The Continuous Joint Replenishment Problem is Strongly $\mathcal{NP}$-Hard

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The Continuous Periodic Joint Replenishment Problem (CPJRP) has been one of the core and most studied problems in supply chain management for the last half a century. Nonetheless, despite the vast effort put into studying the problem, its complexity has eluded researchers for years. Although the CPJRP has one of the tighter constant approximation ratio of 1.02, a polynomial optimal solution to it was never found. Recently, the discrete version of this problem was finally proved to be $\mathcal{NP}$-hard. In this paper, we extend this result and finally prove that the CPJRP problem is also strongly $\mathcal{NP}$-hard. Key words: Computational Complexity, Joint Replenishment Problem, Supply Chain Management.

1. Introduction

The joint replenishment problem (JRP) is a basic problem in the field of inventory management. The JRP aims to synchronize orders of different commodities so as to order them together and save costs. In this research, we refer to the schedule of the replenishment times for each commodity as the ordering policy. Whenever a commodity is ordered, it incurs a fixed ordering cost as well as linear holding costs that are proportional to the amount of the commodity held in storage. Linking all commodities, a joint ordering cost is incurred whenever one or more commodities are ordered. The objective of JRP is to minimize the sum of ordering and holding costs. There are many distinctions between the different JRP models studied in the literature, which we elaborate upon in Section 1.1. In this research, we study the continuous periodic review JRP (CPJRP) model with continuous infinite horizon, and steady (stationary) demand. That is, facing a constant demand, we need to minimize the average periodic holding and setup costs of all

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commodities as well as the joint ordering costs. However, in the CPJRP the orders of each commodity are placed periodically. The cycle times for each commodity’s orders are pre-determined and inflexible. The joint replenishment is continuously reviewed and is paid only at times where at least one commodity is being ordered. Of all the different variations of the JRP model, the CPJRP is by far the most studied one.

Except for the non-stationary demand JRP model, the complexity of all JRP variations has been an open question for more than five decades, until recently when Cohen-Hillel and Yedidsion (2018) finally resolved the complexity of the discrete version of our problem, namely the discrete periodic JRP (DPJRP). In this research we extend the result of Cohen-Hillel and Yedidsion (2018), and prove that the CPJRP is strongly \( \mathcal{NP} \)-hard.

1.1. Literature review

Many variations of the JRP problem have been studied in the literature. Schulz and Telha (2011) distinguished between some characteristics of this problem.

- Commodity order policy constraints: There are three types of order policy constraints for the JRP. The first model requires a periodic ordering policy, also called a periodic review. A periodic review is the policy where for each commodity we must pre-determine a cycle time. An order will occur at each multiple of that cycle time. We refer to this model as the PJRP. The second model does not require a cycle time for each commodity. However, it requires a cyclic ordering policy. We refer to this model as the cyclic JRP. This model has no limits on the ordering policy.

- Joint order policy constraints: The joint ordering cost in the JRP model is a complicated function of the inter-replenishment times, so it is often assumed that joint orders are placed periodically, even if some joint orders are empty and the cycle times of the commodities are always a multiple of the joint order cycle time. This model is often referred to as Strict JRP. We refer to the model with continuous joint replenishment review, where a joint order is placed only at the periods where at least one commodity is ordered, as General JRP, or simply as JRP.
• Demand type: We make a distinction between problems with stationary demand for each commodity and problems with fluctuating demand.

• Time horizon: The time horizon defines the horizon for which one must plan an order policy. We distinguish between the problem with infinite horizon and the problem with finite horizon.

• Solution integrality: The integrality of the solution determines whether the ordering policy will be integral or not.

Our focus in this research is the periodic, general, continuous time JRP with stationary demand and infinite horizon, referred to as the CPJRP. The JRP is a special case of the One-Warehouse-N-Retailers problem (OWNR), which deals with a single warehouse receiving goods from an external supplier and distributing to multiple retailers. The warehouse could also serve as a storage point. JRP in particular is a special case of the OWNR with a very high warehouse holding cost. Arkin et al. (1989) stated that since JRP is a special case of OWNR, proving JRP hardness also proves the hardness of OWNR.

**Strict PJRP.** The problem of Strict PJRP was well covered in the reviews by Goyal and Satir (1989) and Khouja and Goyal (2008). Many research attempts have been made in order to find efficient solutions to the Strict PJRP since the early 1970s. Heuristic approaches were suggested in Shu (1971), Nocturne (1973), Silver (1976), Van Eijs (1993), Kaspi and Rosenblatt (1983), Goyal and Belton (1979), Kaspi and Rosenblatt (1991), Goyal and Deshmukh (1993), Viswanathan (1996), Fung and Ma (2001), Viswanathan (2002), Porras and Dekker (2004), Wildeman et al. (1997), and Olsen (2005).

Since JRP is a special case of OWNR, results regarding the OWNR hold for JRP as well. Hence, the following results are applicable for JRP. A prominent advancement in the study of OWNR, the optimal Power-of-Two policy, was achieved by Roundy (1985). This policy could be computed in $O(n \log n)$ time. Roundy (1985) proved that the cost of the best power-of-two policy achieves 98% of an optimal policy (94% if the base planning period is fixed). In other words, he suggested a 1.02-approximation (1.064 for the fixed-based planning period) for JRP, where a $\rho$-approximation algorithm is an algorithm that is polynomial with respect to the number of elements,
and the ratio between the worst case scenario solution and the optimal solution is bounded by a constant, $\rho$. Based on Roundy (1985), Jackson et al. (1985) proposed an efficient algorithm that offers a replenishment policy in which the cost is within a factor of $\sqrt{2} \approx 1.06$ of the optimal solution. This approximation was later improved to $\frac{1}{\sqrt{2\ln 2}}$ for a non-fixed-based planning period (Muckstadt and Roundy 1993).

Research has been done based on the Power-of-Two policy, including Lee and Yao (2003), Muckstadt and Roundy (1987), Teo and Bertsimas (2001). Teo and Bertsimas (2001) have also noted that finding the optimal lot sizing policies for stationary demand lot sizing problems is still an open issue.

Lu and Posner (1994) presented a fully polynomial time approximation scheme (FPTAS) for the Strict PJRP model with fixed base. Later, Segev (2013) presented a quasi-polynomial-time approximation scheme (QPTAS), which shows that the problem is most likely not $\mathcal{APX}$-hard. In addition, an efficient polynomial time approximation scheme (EPTAS) for JRP with finite time horizon and stationary demand was presented by Nonner and Sviridenko (2013).

This problem was researched in many other different setups, such as JRP under resource constraints (Goyal 1975, Khouja et al. 2000, Moon and Cha 2006), minimum order quantities (Porras and Dekker 2006), and non-stationary holding cost (Levi et al. 2006, Nonner and Souza 2009, Levi et al. 2008).

**General PJRP.** Porras and Dekker (2005) pointed out that adding the correction factor leads to a completely different problem, at least in terms of exact solvability. Porras and Dekker (2004) show that changing the model from Strict PJRP to PJRP significantly changes the joint replenishment cycles and the commodities replenishment cycles. The difference in solvability is evidenced by the sheer number of decision variables. In the Strict PJRP, all commodities’ cycle times are simple functions of the joint replenishment cycle time. Thus, there is actually only a single decision variable. However, this is not the case with the PJRP where we have $n$ decision variables, one for each commodity. In practice, Strict PJRP is much less common than PJRP as it involves paying for
empty deliveries. Strict PJRP may occur only if there is a binding contract with a delivery company. Although such a binding contract may decrease the cost of the joint replenishment significantly, it usually limits the flexibility of choosing the joint replenishment cycles. Schulz and Telha (2011) presented a polynomial time approximation scheme (PTAS) for the PJRP case.

