Anonymous Processors with Synchronous Shared Memory *

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Abstract

We investigate anonymous processors computing in a synchronous manner and communicating via read-write shared memory. This system is known as a parallel random access machine (PRAM). It is parametrized by the number of processors and the number of shared memory cells. We consider the problem to assign a unique integer name from the interval $[1, n]$ to each among the $n$ processors of a PRAM. We develop algorithms for each of the eight specific cases determined by which of the following independent properties hold: (1) concurrently attempting to write distinct values into the same memory cell is allowed or it is not, (2) the number of shared variables is unlimited or it is a constant independent of $n$, and (3) the number of processors $n$ is known or it is unknown. Our algorithms terminate almost surely, they are Las Vegas when $n$ is known, they are Monte Carlo when $n$ is unknown, and they always use the $O(n \log n)$ expected number of random bits. We show lower bounds on time, depending on whether the amounts of shared memory are constant or unlimited. In view of these lower bounds, all the Las Vegas algorithms we develop are asymptotically optimal with respect to their expected time, as determined by the available shared memory. Our Monte Carlo algorithms do not produce duplicate names with probabilities that are $1 - n^{-\Omega(1)}$, which is best possible when terminating almost surely and using $O(n \log n)$ random bits.

Key words: anonymous processors, shared memory, read-write registers, synchrony, randomized algorithm, naming.

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1 Introduction

We consider a distributed system in which some $n$ processors communicate using read-write shared memory. It is assumed that the operations performed on shared memory occur synchronously, in that executions of algorithms are structured as sequences of globally synchronized rounds. Each processor is an independent random access machine with its own private memory.

The $n$ processors are anonymous, in that they do not have any names or other features facilitating identification or distinguishing. The goal we consider is to assign unique integers in the range $[1, n]$ to the given $n$ processors as their identifiers. Algorithms assigning names to anonymous processors in such manner are called naming in this paper.

The algorithmic problems considered. Algorithms for a PRAM are normally designed assuming that processors are equipped with unique identifiers from a contiguous segment of integers. When processors are anonymous then the naming problem, to assign unique identifiers to processors, is key as it turns an anonymous system into one with identifiable processors.

We say that a parameter of an algorithmic problem is known when it can be used in a code of an algorithm. We consider two groups of naming problems, depending on whether the number of processors $n$ is known or not.

Further, we consider two categories of naming problems depending on how much shared memory is available. In one case, a constant number of memory cells is available. This means that the amount of memory is independent from $n$ but as large as needed in an algorithm’s design. In the other case, the number of shared memory cells is unlimited, in the sense that it grows with $n$ but how much is used by an algorithm depends on $n$. When an unlimited amount of memory cells is assumed to be available, then, in the algorithms we develop, actually $O(n)$ memory cells suffice when $n$ is known, and $O(n^b)$ of memory suffices, for a parameter $b > 0$, when $n$ is unknown.

Finally, we consider two kinds of the naming problem determined by a choice among the Arbitrary PRAM and Common PRAM variants.

A summary of results. We consider randomized algorithms executed by anonymous processors that operate in a synchronous manner using read-write shared memory with the goal to assign unique names to the processors. This problem is investigated in eight specific cases, depending on additional assumptions, and we give an algorithm for each case. The three independent assumptions regard: (1) the knowledge of $n$, (2) the amount of shared memory, and (3) the PRAM variant. A list of the naming problems’ specifications and the corresponding algorithms is as follows:

1. Arbitrary PRAM, a constant number of shared memory cells, the number of processors $n$ is known: algorithm ARBITRARY-CONSTANT-KNOWN in Section 4.1
2. Arbitrary PRAM, unlimited number of shared memory cells, the number of processors $n$ is known: algorithm ARBITRARY-UNLIMITED-KNOWN in Section 4.2
3. Common PRAM, a constant number of shared memory cells, the number of processors $n$ is known: algorithm COMMON-CONSTANT-KNOWN in Section 5.1
4. Common PRAM, unlimited number of shared memory cells, the number of processors $n$ is known: algorithm COMMON-UNLIMITED-KNOWN in Section 5.2
5. Arbitrary PRAM, a constant number of shared memory cells, the number of processors \( n \) is unknown: algorithm Arbitrary-Constant-Unknown in Section 6.1.

6. Arbitrary PRAM, unlimited number of shared memory cells, the number of processors \( n \) is unknown: algorithm Arbitrary-Unlimited-Unknown in Section 6.2.

7. Common PRAM, a constant number of shared memory cells, the number of processors \( n \) is unknown: algorithm Common-Constant-Unknown in Section 7.1.

8. Common PRAM, unlimited number of shared memory cells, the number of processors \( n \) is unknown: algorithm Common-Unlimited-Unknown in Section 7.2.

The naming algorithms we give terminate with probability 1. These algorithms are Las Vegas for a known number of processors \( n \) and otherwise they are Monte Carlo. All our algorithms use the optimum number \( O(n \log n) \) of random bits. We show that when the number of processors \( n \) is unknown, then a Monte Carlo naming algorithm that uses \( O(n \log n) \) random bits has to have the property that it fails to assign unique names with a probability that is \( n^{-\Omega(1)} \). All Monte Carlo algorithms that we give for the case of unknown \( n \) have the asymptotically optimum probability of error, which is \( n^{-\Omega(1)} \). We show that naming algorithms with \( n \) processors and \( C > 0 \) shared memory cells need to operate in \( \Omega(n/C) \) expected time on an Arbitrary PRAM and in \( \Omega(n \log n/C) \) expected time on a Common PRAM. We show that any naming algorithm needs to work in the expected time \( \Omega(\log n) \); this bound is meaningful when the amount of shared memory is unlimited. Based on these facts, all our Las Vegas algorithms for the case of known \( n \) operate in the asymptotically optimum time, and when the amount of memory is unlimited, they use only an expected amount that is provably necessary. Two of our Monte Carlo algorithms for unknown \( n \) and a constant number of shared memory cells operate in the optimum expected time. The other two Monte Carlo algorithms, for unknown \( n \) and unlimited amount of shared memory, operate using the optimum expected number of shared memory cells, given their expected time.

Previous and related work. The problem of distributed computing in anonymous networks was first considered by Angluin [2]. That work showed, in particular, that randomization is needed in naming algorithms when executed in environments that are symmetric; more related impossibility results are surveyed in [20]. The work about anonymous networks that followed was either on specific network topologies or on problems in general message-passing systems. Specific topologies included that of a ring, investigated by Attiya et al. [8, 9], Flocchini et al. [21], and Itai and Rodeh [23]. Work on algorithmic problems in anonymous networks of general topologies included that by Afek and Matias [1] and Schieber and Snir [37] on leader election, spanning trees and naming, by Angluin et al. [1] on adversarial communication by anonymous agents, by Angluin et al. [4] on self-stabilizing protocols, by Dereniowski and Pelc [16] on leader-election by anonymous agents, by Kowalski and Malinowski [26] on named agents meeting in anonymous networks, by Metivier et al. [30] on naming anonymous unknown graphs, by Michail et al. [31] on naming and counting nodes in dynamic anonymous networks, and Yamashita and Kameda [38] on properties of networks that allow for deterministic solutions for representative algorithmic problems. General question of computability in anonymous message-passing systems implemented in networks were studied by Boldi and Vigna [11], Emek et al. [18], and Sakamoto [36].

Lipton and Park [28] considered naming in asynchronous systems with read-write shared memory controlled by adaptive schedulers; they proposed a solution that terminates with a positive
probability that can be made arbitrarily close to 1 assuming a known $n$. Eğecioğlu and Singh [17] proposed a polynomial-time Las Vegas naming algorithm for asynchronous systems with known $n$ and weak read-write shared memory with oblivious scheduling of events. Kutten et al. [27] gave a Las Vegas algorithm for asynchronous system of shared read-write memory controlled by an oblivious scheduler and with a known $n$ that works in the expected time $O(\log n)$ while using $O(n)$ shared registers. Additionally, they developed an algorithm for the adaptive scheduler requiring an unbounded collection of shared registers. Paper [27] showed also that $n$ needs to be known for a Las Vegas naming algorithm to exist, that a scheduler needs to be oblivious for a finite-state Las Vegas naming algorithm to exist, and that logarithmic time is required to assign names to anonymous processes. Panconesi et al. [38] gave a randomized wait-free naming algorithm in anonymous systems with processes prone to crashes that communicate by single-writer registers. The model for that work [38] assigns unique registers to nameless processes and so has a potential to defy the impossibility of wait-free naming for general multi-writer registers, which was observed by Kutten et al. [27]. Buhrman et al. [13] considered the relative complexity of naming and consensus problems in asynchronous systems with shared memory that are prone to crash failures, demonstrating that naming is harder that consensus.

Next we review work on problems in anonymous distributed systems different from naming. Aspnes et al. [5] gave a comparative study of anonymous distributed systems with different communication mechanisms, including broadcast and shared-memory objects of various functionalities, like read-write registers and counters. Aspnes et al. [6] considered solving consensus in anonymous systems with infinitely many processes. Attiya et al. [7] and Jayanti and Toueg [25] studied the impact of initialization of shared registers on solvability of tasks like consensus and wakeup in fault-free anonymous systems. Bonnet et al. [12] considered solvability of consensus in anonymous systems with processes prone to crashes but augmented with failure detectors. Guerraoui and Ruppert [22] showed that certain tasks like time-stamping, snapshots and consensus have deterministic solutions in anonymous systems with shared read-write registers prone to process crashes. Rupert [35] studied the impact of anonymity of processes on wait-free computing and mutual implementability of types of shared objects.

## 2 Technical Preliminaries

A synchronous shared-memory system in which some $n$ processors operate concurrently is the assumed model of computation. The essential properties of such systems are as follows: (1) shared memory cells have only reading/writing capabilities, and (2) operations of accessing the shared registers are globally synchronized so that processors work in lockstep.

An execution of an algorithm is structured as a sequence of rounds so that each processor performs either a read from or a write to a shared memory cell, along with local computation. We assume that a processor carries out its private computation in a round in a negligible portion of the round. Processors can generate as many private random bits per round as needed; all these random bits generated in an execution are assumed to be independent.

Each shared memory cell is assumed to be initialized to 0 as a default value. This assumption simplifies the exposition, but it can be removed as the algorithms given could be modified in a relatively straightforward manner to work with dirty memory. A shared memory cell can store any value as needed in algorithms, in particular, integers of magnitude that may depend on $n$; all our
algorithms require a memory cell to store $O(\log n)$ bits. An invocation of either reading from or writing to a memory location is completed in the round of invocation. This model of computation is referred in the literature as Parallel Random Access Machine (PRAM) [24][34]. PRAM is usually defined as a model with unlimited number of shared-memory cells, by analogy with the random-access machine (RAM) model. In this paper, we consider the following two instantiations of the model, determined by the amount of shared memory. In one situation, there is a constant number of shared memory cells, which is independent of the number of processors $n$ but as large as needed in the specific algorithm. In the other case, the number of shared memory cells is unlimited in principle, but the expected number of shared registers accessed in an execution depends on $n$ and is sought to be minimized.

A concurrent read occurs when a group of processors read from the same memory cell in the same round; this results in each of these processors obtaining the value stored in the memory cell at the end of the preceding round. A concurrent write occurs when a group of processors invoke a write to the same memory cell in the same round. Without loss of generality, we may assume that a concurrent read of a memory cell and a concurrent write to the same memory cell do not occur simultaneously. A clarification is needed regarding which value gets written to a memory cell in a concurrent write, when multiple distinct values are attempted to be written; such stipulations determine suitable variants of the model. We will consider algorithms for the following two PRAM variants determined by the concurrent-write semantics.

*Common PRAM* is defined by the property that when a group of processors want to write to the same shared memory cell in a round then all the values any of the processors want to write must be identical, otherwise the operation is illegal. Concurrent attempts to write the same value to a memory cell result in this value getting written in this round.

*Arbitrary PRAM* allows attempts to write any legitimate values to the same memory cell in the same round. When this occurs, then one of these values gets written, while a selection of this value is arbitrary. All possible selections of values that get written need to be taken into account when arguing about correctness of an algorithm.

We will rely on certain standard algorithms developed for PRAMs, as explained in [24][34]. One of them is for prefix-type computations. A typical situation in which it is applied occurs when there is an array of $m$ shared memory cells, each memory cell storing either 0 or 1. This may represent an array of bins where 1 stands for a nonempty bin while 0 for an empty bin. Let the rank of a nonempty bin of address $x$ be the number of nonempty bins with addresses smaller than or equal to $x$. Ranks can be computed in time $O(\log m)$ by using an auxiliary memory of $O(m)$ cells, assuming there is at least one processor assigned to a nonempty bin, while other processors do not participate. The bins are associated with the leaves of a binary tree. The processors traverse a binary tree from the leaves to the root and back to the leaves. When updating information at a node, only the information stored at the parent and the sibling and the children is used. We may observe that the same memory can be used repeatedly when such computation needs to be performed multiple times. A possible approach is to verify if the information at a needed memory cell, representing either a parent, a sibling or a child of a visited node, is fresh or rather stale from previous executions. This could be accomplished in the following three steps by a processor. First, the processor erases a memory cell it needs to read by rewriting its present value by a blank value. Second, the processor writes again the value at it visits, which may have been erased.
in the previous step by other processors that need the value. Finally, the processor reads again the
memory cell it just erased, to see if it stays erased, which means its contents were stale, or not,
which means its contents got rewritten so they are fresh.

