Algorithmic market making: the case of equity derivatives

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Abstract
In this article, we tackle the problem of a market maker in charge of a book of equity derivatives on a single liquid underlying asset. By using an approximation of the portfolio in terms of its vega, we show that the seemingly high-dimensional stochastic optimal control problem of an equity option market maker is in fact tractable. More precisely, the problem faced by an equity option market maker is characterized by a two-dimensional functional equation that can be solved numerically using interpolation techniques and classical Euler schemes, even for large portfolios. Numerical examples are provided for a large book of equity options.

Key words: Market making, Algorithmic trading, Options, Stochastic optimal control.

1 Introduction
The electronification of financial markets started in the seventies on stock exchanges and now affects each and every asset class. Most exchanges and platforms are indeed now fully automated, even for assets that are traditionally traded over the counter (OTC). In banks and in the brokerage industry, automation is also massive as a major proportion of execution is now carried out using algorithms on equity markets. However, automation of trading, and especially market making, is only rarely the norm outside of the equity world, although banks are more and more developing market making algorithms for various asset classes (currencies, bonds, etc.).

In the academic literature, many market making models have been proposed since the eighties. In the early literature on market making, the two main references are the paper of Ho and Stoll [17] and the paper of Grossman and Miller [8]. Ho and Stoll introduced indeed a very relevant framework to tackle the main problem faced by market makers: inventory management. Grossman and Miller, who were more interested in capturing the essence of liquidity, proposed a very simple model with 3 periods that encompassed both market makers and final customers, enabled to understand what happens at equilibrium, and contributed to the important literature on the price formation process. If the latter paper belongs to a strand of literature that is extremely important to go beyond the simple Walrasian view of markets, it is of little help to build market making algorithms. The former paper however, although it took more than 25 years, paved the way to a recent mathematical literature on algorithmic market making.

A seminal reference of the new literature on market making is the paper of Avellaneda and Stoikov [1] who revived the dynamic approach proposed by Ho and Stoll. They indeed showed how the quoting and inventory

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management problems of market makers could be addressed using the tools of stochastic optimal control. Since then, many models have been proposed, most of them to tackle the same problem of single-asset market making as Avellaneda and Stoikov. For instance, Guéant, Lehalle, and Fernandez-Tapia provided in [12] a rigorous analysis of the Avellaneda-Stoikov stochastic optimal control problem and proved that the problem could be simplified into a system of linear ordinary differential equations (ODE) in the case of exponential intensity functions. Cartea, Jaimungal, and coauthors contributed a lot to the literature and added many features to the initial models: market impact, alpha signals, ambiguity aversion, etc. (see [3, 4, 5]). They also considered a different objective function: a risk-adjusted expectation instead of a Von Neumann-Morgenstern expected utility.

The models proposed in the papers of the above literature all share the same characteristics: (i) they are agnostic with respect to the market structure, but are in fact more adapted to OTC markets or limit order book markets with small tick size, (ii) they only deal with single-asset market making, and (iii) they do not deal with the market making of derivative products. However, they have often been extended and specific models have been developed.

Models have been indeed specifically developed by Guilbaud and Pham (see [14, 15]) for assets traded through a limit order book with a large tick size (e.g. most stocks) and for assets traded on platforms with a pro-rata microstructure (e.g. some currency pairs). Interestingly, these models enable the use of aggressive orders by market makers, which is – surprisingly – a standard behavior on equity markets (see [19]).

As far as multi-asset market making is concerned, models have been developed recently to account for the correlation structure of assets. Guéant extended to a multi-asset framework models à la Avellaneda-Stoikov and models à la Cartea-Jaimungal (see [9], [10], and [11]) and showed that the problem boils down, for general intensity functions, to solving a system of (a priori nonlinear) ODEs. The associated question of the numerical methods to approximate the solution of the equations characterizing the optimal quotes of a multi-asset market maker is addressed in [2] using a factorial approach and in [13] using reinforcement learning, both with applications to corporate bond markets.

Finally, as far as asset classes are concerned, there have been few attempts to address market making problems outside of the cash world, because of the intrinsic difficulty of the problem. Market making models for derivatives must account indeed for the strategies on both the market for the underlying asset and the derivatives market, and usually for numerous derivative contracts (e.g. options for lots of strikes and maturities). Option market making is only addressed in a paper by El Aoud and Abergel (see [6]) and in a paper by Stoikov and Sağlam [20]. In the former, the authors consider a single-option market driven by a stochastic volatility model and assume that the position is always $\Delta$-hedged. They provide optimal bid and ask quotes for the option and focus on the risk of model misspecification. In the latter, the authors consider three different settings, but all with only one option: (i) a market maker in a complete market where continuous trading in the perfectly liquid underlying stock is allowed, (ii) a market maker who may not trade continuously in the underlying stock, but rather sets bid and ask quotes in the option and the stock, and (iii) a market maker in an incomplete market with residual risks due to stochastic volatility and overnight jumps – a framework closer to ours.

