(s)(x'(s) + \delta u_n'(s) - f(s, x(s)) - f_x(s, x(s))\delta u_n(s))ds + O((\delta u_n)^2) =$$

$$= \delta u_n(t) + \int_{t_0}^{t} \lambda(s)(\delta u_n'(s) - f_x(s, x(s))\delta u_n(s))ds + O((\delta u_n)^2).$$

After an integration by parts the above equality becomes

$$\delta u_{n+1}(t) = (1+\lambda(t))\delta u_n(t) - \int_{t_0}^{t} \left(\lambda'(s) + f_x(s, x(s))\lambda(s)\right)\delta u_n(s)ds + O((\delta u_n)^2).$$

In order that $u_{n+1}$ be a better approximation than $u_n$, there is required that $\lambda$ to be the solution of the following initial value problem

\begin{align*}
\lambda'(s) &= -f_x(s, x(s))\lambda(s) \quad s \in [t_0, t] \\
\lambda(t) &= -1
\end{align*}

(4) (5)
Because \(x(s)\) is an unknown function, instead of (4)-(5) there is considered the problem

\[
\begin{align*}
\lambda'(s) &= -f_x(s, u_n(s))\lambda(s) \quad s \in [t_0, t] \\
\lambda(t) &= -1
\end{align*}
\] (6)

with the solution denoted \(\lambda_n(s, t)\).

The recurrence formula (3) becomes

\[
u_{n+1}(t) = u_n(t) + \int_{t_0}^t \lambda_n(s, t)(u'_n(s) - f(s, u_n(s)))ds, \quad n \in \mathbb{N}.
\] (8)

The solution of the initial value problem (6)-(7) is

\[
\lambda_n(s, t) = -e^{\int_{t_0}^t f_x(\tau, u_n(\tau))d\tau}.
\] (9)

It results that \(|\lambda_n(s, t)| \leq e^{L(t-s)} \leq e^{LT}, \quad \forall \ t_0 \leq s \leq t \leq t_f \) and \(T = t_f - t_0\).

The following convergence result occurs:

**Theorem 2.1** [5] (Th. 2) If \(f: [t_0, t_f] \times \mathbb{R} \to \mathbb{R}\) is a continuous function such that it has the first order partial derivatives in \(x\) bounded \(|f_x(t, x)| \leq L, \forall (t, x) \in [t_0, t_f] \times \mathbb{R}\) then the sequence \((u_n)_{n \in \mathbb{N}}\) defined by (8) converge uniformly to \(x(t)\), the solution of the initial value problem (1)-(2).

**Proof.** Subtracting the equality

\[
x(t) = x(t) + \int_{0}^{t} \lambda_n(s, t) \left(x'(s) - f(s, x(s))\right) ds
\]

from (3) it results

\[
e_{n+1}(t) = e_n(t) + \int_{t_0}^{t} \lambda_n(s, t) \left(e'_n(s) - (f(s, u_n(s)) - f(s, x(s)))\right) ds,
\]

where \(e_n(t) = u_n(t) - x(t), \quad n \in \mathbb{N}.

Again, after an integration by parts, it results

\[
e_{n+1}(t) = -\int_{t_0}^{t} \left(\lambda'_n(s, t)e_n(s) + (f(s, u_n(s)) - f(s, x(s)))\lambda_n(s, t)\right) ds.
\]

Taking into account (3), the above equality gives

\[
e_{n+1}(t) = \int_{t_0}^{t} \lambda_n(s, t) \left(f_x(s, u_n(s))e_n(s) - (f(s, u_n(s)) - f(s, x(s)))\right) ds.
\]
The hypothesis on $f$ implies the inequality
\[ |f_x(s, u_n(s))e_n(s) - (f(s, u_n(s)) - f(s, x(s)))| \leq 2L|e_n(s)| \]
end consequently
\[ |e_{n+1}(t)| \leq 2Le^{LT} \int_{t_0}^{t} |e_n(s)| ds. \]  

(10)

Let be $M = 2Le^{LT}$. For any continuous function $\varphi$ on $[t_0, t_f]$ we use the notation $\|\varphi\|_\infty = \max_{t_0 \leq t \leq t_f} |\varphi(t)|$.

