On the codescent of étale wild kernels in $p$-adic Lie extensions

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Abstract

Let $F$ be a number field. We estimate the kernels and cokernels of the codescent maps of the étale wild kernels over various $p$-adic Lie extensions. For this, we propose a novel approach of viewing the étale wild kernel as an appropriate fine Selmer group in the sense of Coates-Sujatha. This viewpoint reduces the problem to a control theorem of the said fine Selmer groups, which in turn allows us to employ the strategies developed by Mazur and Greenberg. As applications of our estimates on the kernels and cokernels of the codescent maps, we establish asymptotic growth formulas for the étale wild kernels in the various said $p$-adic Lie extensions. We then relate these growth formulas to Greenberg’s conjecture (and its noncommutative analogue). Finally, we shall give some examples to illustrate our results.

Keywords and Phrases: Étale wild kernels, fine Selmer groups, $p$-adic Lie extensions, Greenberg’s conjecture.

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1 Introduction

Throughout the paper, $p$ will denote a fixed prime. Let $F$ be a number field. In the event that $p = 2$, assume further that $F$ has no real primes. The classical wild kernel $K_w^2(F)$ fits into the Moore exact sequence

$$0 \rightarrow K_w^2(F) \rightarrow K_2(F) \rightarrow \mu(F) \rightarrow \mu(F) \rightarrow 0$$

(cf. [46, p. 157]). Its $p$-primary part can be described as the kernel of the localization map

$$H^2(G_S(F), \mathbb{Z}_p(2)) \rightarrow \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(2))$$

of continuous cohomology groups, where $S$ is a finite set of primes of $F$ containing those primes above $p$ and infinity. Inspired by this latter observation, Kolster [27] and Nguyen Quang Do [49, 51] defined higher analog of the $p$-primary part of the classical wild kernel which they coined as the étale wild kernels $WK_{2i}^G(F)$ (see Subsection 3.3 for the precise definition). We should also remark that this étale wild kernel has been studied in a slightly different guise by Schneider [53]. These étale wild kernels can be thought as analogues of the $(p$-primary) ideal class groups (see [1]) and they are central in many deep questions in algebraic number theory. For instances, the conjecture of Kummer–Vandiver can be reformulated in terms of the wild kernels (see [6]). More importantly, they can be related to the special values of the
Dedekind zeta function which is an equivalent form of Lichtenbaum conjecture (see [4, 5, 28, 36]; also see [53]). We should mention that this latter conjecture is known when \( F \) is either a totally real field or an abelian field (cf. [7, 30]) building on the main conjecture proved by Wiles [64]. Recently, Nickel has investigated annihilation problem of the étale wild kernel and showed that such problem are related to the equivariant Tamagawa number conjecture (see [52]).

A curious observation is that despite étale wild kernel’s intimate relation in Iwasawa theory (see [27, 49, 51, 52]), the asymptotic behaviour of the étale wild kernels (in the spirit of [25]) over a \( p \)-adic Lie extension has not been studied before. In fact, to the best knowledge of the author, even the case of \( \mathbb{Z}_p \)-extension does not seem to be written down in literature. The aim of the paper is therefore to fill in this gap. Namely, we will establish asymptotic growth formulas for the étale wild kernels in a \( p \)-adic Lie extension \( F_\infty \). For this, one is led to analysing the codescent map

\[
\left( \lim_{\leftarrow} W K_{2i}^{et}(F_n) \right)_{G_n} \longrightarrow W K_{2i}^{et}(F_n),
\]

where the \( F_n \)'s are certain appropriate subextensions of \( F_\infty /F \) with \( G_n = \text{Gal}(F_\infty/F_n) \). We note that for a finite \( p \)-extension \( M/F \), the kernel and cokernel of this codescent map has been well studied in [1, 2, 29]. However, despite their explicit nature, it does not seem easy to obtain good enough estimates from these mentioned works on the kernels and cokernels for our problem in hand.

Therefore, we shall take a different approach which we explain briefly here. Firstly, via the Poitou-Tate duality, we identify the Pontryagin dual of the étale wild kernel with the kernel of the following localization maps

\[
H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(-i)) \longrightarrow \bigoplus_{v \in S} H^1(F_v, \mathbb{Q}_p/\mathbb{Z}_p(-i)).
\]

(In fact, this is the form that Schneider [53] works with.) Following Coates-Sujatha [11], we shall call this the fine Selmer group of \( \mathbb{Q}_p/\mathbb{Z}_p(-i) \) over \( F \), which we denote by \( R_i(F) \). Under this identification, the Pontryagin dual of \( \lim_{\leftarrow} W K_{2i}^{et}(L) \) turns out to be isomorphic to the kernel of

\[
H^1(G_{S}(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(-i)) \longrightarrow \bigoplus_{w \in S(F_\infty)} H^1(F_{\infty,w}, \mathbb{Q}_p/\mathbb{Z}_p(-i)),
\]

which will be denoted by \( R_i(F_{\infty}) \). Therefore, the question of studying the kernel and cokernel of the codescent map is equivalent to understanding the kernel and cokernel of the restriction maps

\[
R_i(F_n) \longrightarrow R_i(F_\infty)^{G_n}.
\]

This viewpoint opens up a channel for us to mimic the techniques developed by Mazur [44] and Greenberg [18] for the study of Selmer groups of abelian varieties (also see [31, 40]).

However, we should emphasize that this improvisation is not a direct procedure. Indeed, in tackling these sorts of problems, one is naturally led to the problem of estimating the growth of \( H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(i)(F_\infty)) \) (for \( k = 1, 2 \)) as \( n \) varies (see discussion in the beginning of Section 4), where \( \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty) = (\mathbb{Q}_p/\mathbb{Z}_p(-i))_{G_\infty(F_\infty)} \). The work of Greenberg [18] utilized a Lie algebraic approach which is useful in showing the finiteness of these cohomology groups. However, for our purposes in hand, we also need to
know the growth rate of these cohomology groups which does not seem to be accessible directly from the Lie algebraic approach. In this paper, we shall therefore study these cohomology groups directly. We succeed in obtaining estimates for these cohomology groups when the $p$-adic Lie extension is either a $\mathbb{Z}_p^d$-extension (see Subsection 4.2), a multi-False-Tate extension (see Subsection 4.3), a $GL_2$-extension cut out by an elliptic curve without complex multiplication in (see Subsection 4.4), or a compositum of a $GL_2$-extension with multi-False-Tate extension (see Subsection 4.5). Interestingly, in every of these cases, the growth of the $p$-exponent of the group $H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(i)(F_\infty))$ can be shown to be $O(n)$. Despite this uniform outcome, we do not have a unified elegant way of proving these simultaneously, and have to resort to a case-by-case analysis.

We shall say a little on this analysis and leave the details to Section 4. For a $\mathbb{Z}_p^d$-extension, it is well-known that the $H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(i)(F_\infty))$ are finite (see [18, 58]). Our new insight towards obtaining a growth estimate relies on an observation of Cuoco-Monsky [12] that these cohomology groups have exponents bounded by $p^{dn+c}$ for some constant $c$ independent of $n$. It would seem that the utilization of such observation is absent in the work of Greenberg [18]. Unfortunately, this observation of Cuoco-Monsky does not carry over for noncommutative $p$-adic extensions. Thankfully, for these noncommutative extensions considered, we are able to call upon Tate’s Lemma (see Lemma 3.1) which is possible due to the Pontryagin dual of the étale wild kernel being the kernel of cohomology groups with coefficient in $\mathbb{Q}_p/\mathbb{Z}_p(-i)$. The utilization of Tate’s Lemma is crucial for us in making headway towards obtaining estimates on the cohomology groups $H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(i)(F_\infty))$ as mentioned in the previous paragraph.

As applications of these estimates, we obtain asymptotic growth formulas for the étale wild kernels in the various said $p$-adic Lie extensions. We then relate the growth of étale wild kernels to Greenberg’s conjecture. Note that Greenberg’s conjecture was originally stated over the multiple $\mathbb{Z}_p$-extension (see [17, 60]), but there are natural analogue formulations of it over certain noncommutative $p$-adic Lie extensions (see [11, 20]; also see Subsection 3.5). We shall see that Greenberg’s conjecture placed a constraint on the growth rate of the étale wild kernels in such a $p$-adic Lie extension. For some of these $p$-adic Lie extensions, we give an equivalent characterization of Greenberg’s conjecture in term of the growth of étale wild kernels (see Corollaries 5.4 and 5.6).

We say something briefly on the situation $p = 2$ and the number field $F$ has no real primes. In this case, there are extra complications from technical cohomological considerations. It would seem that the object to consider in this situation is the so-called positive étale wild kernel as developed in [3]. We believe there should be some scope for further work in this aspect, although this will not be pursued here.

We end the introductional section giving an outline of the paper. In Section 2 we collect preliminary facts on uniform $p$-groups and modules over their Iwasawa algebras. We also recall certain standard bounds on the cohomology groups. In Section 3 we introduce the étale wild kernels and describe how they can be interpreted as appropriate fine Selmer groups in the sense of [11, 38]. We also review the Iwasawa $\mu$-conjecture and Greenberg’s conjecture which will be required for our subsequent discussion. Section 4 is where we analyse the codescent of étale wild kernels in $p$-adic Lie extensions. These will be applied in Section 5 to obtain growth formulas for the étale wild kernels. We then discuss the connections between these formulas and Greenberg’s conjecture. Finally, in Section 6 we give several examples to illustrate our results. Interestingly, in some of these examples, we can even obtain better
and unconditional bounds on the growth of the étale wild kernel than those predicted by Greenberg’s conjecture (see Proposition 6.2) by building on certain calculations of Sharifi [55].

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2 Uniform pro-\(p\) groups

We collect here several useful results on uniform pro-\(p\) groups that will be required later in the paper. For a finitely generated pro-\(p\) group \(G\), write \(G^{p^n} = \langle g^{p^n} | g \in G \rangle\), in other words, the group generated by the \(p^n\)-th-powers of elements in \(G\). We also write \(G^{(p^n)} = \{g^{p^n} | g \in G\}\), which is the set consisting of the \(p^n\)-th-powers of elements in \(G\). The pro-\(p\) group \(G\) is said to be powerful if \(G/G^{p}\) is abelian. The lower \(p\)-series of \(G\) is defined by \(P_1(G) = G\), and \(P_{n+1}(G) = P_n(G)p[P_n(G), G]\), for \(n \geq 1\).

For a powerful group \(G\), it follows from [13, Theorem 3.6] that \(G^{p^n} = G^{(p^n)} = P_{n+1}(G)\). Furthermore, the \(p\)-power map

\[
P_n(G)/P_{n+1}(G) \xrightarrow{\text{p}} P_{n+1}(G)/P_{n+2}(G)
\]

is surjective for every \(n \geq 1\). If the \(p\)-power maps are isomorphisms for all \(n \geq 1\), we then say that the group \(G\) is uniformly powerful (abbrv. uniform). Then \([G : P_2(G)] = [P_n(G) : P_{n+1}(G)]\) for every \(n \geq 1\). Consequently, one has \([G : P_{n+1}(G)] = p^{dn}\), where \(d = \text{dim } G\) (see [13, Definition 4.1]). A well-known result of Lazard (cf. [13, Corollary 8.34]) asserts that a compact \(p\)-adic Lie group always contains an open normal uniform subgroup. Therefore, one can always reduce consideration for a general compact \(p\)-adic Lie group to the case of a uniform group, which we will do throughout the paper. In particular, for a uniform group, we have \(G^{p^n} = G^{(p^n)}\). This latter fact and the following lemma will be utilized without further mention.

