The effect of radiation on stability conditions in \(f(R)\) gravity models

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We solve the field equations of modified gravity for \(f(R)\) model in metric formalism. Further, we obtain the fixed points of the dynamical system in phase space analysis of \(f(R)\) models, both with and without the effects of radiation. Stability of these points is studied by invoking perturbations about them. We apply the conditions on the eigenvalues of the matrix obtained in the linearized first-order differential equations for stability of points. Following this, these fixed points are used for the dynamics of different phases of the universe. Certain linear and quadratic forms of \(f(R)\) are determined from the geometrical and physical considerations and the dynamics of the scale factor is found for those forms. Further, we determine the Hubble parameter \(H(t)\), Ricci scalar \(R\) and scalar curvature \(\hat{R}\) for radiation-, matter- and acceleration-dominated phases of the universe, whose time-ordering may explain an arrow of time throughout the cosmic evolution.

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I. INTRODUCTION

Present universe is in the phase of accelerated expansion\(\dagger\). There are many observational evidences, which indicate the presence of hitherto unknown dark energy such as Supernovae Ia, Large-scale structure, Cosmic Microwave Background anisotropies, etc. \(\ddagger\). By all reckoning, the explanation of present accelerated expansion of the universe is a major challenge in cosmology. There are many approaches to explain its dynamics. The simplest candidate for dark energy is the cosmological constant \(\ddagger\). However, there are two main problems associated with cosmological constant – (i) the fine tuning problem and (ii) the coincidence problem. Besides cosmological constant, there exist two basic approaches to explain dark energy. The first approach is based on modified matter models. In this approach \(T_{\mu\nu}\) in the Einstein equations includes an exotic matter component like quintessence, k-essence, Phantom etc. \(\ddagger\). The second approach is through modified gravity models wherein the late-time accelerated cosmic expansion is realized without using the explicit dark energy matter component in the universe. In these models, we have a spectrum of \(f(R)\) gravity \(\ddagger\), scalar-tensor theories, Gauss-Bonnet dark energy model etc. \(\ddagger\). Further, in \(f(R)\) models, one modifies the laws of gravity by replacing Lagrangian density i.e. scalar curvature \(R\) of the Hilbert’s action by an arbitrary function of \(R\). At present, there is no specific functional form of \(R\) which may satisfy all the conditions of cosmological viability. To achieve this we use the stability conditions of respective eras and determine the corresponding forms of \(f(R)\). By solving the field equations for different forms of \(f(R)\) for the corresponding eras, the scale factor of expansion is determined. From here we find the scalar curvature \(R\) and compare them in different eras. This can lead to the determination of a time-ordering of various epochs, dominated by radiation, matter and dark energy, respectively, throughout the evolution of the universe.

In section II, the fixed points of the dynamical system have been determined within the framework of \(f(R)\) models in metric formalism. Sections III and IV properties and stability of the fixed points of the dynamical system is are explained without and with radiation. Section V contains the form of \(f(R)\), scale factor \(a(t)\), Hubble parameter \(H(t)\), and scalar curvature \(R\) in radiation dominated phase. In sections VI and VII we derive the form of \(f(R)\), scale factor \(a(t)\) and scalar curvature \(R\) for matter dominated and the present accelerated expansion dominated phases, respectively. Together, the time ordering may be used further to determine an arrow of time through the cosmic evolution. Finally, we conclude our results in section VIII.

