A non-perturbative study of the interplay of two fermion-boson interactions

Zhao-Kun Yang,1 Xiao-Yin Pan,2 and Guo-Zhu Liu1

1Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China
2Department of Physics, Ningbo University, Ningbo, Zhejiang 315211, China

In condensed-matter systems, electrons are subjected to two different interactions under certain conditions. Even if both interactions are weak, it is difficult to perform perturbative calculations due to the complexity caused by the interplay of two interactions. When one or two interactions are strong, ordinary perturbation theory may entirely break down. Here we consider undoped graphene as an example and provide a non-perturbative quantum-field-theoretic analysis of the interplay of electron-phonon interaction and Coulomb interaction. We treat these two interactions on an equal footing and derive the exact Dyson-Schwinger equation of the full Dirac-fermion propagator. We prove that this exact equation is self-closed by using several identities. After solving this equation, we find that the energy (momentum) dependence of renormalized fermion velocity is dominantly determined by the electron-phonon (Coulomb) interaction. In particular, the renormalized fermion velocity exhibits a logarithmic momentum dependence and a non-monotonic energy dependence.

It is necessary to study the interplay of two interactions in certain condensed matter systems. For instance, disorder scattering leads to Anderson localization in two-dimensional (2D) non-interacting metals, but direct electron-electron interaction tends to destroy localization and restore metallic behavior. The metal-insulator transition found in some 2D dilute systems may result from the interplay of disorder and electron-electron interaction. Another notable example is phonon-mediated superconductivity. While electron-phonon interaction (EPI) favors superconductivity by mediating an effective attraction between electrons, direct Coulomb interaction is repulsive and thus disfavors superconductivity. To gain a refined description of superconductivity, one needs to consider both EPI and Coulomb interaction when the latter is poorly screened by low carrier density.

One can employ a specific Yukawa-type fermion-boson interaction (FBI) to describe each of the interactions mentioned above. The EPI is already a standard FBI by definition. The Coulomb interaction can be transformed into a Yukawa-coupling between charged electrons and an auxiliary boson. Similar manipulation can be applied to treat disorder scattering. In case two interactions are equally important, one has to couple electrons to two kinds of bosons and study the interplay of two FBIs.

The coexistence of two FBIs makes theoretical analysis rather involved. It is difficult enough to study one single FBI, especially when its coupling constant is not small. The traditional approach to investigate one single FBI is to adopt the Migdal-Eliashberg (ME) theory. Although ME theory was originally proposed to treat EPI-mediated superconductivity, in the past sixty years it has already been generalized to study many other sorts of FBIs. The efficiency of ME theory relies crucially on the validity of Migdal theorem, which states that the quantum corrections to the fermion-boson vertex function, denoted by \( \Gamma_v(q,p) \) with \( p \) (\( q \)) being the fermion (boson) energy-momentum, are small and negligible. We emphasize that the Migdal theorem is justified only in the case of weak EPI owing to the existence of a small parameter \( \lambda(\omega_D/E_F) \ll 1 \), where \( \lambda \) is a dimensionless coupling constant, \( \omega_D \) is Debye frequency, and \( E_F \) is Fermi energy. In a large number of unconventional superconductors and strange metals, the reliability of Migdal theorem and the applicability of ME theory are both in doubt.

Recently, a non-perturbative Dyson-Schwinger (DS) equation approach was developed by the authors to determine the full fermion-boson vertex function with the help of several exact identities. The DS equation of the full fermion propagator derived by using this approach is self-closed and free of approximations. We have previously applied this approach to study EPI-induced superconducting transition in metals and the many-body effects caused by unscreened Coulomb interaction in Dirac fermion systems. Here, we generalize this approach to investigate systems in which the fermions are coupled to two different sorts of bosons. Although our approach is generically applicable, for concreteness we consider the interplay of EPI and Coulomb interaction in undoped graphene. We focus on the fermion velocity renormalization induced by such an interplay.