Lemma 2.1. Let \(G\) be a uniform group and \(N\) a closed normal subgroup of \(G\) such that \(R := G/N\) is uniform. Then \(N\) is also uniform. Furthermore, writing \(N_n = N^{p^n}\), \(G_n = G^{p^n}\), and \(R_n = R^{p^n}\), we have \(N_n = G_n \cap N\) and \(G_n/N_n \cong R_n\).

Proof. See [24, Lemma 2.6].
2.1 Iwasawa invariants

Let $G$ denote a uniform pro-$p$ group. The completed group algebra of $G$ over $\mathbb{Z}_p$ is given by

$$\mathbb{Z}_p[[G]] = \lim\limits_{\rightarrow} \mathbb{Z}_p[G/U],$$

where $U$ runs over the open normal subgroups of $G$ and the inverse limit is taken with respect to the canonical projection maps. It is well known that $\mathbb{Z}_p[[G]]$ is a Noetherian Auslander regular ring without zero divisors (cf. [61, Theorem 3.26] or [38, Theorem A.1]; also see [43]). Hence it admits a skew field $Q(G)$ which is flat over $\mathbb{Z}_p[[G]]$ (see [16, Chapters 6 and 10] or [32, Chapter 4, §9 and §10]). As a consequence, one can define the $\mathbb{Z}_p[[G]]$-rank of a finitely generated $\mathbb{Z}_p[[G]]$-module $M$ by setting

$$\text{rank}_{\mathbb{Z}_p[[G]]}(M) = \dim_{Q(G)}(Q(G) \otimes_{\mathbb{Z}_p[[G]]} M).$$

The $\mathbb{Z}_p[[G]]$-module $M$ is said to be torsion if $\text{rank}_{\mathbb{Z}_p[[G]]}(M) = 0$. It is a standard fact that $M$ is torsion over $\mathbb{Z}_p[[G]]$ if and only if $\text{Hom}_{\mathbb{Z}_p[[G]]}(M, \mathbb{Z}_p[[G]]) = 0$ (for instance, see [38, Lemma 4.2]). In the event that the torsion $\mathbb{Z}_p[[G]]$-module $M$ satisfies $\text{Ext}^1_{\mathbb{Z}_p[[G]]}(M, \mathbb{Z}_p[[G]]) = 0$, we shall say that $M$ is a pseudo-null $\mathbb{Z}_p[[G]]$-module.

For a finitely generated $\mathbb{Z}_p[[G]]$-module $M$, denote by $M[p^\infty]$ the $\mathbb{Z}_p[[G]]$-submodule of $M$ consisting of elements of $M$ annihilated by some power of $p$. Howson [23, Proposition 1.11], and independently Venjakob [61, Theorem 3.40], showed that there is a $\mathbb{Z}_p[[G]]$-homomorphism

$$\varphi : M[p^\infty] \rightarrow \bigoplus_{i=1}^s \mathbb{Z}_p[[G]]/p^{\alpha_i},$$

whose kernel and cokernel are pseudo-null $\mathbb{Z}_p[[G]]$-modules, and where the integers $s$ and $\alpha_i$ are uniquely determined. The $\mu_G$-invariant of $M$ is then defined to be $\mu_G(M) = \sum_{i=1}^s \alpha_i$.

2.2 Some estimates on cohomology groups

In this subsection, we record certain basic estimates on the cohomology groups. For an abelian group $M$, denote by $M[p^j]$ the subgroup of $M$ consisting of elements of $M$ annihilated by $p^j$. In particular, we have $M[p^\infty] = \cup_{j \geq 1} M[p^j]$. For a discrete $p$-primary abelian group or a compact pro-$p$ abelian group $M$, its Pontryagin dual is defined by $M^\vee = \text{Hom}_{\text{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$, i.e., the set of continuous group homomorphisms from $M$ to $\mathbb{Q}_p/\mathbb{Z}_p$, where $\mathbb{Q}_p/\mathbb{Z}_p$ is given the discrete topology. If $G$ is a profinite group and $M$ a $G$-module, we let $M^G$ denote the subgroup of $M$ consisting of elements fixed by $G$, and set $M_G$ to be the largest quotient of $M$ on which $G$ acts trivially. In particular, if $M$ is a discrete $G$-module, we denote by $H^k(G, M)$ the $k$-th Galois cohomology group of $G$ with coefficients in $M$. For a finite $p$-group $N$, we write $\text{ord}_p(N)$ for the $p$-exponent of $N$, i.e., $|N| = p^{\text{ord}_p(N)}$.

Lemma 2.2. Let $G$ be a pro-$p$ group. Suppose that $M$ is a discrete $G$-module which is cofinitely generated over $\mathbb{Z}_p$. Write $M_{\text{div}}$ for the maximal $p$-divisible subgroup of $M$. If $h_1(G) = \dim_{\mathbb{Z}/p\mathbb{Z}}(H^1(G, \mathbb{Z}/p\mathbb{Z}))$ is finite, then

$$\dim_{\mathbb{Z}/p\mathbb{Z}}(H^1(G, M)[p]) \leq h_1(G) \left( \text{corank}_{\mathbb{Z}_p}(M) + \text{ord}_p(M/M_{\text{div}}) \right).$$
If $h_2(G) = \dim_{\mathbb{Z}/p\mathbb{Z}} (H^2(G, \mathbb{Z}/p\mathbb{Z}))$ is finite, then
\[
\dim_{\mathbb{Z}/p\mathbb{Z}} (H^2(G, \mathbb{Z}/p\mathbb{Z})[p]) \leq h_2(G) \left( \text{corank}_{\mathbb{Z}/p\mathbb{Z}} (M) + \text{ord}_p(M/M_{\text{div}}) \right).
\]

Proof. The first inequality is proven in [42, Lemma 3.2]. The second inequality is proven similarly. □

**Lemma 3.3.** Let $G$ be a pro-$p$ group. Suppose that $M$ is a finite discrete $G$-module. If $h_1(G) = \dim_{\mathbb{Z}/p\mathbb{Z}} (H^1(G, \mathbb{Z}/p\mathbb{Z}))$ is finite, then $H^1(G, M)$ is finite with
\[
\text{ord}_p(H^1(G, M)) \leq h_1(G) \text{ ord}_p(M).
\]

If $h_2(G) = \dim_{\mathbb{Z}/p\mathbb{Z}} (H^2(G, \mathbb{Z}/p\mathbb{Z}))$ is finite, then $H^2(G, M)$ is finite with
\[
\text{ord}_p(H^2(G, M)) \leq h_2(G) \text{ ord}_p(M).
\]

Proof. This follows from a standard dévissage argument and noting that the only simple discrete $G$-module is $\mathbb{Z}/p\mathbb{Z}$ with trivial $G$-action (cf. [47, Corollary 1.6.13]). □

### 3 Arithmetic preliminaries

#### 3.1 Tate twist

Let $K$ be any field of characteristic $\neq p$. Denote by $\mu_{p^n}$ the group of all the $p$-power roots of unity contained in a fixed separable closure $K^{\text{sep}}$ of $K$. The natural action of $\text{Gal}(K^{\text{sep}}/K)$ on $\mu_{p^n}$ induces a continuous character
\[
\chi : \text{Gal}(K^{\text{sep}}/K) \rightarrow \text{Aut}(\mu_{p^n}) \cong \mathbb{Z}_p^\times.
\]

If $M$ is either a discrete or compact $\text{Gal}(K^{\text{sep}}/K)$-module, we shall write $M(i)$ for the $\text{Gal}(K^{\text{sep}}/K)$-module which is $M$ as an abelian group but with a $\text{Gal}(K^{\text{sep}}/K)$-action given by
\[
\sigma \cdot x = \chi(\sigma)^i \sigma x,
\]
where the action on the right is the original action of $\text{Gal}(K^{\text{sep}}/K)$ on $M$. Plainly, we have $M(0) = M$ and $\mu_{p^n} \cong \mathbb{Q}_p/\mathbb{Z}_p(1)$. One also checks directly that
\[
\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(i), \mathbb{Q}_p/\mathbb{Z}_p(j)) = \mathbb{Q}_p/\mathbb{Z}_p(j - i) \quad \text{and} \quad \text{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p(i), \mathbb{Q}_p/\mathbb{Z}_p(j)) = \mathbb{Z}_p(j - i).
\]

The continuous character
\[
\chi : \text{Gal}(K^{\text{sep}}/K) \rightarrow \text{Aut}(\mu_{p^n}) \cong \mathbb{Z}_p^\times.
\]

naturally induces a group homomorphism
\[
\kappa : \text{Gal}(\mu_{p^n})/K \rightarrow \text{Aut}(\mu_{p^n}) \cong \mathbb{Z}_p^\times
\]

which is sometimes called the $(p$-adic) cyclotomic character. We now record the following well-known lemma (cf. [60]) which will be frequently used in this article.
Lemma 3.1 (Tate’s Lemma). Suppose that $H^0(\text{Gal}(K^{\text{sep}}/K), \mathbb{Q}_p/\mathbb{Z}_p(i))$ is finite and that the Galois group $\text{Gal}(K(\mu_{p^\infty})/K)$ is infinite. Then one has

$$H^1(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0.$$ 

Proof. By base-changing, we may assume that $\Gamma := \text{Gal}(K(\mu_{p^\infty})/K) \cong \mathbb{Z}_p$. Then $\mathbb{Q}_p/\mathbb{Z}_p(i)$ can be viewed as a $\mathbb{Z}_p[[\Gamma]]$-module. Plainly, it is cotorsion over $\mathbb{Z}_p[[\Gamma]]$ and so it follows from [47, Proposition 5.3.20] that

$$\text{corank}_{\mathbb{Z}_p[[\Gamma]]}\left(H^1(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i))\right) = \text{corank}_{\mathbb{Z}_p[[\Gamma]]}\left(H^0(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i))\right).$$

Since $H^0(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^0(\text{Gal}(K^{\text{sep}}/K), \mathbb{Q}_p/\mathbb{Z}_p(i))$ is finite by the hypothesis, so is $H^1(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i))$. But by [47, Proposition 1.7.7], $H^1(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i))$ is a quotient of $\mathbb{Q}_p/\mathbb{Z}_p(i)$ which is a $p$-divisible group, and whence, we must have

$$H^1(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0$$

which is what we want to show. \[\square\]

The lemma in particular applies when $K$ is either a finite extension of $\mathbb{Q}_p$ or $\mathbb{Q}$, and $i \neq 0$.

3.2 Iwasawa cohomology groups

Let $F$ be a number field, and $S$ a finite set of primes of $F$ containing all the primes above $p$ and the infinite primes. Denote by $F_S$ the maximal extension of $F$ unramified outside $S$. For every extension $L$ of $F$ contained in $F_S$, we write $G_S(L) = \text{Gal}(F_S/L)$. For every $k \geq 0$, the continuous cohomology groups $H^k(G_S(L), \mathbb{Z}_p(i + 1))$ is defined to be

$$\lim_{m \to \infty} H^k(G_S(L), \mathbb{Z}/p^m\mathbb{Z}(i + 1)).$$

Since $p$ is odd, we have $H^k(G_S(L), \mathbb{Z}_p(i + 1)) = 0$ for $k \geq 3$ (cf. [47, Proposition 10.11.3]).

