Affine spheres and finite gap solutions of Tzitzèica equation

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Abstract

The purpose of the present paper is to give an explicit form of the finite gap solutions to the Tzitzèica equation (2D Toda equation of type $A_2^{(2)}$) in terms of Riemann theta function. We give explicit expressions of proper affine spheres derived from finite gap solutions to the Tzitzèica equation.

1. Introduction

This paper concerns with the following 2 dimensional Toda lattice (2DTL) of type $A_2^{(2)}$:

$$\partial_x \partial_t u = e^u - e^{-2u},$$

(1.1)

Nowadays, this 2DTL is called the Tzitzèica equation named after a Romanian mathematician G. Tzitzèica. In the context of soliton theory, Tzitzèica equation was discovered by Bullough and Dodd [1], Žiber and Šabat [2] independently (see also Mikhailov [3], p. 74). In addition, Mikhailov [3] pointed out that Tzitzèica equation is obtained as a reduction of periodic 2DTL

$$\partial_x \partial_t u = e^{u_{i-1} - u_{i-1} - u}$$

of period 3 under the condition $u_3 = 0$, $u_4 = -u_2 = u$. This reduction corresponds to the affine root system $A_2^{(2)}$ (or the reduced root system $BC_3$) [4]. Note that this reduction was already known in projective differential geometry, see [5].

Early in the twenty century, Tzitzèica [6] studied non-degenerate surfaces, that is, surfaces with non-degenerate second fundamental form in the Euclidean 3-space $\mathbb{E}^3$ with the property that $K/\bar{d}^4$ is constant over the surface, where $K$ is the Euclidean Gaussian curvature function and $\bar{d}$ is the Euclidean distance function from the origin of $\mathbb{E}^3$ to each point of the surface. Tzitzèica noticed that $K/\bar{d}^4$ is invariant under equiaffine transformations of $\mathbb{E}^3$.

Tzitzèica’s observation initiated affine differential geometry of surfaces. The surfaces discovered by Tzitzèica are now referred as proper affine sphere due to the fact that a proper affine sphere is the set of points where the affine distance from the origin is non-zero constant. In fact the affine distance function $d$ from the origin satisfies $K/\bar{d}^4 = 1/d^4$.

On every proper affine sphere, there exists a unique (up to sign) semi-Riemannian metric invariant under equiaffine transformations of $\mathbb{E}^3$. Such a metric is called the Blaschke metric. Note that the Blaschke metric of a proper affine sphere is conformal to the Euclidean second fundamental form. There are two classes of proper affine spheres and they are said to be indefinite or definite according as the non-degenerate affine metric is indefinite or definite, respectively. In this paper, we study indefinite proper affine spheres.

The Gauss–Codazzi equations for indefinite proper affine spheres are described by the Tzitzèica equation, where $h = e^u$ $dx \wedge dt$ is Blaschke metric on $M$ and the affine mean curvature is normalized to $-1$. Tzitzèica equation arises not only in affine differential geometry but also in many other realms of mathematical physics and differential geometry. For instance elliptic versions of Tzitzèica equation are integrability conditions of the following three kinds of surfaces: (1) Lagrangian minimal surfaces in complex projective plane (cf. [7]), (2) Lagrangian minimal surfaces in complex hyperbolic plane [8] and (3) affine spheres with positive definite Blaschke metric [9]. These elliptic versions are closely related to certain Kähler–Einstein metrics [10].

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For the geometry of surfaces in 3-dimensional spaces using the soliton theory we refer to [11] and [12]. More generally, for the close relation between minimal surfaces (nonlinear sigma models) in symmetric spaces and 2DTL, for example, see [13, 14] and [15].

Some explicit solutions of Tzitzéica equation (1.1) are known. For example, the hexenhut and the Jonas Kelch are realized using the solution of $u = 0$ and $u = \log(1 - 3/2 \cosh^2(\sqrt{3}(x + t)/2))$, respectively. Moreover, the finite gap solutions of (1.1) may be described by Cherdantsev and Sharipov in [16] as follows:

$$e^n = C - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial t} \log \theta(Ux + Vt - e),$$

where $\theta$ is the Prym-theta function for some compact Riemann surface of genus $2g$ and the constant $C$ is determined by some spectral data. They also gave the Baker-Akhiezer function in terms of the spectral data (see [16]). On the other hand, we may expect the existence of the solution of (1.1) in terms of the Jacobi elliptic functions because the elliptic version of (1.1) is solved in terms of the Jacobi elliptic function by Castro-Urbano (see [17]).

In this paper, first of all, we construct the solution of (1.1) in terms of the Jacobi elliptic functions. However, it is not clear what kind of the spectral data produces the solution in terms of the Jacobi elliptic functions as a special case of the finite gap solutions. Therefore, we explicitly give the spectral data for the solution in terms of the Jacobi elliptic functions. Moreover, we can describe the finite gap solutions more explicitly in terms of Riemann theta function. Precisely speaking, we can give the constant $C$ in (1.2) explicitly (see (3.8), (5.9) and theorem 5.5). Moreover, we can give Abelian differentials of second kind on the spectral curve explicitly (see (5.7)) and consequently we obtain a real frame for indefinite proper affine spheres in terms of the Baker-Akhiezer function.

For our purpose stated above, first of all, in section 4, we deduce the solution of the Tzitzéica equation and gives a Blaschke immersion of indefinite proper affine spheres in terms of the Jacobi elliptic functions. Therefore, the well known elliptic function theory hides in the background. We employ the elliptic function theory and reconstruct the whole theory for solutions of the Tzitzéica equation in terms of the Riemann theta function (see (4.17)), and for Blaschke immersions of indefinite proper affine spheres in terms of the meromorphic function $\tilde{\Psi}$ which is a solution of the Schrödinger equation $\partial_x \partial_t \tilde{\Psi} = e^\tilde{\Psi}$ (see theorem 4.9). These reconstructions give a nice story for a general case of the corresponding problems. Since the elliptic curve used in section 4 is the Prym variety of the spectral curve of genus 2, we must consider the spectral curve of genus 2g and the g-dimensional Prym variety for the general case. The standard argument in the integrable systems and the result in section 4 help us to reconstruct the solutions of the problems for the general case. In section 5, we get the solutions of the Tzitzéica equation and give expressions of the immersions of indefinite proper affine spheres of finite type from the points of views of the algebro-geometric approach to the integrable systems.

2. Indefinite proper affine spheres

2.1. Fundamental equations

We start with description of affine spheres in equiaffine differential geometry. For more details we refer to the textbook [18] by Nomizu and Sasaki.

Let $\mathbb{R}^3$ be a Cartesian 3-space. We denote by $\det:\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ the determinant function, which defines a volume element of $\mathbb{R}^3$. Then the triplet $\mathbf{A}^3 = (\mathbf{R}^3, D, \det)$ is an equiaffine 3-space, i.e., it satisfies $D \det = 0$, where $D$ is the canonical covariant differentiation for $\mathbb{R}^3$. An immersion $\psi: M \rightarrow \mathbf{A}^3$ of an oriented 2-manifold is said to be an affine immersion if $(M, \psi)$ admits a complementary subbundle $\mathbf{N}$ of the tangent bundle $TM$ of $M$ in the pull-back bundle $\psi^*T\mathbf{A}^3$, that is, if we have $\psi^*(\mathbf{A}^3) = TM \oplus \mathbf{N}$. A local section $\xi$ of $\mathbf{N}$ is called a transversal vector field of $(M, \psi)$. In the following, we assume that there is a global transversal vector field $\xi$ to $(M, \psi)$ so that $\mathbf{N} = \bigcup_{\xi \subseteq \mathbb{R}^3}$.

For any vector fields $X$ and $Y$ on $M$ we have the Gauss formula

$$D_X \psi \omega Y = \psi (\nabla_X Y) + h(X, Y) \xi.$$

We easily verify that $\nabla$ defines a torsion free linear connection on $M$ and $h$ is a symmetric tensor field on $M$. The symmetric tensor field $h$ is called the affine fundamental form derived from $\xi$. We also have the Weingarten formula

$$D_X \xi = -\psi \omega (SX) + \mathbf{T}(X) \xi.$$

The endomorphism field $S$ of $TM$ is called the affine shape operator with respect to $\xi$. The 1-form $\mathbf{T}$ is called the transversal connection form. For example, an immersion $\psi: M \rightarrow \mathbf{A}^3 \setminus \{0\}$ with the property that the position vector field $\psi$ is transversal to $M$ is an affine immersion and in particular it is called centro-affine immersion. For
an affine immersion \((M, \psi, \xi)\), we may define a volume element \(\vartheta\) on \(M\) by \(\vartheta(X, Y) = \det(X, Y, \xi)\) for any vector fields \(X\) and \(Y\) on \(M\). We address here some known facts about affine immersions.

For an affine immersion, the rank of the affine fundamental form is independent of the choice of the transversal vector field.

The triplet \((M, \nabla, \vartheta)\) is equiaffine, i.e., \(\nabla \vartheta = 0\) if and only if \(T = 0\).

Let \(\psi : M \rightarrow A^1\) be an affine immersion with non-degenerate affine fundamental form \(h\). Then there is a unique (up to sign) transversal vector field \(\xi\) such that \(T = 0\) and the volume element of the semi-Riemannian surface \((M, h)\) coincides with \(\vartheta\). This \(\xi\) is called the Blaschke normal field and the pair \((\psi, \xi)\) is called a Blaschke immersion. The affine fundamental form \(h\) with respect to the Blaschke normal vector field is traditionally called the Blaschke metric of \((M, \psi)\). For a Blaschke immersion \((M, \psi, \xi)\) we have the following fundamental equations.

\[
\begin{align*}
R(X, Y)Z &= h(Y, Z)SX - h(Z, X)SY, & \text{(Gauss equation)} \\
(\nabla h)(Y, Z) &= (\nabla h)(X, Z), & \text{(Codazzi equation)} \\
h(SX, Y) &= h(X, SY). & \text{(Ricci equation)}
\end{align*}
\]

Here \(R\) is the curvature tensor field of the connection \(\nabla\). The Codazzi equation implies that \(C := \nabla h\) is a symmetric covariant tensor field. We call the \(C\) cubic form of \((M, \psi, \xi)\).

### 2.2. Affine spheres

Now let \((M, \psi, \xi)\) be a Blaschke immersion with indefinite Blaschke metric \(h\). Then we can take an asymptotic coordinate system \((x, t)\) on a simply connected domain \(D \subset M\) with respect to the Lorentz conformal structure determined by \(h\). Namely, the Blaschke metric \(h\) is represented as \(h = e^u \, dx \wedge dt\) with respect to \((x, t)\). It then follows from Fact 2 and Fact 3 that the cubic form \(C\) is represented as \(C = Adx^3 + Bdt^3\).

**Definition 2.1.** A Blaschke immersion \((M, \psi, \xi)\) is said to be an affine sphere if \(S = k\) \(I\) for some constant \(k\). Moreover, a affine sphere \((M, \psi, \xi)\) is said to be proper if \(k \neq 0\) and improper if \(k = 0\).

Let \((M, \psi, \xi)\) be a proper affine sphere. Then all the Blaschke normals meet in one point.

### 2.3. Tzitzéica equation

In the following, we consider proper affine spheres with indefinite Blaschke metric \(h\). In particular, for such proper affine spheres we have \(\xi = -H\psi\), where \(H = \frac{1}{2}\)trace \(S\) is the affine mean curvature and it is a negative constant. By scaling the affine metric, without loss of generality, we may assume that \(H = -1\), hence we may take \(\xi = \psi\). Note that \((M, \psi, \xi)\) is centro-affine.

Set \(\partial_x = \frac{\partial}{\partial x}\), \(\partial_t = \frac{\partial}{\partial t}\), then the induced connection \(\nabla\) is given by

\[
\nabla_{\partial_x} \partial_x = u_x \partial_x + Ae^{-u} \partial_t, \quad \nabla_{\partial_t} \partial_t = 0 = \nabla_{\partial_x} \partial_t, \quad \nabla_{\partial_t} \partial_x = Be^{-u} \partial_x + u_t \partial_t.
\]

Set \(\psi_1 = \psi_x(\partial_x)\) and \(\psi_x = \psi_t(\partial_t)\) then the Gauss-Weingarten equations are given by

\[
\psi_{xx} = u_x \psi_x + Ae^{-u} \psi_t, \quad \psi_{xt} = e^u \psi_x = \psi_{tx}, \quad \psi_x = Be^{-u} \psi_x + u_t \psi_t.
\] (2.1)

Here we introduce a matrix valued function (called a framing) \(F = (\psi_1, \psi_x, \psi_t)\). Since \(\psi\) is a Blaschke immersion, we see that \(\det F = e^u \neq 0\). We rewrite (2.1) in the following way \([19, 20]\):

\[
F^{-1} \frac{\partial F}{\partial t} = \left(\begin{array}{ccc}
u & 0 & 1 \\
0 & 0 & 0 \\
e^{-u} & 0 & 0 \\
\end{array}\right) = U_t, \quad F^{-1} \frac{\partial F}{\partial x} = \left(\begin{array}{ccc}
0 & Ae^{-u} & 0 \\
0 & 0 & u_x \\
e^u & 0 & 0 \\
\end{array}\right) = V.
\] (2.2)

The compatibility condition of the system (2.2) is given by

\[
u_x = e^u - AB \, e^{-2u}, \quad A_t = 0, \quad B_x = 0.
\] (2.3)

The first equation is the Gauss equation and the last two equations are Codazzi equations. Throughout this paper we assume that indefinite proper affine spheres are weakly regular, that is, \(AB \neq 0\) \([20]\). Then the Codazzi equation means that \(A = A(x)\) and \(B = B(t)\). Therefore, choosing the coordinate system \((x, t)\) we may assume that the cubic form is represented as \(C = dx^3 + dt^3\) from the first step. In this case, we have \(A = B = 1\) and (2.3) is called the Tzitzéica equation \([6]\).
The compatibility condition (2.3) is invariant under the replacements
\[
U(\nu) = \begin{pmatrix} u_t & 0 & 1 \\ ve^{-u} & 0 & 0 \\ 0 & e^u & 0 \end{pmatrix}, \quad V(\nu) = \begin{pmatrix} 0 & \nu^{-1}e^{-u} & 0 \\ 0 & u_s & 1 \\ e^u & 0 & 0 \end{pmatrix}, \quad \nu \in \mathbb{R}^*,
\]
(2.4)
respectively. We denote by \(F(\nu)\) the solution to the system
\[
\partial_t F(\nu) = F(\nu) U(\nu), \quad \partial_x F(\nu) = F(\nu) V(\nu).
\]
Using \(\lambda\) with \(\lambda^3 = \nu\) and setting
\[
F(\lambda) \equiv F(\nu) \begin{pmatrix} \lambda^{-1}e^{-\frac{\lambda}{2}} & 0 & 0 \\ 0 & \lambda e^{-\frac{\lambda}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
we obtain a new framing \(F(\lambda)\) with
\[
\partial_t F(\lambda) = F(\lambda) U(\lambda), \quad \partial_x F(\lambda) = F(\lambda) V(\lambda), \quad (\nu = \lambda^3)
\]
where
\[
U(\lambda) = \begin{pmatrix} \frac{1}{2}u_t & 0 & \lambda e^\frac{\lambda}{2} \\ \lambda e^{-u} - \frac{1}{2}u_t & 0 & 0 \\ 0 & \lambda e^\frac{\lambda}{2} & 0 \end{pmatrix}, \quad V(\lambda) = \begin{pmatrix} -\frac{1}{2}u_s & \lambda^{-1}e^{-u} & 0 \\ 0 & \frac{1}{2}u_s & \lambda^{-1}e^\frac{\lambda}{2} \\ \lambda^{-1}e^\frac{\lambda}{2} & 0 & 0 \end{pmatrix}
\]
(2.5)
Thus for any indefinite proper affine sphere \(\psi: M \to \mathbb{A}^3\), there exists a map \(F(\lambda): \mathbb{D} \times \mathbb{R}^* \to SL(3, \mathbb{R})\) satisfying (2.5). We call \(F(\lambda)\) a coordinate extended framing of an indefinite proper affine sphere \((M, \psi)\).

Coordinate extended framings are examples of extended framings which will be introduced in later section (definition 3.2).

Before closing this section, we state the assumption on \(u\) throughout the paper. We assume that \(u(x, t) = u(t, x)\) with respect to the coordinate system \((x, t)\) so that the cubic form is given by \(C = dx^3 + dt^3\). Of course, this assumption is independent of introducing the non-zero parameter \(\nu\).