In this paper, specifically dedicated to the market making of equity derivatives, we consider the case of a market maker in charge of a book of options whose prices are driven by a stochastic volatility model. We assume that trading in continuous time can be carried out in the underlying asset so that the residual risk is only that of the vega associated with the inventory. Using a constant-vega approximation and the same factorial method as in [2], we show that the problem of an equity derivatives market maker boils down to solving a two-dimensional functional equation of the Hamilton-Jacobi-Bellman type that can be tackled numerically using interpolation techniques and the same Euler schemes as those classically used for systems of ODEs. In particular, in spite of the large number of assets, the market making problem is tractable.

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In Section 1 we describe the model and present the optimization problem of the equity option market maker. In Section 2, we show how that problem can be simplified under the constant-vega approximation. In particular, we show that solving the high-dimensional stochastic optimal control problem of the market maker boils down to solving a two-dimensional functional equation. In Section 3, we consider the example of a book of
equity options with several strikes and maturities and provide numerical results obtained through interpolation techniques and an explicit Euler scheme.

2 Description of the problem

We fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) satisfying the usual conditions. Throughout the paper, we assume that all stochastic processes are defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})\).

2.1 The market

We consider a stock whose price dynamics is described by a one-factor stochastic volatility model of the form

\[
\begin{align*}
    dS_t & = \mu S_t dt + \sqrt{\nu_t} S_t dW^S_t \\
    d\nu_t & = a_\nu(t, \nu_t) dt + \xi \sqrt{\nu_t} dW^\nu_t,
\end{align*}
\]

where \(\mu \in \mathbb{R}, \xi \in \mathbb{R}^+, (W^S_t, W^\nu_t)_{t \in \mathbb{R}^+}\) is a couple of Brownian motions with quadratic covariation given by \(\rho = \frac{d(W^S_t, W^\nu_t)}{d t} \in (-1, 1)\), and \(a_\nu\) is such that the processes are well defined (in particular, we assume that the process \((\nu_t)_{t \in \mathbb{R}^+}\) stays positive almost surely).

Remark 1. A classical example for the function \(a_\nu\) is that of the Heston model (see [10]), i.e. \(a_\nu : (t, \nu) \mapsto \kappa_\nu (\theta_\nu - \nu)\) where \(\kappa_\nu, \theta_\nu \in \mathbb{R}^+\) satisfy the Feller condition \(2\kappa_\nu \theta_\nu > \xi^2\).

Assuming interest rates are equal to 0, we introduce a risk-neutral / pricing probability measure \(\mathbb{Q}\) under which the price and volatility processes become

\[
\begin{align*}
    dS_t & = \sqrt{\nu_t} S_t d\tilde{W}^S_t \\
    d\nu_t & = a_\nu(t, \nu_t) dt + \xi \sqrt{\nu_t} d\tilde{W}^\nu_t,
\end{align*}
\]

where \((\tilde{W}^S_t, \tilde{W}^\nu_t)_{t \in \mathbb{R}^+}\) is another couple of Brownian motions under \(\mathbb{Q}\) with quadratic covariation given by \(\rho = \frac{d(\tilde{W}^S_t, \tilde{W}^\nu_t)}{d t} \in (-1, 1)\), and where \(a_\nu\) is such that the processes are well defined.

We consider \(N \geq 1\) European equity options written on the above stock (hereafter, the underlying asset). For all \(i = 1, \ldots, N\), we denote by \((\mathcal{O}^i_t)_{t \in \mathbb{R}^+}\) the price process associated with the \(i\)-th option. Its maturity is denoted by \(T^i\).

Remark 2. In applications to equity derivatives, the options under consideration will always be call and/or put options. However, our setting enables to consider any European payoff.

