RELATIVE HARD LEFSCHETZ FOR
SOERGEL BIMODULES

BEN ELIAS AND GEORDIE WILLIAMSON

Abstract. We prove the relative hard Lefschetz theorem for Soergel bimodules. It follows that the structure constants of the Kazhdan-Lusztig basis are unimodal. We explain why the relative hard Lefschetz theorem implies that the tensor category associated by Lusztig to any 2-sided cell in a Coxeter group is rigid and pivotal.

Contents
1. Introduction 1
1.1. Acknowledgements 4
2. Background 5
2.1. Soergel bimodules and duality 5
2.2. Perverse cohomology and graded multiplicity spaces 7
2.3. Polarizations of Soergel bimodules 10
2.4. Forms on multiplicity spaces 11
3. Relative hard Lefschetz and Hodge-Riemann 12
3.1. Statement 12
3.2. A conjecture 14
3.3. Base cases 14
3.4. Structure of the proof 15
4. The proof 16
4.1. Hodge-Riemann implies hard Lefschetz 16
4.2. Signs via limit arguments 19
5. Rigidity 24
References 28

1. Introduction

Let \((W, S)\) denote a Coxeter system and \(\mathcal{H}\) its Hecke algebra. It is an algebra over \(\mathbb{Z}[v^\pm 1]\) with standard basis \(\{H_x \mid x \in W\}\) and Kazhdan-Lusztig basis \(\{H_x \mid x \in W\}\). The Kazhdan-Lusztig positivity conjectures are the statements:

1. (“positivity of Kazhdan-Lusztig polynomials”) if we write \(H_x = \sum h_{y,x} H_y\), then \(h_{y,x} \in \mathbb{Z}_{\geq 0}[v]\);
(2) (“positivity of structure constants”) if we write $H_x H_y = \sum \mu^z_{x,y} H_z$ then $\mu^z_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

These conjectures have been known since the 1980s for Weyl groups of Kac-Moody groups [KL80, Spr82], using sophisticated geometric technology. More recently in [EW14] the authors proved these conjectures algebraically for arbitrary Coxeter systems by establishing Soergel’s conjecture.

Let us briefly recall the setting of Soergel’s conjecture. For a certain reflection representation $\mathfrak{h}$ of $(W, S)$ over the real numbers, Soergel constructed a category $\mathcal{B}$ of Soergel bimodules, which is a full subcategory of the category of graded $R$-bimodules, where $R$ denotes the polynomial functions on $\mathfrak{h}$. The category of Soergel bimodules $\mathcal{B}$ is monoidal under tensor product of bimodules, and is closed under grading shift. Soergel showed that one has a canonical isomorphism $\text{ch} : [\mathcal{B}] \cong \mathcal{H}$ of $\mathbb{Z}[v^{\pm 1}]$-algebras between the split Grothendieck group of Soergel bimodules and the Hecke algebra. (Property (1) follows because the coefficient of $H_y$ in $\text{ch}([B])$ is given by the graded dimension of a certain hom space. Property (2) follows because $\mu^z_{x,y}$ gives the graded multiplicity of $B_z$ as a summand in $B_x \otimes R B_y$.)

The geometric techniques used to understand the Kazhdan-Lusztig basis yield another remarkable property of the structure constants $\mu^z_{x,y}$. Using duality, one can show that $\mu^z_{x,y}$ is preserved under swapping $v$ and $v^{-1}$. The quantum numbers

$$[m] := \frac{v^m - v^{-m}}{v - v^{-1}} = v^{-m+1} + v^{-m+3} + \cdots + v^{m-3} + v^{m-1} \in \mathbb{Z}[v^{\pm 1}]$$

for $m \geq 1$ give a $\mathbb{Z}$-basis for those elements of $\mathbb{Z}[v^{\pm 1}]$ preserved under swapping $v$ and $v^{-1}$. A folklore conjecture states:1

3) (“unimodality of structure constants”) if we write $\mu^z_{x,y} = \sum_{m \geq 1} a_m[m]$, then $a_m \geq 0$ for all $m$.

(In other words, each $\mu^z_{x,y}$ is the character of a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation.)

---

1Unimodality is stated as a question in [dC06, § 5.1], however experts assure the authors that the conjecture is much older. In [dC06] positivity properties (2) and (3) are checked for $W$ a finite reflection group of $H_4$ by computer (almost three trillion polynomials $\mu^z_{x,y}$ need to be computed!). For $H_4$, property (1) had already been checked by Alvis [Alv87] in 1987. In [dC06, § 5.2] it is incorrectly stated that the unimodality conjecture is open for Weyl groups.
In geometric settings unimodality follows from the relative hard Lefschetz theorem of [BBD82]. Recall that the relative hard Lefschetz theorem states that if \( f : X \to Y \) is a projective morphism of complex algebraic varieties and if \( \eta \) is a relatively ample line bundle on \( X \) then for all \( i \geq 0 \), \( \eta \) induces an isomorphism:

\[
\eta^i : p^i\mathcal{H}_{-i}(Rf_*IC_X) \simrightarrow p^i\mathcal{H}(Rf_*IC_X).
\]

(Here \( IC_X \) denotes the intersection cohomology complex on \( X \) and \( p^i \) denotes perverse cohomology.) In this paper we prove unimodality for all Coxeter groups, by adapting the relative hard Lefschetz theorem to the context of Soergel bimodules.

Inside the category of Soergel bimodules we consider the full subcategory \( pB \) consisting of direct sums of the indecomposable self-dual bimodules \( B_x \) without shifts. We call \( pB \) the subcategory of perverse Soergel bimodules. Soergel’s conjecture implies that each \( B \in pB \) admits a canonical isotypic decomposition

\[
B = \bigoplus_{x \in W} V_x \otimes_R B_x
\]

for certain real (degree zero) vector spaces \( V_x \). If a Soergel bimodule is not perverse, its decomposition into indecomposable summands of the form \( B_x(i) \) is not canonical. However, there is a canonical filtration on any Soergel bimodule called the perverse filtration, whose \( i \)-th subquotient has indecomposable summands of the form \( B_x(-i) \) for some \( x \in W \). Taking the subquotients of this filtration and shifting them appropriately, one obtains for each \( i \) the perverse cohomology functor

\[
H^i : B \to pB.
\]

Any degree \( d \) map \( B \to B'(d) \) induces a map \( H^i(B) \to H^{i+d}(B') \) on perverse cohomology.

**Remark 1.1.** The category \( B \) is an analogue of semi-simple complexes, \( pB \) is an analogue of the category of semi-simple perverse sheaves and \( H^i \) is an analogue of the perverse cohomology functor.

This main result of this paper is the following:

**Theorem 1.2.** (Relative hard Lefschetz for Soergel bimodules) Let \( x, y \in W \) be arbitrary and fix \( \rho \in h^* \) dominant regular (i.e. \( \langle \rho, \alpha^*_s \rangle > 0 \) for all \( s \in S \)). The map

\[
\eta : B_x \otimes_R B_y \to B_x \otimes_R B_y[2]
\]

\[
b \otimes b' \mapsto b \otimes \rho b' = b\rho \otimes b'
\]

induces an isomorphism (for all \( i \geq 0 \))

\[
\beta^i : H^{-i}(B_x \otimes_R B_y) \simrightarrow H^i(B_x \otimes_R B_y).
\]
Remark 1.3. A stronger version of the above theorem, involving iterated tensor products of indecomposable Soergel bimodules of arbitrary length is still open (see Conjecture 3.4). It is amusing that establishing Conjecture 3.4 for Bott-Samelson bimodules (i.e., when all $x_i \in S$, in the notation of Conjecture 3.4) was the authors’ original plan of attack to settle Soergel’s conjecture. This remains a very interesting Hodge theoretic statement that we cannot prove!

As was true in our previous work on hard Lefschetz type theorems for Soergel bimodules [EW14, Wil14], the inductive proof we use to establish our main theorem actually requires proving a stronger statement, analogous to the relative Hodge-Riemann bilinear relations [dCM05]. That is, we must calculate the signatures of certain forms on the multiplicity spaces of $H^{-i}(B_x \otimes_R B_y)$, see Theorem 3.3. The following is an immediate consequence of Theorem 1.2:

**Corollary 1.4.** The structure constants $\mu_{x,y}^z$ of multiplication in the Kazhdan-Lusztig basis are unimodal.

Relative hard Lefschetz for Soergel bimodules also has important consequences for certain tensor categories associated to cells in Coxeter groups. Recall that to any two sided cell $c \subset W$ in a finite or affine Weyl group Lusztig has associated a tensor category, which categorifies the $J$-ring of $c$. These categories (for finite Weyl groups) are fundamental for the representation theory of finite reductive groups of Lie type: by results of Bezrukavnikov, Finkelberg and Ostrik [BFO12] and Lusztig [Lus15], their (Drinfeld) centers are equivalent to the braided monoidal category of unipotent character sheaves corresponding to $c$.

Given any two sided cell $c \subset W$ in an arbitrary Coxeter group Lusztig has generalised his construction to yield a monoidal category $\mathcal{J}$. (Note that $\mathcal{J}$ is only “locally unital” unless $c$ contains finitely many left cells, and the existence of a unit relies on a conjecture in general, see Remark 5.1.) In the last section of this paper we explain why Theorem 1.2 implies that $\mathcal{J}$ is rigid and pivotal (see Theorem 5.2). (The rigidity was conjectured by Lusztig [Lus15, § 10] when $W$ is finite). This is an important step towards the study of “unipotent character sheaves” associated to any Coxeter system.

By a theorem of [Müg03, EN05], rigidity of $\mathcal{J}$ implies that the (Drinfeld) center of $\mathcal{J}$ is a modular tensor category. We expect cells in non-crystallographic Coxeter groups to provide many new examples of modular tensor categories (see [Ost14, 5.4]).

1.1. **Acknowledgements.** That relative hard Lefschetz is tied to the rigity of Lusztig’s categorifications of the $J$-ring was suggested to us by Victor Ostrik. We would like to thank him, as well as Roman Bezrukavnikov, George Lusztig and Noah Snyder for useful discussions.
2. Background

2.1. Soergel bimodules and duality. Let $\mathfrak{h}$ be an $\mathbb{R}$-linear realization of the Coxeter system $(W, S)$, as in [Soe07, § 2]. Thus $\mathfrak{h}$ is a finite-dimensional $\mathbb{R}$-vector space, equipped with linearly independent subsets of roots $\{\alpha_s\}_{s \in S} \subset \mathfrak{h}^*$ and coroots $\{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}$, such that
\[
\langle \alpha_s, \alpha_t^\vee \rangle = -2 \cos(\pi/m_{st}),
\]
where $m_{st}$ denotes the order (possibly $\infty$) of $st \in W$.\(^2\) We have an action of $W$ on $\mathfrak{h}$ given by the formula
\[
s(v) = v - \langle \alpha_s, v \rangle \alpha_s^\vee,
\]
for all $s \in S$ and $v \in \mathfrak{h}$. The contragredient action of $W$ on $\mathfrak{h}^*$ is defined by an analogous formula
\[
s(f) = f - \langle f, \alpha_s^\vee \rangle \alpha_s,
\]
for all $s \in S$ and $f \in \mathfrak{h}^*$.

