ON THE EQUIVARIANT COHOMOLOGY OF HYPERPOLAR ACTIONS ON SYMMETRIC SPACES

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Abstract. We show that the equivariant cohomology of any hyperpolar action of a compact and connected Lie group on a symmetric space of compact type is a Cohen-Macaulay ring. This generalizes some results previously obtained by the authors.

1. Introduction

When investigating smooth group actions on manifolds, an insightful topological invariant one associates to it is its equivariant cohomology. (The coefficient ring throughout this paper is always \( \mathbb{Q} \).) This remains an effective tool even in cases when concrete descriptions are hard to obtain. Geometric features of the action are reflected by algebraic properties of its equivariant cohomology. Equivariant formality is an example of such a property, which is relevant due to its simple definition (see Sect. 2) and to the numerous situations when it is satisfied, such as Hamiltonian actions of compact Lie groups on compact symplectic manifolds (cf. [23]) or isotropy actions of compact symmetric spaces (see [9]). A larger class consists of the so-called Cohen-Macaulay actions, which are also defined in Sect. 2. Their relevance in the context of equivariant cohomology was noticed by several authors already in the early 70s, before being thoroughly investigated in the more recent papers [12] and [13]. These investigations bore fruit: using results and techniques developed there, the first- and third-named authors of this paper have shown in [11] that any cohomogeneity-one action of a compact connected Lie group on a closed manifold is Cohen-Macaulay.

Cohomogeneity-one actions are special cases of hyperpolar actions. Recall that an isometric action of a compact connected Lie group on a Riemannian manifold is polar if there exists a submanifold which is intersected by each orbit of the action, the orbit being orthogonal to the submanifold at each intersection point. If the submanifold can be chosen to be flat relative to the induced Riemannian metric, we say that the action is hyperpolar. In this paper we will be interested in the case when the manifold on which the group acts is a Riemannian symmetric space of compact type, that is, a quotient \( G/H \), where \( G \) is a compact connected semisimple Lie group and \( H \) a closed subgroup such that \( G^\sigma_0 \subset H \subset G^\sigma \), where \( \sigma \) is an involutive automorphism of \( G \), \( G^\sigma \) its fixed point set and \( G^\sigma_0 \) the identity component of the latter group. Polar actions on compact symmetric spaces have been extensively investigated by many authors, see for instance [19], [20], [24], [26], [25], and [27]. We will prove as follows:

\[ \text{Date: September 3, 2018.} \]
Theorem 1.1. Any hyperpolar action of a compact connected Lie group on a symmetric space of compact type is Cohen-Macaulay.

The proof relies essentially on the classification of the actions mentioned in the theorem, which was obtained by A. Kollross in \[27\]. Before stating this result, we need to describe an important class of examples of hyperpolar actions on the symmetric space \(G/H\) mentioned above. Let \(\tau\) be another involutive automorphism of \(G\) and consider its fixed point set, \(G_{0}\), along with the identity component \(K := G_{0}\). The action of \(K\) on \(G/H\) by left translations turns out to be hyperpolar, see \[19\] Ex. 3.1. It is known under the generic name of a Hermann action, after R. Hermann, who first investigated this situation in \[21\] (note that originally, in \[21\], the group \(G_{r}\) was assumed to be connected). According to Kollross’ theorem \[27\], any hyperpolar action is orbit equivalent to a direct product of actions of one of the following types: transitive, of cohomogeneity one, or Hermann (for the notion of orbit equivalence, see Sect. 4). We already know that any action of one of the first two types is Cohen-Macaulay, see \[11\] Cor. 1.2. Thus, to prove Thm. 1.1 there are two steps to be performed: show first that any Hermann action is Cohen-Macaulay and second that the Cohen-Macaulay property is preserved under orbit equivalence. Both goals are actually achieved, as follows:

Theorem 1.2. If \(G, K,\) and \(H\) are as above, then the (Hermann) action of \(K\) on \(G/H\) by left translations is Cohen-Macaulay.

A big part of the paper is devoted to the proof of this theorem, see Sect. 3 (also note that the result remains true even if \(G\) is not necessarily semisimple, see Rem. 3.4). It is worth recalling at this point a related result, obtained by the first-named author of this paper in \[9\], which says that the \(K\)-action on \(G/K\) by left translations is equivariantly formal; recently, a conceptually different proof has been obtained by the second-named author in \[17\]. Thm. 1.2 above is a generalization of this (one also takes into account that, in general, a smooth \(G\)-action is equivariantly formal if and only if it is Cohen-Macaulay and there exists at least one isotropy subgroup whose rank is equal to rank \(G\), see Prop. 2.4 below). Pairs \((G, K)\) with the property that the action of \(K\) on \(G/K\) is equivariantly formal have also been investigated in \[10\], \[5\], \[6\], \[7\], and again \[17\].

