The massless thermal field and the thermal fermion bosonization in two dimensions

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Abstract: We come back to the issue of bosonization of fermions in two spacetime dimension and give a new construction in the steady state case where left and right moving particles can coexist at two different temperatures. A crucial role in our construction is played by translation invariant infrared states and the corresponding field operators which are naturally linked to the infrared behaviour of the correlation functions. We present two applications: a simple new derivation in the free relativistic case of a formula by Bernard and Doyon and a full operator solution of the massless Thirring model in the steady state case where the left and right movers have two distinct temperatures.

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Giovanni Morchio (1948–2021). In memoriam.
1 Introduction

The theoretical study of models of fermions in 1+1 space-time dimension has regained interest and importance, mainly because structures which are effectively one-dimensional are nowadays available and play a significant role in the current technological development.

An important notion for their understanding, at both relativistic and non relativistic level, is the so called bosonization. This is indeed an extremely useful mathematical tool and, more importantly, it is a fact deeply rooted in the physical peculiarities of one-dimensional fermionic systems.

In introducing his model [1] Tomonaga gives credit to Bloch [2, 3] for the first observation of the fact that in some approximate sense the behavior of an assembly of Fermi particles can be described by a quantized field of sound waves in the Fermi gas, where the sound field obeys Bose statistics. Tomonaga’s construction later evolved in the Luttinger model, a model of interacting fermions linearized around the Fermi momenta [4]. The zero temperature solution of the Luttinger model was given shortly after by Lieb and Mattis [5]; the bosonization method allowed them to replace the fermionic Hamiltonian by a quadratic expression in the boson collective modes and the charge operators and to diagonalize it by a Bogoliubov transformation. Actually, the description in terms of bosons revealed itself to be appropriate to describe the low energy excitations of a generic interacting electron gas in one dimension — also called a “Luttinger liquid” [6].
The thermal equilibrium correlation functions of the Luttinger model were first constructed in [7] where again the fermion bosonization procedure was a crucial ingredient. Since then, a considerable amount of work has been done to characterize the thermodynamical properties of one dimensional Luttinger liquids. In particular, intense study has been devoted to systems out of equilibrium whose ends are in contact with thermal reservoirs at temperatures $T_l$ and $T_r$: this idea has been successfully applied to obtain exact results for conformal field theories which asymptotically evolve into steady states [8–11] and to the study of the quantum transport of anyons in one space dimension [12–14].

The relativistic counterpart of the Luttinger model is the Thirring model, a model of fermions with current-current interaction introduced a few years earlier than the Luttinger model [15, 16]. The model is actually the prototype of a large family of two-dimensional models which have been theoretical laboratories to discover non-perturbative features also shared by realistic four-dimensional models (see [16] for an old but still good review and [17] for a more recent survey).

The Thirring and the Luttinger models are indeed closely related: the directions of the momenta of the fermions in the Luttinger model correspond to the spin degrees of freedom in the Thirring model. There are however substantial differences: the Thirring model is local and covariant and, as such, has the standard ultraviolet divergences of relativistic quantum field theory; on the other hand, fermions in the Luttinger model interact non-locally and ultraviolet divergences are absent. The Thirring model is also plagued by the infrared divergences typical of any local and covariant gauge theory [18, 19] which arise here from the peculiarities of spacetime dimension two; on the other hand the Luttinger model is defined on a compact space and therefore infrared regular; of course it is not Lorentz covariant but this is not a theme in condensed matter physics.

It took some effort [16] to get a correct vacuum (i.e. zero temperature) solution of the Thirring model; Johnson [20] was the first to compute the $n$-point functions at zero temperature by using the Ward identities; his solution was completed by Klaiber [21] who exhibited a quantum field operator having precisely Johnson’s $n$-point functions. Klaiber’s explicit operatorial solution is written in terms of the two key building blocks that are the massless two-dimensional free scalar field and its dual. Klaiber used a non-covariant but positive definite quantization of the building blocks and recovered Lorentz covariance in the last step. The ab initio fully covariant quantization was given twenty years later in [22]. The bosonization technique was successfully applied also to deal with the massive case [23, 24].

On the other hand, less attention has been paid to the thermal representations of the Thirring model [12, 25–29] in comparison with the Luttinger liquids; furthermore, the relevant features have been somehow buried in irrelevant technical complications; examples are the introduction of fictitious chemical potentials to tame the infrared divergences which at a first glance may seem to be worse than in vacuo or the redundancy of the so called thermofield formalism.

Here we provide some new information about the quantization of two-dimensional massless field and its dual field in the steady states where there are two independent temperatures for the left and right movers. We face the infrared singularities directly,
without introducing artificial chemical potentials for the left and right movers as is done in the literature. We provide the general setup by using the Krein space formalism which is the natural setting when quantizing gauge theories in local and covariant gauges [19, 30]; this constructions allows in particular an intrinsic construction of the relevant left and right charge operators which, as in the zero temperature case [22, 31], belong to the Krein extension of the field algebra and are not introduced by hand.

The charge operators are in turn a crucial ingredient for the bosonization of the free Dirac field in the steady state with two distinct temperatures for the left and right movers. We recover in our construction a formula by Bernard and Doyon [8].

We also construct a general class of interacting models characterized by two temperatures. In particular for the Thirring model [16, 21] we provide the full solution in the steady state case. The results described here may be of relevance for further discussions of integrable models of QFT in 1+1 spacetime dimensions and also for 2-D gravity.

2 Vacuum of the massless field. Left and right movers

Let us first recall the general recipe to obtain a large class of (possibly) inequivalent quantizations of a free massive bosonic field and summarize the main features of the zero temperature massless case. The starting ingredient of any field theory is the covariant commutator:

\[ [\phi(x), \phi(y)] = C(x - y) \quad (2.1) \]

a distribution that encodes the commutation relations of the field operators at different spacetime events; for linear field theories it is a c-number.

Next, the commutation relations (2.1) needs to be represented by operators in a Hilbert space. For linear fields, any relevant Hilbert space structure may be encoded in a two-point function \( W(x, y) \) such that

\[ W(x, y) - W(y, x) = C(x - y). \quad (2.2) \]

If we suppose translation invariance, \( W(x - y) \) may be introduced through its Fourier representation

\[ \bar{W}(k) = \tilde{n}(k) \tilde{C}(k), \quad (2.3) \]

where \( \tilde{n}(k) \) is a multiplier for the distribution \( \tilde{C}(k) \). The commutation relations (2.2) now read \( \bar{W}(k) - \bar{W}(-k) = C(k) \) i.e.

\[ \tilde{n}(k) + \tilde{n}(-k) = 1; \quad (2.4) \]

the above condition must hold on shell, i.e. on a neighborhood of the support of the distribution \( \tilde{C}(k) \).

Any choice of the weight \( \tilde{n}(k) \) such that the distribution (2.3) is a positive measure endows the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) of smooth and rapidly decreasing test functions [32] with a positive semi-definite scalar product which expresses the vacuum-to-vacuum quantum mechanical transition amplitudes encoded in the two-point function:

\[ \langle \Psi_f, \Psi_g \rangle = \langle \Omega, \phi(f)\phi(g)\Omega \rangle = \langle f, g \rangle = \int f(x)W(x - y)g(y)dxdy. \quad (2.5) \]
More specifically, the covariant commutator \( C_m(x) \) of the massive Klein-Gordon field is the unique solution of the Cauchy problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\Box + m^2)C_m(x) = 0, \\
C_m(0, x) = 0, \\
\partial_0 C_m(0, x) = -i\delta(x);
\end{array} \right.
\end{aligned}
\]

(2.6)
in Fourier space\(^1\) the solution is written:

\[
\tilde{C}_m(k) = 2\pi \text{sgn}(k^0)\delta(k^2 - m^2).
\]

(2.10)
The right choice for the Wightman vacuum is the step function \( \tilde{n}(k) = \theta(k^0) \) i.e. only positive energy is allowed for the states in the Hilbert space of the model:

\[
\tilde{W}_m(k) = \theta(k^0)\tilde{C}_m(k) = 2\pi \theta(k^0)\delta(k^2 - m^2).
\]

(2.11)
The massless limit can be taken straightforwardly:

\[
\tilde{W}_0(k) = 2\pi \theta(k^0)\delta(k^2).
\]

There is however one notable very important exception: in spacetime dimension \( d = 2 \) the distribution \( \theta(k^0) \) is not a multiplier for \( \delta(k^2) \); in that case the limit is well-defined only on test functions vanishing in Fourier space at \( k = 0 \)

\[
S_0(\mathbb{R}^2) = \left\{ h \in \mathcal{S}(\mathbb{R}^2), \quad \tilde{h}(0) = \int h(x)dx = 0 \right\}
\]

(2.12)
and need to be extended (i.e. regularized) to general test function of \( \mathcal{S}(\mathbb{R}^2) \).