II. FIELD EQUATIONS AND PHASE SPACE DYNAMICS

In \(f(R)\) gravity we derive field equations in two approaches (i) metric formalism (ii) Palatini formalism. In metric formalism variation of the action is taken with respect to \(g_{\mu\nu}\), while in Palatini formalism connections \(\Gamma^\beta_{\alpha\delta}\) and metric tensor \(g_{\mu\nu}\) are treated as independent variables. In Palatini formalism, there is no usual connection between \(\Gamma^\beta_{\alpha\delta}\) and \(g_{\mu\nu}\) and action is varied w.r.t. \(g_{\mu\nu}\). We consider the field equations in the background of spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime with a metric

\[
ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]
\]

where \(a(t)\) is time dependent scale factor. For this metric the Ricci scalar \(R\) is given by

\[
R = 6(2H^2 + \dot{H})
\]

where \(H\) is the Hubble parameter and overdot represent derivative w.r.t. time. The action with Lagrangian den-
where $\kappa^2 = 8\pi G$ and $g$ is the determinant of metric tensor $g_{\mu\nu}$. Varying the action \[ (3) \] w.r.t. $g_{\mu\nu}$, the field equations obtained are
\[ F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \nabla^2 F(R) = \kappa^2 T_{\mu\nu}, \]
where $F(R) \equiv \frac{\delta f}{\delta R}$ and $T_{\mu\nu}$ is the matter energy-momentum tensor. From the field equations we obtain the following equations
\[ 3FH^2 = \kappa^2 (\rho_m + \rho_r) + \frac{(FR - f)}{2} - 3H \dot{F} \]
\[ -2FH = \kappa^2 (\rho_m + \frac{4}{3} \rho_r) + \dot{F} - H \ddot{F} \]
where $\rho_m$ and $\rho_r$ are energy densities of matter and radiation respectively.

The effective equation of state is defined by
\[ w_{eff} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = -\frac{1}{3} (2x_3 - 1) \]

There are four variables (dimensionless) defined as
\[ x_1 \equiv -\frac{\dot{F}}{FH} \]
\[ x_2 \equiv -\frac{f}{6FH^2} \]
\[ x_3 \equiv \frac{R}{6H^2} \]
\[ x_4 \equiv \frac{\kappa^2 \rho_r}{3FH^2} \]

Differentiation of these variables w.r.t. $\ln a(t)$ gives
\[ \frac{dx_1}{dN} = -1 - x_3 - 3x_2 + x_1^2 - x_1 x_3 + x_4 \]
\[ \frac{dx_2}{dN} = \frac{x_1 x_3}{m} - x_2 (2x_3 - 4 - x_1) \]
\[ \frac{dx_3}{dN} = -\frac{x_1 x_3}{m} - 2x_3 (x_3 - 2) \]
\[ \frac{dx_4}{dN} = -2x_3 x_4 + x_1 x_4 \]
where $N$ stands for $\ln a(t)$ and
\[ m \equiv \frac{d\log F}{d\log R} = \frac{Rf_{,RR}}{f_{,R}}, \]
\[ r \equiv -\frac{d\log f}{d\log R} = -\frac{Rf_{,R}}{f} = \frac{x_3}{x_2} \]
where $f_{,R} \equiv \frac{df}{dR}$ and $f_{,RR} \equiv \frac{df}{dR^2}$. The fixed points of the system are obtained by equating the equations \[ (12) - (16) \] to zero. Thus, the points are given by
\[ P_1 : (x_1, x_2, x_3, x_4) = (0, -1, 2, 0), \quad \Omega_m = 0, w_{eff} = -1 \]
\[ P_2 : (x_1, x_2, x_3, x_4) = (-1, 0, 0, 0), \quad \Omega_m = 2, w_{eff} = \frac{1}{3} \]
\[ P_3 : (x_1, x_2, x_3, x_4) = (1, 0, 0, 0), \quad \Omega_m = 0, w_{eff} = \frac{1}{3} \]
\[ P_4 : (x_1, x_2, x_3, x_4) = (-4, 5, 0, 0), \quad \Omega_m = 0, w_{eff} = \frac{1}{3} \]
\[ P_5 : (x_1, x_2, x_3, x_4) = \left( \frac{3m}{1 + m}, \frac{1 + 4m}{2(1 + m)^2}, \frac{1 + 4m}{2(1 + m)^2}, 0 \right), \quad \Omega_m = \frac{m(7 + 10m)}{2(1 + m)^2}, \quad w_{eff} = -\frac{m}{1 + m} \]
\[ P_6 : (x_1, x_2, x_3, x_4) = \left( \frac{2(1 - m)}{1 + 2m}, \frac{1 - 4m}{m(1 + 2m)}, \frac{(1 - 4m)(1 + m)}{m(1 + 2m)^2}, 0 \right), \quad \Omega_m = 0, \quad w_{eff} = \frac{2}{3} - \frac{5m - 6m^2}{3m(1 + 2m)} \]
\[ P_7 : (x_1, x_2, x_3, x_4) = (0, 0, 0, 1), \quad \Omega_m = 0, \quad w_{eff} = \frac{1}{3} \]
\[ P_8 : (x_1, x_2, x_3, x_4) = \left( \frac{4m}{1 + m}, \frac{2m}{(1 + m)^2}, \frac{2m}{1 + m}, \frac{1 - 2m - 5m^2}{(1 + m)^2} \right), \quad \Omega_m = 0, \quad w_{eff} = \frac{1 - 3m}{3 + 3m} \]
III. FIXED POINTS WITHOUT RADIATION