Our second goal is to show that the Cohen-Macaulay property is preserved under orbit equivalence. This turns out to be true: see Thm. 4.2 which represents the main result of Sect. 4. The proof of Thm. 1.1 is also outlined in full detail in that section.

2. General considerations

2.1. The action of \(G\) on \(M\). Throughout this subsection \(G\) will always be a compact and connected Lie group which acts smoothly on a closed manifold \(M\), although many of the results presented below hold in a larger generality. To any such action one attaches the equivariant cohomology \(H_{G}^{*}(M)\). This can be defined as the usual cohomology of the Borel construction \(EG \times_{G} M\), where \(EG \rightarrow BG\) is the classifying principal bundle of \(G\).
For an introduction to this theory we refer to [22, Ch. III] or [15, Appendix C]. It is worth mentioning here that $H^*_G(M)$ has a canonical structure of an $H^*(BG)$-algebra.

In case $H^*_G(M)$ is free as an $H^*(BG)$-module, we say that the $G$-action is *equivariantly formal*. Here is a well-known class of examples of such actions (cf. [16, Thm. 6.5.3]):

**Example 2.1.** If $H^{\text{odd}}(M) = 0$ then any $G$-action on $M$ is equivariantly formal.

The following criterion will be useful later, see [22, p. 46].

**Proposition 2.2.** Let $T$ be a torus acting on a closed manifold $M$.

(a) The total Betti number of the fixed point set $M^T$ is at most equal to the total Betti number of $M$.

(b) The two numbers mentioned above are equal if and only if the $T$-action is equivariantly formal.

In what follows we will be looking in more detail at the algebraic structure of $H^*_G(M)$. Relative to its structure of an $H^*(BG)$-module, it is a graded module over a graded ring. The latter, $H^*(BG)$, is a commutative Noetherian ring, cf., e.g., [31, Sect. 2]. It is also a *local ring, in the sense that it has a unique graded ideal which is maximal among all graded ideals (namely $H^{>0}(BG)$). The theory of graded modules over *local Noetherian rings is nicely treated in [4, Sect. 1.5]. We also refer to [13, Sect. 5] for a self-contained approach, suitable for applications to equivariant cohomology.

**Definition 2.3.** A smooth action $G \times M \to M$ is called Cohen-Macaulay if $H^*_G(M)$ as a module over $H^*(BG)$ is Cohen-Macaulay.

An immediate observation is that any equivariantly formal action is Cohen-Macaulay. The converse implication is in general not true, as one can easily see in concrete situations (for example, by [11, Prop. 2.6], any transitive action is Cohen-Macaulay, without being in general equivariantly formal). However, the following result is helpful in this context, see [12, Prop. 2.5]:

**Proposition 2.4.** A smooth action $G \times M \to M$ is equivariantly formal if and only if it is Cohen-Macaulay and there exists at least one point in $M$ whose isotropy subgroup has rank equal to rank $G$.

For later use, we also mention:

**Proposition 2.5.** Let $G \times M \to M$ be a smooth action and $T \subset G$ an arbitrary maximal torus.

(a) The $G$-action on $M$ is equivariantly formal if and only if so is the induced $T$-action.

(b) The $G$-action on $M$ is Cohen-Macaulay if and only if so is the induced $T$-action.

Item (a) is a standard result, see for instance [15, Prop. C.26]. Item (b) is the content of [12, Prop. 2.9]. It is worth pointing out in this context that if $H \subset G$ is an arbitrary closed
subgroup, then the property of being equivariantly formal is preserved when passing from $G$ to $H$, whereas it is possible for the $G$-action to be Cohen-Macaulay and the $H$-action not to be like that, see [12, Ex. 2.13].

