On dynamic colouring of cartesian product of complete graph with some graphs

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ABSTRACT
A proper vertex colouring is called a 2-dynamic colouring, if for every vertex v with degree at least 2, the neighbours of v receive at least two colours. The smallest integer k such that G has a dynamic colouring with k colours denoted by \( \chi_2(G) \). We denote the cartesian product of G and H by \( G\Box H \). In this paper, we find the 2-dynamic chromatic number of cartesian product of complete graph with complete graph \( K_r\Box K_r \), complete graph with complete bipartite graph \( K_r\Box K_1 \), and wheel graph with complete graph \( W_k\Box K_r \).

1. Introduction
Graphs in this note are simple and finite [1,2]. We denote the edge set and vertex set of G, by E(G) and V(G), respectively. The number of vertices of G is called order of G. The degree of a vertex \( v_i \) in a graph G denoted by \( d_i \) or \( d(v_i) \), is the minimum number of edges incident in it, where \( \delta(G) \) and \( \Delta(G) \) denotes minimum and maximum vertex degree in a graph G. A proper vertex colouring of G is a function \( c : V(G) \rightarrow L \), with property: if \( u,v \in V(G) \) are adjacent, then \( c(u) \) and \( c(v) \) are different. A vertex k-colouring is a proper vertex colouring with \( |L| = k \). The smallest integer \( k \) such that G has a vertex k-colouring is called the chromatic number of G and denoted by \( \chi(G) \). A proper vertex k-colouring is called dynamic [3,4] if for every vertex v with degree at least 2, the neighbours of v receive at least two different colours. The smallest integer \( k \) such that G has a dynamic k-colouring is called 2-dynamic chromatic number of G and denoted by \( \chi_2(G) \). Moreover, 2-dynamic colouring is generally said to be dynamic colouring.

1. Preliminaries
We shall make use of the following lemmas, in order to prove our results,

Lemma 2.1 ([6]): For any positive integer n,

\[
\chi_2(C_n) = \begin{cases} 
3 & \text{if } n \mid 3 \\
5 & \text{if } n = 5 \\
4 & \text{otherwise.}
\end{cases}
\]

Lemma 2.2 ([7]): For \( i \) and \( j \geq 2 \), \( \chi_2(K_{i,j}) = 4 \) and \( \chi_2(K_{i}) = 3 \).

Lemma 2.3 ([7]): For any positive integer n, \( \chi_2(K_n) = n \).

Lemma 2.4 ([8]): Let G and H be two graphs. If \( \delta(G) \geq 2 \),

\[
\chi_2(G\Box H) \leq \max(\chi_2(G), \chi(H)).
\]
Lemma 2.5 ([9]): Let $G$ be a connected graph then,
\[ \chi_r (G) \geq \chi_{r-1} (G) \geq \cdots \geq \chi_2 (G) \geq \chi (G). \]

Lemma 2.6 ([2]): For any positive integer $n$,
\[ \chi (C_n) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd}.
\end{cases} \]

3. Sub results
For finding the dynamic chromatic number of wheel, we need the dynamic chromatic number of join of two graphs.

Theorem 3.1: For any two connected graphs $G$ and $H$, then
\[ \chi_2 (G + H) = \chi (G) + \chi (H). \]
\[ \text{Proof:} \] Let $V(G) = \{u_i : 1 \leq i \leq r\}$ and $V(H) = \{v_j : 1 \leq j \leq s\}$. By the definition of join of two graphs,
\[ V(G + H) = V(G) \cup V(H) \quad \text{and} \quad E(G + H) = E(G) \cup E(H) \cup \{u_iv_j : u_i \in E(G) \text{ and } v_j \in E(H)\}. \]
Consider the vertex colourings
\[ c_1 : V(G) \rightarrow \{1, 2, \ldots, \chi (G)\} \]
and
\[ c_2 : V(H) \rightarrow \{1, 2, \ldots, \chi (H)\}. \]
Let $k = \chi (G) + \chi (H)$ and define $c : V(G + H) \rightarrow \{1, 2, \ldots, k\}$. For every $u_i \in V(G)$, provide $c(u_i) = c_1(u_i)$ and for every $v_j \in V(H)$,
\[ c(v_j) = c_2(v_j) + \max\{c_1(u_i) : i = 1, 2, \ldots, r\}. \]
Now, we claim that $c$ is a dynamic colouring of $G + H$. Clearly, $c$ is a proper colouring. Moreover, for vertices $w \in V(G + H)$ having $d(w) \geq 2$,
\[ |c(N_{G + H} (w))| \geq 2, \]
even still $G = K_2$ and $H = K_1$ and it completes the proof.

