Existence of Constant Mean Curvature Hypersurfaces in Asymptotically Flat Spacetimes

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Abstract

The problem of existence of spacelike hypersurfaces with constant mean curvature in asymptotically flat spacetimes is considered for a class of asymptotically Schwarzschild spacetimes satisfying an interior condition. Using a barrier construction, a proof is given of the existence of complete hypersurfaces with constant mean curvature which intersect null infinity in a regular cut.

1991 Mathematics Subject Classification. Primary: 53A10 Secondary: 53B30

Key words and phrases. Constant mean curvature, Lorentzian geometry, asymptotic structure

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†Supported by STINT, The Swedish Foundation for International Cooperation in Research and Higher Education
1 Introduction

In this paper we prove existence of complete constant mean curvature (CMC) hypersurfaces asymptotic to a cut of null infinity, \( I^+ \), in a class of asymptotically flat spacetimes.

Entire CMC hypersurfaces in Minkowski space have been classified by Treibergs in [11] where the existence of a CMC hypersurface asymptotic to any \( C^2 \) cut of \( I^+ \) was proved. The method used in [11] is based on the ideas of [5] and is limited to the case when the ambient space is flat.

R. Bartnik and L. Simon [1] have studied the Dirichlet problem for the prescribed mean curvature equation in the case of Minkowski space. The existence and regularity of hypersurfaces with prescribed mean curvature follow once a uniform gradient bound is established. The first apriori gradient estimate in non-flat spacetime was given by C. Gerhardt [6].

R. Bartnik has extended the existence and regularity results for the Dirichlet problem to Lorentzian manifolds in [4] and [3], and in [2] he proved the existence of maximal hypersurfaces asymptotic to spatial infinity in asymptotically flat spacetimes at spatial infinity, satisfying an interior condition which is a uniform bound on the size of domains of dependence.

In this paper we prove existence of complete CMC hypersurfaces asymptotic to a cut of \( I^+ \) in a class of asymptotically Schwarzschild spacetimes (Definition 2.2). The method used in this paper is similar to that used in [2], i.e. we prove existence of a CMC hypersurface by proving convergence of the solutions to a sequence of Dirichlet problems, using barriers at infinity and a condition on the causal structure. In the case studied in this paper, the interior condition used in [2] may be replaced by a future interior condition, see Definition 4.1.

1.1 Overview of this paper

In this paper, in order to keep calculations and notation manageable we deal only with 4-dimensional spacetimes and with mean curvature \( k = 3 \). The results generalize in a straightforward way to general dimension and arbitrary constant mean curvature.

In Section 2 we define asymptotically Schwarzschild spacetimes, (see Definition 2.2). These spacetimes admit a future null infinity \( I^+ \) as an asymptote with conformal factor \( \Omega \) and coordinates \((u, x, y^A)\) in a neighborhood \( \mathcal{N} \) of \( I^+ \) so that on \( I^+ = \{x = 0\} \), \( u \) is the affine parameter of null geodesics, \( \nabla u|_{I^+} = \partial_x \), \( \partial_x \) is null on \( \mathcal{N} \) and \( y^A \) are coordinates on \( S^2 \). It may be shown that any asymptotically flat and empty spacetime admits such a structure locally near \( I^+ \) if the stress energy tensor decays to second order at \( I^+ \) [9, Lemma 4.3].

In Section 3 we construct in terms of \((u, x, y^A)\) a foliation of spacelike hypersurfaces \( \{S_\tau\}_{\tau \in (0, \infty)} \) and prove that this foliation has properties analogous to those of the hyperboloids in Minkowski space. In particular \( \{S_\tau\} \) can be used as upper and lower barriers for CMC hypersurfaces near \( I^+ \) and further, the apriori estimates given by [2, Theorem 3.1] apply.

In Section 4, Theorem 4.2, we prove our main result, namely the existence of complete \( k = 3 \) CMC hypersurfaces in asymptotically Schwarzschild space-
times satisfying the future interior condition (Definition 4.1). This condition is a uniform bound on the size of domains of dependence in the future of a reference hypersurface together with the condition that domains of dependence of compact sets be compact.

Using the maximum principle, the future interior condition and the properties of \{S_\tau\} proved in Section 3, we use an integral estimate to construct a global upper barrier for \(k = 3\) hypersurfaces, see Theorem 4.1.

Using causality and the global upper barrier we prove using an Arzela-Ascoli type argument that the solutions to a sequence of Dirichlet problems has a subsequence which converges to a spacelike hypersurface \(M\) with mean curvature \(k_M = 3\). Using \{S_\tau\} as lower barriers for small \(\tau\) we prove using a gradient estimate that the mean curvature operator is uniformly elliptic on \(M\) which allows us to prove that \(M\) is complete.

Acknowledgements: We are grateful to Robert Bartnik for helpful remarks.

2 Preliminaries

The aim of this section is to present definitions and some simple calculations for later reference.

2.1 Notation

A spacetime \((V, g)\) is a smooth four-dimensional manifold with a smooth Lorentzian metric with signature \((-+,+,+,+)\). We denote the metric pairing by \(\langle \cdot, \cdot \rangle\), the canonical connection by \(\nabla\), and use the summation convention with index ranges \(0 \leq \alpha, \beta \leq 3, \ 1 \leq i, j \leq 3\) where we use lower case latin letters \(i, j, k\ldots\) for spacelike frame indices. Finally we use upper case latin letters \(A, B, C\ldots\) for indices with the range \(2, 3\). We shall use the notation of Hawking and Ellis [8] in describing causal relationships.

We suppose that \(V\) is time-orientable and that \(T^1\) is a \(C^2\) unit timelike vector field on \(V\). Let \(T^*_1\) be the dual of \(T_1\) with respect the metric (in local coordinates \(T^*_\mu = g_{\mu\nu}T^\nu\)) and construct a reference Riemannian metric

\[
ge_E = g + 2T^*_1 \otimes T^*_1. \tag{2.1}
\]

This metric will be used to measure the size of tensors and their covariant derivatives. For any tensor \(B\), we define the norms

\[
\|B\|(x) = (g_E(B, B)(x))^{1/2}, \quad x \in V,
\]

\[
\|B\| = \sup \{\|B\|(x) : x \in V\},
\]

\[
\|B\|_k = \sum_{j=0}^k \|\nabla^j B\|.
\]

In the estimates we will use \(C\) to denote a generic positive constant.

Assume that \((V, g)\) is stably causal, let \(t\) be a \(C^\infty\) time function on \(V\) and let \(S_t\) denote the level sets of \(t\). The foliation \(\{S_t\}\) will be called the reference foliation and the \(S_t\) will be called reference slices.
Using local coordinates \((x^\mu) = (t,x^i)\) where \((x^i)\) are coordinates on the slices \(S_t\), the metric \(g\) takes the form

\[
g_{\mu\nu}dx^\mu dx^\nu = -(\alpha^2 - \beta^2)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j,
\]

where \(\gamma\) is the induced metric on \(S_t\), \(\alpha\) is the lapse function and \(\beta\) is the shift vector.

We write \(\partial_i, \partial_t\) for coordinate tangent vectors and denote partial derivatives by subscripts, so the tangential gradient operator on the slices \(S_t\) is \(D\phi = \gamma^{ij} \phi_j \partial_i\) and the metric pairing \(\gamma(X,Y) = X \cdot Y\), where \(X,Y \in TS_t\). Then the future-directed unit normal \(\hat{T}\) of \(S_t\) can be written as

\[
\hat{T} = -\alpha \nabla t = \alpha^{-1}(\partial_t - \beta).
\]

(2.2)

We will work in the setup used by Bartnik in [4] and [3]. In particular we will often work with the mean curvature of a hypersurface \(M\) expressed in terms of a reference foliation \(S_t\) and a height function \(w\). Note that we use the convention that objects defined w.r.t. \(M\) are distinguished by an upper or lower index \(M\), but objects defined w.r.t. \(S_t\) are not given an upper index 0 as in [3]. In particular, the notation \(k(w)\), where \(w\) is a height function, denotes the mean curvature operator defined w.r.t the reference foliation operating on \(w\).

Choosing an orthonormal frame \(\{z_i\}_{i=1}^3\) on \(S_t\) yields an adapted orthonormal frame \(\{z_\mu\}_{\mu=0}^3\) on \(V\), where \(z_0 = \hat{T}\). The divergence operator, the second fundamental and the mean curvature for the reference slices are given by

\[
\text{div} X = \sum_{i=1}^3 \langle z_i, \nabla_{z_i} X \rangle, \quad X \in \Gamma(TV)
\]

\[
k(z_i, z_j) = \langle z_i, \nabla_{z_j} \hat{T} \rangle,
\]

\[
k = \sum_{i=1}^3 k(z_i, z_i) = \text{div}\hat{T}.
\]

For a spacelike hypersurface \(M \subset V\), we define

\[
\text{div}_M X = \sum_{i=1}^3 \langle e_i, \nabla_{e_i} X \rangle, \quad X \in \Gamma(TV),
\]

\[
\nabla^M \phi = \sum_{i=1}^3 e_i(\phi) e_i, \quad \phi \in C^\infty(M),
\]

where \(\{e_i\}_{i=1}^3\) is any orthonormal frame on \(M\).

Let \(N\) be the future-directed unit normal to \(M\). Then the second fundamental form and mean curvature of \(M\) are

\[
k_M(e_i, e_j) = \langle e_i, \nabla_{e_j} N \rangle,
\]

\[
k_M = k(e_i, e_i) = \text{div}_M N.
\]
The height function $w$ of $M$ is defined as the restriction of the time function to $M$, i.e. $w = t|_M$ and extended to $V$ by the requirement $\partial_t w = 0$. Thus $M$ is a level set of $\Phi = t - w$ and with $\eta^{-2} = -\langle \nabla \Phi, \nabla \Phi \rangle$, $N$ can be written as

$$N = -\eta \nabla \Phi$$

(2.3)

$$= \nu(W + \hat{T}),$$

(2.4)

where $W = (1 + \beta \cdot Dw)^{-1} \alpha Dw$ and $\nu = -\langle \hat{T}, N \rangle = (1 - |W|^2)^{-1/2}$.

In the proof of Theorem 4.2 we will need adapted frames on $TM$ and $TS$. On $TM$ we choose the frame $\{e_1, \{e_A\}_{A=2,3}\}$, where

$$e_1 = |\nabla^M w|^{-1} \nabla^M w, \quad \text{if } |\nabla^M w| \neq 0,$$

(2.5)

and $e_A$ are two common tangent vectors of $TM$ and $TS$ at the intersection point, the frame on $TS$ will be chosen as $\{z_i\}_{i=1}^3 = [z_i, \{e_A\}_{A=2,3}]$, where

$$z_1 = |Dw|^{-1} Dw, \quad \text{if } |Dw| \neq 0.$$

(2.6)

$e_1$ and $z_1$ are related by

$$e_1 = \nu(z_1 + |W|\hat{T}),$$

(2.7)

cf. [2, p. 159].

