Monitorability of $\omega$-regular languages

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Abstract. Arguably, $\omega$-regular languages play an important rôle as a specification formalism in many approaches to systems monitoring via runtime verification. However, since their elements are infinite words, not every $\omega$-regular language can sensibly be monitored at runtime when only a finite prefix of a word, modelling the observed system behaviour so far, is available. The monitorability of an $\omega$-regular language, $L$, is thus a property that holds, if for any finite word $w$, observed so far, it is possible to add another finite word $v$, such that $vw$ becomes a “finite witness” wrt. $L$; that is, for any infinite word $w$, we have that $wvw \in L$, or for any infinite word $w$, we have that $wvw \notin L$. This notion has been studied in the past by several authors, and it is known that the class of monitorable languages is strictly more expressive than, e.g., the commonly used class of so-called safety languages. But an exact categorisation of monitorable languages has, so far, been missing. Motivated by the use of linear-time temporal logic (LTL) in many approaches to runtime verification, this paper first determines the complexity of the monitorability problem when $L$ is given by an LTL formula. Further, it then shows that this result, in fact, transfers to $\omega$-regular languages in general, i.e., whether they are given by an LTL formula, a nondeterministic Büchi automaton, or even by an $\omega$-regular expression.

1 Introduction

In a nutshell, the term runtime verification subsumes many techniques that are used for monitoring systems, i.e., for checking their execution as it is happening. Naturally, there exists a variety of different approaches to runtime verification. In this article, we will focus on those which are based on the theory of formal languages, where a so called monitor checks whether or not a consecutive sequence of observed system actions belongs to a formally specified language. For example, if the language comprises all undesired system behaviours, then a positive outcome of this check would normally lead to the raising of an alarm by the monitor, whereas if the language describes a desired system behaviour, the monitor could be switched off.

As a formalism to describe such languages, many runtime verification approaches (cf. [17,9,14]), use linear-time temporal logic (LTL [15]), whose formulae describe sets (languages) of infinite words (or, $\omega$-languages), meaning that the models of an LTL formula are infinitely long sequences of symbols. The rationale for using LTL to describe properties of systems is that many systems for which formal verification is required (at

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runtime or off-line) are critical and/or reactive; that is, their failure would have catastrophic impact on its users and/or the environment, and consequently one would like to make assertions about the entire lifespan of such systems, some of which are never switched off, unless they are physically broken and can be replaced in a controlled manner. A typical requirement for such systems, that can also easily be formalised in LTL, would be “the system must never enter a bad state.” Although the monitor would require an infinitely long observation to flag satisfaction of the property, it is always able to raise an alarm after finitely many observations, simply due to the fact that a violation of such a property can always be detected in the same instance as the system entering the bad state. Hence, if such a property, formalised as an LTL formula, is monitored, one would expect the monitor to only detect violations. Formal languages which describe properties of this form are therefore referred to as safety languages or safety properties, and they have in common that all sequences of actions that violate them are detectable after finitely many observations. Note that languages belonging to the complementary class of safety properties are known as the co-safety properties, implying that satisfaction (rather than violation) of any such type of property is always detectable by a monitor after finitely many observations, i.e., via a finite “witness.”

Since the languages definable by LTL formulae exceed the expressiveness of safety and co-safety languages, a natural question to ask, given an arbitrary LTL formula, is whether or not the given formula is monitorable at all. This is, arguably, an interesting question in its own right, and ideally, we would like to know the answer prior to any attempts of building a monitor, or starting a monitoring process based on an unmonitorable language. Of course, what we then need is a more general notion of monitorability of an LTL formula: Intuitively, we say that the language given by an LTL formula is monitorable if, after any number of observed actions, the monitor is still able to detect the violation or satisfaction of the monitored property, and after at most finitely many additional observations. As an example of a non-monitorable, LTL-definable language consider a property such as “it is always the case that a request will eventually be answered,” which is a so called liveness property. For this property no finite witnesses of violation or satisfaction exist, since any finite sequence of actions can be extended to satisfy this property. In order to know that some request is, indeed, never answered, a monitor would therefore require an infinite sequence of actions. In consequence, most examples of liveness properties that can be found in the literature violate the intuitive definition of monitorability given above. To determine whether or not an LTL formula specifies a liveness property is a PSpace-complete problem [23]. However, they are not the only types of properties, which can be formally specified in LTL that are not monitorable, and as this paper will show there exists no criterion that allows to answer the monitorability question for any given formula in a simple, syntactic manner.

Pnueli and Zaks [16] were the first to formalise a notion of monitorability, which matches the intuitive account given above: According to [16] a formula is monitorable wrt. a finite sequence of actions, if that finite sequence can be extended to be a finite witness for violation or satisfaction of that formula. However, Pnueli and Zaks did not address the question of deciding monitorability for a given formula (and sequence). In [2] a slightly more general formalisation based on a 3-valued semantics for LTL is given, such that monitorability of an LTL formula becomes a property of only the
formula. Moreover, Falcone et al. \cite{5} have recently shown that the definition given in
\cite{2} is, indeed, a generalisation of the one given earlier in \cite{16}, and termed it “classical
monitorability.” In their paper, they have at first wrongly concluded—but later also
corrected \cite{6}—that the class of monitorable languages, under classical monitorability,
consists exactly of the obligation properties in the hierarchy of safety-progress prop-
nerties (cf. \cite{13}), which is orthogonal to the safety-liveness classification. An obligation
property, for example, is obtained by taking a positive Boolean combination of safety
and co-safety properties. Despite their correction, Falcone et al. left the question re-
garding the complexity of monitorability of an LTL formula (or \(\omega\)-regular language
in general) open. Note that \cite{2} did imply a decision procedure based on the construc-
tion and subsequent analysis of deterministic monitors for LTL formulae, but the given
procedure requires 2ExpSpace (see Sec. 3).

One of the main contributions of this paper is a proof that this upper bound is not
optimal, in that monitorability of an LTL formula can be decided in PSpace. In fact, it
will show that the monitorability problem of LTL, i.e., the decision problem that asks
“is a given LTL formula monitorable?” is PSpace-complete, and that this result even
transfers to \(\omega\)-regular languages in general—regardless as to whether they are given by
an LTL formula, a nondeterministic Büchi automaton, or an \(\omega\)-regular expression. As
such it is also proof that no simple syntactic categorisation of monitorability of an LTL
formula (or \(\omega\)-regular language), which could be checked in polynomial time, exists. On
the other hand, the result implies that checking monitorability is no more complex than
checking safety or co-safety, which have often served as the “monitorable fragment” in
the past (cf. \cite{18,8,9}).

As a special case the paper also considers the monitorability problem of Büchi au-
tomata, where the automaton in question is deterministic, and shows that this restricted
form of the problem is solvable in polynomial time. Finally, it shows that the moni-
torable \(\omega\)-languages are closed under the usual Boolean connectives; that is, they are
closed under finitary application of union, intersection, and complementation.

Outline. The remainder is structured as follows. The next section recalls some prelim-
inary notions and notations used throughout this paper. Sec. 2 gives a formal account
of monitorability of an \(\omega\)-language and phrases the corresponding decision problem(s).
Sec. 4 puts two well-known classifications of \(\omega\)-regular languages, namely the classifi-
cation in terms of the safety-progress hierarchy (cf. \cite{13}) as well as a topological view,
in relation with the notion of monitorability. The main contribution of this paper, which
makes use of these classifications, can be found in sections 5 and 6 and as such they
are also the most technical sections, in that they contain the complexity analyses and
proofs of the monitorability problems of \(\omega\)-regular languages. Sec. 7 details on closure
properties of monitorable \(\omega\)-languages, and Sec. 8 concludes.