In the above one-factor model, we know that for all \(i = 1, \ldots, N\), and all \(t \in [0, T^i]\), \(\mathcal{O}^i_t = O^i(t, S_t, \nu_t)\) where \(O^i\) is solution on \([0, T^i] \times \mathbb{R}^+\) of the following partial differential equation (PDE):

\[
0 = \partial_t O^i(t, S, \nu) + a_\nu(t, \nu) \partial_\nu O^i(t, S, \nu) + \frac{1}{2} \nu S^2 \partial_{SS}^2 O^i(t, S, \nu) + \rho \nu S \partial_{SV}^2 O^i(t, S, \nu) + \frac{1}{2} \xi^2 \nu \partial_{SV}^2 O^i(t, S, \nu). \tag{3}
\]

Remark 3. Options prices are also characterized by a terminal condition corresponding to the payoff. However, we will only consider short-term optimization problems for which the time horizon is before the maturity of all the options under consideration. Therefore, we shall never use the final condition associated with Eq. (3).

\[\text{[For references, see for instance [2].]}\]
2.2 The optimization problem of the market maker

We consider an equity option market maker in charge of providing bid and ask quotes for the N above options over the period [0, T] where T < min_{i=1,\ldots,N} T^i (see Remark 3). For all i = 1, \ldots, N we denote by O_t^i − δ^{i,b}_t and O_t^i + δ^{i,a}_t the bid and ask prices proposed by the market maker for the i-th option, where (δ^{i,b}_t)_{t\in[0,T]} := (δ^{i,b}_t, δ^{i,a}_t)_{t\in[0,T]} is F-predictable and bounded from below by a given constant δ_\infty. Hereafter, we denote by A the set of such admissible control processes. The dynamics of the inventory process (q_t)_{t\in[0,T]} := (q_t^1, \ldots, q_t^N)_{t\in[0,T]} of the market maker is given by

\[ dq_t^i := z^i (dN_t^{i,b} − dN_t^{i,a}), \quad i = 1, \ldots, N, \]

where z^i is the size of transactions for option i, and (N_t^{i,b})_{t\in\mathbb{R}^+}, (N_t^{i,a})_{t\in\mathbb{R}^+} are the transaction processes of the i-th option on the bid and ask side, whose respective intensity processes (λ^{i,b}_t)_{t\in\mathbb{R}^+} and (λ^{i,a}_t)_{t\in\mathbb{R}^+} are given by

\[ λ^{i,b}_t := Λ^{i,b} (δ^{i,b}_t) \mathbf{1}_{\{q_{t−} + z^i e^i < \mathbb{Q}\}}, \quad λ^{i,a}_t := Λ^{i,a} (δ^{i,a}_t) \mathbf{1}_{\{q_{t−} − z^i e^i < \mathbb{Q}\}} \]

with (e^i)_{i=1,\ldots,N} the canonical basis of \mathbb{R}^N and \mathbb{Q} the compact set of authorized inventories for the market maker. For i = 1, \ldots, N, Λ^{i,b}, Λ^{i,a} are positive functions satisfying the following classical hypotheses (see [9, 10] for similar assumptions):

- Λ^{i,b}, Λ^{i,a} are twice continuously differentiable.
- Λ^{i,b}, Λ^{i,a} are strictly decreasing, with Λ^{i,b} < 0, Λ^{i,a} < 0.
- \lim_{\delta \to +\infty} Λ^{i,b}(\delta) = \lim_{\delta \to +\infty} Λ^{i,a}(\delta) = 0.
- \sup_{\delta \in \mathbb{R}} \frac{Λ^{i,b}(\delta)}{Λ^{i,a}(\delta)} < 2, j = a, b.

The above conditions are sufficiently general to allow for several relevant forms of intensities: the exponential intensities initially introduced in [11] and used in most of the literature, logistic intensities as in [2], or many SU Johnson intensities as in [13].

Remark 4. In this paper, we only consider the case of transactions with constant size. A generalization that would account for the distribution of trade size could be obtained by following the same reasoning as in [2].

In addition to quoting prices for the N options, the market maker can buy and sell the underlying stock. We assume that the market for that stock is liquid enough to ensure a perfect Δ-hedging.

Remark 5. In practice, for a portfolio that is not vega-hedged, it is usually suboptimal to perfectly Δ-hedge the portfolio because of the correlation between the spot process and the instantaneous variance process. Nevertheless, we assume here for the sake of simplicity that Δ-hedging is carried out in continuous time. A study of the optimal position in the underlying asset and its consequence on our problem is carried out in the Appendix.

In what follows, we denote by (Δ_t)_{t\in[0,T]} the Δ of the portfolio:

\[ Δ_t := \sum_{i=1}^{N} ∂_\mathcal{S} O^i (t, S_t, ν_t) q_t^i \text{ for all } t \in [0, T]. \]

The resulting dynamics for the cash process (X_t)_{t\in[0,T]} of the market maker is:

\[ dX_t := \sum_{i=1}^{N} (z^i (δ^{i,b}_t dN_t^{i,b} + δ^{i,a}_t dN_t^{i,a}) − O_t^i dq_t^i) + S_t dΔ_t + d\langle Δ, S \rangle_t. \]
We denote by \((V_t)_{t \in [0,T]}\) the process for the Mark-to-Market (MtM) value of the market maker’s portfolio (cash, shares, and options), i.e.,

\[
V_t := X_t - \Delta_t S_t + \sum_{i=1}^{N} q_i^t O_i^t.
\] (8)