It is useful for what follows to recall the simple construction that deals with this problem. By introducing the lightcone variables \( x^\pm = x^0 \pm x^1 \) and \( k^\pm = k^0 \pm k^1 \) so that \( \partial_0 = \partial_+ + \partial_- \) and \( \partial_1 = \partial_+ - \partial_- \) one would write

\[
\theta(k^0)\delta(k^2)dk^0 dk^1 = \frac{1}{2k^+}\theta(k^+)^2\delta(k^-)dk^+ dk^- + \frac{1}{2k^-}\theta(k^-)^2\delta(k^+)dk^+ dk^-.
\]

(2.13)
The standard regularization of the r.h.s. goes as follows [21, 33]. Consider for instance the right-mover (the first term at the rhs) and define its regularization by subtracting the divergent part:

\[
W_r(h) = \pi \int_{2k^+}^{\infty} \frac{1}{k^+} \tilde{h}(k^+, 0)dk^+ + \pi \int_0^{2k^-} \frac{1}{k^-} \left[ \tilde{h}(k^+, 0) - \tilde{h}(0, 0) \right]dk^+.
\]

(2.14)

\(^1\)Conventions about the Fourier transform in spacetime dimension \( d \) are as follows: for test functions

\[
f(x) = \frac{1}{(2\pi)^d} \int e^{ikx} \tilde{f}(k)
\]

and for distributions

\[
W(x) = \frac{1}{(2\pi)^d} \int e^{-ikx} \tilde{W}(k)
\]

so that

\[
\int W(x)f(x)dx = \frac{1}{(2\pi)^d} \int \tilde{W}(k)\tilde{f}(k)dk.
\]

(2.9)
where $\kappa$ is an arbitrary infrared regulator having the dimension of a mass. A similar definition provides the regularized $W_l(h)$ and the complete regularized massless two-point function is the sum $W_0(h) = W_r(h) + W_l(h)$.

To compute the Fourier antitransform of the above distributions we may treat the exponential $\exp(-ikx)$ as a test function by adding to $x$ an imaginary vector in the backward tube [32]; this is possible because of the positivity of the energy spectrum. We get

$$W_r(x) = -\frac{1}{4\pi} \log \left( i\mu(x^- - i\epsilon) \right), \quad \mu = e^{\gamma\kappa}$$

(2.15)

(the infrared regulator $\mu$ is not to be confused with a chemical potential). $W_l(z^-)$ is analytic in the lower half-plane of the complex variable $z^-$. An identical calculation provides the Wightman function of the left-mover

$$W_l(x) = -\frac{1}{4\pi} \log \left( i\mu(x^+ - i\epsilon) \right)$$

(2.16)

which is analytic in the lower half-plane of the complex variable $z^+$. The left and right movers are not local fields: the commutators

$$C_r(x) = -\frac{i}{4} \text{sgn}(x^-), \quad C_l(x) = -\frac{i}{4} \text{sgn}(x^+).$$

(2.17)

do not vanish when $x$ is space-like ($x$ is spacelike if $x^+x^- < 0$). On the other hand their sum is Lorentz invariant and local:

$$W_0(x) = W_l(x^-) + W_r(x^+) = -\frac{1}{4\pi} \log \left( -\mu^2 x^2 + i\epsilon x^0 \right),$$

$$C_0(x) = -\frac{i}{4} \text{sgn}(x^+) - \frac{i}{4} \text{sgn}(x^-) = -\frac{i}{2} \text{sgn}(x^0) \theta(x^2).$$

(2.18)

(2.19)

The distribution $W_0(x)$ actually extends to a maximally analytic function of the Lorentz invariant complex variable $z^2$ with a cut on the real positive axis:

$$W_0(z) = -\frac{1}{4\pi} \log(-\mu^2 z^2).$$

(2.20)

However there is a price to pay: having enforced Lorentz invariance and locality we have renounced to positive-definiteness i.e. to a straightforward quantum mechanical interpretation of the model. This is an unavoidable feature of gauge quantum field theories.

3 Introducing the thermal states

Let us now briefly introduce the discussion with the thermal massive case presented in a way that is suitable for an extension to the massless case. The starting point to construct the two-point function consists in inserting the Bose-Einstein distribution for $\tilde{n}(k)$ in eq. (2.3); in the massive case this choice is alright since $\tilde{n}(k)$ is a well-defined multiplier for the commutator $\tilde{C}_m(k)$:

$$\tilde{W}_{m,\beta}(k) = \frac{2\pi \text{sgn}(k^0) \delta(k^2 - m^2)}{1 - e^{-\beta k^0}}. $$

(3.1)
Moreover, the following manipulations are perfectly meaningful:
\[
W_{m\beta}(x) = \frac{1}{2\pi} \int e^{-ikx} \frac{\theta(k^0)\delta(k^2 - m^2)}{1 - e^{-\beta k^0}} dk + \frac{1}{2\pi} \int e^{-ikx} \frac{e^{\beta k^0} \theta(-k^0)\delta(k^2 - m^2)}{1 - e^{\beta k^0}} dk
\]
\[
= \frac{1}{2\pi} \sum_{n=0}^{\infty} \int e^{-ikx-n\beta k^0} \theta(k^0)\delta(k^2 - m^2) dk + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int e^{-ikx+n\beta k^0} \theta(-k^0)\delta(k^2 - m^2) dk
\]
\[
= \sum_{n=0}^{\infty} W_m(t-in\beta, \bar{x}) + \sum_{n=1}^{\infty} W'_m(t+in\beta, \bar{x})
\]

(3.2)

where \(W'_m(z) = W_m(-z)\). The series at r.h.s. of (3.2) converges in the sense of distributions.

Once more the zero mass limit cannot be taken straightforwardly and it is actually trickier than at zero temperature. For example, following (3.2) one might try to define the thermal massless two-point function as a series constructed in terms of the massless two-point function (2.20) as follows:

\[
W_{0\beta}(t, \bar{x}) = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \log \left( -\mu^2 (t-in\beta - \bar{x})^2 \right) - \frac{1}{4\pi} \sum_{n=1}^{\infty} \log \left( -\mu^2 (t+in\beta - \bar{x})^2 \right).
\]

(3.3)

Every term entering in this series is well defined because of the maximal analyticity of \(W_0\); the series formally satisfies the KMS periodicity condition at temperature \(T = 1/\beta\). But, unfortunately, the series does not converge.

4 Thermal correlators of the left and right movers

Let us proceed \textit{ab initio} as in the zero temperature case by formally defining the thermal two-point function by its Fourier transform as follows:

\[
\overline{W}_\beta(k) = \frac{2\pi \text{sgn}(k^0)\delta(k^2)}{1 - e^{-\beta k^0}}.
\]

(4.1)

Here we avoid the use of a cutoff in the exponential in the form of a fictive chemical potential, as it is done in the literature [12, 25] but directly face the infrared divergence. At a first superficial glance the infrared divergence seems to be worse than in (2): the above distribution appears to be well-defined only on test function vanishing at least quadratically at \(k = 0\); we will clarify below why it is not so.

The left and right moving parts of the r.h.s. of (4.1) have now two contributions, according with the sign of the energy (\(p\) — positive, \(n\) — negative):

\[
\overline{W}_{r\beta}(k) = \overline{W}_{p_{r\beta}}(k) + \overline{W}_{n_{r\beta}}(k) = \frac{2\pi \theta(k^+)\delta(k^-)}{k^+(1 - e^{-\frac{1}{\beta}k^+})} + \frac{2\pi e^{\frac{1}{2}k^+} \theta(-k^+)\delta(k^-)}{|k^+|(1 - e^{\frac{1}{2}k^+})},
\]

\[
\overline{W}_{l\beta}(k) = \overline{W}_{p_{l\beta}}(k) + \overline{W}_{n_{l\beta}}(k) = \frac{2\pi \theta(k^-)\delta(k^+)}{k^- (1 - e^{-\frac{1}{\beta}k^-})} + \frac{2\pi e^{\frac{1}{2}k^-} \theta(-k^-)\delta(k^+)}{|k^-|(1 - e^{\frac{1}{2}k^-})}.
\]

(4.2)

(4.3)

One by one, the distributions at the r.h.s. are well-defined only on test functions vanishing at least quadratically at \(k = 0\).
Let us focus for instance on $W_{r\beta}^p$: given a general test function $h$ we may introduce an infrared regularized distribution as follows:

$$W_{r\beta}^p(h) = \pi \int_{2\kappa}^{\infty} \frac{\tilde{h}(k^+, 0)}{k^+(1 - e^{-\frac{1}{2} \beta k^+})} dk^+ + \pi \int_0^{2\kappa} \frac{\tilde{h}(k^+, 0) - \tilde{h}(0, 0) - \partial_k \tilde{h}(0, 0) k^+}{k^+(1 - e^{-\frac{1}{2} \beta k^+})} dk^+ + A \tilde{h}(0, 0) + B \partial_k \tilde{h}(0, 0). \tag{4.4}$$

The constants $A$ and $B$ are arbitrary and can be adjusted at will. To compute the $x$-space representation of $W_{r\beta}^p$ we set $\tilde{h}(k^+, 0) = e^{-\frac{1}{2} ik^+ x}$, expand the denominators at the r.h.s. of eq. (4.4) and then interchange the integrals and the series:

$$W_{r\beta}^p(x) = A + \frac{i B x^-}{2} + \frac{1}{4\pi} \left( i \kappa x^- - \log(i \kappa e^\gamma x^-) \right) + \frac{1}{4\pi} \sum_{n=1}^{\infty} \left[ \Gamma(0, n \beta \kappa) - \frac{i x^- (e^{-\beta n} - 1)}{\beta n} - \log \left( 1 + \frac{i x^-}{\beta n} \right) \right]. \tag{4.5}$$

The terms proportional to $x^-$ are crucial for the convergence of the series at the rhs; they come precisely from the subtraction of the term proportional to $k^+$ at the r.h.s. of eq. (4.4).

