Remarks on Blow-up of Smooth Solutions to the Compressible Fluid with Constant and Degenerate Viscosities

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Abstract: In this paper, we will show the blow-up of smooth solutions to the Cauchy problem for the full compressible Navier-Stokes equations and isentropic compressible Navier-Stokes equations with constant and degenerate viscosities in arbitrary dimensions under some restrictions on the initial data. In particular, the results hold true for the full compressible Euler equations and isentropic compressible Euler equations and the blow-up time can be computed in a more precise way. It is not required that the initial data has compact support or contain vacuum in any finite regions. Moreover, a simplified and unified proof on the blow-up results to the classical solutions of the full compressible Navier-Stokes equations without heat conduction by Xin [42] and with heat conduction by Cho-Bin [5] will be given.

1 Introduction

The full compressible Navier-Stokes equations with constant viscosities read as

\[
\begin{aligned}
&\partial_t \rho + \text{div}(\rho u) = 0, \\
&\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p = \text{div}(\mathcal{T}), \\
&\partial_t (\frac{1}{2} \rho |u|^2 + \rho e) + \text{div}(\frac{1}{2} \rho |u|^2 + \rho e + p)u) = \text{div}(u\mathcal{T}) + \mathcal{K}\Delta \theta.
\end{aligned}
\] (1.1)

Here \((x, t) \in \mathbb{R}^n \times \mathbb{R}_+\) and \(\rho = \rho(x, t), u = (u_1, u_2, \cdots, u_n), \theta, p\) and \(e\) denote the density, velocity, absolute temperature, pressure and internal energy, respectively. \(\mathcal{T}\) is the stress tensor given by

\[\mathcal{T} = \mu (\nabla u + (\nabla u)^T) + \lambda \text{div}uI,\]

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where $I$ is the identity matrix, and $\mu$ and $\lambda$ are the coefficients of viscosity and the second coefficient of viscosity, respectively, which satisfy

$$\mu \geq 0, 2\mu + n\lambda \geq 0.$$  

We also denote by $\mathcal{K} \geq 0$ the coefficient of heat conduction.

If $\mu = \lambda = \mathcal{K} = 0$, the Navier-Stokes equations (1.1) become the compressible Euler equations. The polytropic gas satisfies the following state equations:

$$p = R\rho\theta, \quad e = c_v\theta, \quad p = Aexp\left(\frac{s}{c_v}\right)\rho^\gamma,$$

where $R > 0$ is the gas constant, $A > 0$ is an absolute constant, $\gamma > 1$ is the specific heat ratio, $c_v = \frac{R}{\gamma - 1}$ and $s$ is the entropy. For simplicity, we take $A = 1$. The pressure can be expressed as

$$p = (\gamma - 1)\rho e.$$  

(1.3)

The initial data to the equations (1.1) are imposed as

$$(\rho, u, s)(x, t)|_{t=0} = (\rho_0(x), u_0(x), s_0(x)).$$

(1.4)

If the entropy is a constant, the full compressible Navier-Stokes equations reduce to isentropic ones which read as

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t \rho u + \text{div}(\rho u \otimes u) + \nabla p = \text{div}(T). 
\end{cases}$$  

(1.5)

The state equation of the isentropic process becomes

$$p = \rho^\gamma, \gamma > 1.$$  

(1.6)

The initial data to the equations (1.5) are imposed as

$$(\rho, u)(x, t)|_{t=0} = (\rho_0(x), u_0(x)).$$

(1.7)

The compressible Navier-Stokes equations with density-dependent viscosity are another kind of important models. When deriving the compressible Navier-Stokes equations by Chapman-Enskog expansions from the Boltzmann equation, the viscosity depends on the temperature and thus on the density for the isentropic flows. The isentropic compressible Navier-Stokes equations with density-dependent viscosity can be written as

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p = \text{div}\left[h(\rho)\left(\frac{\nabla u + (\nabla u)^t}{2}\right)\right] + \nabla (g(\rho)\text{div}u),
\end{cases}$$

where the pressure $p$ is same as in (1.6), $h(\rho)$ and $g(\rho)$ are the Lamé viscosity coefficients satisfying

$$h(\rho) \geq 0, h(\rho) + ng(\rho) \geq 0.$$  

(1.9)

In particular, we consider the case

$$h(\rho) = \rho^\alpha, g(\rho) = (\alpha - 1)\rho^\alpha$$

quadruplet.
for $\alpha > 1 - \frac{1}{n}$ such that (1.9) is satisfied. For more general $h(\rho)$ and $g(\rho) = p\phi'(\rho) - h(\rho)$, it is referred to [2], [18] and references therein for entropy estimates and existence of weak solutions. It should be noted that the viscous Saint-Venant system for the shallow water, derived from the incompressible Navier-Stokes equations with a moving free surface, is expressed exactly as in (1.8) with $h(\rho) = \rho$, $g(\rho) = 0$ and $\gamma = 2$.

There have been extensive studies on the compressible Navier-Stokes equations with constant and degenerate viscosities (see [5], [6], [7], [13], [19], [20], [21], [22], [33], [34], [35] on the case of constant viscosity and [10], [25], [26], [27], [30], [37], [41], [26] on the case of degenerate viscosity). For the Navier-Stokes equations with constant viscosity, the global existence and uniqueness of the strong (classical) solution in one-dimensional case has been well-understood (see [28], [29], [22] and references therein) and it is proved that if the solution has no vacuum initially then it will not appear vacuum later (in any finite time). Moreover, even if vacuum is permitted, the global well-posedness of the solution to the one-dimensional navier-Stokes equations was studied recently under some compatibility conditions (see [9], [24] and references therein). However, in multi-dimensional case, the global well-posedness of the classical solution to these models for large initial data remains completely open. For the Navier-Stokes equations with density-dependent viscosity, the global existence and uniqueness of the classical solution remain open in multi-dimensional case except the periodic or Cauchy problem of the 2D Kazhikhov-Vaigant model in compressible flow (see [11], [26] and references therein). In fact, if vacuum is permitted, the global well-posedness of the classical solution to the one-dimensional Navier-Stokes equations with density-dependent viscosity still remains open.

These will be subtle issues because the classical solution to the compressible Euler equations and Navier-Stokes equations may blow up in general. For the compressible Euler equations, Sideris [39] firstly showed that the life span of the classical solution is finite if the initial velocity is large enough in some region with compact support. Makino, Ukai and Kawashima [38] studied the blow-up of the classical solution if initial density and velocity hold compact support. Chemin [4] investigated the blow-up of the classical solution to the one-dimensional compressible Euler equations. Recently, the 3-D shock formation in general settings was studied by Christodoulou [6]. For the compressible Navier-Stokes equations without heat conduction, Xin [42] firstly obtained the blow-up of the classical solution under assumptions that the initial density has compact support, which was generalized by Cho and Jin to the case with heat conduction in [5] and Razaonova to the case with initial data rapidly decay at far fields in [40]. Luo and Xin [31] studied the blow-up of symmetric smooth solutions to two dimensional isentropic Navier-Stokes equations. Recently, Xin and Yan [43] proved that any classical solutions of viscous compressible fluids without heat conduction will blow up in finite time, as long as the initial data has an isolated mass group.

In this paper, we will prove the blow-up of the classical solution to the Cauchy problem for compressible Navier-Stokes equations in arbitrary dimensions with constant viscosity and degenerate viscosities under some restrictions of the initial data. In particular, the results hold true for the compressible Euler equations and the blow-up time can be computed in a more precise way. It should be noted that it is not required that the initial data has compact support or has fast decay at far fields. And we will give a new and simplified proof of the blow-up results appeared in [42] and [5]. To obtain our main results, some physical quantities such as mass, momentum, momentum of inertia, internal energy, potential energy, total energy and some combined functionals of these quantities are introduced. The basic properties and specific relationships between them are helpful and crucial to prove the main results. More precisely, for the full Navier-Stokes and Euler equations, the upper and lower decay rates of the internal energy will be calculated in a precise way. And for the isentropic Navier-Stokes and Euler equations, the upper and lower decay rates of the potential energy will be presented accordingly.
Then the main results will be proved by comparing the coefficients of the upper and lower decay rates.

