Global firing induced by noise or diversity in excitable media

C.J. Tessone, A. Scirè, R. Toral, P. Colet

Institut Mediterrani d’Estudis Avançats (IMEDEA), CSIC-UIB, Ed.Mateu Orfila, Campus UIB, E-07122 Palma de Mallorca (Spain)

(Dated: February 4, 2022)

PACS numbers: 05.45.-a, 05.45.Xt, 05.40.-a, 02.50.-r

Excitable behavior appears in a large variety of physical, chemical and biological systems \[1, 2\]. Typically this behavior occurs for parameter values close to an oscillation bifurcation, and is characterized by a nonlinear response to perturbations of a stationary state: while small perturbations induce a smooth return to the fixed point, perturbations exceeding a given threshold induce a return through a large phase space excursion (firing), largely independent of the magnitude of the perturbation. Furthermore, after one firing the system cannot be excited again within a refractory period of time. In many situations of interest, the firings are induced by random perturbations or noise \[3\].

In coupled excitable systems, global firing (a macroscopic fraction of the units fires simultaneously) excited by noise has been observed in chemical excitable media \[4\], neuron dynamics \[5\] and electronic systems \[6\], and it has been described through several theoretical approaches \[7, 8\]. This synchronized firing can be considered as a constructive effect induced by the noise. Other examples in which noise actually helps to obtain a more ordered behavior are stochastic resonance \[9\], stochastic coherence (or coherence resonance) \[10\], and noise-induced phase transitions \[11\].

Diversity, the fact that not all units are identical, is an important ingredient in realistic modeling of coupled systems. Ensembles of coupled oscillators with diversity have been paradigmized \[12\] and largely studied \[13, 14\], with the result that synchronized behavior can appear once the disorder induced by the diversity is overcome by the entraining effect of the coupling. It has been shown that in a purely deterministic excitable system diversity may induce global collective firing \[15\] if a fraction of the elements are above the oscillatory bifurcation. So, diversity and noise might be expected to play a similar role.

In this work, we develop an analytical understanding for the emergence of global firing in coupled excitable systems in presence of disorder, either noise or diversity. We show that three different dynamical regimes are possible: sub-threshold motion, where all elements remain confined near the fixed point; coherent pulsations, where a macroscopic fraction fire simultaneously; and incoherent pulsations, where units fire in a disordered fashion. We also show that the mechanism for global firing is generic: it arises from degradation of entrainment originated either by noise or by diversity.

We develop a theory for the emergence of global firings in non-identical excitable systems subject to noise. Three different dynamical regimes arise: sub-threshold motion, where all elements remain confined near the fixed point; coherent pulsations, where a macroscopic fraction fire simultaneously; and incoherent pulsations, where units fire in a disordered fashion. We show that three different dynamical regimes are possible: sub-threshold motion, where all elements remain confined near the fixed point; coherent pulsations, where a macroscopic fraction fire simultaneously; and incoherent pulsations, where units fire in a disordered fashion. Remarkably, the coherent behavior appears through a genuine phase transition when the noise intensity, the coupling or the diversity cross a critical value. A second phase transition to the disordered (incoherent) phase is recovered for large enough noise intensity or diversity, or small enough coupling. The mechanism for global firing is the degradation of entrainment which can be originated either by noise or diversity. This is generic and opens a new scenario for experimental observations.

We consider as a prototypical model an ensemble of globally coupled active-rotators \(\phi_j(t), j = 1, \ldots, N\), whose dynamics is given by

\[
\dot{\phi}_j = \omega_j - \sin \phi_j + \frac{C}{N} \sum_{k=1}^{N} \sin (\phi_k - \phi_j) + \sqrt{D} \xi_j. \tag{1}
\]

The natural frequencies \(\omega_j\) are distributed according to a probability density function \(g(\omega_j)\), with mean value \(\bar{\omega}\) and variance \(\sigma^2\). Notice that \(\omega_j < 1\) (resp. \(\omega_j > 1\)) corresponds to an excitable (resp. oscillatory) \[17\] behavior of the solitary rotator \(j\). Throughout the paper we consider the case \(\bar{\omega} < 1\). \(D\) is the intensity of the Gaussian noises \(\xi_j\) of zero mean and correlations \(\langle \xi_j(t)\xi_k(t) \rangle = \delta(t-t')\delta_{jk}\), and \(C\) is the coupling intensity.