Corollary 3.2: For any positive integer $n$,
\[ \chi_2 (W_n) = \begin{cases} 
3 & \text{if } n \text{ is odd} \\
4 & \text{if } n \text{ is even}.
\end{cases} \]
\[ \text{Proof:} \] From the definition of wheel $W_n = C_{n-1} + K_1$. Using Theorem 3.1,
\[ \chi_2 (W_n) = \chi (C_{n-1}) + \chi (K_1) = \chi (C_{n-1}) + 1 \]
and it completes the proof.

4. Main results
In the next theorem, we obtain the dynamic chromatic number of cartesian product between two complete graphs.

Theorem 4.1: For any two positive integers $r, s$, then
\[ \chi_2 (K_r \square K_s) = \begin{cases} 
4 & \text{if } r = s = 2 \\
\max (r, s) & \text{otherwise.}
\end{cases} \]
\[ \text{Proof:} \] Let $V(K_r) = \{u_i : 0 \leq i \leq r - 1\}$ and $V(K_s) = \{v_j : 1 \leq j \leq s\}$. By the definition of cartesian product, $V(K_r \square K_s) = \bigcup_{i=0}^{r-1} \{u_iv_j : 1 \leq j \leq s\}$. For proving this theorem we need to consider two cases,
Case 1: When $r = s = 2$.
Clearly, $K_2 \square K_2 = C_4$. So, $\chi_2 (K_2 \square K_2) = \chi_2 (C_4) = 4 = r + s$.
Case 2: When both $r, s \neq 2$.
Consider the mapping
\[ f \ni f : V (K_r \square K_s) \rightarrow \{1, 2, \ldots, k\}, \text{ where } k = \max (r, s). \]
Define,
\[ f (u_i v_j) = \begin{cases} 
k & \text{if } i + j \equiv (0 \text{ mod } k) \\
 i + j \mod k & \text{if } i + j \not\equiv (0 \text{ mod } k)
\end{cases} \]
since
\[ f (u_i v_j) \neq f (u_i v_{j+1}) \]
and
\[ f (u_i v_j) \neq f (u_{i+1} v_j) \forall i \in \{0, 1, \ldots, r - 1\}, j \in \{1, 2, \ldots, s\}. \]
so, $f$ produces a proper colouring for $K_r \square K_s$. Next we need to show $f$ is dynamic colouring. Since for every vertex $(u_i v_j) \in V(K_r \square K_s)$,
\[ |f (N_{K_r \square K_s} (u_i v_j))| = \Delta (G) + \Delta (H) \geq 2 \]
even $r$ or $s$ is 1 and the proof is complete.

Generally, consider the $p$-dynamic colouring,
\[ \chi_p (K_r \square K_s) = \chi_2 (K_r \square K_s) = \chi (K_r \square K_s), \]
whenever $p \in \{3, 4, \ldots, \max (r, s) - 1\}$, but it varies when $p \geq \max (r, s)$. This will not suit for $K_2 \square K_2$, since
\[ \chi_2 (G) = \chi_{2-1} (G) = \cdots = \chi_2 (G) > \chi (G) = 2, \]
where $\delta = 2$ in $K_2 \square K_2$.

Corollary 4.2: For positive integers $n \geq 2$ and $3 \leq \text{atleast one } n \geq 4$, then
\[ \chi_2 (K_n \square K_2 \square \cdots \square K_n) = \max \{n : 1 \leq j \leq i\} \]
where $i \in \{2, 3, 4 \ldots\}$. 