In the following, we denote the full Navier-Stokes equations \((1.1)\) and \((1.2)\) as (CNS), the isentropic Navier-Stokes equations \((1.5)\) and \((1.6)\) as (ICNS) and the Navier-Stokes equations with density dependent viscosity \((1.8)\) and \((1.9)\) as (DICNS) for short.

The paper is organized as follows. In section 2, we will introduce some physical quantities and present our main results. In Section 3 and 4, we will give some basic properties of the physical quantities and the proof of the main results.

## 2 Main Results

The following physical quantities will be used in this paper.

\[
\begin{align*}
M(t) &= \int_{\mathbb{R}^n} \rho dx \quad \text{(mass)}, \\
\mathbb{P}(t) &= \int_{\mathbb{R}^n} \rho u dx \quad \text{(momentum)}, \\
F(t) &= \int_{\mathbb{R}^n} \rho u x dx \quad \text{(momentum weight)}, \\
G(t) &= \frac{1}{2} \int_{\mathbb{R}^n} |x|^2 dx \quad \text{(momentum of inertia)}, \\
E(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \rho |u|^2 dx + \int_{\mathbb{R}^n} \rho e dx \triangleq E_k(t) + E_i(t) \quad \text{(total energy)}, \\
IE(t) &= \frac{1}{2} \int_{\mathbb{R}^n} \rho |u|^2 dx + \frac{1}{\gamma - 1} \int_{\mathbb{R}^n} P dx \triangleq E_k(t) + I(t) \quad \text{(energy)}.
\end{align*}
\]

The basic properties and relationships between these quantities will be discussed in Section 3 and 4. To prove our main results, we introduce the following functionals:

\[
\begin{align*}
H(t) &= 2E_k(t) + n(\gamma - 1)E_i(t), \\
IH(t) &= 2E_k(t) + n(\gamma - 1)I(t), \\
J(t) &= G(t) - (t + 1)F(t) + (t + 1)^2 E(t), \\
IJ(t) &= G(t) - (t + 1)F(t) + (t + 1)^2 IE(t).
\end{align*}
\]

We always assume that \(M(0), \mathbb{P}(0), F(0), G(0), E(0), IE(0)\) are finite and \(M(0) > 0, \mathbb{P}(0) \neq 0, E(0) > 0, IE(0) > 0\). Since \(F(t)^2 \leq 4G(t)E_k(t)\) (see Lemma 3.2) and \(E_i(t) \geq 0, I(t) \geq 0\), it follows that \(J(0) > 0\) and \(IJ(0) > 0\) respectively.

For technical reason, we impose decay conditions on the solutions as follows:

\[
|u| \to 0, \rho u = o\left(\frac{1}{|x|^{n+1}}\right), |\nabla u| = o\left(\frac{1}{|x|^n}\right), P = o\left(\frac{1}{|x|^n}\right),
\]

as \(|x| \to \infty\) for any fixed \(t > 0\).

If \(K \neq 0\), we impose

\[
|\nabla \theta| = o\left(\frac{1}{|x|^{n-1}}\right), \quad |x| \to \infty, \quad t > 0.
\]

It should be remarked that the conditions \((2.3)-(2.4)\) guarantee that the integration by parts in our calculations make sense, which is similar to those in \[40\]. In particular, for the classical solution to (CNS) satisfying \((2.3)-(2.4)\), the conservations of the mass \(M(t)\), momentum \(\mathbb{P}(t)\) and energy \(E(t)\) hold true.

Before we state our main results, we give the following definitions.
Definition 2.1  For the Cauchy problem of (CNS) without heat conduction ($K = 0$), we call $(\rho, u, s) \in X(T)$ if $(\rho, u, s)$ is a classical solution in $[0, T]$ satisfying (2.3).

Definition 2.2  For the Cauchy problem of (ICNS), we call $(\rho, u) \in Y(T)$ if $(\rho, u)$ is a classical solution in $[0, T]$ satisfying (2.3).

Definition 2.3  For the Cauchy problem of (DICNS), we call $(\rho, u) \in Z(T)$ if $(\rho, u)$ is a classical solution in $[0, T]$ satisfying (2.3) in which the condition $|\nabla u| = o\left(\frac{1}{|x|^n}\right)$ is replaced by

$$\rho^\alpha |\nabla u| = o\left(\frac{1}{|x|^n}\right).$$

Our main results are stated as follows.

Theorem 2.1  Let $1 < \gamma \leq 1 + \frac{2}{n}$. If the initial values satisfy

$$E(0)\frac{n(\gamma - 1)}{\exp(s_1/\epsilon_0)}J(0)\left(\frac{n}{\gamma - 1}\right)^{\frac{n}{2}} < \left(\frac{\Gamma\left(\frac{n}{2} + 1\right)}{\pi}\right)^{\gamma - 1}\frac{1}{2^{\gamma - n}}\frac{1}{(\gamma - 1)},$$

where $s_1 = \min_x s_0(x)$. Then there exists a $T_1^* > 0$ such that there is no solution in $X(T_1^*)$ to the Cauchy problem of (CNS) without heat conducting. Moreover, the result holds true for the compressible Euler equations.

Theorem 2.2  Let $1 < \gamma \leq 1 + \frac{2}{n}$. If the initial values satisfy

$$E(0)\frac{n(\gamma - 1)}{M(0)}\frac{IJ(0)}{(\gamma - 1)}\left(\frac{n}{\gamma - 1}\right)^{\frac{n}{2}} < \left(\frac{\Gamma\left(\frac{n}{2} + 1\right)}{\pi}\right)^{\gamma - 1}\frac{1}{2^{\gamma - n}}\frac{1}{(\gamma - 1)}.$$  \hspace{1cm} (2.7)

Then there exists a $T_2^* > 0$ such that there is no solution in $Y(T_2^*)$ to the Cauchy problem of (ICNS). In particular, the claim above is true for isentropic compressible Euler equations.

To study the blow-up of the classical solution to the isentropic Navier-Stokes equations with density-dependent viscosity, we assume that the energy in general n-dimensional case and the upper bound of the density in 1-dimensional case are finite.

Assumptions 1. Energy Bounds:

$$\int_{\mathbb{R}^n}\left[\frac{1}{2}\rho u^2 + \frac{1}{\gamma - 1}\rho^\gamma\right]dx + \int_0^t\int_{\mathbb{R}^n}h(\rho)|\nabla u + (\nabla u)^t|^2dxdt + \int_0^t\int_{\mathbb{R}^n}g(\rho)(\text{div}u)^2dxdt$$

$$= \int_{\mathbb{R}^n}\left[\frac{1}{2}\rho_0u_0^2 + \frac{1}{\gamma - 1}\rho_0^\gamma\right]dx \equiv C_{13} < \infty. \hspace{1cm} (2.8)$$

Assumptions 2. Upper Bound of Density: When $n = 1$, we assume that $\max \rho(x, t) = C_{14} < \infty$.

Then we have
Theorem 2.3  Suppose that Assumptions 2 holds. Let \( n = 1, \alpha \geq \gamma \) and \( 1 < \gamma \leq 3 \). If the initial values satisfy

\[
IJ(0)\left\{ \frac{1}{2}\left[ \max(2, (\gamma - 1))IE(0) + C_{18} \right] \right\}^{\frac{\gamma - 1}{\gamma + 1}} < \frac{\exp\left( \frac{1 - \gamma}{4} C_{14}^{\alpha - \gamma} \right)}{M(0)^{\frac{\gamma - 1}{\gamma + 1}}} \tag{2.9}
\]

where \( C_{18} \) is any positive number between 0 and \( \frac{\rho(0)^2}{M(0)} \) and \( C_{14} = \max \rho = C(\rho_0, u_0) \), then there exists a \( T^*_3 > 0 \) such that there is no solution in \( Z(T^*_3) \) to the Cauchy problem of (DICNS).

