Geometrical description of algebraic structures: Applications to Quantum Mechanics

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Abstract

Geometrization of physical theories have always played an important role in their analysis and development. In this contribution we discuss various aspects concerning the geometrization of physical theories: from classical mechanics to quantum mechanics. We will concentrate our attention into quantum theories and we will show how to use in a systematic way the transition from algebraic to geometrical structures to explore their geometry, mainly its Jordan-Lie structure.

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1 Introduction

The two quotations above should make clear why we would like to privilege a geometrical description of physical systems. The geometrical description of physical systems uses more general objects than the traditional Euclidean spaces: differentiable manifolds, sometimes endowed with particular structures. After the formulation of General Relativity by means of the (pseudo-) Riemannian geometry, it is accepted without any doubt that the equations used to describe physical systems should be written in tensorial form. For instance, we may indeed consider classical Gauge Theories as the Theory of Connections.

Quantum theories, due to the superposition rule, are always formulated as theories on complex vector spaces or algebras (the Schrödinger equation on a Hilbert space and the Heisenberg equation on a $C^*$-algebra). It is however convenient to analyse the problem from a more general perspective, which is manifestly necessary when the character of rays rather than vector of pure states is taken into account. We hope that a geometrization of quantum mechanics may be used for a more sound theory of Quantum Gravity. It follows that to ‘geometrize’ quantum theories we should first describe some fundamental algebraic structures in tensorial terms and then apply this procedure to describe quantum theories by means of tensorial entities. The general ideology of this presentation is being elaborated in a book provisionally entitled: Geometrical Theory of Classical and Quantum Dynamical Systems.

We shall proceed by explaining first what we mean by a geometrical description of physical systems by considering Newton’s equations and Maxwell’s equations, then we consider the usual formulation of quantum mechanics and we finally introduce a tensorial description of the algebraic structures emerging in the usual formulation.
1.1 Newton’s equations

We shall start by considering Newton’s equations, this is a second-order differential equation on some connected and simply connected configuration space $Q$:

$$\frac{d^2 x}{dt^2} = F \left( x, \frac{dx}{dt} \right).$$

With this equation we associate a vector field $\Gamma$ on the tangent bundle $TQ$, or velocity phase space, of $Q$, say [19],

$$\Gamma = v \frac{\partial}{\partial x} + F(x,v) \frac{\partial}{\partial v}.$$

Having a vector field on a manifold we can use all transformations on $TQ$ to transform it and find an easier way to integrate it for instance. Usually the existence of additional geometrical structures compatible with the given dynamics will uncover properties of it and will help with its integration. Thus, given a dynamics $\Gamma$ one usually looks for compatible structures, among them and most noticeably, Poisson brackets. In other words, one tries to find out if the given dynamics is Hamiltonian with respect to some (in principle unknown) Poisson brackets. Poisson brackets are encoded into a Poisson bivector field, i.e. a $\Gamma$-invariant, contravariant skew-symmetric 2-tensor field $\Lambda$ such that $\mathcal{L}_\Gamma \Lambda = 0$. This equation has a clear tensorial meaning. This bivector field is required to satisfy $[\Lambda, \Lambda]_S = 0$, where $[\cdot, \cdot]_S$ is the Schouten bracket or, equivalently, the associated Poisson bracket:

$$\{f_1, f_2\} = \Lambda (df_1, df_2)$$

should satisfy the Jacobi identity, which is a quadratic relation. This condition for $\Lambda$ is a nonlinear partial differential equation. It may admit no solution, one solution or many solutions. When it has more than one solution the dynamical system we are describing is called a bi-Hamiltonian system and exhibits some integrability properties [16]. As a matter of fact we have to distinguish the case of degenerate and non-degenerate tensors. Whenever the Poisson tensor is non-degenerate one can define (‘modulo’ an arbitrary and irrelevant additive constant) a Hamiltonian function $H$ via

$$\Lambda(\cdot, dH) = \Gamma,$$

and the dynamics can be written in Hamiltonian form as:

$$\mathcal{L}_\Gamma f = \{f, H\}.$$
In this case we say that $\Gamma$ defines an inner derivation of the Poisson algebra defined by the Poisson bivector field on the space $\mathcal{F}(M)$ of smooth functions on the manifold (the tangent bundle in the case of second-order differential equations). If, instead, $\Lambda$ is degenerate, then $\Gamma$ is still a derivation of the Poisson algebra but it need not to be inner and it may define what is called an outer derivation, i.e. it will not be the image under $\Lambda$ of a 1-form. When it is the image of a 1-form, the 1-form needs to be closed only on vector fields which define inner derivations. We shall not insist on these aspects and refer the reader to the literature [21].

When $\Lambda$ is non-degenerate, the condition

$$\Lambda(df, dh) = 0, \forall f \in \mathcal{F}(M)$$

implies that the function $h$ is a constant function and we may associate a (symplectic) structure $\omega_\Lambda$ to $\Lambda$ and the quadratic condition coming from the Schouten bracket becomes $d\omega_\Lambda = 0$.

In the case of a second-order differential equation, by using $\tau_Q : TQ \to Q$ we may further require the localization property (i.e., the possibility of measuring simultaneously observables depending only on configuration variables):

$$\Lambda(\tau_Q^* dg_1, \tau_Q^* dg_2) = 0, \quad \forall g_1, g_2 \in \mathcal{F}(Q),$$

and then we find (see [5]) that there exists a function $L \in \mathcal{F}(TQ)$ such that

$$\omega_\Lambda = -d\theta_L,$$

where $\theta_L = S^*(dL)$ and $S$ denotes the soldering (1,1)-tensor field (or vertical endomorphism) [21]:

$$S = dx^j \otimes \frac{\partial}{\partial v^j}.$$  

In such a case the dynamics $\Gamma$ may be described in terms of a Lagrangian function $L \in \mathcal{F}(TQ)$ and if we introduce the Liouville vector field [21]

$$\Delta = v^i \frac{\partial}{\partial v^i},$$

which is the infinitesimal generator of dilations along the fibres, and the energy function $E_L = \Delta L - L$, then $(TQ, \omega_\Lambda, E_L)$ is a Hamiltonian system such that $i(\Gamma)\omega_\Lambda = dE_L$.  

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Then starting from Newton’s equations on $Q$ we have defined a second-order vector field on $TQ$ and if there exists a localizable compatible non-degenerate Poisson tensor $\Lambda$ we have defined a symplectic structure and a Lagrangian function such that the original dynamics is both a Hamiltonian and Lagrangian system, completing a geometrization of the original equations of motion.

**Remark:** This approach shows very clearly how we reduce $\text{Diff} (TQ)$ to $\text{Diff} (TQ, \Lambda)$ and further to tangent bundle diffeomorphisms according to Klein’s Erlangen programme, i.e. we may start with the diffeomorphism group and ‘break it’ to appropriate subgroups by means of additional structures. These subgroups in general are enough to identify the manifold along with the additional structures.

### 1.2 Maxwell’s equations

*Maxwell’s equations* for the electric and magnetic fields, $E$ and $B$, in empty space and without sources can be written as:

$$\frac{d}{dt} \begin{pmatrix} B \\ E \end{pmatrix} = \begin{pmatrix} 0 & -\text{rot} \\ \text{rot} & 0 \end{pmatrix} \begin{pmatrix} B \\ E \end{pmatrix},$$

which are evolution or dynamical equations on the space of electric and magnetic fields, and:

$$\nabla \cdot B = 0, \quad \nabla \cdot E = 0,$$

which are *constraint* equations. Here again we may rewrite this system of equations in Hamiltonian form, but the constraints Eqs. (4) will restrict the possible Cauchy data we may evolve with the evolutionary equations. It is possible to argue, and it is often done, that a Lagrangian description for these equations is possible by means of a degenerate (or gauge invariant) Lagrangian written on a bigger carrier space described by vector potentials $A = (A, \phi)$, such that $\text{rot} A = B$, and $E = \dot{A} - \nabla \phi$, The introduction of the vector potential is a way to take into account holonomic constraints given by $\text{div} B = 0$, the constraint on $E$ being on ‘velocities’ is a non-holonomic constraint. Thus we can achieve a geometrization of Maxwell’s equations in empty space without sources as a non-holonomic degenerate Lagrangian system on the space of vector potentials. The geometrization of Maxwell’s equations in empty space is completed in covariant form when considered as a theory of connections on a $U(1)$-principal bundle over space-time.
For completeness we comment on the covariant geometrical formulation of Maxwell’s equations when we consider also sources. A covariant geometrical formulation \cite{18, 20} requires the introduction of the Faraday 2-form
\[ D = E \wedge dt - B, \]
and the Ampère’s odd 2-form
\[ G = H \wedge dt + D, \]
with the odd 3-form
\[ J = \rho + j \wedge dt. \]

The equations
\[ \begin{align*}
  dF &= 0, \\
  dG &= J,
\end{align*} \]
must be supplemented with phenomenological constitutive equations
\[ \mathcal{C}(F, G, J) = 0. \]

The constitutive equations are ‘phenomenological’ relations between the Faraday tensor, the Ampère’s tensor and the sources. They need not be local, i.e.
\[ \mathcal{C}(F, G, J)(x, t) \neq \mathcal{C}(F(x, t), G(x, t), J(x, t)). \]

When they are local, additional geometrical structures may be associated with them.

More general Gauge Theories \cite{2} are also geometrical theories, indeed they can be considered, as Maxwell’s equations, theories of connections in principal bundles, for instance with structural group $SU(3) \times SU(2) \times U(1)$. In this respect also General Relativity may be considered a theory of (pseudo-) Riemannian connections. Both theories have been considered jointly in the framework of Kaluza–Klein theories \cite{6, 7}. Once again we have achieved a purely tensorial description of our physical system.

We will turn now our attention to quantum theories.

2 Geometrical description of Quantum Mechanics

The geometry of quantum mechanics, contrarily to what has happened with other physical theories, has played a minor rôle after its beginning. In fact,
von Neumann’s formulation of quantum mechanics in terms of the theory of Hilbert spaces constituted already a formidable geometrization of quantum mechanics, however, further geometrical analysis of the theory was not pursued.

2.1 Quantum Mechanics

Quantum mechanics [13] is usually described in the realm of Hilbert space theory in different ways called ‘pictures’. We will concentrate here only on the dynamics of quantum systems, i.e. on the equation describing the evolution of quantum states. Thus we will associate a complex separable Hilbert space $\mathcal{H}$ (the set of pure states) with our physical system and the observables of the system are the self-adjoint operators (not necessarily bounded) on $\mathcal{H}$. We will not analyze here other physical aspects of quantum systems like the measure process, etc.

- Schrödinger’s picture: The equation of motion, Schrödinger equation, is written as:
  \[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle, \quad |\psi\rangle \in \mathcal{H}, \] (5)

  where $H$ is a self-adjoint operator, typically unbounded, on the Hilbert space $\mathcal{H}$.

- Heisenberg’s picture: The equations of motion are written as:
  \[ i\hbar \frac{dA}{dt} = [H, A], \] (6)

  where $A$ and $H$ are self-adjoint operators on $\mathcal{H}$.

- Dirac’s interaction picture: The evolution equation is written now as:
  \[ \left( i\hbar \frac{d}{dt}U \right) U^{-1} = H, \] (7)

  where $U$ is a unitary operator on $\mathcal{H}$. This is a generalized Dirac picture written on the group of unitary transformations, in this sense it is a quantum theory written on a group manifold.