We end this Subsection by stating a formula which will be used in Section 4. From the definitions, we have

$$\alpha \nabla^M w = \alpha (\nabla t)\| = \nu N - \hat{T}.$$

Applying the divergence operator to the last equation and using

$$\text{div}_M(\hat{T}) = k + \nu^2 \langle W, \nabla_W \hat{T} \rangle + \nu^2 \langle W, \nabla_T \hat{T} \rangle,$$

cf. [2, Equation (2.18)], we obtain

$$k_M \nu = \text{div}_M(\alpha \nabla^M w) +$$

$$+ k + (\nu^2 - 1)(k(\hat{W}, \hat{W}) - \alpha^{-1} \hat{T}(\alpha)) + \langle \nabla^M w, \nabla^M \alpha \rangle,$$

(2.8)

where $\hat{W} = W/|W|$.

### 2.2 Asymptotically Schwarzschild spacetimes

In this Section we will define asymptotically Schwarzschild spacetimes, which are the class of spacetimes covered by the existence Theorem for CMC hypersurfaces proved in Section 4.

Let $(V, g)$ be a spacetime with future null infinity $\mathcal{I}^+ \approx \mathbb{R} \times S^2$ as an asymptote, with conformal factor $\Omega$ and unphysical metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, cf. [7]. If the stress energy tensor decays to second order at $\mathcal{I}^+$, it is possible to prove,
cf. [1, Lemma 4.3], that there are coordinates \( \{ u, x, y^A \} \) locally at \( I^+ \), so that \( \Omega = x + O(x^3) \) and \( \tilde{g}_{\mu\nu} \) takes the form

\[
(\tilde{g}_{\mu\nu}) = \begin{pmatrix}
-x^2 f & O(x^2) & O(x^2) \\
1 & 0 & 0 \\
O(x^2) & 0 & O(x^2) \\
0 & O(x^2) & \tilde{g}_{AB}
\end{pmatrix},
\]

(2.9)

where \( f \) is a smooth function on \( \tilde{V} = V \cup I^+ \) and for fixed \((x, u)\), \( \tilde{g}_{AB} \) is a smooth Riemannian metric on \( S^2 \).

In the remainder of the paper, we will deal with uniform estimates near \( I^+ \). To this end we make the following definition.

**Definition 2.1** Given coordinates \( x^\mu = (u, x, y^A), A = 2, 3, \) in \( \mathcal{N} \), where \( \mathcal{N} = \mathbb{R} \times [0, x_0) \times S^2 \), we say that a function \( f \) on \( \mathcal{N} \) satisfies

\[
f = O(x^\alpha)
\]

for some \( \alpha \in \mathbb{R} \) if for some constant \( C > 0 \),

\[
|f(u, x, y^A)| < Cx^\alpha \quad \text{for all} \quad (u, x, y^A).
\]

We extend this definition to tensors by working with components in a given frame.

Let \( m > 0 \) be given. The Schwarzschild metric with mass \( m \) can be put on the form \( x^{-2}\tilde{g}_{(0)}^{\mu\nu} \) where the conformal Schwarzschild metric \( \tilde{g}_{\mu\nu}^{(0)} \) is given by

\[
\tilde{g}_{\mu\nu}^{(0)}dx^\mu dx^\nu = -x^2 h(x)du^2 + 2dudx + d\theta^2 + \sin \theta^2 d\phi^2.
\]

(2.10)

Here

\[
h(x) = 1 - 2mx
\]

(2.11)

and \( y^A = (\theta, \phi) \). The unphysical Schwarzschild metric \( \tilde{g}_{\mu\nu}^{(0)} \) is an example of a metric of the form (2.9) on a domain of the form given in Definition 2.1.

We are now ready to introduce the precise notion of asymptotic behavior at \( I^+ \) which we will work with in the rest of the paper.

**Definition 2.2** Let \((V, g)\) be a stably causal spacetime with an asymptote \( I^+ \), with \( \tilde{V} = V \cup I^+ \) the unphysical spacetime, \( \Omega \in C^\infty(\tilde{V}) \) the conformal factor and \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \) the unphysical metric. Let \( m > 0 \) be given and let \( x_0 < 1/2m \). Then \((V, g)\) is said to be asymptotically Schwarzschild if the following holds.

1. There is a coordinate system \((u, x, y^A)\) on a domain \( \mathcal{N} \) as in Definition 2.1, and an embedding \( F : \mathcal{N} \to \tilde{V} \) such that in the coordinates \((u, x, y^A)\) on \( F(\mathcal{N}) \), \( \tilde{g}_{\mu\nu} \) takes the form (2.3) with \( \tilde{g}_{0A} = O(x^2) \) in the sense of Definition 2.1, and \( \Lambda^{-1} g_{AB}^{(0)} \xi^A \xi^B \leq \tilde{g}_{AB} \xi^A \xi^B \leq \Lambda \tilde{g}_{AB}^{(0)} \xi^A \xi^B \) for some constant \( \Lambda > 0 \) and all \( \xi^A \), where \( g_{AB}^{(0)} \) is the standard metric on \( S^2 \).
2. $F(N \cap \{ x = 0 \}) = I^+$ and on $F(N)$, $\partial_x$ is future oriented.

3. On $F(N)$ the conditions

$$\Omega^{-6}|\tilde{g}_{uu}| + \Omega^{-5} \sum_{\mu=0}^{3} |\partial_\mu \tilde{g}_{uu}| + \Omega^{-4} \sum_{\mu,\nu=0}^{3} |\partial_\mu \partial_\nu \tilde{g}_{uu}| < C, \tag{2.12}$$

$$|\tilde{g}_{uA}|\Omega^{-3} + \Omega^{-2} \sum_{\mu=0}^{3} |\partial_\mu \tilde{g}_{uA}| + \Omega^{-1} \sum_{\mu,\nu=0}^{3} |\partial_\mu \partial_\nu \tilde{g}_{uA}| < C, \tag{2.13}$$

$$\Omega^{-2}|\delta \tilde{g}_{AB}| + \Omega^{-1} \sum_{\mu=0}^{3} |\partial_\mu \delta \tilde{g}_{AB}| + \sum_{\mu,\nu=0}^{3} |\partial_\mu \partial_\nu \delta \tilde{g}_{AB}| < C, \tag{2.14}$$

where $\delta \tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu} - \tilde{g}_{\mu\nu}^{(0)}$ with $\tilde{g}^{(0)}$ given by (2.10) and

$$|\Omega^{-2}(\partial_x \Omega - 1)| + \sum_{i=1}^{3} |\Omega^{-3} \partial_i \Omega| < C$$

hold.

4. The components of the Ricci tensor of the physical metric $g$, w.r.t. the coordinate frame are assumed to be bounded in $F(N \setminus \{ x = 0 \})$.

We will use the notation $\tilde{V}_E = F(N \cap \{ u > 0 \})$ and $V_E = \tilde{V}_E \cap V$ and similarly $V_I = V \setminus \text{int}(V_E)$.

If we demand that the metric $g$ satisfies Einstein’s equation and that the stress energy tensor vanishes asymptotically to order 2 in $\Omega$, Point 3 is immediately satisfied. Note that $I^+$ is given by $\{ x = 0 \}$ and that $I^+$ is in the future of $V$, i.e. it is future null infinity. Further, $\tilde{V}_E$ is the subset of $F(N)$ which lies to the future of the backward lightcone of the cut of $I^+$ given by $\{ x = 0, u = 0 \}$.

Some calculations will be done in the unphysical spacetime $(\tilde{V}, \tilde{g})$, in which case all the corresponding geometrical quantities will be denoted with tilde and all the physical ones, i.e. calculated in $(V, g)$ will be denoted without tilde. We remark that our convention for the usage of the tilde here coincides with that in [8] and is opposite of that in eg. [7].

2.3 The reference foliation

The computations done in this Subsection will be used in Section 3 to construct a reference foliation which gives barriers near $I^+$ in asymptotically Schwarzschild spacetimes.

Let $H$ be a $C^1$ function on $(0, x_0) \times (0, \infty)$ which extends as a continuous function to $[0, x_0) \times (0, \infty)$. Further let $(V, g, N)$ and $(u, x, y^A)$ be as in Definition 2.2. Consider the one parameter family of embeddings

$$\Phi_\tau : (0, x_0) \times S^2 \to N, \quad \tau \in (0, \infty),$$

defined by

$$\Phi_\tau : (x, y^2, y^3) \to (H(x, \tau), x, y^2, y^3)$$
and let \( S_\tau = \Phi_\tau((0,x_0) \times S^2) \). Then \( S_\tau \) is the graph of a height function \( H(\cdot, \tau) \).

The pushforward of the canonical frame to \( S_\tau \) is

\[
\begin{align*}
v_1 &= \Phi_\tau(\partial_x) = \partial_x H \partial_u + \partial_x, \\
v_A &= \Phi_\tau(\partial_A) = \partial_A.
\end{align*}
\]  

(2.15)

In terms of this frame, the induced metric \( \gamma \) on \( S_\tau \) has components

\[
\begin{align*}
\gamma_{11} &= (\partial_x H)^2 g_{uu} + 2 \partial_x H g_{ux} + g_{xx} \\
\gamma_{1A} &= \partial_x H g_{uA} + g_{xA} \\
\gamma_{AB} &= g_{AB}.
\end{align*}
\]  

(2.16)

Assuming that \( H \) is such that the foliation \( \{S_\tau\} \) consists of spacelike hypersurfaces, the future directed unit normal of \( S_\tau \) is

\[
\hat{T} = -\eta \left( g^{uu} - 2 g^{ux} \partial_x H + g^{xx}(\partial_x H)^2 \right).
\]

(2.17)

Then the lapse and shift is given by

\[
\begin{align*}
\alpha &= \eta \partial_\tau H \\
\beta_1 &= \partial_\tau H (g_{uu} \partial_x H + g_{ux}) \\
\beta_A &= \partial_\tau H g_{uA},
\end{align*}
\]  

(2.19)

(2.20)

where we have used \( \gamma_{ij} \) to lower the index on \( \beta^i \).

Let \( k_{ij} = \langle v_i, \nabla_{v_j} \hat{T} \rangle \) be the components of the second fundamental form of \( S_\tau \) in the frame \( \{v_i\} \). Then

\[
\begin{align*}
k_{11} &= \eta (\partial_x H)^2 \Gamma^\mu_{uu} T_\mu + 2 \partial_x H \Gamma^\mu_{ux} T_\mu + \\
&\quad + \Gamma^\mu_{xx} T_\mu + \partial^2_x H), \\
k_{1A} &= \eta (\partial_x H \Gamma^\mu_{uA} + \Gamma^\mu_{xA}) T_\mu, \\
k_{AB} &= \eta \Gamma^\mu_{AB} T_\mu.
\end{align*}
\]  

(2.21)

Given the foliation \( \{S_\tau\} \), we define a foliation \( \{\tilde{S}_\tau\} \) of \( N \) by \( \tilde{S}_\tau = S_\tau \cup \{u = H(0, \tau)\} \). The second fundamental form of \( \tilde{S}_\tau \) w.r.t. the unphysical metric \( \tilde{g} \) is

\[
\tilde{k}_{ij} = \Omega (k_{ij} + \lambda \gamma_{ij}),
\]

(2.22)

where \( \lambda = \tilde{T}(\Omega) = \tilde{T}^\alpha \frac{\partial \Omega}{\partial x^\alpha} \). Here \( \tilde{T} = \Omega^{-1} \hat{T} \) is the unit normal of \( \tilde{S}_\tau \) w.r.t. \( \tilde{g} \).