A Galois extension $F_{\infty}$ of $F$ is said to be a uniform $p$-adic Lie extension of $F$ if its Galois group $G = \text{Gal}(F_{\infty}/F)$ is a uniform pro-$p$ group. We shall always assume that our uniform pro-$p$ Lie extension $F_{\infty}$ is contained in $F_S$ for some finite set $S$. The Iwasawa cohomology group $H^k_{\text{Iw},S}(F_{\infty}/F, \mathbb{Z}_p(i + 1))$ is then defined to be

$$H^k_{\text{Iw},S}(F_{\infty}/F, \mathbb{Z}_p(i + 1)) = \lim_{L \to \infty} H^k(G_S(L), \mathbb{Z}_p(i + 1)),$$

where $L$ runs through all the finite extensions of $F$ contained in $F_{\infty}$ and the transition maps are given by corestriction on cohomology. It can be shown that these cohomology groups are finitely generated over $\mathbb{Z}_p[[G]]$ (for instance, see [43, Proposition 4.1.3]).

In this paper, we are mostly interested in the case when $i \geq 1$. Indeed, in this situation, it is by now well known that these cohomology groups are related to algebraic $K$-groups. More precisely, we have an isomorphism

$$K_{2i}(\mathcal{O}_{F,S})[p^\infty] \cong H^2(G_S(F), \mathbb{Z}_p(i + 1))$$
which is induced by the $p$-adic Chern class maps of Soulé [59]. We should mention that this isomorphism is a consequence of Rost and Voevodsky [63] (also see [11] Section 3.2 and references therein for more details). Building on the well known fact that $H^2(G_S(L),\mathbb{Z}_p(i+1))$ is finite for every finite extension $L$ of $F$ (cf. [9] [59]), one can even establish the following for the second Iwasawa cohomology groups.

**Lemma 3.2.** Assume that $i \geq 1$. Let $F_\infty$ be a uniform $p$-adic Lie extension of $F$ contained in $F_S$. Then the module $H^2_{\text{Iw},S}(F_\infty/F,\mathbb{Z}_p(i+1))$ is torsion over $\mathbb{Z}_p[[\text{Gal}(F_\infty/F)]]$.

*Proof.* See [11] Proposition 4.1.1. \hfill \Box

**Remark 3.3.** Although we are concerned with the situation $i \geq 1$, we should mention that Schneider has conjectured that $H^2(G_S(F),\mathbb{Q}_p/\mathbb{Z}_p(i+1))$ is finite for $i < 0$ (see [53] p. 192)). Granted this conjecture, the argument in [11] Proposition 4.1.1 will carry over to obtain the conclusion of the preceding lemma for $i < 0$.

### 3.3 Étale wild kernel

Retain the settings of Subsection 3.2. For each $v \in S$, we have a continuous group homomorphism

$$\text{Gal}(\bar{F}_v/F_v) \to \text{Gal}(\bar{F}/F) \to \text{Gal}(F_S/F).$$

This in turn induces a natural map on cohomology

$$H^k(G_S(F), M) \to H^k(F_v, M)$$

for every $G_S(F)$-module $M$.

Let $i$ be a fixed positive integer. Following [11, 27, 29, 49, 51, 52], the étale wild kernel $WK^{\text{et}}_{2i}(F)$ is defined by

$$\text{ker} \left( H^2(G_S(F),\mathbb{Z}_p(i+1)) \to \bigoplus_{v \mid p} H^2(F_v,\mathbb{Z}_p(i+1)) \right).$$

By Poitou-Tate duality, the Pontryagin dual of the étale wild kernel fits into the following exact sequence

$$0 \to WK^{\text{et}}_{2i}(F)^{\vee} \to H^1(G_S(F),\mathbb{Q}_p/\mathbb{Z}_p(-i)) \to \bigoplus_{v \in S} H^1(F_v,\mathbb{Q}_p/\mathbb{Z}_p(-i)).$$

In other words, the Pontryagin dual of the étale wild kernel can be thought as the fine Selmer group with coefficient in $\mathbb{Q}_p/\mathbb{Z}_p(-i)$ (in the sense of [11] [38]). We shall write $R_i(F) = WK^{\text{et}}_{2i}(F)^{\vee}$. Note that the étale wild kernel is independent of the choice of $S$ (cf. [15] Corollary 6.6)).

Let $F_\infty$ be a uniform $p$-adic Lie extension of $F$ contained in $F_S$. We can define $R_i(L)$ and $WK^{\text{et}}_{2i}(L)$ for every finite extension $L$ of $F$ similarly. We then write $R_i(F_\infty) = \lim L \to R_i(L)$ and $Y_i(F_\infty) = \lim L \to WK^{\text{et}}_{2i}(L)$, where $L$ runs through all the finite extensions of $F$ contained in $F_\infty$. Note that $R_i(F_\infty) = Y_i(F_\infty)^{\vee}$.

**Lemma 3.4.** Let $F_\infty$ be a uniform $p$-adic Lie extension of $F$ contained in $F_S$. Then $Y_i(F_\infty)$ is torsion over $\mathbb{Z}_p[[\text{Gal}(F_\infty/F)]]$.

*Proof.* Since $Y_i(F_\infty)$ is contained in $H^2_{\text{Iw},S}(F_\infty/F,\mathbb{Z}_p(i+1))$, the conclusion follows from Lemma 3.2. \hfill \Box
3.4 Iwasawa $\mu$-conjecture

For a given $p$-adic Lie extension $\mathcal{L}$ of $F$, denote by $K(\mathcal{L})$ the maximal unramified abelian pro-$p$ extension of $\mathcal{L}$ in which every prime above $p$ splits completely.

The cyclotomic $\mathbb{Z}_p$-extension of $F$ will always be denoted by $F^{\text{cyc}}$. A well-known theorem of Iwasawa [26] asserts that $\text{Gal}(K(F^{\text{cyc}})/F^{\text{cyc}})$ is a torsion $\mathbb{Z}_p[\Gamma]$-module, where $\Gamma = \text{Gal}(F^{\text{cyc}}/F)$. In fact, Iwasawa conjectured the following.

**Conjecture 3.5** (Iwasawa $\mu$-conjecture). The group $\text{Gal}(K(F^{\text{cyc}})/F^{\text{cyc}})$ is finitely generated over $\mathbb{Z}_p$, or equivalently, is a torsion $\mathbb{Z}_p[[\Gamma]]$-module with trivial $\mu_\Gamma$-invariant.

**Remark 3.6.** When $F$ is an abelian extension of $\mathbb{Q}$, the conjecture is known to be valid by the theorem of Ferrero-Washington [14].

We now record certain consequences of the Iwasawa $\mu$-conjecture in the form of the following two lemmas. These will be required for our subsequent discussion.

**Lemma 3.7.** Let $F$ be a number field. Suppose that the Iwasawa $\mu$-conjecture holds for $F(\mu_p)^{\text{cyc}}$. Then $Y_i(F^{\text{cyc}})$ is finitely generated over $\mathbb{Z}_p$.

**Proof.** By [38, Lemma 3.2], it suffices to show that $Y_i(F(\mu_p)^{\text{cyc}})$ is finitely generated over $\mathbb{Z}_p$. Thus, we may assume that $F$ contains $\mu_p$. In this situation, the $\mathbb{Z}_p$-finite generation of $Y_i(F^{\text{cyc}})$ is equivalent to the Iwasawa $\mu$-conjecture by [38, Theorem 3.5].

**Lemma 3.8.** Let $F_\infty$ be a pro-$p$ $p$-adic Lie extension of a number field $F$ containing $F^{\text{cyc}}$. Suppose that $Y_i(F^{\text{cyc}})$ is finitely generated over $\mathbb{Z}_p$. Then $Y_i(F_\infty)$ is finitely generated over $\mathbb{Z}_p[[H]]$, where $H = \text{Gal}(F_\infty/F^{\text{cyc}})$.

**Proof.** See [38, Lemma 5.2].

3.5 Greenberg’s conjecture

We now come to recalling the following conjecture of Greenberg [17, Conjecture 3.5] (also see [50, Conjecture 4.7]).

**Conjecture 3.9.** Let $F$ be a number field and $\bar{F}$ the compositum of all $\mathbb{Z}_p$-extensions of $F$. Then $\text{Gal}(K(\bar{F})/\bar{F})$ is a pseudo-null $\mathbb{Z}_p[[\text{Gal}(\bar{F}/F)]]$-module.

Actually, Greenberg’s conjecture is concerned with the pseudo-nullity of a slightly bigger Galois group. For a discussion of the relation between the original conjecture of Greenberg and the slightly weaker version adopted here, we refer readers to [33, Subsection 4.2].

The naive noncommutative analogue of Greenberg’s conjecture is false in general (see [20, Section 5] for counterexamples). However, if $F_\infty$ “comes from algebraic geometry” in the sense of Fontaine-Mazur [15], it seems quite plausible that a direct analog of Greenberg’s conjecture holds, namely, $\text{Gal}(K(F_\infty)/F_\infty)$ is a pseudo-null $\mathbb{Z}_p[[\text{Gal}(F_\infty/F)]]$-module (see [20, Question 1.3]).
We shall primarily be interested in two classes of noncommutative $p$-adic Lie extensions in this paper. The first of which is the False-Tate extension $F(\mu_{p^\infty}, \alpha^{-p^\infty})$. Some cases of pseudo-nullity have been established by Sharifi \cite{SS2}. We remark that Sharifi also has some positive results pertaining to the original Greenberg’s conjecture for $\mathbb{Q}(\mu_p)$ (for $p < 1000$). The other class of noncommutative $p$-adic Lie extensions that will be considered in the paper is the extension obtained by adjoining all the $p$-division points of an elliptic curve $E$ without complex multiplication. In this situation, the analogue of Greenberg’s conjecture turns out to be equivalent to the Conjecture B of Coates-Sujatha which is concerned with the pseudo-nullity of the fine Selmer group of the elliptic curve $E$ in question (see \cite{SS} Section 4).

For our purposes, we have the following analogous observation for the étale wild kernel.

**Lemma 3.10.** Suppose that $F$ contains $\mu_p$ and that $F_\infty$ is a uniform $p$-adic Lie extension containing $F^{\text{cy}}$. Writing $G = \text{Gal}(F_\infty/F)$, we then have that $\text{Gal}(K(F_\infty)/F_\infty)$ is a pseudo-null $\mathbb{Z}_p[[G]]$-module if and only if $Y_i(F_\infty)$ is a pseudo-null $\mathbb{Z}_p[[G]]$-module. In particular, the pseudo-nullity of $Y_i(F_\infty)$ is independent of $i \geq 1$.

**Proof.** We first show that every non-archimedean prime of $F_\infty$ must split completely in $K(F_\infty)$. Indeed this automatically holds for the primes above $p$ from the definition of $K(F_\infty)$. For a non-archimedean prime $w$ of $F_\infty$ outside $p$, write $v$ for the prime of $F$ below $w$. Since $v$ does not divide $p$, it follows from \cite{Kato} Theorem 7.5.3 that the maximal pro-$p$ extension of $F_v$ is of dimension 2, whose inertia subgroup is of dimension 1. Since $F^{\text{cy}} \subseteq F_\infty$, the prime $v$ is finitely decomposed and unramified in $F^{\text{cy}}/F$. In view of the fact that $K(F_\infty)/F_\infty$ is unramified, the prime $w$ must therefore split completely in $K(F_\infty)/F_\infty$. This establishes our claim.

