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To cite this version:
Simon Moulin. High frequency dispersive estimates in dimension two. 11 pages. 2007. <hal-00165166v1>

HAL Id: hal-00165166
https://hal.archives-ouvertes.fr/hal-00165166v1
Submitted on 24 Jul 2007 (v1), last revised 13 Jan 2007 (v2)

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High frequency dispersive estimates in dimension two

SIMON MOULIN

Abstract

We prove dispersive estimates at high frequency in dimension two for both the wave and the Schrödinger groups for a very large class of real-valued potentials.

1 Introduction and statement of results

The purpose of this note is to prove dispersive estimates at high frequency for the wave group $e^{it\sqrt{G}}$ and the Schrödinger group $e^{itG}$, where $G$ denotes the self-adjoint realization of the operator $-\Delta + V$ on $L^2(\mathbb{R}^2)$ and $V$ is a real-valued potential which decays at infinity in a way that $G$ has no real resonances nor eigenvalues in an interval $[a_0, +\infty)$, $a_0 > 0$. In fact, we are looking for as large as possible class of potentials for which we have dispersive estimates similar to those we do for the free operator $G_0$. Hereafter $G_0$ denotes the self-adjoint realization of the operator $-\Delta$ on $L^2(\mathbb{R}^2)$. It turns out that in dimension two one can get such dispersive estimates at high frequency for potentials satisfying

$$\sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|V(x)|dx}{|x-y|^{1/2}} \leq C < +\infty. \tag{1.1}$$

Clearly, (1.1) is fulfilled for potentials $V \in L^\infty(\mathbb{R}^2)$ satisfying

$$|V(x)| \leq C\langle x \rangle^{-\delta}, \quad \forall x \in \mathbb{R}^2, \tag{1.2}$$

with constants $C > 0, \delta > 3/2$. Given any $a > 0$, set $\chi_a(\sigma) = \chi_1(\sigma/a)$, where $\chi_1 \in C^\infty(\mathbb{R})$, $\chi_1(\sigma) = 0$ for $\sigma \leq 1$, $\chi_1(\sigma) = 1$ for $\sigma \geq 2$. Our first result is the following

**Theorem 1.1** Let $V$ satisfy (1.1). Then, there exists a constant $a_0 > 0$ so that for every $a \geq a_0$, $0 < \epsilon \ll 1$, $2 \leq p < +\infty$, we have the estimates

$$\left\| e^{it\sqrt{G}} G^{-3/4-\epsilon} \chi_a(G) \right\|_{L^{1} \rightarrow L^{\infty}} \leq C t|t|^{-1/2}, \quad t \neq 0, \tag{1.3}$$

$$\left\| e^{it\sqrt{G}} G^{-3\alpha/4} \chi_a(G) \right\|_{L^{p'} \rightarrow L^{p}} \leq C |t|^{-\alpha/2}, \quad t \neq 0, \tag{1.4}$$

where $1/p + 1/p' = 1, \alpha = 1 - 2/p$.

The estimate (1.3) is proved in [2] under the assumption (1.2). Moreover, if in addition one supposes that $G$ has no strictly positive resonances, it is shown in [2] that (1.3) holds for any $a > 0$ still under (1.2). In dimension three an analogue of (1.3) is proved in [3], [4] for potentials satisfying (1.2) with $\delta > 2$, and extended in [3] to a large subset of potentials satisfying

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|dx}{|x-y|} \leq C < +\infty. \tag{1.5}$$
In dimensions $n \geq 4$ there are very few results. In [1], an analogue of (1.3) is proved for potentials belonging to the Schwartz class, while in [10] dispersive estimates with a loss of $(n-3)/2$ derivatives are obtained for potentials satisfying (1.2) with $\delta > (n+1)/2$. Recently, in [6] dispersive estimates at low frequency have been proved in dimensions $n \geq 4$ for a very large class of potentials, provided zero is neither an eigenvalue nor a resonance.

Our second result is the following

**Theorem 1.2** Let $V$ satisfy (1.1). Then, there exists a constant $a_0 > 0$ so that for every $a \geq a_0$, we have the estimate

$$\|e^{itG} \chi_a(G)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-1}, \quad t \neq 0.$$  

(1.6)

Note that the estimate (1.6) (for any $a > 0$) is proved in [9] for potentials satisfying (1.2) with $\delta > 2$. In dimension three an analogue of (1.6) (for any $a > 0$) is proved in [0] for potentials satisfying (1.5) with $C > 0$ small enough, and in [8] for potentials $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$, $0 < \epsilon \ll 1$, not necessarily small. In dimensions $n \geq 4$, an analogue of (1.6) (for any $a > 0$) is proved in [1] for potentials satisfying (1.2) with $\delta > n$ as well as the condition $\hat{V} \in L^1$. This result has been recently extended in [3] to potentials satisfying (1.2) with $\delta > n-1$ and $\hat{V} \in L^1$. Note also the work [2], where an analogue of (1.6) (for any $a > 0$) with a loss of $(n-3)/2$ derivatives is obtained for potentials satisfying (1.2) with $\delta > (n+2)/2$. In [4] dispersive estimates at low frequency have been also proved in dimensions $n \geq 4$ for a very large class of potentials, provided zero is neither an eigenvalue nor a resonance.

