The boundedness of some operators with rough kernel on the weighted Morrey spaces

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Abstract

Let $\Omega \in L^q(S^{n-1})$ with $1 < q \leq \infty$ be homogeneous of degree zero and has mean value zero on $S^{n-1}$. In this paper, we will study the boundedness of homogeneous singular integrals and Marcinkiewicz integrals with rough kernel on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $q' \leq p < \infty$(or $q' < p < \infty$) and $0 < \kappa < 1$. We will also prove that the commutator operators formed by a $BMO(\mathbb{R}^n)$ function $b(x)$ and these rough operators are bounded on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $q' < p < \infty$ and $0 < \kappa < 1$.

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1. Introduction

Suppose that $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^q(S^{n-1})$ with $1 < q \leq \infty$ be homogeneous of degree zero and satisfy the cancellation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. The homogeneous singular integral operator $T_\Omega$ is defined by

$$T_\Omega f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(x - y) dy$$

and a related maximal operator $M_\Omega$ is defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y')f(x - y)| \, dy.$$
Let $b$ be a locally integrable function on $\mathbb{R}^n$, the commutator of $b$ and $T_\Omega$ is defined as follows

$$[b, T_\Omega]f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x).$$

The Marcinkiewicz integral of higher dimension $\mu_\Omega$ is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} |f(y)| dy.$$ 

It is well known that the Littlewood-Paley $g$-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley $g$-function. In this paper, we will also consider the commutator $[b, \mu_\Omega]$ which is given by the following expression

$$[b, \mu_\Omega]f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy.$$ 

The classical Morrey spaces $L^{p,\lambda}$ were first introduced by Morrey in [10] to study the local behavior of solutions to second order elliptic partial differential equations. Recently, Komori and Shirai [9] considered the weighted version of Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces.

The main purpose of this paper is to discuss the weighted boundedness of the above operators $M_\Omega$, $T_\Omega$ and $\mu_\Omega$ with rough kernels on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $q' \leq p < \infty$ and $0 < \kappa < 1$, where we set the notation $q' = q/(q-1)$ when $1 < q < \infty$ and $q' = 1$ when $q = \infty$. We shall also show that the commutators $[b, T_\Omega]$ and $[b, \mu_\Omega]$ are bounded operators on the weighted Morrey spaces $L^{p,\kappa}(w)$ for $q' < p < \infty$ and $0 < \kappa < 1$, where the symbol $b$ belongs to $BMO(\mathbb{R}^n)$. Our main results are stated as follows.

**Theorem 1.** Assume that $\Omega \in L^q(S^{n-1})$ with $1 < q < \infty$. Then for every $q' \leq p < \infty$, $w \in A_{p/q'}$ and $0 < \kappa < 1$, there exists a constant $C > 0$ independent of $f$ such that

$$\|M_\Omega(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$
Theorem 2. Assume that $\Omega \in L^q(S^{n-1})$ with $1 < q < \infty$. Then for every $q' \leq p < \infty$, $w \in A_{p/q'}$ and $0 < \kappa < 1$, there exists a constant $C > 0$ independent of $f$ such that

$$\|T\Omega(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$  

Theorem 3. Assume that $\Omega \in L^q(S^{n-1})$ with $1 < q < \infty$ and $b \in BMO(\mathbb{R}^n)$. Then for every $q' < p < \infty$, $w \in A_{p/q'}$ and $0 < \kappa < 1$, there exists a constant $C > 0$ independent of $f$ such that

$$\|[b, T\Omega](f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$  

Theorem 4. Assume that $\Omega \in L^q(S^{n-1})$ with $1 < q \leq \infty$. Then for every $q' < p < \infty$, $w \in A_{p/q'}$ and $0 < \kappa < 1$, there exists a constant $C > 0$ independent of $f$ such that

$$\|\mu\Omega(f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$  

Theorem 5. Assume that $\Omega \in L^q(S^{n-1})$ with $1 < q \leq \infty$ and $b \in BMO(\mathbb{R}^n)$. Then for every $q' < p < \infty$, $w \in A_{p/q'}$ and $0 < \kappa < 1$, there exists a constant $C > 0$ independent of $f$ such that

$$\|[b, \mu\Omega](f)\|_{L^{p,\kappa}(w)} \leq C\|f\|_{L^{p,\kappa}(w)}.$$  

2. Notations and definitions

First let us recall some standard definitions and notations. The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [11]. A weight $w$ is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere, $B = B(x_0, r)$ denotes the ball with the center $x_0$ and radius $r$. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|B|} \int_B w(x) \, dx\right)\left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx\right)^{p-1} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n,$$

where $C$ is a positive constant which is independent of $B$.

