On completeness of the Ribaucour transformations for triply orthogonal curvilinear coordinate systems in $\mathbb{R}^3$*

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Let us consider the following system introduced in [1] and describing triply orthogonal curvilinear coordinate systems in $\mathbb{R}^3$:

\[
\begin{align*}
\frac{\partial \beta}{\partial x_0} &= \lambda^2 \frac{\partial \alpha}{\partial x_0}, \\
\frac{\partial \beta}{\partial x_1} &= -\lambda^2 \frac{\partial \alpha}{\partial x_1}, \\
\frac{\partial (\alpha + \beta)}{\partial x_2} &= (\beta - \alpha) \frac{\partial \log \lambda}{\partial x_2}.
\end{align*}
\]

If we will suppose that $\lambda(x_0, x_1, x_2)$ is a given function then (1) becomes a linear system for the unknowns $\alpha(x_0, x_1, x_2)$, $\beta(x_0, x_1, x_2)$. Its compatibility conditions reduce to a single equation for the function $M = \log \lambda$ (which we will call hereafter a "potential"):

\[
\frac{\partial^3 M}{\partial x_0 \partial x_1 \partial x_2} = \text{cth}(M) \frac{\partial M}{\partial x_0} \frac{\partial M}{\partial x_1} \frac{\partial M}{\partial x_2} + \text{th}(M) \frac{\partial M}{\partial x_1} \frac{\partial M}{\partial x_0} \frac{\partial M}{\partial x_2}.
\]

The system (1) has a remarkable property (discovered by G. Darboux, see [1]): if one has two its solutions $\{\alpha, \beta\}$ and $\{\bar{\alpha}, \bar{\beta}\}$ for a given potential $M(x_0, x_1, x_2)$ then one can obtain (performing a quadrature) a solution $\{\bar{\alpha}_1, \bar{\beta}_1\}$ of (1) with a new potential. The formulas of this transformation are given by:

\[
\begin{align*}
M_1 &= -M_0 + \ln(\beta/\alpha), \\
\bar{\alpha}_1 &= \bar{\alpha} - \frac{\xi}{\beta}, \\
\bar{\beta}_1 &= \bar{\beta} - \frac{\xi}{\alpha},
\end{align*}
\]

where $\xi(\bar{\alpha}, \bar{\beta}, x_2)$ is obtained with a quadrature from the following compatible system

\[
\begin{align*}
\frac{\partial \xi}{\partial \bar{\alpha}} &= \beta, \\
\frac{\partial \xi}{\partial \bar{\beta}} &= \alpha, \\
\frac{\partial \xi}{\partial x_2} &= -\frac{\bar{\alpha}^2 + \bar{\beta}^2}{2} \frac{\partial \bar{\alpha}}{\partial x_2} \left( \frac{\alpha - \beta}{\bar{\alpha} - \bar{\beta}} \right),
\end{align*}
\]

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in which one is to suppose that $\alpha$, $\beta$ are expressed as functions of $\alpha$, $\beta, x_2$. Note that $\alpha_1 = 1/\beta$, $\beta_1 = 1/\alpha$ also give a solution of $(M_1)$ (as we will denote below the system $\alpha$ with the potential $M_1$), and $\xi_1 = -\xi/\alpha\beta$ is the corresponding to $\alpha_1, \beta_1, \alpha, \beta$ solution of $(\alpha)$. This transformation is an analogue of the well known Moutard transformation (see. [1]).

As Darboux had shown (1, 2), $(3), (4)$, correspond to the transformation of the connected to the pair $\{\alpha, \beta\}, \{\alpha, \beta\}$ of solutions of $(\alpha)$ triply orthogonal curvilinear coordinate system by inversion. The variation of the second order $\{\alpha, \beta\}$ correspond to the so called Combescure transformations of the orthogonal coordinate system; composition of inversions and Combescure transformations are equivalent to compositions of the Ribaucour transformations. Hence we will call the transformation $(3), (4)$ the Ribaucour-Darboux transformation.