3. Twisted loop algebras and twisted loop groups

3.1. 6-symmetric spaces

Let \(G = SL(3, \mathbb{R})\) be a real special linear group of degree 3 and \(\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})\) its Lie algebra. We denote by \(\mathfrak{g}^c = \mathfrak{sl}(3, \mathbb{C})\) the complexification of \(\mathfrak{g}\). Define two automorphisms \(\sigma\) and \(\gamma\) of \(\mathfrak{g}^c\) by
\[
\sigma(\xi) = -Ad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \xi, \quad \gamma(\xi) = Ad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega & 0 \end{pmatrix} \cdot \xi, \quad \xi \in \mathfrak{g}^c,
\]
where \(\omega = \exp(2\pi \sqrt{-1}/3)\). We then see that \(\tau = \gamma \sigma = \sigma \gamma\) is an automorphism of order 6 [20]. Let \(\epsilon = -\omega^2\) be the sixth root of unity. Denote by \(\mathfrak{g}^c_j\) be the \(\epsilon^j\)-eigenspace of \(\tau\), where \(j = 0, 1, 2, 3, 4, 5\). Set \(\mathfrak{g}_j = \mathfrak{g}^c_j \cap \mathfrak{g}\). For example, we have \(\mathfrak{g}_0\) consists of the matrices of the form
\[
\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R},
\]
which is a Lie subalgebra of \(\mathfrak{g}\) and is isomorphic to \(\mathfrak{o}(1, 1)\) in \(\mathfrak{sl}(3, \mathbb{R})\). Let \(K\) be a subgroup of \(G\), which consists of the matrices of the form
\[
\begin{pmatrix} s & 0 & 0 \\ 0 & s^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s \in \mathbb{R}^*
\]
and is isomorphic to \(SO(1, 1) \subset SL(3, \mathbb{R})\). We denote by \(SO^0(1, 1)\) the identity component of \(SO(1, 1)\). Then \(SO^0(1, 1)\) is isomorphic to \(\exp(\mathfrak{g}_0)\) = \(K_+\). We denote by \(SO(2, 1)\) the Lorentzian group with Lie algebra \(\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2\) and its identity component by \(SO^0(2, 1)\). We then easily see the following fact.

**Proposition 3.1.** The automorphism \(\tau\) of order 6 gives a semi-Riemannian 6-symmetric space structure on \(SL(3, \mathbb{R})/SO^0(1, 1)\). The involution \(\tau\) gives \(SL(3, \mathbb{R})/SO^0(2, 1)\) the outer semi-Riemannian symmetric space structure. Moreover, we have the homogeneous projection \(\pi: SL(3, \mathbb{R})/SO^0(1, 1) \to SL(3, \mathbb{R})/SO^0(2, 1)\).
3.2. Twisted loop algebras

Let \( \hat{\sigma} \) be an involution on \( g^C \) defined by
\[
\hat{\sigma} = \text{Ad} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We then have \( \hat{\sigma}(g_j) = g_j^{-1} \). We consider a twisted loop algebra
\[
\Lambda g^C_r = \{ \xi : S^1 \to g^C \mid \xi(e^\lambda) = \tau^r(\lambda) \text{ for all } \lambda \in S^1 \text{ and } ||\xi|| < \infty \},
\]
where \( ||\xi|| = \sum_{n \in \mathbb{Z}} ||\xi_n|| \) for \( \xi(\lambda) = \sum_{n \in \mathbb{Z}} \xi_n e^{in\lambda} \) and \( ||\xi_n|| = \max_{j \in \{1,2,3\}} \sum_{i=0}^{5} ||(\xi_n)_i|| \). Then, \( \Lambda g^C_r \) is a Banach Lie algebra (see [21, 22]). Set
\[
\Lambda g^F_r = \left\{ \xi \in \Lambda g^C_r \mid \xi = \sum_{j \in \mathbb{Z}} \xi_j \lambda^j \right\},
\]
where \([j] \in \mathbb{Z}/6\mathbb{Z} \).

We decompose each element \( \xi \in \Lambda g^F_r \) as \( \xi = \xi_+ + \xi_0 + \xi_- \), where \( \xi_+(\lambda) = \sum_{j > 0} \xi_j \lambda^j \), \( \xi_-(\lambda) = \sum_{j < 0} \xi_j \lambda^j \).

Next, we extend the involution \( \hat{\sigma} \) to \( \Lambda g^F_r \) by
\[
(\hat{\sigma} \xi)(\lambda) = \hat{\sigma}(\xi(\lambda^{-1})).
\]

We may write \( \xi \) as
\[
\xi = (\xi_+ + \hat{\sigma}(\xi_-) + (\xi_- - \hat{\sigma}(\xi_+) + \xi_0).
\]

We now define the following Banach Lie subalgebras of \( \Lambda g^F_r \):
\[
\Lambda g^F_r = \{ \xi \in \Lambda g^F_r \mid \hat{\sigma} \xi = \xi \}, \quad \Lambda_{ad} \Lambda g^F_r = \left\{ \xi \in \Lambda g^F_r \mid \xi = \sum_{j \leq 0} \xi_j \lambda^j \right\}.
\]

It then follows from (3.1) that
\[
\Lambda g^F_r = \Lambda g^F_r \oplus \Lambda_{ad} \Lambda g^F_r,
\]
which is a vector space decomposition into Banach Lie subalgebras.

For a positive integer \( d \equiv 1 \pmod{6} \), we define the vector subspace \( \Lambda^d g^F_r \) of \( \Lambda g^F_r \) by
\[
\Lambda^d g^F_r = \left\{ \xi \in \Lambda g^F_r \mid \xi = \sum_{j = -d}^{d} \xi_j \lambda^j \right\}.
\]

3.3. Twisted loop groups

Let \( \Lambda G_r \) be a twisted loop group defined by
\[
\Lambda G_r = \{ g : S^1 \to G^C \mid g(e^\lambda) = \tau g(\lambda), \quad \overline{g(\lambda)} = g(\lambda) \text{ for all } \lambda \in S^1 \text{ and } ||g|| < \infty \}
\]
whose Lie algebra is \( \Lambda g^F_r \). Here we denote the Lie group automorphism corresponding to \( \tau \) by the same letter.

Define subgroup \( \Lambda^d G_r \) of \( \Lambda G_r \) by
\[
\Lambda^d G_r = \{ g \in \Lambda G_r \mid \hat{\sigma} g = g \}.
\]

Consider a map \( g : D \subset \mathbb{R}^2 \ni (x, t) \mapsto g(x, t) \in \Lambda G_r \). We extend the involution \( \hat{\sigma} \) to the above map by the rule
\[
(\hat{\sigma}^* g)(x, t) = \hat{\sigma}(g(t, x)).
\]

Define the mapping spaces \( \Lambda G_r(D) \) and \( \Lambda^d G_r(D) \) by
\[
\Lambda G_r(D) = \{ g : D \to \Lambda G_r \}, \quad \Lambda^d G_r(D) = \{ a \in \Lambda G_r(D) \mid \hat{\sigma}^* a = a \},
\]
\[
\Lambda g^F_r(D) = \{ \xi : D \to \Lambda g^F_r \}, \quad \Lambda^d g^F_r(D) = \{ \xi \in \Lambda g^F_r(D) \mid \hat{\sigma}^* \xi = \xi \}.
\]

3.4. Affines spheres of finite type

Now we return to indefinite proper affine spheres. Let \( \psi : D \subset \mathbb{R}^2 \to A^l \) be an indefinite proper affine sphere parametrized by global asymptotic coordinates \( (x, t) \) as before. The coordinate extended framing \( F(\lambda) \) can be extended analytically on \( \mathbb{C}^\ast \). The (extended) map \( F(\lambda) \) is uniquely determined by the values on \( S^1 \subset \mathbb{C}^\ast \) (cf. [20], p. 234). Hence \( F(\lambda) \) is regarded as a map into \( \Lambda G_r \). In addition, since we assumed that \( u(x, t) = u(t, x) \), it follows from (2.5) that \( \hat{\sigma}^* (U(\lambda)) = \overline{V(\lambda)} \). Therefore, the coordinate extended framing \( F(\lambda) \) can be considered as a map \( F \in \Lambda^d G_r(D) \). Moreover, since
\[
\psi = F \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathcal{F}(\nu)|_{\nu=\lambda} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = F(\lambda)|_{\lambda=\lambda} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

we see that the 3rd column vector of the coordinate extended framing \( F(\lambda)|_{\lambda=\lambda} \) gives a Blaschke immersion \( \psi: D \to \mathbb{A}^3 \). Here we give the following definition (compare with [20, 23]).

**Definition 3.2.** An element \( F \in N^*G_\tau(D) \) which satisfies \( F^{-1}dF = \lambda_0 \lambda dx + \alpha_0 + \lambda^{-1}\alpha_{t}dx \) with \( \alpha_{t} = \delta^*(\alpha_0) \) is said to be an extended framing for indefinite proper affine sphere.

**Remark 3.3.** One can check that every extended framing \( F \in N^*G_\tau(D) \) induces a harmonic map (nonlinear \( \sigma \)-model) \( F: K^i: D \to G/K \) (see proposition 3.1, [24]).

Next we introduce the following notion for affine spheres.

**Definition 3.4.** An indefinite proper affine sphere is said to be of finite type if its extended framing \( a \) is obtained from the following differential equation under certain initial condition:

\[
\begin{align*}
d\xi &= [\xi, \alpha\xi], \\
\xi &= \lambda^d \phi(D) = \left\{ \xi \in \lambda^d \phi(D) | \xi = \sum_{j=0}^d \xi_j \lambda^j \right\}, \\
\alpha &= \left( \lambda \xi_j + \frac{1}{2} \xi_{j-1} \right) dt + \left( \lambda^{-1} \delta^*(\xi_j) + \frac{1}{2} \delta^*(\xi_{j-1}) \right) dx,
\end{align*}
\]

where \( d \equiv 1 \) (mod 6). One can see that \( a \) satisfies Maurer–Cartan equation and hence there is a solution \( a \in \lambda^d \phi(D) \) to the equation \( a^{-1}da = \alpha \). Moreover this \( a \) is an extended framing for affine spheres.

**Example 3.5 (hexenhut).** We present a trivial solution \( u = 0 \) of Tzitzéica equation and corresponding affine sphere.

Take a matrix \( A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \). Note that \( \delta^*(A) = A^{-1} \). Then we easily see that \( a(x, t) = \exp(t\lambda A + x\lambda^{-1}A^{-1}) \) is an extended framing of an indefinite proper affine sphere \( \psi_0 \) of finite type. After some elementary calculation we obtain

\[
\psi_0 = a|_{\lambda=\lambda} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 & -\sqrt{3}/3 & 1/3 \\ 1/3 & \sqrt{3}/3 & 1/3 \\ 2/3 & 0 & 1/3 \end{pmatrix} \cdot \begin{pmatrix} \cos \left( \frac{\sqrt{3}}{2}(x - t) \right) e^{-\frac{i\pi}{3}} \\ \sin \left( \frac{\sqrt{3}}{2}(x - t) \right) e^{-\frac{i\pi}{3}} \\ e^{i(x + t)} \end{pmatrix}
\]

Hence the resulting surface \( \psi_0 \) is affine congruent to the hexenhut \( Z(X^2 + Y^2) = 1 \) in \( \mathbb{C}^3(X, Y, Z) \) (see Figure 1). The hexenhut \( \psi_0 \) corresponds to the trivial solution \( u = 0 \) to the Tzitzéica equation [25].

**Remark 3.6.** One can establish the Symes method (also called AKS-scheme) for constructing indefinite proper affine sphere of finite type. This scheme was established in a separate publication [26].

### 4. Construction of Blaschke immersions of affine spheres

#### 4.1. Blaschke immersions in terms of elliptic functions

In this section, we give a solution of Tzitzéica equation (1.1) in terms of elliptic functions and represent affine spheres in terms of elliptic functions. Although these formulas have been known in some literature, our purpose here is to represent them in terms of Riemann theta function. This achievement gives a nice story for the description of the solutions in terms of the spectral curves of higher genus and Prym-theta functions.

Introducing new coordinates \( \hat{x}, \hat{y} \) by

\[
\hat{x} = \frac{1}{2}(x + t), \quad \hat{y} = \frac{1}{2}(x - t),
\]

we have \( 4\frac{\partial}{\partial \alpha} = \frac{\partial^2}{\partial \hat{x}^2} - \frac{\partial^2}{\partial \hat{y}^2} \).

We now assume that a solution \( u \) depends only on the parameter \( \hat{y} \) and write \( u = u(\hat{y}) \). We suppose that an initial condition \( e^{u(0)} = \alpha, \quad u(0) = 0 \), where \( \alpha \) is a real number with \( \alpha > 2 \). If we set \( Y(\hat{y}) = e^{u(\hat{y})} \), then the equation (1.1) may rewritten as
\[
\left( \frac{dY}{d\eta} \right)^2 = -8\left( Y^3 - aY^2 + 1 \right) = -8(Y - \zeta_1)(Y - \zeta_2)(Y - \zeta_3),
\]

where
\[
a = \frac{a^3 + 4}{2a^2}, \quad \zeta_1 = \frac{1 - \sqrt{a^3 + 1}}{a^2}, \quad \zeta_2 = \frac{a}{2}, \quad \zeta_3 = \frac{1 + \sqrt{a^3 + 1}}{a^2}.
\]

If we set \( Y = \zeta_2 - (\zeta_2 - \zeta_3)X^2 \) and \( p^2 = \frac{a - \zeta_2}{\zeta_2 - \zeta_3} \), we obtain
\[
\left( \frac{dX}{d\eta} \right)^2 = 2(\zeta_2 - \zeta_3)(1 - p^2X^2)(1 - X^2),
\]

hence, setting \( X(\hat{\eta}) = \tilde{X}(\sqrt{2}\zeta_2 - \zeta_3 \hat{\eta}) \) and \( \hat{z} = \sqrt{2}\zeta_2 - \zeta_3 \hat{\eta} \), we have
\[
\left( \frac{d\hat{X}}{d\hat{\eta}} \right)^2 = (1 - p^2\tilde{X}^2)(1 - \tilde{X}^2).
\]

Then, we see that \( \tilde{X}(\hat{\eta}) = \text{sn}(\hat{\eta}, p) \), where \( \text{sn}(\hat{\eta}, p) \) is the Jacobi \( \text{sn} \)-function with modulus \( p \). Therefore, we obtain a solution of \((1.1)\) as follows.

\[
\exp(u(\hat{\eta})) = Y(\hat{\eta}) = \zeta_2 - (\zeta_2 - \zeta_3) \text{sn}^2(\sqrt{2}\zeta_2 - \zeta_3 \hat{\eta}, \hat{p}).
\] (4.1)

The function \( Y(\hat{\eta}) \) can be extended as a function of complex variables. Extending \( \hat{\eta} \) as a complex valued function, we see that \( Y(\hat{\eta}) \) is a doubly-periodic function with the periods \( 2\omega^0 \), \( 2\omega^0 \), where
\[
\omega^0 = \frac{K(p)}{\sqrt{2}\zeta_2 - \zeta_1}, \quad \omega^0 = \frac{\sqrt{-1}K'(p)}{\sqrt{2}\zeta_2 - \zeta_1},
\] (4.2)

and \( K(p) \) is the complete elliptic integral of 1st kind, \( K'(p) = K(p') \) and \( p' = \sqrt{1 - p^2} \). The \( Y \) can be essentially represented by Weierstrass \( \wp \)-function. In fact, if we set \( P(z) = P(\sqrt{2}\sqrt{-1}\hat{\eta}) = Y(\hat{\eta}) - \frac{a}{2} \) then we have
\[
\left( \frac{dP}{dz} \right)^2 = 4(P - \eta_1)(P - \eta_2)(P - \eta_3),
\]

\[\text{where } \eta_j = \zeta_j - \frac{a}{2}, \quad j = 1, 2, 3 \text{ with the property } \eta_1 + \eta_2 + \eta_3 = 0. \]

The solution of this equation is given by \( \text{Weierstrass } \wp \)-function \( \wp(z) \), which is related to the Jacobi \( \text{sn} \)-function by
\[
\wp(z) = \eta_1 + \frac{\eta_2 - \eta_1}{\text{sn}^2(\sqrt{\eta_2 - \eta_1} z, k)},
\]

where \( k^2 = \frac{\eta_2 - \eta_1}{\eta_2 - k \zeta_2} = 1 - p^2 = (p')^2 \), hence we have \( k = p' \) and \( k' = p \). As it is well-known, \( \wp \)-function is doubly-periodic with periods \( 2\omega(k), 2\omega'(k) \), where
\[
\omega(k) = \frac{K(k)}{\sqrt{\eta_2 - \eta_1}}, \quad \omega'(k) = \frac{\sqrt{-1}K'(k)}{\sqrt{\eta_2 - \eta_1}}.
\]

Since we know that \( k = p' \) and \( k' = p \), the above equation yields the following.
\[
\omega(k) = \sqrt{2}\sqrt{-1}\omega^0, \quad \omega'(k) = \sqrt{2}\sqrt{-1}\omega^0.
\] (4.3)

We see that
\[
\wp(\sqrt{2}\sqrt{-1}\hat{\eta}) = Y(\hat{\eta} + \omega^0) - \frac{a}{2},
\] (4.4)

where we used the well-known relations
\[
\text{sn}(\sqrt{-1}v, p') = \sqrt{-1}\frac{\text{sn}(v, p)}{\text{cn}(v, p)}, \quad \text{sn}(v + \sqrt{-1}K'(p), p) = \frac{1}{p} \text{sn}(v, p).
\]

**Remark 4.1.** \( Y(\hat{\eta}) = \zeta_2 + \frac{\zeta_2 - \zeta_3}{\text{sn}^2(\sqrt{2}\zeta_2 - \zeta_3 \hat{\eta}, p)} \) is also a solution of \((1.1)\). The Weierstrass \( \wp \)-function is defined by
\[
\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L, \omega \neq 0} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right), \quad L = Z\omega(k) + Z\omega'(k).
The sum of the right hand side is absolutely and uniformly convergent on a compact subset of $\mathbb{C}$. It is an elliptic function.