**Finite horizon.** Several heuristics were designed to deal with the finite horizon model. Most of the finite time heuristics assume variable demands and run-in time $\Omega(T)$ (Levi et al. 2006, Joneja 1990). Schulz and Telha (2011) presented a polynomial-time $\sqrt{9/8}$-approximation algorithm for the JRP with dynamic policies and finite horizon. As the time horizon $T$ increases, the ratio converges to $\sqrt{9/8}$. Schulz and Telha (2011) also presented an FPTAS for the Strict PJRP case with no fixed base and a finite time horizon.

2. **Model Formulation**

We consider the case of an infinite time horizon, and a system composed of several commodities, for each of which there is an external stationary demand. The demand has to be satisfied in a timely fashion so as to prevent delays. Backlogging and lost sales are not allowed. Each commodity incurs a fixed ordering cost for each time at which an order of the commodity is placed, as well as a linear inventory holding cost for each time unit (referred to as a period) a unit of commodity remains in storage. In addition, a joint ordering cost is incurred for each time where one or more orders are placed. We use the following notations, were the units are given in square brackets:

\[
\begin{align*}
N &- \text{ Number of commodities in the system, } [\text{units}] . \\
\lambda_c &- \text{ Demand rate for commodity } c, [\text{units per period}] . \\
h_c &- \text{ Holding cost for commodity } c \text{ per period, } [$$/\text{units·period}] . \\
K_c &- \text{ Fixed ordering cost for commodity } c, [$] . \\
K_0 &- \text{ Fixed joint ordering cost, } [$] .
\end{align*}
\]

The objective is to find an ordering cycle time, $t_c \in \mathbb{R}$, for each commodity, $c$, so as to minimize the periodic sum of ordering and holding costs of all commodities.
The simple model, in which there is only a single commodity, is known as the Economic Order Quantity (EOQ). While examining commodity $c$, we define its standalone problem as the optimal ordering quantity problem for a single commodity, $c$, with no joint setup cost and an infinite horizon. The standalone problem is a simple EOQ problem.

The EOQ model assumes without loss of generality that there is no on-hand inventory at time 0. Shortage is not allowed, so we must place an order at time 0. The average periodic cost, as a function of the cycle time $t_c$, denoted by $g(t_c)$, is given by

$$g(t_c) = \frac{K_c}{t_c} + \lambda_c h_c \frac{t_c}{2},$$

and the optimal cycle time for $g(t_c)$, denoted by $t_c^*$, is

$$t_c^* = \sqrt{\frac{2K_c}{h_c \lambda_c}}.$$  

See full elaboration and additional analysis in Nahmias (2001) and Zipkin (2000).

3. \textit{NP}-Hardness proof

The \textit{NP}-Hardness proof of the CPJRP is based on the \textit{NP}-Hardness proof of the DPJRP in Cohen-Hillel and Yedidsion (2018). In this proof, we take advantage of the instance of the DPJRP used in Cohen-Hillel and Yedidsion (2018) with slight changes in the instance parameters and without the integrality constraint that defines the DPJRP. We show that the instance we construct for the CPJRP is as hard as the 3SAT instance from which Cohen-Hillel and Yedidsion (2018) constructed their instance, and thus, Strongly \textit{NP}-hard.

The 3SAT is defined as follows:

\textbf{DEFINITION 1.} Given a logical expression, $\varphi$, in a Conjunctive Normal Form (CNF) with $m$ clauses and $n$ variables, $x_1, ..., x_n$, where each clause contains exactly 3 literals, is there a feasible assignment to the variables such that each clause contains at least one true literal?

Before we continue with the \textit{NP}-hardness proof, we would like to show a schematic sketch of the proof in Cohen-Hillel and Yedidsion (2018), as in this paper we meticulously show that each step in this proof holds in the continuous environment for the instance we built.
3.1. Proof sketch

In this section, we explain the steps taken in Cohen-Hillel and Yedidsion (2018). For each step we explain the adjustments required for a continuous environment.

1. **Polynomial time reduction** – Given the instance of 3SAT, three sets of commodities were constructed,

   - **Constant** commodities - commodities, denoted \( c_y^i \), that in a discrete environment would be ordered at \( t_{c_y^i}^* \) regardless of the other commodities.
   - **Variable** commodities - commodities, denoted \( c_x^i \), whose \( t_{c_x^i} \) may change according to other commodities’ order pattern. Each commodity in this set is associated with one variable in the original 3SAT problem. The optimal cycle time of \( c_x^i \in Variables \) is one of two unique prime numbers \( p_i \) and \( \overline{p}_i = p_i + b_i \). Each such option is associated with either the variable or its negation in the original 3SAT problem: \( p_i \) is associated with \( x_i = False \) and \( \overline{p}_i \) with \( x_i = True \).
   - **Clause** commodities - commodities, denoted \( c_z^i \), which, just like the **Constants**, in a discrete environment would be ordered at \( t_{c_z^i}^* \) regardless of the other commodities. However, \( t_{c_z^i}^* \) is associated with a clause in the original 3SAT. \( t_{c_z^i}^* \) is the product of the primes associated with the clause’s three literals. Accordingly, if the cycle time of at least one of the relevant variable commodities were set to the cycle time associated with the right literal, the cycle time of the **Clauses** commodity would be synchronized with it. For example, given a clause \((x_3 \cup \overline{x}_6 \cup x_9)\) we create a commodity \( c_z^i \) with \( t_{c_z^i}^* = p_3 \cdot \overline{p}_6 \cdot p_9 \).

We denote the instance created for the JRP problem by \( \Gamma \) (both for the DPJRP and the CPJRP). In our reduction, we use the same three sets, but adjust the holding and setup costs of commodities **Constants** and **Clauses**.

2. **Constants and Clauses cycle times** – Cohen-Hillel and Yedidsion (2018) showed that the optimal cycle time of **Constants** and **Clauses** are set to \( t_{c_y^i}^* \) and \( t_{c_z^i}^* \), respectively, regardless of other commodities.

In a continuous environment, the optimal cycle time would always be influenced by other commodities. However, we show that it is restricted to a very narrow time interval. Moreover, we show
that in an optimal solution all \textit{Constants} and \textit{Clauses} are synchronized in a way that resembles a discrete environment. That is, for any optimal solution there exists a seed \( \beta \), for which in that optimal solution \( \forall c^y_i \in \text{Constants} : t^*_y = \beta t^*_y \) and \( \forall c^z_i \in \text{Clauses} : t^*_z = \beta t^*_z \).

3. \textbf{Variables cycle time} – Cohen-Hillel and Yedidsion (2018) showed that the optimal cycle time of variable \( t^x_i \) is one of two unique prime numbers \( p_i \) and \( \overline{p}_i = p_i + b_i \). We show that in a continuous environment, the same applies with a small change. The optimal cycle time for a \textit{Variables} commodity applies \( c^x_i \in \{ \beta p_i, \beta \overline{p}_i \} \).

4. \textbf{Optimal solution} – Cohen-Hillel and Yedidsion (2018) show that if there is a solution to the original \textit{3SAT} problem, in an optimal solution of \( \Gamma \), the cycle times of \textit{Variables} are set to synchronize with all of the commodities of type \textit{Clauses}.