2 Basic notions and notation

We encode information about a system’s state in terms of a finite set of atomic pro-
positions, \(AP\), and define an action to be an element of \(2^{AP}\). In a sense, an action can
be seen as a global state that is determined by the individual atomic sub-states encoded
by elements from \( AP \). We will therefore use the terms action and state synonymously. The system behaviour which the monitor observes then consists of a sequence of actions. Therefore, we define an alphabet, \( \Sigma := 2^AP \), and treat consecutive sequences of actions as words over \( \Sigma \). As is common, we define \( \Sigma^* \) as the set of all finite words over \( \Sigma \), including the empty word, and \( \Sigma^\omega \) to be the set of infinite words obtained by concatenating an infinite sequence of nonempty words over \( \Sigma \). Infinite words are of the form \( w = w_0 w_1 \ldots \in \Sigma^\omega \) and are usually abbreviated by \( w, w' \), and so on, whereas finite words are of the form \( u = u_0 \ldots u_n \in \Sigma^* \) and are usually abbreviated by \( u, u', v \), and so on. Let \( w \in \Sigma^\omega \), then \( w^i \) denotes the infinite suffix \( w_iw_{i+1} \ldots \), whereas \( u \preceq w \) denotes a prefix of \( w \), \( u \) is a proper prefix of \( w \) (\( u \prec w \)), if \( u \preceq w \) and \( u \neq w \). For any \( p \in AP \), and a given \( \sigma \in \Sigma \), if \( p \in \sigma \) holds, we also say that “\( p \) holds (or, is true) in the state \( \sigma \)”.

The syntax of LTL formulae, which are given by the set \( LTL(AP) \), is defined as follows: \( \varphi := p \mid \neg \varphi \mid \varphi \lor \psi \mid X \varphi \mid \varphi U \psi \), with \( p \in AP \). If the set of atomic propositions is clear from the context, we write LTL instead of \( LTL(AP) \). LTL formulae are interpreted over elements from \( \Sigma^\omega \) as follows. Let \( i \in N \), and \( \varphi, \psi \in LTL \), then

\[
\begin{align*}
w^i &\models p \iff p \in w_i \\
w^i &\models \neg \varphi \iff w^i \not\models \varphi \\
w^i &\models \varphi \lor \psi \iff w^i \models \varphi \lor w^i \models \psi \\
w^i &\models X \varphi \iff w^{i+1} \models \varphi \\
w^i &\models \varphi U \psi \iff \exists k \geq i. w^k \models \psi \land \forall j < k. w^j \not\models \varphi
\end{align*}
\]

Further, we will make use of the usual syntactic sugar such as \( true \equiv p \lor \neg p \), \( false \equiv \neg true \), \( \varphi \land \psi \equiv \neg(\neg \varphi \lor \neg \psi) \), \( F \varphi \equiv true U \varphi \), and \( G \varphi \equiv \neg(F \neg \varphi) \).

It is well-known that, for any \( \varphi \in LTL \), we can construct a nondeterministic Büchi automaton (NBA), \( A_\varphi = (\Sigma, Q, Q_0, \delta, F) \), where \( \Sigma \) is the alphabet, \( Q \) the set of states, \( Q_0 \subseteq Q \) designated initial states, \( \delta : Q \times \Sigma \rightarrow 2^Q \) the transition relation, and \( F \subseteq Q \) a set of final states, such that the accepted language of \( A_\varphi \) contains exactly all the models of \( \varphi \), i.e., \( L(A_\varphi) = L(\varphi) \). If some language of infinite words, called an \( \omega \)-language, \( L \subseteq \Sigma^\omega \) is such that there exists an NBA, \( A \), such that \( L(A) = L \), then \( L \) is called \( \omega \)-regular. Obviously, the language specified by an LTL formula is always \( \omega \)-regular. The size of \( A_\varphi \), usually measured wrt. \( |Q| \), is, in the worst-case, exponential wrt. the size of \( \varphi \). For details on the construction as well as further properties of \( A_\varphi \), cf. [22].

3 When is an \( \omega \)-language monitorable?

Let us fix an \( L \subseteq \Sigma^\omega \) for the remainder of this section. In accordance with [16] and [2], Falcone et al. [5] formally define the monitorability of an \( \omega \)-language as follows.

**Definition 1.** \( L \) is called

- negatively determined by \( u \in \Sigma^* \), if \( u \Sigma^\omega \cap L = \emptyset \);
- positively determined by \( u \in \Sigma^* \), if \( u \Sigma^\omega \subseteq L \);
- \( u \)-monitorable for \( u \in \Sigma^* \), if \( \exists v \in \Sigma^* \), s.t. \( L \) is positively or negatively determined by \( uv \).
– monitorable, if it is $u$-monitorable for any $u \in \Sigma^*$.

This also lends itself to another, sometimes more intuitive way to think about monitorability of an $\omega$-language, namely in terms of good and bad prefixes.

**Definition 2.** The set of good and bad prefixes for $L$ are defined as $\text{good}(L) := \{ u \in \Sigma^* \mid u\Sigma^\omega \subseteq L \}$ and $\text{bad}(L) := \{ u \in \Sigma^* \mid u\Sigma^\omega \cap L = \emptyset \}$, respectively.

For brevity, we also write $\text{good}(\varphi)$ (respectively, $\text{bad}(\varphi)$) short for $\text{good}(\mathcal{L}(\varphi))$ (respectively, $\text{bad}(\mathcal{L}(\varphi))$), and $\text{good}(A)$ (respectively, $\text{bad}(A)$) short for $\text{good}(\mathcal{L}(A))$ (respectively, $\text{bad}(\mathcal{L}(A))$).

**Proposition 1.** $L$ is monitorable if $\forall u \in \Sigma^* \cdot \exists v \in \Sigma^*, uv \in \text{good}(L) \lor vw \in \text{bad}(L)$.

In other words, $L$ is not monitorable if there exists a finite word $u \in \Sigma^*$ for which we can not find a finite extension $v \in \Sigma^*$, such that $uv$ is either a good or a bad prefix of $L$. Naturally, given some $L$, not every finite word is a good or a bad prefix of $L$, in which case we call such a word undetermined (wrt. $L$). Let $u \in \Sigma^*$ be an undetermined prefix, then, depending on $L$, the following scenarios are possible: we can find a finite extension $v \in \Sigma^*$, such that $uv \in \text{good}(L)$, we can find a finite extension $v$, such that $uv \in \text{bad}(L)$, or there does not exist a finite extension $v$, such that $uv \in \text{good}(L)$ or $uv \in \text{bad}(L)$ would hold. In [2], the latter were called “ugly” prefixes, and $L$ “non-monitorable,” if there exists an ugly prefix for it.

Let us now define the monitorability problem of an $\omega$-language as follows.

**Definition 3.** The monitorability problem for some $L$ is the following decision problem:

Given: A set $L \subseteq \Sigma^\omega$.

Question: Does $\forall u \in \Sigma^* \cdot \exists v \in \Sigma^*, uv \in \text{good}(L) \lor vw \in \text{bad}(L)$ hold?

When $L$ is given in terms of an LTL formula, an NBA, or an $\omega$-regular expression (which are basically defined like ordinary regular expressions, augmented with an operator for infinite repetition of a regular set, cf. [22]), we call this problem the monitorability problem of $\omega$-regular languages, or—more specifically—the monitorability problem of LTL/Büchi automata-$\omega$-regular expressions, respectively.

