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DEDICATION

To

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ABSTRACT

In this thesis, we construct and classify planar noncommutative phase spaces by the coadjoint orbit method on the anisotropic and absolute time kinematical groups. We show that noncommutative symplectic structures can be generated in the framework of centrally extended anisotropic kinematical groups as well as in the framework of noncentrally abelian extended absolute time kinematical groups.

However, noncommutative phase spaces realized with noncentral abelian extensions of the kinematical groups are algebraically more general than those constructed on their central extensions. As the coadjoint orbit construction has not been carried through some of these planar kinematical groups before, physical interpretations of new generators of those extended structures are given. Furthermore, in all the cases discussed here, the noncommutativity is measured by naturally introduced fields, each corresponding to a minimal coupling.

This approach allows to not only construct directly a dynamical system when of course the symmetry group is known but also permits to eliminate the non minimal couplings in that system. Hence, we show also that the planar noncommutative phase spaces arise naturally by introducing minimal coupling. We introduce here new kinds of couplings. A coupling of position with a dual potential and a mixing model (that is minimal coupling of the momentum with a magnetic potential and of position with a dual potential).

Finally we show that this group theoretical discussion can be recovered by a linear deformation of the Poisson bracket. The reason why linear deformation of Poisson bracket is required here is that the noncommutative parameters (which are fields) are constant (they are coming from central and noncentral abelian extensions of kinematical groups).
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INTRODUCTION

0.1 Introduction

This thesis takes place within the framework of the titled domain noncommutative geometry. Noncommutativity appeared in nonrelativistic mechanics first in the work of Peierls [1] on the diamagnetism of conduction electrons. In relativistic quantum mechanics, the suggestions to use noncommutative coordinates goes back to Heisenberg and was firstly formalized in 1947 by Snyder [2] at small length scales. Some time later, Von Neumann [3] introduced the term noncommutative geometry to refer in general to a geometry in which an algebra of functions is replaced by a more general associative algebra called noncommutative algebra. For him, operator algebra theory was a noncommutative outgrowth of measure theory. As in the quantization of classical phase space, coordinates are replaced by generators of the algebra.

The correspondence between “spaces” and “commutative algebras” is familiar in Mathematics and in theoretical Physics. This correspondence allows an algebraic translation of various geometrical concepts on spaces in appropriate algebras of functions on these spaces. Replacing these commutative algebras by noncommutative algebras, i.e. forgetting commutativity, leads then to noncommutative generalizations of geometries where notions of “spaces of points” are not involved.

Interest in Snyder’s idea was revived much later when Mathematicians, notably Connes [4, 5] and Woronowicz [6], succeeded in generalizing the notion of differential structure to noncommutative geometry. Such a noncommutative generalization was a need in Physics for the formulation of quantum theory and the understanding of its relations with classical Physics. Indeed, as the role of symplectic geometry and hence symplectic structures has increased its importance in both Mathematics and Physics to constitute nowadays an essential technique of describing and modeling natural phenomena, then noncommutative symplectic structures offer a novel and promising framework for the construction of physical theories. Particularly, noncommutative phase spaces
provide mathematical backgrounds for the study of magnetic fields in Physics.

As applied to Physics, noncommutative geometry is understood mainly in two approaches. The first one is the spectral triple of A. Connes \cite{7} with the Dirac operator playing a central role in unifying, through the universal action principle, gravitation with standard model of fundamental interactions. The second one is the quantum field theory on noncommutative spaces \cite{8} with Moyal product as main ingredient. Besides these, a proposition by several authors \cite{9, 10} corresponds to space coordinates that no longer commute. This was implemented by an extension of the Poisson structure on the cotangent space such that the brackets satisfy \( \{ x^k, x^l \} \neq 0 \). Upon quantization, the corresponding operators should then also be noncommutative.

One motivation for this work is to demonstrate that this extension of the Poisson structure is achieved when we consider a Lie group \( G \). Indeed, models associated with a given symmetry group can be conveniently constructed using the coadjoint orbit method also called Souriau’s method. His theorem says in fact that when a symmetry group \( G \) acts transitively on a phase space, then the latter is a coadjoint orbit of \( G \) equipped with its canonical symplectic form \cite{11, 12, 13}. In other words, the classical phase spaces of elementary systems correspond to coadjoint orbits of their symmetry groups. Thus, by considering a Lie group \( G \), the problem is to find a symplectic manifold \( X \) whose symmetry group is \( G \). Under some assumptions, this problem has a regular solution according to the Souriau’s approach.

The first applications that Souriau presented in his book \cite{11} concern both the Poincaré and the Galilei groups for which coadjoint orbits represent elementary particles characterized by the invariants \( m \) (mass) and \( s \) (spin). Souriau himself goes one step further as he considers massless particles with spin, \( m = 0, s \neq 0 \) identified as relativistic and nonrelativistic spin respectively. Souriau’s ideas were later extended to larger groups. Taking \( G = Poincare \times H_0 \) where \( H_0 \) is an internal symmetry group (e.g. \( SU(2), SU(3), \ldots \)) yields relativistic particles with internal structure (for more details see \cite{14, 15}).

The nonrelativistic kinematical groups admit nontrivial central extensions by one-dimensional algebra in dimension \( d \geq 3 \). But in the plane, they admit an exotic \cite{16} two-parameter central extension. The one-parameter central extension of the spatial Galilei group has been considered by Souriau in his book \cite{11}, the two-parameter central extension of the planar Galilei group was studied in \cite{9, 16}. Furthermore, the coadjoint orbit method has recently also been applied to smaller spacetime symmetry groups. For example, in \cite{17} a classical “photon” model was constructed, based en-
tirely on the Euclidean group $E(3)$, a subgroup of both the Poincaré and the Galilei groups. An other application of the Souriau’s method is found in [18] where the most general dynamical systems on which the nonrelativistic conformal groups act transitively as symmetries are constructed.

A related motivation comes from a curiosity about a general solution to the above problem, that is, to find more general symplectic manifolds whose symmetry groups are the planar kinematical groups. This takes place in a well known fact that if a free particle is coupled with an external field, this reduces the symmetry group to a subgroup (of the Galilei or Poincaré group) and conversely, this reduced symmetry is consistent with the symmetry subgroup [11].

This thesis is devoted to realize classical dynamical systems associated with anisotropic kinematical groups (kinematical groups without rotation parameters [19]) by use of the reciprocal schema. Precisely, we start with a model with anisotropic kinematical symmetry and by applying Souriau’s method, we obtain models with additional terms (in the symplectic 2-form) interpreted as fields. The latter are linked to the free particle so as to preserve the anisotropic kinematical symmetries. More specifically, we study the maximal coadjoint orbit (all invariants are nonvanishing) of the all planar anisotropic kinematical groups (oscillating and expanding Newton-Hooke Lie groups, Galilei, Para-Galilei, Carroll and Static Lie groups) according to the classification in [20].

Furthermore, equivalently to Souriau’s theorem, the dual $G^*$ of the Lie algebra $G$ has a natural Poisson structure whose symplectic leaves are the coadjoint orbits. Depending on the Lie group, these orbits may provide noncommutative phase spaces. Note that on its general form, a noncommutative phase space allows for nonzero commutator among the coordinates and among the momenta. Thus, by applying the Souriau’s method, we construct and classify noncommutative phase spaces (which are effectively generalized or modified symplectic structures) associated to the extended kinematical groups. As already argued we consider the case where the symmetry groups are the kinematical groups according to the classification in [20] and realize noncommutative phase spaces on their maximal coadjoints orbits by using central extensions of all anisotropic kinematical algebras (by relaxing the isotropy condition or dropping the rotation generators [19]), the latter being the Lie algebras for the kinematical groups.

Note that for the one-parameter centrally extended kinematical algebras, the non-trivial Lie bracket which contains the only central extension parameter $m$ (i.e the mass
of the system) is
\[ [K_i, P_j] = M \delta_{ij} \]
which means that the generators of space translations as well as pure kinematical group transformations commute. One can not then associate noncommutative phase spaces to both planar kinematical groups and their one-fold centrally extended kinematical groups. So, it is necessary to work with the two-fold central extensions of the kinematical algebras and of their corresponding Lie groups and it is then the absence of the symmetry rotations (i.e anisotropy of the space) which guaranties the existence of noncommutative phase space for planar anisotropic kinematical groups.

However, it is possible to associate a noncommutative phase space to absolute time groups by considering their noncentral abelian extensions. Thus, we enlarge this theory by considering also the noncentral abelian extensions of the absolute time kinematical algebras associated to the Lie groups classified in [21]. Explicitly, we show that noncommutative symplectic structures can be generated in the framework of centrally extended anisotropic kinematical algebras as well as in the framework of noncentrally abelian extended isotropic kinematical algebras (rotations included).

Meanwhile, physical theories with noncommuting coordinates have become the focus of recent research (see, e.g., [9, 10, 16, 22, 23], . . . ), the notion of noncommutativity having different physical interpretations. For example, it is well known that in the presence of an electromagnetic field, momenta do not commute. This has been realized in this thesis for the first time as the maximal coadjoint orbits of the centrally and noncentrally abelian extended planar Para-Galilei groups [24, 29]. Also, it has been proved that in the presence of the dual electromagnetic field the positions do not commute and the maximal coadjoint orbits of the extended planar Galilei groups have been shown to be models of noncommutative phase spaces for this case [29].

Also, we realize a phase space with noncommutative positions and momenta by Souriau’s method on the anisotropic Newton-Hooke groups [25]. In [26], the authors have found a similar symmetry in the so-called Hill problem (the latter is studied in celestial mechanics), which is effectively an anisotropic harmonic oscillator in a magnetic field. This system has no rotational symmetry while translations and generalized boosts still act as symmetries. The noncommutative version of the Hill problem has been discussed in [27]. In this work, we study also the classical dynamical systems associated with Aristotle group [28], a subgroup of the Galilei group to highlight the fact that it is always possible to construct noncommutative phase spaces on noncentrally
abelian extended Lie structures.

Furthermore, we classify the obtained nonrelativistic models which are noncommutative phase spaces \[29\]. Through our constructions the coordinates of the phase spaces do not commute due to the presence of naturally introduced fields giving rise to minimal couplings. Thus, this group theoretical method allows not only a direct construction of a dynamical system when of course the symmetry group is known but also permits to eliminate the non minimal couplings in that system. Hence, to be complete in this work, we show also that planar noncommutative phase spaces arise naturally by introducing minimal couplings, each type of noncommutative phase space realized here corresponding to a specific minimal coupling.

As the coadjoint orbit construction has not been carried through some of these planar kinematical groups before, physical interpretations of new generators of those extended structures are given by symplectic realization methods. Note also that noncommutative phase spaces realized with noncentral abelian extensions of the kinematical groups are algebraically more general than those constructed on their central extensions. It is also shown in this work that the noncommutativity of momenta implies some modification of the second Newton law \[24, 30, 31\]. We prove finally that the group theoretical discussion above can be recovered by a linear deformation of Poisson brackets. One simple reason why linear deformation is required here is that the fields which are responsible to the noncommutativity are all constant (because coming from central or noncentral abelian extensions).

In all these approaches, for simplicity in notation and for a clear physical interpretation, we work in two-dimensional space (except for the anisotropic Newton-Hooke groups case where we consider also the three-dimensional space to highlight the way of finding the invariant) although the extension to higher dimensions is straightforward.

This thesis is organized in the following way.

Chapter 1 is devoted to the construction of central extensions of planar anisotropic kinematical groups and noncentral abelian extensions of planar absolute time groups according to the classification in \[20\] and \[21\] respectively. We consider the (maximal) central extensions of kinematical algebras which exponentiate to the corresponding kinematical groups and then appear to have a clear physical interpretation. In the application to the nonrelativistic particles, the central extensions of planar anisotropic kinematical groups considered here all have a common property. They have two central extension generators, one of them is related to the particle’s mass \(m\) and an other can
be related to the particle’s spin ($s$) [32, 33, 34, 35]. Note that the noncentral abelian extensions of absolute time planar kinematical groups have a different number of extension parameters.

In Chapter 2, we construct noncommutative phase spaces by introducing minimal couplings. We start by the usual coupling of momentum with a magnetic potential. Then, we introduce a new kind of coupling. That of position with a dual potential and we finish by a mixing model. In other words, we study the planar mechanics in the following three situations: when a charged massive particle is in an electromagnetic field, when a massless spring is in a dual magnetic field and finally when a pendulum is in an electromagnetic field and in its dual counterpart. It is shown in this Chapter that under the presence of these fields, the charged massive particle acquires an oscillation state of motion with a certain frequency, the massless spring acquires a mass and that the pendulum looks like two synchronized oscillators. The second and third results above are quite new and expressed the formula of minimal couplings in symplectic terms as done by Souriau for the first time [11]. Furthermore, as it will be shown in the next Chapter, each kind of minimal coupling is realized by coadjoint orbit method on a specific kinematical group.

The aim of Chapter 3 is to construct and classify noncommutative phase spaces by coadjoint orbit method in the framework of centrally and noncentrally abelian extended Lie groups we have encountered so far. We study, first of all, the maximal coadjoint orbits of the Aristotle group. We obtain in this case, phase spaces equipped with modified symplectic structures by using a noncentrally abelian extended Aristotle group. This example proves that one can always obtain a noncommutative phase space on the noncentrally abelian extended Lie group by the coadjoint orbit method. We do a similar construction on the nonrelativistic anisotropic kinematical groups (i.e Newton-Hooke groups, Galilei, Para-Galilei groups, Static and Carroll groups). The obtained orbits are physically interpreted as the phase spaces of accelerated particles moving in the respective kinematical spacetimes. Finally, we construct maximal coadjoint orbits on noncentrally abelian extended absolute time kinematical groups as classified in [21]. The noncommutative phase spaces obtained in this case are algebraically general than those obtained in the anisotropic kinematical case.

Through all these constructions, the coordinates of the phase spaces do not commute due to the presence of naturally introduced fields giving rise to minimal couplings defined in the previous Chapter. The main result here is that the extended Galilei groups give rise to phase spaces with noncommuting position coordinates, the extended Para-Galilei groups and noncentrally abelian extended Aristotle group yield noncommuting
momenta, and the remaining groups all determine completely noncommutative phase spaces. By symplectic realization methods, physical interpretations of generators coming from the obtained extended structures are given.

In Chapter 4, we show that the group theoretical construction above leads to similar results as in the case of a linear deformation of Poisson brackets. Under some assumptions, it is shown that the linear deformation of the Poisson bracket gives rise to the same classification of planar noncommutative phase spaces as by the Souriau method. Linear deformation is required here because the noncommutative parameters (fields) introduced in the previous Chapter are all constant (they come from centrally or noncentrally abelian extended structures). Thus, to establish a relationship between the group theoretical construction used previously and the deformation of the Poisson bracket developed in this Chapter, the latter may be a linear one.

Finally, we conclude and briefly list directions for future research.
Chapter One

Planar Kinematical Groups, Their Central and Noncentral Extensions

Lie groups are frequently introduced in Physics as groups of transformations acting on manifolds. In this Chapter we revisit all possible kinematical groups and their corresponding Lie algebras according to the classification in [20]. The former are assumed to be simply connected to single out a Lie group for the given Lie algebra. We then extract those for which rotation generators can be dropped producing anisotropic kinematical algebras (and groups) [19]. Recall that kinematical algebras are Lie algebras that are generated by spatial displacements, time translations, rotations and inertial transformations.

Since the structure of (2 + 1)-dimensional kinematical groups is significantly different from that of the (3 + 1)-dimensional ones [33], then it is interesting to study the problem of finding central extensions of the planar anisotropic kinematical groups and noncentral abelian extensions of planar absolute time kinematical groups according to the classifications in [20] and in [21] respectively.

The aim of this Chapter is to solve the above problem: we compute central (and noncentral) extensions of the planar anisotropic kinematical algebras (and the absolute time kinematical algebras) and their corresponding Lie groups with the assumption that an abelian extension (central or noncentral) of a Lie algebra should integrates to an abelian extension (central or noncentral) of its corresponding Lie group [36].

In other words, we restrict our attention to the central and noncentral abelian extensions of Lie algebras which exponentiate to the corresponding Lie groups, i.e through this thesis, we disregard extensions which correspond to introducing a central generator that measures the noncommutativity between \( J \) and \( H \) (we will consider \([J,H] = 0\)
i.e. rotation invariance). In the nonrelativistic anisotropic case, the central charges that have denoted $M$ and $S$, related respectively to the particle’s mass ($m$) and spin ($s$) \cite{32,33,34,35}, survive the exponentiation to the corresponding groups. Physically, $S$ is interpreted as the intrinsic angular momentum operator representing rotation in the rest-frame.

This Chapter is organized as follows.

In the next section, we review the Bacry and Levy-Leblond approach of classifying kinematical algebras and groups \cite{20} and extract the anisotropic kinematical ones (without rotations). In sections two and three, we determine central extensions of anisotropic kinematical algebras and noncentral abelian extensions of absolute time kinematical algebras (rotations included) respectively using dimensional analysis. Some of these extensions have not appeared previously in the literature.

### 1.1 Possible kinematical groups

Consider a manifold $M$ on which a transformation group $G$ acts transitively (this action is usually the left one) meaning that $M$ is a $G$-homogeneous space. When $M$ describes spacetime, $G$ is the kinematical group of $M$. All $d$-dimensional spacetimes with a constant curvature have a kinematical group of dimension $\frac{1}{2}d(d+1)$ \cite{37}.

Here, spacetime denotes a mathematical model that, of a physical dynamic system, unifies space and time into a manifold of four-dimensions. Note that traditionally in physics, spacetimes are described by (pseudo-) Riemann spaces, i.e.: by smooth manifolds and with a tensor metric $g_{ij}(x)$ and are of the form $T \times \Sigma$ where the one-dimensional space $T$ is the universally defined time line (admitting the preferred coordinate $t$ and its one-form $dt$) while the manifold $\Sigma$ is with possibly punctures and a bounded (this last assumption is identically satisfied for the spacetimes we examine).

Furthermore, in general relativity, it is assumed that spacetime is curved by the presence of matter (energy), this curvature being represented by the Riemann tensor and in nonrelativistic classical mechanics, the use of Euclidean space instead of spacetime is appropriate as the time is treated as universal and constant, being independent of the state of motion of the observer.

It is well known that there are different kinds of kinematics on homogeneous $(3+1)$-dimensional spacetimes \cite{20} and all of them can be contracted from Minkowski, de Sitter and Anti-de Sitter spacetimes under group contraction \cite{38} respectively. The complete kinematical group, whatever it may be, will always have a subgroup account-
Possible kinematical groups

ing for the isotropy of space (rotation group) and the equivalence of inertial transformations (boosts of different kinds for each case). The remaining transformations are translations which may be either commutative or not and are responsible of homogeneity of space and time. Roughly speaking, the point-set of the corresponding spacetime is, in each case, the point-set of these translations. This holds for usual special relativistic kinematics but also for Galilean and other conceivable nonrelativistic kinematics [20], which differ from each other precisely by being grounded on different kinematical groups (depending on the different laws of composition of the transformations groups). The best known examples are the Poincaré group \( P \), a group naturally associated to Minkowski spacetime as its group of motion and the Galilei group \( G \) which represents the classical mechanics.

Therefore, kinematical spacetime is defined as the quotient space of the whole kinematical group by the subgroup including rotations and boosts. This means that local spacetime is always a homogeneous manifold. We shall thus be concerned with such very special kinds of spacetimes: the homogeneous spacetimes of groups which can be called kinematical.

More specifically, let \( G \) denotes the kinematical group and \( \mathfrak{g} \) its Lie algebra. We choose a basis for \( \mathfrak{g} \) in which the infinitesimal generators of rotation, those of boosts and spatial translations along the spatial directions and that of time translation are denoted respectively as \( J_a, K_i, P_i, H \), \( a = 1, 2, \ldots, \frac{(d-2)(d-1)}{2} \), \( i = 1, 2, \ldots, d-1 \).

Before we proceed, we point the reader’s attention to the choice of the above quantities and at the same time recall their physical significances. First of all, note that if a general element of a one-parameter Lie group generated by \( X \) takes the form \( \exp(xX) \) then this means that \( xX \) must be dimensionless and then that the physical dimension of the parameter \( x \) must be the inverse of that of \( X \). For that reason, the parameters \( x^i, v^i \) and \( t \) (that will be used later in this work) associated respectively to \( P_i \) (i.e the generators of spatial translations in the \( i^{th} \) spatial direction), \( K_i \) (i.e generators of boosts in the \( i^{th} \) spatial direction) and \( H \) (i.e generator of time translation) have dimension of a length, a velocity and a duration. Their duals (in the dual Lie algebra) are respectively linear momenta \( p_i \), static momenta \( k_i \) and energy \( E \).

Thus, if \( L \) and \( T \) denote respectively the dimension of a length and of a duration, then the physical dimensions of \( P_i, H \) and \( K_i \) are \( L^{-1}, T^{-1} \) and \( L^{-1}T \) respectively. It is also known that the generators \( J_a \) of rotations are dimensionless. Physically, \( J_a \) correspond to the angular momenta and as said early \( H \) corresponds to the Hamiltonian, \( P_i \) correspond to the components of linear momenta and \( K_i \) correspond to the components
of the static momenta.

Furthermore, the relations between these physical quantities \( P_i, K_i \) and \( H \) depend on the structure of the Lie algebras and can be determined by symplectic realization methods of the corresponding Lie groups [39]. The variables \( p_i \) and \( k_i \) yield the basic canonical variables on the phase space via the the coadjoint orbit method as it will be seen later in this thesis.

At this level, let us summarize the results obtained by Bacry and Lévy-Leblond concerning the possible structures of groups of transformations which relate two inertial systems under quite general physical hypotheses.

**1.1.1 Bacry and Lévy-Leblond approach**

In [20], Bacry and Lévy-Leblond have classified the possible ten-parameters kinematical groups consisting of the spacetime translations, spatial rotations and inertial transformations connecting different inertial frames of reference. Bacry and Lévy-Leblond have shown, under the assumption:

- the space must be isotropic, meaning that the rotation group \( SO(d - 1) \) generated by \( J_a, a = 1, 2, \ldots, \frac{(d-2)(d-1)}{2} \), is a subgroup of the kinematical group,
- the spacetime must be homogeneous, meaning that the space translations group generated by \( P_i, i = 1, 2, \ldots, d - 1 \) and the time translations group generated by \( H \), are subgroups of the kinematical group,
- the inertial transformations group generated by \( K_i, i = 1, 2, \ldots, d - 1 \) is a non-compact subgroup of the kinematical group,
- the parity \((\pi : H \to H, P_i \to -P_i, K_i \to -K_i, J_a \to J_a)\) and the time-reversal \((\theta : H \to -H, P_i \to P_i, K_i \to -K_i, J_a \to J_a)\) are automorphisms of the kinematical group,

that there are eleven kinematical groups.

The corresponding kinematical algebras are characterized by the fact that the inertial transformation generators and the space translation generators behave as vectors under rotations while the time translation generator behaves as a scalar:

\[
[J_j, J_k] = J_l \epsilon_{jkl}, \quad [J_j, K_k] = K_l \epsilon_{jlk}, \quad [J_j, P_k] = P_l \epsilon_{jlk}, \quad [J_j, H] = 0 \quad (1.1)
\]

In the above relation, \( \epsilon_{jlk} \) are the three-dimensional totally antisymmetric Levi-Civita symbols and summation convention for a repeated index up and down is implied (Einstein convention).
As each Lie bracket \([K_i, H], [K_i, P_j] \) and \([P_i, H] \) is invariant under the parity and the time-reversal automorphisms as well as their product (\( \Gamma = \pi\theta : H \rightarrow -H, P_i \rightarrow -P_i, K_i \rightarrow K_i, J_a \rightarrow J_a \)), we also have that:

\[
[K_j, K_k] = \mu J_l \epsilon^l_{jk}, \quad [K_j, P_k] = \gamma \delta_j k H, \quad [K_j, H] = \lambda P_j
\]

where the physical dimension of the parameters \(\mu\) and \(\gamma\) is \(L^{-2}T^2\) while the parameter \(\lambda\) is dimensionless and finally,

\[
[P_j, P_k] = \alpha J_l \epsilon^l_{jk}, \quad [P_j, H] = \beta K_j
\]

where the physical dimension of the parameter \(\alpha\) is \(L^{-2}\) while that of the parameter \(\beta\) is \(T^{-2}\). Only three of the five parameters \(\alpha, \beta, \lambda, \mu, \gamma\) are independent. Effectively the Jacobi identities

\[
[K_i, [P_j, P_k]] + [P_j, [P_k, K_i]] + [P_k, [K_i, P_j]] = 0
\]

and

\[
[K_i, [K_j, P_k]] + [K_j, [P_k, K_i]] + [P_k, [K_i, K_j]] = 0
\]

imply that

\[
\alpha = \beta \gamma, \quad \mu = -\lambda \gamma
\]

If we compute the adjoint representation of the generators \(K_i\), we verify that the noncompactness of the boost transformations implies that \(\mu \leq 0\), meaning that \(\lambda\) and \(\gamma\) are all positive or all negative when one of them is not equal to zero. The brackets (1.2) and (1.3) become

\[
[K_j, K_k] = -\lambda \gamma J_l \epsilon^l_{jk}, \quad [K_j, P_k] = \gamma \delta_j k H, \quad [K_j, H] = \lambda P_j
\]

and

\[
[P_j, P_k] = \beta \gamma J_l \epsilon^l_{jk}, \quad [P_j, H] = \beta K_j
\]

We remain with three parameters \(\beta, \gamma\) and \(\lambda\) constrained by the fact that \(\lambda\) and \(\gamma\) are of the same sign when they are all different from zero. If each generator is multiplied by \(-1\), the three parameters change sign. As \(\lambda\) is dimensionless, we can assume (after normalization) that \(\lambda = 1\) or \(\lambda = 0\) and then that \(\gamma \geq 0\).

Let \(\kappa\) denotes the inverse of the universe radius \(r\) and let \(\omega\) denotes the time curvature (frequency). Then, for each of the two values of \(\lambda, \gamma = \frac{1}{c^2}\) or \(\gamma = 0\) where \(c = \frac{\omega}{\kappa}\) is a velocity. Also for each of the two values of \(\gamma\), there are three Lie algebras corresponding to \(\beta = \pm \omega^2\) and \(\beta = 0\).

Explicitly, we have:
A) the kinematical algebras corresponding to $\lambda = 1$ and defined by the brackets (1.1), (1.4) and

$$[K_j, K_k] = -\gamma J_l \epsilon^l_{jk}, \quad [K_j, P_k] = \gamma \delta_{jk} H, \quad [K_j, H] = P_j$$  \hspace{1cm} (1.5)

A1) Case $\gamma = \frac{1}{c^2}$

In this case the Lie algebra is defined by (1.1) and

$$[K_j, K_k] = -\frac{1}{c^2} J_l \epsilon^l_{jk}, \quad [K_j, P_k] = \frac{1}{c^2} \delta_{jk} H, \quad [K_j, H] = P_j$$  \hspace{1cm} (1.6)

and

A11) Case $\beta = \omega^2$ (the de Sitter algebra $dS_+$)

$$[P_j, P_k] = \kappa^2 J_l \epsilon^l_{jk}, \quad [P_j, H] = \omega^2 K_j$$  \hspace{1cm} (1.7)

A12) Case $\beta = 0$ (the de Poincaré algebra $P$)

$$[P_j, P_k] = 0, \quad [P_j, H] = 0$$  \hspace{1cm} (1.8)

A13) Case $\beta = -\omega^2$ (the anti-de Sitter algebra $dS_-$)

$$[P_j, P_k] = -\kappa^2 J_l \epsilon^l_{jk}, \quad [P_j, H] = -\omega^2 K_j$$  \hspace{1cm} (1.9)

A2) Case $\gamma = 0$

In this case the Lie algebra is defined by (1.1) and

$$[K_j, K_k] = 0, \quad [K_j, P_k] = 0, \quad [K_j, H] = P_j$$  \hspace{1cm} (1.10)

and

A21) Case $\beta = \omega^2$ (the expanding Newton-Hooke algebra $NH_+^2$)

$$[P_j, P_k] = 0, \quad [P_j, H] = \omega^2 K_j$$  \hspace{1cm} (1.11)

A22) Case $\beta = 0$ (the Galilei algebra $G$)

$$[P_j, P_k] = 0, \quad [P_j, H] = 0$$  \hspace{1cm} (1.12)

A23) Case $\beta = -\omega^2$ (the oscillating Newton-Hooke algebra $NH_-^2$)

$$[P_j, P_k] = 0, \quad [P_j, H] = -\omega^2 K_j$$  \hspace{1cm} (1.13)
B) the kinematical algebras corresponding to \( \lambda = 0 \) and defined by the brackets (1.1), (1.4) and

\[
[K_j, K_k] = 0, \ [K_j, P_k] = \gamma \delta_{jk} H, \ [K_j, H] = 0
\]  
(1.14)

B1) Case \( \gamma = \frac{1}{c^2} \)
In this case the Lie algebra is defined by (1.1),

\[
[K_j, K_k] = 0, \ [K_j, P_k] = \frac{1}{c^2} \delta_{jk} H, \ [K_j, H] = 0
\]  
(1.15)

and

B11) Case \( \beta = \omega^2 \) (the Para-Poincaré algebra \( \mathcal{P}_+ \))

\[
[P_j, P_k] = \kappa^2 J_t \epsilon_{jk}^l, \ [P_j, H] = \omega^2 K_j
\]  
(1.16)

B12) Case \( \beta = 0 \) (the Carroll algebra \( \mathcal{C} \))

\[
[P_j, P_k] = 0, \ [P_j, H] = 0
\]  
(1.17)

B13) Case \( \beta = -\omega^2 \) (the Para-Poincaré algebra \( \mathcal{P}_- \))

\[
[P_j, P_k] = -\kappa^2 J_t \epsilon_{jk}^l, \ [P_j, H] = -\omega^2 K_j
\]  
(1.18)

B2) Case \( \gamma = 0 \)
In this case we can assume that \( \beta \geq 0 \), the Lie algebra is then defined by (1.1),

\[
[K_j, K_k] = 0, \ [K_j, P_k] = 0, \ [K_j, H] = 0
\]  
(1.19)

B21) Case \( \beta = \omega^2 \) (the Para-Galilei algebra \( \mathcal{G}_+ \))

\[
[P_j, P_k] = 0, \ [P_j, H] = \omega^2 K_j
\]  
(1.20)

B22) Case \( \beta = 0 \) (the Static algebra \( \mathcal{S}_t \))

\[
[P_j, P_k] = 0, \ [P_j, H] = 0
\]  
(1.21)

The possible planar kinematical algebras are then defined by the brackets (1.1) reduced to:

\[
[J, K_k] = K_t \epsilon_{jk}^l, \ [J, P_k] = P_t \epsilon_{jk}^l, \ [J, H] = 0
\]  
(1.22)

and to summarize the above discussion, we distinguish the case where boosts do not commute with time translation (i.e \( [K_i, H] = P_i \)) from the case where they commute (i.e \( [K_i, H] = 0 \)) as detailed below:
Boosts not commuting with time translations

This case corresponds to $\lambda = 1$. The planar kinematical algebras are then defined by the brackets (1.22), (1.4) and (1.5). According to the values of $\gamma$ and $\beta$ as detailed above, the possible planar kinematical algebras of this form are summarized in the following table where $dS_+\, , \, P\, , \, dS_-\, , \, NH_+\, , \, G\, , \, NH_-$ stand respectively for the de Sitter, Poincaré, the anti de Sitter, the expanding Newton-Hooke, the Galilei and the oscillating Newton-Hooke Lie algebras and where we have omitted the brackets of the form $[J, X_i] = X_j \epsilon^j_i$ and $[J, H] = 0$.

| $\gamma = \frac{1}{c^2}$ | $[K_j, K_k] = -\frac{1}{c^2} J_l \epsilon^l_{jk}, \ [K_j, P_k] = \frac{1}{c^2} \delta_{jk} H, \ [K_j, H] = P_j$ |
| $dS_+$ | $\beta = \omega^2$ | $[P_j, P_k] = \kappa^2 J_l \epsilon^l_{jk}, \ [P_j, H] = \omega^2 K_j$ |
| $\mathcal{P}$ | $\beta = 0$ | $[P_j, P_k] = 0, \ [P_j, H] = 0$ |
| $dS_-$ | $\beta = -\omega^2$ | $[P_j, P_k] = -\kappa^2 J_l \epsilon^l_{jk}, \ [P_j, H] = -\omega^2 K_j$ |
| $\gamma = 0$ | $[K_j, K_k] = 0, [K_j, P_k] = 0, [K_j, H] = P_j$ |
| $NH_+$ | $\beta = \omega^2$ | $[P_j, P_k] = 0, \ [P_j, H] = \omega^2 K_j$ |
| $NH_-$ | $\beta = 0$ | $[P_j, P_k] = 0, \ [P_j, H] = 0$ |
| $NH_-$ | $\beta = -\omega^2$ | $[P_j, P_k] = 0, \ [P_j, H] = -\omega^2 K_j$ |

Table 1.1: Kinematical algebras with $[K_i, H] = P_i$

Boosts commuting with time translations

This case corresponds to $\lambda = 0$. The planar kinematical algebras are then defined by the brackets (1.22), (1.4) and (1.14). In this case, according to the values of $\gamma$ and $\beta$, we obtain the kinematical algebras summarized in the table below where $\mathcal{D}'_+, \ \mathcal{D}'_-\, , \ \mathcal{C}, \ \mathcal{C}'_+, \ \mathcal{C}'_-$ and $\mathcal{S}$ stand respectively for the Para-Poincaré, the anti Para-Poincaré, the Carroll, the Para-Galilei, the Anti-Para-Galilei and the Static Lie algebras and where we have omitted the brackets of the form $[J, X_i] = X_j \epsilon^j_i$ and $[J, H] = 0$. 

Thus, under a natural assumption on the decomposition of the adjoint action into irreducible (corresponding to isotropy of the space and homogeneity of the spacetime) and requiring that time and space reversals are automorphisms, there are exactly 11 kinematical algebras [20].

In the discussion above, according to the link between the three parameters $\beta$, $\gamma$ and $\lambda$, we have obtained an other Lie algebra: the Anti-Para-Galilei Lie algebra denoted $G'_-$ This is consistent with the contraction process. In fact, as the Para-Galilei algebra $G'_+$ is a velocity-space contraction of the Para-Poincaré algebra $P'_+$ or a velocity-time contraction of the Newton-Hooke group $NH_+$ [38], by the same process, the anti Para-Galilei algebra $G'_-$ is a velocity-space contraction of the anti-Para-Poincaré algebra $P'_-$ or a velocity-time contraction of the Newton-Hooke group $NH_-$, the latter correspond to the value $\beta = -\omega^2$.

