ON THE INTERNAL APPROACH TO DIFFERENTIAL EQUATIONS

3. INFINITESIMAL SYMMETRIES

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Abstract. The geometrical theory of partial differential equations in the absolute sense, without any additional structures, is developed. In particular the symmetries need not preserve the hierarchy of independent and dependent variables. The order of derivatives can be changed and the article is devoted to the higher–order infinitesimal symmetries which provide a simplifying "linear approximation" of general groups of higher–order symmetries. The classical Lie’s approach is appropriately adapted.

1. Preface

If the invertible higher–order transformations of differential equations are accepted as a reasonable subject, the common Lie–Cartan’s methods are insufficient for complete solution of the symmetry problem. We recall that even the structure of all higher–order symmetries of the trivial (empty) systems of differential equations (that is, of the infinite–order jet spaces without any differential constrains) is unknown [1, 2, 3]. The same can be said for the “linearized theory” of the higher–order infinitesimal transformations treated in this article.

Let us outline the core of the subject. We start with surfaces

\[ w^1 = w^1(x_1, \ldots, x_n), \ldots, w^m = w^m(x_1, \ldots, x_n) \]

lying in the space \( \mathbb{R}^{m+n} \) with coordinates \( x_1, \ldots, x_n, w^1, \ldots, w^m \). The higher–order transformations are defined by formulae

\[
\begin{align*}
\bar{x}_i &= W_i(\cdots, x_i', w^j_I', \cdots), \\
\bar{w}^j &= W^j(\cdots, x_i', w^j_I', \cdots)
\end{align*}
\]

\[(i, i' = 1, \ldots, n; j, j' = 1, \ldots, m) \quad (1.1)\]

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where the given smooth functions $W_i, W^j$ depend on the independent variables $x_1, \ldots, x_n$ and a finite number of jet variables

$$w^j_i = \frac{\partial^{I|\omega^j}}{\partial x^j} = \frac{\partial^{i_1+\ldots+i_r \omega^j}}{\partial x_{i_1} \cdots \partial x_{i_r}} \quad (j = 1, \ldots, m; i_1, \ldots, i_r = 1, \ldots, n; r = 0, 1, \ldots).$$

The resulting surface

$$\bar{w}^1 = \bar{w}^1(\bar{x}_1, \ldots, \bar{x}_n), \ldots, \bar{w}^m = \bar{w}^m(\bar{x}_1, \ldots, \bar{x}_n)$$

again lying in $\mathbb{R}^{m+n}$ appears as follows. We put

$$\bar{x}_i = W_i(\cdot, \cdot, \cdot, \bar{w}^1(\cdot, \cdot, \cdot, x_n), \cdot) = \bar{x}_i(x_1, \ldots, x_n)$$

and assuming

$$\det \begin{pmatrix} \frac{\partial \bar{x}_i}{\partial x^j} \end{pmatrix} = \det (D_i W_i) \neq 0 \quad (D_i = \frac{\partial}{\partial x^i} + \sum w^j_{i, \cdot} \frac{\partial}{\partial w^j_{i, \cdot}}),$$

there exists the smooth inversion $x_i = \bar{x}_i(\bar{x}_1, \ldots, \bar{x}_n)$ of the implicit function system (1.2) where $i = 1, \ldots, n$. This provides the result

$$\bar{w}^j = \bar{w}^j(\bar{x}_1, \ldots, \bar{x}_n) = W^j(\cdot, \cdot, \cdot, \bar{w}^1(\cdot, \cdot, \cdot, \bar{x}_n), \cdot), \frac{\partial^{I|\omega^j}}{\partial x^j}(\cdot, \cdot, \cdot, \bar{x}_1, \ldots, \bar{x}_n, \cdot) \cdot \cdot \cdot.$$

One can also obtain certain prolongation formulae

$$\bar{w}^j_i = \frac{\partial^{I|\omega^j}}{\partial x^j}(\cdot, \cdot, \cdot, \bar{w}^1(\cdot, \cdot, \cdot, x_n), \cdot) \quad (j, I \text{ as above})$$

for the derivatives by resolving the recurrence

$$\sum W^j_{I, v} D_v W_i = \sum D_i W^j_{I, v}. \quad (1.4)$$

Functions $W_i$ satisfying (1.3) and $W^j$ may be arbitrary here. It is however not easy to describe all invertible transformations (1.1) and even more, to investigate the higher-order symmetries of differential equations. So we recall the ancient infinitesimal version

$$\bar{x}_i = x_i + \varepsilon z_i(\cdot, \cdot, \cdot, w^j_{I, \cdot}, \cdot), \quad \bar{w}_i = w^j_{\cdot, \cdot} + \varepsilon z^j(\cdot, \cdot, \cdot, w^j_{I, \cdot}, \cdot)$$

of formulae (1.1) with a "small parameter $\varepsilon". \text{ Then the invertibility mod } \varepsilon^2 \text{ is trivially ensured by the change of } \varepsilon \text{ into } -\varepsilon. \text{ Alas, if we pass to rigorous exposition, quite other difficulties not occuring in the classical finite-dimensional theory appear.}$

Let us introduce the infinite-dimensional space $\mathbb{M}(m, n)$ with coordinates

$$x_i, \ w^j_i \quad (j = 1, \ldots, m; I = i_1 \cdots i_r; i, i_1, \ldots, i_r = 1, \ldots, n; r = 0, 1, \ldots) \quad (1.5)$$

($I \text{ may be permuted}$) supplied with the module $\Omega(m, n)$ of contact forms

$$\omega = \sum \alpha^{j}_{i} \omega^j_i \quad (\text{finite sum, } \omega^j_i = dw^j_i - \sum w^j_{i, l} dx_l). \quad (1.6)$$
We are interested in the vector fields
\[ Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z^j_I \frac{\partial}{\partial w^j_I} \] (infinite sum, arbitrary coefficients) \quad (1.7)
such that \( L_Z \Omega(m, n) \subset \Omega(m, n) \) holds true for the Lie derivative \( L_Z \). (Roughly saying, the contact forms are preserved after infinitesimal \( Z \)-shifts.) The inclusion is equivalent to the congruence
\[ L_Z \omega^j_I = Z | d \omega^j_I + \omega^j_I(Z) = Z | \sum dx_i \wedge \omega^j_I + d \omega^j_I(Z) \equiv 0 \quad (\text{mod } \Omega(m, n)) \]
which immediately gives the recurrence condition
\[ \omega^j_I(Z) = D_i \omega^j_I(Z) \quad \text{hence} \quad z^j_I = D_i z^j_I - \sum w^j_I D_i z^j_I , \] \quad (1.8)
this is the infinitesimal version of clumsy formulae (1.4). With this preparation, we can eventually turn to the main topic.

The vector fields \( Z \) satisfying \( L_Z \Omega(m, n) \subset \Omega(m, n) \) are called \textit{generalized} (or Lie–Bäcklund) \textit{infinitesimal symmetries} of the jet space \( M(m, n) \) in actual literature. However such \( Z \) need not generate any Lie group which is in contradiction with the congenial classical point of view. So we prefer the shorter term \textit{variation} \( Z \) in this case \cite{4}. The \textit{infinitesimal symmetries} ensure the existence of a \textit{true Lie group} in our conception.