All these images can be seen as different realizations in associated vector bundles of a principal connection (see e.g. [13]).
2.2 Geometrical description of the Schrödinger picture

Previous descriptions of quantum mechanics are given, except for the generalized Dirac picture, on carrier spaces which are complex linear spaces. To describe the equations in tensorial terms [4] we should replace linear spaces with real manifolds and linear operators with tensor fields. We first consider the complex linear space as a real linear space. Then, to this end we may use our experience in going from special relativity to general relativity which replaces the affine Minkowski space with a general Lorentzian manifold. Let us recall what is done to go from special relativity described on some affine space modelled on a Minkowski vector space $V$ with Minkowskian metric (inner product) $\eta_{\mu\nu}x^\mu x^\nu$ to a description on a pseudo-Riemannian manifold $M$ by replacing the Minkowskian inner product with the metric tensor $g = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$.

We may perform a similar trick by formally replacing the inner Hermitean product $\langle \psi | \psi \rangle$ on the complex Hilbert space $\mathcal{H}$ with the Hermitean tensor $\langle d\psi | d\psi \rangle$.

Let us consider an orthonormal Hilbert basis in $\mathcal{H}$, say $\{|e_j\rangle \mid j = 1, 2, \ldots\}$, $\langle e_j | e_k \rangle = \delta_{jk}$, and define complex coordinate functions:

$$\langle e_j | \psi \rangle = z^j = x^j + iy^j, \quad |d\psi\rangle = (dz^j) |e_j\rangle.$$  \hspace{1cm} (8)

Our Hermitean tensor will give rise to:

$$\langle d\psi | d\psi \rangle = (dz^k \otimes dz^l) \langle e_k | e_l \rangle,$$

that written in real coordinates looks like:

$$\langle d\psi | d\psi \rangle = (dx^k \otimes dx^l + dy^k \otimes dy^l)\delta_{kl} + i (dx^k \otimes dy^l - dy^l \otimes dx^k)\delta_{kl}.$$

In this way we obtain a symmetric, Riemannian tensor, and a skew-symmetric symplectic tensor, i.e.

$$g = dx^k \otimes dx^k + dy^k \otimes dy^k, \quad \omega = dx^k \wedge dy^k.$$ \hspace{1cm} (9)

In a more rigorous way, the Hilbert space $\mathcal{H}$ can be seen as a real space and then both, a Riemann and a symplectic structure, are defined by:

$$g(v, w) = \text{Re} \langle v, w \rangle, \quad \omega(v, w) = \text{Im} \langle v, w \rangle.$$

The 2-form $\omega$ can be shown to be an exact 1-form, i.e. there exists a 1-form $\theta$ such that $\omega = -d\theta$. Moreover, there is a complex structure $J$ in $\mathcal{H}$ when
considered as a real space: the $\mathbb{R}$-linear map corresponding to multiply by the complex number $i$, $Jv = iv$, and therefore such that

$$J^2 = -I.$$ 

This complex structure relates the Riemann and the symplectic structures:

$$g(v_1, v_2) = -\omega(Jv_1, v_2), \quad \omega(v_1, v_2) = g(Jv_1, v_2),$$

together with:

$$g(Jv_1, Jv_2) = g(v_1, v_2).$$

By passing to a contravariant form the structures (9) can be substituted by the corresponding contravariant tensors:

$$G = \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^k} + \frac{\partial}{\partial y^k} \otimes \frac{\partial}{\partial y^k}; \quad \Lambda = \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial y^k}.$$ \hspace{1cm} (10)

We may associate with the first tensor a bi-differential operator:

$$(f_1, f_2) \equiv G(df_1, df_2) = [[\Delta, f_1], f_2],$$ \hspace{1cm} (11)

where the Laplacian $\Delta$ is the second-order differential operator:

$$\Delta f = \frac{\partial^2 f}{\partial x^k \partial x^k} + \frac{\partial^2 f}{\partial y^k \partial y^k}.$$ 

With the skew-symmetric tensor we may associate a Poisson bracket defined as:

$$\{f, g\} = \Lambda(df, dg) = \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial y^k} - \frac{\partial f}{\partial y^k} \frac{\partial g}{\partial x^k}.$$ \hspace{1cm} (12)

To the Schrödinger equation, Eq. (5), we associate the linear equation

$$\frac{dz^k}{dt} = A^k_j z^j.$$ \hspace{1cm} (13)

in the coordinate system $z^k$ introduced above, Eq. (8). When $H$ is Hermitean, the matrix $\|A^k_j\|$ is skew-Hermitean, i.e. the infinitesimal generator of a unitary transformation if $H$ defines a self-adjoint operator on $\mathcal{H}$.