One of the main contributions of [2] was a procedure that, given a formula $\varphi \in \text{LTL}$, constructs a deterministic finite-state machine (i.e., a monitor for $\varphi$) whose input is a consecutively growing, finite word $u \in \Sigma^*$, and whose output is $\top$ if $u \in \text{good}(\varphi)$, $\bot$ if $u \in \text{bad}(\varphi)$, and $?$ if $u$ is undetermined. Once this monitor is computed, the monitorability of $\varphi$ can be determined in polynomial time, simply by checking if there exists a state whose output is $?$ with no path leading to a $\top$- or $\bot$-state. If such a state, called a $?$-trap, exists, then $\varphi$ is not monitorable. Notice, however, that this monitor construction (and this decision procedure) requires $2\text{ExpSpace}$: as a first step, it creates two NBAs, one which accepts all models of $\varphi$ and one that accepts all counterexamples of $\varphi$ (i.e., all models of $\neg \varphi$), and then proceeds by examining and transforming the resulting state graphs of these automata. Recall, NBAs accepting the models of an LTL formula are, in the worst case, exponentially larger than the corresponding formula. Since at some point, the two automata are made deterministic, the double exponential “blow up” follows. Moreover, although not explicitly mentioned in [2], by altering the first step of
this procedure, it can be used to decide the monitorability of \( \omega \)-regular languages, in general, i.e., whether given as an LTL formula, as NBA, or as an \( \omega \)-regular expression. For example, if instead of a formula, an NBA is given, one has to explicitly complement this automaton, which also involves a worst-case exponential “blow up” wrt. the number of states of the original NBA. However, then the rest of the procedure described in \([2]\) stays the same. On the other hand, if we are given an \( \omega \)-regular expression instead, we first have to build an NBA, which can occur in polynomial time. Then, in order to get the complementary language, one also needs to complement this automaton. Hence, independent of the concrete representation of an \( \omega \)-regular language, the construction and subsequent analysis of the corresponding monitor can decide monitorability in 2ExpSpace. Therefore, indirectly, \([2]\) shows decidability of the monitorability problem, but whether or not this bound is tight was left open in that paper.

**Examples.** Let us examine some examples to understand how this construction works and what its outcome is. Fig. 1 depicts some finite state machines (i.e., the monitors) for several LTL formulae, which were automatically generated using the LTL3 tool\(^1\), which are written by the author of this paper and implement the above construction. Each monitor is complete in a sense that for every action from the alphabet, there exists a transition. Note that, although not explicitly marked, the initial state is the top-most ?-state, respectively. Any word \( u \in \Sigma^* \) which has a corresponding path in a monitor to a ?-state is undetermined wrt. the \( \omega \)-language being monitored. On the other hand, if \( u \) leads to a state labelled \( \top \) (respectively, \( \bot \)), then \( u \) is a good (respectively, bad) prefix of the \( \omega \)-language being monitored. It is easy to see, that all the formulae give rise to a monitorable language; that is, from any reachable state in the respective monitor, there always exists a path to a state labelled either \( \top \) or \( \bot \). Let us, therefore, also present a language which is not monitorable and whose (practically not very useful) monitor is depicted in Fig. 2. Clearly, the right-most state is a ?-trap; that is, once reached by some

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\(^1\) Available under an open source license at http://LTL3tools.SourceForge.Net/
finite prefix $u \in \Sigma^*$, there exists no extension $v \in \Sigma^*$ for $u$, such that a $\top$- or a $\bot$-state can be reached. Or, in other words, every word $u = u_0 \ldots u_n$, such that $a \in u_0$ is an ugly prefix of $L(a \wedge X(GFb))$.

4 A classification of $\omega$-languages

4.1 The safety-liveness view

Alpern and Schneider [1] were the first to give a formal characterisation of $\omega$-languages in terms of safety and liveness properties. This view was subsequently extended to an entire hierarchy of $\omega$-languages, where languages defining safety properties, or their complement are at the bottom (cf. [13]).

Definition 4. $L$ describes a safety language (also called a safety property), if $\forall w \notin L. \exists u \prec w. u \Sigma^\omega \cap L = \emptyset$. $L$ describes a co-safety language (also called co-safety property), if $\forall w \in L. \exists u \prec w. u \Sigma^\omega \subseteq L$.

In other words, if $L$ specifies a safety language, then all infinite words $w \notin L$, have a bad prefix. On the other hand, if $L$ specifies a co-safety language, then all infinite words $w \in L$, have a good prefix. This also explains why safety and co-safety properties lend themselves so well to runtime verification using monitors: if the specification to be monitored gives rise to a safety language, then all violations of the specification are detectable by the monitor after only finitely many observations of actions emitted by the system under scrutiny. That is, let $u \in \Sigma^*$ be a word, resembling the sequence of actions, then either there exists a $v \in \Sigma^*$, such that $uv \in bad(L)$, or $u \in good(L)$ already holds. On the other hand, if the specification to be monitored gives rise to a co-safety language, then all models of the specification are detectable by the monitor after only finitely many observations. That is, let $u \in \Sigma^*$, then either there exists a $v \in \Sigma^*$, such that $uv \in good(L)$, or $u \in bad(L)$ already holds. It follows that a co-safety language always has the form $B \Sigma^\omega$, where $B \subseteq \Sigma^*$.

Safety and co-safety are dual in a sense that if $L$ is a safety language, then $\Sigma^\omega \setminus L$, from this point forward also abbreviated as $\overline{L}$, is a co-safety language, and vice versa. The following easy to prove proposition makes this duality formal.

\[\text{Fig. 2. "Monitor" for a non-monitorable language given by } a \wedge X(GFb).\]
Proposition 2. $\text{bad}(L) = \text{good}(L)$ and $\text{bad}(L) = \text{good}(L)$.

Definition 5. $L$ describes a liveness language (also called a liveness property) if $\forall u \in \Sigma^* . u \Sigma^w \cap L \neq \emptyset$.

In other words, if $L$ specifies a liveness language, then $\text{bad}(L) = \emptyset$—in which case $L$ may only be monitorable if $\text{good}(L) \neq \emptyset$ also holds. The definition of liveness, however, does not require $\text{good}(L)$ to be empty or non-empty. Hence, from a runtime verification point of view, many liveness languages which are commonly used to describe system properties in the area of formal verification using, say, temporal logic model checking, turn out to be not monitorable.

Examples. Let us look at some example languages, specified in terms of LTL formulae. The formula $\varphi = G \neg \text{bad\_state}$ with $\text{bad\_state} \in \text{AP}$ formalises, in an abstract manner, the requirement from the introduction: the system must never enter a bad state. In other words, $\neg \text{bad\_state}$ must always be true. It is a safety property as any prefix containing a state in which $\text{bad\_state}$ is true is a bad prefix of $\varphi$, e.g., $u = 0000\ldots \{\text{bad\_state}\} \in \text{bad}(\varphi)$. Naturally, $\neg \varphi = \neg (G \neg \text{bad\_state}) = F \text{bad\_state}$ describes a co-safety property. Any finite word containing a state where the proposition $\text{bad\_state}$ is true is a good prefix for $\neg \varphi$. In practical terms, this means that a monitor, checking either language will be able to make a conclusive decision after the first occurrence of $\text{bad\_state}$ in the observed sequence of system actions. In fact, we can postulate the following proposition which, using Proposition 2, is easy to prove formally:

Proposition 3. If $\varphi$ specifies a safety or a co-safety language, then $\varphi$ is monitorable.

It is also easy to verify that the formula $F \text{bad\_state}$ meets the definitions of both co-safety and liveness. However, as we will see, not all co-safety languages are also liveness languages, and vice versa. In fact, unlike $F \text{bad\_state}$ most liveness languages do not lend themselves to runtime verification via monitors, because they may have neither bad nor good prefixes that would eventually lead a monitor to a conclusive answer. The liveness property given in the introduction, and formalised in LTL as $G (\text{request} \rightarrow F \text{answer})$, is such a case. The following proposition is easy to prove:

Proposition 4. If $\varphi$ specifies a liveness language, such that $\text{good}(\varphi) = \emptyset$, then $\varphi$ is not monitorable.

It follows that a formula of the form $\varphi = GF(\psi)$ is not monitorable unless $\psi = true$ or $\psi = false$. In the first case we would get $L(\varphi) = \Sigma^w$, and in the latter $L(\varphi) = \emptyset$, both of which meet the definition of monitorability. In fact, the languages given by the sets $\emptyset$ and $\Sigma^w$ are both safety and co-safety.

As a final example, let us consider obligation languages. In [13], Manna and Pnueli define the class of obligation languages as follows.