An infinitely prolonged system of differential equations
\[ D_{i_1} \cdots D_{i_r} f^k = 0 \quad (k = 1, \ldots, K; i_1, \ldots, i_r = 1, \ldots, n; r = 0, 1, \ldots) \] \quad (1.9)
can be regarded as a subspace \( M \subset M(m, n) \). This is the \textit{external approach}, the reasonings are firmly localized in the ambient space \( M(m, n) \). Every vector field \( Z \) tangent to \( M \) admits the natural restriction \( Y \) to \( M \). If \( Z \) tangent to \( M \) is moreover a variation in the ambient space, we speak of \textit{external variation} \( Y \) of differential equations. Let \( \Omega \) be the restriction of module \( \Omega(m, n) \) to \( M \). If \( Z \) is a variation then clearly \( L_Y \Omega \subset \Omega \) and conversely, a vector field \( Y \) on \( M \) satisfying \( L_Y \Omega \subset \Omega \) can always be extended to a variation \( Z \) (use the recurrence (1.8) restricted to \( M \)). So we may speak of \textit{internal variation} \( Y \) of differential equations as well and there is no essential distinction between external and internal concepts. Quite analogously, the infinitesimal symmetry \( Z \) tangent moreover to \( M \) leads to \textit{external infinitesimal symmetry} \( Y \) of differential equations. Let however \( Y \) be a vector field on \( M \) such that \( L_Y \Omega \subset \Omega \) and let \( Y \) \textit{generate a Lie group on} \( M \). Then we speak of \textit{internal infinitesimal symmetry}. Such \( Y \) can always be extended into a variation \( Z \) but this extension need not generate a Lie group on the ambient space, i.e., \( Y \) \textit{need not be} the external symmetry.

We are interested in the \textit{internal theory} in this article. For this aim, the space \( M \) equipped with module \( \Omega \) will be characterized without any use of the localization in the ambient space \( M(m, n) \).
2. Fundamental concepts

Though we develop [4, 5, 6], the exposition is made self-contained. All fundamental concepts are of the global nature, however, we deal with the local theory, that is, the definition domains are not discussed. No advanced technical tools are needed. We deal with modules of differential forms and vector fields together with the elementary algebra. The existence of bases of various modules to appear is tacitly postulated. A certain novelty lies in the use of the infinite–dimensional manifolds, however, they are of a classical nature without any functional analysis and norm estimates.

Let $M$ be a smooth manifold modelled on $\mathbb{R}^\infty$, that is, there are coordinates $h^i : M \to \mathbb{R}$ ($i = 1, 2, \ldots$) and the structural algebra $\mathcal{F}$ of functions $f : M \to \mathbb{R}$ expressible as the smooth composite $f = f(h^1, \ldots, h^m)$ in terms of coordinates. Then $\Phi$ denotes the $\mathcal{F}$–module of differential forms $\varphi = \sum f^i dg^i$ (finite sum with $f^i, g^i \in \mathcal{F}$) and $T$ denotes the $\mathcal{F}$–module of vector fields $Z$. They are regarded as $\mathcal{F}$–linear functions on $\mathcal{F}$–module $\Phi$, i.e., we have $\mathcal{F}$–linear functions $\varphi(Z) = Z|\varphi = \sum f^i dg^i(Z) = \sum f^i Zg^i \in \mathcal{F}$ ($Z \in \mathcal{T}$ fixed)
of variable $\varphi \in \Phi$.

If $\varphi_1, \varphi_2, \ldots$ is a basis of $\mathcal{F}$–module $\Phi$, then the values $z^i = \varphi^i(Z)$ can be arbitrarily prescribed which is denoted by

$$Z = \sum z^j \frac{\partial}{\partial \varphi^j} \quad \text{(infinite sum, } z^j = \varphi^j(Z)) \quad (2.1)$$

If in particular $\varphi^j = dh^j$ are differentials of coordinates, the well–known series

$$Z = (\sum z^j \frac{\partial}{\partial dh^j} =) \sum z^j \frac{\partial}{\partial h^j} \quad \text{(infinite sum, } z^j = Zh^j)$$

(abbreviation of notation) appears as a particular subcase.

**Definition 2.1.** A submodule $\Omega \subset \Phi$ of a finite codimension $n = n(\Omega)$ is called a diffiety if there exists filtration $\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \bigcup \Omega_l$ by finite-dimensional submodules $\Omega_l \subset \Omega$ ($l = 0, 1, \ldots$) satisfying

$$\mathcal{L}_l \Omega_l \subset \Omega_{l+1} \quad \text{(all } l), \quad \Omega_l + \mathcal{L}_l \Omega_l = \Omega_{l+1} \quad \text{(} l \text{ large enough)} \quad (2.2)$$

the so–called good filtration. Here $\mathcal{H} = \mathcal{H}(\Omega) \subset \mathcal{T}$ is the submodule of all vector fields $Z$ such that $\Omega(Z) = 0$.

Diffieties $\Omega$ exactly correspond to the infinitely prolonged general systems of partial differential equations in the absolute sense, i.e., without any additional structure. They realize the ancient E. Cartan’s dream of the autonomous and coordinate–free theory in surprisingly simple and clear manner.

The technical concept of dependent variables can be related to the choice of the filtration (2.2), see examples below and also the discussion in [4] for the
particular case \( n = n(\Omega) = 1\). The technical concept of independent variables provides the link to the contact forms.

**Definition 2.2.** Functions \( x_1, \ldots, x_n \in \mathcal{F} \) \((n = n(\Omega))\) are called independent variables for the diffiety \( \Omega \) if differentials \( dx_1, \ldots, dx_n \) together with \( \Omega \) generate the module \( \Phi \).

It follows that every form \( \varphi \in \Phi \) admits a unique representation

\[
\varphi = \sum f^i dx_i + \omega \quad (f^i \in \mathcal{F}, \omega \in \Omega).
\]  

(2.3)

The vector fields \( D_1, \ldots, D_n \in \mathcal{H} \) uniquely defined by \( \varphi(D_i) = f^i \) are called total derivatives to the independent variables \( x_1, \ldots, x_n \). If in particular \( \varphi = df \) \((f \in \mathcal{F})\), formula (2.3) provides the well-known contact forms

\[
\omega_f = df - \sum f^i dx_i = \omega_f \in \Omega \quad (f^i = D_i f = df(D_i))
\]  

(2.4)

of the diffiety \( \Omega \).

**Definition 2.3.** A vector field \( Z \in \mathcal{T} \) is a variation of diffiety \( \Omega \) if \( L_Z \Omega \subset \Omega \). If \( Z \) moreover generates a Lie group, we speak of infinitesimal symmetry.

Vector fields will be represented by series (2.1) and appropriate choice of the forms \( \varphi^j \) will simplify the calculation of variations due to the following lemma.

**Lemma 2.1.** A vector field \( Z \in \mathcal{T} \) is a variation of diffiety \( \Omega \) if and only if

\[
(L_{D_i} \omega)(Z) = D_i \omega(Z) \quad (i = 1, \ldots, n; \omega \in \Omega).
\]  

(2.5)

In fact only forms \( \omega \) of a basis of \( \Omega \) are sufficient.