For a more explicit derivation of these formulae see [9, Section 3].

### 3 Barriers near \( I^+ \)

The goal of this section is to construct a foliation \( \{S_\tau\} \) near \( I^+ \) which can be used to get local barriers for the mean curvature operator near \( I^+ \) in asymptotically Schwarzschild spacetimes. This will allow us in Section 4 to construct a global upper barrier which is of central importance in the proof of the existence
result. Using \{S_\tau\} as local lower barriers near \mathcal{I}^+ is essential in proving the completeness of CMC the hypersurface constructed in the existence part of the argument.

In order to be able to use the \{S_\tau\} as barriers we must check that for sufficiently large \tau, the mean curvature \(k\) of \(S_\tau\) satisfies \(k < 3\) and that for sufficiently small \(\tau > 0\), \(k > 3\). Further to be able to apply the gradient estimates [2, Theorem 3.1] we must check that the foliation \{S_\tau\} satisfies the following properties in a subset of \(V_E\).

1. Choose \(T_1 = \hat{T}\), where \(\hat{T}\) is the normal of the foliation, then
   \[
   \max(\|\alpha\|, \|\alpha^{-1}\nabla\alpha\|, \|k(\cdot, \cdot)\|, \|\hat{T}\|_2) < C, \tag{3.23}
   \]
   where the norm is taken with the Euclidean metric \(g_E\) given by (2.1).

2. The Ricci tensor of the spacetime is bounded, i.e.
   \[
   \|\text{Ric}\| < C. \tag{3.24}
   \]

3.1 A foliation of the Schwarzschild spacetime

Consider the Schwarzschild metric in the \((t, r, \theta, \phi)\) coordinate system. Let \(A = 1 - \frac{2m}{r}\). The height function

\[
w(\tau, r) = \sqrt{\tau^2 + (r - 2m)^2 + 2m \log (r - 2m + \sqrt{\tau^2 + (r - 2m)^2}) - 2m(\log 2 - 1)},
\]

is the solution to the radial mean curvature equation in the Schwarzschild spacetime,

\[
k(w) = \left( \frac{w' A r^2}{\sqrt{1 - (A w')^2}} \right)' r^{-2} \sqrt{A}
   = \frac{3}{\tau} \sqrt{A} \left(1 - \frac{4m}{3r}\right), \tag{3.25}
\]

satisfying

\[
\lim_{\tau \to 0} w(\tau, r) = r + 2m \log (r - 2m),
\]

i.e. the level set \(t = w(\tau, r)\) tends to the light cone as \(\tau \to 0\), cf [6, Section 3].

Let \(\{u, x, y^A\}_{A=2,3}\) be a coordinate system for the Schwarzschild spacetime, where \(u\) is the retarded coordinate and \(x = 1/r\) is the conformal factor. In this coordinate system the foliation given by \(t = w(\tau, r)\) is given by \(u = H(\tau, x)\) where

\[
H(x, \tau) = w(\tau, 1/x) - \frac{1}{x} - 2m \log \left(\frac{1}{x} - 2m\right)
   = \frac{1}{x} \left(\sqrt{\tau^2 x^2 + h^2(x)} - h(x)\right)
   + 2m(\log(h(x) + \sqrt{\tau^2 x^2 + h^2(x)}) - \log h(x) - \log 2), \tag{3.26}
\]
where \( h(x) \) is given by (2.11). The partial derivatives of the height function \( H \) are

\[
\partial_x H = -\frac{1}{x^2} \left( \frac{1}{\sqrt{\tau^2 x^2 + h^2(x)}} - \frac{1}{h(x)} \right)
\]  
(3.27)

and

\[
\partial_\tau H = \frac{\tau x}{\sqrt{\tau^2 x^2 + h^2(x)}} \left( 1 + \frac{2mx}{h(x) + \sqrt{\tau^2 x^2 + h^2(x)}} \right).
\]  
(3.28)

In the two following lemmata we summarize the properties of the foliation given by the height function (3.26) in the Schwarzschild spacetime. The calculations are straightforward and they are done in detail in [9, Section 3]. The function given by (3.26) will be used in the construction of the barriers in asymptotically Schwarzschild spacetimes.

**Lemma 3.1** Let \((V, g)\) be the Schwarzschild spacetime and let \(\tilde{V}_E\) be as in Definition 2.2. There is a foliation \(\{\tilde{S}_\tau\}_{\tau \in (0, \infty)}\) of \(\tilde{V}_E\) and a foliation \(\{S_\tau\}_{\tau \in (0, \infty)}\) of \(V_E\), where \(S_\tau = \tilde{S}_\tau \cap V\), such that the following is true.

1. The induced metric \(\gamma\) is Riemannian and conformally related to a metric which is \(C^\infty\) on \(\tilde{S}_\tau\).

2. For each \(\tau\), \(\tilde{S}_\tau \cap \mathcal{I}^+ \approx S^2\).

3. The lapse function satisfies \(\alpha|_{\mathcal{I}^+} = 1\).

4. The shift vector satisfies \(\beta|_{\mathcal{I}^+} = 0\).

5. \(|\hat{T}(\alpha)| \leq \frac{C}{\tau}\) and \(|\hat{T}(v_1(\alpha))| < \frac{C}{\tau}\).

**Proof:** Using (2.16) we get

\[
\Phi^*_t (g^{(0)}) = \frac{\tau^2}{h(x)(\tau^2 x^2 + h^2(x))} dx^2 + d\theta^2 + \sin^2 \theta d\phi^2.
\]  
(3.29)

Further, using (2.19) and (2.20) we have

\[
\alpha = \sqrt{h(x)} \left( 1 + \frac{2mx}{h(x) + \sqrt{\tau^2 x^2 + h^2(x)}} \right),
\]  
(3.30)

\[
\beta = \frac{h^2(x)x}{\tau} \left( 1 + \frac{2mx}{h(x) + \sqrt{\tau^2 x^2 + h^2(x)}} \right) v_1
\]
\[
= \frac{(h(x))^{\frac{3}{2}} \alpha x}{\tau} v_1.
\]  
(3.31)

Using the lapse and the shift we write

\[
\hat{T}(\alpha) = \alpha^{-1}(\partial_\tau \alpha - \beta(\alpha))
\]

\[
\hat{T}(v_1(\alpha)) = \alpha^{-1}(\partial_\tau \partial_x \alpha - \beta(\partial_x \alpha))
\]

and a straightforward calculation shows that

\[
|\partial_\tau \alpha| < \frac{C}{\tau}, \quad |\partial_x \alpha| < C, \quad |\partial_\tau \partial_x \alpha| < \frac{C}{\tau} \quad \text{and} \quad |\partial_x^2 \alpha| < C,
\]  
(3.32)
then
\[ |\hat{T}(\alpha)| < \frac{C}{\tau} \quad \text{and} \quad |\hat{T}(v_1(\alpha))| < \frac{C}{\tau}. \]

Beside the properties given by this Lemma the foliation defined by (3.26) has a second fundamental form with similar behavior to that of the hyperboloids in Minkowski space.

**Lemma 3.2 (9, Lemma 3.2)** Let \((V, g)\) and \(\tilde{S}_\tau,\{S_\tau\}\) be as in Lemma 3.1. The physical and unphysical second fundamental forms \(k_{ij}\) and \(\tilde{k}_{ij}\) of \(S_\tau\) and \(\tilde{S}_\tau\), respectively in the canonical frame given by (2.15), are of the form
\[ k_{ij} = \frac{1}{\tau} (\gamma_{ij} + r_{ij}) \]
where \(|r_{ij}| = O(x)\) and
\[ \tilde{k}_{11} = \frac{2m\sqrt{h(x)}}{\tau} \tilde{\gamma}_{11}, \quad \tilde{k}_{ij} = 0 \text{ for } (i, j) \neq (1, 1). \]

### 3.2 A foliation of asymptotically Schwarzschild spacetimes

Next we will see that the foliation introduced in Subsection 3.1 gives local barriers for the mean curvature equation in asymptotically Schwarzschild spacetimes. Let \(0 < \tau_0^- < \tau_0^+ < \infty\) be given.

**Lemma 3.3** The function given by (3.26) has the following properties:

1. For each fixed \(x\)
\[ \lim_{\tau \to \infty} \partial_x^2 H = \frac{1}{x^2 h(x)}. \]

2. For all \(\tau > \tau_0^+\)
\[ \partial_x H = O(x^{-2}), \quad \partial_x^2 H = x^{-1} \partial_x H + O(x^{-2}). \]

3. For all \(\tau < \tau_0^-\)
\[ \partial_x H = \frac{1}{2} \tau^2 (1 + O(x)). \]

Let \(\epsilon_+ > 0\) be given. We denote by \(V_E^+\) and \(V_E^-\) be given by
\[ V_E^+ = V_E \cap \{ \cup_{\tau \geq \tau_0^+} S_\tau \}, \quad V_E^- = V_E \cap \{ \cup_{\epsilon_+ \tau \leq \tau \leq \tau_0^-} S_\tau \}, \quad (3.33) \]

We claim that in the region given by \(V_E^+ \cup V_E^-\), (see Figure 3), the foliation \(\{S_\tau\}\) behaves similarly in an asymptotically Schwarzschild spacetime as in the standard Schwarzschild spacetime.

**Lemma 3.4** Let \((V, g)\) be an asymptotically Schwarzschild spacetime and let \(V_E^+\) be given by (3.33). There is a foliation \(\{\tilde{S}_\tau\}_{\tau \in (\epsilon_+, \infty)}\) of \(V_E^+\) and a foliation \(\{S_\tau\}_{\tau \in (\epsilon_-, \infty)}\) of \(V_E^-\), where \(S_\tau = \tilde{S}_\tau \cap V\), such that the following is true.
1. The induced metric $\gamma$ is Riemannian and of the form $\gamma_{ij} = x^{-2}(\tilde{\gamma}_{ij}^{(0)} + r_{ij})$, where $\tilde{\gamma}_{ij}^{(0)}$ is the metric given by (3.29) and $|r_{ij}| \leq Cx^2$ and $|\Omega| \leq \Omega^0$.

2. $\tilde{S}_\tau \cap \mathcal{I}^+ \approx S^2$.

3. The lapse function satisfies $\alpha|_{\mathcal{I}^+} = 1$.

4. The shift vector satisfies $\beta|_{\mathcal{I}^+} = 0$ and $|v_j(\beta^i)| < C$.

5. $|\hat{T}(\alpha)| \leq \frac{C_0}{x} + Cx^3$ and $\max(\|\alpha^{-1}\nabla\alpha\|, |\hat{T}(v_i(\alpha))|) < C$.