We now make use of the above claim to establish the following exact sequence

$$0 \to \text{Gal}(K(F_\infty)/F_\infty) \to H^2_{\text{Iw},S}(F_\infty/F, \mathbb{Z}_p(1)) \to \lim_{L \mid F_\infty} \bigoplus_{w_L \in S(L)} H^2(L_w, \mathbb{Z}_p(1)).$$

where $S(L)$ denotes the set of primes of $L$ above $S$. Recall that it follows from the Poitou-Tate sequence (see \cite{Kato}) that for every finite extension $L$ of $F$ contained in $F_\infty$, we have the following exact sequence

$$0 \to \text{Gal}(K_S(L)/L) \to H^2(G_S(L), \mathbb{Z}_p(1)) \to \lim_{L \mid F_\infty} \bigoplus_{w_L \in S(L)} H^2(L_w, \mathbb{Z}_p(1)),$$

where $K_S(L)$ is the maximal unramified abelian pro-$p$ extension of $L$ in which every prime above $S(L)$ splits completely. (Note that $\text{Gal}(K_S(L)/L) \cong \text{Cl}_S(L)[p^\infty]$, where $\text{Cl}_S(L)$ is the $S$-class group of $L$.) Taking inverse limit, we obtain the exact sequence

$$0 \to \text{Gal}(K_S(F_\infty)/F_\infty) \to H^2_{\text{Iw},S}(F_\infty/F, \mathbb{Z}_p(1)) \to \lim_{L \mid F_\infty} \bigoplus_{w_L \in S(L)} H^2(L_w, \mathbb{Z}_p(1)).$$

Since $K_S(F_\infty) = K(F_\infty)$ by the claim proven in the first paragraph, this establishes the exact sequence that we require.

Since $F_\infty$ contains $\mu_{p^\infty}$, we may apply \cite{SB} Lemma 2.5.1(c)] to conclude that

$$H^2_{\text{Iw}}(F_\infty/F, \mathbb{Z}_p(i + 1)) \cong H^2_{\text{Iw}}(F_\infty/F, \mathbb{Z}_p(1)) \otimes \mathbb{Z}_p(i).$$
For each \( v \in S \), we fix a prime \( w_v \) of \( F_\infty \) above \( v \) and write \( G_v \) for the decomposition group of \( G \) at this said prime. Then there is an isomorphism

\[
\lim_L \bigoplus_{w_L \in S(L)} H^2(L_{w_L}, \mathbb{Z}_p(j)) \cong \bigoplus_{v \in S} \text{Ind}^{G_v}_{G} (H^2_{\text{Iw}}(F_\infty, w_v / F_v, \mathbb{Z}_p(j)))
\]

for every \( j \) (for instances, see [57, Lemma 5.3.2]), where the local Iwaswa cohomology group is defined analogously as in the global situation and \( \text{Ind}^{G_v}_{G} \) is the compact induction in the sense of [47, P. 737]. Now, a similar argument to that in [56, Lemma 2.5.1(c)] yields

\[
H^2_{\text{Iw}}(F_\infty, w_v / F_v, \mathbb{Z}_p(i + 1)) \cong H^2_{\text{Iw}}(F_\infty, w_v / F_v, \mathbb{Z}_p(1)) \otimes \mathbb{Z}_p(i).
\]

In view of these observations, upon applying \( - \otimes \mathbb{Z}_p(i) \) to the exact sequence obtained in the preceding paragraph, we have \( Y_i(F_\infty) = \text{Gal}(K(F_\infty) / F_\infty)(i) \). The conclusions of the lemma are now immediate from this.

\[\square\]

## 4 Codescent in \( p \)-adic extensions

We begin setting up notation which will be adhered throughout this section without further mention. Let \( F_\infty \) be a uniform \( p \)-adic Lie extension of \( F \) contained in \( F_S \) for some appropriate set \( S \) of primes. Write \( G = \text{Gal}(F_\infty / F) \) and \( G_n = G^p \). The fixed field of \( G_n \) is in turn denoted by \( F_n \). Note that \( |F_n : F| = p^{dn} \), where \( d \) is the dimension of \( G \). For each \( n \), there is a natural map \( r_n : R_n(F_n) \to R_n(F_\infty)^{G_n} \) induced by the restriction on cohomology. (Alternatively, one can also view this as the Pontryagin dual of the corestriction maps on cohomology.) The goal of this section is to estimate the kernel and cokernel of these maps in various \( p \)-adic Lie extensions. More precisely, we will carry out the analysis for a \( \mathbb{Z}_p \)-extension in Subsection 4.1, a \( \mathbb{Z}_p^d \)-extension in Subsection 4.2, a multi-False-Tate extension in Subsection 4.3, a certain \( GL_2 \)-extension in Subsection 4.4, and a compositum of a \( GL_2 \)-extension with multi-False-Tate extension in Subsection 4.5.

We now give an overview of the general approach, leaving the details to the respective subsections. Throughout the discussion, we shall denote by \( S(F_n) \) the set of primes of \( F_n \) above \( S \). Consider the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & R_n(F_n) & \to & H^1(G_S(F_n), \mathbb{Q}_p / \mathbb{Z}_p(-i)) & \to & \bigoplus_{v \in S(F_n)} H^1(F_{v, w_v}, \mathbb{Q}_p / \mathbb{Z}_p(-i)) \\
\Bigg| & & r_n & & h_n & & g_n \\
0 & \to & R_n(F_\infty)^{G_n} & \to & H^1(G_S(F_\infty), \mathbb{Q}_p / \mathbb{Z}_p(-i))^{G_n} & \to & \bigoplus_{w \in S(F_\infty)} H^1(F_{\infty, w}, \mathbb{Q}_p / \mathbb{Z}_p(-i)) \\
\end{array}
\]

with exact rows. The snake lemma yields an exact sequence

\[
0 \to \ker r_n \to \ker h_n \to C_n \to \text{coker } r_n \to \text{coker } h_n,
\]
where $C_n$ is a subgroup of $\ker g_n$. Therefore, in estimating $\ker r_n$ and $\coker r_n$, one is reduced to studying $\ker h_n$, $\coker h_n$ and $\ker g_n$. From the Hochschild-Serre spectral sequence, we see that

$$\ker h_n = H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)) \quad \text{and} \quad \coker h_n \subseteq H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)),$$

where we write $\mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty) = (\mathbb{Q}_p/\mathbb{Z}_p(-i))^{G_0(F_\infty)}$. This therefore leads us to estimating the groups $H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))$ and $H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))$.

To analyse $\ker g_n$, we first split it into

$$\bigoplus_{v \in S \setminus v_n} \ker g_{v_n},$$

where

$$g_{v_n} : H^1(F_{v_n}, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \to \bigoplus_{w | v_n} H^1(F_{\infty,w}, \mathbb{Q}_p/\mathbb{Z}_p(-i)).$$

Therefore, in estimating $\ker g_n$, we need to bound each $\ker g_{v_n}$ and the number of primes of $F_n$ above each $v$. By Shapiro’s lemma, we see that

$$\ker g_{v_n} = H^1\left( \text{Gal}(F_{\infty,w}/F_{v_n}), \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w}) \right)$$

for a fixed prime $w$ of $F_\infty$ above $v_n$, where $\mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w}) = (\mathbb{Q}_p/\mathbb{Z}_p(-i))^{\text{Gal}(F_{\infty,w}/F_{\infty})}$. Thus, the problem of bounding $\ker g_{v_n}$ is the same as bounding

$$H^1\left( \text{Gal}(F_{\infty,w}/F_{v_n}), \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w}) \right).$$

The estimates for the cohomology groups will be dealt with in each of the respective subsections. Here, we shall take care of the estimate on the number of primes above $v \in S$. If the prime $v$ splits completely in $F_\infty/F$, then $\text{Gal}(F_{\infty,w}/F_{v_n}) = 0$ and so $H^1\left( \text{Gal}(F_{\infty,w}/F_{v_n}), \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w}) \right) = 0$ for all $n$. It therefore suffices to consider the primes $v \in S$ which do not split completely in $F_\infty/F$. This is precisely the content of the next lemma.

**Lemma 4.1.** Let $F_\infty$ be a uniform $p$-adic Lie extension of $F$ of dimension $d$. Suppose that $v$ is a prime of $F$ such that the decomposition group of $G = \text{Gal}(F_\infty/F)$ at $v$ has dimension $t$. Then the number of primes of $F_n$ above $v$ is $O(p^{(d-t)n})$.

**Proof.** Fix a prime of $F_\infty$ above $v$. Let $G_{n,v}$ be the decomposition group of $G_n$ at this said prime. In other words, we have $G_{n,v} = G_n \cap G_v$, where $G_v = G_{v,0}$. By [13 Chap. 4, Exercise 14], there exists a constant $C_v$ such that $|G_n : G_{n,v}| = C_v p^{dn}$ for sufficiently large $n$. Since $|G : G_n| = p^{dn}$, we therefore have

$$|G/G_n : G_v/G_{n,v}| = (1/C_v)p^{(d-t)n} = O(p^{(d-t)n}).$$

But the index $|G/G_n : G_v/G_{n,v}|$ is precisely the number of primes of $F_n$ above $v$. Thus, this proves the lemma. \qed

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4.1 \( \mathbb{Z}_p \)-extension

As a start, we consider the case of a \( \mathbb{Z}_p \)-extension. In compliance with tradition when working over a \( \mathbb{Z}_p \)-extension, we shall write \( \Gamma = \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p \) and \( \Gamma_n = \Gamma^{p^n} \) here. Since \( \mathbb{Z}_p \)-extensions are unramified outside \( p \) (cf. [26, Theorem 1]), we may take \( S \) in the definition of the étale wild kernel to consist precisely the set of primes above \( p \) and infinite primes.

**Proposition 4.2.** Let \( i \geq 1 \) be given. The kernel and cokernel of the map \( r_n : R_i(F_n) \rightarrow R_i(F_\infty)^{\Gamma_n} \) are finite and bounded independently of \( n \).

**Proof.** We begin with the global cohomology groups

\[
H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)) \quad \text{and} \quad H^2(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)).
\]

Since \( \Gamma_n \cong \mathbb{Z}_p \), we have \( H^2(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)) = 0 \). Now, if \( \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty) \) is finite. Since \( \Gamma_n \) is procyclic, it follows from [47, Proposition 1.7.7] that \( H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)) \cong \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)^{\Gamma_n} \). Since the latter is a quotient of \( \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty) \), we have

\[
|H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))| \leq |\mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)|.
\]

On the other hand, if \( \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty) \) is infinite, then necessarily, \( \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty) = \mathbb{Q}_p/\mathbb{Z}_p(-i) \). In this case, we have \( H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) = 0 \) by Tate’s Lemma. Either way, we see that \( H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)) \) is finite and bounded independently of \( n \).

We now analyze \( H^1(\Gamma_{n,w}, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w})) \) for a fixed prime \( w \) of \( F_\infty \) above \( v_n \). Let \( v \) be the prime of \( F \) below \( v_n \). As seen in the discussion before Subsection 4.1, it suffices to consider the primes \( v \in S \) which are finitely decomposed in \( F_\infty/F \). By Lemma 4.1, the number of primes of \( F_n \) above each such \( v \) is bounded independently of \( n \). Hence it remains to show that \( H^1(\Gamma_{n,w}, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w})) \) is finite and bounded independently of \( n \) which in turn follows from a similar argument to that for the global cohomology group \( H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)) \). Hence this concludes the proof of the proposition. \( \square \)

4.2 \( \mathbb{Z}_p^d \)-extension

We now come to the case of a \( \mathbb{Z}_p^d \)-extension, where \( d \geq 2 \). Since a \( \mathbb{Z}_p^d \)-extension is unramified outside \( p \) (cf. [26, Theorem 1]), we may take \( S \) to consist precisely the set of primes above \( p \) and infinite primes as in the preceding subsection.

