Unifying the theory of Integration within normal-, Weyl- and antinormal-ordering of operators and the s–ordered operator expansion formula of density operators

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By introducing the $s$–parameterized generalized Wigner operator into phase-space quantum mechanics we invent the technique of integration within $s$–ordered product of operators (which considers normal ordered, antinormally ordered and Weyl ordered product of operators as its special cases). The $s$–ordered operator expansion (denoted by $\§ \cdots \§$) formula of density operators is derived, which is

$$\rho = \frac{2}{1 - s} \int \frac{d^2 \beta}{\pi} \langle - \beta | \rho | \beta \rangle \exp \left\{ \frac{2}{s - 1} \left( |s\beta|^2 - \beta^* a + \beta a^\dagger - a^\dagger a \right) \right\},$$

The $s$–parameterized quantization scheme is thus completely established.

**Keywords:** $s$–parameterized generalized Wigner operator, technique of integration within $s$–ordered product of operators, $s$–ordered operator expansion formula, $s$–parameterized quantization scheme

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I. INTRODUCTION

The subject about operators and their classical correspondence has been a hot topic since the birth of quantum mechanics (QM) and now becomes a field named QM in phase space. Because Heisenberg’s uncertainty principle prohibits the notion of a system being described by a point in phase space, only domains of minimum area $2\pi \hbar$ in phase space is allowed. Wigner [1] introduced a function whose marginal distribution gives probability of a particle in coordinate space or in momentum space, respectively. The Wigner distribution is related to operators’ Weyl ordering (or Weyl quantization scheme) [2]. We notice that each phase space distribution is associated with a definite operator ordering for quantizing classical functions. For examples, P-representation (as a density operator $\rho$’s classical correspondence) is actually $\rho$’s antinormally ordered expansion in terms of the completeness of coherent state $|z\rangle = \exp[-\frac{|z|^2}{2} + za^\dagger] |0\rangle$ [3].

$$\rho = \int \frac{d^2 z}{\pi} P(z) |z\rangle \langle z|$$

because the coherent states compose a complete set $\int \frac{d^2 z}{\pi} |z\rangle \langle z| = 1$ [3]. The Wigner distribution function $W(p, x)$ of $\rho$, defined as $Tr [\rho \Delta (p, x)]$, is proportional to the classical Weyl correspondence $h(p, x)$ of $\rho$ ($\rho$’s Weyl ordered expansion), i.e.,

$$\rho = \int_{-\infty}^{\infty} dp dx \Delta(p, x) h(p, x),$$

$$Tr [\rho \Delta (p, x)] = (2\pi)^{-1} h(p, x) = W(p, x).$$

since the Wigner operator $\Delta(p, x)$ is complete too, $\int_{-\infty}^{\infty} dp dx \Delta(p, x) = 1$. The original form of $\Delta(p, x)$ defined in the coordinate representation is [6]

$$\Delta(x, p) = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{iup} \left| x + \frac{u}{2} \right\langle x - \frac{u}{2} \right|,$$
for the Wigner operator in the entangled state representation we refer to [2]. When \( \rho = \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) X^{m-l} P^n X^l \), \([X, P] = i, \ h = 1\), according to Eqs. (3)-(4), the classical correspondence of \( \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) X^{m-l} P^n X^l \) is

\[
2\pi \text{Tr} \left[ \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) X^{m-l} P^n X^l \Delta (x, p) \right]
\]

\[
= \int_{-\infty}^{\infty} du e^{ipu} \left\langle x - \frac{u}{2} \right| \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \frac{m!}{l!(m-l)!} X^{m-l} P^n X^l | x + \frac{u}{2} \rangle
\]

\[
x^m \int_{-\infty}^{\infty} du e^{ipu} \int_{-\infty}^{\infty} dp' e^{-ip'u} P^n
\]

\[
x^m \int_{-\infty}^{\infty} dp' \delta (p - p') P^n
\]

\[
x^m P^n,
\]

this is the original definition of Weyl quantization scheme (quantizing classical coordinate and momentum quantity \( x^m p^n \) as the corresponding operators) as [2]

\[
x^m p^n \rightarrow \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) X^{m-l} P^n X^l ,
\]

its right-hand side is in Weyl ordering, so we introduce the symbol \( \vdash \vdash \) to characterize it [8], i.e.,

\[
\left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) X^{m-l} P^n X^l = \vdash \vdash \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) X^{m-l} P^n X^l \vdash \vdash .
\]

It is worth emphasizing that the order of operators \( X \) and \( P \) are permuted within the Weyl ordering symbol [8], a useful property which has been overlooked for a long time. Based on this fact a useful method called integration within Weyl ordered product of operators has been invented [8].