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) \, dx \leq C \text{ess inf}_{x \in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$
A weight function $w$ is said to belong to the reverse Hölder class $RH_r$ if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left( \frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right)$$

for every ball $B \subseteq \mathbb{R}^n$.

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. If $w \in A_p$ with $1 \leq p < \infty$, then there exists $r > 1$ such that $w \in RH_r$.

We give the following results that we will use frequently in the sequel.

**Lemma A** ([6]). Let $w \in A_p, p \geq 1$. Then, for any ball $B$, there exists an absolute constant $C$ such that

$$w(2B) \leq Cw(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C\lambda^{np}w(B),$$

where $C$ does not depend on $B$ nor on $\lambda$.

**Lemma B** ([7]). Let $w \in RH_r$ with $r > 1$. Then there exists a constant $C$ such that

$$\frac{w(E)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^{(r-1)/r}$$

for any measurable subset $E$ of a ball $B$.

A locally integrable function $b$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx < \infty,$$

where $b_B = \frac{1}{|B|} \int_B b(y) \, dy$ and the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

**Theorem C** ([5,8]). Assume that $b \in BMO(\mathbb{R}^n)$. Then for any $1 \leq p < \infty$, we have

$$\sup_B \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p \, dx \right)^{1/p} \leq C\|b\|_*.$$

Next we shall define the weighted Morrey space and give one of the results relevant to this paper. For further details, we refer the readers to [9].
Definition 1. Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w$ be a weight function. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(w) = \{ f \in L^p_{\text{loc}}(w) : \| f \|_{L^{p,\kappa}(w)} < \infty \},$$

where

$$\| f \|_{L^{p,\kappa}(w)} = \sup_B \left( \frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) \, dx \right)^{1/p}$$

and the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

In [9], the authors obtained the following result.

Theorem D. If $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$, then the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p,\kappa}(w)$.

We are going to conclude this section by giving several results concerning the weighted boundedness of rough operators $M_\Omega$, $T_\Omega$ and $\mu_\Omega$ on the weighted $L^p$ spaces. Given a Muckenhoupt’s weight function $w$ on $\mathbb{R}^n$, for $1 \leq p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the space of all functions satisfying

$$\| f \|_{L^p_w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

Theorem E ([4]). Suppose that $\Omega \in L^q(S^{n-1})$, $1 < q < \infty$. Then for every $q' \leq p < \infty$ and $w \in A_{p/q'}$, there is a constant $C$ independent of $f$ such that

$$\| M_\Omega(f) \|_{L^p_w} \leq C \| f \|_{L^p_w}$$

$$\| T_\Omega(f) \|_{L^p_w} \leq C \| f \|_{L^p_w}.$$

Theorem F ([2]). Suppose that $\Omega \in L^q(S^{n-1})$, $1 < q \leq \infty$. Then for every $q' < p < \infty$ and $w \in A_{p/q'}$, there is a constant $C$ independent of $f$ such that

$$\| \mu_\Omega(f) \|_{L^p_w} \leq C \| f \|_{L^p_w}.$$

Theorem G ([3]). Suppose that $\Omega \in L^q(S^{n-1})$ with $1 < q \leq \infty$ and $b \in BMO(\mathbb{R}^n)$. Then for $q' < p < \infty$ and $w \in A_{p/q'}$, there is a constant $C > 0$ independent of $f$ such that

$$\| [b, \mu_\Omega](f) \|_{L^p_w} \leq C \| f \|_{L^p_w}.$$

Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq \frac{A}{B} \leq C.$
3. Proof of Theorem 1

First, by using Hölder’s inequality, we can easily see that

\[ M_\Omega f(x) \leq C \cdot \|\Omega\|_{L_q(S^{n-1})} M_{q'}(f)(x), \]

where \( M_{q'}(f)(x) = M(|f|^{q'})(x)^{1/q'} \). Then for \( q' < p < \infty \) and \( w \in A_{p/q'} \), it follows immediately from Theorem D that

\[ \|M_{q'}(f)\|_{L_p,w} = \|M(|f|^{q'})\|_{L_{p,q',w}}^{1/q'} \leq C \|f\|_{L_{p,q',w}}. \]