**Proof.** If \( \omega \in \Omega \) then \( d\omega \cong \sum dx_i \wedge \omega_i \) (mod \( \Omega \wedge \Omega \)) for appropriate forms \( \omega_i \in \Phi \), however we infer that \( \omega_i = L_{D_i} \omega \in \Omega \). If \( Z \in \mathcal{T} \) is a vector field, then

\[
L_Z \omega = Z|d\omega + d\omega(Z) \cong - \sum \omega_i(Z) dx_i + \sum D_i \omega(Z) dx_i \quad \text{(mod } \Omega)\)
\]

by applying (2.4) to the function \( f = \omega(Z) \) which implies (2.5). The last assertion of Lemma 2.1 is trivial. \( \square \)

Infinitesimal symmetries cause more difficulties. We can state the following general result [7, Lemma 5.4, Theorems 5.6 and 11.1] without proof. In examples to follow, simplified arguments will be enough.

**Lemma 2.2.** Let \( \Gamma \subset \Omega \) be a finite-dimensional submodule of diffiety \( \Omega \) such that \( \Omega = \Gamma + L_Z \Gamma + L_Z^2 \Gamma + \cdots \) and \( Z \) be a variation of \( \Omega \). Then \( Z \) generates a Lie group \((\text{i.e., } Z \text{ is the infinitesimal symmetry of } \Omega)\) if and only if

\[
L_Z^{k+1} \Gamma \subset \Gamma + L_Z \Gamma + \cdots + L_Z^k \Gamma
\]  

(2.6)

for appropriate \( k \) large enough.
Recalling good filtration (2.2), one can choose \( \Gamma = \Omega_l \) with \( l \) large enough. We have introduced all fundamental concepts and technical tools for the subsequent exposition, however, three Remarks are still necessary for better clarity.

**Remark 1.** Lemma 2.1 replaces the vague condition \( L_Z \Omega \subset \Omega \) for the variation \( Z \) with more effective condition (2.5). Though it is quite simple, the condition

\[
(\mathcal{L}_D \omega)(Z) = D\omega(Z) \quad (\omega \in \Omega, D \in \mathcal{H})
\]

clearly equivalent to (2.5) is still better. Paradoxically, it is only latently occurring in actual literature and we can refer to the ambitious exposition [8, p. 107–113] which rests on rather special mechanism of "\( \ell_\phi \)-linearization". The rule (2.7) involves this mechanism as a particular subcase when \( \omega = \omega_f \in \Omega \) is a contact form and \( Z = 3 \) is the *evolutionary operator* satisfying the additional requirement \( 3x_i = 0 \) (\( i = 1, \ldots, n \)). It is also highly interesting to compare diffieties and symmetries [8] with our definitions 2.1–2.3.

**Remark 2.** The distinction between variations and infinitesimal symmetries is neglected in actual literature. For instance, clearly \( \mathcal{H} \subset \text{Var} \) where \( \text{Var} \) is the module of all variations, however, the factormodule \( \text{Sym} = \text{Var}/\mathcal{H} \) introduced in [8, pp. 9, 107] is a confusing object in this respect since the class \([Z] \in \text{Sym}\) may involve both the true variations and the true infinitesimal symmetries. Moreover the frequent use of the evolutionary operator \( 3 \in [Z] \) is not a lucky measure since \( 3 \) need not be the "best possible" element of the class \([Z]\) in the sense that only appropriate improvement \( 3 + D \in [Z] \) \( (D \in \mathcal{H}) \) may ensure a true Lie group.

**Remark 3.** If the underlying space \( M \) is of a finite dimension, the above theory simplifies. The Frobenius theorem can be applied to the submodule \( \Omega \subset \Phi \) and such a diffiety \( \Omega \) represents a system of partial differential equations where the solution depends on a finite number of constants.

### 3. One Independent Variable

The particular case \( n = n(\Omega) = 1 \) of ordinary differential equations was treated in [4]. Then all variations can be determined by a mere linear algebra but no finite algorithm is as yet available for the totality of all infinitesimal symmetries except the systems of \( m \) equations with \( m + 1 \) unknown functions at most. We discuss the differential equation

\[
\frac{d^2 u}{dx^2} = F(x, u, v, \frac{du}{dx}, \frac{dv}{dx}, \frac{d^2 v}{dx^2}) \quad (u = u(x), v = v(x))
\]

(3.1) as a transparent example here.

Passing to the diffiety, let us introduce the space \( M \) with coordinates

\[
x, u_0, u_1, v_r \quad (r = 0, 1, \ldots)
\]
and diffeiety $\Omega$ with the basis

$$\alpha_0 = du_0 - u_1 dx, \quad \alpha_1 = du_1 - F dx, \quad \beta_r = dv_r - v_{r+1} dx \quad (r = 0, 1, \ldots)$$  \hspace{1cm} (3.2)$$

where $F = F(x, u_0, v_0, u_1, v_1, v_2)$. The total derivative

$$D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u_0} + F \frac{\partial}{\partial u_1} + \sum v_{r+1} \frac{\partial}{\partial v_r} \in \mathcal{H}$$

clearly satisfies $\mathcal{L}_D \alpha_0 = \alpha_1$ and

$$\mathcal{L}_D \alpha_1 = dF - D F dx = F_{u_0} \alpha_0 + F_{v_0} \beta_0 + F_{u_1} \alpha_1 + F_{v_1} \beta_1 + F_{v_2} \beta_2,$$  \hspace{1cm} (3.3)$$

moreover $\mathcal{L}_D \beta_r = \beta_{r+1}$. The order–preserving filtration $\Omega_\ast$ consists of submodules $\Omega_l \subset \Omega$ ($l = 0, 1, \ldots$) generated by the forms $\alpha_r, \beta_r$ where $r \leq l$.

### 3.1 The direct approach to variations.

Using (2.4) with $n = 1$ and $D_1 = D$, we conclude that $Z$ is a variation if and only if $\alpha_1(Z) = D_0 \alpha_0(Z)$ and

$$D_0 \alpha_1(Z) = F_{u_0} \alpha_0(Z) + F_{v_0} \beta_0(Z) + F_{u_1} \alpha_1(Z) + F_{v_1} \beta_1(Z) + F_{v_2} \beta_2(Z),$$

moreover $\beta_{r+1}(Z) = D \beta_r(Z)$. Assuming the development

$$Z = z \frac{\partial}{\partial x} + z^0 \frac{\partial}{\partial u_0} + z^1 \frac{\partial}{\partial u_1} + \sum z_r \frac{\partial}{\partial v_r} \quad (z = Z x, z^r = \alpha_r(Z), z_r = \beta_r(Z))$$

of the kind (2.1), we have the recurrences $z^1 = D z^0, z^r+1 = D z_r$ ($r = 0, 1, \ldots$) together with the crucial requirement

$$D^2 z^0 = F_{u_0} z^0 + F_{v_0} z_0 + F_{u_1} D z^0 + F_{v_1} D z_0 + F_{v_2} D^2 z_0$$  \hspace{1cm} (3.4)$$

for the initial coefficients $z_0$ and $z^0$ (coefficient $z$ is arbitrary). It is not easy to resolve equation (3.4) where the functions $z_0, z^0$ depend on finite but uncertain number of coordinates.