**Balls into bins.** Assigning names to processors can be visualized as throwing balls into bins.
Imagine that balls are handled by processors and bins are represented by either memory addresses
or rounds in a segment of rounds. Throwing a ball means either writing into some memory address
a value that represents a ball or choosing a round from a segment of rounds. A collision occurs
when two balls end up in the same bin; this means that two processors wrote to the same memory
address, not necessarily in the same round, or they selected the same round. The rank of a bin
containing a ball is the number of bins with smaller or equal positions that contain balls. When
each in a group of processors throws a ball and there is no collision then this in principle breaks
symmetry in a manner that allows to assign unique names in the group, namely, ranks of selected
bins may serve as names.

The following terms refer to the status of a bin in a given round. A bin is called empty where
there are no balls in it. A bin is singleton when it contains a single ball. A bin is multiple when
there are at least two balls in it. Finally, a bin with at least one ball is occupied.

The idea of representing attempts to assign names as throwing balls into bins is quite generic.
In particular, it was applied by Eğecioğlu and Singh [17], who proposed a synchronous algorithm
that repeatedly throws all balls together into all available bins, the selections of bins for balls made
independently and uniformly at random. In their algorithm for \( n \) processors, we can use \( b \cdot n \)
memory cells, where \( b > 1 \). Let us choose \( b = 3 \) for the following calculations to be specific. This
algorithm has an exponential expected-time performance. To see this, we estimate the probability
that each bin is either singleton or empty. Let the balls be thrown one by one. After the first \( n/2 \)
balls are in singleton bins, the probability to hit an empty bin is at most \( \frac{2^5n}{3^1} = \frac{5}{6} \); we treat this
as a success in a Bernoulli trial. The probability of \( n/2 \) such successes is at most \( \left( \frac{5}{6} \right)^{n/2} \), so the
expected time for a success to happen is at least \( \left( \sqrt[6]{\frac{5}{6}} \right)^n \), which is exponential in \( n \). We consider
related processes that could be as fast as \( O(\log n) \) expected time, given sufficiently many shared
memory cells, see Section 5.2. The idea is to let balls in singleton bins stay put and only move
those that collided with other balls by landing in bins that became thereby multiple. To implement
this in a Common PRAM, we need a way to detect collisions, which we explain next.

**Collisions among balls.** We will use a randomized procedure for Common PRAM to verify if a
collision occurs in a bin, say, a bin \( x \), which is executed by each processor that selected bin \( x \). This
procedure VERIFY-COLLISION is represented in Figure 1. There are two arrays TAILS and HEADS
of shared memory cells. Bin \( x \) is verified by using memory cells TAILS\([x]\) and HEADS\([x]\). First,
the memory cells TAILS\([x]\) and HEADS\([x]\) are set to false each, and next one of these memory cells
is selected randomly and set to true.

**Lemma 1** For an integer \( x \), procedure VERIFY-COLLISION\((x)\) executed by one processor never
detects a collision, and when multiple processors execute this procedure then collision is detected
with a probability that is at least \( \frac{1}{2} \).

**Proof:** When only one processor executes the procedure, then first the processor sets both HEADS\([x]\)
and TAILS\([x]\) to false and next only one of them to true. This guarantees that HEADS\([x]\) and TAILS\([x]\)
store different values and so collision is not detected. When some \( m > 1 \) processors execute the
Procedure VERIFY-COLLISION ($x$)

\begin{verbatim}
initialize Heads[$x$] ← Tails[$x$] ← false
toss_v ← outcome of tossing a fair coin
if toss_v = tails then Tails[$x$] ← true else Heads[$x$] ← true
if Tails[$x$] = Heads[$x$] then return true else return false
\end{verbatim}

Figure 1: A pseudocode for a processor $v$ of a Common PRAM, where $x$ is a positive integer. Heads and Tails are arrays of shared memory cells. When the parameter $x$ is dropped in a call then this means that $x = 1$. The procedure returns true when a collision has been detected.

procedure, then collision is not detected only when either all processors set Heads[$x$] to true or all processors set Tails[$x$] to true. This means that the processors generate the same outcome in their coin tosses. This occurs with the probability $2^{-m+1}$, which is at most $\frac{1}{2}$. □

Pseudocode conventions. We give pseudocode representations of algorithms, like this in Figure 1. We apply some conventions that we summarize next.

When an instruction is conditional on a statement then a processor that does not meet the condition pauses as long as it would be needed for all the processors that meet the condition complete their instructions, even when there are no such processors. This means that, at any round of an execution, all the processors that have not terminated yet are at the same line of the pseudocode.

A pseudocode for a processor $v$ has a number of variables referred to. To emphasize that a variable $x$ is a private variable for processor $v$, we may denote $x$ by $x_v$. Typically, shared variables have names starting with a capital while private variables have names all in small letters. Certain variables could be designated either as private or shared, like variables controlling for-loops; in such situations the variables are considered private. An instruction of the form $x ← y ← \ldots ← z ← \alpha$, where $x, y, \ldots, z$ are variables and $\alpha$ is a value, means assigning $\alpha$ as the value stored in all the listed variables $x, y, \ldots, z$.

Properties of naming algorithms. Naming algorithms in distributed environments like multi-writer read-write shared memory have to be randomized to break symmetry. An eventual assignment of proper names cannot be a sure event, because, in principle, two processors can generate the same strings of random bits in the course of an execution. We say that an event is almost sure, or occurs almost surely, when it occurs with probability 1. When $n$ processors generate their private strings of random bits then it is an almost sure event that all these strings are eventually distinct. Therefore the most we could expect of a randomized naming algorithm executed by a set of $n$ processors is that the algorithm eventually terminates almost surely and that at the moment of termination the output is correct in that the assigned names are without duplicates and fill the whole interval $[1, n]$.

We will deal with two kinds of randomized (naming) algorithms, called Monte Carlo and Las
Las Vegas, which are defined as follows. A randomized algorithm is Las Vegas when it terminates almost surely and the algorithm returns a correct output upon termination. A randomized algorithm is Monte Carlo when it terminates almost surely and an incorrect output may be produced upon termination, but the probability of error converges to zero with the size of input growing unbounded.

The naming algorithms we develop have qualities that depend on whether \( n \) is known or not, according to the following simple rule: each algorithm for \( n \) known is Las Vegas while each algorithm for \( n \) unknown is Monte Carlo. Our Monte Carlo algorithms have the probability of error converging to zero with a rate that is polynomial in \( n \). Moreover, when incorrect (duplicate) names are assigned, the set of integers used as names makes a contiguous segment starting at the smallest name 1.

A naming algorithm cannot be Las Vegas when \( n \) is unknown, as was observed by Kutten et al. \[27\] in a more general case of asynchronous computations against an oblivious adversary. We give a short proof of this fact for synchronous computations, for completeness' sake.

**Proposition 1** \[27\] There is no Las Vegas naming algorithm when \( n > 1 \) is unknown.

**Proof:** Let \( n > 0 \) be a possible size of the system. If a naming algorithm does not terminate with a positive probability for \( n \) processors then it is not Las Vegas by definition. Let us suppose that a naming algorithm terminates almost surely when executed by \( n \) processors. Consider an execution \( E \) on \( n \) processors that uses specific strings of random bits such that the algorithm terminates in \( E \) with these random bits. Such strings of random bits exist because the algorithm terminates with a positive probability. Take a random string \( \alpha_E \) generated by a processor by the time of termination in \( E \), say, a shortest one. Consider an execution \( E' \) on \( n + 1 \) processors such that \( n \) processors obtain the same strings of random bits as in \( E \) and the extra processor obtains \( \alpha_E \) as its random bits. Then the executions \( E \) and \( E' \) are indistinguishable for the \( n \) processors participating in \( E \), because \( n \) is not a part of code. Execution \( E' \) results in a name being duplicated. The probability of duplication for \( n + 1 \) processors is at least as large as the probability to generate the finite random strings for \( n \) processors as in \( E \), and additionally to generate \( \alpha_E \) for the extra processor, and so this probability is positive. \( \square \)

We give algorithms that use the expected number of \( O(n \log n) \) random bits with large probability. This amount of random information is necessary if an algorithm is to terminate almost surely. The following fact is folklore, we give a proof for completeness' sake, by resorting to the notions of information theory \[15\].

**Proposition 2** If a randomized naming algorithm is correct with some probability \( p_n \) then it requires \( \Omega(n \log n) \) random bits with the same probability \( p_n \), when executed by \( n \) anonymous processors. In particular, a Las Vegas naming algorithm uses \( \Omega(n \log n) \) random bits almost surely.

**Proof:** Let us assign conceptual identifiers to the processors, for the sake of argument. These identifiers are known only to an external observer and not to algorithms.

A distribution of correct names among the \( n \) anonymous processors, which results from executing the algorithm, is a random variable with a uniform distribution on the set of all permutations of the conceptual identifiers, conditional on all the names being distinct and in the interval \([1, n]\). This is because of symmetry: all processors execute the same code without explicit private identifiers. There are \( n! \) elementary events, each determined by a permutation of the conceptual identifiers, so
each occurs with probability $1/n!$. The Shannon entropy of this random variable is thus $\log(n!) = \Theta(n \log n)$.

The property that all assigned names are distinct and in the interval $[1, n]$ holds with probability $p_n$. An execution needs to generate a total of $\Omega(n \log n)$ random bits with this probability $p_n$. A Las Vegas algorithm terminates almost surely, and returns correct names upon termination. This means that $p_n = 1$, and so $\Omega(n \log n)$ random bits are used almost surely. □

When $n$ is unknown, then the restriction $\mathcal{O}(n \log n)$ on the number of random bits makes it inevitable that the probability of error is at least polynomially bounded from below.

**Proposition 3** For unknown $n$, if a randomized naming algorithm is executed by $n$ anonymous processors, then duplicate names may be assigned to distinct processors with a probability that is $n^{-\Omega(1)}$, if only the algorithm uses $\mathcal{O}(n \log n)$ random bits with a probability that is $1 - n^{-\Omega(1)}$.

**Proof:** Suppose the algorithm uses at most $cn \log n$ random bits with a probability $p_n$ when executed by a system of $n$ processors, for some constant $c > 0$. Then one of these processors uses at most $c \log n$ bits with a probability $p_n$, by the pigeonhole principle.

Consider an execution for $n + 1$ processors. Let us distinguish one processor $v$ and consider the actions of the remaining $n$ processors. One of these processors, say $w$, uses at most $c \log n$ bits with the probability $p_n$. Processor $v$ generates the same string of bits with the probability $2^{-c \log n} = n^{-c}$. The random bits generated by $w$ and $v$ are independent. Therefore duplicate names occur with the probability $n^{-c} \cdot p_n$. When $p_n = 1 - n^{-\Omega(1)}$, then the probability of duplicate names is at least $n^{-c}(1 - n^{-\Omega(1)}) = n^{-\Omega(1)}$. □

# 3 Lower Bounds on Time

We consider two kinds of algorithmic naming problems, determined by the amount of shared memory. One case is for a constant number of shared memory cells, for which we give an optimal lower bound on time for $\mathcal{O}(1)$ shared memory. The other case is when the number of shared memory cells and their functionality are unlimited, for which we give an “absolute” lower bound on time. We begin with lower bounds that reflect the amount of available shared memory.

Intuitively, as the processors generate random bits, these need to be made common knowledge through some implicit process that assigns explicit names. There is an underlying flow of information spreading knowledge among the processors through the available shared memory. The time is lower bounded by the rate of flow of information and the total amount of bits that need to be shared. These intuitions are made formal in the proofs of Theorems 1 and 2.

**Theorem 1** Let a naming algorithm be given, for a Common PRAM with $n$ processors and $C > 0$ shared memory cells, such that when it terminates then correct names are assigned to all processors with a probability $p_n$. This algorithm has to operate in $\Omega(n \log n / C)$ time with the same probability $p_n$. In particular, if this algorithm is Las Vegas then it operates almost surely in $\Omega(n \log n / C)$ time.

**Proof:** An algorithm needs to make the processors share $\Omega(n \log n)$ random bits with the probability $p_n$, by Proposition 2 using the available $C > 0$ shared memory cells. We may assume without
loss of generality that an execution is structured into the following phases, each consisting of $C + 1$ rounds. In the first round of a phase, each processor either writes into a shared memory cell or pauses. In the following rounds of a phase, every processor learns the current values of each among the $C$ memory cells. This may take $C$ rounds for every processors to scan the whole shared memory, but we do not include this reading overhead as contributing to the lower bound. Instead, since this is a simulation anyway, we conservatively assume that the process of learning all the contents of shared memory cells at the end of a phase is instantaneous and complete.