Let $R$ be the ring of polynomial functions on $\mathfrak{h}$, graded so that the linear terms $\mathfrak{h}^*$ have degree 2. It comes equipped with an action of $W$. Define a graded $R$-bimodule $B_s = R \otimes_{R^s} R(1)$ for each $s \in S$, where $R^s$ denotes the $s$-invariant polynomial subring. We use the standard convention for grading shifts, so that the $(1)$ above indicates that the minimal degree element $1 \otimes 1$ lives in degree $-1$. Given two graded $R$-bimodules $B, B'$ their tensor product over $R$ is denoted $BB' := B \otimes_R B'$. For a sequence $w = (s_1, s_2, \ldots, s_d)$ with $s_i \in S$, the tensor product
\[
BS(w) = B_{s_1}B_{s_2} \ldots B_{s_d}
\]
is called a Bott-Samelson bimodule.

Soergel proved in [Soe07] that, when $x$ is a reduced expression for an element $x \in W$, there is a unique indecomposable direct summand $B_x \cong BS(x)$ which is not isomorphic to a summand of a shift of any Bott-Samelson bimodule corresponding to a shorter reduced expression. Moreover, this summand does not depend on the reduced expression of $x$, up to non-canonical isomorphism. (Using the main theorem of [EW14] one can make this isomorphism canonical.) Note that the two notations for $B_s$ agree.

Let $\mathcal{B}$ denote the full subcategory of graded $R$-bimodules whose objects are finite direct sums of grading shifts of summands of Bott-Samelson bimodules. The objects in this category $\mathcal{B}$ are known as Soergel bimodules, and the bimodules $\{B_x\}_{x \in W}$ give a complete list of non-isomorphic indecomposable objects up to grading shift. Because Bott-Samelson bimodules are

\(^2\)The choice of roots and coroots plays a significant role in this paper, but only up to positive rescaling; what is important (in order that we may cite certain results from [Soe07] and [EW14]) is that our representations is reflection faithful [Soe07] and that there be a well-defined notion of positive roots. If the reader prefers, they may also take the representation given by a realisation of a generalised Cartan matrix.
closed under tensor product, $\mathcal{B}$ is as well, and inherits its monoidal structure from $R$-bimodules.

If $B$ is a Soergel bimodule we will often use the symbol $B$ to denote the identity morphism on $B$. For example, if $f : B' \to B''$ is a morphism then $Bf : BB' \to BB''$ denotes the tensor product of the identity on $B$ with $f$. Similarly, given $r \in R$ of degree $m$, $rB$ (resp. $Br$) denotes the morphism $B \to B(m)$ given by left (resp. right) multiplication by $r$.

For two Soergel bimodules $B$ and $B'$, we let $\text{Hom}(B, B')$ denote the degree zero homomorphisms of $R$-bimodules, and write

$$\text{Hom}^\bullet(B, B') = \bigoplus_{m \in \mathbb{Z}} \text{Hom}(B, B'(m))$$

for the graded vector space of bimodule homomorphisms of all degrees. A morphism $f \in \text{Hom}(B, B'(m))$ is said to be a degree $m$ morphism from $B$ to $B'$. By a theorem of Soergel [Soe07, Theorem 5.15], $\text{Hom}^\bullet(B, B')$ is free of finite rank as a left or right $R$-module.

Given a Soergel bimodule $B \in \mathcal{B}$ its dual is

$$\mathcal{D}B := \text{Hom}^\bullet_R(B, R)$$

where $\text{Hom}^\bullet_R$ denotes the graded vector space of right $R$-module homomorphisms of all degrees. We make $\mathcal{D}B$ into an $R$-bimodule via $r_1 f r_2(b) = f(r_1 b r_2)$. Because $\mathcal{D}BS(w) \cong BS(w)$, the functor $\mathcal{D}$ descends to a contravariant equivalence of $\mathcal{B}$. By the defining property of the indecomposable bimodule $B_x$, we must also have $\mathcal{D}B_x \cong B_x$. As usual, $\mathcal{D}B \cong B$ canonically, for any Soergel bimodule $B$.

A pairing on two Soergel bimodules $B, B'$ is a homogeneous bilinear form

$$\langle \cdot, \cdot \rangle : B \times B' \to R$$

such that $\langle rb, b' \rangle = \langle b, rb' \rangle$ and $\langle br, b' \rangle = \langle b, b' r \rangle = \langle b, b' \rangle r$ for all $b \in B, b' \in B$ and $r \in R$. (Note the asymmetry in the conditions on the left and right $R$-actions.) The homogeneous condition states that $\text{deg} b + \text{deg} b' = \text{deg} \langle b, b' \rangle$. A pairing induces bimodule morphisms $B \to \mathcal{D}B'$ and $B' \to \mathcal{D}B$. We say that a pairing is non-degenerate if one (or equivalently both) of these morphisms is an isomorphism.

A (non-degenerate) form on a Soergel bimodule is a (non-degenerate) pairing

$$\langle \cdot, \cdot \rangle : B \times B \to R$$

which is in addition symmetric: $\langle b, b' \rangle = \langle b', b \rangle$ for all $b, b' \in B$. A polarized Soergel bimodule is a pair $(B, \langle \cdot, \cdot \rangle_B)$ where $B \in \mathcal{B}$ is a Soergel bimodule and $\langle \cdot, \cdot \rangle_B$ is a non-degenerate form, in which case $\langle \cdot, \cdot \rangle_B$ is the polarization.

---

3This is the convention used in [EW14]. The opposite convention is used in [Wil14].

4Warning: this is stronger than the condition $\langle b, B \rangle = 0 \Rightarrow b = 0$. 
Given a map \( f : B \rightarrow B'(m) \) between polarized Soergel bimodules its
adjoint is the unique map \( f^* : B' \rightarrow B(m) \) such that
\[
\langle f(b), b' \rangle_B = \langle b, f^*(b') \rangle_{B'}
\]
for all \( b \in B, b' \in B' \).

Equivalently \( f^* = \mathcal{D} f \mathcal{D} (B'(m)) \rightarrow \mathcal{D} B \), where we use the polarizations to
identify \( B = \mathcal{D} B, B'(-m) = \mathcal{D} (B'(m)) \).

2.2. Perverse cohomology and graded multiplicity spaces. All mor-
phisms between indecomposable self-dual Soergel bimodule s are of non-
negative degree, and those of degree zero are isomorphisms. That is:

\[
\text{Hom}(B_x, B_y) = \begin{cases} R & \text{if } x = y, \\ 0 & \text{otherwise}. \end{cases}
\] (2.1)

\[
\text{Hom}(B_x, B_y(m)) = 0 \quad \text{for } x, y \in W \text{ and } m < 0, \] (2.2)

These fundamental Hom-vanishing statements are equivalent to Soergel’s
conjecture (see the paragraph following [EW14, Theorem 3.6]).

A Soergel bimodule \( B \) is perverse if it is isomorphic to a direct sum of
indecomposable bimodules \( B_x \) without shifts. We denote by \( pB \) the full
subcategory of perverse Soergel bimodules. As a consequence of (2.2), any
perverse Soergel bimodule admits a canonical decomposition
\[
B = \bigoplus_{x \in W} V_x \otimes_R B_x
\] (2.3)

for some finite dimensional real vector spaces \( V_x \). (Concretely, one has \( V_x = \text{Hom}(B_x, B) \).) The rest of this section is dedicated to understanding what
replaces this multiplicity space \( V_x \) in case the bimodule \( B \) in question is not
perverse.

By the classification of indecomposable bimodules, every Soergel bimodule
splits into a direct sum of shifts of perverse Soergel bimodules, but this splitting is
not canonical. However, it is a consequence of (2.1) and (2.2) that \( B \) admits
a unique functorial (non-canonically split) filtration, the perverse filtration,
whose subquotients isomorphic to a shift of a perverse Soergel bimodule,
see [EW14, §6.2]. Before discussing the details, it is worth illustrating this
subtle point in examples.

**Example 2.1.** The Bott-Samelson bimodule \( B_s B_s \) is isomorphic to \( B_s(+1) \oplus B_s(-1) \). The degree \(-1\) projection map, that is, the map \( B_s B_s \rightarrow B_s(-1) \), is canonical up to a scalar. After all, it is easy to confirm from (2.1) and
(2.2) that \( \text{Hom}^*(B_s B_s, B_s) \cong \text{Hom}^*(B_s(+1) \oplus B_s(-1), B_s) \) is zero in degrees
\( \leq -2 \), and is one-dimensional in degree \(-1\). The same can be said about
the degree \(-1\) inclusion map, that is, the map \( B_s(+1) \rightarrow B_s B_s \). However,
the degree \(+1\) projection map \( B_s B_s \rightarrow B_s(+1) \) (resp. the degree \(+1\) inclusion
map \( B_s(-1) \rightarrow B_s B_s \)) is not canonical; adding to it an \( R\)-multiple of
the degree \(-1\) projection map will give another valid projection map. Said
another way, \( B_s(+1) \) is a canonical submodule, and \( B_s(-1) \) a canonical
quotient, and this filtration of \( B_s B_s \) splits, but not canonically.
Example 2.2. Suppose that $W$ is of type $A_2$ with simple reflections $\{s, t\}$. Then $BS(stst)$ is isomorphic to $B_{sts}(-1) \oplus B_{sts}(+1) \oplus B_{st}$. The degree $-1$ projection map to $B_{sts}(-1)$ is canonical. However, the degree 0 projection map to $B_{st}$ is not canonical! The morphism space $\text{Hom}(BS(stst), B_{st})$ is two-dimensional; it has a one-dimensional subspace arising as the composition of the canonical projection to $B_{sts}(-1)$ followed by a non-split map $B_{sts}(-1) \to B_{st}$, and any morphism not in this one-dimensional subspace will serve as a projection map to $B_{st}$. This example is meant to loudly proclaim that even what appears to be an “isotypic component,” such as the summand $B_{st}$ which is the only one of its kind, is not canonically a direct summand, owing to the presence of other summands with lower degree shifts.

For any $i \in \mathbb{Z}$, define $B \leq i$ (resp. $B > i$) to be the full additive subcategory of $B$ consisting of bimodules which are isomorphic to direct sums of $B_x(m)$ with $m \geq -i$ (resp. $m < -i$). In formulas:

$$B \leq i := \langle B_x(m) \mid x \in W, m \geq -i \rangle_{\oplus, \simeq},$$

$$B > i := \langle B_x(m) \mid x \in W, m < -i \rangle_{\oplus, \simeq}.$$  

Similarly we define $B < i$ and $B \geq i$. We have $pB = B \geq 0 \cap B \leq 0$. We can rephrase (2.2) as the statement:

(2.4) \quad $\text{Hom}(B \leq i, B > i) = 0$.

Any Soergel bimodule $B$ admits a unique \textit{perverse filtration}

$$\cdots \subset \tau_{\leq i}B \subset \tau_{\leq i+1}B \subset \cdots$$

by split inclusions such that $\tau_{\leq i}B \subset B \leq i$ and $B/\tau_{\leq i}B \in B > i$, see [EW14, §6.2]. This is a direct consequence of (2.4). If $f : B \to B'$ is a morphism then $f(\tau_{\leq i}B) \subset \tau_{\leq i}B'$. We have:

(2.5) \quad $\tau_{\leq i}(B(m)) = (\tau_{\leq i+m}B)(m)$.