A similar expression holds for the negative energy part with the noticeable difference that here the expansion starts with $n = 1$:

$$W_{r\beta}^n(x) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \left[ \Gamma(0, n \beta \kappa) + \frac{i x^- (e^{-\beta n} - 1)}{\beta n} - \log \left( 1 - \frac{i x^-}{\beta n} \right) \right]. \tag{4.6}$$

All in all

$$W_{r\beta}(x) = W_{r\beta}^p(x) + W_{r\beta}^n(x) = A + \frac{i B x^-}{2} + \frac{1}{4\pi} \left( i \kappa x^- - \log(i \kappa e^\gamma x^-) \right) + \frac{1}{4\pi} \sum_{n=1}^{\infty} \left[ 2 \Gamma(0, n \beta \kappa) - \log \left( 1 + \left( \frac{x^-}{\beta n} \right)^2 \right) \right]. \tag{4.7}$$

The terms linear in $x^-$ that guarantee the convergence of the series (4.5) and (4.6) have disappeared at the r.h.s. of eq. (4.7). The remaining contribution proportional to $x^-$ is a regular solution of the wave equation that does not depend on the temperature but gives an anomalous contribution to the commutator. We may remove it by choosing $B = -\kappa/2\pi$. Similarly we might dispose of all the constant terms in the series but it is wiser not to do so; we simply take $A = 0$. In the end, setting $\beta = \beta_r$, $\kappa = \kappa_r$ and $\mu_r = \kappa_r e^\gamma$

$$W_{r\beta_r}(x) = -\frac{1}{4\pi} \log(i \mu_r x^-) - \frac{1}{4\pi} \log \left( q_r \beta_r \frac{\sinh(\frac{x^-}{\beta_r})}{\pi x^-} \right) = W_r(x) + T_{r\beta_r}(x) \tag{4.8}$$

where

$$\log q_r = \sum_{n=1}^{\infty} 2 \Gamma(0, n \beta_r \kappa_r). \tag{4.9}$$

The structure of the final result is quite interesting. The first term $W_r(x)$ does not depend on the temperature and is the only one contributing to the commutator; it coincides
with (2.15) and as such it reproduces (2.17). The second term \( T_{r\beta_r}(x) \) does depend on the inverse temperature \( \beta_r \). It is a regular symmetric function of the variable \( x^- \) and therefore has zero commutator:

\[
T_{r\beta_r}(x) - T_{r\beta_r}(-x) = 0. \tag{4.10}
\]

In this sense \( T_{r\beta_r}(x) \) may be understood as a classical correction to the quantum zero-temperature two-point function \( W_r(x) \). Note also that \( T_{r\beta_r} \) tends to zero in the sense of distributions when the temperature vanishes.

The argument of the log in the second term \( T_{r\beta_r}(x) \) is an entire function of \( x^- \). It vanishes only if \( x^- \in i\beta(\mathbb{Z} \setminus \{0\}) \). In particular it is holomorphic and has no zeros in the set \( \{x^- : |\text{Im } x^-| < \beta_r, \} \). This is to be intersected by the domain of the first term, namely \( \{x^- : ix^- \notin \mathbb{R}_-\} \) which contains \( \{x^- : \text{Im } x^- < 0\} \). Hence the whole expression in (4.8) is analytic in the strip

\[
\{x^- : -\beta_r < \text{Im } x^- < 0\} \tag{4.11}
\]

and has boundary values in the sense of tempered distributions at the boundary of the strip. As a very important consequence the Wick powers \( W_{r\beta_r}(x)^n \) are well-defined for any integer \( n \geq 0 \).

An identical construction provides the thermal equilibrium two-point function for the left mover. Of course there is no necessity to take the same temperature; in the following we will denote by \( \beta_l \) the inverse left temperature; similarly we will denote by \( \kappa_l \) the left infrared cutoff:

\[
W_{l\beta_l}(x) = -\frac{1}{4\pi} \log(i\mu_l x^+) - \frac{1}{4\pi} \log \left( \frac{q_l\beta_l \sinh(\frac{\pi x^+}{\beta_l})}{\pi x^+} \right). \tag{4.12}
\]

Equations (4.8) and (4.12) show that \( W_{r\beta_r} \) and \( W_{l\beta_l} \) behave in the infrared better than their positive and negative energy parts and only one subtraction is needed to regularize either \( W_{r\beta_r} \) or \( W_{l\beta_l} \). This may also be seen by summing eq. (4.4) with a similar expression that may readily be written for \( W_{r\beta_r}^n(h) \); the second subtraction goes away and we get a formula that corresponds to the separation at the r.h.s. of eq. (4.8):

\[
W_{r\beta_r}(h) = W_r(h) + T_{r\beta_r}(h) = \pi \int_{2\kappa_r}^{\infty} \frac{\tilde{h}(k)|_{k^- = 0}}{2k^+} dk^+ + \pi \int_{0}^{2\kappa_r} \frac{\tilde{h}(k) - \tilde{h}(0)|_{k^- = 0}}{2k^+} dk^+ + \\
+ \pi \int_{2\kappa_r}^{\infty} \frac{e^{-\frac{1}{2}\beta_r k^+} \tilde{h}(k) + \tilde{h}(-k)|_{k^- = 0}}{2k^+(1 - e^{-\frac{1}{2}\beta_r k^+})} dk^+ + \\
+ \pi \int_{0}^{2\kappa_r} e^{-\frac{1}{2}\beta_r k^+} \frac{[\tilde{h}(k) + \tilde{h}(-k) - 2\tilde{h}(0)|_{k^- = 0}]}{2k^+(1 - e^{-\frac{1}{2}\beta_r k^+})} dk^+. \tag{4.13}
\]

This formula shows that both \( W_r \) and \( T_{r\beta_r} \) violate the positive-definiteness unless \( h \in \mathcal{S}_0(\mathbb{R}^2) \).
4.1 The steady state representation of the massless scalar field and its dual

At this point we are in position to consider the steady state two-point functions for the massless field and its dual

\[ \phi(x) = \phi_r(x^-) + \phi_l(x^+), \quad \tilde{\phi}(x) = \phi_r(x^-) - \phi_l(x^+) : \]  

(4.14)

\[ \langle \Omega, \phi(x)\phi(0)\Omega \rangle = \langle \Omega, \tilde{\phi}(x)\tilde{\phi}(0)\Omega \rangle = W_{r\beta_r}(x) + W_{l\beta_l}(x) = \]  

\[ = - \frac{1}{4\pi} \log(-\mu_r\mu_l x^2) - \frac{1}{4\pi} \log \left( \frac{q_r \beta_r \sinh(\frac{\pi x^-}{\beta})}{q_l \beta_l \sinh(\frac{\pi x^+}{\beta})} \right), \]  

(4.15)

\[ \langle \Omega, \phi(x)\tilde{\phi}(0)\Omega \rangle = \langle \tilde{\phi}(x)\phi(0)\Omega \rangle = W_{r\beta_r}(x) - W_{l\beta_l}(x) = \]  

\[ = - \frac{1}{4\pi} \log \left( \frac{\mu_r x^-}{\mu_l x^+} \right) - \frac{1}{4\pi} \log \left( \frac{q_r \beta_r x^+ \sinh(\frac{\pi x^-}{\beta})}{q_l \beta_l x^- \sinh(\frac{\pi x^+}{\beta})} \right). \]  

(4.16)

Here we denoted by \( \Omega \) the cyclic fundamental state in the Fock space constructed out of the two-point function. Because of the lack of positivity the reconstruction requires an additional structure described below.

The two-point function is translation invariance is preserved but of course not Lorentz invariant. The fields \( \phi \) and \( \tilde{\phi} \) are local but not relatively local i.e. they do not commute with each other at spacelike separations. The reason behind this failure [18, 19] is the local Gauss law

\[ \partial_\mu \tilde{\phi} = \epsilon_{\mu
u} \partial^\nu \phi, \]  

(4.17)

\[ \epsilon^{\mu\nu} \partial_\nu \tilde{\phi} = \epsilon^{\mu\nu} \epsilon_{\mu\nu} \partial^\rho \phi = \partial^\rho \phi. \]  

(4.18)

Let us briefly mention the thermal equilibrium case \( \beta_l = \beta_r = \beta \):

\[ W_\beta(x) = - \frac{1}{4\pi} \log(-\mu^2 x^2) - \frac{1}{4\pi} \log \left( q^2 \beta^2 \sinh(\frac{\pi x^-}{\beta}) \sinh(\frac{\pi x^+}{\beta}) \right). \]  

(4.19)

Here the whole expression in (4.19) is analytic in the tube

\[ \{ (x^-, x^+) : -\beta < \text{Im} x^- < 0, \quad -\beta < \text{Im} x^+ < 0 \} \]  

(4.20)

and has boundary values in the sense of tempered distributions at the boundary of the tube. The distributions \( W_\beta(x)^n \) are well-defined for all integers \( n \geq 0 \) i.e. the Wick powers of the field whose two-point function is (4.19) are well-defined. Note that (4.19) can also be rewritten as

\[ W_\beta(x) = \frac{1}{4\pi} \log \left( \frac{\pi^2}{\mu^2 q^2 \beta^2} \right) - \frac{1}{4\pi} \log \left( - \sinh \left( \frac{\pi x^-}{\beta} \right) \sinh \left( \frac{\pi x^+}{\beta} \right) \right). \]  

(4.21)

In particular if we set \( x^1 = 0, \ x^+ = x^- = t, \) we find

\[ W_\beta(t, 0) = \frac{1}{4\pi} \log \left( \frac{\pi^2}{\mu^2 q^2 \beta^2} \right) - \frac{1}{2\pi} \log \left( - \sinh \left( \frac{\pi (t - ie)}{\beta} \right) \right). \]  

(4.22)
5 Krein-Hilbert spaces of the left and right movers

Here we consider, as usual in quantum field theory, \( W_{r,\beta_r} \) and \( W_{l,\beta_l} \) as two-point distributions on \( S(\mathbb{R}^2) \), the Schwartz space of smooth and rapidly decreasing test functions. As such, they define two inner products on \( S(\mathbb{R}^2) \) that must be used to construct the left and right one-particle spaces and the Fock spaces associated to them:

\[
\langle f, g \rangle_r = \int \bar{f}(x) W_{r,\beta_r} (x-y) g(y) dx dy,
\]

\[
\langle f, g \rangle_l = \int \bar{f}(x) W_{l,\beta_l} (x-y) g(y) dx dy
\]

(5.1)

(5.2)

(we left the dependence on the temperature implicit at the lhs).