If we associate a vector field $\Gamma$ with our linear equation, Eq. (13), as follows:

$$\Gamma = A^k_j z^j \frac{\partial}{\partial z_k},$$
we find that $\mathcal{L}_\Gamma \langle \cdot | \cdot \rangle = 0$, then $\Gamma$ preserves the Hermitean product. The vector field $\Gamma$ is at the same time Hamiltonian and Killing, i.e., it preserves both the symmetric and the skew-symmetric part separately.

We observe that for any Hermitean operator $A$ we may define an evaluation function which is real valued:

$$f_A(\psi) = \langle \psi | A | \psi \rangle,$$

and an expectation value function:

$$e_A(\psi) = \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle}.$$

We can compute the symmetric bracket defined before, Eq. (11), for pairs of evaluation functions and we find that:

$$(f_A, f_B) = G(df_A, df_B) = f_{AB+BA}.$$

Similarly we can compute the Poisson bracket of two evaluation functions, Eq. (12), and we get:

$$\{f_A, f_B\} = \Lambda(df_A, df_B) = f_{(AB-BA)}.$$

The function $f_A$ defines a Hamiltonian vector field that with the natural identification of $T\mathcal{H}$ with $\mathcal{H} \oplus \mathcal{H}$, can be seen to be given by $X_A(\psi) = -iA|\psi\rangle$, and whose integral curves are the solutions of the equation:

$$\frac{d}{dt} |\psi\rangle = -iA |\psi\rangle,$$

therefore the dynamical evolution corresponding to a given Hamiltonian $H$ is given by Schrödinger equation Eq. (5). Moreover, the expectation value function is such that:

$$(e_A, e_A) = \frac{\langle \psi | A^2 | \psi \rangle}{\langle \psi | \psi \rangle} - \left( \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle} \right)^2,$$

i.e. the physical interpretation of such a function is clear: $(e_A, e_A)$ is the square of the standard deviation.

**Remark:** More precisely, the space of pure states of a quantum system is not associated with a Hilbert space $\mathcal{H}$ but to the manifolds of rays of the
Hilbert space $\mathcal{H}$, this is to the projective space $\mathbb{P}\mathcal{H}$. For instance, if $\mathcal{H} = \mathbb{C}^2$, we have that $\mathbb{C}^2$ is a principal bundle with structural group $\mathbb{C}^*$ and base the projective space $\mathbb{CP}^1$ that can be identified with the two-dimensional sphere $S^2$, that is the space of pure states the system. The bivector field $\Lambda$ on $\mathbb{C}^2$ given by Eq. (12) is not projectable on $S^2$, but at each $|\psi\rangle$ we define two new tensor fields:

$$\tilde{\Lambda}_{|\psi\rangle} = \langle \psi | \psi \rangle \Lambda_{|\psi\rangle}, \quad \tilde{G}_{|\psi\rangle} = \langle \psi | \psi \rangle G_{|\psi\rangle}.$$  

These two tensor fields are now projectable. They define corresponding bi-differential operators on $S^2$. Note that neither the function $f_A$ is projectable onto the quotient, however $e_A$ is projectable. Furthermore, even if the symplectic structure $\omega$ is projectable, the corresponding potential function $\theta$ is not projectable and then the projected symplectic form is not exact anymore [3].

Now we are in the position of defining observables from elements of $\mathcal{F}(S^2, \mathbb{C})$ by requiring that $f \in \mathcal{F}(S^2, \mathbb{R})$ and moreover the Hamiltonian vector field associated with $f$ preserves the projected symmetric tensor on $S^2$. Thus, observables are intrinsically defined without any reference to the original Hilbert space.