To do so, they formulated the total cost of solution \( S \), denoted by \( TC(S) \), as a sum of three cost functions: The first cost function, \( TC_{\text{Constants}}(S) \), sums all the costs that are associated with the commodities \( c^y_i \in \text{Constants} \), including all the joint replenishment costs induced by \( c^y_i \in \text{Constants} \). The second cost function, \( TC_{\text{Variables}}(S) \), sums all the costs that are associated with the commodities \( c^x_i \in \text{Variables} \), including only the marginal addition to the joint replenishment costs induced by \( c^x_i \in \text{Variables} \), assuming all the joint replenishment costs induced by \( c^y_i \in \text{Constants} \) are already paid for. The third cost function, \( TC_{\text{Clauses}}(S) \), sums all the costs that are associated with the commodities \( c^z_i \in \text{Clauses} \), including only the marginal addition to the joint replenishment costs induced by \( c^z_i \in \text{Clauses} \) assuming all the joint replenishment costs induced by \( c^y_i \in \text{Constants} \) and \( c^x_i \in \text{Variables} \) are already paid for.

To show the equivalence to the \textit{3SAT}, they proved the following two steps:

(a) Ignoring commodities of type \textit{Clauses}, for each \( c^x_i \in \text{Variables} \) the marginal cost of setting \( t^x_i = p_i \) is lower than setting it to \( t^x_i = \overline{p}_i \); thus showing that setting \( t^x_i = p_i \) for all \( c^x_i \in \text{Variables} \) gives a lower bound on the marginal cost of \textit{Variables}, denoted \( LB(TC_{\text{Variables}}) \) and setting \( t^x_i = \overline{p}_i \) for all \( c^x_i \in \text{Variables} \) gives an upper bound on the marginal cost of \textit{Variables}, denoted \( UB(TC_{\text{Variables}}) \).
(b) Not synchronizing even one commodity of type *Clauses* with at least one of the commodities of type *Variables* costs more than $UB(\text{TC}_{\text{Variables}}) - LB(\text{TC}_{\text{Variables}})$ by proving that an optimal solution to $\Gamma$ is equivalent to at least one true assignment to the original 3SAT instance, if such exists.

We show that both proofs still hold even when we set $t_{c_i} \in \{\beta t_{c_i}^*, \beta t_{c_i}^*\}$ and $t_{c_i}^* = \beta t_{c_i}$ instead of $t_{c_i} \in \{t_{c_i}^*, t_{c_i}^*\}$ and $t_{c_i}^* = t_{c_i}$, respectively.

In the following we use the titles of the steps in §3.1 as subsections associated with each step.

### 3.2. Polynomial time reduction

The total cost of types $c_i^y \in \text{Constants}$ and $c_i^z \in \text{Clauses}$ commodities in Cohen-Hillel and Yedidsion (2018) is a constant. We change the cost parameters of these commodities to adjust them to a continuous environment with as little change to their respective optimal cycle times as possible.

Accordingly, we create a commodity for which the optimal solution is $\beta t_{c_i}^* \left(\beta t_{c_i}^*\right)$, where the seed $\beta$ is close to 1, regardless of the cycle times of the other commodities in the problem.

In Cohen-Hillel and Yedidsion (2018), each commodity $c_i^y \in \text{Constants}$ ($c_i^z \in \text{Clauses}$) is associated with a time $t_{c_i}^* \left(t_{c_i}^*\right)$. This cycle time is the optimal cycle time for that commodity regardless of the solution to other commodities. For our reduction, we denote an auxiliary parameter

$$\delta = \frac{1}{6n(\beta n)^6}.$$  \hfill (3)

We set the holding cost ($h_{c_i}$) and ordering cost ($K_{c_i}$) for each commodity, $c_i^y \in \text{Constants}$ as follows:

$$h_{c_i} = \frac{1}{(\delta^2 + 2\delta)(t_{c_i}^* )^2},$$  \hfill (4)

$$K_{c_i} = \frac{1}{(\delta^2 + 2\delta)}.$$  \hfill (5)

Similarly, for each commodity, $c_i^z \in \text{Clauses}$:

$$h_{c_i} = \frac{1}{(\delta^2 + 2\delta)(t_{c_i}^* )^2}, K_{c_i} = \frac{1}{(\delta^2 + 2\delta)}.$$
We make no changes in the holding and setup costs of \( c_i^x \in Variables \):

\[
\begin{align*}
    h_{c_i^x} &= \alpha_c \frac{\bar{p}_i^2 - b_i^2}{\bar{p}_i (\bar{p}_i + \frac{b_i}{2}) \frac{b_i}{2}}, \\
    K_{c_i^x} &= h_{c_i^x} \cdot \bar{p}_i \left( \bar{p}_i + b_i \right) - \frac{\bar{p}_i + b_i}{\bar{p}_i + b_i - 1} \alpha_c \overline{\alpha}_v,
\end{align*}
\]

where \( \alpha_c, \overline{\alpha}_v, \bar{p}_i \), and \( b_i \) are constants taken from Cohen-Hillel and Yedidsion (2018), such that \( \bar{p}_i \) and \( \overline{\alpha}_v \) are prime numbers that are unique to commodity \( c_i^x \). Without loss of generality, we assume that the primes are sorted in ascending order, which makes

\[
\bar{p}_n = \max \{ \bar{p}_i \}
\]

Moreover, according to Cohen-Hillel and Yedidsion (2018), \( b_i \) is small enough so that

\[
\forall i : \bar{p}_i = \bar{p}_i + b_i < \bar{p}_{i+1}.
\]

Just like Cohen-Hillel and Yedidsion (2018), We set the joint ordering cost to be:

\[
K_0 = 1,
\]

and the demand for each commodity per time unit to be:

\[
\forall c : \lambda_c = 2.
\]

### 3.3. Constants and Clauses cycle times

In this section, we refer only to \( c_i^y \in Constants \) and note that everything applies to \( c_i^z \in Clauses \) as well due to their similar cost functions.

For each commodity \( c_i^y \in Constants \), we define two EOQ problems. In the first EOQ problem, denoted \( \theta_1 \), we define: \( h_1 = h_{c_i^y} \) and \( K_1 = K_{c_i^y} \). The solution for this problem defines a lower bound on the marginal average periodic cost of commodity \( c_i^y \). This problem is in fact the standalone cost of commodity \( c_i^y \). In the second EOQ problem, denoted \( \theta_2 \), we define: \( h_2 = h_{c_i^y} \) and \( K_2 = K_{c_i^y} + K_0 \).

That is, we pay \( K_0 \) for each order of commodity \( c_i^y \). The solution for this problem defines an upper bound on the marginal average periodic cost of commodity \( c_i^y \). Let us define the optimal solution for \( \theta_i \) by \( t_i \) for \( i = 1, 2 \) Substituting for \( h_{c_i^y}, K_{c_i^y}, \) and \( \lambda_{c_i^y} \) using Eqs. (4), (5), and (9) into Eq. (2),

we get:

\[
t_1 = t_{c_i^y}^*.
\]

\[
t_1 = t_{c_i^y}^*.
\]
Similarly, substituting for \( h_c, K_c, k_0, \) and \( \lambda_c \) using Eqs. (4), (5), (8), and (9) into Eq. (2), we get:

\[
t_2 = (1 + \delta)t^*_c y_i.
\]

**Corollary 1.** In any optimal solution to \( \Gamma \), \( t^*_c y_i \leq t_{c y} \leq (1 + \delta)t^*_c y_i \) for any \( c \in \text{Constants} \).

Let us consider a non-discrete cycle time \( t_{c y} \) to be the product of a discrete number \( t^*_c y_i \) and a seed \( \beta_{c y} \), i.e., \( t_{c y} = t^*_c y_i \cdot \beta_{c y} \), where \( 1 \leq \beta_{c y} \leq 1 + \delta \).