**Proposition 4.3.** Let \( i \geq 1 \) be given. Let \( F_\infty \) be a \( \mathbb{Z}_p^d \)-extension of \( F \), where \( d \geq 2 \). The kernel and cokernel of the map \( r_n : R_i(F_n) \rightarrow R_i(F_\infty)^{G_n} \) are finite with

\[
\text{ord}_p(\ker r_n) = O(n) \quad \text{and} \quad \text{ord}_p(\text{coker } r_n) = O(p^{(d-1)n}).
\]

If the dimension of the decomposition group of \( G \) at every prime of \( F \) above \( p \) is at least 2, then one has \( \text{ord}_p(\text{coker } r_n) = O(np^{(d-2)n}) \).
Proof. Since $H^0(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))$ is finite, it follows from \[58\] that $H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))$ is also finite for $k = 1, 2$. By Lemma \[2.2\] dim$_{\mathbb{Z}/p^2} H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))[p]$ is bounded independent of $n$ for $k = 1, 2$. On the other hand, by \[35\] Lemma 2.1.1 (also see \[12\] Theorem 2.8), there exists a constant $c$ independent of $n$ such that $p^{d_{n+1}}$ annihilates $H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))$. Combining these observations, we have ord$_p(\ker h_n) = O(n)$ and ord$_p(\text{coker } h_n) = O(n)$. Now, let $v \in S$ and consider

$$\bigoplus_{v_n | v} \ker g_{v_n} = \bigoplus_{v_n | v} H^1(\text{Gal}(F_{\infty,w}/F_{v_n}), \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w})).$$

As seen in the proof of Proposition \[4.2\] we may assume that the prime $v$ does not decompose completely in $F_{\infty}/F$. If the decomposition group of $v$ is of dimension one, then a similar argument to that in the proof of Proposition \[4.2\] shows that

$$\text{ord}_p \left( H^1(\text{Gal}(F_{\infty,w}/F_{v_n}), \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_{\infty,w})) \right) = O(1).$$

Since the decomposition group is of dimension one, the number of primes of $F_n$ above $v$ is $O(p^{d-1}n)$ by Lemma \[4.1\] and so we have

$$\text{ord}_p \left( \bigoplus_{v_n | v} \ker g_{v_n} \right) = O(p^{d-1}n).$$

If the decomposition group of $v$ has dimension $\geq 2$, we can apply a similar argument to that for the global cohomology groups $H^k(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty))$ to obtain ord$_p(\ker g_{v_n}) = O(n)$. Since the number of primes of $F_n$ is at most $O(n p^{d-2}n)$ by Lemma \[4.1\] we have

$$\text{ord}_p \left( \bigoplus_{v_n | v} \ker g_{v_n} \right) = O(np^{d-2}n).$$

Combining these estimates, we obtain the conclusion of the proposition. \[\square\]

4.3 Multi-False-Tate extensions

In this subsection, we shall always suppose that our number field $F$ contains a primitive $p$-th root of unity. Let $d \geq 2$. Consider $F_\infty = F \left( \mu_{p^{\infty}}, \sqrt[p]{\alpha_1}, \ldots, \sqrt[p]{\alpha_{d-1}} \right)$, where $\alpha_1, \ldots, \alpha_{d-1} \in F^{\times}$, whose image in $F^{\times}/(F^{\times})^p$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. Then $G = \text{Gal}(F_\infty/F) \cong \mathbb{Z}_{p^{d-1}} \times \mathbb{Z}_p$. Following Coates (see \[62\] Section 8), we call this a multi-False-Tate extension. Here, the set $S$ is taken to comprise precisely of all the primes of $F$ above $p$, the infinite primes and the primes that ramify in $F_{\infty}/F$.

**Proposition 4.4.** Let $i \geq 1$ be given. Let $F_{\infty}$ be the multi-False-Tate extension of $F$ defined as above. Then the kernel and cokernel of the map $r_n : R_i(F_n) \rightarrow R_i(F_\infty)^{G_n}$ are finite with

$$\text{ord}_p(\ker r_n) = O(n) \quad \text{and} \quad \text{ord}_p(\text{coker } r_n) = O(np^{d-2}n).$$

**Proof.** Write $H = \text{Gal}(F_\infty/F(\mu_{p^{\infty}}))$ and $\Gamma = \text{Gal}(F(\mu_{p^{\infty}})/F)$. As before, we begin by examining the kernel and cokernel of the maps $h_n$ which reduces us to estimating the cohomology groups $H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$ and $H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$. We shall show that

$$\text{ord}_p \left( H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \right) = O(n) \quad \text{and} \quad \text{ord}_p \left( H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \right) = O(n)$$

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by induction on \( d \). Suppose first that \( d = 2 \). Then one has \( H \cong \Gamma \cong \mathbb{Z}_p \). Since these groups have \( p \)-cohomological dimension one, it follows from the Hochschild-Serre spectral sequence

\[
H^r(\Gamma_n, H^s(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) \Rightarrow H^{r+s}(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))
\]

that we have a short exact sequence

\[
0 \rightarrow H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{\Gamma_n} \rightarrow 0
\]

and an isomorphism

\[
H^1(\Gamma_n, H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) \cong H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)).
\]

Tate’s Lemma tells us that \( H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) = 0 \), and so it follows from the short exact sequence that \( H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \cong H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{\Gamma_n} \). Since \( H_n \) acts trivially on \( \mathbb{Q}_p/\mathbb{Z}_p(-i) \) and \( H_n \cong \mathbb{Z}_p(1) \) as \( \Gamma \)-modules by Kummer theory, we have

\[
H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{\Gamma_n} \cong \text{Hom}_{\mathbb{Z}_p}((\mathbb{Z}_p(1), \mathbb{Q}_p/\mathbb{Z}_p(-i))^{\Gamma_n} = \mathbb{Q}_p/\mathbb{Z}_p(-1-i))^{\Gamma_n}
\]

which can be seen to be finite with ord\(_p\)-growth \( O(n) \). On the other hand, we have

\[
H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \cong H^1(\Gamma_n, H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) \cong H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-1-i)) = 0,
\]

where the final zero follows from Tate’s Lemma noting that \(-1-i \neq 0 \) for \( i \geq 1 \). This completes the proof of the required estimates for \( d = 2 \). Now suppose that \( d \geq 3 \). We set

\[
N = \text{Gal} \left( F_{\infty}/F \left( \mu_{p^\infty}, \sqrt[p]{\alpha_1}, \ldots, \sqrt[p]{\alpha_{d-2}} \right) \right) \quad \text{and} \quad V = \text{Gal} \left( F \left( \mu_{p^\infty}, \sqrt[p]{\alpha_1}, \ldots, \sqrt[p]{\alpha_{d-2}} \right) / F \right).
\]

Since \( N_n \cong \mathbb{Z}_p \), the Hochschild-Serre spectral sequence

\[
H^r(V_n, H^s(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) \Rightarrow H^{r+s}(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))
\]

degenerates to yield an exact sequence

\[
0 \rightarrow H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{V_n}
\]

\[
\rightarrow H^2(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(V_n, H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))).
\]

By induction, both \( H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \) and \( H^2(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \) are finite with

\[
\text{ord}_p(H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = O(n) \quad \text{and} \quad \text{ord}_p(H^2(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = O(n).
\]

Now writing \( U = \text{Gal} \left( F \left( \mu_{p^\infty}, \sqrt[p]{\alpha_1}, \ldots, \sqrt[p]{\alpha_{d-2}} \right) / F(\mu_{p^\infty}) \right) \), we see that \( U \) acts trivially on \( N \), and so the action of \( V \) on \( N \) factors through \( \Gamma \). Consequently, we have \( N \cong \mathbb{Z}_p(1) \) as \( V \)-modules which in turn implies that

\[
H^1(N, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \cong \mathbb{Q}_p/\mathbb{Z}_p(-1-i)
\]

as \( V \)-modules. Similarly, we have \( H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \cong \mathbb{Q}_p/\mathbb{Z}_p(-1-i) \). This in turn implies that

\[
H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{V_n} \cong \left( \mathbb{Q}_p/\mathbb{Z}_p(-1-i) \right)^{V_n}
\]

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where we make use of the facts that $h$ for every $j$ degenerates to yield $H$ Since $\sim$ For each $\mathbb{F}$ are finite with $\mathbb{O}$ each $G$ without complex multiplication. The following will be the main result of this subsection.

We now come to the case of a $GL$ has no complex multiplication. Suppose that $E$ is finite with ord$\mathbb{P}$, $w$ $\infty$ is a multi-False-Tate extension. Therefore, we may apply a similar argument as above to show which again is finite with ord$\mathbb{P}$ --adic Lie extension of $\mathbb{F}$ $\mathbb{G}$ $\mathbb{Q}/\mathbb{Z}$, $\mathbb{P}$ $\mathbb{P}$-division points of an elliptic curve $E$ which is finite, and it follows from Tate's Lemma that the number of primes of $F_n$ above each $v$ is $O(p^{(d-2)n})$. Combining these observations, we obtain ord$_p(ker g_n) = O(np^{2n})$. The proof of the proposition is now completed.

4.4 $GL_2$-extension

We now come to the case of a $GL_2$-extension cut out by the $p$-division points of an elliptic curve $E$ without complex multiplication. The following will be the main result of this subsection.

Proposition 4.5. Let $i \geq 1$ be given. Let $E$ be an elliptic curve defined over a number field $F$ and suppose that $E$ has no complex multiplication. Suppose that $E(F)[p] = E[p]$ and that $F_\infty = F(E[p^{\infty}])$ is a uniform $p$-adic Lie extension of $F$. Then the kernel and cokernel of the map $r_n : R_i(F_n) \to R_i(F_\infty)^G_n$ are finite with

$$\text{ord}_p(\ker r_n) = O(n) \quad \text{and} \quad \text{ord}_p(\coker r_n) = O(np^{2n}).$$

For the proof, we require three lemmas. The first of which is as follows.

Lemma 4.6. For $j = 1, 2$, the group $H^j(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$ is finite with

$$\text{ord}_p(H^j(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = O(n).$$

Proof. Since $E$ is an elliptic curve without complex multiplication, the group $G$ has dimension 4. By the discussion in [61] pp. 302, and enlarging $F$ if necessary, we may assume that $G = Z \times H$, where $Z \cong \mathbb{Z}_p$ and $H = \text{Gal}(F_\infty/F^{\text{cycc}})$. Then $\mathbb{Q}_p/\mathbb{Z}_p(-i)Z_n$ is finite, and it follows from Tate’s Lemma that $H^1(Z_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) = 0$. Therefore, the spectral sequence

$$H^r(H_n, H^s(Z_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) \Longrightarrow H^{r+s}(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$$

degenerates to yield

$$H^j(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)\mathbb{Z}_n) \cong H^j(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$$

for every $j$. Now, applying Lemma 2.3 we see that

$$\text{ord}_p(H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = \text{ord}_p(H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)\mathbb{Z}_n)) \leq 3 \text{ ord}_p(\mathbb{Q}_p/\mathbb{Z}_p(-i)\mathbb{Z}_n) = O(n);$$

$$\text{ord}_p(H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = \text{ord}_p(H^2(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)\mathbb{Z}_n)) \leq 6 \text{ ord}_p(\mathbb{Q}_p/\mathbb{Z}_p(-i)\mathbb{Z}_n) = O(n).$$

where we make use of the facts that $h_1(H_n) = 3$ and $h_2(H_n) = 6$. This establishes the lemma. \qed
The remaining two lemmas are concerned with estimating the local cohomology groups. As a start, we consider the case, where the prime is above \( p \).

