Ground state solutions for non-autonomous fractional Choquard equations

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Abstract

We consider the following nonlinear fractional Choquard equation,
\[
\begin{cases}
(\Delta y)u + u = (1 + a(x))(I_{\alpha}(|u|^p))|u|^{p-2}u & \text{in } \mathbb{R}^N, \\
u(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]
\[(0.1)\]

here \(s \in (0, 1), \alpha \in (0, N), p \in [2, \infty)\) and \(\frac{N-2s}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}\). Assume \(\lim_{|x| \to \infty} a(x) = 0\) and satisfying suitable assumptions but not requiring any symmetry property on \(a(x)\), we prove the existence of ground state solutions for \((0.1)\).

Keywords: stationary Choquard equation, stationary nonlinear Schrödinger–Newton equation, stationary Hartree equation, Riesz potential, concentration compactness

Mathematics Subject Classification numbers: 35J20, 35J60, 35Q40

1. Introduction

The nonlinear Choquard or Choquard–Pekar equations are of form
\[
\begin{cases}
(\Delta)u + u = (I_{\alpha}(|u|^p))|u|^{p-2}u & \text{in } \mathbb{R}^N, \\
u(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]
\[(1.1)\]

Here \(\alpha \in (0, N), p \in (1, \infty), I_{\alpha} : \mathbb{R}^N \to \mathbb{R}\) is the Riesz potential defined by

\[I_{\alpha}(f)(x) = \frac{1}{\omega_{n-\alpha}} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n,\]

where \(\omega_{n-\alpha}\) is the surface measure of the unit \((n-\alpha)\)-dimensional sphere.

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and $\Gamma$ is the Gamma function, see [36]. It is well known that if $u$ solves (1.1) when $\alpha = 2$, $N \geq 3$, then $(u, v) = (u, I_\alpha * |u|^p)$ satisfies the system

\begin{align*}
-\Delta u + u &= v |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \\
-\Delta v &= |u|^p \quad \text{in } \mathbb{R}^N, \\
u(x) &\to 0 \quad \text{as } |x| \to \infty, \\
v(x) &\to 0 \quad \text{as } |x| \to \infty.
\end{align*}

Equation (1.1) has several physical origins. In the case $N = 3, p = 2$ and $\alpha = 2$, the problem

\begin{align*}
-\Delta u + u &= (I_2 * |u|^2)u \quad \text{in } \mathbb{R}^3, \\
u(x) &\to 0 \quad \text{as } |x| \to \infty
\end{align*}

appeared in [35] by Pekar when he described the quantum mechanics of a polaron. In 1976, Choquard used (1.3) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma, see [23]. In [30], Moroz, Penrose and Tod proposed (1.3) as a model of self-gravitating matter in a programme in which quantum state reduction was understood as a gravitational phenomenon. Equations of type (1.1) are usually called the nonlinear Schrödinger-Newton equation. If $u$ solves (1.1), then the function $\psi$ defined by

$$\psi(t, x) = e^{it} u(x)$$

is a solitary wave solution of the focusing time-dependent Hartree equation

$$i\psi_t = -\Delta \psi - (I_\alpha * |\psi|^p)\psi |\psi|^{p-2} \psi \text{ in } \mathbb{R}_+ \times \mathbb{R}^N.$$ 

So (1.1) is also known as the stationary nonlinear Hartree equation.

In [23], Lieb proved that the ground state of (1.3) is radial and unique up to translations; later, in [25], Lions proved the existence of infinitely many radially symmetric solutions to (1.3); in [40], Wei and Winter showed the nondegeneracy of the ground state and studied the multi-bump solutions for (1.3); in [29], Ma and Zhao proved, under some assumptions on $N$, $\alpha$ and $p$, that every positive solution of (1.1) is radially symmetric and monotone decreasing about some point by the method moving planes in an integral form developed in [8]; in [9], Cingolani, Clapp and Secchi proved some existence and multiplicity results, and established the regularity and some decay asymptotics at infinity of the ground states for (1.1) in the electromagnetic case. In [33], Moroz and Schaftingen considered problem (1.1). They eliminated the restriction of [29], proved the regularity, positivity and radial symmetry of the ground states for optimal range of parameters. They also derive the decay asymptotics at infinity of the ground states. In [34], Moroz and Schaftingen showed that for some values of the parameters, (1.1) does not have nontrivial nonnegative super solutions in exterior domains. The same techniques yield optimal decay rates when super solutions exist. In [31], Moroz and Schaftingen proved the existence of ground state solutions to the nonlinear Choquard equation

\begin{align*}
-\Delta u + u &= (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^N, \\
u(x) &\to 0 \quad \text{as } |x| \to \infty
\end{align*}

under almost necessary conditions on the nonlinearity $F(u)$ in the spirit of Berestycki and Lions in [5]. In [15], Clapp and Salazar considered the following equation in exterior domains,

$$-\Delta u + W(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad u \in H^1_0(\Omega).$$

\begin{equation}
(1.2)
\end{equation}
They established the existence of a positive solution and multiple sign changing solutions for (1.5). Recently, Moroz and Schaftingen the following equation

\[-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{-\alpha}(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^d\]

and proved the existence of semi-classical solutions, see [32]. For more result on Choquard equations, we limit ourselves to citing a few recent papers [1, 2, 6, 10, 11, 14, 17, 28, 37, 38, 39, 42], referring to their bibliography for a broader list of works, although still not exhaustive.

