Separation of variables for the classical and quantum Neumann model.

O. Babelon * M. Talon *

Abstract

The method of separation of variables is shown to apply to both the classical and quantum Neumann model. In the classical case this nicely yields the linearization of the flow on the Jacobian of the spectral curve. In the quantum case the Schrödinger equation separates into one-dimensional equations belonging to the class of generalized Lamé differential equations.
1 Introduction

The classical Neumann model describes the motion of a particle constrained to lie on an \((N-1)\)-sphere in \(N\) dimensional space and submitted to harmonic forces [1]. Originally this model has been solved by Neumann [1] for \(N = 3\) by separation of variables in the Hamilton-Jacobi equation, and subsequently K. Uhlenbeck has shown that the model is Liouville integrable by exhibiting \((N-1)\) conserved quantities in involution [2]. The classical solution of J. Moser [3] was shown to be deeply related to the theory of hyperelliptic curves by Mumford [4] emphasizing the \textit{algebraic} integrability of the model.

Recently, new methods for solving integrable models have been introduced [5, 6, 7, 8, 9], based on underlying Lie algebraic structures. The relation between the group-theoretical and the algebraic-geometrical approaches has been explained in [10]. However the old methods of resolution by separation of variables begin to attract new interest [11]. As a matter of fact the solution of the quantum Toda chain was given in Gutzwiller [12] and Sklyanin [13] by a method closely connected to this procedure.

We show in this article that the general Neumann model (any \(N\)) admits such a resolution both at the classical and \textit{quantum} level. This is achieved by choosing appropriate coordinates on the sphere, generalizing Neumann’s coordinates for \(N = 3\) [1, 3]. Considering what we know about the Lax formulation of the model [14], this choice of coordinates is also strongly reminiscent of the one advocated by Sklyanin [11]. The simplifying feature of this model as compared to the general case, is that the separation of variables is achieved by a transformation involving only the space coordinates and not the momenta [11].

We first recall how the separation of variables in the Hamilton-Jacobi equation works in the classical case. This is a straightforward generalization of the work of Neumann [1, 3]. One gets \((N-1)\) one-dimensional ordinary differential equations, which are readily solved. We will see that the resolution of the Hamilton-Jacobi equation in this situation of a Liouville integrable system provides a canonical transformation which allows to linearize at one stroke the flows of the complete family of hamiltonians in involution. The separation of variables naturally lead to Abel sums and provide a simple proof of the linearization of the flow on the Jacobian of the Riemann surface associated to this dynamical problem by the Lax spectral theory [5, 4, 15]. In the quantum case, the Schrödinger equation similarly separates itself into \((N-1)\) one-dimensional equations. The quasi-classical quantization conditions are readily obtained. The implications of the exact quantization conditions will be studied elsewhere.

2 The Neumann model

The Neumann model has two equivalent formulations, one in terms of constrained dynamical variables with Dirac brackets, and an unconstrained one with canonical Poisson brackets. These formulations remain equivalent at the quantum level [15] so we shall stick to the unconstrained one which is more convenient for our present purposes.

We start from a \(2N\)-dimensional free phase space \(\{x_n, y_n, \ n = 1 \cdots N\}\) with canonical Poisson brackets: \(\{x_n, y_m\} = \delta_{nm}\) and introduce the “angular momentum” antisymmetric ma-
trix: $J_{kl} = x_k y_l - x_l y_k$ and the Hamiltonian:

$$H = \frac{1}{4} \sum_{k \neq l} J_{kl}^2 + \frac{1}{2} \sum_k a_k x_k^2$$

(1)

We shall assume in the following that: $a_1 < a_2 < \cdots < a_N$. The Hamiltonian equations are,

with $X = (x_k)$, $Y = (y_k)$ and the diagonal constant matrix $A = (a_k \delta_{kl})$:

$$\dot{X} = -J X \quad \dot{Y} = -J Y - A X$$

They automatically ensure that $\sum x_k^2$ remains constant and lead to the non-linear Newton equations for the particle:

$$\dot{x}_k = -a_k x_k - x_k \sum_l (\dot{x}_l^2 - a_l x_l^2)$$

Let us note that the above Hamiltonian is invariant under the canonical transformations:

$$(x_k, y_k) \rightarrow (x_k, y_k + \phi x_k)$$

(2)

where $\phi$ may be any function of $\sum x_k^2$. Classically the constrained system can be viewed as the Hamiltonian reduction of the above unconstrained one under this symmetry.

The Liouville integrability of this system is a consequence of the existence of $(N - 1)$ independent quantities in involution, first found by K. Uhlenbeck:

$$F_k = x_k^2 + \sum_{l \neq k} \frac{J_{kl}^2}{a_k - a_l}$$

(3)

Notice that $H = 1/2 \sum a_k F_k$ and $\sum_k F_k = 1$.

3 Neumann’s coordinates on $S^{(N-1)}$.

Generalizing Neumann’s choice in [1], we shall introduce $(N - 1)$ parameters $t_1, \cdots, t_{N-1}$ on the sphere. They are the roots of the equation:

$$u(t) \equiv \sum_k \frac{x_k^2}{t - a_k} = 0$$

Notice that this equation is invariant by $x_k \rightarrow \lambda x_k$ so that $t_1 < t_2 < \cdots < t_{N-1}$ are indeed defined on the sphere. Conversely, by definition of the $t_j$ we have for $x \in S^{(N-1)}$:

$$u(t) = \frac{\prod_j (t - t_j)}{\prod_k (t - a_k)} \quad \Rightarrow \quad x_k^2 = \frac{\prod_j (a_k - t_j)}{\prod_{l \neq k} (a_k - a_l)}.$$

Considering the graph of $u(t)$ it is easy to see that:

$$a_1 < t_1 < a_2 < t_2 < a_3 < \cdots < t_{N-1} < a_N$$

and we have a bijection of this domain $D$ of the $t_j$’s on the “quadrant” $x_k > 0 \ \forall k$ of the sphere. In general the sphere appears as a $2^N$–fold covering of the domain $D$, ramified on the edges of $D$. For instance $x_1 = \sqrt{t_1 - a_1} \varphi(t_2, \cdots, t_{N-1})$ so that $x_1$ changes its sign when we turn around the ramification point $t_1 = a_1$ in the complex plane. By similar analytic continuations we can cover the whole sphere.
**Proposition.** The \((t_j)’s\) define an orthogonal system of coordinates on the sphere.