Let us construct starting from some fixed potential $M_0(x_0, x_1, x_2)$ for which one can find all solutions of $(\alpha)$ (e.g. $M_0 = \text{const}$) a chain of Ribaucour-Darboux transformations $(M_0) \rightarrow (M_1) \rightarrow (M_2) \rightarrow \ldots (M_k) \rightarrow \ldots$. The potential $M_k$ will depend (apriori) on $3k$ functions of 1 variable — initial data for the solutions $(\alpha_s, \beta_s)$ of $(M_s)$, $s = 0, \ldots, k - 1$. Indeed if we will define $u = \log(\beta/\alpha)$ then $M_{s+1} = -M_s + u$ and the equations for $u$ are given by the elimination of $\beta$ from $(\alpha)$ with $\beta = \alpha \exp(u)$:

\[
\begin{align*}
\partial_0 \partial_1 u &= -\frac{2M - e^{2u}}{e^{2M} + e^{2u}} \partial_1 u \partial_0 u + \frac{2M - e^{2u}}{e^{2M} + e^{2u}} \partial_1 u \partial_0 M + \frac{2M - e^{2u}}{e^{2M} + e^{2u}} \partial_0 u \partial_1 M, \\
\partial_0 \partial_2 u &= \frac{e^{2u}}{e^{2M} + e^{2u}} \partial_0 \partial_2 M + \frac{2e^{2u}}{e^{2M} + e^{2u}} \partial_1 u \partial_2 M - \frac{e^{2u}}{e^{2M} + e^{2u}} \partial_0 u \partial_2 u, \\
\partial_1 \partial_2 u &= \frac{e^{2u}}{e^{2M} + e^{2u}} \partial_1 \partial_2 M + \frac{2e^{2u}}{e^{2M} + e^{2u}} \partial_1 u \partial_2 M - \frac{e^{2u}}{e^{2M} + e^{2u}} \partial_1 u \partial_2 u,
\end{align*}
\]

$\partial_i = \partial/\partial x_i$. As one can easily check, $(\alpha)$ give the necessary and sufficient compatibility conditions for $(\beta)$. Then the known theorem by Darboux [1] p. 335 states that if one will fix in a neighborhood of $(x_0(0), x_1(0), x_2(0))$ the initial values $\varphi_0(x_0) = u(x_0, x_1(0), x_2(0)), \varphi_1(x_1) = u(x_0(0), x_1, x_2(0)), \varphi_2(x_2) = u(x_0(0), x_1(0), x_2)$ (supposed to be continuously differentiable) then there exists the unique solution of $(\beta)$ with such initial data. So for the Ribaucour-Darboux transformations the potential $M_{s+1}$ depends on $M_s$ and 3 functions of 1 variable — the initial data for $u_s$.

As we have shown in $(\alpha)$, the necessary quadratures $(\beta)$ may be avoided for $(M_s), s \geq 2$ ("the Bäcklund cube"). For the aforementioned case of the Moutard transformations the so called "pfaffian formulas" $(\gamma)$ are known as an analogue of the "wronskian formulas" for $(1 + 1)$-dimensional integrable systems $(\zeta)$.

The theory of Ribaucour transformations was extensively studied in the monographs by Darboux [1, 2] and in numerous papers of his contemporaries [3, 4, 6, 8] for the general case of the $n$-dimensional euclidean space. For example L.Bianchi constructed the traditional ( $(1 + 1)$-dimensional) Bäcklund transformation for the system of the resonant 3-wave interaction using Egorov reduction of the equations of triply orthogonal curvilinear coordinate systems (see [3, 13]). One may ask how wide is the class of potentials for $(\alpha)$ obtainable (via algebraic operations!) from a given potential $M_0$. In the theory of $(1 + 1)$-dimensional integrable equations the question of density of the finite gap solutions of the Korteweg-de Vries equation in the class of all quasiperiodic functions was answered positively. In [1] we have shown that for the case of the Moutard transformations (such transformations possess the typical properties of $(2 + 1)$-
dimensional Bäcklund transformations while the Moutard equation itself \( u_{xy} = M(x, y)u \) is formally \((1+1)\)-dimensional) the set of the potentials \( M_k \) obtainable form any fixed \( M_0 \) is "locally dense" in the space of all smooth functions in a sense to be detailed below.