The Common variant of PRAM requires that if a memory cell is written concurrently into then there is a common value that is written by all the writers. All these values need to be determined by the code and the addresses of the memory cells. This means, that for each phase and any memory cell, any processors choosing to write into this memory cell know the common value to be written. By extension, and the structure of the execution through which all processors read all the registers after a round of writing, we conclude that all the processors know what value gets written into each available memory cell in a phase, if any is written into a particular cell. This implies that the contents written into shared memory cells do not convey any new information but are already implicit in the states of the processors represented by their private memories.

When a processor reads all the shared memory cells in a phase, then the only new information it may learn is where writes were performed and where not. This allows to obtain at most $C$ bits of information per phase with probability $p_n$, because each register either was written to or not. It follows that $\Omega(n \log n/C)$ phases are needed to settle on $\Omega(n \log n)$ bits of information that determine proper names, with the corresponding probability $p_n$. When the algorithm is Las Vegas then $p_n = 1$. □

For Arbitrary PRAM, written contents can spread information. Obviously, the amount of such information is bounded above by how many random bits are written, if few are chosen to be written. Intuitively, this is not a real limitation because an algorithm can make processors write suitably many bits. A true limit is determined by the size of a group of processors choosing to write to the same register. These intuitions are made formal in the proof of the following Theorem.

**Theorem 2** Let a naming algorithm be given, for an Arbitrary PRAM with $n$ processors and $C > 0$ shared memory cells, such that when it terminates then correct names assigned with a probability $p_n$. This algorithm has to operate in $\Omega(n/C)$ time with the probability $p_n$. In particular, if this algorithm is Las Vegas then it operates almost surely in $\Omega(n/C)$ time.

**Proof:** The algorithm needs to make the processors learn $\Omega(n \log n)$ bits with the probability $p_n$, by Proposition using the available $C > 0$ shared memory cells. As in the proof of Theorem we may assume that an execution is structured into phases, such that each processor performs at most one write in a phase and then reads all the registers.

Consider a register and a group of processors that attempt to write their values into this register in a phase. When such a group is empty, then this provides merely one bit of information. Otherwise, the values attempted to be written are represented as strings of bits. If some of these values have 0 and some have 1 at some bit position among the strings, then this bit position has a potential to provide at least one bit of information. The maximum amount of information is provided by a write when the written string of bits allows to identify the writer by comparing its written value to the other values attempted to be written concurrently to the same memory cell.
It follows that this amount is at most the binary logarithm of the size of this group of processors. Each memory cell written to in a round generates at most \( \lg n \) bits of information because there may be at most \( n \) writers to it.

So the maximum number of bits of information learnt by the processors in a phase is \( C \lg n \). We conclude that the number of rounds needs to be \( \Omega(n \lg n / (C \lg n)) = \Omega(n/C) \).

Next we consider “absolute” requirements on time for a PRAM to assign unique names to its \( n \) processors. The “absoluteness” stems from the weakness of assumptions. First, it is not assumed that \( n \) is either known or unknown. Second, concurrent writing is not constrained in any way. Third, shared memory cells are unlimited in their number and functionality. Kutten et al. [27] observed that any Las Vegas naming algorithm for asynchronous systems against an oblivious scheduler requires time \( \Omega(\log n) \) almost surely in the big-step model, under similar assumptions. That fact subsumes the case of a synchronous PRAM, which can be interpreted as controlled by a specialized oblivious scheduler. We give a concise proof of this lower bound for completeness sake. The argument is in the spirit of similar arguments given by Cook et al. [14], Beame [10] and Kutten et al. [27].

We formalize the notion of flow of information during an execution of an algorithm. Intuitively, for such a formalization to be applicable in an argument about a lower bound, it needs to be liberal enough in not exhibiting any actual flow of information or learning but rather in exhibiting of a mere possibility to occur.

A relation \( \text{processor } v \text{ knows processor } w \text{ in round } t \) is defined recursively as follows. First, for any processor \( v \), we have that \( v \) knows \( v \) in any round \( t > 0 \). Second, if a processor \( v \) writes to a shared memory cell \( R \) in a round \( t_1 \) and processor \( w \) reads from \( R \) in a round \( t_2 > t_1 \) such that there was no other write into this memory cell after \( t_1 \) and prior to \( t_2 \) then processor \( w \) knows in round \( t_2 \) each processor that \( v \) knows in round \( t_1 \). Finally, the relation is the smallest transitive relation that satisfies the two postulates formulated above. This means that it is the smallest relation such that if processor \( v \) know processor \( w \) in round \( t_1 \) and \( z \) knows \( v \) in round \( t_2 \) such that \( t_2 > t_1 \) then processor \( z \) knows \( w \) in round \( t_2 \). In particular, the knowledge accumulates with time, in that if a processor \( v \) knows processor \( z \) in round \( t_1 \) and round \( t_2 \) satisfies \( t_2 > t_1 \) then \( v \) knows \( z \) in round \( t_2 \) as well.

**Lemma 2** When a naming algorithm terminates and names have been assigned correctly then each processor knows all processors.

**Proof:** The names are determined by random bits generated uniformly and independently by the processors. Suppose an external observer assigns “hidden” conceptual names that are unknown to an algorithm, in that they are not referred to in the code. Then the distribution of names is uniform on all \( n! \) permutations of the hidden names, because of symmetry and anonymity, assuming we disregard the number of random bits that need to be generated. Suppose we associate strings of random bits (of potentially unbounded length) with all but one processor \( v \). Then the distribution of names for \( v \) is also uniform on \([1, n]\), as determined by the random bits that \( v \) uses, conditional on the assignment of random bits to the other processors. This means that each processor different from \( v \) needs to know \( v \). We conclude that every processor has to know any other processor.

The fact stated in Lemma 2 allows to argue about time when we estimate how much information is shared per round of an algorithm. The amount may depend on rounds because we do not restrict
The shared memory, unlike in Theorems 1 and 2.

**Theorem 3** ([27]) *Let a naming algorithm be given, for a PRAM with* $n$ *processors, such that when it terminates then unique names in the interval* $[1,n]$ *are assigned to all among its* $n$ *processors with a probability* $p_n$. *This algorithm requires* $\Omega(\log n)$ *time with the probability* $p_n$. *In particular, a Las Vegas naming algorithms operates in time* $\Omega(\log n)$ *almost surely.*

**Proof:** We may not mention probability for conciseness sake, but the argument is understood to refer to an event that holds with the probability $p_n$, because we consider executions that terminate with unique names in the interval $[1,n]$ assigned to all among its $n$ processors.

We may structure an execution of the algorithm into *phases* as follows. A phase consists of two rounds. In the first round of a phase, each processor either writes to a shared memory cell or pauses. In the second round of a phase, each processor either reads from a shared memory cell or pauses. Such structuring can be done without loss of generality at the expense of slowing down an execution by a factor of at most 2. Observe that the knowledge in the first round of a phase is the same as in the last round of the preceding phase.

Phases are numbered from 1 to infinity, comprising pairs of rounds $\{2i - 1, 2i\}$, for integers $i \geq 1$. In particular, the first phase consists of rounds 1 and 2. We also add phase 0 that represents the knowledge before any reads or writes were performed.

We show the following invariant, for $i \geq 0$: with the probability $p_n$, at the end of phase $i$, a processor knows at most $2^i$ processors. The proof is by induction on $i$.

The base if for $i = 0$. It follows from the fact that a processor knows only itself in phase 0. To show the inductive step, suppose the invariant holds for phase $i \geq 0$ and consider the next phase $i + 1$. A processor $v$ may increase its knowledge by reading in the second round of phase $i + 1$. The read is of a shared memory cell $R$. The latest write into this memory cell occurred by the first round of phase $i + 1$. This means that the processor $w$ that wrote to $R$ by phase $i + 1$ as the last one knew at most $2^i$ processors in the round of writing, by the inductive assumption and the fact that what is written in phase $i + 1$ was learnt by the immediately preceding phase $i$. Moreover, by the semantics of writing, the value written to by $w$ in that round removed any previous information stored in $R$. Processor $v$ starts phase $i + 1$ knowing at most $2^i$ processors, and also learns of at most $2^i$ other processors by reading in phase $i + 1$, namely, those values known by the latest writer of the contents read. It follows that processor $v$ knows at most $2^i + 2^i = 2^{i+1}$ processors by the end of phase $i + 1$.

When the algorithm terminate then each processor knows all the processors with probability $p_n$, by Lemma 2. This means that, with probability $p_n$, the number of phases $j$ is at least such that $2^j \geq n$. This inequality holds almost surely when the algorithm is Las Vegas.

The three lower bounds on time given in this Section may be applied in two ways. One is to infer optimality of time for a given amount of shared memory used. Another is to infer optimality of shared memory use given a time performance. For example, with $O(1)$ shared memory, time is optimal when it is $O(n)$ on Arbitrary PRAM and when it is $O(n \log n)$ on Common PRAM. On the other hand, when an algorithm operates in the optimum time $O(\log n)$ then the amount of shared memory used is also at optimum when it is $O(n/\log n)$ on Arbitrary PRAM and when it is $O(n)$ on Common PRAM.
Algorithm Arbitrary-Constant-Known

repeat
    initialize Counter ← name$_v$ ← 0
    $x ← $ random integer in $[1, n^\beta]$ 
    for $i ← 1$ to $n$ do if name$_v$ ≤ 0 then
        Pad ← $x$
        if Pad = $x$ then
            Counter ← Counter + 1 ; name$_v$ ← Counter
    until Counter ≥ $n$

Figure 2: A pseudocode for a processor $v$ of an Arbitrary PRAM, where $n$ is known and the number of shared memory cells is a constant independent of $n$. The variables Counter and Pad are shared. The private variable name stores the acquired name. The constant $\beta > 0$ is parameter to be determined by analysis.

4 Arbitrary PRAM of Known Size

Let PRAM be Arbitrary and $n$ be known. We consider two cases. In one case, there are a constant number shared memory cells. In the other case, we want time to be as small as possible with as many shared memory cells as needed, and then we minimize the number of shared memory cells.

4.1 Arbitrary with Constant Memory

The algorithm for Arbitrary PRAM in the case when $n$ known and there are a constant number of shared memory cells is called Arbitrary-Constant-Known. The pseudocode is given in Figure 2.

An execution proceeds so that the processors repeatedly write random strings of bits representing integers to the shared memory cell Pad and next read it to verify the outcome of writing. A processor that reads the same value as it attempted to write increments Counter and uses the obtained number as a tentative name stored at the private variable name$_v$. The values of Counter could get incremented less than $n$ times, which occurs when some two processors chose the same random integer to write. The correctness of the assigned names is verified by the condition controlling the repeat-loop. The verification results either in performing another iteration or terminating, in the latter case the value stored at name$_v$ becomes the final name of processor $v$.

Balls into bins. The selection of random integers in the range $[1, n^\beta]$ by $n$ processors can be interpreted as throwing $n$ balls into $n^\beta$ bins. A collision represents two processors assigning themselves the same name. Therefore an execution of the algorithm can be interpreted as performing such ball placements repeatedly until there is no collision.

Lemma 3 For each $a > 0$ there exists $\beta > 0$ such that when $n$ balls are thrown into $n^\beta$ bins then
the probability of a collision is at most $n^{-a}$.

**Proof:** Consider the balls thrown one by one. When a ball is thrown, then at most $n$ bins are already occupied, so the probability of the ball ending in an occupied bin is at most $n/n^\beta = n^{-\beta+1}$. No collisions occur with a probability that is at least

$$\left(1 - \frac{1}{n^{\beta-1}}\right)^n \geq 1 - \frac{n}{n^\beta} = 1 - n^{-\beta+2},$$

by the Bernoulli’s inequality. If we take $\beta \geq a + 2$ then just one iteration of the repeat-loop is sufficient with a probability that is at least $1 - n^{-a}$.

**Theorem 4** Algorithm Arbitrary-Constant-Known is Las Vegas. For some $c > 0$ and any $a > 0$, there exists $\beta > 0$ such that the algorithm terminates within time $cn$ using at most $cn \ln n$ random bits with a probability that is at least $1 - n^{-a}$.

**Proof:** The algorithm terminates only when $n$ different names have been assigned, by the condition controlling the repeat loop. Lemma 3 estimates the probability of the event that the $n$ processors select different numbers in the interval $[1, n^\beta]$ as their $x$ values in one iteration of the repeat-loop. It implies that just one iteration of the repeat-loop is sufficient with a probability that is at least $1 - n^{-a}$, and moreover, that the algorithm terminates almost surely. One iteration of the repeat-loop takes $\mathcal{O}(n)$ rounds and it requires $\mathcal{O}(n \log n)$ random bits.

Algorithm Arbitrary-Constant-Known is optimal with respect to the following performance measures: the expected time $\mathcal{O}(n)$, by Theorem 2, and the expected number of random bits $\mathcal{O}(n \log n)$, by Proposition 2.

### 4.2 Arbitrary with Unlimited Memory

The algorithm for Arbitrary PRAM in the case when $n$ known and there is an unlimited supply of initialized shared memory cells is called Arbitrary-Unlimited-Known. Figure 3 contains its pseudocode.