A simple proof is obtained by considering, for each root \(t_j\) the vector:

\[
\vec{v}_j = \left( \frac{x_1}{t_j - a_1}, \cdots, \frac{x_N}{t_j - a_N} \right)
\]

We easily check that \(\vec{x} \cdot \vec{v}_j = 0\), \(\vec{v}_j \cdot \vec{v}_j' = 0\) for \(j \neq j'\) and therefore \((\vec{v}_j)\) is an orthogonal basis to the tangent space to \(S^{(N-1)}\) at \(\vec{x}\). On the other hand, taking the derivatives of the equations \(u(t_j) = 0\) with respect to \(t_1, \cdots, t_{N-1}\), where the \(x_k\) are considered as functions of these variables, we obtain:

\[
\vec{v}_i \cdot \frac{\partial \vec{x}}{\partial t_i} = 0 \quad \text{if} \quad i \neq j, \quad 2\vec{v}_j \cdot \frac{\partial \vec{x}}{\partial t_j} = \vec{v}^2_j
\]

Therefore:

\[
\frac{\partial \vec{x}}{\partial t_j} = \frac{1}{2} \vec{v}_j.
\]  \hspace{1cm} (4)

As a byproduct, since \(\vec{v}^2_j = -u'(t_j)\) we get the metrical tensor:

\[
g_{jj'} = \frac{\partial \vec{v}_j}{\partial t_j}, \frac{\partial \vec{v}_j}{\partial t_j'} = -\frac{1}{4} \delta_{jj'} \frac{\prod_{n \neq j} (t_j - t_n)}{\prod_{k} (t_j - a_k)}.\]

4 **The Hamilton–Jacobi equation**

It is convenient to start from the Lagrangian formalism since this is well suited to the change of coordinates on the sphere.

\[
\mathcal{L} = \frac{1}{2} \dot{\vec{x}}^2 - \mathcal{U} = \frac{1}{2} \sum_{jj'} g_{jj'} \dot{t}_j \dot{t}_{j'} - \mathcal{U}
\]

Considering the polynomial:

\[
\prod_k (t - a_k) \ u(t) = \sum_k x_k^2 \prod_{l \neq k} (t - a_l) = \prod_j (t - t_j)
\]

we see that:

\[
\mathcal{U} = \frac{1}{2} \sum_k a_k x_k^2 = \frac{1}{2} \left( \sum_k a_k - \sum_j t_j \right)
\]

The conjugate momentum to \(t_j\) is: \(p_j \equiv \partial \mathcal{L}/\partial \dot{t}_j = g_{jj} \dot{t}_j\) leading to the hamiltonian:

\[
H = \sum_j p_j \dot{t}_j - \mathcal{L} = \frac{1}{2} \sum_j \ g^{jj} p_j^2 + \mathcal{U}
\]

where \(g^{jj} = (g^{-1})_{jj} = 1/g_{jj}\). The Hamilton–Jacobi equation is a first–order non–linear partial differential equation obtained by substituting in \(H\): \(p_j \to \partial S/\partial t_j\). The action \(S\) is function of the space coordinates \(t_j’\)’s. For a fixed energy \(E\) the Hamilton–Jacobi equation reads:

\[
-2 \sum_j \frac{\prod_k (t_j - a_k)}{\prod_{n \neq j} (t_j - t_n)} \left( \frac{\partial S}{\partial t_j} \right)^2 + \frac{1}{2} \left( \sum_k a_k - \sum_j t_j \right) - E = 0 \quad (5)
\]
The method of separation of variables consists in looking for a so-called complete solution of eq. (5), i.e. depending on \((N - 1)\) arbitrary constants, of the form:

\[ S(t_1, \cdots, t_{N-1}) = S_1(t_1) + \cdots + S_{N-1}(t_{N-1}). \]

For this purpose we need to consider the following Vandermonde determinant:

\[
D \equiv D(t_1, \cdots, t_{N-1}) = \begin{vmatrix}
1 & \cdots & 1 \\
t_1 & \cdots & t_{N-1} \\
\vdots & \ddots & \vdots \\
t_1^{N-2} & \cdots & t_{N-1}^{N-2}
\end{vmatrix} = \prod_{m>n} (t_m - t_n)
\]

We shall also need similar determinants \(D_j\) obtained by removing the \(j\)th column and the last row in \(D\). One has the useful identities:

\[
\frac{D}{D_j} = (-1)^{N-j-1} \prod_{n\neq j} (t_j - t_n) \quad (6)
\]

Moreover considering the determinant obtained by replacing the last row by \(t_j^k\) and expanding it over this row, we find:

\[
\sum_j (-1)^{N-1-j} t_j^k D_j = \begin{cases}
0 & k = 0, \ldots, N - 3 \\
D & k = N - 2 \\
(\sum_j t_j) D & k = N - 1
\end{cases}
\quad (7)
\]