In conclusion, the Lie brackets for the possible planar kinematical algebras according to the classification in [20] are summarized in the following table where the brackets of the form $[J, X_i] = X_j e^i_j$ and $[J, H] = 0$ are omitted and where we have also considered the Anti-Para-Galilei Lie algebra $G'_-$.

| $\gamma$ | $\beta$ | $[K_j, K_k] = 0$, $[K_j, P_k] = \frac{1}{\omega^2} \delta_{jk} H$, $[K_j, H] = 0$ |
|----------|---------|-------------------------------------------------|
| $P'_+$   | $\omega^2$ | $[P_j, P_k] = \kappa^2 J_i e^i_j$, $[P_j, H] = \omega^2 K_j$ |
| $G'_+$   | $0$     | $[P_j, P_k] = 0$, $[P_j, H] = 0$ |
| $P'_-$   | $-\omega^2$ | $[P_j, P_k] = -\kappa^2 J_i e^i_j$, $[P_j, H] = -\omega^2 K_j$ |

Table 1.2: Kinematical algebras with $[K_i, H] = 0$
Table 1.3: Lie brackets for the possible kinematical algebras

| Lie algebra | \([K_i, H]\) | \([K_i, K_j]\) | \([K_i, P_j]\) | \([P_i, P_j]\) | \([P_i, H]\) |
|-------------|-------------|-------------|-------------|-------------|-------------|
| \(dS_+\)   | \(P_i\)     | \(-\frac{1}{c^2} J_k \epsilon_{ij}^k\) | \(\frac{1}{c^2} H \delta_{ij}\) | \(\kappa^2 J_k \epsilon_{ij}^k\) | \(\omega^2 K_i\) |
| \(\mathcal{P}\) | \(P_i\)     | \(-\frac{1}{c^2} J_k \epsilon_{ij}^k\) | \(\frac{1}{c^2} H \delta_{ij}\) | \(0\) | \(0\) |
| \(dS_-\)   | \(P_i\)     | \(-\frac{1}{c^2} J_k \epsilon_{ij}^k\) | \(\frac{1}{c^2} H \delta_{ij}\) | \(-\kappa^2 J_k \epsilon_{ij}^k\) | \(-\omega^2 K_i\) |
| \(\mathcal{N}\) | \(P_i\)     | \(0\) | \(0\) | \(0\) | \(\omega^2 K_i\) |
| \(\mathcal{N}^\perp\) | \(P_i\)     | \(0\) | \(0\) | \(0\) | \(0\) |
| \(\mathcal{P}'_+\) | \(0\) | \(0\) | \(\frac{1}{c^2} H \delta_{ij}\) | \(\kappa^2 J_k \epsilon_{ij}^k\) | \(\omega^2 K_i\) |
| \(\mathcal{C}\) | \(0\) | \(0\) | \(\frac{1}{c^2} H \delta_{ij}\) | \(0\) | \(0\) |
| \(\mathcal{P}'_-\) | \(0\) | \(0\) | \(\frac{1}{c^2} H \delta_{ij}\) | \(-\kappa^2 J_k \epsilon_{ij}^k\) | \(-\omega^2 K_i\) |
| \(\mathcal{G}'_+\) | \(0\) | \(0\) | \(0\) | \(0\) | \(\omega^2 K_i\) |
| \(\mathcal{S}^\prime_-\) | \(0\) | \(0\) | \(0\) | \(0\) | \(-\omega^2 K_i\) |
| \(\mathcal{S}\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |

Furthermore, through the classification in [20], to each kinematical algebra is associated a corresponding Lie group. Each of these associated groups is either the de Sitter or the anti-de Sitter or one of their contractions. Recall that there are three fundamental types of contraction: velocity-space, velocity-time and space-time contraction corresponding respectively to contracting to the subgroup generated by \(H\), \(P\) and \(K\).

Geometrically, the velocity-space contraction (i.e making the substitution \(K \rightarrow \epsilon K\), \(P \rightarrow \epsilon P\) into the Lie algebra and calculating the singular limit of the Lie brackets as \(\epsilon \rightarrow 0\)) means to describe spacetime near a timelike geodesic equivalent to passing from relativistic to absolute time as stated in [21]. So, \(dS_+\) and \(dS_-\) contract to \(NH_\pm\) respectively, Poincaré group \(P\) contracts to Galilei group \(G\), Para-Poincaré groups \(P'_\pm\) contract to Para-Galilei groups \(G'_\pm\) and Carroll group \(C\) contracts to Static group \(S\) through a velocity-space contraction. In other words, the former are relativistic while the latter are absolute time groups.

Similarly, the velocity-time contraction (i.e making the substitution \(K \rightarrow \epsilon K\), \(H \rightarrow \epsilon H\) into the Lie algebra and calculating the singular limit of the Lie brackets as \(\epsilon \rightarrow 0\)) geometrically means to describe spacetime near a spacelike geodesic equivalent to passing from relativistic to absolute space. So, \(dS_+\) and \(dS_-\) contract to \(P'_\pm\) respectively, \(NH_\pm\) contract to \(G'_\pm\), Poincaré groups \(P\) contract to Carroll \(C\) and the Galilei group \(G\) contracts to Static group \(S\) through a velocity-time contraction. In other words, the former are relativistic while the latter are absolute space groups.
Finally, the space-time contraction (i.e. making the substitution $P \rightarrow \epsilon P$, $H \rightarrow \epsilon H$ into the Lie algebra and calculating the singular limit of the Lie brackets as $\epsilon \rightarrow 0$) geometrically means to describe spacetime near an event (physically the spacelike and the timelike intervals are small but the boosts are not restricted). The corresponding group is called local group as opposed to a cosmological group. So, $dS_+$ and $dS_-$ contract to Poincaré group $P$, $NH$ contract to Galilei group $G$, Para-Galilei groups $G' \pm$ contracts to Static group $S$ and Para-Poincaré groups $P' \pm$ contract to Carroll group $C$ through a space-time contraction. In other words, the former are cosmological groups while the latter are local groups.

The kinematical groups are then distributed according to the table below:

| Relative time groups | de Sitter, Poincaré, Para-Poincaré, Carroll |
|----------------------|---------------------------------------------|
| Absolute time groups | Newton-Hooke, Galilei, Para-Galilei, Static |
| Relative space groups | de Sitter, Newton-Hooke, Poincaré, Galilei |
| Absolute space groups | Para-Poincaré, Para-Galilei, Carroll, Static |
| Cosmological groups  | de Sitter, Newton-Hooke, Para-Poincaré, Para-Galilei |
| Local groups         | Poincaré, Galilei, Carroll, Static |

Table 1.4: Kinematical groups classification according [21]

### 1.1.2 Planar anisotropic kinematical algebras

Let us consider the planar kinematical algebras of the table (1.3). Their corresponding planar anisotropic Lie algebras are obtained by dropping the generators $J_i$ of rotations, meaning that they are generated by $\{K_i, P_i, H\}$. This is only possible for the two Newton-Hooke Lie algebras, the Galilei and the Para-Galilei Lie algebras, the Carroll Lie algebra and the Static Lie algebra where the rotation generators do not appear in the right hand side of the brackets $[K_i, K_j]$ and $[P_i, P_j]$. The planar anisotropic kinematical algebras are summarized in the following table (for $i, j = 1, 2$):

| Lie algebra | $[K_i, H]$ | $[K_i, K_j]$ | $[K_i, P_j]$ | $[P_i, P_j]$ | $[P_i, H]$ |
|-------------|------------|--------------|--------------|--------------|------------|
| $NH_+$      | $P_i$      | 0            | 0            | 0            | $\omega^2 K_i$ |
| $G$         | $P_i$      | 0            | 0            | 0            | 0          |
| $NH_-$      | $P_i$      | 0            | 0            | 0            | $-\omega^2 K_i$ |
| $\mathcal{E}$ | 0          | 0            | $\frac{1}{2} H \delta_{ij}$ | 0            | 0          |
| $G' \pm$    | 0          | 0            | 0            | $\pm \omega^2 K_i$ | 0          |
| $S$         | 0          | 0            | 0            | 0            | 0          |

Table 1.5: Lie brackets for the planar anisotropic kinematical algebras
Next to the above anisotropic kinematical algebras we consider in this thesis, also the absolute time Lie algebras which correspond to the absolute time groups given in the table (1.4) with respect to the isotropy of the two-dimensional space. One simple reason we consider the two types of kinematical algebras (namely anisotropic and absolute time) is that they both admit central and noncentral abelian extensions. Hereafter, absolute (relative) time groups will be called nonrelativistic (relativistic) kinematical groups. In this thesis, as a relative-time group [21, 40], the “Carroll ” group is considered in a relativistic theory: i.e that incorporates the speed of light as a parameter. In other words, in its corresponding Lie algebra, the energy $H$ is not considered as a central charge contrary to the assumption given in [41] where the contracted Lie algebra contains a Heisenberg sub-algebra in which energy $H$ appears as a central charge. In this context, all possible anisotropic kinematical groups considered here are absolute time groups except the “Carroll ” group.

In the following sections, we determine central and noncentral abelian extensions of the above planar kinematical algebras.

1.2 Extensions of planar anisotropic kinematical algebras

It is a well known fact in physics (particularly in quantum mechanics) that a projective representation of a group is essentially equivalent to a regular representation of a central extension of the group. For this reason, central extensions are of special importance in physics. In general, a kinematical group should have or not a central extension in a certain dimension. However some of them still have plenty of noncentral extensions, some of which are interesting from the physical point of vue.

We restrict our study to the planar kinematical algebras which are six-dimensional and we are interested in the central extensions of anisotropic kinematical algebras (without rotation parameters) and the noncentral abelian extensions of the absolute time kinematical algebras (rotations included). Note also that the passage from the extended Lie algebras to their corresponding extended Lie groups goes via the exponential function as usually and the group multiplication laws are defined by the Baker-Campbell-Hausdorff formulas [42] as we will see it later through our constructions.

To facilitate reading, let us first of all begin this section with some fundamental definitions.
1.2.1 mathematical preliminaries: abelian extensions of Lie algebras

**Definition 1.1.** A Lie algebra \( \hat{\mathfrak{g}} \) is an extension of a Lie algebra \( \mathfrak{g} \) by an abelian Lie algebra \( \mathcal{A} \) if there is an exact sequence of Lie algebras

\[
0 \rightarrow \mathcal{A} \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0
\]

i.e with continuous homomorphisms: such that \( \pi \) admit a continuous global section \( s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}} \), \( \pi \circ s = \text{id}_\mathfrak{g} \)

**Definition 1.2.** Let \( \hat{\mathfrak{g}}_1 \) and \( \hat{\mathfrak{g}}_2 \) be two extensions of \( \mathfrak{g} \) by an abelian Lie algebra \( \mathcal{A} \) i.e

\[
0 \rightarrow \mathcal{A} \rightarrow \hat{\mathfrak{g}}_1 \xrightarrow{\pi_1} \mathfrak{g} \rightarrow 0
\]

and

\[
0 \rightarrow \mathcal{A} \rightarrow \hat{\mathfrak{g}}_2 \xrightarrow{\pi_2} \mathfrak{g} \rightarrow 0
\]

\( \hat{\mathfrak{g}}_1 \) and \( \hat{\mathfrak{g}}_2 \) are equivalent if there exists a Lie isomorphism \( \Phi : \hat{\mathfrak{g}}_1 \rightarrow \hat{\mathfrak{g}}_2 \) such that

\[
\pi_2 \circ \Phi = \pi_1
\] (1.23)

There is a cocycle construction of abelian Lie algebra extensions. In order to define the cohomology of \( \mathfrak{g} \) with coefficients in a \( \varrho(\mathfrak{g}) \)-module \( \mathcal{A} \) (we shall take \( \mathcal{A} \) here to be the representation space of a finite-dimensional representation \( \varrho \) of \( \mathfrak{g} \)), the coboundary operator is introduced:

\[
(\delta \alpha_n)(X_1, X_2, ..., X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \varrho(X_i)\alpha_n(X_1, X_2, ..., \hat{X}_i, ..., X_{n+1}) + \sum_{i,j=1, i<j}^{n+1} \alpha_n([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{n+1})
\] (1.24)

where \( \alpha_n \) is a one \( n \)-cochain on \( \mathfrak{g} \) with values in \( \mathcal{A} \) (i.e an alternate multilinear map), \( X_1, X_2, ..., X_{n+1} \in \mathfrak{g} \) while the symbol “:” means that the variable under it has been deleted.

By using the fact that \( \varrho \) is a Lie algebra homomorphism, we verify that \( \delta^2 = 0 \). This permits us to obtain by the usual procedure the cohomology group

\[
H^n_\varrho(\mathfrak{g}, \mathcal{A}) = \frac{Z^n_\varrho(\mathfrak{g}, \mathcal{A})}{B^n_\varrho(\mathfrak{g}, \mathcal{A})}
\]

of the Lie algebra \( \mathfrak{g} \) with values in \( \mathcal{A} \) with respect to the representation \( \varrho \) where the group \( Z^n_\varrho(\mathfrak{g}, \mathcal{A}) = \text{Ker}(\delta : C^n(\mathfrak{g}, \mathcal{A}) \rightarrow C^{n+1}(\mathfrak{g}, \mathcal{A})) \) is the group of \( n \)-cocycles and \( B^n_\varrho(\mathfrak{g}, \mathcal{A}) = \text{Im}(\delta : C^{n-1}(\mathfrak{g}, \mathcal{A}) \rightarrow C^n(\mathfrak{g}, \mathcal{A})) \) is the group of \( n \)-coboundaries. It is well known that the second cohomology group \( H^2(\mathfrak{g}, \mathcal{A}) \) is related to the extensions of \( \mathfrak{g} \) by \( \mathcal{A} \) [43].
Note that the semi-direct sum of $G$ and $A$ consists of pairs $(X, A) \in G \times A$ with the commutator
\[
[(X_i, A_a), (X_j, A_b)] = ([X_i, X_j], g(X_i)A_b - g(X_j)A_a)
\] (1.25)

Let $\alpha_2 \in Z^2(G, A)$ be a two-cocycle. Define the following modified commutator in the sum vector space $G \oplus \alpha_2 A$
\[
[(X_i, A_a), (X_j, A_b)]_{\alpha_2} = ([X_i, X_j], g(X_i)A_b - g(X_j)A_a + \alpha_2(X_i, X_j))
\] (1.26)

**Proposition 1.3.** The modified bracket $[\cdot, \cdot]_{\alpha_2}$ is a Lie bracket.

- The bilinearity and antisymmetry of the above modified bracket follow from the definition of the two-cocycle and from that of the Lie bracket of $G$.
- The Jacobi identity in $\hat{G}$ is equivalent to the cocycle condition for $\alpha_2$:
\[
(\delta \alpha_2)(X_i, X_j, X_k) = g(X_i)\alpha_2(X_j, X_k) - g(X_j)\alpha_2(X_i, X_k)
+ g(X_k)\alpha_2(X_i, X_j) + \alpha_2([X_i, X_j], X_k) + \alpha_2([X_i, X_k], X_j) = 0
\] (1.27)

Then $G \oplus \alpha_2 A$ endowed with the Lie bracket (1.26) defines an abelian Lie algebra extension $\hat{G} = G \oplus \alpha_2 A$.

Let $\alpha_1$ and $\alpha_2 \in Z^2(G, A)$. The Lie algebras formed from these two-cocycles are isomorphic through a map of the type:
\[
\Phi(X, A) = (X, A + \beta(X)), X \in G, A \in A
\]
where $\beta \in H^1_G(G, A)$.

**Proposition 1.4.** The condition
\[
[\Phi(X_i, A_a), \Phi(X_j, A_b)]_{\hat{G}_2} = \Phi([([X_i, A_a], (X_j, A_b)]_{\hat{G}_1})
\] (1.28)
is the same as
\[
\alpha_2 - \alpha'_2 = \delta \beta
\] (1.29)

**Proof.**
\[
\Phi([([X_i, A_a], (X_j, A_b)]_{\hat{G}_1}) = \Phi([([X_i, A_a], (X_j, A_b)]_{\hat{G}_2})
= ([X_i, X_j], g(X_i)(A_b + \beta(X_j)) - g(X_j)(A_a + \beta(X_i)) + \alpha'_2(X_i, X_j))
\]
Extensions of planar anisotropic kinematical algebras

\[ \Phi([X_i, A_a], (X_j, A_b)]_{\hat{G}}) = ([X_i, X_j], \varrho([X_i, X_j]+\alpha_2', X_i, X_j))) = (\varrho(X_i) beta(X_j) - \varrho(X_j)\beta(X_i) + \alpha_2'(X_i, X_j), \delta) \]

\[ (\alpha_2 - \alpha_2') = \delta \beta \]

i.e \( \alpha_2 \) and \( \alpha_2' \) are cohomologous. Thus, up to an isomorphism of the above type, the Lie algebra extensions are parametrized by elements of \( H^2_\varrho(G, A) \).

So, each \( \alpha_2 \in H^2_\varrho(G, A) \) defines a new Lie algebra abelian extension. Then, the second cohomology group \( H^2_\varrho(G, A) \) is related to the abelian extensions of \( G \) by an abelian Lie algebra \( A \) for a given representation \( \varrho \).

The extension \( \hat{G} \) is called central if \( A \) lies in the center of \( \hat{G} \), i.e \([A, \hat{G}] = 0\), this is the case when the representation \( \varrho \) of \( G \) is trivial \([44]\). In this case the commutator (1.26) is reduced to

\[ [(X_i, A_a), (X_j, A_b)]_{\alpha_2} = ([X_i, X_j], \alpha_2(X_i, X_j)) \quad (1.30) \]

where the two-cocycle \( \alpha_2 \) satisfies:

\[ \alpha_2(X_i, X_j) = -\alpha_2(X_j, X_i) \quad (1.31) \]

and

\[ \alpha_2(X_i, [X_j, X_k]) + \alpha_2(X_j, [X_k, X_i]) + \alpha_2(X_k, [X_i, X_j]) = 0 \quad (1.32) \]

from the properties of the Lie brackets in \( G \) and \( \hat{G} \).

As it has been argued in the introduction, we will consider also the general case: noncentral abelian extension of \( G \) by an abelian Lie algebra \( A \) for a nontrivial action \( \varrho \), the word noncentral meaning that \([A, \hat{G}] \neq 0\).

Explicitly, let us consider that \( G \) is a \( n \)-dimensional real Lie algebra generated by \( X_i \) such that:

\[ [X_i, X_j] = X_k C^k_{ij} \quad (1.33) \]

We know that the \( n^3 \) structure constants \( C^k_{ij} \) satisfy the antisymmetry character:

\[ C^k_{ij} = -C^k_{ji} \quad (1.34) \]

and the Jacobi identity:

\[ C^l_{ir} C^r_{jk} + C^l_{jr} C^r_{ki} + C^l_{kr} C^r_{ij} = 0 \quad (1.35) \]
Let us now consider that the Lie algebra $\mathcal{A}$ is generated by $Y_\alpha$ and $Z_\alpha, \alpha = 1, \ldots, \dim \mathcal{A}$ such that the non trivial Lie brackets are:

$$[Y_\alpha, Y_\beta] = Z_\alpha C^\alpha_{\alpha\beta} \quad (1.36)$$

- A central abelian extension $\hat{\mathcal{G}}$ of $\mathcal{G}$ by $\mathcal{A}$ is the Lie algebra generated by $X_i$ and $Y_\alpha$ and defined by the non trivial Lie brackets

$$[X_i, X_j] = X_k C^k_{ij} + Y_\alpha d^\alpha_{ij} \quad (1.37)$$

or by $(1.30)$ in terms of 2-cocycle. It follows that

$$\alpha_2(X_i, X_j) = Y_\alpha d^\alpha_{ij} \quad (1.38)$$

and we verify from the properties $(1.31)$ and $(1.32)$ of the two-cocycle that:

$$d^\alpha_{ij} = -d^\alpha_{ji} \quad (1.39)$$

and

$$d^\alpha_{ij} C^r_{jk} + d^\alpha_{jr} C^r_{ki} + d^\alpha_{kr} C^r_{ij} = 0 \quad (1.40)$$

Then the structures constant $d^\alpha_{ij}$ satisfy also the antisymmetry character $(1.39)$ and the closure condition $(1.40)$ in $\hat{\mathcal{G}}$.

Looking for solutions of the system $(1.39)-(1.40)$, we obtain the maximal central extension $\hat{\mathcal{G}}$ of the Lie algebra $\mathcal{G}$ by the abelian Lie algebra $\mathcal{A}$. All central extensions of a Lie algebra by an abelian Lie algebra are obtained in this manner.

- A noncentral nonabelian extension $\hat{\mathcal{G}}$ of $\mathcal{G}$ by $\mathcal{A}$ is the Lie algebra generated by $X_i, Y_\alpha, Z_\alpha$ and $T_s$ and defined by the non trivial Lie brackets $(1.36), (1.37)$ and

$$[X_i, Y_\beta] = T_s C^s_{ij} \quad (1.41)$$

with Jacobi identities and the skew-symmetry character of the Lie bracket in $\hat{\mathcal{G}}$. We have the following relations between the structure constants:

$$d^\alpha_{ij} = -d^\alpha_{ji}$$

$$d^\alpha_{ij} C^r_{jk} + d^\alpha_{jr} C^r_{ki} + d^\alpha_{kr} C^r_{ij} = 0$$

$$C^\nu_{\alpha\nu} d^\nu_{ij} + C^\nu_{\beta\nu} d^\nu_{ki} + C^\nu_{\gamma\nu} d^\nu_{ij} = 0$$

$$C^s_{\alpha k} C^k_{ij} = 0$$

$$C^s_{\alpha \beta} d^s_{ij} = 0 \quad (1.42)$$
Particularly, the Lie algebra $\hat{G}$ will be called noncentral abelian extension of $G$ by $A$ if $Z_a = 0$ (i.e. $A$ is abelian). Its Lie structure is given by the system of brackets \((1.37)\) and \((1.41)\). In this case the system \((1.42)\) is reduced to

\[
\begin{align*}
    d_{i j}^a = -d_{j i}^a \\
    d_{i l}^a C_{jk}^r + d_{j l}^a C_{ki}^r + d_{k l}^a C_{ij}^r = 0 \\
    C_{i l}^a d_{ij}^r + C_{j l}^a d_{ki}^r + C_{k l}^a d_{ij}^r = 0 \\
    C_{\alpha k} C_{ij}^k = 0 
\end{align*}
\]  

\[(1.43)\]

Looking for solutions of the system \((1.43)\), we obtain the maximal noncentral abelian extension $\hat{G}$ of the Lie algebra $G$ by the abelian Lie algebra $A$. All noncentral abelian extensions of a Lie algebra by an abelian Lie algebra are obtained in the same way.

Furthermore, central and noncentral abelian extensions of Lie algebras can also be found by looking at the second Lie algebra cohomology group (determining the two-cocycles which define extensions) but this method is not analyzed in this thesis. Note that the facts cited above are standard knowledge and can be found in [36, 43, 45, 46].

\[1.2.2\] Central extensions of planar anisotropic kinematical algebras

In the following, we use the discussion detailed above to determine the central extensions of the possible planar anisotropic kinematical algebras namely: the Carroll algebra, the Galilei algebra, the Para-Galilei algebras, the two Newton-Hooke algebras and the Static algebra.

i) Carroll algebra

The Carroll algebra was first introduced in [47] as a velocity-time contraction [38] of the Poincaré algebra through a rescaling of the boosts and the time translations. Although appearing naturally in the classification of kinematical groups, as an alternative intermediate algebra in the contraction of the Poincaré group onto the Static group, and therefore as another nonrelativistic limit (the other being the Galilei algebra), the Carroll algebra has played no distinguish role in Kinematics. Recently it has been analyzed whether the Carroll algebra (considered as a nonrelativistic limit of the Poincaré algebra: i.e the energy $H$ is considered as a central extension parameter) constitutes an object in the study of the problem of tachyon condensation in string theory [41].
In the classification by Mcrae [21], Carroll group appeared as a relative-time group and its Lie algebra can be considered therefore as a relativistic kinematical algebra whose only nontrivial bracket: 
\[ [K_i, P_j] = \frac{1}{c^2} \delta_{ij} H \] is such that \( H \) is not a central extension parameter. Furthermore, in the classification detailed in [40], when the Lie bracket 
\[ [K_i, P_j] \neq 0 \], it is stated that the considered algebra is a relativistic one. As pointed out previously, we consider the “Carroll” algebra at least relativistically and it is in this context where a possible cosmological interpretation and the noncommutative phase spaces on the Carroll group recover some interest.

The only nontrivial Lie bracket of the anisotropic Carroll algebra is given by:
\[ [K_i, P_j] = \frac{1}{c^2} H \delta_{ij} \] (1.44)

By using standard methods [12, 13, 20, 39, 46, 48], i.e looking for the solutions of the system (1.39)-(1.40), we obtain that the central extension of this Lie algebra is defined by the following Lie brackets
\[ [K_i, K_j] = \frac{1}{c^2} S \epsilon_{ij} \] (1.45)
\[ [K_i, P_j] = \frac{1}{c^2} H \delta_{ij}, \ [P_i, P_j] = \kappa^2 S \epsilon_{ij} \]
\[ [K_i, H] = 0, \quad [P_i, H] = 0 \]

where \( \kappa \) is the inverse of the universe radius \( r \) while the dimensionless generator \( S \) spans the center of the extended Lie algebra.

ii) Absolute time anisotropic Lie algebras

The cohomological structure which determines the existence of central extensions of absolute time kinematical groups originates in their invariant subgroup spanned by translations and boosts [11].

The planar nonrelativistic anisotropic Lie algebras are collectively defined by the following Lie structure:
\[ [K_j, K_k] = 0 \] (1.46)
\[ [K_j, P_k] = 0, \quad [P_j, P_k] = 0 \]
\[ [K_j, H] = \lambda P_j, \quad [P_j, H] = \beta K_j \]

for \( j, k = 1, 2 \) and with \( \lambda = 1 \) or zero, \( \beta = \pm \omega^2 \) or zero.

**Theorem 1.5.** The vector space of central extensions of the planar nonrelativistic anisotropic kinematical algebra (1.46) is two-dimensional.
Proof. Dimensional analysis permits us to set that a priori the possible central extensions of (1.46) are defined by the following Lie brackets:

\[
\begin{align*}
[K_j, K_k] &= \frac{\mu}{c^2} S\epsilon_{jk} \\
[K_j, P_k] &= \gamma M\delta_{jk}, \quad [P_j, P_k] = \kappa^2 \alpha S\epsilon_{jk} \\
[K_j, H] &= \lambda P_j, \quad [P_j, H] = \beta K_j
\end{align*}
\]

where \(S\), \(\alpha\), \(\gamma\) and \(\mu\) are dimensionless while the dimension of \(M\) is \(L^{-2}T\) and that of the parameter \(\beta\) is \(L^{-2}\).

The Lie brackets (1.47) will form Lie algebras if every triplet satisfies the Jacobi identities

\[
[K_i, [P_j, H]] + [P_j, [H, K_i]] + [H, [K_i, P_j]] = 0
\]

which imply that \(\mu\) and \(\alpha\) are related by the following relation:

\[
\frac{\mu\beta}{c^2} = -\kappa^2 \lambda \alpha
\]

By using relation \(c = \frac{\omega}{c}\) in (1.48) for the possible values of \(\lambda\) and \(\beta\) (in section 1.1), we get the following cases:

- \(\mu = \mp \alpha\) for \(\lambda = 1\) and \(\beta = \pm \omega^2\)
- \(\mu \in \mathbb{R}\) while \(\alpha = 0\) when \(\lambda = 1\) and \(\beta = 0\)
- \(\alpha \in \mathbb{R}\) while \(\mu = 0\) when \(\lambda = 0\) and \(\beta = \pm \omega^2\)
- \(\mu, \alpha \in \mathbb{R}\) when \(\lambda = 0\) and \(\beta = 0\)

Thus, we are left with two independent central generators \(M\) and \(S\) and we can assume after normalization (\(\mu = 1\) and \(\gamma = 1\)) that the central extended nonrelativistic planar anisotropic Lie algebras are given by the Lie brackets summarized in the following table (for \(i, j = 1, 2\)):

| Lie algebra | \([P_i, H]\) | \([K_i, H]\) | \([P_i, P_j]\) | \([K_i, K_j]\) | \([K_i, P_j]\) |
|-------------|--------------|--------------|---------------|---------------|---------------|
| Extended \(\mathcal{NH}_+\) | \(\omega^2 K_i\) | \(P_i\) | \(-\kappa^2 S\epsilon_{ij}\) | \(\frac{1}{c^2} S\epsilon_{ij}\) | \(M\delta_{ij}\) |
| Extended \(\mathcal{NH}_-\) | \(-\omega^2 K_i\) | \(P_i\) | \(\kappa^2 S\epsilon_{ij}\) | \(\frac{1}{c^2} S\epsilon_{ij}\) | \(M\delta_{ij}\) |
| Extended \(\mathcal{G}\) | \(0\) | \(P_i\) | \(0\) | \(\frac{1}{c^2} S\epsilon_{ij}\) | \(M\delta_{ij}\) |
| Extended \(\mathcal{G}'_\pm\) | \(\pm \omega^2 K_i\) | \(0\) | \(\kappa^2 S\epsilon_{ij}\) | \(0\) | \(M\delta_{ij}\) |
| Extended \(\mathcal{S}\) | \(0\) | \(0\) | \(\kappa^2 S\epsilon_{ij}\) | \(\frac{1}{c^2} S\epsilon_{ij}\) | \(M\delta_{ij}\) |

Table 1.6: Central extensions of nonrelativistic planar anisotropic kinematical algebras
The two-fold central extensions summarized in the above table (1.6) are then the maximal extensions of the nonrelativistic anisotropic Lie algebras that can exponentiate to the corresponding planar kinematical groups.

Furthermore, these centrally extended anisotropic Lie algebras are new except for the Newton-Hooke groups case [25, 26, 49].

1.2.3 Noncentral abelian extensions of the absolute time planar kinematical algebras

Many authors have had a big interest in studying three absolute time Lie algebras namely Galilei algebra and the two Newton-Hooke algebras as well as their centrally extended structures [33, 34, 49, 50, 51, 52, 53]. In the following, we revisit them and see three others namely Para-Galilei algebras and the Static one. But we are interested here in their noncentrally abelian extended Lie structures.

The six planar absolute time Lie algebras are defined by the following nontrivial Lie brackets

\[ [J, K_j] = K_i e^i_j, \quad [J, P_j] = P_i e^i_j, \quad [K_i, H] = \lambda P_i, \quad [P_i, H] = \beta K_i, \quad i, j = 1, 2 \]

with \( \lambda = 1 \) or zero, \( \beta = \pm \omega^2 \) or zero.

We distinguish four cases:

i) Galilei algebra

In the discussion above (in section 1.1), the Galilei algebra corresponds to the case \( \lambda = 1 \) and \( \beta = 0 \). The corresponding \((2 + 1)\)-Galilei group \( G \) is a six-parameter Lie group. It is the kinematical group of a classical, nonrelativistic spacetime having two spatial and one time dimensions. It consists of translations of time and space, rotations in the two-dimensional space and velocity boosts. Explicitly, the Galilei group \( G \) in two-dimensional space is defined by the multiplication law

\[
(\theta, \vec{v}, \vec{x}, t)(\theta', \vec{v}', \vec{x}', t') = (\theta + \theta', R(\theta)\vec{v}' + \vec{v}, R(\theta)\vec{x}' + \vec{v}t' + \vec{x}, t + t')
\]

(1.49)

where \( \theta \) is an angle of rotations, \( \vec{v} \) is a boost vector, \( \vec{x} \) is a space translation vector and \( t \) is a time translation parameter. Its Lie algebra \( \mathfrak{g} \) is then generated by the left invariant vector fields

\[
J = \frac{\partial}{\partial \theta}, \quad \vec{K} = R(-\theta)\frac{\partial}{\partial \vec{v}}, \quad \vec{P} = R(-\theta)\frac{\partial}{\partial \vec{x}}, \quad H = \frac{\partial}{\partial t} + \vec{v}. \frac{\partial}{\partial \vec{x}}
\]

(1.50)
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that satisfy the nontrivial Lie brackets

\[ [J, K_j] = K_i \epsilon^i_j, \quad [J, P_j] = P_i \epsilon^i_j, \quad [K_i, H] = P_i \quad ; i, j = 1, 2 \quad (1.51) \]

The Galilei algebra can also be obtained from the Poincaré algebra \( \mathfrak{so}(1, 3) \) through the velocity-space contraction determined by rescaling the boosts and the space translations \[38\]. Furthermore, the planar Galilei group \( G \) and its Lie algebra \( \mathfrak{g} \) have been defined in \[24\].

More specifically, the Lie algebra \( \mathfrak{g} \) of the Galilei group in two-dimensional space is given by the Lie brackets:

\[
\begin{align*}
[J, K_1] &= K_2 \\
[J, K_2] &= -K_1, \quad [K_1, K_2] = 0 \\
[J, P_1] &= P_2, \quad [K_1, P_1] = 0, \quad [K_2, P_1] = 0 \\
[J, P_2] &= -P_1, \quad [K_1, P_2] = 0, \quad [K_2, P_2] = 0, \quad [P_1, P_2] = 0 \\
[J, H] &= 0, \quad [K_1, H] = P_1, \quad [K_2, H] = P_2, \quad [P_1, H] = 0, \quad [P_2, H] = 0
\end{align*}
\]

Note that the planar Galilei algebra admits three central extensions overall. This has been demonstrated by looking at the second Lie algebra cohomology group \( H^2(\mathfrak{g}, \mathbb{R}) \).

As noted in \[33, 54\], \( H^2(\mathfrak{g}, \mathbb{R}) \) is three-dimensional which in turn means that the Galilei algebra admits three central extensions in total. However, only two of those extensions containing the parameters \( M \) and \( S \) exponentiate to the corresponding groups. The third extension whose central parameter extension measures the noncommutativity of \( [J, H] \) does not survive the exponentiation and is ignored. Here we are interested in the noncentral extensions of the planar Galilei algebra (with the assumption that \( [J, H] = 0 \) meaning that rotation invariance is taking into account).

Using dimensional analysis and considering the extended Lie algebra which exponentiate in the Lie group and then appear to have a clear physical interpretation, we then find the following particular noncentral extended Lie algebra structure:

\[
\begin{align*}
[J, K_j] &= K_i \epsilon^i_j, \quad [K_j, K_k] = \frac{1}{c^2} S \epsilon_{jk} \\
[J, P_j] &= P_i \epsilon^i_j, \quad [K_j, P_k] = M \delta_{jk}, \quad [P_j, P_k] = 0 \\
[J, F_j] &= F_i \epsilon^i_j, \quad [K_i, F_j] = 0, \quad [P_i, F_j] = 0 \\
[J, H] &= 0, \quad [K_j, H] = P_j, \quad [P_j, H] = F_j
\end{align*}
\]

where \( F_i, M \) and \( S \) are new generators such that \( F_i \) behave as vectors under rotations while \( M \) and \( S \) behave as scalars. This noncentrally abelian extended Galilei algebra has been used in \[24\].
ii) Newton-Hooke algebras

As already argued, the two nonrelativistic Newton-Hooke cosmological groups were introduced by Bacry and Lévy-Leblond in [20] who classified all kinematical groups in $(3 + 1)$-dimensional spacetime. The Newton-Hooke symmetries can be obtained through velocity-space contraction from the de Sitter and anti-de Sitter geometries and they describe respectively the nonrelativistic expanding (with the symmetry described by $NH_+$ algebra) and oscillating (with the symmetry described by $NH_-$ algebra) universes. They contract themselves through a space-time contraction in Galilei group.

In the discussion detailed above (in section 1.1), the Newton-Hooke algebras correspond to the case $\lambda = 1$ and $\beta = \pm \omega^2$. Here, we determine the noncentral abelian extension of the Newton-Hooke algebras in the same manner as we did in the discussion for the Galilei algebra. Dimensional analysis permits us to set that after normalization the noncentral abelian extensions of $NH_{\pm}$ coincides with their two-fold centrally extended Lie algebras whose nontrivial Lie brackets are given by:

\[
\begin{align*}
[J, K_j] &= K_i \epsilon^i_j, \\
[J, P_j] &= P_i \epsilon^i_j, \\
[K_j, K_k] &= \frac{1}{c^2} S \epsilon_{jk}, \\
[K_j, P_k] &= M \delta_{jk} \\
[P_j, P_k] &= \kappa^2 S \epsilon_{jk}, \\
[K_j, H] &= P_j, \\
[P_j, H] &= \pm \omega^2 K_j
\end{align*}
\]

(1.53)

where $S$ and $M$ generate the center of these extended algebras, $\kappa$ is a constant whose dimension is inverse of that of a length while $\omega$ is the time curvature (frequency).

iii) Para-Galilei algebras

The nonrelativistic Para-Galilei groups (i.e namely Para-Galilei and Anti-Para-Galilei) are obtained through a velocity-time contraction of the Newton-Hooke groups or through a velocity-space contraction of the Para-Poincaré groups (i.e namely Para-Poincaré and Anti-Para-Poincaré). They contract themselves by a space-time contraction in the Static group [20, 38].

For the Para-Galilei algebras, which correspond to the case $\lambda = 0$ and $\beta = \pm \omega^2$ in the discussion (of kinematical algebras) encountered so far (in section 1.1), we proceed in a similar way to search for their associated noncentral abelian extensions.

We find that their corresponding noncentrally abelian extended Lie algebras has the
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following structure

\[
\begin{align*}
[J, K_j] &= K_i \epsilon^i_j, \quad [K_j, K_k] = 0 \\
[J, P_j] &= P_i \epsilon^i_j, \quad [K_j, P_k] = M \delta_{jk}, \quad [P_j, P_k] = \kappa^2 S \epsilon_{jk} \\
[J, H] &= 0, \quad [K_j, H] = \Pi_j, \quad [P_j, H] = \pm \omega^2 K_j \\
[J, \Pi_j] &= \Pi_i \epsilon^i_j
\end{align*}
\] (1.54)

where \( \vec{\Pi} \) behaves as a vector under rotation and has the same dimension as \( \vec{\bar{P}} \) while \( S \) and \( M \) commute with other generators of this noncentrally abelian extended Lie algebras. This case corresponds to the planar noncentrally abelian extended Para-Galilei algebras [24].

iv) Static algebra

The planar Static algebra is an abelian Lie algebra generated by \( J \) for rotation, \( K^i \) for boosts, \( \bar{P} \) for space translations and \( H \) for time translations. The general element of the connected Static group can be written as

\[
g = \exp(\vec{v} \vec{K} + \vec{x} \bar{P} + tH) \exp(\theta J)
\]

where the parameter \( \vec{v}, \vec{x} \) and \( t \) are respectively the velocity parameter, the space and time translation parameters while \( \theta \) is an angle. As already argued, we restrict our study to the planar anisotropic Static group.