The algorithm uses an array $\text{Counter}[1, \frac{n}{\ln n}]$ of $\frac{n}{\ln n}$ shared memory cells. Each time an iteration of the outer repeat-loop begins, a new segment of memory cells initialized to 0s is used. The memory cell $\text{Counter}[x]$ stores the number of writes to shared memory cell $\text{Bin}[x]$. After the value attempted by a processor $v$ to be written gets written, $v$ assigns the pair $(x_v, \text{Counter}[x_v])$ as a value of position $v$. The inner loop terminates when each processor $v$ assigned itself a new value for position $v$. The rank of position $v$ is defined as follows: we arrange all such pairs in the lexicographic order, comparing first on $x$ and then on $\text{Counter}[x]$, and the rank is the position of this entry in the resulting list, where the first entry has position 1, the second 2, and so on. Ranks are computed using a prefix-type algorithm, which operates in time $\mathcal{O}(\log n)$.

**Balls into bins.** We consider throwing $n$ balls into $\frac{n}{\ln n}$ bins. Each ball has a label assigned randomly from the range $[1, n^\beta]$, for $\beta > 0$. We say that a labeled collision occurs when there are two balls with the same labels in the same bin.
Algorithm Arbitrary-Unlimited-Known

repeat
    allocate Counter[1, \( \frac{n}{\ln n} \)]  /* array of fresh memory cells initialized to 0s */
    initialize position, \( (0,0) \)
    \( x \leftarrow \) a random integer in \([1, \frac{n}{\ln n}]\)  /* choose a bin */
    \( y \leftarrow \) a random integer in \([1, n^\beta]\)  /* choose a label for the ball */
    repeat
        initialize All-Named \( \leftarrow \) true
        if position = \( (0,0) \) then
            Bin \([x]\) \( \leftarrow y \)
            if Bin\([x]\) = \( y \) then
                Counter\([x]\) \( \leftarrow \) Counter\([x]\) + 1 ; position \( \leftarrow (x, \text{Counter}[x]) \)
            else All-Named \( \leftarrow \) false
        until All-Named
    /* each processor has name determined */
    name, \( \leftarrow \) rank of position
    until the maximum name is at least \( n \)  /* no duplicates among names */

Figure 3: A pseudocode for a processor \( v \) of an Arbitrary PRAM, where \( n \) is known and the number of shared memory cells is unlimited. The variables Counter and Bin denote arrays of \( \frac{n}{\ln n} \) shared memory cells, the variable All-Named is also shared. The private variable name stores the acquired name. The constant \( \beta > 0 \) is a parameter to be determined by analysis.

Lemma 4 For each \( a > 0 \) there exists \( \beta > 0 \) and \( c > 0 \) such that when \( n \) balls are labeled with random integers in \([1, n^\beta]\) and next are thrown into \( \frac{n}{\ln n} \) bins then there are at most \( c \ln n \) balls in every bin and no labeled collision occurs with a probability that is at least \( 1 - n^{-a} \).

Proof: We estimate from above the probabilities of the event that there are more than \( c \ln n \) balls in some bin and that there is a labeled collision. We show that each of them can be made to be at most \( n^{-a}/2 \), from which it follows that some of these two events occurs with a probability that is at most \( n^{-a} \).

Let \( p \) denote the probability of selecting a specific bin when throwing a ball, which is \( p = \frac{\ln n}{n} \). Let \( S_n \) be the number of successes in \( n \) independent Bernoulli trials, with \( p \) as the probability of success. Let \( b(i;n,p) \) be the probability of \( i \) successes. For \( r > np \), the following bound holds

\[
\Pr(S_n \geq r) \leq b(r; n, p) \cdot \frac{r(1-p)}{r - np},
\]

see Feller [19]. When we set \( r = c \ln n \), for a sufficiently large \( c > 0 \), then

\[
b(r; n, p) = \binom{n}{c \ln n} \left( \frac{\ln n}{n} \right)^{c \ln n} \left( 1 - \frac{\ln n}{n} \right)^{n - c \ln n}.
\]
Formula (3) translates (2) into the following bound
\[
\Pr(S_n \geq r) \leq \left( \frac{n}{c \ln n} \right)^{c \ln n} \left( 1 - \frac{\ln n}{n} \right)^{n - c \ln n} \cdot \frac{c \ln n (1 - \ln n)}{c \ln n - \ln n}.
\] (4)
The right-hand side of (4) can be upper bounded as follows:
\[
\left( \frac{e n}{c \ln n} \right)^{c \ln n} \left( \frac{\ln n}{n} \right)^{c \ln n} \left( 1 - \frac{\ln n}{n} \right)^{n - c \ln n} \cdot \frac{c}{c - 1} \leq \left( \frac{e}{c} \right)^{c \ln n} \left( 1 - \frac{\ln n}{n} \right)^{n} \left( \frac{n}{n - \ln n} \right)^{c \ln n} \cdot \frac{c}{c - 1} \leq n^{-c \ln c + c - 1},
\]
for each sufficiently large \( n > 0 \). This is because
\[
\left( \frac{n}{n - \ln n} \right)^{c \ln n} = \left( 1 + \frac{\ln n}{n - \ln n} \right)^{c \ln n} \leq \exp \left( \frac{c \ln^2 n}{n - \ln n} \right),
\]
which converges to 1. The probability that the number of balls in some bin is greater than \( c \ln n \) is therefore at most \( n \cdot n^{-c \ln c + c - 1} = n^{-c(\ln c - 1)} \), by the union bound. This probability can be made less than \( n^{-a/2} \) for a sufficiently large \( c > e \).

The probability of a labeled collision is at most that of a collision when \( n \) balls are thrown into \( n^\beta \) bins. This probability is at most \( n^{-\beta + 2} \) by (1) in the proof of Lemma 4. This number can be made at most \( n^{-a/2} \) for a sufficiently large \( \beta \).

**Theorem 5** Algorithm Arbitrary-Unlimited-Known is Las Vegas. For any \( a > 0 \), there exists \( \beta > 0 \) and \( c > 0 \) such that the algorithm assigns names within \( c \ln n \) time and generates at most \( c n \ln n \) random bits with a probability that is at least \( 1 - n^{-a} \).

**Proof:** The algorithm terminates only when \( n \) different names have been assigned, which is provided by the condition that controls the main repeat-loop. The main repeat-loop ends after an iteration in which each group of processors that select the same value for \( x \) next select distinct values for \( y \). We interpret the random selections in an execution as throwing \( n \) balls into \( \frac{n}{\ln n} \) bins, where a number \( x \) determines a bin. The number of iterations of the inner repeat-loop equals the maximum number of balls in a bin. It follows that one iteration of the main repeat-loop suffices with a probability that is at least \( 1 - n^{-a} \), for a suitable \( \beta > 0 \), by Lemma 4. It follows that the algorithm terminates almost surely. Having fixed \( \beta > 0 \), take \( c > 0 \) such that an iteration of the main repeat-loop never takes more than \( c \ln n \) steps and one processor uses at most \( c \ln n \) random bits in it.

Algorithm Arbitrary-Unlimited-Known is optimal with respect to the following performance measures: the expected time \( O(\log n) \), by Theorem 3, the expected number of used random bits \( O(n \log n) \), by Proposition 2, and the expected number of shared memory cells \( O(n / \log n) \), by Theorem 2.

## 5 Common PRAM of Known Size

We consider a Common PRAM when the number of processors \( n \) is known. This is broken into two sub-cases of either a constant amount of shared memory or of amount that is unlimited in principle.
Algorithm **Common-Constant-Known**

```plaintext
repeat
    initialize K ← n ; name_v ← Counter ← 0 ; no-collision_v ← true
    repeat
        initialize All-Named ← true
        if name_v ≤ 0 then
            slot_v ← random integer in [1, K lg n] /* throw a ball into a bin */
            for i ← 1 to K lg n do if slot_v = i then
                for j ← 1 to β lg n do if Verify-Collision then
                    All-Named ← no-collision_v ← false
                if no-collision_v then
                    Counter ← Counter + 1 ; name_v ← Counter
        if n − Counter > lg^4 n then K ← (n − Counter) else K ← n/lg n
    until All-Named
until Counter ≥ n
```

Figure 4: A pseudocode for a processor v of a Common PRAM, where n is known and there is a constant number of shared memory cells. Procedure Verify-Collision has its pseudocode in Figure 1. The variables All-Named, K, and Counter are shared. The private variable name stores the acquired name. The constant β is a parameter to be determined by analysis.

### 5.1 Common with Constant Memory

First we consider the case when the number of available shared memory cells is constant. We propose an algorithm **Common-Constant-Known**, whose pseudocode is in Figure 4.

The algorithm is structured as a repeat-loop. It begins by an initialization of some of variables and is followed by an inner repeat-loop. Choosing a random integer slot_v in [1, K lg n] can be interpreted as throwing a ball into a bin. The inner repeat loop terminates when all processors obtain some names, possibly with duplicates. Procedure Verify-Collision has its argument omitted, because only one bin is checked for collision at a time. Names are assigned in the inner repeat-loop only to these processors that withstood the test for collisions in the inner for-loop. The shared variable Counter stores the last assigned name. The number n − Counter is an upper bound on the number of processors that still need names. The shared variable K is set to n − Counter as long as this number is greater than lg^4 n, otherwise K is set to n/lg n, which serves to facilitate the probabilistic analysis.

**Balls into bins.** As a preparation of analysis, we consider a related process of repeatedly throwing balls into bins. The process proceeds through stages, each representing one iteration of the inner repeat-loop. A stage results in some balls removed and some transitioning to the next stage, so that eventually no balls remain and the process terminates. The balls that participate in a stage
are called eligible for the stage. In the first stage, all balls are eligible, so we throw $n$ balls into $n \lg n$ bins. When balls are thrown for a stage, the balls from singleton bins are removed and those from multiple bins become eligible for the next stage. Let $k$ denote the number of balls eligible for the stage. If $k > \lg^4 n$ then the eligible balls are thrown into $k \lg n$ bins, otherwise there are $n$ bins to throw balls into.

**Lemma 5** The number of times a ball is thrown summed over all stages is $O(n)$ with a probability that is $1 - n^{-\Omega(\log n)}$.

**Proof:** We consider two cases, depending on two types of stages. The number of balls thrown, summed over all stages, is the same as the number of balls eligible for all stages.

The first case is when we throw $k$ balls into $k \lg n$ bins and $k > \lg^4 n$. The probability that a ball ends up single in a bin is as follows:

$$k \lg n \cdot \frac{1}{k \lg n} \left(1 - \frac{1}{k \lg n}\right)^{k-1} = (1 - \frac{1}{k \lg n})^{k-1} \geq 1 - \frac{k-1}{k \lg n} \geq 1 - \frac{1}{\lg n},$$

where we used the Bernoulli’s inequality. Let $Z_k$ be the number of singleton balls. The expectancy of $Z_k$ is

$$\mu_k \geq k \left(1 - \frac{1}{\lg n}\right).$$

To estimate the deviation of $Z_k$ from its expected value we use the bounded differences inequality [29, 32]. Let $B_j$ be the bin of ball $b_j$, for $1 \leq j \leq k \lg n$. Then $Z_k$ is of the form $Z_k = h(B_1, \ldots, B_{k \lg n})$ where $h$ satisfied the Lipschitz condition with constant 2, because moving one ball to a different bin results in changing the value of $h$ by at most 2 with respect to the original value. The bounded-differences inequality specialized to this instance is as follows, for any $d > 0$:

$$\Pr(Z_k \leq \mathbb{E}[Z_k] - d\sqrt{k}) \leq \exp(-d^2/8). \quad (5)$$

We use this inequality for $d = \lg n$, which makes the right-hand side of (5) to be $n^{-\Omega(\log n)}$. As long as $k \geq \lg^4 n$, we have that

$$\mathbb{E}[Z_k] \geq k \left(1 - \frac{1}{\lg n}\right) = \Omega(\log^4 n),$$

which implies that $d\sqrt{k} = o(\mathbb{E}[Z_k])$. This means that the number of balls that transition to the next stage is $o(k)$ with a probability that is $1 - n^{-\Omega(\log n)}$. We conclude that if there are $k = n$ balls at the start then there remain $O(\log^4 n)$ balls eligible for next stages, while in each stage the number of eligible balls decreases at least geometrically with a probability that is at least $1 - n^{-\Omega(\log n)}$.

The second case occurs when $k \leq \lg^4 n$. Then we throw at most $\lg^4 n$ balls into $n$ bins. They all end up in singleton bins with a probability that is at least

$$\left(\frac{n - \lg^4 n}{n}\right)^{\lg^4 n} \geq \left(1 - \frac{\lg^4 n}{n}\right)^{\lg^4 n} \geq 1 - \frac{\lg^8 n}{n},$$

by the Bernoulli’s inequality. So the probability of a collision is at most $\frac{\lg^8 n}{n}$. One stage without any collision terminates the process. If we repeat such stages $\lg n$ times, without even removing singleton balls, then the probability of collisions occurring in all these stages is at most $(\frac{\lg^8 n}{n})^{\lg n} = n^{-\Omega(\log n)}$. The number of eligible balls summed over these stages is at most $\lg^5 n = o(n)$.