To characterize collective behavior we use the time-dependent global amplitude, \(\rho(t)\), and phase, \(\Psi(t)\) \[12\] \[18\].

\[
\rho(t) e^{i\Psi(t)} = \frac{1}{N} \sum_{k=1}^{N} e^{i\phi_k(t)}. \tag{2}
\]

The Kuramoto order parameter \(\rho \equiv \langle \rho(t) \rangle\), where \(\langle \cdot \rangle\) denotes the time average, is known to be a good measure of collective synchronization in coupled oscillators systems, i.e. \(\rho = 1\) when oscillators synchronize \(\phi_j(t) = \phi_k(t), \forall j, k\) and \(\rho \rightarrow 0\) for desynchronized behavior. Notice, however, that the Kuramoto parameter adopts a non-zero value even when all the variables \(\phi_j\), being equal...
to each other, are at rest. To discriminate between this static entrainment from the dynamic entrainment (time synchronization) of excitable systems when all units fire synchronously, we use the order parameter introduced by Shinomoto and Kuramoto \[13\]

$$\zeta = \left\langle \left| \rho(t) e^{i \Psi(t)} - \left\langle \rho(t) e^{i \Psi(t)} \right\rangle \right| \right\rangle,$$  

(3)

which differs from zero only in the case of synchronous firing. Finally, a measure for the activity of the units, widely used in problems of stochastic transport in non-symmetric potentials is the current

$$J = \frac{1}{N} \sum_{k=1}^{N} \left\langle \dot{\phi}_k(t) \right\rangle.$$  

(4)

A non-zero current \(J\) describes a situation in which the systems are firing (not necessarily synchronized).

We now provide an analytical theory to understand the behavior of \(\rho\), \(\zeta\) and \(J\) as a function of the control parameters, \(C\), \(D\) and \(\sigma\). The theory proceeds in three steps. First, under the assumption of entrainment, we derive a dynamical equation for the global phase \(\Psi\), depending on the value of the Kuramoto parameter \(\rho\). Second, using the solution of that equation, we obtain expressions for \(\zeta\) and \(J\) which depend on \(\rho\). Finally, we calculate self-consistently the value of \(\rho\).

Averaging equation \(\text{(2)}\) over the whole ensemble and using the definition of global amplitude and phase of Eq. \(\text{(2)}\) we have

$$\frac{1}{N} \sum_{k=1}^{N} \dot{\phi}_j = \omega - \rho(t) \sin \Psi(t) + \sqrt{\frac{D}{N}} \xi(t).$$  

(5)

where \(\xi(t)\) is a Gaussian noise of zero mean and correlations \(\langle \xi(t) \xi(t') \rangle = \delta(t - t').\) Taking the time-derivative of Eq. \(\text{(2)}\) and introducing \(\dot{\delta}_j(t) = \phi_j(t) - \Psi(t)\), we obtain:

$$\dot{\rho}(t) + i \rho(t) \dot{\Psi}(t) = \frac{i}{N} \sum_{k=1}^{N} \phi_k e^{i \delta_k(t)}.$$  

(6)

We consider now that the rotators are synchronized in the sense that \(\delta_j(t) \ll 1\) and substitute the expansion \(e^{i \delta_k} = 1 + i \delta_k + \mathcal{O}(\delta_k^2)\) in the previous expression. Equating real and imaginary parts, we obtain

$$\rho(t) \dot{\Psi}(t) = \frac{1}{N} \sum_{k=1}^{N} \phi_k + \mathcal{O}(\delta_k^2).$$  

(7)

The definition of \(\delta_i\) leads to \(\rho(t) = N^{-1} \sum_k e^{i \delta_k}.\) Hence \(\dot{\rho}(t) = O(\delta_k^2)\) and, consistently with the order of the approximation, we can replace in equation \(\text{(7)}\) the time dependent \(\rho(t)\) by the constant value \(\rho\). Therefore, Eq. \(\text{(7)}\) can be approximated by \(\rho \dot{\Psi}(t) = \omega - \rho \sin \Psi(t) + \sqrt{D}/N \xi(t)\), which in the limit \(N \to \infty\), reduces to

$$\dot{\Psi}(t) = \frac{\omega}{\rho} - \sin \Psi(t).$$  

(8)