The dynamics of that process is given by

\[
dV_t = \sum_{i=1}^{N} z^i \left( \delta_t^{i,a} dN_t^{i,a} + \delta_t^{i,b} dN_t^{i,b} \right) + q_i^t dO_i^t - \Delta_t dS_t
\]

\[
= \sum_{i=1}^{N} z^i \left( \delta_t^{i,a} dN_t^{i,a} + \delta_t^{i,b} dN_t^{i,b} \right) + q_i^t \partial_v O_i^t(t, S_t, \nu_t) \left( a_p(t, \nu_t) - a_Q(t, \nu_t) \right) dt + \sqrt{\gamma} \xi q_i^t \partial_v O_i^t(t, S_t, \nu_t) dW_t^v. \] (9)

For all \(i = 1, \ldots, N\), the vega of the \(i\)-th option is defined as

\[
V_i^v := \partial_v \sqrt{\gamma} O_i^v(t, S_t, \nu_t) = 2 \sqrt{\gamma} \nu_v \partial_v O_i^v(t, S_t, \nu_t) \text{ for all } t \in [0, T].
\] (10)

Hence, we can rewrite the dynamics of the portfolio as

\[
dV_t = \sum_{i=1}^{N} z^i \left( \delta_t^{i,a} dN_t^{i,a} + \delta_t^{i,b} dN_t^{i,b} \right) + q_i^t \left( a_p(t, \nu_t) - a_Q(t, \nu_t) \right) \frac{1}{2 \sqrt{\gamma}} dt + \xi \frac{1}{2} q_i^t V_i^v dW_t^v. \] (11)

Following the academic literature on market making, we can consider two objective functions:

- As in the initial Avellaneda and Stoikov setting \([1]\) (see also \([9, 10, 12]\)), we can consider the following expected utility objective function:

\[
\sup_{\delta \in A} \mathbb{E} \left[ - \exp \left( - \gamma V_T \right) \right], \] (12)

where \(\gamma > 0\) is the risk-aversion parameter of the market maker.

- As in \([3, 4, 5]\), but also in \([10]\), we can consider a risk-adjusted expectation for the objective function, i.e.

\[
\sup_{\delta \in A} \mathbb{E} \left[ V_T - \frac{\gamma}{2} \int_0^T \left( \sum_{i=1}^{N} \xi q_i^t V_i^v \right)^2 dt \right]. \] (13)

This second objective function in our case writes

\[
\sup_{\delta \in A} \mathbb{E} \left[ \int_0^T \sum_{i=1}^{N} \left( \sum_{j=a,b} z^{i,j} A^{i,j} (\delta_t^{i,j})_{Q,t} \psi(j) z^i e^i \in \mathcal{Q} \right) + q_i^t \left( a_p(t, \nu_t) - a_Q(t, \nu_t) \right) \frac{1}{2 \sqrt{\gamma}} dt - \frac{7 \xi^2}{8} \int_0^T \left( \sum_{i=1}^{N} q_i^t V_i^v \right)^2 dt \right], \] (14)

where

\[
\psi(j) := \begin{cases} +1 & \text{if } j = a \\ -1 & \text{if } j = b. \end{cases}
\]

These two objective functions are close to one other in practice. Guéant showed in \([10]\) that they give similar optimal quotes in practical examples. Furthermore, in many cases, the expected utility framework with exponential utility function can be reduced to the maximization of the expected PnL minus a quadratic penalty of the above form, up to a change in the intensity functions (see \([18]\)).

In what follows, we consider the second framework. Therefore, we define the value function

\[
u : (t, S, \nu, q) \in [0, T] \times \mathbb{R}^{+2} \times \left( \mathcal{Q} \cap \prod_{i=1}^{N} z^i \mathbb{Z} \right) \mapsto u(t, S, \nu, q)
\]
associated with (14) as

\[ u(t, S, \nu, q) = \sup_{(\delta_x)_{x \in \mathbb{x}, \tau} \in \mathcal{A}_t} \mathbb{E}_{(t, S, \nu, q)} \left[ \int_t^T \sum_{i=1}^N \left( \left( \sum_{j=a,b} z^{i,j} \Lambda^{i,j}(\delta_x^i) \mathbb{1}_{\left\{ q_i - \psi(j)z^i e^j \in \mathcal{Q} \right\}} \right) \right) \right. \]
\[
+ q_s^i \nu^i s \frac{a_P(s, \nu_s) - a_Q(s, \nu_s)}{2 \sqrt{\nu_s}} ds - \frac{\gamma \xi^2}{8} \int_t^T \left( \sum_{i=1}^N q_s^i \nu^i \right)^2 ds \left. \right], \tag{15} \]

where \( \mathcal{A}_t \) is the set of admissible controls starting from \( t \).