Therefore, from Eq. (6) and Eq. (7)

\[
x^m p^n \rightarrow \vdash \vdash \left( \frac{1}{2} \right)^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) X^{m-l} P^n X^l \vdash \vdash = \vdash \vdash X^m P^n \vdash \vdash .
\]

Following Eq. (11) we have

\[
\vdash \vdash X^m P^n \vdash \vdash = \int_{-\infty}^{\infty} dp dx \Delta (x, p) x^m p^n,
\]

which implies \( \Delta (x, p) = \vdash \delta (x - X) \delta (p - P) \vdash \), or \( \Delta (\alpha) = \vdash \delta (\alpha + a^\dagger) \delta (\alpha - a) \vdash \), \( \alpha = (x + ip) / \sqrt{2} \), a delta operator-function form in Weyl ordering.

Having realized that each phase space distribution accompanies a definite operator ordering for quantizing classical functions, we may think of that each complete set of operators corresponds to an operator-ordering rule. In this work we shall introduce a complete set of operators characteristic of a \( s \)-parameter (the generalized Wigner operator) and then introduce a generalized quantization scheme with the \( s \)-parameter operator ordering. Historically, Cahill and Glauber [9] have introduced the \( s \)-parameterized quasiprobability distribution according to which the coherent state expectation of \( \rho \), the Wigner function of \( \rho \), and the P-representation of \( \rho \) respectively corresponds to three distinct values of \( s \), i.e., \( s = 1, 0, -1 \). However, the \( s \)-parameterized quantization scheme associated with the \( s \)-parameterized quasiprobability distribution has not been completely established, as the fundamental problem of what is \( \rho \)'s \( s \)-ordered operator expansion has not been touched yet. In another word, the problem of how to arrange any given operator as its \( s \)-ordered form has been unsolved, say for instance, no references has ever reported what is the \( s \)-ordered operator expansion of \( e^{\text{exp} (\lambda a^\dagger a)} \)? \( (a, a) = 1 \) In this work we shall solve this important problem by introducing the technique of integration within \( s \)-ordering of operators, which in the cases of

\[\vdash \vdash X^m P^n \vdash \vdash = \int_{-\infty}^{\infty} dp dx \Delta (x, p) x^m p^n,\]
theory. By analogy with the usual Wigner operator \[6\] we introduce a generalized Wigner operator for distributions. For this purpose we should introduce a generalized Wigner operator for the $s$–parameterized phase space correspondence between an operator and its classical correspondence in the sense of $s$–parameterized quasiprobability $s$-follows: In Sec. 2 we introduce the explicit which is the pure coherent state, $|\alpha\rangle$, $\langle x| = (x + ip)/\sqrt{2}$. In Sec. 4 we introduce the symbol $\delta \cdots \delta$ denoting $s$–ordering of operators and the technique of integration within $s$–ordered product of operators. In Sec. 5-6 we derive density operator’s expansion formula in terms of $s$–ordered quantization scheme, such that the $s$-ordered expansion of $\exp(\lambda a^\dagger a)$ is obtained. In this way we develop and enrich the theory of phase space quantum mechanics.

II. THE $S$–PARAMETERIZED WIGNER OPERATOR AND QUANTIZATION SCHEME

Our aim is to construct $s$–parameterized quantization scheme, in another word, we want to construct a one-to-one correspondence between an operator and its classical correspondence in the sense of $s$–parameterized quasiprobability distribution. For this purpose we should introduce a generalized Wigner operator for the $s$–parameterized phase space theory. By analogy with the usual Wigner operator \[6\] we introduce a generalized Wigner operator for $s$–parameterized distributions,

$$\Delta_s (\alpha) = \int \frac{d^2 \beta}{2\pi^2} \exp \left( \frac{s|\beta|^2}{2} + \beta a^\dagger - \beta^* a - \beta^* \alpha + \beta^* \alpha \right).$$

(10)

Using the Baker-Hausdorff formula to put the exponential in normally ordered form, and using the technique of integration within normal product of operators \[10, 11\], for $s < 1$, we obtain

$$\Delta_s (\alpha) = \int \frac{d^2 \beta}{2\pi^2} : \exp \left[ -\frac{(1-s)|\beta|^2}{2} + \beta a^\dagger - \beta^* a - \beta^* \alpha + \beta^* \alpha \right] :$$

$$= \frac{1}{(1-s)\pi} : \exp \left[ -\frac{2}{1-s} (a^\dagger - \alpha^*)(a - \alpha) \right] : ,$$