In this paper we solve the problem of (local) density of \( M_k(x_0, x_1, x_2) \) obtainable from a given initial \( M_0(x_0, x_1, x_2) \) in the space of all solutions of \((\mathbb{2})\). Hence we have proved that the Ribaucour transformation lets us to construct "almost all" orthogonal curvilinear coordinate system in \( \mathbb{R}^3 \), consequently the systems of equations describing such coordinate systems \((\mathbb{2})\) in particular) shall be considered as "true" integrable \((2+1)\)-dimensional nonlinear systems.

Note that due to the known theorem by Darboux \([1, p. 335]\) the initial data for \((\mathbb{2})\) are given by \( \Phi^{(0,1)}(x_0, x_1) = M(x_0, x_1, 0) \), \( \Phi^{(0,2)}(x_0, x_2) = M(x_0, 0, x_2) \), \( \Phi^{(1,2)}(x_1, x_2) = M(0, x_1, x_2) \) in a neighborhood of \( (x_0(0), x_1(0), x_2(0)) = (0, 0, 0) \).

**Theorem 1** Let us fix some \( M_0(x_0, x_1, x_2) \in \mathcal{C}^{\infty} \) in a neighborhood of \((0,0,0)\). Then for any \( N = 0, 1, 2, \ldots \) one can find \( K \) such that for an arbitrary triple system of numbers \( P_x^{(0,1)}, P_x^{(0,2)}, P_x^{(1,2)} ; t_i \in \{x_0, x_2\}, r_i \in \{x_1, x_2\}, 0 \leq k \leq N \), the corresponding derivatives of the potential \( M_k \) from the chain of the Ribaucour-Darboux transformations coincide with \( P_x^{(i,j)} \):

\[
\partial_{x_0}^\alpha \partial_{x_1}^\beta \partial_{x_2}^\gamma M_K = P_x^{(0,1)} \overset{\text{def}}{=} P_{x_0}^{(0,1)} x_0 \ldots x_1 \ldots x_1, \\
\partial_{x_0}^\alpha \partial_{x_2}^\beta \partial_{x_1}^\gamma M_K = P_x^{(0,2)} \overset{\text{def}}{=} P_{x_0}^{(0,2)} x_0 \ldots x_2 \ldots x_2, \\
\partial_{x_1}^\alpha \partial_{x_2}^\beta \partial_{x_0}^\gamma M_K = P_x^{(1,2)} \overset{\text{def}}{=} P_{x_1}^{(1,2)} x_1 \ldots x_2 \ldots x_2.
\]

Obviously we suppose

\[
P \overset{\text{def}}{=} P^{(0,1)} = P^{(0,2)} = P^{(1,2)} = M(0, 0, 0), \quad P_{x_0}^{(0,1)} = P_{x_1}^{(0,1)} = P_{x_2}^{(0,2)} = \partial_0^1 M_K(0, 0, 0),
\]

\[
P_{x_1}^{(1)} = P_{x_1}^{(0,1)} = P_{x_1}^{(1,2)} = \partial_1^1 M_K(0, 0, 0), \quad P_{x_2}^{(2)} = P_{x_2}^{(0,2)} = P_{x_2}^{(1,2)} = \partial_2^2 M_K(0, 0, 0).
\]

