On the completeness of looped-functionals arising in the analysis of periodic, sampled-data and hybrid systems

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Abstract

We prove that the looped-functional considered in [1,2,3] for the analysis of sampled-data, impulsive and switched systems is “complete”, in the sense that it can be used to equivalently characterize a certain discrete-time stability criterion involving matrix exponentials. This result is derived for a wider class of linear systems, referred to as pseudo-periodic systems with impulses, encompassing periodic, impulsive, sampled-data and switched systems.

Key words: Looped functionals, periodic systems, sampled-data systems, hybrid systems.

1 Introduction

Looped-functionals have been introduced in [4,1] for the analysis of sampled-data systems. The main rationale was to reformulate a discrete-time condition into another condition devoid of exponential terms, allowing then for the consideration of uncertain time-varying systems and nonlinear systems [5]. They have been further considered for the analysis of impulsive systems (see e.g. [6,7]) in [8,9] and switched systems (see e.g. [10,11,12]) in [3]. The key idea behind the use of looped-functionals is to encode a discrete-time stability condition in a condition that is convex in terms of the matrices of the systems. Due to the convexity property, the resulting conditions are readily extendable to uncertain systems and linear time-varying systems, unlike the discrete-time stability conditions that are nonconvex in the matrices of the system due to the presence of exponential terms; see e.g. [8,9,3].

In the papers [2,3], the considered looped-functional led to sufficient conditions for the feasibility of a certain discrete-time stability criterion characterizing the stability of impulsive and switched systems. We show here that this very same looped-functional is “complete” in the sense that the resulting criterion is actually equivalent to the discrete-time stability condition aimed to be represented in a convex way. This result is proved for a larger class of systems, referred to as pseudo-periodic systems, encompassing periodic systems, impulsive systems, sampled-data systems and switched systems, proving then the sufficiency and the necessity of the conditions obtained in [3,9].

Notations: The sets of symmetric and positive definite matrices of dimension $n$ are denoted by $\mathbb{S}^n$ and $\mathbb{S}^n_+$ respectively. Given two symmetric matrices $A, B$, the expression $A \succ (\succeq) B$ means that $A - B$ is positive (semi)definite. Given a square matrix $A$, the operator $\text{He}(A)$ stands for the sum $A + A^T$. The set of natural numbers $\{1, \ldots\}$ is denoted by $\mathbb{N}$ where the set of whole numbers is denoted by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2 Definitions

Some definitions and preliminary results are introduced below. In particular, pseudo-periodic functions and pseudo-periodic systems are formally characterized. Discrete-time state-transition matrices for pseudo-periodic systems are also defined.

Definition 2.1 (Pseudo-periodic functions) Given scalars $0 < \delta \leq T_{\min} \leq T_{\max} < \infty$, let $f :$
for some seed matrix function $A$ the unique matrix function state-transition matrix of system (3)-(4) is defined as

$$ \Phi(t, T) = A(t, T) \Phi(t, T), \quad \eta \in [0, T] $$

(6)

for all $T \in [T_{\text{min}}, T_{\text{max}}]$.

Using the state-transition matrix above, it is possible to define the transition map associated to the system (3)-(4). This operator plays a key role in the paper.

Definition 2.4 (Discrete-time transition map) Given a family of transition times, the discrete-time transition map corresponding to system (3)-(4) is defined by $\Psi(T_k) := \Phi(T_k, T_k)$ and we have that

$$ x(t_{k+1}) = \Psi(T_k)x(t_k), \quad k \in \mathbb{N} $$

(7)

where $\{t_k\}_{k \in \mathbb{N}} \in I$.

The discrete-time transition map hence defines the sequence of state values for any sequence of transition times in $I$.

3 Main results

3.1 Preliminaries

The following result is the main result of the paper that will allow us to prove that the conditions based on looped-functionals considered in [2,3] equivalently characterize a certain discrete-time stability condition.

Theorem 3.1 Let $A, J \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{S}^n$ be given matrices, and let $T$ be a positive scalar. Then, the following statements are equivalent:

(a) There exists a matrix $P \in \mathbb{S}^n_{>0}$ such that the matrix inequality

$$ J^T \Psi(T)^T P \Psi(T) J - J^T P J + M + \varepsilon I \preceq 0 $$

(8)

holds where $\Psi(\cdot)$ is defined in (7).