In [16], d’Aienia, Siciliano and Squassina considered the fractional Choquard equation

\[-\Delta_y u + \omega u = (I_\alpha * |u|^2)u, \quad u \in H^{1/2}(\mathbb{R}^d), \quad u > 0,\]

and obtained regularity, existence, nonexistence, symmetry and decay properties of the corresponding solutions. For \(s = \frac{1}{2}\), problem (1.6) has been used to model the dynamics of pseudo-relativistic boson stars. Indeed, in [20] the following equation was studied,

\[-\Delta u + u = (I_2 * |u|^2)u, \quad u \in H^{1/2}(\mathbb{R}^2), \quad u > 0,\]

and in [19] it was shown that the dynamical evolution of boson stars is described by the nonlinear evolution equation

\[i\partial_t \psi = \sqrt{-\Delta + m^2} \psi - (I_2 * |\psi|^2)\psi \quad (m \geq 0),\]

for a field \(\psi : [0, T) \times \mathbb{R}^2 \to \mathbb{C}\), which has interesting applications in the quantum theory for large systems of self-interacting. In [13], the existence and concentration of a single-spike solution for the generalized pseudo-relativistic Hartree equation

\[\sqrt{-\varepsilon^2 \Delta + m} u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N\]

was obtained. We refer the interested readers to [12, 13, 21] and the references therein for more results on the pseudo-relativistic Hartree equation. The square root of the Laplacian also appears in the semi-relativistic Schrödinger-Poisson-Slater systems, see, for example, [4].

So motivated by the above-cited works, in this paper we study the following non-autonomous nonlinear fractional Choquard equation

\[
\begin{cases}
(-\Delta u + u = (1 + a(x)(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\
u(x) \to 0 \quad \text{as } |x| \to \infty.
\end{cases}
\]

Here we assume that \(s \in (0, 1), \alpha \in (0, N), \ p \in [2, \infty), \ \frac{N - 2s}{N + \alpha} < \frac{1}{p} < \frac{N}{N + \alpha}, \ a = a(x)\) is a scalar function and satisfies the following conditions:

(a1) \(a(x) \in L^\infty(\mathbb{R}^N), \lim_{|x| \to +\infty} a(x) = 0;\)

(a2) \(a(x) \geq 0, \ a(x) > 0\) on a positive measure set and \(a(x) \in L^{\frac{2N}{N + 2p + \frac{2p}{p} + \alpha}}(\mathbb{R}^N).\)

Our main theorem is

**Theorem 1.1.** Assume \(s \in (0, 1), \alpha \in (0, N), \ p \in [2, \infty). \ If \ \frac{N - 2s}{N + \alpha} < \frac{1}{p} < \frac{N}{N + \alpha}, \ a(x)\) satisfies conditions (a1) and (a2), then problem (1.10) has at least one positive ground state solution.
Remark 1.2. If $s=1$, theorem 1.1 was proved by Lions, see [25] and [26]. In [42], the authors gave an extension of Lions’s result by a Min-Max method argument (also in the case $s=1$).

(1.10) has a variational structure: critical points of the functional $E_{\alpha,p} \in C^0(H^s(\mathbb{R}^N) \cap L^{N+\alpha}(\mathbb{R}^N); \mathbb{R})$ defined by

$$E_{\alpha,p}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (1 + a(x))(I_n \ast |u|^p)|u|^p \, dx$$

are weak solutions of (1.10). This functional is well defined by the Hardy–Littlewood–Sobolev inequality which states that if $t \in (1, \frac{N}{N+\alpha})$, then for every $v \in L^t(\mathbb{R}^N)$, $I_n \ast v \in L^{\frac{N}{N+\alpha}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_n \ast v|^{\frac{N}{N-\alpha}} \, dx \leq C \left( \int_{\mathbb{R}^N} |v|^t \, dx \right)^{\frac{N}{N-\alpha}},$$

where $C > 0$ depends only on $\alpha$, $N$ and $t$. Note also that by the Sobolev embedding, $H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{N}{N+\alpha}}(\mathbb{R}^N)$ if and only if $\frac{N-2s}{N+\alpha} \leq \frac{1}{p} \leq \frac{N}{N+\alpha}$.

To prove theorem 1.1, we use the idea of [7] which studied the positive solutions for some non-autonomous Schrödinger–Poisson systems. Similar ideas were also used in [3]. The rest of this paper is organized as follows. In section 2, we study some properties of $E_{\alpha,p}$ under a natural constraint, the Nehari manifold. In section 3, a crucial compactness theorem by the concentration compactness argument will be given. In section 4, we prove theorem 1.1.