From (10) we obtain successively:

For $n = 0$
\[ |e_1(t)| \leq M \int_{t_0}^{t} |e_0(s)| ds \leq M(t - t_0)\|e_0\|_\infty \Rightarrow \|e_1\|_\infty \leq MT\|e_0\|_\infty. \]

For $n = 1$
\[ |e_2(t)| \leq M \int_{t_0}^{t} |e_1(s)| ds \leq \frac{M^2(t - t_0)^2}{2}\|e_0\|_\infty \Rightarrow \|e_2\|_\infty \leq \frac{M^2T^2}{2}\|e_0\|_\infty. \]

Inductively, it results that
\[ |e_n(t)| \leq M \int_{t_0}^{t} |e_{n-1}(s)| ds \leq \frac{M^n(t - t_0)^n}{n!}\|e_0\|_\infty \Rightarrow \|e_n\|_\infty \leq \frac{M^nT^n}{n!}\|e_0\|_\infty. \]

and hence $\lim_{n \to \infty} \|e_n\|_\infty = 0$. ■

3 The case of a system of ordinary differential equations

Let be the system of ordinary differential equation
\[
\begin{align*}
x'_1(t) &= f_1(t, x_1(t), \ldots, x_m(t)) & x_1(t_0) &= x^0_1 \\
\vdots \\
x'_m(t) &= f_m(t, x_1(t), \ldots, x_m(t)) & x_m(t_0) &= x^0_m
\end{align*}
\]  

(11)

where $t \in [t_0, t_f]$ with $t_0 < t_f < \infty$.

We shall use the notations
\[
\mathbf{x} = (x_1, \ldots, x_m) \\
\|\mathbf{x}\|_1 = \sum_{j=1}^{m} |x_j| \\
\|\mathbf{x}\|_\infty = \max_t \|\mathbf{x}(t)\|_1
\]
Thus, any equation of (11) may be rewritten as

\[ x_i'(t) = f_i(t, x(t)), \quad i \in \{1, \ldots, m\}. \]

The following hypothesis are introduced:

- The functions \( f_1, \ldots, f_m \) are continuous and have first and second order partial derivatives in \( x_1, \ldots, x_m \).
- There exists \( L > 0 \) such that for any \( i \in \{1, \ldots, m\} \)

\[ |f_i(t, x) - f_i(t, y)| \leq L \sum_{j=1}^{m} |x_j - y_j| = L \|x - y\|_1, \forall x, y \in \mathbb{R}^m. \]

As a consequence

\[ \left| \frac{\partial f_i(t, x)}{\partial x_j} \right| = \left| f_{i,j}(t, x) \right| \leq L, \forall (t, x) \in [t_0, t_f] \times \mathbb{R}^m, \forall i, j \in \{1, \ldots, m\}. \]

According to the VIM there are considered the sequences

\[ u_{n+1,i}(t) = u_{n,i}(t) + \int_{t_0}^{t} \lambda_i(s)(u_{n,i}'(s) - f_i(s, u_n(s)))ds, \quad n \in \mathbb{N}, \quad (12) \]

\( i \in \{1, \ldots, m\} \) and where \( u_n = (u_{n,1}, \ldots, u_{n,m})^T \).

There are supposed that \( u_{n,i}(t_0) = x_{i,0} \) and that \( u_{n,i} \) is a continuous derivable function for any \( i \in \{1, \ldots, m\} \).

The VIM in this case is little tricky, [4]. The Lagrange multiplier attached to the \( i \)-th equation will act only on \( x_i \).

Denoting \( x(t) \) the solution of the initial value problem (11), if \( u_{n,i}(t) = x_i(t) + \delta u_{n,i}(t) \) and \( u_{n+1,i}(t) = x_i(t) + \delta u_{n+1,i}(t) \) but for \( j \neq i \), \( u_{n,j}(t) = x_j(t) \) then (12) implies

\[ \delta u_{n+1,i}(t) = \delta u_{n,i}(t) + \int_{t_0}^{t} \lambda_i(s) \left( x_i'(s) + \delta u_{n,i}'(s) - f_i(s, x(s)) \right) ds = \]

\[ = \delta u_{n,i}(t) + \int_{t_0}^{t} \lambda_i(s) \left( x_i'(s) + \delta u_{n,i}'(s) - f_i(s, x(s)) - f_{i,j}(s, x(s)) \delta u_{n,i}(s) \right) ds + O((\delta u_{n,i})^2) = \]
$$= \delta_{n,i}(t) + \int_{t_0}^{t} \lambda_i(s) \left( \delta u_{n,i}'(s) - f_{ix_i}(s, x(s)) \delta u_{n,i}(s) \right) ds + O((\delta u_{n,i})^2).$$