On the other hand, rewriting (2.1) by the new coordinates $\hat{x}$, $\hat{y}$ and differentiating the rewritten equation one more time with respect to $\hat{x}$ we have $\psi_{\text{xxx}} = 2n^2 \psi + 2\psi$. Let $\mu_1, \mu_2, \mu_3$ be the solutions of the characteristic equation $\mu^3 - 2n\mu - 2 = 0$, that is,

$$
\mu_1 = 1 + \frac{1}{\sqrt{\alpha^3 + 1}}, \quad \mu_2 = -2, \quad \mu_3 = 1 - \frac{1}{\sqrt{\alpha^3 + 1}}.
$$

(4.5)

We then may write $\psi(\hat{x}, \hat{y}) = (e^{\mu_1 \hat{x}}) C_1(\hat{y}), e^{\mu_2 \hat{x}} C_2(\hat{y}), e^{\mu_3 \hat{x}} C_3(\hat{y}))$. Substituting this expression into the rewritten equation of (2.1), in particular, into $\psi_{\hat{y}\hat{y}} = \frac{1}{2} \psi_{\hat{y}^2} - e^{\beta \hat{y}}$, we obtain $(2e^{\beta} + \mu_1^2 - 2n)C_1(\hat{y}) = u_\beta e^{\alpha(\hat{x})}$. Using the relations

$$
ue^{\beta} = -2(\zeta_2 - \zeta_3) \text{sn} (v) \frac{d}{d\hat{y}} \text{sn} (v) = 2(\zeta_2 - \zeta_3) \text{cn} (v) \frac{d}{d\hat{y}} \text{cn} (v) 
$$

with $v = \sqrt{\frac{1}{\beta}} \sqrt{\zeta_2 - \zeta_3}$, we obtain

which, together with $\det F = e^{\beta}$, yield the explicit parametrization

$$
\psi(\hat{x}, \hat{y}) = \begin{cases} 
\epsilon_1 \exp(\mu_1 \hat{x}) \text{dn}(\sqrt{2} \sqrt{\zeta_2 - \zeta_3}, \hat{y}, p), \\
\epsilon_2 \exp(\mu_2 \hat{x}) \text{sn}(\sqrt{2} \sqrt{\zeta_2 - \zeta_3}, \hat{y}, p), \\
\epsilon_3 \exp(\mu_3 \hat{x}) \text{cn}(\sqrt{2} \sqrt{\zeta_2 - \zeta_3}, \hat{y}, p),
\end{cases}
$$

(4.6)

where $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ are non-zero real numbers with

$$
\epsilon_1 \epsilon_2 \epsilon_3 = (\sqrt{2} \sqrt{\zeta_2 - \zeta_3})^{-2}.
$$

This is a rotational surface as an orbit of the group $SO(1, 1)$ of hyperbolic rotations (Lorentz boosts) up to affine transformations. When $\alpha$ tends to 2, the solution converges to a trivial solution $u = 0$.

**Remark 4.2.** Analogously, we may get a solution of (1.1) which depends only on the variable $\hat{x}$ as follows.

$$
e^{\mu(\hat{x})} = \zeta_1 + (\zeta_3 - \zeta) \text{sn}^2 (\sqrt{2} \sqrt{\zeta_2 - \zeta_3}, \hat{x}, q) \quad \text{with} \quad q^2 = \frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}.
$$
The corresponding affine sphere is obtained as follows.

\[
\psi(\hat{x}, \hat{y}) = \begin{pmatrix}
\epsilon_1 \sqrt{e^u - a} \exp\left(- \int \frac{d\hat{x}}{e^u - a}\right) \\
\epsilon_2 \cos(\sqrt{2a} \hat{y}) \exp\left(\frac{1}{2} u - \int e^{-u} d\hat{x}\right) \\
\epsilon_3 \sin(\sqrt{2a} \hat{y}) \exp\left(\frac{1}{2} u - \int e^{-u} d\hat{x}\right)
\end{pmatrix}
\]

Therefore, this is a rotational surface as an orbit of the rotational group \(SO(2)\) of elliptic type up to affine transformations. When \(\alpha\) tends to 2, the above solution converges to the 1-soliton solution

\[
e^{\alpha(\hat{x})} = 1 - \frac{3}{2 \cosh^2(\sqrt{3} \hat{x})},
\]

of Tzitzéica equation. The rotational surface corresponding to the 1-soliton is called the Jonas Kelch (Figure 2, see also [25]).

4.2. In the rest of the section 4, we represent the formulas (4.1) and (4.6) in terms of the Riemann theta function.

We consider an elliptic curve \(C\) defined by \(B^2 = 4(\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)\). The curve is parametrized by \((B, \eta) \equiv (\psi'(z), \psi(z))\). We may choose a cycle \(\{a_0, b_1\}\) of \(C\) so that the following holds:

\[
\begin{aligned}
\int a_0 \frac{d\eta}{B} &= \int a_0 \frac{d\psi(z)}{\psi(z)} = \int a_0 \frac{d\bar{w}}{\bar{w}} = 2\omega(k), \\
\int b_1 \frac{d\eta}{B} &= \int b_1 \frac{d\psi(z)}{\psi(z)} = \int b_1 \frac{d\bar{w}}{\bar{w}} = 2\omega'(k).
\end{aligned}
\]

If we take \(w_1 = \frac{\pi \sqrt{-1} d\eta}{\omega(k)}\), then we easily see that

\[
\int_{a_0} w_1 = 2\pi \sqrt{-1}, \quad \int_{b_1} w_1 = 2\pi \sqrt{-1} \frac{\omega'(k)}{\omega(k)} = \Pi(k).
\]

Then, the elliptic curve has a Riemann period matrix \((2\pi \sqrt{-1} \quad \Pi(k))\). Setting \(\bar{B} = \sqrt{2} \sqrt{-1} B, \eta = \zeta - \frac{a}{3}\), we obtain \(\bar{B}^2 = -8(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)\), which is denoted by \(C^0\). We may choose a cycle \(\{a_0, b_1\}\) of \(C^0\) by \(a_0^0 = b_1\) and \(b_1^0 = -a_0\), so that the following relations hold:
\[
\left\{ \begin{aligned}
\int_{a_1} d\zeta_k^0 &= -\int_{b_1} \frac{d\eta}{\sqrt{2}\sqrt{-1}B} = 2\omega^0, \\
\int_{b_1} d\zeta_k^0 &= -\int_{a_1} \frac{d\eta}{\sqrt{2}\sqrt{-1}B} = -2\omega^0.
\end{aligned} \right.
\]

If we take \( w_1^0 = \frac{\pi\sqrt{-1}}{\omega^0} \), then we see that
\[
\int_0 a_1 w_1^0 = 2\pi\sqrt{-1}, \quad \int_0 b_1 w_1^0 = 2\pi\sqrt{-1} \frac{\omega^0}{\omega^0} = -2\pi \frac{K'(p)}{K(p)} = \Pi(p).
\]

(4.9)

Thus, the curve \( C^0 \) has a Riemann period matrix \((2\pi\sqrt{-1} \Pi(p))\). Let \( \zeta_w \) be the Weierstrass \( \zeta \)-function defined by
\[
\zeta_w(z) = \frac{1}{z} + \sum_{\omega \in \mathbb{L}, \omega \neq 0} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).
\]

It is a meromorphic function on \( \mathbb{C} \) and an odd function, but not an elliptic function. In fact, we have the following properties:
\[
\left\{ \begin{aligned}
\zeta'_w(z) &= -\phi(z), \\
\zeta_w(z + 2\omega) &= \zeta_w(z) + 2\zeta_w(\omega), \\
\zeta_w(z + 2\omega') &= \zeta_w(z) + 2\zeta_w(\omega'), \\
\zeta_w(\omega + \omega') &= \zeta_w(\omega) + \zeta_w(\omega'), \\
\zeta_w(\omega) - \zeta_w(\omega') &= \frac{\pi\sqrt{-1}}{2}, \quad (\text{Legendre's relation})
\end{aligned} \right.
\]

(4.10)

where \( \omega = \omega(k) \) and \( \omega' = \omega'(k) \). In the sequel, we keep on using this notation when there is no confusion. The last equation in (4.10) follows from the 2nd and 3rd equations in (4.10) and from the fact \( \int_\Gamma \zeta_w(z) \, dz = 2\pi\sqrt{-1} \), where \( \Gamma \) is a fundamental parallelogram which contains a pole of \( \zeta_w(z) \) inside. Let \( \Omega^0_\infty \) be an Abelian differential of 2nd kind defined by
\[
\Omega^0_\infty = -\left( \phi(z) + \frac{\zeta_w(\omega)}{\omega} \right) \frac{d\zeta}{B}.
\]

(4.11)

Set \( U^0 = \int_{b_1} \Omega^0_\infty \). It follows from (4.10) and \( b_1^0 = -a_1 \) that
\[
\begin{aligned}
\int_{b_1} \Omega^0_\infty &= -\int_0^{z+2\omega} \left( \phi(z) - \frac{\zeta_w(\omega)}{\omega'} \right) \frac{dz}{\sqrt{2}\sqrt{-1}} \\
&= -\frac{1}{\sqrt{2}\sqrt{-1}} \left[ \zeta_w(z) - \frac{\zeta_w(\omega)}{\omega'} \right]_z^{z+2\omega} \\
&= -\frac{1}{\sqrt{2}\sqrt{-1}} \left( 2\zeta_w(\omega) - 2\zeta_w(\omega') \frac{\omega}{\omega'} \right) \\
&= -\frac{\pi\sqrt{-1}}{2\omega^0},
\end{aligned}
\]

hence we obtain
\[
U^0 = \frac{\pi\sqrt{-1}}{2\omega^0}.
\]

(4.12)

Similarly, we see that \( \int_a \Omega^0_\infty = 0 \). Therefore, \( \Omega^0_\infty \) is the normalized Abelian differential of 2nd kind.

Remark 4.3. Alternatively, since we have \( \Omega^0_\infty = -\left( \frac{z^2}{\sqrt{2}\sqrt{-1}} + O(1) \right) \frac{dz}{\sqrt{2}\sqrt{-1}} \), it follows from the reciprocity laws for differentials of 1st and 2nd kind that
\[
U^0 = -\frac{1}{\sqrt{2}\sqrt{-1}} \frac{d}{dz} \bigg|_{z=0} \int_{a_0}^{b_0} w_1^0 = -\frac{1}{\sqrt{2}\sqrt{-1}} \frac{\pi\sqrt{-1}}{2\sqrt{2}\sqrt{-1} \omega^0} = \frac{\pi\sqrt{-1}}{2\omega^0},
\]

\[
4.3. \text{Riemann theta function and Jacobi theta functions}
\]

For \( \Pi = \Pi(p) \), set \( \tau = \frac{\Pi}{2\pi\sqrt{-1}} \), \( \nu = \frac{z}{2\pi\sqrt{-1}} \). The Riemann theta function \( \theta(z; \Pi) \) for elliptic curve with the period matrix \((2\pi\sqrt{-1} \Pi)\) is defined by
\[ \theta(z; \Pi) = \sum_{n \in \mathbb{Z}} \exp \left( \frac{1}{2} \pi in^2 + mnz \right), \]

which absolutely and uniformly converges on a compact subset of \( \mathbb{C} \). The Jacobi theta functions are described in terms of the Riemann theta function as follows (see [27] and [28]).

\[
\begin{align*}
\theta_0(v|\tau) &= \theta(z + \pi i; \Pi), \\
\theta_1(v|\tau) &= -\exp\left(\frac{1}{8} \mathcal{P} + \frac{1}{2} (z + \pi i) \right) \theta\left(z + \pi i; \Pi + \frac{1}{2} \mathcal{P}; \Pi \right), \\
\theta_2(v|\tau) &= \exp\left(\frac{1}{8} \mathcal{P} + \frac{1}{2} z \right) \theta\left(z + \frac{1}{2} \mathcal{P}; \Pi \right), \\
\theta_3(v|\tau) &= \theta(z; \Pi).
\end{align*}
\] (4.13)

We then see that \( \theta_3(v + \frac{1}{2}) = \theta_0(v), \theta_3(v + \frac{1}{2}) = -\theta_1(v) \) (see [28]). Although the Jacobi theta functions are not elliptic functions, there are some relations between the Jacobi theta functions and the Jacobi elliptic functions as follows.

\[
\begin{align*}
\text{sn}(u, p) &= \frac{\theta_3(0) \theta_3(v)}{\theta_3(0) \theta_3(v)}, \\
\text{cn}(u, p) &= \frac{\theta_0(0) \theta_3(v)}{\theta_3(0) \theta_3(v)}, \\
\text{dn}(u, p) &= \frac{\theta_0(0) \theta_3(v)}{\theta_3(0) \theta_3(v)}, \\
\text{where } u &= 2K(p) v.
\end{align*}
\] (4.14)

Moreover, the following beautiful formula is known

\[
\frac{d^2}{du^2} \log \theta_0 \left( \frac{u}{2K(p)} \right) = \frac{E(p)}{K(p)},
\] (4.15)

where \( E(p) = \int_0^{\frac{\mathcal{P}}{2}} \text{dn}^2(u, p) \ du \) is the complete elliptic integral of 2nd kind. We now calculate

\[
\begin{align*}
-2 \frac{\partial}{\partial x} \frac{\partial}{\partial t} \log \theta_0 \left( \frac{1}{2\omega^0} \right) \tau &= \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \log \theta_0 \left( \frac{1}{2\omega^0} \right) \tau \\
&= \frac{1}{2} \frac{\partial^2}{\partial \mathcal{P}^2} \log \theta_0 \left( \frac{1}{2\omega^0} \right) \tau \\
&= \frac{1}{2} \left( \frac{1}{2\omega^0} \right)^2 (2K(p))^2 \left( \text{dn}^2 \left( \frac{K(p)}{\omega^0} \mathcal{P}, p \right) - \frac{E(p)}{K(p)} \right) \\
&= (\zeta_2 - \zeta_4) \left( 1 - p^2 \text{sn}^2(\sqrt{2} \sqrt{\zeta_2 - \zeta_4} \mathcal{P}, p) - \frac{E(p)}{K(p)} \right) \\
&= (\zeta_2 - \zeta_4) \left( 1 - \frac{E(p)}{K(p)} \right) - (\zeta_2 - \zeta_4) \text{sn}^2(\sqrt{2} \sqrt{\zeta_2 - \zeta_4} \mathcal{P}, p),
\end{align*}
\]

where we have used (4.2) and \( p^2 = \frac{\zeta_4 - \zeta_2}{\zeta_2 - \zeta_4} \). If we set \( C = \zeta_4 + \frac{E(p)}{K(p)} (\zeta_2 - \zeta_4) \) then we have

\[
C - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial t} \log \theta_0 \left( \frac{1}{2\omega^0} \right) \tau = \zeta_4 - (\zeta_2 - \zeta_4) \text{sn}^2(\sqrt{2} \sqrt{\zeta_2 - \zeta_4} \mathcal{P}, p) \\
= Y(\mathcal{P}) = \exp(u(\mathcal{P})),
\]

which is a solution of (1.1) (see (4.1)). On the other hand, for \( e = \pi \sqrt{-1} \) we see that

\[
\theta(U^0(x - t) + e; \Pi(p)) = \theta \left( \frac{\pi i}{\omega^0} \mathcal{P} + \sqrt{-1}; \Pi(p) \right) = \theta_0 \left( \frac{1}{2\omega^0} \mathcal{P} \right) \tau.
\]

Thus, we obtain the following formula.

\[
\exp(u(\mathcal{P})) = C - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial t} \log \theta(U^0(x - t) + e; \Pi(p)),
\] (4.17)

where \( C = \zeta_4 + \frac{E(p)}{K(p)} (\zeta_2 - \zeta_4) \) and \( e = \pi \sqrt{-1} \).