It is remarkable that the global phase obeys the same dynamics than the individual units but with a natural frequency scaled with \(\rho\), the Kuramoto parameter measuring the entrainment degree. Therefore, a decrease in the entrainment lowers the global excitability threshold from \(\omega = 1\) to \(\omega = \rho\) and the system can start firing synchronously. The effect can be understood as a broadening of the distribution of the phases \(\phi\), so that a fraction of the rotators crosses over the threshold and, if the coupling is large enough, they pull a macroscopic fraction of the oscillators. Thus degradation of the entrainment has the paradoxical effect of increasing the coherent firing. It is essential to realize that Eq. \(\text{(8)}\) depends only on the value of \(\rho\) and not in the specific way the degradation of \(\rho\) is achieved, so that similar effects can be achieved either increasing the noise, either decreasing the coupling, or increasing the diversity in the natural frequencies; a significantly insightful result not previously understood nor discussed.

Let us now express \(\zeta\) and \(J\) as a function of the Kuramoto order parameter \(\rho\). In the case \(\rho < \omega\), the solution of \(\text{(8)}\) is given by \[14\]

$$\omega - \rho \sin \Psi(t) = \frac{\omega^2 - \rho^2}{\omega - \rho \cos \Omega t},$$  

(9)

where \(\Omega = \sqrt{\langle \omega / \rho \rangle^2 - 1}\) is the frequency of the global phase oscillations. The current is obtained from Eq. \(\text{(9)}\), \(J = \omega - \langle \rho \sin \Psi(t) \rangle\). Time averages are computed over a period \(T = 2 \pi / \Omega\) using Eq. \(\text{(9)}\):

$$J = \frac{\omega^2 - \rho^2}{T} \int_0^T \frac{dt}{\omega - \rho \cos \Omega t} = \sqrt{\omega^2 - \rho^2}.$$  

(10)

For \(\rho > \omega\), \(J = 0\).

Approximating \(\rho(t)\) by a constant value, the Shinomoto-Kuramoto parameter \(\zeta \approx \rho \left( \langle e^{i \Psi(t)} \rangle - \langle e^{i \Psi(t)} \rangle \right)\) can be computed performing again the time averages over a period \(T\) using Eq. \(\text{(9)}\):

$$\zeta = \frac{2}{\pi} \sqrt{2(\omega - \sqrt{\omega^2 - \rho^2})(\omega + \rho)} K \left( \frac{2 \rho}{\rho - \omega} \right),$$  

(11)

where \(K(m)\) is the complete elliptic integral of the first kind \[20\]. If \(\rho > \omega\) we get \(\zeta = 0\).

As a final step, we derive a equation for \(\rho\) using a self-consistent, Weiss-like, mean field approximation, which assumes constant values for the global magnitudes and then averages over their probability distribution \[11\,12\]. For our particular case, we start by rewriting Eq. \(\text{(11)}\) as

$$\dot{\phi}_i(t) = - \frac{V(\phi_i; \Psi, \rho, \omega)}{\partial \phi_i} + \sqrt{D} \xi_i(t),$$  

(12)

where we have defined the potential

$$V(\phi; \Psi, \rho, \omega) = - \omega \phi - \cos(\phi) - C \rho \cos(\Psi - \phi).$$  