(11)

this is named $s$–parameterized Wigner operator. In particular, when $s = 0$, Eq. (11) reduces to the usual normally ordered Wigner operator \[12\]

$$\Delta_s (\alpha) \rightarrow \Delta (\alpha) = \frac{1}{\pi} : \exp \left[ -2 (a^\dagger - \alpha^*) (a - \alpha) \right] :$$

$$= \frac{1}{\pi} : \exp \left[ -(x - X)^2 + (p - P)^2 \right] : ,$$

(12)

where $X = \frac{a^\dagger + a}{\sqrt{2}}$, $P = \frac{i(a^\dagger - a)}{\sqrt{2}}$. On the other hand, by putting the exponential in (11) within antinormal ordering symbol $\hat{\cdots}$, we have for $s < -1$,

$$\Delta_{s=-1} (\alpha) = : \delta (a^\dagger - \alpha^*) \delta (a - \alpha) : = \delta (a - \alpha) \delta (a^\dagger - \alpha^*)$$

$$= |\alpha\rangle \langle \alpha| ,$$

(14)

which is the pure coherent state, $|\alpha\rangle = \exp \left( -\frac{1}{2} |\alpha|^2 + \alpha a^\dagger \right) |0\rangle$. Using

$$: \exp \left[ (e^\lambda - 1) a^\dagger a \right] : = e^{\lambda a^\dagger a} ,$$

(15)

we can convert (11) to the form

$$\Delta_s (\alpha) = \frac{1}{(1-s)\pi} e^{\frac{s}{s-1} a^\dagger a} : \exp \left[ \left( \frac{s+1}{s-1} - 1 \right) a^\dagger a \right] : e^{\frac{s}{s-1} \alpha^* a - \frac{s}{s-1} |\alpha|^2}$$

$$= \frac{1}{(1-s)\pi} e^{\frac{s}{s-1} a^\dagger a} e^{a^\dagger a \ln \frac{s}{s-1}} e^{\frac{s}{s-1} \alpha^* a - \frac{s}{s-1} |\alpha|^2} .$$

(16)
III. THE $s$–PARAMETERIZED QUANTIZATION SCHEME

It follows from (11) that

$$2 \int d^2 \alpha \Delta_s (\alpha) = \frac{2}{(1-s) \pi} \int d^2 \alpha : \exp \left[ \frac{-2}{1-s} (a^\dagger - \alpha^*) (a - \alpha) \right] : = 1,$$

so $\Delta_s (\alpha)$ is complete and $\rho$ can be expanded as

$$\rho = 2 \int d^2 \alpha \Delta_s (\alpha) \Psi (\alpha),$$

which is a new classic–quantum mechanical correspondence between $\Psi (\alpha)$ and $\rho$, when $s = 0$, yields the Weyl correspondence. By noting the form of $\Delta_s (\alpha')$ and using $\int \frac{d^2 z}{\pi} |z| = 1$ we calculate

$$Tr [\Delta_s (\alpha') \Delta_s (\alpha)] = G \int \frac{d^2 z}{\pi} |z| e^{\frac{2}{1+s} a^\dagger a} e^{\frac{2}{1+s} a^\dagger a} e^{\frac{2}{1+s} a^\dagger a} e^{\frac{2}{1+s} a^\dagger a} |z\rangle$$

$$= G e^{\frac{4s'}{1+s} (a^\dagger a)} \frac{d^2 z}{\pi} \exp \left[ \frac{2s}{1+s} (\alpha' - \alpha) - \frac{2s}{1-s} (\alpha'^* - \alpha^*) \right]$$

$$= \frac{1}{4\pi} \delta (\alpha' - \alpha) (\alpha'^* - \alpha^*) e^{-\left( \frac{2s}{1+s} + \frac{2s}{1-s} \right) |\alpha|^2}$$

$$= \frac{1}{2\pi} \delta (q' - q) (p' - p),$$

where $G \equiv \frac{\pi}{(1+s)(1-s)^2 \pi^2 \alpha' \alpha''}$. Therefore, the classical function corresponding to $\rho$ (in the context of the $s$–parameterized quantization scheme) is given by

$$2\pi Tr [\Delta_s (\alpha') \rho] = 4\pi \int d^2 \alpha' Tr [\Delta_s (\alpha) \Delta_s (\alpha')] \Psi (\alpha', s)$$

$$= \int d^2 \alpha' \delta (\alpha - \alpha') (\alpha'^* - \alpha'^*) \Psi (\alpha', s)$$

$$= \Psi (\alpha, s).$$

Eq. (20) is the reciprocal relation of (18). Thus we have established one-to-one mapping between operators and their $s$–parameterized classical correspondence. The $s$–parameterized quantization scheme is completed, of which the Weyl quantization is its special case.