We will need an equivalent characterization of the Cohen-Macaulay condition above, this time exclusively in terms of the ring structure of $H^*_G(M)$. The latter is in general not a commutative ring, but it is nevertheless graded commutative, in the sense that $x \cdot y = (-1)^{(\deg x)(\deg y)} y \cdot x$, for all $x, y \in H^*_G(M)$. Although a self-contained treatment of graded commutative rings is not immediately available in the literature, there is no essential difference relative to the (usual) commutative case. A systematic and thorough study of graded commutative rings has been undertaken by M. Poulsen in Appendix A of his Master Thesis [30]. For instance, by using the remark following Prop. A.5 in [30] and also taking into account that $H^*_G(M)$ is finitely generated as an algebra over its degree zero component $H^*_0G(M) \simeq \mathbb{Q}$ (see for instance [31, Thm. 2.1]), we deduce that $H^*_G(M)$ is a Noetherian ring.

For graded commutative Noetherian rings of the type $R = \bigoplus_{i \geq 0} R^i$ whose degree zero component $R^0$ is a field, one can see in [30] that the concepts of Krull dimension, depth (relative to the ideal $\bigoplus_{i > 0} R^i$), and Cohen-Macaulay can be defined in the same way as for commutative rings. This enables us to prove the following result. It is obtained by adapting [32, Prop. 12, Sect. IV.B] to our current set-up.

**Lemma 2.6.** (J.-P. Serre) Let $R = \bigoplus_{i \geq 0} R^i$ and $S = \bigoplus_{i \geq 0} S^i$ be two graded commutative Noetherian rings with $R^0 = S^0 = \mathbb{Q}$ and $\varphi : R \to S$ a homomorphism of graded rings which makes $S$ into an $R$-module which is finitely generated. Let also $A$ be a finitely generated $S$-module. Then $A$ is Cohen-Macaulay as an $S$-module if and only if so is $A$ as an $R$-module.

**Corollary 2.7.** The $G$-action on $M$ is Cohen-Macaulay if and only if the ring $H^*_G(M)$ is Cohen-Macaulay.

2.2. The action of $K$ on $G/H$. Let $G$ be again a compact and connected Lie group and let $K, H \subset G$ be closed subgroups. In this section we list some results concerning the three group actions mentioned in the following proposition.

**Proposition 2.8.** If $K$ and $H$ are connected, the following assertions are equivalent:

(a) the action of $K$ on $G/H$ by left translations is Cohen-Macaulay;

(b) the action of $H$ on $G/K$ by left translations is Cohen-Macaulay;

(c) the action of $H \times K$ on $G$ given by $(h, k).g = h g k^{-1}$ is Cohen-Macaulay.

**Proof.** Observe that we have the ring isomorphisms

$$H^*_K(G/H) \simeq H^*_H \times_H (G) \simeq H^*_H(G/K),$$

which are due to the fact that both factors of the product $H \times K$ act freely on $G$. The equivalences now follow from Cor. 2.7.

In the light of item (c) above, the following proposition appears to be useful. It is a consequence of [6, Thm. 10.2.2]. Denote by $g, t, h$ the Lie algebras of $G, K$, and $H$. 


Proposition 2.9. (J. Carlson) If $K$ and $H$ are connected, then the equivariant cohomology of the action of $H \times K$ on $G$ given by $(h, k).g = h g k^{-1}$ depends only on $g$, $h$, $k$, and the inclusions $h \hookrightarrow g$ and $k \hookrightarrow g$.

Finally, we mention a result that shows how to deal with the situation when $H$ is not connected.

Proposition 2.10. Assume $K$ is connected and let $H_0$ be the identity component of $H$. If the $K$-action on $G/H_0$ is Cohen-Macaulay, then so is the $K$-action on $G/H$.

Proof. Use the characterization given by Prop. 2.8 (c). Note that $H_0 \times K$ is the identity component of $H \times K$, hence $H^*_{H \times K}(G) \simeq H^*_{H_0 \times K}(G)^{H/H_0}$. Since $H^*_{H_0 \times K}(G)$ is a Cohen-Macaulay ring and the order of $H/H_0$ is invertible in this ring, one can use [4, Cor. 6.4.6] and deduce that the ring of invariants is Cohen-Macaulay too.

3. Hermann actions

In this section we will prove Thm. 1.2. The meaning of $G$, $H$, and $K$ is like in Sect. 1. In particular, the Lie group $G$ is compact, connected, and semisimple (the case when $G$ is not necessarily semisimple is discussed in Rem. 3.4). Due to Propositions 2.8, 2.9, and 2.10, we do not lose any generality by assuming that $G$ is simply connected and only prove Thm. 1.2 in this special situation. Under this assumption, the groups $H$ and $K$ mentioned in the introduction are equal to the fixed point sets of the involutive automorphisms $\sigma$ and $\tau$ respectively. The induced automorphisms of $g$ will also be denoted by $\sigma$ and $\tau$. In general, we will not make any notational distinction between an automorphism of $G$ and the induced Lie algebra automorphism.