2. Preliminaries and variational setting

For $s \in (0, 1)$ and $N \geq 2$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ can be defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

which is endowed with the norm

$$\|u\|_s := \left( \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.$$

The Gagliardo semi-norm of $u$ is defined by

$$[u]_{H^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.$$

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying smooth functions on $\mathbb{R}^N$ and the topology of $\mathcal{S}$ is generated by

$$p_m(\varphi) = \sup_{x \in \mathbb{R}^N} (1 + |x|)^m \sum_{|\gamma| \leq m} |D^\gamma \varphi(x)|, \quad m = 0, 1, 2, \cdots,$$

where $\varphi \in \mathcal{S}$. Denote the topological dual of $\mathcal{S}$ by $\mathcal{S}'$, then for any $\varphi \in \mathcal{S}$, the usual Fourier transformation of $\varphi$ is given by
\[ \mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(x) \, dx \]

and one can extend \( \mathcal{F} \) from \( S \) to \( S' \). Furthermore, it holds that

\[ [u]_r = C \left( \int_{\mathbb{R}^n} |\xi|^{2\gamma} |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} = C \|(-\Delta)^{\frac{\gamma}{2}}u\|_{L^2(\mathbb{R}^n)} \]

for a suitable positive constant \( C = C(N, \gamma) \). Hence we have

\[ \|u\|_r = \left( \int_{\mathbb{R}^n} |u|^2 \, dx + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\gamma}{2}}u|^2 \, dx \right)^{\frac{1}{2}}. \]

From the fractional Sobolev embedding theorem, \( H^s(\mathbb{R}^N) \) embeds continuously into \( L^q(\mathbb{R}^N) \) for all \( q \in \left[ 2, \frac{2N}{N-s} \right] \) and compactly into \( L^q_{\text{loc}}(\mathbb{R}^N) \) for all \( q \in \left[ 2, \frac{2N}{N-s} \right] \), see [18].

The functional \( E_{\alpha, p} \) is bounded neither from below nor from above. So it is not convenient to consider \( E_{\alpha, p} \) restricted to a natural constraint, the Nehari manifold, that contains the critical points of \( E_{\alpha, p} \) and on which \( E_{\alpha, p} \) turns out to be bounded from below. Define

\[ \mathcal{N} := \{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : G(u) = 0 \}, \tag{2.1} \]

here

\[ G(u) = E'_{\alpha, p}(u)[u] = \|u\|^2 - \int_{\mathbb{R}^n} (1 + a(x))(I_0 * |u|^p)|u|^p \, dx. \tag{2.2} \]

Note that

\[ E_{\alpha, p}|_{\mathcal{N}}(u) = \left( \frac{1}{2} - \frac{1}{2p} \right) \|u\|^2. \tag{2.3} \]

**Lemma 2.1.**

1. \( \mathcal{N} \) is a \( C^1 \) regular manifold which is diffeomorphic to the standard sphere of \( W^{1,2}(\mathbb{R}^N) \);
2. \( E_{\alpha, p} \) is bounded from below by a positive constant on \( \mathcal{N} \);
3. \( u \) is a nonzero free critical point of \( E_{\alpha, p} \) if and only if \( u \) is a critical point of \( E_{\alpha, p} \) constrained on \( \mathcal{N} \).

**Proof.**

1. Let \( u \in H^s(\mathbb{R}^N) \setminus \{0\} \) with \( \|u\| = 1 \). Then there exists a unique \( t \in \mathbb{R}^+ \setminus \{0\} \) such that \( tu \in \mathcal{N} \). Indeed, such \( t \) must satisfy

\[ 0 = E'_{\alpha, p}(tu)[tu] = t^2 \|u\|^2 - t^{2p} \int_{\mathbb{R}^n} (1 + a(x))(I_0 * |u|^p)|u|^p \, dx. \]

Define

\[ A := \int_{\mathbb{R}^n} (1 + a(x))(I_0 * |u|^p)|u|^p \, dx, \]

then we are led to find a positive solution of equation \( t^2(1 - At^{2p-2}) = 0 \) with \( A > 0 \).

Since \( p > 1 \), the equation \( 1 - At^{2p-2} = 0 \) has an unique solution \( t = t(u) > 0 \). The corresponding point \( t(u)u \in \mathcal{N} \), which is called the projection of \( u \) on \( \mathcal{N} \). Moreover,

\[ E_{\alpha, p}(t(u)u) = \max_{t > 0} E_{\alpha, p}(tu). \]
Suppose $u \in \mathcal{N}$, then
$$
\left| \int_{\mathbb{R}^N} (1+a(x))(I_\alpha * |u|^p)|u|^p \, dx \right| \leq C \left| \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx \right| \leq C\|u\|^{2p}
$$
and then
$$
0 = \|u\|_{2}^2 - \int_{\mathbb{R}^N} (1 + a(x))(I_\alpha * |u|^p)|u|^p \, dx
\geq \|u\|_{2}^2 - C\|u\|^{2p}.
$$
From which we have
$$
\|u\|_{2} \geq C_1 > 0 \quad u \in \mathcal{N}.
$$
Since $E_{\alpha,p}$ a $C^2$ functional and
$$
E_{\alpha,p}[v,w] = \langle v, w \rangle - (p-1) \int_{\mathbb{R}^N} (1 + a(x))(I_\alpha * |u|^p)|u|^{p-2} vw \, dx
- p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^{p-2} u v}{|x-y|^{N-\alpha}} dy (1 + a(x))|u|^{p-1} v \, dx,
$$
$G$ is a $C^1$ functional and
$$
G'(u)[u] = E_{\alpha,p}'[u,u]
= \|u\|_{2}^2 - (2p-1) \int_{\mathbb{R}^N} (1+a(x))(I_\alpha * |u|^p)|u|^p \, dx
= (2 - 2p)\|u\|_{2}^2 \leq (2 - 2p)C < 0.
$$
(2) From the above argument, we have
$$
E_{\alpha,p}|_{\mathcal{N}}(u) = \left( \frac{1}{2} - \frac{1}{2p} \right)\|u\|_{2}^2 > C > 0.
$$
(3) Clearly, if $u \not= 0$ is a critical point of $E_{\alpha,p}$, then $E_{\alpha,p}'(u) = 0$ and $u \in \mathcal{N}$. On the other hand, let $u$ be a critical point of $E_{\alpha,p}$ constrained on $\mathcal{N}$, then there exists $\lambda \in \mathbb{R}$ such that $E_{\alpha,p}'(u) = \lambda G'(u)$. Therefore,
$$
0 = G(u) = E_{\alpha,p}'(u)[u] = \lambda G'(u)[u].
$$
Since $G'(u)[u] < 0$, $\lambda = 0$ and $E_{\alpha,p}'(u) = 0$.