**Lemma 4.7.** Let \( K \) be a finite extension of \( \mathbb{Q}_p \), and let \( E \) be an elliptic curve defined over \( K \) without complex multiplication. Suppose that \( E(K)[p] = E[p] \) and \( K_\infty = K(E[p^{\infty}]) \) is a uniform \( p \)-adic Lie extension of \( K \). Write \( \mathcal{G} = \text{Gal}(K_\infty/K) \). Then the group \( H^1(\mathcal{G}_\mathbb{Z}/\mathbb{Z}_p(-i)) \) is finite with

\[
\text{ord}_p(H^1(\mathcal{G}_\mathbb{Z}/\mathbb{Z}_p(-i))) = O(n).
\]

**Proof.** Enlarging \( K \) if necessary, we may assume that \( E \) has no additive reduction over \( K \). Now, suppose that \( E \) has split multiplicative reduction. By the theory of Tate curves, there exists \( q \in K^\times \) satisfying \( |q| < 1 \) such that \( E(K) \cong \mathbb{K}^\times/\sqrt[p^2]{q} \) as \( \text{Gal}(K/K) \)-modules. Since \( K \) is assumed to contain \( K(E[p]) \), it also contains \( \mu_p \). Therefore, \( K(\mu_p) \) is a \( \mathbb{Z}_p \)-extension of \( K \). Write \( H = \text{Gal}(K_\infty/K(\mu_p)) \) and \( G = \text{Gal}(K(\mu_p)/K) \). By the theory of Tate curves, \( K_\infty \) is obtained from \( K(\mu_p) \) by adjoining all the \( p \)-power roots of \( q \). Therefore, \( K_\infty \) is a False-Tate extension of \( K \). The required estimate follows from a similar argument to that in the proof of Proposition 4.2.

When the elliptic curve \( E \) has good supersingular reduction, the dimension of \( \mathcal{G} \) is either 2 or 4 (cf. \cite[IV A.2.2]{Sil}). When the dimension of \( \mathcal{G} \) is 4, one may now imitate the argument in the proof of Lemma 4.6 to prove the asserted estimate. For the dimension 2 case, we note that the Lie algebra of \( \mathcal{G} \) is commutative (see loc. cit.). Therefore, in this context, we may apply a similar argument to that in Proposition 4.3.

Now, suppose that \( E \) has good ordinary reduction. Then the dimension of \( \mathcal{G} \) is 3 (cf. \cite[Proposition 2.8]{Sil}). For our purposes, we need to delve into a more detailed analysis of this group \( \mathcal{G} \) following the discussion in \cite[Lemmas 2.8 and 3.15]{Sil}. As \( E \) has good ordinary reduction, there is a short exact sequence of \( \mathcal{G} \)-modules

\[
0 \longrightarrow \widehat{E}[p^{\infty}] \longrightarrow E[p^{\infty}] \longrightarrow \widehat{E}[p^{\infty}] \longrightarrow 0,
\]

where \( \widehat{E} \) (resp., \( \widehat{E} \)) denotes the formal group (resp., the maximal étale quotient of the \( p \)-divisible group) of \( E \). It follows from the definition that \( \text{Gal}((K/K^{ur}) \) acts trivially on \( \widehat{E}[p^{\infty}] \), where \( K^{ur} \) is the maximal unramified extension of \( K \). Therefore, the extension \( L_\infty := K(\widehat{E}[p^{\infty}]) \) is an unramified one dimensional extension of \( K \) contained in \( K_\infty \), which upon enlarging \( K \), we may assume it to be a \( \mathbb{Z}_p \)-extension. Then \( M_\infty = L_\infty(\mu_p) \) is a \( \mathbb{Z}_p \)-extension of \( K(\widehat{E}[p^{\infty}]) \). Again, enlarging \( K \), if necessary, we may assume that \( \text{Gal}(K_\infty/M_\infty) \cong \mathbb{Z}_p \) and \( \text{Gal}(M_\infty/K) \cong \mathbb{Z}_p^2 \). Write \( U = \text{Gal}(K_\infty/M_\infty) \) and \( V = \text{Gal}(M_\infty/K) \).

Plainly, \( V \) acts on \( U \) by conjugation, and we like to understand this action more concretely. Let \( \rho \) be the representation of \( \mathcal{G} \) on \( T_p E \). Fix a basis of \( T_p E \) which is chosen in such a way that the first element is a basis of \( T_p \widehat{E} \). Under this choice of basis, the group \( \mathcal{G} \) can be viewed as a subgroup of \( GL_2(\mathbb{Z}_p) \) consisting of elements of the form

\[
\begin{pmatrix}
\eta(\sigma) & a(\sigma) \\
0 & \varepsilon(\sigma)
\end{pmatrix}, \quad \sigma \in \mathcal{G}
\]

where \( \eta \) (resp., \( \varepsilon \)) is the representation of \( \mathcal{G} \) on \( T_p \widehat{E} \) (resp., \( T_p \widehat{E} \)). Note that \( a(\sigma) \) is not identically zero. Else \( T_p E \otimes \mathbb{Q}_p \) will decompose into a direct sum of two one-dimensional subspaces but this contradicts the underlying assumption that \( E \) has no complex multiplication (see \cite[IV A.2.4]{Sil}).
Now, since the group $U$ acts trivially on both $\tilde{E}[p^\infty]$ and $\mu_{p^\infty}$, it identifies with the subgroup of elements of the form
\[
\begin{pmatrix}
1 & a(\sigma) \\
0 & 1
\end{pmatrix}, \quad \sigma \in G.
\]
Under this identification, a straightforward calculation shows that $U \cong \mathbb{Z}_p(\eta^{-1})$ as a $V$-module. Consequently, we have
\[
H^1(U, \mathbb{Q}_p/\mathbb{Z}_p(-i)) = \text{Hom}_{\mathbb{Z}_p}(U, \mathbb{Q}/\mathbb{Z}_p(-i)) = \mathbb{Q}/\mathbb{Z}_p(\eta^{-1}\varepsilon)(-i) = \mathbb{Q}/\mathbb{Z}_p(\varepsilon^2)(-1 - i)
\]
as $V$-modules, where in the final equality, we have made used of the fact that $\mathbb{Q}_p/\mathbb{Z}_p(\eta\varepsilon) = \mathbb{Q}_p/\mathbb{Z}_p(1)$ which in turn is a consequence of the Weil-pairing. One has similar conclusion for $H^1(U_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$.

We can now estimate the cohomology group $H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(i))$ building on the above discussion. The inflation-restriction sequence yields an exact sequence
\[
0 \longrightarrow H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \longrightarrow H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \longrightarrow H^1(U_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{V_n}.
\]
Since $V \cong \mathbb{Z}_p^2$, we can apply a similar argument to that in Proposition 4.3 to conclude that $H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$ is finite with ord$_p(H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = O(n)$. On the other hand, it follows from the above discussion that
\[
H^1(U_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{V_n} = (\mathbb{Q}/\mathbb{Z}_p(\varepsilon^2)(-1 - i))^{V_n} \cong (\tilde{E}[p^\infty] \otimes \tilde{E}[p^\infty](-1 - i))^{V_n},
\]
where the latter can be easily seen to be finite with ord$_p$-growth $O(n)$. This establishes the required estimates for the case when $E$ has good ordinary reduction. The proof of the lemma is thus completed. 

We now consider the primes which do not lie above $p$.

**Lemma 4.8.** Let $K$ be a finite extension of $\mathbb{Q}_l$, where $l \neq p$. Suppose that $E$ is an elliptic curve defined over $K$ and has potential multiplicative reduction. Suppose that $E(K)[p] = E[p]$ and $K_{\infty} = K(E[p^\infty])$. Write $G = \text{Gal}(K_{\infty}/K)$. Then $H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$ is finite with ord$_p(H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = O(n)$.

**Proof.** Since $E$ has potential multiplicative reduction, the group $G$ is of dimension 2 (cf. 10 Lemma 2.8(i)). By a result of Iwasawa [47, Theorem 7.5.3], $K_{\infty}$ has no nontrivial $p$-extension, and so $H^1(K_{\infty}, \mathbb{Q}_p/\mathbb{Z}_p(-i)) = 0$. It now follows from this and the inflation-restriction sequence that
\[
H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \cong H^1(K_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)).
\]
Taking [47 Proposition 7.3.10(ii)] into account, we see that the latter is finite with ord$_p$-growth $O(n)$. 

We can now prove Proposition 4.5.

**Proof of Proposition 4.5** Enlarging $F$ if necessary, we may assume that $E$ has no additive reduction over $F$. Therefore, we can take $S$ to be the set of primes consisting of primes above $p$ and the multiplicative primes of $E$. Lemma 4.6 tells us that the maps $h_n$ have finite kernel and cokernel with ord$_p$-growth $O(n)$. We now consider the local kernels
\[
\bigoplus_{v|p} \ker g_{v_n}.
\]
If $v$ is a prime above $p$, we see that $\text{ord}_p(\text{ker} g_{v_n}) = O(n)$ by Lemma 4.7. Since the decomposition group of $G$ at such a prime has dimension at least 2 as seen in the proof of Lemma 4.7 we always have

$$\text{ord}_p \left( \bigoplus_{v_n \mid v} \ker g_{v_n} \right) = O(np^{2n}).$$

For the primes outside $p$, since these primes are multiplicative primes by our choice of $S$, we may apply the above argument in conjunction with Lemma 4.8 to obtain the required local estimates. Combining these estimates, we obtain the conclusion of the proposition.

### 4.5 Compositum of $GL_2$-extension and multi-False-Tate extension

We come to the case of the compositum of a $GL_2$-extension carved out by an elliptic curve without complex multiplication and a multi-False-Tate extension.

**Proposition 4.9.** Let $E$ be an elliptic curve defined over $F$ without complex multiplication. Suppose that $E(F)[p] = E[p]$ and that $F(E[p^\infty])$ is a uniform $p$-adic Lie extension of $F$. Let $\alpha_1, \ldots, \alpha_r \in F^\times$, whose image in $F^\times/(F^\times)^p$ are linearly independent over $\mathbb{F}_p$. Set $F_\infty = F \left( E[p^\infty], r\sqrt[p]{\alpha_1}, \ldots, r\sqrt[p]{\alpha_r} \right)$ which is a uniform $p$-adic Lie extension of dimension $r + 4$. Then the kernel and cokernel of the map $r_n : R_i(F_n) \rightarrow R_i(F_\infty)^{G_n}$ are finite with

$$\text{ord}_p(\ker r_n) = O(n) \quad \text{and} \quad \text{ord}_p(\text{coker} r_n) = O(np^{(r+2)n}).$$

**Proof.** We begin showing that

$$\text{ord}_p(H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = O(n) \quad \text{and} \quad \text{ord}_p(H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = O(n)
$$

by induction on $r$. When $r = 0$, this is Lemma 4.6. Suppose that $r \geq 1$. Write

$$N = \text{Gal}(F_\infty/F \left( E[p^\infty], r\sqrt[p]{\alpha_1}, \ldots, r\sqrt[p]{\alpha_r} \right))$$

and $V = \text{Gal}(F \left( E[p^\infty], r\sqrt[p]{\alpha_1}, \ldots, r\sqrt[p]{\alpha_r} \right)/F)$. As $N_n \cong \mathbb{Z}_p$, the Hochschild-Serre spectral sequence

$$H^r(V_n, H^s(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) \Rightarrow H^{r+s}(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$$

yields an exact sequence

$$0 \rightarrow H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) V_n$$

$$\rightarrow H^2(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^2(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))).$$

By induction, both $H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$ and $H^2(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))$ are finite with $\text{ord}_p$-growth $O(n)$. Now, if we write $U = \text{Gal}(F \left( E[p^\infty], r\sqrt[p]{\alpha_1}, \ldots, r\sqrt[p]{\alpha_r} \right)/F(p^{\infty}))$, we then see that $U$ acts trivially on $N$. Thus, the action of $V$ on $N$ factors through $\Gamma$, and consequently, one has $N \cong \mathbb{Z}_p(1)$ as $V$-modules. This in turn implies that

$$H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \cong \mathbb{Q}_p/\mathbb{Z}_p(-1 - i)$$

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as $V$-modules. Hence we have that $H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^V_n = (\mathbb{Q}_p/\mathbb{Z}_p(-1-i))^V_n$ is finite with $\text{ord}_p$-growth $O(n)$. On the other hand,

$$H^1(V_n, H^1(N_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) = H^1(V_n, \mathbb{Q}_p/\mathbb{Z}_p(-1-i))$$

which again is finite with $\text{ord}_p$-growth $O(n)$ by our induction hypothesis. Finally, the local cohomology groups can be estimated via the above induction procedure, and building on Lemmas 4.7 and 4.8.