Let us focus on \( W_{r,\beta_r} \). As in the zero temperature case [31] \( W_{r,\beta_r} \) is not positive-semidefinite on \( S(\mathbb{R}^2) \) but it is so when restricted on the chargeless subspace \( S_0(\mathbb{R}^2) \). Indeed, when either \( f \) or \( g \) belongs to \( S_0(\mathbb{R}^2) \) the scalar product (5.1) computed by using eq. (4.13) with \( h(k) = \frac{f(k)g(k)}{\pi} \) simplifies and no subtraction is needed anymore:

\[
\langle f, g \rangle_r = \pi \int_0^{\infty} \frac{f(k)g(k)k^{-\alpha}}{2k^+} dk^+ + \pi \int_0^{\infty} \frac{e^{-\frac{1}{2}\beta_r k^+} [f(k)g(k)]_{k=0}^{k^{-\alpha}}}{2k^+ (1-e^{-\frac{1}{2}\beta_r k^+})} dk^+.
\]

(5.3)

An analogous formula holds for \( W_{l,\beta_l} \). The non-negativity of \( \langle f, f \rangle_r \) when \( f \) belongs to \( S_0(\mathbb{R}^2) \) is self-evident.

Consider now two general test functions \( f, g \in S(\mathbb{R}^2) \). A simplification occurs by choosing a real test function \( \tilde{\chi}_r \in S(\mathbb{R}^2) \) such that

\[
\langle \chi_r, \chi_r \rangle_r = 0, \quad \tilde{\chi}_r(0) = 1.
\]

(5.4)

It easy to convince oneself that a function with the above properties exists by considering for instance \( \chi_\alpha = \exp[-\frac{1}{2} \alpha (k^2 + k^{-2})] \); whatever be the values of the inverse temperature \( \beta_r \) and the infrared regulator \( \kappa_r \) the expression \( \langle \chi_\alpha, \chi_\alpha \rangle_r \) is positive for \( \alpha \) sufficiently close to zero and negative for \( \alpha \) sufficiently large. Thus, there exists an intermediate value \( \alpha_r \) depending on \( \beta_r \) and \( \kappa_r \) such that the conditions (5.4) are satisfied.

Given general \( f \) and \( g \) let us introduce their chargeless projections

\[
f_0 = f - \tilde{f}(0) \chi_r, \quad g_0 = g - \tilde{g}(0) \chi_r,
\]

(5.5)

so that the pseudo-scalar product (5.1) may be rewritten as follows:

\[
\langle f, g \rangle_r = \langle f_0, g_0 \rangle_r + \tilde{f}(0) \langle \chi_r, g_0 \rangle_r + \tilde{g}(0) \langle f_0, \chi_r \rangle_r;
\]

(5.6)

in the above formula \( \langle \chi_r, g_0 \rangle_{r,\beta_r} \) and \( \langle f_0, \chi_r \rangle_{r,\beta_r} \) are computed by using eq. (5.3). Following step by step the construction displayed in [31] one introduces a Krein-Hilbert majorant topology defined by the scalar product

\[
\langle f, g \rangle_r = \langle f_0, g_0 \rangle_r + \tilde{f}(0) \tilde{g}(0) + \langle f_0, \chi_r \rangle_r \langle \chi_r, g_0 \rangle_r
\]

(5.7)
and constructs the one-particle Krein-Hilbert space $H_{r\beta}^{(1)}$ of the right mover by completing $S(R^2)$ w.r.t. the Hilbertian norm (5.7).

There exist a special translation invariant vector $v_r \in H_{r\beta}^{(1)}$ such that

$$\langle \chi_r, f \rangle_r = (v_r, f)_r.$$  \hspace{1cm} (5.8)

$v_r$ has Hilbert norm equal to one and, at the same time, zero expectation value:

$$(v_r, v_r)_r = 1, \quad \langle v_r, v_r \rangle_r = 0.$$  \hspace{1cm} (5.9)

Actually, $v_r$ is the strong limit of a sequence of regular test functions converging pointwise to zero;\footnote{As an example consider the sequence $v_n$ constructed as follows:}

Summarizing, $H_{r\beta}^{(1)}$ admits the following orthogonal decomposition

$$H_{r\beta}^{(1)} = L^2_{r\beta}(R) \oplus v_r \oplus \chi_r;$$  \hspace{1cm} (5.11)

the metric operator $\eta$ representing the two-point function in the Hilbert scalar product

$$\langle \Psi_1, \Psi_2 \rangle_r = (\Psi_1, \eta \Psi_2)_r$$  \hspace{1cm} (5.12)

acts like the identity operator when restricted to $L^2_{r\beta}$ and interchanges $\chi_r$ and $v_r$:

$$\eta = \begin{pmatrix} I_{L^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \eta^2 = I.$$  \hspace{1cm} (5.13)

The Fock-Segal quantization procedure may then be used to obtain the symmetric Fock space of the right mover $F(H_{r\beta}^{(1)})$. Similarly one constructs the symmetric Fock space of the left mover $F(H_{l\beta}^{(1)})$ and the tensor product $F(H_{l\beta}^{(1)}) \otimes F(H_{r\beta}^{(1)})$. Finally the field operators

$$\phi_l(f) = \phi_l^{(+)}(f) + \phi_l^{(-)}(f), \quad \phi_r(f) = \phi_r^{(+)}(f) + \phi_r^{(-)}(f),$$  \hspace{1cm} (5.14)

operate on the corresponding factors as follows:

$$(\phi_r^{(+)}(f)\Psi)_n(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\Psi^{n-1}_r(x_1, \ldots, \hat{x}_j, \ldots, x_n),$$  \hspace{1cm} (5.15)

$$(\phi_r^{(-)}(f)\Psi)_n(x_1, \ldots, x_n) = \sqrt{n + 1} \int f(x)W_{r\beta}(x, x')\Psi_n^{n+1}(x', x_1, \ldots, x_n)dx dx'.$$  \hspace{1cm} (5.16)

where $\phi$ is a smooth positive function such that $0 \leq \phi \leq 1$, $\phi(0) = 1$ and $\phi(k) = 0$ for $|k| > 1$. The strong limit $||v_n - v_r||_r \rightarrow 0$ can be shown as follows:

$$(v_n, f)_r = \langle v_n, f_0 \rangle_r + \langle v_n, \chi_r \rangle_r \langle \chi_r, f_0 \rangle_r = \langle v_n, f_0 \rangle_r + \langle \chi_r, f_0 \rangle_r \quad \rightarrow \quad \langle \chi_r, f_0 \rangle_r \rightarrow 0,$$

$$(v_n, v_n)_r = 1 + \frac{\langle \chi_n, \chi_n \rangle_r}{\langle \chi_n, \chi_r \rangle_r} \leq 1 + \frac{1}{\langle \chi_n, \chi_r \rangle_r} \rightarrow 1.$$
The above standard formulae may be used to extend the field algebra to include operators such as \( \phi_\gamma(f) \) with \( f \in H^{(1)}_{1/\beta} \) and \( \phi_i(f) \) with \( f \in H^{(1)}_{1/\delta} \). Particularly important in the following will be the role of the fields \( \phi^{-}_r(v_r) \) and \( \phi^{+}_i(v_i) \): a simple calculation shows that
\[
[\phi^{-}_r(v_r), \phi_r(f)] = [\phi^{+}_i(v_i), \phi^{+}_l(f)] = (v_r, f)_r = (\chi_r, f)_r = \tilde{f}(0) 
\]
so that
\[
[\phi^{+}_r(v_r), \phi_r(x)] = [\phi^{+}_i(v_i), \phi_i(x)] = \mp 1. 
\]

6 Thermal fermion bosonization (two temperatures)

Here we consider the field algebra \( \mathcal{F} \) generated by the fields \( \phi_i, \phi_r \) and their Wick-ordered exponentials \( \exp(z\phi_i) : \) and \( \exp(z\phi_r) : \). One way to define the Wick exponentials is to ensure the strong convergence of the series
\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} : \phi_i^n : (f), \quad \sum_{n=0}^{\infty} \frac{z^n}{n!} : \phi_r^n : (f), 
\]
in the Fock-Krein space that we constructed. The infrared behaviour of the norm associated to the scalar product (5.7) shows that the above series cannot converge for arbitrary tempered test functions in \( S(\mathbb{R}^2) \). In the zero temperature case the Wick-exponentials are Jaffe-type fields \([22, 34, 35]\); a similar restriction works also in our case but we do not further dwell on this point here and we limit ourselves to test functions having compact support in \( x \)-space where the proof of the strong convergence of the above series has no substantial difficulty. We do not reproduce the proof here.