### 3.2 Better approach.

We introduce the alternative basis $\pi_r$ ($r = 0, 1, \ldots$) of diffeiety $\Omega$ such that $\mathcal{L}_D \pi_r = \pi_{r+1}$, the standard basis \[.\] Then $Z$ is a variation if and only if $\pi_{r+1}(Z) = D \pi_r(Z)$ and therefore the formula

$$Z = z \frac{\partial}{\partial x} + \sum D^r p \frac{\partial}{\partial \pi_r} \quad (p = \pi_0(Z) \text{ hence } D^r p = \pi_r(Z))$$  \hspace{1cm} (3.5)$$

with arbitrary functions $z$ and $p$ provides all variations $Z$. (Some "degenerate cases" are omitted here, see below.)

In order to obtain the standard basis, identity (3.3) will be reduced. Clearly

$$\mathcal{L}_D(\alpha_1 - F_{v_2} \beta_1) = F_{u_0} \alpha_0 + F_{v_0} \beta_0 + F_{u_1} \alpha_1 + (F_{v_1} - DF_{v_2}) \beta_1.$$

Denoting $\alpha = \alpha_1 - F_{v_2} \beta_1$, the form $\alpha_1$ can be replaced with $\alpha$ in the original basis (3.2) and identity (3.3) simplifies as

$$\mathcal{L}_D \alpha = F_{u_0} \alpha_0 + F_{v_0} \beta_0 + F_{u_1} \alpha + A \beta_1 \quad (A = F_{v_1} + F_{u_1} F_{v_2} - DF_{v_2}).$$
The last summand can be deleted as well. Clearly
\[ L_D(\alpha - A\beta_0) = F_{u_0}\alpha_0 + (F_{v_0} - DA)\beta_0 + F_{u_1}\alpha, \]

\[ L_D\alpha_0 = \alpha_1 = \alpha + F_{v_2}\beta_1 \text{ hence } L_D(\alpha_0 - F_{v_2}\beta_0) = \alpha - DF_{v_2}\beta_0. \]

Denoting \( \beta = \alpha - A\beta_0, \gamma = \alpha_0 - F_{v_2}\beta_0, \) both forms \( \alpha \) and \( \alpha_0 \) can be replaced with \( \beta \) and \( \gamma \).

We have the basis

\[ \beta = \alpha - A\beta_0, \gamma = \alpha_0 - F_{v_2}\beta_0, \beta_r (r = 0, 1, \ldots) \quad (3.6) \]

satisfying

\[ L_D\beta = F_{u_0}(\gamma + F_{v_2}\beta_0) + F_{u_1}(\beta + A\beta_0) + (F_{v_0} - DA)\beta_0 = F_{u_0}\gamma + F_{u_1}\beta + B\beta_0, \]

\[ L_D\gamma = \alpha - DF_{v_2}\beta_0 = \beta + A\beta_0 - DF_{v_2}\beta_0 = \beta + C\beta_0 \]

with certain coefficients \( B \) and \( C \). Finally

\[ L_D(C\beta - B\gamma) = DC\beta - DB\gamma + C L_D\beta - B L_D\gamma = M\beta + N\gamma \]

with certain \( M, N \). We are done.

3.3. **The controllable subcase.** In general

\[ \det \begin{pmatrix} C & -B \\ M & N \end{pmatrix} \neq 0. \]

Then we have the standard basis

\[ \pi_0 = C\beta - B\gamma, \pi_1 = L_D\pi_0 = M\beta + N\gamma, \pi_r = L_D^r\pi_0 \quad (r = 2, 3, \ldots). \]

Indeed, the forms \( \beta, \gamma \) are linear combinations of \( \pi_0, \pi_1 \). Either \( B \neq 0 \) or \( C \neq 0 \) therefore \( \beta_0 \) can be expressed in terms of \( \beta, \gamma, L_D\beta, L_D\gamma \) and therefore in terms of \( \pi_0, \ldots, \pi_4 \). Then the forms \( \beta_r = L_D^r\beta_0 \) are involved, too.

3.4. **On the noncontrollable subcase.** If \( B = C = 0 \) identically then \( d\beta = d\gamma \equiv 0 \text{ (mod } \beta, \gamma) \) and \( F = D^2G \) for appropriate \( G = G(x, u_0, v_0) \). If either \( B \neq 0 \) or \( C \neq 0 \) but \( \det \begin{pmatrix} C & -B \\ M & N \end{pmatrix} = 0 \) then the form \( C\beta - B\gamma \) is a multiple of a total differential and \( F = DG \) where \( G = G(x, u_0, v_0, u_1, v_1) \). We refer to [4] for this "degenerate" case.

3.5. **Towards the infinitesimal symmetries.** While variations were obtained in full generality, the additional requirements of Lemma 2.2 cannot be completely analyzed in full generality here due to a limited space. We may refer to [4] for the "simple" function \( F = F(v_1) \). So we shall deal with analogous "easier" function.
3.6. **Survey of explicit formulae.** Let us suppose \( F = u_0 v_1 \) from now on. In the direct approach, equation (3.4) simplifies as
\[
D^2 z^0 = v_1 z^0 + u_0 D z_0 \quad (D = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u_0} + u_0 v_1 \frac{\partial}{\partial u_1} + \sum v_{r+1} \frac{\partial}{\partial v_r}).
\] (3.7)
We shall soon state the explicit solution.

Turning to the standard basis, one can see that
\[
\alpha = \alpha_1, \beta = \alpha_1 - u_0 \beta_0, \gamma = \alpha_0, \pi_0 = u_0 \beta + u_1 \gamma, \pi_1 = 2u_1 \beta + 2u_0 v_1 \gamma.
\]
This implies
\[
\Delta \beta = 2u_0 v_1 \pi_0 - u_1 \pi_1, \Delta \gamma = -2u_1 \pi_0 + u_0 \pi_1 \quad (\Delta = 2u_0^2 v_1 - 2u_1^2)
\]
and therefore
\[
\Delta \alpha_0 = -2u_1 \pi_0 + u_0 \pi_1, u_0 \Delta \beta_0 = \Delta \alpha_1 - 2u_0 v_1 \pi_0 + u_1 \pi_1.
\]
So we have the solution
\[
\begin{align*}
  z^0 &= \alpha_0(Z) = \frac{1}{\Delta} (-2u_1 p + u_0 D p), \\
  z_0 &= \beta_0(Z) = \frac{1}{u_0} D z^0 - \frac{2u_1}{\Delta} p + \frac{u_1}{u_0 \Delta} D p
\end{align*}
\] (3.8)
of equation (3.7) with arbitrary function \( p \). As yet we have dealt with variations only.