Finally denoting \(\Delta(t) = \prod_k (t - a_k)\) we have in view of (5):

\[
g^{jj} = -4(-1)^{N-j-1} \frac{D_j}{D} \Delta(t_j)
\]

and we can rewrite equation (5) as:

\[
-4 \sum_j (-1)^{N-j-1} D_j \Delta(t_j) \left( \frac{\partial S_j}{\partial t_j} \right)^2 + D(- \sum_j t_j + \sum_k a_k - 2E) = 0
\]

Using identities (7) one can separate the variables, getting:

\[
4 \Delta(t_j) \left( \frac{\partial S_j}{\partial t_j} \right)^2 + \sum_{k=0}^{N-1} c_k t_j^k = 0
\]

where:

\[
c_{N-1} = 1, \quad c_{N-2} = 2E - \sum_k a_k, \quad \text{and the other } c_k \text{ arbitrary.}
\]

Let us remark that all \(S_j\)'s verify the same equation which we can write as:

\[
4 \Delta(t) \left( \frac{dS}{dt} \right)^2 + \prod_{n=1}^{N-1} (t - b_n) = 0 \quad (8)
\]

where the \(b_n\)'s are \((N - 1)\) independent constants. The energy is obtained in terms of the \(b_n\)'s as: \(E = 1/2 (\sum_k a_k - \sum_n b_n)\).
The general linear flow on the Liouville torus.

We have obtained the complete solution of the Hamilton–Jacobi equation:

$$S(t_1, \ldots, t_{N-1}; b_1, \ldots, b_{N-1}) = \sum_j S_j(t_j; b_1, \ldots, b_{N-1}).$$

One can interpret this action as a generating function for a canonical transformation from the original variables \((t_j, p_j)\) to new variables \((b_n, \psi_n)\) linearizing the equations of motion. As usual, one has:

$$p_j = \frac{\partial S}{\partial t_j}, \quad \psi_n = \frac{\partial S}{\partial b_n}$$

This transformation is at the level of the symplectic structure and does not refer to any specific hamiltonian. Since all the \(F_k\) in eq. (3) play the same role it is natural to consider the Hamilton–Jacobi equations with respect to any one of the hamiltonians in involution \(F_k\). We show in this section that they all lead to the same separated equations for \(S\).

In order to do that, we first express \(J_{kl}\) in terms of the coordinates \(t_j\) and their conjugate momenta \(p_j\). Consider the canonical 1–form \(\alpha = \sum y_k dx_k\) which reduces on the sphere to \(\alpha = \sum p_j dt_j\). So we have in view of eq. (4)

$$p_j = \sum_k y_k \frac{\partial x_k}{\partial t_j} \quad \text{whence} \quad \vec{y} \cdot \vec{v}_j = 2 p_j.$$

This determines \(\vec{y}\) modulo a vector proportional to \(\vec{x}\) which does not affect the value of \(J_{kl}\). We find \(\vec{y} = 1/2 \sum g^{ij} p_j \vec{v}_j\) and:

$$J_{kl} = -\frac{1}{2} \sum_j (a_k - a_l) v^k_j v^l_j g^{ij} p_j.$$

With the help of this formula, \(F_k\) may be expressed as:

$$F_k = x_k^2 \left( 1 - \sum_j \frac{g^{jj} p_j^2}{t_j - a_k} \right)$$

It is convenient \[14\] to introduce the generating function for the \(F_k\):

$$\mathcal{H}(\lambda) \equiv \sum_k \frac{F_k}{\lambda - a_k} = \frac{\Pi_n(\lambda - b_n)}{\Pi_k(\lambda - a_k)}$$

for appropriate \(b_n\)‘s and we have used \(\sum F_k = 1\). By a simple calculation we find:

$$\mathcal{H}(\lambda) = u(\lambda) \left( 1 - \sum_j \frac{g^{jj} p_j^2}{t_j - \lambda} \right)$$

where \(u(\lambda)\) has been introduced in section 3. Let us consider the Hamilton–Jacobi equation associated to the hamiltonian \(\mathcal{H}(\lambda)\). It reads:

$$\sum_j \left\{ g^{jj} \left( \frac{\partial S}{\partial t_j} \right)^2 \frac{1}{t_j - \lambda} \right\} + \frac{\Pi_n(\lambda - b_n)}{\Pi_j(\lambda - t_j)} - 1 = 0$$
Proposition. For any \( k = 0, \ldots, N - 2 \) we have:

\[
\sum_j (-1)^{N-j-1} \frac{D_j t_j^k}{t_j - \lambda} = (-1)^N \frac{D^k \lambda}{\prod_j (t_j - \lambda)}
\]  

(12)

This is proven similarly to eq. (7) by considering the determinant obtained by replacing the last row of \( D \) by \( t_j^k / (t_j - \lambda) \). Using now the identity (12) for \( k = 0 \) one can rewrite eq. (11) in the form:

\[
\sum_j (-1)^{N-1-j} \frac{D_j}{D} \frac{1}{t_j - \lambda} \left[ 4 \Delta(t_j) \left( \frac{\partial S}{\partial t_j} \right)^2 + t_j^{N-2} (t_j - \lambda) + \prod_n (\lambda - b_n) \right] = 0
\]

This leads to the separated equations:

\[
4 \Delta(t_j) \left( \frac{\partial S}{\partial t_j} \right)^2 + \prod_n (\lambda - b_n) + (t_j - \lambda) \left( t_j^{N-2} + \sum_{l=0}^{N-3} c_l(\lambda)t_j^l \right) = 0
\]

We look for a solution of the form: \( S = \sum_j S_j(t_j) \) independent of \( \lambda \). This is indeed possible and is achieved by setting:

\[
t^{N-2} + \sum_{l=0}^{N-3} c_l(\lambda)t^l = \frac{\prod_n (t - b_n) - \prod_n (\lambda - b_n)}{t - \lambda}
\]

and we recover exactly eq. (8). In addition we have obtained an interpretation of the \( b_n \)'s in eq. (8): they are the roots of \( H(\lambda) \). So we have the expression of K. Uhlenbeck’s conserved quantities \( F_k \) in terms of the integration constants \( b_n \) of the Hamilton–Jacobi equation:

\[
F_k = \frac{\prod_n (a_k - b_n)}{\prod_{l \neq k} (a_k - a_l)}
\]