The constants depend only on the choice of $\tau_0^+$ and $x_0$, and the frame used on $\tilde{S}_\tau$ and $S_\tau$ is the canonical one given by (2.17).

**Proof:** As in Lemma 3.1 we compute

$$\bar{\gamma}_{11} = -x^2 h(x)(\partial_x H)^2 + 2(\partial_x H)^2 \delta g_{uu}$$

$$= \frac{\tau^2}{(\tau^2 x^2 + h^2(x))h(x)} + \delta g_{uu}(\partial_x H)^2, \quad (3.34)$$

from (3.27) we calculate

$$(\partial_x H)^2 = \frac{\tau^2 \epsilon_0(x, \tau)}{x^4 h(x)(\tau^2 x^2 + h^2(x))}, \quad (3.35)$$

where

$$\epsilon_0(x, \tau) = \frac{1}{h(x)} \left( x^2 + 2\frac{h(x)}{\tau} - 2\frac{h(x)}{\tau} \sqrt{x^2 + \left(\frac{h(x)}{\tau}\right)^2} \right), \quad (3.36)$$

Figure 1: The unphysical and the physical picture ($r = 1/x$)
For each $\tau$, $\epsilon_0|_{x^+} = 0$. Now (3.34) can be written in the form

$$\tilde{\gamma}_{11} = \frac{\tau^2(1 + \epsilon_1)}{(\tau^2x^2 + h^2(x))h(x)},$$

(3.37)

where

$$\epsilon_1(x^\mu, \tau) = \epsilon_0(x, \tau)\delta\tilde{g}_{uu}x^{-4}.$$ (3.38)

It follows from (2.12) that $\epsilon_1 = O(x^2)$.

A calculation shows that the other components of the metric satisfy

$$\tilde{\gamma}_{1A} = O(x),$$

$$\tilde{\gamma}_{AB} = \tilde{g}_{AB},$$

(3.39)

and Point 1 follows.

It follows from Definition 2.2 that the inverse of metric takes the form

$$(\tilde{g}^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & x^2h(x) + O(x^6) & O(x^3) & O(x^3) \\ 0 & O(x^3) & O(x^3) & \tilde{g}^{AB} \\ 0 & O(x^3) & O(x^3) & \end{pmatrix}.$$ (3.40)

Let $\delta\tilde{g}^{\mu\nu} = \tilde{g}^{\mu\nu} - \tilde{g}^{(0)\mu\nu}$. Using equation (2.18) we have

$$\tilde{\eta}^{-2} = \frac{\tau^2}{(\tau^2x^2 + h^2(x))h(x)} + \delta\tilde{g}^{xx}(\partial_xH)^2.$$

Letting

$$\epsilon_2(x^\mu, \tau) = \epsilon_0(x, \tau)\delta\tilde{g}^{xx}x^{-4},$$

we get

$$\tilde{\eta}^{-2} = \frac{\tau^2(1 + \epsilon_2)}{(\tau^2x^2 + h^2(x))h(x)}.$$ (3.41)

By Equation (3.40), $\epsilon_2 = O(x^2)$. In terms of the physical metric, $\eta$ becomes

$$\eta = \frac{\sqrt{(\tau^2x^2 + h^2(x))h(x)}}{\tau x\sqrt{1 + \epsilon_2}}(1 + O(x^2))$$

$$= \frac{\eta^{(0)}}{\sqrt{1 + \epsilon_2}}(1 + O(x^2)),$$

where $\eta^{(0)}$ is the quantity corresponding to $\eta$ in the Schwarzschild metric.

Using (3.28) and (2.13) we get

$$\alpha = \frac{\alpha^{(0)}}{\sqrt{1 + \epsilon_2}}(1 + O(x^2)),$$

where $\alpha^{(0)}$ is given by (3.31). This proves Point 3. Further, using (2.20) we get

$$\beta_1 = \beta_1^{(0)} + O(x^2)$$

$$\beta_A = \partial_\tau H g_{uA} = O(x),$$

where $\beta_1$ and $\beta_A$ are the other components of the metric.
where $\beta_i^{(0)}$ denotes the components of the dual w.r.t. the metric $x^{-2}\gamma^{(0)}$ of the vector given by (3.31). Thus raising the index with the metric $\gamma$ we get $\lim_{x \to 0} \beta = 0$. Now we will estimate $v_1(\beta^i)$. Observing that $\beta_i = \tilde{\beta}_i$, we find

$$v_1(\tilde{\beta}^i) = \gamma^{ij} v_l(\tilde{\beta}_j) + v_l(\tilde{\gamma}^{ij}) \tilde{\beta}_j.$$  

The most delicate calculation is $v_1(\tilde{\beta}_1)$ and $v_1(\tilde{\gamma}_{11})$, due to the fact that the height function $H$ is independent of the coordinates $(y^4)$.

By (3.37) we can estimate these derivatives if we can estimate $v_1(\epsilon_1)$. Note that $v_1 = \partial_x H \partial_u + \partial_x$ and thus, if $f = f(\tau, x)$ then $v_1(f) = \partial_x f$ because $v_1(\tau) = 0$. Therefore

$$v_1(\epsilon_1) = \partial_x \epsilon_0(x, \tau) \delta g_{uu} x^{-4} + \epsilon_0(x, \tau) v_1(\delta g_{uu})x^{-4} - 4\epsilon_0(x, \tau) \delta g_{uu} x^{-5}. \quad (3.41)$$

A straightforward calculation shows that

$$\left| \partial_x \epsilon_0 \right| < C. \quad (3.42)$$

Using this and the decay of the metric we obtain $|v_j(\beta^i)| < C$. This proves Point 4.

It remains to prove Point 5. Observe that the normal of the foliation can be written using different frames as follows

$$\hat{T} = -\Omega \tilde{h} \left( - \partial_x H \partial_u + \left( \frac{h(x)}{\sqrt{h^2(x) + x^2 \tau^2}} + \mathcal{O}(x^4) \right) \partial_x + \mathcal{O}(x) \partial_A \right) \quad (3.43)$$

$$= \alpha^{-1} (\partial_x - \beta^i v_i). \quad (3.44)$$

If $\hat{T}$ acts on a function $f = f(\tau, x)$ we use (3.44) while if $\hat{T}$ acts on a function $f = f(u, x)$ we use (3.43). Note that

$$\alpha = \frac{\alpha(0)x^{-1}\Omega}{\sqrt{1 + \epsilon_2}}. \quad (3.45)$$

Therefore

$$\hat{T}(\alpha) = \hat{T}(\alpha(0)) \frac{x^{-1}\Omega}{\sqrt{1 + \epsilon_2}} - \frac{\alpha}{2(1 + \epsilon_2)} \hat{T}(\epsilon_2) + \frac{\alpha(0)}{\sqrt{1 + \epsilon_2}} \hat{T}(\Omega x^{-1})$$

and

$$\hat{T}(\epsilon_2) = \hat{T}(\epsilon_0) \delta g^{mx} x^{-4} + \epsilon_0(x, \tau) \hat{T}(\delta g^{mx})x^{-4} - 4\epsilon_0(x, \tau) \delta g^{mx} \hat{T}(x^{-4}). \quad (3.46)$$

A straightforward calculation gives

$$\left| \partial_\tau \epsilon_0 \right| < \frac{C x^3}{\tau^3}. \quad (3.47)$$

Using (3.46) and (3.42) we estimate

$$\hat{T}(\epsilon_0) = \alpha^{-1} (\partial_\tau \epsilon_0 - \beta(\epsilon_0)) \left\langle \frac{\mathcal{O}(x)}{\tau} \right\rangle.$$
On the other hand
\[ \Omega \hat{T}(x^{-1}) = -(x^{-1} \Omega)^2 \hat{T}(x) \]
\[ = (x^{-1} \Omega)^2 \tilde{\eta} \left( \frac{h(x)}{\sqrt{h^2(x) + x^2 \tau^2}} + \mathcal{O}(x^4) \right) \]
and using (3.44) and the decay of the derivatives of the conformal factor, we get the estimate
\[ x^{-1} \hat{T}(\Omega) = -x^{-1} \Omega \tilde{\eta} \left( \frac{h(x)}{\sqrt{h^2(x) + x^2 \tau^2}} + \mathcal{O}(x^4) \right). \]
(3.47)

Thus
\[ |\hat{T}(x^{-1} \Omega)| < C \frac{x}{\tau} + Cx^3 \]
and hence
\[ |\hat{T}(\alpha)| < C \frac{x}{\tau} + Cx^3. \]
(3.48)

Let \( \{\hat{T}, \hat{v}_i\} \) be the frame given by the normal of the foliation and \( \hat{v}_i = \frac{v_i}{\sqrt{\langle v_i, v_i \rangle}} \). Then
\[ \|\alpha^{-1} \nabla \alpha\|^2 = \alpha^{-2} g_E(\nabla \alpha, \nabla \alpha) \]
\[ = \alpha^{-2} \left( \langle \hat{T}(\alpha) \rangle^2 + \sum_{i,j=1}^{3} \frac{v_i(\alpha)v_j(\alpha)}{\sqrt{\langle v_i, v_i \rangle \langle v_j, v_j \rangle}} \langle v_i, v_j \rangle \right). \]

Therefore we need to estimate \( v_i(\alpha) \). The delicate step is to get a bound for
\[ v_1(\epsilon_2) = \partial_x \epsilon_0(x, \tau) \delta \ddot{g}^{xx} x^{-4} + \epsilon_0(x, \tau) v_1(\delta \ddot{g}^{xx}) x^{-4} - 4 \epsilon_0(x, \tau) \delta \ddot{g}^{xx} x^{-5}. \]
Using (3.42), (3.48) and the decay of the metric gives the desired estimate.

Finally, a straightforward calculation shows that
\[ |\partial_\tau \partial_x \epsilon_0| < C \quad \text{and} \quad |\partial_x^2 \epsilon_0| < C. \]
Using this, the decay of the metric and Lemma 3.1, we obtain that \( |\hat{T}(v_i(\alpha))| \) is bounded. \( \square \)

In order to compute the second fundamental form of \( S_\tau \) we will first do the calculations in the unphysical spacetime and then using the conformal transformation given by (2.22), we obtain \( k_{ij} \).

**Lemma 3.5** Let \( (V, g) \) and \( \{\tilde{S}_\tau\}, \{S_\tau\} \) be as in Lemma 3.4. The unphysical and physical second fundamental forms \( \tilde{k}_{ij} \) and \( k_{ij} \) of \( \tilde{S}_\tau \) and \( S_\tau \), respectively, satisfy the estimates
\[ |\tilde{k}_{11}| \leq C \tilde{\gamma}_{11} \left( \frac{1}{\tau} + x^2 \right), \quad |\tilde{k}_{1A}| \leq C \left( \frac{1}{\tau} + x \right), \quad |\tilde{k}_{AB}| \leq C \left( \frac{1}{\tau} + x \right), \]
(3.49)

where \( \tilde{k}_{ij} \) is computed in the coordinate frame on \( T\tilde{S}_\tau \) and \( C \) is some constant depending on \( \tau_0^+ \) and \( x_0 \), and
\[ |k(\hat{v}_i, \hat{v}_j)| \leq \frac{1 + \epsilon}{\tau} + Cx^2, \]
(3.50)

where \( \hat{v}_i = \frac{v_i}{\sqrt{\langle v_i, v_i \rangle}} \), and \( \epsilon \) and \( C \) are constants depending on \( \tau_0^+ \) and \( x_0 \).
**Proof:** Using the corresponding formula to (2.21) for the unphysical metric, we calculate the components of $\tilde{k}_{ij}$. The most complicated calculation will be in the $\{v_1, v_1\}$ direction.