Setting
$$
m := \inf\{E_{\alpha,p}(u) : u \in \mathcal{N}\},
$$
as a consequence of lemma 2.1, $m$ is a positive number.

When $a(x) \equiv 0$, equation (1.10) becomes
$$
(-\Delta)^\gamma u + u = (I_\alpha * |u|^p)|u|^{p-2} u \quad \text{in } \mathbb{R}^N.
$$
(2.4)
In this case, we use the notation $E_{\alpha,p}^{\infty}(u)$ and $\mathcal{N}_{\infty}$, respectively, for the functional and the natural constraint. Namely,
\[ E_{\alpha,p}^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 + |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \, dx, \]

\[ \mathcal{N}_\infty := \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \|u\|_0^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \, dx = 0 \}. \]

In the following lemma, we state some known results from [16] about the positive solutions of (2.4), which are crucial in our proof.

**Lemma 2.2** ([16]). Let \( N \in \mathbb{N}^+, \ s \in (0, 1), \ \alpha \in (0, N) \) and \( p \in (1, \infty) \). Assume that \( \frac{N - 2s}{N + \alpha} < \frac{1}{p} < \frac{N}{N + \alpha} \), then equation (2.4) has a positive, ground state solution \( w \in H^1(\mathbb{R}^N) \) which is radially symmetric about the origin and decaying to zero as \(|x| \to +\infty|\).

Since \( w \) is the ground state solution, setting

\[ m_\infty := \inf \{ E_{\alpha,p}(u) : u \in \mathcal{N}_\infty \}, \]

we have \( E_{\alpha,p}(v) \geq E_{\alpha,p}(w) \), for all \( v \) solution of (2.4). Note also that

\[ m_\infty = E_{\alpha,p}(w) = \left( \frac{1}{2} - \frac{1}{2p} \right) \|w\|_0^2. \]

### 3. A compactness lemma

In this section, we study the compactness of the Palais–Smale sequence of \( E_{\alpha,p} \). We follow the ideas of [7].

**Theorem 3.1.** Let \( \{u_n\} \) be a Palais–Smale sequence of \( E_{\alpha,p} \) constrained on \( \mathcal{N} \), that is to say, \( u_n \in \mathcal{N} \) and

\[ E_{\alpha,p}(u_n) \text{ is bounded}, \quad E_{\alpha,p}'(u_n) \to 0 \text{ strongly in } H^1(\mathbb{R}^N). \]

Then, up to a subsequence, there exist a solution \( \bar{u} \) of (1.10), a number \( k \in \mathbb{N} \cup \{0\} \), \( k \) functions \( u^1, \cdots, u^k \) of \( H^1(\mathbb{R}^N) \) and \( k \) sequence of points \( \{ y^i_n \}_{i=1}^k, y^i_n \in \mathbb{R}^N, \ 0 < i \leq k \), such that

(i) \[ |y^i_n| \to \infty, \quad |y^i_n - y^j_n| \to \infty, \quad \text{if} \quad i \neq j, \quad n \to \infty; \]

(ii) \[ u_n - \frac{1}{k} \sum_{i=1}^k u^i_n \to \bar{u} \text{ in } H^1(\mathbb{R}^N); \]

(iii) \[ E_{\alpha,p}(u_n) \to \sum_{i=1}^k E_{\alpha,p}(u^i) + E_{\alpha,p}(\bar{u}); \]

(iv) \( \bar{u} \) are nontrivial weak solutions of (2.4).

**Here, we agree that in the case** \( k = 0 \) **the above holds without** \( u^i \).
Proof. Since $E_{\alpha,p}(u_n)$ is bounded, from the fact that

$$E_{\alpha,p}|_{X}(u) = \left(\frac{1}{2} - \frac{1}{2p}\right)\|u\|_{p}^{2},$$

we have that $\{u_n\}$ is bounded, too.

Now, we claim that

$$E'_{\alpha,p}(u_n) \to 0 \quad \text{in } H'(\mathbb{R}^N).$$

In fact, we have

$$o(1) = E'_{\alpha,p}(u_n) = E'_{\alpha,p}(u_n) - \lambda_G(G(u_n))$$

for some $\lambda_G \in \mathbb{R}$. Taking the scalar product with $u_n$, we obtain

$$o(1) = \langle E'_{\alpha,p}(u_n), u_n \rangle - \lambda_G(G(u_n), u_n).$$

Since $u_n \in N$, $(E'_{\alpha,p}(u_n), u_n) = 0$ and $(G(u_n), u_n) < C < 0$. Thus $\lambda_G \to 0$ for $n \to +\infty$. Moreover, by the boundedness of $\{u_n\}$, $G'(u_n)$ is bounded and this implies that $\lambda_G G'(u_n) \to 0$, so we have the assertion.