For the analysis that follows we make use of two functions that quantify the average number of joint replenishments per time unit. Let us assume that all the commodities are ordered at time 0. If the ratios between all \( \beta_{c y} \) are rational, there is a time \( T \), which is the least common multiplier of all cycle times, in which all commodities will be ordered together again. Hence, it is sufficient to find the average number of joint replenishments within \( T \) only. If, on the other hand, the ratios between some \( \beta_{c y} \) are irrational, then they will never be ordered together again. In that case we can calculate the average number of joint replenishments per period for each set of commodities with a rational ratio between them separately and sum these averages. Accordingly, we refer to a finite time horizon \( T \). Each cycle time \( t_{c y} \) represents a series of orders \( t_{c y} \) time apart. Let us denote such a series by \( F_{t_{c y}} \), where \( F_{t_{c y}} = \{t_{c y}, 2t_{c y}, \ldots, T\} \). A union over all these series gives us the set of all joint orders in time horizon \( T \). The average number of joint replenishments per time period is the cardinality of the set of order points within the time horizon divided by \( T \). That is:

\[
\frac{\left| \bigcup_{t_{c y} \in S} F_{t_{c y}} \right|}{T},
\]

where the absolute value (within the vertical bars) defines the cardinality of a set.

Comparing the average number of joint replenishments, we define two functions over general sets of time points \( F_1, F_2, \ldots, F_m \) (all bounded by \( T \)).
Definition 2 (Union of Joint Replenishment (UJR)). Function $UJR: F^m \mapsto \mathbb{R}^+$, where $m \leq n$ is the number of time series unionized, represents the average number of joint replenishments per time unit \emph{where at least one commodity belonging to one of the series $F_i$ for $i = 1, \ldots, m$ is ordered.}

$$UJR(F_1, \ldots, F_m) = \frac{\left| \bigcup_{i=1}^{m} F_i \right|}{T}.$$ 

Definition 3 (Intersection of Joint Replenishment (IJR)). Function $IJR: (F^m) \mapsto \mathbb{R}^+$ represents the average number of joint replenishments per time unit \emph{where at least one commodity from each series $F_i$ for $i = 1, \ldots, m$ is ordered.}

$$IJR(F_1, \ldots, F_m) = \frac{\left| \bigcap_{i=1}^{m} F_i \right|}{T}.$$ 

For only two cycle times, $IJR$ could be calculated explicitly by using the least common multipliers (LCM). LCM could be used to define a tight upper bound on the frequency of intersection between two (or more) arithmetic sequences of numbers. This is an upper bound because it reflects the frequency of the intersection between the sequences that are in-phase, and if they’re out of phase, the number of intersections would be zero (consider for example a commodity ordered every even time period and a commodity ordered every odd time period). However, an optimal solution will always strive to synchronize commodities orders to minimize the number of joint replenishments. Hence, 

$$IJR(F_{t_1}, F_{t_2}) = \frac{1}{\text{LCM}(t_1, t_2)}. \quad (11)$$

Note that LCM satisfies:

$$\text{LCM}(a \cdot b, a \cdot c) = a \cdot \text{LCM}(b, c).$$

$UJR$ and $IJR$ are in fact the cardinality of union and intersection of sets normalized by a constant $T$. Accordingly, $UJR$ and $IJR$ hold the cardinality characteristics of union and intersection, some of which we make use of in our proof.

1. $UJR(F_1, F_2) = UJR(F_1) + UJR(F_2) - IJR(F_1, F_2)$
2. $\text{IJR}(F_1, F_2, F_3) \leq \text{IJR}(F_1, F_2)$

3. $\text{IJR}(F_1, (F_2 \cup F_3)) = \text{IJR}(F_1, F_2) + \text{IJR}(F_1, F_3) - \text{IJR}(F_1, F_2, F_3)$

4. $\text{IJR}(F_1, F_2) \leq \text{IJR}(F_3, F_4)$ for $F_1 \subseteq F_3$ and $F_2 \subseteq F_4$.

Note that we do not always calculate $UJR$ or $IJR$ explicitly, but rather make use of the bounds on those functions in the analysis that follows.

**Theorem 1.** In any optimal solution all seeds to the set $\text{Constants}$ will be identical, i.e., \( \exists \beta, \forall c_i \in \text{Constants} : \beta_{c_i} = \beta \).

We will prove the Theorem 1 by contradiction. To those means, we will assume that there exists an optimal solution $S$ that has at least two different seeds ($\beta$’s).

Let $S$ be such a solution, i.e., an optimal solution where the commodities are centered around $k$ different seeds $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_k$. We denote a subset of the commodities that are centered around the same seed $\beta_i$ in $S$ as $A_i$. That is, $c_i \in A_i$ iff $t_{c_i} = t^*_{c_i} \beta_i$. In what follows, we compare solution $S$ with some solution $S'$, which is similar to $S$ with one difference: the cycle times of the commodities in set $A_2$ in $S'$ are changed to be centered around seed $\beta_1$ instead of $\beta_2$, which means that $\forall c_i \in A_2 : t_{c_i} = t^*_{c_i} \beta_1$. We proceed to show that $S'$ is cheaper than $S$, which contradicts the optimality of $S$.

We separate the cost functions of $S$ and $S'$ to the total standalone costs and the joint replenishment costs and compare them separately.

According to Eq. (10), the optimal solution to $\theta_1$, (which is the standalone cost of commodity $c_i \in \text{Constants}$) is: $t_{c_i} = t^*_{c_i}$. Since the standalone cost is convex, the closer the seed $\beta_c$ is to 1, the cheaper the standalone cost is. The difference between $S$ and $S'$ is in set $A_2$ whose seed was changed from $\beta_2$ to $\beta_1$. Since $1 \leq \beta_1 < \beta_2$ by definition, we have the following corollary:

**Corollary 2.** The cumulative standalone cost of all commodities in $S'$ is smaller than that in $S$.

Next, we show that the cumulative joint replenishment cost in $S'$ is smaller than in $S$.

The cost of joint replenishment is linearly dependent on the average number of joint replenishments per time unit, or, in terms of $UJR$ functions, we’d like to show that $UJR_{S'} - UJR_S \leq 0,$
where $UJR_x$ is short for the $UJR$ function on all the commodities of an arbitrary solution $x$. Let us represent the time series of commodities that share the same seed, $\beta_i$, as a pair $(A_i, \beta_i)$. That is, $(A_i, \beta_i) = \bigcup_{c_i^q \in A_i} F_{\beta_i}, c_i^q$. The expressions for the $UJR_{S'}$ and $UJR_S$, respectively, are:

$$UJR_{S'} = UJR((A_1, \beta_1), (A_2, \beta_2), (A_3, \beta_3), ..., (A_n, \beta_n)) =$$

$$= UJR((A_1, \beta_1), (A_3, \beta_3), (A_4, \beta_4), ..., (A_n, \beta_n)) + UJR((A_2, \beta_1))_{s'_1} - IJR((A_2, \beta_1), \{(A_1, \beta_1) \cup (A_3, \beta_3) \cup (A_4, \beta_4) \cup ... \cup (A_n, \beta_n)\})_{s'_{1}} \quad (12)$$

$$UJR_S = UJR((A_1, \beta_1), (A_2, \beta_2), (A_3, \beta_3), ..., (A_n, \beta_n)) =$$

$$= UJR((A_1, \beta_1), (A_3, \beta_3), (A_4, \beta_4), ..., (A_n, \beta_n)) + UJR((A_2, \beta_2))_{s_2} - IJR((A_2, \beta_2), \{(A_1, \beta_1) \cup (A_3, \beta_3) \cup (A_4, \beta_4) \cup ... \cup (A_n, \beta_n)\})_{s_{1}} \quad (13)$$

To simplify the tractability, we refer to the elements of $UJR_{S'}$ and $UJR_S$ in Eqs. (12) and (13) by their underlined notations, $S'_i$ and $S_i$ for $i = 1, 2, 3$. In what follows, we examine the difference $UJR_{S'} - UJR_S = \sum_{i=1}^{3} (S'_{i} - S_{i})$, and show that it is negative.

$S'_i = S_1$ and can both be omitted.