By (A.74) we get that $\tilde{\Gamma}_{xq}^{u}$, $\tilde{\Gamma}_{xx}$ and $\tilde{\Gamma}_{ux}$ are zero, hence we obtain

$$\tilde{k}_{11} = \tilde{k}_{11}^{(0)} + \tilde{\eta}(\partial_x H)^2(\delta\tilde{\Gamma}_{uu}^{u} - \partial_x H\delta\tilde{\Gamma}_{uu}^{x} - 2\delta\tilde{\Gamma}_{ux}^{x}), \quad (3.51)$$

where $\tilde{k}_{ij}^{(0)}$ is the second fundamental form for $\tilde{S}_{\tau}$ in the unphysical Schwarzschild spacetime given in Lemma 3.2 and $\delta\tilde{\Gamma} = \tilde{\Gamma} - \tilde{\Gamma}^{(0)}$. Using (3.35) and Lemma 3.4, we write

$$\partial_x H)^2 = \frac{\tau^2\epsilon_0(x, \tau)}{x^4h(x)(x^2 + h^2(x))} \quad \text{and} \quad \tilde{\eta} = \frac{\sqrt{(\tau^2x^2 + h^2(x))h(x)}}{\tau\sqrt{1 + \epsilon_2}},$$

hence in order to estimate the second term in (3.51) we use the triangle inequality to get

$$\epsilon_0\sqrt{x^2 + \frac{(h(x))}{\tau} \leq \frac{x}{\tau} \left(\frac{2}{\tau} + O(x)\right) + x^3(1 + O(x)).$$

Now we obtain

$$(\partial_x H)^2\tilde{\eta} < \frac{\tilde{\gamma}_{11}}{x^3(1 + \epsilon_1)\sqrt{1 + \epsilon_2}} \left(\frac{2}{\tau} + O(x)\right) + \frac{\tilde{\gamma}_{11}}{x(1 + \epsilon_1)\sqrt{1 + \epsilon_2}}, \quad (3.52)$$

and using (A.74) we estimate

$$|\delta\tilde{\Gamma}_{uu}^{u} - \partial_x H\delta\tilde{\Gamma}_{uu}^{x} - 2\delta\tilde{\Gamma}_{ux}^{x}| < Cx^3. \quad (3.53)$$

Inserting (3.52) and (3.52) in the second term of (3.51), we have

$$|\tilde{\eta}(\partial_x H)^2(\delta\tilde{\Gamma}_{uu}^{u} - \partial_x H\delta\tilde{\Gamma}_{uu}^{x} - 2\delta\tilde{\Gamma}_{ux}^{x})| \leq \frac{\tilde{\gamma}_{11}}{\tau} \left(\frac{2C}{\tau} + O(x)\right) + Cx^2\tilde{\gamma}_{11}. \quad (3.54)$$

On the other hand, $\tilde{k}_{11}^{(0)}$ can be written in terms of $\tilde{\gamma}_{11}$ as

$$\tilde{k}_{11}^{(0)} = \frac{\tilde{\gamma}_{11}}{\tau} \frac{2m\sqrt{h(x)}}{1 + \epsilon_1},$$

inserting this and (3.54) in (3.51) we get

$$|\tilde{k}_{11}| \leq C\tilde{\gamma}_{11}(\frac{1}{\tau} + x^2).$$

In the other directions we easily estimate

$$\tilde{k}_{1A} = \tilde{\eta}(\partial_x H\tilde{\Gamma}_{uA}^{u} + \tilde{\Gamma}_{xA}^{u})T_{\mu} = \tilde{\eta}\partial_x H(\partial_x H\tilde{\Gamma}_{uA}^{x} + \tilde{\Gamma}_{xA}^{x} + \tilde{\Gamma}_{uA}^{x}) = \tilde{\eta}O(1)$$
$$\tilde{k}_{AB} = \tilde{\eta}\tilde{\Gamma}_{A}^{u}\tilde{\Gamma}_{uB}^{u}T_{\mu} = \tilde{\eta}O(1),$$
which proves (3.49).

The relation between the physical and unphysical second fundamental forms after a conformal transformation is given by equation (2.22). \( \lambda \) can be estimated using (3.47) by

\[
\lambda = \tilde{T}(\Omega) = -\tilde{\eta}(\frac{h(x)}{\sqrt{h^2(x) + x^2 \tau^2}} + O(x)) = -\frac{1 + O(x)}{\tau \sqrt{1 + \epsilon^2}} + O(x^2).
\]

Using the estimates (3.49) completes the proof. \( \square \)

In order to obtain an apriori estimate of \( \nu \) in \( V^+_E \), we need to prove that \( \| \hat{T} \|_2 \) is uniformly bounded. For this purpose we will calculate the components of the commutator \([\hat{v}_i, \hat{T}]\).

**Lemma 3.6** Let \((V, g)\) satisfy the conditions of the last Lemma. Then the components of the Lie-brackets \([\hat{v}_i, \hat{T}]\) are bounded in the frame \(\{\hat{T}, \hat{v}_i\}\).

**Proof:** From

\[
[v_i, \hat{T}] = -\alpha^{-1} v_i(\alpha) \hat{T} - \alpha^{-1} v_i(\beta^j) v_j,
\]

we get

\[
[\hat{v}_i, \hat{T}] = \frac{1}{\langle v_i, v_i \rangle} [v_i, \hat{T}] + \frac{\hat{T}(\langle v_i, v_i \rangle)}{\langle v_i, v_i \rangle} \hat{v}_i.
\] (3.55)

Computing the Lie derivative of the metric \( \gamma \), we get

\[
\frac{\hat{T}(\langle v_i, v_i \rangle)}{\langle v_i, v_i \rangle} = 2k(\tilde{v}_i, \tilde{v}_i) + 2 \gamma_{il} \nabla_i \hat{T}^l
\]

and inserting this in (3.55) we obtain

\[
\langle [\hat{v}_i, \hat{T}], \hat{T} \rangle = \alpha^{-1} \tilde{v}_j(\alpha)
\]

\[
\langle [\hat{v}_i, \hat{T}], \hat{v}_j \rangle = \left( v_i(\beta^j) + 2k(\tilde{v}_i, \tilde{v}_i) + \frac{2 \gamma_{il} \nabla_i \hat{T}^l}{\langle v_i, v_i \rangle} \right) \langle \tilde{v}_j, \tilde{v}_k \rangle.
\] (3.56)

The proof is completed by referring to Lemma 3.4 and Lemma 3.3. \( \square \)

**Lemma 3.7** Let \((V, g)\) be an asymptotically Schwarzschild spacetime at null infinity, then the normal of the foliation given by (3.26), satisfies

\[
\| \hat{T} \|_2 < C, \quad \text{in} \ V^+_E.
\]

**Proof:** We choose on \( TS_r \) a ON-frame given by \( z_i = A_{ij} \tilde{v}_i \), where \( A \) is the matrix which orthogonalizes \( \tilde{v}_i \). It should be pointed out that by Lemma 3.4 in a first order approximation this matrix is the identity. Because \( \langle T, \hat{T} \rangle = -1 \), we can write

\[
\nabla_{z_i} \hat{T} = \langle z_j, \nabla_{z_i} \hat{T} \rangle z_j,
\]

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and by (3.56)
\[ \nabla^\top_\tau \hat{T} = \langle z_j, \nabla^\top_\tau \hat{T} \rangle z_j \]
\[ = -(\nabla^\top_\tau z_j, \hat{T}) z_j \]
\[ = \langle [z_i, \hat{T}], \hat{T} \rangle z_j \]
\[ = \alpha^{-1} z_j(\alpha) z_j, \]

hence using Lemma 3.4 and Lemma 3.6 we get
\[ g_E(\nabla^\top_\tau \hat{T}, \nabla^\top_\tau \hat{T}) < C. \]  
\[ (3.57) \]

We see from the above that we only need to estimate \( g_E(\nabla^\top_\tau \hat{T}, \nabla^\top_\tau \hat{T}) \). Let \( \{\hat{T}, z_i\} \) be an ON-frame. We shall estimate the different components of the tensor \( \nabla^\top_\tau \hat{T} \), namely

\[ \langle \hat{T}, \nabla^\tau_\tau \hat{T} \rangle = -(\nabla^\top_\tau \hat{T}, \nabla^\top_\tau \hat{T}) \]
\[ \langle z_i, \nabla^\tau_\tau \hat{T} \rangle = \nabla^\top_\tau \langle z_i, \nabla^\top_\tau \hat{T} \rangle - \langle \nabla^\top_\tau z_i, \nabla^\top_\tau \hat{T} \rangle \]
\[ = -\alpha^{-2} \hat{T}(\alpha) z_j(\alpha) + \alpha^{-1} \hat{T}(z_i(\alpha)) + \]
\[ = \langle [z_i, \hat{T}], \nabla^\top_\tau \hat{T} \rangle - \langle \nabla^\top_\tau \hat{T}, \nabla^\top_\tau \hat{T} \rangle \]
\[ \langle \hat{T}, \nabla_z \nabla^\top_\tau \hat{T} \rangle = -(\nabla_z \hat{T}, \nabla^\top_\tau \hat{T}) \]
\[ \langle \hat{T}, \nabla_{z_j} \nabla^\top_\tau \hat{T} \rangle = -(\nabla_{z_j} \hat{T}, \nabla^\top_\tau \hat{T}) \]
\[ \langle z_i, \nabla_{z_i} \nabla^\top_\tau \hat{T} \rangle = \nabla^\top_\tau \langle z_i, \nabla_{z_i} \hat{T} \rangle - \langle \nabla^\top_\tau z_i, \nabla_{z_i} \hat{T} \rangle \]
\[ = \nabla^\top_\tau \langle z_i, \nabla^\top_\tau \hat{T} \rangle - (\nabla^\top_\tau z_i, \nabla^\top_\tau \hat{T} \rangle - \langle \nabla^\top_\tau \hat{T}, \nabla_{z_i} \hat{T} \rangle \]
\[ \langle z_i, \nabla_{z_k} \nabla_{z_j} \hat{T} \rangle = \nabla_{z_k} \langle z_i, \nabla_{z_j} \hat{T} \rangle - \langle \nabla_{z_k} z_i, \nabla_{z_j} \hat{T} \rangle \]
\[ = \nabla_{z_k} \langle z_i, \nabla_{z_j} \hat{T} \rangle - (D_{z_k} z_i, \nabla_{z_j} \hat{T} \rangle). \]

By Lemma 3.4, Lemma 3.6 and the inequality (3.57) and because the function \( \epsilon_1 \) which appears in the induced metric has bounded second derivative, we can estimate all but the last three components. They can be estimated if we show that \( \nabla^\top_\tau \langle z_i, \nabla_{z_j} \hat{T} \rangle \) and \( \nabla_{z_k} \langle z_i, \nabla_{z_j} \hat{T} \rangle \) are bounded.