Theorem 2.4  Suppose that Assumptions 1 holds. Let \( n = 1, \frac{\gamma + 1}{2} < \alpha \leq \gamma \) and \( 1 < \gamma < 3 \). If the initial values satisfy

\[
IJ(0)\left\{ \frac{1}{2}\left[ \max(2, (\gamma - 1))IE(0) + C_{18} \right] \right\}^{\frac{\gamma - 1}{\gamma + 1}} < \frac{(\Gamma(\frac{\gamma}{2}))^{\gamma - 1}}{(\gamma - 1) \pi^{\gamma - 1} 2^{\frac{3 + \gamma}{2}}}, \tag{2.10}
\]

where \( C_{18} \) is any positive constant between 0 and \( \frac{\rho(0)^2}{M(0)} \), then there exists a \( T^*_4 > 0 \) such that there is no solutions in \( Z(T^*_4) \) to the Cauchy problem of (DICNS).

Theorem 2.5  Suppose that Assumptions 1 holds. Let \( n \geq 2, \frac{\gamma + 1}{2} < \alpha \leq \gamma \) and \( 1 < \gamma < 1 + \frac{2}{n} \). If the initial values satisfy

\[
IJ(0)\left\{ \frac{1}{2}\left[ \max(2, n(\gamma - 1))IE(0) + C_{22} \right] \right\}^{\frac{\gamma - 1}{\gamma + 1}} < \frac{1}{M(0)^{\frac{\gamma - 1}{\gamma + 1}}} \exp\left( \frac{1 - \gamma}{4(\alpha - 1)(2 \alpha - \gamma - 1)} \right) \left[ 1 + n(\alpha - 1)^2(\gamma - 1)^{\gamma - 1}(1 + n(\alpha - 1)^2(\gamma - 1)^{\gamma - 1}) \right] \tag{2.11}
\]

where \( C_{22} \) is any positive constant between 0 and \( \frac{\rho(0)^2}{M(0)} \), then there exists a \( T^*_5 > 0 \) such that there is no solutions in \( Z(T^*_5) \) to the Cauchy problem of (DICNS).

A few remarks are in order.

Remark 2.1  In comparison with results obtained by Rozanova [40], the conditions imposed on the initial data are different from those in [40] and the blow-up of the classical solution to the Navier-Stokes equations with density-dependent viscosities is addressed here.

Remark 2.2  For the one-dimensional Cauchy problem of (DICNS) with \( \mu(\rho) = \rho^\alpha \), Mellet-Vasseur [27] proved the global existence of the strong solution \( \rho \in L^\infty(0, T; H^1(\mathbb{R})) \), \( u \in L^\infty(0, T; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})) \) under the assumption \( 0 < \alpha < \frac{1}{2} \) and uniqueness under assumptions \( \mu(\rho) \geq \nu > 0 \) and \( \gamma \geq 2 \) additionally. Under the jump free boundary conditions, Jiang-Xin-Zhang [23] established the global well-posedness of the strong solutions for \( 0 < \alpha < 1 \). If \( \mu(\rho) = 1 + \rho^\beta \) with \( \beta \geq 0 \), Jiu-Li-Ye [24] obtained the global well-posedness of the classical solution permitting vacuum. However, Theorem 2.3 shows that, if \( \mu(\rho) = \rho^\alpha \) with \( \alpha \geq \gamma, 1 < \gamma \leq 3 \), the classical solution to the 1D Cauchy problem of (DICNS) will blow up in general even though the density has upper bound. While if \( \mu(\rho) = \rho^\alpha \) with \( \frac{\gamma + 1}{2} < \alpha < \gamma, 1 < \gamma < 3 \), Theorem 2.4 shows that the classical solution to the 1D Cauchy problem of (DICNS) will blow up in general even though the energy is bounded. The blow-up result in the multi-dimensional case is presented in Theorem 2.5.
Remark 2.3 In (2.6)–(2.11), we do not require that the initial data has compact support or contain vacuum in any finite region.

Remark 2.4 It is clear that the equality (2.6) holds as long as initial entropy is sufficient large.

Remark 2.5 The time $T_1^* - T_5^*$ can be computed precisely, for example, we can solve out $T_1^*$ in Theorem 2.2 by (3.32). In other words, we can find out the "last" blow-up time.

Remark 2.6 It should be noted that Assumption 1 is the usual energy estimate and can be verified under the condition $\int_{\mathbb{R}^n} \left[ \frac{1}{2} \rho_0 u_0^2 + \frac{1}{\gamma - 1} \rho_0^\gamma \right] dx < \infty$. The Assumptions 2 can also be verified in one-dimensional case under suitable conditions of the initial data (see [17], [23], [27], [10] and references therein).

Remark 2.7 Our results show that if the conditions in Theorem 2.2 are satisfied, the weak solutions obtained in [13] and [33] cannot belong to $Y(T_2^*)$.

3 The Proof of Theorem 2.1–2.2

Let $(\rho, u, s) \in X(T)$ be a classical solution to the Cauchy problem of (CNS). And $(\rho, u) \in Y(T)$ is a classical solution to the Cauchy problem of (ICNS). In this subsection, we will first present some basic relationships between the quantities defined in Section 2. Then we will give the proof of Theorem 2.1 and Theorem 2.2.

Lemma 3.1 For (CNS) and (ICNS), we have

\[
\frac{d}{dt} M(t) = 0, \quad \frac{d}{dt} \mathcal{P}(t) = 0, \quad \frac{d}{dt} G(t) = F(t).
\] (3.1)

For (CNS), we have

\[
\frac{d}{dt} E(t) = 0, \quad \frac{d}{dt} F(t) = H(t).
\] (3.2)

For (ICNS), we have

\[
\frac{d}{dt} IE(t) = -\int_{\mathbb{R}^n} \left[ 2\mu \sum_{j=1}^{n} (\partial_j u_j)^2 + \lambda (\text{div} u)^2 \right. \\
\left. + \mu \sum_{i \neq j} (\partial_j u_i)^2 + 2\mu \sum_{i > j} (\partial_j u_i)(\partial_i u_j) \right] dx,
\] (3.3)

\[
\frac{d}{dt} F(t) = IH(t).
\] (3.4)

Proof. Using (CNS) and (ICNS), applying integration by parts, one can verify (3.1)–(3.4).

Lemma 3.2 For (CNS) and (ICNS), we have

\[
F(t)^2 \leq 4G(t)E_k(t),
\] (3.5)

and

\[
\mathcal{P}(0)^2 \leq 2M(0)E_k(t).
\] (3.6)
**Proof.** Using Lemma 3.1 and Hölder’s inequality, one can verify (3.5)–(3.6).

Based on Lemma 3.1 and Lemma 3.2, we can obtain two-sided estimates of $G(t)$ (see 40).