Corollary 4.3: \( \chi_2(K_2 \square K_2 \square \cdots \square K_2) = 4 \), where we can take cartesian product in any number of times.

In the next two theorems, we obtain the dynamic chromatic number of cartesian product between complete graph and complete bipartite graphs.

Theorem 4.4: For any two positive integers \( s \geq 2 \) and \( n \), then

\[
\chi_2(K_n \square K_{1,s}) = \begin{cases} 
3 & \text{if } n = 1 \\
4 & \text{if } n = 2 \\
n & \text{otherwise.}
\end{cases}
\]

Proof: Let

\[ V(K_n) = \{ u_i : 1 \leq i \leq n \} \]

and

\[ V(K_{1,s}) = \{ w_j : 1 \leq j \leq s \} \]

By the definition of cartesian product,

\[
V(K_n \square K_{1,s}) = \bigcup_{i=1}^{n} \{ u_iw_j : 1 \leq j \leq s \} \cup \bigcup_{i=1}^{n} \{ uiw : 1 \leq j \leq s \}.
\]

Consider a mapping \( f \in f : V(K_n \square K_{1,s}) \rightarrow \{ 1, 2, \ldots, k \} \).

We prove the theorem on the number of vertices in complete graph,

Case 1: When \( n = 1 \).

Clearly, \( K_1 \square K_{1,s} = K_{1,s} \). So, \( \chi_2(K_1 \square K_{1,s}) = \chi_2(K_{1,s}) = 3 \) where \( k = 3 \) in this case.

Case 2: When \( n = 2K_2 \square K_{1,s} \) contains a subgraph of \( K_2 \square K_2 \). So,

\[
\chi_2(K_2 \square K_{1,s}) \geq \chi_2(K_2 \square K_2) = 4.
\]

To prove the reverse inequality, let us define a map \( f \) in such a way that \( f(u_i) = i \) and \( f(u_iw) = i + 2 \) if \( i \in \{ 1, 2 \} \). Clearly \( f \) preserves dynamic colouring and using this \( K_2 \square K_{1,s} \) requires at most 4 colours. It clearly says \( \chi_2(K_2 \square K_{1,s}) = 4 \).

Case 3: When \( n \geq 3 \)

Using Lemma 2.4,

\[
\chi_2(K_n \square K_{1,s}) \leq \max (\chi_2(K_n), \chi(K_{1,s})) = \max (n, 2) = n.
\]

For, the reverse inequality, \( K_0 \square K_{1,s} \) contains \( K_n \), so, it requires at least \( n \) colours for dynamic colouring and this proves the theorem.

From the Theorem 4.4, when \( r = 1 \) then

\[ K_2 \square K_{1,1} = K_2 \square K_2 = C_4. \]

Using Theorem 4.1, \( \chi_2(K_2 \square K_{1,1}) = \chi_2(K_2 \square K_2) = 4. \)

Theorem 4.5: For any positive integers \( r, s \geq 2 \) and \( n \), then

\[
\chi_2(K_n \square K_{r,s}) = \begin{cases} 
4 & \text{if } n = 1, 2 \\
n & \text{if } n \geq 3.
\end{cases}
\]

Proof: Let

\[ V(K_n) = \{ u_i : 1 \leq i \leq n \} \]

and

\[ V(K_{r,s}) = \{ w_j : 1 \leq j \leq r \} \cup \{ v_l : 1 \leq l \leq s \}. \]

By the definition of cartesian product,

\[
V(K_n \square K_{r,s}) = \bigcup_{i=1}^{n} \{ u_iw_j : 1 \leq j \leq r \} \cup \bigcup_{i=1}^{n} \{ u_iw_j : 1 \leq j \leq r \}.
\]

Consider a mapping

\[ f \ni f : V(K_n \square K_{r,s}) \rightarrow \{ 1, 2, \ldots, k \}. \]

We prove the theorem on the number of vertices in complete graph,

Case 1: When \( n = 1 \).

Clearly, \( K_1 \square K_{r,s} = K_{r,s} \). So, \( \chi_2(K_1 \square K_{r,s}) = \chi_2(K_{r,s}) = 4 \), where \( k = 4 \) in this case.