The impact of Coulomb interaction on the properties of graphene has been extensively studied. Gonzalez et al. carried out a first-order renormalization group (RG) analysis of the Coulomb interaction between Dirac fermions by using weak-coupling perturbation theory and revealed a logarithmic renormalization of Fermi velocity, described by \( v \propto \ln (\Lambda/|p|) \), where \( \Lambda \) is an ultraviolet cutoff of fermion momentum \( |p| \). Experiments observed a logarithmic velocity renormalization, which is consistent with first-order RG result. Barnes et al. calculated some higher-order (two-loop and three-loop) corrections and concluded that the logarithmic behavior obtained at first-order is qualitatively altered by such corrections, which signals the breakdown of weak-coupling perturbation theory. In Ref., we revisited this problem by employing our DS equation approach and found that the Dirac fermion velocity does exhibit a logarithmic...
mic renormalization if all the contributions to fermion self-energy are incorporated in a non-perturbative way.

In actual graphene materials, there are other types of interactions than Coulomb interaction. In particular, phonons are always present and interact with fermions. In principle, the renormalized velocity $v(p)$ observed in experiments should receive contributions not only from Coulomb interaction but also from EPI. It is therefore important to consider both of these two interactions so as to make a more direct comparison between theoretical calculations and experimental results.

The interplay of EPI and Coulomb interaction can be described by coupling Dirac fermions to two boson fields. We first write down an effective model for such an interplay and then obtain the DS equation of the full fermion propagator $G(p)$. This equation is formally much more complicated than that for one single FBI. We obtain two exact identities satisfied by several two- and three-point correlation functions, and then use these two identities to prove that the exact DS equation of $G(p)$ can still be made self-closed without invoking any approximation. After solving this equation, we obtain the energy-momentum dependence of renormalized velocity $v(\epsilon, p)$. Our results show that EPI leads to an obvious non-monotonic energy dependence of $v(\epsilon)$ at a fixed $|p|$ and that $v(p)$ exhibits a logarithmic behavior in the small-$|p|$ region at a fixed $\epsilon$ due to the interplay of EPI and Coulomb interaction.

We consider the Lagrangian density $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_p + \mathcal{L}_A + \mathcal{L}_{fp} + \mathcal{L}_{fA}$ defined as follows

$$\mathcal{L}_f = \psi^\dagger(x)(i\partial_0 \sigma_0 - i\partial_1 \sigma_1 - i\partial_2 \sigma_2) \psi(x),$$

$$\mathcal{L}_p = \frac{1}{2}\phi^\dagger(x)D(x)\phi(x),$$

$$\mathcal{L}_A = \frac{1}{2}A(x)\mathcal{F}(x)A(x),$$

$$\mathcal{L}_{fp} = -g\phi(x)\psi^\dagger(x)\sigma_0 \psi(x),$$

$$\mathcal{L}_{fA} = -A(x)\psi^\dagger(x)\sigma_0 \psi(x).$$

Here, spinor field $\psi$ represents Dirac fermion, scalar field $\phi$ represents phonon, and $A$ is another scalar field that describes the Coulomb interaction. $\sigma_0$ is unit $2 \times 2$ matrix and $\sigma_{1,2}$ are Pauli matrices. For notational simplicity, the bare velocity $v_F$ is already absorbed into $\partial_0$. Two operators $\mathcal{D}$ and $\mathcal{F}$ are used to define the equations of free motion of $\phi$ and $A$: $\mathcal{D}\phi = 0$ and $\mathcal{F}A = 0$. There are two FBI terms, namely $\mathcal{L}_{fp}$ and $\mathcal{L}_{fA}$. $\mathcal{L}_{fp}$ describes EPI with coupling constant $g$ and $\mathcal{L}_{fA}$ describes Coulomb interaction. Notice that $\phi$ and $A$ both couple to the fermion density operator $\rho(x) = \psi^\dagger(x)\sigma_0 \psi(x)$. Adding external sources to the above Lagrangian density yields

$$\mathcal{L}_T = \mathcal{L} + J\phi + KA + \psi^\dagger \eta + \eta^\dagger \psi,$$

where $J$, $K$, $\eta$, and $\eta^\dagger$ are external sources for fields $\phi$, $A$, $\psi^\dagger$, and $\psi$, respectively. The partition function is

$$Z[J, K, \eta^\dagger, \eta] = \int D\phi DAD\psi^\dagger D\psi e^{\int dx \mathcal{L}_T},$$