(b) There exist a matrix $P \in \mathbb{S}^n_{>0}$ and continuous matrix-valued functions $Z_1, Z_2 : [0, T] \to \mathbb{S}^n_-$, $Q_1, Q_2 : [0, T] \to \mathbb{S}^n_-$ such that the conditions

$$ Z_1(T) = 0 $$

$$ Z_2(T) - J^T Z_1(0) J - Z_2(0) = 0 $$

(9)
Theorem 3.3 The periodic system (3)-(4) with period $\bar{T}$ is asymptotically stable if and only if one of the following equivalent statements hold:

(a) The matrix $\Psi(\bar{T})J$ is Schur stable.
(b) There exists a matrix $P \in \mathbb{S}_{>0}^n$ such that the LMI

$$ J^T \Psi(\bar{T})^T P \Psi(\bar{T}) J - P \preceq 0 $$

(11)

holds.

(c) There exists a constant matrix $P \in \mathbb{S}_{>0}^n$, a scalar $\varepsilon > 0$ and a continuously differentiable matrix function $Z : [0, \bar{T}] \to \mathbb{S}^{3n}$ verifying

$$ Y_2^T Z(\bar{T}) Y_2 - Y_1^T Z(0) Y_1 = 0 $$

(12)

where

$$ Y_1 = \begin{bmatrix} J & 0_n \\ I_n & 0_n \\ 0_n & I_n \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0_n & I_n \\ I_n & 0_n \\ 0_n & I_n \end{bmatrix} $$

(13)

and such that the LMI

$$ \Theta(\tau) + \text{He} \left[ \begin{bmatrix} A(\tau, \bar{T}) & 0_n & 0_n \\ 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}^T Z(\tau) \right] + \dot{Z}(\tau) \preceq 0 $$

(14)

holds for all $\tau \in [0, \bar{T}]$ where

$$ \Theta(\tau) := \begin{bmatrix} \bar{T} \text{He}[PA(\tau, \bar{T})] & 0_n & 0_n \\ \varepsilon I_n + J^T PJ - P & 0_n \\ * & * & 0_n \end{bmatrix}. $$

(15)

Proof: The proof is given in the Appendix.

Remark 3.2 The above result straightforwardly extends to the case when matrices $\Psi, J$ and $M$ depend on some additional time-invariant parameters, say $\rho$. In such a case, the matrix-valued functions $Z_1, Z_2, Q_1$ and $Q_2$ also need to depend on this additional parameter in order to preserve necessity.

3.3 Pseudo-periodic systems with impulses

This section follows the same lines as the previous one, with the difference that pseudo-periodic systems are considered here. We have the following result:

Theorem 3.4 The pseudo-periodic system (3)-(4) with pseudo-period $T_k \in [T_{min}, T_{max}], 0 < \delta < T_{min} \leq T_{max} < \infty$, is asymptotically stable if one of the following equivalent statements hold:

(a) There exists a matrix $P \in \mathbb{S}_{>0}^n$ such that the LMI

$$ J^T \Psi(\bar{T})^T P \Psi(\bar{T}) J - P \preceq 0 $$

(16)

holds for all $\theta \in [T_{min}, T_{max}]$.

(b) There exists a constant matrix $P \in \mathbb{S}_{>0}^n$, a scalar $\varepsilon > 0$ and a matrix function $Z : [0, T_{max}] \times [T_{min}, T_{max}] \to \mathbb{S}^{3n}$ differentiable with respect to the first variable and verifying

$$ Y_2^T Z(\theta, \theta) Y_2 - Y_1^T Z(0, \theta) Y_1 = 0 $$

(17)

for all $\theta \in [T_{min}, T_{max}]$, where $Y_1$ and $Y_2$ are defined in (13) and such that the LMI

$$ \Theta(\tau, \theta) + \text{He} \left[ \begin{bmatrix} A(\tau, \theta) & 0_n & 0_n \\ 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix}^T Z(\tau, \theta) \right] + \frac{\partial}{\partial \tau} Z(\tau, \theta) \preceq 0 $$

(18)

hold for all $\tau \in [0, \theta]$ and $\theta \in [T_{min}, T_{max}]$ where

$$ \Theta(\tau, \theta) := \begin{bmatrix} \theta \text{He}[PA(\tau, \theta)] & 0_n & 0_n \\ \varepsilon I_n + J^T PJ - P & 0_n \\ * & * & 0_n \end{bmatrix}. $$

(19)
Proof: The proof follows from the fact that the conditions stated in Theorem 3.3 explicitly depend on the period of the system. Following Remark 3.2, the conditions then readily generalize to the case where $T_k$ is not fixed anymore but belongs to a certain range of values. In such a case, the matrix function $Z$ must be extended to also depend on $T_k$ so that the equality constraint (17) can be satisfied; see e.g. [2,3].