**Lemma 4.4.** The \( C \) in (4.17) is given by

\[
C = \frac{a}{S} - \frac{\zeta_2(\omega')}{\omega'}.
\] (4.18)
Proof. First of all, we have from the definition of $E(p)$ that

$$\mathfrak{P}(\sqrt{2} \sqrt{-1} \gamma) + \frac{a}{3} = Y(\gamma + \omega^0) = \zeta_2 - (\zeta_2 - \zeta_3) \sin^2 (\sqrt{2} \sqrt{-1} \gamma + \sqrt{-1} K'(p), p).$$

We calculate

$$\int_0^{K(p)} \sin^2 u \ du = \sqrt{2} \sqrt{-1} \zeta_2 - \zeta_1 \int_{-\omega^0}^{\omega^0} \sin^2 (\sqrt{2} \sqrt{-1} \gamma + \sqrt{-1} K'(p), p) \ dy = \frac{2}{3} (\zeta_2 - \zeta_1) \int_{-\omega^0}^{\omega^0} \mathfrak{P}(\sqrt{2} \sqrt{-1} \gamma) + \frac{a}{3} - \zeta_2) \ dy = \frac{2}{3} \zeta_2 \left( \frac{1}{\sqrt{2}} \zeta_0 (\omega^0) + \left( \frac{a}{3} - \zeta_2 \right) \omega^0 \right) = \frac{K(p)}{3} \left( \frac{\zeta_0 (\omega^0)}{\omega^0} + \frac{a}{3} - \zeta_2 \right),$$

where we used (4.2), (4.3) and (4.10). \hspace{1cm} \Box

4.4. Expression of Blaschke immersions in terms of Riemann theta function

We gave a formula of Blaschke immersion in terms of elliptic functions in (4.6). In this section, we rewrite the formula (4.6) in terms of Riemann theta function. For this purpose, we consider the elliptic curve $\mathcal{C}$ as a Prym variety of a compact Riemann surface $\hat{\mathcal{C}}$ of genus 2. When $u = u(\gamma)$ is given by (4.1), the spectral curve is defined by the equation $\det (U(\lambda) + V(\lambda) - \mu I) = 0$, where $U(\lambda)$ and $V(\lambda)$ are as in (2.5). Since the spectral curve is independent of the parameter $\gamma$, if we take $\gamma = 0$ then the initial condition of $u = u(\gamma)$ implies that the defining equation of the spectral curve is $\mu^2 = 2\mu = \lambda^2 - \lambda^3$. The unraveled covering map $\lambda \mapsto \nu = \lambda^3$ yields $\mu^3 = 2\mu = \nu + \nu^{-1}$. Projectivizing this affine plane curve, we obtain a compact Riemann surface $\hat{\mathcal{C}}$ of genus 2. Since $(\nu - \nu^{-1})^2 = (\nu + \nu^{-1})^2 - 4 = (\mu^3 - 2\mu) - 4 = \nu^2 - 4$, we see that $\hat{\mathcal{C}}$ may be regarded as a hyperelliptic curve $(\mu^3 - 2\mu)^2 = 4 = \nu^2$. We may observe that $\hat{\mathcal{C}}$ admits holomorphic involution $\sigma$ and an anti-holomorphic involution $\rho$ defined by $\sigma(\mu, \nu) = (\mu, -\nu)$ and $\rho(\mu, \nu) = (\bar{\mu}, \sigma^{-1}).$ If we transform the parameters by $\nu = 2^2 - 2\zeta + 2a, \nu = \nu^{-1} = \bar{b}$,

then we obtain the elliptic curve $\hat{\mathcal{C}}^0: \beta^2 = 8(\zeta - \zeta_2)(\zeta - \zeta_3)$. We may use $\hat{\mathcal{C}}^0$ as a Prym variety of $\hat{\mathcal{C}}$.

Let $\varphi: \hat{\mathcal{C}} \to \hat{\mathcal{C}}^0$ be a covering map defined as above. We may choose the canonical homology basis \{\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2\} of $\hat{\mathcal{C}}$ so that

$$\varphi \hat{a}_1 = \hat{a}_1 + \hat{a}_2, \varphi \hat{b}_1 = \hat{b}_1 + \hat{b}_2, \sigma \hat{a}_1 = -\hat{a}_2, \sigma \hat{b}_1 = -\hat{b}_2$$

hold (see Figure 3). Let $\{w_1, w_2\}$ be the basis of $H(\hat{\mathcal{C}}, \mathbb{C})$ such that $(2\sqrt{-1} \ 1 \ \hat{T})$ is the Riemann period matrix for $\hat{\mathcal{C}}$, where $\hat{T} = (\hat{T}_{ij})$ and $\hat{T}_{ij} = \int_{\hat{a}_i}^{\hat{a}_j} w_i$ for $i, j = 1, 2$. It is well known that the Jacobian variety $\mathcal{J}(\hat{\mathcal{C}})$ of $\hat{\mathcal{C}}$ is a complex torus $\mathbb{C}^2/\Lambda$, where $\Lambda = \text{Span}_{\mathbb{Z}}(2\sqrt{-1} \ 1 \ \hat{T})$. The basis $\{w_1, w_2\}$ can be constructed as follows. Set $U = \frac{d\hat{T}}{d\hat{T}}$, $U_2 = \frac{dU}{d\hat{T}}$. We then see that $\varphi^* w_1 = -\frac{\sqrt{-1} \hat{T}}{\hat{T}} u_2$. We want to look for $w_1, w_2$ which satisfy $w_1 + w_2 = -\frac{\sqrt{-1} \hat{T}}{\hat{T}} u_2$. Note that $\{u_1, u_2\}$ is a basis of $H(\hat{\mathcal{C}}, \mathbb{C})$ and $\varphi^* u_1 = u_1$ and $\varphi^* u_2 = -u_2$. If we set $G_{ij} = \int_{\hat{a}_i}^{\hat{a}_j} u_i$ for $i, j = 1, 2$ then we have $G_{11} = -G_{21}$ and $G_{12} = -G_{22} = 2\omega^0$. The last equation follows from $\int_{\hat{a}_i}^{\hat{a}_j} \varphi^* w_0 = \int_{\hat{a}_i}^{\hat{a}_j} w_0 = 2\pi \sqrt{-1}$ for $j = 1, 2$. The Riemann bilinear relation means that $G_{11} = 0$. Thus, we find $w_1, w_2$:

$$w_1 = \frac{\sqrt{-1}}{G_{11}} \left( u_1 - \frac{G_{11}}{2\omega^0} u_2 \right), \quad w_2 = \frac{\sqrt{-1}}{G_{11}} \left( -u_1 - \frac{G_{11}}{2\omega^0} u_2 \right),$$

with the properties $\varphi^* w_0 = w_1 + w_2$ and $\sigma^* w_1 = -w_2$. If we set $T_{ij} = \int_{\hat{a}_i}^{\hat{a}_j} (w_1 - w_2)$ then we have $T_{11} = -T_{21} = T$. We then know that
\[ \hat{\mathcal{T}}_{11} = \hat{\mathcal{T}}_{22} = \frac{1}{2}(\Pi + T), \quad \hat{\mathcal{T}}_{12} = \hat{\mathcal{T}}_{21} = \frac{1}{2}(\Pi - T) \] (4.19)

for \( \Pi = \Pi(p) \), which is as in (4.9). Moreover, since \( \hat{\rho}^* u_j = -u_j \) for \( j = 1, 2 \) we have that
\[ \rho(\hat{a}_j) = -\hat{a}_j, \quad \rho(\hat{b}_j) = \hat{b}_j \quad \text{for} \quad j = 1, 2. \] (4.20)

In fact, since \( \mathcal{C} \) is an \( M \)-curve (see [27]), there are 3 real ovals, which are connected components of fixed points of \( \rho \), and we may take \( \hat{b}_j \) (\( j = 1, 2 \)) as these ovals. In this case, the number \( C_{11} \) above is real and \( \hat{\rho}^* u_j = u_j \) for \( j = 1, 2 \) hold, which is compatible with the reality of \( \Pi \). We define a map \( B : \hat{\mathcal{C}} \longrightarrow \text{Prym}(\hat{\mathcal{C}}) \) by \( B(\hat{P}) = \int_{\hat{a}_j}^{\hat{b}_j} (w_1 + w_2) \). If we change \( \hat{P} \to \hat{P} + \sum_{j=1}^{2} m_j \hat{a}_j + n_j \hat{b}_j \), where \( m_j, n_j \in \mathbb{Z} \) and \( j = 1, 2 \), then \( B(\hat{P}) \) changes into \( B(\hat{P}) + 2\pi \sqrt{-1} m_j + \Pi n_j \), hence it is a well-defined map into \( \text{Prym}(\hat{\mathcal{C}}) \cong \mathcal{C}^0 / \Gamma \), where \( \Gamma = \text{Span}_\mathbb{Z} \{2\pi \sqrt{-1}, \Pi\} \). We have the relations as follows.
\[ \overline{B(\rho(P))} = B(\hat{P}), \quad B(\sigma(\hat{P})) = -B(\hat{P}) \pmod{\Gamma} \] (4.21)

We may consider the function \( f \) on \( \hat{\mathcal{C}} \) defined by \( f((\hat{P}) = \theta(B(\hat{P}) - e) \) for \( e \in \mathcal{C} \), where \( \theta \) is the Riemann theta function \( \theta(z) = \theta(z; \Pi(p)) \) as in section 4.3. Let \( \hat{\mathcal{C}}_0 \) be the simply-connected domain obtained by removing all \( \hat{a} \) and \( \hat{b} \)-cycles from \( \hat{\mathcal{C}} \). The boundary of \( \hat{\mathcal{C}}_0 \) is denoted by \( \partial \hat{\mathcal{C}}_0 = \hat{a}_1 \hat{a}_2^{-1} \hat{b}_1^{-1} \hat{a}_2 \hat{a}_2^{-1} \hat{b}_2^{-1} \). We then know that \( f \) is single-valued on \( \hat{\mathcal{C}}_0 \). However, since the theta function satisfies the relations \( \theta(z + 2\pi \sqrt{-1}) = \theta(z), \theta(z + \Pi) = \exp \left( -\frac{\pi}{4} \Pi \right) \theta(z) \), it is no problem to consider \( f \) on \( \hat{\mathcal{C}} \) when we search for the zeros of the function \( f \). With respect to this problem, there is a study of J. Fay.

**Lemma 4.5 (cf [29])**. If \( f(\hat{P}) = \theta(B(\hat{P}) - e) \neq 0 \) then the zeros of \( f \) is a degree 2 divisor \( \hat{\mathcal{D}} \). Moreover, we have \( B(\hat{\mathcal{D}}) \equiv K + 2e \pmod{\Gamma} \), where \( K \) is given by
\[ K = \frac{1}{2\pi \sqrt{-1}} \sum_{j=1}^{2} \int_{\hat{a}_j}^{\hat{b}_j} B(w_1 + w_2) - \sum_{j=1}^{2} B(\hat{b}_j(0)), \]
and \( \hat{b}_j(0) \) is the initial point of the path \( \hat{b}_j \) in the boundary \( \partial \hat{\mathcal{C}}_0 \).

This is, of course, a special one of the higher genus case, which is stated in the next section. Fay states the property of the divisor \( \hat{\mathcal{D}} \) in terms of the Abelian map \( A : \hat{\mathcal{C}} \longrightarrow j(\hat{\mathcal{C}}) \). Therefore, we here address the outline of the proof. When \( \hat{P} \in \hat{a}_j(\hat{b}_j) \), we denote by \( \hat{P}^- \) the corresponding point of \( \hat{a}_j^{-1}(\hat{b}_j^{-1}) \). Set \( f^-(\hat{P}) = f(\hat{P}^-) \). If \( f \) is not identically zero, then the number \( n \) of the zeros of \( f \) is given by
\[ n = \frac{1}{2 \pi \sqrt{-1}} \int_{\partial \Omega} d \log f \]
\[ = \frac{1}{2 \pi \sqrt{-1}} \left\{ \sum_{j=1}^{2} \int_{\partial \Omega_{j}} (d \log f - d \log f^{-}) - \int_{\partial \Omega} (d \log f - d \log f^{-}) \right\}. \]

But, since we may observe that \( B(\hat{\varphi}) = B(\hat{\varphi}) + \Pi \) when \( \hat{\varphi} \in \hat{\partial} \), the property of \( \theta \) yields
\[ n = \frac{1}{2 \pi \sqrt{-1}} \int_{\partial \Omega} B(\hat{\varphi}) d \log f. \]
Using the same calculation as in above, we obtain
\[ B(\hat{\varphi}) = \frac{1}{2 \pi \sqrt{-1}} \sum_{j=1}^{2} \int_{\partial \Omega_{j}} B(w_{1} + w_{2}) + \sum_{j=1}^{2} \int_{\partial \Omega} d \log f, \quad (\text{mod } \Gamma). \]

Finally, if we denote by \( \hat{\partial}_{j} \) the end point of the path \( \hat{\partial} \) in the boundary \( \partial \hat{\Omega}_{0} \), the property of \( \theta \) yields
\[ f(\hat{\partial}_{j}(1)) = \exp(-\frac{1}{2}\Pi - B(\hat{\partial}_{j}(0)) + e(\hat{\partial}_{j}(0)), \quad \text{which implies the formula of } B(\hat{\varphi}) \text{ stated in Lemma 4.25.} \]

In our case, take \( e = \pi \sqrt{-1} \). Then, since the zeros of \( \theta(x) \) are given by \( z = \pi \sqrt{-1} \pm \frac{1}{2} \Pi \) (mod \( \Gamma \)), it follows from (4.32) below that
\[ \hat{\varphi} = \hat{\varphi}_{\infty} \quad \text{and } (\text{the points on } \hat{\Omega} \text{ which corresponds to the values } \hat{\varphi} = 0 \text{ and } \hat{\varphi} = \infty), \]
respectively, where \( \hat{\varphi} \) and \( \hat{\varphi}^{-1} \) are the local coordinates around \( \hat{\Omega}_{0} \) and \( \hat{\Omega}_{\infty} \) of \( \hat{\Omega} \), respectively.

(1) \( \hat{\varphi} \) is a meromorphic function on \( \hat{\Omega} \setminus \{ \hat{\Omega}_{0}, \hat{\Omega}_{\infty} \} \) and the divisor of the poles is non-special and given by \( \hat{\partial}_{\infty} = \{ \hat{\partial}_{0}, \hat{\partial}_{\infty} \} \), which is independent of the parameters \( x, t \), and \( \hat{\varphi} \).

(2) \( \hat{\varphi} \) has the following asymptotic expansions.
\[ \hat{\varphi} = \begin{cases} \exp(x \hat{\varphi}^{-1}) \left( 1 + \sum_{j=1}^{2} \xi_{j} \hat{\varphi}^{-1} \right) & \text{near } \hat{\Omega}_{0} \\ \exp(-t \hat{\varphi}) \left( 1 + \sum_{j=1}^{2} \eta_{j} \hat{\varphi}^{-1} \right) & \text{near } \hat{\Omega}_{\infty}. \end{cases} \]

Consider the Abelian differential \( \omega_{\hat{\varphi}} \) defined by \( \omega_{\hat{\varphi}} = d \log \hat{\varphi} \). Then, the principal parts of \( \omega_{\hat{\varphi}} \) around \( \hat{\varphi} = 0 \) and \( \hat{\varphi} = \infty \) are given by \( -x \hat{\varphi}^{-2} d\hat{\varphi} \) and \( -t \hat{\varphi}^{-2} d\hat{\varphi}^{-1} \), respectively. Since the residue of \( \omega_{\hat{\varphi}} \) is zero, it follows from the condition (1) that there are two points of zeros, which are denoted by \( \hat{\varphi}_{j}(x, t) \) and \( \hat{\varphi}_{j}(x, t) \). Therefore, \( \omega_{\hat{\varphi}} \) may be described as follows.
\[ \omega_{\hat{\varphi}} = x \hat{\Omega}_{\infty} + t \hat{\Omega}_{0} + \sum_{j=1}^{2} \omega(\hat{\partial}_{j}, \hat{\varphi}) + \sum_{j=1}^{2} m_{j} w_{j}, \quad (4.22) \]
where \( \omega(\hat{\partial}_{j}, \hat{\varphi}) \) is the normalized Abelian differential of 3rd kind with the principal part \( (\hat{\varphi} - \hat{\varphi}_{j})^{-1} d\varphi \) and \( - (\hat{\varphi} - \hat{\varphi}_{j})^{-1} d\varphi \), and \( \hat{\Omega}_{\infty} \) and \( \hat{\Omega}_{0} \) are the normalized Abelian differentials of second kind of the forms
\[ \hat{\Omega}_{\infty} = (-\hat{\varphi}^{-2} + O(1)) d\hat{\varphi} \quad (\text{near } \hat{\varphi} = 0) \quad \text{and} \quad \hat{\Omega}_{0} = (\hat{\varphi}^{-2} + O(1)) d\hat{\varphi}^{-1} \quad (\text{near } \hat{\varphi} = \infty), \]
respectively. We may define \( \hat{\Omega}_{\infty} \) and \( \hat{\Omega}_{0} \) as follows.
\[ \hat{\Omega}_{\infty} = -\varphi^{2} \hat{\Omega}_{\infty} + \frac{1}{2} d\mu, \quad \hat{\Omega}_{0} = \varphi^{2} \hat{\Omega}_{0} + \frac{1}{2} d\mu. \quad (4.23) \]

If we set \( \mu = \hat{\varphi}^{-1} \), then we see that
\[ \hat{\Omega}_{\infty} = (-\hat{\varphi}^{-2} + O(1)) d\hat{\varphi}, \quad \hat{\Omega}_{0} = (C + O(\hat{\varphi}^{2})) d\hat{\varphi}, \]
ne \( \hat{\varphi} = 0 \), where \( C \) is as that in (4.18). It then follows from (4.11) that \( \sigma \hat{\Omega}_{\infty} = -\hat{\Omega}_{\infty} \) and \( \sigma \hat{\Omega}_{0} = -\hat{\Omega}_{0} \). Now, since \( \int_{\partial_{1}} \omega_{\hat{\varphi}} = 2\pi \sqrt{-1} m_{j} \), the integration of (4.22) over the cycle \( \hat{\partial}_{1} \) yields that \( m_{j} \in \mathbb{Z} \). Set
\[ U = (U_{1}, U_{2}) = \left( \int_{\hat{\partial}_{1}} \hat{\Omega}_{\infty}, \int_{\hat{\partial}_{0}} \hat{\Omega}_{\infty} \right) \quad \text{and} \quad V = (V_{1}, V_{2}) = \left( \int_{\hat{\partial}_{1}} \hat{\Omega}_{0}, \int_{\hat{\partial}_{0}} \hat{\Omega}_{0} \right). \]
Then, the various properties of
\[ \hat{\Omega}_\infty, \hat{\Omega}_0 \text{ under the involutions } \sigma, \rho, \text{ and } \int_{\hat{\gamma}} \hat{\Omega}_\infty = -\int_{\hat{\gamma}} \hat{\Omega}_0 = -\int_{\hat{\gamma}} \hat{\Omega}_0 = -U_0 \text{ imply that} \]
\[ U_1 = U_2 = -U_0, \quad V_1 = V_2 = -U_0 = U_0. \]  
(4.24)

**Remark 4.6.** Since \( w_1 + w_2 = \frac{\pi \sqrt{-1}}{\omega^0} (1 + O(\hat{\nu}^2))d\bar{v}, \) we have
\[ \frac{d}{\bar{v}} \left|_{\bar{v}=0} \right. B(\hat{P}) = \frac{\pi \sqrt{-1}}{\omega^0} = 2U_0^0 = -2U_j \quad (j = 1, 2) \]
which is nothing but the consequence of the reciprocity law for differentials of 1st and 2nd kinds.