Let us consider now the left and right global (or rigid) gauge transformations
\[
\gamma_\lambda : \phi_l(x^+) \to \phi_l(x^+) + \lambda_l, \\
\gamma_\lambda : \phi_r(x^-) \to \phi_r(x^-) + \lambda_r, 
\]
which extend to the Wick exponentials as follows:
\[
\gamma_\lambda : \exp(z\phi_l) : (x^+) \to \exp(z\lambda_l) : \exp(z\phi_l) : (x^+), \\
\gamma_\lambda : \exp(z\phi_r) : (x^-) \to \exp(z\lambda_r) : \exp(z\phi_r) : (x^-). 
\]
These transformations act trivially when the field operators are smeared with chargeless test functions (i.e. such that \( \tilde{f}(0) = f f(x)dx = 0 \)). More generally we may consider the local gauge transformations
\[
\gamma_f : \phi_l(x^+) \to \phi_l(x^+) + f_l(x^+), \\
\gamma_f : \phi_r(x^-) \to \phi_r(x^-) + f_r(x^-), 
\]
which also extend to the Wick exponentials under suitable technical conditions on the growth of the functions \( f_l(x^+) \) and \( f_r(x^-) \) at infinity (which we do not specify here). The Hilbert-Krein construction allows for the existence of charge operators
\[
Q_l = \frac{i\phi^{(+)}_l(v_l) - i\phi^{(-)}_l(v_l)}{2}, \quad Q_r = \frac{i\phi^{(+)}_r(v_r) - i\phi^{(-)}_r(v_r)}{2}; 
\]
that implement the left and right gauge transformations: eq. \( \text{(5.18)} \) indeed is equivalent to

\[
\frac{d}{d\lambda_l} \gamma_l(\phi_l(x)) = i[Q_l, \phi_l(x)] = 1, \quad \frac{d}{d\lambda_r} \gamma_r(\phi_r(x)) = i[Q_r, \phi_r(x)] = 1. \tag{6.9}
\]

The charge operators may be used to introduce the dressed Wick exponentials:

\[
\varphi_1(x) = e^{i a Q_r} : \exp z \phi_1 : (x), \tag{6.10}
\]

\[
\varphi_2(x) = e^{i b Q_l} : \exp z \phi_2 : (x). \tag{6.11}
\]

Part of what follows goes as in the zero temperature case \[22\]. Indeed, set aside the infrared convergence problems mentioned above, the possibility to define Wick powers and Wick exponentials of the left and right movers depends only on the ultraviolet behaviour of their correlation functions and at short distances the two temperatures play no role. In particular the values of the constants \( a, b \) and \( z \) that give fermionic anticommutation rules are the same as in the zero temperature. We briefly sketch the construction for the reader’s convenience: since

\[
\varphi_1(x) \varphi_1(y) = \varphi_1(y) \varphi_1(x) \exp \left[ -\frac{i z^2}{4} \text{sgn}(x^+ - y^+) \right], \tag{6.12}
\]

\[
\varphi_1(x) \varphi_1(y) = \varphi_1(y) \varphi_1(x) \exp \left[ -\frac{i |z|^2}{4} \text{sgn}(x^+ - y^+) \right], \tag{6.13}
\]

the fields anticommute at spacelike intervals if there hold the necessary conditions \( z^2 = \pm 4\pi \) and \( |z|^2 = 4\pi \). Also, the identities

\[
\varphi_1(x) \varphi_2(y) = \exp[-(b - a) z] \varphi_2(y) \varphi_1(x), \tag{6.14}
\]

\[
\varphi_1(x) \varphi_2^*(y) = \exp[(b z + a \overline{z})] \varphi_2^*(y) \varphi_1(x), \tag{6.15}
\]

require \( z(b - a) = \pm i\pi \) and \( b z + a \overline{z} = \pm i\pi \). All in all we get

\[
z = 2i\sqrt{\pi}, \quad b = a \pm \frac{1}{2} \sqrt{\pi}. \tag{6.16}
\]

Let us therefore introduce the fields

\[
\psi_1(x) = \psi_l(x^+) = A \exp \left( \frac{\sqrt{\pi}}{4i} Q_r \right) : \exp 2i\sqrt{\pi} \phi_l : (x), \tag{6.17}
\]

\[
\psi_2(x) = \psi_r(x^-) = B \exp \left( \frac{i\sqrt{\pi}}{4} Q_l \right) : \exp 2i\sqrt{\pi} \phi_r : (x), \tag{6.18}
\]

and compute their anticommutators:

\[
\psi_1(x) \psi_1^*(y) + \psi_1^*(y) \psi_1(x) = \frac{|A|^2 \pi (x^+ - y^+)}{q_l \mu_l \beta_l \sinh \left( \frac{\pi (x^+ - y^+)}{\beta_l} \right)} : \exp 2i\sqrt{\pi} \phi_l : e^{-2i\sqrt{\pi} \phi_l} : \times \\
\times \left[ \frac{1}{i(x^+ - y^+) + \varepsilon} + \frac{1}{i(y^+ - x^+) + \varepsilon} \right]. \tag{6.19}
\]
The last factor equals \(2\pi \delta(x^+ - y^+)^2\). At \((x^+ - y^+) = 0\) the Wick product becomes equal to 1 and the first factor is regular:

\[
\psi_1(x)\psi^*_1(y) + \psi^*_1(y)\psi_1(x) = \frac{2\pi |A|^2}{q_l \mu_l} \delta(x^+ - y^+). \tag{6.20}
\]

Similarly

\[
\psi_2(x)\psi^*_2(y) + \psi^*_2(y)\psi_2(x) = \frac{2\pi |B|^2}{q_r \mu_r} \delta(x^- - y^-), \tag{6.21}
\]

while all the remaining anticommutators are zero. By choosing

\[
A = \sqrt{\frac{q_l \mu_l}{2\pi}}, \quad B = \sqrt{\frac{q_r \mu_r}{2\pi}}, \tag{6.22}
\]

the above construction reproduces the canonical commutation relations of a massless free Dirac field.

The Dirac equation\(^3\) is satisfied by construction:

\[
\gamma^\mu \partial_\mu \psi = \begin{bmatrix} \partial_+ \psi_r(x^-) \\ \partial_- \psi_l(x^+) \end{bmatrix} = 0. \tag{6.24}
\]

The two-point expectation value in the fundamental state

\[
\langle \Omega, \psi(x)\psi(y)\Omega \rangle = \begin{pmatrix} 0 & |A|^2 e^{4\pi W_0 \beta_l (x-y)} \rho_l & 1 \\ 0 & e^{-4\pi W_0 \beta_l (x-y)} & 0 \\ 2 i \beta_r \sinh \frac{2(x-\bar{x})}{\rho_r} & 0 & 0 \end{pmatrix} \tag{6.25}
\]

does not depend on the cutoffs but only on the left and right temperatures, as it must be, because a massless Dirac field in 1+1 spacetime dimension is not infrared singular (as opposed to the scalar field \(\phi\)).

On the other hand the \(n\)-point expectation values do not coincide with those of a free Dirac spinor field: many of them should be zero but they are not. As in the zero temperature case, one may impose by hand \(ad\) \(hoc\) selection rules to kill all the unwanted contributions [16] or introduce an extra free fermion [21]. Fortunately, in the present framework these steps are not necessary: in the Hilbert-Krein-Fock space displayed in the previous section there exist special states, called fermionic vacua [22] (see appendix A) which automatically implement the necessary selection rules.

In particular, for the free Dirac fields (6.17) and (6.18) the fermionic states we are referring to are labelled by two angles \(\theta_l\) and \(\theta_r\) as follows:

\[
\Omega^{free}_{\theta_l \theta_r} = \frac{1}{4\pi} \int_{-\sqrt{\pi}}^{\sqrt{\pi}} d\lambda_l \int_{-\sqrt{\pi}}^{\sqrt{\pi}} d\lambda_r e^{i(\lambda_r+\theta_r)Q_r+i(\lambda_l+\theta_l)Q_l} \Omega. \tag{6.26}
\]

\(^3\)Gamma matrices are as follows:

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.23}
\]
The two-point expectation value in the state $\Omega_{\theta_0\bar{\theta}_e}^{free}$ coincides with the expectation value in the fundamental state

$$\langle \Omega_{\theta_0\bar{\theta}_e}^{free}, \psi(x) \bar{\psi}(y) \Omega_{\theta_0\bar{\theta}_e}^{free} \rangle = \langle \Omega, \psi(x) \bar{\psi}(y) \Omega \rangle$$

(6.27)

and the truncated n-point functions in the state $\Omega_{\theta_0\bar{\theta}_e}^{free}$ vanish. The steady state fermion bosonization is thus achieved.