**Lemma 3.3** For (CNS), if $1 < \gamma \leq 1 + \frac{2}{n}$, we have

$$\frac{n(\gamma - 1)}{2} E(0)t^2 + F(0)t + G(0) \leq G(t) \leq E(0)t^2 + F(0)t + G(0),$$

(3.7)

if $\gamma > 1 + \frac{2}{n}$, we have

$$E(0)t^2 + F(0)t + G(0) \leq G(t) \leq \frac{n(\gamma - 1)}{2} E(0)t^2 + F(0)t + G(0).$$

(3.8)

For (ICNS), if $1 < \gamma \leq 1 + \frac{2}{n}$, we have

$$\frac{\mathbb{P}(0)^2}{2M(0)} t^2 + F(0)t + G(0) \leq G(t) \leq IE(0)t^2 + F(0)t + G(0),$$

(3.9)

if $\gamma > 1 + \frac{2}{n}$, we have

$$\frac{\mathbb{P}(0)^2}{2M(0)} t^2 + F(0)t + G(0) \leq G(t) \leq \frac{n(\gamma - 1)}{2} IE(0)t^2 + F(0)t + G(0).$$

(3.10)

**Proof.** For (CNS), in view of Lemma 3.1, if $1 < \gamma \leq 1 + \frac{2}{n}$, we have

$$\frac{d^2}{dt^2} G(t) = H(t) = 2E(t) + (n(\gamma - 1) - 2)E_i(t)$$

$$\leq 2E(t) = 2E(0),$$

(3.11)

and

$$\frac{d^2}{dt^2} G(t) = H(t) = n(\gamma - 1)E(t) + (2 - n(\gamma - 1))E_k(t)$$

$$\geq n(\gamma - 1)E(t) = n(\gamma - 1)E(0).$$

(3.12)

Integrating (3.11) and (3.12) over $[0, t]$, we get (3.7). The proof of (3.8) is similar.

For (ICNS), in view of Lemma 3.1 and Lemma 3.2, if $1 < \gamma \leq 1 + \frac{2}{n}$, we have

$$\frac{d^2}{dt^2} G(t) = IH(t) = 2IE(t) + (n(\gamma - 1) - 2)I(t)$$

$$\leq 2IE(t) \leq 2IE(0),$$

(3.13)

and

$$\frac{d^2}{dt^2} G(t) = IH(t) \geq 2E_k(t) \geq \frac{\mathbb{P}(0)^2}{M(0)},$$

(3.14)

where (3.6) has been used. Integrating (3.13) and (3.14) over $[0, t]$, we get (3.9). The proof of (3.10) is similar. We end the proof of the lemma.

The following lemma is due to Chemin 4.
Lemma 3.4 For any \( f \in L^1(\mathbb{R}^n, dx) \cap L^\gamma(\mathbb{R}^n, dx) \cap L^1(\mathbb{R}^n, |x|^2 dx) \), it holds that

\[
\| f \|_{L^1(\mathbb{R}^n, dx)} \leq C_1 \| f \|_{L^\gamma(\mathbb{R}^n, dx)}^{\frac{2(\gamma-1)}{(n+2)\gamma-n}} \| f \|_{L^1(\mathbb{R}^n, |x|^2 dx)}^{\frac{n(\gamma-1)}{2}},
\]

where \( C_1 = 2B_1 \left( \frac{\pi^\frac{n}{2}}{\Gamma\left(\frac{n+2}{2}\right)} \right)^\frac{2(\gamma-1)}{(n+2)\gamma-n} = 2\left( \frac{\pi^\frac{n}{2}}{\Gamma\left(\frac{n+2}{2}\right)} \right)^\frac{2(\gamma-1)}{(n+2)\gamma-n} \).

**Proof.** For any \( r > 0 \), it follows from the Hölder’s inequality that

\[
\int_{\mathbb{R}^n} |f(x)| dx = \int_{|x| \leq r} |f(x)| dx + \int_{|x| \geq r} |f(x)| dx \leq |B_r|^{1 - \frac{1}{\gamma}} \left( \int_{|x| \leq r} |f(x)|^\gamma dx \right)^\frac{1}{\gamma} + r^{-2} \left( \int_{|x| \geq r} |f(x)| dx \right)^\gamma.
\]

Choosing

\[
r = \left( \frac{\| f \|_{L^1(\mathbb{R}^n, |x|^2 dx)}}{\| f \|_{L^\gamma(\mathbb{R}^n, dx)} |B_1|^{1 - \frac{1}{\gamma}}} \right)^\frac{\gamma}{(n+2)\gamma-n},
\]

we obtain (3.15). The proof of the lemma is finished.

Taking \( f = \rho \) in Lemma 3.4, we arrive at the lower bound of \( E_i(t) \) and \( I(t) \), which is

**Proposition 3.1** For (CNS) and (ICNS), we have

\[
E_i(t) \geq \frac{C_2}{G(t)^{\frac{n(\gamma-1)}{2}}},
\]

and

\[
I(t) \geq \frac{C_3}{G(t)^{\frac{n(\gamma-1)}{2}}},
\]

respectively, where \( C_2 = \left( \frac{\Gamma\left(\frac{n+2}{2}\right)}{(\pi)^\frac{n}{2}} \right)^{\gamma-1} \frac{\exp\left( \frac{n+2}{2}\right) M(0) \gamma-n}{\frac{(n+2)\gamma-n}{2}} \) and \( C_3 = \left( \frac{\Gamma\left(\frac{n+2}{2}\right)}{(\pi)^\frac{n}{2}} \right)^{\gamma-1} \frac{M(0) \gamma-n}{\frac{(n+2)\gamma-n}{2}} \).

The following lemma was shown by Xin [42]. Here we give a new proof of it.

**Lemma 3.5** For (CNS) and the (ICNS), the following estimates hold:

\[
\frac{d}{dt} J(t) \leq \begin{cases} \frac{2-n(\gamma-1)}{\gamma} J(t), & 1 < \gamma \leq 1 + \frac{2}{n}, \\ 0, & \gamma > 1 + \frac{2}{n}. \end{cases}
\]

and

\[
\frac{d}{dt} IJ(t) \leq \begin{cases} \frac{2-n(\gamma-1)}{\gamma} IJ(t), & 1 < \gamma \leq 1 + \frac{2}{n}, \\ 0, & \gamma > 1 + \frac{2}{n}. \end{cases}
\]

respectively.
Proof. Due to Lemma \ref{lem:3.2}, if we regard
\[ G(t) - (t + 1)F(t) + (t + 1)^2E_k(t) \]
as a quadratic function of \((t + 1)\), since
\[ \Delta = (F(t)^2 - 4G(t)E_k(t)) \leq 0, \]
we have
\[ G(t) - (t + 1)F(t) + (t + 1)^2E_k(t) \geq 0. \]
Consequently,
\[ E_i(t) \leq \frac{1}{(t + 1)^2}J(t), \quad I(t) \leq \frac{1}{(t + 1)^2}IJ(t). \]
(3.23)

This, together with Lemma \ref{lem:3.1} shows
\[ \frac{d}{dt}J(t) = (2 - n(\gamma - 1))(t + 1)E_i(t) + (t + 1)^2 \frac{d}{dt}E(t) \]
\[ = (2 - n(\gamma - 1))(t + 1)E_i(t) \]
\[ \leq \begin{cases} \frac{2-n(\gamma-1)}{t+1}J(t), & 1 < \gamma \leq 1 + \frac{2}{n}, \\ 0, & \gamma > 1 + \frac{2}{n}, \end{cases} \]
(3.24)

and
\[ \frac{d}{dt}IJ(t) = (2 - n(\gamma - 1))(t + 1)I(t) + (t + 1)^2 \frac{d}{dt}IE(t) \]
\[ \leq (2 - n(\gamma - 1))(t + 1)I(t) \]
\[ \leq \begin{cases} \frac{2-n(\gamma-1)}{t+1}IJ(t), & 1 < \gamma \leq 1 + \frac{2}{n}, \\ 0, & \gamma > 1 + \frac{2}{n}. \end{cases} \]
(3.25)

The proof of the proposition is finished.