3.1. The case when $G$ is simple. We will prove Thm. 1.2 under the assumption that $G$ is simple and simply connected, which is valid throughout the whole subsection.

Lemma 3.1. Assume $G$ is not equal to $\text{Spin}(8)$. If rank $K \leq$ rank $H$, then a maximal torus in $K$ is group-conjugate with a subgroup of $H$.

Proof. If one of $\sigma$ or $\tau$ is an inner automorphism, the claim in the lemma is clear by [18, Thm. 5.6, p. 424]. From now on we will assume that both $\sigma$ and $\tau$ are outer automorphisms of $G$. Let $T_K \subset K$ be an arbitrary maximal torus. There is a unique maximal torus $T$ in $G$ such that $T_K \subset T$; furthermore, $T$ is $\tau$-invariant and there is a Weyl chamber $C$ in $t := \text{Lie}(T)$ which is $\tau$-invariant as well (see [28, Prop. 3.2, p. 125]). Let $c : G \rightarrow G$ be an inner automorphism such that the torus $S := c(T)$ contains $T_K$, the latter being a maximal torus in $H$. As before, there exists a Weyl chamber $C' \subset c(t)$ which is invariant under $\sigma$. On the other hand, $c c^{-1}$ is an involutive automorphism of $G$ which leaves $S$ invariant; it even leaves the chamber $c(C)$ (inside the Lie algebra of $S$) invariant. But the chambers
$c(C)$ and $C'$ are conjugate under the Weyl group of $(G,S)$; that is, there exists an inner automorphism, say $c'$, such that

$$c'(S) = S \text{ and } c'(C) = c(C).$$

We now compare the involutions $c\tau c^{-1}$ and $c'\sigma c'^{-1}$ of $G$: they both leave the torus $S$ invariant, and along with it, its Lie algebra and the chamber $c(C)$ inside it: both are realized in terms of permuting the simple roots that determine the chamber, the permutation being necessarily a Dynkin diagram automorphism.

Since $G$ is different from $Spin(8)$, there is a unique such (involutive) permutation which is not the identity map. But none of the automorphisms $c\tau c^{-1}$ and $c'\sigma c'^{-1}$ is inner, thus by composing them and then restricting the result to the chamber $c(C)$, one gets the identity map; as an automorphism of $G$, this composition must consequently be an inner automorphism $c_g$ defined by some $g \in S$. That is,

$$c\tau c^{-1} = c'\sigma c'^{-1}c_g.$$ 

Notice that the automorphism in the left hand side of the previous equation leaves $c(T_K) \subset S$ pointwise fixed. Consequently, $c'\sigma c'^{-1}$ does the same. In other words, $c(T_K)$ is contained in the fixed point set of $c'\sigma c'^{-1}$, which is just $c'(H)$. This finishes the proof. □

**Proof of Thm. 1.2 in the case $G$ simple.** First assume that $G$ is different from $Spin(8)$. By Lemma 3.1, there exists $T \subset K$ a maximal torus and $g_0 \in G$ such that $g_0Tg_0^{-1} \subset H$. The actions $K \times G/H \to G/H$ and $g_0Kg_0^{-1} \times G/H$ given by left translation are equivalent, relative to the map

$$(K,G/H) \to (g_0Kg_0^{-1},G/H), \quad (k,gH) \mapsto (g_0kg_0^{-1},g_0gH).$$

Thus it is sufficient to show that the $g_0Kg_0^{-1}$-action on $G/H$ is Cohen-Macaulay. Equivalently, by means of Prop. 2.5 above, that the action of $g_0Tg_0^{-1}$ on $G/H$ is so. But this is clear, because, by the main result in [9], the action of $H$ on $G/H$ is equivariantly formal.

If $G = Spin(8)$, then, like in the proof of Lemma 3.1 we can assume that both $\sigma$ and $\tau$ are outer automorphisms. Their fixed point sets are thus isomorphic to Spin(7) and Thm. 1.2 is a consequence of Lemma 3.2 below. □

**Lemma 3.2.** Any smooth action of a compact connected Lie group on a closed manifold which is an odd-dimensional rational cohomology sphere is Cohen-Macaulay.