Dually, every Soergel bimodule has a unique \textit{perverse cofiltration}

$$\cdots \twoheadrightarrow \tau_{\geq i}B \twoheadrightarrow \tau_{\geq i+1}B \twoheadrightarrow \cdots$$

where every arrow is a split surjection, each $\tau_{\geq i}B \in B \geq i$ and the kernel of $B \twoheadrightarrow \tau_{\geq i}B$ belongs to $B < i$. We have:

(2.6) \quad $D(\tau_{\geq i}B) = \tau_{\leq -i}(DB)$.

The \textit{perverse cohomology} of a Soergel bimodule $B$ is

$$H^i(B) := (\tau_{\leq i}B/\tau_{< i}B)(i).$$

(The shift $(i)$ is included so that $H^i(B)$ is perverse.) Applying (2.3) we obtain canonical isotypic decompositions

$$H^i(B) = \bigoplus_{z \in W} H^i_z(B) \otimes B_z.$$
for certain finite dimensional vector spaces $H^i(B)$. We have a non-canonical isomorphism

$$B \cong \text{gr} B := \bigoplus_{i \in \mathbb{Z}} H^i(B)(-i)$$

and canonical isomorphisms

$$\text{gr} B = \bigoplus_{i, z} H^i_z(B) \otimes B_z(-i) = \bigoplus_{z} H^\bullet_z(B) \otimes_{\mathbb{R}} B_z$$

where $H^\bullet_z(B)$ denotes the graded vector space $\bigoplus H^i_z(B)$. Below we call the graded vector spaces $H^\bullet_z(B)$ multiplicity spaces.

Remark 2.3. To reiterate the point made in Example 2.2: in general, it is not possible to produce separate multiplicity spaces $H^\bullet_z(B)$, for different $z \in W$, without first passing to the associated graded of the perverse filtration.

Let $B, B'$ be Soergel bimodules and $f : B \to B'(m)$ a morphism. Then by (2.4) and (2.5) we have

$$f(\tau \leq i B) \subset \tau \leq i - 1 (B'(m)) = (\tau \leq i + m B')(m).$$

Thus $f$ induces a map

$$f : H^i(B) \to H^{i+m}(B')$$

of Soergel bimodules, and hence a degree $m$ map $\text{gr} f$ from $\text{gr} B$ to $\text{gr} B'$. For any $z \in W$ this induces a map

$$\text{gr}_z f : H^\bullet_z(B) \to H^\bullet_{z+m}(B')$$

of graded vector spaces. To simplify notation, we use $f$ to denote all these maps: $f, \text{gr} f, \text{gr}_z f$ for all $z \in W$. We refer to the maps $\text{gr} f$ and $\text{gr}_z f$ as the maps induced on perverse cohomology.

The following triviality is important later:

Lemma 2.4. If $f : B \to B'(m)$ is a map such that, for all $i \in \mathbb{Z}$,

$$f(\tau \leq i B) \subset \tau \leq i - 1 (B'(m))$$

then $f$ induces the zero map on perverse cohomology. In particular, this applies to the map given by left or right multiplication by any positive-degree polynomial in $R$ on a Soergel bimodule $B$.

Proof. Only the second sentence requires proof. The perverse filtration is a filtration by $R$-bimodules. If $r \in R$ is homogenous of degree $d > 0$ then multiplication by $r$ on the left (resp. right) induces a map (see (2.5))

$$\tau \leq i B \to (\tau \leq i B)(d) = (\tau \leq i-d(B))(d).$$

Therefore, the hypothesis of the lemma applies to multiplication by $r$. \hfill $\square$
2.3. Polarizations of Soergel bimodules. In [EW14, §3.4, see also Corollary 3.9], the Bott-Samelson bimodule $BS(w)$ was equipped with a non-degenerate form called the intersection form. By restriction, one obtains a form on any summand of a Bott-Samelson bimodule. By [EW14, Lemma 3.7], there is, up to an invertible scalar, a unique non-zero form on an indecomposable Soergel bimodule $B_x$ (this statement is equivalent to Soergel’s conjecture), and it is non-degenerate. Thus, letting $\omega$ be any reduced expression for $x$, the restriction of the intersection form to $B_x \oplus \subset BS(x)$ is non-zero, hence is non-degenerate and hence is a polarization of $B_x$. For all $x \in W$ we fix a reduced expression $\omega$ of $x$ and an embedding $B_x \subset BS(x)$, and hence a polarization $\langle -, - \rangle_{B_x}$ on $B_x$. We refer to $\langle -, - \rangle_{B_x}$ as the intersection form on $B_x$. The intersection form has the following important positivity property:

**Lemma 2.5** ([EW14], Lemma 3.10). If $\rho \in h^*$ dominant regular (i.e. $\langle \rho, \alpha_s^\vee \rangle > 0$ for all $s \in S$) and $b \in B_x$ is any non-zero element of degree $-\ell(x)$ then

$$\langle b, \rho^{\ell(x)}b \rangle > 0.$$  

**Remark 2.6.** This lemma and the discussion of the previous paragraph implies that the intersection form on $B_x$ does not depend on the choice of reduced expression $\omega$ or the choice of embedding $B_x \subset BS(x)$, up to multiplication by a positive scalar.

Given any polarized Soergel bimodule $B$, it is explained in [EW14, §3.6] how to produce a polarization on $BB_s$, called the induced form. Moreover, if $B = B_x$ is given its intersection form (i.e. the form restricted from our fixed inclusion $B_x \subset BS(x)$ for a reduced expression) then the induced form on $BB_s$ agrees with the form restricted from the inclusion $B_x B_s \subset BS(x s)$. This is because the intersection form on any Bott-Samelson bimodule $BS(w)$ is constructed by being repeatedly induced from the canonical form on $BS(\emptyset) = R$. Let us generalize this notion of induced forms.

If $B$ and $B'$ are two polarized Soergel bimodules, we define a form on $BB'$ by the formula

$$\langle b \otimes b', c \otimes c' \rangle_{BB'} := \langle (\langle b, c \rangle_B) \cdot b', c' \rangle_{B'} = \langle b', (\langle b, c \rangle_B) \cdot c' \rangle_{B'}.$$  

It is an exercise to confirm that the induced form on $BB_s$ is defined precisely in this fashion.

**Lemma 2.7** ([Wil14], §6.4). The induced form on $BB'$ is non-degenerate, and thus is a polarization of $BB'$.

By iteration, we have an induced form on any tensor product of the form $B_{x_1}B_{x_2} \cdots B_{x_m}$, which we continue to call the intersection form. One could also view $B_{x_1} \cdots B_{x_m}$ as a summand (via the tensor products of our fixed

---

5The reader should not forget that $\langle b, b \rangle$ is, in general, an element of the ring $R$. Here, for degree reasons, one obtains a degree zero element of $R$, hence an element of $\mathbb{R}$.
embeddings) of a Bott-Samelson bimodule $BS(w)$, where $w$ is a concatenation of our chosen reduced expression for each $x_i$. The induced form agrees with the restriction of the intersection form on $BS(w)$ to this summand. All tensor products of the form $B_{x_1}B_{x_2} \ldots B_{x_m}$ are always assumed to be polarized with respect to their intersection form.

Let $(B, \langle -,- \rangle_B)$ be a polarized Soergel bimodule. If $B$ is also perverse then by considering the isotypic decomposition (see (2.3))

\[ B = \bigoplus_{x \in W} V_x \otimes_R B_x \]

and the associated map $B \rightarrow \mathcal{D}B$, we see that $\langle -,- \rangle_B$ is orthogonal for this decomposition. Moreover, $\langle -,- \rangle_B$ is determined by symmetric forms $\langle -,- \rangle_{V_x}$ on each vector space $V_x$ (i.e. if $v,v' \in V_x$ and $b,b' \in B_x$ then $\langle v \otimes b, v' \otimes b' \rangle = \langle v,v' \rangle_{V_x} \langle b,b' \rangle_{B_x}$). We say that $B$ is positively polarized if $B = 0$ or the following conditions are satisfied:

1. $B$ is perverse and vanishes in even or odd degree (because $B_x$ is non-zero in degree $-\ell(x)$, the second condition is equivalent to the existence of $q \in \{0,1\}$ such that $V_x = 0$ for all $x$ with $\ell(x)$ of the same parity as $q$);
2. Let $z \in W$ denote the element of maximal length in $W$ such that $V_z \neq 0$. If $V_y \neq 0$ then $\langle -,- \rangle_{V_y}$ is $(-1)^{(\ell(z)-\ell(y))/2}$ times a positive definite form, for all $y \in W$.

The canonical example of a positively polarized Soergel bimodule is given by the following lemma:

**Lemma 2.8** ([EW14], Proposition 6.12). Suppose that $y \in W$ and $s \in S$ with $ys > y$ (resp. $sy > y$). Then $B_yB_s$ (resp. $B_sB_y$), equipped with its intersection form, is positively polarized.

### 2.4. Forms on multiplicity spaces.

Assume that $(B, \langle -,- \rangle)$ is a polarized Soergel bimodule. If we interpret $\langle -,- \rangle$ instead as an isomorphism

\[ f : B \sim \rightarrow \mathcal{D}(B) \]

then we deduce from the functoriality of the perverse filtration that:

\[ f(\tau_{\leq i}) \subset \tau_{\leq i}(\mathcal{D}B) \overset{(2.6)}{=} \mathcal{D}(\tau_{\geq-i}B), \]

\[ f \text{ induces an isomorphism } H^i(B) \sim \rightarrow H^i(\mathcal{D}B) = \mathcal{D}H^{-i}(B). \]

Statement (2.8) is equivalent to saying that $\langle \tau_{\leq i}B, \tau_{< -i}B \rangle = 0$ and hence that $\langle -,- \rangle$ induces a pairing of Soergel bimodules

\[ \langle -,- \rangle : H^i(B) \times H^{-i}(B) \rightarrow R, \]

and (2.9) tells us that this pairing is non-degenerate. By (2.1) the canonical decompositions

\[ H^i(B) = \bigoplus_{z \in W} H^i_z(B) \otimes_R B_z \quad \text{and} \quad H^{-i}(B) = \bigoplus_{z \in W} H^{-i}_z(B) \otimes_R B_z \]
are orthogonal with respect to \(\langle -, - \rangle\) (i.e. \(\langle \gamma \otimes b, \gamma' \otimes b' \rangle = 0\) for \(\gamma \otimes b \in H_z^i(B) \otimes_R B_z\) and \(\gamma' \otimes b' \in H_{z'}^{-i}(B) \otimes_R B_{z'}\) if \(z \neq z'\)). Applying (2.1) again we conclude that (2.10) is completely determined by the non-degenerate bilinear pairing on the vector spaces

\[
(2.11) \quad H_z^i(B) \times H_{z'}^{-i}(B) \to \mathbb{R}
\]

for all \(z \in W\). To be precise, given \(v \in H_z^i(B)\) and \(v' \in H_{z'}^{-i}(B)\), this pairing (2.11) is defined so that, for all \(b, b' \in B_z\), one has

\[
(2.12) \quad \langle v \otimes b, v' \otimes b' \rangle = \langle v, v' \rangle \langle b, b' \rangle.
\]

The left hand side is (a summand of) the pairing in (2.10) between \(H^i(B)\) and \(H^{-i}(B)\), and the right hand side is the pairing in (2.11) multiplied by the intersection form on \(B_z\).