Definition 6. $L$ describes an obligation language (also called an obligation property) if $L$ either consists of an unrestricted Boolean combination of safety languages, or an unrestricted Boolean combination of co-safety languages, or a positive Boolean combination of safety and co-safety languages.
Falcone et al. [6] have shown that

**Proposition 5.** If \( \varphi \) specifies an obligation language, then \( \varphi \) is monitorable.

To see that the other direction is not true, consider the counterexample given by the formula in Fig. 1(d): it does not specify an obligation language, yet it is monitorable.

4.2 The corresponding topological view

Alpern and Schneider showed in [1] that

**Proposition 6.** Every language \( L \) can be represented as the intersection \( L = L_S \cap L_L \), where \( L_S \) is a safety language, and \( L_L \) is a liveness language.

Their proof is based on the observation that safety languages (over some alphabet \( \Sigma \)) correspond to closed sets in the Cantor topology over \( \Sigma^\omega \) (cf. [13]), and liveness languages to dense sets. It follows that co-safety languages correspond to open sets in that topology. Sets which are both closed and open, are referred to as clopen. It is worth pointing out, and easy to prove, that both \( \emptyset \) and \( \Sigma^\omega \) are clopen. Given a set \( L \subseteq \Sigma^\omega \) and element \( w \in \Sigma^\omega \), \( w \) is a limit point of \( L \), if there exists an infinite sequence of words \( w_1, w_2, \ldots \), all of which are in \( L \), which converges to \( w \). Clearly, any \( w \in L \) is a limit point of \( L \), since \( w, w, \ldots \) converges to \( w \). The topological closure of \( L \), written \( \text{cl}(L) \), is then defined as the set of all limit points of \( L \). Then, obviously, \( L \subseteq \text{cl}(L) \).

The following gives a direct definition of \( \text{cl}(L) \):

**Definition 7.** \( \text{cl}(L) := \{ w \in \Sigma^\omega \mid \forall u < w. \exists w' \in L. u < w' \} \).

From a basic result of topology, a topological closure operator on \( \Sigma^\omega \) defines a topology, where a set \( L \subseteq \Sigma^\omega \) is closed (i.e., a safety language) if and only if \( \text{cl}(L) \subseteq L \) also holds. Moreover, \( L \) is dense (i.e., a liveness language), if and only if \( \text{cl}(L) = \Sigma^\omega \).

This alternative classification of \( \omega \)-languages proved useful as many important results from topology transfer to the commonly used classification in terms of safety and liveness properties. For example, due to Alpern and Schneider [1] it is well-known that the topological closure of a language that is given by an NBA, \( A \), where non-reachable and dead-end states have been eliminated, can be determined by an NBA, \( A' \), which is like \( A \) except that all states are made final. Now, \( \mathcal{L}(A) \) gives rise to a safety language if and only if \( \mathcal{L}(A) = \mathcal{L}(A') \) as \( \mathcal{L}(A') = \text{cl}(\mathcal{L}(A)) \). We will make use of this and similar results in the remainder. For a comprehensive overview on this topology, cf. [1,13].

5 The monitorability problem of LTL

The results of this section will show that the monitorability problem of LTL is PSpace-complete. In order to show this, we will make use of a well-known construction of a tableau for an LTL formula, which has been given many times before in the literature (cf. [20,19]). For reasons of self-containedness, we briefly summarise its most important properties for our purposes.

Let us first fix a formula \( \varphi \in \text{LTL} \) over some alphabet \( \Sigma \). \( SF(\varphi) \) is the set consisting of the subformulae of \( \varphi \) or the negations of subformulae of \( \varphi \). A set \( c \subseteq SF(\varphi) \)
is complete if the following two conditions are met: 1. Boolean consistency of \( c \); 2. for
\( \varphi' = \mu \land \nu \in SF(\varphi) \), \( \varphi' \in c \) if and only if \( \mu \in c \) and \( \nu \in c \). Let \( tab(\varphi) = (V, E) \)
be a (directed) graph, where \( V \) is the set of all complete subsets of \( SF(\varphi) \), and elements
\( (c, d) \in E \) defined as follows:

- for any \( \varphi' = \mu \lor \nu \in SF(\varphi) \): \( \varphi' \in c \) if and only if \( \mu \in c \) or \( \nu \in c \).
- for any \( \varphi' = \exists \psi \in SF(\varphi) \): \( \varphi' \in c \) if and only if \( \psi \in d \).

Let for any \( c \in V \), \( \pi(c) \) the state such that for any atomic proposition \( p \in SF(\varphi) \),
\( \pi(c)(p) = \text{true} \) if and only if \( p \in c \). An infinite path through \( tab(\varphi) \) is called accepting
if for every node \( c \) on that path with \( \varphi' = \mu \lor \nu \in c \), either \( \nu \in c \) or there exists a (not necessarily immediate) successor node \( d \), such that \( \nu \in d \). For any \( c \in V \),
we say that \( c \) is a good node, if the conjunction of all subformulae in \( c \) is satisfiable;
otherwise \( c \) is called a bad node. Notably, it holds that for any \( w \in \Sigma^\omega \) with the property
\( \forall i \geq 0. \exists w' \in \Sigma^\omega \) such that \( w_0 \ldots w_i w' \models \varphi \), there exists an infinite path \( \rho \) of good nodes in \( tab(\varphi) \) starting from a node that contains \( \varphi \), such that \( \pi(\rho) = w \). Moreover
for \( \varphi, \psi \in LTL \), we denote by \( tab(\varphi) \times tab(\psi) \) the cross-product of the tableaux for \( \varphi \)
and \( \psi \), respectively.

**Lemma 1.** Let \( \varphi \) not be monitorable. Then there exists a pair of nodes, \( (q, q') \in tab(\varphi) \times tab(-\varphi) \), reachable on some \( u \in \Sigma^* \) and where \( q, q' \) are conjunctions of subformulae of \( \varphi \), respectively, such that \( L(q) \) and \( L(q') \) are dense.

**Proof.** Following Proposition 1 the non-monitorability of \( \varphi \) is defined as follows
\[
\exists u \in \Sigma^*. \forall v \in \Sigma^*. \text{bad}(\varphi) = \emptyset \land uv\Sigma^\omega \not\subseteq L(\varphi).
\]

In other words, there exists a \( u \in \Sigma^* \), such that none of the finite continuations \( v \) of \( u \) is
(i) a bad or (ii) a good prefix of \( \varphi \). Let us fix such a particular \( u \). From the construction
of \( tab(\varphi) \) it follows that in order for (i) to be true, there must exist a node \( q \in V_{\varphi} \),
reachable on \( u \) (i.e., \( tab(\varphi) \) has a path on \( u \)), such that \( \forall v \in \Sigma^*. \text{bad}(\varphi) \) must be true, which is equivalent to \( \forall v \in \Sigma^*. \text{bad}(\varphi) \) being equivalent to \( \forall v \in \Sigma^*. \text{bad}(\varphi) \), which by Proposition 2 is equivalent to
\[
\forall v \in \Sigma^*. \text{bad}(\varphi).
\]

Let \( q' \in V_{\neg\varphi} \) be a node in \( tab(-\varphi) \), reached on \( u \). Now, for (1) to be true, \( \forall v \in \Sigma^*. \text{bad}(\varphi) \) must be true, which is equivalent to \( \forall v \in \Sigma^*. \text{bad}(\varphi) \). It is easy to see that \( L(q') \) is dense. Requirement (ii), i.e., \( \forall v \in \Sigma^*. \text{bad}(\varphi) \), is equivalent to \( \forall v \in \Sigma^*. \text{bad}(\varphi) \), which by Proposition 2 is equivalent to
\[
\forall v \in \Sigma^*. \text{bad}(\varphi).
\]

**Lemma 2.** If there exists a pair \( (q, q') \in tab(\varphi) \times tab(-\varphi) \), reachable on some \( u \in \Sigma^* \), such that \( L(q) \) and \( L(q') \) are dense, then \( \varphi \) is not monitorable.