where $\int dx \equiv \int dt d^2x$. The generating functional for connected correlation functions is defined via $Z$ as $W = -i\ln Z$. The full propagators of Dirac fermion, phonon, and boson $A$ are defined in order as follows

$$G(x - y) = -i\langle \psi(x)\psi^\dagger(y) \rangle = \frac{\delta^2 W}{\delta \eta^\dagger(x)\delta \eta(y)} \big|_{J=0},$$

$$D(x - y) = -i\langle \psi(x)\psi^\dagger(y) \rangle = -\frac{\delta^2 W}{\delta J(x)\delta J(y)} \big|_{J=0},$$

$$F(x - y) = -i\langle A(x)A(y) \rangle = -\frac{\delta^2 W}{\delta K(x)\delta K(y)} \big|_{J=0}.\ (10)$$

Here, the notation $J = 0$ is an abbreviation of $J = K = \eta^\dagger = \eta = 0$. There are two additional correlation functions that convert $\phi$ and $A$ into each, defined by

$$D_F(x - y) = -i\langle \phi(x)A(y) \rangle = -\frac{\delta^2 W}{\delta J(x)\delta K(y)} \big|_{J=0},$$

$$F_D(x - y) = -i\langle A(x)\phi(y) \rangle = -\frac{\delta^2 W}{\delta K(x)\delta J(y)} \big|_{J=0}.\ (12)$$

Each FBI has its own vertex function. Two FBI s generate two vertex functions. We adopt the scalar fields $\phi$ and $A$ to define two vertex functions $\Gamma_p$ and $\Gamma_A$ as follows

$$\langle \phi\psi\psi^\dagger \rangle = -D(p)\Gamma_p(q,p)G(p + q)G(p),$$

$$\langle A\psi\psi^\dagger \rangle = -F(q)\Gamma_A(q,p)G(p + q)G(p),$$

$$\langle \phi\psi\psi^\dagger \rangle = -D(p)\Gamma_p(q,p)G(p + q)G(p),$$

$$\langle A\psi\psi^\dagger \rangle = -F(q)\Gamma_A(q,p)G(p + q)G(p).\ (13)$$

Here, $G(p)$, $D(q)$, $F(q)$, $D_F(q)$, and $F_D(q)$ are Fourier transformed from $G(x - y)$, $D(x - y)$, $F(x - y)$, $D_F(x - y)$, and $F_D(x - y)$, respectively.

In the framework of quantum field theory, all the $n$-point correlation functions are connected to each other by an infinite number of DS integral equations. We focus on the DS equation of the full fermion propagator $G(p)$ as it embodies many important properties of fermions. To derive the equation of $G(p)$, we use the fact that the partition function $Z$ is invariant under an arbitrary infinitesimal change $\delta^\dagger \psi$ to obtain an identity

$$0 = \langle (i\partial_0 \sigma_0 - i\partial_1 \sigma_1 - i\partial_2 \sigma_2)\psi(x) - g\phi(x)\sigma_0 \psi(x) - A(x)\sigma_0 \psi(x) + \eta(x) \rangle.\ (15)$$

After performing functional derivative of this identity with respect to external source $\eta(y)$, setting all sources to zero, and finally making Fourier transformations, we find that the full propagator $G(p)$ and the free propagator $G_0(p)$ satisfy the following DS integral equation