**Proof** will be given inductively. For \( N = 0 \) choose \( K = 1 \); from \((\mathbb{3})\) \( M_1(0, 0, 0) = -M_0(0, 0, 0) + u_0(0, 0, 0) \). Since (for a given \( M_0 \)) one may set the initial value \( u_0(0, 0, 0) \) to be arbitrary, and all \( \partial_i^k u_0(0, 0, 0) \) set to be 0 (the initial Goursat data for \((\mathbb{3})\), one can easily show (differentiating \( M_1 = -M_0 + u_0 \)) that for \( N = 0 \), \( K = 1 \), the following basic induction proposition holds:

for any \( N \) one can find such \( K \) that the corresponding derivatives \( \partial_i^k \partial_j^l M_K \), \( i, j \in \{0, 1, 2\} \), \( m+n \leq N \), of a conveniently chosen \( M_K \) in \( (0, 0, 0) \) coincide with the given \( P_{x_i}^{(i,j)} \) for \( m+n \leq N \):

\[
P_{x_i}^{(i,j)} = \partial_i^m \partial_j^l M_K(0, 0, 0).
\]

Additionally the higher derivatives \( \partial_i^m \partial_j^l M_K \), \( m+n > N \), depend only on \( P_{x_i}^{(k,l)} \), \( p+q \leq N \), in the following way: \( \partial_i^m M_K \), \( m > N \), coincide with \( \partial_i^m M_0 \) (they do not depend on the lower derivatives \( \partial_i^m \partial_j^l M_K \), \( m+n \leq N \); \( \partial_0^m \partial_1^l M_K \), \( m+n > N \), \( m,n > 0 \),
depend LINEARLY only upon \( P_{x_i x_j}^{(1)} = \partial_{i}^{m} \partial_{j}^{n} M_{K}, \ p + q \leq N, \ p \leq m; \ \partial_{i}^{m} \partial_{j}^{n} M_{K}, \ \partial_{i}^{m} \partial_{j}^{n} M_{K}, \ m + n > N, \ m, n > 0, \) depend upon \( P_{x_i x_j}^{(0)} = \partial_{i}^{m} \partial_{j}^{n} M_{K}, \ p + q \leq N, \) as well as upon \( P_{x_i x_j}^{(0)}, P_{x_i x_j}^{(0)} \) for \( k + s \leq N, \ s \leq n. \)

**Inductive step.** Suppose that the basic induction proposition is true for the derivatives of \( M_{K} \) of orders \( \leq N = N_{0} \), let us prove it for \( N = N_{0} + 1 \). Let \( K_{0} \) be the corresponding to \( N = N_{0} \) number of \( M_{K_{0}} \). Performing \( 3N_{0} + 3 \) consecutive Ribaucour-Darboux transformations \( M_{K_{0}} \xrightarrow{w_{1} = 0} M_{K_{0}+1} \xrightarrow{w_{2} = 0} \ldots \xrightarrow{w_{N_{0}} = 0} M_{K_{0}+N_{0}} \xrightarrow{w_{N_{0}+1}} M_{K_{0}+N_{0}+1} \xrightarrow{w_{N_{0}+2}} \ldots \xrightarrow{w_{N_{0}+N_{0}}} M_{K_{0}+2N_{0}} \xrightarrow{w_{2N_{0}+1}} \ldots \xrightarrow{w_{2N_{0}+N_{0}}} M_{K_{0}+2N_{0}+1} \) and denoting for simplicity \( Q(x_{0}, x_{1}, x_{2}) = M_{K_{0}+3N_{0}+3} \) we have

\[
Q = (-1)^{N_{0}+1}(M_{K_{0}} - u_{1} + u_{2} - \ldots + w_{N_{0}} \mp u(0) \pm u(1) \mp u(2)).
\]