We have shown that the \( b_n \)'s are equivalent to the \( F_k \)'s as action variables. Since the symplectic form is:

\[
\omega = d\alpha = \sum_j dp_j \wedge dt_j = \sum_n db_n \wedge d\psi_n
\]

we see that the \( b_n \)'s are in involution. This is still another proof of the involution property of the \( F_k \)'s. Moreover their conjugate variables \( \psi_n \)'s have a linear time evolution under \( H(\lambda) \) for any \( \lambda \). We have proven:

Proposition. The complete solution of the family of Hamilton–Jacobi equations \( H(\lambda) \) is \( \sum_j S(t_j) \) for \( S \) solution of (8) and \( H(\lambda) \) is expressed in terms of the \( b_n \)'s only.

6 Discussion of the classical solution.

As an application let us solve the equations of motion for the coordinates \( t_k \) for any hamiltonian \( H(\lambda) \). These equations are readily obtained:

\[
i_j = \{ H(\lambda), t_j \} = -2 u(\lambda) \frac{g^{jj} p_j}{t_j - \lambda} \quad \text{with} \quad p_j \rightarrow \frac{\partial S}{\partial t_j}.
\]
Introducing the polynomial:

\[ P(t) = \prod_{k=1}^{N} (t - a_k) \prod_{n=1}^{N-1} (t - b_n) \]

and denoting by \( \tau \) the time variable we get:

\[ \frac{dt_j}{\sqrt{-P(t_j)}} = \frac{4u(\lambda)d\tau}{\prod_{n \neq j}(t_j - t_n)(t_j - \lambda)} \tag{13} \]

Now it is natural to introduce the hyperelliptic Riemann surface of genus \((N - 1)\) defined in the complexified plane \( \mathbb{C}^2 \) by the equation:

\[ s^2 + P(t) = 0 \tag{14} \]

As a matter of fact this curve also appears as the spectral curve, in the Lax pair approach \[14\]. The independent abelian differentials of first class on this surface are the \( t^k dt/s \) for \( k = 0, \cdots, N - 2 \). Recalling the identity \[12\] we get, for precisely these values of \( k \):

\[ \sum_j t^k_j \frac{dt_j}{s_j} = 4d\tau \frac{u(\lambda)}{\prod_i (\lambda - a_i)} \sum_j (-1)^{N-1-j} \frac{t^k_j}{t_j - \lambda} \frac{D_j}{D} = -4 \frac{\lambda^k}{\prod_i (\lambda - a_i)} d\tau \tag{15} \]

Let us interpret geometrically this remarkable relation: to the point \( \vec{x} \) on the sphere are associated \((N - 1)\) unordered points \((t_j, s_j)\) on the Riemann surface of equation \[14\], the \( t_j \)'s being the roots of \( u(t) = 0 \) and \( s_j = \sqrt{-P(t_j)} \). Such an unordered set defines a divisor on the surface, and since \((N - 1)\) is the genus of this surface the path integrals of the Abel sums appearing in the left–hand side of equation \[15\] provide an analytic bijection into the Jacobian torus of the Riemann curve. The right–hand side then shows that this point on the Jacobian evolves linearly with time. Of course the physical quantities must be real, so the divisor moves on a connected component of the real slice of the Jacobian, see \[4\], which can be identified with the Liouville torus.

Let us illustrate this geometry for Neumann’s case of \( N = 3 \). The Riemann surface is of genus 2, and may be pictured as:
There are two abelian differentials: $dt/s$ and $tdt/s$ leading to two path integrals ($P_0$ is any origin):

$$
\begin{align*}
\Omega_1 &= \int_{P_0}^{(t_1,s_1)} \frac{dt}{s} + \int_{P_0}^{(t_2,s_2)} \frac{dt}{s} \\
\Omega_2 &= \int_{P_0}^{(t_1,s_1)} \frac{tdt}{s} + \int_{P_0}^{(t_2,s_2)} \frac{tdt}{s}
\end{align*}
$$

Obviously $\vec{\Omega} = (\Omega_1, \Omega_2)$ has 4 periods corresponding to loops around a homology basis of the Riemann surface. The application $((t_1,s_1), (t_2,s_2)) \rightarrow \vec{\Omega}$ maps the Jacobian of the surface (analytic space of divisors modulo equivalence) to a 2–dimensional complex torus.

Finally we notice that the action itself is given by a similar integral:

$$
S(t_1, \cdots, t_{N-1}) = \frac{1}{2} \sum_j \int_{P_0}^{P_j = (t_j, s_j)} \frac{\prod_n (t - b_n)}{s} dt
$$

Remark that the terms $t^k$ for $k = 0, \cdots, N-2$ in $\prod_n (t - b_n) = t^{N-1} + \sum_{k=0}^{N-2} c_k t^k$ lead to abelian integrals of first kind, while the term $t^{N-1}$ leads to an integral of second kind, having a double pole at $\infty$. So the action may be seen as a multivalued meromorphic function on the Jacobian. More precisely, with similar notations, one can write:

$$
S = \frac{1}{2} \sum_{k=1}^{N-1} c_{k-1} \Omega_k + S_0
$$

where the $c_k$’s are linear functions of $F_k$’s with coefficients polynomials in the $a_k$’s, since $\prod_n (t - b_n) \sim \prod_k (t - a_k) \sum_k F_k/(t - a_k)$ and $S_0$ is:

$$
S_0(P_1 + \cdots + P_{N-1}) = \frac{1}{2} \sum_j \int_{P_0}^{P_j} \frac{t^{N-1} dt}{s}.
$$

As a function of the divisor $P_1 + \cdots + P_{N-1}$, $S_0$ has a simple pole on a variety of codimension 1 obtained when one of the $P_j$’s goes to $\infty$, which is well known to be the divisor of a theta function on the Jacobian torus \[4\]. By restriction to real variables, $S$ becomes a real multivalued analytic function on the $(N-1)$–dimensional real Liouville torus.