$$G^{-1}(p) = G_0^{-1}(p) - i\sigma_0 \int_q G(p + q)D(q)\Gamma_p(q,p)$$

$$-i\sigma_0 \int_q G(p + q)F(q)\Gamma_A(q,p)$$

$$-i\sigma_0 \int_q G(p + q)\Gamma_D(q,p)\Gamma_A(q,p)$$

$$-i\sigma_0 \int_q G(p + q)F_D(q)\Gamma_p(q,p).\ (16)$$
Here we use the abbreviation $\int_q \equiv \frac{\delta^4 q}{(2\pi)^4}$. The fermion self-energy function $\Sigma(p) = G^{-1}(p) - G_0^{-1}(p)$ contains four terms. The first two terms originate from pure EPI and pure Coulomb interaction, respectively. The last two terms embody the contributions produced by the mixing of two bosons. In previous theoretical works, the last two terms are often naively neglected. In this integral equation there are four two-point correlation functions, namely $D(q)$, $F(q)$, $D_F(q)$, and $F_D(q)$, and two FBI vertex functions, including $\Gamma_p(q,p)$ and $\Gamma_A(q,p)$. These six functions are all unknown and each of them satisfies a specific DS integral equation. According to the analysis presented in Refs. [10, 11], the DS equations of $\Gamma_p(q,p)$ and $\Gamma_A(q,p)$ are extremely complicated since they are coupled to an infinite number of DS equations of all the higher-point correlation functions. It seems impossible to solve the equation of $G(p)$ because it is too complicated and depends on many unknown correlation functions.

Below we show that the above DS equation of $G(p)$ can still be made self-closed by generalizing the strategy developed recently by us [10, 11]. To illustrate this, consider the property that $Z$ should not be changed by an arbitrary infinitesimal change $\delta \phi$. This then leads to an equation of the motion of $\phi$: $0 = (D(x)\delta \phi(x) - g\psi^{\dagger}(x)\sigma_0\psi(x) + J)$. Such an equation can be re-written as $D(x) \frac{\delta}{\delta J(x)} = g(\psi^{\dagger}(x)\sigma_0\psi(x)) + J$. After performing functional derivative $\frac{\delta^2}{\delta \psi^{\dagger}(y)\psi^{\dagger}(z)}$, we get

$$D(x) \frac{\delta^3 W}{\delta J(x)\delta \psi^{\dagger}(y)\delta \psi^{\dagger}(z)} = g(\psi^{\dagger}(x)\sigma_0\psi(x)\psi(y)\psi^{\dagger}(z)).$$

This equation is Fourier transformed into

$$D(q)\Gamma_p(q,p) + D_F(q)\Gamma_A(q,p) = D_0(q)g\Gamma_0(q,p),$$

where $D_0(q)$ is the free phonon propagator and $\Gamma_0(q,p)$ is a special vertex function that is originally defined in the coordinate space as

$$(\psi^{\dagger}(x)\sigma_0\psi(y)\psi^{\dagger}(z)) = -\int d\xi d\xi' G(y - \xi)\Gamma_0(\xi - x, x - \xi') G(\xi' - z).$$

The above derivation can also be carried out if the field $\phi$ is replaced with $A$. Repeating the same calculational steps leads us to another important identity

$$F_D(q)\Gamma_p(q,p) + F(q)\Gamma_A(q,p) = F_0(q)\Gamma_0(q,p),$$

where $F_0(q)$ is the free propagator of $A$ boson. Using two identities Eqs. [10, 11], the originally complicated DS equation [10] can be greatly simplified to

$$G^{-1}(p) = G_0^{-1}(p) - i \int_q [g^2 D_0(q) + F_0(q)] \times \sigma_0 G(p + q) \Gamma_0(q,p).$$

This equation looks much simpler, but $\Gamma_0(q,p)$ remains unknown. The equation of $G(p)$ could be entirely self-closed if and only if $\Gamma_0(q,p)$ depends solely on $G(p)$.

Now make a U(1) transformation: $\psi(x) \rightarrow e^{i\theta^{\sigma_m}}\psi(x)$. Here, $\sigma_m$ is a generic $2 \times 2$ matrix and $\theta$ is an infinitesimal constant. $Z[J, K, y^{\dagger}, y]$ should be invariant under such a transformation [10, 11] since it is obtained by integrating out all the configurations of $\psi$ and $\psi^{\dagger}$. Making use of this fact, we derive the following identity

$$\langle [(i\partial_0\psi^{\dagger}\sigma_0 - i\partial_1\psi^{\dagger}\sigma_1 - i\partial_2\psi^{\dagger}\sigma_2)] \sigma_m \psi^{\dagger} \psi \rangle_{0} = \delta(x - y)\sigma_m G(x - z) - G(y - x)\sigma_m^\dagger \delta(x - z).$$