Now we consider the case \( p = q' \). Fix a ball \( B = B(x_0,r_B) \subseteq \mathbb{R}^n \) and decompose \( f = f_1 + f_2 \), where \( f_1 = f\chi_{2B} \), \( \chi_{2B} \) denotes the characteristic function of \( 2B \). Since \( M_\Omega \) is a sublinear operator, then we have

\[ \frac{1}{w(B)^{\kappa/p}} \left( \int_B |M_\Omega f(x)|^p w(x) \, dx \right)^{1/p} \leq C \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \left( \int_B |M_\Omega f_1(x)|^p w(x) \, dx \right)^{1/p} \]

\[ + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |M_\Omega f_2(x)|^p w(x) \, dx \right)^{1/p} = I_1 + I_2. \]

Theorem E and Lemma A imply

\[ I_1 \leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) \, dx \right)^{1/p} \leq C \|f\|_{L_{p,w}} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \]

\[ \leq C \|f\|_{L_{p,w}}. \quad (1) \]

We turn to estimate the term \( I_2 \). For any given \( r > 0 \) and \( x \in B \), by Hölder’s inequality and the \( A_1 \) condition, we thus obtain

\[ \frac{1}{r^n} \int_{|y|<r} |\Omega'(y) f_2(x-y)| \, dy \]

\[ \leq \frac{1}{r^n} \left( \int_{|y|<r} |\Omega'(y)|^q \, dy \right)^{1/q} \left( \int_{|y|<r} |f_2(x-y)|^p \, dy \right)^{1/p} \]

\[ \leq C \cdot \|\Omega\|_{L_q(S^{n-1})} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_2(y)|^p \, dy \right)^{1/p} \]
A simple geometric observation shows that when \( x \in B(x_0, r_B) \) and \( y \in B(x, r) \cap (2B(x_0, r_B))^c \), then we have \( B(x_0, r_B) \subseteq 3B(x, r) \). Hence

\[
\frac{1}{r^n} \int_{|y| < r} |\Omega(y')f_2(x - y)| \, dy \leq C \|f\|_{L^p,\kappa(w)} \cdot \frac{1}{w(B(x, r))^{(1-\kappa)/p}} \leq C \|f\|_{L^p,\kappa(w)} \cdot \frac{1}{w(B(x_0, r_B))^{(1-\kappa)/p}}.
\]

Taking the supremum over all \( r > 0 \), we can get

\[
|M_\Omega(f_2)(x)| \leq C \|f\|_{L^p,\kappa(w)} \cdot \frac{1}{w(B(x_0, r_B))^{(1-\kappa)/p}},
\]

which implies

\[
I_2 \leq C \|f\|_{L^p,\kappa(w)}.
\]  

Combining the above inequality (2) with (1) and taking the supremum over all balls \( B \subseteq \mathbb{R}^n \), we obtain the desired result.

4. Proofs of Theorems 2 and 3

**Proof of Theorem 2.** Fix a ball \( B = B(x_0, r_B) \) and decompose \( f = f_1 + f_2 \), where \( f_1 = f \mathcal{X}_{2B} \). Then we have

\[
\frac{1}{w(B)^{\kappa/p}} \left( \int_B |T_\Omega f(x)|^p w(x) \, dx \right)^{1/p} \leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |T_\Omega f_1(x)|^p w(x) \, dx \right)^{1/p} + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |T_\Omega f_2(x)|^p w(x) \, dx \right)^{1/p} = J_1 + J_2.
\]

Theorem E and Lemma A give

\[
J_1 \leq C \cdot \frac{1}{w(2B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) \, dx \right)^{1/p}
\leq C \|f\|_{L^p,\kappa(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \leq C \|f\|_{L^p,\kappa(w)}.
\]
In order to estimate $J_2$, we first deduce from Hölder’s inequality that

$$|T_\Omega(f_2)(x)| = \left| \int_{(2B)^c} \frac{\Omega((x-y))}{|x-y|^\kappa} f(y) \, dy \right| \leq \sum_{j=1}^\infty \left( \int_{2^{j+1}B \setminus 2^j B} |\Omega((x-y))|^q \, dy \right)^{\frac{1}{q'}} \left( \int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|^{q'}}{|x-y|^\kappa q'} \, dy \right)^{\frac{1}{q'}}.$$