$\square$
Theorem 6 Algorithm Common-Constant-Known is Las Vegas. For any \( a > 0 \) there exist \( \beta > 0 \) and \( c > 0 \) such that the algorithm terminates within time \( cn \ln n \) rounds using at most \( cn \ln n \) random bits with a probability that is at least \( 1 - n^{-a} \).

Proof: The condition controlling the main repeat-loop guarantees that it terminates only when the assigned names are distinct. We consider the process of throwing balls into bins as analyzed in Lemma 5. There are \( \mathcal{O}(n) \) times a ball is thrown with the suitably high probability. Each such a ball is verified for a collision \( \beta \lg n \) times. It follows that one iteration of the outer repeat-loop suffices, conditional on all collisions detected. The probability that a collision of two balls is not detected is \( 2^{-\beta \lg n} = n^{-\beta} \), by Lemma 4. When \( \mathcal{O}(n) \) balls are thrown and each results in a collision then the probability to detect all these collisions is at least \( (1 - n^{-\beta}) \mathcal{O}(n) \geq 1 - n^{-\beta + \mathcal{O}(1)} \), for a specific constant under the big-Oh notation in the exponents. It follows that one iteration of the outer repeat-loop is sufficient with a probability that is at least \( 1 - n^{-\beta + \mathcal{O}(1)} \). Things go wrong with a probability that is at most \( n^{-\Omega(\log n)} \) by choosing a suitably large \( \beta \).

We observe that one iteration of the inner repeat-loop takes either \( \mathcal{O}(k \log n) \) steps, when \( k > \log^4 n \), or \( \mathcal{O}(n) \) steps otherwise. In the former case, the time of all such iterations sums to be \( \mathcal{O}(n \log n) \), because of the at-least-geometric decrease of the consecutive values of \( k \). In the latter case, the time of all iterations sums to be \( \mathcal{O}(n \log n) \) because each iteration takes \( \mathcal{O}(n) \) rounds and there are \( \mathcal{O}(\log n) \) iterations.

Algorithm Common-Constant-Known is optimal with respect to the following performance measures: the expected time \( \mathcal{O}(n \log n) \), by Theorem 1 and the expected number of random bits \( \mathcal{O}(n \log n) \), by Proposition 2.

5.2 Common with Unlimited Memory

Now we consider the case when PRAM is Common, the number of processors \( n \) is known, and the amount of available shared memory cells is unlimited in principle. We propose an algorithm called Common-Unlimited-Known, whose pseudocode is given in Figure 5. The algorithm invokes procedure Verify-Collision, whose pseudocode is in Figure 1.

An execution of the algorithm proceeds as a sequence of iterations of an unrestricted repeat-loop. One iteration begins with \( \lg n \) trials to place \( n \) balls into \((\beta + 1)n\) bins and verify for collisions. Once a collision is detected by a processor, another attempt to place its ball is made. A processor may proceed through a series of such selections. After the trials are over, ranks of nonempty bins are computed. This is accomplished by a prefix-type computation. If the number of ranks equals the number of balls \( n \), then the algorithm assigns ranks of bins as names to processors who own the singleton balls in these bins and terminates, otherwise the outer repeat loop is iterated again.

We consider auxiliary processes of placing balls into bins that abstracts operations on shared memory as performed by algorithm Common-Unlimited-Known.

Balls into bins. The ball \( \beta \)-process is about placing \( n \) balls into \((\beta + 1)\) bins. The process is structured as a sequence of stages. A stage represents an abstraction of one iteration of the inner for-loop in Figure 5 performed as if collisions were detected instantaneously and with certainty. When a ball is moved then it is placed in a bin selected uniformly at random, all such selections
Algorithm COMMON-UNLIMITED-KNOWN

\[ x \leftarrow \text{random integer in } [1, (\beta + 1)n] \]  \hspace{1cm} /* \text{throw a ball into bin } x \*/

repeat
  \text{for } i \leftarrow 1 \text{ to } \lg n \text{ do}
    \begin{align*}
      \text{if } \text{VERIFY-COLLISION}(x) \text{ then } x & \leftarrow \text{random integer in } [1, (\beta + 1)n] \\
      \text{Number-Occupied-Bins} & \leftarrow \text{the total number of currently selected values for } x
    \end{align*}
  \end{align*}
  \text{until } \text{Number-Occupied-Bins} = n

\text{name}_v \leftarrow \text{the rank of bin } x

Figure 5: A pseudocode for a processor \( v \) of a Common PRAM, where \( n \) is known and the number of shared memory cells is unlimited. The constant \( \beta \) is a parameter that satisfies the inequality \( \beta > 1 \). The private variable \text{name} stores the acquired name.

independent from one another. The stages are performed as follows. In the first stage, \( n \) balls are placed into \((\beta + 1)n\) bins. When a bin is singleton in the beginning of a stage then the ball in the bin stays put through the stage. When a bin is multiple in the beginning of a stage, then all the balls in this bin participate actively in this stage: they are removed from the bin and placed in randomly-selected bins. The process terminated after a stage in which all balls reside in singleton bins. It is convenient to visualize a stage as occurring by first removing all balls from multiple bins and then placing the removed balls in randomly selected bins one by one.

We associate the mimicking walk to each execution of the ball \( \beta \)-process. Such a walk is performed on points with integer coordinates on a line. The mimicking walk proceeds through stages, similarly as the ball process. When we are to relocate \( k \) balls in a stage of the ball process then this is represented by the mimicking walk starting the corresponding stage at coordinate \( k \). Suppose that we process a ball in a stage and the mimicking walk is at some position \( i \). Placing this ball in an empty bin decreases the number of balls for the next stage; the respective action in the mimicking walk is to decrement its position from \( i \) to \( i - 1 \). Placing this ball in an occupied bin increases the number of balls for the next stage; the respective action in the mimicking walk is to increment its position from \( i \) to \( i + 1 \). The mimicking walk gives a conservative estimates on the behavior of the ball process, as we show next.

Lemma 6 If a stage of the mimicking walk ends at a position \( k \), then the corresponding stage of the ball \( \beta \)-process ends with at most \( k \) balls to be relocated into bins in the next stage.

Proof: The argument is broken into three cases, in which we consider what happens in the ball process and what are the corresponding actions in the mimicking walk. A number of balls in a bin in a stage is meant to be the final number of balls in this bin at the end of the stage.

In the first case, just one ball is placed in a bin that begins the stage as empty. Then this ball will not be relocated in the next stage. This means that the number of balls for the next stage decreases by 1. At the same time, the mimicking walk decrements its position by 1.
In the second case, some \( j \geq 1 \) balls land in a bin that is singleton at the start of this stage. Then the number of balls in the bin becomes \( j + 1 \) and these many balls will need to be relocated in the next stage. Observe that this contributes to incrementing the number of the eligible balls in the next stage by 1, because only the original ball residing in the singleton bin will be added, while the other balls participate in both stages. At the same time, the mimicking walk increments its position \( j \) times by 1.

In the third and final case, some \( j \geq 2 \) balls land in a bin that is empty at the start of this stage. Then this does not contribute to a change in the number of balls eligible for relocation in the next stage, as these \( j \) balls participate in both stages. Let us consider these balls as placed in the bin one by one. The first ball makes the mimicking walk decrement its position. The second ball makes the walk increment its position, so that it returns to the original position as at the start of the stage. The following ball placements, if any, result in the walk incrementing its positions. □

Random walks. Next we consider a random walk which will estimate the behavior of a ball process. One component of estimation is provided by Lemma 6, in that we will interpret a random walk as a mimicking walk for the ball process.

The random walk is represented as movements of a marker placed on the non-negative side of the integer number line. The movements of the marker are by distance 1 and they are independent. The random \( \beta \)-walk has the marker’s position incremented with probability \( \frac{1}{\beta+1} \) and decremented with probability \( \frac{\beta}{\beta+1} \). This may be interpreted as a sequence of independent Bernoulli trials, in which \( \frac{\beta}{\beta+1} \) is chosen to be the probability of success. We will consider \( \beta > 1 \), for which \( \frac{\beta}{\beta+1} > \frac{1}{\beta+1} \), which means that the probability of success is greater than the probability of failure.

Such a random \( \beta \)-walk proceeds through stages, which are defined as follows. The first stage begins at position \( n \). When a stage begins at a position \( k \) then it ends after \( k \) moves, unless it reaches the zero coordinate in the meantime. The zero point acts as an absorbing barrier, and when the walk’s position reaches it then the random walk terminates. This is the only way in which the walk terminates.

**Lemma 7** For any numbers \( a > 0 \) and \( \beta > 1 \), there exists \( b > 0 \) such that the random \( \beta \)-walk starting at position \( n > 0 \) terminates within \( b \ln n \) stages with a probability that is at least \( 1 - n^{-a} \).

**Proof:** Let the random walk start at position \( k > 0 \) when a stage begins. Let \( X_k \) be the number of moves towards 0 and \( Y_k = k - X_k \) be the number of moves away from 0 in such a stage. The total distance covered towards 0 is

\[
L(k) = X_k - Y_k = X_k - (k - X_k) = 2X_k - k .
\]

The expected value of \( X_k \) is \( \mathbb{E}[X_k] = \frac{\beta k}{\beta + 1} = \mu_k \). The event \( X_k < (1 - \varepsilon)\mu_k \) holds with a probability at most \( \exp(-\frac{\varepsilon^2}{2}\mu_k) \), by the Chernoff bound \([32]\), so that \( X_k \geq (1 - \varepsilon)\mu_k \) occurs with the respective high probability. We say that such a stage is conforming when the event \( X_k \geq (1 - \varepsilon)\mu_k \) holds. If a stage is such then the following inequality holds:

\[
L(k) \geq 2(1 - \varepsilon)\frac{\beta k}{\beta + 1} - k = \frac{\beta - 2\beta\varepsilon - 1}{\beta + 1}k .
\]
We want the inequality $\frac{\beta - 2\beta \varepsilon - 1}{\beta + 1} > 0$, which holds when $\varepsilon < \frac{\beta - 1}{2\beta}$. Let us fix such $\varepsilon > 0$. Now the distance from 0 after $k$ steps starting at $k$ is

$$k - L(k) = \left(1 - \frac{\beta - 2\beta \varepsilon - 1}{\beta + 1}\right) \cdot k = \frac{2(1 + \beta \varepsilon)}{\beta + 1} \cdot k,$$

where $\frac{2(1 + \beta \varepsilon)}{\beta + 1} < 1$ for $\varepsilon < \frac{\beta - 1}{2\beta}$. Let $\rho = \frac{\beta + 1 + 2(1 + \beta \varepsilon)}{\beta + 1} > 1$.

When we start the first stage at position $n$ and the next $\log_{\rho} n$ stages are conforming then after these many stages the random walk ends up at a position that we claim only to be of distance at most $s \ln n$ from 0, for some $s > 0$. The event that all these stages are conforming and the bound $s \ln n$ on distance from 0 holds occurs with a probability that is at least

$$1 - \log_{\rho} n \cdot \exp\left(-\frac{\varepsilon^2}{2} \frac{\beta}{\beta + 1} \ln n\right) \geq 1 - \log_{\rho} n \cdot n^{-\frac{\varepsilon^2}{2} \frac{\beta}{\beta + 1} s}.$$

Let us choose $s > 0$ such that

$$\log_{\rho} n \cdot n^{-\frac{\varepsilon^2}{2} \frac{\beta}{\beta + 1} s} \leq \frac{1}{2n^a},$$

for sufficiently large $n$.

Having fixed $s$, let us take $t > 0$ such that the distance covered towards 0 is at least $s \ln n$ when starting from $k = t \ln n$ and performing $k$ steps. We interpret these movements as if this was a single conceptual stage for the sake of an argument, but its duration comprises all stages when we start from $s \ln n$ until we terminate at 0. It follows that the conceptual stage comprises at most $t \ln n$ real stages, because a stage takes at least one round.

If this last conceptual stage is conforming then the distance covered towards 0 is bounded by

$$L(k) \geq \frac{\beta - 2\beta \varepsilon - 1}{\beta + 1} \cdot k.$$

We want this to be at least $s \ln n$ for $k = t \ln n$, which is equivalent to

$$\frac{\beta - 2\beta \varepsilon - 1}{\beta + 1} \cdot t > s.$$

Now it is sufficient to take $t > s \cdot \frac{\beta + 1}{\beta - 2\beta \varepsilon - 1}$. This last conceptual stage is not conforming with a probability that is at most $\exp\left(-\frac{\varepsilon^2}{2} \frac{\beta}{\beta + 1} t \ln n\right)$. Let us take $t$ that is additionally big enough for the following inequality

$$\exp\left(-\frac{\varepsilon^2}{2} \frac{\beta}{\beta + 1} t \ln n\right) = n^{-\frac{\varepsilon^2}{2} \frac{\beta}{\beta + 1} t} \leq \frac{1}{2n^a}$$

to hold.