The integration of (4.22) over the cycle \( \hat{\gamma} \) and the reciprocity law gives
\[ \sum_{j=1}^2 \int_{\hat{\gamma}} (w_1 + w_2) = 2(x - t)U_0^0 \mod \Gamma. \]  
(4.25)

Define \( F(z) \) by
\[ F(z) = -\frac{1}{\sqrt{2} \sqrt{-1}} \zeta_w(z) + \frac{z}{\sqrt{2} \sqrt{-1}} \zeta_w(\omega') \zeta_w(\omega'). \]

Let \( \{z_1, z_2, z_3\} \) be the set of points where \( \Psi'/(z) = 0 \). We then assume that \( z_1 = \omega', \; z_2 = \omega, \; z_3 = \omega + \omega' \). Note that \( \Psi(z_1) = \eta_1, \Psi(z_2) = \eta_2, \Psi(z_3) = \eta_3 \). We denote by \( \hat{P}_1, \hat{P}_2, \hat{P}_3 \) the points on \( \hat{\mathcal{C}} \) corresponding to \( \eta_1, \eta_2, \eta_3 \), respectively. Precisely, \( \hat{P}_1 = (\mu_1, 1), \hat{P}_2 = (\mu_2, 1), \hat{P}_3 = (\mu_3, 1) \) by the affine coordinate \( \hat{P} = (\mu, \nu) \). We choose a path \( \gamma \) from \( \hat{P}_0 \) to \( \hat{P}_3 \) as in Figure 3 and we fix a path from \( \hat{P}_0 \) to \( \hat{P}_1 \) in the following. Since we have \( \int_{\hat{\gamma}} (w_1 + w_2) \equiv 0 \mod \Gamma \) and we also have \( \int_{\hat{\gamma}} (w_1 + w_2) \) is purely imaginary by \( \rho(\gamma) = -\gamma \) and \( \hat{P}_0 w_j = w_j \), we may fix a path from \( \hat{P}_1 \) to \( \hat{P}_0 = \sigma(\hat{P}_1) \) which passes through only \( \hat{\gamma} \)-cycles. It then follows from
\[ \int_{\hat{\gamma}} (w_1 + w_2) = -\int_{\hat{\gamma}}^{\hat{\gamma}_1} (w_1 + w_2), \]
\[ \int_{\hat{\gamma}_1}^{\hat{\gamma}_2} (w_1 + w_2) = \int_{\hat{\gamma}_1}^{\hat{\gamma}_2} w_1^0 + \int_{\hat{\gamma}_1}^{\hat{\gamma}_2} w_2^0 = 2\pi \sqrt{-1} + \int_{\hat{\gamma}} (w_1 + w_2), \]
that
\[ \int_{\hat{\gamma}} (w_1 + w_2) \equiv \pi \sqrt{-1} \mod 2\pi \sqrt{-1} \mathbb{Z}. \]

In fact, this is verified directly in (4.32) below. We have
\[ \int_{\hat{\gamma}} \hat{\Omega}_\infty + \frac{1}{2} \mu_1 \equiv F(z) + \frac{1}{2} \mu_1, \quad \int_{\hat{\gamma}} \hat{\Omega}_0 + \frac{1}{2} \mu_1 \equiv -F(z) + \frac{1}{2} \mu_1 \mod \Gamma. \]

We have \( \int_{\hat{\gamma}} \hat{\Omega}_\infty + \frac{1}{2} \mu_1 = \hat{\nu}^{-1} + O(\hat{\nu}) \) and \( \int_{\hat{\gamma}} \hat{\Omega}_0 + \frac{1}{2} \mu_1 = C \hat{\nu} + O(\hat{\nu}^2) \) (near \( \hat{\nu} = 0 \)). We put
\[ \begin{cases} \Phi_0(x, t, \hat{P}) = \frac{\theta(B(\hat{P}) - (x - t)U^0 - \epsilon)}{\theta(B(\hat{P}) - \epsilon)}, \\ \Phi_0(x, t, \hat{P}) = \exp \left( x \int_{\hat{\gamma}} \hat{\Omega}_\infty + \frac{1}{2} \mu_1 \right) + t \left( \int_{\hat{\gamma}} \hat{\Omega}_0 + \frac{1}{2} \mu_1 \right) \end{cases} \]  
(4.26)

where \( \epsilon \in \mathbb{C} \) is chosen so that \( f(\hat{P}) \not\equiv 0 \) and the divisor of the poles of \( \Phi_0 \) is \( \hat{D} = \{ \hat{P}_0, \hat{P}_j \} \). We observe that \( \Phi_0 \Phi_0 \) is invariant under the translation \( \hat{P} \rightarrow \hat{P} + m_1 \hat{\gamma}_1 + n_1 \hat{\gamma}_2 \) by the property of \( \theta \) and (4.24). Therefore, it is a meromorphic function on \( \hat{\mathcal{C}} \). It then follows from lemma 4.5 and (4.25) that \( \hat{\Psi} \Phi_0^{-1} \Phi_0^{-1} \) is a holomorphic function on \( \hat{\mathcal{C}} \), hence it is a constant. Evaluating it at \( \hat{\nu} = 0 \) we see that the constant is equal to \( \frac{\theta(\epsilon)}{\theta((x - t)U^0 + \epsilon)} \). We thus obtain the following.

**Lemma 4.7.** \( \hat{\Psi}(x, t, \hat{P}, \epsilon) \) is given by
\[ \hat{\Psi}(x, t, \hat{P}, \epsilon) = \frac{\theta(B(\hat{P}) - (x - t)U^0 - \epsilon)}{\theta(B(\hat{P}) - \epsilon)} \frac{\theta(\epsilon)}{\theta((x - t)U^0 + \epsilon)} \Phi_0(x, t, \hat{P}), \]
where \( \Phi_0(x, t, \hat{P}) \) is as that in (4.26), and we fix the path from \( \hat{P}_0 \) to \( \hat{P}_1 \).
Moreover, using the expression of $\hat{\Psi}$ we see that the following reality condition holds:

$$(\bar{\sigma}^*\hat{\Psi})(x, t, \hat{P}, e) = \hat{\Psi}(t, x, \sigma(\hat{P})) = \overline{\hat{\Psi}(x, t, \sigma(\hat{P}), e)}$$

(4.27)

As we see later, we may prove that $\hat{\Psi}$ satisfies the Schrödinger equation $\partial_t \hat{\Psi} = e^{i\hat{V}}\hat{\Psi}$. For $U(\lambda)$, $V(\lambda)$ given in (2.5), let $F_\lambda: \mathbb{R} \rightarrow \mathbb{C}$ be a solution of the differential equation

$$dF_\lambda = F_\lambda (U(\lambda)dt + V(\lambda)dx).$$

To simplify this equation, define $\hat{F}$ by

$$\hat{F}_\lambda = F_\lambda \left( \begin{array}{ccc} \lambda^2 e^t & 0 & 0 \\ 0 & \lambda^2 e^{-t} & 0 \\ 0 & 0 & 1 \end{array} \right).$$

We then obtain

$$d\hat{F} = \hat{F}_\lambda (U(\nu)dt + V(\nu)dx),$$

where

$$\hat{U}(\nu) = \left( \begin{array}{ccc} u_t & 0 & \nu \\ 1 & -u_t & 0 \\ 0 & 1 & 0 \end{array} \right), \quad \hat{V}(\nu) = \left( \begin{array}{ccc} 0 & e^{-2\nu} & 0 \\ 0 & 0 & e^\nu \\ e^{-\nu} & 0 & 0 \end{array} \right).$$

(4.28)

If we express $\hat{F}_\lambda$ as $\hat{F}_\lambda = (\psi_0, \psi_1, \psi_2)$ then we write down the system of differential equations which we must solve as follows.

$$\begin{cases}
\partial_t \psi_0 = u_t \psi_0 + \psi_1 \\
\partial_t \psi_1 = -u_t \psi_1 + \psi_2 \\
\partial_t \psi_2 = \nu \psi_0
\end{cases} \quad \begin{cases}
\partial_t \psi_0 = e^{-\nu} \psi_1 \\
\partial_t \psi_1 = e^{2\nu} \psi_0 \\
\partial_t \psi_2 = e^{\nu} \psi_1
\end{cases}$$

(4.29)

Lemma 4.8. $\{\psi_0 = \nu^{-1} \partial_0 \hat{\Psi}, \psi_1 = e^{-\nu} \partial_0 \hat{\Psi}, \psi_2 = \hat{\Psi}\}$ is a solution of the system of differential equations (4.29) above.

Set $\hat{\psi}_0 = \nu^{-1} \partial_0 \hat{\Psi}, \hat{\psi}_1 = e^{-\nu} \partial_0 \hat{\Psi}, \hat{\psi}_2 = \hat{\Psi}$ with $\nu = \nu^3$. We define $\hat{W}(x, t, \hat{P})$ by

$$\hat{W}(x, t, \hat{P}) = \sum_{j=0}^{2} \hat{\psi}_j(\hat{P}) \cdot (\hat{\sigma}^*\hat{\psi}_j)(\rho(\hat{P})).$$

(4.30)

It follows from (4.29) that $\hat{W}$ is independent of the parameters $x$ and $t$. Thus, we may write $\hat{W}(x, t, \hat{P}) = \hat{W}(\hat{P})$.

We will arrive at the following conclusion.

Theorem 4.9. For $\hat{\Psi}$ in lemma 4.7, the Blaschke immersion $\psi(\hat{x}, \hat{y})$ given in (4.6) can be described in terms of Riemann theta functions as

$$\psi(\hat{x}, \hat{y}) = \left\{ \begin{array}{l}
\hat{\psi}(x, t, \hat{P}_1, \sqrt{W}(\hat{P}_1)) \\
\hat{\psi}(x, t, \hat{P}_2, \sqrt{W}(\hat{P}_2)) \\
\hat{\psi}(x, t, \hat{P}_3, \sqrt{W}(\hat{P}_3))
\end{array} \right\},$$

which is a solution of (2.1), where $\hat{\zeta}_1 \hat{\zeta}_2 \hat{\zeta}_3 = 1$.

Moreover, the equation $\partial_0 \hat{\psi}, \partial_0 \hat{\psi}, \hat{\psi} = e^{\nu^3} \hat{\psi}$ holds, where $e^{\nu^3}$ coincides with one given by (4.17), that is, a solution of the Tzitzéica equation (1.1).

Proof. Since $\hat{\Psi}$ is single-valued, we carry out the calculations of integrations using the paths $\gamma_+, \hat{B}_+, \hat{A}_+$ by the parts of the disjoint unions $\gamma = \gamma_+ \cup \gamma_-, \hat{B}_1 = \hat{B}_+ \cup \hat{B}_-, \hat{A}_1 = \hat{A}_+ \cup \hat{A}_-$

$$\hat{P}_0 \xrightarrow{\gamma} \hat{P}_1 \xrightarrow{\hat{B}_1} \hat{P}_3 \xrightarrow{\hat{A}_1} \hat{P}_2$$

Let $\{\hat{P}_1, \hat{P}_2, \hat{P}_3\}$ be the points of $\hat{C}$ as above, which are also points of $\nu = 1$. It follows from (4.10) and (4.11) that

$$\begin{cases}
\int_{\hat{P}_1}^{\hat{P}_1} \hat{\Omega}_\infty + \frac{1}{2} \mu_1 = F(z_1) + \frac{1}{2} \mu_2 \\
\int_{\hat{P}_1}^{\hat{P}_1} \hat{\Omega}_0 + \frac{1}{2} \mu_1 = -F(z_1) + \frac{1}{2} \mu_2 \\
F(z_1) = 0, \quad F(z_2) = \frac{1}{2} U^0, \quad F(z_3) = \frac{1}{2} U^0,
\end{cases}$$

where
where \( j = 1, 2, 3 \). Therefore, we obtain
\[
\Phi_\alpha(x, t, \hat{p}) = \begin{cases} 
\exp\left(\frac{x + t}{2} \mu_1\right) & \text{for } j = 1, \\
\exp\left(\frac{1}{2} U^0(x - t) + \frac{x + t}{2} \mu_2\right) & \text{for } j = 2, \\
\exp\left(\frac{1}{2} U^0(x - t) + \frac{x + t}{2} \mu_3\right) & \text{for } j = 3.
\end{cases}
\]  
(4.31)

On the other hand, since \( \varphi^* \frac{d \zeta}{B} = \varphi^* \frac{dz}{\sqrt{2 \sqrt{-1}}} \), we obtain
\[
B(\hat{p}) = \pi \frac{1}{\omega^0} \int_{\hat{p}_j}^\infty \varphi^* \frac{dz}{\sqrt{2 \sqrt{-1}}} = \pi \frac{1}{\sqrt{2 \sqrt{-1}} \omega^0} \sqrt{z_j}
\]
\[
= \begin{cases} 
\frac{1}{\omega^0} \Phi(x, \epsilon, \hat{p}, j) & \text{for } j = 1, \\
\frac{1}{2} \Pi & \text{for } j = 2, \\
\frac{1}{2} \Pi + \frac{1}{\omega^0} \Phi(x, \epsilon, \hat{p}, j) & \text{for } j = 3.
\end{cases}
\]  
(4.32)

We here remark that if \( u = \sqrt{2} \sqrt{-1} \frac{\zeta}{\sqrt{2} \sqrt{-1}} \hat{p} \) then \( v = \frac{u}{2K(p)} = \frac{U^0(x - t)}{2 \pi \sqrt{-1}} \) by (4.2) and (4.12). We now set \( u = \sqrt{2} \sqrt{-1} \frac{\zeta}{\sqrt{2} \sqrt{-1}} \hat{p} \). It then follows from (4.13), (4.14), (4.31) and (4.32) that
\[
d_n(u, p) = \frac{\theta(\pi \sqrt{-1}; \Pi) \theta(U^0(x - t); \Pi)}{\theta(0; \Pi) \theta(U^0(x - t) + \pi \sqrt{-1}; \Pi)} = \frac{1}{\sqrt{2 \sqrt{-1}}} \Phi(x, \epsilon, \hat{p}, j) \exp\left(-\frac{x + t}{2} \mu_1\right),
\]
\[
s_n(u, p) = \frac{\theta(\pi \sqrt{-1}; \Pi) \theta(U^0(x - t) + \pi \sqrt{-1} + \frac{1}{2} \Pi; \Pi)}{\theta(\frac{1}{2} \Pi; \Pi) \theta(U^0(x - t) + \pi \sqrt{-1}; \Pi)} \exp\left(\frac{1}{2} U^0(x - t)\right) = \frac{1}{\sqrt{2 \sqrt{-1}}} \Phi(x, \epsilon, \hat{p}, j) \exp\left(-\frac{x + t}{2} \mu_2\right),
\]
\[
c_n(u, p) = \frac{\theta(\pi \sqrt{-1}; \Pi) \theta(U^0(x - t) + \frac{1}{2} \Pi; \Pi)}{\theta(\frac{1}{2} \Pi; \Pi) \theta(U^0(x - t) + \pi \sqrt{-1}; \Pi)} \exp\left(\frac{1}{2} U^0(x - t)\right) = \frac{1}{\sqrt{2 \sqrt{-1}}} \Phi(x, \epsilon, \hat{p}, j) \exp\left(-\frac{x + t}{2} \mu_3\right).
\]

Take a function \( C(\hat{p}) \) on \( \hat{c} \) and define \( \Phi \) by
\[
\Phi(x, \epsilon, \hat{p}, j, \pi \sqrt{-1}) = C(\hat{p}) \Phi(x, \epsilon, \hat{p}, j, \pi \sqrt{-1}),
\]
with \( C(\varphi(p) \hat{c}) = C(\hat{p}) \), where \( C(\hat{p}_j) = C(\hat{p}_j) = 1, C(\hat{p}_2) = (\hat{p}(\omega^0, -\omega^0; \hat{p}, \pi \sqrt{-1}) \hat{p})^{-1} = 0 \). Let \( \hat{p} \) be the point near \( \hat{p}_3 \) along \( \hat{a}_3 \).