### 7 The quantum case

The quantization of the Neumann system will be performed in the unconstrained formalism. The constraint will appear at the level of the wave functions. We consider $2N$ self–adjoint operators \( \{x_n, y_n, \ n = 1, \cdots, N\} \) with canonical commutation relations:

\[
[x_k, x_l] = [y_k, y_l] = 0, \ [x_k, y_l] = i\hbar \delta_{kl}.
\]

We define $J_{kl} = x_k y_l - x_l y_k$ so that $J_{kk} = 0$. Notice that there is no ordering ambiguity for $k \neq l$. Finally the quantum analog of K. Uhlenbeck’s operators:

$$
F_k = x_k^2 + \sum_{l \neq k} \frac{J_{kl}^2}{a_k - a_l}
$$

are also defined unambiguously and are self–adjoint.
Proposition. The quantum theory is formally integrable, i.e.

\[ [F_k, F_l] = 0 \]

Proof. A proof of this fact was given in [14] using the R-matrix formalism. Since this is a fundamental result we give here a more direct proof. One checks immediately that the \( J_{kl} \) and \( x_k \) obey the algebra, similar to the classical Poisson algebra:

\[
[J_{ij}, J_{kl}] = i\hbar (\delta_{jk} J_{li} + \delta_{il} J_{kj} + \delta_{jl} J_{ik} + \delta_{ik} J_{jl})
\]

First one then shows that:

\[
\left[ \sum_{p \neq k} \frac{J_{kp}^2}{a_k - a_p}, \sum_{q \neq l} \frac{J_{lp}^2}{a_l - a_q} \right] = 0
\]

We can assume \( k \neq l \). Therefore in the commutator \([J_{kp}, J_{lq}]\) we have three possibilities; \( p = q \), \( q = k \), and \( p = l \). The case \( p = q \) produces the term:

\[
\sum_{p \neq k, l} \frac{1}{(a_k - a_p)(a_l - a_p)} [J_{kp} J_{kl} J_{lp} + J_{kp} J_{lp} J_{kl} + J_{kl} J_{lp} J_{kp} + J_{lp} J_{kl} J_{kp}]
\]

The case \( q = k \) gives:

\[
\sum_{p \neq l} \frac{1}{(a_k - a_p)(a_k - a_l)} [J_{kp} J_{lp} J_{lk} + J_{kp} J_{lk} J_{lp} + J_{lp} J_{lk} J_{kp} + J_{lk} J_{lp} J_{kp}]
\]

Notice that this term vanishes for \( p = k \) so that one can assume \( p \neq k, l \) in the sum over \( p \). Finally the case \( p = l \) similarly produces (with \( q \to p \)):

\[
\sum_{p \neq l, k} \frac{1}{(a_l - a_p)(a_k - a_l)} [J_{kl} J_{pk} J_{lp} + J_{kl} J_{lp} J_{pk} + J_{pk} J_{lp} J_{kl} + J_{lp} J_{pk} J_{kl}]
\]

In this last expression, in the first and last terms one can bring \( J_{kl} \) in the middle position. In doing so the commutators cancel. Then the three terms have the same \( J \) factors and add to zero. Similarly one shows that:

\[
\left[ x_k^2, \sum_{q \neq l} \frac{J_{lq}^2}{a_l - a_q} \right] - \left[ x_l^2, \sum_{p \neq k} \frac{J_{kp}^2}{a_k - a_p} \right] = 0
\]

so that finally \([F_k, F_l] = 0\).

The canonical commutation relations are realized by setting:

\[
y_k = -i\hbar \frac{\partial}{\partial x_k}.
\]

We look for a simultaneous diagonalization of the \( F_k \)'s, i.e.: \( F_k \Psi = f_k \Psi \). As in the classical case they are invariant under the symmetry \([3]\). This symmetry translates on the wave function by:

\[
\Psi \rightarrow e^{i\varphi(r)} \Psi.
\]
Here $\varphi$ is simply related to $\phi$ by $\phi = \hbar/r \varphi'(r)$, and $r^2 = \sum_k x_k^2$. This shows that the symmetry is implemented by restricting oneself to wave functions independent of $r$. Consequently we will look for a wave function $\Psi$ depending on Neumann’s variables $t_j$ defined on $S^{(N-1)}$.

As a preparation to the general calculation, we consider first the Schrödinger equation associated to Neumann’s hamiltonian: $H = 1/2 \sum a_k F_k$. We find:

$$H = \frac{1}{4} \sum_{k,l} J_{kl}^2 + \frac{1}{2} \sum_{k} a_k x_k^2.$$ 

It is easy to compute:

$$\sum_{k,l} J_{kl}^2 = -\hbar^2 \left( 2 r^2 \bar{\nabla}^2 + (4 - 2N) \bar{x}.\bar{\nabla} - 2 (\bar{x}.\bar{\nabla})^2 \right)$$

For wave functions independent of $r$ we have $\bar{x}.\bar{\nabla} \Psi = 0$ and $r^2 \bar{\nabla}^2 \Psi$ reduces to the orbital part of the Laplacian, i.e. the Laplacian on the sphere. So we can write without further ado the Schrödinger equation:

$$\left( -\frac{\hbar^2}{2} \frac{\Delta}{\Delta(t_j)} + U \right) \Psi = E \Psi \quad (16)$$

and express the Laplacian on the sphere using Neumann’s coordinates as:

$$\Delta = \frac{1}{\sqrt{g}} \sum_{j,j'} \frac{\partial}{\partial t_j} \left( \sqrt{g} g^{jj'} \frac{\partial}{\partial t_{j'}} \right)$$ 