**Proof.** By Prop. 2.5 (b), it is sufficient to consider the action of a torus $T$ on $X$, where $X$ is a closed manifold with $H^*(X) \cong H^*(S^{2n+1})$ as vector spaces, for some $n \geq 0$. In case the fixed point set $X^T$ is non-empty, this is a union of odd-dimensional smooth orientable submanifolds whose total Betti number is at least 2. On the other hand, by Prop. 2.2 (a), $\dim H^*(X^T) \leq \dim H^*(X)$. Since the latter number is equal to 2, the two sides of the inequality above turn out to be equal. By Prop. 2.2 (b), the action is equivariantly formal and consequently Cohen-Macaulay. Let us now consider the case when the set $X^T$ is empty. There exists a one-dimensional subtorus $S \subset T$ whose action on $X$ is locally free. One can
compute the cohomology of the orbit space \( X/S \) by means of the following version of the Gysin sequence (cf. \cite{S}, Lemma 2.2):

\[
\ldots \rightarrow H^3(X/S) \rightarrow H^{3+2}(X/S) \rightarrow H^{3+2}(X) \rightarrow H^{3+1}(X/S) \rightarrow \ldots
\]

It follows that \( H^*(X/S) \simeq H^*(\mathbb{C}P^n) \) by an isomorphism of vector spaces. Consequently, by Ex. 2.1, the canonical action of \( T/S \) ring \( H \)

we have the ring isomorphism

\[
H_\tau^*(X) \simeq H_{T/S}^*(X/S).
\]

\[\square\]

3.2. The case when \( G \) is not simple. From now on, \( G \) is just simply connected, without being simple. Our main tool in dealing with this situation is the following result, see \cite{27}, Prop. 5.4:

**Proposition 3.3.** (A. Kollross) If \( G \) is simply connected then the \( H \)-action on \( G/K \) is a direct product of actions of one the following types:

(i) the action of \( H' \times L^{n-1} \times K' \) on \( L^n \) defined by

\[
(h, g_1, \ldots, g_{n-1}, k) \cdot (x_1, \ldots, x_n) = (hx_1g_1^{-1}, g_1x_2g_2^{-1}, \ldots, g_{n-2}x_{n-1}g_{n-1}^{-1}, g_{n-1}x_nk^{-1}),
\]

(ii) the action of \( H' \times L^{n-1} \) on \( L^{n-1} \times L/K' \) defined by

\[
(h, g_1, \ldots, g_{n-1}) \cdot (x_1, \ldots, x_{n-1}, x_nK') = (hx_1g_1^{-1}, g_1x_2g_2^{-1}, \ldots, g_{n-2}x_{n-1}g_{n-1}^{-1}, g_{n-1}x_nK'),
\]

(iii) the action of \( L^{n-1} \) on \( H' \times L^{n-2} \times L/K' \) defined by

\[
(g_1, \ldots, g_{n-1}) \cdot (H'x_1, x_2, \ldots, x_{n-1}, x_nK') = (H'x_1g_1^{-1}, g_1x_2g_2^{-1}, \ldots, g_{n-2}x_{n-1}g_{n-1}^{-1}, g_{n-1}x_nK'),
\]

(iv) the action of \( L^n \) on \( L^n \) defined by

\[
(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_n) = (g_1x_1g_2^{-1}, g_2x_2g_3^{-1}, \ldots, g_{n-1}x_{n-1}g_n^{-1}, g_nx_n\alpha(x_1)^{-1}),
\]

where \( L \) is a simply connected, simple and compact Lie group, \( H', K' \subset L \) are fixed points of involutions of \( L \), and \( \alpha \) is an outer or trivial automorphism of \( L \).

To prove Thm. 1.2 for \( G \) semisimple we only need to show that the actions (i)-(iv) are Cohen-Macaulay.

Let us start with (i). We use an inductive argument. Note that the first factor \( L \) in \( H' \times L^{n-1} \times K' \) acts freely on \( L^n \), the orbit space being diffeomorphic to \( L^{n-1} \) via

\[
L^n/L \rightarrow L^{n-1}, \quad (x_1, x_2, \ldots, x_n) \mapsto (x_1x_2, x_3 \ldots, x_n).
\]
Thus it is sufficient to prove that the action of $H' \times L^{n-2} \times K'$ on $L^{n-1}$ given by

$$(h, g_2, \ldots, g_{n-1}, k) \cdot (x_2, \ldots, x_n) = (hx_2g_2^{-1}, g_2x_3g_3^{-1}, \ldots, g_{n-2}x_{n-1}g_{n-1}^{-1}, g_{n-1}x_nk^{-1}),$$

is Cohen-Macaulay. We continue the procedure and gradually drop out the $L$-factors in $H' \times L^{n-1} \times K'$ until we finally obtain the action of $H' \times K'$ on $L$ given by $(h, k) \cdot x = hxk^{-1}$. But this action is Cohen-Macaulay by the result already proved in Subsect. 3.1 (see also Prop. 2.8).