In the discussion of section (1.1), we have seen that the Static algebra corresponds to the case \( \lambda = 0 \) and \( \beta = 0 \). Similar computations as in previous cases yield to the noncentral abelian extension of the Static algebra defined by the following Lie brackets:

\[
\begin{align*}
[J, K_j] &= K_i \epsilon^i_j, \quad [K_j, K_k] = 0 \\
[J, P_j] &= P_i \epsilon^i_j, \quad [K_j, P_k] = M \delta_{jk}, \quad [P_j, P_k] = 0 \\
[J, F_j] &= F_i \epsilon^i_j, \quad [K_j, F_k] = B \delta_{jk}, \quad [P_j, F_k] = \Lambda \delta_{jk} \\
[J, \Pi_j] &= \Pi_i \epsilon^i_j, \quad [K_j, \Pi_k] = M' \delta_{jk}, \quad [P_j, \Pi_k] = B \delta_{jk} \\
[J, H] &= 0, \quad [K_j, H] = \Pi_j, \quad [P_j, H] = F_j
\end{align*}
\] (1.55)

This result is quite new and corresponds to the planar noncentrally abelian extended Static algebra.
Recall that the main focus for this work is the construction of noncommutative phase spaces on the above extended structures by the coadjoint orbit method in a two-dimensional space. As it will be established later in the text, each case treated here corresponds to a minimal coupling in term of symplectic structures. Thus, to be complete in this thesis, let us first of all study the minimal couplings in detail own to construct noncommutative phase spaces and then establish a relationship with the coadjoint orbit method (in Chapter 3).
As it is well known \[55, 57\], once the degrees of freedom of a physical system are identified, the dynamics is determined by two elements: the Hamiltonian and the symplectic structure determined by the canonical Poisson brackets. It is also well known in symplectic mechanics \[11, 57, 58\] that the interaction of a charged particle with a magnetic field can be described in a Hamiltonian formalism by means of a modified symplectic two-form

\[
\sigma = \sigma_0 + eB
\]  
(2.1)

where \(\sigma_0\) is the canonical symplectic form

\[
\sigma_0 = dp_i \wedge dq^i
\]  
(2.2)

while \(e\) is the particle’s charge and the time-independent magnetic field \(B\) is closed (i.e \(dB = 0\)). With this symplectic structure, the canonical momentum variables acquire nonvanishing Poisson brackets \(\{p_i, p_j\} = -eB_{ij}\).

Similarly, a closed two-form which interacts with the particle’s dual charge \(e^*\) may be introduced \[59\]. This provides a new modified symplectic two-form

\[
\sigma = \sigma_0 + eB + e^*B^*
\]  
(2.3)

With this, the canonical momentum and position variables acquire nonvanishing Poisson brackets: \(\{p_i, p_j\} = -eB_{ij}\) and \(\{q^i, q^j\} = -e^*B^*_{ij}\).

Explicitly in noncommutative version, one generalizes the symplectic structures by allowing further nonvanishing Poisson brackets among coordinates called generalized Poisson brackets \[9, 16, 59\]:

\[
\{q^i, q^j\} = G^{ij}, \quad \{p_j, q^i\} = \delta_j^i, \quad \{p_i, p_j\} = F_{ij}
\]  
(2.4)
where $\delta^i_j$ is a unit matrix, whereas $G^{ij} (= -e^* B^*_{ij})$ and $F_{ij} (= -e B_{ij})$ are functions of positions and momenta. Note that the respective physical dimensions of $G^{ij}$ and $F_{ij}$ are $M^{-1}T$ and $MT^{-1}$, $M$ representing a mass while $T$ represents a time.

The fields $F$ and $G$ are responsible of the noncommutativity respectively of momenta and positions. This result is very general in the sense that it takes into account the noncommutativity on the whole phase space. In this thesis, we will see in Chapter 3 that this can be performed by a group theoretical method (i.e the coadjoint orbit method) applied on kinematical groups with the assumptions that $F$ is only momentum-dependent while $G$ is only position-dependent, of course, the phase space coordinates (positions and momenta) are determined through the group theoretical method itself. This method modifies the symplectic 2-form, the corresponding Hamiltonian and the equations of motion.

The aim of this Chapter is to introduce the construction of noncommutative spaces by using different minimal couplings:

- A coupling of momenta with magnetic potentials (the usual one)
- A coupling of positions with dual magnetic potentials (the dual one)
- A coupling with a magnetic field and with a dual field [59] (mixing the two couplings)

In other words, we study the planar mechanics in the following three situations:

- when a charged massive particle is in an electromagnetic field,
- when a massless spring is in a dual magnetic field,
- when a pendulum is in an electromagnetic field and in a dual one

It is shown in the second section that under the presence of these fields

- charged massive particle acquires an oscillation state of motion with a certain frequency,
- massless spring acquires a mass,
- pendulum looks like two synchronized oscillators.

The second and third results above are quite new and have been published in [25]. They provide the formula of minimal couplings in symplectic terms as done by Souriau [11] for the first time (i.e in the case of minimal coupling of momentum with magnetic potential). Furthermore, as already pointed out above, it will be seen in Chapter 3 that
each kind of minimal coupling can be realized group theoretically by the coadjoint orbit method on a specific planar kinematical group.

Note that the word “dual” magnetic field has been also used in [59] and stands for a magnetic field which interacts with the particle’s “dual” charge \( e^* \), \( e \) being the particle’s charge. Furthermore such a dual charge has been interpreted in [60] as the anyon spin in a two-dimensional space \( (n = 2) \). But considering an arbitrary number of dimensions, no such interpretation of the dual charge is assumed.

### 2.1 Noncommutative Phase Spaces

In this section, we review Hamiltonian mechanics in both Darboux’s coordinates and noncommutative coordinates, the noncommutativity coming from the presence of two fields \( F_{ij} \) and \( G^{ij} \). We will distinguish three cases of noncommutative coordinates corresponding respectively to the presence of the magnetic field only, of the dual magnetic field only and of both fields simultaneously.

#### 2.1.1 Symplectic mechanics on Lie groups in commutative coordinates

Symplectic geometry is the mathematical counterpart of the Hamiltonian formalism in classical mechanics. There are three main sources of symplectic manifold: cotangent bundles, complex algebraic manifolds and coadjoint orbits [61]. In our study, we are interested in this last source of symplectic manifolds as it will be seen later in this work.

As it has already said in the introduction of this Chapter, the main ingredients entering in the description of a classical mechanical system are a symplectic manifold \((V, \sigma)\) (i.e a \(2n\) dimensional manifold equipped with a closed non degenerate 2-form \( \sigma \)) called phase space and containing all the states of the system and a smooth function \( H \) defined on \( V \) called the Hamiltonian representing energy and determining the time-evolution of the system [50].

If \( \sigma_{ab} \) are the matrix elements of the matrix representing the symplectic form \( \sigma \) and if \( \sigma^{ab} \) are solutions of \( \sigma_{bc} \sigma^{ca} = \delta^a_b \), then a Poisson bracket of \( f, g \in C^\infty(V, \mathbb{R}) \) is given by

\[
\{ f, g \} = \sigma_{ab} \frac{\partial f}{\partial z^a} \frac{\partial g}{\partial z^b}
\]  

(2.5)

where \( a, b = 1, 2, \ldots 2n \) and where we have introduced the definition:

\[
\sigma_{ab} = \{ x^a, x^b \}
\]  

(2.6)
Here and in what follows, we use summation over repeated indices.

The space $C^\infty(V, \mathbb{R})$ endowed with the Poisson brackets given by (2.5) is an infinite Lie algebra [56]. If $z^a = (p_i, q^i)$ are the canonical coordinates (Darboux’s coordinates) on $V$, the symplectic form on $V$ takes the form (2.2) meaning that there is no coupling to a gauge field. These canonical coordinates are defined up to canonical transformations preserving the basic form $\sigma$ called symplectomorphisms or symplectic realizations.

Moreover the Poisson brackets (2.5) becomes

$\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i}$ \hfill (2.7)

and

$\{p_k, p_i\} = 0, \quad \{p_k, q^i\} = \delta^i_k, \quad \{q^k, q^i\} = 0.$ \hfill (2.8)

This means that momenta $p_i$ as well as the positions $q^i$ are commutative. It is also known that if $X_f$ is the Hamiltonian vector field associated to $f$, then $L_{X_f}(\sigma) = 0$ where $L_X$ is the Lie derivative in the direction of $X$. This property of $X_f$ suggests the following general definition of a vector field.

A vector field $X \in F(V)$, the space of vector fields defined on $V$, is said locally Hamiltonian if $i_X(\sigma)$ is closed. As $\sigma$ is closed, this definition is equivalent to the following property: $X$ is locally Hamiltonian if and only if

$L_X(\sigma) = 0$

In this case, the vector field $X$ is also called symplectic because it defines an infinitesimal automorphism of $(V, \sigma)$. In other words, the flow generated on $V$ by a symplectic vector field consists of canonical transformations of $V$.

Recall that all Hamiltonian vector fields are symplectic and if the manifold $V$ is simply connected, the converse is also true (for more details see [62]).

Equivalently $X_f(g) = \{f, g\}$ and the evolution equations under the flow $\Phi_{exp(sX_f)}$ on $V$ generated by $X_f$ are

$\frac{dz^a}{ds} = X_f(z^a)$

which are exactly the usual Hamiltonian equations when $f$ is the energy.
2.1.2 Lie-Poisson structure

Let us now turn to the algebraic approach and characterize a manifold $V$ by the algebra $C^\infty(V)$. On any symplectic manifold we have an additional algebraic operation on $C^\infty(V)$. It is the so-called Poisson bracket $\{\cdot,\cdot\}$ which is bilinear, skew-symmetric and satisfies the Jacobi identity and the Leibniz rule. It is defined in the following three equivalent ways:

$$\{f_1, f_2\} = X_{f_1}(f_2) = -X_{f_2}(f_1) = \sigma(X_{f_1}, X_{f_2})$$ (2.9)

In any canonical local coordinates we have relations (2.7) yielding (2.8). Note that the functions $1, q^i, p_i$, $1 \leq i \leq n$ satisfy the following commutator relations

$$\{p_i, p_j\} = \{q^i, q^j\} = \{p_i, 1\} = \{q^i, 1\} = 0, \quad \{p_i, q^j\} = \delta^j_i$$

These functions constitute the basis of the $(2n+1)$-dimensional Heisenberg Lie algebra.

Later in the text, we will see that one can generalize the above canonical relations when the cotangent bundle of the space is the dual of the Lie algebra of a symmetry group assuming that the latter is simply connected.

Let $G$ be a Lie group whose the Lie algebra $\mathfrak{g}$ is defined by the structure constants $C^k_{ij}$ satisfying (1.34) and (1.35). Let $V$ be a differential manifold whose the dimension is the same as that of $G$. In a basis $e^i$ of $V$, we associate to each $a \in V$ the matrix defined by

$$K_{ij}(a) = -a_k C^k_{ij}$$ (2.10)

**Proposition 2.1.** The bracket $\{\cdot, \cdot\} : C^\infty(V, \mathbb{R}) \times C^\infty(V, \mathbb{R}) \to C^\infty(V, \mathbb{R})$ defined by

$$\{f, g\} = K_{ij}(a) \frac{\partial f}{\partial a_i} \frac{\partial g}{\partial a_j}$$ (2.11)

is a Poisson bracket.

It suffices to demonstrate that the four properties of a Poissonian structure (bilinearity, skew-symmetry, Jacobi identity and Leibniz rule) are verified. For this, we use the two properties of the structure constants given by relations (1.34) and (1.35).

As this Poisson bracket is defined by the structure constants of the Lie algebra of $G$, we call it a Poisson-Lie bracket also known as the Kirillov-Kostant-Souriau bracket.

Thus, $(C^\infty(V, \mathbb{R}), \{\cdot, \cdot\})$ is a Lie algebra (or a Lie-Poisson algebra).
Equivalently, one can define a Poisson structure $\mathcal{P}$ on a manifold $V$ as a bi-vector field $\mathcal{P} : \Lambda^2(T^*V) \to \mathbb{R}$ so that $\{f, g\} = \mathcal{P}(df, dg)$ is a Poisson bracket on $C^\infty(V)$. In that sense, the data $(V, P)$ define a Poisson manifold.

Furthermore, the Poisson structure $\mathcal{P}$ can be used to identify the tangent and the cotangent spaces of manifold. Indeed, for any one-form $\omega$, $\mathcal{P} : T^*V \to TV$, $\omega \mapsto \mathcal{P}(\omega, .)$ since $\mathcal{P}(\omega, .) \in TV$. Unfortunately no assumption has been made on the non-degeneracy of $\mathcal{P}$ and it cannot always be assumed invertible but when the manifold is symplectic the degeneracy for $\mathcal{P}$ is guaranteed. Thus, every symplectic manifold is also Poisson since the symplectic 2-form defines a Poisson bracket via the relation (2.9).

In this thesis, the definition of Poisson structure is given in terms of a 2-form $\sigma$ rather than a bivector $\mathcal{P}$.

Also, it has been shown that Poisson manifolds can be partitioned into symplectic leaves: Poisson sub-manifolds which are equipped with a symplectic structure. In fact, the symplectic leaves of a Poisson manifold $V$ are defined via the Hamiltonian vector field since their integral curves stay on the symplectic leaves (see [57] for more details ).

When we replace $V$ by the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, then the $K_{ij}(a)$ in relation (2.10) are elements of the Kirillov form [63]. It defines a Poisson bracket on $C^\infty(\mathfrak{g}^*, \mathbb{R})$ of the form (2.11). As the Kirillov form is generally degenerate, $\mathfrak{g}^*$ is a presymplectic manifold. It is regular on the coadjoint orbits. Thus, the dual of a Lie algebra has a natural Poisson structure whose symplectic leaves are the coadjoint orbits. Depending on the kinematical group, these orbits may provide noncommutative phase spaces.

In this thesis, we will generalize this result by using the Lie algebra cohomology with central and noncentral abelian extensions of kinematical algebras as detailed in Chapter 1.

Let us introduce now noncommutative coordinates by coupling the momentum $p_i$ with a magnetic potential $A_i$, the position $q^i$ with a dual magnetic potential $A^*i$ and by mixing the two couplings.

### 2.1.3 Noncommutative coordinates

Let us consider the change of coordinates $(p_i, q^i) \to (\pi_i, x^i)$ with

$$\pi_i = p_i - \frac{1}{2} F_{ik} q^k, \quad x^i = q^i + \frac{1}{2} p_k G^{ki}$$

(2.12)
The matrix form of (2.12) is
\[
\begin{pmatrix}
\pi_i \\
x^i
\end{pmatrix} = \begin{pmatrix}
\delta^k_i & -\frac{1}{2} F_{ik} \\
\frac{1}{2} G^{ik} & \delta^i_k
\end{pmatrix} \begin{pmatrix}
p_k \\
q^k
\end{pmatrix}
\]
As
\[
\begin{pmatrix}
\frac{1}{2} G^{ik} & \delta^i_k \\
-\frac{1}{2} F_{ik} & \delta^k_i
\end{pmatrix}
\]
the transformation (2.12) is a change of coordinates if \(\text{det}(\delta^s_j - \frac{1}{4} F_{jm} G^{ms}) \neq 0\). It follows that (equation (5))
\[
\{\pi_i, \pi_k\} = F_{ik}, \quad \{\pi_i, x^k\} = \delta^k_i, \quad \{x^i, x^k\} = G^{ik}
\]
(2.13)
i.e. the new momenta as well as the new configuration coordinates are noncommutative.

By the Jacobi identity, \(F_{ij}\) and \(G^{ij}\) satisfy the following conditions
\[
\frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} + \frac{\partial F_{ij}}{\partial x^k} = 0, \quad \frac{\partial F_{ij}}{\partial \pi_k} = 0
\]
\[
\frac{\partial G^{jk}}{\partial \pi_i} + \frac{\partial G^{ki}}{\partial \pi_j} + \frac{\partial G^{ij}}{\partial \pi_k} = 0, \quad \frac{\partial G^{ij}}{\partial x^k} = 0
\]
Thus the Jacobi identity implies that the \(F_{ij}\)'s depend only on positions, that the \(G^{ij}\)'s depend only on momenta and that the two 2-forms \(\sigma_1 = F_{ij}(x) dx^i \wedge dx^j\) and \(\sigma_2 = G^{ij}(\pi) d\pi^i \wedge d\pi^j\) are closed.

Let the Poisson brackets of two functions \(f, g\) in the new coordinates be given by
\[
\{f, g\}_{\text{new}} = \frac{\partial f}{\partial \pi_i} \frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \pi_i} = Y_f(g)
\]
It follows that
\[
Y_H = X_H + G^{ij} \frac{\partial H}{\partial q^i} \frac{\partial}{\partial q^j} + F_{ij} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial p_j}
\]
The derivative of any function \(f\) with respect to time \(t\), in terms of \(F\) and \(G\), is then given by
\[
\frac{df}{dt} = X_H(f) + G^{ij} \frac{\partial H}{\partial q^i} \frac{\partial f}{\partial q^j} + F_{ij} \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial p_j}
\]
(2.14)
and the equations of motion are then given by

\[ \frac{dq^k}{dt} = \frac{\partial H}{\partial p_k} + G^{ki} \frac{\partial H}{\partial q^i} \]
\[ \frac{dp^k}{dt} = -\frac{\partial H}{\partial q^k} + F_{ik} \frac{\partial H}{\partial p_i} \]

If for example

\[ H = \frac{\delta^{ij} p_i p_j}{2m} + V_0 \]

is the Hamiltonian with the potential energy \( V_0 \) depending only on the configuration coordinates \( q^i \), the equations of motion are then

\[ \frac{dq^k}{dt} = p^k + G^{ki} \frac{\partial V_0}{\partial q^i} \]
\[ \frac{dp^k}{dt} = -\frac{\partial V_0}{\partial q^k} + F_{ik} \frac{p^i}{m} \]  (2.15)

They are equivalent to the modified second Newton’s law \[30, 31\]

\[ m \frac{d^2 q^k}{dt^2} = -\frac{\partial V_0}{\partial q^k} + F_{ik} \frac{p^i}{m} + mG^{ki} \frac{d}{dt} (\frac{\partial V_0}{\partial q^i}) \]

In the absence of the potential \( V_0 \), the equations (2.15) become

\[ \frac{dq^k}{dt} = p^k \]
\[ \frac{dp^k}{dt} = F_{ik} \frac{p^i}{m} \]  (2.16)

or equivalently

\[ m \frac{d^2 q^k}{dt^2} = F_{ik} \frac{p^i}{m} \]  (2.17)

This means that the noncommutativity of the momenta (first equation of (2.13)) implies that the particle is accelerated and is not free even if the potential \( V_0 \) vanishes identically. The particle will be free if the momenta commute even if positions do not commute.

### 2.2 Couplings in Planar Mechanics

In this section we construct explicitly the noncommutative phase spaces by introducing couplings. We start by the usual coupling of momentum with a magnetic potential. Then, we introduce a new kind of coupling: of position with a dual magnetic potential and we finish by a mixing model.

#### 2.2.1 Coupling of momentum with a magnetic Field

i) commutative coordinates

Consider a four-dimensional phase space (a cotangent space to a plane) equipped with the Darboux’s coordinates \((p_i, q^i), i = 1, 2\). This means that the momenta as well as
the positions commute. Consider also an electron with a mass $m$ and an electric charge $e$, moving on a plane with the electromagnetic potential $A_\mu = (A_i = -\frac{1}{2}B\epsilon_{ik}q^k, \phi = E_iq^i)$ where a symmetric gauge has been chosen, $E$ being an electric field while $B$ is a magnetic field perpendicular to the plane. It is known that the dynamics of the particle is governed by the Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - e\phi$$

(2.18)

where the kinetic energy is minimally coupled to the electric potential. It is also known that if we adapt the symmetric gauge, the magnetic potential is given by

$$A_i = -\frac{1}{2}B\epsilon_{ik}q^k$$

(2.19)

where $B$ is the magnetic field strength, source of the potential $A$ while the electric potential is given by

$$\phi = E_iq^i$$

(2.20)

the electric field $\vec{E}$ being the source of the potential $\phi$. The Hamiltonian is then

$$H = \frac{\vec{p}^2}{2m} - e\vec{E}.\vec{q}$$

(2.21)

and the equations of motion are

$$\frac{dq^i}{dt} = \frac{p^i}{m}, \quad \frac{dp_i}{dt} = eE_i$$

(2.22)

or equivalently

$$m\frac{d^2\vec{q}}{dt^2} = e\vec{E}$$

(2.23)

where the right hand side is the electric force. In this case we have commutativity of momenta as well as positions.

ii) Noncommutative Coordinates

Let us construct noncommutative phase spaces through the minimal coupling process. From the classical electromagnetism, it is known that the coupling of the momentum with the magnetic potential is given by the relations

$$\pi_i = p_i - eA_i, \quad x^i = q^i$$

(2.24)

or equivalently

$$\pi_i = p_i + \frac{eB}{2}\epsilon_{ik}q^k, \quad q^i = x^i$$

(2.25)
The coordinates $\pi_i$ and $x^i$ satisfy
\[
\{x^i, x^k\} = 0, \quad \{\pi_i, x^k\} = \delta^k_i, \quad \{\pi_i, \pi_k\} = -eB\epsilon_{ik}
\]
Then in the presence of an electromagnetic field, the momenta do not commute while
the positions commute. Using the equation (2.13), we have
\[
F_{ij} = -eB\epsilon_{ij}, \quad G^{ij} = 0
\]
Use of (2.18) and (2.26) into (2.14) gives rise the equations of motion
\[
m\frac{d^2\vec{q}}{dt^2} = e(\vec{E} + \frac{\vec{p}}{m} \times \vec{B}),
\]
where the right hand side represents the Lorentz force. Moreover, in the noncommuta-
tive coordinates, the Hamiltonian (2.18) becomes
\[
H = \frac{\pi^2}{2m} - e\vec{E}.\vec{x} + m\omega^2\vec{x}^2 + \vec{\omega}.\vec{L}
\]
where $\omega$ is the cyclotron frequency, $\vec{L} = \vec{x} \times \vec{p}$ is the orbital angular momentum and

\[
\vec{\omega} = \frac{eB}{2m}\vec{n}
\]
with $\vec{n}$ the unit vector in the direction perpendicular to the plane. In the presence of a
magnetic field, the massive charged particle has became an oscillator with frequency $\omega$
given above and the equations of motion are
\[
\frac{d\vec{x}}{dt} = \frac{\pi}{m} + \vec{\omega} \times \vec{x}, \quad \frac{d\vec{p}}{dt} = e\vec{E} + \vec{\omega} \times \vec{\pi} - m\omega^2\vec{x}
\]
or equivalently
\[
m\frac{d^2\vec{x}}{dt^2} = e(\vec{E} + \frac{\vec{\pi}}{m} \times \vec{B}),
\]
where we recognize again the Lorentz force $\vec{f}_{\text{Lorentz}} = e\vec{E} + e\vec{\pi}/m \times \vec{B}$ and where
$\vec{B} = B\vec{n}$. Note that the relations (2.27) and (2.30) have the same form. The New-
ton’s equations are then covariant under the coupling (2.25).

In the next two subsections, we present quite new theories associated to an unusual
coupling of position with a dual magnetic [59] field.

### 2.2.2 Coupling of position with a dual field

#### i) Commutative Coordinates

Consider a massless spring with $k$ as a Hooke’s constant and a dual charge $e^*$ in a
dual magnetic field $B^*$. Suppose that the dynamics of the spring is governed by the
Hamiltonian

\[ H = \frac{k\vec{q}^2}{2} - e^*\vec{p} \cdot \vec{E}^* \]  

(2.31)

where we have used the symmetric gauge

\[ A^*i = -\frac{B^*}{2}p_k\epsilon^{ki}, \quad \phi^* = p_i E^*i \]

\( B^* \) and \( \vec{E}^* \) being respectively the sources of the dual magnetic potential \( A^* \) and \( \phi^* \). The equations of motion are

\[ \frac{d\vec{q}}{dt} = -e^*\vec{E}^*, \quad \frac{d\vec{p}}{dt} = -k\vec{q} \]  

(2.32)

where \( e^*\vec{E}^* \) is a velocity.

It is well known that the first and the second derivative of position are respectively called velocity and acceleration but it is less known that the third derivative is technically called jerk or jolt \[64, 65\]. As its name suggests, jerk is applied when evaluating the destructive effect of motion on a mechanism or discomfort caused to the passenger in a vehicle. It is also known that momentum equals mass times velocity and force equals mass times acceleration. Similarly, mass times jerk equals a quantity called yank, \( \vec{y} \). Equivalently, yank can be defined as the second derivative of the momentum with respect to the time variable \( t \) by the relations:

\[ \vec{y} = m \frac{d\vec{a}}{dt} = \frac{d}{dt} \left( \frac{d(m\vec{v})}{dt} \right) = \frac{d^2\vec{p}}{dt^2}, \quad m = \text{const} \]

or the rate of change of a force with time.

The relations (2.32) can take the form

\[ C\frac{d^2\vec{p}}{dt^2} = e^*\vec{E}^* \]  

(2.33)

where \( C = \frac{1}{k} \) is the spring compliance while \( \frac{d^2\vec{p}}{dt^2} \) is a yank (as defined above). Furthermore, (2.33) is the analogue (dual) of the Newton’s second equation for this case. Note that the spring compliance plays, for the dual equations, a similar role as the mass for the usual equations.

ii) Noncommutative coordinates

Let us consider the coupling of the position with the dual potential \( A^*i \) depending on the momenta \( p_i \):

\[ \pi_i = p_i, \quad q^i = x^i + \frac{e^*B^*}{2}p_k\epsilon^{ki} \]  

(2.34)
In this case, the Poisson brackets become
\[
\{ x^i, x^j \} = -e^* B^* \epsilon^{ij}, \quad \{ p_k, x^i \} = \delta^i_k, \quad \{ p_k, p_i \} = 0
\] (2.35)

Therefore, in the presence of the dual field, positions do not commute while the momenta commute. It follows that
\[
F_{ij} = 0_{ij}, \quad G^{ij} = -e^* B^* \epsilon^{ij}
\] (2.36)

Use (2.31) and (2.36) into (2.14) gives rise to the Newton’s analogue equation
\[
C \frac{d^2 \vec{p}}{dt^2} = e^* (\vec{E}^* + \frac{1}{C} \vec{q} \times \vec{B}^*)
\] (2.37)

the right hand side being a velocity. In noncommutative coordinates, the Hamiltonian is
\[
H = \vec{x}^2 \frac{1}{2C} - e^* \vec{\pi} \cdot \vec{E}^* + \frac{\vec{\pi}^2}{2m_s} - \vec{\omega} \cdot \vec{L}
\] (2.38)

where the spring mass \( m_s \) is defined by
\[
\frac{1}{m_s} = \frac{e^{*2} B^{*2}}{4C}
\] (2.39)

while the vector \( \vec{\omega} \) is given by
\[
\vec{\omega} = \frac{e^* B^*}{2C} \vec{n}
\] (2.40)

The Hooke’s compliance \( C \) can be written as
\[
C = \frac{1}{m_s \omega^2}
\] (2.41)

In the presence of the dual field, the spring then acquires a mass \( m_s \) and the equation of motion is given by
\[
C \frac{d^2 \vec{\pi}}{dt^2} = e^* (\vec{E}^* + \frac{\vec{x}}{C} \times \vec{B}^*)
\] (2.42)

The vector \( \vec{f}^* = e^* (\vec{E}^* + \frac{\vec{x}}{C} \times \vec{B}^*) \) can be considered as a dual Lorentz force with the dimension of velocity. It represents for the spring what the Lorentz force represents for a charged particle. Here also the coupling (2.34) preserves the covariance of the Newton’s analogue equations. Comparing (2.37) and (2.42) and using (2.41) into (2.42) we can conclude that \( e^* \omega^2 (\vec{E}^* + \frac{\vec{x}}{C} \times \vec{B}^*) \) is a kind of jerk [65].
2.2.3 Coupling with a magnetic field and with a dual magnetic field

Now consider the case of a massive pendulum with mass \( m \) and Hooke’s compliance \( C \) under the action of an electromagnetic potential \( A_\mu = (A_i, \phi) \) and a dual electromagnetic potential \( A^*_\mu = (A^*_i, \phi^*) \) with \( A_i = -\frac{1}{2}B\epsilon_{ik}q^k \), \( \phi = E_q \), \( A^*_i = -\frac{1}{2}B^*p_k\epsilon_{ki} \) and \( \phi^* = p_iE^*_i \), where \( \vec{E} \) is an electric field and \( \vec{E}^* \) its dual electric field while \( \vec{B} \) is a magnetic field and \( \vec{B}^* \) its corresponding dual magnetic field.

The corresponding motion is governed by the Hamiltonian

\[
H = \frac{\vec{p}^2}{2m} + \frac{\vec{q}^2}{2C} - e\phi - e^*\phi^* \tag{2.43}
\]

where the extra term \( e^*\phi^* \) is due to the presence of the dual magnetic field in the model. Let

\[
x^i = q^i + \frac{e^*B^*}{2}p_k\epsilon_{ki}, \quad \pi_i = p_i + \frac{eB}{2}\epsilon_{ik}q^k \tag{2.44}
\]

be the minimal coupling in the symmetric gauge; that is

\[
G^{ij} = -e^*B^*\epsilon^{ij}, \quad F_{ij} = -eB\epsilon_{ij}
\]

We assume that the cyclotron frequency acquired by the massive charged particle is equal to the frequency of the massless spring:

\[
\frac{eB}{2} = m\omega, \quad \frac{e^*B^*}{2} = \frac{1}{m_s\omega}
\]

where \( m_s \) is the acquired mass by the spring while

\[
\mu = \frac{m.m_s}{m + m_s}
\]

is the reduced mass of the two synchronized massive oscillators. It follows that

\[
\{x^i, x^j\} = -e^*B^*\epsilon^{ij}, \quad \{\pi_k, x^i\} = \gamma\delta^i_k, \quad \{\pi_k, \pi_i\} = -eB\epsilon_{ki} \tag{2.45}
\]

with \( \gamma = 1 + \frac{m}{m_s} \) and \( m = \mu\gamma \).

In the presence of the two kind of fields, the positions as well as the momenta do not commute. The Hamiltonian in noncommutative coordinates is written as

\[
H = \frac{\vec{\pi}^2}{2\mu} + \frac{M\omega^2\vec{x}^2}{2} - e\phi - e^*\phi^* \tag{2.46}
\]

where \( M = m + m_s \) is the total mass, \( \phi = \vec{E} \cdot \vec{x} + \vec{n} \cdot \vec{E} \times \frac{\vec{x}}{m_s\omega} \)

and \( \phi^* = \vec{\pi} \cdot \vec{E}^* + \vec{n} \cdot m\omega \vec{x} \times \vec{E}^* \). Note that \( M = m_s\gamma \).
The equations of motion in noncommutative coordinates are then
\[
\frac{d\vec{x}}{dt} = \frac{\vec{\pi}}{\mu} + e^*[\gamma \vec{E}^* + \vec{B}^* \times \frac{\vec{x}}{C} - e\vec{B}^* \times \vec{E}]
\]
\[
\frac{d\vec{\pi}}{dt} = -\frac{\gamma \vec{x}}{C} + e[\gamma \vec{E} + \vec{B} \times \frac{\vec{\pi}}{m} - e^* \vec{B} \times \vec{E}^*]
\]
where the Hooke’s compliance \( C \) is given by (2.41).

If the mass \( m \) of the particle is much smaller than the mass \( m_s \) acquired by the spring, i.e. \( m << m_s \), then \( \gamma \) becomes 1 and \( \mu = m << M = m_s \). In that limit, the brackets (2.45) become
\[
\{x^i, x^j\} = e^* B^* e^{ij}, \quad \{\pi_k, x^i\} = \delta^i_k, \quad \{\pi_k, \pi_i\} = eB \epsilon_{ki}
\]
and the Hamiltonian becomes
\[
H = \frac{\vec{\pi}^2}{2m} + \frac{\vec{x}^2}{2C} - e[\vec{E}.\vec{x} + \vec{n}.\vec{E} \times \omega C \vec{\pi}] - e^*[\vec{\pi}.\vec{E}^* + \vec{n}.m \omega \vec{x} \times \vec{E}^*]
\]
and the equations of motion are given by
\[
\frac{d\vec{x}}{dt} = \frac{\vec{\pi}}{m} + e^*[\vec{E}^* + \vec{B}^* \times \frac{\vec{x}}{C} - e\vec{B}^* \times \vec{E}]
\]
\[
\frac{d\vec{\pi}}{dt} = -\frac{\vec{x}}{C} + e[\vec{E} + \vec{B} \times \frac{\vec{\pi}}{m} - e^* \vec{B} \times \vec{E}^*]
\]
The velocity \( ee^* \vec{B}^* \times \vec{E} \) and the force \( ee^* \vec{E}^* \times \vec{B} \) result from the coexistence of the two fields. Thus, in the presence of the two kinds of fields, the positions do not commute as well as the momenta (see equations (2.47)).

In the following Chapter, we determine the maximal coadjoint orbits of the extended structures determined in the first Chapter. These orbits are physically interpreted as noncommutative phase spaces of accelerated particles moving in the respective kinematical spacetimes. The noncommutativity is due to the presence of naturally introduced magnetic fields defining minimal couplings as already been reported.
One of the aims of this thesis is the study of classical dynamical systems associated with the planar kinematical groups. This constitutes the main idea of this Chapter: to construct noncommutative phase spaces by coadjoint orbit method and classify all the possible noncommutative symplectic structures in a two-dimensional space when the symmetry groups are kinematical groups.

Historically, the orbit method was proposed in [62] for the description of the unitary dual (i.e. the set of equivalence classes of unitary irreducible representations) of nilpotent Lie groups. It turned out that the method not only solves this problem but also gives simple and visual solutions to all other principal questions in representation theory: topological structure of the unitary dual, the explicit description of the restriction and induction functors, the formulas for generalized and infinitesimal characters, the computation of the Plancherel measure, . . . [61]. The theory was extended later to the case of solvable groups in [12] and has also paved the way for the geometric quantization which constitutes its natural generalization.

Furthermore, while the idea that symmetry would determine the system which carries it goes back to Wigner [66], it has been Souriau [11] who put it into precise form. As already mentioned in the introduction, his important theorem says that when a symmetry group $G$ acts transitively on the phase space then the latter is a coadjoint orbit of $G$ endowed with its canonical symplectic structure (Kirillov-Kostant-Souriau).

The notion of coadjoint orbits (orbits of the action of a group on the dual of its Lie algebra) is the main ingredient of the orbit method. Moreover it is the most important mathematical object that has been brought into consideration in connection with the or-
bit method hence the title: coadjoint orbit method. Thus, the Souriau’s method relates the coadjoint orbit of a group $G$ to the phase space (where quantization yields the irreducible unitary representation of that group $[13]$).

In this Chapter, the construction of the maximal (i.e invariants are all nonvanishing) coadjoint orbits of centrally and noncentrally extended kinematical groups and their modified symplectic structures is presented in detail own to describe and classify noncommutative phase spaces.

More specifically, we realize in this Chapter the Poisson brackets $\text{(2.4)}$ on the maximal coadjoint orbits of a noncentrally abelian extended Aristotle group (i.e a subgroup of Galilei group), of the centrally extended planar nonrelativistic anisotropic kinematical groups (Newton-Hooke groups, Galilei and Para- Galilei groups, Static and Carroll groups) and of the noncentrally abelian extended planar absolute time kinematical groups as classified in $[21]$. The main characteristic features are the following:

- there exists generators (namely momenta and boosts) which yield the basic canonical variables through the coadjoint orbit method,
- the Hamiltonian belongs to the Lie algebra itself (or to its central or noncentral abelian extension) and acts linearly on the canonical variables,
- the equations of motion are linear to the latter variables.