The detailed calculational procedure that leads to this identity has been demonstrated in Refs. [10, 11]. There are four choices for $\sigma_m$, namely $\sigma_m = \sigma_0$, $\sigma_m = \sigma_1$, $\sigma_m = \sigma_2$, and $\sigma_m = \sigma_3$. Substituting these four matrices into the above identity, we obtain four generalized Ward-Takahashi identities (WTIs):

$$\begin{pmatrix}
\Gamma_0(q,p) \\
\Gamma_1(q,p) \\
\Gamma_2(q,p) \\
\Gamma_3(q,p)
\end{pmatrix} =
\begin{pmatrix}
G_0^{-1}(p + q)\sigma_0 - g\sigma_0 G^{-1}(p) \\
G_0^{-1}(p + q)\sigma_1 - g\sigma_1 G^{-1}(p) \\
G_0^{-1}(p + q)\sigma_2 - g\sigma_2 G^{-1}(p) \\
G_0^{-1}(p + q)\sigma_3 - g\sigma_3 G^{-1}(p)
\end{pmatrix},$$

where the matrix $M$ is given by

$$M = \begin{pmatrix}
q_0 & -q_1 & -q_2 & 0 \\
-q_1 & q_0 & 0 & i(2p_2 + q_2) \\
-q_2 & 0 & q_0 & -i(2p_1 + q_1) \\
0 & -i(2p_2 + q_2) & i(2p_1 + q_1) & q_0
\end{pmatrix}.$$

The generalization WTIs contain four current vertex functions $\Gamma_m(q,p)$, where $m = 0, 1, 2, 3$, that are defined via a generalized current $j_m(x) = \psi^{\dagger}(x)\sigma_m\psi(x)$ as

$$\langle j_m(x)\psi^{\dagger}(x)\psi^{\dagger}(z) \rangle = -\int d\xi d\xi' G(y - \xi)\Gamma_m(\xi - x, x - \xi') G(\xi' - z).$$

The four unknown functions $\Gamma_{0,1,2,3}$ satisfy four coupled WTIs. Each of them can be determined by solving Eq. [22]. According to Eq. [20], we only need $\Gamma_0(q,p)$. 

![FIG. 1: The four diagrams (a)-(d) correspond to the four terms of the fermion self-energy given by Eq. (10). Dashed (wavy) line represents the propagation of the boson $\phi$ (A).](image-url)
are characterized by a fine structure constant \( \alpha = \frac{e^2}{4\pi\varepsilon_0} \), where \( e \) is electric charge and \( \varepsilon \) is dielectric constant. \( \alpha = 0.8 \) for graphene on SiO\(_2\) substrate and \( \alpha = 2.2 \) for graphene suspended in vacuum. EPI strength is measured by a dimensionless parameter \( \lambda \), which is related to \( q \) as \( q^2 = \frac{2\lambda}{c_\varepsilon} \). The free fermion propagator \( G_0(p) = \frac{1}{\varepsilon_\alpha \varepsilon_0 - \sigma \cdot p} \) is renormalized into a full propagator

\[
G(p) = \frac{1}{A_0(\epsilon_\alpha, p)\varepsilon_\alpha\varepsilon_0 - A_1(\epsilon_\alpha, p)\sigma \cdot p} \tag{26}
\]
due to interactions. Inserting these four propagators along with Eq. (25) into Eq. (21) would give rise to two self-consistent equations of the renormalization functions \( A_0(\epsilon_\alpha, p) \) and \( A_1(\epsilon_\alpha, p) \). The energy-momentum dependence of renormalized fermion velocity \( v \) can be easily determined by the ratio

\[
v(\epsilon_\alpha, p) = \frac{A_1(\epsilon_\alpha, p)}{A_0(\epsilon_\alpha, p)} \tag{27}
\]

The numerical results of \( v(\epsilon_\alpha, p) \) are plotted in Fig. 2. The energy (momentum) is in unit of Fermi energy \( E_F \) (Fermi momentum \( p_F \)). One observes from Fig. 2(a-c) that \( v(\epsilon_\alpha, p) \) exhibits a clear non-monotonic dependence on energy for any fixed \( |p| \) due to the interplay of two interactions. In comparison, as shown in Ref. 11, \( v \) is energy independent if there is only Coulomb interaction. Thus the non-monotonic energy-dependence of \( v \) is dominantly induced by EPI. To see this fact more explicitly, we plot \( v(\epsilon) \) in Fig. 2(e) in the \( |p| \to 0 \) limit. As the EPI