One can check that $S^2$ is a Lie-Jordan manifold, i.e., both the Lie and the Jordan product on observables are mutually compatible and

$$f \ast g = (f, g) + i \{f, g\} - fg$$

defines a $\mathbb{C}^*$-algebra structure when the brackets are extended to complex-valued functions whose real and imaginary parts are observables. We will come back to this point in the following section.

### 3 Geometrical description of algebraic structures

In the previous section we have seen an explicit example where algebraic structures (Hermitian products) are promoted to geometric objects (Riemannian and symplectic structures) and their properties analyzed from that perspective. Such procedure is only an instance of a general procedure. Let us consider a few more examples of this mechanism of interest not only for quantum theories.
3.1 Bilinear maps and Frobenius manifolds

To convey the general ideas we consider real vector spaces. Let us consider bilinear (or multilinear) maps like:

\[ B : V \times V \rightarrow V, \quad \text{or} \quad b : V \rightarrow V^*. \]

It is clear that \( B \in V^* \otimes V^* \otimes V \) and \( b \in V^* \otimes V^* \). The linear space \( V \) itself can be immersed into \( \mathcal{F}(V^*) \), by means of the canonical map:

\[ v \mapsto \hat{v}, \quad \hat{v}(\alpha) = \alpha(v), \quad (14) \]

and each vector \( v \in V \) can be regarded as a linear map in \( V^* \); hence we may define polynomial functions out of them.

By introducing a basis for \( V \), \( \{e_j\} \), and the dual basis \( \{\alpha^k\} \) for \( V^* \), we will have:

\[ B = b^l_{jk} \alpha^k \otimes \alpha^j \otimes e_l \]

and now we can promote \( B \) to define a tensor field \( c_B \) on \( V \) by replacing the basis vectors \( e_j \) and \( \alpha^k \) by \( dx_j \) and \( \partial/\partial x_k \) respectively, this is:

\[ c_B = b^l_{jk} \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_k} \otimes dx_l, \quad (15) \]

(clearly \( x_k \) denote linear coordinates on \( V^* \) with respect to the basis \( \alpha^k \)). A first application of this observation lies in considering the structure constants \( b^l_{jk} \), or the tensor \( B \), as the components of an affinely constant connection \( \nabla_B \). We can also imagine that the tensor \( B \) defines a composition law \( \circ \) on the algebra of differential operators by means of:

\[ \partial_j \circ \partial_k = b_{jk}^l \partial_l. \]

Then the associativity condition for the product is equivalent to the vanishing of the curvature. Finally, if we allow \( b_{jk}^l \) to depend on the point we get the notion of a Frobenius manifold, as introduced by Dubrovin [9, 10, 11, 12]. Even more, substituting the differential operators \( \partial_k \) by their corresponding symbols \( p_k \) on the cotangent bundle \( T^*V \), we can define the quadratic functions:

\[ F_{jk} = p_j p_k - \Gamma_{jk}^l p_l, \]
and if \( \mathcal{J} \) denotes the ideal generated by them, the associativity condition for
the product \( \circ \) defined by \( B \), becomes
\[
\{ \mathcal{J}, \mathcal{J} \} \subset \mathcal{J}.
\]

This result constitutes the key observation of Magri and Konopelchenko \cite{14,15} to relate Frobenius manifolds and important hierarchies of completely integrable systems.

### 3.2 The Jordan–Lie manifold structure of the space of endomorphisms

Apart from the association of linear functions on \( V^* \) to vectors in \( V \) discussed above, Eq. \((14)\), to any vector \( v \in V \) we can associate the constant vector field \( X_v : V \to TV \) on \( V \) defined by \( X_v(w) = (w,v) \). The Liouville vector field \( \Delta : v \mapsto (v,v) \) generating infinitesimal dilations, induces a linear structure on the base manifold from the one on the fiber.