But before go on to determine the maximal coadjoint orbits of planar kinematical groups (or their corresponding group extensions) or for a more pedagogical discussion of the applications to construct noncommutative phase spaces, let us give here some formal definitions, keeping in mind that we can apply them to the noncommutative algebraic structures as well as their physical interpretations we need in this thesis.

This Chapter is organized as follows.

In the next section, we give some formal definitions namely the coadjoint orbits and symplectic realization. However, it should not be assumed that the following analysis is complete.

In section two, we study the classical dynamical systems associated with the Aristotle group. We obtain phase spaces which do not commute in momentum sector due to the presence of a naturally introduced magnetic field, i.e the obtained cases correspond to the minimal coupling of the momentum with a magnetic potential.
In section three and four, we construct noncommutative phase spaces as coadjoint orbits of centrally extended planar anisotropic kinematical groups (oscillating and expanding Newton-Hooke, Galilei, Para-Galilei, Carroll and Static Lie groups) and of noncentrally abelian extended absolute time planar kinematical groups (the same groups enumerated above when rotation parameters are taken into account except the Carroll group) respectively. Through these constructions the coordinates of the phase spaces do not commute due to the presence of naturally introduced fields giving rise to minimal couplings. Finally in the section five, we conclude and classify the obtained nonrelativistic models which are noncommutative phase spaces.

Some of the results of this Chapter have been published in [24, 25, 28] and others in [29] (submitted to Journal of Mathematical Physics, August 2013).

3.1 Coadjoint orbit and symplectic realization methods

3.1.1 Coadjoint orbit method

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $Ad : G \to Aut(\mathfrak{g})$ be the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$ such that the automorphism $Ad_g$ associated to $g \in G$ is defined by

$$Ad_g(X) = gXg^{-1}, \quad X \in \mathfrak{g}$$

If $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, it is well known that the coadjoint action of $G$ on $\mathfrak{g}^*$ $Ad^* : \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g}$ is defined as the dual of the map $Ad$:

$$\langle Ad^*_g \alpha, X \rangle = \langle \alpha, Ad_g^{-1}X \rangle, \alpha \in \mathfrak{g}^*$$

where we have denoted the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ as $\langle \cdot, \cdot \rangle$. Then for a given element $\alpha \in \mathfrak{g}^*$, the coadjoint orbits through $\alpha$ are obtained as the images of the map:

$$g \to Ad^*_g \alpha$$

where the usual notation is

$$O_\alpha = \{Ad^*_g(\alpha), \alpha \in \mathfrak{g}^*, g \in G\}$$

Consider now the derivative (push-forward) of $Ad^*$. This can be defined as the dual of the derivative of $Ad$. It is normally denoted $ad$ and it defines a map:

$$(Ad_g)_* \equiv ad_X : \mathfrak{g} \to \mathfrak{g}$$

such that

$$ad_X Y = [X, Y]$$
where by $X$ we have denoting the Lie algebra element corresponding to $g \in G$. Similarly, we define $ad^*$ as the derivative of the map $Ad^*$ i.e:

$$(Ad^*_g)_* = ad^*_X$$

such that:

$$(ad^*_X(\alpha), Y) = \langle \alpha, [X, Y] \rangle$$

If $\alpha = \alpha_i \epsilon^i \in \mathfrak{g}^*$, $X = e_i X^i$, $Y = e_i Y^i \in \mathfrak{g}$ then

$$\langle ad^*_X(\alpha), Y \rangle = K_{ij}(\alpha) X^i Y^j$$

where $K_{ij}(\alpha)$ is of the form (2.10), the Kirillov 2-form [63] on $\mathfrak{g}^*$. The representation $\rho : \mathfrak{g} \to \mathfrak{f}(\mathfrak{g}^*)$ of $\mathfrak{g}$ on the space of vector fields on $\mathfrak{g}^*$ defined by

$$\rho(X_i) = K_{ij}(\alpha) \frac{\partial}{\partial \alpha_j}$$

is a Lie algebra homomorphism such that

$$\text{Ker}(K(\alpha)) = \{ f \in C^\infty(\mathfrak{g}^*, \mathbb{R}) : \rho(X)f = 0, X \in \mathfrak{g} \}.$$ 

This means that $\text{Ker}(K(\alpha))$ is the set of all invariants $f$ of $\mathfrak{g}$ in $\mathfrak{g}^*$ satisfying the following relation:

$$K_{ij}(\alpha) \frac{\partial f}{\partial \alpha_j} = 0$$

(3.1)

The quotient space

$$\mathcal{O}_\alpha = \mathfrak{g}^*/\text{Ker}(K(\alpha)),$$

called the coadjoint orbit of $G$ in $\mathfrak{g}^*$, is a symplectic manifold [67] whose symplectic form $\sigma^{ij}$ is obtained from

$$\Omega_{ij} \sigma^{jk} = \delta_i^k$$

where $\Omega_{ij} = K_{ij}|_{\mathcal{O}_\alpha}$, i.e the restriction of the Kirillov form to the orbit. Explicitly, the two-symplectic form is given by the following relation

$$\sigma = (\Omega^{-1})^{ab} dx_b \wedge dx_a$$

(3.2)

which takes the form $\sigma = dp_i \wedge dq^i$ in the canonical coordinates. The fact that $\alpha$ is arbitrary means that one can choose the invariants under the action of $G$ to label the orbit (as we have silently done above). The invariants which labeled the orbit are generally the cohomology classes as we will see it later in the Chapter 3.
If \((p_i, q^j)\) are denoted collectively by \(x_a\), the Poisson bracket implied by the Kirillov symplectic structure

\[
\{H, f\} = -\Omega_{ab} \frac{\partial H}{\partial x_a} \frac{\partial f}{\partial x_b}
\]  
leads to relations (2.8) where \(p_i, q^j\) represent the generalized canonical coordinates and momenta of the system. Relations (2.8) mean that the momenta commute within themselves as well as the positions.

Interesting consequences arise by considering central and noncentral abelian extensions of Lie algebras and this provides a more general symplectic two-form (3.2) whose extended Poisson brackets of generalized coordinates are defined by

\[
\{x_a, x_b\} = \Theta_{ab}
\]

where

\[
\Theta = \begin{pmatrix}
0 & G & 1 & 0 \\
-G & 0 & 0 & 1 \\
-1 & 0 & 0 & F \\
0 & -1 & -F & 0
\end{pmatrix}
\]

is the inverse of the matrix of the symplectic form

\[
\Omega = \frac{1}{1 - GF} \begin{pmatrix}
0 & F & -1 & 0 \\
-F & 0 & 0 & -1 \\
1 & 0 & 0 & G \\
0 & 1 & -G & 0
\end{pmatrix}
\]

The fields \(F\) and \(G\) are constant because they are coming from central or noncentral abelian extensions of Lie algebras. But cases where they are not constant have been considered [68]. Moreover the respective physical dimensions of \(G^{ij}\) and \(F_{ij}\) are \(M^{-1}T\) and \(MT^{-1}\), \(M\) representing a mass while \(T\) represents a time.

The noncommutative phase space is then defined as a space on which variables satisfy the commutation relations (2.4). The equations of motion corresponding to the above symplectic structure are given by:

\[
\frac{dx^i}{dt} = \{H, x^i\} = \Theta^{ij} \frac{\partial H}{\partial x^j}
\]

more explicitly:

\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} + G \epsilon^{ij} \frac{\partial H}{\partial x^j} \\
\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} + F \epsilon_{ij} \frac{\partial H}{\partial p_j}
\]  

If \(G = F = 0\), then (3.4) are the usual Hamiltonian equations.
3.1.2 Symplectic realizations

Let \((V, \sigma)\) be a symplectic manifold and let \(G\) be a Lie group. The application \(D : G \rightarrow Aut(V)\) is a symplectomorphism if

\[ D^*_g \sigma = \sigma \]  

(3.5)

It is also called a symplectic realization of \(G\) on \((V, \sigma)\).

**Proposition 3.1.** If \(X\) is an element of \(\mathfrak{g}\) and if \(D_{\exp(tX)}\) is a symplectomorphism of \((V, \sigma)\), then \(X\) is locally Hamiltonian vector field.

In fact, the fact that \(D_{\exp(tX)}\) is a symplectomorphism of \((V, \sigma)\), means that

\[ D_{\exp(tX)}^* \sigma = \sigma \].

It follows that \(L_X \sigma = 0\) hence \(i_X(\sigma)\) is closed, which means effectively that \(X\) is locally Hamiltonian. As a Hamiltonian vector field is locally Hamiltonian, we can conclude that to a Hamiltonian vector field corresponds a symplectic realization.

With these preliminaries out of the way, we can now proceed to tackle the problem of constructing and classifying noncommutative phase spaces group theoretically on planar kinematical groups and compare the results obtained with those found with others methods. Note that, in this thesis, the maximal coadjoint orbits are constructed by considering only the evolution of time.

### 3.2 Noncommutative phase space on Aristotle group

Aristotle group is an intermediate group between the Euclidean and the Galilei groups dubbed by Souriau [69]. It contains both Euclidean displacements and time translation but not boosts. In this section, we use the coadjoint orbit method to construct phase spaces endowed with modified symplectic structure on this intermediate group. This method allows therefore the construction of the classical dynamical systems associated with this Lie group.

More specifically, we demonstrate that such deformed objects can be generated in the framework of a noncentrally abelian extended Aristotle algebra. We also realize symplectically both the centrally and noncentrally abelian extended Aristotle Lie group. The obtained in such a way phase spaces (in the noncentrally abelian extended group case only) do not commute in momentum sector due to the presence of a naturally introduced magnetic field corresponding to the minimal coupling of the momentum with a magnetic potential.

Thus, this case corresponds to the minimal coupling of the momentum with the magnetic potential. As the coadjoint orbit construction has not been carried through this Lie
group before, physical interpretations of new generators of the extended corresponding Lie algebras are also given. The results of this section have been published in [28].

### 3.2.1 Aristotle group

Let $V = M \times \mathbb{R}$ be a $(n + 1)$-dimensional nonrelativistic spacetime where $\mathbb{R}$ supports the absolute time coordinate $t$. Let

$$x^\mu = \begin{pmatrix} x^i \\ t \end{pmatrix}, \quad i = 1, ..., n$$

(3.6)

be the coordinate of an arbitrary point of $V$. We use the Greek alphabet for the spacetime indices and the Latin alphabet for the space indices. It is well known that the displacement group with respect to the isotropy and homogeneity of the spacetime $V$ together with the Galilean nonrelativistic principle of motion admits the following transformations:

- **spatial rotation** characterized by three parameters:
  $$x^i' = a^i_j x^j, \quad t' = t$$
  (3.7)
  where $(a^i_j)$ is an orthogonal matrix whose determinant is $+1$, that is an element of $SO(3)$.

- **Galilean boosts** characterized by the three component vector $v^j$:
  $$x^i' = x^i + v^i t, \quad t' = t$$
  (3.8)

- **spatial translations** characterized by three parameters $a^i$:
  $$x^i' = x^i + a^i, \quad t' = t$$
  (3.9)

- **temporal translations** characterized by only one parameter:
  $$x^i' = x^i, \quad t' = t + t_0$$
  (3.10)

By composing the above transformations and under the usually composition of matrices, it has been verified that the above transformations form a group whose multiplication is:

$$(x^i, t, v^i, a^i_j)(x^j', t', v^j', a^j_k) = (x^i + a^i_j x^j' + v^i t', t + t', v^i + a^i_j v^j' + a^i_j a^j_k)$$

(3.11)

This is the Galilei group multiplication law. When $n = 2$, (3.11) becomes (1.49).
Let us now define a subgroup of the Galilei group by dropping the Galilean boosts. We obtain the Aristotle group. The latter is the main group of the less known Aristotle mechanics and is formed by the composition of the transformations (3.7) (3.9) and (3.10). It has order 7 expresses Aristotle’s relativity principle. By composing the above transformations, we obtain the Aristotle multiplication law given by:

\[(x^i, t, a^i_j)(x^j, t', a^j_k) = (x^i + a^i_j x^j, t + t', a^i_j a^j_k)\]  

(3.12)

Thus, the Aristotle group is the group of both Euclidean displacements and time translations (there is no boosts for this group). In a two-dimensional space, the multiplication law (3.12) of this general Lie group is given by

\[(\theta, \vec{x}, t)(\theta', \vec{x}', t') = (\theta + \theta', R(\theta)\vec{x}' + \vec{x}, t + t')\]  

(3.13)

where \(\vec{x}\) is a space translation vector, \(t\) is a time translation parameter and \(\theta\) is a rotation parameter.

In the following subsections, we are going to prove that one can not construct noncommutative phase spaces by the coadjoint orbit method with the first central extension of the two-dimensional Aristotle group because the symplectic structure obtained is canonical. This is due to the fact that in the centrally extended Aristotle Lie algebra, the generators of spatial translations remain commutative. But by considering a noncentral abelian extension of Aristotle group, we will realize a partially noncommutative phase spaces (only momenta do not commute). Furthermore, the noncommutativity of momenta is measured by a term which is associated to the naturally introduced magnetic field. Moreover, this case corresponds to the minimal coupling of the momentum with the magnetic potential [25] as already argued previously.

### 3.2.2 Central extension of the Aristotle group and its maximal coadjoint orbit

The Aristotle Lie algebra \(\mathcal{A}\) is by definition generated by the left invariant vector fields given by the formula

\[X_i = \left. \frac{\partial (gg')^j}{\partial \nu_i'} \right|_{g' = e} \left( \frac{\partial}{\partial \nu_j} \right)\]

where \(X_i\) is the left invariant field corresponding to the parameter \(\nu_i\), \(e\) is the identity element of \(G\) while \(gg'\) is the group multiplication law.

Explicitly, the generators of the Aristotle Lie algebra are given by:

\[J = \frac{\partial}{\partial \theta}, \quad \vec{P} = R(-\theta) \frac{\partial}{\partial \vec{x}}, \quad H = \frac{\partial}{\partial t}\]
such that the only nontrivial Lie brackets are

\[ [J, P_i] = P_j \epsilon_{ij}, \; i, j = 1, 2. \] (3.14)

The multiplication law (3.13) implies that the element \( g \) of this group can be written as:

\[
g = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 & x^1 \\
\sin \theta & \cos \theta & 0 & x^2 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

or equivalently

\[
g = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 & x^1 \\
\sin \theta & \cos \theta & 0 & x^2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and then we can parametrize the element \( g \) of the group in the following manner:

\[
g = \exp(\vec{x}\vec{P} + t\tilde{H}) \exp(\theta J) \] (3.15)

We are now going to employ the formulation we described in section (3.1.1) to construct the classical dynamical systems associated to the Aristotle group. We construct indeed the maximal coadjoint orbits associated to the extensions of this Lie group when evolution of time is considered. Following the Souriau prescription, these coadjoint orbits are identified as elementary systems associated to the extended Aristotle groups. Here, the word elementary means that the action of the considered group should be transitive, at the quantum level, this means that the representation be irreducible [35].

From the relation \( \exp(2\pi J)H \exp(-2\pi J) = H \) and by use of the standard methods, we obtain the following nontrivial Lie brackets for the first central extension Lie algebra \( \hat{A} \) of \( A(2) \)

\[
[J, P_i] = P_j \epsilon_{ij}, \; [P_i, P_j] = \kappa^2 S \epsilon_{ij}, \; i, j = 1, 2
\] (3.16)

where \( S \) generates the center of \( \hat{A} \) and is dimensionless. Dimensional analysis implies that the parameter \( \kappa \) has dimension inverse of that of a length.

Let us look now at the group associated to the above centrally extended Lie algebra.

Let \( g \) be given by (3.15) and \( \hat{g} = \exp(\varphi S)g \) be the corresponding element in the connected Lie group associated to the extended Lie algebra \( \hat{A} \). By use of the Baker-Campbell-Hausdorff formulas [42] and by identifying \( \hat{g} \) with \( (\varphi, \theta, \vec{x}, t) \), we find that
the multiplication law of the connected extended Lie group $\hat{A}$ is:

$$(\varphi, \theta, \vec{x}, t)(\varphi', \theta', \vec{x}', t') = (\varphi' + \frac{1}{2}\kappa^2 R(-\theta)\vec{x} \times \vec{x}' + \varphi, \theta + \theta', R(\theta)\vec{x}' + \vec{x}, t + t')$$

or equivalently

$$(\alpha, g)(\alpha', g') = (\alpha + \alpha' + c(g, g'), gg')$$

where $c(g, g') = \frac{1}{2}\kappa^2 R(-\theta)\vec{x} \times \vec{x}'$ is the two-cocycle defining the central extension while $gg'$ is the multiplication law (3.13).

The adjoint action $\text{Ad}_g(\delta\hat{g}) = g(\delta\hat{g})g^{-1}$ of $A$ on the Lie algebra $\hat{A}$ is given by:

$$\text{Ad}_{(\theta, \vec{x}, t)}(\delta\varphi, \delta\theta, \delta\vec{x}, \delta t) = (\delta\varphi + \kappa^2 R(-\theta)\vec{x} \times \delta\vec{x}
- \frac{1}{2}\kappa^2 \vec{x}^2 \delta\theta, \delta\theta, R(\theta)\delta\vec{x} + \epsilon(\vec{x})\delta\theta, \delta t) \quad (3.17)$$

with

$$\epsilon(\vec{x}) = \begin{pmatrix} 0 & x^2 \\ -x^1 & 0 \end{pmatrix} \quad (3.18)$$

If the duality between the centrally extended Lie algebra and its dual is given by the action

$$\langle (j, \vec{p}, l, E), (\delta\theta, \delta\vec{x}, \delta\varphi, \delta t) \rangle = j\delta\theta + \vec{p}.\delta\vec{x} + l\delta\varphi + E\delta t \quad (3.19)$$

where $j$ is an angular momentum, $\vec{p}$ is a linear momentum, $l$ is an action while $E$ is an energy, then the coadjoint action of the Aristotle Lie group is

$$\text{Ad}^*(\vec{x}, \theta, t)(j, \vec{p}, l, E) = \left(j + \frac{m\omega}{2}(\vec{x}^2) + \vec{x} \times R(\theta)\vec{p}, R(\theta)\vec{p} - m\omega\epsilon(\vec{x}), l, E \right) \quad (3.20)$$

where we have used the relation $l\omega = mc^2$: a relation remembering us the wave-particle duality, the left hand side being an energy associated to a frequency $\omega$, the right hand side being an energy associated to a mass, and the relation $c = \frac{\omega}{\kappa}$ linking the velocity $c$, the frequency $\omega$ and $\kappa$ whose dimension is the inverse of that of a length.

The parameters $l$ and $E$ in (3.20) corresponding to the central generators remain fixed (i.e are $\text{Ad}^*$-invariants): they are called trivial invariants of the coadjoint action of the group on the dual of its centrally extended Lie algebra.

The Kirillov form in the basis $(J, P_1, P_2, H, S)$ is

$$K(a) = \begin{pmatrix} 0 & p_2 & -p_1 & 0 & 0 \\ -p_2 & 0 & m\omega & 0 & 0 \\ p_1 & -m\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.21)$$
The maximal coadjoint orbit of the centrally extended Lie group on the dual of its Lie algebra is characterized by the two trivial invariants \( l \) and \( E \) and a nontrivial invariant, solution of the Kirillov system (3.1) and given by:

\[
s = j + \frac{\vec{p}^2}{2m\omega}.
\] (3.22)

interpreted as an internal angular momentum or spin with \( j \), the angular momentum.

As already argued early, there must exist generators (namely momenta and boosts) which yield the basic canonical variables through the coadjoint orbit method. Thus, to define the canonical (Darboux’s) coordinates on the orbit in this particular group case (where there is no boosts), let us consider the following new variables:

\[
q = -\frac{p_2}{m\omega}, \quad p = p_1 \tag{3.23}
\]

where dimensional analysis has been taking into account. Use of (3.23) in (3.22) gives rise to the expression of the spin \( s \) in function of the canonical coordinates

\[
s = j + \frac{p^2}{2m\omega} + m\omega q^2 \tag{3.24}
\]

Let us denote by \( O(s, l, E) \) the two-dimensional maximal coadjoint orbit of the Aristotle group \( A(2) \) on the dual of its central extended Lie algebra.

The restriction \( \Omega = (\Omega_{ab}) \) of the Kirillov form to the orbit is then

\[
\Omega = \begin{pmatrix}
0 & m\omega \\
-m\omega & 0
\end{pmatrix}
\]

It follows that the symplectic form (3.2) is in this case given by \( \sigma = dp \wedge dq \) where we have used relations (3.23).

The symplectic realization of the Aristotle Lie group is

\[
D(\theta,x,t)(p,q) = (\cos \theta \ p + m\omega q \sin \theta - m\omega x^2, -\frac{p}{m\omega} \sin \theta + q \cos \theta - x^1) \tag{3.25}
\]

The Poisson bracket (3.3) corresponding to this symplectic structure is then the canonical one and the time translation subgroup acts trivially on the orbit (the position and the linear momentum do not depend on time). To overcome this fact and to obtain a noncommutative phase space, let us study the symplectic realization of a noncentral extension of the Aristotle Lie group.

We prove, in the following paragraph, that noncommutative phase spaces can be obtained by considering a noncentral abelian extension of the two-dimensional Aristotle group.
3.2.3 Noncentrally abelian extended group and its maximal coadjoint orbit

In the previous section, we have found that one can not construct noncommutative phase spaces by coadjoint orbit method on the central extension of the Aristotle group because symplectic structure obtained is canonical which means that positions commute as well as momenta. In the following subsection, we see that this construction is possible when we consider a noncentral extension of this Lie group.

Consider the following noncentral abelian extension of the Aristotle Lie algebra defined by the nontrivial Lie brackets

\[
\begin{align*}
[J, P_j] &= P_i \epsilon_j^i, \\
[P_i, P_j] &= \kappa^2 S \epsilon_{ij} \\
[J, F_j] &= F_i \epsilon_j^i, \\
[H, P_i] &= F_i, \\
[P_i, F_j] &= K \delta_{ij}
\end{align*}
\] (3.26)

where \(F_i\) has the dimension of a force while \(S\) and \(K\) are dimensionless.

We recover the Lie algebra defined by (3.14) when \(F_i = 0\), \(K = 0\) and \(S = 0\), the Lie algebra defined by (3.16) when \(F_i = 0\), \(K = 0\).

Consider now the general Lie algebra defined by (3.26) and let \(\hat{g} = \exp(\varphi S + \gamma K) \exp(tH) \exp(\tilde{\eta} F^i + \tilde{x} P_i) \exp(\theta J)\) be the general element of the corresponding connected extended Aristotle group. By identifying \(\hat{g}\) with \((\varphi, \gamma, t, \tilde{\eta}, \tilde{x}, \theta)\) and by explicit calculation, we obtain the multiplication law \(\hat{g}'' = \hat{g}\hat{g}'\) in the extended Lie group. It is such that:

\[
\begin{align*}
\varphi'' &= \varphi' + \frac{1}{2} \kappa^2 \tilde{x} \times R(-\theta) \tilde{x}' + \varphi, \\
\gamma'' &= \gamma' + \frac{1}{2} \tilde{x} R(\theta) \tilde{\eta}' - \frac{1}{2} (\tilde{\eta} + \tilde{x} t') R(\theta) \tilde{x}' + \gamma, \\
\tilde{\eta}'' &= R(\theta) \tilde{\eta}' + \tilde{\eta} + \tilde{x} t', \\
\tilde{x}'' &= R(\theta) \tilde{x}' + \tilde{x}, \\
\theta'' &= \theta' + \theta, \\
t'' &= t + t'
\end{align*}
\]

The adjoint action \(Ad_g(\delta \hat{g}) = g(\delta \hat{g})g^{-1}\) of the Aristotle group \(A\) on the noncentrally abelian extended Lie algebra is given by:

\[
Ad_{(\tilde{x}, \tilde{\eta}, \theta, t)}(\delta \gamma, \delta \varphi, \delta \tilde{x}, \delta \tilde{\eta}, \delta \theta, \delta t) = (\delta \gamma', \delta \varphi', \delta \tilde{\eta}', \delta \tilde{x}', \delta \theta', \delta t')
\]
with
\[
\begin{align*}
\delta \gamma' &= \delta \gamma + \bar{x} \times R(\theta)\delta \bar{\eta} - \bar{\eta} \times R(\theta)\delta \bar{x} - \bar{\eta} \times \bar{x} \delta \theta + \frac{1}{2} \bar{x}^{2} \delta t \\
\delta \bar{\eta}' &= R(\theta)\delta \bar{\eta} + \epsilon (\bar{\eta} - \bar{x} t) \delta \theta - t R(\theta) \delta \bar{x} + \bar{x} \delta t \\
\delta \varphi' &= \delta \varphi + \kappa^{2} R(-\theta) \bar{x} \times \delta \bar{x} - \frac{\bar{x}^{2}}{2} \kappa^{2} \delta \theta \\
\delta \bar{x}' &= R(\theta) \delta \bar{x} + \epsilon (\bar{x}) \delta \theta \\
\delta \theta' &= \delta \theta \\
\delta t' &= \delta t
\end{align*}
\]

where \(\epsilon (\bar{x})\) is given by the relation (3.18).

If the duality between the noncentrally abelian extended Lie algebra and its dual Lie algebra gives rise to the action
\[
\langle (j, \bar{f}, \tilde{p}, h, k, E), (\delta \theta, \delta \bar{\eta}, \delta \bar{x}, \delta \varphi, \delta \gamma, \delta t) \rangle = j \delta \theta + \bar{f} \cdot \delta \bar{\eta} + \tilde{p} \cdot \delta \bar{x} + h \delta \varphi + E \delta t + k \delta \gamma
\]
then the coadjoint action is such that
\[
h' = h, \quad k' = k \tag{3.27}
\]
and
\[
\begin{align*}
\bar{p}' &= R(\theta) \bar{p} + R(\theta) \bar{f} t + k(\bar{\eta} - \bar{x} t) + h \kappa^{2} \epsilon (\bar{x}) , \bar{f}' &= R(\theta) \bar{f} - k \bar{x} \tag{3.28} \\
j' &= j + \bar{x} \times R(\theta) \bar{p} + \bar{\eta} \times R(\theta) \bar{f} - \frac{h}{2} \kappa^{2} \bar{x}^{2} \tag{3.29} \\
E' &= E - \bar{x} \cdot R(\theta) \bar{f} + \frac{1}{2} k \bar{x}^{2} \tag{3.30}
\end{align*}
\]
where \(\bar{p}\) is a linear momentum, \(h\) is an action, \(\bar{f}\) is a force, \(k\) is a Hooke’s constant, \(E\) is an energy while \(j\) is an angular momentum.

The coadjoint orbit is, in this case, characterized by the two trivial invariants \(h\) and \(k\) (3.27), and by the nontrivial invariants \(s\) and \(U\) given by:
\[
\begin{align*}
s &= j - \bar{p} \times \bar{q} + \frac{1}{2} m \omega \bar{q}^{2} , \quad U = E - \frac{1}{2} k \bar{q}^{2} \tag{3.31}
\end{align*}
\]
where we have used the notation
\[
\bar{q} = - \frac{\bar{f}}{k} \tag{3.32}
\]
which defines the basic position variables on the orbit.

Note that the quantities in (3.31) are interpreted as internal angular momentum and internal energy respectively. By considering the coadjoint action, we see that the coadjoint orbit is 4—dimensional. Let us denote it by $\mathcal{O}_{(h,k,s,U)}$.

The restriction $\Omega = (\Omega_{ab})$ of the Kirillov form to the orbit is then

$$
\Omega = \begin{pmatrix}
0 & m\omega & k & 0 \\
-m\omega & 0 & 0 & k \\
-k & 0 & 0 & 0 \\
0 & -k & 0 & 0
\end{pmatrix}
$$

The modified symplectic form is explicitly given by

$$
\sigma = dp_i \wedge dq_i + \frac{1}{2} m\omega \epsilon_{ij} dq^i \wedge dq^j \quad (3.33)
$$

where $\vec{q}$ is given by relation (3.32).

If $(y_a) = (p_1, p_2, q^1, q^2)$, the Poisson brackets are then explicitly given by

$$
\{H, g\} = \frac{\partial H}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial g}{\partial p_i} + F_{ij} \frac{\partial H}{\partial p_i} \frac{\partial g}{\partial p_j} \quad (3.34)
$$

where

$$
F_{ij} = -m\omega \epsilon_{ij} \quad (3.35)
$$

This implies that

$$
\{p_i, p_j\} = F_{ij}, \quad \{p_i, q^j\} = \delta_i^j, \quad \{q^i, q^j\} = 0 \quad (3.36)
$$

Let us now use the symplectic realizations methods defined in section (3.1.2) to find the equations of motion in the dynamical system constructed on this extension of Aristotle group.

Let the symplectic realization of the extended Aristotle Lie group on its coadjoint orbit be given by $(\vec{p'}, \vec{q'}) = L_{(\theta, \eta, \vec{x}, t)}(\vec{p}, \vec{q})$. By using relations (3.28), we have

$$
\vec{p'} = R(\theta)\vec{p} - k[(R(\theta)\vec{q} + \vec{x})t - \vec{\eta}] + h\epsilon(\vec{x}), \quad \vec{q'} = R(\theta)\vec{q} + \vec{x} \quad (3.37)
$$

It follows that $(\vec{p}(t), \vec{q}(t)) = D_{(0,0,0,0,0,t)}(\vec{p}, \vec{q})$ gives rise to

$$
\vec{p}(t) = \vec{p} - k\vec{q} t, \quad \vec{q}(t) = \vec{q}
$$
The equations of motion are then
\[ \frac{d\vec{p}}{dt} = -k\vec{q}, \quad \frac{d\vec{q}}{dt} = 0 \]
(3.38)

From (3.31), the angular momentum is
\[ j = -\vec{q} \times \vec{p} + s - eB\vec{q}^2/2 \]
(3.39)
i.e. the sum of the orbital angular momentum \( L = \vec{q} \times \vec{p} \), the internal angular momentum \( s \) and an extra term \(-eB\vec{q}^2/2\) associated to the magnetic field \( B \) [70].

So with the above noncentral abelian extension of the two-dimensional Aristotle group, we have realized a phase space where momenta do not commute and this noncommutativity is due to presence of a naturally introduced magnetic field given by relation (3.35) or equivalently by
\[ F_{ij} = -eB\epsilon_{ij}. \]
(3.40)

Let us proceed similarly to construct noncommutative phase spaces on the centrally extended anisotropic kinematical groups. They are maximal coadjoint orbits identified as elementary systems associated with those extended Lie structures.

### 3.3 Noncommutative phase spaces constructed on anisotropic kinematical groups

Through this section, we follow the approach we have described above to construct dynamical systems and hence modified symplectic structures (noncommutative phase spaces) associated to the extended groups corresponding to the centrally extended Lie algebras described in section (1.2.2). As we have done for the extended Aristotle groups, we proceed similarly for the centrally extended anisotropic planar Newton-Hooke groups, Galilei group, Para-Galilei, Static and Carroll groups.

Three types of noncommutative phase spaces are obtained: noncommutative phase spaces in momenta and positions sectors in the Newton-Hooke, Static and Carroll groups cases, noncommutative phase spaces in momenta sector only in the Para-Galilei groups case and noncommutative phase spaces in positions sector only in the Galilei group case.

#### 3.3.1 Newton-Hooke noncommutative phase space

Nonrelativistic particle models have been constructed following Souriau’s method for the two-parameter centrally extended anisotropic Newton-Hooke groups in a two-dimensional space [25] yielding similar results as in [26] (for the oscillating case).
Indeed, by considering the planar oscillating Newton-Hooke group $NH_-$, the authors in [26] have found a similar symmetry in the so-called Hill problem (the latter is studied in celestial mechanics), which is effectively an anisotropic harmonic oscillator in a magnetic field. The peculiarity is that this system has no rotational symmetry while translations and boosts still act as symmetries. Note also that, as already said in the introduction, the noncommutative version of the Hill problem has been discussed in [27]. To be complete in this work, let us review the coadjoint orbit construction of the planar anisotropic Newton-Hooke groups $NH_\pm$.

The anisotropic Newton-Hooke groups $ANH_\pm$ being Newton-Hooke groups $NH_\pm$ without the rotation parameters [19], their multiplication laws are given by:

\[
(x^i, v^i, t)(x'^i, v'^i, t') = (x^i \cos \omega t' + \frac{v^i}{\omega} \sin \omega t' + x'^i, \\
-\omega x^i \sin \omega t' + v^i \cos \omega t' + v'^i, t + t')
\] (3.41)

for $ANH_-$ and

\[
(x^i, v^i, t)(x'^i, v'^i, t') = (x^i \cosh \omega t' + \frac{v^i}{\omega} \sinh \omega t' + x'^i, \\
\omega x^i \sinh \omega t' + v^i \cosh \omega t' + v'^i, t + t')
\] (3.42)

for $ANH_+$ with $i = 1, 2, 3$.

The multiplication law (3.41) implies that the general element $g$ of the group $ANH_-$ can be written as:

\[
g = \begin{pmatrix}
\cos \omega t & -\omega \sin \omega t & 0 \\
\frac{1}{\omega} \sin \omega t & \cos \omega t & 0 \\
x^i & v^i & 1
\end{pmatrix}
\]
or equivalently

\[
g = \begin{pmatrix}
\cos \omega t & -\omega \sin \omega t & 0 \\
\frac{1}{\omega} \sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
x^i & v^i & 1
\end{pmatrix}
\]

Similarly, the multiplication law (3.42) implies that the general element $g$ of the group $ANH_+$ can be written as:

\[
g = \begin{pmatrix}
\cosh \omega t & \omega \sinh \omega t & 0 \\
\frac{1}{\omega} \sinh \omega t & \cosh \omega t & 0 \\
x^i & v^i & 1
\end{pmatrix}
\]
or equivalently

\[
g = \begin{pmatrix}
\cosh \omega t & \omega \sinh \omega t & 0 \\
\frac{1}{\omega} \sinh \omega t & \cosh \omega t & 0 \\
0 & 0 & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
x^i & v^i & 1
\end{pmatrix}
\]
Noncommutative phase spaces constructed on anisotropic kinematical groups

Then we can parametrize the element $g$ of the group $ANH_{\pm}$ in the following manner:

$$g = \exp(tH) \exp(\vec{x}\vec{P} + \vec{v}\vec{K})$$

(3.43)

where $\vec{K}, \vec{P}, H$ are generators (i.e. the left invariant vector fields) of the corresponding Lie algebras $ANH_{\pm}$ and are respectively given by:

$$\vec{K} = \frac{\partial}{\partial \vec{v}}, \quad \vec{P} = \frac{\partial}{\partial \vec{x}}, \quad H = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} \pm \omega^2 \vec{x} \cdot \frac{\partial}{\partial \vec{v}}$$

(3.44)

In contrast to the Galilei case, the Newton-Hooke Lie algebras are not contraction of the Poincaré algebra, but can be obtained directly from the de Sitter algebras by velocity-space contractions [20]. They contract themselves onto the Galilei algebra through the space-time contractions.

Standard methods [12, 13, 39, 46] show that the structure of the central extensions of the Lie algebras $ANH_{\pm}$ is

- in one-dimensional space

$$[K, H] = P, \quad [P, H] = \pm \omega^2 K, \quad [K, P] = M$$

- in two-dimensional spaces

$$[K_i, K_j] = \frac{1}{c^2} J_{\beta} \epsilon_{ij}, \quad [K_i, H] = P_i, \quad [K_i, P_j] = M \delta_{ij}$$

$$[P_i, P_j] = \pm \kappa^2 J_{\alpha} \epsilon_{ij}, \quad [P_i, H] = \pm \omega^2 K_i$$

(3.45)

- in three-dimensional spaces

$$[K_i, K_j] = \frac{1}{c^2} J_{k} \epsilon_{ij}^k, \quad [K_i, H] = P_i, \quad [K_i, P_j] = M \delta_{ij}$$

$$[P_i, P_j] = \pm \kappa^2 J_{k} \epsilon_{ij}^k, \quad [P_i, H] = \pm \omega^2 K_i$$

(3.46)

where $\kappa$ is a constant whose dimension is inverse of that of a length, $c$ is a constant with the dimension of a speed while $J_k$ is an internal rotation parameter around the $k^{th}$ axis.