$S'_2 - S_2$ is positive, since in $S'$ we allow the commodities in $A_2$ to be ordered more often. However, since $(A_2, \beta_2)$ and $(A_1, \beta_1)$ represent the same time series multiplied by different constants, we have $UJR((A_2, \beta_2)) = UJR((A_2, \beta_1)) \frac{\beta_1}{\beta_2}$. Therefore,

$$S'_2 - S_2 = UJR((A_2, \beta_1)) \frac{\beta_1}{\beta_2} \leq UJR((A_2, \beta_1)) (\beta_1 - \beta_2) \leq \delta \cdot UJR((A_2, \beta_1)) \leq \delta, \quad (14)$$

where the first inequality comes from both $\beta_1 \geq 1$ and $\beta_2 > 1$, the second inequality comes from the fact that $1 \leq \beta_1 < \beta_2 \leq 1 + \delta$ , and the last inequality comes from the fact that all the elements of $A_2$ are integers and $\beta_1 \geq 1$ and thus, there could not be more than one order per period.
Now we shall decompose $S'_3$ and $S_3$ parts of Eqs. (12) and (13), respectively, by the rule introduced in the third characteristic of $UJR$ and $IJR$ we discussed above. Also, remember that an upper bound on $S_3$ and a lower bound on $S'_3$ are good enough:

$$S'_3 = IJR((A_2, \beta_1), \{(A_1, \beta_1), (A_3, \beta_3), (A_4, \beta_4), \ldots, (A_n, \beta_n)\})$$

$$= IJR((A_2, \beta_1), (A_1, \beta_1)) + IJR((A_2, \beta_1), \{(A_3, \beta_3), (A_4, \beta_4), \ldots, (A_n, \beta_n)\})$$

$$- IJR((A_2, \beta_1), (A_1, \beta_1), \{(A_3, \beta_3), (A_4, \beta_4), \ldots, (A_n, \beta_n)\})$$

$$\geq IJR((A_2, \beta_1), (A_1, \beta_1)),$$  \hspace{1cm} (15)

where the inequality follows the second characteristic of $UJR$ and $IJR$ we discussed above.

An upper bound on $S_3$ could be found by decomposing it according to the third characteristic of the $UJR$ and $IJR$ functions, and ignoring the negative elements. That is:

$$S_3 = IJR((A_2, \beta_2), \{(A_1, \beta_1), (A_3, \beta_3), (A_4, \beta_4), \ldots, (A_n, \beta_n)\})$$

$$\leq \sum_{i=1,3,4,\ldots,n} IJR((A_2, \beta_2), (A_i, \beta_i)).$$  \hspace{1cm} (16)

Considering Eqs. (12)-(16), we have:

$$UJR_{S'} - UJR_S = -IJR((A_2, \beta_1), (A_1, \beta_1)) + \delta + \sum_{i=1,3,4,\ldots,n} IJR((A_2, \beta_2), (A_i, \beta_i)).$$  \hspace{1cm} (17)

We’ll divide the expression into its composing parts and bound each of them separately. Note that $(A_2, \beta_1)$ and $(A_1, \beta_1)$ share the same seed. This means we can create a lower bound on their $IJR$ that is independent of $\delta$. The worst case scenario would be both groups consisting of exactly one commodity each (since adding any commodities would only increase the $IJR$), and those two commodities having the longest cycle time. According to Cohen-Hillel and Yedidsion (2018) the longest cycle time is associated with commodities of type $Clauses$, where for $c^*_i \in Clauses$, $t^*_{c^*_i}$ is the product of the primes associated with the clause’s three literals. Hence, $\forall i: t^*_{c^*_i} < \max \left\{ (\overline{p}_1)^3 \right\} = (\overline{p}_n)^3$. This means,

$$IJR((A_2, \beta_1), (A_1, \beta_1)) > \frac{1}{\beta_1 (\overline{p}_n)^9} > \frac{1}{(1+\delta) (\overline{p}_n)^9} > \frac{1}{2 (\overline{p}_n)^9}.$$  \hspace{1cm} (18)

To bound a term $IJR((A_2, \beta_2), (A_i, \beta_i))$ we would first need to prove an additional lemma,
Lemma 1. $\text{IJR}((A_i, \beta_i), (A_j, \beta_j)) \leq \delta$

We distinguish between two cases:

- $\frac{\beta_j}{\beta_i}$ is irrational.

In this case, $\text{IJR}((A_i, \beta_i), (A_j, \beta_j)) \to 0$, since there will be a maximum of one instance of joint replenishment in an infinite time horizon, and the replenishments will never coincide again.

- $\frac{\beta_j}{\beta_i}$ is rational.

According to the fourth characteristic of $\text{UJR}$ and $\text{IJR}$, we have $\text{IJR}((A_i, \beta_i), (A_j, \beta_j)) \leq \text{IJR}(F_{\beta_i}, F_{\beta_j})$, as the series covered by a single commodity with a cycle time of $\beta_i$ is a superset of the series $(A_i, \beta_i)$. Hence, we consider $\text{IJR}(F_{\beta_i}, F_{\beta_j})$ as an upper bound on $\text{IJR}((A_i, \beta_i), (A_j, \beta_j))$.

Since $\frac{\beta_j}{\beta_i}$ is rational, we can express it as an irreducible fraction $1 + \frac{q}{r}$, where $q$ and $r$ are positive integers.

Using the $\text{LCM}$ function to quantify $\text{IJR}(F_{\beta_i}, F_{\beta_j})$, we can observe that

$$\text{LCM} (\beta_i, \beta_j) = \beta_i \text{LCM} \left( 1, 1 + \frac{q}{r} \right) = \frac{\beta_i}{r} \text{LCM} (r, r + q) = \frac{\beta_i}{r} (r(r + q)) = \beta_i (r + q),$$

where the third equality is true because $r$ and $q$ share no common multipliers.

According to Eq. (11), $\text{IJR}(F_{\beta_i}, F_{\beta_j}) = \frac{1}{\beta_i (r + q)} \leq \frac{1}{r + q}$.

Now, let us consider the largest integer $a$, such that $\frac{1}{a} \geq \delta$. Since $\frac{\beta_j}{\beta_i} = 1 + \frac{q}{r} < 1 + \delta \leq 1 + \frac{1}{a}$, we have $a \cdot q < r$ and thus,

$$\text{IJR}((A_i, \beta_i), (A_j, \beta_j)) \leq \text{IJR}(F_{\beta_i}, F_{\beta_j}) \leq \frac{1}{(1 + a)q} \leq \frac{1}{a + 1} < \delta,$$

where the last inequality comes from the definition of $a$ as the largest integer for which $\delta \leq \frac{1}{a}$.

This completes the proof of Lemma 1.

According to Lemma 1, $\sum_{i=1,3,4,...,n} \text{IJR}((A_2, \beta_2i), (A_j, \beta_j)) \leq (n - 1)\delta$.

Substituting this along with Eqs. (18) and (3) into Eq. (17) we get:

$$\text{UJR}_{S'} - \text{UJR}_S \leq \frac{-1}{2 (\overline{P}_n)^p} + n\delta = \frac{-3n}{6n (\overline{P}_n)^p} + \frac{n}{6n (\overline{P}_n)^p} = \frac{-1}{3 (\overline{P}_n)^p} < 0.$$

This contradicts the optimality assumption of $S$ and proves Theorem 1.
3.4. Variables cycle times

Theorem 1 establishes that the processing of all commodities of type *Constants* share the same seed $\beta$. Normalizing the time by $\beta$, all these commodities are ordered at the same discrete cycle times as in Cohen-Hillel and Yedidsion (2018). Next, we show that the cycle times of commodities of type *Variables* are also restricted to the same discrete cycle times (when time is normalized by $\beta$) as in Cohen-Hillel and Yedidsion (2018).