To prove (1.3) we use the same idea we have already used in [7] to prove low frequency dispersive estimates in dimensions $n \geq 4$. The key point is the following estimate which holds in all dimensions $n \geq 2$:

$$h \int_{-\infty}^{\infty} \left\|Ve^{it\sqrt{V}} \psi(h^2 G_0) f \right\|_{L^1} \, dt \leq \gamma_n C_n(V) h^{-(n-3)/2} \|f\|_{L^1}, \quad h > 0,$$

(1.7)

where $\psi \in C^\infty_c((0, +\infty))$, $\gamma_n > 0$ is a constant independent of $V$, $h$ and $f$, and

$$C_n(V) := \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(x)| dx}{|x - y|^{(n-1)/2}} < +\infty. \quad (1.8)$$

Our approach is based on the observation that if

$$C_n(V) h^{-(n-3)/2} \ll 1,$$

(1.9)

then (1.7) implies (under reasonable assumptions on the potential) a similar estimate for the perturbed wave group, namely

$$h \int_{-\infty}^{\infty} \left\|Ve^{it\sqrt{V}} \psi(h^2 G) f \right\|_{L^1} \, dt \leq \tilde{C}_n(V) h^{-(n-3)/2} \|f\|_{L^1}. \quad (1.10)$$

When $n = 3$, (1.9) is fulfilled for small potentials and all $h$, when $n \geq 4$, (1.9) is fulfilled for large $h$ (i.e. at low frequency) without extra restrictions on the potential, while for $n = 2$, (1.9) is fulfilled for small $h$ (i.e. at high frequency) again without restrictions on the potential others than (1.1). Note that (1.10) may hold without (1.9). Indeed, when $n = 3$, (1.10) is proved in [3] for potentials $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$ and all $h > 0$, and then used to prove the three dimensional analogue of (1.6). In the present paper we adapt this approach to the case of dimension two, and show that (1.6) follows from (1.10) for potentials satisfying (1.1) only, provided the parameter $a$ is taken large enough (see Section 3).
2 Proof of Theorem 1.1

Let \( \psi \in C_0^\infty((0, +\infty)) \) and set

\[
\Phi(t; h) = e^{it\sqrt{G}}\psi(h^2G) - e^{it\sqrt{G}}\psi(h^2G_0).
\]

We will first show that (1.3) and (1.4) follow from the following

Proposition 2.1 Let \( V \) satisfy (1.1). Then, there exist positive constants \( C \) and \( h_0 \) so that for \( 0 < h \leq h_0 \) we have

\[
\|\Phi(t; h)\|_{L^1 \to L^\infty} \leq C|h|^{-1/2}, \quad t \neq 0.
\]  

(2.1)

Writing

\[
\sigma^{-3/4 - \epsilon} \chi_a(\sigma) = \int_0^a \psi(\sigma \theta) \theta^{-1/4 + \epsilon} d\theta,
\]

where \( \psi(\sigma) = \sigma^{1/4 - \epsilon} \chi'_1(\sigma) \in C_0^\infty((0, +\infty)) \), and using (2.1) we get

\[
\left\|e^{it\sqrt{G}} G^{-3/4 - \epsilon} \chi_a(G) - e^{it\sqrt{G}} G_0^{-3/4 - \epsilon} \chi_a(G_0)\right\|_{L^1 \to L^\infty}
\leq \int_0^a \left\|\Phi(t; \sqrt{\theta})\right\|_{L^1 \to L^\infty} \theta^{-1/4 + \epsilon} d\theta \leq C|t|^{-1/2} \int_0^a \theta^{-3/4 - \epsilon} d\theta \leq C|t|^{-1/2},
\]  

(2.2)

provided \( a \) is taken large enough. Clearly, (1.3) follows from (2.2) and the fact that it holds for \( G_0 \). To prove (1.4), observe that an interpolation between (2.1) and the trivial bound

\[
\|\Phi(t; h)\|_{L^2 \to L^2} \leq C
\]

yields

\[
\|\Phi(t; h)\|_{L^{p'} \to L^p} \leq C h^{-\alpha}|t|^{-\alpha/2}, \quad t \neq 0,
\]  

(2.3)

for every \( 2 \leq p \leq +\infty, p' \) and \( \alpha \) being as in Theorem 1.1. Now we write

\[
\sigma^{-3\alpha/4} \chi_a(\sigma) = \int_0^a \psi(\sigma \theta) \theta^{-1 + 3\alpha/4} d\theta,
\]

and use (2.3) to obtain (for \( 0 < \alpha \leq 1 \))

\[
\left\|e^{it\sqrt{G}} G^{-3\alpha/4} \chi_a(G) - e^{it\sqrt{G}} G_0^{-3\alpha/4} \chi_a(G_0)\right\|_{L^{p'} \to L^p}
\leq \int_0^a \left\|\Phi(t; \sqrt{\theta})\right\|_{L^{p'} \to L^p} \theta^{-1 + 3\alpha/4} d\theta \leq C|t|^{-\alpha/2} \int_0^a \theta^{-1 + \alpha/4} d\theta \leq C|t|^{-\alpha/2},
\]  

(2.4)

provided \( a \) is taken large enough. Now, (1.4) follows from (2.4) and the fact that it holds for \( G_0 \).