We are now going to employ the formulation we described in section (3.1.1) to construct noncommutative phase space associated to the one, two and three-dimensional anisotropic Newton-Hooke groups. We construct indeed the maximal coadjoint orbits associated to the central extensions of these Lie groups when evolution of time is considered. Following the Souriau’s prescription, these coadjoint orbits are identified as elementary systems associated to the extended anisotropic Newton-Hooke groups. Recall that the word elementary means that the action of the groups should be transitive and at the quantum level, this means that the representation should be irreducible [35].
i) One-dimensional space case

Let \( g \) be given by (3.43) (i.e \( g = (x, v, t) \) for the short) and \( \hat{g} = \exp(\varphi M)g \) be the corresponding element in the connected Lie groups associated to the extended Lie algebra \( AN\mathcal{H}_\pm(1) \) in one-dimensional space. By use of the Baker-Campbell-Hausdorff formulas [42] and by identifying \( \hat{g} \) with \( (\varphi, g) \), we find that the multiplication laws of the connected extended Lie groups in one-dimensional space are:

\[
(\varphi, g)(\varphi', g') = (\varphi + \varphi' + c(g, g'), gg')
\]

where

\[
c(g, g') = -\frac{1}{2}(x \cos \omega t' + \frac{v}{\omega} \sin \omega t')v' + \frac{1}{2}(v \cos \omega t' - \omega x \sin \omega t')x'
\]  

(3.47)

for \( ANH_- (1) \) and

\[
c(g, g') = -\frac{1}{2}(x \cosh \omega t' + \frac{v}{\omega} \sinh \omega t')v' + \frac{1}{2}(v \cosh \omega t' + \omega x \sinh \omega t')x'
\]  

(3.48)

for \( ANH_+ (1) \)

are the two-cocycles defining the central extensions of these groups while \( gg' \) is the multiplication law (3.41) (for the group \( ANH_- \)) or (3.42) (for \( ANH_+ \)) for \( i = 1 \).

The adjoint actions \( Ad_g(\delta\hat{g}) = g(\delta\hat{g})g^{-1} \) of \( ANH_\pm (1) \) on the Lie algebras \( AN\mathcal{H}_\pm (1) \) are given by:

\[
Ad_{(v, x, t)}(\delta\varphi, \delta x, \delta v, \delta t) = (\delta\varphi', \delta x', \delta v', \delta t')
\]

with

\[
\delta\varphi' = \delta\varphi + v \delta x - x \delta v + \frac{1}{2}(v^2 - \omega^2 x^2) \delta t
\]

\[
\delta x' = \cos \omega t \delta x - \frac{1}{\omega} \sin \omega t \delta v + (v \cos \omega t + \omega x \sin \omega t) \delta t
\]

\[
\delta v' = \cos \omega t \delta v + \omega \sin \omega t \delta x + (v \sin \omega t - \omega x \cos \omega t) \delta t
\]

\[
\delta t' = \delta t
\]

in the case of \( ANH_- (1) \) and

\[
\delta\varphi' = \delta\varphi + v \delta x - x \delta v + \frac{1}{2}(v^2 + \omega^2 x^2) \delta t
\]

\[
\delta x' = \cosh \omega t \delta x - \frac{1}{\omega} \sinh \omega t \delta v - (v \cosh \omega t - \omega x \sinh \omega t) \delta t
\]

\[
\delta v' = \cosh \omega t \delta v - \omega \sinh \omega t \delta x - \omega (v \sinh \omega t - \omega x \cosh \omega t) \delta t
\]

\[
\delta t' = \delta t
\]
in the case of $ANH_{+(1)}$.

If the duality between the extended Lie algebras and their duals is given by the action:

$$\langle (m, k, p, E), (\delta \varphi, \delta v, \delta x, \delta t) \rangle = m \delta \varphi + k \delta v + p \delta x + E \delta t \quad (3.49)$$

where $p$ is a linear momentum, $k$ is static momentum, $m$ is a mass while $E$ is an energy (in the dual Lie algebras), then the coadjoint actions of the anisotropic Newton-Hooke groups on the dual of their centrally extended Lie algebras are

$$Ad^*_{{(x,v,t)}}(m, p, k, E) = (m', p', k', E')$$

with

$$m' = m$$
$$p' = -\omega k \sin \omega t + p \cos \omega t - m(v \cos \omega t + \omega x \sin \omega t)$$
$$k' = k \cos \omega + \frac{p}{\omega} \sin \omega t + m(x \cos \omega t - \frac{v}{\omega} \sin \omega t) \quad (3.50)$$
$$E' = E - p v + \frac{m v^2}{2} + \frac{m \omega^2 x^2}{2} + \omega^2 k x$$

in the $ANH_{-(1)}$ case and

$$m' = m$$
$$p' = \omega k \sinh \omega t + p \cosh \omega t - m(v \cosh \omega t - \omega x \sinh \omega t)$$
$$k' = k \cosh \omega + \frac{p}{\omega} \sinh \omega t - m(x \cosh \omega t + \frac{v}{\omega} \sinh \omega t) \quad (3.51)$$
$$E' = E - p v + \frac{m v^2}{2} - \frac{m \omega^2 x^2}{2} - \omega^2 k x$$

in the $ANH_{+(1)}$ case.

The parameter $m$ in (3.50) and (3.51) corresponding to the central generator remain fixed (i.e is $Ad^*$-invariant): it is called trivial invariant of the coadjoint actions of the anisotropic Newton-Hooke groups $ANH_{\pm}(1)$ on the dual of their centrally extended Lie algebras.

The Kirillov form in the basis $(K, P, H)$ is

$$K_{ij}(k, p, E) = \begin{pmatrix} 0 & m & p \\ -m & 0 & \pm \omega^2 k \\ -p & \mp \omega^2 k & 0 \end{pmatrix} \quad (3.52)$$
We verify that the other invariant, solution of the Kirillov’s system (3.1) is in this case,
\[ U_\pm = E - \frac{p^2}{2m} \pm \frac{m \omega^2 q^2}{2} \]  
(3.53)
where \( q = \frac{k}{m} \). Note that (3.53) can be written as:
\[ E = \frac{p^2}{2m} \pm \frac{m \omega^2 q^2}{2} + U_\pm \]  
(3.54)
meaning that the total energy \( E \) is the sum of the kinetic energy \( \frac{p^2}{2m} \), the potential energy \( \frac{m \omega^2 q^2}{2} \) and the internal energy \( U_\pm \) of the oscillating system in the case of ANH\(_-\)(1) or \( U_+ \) in the case of the expanding system with ANH\(_+\)(1).

The inverse of the restriction of the Kirillov form to the orbit is in both cases:
\[ \Omega^{-1} = \frac{1}{m} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]  
(3.55)
Then, the symplectic two-form (3.2) takes the form (2.2) in canonical coordinates (where we have used the notation \( q = \frac{k}{m} \)).

Denote the two-dimensional orbit by \( O_{m,U_\pm} \). It is a symplectic manifold endowed with the canonical symplectic two-form (2.2) and therefore not interesting for our study because there are one momentum and one position.

Note that the symplectic realizations of ANH\(_-\)(1) and ANH\(_+\)(1) are respectively given by
\[
L_{(v,x,t)}(p,q) = \begin{pmatrix} p \cos \omega t - m \omega q \sin \omega t - m v \cos \omega t, \\ \frac{p}{m \omega} \sin \omega t + (q + x) \cos \omega t - \frac{v}{\omega} \sin \omega t \end{pmatrix}
\]  
(3.56)
and
\[
L_{(v,x,t)}(p,q) = \begin{pmatrix} p \cosh \omega t + m \omega q \sinh \omega t - m(v \cosh \omega t - \omega x \sinh \omega t), \\ \frac{p}{m \omega} \sinh \omega t + (q + x) \cosh \omega t - \frac{v}{\omega} \sinh \omega t \end{pmatrix}
\]  
(3.57)
Let \( (p(t), q(t)) = L_{(0,0,t)}(p,q) \), it follows that
\[
p(t) = p \cos \omega t - m \omega q \sin \omega t , \quad q(t) = \frac{p}{m \omega} \sin \omega t + q \cos \omega t
\]  
(3.58)
for ANH\(_-\)(1) and
\[
p(t) = p \cosh \omega t + m \omega q \sinh \omega t , \quad q(t) = \frac{p}{m \omega} \sinh \omega t + q \cosh \omega t
\]  
(3.59)
The equations of motion are then given by
\[ \frac{dp}{dt} = \pm m\omega^2 q, \quad \frac{dq}{dt} = \frac{p}{m} \quad (3.60) \]
or equivalently \[ \frac{dq}{dt} = \pm \omega^2 q; \] which is a second order differential equation whose solutions are trigonometric functions for \( ANH_-(1) \) case and hyperbolic ones in \( ANH_+ (1) \) case. It is for this reason that \( ANH_- \) describes a universe in oscillation while \( ANH_+ \) describes a universe in expansion.

ii) Two-dimensional spaces case

Note that the nontrivial Lie brackets of the centrally extended Newton-Hooke Lie algebras \( NH_{\pm} \) (rotation included) in two-dimensional space are given by
\[
\begin{align*}
[J, K_i] &= K_j \epsilon_i^j, \\
[J, P_i] &= P_j \epsilon_i^j \\
[K_i, P_j] &= M \delta_{ij}, \\
[K_i, H] &= P_i, \\
[P_i, H] &= \pm \omega^2 K_i
\end{align*}
\]
which means that the generators of space translations as well as pure Newton-Hooke transformations commute. One can not then associate a noncommutative phase space to the Newton-Hooke group. It is then the absence of the symmetry rotations (anisotropy of the plane) which guaranties the noncommutative phase space for the anisotropic Newton-Hooke group.

In this section, we prove that, with the anisotropic Newton-Hooke groups \( ANH_{\pm} (2) \) in two-dimensional spaces, the phase spaces obtained by the coadjoint orbit method are completely noncommutative (i.e momenta as well as positions of the phase spaces do not commute). This is due to the fact that both generators of the pure kinematical (Newton-Hooke) transformations as well as generators of spatial transformations do not commute in the centrally extended Lie algebras.

Indeed, let \( g \) be given by (3.43) (i.e \( g = (\vec{x}, \vec{v}, t) \) for the short) and \( \hat{g} = \exp(\varphi M + \psi S)g \) be the corresponding element in the connected Lie groups associated to the extended Lie algebra \( NH_{\pm} (2) \) in two-dimensional space. By use of the Baker-Campbell-Hausdorff formulas \([42]\) and by identifying \( \hat{g} \) with \( (\varphi + \psi, g) \), we find that the multiplication laws of the connected extended Lie groups \( ANH_{\pm} (2) \) are respectively:
\[
(\varphi + \psi, g)(\varphi' + \psi', g') = (\varphi + \varphi' + \psi + \psi' + c(g, g'), gg')
\]
Noncommutative phase spaces constructed on anisotropic kinematical groups

where

\[
c(g, g') = \frac{1}{2} \left[ \delta_{ij} (v^i \cos \omega t' - \omega x^i \sin \omega t') x^j - (x^i \cos \omega t' + \frac{v^i}{\omega} \sin \omega t') v^j \right] \\
+ \frac{1}{2} \kappa^2 \epsilon_{ij} (x^i \cos \omega t' + \frac{v^i}{\omega} \sin \omega t') x^j + \frac{1}{2c^2} \epsilon_{ij} (v^i \cos \omega t' - \omega x^i \sin \omega t') v^j
\]

for \( ANH_-(2) \) and

\[
c(g, g') = \frac{1}{2} \left[ \delta_{ij} (v^i \cosh \omega t' + \omega x^i \sinh \omega t') x^j - (x^i \cosh \omega t' + \frac{v^i}{\omega} \sinh \omega t') v^j \right] \\
+ \frac{1}{2} \kappa^2 \epsilon_{ij} (x^i \cosh \omega t' + \omega x^i \sinh \omega t') x^j + \frac{1}{2c^2} \epsilon_{ij} (v^i \cosh \omega t' + \omega x^i \sinh \omega t') v^j
\]

for \( ANH_+(2) \)

are the two-cocycles defining the centrally extended anisotropic Newton-Hooke groups in two-dimensional space while \( gg' \) is the multiplication law (3.41) in the \( ANH_- \) case or (3.42) in the \( ANH_+ \) case respectively.

The adjoint actions \( Ad_g(\delta \tilde{g}) = g(\delta \tilde{g}) g^{-1} \) of \( ANH_\pm(2) \) on the Lie algebras \( \tilde{ANH}_\pm(2) \) are given by:

\[
Ad_{(\vec{v}, \vec{x}, t)}(\delta \varphi, \delta \psi, \delta \vec{x}, \delta \vec{v}, \delta t) = (\delta \varphi', \delta \vec{x}', \delta \vec{v}', \delta t')
\]

with

\[
\begin{align*}
\delta \varphi' &= \delta \varphi + \delta_{ij} v^i \delta x^j - \delta_{ij} x^i \delta v^j \\
\delta \psi' &= \delta \psi + \frac{\epsilon_{ij}}{2c^2} v^i \delta v^j + \frac{\epsilon_{ij}}{2} \kappa^2 x^i \delta x^j - \kappa^2 \epsilon_{ij} v^i \delta \vec{v} \\
\delta x^i' &= \cosh \omega t \delta x^i - \frac{1}{\omega} \sinh \omega t \delta v^i - \omega (x^i \cosh \omega t - \frac{v^i}{\omega} \sinh \omega t) \delta t \\
\delta v^i' &= \cosh \omega t \delta v^i + \omega \sinh \omega t \delta x^i - \omega (v^i \cosh \omega t + \omega x^i \sinh \omega t) \delta t \\
\delta t' &= \delta t
\end{align*}
\]

in the case of \( ANH_-(2) \) and

\[
\begin{align*}
\delta \varphi' &= \delta \varphi + \delta_{ij} v^i \delta x^j - \delta_{ij} x^i \delta v^j \\
\delta \psi' &= \delta \psi + \frac{\epsilon_{ij}}{2c^2} v^i \delta v^j + \frac{\epsilon_{ij}}{2} \kappa^2 x^i \delta x^j - \kappa^2 \epsilon_{ij} v^i \delta \vec{v} \\
\delta x^i' &= \cosh \omega t \delta x^i - \frac{1}{\omega} \sinh \omega t \delta v^i + \omega (x^i \cosh \omega t - \frac{v^i}{\omega} \sinh \omega t) \delta t \\
\delta v^i' &= \cosh \omega t \delta v^i - \omega \sinh \omega t \delta x^i + \omega (v^i \cosh \omega t - \omega x^i \sinh \omega t) \delta t \\
\delta t' &= \delta t
\end{align*}
\]

in the case of \( ANH_+(2) \).
If the duality between the extended Lie algebras and their duals is given by the action
\[ \vec{p}.\delta \vec{x} +  \vec{k}.\delta \vec{v} + m.\delta \varphi + h.\delta \psi + E.\delta t, \]
where the linear momentum \( \vec{p} \), the static momentum \( \vec{k} \), the mass \( m \), the action \( h \) and the energy \( E \) are elements of the dual Lie algebras, then the coadjoint actions of the anisotropic Newton-Hooke groups \( ANH_\pm(2) \) on the dual of their centrally extended Lie algebras are
\[
\text{Ad}^*_{(x^i,v^i,t)}(m, h, p^i, k^i, E) = (m', h', p'^i, k'^i, E')
\]
with
\[ m' = m \]
\[ h' = h \]
\[ p'_i = p_i \cos \omega t - \omega k_i \sin \omega t - m\delta_{ij}(v^j \cos \omega t + \omega x^j \sin \omega t) + \frac{h\omega}{2c^2} \epsilon_{ij}(\omega x^j \cos \omega t + v^j \sin \omega t) \] (3.64)
\[ k'_i = k_i \cos \omega t + \frac{p_i}{\omega} \sin \omega t + m\delta_{ij}(x^j \cos \omega t - \frac{v^j}{\omega} \sin \omega t) + \frac{h}{2} \epsilon_{ij}(\frac{1}{\omega} \kappa^2 x^j \sin \omega t - \frac{1}{c^2} v^j \cos \omega t) \]
\[ E' = E + \omega p_i x^i + \omega k_i v^i - h\kappa^2 \epsilon_{ij} v^i x^j \]
in the \( ANH_-(2) \) case and
\[ m' = m \]
\[ h' = h \]
\[ p'_i = p_i \cosh \omega t + \omega k_i \sinh \omega t - m\delta_{ij}(v^j \cosh \omega t - \omega x^j \sinh \omega t) - \frac{h\omega}{2c^2} \epsilon_{ij}(\omega x^j \cosh \omega t + v^j \sinh \omega t) \] (3.65)
\[ k'_i = k_i \cosh \omega t + \frac{p_i}{\omega} \sinh \omega t + m\delta_{ij}(x^j \cosh \omega t - \frac{v^j}{\omega} \sinh \omega t) - \frac{h}{2} \epsilon_{ij}(x^j \sinh \omega t + \frac{1}{v^j} \cosh \omega t) \]
\[ E' = E - \omega p_i x^i + \omega k_i v^i + h\kappa^2 \epsilon_{ij} v^i x^j \]
in the \( ANH_+(2) \) case.

The parameters \( m \) and \( h \) in (3.64) and (3.65) corresponding to the central generators remain fixed. They are trivial invariants under the coadjoint action of \( ANH_\pm(2) \) on the dual of their centrally extended Lie algebras in two-dimensional spaces.
The Kirillov form in the basis \((K_i, P_i, H)\) is
\[
(K_{ij}(k_i, p_i, E, m, h)) = \begin{pmatrix}
0 & \frac{\hbar}{\omega} & m & 0 & p_1 \\
-\frac{\hbar}{\omega} & 0 & 0 & m & p_2 \\
-m & 0 & 0 & \pm \hbar k^2 & \pm \omega^2 k_1 \\
0 & -m & \mp \hbar k^2 & 0 & \pm \omega^2 k_2 \\
-p_1 & -p_2 & \mp \omega^2 k_1 & \mp \omega^2 k_2 & 0
\end{pmatrix}
\] (3.66)

The other invariant, solution of the Kirillov’s system (3.1) is explicitly given by
\[
U = E - \frac{\vec{p}^2}{2\mu_e} \pm \frac{\mu_e \omega^2 \vec{q}^2}{2}
\] (3.67)

with
\[
\mu_e = m \pm \frac{\hbar k^2}{\omega}, \quad \vec{q} = \frac{k}{\mu_e}
\] (3.68)

where \(h\omega_0 = mc^2\) denotes the wave-particle duality, \(\mu_e\) being an effective mass.

The inverse of the restriction of the Kirillov’s matrix on the orbit is given by
\[
\Omega^{-1} = \begin{pmatrix}
0 & \pm \frac{\omega}{\mu_e} & -\frac{1}{\mu_e} & 0 \\
\pm \frac{\omega}{\mu_e} & 0 & 0 & -\frac{1}{\mu_e} \\
\frac{1}{\mu_e} & 0 & 0 & \frac{1}{\mu_e \omega_0} \\
0 & \frac{1}{\mu_e} & -\frac{1}{\mu_e \omega_0} & 0
\end{pmatrix}
\] (3.69)

where we have used the wave-particle duality and (3.68).

The orbit is then equipped with the symplectic form
\[
\sigma = dp_i \wedge dq^i + \frac{1}{\mu_e \omega_0} \epsilon^{ij} dp_i \wedge dp_j \pm \mu_e \omega \epsilon_{ij} dq^i \wedge dq^j
\] (3.70)

The Poisson brackets corresponding to the above symplectic structure are given by:
\[
\{h, f\} = \frac{\partial h}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial f}{\partial p_i} + G_{ij} \frac{\partial h}{\partial q^i} \frac{\partial f}{\partial q^j} + F_{ij} \frac{\partial h}{\partial p_i} \frac{\partial f}{\partial p_j} \quad ; \quad i, j = 1, 2
\] (3.71)

This implies that
\[
\{p_i, p_j\} = F_{ij}, \quad \{p_i, q^j\} = \delta^j_i, \quad \{q^i, q^j\} = G^{ij}
\] (3.72)

with
\[
G^{ij} = -\frac{\epsilon^{ij}}{m \omega_0}, \quad F_{ij} = -(m - \mu_e) \omega \epsilon_{ij}
\] (3.73)
It follows that the magnetic field $B$ and its dual field $B^*$ are such that

$$e^*B^* = -\frac{1}{m\omega_0}, \quad eB = (m - \mu_e)\omega$$

(3.74)

The effective mass is then given in function of the magnetic field by

$$\mu_e = m - \frac{eB}{\omega}$$

(3.75)

The Hamilton’s equations are then

$$\frac{d\pi_i}{dt} = -\frac{\partial H}{\partial q^i} \pm (m - \mu_e)\omega \epsilon_{ik} \frac{\partial H}{\partial p_k}, \quad \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} + \frac{e^i}{2m\omega_0} \frac{\partial H}{\partial \pi^k}$$

(3.76)

As already argued, similar results (for the oscillating case) have been found in [26] in the so-called Hill problem, which is effectively an anisotropic harmonic oscillator in a magnetic field. This system has no rotational symmetry while translations and generalized boosts still act as symmetries. It also has a two-parameter central extension and has been applied to the 3-body problem and galactic dynamics. Furthermore, the noncommutative version of the Hill problem was discussed in [27].

**iii) Three-dimensional spaces case**

Similarly, let $g$ be given by (3.44) (i.e $g = (\vec{x}, \vec{v}, t)$ for the short) and $\hat{g} = \exp(\varphi M + \theta^i J_i) g$ be the corresponding element in the connected Lie groups associated to the extended Lie algebra $\mathcal{AH}_{\pm}(3)$ in three-dimensional space. By use of the Baker-Campbell-Hausdorff formulas [42] and by identifying $\hat{g}$ with $(\varphi + \theta^i, g)$, we find that the multiplication laws of the connected extended Lie groups $\mathcal{AH}_{\pm}(3)$ are respectively:

$$(\varphi + \theta^i, g)(\varphi' + \theta^i', g') = (\varphi + \varphi' + \theta^i + \theta^i' + c(g, g'), gg')$$

where

$$c(g, g') = \frac{\delta_{ij}}{2}[(v^i \cos \omega t' - \omega x^i \sin \omega t')x^j - (x^i \cos \omega t' + \frac{v^i}{\omega} \sin \omega t')v^j]
+ \epsilon_{ij}[(x^i \cos \omega t' + \frac{v^i}{\omega} \sin \omega t')^2 + (v^i \cos \omega t' - \omega x^i \sin \omega t')^2 - \frac{2c^2}{2c^2}]$$

for $\mathcal{AH}_{-}(3)$

and

$$c(g, g') = \frac{\delta_{ij}}{2}[(v^i \cosh \omega t' - \omega x^i \sinh \omega t')x^j - (x^i \cosh \omega t' + \frac{v^i}{\omega} \sinh \omega t')v^j]
+ \epsilon_{ij}[(x^i \cosh \omega t' + \frac{v^i}{\omega} \sinh \omega t')^2 + (v^i \cosh \omega t' + \omega x^i \sinh \omega t')^2 - \frac{2c^2}{2c^2}]$$

for $\mathcal{AH}_{+}(3)$.
for $ANH_+(3)$
are the two-cocycles defining the centrally extended anisotropic Newton-Hooke groups in three-dimensional space while $gg'$ is the multiplication law (3.41) in the $ANH_-(3)$ case or (3.42 in the $ANH_+(3)$ case respectively.

The adjoint actions $Ad_g(\delta \tilde{g}) = g(\delta \tilde{g})g^{-1}$ of $ANH_\pm(3)$ on the Lie algebras $\tilde{ANH}_\pm(3)$ are given by:

$$Ad_{(x,v,t)}(\delta \varphi, \delta \theta^i, \delta \bar{x}, \delta \bar{v}, \delta t) = (\delta \varphi', \delta \bar{x}', \delta \bar{v}', \delta t')$$

with

$$\delta \varphi' = \delta \varphi + \delta_{ij}(v^i \delta x^j - x^i \delta v^j)$$
$$\delta \theta'^i = \delta \theta^i + \epsilon_{ij}^k \frac{v^i}{2\epsilon^2} \delta v^j - \frac{\kappa^2}{2} x^i \delta x^j + \epsilon_{ij}^k \kappa^2 v^i x^j \delta t$$

$$\delta x'^i = \cos \omega t \delta x^i - \frac{1}{\omega} \sin \omega t \delta v^i - \omega (x^i \cos \omega t - \frac{v^i}{\omega} \sin \omega t) \delta t$$
$$\delta v'^i = \cos \omega t \delta v^i + \omega \sin \omega t \delta x^i - \omega (v^i \cos \omega t + \omega x^i \sin \omega t) \delta t$$

in the case of $ANH_-(3)$ and

$$\delta \varphi' = \delta \varphi + \delta_{ij}(v^i \delta x^j - x^i \delta v^j)$$
$$\delta \theta'^i = \delta \theta^i + \epsilon_{ij}^k \frac{v^i}{2\epsilon^2} \delta v^j + \frac{\kappa^2}{2} x^i \delta x^j - \epsilon_{ij}^k \kappa^2 v^i x^j \delta t$$

$$\delta x'^i = \cosh \omega t \delta x^i - \frac{1}{\omega} \sinh \omega t \delta v^i + (\omega x^i \cosh \omega t - v^i \sinh \omega t) \delta t$$
$$\delta v'^i = \cosh \omega t \delta v^i - \omega \sinh \omega t \delta x^i + \omega (v^i \cosh \omega t - \omega x^i \sinh \omega t) \delta t$$

in the case of $ANH_+(3)$.

If the duality between the extended Lie algebras and their duals is given by the action:

$$\langle (m, h_i, k_i, p_i, E), (\delta \varphi, \delta \theta^i, \delta v^i, \delta x^i, dt) \rangle = m.\delta \varphi + h.\delta \theta^i + k_i.\delta v^i + p_i.\delta x^i + E.\delta t$$

where the linear momentum $\vec{p}$, the static momentum $\vec{k}$, the mass $m$, the action $h$ and the energy $E$ are elements of the dual Lie algebras, then the coadjoint actions of the anisotropic Newton-Hooke groups $ANH_\pm(3)$ on the dual of their centrally extended Lie algebras are

$$Ad^*_{(x',v',t)}(m, h, p_i, k_i, E) = (m', h', p_i', k_i', E')$$
with
\[ m' = m, \]
\[ h' = h. \]

\[ p'_i = p_i \cos \omega t - \omega k_i \sin \omega t - m\delta_{ij}(v^j \cos \omega t + \omega x^j \sin \omega t) \]
\[ + \frac{h_k\omega}{2c^2} \epsilon^k_{ij}(\omega x^j \cos \omega t + v^j \sin \omega t) \]
\[ k'_i = k_i \cos \omega t + \frac{p_i}{\omega} \sin \omega t + m\delta_{ij}(x^j \cos \omega t - \frac{v^j}{\omega} \sin \omega t) \]
\[ + \frac{h_k}{2c^2} \epsilon^k_{ij}(\frac{1}{\omega} \kappa^2 x^j \sin \omega t - \frac{1}{c^2} v^j \cos \omega t) \]
\[ E' = E + \omega p_i x^i + \omega k_i v^i - \kappa^2 h_k \epsilon^k_{ij} v^i x^j \]

in the \( \text{ANH}_-(3) \) case and
\[ m' = m, \]
\[ h' = h. \]

\[ p'_i = p_i \cosh \omega t - \omega k_i \sinh \omega t - m\delta_{ij}(v^j \cosh \omega t - \omega x^j \sinh \omega t) \]
\[ - \frac{h_k\omega}{2c^2} \epsilon^k_{ij}(\omega x^j \cosh \omega t + v^j \sinh \omega t) \]
\[ k'_i = k_i \cosh \omega t + \frac{p_i}{\omega} \sinh \omega t + m\delta_{ij}(x^j \cosh \omega t - \frac{v^j}{\omega} \sinh \omega t) \]
\[ - \frac{h_k}{2c^2} \epsilon^k_{ij}(x^j \sinh \omega t + \frac{1}{v^j} \cosh \omega t) \]
\[ E' = E - \omega p_i x^i - \omega k_i v^i + \kappa^2 h_k \epsilon^k_{ij} v^i x^j \]

in the \( \text{ANH}_+(3) \) case.

The parameters \( m \) and \( h_k \) in \((3.79)\) and \((3.80)\) corresponding to the central generators remain fixed. They are trivial invariants under the coadjoint action of \( \text{ANH}_\pm(3) \) on the dual of their centrally extended Lie algebras in three-dimensional spaces.

The Kirillov form in the basis \( (K_i, P_i, H) \) is
\[ (K_{ij}(k_i, p_i, E, m, h_k)) = \begin{pmatrix} \frac{h_k}{c^2} \epsilon^k_{ij} & m\delta_{ij} & p_i \\ -m\delta_{ij} & \pm \kappa^2 h_k \epsilon^k_{ij} & \pm \omega^2 k_i \\ -p_j & \mp \omega^2 k_j & 0 \end{pmatrix} \]

(3.81)

There is another invariant. Assuming that \( \frac{\partial U}{\partial E} = 1 \), the latter is solution of the Kirillov’s system
\[ \begin{cases} A_{ij} \frac{\partial U}{\partial k_j} + \delta_{ij} \frac{\partial U}{\partial p_j} = -\frac{p_i}{m} \\ -\delta_{ij} \frac{\partial U}{\partial k_j} + B_{ij} \frac{\partial U}{\partial p_j} = \mp \frac{\omega^2 k_i}{m} \end{cases} \]

(3.82)
where
\[ A_{ij} = \frac{h_k \epsilon_{k}^{ij}}{mc^2}, \quad B_{ij} = \pm \kappa^2 \frac{h_k \epsilon_{k}^{ij}}{m} \]

Using the relation \( c = \frac{\omega}{\kappa} \) in the above relations, we obtain
\[ B_{ij} = \pm \omega^2 A_{ij} \]

Then the system \((3.82)\) takes the following form:
\[
\begin{align*}
A_{ij} \frac{\partial U}{\partial k_j} + \delta_{ij} \frac{\partial U}{\partial p_j} &= -\frac{p_i}{m} \\
-\delta_{ij} \frac{\partial U}{\partial k_j} \pm \omega^2 A_{ij} \frac{\partial U}{\partial p_j} &= \mp \frac{\omega^2 k_i}{m}
\end{align*}
\]

or equivalently
\[
\begin{align*}
A_{\partial U/\partial k} + \frac{\partial U}{\partial p} &= -\frac{\vec{p}}{m} \\
\frac{\partial U}{\partial k} \pm \omega^2 A_{\partial U/\partial \vec{p}} &= \mp \frac{\omega^2 \vec{k}}{m}
\end{align*}
\] (3.84)

Moreover,
\[
\begin{align*}
(I \pm \omega^2 A^2) \frac{\partial U}{\partial \vec{p}} &= -\frac{\vec{p}}{m} \mp \omega^2 \vec{k} \\
-(I \pm \omega^2 A^2) \frac{\partial U}{\partial k} &= \mp \frac{\omega^2 \vec{k}}{m} \pm \omega^2 A \vec{p}
\end{align*}
\] (3.85)

The solution of the above system is
\[
U = E - \frac{p_i p_j (\Phi_{\pm}^{-1})_{ij}}{2m} - \frac{m \omega^2 q^i q^j (\Phi_{\pm}^{-1})_{ij}}{2} + \omega^2 p_i q^j (\Phi_{\pm}^{-1} A)_{ij} \tag{3.86}
\]

where
\[
\Phi_{\pm} = I \pm \omega^2 A^2 \tag{3.87}
\]

and
\[
q_i = \frac{k_i}{m} \tag{3.88}
\]

We see that \( \Phi_{\pm} \) is a metric for \( \mathbb{R}^3 \). Note also that if \( \Phi_{\pm} = I \) or equivalently if \( A = 0 \), then the nontrivial invariant takes the form:
\[
U = E - \frac{\vec{p}^2}{2m} - \frac{m \omega^2 \vec{q}^2}{2} \tag{3.89}
\]

and is interpreted as the internal energy of a free oscillator.

Let us denote the maximal coadjoint orbit by \( O_{(m, k, U)} \).

The restriction of the Kirillov form on the orbit is then
\[
\Omega = m \begin{pmatrix} A_{ij} & \delta^i_j \\
-\delta_i^j & \pm \omega^2 A_{ij} \end{pmatrix} \tag{3.90}
\]
and its inverse is
\[
\Omega^{-1} = \frac{1}{m} \begin{pmatrix}
\pm \omega^2 (A\Phi^{-1}_{\pm})_{ij} & - (\Phi_{\pm}^{-1})_i^j \\
(\Phi_{\pm}^{-1})_j^i & (A\Phi_{\pm}^{-1})_{ij}
\end{pmatrix}
\] (3.91)

The maximal orbit is then equipped with the symplectic structure
\[
\sigma = (\Phi_{\pm}^{-1})_j^i dp_i \wedge dq^j + \frac{1}{m} (A\Phi_{\pm}^{-1})_{ij} dp_i \wedge dp_j \pm m\omega^2 (A\Phi_{\pm}^{-1})_{ij} dq^i \wedge dq^j
\] (3.92)

and it follows that the Poisson brackets of two functions defined on the orbit is then
\[
\{ f, g \} = (\Phi_{\pm}^{-1})_j^i \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_j} \right) + F_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} + G^{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j}
\] (3.93)

This implies that
\[
\{ p_i, p_j \} = F_{ij}, \quad \{ p_i, q^j \} = (\Phi_{\pm}^{-1})_j^i, \quad \{ q^i, q^j \} = G^{ij}
\] (3.94)

where the magnetic field \( F_{ij} \) and the dual magnetic field \( G^{ij} \) are given by
\[
F_{ij} = \frac{1}{C} (A\Phi_{\pm}^{-1})_{ij} \quad \text{and} \quad G^{ij} = \frac{1}{m} (A\Phi_{\pm}^{-1})^{ij}
\] (3.95)

Moreover the Hamilton’s equations are
\[
\begin{align*}
\frac{dp_k}{dt} &= -(\Phi_{\pm}^{-1})_k^i \frac{\partial H}{\partial q^i} + \frac{1}{C} (A\Phi_{\pm}^{-1})_{ik} \frac{\partial H}{\partial p_i} \\
\frac{dq^k}{dt} &= (\Phi_{\pm}^{-1})_k^i \frac{\partial H}{\partial p_i} + \frac{1}{m} (A\Phi_{\pm}^{-1})^{ik} \frac{\partial H}{\partial q^i}
\end{align*}
\] (3.96)

With the anisotropic Newton-Hooke groups \( ANH_{\pm} \) in three-dimensional spaces, we also have realized phase spaces where the momenta as well as the positions do not commute.

Note that in all these three cases, the fields are constant because they are coming from central extensions of Lie algebras.

Let us follow the similar fashion to construct noncommutative phase spaces on the other nonrelativistic anisotropic planar kinematical groups as announced. But, it is interesting to mention that we will not address explicit calculation at some steps for the following kinematical groups cases as we have done previously.

### 3.3.2 Galilean noncommutative phase space

The planar Galilei group admits a nontrivial two-parameter central extension leading to an exotic model [35]. In this section, we prove that the planar anisotropic Galilei group which admits also a nontrivial central extension provides an exotic structure (noncommutativity in position sector).
The Galilei group $G$ in two-dimensional space is defined by the multiplication law (1.49). Its Lie algebra $\mathfrak{g}$ is then generated by the left invariant vector fields $J, \vec{K}, \vec{P}$ and $H$ defined by (1.50) that satisfy the Lie brackets (1.51).

The central extension of the corresponding anisotropic Galilei Lie algebra is defined by the Lie brackets given in the table (1.5). Explicitly, by standard methods, we verify that the nontrivial brackets for the central extension \( \hat{\mathfrak{g}} \) are (1.51) plus

\[
\left[ K_i, K_j \right] = \frac{1}{c^2} S\epsilon_{ij}, \quad \left[ K_i, P_j \right] = M\delta_{ij} ; i, j = 1, 2
\]

(3.97)

where $M$ and $S$ generate the center of $\hat{\mathfrak{g}}$, $c$ being a constant velocity.

Let $k_i K^i + p_i P^i + EH^i + m M^i + h S^i$ be the general element of the dual of the planar centrally extended Lie algebra where $\vec{k}$ is a kinematic momentum, $\vec{p}$ a linear momentum, $E$ is an energy, $m$ is a mass and $h$ is an action. Then $m$ and $h$ are trivial invariants under the coadjoint action of the planar anisotropic Galilei group. The other invariant, the solution of the Kirillov’s system (3.1), is explicitly given by:

\[
U = E - \frac{\vec{p}^2}{2m}
\]

interpreted as the internal energy of the considered model.