To deal with (ii), we start by modding out the action of the first factor $L$ in $H' \times L^{n-1}$, which is clearly a free action. In this way, we reduce the problem to showing that the action of $H' \times L^{n-2}$ on $L^{n-2} \times L/K'$ described by

$$(h, g_2, \ldots, g_{n-1}) \cdot (x_2, \ldots, x_{n-1}, x_K) = (hx_2g_2^{-1}, g_2x_3g_3^{-1}, \ldots, g_{n-2}x_{n-1}g_{n-1}^{-1}, g_{n-1}x_Kk^{-1}),$$

is Cohen-Macaulay. We continue the procedure until we are led to the action of $H'$ on $L/K'$ given by left translations. Again, this action is Cohen-Macaulay by the result we proved in Subsect. 3.1.

Similarly, in case (iii) we reduce the problem to the action of $L$ on $H' \setminus L \times L/K'$ described by $g \cdot (H'x_1, x_2K') = (H'x_1g^{-1}, gx_2K')$. The map $H' \setminus L \times L/K' \to L/H' \times L/K'$, $H'x_1, x_2K') \mapsto (x_1^{-1}H', K'x_2)$ is an $L$-equivariant diffeomorphism. This allows us to change our focus to the action of $\Delta(L) := \{(g, g) \mid g \in L\}$ on $(L \times L)/(H' \times K')$ by left translations. By Prop. 2.8 this is Cohen-Macaulay if and only if so is the action of $H' \times K'$ on $(L \times L)/\Delta(L)$. But the latter is just the action of $H' \times K'$ on $L$ given by $(h, k) \cdot x = hxk^{-1}$, which was discussed in Subsect. 3.1.

As about (iv), the recursive procedure already used in each of the previous cases leads us to the action of $L$ on itself given by $g \cdot x = gx\alpha(g)^{-1}$. It was proved by Baird in [2, p. 212] that all isotropy groups of this action have the same rank. By [12, Cor. 4.3], the action is thus Cohen-Macaulay.

**Remark 3.4.** Thm. 1.2 holds even when $G$ is not necessarily semisimple. To prove this, consider a finite cover of $G$ of the type $T \times G_s$, where $T$ is a torus and $G_s$ is compact, connected, and simply connected. In view of Propositions 2.8, 2.9, and 2.10 it is sufficient to consider the case when $G$ is equal to such a direct product and $H$ is the identity component of $G^\sigma$. But then both $\sigma$ and $\tau$ leave the factors $T$ and $G_s$ invariant. Their fixed point sets split as direct products of subgroups of $T$ and $G_s$ respectively; that is, $H_0 = T_1 \times H_s$ and $K = T_2 \times K_s$, where $T_1, T_2 \subset T$ are subtori and $H_s, K_s \subset G_s$. The $H_0 \times K$-action on $G$ can be described as follows:

$$((t_1, h), (t_2, k)) \cdot (t, g) = (t_1tt_2^{-1}, hgtk^{-1}).$$

Consequently, the corresponding equivariant cohomology ring is the direct product of the rings $H^*_{T_1 \times T_2}(T)$ and $H^*_{H_s \times K_s}(G_s)$. This is a Cohen-Macaulay ring, since both factors are
that (for the first factor, one takes into account that all isotropy subgroups of the $T_1 \times T_2$-action on $T$ are equal to $T_1 \cap T_2$ and uses [12, Cor. 4.3]; for the second factor, one uses Thm. 1.2).

By Prop. 2.4 above a smooth action of a compact connected Lie group on a closed manifold is equivariantly formal if and only if it is Cohen-Macaulay and at least one of the isotropy subgroups has maximal rank. We deduce as follows:

**Corollary 3.5.** Let $G, K,$ and $H$ be like in Thm. 1.2. The action of $K$ on $G/H$ is equivariantly formal if and only if a maximal torus in $K$ is conjugate with a subgroup of $H$.