On the other hand, since $u_n$ is bounded in $H'(\mathbb{R}^N)$, there exists $\bar{u} \in H'(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n \rightharpoonup \bar{u} \quad \text{in } H'(\mathbb{R}^N)\text{and in } L^{2n/p}(\mathbb{R}^N),$$

$$u_n(x) \to \bar{u}(x) \text{ a.e. on } \mathbb{R}^N.$$
Since $H'(\mathbb{R}^N) \subset L^{\frac{2N}{2N-2\alpha}}(\mathbb{R}^N)$ compactly, we have $|u_n - \bar{u}|_{\frac{2N}{2N-2\alpha}} \to 0$. By lemma 1.20 in [43],
\[
\|u_n\|^{p-2}u_n - |\bar{u}|^{p-2}\bar{u} \xrightarrow{L^{\frac{2N}{2N-2\alpha}}(\Omega)} 0 \quad \text{and} \quad \|u_n\|^{p} - |\bar{u}|^{p} \xrightarrow{L^{\frac{2N}{2N-2\alpha}}(\Omega)} 0.
\]
So $E_{\alpha,\rho}(\bar{u}) = 0$.

If $u_n \rightharpoonup \bar{u}$ in $H'(\mathbb{R}^N)$, we are done. So we can assume that $\{u_n\}$ does not converge strongly to $\bar{u}$ in $H'(\mathbb{R}^N)$. Set
\[
z_n = u_n(x) - \bar{u}(x).
\]
Obviously, we have $z_n \rightharpoonup 0$ in $H'(\mathbb{R}^N)$, but not strongly. Then by direct computation we have
\[
\|u_n\|^2 = \|z_n + \bar{u}\|^2 = \|z_n\|^2 + \|\bar{u}\|^2 + o(1). \quad (3.2)
\]
Now, we claim that
\[
\int_{\mathbb{R}^N} a(x)(I_\alpha * |u_n|^p)(x)|u_n(x)|^{p-2}u_n(x)h(x)dx
\]
\[
= \int_{\mathbb{R}^N} a(x)(I_\alpha * |\bar{u}|^p)(x)|\bar{u}(x)|^{p-2}\bar{u}(x)h(x)dx + o(1), \quad (3.3)
\]
\[
\int_{\mathbb{R}^N} a(x)(I_\alpha * |u_n|^p)(x)|u_n(x)|^p dx = \int_{\mathbb{R}^N} a(x)(I_\alpha * |\bar{u}|^p)(x)|\bar{u}(x)|^p dx + o(1), \quad (3.4)
\]
\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)(x)|u_n(x)|^{p-2}u_n(x)h(x)dx = \int_{\mathbb{R}^N} (I_\alpha * |\bar{u}|^p)(x)|\bar{u}(x)|^{p-2}\bar{u}(x)h(x)dx
\]
\[
+ \int_{\mathbb{R}^N} (I_\alpha * |z_n|^p)(x)|z_n(x)|^{p-2}z_n(x)h(x)dx + o(1). \quad (3.5)
\]
Let us observe that in view of the Sobolev embedding theorems, $u_n \rightharpoonup \bar{u}$ in $H'(\mathbb{R}^N)$ implies
\[
u_n \to \bar{u} \quad \text{in} \quad L^{\frac{2N}{2N-2\alpha}}(\mathbb{R}^N), \quad \|u_n\|^p \to |\bar{u}|^p \quad \text{in} \quad L^{\frac{2N(2N-2\alpha)}{2N}}(\mathbb{R}^N),
\]
\[
\phi_{u_n} \to \phi_{\bar{u}} \quad \text{in} \quad H^{\alpha}(\mathbb{R}^N), \quad \phi_{u_n} \to \phi_{\bar{u}} \quad \text{in} \quad L^{\frac{2N}{2N-2\alpha}}(\mathbb{R}^N).
\]

Proof of (3.3). We estimate
\[
\left| \int_{\mathbb{R}^N} a(x)(I_\alpha * |u_n|^p)(x)|u_n(x)|^{p-2}u_n(x)h(x)dx - \int_{\mathbb{R}^N} a(x)(I_\alpha * |\bar{u}|^p)(x)|\bar{u}(x)|^{p-2}\bar{u}(x)h(x)dx \right|
\]
\[
\leq \left| \int_{\mathbb{R}^N} a(x)(I_\alpha * |u_n|^p)(x) - (I_\alpha * |\bar{u}|^p)(x)|u_n(x)|^{p-2}u_n(x)h(x)dx \right|
\]
\[
+ \left| \int_{\mathbb{R}^N} a(x)(I_\alpha * |\bar{u}|^p)(x)|u_n(x)|^{p-2}u_n(x) - |\bar{u}(x)|^{p-2}\bar{u}(x)h(x)dx \right|.
\]
Since
Thus we have (3.4).