The restriction $\Omega$ of the Kirillov’s matrix on the orbit is given by

\[
\Omega = \begin{pmatrix}
0 & \frac{h}{c^2} & m & 0 \\
-\frac{h}{c^2} & 0 & 0 & m \\
-m & 0 & 0 & 0 \\
0 & -m & 0 & 0
\end{pmatrix}
\]

Then by using the relations (3.88) and

\[
h\omega_0 = mc^2
\]

(3.98)

the latter remembering us the wave-particle duality, the left hand side being an energy associated to a frequency $\omega_0$, the right hand side being an energy associated to a mass, we obtain that, in the canonical coordinates (Darboux coordinates), the Poisson bracket (3.3) takes the form:

\[
\{ f, g \} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} + G^{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j}
\]

(3.99)

with

\[
G^{ij} = -\frac{\epsilon^{ij}}{m\omega_0}
\]

(3.100)
where $\omega_0$ is a frequency.

The inverse of $\Omega$ is given by

$$\Omega^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{m} & 0 \\ 0 & 0 & 0 & -\frac{1}{m} \\ \frac{1}{m} & 0 & 0 & \frac{1}{m\omega_0} \\ 0 & \frac{1}{m} & \frac{1}{m\omega_0} & 0 \end{pmatrix}$$

Furthermore, the maximal coadjoint orbit denoted by $\mathcal{O}_{(m,h,U)}$ is equipped with the symplectic 2-form:

$$\sigma = dp_i \wedge dq^i + \frac{\epsilon_{ij}}{m\omega_0} dp_i \wedge dp_j \quad (3.101)$$

It follows that the corresponding minimal coupling is

$$\pi_i = p_i, \quad x^i = q^i + \frac{p_k}{2m\omega_0} \epsilon^{ki} \quad (3.102)$$

and then that

$$\{p_i, p_j\} = 0, \quad \{p_i, x^k\} = \delta_i^k, \quad \{x^i, x^j\} = G^{ij} \quad (3.103)$$

So following the coadjoint orbit method, we have constructed a nonrelativistic particle model for the two-parameter centrally extended anisotropic Galilei group in a two-dimensional space, recovering the exotic model described in [16]. It is a noncommutative phase space whose positions do not commute, i.e. where only the dual magnetic field $B^*$ is present. It is such that

$$e^* B^* = -\frac{1}{m\omega_0} \quad (3.104)$$

Moreover the equations of motion corresponding to the above symplectic structure are given by:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} + \frac{\epsilon_{ki}}{2m\omega_0} \frac{\partial H}{\partial q^k} \quad (3.105)$$

i.e.

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} - \frac{\epsilon_{ki}}{2m\omega_0} \frac{dp_k}{dt} \quad (3.106)$$

Comparing the results obtained for the anisotropic Newton-Hooke and Galilei groups cases in a two-dimensional space, we observe that with the anisotropic Galilei group, the phase space obtained is only partially noncommutative while the phase spaces obtained with the anisotropic Newton-Hooke groups (as detailed in the previous section) are completely noncommutative. In fact, with the coadjoint orbit method applied to the
centrally extended Lie algebra of the Galilei group, the phase space obtained is such that positions are noncommutative due to the noncommutativity of the generators of the pure Galilei transformations. Although, in the anisotropic Newton-Hooke groups cases, generators of space translations and pure Newton-Hooke transformations do not commute in centrally extended Lie algebras.

Note that it has been proved in [71] that for Hamiltonian function of the form:

\[
H = \frac{1}{2m}(p_1^2 + p_2^2) + V(x^1, x^2), \quad V(x^1, x^2) = \sum_i F_i x^i, \quad F_i = \text{const.}
\]

the corresponding Newton equation

\[
m\frac{d^2 x_i}{dt^2} = F_i
\]

remains undeformed.

In the following section, we realize the Poisson brackets of the form:

\[
\{p_k, p_i\} = F_{ki}, \quad \{p_k, q^i\} = \delta^i_k, \quad \{q^k, q^i\} = 0
\]

by the coadjoint orbit method on the planar anisotropic Para-Galilei groups and prove that the noncommutativity of momenta implies the modification of the second Newton law (3.108) in the sense of [30] [31].

### 3.3.3 Para-Galilean noncommutative phase spaces

As already argued, the Para-Galilei non-relativistic algebra \( G'_+ \) was introduced in the framework of classification of all kinematical groups [20]. For radius parameter \( r \) approaching infinity, the corresponding Para-Galilei group contracts in the Static group acting on the standard (flat) nonrelativistic spacetime. In this thesis, we have defined the other Para-Galilei algebra \( G'_- \) whose Lie bracket \( [P_i, H] = -\omega^2 K_i \). The two Para-Galilei algebras are obtained by a velocity-time contraction of the nonrelativistic and cosmological Newton-Hooke algebras \( \mathcal{NH}_\pm \).

With the planar anisotropic Para-Galilei groups, we construct in this subsection, noncommutative phase spaces whose momenta do not commute. This noncommutativity is due to the presence of naturally introduced magnetic fields \( B_\pm \).

The planar Para-Galilei groups and their Lie algebras have been defined in [24]. Explicitly, the Para-Galilei groups \( G'_\pm \) in two-dimensional spaces are defined by the multiplication
Noncommutative phase spaces constructed on anisotropic kinematical groups

\[ gg' = (\theta, \vec{v}, \vec{x}, t)(\theta', \vec{v}', \vec{x}', t') = (\theta + \theta', R(\theta)\vec{v}') + \vec{v}' + \omega^2 \vec{x}' + R(\theta)\vec{x} + \vec{x} + t + t' \]  
(3.109)

where \( \theta \) is an angle of rotations, \( \vec{v} \) is a boost vector, \( \vec{x} \) is a space translation vector and \( t \) is a time translation parameter.

Their Lie algebras \( \mathcal{G}_\pm \) are then generated by the left invariant vector fields

\[ J = \frac{\partial}{\partial \theta}, \quad \vec{K} = R(-\theta)\frac{\partial}{\partial \vec{v}}, \quad \vec{P} = R(-\theta)\frac{\partial}{\partial \vec{x}}; \quad H = \frac{\partial}{\partial t} \pm \omega^2 \vec{x} \frac{\partial}{\partial \vec{v}} \]  
(3.110)

satisfying the nontrivial Lie brackets

\[ \left[ J, K_j \right] = K_i \epsilon_i^j, \quad \left[ J, P_j \right] = P_i \epsilon_i^j, \quad \left[ P_i, H \right] = \pm \omega^2 K_i \]  
(3.111)

Their corresponding anisotropic Lie algebras generated by \( K_i, P_i, H \) are defined in the table (1.5) by the nontrivial Lie brackets:

\[ [P_i, H] = \pm \omega^2 K_i \]  
(3.112)

The central extensions of the anisotropic Para-Galilei Lie algebras are given in the table (1.6) where \( M \) and \( S \) are the central generators of the extended Lie algebras.

Let \( k_i K^{*i} + p_i P^{*i} + E H^{*} + mM^{*} + hS^{*} \) be a general element of the dual of the centrally extended anisotropic Para-Galilei Lie algebras where \( \vec{k} \) is a kinematic momentum, \( \vec{p} \) is a linear momentum, \( E \) is an energy, \( m \) is a mass and \( h \) is an action.

Then the Kirillov form (2.10) in the basis \( (K_1, K_2, P_1, P_2, H) \), is in this case given by:

\[ (K_{ij}) = \begin{pmatrix}
0 & 0 & m & 0 & 0 \\
0 & 0 & 0 & m & 0 \\
-m & 0 & 0 & \kappa^2 h & \pm \omega^2 k_1 \\
0 & -m & -\kappa^2 h & 0 & \pm \omega^2 k_2 \\
0 & 0 & \mp \omega^2 k_1 & \mp \omega^2 k_2 & 0
\end{pmatrix} \]  
(3.113)

The coadjoint orbits of the centrally extended anisotropic Para-Galilei Lie groups on the dual of their Lie algebras are then characterized by the two trivial invariants \( m \) and \( h \), and a nontrivial invariant \( U_\pm \), solution of the system (3.1) interpreted as internal energy and given by:

\[ U_\pm = E \pm \frac{\vec{q}^2}{2C} \]  
(3.114)
Noncommutative phase spaces constructed on anisotropic kinematical groups

where we have used relation $c = \frac{\omega}{\kappa}$, (3.88) and (3.98) where

$$C = \pm \frac{1}{m\omega^2}$$  \hspace{1cm} (3.115)

Let us denote by $\mathcal{O}_{(m,h,U_{\pm})}$ the coadjoint orbits.

The restriction $\Omega = (\Omega_{ab})$ of the Kirillov form (3.113) on $\mathcal{O}_{(m,h,U_{\pm})}$ is then

$$\Omega = \begin{pmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \\ -m & 0 & 0 & \kappa^2 h \\ 0 & -m & -\kappa^2 h & 0 \end{pmatrix}$$

It follows that

$$\Omega^{-1} = \begin{pmatrix} 0 & \pm \frac{\omega^2}{m\omega_0} & -\frac{1}{m} & 0 \\ \mp \frac{\omega^2}{m\omega_0} & \frac{1}{m} & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 \\ 0 & \frac{1}{m} & 0 & 0 \end{pmatrix}$$

where we have used the wave-particle duality (3.98) and the relation $c = \frac{\omega}{\kappa}$.

The Poisson bracket (3.3) is in this case given by:

$$\{H, f\} = \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial f}{\partial p_i} + F_{ij} \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial p_j}$$  \hspace{1cm} (3.116)

with

$$F_{ij} = \mp \frac{\epsilon^{ij} m\omega^2}{\omega_0}$$  \hspace{1cm} (3.117)

where we have used (3.88) and the wave-particle duality.

Moreover, the symplectic form (3.2) takes the form:

$$\sigma = dp_i \wedge dq^i \pm \frac{\epsilon^{ij} m\omega^2}{\omega_0} dq^i \wedge dq^j$$  \hspace{1cm} (3.118)

If the frequency of the charged particle still unchanged; that is $\omega_0 = \omega$, then the symplectic structure (3.118) becomes equivalent to the one obtained in [26] in the case of Para-Galilei group $G_{+}^{''}$.

The corresponding minimal coupling [25] is

$$\pi_i = p_i \pm \frac{\epsilon^{ij} m\omega^2}{2\omega_0} q^k, \quad q^i = x^i$$  \hspace{1cm} (3.119)
The coordinates $\pi_i$ and $x^i$ are such that
\[
\{x^i, x^k\} = 0, \quad \{\pi_i, x^k\} = \delta_i^k, \quad \{\pi_i, \pi_k\} = F_{ik}
\] (3.120)
So with the planar anisotropic Para-Galilei groups, we have obtained noncommutative phase spaces in momenta sector. This noncommutativity is due to the presence of naturally introduced magnetic fields $B_\pm$ given by the relation:
\[
e B_\pm \epsilon^{ij} = F_{ij}
\] (3.121)
The corresponding equations of motion take the form:
\[
\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} + F_{ij} \frac{\partial H}{\partial p_j} \delta^i_j, \quad \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}
\] (3.122)
i.e.
\[
\frac{dp_i}{dt} = -\frac{\partial V}{\partial q^i} \mp \frac{eB_\pm \epsilon^{ij} p_j}{\omega_0} \frac{dx^j}{dt}, \quad \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}
\] (3.123)
or equivalently
\[
m \frac{d^2x^i}{dt^2} = -\frac{\partial V}{\partial x^i} + eB_\pm \epsilon^{ij} \frac{p_j}{m}
\] (3.124)
for Hamiltonian of the form (3.107) and where we have used (3.121). We interpret the equation (3.124) as the modified Newton’s second law [30, 31]. The second term in this equation is a correction due to the noncommutativity of momenta. It is a damping force which depends on the space through the factor of noncommutativity $F_{ij}$. For $F_{ij} = 0$ or $B = 0$, equation (3.124) leads to the usual Newton’s second law.

### 3.3.4 Static noncommutative phase spaces
We are now going to employ the formulation we described in the above sections to construct noncommutative phase spaces on anisotropic planar Static group.

Let us consider the central extension of the planar anisotropic Static Lie algebra whose Lie algebra structure is given in the table (1.5) by:
\[
[K_j, K_k] = \frac{1}{c^2} S \epsilon_{jk}, \quad [K_j, P_k] = M \delta_{jk}, \quad [P_j, P_k] = \kappa^2 S \epsilon_{jk}, \quad [K_j, H] = 0, \quad [P_j, H] = 0
\] (3.125)
Let $hS^* + k_i K^{*i} + p_i P^{*i} + EH^* + mM^*$ be the general element of the dual of the planar anisotropic centrally extended Static Lie algebra. Then $E$, $m$ and $h$ are trivial.
invariants under the coadjoint action of the planar anisotropic Static Lie group.

The restriction of the Kirillov’s matrix on the orbit is given by

\[
\Omega = \begin{pmatrix}
0 & \frac{h}{c^2} & m & 0 \\
-\frac{h}{c^2} & 0 & 0 & m \\
-m & 0 & 0 & \kappa^2 h \\
0 & -m & -\kappa^2 h & 0
\end{pmatrix}
\] (3.126)

By using the wave-particle duality (3.98) and the equality \( c = \frac{\omega}{\kappa} \), where \( \kappa \) is a constant whose dimension is the inverse of that of a length, we obtain that the Poisson bracket of two functions \( f \) and \( g \) implied by the Kirillov symplectic structure is given by

\[
\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} + G^{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} + F_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j}; \quad i, j = 1, 2
\] (3.127)

with

\[
G^{ij} = -\frac{\epsilon^{ij}}{m \omega_0}, \quad F_{ij} = -(m - \mu_e) \omega \epsilon^{ij}
\] (3.128)

and where

\[
\mu_e = m - \frac{\kappa^2 h}{\omega}, \quad \vec{q} = \frac{\vec{k}}{\mu_e}
\] (3.129)

\( \mu_e \) being an effective mass. It follows that the magnetic fields \( B \) and \( B^* \) are such that

\[
e^* B^* = -\frac{1}{m \omega_0}, \quad eB = (m - \mu_e) \omega
\] (3.130)

The effective mass is then given in function of the magnetic field by

\[
\mu_e = m - \frac{eB}{\omega}
\] (3.131)

The Hamilton’s equations are then

\[
\frac{d\pi_i}{dt} = -\frac{\partial H}{\partial q^i} - (m - \mu_e) \omega \epsilon^{ik} \frac{\partial H}{\partial p_k}, \quad \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} + \frac{e k}{2m \omega_0} \frac{\partial H}{\partial q^k}
\] (3.132)

The inverse of \( \Omega \) is

\[
\Omega^{-1} = \begin{pmatrix}
0 & -\frac{\omega}{\mu_e} & -\frac{1}{\mu_e} & 0 \\
\frac{\omega}{\mu_e} & 0 & 0 & -\frac{1}{\mu_e} \\
\frac{1}{\mu_e} & 0 & 0 & \frac{1}{\mu_e \omega_0} \\
0 & -\frac{1}{\mu_e} & -\frac{1}{\mu_e \omega_0} & 0
\end{pmatrix}
\] (3.133)

where we have used the wave-particle duality and (3.129).
Finally the orbit is equipped with the symplectic form
\[ \sigma = dp_i \wedge dq^i + \frac{1}{\mu_e \omega_0} \epsilon^{ij} dp_i \wedge dp_j - \mu_e \omega \epsilon_{ij} dq^i \wedge dq^j \] (3.134)

We observe that with the planar anisotropic Static group, the phase space obtained is completely noncommutative. In fact, with the coadjoint orbit method applied to the centrally extended anisotropic Static Lie algebra, the phase space obtained is such that positions as well as momenta are noncommutative due to the noncommutativity of both the generators of the pure Static transformations and the generators of space transformations. This noncommutativity is measured by two naturally introduced magnetic fields expressed by relations (3.130). The same results have been obtained in the planar anisotropic oscillating Newton-Hooke group case [25].

### 3.3.5 Carroll noncommutative phase spaces

The planar anisotropic Carroll group is the Carroll group \( C(2) \) without the rotations parameters [19]. Its Lie algebra has the only nontrivial Lie bracket given by (1.44) where the left invariant vector fields are given by:
\[ \vec{K} = \frac{\partial}{\partial v^i}, \quad \vec{P} = \vec{v} \frac{\partial}{\partial t} + \frac{\partial}{\partial \vec{x}^i}, \quad H = \frac{\partial}{\partial t} \]

By using standard methods, we have obtained that the central extension of the planar anisotropic Carroll Lie algebra is given by (1.45).

Let \( hS^* + k_i K^{*i} + p_i P^{*i} + EH^* \) be the general element of the dual of the centrally extended planar Carroll Lie algebra. Then \( E \) and \( h \) are trivial invariants under the coadjoint action.

The restriction of the Kirillov form on the orbit in the basis \( (K_1, K_2, P_1, P_2, H) \) is in this case
\[ \Omega = \begin{pmatrix}
0 & \frac{h}{c^2} & \frac{E}{c^2} & 0 & 0 \\
-\frac{h}{c^2} & 0 & 0 & \frac{E}{c^2} & 0 \\
-\frac{E}{c^2} & 0 & 0 & \kappa^2 h & 0 \\
0 & -\frac{E}{c^2} & -\kappa^2 h & 0 & 0 \\
\end{pmatrix} \] (3.135)

By using relations \( h \omega_0 = mc^2 = E, c = \frac{\kappa}{\kappa} \) and (3.129), \( \kappa \) being a constant whose dimension is the inverse of that of a length, we obtain that the Poisson brackets of two functions defined on the orbit are given by
\[ \{h, f\} = \frac{\partial h}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial f}{\partial p_i} + G^{ij} \frac{\partial h}{\partial q^j} \frac{\partial f}{\partial q^i} + F_{ij} \frac{\partial h}{\partial p_i} \frac{\partial f}{\partial p_j} ; \quad i, j = 1, 2 \] (3.136)

with
\[ G^{ij} = -\frac{\epsilon^{ij}}{m \omega_0}, \quad F_{ij} = -(m - \mu_e) \omega \epsilon_{ij} \] (3.137)
and where we have used the relations (3.129), $\mu_e$ being an effective mass. It follows that the magnetic field $B$ and its dual field $B^*$ are such that

$$e^* B^* = -\frac{1}{m\omega_0}, \quad eB = (m - \mu_e)\omega$$

(3.138)

The effective mass is then given in function of the magnetic field by

$$\mu_e = m - \frac{eB}{\omega}$$

(3.139)

The Hamilton’s equations are then

$$\frac{d\pi_i}{dt} = -\frac{\partial H}{\partial q^i} - (m - \mu_e)\omega \epsilon_{ik} \frac{\partial H}{\partial p_k}$$

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} + \frac{e^k}{2m\omega_0} \frac{\partial H}{\partial q^k}$$

(3.140)

The inverse of $\Omega$ is

$$\Omega^{-1} = \begin{pmatrix}
0 & -\omega & -\frac{1}{\mu_e} & 0 \\
\omega & \mu_e & 0 & -\frac{1}{\mu_e} \\
-\frac{1}{\mu_e} & 0 & 0 & \mu_e \\
0 & \mu_e & -\frac{1}{\mu_e\omega_0} & 0
\end{pmatrix}$$

(3.141)

where we have used the duality wave-particle and (3.129). Finally the orbit is equipped with the symplectic form

$$\sigma = dp_i \wedge dq^i + \frac{1}{\mu_e\omega_0} \epsilon_{ij} dp_i \wedge dp_j - \mu_e \omega \epsilon_{ij} dq^i \wedge dq^j$$

(3.142)

We observe that with the anisotropic Carroll group in two-dimensional spaces, we obtain similar results as in the Static group case. Thus, by applying coadjoint orbit method to the extended planar anisotropic Carroll group, the structure of the phase space obtained is the same as in the Static group case: positions as well as momenta do not commute due to the noncommutativity of both the generators of the pure Carroll transformations and the generators of space transformations.

If we apply the coadjoint orbit method to the centrally extended anisotropic kinematical algebras, we obtain three kinds of noncommutative phase spaces:

- A phase space whose only positions do not commute, i.e: generators of pure kinematical group transformations are noncommutative in extended Lie algebras. This kind of noncommutative phase spaces is realized on the Galilei algebra $\mathcal{G}$.

- A phase space whose only momenta are noncommutative due to the noncommutativity of generators of space translations in centrally extended Lie algebras. This case is realized on the Para-Galilei algebras $\mathcal{G}'$. It arrives also for the Aristotle Lie algebra.
• phase spaces which are completely noncommutative: meaning that both
generators of pure kinematical group transformations and generators of space transla-
tions do not commute. These are obtained by considering the case of the Newton-
Hooke Lie algebras $N_{\pm}$, the Static Lie algebra and the Carroll Lie algebra.

In the following section, we realize noncommutative phase spaces on the planar absolute
time groups by considering their noncentral extensions.

### 3.4 Noncommutative phase spaces constructed on absolute time
groups

For the planar absolute time groups, the noncommutative phase spaces are obtained by
working with noncentral abelian extensions which have been determined in the Chapter 1. In this section, we see in detail that noncommutative phase spaces can be also con-
structed in the framework of noncentrally abelian extended kinematical groups (when
rotations are taking into account) by the coadjoint orbit method. It will be highlighted
that the deformed structures obtained in such way are algebraically more general than
those obtained in the anisotropic case.

#### 3.4.1 Galilei noncommutative phase spaces

With the coadjoint orbit method applied to the noncentrally abelian extended Galilei
group, the phase space obtained is a six-dimensional phase space where one coordinates
commute with the others and where the positions do not commute due to the noncom-
mutativity of the generators of pure Galilei transformations. This noncommutativity is
measured by the presence of the dual magnetic filed $B^\ast$. With the Galilei group, same
results have been obtained in [70] but by using the two- centrally extended Lie algebra
of this group as it has been done also in the section (3.3.3).

Let us consider the Galilei group in two-dimensional space as described previ-
ously. Its Lie algebra $\mathfrak{g}$ is generated by the left invariant vector fields given by relation (1.50) and satisfy the nontrivial Lie brackets (1.51) as already pointed out.

**i) Noncentrally abelian extended group and its maximal coadjoint orbit**

Let $\hat{\mathfrak{g}}$ be the noncentrally abelian extended Galilei Lie algebra satisfying relations (1.51) and

\[
[J, F_i] = F_k \epsilon_i^k, \quad [P_i, H] = F_i, \quad [K_i, K_j] = \frac{1}{c^2} S_{ij}, \quad [K_i, P_j] = M \delta_{ij}
\] (3.143)
If the duality between the extended Lie algebra with \( c \) are two
\( \delta \vec{\eta} \),
we obtain that the multiplication law of using relations (1.51), (3.143) and the Baker-Campbell-Hausdorff formulas [42], we have
\( \mathcal{Q} \)
be the general element of the connected extended Galilei group \( \hat{G} \).
Let
\[
g = e^{\varphi S + \xi M} e^{\eta F_\xi} e^{x^i P_i + tH} e^{v^i K_i} e^{\theta J} \]
be the general element of the connected extended Galilei group \( \hat{G} \). Assume also that \( \hat{g} = (\beta, \vec{\eta}, g) \in \hat{G} \) with \( \beta = (\varphi, \xi) \) and \( g \) the general element of the Galilei group. By using relations (1.51), (3.143) and the Baker-Campbell-Hausdorff formulas [42], we obtain that the multiplication law of \( \hat{G} \) is given by
\[
(\beta, \vec{\eta}, g)(\beta', \vec{\eta}', g') = (\beta + \beta' + c(g, g'), R(\theta)\vec{\eta}' + \vec{\eta} + \vec{c}(g, g'), gg') \tag{3.144}
\]
with \( gg' \) given by (1.49) and where
\[
c(g, g') = \left( \frac{1}{2c^2} R(-\theta)\vec{v} \times \vec{v}', R(-\theta)\vec{v}.\vec{x}' + \frac{\vec{v}^2}{2} t' \right) \tag{3.145}
\]
and
\[
\vec{c}(g, g') = \frac{1}{2}[\vec{x} - \vec{v}t][t' - tR(\theta)\vec{x}']. \tag{3.146}
\]
are two 2—cocycles defining the extended structure.

It follows that the adjoint action \( Ad_g(\delta \hat{g}) = g(\delta \hat{g})g^{-1} \) of the quotient group \( Q = \hat{G}/Z(\hat{G}) \) on the Lie algebra \( \hat{\mathfrak{g}} \) is given by:
\[
Ad_{(x, \vartheta, \theta, \xi)}(\delta \theta, \delta \varphi, \delta \vec{x}, \delta \vec{v}, \delta \vec{\eta}, \delta \xi, \delta t) = (\delta \theta', \delta \varphi', \delta \vec{x}', \delta \vec{v}', \delta \vec{\eta}', \delta \xi', \delta t')
\]
with
\[
\delta \theta' = \delta \theta, \quad \delta t' = \delta t, \quad \delta \vec{v}' = R(\theta)\delta \vec{v} + \epsilon(\vec{v})\delta \theta,
\]
\[
\delta \varphi' = \delta \varphi + \frac{1}{2} R(-\theta)\vec{v} \times \delta \vec{v} - \frac{\vec{v}^2}{2c} \delta \theta,
\]
\[
\delta \vec{x}' = R(\theta)\delta \vec{x} + \vec{v}\delta t + \epsilon(\vec{x} - \vec{v}t)\delta \theta - tR(\theta)\delta \vec{v},
\]
\[
\delta \vec{\eta}' = \delta \vec{\eta} + R(-\theta)\vec{v}.\delta \vec{x} - R(-\theta)\vec{x}.\delta \vec{v} + \frac{\vec{v}^2}{2} \delta t + \vec{v} \times \vec{x} \delta \theta,
\]
\[
\delta \xi' = \delta \xi + R(-\theta)\vec{v}.\delta \vec{x} - \vec{v} \times \vec{x} \delta \theta,
\]
\[
\delta \vec{\eta}' = R(\theta)\delta \vec{\eta} - tR(\theta)\delta \vec{x} + (\vec{x} - \vec{v}t)\delta t + \epsilon(\vec{\eta} - \frac{\vec{v}^2}{2} + \frac{\vec{v}^2}{2}) \delta \theta + \frac{\vec{v}^2}{2} R(\theta)\delta \vec{v}
\]
with
\[
\epsilon(\vec{v}) = \left( \begin{array}{c} v^2 \\ -v^1 \end{array} \right) \tag{3.147}
\]
If the duality between the extended Lie algebra \( \hat{\mathfrak{g}} \) and its dual \( \hat{\mathfrak{g}}^* \) is defined by the action \( j\delta \theta + \vec{k}.\delta \vec{v} + \vec{p}.\delta \vec{x} + E \delta t + \vec{f}.\delta \vec{\eta} + m\delta \xi + h\delta \varphi \), then the coadjoint action of the quotient group \( Q \) on \( \hat{\mathfrak{g}}^* \) is
\[
(h', m', \vec{f}', j', \vec{k}', \vec{p}', E') = Ad^\ast_{(\eta, \vartheta, \theta, \xi)}(h, m, \vec{f}, j, \vec{k}, \vec{p}, E) \tag{3.148}
\]
with

\[ m' = m, \quad h' = h, \quad \vec{f}' = R(\theta)\vec{f}, \quad \vec{p}' = R(\theta)\vec{p} + tR(\theta)\vec{f} - m\vec{v} \]  
(3.149)

\[ \vec{k}' = R(\theta)\vec{k} + tR(\theta)\vec{p} + \frac{t^2}{2} R(\theta)\vec{f} + m(\vec{x} - \vec{v}t) + \frac{h}{c^2} \epsilon(\vec{v}) \]  
(3.150)

\[ E' = E - \vec{v}.R(\theta)\vec{p} - \vec{x}.R(\theta)\vec{f} + \frac{m\vec{v}^2}{2} \]  
(3.151)

\[ j' = j + \vec{x} \times R(\theta)\vec{p} + \vec{v} \times R(\theta)\vec{k} + \vec{\eta} \times R(\theta)\vec{f} + \frac{\vec{x}t}{2} \times R(\theta)\vec{f} + m\vec{v} \times \vec{x} \]  
(3.152)

The observables \( j, \vec{k}, \vec{p}, E, \vec{f}, m \) and \( h \) have respectively the physical dimensions of an angular momentum, a static momentum, a linear momentum, an energy, a force, a mass and an action.

Let the position vector \( \vec{q} \) and the dual magnetic field \( B^* \) be defined by

\[ \vec{q} = \frac{\vec{k}}{m}, \quad B^* = \frac{1}{e^* m\omega} \]  
(3.153)

where \( e^* \) is a dual charge.

The maximal coadjoint orbit of \( Q \) on \( \hat{\mathcal{G}}^* \) denoted by \( O_{(m, B^*, f, U)} \) is then characterized by the two trivial invariants \( m \) and \( h \) and by two nontrivial invariants: the force intensity \( f \) and the (internal) energy \( U \) given respectively by

\[ f = ||\vec{f}||, \quad U = E - \frac{\vec{p}^2}{2m} + \vec{f}.\vec{q} + e^* B^* \vec{f} \times \vec{p} \]  
(3.154)

where the relation \( h\omega_0 = mc^2 \) has been used.

In the basis \((J, F_1, K_1, P_1, K_2, P_2, F_2, H, M, S)\) of the noncentrally abelian extended Galilei Lie algebra, the restriction of the Kirillov’s matrix on the orbit is

\[
\Omega = \begin{pmatrix}
0 & f \sin \alpha & mq^2 & p_2 & -mq^1 & -p_1 \\
-f \sin \alpha & 0 & 0 & 0 & 0 & 0 \\
-mq^2 & 0 & 0 & m & e^* m^2 B^* & 0 \\
-p_2 & 0 & -m & 0 & 0 & 0 \\
mq^1 & 0 & -e^* m^2 B^* & 0 & 0 & m \\
p_1 & 0 & 0 & -m & 0 & 0 \\
\end{pmatrix}
\]  
(3.155)

where \( f_1 = f \cos \alpha, \ f_2 = f \sin \alpha. \)
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Its inverse is

$$\Omega^{-1} = \frac{1}{f \sin \alpha} \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -\frac{p_2}{m} & q - e^* B^* p_1 & 0 & -q - e^* B^* p_2 \\
0 & \frac{p_2}{m} & 0 & -\frac{f \sin \alpha}{m} & 0 & 0 \\
0 & -\frac{p_2}{m} & 0 & 0 & 0 & e^* B^* f \sin \alpha \\
0 & q + e^* B^* p_2 & 0 & -e^* B^* f \sin \alpha & \frac{f \sin \alpha}{m} & 0
\end{pmatrix} \quad (3.156)$$

where $\vec{A}^* = \frac{1}{2} \vec{B}^* \times \vec{p}$ is the dual magnetic potential \[25\] while $\vec{B}^* = B^* \vec{n}$ with $\vec{n}$ the unit vector perpendicular to the plane.

The orbit is then equipped with the symplectic form

$$\sigma_1 = \sigma_0 + G^{ij} dp_i \wedge dq_j \quad (3.157)$$

where $\sigma_0 = ds \wedge d\alpha + dp_i \wedge dq^i$, $G^{ij} = e^* B^* \epsilon^{ij}$ and $s = j - \vec{p} \times (\vec{q} - e^* \vec{A}^*)$ is the sum of the orbital momentum $\vec{L} = \vec{q} \times \vec{p}$, the angular momentum $j$ and an extra term $\vec{p} \times e^* \vec{A}^*$ associated to the dual magnetic field $B^*$ \[9\].

The Poisson brackets (of two functions $H$ and $f$ defined on the orbit) corresponding to the symplectic structure (3.157), are given by

$$\{H, f\} = \frac{\partial H}{\partial s} \frac{\partial f}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \frac{\partial f}{\partial s} + \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial f}{\partial p_i} - e^* B^* \epsilon^{ij} \frac{\partial H}{\partial q^i} \frac{\partial f}{\partial q^j} \quad (3.158)$$

Then

$$\{s, p_j\} = p_j \epsilon^i_j, \quad \{s, q^i\} = \epsilon^i_j (q^j - e^* A^* i)$$

$$\{q^i, q^j\} = -e^* B^* \epsilon^{ij}, \quad \{p_i, q^j\} = \delta^j_i \quad (3.159)$$

are the Poisson brackets within the coordinates on the orbit. We observe that $\alpha$ commute with all the other coordinates, that the momenta commute and form a vector, that positions do not commute and do not form a vector due to the presence of the dual magnetic field $B^*$.

Note also that $(s, p_i, \alpha, q^i) = (q^i - e^* A^* i)$ are canonical phase coordinates on the coadjoint orbit and that

$$\{s, A^*_j\} = A^*_i \epsilon^i_j, \quad \{q^i, A^*_j\} = \frac{B^*}{2} \epsilon^i_j \quad (3.160)$$
ii) Symplectic realization and equations of motion

Let \((s', \vec{p}', \alpha', \vec{q}') = D_{(\vec{q}, \vec{x}, \vec{v}, \vec{t})}(s, \vec{p}, \alpha, \vec{q})\) be the symplectic realization of the extended Galilei group on its coadjoint orbit. Use of the relations (3.149) to (3.152) gives rise to

\[
\alpha' = \alpha + \theta, \quad \vec{q}' = \vec{q} + \frac{1}{m}(R(\theta)\vec{p} - m\vec{v})t + \frac{R(\theta)\vec{f} t^2}{m} + \vec{x} - e^* m \vec{v} \times \vec{B}(3.161)
\]

\[
s' = s + \frac{K^*}{m}t + \frac{N^* t^2}{m}, \quad \vec{q}' = \vec{q} + \frac{\vec{p} t}{m} + \frac{\vec{f} t^2}{2} \times R(\theta)\vec{f} - \frac{e^* B^* m^2 \vec{v}^2}{2}
\]

(3.162)

\[
\vec{p}' = R(\theta)\vec{p} + R(\theta)\vec{f} t - m\vec{v}
\]

(3.163)

where \(\frac{K^*}{m}\) and \(\frac{N^*}{m}\) are respectively an energy and a power with \(K^*\) and \(N^*\) given by:

\[
K^* = m\vec{q} \times \vec{f} + e^* m B^* \vec{p} \cdot \vec{f}, \quad N^* = \vec{f} \times \vec{p} - e^* m B^* \vec{f}^2
\]

(3.164)

It follows that the evolution with respect to the time \(t\) is given by

\[
\alpha(t) = \alpha, \quad \vec{q}(t) = \vec{q} + \frac{\vec{p}}{m}t + \frac{\vec{f} t^2}{2}
\]

(3.165)

and

\[
s(t) = s + \frac{K^*}{m}t + \frac{N^* t^2}{m}, \quad \vec{p}(t) = \vec{p} + \vec{f} t
\]

(3.166)

The corresponding Hamiltonian vector field is

\[
X_H = \frac{\vec{p}}{m} \frac{\partial}{\partial \vec{q}} + \vec{f} \frac{\partial}{\partial \vec{p}} + \frac{K^*}{m} \frac{\partial}{\partial s}
\]

(3.167)

and then the Hamiltonian function, given by

\[
H = \frac{\vec{p}^2}{2m} + V_0(\vec{q}, \alpha) + e^* \vec{p} \cdot (\vec{f} \times \vec{B}^*)
\]

(3.168)

is the sum of a kinetic energy \(T(\vec{p}) = \frac{\vec{p}^2}{2m}\), of a potential energy \(V_0(\vec{q}, \alpha) = -\vec{f} \cdot \vec{q} - \frac{K^*}{m} \alpha\) depending on the position-angle variables \((\vec{q}, \alpha)\) and of an exotic energy \(E_{\text{exotic}}^* = e^* \vec{p} \cdot (\vec{f} \times \vec{B}^*)\) depending on the dual magnetic field.

The equations of motion are then given by

\[
\begin{pmatrix}
\vec{p}(t) \\
K^*(t)
\end{pmatrix} = m \frac{d}{dt} \begin{pmatrix}
\vec{q}(t) \\
s(t)
\end{pmatrix}, \quad \frac{d}{dt} \begin{pmatrix}
\vec{p}(t) \\
\alpha(t)
\end{pmatrix} = \begin{pmatrix}
\vec{f} \\
0
\end{pmatrix}
\]

(3.169)
The equations (3.169) define the linear momentum \( \vec{p} (t) = m \frac{d\vec{q}(t)}{dt} \), the force \( \vec{f} = \frac{d\vec{p}}{dt} \) and the quantity \( K^*(t) = m \frac{ds}{dt} \) whose dimension is that of a linear momentum squared. Moreover

\[
\frac{d^2}{dt^2} \begin{pmatrix} \vec{p} \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \frac{d^2}{dt^2} \begin{pmatrix} \vec{q} \\ s \end{pmatrix} = \frac{1}{m} \begin{pmatrix} \vec{f} \\ N^* \end{pmatrix}
\]

We interpret the equations (3.170) as the Euler-Newton’s second law [31] associated to the above noncentral extended Galilei group.