We may write \( B(\hat{p}) = -\frac{1}{2} \Pi - (a - \pi) \sqrt{-1} \), where \( a \in \mathbb{R} \). We then have \( B(\hat{p}_3) = \lim_{a \to \pi} B(\hat{p}) \). We define \( \tilde{W} \) using \( \Phi \) as well as the way we defined \( \tilde{W} \) using \( \Phi \). We then see that
\[
\sqrt{W(\hat{p})} = \sqrt{C(\hat{p}) \Phi(p) \Phi(\hat{p})} = \left| \frac{\theta(\frac{1}{2} \Pi + (a - \pi) \sqrt{-1}; \Pi) \theta(0)}{\theta(\frac{1}{2} \Pi + (a - \pi) \sqrt{-1}; \Pi) \theta(\pi \sqrt{-1})} \right| \sqrt{\tilde{W}(\hat{p})}.
\]

Here, we note that
\[
\left( \frac{\theta(\frac{1}{2} \Pi + (a - \pi) \sqrt{-1})}{\theta(\frac{1}{2} \Pi + a \sqrt{-1})} \right) = \frac{\theta(-\frac{1}{2} \Pi - (a - \pi) \sqrt{-1} + \Pi)}{\theta(-\frac{1}{2} \Pi - a \sqrt{-1} + \Pi)}
\]
\[
= -\frac{\theta(\frac{1}{2} \Pi + (a - \pi) \sqrt{-1})}{\theta(\frac{1}{2} \Pi + a \sqrt{-1})}.
\]
Therefore, we see that
\[
\tilde{\Psi}(x, t, \hat{P}_j) = \lim_{a \to x} C(\hat{P}_j)\tilde{\Psi}(x, t, \hat{P}_j) \left| \begin{array}{c}
\theta\left(\frac{1}{2} \Pi + (a - \pi)\sqrt{-1}\right)\theta(\pi\sqrt{-1})
\frac{\theta\left(\frac{1}{2} \Pi + a\sqrt{-1}\right)\theta(0)}{
\theta\left(\frac{1}{2} \Pi + a\sqrt{-1}\right)\theta(0)}
\end{array} \right|
\]
\[
= \lim_{a \to x} \tilde{\Psi}(x, t, \hat{P}_j) \exp(-\pi\sqrt{-1}/2)\theta\left(\frac{1}{2} \Pi + a\sqrt{-1}\right)\theta(0)
\frac{\theta\left(\frac{1}{2} \Pi + (a - \pi)\sqrt{-1}\right)\theta(\pi\sqrt{-1})}{\theta\left(\frac{1}{2} \Pi + a\sqrt{-1}\right)\theta(0)}
\]
\[
= \pm \lim_{\hat{P}_j \to \hat{P}_j} \tilde{\Psi}(x, t, \hat{P}_j).
\]

Since we may obtain the explicit expression of \(\tilde{\Psi}(\hat{P}_j)\), \(j = 1, 2, 3\) using the Jacobi elliptic functions, we obtain the following.
\[
\tilde{W}(\hat{P}_1) = 2 + \frac{\zeta_1}{\zeta_2}, \quad \tilde{W}(\hat{P}_2) = 1 - \frac{\zeta_1}{\zeta_2}, \quad \tilde{W}(\hat{P}_3) = 2 + \frac{\zeta_1}{\zeta_2},
\]
which implies that \(\tilde{W}(\hat{P}_1)\tilde{W}(\hat{P}_2)\tilde{W}(\hat{P}_3) = \sqrt{2}\left(\zeta_2 - \frac{\zeta_1}{\zeta_2}\right)\).

Therefore, \(\psi(\hat{x}, \hat{y})\) in (4.6) can be described as those forms stated in the theorem. Next, we show that the \(\tilde{\Psi}\) satisfies the Schrödinger equation \(\partial_t \partial_y \tilde{\Psi} = e^{\mu t}\tilde{\Psi}\). Near \(\hat{P}_0\), \(\tilde{\Psi}(x, t, \nu) = \exp(x\nu^{-1})(1 + \sum_{j=0}^{\infty} \xi_j \nu^j)\). We set \(e^u = \partial_k \tilde{\xi}_k\). We then see that
\[
\hat{\Phi} := (\partial_x \partial_y \tilde{\Psi} - e^{\nu t}\tilde{\Psi})^{-1} \rightarrow 0 \quad \text{as} \quad \nu \to 0.
\]

Since the poles of \(\tilde{\Psi}\) are independent of the parameters \(x, t\), we see that the poles of \((\partial_x \partial_y \tilde{\Psi} - e^{\nu t}\tilde{\Psi})\) coincides with the zeros of \(\tilde{\Psi}^{-1}\). But, since the zeros of \((\partial_x \partial_y \tilde{\Psi} - e^{\nu t}\tilde{\Psi})\) changes with \(x, t\), if \(\tilde{\Psi} \neq 0\) then it follows from lemma 4.5 that the zeros of \(\tilde{\Psi}\) are the degree 2 divisor \(\hat{q}_k(x, t), \hat{q}_k(x, t)\). We then have \(\hat{\Phi} \in H^0(\hat{\mathcal{C}}, \mathcal{O}_\mathcal{C}(\hat{q}_k(x, t) + \hat{q}_k(x, t)))\), since \(\hat{q}_k(x, t), \hat{q}_k(x, t)\) is a non-special divisor at the origin \((x, t) = (0, 0)\), it remains non-special near the origin.

Thus, \(\hat{\Phi}\) must be a constant on \(\hat{\mathcal{C}}\). Evaluating it at \(\nu = 0\) we obtain \(\hat{\Phi} \equiv 0\), hence we have \(\hat{\partial}_x \hat{\partial}_y \hat{\Psi} - e^{\nu t}\hat{\Psi} \equiv 0\).

Finally, we show that the above coincides with one in (4.17). Near \(\hat{P}_0\), we may write \(\int_{\hat{P}_0} \hat{\Omega}_0 + \frac{1}{2} \mu_1 = C\nu + O(\nu^3)\), where \(C = \frac{a}{3} - \frac{\zeta_1\omega_j}{\zeta_2}\) as in (4.18). Differentiating \(\log \hat{\Psi}\) by \(\nu\) and setting \(\nu = 0\), we obtain from lemma 4.7.
\[
\hat{\xi}_1 = \frac{d}{d\nu} \left. \log \theta(B(\hat{P}) - (x - t)U^0 - e) \right|_{\nu=0} \right|_{\nu=0} \log \theta(B(\hat{P}) - (x - t)U^0 - e) = C + C_0(x),
\]
where \(C_0(x)\) is a function of the parameter \(x\) only. It then follows from \(e^u = \partial_x \hat{\xi}_k\) that \(e^u = C + \partial_x \frac{d}{d\nu} \left. \log \theta(B(\hat{P}) - (x - t)U^0 - e) \right|_{\nu=0}
\]
\[
\frac{d}{d\nu} \left. \log \theta(B(\hat{P}) - (x - t)U^0 - e) \right|_{\nu=0} = \sum_{j=1}^{2} \int_{\hat{B}_j} \Omega_{\infty} = 2U^0
\]
which, together with \(e^u = \partial_x \hat{\xi}_k\), yields
\[
e^u = C - 2 \partial_x \log \theta((x - t)U^0 + e).
\]

\[\square\]

5. Blaschke immersions of finite type in terms of Prym-theta functions

Let \(d \equiv 1 \mod 6\) and \(\xi(x, t, \lambda) \in \Lambda^d_{\mathbb{D}}(\mathbb{D})\) be a solution of \(d\xi = [\xi, a^{-1}da]\). For \(\lambda, \mu \in \mathbb{C}\), the spectral curve is defined by the equation \(\det(\xi(x, t, \lambda) - \mu I) = 0\). However, since \(\xi = \text{Ad}^{-1}(\lambda^{-1}\xi\lambda)\) for some initial data \(\xi(\lambda) \in \Lambda^d_{\mathbb{D}}\), we see that the spectral curve is independent of the parameters \(x, t\) given by the equation \(\det(\xi(\lambda) - \mu I) = 0\), which becomes \(\mu^3 - \frac{1}{4}(\text{trace}(\xi(\lambda)^2))\mu = -\det(\frac{1}{\xi(\lambda)})\). Since the involution \(\sigma\) given in section 3 has a property that \(\sigma(\xi(\lambda)) = \xi(-\lambda)\) and hence
\[
\det(\xi(\lambda) - \mu I) = \det(\sigma(\xi(-\lambda) - \mu I)
\]
\[
= \det(\text{Ad}^{-1}(\lambda^{-1}\xi(-\lambda) - \mu I) = -\det(\xi(\lambda) - \mu I)
\]

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holds, the spectral curve has a holomorphic involution $\sigma$ defined by $\sigma(\mu, \lambda) = (-\mu, -\lambda)$, where

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Moreover, since $\xi(\lambda) \in \mathbb{Z} X^{-1}$, the spectral curve also admits an anti-holomorphic involution $\rho$ defined by $\rho(\mu, \lambda) = (\overline{\mu}, -\lambda^{-1})$. Set $P(\lambda) = \frac{1}{2} \text{trace } \xi(\lambda)^2$, $Q(\lambda) = \det \xi(\lambda)$. The existence of the involutions $\sigma, \rho$ and the special forms of elements of each $\tilde{g}_j$ $(j = 0, 1, 2, 3, 4, 5)$ implies that $P(\lambda)$ and $Q(\lambda)$ have the expansions of the forms

$$P(\lambda) = \sum_{j=-2k}^{2k} p_j(\xi) \lambda^j, \quad \text{with } p_j = p_{j'}, \quad p_{j'} = p_j,$$

$$Q(\lambda) = \sum_{j=-(6k+1)}^{6k+1} q_j(\xi) \lambda^j, \quad \text{with } q_j = q_{j'}, \quad p_{j'} = q_j,$$

where we have expressed $d$ as $d = 6k + 1$. Therefore, we see that the spectral curve has $12d$ branch points. Projectivizing this affine plane curve, we have a compact Riemann surface $\mathcal{C}$, whose genus is given by $(6d - 2)$. A three-fold covering map given by $\mathcal{C} \longrightarrow \hat{\mathcal{C}}$, $(\mu, \lambda) \longrightarrow (\mu, \nu, \nu = \lambda^3)$, is unramified, hence the genus $\tilde{g}$ of $\hat{\mathcal{C}}$ is $2d$. The curve $\mathcal{C}$ also admits the involutions $\sigma$ and $\rho$. We assume that $C$ has no branch points over $\nu \in S$. We denote by $P(\nu)$ and $Q(\nu)$ the corresponding Laurent polynomials obtained by setting $\nu = \lambda^j$ in $P(\lambda)$ and $Q(\lambda)$, respectively. Let $\{\nu_0, \nu_2, \nu_4, \nu_{4d+1}\}$ be the set of $(4k + 1)$ points of $S$ which are distinct each other. For example, we take $\nu_j = \exp(2\pi \sqrt{-1} j/(4k + 1))$. For each $j$, the equation $(\mu^2 - P(\nu_j)\mu - Q(\nu_j))^2 = 0$ has solutions $\mu = \pm \mu_j, \pm \mu_{j+2}, \pm \mu_{j+3}$ with $\mu_{j+1} < \mu_{j+2} < \mu_{j+3}$ and $\mu_{j+1} + \mu_{j+2} + \mu_{j+3} = 0$ because there is no branch points on $S$. Therefore, we may assume that our spectral curve is a hyperelliptic curve defined by

$$\prod_{j=1}^{4k+1} (\mu^2 - P(\nu_j)\mu - Q(\nu_j)^2) = \bar{v}^2.$$

Note that both of $P(\nu)$ and $Q(\nu)$ are real for each $j = 1, 2, \ldots, 4k + 1$. Keeping these in mind, we consider a compact Riemann surface $\hat{\mathcal{C}}$ of genus $\tilde{g}$ obtained by projectivizing hyperelliptic curve defined by

$$\prod_{j=1}^{6k+1} (\mu - \mu_j)(\mu + \mu_j) = \bar{v}^2 \quad \text{with } \mu_j \in \mathbb{R} \text{ and } \mu_1 > \mu_2 > \cdots > \mu_{6k+1} > 0.$$

This is an $M$-curve (see [27]). We see that the involutions $\sigma$ and $\rho$ act on $\hat{\mathcal{C}}$ by $\sigma(\mu, \nu) = (-\mu, -\nu)$ and $\rho(\mu, \nu) = (\overline{\mu}, -\overline{\nu})$.

We may choose the canonical holomorphy basis $\{\hat{a}_1, \ldots, \hat{a}_{6k}, \hat{b}_1, \ldots, \hat{b}_{6k}\}$ of $\hat{\mathcal{C}}$ so that $\sigma(\hat{a}_1) = -\hat{a}_{6k+1}, \sigma(\hat{b}_1) = -\hat{b}_{6k+1}$, hold for $j = 1, 2, \ldots, 6k$ (see Figure 4).

Let $\{u_0, \ldots, u_{6k}\}$ be the basis of $H^1(\hat{\mathcal{C}}, \mathbb{C})$ given by

$$u_j = \frac{\mu^{j-1} - \mu^{j+1}}{\bar{v}}, \quad u_{6k+j} = \frac{\mu^{j-1} - \mu^{j+1}}{\bar{v}} \quad \text{for } j = 1, 2, \ldots, d.$$ 

We find the properties $\sigma^* u_j = u_j, \sigma^* u_{d+j} = -u_{d+j}$ for $j = 1, 2, \ldots, d$. Define two $d \times d$-matrices $K = (K_{ij})$ and $\check{K} = (\check{K}_{ij})$ by $K_{ij} = \int_{\hat{a}_i} u_j$ and $\check{K}_{ij} = \int_{\hat{b}_i} u_j$. Note that $\int_{\hat{a}_i} u_j = -\int_{\hat{b}_i} u_j$ and $\int_{\hat{a}_{d+i}} u_{d+j} = \int_{\hat{b}_{d+i}} u_{d+j}$. Now, we may construct a new basis $\{w_1, \ldots, w_{d}\}$ of $H^1(\hat{\mathcal{C}}, \mathbb{C})$ as follows. If we write $w = (w_1, \ldots, w_d), \check{w} = (\check{w}_1, \ldots, \check{w}_d)$, $u = (u_0, \ldots, u_d)$ and $u = (u_{6k+1}, \ldots, u_{6k})$, we put

$$\check{w} = \pi \sqrt{-1} (\check{K}^{-1} u + \check{K}^{-1} \check{w}), \quad \check{w} = \pi \sqrt{-1} (\check{K}^{-1} u + \check{K}^{-1} \check{w}).$$