Reassembling this data, we conclude that \(\langle -, - \rangle\) descends to a symmetric non-degenerate form

\[
\langle -, - \rangle : \text{gr}B \times \text{gr}B \to \mathbb{R}
\]

and that this form is determined by the symmetric non-degenerate graded bilinear forms

\[
\langle -, - \rangle : H^*_z(B) \times H^*_z(B) \to \mathbb{R}
\]

on multiplicity spaces for all \(z \in W\).

Here is another important triviality:

**Lemma 2.9.** Let \(B = BS(x)\) be a Bott-Samelson bimodule associated to a reduced expression \(x\) for an element \(x \in W\), polarized with respect to its intersection form. The summand \(B_z\) appears with multiplicity one having no grading shift, so that \(H^*_z(B) = \mathbb{R}\) in degree zero. Up to a positive scalar, the form \(H^0(B) \times H^0(B) \to \mathbb{R}\) is just the standard form, with \(\langle 1, 1 \rangle = 1\).

**Proof.** This follows immediately from (2.12), because the intersection form on \(BS(x)\) restricts to a positive multiple of the intersection form on \(B_x\) (see Remark 2.6). \(\square\)

### 3. Relative hard Lefschetz and Hodge-Riemann

#### 3.1. Statement

We fix once and for all a dominant regular \(\rho \in h^*\), that is, an element such that \(\langle \rho, a^\vee_s \rangle \geq 0\) for all \(s \in S\).

Let \(x := (x_1, \ldots, x_m)\) be a sequence of elements in \(W\), and fix scalars \(a := (a_1, \ldots, a_{m-1}) \in \mathbb{R}^{m-1}\). Consider the operator

\[
L_a : B_{x_1} B_{x_2} \cdots B_{x_m} \to B_{x_1} B_{x_2} \cdots B_{x_m} (2)
\]

\[
L_a = a_1 B_{x_1} \rho B_{x_2} \cdots B_{x_m} + a_2 B_{x_1} B_{x_2} \rho \cdots B_{x_m} + \cdots + a_{m-1} B_{x_1} B_{x_2} \cdots \rho B_{x_m}.
\]

In words, \(L_a\) is the sum of the operators of multiplication by \(a_i \rho\) in the gap between \(B_{x_i}\) and \(B_{x_{i+1}}\).

We have explained that to any \(z \in W\) we may associate a graded vector space

\[
V^* := H^*_z(B_{x_1} B_{x_2} \cdots B_{x_m})
\]
equipped with

1. a symmetric graded non-degenerate form \( \langle -, - \rangle_{V^\bullet} \) obtained from the intersection form on \( B_{x_1} \ldots B_{x_m} \);
2. a degree two Lefschetz operator \( L_a : V^\bullet \to V^{\bullet+2} \) obtained by taking perverse cohomology of \( L_a \).

**Remark 3.1.** The operator \( L_a \) involves only internal multiplication by polynomials. One could also consider the Lefschetz operator \( L_a + a_0 \rho \cdot (-) + a_m(-) \cdot \rho \) which includes multiplication on the left and right. However, as observed in Lemma 2.4, left and right multiplication by polynomials act trivially on perverse cohomology, so this does not affect the degree 2 operator on \( V^\bullet \).

We say that \( L_a \) satisfies relative hard Lefschetz if for any \( d \geq 0 \), \( L_a \) induces an isomorphism:

\[
L_a^d : V^{-d} \cong V^d.
\]

We say that \( L_a \) satisfies relative Hodge-Riemann if \( L_a \) satisfies relative hard Lefschetz and the restriction of the Lefschetz form \( \langle v, v' \rangle := \langle v, L_a^d v' \rangle_{V^\bullet} \) on \( V^{-d} \) to

\[
P^{-d} := \ker L_a^{d+1} : V^{-d} \to V^{d+2}
\]

is \((-1)^{\varepsilon(x, z, d)}\)-definite, for all \( d \geq 0 \), where

\[
\varepsilon(x, z, d) := \frac{1}{2} \left( \sum_{i=1}^m \ell(x_i) - \ell(z) - d \right).
\]

Note that relative hard Lefschetz and relative Hodge-Riemann are both statements about \( H^\bullet_z \) which are required to hold for all \( z \in W \).

**Remark 3.2.** The sign \((-1)^{\varepsilon(x, z, d)}\) might appear mysterious. The following is a useful mnemonic. Set \( B := B_{x_1} \ldots B_{x_m} \) and consider the finite dimensional graded vector space

\[
\overline{B} := B \otimes_R \mathbb{R}.
\]

We have a non-canonical isomorphism

\[
\overline{B} \cong \bigoplus_{z \in W} H^\bullet_z(B) \otimes \overline{B}_z.
\]

Now \( \varepsilon(x, z, d) \) has the following meaning: it is half the difference between the smallest non-zero degree in \( H^{-d}_z(B) \otimes_R \overline{B}_z^{-\ell(z)} \) on the right hand side (i.e. \( -\ell(z) - d \)) and the smallest non-zero degree in \( \overline{B} \) (i.e. \( -\sum \ell(x_i) \)). In this way one may see that the above definition is compatible with the signs predicted by Hodge theory in the geometric setting (see [dCM05] and [Wil, Theorem 3.12], where the signs are made explicit).
For $x_1, \ldots, x_m \in W$ as above we introduce the following abbreviations:

\[
RHL(x_1, \ldots, x_m) : \quad \text{$L_a$ satisfies relative hard Lefschetz for all } a := (a_1, \ldots, a_{m-1}) \in \mathbb{R}_{>0}^{m-1}.
\]

\[
RHR(x_1, \ldots, x_m) : \quad \text{$L_a$ satisfies relative Hodge-Riemann for all } a := (a_1, \ldots, a_{m-1}) \in \mathbb{R}_{>0}^{m-1}.
\]

As always, it is implicitly assumed in these statements that all tensor products of the form $B_{x_1} \cdots B_{x_m}$ are equipped with their intersection form.

The main theorem of this paper is:

**Theorem 3.3.** For any $x, y \in W$, $RHR(x, y)$ holds.

### 3.2. A conjecture.

**Conjecture 3.4.** For any $x_1, \ldots, x_m \in W$, $RHR(x_1, \ldots, x_m)$ holds.

More generally, relative Hodge-Riemann should hold for any operator of the form

\[
B_{x_1}\rho_1B_{x_2} \cdots B_{x_m} + B_{x_2}B_{x_2}\rho_2 \cdots B_{x_m} + \cdots + B_{x_1}B_{x_2} \cdots \rho_{m-1}B_{x_m},
\]

where $\rho_1, \ldots, \rho_{m-1}$ is any sequence of dominant regular elements. (Such elements span the cone of relatively ample classes in the Weyl group case.) For the conjecture above, one sets $\rho_i = a_i\rho$.

### 3.3. Base cases.

**Lemma 3.5.** $RHL(x)$ and $RHR(x)$ hold, for any $x \in W$.

*Proof.* The only nonvanishing $H^*_z(B_x)$ occurs when $z = x$, and this multiplicity space is concentrated in degree zero. Thus $RHL(x)$ is trivial, and $RHR(x)$ is equivalent to the statement that the form $H^0_z(B_x) \times H^0_z(B_x) \to \mathbb{R}$ is positive definite, which holds by Lemma 2.9. \hfill $\Box$

**Lemma 3.6.** If $RHL(x_1, x_2, \ldots, x_m)$ holds, then so does $RHL(x_1, \ldots, x_m, id)$ and $RHL(id, x_1, \ldots, x_m)$. The same statement can be made for $RHR$.

*Proof.* Let us compare $RHL(x_1, x_2, \ldots, x_m)$ and $RHL(x_1, \ldots, x_m, id)$. Because $B = B_{x_1} \cdots B_{x_m} = B_{x_1} \cdots B_{x_m}B_{id}$, the multiplicity spaces $H^*_z(B)$ being studied are the same. The operator $L_a$ on $B$ is different, because in the latter case, one is also permitted to multiply by $a_m\rho$ in the slot before the final $B_1$. However, this is equal to right multiplication by $a_m\rho$, which acts trivially on perverse cohomology. See Lemma 2.4 and Remark 3.1. Thus the Lefschetz operators on $H^*_z(B)$ are the same. \hfill $\Box$

To warm up, we consider the first interesting case: $RHR(x, s)$, for $s \in S$. This splits into two subcases: $xs < x$ and $xs > x$. Suppose that $xs > x$. Then $B_xB_s$ is perverse, and so each $H^*_z(B_xB_s)$ is concentrated in degree 0 and $RHL(x, s)$ holds automatically. In this case $RHR(x, s)$ is equivalent to Lemma 2.8.

Suppose now that $xs < x$. Then $B_xB_s \cong B_x(+1) \oplus B_x(-1)$. The action of $B_x\rho B_s$ on the multiplicity spaces $H^*_z(B_xB_s)$ is independent of $x$ (see Lemma
4.15 below), and can be computed when $x = s$, where it is a simple exercise. (We have been brief here because this computation, expanded upon and in further generality, comprises the bulk of §4.2.)

3.4. Structure of the proof. Let us outline the major steps in the proof of Theorem 3.3, which will be carried out in the rest of this paper. The proof is by induction on $\ell(x) + \ell(y)$ and then on $\ell(y)$. More precisely, for integers $M$ and $N$, consider the statements:

$X_{M,N}^\ell$:

1. $\ell(x') + \ell(y') < M$,
2. $\ell(x') + \ell(y') = M$ and $\ell(y') \leq N$.

$RHR(x', y')$ holds whenever either

$Y_{M,N}^\ell$:

1. $\ell(x') + \ell(y') + 1 < M$,
2. $\ell(x') + \ell(y') + 1 = M$ and $\ell(y') \leq N$.

(3.1) $RHR(< x, y) + RHR(x, < y) \Rightarrow RHL(x, y)$.

Let us fix $s \in S$ with $sy < y$ and set $\dot{y} := sy$. Again weak Lefschetz style arguments yield (Proposition 4.9):

(3.2) $RHR(< x, s, \dot{y}) + RHR(x, s, < \dot{y}) \Rightarrow RHL(x, s, \dot{y})$.

We now distinguish two cases. If $xs > x$ then an easy limit argument (Proposition 4.11) gives:

(3.3) $RHR(\leq xs, \dot{y}) + RHL(x, s, \dot{y}) \Rightarrow RHR(x, s, \dot{y})$.

If $xs < x$ then a more complicated limit argument (Proposition 4.13) allows us to reach essentially the same conclusion:

(3.4) $RHR(x, \dot{y}) + RHL(x, s, \dot{y}) \Rightarrow RHR(x, s, \dot{y})$.

Another limit argument (Proposition 4.12) yields:

(3.5) $RHR(x, s, \dot{y}) + RHL(x, \leq y) \Rightarrow RHR(x, y)$.

Thus assuming $X_{M,N}$ and $Y_{M,N-1}$ we have concluded that $X_{M,N+1}$ holds.