So with the noncentrally abelian extended Galilei group, we obtain also an exotic model [70] i.e with noncommutativity in positions sector.

In the following paragraph, we study the case of the absolute time Para-Galilei groups and as in the anisotropic case, we show that the noncommutativity of the momenta induces some modification of the Newton’s second law [30, 31]. Comparative summary between results obtained on both the absolute time Galilei (see previous section) and Para-Galilei groups will be given.

### 3.4.2 Para-Galilei noncommutative phase spaces

In this case, the phase space obtained is a six-dimensional phase space where one coordinates commute with the others and where the momenta do not commute due to the noncommutativity of the generators of space translations in the extended Para-Galilei groups. This noncommutativity is measured by the presence of a magnetic field \( B \).

More specifically, let us consider the planar Para-Galilei groups as described in the previous section. Their Lie algebras \( \mathfrak{g}'_\pm \) are generated by the left invariant vector fields given by relation (3.110) and satisfy the nontrivial Lie brackets (3.111).

i) Noncentrally abelian extended groups and their maximal coadjoint orbits

The noncentrally abelian extended Para-Galilei Lie algebras \( \mathfrak{g}'_\pm \) satisfy the nontrivial Lie brackets (3.111) and

\[
[J, \Pi_i] = \Pi_k \epsilon_i^k, \quad [K_i, H] = \Pi_i, \quad [P_i, P_j] = \kappa^2 S \epsilon_{ij}, \quad [K_i, P_j] = M \delta_{ij}
\]

where \( \kappa \) is a constant whose dimension is the inverse of that of a length. As in the Galilei case, \( S \) and \( M \) generate the center of \( \mathfrak{g}'_\pm \) and \( \Pi_i \) are new generators which do not commute with \( J \), hence the term noncentral abelian extension.
Let
\[ \hat{g} = \exp(\varphi S + \xi M) \exp(\theta J) \]
be the general element of the connected extended Para-Galilei groups \( \hat{G}_{\pm} \). As in the case of the Galilei group, the relations (3.111), (3.171) and the Baker-Campbell-Hausdorff formulas give rise to the group multiplication laws
\[ (\beta, \bar{I}, g)(\beta', \bar{I}', g') = (\beta + \beta' + c(g, g'), \bar{I} + R(\theta)\bar{I}' + \bar{c}(g, g'), gg') \] (3.172)
with \( gg' \) given by (3.109) and where
\[ c(g, g') = \left( \frac{1}{2} \kappa^2 R(-\theta)\vec{x} \times \vec{x}', -R(-\theta)\vec{x} \cdot \vec{v}' \mp \frac{\omega^2 \vec{x}^2}{2} t' \right) \] (3.173)
and
\[ \bar{c}(g, g') = \frac{1}{2} \left[ (\vec{v} \mp \omega^2 \vec{x} t) t' - t R(\theta) \vec{v}' \right] . \] (3.174)
are two 2-cocycles.

It follows that the adjoint action \( Ad_g(\delta \hat{g}) = g(\delta \hat{g})g^{-1} \) of the quotient groups \( Q'_{\pm} = \hat{G}'_{\pm} / Z(\hat{G}'_{\pm}) \) on the extended Para-Galilei Lie algebras \( \hat{G}'_{\pm} \) is
\[ Ad_{(\vec{x}, \vec{v}, \theta, t)}(\delta \theta, \delta \varphi, \delta \vec{x}, \delta \vec{v}, \delta \bar{I}, \delta \xi, \delta t) = (\delta \theta', \delta \varphi', \delta \vec{x}', \delta \vec{v}', \delta \bar{I}', \delta \xi', \delta t') \]
with
\[ \delta \theta' = \delta \theta , \delta t' = \delta t , \delta \vec{x}' = R(\theta) \delta \vec{x} + \epsilon(\vec{x}) \delta \theta \]
\[ \delta \vec{v}' = R(\theta) \delta \vec{v} \pm \omega^2 \vec{x} \delta t + \epsilon(\vec{v} \mp \omega^2 \vec{x} t) \delta \theta \mp \omega^2 t R(\theta) \delta \vec{x} \]
\[ \delta \bar{I}' = R(\theta) \bar{I}' - t R(\theta) \delta \vec{x} \mp \frac{\omega^2 \vec{x}^2}{2} R(\theta) \delta \vec{v} + (\vec{v} \mp \omega^2 \vec{x} t) \delta t + \epsilon(\bar{I} - \frac{1}{2}(\vec{v} \mp \omega^2 \vec{x} t)) \delta \theta \]
\[ \delta \varphi' = \delta \varphi + \kappa^2 R(-\theta) \vec{x} \times \vec{x} - \vec{v}^2 \kappa^2 \delta \theta \]
\[ \delta \xi' = \delta \xi - R(-\theta) \vec{x} \cdot \vec{v} + R(-\theta) \vec{v} \cdot \vec{x} \mp \frac{\omega^2 \vec{x}^2}{2} \delta t + \vec{v} \times \vec{x} \delta \theta \]

where \( \epsilon(\vec{v}) \) is given by (3.147).

If the duality between the extended Lie algebras \( \hat{G}'_{\pm} \) and their duals \( \hat{G}'_{\pm}^* \) gives rise to the action \( j \delta \theta + k. \delta \vec{v} + \bar{I} \delta \vec{x} + E \delta t + \vec{p} \delta \bar{I} + m \delta \xi + h \delta \varphi \), then the coadjoint actions are such that
\[ m' = m , \ h' = h \] (3.175)
\[ \vec{p}' = R(\theta) \vec{p} , \ \vec{k}' = R(\theta) \vec{k} + t R(\theta) \vec{p} + m \vec{x} \] (3.176)
\[ \bar{I}' = R(\theta) \bar{I} \pm \omega^2 t R(\theta) \vec{k} \mp \frac{\omega^2 \vec{x}^2}{2} \left( t R(\theta) \vec{p} + m \omega \epsilon(\vec{x}) \right) - m(\vec{v} \mp \omega^2 \vec{x} t) \] (3.177)
Noncommutative phase spaces constructed on absolute time groups

\[ E' = E - \omega^2 \vec{x} \cdot \mathbf{R}(\theta) \vec{k} - \vec{v} \cdot \mathbf{R}(\theta) \vec{p} - \frac{\vec{\omega} \times \vec{x}}{2C} \]  

(3.178)

where \( C \) is given by relation (3.115)

\[ j' = j + \vec{x} \times \mathbf{R}(\theta) \vec{l} + \vec{v} \times \mathbf{R}(\theta) \vec{k} + \vec{l} \times \mathbf{R}(\theta) \vec{p} + \frac{\vec{\omega} t}{2} \times \mathbf{R}(\theta) \vec{p} + \vec{m} \vec{v} \times \vec{x} \]  

(3.179)

Define the vector \( \vec{q} \) by (3.153) and the magnetic field \( B \) by

\[ B = \frac{m \omega}{e} \]  

(3.180)

where \( e \) is the electric charge. The coadjoint orbits denoted by \( \mathcal{O}(m, B, p, U_\pm) \) are characterized by two trivial invariants \( m \) and \( B \) and by two nontrivial invariants \( p \) and \( U_\pm \) respectively given by:

\[ p = ||\vec{p}||, \quad U_\pm = E + \frac{\vec{q}^2}{2C} - \frac{\vec{\omega} \cdot \vec{l}}{m} + eB\vec{p} \times \vec{q} \]  

(3.181)

where we have used the relation \( h\omega_0 = mc^2 \) and (3.115).

In the basis \((J, P_1, K_1, \Pi_1, K_2, \Pi_2, P_2, H, M, S)\) of the extended Para-Galilei Lie algebras, the restriction of the Kirillov form on the coadjoint orbit is then

\[
\Omega = \begin{pmatrix}
0 & p \sin\alpha & mq^2 & I_2 & -mq^1 & -I_1 \\
-p \sin\alpha & 0 & 0 & 0 & 0 & 0 \\
-mq^2 & 0 & 0 & m & 0 & 0 \\
-p_2 & 0 & -m & 0 & 0 & eB \\
mq^1 & 0 & 0 & 0 & m & 0 \\
I_1 & 0 & 0 & -eB & -m & 0
\end{pmatrix}
\]  

(3.182)

where \( p_1 = p \cos\alpha \), \( p_2 = p \sin\alpha \).

The inverse of \( \Omega \) is

\[
\Omega^{-1} = \frac{1}{p \sin\alpha} \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -\frac{I_2}{m} - \frac{eBq^1}{m} & 0 & \frac{I_1}{m} - \frac{eBq^2}{m} & -q^1 \\
0 & \frac{I_2}{m} + \frac{eBq^1}{m} & 0 & \frac{p \sin\alpha}{m} & 0 & 0 \\
0 & -q^2 & \frac{p \sin\alpha}{m} & 0 & 0 & 0 \\
0 & -\frac{I_1}{m} + \frac{eBq^2}{m} & -\frac{eBp \sin\alpha}{m^2} & 0 & 0 & \frac{-p \sin\alpha}{m} \\
0 & q^1 & \frac{eBp \sin\alpha}{m^2} & 0 & 0 & 0
\end{pmatrix}
\]  

(3.183)

We then verify that the symplectic form on the orbit \( \mathcal{O}(m, B, p, U_\pm) \) is

\[ \sigma_1' = \sigma_0' + F_{ij} dq^i \wedge dq^j \]  

(3.184)
where \( \sigma'_0 = ds \wedge d\alpha + dI_i \wedge dq^i \), \( F_{ij} = eB\epsilon_{ij} \) and \( s = j - \vec{q} \times (\vec{I} + e\vec{A}) \) is the sum of the angular momentum \( j \), the orbital angular momentum \( \vec{L} = \vec{q} \times \vec{I} \) and an extra term \( eB\frac{\vec{x}^2}{2} \) associated to the magnetic field \( B \), \( \vec{A} = \frac{1}{2}\vec{B} \times \vec{q} \) being the magnetic potential \[25\] with \( \vec{B} = B\vec{n} \).

The Poisson brackets of two functions \( H \) and \( f \) on the orbit corresponding to the symplectic form (3.184) is

\[
\{H, f\} = \frac{\partial H}{\partial s} \frac{\partial f}{\partial \alpha} - \frac{\partial H}{\partial \alpha} \frac{\partial f}{\partial s} + \frac{\partial H}{\partial I_i} \frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial f}{\partial I_i} - eB\epsilon_{ij} \frac{\partial H}{\partial I_i} \frac{\partial f}{\partial I_j} \tag{3.185}
\]

and the nontrivial Poisson brackets within the coordinates are

\[
\{s, I_j\} = (I_i - eA_i)\epsilon^i_j, \quad \{s, q^i\} = \epsilon^i_j q^j, \quad \{I_i, I_j\} = -eB\epsilon_{ij}, \quad \{I_i, q^j\} = \delta^j_i \tag{3.186}
\]

This means that \( \alpha \) commute with all the other coordinates, that the positions coordinates commute and form a vector while momenta do not commute and do not form a vector due to the presence of the magnetic field \( B \).

Note that \( (s, \vec{I} = I_i - eA_i, \alpha, q^i) \) are the canonical phase coordinates on the coadjoint orbit and that we have moreover these Poisson brackets

\[
\{s, A_j\} = -A_i\epsilon^i_j, \quad \{I_i, A_j\} = \frac{B}{2}\epsilon_{ij} \tag{3.187}
\]

**ii) Symplectic realizations and equations of motion**

Let the symplectic realizations of the extended Para-Galilei groups on their coadjoint orbits be given by \( (s', \vec{p}', \alpha', \vec{q}') = D_{(\theta, \vec{x'}, \vec{x}, t)}(s, \vec{p}, \alpha, \vec{q}) \). By using relations (3.176) to (3.179), we obtain

\[
\alpha' = \alpha + \theta, \quad \vec{q}' = R(\theta)\vec{q} + \frac{R(\theta)\vec{p}}{m} t + \vec{x} \tag{3.188}
\]

\[
s' = s + \frac{K}{m} t + \frac{N t^2}{m} \pm (\vec{I} - \frac{\vec{v} t}{2} \pm \frac{\omega^2 \vec{x}^2}{2}) \times R(\theta)\vec{p} + eB\frac{\vec{x}^2}{2} \tag{3.189}
\]

\[
\vec{I}' = R(\theta)\vec{I} + \frac{1}{C}[R(\theta)\vec{q} + \vec{x}]t \pm \omega^2 R(\theta)\vec{p} \frac{t^2}{2} + eB\epsilon(\vec{x}) + m\vec{v} \tag{3.190}
\]

where \( \frac{K}{m} \) and \( \frac{N}{m} \) are respectively an energy and a power and where \( K \) and \( N \) are given by

\[
K = \vec{I} \times \vec{p} - eB\vec{p} \times \vec{q}, \quad N = \frac{1}{C} \vec{q} \times \vec{p} \mp eB\frac{\vec{p}^2}{2m} \tag{3.191}
\]
where we have used relation (3.115).

The evolution with respect to the time \( t \) is

\[
\alpha(t) = \alpha, \quad \vec{q}(t) = \vec{q} + \frac{\vec{p}}{m} t
\]

and

\[
s(t) = s + \frac{K}{m} t + \frac{N}{m^2} t^2, \quad \vec{I}(t) = \vec{I} + \frac{1}{C} \vec{q} t + \frac{1}{C m} \vec{p} \cdot \vec{I} ^2
\]

where we have used relation (3.115).

It follows that the corresponding Hamiltonian vector field is

\[
X_H = \frac{\vec{p} \cdot \partial}{m} + \frac{\vec{q} \cdot \partial}{C} + \frac{K \partial}{m} \partial s
\]

and then the Hamiltonian function (energy) is

\[
H = \vec{I} \cdot \vec{p} - \frac{\vec{q}^2}{2C} - \frac{K}{m} \alpha + e \vec{p} \cdot (\vec{B} \times \vec{q})
\]

that is a sum of a kinetic term \( T(\vec{p}) = \vec{I} \cdot \frac{\vec{p}}{m} \), of a potential energy \( V_0(\vec{q}, \alpha) = -\frac{\vec{q}^2}{2C} - \frac{K}{m} \alpha \) depending on the position-angle variables \((\vec{q}, \alpha)\) and of an exotic energy \( E_{\text{exotic}} = e \vec{p} \cdot (\vec{B} \times \vec{q}) \) depending on the magnetic field.

The equations of motion are then given by

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} \vec{I}(t) \\ s(t) \end{pmatrix} &= \begin{pmatrix} 1/C \\ K \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \vec{q}(t) \\ \alpha(t) \end{pmatrix} = \begin{pmatrix} 1/m \vec{p} \\ 0 \end{pmatrix}
\end{align*}
\]

The equations (3.196) define the linear momentum \( \vec{p}(t) = m \frac{d\vec{q}(t)}{dt} \), the position \( \vec{q} = C \frac{d\vec{I}}{dt} \) and the quantity \( K(t) = m \frac{ds}{dt} \) whose dimension is that of a linear momentum squared.

Moreover,

\[
\frac{d^2}{dt^2} \begin{pmatrix} \vec{I} \\ s \end{pmatrix} = \frac{1}{m} \begin{pmatrix} 1/C \vec{p} \\ N \end{pmatrix}, \quad \frac{d^2}{dt^2} \begin{pmatrix} \vec{q} \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

We interpret the equations (3.197) as the modified Euler-Newton’s second law [31] associated to the Para-Galilei groups.

So with the noncentrally abelian extended Para-Galilei group, we obtain a six-dimensional phase space where one coordinates commute with the others and where the momenta do not commute. This noncommutativity is measured by the presence of the magnetic field \( B \) through the Poisson brackets (3.186).
Comparative analysis between noncentrally Galilei and Para-Galilei cases

In both cases the coadjoint orbit is a six-dimensional symplectic manifold i.e the product of a two-dimensional cylinder (parametrized by an angle $\alpha$ and a conjugate action $s$) with a four-dimensional phase space parametrized by the position-momentum coordinates. These coadjoint orbits are denoted $\mathcal{O}_{(m,f,B^*,U)}$ for the Galilei group and $\mathcal{O}_{(m,p,B,U_\pm)}$ for the Para-Galilei groups where $B$ and $B^*$ are respectively a magnetic and a dual magnetic field. The polar coordinates of the points on the basis circle of the cylinder are $(f, \alpha)$ for the orbit $\mathcal{O}_{(m,f,B^*,U)}$ and $(p, \alpha)$ for $\mathcal{O}_{(m,p,B,U_\pm)}$ where $f$ is a constant force intensity while $p$ is a constant linear momentum. The orbit $\mathcal{O}_{(m,f,B^*,U)}$ is equipped with the symplectic form

$$\sigma_1 = ds \wedge d\alpha + dp_i \wedge dq^i + e^* B^* dp_i \wedge dp_j$$

and is then a noncommutative symplectic manifold where the positions $q^i$ do not commute while the orbit $\mathcal{O}_{(m,p,B,U_\pm)}$ endowed with the symplectic form

$$\sigma'_1 = ds \wedge d\alpha + dI_i \wedge dq^i + eB dq^i \wedge dq^j$$

is a noncommutative symplectic manifold where the momenta $I_i$ do not commute.

Kinematically, the positions $q^i$ on $\mathcal{O}_{(m,f,B^*,U)}$ behave as the momenta $I_i$ on $\mathcal{O}_{(m,p,B,U_\pm)}$ whereas the momenta $p_i$ on $\mathcal{O}_{(m,f,B^*,U)}$ behave as the positions $q^i$ on $\mathcal{O}_{(m,p,B,U_\pm)}$. The former are quadratic in time while the last ones are linear in time. The action $s$ conjugated to the angle is quadratic in time in the two cases. Note that the angle $\alpha$ is constant in time in the two cases.

The dynamic equations for the system described by the Galilean orbit are

$$\frac{d^2 s}{dt^2} = \frac{N}{m}, \quad \frac{d\vec{p}}{dt} = \frac{m d^2 \vec{q}}{dt^2}$$ \hspace{1cm} (3.198)$$

while they are

$$\frac{d^2 s}{dt^2} = \frac{N}{m}, \quad \frac{d\vec{q}}{dt} = C \frac{d^2 \vec{I}}{dt^2}$$ \hspace{1cm} (3.199)$$

for the system described by the Para-Galilean orbits, where $C$ is given by relation (3.115), the inverse of the spring constant.

As the equation $\frac{d\vec{q}}{dt} = \frac{m d^2 \vec{q}}{dt^2}$ is called the Galilei-Newton law [39] for a massive particle with mass $m$, the law $\frac{d\vec{I}}{dt} = C \frac{d^2 \vec{I}}{dt^2}$ can be called the Para-Galilei-Newton law for a spring whose constant $C$ is the inverse of the usual Hooke’s constant.

The following table summarizes these constructions in both cases.

---

Noncommutative phase spaces constructed on absolute time groups
### Comparative analysis between noncentrally Galilei and Para-Galilei cases

| Group | Galilei group | Para-Galilei groups |
|-------|---------------|---------------------|
| Invariants (non trivial) | $f, U = le^{-\frac{\tilde{q}^2}{2m}} + \int \tilde{q} \, dq + e^* B^* \tilde{f} \times \tilde{\alpha}$ | $p, U = E + \frac{\tilde{q}^2}{2m}$ |
| magnetic fields | $e^* B^* = \frac{1}{m\omega}, B^* = B^* \tilde{e}_3$ | $e B = m \omega, B = B \tilde{e}_3$ |
| | $A^* = \frac{1}{2} B^* \times \tilde{p}$ (dual potential) | $\tilde{A} = \frac{1}{2} \tilde{B} \times \tilde{q}$ (potential) |
| noncommutative phase coordinates | $\vec{q} = \frac{\tilde{q}}{m}, \alpha = \arctg \left( \frac{\tilde{q}}{\tilde{f}} \right)$ | $\vec{q} = \frac{\tilde{q}}{m}, \alpha = \arctg \left( \frac{\tilde{q}}{\tilde{f}} \right)$ |
| | $s = j + \tilde{p} \times \tilde{q} - e^* B^* \tilde{e}_2$ | $s = j + \tilde{I} \times \tilde{q} - e B \tilde{e}_2$ |
| symplectic form | $\sigma_1 = \sigma_0 + G^{ij} dp_i \wedge dp_j$ | $\sigma'_1 = \sigma'_0 + F_{ij} dq^i \wedge dq^j$ |
| Poisson brackets | $\{ s, p_i \}, \alpha, q^i \} = e^* B^* \epsilon^{ij}$ | $\{ s, I_j \} = (I_i - e A_i) \epsilon^{ij}$ |
| | $\{ s, q^i \} = e^i_j (q^j - e^* A^k \tilde{p}^k)$ | $\{ s, q^i \} = e^i_j$ |
| | $\{ p_i, p_j \} = 0$ | $\{ I_i, I_j \} = - e B \epsilon_{ij}$ |
| | $\{ p_i, q^j \} = \delta^j_i$ | $\{ I_i, q^j \} = \delta^j_i - e B \epsilon_{ij}$ |
| | $\{ q^i, q^j \} = - e^* B^* \epsilon^{ij}$ | $\{ q^i, q^j \} = 0$ |
| Hamiltonian function | $H = \frac{\tilde{p}^2}{2m} + V(\tilde{q}, \alpha) + e^* f(\tilde{p} \times \tilde{B}^*)$ | $H = \tilde{I} \frac{\tilde{p}^2}{2m} + V(\tilde{q}, \alpha) + e \tilde{p} (\tilde{B} \times \tilde{q})$ |
| potential energy | $V(\tilde{q}, \alpha) = - \tilde{f} \tilde{q} - \frac{K^* \alpha}{m}$ | $V(\tilde{q}, \alpha) = \frac{\tilde{q}^2}{2m} - \frac{K \alpha}{m}$ |
| equations of motion | $\frac{d\tilde{q}}{dt} = \tilde{p} \tilde{m}, \frac{d\tilde{p}}{dt} = \tilde{f} (\text{constant})$ | $\frac{d\tilde{q}}{dt} = \tilde{p} \tilde{m} (\text{constant}), \frac{d\tilde{p}}{dt} = \tilde{q}$ |
| | with | with |
| | $K^* (t) = K^* + N^* t$ | $K(t) = K + N t$ |
| | $K^* = m \tilde{q} \tilde{f} + e^* m B^* \tilde{p} \tilde{f}$ | $K = \tilde{I} \times \tilde{p} - e B \tilde{q} \tilde{p}$ |
| | $N^* = \tilde{f} \times \tilde{p} - e^* B^* \tilde{m}$ | $N = \tilde{q} \times \tilde{p} - e B \tilde{m}$ |
| "Newton’s equations" | $\frac{d^2 s(t)}{dt^2} = \frac{N^*}{m}, \frac{d^2 \tilde{q}(t)}{dt^2} = \frac{\tilde{f}}{m}$ | $\frac{d^2 s(t)}{dt^2} = \frac{N}{m}, \frac{d^2 \tilde{q}(t)}{dt^2} = \frac{\tilde{f}}{me}$ |
| canonical phase coordinates | $\{ s, p_i, \alpha, q^j \} \alpha$ | $\{ s, I_i, \alpha, q^j \} \alpha$ |
| | with $\tilde{q}^i = q^i - e^* A^i \tilde{q}$ | with $\tilde{I} = I_i - e A_i$ |

Table 3.1
3.4.3 Newton-Hooke noncommutative phase spaces

As it has been already said, we have found that noncentral abelian extensions of absolute time Newton-Hooke Lie algebras lead to nonvanishing commutator of two boosts and two momenta as their corresponding two-parameter centrally extended Lie algebras as well as their corresponding centrally extended anisotropic Lie algebras as detailed in [25]. This means that totally noncommutative phase spaces can be realized with the Newton-Hooke groups in both cases (i.e anisotropic and absolute time cases).

However, noncommutative phase spaces obtained with noncentral abelian extensions of Newton-Hooke groups are algebraically more general than those obtained with central extensions of the same groups: their Poissonian structures (and hence their symplectic two-forms on the orbits) contain additional terms. This case is not reviewed here.

3.4.4 Static noncommutative phase space in the absolute time case

With the planar noncentrally abelian extended Static group, we realize a completely noncommutative phase space equipped with a modified symplectic structure.

Indeed, the noncentrally abelian extended Static Lie algebra of the planar Static group satisfies the nontrivial Lie brackets (1.55).

Let
\[ g = \exp(v^t K_i + x^i P_i + tH)\exp(\theta J) \]
be the general element of the Static group and
\[ \hat{g} = \exp(\xi M + bS + \varphi M' + a\Lambda)\exp(\eta^t F_i + ^t\Pi_i)g \]
be the general element of the connected Lie group associated to the noncentrally extended Static Lie algebra $\mathcal{G}$. By using the Baker-Hausdorff formulas [42] and by identifying $\hat{g}$ with $(\beta, \vec{\nu}, g)$ where $\beta = (\xi, \varphi, b, a)$, $\vec{\nu} = (\vec{\eta}, \vec{I})$, we obtain that the multiplication law of the corresponding extended Lie group is

\[ (\beta, \vec{\nu}, g)(\beta', \vec{\nu}', g') = (\beta + \beta' + c(g, g'), R(\theta)\vec{\nu}' + \vec{\nu} + \vec{c}(g, g'), gg') \]

with
\[ gg' = (\theta, \vec{v}, \vec{x}, t)(\theta', \vec{v}', \vec{x}', t') = (\theta + \theta', R(\theta)\vec{v}' + \vec{v}, R(\theta)\vec{x}' + \vec{x}, t + t') \tag{3.200} \]
and where

\[ c(g, g') = \left( -\frac{1}{2} R(-\theta) \bar{v} \cdot \bar{x}' + \frac{1}{2} R(-\theta) \bar{x} \cdot \bar{v}', -\frac{1}{2} R(-\theta) \bar{v} \cdot \bar{I}' + \frac{1}{2} R(-\theta) \bar{I} \cdot \bar{v}' \right) - \frac{1}{2} \bar{I} \cdot \bar{v} - \frac{1}{2} R(-\theta) \bar{I}' \cdot R(-\theta) \bar{v}', \]

\[ \left( -\frac{1}{2} R(-\theta) \bar{v} \cdot \bar{v}' + \frac{1}{2} R(-\theta) \bar{v} \cdot \bar{v}' - \frac{1}{2} R(-\theta) \bar{x} \cdot \bar{l}' + \frac{1}{2} R(-\theta) \bar{l} \cdot \bar{x}' \right) - \frac{1}{2} \bar{l} \cdot \bar{x} - \frac{1}{2} R(-\theta) \bar{l} \cdot \bar{x}' - \frac{1}{2} R(-\theta) \bar{v} \cdot \bar{l}', \]

while

\[ \bar{c}(g, g') = \left( \frac{1}{2}[\bar{x} l' - t R(\theta) \bar{x}'], \frac{1}{2}[\bar{v} t' - t R(\theta) \bar{v}'] \right) \]

\( \theta \) being an angle of rotations, \( \bar{v} \) a boost vector, \( \bar{x} \) a space translation vector and \( t \) being a time translation parameter.

Let \( j J^i + m M^i + \beta S^i + \mu M^i + k \Lambda^i + f_i F^{*i} + I_i \Pi^{*i} + k_i K^{*i} + p_i P^{*i} + E H^i, \)

\((i = 1, 2)\) be the general element of the dual of the noncentrally abelian extended Static Lie algebra where the observables are an angular momentum \( j \), a static momentum \( \bar{p} \), an energy \( E \), a force \( \bar{f} \), two linear momenta \( \bar{p} \) and \( \bar{I} \), two masses \( m \) and \( \mu \), a frequency \( \omega = \frac{\bar{b}}{\mu} \) and a Hooke’s constant \( k \).

It follows that the adjoint action of the quotient groups \( Q = \mathbb{G}/Z(\mathbb{G}) \) on the above extended Lie algebra \( \mathbb{G} \) is such that:

\[ \delta \theta' = \delta \theta, \quad \delta t' = \delta t, \quad \delta \bar{x}' = R(\theta) \delta \bar{x} + \epsilon(\bar{x}) \delta \theta \]

\[ \delta \bar{v}' = R(\theta) \delta \bar{v} + \epsilon(\bar{v}) \delta \theta \]

\[ \delta \bar{v}' = R(\theta) \delta \bar{v} - t R(\theta) \delta \bar{x} + \bar{x} \delta t + \epsilon(\bar{x} - \frac{t}{2}) \delta \theta \]

\[ \delta \bar{l}' = R(\theta) \delta \bar{l} - t R(\theta) \delta \bar{v} + \bar{v} \delta t + \epsilon(\bar{l} - \frac{t}{2}) \delta \theta \]

\[ \delta \phi' = \delta \phi + R(-\theta) \bar{v} \times \delta \bar{l} - R(-\theta) \bar{l} \times \delta \bar{v} + \bar{v} \times \bar{l} \delta \theta \]

\[ \delta \xi' = \delta \xi - R(-\theta) \bar{x} \cdot \delta \bar{v} + R(-\theta) \bar{v} \cdot \delta \bar{x} + \bar{x} \times \delta \theta \]

\[ \delta b' = R(-\theta) \bar{v} \cdot \delta \bar{v} + R(-\theta) \bar{x} \cdot \delta \bar{x} - R(-\theta) \bar{l} \cdot \delta \bar{v} - R(-\theta) \bar{v} \cdot \delta \bar{l} \]

\[ \delta a' = \delta a + R(-\theta) \bar{x} \cdot \delta \bar{l} + R(-\theta) \bar{l} \cdot \delta \bar{x} + \bar{x} \times \bar{l} \delta \theta \]
where \( \varepsilon(\vec{v}) \) is given by the relation (3.147).

If the duality between \( \mathfrak{g} \) and its dual \( \mathfrak{g}^* \) gives rise to the action
\[
j \delta \theta + \vec{k}. \delta \vec{v} + \vec{f}. \delta \vec{l} + E \delta t + \vec{1}. \delta \vec{1} + m \delta \xi + \mu \delta \varphi + \beta \delta b + k \delta a,
\]
then the coadjoint action
\[
(m', \mu', \beta', k', \vec{f}', \vec{k}', \vec{p}', E', \vec{l}') = \text{Ad}^*_{(\vec{\eta}, \vec{e}, g)}(m, \mu, \beta, k, \vec{f}, \vec{k}, \vec{p}, E, \vec{l})
\]
is such that
\[
m' = m, \quad \mu' = \mu, \quad \beta' = \beta, \quad k' = k
\]
\[
\vec{f}' = R(\theta) \vec{f} - k \vec{x} - \beta \vec{v}
\]
\[
\vec{p}' = R(\theta) \vec{p} + t R(\theta) \vec{f} - tk \vec{x} - t \beta \vec{v} - m \vec{v} + k \vec{l} + \beta \vec{l}
\]
\[
\vec{l}' = R(\theta) \vec{l} - \mu \vec{v} - \beta \vec{x}
\]
\[
\vec{k}' = R(\theta) \vec{k} + t R(\theta) \vec{l} + m \vec{x} - t \mu \vec{v} - t \beta \vec{x} + m \vec{l} + \beta \vec{l}
\]
\[
E' = E - \vec{v}. R(\theta) \vec{l} - \vec{x}. R(\theta) \vec{f} + \frac{1}{2} k (\vec{x}^2) + \frac{1}{2} \mu (\vec{v}^2) + \beta \vec{x}. \vec{v}
\]
\[
\vec{j}' = j + \vec{x} \times R(\theta) \vec{p} + \vec{v} \times R(\theta) \vec{k} + \vec{l} \times R(\theta) \vec{l} + \vec{q} \times R(\theta) \vec{f}
\]
\[
+ \frac{\vec{v} \times R(\theta) \vec{l}}{2} + \frac{\vec{x} \times R(\theta) \vec{f}}{2} + m \vec{v} \times \vec{x} + \beta \vec{l} \times \vec{x} + \mu \vec{l} \times \vec{v}
\]

Then \( m, \mu, \beta \) and \( k \) are trivial invariants under coadjoint action of the Static group in two-dimensional space. The nontrivial invariants \( s \) and \( U \), solutions of (3.1), are explicitly given by:
\[
s = j - (\vec{k} - \vec{b} \vec{p}) \times \vec{u} + (\vec{p} - \frac{\beta}{k} \vec{k}) \times \vec{q}
\]
\[
U = E - \frac{\mu_e \vec{u}^2}{2} - \frac{k_e \vec{q}^2}{2} - \frac{\beta \mu_e}{\mu} \vec{q}. \vec{u} - \nu h
\]
where \( \nu \) is a frequency and where we have used the relations
\[
\vec{l} = \mu_e \vec{u}, \quad \vec{f} = -k_e \vec{q}, \quad h = j - s
\]
defining the velocity vector \( \vec{u} \), the position vector \( \vec{q} \) and the action \( h \) where
\[
k_e = k - \frac{\beta^2}{\mu}, \quad \mu_e = \mu - \frac{\beta^2}{k}
\]
are the effective Hooke’s constant and the effective mass respectively.

The the restriction of the Kirillov form in the basis 
\( (J, P_1, P_2, K_1, K_2, F_1, F_2, \Pi_1, \Pi_2, H, M, M', S, \Lambda) \) to the orbit is

\[
\Omega = \begin{pmatrix}
0 & 0 & -m & 0 & k & 0 & \beta & 0 \\
0 & 0 & 0 & -m & 0 & k & 0 & \beta \\
m & 0 & 0 & 0 & \beta & 0 & \mu & 0 \\
0 & m & 0 & 0 & \beta & 0 & \mu & 0 \\
-k & 0 & -\beta & 0 & 0 & 0 & 0 & 0 \\
0 & -k & 0 & -\beta & 0 & 0 & 0 & 0 \\
-\beta & 0 & -\mu & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta & 0 & -\mu & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(3.213)

Its inverse is given by:

\[
\Omega^{-1} = \frac{1}{\beta^2 - \mu k} \begin{pmatrix}
0 & 0 & 0 & 0 & \mu & 0 & -\beta & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & -\beta \\
0 & 0 & 0 & 0 & -\beta & 0 & k & 0 \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & k \\
-\mu & 0 & \beta & 0 & 0 & 0 & m & 0 \\
0 & -\mu & 0 & \beta & 0 & 0 & 0 & m \\
-\beta & 0 & -k & 0 & -m & 0 & 0 & 0 \\
0 & \beta & 0 & -k & 0 & -m & 0 & 0
\end{pmatrix}
\]

(3.214)

The symplectic form on the orbit \( O_{(m, \mu, \beta, k, s, U)} \) is then given by

\[
\sigma = d\vec{p} \wedge d\vec{q} + d\vec{k} \wedge d\vec{u} + \frac{\beta}{k} d\vec{p} \wedge d\vec{u} - \frac{\beta}{\mu} d\vec{k} \wedge d\vec{q}
\]

(3.215)

The Poisson brackets of two functions \( H \) and \( f \) on the orbit corresponding to the symplectic form (3.215) is

\[
\{H, f\} = \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial f}{\partial p_i} + \frac{\partial H}{\partial k_i} \frac{\partial f}{\partial u^i} - \frac{\partial H}{\partial u^i} \frac{\partial f}{\partial k_i} - \frac{\beta}{k} \epsilon^{ij} \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial u^j} + \frac{\beta}{\mu} \epsilon^{ij} \frac{\partial H}{\partial q^i} \frac{\partial f}{\partial k_j}
\]

(3.216)

and the nontrivial Poisson brackets within the coordinates are

\[
\{p_j, q^i\} = \delta^i_j, \quad \{k_j, u^i\} = \delta^i_j, \quad \{p_j, u^i\} = \frac{\beta}{k} \epsilon^{ij}, \quad \{q^i, k_j\} = \frac{\beta}{\mu} \epsilon^{ij}, \quad \{p_i, k_j\} = 0, \quad \{q^i, u^j\} = 0
\]

(3.217)

This means that the linear momentum \( \vec{p} \) is canonically conjugate to the position \( \vec{q} \) as well as the static momentum \( \vec{k} \) is canonically conjugate to the the velocity \( \vec{u} \). Moreover the
linear momentum does not commute with the velocity as well as the static momentum does not commute with the position. Note that in this case, \((\tilde{q}', \tilde{u}')\) is an element of the tangent space (evolution space) while \((\tilde{p}, \tilde{k})\) is its dual.