Having selected $s$ and $t$, we can conclude that there are at most $(s + t) \ln n$ stages with a probability that is at least $1 - n^{-a}$.

\[\square\]

**Lemma 8** For any numbers $a > 0$ and $\beta > 1$, there exists $b > 0$ such that the ball $\beta$-process starting at position $n > 0$ terminates within $b \ln n$ stages with a probability that is at least $1 - n^{-a}$.
Proof: We estimate the behavior of the ball $\beta$-process on $n$ balls by the behavior of the random $\beta$-walk starting at position $n$. The justification of the estimation is in two steps. One is the property of mimicking walks given as Lemmas 6. The other is provided by Lemma 7 and is justified as follows. The probability of decrementing and incrementing position in the random $\beta$-walk are such that they reflect the probabilities of landing in an empty bin or in an occupied bin. Namely, we use the facts that during executing the ball $\beta$-process, there are at most $n$ occupied bins and at least $\beta \cdot n$ empty bins in any round. In the ball $\beta$-process, a probability of landing in an empty bin is at least $\frac{\beta n}{(\beta + 1)m} = \frac{\beta}{\beta + 1}$, and a probability of landing in an occupied bin is at most $\frac{n}{(\beta + 1)m} = \frac{1}{\beta + 1}$. This means that the random $\beta$-walk is consistent with Lemma 6 in providing estimates on the time of termination of the ball $\beta$-process from above.

Incorporating verifications. We consider the random $\beta$-walk with verifications, which is defined as follows. The process proceeds through stages, similarly as the regular random $\beta$-walk. For any round of the walk and a position at which the walk is at, we first perform a Bernoulli trial with the probability $\frac{1}{2}$ of success. Such a trial is referred to as a verification, which is positive when a success occurs otherwise it is negative. After a positive verification a movement of the marker occurs as in the regular $\beta$-walk, otherwise the walk pauses at the given position for this round.

Lemma 9 For any numbers $a > 0$ and $\beta > 1$, there exists $b > 0$ such that the random $\beta$-walk with verifications starting at position $n > 0$ terminates within $b \ln n$ stages with a probability that is at least $1 - n^{-\alpha}$.

Proof: We provide an extension of the proof of Lemma 7 which states a similar property of regular random $\beta$-walks. That proof estimated times of stages. Suppose the regular random $\beta$-walk starts at a position $k$, so that the stage takes $k$ moves. There is a constant $d < 1$ such that the walk ends at a position at most $dk$ with a probability that is exponential in $k$. Moreover, the proof of Lemma 7 is such that all the values of $k$ considered are at least logarithmic in $n$, which provides at most a polynomial bound on error. A random walk with verifications is slowed down by negative verifications. Observe that a random walk with verifications that is performed $3k$ times undergoes at least $k$ positive verifications with a probability exponential in $k$ by the Chernoff bound [32]. This means that the proof of Lemma 7 can be adapted to the case of random walks with verifications almost verbatim, with the modifications contributed by polynomial bounds on error of estimates of the number of positive verifications in stages.

Next, we consider the ball $\beta$-process with verifications, which is defined as follows. The process proceeds through stages, similarly as the regular ball process. The first stage starts with placing $n$ balls into $(\beta + 1)n$ bins. For any following stage, we first go through multiple bins and for each ball in such a bin, except for the ball that landed first in this bin, we perform a Bernoulli trial with the probability $\frac{1}{2}$ of success, which we call a verification. A success in a trial is referred to as a positive verification otherwise it is a negative one. If at least one positive verification occurs for a ball in a multiple bin then all the balls in this bin are relocated in this stage to bins selected uniformly at random and independently for each such a ball, otherwise the balls stay put in this bin until the next stage.

Lemma 10 For any numbers $a > 0$ and $\beta > 1$, there exists $b > 0$ such that the ball $\beta$-process with verifications starting at position $n > 0$ terminates within $b \ln n$ stages with a probability that is at least $1 - n^{-\alpha}$. 22
Proof: The argument proceeds by combining Lemma 6 with Lemma 9, similarly as the proof of Lemma 8 is proved by combining Lemma 6 with Lemma 7. The details follow.

For any execution of a ball process with verifications, we consider a “mimicking random walk,” also with verifications, defined such that when a ball from a multiple bin is handled then the outcome of a random verification for this ball is mapped on a verification for the corresponding random walk. Observe that for a ball \( \beta \)-process with verifications just one positive verification is sufficient among \( j - 1 \) trials when there are \( j > 1 \) balls in a multiple bin, so a random \( \beta \)-walk with verifications provides an upper bound on time of termination of the ball \( \beta \)-process with verifications. The probabilities of decrementing and incrementing position in the random \( \beta \)-walk with verifications are such that they reflect the probabilities of landing in an empty bin or in an occupied bin, similarly as without verifications. All this gives a consistency of a \( \beta \)-walk with verifications with Lemma 6 in providing estimates on the time of termination of the ball \( \beta \)-process from above.

\[\square\]

Theorem 7 Algorithm Common-Unlimited-Known is Las Vegas. For each \( a > 0 \) and any \( \beta > 1 \) in the pseudocode, there exists \( c > 0 \) such that the algorithm assigns proper names within time \( c \ln n \) and using at most \( cn \lg n \) random bits with a probability that is at least \( 1 - n^{-a} \).

Proof: The repeat-loop is executed \( O(1) \) times with a probability that is at least \( 1 - n^{-a} \), by Lemma 10. The algorithm terminates when there are \( n \) different ranks, by the condition controlling the repeat-loop. As ranks are distinct and each in the interval \([1, n]\), by their definition, each name is unique. It follows that the algorithm terminates almost surely and so it is Las Vegas.

An iteration of the repeat-loop in Figure 5 takes \( O(\log n) \) steps. This is because of the following two facts. First, it consists of \( \lg n \) iterations of the for-loop, each taking \( O(1) \) rounds. Second, it concludes with verifying the until-condition, which is carried out by counting nonempty bins by a prefix-type computation. It follows that time until termination is \( O(\log n) \) and the number of used random bits is \( O(n \log n) \), all with a probability that is at least \( 1 - n^{-a} \).

Algorithm Common-Unlimited-Known is optimal with respect to the following performance measures: the expected number of random bits \( O(n \log n) \), by Proposition 2, the expected time \( O(\log n) \), by Theorem 3, and the expected number of shared memory cells \( O(n) \), by Theorem 1.

6 Arbitrary PRAM of Unknown Size

We consider the case of Arbitrary PRAM when \( n \) is unknown. This is broken into two sub-cases of either a constant amount of shared memory or of an unlimited supply of shared memory.

6.1 Arbitrary with Constant Memory

Algorithm Arbitrary-Constant-Unknown is for Arbitrary PRAM when \( n \) is unknown and there are a constant number of shared variables. Its pseudocode is given in Figure 6.

The algorithm is structured as a repeat-loop, each iteration for a specific integer \( k \) where \( k \) goes through the consecutive powers of 2. In an iteration, processors choose random integers in the range \([1, 2^k]\). The inner repeat-loop assigns names to processors. The shared variable Counter is used to go through consecutive integers when assigning names. A next number is assigned by
**Algorithm** **Arbitrary-Constant-Unknown**

initialize $k \leftarrow 1$  
/* initial approximation of $\lg n$ */

repeat
    initialize $\text{Counter} \leftarrow \text{name}_v \leftarrow 0$
    $k \leftarrow 2k$
    $x \leftarrow \text{random integer in } [1, 2^k]$  
    /* throw a ball into bin $x$ */
    repeat
        $\text{All-Named} \leftarrow \text{true}$
        if $\text{name}_v \leq 0$ then
            $\text{Pad} \leftarrow x$
            if $x = \text{Pad}$ then
                $\text{Counter} \leftarrow \text{Counter} + 1$ ; $\text{name}_v \leftarrow \text{Counter}$
            else $\text{All-Named} \leftarrow \text{false}$
        until $\text{All-Named}$
    until $\text{Counter} \leq 2^{k/\beta}$

Figure 6: A pseudocode for a processor $v$ of an Arbitrary PRAM, when the number of processors $n$ is unknown and there is a constant number of shared memory cells. The variables $\text{Counter}$, $\text{All-Named}$ and $\text{Pad}$ are shared. The private variable $\text{name}$ stores the acquired name. The constant $\beta > 0$ is a parameter to be determined by analysis.

all processors writing to shared $\text{Pad}$ and next reading if the same value as attempted to write is stored. The shared variable $\text{All-Named}$ is used to verify if all processors have names. The outer loop terminates when the number of the assigned names is less than or equal to $2^{k/\beta}$, where $\beta > 0$ is a parameter.

**Balls into bins.** We consider the following auxiliary ball $\beta$-process of throwing balls into bins, for a parameter $\beta > 0$. The process proceeds through stages. The $i$th stage has the number $k$ equal to the value of $k$ during the $i$th iteration of outer repeat-loop in algorithm **Arbitrary-Constant-Unknown**. During a stage $i$, with the corresponding $k = 2^i$, we first throw $n$ balls into $2^k$ bins and next count the number of occupied bins. A stage is last in an execution of the ball process when the number of occupied bins is less than or equal to $2^{k/\beta}$. The ball $\beta$-process terminates by the stage in which the inequality $n \leq 2^{k/\beta}$ holds, because $n$ is an upper bound on the number of occupied bins in a stage. The inequality $n^\beta \leq 2^k$ is equivalent to $\beta \lg n \leq k$. Since $k$ goes through consecutive powers of 2, we obtain that the number of stages is at most $\lg(\beta \lg n) = \lg \beta + \lg \lg n$.

**Lemma 11** For any $a > 0$ and a sufficiently large $\beta > 0$, the event that either the number of occupied bins is at most than $2^{k/\beta}$ or there are collisions, in a $i$th stage of the ball $\beta$-process such that $2^i = k$, holds with a probability that is at most $n^{-a}$, for all sufficiently large $n$.  

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Proof: The probability that there are at most $2^{k/\beta}$ occupied bins is at most

$$\left(\frac{2^k}{2^{k/\beta}}\right)^n \leq \left(\frac{e^{2^k}}{2^{k/\beta}}\right)^{2^k/\beta} 2^{k(\beta^{-1}-1)n}$$

$$\leq e^{2^{k/\beta}} \cdot 2^{k(1-\beta^{-1})2^{k/\beta}} \cdot 2^{k(\beta^{-1}-1)n}$$

$$\leq e^{2^{k/\beta}} \cdot 2^{k(\beta^{-1}-1)(n-2^{k/\beta})}. \quad (6)$$

Next we estimate from above the natural logarithm of the right-hand side of (6). We obtain the following upper bound:

$$2^{k/\beta} + k(\beta^{-1} - 1)(n - 2^{k/\beta}) \ln 2 < 2^{k/\beta} - \frac{1}{2}(n - 2^{k/\beta}) \ln 2$$

$$= 2^{k/\beta} - \frac{\ln 2}{2} n + \frac{\ln 2}{2} 2^{k/\beta}$$

$$= -\frac{\ln 2}{2} n + 2^{k/\beta} \cdot 2 + \ln 2 \cdot 2. \quad (7)$$

The estimate (7) is at most $-n \cdot \frac{\ln 2}{4}$ when $2^{k/\beta} < n \cdot \delta$, for $\delta = \frac{\ln 2}{2(2+\ln 2)}$, by a direct algebraic verification. The condition on $k$ can be restated as

$$k < \beta \lg(n\delta) \ . \quad (8)$$

When this condition (8) is satisfied, then the probability of at most $2^{k/\beta}$ occupied bins is at most

$$\exp\left(-n \cdot \frac{\ln 2}{4}\right) \leq \frac{n^{-a}}{2}$$

for sufficiently large $n$.

Next let us consider the probability that collisions do not occur, when $k \geq \lg n$. This probability is at least

$$\left(1 - \frac{n}{2^k}\right)^n \geq 1 - \frac{n^2}{2^k} ,$$

by the Bernoulli’s inequality. It follows that the probability of a collision can be bounded from above by $\frac{n^2}{2^k}$. This bound in turn is at most $\frac{n^a}{2^k}$ when

$$k > (2 + a) \lg n \ . \quad (9)$$

In order to have the inequalities (8) and (9) cover all possible $k$, it is sufficient to have

$$(2 + a) \lg n < \beta \lg(n\delta) ,$$

which means $\beta > \frac{(2 + a) \lg n}{\lg(n\delta)}$. Since $\frac{\lg n}{\lg(n\delta)} < 2$ for sufficiently large $n$, the inequality $\beta > 2(2 + a)$ suffices when $n$ is large enough. \quad \square

When we sum up the numbers of occupied bins over all the stages then the result is the number of bins occupied ever in an execution.

**Lemma 12** For each $a > 0$ there exists $\beta > 0$ and $c > 0$ such that when the ball $\beta$-process terminates then the number of bins occupied ever is at most $cn$ and the number of random bits used is at most $cn \ln n$, all this with a probability that is at least $1 - n^{-a}$.  

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Proof: As \( k \) ranges from 2 to \( \log n \) through the stages, then the numbers of available bins increase quadratically through the stages because \( k \) is doubled when we transition to the next stage. This means that the total number of all these bins is \( O(n) \). At the same time, the number of random bits increases geometrically through the stages, so the total number of random bits a processor uses is \( O(\log n) \). As \( k \) ranges from \( \log n \) to \( \beta \log n \), the number of occupied bins is at most \( n \) in each stage. There are only \( \log(\beta + 1) \) such stages so the total number of all these bins is \( \log(\beta + 1)n \). At the same time, a processor uses at most \( \log(\beta + 1)\log n \) random bits in all these stages. \( \square \)

We map an execution of the algorithm into an execution of the ball \( \beta \)-process in order to apply Lemmas 11 and 12 to the proof of the following fact.