**Proof.** Let \( (q, q') \in tab(\varphi) \times tab(-\varphi) \) be reached via some \( u \in \Sigma^* \), such that
\[
\forall v \in \Sigma^*. \text{bad}(\varphi) \land v \not\in \text{bad}(\varphi).
\]

Since \( q \) is reached on \( u \), and by the construction of \( tab(\varphi) \) it follows that \( \forall v \in \Sigma^*. \text{bad}(\varphi) \) is equivalent to \( \forall v \in \Sigma^*. \text{bad}(\varphi) \) (and, accordingly, for \( \varphi' \) and \( \neg\varphi \)).
Thus, together with Proposition 2 we get \( \exists u \in \Sigma^*. \forall v \in \Sigma^*. uv \not\in \text{good}(\varphi) \), which corresponds to the definition of non-monitorability of \( \varphi \), used in the
previous lemma.
Theorem 1. The monitorability problem of LTL is decidable in PSpace.

Proof. By Lemma 1 and 2, \( \varphi \) is not monitorable if and only if there exists a word, corresponding to a path through \( \text{tab}(\varphi) \times \text{tab}(\neg \varphi) \) that contains a pair \((q, q')\), such that \( \mathcal{L}(q) \) and \( \mathcal{L}(q') \) are dense. As \( \text{tab}(\varphi) \) and \( \text{tab}(\neg \varphi) \) are of exponential size wrt. \(|\varphi|\), we cannot construct either explicitly. Instead, we will guess, in a step-wise manner, a path through \( \text{tab}(\varphi) \times \text{tab}(\neg \varphi) \) to some pair \((q, q')\), and check if both \( \mathcal{L}(q) \) and \( \mathcal{L}(q') \) are dense. To check whether or not an LTL formula specifies a dense set is equivalent to checking whether or not it specifies a liveness language (cf. Sec. 4.2). It follows from Ultes-Nitsche and Wolper’s work [23] (Remark 4.3 and Theorem 4.6, if we replace \( L_\omega \) to correspond to \( \Sigma_\omega \)) that this problem can be decided in PSpace. So, if the answer to this check is “yes”, then \( \varphi \) is not monitorable.

Since due to Savitch’s theorem we know that NPSPACE is equal to PSPACE (cf. [14]), we have thus shown that the “non-monitorability problem of LTL” is, in fact, decidable in PSpace. □

Theorem 2. The monitorability problem of LTL is PSPACE-complete.

Proof. It is sufficient to show PSPACE-hardness. We will reduce the PSPACE-complete problem of determining whether or not a formula \( \varphi \in \text{LTL} \) is satisfiable [20] to the monitorability problem of LTL. Let us construct, in constant time, a formula \( \psi \in \text{LTL}(\text{AP}') := \text{Ga} \lor \text{GF}(a' \land \varphi) \), where \( a \in \text{AP} \), \( \text{AP}' := \text{AP} \cup \{a'\} \) and \( a' \notin \text{AP} \). We now claim that \( \psi \) is monitorable if and only if \( \mathcal{L}(\varphi) = \emptyset \).

If \( \mathcal{L}(\varphi) = \emptyset \), then \( \psi = \text{Ga} \), which can easily be seen monitorable.

For the other direction, assume that \( \psi \) is monitorable, but that \( \mathcal{L}(\varphi) \neq \emptyset \). Let \( \Sigma' := 2^{\text{AP}'} \) and \( u \in (2^{\text{AP}' \setminus \{a\}})^* \). It is easy to see that \( u \Sigma_\omega \cap \mathcal{L}(\text{Ga}) = \emptyset \), but \( u \Sigma_\omega \cap \mathcal{L}(\psi) \neq \emptyset \). Hence, for \( \psi \) to be monitorable, \( u \) has to be extensible with some \( v \in \Sigma_* \), such that \( uv \Sigma_\omega \subseteq \mathcal{L}(\text{GF}(a' \land \varphi)) \). Now, observe that irrespective of our choice of \( \varphi \) (including the case \( \varphi = \text{true} \)), so long as \( \mathcal{L}(\varphi) \neq \emptyset \), the set \( \mathcal{L}(\text{GF}(a' \land \varphi)) \) neither has a bad nor a good prefix. This means that \( \exists u \in \Sigma_* \), \( \forall v \in \Sigma_* \), \( uv \Sigma_\omega \cap \mathcal{L}(\psi) \neq \emptyset \land uv \Sigma_\omega \not\subseteq \mathcal{L}(\psi) \); that is, \( \psi \) is not monitorable. Contradiction. □

6 The monitorability problem of Büchi automata

Let for the rest of this section \( \mathcal{A} = (\Sigma, Q, Q_0, \delta, F) \) be a fixed NBA with \( \mathcal{L}(\mathcal{A}) \subseteq \Sigma_\omega \). As pointed out in Sec. 4.2 a topologically closed set can be obtained from \( \mathcal{L}(\mathcal{A}) \) via an automaton, referred to as \( \text{safe}(\mathcal{A}) \), whose accepted \( \omega \)-language will always be closed, irrespective of \( \mathcal{L}(\mathcal{A}) \). What is more, \( \text{safe}(\mathcal{A}) \) can be constructed in polynomial time wrt. the size of \( \mathcal{A} \). Moreover for the next results, we also need an explicit representation of \( \text{live}(\mathcal{A}) \), whose accepted \( \omega \)-language will always be dense. Like its counterpart \( \text{safe}(\mathcal{A}) \), it can be constructed in polynomial time wrt. the size of \( \mathcal{A} \). For details on these constructions, see [1] Sec. 4. Finally, we need to introduce the notion of a tight automaton, as a finite-state acceptor for good prefixes, as follows.
6.1 Tight automata

In preparation for the main results of this section, let us discuss how to obtain a tight automaton over \(\Sigma^*\) that, given some NBA \(A\), accepts \(good(A)\). The construction can be described by a two-stage process.

First, we construct a nondeterministic finite automaton (NFA) that accepts the potentially good prefixes of \(A\), where, without loss of generality, all unreachable and dead-end states have been eliminated. From \(A\), we can easily derive the NFA \(G_A^p = (\Sigma, Q, Q_0, \delta, F)\), where \(F := Q\) is the set of accepting states, and the rest defined as for \(A\). For this NFA it holds that

**Proposition 7.** \(L(G_A^p) = \{u \in \Sigma^* \mid \exists w \in \Sigma^*. uw \in L(A)\}\).

From this point forward, let as a notational convention, \(A(q)\) be like \(A\), except that \(Q_0 = \{q\}\).

**Proof.** (\(\subseteq\)): Take any \(u \in L(G_A^p)\). Obviously, there exists an accepting run on \(u\) in \(G_A^p\) to some state \(q \in Q\) and by construction also a (finite) run in \(A\) reaching the same \(q\) as both automata share the same \(\delta\). As by assumption \(A\) is non-empty, and all unreachable and dead-end states have been eliminated, it then follows that \(L(A(q)) \neq \emptyset\), i.e., \(\exists w \in \Sigma^*. w \in L(A(q))\). Moreover, as \(q\) was reached on \(u\), it then follows that \(uw \in L(A)\).

(\(\supseteq\)): Let \(w \in \Sigma^*\) be such that there exists an accepting run in \(A\), i.e., \(w \in L(A)\). As \(\delta\) is the same for both automata, it follows that there also exists a run on the state space of \(G_A^p\), in a sense that for each symbol in \(w\) there always exists a successor state in \(G_A^p\). Now, pick any \(u \prec w\), then \(u \in L(G_A^p)\) as all states in \(G_A^p\) are accepting. \(\square\)

Second, to obtain an NFA that contains only the good prefixes, but no other words, we proceed as follows. As \(G_A^p\) is but an ordinary NFA, we can apply the standard subset construction to obtain a deterministic finite automaton (DFA) accepting the same language, and whose states consist of a subset of states of \(G_A^p\), respectively. Let \(G_A = (\Sigma, Q', (Q_0'), \delta', F')\) be this DFA, defined as expected, except that we set the accepting states to be

\[F' := \{\langle q_0, \ldots, q_n \rangle \in Q' \mid L(A(q_0)) \cup \ldots \cup L(A(q_n)) = \Sigma^*\}\.