When $x \in B$ and $y \in 2^{j+1}B \setminus 2^j B$, then by a direct calculation, we can see that $2^{j-1}r_B \leq |y-x| < 2^{j+2}r_B$. Hence

$$\left( \int_{2^{j+1}B \setminus 2^j B} |\Omega((x-y))|^q \, dy \right)^{\frac{1}{q'}} \leq C \cdot \|\Omega\|_{L^q(S^{n-1})} |2^{j+1}B|^{\frac{1}{q'}}. \quad (3)$$

We also note that if $x \in B$, $y \in (2B)^c$, then $|y-x| \sim |y-x_0|$. Consequently

$$\left( \int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|^{q'}}{|x-y|^\kappa q'} \, dy \right)^{\frac{1}{q'}} \leq \frac{1}{|2^{j+1}B|} \left( \int_{2^{j+1}B} |f(y)|^{q'} \, dy \right)^{\frac{1}{q'}}.$$

So we have

$$|T_\Omega(f_2)(x)| \leq C \sum_{j=1}^\infty \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|^{q'} \, dy \right)^{\frac{1}{q'}}.$$

We shall consider two cases. When $p = q'$, then by the $A_1$ condition, we get

$$|T_\Omega(f_2)(x)| \leq C \sum_{j=1}^\infty \left( \frac{1}{w(2^{j+1}B)} \int_{2^{j+1}B} |f(y)|^{q'} w(y) \, dy \right)^{\frac{1}{q'}} \leq C \|f\|_{L^{q'}(w)} \sum_{j=1}^\infty \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}}. \quad (4)$$

When $p > q'$, set $s = p/q' > 1$. Then it follows from the Hölder’s inequality and the $A_s$ condition that

$$|T_\Omega(f_2)(x)| \leq C \sum_{j=1}^\infty \left( \frac{1}{|2^{j+1}B|^{1/q'}} \left( \int_{2^{j+1}B} |f(y)|^{q'} w(y) \, dy \right)^{\frac{1}{q'}} \right) \times \left( \int_{2^{j+1}B} w^{s'/s}(y) \, dy \right)^{1/s'q'} \leq C \sum_{j=1}^\infty \left( \frac{1}{w(2^{j+1}B)} \int_{2^{j+1}B} |f(y)|^{q'} w(y) \, dy \right)^{\frac{1}{p'}} \leq C \|f\|_{L^{p'}(w)} \sum_{j=1}^\infty \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}}. \quad (5)$$
Hence, for every $q' \leq p < \infty$, by the estimates (4) and (5), we obtain

$$J_2 \leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \left( \frac{w(B)}{w(2^{j+1}B)} \right)^{(1-\kappa)/p}.$$ 

Since $w \in A_{p/q}$, then there exists $r > 1$ such that $w \in RH_r$. By using Lemma B, we thus get

$$\frac{w(B)}{w(2^{j+1}B)} \leq C \left( \frac{|B|}{|2^{j+1}B|} \right)^{(r-1)/r}. \quad (6)$$

Therefore

$$J_2 \leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \left( \frac{1}{2^{jn}} \right)^{(1-\kappa)(r-1)/pr} \leq C \|f\|_{L^{p,\kappa}(w)},$$

where the last series is convergent since $(1-\kappa)(r-1)/pr > 0$. Using the estimates for $J_1$ and $J_2$ and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we complete the proof of Theorem 2.

**Proof of Theorem 3.** As in the proof of Theorem 2, we can write

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B \left| [b, T_{\Omega}] f(x) \right|^p w(x) \, dx \right)^{1/p} \leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B \left| [b, T_{\Omega}] f_1(x) \right|^p w(x) \, dx \right)^{1/p} + \frac{1}{w(B)^{\kappa/p}} \left( \int_B \left| [b, T_{\Omega}] f_2(x) \right|^p w(x) \, dx \right)^{1/p} = J'_1 + J'_2.$$ 