As is well-known [36, 37] the classical currents

$$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x), \quad j_\mu(x) = \bar{\psi}(x) \gamma^5 \gamma^\mu \psi(x) = \epsilon^{\mu\nu} j_\nu$$

(6.28, 6.29)

require an ultraviolet regularization at the quantum level.

$$j_+(x) = j_0(x) + j_1(x) = 2 \lim_{\epsilon \to 0} \left[ \psi_+(x) - \psi_+(x + \epsilon) \psi_-(x + \epsilon) \right] = -2 \sqrt{\pi} \epsilon^+ \partial_+ \phi_+ (x^+) \frac{1}{\sqrt{\beta_l \sinh \frac{\pi x^+}{\beta_l}}} \to - \frac{2}{\sqrt{\pi}} \partial_+ \phi_+ (x^+).$$

(6.30)

Similarly

$$j_-(x) = - \frac{2}{\sqrt{\pi}} \partial_- \phi_+ (x^-).$$

(6.31)

There follow the commutation relations [36–38]

$$[j_+(x), j_+(y)] = 2i \xi \delta'(x^+ - y^+), \quad [j_+(x), j_-(y)] = 2i \xi \delta'(x^- - y^-), \quad \xi = \frac{2}{\pi}.$$

(6.32)

An elementary computation gives (see eq. (4.12))

$$\langle \Omega, j_+(x) j_+(0) \Omega \rangle = - \frac{4}{\pi} \beta_l^2 W_l \left( \frac{4 \pi x^+}{\beta_l} \right) = \frac{1}{\beta_l^2 \sinh^2 \left( \frac{4 \pi x^+}{\beta_l} \right)}.$$  

(6.33)

By subtracting the vacuum (i.e. zero temperature) expectation value we get the normal ordered two-point expectation value of the current

$$\langle \Omega, j_+(x) j_+(0) : \Omega \rangle = - \frac{4}{\pi} \beta_l^2 \left( W_l \left( \frac{4 \pi x^+}{\beta_l} \right) - W_l \left( \frac{4 \pi x^+}{\beta_l} \right) \right)$$

$$= \frac{1}{\pi^2} \beta_l^2 \log \left( \frac{q_l \beta_l \sinh \left( \frac{4 \pi x^+}{\beta_l} \right)}{4 \pi x^+} \right) = - \frac{1}{\beta_l^2 \sinh^2 \left( \frac{4 \pi x^+}{\beta_l} \right)} + \frac{1}{\pi^2} \frac{4 \pi x^+}{\beta_l^2}. \quad (6.34)$$

We can now take the limit where the two points coincide and see that there are persistent squared currents

$$\langle \Omega, j_+^2(x) : \Omega \rangle = \frac{1}{3 \beta_l^2}, \quad \langle \Omega, j_-^2(x) : \Omega \rangle = \frac{1}{3 \beta_l^2}. \quad (6.35)$$

and non-vanishing expectation values of the energy momentum tensor [36–38]

$$\Theta_{\mu\nu}(x) = \frac{1}{2\xi} : (2j_\mu j_\nu - g_{\mu\nu} j^a j_a) : (x),$$

(6.36)

$$\langle \Omega, \Theta_+(x) : \Omega \rangle = \frac{\pi}{12 \beta_l^2}, \quad \langle \Omega, \Theta_-(x) : \Omega \rangle = \frac{\pi}{12 \beta_l^2}. \quad (6.37)$$

(6.38)
By considering the total energy flow we are able to reproduce a formula by Bernard and Doyon [8] in the special case where the central charge is equal to one:

$$\langle \Omega, : (\Theta_+(x) - \Theta_-(x)) : \Omega \rangle = \frac{\pi}{12\beta^2_I} - \frac{\pi}{12\beta^2_T}. \quad (6.38)$$

## 7 Interacting fermions: the steady-state Thirring model

Here we construct the full operator solution of the Thirring model in the general steady state with a left and right temperature. Following [31], let us at first introduce generic non-free fermion operators in terms of five real parameters $a, b, c, d, \sigma$ as follows:

$$\psi_1(x) = A \exp \left( \frac{\sigma \pi}{4ic} Q_l + \frac{\sigma \pi}{4id} Q_r \right) : \exp (ia \phi_l + i b \phi_r) : (x), \quad (7.1)$$

$$\psi_2(x) = B \exp \left( \frac{i \sigma \pi}{4a} Q_l + \frac{i \sigma \pi}{4b} Q_r \right) : \exp (ic \phi_l + i d \phi_r) : (x). \quad (7.2)$$

Their commutation relations are easily computed:

$$\psi_1(x)\psi_1(y) = \psi_1(y)\psi_1(x) e^{\frac{\pi}{4} \operatorname{sgn}(x+y) + \frac{\pi}{4} \operatorname{sgn}(x-y)}, \quad (7.3)$$

$$\psi_2(x)\psi_2(y) = \psi_2(y)\psi_2(x) e^{\frac{\pi}{4} \operatorname{sgn}(x+y) + \frac{\pi}{4} \operatorname{sgn}(x-y)}, \quad (7.4)$$

$$\psi_1(x)\psi_2(y) = \psi_2(y)\psi_1(x) e^{-i\sigma \pi} e^{\frac{\pi}{4} \operatorname{sgn}(x+y) + \frac{\pi}{4} \operatorname{sgn}(x-y)}. \quad (7.5)$$

At spacelike separated events $x$ and $y$ the fields should anticommute (see [14] for the anyonic case); for that to happen the following conditions must hold:

$$a^2 - b^2 = 4(2n + 1)\pi, \quad c^2 - d^2 = 4(2m + 1)\pi,$$

$$ac - bd = 4k\pi, \quad \sigma = \frac{1}{2}[1 + (-1)^k], \quad (7.6)$$

where $n, m$ and $k$ are integers. These condition are algebraic, do not and cannot depend on the temperatures. As regards the currents, at first order in $\epsilon$ we have the following

$$\psi_1^*(x + \epsilon)\psi_1(x) - (\psi_1^*(x + \epsilon)\psi_1(x)) \simeq -iA^2 \left( a e^+ \partial_+ \phi_l(x^+) + b e^- \partial_- \phi_r(x^-) \right) \times$$

$$\times \left( \frac{i \mu q_l}{\beta_l} \sinh \left( \frac{\pi x}{\beta_l} \right) \right) - \frac{e^2}{4\pi} \left( \frac{i \mu r q_r}{\beta_r} \sinh \left( \frac{\pi x}{\beta_r} \right) \right) - \frac{\epsilon^2}{4\pi}. \quad (7.7)$$

An obvious multiplicative renormalization produces a quantum current which explicitly breaks Lorentz covariance:

$$e^{i(4\pi - a^2)W_{l_0}(\epsilon) - b^2W_{r_0}(\epsilon)} [\psi_1^*(x + \epsilon)\psi_1(x) - (\psi_1^*(x + \epsilon)\psi_1(x))] \rightarrow$$

$$\rightarrow -\frac{a}{2\pi} \partial_+ \phi_l(x^+) - \frac{b}{2\pi} \frac{e^+}{e^+} \partial_- \phi_r(x^-). \quad (7.8)$$
This shortcoming may be corrected along the lines indicated in [21, 31] by introducing the currents \( J_+ \) and \( J_- \) as follows:

\[
J_+ = \frac{2\psi_1^*(x + e)\psi_1(x) - 2(\psi_1^*(x + e)\psi_1(x))(1 + ig_+e^+J_+ + ig_-e^-J_-)}{e^{i(a^2 + 4e)W_{\beta\beta}(e) + b^2 W_{\beta\gamma}(e)}}
\]

\[
\simeq - \frac{1}{2\pi} \left( a \partial_+ \phi_1(x^+) + b \frac{e^-}{e^+} \partial_- \phi_1(x^-) \right) - \frac{g_+J_+}{2\pi} - \frac{g_-e^-J_-}{2\pi e^+}.
\]  

(7.9)

\[
J_- = \frac{[\psi_2^*(x + e)\psi_2(x) - \langle \psi_2^*(x + e)\psi_2(x) \rangle (1 + g_+e^+J_+ + g_-e^-J_-)]}{e^{i\beta W_{\beta\beta}(e) + (d^2 - 4\pi)^2 W_{\beta\gamma}(e)}}
\]

\[
\simeq - \frac{1}{2\pi} \left( \frac{e^+}{e^-} \partial_+ \phi_1(x^+) + d \partial_- \phi_1(x^-) \right) - \frac{g_+e^+J_+}{2\pi e^-} - \frac{g_-J_-}{2\pi}.
\]  

(7.10)

At the algebraic level Lorentz covariance is maintained if

\[
g_+ \pi = \frac{c}{a - c}, \quad g_- \pi = \frac{b}{d - b},
\]

(7.11)

which imply

\[
J_+ = -\frac{1}{\pi}(a - c) \partial_+ \phi_1(x^+), \quad J_- = -\frac{1}{\pi}(d - b) \partial_- \phi_1(x^-),
\]

(7.12)

or equivalently

\[
J_\mu = -\frac{1}{4\pi}(a - c + d - b) \partial_\mu \phi - \frac{1}{4\pi}(c - a + d - b) \partial_\mu \tilde{\phi},
\]

(7.13)

\[
\tilde{J}_\mu = \epsilon_{\mu\nu} J_\nu = -\frac{1}{4\pi}(a - c + d - b) \partial_\mu \tilde{\phi} - \frac{1}{4\pi}(c - a + d - b) \partial_\mu \phi.
\]

(7.14)

The condition

\[
d - b = a - c
\]

(7.15)

is imposed to preserve the vector and pseudovector characters of the currents \( J_\mu \) and respectively \( \tilde{J}_\mu \). Of course in the steady-state representation Lorentz symmetry is not implemented; the correlation functions depend on the two temperatures and only translation symmetry is unbroken.