**Remark 3.6.** If $G$ is a compact connected Lie group and $K, H \subset G$ are two connected and closed subgroups such that the $K$-action on $G/H$ is equivariantly formal, then rank $K \leq$ rank $H$ (this follows readily from Prop. 2.4). The converse is not true, even for Hermann actions. Take for instance two Dynkin diagram involutions of $Spin(8)$ which are not conjugate with each other. Their fixed point sets, $H$ and $K$ respectively, are both isomorphic to $Spin(7)$. It turns out that the action of $H$ on $Spin(8)/K \simeq S^7$ by left translations is transitive, with isotropy subgroups isomorphic to the exceptional compact Lie group of type $G_2$, see e.g. [33, Thm. 3]. Thus this action is not equivariantly formal. It is interesting to notice, however, that if $G$ is simple and simply connected, $G \neq Spin(8)$, then the condition rank $K \leq$ rank $H$ is sufficient for the $K$-action on $G/H$ to be equivariantly formal: this follows from Lemma 3.1 above.

### 4. Orbit equivalent actions

We start by proving the following result, whose relevance will become clear immediately:

**Lemma 4.1.** Let $M$ be a closed manifold and $K, K'$ two compact and connected Lie groups that act on $M$ such that $K \subseteq K'$ and $Kp = K'p$ for all $p \in M$. Then the $K$-action on $M$ is Cohen-Macaulay if and only if so is the $K'$-action.

**Proof.** Let $b$ denote the maximal rank of an isotropy subgroup of the $K$-action and $M_{b,K}$ the subspace of $M$ consisting of all points whose isotropy group has rank equal to $b$. In the same way, to the action of $K'$ one assigns the number $b'$ and the subspace $M_{b',K'}$. For any $p \in M$ we have $Kp = K'p$. By using [29, Prop. 10, p. 207] (see also [14, Prop. 7]) we deduce that the $K$- and $K'$-isotropy subgroups at $p$ have equal corank. This implies:

$$M_{b,K} = M_{b',K'}.$$ 

(1)

Pick maximal tori $T$ in $K$ and $T'$ in $K'$ such that $T \subset T'$. The minimal dimension of a $T$-orbit in $M$ is rank $K - b$, cf. e.g. [12, Lemma 4.1]. Similarly, the minimal dimension of a $T'$-orbit in $M$ is rank $K' - b'$. But the two aforementioned numbers are the $K$-, respectively $K'$-coranks of any point in $M_{b,K}$ respectively $M_{b',K'}$, hence, by eq. (1), they are equal. Thus, there exists a subtorus $S \subset T$ of rank equal to the two numbers which acts locally freely on $M$. 

The sets $M_{b,T}$ and $M_{b',T'}$ defined in the same way as before are non-empty, clearly contained in $M_{b,K}$ and $M_{b',K'}$ respectively. The $K$-action on $M_{b,K}$ is Cohen-Macaulay, see [12, Cor. 4.3]. Consequently, this time by Prop. 2.5 (b) and Cor. 2.7, $H_T^*(M_{b,K})$ is a Cohen-Macaulay ring. Since $S \subset T$ acts locally freely, the latter ring is isomorphic to $H^{*}_{T/S}(M_{b,K}/S)$.

On the other hand, the $T/S$-action on $M_{b,K}/S$ admits points that are fixed. Namely, they are orbits of the form $Sp$ such that $Sp = Tp$, which implies that

\[(2) \quad \text{corank}_T T_p = \text{rank} S.\]

But if $p \in M$ satisfies the latter condition, then $Sp$ is a connected and closed submanifold of $Tp$, of the same dimension as the latter, hence $Sp = Tp$. Thus condition (2) characterizes the fixed points. Since $\text{rank} S = \text{rank} T - b$, that condition is actually equivalent to $p \in M_{b,T}$.

We have actually proved as follows:

\[(3) \quad (M/S)^{T/S} = (M_{b,K}/S)^{T/S} = M_{b,T}/S.\]

From the previous considerations, the $T/S$-action on $M_{b,K}/S$ is equivariantly formal. Consequently, by eq. (3),

\[(4) \quad \dim H^*(M_{b,K}/S) = \dim H^*(M_{b,T}/S).\]

In the same way, one analyzes the $K'$-action on $M$ and obtains

\[(5) \quad (M/S)^{T'/S} = (M_{b',K'}/S)^{T'/S} = M_{b',T'}/S.\]

as well as

\[(6) \quad \dim H^*(M_{b',K'}/S) = \dim H^*(M_{b',T'}/S).\]