Proof of Proposition 2.1. We will first prove the following

Lemma 2.2 Let \( V \) satisfy (1.1). Then, there exist positive constants \( C \) and \( h_0 \) so that for \( 0 < h \leq h_0 \) we have

\[
\|\psi(h^2G) - \psi(h^2G_0)\|_{L^1 \to L^1} \leq Ch^{1/2}.
\]  

(2.5)
Proof. We will make use of the formula
\[
\psi(h^2G) = \frac{2}{\pi} \int_C \frac{\partial \wedge}{\partial z}(z)(h^2G - z^2)^{-1}zL(dz),
\]  
(2.6)
where \(L(dz)\) denotes the Lebesgue measure on \(C\), \(\wedge \in C_0^\infty(\mathbb{C})\) is an almost analytic continuation of \(\varphi(\lambda) = \psi(h^2)\) supported in a small complex neighbourhood of \(\text{supp} \varphi\) and satisfying
\[
\left| \frac{\partial \wedge}{\partial z}(z) \right| \leq C_N|\text{Im} z|^N, \quad \forall N \geq 1.
\]

For \(\pm \text{Im} \lambda \geq 0, \text{Re} \lambda > 0\), set
\[
R_0^\pm(\lambda) = (G_0 - \lambda^2)^{-1}, \quad R^\pm(\lambda) = (G - \lambda^2)^{-1}.
\]
We have the identity
\[
R^\pm(\lambda) \left(1 + VR_0^\pm(\lambda)\right) = R_0^\pm(\lambda).
\]
(2.7)
It is well known that the kernels of the operators \(R_0^\pm(\lambda)\) are given in terms of the zero order Hankel functions by the formula
\[
[R_0^\pm(\lambda)](x, y) = \pm i4^{-1}H_0^\pm(\lambda|x - y|).
\]
Moreover, the functions \(H_0^\pm\) satisfy the bound
\[
|H_0^\pm(\lambda)| \leq C|\lambda|^{-1/2}e^{-|\text{Im} \lambda|}, \quad |\lambda| \geq 1, \pm \text{Im} \lambda \geq 0,
\]
(2.8)
while near \(\lambda = 0\) they are of the form
\[
H_0^\pm(\lambda) = a_{0,1}(\lambda) + a_{0,2}(\lambda) \log \lambda,
\]
(2.9)
where \(a_{0,j}^\pm\) are analytic functions. In particular, we have
\[
|H_0^\pm(\lambda)| \leq C|\lambda|^{-1/2}, \quad \text{Re} \lambda > 0, \pm \text{Im} \lambda \geq 0.
\]
(2.10)
Using these bounds we will prove the following

**Lemma 2.3** Let \(V\) satisfy (1.1). Then, there exist constants \(C > 0\) and \(0 < h_0 \leq 1\) so that for \(z \in C^\pm := \{z \in \text{supp} \wedge, \pm \text{Im} z \geq 0\}\), we have the estimates
\[
\|VR_0^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq Ch^{1/2}, \quad 0 < h \leq 1,
\]
(2.11)
\[
\|VR^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq Ch^{1/2}, \quad 0 < h \leq h_0,
\]
(2.12)
\[
\|R_0^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq Ch^2|\text{Im} z|^{-2}, \quad 0 < h \leq 1, \text{Im} z \neq 0,
\]
(2.13)
\[
\|R^\pm(z/h)\|_{L^1 \rightarrow L^1} \leq Ch^2|\text{Im} z|^{-2}, \quad 0 < h \leq h_0, \text{Im} z \neq 0.
\]
(2.14)

**Proof.** By (1.1) and (2.10), the norm in the LHS of (2.11) is upper bounded by
\[
\sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} |V(x)||H_0^\pm(z|x - y|/h)|dx \leq Ch^{1/2} \int_{\mathbb{R}^2} \frac{|V(x)dx}{|x - y|^1/2} \leq C'h^{1/2}.
\]
Similarly, the norm in the LHS of (2.13) is upper bounded by
\[
\sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} |H_0^\pm(z|x - y|/h)|dx = h^2 \sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} |H_0^\pm(z|x - y|)|dx
\]
\[
\leq C h^2 |\text{Im} z|^{-2} \int_{\mathbb{R}^2} \langle x - y \rangle^{-3/2} |x - y|^{-1} dx = C' h^2 |\text{Im} z|^{-2} \int_0^\infty (\sigma)^{-3/2} d\sigma.
\]

To prove (2.12) and (2.14) we will make use of the identity (2.7). It follows from (2.11) that there exists a constant \(0 < h_0 \leq 1\) so that for \(0 < h \leq h_0\) the operator \(1 + VR_0^\pm (z/h)\) is invertible on \(L^1\) with an inverse satisfying

\[
\| (1 + VR_0^\pm (z/h))^{-1} \|_{L^1 \to L^1} \leq C, \quad z \in \mathbb{C}_\nu^\pm,
\]

with a constant \(C > 0\) independent of \(z\) and \(h\). Clearly, (2.12) follows from (2.11) and (2.15), while (2.14) follows from (2.13) and (2.15).