Let us focus now on the field equation:

\[
i\gamma^\mu \partial_\mu \psi = -\begin{pmatrix} 0 & c \partial_+ \phi_1 \\ b \partial_- \phi_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}.
\]

(7.16)

After some manipulations and the necessary ultraviolet regularization it may be rewritten as follows:

\[
i\gamma^\mu \partial_\mu \psi = -\frac{1}{2} (b + c) \gamma^\mu : \partial_\mu \phi \psi : (x) - \frac{1}{2} (b - c) \gamma^\mu : \partial_\mu \tilde{\phi} \psi : (x)
\]

(7.17)

\[
= \pi \frac{b + c}{a - c} \gamma^\mu : J_\mu \psi : (x) + \pi \frac{b - c}{a - c} \gamma^\mu : \tilde{J}_\mu \psi : (x).
\]

(7.18)

The special choice \( b = c \) (which implies \( a = d \)) corresponds to the Thirring model:

\[
i\gamma^\mu \partial_\mu \psi(x) = \frac{2\pi b}{a - b} : \gamma^\mu J_\mu \psi : (x),
\]

(7.19)
The operator solution is then written as follows

\[ \psi_1(x) = A \exp \left( \frac{\pi}{4ib} Q_l + \frac{\pi}{4a} Q_r \right) : \exp (ia \phi_l + ib \phi_r) : (x), \]

\[ \psi_2(x) = B \exp \left( \frac{i\pi}{4a} Q_l + \frac{i\pi}{4b} Q_r \right) : \exp (ib \phi_l + ia \phi_r) : (x), \]

with \(a^2 - b^2 = 4(2n + 1) \pi\). Note that in this case \(\sigma = 1\) and the occurrence of the infrared operators in eqs. (7.20) and (7.21) is unavoidable; they play the role of Coleman’s spurions [24] but here, as in [22], they emerge naturally in the construction. Here the global gauge transformations (6.3) now read

\[ \gamma_\lambda : \psi_1(x) \rightarrow \exp(ia \lambda_l + ib \lambda_r) \psi_1(x), \]

\[ \gamma_\lambda : \psi_2(x) \rightarrow \exp(ib \lambda_l + ia \lambda_r) \psi_2(x). \]

Define

\[ \Omega_{\lambda \lambda} = e^{i\lambda (Q_l + Q_r)} e^{i\bar{\lambda} (Q_l - Q_r)} \Omega. \]

Since

\[ \langle \Omega_{\lambda \lambda} \rangle = e^{im_1(a \phi_l + b \phi_r)} : (x_1) \ldots \cdot e^{im_n(a \phi_l + b \phi_r)} : (x_k) \cdot e^{im_1(b \phi_l + a \phi_r)} : (y_1) \ldots \times \]

\[ \times : e^{im_1(b \phi_l + a \phi_r)} : (y_k) \rangle \Omega_{\lambda \lambda} \rangle = \]

\[ = e^{-\frac{i}{2} (\lambda + \lambda')(a + b)(n_1 + \ldots + n_k + m_1 + \ldots + m_n)} e^{-\frac{i}{2} (\lambda + \lambda')(a - b)(n_1 + \ldots + n_k - m_1 + \ldots - m_n)} \times \]

\[ \times \langle \Omega : e^{im_1(a \phi_l + b \phi_r)} : (x_1) \ldots \cdot e^{im_n(a \phi_l + b \phi_r)} : (x_k) \cdot e^{im_1(b \phi_l + a \phi_r)} : (y_1) \ldots \times \]

\[ \times : e^{im_1(b \phi_l + a \phi_r)} : (y_k) \rangle \]

the relevant fermionic vacuum \( \Omega_T \) is given by (see appendix A)

\[ \Omega_T = \frac{a^2 - b^2}{16 \pi^2} \int_{\frac{2\pi}{a}}^{\frac{2\pi}{a}} d\lambda \int_{\frac{2\pi}{a}}^{\frac{2\pi}{a}} d\bar{\lambda} \Omega_{\lambda \lambda}. \]

It provides the necessary selection rules. In particular the non zero two-point functions are

\[ \langle \Omega_T, \psi_1^\dagger(x) \psi_1(y) \Omega_T \rangle = \left( 2i \beta_l \sinh \left( \frac{\pi (x^+ - y^+)}{\beta_l} \right) \right)^{-\frac{\beta_l^2}{4\pi}} \left( 2i \beta_r \sinh \left( \frac{\pi (x^- - y^-)}{\beta_r} \right) \right)^{-\frac{\beta_r^2}{4\pi}} \]

\[ \langle \Omega_T, \psi_2^\dagger(x) \psi_2(y) \Omega_T \rangle = \left( 2i \beta_l \sinh \left( \frac{\pi (x^+ - y^+)}{\beta_l} \right) \right)^{\frac{\beta_l^2}{4\pi}} \left( 2i \beta_r \sinh \left( \frac{\pi (x^- - y^-)}{\beta_r} \right) \right)^{\frac{\beta_r^2}{4\pi}} \]

where

\[ A^2 = \left( \frac{q \mu_l}{2\pi} \right)^\frac{\beta_l^2}{4\pi} \left( \frac{q \mu_r}{2\pi} \right)^ \frac{\beta_r^2}{4\pi}, \quad B^2 = \left( \frac{q \mu_l}{2\pi} \right)^\frac{\beta_l^2}{4\pi} \left( \frac{q \mu_r}{2\pi} \right)^ \frac{\beta_r^2}{4\pi}. \]

We see that in the interacting fields the left and right components are no more separated; the correlation functions are products of factors which are functions of \(x^+\) and \(x^-\). The two inverse temperatures appears in every correlator. Finally, proceeding as before we get

\[ \langle \Omega, : J_+^2(x) : \Omega \rangle = \frac{(a - b)^2}{4\pi} \frac{1}{3 \beta_l^2} \quad \langle \Omega, : J_-^2(x) : \Omega \rangle = \frac{(a - b)^2}{4\pi} \frac{1}{3 \beta_r^2}. \]
But, since

\[[J_+(x), J_+(y)] = 2i\xi \delta'(x^+ - y^+), \quad [J_+(x), J_-(y)] = 2i\xi \delta'(x^- - y^-), \quad \xi = \frac{(a - b)^2}{2\pi}\]

the expectation values of the energy-momentum tensor remain the same as before:

\[\langle \Omega, : \Theta_+(x) : \Omega \rangle = \frac{1}{2\xi} \langle \Omega, : J^2_+(x) : \Omega \rangle = \frac{\pi}{12\beta_l^2}, \quad (7.31)\]

\[\langle \Omega, : \Theta_-(x) : \Omega \rangle = \frac{1}{2\xi} \langle \Omega, : J^2_-(x) : \Omega \rangle = \frac{\pi}{12\beta_r^2}. \quad (7.32)\]

8 Conclusions

Krein space techniques provide not-so well-known and yet powerful tools to deal with gauge QFT’s; they are in particular very effective to construct operator solutions of models of QFT in spacetime dimension two.

The great advantage of Krein space techniques is that they are directly related to the infrared behaviour of the correlation functions of the quantum fields; some of the ingredients that are introduced ad hoc in other constructions such as the infrared operators, here emerge naturally.

In this paper we have used the above techniques to construct thermal representations for the massless boson and spinor field. We have treated the left and right degrees of freedom independently. As an application we have discussed a full operator solution of the Thirring field in the steady state where there are two independent left and right temperatures. It is seen in the construction the importance of the infrared degrees of freedom that naturally emerge in our context.

The methods and the results discussed in this paper may prove to be useful also in other contexts where the massless two-dimensional field is relevant such as conformal field theory and string theory.

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A Fermionic vacua

An element \( \Phi \) of the field algebra \( \mathcal{F} \) has left charge \( q_l \) and right charge \( q_r \) if

\[\gamma_l^{\lambda_l}(\Phi) = e^{i\lambda_l q_l} \Phi, \quad \gamma_r^{\lambda_r}(\Phi) = e^{i\lambda_r q_r} \Phi. \quad (A.1)\]

Examples are the Wick exponentials

\[\exp(iq_l \phi_l) : (x) \rightarrow \exp(i\lambda_l q_l) : \exp(iq_l \phi_l) : (x) \quad (A.2)\]

\[\exp(iq_r \phi_r) : (x) \rightarrow \exp(i\lambda_r q_r) : \exp(iq_r \phi_r) : (x). \quad (A.3)\]
Fields having left and right charges which are integer multiples of \(q_l\) and \(q_r\) constitute a subalgebra of \(\mathcal{F}\) which we denote \(\mathcal{F}_{q_l q_r}\). The following simple observation is nonetheless crucial: formula (6.9) implies that

\[
\langle \Omega, e^{-i\lambda_l Q_l} : e^{iq_l \phi_l} : (x) \ e^{i\lambda'_r Q_r} \Omega \rangle = \\
= \langle \Omega, e^{-\frac{1}{2}\lambda_l \phi_l^{-1}} (v_l) : e^{iq_l \phi_l} : (x) \ e^{-\frac{1}{2}\lambda'_r \phi_r^{-1}} (v_r) \Omega \rangle = e^{-\frac{1}{2} (\lambda_l + \lambda'_r) q_{lr}}.
\] (A.4)