Note that as a notational convention we let \(\langle q_0, \ldots, q_n \rangle\) be the single DFA-state whose label is made up of the individual state labels \(q_0, \ldots, q_n\) of \(G_A^p\).

**Proposition 8.** \(L(G_A) = good(A)\).

**Proof.** (\(\subseteq\)): Take any \(u \in L(G_A)\). By the subset construction and the fact that all states in \(L(G_A^p)\) are accepting, it follows that \(u \in L(G_A)\) must hold. Hence, \(u\) is a potentially good prefix of \(L(A)\). Now, recall that for \(u \in L(G_A)\) to hold, \(G_A\) must be in some state \(\langle q_0, \ldots, q_n \rangle\) such that \(L(A(q_0)) \cup \ldots \cup L(A(q_n)) = \Sigma^*\) holds. Moreover, by the construction of \(L(G_A^p)\), we know that there exist \(n + 1\) runs in \(A\) on \(u\) to the individual states \(q_0, \ldots, q_n\), i.e., each of these state can be reached on \(u\). Now, if the union of these states’ individual languages corresponds to the universal language, \(\Sigma^*\), then clearly \(u \in good(A)\).
(2): Take any \( u \in \text{good}(\mathcal{A}) \). By the previous proposition and the construction of \( G_A^p \), there exist runs on \( u \) in \( G_A^p \) to states \( q_0, \ldots, q_n \), each of which is accepting, i.e., there is at least one such run. Moreover from the construction of \( G_A \), whose state graph corresponds to the deterministic variant of \( G_A^p \), it follows that there has to be a state \( (q_0, \ldots, q_n) \in Q' \) which can be reached on \( u \). We now have to show that \( (q_0, \ldots, q_n) \) is an accepting state of \( G_A \). For assume not, i.e., \( \mathcal{L}(\mathcal{A}(q_0)) \cup \ldots \cup \mathcal{L}(\mathcal{A}(q_n)) \neq \Sigma^\omega \) holds, then there exists a word \( w \in \Sigma^\omega \), such that \( w \not\in \mathcal{L}(\mathcal{A}(q_0)) \cup \ldots \cup \mathcal{L}(\mathcal{A}(q_n)) \), and consequently \( uw \not\in \mathcal{L}(\mathcal{A}) \). Clearly, then \( u \not\in \text{good}(\mathcal{L}(\mathcal{A})) \). Contradiction.

Remark 1. In [11], Kupferman and Lampert discuss properties of an NFA, referred to as a “tight automaton,” that accepts all the good prefixes of some NBA, \( \mathcal{A} \). The name stems from the fact that their paper is more concerned with the construction of so called “fine automata,” which accept only some good prefixes, but not all. Although they do not explicitly give details on how to obtain a tight automaton, and only consider the special case where the NBA describes a co-safety language, they conclude, using a language-theoretic argument, that such an automaton must, in the worst-case, be of exponential size wrt. \( \mathcal{A} \)—which agrees with our procedure above. Their restriction to only examine NBAs which describe co-safety languages seems motivated solely by their application of model checking (co-) safety languages. Consequently, they discuss how to obtain tight automata for NBAs describing safety languages, then accepting all the good prefixes, and tight automata for for NBAs describing co-safety languages, then accepting all the good prefixes. However, it is easy to see that the constructions outlined on an abstract level by Kupferman and Lampert easily transfer to general NBAs, and result in the above described procedure when an acceptor for good prefixes is needed. Hence, \( G_A \) can be considered as a general form of a tight automaton capturing good prefixes, regardless as to whether \( \mathcal{A} \) describes a co-safety language, or not.

6.2 Deciding monitorability—The general case

Now that we have all the required tools at hand, let us continue to prove this section’s main result, namely the complexity of the monitorability problem of Büchi automata. We will do this by way of the following lemmas, which provide sufficient and necessary conditions for deciding the monitorability of a language defined by some NBA.

Lemma 3. If \( \forall u \in \text{good}(\text{safe}(\mathcal{A})) \). \( \exists v \in \Sigma^* \). \( uv \Sigma^\omega \subseteq \mathcal{L}(\text{live}(\mathcal{A})) \), then \( \mathcal{A} \) is moni-
torable.

Proof. For any \( u \in \text{good}(\text{safe}(\mathcal{A})) \) there does not exist a \( v \in \Sigma^* \), such that \( uv \Sigma^\omega \cap \mathcal{L}(\mathcal{A}) = \emptyset \) as \( u \) is a good prefix of \( \mathcal{L}(\text{safe}(\mathcal{A})) \), and \( \mathcal{L}(\text{live}(\mathcal{A})) \) does not, by definition of \( \text{live}(\mathcal{A}) \), have any bad prefixes. Now, if the assumption \( \forall u \in \text{good}(\text{safe}(\mathcal{A})) \). \( \exists v \in \Sigma^* \). \( uv \Sigma^\omega \subseteq \mathcal{L}(\text{live}(\mathcal{A})) \) holds, then any such \( u \) is extensible to to be a good prefix of \( \mathcal{L}(\text{live}(\mathcal{A})) \) and thus \( \mathcal{L}(\mathcal{A}) \).

On the other hand, if \( u \not\in \text{good}(\text{safe}(\mathcal{A})) \), then by the definition of a closed set, this \( u \) is extensible to be a bad prefix of \( \mathcal{L}(\text{safe}(\mathcal{A})) \) and thus \( \mathcal{L}(\mathcal{A}) \).

As for any \( u \in \Sigma^* \), either \( u \in \text{good}(\text{safe}(\mathcal{A})) \), or not, any \( u \) can be extended to be either a good or a bad prefix of \( \mathcal{L}(\mathcal{A}) \) under the lemma’s assumption. \( \square \)
Corollary 1. If there exists no good prefix of $L(safe(A))$, then $A$ is monitorable.

Lemma 4. Let $A$ be monitorable, then $\forall u \in good(safe(A)). \exists v \in \Sigma^* . uv\Sigma^* \subseteq L(live(A))$.

Proof. We are going to show the contrapositive of the lemma’s statement; that is, $\exists u \in good(safe(A)). \forall v \in \Sigma^* . uv \notin good(live(A))$ implies that $A$ is not monitorable.

Let us now fix such a prefix $u$. Since $u \in good(safe(A))$, for $A$ to be monitorable after $u$, there would have to exist some $v \in \Sigma^*$, such that $uv \in good(live(A))$, thus $uv \in good(safe(A)) \cap live(A)$, and therefore $uv \in good(A)$. However, by assumption this is not possible. Hence, $u$ is an “ugly prefix” of $L(A)$ and, consequently, $A$ not monitorable.

Theorem 3. The monitorability problem of Büchi automata is decidable in PSpace.