It follows from Lemma \ref{lem:3.5} that

Proposition 3.2 For \((CNS)\) and \((ICNS)\), the following estimates hold:
\[ E_i(t) \leq \begin{cases} \frac{C_4}{(t+1)^{n(\gamma-1)}}, & 1 < \gamma \leq 1 + \frac{2}{n}, \\ \frac{C_4}{(t+1)^2}, & \gamma > 1 + \frac{2}{n}, \end{cases} \]
(3.26)

and
\[ I(t) \leq \begin{cases} \frac{C_5}{(t+1)^{n(\gamma-1)}}, & 1 < \gamma \leq 1 + \frac{2}{n}, \\ \frac{C_5}{(t+1)^2}, & \gamma > 1 + \frac{2}{n}, \end{cases} \]
(3.27)

respectively, where \(C_4 = J(0), C_5 = IJ(0)\).

Now we are ready to prove Theorem \ref{thm:2.1} and Theorem \ref{thm:2.2}.
Proof of Theorem 2.1. Suppose that the life span of the classical solution \( t = +\infty \). Then by Proposition 3.1 and Proposition 3.2 if \( 1 < \gamma \leq 1 + \frac{2}{n} \), we have
\[
\frac{C_2}{G(t)^{n(\gamma-1)/2}} \leq E_i(t) \leq \frac{C_4}{(t+1)^{n(\gamma-1)/2}},
\]
for all \( t \geq 0 \). In view of (3.7), one has
\[
G(t) \leq E(0)t^2 + F(0)t + G(0).
\]
(3.29)
Substituting (3.29) to (3.28) yields
\[
\frac{C_2}{(E(0)t^2 + F(0)t + G(0))^{n(\gamma-1)/2}} \leq \frac{C_4}{(t+1)^{n(\gamma-1)/2}}.
\]
(3.30)
Let \( t \) goes to infinity, we get
\[
\frac{E(0)^{n(\gamma-1)/2}J(0)}{\exp(\frac{n}{2\nu})M(0)} \geq \left( \frac{\Gamma(\frac{n}{2}+1)}{(\pi)^{\frac{n}{2}}} \right)^{\gamma-1} \frac{1}{2^{\frac{n+2}{2}n-n}(\gamma-1)},
\]
(3.31)
which contradicts (2.6). Hence if (2.6) holds, then there exists a time \( T_1^* < \infty \), satisfying
\[
\frac{C_2}{(E(0)T_1^*t^2 + F(0)T_1^*t + G(0))^{n(\gamma-1)/2}} > \frac{C_3}{(T_1^*+1)^{n(\gamma-1)/2}},
\]
(3.32)
such that \([0, T_1^*)\) is the life span of the classical solution. Indeed, one can solve out \( T_1^* \) by (3.32).
The proof of the theorem is finished.

Proof of Theorem 2.2. The proof is similar to Theorem 2.1 and we omit the details here.

In [42], Xin investigated the blow-up of the classical solution \((\rho, u, s) \in C^1([0, T]; H^m(\mathbb{R}^n))\) with \( m > \left\lceil \frac{n}{2} \right\rceil + 2 \) and \( n \geq 1 \) to the Cauchy problem of (CNS) without heat conduction if the initial density has compact support, where \( s \) is the entropy of the solution. This was generalized by Cho and Jin to the case with heat conduction in [5], by Razanova to the case that initial data rapidly decays at far fields in [40] and recently by Xin and Yan [43] to the case that initial data has an isolated mass group.

The following is a key lemma due to Xin [42] which says that if the initial density has compact support, then the compact support will keep unchanged for all time.

Lemma 3.6 ([42]) For the viscous compressible Navier-Stokes equations (CNS) with
\[
\mu > 0, 2\mu + n\lambda > 0, \mathcal{K} \geq 0,
\]
(3.33)
Suppose that \((\rho, u, s) \in C^1([0, T]; H^m(\mathbb{R}^n))\) with \( m > \left\lceil \frac{n}{2} \right\rceil + 2 \) is a classical solution of (CNS) and the initial density has compact support. Then the support of the density \( \rho(x, t) \) will not grow in time.

Now we give an uniform proof of blow-up results obtained by Xin [42] for \( \mathcal{K} = 0 \) and by Cho and Jin [5] for \( \mathcal{K} > 0 \).

Proposition 3.3 Suppose that the assumptions of Lemma 3.6 and (2.1) hold true. Then any smooth solution to the Cauchy problem of (CNS) will blow up in finite time.
Proof. It follows from Lemma 3.6 that the support of the density \( \text{supp}_x \rho(x, t) = \text{supp}_x \rho_0(x) \triangleq D \). Adopting the arguments in [42], we have \( u(x, t) = 0 \) in the outside of the support of density. Hence the assumptions (2.3) are satisfied. Then

\[
G(t) = \frac{1}{2} \int_{\text{supp}_x \rho(x, t)} \rho |x|^2 dx = \frac{1}{2} \int_D \rho |x|^2 dx \\
\leq \frac{1}{2} |D|^2 \int_{\mathbb{R}^n} \rho dx = \frac{1}{2} M(0) |D|^2.
\]

(3.34)

According to Lemma 3.1 if \( 1 < \gamma \leq 1 + \frac{2}{n} \), we get

\[
\frac{n(\gamma - 1)}{2} E(0) t^2 + F(0) t + G(0) \leq \frac{1}{2} M(0) |D|^2,
\]

(3.35)

and, if \( \gamma > 1 + \frac{2}{n} \), we have

\[
E(0) t^2 + F(0) t + G(0) \leq G(t) \leq \frac{1}{2} M(0) |D|^2,
\]

(3.36)

respectively. The inequalities (3.35) and (3.36) imply that the life span is finite. The proof of the proposition is finished.

Remark 3.1 If the temperature \( \theta(x, t) = 0 \) in the outside of the support of the density, the assumption (2.4) is satisfied automatically.

4 The Proof of Theorems 2.3-2.5

In this subsection, we will prove Theorems 2.3-2.5 which is on the blow-up of the classical solution to the Navier-Stokes equations with density-dependent viscosities (DICNS). In the following, the notations \( M(t), \bar{P}(t), G(t), F(t), IE(t), IJ(t) \) are same as in Section 2. However, \( IH(t) \) should be modified as

\[
DIH(t) = 2E_k(t) + n(\gamma - 1)I(t) - \int_{\mathbb{R}^n} [h(\rho) + ng(\rho)](\text{div} u) dx,
\]

(4.1)

where \( h(\rho) = \rho^\alpha, g(\rho) = (\alpha - 1)\rho^\alpha \). To prove Theorems 2.3-2.5, the key is to obtain the upper and lower decay rates of the potential energy \( I(t) \).

The following basic relationships of the quantities defined in Section 2 hold true.

Lemma 4.1 For (DICNS), we have

\[
\frac{d}{dt} M(t) = \frac{d}{dt} \bar{P}(t) = 0, \quad \frac{d}{dt} G(t) = F(t), \quad \frac{d}{dt} F(t) = DIH(t),
\]

(4.2)

and

\[
\frac{d}{dt} IE(t) = \begin{cases} 
-\alpha \int_{\mathbb{R}^n} \rho^\alpha u_x^2 dx, & n = 1; \\
- \int_{\mathbb{R}^n} [h(\rho) \sum_{j=1}^{n} (\partial_j u_j)^2 + g(\rho)(\text{div} u)^2 \\
+ \frac{h(\rho)}{2} \sum_{i \neq j} (\partial_j u_i)^2 + h(\rho) \sum_{i > j} (\partial_i u_i)(\partial_i u_j)] dx, & n \geq 2.
\end{cases}
\]
The estimates of the decay rate of $I(t)$ is divided into two steps.

**Step 1. Lower Bounds of $I(t)$**

Similar to Lemma 4.3 based on Lemma 4.1 and Lemma 3.2, we can get two-sided estimates of $F(t)$ and $G(t)$.