**Remark 6.** It is important to notice that we defined the value function for values of the inventories on the set \( \mathcal{Q} \cap \prod_{i=1}^N z^i \mathbb{Z} \). In fact, we could define it more generally for \( q \in \mathcal{Q} \) but nothing would guarantee that \( u \) is continuous with respect to the inventory variable in that case\(^4\) because of the presence of indicator functions in the definition of intensity functions. An interesting trick (proposed by Bruno Bouchard) to define a continuous value function on the entire set \( \mathcal{Q} \) is to slightly change the model by smoothing the indicator functions. This could mean for instance that when a trade will bypass a risk limit, we choose randomly to accept it or not, with a probability that depends on the distance to the risk limit.

### 2.3 Assumptions and approximations

The above stochastic optimal control problem can be addressed from a theoretical point of view using an approach similar to that of [10]. However, when it comes to approximating the optimal quotes a market making should set for the \( N \) options, classic numerical methods are of no help because the value function \( u \) has \( N + 2 \) variables (in addition to the time variable). In order to beat the curse of dimensionality and be able to approximate the solution of (15) we propose a method based on the following assumptions / approximations:

**Assumption 1.** We approximate the vega of each option over \([0, T]\) by its value at time \( t = 0 \), namely

\[ \mathbb{V}_i^i = \mathbb{V}_0^i =: \mathbb{V}_i \in \mathbb{R}, \text{ for all } i = 1, \ldots, N. \] (16)

**Assumption 2.** We assume that \( a_P = a_Q \).

**Assumption 3.** We assume that the set of authorized inventories is associated with vega risk limits, i.e.

\[ \mathcal{Q} = \left\{ q \in \mathbb{R}^N \left| \sum_{i=1}^N q^i \mathbb{V}_i^i \in \left[ -\mathbb{V}, \mathbb{V} \right] \right. \right\}, \]

where \( \mathbb{V} \in \mathbb{R}^{++} \) is the vega risk limit of the market maker.

The first assumption is acceptable if \( T \) is not too large. This raises in fact the deep question of the reasonable value of \( T \), as there is no natural choice for the horizon of the optimization problem. In practice, \( T \) has to be sufficient large to allow for several transactions in each option and small enough for the constant-vega approximation to be relevant (and smaller than the maturities of the options). It is also noteworthy, although it is time-inconsistent, that one can use the proceed of the model (with the constant-vega approximation) over a short period of time and then run the model again with updated vegas. This is a classical practice in applied optimal control when the parameters are estimated online.

The second assumption is acceptable as stochastic volatility models are often calibrated using historical data. This assumption also states that the market maker has no personal view on the future behavior of volatility.

\(^4\)In the case where the distribution of trade size is absolutely continuous, and not a Dirac mass as in the present paper, this problem does not appear (see [2]).
The third assumption states that risk limits are related to the only source of risk (as the portfolio is Δ-hedged). This is a natural assumption. The only drawback is that no risk limit can be set to individual options.

3 An approximate solution to the problem

3.1 Change of variables: beating the curse of dimensionality

Under the above assumptions, the $N + 2$ state variables can be replaced by a single one: the vega of the portfolio $\mathcal{V}^π := \sum_{i=1}^N q_i \mathcal{V}_i$ whose dynamics is

$$d\mathcal{V}_i^π = \sum_{i=1}^N z^i (dN_{i}^{t-b} - dN_{i}^{t-a}).$$

It is clear indeed that the value function $u$ verifies

$$v(t, s, \nu, q) \in [0, T] \times \mathbb{R}^2 \times \left( Q \cap \prod_{i=1}^N z^i \mathbb{Z} \right),$$

where

$$v(t, \mathcal{V}^π) = \sup_{(\delta_s)_{s \in [t, T]} \in \mathcal{A}_t} \mathbb{E}(v(t, \mathcal{V}^π)) \left[ \int_t^T \left( N \sum_{i=1}^N z^i \delta_{i,j} \Lambda_{i-j} (\delta_{i-j}^s) \mathbb{I}_{|\mathcal{V}^π - \psi(j)z^i \mathcal{V}^π| \leq \mathcal{V}^π} - \frac{\gamma^2}{8} \mathcal{V}^π \right) ds \right].$$

In other words, the problem boils down, under the three above assumptions, to a low-dimensional optimal control problem where the unique state variable is driven by $2N$ controlled point processes.