Finally, if $x, y \in W$ and $t \in S$ is such that $\ell(x) + \ell(y) + 1 = M$ and $\ell(y) = N$ then as in (3.2) we deduce:

(3.6) $RHR(< x, t, y) + RHR(x, t, < y) \Rightarrow RHL(x, t, y)$. 

(Relative Hard Lefschetz for Soergel Bimodules 15)
If \( xt < x \) then we have

\[
RHR(x, y) + RHL(x, t, y) \Rightarrow RHR(x, t, y).
\]

If \( xt > x \) then we have

\[
RHR(\leq xt, y) + RHL(x, t, y) \Rightarrow HR(x, t, y).
\]

Thus assuming \( X_{M,N} \) and \( Y_{M,N-1} \) we have deduced that \( Y_{M,N} \) holds. Putting these two steps together we deduce:

\[
X_{M,N} + Y_{M,N-1} \Rightarrow X_{M,N+1} + Y_{M,N}.
\]

We conclude by induction that \( X_{M,M}, Y_{M,M} \) hold for all \( M \). This reduces the proof of the theorem to the propositions listed above.

4. The proof

4.1. Hodge-Riemann implies hard Lefschetz. In [EW14] it was observed that homological algebra in the homotopy category of Soergel bimodules can be used to imitate the weak Lefschetz theorem. This is the key step to deduce the hard Lefschetz theorem by induction. In this section we show that the same idea is useful for studying relative hard Lefschetz.

Recall that \( B \) denotes the category of Soergel bimodules. Let

\[
K := K^b(B)
\]

denote its bounded homotopy category. As in [EW14, §6.1] we denote the cohomological degree of an object by an upper left index, so as not to get confused with the grading. Thus, an object in \( K \) is a complex

\[
\ldots \rightarrow iF \rightarrow i+1F \rightarrow \ldots
\]

with each \( iF \in B \). We denote by \((K^{\leq 0}, K^{\geq 0})\) the perverse \( t \)-structure on \( K \) (see [EW14, §6.3]).

**Lemma 4.1.** Let \( F = (0 \rightarrow 0F \xrightarrow{d_0} 1F \rightarrow \ldots) \) be a complex supported in non-negative homological degrees, and suppose that \( F \in K^{\geq 0} \). Then the induced map

\[
d_0 : H^i(0F) \rightarrow H^i(1F)
\]

is split injective for all \( i < 0 \).

**Proof.** Because \( F \in K^{\geq 0} \) then by definition we can find an isomorphism of complexes

\[
F \cong F_p \oplus F_c
\]

with \( F_c \) contractible and \( F_p \) such that \( H^i(jF_p) = 0 \) if \( i < -j \). Only the summand \( F_c \) contributes to \( H^i(0F) \) for \( i < 0 \), but the first differential in a contractible complex is a split injection. \( \square \)
Given any \( x \in W \) we denote by
\[
F_x = (\ldots \rightarrow -1 F_x = 0 \rightarrow 0 F_x = B_x \rightarrow 1 F_x \rightarrow \ldots)
\]
a fixed choice of minimal complex for the Rouquier complex (unique up to isomorphism), see [EW14, §6.4]. The following lemma shows that tensor product with \( F_x \) is left \( t \)-exact.

**Lemma 4.2.** For any \( x \in W \), \((K \geq 0)F_x \subset K \geq 0\) and \( F_x(K \geq 0) \subset K \geq 0\).

**Proof.** Because \( F_x \) is a tensor product of various \( F_s \), \( s \in S \), it is enough to prove the lemma for \( x = s \). That \((-) \otimes F_s \) preserves \( K \geq 0\) is proven in [EW14, Lemma 6.6]; the proof deduces the general statement from [EW14, Lemma 6.5], which states that \( B_x F_s \in K \geq 0\) for all \( x \in W \) and \( s \in S \). The same proof shows that \( F_s B_x \in K \geq 0\), and consequently that \( F_s \otimes (-) \) preserves \( K \geq 0\). \(\Box\)

The following proposition is fundamental for what follows. (In rough form it appears first in [EW14] as Theorem 6.9, Lemma 6.15 and Theorem 6.21.)

**Proposition 4.3.** For any \( x \) there exists a map
\[
d_x : B_x \rightarrow F(1)
\]
between positively polarized Soergel bimodules such that

1. all summands of \( F \) are isomorphic to \( B_z \) with \( z < x \);
2. \( d_x \) is isomorphic to the first differential on a Rouquier complex;
3. if \( d_x^* : F \rightarrow B_x(1) \) denotes the adjoint of \( d \), then
\[
d_x^* \circ d_x = B_x \rho - (x \rho) B_x.
\]

**Proof.** Except for part (2) this proposition is [Wil14, Proposition 7.14]. However the reader may easily check that the inductive proof of [Wil14, Proposition 7.14] goes through if one adds the inductive assumption “\( d_x \) is isomorphic to the first differential on a Rouquier complex”. (Indeed, the proof mimics tensoring with a complex isomorphic to the Rouquier complex \( F_s \) to carry out the induction.) \(\Box\)

Exchanging left and right actions gives:

**Proposition 4.4.** For any \( y \) there exists a map
\[
d_y : B_y \rightarrow G(1)
\]
between positively polarized Soergel bimodules such that:

1. all summands of \( G \) are isomorphic to \( B_z \) with \( z < y \);
2. \( d_y \) is isomorphic to the first differential on a Rouquier complex;
3. if \( d_y^* : G \rightarrow B_y(1) \) denotes the adjoint of \( d \), then
\[
d_y^* \circ d_y = \rho B_y - B_y(y^{-1} \rho).
\]

Putting these three statements together gives:
Proposition 4.5. Consider the map
\[
f := \left( \begin{array}{c} d_x B_y \\ B_x d_y \end{array} \right) : B_x B_y \to E(1) := FB_y(1) \oplus B_x G(1).
\]
Here, \(d_x\) and \(F\) are as in Proposition 4.3, and \(d_y\) and \(G\) are as in Proposition 4.4. Then
(1) the induced map
\[
f : H^i(B_x B_y) \to H^{i+1}(E)
\]
is split injective for \(i < 0\);
(2) if \(f^* : E \to B_x B_y(1)\) denotes the adjoint of \(f\) then
\[
f^* \circ f = B_x(2\rho)B_y - x(\rho)B_x B_y - B_x B_y(y^{-1}\rho).
\]
Proof. The first claim follows by noticing that \(f\) is isomorphic to the first differential on a Rouquier complex representing
\[
F_x F_y \cong (B_x \to E(1) \to \ldots)(B_y \to F(1) \to \ldots).
\]
Because \(F_x F_y \in K^\geq 0\) the first claim in the lemma follows from Lemma 4.1.

The adjoint of \(f\) is given by the matrix
\[
\left( \begin{array}{c} d_x^* B_y & B_x d_y^* \end{array} \right)
\]
and hence
\[
f^* \circ f = (d_x^* \circ d_x)B_y + B_x(d_y^* \circ d_y) = B_x(2\rho)B_y - x(\rho)B_x B_y - B_x B_y(y^{-1}\rho)
\]
which is the second claim in the lemma.

Similarly we have:
Proposition 4.6. Fix \(a, b > 0\) and consider the map
\[
g_{a,b} := \left( \begin{array}{c} \sqrt{a} \cdot d_x B_s B_y \\ \sqrt{b} \cdot B_x B_s d_y \end{array} \right) : B_x B_s B_y \to E(1) := FB_s B_y(1) \oplus B_x B_s G(1).
\]
Then
(1) the induced map
\[
g_{a,b} : H^i(B_x B_s B_y) \to H^{i+1}(E)
\]
is split injective for \(i < 0\);
(2) if \(g_{a,b}^* : E \to B_x B_y(1)\) denotes the adjoint of \(g_{a,b}\) then
\[
g_{a,b}^* \circ g_{a,b} = aB_x(\rho)B_s B_y + bB_x B_s(\rho)B_y - a(x\rho)B_x B_s B_y - bB_x B_y(y^{-1}\rho).
\]
Proof. The argument for (2) is the same as for the previous proposition.

It remains to show part (1). Note that \(g_{a,b}\) is the first differential on a complex representing
\[
F_x B_s F_y \cong (B_x \to E(1) \to \ldots) B_s(B_y \to F(1) \to \ldots).
\]
and so \(F_x B_s F_y \in K^\geq 0\) by Lemma 4.2. Now (1) follows from Lemma 4.1.

The following two propositions explain the title of this section.
Proposition 4.7. Fix \( x, y \in W \) and suppose \( RHR(x', y) \) and \( RHR(x, y') \) hold for all \( x' < x, y' < y \). Then \( RHL(x, y) \) holds.

Remark 4.8. This proposition is an instance of the philosophy that \( HR \) in dimension \( \leq n - 1 \) implies \( HL \) in dimension \( n \).

Proof. Let us keep the notation in the statement of Proposition 4.5. We assume that \( B_x B_y \) is standardly polarized and \( E \) is polarized with the induced form. Fix \( z \in W \) and consider the graded vector spaces

\[
V := H^\bullet_z(B_x B_y) \quad \text{and} \quad U := H^\bullet_z(E).
\]

These have operators \( L : V^i \to V^{i+2} \) and \( L : U^i \to U^{i+2} \) obtained by applying \( H^\bullet_z(-) \) to the maps

\[
B_x B_y \to B_x B_y(2) : bb' \mapsto b(\rho)b',
\]

\[
E \to E(2) : (bb', bb') \mapsto (b\rho b', b\rho b').
\]

Also, the maps \( f, f^\ast \) of Proposition 4.5 induce maps (again by taking perverse cohomology)

\[
U^i \xrightarrow{f} V^i \xrightarrow{f^\ast} U^{i+2}.
\]

These maps are morphisms of \( \mathbb{R}[L] \)-modules. We have:

1. \( f \) is injective in degrees \( < 0 \), by Proposition 4.5(1).
2. \( \langle f(v), f(v') \rangle = \langle v, f^\ast(f(v')) \rangle = \langle v, 2Lv' \rangle \) for all \( v, v' \in V^\bullet \). The first equality holds because \( f^\ast \) is the adjoint of \( f \). The second equality holds by Proposition 4.5(2), and by Lemma 2.4.
3. \( U \) satisfies the Hodge-Riemann bilinear relations. This is because \( E \) is a positively polarized direct sum (of tensor products), and we have assumed \( RHR(x', y) \) and \( RHR(x, y') \), one of which applies to each direct summand of \( E \).

Now we may deduce from [EW14, Lemma 2.3] that \( L^i_V : V^{-i} \to V^i \) is injective and hence is an isomorphism by a comparison of dimension. The property \( HL(x, y) \) follows.

Proposition 4.9. Fix \( x, y \in W \) and \( s \in S \) and suppose \( HR(x', s, y) \) and \( HR(x, s, y') \) hold for all \( x' < x, y' < y \). Then \( HL(x, y) \) holds.

Proof. The proof is the same as that of the previous proposition, replacing Proposition 4.5 with Proposition 4.6.

4.2. Signs via limit arguments. In this section we will repeatedly appeal to the principle of conservation of signs, which states that a continuous family of non-degenerate symmetric forms on a real vector space has constant signature. The following lemma, which was one of the key techniques used by de Cataldo and Migliorini in their proof of the Hodge-Riemann bilinear relations in geometry [dCM02], is an immediate consequence.
**Lemma 4.10.** Consider a polarized graded vector space and a continuous family of operators $L_t$ parametrized by a connected set. Assume all the operators in the family satisfy hard Lefschetz. If any member of the family satisfies the Hodge-Riemann bilinear relations, then they all do.