**Lemma 4.2** Let $n = 1$ and suppose that Assumption 2 holds. If $\alpha \geq 1$, then there exist two positive numbers $C_{16}$ and $C_{18}$ such that

$$\begin{align*}
\left(\frac{\mathbb{P}(0)^2}{M(0)} - C_{18}\right) t + C_{16}\left(\frac{\mathbb{P}(0)^2}{2M(0)} - IE(0)\right) + F(0) &\leq F(t), \\
&\leq \max(2, \gamma - 1)IE(0) + C_{18}\left(\frac{\mathbb{P}(0)^2}{2M(0)} - IE(0)\right) + F(0),
\end{align*}$$

(4.3)

and

$$\begin{align*}
\frac{1}{2}\left(\frac{\mathbb{P}(0)^2}{M(0)} - C_{18}\right) t^2 + \left(C_{16}\left(\frac{\mathbb{P}(0)^2}{2M(0)} - IE(0)\right) + F(0)\right) t + G(0) &\leq G(t), \\
&\leq \frac{1}{2}\max(2, \gamma - 1)IE(0) + C_{18}t^2 + \left[-C_{16}\left(\frac{\mathbb{P}(0)^2}{2M(0)} - IE(0)\right) + F(0)\right] t + G(0),
\end{align*}$$

(4.4)

where $C_{18}$ is any positive number between 0 and $\frac{\mathbb{P}(0)^2}{2M(0)}$ satisfying $C_{16}C_{18} = \frac{\alpha}{2}M(0)C_{14}^{\alpha - 1}$.

**Proof.** Using the Cauchy’s inequality, we have

$$\begin{align*}
|\int_{\mathbb{R}} [h(\rho) + g(\rho)] \text{div}$dx$| &\leq \alpha|\int_{\mathbb{R}} \rho^\alpha u_x dx| \leq C_{15}\alpha|\int_{\mathbb{R}} \rho^\alpha dx| + C_{16}|\int_{\mathbb{R}} \rho^\alpha u_x^2 dx| \\
&\leq C_{17}\int_{\mathbb{R}} \rho dx - C_{16} \frac{d}{dt}IE(t) = C_{18} - C_{16} \frac{d}{dt}IE(t),
\end{align*}$$

(4.5)

where $C_{16}C_{18} = \frac{\alpha}{2}M(0)C_{14}^{\alpha - 1}$ and $C_{14} = \max \rho$. Choose $C_{16}$ large enough such that $0 < C_{18} < \frac{\mathbb{P}(0)^2}{M(0)}$. From Lemma 4.1 we have

$$\begin{align*}
\frac{d^2}{dt^2} G(t) = \frac{d}{dt} F(t) = DIH(t) &\geq 2E_k(t) + (\gamma - 1)I(t) - C_{18} + C_{16} \frac{d}{dt}IE(t) \\
&\geq \frac{\mathbb{P}(0)^2}{M(0)} - C_{18} + C_{16} \frac{d}{dt}IE(t),
\end{align*}$$

(4.6)

and

$$\begin{align*}
\frac{d^2}{dt^2} G(t) &\leq \frac{d}{dt} F(t) = DIH(t) \leq 2E_k(t) + (\gamma - 1)I(t) + C_{18} - C_{16} \frac{d}{dt}IE(t) \\
&\leq \max(2, \gamma - 1)IE(0) + C_{18} - C_{16} \frac{d}{dt}IE(t).
\end{align*}$$

(4.7)

Integrating (4.6) and (4.7) with respect to $t$ over $[0, t]$, we finish the proof of the lemma.

**Lemma 4.3** Let $n \geq 1$ and suppose that Assumption 1 hold. If $1 \leq \alpha \leq \gamma$, then there exist two positive numbers $C_{21}$ and $C_{22}$ such that

$$\begin{align*}
\left(\frac{\mathbb{P}(0)^2}{M(0)} - C_{22}\right) t + C_{21}\left(\frac{\mathbb{P}(0)^2}{2M(0)} - IE(0)\right) + F(0) &\leq F(t), \\
&\leq \max(2, n(\gamma - 1))IE(0) + C_{22}t - C_{21}\left(\frac{\mathbb{P}(0)^2}{2M(0)} - IE(0)\right) + F(0),
\end{align*}$$

(4.8)
\[
\frac{1}{2} (\mathbb{P}(0)^2 - C_{22})^2 + (C_{21}(\mathbb{P}(0)^2 - IE(0)) + G(0)) \leq G(t)
\]

\[
\leq \frac{1}{2} \left[ \max(2, n(\gamma - 1))IE(0) + C_{22} \right]^2 + [-C_{21}(\mathbb{P}(0)^2 - IE(0)) + F(0)] + G(0),
\]

(4.9)

where where \( C_{22} \) is any positive number between 0 and \( \frac{\mathbb{P}(0)^2}{2M(0)} \) satisfying \( C_{21}C_{22} = \frac{1}{4} n[1 + n(\alpha - 1)]^2 (\gamma - 1)^n M(0)^{\frac{n-1}{2}} C_{13} \).

**Proof.** By interpolation inequality, we have

\[
\int_{\mathbb{R}^n} \rho^\alpha dx \leq \left( \int_{\mathbb{R}^n} \rho dx \right)^{\frac{\alpha}{n}} \left( \int_{\mathbb{R}^n} \rho^{\alpha} dx \right)^{\frac{1}{n}} \leq (\gamma - 1)^{\frac{\alpha}{n}} M(0)^{\frac{n-1}{2}} C_{19}^\frac{\alpha - 1}{n}. \tag{4.10}
\]

Here we used the Assumption 1 which means that the energy is finite. Using Young’s inequality yields

\[
| \int_{\mathbb{R}^n} [h(\rho) + ng(\rho)](divu) dx | = [1 + n(\alpha - 1)] \int_{\mathbb{R}^n} \rho^\alpha (divu) dx | \leq \frac{n^2}{\alpha} [1 + n(\alpha - 1)] \left( \int_{\mathbb{R}^n} \rho^\alpha dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \rho^\alpha |\nabla u|^2 dx \right)^{\frac{1}{2}}
\]

\[
\leq C_{20} \int_{\mathbb{R}^n} \rho^\alpha dx + C_{21} \int_{\mathbb{R}^n} \rho^\alpha |\nabla u|^2 dx
\]

\[
\leq \frac{C_{22} - C_{21}}{dt} IE(t), \tag{4.12}
\]

where \( C_{21}C_{22} = \frac{1}{4} C_{19} n[1 + n(\alpha - 1)]^2 \) and we choose \( C_{21} \) is large enough such that \( C_{22} < \frac{\mathbb{P}(0)^2}{2M(0)} \).

Similarly to (4.6) and (4.7), one can obtain

\[
\frac{d^2}{dt^2} G(t) = \frac{d}{dt} F(t) \geq \frac{\mathbb{P}(0)^2}{M(0)} - C_{22} + C_{21} \frac{d}{dt} IE(t), \tag{4.13}
\]

and

\[
\frac{d^2}{dt^2} G(t) = \frac{d}{dt} F(t) \leq \max(2, n(\gamma - 1))E(0) + C_{22} - C_{21} \frac{d}{dt} IE(t). \tag{4.14}
\]

Integrating (4.13) and (4.14) with respect to \( t \) over \([0, t]\), we finish the proof of the lemma.

On the other hand, it follows from Lemma 3.4 that

**Proposition 4.1** For (DICNS), we have

\[
I(t) \geq \frac{C_{23}}{G(t)^{\frac{\alpha - 1}{2}}} \tag{4.15}
\]

where \( C_{23} = C_3 \).

Hence Lemma 4.2, Lemma 4.3 and Proposition 4.1 give the lower bounds of \( I(t) \). In the next step, we will give the upper bound of \( I(t) \).