**Theorem 8** Algorithm Arbitrary-Constant-Unknown is Monte Carlo. For each \( a > 0 \) there exists \( \beta > 0 \) and \( c > 0 \) such that the algorithm assigns unique names, works in time at most \( cn \), and uses at most \( cn \log n \) random bits, all this with a probability that is at least \( 1 - n^{-a} \).

Proof: There is a direct correspondence between iterations of the outer repeat-loop and stages of the ball process. An error in the ball process represents assigning the same name to two different processors. The number of bins occupies in a stage corresponds to the number of times the inner repeat-loop is iterated, because executing instruction \( \text{Pad} \leftarrow x \) eliminates one bin. When the algorithm terminates then Counter stores the number of the assigned names. It follows that the termination condition of the algorithm corresponds to the termination condition of the ball \( \beta \)-process. We conclude that Lemmas 11 and 12 apply. This implies the following: the names are correct and execution terminates in \( O(n) \) time while \( O(n \log n) \) bits are used with a probability that is at least \( 1 - n^{-a} \). \( \square \)

Algorithm Arbitrary-Constant-Unknown is optimal with respect to the following performance measures: the expected time \( O(n) \), by Theorem 2, the expected number of random bits \( O(n \log n) \), by Proposition 2, and the probability of error \( n^{-O(1)} \), by Proposition 3.

### 6.2 Arbitrary with Unlimited Memory

Algorithm Arbitrary-Unlimited-Unknown is designed for Arbitrary PRAM when there is an unlimited amount of shared registers and the number of processors \( n \) is unknown. The pseudocode for this algorithm is given in Figure 7.

The algorithm is structured as a repeat-loop. The parameter \( k \) equals the iteration’s number. In one iteration, a processor uses two numbers \( x \in [1, 2^k/k] \) and \( y \in [1, 2^{\beta k}] \) selected uniformly at random. For a given \( k \), the selections among the processors are independent but we do not require independence of selections for different values of \( k \). This allows to recycle the random bits for \( k \) and add only the needed amount of new random bits when moving to the next iteration \( k + 1 \). We interpret \( x \) as a bin’s number and \( y \) as a label for a ball. Processors write their values \( y \) into the respective bin by instruction \( \text{Bin}[x] \leftarrow y \) and verify what value got written. After a successful write, a processor increments \( \text{Counter}[x] \) and assigns \( (x, \text{Counter}[x]) \) as its position. This is repeated \( \beta k \) times. The main repeat-loop terminates when all values attempted to be written actually got written. Then processors assign themselves names according to the ranks of positions.

**Balls into bins.** We consider a related process of placing labeled balls into bins. The process is referred to as ball \( \beta \)-process and proceeds through stages. In the \( k \)th stage, we place \( n \) balls...
Algorithm **ARBITRARY-UNLIMITED-UNKNOWN**

```
initialize $k \leftarrow 0$ /* initial approximation of $\lg n$ */
repeat
    initialize All-Named $\leftarrow$ true ; position$_v$ $\leftarrow (0,0)$
    $k \leftarrow k + 1$
    $x \leftarrow$ random integer in $[1, 2^k/k]$ /* choose a bin */
    $y \leftarrow$ random integer in $[1, 2^{\beta k}]$ /* choose a label for the ball */
    for $i \leftarrow 1$ to $\beta k$ do if position$_v$ $= (0,0)$ then
        Bin$_[x]$ $\leftarrow y$
        if Bin$_[x] = y$ then
            Counter$_[x]$ $\leftarrow$ Counter$_[x]$ + 1 ; position$_v$ $\leftarrow (x, Counter$_[x]$)$
        if position$_v$ $= (0,0)$ then All-Named $\leftarrow$ false
    until All-Named
name$_v$ $\leftarrow$ the rank of position$_v$
```

Figure 7: A pseudocode for a processor $v$ of an Arbitrary PRAM, when the number of processors $n$ is unknown and the number of shared memory cells is unlimited. The variable Bin is an array of shared memory cells, the variables All-Named and NumberOfOccupiedBins are also shared. The private variable name stores the acquired name. The constant $\beta > 0$ is a parameter to be determined by analysis.

into $2^k/k$ bins, with labels taken from $[1, 2^{\beta k}]$. A stage terminates the process when there are at most $\beta k$ labels of balls in each bin. We say that a labeled collision occurs, in a configuration produced by the process, if some bin contains two balls with the same label. Random bits used to make selections in a stage $k$ are reused in the next stage $k + 1$ and augmented only by the minimum number of fresh bits needed, when all the new bits are selected uniformly at random and independently.

**Lemma 13** For each $a > 0$ there exists $\beta > 0$ and $c > 0$ such that the ball $\beta$-process terminates by stage number $c \ln n$, at most $cn \ln n$ random bits were used, and no labeled collision occurs with a probability that is at least $1 - n^{-a}$.

**Proof:** First, we show that when $k$ is suitable small then the probability of at most $\beta k$ different labels in all bins is small. There are at least $\frac{nk}{2^k}$ balls in some bin, by the pigeonhole principle. We consider these balls and their labels. The probability that all these balls have at most $\beta k$ labels is
at most
\[
\left(\frac{2^{\beta k}}{\beta k}\right)^{\frac{n_k}{2k}} \leq \left(\frac{e^{2^{\beta k}}}{\beta k}\right)^{\beta k} \cdot \left(\frac{\beta k}{2^{\beta k}}\right)^{\frac{n_k}{2k}} \\
\leq e^{\beta k} \cdot 2^{\beta k (\beta k - \frac{n_k}{2k})} \cdot (\beta k)^{\frac{n_k}{2k} - \beta k} \\
\leq e^{\beta k} \cdot \left(\frac{\beta k}{2^{\beta k}}\right)^{\frac{n_k}{2k} - \beta k}
\]
(10)

We want to show that this is at most \(n^{-a}/3\). We compare natural logarithms of \(n^{-a}/3\) and the right-hand side of (10). We want the following equivalent in equalities:
\[
\beta k + \left(\frac{n_k}{2k} - \beta k\right) (\ln(\beta k) - \beta k \ln 2) \leq -a \ln n - \ln 3 \]
\[
\beta k + a \ln n + \ln 3 \leq k \left(\frac{n_k}{2k} - \beta\right) (\beta k \ln 2 - \ln(\beta k))
\]

We use the inequality \(\beta k \ln 2 - \ln(\beta k) \geq \frac{\beta k}{2},\) which holds for all \(k\) and sufficiently large \(\beta\). We continue with the following sufficient inequalities
\[
\beta k + a \ln n + \ln 3 \leq \beta k^2 \left(\frac{n_k}{2k} - \beta\right)
\]
\[
\frac{2}{k} + \frac{2(a \ln n + \ln 3)}{\beta k^2} \leq \frac{n_k}{2k} - \beta.
\]
(11)

Observe that the left-hand side of (11) is less than 1 for sufficiently large \(k\) such that \(k = O(\sqrt{\log n})\), while the right-hand side is at least 1 when \(2^k \leq \frac{n}{2k^4}\).

Second, we show that there are at most \(\beta k\) balls in each bin with a high probability when \(k\) is suitably large. Let us throw balls one by one. Take a specific bin, and let \(p_k = \frac{k}{2^k}\) be the probability of a ball falling into this bin. Let \(S_n\) be the number of successes. We will use (2) for \(p = p_k\) and \(r = \beta k\); for its applicability we need \(r > np = \frac{n_k}{2k}\), which holds when \(2^k > n\), and which we assume.

The formula (2) takes the form
\[
\Pr(S_n \geq \beta k) \leq b(r; n, p) \cdot \frac{\beta k (1 - \frac{k}{2^k})}{\beta k - \frac{n_k}{2k}}
\]
\[
= b(r; n, p) \cdot \frac{\beta (1 - \frac{k}{2^k})}{\beta - \frac{n_k}{2k}}
\]
\[
= b(r; n, p) \cdot \beta \cdot \frac{2^k - k}{\beta 2^k - n},
\]
which is at most \(2 \beta b(r; n, p)\). The formula \(\beta b(r; n, p)\) reads as follows
\[
b(\beta k; n, p) = \left(\frac{n_k}{\beta k}\right) \left(\frac{k}{2^k}\right)^{\beta k} \left(1 - \frac{k}{2^k}\right)^{n-\beta k}.\]
(12)
We substitute to further estimate

\[ 2\beta b(r; n, p) \leq 2\beta \left( \frac{n}{\beta k} \right) \left( \frac{k}{2^k} \right)^\beta k \left( 1 - \frac{k}{2^k} \right)^{n - \beta k} \]

\[ \leq 2\beta \left( \frac{en}{\beta k} \right)^{\beta k} \left( \frac{k}{2^k} \right)^{\beta k} \left( 1 - \frac{k}{2^k} \right)^n \left( \frac{2^k}{2^k - k} \right)^\beta k \]

\[ < 2\beta \left( \frac{en}{\beta (2^k - k)} \right)^{\beta k} e^{-\frac{k^2}{2^k}} \]

which is at most \( 2\beta 2^{-\beta k} \) when \( \beta(2^k - k) \geq 2en \); the smallest such \( k \) satisfies \( k = \lg n + \mathcal{O}(1) \). It follows that some bin has at least \( \beta k \) balls with a probability that is at least \( 2\beta n^{2 - \beta k} \leq \frac{n^a}{3} \). This holds for \( k \) such that \( 2^k > n \) but \( k = \lg n + \mathcal{O}(1) \).

Third, the probability of all balls obtaining different labels is at least

\[ \left( 1 - \frac{n}{2^\beta k} \right)^n \geq 1 - \frac{n^2}{2^\beta k}, \]

by the Bernoulli’s inequality. So the probability of two different balls obtaining the same label is at most \( \frac{n^2}{2^\beta k} \). This can be made at most \( \frac{n^a}{3} \) for \( k = \lceil \lg n \rceil + 1 \) and a sufficiently large \( \beta \). \( \square \)

**Theorem 9** Algorithm Arbitrary-Unlimited-Unknown is Monte Carlo. For any \( a > 0 \), there exist \( \beta > 0 \) and \( c > 0 \) such that the algorithm works within time \( c \ln^2 n \), uses at most \( cn \ln n \) random bits, writes to a range of at most \( cn/\ln n \) shared memory cells, and assigns proper names, all this with a probability that is at least \( 1 - n^{-a} \).

**Proof:** We resort to Lemma [13] which summarizes the needed facts about the ball \( \beta \)-process. It implies that \( \mathcal{O}(\log n) \) iterations of the main repeat-loop suffice with a probability that is at least \( 1 - n^{-a} \). One iteration takes time that is proportional to \( k \), which is \( \mathcal{O}(\log n) \). It follows that the total time is \( \mathcal{O}(\log^2 n) \) with the respective probability. The number of random bits used per processor in the last iteration is \( \mathcal{O}(\log n) \). These random bits accumulate through all the iterations, so this bound applies throughout. \( \square \)

Algorithm Arbitrary-Unlimited-Unknown is optimal with respect to the following performance measures: the expected number of random bits \( \mathcal{O}(n \log n) \), by Proposition 2 and the probability of error \( n^{-\mathcal{O}(1)} \), by Proposition 3. The expected time of algorithm Arbitrary-Unlimited-Unknown is quadratic in \( \log n \) so we cannot claim time optimality by Theorem 3. We develop a Monte Carlo algorithm for Common PRAM with unlimited shared memory and \( n \) unknown in Section 7.2 which operates in the expected optimal time \( \mathcal{O}(\log n) \) and can be performed on Arbitrary PRAM with unlimited memory. Given the expected time \( \mathcal{O}(\log^2 n) \) of algorithm Arbitrary-Unlimited-Unknown, the expected number of shared memory optimal for this time would need to be \( \mathcal{O}(n/\log^2 n) \), if Theorem 2 were to be used, while it is only \( \mathcal{O}(n/\log n) \). All this means that the algorithm misses both time optimality and shared-space optimality by factors that are \( \mathcal{O}(\log n) \) each.
Algorithm **COMMON-CONSTANT-UNKNOWN**

```plaintext
initialize Settled ← 0
repeat
  (N,Tentative) ← ESTIMATE-SIZE
  CHECK-NAMES(N,Settled,Tentative,N\lg N)
  initialize ℓ ← 1
  while ℓ < β\lg N and CHECK-NAMES(N,Settled,Tentative,N)
    do ℓ ← ℓ + 1
  until ℓ < β\lg N /* last CHECK-NAMES did not detect any collision */
```

Figure 8: A pseudocode for a processor $v$ of a Common PRAM, where $n$ is unknown and there is a constant number of shared memory cells. Procedures ESTIMATE-SIZE and CHECK-NAMES have their pseudocodes in Figures 9 and 10 respectively. The private variable `name` stores the acquired name. The constant $\beta > 0$ is a parameter to be determined by analysis, which is inherited by procedure CHECK-NAMES.