Let us call the **principal** derivative of the function \( u_{i}, \ 1 \leq i \leq N_{0} \), its derivative \( \partial_{i}^{m} u_{i} \) at \((0, 0, 0)\), and its **auxiliary** derivative the derivative \( \partial_{i}^{m+1-i} u_{i} \) at the same point. For \( v_{i} \) and \( w_{i} \), \( 1 \leq i \leq N_{0} \), respectively we suppose \( \partial_{i}^{2} v_{i}, \partial_{i}^{2} w_{i} \) to be the principal and \( \partial_{i}^{m+1-i} v_{i}, \partial_{i}^{m+1-i} w_{i} \), to be the auxiliary derivatives at \((0, 0, 0)\). All the other (nonmixed) derivatives of arbitrary orders for these functions will be set to 0 at \((0, 0, 0)\). For \( u(0), u(1), u(2) \) we suppose \( \partial_{i}^{m+1-i} u(0), \partial_{i}^{m+1-i} u(1), \partial_{i}^{m+1-i} u(2) \) at \((0, 0, 0)\) to be the principal derivatives, all the other (nonmixed) derivatives of arbitrary orders for these functions are set to 0 at \((0, 0, 0)\) (no auxiliary derivatives). Let us fix the values of \( u_{i}, v_{i}, w_{i}, u(k) \) and the values of their auxiliary derivatives at \((0, 0, 0)\) to be ”generic”: more precisely the respective inequalities for them will be specified below, such inequalities may be obtained using the values of \( P_{x_i x_j}^{(1)}, k + s \leq N_{0} + 1 \).

Inside the given step of the main induction (for \( N = N_{0} + 1 \)) we will perform an auxiliary induction over \( m \) in order to show the validity of the main inductive proposition for \( \partial_{i}^{m+1} Q \) for all \( m, n \) for conveniently chosen principal derivatives of \( u_{i}, v_{i}, w_{i}, u(k) \). For all \( m = 0 \partial_{i}^{N_{0}+1} Q = (-1)^{N_{0}+1}(\partial_{i}^{N_{0}+1} M_{K_{0}} \pm \partial_{i}^{N_{0}+1} u(1)). \) The value of \( \partial_{i}^{N_{0}+1} M_{K_{0}} \) at \((0, 0, 0)\) as we know from the main induction proposition is equal to \( \partial_{i}^{N_{0}+1} M_{0}. \) Consequently one may unambiguously determine the principal derivative \( \partial_{i}^{N_{0}+1} u(1) \) in such a way that \( \partial_{i}^{N_{0}+1} Q = P_{x_i x_j}^{(1)} \). Since \( \partial_{i}^{1} Q = (-1)^{N_{0}+1}(\partial_{i}^{1} M_{K_{0}} \pm \partial_{i}^{1} u_{N_{0}+1} \pm \partial_{i}^{1} w_{N_{0}+1-n}) \), \( 0 < n \leq N_{0} \), one may uniquely determine the derivatives \( \partial_{i}^{1} M_{K_{0}} \) (they may be chosen arbitrarily according to the same proposition) using the fixed above auxiliary derivatives of \( u_{i}, v_{i}, w_{i} \) in such a way that \( \partial_{i}^{1} Q \) be equal to the desired value \( P_{x_i x_j}^{(1)}. \) Note that due to

\[
M_{K_{0}+s} = (-1)^{s}(M_{K_{0}} - u_{1} + \ldots)
\]

the values of \( M_{K_{0}+s} \) and all their nonmixed derivatives at the origin may be consecutively (using induction on \( s \)) found from the derivatives of \( M_{K_{0}} \) and \( u_{i}, v_{i}, w_{i}, u^{(s)} \) with respect to \( x_{s} = 1 \). So they depend only on \( P_{x_i x_j}^{(0)} \). Applying \( \partial_{i}^{n} \), \( n > N_{0} + 1 \), to (14) we see that \( \partial_{i}^{n} Q \equiv \partial_{i}^{n} M_{K_{0}} \equiv \partial_{i}^{n} M_{0} \) due to the choice of the nonmixed derivatives of \( v_{i}, w_{i}, u^{(s)} \) and the inductive proposition.