Proof. Observe that due to Lemma 1 and 4, the monitorability of $A$ is decidable in PSpace if and only if it can be checked in PSpace, whether the following is true:

$$\forall u \in good(safe(A)). \exists v \in \Sigma^* . uv\Sigma^* \subseteq L(live(A)). \quad (2)$$

However, instead of giving an algorithm for checking if, for some $A$, this property holds, we devise an algorithm that returns $true$ if the complementary statement holds, i.e., if

$$\exists u \in good(safe(A)). \forall v \in \Sigma^* . uv\Sigma^* \nsubseteq L(live(A))$$

is true. For some $A$ this is the case if there exists some finite word $u \in good(safe(A))$, such that $u$ cannot be extended to be a good prefix of $L(live(A))$. Let, therefore, $G$ be the tight automaton over $\Sigma$, such that $L(G) = good(safe(A))$. As pointed out in Remark 1, $G$ may, in the worst-case be of exponential size wrt. $safe(A)$, which stems from the fact that a standard subset construction needs to be applied. In our case this means that the states of $G$ are the exponentially many sets of states of $safe(A)$. Therefore, we can only guess, in a step-wise manner, a path through $G \times live(A)$, corresponding to a word $u \in \Sigma^*$, to a pair of states $(q, q')$, where $q$ is now a set of states of $safe(A)$. Note that using $\delta$ of $A$, we can easily check the connectedness of two states in $G \times live(A)$ in PSpace. Next, we check if $q$ is an accepting state in $G$, which, by Proposition 3 is the case if and only if the states $q_0, \ldots, q_n \in q$ are such that $L(safe(A))(q_0)) \cup \ldots \cup L(safe(A)(q_n)) = \Sigma^*$. This property can be checked in PSpace in the size of $safe(A)$, because the union of two NBAs is of polynomial size and determining language equivalence of two NBAs is a PSpace-complete problem [21]. Moreover, as $q'$ was reached on $u$, we have $uv\Sigma^* \not\subseteq L(live(A))$ for all possible extensions $v \in \Sigma^*$ if and only if $live(A)$ does not contain a state $p$ that is reachable from $q'$, such that $L(live(A)(p))$ is open. Using the algorithm presented in 3, which we have employed before, this can be checked in PSpace as well. So, if no such $p$ exists and $q$ is an accepting state in $G$, then obviously $uv\Sigma^* \not\subseteq L(live(A))$ for some $u$ and all its possible extensions $v \in \Sigma^*$ and, consequently, our algorithm returns $true$.

This procedure for checking if the complement of (2) holds is nondeterministic and does not use more than polynomial space wrt. the size of $A$, and hence is in NPSpace. Again, as NPSpace = PSpace = co-PSpace, the statement follows. □
Theorem 4. The monitorability problem of Büchi automata is PSpace-complete.

Proof. It is sufficient to show PSpace-hardness. We proceed by reducing the PSpace-complete problem of checking if some NFA \( B \) over some alphabet \( \Sigma = \{a_1, \ldots, a_n\} \), is such that \( L(B) = \Sigma^* \) \(^7\). In other words, we construct for \( B \) in at most polynomial time an NBA, \( A \), such that \( A \) is monitorable if and only if \( L(B) = \Sigma^* \).

Let us first check if \( L(B) = \emptyset \) is true. It is well known that this can be done in polynomial time (cf. \(^7\)). If the answer is “yes”, we return the NBA, \( A \), which corresponds to the models of the LTL formula \( GFa \) over the alphabet \( \Sigma' := \{a, b\} \), which by Proposition\(^3\) is non-monitorable.

If \( L(B) \neq \emptyset \), we proceed as follows. Let \( \Sigma_1 := \{a_1^1, \ldots, a_n^1\} \), \( \Sigma_2 := \{a_1^2, \ldots, a_n^2\} \) be alphabets. Let us construct, in linear time, an NFA, \( B_1 \), respectively \( B_2 \), which is like \( B \), except that it accepts \( L(B) \) projected onto \( \Sigma_1 \), respectively onto \( \Sigma_2 \). Let \( B_1 \), respectively \( B_2 \), be the language accepted by \( B_1 \), respectively \( B_2 \). We now construct an NBA, \( A \), such that it accepts the following language, split into three parts for readability:

\[
(i) \quad ((\Sigma_1 \cup \Sigma_2)^* (B_1 B_2 \cup B_2 B_1))^\omega \\
(ii) \quad \cup ((\Sigma_1 \cup \Sigma_2)^* B_1)^\omega \\
(iii) \quad \cup ((\Sigma_1 \cup \Sigma_2)^* B_2)^\omega.
\]

It is easy to see that \( A \) can be constructed in time no more than polynomial wrt. the size of \( B \). We now prove the following two claims.

Let \( L(B) = \Sigma^* \), then \( A \) is monitorable: Notice first that a word \( w \in (\Sigma_1 \cup \Sigma_2)^\omega \) either is

- an alternation of finite words over \( \Sigma_1 \) and \( \Sigma_2 \),
- entirely over \( \Sigma_1 \) (respectively, \( \Sigma_2 \)),
- an alternation of finite words over \( \Sigma_1 \) and \( \Sigma_2 \), followed by an infinite word over \( \Sigma_1 \) or \( \Sigma_2 \).

One can easily verify that all these cases are covered by the language accepted by \( A \). Hence, if \( L(B) = \Sigma^* \), then \( L(A) = \Sigma^\omega \), and therefore \( A \) is monitorable.

Let \( A \) be monitorable, then \( L(B) = \Sigma^* \): For assume not, that is, we assume \( A \) is monitorable, but that \( L(B) \neq \Sigma^* \) holds. From the latter it follows that there must exist a finite word \( u \in \Sigma^* \), corresponding to some word \( u' \in \Sigma_1^* \), such that \( u \notin L(B) \), and consequently \( u' \notin B_1 \) (respectively, for \( B_2 \)). Due to way we have chosen the \( \omega \)-regular expression above, this \( u' \) implies the existence of the language \( L \subseteq (\Sigma_1 \cup \Sigma_2)^\omega \), such that \( L \nsubseteq L(A) \); for example, we can easily prove that \( L := (u' (\Sigma_1 \cup \Sigma_2))^\omega \) is not a subset of any of the three sets given by (i) – (iii) above, and hence \( L \nsubseteq L(A) \). Therefore, \( A \) is not universal over \( (\Sigma_1 \cup \Sigma_2)^\omega \). Notice further that all words \( w \in L(A) \) are such that they require infinitely often the occurrence of a finite word \( u \) either in \( (i) \ B_1B_2 \) and interchangeably with \( B_2B_1 \), \( (ii) \ B_1 \), or \( (iii) \ B_2 \). More concretely, all infinite \( w \) are such that finite words of the form

\[
(i) \quad a_1^1 \ldots a_k^1 a_2^2 \ldots a_2^2 \text{ and optionally the mirrored version occur infinitely often, or} \\
(ii) \quad a_2^1 \ldots a_2^1 \text{ occurs infinitely often, or} \\
(iii) \quad a_2^2 \ldots a_2^2 \text{ occurs infinitely often,}
\]
where, for all indices \( g \), we have \( a_1^g \in \Sigma_1 \) and \( a_2^g \in \Sigma_2 \). In what follows, let \( L_i, L_{ii}, \) and \( L_{iii} \) be the languages corresponding to the sets given by \((i), (ii), \) and \((iii)\), respectively. It is obvious that \( L_i, L_{ii} \) and \( L_{iii} \) each define a dense but not open set over the words in \((\Sigma_1 \cup \Sigma_2)^\omega\) as the infinite repetition of a finite word is required in each case. Moreover, as dense sets are closed under union, \( L_i \cup L_{ii} \cup L_{iii} = L(A) \), it follows that \( A \) defines a dense but not open set. Together with the fact that \( A \) is not universal, it follows that \( A \) defines a classical liveness property, i.e., is not monitorable. Contradiction. \( \square \)

6.3 Deciding monitorability—The deterministic case

It is well-known that languages expressible by deterministic Büchi automata (DBAs) are strictly less expressive than the ones accepted by general (or, nondeterministic) NBAs: For example, one cannot express the language given by the \( \omega \)-regular expression \((a + b)^*a^\omega\) over \( \Sigma = \{a, b\} \) as can be easily proven. On the other hand, it is possible to represent all safety and co-safety languages using DBAs (cf. [10]), although not every DBA-representable language is necessarily monitorable as the example over \( \Sigma = \{a, b\} \), depicted in Fig. 3, illustrates: obviously, the language has neither good nor bad prefix. Hence, it is reasonable wanting to be able to examine DBAs for their monitorability as well. Not surprisingly though, if we know that the automaton in question

![Fig. 3. Deterministic Büchi automaton over \( \Sigma = \{a, b\} \) describing a non-monitorable language.](image)

is deterministic, we can check its monitorability more efficiently than before, using the criterion defined in Lemma 5.