We then see that $\{w_1, \ldots, w_d\}$ is the normalized basis of $H^1(\hat{\mathcal{C}}, \mathbb{C})$ with the property $\sigma^* \check{w} = -\check{w}$. Define a matrix $\tilde{T} = (\tilde{T}_{ij})$ by $\tilde{T}_{ij} = \int_{\hat{a}_i} (w_j + w_{d+j})$ and $\tilde{T}_{ij} = \int_{\hat{b}_i} (w_j - w_{d+j})$, respectively. We then see that

$$\tilde{T}_{ij} = \frac{1}{2} (\Pi_{ij} + \Pi_{j'i'}), \quad \tilde{T}_{ij'} = \frac{1}{2} (\Pi_{ij} - \Pi_{j'i'}),$$

$$\Pi_{ij} = \int_{\hat{a}_i} (w_j + w_{d+j}), \quad \Pi_{ij'} = \int_{\hat{b}_i} (w_j + w_{d+j}) \quad \text{for } i, j = 1, 2, \ldots, d,$$

where $i' = d + i, j' = d + j$. From this, we may find a compact Riemann surface $C^0$ of genus $d$ with the Riemann period matrix $\begin{pmatrix} 2\pi \sqrt{-1} & 0 \\ 0 & 2\pi \sqrt{-1} \end{pmatrix}$ by Torelli’s theorem. The Jacobian variety of $C^0$ is nothing but the Prym variety $\text{Prym}(\check{C}) \simeq C^0/\Gamma$ of $\hat{\mathcal{C}}$, where $\Gamma = \text{Span}_{\mathbb{Z}}(2\pi \sqrt{-1}, 1, 1)$. A map $B: \hat{\mathcal{C}} \rightarrow \text{Prym}(\check{C})$ is defined by

$$B(\hat{P}) = \big( B_1(\hat{P}), B_2(\hat{P}), \ldots, B_d(\hat{P}) \big) \text{ and } B_j(\check{P}) = \int_{\hat{a}_j} (w_j + w_{d+j}) \text{ for } j = 1, 2, \ldots, d.$$
Consider the Riemann theta function \( \theta \) on Prym(\( \hat{C} \)), which is defined by

\[
\theta(z) = \sum_{m \in \mathbb{C}^d} \exp \left( \frac{1}{2} \langle m, \Pi m \rangle + \langle m, z \rangle \right),
\]

where \( z \in \mathbb{C}^d \) and \( \langle \cdot, \cdot \rangle \) is the standard inner product of \( \mathbb{C}^d \). The theta function has the quasi-periodic properties:

\[
\begin{cases}
\theta(z + 2\pi \sqrt{-1} e_j) = \theta(z), \\
\theta(z + \Pi e_j) = \exp \left( -\frac{1}{2} \Pi_{jj} - z_j \right) \theta(z) & 1 \leq j \leq d,
\end{cases}
\]

where \( e_1, \cdots, e_d \) is a standard basis of \( \mathbb{C}^d \) and \( z = (z_1, \cdots, z_d) \). Since \( \hat{C} \) is an \( M \)-curve, we may take \( \hat{b}_j \)'s \((j = 1, 2, \cdots, \hat{g})\) as real ovals. Therefore, we have \( \rho(\hat{a}_j) = -\hat{a}_j, \rho(\hat{b}_j) = \hat{b}_j \) for \( j = 1, 2, \cdots, \hat{g} \), and thus we have \( \rho \cdot w_j = w_j \) for \( \alpha = 1, 2, \cdots, \hat{g} \). From this, we see that \( \Pi \) is real and \( \theta(z) = \theta(\bar{z}) \). Moreover, since

\[
\rho \cdot u_{d+i} = -u_{d+i},
\]

we see that \( \hat{\mathbf{k}} \) is a real matrix. Note that

\[
\int_{\hat{P}_0} B_j(w_j + w_{d+j}) \equiv 0 \pmod{\Gamma} \text{ for } j = 1, \cdots, d
\]

because \( \sigma^* w_j = -w_{d+j} \). Therefore, we obtain

\[
B(\rho(\hat{P})) \equiv B(\hat{P}), \quad B(\sigma(\hat{P})) \equiv -B(\hat{P}) \pmod{\Gamma}.
\]

Consider a function \( f \) on \( \hat{C} \) defined by

\[
f(\hat{P}) = \theta(\hat{B}(\hat{P}) - e) \text{ for } e \in \mathbb{C}^d.
\]

The method similar to those in section 4.4 yield the following.

**Lemma 5.1 (cf. [29]).** If \( f(\hat{P}) = \theta(\hat{B}(\hat{P}) - e) \equiv 0 \) for some \( e \in \mathbb{C}^d \) then the zeros of \( f \) is a degree \( \hat{g} (=2d) \) divisor \( \hat{D} \). Moreover, we have \( B(\hat{D}) \equiv K + 2e \pmod{\Gamma} \), where \( K = (K_{1}, \cdots, K_{d}) \in \mathbb{C}^d \) and each \( K_i \) is given by

\[
K_i = \frac{1}{2\pi \sqrt{-1}} \sum_{\alpha = 1}^{\hat{g}} \int_{\hat{P}_0} B_j(w_{[\alpha]} + w_{[d+\alpha]}) \cdot B_j(\hat{b}_i(0)) - B_j(\hat{b}_{d+i}(0)) \pmod{\Gamma}.
\]
where \( j \) is a point of zeros, which are denoted by \( j \). For a given \( j \), the points of zeros, which are denoted by \( j \), follow. Consider the Abelian differential \( \psi(x, t, \tilde{\nu}) \) of the Schrödinger equation \( \partial_t \psi = -\psi \) on \( \mathbb{C} \) of which the integrability condition ensures that \( \psi \) is a solution of the Tzitzéica equation. Moreover, the description of \( \psi \) in terms of the Riemann theta function yields the formula of the solution of Tzitzéica equation in terms of the Riemann theta function. For this purpose, we look for \( \psi \) by the following conditions: Denote by \( \hat{P}_0 \) and \( \hat{P}_\infty \) the points on \( \mathbb{C} \) which corresponds to the values \( \tilde{\nu} = 0 \) and \( \tilde{\nu} = \infty \), respectively, where \( \tilde{\nu} \) and \( \tilde{\nu}^{-1} \) are the local coordinates around \( \hat{P}_0 \) and \( \hat{P}_\infty \) of \( \mathbb{C} \), respectively.

1. \( \psi \) is a meromorphic function on \( \mathbb{C} \backslash \{ \hat{P}_0, \hat{P}_\infty \} \) and the divisor of the poles is given by \( \hat{D}_\infty = \{ \hat{P}_1, \cdots, \hat{P}_\ell \} \) which is independent of the parameters \( x \) and \( t \).
2. \( \psi \) has the following asymptotic expansions.

\[
\psi = \begin{cases} 
\exp(x\tilde{\nu}^{-1})\left(1 + \sum_{j=1}^{\ell} \hat{\nu}_j \hat{P}_j \right) & \text{near } \hat{P}_0, \\
\exp(-t\tilde{\nu})\left(1 + \sum_{j=1}^{\ell} \hat{\nu}_j \hat{P}_j \right) & \text{near } \hat{P}_\infty.
\end{cases}
\]

Consider the Abelian differential \( \omega_{\psi} \) defined by \( \omega_{\psi} = d \log \psi \). It follows from the condition (1) that there are \( \hat{\ell} \) points of zeros, which are denoted by \( \{ \hat{q}_j(x, t), \hat{q}_j(x, t), \cdots, \hat{q}_j(x, t) \} \). Therefore, \( \omega_{\psi} \) may be described as follows.

\[
\omega_{\psi} = x\Omega_\infty + t\hat{\Omega}_0 + \sum_{j=1}^{\hat{\ell}} \omega(\hat{q}_j, \hat{p}_j) + \sum_{j=1}^{\hat{\ell}} m_j \omega_j,
\]

where \( \omega(\hat{q}_j, \hat{p}_j) \) is the normalized Abelian differential of third kind with the principal part \((\tilde{\nu} - \hat{q}_j)^{-1}d\tilde{\nu}\) and \(-((\tilde{\nu} - \hat{p}_j)^{-1}d\tilde{\nu})\), and \( \Omega_\infty \) and \( \hat{\Omega}_0 \) are the normalized Abelian differential of second kind with the properties \( \sigma^* \Omega_\infty = -\Omega_\infty \) and \( \sigma^* \hat{\Omega}_0 = -\hat{\Omega}_0 \). Set \( \hat{U} = \{ U_1, U_2, \cdots, U_{\hat{\ell}} \} \) and \( \hat{V} = \{ V_1, V_2, \cdots, V_{\hat{\ell}} \} \), where \( U_j = \int_{\hat{q}_j}^{\hat{p}_j} \Omega_\infty \) and \( V_j = \int_{\hat{p}_j}^{\hat{q}_j} \hat{\Omega}_0 \) for \( j = 1, 2, \cdots, \hat{\ell} \). It follows from \( \rho(\hat{b}_j) = \hat{b}_j \) that \( \hat{V} = \hat{U} \). The integration of (5.5) over the cycle \( \hat{b}_j \) and the reciprocity law gives

\[
\sum_{j=1}^{\hat{\ell}} \int_{\hat{b}_j}^{\hat{b}_j} w_k = -xU_k - tV_k \mod (\Gamma) \quad \text{for } k = 1, 2, \cdots, \hat{\ell}.
\]
Since $\sigma^*\hat{\Omega}_\infty = -\hat{\Omega}_\infty$ and $\sigma^*(\hat{b}_j) = -\hat{b}_{d+j}$ we have $U_{d+j} = U_j$ for $j = 1, 2, \cdots, d$. Hence, it follows from
\[
\frac{d}{d\hat{\nu}} \Big|_{\hat{\nu}=0} B_j(\hat{\nu}) = \big((U_j + U_{d+j}) = -2U_j, \text{the reality of the matrix } \hat{K} \text{ and } (5.1) \text{ that } U_j = U_{d+j} \text{ is purely imaginary for } j = 1, 2, \cdots, d. \text{ Therefore, we obtain}
\]
\[
\sum_{j=1}^{d_m} \int_{\hat{\nu}^j} (w_k + w_{d+k}) \equiv 2(t - x)U_k \mod \Gamma \text{ for } k = 1, 2, \cdots, d. \tag{5.6}
\]
We may give $\hat{\Omega}_\infty, \hat{\Omega}_0$ explicitly as follows.

**Lemma 5.2.** Define $\hat{\Omega}_\infty, \hat{\Omega}_0$ by

\[
\begin{align*}
\hat{\Omega}_\infty &= \left( \frac{1}{2} \hat{\mu}^{d+1} + \sum_{j=1}^{d} c_j \hat{\mu}^{2j-1} \right) d\hat{\mu} + \frac{1}{2} d\hat{\mu} \\
\hat{\Omega}_0 &= -\left( \frac{1}{2} \hat{\mu}^{d+1} + \sum_{j=1}^{d} c_j \hat{\mu}^{2j-1} \right) d\hat{\mu} + \frac{1}{2} d\hat{\mu} \tag{5.7}
\end{align*}
\]

where $c_j \ (j = 1, 2, \cdots, d)$ is given by

\[
c_j = -\sum_{i=1}^{d} (K^{-1})_{ji} \int_{\hat{\nu}} \frac{1}{2} \hat{\mu}^{d+1} d\hat{\mu}. \tag{5.8}
\]

We then have the following.

(1) $c_j$ is a real number for $j = 1, 2, \cdots, d$,

(2) $\hat{\Omega}_\infty$ and $\hat{\Omega}_0$ are the normalized Abelian differentials of 2nd kinds with the properties $\sigma^*\hat{\Omega}_\infty = -\hat{\Omega}_\infty$, $\sigma^*\hat{\Omega}_0 = -\hat{\Omega}_0$, $\rho^*\hat{\Omega}_\infty = \hat{\Omega}_0$ and $\rho^*\hat{\Omega}_0 = (-\hat{\nu}^{-2} + O(1))d\hat{\nu}$, $\hat{\Omega}_0 = (C + O(\hat{\nu}^2))d\hat{\nu}$ near $\hat{\nu} = 0$, where

\[
C = c_d + \frac{1}{4} \sum_{j=1}^{d} \mu_j^4. \tag{5.9}
\]

**Proof.**

(1) Since $\rho(\hat{a}_i) = -\hat{a}_i$ for $i = 1, 2, \cdots, d$, we have

\[
\int_{\hat{\nu}} \frac{1}{2} \hat{\mu}^{d+1} d\hat{\mu} = -\int_{\hat{\nu}} \frac{1}{2} \rho^*(\hat{\mu})^{d+1} d\hat{\mu} = -\int_{\rho(\hat{a}_i)} \frac{1}{2} \hat{\mu}^{d+1} d\hat{\mu} = \int_{\hat{\nu}} \frac{1}{2} \hat{\mu}^{d+1} d\hat{\mu},
\]

which, together with the reality of the matrix $\hat{K}$ and (5.8), implies that each $c_j$ is a real number.

(2) The properties $\sigma^*\hat{\Omega}_\infty = -\hat{\Omega}_\infty$ and $\sigma^*\hat{\Omega}_0 = -\hat{\Omega}_0$ are clear, and the property $\rho^*\hat{\Omega}_\infty = \hat{\Omega}_0$ follows from the reality of $c_j$. The choice of $c_j$ by (5.8) implies that all the $\hat{a}_j$- and $\hat{a}_{d+j}$-cycles of $\hat{\Omega}_\infty, \hat{\Omega}_0$ are zero. Finally, for $\hat{\nu} = \mu^{-1}$ we have $\hat{\nu}^{-1} = \hat{\nu}^{d+1}(1 - \sum_{j=1}^{d} \mu_j^2 \hat{\nu}^2 + O(\hat{\nu}^4)) \approx \hat{\nu}^{d+1}(1 + \frac{1}{2} \sum_{j=1}^{d} \mu_j^4 \hat{\nu}^4 + O(\hat{\nu}^4))$, which, together with the definition of $\hat{\Omega}_\infty, \hat{\Omega}_0$ in (5.7), yields

\[
\hat{\Omega}_\infty = \left( \frac{1}{2} \hat{\nu}^{-2} - c_d - \frac{1}{4} \sum_{j=1}^{d} \mu_j^2 + O(\hat{\nu}^2) \right) d\hat{\nu} - \frac{1}{2} \hat{\nu}^{-2} d\hat{\nu},
\]

\[
\hat{\Omega}_0 = \left( c_d + \frac{1}{4} \sum_{j=1}^{d} \mu_j^4 + O(\hat{\nu}^2) \right) d\hat{\nu},
\]

near $\hat{\nu} = 0$. \hfill \Box
Set $U = \{U_0, U_2, \cdots, U_d\}$. We choose three points $\{\hat{P}_1, \hat{P}_2, \hat{P}_3\}$ on $\hat{C}$, which are expressed as $\hat{P}_1 = (\mu_1, 0)$, $\hat{P}_2 = (-\mu_2, 0)$, $\hat{P}_3 = (-\mu_3, 0)$ using the coordinate function $(\mu, \nu)$. It follows from lemma 5.2 that
\[
\int_{\hat{P}_1}^\theta \Omega_\infty + \frac{1}{2} \mu_1 = \nu^{-1} + O(\nu), \quad \int_{\hat{P}_2}^\theta \Omega_0 + \frac{1}{2} \mu_1 = C\nu + O(\nu^3) \quad \text{(near } \nu = 0). \]
We put
\[
\begin{align*}
\Phi_0(x, t, \hat{P}) &= \frac{\theta(B(\hat{P}) + (x - t)U - e)}{\theta(B(\hat{P}) - e)}, \\
\Phi_1(x, t, \hat{P}) &= \exp \left( x \left( \int_{\hat{P}_1}^\theta \Omega_\infty + \frac{1}{2} \mu_1 \right) + t \left( \int_{\hat{P}_2}^\theta \Omega_0 + \frac{1}{2} \mu_1 \right) \right),
\end{align*}
\tag{5.10}
\]
where $e \in \mathbb{C}$ is chosen so that $f(\hat{P}) \neq 0$ and the divisor of the poles of $\Phi_0$ is $\hat{D} = \{\hat{P}_1, \hat{P}_2, \cdots, \hat{P}_3\}$. We observe that $\Phi_0\Phi_1$ is invariant under the translation $\hat{P} \mapsto \hat{P} + m_1\hat{a}_1 + n_1\hat{b}_1$ by the property of $\theta$. Therefore, it is a meromorphic function on $\hat{C}$. It follows from lemma 5.1 and (5.6) that $\hat{\Psi}\Phi_0^{-1}\Phi_1^{-1}$ is a holomorphic function on $\hat{C}$, hence a constant. Evaluating it at $\nu = 0$ we see that the constant is equal to $\frac{\theta(e)}{\theta((x - t)U - e)}$. We thus obtain the following.

**Lemma 5.3.** $\hat{\Psi}(x, t, \hat{P}, e)$ is given by
\[
\hat{\Psi}(x, t, \hat{P}, e) = \frac{\theta(B(\hat{P}) + (x - t)U - e)}{\theta(B(\hat{P}) - e)} \theta(e) \Phi_0(x, t, \hat{P}),
\]
where $\Phi_0(x, t, \hat{P})$ is as that in (5.10), and we fix the path from $\hat{P}_0$ to $\hat{P}_1$.

For our chosen canonical homology basis of $\hat{C}$ in Figure 4, we decompose $\hat{a}_i$ and $\hat{b}_i$ into the disjoint unions $\hat{a}_i = \hat{a}_{i+} \cup \hat{a}_{i-}$ and $\hat{b}_i = \hat{b}_{i+} \cup \hat{b}_{i-}$, where $\hat{a}_{i+}$ is the part of $\hat{a}_i$ lying on $\text{Im } (\mu) \geq 0$ and $\hat{b}_{i-}$ is the part of $\hat{b}_i$ lying on the upper sheet over $\nu = 0$. We also decompose $\gamma_i$, which is the fixed path from $\hat{P}_0$ to $\hat{P}_\infty$, into the disjoint union $\gamma_i = \gamma_{i+} \cup \gamma_{i-}$, where $\gamma_{i+}$ is the part of $\gamma_i$ lying on the upper sheet over $\nu = 0$ and it is nothing but the fixed path from $\hat{P}_0$ to $\hat{P}_i$.

We here prepare the following lemma.