We are now in a position to prove the equivalence stated in the theorem. First, the $K$-action on $M$ is Cohen-Macaulay if and only if so is the induced $T$-action, see Prop. 2.5 (b). But $H_T^*(M) = H^*_{T/S}(M/S)$ and the $T/S$-action on $M/S$ admits fixed points, thus the latter Cohen-Macaulay condition is equivalent to: the $T/S$-action on $M/S$ is equivariantly formal. Equivalently, by eqs. (3) and (4),

\[\dim H^*(M/S) = \dim H^*(M_{b,K}/S).\]

In exactly the same way, this time by using eqs. (5) and (6), the $K'$-action on $M$ is Cohen-Macaulay if and only if

\[\dim H^*(M/S) = \dim H^*(M_{b',K'}/S).\]

The proof is completed by taking into account eq. (1). □

Let us now recall that two isometric actions of two connected compact Lie groups on a Riemannian manifold are orbit equivalent if there is an isometry of the manifold to itself which maps each orbit of the first group action to an orbit of the second one. From Lemma 4.1 we deduce as follows:
**Theorem 4.2.** Let $M$ be a compact Riemannian manifold and $K$ a connected and closed subgroup of the isometry group of $M$. Assume that the $K$-action on $M$ is orbit equivalent to a Cohen-Macaulay action on $M$. Then the $K$-action is Cohen-Macaulay as well.

**Proof.** Let $G$ be the isometry group of $M$. By hypothesis, there exists a connected and closed subgroup $K' \subset G$ whose canonical action on $M$ is Cohen-Macaulay such that the actions of $K$ and $K'$ on $M$ are orbit equivalent. Thus there exists an isometry $f : M \to M$ which maps any $K$-orbit to a $K'$-orbit. Consider the closed subgroup $K''$ of $G$ generated by $f^{-1}K'f$ and $K$. Note that $K''$ is connected. The key-observation is that for any $p \in M$, we have

$$Kp = (f^{-1}K'f)p = K''p.$$  

The first equality is clear and immediately implies the second. Due to eq. (7), the result stated in the proposition is a direct consequence of Lemma 4.1, used twice. 

We finish this section with a result which is related to Lemma 4.1 and will be needed later.

**Lemma 4.3.** Let $M$ be a closed manifold and $K$ a compact and connected Lie group that acts smoothly on $M$. Let also $H \subset K$ be the kernel of the action. Then the $K$-action on $M$ is Cohen-Macaulay if and only if so is the $K/H$-action.

**Proof.** Let $b$ and $M_{b,K}$ be like in the proof of Lemma 4.1. Again, let $S \subset T$ be a subtorus of rank equal to rank $T - b$ whose action on $M$ is locally free. Pick maximal tori $T$ and $T_1$ in $G$ and $H$ respectively such that $T_1 \subset T$. We already noticed that the $K$-action on $M$ is Cohen-Macaulay if and only if

$$\dim H^*(M/S) = \dim H^*(M_{b,T}/S).$$

Observe that $T/T_1$ is a maximal torus in $K/H$, cf. [3, Ch. 9, Prop. 2 (c)]. Furthermore, for the $K/H$-action on $M$, the maximal rank of an isotropy subgroup is $b_1 := b - \text{rank } H$. Since the intersection $T_1 \cap S$ is finite, the image $S_1$ of $S$ under the projection $T \to T/T_1$ is a subtorus of rank equal to rank $S = \text{rank } T/T_1 - b_1$ which acts locally freely on $M$. As before, the $K/H$-action on $M$ is Cohen-Macaulay if and only if

$$\dim H^*(M/S_1) = \dim H^*(M_{b_1,T/T_1}/S_1).$$

But $M/S_1 = M/S$ and $M_{b_1,T/T_1} = M_{b,T}$, thus the equivalence stated in the lemma is clear. 

We are now in a position to prove the main result of the paper:

**Proof of Thm. 1.1.** As already mentioned in the introduction, Kollros has shown in [27] that any hyperpolar action is orbit equivalent to a direct product of actions of one of the following types: transitive, of cohomogeneity one, or Hermann. We know that an action of each of these three types is Cohen-Macaulay: for the first two, see [11, Cor. 1.2], for the last, Thm. 1.2. We apply Thm. 4.2 even though the acting group is not necessarily a subgroup of $\text{Iso}(M)$, we can mod out the kernel of the action and use Lemma 4.3.
Remark 4.4. It would be interesting to find a classification-free proof of Thm. 1.1, using the very definition of a hyperpolar action.

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