IV. EXPANSION FORMULA OF $|z\rangle \langle z|$ IN TERMS OF $s$–PARAMETERIZED QUANTIZATION SCHEME

When $\rho = |z\rangle \langle z|$, using (20) we have

$$2\pi Tr [\Delta_s (\alpha) |z\rangle \langle z|] = \frac{2}{1+s} |z| : \exp \left[ \frac{-2}{1+s} (a^\dagger - \alpha^*) (a - \alpha) \right] : |z\rangle$$

$$= \frac{2}{1+s} \exp \left[ \frac{-2}{1+s} (z^* - \alpha^*) (z - \alpha) \right],$$

this is the $s$–parameterized classical correspondence of $|z\rangle \langle z|$ in phase space. Eq. (21) represents a kind of phase space distribution, since the integration over it leads to the completeness

$$\int \frac{d^2 z}{\pi} |z\rangle \langle z| \rightarrow \frac{2}{1+s} \int \frac{d^2 z}{\pi} \exp \left[ \frac{-2}{1+s} (z^* - \alpha^*) (z - \alpha) \right] = 1. (22)$$
For this s-parameterized distribution we can define s-ordered form of $|z\rangle \langle z|$ through the following formula

$$
|z\rangle \langle z| = \frac{2}{1+s} \frac{\xi}{\xi} \exp \left[ -\frac{2}{1+s} (z^*-a^\dagger) (z-a) \right],
$$

(23)

where $\xi \cdot \cdot \cdot \xi$ means s-ordering symbol. This definition is consistent with those well-known ordered formulas of $|z\rangle \langle z|$. Indeed, when in (23) $s = 0$, $\xi \cdot \cdot \cdot \xi$ converts to Weyl ordering, so (23) reduces to

$$
|z\rangle \langle z| = 2^\frac{z}{\xi} \exp \left[ -2 (z^*-a^\dagger) (z-a) \right],
$$

(24)

as expected; when in (23) $s = 1$, $\xi \cdot \cdot \cdot \xi$ becomes normal ordering, so (23) reduces to

$$
|z\rangle \langle z| = \exp \left[ -(z^*-a^\dagger) (z-a) \right],
$$

(25)

which is as expected too; when $s = -1$, $\xi \cdot \cdot \cdot \xi$ becomes antinormal ordering,

$$
|z\rangle \langle z| = \lim_{\epsilon \to 0} \frac{2}{\epsilon} \xi \exp \left[ \frac{2}{\epsilon} (z^*-a^\dagger) (z-a) \right],
$$

(26)

still is as expected.

V. THE TECHNIQUE OF INTEGRATION WITHIN $s$–ORDERED PRODUCT OF OPERATORS

Let us introduce the technique of integration within s-ordered product of operators (IWSOP) by listing some properties of the s-ordered product of operators which is defined through (23):

1. The order of Boson operators $a$ and $a^\dagger$ within a s-ordered symbol can be permuted, even though $[a,a^\dagger] = 1$.
2. $c$-numbers can be taken out of the symbol $\xi \cdot \cdot \cdot \xi$ as one wishes.
3. An s-ordered product of operators can be integrated or differentiated with respect to a $c$-number provided the integration is convergent.
4. The vacuum projection operator $|0\rangle \langle 0|$ has the s-ordered product form (see (23))

$$
|0\rangle \langle 0| = \frac{2}{1+s} \frac{\xi}{\xi} \exp \left( -\frac{2}{1+s} a^\dagger a \right) \xi.
$$

(27)

5. the symbol $\xi \cdot \cdot \cdot \xi$ becomes $:\cdot $ for $s = 1$, becomes $\cdot :$ for $s = 0$, and becomes $\cdot \cdot :$ for $s = -1$.

VI. DENSITY OPERATOR’S EXPANSION FORMULA IN TERMS OF $s$–ORDERED QUANTIZATION SCHEME

Using (11) and (23) we have the expansion within $\xi \cdot \cdot \cdot \xi$,

$$
\rho = \int \frac{d^2z}{\pi} P(z) \xi z \langle z | = \frac{2}{1+s} \int \frac{d^2z}{\pi} P(z) \xi \exp \left[ -\frac{2}{1+s} (z^*-a^\dagger) (z-a) \right] \xi.
$$

(28)

Substituting Mehta’s expression of $P(z)$ [13]

$$
P(z) = e^{\beta^2} \int \frac{d^2\beta}{\pi} (-\beta | \beta \rangle \langle \beta | e^{\beta^2+\beta^* z-\beta z^*},
$$