3.7. **The realm of true symmetries.** Let us consider variations \( Z \) such that moreover
\[
\mathcal{L}_Z \pi_0 = \lambda \pi_0
\]
(3.9) for an appropriate factor \( \lambda \in \mathcal{F} \). In more detail, we have the requirement
\[
Z | d \pi_0 + dp = \lambda \pi_0 \quad (p = \pi_0(Z))
\]
where
\[
d \pi_0 = dx \wedge \pi_1 + \alpha_0 \wedge \beta + u_0 (-\alpha_0 \wedge \beta_0) + \alpha_1 \wedge \gamma = dx \wedge \pi_1 + 2u_0 \beta_0 \wedge \gamma,
\]
\[
dp = Dp dx + pu_0 \alpha_0 + pu_1 \alpha_1 + \sum p_v \beta_r.
\]
We insert \( \alpha_0 = \gamma, \alpha_1 = \beta + u_0 \beta_0 \) in order to use the advantageous intermediate basis (3.6). Then (3.9) is expressed by the identity
\[
(2u_1 z + p_{u_1}) \beta + (2u_0 v_1 z + p_{u_0}) \gamma + (-2u_0 c + u_0 p_{u_1} + p_{u_0}) \beta_0 + p_{v_1} \beta_1 + p_{v_2} \beta_2 + \cdots = \lambda (u_0 \beta + u_1 \gamma)
\]
where \( c = \gamma(Z) = \alpha_0(Z) = z^0 \). The conditions
\[
2u_1 z + p_{u_1} = \lambda u_0, \ 2u_0 v_1 z + p_{u_0} = \lambda u_1
\]
(3.10)
determine \( z \) and the useless factor \( \lambda \) in terms of function \( p \). Function \( p \) is subjected to the remaining conditions
\[
-2u_0 c + u_0 p_{u_1} + p_{v_0} = 0, p_{v_1} = p_{v_2} = \cdots = 0.
\]
It follows that \( p = p(x, u_0, u_1, v_0) \) and inserting (3.8) for \( c \), only one differential equation
\[
u_0^2(p_x + u_1 p_{u_0}) + 2u_1^2(p_{v_0} + u_1 p_{u_1}) = 2u_0 u_1 p
\]
for this function appears. Some particular solutions can be explicitly found.

**Remark 4.** In fact we have obtained all the infinitesimal symmetries \( Z \) and the reasons are as follows. In one direction, one can observe that only the forms \( \omega = \lambda \pi_0 \) (\( \lambda \neq 0 \)) have the property that the family \( \omega, L\omega, L^2\omega, \ldots \) generates the diffiety \( \Omega \). Since every infinitesimal symmetry \( Z \) preserves this property, it does satisfy (3.9). In the opposite direction, (3.9) means that the vector field \( Z \) preserves the Pfaffian equation \( \pi_0 = 0 \) and therefore preserves the adjoint space to this equation. This is a finite-dimensional space and therefore \( Z \) generates a Lie group. (We recall that the adjoint space consists of the most economical family of functions such that the equation \( \pi_0 = 0 \) can be expressed in terms of these functions. In our case, three functions
\[
\frac{u_0}{u_1}, u_0 v_0 - u_1 \ln u_0, 2x - u_0 u_1
\]
are enough. It follows that vector fields \( Z \) generate the Lie contact group preserving the equation \( \pi_0 = 0 \) in three-dimensional underlying space.)

3.8. **On the evolutionary operators.** Assuming \( z = 0 \) in equations (3.10), then the resulting conditions \( p_{u_1} = \lambda u_0, p_{v_0} = \lambda u_1 \) imply \( p = p(x, u_0 u_1, v_0) \) and equation (3.11) admits only trivial solution \( p = 0 \) of this kind. It follows that the evolutionary operators generating a group do not exist.

4. **Several independent variables**

For the partial differential equations, the simplified requirements on the variations can be found as well, however, they remain complicated so that no general existence or non-existence results are actually available. The finite algorithm can be found only for the higher-order symmetries of (roughly saying) \( m \) equations with \( m + 1 \) unknown functions at most. The differential equation
\[
\frac{\partial u}{\partial x_n} = F(x_1, \ldots, x_n, u, v, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{n-1}}, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n})
\]
where \( u = u(x_1, \ldots, x_n), v = v(x_1, \ldots, x_n) \) may serve for an example. We however suppose \( n = 2 \) for technical reasons.

Let us introduce space \( M \) with coordinates
\[
x, y, u_r, v_{rs} \quad (r, s = 0, 1, \ldots),
\]
where $F_c$ for the initial terms $A = 0$ and $B = 0$ identically.

4.1. Towards the variations. We suppose

$$Z = z_1 \frac{\partial}{\partial x} + z_2 \frac{\partial}{\partial y} + \sum_{r} z_r \frac{\partial}{\partial \alpha_r} + \sum_{s} z_s \frac{\partial}{\partial \beta_s} \quad (z_r = \alpha_r(Z), z_s = \beta_s(Z))$$

in the direct approach. Identity (4.1) immediately gives the requirement

$$D_y z = F_v z + F_u D_x z = 0$$

for the initial terms $z = \alpha(Z)$ and $z_0 = \beta(Z)$ of the recurrences

$$z_{r+1} = D_x z_r, \quad z_{s+1} = D_x z_s, \quad z_{r,s+1} = D_y z_{r,s}.$$ 

Passing to the better approach, identity (4.1) is simplified as

$$L_{D_x} \gamma = F_v \gamma + A \beta + B \beta_x + F_u L_{D_x} \gamma \quad (\gamma = \alpha - F_v \beta)$$

where

$$A = F_v + F_u F_v - D_y F_v + F_u D_x F_v, \quad B = F_v + F_u D_v.$$ 

The forms $\alpha_r$ can be replaced with $\gamma_r = \gamma_{D_x} (r = 0, 1, \ldots)$ in the original basis of $\Omega$. So we obtain

$$Z = z_1 \frac{\partial}{\partial x} + z_2 \frac{\partial}{\partial y} + \sum_{r} c_r \frac{\partial}{\partial \gamma_r} + \sum_{s} z_s \frac{\partial}{\partial \gamma_s} \quad (c_r = \gamma_r(Z))$$

together with the recurrence $c_{r+1} = D_x c_r$ ($r = 0, 1, \ldots$) and the requirement

$$D_y c_0 = F_v c_0 + A z_0 + B D_x z_0 + F_v D_x c_0$$

for the initial terms $c_0$ and $z_0$. Alas, the existence of a solution is still ambiguous. Explicit solution can be found if $A \neq 0$ but $B = 0$ identically.
4.2. **Infinitesimal symmetries.** Variations $Z$ such that moreover
\[ \mathcal{L}_Z \beta = \lambda^1 \beta + \lambda^2 \gamma, \quad \mathcal{L}_Z \gamma = \mu^1 \beta + \mu^2 \gamma \] (4.5)
are just the infinitesimal symmetries. In more detail, let us abbreviate $b = z_{00} = \beta(Z), c = c_0 = \gamma(Z)$. Then
\[ \mathcal{L}_Z \beta = Z \frac{d\beta}{dB} + db, \quad \mathcal{L}_Z \gamma = Z \frac{d\gamma}{D\gamma} + dc \]
should be inserted into (4.5) with
\[ \frac{d\beta}{d\beta} = dx \wedge \beta_x + dy \wedge \beta_y, \quad \frac{d\gamma}{d\gamma} = d\alpha - F_v \beta_x + dF_v \beta_y \]
and
\[ db \equiv b_u \alpha_1 + b_v \beta_x + b_v \beta_y + (\cdot), \quad dc \equiv c_u \alpha_1 + c_v \beta_x + c_v \beta_y + (\cdot) \]
(mod $\beta, \gamma$) where $(\cdot)$ denotes the higher-order summands. The final result follows by a mere routine. By looking at the isolated terms $(\cdot)$, one can infer that $b, c$ depend only on the lower-order variables $x, y, u, v, u_1, v_x, v_y$. So the requirement (4.4) is reduced to a finite dimension. Moreover
\[ z^1 + b_v = z^2 + b_v = b_u = 0 \]
easily follows from the first equation (4.5) and the second equation implies
\[ bF_v v_y + c_v = 0, \quad z^1 F_v u_1 + bF_v u_1 = 0, \quad z^2 F_u x + bF_v v_x + c_v = 0. \] (4.6)
It follows that the symmetries are subjected to very strong additional requirements, however, they are of the classical finite-dimensional nature. Assuming $z^1 = z^2 = 0$, then $Z = 3$ becomes the evolitional operator and the system (4.6) is compatible: the Frobenius theorem comfortably applies.