Proceeding as in the previous section we find

$$\lambda_i(s) := \lambda_{n,i}(s, t)$$

as the solution of the initial value problem

$$\lambda'(s) = -f_{ix_i}(s, u_n(s)) \lambda(s) \quad s \in [t_0, t]$$

$$\lambda(t) = -1$$

Then

$$\lambda_{n,i}(s, t) = -e^{\int_{t_0}^{t} f_{ix_i}(\tau, u_n(\tau)) d\tau}.$$ 

and

$$|\lambda_{n,i}(s, t)| \leq e^{L(t-s)} \leq e^{LT}, \quad \forall \; t_0 \leq s \leq t \leq t_f \text{ and } T = t_f - t_0.$$ 

The recurrence formula (12) becomes

$$u_{n+1,i}(t) = u_{n,i}(t) + \int_{t_0}^{t} \lambda_{n,i}(s, t) (u_{n,i}'(s) - f_i(s, u_n(s))) ds, \quad n \in \mathbb{N}, \quad (13)$$

for any \( i \in \{1, \ldots, m\} \).

The convergence result is:

**Theorem 3.1** If the hypothesis stated above are valid then the sequence \((u_n)_{n \in \mathbb{N}}\) defined by (13) converge uniformly to \(x(t)\), the solution of the initial value problem (11).

**Proof.** The proof is similar to the proof of Theorem 2.1. Subtracting the equality

$$x_i(t) = x_i(t) + \int_{t_0}^{t} \lambda_{n,i}(s, t) \left( x_i'(s) - f_i(s, x(s)) \right) ds$$

from (13) it results

$$e_{n+1,i}(t) = e_{n,i}(t) + \int_{t_0}^{t} \lambda_{n,i}(s, t) \left( e_{n,i}'(s) - (f_i(s, u_n(s)) - f_i(s, x(s))) \right) ds,$$

where \( e_n(t) = u_n(t) - x(t), \; n \in \mathbb{N}. \)

Again, after an integration by parts, it results

$$e_{n+1,i}(t) = \int_{t_0}^{t} \lambda_{n,i}(s, t) \left( f_{ix_i}(s, u_n(s)) e_{n,i}(s) - (f_i(s, u_n(s)) - f_i(s, x(s))) \right) ds.$$
The hypothesis on $f_i$ implies the inequality
\[
|f_{i+1}(s, u_n(s)) - f_i(s, u_n(s)) - f_i(s, x(s))| \leq L|e_{n,i}(s)| + L\|e_n(s)\|_1
\]
end consequently
\[
|e_{n+1,i}(t)| \leq Le^{LT} \int_{t_0}^{t} (|e_{n,i}(s)| + \|e_n(s)\|_1) ds.
\]
Summing these inequalities, for $i = 1 : m$, it results
\[
\|e_{n+1}(t)\|_1 \leq (m + 1)Le^{LT} \int_{t_0}^{t} \|e_n(t)\|_1 ds.
\] (14)

Let $M = (m + 1)Le^{LT}$. From (14) we obtain successively:

For $n = 0$
\[
\|e_1(t)\|_1 \leq M \int_{t_0}^{t} \|e_0(s)\|_1 ds \leq M(t - t_0)\|e_0\|_\infty \Rightarrow \|e_1\|_\infty \leq MT\|e_0\|_\infty.
\]

For $n = 1$
\[
\|e_2(t)\|_1 \leq M \int_{t_0}^{t} \|e_1(s)\|_1 ds \leq \frac{M^2(t - t_0)^2}{2}\|e_0\|_\infty \Rightarrow \|e_2\|_\infty \leq \frac{M^2T^2}{2}\|e_0\|_\infty.
\]

Inductively, it results that
\[
\|e_n(t)\|_1 \leq M \int_{t_0}^{t} \|e_{n-1}(s)\|_1 ds \leq \frac{M^n(t - t_0)^n}{n!}\|e_0\|_\infty \Rightarrow \|e_n\|_\infty \leq \frac{M^nT^n}{n!}\|e_0\|_\infty.
\]
and hence $\lim_{n \to \infty} \|e_n\|_\infty = 0$. ■

4 Computational results

Although the VIM may be implemented for symbolic computation our experiments are disappointing for nonlinear equations. As an example the Mathematica procedure

```mathematica
In[1]:= VIM[f_, U0_, m_] := Module[{V, U = U0, df, Lambda},
    df[t_, x_] := D[f[t, x], x];
    Lambda[U_] := -Exp[Integrate[df[w, x] /. {x -> U, t -> w}, {w, s, t}]];
    For[i = 0, i < m, i++,
        V = U +
        Integrate[Lambda[U] ((D[U, t] - f[t, U]) /. t -> s), {s, 0, t}];
        U = V;
    ]
```
solves the initial value problem (4)-(5).