Case 2: When \( n = 2 \)

\( K_2 \square K_{r,s} \) contains a subgraph of \( K_2 \square K_2 \). So,

\[
\chi_2(K_2 \square K_{r,s}) \geq \chi_2(K_2 \square K_2) = 4.
\]

To prove the reverse inequality, by Lemma 2.4,

\[
\chi_2(K_{r,s} \square K_2) \leq \max (\chi_2(K_{r,s}), \chi(K_2)) = \max (4, 2) = 4.
\]

It clearly says, \( \chi_2(K_{n,1}) = 4 \), where \( k = 4 \) in this case.

Case 3: When \( n \geq 3 \)

Define the mapping \( f \) by \( f(u_iw_j) = i \) and

\[
f(u_iw_j) = \begin{cases} 
n & \text{if } i + 1 \equiv 0 \text{ (mod } n) \\
i + 1 \text{ (mod } n) & \text{if } i + 1 \not\equiv 0 \text{ (mod } n)\end{cases}
\]

For every \( i = 1, 2, \ldots, n \), so, \( f \) produces a proper colouring for \( K_n \square K_{r,s} \) with \( n(=k) \) colours. Next we need to show \( f \) is dynamic colouring. Since for every vertex

\[ (u, w) \in V(K_n \square K_{r,s}), \quad |f(N_{K_n \square K_{r,s}}(u, w))| = n - 1 \geq 2 \]

where \( n \) is minimum 3 in this case. Thus \( f \) contributes dynamic colouring and the proof is complete.

In the next theorem, we obtain the dynamic chromatic number of cartesian product of wheel graph with complete graph.

Theorem 4.6: For any positive integer \( l \geq 4 \) and \( n \), then

\[
\chi_2(W_l \square K_n) = \max (\chi_2(W_l), \chi_2(K_n)).
\]
Proof: Let

\[ V(W_l) = \{ u_i : 0 \leq i \leq l - 1 \} \]

and

\[ V(K_n) = \{ v_j : 0 \leq j \leq n - 1 \}, \]

where \( u_0 \) is the centre vertex in the wheel \( W_l \) i.e. \( u_0 \) is adjacent to remaining \( l - 1 \) vertices. By the definition of cartesian product,

\[ V(W_l \square K_n) = \bigcup_{i=0}^{l-1} \{ u_i v_j : 0 \leq j \leq n - 1 \}. \]

Consider a mapping

\[ f : V(K_r \square K_1, s) \rightarrow \{1, 2, \ldots, k\}, \]

where \( k = \max\{\chi_2(W_l), \chi_2(K_n)\} \).

For proving the theorem, we need to consider two cases,

Case 1: When \( l - 1 \) is even.

Define,

\[ f(u_i v_j) = \begin{cases} 
  j \pmod{k} & \text{if } i = 0 \\
  j + 1 \pmod{k} & \text{if } i = 1, 3, \ldots, l - 2 \\
  j + 2 \pmod{k} & \text{if } i = 2, 4, \ldots, l - 1 
\end{cases} \]

Case 2: When \( l - 1 \) is odd.

Define,

\[ f(u_i v_j) = \begin{cases} 
  j \pmod{k} & \text{if } i = 0 \\
  j + 1 \pmod{k} & \text{if } i = 1, 3, \ldots, l - 3 \\
  j + 2 \pmod{k} & \text{if } i = 2, 4, \ldots, l - 2 \\
  j + 3 \pmod{k} & \text{if } i = l - 1 
\end{cases} \]

For the above two cases, we claim that \( f \) is a dynamic colouring of \( W_l \square K_n \). Clearly, \( f \) is a proper colouring. Moreover, for every vertex

\[ u_i \in V(W_l), \quad |c(N_{W_l}(u_i))| \geq 2, \]

where \( c \) is dynamic colouring for \( W_l \). Since for every vertex \( (u_i, v_j) \in V(W_l \square K_n) \), \( |f(N_{W_l \square K_n}(u_i, v_j))| \geq 2 \) and the proof is complete.

\[ \square \]

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