This promptly generalizes to

\[
\langle \Omega, e^{-i\lambda_l Q_l} : e^{i(qn_1 \phi_1)} : (x_1) \ldots : e^{iqn_k \phi_k} : (x_k) \ e^{i\lambda'_r Q_r} \Omega \rangle = \\
= e^{-\frac{1}{2} (\lambda_l + \lambda'_r) q_{lr}} (\Omega, : e^{i(qn_1 \phi_1)} : (x_1) \ldots : e^{iqn_k \phi_k} : (x_k) \Omega) (A.5)
\]

so that

\[
\int_{-\frac{2\pi}{q_l}}^{\frac{2\pi}{q_l}} \delta_{0,n_1+\ldots+n_k} (\Omega, : e^{i(qn_1 \phi_1)} : (x_1) \ldots : e^{iqn_k \phi_k} : (x_k) \Omega).
\] (A.6)

It follows the correlation function \(\langle \Omega^{q_l q_r}, \Phi \Omega^{q_l q_r} \rangle\) of an operator \(\Phi\) belonging to \(\mathcal{F}_{q_l q_r}\) on the state

\[
\Omega^{q_l q_r} = \frac{q_l q_r}{16\pi^2} \int_{-\frac{2\pi}{q_l}}^{\frac{2\pi}{q_l}} d\lambda_l \int_{-\frac{2\pi}{q_r}}^{\frac{2\pi}{q_r}} d\lambda_r e^{i\lambda_l Q_l e^{i\lambda_r Q_r}} \Omega
\] (A.7)

vanishes unless his charges are equal to zero: \(\gamma_l^\lambda(\Phi) = \gamma_r^\lambda(\Phi) = 0\). More generally we may introduce two angular parameters and define the so called \(\theta\)-vacua states:

\[
\Omega_{q_l q_r}^{\theta_l \theta_r} = \frac{q_l q_r}{16\pi^2} \int_{-\frac{2\pi}{q_l}}^{\frac{2\pi}{q_l}} d\lambda_l \int_{-\frac{2\pi}{q_r}}^{\frac{2\pi}{q_r}} d\lambda_r e^{i(\lambda_l + \theta_l) Q_l e^{i(\lambda_r + \theta_r) Q_r}} \Omega
\] (A.8)

such that the correlation functions of fields belonging to \(\mathcal{F}_{q_l q_r}\) vanish unless they are chargeless.

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**References**

[1] S. Tomonaga, *Remarks on Bloch’s Method of Sound Waves Applied to Many-Fermion Problems*, *Prog. Theor. Phys.* 5 (1950) 544.

[2] F. Bloch, *Bremsvermögen von Atomen mit mehreren Elektronen*, *Z. Phys.* 81 (1933) 363.

[3] F. Bloch, *Inkohärente Röntgenstreueung und Dichteschwankungen eines entarteten Fermigases*, *Helv. Phys. Acta* 7 (1934) 385.
[4] J.M. Luttinger, An Exactly Soluble Model of a Many-Fermion System, J. Math. Phys. 4 (1963) 1154 [SPIRE].
[5] D.C. Mattis and E.H. Lieb, Exact solution of a many fermion system and its associated boson field, J. Math. Phys. 6 (1965) 304 [SPIRE].
[6] F.D.M. Haldane, Luttinger liquid theory of one-dimensional quantum fluids. I. Properties of the Luttinger model and their extension to the general 1D interacting spinless Fermi gas, J. Phys. C 14 (1981) 2585 [SPIRE].
[7] R. Heidenreich, R. Seiler, D.A. Uhlenbrock, The Luttinger model, J. Stat. Phys. 22 (1980) 27.
[8] D. Bernard and B. Doyon, Energy flow in non-equilibrium conformal field theory, J. Phys. A 45 (2012) 362001 [arXiv:1202.0239] [SPIRE].
[9] D. Bernard and B. Doyon, Non-equilibrium steady-states in conformal field theory, Ann. Henri Poincaré 16 (2015) 113 [arXiv:1302.3125].
[10] D. Bernard and B. Doyon, Conformal field theory out of equilibrium: a review, J. Stat. Mech. 1606 (2016) 064005 [arXiv:1603.07765] [SPIRE].
[11] S. Hollands and R. Longo, Non-Equilibrium Thermodynamics and Conformal Field Theory, Commun. Math. Phys. 357 (2018) 43 [arXiv:1605.01581] [SPIRE].
[12] A. Liguori, M. Mintchev and L. Pilo, Bosonization at finite temperature and anyon condensation, Nucl. Phys. B 569 (2000) 577 [hep-th/9906205] [SPIRE].
[13] M. Mintchev and P. Sorba, Luttinger Liquid in Non-equilibrium Steady State, J. Phys. A 46 (2013) 095006 [arXiv:1210.5409] [SPIRE].
[14] M. Mintchev and P. Sorba, Anyon Quantum Transport and Noise away from Equilibrium, Ann. Phys. 532 (2020) 2000276 [arXiv:2006.13323] [SPIRE].
[15] W. Thirring, A soluble relativistic field theory, Annals Phys. 3 (1958) 91.
[16] A.S. Wightman, Introduction to some aspects of the relativistic dynamics of quantized fields, in Cargèse Lectures in Theoretical Physics: Summer School on Theoretical Physics, Cargese, France (1964), M. Levy eds., Gordon and Breach, New York, U.S.A. (1967), pg. 171.
[17] E. Abdalla, M.C.B. Abdalla and K.D. Rothe, Non Perturbative Methods in 2-Dimensional Quantum Field Theory, World Scientific, Singapore (2001).
[18] F. Strocchi and A.S. Wightman, Proof of the Charge Superselection Rule in Local Relativistic Quantum Field Theory, J. Math. Phys. 15 (1974) 2198 [Erratum ibid. 17 (1976) 1930] [SPIRE].
[19] F. Strocchi, Local and Covariant Gauge Quantum Field Theories. Cluster Property, Superselection Rules and the Infrared Problem, Phys. Rev. D 17 (1978) 2010 [SPIRE].
[20] K. Johnson, Solution of the equations for the Green’s functions of a two-dimensional relativistic field theory, Nuovo Cim. 21 (1961) 773.
[21] B. Klaiber, The thirring model, Lect. Theor. Phys. A 10 (1968) 141 [SPIRE].
[22] G. Morchio, D. Pierotti and F. Strocchi, Infrared and Vacuum Structure in Two-dimensional Local Quantum Field Theory Models. 2. Fermion Bosonization, J. Math. Phys. 33 (1992) 777 [SPIRE].
[23] S. Mandelstam, Soliton Operators for the Quantized sine-Gordon Equation, Phys. Rev. D 11 (1975) 3026 [SPIRE].
[24] S.R. Coleman, *The Quantum sine-Gordon Equation as the Massive Thirring Model*, Phys. Rev. D 11 (1975) 2088 [inSPIRE].

[25] H. Yokota, *The Thirring Model at Finite Temperature and Density: Analysis Based on a Derivative Coupling Model*, Prog. Theor. Phys. 77 (1987) 1450 [Erratum ibid. 81 (1989) 725] [inSPIRE].

[26] I. Sachs, A. Wipf and A. Dettki, *Conformal and thermodynamic properties of a family of Thirring like models*, Phys. Lett. B 317 (1993) 545 [hep-th/9308130] [inSPIRE].

[27] A.K. Das and A.J. da Silva, *Exact effective action for (1 + 1)-dimensional fermions in an Abelian background at finite temperature*, Phys. Rev. D 59 (1999) 105011 [hep-th/9808027] [inSPIRE].

[28] R.L.P.G. Amaral, L.V. Belvedere and K.D. Rothe, *Two-dimensional thermofield bosonization*, Annals Phys. 320 (2005) 399 [hep-th/0504012] [inSPIRE].

[29] S.E. Korenblit and V.V. Semenov, *On thermofield bosonization for the Thirring model, tilde conjugation rules, and thermofield vacuum averages*, Russ. Phys. J. 55 (2013) 1011 [inSPIRE].

[30] F. Strocchi, *An introduction to non-perturbative foundations of quantum field theory*, International Series of Monographs on Physics. Vol. 158, Oxford University Press, Oxford, U.K. (2013).

[31] G. Morchio, D. Pierotti and F. Strocchi, *Infrared and Vacuum Structure in Two-dimensional Local Quantum Field Theory Models. The Massless Scalar Field*, J. Math. Phys. 31 (1990) 1467 [inSPIRE].

[32] R.F. Streater and A.S. Wightman, *PCT, spin and statistics, and all that*, Princeton University Press, Princeton, U.S.A. (2000).

[33] I.M. Gel’fand and G.E. Shilov, *Generalized Functions. Volume 1*, AMS Chelsea Publishing, Providence, U.S.A. (1964).

[34] A. Jaffe, *Entire functions of the free field*, Annals Phys. 32 (1965) 127.

[35] D. Pierotti, *The exponential of the two-dimensional massless scalar field as infrared jaffe field*, Lett. Math. Phys. 15 (1988) 219 [inSPIRE].

[36] C.G. Callan, R.F. Dashen and D.H. Sharp, *Solvable Two-Dimensional Field Theory Based on Currents*, Phys. Rev. 165 (1968) 1883 [inSPIRE].

[37] H. Sugawara, *A Field theory of currents*, Phys. Rev. 170 (1968) 1659 [inSPIRE].

[38] G.F. Dell-Antonio, Y. Frishman and D. Zwanziger, *Thirring model in terms of currents: Solution and light cone expansions*, Phys. Rev. D 6 (1972) 988 [inSPIRE].