To spell out this general argument in slightly more detail: one is given a finite-dimensional polarized graded vector space $V^\bullet$. A degree 2 Lefschetz operator induces a symmetric form on each $V^{-i}$, $i \in \mathbb{Z}_{\geq 0}$, which collectively are non-degenerate if and only if $L$ satisfies hard Lefschetz. If $L$ does satisfy hard Lefschetz, then $L$ satisfies the Hodge-Riemann bilinear relations if and only if the signature of the Lefschetz form on each $V^{-i}$ agrees with a certain formula, which depends only on the graded dimension of $V$. From this, one deduces the lemma above. The applications will become clear immediately.

**Proposition 4.11.** Suppose $x, y \in W$, $s \in S$ and $xs > x$. Assume $RHL(x,s,y)$ and $RHR(\leq xs, y)$. Then $RHR(x,s,y)$ holds.

**Proof.** For $a, b \in \mathbb{R}$, consider the Lefschetz operator
\[ L_{a,b} := B_x(a\rho)B_sB_y + B_xB_s(b\rho)B_y : B_xB_sB_y \to B_xB_sB_y(2). \]
Recall that $HR(x,s,y)$ means that $L_{a,b}$ induces an operator on $H^\bullet_x(B_xB_sB_y)$ which satisfies hard Lefschetz and Hodge-Riemann, for any $a > 0$, $b > 0$.

However $B_xB_s$ is perverse, and by $RHR(x,s)$ (see Lemma 2.8 above) the restriction of the intersection form on $B_xB_s$ to each summand $B_xB_s$ is a multiple of the intersection form on $B_x$ with sign $(-1)^{\ell(x)+1-\ell(z)}/2$. By $RHR(\leq xs, y)$, $L_{0,b}$ satisfies relative Hodge-Riemann on $B_xB_sB_y$ for any $b > 0$ (it is an exercise to confirm that the signs are correct). Thus $L_{a,b}$ satisfies relative hard Lefschetz for all $a \geq 0$ and $b > 0$ and satisfies relative Hodge-Riemann for $a = 0$, $b > 0$. We can now appeal to the principle of conservation of signs to conclude that relative Hodge-Riemann is satisfied for all $a \geq 0, b > 0$. Thus $RHR(x,s,y)$ holds. \[ \square \]

The previous proof uses the case special case $a = 0, b > 0$ to deduce the general case $a > 0, b > 0$. Here we go the other way:

**Proposition 4.12.** Suppose $x, y \in W$, $s \in S$ and that $sy > y$. Assume $RHR(x,s,y)$ and $RHL(x,\leq sy)$. Then $RHR(x,sy)$ holds.

**Proof.** Let $L_{a,b}$ denote the Lefschetz operator considered in the previous proof. By our assumptions, $L_{a,b}$ satisfies Hodge-Riemann for $a > 0, b > 0$ and hard Lefschetz for $a > 0, b = 0$. By the principle of conservation of signs, Hodge-Riemann is also satisfied for $a > 0, b = 0$. Now $B_xB_{sy}$ is a summand of $B_xB_sB_y$ and the intersection form on $B_xB_sB_y$ restricts to a positive multiple of the intersection form on $B_xB_{sy}$. We conclude\(^6\) that $L_{a,0}$ satisfies Hodge-Riemann on $B_xB_{sy}$, which is what we wanted. \[ \square \]

---

\(^6\)We are using the fact that relative Hodge-Riemann is preserved under taking polarized direct summands. See [Wil14, Lemma 4.5] for a related situation.
Proposition 4.13. Let $x, y \in W$ and $s \in S$ be such that $xs < x$. Assume $HL(x, s, y)$, $HR(x, y)$. Then $HR(x, s, y)$ holds.

The proof of Proposition 4.13 is more complicated than that of Proposition 4.11, and will occupy the rest of this section. Here is a sketch of our approach. We fix a decomposition $B_x B_s = B_x(1) \oplus B_x(-1)$ and explicitly calculate the Lefschetz operator and forms in the decomposition $B_x B_s B_y = B_x B_y(1) \oplus B_x B_y(-1)$ in terms of the corresponding operators on $B_x B_y$. Appealing to $RHR(x, y)$ we will see that the signs are correct for $b \gg a > 0$. By the principle of conservation of signs (which is applicable by our $RHL(x, s, y)$ assumption) we deduce that $RHR(x, s, y)$ holds, which is what we wanted to show.

For simplicity we assume $\rho(\alpha^s) = 1$ for all $s \in S$.

Lemma 4.14. The map $r \mapsto (\partial_s(-rs(\rho)), \rho \partial_s(r))$ gives an isomorphism

$$R = R^s \oplus \rho R^s$$

of $R^s$-bimodules.

Proof. $R$ is free as an $R^s$-module with basis $\{1, \gamma\}$ where $\gamma \in R^2$ is any degree two element which is not $s$-invariant. In particular we can take $\gamma = \rho$. Under the map as in the statement of the lemma we have

$$1 \mapsto (\partial_s(-s\rho), \rho \partial_s(1)) = (1, 0)$$
$$\rho \mapsto (\partial_s(-s\rho), \rho \partial_s(1)) = (0, \rho).$$

and so our map sends a basis to a basis, and the lemma follows. \qed

By [Wil11, Proposition 7.4.3] there exists a $(R, R^s)$-bimodule $B^s_x$ (a “singular Soergel bimodule”) and a canonical isomorphism

$$B^s_x \otimes_{R^s} R = B_x.$$ (4.2)

Our choice of isomorphism (4.1) yields a decomposition

$$B_x B_s = B^s_x \otimes_{R^s} R \otimes_{R^s} R(1) = B_x(1) \oplus B_x(-1).$$ (4.3)

Now consider the endomorphism $B_x \rho B_s : B_x B_s \to B_x B_s(2)$.

Lemma 4.15. With respect to the decomposition (4.3) the degree 2 endomorphism $B_x \rho B_s$ is given by the matrix:

$$\begin{pmatrix} 0 & B_x(-\rho(s)) \\ B_x & B_x(\rho + s) \end{pmatrix} : B_x(1) \oplus B_x(-1) \to B_x(3) \oplus B_x(1).$$ (4.4)

Proof. We identify $B_x$ with $B^s_x \otimes_{R^s} R$, and write an element of it as $b \otimes f$ for $b \in B^s_x$ and $f \in R$. Similarly, we identify $B_x B_s$ with $B^s_x \otimes_{R^s} R \otimes_{R^s} R(1)$. 

Consider an element of the form $b \otimes 1 \in B_x$. We calculate the action of $B_x \rho B_s$ on the summand $B_x(1)$:

\[
B_x(1) \xrightarrow{(4.2)} B_x B_s \xrightarrow{B_x \rho B_s} B_x B_s \xrightarrow{(4.2)} B_x(1) \oplus B_x(-1)
\]

\[
b \otimes 1 \longmapsto b \otimes 1 \otimes 1 \longmapsto b \otimes \rho \otimes 1 \longmapsto (0, b \otimes 1)
\]

Similarly we calculate the action on the summand $B_x(-1)$:

\[
B_x(-1) \xrightarrow{(4.3)} B_x B_s \xrightarrow{B_x \rho B_s} B_x B_s \xrightarrow{(4.3)} B_x(1) \oplus B_x(-1)
\]

\[
b \otimes 1 \longmapsto b \otimes \rho \otimes 1 \longmapsto b \otimes \rho^2 \otimes 1 \longmapsto (b \otimes (-\rho s(\rho)), b \otimes (\rho + s\rho))
\]

The lemma follows. $\square$

**Lemma 4.16.** The singular Soergel bimodule $B_x^s$ admits a unique invariant form

\[
\langle -, - \rangle_{B_x^s} : B_x^s \times B_x^s \to R^s
\]

such that $\langle -, - \rangle \otimes_{R^s} R$ agrees with the intersection form under the identification (4.2).

Here and in the following proof, an invariant form on an $(R, R^s)$-bimodule means a graded bilinear form $\langle -, - \rangle : B_x^s \times B_x^s \to R^s$ which satisfies $\langle rb, b' \rangle = \langle b, rb' \rangle$ and $\langle br', b' \rangle = \langle b, br' \rangle = \langle b, b' r' \rangle$ for all $b, b' \in B_x^s, r \in R, r' \in R^s$.

**Proof.** Let $B_{x-1}^s$ denote the $(R^s, R)$-bimodule obtained from $B_x^s$ by interchanging left and right actions. Then $B_{x-1}^s$ agrees with the indecomposable singular Soergel bimodule parametrized by the coset of $x^{-1}$ in $\langle s \rangle \backslash W$, as described in [Wil11, Theorem 7.4.2]. Soergel’s conjecture and [Wil11, Theorem 7.4.1] implies that $\text{Hom}(B_{x-1}^s, \mathcal{D}(B_{x-1}^s))$ is one dimensional. (We denote by $\mathcal{D}$ the duality functor on singular Soergel bimodules defined in [Wil11, §6.3].) We can regard elements in this Hom space as maps $B_x^s \to \text{Hom}_{-R^l}(B_x^s, R^l)$ and hence as invariant forms

\[
\langle -, - \rangle : B_x^s \times B_x^s \to R^l.
\]

We conclude that $B_x^s$ admits an invariant form which is unique up to scalar. Given any such form $\langle -, -, - \rangle \otimes_{R^s} R$ is a non-degenerate form on $B_x$, and hence agrees with the intersection form on $B_x$ up to scalar. The lemma follows. $\square$

Our fixed decomposition (4.3) gives the basic identification:

\[
B_x B_s B_y = B_x B_y(1) \oplus B_x B_y(-1)
\]

(4.5)

The following is immediate from the definitions:

**Lemma 4.17.** Under (4.5) the invariant form is given by:

\[
\langle (b_1, b_2), (b'_1, b'_2) \rangle = \langle b_1, b'_2 \rangle + \langle b_2, b'_1 \rangle + \langle \rho b_2, b'_2 \rangle.
\]
We now put the above calculations together. Until the end of the section let us in addition fix \( z \in W \) and set

\[
V^\bullet := H^\bullet_x(B_x B_y).
\]

Then \( V^\bullet \) is equipped with a symmetric form \( \langle -,- \rangle_{V^\bullet} \) and a Lefschetz operator \( L : V^\bullet \to V^{\bullet+2} \). This data satisfies Hodge-Riemann, by our assumption \( HR(x,y) \). Our identification (4.5) fixes an isomorphism

\[
(4.6) \quad H^\bullet_x(B_x B_s B_y) = V^\bullet(1) \oplus V^\bullet(-1).
\]

**Proposition 4.18.** Under the identification (4.6):

1. The invariant form is given by

\[
(4.7) \quad \langle (v_1,v_2), (v'_1,v'_2) \rangle = \langle v_1,v'_2 \rangle + \langle v_2,v'_1 \rangle + \langle v_2,Lv_2 \rangle.
\]
   for \( v_1,v'_1 \in V^\bullet(1) \) and \( v_2,v'_2 \in V^\bullet(-1) \).