The equality \( P_{x_i x_j}^{(0) + 1} = \partial_{i}^{N_{0}+1} Q \) is easy to obtain as before (for \( P_{x_i x_j}^{(1)} = \partial_{i}^{N_{0}+1} Q \) choosing the appropriate principal derivative \( \partial_{i}^{N_{0}+1} u(0) \). The choice of the lower derivatives \( \partial_{i}^{N_{0}} M_{0}, n \leq N_{0} \) is
analogous to the above case. The proposition about the independence of the higher derivatives on the lower ones is obvious as well.

Let us choose the principal derivative \( \partial_{x_0+1}^{N_0+1} u(0, 0, 0) \) in such a way that \( P_{x_0+1}^{(2)} = \partial_{x_0+1}^{N_0+1} Q \). The required dependence of \( \partial_{x_0+1}^{m} Q \), \( m > N_0 + 1 \) is obvious.

Thus for \( m = 0 \) and \( m = N_0 + 1 \) we have:

a) the principal derivatives \( \partial_{x_0}^{N_0+1} u(0), \partial_{x_i}^{N_0+1} u(1) \) and \( \partial_{x_i} u_i, i \leq m \), are already chosen in such a way that the respective equalities \( p + q \leq N_0 + 1, p \leq m \), are achieved (for \( Q = M_K \));

b) the higher derivatives \( \partial_{x_0}^{m} \partial_{x_i}^{q} Q, p + q \leq N_0 + 1, p \leq m \), and the already defined principal derivatives depend (according to the construction of \( Q \)) only on the values of \( P_{x_0+1}^{(0,1)} \).

\( \sigma + t \leq N_0 + 1, \sigma \leq m \).

We will show the validity of a), b) for \( 0 < m < N_0 + 1 \) by an auxiliary induction over \( m \), \( m \leq N_0 \). In order to perform its step \( (m = m_0 + 1) \) apply \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{N_0-m_0} \) to \( \{I\} \) taking into consideration the equations \( \{I\}, \{II\} \). We will obtain an expression including in its right hand side (besides the already defined values of the principal derivatives and \( \partial_{x_0}^{m} \partial_{x_i}^{q} M_{K_0} \), \( p \leq m_0 \)) the values of \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{N_0-m_0} M_{K_0}, \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} M_{K_0} \), \( q < N_0 - m_0 \), as well as the only (so far undefined) principal derivative \( \partial_{x_0}^{m_0+1} u_{m_0+1}(0, 0, 0) \) with the coefficient

\[
\begin{align*}
c_{m_0+1} = & - \frac{\exp(4M_L) + \exp(2u_{m_0+1})}{\exp(4M_L) - \exp(2u_{m_0+1})} \partial_{x_0}^{N_0-m_0} u_{m_0+1} + F,
\end{align*}
\]

\( L = K_0 + m_0 + 1 \), where \( F \) comprises only the defined above values of \( u_i, v_i \), their auxiliary derivatives at \( (0, 0, 0) \), the defined during the previous steps of the auxiliary induction principal derivatives \( \partial_{x_0}^{0} u_i, i \leq m_0 \), \( i \neq m_0 - 1 \), as well as \( \partial_{x_i}^{q} M_{K_0}(0, 0, 0) \) with \( q \leq N_0 \).