We consider the properties and stability of these fixed points in the absence of radiation. For stability about the fixed points \( \{x_1, x_2, x_3\} \) we invoke linear perturbations \( \delta x_i(i = 1, 2, 3) \) around that specific point. Linearization of the equations \([12, 15]\) gives first order differential equations

\[
\frac{d}{dN} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} = M \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix},
\]

where \( M \) is a \( 3 \times 3 \) matrix whose components depend upon \( x_1, x_2 \) and \( x_3 \). Stability of each fixed point depends upon the eigenvalues of the matrix \( M \) obtained by taking linear perturbations around that specific point. In the absence of radiation, we have only six fixed points \( P_1-P_6 \) as below.

(1) Point \( P_1: (0, -1, 2) \) corresponds to de-Sitter point. Here \( w_{eff} = -1 \) and eigenvalues corresponding to this point are

\[
-3, \quad \frac{3}{2} \pm \frac{\sqrt{25 - \frac{16}{m}}}{2}
\]

\( P_1 \) is stable when real parts of all the eigenvalues is negative. Hence condition for stability is \( 0 < m/r = -2 < 1 \) otherwise it is a saddle point. So this point can be taken as an acceleration point.

(2) Point \( P_2: (-1, 0, 0) \) is denoted by \( \phi \)-matter-dominated (\( \phi \) MDE) epoch. The eigenvalues of the \( 3 \times 3 \) matrix of perturbations about \( P_2 \) are given by

\[
-2, \quad \frac{1}{2} \left[ 7 + \frac{1}{m} - \frac{m'}{m^2} r(1 + r) \right] \pm \sqrt{\left( 7 + \frac{1}{m} - \frac{m'}{m^2} r(1 + r) \right)^2 - 4 \left( 12 + \frac{3}{m} - \frac{m'}{m^2} r(3 + 4r) \right)}
\]

where \( m' \) is derivative of \( m \) w.r.t. \( r \). If \( m \) is constant, then eigenvalues are \(-2, 3, 4 + \frac{1}{m} \). In this case \( P_2 \) is a saddle point because eigenvalues are negative and positive.

\( P_2 \) can not be a matter dominated point because \( \Omega_m = 2 \) and \( w_{eff} = \frac{1}{3} \).

(3) Point \( P_3: (1, 0, 0) \) is the kinetic point. The eigenvalues corresponding to this point are

\[
\frac{1}{2} \left[ 9 + \frac{1}{m} - \frac{m'}{m^2} r(1 + r) \right] \pm \sqrt{\left( 9 - \frac{1}{m} + \frac{m'}{m^2} r(1 + r) \right)^2 - 4 \left( 20 - \frac{5}{m} - \frac{m'}{m^2} r(5 + 4r) \right)}
\]

If \( m \) is constant, the eigenvalues are \( 2, 5, 4 - \frac{1}{m} \). In this case \( P_3 \) is unstable for \( m < 0 \) and \( m > \frac{1}{4} \) and a saddle otherwise.