(17)

Here $g = \prod_j g_{jj}$ is the determinant of the metrical tensor. It involves $\prod_j \prod_{n \neq j} (t_j - t_n)^2 = \pm D^2$ so neglecting numerical factors that cancel themselves between $\sqrt{g}$ and $1/\sqrt{g}$ we can set $\sqrt{g} = D/\sqrt{\prod_n \Delta(t_n)} > 0$. From this we get:

$$\sqrt{g} g^{jj} = -4(-1)^{N-j-1} \frac{D_j}{\sqrt{\prod_{n \neq j} \Delta(t_n)}} \sqrt{\Delta(t_j)}$$

and we remark that $D_j/\sqrt{\prod_{n \neq j} \Delta(t_n)}$ is independent of $t_j$ and can be commuted with $\partial/\partial t_j$ to the left of $\Delta$.

Finally the equation (16) reduces to the following simple form:

$$\sum_{j=1}^{N-1} (-1)^{N-j-1} \frac{D_j}{D} \left\{ 4 \hbar^2 \sqrt{\Delta(t_j)} \frac{\partial}{\partial t_j} \sqrt{\Delta(t_j)} \frac{\partial}{\partial t_j} - t_j^{N-1} + t_j^{N-2} \left( \sum_k a_k - 2E \right) \right\} \Psi = 0.$$

It is now easy to separate the variables, i.e. to look for a solution of the form:

$$\Psi(t_1, \ldots, t_{N-1}) = \Psi_1(t_1) \Psi_2(t_2) \cdots \Psi_{N-1}(t_{N-1})$$

and we find that the $\Psi_j$’s all obey the same ordinary differential equation:

$$\left\{ 4 \hbar^2 \sqrt{\Delta(t)} \frac{d}{dt} \sqrt{\Delta(t)} \frac{d}{dt} - t^{N-1} + t^{N-2} \left( \sum_k a_k - 2E \right) + \sum_{k=1}^{N-2} c_k t^{N-k-2} \right\} \Psi(t) = 0.$$
Writing \( t^{N-1} - t^{N-2}(\sum_k a_k - 2E) - \sum_{k=1}^{N-2} c_k t^{N-k-2} = \Pi_n(t - b_n) \) and:

\[
\frac{\Pi_n(t - b_n)}{\Pi_k(t - a_k)} = \sum_k \frac{f_k}{t - a_k}
\]

we get the separated Schrödinger equation in the form:

\[
\left[ \frac{d^2}{dt^2} + \frac{1}{2} \sum_k \frac{1}{t - a_k} \frac{d}{dt} - \frac{1}{4\hbar^2} \sum_k \frac{f_k}{t - a_k} \right] \Psi(t) = 0 \tag{18}
\]

This equation has been studied in the litterature and is known as the generalized Lamé equation [16].

8 The general Schrödinger equation.

Since all the \( F_k \)'s commute and are self–adjoint, there exists a complete set of common eigenvectors. We therefore expect that similarly to the classical case the same separation of variables occur for the whole family of hamiltonians \( \mathcal{H}(\lambda) \). We shall show that it is indeed the case. As a byproduct, this is an alternative proof that the \( F_k \)'s in equation (18) are their eigenvalues.

For wave functions independent of \( r \) we have by (4), denoting \( v^k_j = x_k/(t_j - a_k) \):

\[
\frac{\partial}{\partial x_k} = \frac{1}{2} \sum_j g^{ij} v^k_j \frac{\partial}{\partial t_j} \quad \Rightarrow \quad J_{kl} = \frac{i\hbar}{2} \sum_j (a_k - a_l) v^k_j v^l_j g^{ij} \frac{\partial}{\partial t_j}.
\]

Let us compute the generic hamiltonian:

\[
\mathcal{H}(\lambda) = \sum_k \frac{F_k}{\lambda - a_k} = u(\lambda) + \frac{1}{2} \sum_{k,l} \frac{J_{kl}^2}{(\lambda - a_k)(\lambda - a_l)}
\]

Proposition. We have (compare with the classical formula (10)):

\[
\mathcal{H}(\lambda) = u(\lambda) \left[ 1 - \hbar^2 \sum_i \frac{1}{\lambda - t_i} \frac{1}{\sqrt{g}} \frac{\partial}{\partial t_i} \left( \sqrt{g} g^{ii} \frac{\partial}{\partial t_i} \right) \right]
\]

Proof. We only sketch the main steps of the calculation. We first have:

\[
\sum_{k,l} \frac{J_{kl}^2}{(\lambda - a_k)(\lambda - a_l)} = -\frac{\hbar^2}{4} \sum_{k,l,i,j} \frac{(a_k - a_l)^2}{(\lambda - a_k)(\lambda - a_l)} v^k_j v^l_i g^{ij} \left[ v^l_i v^k_j \frac{\partial}{\partial t_j} + 2 v^l_i \frac{\partial v^k_j}{\partial t_j} \right] g^{ii} \frac{\partial}{\partial t_i}
\]