### 7 Common PRAM of Unknown Size

We consider Common PRAM when $n$ is unknown. This is broken into two sub-cases depending on whether the amount of shared memory is $\mathcal{O}(1)$ or it is unlimited in principle.

#### 7.1 Common with Constant Memory

Algorithm **COMMON-CONSTANT-UNKNOWN** has its pseudocode in Figure 8. It calls two procedures ESTIMATE-SIZE and CHECK-NAMES, whose pseudocodes are given in Figures 9 and 10 respectively. The shared variables have the following meaning: $N$ is an approximation of $n$, Settled is an upper bound on those names that have been settled, and Tentative is the maximum tentative name assigned.

The purpose of procedure ESTIMATE-SIZE is to estimate the unknown number of processors $n$, which it returns as $N$, and assign tentative names up to a maximum value, which it returns as Tentative. The procedure tries consecutive values of $k$ as approximations of $\lg n$. For a given $k$, an experiment is carried out to throw $n$ balls into $k2^k$ bins. The execution stops when the number of occupied bins is at most $2^k$, which is then treated as an approximation of $n$. Tentative names are assigned to processors in the order of bins they selected. It is expected that the produced values of $N$ and Tentative are different, as no verifications for collisions are performed.

**Lemma 14** Procedure ESTIMATE-SIZE returns an estimate of $n \geq 24$ in the form $2^k$ for integer $k > 0$ such that the inequalities $\frac{n}{3} < 2^k < 2n$ hold with a probability that is $1 - 2^{-\Omega(n)}$.

**Proof:** The inequality $2^k < 2n$ occurs with certainty, as it follows from the observation that $n$ is an upper bound on the number of occupied bins, which implies a general estimate on the maximum
**procedure** \texttt{Estimate-Size}

\begin{verbatim}
initialize \( k \leftarrow 0 \)  
/\* initial approximation of \( \lg n \) /\*
repeat
\( k \leftarrow k + 1 \)
slot_v \leftarrow \text{random integer in} \ [1, k2^k] 
initialize \( \text{Counter} \leftarrow 0 \)
for \( i \leftarrow 1 \) \text{to} \ k2^k \text{do if} \ slot_v = i \text{then} 
\( \text{Counter} \leftarrow \text{Counter} + 1 \); \ slot_v \leftarrow \text{Counter} 
until \text{Counter} \leq 2^k 
/\* \text{Counter} \text{ equals maximum assigned tentative name} /\*
return \((2^k, \text{Counter})\)  
/\* 2^k \text{ is an approximation of } n /\*
\end{verbatim}

Figure 9: A pseudocode for a processor \( v \) of a Common PRAM. This procedure is invoked by algorithm \texttt{Common-Constant-Unknown} in Figure 8. The variable \( \text{Counter} \) is shared.

value of \( k \) that is possible to reach. We consider two cases. If \( n \) is a power of 2, say \( n = 2^i \), then the procedure terminates by the time \( i = k \), so that \( 2^k < 2^{i+1} = 2n \). Otherwise, the maximum possible \( k \) equals \( \lceil \lg n \rceil \), because \( 2^{\lfloor \lg n \rfloor} < n < 2^{\lceil \lg n \rceil} \). This gives \( 2^{\lceil \lg n \rceil} = 2^{\lfloor \lg n \rfloor + 1} < 2n \).

Now we estimate the lower bound on \( 2^k \). Let us assume that \( 2^k \leq \frac{n}{3} \). Then \( n \) balls fall into at most \( 2^k \) bins with a probability that is at most

\[
\binom{k2^k}{2^k} \left( \frac{2^k}{k2^k} \right)^n \leq \left( \frac{ek2^k}{2^k} \right)^{2^k} \cdot \frac{1}{k^n} = (ek)^{2^k} k^{-n} = e^{2^k} k^{-n} \leq e^{n/3} k^{2k - n} < e^{-n/3},
\]

which holds when integer \( k \) satisfies \( k > e \), so that \( n \geq 24 \). \( \square \)

Procedure \texttt{Check-Names} verifies tentative names for collisions, and resets such names when a collision is detected. In its list of parameters \((N, \text{Settled}, \text{Tentative}, \text{Offset})\), the first three, namely, \( N \), \text{Settled}, and \text{Tentative} are shared variables with the same names, while \text{Offset} is not. The parameter \text{Offset} determines the size of an interval from which integers are drawn randomly, and is set either to \( N \) or to \( N \lg N \) in the algorithm, see Figure 8. An execution resets the shared variable \text{Settled} to the upper boundary of the range of names that have been positively verified for lack of collisions and \text{Tentative} to the maximum name assigned tentatively. When \( v \) detects a collision then it first sets \( \text{slot}_v \) to a randomly selected integer from an interval of \text{Offset} many integers otherwise to a temporary name in the range between \text{Settled} + 1 and \text{Tentative}.

**Balls into bins.** We consider a balls-into-bins process, which is similar to that in Section 5.1 where the problem of assigning names on a Common PRAM with constant memory and for a known size \( n \) was addressed. We proceed through stages. In a stage, we have a number of eligible balls which we throw into a number of bins. In each stage, after throwing balls into bins, we remove balls from singleton bins and those that are in multiple bins become eligible for the next stage. We use \( N \) as an estimate of \( n \) produced by \texttt{Estimate-Size}. In the first stage, we throw \( n \) eligible balls
procedure Check-Names(N, Settled, Tentative, Offset)

    initialize Some-Collision ← collision_v ← false; Counter ← Settled
    for i ← Settled + 1 to Tentative do if slot_v = i then
        for j ← 1 to β lg N do if VERIFY-COLLISION then
            Some-Collision ← collision_v ← true
        if collision_v then
            slot_v ← random integer in [Tentative + 1, Tentative + Offset]
        else
            Counter ← Counter + 1; name_v ← slot_v ← Counter
    Settled ← Counter
    for ℓ ← Tentative + 1 to Tentative + Offset do if ℓ = slot_v then
        Counter ← Counter + 1; slot_v ← Counter
    Tentative ← Counter
    return Some-Collision /* true only when some collision detected */

Figure 10: A pseudocode for a processor v of a Common PRAM. This procedure invokes procedure VERIFY-COLLISION, whose pseudocode is in Figure 11 and is invoked by algorithm COMMON-CONSTANT-UNKNOWN in Figure 8. The variables N, Settled, Tentative, Counter, and Some-Collision are shared. The private variable name stores the acquired name. The constant β > 0 is the same as in Figure 8.

into N lg N bins. In the following stages, we use N bins only. The first stage in which no multiple bins are produced terminates the process.

Lemma 15 The number of times a ball is thrown summed over all stages is $O(n)$ with a probability that is $1 - n^{-\Omega(log n)}$.

Proof: The proof is similar to that of Lemma 5. The key element of the argument is that in each stage the number of bins exceeds the number of balls by a logarithmic factor. In particular, in the first stage, we throw $n$ balls into at least $\frac{2}{3}N$ bins, by Lemma 14. Then, in each of the subsequent stages, we throw $O(n/log n)$ balls into at least $\frac{2}{3}N$ bins, with the respective high probability. The probabilistic arguments as given in the proof of Lemma 5 apply, with the suitable small modifications. □

Theorem 10 Algorithm COMMON-CONSTANT-UNKNOWN is Monte Carlo. For any $a > 0$, there exist $\beta > 0$ and $c > 0$ such that the algorithm works within time $cn \ln n$ using at most $cn \ln n$ random bits and assigns proper names, all this with a probability that is at least $1 - n^{-a}$. 32
Algorithm **Common-Unlimited-Unknown**

```plaintext
initialize \( k \leftarrow 1 \)  
/* initial approximation of \( \lg n \) */
repeat
    \( k \leftarrow 2k \)
    \( x \leftarrow \text{random integer in } [1, 2^k] \)  
    /* choose a bin */
    \( \text{Number-Occupied-Bins} \leftarrow \text{the total number of selected values for } x \)
until \( \text{Number-Occupied-Bins} \leq 2^{k/\beta} \)
\( \text{name}_v \leftarrow \text{the rank of bin } x \)
```

Figure 11: A pseudocode for a processor \( v \) of a Common PRAM, when the number of processors \( n \) is unknown and the number of shared memory cells is unlimited. The private variable \( \text{name} \) stores the acquired name. The constant \( \beta > 0 \) is a parameter to be determined by analysis.

**Proof:** We assume the inequalities \( \frac{n}{3} < N < 2n \), in accordance with Lemma 14. Procedure **Estimate-Size** takes \( \mathcal{O}(n \log n) \) time because

\[
\sum_{k=1}^{\lg n + \mathcal{O}(1)} k2^k = \mathcal{O}(\log n) \quad \sum_{k=1}^{\lg n + \mathcal{O}(1)} 2^k = \mathcal{O}(n \log n) .
\]

When **Check-Names** is called with parameters \((N, \text{Settled}, \text{Tentative}, \text{Offset})\) then \( \text{Settled} \) is at most \( \text{Tentative} \), which is at most \( n \), and \( N \) is at most \( 2n \). It follows that the procedure operates in time \( \mathcal{O}((\text{Tentative} - \text{Settled}) \cdot \log N + \text{Offset}) \). Observe that the contributions of the part \((\text{Tentative} - \text{Settled}) \cdot \log N\) over all executions of this procedure contribute \( \mathcal{O}(n \log n) \) to time bound. This is because their sum is upper bounded by the total number of ball throws times \( \log N \), where the number of ball throws is bounded as in Lemma 15. Suppose there are at most \( dn \) balls thrown, with the respective high probability, so at most these many times balls participate in collisions. Some collision is not detected with a probability that is at most \( dn2^{-\beta \lg n} = dn^{-\beta+1} \), which can be made smaller than \( n^{-a} \) for \( \beta > a + 1 \) and sufficiently large \( n \).

Next we consider the contributions of the part \( \text{Offset} \) to time. The first call of **Check-Names** contributes \( \mathcal{O}(n \log n) \) because \( \text{Offset} = N \lg N \). The while-loop also contributes \( \mathcal{O}(n \log n) \) in total, because there are \( \mathcal{O}(\log n) \) iterations, each contribution suitably bounded by \( \text{Offset} = N \).  

### 7.2 Common with Unlimited Memory

We assume that the amount of shared memory of a Common PRAM is not restricted and \( n \) is unknown. Algorithm **Common-Unlimited-Unknown** has its pseudocode given in Figure 11. This is the only algorithm for the Common PRAM among those we give, in which procedure **Verify-Collision** is not used.
The algorithm is structured as a repeat loop. In each iteration, we double the integer parameter $k$, so we start from $k = 2$. We interpret an iteration as beginning with throwing $n$ balls into $2^k$ bins. Once in an iteration of the repeat-loop, we count the number values for $x$ randomly selected in this iteration, which is stored in the shared variable $Number$-$Occupied$-$Bins$. The ranks of occupied bins are assigned as names to the processors that own the balls in these bins. We consider the same ball $\beta$-process to throw balls into bins as in Section 6.1, where $\beta > 0$ is a parameter.

**Theorem 11** Algorithm **Common-Unlimited-Unknown** is Monte Carlo. For any $a > 0$ there exist $\beta > 0$ and $c > 0$ such that the algorithm works within time $cn \ln n$ and uses at most $cn \ln n$ random bits, and it assigns proper names with a probability that is at least $1 - n^{-a}$.

**Proof:** Take a $\beta > 0$ that exists by Lemma 11. For this $\beta$, the probability of termination with duplicate names is at most $n^{-a}$. The repeat-loop terminates by the time $k$ satisfies the inequality $k \geq \beta \lg n$. For a given $k$, the time to count the number of selected values of $x$ is $O(k)$. As $k$ grows geometrically up to $\beta \lg n$, we obtain that the total time spent to count such selections and to assign final ranks is $O(\log n)$, where the big-Oh constant depends on $\beta$. The estimate on the number of random bits follows from Lemma 12. \qed

Algorithm **Common-Unlimited-Unknown** is optimal with respect to the following performance measures: the expected number of random bits $O(n \log n)$, by Proposition 2, and the probability of error $n^{-O(1)}$, by Proposition 8. It requires $n^\beta$ shared memory cells, for a suitably large $\beta > 0$, so we cannot claim it is optimal with respect to the amount of shared memory by resorting to Theorem 11 and the algorithm may miss shared space optimality by a factor that is polynomial in $n$. Algorithm **Common-Unlimited-Unknown** assumes that a range $[1, n^\beta]$ of shared memory cells is available, while the expected number of cells written-to is $O(n)$, by Lemma 12, so a majority of these $n^\beta$ cells are not accessed with a large probability. A related algorithm **Arbitrary-Unlimited-Unknown** given in Section 6.2 operates in $O(\log^2 n)$ time and misses shared-space optimality by a factor that is $O(\log n)$.

**8 Conclusion**

It is an open problem to develop Monte Carlo algorithms for Arbitrary and Common PRAMs, for the case when $n$ is unknown and there is unlimited amount of shared memory, such that they are asymptotically optimal with respect to the following three metrics: the expected time, the expected amount of shared memory that may be referred to, and the expected number of random bits. Here “referring to” a shared memory means that the memory may be written to, depending on selections of random bits, so that a lack of memory cells of some addresses may result in a runtime error.

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