2. The operator induced by \( L_{a,b} := B_x(a \rho) B_s B_y + B_x B_s(b \rho) B_y \) is given by

\[
(4.8) \quad a \begin{pmatrix} 0 & X \\ \text{id} & Y \end{pmatrix} + b \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}
\]
   for certain (unspecified) maps \( X : V(-1) \to V(1) \) and \( Y : V(-1) \to V(-1) \).

**Proof.** (1) (resp. (2)) is an immediate consequence of Lemma 4.17 (resp. Lemma 4.15).

**Proposition 4.19.** Assume \( HR(x,y) \). Then for \( b \gg a > 0 \) the operator \( L_{a,b} \) satisfies \( HR \) on \( V^\bullet(1) \oplus V^\bullet(-1) \).

**Proof.** We roll up our sleeves and calculate everything in a basis.

Fix a degree \(-d \leq 0 \). By [EW14, Lemma 5.2] it is enough to show that for \( b \gg a > 0 \) the signature of the Lefschetz form on the degree \(-d \) piece of \( V^\bullet(1) \oplus V^\bullet(-1) \) is equal to the signature of the Lefschetz form on the primitive subspace

\[
P^{-d+1} := \ker L^d : V^{-d+1} \to V^{d+1}.
\]

To this end let us fix bases:

\[
x_1, \ldots, x_m \quad \text{for } V^{-d-1};
\]
\[
p_1, \ldots, p_n \quad \text{for } P^{-d+1}.
\]

Because \( L \) satisfies hard Lefschetz on \( V \) we deduce that

\[
Lx_1, \ldots, Lx_m, p_1, \ldots, p_n \quad \text{is a basis for } V^{-d+1}.
\]

Thus a basis for \( (V^\bullet(1) \oplus V^\bullet(-1))^d = V^{d+1} \oplus V^{d-1} \) is given by

\[
(0,x_1), \ldots, (0,x_m), (Lx_1,0), \ldots, (Lx_m,0), (p_1,0), \ldots, (p_n,0).
\]

Let us write

\[
L_{a,b} = aA + bB
\]
where
\[
A = \begin{pmatrix} 0 & X \\ \text{id} & Y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}
\]
are the matrices appearing in Proposition 4.8. We calculate the leading terms of the Lefschetz form \((v, w) \mapsto \langle v, L_{a,b}^d w \rangle\) in the above basis with respect to the parameter \(b\). We have:
\[
\begin{align*}
\langle (0, x_i), L_{a,b}^d(0, x_j) \rangle &= b^d \langle Lx_i, L^d x_j \rangle + O(b^{d-1}) = b^d \langle x_i, L^{d+1} x_j \rangle + O(b^{d-1}) \\
\langle (Lx_i, 0), L_{a,b}^d(0, x_j) \rangle &= b^d \langle Lx_i, L^d x_j \rangle + O(b^{d-1}) = b^d \langle x_i, L^{d+1} x_j \rangle + O(b^{d-1}) \\
\langle (Lx_i, 0), L_{a,b}^d(Lx_i, 0) \rangle &= b^d \langle (Lx_i, 0), (L^{d+1} x_i, 0) \rangle + O(b^{d-1}) = O(b^{d-1})
\end{align*}
\]
where \(O(b^k)\) denotes a polynomial in \(b\) and \(a\) in which all powers of \(b\) are bounded by \(k\). Using that \(L^d p_i = 0\) we have
\[
\begin{align*}
\langle (0, x_i), L_{a,b}^d(p_i, 0) \rangle &= dab^{-1} \langle Lx_i, L^{d-1} p_i \rangle + O(b^{d-2}) = O(b^{d-2}) \\
\langle (Lx_i, 0), L_{a,b}^d(p_i, 0) \rangle &= dab^{-1} \langle Lx_i, L^{d-1} p_i \rangle + O(b^{d-2}) = O(b^{d-2}) \\
\langle (p_i, 0), L_{a,b}^d(p_i, 0) \rangle &= dab^{-1} \langle p_i, L^{d-1} p_i \rangle + O(b^{d-2}).
\end{align*}
\]
Thus if we define matrices
\[
R := ((L^d x_j)_1 \leq i, j \leq n) \quad \text{and} \quad Q := ((p_i, L^d p_j)_1 \leq i, j \leq n)
\]
then we can write the Gram matrix of the Lefschetz form \((v, w) \mapsto \langle v, L_{a,b}^d w \rangle\) as a block matrix with entries:
\[
\begin{pmatrix}
b^d R + O(b^{d-1}) & b^d R + O(b^{d-1}) & O(b^{d-2}) \\
b^d R + O(b^{d-1}) & O(b^{d-2}) & O(b^{d-2}) \\
O(b^{d-2}) & O(b^{d-2}) & dab^{-1} Q + O(b^{d-2})
\end{pmatrix}
\]
For \(b \gg a > 0\) this matrix has the same signature of the matrix
\[
\begin{pmatrix}
R & R & 0 \\
R & 0 & 0 \\
0 & 0 & Q
\end{pmatrix}
\]
Now the submatrix \(\begin{pmatrix} R & R \\ R & 0 \end{pmatrix}\) is easily seen to be non-degenerate with signature 0. Thus for \(b \gg a > 0\) our matrix has the same signature of \(Q\). We have already remarked that by [EW14, Lemma 5.2] this is what we wanted to know. \(\square\)

Thus Proposition 4.13 holds (see the remarks immediately after the statement of the proposition).

5. Ridigity

Let \(c \subset W\) be a two-sided cell, \(a\) its \(a\)-value\(^7\) and \(J = \bigoplus_{x \in c} \mathbb{Z}j_x\) the \(J\)-ring associated to \(c\) (\(J\) is denoted \(J^c\) in [Lus14, § 18.3]). Following Lusztig

\(^7\)It is a non-trivial fact (a consequence of Soergel’s conjecture) that the \(a\)-function is well-defined for any Coxeter group, see [Lus15, §10.1].
[Lus15, § 10], we define a semi-simple monoidal category $J$ ($J$ is denoted $C^c$ in [Lus14, § 18.5]).

We first consider the subcategory $B_{<c} \subset B$, consisting of all direct sums of shifts of $B_z$ with $z <_{LR} c$ ($<_{LR}$ denotes the two-sided preorder). Let $I_c$ denote the ideal in $B$ consisting of all morphisms which factor through objects in $B_{<c}$. Because $B_{<c}$ is closed under tensor products with arbitrary objects of $B$, $I_c$ is a tensor ideal in $B$, and we can form the quotient of additive categories $B'_c := B/I_c$. Then $B'_c$ is a graded additive monoidal category and we set $B_c$ to be the full graded additive subcategory generated by $B_x$ with $x \in c$. We denote the image of $B_x$ in $B'_c$ by $B^c_x$. The objects $B^c_x(m)$ with $x \in W$ and $x \not\in c$ (resp. $x \in c$) give representatives for the isomorphism classes of the indecomposable objects in $B'_c$ (resp. $B_c$). Moreover $B_c$ is a graded additive monoidal category (without unit unless $c = \{\text{id}\})$.

The (obvious analogues of the) crucial vanishing statements (2.1) and (2.2) still hold in $B'_c$ and $B_c$, and hence the perverse filtration and perverse cohomology functors descend to $B'_c$ and $B_c$. We denote them by the same symbols. It is immediate from the definition of the $a$-function that, for all $x, y \in c$,

$$(5.1) \quad H^i(B^c_x B^c_y) = 0 \quad \text{if } |i| > a.$$  

We now come to the definition of $J$. It is a full subcategory of $B_c$, although with a different monoidal structure. The objects of $J$ are given by direct sums (without shifts) of $B^c_x$ with $x \in c$, and thus by (2.1) the category is semi-simple. The monoidal product is given by

$$B \ast B' := H^{-a}(BB') \in B_c$$

(the lowest potentially non-zero degree, by (5.1)). Lusztig proves that $J$ is a semi-simple monoidal category (this result relies in an essential way on [EW14]), and that the map $j_x : [B^c_x] \rightarrow [J]$ induces an isomorphism $J \xrightarrow{\sim} [J]$, where $[J]$ denotes the Grothendieck group of $J$.

**Remark 5.1.** The reader is warned that in general $J$ is a “monoidal category without unit”, i.e. it has an associator but no unit. In general, Lusztig conjectures [Lus14, §13.4] that the $a$-function is bounded (i.e. $a(z) \leq N$ for all $z \in W$ and some fixed constant $N$, which he describes explicitly). This boundedness is known to hold for finite and affine Weyl groups. Under the assumption of this conjecture, it turns out that $J$ has a unit if and only if $c$ contains finitely many left cells (as is always the case in finite and affine type). In this case Lusztig proves [Lus14, §18.5] that the object $\bigoplus_{x \in D \cap c} B^c_x$ is a unit for $J$ (here $D \subset W$ denotes the set of distinguished involutions). Even when $c$ contains infinitely many left cells $J$ is “locally unital” (and still under the boundedness assumption). For any given object $B \in J$, only finitely many $B^c_x$ with $x \in D \cap c$ satisfy $B^c_x \ast B \neq 0$. The formal direct sum $\bigoplus_{x \in D \cap c} B^c_x$, while not an object in $J$ when $D \cap c$ is infinite, acts on any object, and it will act as a monoidal identity would.
Our aim in this chapter is to show that the relative hard Lefschetz theorem for Soergel bimodules implies

**Theorem 5.2.** \( \mathcal{J} \) is a rigid, pivotal monoidal category.

**Remark 5.3.** For finite and affine Weyl groups the rigidity of \( \mathcal{J} \) has been proved by Bezrukavnikov, Finkelberg and Ostrik [BFO09, §4.3] (using the geometric Satake equivalence). Lusztig has also proven rigidity for Weyl groups (see [Lus15, §9.3] and [Lus14, §18.19]). His techniques probably extend to crystallographic Coxeter groups. Lusztig also conjectured the rigidity to hold for any finite Coxeter group [Lus15, §10], in which case he expects the Drinfeld center \( Z(\mathcal{J}) \) to be related to the “unipotent characters” of \( W \). Ostrik has informed us that for the interesting case of the two-sided cell in \( H_4 \) with \( a \)-value 6, he has been able to verify the rigidity of \( \mathcal{J} \) by other means.

**Remark 5.4.** As we will see, the pivotal structure on \( \mathcal{J} \) will depend on our fixed choice of regular dominant element \( \rho \in h^* \). We do not know if the structure varies in an interesting way with \( \rho \). It is possible that the Hodge-Riemann relations might allow one to show that \( \mathcal{J} \) is unitary, and hope to address this question in future work.

Because \( \mathcal{J} \) does not have a unit in general the standard definition of rigidity does not make sense. We will prove the following (which is equivalent to the usual notion of rigidity if \( \mathcal{J} \) has a unit, see Remark 5.6 below):

**Proposition 5.5.**

1. For \( B, X, Y \in \mathcal{J} \) we have canonical isomorphisms

\[
\begin{align*}
\text{Hom}_\mathcal{J}(X, B \ast Y) & \xrightarrow{\phi_{X,Y}} \text{Hom}_\mathcal{J}(B^\vee \ast X, Y) \\
\text{Hom}_\mathcal{J}(X, Y \ast B) & \xrightarrow{\chi_{X,Y}} \text{Hom}_\mathcal{J}(X \ast B^\vee, Y)
\end{align*}
\]

functorial in \( X \) and \( Y \).