**Step 2. Upper Bound of \( I(t) \)**
Lemma 4.4  Let \( n = 1 \). Then we have the following inequality:

\[
\frac{d}{dt} IJ(t) + \alpha(t + 1)^2 \int_{\mathbb{R}} \rho^\alpha u_x^2 dx \leq \begin{cases} 
\frac{3}{t+1} IJ(t) + \alpha(t+1) \int_{\mathbb{R}} \rho^\alpha u_x dx, & 1 < \gamma \leq 3, \\
\alpha(t + 1) \int_{\mathbb{R}} \rho^\alpha u_x dx, & \gamma > 3.
\end{cases} \tag{4.16}
\]

**Proof.** By Lemma 4.1, one can compute

\[
\frac{d}{dt} IJ(t) = (3 - \gamma)(t + 1) I(t) + \alpha(t + 1) \int_{\mathbb{R}} \rho^\alpha u_x dx - \alpha(t + 1)^2 \int_{\mathbb{R}} \rho^\alpha u_x^2 dx. \tag{4.17}
\]

The case of \( \gamma > 3 \) holds obviously. The other case of \( 1 < \gamma \leq 3 \) follows from \( (3.23) \).

Similar to Lemma 4.4, we can get

**Lemma 4.5**  Let \( n \geq 2 \). Then we have the following inequality:

\[
\frac{d}{dt} IJ(t) + (t + 1)^2 \left\{ \int_{\mathbb{R}^n} \left[ \rho^\alpha \sum_{i=1}^n (\partial_j u_i)^2 + (\alpha - 1) \rho^\alpha (\text{div} u)^2 \right] dx \right\}
\]

\[
+ \frac{\rho^\alpha}{2} \sum_{i \neq j} (\partial_j u_i)^2 + \rho^\alpha \sum_{i \neq j} (\partial_j u_i)(\partial_i u_j) dx
\]

\[
\leq \begin{cases} 
\frac{2 - n(\gamma - 1)}{t + 1} IJ(t) + [1 + n(\alpha - 1)](t + 1) \int_{\mathbb{R}^n} \rho^\alpha (\text{div} u) dx, & 1 < \gamma \leq 1 + \frac{2}{n}, \\
[1 + n(\alpha - 1)](t + 1) \int_{\mathbb{R}^n} \rho^\alpha (\text{div} u) dx, & \gamma > 1 + \frac{2}{n}.
\end{cases} \tag{4.18}
\]

In view of Lemma 4.4 and Lemma 4.5, we can obtain the upper bound of \( I(t) \).

**Proposition 4.2**  Let \( n = 1 \). If \( \alpha \geq \gamma \), we have the following estimates:

\[
IJ(t) \leq \begin{cases} 
C_{26} (1 + t)^{1-\gamma} \exp(-\frac{C_{25}}{t+1}), & 1 < \gamma \leq 3, \\
C_{26} (1 + t)^{-2} \exp(-\frac{C_{25}}{t+1}), & \gamma > 3.
\end{cases} \tag{4.19}
\]

where \( C_{25} = \frac{\alpha(\gamma-1)}{4} C_{14}^{\alpha-\gamma} \) and \( C_{26} = IJ(0) \exp(\frac{\alpha(\gamma-1)}{4} C_{14}^{\alpha-\gamma}) \).

**Proof.** It follows from Young’s inequality and \( (3.23) \) that

\[
\alpha(t + 1) \int_{\mathbb{R}} \rho^\alpha u_x dx \leq \frac{\alpha}{4} \int_{\mathbb{R}} \rho^\alpha dx + \alpha(t + 1)^2 \int_{\mathbb{R}} \rho^\alpha u_x^2 dx
\]

\[
\leq C_{24} \int_{\mathbb{R}} \rho^\gamma dx + \alpha(t + 1)^2 \int_{\mathbb{R}} \rho^\alpha u_x^2 dx
\]

\[
\leq \frac{C_{25}}{(t+1)^2} IJ(t) + \alpha(t + 1)^2 \int_{\mathbb{R}} \rho^\alpha u_x^2 dx. \tag{4.20}
\]

Here we used the fact that \( \alpha \geq \gamma \). Substituting \( (4.20) \) into \( (4.16) \) yields

\[
\frac{d}{dt} I(t) \leq \begin{cases} 
\frac{C_{25}}{(t+1)^2} IJ(t) + \frac{3-\gamma}{t+1} IJ(t), & 1 < \gamma \leq 3, \\
\frac{C_{25}}{(t+1)^2} IJ(t), & \gamma > 3.
\end{cases} \tag{4.21}
\]

Using Gronwall’s inequality, we obtain

\[
IJ(t) \leq \begin{cases} 
C_{26} (1 + t)^{3-\gamma} \exp(-\frac{C_{25}}{t+1}), & 1 < \gamma \leq 3, \\
C_{26} \exp(-\frac{C_{25}}{t+1}), & \gamma > 3.
\end{cases} \tag{4.22}
\]
This, together with (4.23), shows

\[ I(t) \leq \begin{cases} 
  C_{26}(1 + t)^{1-\gamma} \exp\left(-\frac{C_{26}}{t+1}\right), & 1 < \gamma \leq 3, \\
  C_{26}(1 + t)^{-2} \exp\left(-\frac{C_{26}}{t+1}\right), & \gamma > 3. 
\end{cases} \tag{4.23} \]

Here \( C_{24} = \frac{\alpha}{4} C_{14}^{\alpha-\gamma}, C_{25} = \frac{\alpha(\gamma-1)}{4} C_{14}^{\alpha-\gamma} \) and \( C_{26} = IJ(0) \exp\left(\frac{\alpha(\gamma-1)}{4} C_{14}^{\alpha-\gamma}\right) \). The proof of the proposition is finished.

**Proposition 4.3** Let \( n \geq 1 \). If \( \frac{\gamma+1}{2} < \alpha \leq \gamma \) and \( 1 < \gamma < 1 + \frac{2}{n} \), then we have the following estimates:

\[ I(t) \leq \begin{cases} 
  C_{29}(1 + t)^{-n(\gamma-1)} \exp\left(-\frac{C_{27}}{(t+1)^{\frac{\gamma}{n+1}}}\right) + \frac{1}{(1+t)^2}, & n \geq 2, \\
  C_{31}(1 + t)^{1-\gamma} \exp\left(-\frac{C_{28}}{(t+1)^{\frac{\gamma}{n+1}}}\right) + \frac{1}{(1+t)^2}, & n = 1. 
\end{cases} \tag{4.24} \]

Here

\[ C_{27} = \exp\left(\frac{\gamma-1}{4(\alpha-1)(2\alpha - \gamma - 1)}[1 + n(\alpha - 1)]^2(\gamma - 1)^{\frac{\alpha-1}{\alpha-1}} M(0) \frac{2-\alpha}{2-\alpha}\right); \]
\[ C_{28} = \frac{2\alpha - \gamma - 1}{\gamma - 1}; \]
\[ C_{29} = IJ(0) \exp\left(\frac{\gamma-1}{4(\alpha-1)(2\alpha - \gamma - 1)}[1 + n(\alpha - 1)]^2(\gamma - 1)^{\frac{\alpha-1}{\alpha-1}} M(0) \frac{2-\alpha}{2-\alpha}\right); \]
\[ C_{30} = \exp\left(\frac{\alpha(\gamma-1)}{4(2\alpha - \gamma - 1)}(\gamma - 1)^{\frac{\alpha-1}{\alpha-1}} M(0) \frac{2-\alpha}{2-\alpha}\right); \]
\[ C_{32} = IJ(0) \exp\left(\frac{\alpha(\gamma-1)}{4(2\alpha - \gamma - 1)}(\gamma - 1)^{\frac{\alpha-1}{\alpha-1}} M(0) \frac{2-\alpha}{2-\alpha}\right). \]