Moreover, by using the identification above of vectors on \( V \) with tangent vectors to \( TV \), any linear transformation \( A : V \to V \) induces a transformation \( T_A \) on tangent vectors:
\[
T_A : TV \to TV, \quad (w,v) \mapsto (w,Av).
\]

For instance, \( T_I = \Delta \). In this way the algebra \( \text{End}(V) \) is mapped into the algebra of linear endomorphisms on \( TV \) preserving the base. The map \( T_A \mapsto X_A = T_A(\Delta) \) is injective and it therefore allows us to induce a composition law on vector fields as:
\[
X_A \cdot X_C = T_{AC}(\Delta) = X_{AC}.
\]

If we consider now the Jordan product \( \circ \) defined in the space of endomorphisms of the linear space \( V \) by \( A \circ C = \frac{1}{2}(AC + CA) \), a similar product is induced on the corresponding vector fields.
\[
X_A \circ X_C = T_{AC}(\Delta) = X_{AC}.
\]

Now, to any bilinear map \( B : V \times V \to V \) we can associate in addition to the tensor field \( c_B \) given by Eq. \((15)\) a 2-contravariant tensor field \( \tau_B \) on \( V^* \) given by \( i_\Delta c_B \), or more explicitly:
\[
\tau_B(df_1, df_2)(\alpha) = \alpha(B(df_1(\alpha), df_2(\alpha))).
\]
Hence, if we consider the symmetric bilinear composition \( B(A, C) = A \circ C \), defined by the Jordan bracket above, the dual space \( E^* \) of the space of endomorphisms \( E \) of the linear space \( V \) inherits a symmetric contravariant 2-tensor \( \tau_B \) and the corresponding (Jordan) symmetric bracket \( \langle \cdot, \cdot \rangle \) on the algebra of functions \( \mathcal{F}(E^*) \).

The space of endomorphisms \( E \) also carries the Lie algebra bracket:

\[
L(A, C) = \frac{1}{2} [A, C] = \frac{1}{2} (AC - CA),
\]

inducing the corresponding skew-symmetric contravariant 2-tensor \( \tau_L \) on \( E^* \) that defines a Poisson bracket \( \{\cdot, \cdot\} \) on \( \mathcal{F}(E^*) \). Thus the space \( E^* \) has the structure of a Jordan-Lie manifold. The two brackets above are compatible in a trivial way because the two tensors add to the canonical tensor induced by the obvious bilinear map \( B_0(A, C) = AC \) on \( E \).

Let us consider the simple example of a complex linear space \( V \) of dimension 2. The space \( E \) of endomorphisms of \( V \) has complex dimension 4. A basis for \( E \) can be chosen as the set of the \( 2 \times 2 \) (Hermitean) matrices:

\[
\begin{align*}
\sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

The corresponding (complex) coordinate functions on matrices are given by:

\[
z_\mu(A) = \frac{1}{2} \text{Tr} (\sigma_\mu A),
\]

and for any \( 2 \times 2 \) matrix \( A \) we have: \( A = z_\mu \sigma_\mu \). Clearly now the skew-symmetric tensor \( \tau_L \) becomes:

\[
I = \varepsilon_{jkl} z_j \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l},
\]

while the symmetric tensor \( \tau_B \) has the form:

\[
R = \frac{\partial}{\partial z_0} \otimes \left( z_j \frac{\partial}{\partial z_j} \right) + z_0 \left( \frac{\partial}{\partial z_0} \otimes \frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1} \otimes \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \otimes \frac{\partial}{\partial z_2} \right).
\]

Both tensors define a \((1, 1)\) tensor field \( J \) such that \( J^3 = -J \), which is a generalisation of the complex structure.
3.3 The $\mathbb{C}^*$-algebra approach to Quantum mechanics

The connection of the Schrödinger picture with the Heisenberg picture is provided by the momentum map associated with the symplectic action of the unitary group on the Hilbert space or the complex projective Hilbert space of the Schrödinger picture. The inverse connection is provided by the GNS construction which generalizes to Quantum Mechanics the concept of symplectic realization of a Poisson manifold.