To prove (2.5) we rewrite the identity (2.7) in the form

\[
R^\pm (z/h) - R_0^\pm (z/h) = R_0^\pm (z/h) VR_0^\pm (z/h) (1 + VR_0^\pm (z/h))^{-1},
\]

and hence, using Lemma 2.3 and (2.15), we get

\[
\| h^{-2} R^\pm (z/h) - h^{-2} R_0^\pm (z/h) \|_{L^1 \to L^1} \leq C h^{1/2} |\text{Im} z|^{-2}, \quad 0 < h \leq h_0, \quad z \in \mathbb{C}_\nu^\pm, \quad \text{Im} z \neq 0.
\]

It is easy now to see that (2.5) follows from (2.6) and (2.16).

We will now derive (2.1) from the following

**Proposition 2.4** Let \(V\) satisfy (1.1). Then, there exist positive constants \(C\) and \(h_0\) so that we have, for \(0 \leq s \leq 1/2, 0 < \epsilon \ll 1, f, g \in L^1\),

\[
\| e^{it\sqrt{G_0}} \psi(h^2 G_0) f \|_{L^\infty} \leq C h^{-3/2} |t|^{-1/2} \| f \|_{L^1}, \quad h > 0, t \neq 0,
\]

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^\infty |t|^s |x - y|^{-s} |Ve^{it\sqrt{G_0}} \psi(h^2 G_0) f(x)| |g(y)| dt dx dy
\]

\[
\leq C h^{-1/2} \| f \|_{L^1} \| g \|_{L^1}, \quad h > 0,
\]

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^\infty |t|^s |(x - y)/h|^{-s} |Ve^{it\sqrt{G_0}} \psi(h^2 G_0) f(x)| |g(y)| dt dx dy
\]

\[
\leq C h^{-1/2 - \epsilon s} \| f \|_{L^1} \| g \|_{L^1}, \quad 0 < h \leq h_0.
\]

As in [12], using Duhamel’s formula

\[
e^{it\sqrt{G}} = e^{it\sqrt{G_0}} + i \frac{\sin ((t - \tau)\sqrt{G_0})}{\sqrt{G_0}} \left( \sqrt{G} - \sqrt{G_0} \right) - \int_0^t \frac{\sin ((t - \tau)\sqrt{G_0})}{\sqrt{G_0}} Ve^{i\tau\sqrt{G}} d\tau
\]

we get the identity

\[
\Phi(t; h) = \sum_{j=1}^2 \Phi_j(t; h),
\]

where

\[
\Phi_1(t; h) = \left( \psi_1(h^2 G) - \psi_1(h^2 G_0) \right) e^{it\sqrt{G}} \psi(h^2 G)
\]

\[
+ \psi_1(h^2 G_0) e^{it\sqrt{G_0}} \left( \psi(h^2 G) - \psi(h^2 G_0) \right)
\]

\[
- i \psi_1(h^2 G_0) \sin \left( t\sqrt{G_0} \right) \left( \psi(h^2 G) - \psi(h^2 G_0) \right)
\]

\[
+ i \tilde{\psi}_1(h^2 G_0) \sin \left( t\sqrt{G_0} \right) \left( \tilde{\psi}(h^2 G) - \tilde{\psi}(h^2 G_0) \right),
\]

\[
\Phi_2(t; h) = \left( \psi_2(h^2 G) - \psi_2(h^2 G_0) \right) e^{it\sqrt{G}} \psi(h^2 G)
\]

\[
+ \psi_2(h^2 G_0) e^{it\sqrt{G_0}} \left( \psi(h^2 G) - \psi(h^2 G_0) \right)
\]

\[
- i \psi_2(h^2 G_0) \sin \left( t\sqrt{G_0} \right) \left( \psi(h^2 G) - \psi(h^2 G_0) \right)
\]

\[
+ i \tilde{\psi}_2(h^2 G_0) \sin \left( t\sqrt{G_0} \right) \left( \tilde{\psi}(h^2 G) - \tilde{\psi}(h^2 G_0) \right),
\]
\[ \Phi_2(t; h) = -h \int_0^t \tilde{\psi}_1(h^2 G_0) \sin \left( (t - \tau) \sqrt{G_0} \right) V e^{i \tau \sqrt{G_0}} \psi(h^2 G) d\tau, \]

where \( \psi_1 \in C_0^\infty((0, +\infty)), \psi_1 = 1 \) on supp \( \psi \), \( \tilde{\psi}(\sigma) = \sigma^{1/2} \psi(\sigma), \tilde{\psi}_1(\sigma) = \sigma^{-1/2} \psi_1(\sigma) \). By Proposition 2.4 and (2.5), we have

\[ \| \Phi_1(t; h)f \|_{L^\infty} \leq C h^{-1} |t|^{-1/2} \| f \|_{L^1} + C h^{1/2} \| \Phi(t; h)f \|_{L^\infty}, \quad (2.21) \]

\[ t^{1/2} \| [\Phi_2(t; h)f, g] \| \leq h \int_0^{t/2} (t - \tau)^{1/2} \| \sin \left( (t - \tau) \sqrt{G_0} \right) \tilde{\psi}_1(h^2 G_0) g \|_{L^\infty} \| V e^{i \tau \sqrt{G_0}} \psi(h^2 G) f \|_{L^1} d\tau \]

\[ + h \int_{t/2}^t \| V \sin \left( (t - \tau) \sqrt{G_0} \right) \tilde{\psi}_1(h^2 G_0) g \|_{L^1} \tau^{1/2} \| e^{i \tau \sqrt{G_0}} \psi(h^2 G) f \|_{L^\infty} d\tau \]