It is important to recall (see Remark [6]) that because $u$ is defined for inventories on the grid $Q \cap \prod_{i=1}^N z^i \mathbb{Z}$, the function $v$ is a priori defined only for $\mathcal{V}^π \in \left\{ \sum_{i=1}^N q_i \mathcal{V}_i \mid q \in \prod_{i=1}^N z^i \mathbb{Z} \right\} \cap [-\mathcal{V}, \mathcal{V}].$

3.2 Hamilton-Jacobi-Bellman equation, optimal control, and extension to the full vega domain

The Hamilton-Jacobi-Bellman equation associated with (17) is given by

$$0 = \partial_t v(t, \mathcal{V}^π) - \frac{\gamma^2}{8} \mathcal{V}^π + \sum_{i,j=1}^N z^i \mathbb{I}_{|\mathcal{V}^π - \psi(j)z^i \mathcal{V}^π| \leq \mathcal{V}^π} H^{i,j} \left( \frac{v(t, \mathcal{V}^π) - v(t, \mathcal{V}^π - \psi(j)z^i \mathcal{V}^π)}{z^i} \right),$$

(18)

with final condition $v(T, \mathcal{V}^π) = 0,$ where

$$H^{i,j}(p) := \sup_{\delta_{i,j} \geq \delta_{\infty}} \Lambda_{i,j} (\delta_{i,j}^s) (\delta_{i,j}^s - p), \quad i = 1, \ldots, N, \quad j = a, b.$$ (19)

We end up therefore with a two-dimensional functional equation of the Hamilton-Jacobi-Bellman type.

Once the value function is known, the optimal controls, which are the optimal mid-to-bid and ask-to-mid associated with the $N$ options, are given by the following formula (see [10]):

$$\delta_{i,j}^{t,t} = \max \left( \delta_{\infty}, (\Lambda_{i,j})^{-1} \left( -H^{i,j} \left( \frac{v(t, \mathcal{V}^π) - v(t, \mathcal{V}^π - \psi(j)z^i \mathcal{V}^π)}{z^i} \right) \right) \right), \quad i = 1, \ldots, N, \quad j = a, b.$$ (19)

The question that remains is therefore the approximation of $v$. Of course, because $v$ is only defined on $[0, T] \times \left\{ \sum_{i=1}^N q_i \mathcal{V}_i \mid q \in \prod_{i=1}^N z^i \mathbb{Z} \right\} \cap [-\mathcal{V}, \mathcal{V}],$ designing a numerical method is not straightforward.
In order to approximate \( v \), a strategy could consist in extending the definition of \( v \) and therefore the Hamilton-Jacobi-Bellman equation (20) to \([0, T] \times [-\overline{\nu}, \overline{\nu}]\). As discussed in Remark 6, the danger of this extension is however that \( v \) may have discontinuities unless we smooth the indicator functions in the definition of the intensity functions or equivalently in the Hamilton-Jacobi-Bellman equation.

Our strategy consists in fact in smoothing the indicator functions in the Hamilton-Jacobi-Bellman equation in order to approximate, instead of \( v \), the continuous viscosity solution \( \tilde{v} \) of the following equation:

\[
0 = \partial_t \tilde{v}(t, \nu) - \frac{\gamma^2}{8} \nu^2 + \sum_{i=1}^{N} \sum_{j=a,b} \nu^i \Phi \left( \left| \nu^i - \psi(j) \nu^j \right| - \overline{\nu} \right) H^{i,j} \left( \frac{\tilde{v}(t, \nu^i) - \tilde{v}(t, \nu^i - \psi(j) \nu^j)}{\nu^i} \right),
\tag{20}
\]

with final condition \( \tilde{v}(T, \nu) = 0 \), where \( \Phi \) is a logistic function that approximates the indicator function.

Then, in order to approximate \( \tilde{v} \), we consider a regular grid of \([0, T] \times [-\overline{\nu}, \overline{\nu}]\) and use an explicit Euler scheme on that grid using linear interpolation whenever the required point in the functional equation is not on the grid.

\section{Numerical results}

\subsection{Model parameters}

In this section we consider a book of options and derive the optimal quotes using the above approach.

For this purpose, we consider an underlying stock with the following characteristics:

- Stock price at time \( t = 0 \): \( S_0 = 10 \) €.
- Instantaneous variance at time \( t = 0 \): \( \nu_0 = 0.04 \) year\(^{-1}\).
- Heston model, i.e. \( a_p(t, \nu) = a_Q(t, \nu) = \kappa(\theta - \nu) \) where \( \kappa = 2 \) year\(^{-1}\) and \( \theta = 0.04 \) year\(^{-1}\).
- Volatility of volatility parameter: \( \xi = 0.2 \) year\(^{-1}\).
- Spot-variance correlation: \( \rho = -0.5 \).