By Theorem E and the well-known boundedness criterion for the commutators of linear operators, which was obtained by Alvarez, Bagby, Kurtz and Pérez (see [1]), we see that $[b, T_{\Omega}]$ is bounded on $L^p_w$ for all $q' < p < \infty$ and $w \in A_{p/q}$. This together with Lemma A yield

$$J'_1 \leq C \|b\|_{\ast} \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) \, dx \right)^{1/p} \leq C \|b\|_{\ast} \|f\|_{L^{p,\kappa}(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \leq C \|b\|_{\ast} \|f\|_{L^{p,\kappa}(w)}. \quad (7)$$
We now turn to deal with the term $J_2'$. For any given $x \in B$, we have
\begin{equation}
\left| [b, T_\Omega] f_2(x) \right| \leq |b(x) - b_B| \cdot \left| \Omega((x - y)'_\Omega) \frac{|f(y)|}{|x - y|^n} \right| \cdot \int_{(2B)^c} |\Omega((x - y)'_\Omega)| |b(y) - b_B| |f(y)| dy
\end{equation}
\begin{equation}
= \left| \int_{(2B)^c} |\Omega((x - y)'_\Omega)| |b(y) - b_B| |f(y)| dy \right|
\end{equation}
\begin{equation}
= I + \Pi.
\end{equation}

In the proof of Theorem 2, for any $q' < p < \infty$, we have already showed
\begin{equation}
I \leq C |b(x) - b_B| \cdot \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}}.
\end{equation}

Consequently
\begin{equation}
\frac{1}{w(B)^{\kappa/p}} \left( \int_B |f|^{p} w(x) dx \right)^{1/p}
\leq C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{1}{w(2^{j+1}B)^{(1-\kappa)/p}} \cdot \left( \int_B |b(x) - b_B|^{p} w(x) dx \right)^{1/p}
\end{equation}
\begin{equation}
= C \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}} \cdot \left( \frac{1}{w(B)} \int_B |b(x) - b_B|^{p} w(x) dx \right)^{1/p}.
\end{equation}

Using the same arguments as that of Theorem 2, we can see that the above summation is bounded by a constant. Hence
\begin{equation}
\frac{1}{w(B)^{\kappa/p}} \left( \int_B |f|^{p} w(x) dx \right)^{1/p} \leq C \|f\|_{L^{p,\kappa}(w)} \left( \frac{1}{w(B)} \int_B |b(x) - b_B|^{p} w(x) dx \right)^{1/p}.
\end{equation}

Since $w \in A_{p'/q'}$, then $w \in A_{p'}$. As before, there exists a number $r > 1$ such that $w \in RH_r$. By the reverse Hölder’s inequality and Theorem C, we get
\begin{equation}
\left( \frac{1}{w(B)} \int_B |b(x) - b_B|^{p} w(x) dx \right)^{1/p}
\leq C \cdot \left( \frac{1}{w(B)} \int_B \left| b(x) - b_B \right|^{p^{r'}} w(x) dx \right)^{1/p^{r'}}
\leq C \cdot \left( \frac{1}{|B|} \int_B \left| b(x) - b_B \right|^{p^{r'}} dx \right)^{1/p^{r'}}
\leq C \|b\|_\ast.
\end{equation}
Hence we can get $v^{[6]}$, which implies $v^{[1]}$. Let

Substituting the above inequality (11) into (10), we thus have

On the other hand, it follows from Hölder’s inequality and (3) that

So we have

Set $s = p/q' > 1$. Then by using Hölder’s inequality, we thus obtain

Let $v(y) = w^{-s'/s}(y) = w^{1-s'}(y)$. Then we have $v \in A_{s'}$ because $w \in A_s$ (see [6]), which implies $v \in A_{q',s'}$. Following along the same lines as that of (8), we can get

Substituting the above inequality (11) into (10), we thus have

Hence

$$\frac{1}{w(B)^{1/p}} \left( \int_B w(x) \, dx \right)^{1/p} \leq C \|b\|_* \|f\|_{L^{p,s}(w)}.$$ (9)

$$II \leq C \sum_{j=1}^{\infty} \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} \, dy \right)^{1/q'}$$

$$\leq C \sum_{j=1}^{\infty} \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} \, dy \right)^{1/q'}$$

$$= III + IV.$$ (10)

$$\left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} \, dy \right)^{1/q'}$$

$$\leq C \|f\|_{L^{p,s}(w)} \cdot w(2^{j+1}B)^{\kappa/p} \left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q's'} w^{-s'/s}(y) \, dy \right)^{1/q's'}.$$ (11)

$$\left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q's'} v(y) \, dy \right)^{1/q's'} \leq C \|b\|_*.$$ (12)

$$\left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} f(y) \, dy \right)^{1/q'}$$

$$\leq C \|b\|_* \|f\|_{L^{p,s}(w)} \cdot w(2^{j+1}B)^{\kappa/p} \cdot v(2^{j+1}B)^{1/q's'}.$$ (13)
Now let’s deal with the last term IV. Since $b \in BMO(\mathbb{R}^n)$, then a simple computation shows that
\[
|b_{2j+1B} - b_B| \leq C \cdot j \|b\|_{*}.
\] (14)