3.4 Impulsive and sampled-data systems

The case of impulsive systems is obtained by simply choosing the matrix $\Sigma(t)$ in (3) to be a constant matrix $A \in \mathbb{R}^{n \times n}$. This leads to the following system

\[
\dot{x}(t) = Ax(t), \quad t \notin \{t_k\}_{k \in \mathbb{N}} \\
x(t) = Jx(t^-), \quad t \in \{t_k\}_{k \in \mathbb{N}} \\
x(t_0) = x_0.
\]

and the following result completing the one derived in [2]:

**Corollary 3.5** The impulsive system (20) with interimpulses times $T_k := t_{k+1} - t_k \in [T_{\min}, T_{\max}]$, $0 < \delta \leq T_{\min} \leq T_{\max} < \infty$, is asymptotically stable if one of the following equivalent statements hold:

(a) There exists a matrix $P \in \mathbb{S}^n_{> 0}$ such that the LMI

\[
J^T e^{A^T \theta} Pe^{A \theta} J - P < 0
\]

holds for all $\theta \in [T_{\min}, T_{\max}]$.

(b) There exist constant matrices $P \in \mathbb{S}^n_{> 0}$, a scalar $\varepsilon > 0$ and a matrix function $Z : [0, T_{\max}] \times [T_{\min}, T_{\max}] \rightarrow \mathbb{S}^n$, differentiable with respect to the first variable and verifying

\[
Y_2^T Z(\theta, 0) Y_2 - Y_1^T Z(0, \theta) Y_1 = 0
\]

for all $\theta \in [T_{\min}, T_{\max}]$, where $Y_1$ and $Y_2$ are defined in (13) and such that the LMI

\[
\Theta(\theta) + \Theta(0) \leq 0
\]

hold for all $\tau \in [0, \theta]$ and $\theta \in [T_{\min}, T_{\max}]$ where $\Theta(\theta) := \Theta(\theta) = [\theta \text{He}[PA] 0_n 0_n]$

3.5 Switched systems

Let us consider now switched systems of the form

\[
\dot{x}(t) = A_{\sigma(t)}x(t) \\
x(0) = x_0
\]

where $\sigma : \mathbb{R}_+ \rightarrow \{1, \ldots, N\}$ and assume that the switching times are given by the sequence $\{t_k\}_{k \in \mathbb{N}}$. We then have the following result completing the one obtained in [3]:

**Corollary 3.7** The switched system (27) is asymptotically stable for any dwell-time $t_{k+1} - t_k =: T_k \in [T_{\min}, T_{\max}]$, $0 < \delta \leq T_{\min} \leq T_{\max} < \infty$, $k \in \mathbb{N}$, if one of the following equivalent statements hold:

(a) There exist matrices $P_i \in \mathbb{S}^n_{> 0}$, $i = 1, \ldots, N$ such that the LMIs

\[
e^{A_{i,0} \theta} Pe^{A_{i,0} \theta} - P_i < 0
\]

holds for all $\theta \in [T_{\min}, T_{\max}]$ and for all $i, j = 1, \ldots, N$, $i \neq j$.

(b) There exist constant matrices $P_i \in \mathbb{S}^n_{> 0}$, $i = 1, \ldots, N$, a scalar $\varepsilon > 0$ and matrix functions $Z_{ij} : [0, T_{\max}] \times [T_{\min}, T_{\max}] \rightarrow \mathbb{S}^n$, $i, j = 1, \ldots, N$, $i \neq j$, differentiable with respect to the first variable and verifying

\[
W_2^T Z_{ij}(\theta, 0) W_2 - W_1^T Z_{ij}(0, \theta) W_1 = 0
\]

for all $\theta \in [T_{\min}, T_{\max}]$ where

\[
W_1 = \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix}, \quad W_2 = \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix}
\]
and such that the LMI
\[
\Theta_{ij}(\theta) + \text{He} \left( Z_{ij}(\tau, \theta) \begin{bmatrix} A_i & 0_n & 0_n \\ 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n \end{bmatrix} \right) + Z'_{ij}(\tau, \theta) \leq 0
\]
hold for all \( \tau \in [0, \theta] \), all \( \theta \in [T_{\min}, T_{\max}] \) and for all \( i, j = 1, \ldots, N \), \( i \neq j \), where
\[
\Theta_{ij}(\theta) := \begin{bmatrix} \theta [PA_i] & 0_n & 0_n \\ * & \varepsilon I_n + P_i - P_j & 0_n \\ * & * & 0_n \end{bmatrix}.
\]