2. For \( B, X, Y, Z \in \mathcal{J} \) the following diagrams commute:

\[
\begin{align*}
\text{Hom}_\mathcal{J}(X, B \ast Y) & \xrightarrow{(-) \ast Z} \text{Hom}_\mathcal{J}(X \ast Z, B \ast Y \ast Z) \\
\text{Hom}_\mathcal{J}(B^\vee \ast X, Y) & \xrightarrow{(-) \ast Z} \text{Hom}_\mathcal{J}(B^\vee \ast X \ast Z, Y \ast Z) \\
\text{Hom}_\mathcal{J}(X, Y \ast B) & \xrightarrow{Z \ast (-)} \text{Hom}_\mathcal{J}(Z \ast X, Z \ast Y \ast B) \\
\text{Hom}_\mathcal{J}(X \ast B^\vee, Y) & \xrightarrow{Z \ast (-)} \text{Hom}_\mathcal{J}(Z \ast X \ast B^\vee, Z \ast Y)
\end{align*}
\]

We make some remarks before turning to the proof. It is easy to see that \( B_\delta \in \mathcal{B} \) is self-dual (this is immediate in the language of [EW13], where the cup and cap maps provide the unit and counit). It follows that any Bott-Samelson module is rigid. Hence \( \mathcal{B} \) is rigid (taking the Karoubi envelope
preserves rigidity). Let us denote by $B \mapsto \overline{B}$ the duality on $\mathcal{B}$. It is easy to see that $B$ is even pivotal (i.e. we have a canonical isomorphism $B \sim (\overline{B})^\vee$).

As quotients of a rigid, pivotal monoidal category, the monoidal categories $\mathcal{B}_c$ and $\mathcal{B}_c'$ are rigid and pivotal. We abuse notation and also denote the duality on $\mathcal{B}_c$ by $B \mapsto \overline{B}$.

**Proof.** We first establish (1). We will construct the isomorphism $\phi_{X,Y}$, the proof for $\chi_{X,Y}$ is similar. Let $X, Y, B \in \mathcal{J}$. We have canonical identifications (by definition and the analogue for $\mathcal{B}_c$ of (2.4))

$$\text{Hom}_\mathcal{J}(X, B \ast Y) = \text{Hom}_{\mathcal{B}_c}(X, H^{-a}(BY)) = \text{Hom}_{\mathcal{B}_c}(X, BY(-a)) = \text{Hom}_{\mathcal{B}_c}(B^\vee X, Y(-a)) = \text{Hom}_{\mathcal{B}_c}(H^a(B^\vee X), Y).$$

Precomposing with the isomorphism $H^{-a}(B^\vee X) \sim H^a(B^\vee X)$ given by relative hard Lefschetz gives an isomorphism

$$\text{Hom}_{\mathcal{B}_c}(H^a(B^\vee X), Y) \sim \text{Hom}_{\mathcal{B}_c}(H^{-a}(B^\vee X), Y) = \text{Hom}_\mathcal{J}(B^\vee \ast X, Y).$$

The composition of these isomorphisms defines our isomorphism $\phi_{X,Y}$. It is immediate to check that this isomorphism is natural in $X$ and $Y$.

We now turn to (2). As before we only establish the commutativity of (5.2), with (5.3) being similar. Choose $f \in \text{Hom}_\mathcal{J}(X, B \ast Y)$ and let $f_{NE}$ (resp. $f_{SW}$) denote the image of $f$ in $\text{Hom}_\mathcal{J}(B^\vee \ast X \ast Z, Y \ast Z)$ obtained by passing through the north-east (resp. south-west) corner of (5.2). We must prove that $f_{SW} = f_{NE}$.

Via $\text{Hom}_\mathcal{J}(X, BY) = \text{Hom}_{\mathcal{B}_c}(X, BY(-a))$ we may regard $f$ as a map

$$f : X \to BY(-a).$$

From $f$ we obtain the following maps:

$$f' : B^\vee X \to Y(-a), \quad \varphi : H^a(f') : H^a(B^\vee X) \to Y,$$

$$g := fZ : XZ \to BYZ(-a),$$

$$g' := f'Z : B^\vee XZ \to YZ(-a), \quad \gamma := H^{2a}(g') : H^{2a}(B^\vee XZ) \to H^a(YZ),$$

$$h : H^{-a}(XZ) \to BH^{-a}(YZ)(-a), \quad h' : B^\vee H^{-a}(XZ) \to H^{-a}(YZ)(-a).$$

(Here $f'$ (resp. $g'$, $h'$) are obtained from $f$ (resp. $g$, $h$) using the dual pair $(B, B^\vee)$ in $\mathcal{B}_c$, and $h$ is uniquely determined by $H^0(h) = H^{-a}(g)$.)

Consider the diagram given in Figure 1. The maps which have not been defined above are given as follows:

1. All maps labelled $\sim$ are relative hard Lefschetz isomorphisms (given by our fixed choice of $\rho \in \mathfrak{h}^*$). At the top and bottom of the middle square we use the canonical identifications:

$$H^{2a}(B^\vee XZ) = H^a(H^a(B^\vee X)Z) = H^a(B^\vee H^a(XZ)),$$

$$H^{-2a}(B^\vee XZ) = H^{-a}(H^{-a}(B^\vee X)Z) = H^{-a}(B^\vee H^{-a}(XZ))$$
(2) We set \( l := H^{-a}(\varphi Z) \) and \( r := H^a(h') \).

It is straightforward but tedious to check that all squares and triangles in Figure 1 commute. If \( q \) denotes the relative hard Lefschetz isomorphism \( q : H^{-a}(YZ) \to H^a(YZ) \) we deduce from the commutativity of the diagram that \( q \circ f_{NE} = q \circ f_{SW} \), and hence that \( f_{NE} = f_{SW} \), which is what we wanted to show. \( \square \)

**Remark 5.6.** Suppose that \( c \) contains finitely many left cells. Then \( \mathcal{J} \) has a unit (see Remark 5.1), which we denote by \( \mathbb{1} \). Applying the isomorphisms of Proposition 5.5 to the identity maps in \( \text{Hom}_\mathcal{J}(B, B) \) and \( \text{Hom}_\mathcal{J}(B^\vee, B^\vee) \), we obtain morphisms \( \varepsilon : \mathbb{1} \to B * B^\vee \) and \( \mu : B^\vee * B \to \mathbb{1} \). Using the naturality of Proposition 5.5(1) and the commutativity of Proposition 5.5(2) one may check that for \( f : X \to B * Y \), \( \phi_{X,Y}(f) \) is given by the composition\(^8\)

\[
B^\vee * X \xrightarrow{B^\vee \ast f} B^\vee * B * Y \xrightarrow{\mu * Y} \mathbb{1} * Y = Y.
\]

Similarly, the inverse of \( \phi_{X,Y} \) sends \( g : B^\vee * X \to Y \) to

\[
X = \mathbb{1} * X \xrightarrow{\varepsilon * X} B * B^\vee \ast X \xrightarrow{B \ast g} B * Y.
\]

From this one easily deduces that \( B^\vee \) (and \( \varepsilon, \mu \)) is left dual to \( B \). Similarly, one deduces that \( B^\vee \) is right dual to \( B \). Hence \( \mathcal{J} \) is rigid in the usual sense.

**References**

[Alv87] D. Alvis. The left cells of the Coxeter group of type \( H_4 \). *J. Algebra*, 107(1):160–168, 1987.

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analyse et topologie sur les espaces singuliers, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

---

\(^8\)Here are more details: by Proposition 5.5(2) one can show that \( \phi_{B^\vee Y,Y}(\text{id}_{B^\vee Y}) = \mu * Y \).

By naturality of Proposition 5.5(1) under precomposition with \( f \), one obtains the desired equality.
[BFO09] R. Bezrukavnikov, M. Finkelberg, and V. Ostrik. On tensor categories attached
to cells in affine Weyl groups. III. Israel J. Math., 170:207–234, 2009.
[BFO12] R. Bezrukavnikov, M. Finkelberg, and V. Ostrik. Character $D$-modules via Drin-
feld center of Harish-Chandra bimodules. Invent. Math., 188(3):589–620, 2012.
[dC06] F. du Cloux. Positivity results for the Hecke algebras of noncrystallographic
finite Coxeter groups. J. Algebra, 303(2):731–741, 2006.
[dCM02] M. A. A. de Cataldo and L. Migliorini. The hard Lefschetz theorem and
the topology of semismall maps. Ann. Sci. École Norm. Sup. (4), 35(5):759–772,
2002.
[dCM05] M. A. A. de Cataldo and L. Migliorini. The Hodge theory of algebraic maps.
Ann. Sci. École Norm. Sup. (4), 38(5):693–750, 2005.
[ENO05] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. Ann. of Math. (2),
162(2):581–642, 2005.
[EW13] B. Elias and G. Williamson. Soergel calculus. to appear in Representation Theory,
2013. arXiv:1309.0865.
[EW14] B. Elias and G. Williamson. The Hodge theory of Soergel bimodules. Ann. of
Math. (2), 180(3):1089–1136, 2014.
[KL80] D. Kazhdan and G. Lusztig. Schubert varieties and Poincaré duality. In Geome-
try of the Laplace operator (Univ. Hawaii, 1979), Proc. Sympos. Pure Math.,
XXXVI, pages 185–203. Amer. Math. Soc., Providence, R.I., 1980.
[Lus14] G. Lusztig. Hecke algebras with unequal parameters, 2014. updated version of
book, at arXiv:math/0208154v2.
[Lus15] G. Lusztig. Truncated convolution of character sheaves. Bull. Inst. Math. Acad.
Sin. (N.S.), 10(1):1–72, 2015.
[Müg03] M. Müger. From subfactors to categories and topology. II. The quantum double
tensor categories and subfactors. J. Pure Appl. Algebra, 180(1-2):159–219, 2003.
[Ost14] V. Ostrik. Multi-fusion categories of harish-chandra bimodules, 2014. Proceed-
ing of the ICM, to appear.
[Soe07] W. Soergel. Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Poly-
nomringen. J. Inst. Math. Jussieu, 6(3):501–525, 2007.
[Spr82] T. A. Springer. Quelques applications de la cohomologie d’intersection. In Bour-
baki Seminar, Vol. 1981/1982, volume 92 of Astérisque, pages 249–273. Soc.
Math. France, Paris, 1982.
[Wil] G. Williamson. The Hodge theory of the Decomposition Theorem (after de
Cataldo and Migliorini). Astérisque, to appear. Séminaire Bourbaki. Exp. No.
1115, viii, 283–307.
[Wil11] G. Williamson. Singular Soergel bimodules. Int. Math. Res. Not. IMRN,
(20):4555–4632, 2011.
[Wil14] G. Williamson. Local Hodge theory of Soergel bimodules. to appear in Acta.
Math., arXiv:1410.2028, 2014.

University of Oregon, Eugene, Oregon, USA
E-mail address: belias@uoregon.edu

Max-Planck-Institut für Mathematik, Bonn, Germany
E-mail address: geordie@mpim-bonn.mpg.de