**Theorem 2.** In an optimal solution to $\Gamma$, $\forall c_{xi}^r \in \text{Variables}: t_{c_{xi}^r} \in \{p_i, \overline{p}_i\}$.

Our proof makes use of Cohen-Hillel and Yedidsion (2018), who proved that the cycle time of $c_{xi}^r \in \text{Variables}$ is one of two unique prime numbers $p_i$ and $\overline{p}_i$. The standalone cost of these commodities, $t_{c_{xi}^r}^*$, satisfies $p_i < t_{c_{xi}^r}^* < \overline{p}_i$. Their proof was constructed of two parts

1. The cycle time of $c_{xi}^r \in \text{Variables}$ satisfies $t_{c_{xi}^r} > p_i - 1$ and $t_{c_{xi}^r} < \overline{p}_i + 1$.

2. The cost of setting $t_{c_{xi}^r} \in \{p_i, \overline{p}_i\}$ dominates any other solution where $t_{c_{xi}^r} \in (p_i - 1, \overline{p}_i + 1)$. The proof of the first part simply showed that just the standalone cost (without paying any joint replenishment costs) of a solution with $t_{c_{xi}^r} \in \{p_i - 1, \overline{p}_i + 1\}$ is more expensive than the worst case solution for $t_{c_{xi}^r} = \overline{p}_i$. This proof is independent of the discrete environment and holds in a continuous environment as well, and due to the convexity of the standalone cost, that means that in an optimal solution of $\Gamma$, $t_{c_{xi}^r} \in (p_i - 1, \overline{p}_i + 1)$.

The second part of their proof refers to the very high synchronization between the cycle times of commodities of type *Constants* with both $p_i$ and $\overline{p}_i$. Cohen-Hillel and Yedidsion (2018) constructed their $\mathcal{NP}$-hard instance by setting the optimal cycle times of $c_{yj}^r \in \text{Constants}$ to be a multiplication of each of the primes $p_i$ and $\overline{p}_i$ with a very large set of other primes (for each such multiplication, a commodity of type *Constants* was created). Accordingly, when setting $t_{c_{xi}^r} \in \{p_i, \overline{p}_i\}$ the marginal cost added to the total cost due to joint replenishments is very small. They show that even when assuming maximum synchronization between the commodities of type *Variables*, for any solution that is not $p_i$ or $\overline{p}_i$ but still within the range $t_{c_{xi}^r} \in (p_i, \overline{p}_i)$, and assuming the optimal standalone costs for these commodities, the marginal cost incurred by these commodities is still greater than...
the marginal cost of setting $t_{c_i^x} \in \{p_i, \overline{p}_i\}$. Next, we show that their bound on the synchronization between the cycle time assuming $t_{c_i^x} \in (p_i, \overline{p}_i)$ still holds in a continuous setting. Using our representation of continuous times as a multiplication of a discrete time and a seed, we refer to a continuous cycle time as $t_{c_i^x} \beta_i$. We may assume $1 \leq \beta_i < \frac{1}{t_{c_i^x}}$ (this is enough to allow us to cover any continuous value).

Considering two cycle times, $t_{c_i^x} \beta_i$ and $t_{c_j^x} \beta_j$, where $\beta_i \leq \beta_j$ (without loss of generality). The rate of synchronization of their cycle times $IJR\left(\left\{t_{c_i^x}\right\}, \beta_i, \left\{t_{c_j^x}\right\}, \beta_j\right)$, is maximized by minimizing $\text{LCM}(t_{c_i^x} \beta_i, t_{c_j^x} \beta_j)$. The LCM function is minimized by finding the largest common multiplier of the two numbers. For any seed, the largest common multiplier is itself, and the smaller the seed, the smaller the LCM. Thus,

$$\text{LCM} \left( t_{c_i^x} \beta_i, t_{c_j^x} \beta_j \right) \leq \text{LCM} \left( t_{c_i^x} \beta_i, t_{c_j^x} \beta_i \right) \leq \text{LCM} \left( t_{c_i^x}, t_{c_j^x} \right).$$

In other words, a common minimal seed gives us a lower bound to the synchronization of any two numbers. The lower bound used in Cohen-Hillel and Yedidsion (2018) is in fact a lower bound with a common seed of 1 and thus, the bound holds for a continuous setting as well.

Moreover, it applies also to any cycle time in the continuous ranges $(p_i - 1, p_i]$ and $[\overline{p}_i, \overline{p}_i + 1)$ not including $\beta \overline{p}_i$. The upper bound on the marginal cost of a solution with $t_{c_i^x} \in \{\beta p_i, \overline{p}_i\}$ also holds as all commodities share the same seed $\beta$ and are practically discrete under a time normalized by $\beta$.

To simplify our analysis, we define the function $\Delta_c(t_c)$ that describes the marginal average cost per time unit associated with commodity $c$’s cycle time, $t_c$, to the other commodities in the system. We denote a lower and an upper bound on any general function $f$ by $\text{LB}(f)$ and $\text{UB}(f)$, respectively. Accordingly, $\text{LB}(\Delta_c(t_c))$ and $\text{UB}(\Delta_c(t_c))$ are lower and upper bounds on $\Delta_c(t_c)$, respectively. Thus, the marginal contribution of a commodity $c_i^\ell \in \text{Variables}$ to the commodities $c_i^y \in \text{Constants}$ is composed of its standalone cost given by Eq. (1) and its marginal contribution to the joint replenishment cost, denoted:

$$jr_{c_i^\ell}^{c_i^x}(t_{c_i^x}) := UJR\left( F_{c_i^x}^{c_i^x}, (\text{Constants}, \beta) \right) - UJR\left( ((\text{Constants}, \beta)) \right).$$
According to the first characteristic of UJR and IJR:

\[ jr^{c_t}(t_{c_t}) = UJR(\hat{F}_{t_{c_t}}) - IJR(\hat{F}_{t_{c_t}}, (\text{Constants}, \beta)) \]

According to Cohen-Hillel and Yedidsion (2018), for \( t_{c_t} \neq \beta_{\text{p}_i} \)

\[ jr^{c_t}(t_{c_t}) \geq \frac{K_0(\alpha_v, \alpha_n)}{t_{c_t}} \geq \frac{K_0(\alpha_v, \alpha_n)}{\beta t_{c_t}} \]  

(19)

and for \( t_{c_t} \in \{\beta_{\text{p}_i}, \beta_{\text{p}_i} \} \)

\[ jr^{c_t}(t_{c_t}) \leq \frac{K_0(\alpha_c)}{\beta t_{c_t}} \]  

(20)

where \( \alpha_v, \alpha_n, \) and \( \alpha_c \) are constants taken from Cohen-Hillel and Yedidsion (2018), which we decided to leave as is for ease of validation.