**Remark 5.** Variations $Z$ preserving the Pfaffian system $\beta = \gamma = 0$ preserve the adjoint variables and therefore generate a group. The converse assertion will be mentioned in broader context later on.

5. **Trivial differential equations**

We recall the space $M = M(m, n)$ with jet coordinates (1.5), the diffiety $\Omega = \Omega(m, n)$ of the forms (1.6) and the total derivatives $D_i$ in (1.3). They correspond to the trivial (empty) system (1.9). The variations
\[ Z = \sum z_i D_i + \sum z^j \frac{\partial}{\partial \omega^j_i} \quad (z_i = Z x_i, z^j = \omega^j_i(Z)) \] (5.1)
are given by the formulae
\[ z^j_i = \omega^j_i(Z) = D_I \omega^j_i(Z) = D_I z^j \quad (I = i_1 \cdots i_r, D_I = D_{i_1} \cdots D_{i_r}) \] (5.2)
already stated in [1.8].

We recall the triviality
\[ df = \sum \frac{\partial f}{\partial x_i} dx_i + \sum \frac{\partial f}{\partial w^i_j} dw^i_j = \sum D_i f dx_i + \sum \frac{\partial f}{\partial w^i_j} \omega^i_j \quad (f \in F) \quad (5.3) \]

which implies the equalities \( \partial f / \partial w^i_j = \partial f / \partial \omega^i_j \). The inclusion \( L_Z \Omega \subset \Omega \) is clearly equivalent to \( L_Z \mathcal{H} \subset \mathcal{H} \) whence the rules
\[ L_Z D_i = [Z, D_i] = -D_i z_i \cdot D_i, \quad L_Z \mathcal{L}_D, L_Z = -D_i z_i \cdot L_D, \]
\[ L_Z \omega^i_j = L_Z \mathcal{L}_D, \omega^i_j = L_D \mathcal{L}_Z, \omega^i_j - D_i z_i \omega^i_j \quad (5.4) \]
easily follow. If \( \Omega(m, n)_l \subset \Omega(m, n) \) is the submodule generated by all forms \( \omega^i_j \) \( (j = 1, \ldots, m; I = i_1 \cdots i_r; |I| = r \leq l) \) then \( \Omega_* : \Omega(m, n)_0 \subset \Omega(m, n)_1 \subset \cdots \subset \Omega(m, n) \) is a good "order–preserving" filtration.

**Theorem 5.1** (Lie–Bäcklund). Let \( m > 1 \) and \( Z \) be a variation such that \( L_Z \Omega(m, n)_l \subset \Omega(m, n)_{l+1} \) for appropriate \( l \). Then the inclusion is satisfied for all \( l = 0, 1, \ldots \) and \( Z \) is the infinitesimal point symmetry in the sense that all functions \( Zx_i, Zw^j \) \( (i = 1, \ldots, n; j = 1, \ldots, m) \) depend only on coordinates \( x_1, \ldots, x_n, w^1, \ldots, w^m \).

**Proof.** Abbreviating \( \Omega_l = \Omega(m, n)_l \) for now, we assume \( L_Z \Omega_l \subset \Omega_l \) with a certain \( l \). Then \( L_Z \Omega_{l+1} \subset \Omega_{l+1} \) follows by applying (5.4) to the equality \( \Omega_{l+1} = \Omega_l + L_Z \Omega_l \). Let \( l = 0 > 0 \) and \( \omega \in \Omega_{l-1} \). Then \( L_D \omega \in \Omega_{l-1} \) hence \( L_Z L_D \omega \in \Omega_l \) and (5.3) implies \( L_D, \omega \in \Omega_l \) whence \( L_Z \omega \in \Omega_{l-1} \) and \( L_Z \Omega_{l-1} \subset \Omega_{l-1} \). So we may assume \( L_Z \Omega_0 = \Omega_0 \). In more detail
\[ L_Z \omega^j = [Z, \omega^j] = \sum \lambda^j_i \omega^k = \sum \lambda^j_i \omega^k = \sum \lambda^j_i \omega^k \quad (j = 1, \ldots, m) \]
where \( d \omega^j = \sum dx_i \wedge \omega^i_j \). Applying moreover (5.3) to the functions \( f = \omega^j(Z) = z^j \), one can directly obtain the identities
\[ z_i + \sum \frac{\partial z^j_i}{\partial w^i_j} = 0, \quad \frac{\partial z^j_i}{\partial w^i_k} = 0 \quad (j \neq k), \]
\[ \frac{\partial z^j_i}{\partial w^i_j} = 0 \quad (i = 1, \ldots, n; j, k = 1, \ldots, m; |I| > 1). \]
Since \( m > 1 \), we infer that
\[ z_i = z_i(\cdot, x_i, w^j, \cdot), \quad z^j = -\sum z_i w^i_j + F^j(\cdot, x_i, w^j, \cdot) \]
for appropriate functions \( F^j \) and therefore
\[ Z w^j = dw^j(Z) = \omega^j(Z) + \sum w^i_j dx_i(Z) = z^j + \sum w^i_j z_i = F^j. \]
This concludes the proof. \( \square \)
Remark 6. The original Lie–Bäcklund theorem concerns the symmetries of the finite–order jet spaces. It was rigorously proved much later and we refer to the extensive discussion [8, pp. 66–80]. Alas, this rather tedious method fails for the diffieties where we refer to short tricky proof [3].

The above result for the infinitesimal symmetries can be completed as follows.

**Theorem 5.2.** All infinitesimal symmetries of diffieties $\Omega(1, n)$ are only the classical Lie contact transformations.

**Proof.** Lemma 2.2 can be applied with module $\Gamma = \Omega(1, n)$ which is generated by single form $\omega^1$. It follows that all forms $\omega^1, LZ\omega^1, L^2Z\omega^1, \ldots$ must be of a limited order and this is possible if and only if $LZ\omega^1 = \lambda \omega^1$. □

Remark 7. We note on this occasion that the classical "wave mechanism" of Lie’s contact transformations of diffiety $\Omega(1, n)$ can be generalized to obtain many higher–order symmetries of diffieties $\Omega(m, n)$ where $m > 1$ [9, 10]. They destroy the order–preserving filtration $\Omega(m, n)_*$. Let us conclude with the infinitesimal symmetries $Z$ such that

$$\mathcal{L}_Z^2 \Omega(m, n)_0 \subset \Omega(m, n)_0 + \mathcal{L}_Z \Omega(m, n)_0 \subset \Omega(m, n)_1.$$  \hfill (5.5)

Alas, even this "simple" symmetry problem cannot be resolved in full generality here. So we suppose $m = 2$ and the more explicit condition

$$\mathcal{L}_Z \pi = \lambda \pi, \mathcal{L}_Z \omega^2 = \sum \mu_j \omega^j + \sum \lambda_i \mathcal{L}_D_i \pi \quad (\pi = \omega^1 + a \omega^2)$$  \hfill (5.6)

where $j = 1, 2$ and $a$ is a given function. Requirement (5.3) is satisfied since

$$\mathcal{L}_Z \mathcal{L}_D_i \pi = \mathcal{L}_D_i \mathcal{L}_Z \pi - D_i z_i \mathcal{L}_D_i \pi = D_i \lambda \pi + (\lambda - D_i z_i) \mathcal{L}_D_i \pi \in \Omega(2, n)_1$$

by applying (5.4).