(13)
Note that the coupling appears only through the global parameters $\rho$ and $\Psi$. For fixed $\rho$ and $\Psi$, the stationary probability distribution function reads \[ P_{\text{st}}(\phi; \Psi, \rho, \omega) = Z^{-1} e^{-2V(\phi)/D} \int_0^{2\pi} d\phi' e^{2V(\phi' + \phi)/D}, \] where $Z$ is a normalizing constant. From its definition, we have $\rho = (1/N) \sum_{k=1}^{N} (\cos(\phi_k - \Psi))$, and we obtain \[ \rho = \int d\omega g(\omega) \int_0^{2\pi} d\Psi P(\Psi; \rho) \int_0^{2\pi} d\phi P_{\text{st}}(\phi; \Psi, \rho, \omega) \cos(\phi - \Psi) \] where we have performed a triple average: with respect to the distribution (14), with respect to the distribution $g(\omega)$ of natural frequencies and with respect to the distribution $P(\Psi; \rho)$ of the global phase which is inversely proportional to the instantaneous velocity given by the dynamics (3), namely $P(\Psi; \rho) = (1/2\pi) \sqrt{\omega^2 - \rho^2}/(\omega - \rho \sin \Psi)$ for $\rho < \omega$ and $P(\Psi; \rho) = \delta(\Psi - \arcsin(\omega/\rho))$ otherwise. The self-consistent equation (15) for $\rho$ needs to be solved numerically.

In the following, we discuss the theoretical results and compare them with the numerical results obtained from a numerical integration of Eqs. (11). Fig. 1 shows $\rho$, $\zeta$ and $J$ as function of the noise intensity $D$ in absence of diversity. The solid lines correspond to the theoretical results while symbols show the numerical results for different system sizes. In this figure, we can observe the three aforementioned behaviors: For small noise intensity (regime I) each rotator fluctuates around its fixed point. Although for un-coupled rotators noise would eventually excite some spontaneous random firings, the coupling of a large number of units suppresses these individual firings. The Kuramoto parameter $\rho$ close to 1 and the deviations from unity are due to the small dispersion induced by noise. Region I is, in fact, characterized by $\rho > \omega$ for which our theory predicts that the Shinomoto-Kuramoto parameter $\zeta$ and the current $J$ vanish which physically reflects the nonexistence of collective movement. In this region, the numerical results for $\rho$, $\zeta$ and $J$ are in excellent agreement with the theoretical predictions.

Our theory predicts that a transition to a dynamical state characterized by synchronized firing behavior (regime II) takes place when $\rho = \omega$, in very good agreement with the numerical results. This transition is clearly signaled by non-vanishing values of $\zeta$ and $J$. The prediction of $\rho$ is good for a large part of region II (up to values of $\rho = 0.7$). Later it underestimates its value. For very large noise intensity, the rotators desynchronize while keeping a non-zero current value (regime III). Hence, the synchronized activity, as measured by $\zeta$ goes though a maximum as noise amplitude increases. Our theory predicts that the transition between regions II and III occurs for $\rho = 0$ where $\zeta = 0$ and the current takes the maximum possible value $J = \omega$. Surprisingly, since the small values of $\rho$ in this transition are beyond the assumptions of the theory, the location of the second
transition is also well predicted. Moreover, the whole shape of the Shinomoto-Kuramoto parameter $\zeta$ is well reproduced over the whole range. The maximum of Eq. 11 occurs for $\rho \approx 0.821\omega$, which is well confirmed by the numerical results. The theoretically predicted current $J$ fits the numerical values in the same range than $\rho$. Note, however, the numerical simulations show a local maximum for the current $J$ which indicates a local increasing in the total transport due to the coherent dynamics in the regime II. This local maximum is not present in the theoretical approximation.

Some of these states were already described by Kuramoto and Shinomoto 18. By looking at the probability distribution of $\phi_i$, these authors identify two regions in parameter space: the time-periodic regime (P) and the stationary regime (S). Region P corresponds to our regime II where the order parameter $\zeta$ is different from zero and there is collective motion of the oscillators. Our findings allow us to split region S of these authors in our distinct regions I and III: while region I is a fluctuating regime around the steady state, region III has a high activity as characterized by a non-zero current $J$.

These results indicate that noise acts in two antagonistic ways: while a given noise intensity can excite the sub-threshold units, forcing a synchronized firing, large amplitude noise deteriorates the synchronization properties of the ensemble. This scenario resembles the so-called noise induced phase transitions 11 in which a transition to an ordered ferromagnetic-like state is induced by increasing the noise intensity; the order is destroyed again for large enough noise. Here, the transition is towards an organized collective motion of the active rotators.