(4) Point \( P_4: (-4, 5, 0) \) has eigenvalues:

\[
-5, -3, 4(1 + \frac{1}{m})
\]

It is stable for \(-1 < m < 0 \) and saddle otherwise. This point cannot be used as a radiation or a matter dominated point.

(5) Point \( P_5: (3m - \frac{1 + 4m}{2(1 + m)^2}, \frac{1 + 4m}{2(1 + m)}) \) can be regarded as a standard matter point in the limit \( m \to 0 \). In this limit \( \Omega_m = 1 \) and \( a \propto t^2 \). Hence necessary condition for this point to be a standard matter point is

\[
m(r = -1) = 0.
\]

Eigenvalues corresponding to point \( P_5 \) are given by

\[
3(1 + m'),
\]

\[
-3m \pm \sqrt{m(256m^3 + 160m^2 - 31m - 16) \over 4m(m + 1)}
\]

For a cosmologically viable trajectory, we want a saddle matter point. Hence, the condition for a saddle matter epoch is given by

\[
m(r \leq -1) > 0, m'(r \leq -1) > -1,
\]

\[
m(r = -1) = 0
\]

(6) Point \( P_6: (2(1-m), -\frac{1-4m}{m(1+2m)}, -\frac{(1-4m)(1+m)}{m(1+2m)}) \) can also be an acceleration dominated point. The eigenvalues corresponding to this point are:

\[
-4 + \frac{1}{m}, \quad \frac{2 - 3m - 8m^2}{m(1 + 2m)}, -\frac{2(m^2 - 1)(1 + m')}{m(1 + 2m)}
\]

Stability of this point depends on both \( m \) and \( m' \). The condition of acceleration \( (w_{eff} < -\frac{1}{3}) \) depends on the value of \( m \).

IV. FIXED POINTS WITH RADIATION

Next, we consider the radiation with other components of universe. In this case we have eight fixed points. Stability about the fixed points \( \{x_1, x_2, x_3, x_4\} \) is determined in the same way as in absence of radiation. Here we have \( 4 \times 4 \) matrix of perturbations about each fixed point and four eigenvalues.
(1) Point $P_1$ corresponds to de-Sitter point. Here $w_{e,ff} = -1$ and eigenvalues corresponding to this point are

$$-4, -3, -\frac{3}{2} \pm \sqrt{\frac{25 - 16}{2}}$$

(35)

In the presence of radiation, we have an eigenvalue $-4$ in addition to those in the absence of radiation. Since this eigenvalue is negative, therefore the condition of stability is the same in both cases. $P_1$ is stable when $0 < m(r = -2) < 1$. This point may be taken as an acceleration point. The condition of stability for this point is same as in the case of without radiation because here we have only an extra eigenvalue $-4$, which is negative.

(2) Point $P_2$ is denoted by $\phi$-matter-dominated ($\phi$ MDE) epoch. The eigenvalues corresponding to this point are given by

$$-2, -1, \frac{1}{2} [7 + \frac{1}{m} - \frac{m'}{m^2} r(1 + r) \pm \sqrt{(7 + \frac{1}{m} - \frac{m'}{m^2} r(1 + r))^2 - \frac{1}{4}(12 + \frac{3}{m} - \frac{m'}{m^2} r(3 + 4r))}]$$

(3) Point $P_3$ is known as kinetic point. The eigenvalues for the $4 \times 4$ matrix of perturbations about this point are

$$1, 2, \frac{1}{2} [9 + \frac{1}{m} - \frac{m'}{m^2} r(1 + r) \pm \sqrt{(9 - \frac{1}{m} + \frac{m'}{m^2} r(1 + r))^2 - \frac{1}{4}(20 - \frac{5}{m} - \frac{m'}{m^2} r(5 + 4r))}]$$

(37)

If $m$ is constant, the eigenvalues corresponding to this point are $2, 5, 4 - \frac{1}{m}$. In this case $P_3$ is unstable for $m < 0$ and $m > \frac{1}{4}$ and a saddle otherwise.