Passing to the symmetry problem, the obvious identities

$$\mathcal{L}_Z \pi = \mathcal{L}_Z \omega^1 + a \mathcal{L}_Z \omega^2 + Za \omega^2, \quad \mathcal{L}_D_i \pi = \omega^1_i + a \omega^2_i + D_i a \omega^2$$

$$\mathcal{L}_Z \omega^j = Z \sum dx_i \wedge \omega^j_i + dz^j, \quad dz^j = \sum D_i z^j dx_i + \sum \frac{\partial z^j}{\partial w^i} \omega^i$$

directly give the following result. The coefficients $z^j = z^j(\cdot, x_i, w^j, w^j_i, \cdot)$ are of the first order at most and satisfy the conditions

$$Dz^1 + aDz^2 = Za, \quad D_i z^1 = aD_i z^2, \quad z_i = D_i z^2$$

where

$$D = a \frac{\partial}{\partial w^i} - \frac{\partial}{\partial w^2}, \quad D_i = a \frac{\partial}{\partial w^i} - \frac{\partial}{\partial w^2}, \quad Za = \sum z_i D_i a + \sum D_i z^j \frac{\partial a}{\partial w^i}.$$
Explicit solution with $\lambda_i \neq 0$ could be stated in the particular case when $a = a(x_1, \ldots, x_n)$. In any case, we conclude that the higher-order symmetries of trivial equations are highly nontrivial.

6. Further perspectives

We have employed only elementary tools as yet. The general theory however rests on more advanced principles.

6.1. The Lie–Poisson bracket. A variation $Z$ of diffiety $\Omega$ is determined if the initial terms of certain recurrences, namely some functions $\omega(Z)$ with special $\omega \in \Omega$, are known. It follows that the familiar identity

$$\omega([X, Y]) = X\omega(Y) - Y\omega(X) - d\omega(X, Y)$$

(6.1)

determines the bracket $[X, Y]$ in terms of variations $X, Y$. If in particular $X = \mathcal{X}$, $Y = \mathcal{Y}$ are the evolutional symmetries (hence $\mathcal{X}x_i = \mathcal{Y}x_i = 0$) and $\omega = \omega_f = df - \sum D_i f dx_i$ is a contact form, we obtain the triviality

$$\mathcal{X}\mathcal{Y}f = \mathcal{Y}\mathcal{X}f = \mathcal{XY}f - \mathcal{YX}f.$$

However, nontrivial interpretation is possible [8]. Denoting

$$\{F, G\} = \mathcal{X}F - \mathcal{Y}G \quad (F = \omega_f(\mathcal{X}), G = \omega_f(\mathcal{Y})), \quad (6.2)$$

we have the higher-order Poisson bracket $\{\cdot\}$. The sense of the construction lies in the schema

$$\mathcal{X} \leftrightarrow \omega_f(\mathcal{X}), \quad \mathcal{Y} \leftrightarrow \omega_f(\mathcal{Y}), \quad [\mathcal{X}, \mathcal{Y}] \leftrightarrow \{\omega_f(\mathcal{X}), \omega_f(\mathcal{Y})\}$$

which means that the Lie bracket of vector fields corresponds to the Poisson bracket of (families of) certain functions. Using (6.1), analogous "bracket" can be introduced for general variations $X, Y$ and forms $\omega \in \Omega$ as well.

6.2. The role of involutivity [5,11]. Let $\Omega_l \subset \Omega$ be a term of a good filtration $\Omega_\ast$ of a diffiety $\Omega$ and $x_1, \ldots, x_n$ be "not too special" independent variables. We introduce the following construction.

1: Let $\omega^1, \ldots, \omega^{\sigma_1} \in \Omega_l$ be a maximal family such that

$$\mathcal{L}_{D_1} \omega^1, \ldots, \mathcal{L}_{D_1} \omega^{\sigma_1}$$

are linearly independent forms mod $\Omega_l$.

2: Let $\omega^{\sigma_1+1}, \ldots, \omega^{\sigma_1+\sigma_2} \in \Omega_l$ be a maximal family such that

$$\mathcal{L}_{D_2} \omega^{\sigma_1+1}, \ldots, \mathcal{L}_{D_2} \omega^{\sigma_1+\sigma_2}$$

are linearly independent forms mod $\Omega_l$ and the previous $\mathcal{L}_{D_1} \omega^1, \ldots, \mathcal{L}_{D_1} \omega^{\sigma_1}$. :
\( n - 1 \): Let \( \omega^{\sigma_1 + \cdots + \sigma_n - 2 + 1}, \ldots, \omega^{\sigma_1 + \cdots + \sigma_n - 1} \in \Omega_l \) be a maximal family such that
\[
\mathcal{L}_{D_{n-1}} \omega^{\sigma_1 + \cdots + \sigma_n - 2 + 1}, \ldots, \mathcal{L}_{D_{n-1}} \omega^{\sigma_1 + \cdots + \sigma_n - 1}
\]
are linearly independent forms mod \( \Omega_l \) and the previous \( \mathcal{L}_{D_1} \omega^1, \ldots, \mathcal{L}_{D_{n-2}} \omega^{\sigma_1 + \cdots + \sigma_n - 2} \).

\( n \): Let \( \omega^{\sigma_1 + \cdots + \sigma_n - 1 + 1}, \ldots, \omega^{\sigma_1 + \cdots + \sigma_n} \in \Omega_l \) be a maximal family such that
\[
\mathcal{L}_{D_n} \omega^{\sigma_1 + \cdots + \sigma_n - 1 + 1}, \ldots, \mathcal{L}_{D_n} \omega^{\sigma_1 + \cdots + \sigma_n}
\]
are linearly independent forms mod \( \Omega_l \) and the previous \( \mathcal{L}_{D_1} \omega^1, \ldots, \mathcal{L}_{D_{n-1}} \omega^{\sigma_1 + \cdots + \sigma_n - 1} \).