Example 4.1 The problem

\[ x'(t) = 2x(t) + t \]
\[ x(0) = 0 \]

with the solution \( x(t) = \frac{1}{4}(e^{2t} - 2t - 1) \) is solved in an iteration,

\begin{verbatim}
In[1]:= f[t_, x_] := 2 x + t
In[2]:= U0 = 0;
In[3]:= VIM[f, U0, 1]
Out[3]= (1/4)*(-1 + E^(2*t) - 2*t)
\end{verbatim}

but for a nonlinear differential equation it does not give an acceptable result in a reasonable time.

Much better results we have obtained with numerical computation. The relation (8) is transformed into

\[ u_{n+1}(t) = \int_{t_0}^{t} f(s, u_n(s)) - f(x, u_n(s))u_n(s) e^{\int_{t_0}^{s} f_x(\tau, u_n(\tau))d\tau} ds + 
\]
\[ + e^{\int_{t_0}^{t} f_x(\tau, u_n(\tau))d\tau} x_0. \]

\( u_n \) will be computed as a first order spline function defined by the points \( (t_i, u_n(t_i))_{0 \leq i \leq m} \). \( (t_i)_{0 \leq i \leq m} \) is an equidistant grid on \([t_0, t_f]\). The integrals are computed with the trapezoidal rule.

The following Scilab, [8], code solves the initial value problem (4)-(5)

\begin{verbatim}
function [t, u_old, er, iter]=vim (f, df, x0, t0, tf, m, nmi, tol)
    t=linspace (t0, tf, m)
    h=(tf-t0)/(m-1)
    u_old=x0*ones (1,m)
    sw=0
    iter=0
    while sw do
        iter=iter+1
        u_new=x0*ones (1,m)
        f0=zeros (1,m)
        df0=zeros (1,m)
        for j=1:m do
            f0(j)=f(t(j), u_old(j))
            df0(j)=df(t(j), u_old(j))
        end
        for i=2:m do
            z=zeros (1,m)
            z(i)=0
            for j=i-1:-1:1 do
                z(j)=0.5*h*(df0(j)+df0(j+1))+z(j+1)
            end
        end
        sw=(norm(u_new-2*u_old))/norm(u_old)
        u_old=u_new
    end
end
\end{verbatim}
\[ w = (f_0 - df_0 \cdot u_{\text{old}}) \cdot \exp(z) \]
\[ s = w(1) + w(i) \]
\[ \text{if } i > 2 \text{ then} \]
\[ \quad \text{for } j = 2: i - 1 \text{ do} \]
\[ \quad s = s + 2 \cdot w(j) \]
\[ \quad \text{end} \]
\[ \text{end} \]
\[ u_{\text{new}}(i) = 0.5 \cdot h \cdot s + \exp(z(1)) \cdot x_0 \]
\[ nrm = \text{norm}(u_{\text{new}} - u_{\text{old}}, \infty) \]
\[ u_{\text{old}} = u_{\text{new}} \]
\[ \text{if } nrm < tol \text{ then} \]
\[ \quad sw = \%f \]
\[ \text{end} \]
\[ \text{end} \]
\[ \text{if } nrm < tol \text{ then} \]
\[ \quad er = 0 \]
\[ \text{else} \]
\[ \quad er = 1 \]
\[ \text{end} \]
\[ \text{end function} \]

For the above example

\begin{verbatim}
1     def('y=f(t,x)', 'y=2*x+t')
2     def('y=df(t,x)', 'y=2')
3     x0=0
4     t0=0
5     tf=1
6     m=100
7     nmi=5
8     tol=1e-5
\end{verbatim}

we obtained \( \max_{0 \leq i \leq m} |u_2(t_i) - x(t_i)| \approx 0.0000713 \) and \( \max_{0 \leq i \leq m} |u_2(t_i) - u_1(t_i)| \leq tol = 10^{-5} \).

**Example 4.2** The problem

\[ x'(t) = x^2(t) + 1 \]
\[ x(0) = 0 \]

has the solution \( x(t) = \frac{1-e^{-2t}}{1+e^{-2t}} \). For \( m = 100 \) the results were \( \max_{0 \leq i \leq m} |u_4(t_i) - x(t_i)| \approx 0.0000293 \) and \( \max_{0 \leq i \leq m} |u_4(t_i) - u_3(t_i)| \leq tol = 10^{-5} \).

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