\[ \leq C h^{-1/2} \| g \|_{L^1} \int_{-\infty}^\infty \| V e^{i \tau \sqrt{G_0}} \psi(h^2 G) f \|_{L^1} d\tau \]

\[ + h \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \| e^{i \tau \sqrt{G_0}} \psi(h^2 G) f \|_{L^1} \int_{-\infty}^\infty \| V \sin \left( (t - \tau) \sqrt{G_0} \right) \tilde{\psi}_1(h^2 G_0) g \|_{L^1} d\tau \]

\[ \leq C h^{-1} \| g \|_{L^1} \| f \|_{L^1} + C h^{1/2} \| g \|_{L^1} \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \| e^{i \tau \sqrt{G_0}} \psi(h^2 G) f \|_{L^\infty}, \]

for \( t > 0 \), which clearly implies

\[ t^{1/2} \| \Phi_2(t; h)f \|_{L^\infty} \leq C h^{-1} \| f \|_{L^1} + C h^{1/2} \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \| e^{i \tau \sqrt{G_0}} \psi(h^2 G) f \|_{L^\infty}. \quad (2.22) \]

By (2.20)-(2.22), we conclude

\[ t^{1/2} \| \Phi(t; h)f \|_{L^\infty} \leq C h^{-1} \| f \|_{L^1} + C h^{1/2} t^{1/2} \| \Phi(t; h)f \|_{L^\infty} \]

\[ + C h^{1/2} \sup_{t/2 \leq \tau \leq t} \tau^{1/2} \| \Phi(\tau; h)f \|_{L^\infty}. \quad (2.23) \]

Taking \( h \) small enough we can absorb the second and the third terms in the RHS of (2.23), thus obtaining (2.1). Clearly, the case of \( t < 0 \) can be treated in the same way. \( \square \)

**Proof of Proposition 2.3.** The kernel of the operator \( e^{i t \sqrt{G_0}} \psi(h^2 G_0) \) is of the form \( K_h(|x - y|, t) \), where

\[ K_h(\sigma, t) = (2\pi)^{-1} \int_0^\infty e^{it\lambda} J_0(\sigma \lambda) \psi(h^2 \lambda^2) \lambda d\lambda = h^{-2} K_1(\sigma h^{-1}, th^{-1}), \quad (2.24) \]

where \( J_0(z) = (H_0^0(z) + H_0^0(z)) / 2 \) is the Bessel function of order zero. It is shown in [12] (Section 2) that \( K_h \) satisfies the estimates (for all \( \sigma, h > 0, t \neq 0 \))

\[ |K_1(\sigma, t)| \leq C |t|^{-s} \sigma^{s-1/2}, \quad \forall s \geq 0, \quad (2.25) \]

\[ |K_h(\sigma, t)| \leq C h^{-3/2} |t|^{-s} \sigma^{s-1/2}, \quad 0 \leq s \leq 1/2. \quad (2.26) \]

Clearly, (2.17) follows from (2.26) with \( s = 1/2 \). We also have

\[ \int_{-\infty}^\infty \langle t \rangle^s |K_1(\sigma, t)| dt \leq C(\sigma)^{s-1/2}, \quad 0 \leq s \leq 1/2. \quad (2.27) \]
This estimate is proved in [3] (see the proof of Lemma 3.1) in all dimensions with \(|t|^s\) replaced by \(|t|^s\), and it is easy to see that this implies (2.27). By (2.24) and (2.27), we get
\[
\int_{-\infty}^{\infty} |(t/h)^s| |K_h(\sigma, t)| dt \leq Ch^{-1} \langle \sigma/h \rangle^s - 1/2, \quad 0 \leq s \leq 1/2, \tag{2.28}
\]
which in turn implies
\[
\int_{-\infty}^{\infty} |t|^s |K_h(\sigma, t)| dt \leq Ch^{-1/2} \langle \sigma/h \rangle^{s-1/2}, \quad 0 \leq s \leq 1/2. \tag{2.29}
\]
It is easy to see that (2.18) follows from (2.29) together with (1.1).