The involutivity theorem reads: denoting
\[
\omega^j_{r_1 \ldots r_n} = \mathcal{L}_{D_1}^{r_1} \cdots \mathcal{L}_{D_n}^{r_n} \omega^j,
\]
and assuming \( l \) large enough, then the forms
\[
\omega^j_{r_1 \ldots r_n} (j = 1, \ldots, \sigma_1), \quad \omega^j_{0 r_2 \ldots r_n} (j = \sigma_1 + 1, \ldots, \sigma_2), \quad \ldots \quad \omega^j_{0 \ldots 0 r_{n-1} r_n} (j = \sigma_n - 1 + 1, \ldots, \sigma_n)
\]
where \( r_1, \ldots, r_n = 1, 2, \ldots \) are linearly independent. It follows that together with a basis of \( \Omega_l \) they provide a basis of total diffiety \( \Omega \). The result is of a fundamental importance. Variations \( Z \) are determined by certain values \( \omega(Z) \) where \( \omega \in \Omega \). If \( \omega \in \Omega_l \), these functions \( \omega(Z) \) are subjected to a certain \textit{finite number} of requirements arising from the differential equations under consideration. Assuming \( \omega \notin \Omega_l \), it is sufficient to consider only forms \( (6.3) \) of the basis and they are subjected only to the recurrences
\[
\omega^j_{r_1+1 \ldots r_n} (Z) = D_1 \omega^j_{r_1 \ldots r_n} (Z), \ldots, \omega^j_{r_1 \ldots r_{n-1} r_n+1} (Z) = D_n \omega^j_{r_1 \ldots r_n} (Z).
\]

It should be however noted that this deep achievement does not much affect the earthly practice of routine calculations.

6.3. \textbf{Adjustment of ordinary differential equations.} The case of one independent variable and one total derivative \( D = D_1 \) is simple \[115\]. If \( \Omega_n \) is a good filtration, the forms \( \omega \in \Omega \) are organized into several ramifications.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1a.png}
\caption{Figure 1a)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1b.png}
\caption{Figure 1b)}
\end{figure}

After an appropriate change of lower–order terms, the strings become rectified and this provides the standard basis. If there is only one string, it does not
change after any symmetry \[4\] Theorem 26]. We have omitted some "noncontrollability" submodules \( R \subset \Omega \) of a finite dimension here.

6.4. **Adjustment of partial differential equations.** The arrangements are not so easy (see \[5\]) and we can mention only the case of two independent variables. If \( \Omega \) is a good filtration, the forms \( \omega \in \Omega \) are organized into two-dimensional sheets which may be ramified. The sheets can be made flat and each of them then consists of a finite family of overlapping infinite "triangles". We present the case of only one ramified sheet and the corresponding adjustment with two overlapping triangles. In this particular case, the symmetries \( Z \) preserve the vertices \( \beta, \gamma \).

![Figure 2a)

\[ \mathcal{L}_{D_1}, \mathcal{L}_{D_2} = \mathcal{L}_{D_2}, \mathcal{L}_{D_1} \]

We omit the proof which rests on Lemma \[2.2\] and also the mention of the separated "noncontrability" submodules \( R \subset \Omega \) consisting of strings.

6.5. **Isospectral solutions and solitons.** We briefly reinterpret the calculation \[12\] of the KdV–hierarchy in order to point out some general aspects. The original problem reads: let

\[
v_{xx} + (\lambda + q(x, t))v = 0 \quad (\lambda \in \mathbb{R}, v = v(x, t)) \quad (6.4)
\]

be the eigenvalue problem depending on a parameter \( t \) and our task is to determine such evolution equations

\[
v_t = P \quad (P = A(\lambda; \cdot, q_r, \cdot) + B(\lambda; \cdot, q_r, \cdot) + B(\lambda; \cdot, q_r, \cdot) v_x) \quad (q_r = \frac{\partial q}{\partial x^r}) \quad (6.5)
\]

that the compatibility conditions of the system (6.4) (6.5) are of the special kind

\[
q_t = Q(\cdot, q_r, \cdot). \quad (6.6)
\]

The eigenvalue \( \lambda \) is preserved and does not affect the evolution of the function \( q \).

The reinterpretation is as follows. We start with the ordinary differential equation

\[
\frac{d^2 v}{dx^2} + (\lambda + q) v = 0 \quad (\lambda \in \mathbb{R}, v = v(x), q = q(x))
\]
with two unknown functions. In terms of diffieties, we have space $\mathbf{M}$ with coordinates $x, \lambda, v, v_x, q, q_0, \ldots$ and the module $\Omega$ with the basis

$$
d\lambda, \alpha = dv - v_x dx, \alpha_x = dv_x + (\lambda + q_0)dx, \beta_r = dq_r - q_{r+1}dx \quad (r = 0, 1, \ldots).
$$

Our task is to determine variations $Z$ such that

$$Zx = 0 \quad (Z = \text{the evolutional operator}), \ Z\lambda = 0, \ Zq = Q(\cdot, q_r, \cdot). \quad (6.7)
$$

Roughly saying, function $q$ is "autonomous" in the evolution. Quite general variations $Z$ can be determined with the use of the standard basis

$$
\pi_0 = \alpha, \pi_1 = \mathcal{L}_D \pi_0 = \alpha_1, \pi_2 = \mathcal{L}_D \pi_1 = -(\lambda + q)\alpha - v(d\lambda + \beta_0), \ldots
$$

where

$$
\mathcal{L}_D = \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} - (\lambda + q)\frac{\partial}{\partial v_x} + \sum q_r \frac{\partial}{\partial q_r}
$$

is the total derivative. Assuming moreover $Zx = Z\lambda = 0$, we obtain

$$Z = \sum D^r P \frac{\partial}{\partial \pi_r} \quad (P = \pi_0(Z) = \alpha(Z) = Zv)
$$

where $P$ may be an arbitrary function. Then

$$D^2 P = \pi_2(Z) = -(\lambda + q)Zv - vZq = -(\lambda + q)P - vQ
$$

and the last requirement (6.7) is expressed by

$$Q(\cdot, q_r, \cdot) = -\frac{1}{v}(D^2 P + (\lambda + q)P) \quad (6.8)
$$

which provides very strong additional conditions on the function $P$. First of all, we infer that

$$P_{vv} = P_{vvx} = P_{v_x v} = 0
$$

by looking at the higher–order term $D^2 P$ in (6.8). So we may suppose

$$P = A(\lambda; \cdot, q_r, \cdot)v + B(\lambda; \cdot, q_r, \cdot)v_x + C(\lambda; \cdot, q_r, \cdot).
$$

Substitution into (6.8) then yields

$$2DA + D^2 B = 0, \ C = 0
$$

and finally

$$Q(\cdot, q_r, \cdot) = \frac{1}{2}D^3 B + 2(\lambda + q_0)DB + q_1 B \quad (D = \sum q_{r+1} \frac{\partial}{\partial q_r}). \quad (6.9)
$$

It is not easy to discuss this equation in full generality, however, there are particular solutions

$$B = B_0(\cdot, q_r, \cdot)\lambda^n + \cdots + B_{n-1}(\cdot, q_r, \cdot)\lambda + B_n(\cdot, q_r, \cdot)
$$

for any $n = 0, 1, \ldots$. We recall the final result [12]

$$B_0 = 1, B_1 = -\frac{1}{2}q_0, B_2 = \frac{1}{8}(q_2 + 3q_0^2), \ldots$$
which moreover provides the famous KdV–hierarchy

\[ Zq = Q = (q_1 =) q_x, (q_2 + 3q_0^2)_x, (q_4 + 5q_1^2 + 10qq_2 + 10q_0^3)_{xx}, \ldots \]

as the final result.

The reinterpretation of the problem indicates some delicate features of the theory which were not yet discussed in actual literature and incorporate the original problem in much broader context. Analogous approach can be applied to nonlinear and partial differential equations.

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