It follows immediately from the inequalities (5) and (14) that
\[
IV \leq C \|b\|_{*} \sum_{j=1}^{\infty} j \cdot \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|^{q'} \, dy \right)^{1/q'}
\]
\[
\leq C \|b\|_{*} \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} j \cdot w(2^{j+1}B)^{(\kappa-1)/p}.
\]

Therefore, by the estimate (6), we obtain
\[
\frac{1}{w(B)^{\kappa/p}} \left( \int_{B} IV^p w(x) \, dx \right)^{1/p} \leq C \|b\|_{*} \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{j \cdot w(B)^{(1-\kappa)/p}}{w(2^{j+1}B)^{(1-\kappa)/p}}
\]
\[
\leq C \|b\|_{*} \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} \frac{j}{2^{jn\theta}}
\]
\[
\leq C \|b\|_{*} \|f\|_{L^{p,\kappa}(w)},
\] (15)

where $w \in RH_r$ and $\theta = (1 - \kappa)(r - 1)/pr$. Summarizing the estimates (13) and (15) derived above, we can get
\[
\frac{1}{w(B)^{\kappa/p}} \left( \int_{B} II^p w(x) \, dx \right)^{1/p} \leq C \|b\|_{*} \|f\|_{L^{p,\kappa}(w)}.
\] (16)

Combining the inequalities (7), (9) with the inequality (16) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we conclude the proof of Theorem 3.

5. Proofs of Theorems 4 and 5

Proof of Theorem 4. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$. Let $f = f_1 + f_2$, where $f_1 = f \chi_{2B}$. Then we have
\[
\frac{1}{w(B)^{\kappa/p}} \left( \int_{B} |\mu_{\Omega} f(x)|^p w(x) \, dx \right)^{1/p}
\]
\[
\leq \frac{1}{w(B)^{\kappa/p}} \left( \int_{B} |\mu_{\Omega} f_1(x)|^p w(x) \, dx \right)^{1/p}
\]
\[
+ \frac{1}{w(B)^{\kappa/p}} \left( \int_{B} |\mu_{\Omega} f_2(x)|^p w(x) \, dx \right)^{1/p}
\]
\[
= K_1 + K_2.
\]
Theorem F and Lemma A imply
\[ K_1 \leq C \cdot \frac{1}{w(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x) \, dx \right)^{1/p} \]
\[ \leq C \|f\|_{L^p(w)} \cdot \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \]
\[ \leq C \|f\|_{L^p(w)}. \]

To estimate \( K_2 \), observe that when \( x \in B \) and \( y \in 2^{j+1}B \setminus 2^jB (j \geq 1) \), then
\[ t \geq |x - y| \geq |y - x_0| - |x - x_0| \geq 2^{j-1}r_B. \]
Therefore
\[ |\mu(f_2)(x)| = \left( \int_0^\infty \left( \int_{(2B)^c \cap \{y:|x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} |f(y)| \, dy \right)^{2/p} \, dt \right)^{1/2} \]
\[ \leq \sum_{j=1}^\infty \left( \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \, dy \right) \cdot \left( \int_{2^{j-1}r_B}^\infty \frac{dt}{t^2} \right)^{1/2} \]
\[ \leq C \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^{1/n}} \cdot \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \, dy. \]

When \( \Omega \in L^\infty(S^{n-1}) \), then by assumption, we have \( w \in A_p, 1 < p < \infty \). It follows from the Hölder’s inequality and the \( A_p \) condition that
\[ |\mu(f_2)(x)| \leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^{1/n}} \cdot \frac{1}{|2^{j+1}B|^{(n-1)/n}} \int_{2^{j+1}B} |f(y)| \, dy \]
\[ \leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \left( \int_{2^{j+1}B} |f(y)|^p w(y) \, dy \right)^{1/p} \]
\[ \times \left( \int_{2^{j+1}B} w(y)^{-p'/p} \, dy \right)^{1/p'} \]
\[ \leq C \|\Omega\|_{L^\infty(S^{n-1})} \|f\|_{L^p(w)} \sum_{j=1}^\infty w(2^{j+1}B)^{\kappa-1/p}. \] (17)