5 Growth of étale wild kernels in $p$-adic Lie extensions

In this section, we will study the growth of étale wild kernel in the various $p$-adic Lie extensions as considered in Section 4.

5.1 Growth in $\mathbb{Z}_p$-extension

Proposition 5.1. Let $i \geq 1$ be given. Let $F_\infty$ be a $\mathbb{Z}_p$-extension of $F$. Then we have

$$\text{ord}_p(WK_{i}^{\text{ét}}(F_n)) = \mu(Y_i(F_\infty))p^n + \lambda(Y_i(F_\infty))n + O(1).$$

Proof. Proposition 4.2 tells us that $Y_i(F_\infty)_{G_n}$ is finite and

$$|\text{ord}_p(WK_{i}^{\text{ét}}(F_n)) - \text{ord}_p(Y_i(F_\infty))| = O(1).$$

The required conclusion follows from this and [47, Proposition 5.3.17].

5.2 Growth in $\mathbb{Z}_p^d$-extensions

We begin with a general result on the growth of étale wild kernel in a $\mathbb{Z}_p^d$-extension.

Proposition 5.2. Let $i \geq 1$ be given. Let $F_\infty$ be a $\mathbb{Z}_p^d$-extension of $F$. Then we have

$$\text{ord}_p(WK_{i}^{\text{ét}}(F_n)) = \mu_G(Y_i(F_\infty))p^{dn} + l_0(Y_i(F_\infty))np^{(d-1)n} + O(p^{(d-1)n})$$

for a certain integer $l_0(Y_i(F_\infty)) \geq 0$.

Proof. It follows from Proposition 4.2 that $Y_i(F_\infty)_{G_n}$ is finite with

$$|\text{ord}_p(WK_{i}^{\text{ét}}(F_n)) - \text{ord}_p(Y_i(F_\infty))| = O(p^{(d-1)n}).$$

On the other hand, the finiteness of $Y_i(F_\infty)_{G_n}$ in particular implies that $\text{rank}_{\mathbb{Z}_p} Y_i(F_\infty)_{G_n} = 0$, and therefore, the hypothesis of [12, Theorem 3.4] is satisfied. Hence we may combine the said theorem with the above estimate to obtain the required conclusion. (For the precise description of the invariant $l_0(Y_i(F_\infty))$, we refer the readers to [12, Definition 1.2].)

We now specialize to the situation, where the $\mathbb{Z}_p^d$-extension contains the cyclotomic $\mathbb{Z}_p$-extension. We shall write $G = \text{Gal}(F_\infty/F)$ and $H = \text{Gal}(F_\infty/F^{\text{cy}})$.  

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Theorem 5.3. Let \( F_\infty \) be a \( \mathbb{Z}_p^d \)-extension of \( F \) which contains \( F^{\text{cy}} \). Suppose that \( Y_i(F^{\text{cy}}) \) is finitely generated over \( \mathbb{Z}_p \). Then \( Y_i(F_\infty) \) is a pseudo-null \( \mathbb{Z}_p[[G]] \)-module if and only if
\[
\text{ord}_p(WK_{2i}^{\text{et}}(F_n)) = O(p^{(d-1)n}).
\]

Proof. By Lemma \( \underline{54} \) we have that \( Y_i(F_\infty) \) is a finitely generated \( \mathbb{Z}_p[[H]] \)-module. Since each \( Y_i(F_\infty)_G \) is finite, we may apply \( \underline{41} \) Proposition 2.4.1 to obtain
\[
\text{ord}_p(Y_i(F_\infty)_G) = \text{rank}_{\mathbb{Z}_p[[H]]} (Y_i(F_\infty)) np^{(d-1)n} + O(p^{(d-1)n}).
\]

On the other hand, again building on the fact that \( Y_i(F_\infty) \) is a finitely generated \( \mathbb{Z}_p[[H]] \)-module, a well-known result of Venjakob \( \underline{32} \) Example 2.3 and Proposition 5.4 tells us that \( Y_i(F_\infty) \) is a pseudo-null \( \mathbb{Z}_p[[G]] \)-module if and only if \( Y_i(F_\infty) \) is a torsion \( \mathbb{Z}_p[[H]] \)-module. In view of the above estimate, the latter is then equivalent to saying that
\[
\text{ord}_p(Y_i(F_\infty)_G) = O(p^{(d-1)n}).
\]

Combining this with Proposition \( \underline{53} \) this is the same as saying that
\[
\text{ord}_p(WK_{2i}^{\text{et}}(F_n)) = O(p^{(d-1)n}),
\]
which concludes the proof of the theorem. \( \blacksquare \)

Corollary 5.4. Assume that \( \mu_p \subseteq F \). Let \( \bar{F} \) be the compositum of all \( \mathbb{Z}_p \)-extensions of \( F \) and write \( G = \text{Gal}(\bar{F}/F) \cong \mathbb{Z}_p^d \). Suppose that the Iwasawa \( \mu \)-conjecture is valid for \( F^{\text{cy}} \). Then the following statements are equivalent.

(a) Greenberg’s conjecture is valid. i.e., \( \text{Gal}(K(\bar{F})/\bar{F}) \) is a pseudo-null \( \mathbb{Z}_p[[G]] \)-module.
(b) \( \text{ord}_p(WK_{2i}^{\text{et}}(F_n)) = O(p^{(d-1)n}) \) for some \( i \geq 1 \).
(c) \( \text{ord}_p(WK_{2i}^{\text{et}}(F_n)) = O(p^{(d-1)n}) \) for all \( i \geq 1 \).

Proof. This follows from a combination of Lemma \( \underline{51}10 \) and Theorem \( \underline{53} \) \( \blacksquare \)

5.3 Growth in multi-False-Tate extensions

We come to the situation of a multi-False-Tate extension \( F_\infty = F(\mu_p^\infty, \sqrt[p]{\alpha_1}, \ldots, \sqrt[p]{\alpha_{d-1}}) \), where \( \alpha_1, \ldots, \alpha_{d-1} \in F^x \) whose image in \( F^x/(F^x)^p \) are linearly independent over \( \mathbb{F}_p \). In this subsection, \( F \) is always assumed to contain \( \mu_p \) and so \( \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p^{d-1} \times \mathbb{Z}_p \). We shall write \( G = \text{Gal}(F_\infty/F) \) and \( H = \text{Gal}(F_\infty/F^{\text{cy}}) \).

Theorem 5.5. Let \( i \geq 1 \) be given. Let \( F_\infty \) be the multi-False-Tate extension of \( F \) as above. Suppose that the Iwasawa \( \mu \)-conjecture is valid for \( F^{\text{cy}}/F \). Then we have
\[
\text{ord}_p(WK_{2i}^{\text{et}}(F_n)) = \text{rank}_{\mathbb{Z}_p[[H]]} (Y_i(F_\infty)) np^{(d-1)n} + O(p^{(d-1)n}).
\]
Proof. In view of the hypothesis of the theorem and Lemmas 3.7 and 3.8, we have that $Y_i(F_{\infty})$ is finitely generated over $\mathbb{Z}_p[[H]]$. Proposition 4.3 tells us that each $Y_i(F_{\infty})_{G_n}$ is finite. It then follows from an application of Proposition 2.4.1 (of which the idea originates from [44]) that

$$\text{ord}_p(Y_i(F_{\infty})_{G_n}) = \text{rank}_{\mathbb{Z}_p[[H]]}(Y_i(F_{\infty}))np^{(d-1)n} + O(p^{(d-1)n}).$$

The required conclusion now follows from this and Proposition 4.4.

Corollary 5.6. Retain the setting of Theorem 5.5. Then the following statements are equivalent.

(a) The noncommutative analog of Greenberg’s conjecture is valid for $F_{\infty}$. In other words, the module $\text{Gal}(K(F_{\infty})/F_{\infty})$ is pseudo-null over $\mathbb{Z}_p[[G]]$.

(b) $\text{ord}_p(WK_{2i}^{et}(F_n)) = O(p^{(d-1)n})$ for some $i \geq 1$.

(c) $\text{ord}_p(WK_{2i}^{et}(F_n)) = O(p^{(d-1)n})$ for all $i \geq 1$.

Proof. This follows from a combination of Lemma 5.10 and Theorem 5.5.

5.4 Growth in $GL_2$-extension

We now come to the $GL_2$-extension situation. Unfortunately, the structure theory for modules over such an Iwasawa algebra is not as refined as in the $\mathbb{Z}_p^2$-case or $\mathbb{Z}_p \times \mathbb{Z}_p$-case. We at least are able to rely on [39] Proposition 2.4 to obtain an asymptotic upper bound. However, since [39] Proposition 2.4 is an estimate on the quantity $\text{ord}_p(M_{G_n}/p^n)$ for a $\mathbb{Z}_p[[G]]$-module $M$, we can only obtain an upper bound for $WK_{2i}^{et}(F_n)[p^n]$ rather than the whole group $WK_{2i}^{et}(F_n)$.