We need to show that \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} M_{K_0} \), \( q < N_0 - m_0 \), and the principal derivative \( \partial_{x_0}^{m_0+1} u_{m_0+1}(0, 0, 0) \) may be determined uniquely by the equality to be achieved and by \( \{I\} \) with \( p + q \leq N_0 + 1, p \leq m_0 + 1 \): due to \( q < N_0 - m_0 \) the coefficient of the principal derivative in \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} Q \) includes only the initial values of the functions at \( (0, 0, 0) \) and \( \partial_{x_i}^{q} M_{K_0} \) which were determined above. We remark now that the value of \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} M_{K_0+s}(0, 0, 0) \) depend linearly on the values of \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} M_{K} \) and \( \partial_{x_0}^{m_0+1} u_{m_0+1}(0, 0, 0) \). Expressing the values of \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} M_{K_0} \) linearly into the aforementioned quantities and substituting into the equality we are to establish on this inductive step, we may use the condition of genericity of \( u_{m_0+1}(0, 0, 0) \) and its auxiliary derivative \( \partial_{x_i}^{N_0-m_0} u_{m_0+1} \) and conclude that the resulting coefficient of the principal derivative is not equal to 0 (this is one of the inequalities constraining them) and consequently we can determine the principal derivative \( \partial_{x_0}^{m_0+1} u_{m_0+1}(0, 0, 0) \) and afterwards \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} M_{K_0} \), \( p \leq N_0 - m_0 \), uniquely in such a way that \( P_{x_0+1}^{(0,1)} \).

The proposition about the dependence of \( \partial_{x_0}^{m_0+1} \partial_{x_i}^{q} Q, m_0 + 1 + q \leq N_0 + 1 \), only upon the used \( P_{x_0+1}^{(0,1)} \), \( s \leq m_0 + 1 \), can be easily obtained from \( \{I\} \) taking into consideration \( \{I\} \) and \( \{II\} \). The auxiliary induction is therefore completed.

Consider now the equations

\[
P_{x_0+1}^{(0,2)} = \partial_{x_0+1}^{N_0+1-m} \partial_{x_2}^{m} Q(0, 0, 0),
\]
simultaneously. We will show the possibility to satisfy them by an appropriate choice of the principal derivatives \( \partial_2 v_i, \partial_2 w_i \), performing the second auxiliary induction over \( m \).

Indeed for \( m = 0 \) the above equations are already true: \( P^{(0)}_{x_0 N_0+1} = P^{(0)}_{x_0 N_0+1} = P^{(0)}_{x_0 N_0+1} \), as we established above. The proposition about the dependence of the higher order derivatives is trivial in this case. Suppose all these facts to be true for \( m \leq m_0 \), we will show their validity for \( m = m_0 + 1 \). We have (cf. (9), (11))

\[
\begin{align*}
P^{(0, 2)}_{x_0 N_0-m_0 x_2} = & \partial_1^{N_0-m_0} \partial_2^{m_0+1} Q = (-1)^{N_0+1} \left( k_1 \partial_1^{N_0-m_0} \partial_2^{m_0+1} M_{K_0} \right. \\
+ & a \partial_2^{m_0+1} v_{m_0+1} + b \partial_2^{m_0+1} w_{m_0+1} + F, \\
P^{(1, 2)}_{x_1 N_0-m_0 x_2} = & \partial_1^{N_0-m_0} \partial_2^{m_0+1} Q = (-1)^{N_0+1} \left( k_2 \partial_1^{N_0-m_0} \partial_2^{m_0+1} M_{K_0} \right. \\
+ & c \partial_2^{m_0+1} v_{m_0+1} + d \partial_2^{m_0+1} w_{m_0+1} + G,
\end{align*}
\]