When \( \Omega \in L^q(S^{n-1}), 1 < q < \infty \), by the inequalities (3) and (5), we get
\[ |\mu(f_2)(x)| \leq C \|\Omega\|_{L^q(S^{n-1})} \sum_{j=1}^\infty \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(y)|^{q'} \, dy \right)^{1/q'} \]
\[ \leq C \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(w)} \sum_{j=1}^\infty w(2^{j+1}B)^{\kappa-1/p}. \] (18)
Hence, for $1 < q \leq \infty$, $q' < p < \infty$, by the estimates (17) and (18), we have

$$K_2 \leq C \|f\|_{L^p, \kappa'(w)} \sum_{j=1}^{\infty} \left( \frac{w(B)}{w(2^{j+1}B)} \right)^{(1-\kappa)/p} \leq C \|f\|_{L^p, \kappa'(w)}.$$

Using the above estimates for $K_1$ and $K_2$ and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we get our desired result.

**Proof of Theorem 5.** As before, we can write

$$
\frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mu] f(x) w(x) dx \right)^{1/p} \leq \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mu_1] f_1(x) w(x) dx \right)^{1/p} + \frac{1}{w(B)^{\kappa/p}} \left( \int_B |[b, \mu_2] f_2(x) w(x) dx \right)^{1/p} = K'_1 + K'_2.
$$

Theorem G and Lemma A yield

$$K'_1 \leq C \|b\|_* \|f\|_{L^p, \kappa'(w)} \frac{w(2B)^{\kappa/p}}{w(B)^{\kappa/p}} \leq C \|b\|_* \|f\|_{L^p, \kappa'(w)}.$$

Finally, let us deal with the term $K'_2$. For any fixed $x \in B$, we have

$$
[b, \mu] f_2(x) \leq |b(x) - b_B| \left( \int_0^\infty \left( \int_{(2B)^c \cap \{y:|x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right)^2 \frac{dt}{t^3} \right)^{1/2} + \left( \int_0^\infty \left( \int_{(2B)^c \cap \{y:|x-y| \leq t\}} \frac{\Omega(x-y)}{|x-y|^{n-1}} |b(y) - b_B| f(y) dy \right)^2 \frac{dt}{t^3} \right)^{1/2} = I + II.
$$

In the proof of Theorem 4, for any $q' < p < \infty$, we have already proved

$$I \leq C |b(x) - b_B| \cdot \|f\|_{L^p, \kappa'(w)} \sum_{j=1}^{\infty} w(2^{j+1}B)^{(\kappa-1)/p}.$$
Following the same lines as in the proof of Theorem 3, we obtain

\[
\frac{1}{w(B)^{\kappa/p}} \left( \int_B |f|^p \, dx \right)^{1/p} \leq C \|b\|_\infty \|f\|_{L^p(X)}.
\]

On the other hand, we note that when \( x \in B \) and \( y \in 2^{j+1}B \setminus 2^jB \) (\( j \geq 1 \)), then we have \( t \geq 2^{j-1}r_B \). Consequently

\[
\begin{align*}
II & \leq C \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^{1/n}} \int_{2^{j+1}B \setminus 2^jB} \frac{\Omega(x-y)|b(y)-b_B| f(y)}{|x-y|^{n-1}} \, dy \\
& \leq C \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|^{1/n}} \int_{2^{j+1}B \setminus 2^jB} \frac{\Omega(x-y)|b(y)-b_{2^{j+1}B}| f(y)}{|x-y|^{n-1}} \, dy \\
& \quad + C \sum_{j=1}^\infty \frac{|b_{2^{j+1}B}-b_B|}{|2^{j+1}B|^{1/n}} \int_{2^{j+1}B \setminus 2^jB} \frac{\Omega(x-y)|b(y)| f(y)}{|x-y|^{n-1}} \, dy \\
& = III + IV.
\end{align*}
\]