Next, we show that \( UB\left(\Delta_{c_t}\left(\beta_{\text{p}_i}\right)\right) < LB\left(\Delta_{c_t}\left(\beta_{\text{p}_i} + y\right)\right) \), where \( y \neq 0 \). Using Eq. (19) with \( t_{c_t} = \beta(p_i + y) \), Eq. (20) with \( t_{c_t} = \beta_p \), and Eq. (1), we get:

\[
\begin{align*}
LB\left(\Delta_{c_t}\left(p_i + y\right)\right) - UB\left(\Delta_{c_t}\left(p_i\right)\right) &= K_{c_t} + K_0(\alpha_v, \alpha_n) + \beta h_{c_t}\left(p_i + y\right) - \left(\frac{K_{c_t}}{\beta_{\text{p}_i}} + \beta_{\text{p}_i} h_{c_t} + K_0 \cdot \frac{\alpha_c}{\beta_{\text{p}_i}}\right) \\
&= -\frac{yK_{c_t}}{\beta_{\text{p}_i}} + \beta y h_{c_t} + K_0 \left(\frac{\alpha_v, \alpha_n}{\beta\left(p_i + y\right)} - \frac{\alpha_c}{\beta_{\text{p}_i}}\right).
\end{align*}
\]

(21)

Substituting for \( h_{c_t}, K_{c_t} \) and \( K_0 \) using Eqs. (6), (7) and (8) into Eq. (21) we get:

\[
\begin{align*}
LB\left(\Delta_{c_t}\left(p_i + y\right)\right) - UB\left(\Delta_{c_t}\left(p_i\right)\right) &= -\alpha_c y \frac{p^2 - b_i}{\beta\left(p_i + y\right)} \left(\frac{p + b_i}{\beta\left(p_i + y\right)} - \beta\right) + y \frac{p + b_i}{\beta\left(p_i + y\right)} \frac{\alpha_v, \alpha_n}{\beta\left(p_i + y\right)} + \frac{\alpha_v, \alpha_n}{\beta_{\text{p}_i}} - \frac{\alpha_c}{\beta_{\text{p}_i}} \\
&= -\alpha_c y \frac{p^2 - b_i}{\beta\left(p_i + y\right)} \left(\frac{p + b_i}{\beta\left(p_i + y\right)} - \beta\right) + y \frac{p + b_i}{\beta\left(p_i + y\right)} \frac{\alpha_v, \alpha_n}{\beta\left(p_i + y\right)} + \frac{\alpha_v, \alpha_n}{\beta_{\text{p}_i}} - \frac{\alpha_c}{\beta_{\text{p}_i}} \\
&= -\alpha_c y \frac{p^2 - b_i}{\beta\left(p_i + y\right)} \left(\frac{p + b_i}{\beta\left(p_i + y\right)} - \beta\right) + y \frac{p + b_i}{\beta\left(p_i + y\right)} \frac{\alpha_v, \alpha_n}{\beta\left(p_i + y\right)} + \frac{\alpha_v, \alpha_n}{\beta_{\text{p}_i}} - \frac{\alpha_c}{\beta_{\text{p}_i}}.
\end{align*}
\]
The last term is identical to the term for $LB \left( \Delta_{c_i} \left( p_i + y \right) \right) - UB \left( \Delta_{c_i} \left( p_i \right) \right)$ in Cohen-Hillel and Yedidsion (2018)\textsuperscript{2} multiplied by a positive constant $\frac{1}{\beta}$, which does not affect the remainder of the proof in Cohen-Hillel and Yedidsion (2018), showing that this term is positive.

3.5. Optimal solution

In this section, we trace the two proofs in Cohen-Hillel and Yedidsion (2018) mentioned in §3.1 and show that they are valid for $t_{c_i} \in \left\{ \beta p_i, \beta^* p_i \right\}$ and $t_{c_i}^* = \beta t_{c_i}^*$ instead of $t_{c_i} \in \left\{ p_i, \overline{p}_i \right\}$ and $t_{c_i}^* = \overline{t}_{c_i}^*$, respectively.

We denote $\Delta_{c_i}^{TC_{Variables}} \left( t_{c_i}^*, S \right)$ as the marginal average periodic cost added to the function $TC_{Variables} \left( S \right)$ associated with commodity $c_i$’s cycle time $t_{c_i}^*$, where $c_i \in Variables$, $t_{c_i}^* \in \left\{ \beta p_i, \beta^* p_i \right\}$ and a solution $S$ that applies the characteristics of an optimal solution in §3.3 and §3.4 to the other commodities in the system. In the next lemma, we formulate bounds on $\Delta_{c_i}^{TC_{Variables}} \left( t_{c_i}^*, S \right)$.

**Lemma 2.** For any solution $S$ that satisfies the condition of Theorem 2, $\Delta_{c_i}^{TC_{Variables}} \left( \beta \overline{p}_i, S \right) < \Delta_{c_i}^{TC_{Variables}} \left( \beta p_i, S \right)$.

The function $\Delta_{c_i}^{TC_{Variables}} \left( t_{c_i}^*, S \right)$ is composed of two parts: the standalone cost of $c_i^*$, denoted $SAC_{c_i}^* \left( t_{c_i}^* \right)$ and $SAC_{c_i}^* \left( \beta p_i \right)$, and the marginal contribution to the joint replenishment cost, $jr_{c_i}^* \left( t_{c_i}^* \right)$. Thus:

$$\Delta_{c_i}^{TC_{Variables}} \left( t_{c_i}^*, S \right) = SAC_{c_i}^* \left( t_{c_i}^* \right) + jr_{c_i}^* \left( t_{c_i}^* \right).$$

We base our proof on Cohen-Hillel and Yedidsion (2018),\textsuperscript{3} which states that $\Delta_{c_i}^{TC_{Variables}} \left( p_i, S \right) < \Delta_{c_i}^{TC_{Variables}} \left( \overline{p}_i, S \right)$. Since for $c_i$, $p_i < \beta p_i < t_{c_i}^* < \overline{p}_i < \beta \overline{p}_i$, and due to the convexity of the cost function, we may infer that the standalone costs of $c_i^*$ satisfies:

$$SAC_{c_i}^* \left( p_i \right) \geq SAC_{c_i}^* \left( \beta p_i \right),$$

(23)
and

\[ SAC^c_i (\overline{p}_i) \leq SAC^c_i (\beta \overline{p}_i) \]  \hspace{1cm} (24)

\[ j^c_i (p) - j^c_i (\overline{p}_i) > 0. \]

Assuming all commodities share the same \( \beta \), \( j^c_i (t) \) is linearly increasing in \( \frac{1}{\beta} \). Cohen-Hillel and Yedidsion (2018) showed that \( j^c_i (p_i) - j^c_i (\overline{p}_i) > 0 \). Hence,

\[ j^c_i (p) - j^c_i (\overline{p}_i) \geq \frac{1}{\beta} \left( j^c_i (p) - j^c_i (\overline{p}_i) \right) = j^c_i (\beta p_i) - j^c_i (\beta \overline{p}_i). \]  \hspace{1cm} (25)

According to (22), and substituting Eqs. (23), (24), and (25):

\[ \Delta^c_{i \in \text{Variables}} (\beta \overline{p}_i, S) - \Delta^c_{i \in \text{Variables}} (\beta p_i, S) > \Delta^c_{i \in \text{Variables}} (\overline{p}_i, S) - \Delta^c_{i \in \text{Variables}} (p_i, S) > 0, \]

where the second inequality follows from Cohen-Hillel and Yedidsion (2018). This completes the proof of Lemma 2.