To prove (2.19) we will use the formula
\[
e^{it\sqrt{G}} \psi(h^2G) = (i\pi h)^{-1} \int_{0}^{\infty} e^{it\varphi_h(\lambda)} \left(R^+(\lambda) - R^-(\lambda)\right) d\lambda, \tag{2.30}
\]
where \(\varphi_h(\lambda) = \varphi_1(h\lambda), \varphi_1(\lambda) = \lambda \psi(\lambda^2)\). Combining (2.30) together with (2.7), we get
\[
Ve^{it\sqrt{G}} \psi(h^2G) = (i\pi h)^{-1} \sum_{\pm} \int_{-\infty}^{\infty} VP_h^\pm (t-\tau) U_h^\pm (\tau) d\tau, \tag{2.31}
\]
where
\[
P_h^\pm (t) = \int_{0}^{\infty} e^{it\varphi_h(\lambda)} R_h^\pm (\lambda) d\lambda,
\]
\[
U_h^\pm (t) = \int_{0}^{\infty} e^{it\varphi_h(\lambda)} \left(1 + VR_h^\pm (\lambda)\right)^{-1} d\lambda,
\]
where \(\varphi_h(\lambda) = \varphi_1(h\lambda), \varphi_1(\lambda) \in C_{\infty}^0((0, +\infty))\) is such that \(\varphi_1 = 1\) on supp \(\varphi_1\). The kernel of the operator \(P_h^\pm (t)\) is of the form \(A_h^\pm(|x - y|, t/h)\), where
\[
A_h^\pm (\sigma, t) = \pm i4^{-1} \int_{0}^{\infty} e^{it\varphi_h(\lambda)} H_0^\pm (\sigma\lambda) d\lambda = h^{-1} A_1^\pm (\sigma/h, t/h). \tag{2.32}
\]
In the same way as in the proof of (2.27) one can see that the functions \(A_h^\pm\) satisfy the estimate
\[
\int_{-\infty}^{\infty} |t|^s |A_h^\pm (\sigma, t)| dt \leq C \sigma^{-\epsilon} (\sigma)^{s-1/2}, \quad 0 \leq s \leq 1/2, \quad 0 < \epsilon \ll 1. \tag{2.33}
\]
By (2.32) and (2.33), we have
\[
\int_{-\infty}^{\infty} |t|^s |A_h^\pm (\sigma, t)| dt \leq Ch^{1/2} \langle \sigma/h \rangle^{s-1/2} (1 + \sigma^{-\epsilon}), \quad 0 \leq s \leq 1/2, \quad 0 < h \leq 1, \tag{2.34}
\]
where \(\epsilon = 0\) if \(0 \leq s < 1/2, \epsilon = \epsilon\) if \(s = 1/2\).

Clearly, it suffices to prove (2.19) with \(s = 0\) and \(s = 1/2\). For these values of \(s\), using (1.1), (2.31) and (2.34), we obtain
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |t|^s \langle |x - y|/h \rangle^{-s} |Ve^{it\sqrt{G}} \psi(h^2G) f(x) | g(y) | dt dx dy
\]
\[
\leq Ch^{-1} \sum_{\pm} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle |x - y|/h \rangle^{-s} (|t - \tau|^s + |\tau|^s)
\times |VP_h^\pm (t-\tau) U_h^\pm (\tau) f(x) | g(y) | d\tau dtdx dy
\]
7
\[
\begin{aligned}
&\leq C h^{-1} \sum_{\pm} \int_{R^2} \int_{R^2} \int_{R^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |V(x)| \langle |x-y|/h \rangle^{-s} \langle |t-\tau|^{s} + |\tau|^{s} \rangle \\
&\quad \times |A_h^\pm(\langle x-x'\rangle, t-\tau) | \langle |U_h^\pm(\tau) f(x')| \rangle \langle |g(y)| \rangle \ dr dt dx' dy \\
&\leq C h^{-1} \sum_{\pm} \int_{R^2} \int_{R^2} \int_{R^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |V(x)| \langle |x-y|/h \rangle^{-s} \langle |x-x'|^{s-1/2} (1 + |x-x'|^{-s}) \rangle |g(y)| \\
&\quad \times \left( \int_{-\infty}^{\infty} \langle |U_h^\pm(\tau) f(x')| \rangle \ dr \right) dx' dx dy \\
&\quad \quad + C h^{-1/2} \sum_{\pm} \int_{R^2} \int_{R^2} \int_{R^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |V(x)| \langle |x-y|/h \rangle^{-s} \langle |x-x'|^{s-1/2} \rangle |g(y)| \\
&\quad \quad \times \left( \int_{-\infty}^{\infty} \langle |U_h^\pm(\tau) f(x')| \rangle \ dr \right) dx' dx dy := I_1 + I_2. \quad (2.35)
\end{aligned}
\]

To estimate $I_1$ when $s = 1/2$, set $q = (2\epsilon_{1/2})^{-1}, 1/p + 1/q = 1$, and observe that in view of (1.1) we have the bound

\[
\int_{R^2} |V(x)| \langle |x-y|/h \rangle^{-1/2} |x-x'|^{-\epsilon_{1/2}} dx
\]

\[
\leq \left( \int_{R^2} |V(x)| \langle |x-y|/h \rangle^{-p/2} dx \right)^{1/p} \left( \int_{R^2} |V(x)||x-x'|^{-1/2} dx \right)^{1/q}
\]

\[
\leq C_1 \left( \int_{R^2} |V(x)| \langle |x-y|/h \rangle^{-1/2} dx \right)^{1/p}
\]

\[
\leq C_1 h^{1/(2p)} \left( \int_{R^2} |V(x)||x-y|^{-1/2} dx \right)^{1/p} \leq C_2 h^{1/2-\epsilon_{1/2}}.
\]

Thus, we obtain

\[
I_1 \leq C' h^{s-1/2-\epsilon_s} \sum_{\pm} \int_{R^2} \int_{R^2} \int_{-\infty}^{\infty} |U_h^\pm(\tau) f(x')| |g(y)| \ dr dx' dy. \quad (2.36)
\]

To estimate $I_2$ when $s = 1/2$, we use the inequality

\[
\langle |x-y|/h \rangle^{-1/2} |x-x'|^{-1/2} \leq \langle |x'-y|/h \rangle^{-1/2} \left( |x-y|^{-1/2} + |x-x'|^{-1/2} \right).
\]