**Lemma 5.4.** Choose $e = \pi \sqrt{-1} \Delta$, where $\Delta = \{1, 1, \cdots, 1\} \in \mathbb{R}^d$. When we choose the paths
\[
\hat{P}_0 \xrightarrow{\gamma_{j+}} \hat{P}_j \xrightarrow{\hat{b}_{j+}} \hat{P}_{j+} \xrightarrow{\hat{a}_{j+}} \hat{P}_j,
\]
we have
\[
B(\hat{P}_j) = \begin{cases} 
\varepsilon & (j = 1), \\
-\frac{1}{2}I_d + e & (j = 2), \\
-\frac{1}{2}I_d + \pi \sqrt{-1} e_d + e & (j = 3), 
\end{cases}
\tag{5.11}
\]
(mod $2\pi \sqrt{-1} \mathbb{Z}^d$) and
\[
\int_{\hat{P}_j}^\theta \Omega_\infty = \begin{cases} 
0 & (j = 1), \\
-\frac{1}{2}U_d + \frac{1}{2}(\mu_2 - \mu_1) & (j = 2), \\
-\frac{1}{2}U_d + \frac{1}{2}(\mu_3 - \mu_1) & (j = 3), 
\end{cases}
\tag{5.12}
\]
Moreover, each $\hat{\Psi}(x, t, \hat{P}_j, e)$ is real for $j = 1, 2, 3$.

**Proof.** Set $a_j^\alpha = \varphi(\hat{a}_j) = \varphi(\hat{a}_{j+})$ and $b_j^\alpha = \varphi(\hat{b}_j) = \varphi(\hat{b}_{j+})$ for $j = 1, 2, \cdots, d$, where $\varphi : \hat{C} \to \text{Prym}(\hat{C})$ is a double covering map. Moreover, set $P_0 = \varphi(\hat{P}_0)$ and $P_k = \varphi(\hat{P}_k) = \varphi(\hat{P}_{\infty})$ for $\alpha = 1, 2, 3$, where $P_{\infty} = \sigma(\hat{P}_0)$. There exists a normalized basis $w^\alpha$ of $H^1(\text{Prym}(\hat{C}), \mathbb{C})$ such that $\varphi^*w^\alpha = (w + \hat{w})$. It follows from $\sigma(P_0) = P_0$, $\sigma(P_\infty) = P_\infty$, $\rho(\hat{P}_0) = P_0^\alpha$, $\rho^*(w + \hat{w}) = -(w + \hat{w})$ and $\rho^*(w + \hat{w}) = (w + \hat{w})$ that $\int_{P_0}^P (w + \hat{w}) \equiv 0 \text{ mod } 2\pi \sqrt{-1} \mathbb{Z}^d$. Hence, we may fix a path from $P_0$ to $P_1 = \sigma(\hat{P}_1)$ which passes through only $\hat{a}$-cycles. It then follows from
\[ \int_{\tilde{P}_1} (w + \tilde{w}) = - \int_{\tilde{P}_1} (w + \tilde{w}), \]
\[ \int_{\tilde{P}_2} (w + \tilde{w}) = - \int_{\tilde{P}_2} (w + \tilde{w}), \]
\[ \int_{\tilde{P}_3} (w + \tilde{w}) = - \int_{\tilde{P}_3} (w + \tilde{w}), \]
that \( \int_{\tilde{P}_1} (w + \tilde{w}) = - \frac{1}{2} \Pi_d. \) Similarly, it follows from
\[ \int_{\tilde{P}_1} (w + \tilde{w}) = - \int_{\tilde{P}_2} (w + \tilde{w}), \]
\[ \int_{\tilde{P}_2} (w + \tilde{w}) = - \int_{\tilde{P}_3} (w + \tilde{w}), \]
\[ \int_{\tilde{P}_3} (w + \tilde{w}) = - \int_{\tilde{P}_1} (w + \tilde{w}), \]
that \( \int_{\tilde{P}_1} (w + \tilde{w}) = \frac{1}{2} \Pi_d. \) It then follows from
\[ \int_{\tilde{P}_1} \Omega_\infty = - \int_{\tilde{P}_1} \Omega_\infty, \]
\[ \int_{\tilde{P}_2} \Omega_\infty = \int_{\tilde{P}_2} \Omega_\infty, \]
\[ \int_{\tilde{P}_3} \Omega_\infty = \int_{\tilde{P}_3} \Omega_\infty, \]
that
\[ \int_{\tilde{P}_1} \Omega_\infty = - \frac{1}{2} U_d + \frac{1}{2} (\mu_2 - \mu_1). \]
Similarly, we obtain \( \int_{\tilde{P}_1} \Omega_\infty = - \frac{1}{2} U_d + \frac{1}{2} (\mu_3 - \mu_1). \)

For any function \( \hat{\psi} = \hat{\psi}(x, t, \tilde{P}) \) on \( D \times \tilde{C} \), we define an action \( \hat{\sigma} * \) by
\[ (\hat{\sigma} * \hat{\psi})(x, t, \tilde{P}) = \hat{\psi}(x, t, \sigma(\tilde{P})), \]
where \( \sigma \) is the involution of \( \tilde{C} \).

We may verify that the following reality condition of \( \hat{\Psi} \) holds:
\[ (\hat{\sigma} * \hat{\Psi})(x, t, \tilde{P}, e) = \hat{\Psi}(x, t, \sigma(\tilde{P}), e). \]

We define \( \hat{W}(x, t, \tilde{P}) \) as in (4.30). We then see from (5.4) that \( \hat{W}(x, t, \tilde{P}) \) is independent of the parameters \( x \) and \( t \). Therefore, we may write it as \( \hat{W}(\tilde{P}) \).

We now obtain the following.

**Theorem 5.5.** For \( e = \pi /\sqrt{-1} \Delta \), the \( \hat{\Psi} \) in lemma 5.3 satisfies \( \partial_t \partial_{\tilde{P}} \hat{\psi} = e^u \hat{\psi} \) and \( e^u \) is a solution of the Tzitzeica equation and may be described as
\[ e^u = C - 2 \partial_t \partial_{\tilde{P}} \log \theta((x - t)U - e), \]
where \( C \) is as in (5.9). A Blaschke immersion \( \psi \) of an indefinite proper affine sphere with \( \det F = e^u \) may be described as
\[ \psi(\tilde{\xi}, \tilde{\eta}) = \left\{ \hat{\xi} \frac{\hat{\Psi}(x, t, \tilde{P}_1, e)}{\sqrt{|\hat{W}(\tilde{P}_1)|}}, \hat{\xi} \frac{\hat{\Psi}(x, t, \tilde{P}_2, e)}{\sqrt{|\hat{W}(\tilde{P}_2)|}}, \hat{\xi} \frac{\hat{\Psi}(x, t, \tilde{P}_3, e)}{\sqrt{|\hat{W}(\tilde{P}_3)|}} \right\}, \]
where $\hat{c}_2 \hat{c}_3 = 1$.

**Proof.** We choose a function $e^u$ so that the following estimate holds:

$$
(\partial_\xi \hat{\eta} - e^u) \hat{\Psi} = \begin{cases} O(\hat{v}) \exp(\hat{x}\hat{v}) & \text{near } \hat{P}_0, \\ O(\hat{v}^{-1}) \exp(-\hat{v}) & \text{near } \hat{P}_\infty. \end{cases}
$$

(5.13)

Then, $e^u$ must be given as $e^u = \partial_\xi \hat{\eta} = -\partial_\xi \hat{\eta}_0$. We then see that

$$
\hat{\Phi} := (\partial_\xi \hat{\eta}_0 - e^u \hat{\Psi}) \hat{\Psi}^{-1} \longrightarrow 0 \quad \text{as } \hat{v} \to 0.
$$

Since the poles of $\hat{\Psi}$ are independent of the parameters $x, t$, we see that the poles of $(\partial_\xi \hat{\eta}_0 - e^u \hat{\Psi})$ coincides with the zeros of $\hat{\Psi}^{-1}$. However, since the zeros of $(\partial_\xi \hat{\eta}_0 - e^u \hat{\Psi})$ changes with $x, t$, if $\hat{\Psi} \neq 0$ then it follows from lemma 5.1 and lemma 5.3 that the zeros of $\hat{\Psi}$ are the degree $\hat{g}$ divisor $\hat{D}$ given by $\hat{D} = (\hat{q}_1(x, t), \hat{q}_2(x, t), \cdots, \hat{q}_k(x, t))$. We then have $\hat{\Phi} \in I^H(\hat{C}, \mathcal{O}_x(\hat{D}))$. Since $\hat{D}$ is non-special at the origin $(x, t) = (0, 0)$, it remains non-special near the origin. Thus, $\hat{\Phi}$ must be a constant on $\hat{C}$. Evaluating it at $\hat{v} = 0$ we obtain $\hat{\Phi} \equiv 0$, which implies that $\partial_\xi \partial_\xi \hat{\Psi} - e^u \hat{\Psi} \equiv 0$. We show $e^u$ is a solution of the Tzitzéica equation. Since $\int_{\hat{P}_0}^\hat{P} \hat{\Omega}_0 + \frac{1}{2} \mu_1 = C \hat{v} + O(\hat{v}^3)$ and

$$
\left. \frac{d}{d \hat{v}} \right|_{\hat{v}=0} B(\hat{P}) = -2U,
$$

the same method as those in the proof of theorem 4.9 yields that

$$
e^u = C - 2\partial_\xi \partial_\xi \log \theta((x - t) U - e) .
$$

Set $\hat{\psi}_2 = \hat{\Psi}$ and define $\psi_0, \psi_1$ by $\psi_0 = e^u \partial_\xi \psi_2$ and $\psi_1 = e^{-u} \partial_\xi \psi_2$, where $\nu$ is the coordinate function on $\hat{C}$. We show these satisfies all the equations in (5.4). First of all, it follows from $\partial_\xi \partial_\xi \hat{\psi}_2 = e^u \psi_2$ that $\partial_\xi \psi_0 = \nu^{-1} e^u \psi_2$ and $\partial_\xi \psi_1 = -u_1 \psi_1 + \psi_2$. Next, note that we may write $\nu = \hat{v}^3$ locally near $\hat{P}_0$ or $\hat{P}_\infty$. This, together with the definitions of $\psi_0$ and $\psi_1$, implies that

$$
\partial_\xi \psi_0 = u_1 \psi_0 - \psi_1 = O(\hat{v}^{-1}) \exp(-\hat{v}) \quad \text{near } \hat{P}_\infty .
$$

Thus, we have $(\partial_\xi \psi_0 = u_1 \psi_0 - \psi_1) \psi_2^{-1} \longrightarrow 0$ as $\hat{v} \to \infty$, hence we have

$$
\partial_\xi \psi_0 = u_1 \psi_0 - \psi_1 \equiv 0
$$

(5.14)

as above. Next, we show $\partial_\xi \psi_1 = e^{-2u}\psi_0$ holds using (5.10) and (5.14). It follows from (5.2) and (5.10) that

$$
\hat{\psi}_2(\hat{P}) = (\hat{\sigma}_1(\hat{P}))(\sigma(\hat{P})).
$$

We calculate

$$
\hat{v}^{-3} \hat{\psi}_0(\hat{P}) = \hat{\psi}_2(\hat{P}) = (\hat{\sigma}_1(\hat{P}_0))(\sigma(\hat{P})) = (\hat{\sigma}_1(\hat{u}(\hat{v})))((\hat{\ sigma}_1(\hat{\psi}_1)))(\sigma(\hat{P})),
$$

hence we obtain

$$
\hat{\sigma}_1(\hat{u}(\hat{v})))((\hat{\ sigma}_1(\hat{\psi}_1)))(\sigma(\hat{P})) = \hat{v}^{-3} \hat{\psi}_0(\hat{P}).
$$

(5.15)

Differentiating the first equation in (5.15) by the parameter $t$ we have

$$
(\hat{\sigma}_1(\hat{u}(\hat{v})))((\hat{\ sigma}_1(\hat{\psi}_1)))(\sigma(\hat{P})) + (\hat{\sigma}_1(\hat{v}(\hat{\psi}_1)))(\sigma(\hat{P})) = \hat{v}^{-3} \partial_\xi \hat{\psi}_0(\hat{P}).
$$

Now, since

$$
(\hat{\sigma}_1(\hat{u}(\hat{v})))((\hat{\sigma}_1(\hat{\psi}_1)))(\sigma(\hat{P})) = u_1(\hat{\sigma}_1(\hat{v}(\hat{\psi}_1)))(\sigma(\hat{P})) = \hat{v}^{-3} \hat{\psi}_0(\hat{P}),
$$

and

$$
\partial_\xi \hat{\psi}_0(\hat{P}) = \partial_\xi \hat{\psi}_0(\hat{P}) + \hat{\psi}_1(\hat{P})(\sigma(\hat{P})),
$$

(5.14)

where the last equation follows from (5.14), we have

$$
(\hat{\sigma}_1(\hat{v}(\hat{\psi}_1)))(\sigma(\hat{P})) = \hat{v}^{-3} \hat{\psi}_1(\hat{P}),
$$

hence

$$
e^u(\partial_\xi \hat{\psi}_1(\hat{P}) = \hat{v}^{-3} \hat{\sigma}_1(\hat{\hat{\psi}_1})(\sigma(\hat{P})) = e^{-u}\hat{\psi}_0(\hat{P})
$$

by the last equation in (5.15). Therefore we obtain $\partial_\xi \psi_1 = e^{-2u}\hat{\psi}_0$. Thus, we have proved (5.4). The integrability condition for (5.4) means that $e^u$ is the solution of the Tzitzéica equation.

On the other hand, we find that $W(\hat{v})$ be expressed as

$$
\hat{W}(\hat{P}) = -e^{-u}(\partial_\xi \hat{\psi}(\hat{P}) \cdot \partial_\xi \hat{\Psi}(\sigma(\hat{P}))) + \partial_\xi \hat{\psi}(\sigma(\hat{P})).
$$

We here prove

$$
\hat{\Psi}(\hat{P}) \cdot \hat{\Psi}(\sigma(\hat{P})) = |\hat{\Psi}(\hat{P})|^2 / |\hat{\Psi}(x, t, \hat{P}_0)|^2 \quad \text{for } j = 1, 2, 3.
$$

(5.16)
For this, we calculate

\[
\int_{\hat{P}_1} \hat{\Omega}_\infty = \int_{\hat{P}_1} \hat{\Omega}_\infty = -\int_{\hat{P}_1} \hat{\Omega}_\infty - \int_{\hat{P}_1} \hat{\Omega}_\infty = -\int_{\hat{P}_1} \hat{\Omega}_\infty - \int_{\hat{P}_1} \hat{\varphi}^\ast \hat{\Omega}_\infty = -\int_{\hat{P}_1} \frac{1}{2} \, d\mu = -\int_{\hat{P}_1} \hat{\Omega}_\infty - \mu_1,
\]

which implies that \(\int_{\hat{P}_1} \hat{\Omega}_\infty + \frac{1}{2} \mu_1 = -\left(\int_{\hat{P}_1} \hat{\Omega}_\infty + \frac{1}{2} \mu_1\right)\). Therefore we have \(\Phi(x, t, \hat{P}_1)\Phi(x, t, \sigma(\hat{P}_1)) = 1\), from which and (5.2) and (5.11), we obtain (5.16). Set \(V(x, t, \hat{P}_1) = \hat{\Psi}(x, t, \hat{P}_1, e) \cdot \hat{\Psi}(x, t, \sigma(\hat{P}_1), e)\), which is non-zero and real for \(j = 1, 2, 3\) by (5.16). Calculating \(\partial_x \partial_y V(x, t, \hat{P}_1)\) and using \(\partial_x \partial_y \hat{\Psi} = e^x \hat{\Psi}\) we have

\[
\hat{W}(\hat{P}_1) = 3V(x, t, \hat{P}_1) = e^{-x} \partial_x \partial_y V(x, t, \hat{P}_1),
\]

which implies that each \(\hat{W}(\hat{P}_1)\) is real. We now assume that \(\hat{P}_j(j = 1, 2, 3)\) is a zero of \(f(\hat{P}_1) = \theta(B(\hat{P}_1) - e)\). Then \(\hat{P}_1\) is also a zero of \(f(\sigma(\hat{P}_1))\) by (5.2) and the properties of the theta function. Therefore, \(\hat{P}_j\) is a pole of \(\sqrt{\hat{W}(\hat{P}_1)}\). Thus, the pole of \(\hat{\Psi}(x, t, \sigma(\hat{P}_1), e)\) cancel with the pole of \(\sqrt{\hat{W}(\hat{P}_1)}\) each other.

Therefore, as in the proof of theorem 4.9, introducing \(\hat{\Psi}(x, t, \hat{P}_1)\) and \(\hat{W}(\hat{P}_1)\) with \(C(\hat{P}_1) = C(\hat{P}_2) = 1\) and \(C(\hat{P}_3)\) is \(\left(\frac{\pi \sqrt{1 - \frac{1}{2U_d}}}{2U_d}, \frac{\pi \sqrt{1 - \frac{1}{2U_d}}}{2U_d}, \hat{P}_3\right)^{-1}\), which is zero by lemma 5.4, we obtain \(\hat{\psi}(\hat{x}, \hat{y})\) as those forms stated in the theorem. \(\square\)

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