(29)

where $|\beta\rangle$ is also a coherent state, $(-\beta | \beta \rangle = e^{-2|\beta|^2}$, into (28) we have

$$
\rho = \frac{2}{1+s} \int \frac{d^2\beta}{\pi} (-\beta | \beta \rangle \langle \beta | e^{\beta^2} \int \frac{d^2z}{\pi} \xi \exp \left[ |z|^2 + \beta^* z - \beta z^* - \frac{2}{1+s} (z^*-a^\dagger) (z-a) \right] \xi
$$

\[ = \frac{2}{1-s} \int \frac{d^2\beta}{\pi} (-\beta | \beta \rangle \xi \exp \left[ \frac{2}{s-1} (s|\beta|^2 - \beta^* a + a^\dagger a) \right] \xi \] 

(30)
this is density operator’s expansion formula in terms of \( s \)-ordered quantization scheme. In particular, when \( s = 0 \), (30) becomes

\[
\rho = 2 \int \frac{d^2 \beta}{\pi} (-\beta) \rho |\beta\rangle \langle \beta| \exp \left[ \frac{2}{\pi} (\beta^* a - \beta a^\dagger + a^\dagger a) \right] ;
\]

which is the formula converting \( \rho \) into its Weyl ordered form \([7,8]\); while for \( s = -1 \), (30) becomes

\[
\rho = 2 \int \frac{d^2 \beta}{\pi} (-\beta) \rho |\beta\rangle \langle \beta| \exp \left[ - (|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a) \right] ;
\]

which is the formula converting \( \rho \) into its antinormally ordered form \([14]\), as expected.

\[\text{VII. APPLICATION}\]

We now use (30) to derive the \( s \)-ordered expansion of \( e^{\lambda a^\dagger a} \), using \([15]\) and the IWSOP technique we have

\[
e^{\lambda a^\dagger a} = \frac{2}{1 + e^{\lambda}} \exp \left[ \frac{2e^{\lambda} - 2}{1 + e^{\lambda}} a a^\dagger \right] ;
\]

and for \( s = -1 \), (33) becomes \([10]\)

\[
e^{\lambda a^\dagger a} = e^{-\lambda} \exp \left[ (1 - e^{-\lambda}) aa^\dagger \right] ;
\]

which is also correct. Further, we consider the \( s \)-ordered expansion of the generalized Wigner operator itself, using \([10]\) and (30) we have

\[
\Delta_s (\alpha^*, \alpha) = \frac{2}{(1-s)^2} \pi \int \frac{d^2 \beta}{\pi} (-\beta) : \exp \left[ \frac{2}{1-s} (a^\dagger - \alpha^*) (a + \alpha) \right] : \langle \beta| \exp \left[ \frac{2}{s-1} (s|\beta|^2 - \alpha^* a + \alpha a^\dagger - a^\dagger a) \right] \quad \text{§}
\]

\[
= \frac{2}{(1-s)^2} \pi \int \frac{d^2 \beta}{\pi} e^{-2|\beta|^2} \exp \left\{ \frac{2}{s-1} \left[ (-\beta^* - \alpha^*) (\beta - \alpha) \ight. \right. \\
\left. \left. + s|\beta|^2 - \alpha^* a + \alpha a^\dagger - a^\dagger a \right] \right\} \quad \text{§}
\]

\[
= \frac{2}{(1-s)^2} \delta \left[ \frac{2}{s-1} (a - \alpha) \right] \delta \left[ \frac{2}{s-1} (\alpha^* - a^\dagger) \right] \quad \text{§}
\]

which in the case of \( s = 0 \) becomes the Weyl ordered form of the usual Wigner operator \( \Delta (\alpha) = \frac{1}{2} \delta (\alpha^* - a^\dagger) \delta (\alpha - a) \). \quad \text{○}

In summary, by introducing the \( s \)-parameterized generalized Wigner operator into phase-space quantum mechanics we have proposed the technique of integration within \( s \)-ordered product of operators (which considers normal ordered, antinormally ordered and Weyl ordered product of operators as its special cases). The \( s \)-ordered operator expansion (denoted by \( \text{§} \cdots \text{§} \)) formula of density operators is derived. The theory of Integration within normal-, Weyl- and antinormal-ordering of operators can now be tackled in a unified way. The \( s \)-parameterized quantization scheme is completely established, of which the Weyl quantization is its special case. For the mutual transformation between the Weyl ordering and \( X \sim P \) (or \( P \sim X \)) ordering of operators we refer to \([15]\).

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