where \( F \) and \( G \) comprise all the terms including only the already found during the previous inductive steps quantities as well as \( \partial_0 \partial_1^{m_0+1} M_{K_0} \), \( s < N_0 - m_0 \), \( \partial_1 \partial_2^{m_0+1} M_{K_0} \), \( s < N_0 - m_0 \). The coefficients \( k_i \) include only the values \( u_i, v_i, w_i, u^{(i)} \) at \((0, 0, 0)\). The coefficients \( a, d \) include respectively the auxiliary derivatives \( \partial_0 \partial_1^{m_0+1} v_{m_0+1}, \partial_0 \partial_1^{m_0+1} w_{m_0+1} \) with the coefficients \(-((\exp(2M_L) + \exp(2v_{m_0+1}))/((\exp(2M_L) - \exp(2w_{m_0+1})))((\exp(2v_{m_0+1}) + 1)), -(\exp(2M_T) - \exp(2w_{m_0+1})))/((\exp(2M_T) + \exp(2w_{m_0+1}))((\exp(2w_{m_0+1}) + 1)), L = K_0 + N_0 + m_0 + 1, T = K_0 + 2N_0 + m_0 + 1, \) definable via the fixed quantities, these coefficient do not vanish due to the genericity condition (one can find the corresponding inequalities from \( P^{(i,j)}_{x^i x^j} \)). Again from the
genericity of the auxiliary derivatives one can suppose the determinant \( \begin{vmatrix} a & b \\ c & d \end{vmatrix} \) to be nonzero at \((0, 0, 0)\) (again the inequalities for the corresponding auxiliary derivatives expressible in terms of \( P^{(i,j)}_{x^i x^j} \). \( \partial_0 \partial_1^{m_0+1} M_{K_0}, \partial_0 \partial_1^{m_0+1} M_{K_0} \) according to the main inductive proposition may be found from \( P^{(0, 1)}_{x_0 N_0-m_0} P^{(0, 2)}_{x_0 N_0-m_0} P^{(1, 2)}_{x_1 N_0-m_0}, s \leq m_0 + 1 \).

The quantities \( \partial_0 \partial_1^{m_0+1} M_{K_0}, s < N_0 - m_0, \partial_0 \partial_1^{m_0+1} M_{K_0}, s < N_0 - m_0 \) may be found from (8), (9) with \( q = m_0 + 1 \). Indeed applying \( \partial_0 \partial_1^{m_0+1} \), \( s < N_0 - m_0, \partial_0 \partial_1^{m_0+1} \), \( s < N_0 - m_0 \), to (9), one can conclude from (8) that the coefficients of the undefined so far principal derivatives \( \partial_2^{m_0+1} v_{m_0+1}, \partial_2^{m_0+1} w_{m_0+1} \) due to \( s < N_0 - m_0 \) vanish, consequently we (via an induction over \( s \)) unambiguously find \( \partial_0 \partial_1^{m_0+1} M_{K_0}, s < N_0 - m_0, \partial_0 \partial_1^{m_0+1} M_{K_0}, s < N_0 - m_0 \) (their coefficients do not vanish due to the genericity condition of the values of \( u_i \) and the other functions at \((0, 0, 0)\)) in such a way that (8), (9) hold for \( q = m_0 + 1, p + q < N_0 + 1 \).

Hence by an appropriate (and unique) choice of the principal derivatives \( \partial_2^{m_0+1} v_{m_0+1}, \partial_2^{m_0+1} w_{m_0+1} \) one can satisfy the equalities marked by the question sign in (2). Applying to (10) the operators \( \partial_0^{n} \partial_2, \partial_1^{n} \partial_2, m + n > N_0 + 1, n \leq m_0 + 1 \), we get the necessary result on the dependence of the higher derivatives \( \partial_0^{m} \partial_2^{Q}, \partial_1^{m} \partial_2^{Q} \) upon \( P^{(i,j)}_{x^i x^j} \).

Consequently the main inductive proposition and the Theorem are proved.

Remark. One may consider of interest to generalize this result for the case of the Ribaucour transformations of the orthogonal curvilinear coordinate systems in the \( n \)-dimensional euclidean
space (\([\mathbf{I}], \mathbf{II}\)). We shall note that the system describing such coordinates is an overdetermined system with the initial (Goursat like) data given by \(n(n-1)/2\) functions of two variables for arbitrary \(n\). Consequently we have an \((2+1)\)-dimensional system possessing a \((2+1)\)-dimensional Bäcklund transformation — the Ribaucour transformation. Unfortunately no representation analogous to (II) is known for this case. Nevertheless one may conjecture the hypothesis about the completeness of the Ribaucour transformations also in the \(n\)-dimensional case.

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