The summation over \( k \) can be performed with the help of the following identities:

\[
\sum_k v^k_j v^k_i = 4 g_{ii} \delta_{ij}
\]
\[
\sum_k a_k v^k_j v^k_i = 4 t_i g_{ii} \delta_{ij}
\]
\[
\sum_k \frac{v^k_j v^k_i}{\lambda - a_k} = 4 g_{ii} \delta_{ij} + \frac{u(\lambda)}{(t_i - \lambda)(t_j - \lambda)}
\]
leading to the result:

$$\mathcal{H}(\lambda) = u(\lambda) - \frac{\hbar^2}{8} \sum_{l,i,j} \left[ \frac{(\lambda - a_i)u(\lambda)}{(t_i - \lambda)(t_j - \lambda)} + \frac{4g_{ij}(t_i - a_i)^2\delta_{ij}}{(t_i - \lambda)(a_i - \lambda)} \right] g^{ij} \left[ v^i_j v^i_j \frac{\partial}{\partial t_j} + 2v^j_i \frac{\partial v^i_j}{\partial t_j} \right] g^{ii} \frac{\partial}{\partial t_i}$$

To perform the summation over \(l\) in addition to the previous identities we need the following ones:

$$\sum_l v^i_j \frac{\partial v^j_l}{\partial t_j} = 2 \frac{\partial g_{ii}}{\partial t_i} \delta_{ij} + 2 \frac{g_{ij}}{t_i - t_j} (1 - \delta_{ij})$$

$$\sum_l a_l v^i_j \frac{\partial v^j_l}{\partial t_j} = 2 \frac{\partial(t_l g_{ii})}{\partial t_i} \delta_{ij} + 2 \frac{t_l g_{ij}}{t_i - t_j} (1 - \delta_{ij})$$

$$\sum_l \frac{1}{\lambda - a_l} v^i_j \frac{\partial v^j_l}{\partial t_i} = \frac{1}{2} \frac{\partial}{\partial t_i} \left[ \frac{u(\lambda)}{(t_i - \lambda)^2} + 4 \frac{g_{ii}}{\lambda - t_i} \right]$$

The following result then emerges:

$$\mathcal{H}(\lambda) = u(\lambda) \left[ 1 + \frac{\hbar^2}{2} \sum_i \frac{1}{\lambda - t_i} \left\{ -2 \frac{\partial}{\partial t_i} - g^{ii} \frac{\partial g_{ii}}{\partial t_i} + \sum_{j \neq i} \frac{1}{t_j - t_i} \right\} g^{ii} \frac{\partial}{\partial t_i} \right]$$

The result now follows by noticing that:

$$\frac{\partial \log \sqrt{g}}{\partial t_i} = \frac{1}{2} \left( g^{ii} \frac{\partial g_{ii}}{\partial t_i} - \sum_{j \neq i} \frac{1}{t_j - t_i} \right)$$

The corresponding Schrödinger equation is:

$$\mathcal{H}(\lambda) \Psi = \left( \sum_k \frac{f_k}{\lambda - a_k} \right) \Psi$$

and defining the \(b_n\)'s such that \(\prod_n (\lambda - b_n) / \Delta (\lambda) = \sum_k f_k / (\lambda - a_k)\) it can be rewritten as:

$$\sum_{j=1}^{N-1} \frac{(-1)^{N-j-1}D_j}{D(\lambda - t_j)} \left\{ 4\hbar^2 \frac{\sqrt{\Delta(t_j)}}{\sqrt{\Delta(t_j)}} \frac{\partial}{\partial t_j} \sqrt{\Delta(t_j)} \frac{\partial}{\partial t_j} - t_j^{N-2} (t_j - \lambda) - \prod_n (\lambda - b_n) \right\} \Psi = 0$$

As in the classical case this leads to the separated equations:

$$\left\{ 4\hbar^2 \frac{\sqrt{\Delta(t_j)}}{\sqrt{\Delta(t_j)}} \frac{\partial}{\partial t_j} \sqrt{\Delta(t_j)} \frac{\partial}{\partial t_j} - \prod_n (\lambda - b_n) - (t_j - \lambda) \left( t_j^{N-2} + \sum_{k=0}^{N-3} c_k(\lambda) t_j^k \right) \right\} \Psi_j(t_j) = 0$$

We obtain equations independent of \(\lambda\) by choosing:

$$t_j^{N-2} + \sum_{k=0}^{N-3} c_k(\lambda) t_j^k = \frac{\prod_n (t - b_n) - \prod_n (\lambda - b_n)}{t - \lambda}$$

and we get back eq. (18) with the \(f_k\)'s identified with the eigenvalues of the \(F_k\)'s.
9 The semi–classical quantization conditions

The solution of the quantum Neumann model has been reduced to the study of the Lamé equation (18). We shall not dwell further on this question in this paper and content ourselves with a discussion of the semi–classical quantization. The semi–classical wave function is of the form:

\[ \Psi_{f_1 \cdots f_N}(t_1, \ldots, t_{N-1}) = \exp \left( \frac{i}{\hbar} S(t_1, \ldots, t_{N-1}) \right). \]

Notice that the semi–classical quantization conditions:

\[ \oint p_k dq_k = 2\pi \hbar n \]

are canonical invariants and mean that \( \Psi \) is univalued on the Liouville torus since \( \oint pdq \) is the variation of \( S \) when one describes a non–trivial loop around the Liouville torus.

In order to understand further these conditions, it is necessary to describe the corresponding cycles on the curve (14). Since \( P(t) = \prod_k (t - a_k) \prod_n (t - b_n) \) one has to worry about the relative disposition of the \( a_k \)'s and \( b_n \)'s which depends on the signs of the \( f_k \)'s. This introduces several cases as first noted by Neumann [1]. In order to simplify the discussion we shall assume for example that all \( f_k \)'s are positive, implying:

\[ a_1 < b_1 < a_2 < b_2 < \cdots < b_{N-1} < a_N \]

as may be seen immediately by considering the graph of \( \sum_k f_k / (t - a_k) \). Moreover we know that:

\[ a_1 < t_1 < a_2 < t_2 < a_3 < \cdots < t_{N-1} < a_N \]

so that classically each \( t_j \) is constrained in an interval \([a_j, b_j]\) or \([b_j, a_{j+1}]\) in view of equation (8) i.e. each \( P(t_j) \) must be \( \leq 0 \). Since \( P(t) > 0 \) for \( t > a_N \) one sees that \( t_j \in [b_j, a_{j+1}] \).