The reverse scenario can be observed varying the coupling strength $C$, see Fig. 1. The Kuramoto parameter $\rho$ increases with $C$ (notice, however, the existence of a small bump in the numerical results), indicating that the degree of synchronization increases with coupling, as expected. A large coupling suppresses noise-induced firings, and the system is macroscopically at rest, regime I, as indicated by the vanishing of $\zeta$ and $J$. For weak coupling the noise induces desynchronized individual firings (regime III) characterized by a macroscopic current $J$ and again a zero value for $\zeta$. For intermediate values of the coupling (regime II) the interplay of noise and coupling leads to the largest degree of synchronized firing with a large value for $\zeta$.

Finally Fig. 11 shows $\rho$, $\zeta$ and $J$ as a function of the diversity $\sigma$. It is clear in the figure the existence of the same three regimes that were obtained by varying the noise intensity or the coupling. Altogether Fig. 11 clearly illustrates the fact that similar effects can be achieved increasing the noise, decreasing the coupling or increasing the diversity in the natural frequencies as theoretically predicted.

In summary we have developed a theory for the emergence of coherent firing in subthreshold excitable systems. It arises as a consequence of a phase transition induced by noise or diversity. The underground mechanism is the degradation of entrainment originated by disorder. This leads to the somehow paradoxical result of establishing a lower effective threshold for collective firing. It does not matter the specific source of disorder responsible for entrainment degradation, either noise or diversity when nonlinearly combined with coupling lead to similar results. Coherent firing is thus achieved by a global saddle-node bifurcation, triggered by the degree of entrainment of the ensemble, $\rho$. This mechanism is not restricted to the model we considered, it will exist in any physical, chemical or biological system with the necessary generic ingredients.

We acknowledge financial support by the Ministerio de Educaci´on y Ciencia (Spain), FEDER projects FIS2004-5073, FIS2004-953, BFM2001-0341 and the EU NoE BioSim (LSHB-CT-2004-005137).

[1] J. Murray, Mathematical Biology (Springer-Verlag, New York, 2002), 3rd ed.
[2] E. Meron, Phys. Rep. 218, 1 (1992).
[3] B. Lindner, J. García-Ojalvo, A. Neiman, and L. Schimansky-Geier, Phys. Rep. 392, 321 (2004).
[4] S. Kadar, J. Wang, and K. Showalter, Nature 391, 770 (1998).
[5] S. Tanabe and K. Pakdaman, Biol Cybern. 85, 269 (2001).
[6] D. Potsnov, S. Han, T. Yim, and O. Soronvsteva, Phys. Rev. E 59, R3791 (1999).
[7] M. Zaks, A. Neiman, S. Feistel, and L. Schimansky-Geier, Phys. Rev. E 68, 066206 (2003).
[8] S. Park and S. Kim, Phys. Rev. E 53, 3425 (1996).
[9] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. 70, 223 (1998).
[10] A. Pikovsky and J. Kurths, Phys. Rev. Lett. 78, 775 (1997).
[11] C. van den Broeck, J. Parrondo, and R. Toral, Phys. Rev. Lett. 73, 3395 (1994).
[12] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence (Springer-Verlag, New York, 1984).
[13] S. Strogatz, Physica D 143, 1 (2000).
[14] S. Strogatz, Nonlinear dynamics and chaos (Perseus Books, Cambridge MA, 1994).
[15] J. Cartwright, Phys. Rev. E 62, 1149 (2000).
[16] Y. Kuramoto, in International Symposium on Mathematical Problems in Theoretical Physics, edited by H. Araki (Springer-Verlag, New York, 1975), vol. 39.
[17] In this case, note that actual frequency is $\sqrt{\omega^2 - 1}$.
[18] S. Shimomoto and Y. Kuramoto, Prog. Theor. Phys. 75, 1105 (1986).
[19] This form of the solution assumes a particular initial condition $\Psi(t = 0) = -\pi/2$, which is otherwise irrelevant when taking the time average.
[20] M. Abramowitz and I. Stegun, eds., Handbook of mathematical functions (Dover Pub., New York, 1964).
[21] C. Gardiner, Handbook of Stochastic Methods (Springer-Verlag, Berlin, 1985), 2nd ed.