Using the estimates for \( \hat{T} \) in the proof of Lemma 3.4, the estimates for \( k_{ij} \) in Lemma 3.5 and the estimates for the Christoffel symbols in (A.74) and (A.75) one finds after a lengthy calculation that
\[ |\hat{T}(k(z_i, z_j))| \leq C \]
which is gives the required estimate for \( \nabla^\top_\tau \langle z_i, \nabla_{z_j} \hat{T} \rangle \).

To estimate \( \nabla_{z_k} \langle z_i, \nabla_{z_j} \hat{T} \rangle \), we calculate the tangential derivative of \( k_{ij} \) from the tangential derivative of the unphysical \( \hat{k}_{ij} \) as follows.
\[ \nabla_{v_l} k_{ij} = \nabla_{v_l}(\Omega^{-1} \hat{k}_{ij} - \lambda \gamma_{ij}) \]
\[ = \Omega^{-2} v_l(\Omega) \hat{k}_{ij} - \Omega^{-1} \nabla_{v_l} \hat{k}_{ij} - v_l(\lambda) \gamma_{ij} - \lambda \nabla_{v_l} \gamma_{ij} \]
\[ = \Omega^{-2} v_l(\Omega) \hat{k}_{ij} - \Omega^{-1} \nabla_{v_l} \hat{k}_{ij} - \Omega^{-2} C^l_{ij} \hat{k}_{ij} - v_l(\lambda) \gamma_{ij} - \lambda v_l C_{ij} \hat{T}_j - \lambda k_{il} \hat{T}_j, \]
Once again because the second derivatives of the functions $\epsilon_1$ and $\epsilon_2$ which appear in the induced metric and the unphysical second fundamental form are bounded we can estimate $\nabla \tilde k_{ij}$ and $\nu_l(\lambda)$. From this we obtain the desired estimate of $\nabla_{z_k}(z_i, \nabla_{z_j} T)$.

\[ \square \]

**Remark 3.1** By Lemma 3.3, we have for $\epsilon_- < \tau < \tau_0^-$

\[ \partial_x H = \frac{1}{2} \tau^2 (1 + O(x)), \]

which implies that in $V_E^-$ we can prove results similar to Lemma 2.4, Lemma 2.5 and Lemma 3.7 under much weaker conditions in the decay of the metric than those of the Definition 2.2.

The following Lemma gives a lower bound for the height function of a spacelike hypersurface in $V_E^+$, assuming a lower bound at $x = x_0$. The proof uses an estimate for the backward lightcones.

**Lemma 3.8** Let $(V, g)$ be asymptotically Schwarzschild. Then $\tau_0^+$ can be chosen sufficiently large so that the following holds. Let $S = S_0^+$ and suppose that $M^+$ is a compact hypersurface with height function $w = \tau|_{M^\tau_0^+}$ over $V_E^+ \cap S$ satisfying $w(p) \geq \tau_+ - \tau_0^+$ when $x(p) = x_0$. Then if $\tau_+$ is sufficiently large, the lower bound

\[ w(c(x)) \geq \tau_+ \sqrt{1 - \frac{2}{x_{\tau_+}} - \tau_0^+} \geq 0 \]

holds for $x_0 \geq x(p) \geq x_1$, where $c(x)$ is an integral curve of $\dot v_1 = v_1/|v_1|$ and $x_1$ is the root of the equation $\tau_+ \sqrt{1 - \frac{2}{x_{\tau_+}} - \tau_0^+} = 0$.

**Proof:** For motivation, consider the foliation in the Minkowski spacetime given by

\[ \tau^2 = t^2 - r^2, \quad \tau > 0. \]

The height function of the backward light cone at $\tau = \tau_+$ with respect to the slice $\tau = \tau_0^+$ satisfies

\[ v(x) \geq \tau_+ \sqrt{1 - \frac{2}{x_{\tau_+}} - \tau_0^+} \geq 0 \]

when $2/\tau_+ \approx x_1 \leq x \leq x_0$.

Now we consider the general case. Let $w$ be the height function of a spacelike hypersurface. Then it satisfies

\[ (\alpha - |\beta|)|Dw| \leq 1. \]

Let $c$ be a curve with $\dot c = v_1/|v_1|$. Then

\[ \frac{dw(c)}{dt} = \langle Dw, \dot c \rangle = \frac{w_x}{\sqrt{\langle v_1, v_1 \rangle}} \leq |Dw|, \]
and inserting this in (3.59), we obtain

\[ f(w(x), x, y^A)w_x \leq 1. \]  \hspace{1cm} (3.60)

where

\[ f(w(x), x, y^A) = (\alpha - |\beta|) \frac{1}{\sqrt{(v_1, v_1)}}. \]  \hspace{1cm} (3.61)

By Lemma 3.4, spacetimes which are asymptotically Schwarzschild satisfy

\[ \alpha = \frac{1}{\sqrt{1 + \epsilon_2}} + O(x^2) \]
\[ \beta_1 = \frac{x}{\tau \sqrt{1 + \epsilon_1}} + O(x^2) \]
\[ \sqrt{\gamma} = \frac{x}{\tau \sqrt{1 + \epsilon_1}} + O(x^2). \]

Using this in Equation (3.61), we obtain

\[ f(\tau, x, y^A) = \frac{x}{\tau \sqrt{1 + \epsilon_2}}(\sqrt{1 + \tau^2 x^2} - \sqrt{1 + \epsilon_2} + O(x)). \]

Hence the differential inequality (3.60) becomes

\[ (f_0(w(x), x, y^A) + f_1(w(x), x, y^A))xw_x \leq w(x) + \tau_0^+ \]

where \( f_0 \) is given by

\[ f_0(w(x), x, y^A) = \sqrt{1 + (w(x) + \tau_0^+)^2 x^2} - 1. \]

If \( \tau_0^+ \) is chosen sufficiently large, then \( |f_1| < \epsilon \ll 1 \). The differential equation

\[ (\sqrt{1 + (w(x) + \tau_0^+)^2 x^2} - 1 + \epsilon)xw'(x) = w(x) + \tau_0^+, \]

has the solution

\[ \left( \frac{y}{(\sqrt{y^2 + 1 + 1})(\sqrt{y^2 + 1 + \epsilon})} \right)^{\epsilon} \left( \sqrt{y^2 + 1 + 1} \right) = Ax^{1+\epsilon}, \]

with \( y = (w(x) + \tau_0^+)x \) and initial condition \( x = x_0, \ y = \tau_+ x_0 \). Using this we get the estimate

\[ \sqrt{y^2 + 1 + 1} \geq \tau_+ x. \]

Hence a lower bound of the height function \( w \) along \( x \) is given by

\[ w(c(x)) \geq \tau_+ \sqrt{1 - \frac{2}{|b|\tau_+} - \tau_0^+} \]

for \( x_0 \geq x(p) \geq x_1 \approx \frac{2}{\tau_+} \) and \( \tau_+ \) sufficiently large.

We are now ready to find the setup where the existence proof in Section 4 can be carried out.
Theorem 3.9 Let \((V, g)\) be an asymptotically Schwarzschild spacetime. Then there exist \(\tau_0^-, \tau_0^+, \epsilon^- \leq \tau_0^- < \tau_0^+ < \infty\) and \(x_0 > 0\) so that with \(V_E^-\) and \(V_E^+\) defined as in (3.33) with the given \(x_0\) used in defining \(V_E^\pm\) in Definition 2.2 so that the following holds.

1. In \(V_E^+\), \(S_\tau\) has mean curvature \(k < 3\), the foliation \(\{S_\tau\}\) satisfies (3.23) and (3.24), and the lower bound on the height function \(w\) given by (3.58) holds.

2. In \(V_E^-\), \(S_\tau\) has mean curvature \(k > 3\) and the foliation \(\{S_\tau\}\) satisfies (3.23) and (3.24).

Proof: Point 1 is a direct consequence of Lemmata 3.4, 3.5, 3.7 and 3.8. By Lemma 3.3, we can write for \(\epsilon^- \tau < \tau_0^-\)
\[
\partial_u H = \frac{1}{2} \tau^2 (1 + O(x)),
\]
which implies that in \(V_E^-\) we can prove results similar to Lemmata 3.4, 3.5 and 3.7 under much weaker conditions in the decay of the metric than those of the Definition 2.2. Point 2 follows.

This shows that in \(V_E^+\) and \(V_E^-\), \(\{S_\tau\}\) can be used as local upper respectively lower barriers for CMC hypersurfaces with mean curvature \(k = 3\) and that the gradient estimates [2, Theorem 3.1] can be applied.

4 Proof of existence of CMC hypersurfaces

The main difficulty in proving the existence of CMC surfaces is in obtaining an apriori height bound. This can be done for spacetimes which are asymptotically Schwarzschild and satisfy the following interior condition on the causal structure.

Definition 4.1 Let \((V, g)\) be an asymptotically Schwarzschild spacetime and let \(0 < \tau_0^- < \tau_0^+ < \infty\) be given. We say that \((V, g)\) satisfies the future interior condition if

1. There exists a family of complete spacelike hypersurfaces \(\{S_\tau\}_{\tau \in (0, \infty)}\), such that
   (a) In the exterior region \(V_E\), \(S_\tau\) is as in Lemma 3.4.
   (b) \(\{S_\tau\}_{\tau \in [\tau_0^+, \infty)}\) is a foliation of \(D^+(S_{\tau_0^+})\).

2. Let \(V_I^- = D^+(S_{\tau_0^+}) \cap V_I\) denote the future interior region. Then there is a constant \(C\) such that for all \(q \in V\) with \(x(q) = x_0\),
\[
\sup_{p \in V_I^- \setminus I^+(q)} (\tau(p) - \tau(q)) \leq C_I, \quad (4.62)
\]
where \(I^+(q)\) denotes the chronological future of \(q\).
Remark 4.1 1. The future interior condition is satisfied in Minkowski space with \( \{ S_\tau \} \) given by the hyperboloids
\[
t^2 - \tau^2 = \tau^2, \quad \tau \in (\tau_0, \infty), \ t > 0,
\]
for some \( \tau_0 > 0 \).

2. The compactness of the domain of dependence of compact sets is fundamental in order to be able to apply the existence and regularity results for the Dirichlet problem for CMC hypersurfaces.

In the rest of this section \((V, g)\) we will be assumed to be an asymptotically Schwarzschild spacetime, cf. Definition 2.2, which satisfies the future interior condition introduced in Definition 4.1 and \( \tau_0^+, \tau^+_0, x_0 \) and \( V^+_E \) will be as in Theorem 3.9. In the following also let \( S = S_{\tau_0^+} \).