Let us denote the following four solutions

- \( \underline{S} \) - \( \forall c \in \text{Variables}: t_{c,i} = p_i \)
- \( \overline{S} \) - \( \forall c \in \text{Variables}: t_{c,i} = \overline{p}_i \)
- \( \beta \underline{S} \) - \( \forall c \in \text{Variables}: t_{c,i} = \beta p_i \)
- \( \beta \overline{S} \) - \( \forall c \in \text{Variables}: t_{c,i} = \beta \overline{p}_i \)

Using Lemma 2 we get that in a continuous environment:

\[ \text{LB} (TC_{\text{Variables}} (S)) = TC_{\text{Variables}} (\beta \underline{S}) \]

and

\[ \text{UB} (TC_{\text{Variables}} (S)) = TC_{\text{Variables}} (\beta \overline{S}). \]

We consider the time series induced by the entire set of Variables in a given solution \( S \) and denote it \( F(S) \). Let us extend the notation of \( j^c_i (t_{c,i}) \) to this set:

\[ j^S(\beta) = UJR (F(S)) - IJR (F(S), (\text{Constants}, \beta)). \]
In addition, we denote $jr^{c_i}(S, \beta)$ to be the marginal addition of a single commodity $c_i$ to the joint replenishment cost assuming solution $S$.

$$jr^{c_i}(S, \beta) = UJR\left(F_{t_{c_i}}^{*}\right) - IJR\left(F_{t_{c_i}}^{*}, (\text{Constants}, \beta)\right).$$

Cohen-Hillel and Yedidsion (2018) proved that the lower bound on a solution with even one unsynchronized commodity of type $\text{Clauses}$ is greater than the upper bound on a solution where all commodities of type $\text{Clauses}$ are satisfied. That is, given a solution $S$ with a commodity $C_i \in \text{Clauses}$ that is unsynchronized with any of its associated variable commodities:

$$LB\left(TC_{\text{Variables}}(S)\right) + LB\left(jr^{c_i}(S, 1)\right) > UB\left(TC_{\text{Variables}}(S)\right)$$

And according to Lemma $\text{[2]}$

$$TC_{\text{Variables}}(S) + LB\left(jr^{c_i}(S, 1)\right) > TC_{\text{Variables}}(\overline{S}).$$

Based on this, we shall prove the following Lemma:

**Lemma 3.** $TC_{\text{Variables}}(\beta S) + LB\left(jr^{c_i}(S, \beta)\right) - TC_{\text{Variables}}(\overline{\beta S}) > 0$

The cost $TC_{\text{Variables}}(\beta S)$ is constructed of two elements, the standalone cost and the marginal addition to the number of joint replenishments per period:

$$TC_{\text{Variables}}(\beta S) = \sum_{c_i \in \text{Variables}} SAC^{c_i}(t_{c_i}) + K_0 \cdot jr^S(\beta)$$

$$= \sum_{c_i \in \text{Variables}} \left(\frac{K_{r_c}}{t_{c_i}} + t_{c_i} \cdot h_{c_i}\right) + K_0 \cdot jr^S(\beta).$$

Accordingly,

$$TC_{\text{Variables}}(\beta S) - TC_{\text{Variables}}(\overline{\beta S}) + LB\left(jr^{c_i}(S, \beta)\right)$$

$$= \sum_{c_i \in \text{Variables}} \left(\frac{K_{r_c}}{\beta p} + \beta p \cdot h_{c_i} - \frac{K_{r_c}}{\beta p} - \beta p \cdot h_{c_i}\right) + K_0 \left(jr^{\beta S}(\beta) - jr^{\overline{\beta S}}(\beta) + LB\left(jr^{c_i}(S, \beta)\right)\right).$$
Substituting for $p_i = (p_i + b_i)$

$$TC_{\text{Variables}}(\beta S) - TC_{\text{Variables}}(\beta \overline{S}) + LB \left( j r_c^i (S, \beta) \right)$$

$$= \sum_{c_i \in \text{Variables}} \left( \frac{b_i K_{c_i}^x}{\beta p_i (p_i + b_i)} - \beta \cdot b_i \cdot h_{c_i} \right) + K_0 \left( j r_{\beta S}^i(\beta) - j r_{\beta \overline{S}}^i(\beta) + LB \left( j r_c^i (S, \beta) \right) \right), \quad (26)$$

Substituting for $K_{c_i}^x$ and $K_0$ using Eqs. (7) and (8) into Eq. (26) we get:

$$TC_{\text{Variables}}(\beta S) - TC_{\text{Variables}}(\beta \overline{S}) + LB \left( j r_c^i (S, \beta) \right)$$

$$= \sum_{c_i \in \text{Variables}} \left( b_i \cdot h_{c_i} \left( \frac{1}{\beta} - \beta \right) - \frac{\alpha_c \overline{a}_v b_i}{\beta p_i (p_i + b_i - 1)} \right) + \left( j r_{\beta S}^i(\beta) - j r_{\beta \overline{S}}^i(\beta) + LB \left( j r_c^i (S, \beta) \right) \right)$$

$$> \sum_{c_i \in \text{Variables}} \left( \frac{1}{\beta} - \beta \right) - \frac{\alpha_c \overline{a}_v b_i}{\beta p_i (p_i + b_i - 1)} + \left( j r_{\beta S}^i(\beta) - j r_{\beta \overline{S}}^i(\beta) + LB \left( j r_c^i (S, \beta) \right) \right),$$

where the inequality comes from $b_i \cdot h_{c_i} < 1$ and $\beta \geq 1$ (which makes $\left( \frac{1}{\beta} - \beta \right) \leq 0$).

The joint replenishment frequency scales linearly with the $\beta$, hence the cost is linear to $\frac{1}{\beta}$. Thus:

$$TC_{\text{Variables}}(\beta S) - TC_{\text{Variables}}(\beta \overline{S}) + LB \left( j r_c^i (S, \beta) \right)$$

$$\geq \frac{1}{\beta} \sum_{c_i \in \text{Variables}} \left( 1 - \beta^2 \right) - \frac{\alpha_c \overline{a}_v b_i}{p_i (p_i + b_i - 1)} + \frac{1}{\beta} \left( j r_{\overline{S}}^i(1) - j r_{\overline{S}}^i(1) + LB \left( j r_c^i (S, 1) \right) \right)$$

$$\geq \frac{1}{\beta} \sum_{c_i \in \text{Variables}} \left( 1 - \beta^2 \right) (1 + \beta) - \frac{\alpha_c \overline{a}_v b_i}{p_i (p_i + b_i - 1)} + \frac{1}{\beta} \left( j r_{\overline{S}}^i(1) - j r_{\overline{S}}^i(1) + LB \left( j r_c^i (S, 1) \right) \right)$$

$$\geq \frac{1}{\beta} \sum_{c_i \in \text{Variables}} \left( -3 \delta - \frac{\alpha_c \overline{a}_v b_i}{p_i (p_i + b_i - 1)} \right) + \frac{1}{\beta} \left( j r_{\overline{S}}^i(1) - j r_{\overline{S}}^i(1) + LB \left( j r_c^i (S, 1) \right) \right)$$

$$= -3n \cdot \delta \frac{1}{\beta} + \frac{1}{\beta} \left[ \sum_{c_i \in \text{Variables}} \left( -\frac{\alpha_c \overline{a}_v b_i}{p_i (p_i + b_i - 1)} \right) + \left( j r_{\overline{S}}^i(1) - j r_{\overline{S}}^i(1) + LB \left( j r_c^i (S, 1) \right) \right) \right], \quad (27)$$

where the last inequality comes from $(1 - \beta > -\delta)$ and $\beta < 2$. The term in square brackets in Eq. (27) is, in fact, the term $TC_{\text{Variables}}(\overline{S}) - TC_{\text{Variables}}(S) + LB \left( j r_c^i (S, 1) \right)$. Cohen-Hillel and Yedidsion (2018) showed that:

$$TC_{\text{Variables}}(\overline{S}) - TC_{\text{Variables}}(S) + LB \left( j r_c^i (S, 1) \right) \geq \frac{1}{(p_n)^x}. \quad (28)$$
Substituting for $\delta$ using Eq. (3) and using Eq. (28) to replace the square brackets in Eq. (27) we get:

$$TC_{\text{Variables}}(\beta S) - TC_{\text{Variables}}(\beta \overline{S}) + LB\left(jr^\overline{S}S, \beta\right)$$

$$> \frac{1}{\beta} \left(\frac{-3n}{6n(\overline{p}_n)^6} + \frac{1}{(\overline{p}_n)^6}\right)$$

$$> \frac{1}{2} \cdot \frac{1}{2(\overline{p}_n)^6} = \frac{1}{4(\overline{p}_n)^6} > 0,$$

where the second inequality is true since $\beta < 2$. This completes our proof.

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