Let us forget momentarily \( t_1, \ldots, t_{N-1} \) and concentrate on the motion of \( t_j \). Recalling equation (13) we see that \( dt_j / d\tau \) vanishes when \( t_j \to b_j \) and \( t_j \to a_{j+1} \) and is of a fixed sign in this interval. So \( t_j \) oscillates between its bounds. Now when \( t_j = a_{j+1} \) in view of section 3, \( x_{j+1} = 0 \) but \( \dot{x}_{j+1} \neq 0 \). So the point on the sphere \( S^{(N-1)} \) crosses the limit \( x_{j+1} = 0 \) of its quadrant when \( t_j \to a_{j+1} \) and continues its motion on the symmetric quadrant with respect to the hyperplane \( x_{j+1} = 0 \). In the space of the \( t \) variables this may be seen as an analytic continuation around the branch point \( t_j = a_{j+1} \). To sum up our discussion one loop on the sphere corresponds to two oscillations on the interval \([b_j, a_{j+1}]\). Writing that the wave function \( \Psi \) resumes its initial value at the end of the loop we get the semi–classical quantization conditions:

\[ 4 \int_{b_j}^{a_{j+1}} \frac{1}{2} \sqrt{\frac{\prod_n (t - b_n)}{\prod_k (t - a_k)}} dt = 2\pi \hbar n_j, \quad n_j \in \mathbb{Z} \]

These \((N-1)\) quantization conditions determine the \( b_j \)'s or equivalently the \( f_k \)'s. Of course the range of the \( f_j \)'s should be restricted so that the previously mentioned constraints on the \( f_k \)'s be obeyed (including in particular the positivity of the energy for the case of original Neumann's hamiltonian). Finally let us notice that the integrals appearing in these conditions express themselves in terms of some of the periods of abelian integrals on the Riemann surface (14).
10 Conclusion.

In this article we have shown that the method of separation of variables which was known to apply to the classical Neumann model applies equally well to the quantum Neumann model. This model is presumably one of the simplest non trivial integrable model which can be solved in this way. This is because the necessary change of variables to separate the equations is only a change of coordinates on the sphere. Moreover we have treated on the same footing the whole family of integrals of motion in involution discovered by K. Uhlenbeck in the classical case as well as in the quantum case. This method of resolution introduces a Riemann surface and the associated Jacobian torus in a particularly simple and elegant way.

In the classical case this leads to the linearization of the flow on the Jacobian, while in the quantum case the quantization conditions involve the real periods of the underlying Riemann surface. We will promote our semi–classical analysis to an exact one in a coming work.

Recently Sklyanin has reconsidered this method of separation of variables in various integrable systems [11]. It is interesting to remark that in such system as the Toda chain the separated equations appear to be finite difference equations (Baxter equation) while in our case we end up with ordinary differential equations. We expect this to be related to the fact that in the Lax pair formulation the Neumann model involves a linear Poisson structure [14] while in the Toda chain the Poisson structure is quadratic [3].

Acknowledgements We thank M. Dubois-Violette and R. Kerner for helpful discussions.

References

[1] C. Neumann, De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur. Crelle Journal 56 (1859), p. 46.
[2] K. Uhlenbeck, Equivariant harmonic maps into spheres. Springer Lecture Notes in Mathematics 949 (1982), p. 146.
[3] J. Moser. Various aspects of hamiltonian systems. In Proceedings of the C.I.M.E. Bressanone. Progress in Math., volume 8. Birkhauser, (1978).
[4] D. Mumford. Tata lectures on Theta II. Birkhauser Boston, (1984).
[5] M. Adler and P. van Moerbeke, Completely integrable systems, euclidean Lie algebras and curves. Adv. Mat. 38 (1980), p. 267.
[6] B. Kostant, The solution to a generalized Toda lattice and representation theory. Adv. Mat. 34 (1979), p. 195.
[7] M. Adler, On a trace functional for formal pseudodifferential operators and symplectic structure of the Korteweg de Vries type equations. Inv. Math. 50 (1979), p. 219.
[8] W. Symes, Systems of Toda type, inverse spectral problems and representation theory. Inv. Math 59 (1980), p. 13.
[9] E. K. Sklyanin. *On complete integrability of the Landau-Lifschitz equation*. Preprint LOMI E3-79, Leningrad 1979.

[10] A. G. Reyman and M. A. Semenov-Tjan-Shanskii, *Group-theoretical methods in the theory of finite dimensional integrable systems*. Encyclopedia of Mathematical Science **16** (1991), p. 190.

[11] E.K. Sklyanin. *Quantum inverse scattering method. Selected topics*. Preprint HU-TFT-91-51 University of Helsinki, (1991).

[12] M. Gutzwiller, *The quantum mechanical Toda lattice, part II*. Ann. Phys. N.Y. **133** (1981), p. 304.

[13] E.K. Sklyanin. The quantum Toda chain. In *Non-linear equations in classical and quantum field theory*. Springer Notes in Physics, vol.226, (1985).

[14] J. Avan and M. Talon, *Alternative Lax structures for the classical and quantum Neumann model*. Phys. Lett. B **268** (1991), p. 209.

[15] J. Avan and M. Talon, *Poisson structure and integrability of the Neumann-Moser-Uhlenbeck model*. Intern. Journ. Mod. Phys. A **5** (1990), p. 4477.

[16] E.T. Whittaker and G.N. Watson. *A course of modern analysis*. Cambridge University Press, (1902).