The construction of the global upper barrier is based on a height estimate. This is obtained using again the foliation given by Lemma 3.4 and then applying a test function argument to the mean curvature formula (2.8).

Theorem 4.1 Suppose that \( M \) is a compact hypersurface with mean curvature \( k = 3 \), such that \( \partial M \subset V^+_E \cap S \) and let \( w \) be the height function of \( M^+ = M \cap D^+(S) \) over \( S \), defined by \( w = \tau \mid_{M^+} - \tau_0^+ \). Then there is a constant \( C \) such that \( \sup w \leq C \).

Proof: The outline of the proof is similar to that of [2, Theorem 5.3]. By Theorem 3.9 the mean curvature \( k \) of \( S_\tau \) satisfies \( k < 3 \) in \( V^+_E \) for \( \tau > \tau_0^+ \). Therefore by the maximum principle (cf. eg. [10, Chapter II]), the height function \( w \) has no local maximum in \( \text{int}(V^+_E) \). Thus we may assume that \( w \) takes its supremum in \( M^+ \cap V^+_E \).

Let \( M^+_E = M \cap V^+_E \). The boundary of \( M^+_E \) is the union of components where \( w = 0 \) and a set \( B^+ = M^+ \cap B \) where \( B = \{ p \in V^+_E : x(p) = x_0 \} \).

Let \( \phi \) be a continuously differentiable function on \( \mathbb{R}^+ \), such that \( \phi'(s) > 0, \phi(s) > 0 \) for \( s > 0 \). Using Equation (2.8), we have on \( M^+ \),
\[
\text{div}_M (\alpha^2 \phi(w) \nabla^M w) = \alpha \phi(w)(k \nu - \text{div}_M \tilde{T} + \langle \nabla^M w, \nabla^M \alpha \rangle) + (\nu^2 - 1)\phi'(w)
\]
\[
= \phi(w) \left( (\nu^2 - 1) \left( \frac{\alpha k}{\nu + 1} - \alpha k(\tilde{W}, \tilde{W}) + \tilde{T}(\alpha) + \frac{\phi'(w)}{\phi(w)} \right) + \alpha \nu (k_M - k) \right). \tag{4.63}
\]

Integrating (4.63) by parts over \( M^+_E \) gives
\[
\int_{B^+} \alpha^2 \phi(w) \langle \nabla^M w, \sigma \rangle d\mathcal{B}^+ = \int_{M^+_E} \phi(w) (\nu^2 - 1) \left( \frac{\phi'(w)}{\phi(w)} + \tilde{T}(\alpha) - \alpha k(\tilde{W}, \tilde{W}) + \frac{\alpha}{\nu + 1} k \right) d\nu_{M^+_E} + \int_{M^+_E} \alpha \phi(w) (k_M - k) \nu d\nu_{M^+_E} \tag{4.64}
\]
where $\sigma$ is the outer normal of $B^+$, $dv_{M_E^+}$ is the volume form on $M_E^+$ and $B^+$ is the induced area on $B^+$.

Using Lemmata 3.4 and 3.5 we estimate

$$|\hat{T}(\alpha) - \alpha k(\hat{W}, \hat{W}) + \frac{\alpha}{\nu + 1}k| \leq \frac{1 + \epsilon}{\tau} + (1 - 2\epsilon)x, \text{ in } M_E^+. \quad (4.65)$$

Let $\tau_+ - \tau_0^+ = \sup_{M^+} w - C_I$. Using the interior condition (4.62), we have that

$$\inf_{p \in B^+} w(p) \geq \tau_+ - \tau_0^+. \quad (4.66)$$

Thus for all $p \in B^+$,

$$\tau_+ - \tau_0^+ + C_I \geq w(p) \geq \tau_+ - \tau_0^+. \quad (4.67)$$

Since we want a height estimate of $w$, we may suppose that $\tau_+ - \tau_0^+ > \frac{1}{x_0}$. Then $w \geq 1$ for all $w(p) \in B^+$. The gradient estimate [2, Theorem 3.1] applies and gives $\nu(p) \leq C$ for all $p \in B^+$, where $C$ is a constant does not depend on $\tau_+$. On the other hand, by assumption $V$ is asymptotically Schwarzschild and using (3.37) we have

$$dB^+ = \sqrt{\det g_{AB}(\tau)}dy^2dy^3 \leq C\sqrt{\det g_{AB}(\tau_0^+)}dy^2dy^3,$$

where $g_{AB}(\tau)$ is the induced metric on $B^+$ and

$$\langle \nabla M w, \sigma \rangle = |\nabla M w|\langle e_1, \sigma \rangle = \alpha^{-1}\sqrt{\nu^2 - 1}\langle e_1, \sigma \rangle.$$

The boundary term in (4.64) is now estimated by

$$\left| \int_{B^+} \alpha^2 \phi(w)\langle \nabla M w, \sigma \rangle dB^+ \right| \leq C \sup_M \phi(w). \quad (4.66)$$

The volume form on $M$ can be written [2]

$$dv_{M_E^+} = |1 + \beta \cdot Dw|\nu^{-1}\sqrt{\det \gamma}dx^2dy^3,$$

but by (3.37) we estimate in $V_E$

$$\sqrt{\det \gamma} \geq \sqrt{\det \gamma(\tau_0^+)}$$

and

$$|1 + \beta \cdot Dw| \geq \epsilon > 0.$$

Therefore considering $M_E^+$ as a graph over $S' \subset S \cap V_E^+$ and using Theorem 3.9, we can estimate

$$\int_{M_E^+} \alpha \phi(w)(k_M - k)\nu dv_{M_E^+} \geq C \int_{S'} \phi(w)d\mu(\gamma). \quad (4.67)$$
Inserting (4.65), (4.66), (4.67) into (4.64) gives
\[
\int_{M^+} \phi(w)(\nu^2 - 1) \left( \frac{\phi'}{\phi}(w) - \frac{1 + \epsilon}{w + \tau_0} - (1 - 2\epsilon)x \right) dv_{M^+} + \\
+ \int_{S'} \phi(w)d\mu(\gamma) \leq C \sup_{M^+} \phi(w). \tag{4.68}
\]
We will choose \( \phi \) such that the first term of (4.68) becomes positive and in consequence may be discarded, obtaining in this way
\[
\int_{S'} \phi(w)d\mu(\gamma) \leq C \sup_{M} \phi(w).
\]

On the other hand we want to find \( \phi \) such that \( \sup_{M^+} w \) is bounded, in other words so that \( \tau_+ \) is bounded. In order to do this, observe that by Lemma 3.8 (see Figure 2)
\[
w + \tau_0 \geq \tau_+ \sqrt{1 - \frac{2}{b|x(p)|\tau_+}},
\]
for \( 2/\tau_+ \leq x(p) \leq x_0 \), or
\[
x \leq \frac{2}{\tau_+(1 - ((w + \tau_0)/\tau_+)^2)}, \quad \text{for } \tau_0 \leq w + \tau_0 \leq \tau_+ - 1. \tag{4.69}
\]

To construct \( \phi \) such that the first integral in (4.68) is positive, we solve the following differential equation
\[
\frac{\phi'}{\phi}(s) = \frac{1 + \epsilon}{s + \tau_0} + \frac{2(1 - 2\epsilon)}{\tau_+^2(1 - ((s + \tau_0)/\tau_+)^2)}. \tag{4.70}
\]
The solution is
\[ \phi(s) = \left( \frac{\tau^+ + \tau^+_0 + s}{\tau^+ - \tau^+_0 - s} \right)^{1-2\epsilon} (s + \tau^+_0)^{1+\epsilon}. \]

It is easily seen using (4.69) that this satisfies the required condition in the region \( s \leq \tau^+ - \tau^+_0 - 1 \). In order to get a function which satisfies \( \phi(0) = 0 \) and is defined for all \( s \in \mathbb{R}^+ \), positive and continuously differentiable, we construct \( \phi \) as follows:

\[
\phi(s) = \begin{cases} 
D_1 s^{D_2} & 0 \leq s \leq 1 \\
\left( \frac{\tau^+ + \tau^+_0 + s}{\tau^+ - \tau^+_0 - s} \right)^{1-2\epsilon} (s + \tau^+_0)^{1+\epsilon} & 1 \leq s \leq \tau^+ - \tau^+_0 - 1 \\
(2\tau^+ - 1)^{1-2\epsilon}(\tau^+ - 1)^{1+\epsilon} e^{D_3(s+\tau^+_0+2)} & \tau^+ - \tau^+_0 - 1 \leq s,
\end{cases}
\]

where
\[
D_1 = \left( \frac{\tau^+ + \tau^+_0 + 1}{\tau^+ - \tau^+_0 - 1} \right)^{1-2\epsilon} (1 + \tau^+_0)^{1+\epsilon},
D_2 = \frac{1 + \epsilon}{1 + \tau^+_0} + \frac{2(1 - 2\epsilon)}{\tau^+(1 - ((1 + \tau^+_0)/\tau^+)^2)},
D_3 = \frac{1 + \epsilon}{\tau^+ - 1} + \frac{1 - 2\epsilon}{1 - 1/2\tau^+}.
\]

With \( \phi \) chosen as above, the first term in (4.68) is positive (cf. Theorem 5.3) for details) and may be discarded, so (4.68) becomes
\[
\int_{S'} \phi(w) d\mu(\gamma) \leq C(\tau^+_0)^{2-\epsilon}. \tag{4.71}
\]

By construction \( \phi(w) \geq 1 \) when \( w \geq 1 \). By Lemma 3.8, \( w \geq 1 \) when
\[
\frac{2(1 + \delta)}{\tau^+} \leq x \leq x_0,
\]
where
\[
1 + \delta = \frac{1}{1 - ((1 + \tau^+_0)/\tau^+)^2}.
\]

Thus, from the inequality (4.71), we get
\[
\int_{2(1+\delta)/\tau^+}^{x_0} x^{-3} \int_{S^2} d\mu(\tilde{\gamma}) \leq C(\tau^+_0)^{2-\epsilon}
\]
and integrating shows that \( \tau^+ \leq C \). \( \square \)

The main existence theorem follows from this estimate.

**Theorem 4.2** There is a complete CMC hypersurface \( M, C^\infty \) on each compact set, with \( k_M = 3 \) and with the same cut at \( \mathcal{T}^+ \) as \( S \). Furthermore there is a constant \( C \) such that in \( V_E, \nu = -\langle \tilde{T}, N \rangle < C \), where \( \tilde{T} \) and \( N \) are the normals of \( S_\tau \) and \( M \) respectively.
Proof: Consider the region of the spacetime given by

\[ V' = \{ p \in V : p \in D^{-}(S_{\tau_{\text{max}}}) \}, \]

where \( \tau_{\text{max}} \) is the height estimate given by Theorem 1.1, see Figure 3. Let \( \{x_n\}_{n=1}^{\infty} \) be a decreasing sequence in \((0, x_0)\) such that \( x_n \to 0 \) as \( n \to \infty \). By Point 3 of Definition 4.1 and the fact that the mean curvature equation \( k(w) = 3 \) satisfies the mean curvature structure condition with structure function \( F \in C^\infty \), [4, Theorem 4.1] ensures that the Dirichlet problem

\[
\begin{align*}
    k_{M_n} &= 3, \quad \text{in } S_n = S \setminus S \cap \{ p \in V : x(p) < x_n \} \\
    \partial M_n &= \partial S_n
\end{align*}
\]

has solution \( M_n \) which is a \( C^\infty \) regular spacelike hypersurface satisfying

\[ M_n \subset V'. \]

Note that \( \Sigma(S_n) = \emptyset \) (cf. [4, (3.13)]) since \( S \) is spacelike.