Proposition 5.7. Let $i \geq 1$ be given. Let $E$ be an elliptic curve defined over $F$ which has no complex multiplication. Let $F_\infty = F(E[p^\infty])$ and assume that $G = \text{Gal}(F_\infty/F)$ is a uniform pro-$p$ group. Write $H = \text{Gal}(F_\infty/F^{cyc})$. Suppose that the Iwasawa $\mu$-conjecture is valid for $F^{cyc}$. Then we have

$$\text{ord}_p(WK_{2i}^{et}(F_n)[p^n]) \leq \text{rank}_{\mathbb{Z}_p[[H]]}(Y_i(F_{\infty}))np^{3n} + O(p^{3n}).$$

Proof. By Lemmas 3.7 and 3.8, we have that $Y_i(F_{\infty})$ is finitely generated over $\mathbb{Z}_p[[H]]$. Therefore, it follows from an application of [39] Proposition 2.4 that

$$\text{ord}_p(Y_i(F_{\infty})_{G_n}/p^n) \leq \text{rank}_{\mathbb{Z}_p[[H]]}(Y_i(F_{\infty}))np^{3n} + O(p^{3n}).$$

Since

$$\left( Y_i(F_{\infty})_{G_n}/p^n \right)^{\vee} \cong \left( Y_i(F_{\infty})_{G_n} \right)^{\vee}[p^n] \cong \left( Y_i(F_{\infty})^{\vee} \right)^{G_n}[p^n] = R_i(F_{\infty})^{G_n}[p^n],$$

we have

$$\text{ord}_p(R_i(F_{\infty})^{G_n}[p^n]) \leq \text{rank}_{\mathbb{Z}_p[[H]]}(Y_i(F_{\infty}))[p^n]np^{3n} + O(p^{3n}).$$

On the other hand, from the restriction maps of the fine Selmer groups, we have the following exact sequence

$$0 \rightarrow (\ker r_n)[p^n] \rightarrow R_i(F_n)[p^n] \rightarrow (R_i(F_{\infty})^{G_n})[p^n].$$
By virtue of Proposition 4.5, one has ord$_p((\ker r_n)[p^n]) = O(n)$. Combining this with the above estimate, we obtain ord$_p(R_i(F_n)[p^n]) \leq \text{rank}_{Z_p[[H]]} (Y_i(F_\infty))np^{3n} + O(p^{3n})$, or equivalently,

$$\text{ord}_p(WK^{\text{et}}_{2i}(F_n)/p^n) \leq \text{rank}_{Z_p[[H]]} (Y_i(F_\infty))np^{3n} + O(p^{3n}).$$

But since $WK^{\text{et}}_{2i}(F_n)$ is finite, we have an equality $\text{ord}_p(WK^{\text{et}}_{2i}(F_n)/p^n) = \text{ord}_p(WK^{\text{et}}_{2i}(F_n)/p^n)$ and so this proves the proposition.

As the estimate in Proposition 5.7 is not as precise as those in Theorems 5.2 and 5.5, we are not able to prove an analogue of Corollaries 5.4 and 5.6. We do have enough to at least establish the following implication.

**Corollary 5.8.** Retain the setting of Proposition 5.7. Suppose that $\text{Gal}(K(F_\infty)/F_\infty)$ is pseudo-null over $Z_p[[G]]$. Then

$$\text{ord}_p(WK^{\text{et}}_{2i}(F_n)/p^n) = O(p^{3n}).$$

**Proof.** By Lemma 3.10 Greenberg’s conjecture is equivalent to $Y_i(F_\infty)$ being pseudo-null over $Z_p[[G]]$. Since $Y_i(F_\infty)$ is also finitely generated over $Z_p[[H]]$, the result of Venjakob [62, Example 2.3 and Proposition 5.4] tells us that $Y_i(F_\infty)$ is a torsion $Z_p[[H]]$-module, or equivalently, $\text{rank}_{Z_p[[H]]} (Y_i(F_\infty)) = 0$. The estimate of the corollary is now immediate from this and the preceding theorem.

### 5.5 Growth in compositum of $GL_2$-extension and multi-False-Tate extension

In this subsection, we consider the case of the compositum of a $GL_2$-extension coming from an elliptic curve without complex multiplication and a multi-False-Tate extension. As the proofs are similar to those in Proposition 5.7 and Corollary 5.8 we shall merely state the conclusions and omit their proofs.

**Proposition 5.9.** Let $E$ be an elliptic curve defined over $F$ without complex multiplication. Suppose that $E(F)[p] = E[p]$ and that $F(E[p^\infty])$ is a uniform $p$-adic Lie extension of $F$. Let $\alpha_1, \ldots, \alpha_r \in F^\times$, whose image in $F^\times/(F^\times)^p$ are linearly independent over $F_p$. Set $F_\infty = F(E[p^\infty], r\sqrt[p]{\alpha_1}, \ldots, r\sqrt[p]{\alpha_r})$ which is a uniform $p$-adic Lie extension of dimension $r + 4$. Suppose that the Iwasawa $\mu$-conjecture is valid for $F^\text{cyc}$. Then for $i \geq 1$, we have

$$\text{ord}_p(WK^{\text{et}}_{2i}(F_n)/p^n) \leq \text{rank}_{Z_p[[H]]} (Y_i(F_\infty))np^{(r+3)n} + O(p^{(r+3)n}),$$

where $H = \text{Gal}(F_\infty/F^\text{cyc})$.

**Corollary 5.10.** Retain the setting of Proposition 5.9. Suppose that $\text{Gal}(K(F_\infty)/F_\infty)$ is pseudo-null over $Z_p[[G]]$. Then

$$\text{ord}_p(WK^{\text{et}}_{2i}(F_n)/p^n) = O(p^{(r+3)n}).$$

### 6 Examples

We end the paper with some examples to illustrate our results.
(1) Let $F = \mathbb{Q}(\mu_p)$. Now, if $p$ is an irregular prime $< 1000$, a result of Sharifi \cite[Theorem 1.3]{55} asserts that $\text{Gal}(K(F)/\bar{F})$ is pseudo-null over $\mathbb{Z}_p[[\text{Gal}(F/\bar{F})]]$. We can now apply Corollary \cite[5.4]{41} to conclude that $\text{ord}_p(WK_{2i}^d(F_n)) = O(p^{(p-3)n/2})$ for all $i \geq 1$.

(2) Let $F = \mathbb{Q}(\mu_p)$ and $F_\infty = \mathbb{Q}(\mu_p, p^{-\infty})$. If $p$ is an irregular prime $< 1000$, a result of Sharifi \cite[Propositions 3.3 and 2.1a]{55} tells us that $\text{Gal}(K(F_\infty)/F)$ is pseudo-null over $\mathbb{Z}_p[[\text{Gal}(F_\infty)/F]]$. Corollary \cite[6.6]{35} then tells us that $\text{ord}_p(WK_{2i}^d(F_n)) = O(p^n)$ for all $i \geq 1$. We shall see below that we actually have a better estimate.

(3) Take $p = 5$ and $F = \mathbb{Q}(\mu_5)$. Let $E$ be the elliptic curve 150A1 of Cremona’s table which is given by

$$y^2 + xy = x^3 - 3x - 3.$$  

When $F_\infty$ is one of the following 5-adic Lie extensions

$$\mathbb{Q}(E[5^\infty], 35^\infty), \mathbb{Q}(E[5^\infty], 25^\infty, 35^\infty), \mathbb{Q}(E[5^\infty], 35^\infty, 55^\infty), \mathbb{Q}(E[5^\infty], 25^\infty, 35^\infty, 55^\infty),$$

the Pontryagin dual of the fine Selmer group of $E$ over such $p$-adic Lie extension is known to be pseudo-null (cf. \cite[Example 23]{38} and \cite[Section 6, Example (b)]{35}) which in turn implies that $\text{Gal}(K(F_\infty)/F)$ is pseudo-null. Hence Corollary \cite[5.10]{35} applies to these examples.

We now come back to the second example and derive a better (and unconditional) estimate than that predicted by Greenberg’s conjecture. For this, we require the following preparatory lemma.

**Lemma 6.1.** Let $G$ be a compact pro-$p$ $p$-adic Lie group which contains a closed normal subgroup $H \cong \mathbb{Z}_p^{d-1}$ such that $G/H \cong \mathbb{Z}_p$. Let $M$ be a $\mathbb{Z}_p[[G]]$-module which is finitely generated torsion over $\mathbb{Z}_p[[H]]$ with $\mu_H(M) = 0$. Suppose further that $M_{G_n}$ is finite for every $n$. Then we have

$$\text{ord}_p(M_{G_n}) = O(np^{(d-2)n}).$$

**Proof.** Fix a subgroup $\Gamma$ of $G$ such that $\Gamma$ maps isomorphically to $G/H$ under the natural quotient map $G \rightarrow G/H$. For every subgroup $U$ of $G$, we write $M(U)$ for the $\mathbb{Z}_p[[G]]$-submodule of $M$ generated by all elements of the form $(u - 1)x$, where $u \in U$ and $x \in M$. From the proof of \cite[Proposition 2.4.1]{35}, there exists an $n_0$ such that whenever $n \geq n_0$, we have

$$\left|\text{ord}_p(M_{G_n}) - (n - n_0)\text{rank}_{\mathbb{Z}_p}(M_{H_n})\right| \leq \text{ord}_p\left((M/M(\Gamma_{n_0}))_{H_n}\right) + 2\text{ord}_p(M_{H_n}[p^\infty]).$$

By \cite[Theorem 2.4.1]{35} and the hypothesis that $\mu_H(M) = 0$, we have

$$\text{ord}_p(M_{H_n}[p^\infty]) = O(np^{(d-2)n}).$$

Since $\mu_H(M) = 0$, we also have $\mu_H(M/M(\Gamma_{n_0})) = 0$, and so \cite[Theorem 2.4.1]{35} applies to yield

$$\text{ord}_p\left((M/M(\Gamma_{n_0}))_{H_n}\right) = O(np^{(d-2)n}).$$

Finally, by a result of Harris \cite[Theorem 1.10]{21} and the hypothesis that $M$ is torsion over $\mathbb{Z}_p[[H]]$, we have

$$(n - n_0)\text{rank}_{\mathbb{Z}_p}(M_{H_n}) = O(np^{d-2}).$$

Combining all of the above estimates, we obtain the required conclusion of the lemma. \qed
We can now prove the following.

**Proposition 6.2.** Let $F = \mathbb{Q}(\mu_p)$ and $F_\infty = \mathbb{Q}(\mu_p, p^{-p^\infty})$, where $p$ is an irregular prime $< 1000$. Then

$$\text{ord}_p(WK^{et}_{2i}(F_n)) = O(n)$$

for every $i \geq 1$.

**Proof.** The results of Sharifi [55, Propositions 3.3 and 2.1a] actually show that $\text{Gal}(K(F_\infty)/F_\infty)$ is finitely generated over $\mathbb{Z}_p$. In particular, it is torsion over $\mathbb{Z}_p[[H]]$ with $\mu_H(\text{Gal}(K(F_\infty)/F_\infty)) = 0$, where $H = \text{Gal}(F_\infty/F_{\text{cyc}})$. From the proof of Lemma 3.10 we see that $Y_i(F_\infty) = \text{Gal}(K(F_\infty)/F_\infty)(i)$, and so $Y_i(F_\infty)$ is also torsion over $\mathbb{Z}_p[[H]]$ with trivial $\mu_H$-invariant. Proposition 4.4 tells us that $Y_i(F_\infty)G_n$ is finite for every $n$. Hence we may apply Lemma 6.1 to conclude that

$$\text{ord}_p(Y_i(F_\infty)G_n) = O(n).$$

Now combining this with Proposition 4.4 again, we have the required estimates of the proposition.

**Remark 6.3.** The referee has raised the question of whether the stronger estimate of the preceding proposition is expected to be true in general. We shall say a bit on this aspect here. First of all, we like to mention that Sharifi’s calculations rest on two components: (1) the validity of Kummer-Vandiver’s conjecture and (2) certain delicate cup products computations (see [54, Theorem 5.7, and Corollaries 5.8 and 5.9]). For (1), large amount of computation works have been done in verifying Kummer-Vandiver’s conjecture (see [22] and the reference therein). In view of these computations and general belief, one would expect Kummer-Vandiver’s conjecture to hold (though a proof remains elusive). For the cup products computations, to the best knowledge of the author, although there has been some further computation works done, the details do not seem to have appeared in literature as yet. In conclusion, it would seem quite plausible that the estimate in Proposition 6.2 might hold for all irregular primes $p$, although the author has to confess that he does not know of any means (other than Sharifi’s approach) to tackle this problem.

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