The linear transformations preserving both $g$ and $\omega$ as defined in Eq. (9) constitute the unitary group $\mathcal{U}(\mathcal{H})$. If we denote by $\mathfrak{u}(\mathcal{H})$ its Lie algebra, the set of skew-Hermitean operators acting on $\mathcal{H}$, and identify the set of all Hermitean operators with the dual $\mathfrak{u}^*(\mathcal{H})$ via the pairing (in the infinite dimensional case we should restrict to Hilbert-Schmidt operators):

$$\langle A, T \rangle = \frac{1}{2} \text{Tr} (AT), \quad A \in \mathfrak{u}^*(\mathcal{H}), T \in \mathfrak{u}(\mathcal{H}),$$

we can consider $\hat{T}$ as the linear map associated with $T$, $\hat{T}(A) = \langle A, T \rangle$. A bracket can then be defined as before by:

$$\{\hat{T}_1, \hat{T}_2\} = [T_1, T_2]^\sim,$$

and similarly a Jordan bracket is introduced by means of:

$$(\hat{T}_1, \hat{T}_2) = (T_1T_2 + T_2T_1)^\sim.$$

These two brackets are compatible in the sense that they define a Lie–Jordan algebra in $\mathfrak{u}^*(\mathcal{H})$. If we consider the inner product on $\mathfrak{u}^*(\mathcal{H})$:

$$\langle A, B \rangle_{\mathfrak{u}^*} = \frac{1}{2} \text{Tr} (AB),$$

we find that this inner product is preserved by the Hamiltonian vector fields associated with $\hat{T}$ for any $T \in \mathfrak{u}(\mathcal{H})$. These vector fields are related with corresponding vector fields on $\mathcal{H}$, namely,

$$\frac{d}{dt} (e^{-itA} \mid \psi \rangle)_{t=0} = -i A \mid \psi \rangle = X_A (\mid \psi \rangle),$$

with $i A \in \mathfrak{u}(\mathcal{H})$. The vector field $X_A$ on $\mathcal{H}$ is Hamiltonian with Hamiltonian function $f_A(\mid \psi \rangle) = \frac{1}{2} \langle \psi \mid A \mid \psi \rangle$. The momentum map which relates $X_A$
with the Hamiltonian vector field on \( u^*(\mathcal{H}) \) associated with \((iA)^{-}\) is given by
\[
\mu: \mathcal{H} \rightarrow u^*(\mathcal{H}), \quad \mu(\ket{\psi}) = \ket{\psi}\bra{\psi}.
\]
The symmetric tensor associated with the Jordan bracket:
\[
R(d\hat{T}_1, d\hat{T}_2) = (\hat{T}_1, \hat{T}_2) = (T_1T_2 + T_2T_1)^{-},
\]
is a contravariant symmetric 2-tensor as we discussed earlier. Similarly, the skew-symmetric tensor defined by:
\[
I(d\hat{T}_1, d\hat{T}_2) = \{\hat{T}_1, \hat{T}_2\},
\]
is the Poisson tensor associated with the Lie algebra \( u(\mathcal{H}) \). These two tensor fields are \( \mu \)-related with the tensors \( G \) and \( \Lambda \) defined on \( \mathcal{H} \), respectively, by Eq. (10).

By considering the complex contravariant tensor \( R+iI \) we obtain a tensor field which allows us to consider the algebra of linear functions on \( u^*(\mathcal{H}) \) of the form \( \hat{T} + i \hat{S}, T, S \in u(\mathcal{H}) \), as a \( \mathbb{C}^* \)-algebra of complex valued functions. In this setting the momentum map relates the Schrödinger picture with the Heisenberg picture. To go from the Heisenberg picture to the Schrödinger picture we consider an Hermitean realization of the Lie–Jordan algebra on \( u^*(\mathcal{H}) \). This is a generalisation of the symplectic realisation of the Poisson structure on \( u^*(\mathcal{H}) \). The existence of these Hermitean realizations for the Lie–Jordan algebra structure on \( u^*(\mathcal{H}) \), the real part of the \( \mathbb{C}^* \)-algebra we are considering, is the essential content of the so called Gelfand–Naimark–Segal (GNS) construction. We refer to [8] for further details on these interesting aspects.

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