The second term on the right hand of (3.6) satisfies

\[ \int_{\mathbb{R}^N} |u_a(p)(x)(|u_a(x)|^{p-2}u_a(x) - |\bar{u}(x)|^{p-2}\bar{u}(x))|h(x)dx \]

\[ \leq C |a| |u_a(p)(x)(|u_a(x)|^{p-2}u_a(x) - |\bar{u}(x)|^{p-2}\bar{u}(x))|h(x)dx \]

we have the following estimate,

\[ \int_{\mathbb{R}^N} a(x)(|u_a(p)(x)|^{p-2}u_a(x)h(x)dx - \int_{\mathbb{R}^N} a(x)(|u_a(p)(x)|^{p-2}u_a(x)h(x)dx \]

Then we have (3.3).

**Proof of (3.4).** We estimate

\[ \int_{\mathbb{R}^N} a(x)(|u_a(p)|^{p}(x)|u_a(x)|^{p}dx - \int_{\mathbb{R}^N} a(x)(|u_a(p)(x)|^{p}|\bar{u}(x)|^{p}dx \]

\[ \leq \int_{\mathbb{R}^N} a(x)(|u_a(p)(x)|^{p}(x)|u_a(x)|^{p}dx - \int_{\mathbb{R}^N} a(x)(|u_a(p)(x)|^{p}|\bar{u}(x)|^{p}dx \]

\[ + \int_{\mathbb{R}^N} a(x)(|u_a(p)(x) - (u_a + |\bar{u}(x)|)|a(x)|^{p}dx \]  

(3.6)

The first term on the right hand of (3.6) satisfies

\[ \int_{\mathbb{R}^N} a(x)(|u_a(p)(x)|^{p} - |a(x)|^{p})dx \]

\[ \leq C |a| \int_{\mathbb{R}^N} |u_a(p) - |\bar{u}(x)|^{p}dx \]

\[ + C |a| \int_{\mathbb{R}^N} |u_a(p) - |\bar{u}(x)|^{p}dx \]

(3.6)

The second term on the right hand of (3.6) satisfies

\[ \int_{\mathbb{R}^N} a(x)(|u_a(p)(x) - (u_a + |\bar{u}(x)|)|a(x)|^{p}dx \]

\[ \leq C |a| \int_{\mathbb{R}^N} |u_a(p)(x) - (u_a + |\bar{u}(x)|)|a(x)|^{p}dx \]

\[ + C |a| \int_{\mathbb{R}^N} |u_a(p)(x) - (u_a + |\bar{u}(x)|)|a(x)|^{p}dx \]

Thus we have (3.4).
Proof of (3.5). Indeed, we have

\[
\begin{aligned}
&\int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^p)(x) |u_n(x)|^{p-2} u_n(x) h(x) dx \\
&- \int_{\mathbb{R}^N} (I_{\alpha} * |u_n - \bar{u}|^p)(x) |u_n(x) - \bar{u}(x)|^{p-2} (u_n - \bar{u})(x) h(x) dx \\
&= \int_{\mathbb{R}^N} (I_{\alpha} * (|u_n|^p - |u_n - \bar{u}|^p))(x) \left(|u_n(x)|^{p-2} u_n(x) \right) dx \\
&- \int_{\mathbb{R}^N} |u_n(x) - \bar{u}(x)|^{p-2} (u_n - \bar{u})(x) h(x) dx \\
&+ \int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^p)(x) |u_n(x) - \bar{u}(x)|^{p-2} (u_n - \bar{u})(x) h(x) dx \\
&+ \int_{\mathbb{R}^N} (I_{\alpha} * |u_n - \bar{u}|^p)(x) |u_n(x)|^{p-2} u_n(x) h(x) dx \\
&- 2 \int_{\mathbb{R}^N} (I_{\alpha} * |u_n - \bar{u}|^p)(x) |u_n(x) - \bar{u}(x)|^{p-2} (u_n - \bar{u})(x) h(x) dx.
\end{aligned}
\]

From the proof of lemma 4.3 in [16], we have \(I_{\alpha} * (|u_n|^p - |u_n - \bar{u}|^p) \to I_{\alpha} * |u|^p\) in \(L^{2N/n}(\mathbb{R}^N)\) and \(|u_n - \bar{u}|^p \to 0\) weakly in \(L^{N/n}(\mathbb{R}^N)\) as \(n \to \infty\). So we have (3.5).

Combining the above estimates and lemma 4.3 in [16], we obtain

\[
E_{\alpha,p}(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 dx - \frac{1}{2p} \int_{\mathbb{R}^N} (1 + a(x))(I_{\alpha} * |u_n|^p)|u_n|^p dx \\
= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |u_n|^2 dx - \frac{1}{2p} \int_{\mathbb{R}^N} a(x)(I_{\alpha} * |u_n|^p)(x) |u_n(x)|^p dx \\
- \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^p)(x) |u_n(x)|^2 dx
\]

and for all \(h \in H'(\mathbb{R}^N)\),

\[
\alpha(1) = (E'_{\alpha,p}(u_n), h) \\
= (u_n, h) - \int_{\mathbb{R}^N} (1 + a(x))(I_{\alpha} * |u_n|^p)(x) |u_n(x)|^{p-2} u_n(x) h dx
\]