However, before examining this condition, let us first make the following assumption without loss of generality: let \( A \) be a complete automaton; that is, for each symbol \( a \in \Sigma \) and each state \( q \in Q \), there exists a state \( q' \in Q \), such that \( \delta(q, a) = q' \). It is easy to see that completing a deterministic automaton takes time linear in the size of the automaton: one merely has to add a “trap”-state, the corresponding transitions, and self-loops to it as necessary. Let us use the symbol \( \dagger \) to denote this special state. Moreover as a further notational convention, if there exists a path in \( A \), i.e., a state-action sequence, from state \( q \) to \( q' \), we also write \( q \rightsquigarrow q' \), or \( q \rightsquigarrow^u q' \) to denote the fact that the sequence of actions in this path corresponds to the finite word \( u \).

Lemma 5. A deterministic Büchi automaton \( A \), defined as expected, is monitorable if and only if for every state \( q \in Q \), it holds that

- a path exists such that \( q \rightsquigarrow \dagger \), or
- a path exists, \( q \leadsto q' \), with \( \mathcal{L}(A(q')) = \Sigma^\omega \).

**Proof.** As \( A \) is deterministic, let, in what follows, \( Q_0 = \{ q_0 \} \).

If \( q \leadsto \dagger \) holds then there exists a prefix \( uv \in \Sigma^* \), such that \( uv \in \text{bad}(A) \) with \( q_0 \leadsto u q \) and \( q \leadsto v \dagger \). On the other hand, if a path exists, \( q \leadsto q' \), such that \( \mathcal{L}(A(q')) = \Sigma^\omega \), then there exists a prefix \( uv \in \Sigma^* \), such that \( uv \in \text{good}(A) \) with \( q_0 \leadsto u q \) and \( q \leadsto v q' \). Obviously, if every state implies the existence of either a bad or a good prefix, then \( A \) is monitorable.

For the other direction, assume the opposite, i.e., that \( A \) is monitorable and that there exists a state \( q \), such that there do not exist paths \( (i) q \leadsto \dagger \) and \( (ii) q \leadsto q' \), where \( \mathcal{L}(A(q')) = \Sigma^\omega \). Let \( u \in \Sigma^* \) be the word defined by \( q_0 \leadsto u q \). From \( (i) \) it follows that \( u \) cannot be extended to be a bad prefix for \( A \). From \( (ii) \) it follows that \( u \) cannot be extended to be a good prefix for \( A \). Hence, \( u \) is an “ugly prefix”, and \( A \) not monitorable. Contradiction. \( \square \)

**Theorem 5.** The monitorability problem of Büchi automata, when the automata are deterministic, can be solved in polynomial time.

**Proof.** Recall, completion of \( A \) takes linear time wrt. the size of \( A \). So, without loss of generality, we assume the input automaton \( A \) complete already. Checking the condition of Lemma 5 for \( A \) means iterating through the \( |Q| \) states of \( A \) and checking for each \( q \in Q \) whether any of the two sub-conditions holds.

Using depth-first search, it is easy to see that the first condition can be checked in polynomial time (in fact, in time \( O(|Q| + |Q| \cdot |Q|) = O(|Q| \cdot |Q|) \) as there are \( |Q| \cdot |Q| \) transitions in a complete automaton).

The second condition involves checking for each reachable state, \( q' \), from state \( q \), whether or not \( \mathcal{L}(A(q')) = \Sigma^\omega \). In the general case, i.e., when \( A \) is nondeterministic, the latter problem is known to be PSpace-complete in the size of \( A \). However, as \( A \) is deterministic, this condition can, in fact, be checked in time linear wrt. the size of \( A \): As Kurshan outlines a construction for a DBA \( A' \), such that \( \mathcal{L}(A') = \overline{\mathcal{L}(A)} \), where \( A' \) has only \( 2|Q| \) states. Now, checking if \( \mathcal{L}(A') = \emptyset \) holds is known to be LogSpace-complete for NLogSpace [24] and clearly the case if and only if \( \mathcal{L}(A) = \Sigma^\omega \) holds.

From these two observations it now easily follows that checking both conditions of Lemma 5 can be done in no more than polynomial time wrt. the size of \( A \). \( \square \)

Finally, observe that the non-monitorable DBA depicted in Fig. 3 is complete for \( \Sigma = \{a, b\} \), but incomplete and monitorable for \( \Sigma = \{a, b, c\} \).

### 7 Closure of the monitorable \( \omega \)-languages

We now examine closure properties of monitorable \( \omega \)-languages. Let us fix two languages \( L, M \subseteq \Sigma^\omega \) for the remainder of this section.

**Proposition 9.** Let \( L \) and \( M \) be monitorable, then \( L \cap M \) is monitorable.
Proof. Since \( L \) is monitorable two cases arise: every \( u \in \Sigma^* \) is extensible to be a good prefix of \( L \) or to a bad prefix of \( L \) (or both, but this case is covered in the following):

(i) Let us fix some \( u \in \Sigma^* \), such that \( u\Sigma^\omega \cap L = \emptyset \) holds. Then, irrespective of \( M \), \( u\Sigma^\omega \cap L \cap M = \emptyset \), and hence \( u \in \text{bad}(L \cap M) \). 

(ii) Let us fix some \( u \in \Sigma^* \), such that \( u\Sigma^\omega \subseteq L \) holds. Then, by the monitorability of \( M \),

\[
\exists v \in \Sigma^*. uv\Sigma^\omega \cap M = \emptyset \lor uv\Sigma^\omega \subseteq M.
\]

As before, if \( uv\Sigma^\omega \cap M = \emptyset \), then \( uv\Sigma^\omega \cap M \cap L = \emptyset \) and hence \( uv \in \text{bad}(L \cap M) \).

On the other hand, if \( uv\Sigma^\omega \subseteq M \), then \( uv\Sigma^\omega \subseteq L \cap M \) and, consequently, \( uv \in \text{good}(L \cap M) \).

As all finite words are extensible to be either good or bad prefixes of \( L \), and in either case it is possible to find a good or a bad prefix of \( L \cap M \), we conclude that \( L \cap M \) is monitorable as well.

Proposition 10. Let \( L \) be monitorable, then \( L \) is monitorable.

Proof. Follows directly from applying Proposition 2 to Proposition 1.

Theorem 6. The monitorable \( \omega \)-languages are closed under (finitary) application of intersection, complement, and union.

Proof. Follows now easily from the fact that \( L \cup M = \overline{L \cap M} \).

8 Conclusions

The formal concept of monitorability of an \( \omega \)-regular language was first introduced by Pnueli and Zaks [16]. A subsequent result of [2] implies that the monitorability problem of LTL and NBAs as laid out in Sec. 3 is, in fact, decidable using a 2ExpSpace algorithm, whereas [6] recently could show that the monitorable \( \omega \)-languages are strictly more expressive than the commonly used set of safety properties (and, in fact, an unrestricted Boolean combination thereof), known to be PSpace-complete when the language is given by an LTL formula or an NBA. The present paper closes the exponential “gap” that lies between these observations, in that it shows that the monitorability problem of LTL and NBAs are both, in fact, PSpace-complete (unless, of course, the NBAs are, in fact, deterministic).

Besides being of theoretical merit in order to being able to classify the monitorable \( \omega \)-regular languages wrt. existing classifications such as the safety-progress hierarchy, a practical interpretation of this result is that checking the monitorability of a formal specification, given as LTL formula or by an NBA, is computationally as involved as checking, say, if the specification defines a safety language. Moreover, knowing the upper bound of the problem, we can devise new and probably faster algorithms than [2] for checking the monitorability of specifications, such that users can determine—prior to the actual runtime verification process, or any attempts to build a monitor—whether or not their specifications are monitorable at all.

As a final theoretical contribution, it is worth pointing out that, using the results of this paper, one can easily show that the monitorability problem of \( \omega \)-regular expressions
is also PSpace-complete, since there exists a polynomial time transformation from $\omega$-regular expressions to NBAs, which yields membership in PSpace. Together with the proof of Theorem \cite{Garey79} where we used $\omega$-regular expressions, we then obtain completeness for this problem.

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