**Symplectic realizations and equations of motion**

Let the symplectic realizations of the noncentrally abelian extended Static group on its coadjoint orbit be given by \((\tilde{p}', \tilde{k}', \tilde{q}', \tilde{u}') = D_{(q,\ell,\theta,\dot{\theta},\ddot{\theta},x,t)}(\tilde{p}, \tilde{k}, \tilde{q}, \tilde{u})\). By using coadjoint action, we verify that

\[
\begin{align*}
\tilde{q}' &= R(\theta)\tilde{q} + \tilde{v}\tau + \frac{k}{k_e}\tilde{x} \\
\tilde{u}' &= R(\theta)\tilde{u} - \nu_e\tilde{x} - \frac{\mu}{\mu_e}\tilde{v} \\
\tilde{p}' &= R(\theta)\tilde{p} - tk_e[R(\theta)\tilde{q} - \tilde{v}\tau - \frac{k}{k_e}\tilde{x}] - m\tilde{v} + \kappa\tilde{q} + \beta\tilde{\tau} \\
\tilde{k}' &= R(\theta)\tilde{k} + t\mu_e[R(\theta)\tilde{u} - \nu_e\tilde{x} - \frac{\mu}{\mu_e}\tilde{v}] + m\tilde{x} + \mu\tilde{u} + \beta\tilde{q}
\end{align*}
\]

\(\tau = \frac{\beta}{k_e}, \nu_e = \frac{\beta}{\mu_e}\) being a duration and a frequency respectively.

The evolution with respect to the time \(t\) is

\[
\tilde{q}(t) = q, \quad \tilde{u}(t) = \tilde{u}
\]

and

\[
\begin{align*}
\tilde{p}(t) &= \tilde{p} - tk_e\tilde{q} \\
\tilde{k}(t) &= \tilde{k} + t\mu_e\tilde{u}
\end{align*}
\]

The equations of motion are then given by

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} \tilde{p}(t) \\ \tilde{k}(t) \end{pmatrix} &= \begin{pmatrix} -k_e\tilde{q} \\ \mu_e\tilde{u} \end{pmatrix}, \quad \frac{d}{dt} \begin{pmatrix} \tilde{q}(t) \\ \tilde{u}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\]

With the planar noncentrally abelian extended Static group, we have obtained a completely noncommutative phase space equipped with modified symplectic structure defined by relation (3.215) and dynamics given by equations (3.221).
3.5 Conclusions and Classification

We can summarize our findings. From the group theoretical discussion above, we see that the coadjoint orbit method applied to the two-parameter central extensions of anisotropic Lie groups and to the noncentral abelian extensions of absolute time Lie groups gives rise to three kinds of noncommutative phase spaces:

- A phase space whose only positions do not commute, i.e: generators of pure kinematical group transformations are noncommutative in extended algebras. This arrives with the Galilei algebra $G$ by considering both the anisotropic and the absolute time cases, recovering the so called exotic model [35].

- A phase space whose only momenta are noncommutative due to the noncommutativity of generators of space translations in extended Lie algebras. This is the case of the Para-Galilei group $G'$, by considering both the anisotropic and the absolute time cases. The same result can be obtained also with the Aristotle group (which is a subgroup of the Galilei group) by considering one of it noncentral abelian extensions.

- A phase space which is completely noncommutative, meaning that both generators of pure kinematical group transformations and generators of space translations do not commute. This is obtained with the Newton-Hooke groups $NH_{\pm}$, the Static group and the Carroll group in both anisotropic and absolute time cases.

Note also that the noncommutative phase spaces realized with noncentral abelian extensions are algebraically more general than those obtained with central extensions. Indeed, we have remarked that with noncentral abelian extensions, it is always possible to realize a noncommutative phase space by the coadjoint orbit method while when uses a central extension, noncommutative phase space shall not exist. This remark has been well explained with the Aristotle group. This will be also clarified in the next Chapter where a comparative analysis between the centrally and the noncentrally abelian extended results will be given. Furthermore, in the noncentrally abelian extended results, we have additional terms in the two-form symplectic structures.

Through these constructions, we have found that each type of noncommutative phase space corresponds to a minimal coupling of:

- the position with a group theoretically (naturally) introduced dual magnetic field in the position noncommutativity sector,

- the momentum with a group theoretically (naturally) introduced magnetic field in the momentum noncommutativity sector,
• the momentum with a group theoretically (naturally) introduced magnetic field and the position with a naturally introduced dual magnetic field in the noncommutativity of positions as well as momenta cases.
The mathematical background for the physics on noncommutative spacetimes is the noncommutative geometry [5]. The idea of noncommutative spacetimes has been used extensively in general classical and quantum field theory [8, 74, 75]. In canonical quantization for instance, one would replace the spacetime coordinates by noncommuting operators and so consider a noncommutative geometry. It was shown how the passage from classical to quantum mechanics or from Newtonian mechanics to special relativity can be understood as a deformation of algebraic structures [2].

A generalization of this concept is to consider deformation of Poissonian structure of spaces. Under some assumptions this generalization yields noncommutative spaces and in the particularly case of phase spaces, i.e described as symplectic manifolds, we obtain noncommutative symplectic structures. Furthermore, as it was demonstrated in [71], the spacetime noncommutativity can be distinguished into three kinds in accordance with the Hopf-algebraic classification [76], that is, there exists the canonical, the Lie-algebraic and quadratic noncommutativity respectively.

In this Chapter, we see in detail the linear deformation of the Poisson bracket as a particular case of the Lie-algebraic noncommutativity and demonstrate that under some conditions, this gives rise to the noncommutative phase spaces constructed group theoretically (i.e by the coadjoint orbit method) in the previous Chapter.

Our generalization focuses on the Poisson brackets between different spatial coordinates and momenta. We point out that the planar systems with (anisotropic) kinemati-
cal group type symmetries are (anisotropic) noncommutative phases spaces in constant magnetic backgrounds.

This Chapter is organized as follows.

In the next section, we determine the Poissonian structures associated to the centrally extended planar anisotropic kinematical groups defined in the first Chapter. By using the Kirillov-Kostant-Souriau bracket, the noncommutative spaces obtained are reduced to noncommutative phase spaces on the maximal coadjoint orbits of the corresponding kinematical groups.

In section two, we determine the Poissonian structures associated to the noncentrally abelian extended absolute time kinematical groups. Through these constructions, it is also highlighted that the noncommutative spaces obtained are algebraically general than those obtained in the centrally extensions framework.

The comparative analysis between the centrally and the noncentrally abelian extended results is given in the next section. Section four is devoted to the linear deformation of the Poisson bracket in a four-dimensional space. With some assumptions, we prove that this gives rise to the same noncommutative phase spaces as those obtained group theoretically, i.e. by the coadjoint orbit method. Finally, section five is devoted to the general case, that is the linear deformation of a $2n$-dimensional space and this provides the fundamental commutation relations and equations of motion of a general noncommutative space.

4.1 Poisson manifolds associated to the planar anisotropic kinematical groups

In Chapter 2, it has already mentioned that the dual of the Lie algebra of a Lie group has a natural Poisson structure (called Kirilov-Kostant-Souriau bracket) given by the relation (2.11). In the following section, we determine Poisson manifolds or Poisson-Lie structures associated to the centrally extended planar anisotropic kinematical groups considered in this thesis.

4.1.1 Poisson-Lie structure associated to Carroll group

Consider the centrally extended Carrollian algebra defined by the brackets (1.45). The dual (Lie algebra) $\mathfrak{g}^*$ of the centrally extended Carrollian Lie algebra equipped with the Poisson structure (2.11) is a Poisson manifold. It is a presymplectic manifold which
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turns in a symplectic structure on the coadjoint orbit where the Poisson structure corresponding to the symplectic structure (3.142) takes the form (3.136).

More specifically, let $X = (\delta v^i, \delta x^i, \delta t, \delta \phi)$ and $\mu = (k_i, p_i, E, h)$ be respectively an infinitesimal displacement on Carrollian centrally extended Lie algebra and an element of its dual Lie algebra such that

$$\langle \mu, X \rangle = k_i \delta v^i + p_i \delta x^i + E \delta t + h \delta \phi$$

(4.1)

is the associated scalar whose physical dimension is action where $\langle ., . \rangle$ stands for the pairing between the Lie algebra and its dual.

The Poisson bracket (3.3) of two functions $f$ and $g$ in $C^\infty(\hat{\mathbb{G}}^*, \mathbb{R})$ is in this case given by:

$$\{f, g\} = -\frac{\hbar}{m^2 c^2} \epsilon_{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} + \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) - \kappa^2 \hbar \epsilon_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j}$$

(4.2)

Then, we obtain a noncommutative space whose coordinates satisfy the following Poisson brackets:

$$\{p_i, p_j\} = -\kappa^2 \hbar \epsilon_{ij}$$
$$\{p_i, q^j\} = \delta_i^j, \{q^i, q^j\} = -\frac{\hbar}{m^2 c^2} \epsilon_{ij}$$
$$\{p_i, E\} = 0, \{q^i, E\} = 0$$

(4.3)

As it has been already concluded, with the Carroll group, a dynamical system which corresponds to a noncommutative space can be constructed and its restriction to the coadjoint orbit permits us to recover the noncommutative phase space obtained in section (3.3.5) and equipped with the symplectic form (3.142).

4.1.2 Poisson-Lie structures associated to centrally extended absolute time anisotropic kinematical groups

The centrally extended absolute time anisotropic kinematical algebras are defined by the nontrivial Lie brackets (1.47) where the parameters are related by the relation (1.48).

The Kirillov form, in the basis $(K_j, P_j, H, M, S)$, is in this case given by:

$$K_{ij}(a) = \begin{pmatrix}
-\frac{\hbar}{m^2 c^2} \epsilon_{ij} & \gamma m \delta_{ij} & \lambda p_j & 0 & 0 \\
-\gamma m \delta_{ij} & \kappa^2 \alpha \hbar \epsilon_{ij} & \beta k_i & 0 & 0 \\
-\lambda p_j & -\beta k_j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
where \( k_i, p_i, E, m, h \) span the corresponding dual Lie algebras.

The Poisson bracket (3.3) of two functions \( f \) and \( g \) on dual Lie algebras is then given collectively by:

\[
\{ f, g \} = -\frac{h}{m^2 c^2 \mu \epsilon_{ij}} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_j} + \gamma \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) - \kappa^2 h \epsilon_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} + \lambda \frac{p_i}{m} \frac{\partial f}{\partial E} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial E} \right)
\]

(4.4)

where we have used the relation (3.88).

We then obtain a noncommutative space whose coordinates satisfy the following Poisson brackets:

\[
\{ p_i, p_j \} = -\kappa^2 \alpha h \epsilon_{ij}
\]

\[
\{ p_i, q^j \} = \gamma \delta^j_i, \quad \{ q^i, q^j \} = -\frac{h}{m^2 c^2 \mu \epsilon_{ij}}
\]

(4.5)

\[
\{ p_i, E \} = -\beta m q^i, \quad \{ q^i, E \} = -\lambda \frac{p_i}{m}
\]

Using the normalization \( \gamma = 1 \), we distinguish the following cases:

- \( \alpha = -1, \beta = \pm \omega^2 \) and \( \lambda = \mu = 1 \):

In this case, (4.4) becomes

\[
\{ f, g \} = -\frac{h}{m^2 c^2 \mu \epsilon_{ij}} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q^j} + \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} + \kappa^2 h \epsilon_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} + \lambda \frac{p_i}{m} \frac{\partial f}{\partial E} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial E} \right)
\]

(4.6)

and then

\[
\{ p_i, p_j \} = -\kappa^2 h \epsilon_{ij}
\]

\[
\{ p_i, q^j \} = \delta^j_i, \quad \{ q^i, q^j \} = -\frac{h}{m^2 c^2 \epsilon_{ij}}
\]

(4.7)

\[
\{ p_i, E \} = \mp \omega^2 q^i, \quad \{ q^i, E \} = -\frac{p_i}{m}
\]

The system above provides planar noncommutative spaces on Newton-Hooke groups and their restrictions to the coadjoint orbits give rise to the noncommutative phase spaces obtained in section (3.3.3) i.e equipped with the modified symplectic structures given by (3.70).
\[ \{ f, g \} = -\frac{\hbar}{m^2 c^2} \epsilon_{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} + \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) + \frac{p_i}{m} \left( \frac{\partial f}{\partial E} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial E} \right) \]

and then

\[ \{ p_i, p_j \} = 0 \]
\[ \{ p_i, q^j \} = \delta^j_i \]
\[ \{ q_i, q^j \} = 0 \]
\[ \{ p_i, E \} = 0 \]

The system above provides planar noncommutative space on Galilei group and its restriction to the orbit gives rise to the noncommutative phase space obtained in section (3.3.1) i.e. equipped with the modified symplectic structure given by (3.101).

\[ \{ f, g \} = \{ p_i, q^j \} = 0 \]
\[ \{ q_i, q^j \} = \frac{\hbar}{m^2 c^2} \epsilon_{ij} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} \]

The system above provides planar noncommutative spaces on Para-Galilei groups and their restrictions to the orbit give rise to the noncommutative phase spaces obtained in section (3.3.2) i.e. equipped with the modified symplectic structures given by (3.118).

\[ \{ f, g \} = \{ p_i, p_j \} = \frac{\hbar}{m^2 c^2} \epsilon_{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} \]

The system above provides planar noncommutative spaces on Para-Galilei groups and their restrictions to the orbit give rise to the noncommutative phase spaces obtained in section (3.3.2) i.e. equipped with the modified symplectic structures given by (3.118).

\[ \{ f, g \} = \{ p_i, E \} = \frac{\hbar}{m^2 c^2} \epsilon_{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} \]

The system above provides planar noncommutative spaces on Para-Galilei groups and their restrictions to the orbit give rise to the noncommutative phase spaces obtained in section (3.3.2) i.e. equipped with the modified symplectic structures given by (3.118).

\[ \{ f, g \} = \{ p_i, p_j \} = \frac{\hbar}{m^2 c^2} \epsilon_{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} \]

The system above provides planar noncommutative spaces on Para-Galilei groups and their restrictions to the orbit give rise to the noncommutative phase spaces obtained in section (3.3.2) i.e. equipped with the modified symplectic structures given by (3.118).
and then

\[ \{ p_i, p_j \} = -\kappa^2 \hbar \epsilon_{ij} \]
\[ \{ p_i, q^j \} = \delta_i^j, \quad \{ q^i, q^j \} = -\frac{\hbar}{m^2 c^2} \epsilon^{ij} \]
\[ \{ p_i, E \} = 0, \quad \{ q^i, E \} = 0 \]  

(4.12)

The system above provides planar noncommutative space on Static group and its restriction to the orbit gives rise to the noncommutative phase space obtained in section (3.3.4) i.e equipped with the modified symplectic structures given by (3.134).

### 4.2 Poisson-Lie structures associated to noncentrally abelian extended absolute time kinematical groups

Noncentrally extensions of absolute time kinematical groups have been defined in the last section of the first Chapter.

In this section, we determine their associated Poisson structures and hence the corresponding noncommutative spaces. In general, the obtained structures are those obtained in the previous sections i.e with central extensions of anisotropic kinematical groups with additional terms due to the presence of rotational generator (isotropy case).

For example, for the noncentrally abelian extended Newton-Hooke absolute time Lie algebras defined by the Lie brackets (1.53), the associated Poissonian structures are given by

\[ \{ f, g \} = \{ f, g \} + p_j \epsilon^{ij}(\frac{\partial f}{\partial j} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial j}) + q^i \epsilon_{ij}(\frac{\partial f}{\partial j} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial j}) \]

where the Kirillov form in the basis \((J, K_i, P_i, H, M, S)\) is given by

\[ (K_{ij}) = \begin{pmatrix}
0 & k_i \epsilon_{ij} & p_i \epsilon_{ij} & 0 & 0 & 0 \\
-\kappa^2 \epsilon_{ij} & m \delta_{ij} & p_i & 0 & 0 & 0 \\
\kappa^2 \epsilon_{ij} & -p_j & -m \delta_{ij} & \pm \omega^2 k_i & 0 & 0 \\
0 & p_j & \mp \omega^2 k_j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

and where we have used the relation \((3.88)\).
Poisson-Lie structures associated to noncentrally abelian extended absolute time kinematical groups

It holds that:
\[
\{j, p_k\} = p_i \epsilon^i_k, \quad \{p_i, p_j\} = -\kappa^2 h \epsilon_{ij} \\
\{j, q^k\} = q^i \epsilon^i_k, \quad \{p_i, q^j\} = \delta^i_j, \quad \{q^i, q^j\} = -\frac{h}{m^2 c^2} \epsilon^{ij} \\
\{j, E\} = 0, \quad \{p_i, E\} = \mp m \omega q^i, \quad \{q^i, E\} = -\frac{p_i}{m}
\] (4.13)

For the noncentrally abelian extended Galilei absolute time Lie algebra defined by the Lie brackets (1.52), the associated Poissonian structure is
\[
\{f, g\} = p_i \epsilon^i_j \frac{\partial f}{\partial p_j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_j} + q^i \epsilon^j_i \left( \frac{\partial f}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial q^j} \right) + \pi_i \epsilon^j_i \left( \frac{\partial f}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} \frac{\partial g}{\partial \pi_j} \right)
\]
where the Kirillov form in the basis \((J, K_i, P_i, F_i, H, M, S)\) is given by
\[
(K_{ij}) = \left(\begin{array}{ccccccc}
0 & k_i \epsilon^i_j & p_i \epsilon^i_j & f_i \epsilon^i_j & 0 & 0 & 0 \\
-k_j \epsilon^j_i & \frac{h}{c^2} \epsilon^{ij} & m \delta_{ij} & 0 & p_i & 0 & 0 \\
-p_j \epsilon^j_i & -m \delta_{ij} & 0 & 0 & f_i & 0 & 0 \\
-f_j & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -p_j & -f_j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\]
and where we have used the relation (3.88).

It holds that:
\[
\{j, p_i\} = p_k \epsilon^k_i, \quad \{p_i, p_j\} = 0 \\
\{j, q^i\} = q^k \epsilon^k_i, \quad \{p_i, q^j\} = \delta^i_j, \quad \{q^i, q^j\} = -\frac{h}{m^2 c^2} \epsilon^{ij} \\
\{j, f_i\} = f_k \epsilon^k_i, \quad \{p_i, f_j\} = 0, \quad \{q^i, f_j\} = 0, \quad \{f_i, f_j\} = 0 \\
\{j, E\} = 0, \quad \{p_i, E\} = -f_i, \quad \{q^i, E\} = -\frac{p_i}{m}, \quad \{f_i, E\} = 0
\] (4.14)

For the noncentrally abelian extended Para-Galilei absolute time Lie algebras defined by the Lie brackets (1.54), the associated Poissonian structures are given by
\[
\{f, g\} = p_i \epsilon^i_j \frac{\partial f}{\partial p_j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_j} + q^i \epsilon^j_i \left( \frac{\partial f}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial q^j} \right) + \pi_i \epsilon^j_i \left( \frac{\partial f}{\partial \pi_j} - \frac{\partial f}{\partial \pi_j} \frac{\partial g}{\partial \pi_j} \right)
\]
where the Kirillov form in the basis \((J, K_i, P_i, \Pi_i, H, M, S)\) is given by

\[
(K_{ij}) = \begin{pmatrix}
0 & k_i \epsilon_j^i & p_i \epsilon_j^i & \pi_i \epsilon_j^i & 0 & 0 & 0 \\
-k_i \epsilon_j^i & 0 & m \delta_{ij} & 0 & \pi_i & 0 & 0 \\
-\pi_i \epsilon_j^i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\pi_j & \mp \omega^2 k_i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and where we have used the relation (3.88).

It holds that:

\[
\{j, p_i\} = p_i \epsilon_j^k, \quad \{p_i, p_J\} = -\kappa^2 \epsilon_{ij}
\]

\[
\{j, q^i\} = q^k \epsilon_j^k, \quad \{p_i, q^i\} = \delta^i_J, \quad \{q^i, q^j\} = 0
\]

\[
\{j, \pi_i\} = \pi_k \epsilon_j^k, \quad \{p_i, \pi_j\} = 0, \quad \{q^i, \pi_j\} = 0, \quad \{\pi_i, \pi_j\} = 0
\]

\[
\{j, E\} = 0, \quad \{p_i, E\} = \mp m \omega^2 q^i, \quad \{q^i, E\} = -\frac{\pi_i}{m}, \quad \{\pi_i, E\} = 0
\]

For the noncentrally abelian extended Static absolute time Lie algebra defined by the Lie brackets (1.55), the associated Poissonian structures are given by

\[
\{f, g\} = p_i \epsilon_j^i \left( \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial q_j} \right) + q^i \epsilon_j^j \left( \frac{\partial f}{\partial \pi_j} \frac{\partial g}{\partial \pi_j} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^i} \right) + \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^i} \right)
\]

\[
+ \kappa \left( \frac{\partial f}{\partial \Pi_i} \frac{\partial g}{\partial \Pi_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_i} \right) + B \left( \frac{\partial f}{\partial \Pi_i} \frac{\partial g}{\partial \Pi_i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^i} \right) + m' \left( \frac{\partial f}{\partial \Pi_i} \frac{\partial g}{\partial \Pi_i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^i} \right)
\]

\[
+ \frac{\pi}{m} \left( \frac{\partial f}{\partial E} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial E} \right) + f_i \left( \frac{\partial f}{\partial E} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial E} \right)
\]

where the Kirillov form in the basis \((J, K_i, P_i, F_i, \Pi_i, H, M, S, \Lambda, M')\) is given by

\[
(K_{ij}) = \begin{pmatrix}
0 & k_i \epsilon_j^i & p_i \epsilon_j^i & f_i \epsilon_j^j & \pi_i \epsilon_j^i & 0 & 0 & 0 & 0 & 0 \\
-k_i \epsilon_j^i & 0 & m \delta_{ij} & B \delta_{ij} & m' \delta_{ij} & \pi_i & 0 & 0 & 0 & 0 \\
f_i \epsilon_j^i & -B \delta_{ij} & -\kappa \delta_{ij} & B \delta_{ij} & f_i & 0 & 0 & 0 & 0 & 0 \\
-\pi_i \epsilon_j^i & -m' \delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\pi_j & -f_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Comparative analysis of the centrally and the noncentrally abelian extended results

and where we have used the relation (3.88)

It holds that:

\[
\begin{align*}
\{j, p_i\} &= p_k \epsilon^k_i, \quad \{p_i, p_j\} = -\kappa^2 \hbar \epsilon_{ij} \\
\{j, q^i\} &= q^k \epsilon^k_i, \quad \{p_i, q^j\} = \delta^j_i, \quad \{q^i, q^j\} = 0 \\
\{j, f_i\} &= f_k \epsilon^k_i, \quad \{p_i, f_j\} = -\kappa \delta_{ij}, \quad \{q^i, f_j\} = -\frac{B}{m} \delta^j_i, \quad \{f_i, f_j\} = 0 \\
\{j, \pi_i\} &= \pi_k \epsilon^k_i, \quad \{p_i, \pi_j\} = -B \delta_{ij}, \quad \{q^i, \pi_j\} = -\frac{m}{m} \delta^j_i, \quad \{f_i, \pi_j\} = 0, \quad \{\pi_i, \pi_j\} = 0 \\
\{j, E\} &= 0, \quad \{p_i, E\} = -f_i, \quad \{q^i, E\} = -\frac{\pi_i}{m}, \quad \{f_i, E\} = 0, \quad \{\pi_i, E\} = 0
\end{align*}
\] (4.16)

Let us compare the results of the two previous sections in order to highlight the fact that noncommutative spaces constructed on kinematical groups by considering their noncentral abelian extensions are algebraically more general than those obtained with central extensions.

4.3 Comparative analysis of the centrally and the noncentrally abelian extended results

Note that the motion equations are given by the last line in each system of the Poisson brackets (4.13), (4.14), (4.15) and (4.16).

First of all in the centrally extended cases, the momentum \(p_i\) changes due to a position dependent force while the position \(q^i\) changes due to a momentum dependent velocity in the Newton-Hooke groups case, it is only the position \(q^i\) which changes due to a momentum dependent velocity in the Galilei group case, it is only the momentum \(p_i\) which changes due to a position dependent force in the Para-Galilei groups case and the position \(q^i\) as well the momentum \(p_i\) are constant in the Static and Carroll groups cases.

In the noncentrally abelian extended groups cases, the momentum \(p_i\) which was constant in the Galilei and Static cases is now changing due to a constant force \(f_i\), the position \(q^i\) which was constant in the Para-Galilei and Static cases is now changing due to a constant momentum \(\pi_i\). Moreover the first column of (4.13), (4.14), (4.15) and (4.16) shows that the momenta \(p_i\) and \(\pi_i\), the positions \(q^i\) and the forces \(f_i\) behave as components of vectors while the energy \(E\) behaves as a scalar under rotation through the Poisson brackets with an angular momentum \(j\).
Furthermore, comparing relations: \((4.7)\) and \((4.13)\) (for Newton-Hooke groups case), \((4.8)\) and \((4.14)\) (for Galilei group case), \((4.10)\) and \((4.15)\) (for Para-Galilei groups case), \((4.12)\) and \((4.16)\) (for Static group case), we can conclude that noncentrally abelian extended results are algebraically more general than those obtained with centrally extended structures.

### 4.4 Possible four-dimensional noncommutative phase spaces by linear deformation

Let us consider the phase space with coordinates \((p_1, p_2, q_1, q_2)\) such that the Poisson brackets be given by:

\[
\{p_i, p_j\} = (a_k q^k) \epsilon_{ij} + n \epsilon_{ij}, \quad \{q^i, q^j\} = (p_k b^k) \epsilon^{ij} + d \epsilon^{ij}, \quad \{p_j, q^i\} = \delta^i_j
\]  

and such that the equations of motion are

\[
\{E, p_i\} = w_{ik} q^k, \quad \{E, q^i\} = p_k r^{ki}
\]

It is a linear deformation (of the form Lie-algebraic deformation given in \([71]\)) of the canonical Poisson brackets of phase space coordinates.

The dimensions of the structure constants also called the noncommutative parameters are fixed: \([a_k] = M L^{-1} T^{-1}, [b^k] = M^{−2} L T^2, [w_{ij}] = M T^{-2}, [r^{ij}] = M^{-1}, [n] = M T^{-1}, [d] = M^{-1} T\) where \(M, L\) and \(T\) stand for mass, length and time respectively.

The Jacobi identities with two \(p\)’s and one \(q\) give rise to the constraints \(a_k = 0\). The Jacobi identities with two \(q\)’s and one \(p\) give rise to the constraints \(b^k = 0\). The Jacobi identity with two \(p\)’s and \(H\) adds that the matrix \((W = (w_{ij}))\) is symmetric while that with two \(q\)’s and \(E\) adds that the matrix \((R = (r^{ij}))\) is also symmetric.

Lastly the Jacobi identities with one \(p\), one \(q\) and \(E\) imply that the two matrices \((w_{ij})\) and \((r^{ij})\) are related by

\[
dw_{kl} = n \epsilon_{kl} \epsilon_{ij} r^{ij}
\]  

or equivalently:

\[
d \begin{pmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{pmatrix} = n \begin{pmatrix} -r^{22} & r^{12} \\ r^{12} & -r^{11} \end{pmatrix}
\]

As the matrices \(W\) and \(R\) are diagonalizable, let us suppose that \(W = diag(w, w)\) and \(R = diag(\frac{1}{m}, \frac{1}{m})\) respectively.
Then the Poisson brackets (4.17) become

\[
\{ p_i, p_j \} = n \epsilon_{ij}, \quad \{ q^i, q^j \} = d \epsilon^{ij}, \quad \{ p_j, q^i \} = \delta^i_j
\]

(4.20)

while the equations of motion (4.18) are written as:

\[
\{ E, p_i \} = -\frac{n}{md} \delta_{ij} q^j, \quad \{ E, q^i \} = p_j \frac{m}{d} \delta_{ij}
\]

(4.21)

We distinguish five interesting cases according to the possible values of the nontrivial noncommutative parameters, i.e: case I stands for \( n \neq 0, d \neq 0, r \neq 0 \), case II stands for \( n \neq 0, d \neq 0, r = 0 \), case III stands for \( n \neq 0, d = 0, r \neq 0 \), case IV stands for \( n = 0, d \neq 0, r \neq 0 \) and case V stands for \( n = 0, d = 0, r = 0 \), as summarized in the following table:

|        | case I | case II | case III | case IV | case V |
|--------|--------|---------|----------|---------|--------|
| \{ p_i, p_j \} | n \epsilon_{ij} | n \epsilon_{ij} | n \epsilon_{ij} | 0 | 0 |
| \{ q^i, q^j \} | d \epsilon^{ij} | d \epsilon^{ij} | 0 | d \epsilon^{ij} | 0 |
| \{ p_j, q^i \} | \delta^i_j | \delta^i_j | \delta^i_j | \delta^i_j | \delta^i_j |
| \{ E, p_i \} | \frac{p_j}{m} \delta^{ij} | 0 | 0 | \frac{p_j}{m} \delta^{ij} | \frac{p_j}{m} \delta^{ij} |
| \{ E, q^i \} | -\frac{n}{md} \delta_{ij} q^j | 0 | w q^j | 0 | w q^j |

Table 4.1: Lie-algebraic noncommutative phase spaces in a two-dimensional space

Setting \( n = eB, d = e^* B^* \) and \( w = \pm m \omega^2 \) where \( B \) and \( B^* \) are the magnetic and dual magnetic fields, \( e \) is the electric charge while \( e^* \) is its dual charge, then the case I corresponds to the two anisotropic Newton-Hooke noncommutative phase spaces (i.e by considering the central extensions of the two Newton-Hooke groups), the case II corresponds to the both anisotropic Static and Carrollian noncommutative phase spaces with the assumption that \( E = mc^2 \) (i.e Einstein’s formula) in the Carroll group case. The case III corresponds to the anisotropic Para-Galilean noncommutative phase spaces while case IV corresponds to the anisotropic Galilean noncommutative phase space. Finally, the case V corresponds to the canonical one (commutative case).

Furthermore, using the wave-particle duality (3.98) and the equality \( c = \frac{\hbar}{\kappa} \) in relations (4.5), (4.7), (4.8), (4.10), (4.12), the above results become equivalent to the Poissonian structures constructed in the previous section and their restrictions to the corresponding coadjoint orbits give rise to the same symplectic structures as those constructed group theoretically.
We have thus proved the following theorem:

**Theorem 4.1.** *The planar systems with anisotropic kinematical group type symmetries are noncommutative anisotropic phase spaces in uniform magnetic backgrounds. Full planar kinematical group symmetries (with rotation) give rise to noncommutative phase spaces in the isotropic case.*

As we have already seen, we can realize these noncommutative phase spaces by the coadjoint orbit method or, in the anisotropic case, by linear deformation of the Poisson bracket. Every type corresponds to a minimal coupling.

Among them, the most general ones are the anisotropic noncommutative Newton-Hooke phase spaces (case I in the above table) which model effectively an anisotropic oscillator. This result recovers the result obtained in [52] in the oscillating Newton-Hooke group case.

### 4.5 2n-dimensional possible noncommutative phase spaces by linear deformation

We start with Poisson brackets of coordinates on a $2n$-dimensional noncommutative phase space ($n \geq 3$) defined by

$$\{p_i, p_j\} = a_{ijk} q^k + \alpha_{ij} , \{p_j, q^i\} = \delta_j^i , \{q^j, q^j\} = p^k b^{kij} + \beta^{ij}$$

(4.22)

where the structure constants $a_{ijk}, b^{kij}, \alpha_{ij}, \beta^{ij}$ also called the noncommutative parameters are characterized by

$$a_{ijk} = -a_{jik} , b^{kij} = -b^{kji} , \alpha_{ij} = -\alpha_{ji} , \beta^{ij} = -\beta^{ji}$$

(4.23)

We also suppose that the motion equations are

$$\{E, p_i\} = \lambda_{ik} q^k , \{E, q^i\} = p_k r^{ki}$$

(4.24)

$\lambda_{ij}$ and $r^{ij}$ being also constants. Note that $E$ stands for the Hamiltonian function or the total energy.

The Jacobi identity with three $p$ implies the constraint equations

$$a_{ijk} + a_{jki} + a_{kij} = 0$$

(4.25)

while that with three $q$ implies

$$b^{ijk} + b^{jki} + b^{kij} = 0$$

(4.26)
The Jacobi identity with two $p$ and one $q$ implies that
\[ a_{ijkl} \beta^{lk} = 0 , \ a_{ijkl} b^{mk} = 0 \] (4.27)
for each fixed value of $m$. Similarly the Jacobi identity with two $q$ and one $p$ implies that
\[ \alpha_{il} b^{lk} = 0 , \ a_{ilm} b^{jk} = 0 \] (4.28)
for each fixed value of $m$.

The Jacobi identity with two $p$ and $E$ implies that
\[ \lambda_{ij} = \lambda_{ji} , \ a_{ijk} r^{kl} = 0 \] (4.29)
for each fixed value of $l$. Similarly, the Jacobi identity with two $q$ and $E$ implies that
\[ r^{ij} = r^{ji} , \ \lambda_{kl} b^{kj} = 0 \] (4.30)
for each fixed value of $l$.

Finally the Jacobi identity with one $p$, one $q$ and $E$ implies that
\[ \alpha_{ik} r^{kj} = \lambda_{ik} \beta^{kj} , \ a_{ijk} r^{kl} = 0 , \ \lambda_{ik} b^{kj} = 0 \] (4.31)
for each fixed value of $l$. From the symmetry of $r^{ij}$ and $\lambda_{ij}$, we can set that
\[ r^{ij} = \frac{1}{m} \delta^{ij} , \ \lambda_{ij} = w \delta_{ij} . \]

Then, it follows that $\alpha_{ij} = mw \beta^{ij}$ and that $a_{ijk} = b^{ijk} = 0$.

The Poisson brackets (4.22) become in this case
\[ \{ p_i , p_j \} = mw \beta_{ij} , \ \{ p_j , q^i \} = \delta^i_j , \ \{ q^i , q^j \} = \beta^{ij} \] (4.32)
while the motion equations (4.24) become
\[ \{ E, p_i \} = w q^i , \ \{ E, q^i \} = p_i \] (4.33)

The Poisson brackets (4.32) realize the case where the momenta as well the positions do not commute, the momenta change due to a position dependent force while the positions change due to a momenta dependent velocity. This provides the general commutation relations for the $2n$-dimensional noncommutative space where $mw \beta_{ij}$ and $\beta^{ij}$ are the noncommutative parameters.
Conclusion and outlook

To conclude, let us revisit the main results we can across in this thesis. Recall that the main focus was to provide a detailed investigation on noncommutative phase spaces by group theory in a universe with two space and one time dimensions. We have first provided a detailed account of the algebraic structures of the centrally and noncentrally abelian extended planar kinematical groups (which act transitively on manifolds). We then went on to describe the construction of noncommutative phase spaces by introducing minimal couplings and turning to the extended structures above, we applied the coadjoint orbit method to construct and classify noncommutative phase spaces of planar anisotropic and absolute time kinematical Lie groups. We have distinguished between the following three cases: noncommutative phase spaces whose only positions do not commute, noncommutative phase spaces whose only momenta do not commute and noncommutative phase spaces whose both positions and momenta do not commute. Each case was realized group theoretically and corresponded to a specific minimal coupling. We have proved also that the group theoretical discussion, which allows the study of dynamical systems and in the same time eliminates the non minimal couplings in them, can be compared with the linear deformation of the Poisson bracket in a $2n$-dimensional phase space.

We have therefore described the methodology and the details relative to noncommutative phase spaces in $(2 + 1)$—dimensional spaces.

While we hope this analysis is complete, it would be unfair to label it fully conclusive. There are some open questions that the author feels are left unanswered and therefore deserve further researching. In particular, we would like to mention three of them:

- Since the kinematical groups discussed here are contractions from the de Sitter or the anti-de Sitter group and therefore still related themselves through the contraction process [38], it would be interesting to verify if the extended structures and hence the obtained noncommutative phase spaces are also related via contractions;
• It would be also a great step to find a star product corresponding to the noncommutative momenta sector. Indeed, it has been argued that when the momenta of the phase space do not commute, one would have to envisage the corresponding star-product which is still unknown [77].

• Noncommutative phase spaces constructed on some kinematical groups in this thesis provide an interpretation of the modified Newton’s equations (e.g: as a damping force [51]). It would also be important to see a more in depth analysis of the associated dynamics and possible physical applications, as it has been done with the Hill’s problem for the Newton-Hooke groups [27].
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