We get

\[
I_2 \leq C'' h^{-1/2} \sum_{\pm} \int_{R^2} \int_{R^2} \int_{-\infty}^{\infty} |\tau|^{s} \langle |x'-y|/h \rangle^{-s} |U_h^\pm(\tau) f(x')| |g(y)| \ dr dx' dy. \quad (2.37)
\]
On the other hand, by the identity
\[(1 + VR_0^\pm(\lambda))^{-1} = 1 - VR_0^\pm(\lambda) (1 + VR_0^\pm(\lambda))^{-1},\]
we obtain
\[U_h^\pm(t) = \hat{\varphi}_h(t) - \int_{-\infty}^{\infty} VP_h^\pm(t - \tau) U_h^\pm(\tau) d\tau. \tag{2.38}\]
Since
\[\hat{\varphi}_h(t) = h^{-1}\hat{\varphi}_1(t/h),\]
we have
\[\int_{-\infty}^{\infty} |t|^s|\hat{\varphi}_h(t)| dt \leq Ch^s. \tag{2.39}\]
Using (2.38) and (2.39), in the same way as in the proof of (2.35)-(2.37), we obtain with \(s = 0\) or \(s = 1/2\),
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |t|^s |\langle x - y/h \rangle^{-s} |U_h^\pm(t) f(x)| |g(y)| dt dx dy \leq Ch^{s} \|f\|_{L^1} \|g\|_{L^1} \\
+ Ch^{s+1/2 - \epsilon} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left|U_h^\pm(\tau) f(x')\right| |g(y)| d\tau dx' dy \\
+ Ch^{1/2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |\tau|^s |\langle x' - y/h \rangle^{-s} |U_h^\pm(\tau) f(x')| |g(y)| d\tau dx' dy. \tag{2.40}
\]
Taking \(h\) small enough we can absorb the second and the third terms in the RHS of (2.40) and get the estimate
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |t|^s |\langle x - y/h \rangle^{-s} |U_h^\pm(t) f(x)| |g(y)| dt dx dy \leq C'h^s \|f\|_{L^1} \|g\|_{L^1}. \tag{2.41}
\]
Now (2.19) follows from (2.35)-(2.37) and (2.41).

\section{Proof of Theorem 1.2}

Set
\[\Psi(t; h) = e^{itG} \psi(h^2 G) - e^{itG_0} \psi(h^2 G_0).\]
As in the previous section, one can derive (1.6) from the following

\begin{proposition}
Let \(V\) satisfy (1.1). Then, there exist positive constants \(C\) and \(h_0\) so that for \(0 < h \leq h_0, 0 < \epsilon \ll 1\), we have
\[\|\Psi(t; h)\|_{L^1 \to L^\infty} \leq C'h^{1/2 - \epsilon} |t|^{-1}, \quad t \neq 0. \tag{3.1}\]

\end{proposition}

\begin{proof}
We will derive (3.1) from (2.19). To this end, we will use the identity
\[e^{it\lambda^2} \varphi(h^2 \lambda^2) = \int_{-\infty}^{\infty} e^{i\tau \lambda} \zeta_h(t, \tau) d\tau, \tag{3.2}\]
where \(\varphi \in C_c^\infty((0, +\infty)), \varphi = 1\) on \(\text{supp} \psi_1\), the functions \(\psi\) and \(\psi_1\) being as in the previous section, and
\[\zeta_h(t, \tau) = (2\pi)^{-1} \int_{0}^{\infty} e^{it\lambda^2 - i\tau \lambda} \varphi(h^2 \lambda^2) d\lambda = h^{-1} \zeta_1(th^{-2}, \tau h^{-1}). \tag{3.3}\]

\end{proof}
We deduce from (3.2) the formula

$$e^{itG} \psi(h^2 G) = \int_{-\infty}^{\infty} \phi_0(t, \tau) e^{i\tau \sqrt{G} \psi(h^2 G)} d\tau.$$  \hfill (3.4)

Given any integer $m \geq 0$, integrating by parts $m$ times and using the well known bound

$$\left| \int_{-\infty}^{\infty} e^{it\lambda^2 - i\tau \lambda} \phi(\lambda) d\lambda \right| \leq C|t|^{-1/2}, \quad \forall t \neq 0, \tau \in \mathbb{R},$$

where $\phi \in C_0^\infty(\mathbb{R})$, one easily obtains the bound

$$|\zeta_1(t, \tau)| \leq C_m |t|^{-m-1/2} (\tau)^m, \quad \forall t \neq 0, \tau \in \mathbb{R}. \hfill (3.5)$$

By (3.3) and (3.5),

$$|\zeta_h(t, \tau)| \leq C_m h^{2m} |t|^{-m-1/2} (\tau/h)^m, \quad \forall t \neq 0, \tau \in \mathbb{R}, h > 0, \hfill (3.6)$$

for every integer $m \geq 0$, and hence for all real $m \geq 0$. By (2.5), (2.20) and (3.4), we get