(4) Point $P_4$ has eigenvalues

$$-5, -4, -3, 4(1 + \frac{1}{m})$$

(38)

It is stable for $-1 < m < 0$ and saddle otherwise. This point can not be use as a radiation or a matter dominated point.

(5) Point $P_5$ can be regarded as a standard matter point in the limit $m \to 0$. Eigenvalues for point $P_5$ are given by

$$-1, 3(1 + m'), \frac{-3m \pm \sqrt{m(256m^3 + 160m^2 - 31m - 16)}}{4m(m + 1)}$$

(39)

where $m'$ is derivative of $m$ w.r.t. $r$. For a cosmologically viable trajectory, we want a saddle matter point. The condition for a saddle matter epoch is given by

$$m(r \leq -1) > 0, m'(r \leq -1) > -1,$$

$$m(r = -1) = 0$$

(40)

(6) Point $P_6$ can also be an acceleration dominated point. The eigenvalues corresponding to this point are given by

$$-\frac{2(-1 + 2m + 5m^2)}{m(1 + 2m)}, -4 + \frac{1}{m}, \frac{2 - 3m - 8m^2}{m(1 + 2m)}, -\frac{2(2m^2 - 1)(1 + m')}{m(1 + 2m)}$$

(41)

Stability of this point depends on both $m$ and $m'$. Condition of acceleration ($w_{e,ff} < -\frac{1}{3}$) depends on the value of $m$.

(7) Point $P_7$ corresponds to a standard radiation point. The eigenvalues of $P_7$ for constant $m$ are $4, 1, -1$. Thus, $P_7$ is a saddle point.

(8) Point $P_8$ also is a radiation point. In this case dark energy is non-zero, therefore $P_8$ is acceptable as a radiation point. The eigenvalues of $P_8$ are given by

$$1, 4(1 + m'), \frac{m - 1 \pm \sqrt{81m^2 + 30m - 15}}{2(m + 1)}$$

(42)

Point $P_8$ is a saddle point in the limit $m \to 0$. The acceptable radiation dominated point $P_8$ lies at point $(0, -1)$ in the $(m, r)$ plane.

V. DYNAMICS OF RADIATION DOMINATED PHASE

For radiation dominated era, phase space analysis shows that we can find a radiation point in the limit $m \to 0$ at point $P_8$. This point lies on the line $m = -r - 1$ in the $(m, r)$ plane. Hence, the necessary condition for
this point to exist as an exact standard radiation point is given by
\[ m(r = -1) \approx 0. \]  
(43)

From definition of \( r \) and the above condition, the form of \( f(R) \) for radiation dominated era is given by
\[ f(R) = \alpha R \]  
(44)

where \( \alpha \) is an integration constant. Standard radiation point is obtained by substitution of \( m \approx 0 \) in the radiation point of \( m(r) \) curve. In this condition, the effective equation of state is
\[ w_{\text{eff}} = \frac{1}{3} \]  
(45)

Using equations (7) and (45), the Hubble parameter \( H(t) \) is given by
\[ H(t) = \frac{1}{(2t - c_1)} \]  
(46)

where \( c_1 \) is an integration constant.

VI. DYNAMICS OF MATTER DOMINATED ERA

From the field equations (5) and (6) we obtain the following equations
\[-\frac{\kappa^2 \rho_r}{3} + 3FH^2 + F\dot{H} - \frac{\dot{F}}{2} - 2HF - \ddot{F} = 0 \]  
(49)

In phase space analysis of dynamical system, there is a point \( P_5 \) which represents a standard matter era in the limit \( m \to 0 \). In matter dominated phase of the Universe
\[ m(r = -1) \approx 0 \]  
(50)

Using the definition of \( r \) or \( m \), the form of \( f(R) \) is given by
\[ f(R) = \beta R \]  
(51)

where \( \beta \) is a integration constant. Thus, in matter dominated phase the form of \( f(R) \) is similar as in the case of radiation dominated phase.