Proof: Based on Theorem 3.4, we can define \( A(\tau, T) = A_i, Z = Z_{ij} \) and \( M = M_{ij} := P_i - P_j \) for all \( i, j = 1, \ldots, N \), \( i \neq j \). The result then follows. \( \diamond \)

4 Conclusion

A proof for the necessity of certain conditions obtained from the use of certain looped-functional has been obtained. It is shown that the conditions obtained for the analysis of switched, sampled-data and impulsive systems are necessary and sufficient for the characterization of the feasibility of a certain family of discrete-time stability conditions.

A Proof of Theorem 3.1

A.1 Preliminaries

Lemma A.1 ([14]) Let \( U : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n} \), \( Q : \mathbb{R}_{\geq 0} \to \mathbb{S}^n \) and \( S_0 \in \mathbb{S}^n \). The symmetric solution \( S : \mathbb{R}_{\geq 0} \to \mathbb{S}^n \) of the matrix equality
\[
\dot{S}(t) + U(t)^T S(t) + S(t) U(t) = Q(t), \quad t \geq 0
\]
is given by
\[
S(t) = X_U(t, 0)^T S_0 X_U(t, 0) + \int_0^t X_U(t, s)^T Q(s) X_U(t, s) ds
\]
(31)

where \( X_U(t, s) = Y_U(s) Y_U(t)^{-1} \) where \( Y_U(t) = U(t) Y_U(0), Y_U(0) = I \). When \( U(t) \equiv U \) is a constant matrix, then \( X_U(t, s) = e^{-U(t-s)} \).

Lemma A.2 Let \( T \) be a positive scalar, \( M \in \mathbb{S}^n \) be a symmetric matrix and \( Q_1, Q_2 : [0, T] \to \mathbb{S}_{\geq 0}^n \) be continuous matrix-valued functions. The solutions \( Z_1, Z_2 : [0, T] \to \mathbb{S}^n \) to the equations
\[
\text{He} [(TP + Z_1(\tau))A(\tau)] + \dot{Z}_1(\tau) = -TQ_1(\tau)
\]
\[
M + \varepsilon I_n + \dot{Z}_2(\tau) = -TQ_2(\tau)
\]
where \( Z_2(\tau) = Z_2(0) - (M + \varepsilon I_n) \tau - T\bar{Q}_2(\tau) \)
\[
Z_1(\tau) = X_A(\tau, 0)^T [TP + Z_1(0)] X_A(\tau, 0) - T[P + \bar{Q}_1(\tau)]
\]
are given by
\[
\bar{Q}_2(\tau) = \int_{\tau}^T Q_2(s) ds,
\]
\[
\bar{Q}_1(\tau) = \int_0^T X_A(\tau, s)^T Q_1(s) X_A(\tau, s) ds.
\]

Proof: The proof of the solution for \( Z_2 \) is immediate. The solution \( Z_1 \) is directly obtained from Lemma A.1. \( \diamond \)

A.2 Proof of b) \Rightarrow a.)

Assume that the conditions of Statement b) hold and let us define the vector \( \xi_k(\tau) = \text{col}(x(t_k + \tau), x(t_k^-)) \) where \( x(t_k + \tau) \) is a right-continuous function at \( \tau = 0 \) with left-limit \( x(t^-_k) := \lim_{\tau \uparrow 0} x(t_k + \tau) \) and right-limit \( x(t_k) = J x(t^-_k) \). Let us define the matrix \( Z(\tau) = \text{diag}(Z_1(\tau), Z_2(\tau)) \) and rewrite (10) as
\[
\Omega(\tau, \bar{T}) + \text{He} \left( Z(\tau) \begin{bmatrix} A(\tau) & 0 \\ 0 & 0 \end{bmatrix} \right) + \dot{\bar{Z}}(\tau) \leq 0
\]
(32)

where
\[
\Omega(\tau, \bar{T}) := \begin{bmatrix} \bar{T} \text{He}[PA(\tau)] & 0 \\ 0 & M + \varepsilon I_n \end{bmatrix}.
\]