$$\left| \langle \Psi(t; h) f, g \rangle \right| \leq C h^{1/2} \| \Psi(t; h) f \|_{L^\infty} \| g \|_{L^1}
+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle e^{i\tau \sqrt{G} \psi(h^2 G)} f, (\psi_1(h^2 G) - \psi_2(h^2 G)) g \right\rangle \right| d\tau
+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle e^{i\tau \sqrt{G} \psi(h^2 G)} \psi_1(h^2 G) (\psi(h^2 G) - \psi(h^2 G)) f, g \right\rangle \right| d\tau
+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle \sin \left( \tau \sqrt{G_0} \right) \psi_1(h^2 G) (\psi(h^2 G) - \psi(h^2 G)) f, g \right\rangle \right| d\tau
+ \int_{-\infty}^{\infty} |\zeta_h(t, \tau)| \left| \left\langle \sin \left( \tau \sqrt{G_0} \right) \psi(h^2 G) (\psi(h^2 G) - \psi(h^2 G)) f, g \right\rangle \right| d\tau
+ h \int_{-\infty}^{\infty} \int_{0}^{\tau} |\zeta_h(t, \tau)| \left| \left\langle \Psi e^{i\tau \sqrt{G} \psi(h^2 G)} f, \sin \left( (\tau - \tau') \sqrt{G_0} \right) \psi_1(h^2 G) g \right\rangle \right| d\tau' d\tau. \hfill (3.7)$$

Using (3.6) with $m = 1/2$ and (2.28) with $s = 1/2$ together with (2.5), we obtain that the first integral in the RHS of (3.7) is upper bounded by

$$C h |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |\tau/h|^{1/2} \left| K_h(|x - y|, \tau) \right| |f(x)| \left| \left( \psi(h^2 G) - \psi(h^2 G) \right) g(y) \right| d\tau dxdy$$

$$\leq C |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(x)| \left| \left( \psi(h^2 G) - \psi(h^2 G) \right) g(y) \right| dxdy \leq C h^{1/2} |t|^{-1} \| f \|_{L^1} \| g \|_{L^1},$$

and similarly for the next three integrals. The last term is upper bounded by

$$C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\tau} \left| \tau'/h | \right| |(\tau - \tau')/h|^{1/2} \left| K_h(|x - y|, \tau') \right|$$

$$\times \left| \Psi e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau' d\tau dxdy$$

$$\leq C h^{3/2} |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\tau|^{1/2} \left| \Psi e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy$$

$$+ C h^2 |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| e^{i\tau \sqrt{G} \psi(h^2 G)} f(x) \right| |g(y)| d\tau \int_{-\infty}^{\infty} \left| K_h(|x - y|, \tau) \right| d\tau dxdy.$$
\[
\leq C h^{1/2} |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |\tau|^{1/2} \langle |x - y|/h \rangle^{-1/2} \left| V_e^{i\tau \sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| \, d\tau \, dxdy
\]
\[
+ C h |t|^{-1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left| V_e^{i\tau \sqrt{G}} \psi(h^2 G) f(x) \right| |g(y)| \, d\tau \, dxdy
\]
\[
\leq C h^{1/2} |t|^{-1} \| f \|_{L^1} \| g \|_{L^1},
\]
where \( \tilde{K}_h(|x - y|, t) \) denotes the kernel of the operator \( \sin(t\sqrt{G_0}) \tilde{\psi}_1(h^2 G_0) \), and we have used (2.19) together with the fact that the function \( \tilde{K}_h(\sigma, t) \) satisfies (2.28). Thus, we obtain
\[
|\langle \Psi(t; h) f, g \rangle| \leq C h^{1/2} \| \Psi(t; h) f \|_{L^\infty} \| g \|_{L^1} + C h^{1/2-t} |t|^{-1} \| f \|_{L^1} \| g \|_{L^1},
\]
which clearly implies (3.1), provided \( h \) is taken small enough.

References

[1] M. Beals, *Optimal \( L^\infty \) decay estimates for solutions to the wave equation with a potential*, Commun. Partial Diff. Equations 19 (1994), 1319-1369.

[2] F. Cardoso, C. Cuevas and G. Vodev, *Dispersive estimates of solutions to the wave equation with a potential in dimensions two and three*, Serdica Math. J. 31 (2005), 263-278.

[3] P. D’Ancona and V. Pierfelice, *On the wave equation with a large rough potential*, J. Funct. Analysis 227 (2005), 30-77.

[4] V. Georgiev and N. Visciglia, *Decay estimates for the wave equation with potential*, Commun. Partial Diff. Equations 28 (2003), 1325-1369.

[5] M. Goldberg, *Dispersive bounds for the three dimensional Schrödinger equation with almost critical potentials*, GAFA 16 (2006), 517-536.

[6] J.-L. Journé, A. Sofer and C. Sogge, *Decay estimates for Schrödinger operators*, Commun. Pure Appl. Math. 44 (1991), 573-604.

[7] S. Moulin, *Low frequency dispersive estimates for the wave equation in higher dimensions*, submitted.

[8] S. Moulin and G. Vodev, *Low frequency dispersive estimates for the Schrödinger group in higher dimensions*, submitted.

[9] W. Schlag, *Dispersive estimates for Schrödinger operators in two dimensions*, Commun. Math. Phys. 257 (2005), 87-117.

[10] I. Rodnianski and W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. 155 (2004), 451-513.

[11] G. Vodev, *Dispersive estimates of solutions to the Schrödinger equation in dimensions \( n \geq 4 \)*, Asymptot. Anal. 49 (2006), 61-86.

[12] G. Vodev, *Dispersive estimates of solutions to the wave equation with a potential in dimensions \( n \geq 4 \)*, Commun. Partial Diff. Equations 31 (2006), 1709-1733.

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