We consider the case of a market maker dealing with 20 European call options written on that stock where the strike \( \times \) maturity couples are the elements \((K^i, T^i)_{i=1,...,20}\) of the set \( \mathcal{K} \times \mathcal{T} \), where

\[
\mathcal{K} = \{8 \text{ €}, 9 \text{ €}, 10 \text{ €}, 11 \text{ €}, 12 \text{ €}\} \quad \text{and} \quad \mathcal{T} = \{1 \text{ year, 1.5 years, 2 years, 3 years}\}.
\]

![Implied volatility surface associated with the above parameters](image-url)
The associated implied volatility surface is plotted in Figure 1.\(^5\)

The liquidity parameters of these options are the following:

- **Intensity functions:**
  \[ \Lambda_{i,j}(\delta) = \frac{\lambda}{1 + e^{a+b\delta_{i,j}}}, \quad i = 1, \ldots, N, \quad j = a, b. \]
  where \(\lambda = 37,800 = 252 \times 150 \text{ year}^{-1}, \quad a = 0.7, \quad \text{and} \quad b = 150 \text{ year}^2.\)

  The choice of \(\lambda\) corresponds to 150 requests per day. The choice of \(a\) corresponds to a probability of \(\frac{1}{1+e^{a}} \approx 33\%\) to trade when the answered quote is the mid-price. The choice of \(b\) corresponds to a probability of \(\frac{1}{1+e^{-b}} \approx 69\%\) to trade when the answered quote corresponds to an implied volatility 1\% better for the client and a probability of \(\frac{1}{1+e^{-2b}} \approx 10\%\) to trade when the answered quote corresponds to an implied volatility 1\% worse for the client.

- **Size of transactions:** \(z_i = 5 \cdot 10^5\) assets. This corresponds approximately to 500,000 € per transaction.\(^6\)

Regarding the risk limits and the objective function, we consider the following:

- **Vega risk limit:** \(\mathbb{V} = 10^7\) € · year\(^{\frac{1}{2}}\).

- **Time horizon given by** \(T = 0.008\) year (i.e. 2 days). This horizon ensures convergence towards stationary quotes at time \(t = 0\) (see Figure 3 below).

- **Risk aversion given by** \(\gamma = 3 \cdot 10^{-6}\) €\(^{-1}\).

4.2 Optimal quotes

Using a monotone explicit Euler scheme with linear interpolation on a grid of size 1500 × 60, we approximate the value function solution to (20) on the domain \([0, T] \times [-\mathbb{V}, \mathbb{V}]\). This value function is plotted in Figure 2.

![Figure 2: Value function as a function of the portfolio vega.](image)

From that value function, we deduce the optimal bid and ask quotes of the market maker for each option as a function of the portfolio vega. As mentioned above, we chose \(T = 0.008\) year (i.e. 2 days) to ensure convergence of the optimal quotes to their stationary values (see Figure 3).

\(^5\)This plot has been computed using \(10^5\) Monte-Carlo simulations for each option.

\(^6\)This is only an approximation as trade sizes are in number of options and option prices move.
Focusing on the asymptotic values, we now present in Figures 4, 5, 6, 7, and 8 the optimal bid quotes as a function of the portfolio vega for each strike and maturity. More precisely, as the options we consider can have very different prices, we consider instead of the optimal bid quotes themselves the ratio between each optimal mid-to-bid quote and the price (at time $t = 0$) of the corresponding option.

Figure 3: Optimal mid-to-bid quotes as a function of time for option 1: $(K^1, T^1) = (8, 1)$.

Figure 4: Optimal mid-to-bid quotes divided by option price as a function of the portfolio vega for $K = 8$.

Figure 5: Optimal mid-to-bid quotes divided by option price as a function of the portfolio vega for $K = 9$. 
The results are in line with what was expected: the mid-to-bid quotes increase with the portfolio vega. The incentive to buy options with positive vega decreases indeed with the vega of the portfolio.

For equity derivatives traders, it is often more convenient to work in terms of implied volatility. In Figures 9, 10, 11, 12, and 13 we plot the optimal bid quotes for the 20 options in terms of implied volatility (divided
by the implied volatility at time $t = 0$). The results are in line with our expectations: the implied volatility proposed by a market maker to buy a positive-vega option decreases with the vega of the portfolio.

Figure 9: Optimal bid quotes as a function of the portfolio vega for K=8.

Figure 10: Optimal bid quotes as a function of the portfolio vega for K=9.