From Eq. (22), it is easy to obtain

\[
\Gamma_0 = \frac{1}{|M|} \left[ M_{11} \left( G^{-1}(p + q)\sigma_0 - \sigma_0 G^{-1}(p) \right) 
- M_{21} \left( G^{-1}(p + q)\sigma_1 - \sigma_1 G^{-1}(p) \right) 
+ M_{31} \left( G^{-1}(p + q)\sigma_2 - \sigma_2 G^{-1}(p) \right) 
- M_{41} \left( G^{-1}(p + q)\sigma_3 - \sigma_3 G^{-1}(p) \right) \right], \tag{25}
\]

where

\[
|M| = q_0^2(q_0^2 - q_1^2 - q_2^2 - (2p_1 + q_1)^2 - (2p_2 + q_2)^2) 
+ (q_1(2p_1 + q_1) + q_2(2p_2 + q_2))^2,
\]

\[
|M_{11}| = q_0(q_0^2 - (2p_1 + q_1)^2 - (2p_2 + q_2)^2),
\]

\[
|M_{21}| = q_1(2p_1 + q_1)^2 - q_1q_0^2 + q_2(2p_1 + q_1)(2p_2 + q_2),
\]

\[
|M_{31}| = q_2q_0^2 - q_2(2p_2 + q_2)^2 - q_1(2p_1 + q_1)(2p_2 + q_2),
\]

\[
|M_{41}| = iq_0q_1(2p_2 + q_2) - iq_0q_2(2p_1 + q_1).
\]

Since \( \Gamma_0(q, p) \) depends only on fermion propagator \( G(p) \), the DS equation of \( G(p) \) given by Eq. (23) is completely self-closed and can be solved by iteration method 10.

We adopt the Matsubara formalism and express momentum as \( p = (\epsilon_\alpha, \mathbf{p}) \). The two free boson propagators are \( D_0(q) = \frac{2\lambda}{\omega_n^2 + \omega_q^2 - \omega_{\mathbf{p}}^2} \) and \( F_0(q) = \frac{2\lambda}{\omega_n^2 - \omega_q^2} \). The phonon dispersion is \( \Omega_\mathbf{q} = c_\varepsilon |\mathbf{q}| \), where \( c_\varepsilon \) is phonon velocity. The effective strength of Coulomb interaction is characterized by a fine structure constant \( \alpha = \frac{e^2}{4\pi\varepsilon_0} \), where \( e \) is electric charge and \( \varepsilon \) is dielectric constant. \( \alpha = 0.8 \) for
strength parameter $\lambda$ increases, the non-monotonicity becomes more pronounced, which can be seen by comparing the results of Fig. 2(a-c). Moreover, we find that $v$ is a decreasing function of $|p|$ at any fixed energy for $\alpha = 0.8$, no matter whether the effects of EPI are taken into account. For $\alpha = 2.2$, $v$ first decreases with growing $|p|$, but tends to increase as $|p|$ approaches its ultraviolet cutoff. This upturn behavior is shown in Fig. 2(d). According to Fig. 2(f), EPI makes little contribution to the $|p|$-dependence of $v$. In particular, adding EPI to the system does not change the logarithmic $|p|$-dependence of $v(p)$ in the small-$|p|$ region caused purely by the Coulomb interaction. This result provides a natural explanation of the surprisingly good agreement between the experimental result of $v(p)$ measured in realistic graphene materials [14-16] and the theoretical result of $v(p)$ calculated without taking into account the impact of EPI. When the Coulomb interaction becomes strong (e.g., $\alpha = 2.2$), EPI can drive $v(p)$ to deviate from the logarithmic behavior as $|p|$ is approaching $p_F$. The velocity $v$ seems to increase abruptly if $\epsilon \to 0$ and $|p| \to 0$. As discussed in Ref. [11], this is an artifact caused by infrared cutoffs and the logarithmic $|p|$-dependence of fermion velocity is actually robust as the infrared cutoffs of $\epsilon$ and $|p|$ are decreased.

In the future, we wish to generalize our DS equation approach further to investigate the interplay of two FBIs in other condensed matter systems, such as those mentioned at the beginning of this paper.

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