In matter dominated phase, we neglect the energy density of radiation i.e. \( \rho_r = 0 \). For \( f(R) = \beta R \), \( F = \beta \) and therefore \( \ddot{F} = 0 \). Using equations (49) and (2) and these values of \( F \) and \( \dot{F} \) the Hubble parameter is given as
\[ H(t) = \frac{1}{(\frac{3}{2}t - c_3)} \]  
(52)

where \( c_3 \) is an integration constant.

Scale factor in this phase is given by the expression
\[ a(t) = c_4 \left( \frac{3}{2}t - c_3 \right)^{\frac{2}{3}} \]  
(53)
The variation of Hubble parameter $H(t)$, scale factor $a(t)$ and Ricci scalar $R$, with time is plotted in figures 3, 4, and 5, respectively. Hubble parameter $H(t)$, scale factor $a(t)$, and Ricci scalar $R$ in this phase can also be calculated by the same procedure as we followed in the radiation era. Expressions for these parameters are same in both approaches. For $m \approx 0$, the effective equation of state is given by

$$w_{eff} = 0$$  \hspace{1cm} (55)$$

These expressions of scale factor $a(t)$, Hubble parameter $H(t)$, and Ricci scalar $R$ in matter dominated phase are similar to the expressions of standard ($\Lambda$CDM) model.

VII. DYNAMICS OF ACCELERATED EXPANSION DOMINATED PHASE

In the phase space analysis, there is a point $P_1$, for which effective equation of state is

$$w_{eff} = -1$$  \hspace{1cm} (56)$$

This point is called de Sitter point. If we take de Sitter expansion, this point is stable when $0 < m < 1$ at $r = -2$. Now from the definition of $r$, the form of $f(R)$ in this phase is given by

$$f(R) = \alpha R^2$$  \hspace{1cm} (57)$$

We have the effective equation of state

$$w_{eff} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}$$  \hspace{1cm} (58)$$

Now using equations (23) and (25), we get the Hubble parameter

$$H(t) = c_5$$  \hspace{1cm} (59)$$

where $c_5$ is an integration constant. Therefore Ricci scalar in this phase is given by

$$R = 12c_5^2$$  \hspace{1cm} (60)$$

Using the expression of Hubble parameter $H(t)$, the scale factor is given by

$$a(t) = e^{c_5 t + c_6}$$  \hspace{1cm} (61)$$
FIG. 6. Plot for variation of scale factor $a(t)$ with cosmic time $t$ in acceleration dominated phase. The red, green and blue curves correspond to $(c_5, c_6) \equiv (1, 0); (c_5, c_6) \equiv (2, 1); (c_5, c_6) \equiv (3, 2)$, respectively.

where $c_6$ is another integration constant. We can also find out these parameters using equations (49) and (2) in spatially flat universe.

Here, figure 6 shows the variation of scale factor $a(t)$ with time. It is clear that expansion in this phase is exponential. This behaviour is found to be similar to the case of the standard (ΛCDM) model.

VIII. CONCLUSION

While describing the $f(R)$ models of modified gravity, we have studied the properties and stability of the fixed points of dynamical system. We have considered radiation in our analysis and compared it with the case without radiation. It is found that the nature of the fixed points with radiation remains unaltered as that without radiation (except that with radiation we have the emergence of an extra eigenvalue for each point). The forms of $f(R)$ for different phases have been determined by using the conditions of phase space analysis for a cosmologically viable model. The Hubble parameter $H(t)$, Ricci scalar $R$ have been determined for radiation-, matter- and acceleration-dominated phases of the universe, with a view that their time-ordering may explain an arrow of time throughout the cosmic evolution in a future study. These parameters are found to be consistent with Λ cold dark matter (ΛCDM) model.

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