1837
\[
\begin{align*}
\langle \mu_h, h \rangle &= \int_{\mathbb{R}^N} a(x)(I_\alpha * |u_n|^p)(x)|u_n(x)|^{p-2} u_n(x) h(x) \, dx \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)(x)|u_\alpha(x)|^{p-2} u_\alpha(x) h(x) \, dx.
\end{align*}
\]
Thus
\[
\begin{align*}
o(1) &= (E_{\alpha,p}'(u_n), h) \\
&= \langle \tilde{u}, h \rangle + \langle z_n^1, h \rangle - \int_{\mathbb{R}^N} a(x)(I_\alpha * |\tilde{u}|^p)(x)|\tilde{u}(x)|^{p-2} \tilde{u}(x) h(x) \, dx \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}|^p)(x)|\tilde{u}(x)|^{p-2} \tilde{u}(x) h(x) \, dx \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * |z_n^1|^p)(x)|z_n^1(x)|^{p-2} z_n^1(x) h(x) \, dx + o(1) \\
&= \langle \tilde{u}, h \rangle + \langle z_n^1, h \rangle - \int_{\mathbb{R}^N} (1 + a(x))(I_\alpha * |\tilde{u}|^p)(x)|\tilde{u}(x)|^{p-2} \tilde{u}(x) h(x) \, dx \\
&\quad - \int_{\mathbb{R}^N} (I_\alpha * |z_n^1|^p)(x)|z_n^1(x)|^{p-2} z_n^1(x) h(x) \, dx + o(1) \\
&= (E_{\alpha,p}'(\tilde{u}), h) + (E_{\alpha,p}'(z_n^1), h) + o(1) \\
&= (E_{\alpha,p}'(z_n^1), h) + o(1).
\end{align*}
\]
Hence
\[
E_{\alpha,p}'(z_n^1) = o(1) \quad \text{in } H'(\mathbb{R}^N). \tag{3.7}
\]
Furthermore,
\[
0 = \langle E_{\alpha,p}'(u_n), u_n \rangle = (E_{\alpha,p}'(\tilde{u}), \tilde{u}) + (E_{\alpha,p}'(z_n^1), z_n^1) + o(1)
\]
\[
= (E_{\alpha,p}'(z_n^1), z_n^1) + o(1).
\]
Setting
\[
\delta := \lim_{n \to +\infty} \sup \left( \sup_{y \in \mathbb{R}^N} \int_{B(\delta, y)} \left| z_n^1 \right|^{2Np/(N+\sigma)} \, dx \right),
\]
we have that \( \delta > 0 \). Otherwise, if \( \delta = 0 \), then by [40, lemma 1.21], \( z_n^1 \to 0 \) in \( L^{2Np/(N+\sigma)}(\mathbb{R}^N) \). This is a contradiction to the fact that \( u_n \) does not converge strongly to \( \tilde{u} \) in \( L^{2Np/(N+\sigma)}(\mathbb{R}^N) \).

Then we may assume there exists a sequence of \( \{ y_n^1 \} \subset \mathbb{R}^N \) such that
\[
\int_{B(\delta, y_n^1)} \left| z_n^1 \right|^{2Np/(N+\sigma)} \, dx > \delta/2.
\]
Now we consider \( z_n^1(y_n^1) \). We assume \( z_n^1(y_n^1) \to u^1 \) in \( H'(\mathbb{R}^N) \). Therefore, \( z_n^1(y_n^1) \to u^1 \) a.e. on \( \mathbb{R}^N \). Since
\[
\int_{B(0)} \left| z_n^1(x + y_n^1) \right|^{2Np/(N+\sigma)} \, dx > \delta/2,
\]
from the Rellich theorem it follows that
\[ \int_{B(0)} |u^1(x)|^{2Np/(N+\sigma)} \, dx > \delta/2. \]
Thus, \( u^1 \equiv 0 \). Since \( \psi_n^1 \to 0 \) in \( W^{1,2}(\mathbb{R}^N) \), \( (\psi_n^1) \) must be unbounded, and up to a subsequence, we can assume that \( |\psi_n^1| \to +\infty \). Furthermore, (3.7) implies \( E_{\alpha,p}^\infty(u^1) = 0 \). Finally, let us set
\[ z_n^2(x) = \psi_n^1(x) - u^1(x - \psi_n^1). \]

Then, we have
\[ \|z_n^2\|^2 = \|u_n\|^2 - \|u^1\|^2 - \|a\|^2 + o(1) \]
and
\[ \int_{\mathbb{R}^N} (I_\alpha + |z_n^2|^p)(x)|z_n^2(x)|^p \, dx = \int_{\mathbb{R}^N} (I_\alpha + |u_n|^p)(x)|u_n(x)|^p \, dx - \int_{\mathbb{R}^N} (I_\alpha + |\tilde{a}|^p)(x)|\tilde{a}(x)|^p \, dx - \int_{\mathbb{R}^N} (I_\alpha + |u^1|^p)(x)|u^1(x)|^p \, dx + o(1). \]
This implies
\[ E_{\alpha,p}^\infty(z_n^2) = E_{\alpha,p}^\infty(a^1) - E_{\alpha,p}^\infty(u^1) + E_{\alpha,p}^\infty(a^1) + E_{\alpha,p}^\infty(u^1) + o(1), \]
and hence we obtain
\[ E_{\alpha,p}(u_n) = E_{\alpha,p}(a) + E_{\alpha,p}^\infty(z_n^2) + E_{\alpha,p}^\infty(u^1) + o(1). \]
As before, one can prove that
\[ E_{\alpha,p}^\infty(z_n^2) = o(1) \quad \text{in} \quad H^\prime(\mathbb{R}^N). \]