Pre- and post-multiplying the above matrix inequality by \( \xi_k(\tau)^T \) and \( \xi_k(\tau) \), and integrating w.r.t. \( \tau \) over \( [0, \bar{T}] \) leads to
\[
\bar{T} \int_{t_k}^{t_k + \bar{T}} x(t_k + \tau)^T \bar{A}(\tau)^T P + P A(\tau) x(t_k + \tau) d\tau + \bar{T} x(t^-_k)^T (M + \varepsilon I_n) x(t^-_k) + \varphi(\bar{T}) \leq 0
\]
(33)

where \( \varphi(\bar{T}) := \xi_k(\bar{T})^T Z(\bar{T}) \xi_k(\bar{T}) - \xi_k(0)^T Z(0) \xi_k(0) \). Letting \( V(x) = x^T P x, P \in \mathbb{S}^n_{\geq 0}, \) the above inequality
becomes
\[\bar{T} \left[ V(x(t_k + \bar{T})) - V(Jx(t_k)) \right] + \bar{T}x(t_k^{-})^\top Mx(t_k^{-}) + \varphi(\bar{T}) \leq -\bar{T} \varepsilon \|x(0)\|_2^2.\] (A.8)

Noting that
\[\varphi(\bar{T}) = x(t_k + \bar{T})^\top Z_1(\bar{T})x(t_k + \bar{T}) + x(t_k^{-})^\top Z_2(T) - J^\top Z_1(0)J - Z_2(0)x(t_k^{-})\]
then, from (9), we have that \(\varphi(\bar{T}) = 0\). Using finally (7), we get that
\[x(0)^\top [J^\top \Psi(T)^\top P \Psi(T)J - J^\top PJ + M] x(0) \leq -\varepsilon \|x(0)\|_2^2\] (A.9)
or equivalently
\[J^\top \Psi(T)^\top P \Psi(T)J - J^\top PJ + M + \varepsilon I_n \leq 0.\] (A.10)
This proves the result.

### A.3 Proof of (a) \(\Rightarrow\) (b)

This case is more involved. The key idea is to show that we can always solve for the LMIs (10) regardless of the stability of the system. We then show that, under the assumption the LMI condition (8), we can actually construct matrix-valued functions \(Z_1, Z_2, Q_1, Q_2\) that satisfy the equality condition (9).

Using the explicit solution for \(Z_1\) derived in Lemma A.2, the first condition in (9) given by \(Z_1(\bar{T}) = 0\) reformulates as
\[X_A(\bar{T},0)^\top (Z_1(0) + \bar{T}P)X_A(\bar{T},0) - \bar{T}(P + \bar{Q}_1(\bar{T})) = 0\] (A.12)
and hence we have that
\[Z_1(0) = \bar{T} \left[ -P + X_A(\bar{T},0)^\top (P + \bar{Q}_1(\bar{T}))X_A(\bar{T},0)^{-1} \right].\] (A.13)

The second condition in (9) rewrites
\[\bar{T} \left[ J^\top [P - X_A(\bar{T},0)^{-1} (P + \bar{Q}_1(\bar{T}))X_A(\bar{T},0)^{-1}]J \right] - \bar{T} \left[ M + \varepsilon I_n + \bar{Q}_2(\bar{T}) \right] = 0.\] (A.14)

Dividing by \(\bar{T}\) and reorganizing the terms yield
\[J^\top [X_A(\bar{T},0)^{-1} \tau PX_A(\bar{T},0) - P] J + M + \varepsilon I_n = -\bar{Q}_2(\bar{T}) - J^\top X_A(\bar{T},0)^{-1} \tau \bar{Q}_1(\bar{T})X_A(\bar{T},0)^{-1} J.\] (A.15)

Now noting that \(\Psi(\bar{T}) = X_A(\bar{T},0)^{-1}\), we get that
\[J^\top [\Psi(\bar{T})^\top P \Psi(\bar{T}) - P] J + M + \varepsilon I_n = -\bar{Q}_2(\bar{T}) - J^\top \Psi(\bar{T})^\top \bar{Q}_1(\bar{T}) \Psi(\bar{T}) J.\] (A.16)
Note that by assumption the right-hand side is negative semidefinite, hence we can clearly find \(Q_1, Q_2 \geq 0\) and \(Q_2\) such that the above equality holds. An obvious choice is \(Q_1(\tau) = 0\) for all \(\tau \in [0, \bar{T}]\) and
\[Q_2(s) = \frac{J^\top [\Psi(\bar{T})^\top P \Psi(\bar{T}) - P] J + M + \varepsilon I_n}{\bar{T}} \geq 0.\] (A.17)
This proves the result.

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