When \( \Omega \in L^\infty(S^{n-1}) \), then it follows from Hölder’s inequality that

\[
\begin{align*}
III & \leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(y)-b_{2^{j+1}B}| \|f\| \, dy \\
& \leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \left( \int_{2^{j+1}B} |b(y)-b_{2^{j+1}B}|^{p'} w^{-p'/p} \, dy \right)^{1/p'} \\
& \quad \times \left( \int_{2^{j+1}B} |f(y)|^p w(y) \, dy \right)^{1/p} \\
& \leq C \|\Omega\|_{L^\infty(S^{n-1})} \|f\|_\infty \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \cdot w(2^{j+1}B)^{\kappa/p} \\
& \quad \times \left( \int_{2^{j+1}B} |b(y)-b_{2^{j+1}B}|^{p'} w^{-p'/p} \, dy \right)^{1/p'}.
\end{align*}
\]

Set \( u(y) = w^{-p'/p}(y) = w^{1-p'}(y) \). In this case, since \( w \in A_p \), then we have \( u \in A_{p'} \), it follows from the inequality (8) and the \( A_p \) condition that

\[
\begin{align*}
III & \leq C \|\Omega\|_{L^\infty(S^{n-1})} \|b\|_\infty \|f\|_\infty \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} w(2^{j+1}B)^{\kappa/p} \cdot u(2^{j+1}B)^{1/p'} \\
& \leq C \|\Omega\|_{L^\infty(S^{n-1})} \|b\|_\infty \|f\|_\infty \sum_{j=1}^\infty w(2^{j+1}B)^{(\kappa-1)/p}. \quad (19)
\end{align*}
\]
When $\Omega \in L^q(S^{n-1})$, then by using Hölder’s inequality, the inequalities (3) and (12), we can deduce

$$III \leq C\|\Omega\|_{L^q(S^{n-1})} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1/q'}} \left( \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^{q'} |f(y)|^{q'} dy \right)^{1/q'}$$

$$\leq C\|\Omega\|_{L^q(S^{n-1})} \|b\|_* \|f\|_{L^{p,\kappa}(w)} \cdot \sum_{j=1}^{\infty} w(2^{j+1}B)^{(\kappa-1)/p}.$$  \hfill (20)

Hence, for $1 < q \leq \infty$, $q' < p < \infty$, by the estimates (19) and (20), we get

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B II \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} w(B)^{(1-\kappa)/p} \sum_{j=1}^{\infty} w(2^{j+1}B)^{(1-\kappa)/p} \right) \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w)}.$$  \hfill (19)

Again, in Theorem 4, we have already obtained the following inequality

$$\frac{1}{|2^{j+1}B|^{1/n}} \int_{2^{j+1}B \setminus 2^jB} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| dy \leq C\|f\|_{L^{p,\kappa}(w)} \cdot w(2^{j+1}B)^{(\kappa-1)/p}.$$  \hfill (14)

From (14), it follows immediately that

$$IV \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w)} \sum_{j=1}^{\infty} j \cdot w(2^{j+1}B)^{(\kappa-1)/p}.$$  \hfill (15)

The rest of the proof is exactly the same as that of (15), we finally obtain

$$\frac{1}{w(B)^{\kappa/p}} \left( \int_B IV \|f\|_{L^{p,\kappa}(w)} \right)^{1/p} \leq C\|b\|_* \|f\|_{L^{p,\kappa}(w)}.$$  \hfill (20)

Therefore, by combining the above estimates and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we conclude the proof of Theorem 5.

\[ \square \]

References

[1] J. Alvarez, R. J. Bagby, D. S. Kurtz and C. Pérez, Weighted estimates for commutators of linear operators, Studia Math, 104(1993), 195-209.

[2] Y. Ding, D. Fan, Y. Pan, Weighted boundedness for a class of rough Marcinkiewicz integrals, Indiana Univ. Math. J, 48(1999), 1037–1055.
[3] Y. Ding, S. Lu, K. Yabuta, On commutators of Marcinkiewicz integrals with rough kernel, J. Math. Anal. Appl, 275(2002), 60–68.

[4] J. Duoandikoetxea, Weighted norm inequalities for homogeneous singular integrals, Trans. Amer. Math. Soc, 336(1993), 869–880.

[5] J. Duoandikoetxea, Fourier Analysis, American Mathematical Society, Providence, Rhode Island, 2000.

[6] J. Garcia-Cuerva and J. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.

[7] R. F. Gundy and R. L. Wheeden, Weighted integral inequalities for nontangential maximal function, Lusin area integral, and Walsh-Paley series, Studia Math, 49(1974), 107–124.

[8] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math, 14(1961), 415-426.

[9] Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, Math. Nachr, 282(2009), 219–231.

[10] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc, 43(1938), 126–166.

[11] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc, 165(1972), 207–226.