Now, if \( z_n^2 \to 0 \) in \( H^\prime(\mathbb{R}^N) \), then we are done. Otherwise, \( z_n^2 \to 0 \) and not strongly and we repeat the above argument. Then we obtain a sequence of points \( \{y_n^2\} \subseteq \mathbb{R}^N \) such that \( |y_n^2| \to \infty \), \( |y_n^2 - y_n^1| \to +\infty \) if \( i \neq j \) as \( n \to +\infty \) and a sequence of functions \( z_n^2(x) = z_n^{j-1} - u^{j-1}(x - y_n^{j-1}) \) with \( j \geq 2 \) such that
\[ z_n^j(x + y_n^j) \to u^j(x) \quad \text{in} \quad H^\prime(\mathbb{R}^N), \quad E_{\alpha,p}^\infty(u^j) = 0 \]
and
\[ E_{\alpha,p}(u_n) = E_{\alpha,p}(a) + \sum_{j=1}^{k-1} E_{\alpha,p}(u^j) + E_{\alpha,p}^\infty(z_n^2) + o(1). \]

Then, since \( E_{\alpha,p}^\infty(u^j) \geq m_\infty \) for all \( j \) and \( E_{\alpha,p}(u_n) \) is bounded, the iteration must stop at some finite index \( k \). Thus we have completed the proof of theorem 3.1. \( \square \)

**Lemma 3.2.** Let \( \{u_n\} \) be a (PS)\(_d\) sequence. Then \( \{u_n\} \) is relatively compact for all \( d \in (0, m_\infty) \).
Moreover, if \( E_{\alpha,p}(u_n) \to m_\infty \), then either \( \{u_n\} \) is relatively compact or the statement of theorem 3.1 holds with \( k = 1 \), and \( u^1 = w \), the ground state solution of (2.4).
Proof. Let us consider a \((PS)_j\) sequence \(\{u_n\}\) and apply theorem 3.1. Specially, note that \(E_{\alpha,p}(u^j) \geq m_\infty\) for all \(j\).

When \(E_{\alpha,p}(u^j) \to d < m_\infty\), then \(k = 0\), and then \(u_n \to \bar{u}\) in \(H'(\mathbb{R}^N)\). When \(E_{\alpha,p}(u^j) \to m_\infty\), if \(\{u_n\}\) is not compact, then \(k = 1\), and \(\bar{u} = 0\), \(u^1 = w\). \(\square\)

4. Proof of theorem 1.1

Proof. To prove the existence of a ground state solution of (1.10), we just need to show that

\[ m < m_\infty. \]  

(4.1)

If this is the case, using lemma 3.2 and standard arguments, it is easy to see that \(m\) is achieved by a function \(u\) which solves (1.10). Furthermore, \(u\) is positive. Indeed, let \(\{u_n\} \subseteq \mathcal{N}\) be a minimizing sequence, \(E_{\alpha,p}(u_n) \to m\). By theorem 6.17 of [23], we have \(\|u_n\|^p \leq \|u_n\|^p\) \(= \int_{\mathbb{R}^N} (1 + a(x))(I_\alpha |w|^p)|w|^p \, dx\). So \(t_0|u_n| \in \mathcal{N}\) for some \(t_0 \in (0, 1]\). Thus by (2.3), we have \(E_{\alpha,p}(t_0|u_n|) = (\frac{1}{2} - \frac{1}{2p})\|u_n\|^p \leq (\frac{1}{2} - \frac{1}{2p})\|u_n\|^p = E_{\alpha,p}(u_n)\). This shows that \(\{t_0|u_n|\}\) is also a minimizing sequence and the minimizer \(u \geq 0\). By minor modification of theorem 3.2 in [16], \(u \in C^0(\mathbb{R}^N)\). Finally, from the maximum principle for fractional Laplacian (see [18]), we have \(u > 0\).

To verify condition (4.1), we consider the projection \(tw\) on \(\mathcal{N}\) of the minimizer \(w\) of \(E_{\alpha,p}^\infty\) on \(\mathcal{N}_\infty\). First, let us show that \(t < 1\). Indeed, if \(t \geq 1\) would be true, then we have

\[
0 = t^2\|w\|^2 - t^2p \int_{\mathbb{R}^N} (1 + a(x))(I_\alpha |w|^p)|w|^p \, dx \\
< t^2\|w\|^2 - t^2p \int_{\mathbb{R}^N} (I_\alpha |w|^p)|w|^p \, dx \\
= (t^2 - t^2p)\|w\|^2 \leq 0,
\]

a contradiction. So \(t < 1\).

Then we have

\[
m \leq E_{\alpha,p}(tw) = \frac{1}{2}t^2\|w\|^2 - \frac{t^2p}{2p} \int_{\mathbb{R}^N} (1 + a(x))(I_\alpha |w|^p)|w|^p \, dx \\
< \frac{1}{2}t^2\|w\|^2 - \frac{t^2p}{2p} \int_{\mathbb{R}^N} (I_\alpha |w|^p)|w|^p \, dx \\
= \left(\frac{t^2}{2} - \frac{t^2p}{2p}\right)\|w\|^2 \\
< \left(\frac{1}{2} - \frac{1}{2p}\right)\|w\|^2 = m_\infty,
\]

Hence, \(m < m_\infty\) and the proof is completed. \(\square\)
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