Figure 11: Optimal bid quotes as a function of the portfolio vega for K=10.
1. Conclusion

In this article, we tackled the problem of an equity derivatives market maker dealing with options on a single underlying asset. Using a constant-vega approximation, we showed how to reduce the problem to a two-dimensional functional equation whose solution can easily be approximated using an explicit Euler scheme and linear interpolation. Furthermore, our method scales linearly in the number of options and can therefore be used with large books of equity derivatives. Our method is illustrated by an example involving 20 European calls, but our model can be used with any European derivative contract.

Appendix: an alternative to the $\Delta$-hedging assumption

Throughout the body of this paper, we assumed that the market maker ensured $\Delta$-hedging. In this appendix, we show that this assumption can be relaxed without much change in the reasoning.

Let us introduce the process $(q^S_t)_{t\in[0,T]}$ representing the inventory of the market maker in the underlying asset. The cash process of the market maker $(X_t)_{t\in[0,T]}$ rewrites as

$$dX_t = \sum_{i=1}^{N} \left( z_i \left( \delta_i^{i,b} dN_i^{i,b} + \delta_i^{i,a} dN_i^{i,a} \right) - O_i^{i,a} dq_i^a \right) - q_i^S dS_t - d(q^S, S)_t.$$

As noted while publishing this paper, our method can easily be extended to the case of multiple underlying assets using the same method as in [2] if the instantaneous variance processes of the different assets are driven by a few factors.
The Mark-to-Market value of the portfolio writes
\[ V_t = X_t + q_t^S S_t + \sum_{i=1}^{N} q_i^t O_i^t. \]
and its dynamics is
\[ dV_t = \sum_{i=1}^{N} \left( z_i^t \left( \delta_i^i b dN_i^b + \delta_i^i a dN_i^a \right) + \frac{\xi}{2} q_i^t V_t^\sigma dW_t^\nu \right) + \sqrt{\nu_t} S_t \left( \sum_{i=1}^{N} q_i^t \partial_S O_i(t, S_t, \nu_t) + q_i^t \right) dW_t^S. \]
Denoting by \( \Delta_t^\pi := \sum_{i=1}^{N} q_i^t \partial_S O_i(t, S_t, \nu_t) \) the \( \Delta \) of the market maker’s portfolio at time \( t \), our mean-variance optimization problem becomes
\[
\sup_{(\delta, q^S) \in \mathcal{A}'} \mathbb{E} [V_T] - \frac{\gamma}{2} \left[ \int_0^T \frac{\xi}{2} V_t^\sigma dW_t^\nu + \sqrt{\nu_t} S_t \left( \Delta_t^\pi + q_t^S \right) dW_t^S \right],
\]
where
\[
\mathcal{A}' = \left\{ (\delta_t, q_t^S)_{t \in [0, T]} \mid \delta \text{ is an } \mathbb{R}^{2N}\text{-valued predictable process bounded from below by } \delta_\infty \right. 
\left. \quad \text{ and } q^S \text{ is an } \mathbb{R}\text{-valued adapted process with } \mathbb{E} \left[ \int_0^T \nu_t S_t^2 \left( \Delta_t^\pi + q_t^S \right)^2 dt \right] < +\infty \right\}.
\]
Noticing that
\[
\forall \left( \int_0^T \frac{\xi}{2} V_t^\sigma dW_t^\nu + \sqrt{\nu_t} S_t \left( \Delta_t^\pi + q_t^S \right) dW_t^S \right) = \int_0^T \left( \frac{\xi^2}{4} V_t^\sigma + \nu_t S_t^2 \left( \Delta_t^\pi + q_t^S \right)^2 + \rho \xi V_t^\sigma \sqrt{\nu_t} S_t \left( \Delta_t^\pi + q_t^S \right) \right) dt,
\]
we easily see that the variance term is minimized for \( q^S = q^{S*} \) where
\[
\forall t \in [0, T], \quad q^{S*}_t = -\Delta_t^\pi - \frac{\rho \xi V_t^\sigma}{2 \sqrt{\nu_t} S_t},
\]
and that its minimum value is
\[
(1 - \rho^2) \int_0^T \frac{\xi^2}{4} V_t^\sigma dt.
\]
Therefore, the optimization problem boils down to
\[
\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[ \int_0^T \left( \sum_{i=1}^{N} \sum_{j=a,b} z_i^j \delta_i^{(i,j)} A^{(i,j)} \mathbb{I} \left( \left| V_t - \psi(j) z_i \right| \leq \psi \right) \right) - \frac{\gamma \xi^2}{8} (1 - \rho^2) V_t^\sigma \right] dt,
\]
and we recover the same optimization problem as in the body of the paper, except that the risk aversion parameter is multiplied by \( 1 - \rho^2 \) to account for the reduction of risk made possible by the optimal trading strategy in the underlying asset in presence of vol-spat correlation.

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