We will now prove that the sequence \( \{M_n\} \) has a convergent subsequence. Construct a compact set \( K \) as follows. Let \( x_K \leq x_0 \) be given and define compact sets \( S_K, L_K \) by

\[
\begin{align*}
    S_K &= S_{\tau_{\text{max}}} \cap \{ p \in V' : x(p) \geq x_K \}, \\
    L_K &= \{ p \in V : x(p) = x_K \text{ and } 0 \leq u \leq H(x_K, \tau_{\text{max}}) \},
\end{align*}
\]

where \( H \) is the height function used in the definition of the foliation \( \{S_{\tau}\} \), cf. Section 3. Note that the “past boundary” of \( L_K \) is the intersection with \( \{x = x_K\} \) of the backward light cone of the cut of \( S \) with \( \mathcal{I}^+ \), and the “future boundary” of \( L_K \) is the intersection of the surface \( S_{\tau_{\text{max}}} \) with \( \{x = x_K\} \).
Defining $S_K = L_K \cup S_K$, we have $S_K$ compact. Hence in view of Point 3 of Definition 4.1, the set

$$K = D^-(S_K)$$

is compact. Let now

$$M^K_n = M_n \cap K.$$ 

By construction of $M_n$, it follows using the height estimate from Theorem 4.1 and causality, that $\partial M^K_n \subset L_K$. By the compactness of $K$ and using [4, Theorem 3.8], there is a subsequence $M^K_n$ and a weakly spacelike hypersurface $M^K$ such that if $p$ is an accumulation point of $M^K_n$, $p \in M^K$ and $M^K - \Sigma(M^K)$ is a $C^{\infty}$ regular hypersurface with mean curvature 3 where $\Sigma(M^K)$ is the singular set consisting of a union of null geodesics beginning and ending in $\partial M^K$.

Because this holds for each $x_K \leq x_0$, by letting $x_K \to 0$, we find a hypersurface $M$ and a singular subset $\Sigma(M)$ such that $M - \Sigma(M)$ is a $C^{\infty}$ regular hypersurface with mean curvature 3 and $\Sigma(M)$ consists of a union of null geodesics beginning and ending at $I^+$. We claim that $\Sigma(M) = \emptyset$. To see this it is sufficient to recall that $I^+$ is in the future of $V$ and noting that the existence of a null geodesic beginning and ending at $I^+$ would contradict this fact.

It follows that $M$ is $C^{\infty}$ regular with $k_M = 3$. By construction $M \subset D^-(S_{r_{\text{max}}})$ and therefore has the same cut at $I^+$ as $S_{r_{\text{max}}}$ and hence the same cut as $S$. This proves the existence of $M$.

**Claim:** There is a constant $C$ such that in $V_E$,

$$\nu = -\langle \hat{T}, N \rangle < C. \quad (4.72)$$

To prove the claim, we consider $M^{-}_n = M_n \cap V^{-}_E$ and $M^- = M \cap V^{-}_E$ as given by the height functions $w_n$ respectively $w$.

The $S_{\tau}$ for $\tau < \tau_0$ are lower barriers for CMC hypersurfaces with $k = 3$ and therefore the maximum principle argument applied as in the proof of Theorem 4.1 implies that the infimum of $w_n$ must be attained at a point $p$ satisfying $p \in M_n \cap B$ or $p \in \partial M_n$. We will use this to prove a lower bound for $w$.

By construction, if $p \in \partial M_n$, $w(p) = \tau^{-}_0$. It remains to consider the case when the infimum of $w_n$ is attained in $M_n \cap B$. By the above, $w_n(p) \to w(p)$ and $M$ is strictly spacelike on compacts. As a consequence there exists $N_0 > 0$ and $\tau_\epsilon \in (0, \tau^-_0)$ such that $w_n(p) \geq \tau_\epsilon > 0$, when $n \geq N_0$ and $p \in B$, which in turn implies that $w_n(p) \geq \tau_\epsilon$, when $p \in V^{-}_E$. It follows that

$$w \geq \tau^{-}_\epsilon, \quad \text{on } V^{-}_E.$$ 

Choose $\epsilon > 0$ in (3.33) such that $0 < \epsilon \leq \tau_\epsilon$. By Point 2 of Theorem 3.9, the foliation $\{S_{\tau}\}$ satisfies (3.23) and (3.24) and therefore the gradient estimate [4, Corollary 3.4] implies

$$\nu < C, \quad \text{on } M$$

which completes the proof of the claim.

It remains to prove that $M$ is complete. Let $M_E = M \cap V_E$ and let $p \in M_E$ be given. Let $c : [0, \infty) \to M_E$ be an arclength parametrized curve in $M_E$ with $c(0) = p$. To prove that $M$ is complete, it is sufficient to prove that $x(c(t)) > 0$
for $t < \infty$, since this implies that any geodesic in $M$ reaching infinity must have infinite length.

Let $\sigma$ be the metric induced on $M$ from $g$. Choose the adapted ON frame $\{e_i\}$ on $M$ as in Section 2. Let $\tilde{e}_i = \Omega^{-1}e_i$ be the corresponding ON frame w.r.t. the conformally related metric $\tilde{\sigma} = \Omega^2\sigma$ and let $\tilde{\nabla}^M x = \tilde{e}_i(x)\tilde{e}_i$ be the gradient of $x$ in $M$ w.r.t. $\tilde{\sigma}$. We compute

$$\left| \frac{d}{dt} x(c(t)) \right| \leq \Omega(c(t))\|\tilde{\nabla}^M x\|_{\tilde{\sigma}} = \Omega(c(t))\|\tilde{\nabla}^M x\|_{\tilde{\sigma}}, \quad (4.73)$$

where we used the assumption that $c$ is arclength parametrized. Using the $\nu$ estimate in (4.72), the definition of the adapted frame and (2.7) together with the fact that $x \in C^\infty(\tilde{V})$, we find that $\|\tilde{\nabla}^M x\|_{\tilde{\sigma}} \leq C$ in $V_E$ for some constant $C < \infty$. From (4.73) and the assumptions on $\Omega$ we now get

$$\left| \frac{d}{dt} \ln(x(c(t))) \right| \leq C$$

which after integrating gives

$$\frac{x(p)}{x(c(t))} \leq C' e^{Ct},$$

which completes the proof. □

This result remains valid for hypersurfaces with arbitrary constant mean curvature. We have chosen $k = 3$ for simplicity.

### A The estimate of $\delta \Gamma^\mu_{\nu\lambda}$

By definition the unphysical metric can be written as

$$(\tilde{g}_{\mu\nu}) = \begin{pmatrix} -x^2 h(x) + \mathcal{O}(x^6) & 1 & \mathcal{O}(x^3) & \mathcal{O}(x^3) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{g}_{AB} & \\ 0 & 0 & 0 & \tilde{g}_{AB} \end{pmatrix}$$

from which it follows that the inverse metric takes the form

$$(\tilde{g}^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ x^2 h(x) + \mathcal{O}(x^6) & \mathcal{O}(x^3) & \mathcal{O}(x^3) & \\ 0 & 0 & \tilde{g}^{AB} \end{pmatrix}.$$ 

From this we can obtain the Christoffel symbols needed to compute $\tilde{k}_{11}$.

$\tilde{\Gamma}^\mu_{xx}$ are zero because $x$ is the affine parameter of null geodesics. The other components can easily be computed from the above expression of the metric and are of the form

$$\tilde{\Gamma}_u^x = \frac{1}{2} \tilde{g}^{ux}(2\tilde{g}_{ux,u} - \tilde{g}_{uu,u}) = \tilde{\Gamma}_u^x + \mathcal{O}(x^3),$$

$$\tilde{\Gamma}_u^x = \frac{1}{2} \tilde{g}^{ux} \tilde{g}_{uu,x} + \frac{1}{2} \tilde{g}^{xA} \tilde{g}_{Au,x} = \tilde{\Gamma}_u^x + \mathcal{O}(x^3), \quad \tilde{\Gamma}_{ux}^u = 0,$$

$$\tilde{\Gamma}_u^x = \frac{1}{2} \tilde{g}^{ux} \tilde{g}_{uu,u} + \frac{1}{2} \tilde{g}^{xx}(2\tilde{g}_{ux,u} - \tilde{g}_{uu,u}) + \frac{1}{2} \tilde{g}^{xA}(2\tilde{g}_{A,u,u} - \tilde{g}_{uu,u})$$

$$= \tilde{\Gamma}_u^x + \mathcal{O}(x^5), \quad (A.74)$$

27
where the $\hat{\Gamma}$ are the Christoffel symbols for the conformal Schwarzschild metric. The last equality requires that $\tilde{g}_{uu,u} = O(x^5)$. In the other directions we need to estimate

\[
\begin{align*}
\hat{\Gamma}^u_{uA} &= -\frac{1}{2} \tilde{g}^{xu} \tilde{g}_{uA,x} = O(x^2), \\
\hat{\Gamma}^x_{xA} &= \frac{1}{2} \tilde{g}^{ux} \tilde{g}_{uA,x} + \frac{1}{2} \tilde{g}^{xC} \tilde{g}_{AC,x} = O(x^2), \\
\hat{\Gamma}^x_{uA} &= \frac{1}{2} \tilde{g}^{ux} \tilde{g}_{uu,A} - \frac{1}{2} \tilde{g}^{xx} \tilde{g}_{uA,x} + \frac{1}{2} \tilde{g}^{xC} (\tilde{g}_{uC,A} + \tilde{g}_{AC,u} - \tilde{g}_{uA,C}) = O(x^4), \\
\hat{\Gamma}^u_{AB} &= -\frac{1}{2} \tilde{g}^{ux} \tilde{g}_{AB,x} = O(1), \\
\hat{\Gamma}^x_{AB} &= \frac{1}{2} \tilde{g}^{ux}(\tilde{g}_{uB,A} + \tilde{g}_{Au,B} - \tilde{g}_{AB,u}) - \frac{1}{2} \tilde{g}^{xx} \tilde{g}_{AB,x} + \\
&\quad + \frac{1}{2} \tilde{g}^{xC} (\tilde{g}_{BC,A} + \tilde{g}_{AC,B} - \tilde{g}_{AB,C}) = O(x^2).
\end{align*}
\]

(A.75)

Once again the last equality requires that $\tilde{g}_{AB,u} = O(x^2)$.

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