**Proof of Proposition 4.3.** It follows from (4.10) that

\[ \int_{\mathbb{R}^n} \rho^\alpha dx \leq \left( \int_{\mathbb{R}^n} \rho dx \right)^{\frac{\alpha-1}{\gamma-1}} \int_{\mathbb{R}^n} \rho^\alpha dx \leq (\gamma - 1)^{\frac{\alpha-1}{\gamma-1}} M(0) \frac{2-\alpha}{2-\alpha} (t+1)^{-\frac{2(\alpha-1)}{\gamma-1}} IJ(t)^{\frac{\alpha-1}{\gamma-1}} \tag{4.25} \]

for \( 1 \leq \alpha \leq \gamma \). Consequently, when \( n \geq 2 \), one has

\[ [1 + n(\alpha - 1)](t+1) \int_{\mathbb{R}^n} \rho^\alpha div u dx \]
\[ \leq \frac{1}{4(\alpha - 1)}[1 + n(\alpha - 1)]^2 \int_{\mathbb{R}^n} \rho^\alpha dx + (\alpha - 1)(t+1)^2 \int_{\mathbb{R}^n} \rho^\alpha (div u)^2 dx \]
\[ \leq \frac{1}{4(\alpha - 1)}[1 + n(\alpha - 1)]^2(\gamma - 1)^{\frac{\alpha-1}{\gamma-1}} M(0) \frac{2-\alpha}{2-\alpha} (t+1)^{-\frac{2(\alpha-1)}{\gamma-1}} IJ(t)^{\frac{\alpha-1}{\gamma-1}} + (\alpha - 1)(t+1)^2 \int_{\mathbb{R}^n} \rho^\alpha (div u)^2 dx. \tag{4.26} \]

When \( n = 1 \), one has

\[ \alpha(t + 1) \int_{\mathbb{R}} \rho^\alpha u dx \leq \frac{\alpha}{4} \int_{\mathbb{R}} \rho^\alpha dx + \alpha(t+1)^2 \int_{\mathbb{R}} \rho^\alpha u^2 dx \]
\[ \leq \frac{\alpha}{4} (\gamma - 1)^{\frac{\alpha-1}{\gamma-1}} M(0) \frac{2-\alpha}{2-\alpha} (t+1)^{-\frac{2(\alpha-1)}{\gamma-1}} IJ(t)^{\frac{\alpha-1}{\gamma-1}} + \alpha(t+1)^2 \int_{\mathbb{R}} \rho^\alpha u^2 dx. \tag{4.27} \]
Putting (4.26) and (4.27) into (4.18), we have
\[
\frac{d}{dt} IJ(t) \leq \begin{cases} 
\frac{1}{4(\alpha - 1)} [1 + n(\alpha - 1)]^2 (\gamma - 1) \frac{\alpha - 1}{\gamma - 1} \cdot M(0) \frac{\gamma - \alpha}{\gamma - 1} + \frac{2(\alpha - 1) - 1}{\gamma - 1} IJ(t) \frac{\alpha - 1}{\gamma - 1}, \\
+ \frac{2 - n(\gamma - 1)}{\gamma - 1} IJ(t), \\
\frac{a}{2} (\gamma - 1) \frac{\alpha - 1}{\gamma - 1} M(0) \frac{\gamma - \alpha}{\gamma - 1} (t + 1)^{\frac{2(\alpha - 1) - 1}{\gamma - 1}} IJ(t) \frac{\alpha - 1}{\gamma - 1} + \frac{2 - \gamma}{\gamma + 1} IJ(t), 
\end{cases} 
\quad n \geq 2, 
\tag{4.28}
\]

for \( 1 < \gamma \leq 1 + \frac{2}{n} \).

If \( IJ(t) \leq 1 \), one deduces that \( I(t) \leq \frac{1}{(1 + t)^2} IJ(t) \leq \frac{1}{(t + 1)^2} \).

If \( IJ(t) > 1 \), since \( \alpha \leq \gamma \), one has
\[
\frac{d}{dt} IJ(t) \leq \begin{cases} 
\left(2 - n(\gamma - 1) + \frac{1}{4(\alpha - 1)} [1 + n(\alpha - 1)]^2 (\gamma - 1) \frac{\alpha - 1}{\gamma - 1} \right) \cdot M(0) \frac{\gamma - \alpha}{\gamma - 1} (t + 1)^{\frac{2(\alpha - 1) - 1}{\gamma - 1}} IJ(t), \\
\frac{3 - \gamma}{t + 1} + \frac{a}{4} (\gamma - 1) \frac{\alpha - 1}{\gamma - 1} M(0) \frac{\gamma - \alpha}{\gamma - 1} (t + 1)^{\frac{2(\alpha - 1) - 1}{\gamma - 1}} IJ(t). 
\end{cases} 
\quad n = 1, 
\tag{4.29}
\]

Thanks to \( \gamma < 1 + \frac{2}{n} \) and \( \frac{\gamma - 1}{2} < \alpha \), we obtain by Gronwall’s inequality that
\[
IJ(t) \leq \begin{cases} 
C_{29} (1 + t)^{2 - n(\gamma - 1)} \exp(-\frac{C_{27}}{(t + 1)^{2 + 2n}}), \\
C_{31} (1 + t)^{3 - \gamma} \exp(-\frac{C_{30}}{(t + 1)^{2 + 2n}}), 
\end{cases} 
\quad n \geq 2, 
\tag{4.30}
\]

which implies
\[
I(t) \leq \frac{1}{(1 + t)^2} IJ(t) \leq \begin{cases} 
C_{29} (1 + t)^{-n(\gamma - 1)} \exp(-\frac{C_{27}}{(t + 1)^{2 + 2n}}), \\
C_{31} (1 + t)^{1 - \gamma} \exp(-\frac{C_{30}}{(t + 1)^{2 + 2n}}), 
\end{cases} 
\quad n \geq 2, 
\tag{4.31}
\]

Here
\[
C_{27} = \exp(\frac{\gamma - 1}{4(\alpha - 1)(2\alpha - \gamma - 1)} [1 + n(\alpha - 1)]^2 (\gamma - 1) \frac{\alpha - 1}{\gamma - 1}) M(0) \frac{\gamma - \alpha}{\gamma - 1}; \\
C_{28} = \frac{2\alpha - \gamma - 1}{\gamma - 1}; \\
C_{29} = IJ(0) \exp(\frac{\gamma - 1}{4(\alpha - 1)(2\alpha - \gamma - 1)} [1 + n(\alpha - 1)]^2 (\gamma - 1) \frac{\alpha - 1}{\gamma - 1}) M(0) \frac{\gamma - \alpha}{\gamma - 1}; \\
C_{30} = \exp(\frac{\alpha(\gamma - 1)}{4(2\alpha - \gamma - 1)} (\gamma - 1) \frac{\alpha - 1}{\gamma - 1}) M(0) \frac{\gamma - \alpha}{\gamma - 1}; \\
C_{31} = IJ(0) \exp(\frac{\alpha(\gamma - 1)}{4(2\alpha - \gamma - 1)} (\gamma - 1) \frac{\alpha - 1}{\gamma - 1}) M(0) \frac{\gamma - \alpha}{\gamma - 1}).
\]

Combining the cases \( IJ(t) \leq 1 \) and \( IJ(t) > 1 \), we finish the proof of the proposition.

Now, we are ready to prove Theorem 2.3 and Theorem 2.4. The key idea is to compare the coefficients in the lower bounds and the upper bounds of the potential energy \( I(t) \), which is similar to the proof of Theorem 2.1 and we omit the details here.

**Proof of Theorem 2.3.** Theorem 2.3 follows from Lemma 4.1, Proposition 4.1 and Proposition 4.2.

**Proof of Theorem 2.4.** Theorem 2.4 follows from Lemma 4.3, Proposition 4.1 and Proposition 4.3.

**Proof of Theorem 2.5.** Theorem 2.5 follows from Lemma 4.3, Proposition 4.1 and Proposition 4.3.
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