The role of the geometric mean in case-control studies

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Abstract

Historically used in settings where the outcome is rare or data collection is expensive, outcome-dependent sampling is relevant to many modern settings where data is readily available for a biased sample of the target population, such as public administrative data. Under outcome-dependent sampling, common effect measures such as the average risk difference and the average risk ratio are not identified, but the conditional odds ratio is. Aggregation of the conditional odds ratio is challenging since summary measures are generally not identified. Furthermore, the marginal odds ratio can be larger (or smaller) than all conditional odds ratios. This so-called non-collapsibility of the odds ratio is avoidable if we use an alternative aggregation to the standard arithmetic mean. We provide a new definition of collapsibility that makes this choice of aggregation method explicit, and we demonstrate that the odds ratio is collapsible under geometric aggregation. We describe how to partially identify, estimate, and do inference on the geometric odds ratio under outcome-dependent sampling. Our proposed estimator is based on the efficient influence function and therefore has doubly robust-style properties.

Keywords — collapsibility, odds ratio, outcome-dependent sampling, partial identification, efficient influence function, geometric aggregation

1 Introduction

Outcome-dependent sampling schemes like case-control sampling\(^1\) have long been used in medical, epidemiological, and survey research when the outcome is rare or the data are expensive to collect. For example, pancreatic cancer is a particularly deadly but rare cancer whose risk factors have been analyzed under case-control designs [Hassan et al., 2007, Lucenteforte et al., 2012]. Case-control methods are also often used in criminal justice settings to understand risk factors for crime and to audit for racial bias in police brutality [Bogstrand and Gjerde, 2014, Campbell et al., 2003, Loftin and McDowall, 1988, Ridgeway, 2020, Wheeler et al., 2017].

A case-control design samples from two distributions: the population where the outcome occurred (‘cases’) and the population where the outcome did not occur (‘controls’) [Breslow, 1996, Cornfield et al., 1951, Miettinen, 1976]. An alternative outcome-dependent sampling scheme, known as case-population sampling, samples from the general population instead of the control distribution [Jun and Lee, 2020, Lancaster and Imbens, 1996]. The broader class of biased sampling designs has a vast literature including choice-based sampling [Heckman and

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\(^{1}\)In case-control sampling, ‘cases’ are observations where the outcome occurred and ‘controls’ are observations where the outcome did not occur. This is also known as case-referent sampling, case-noncase sampling, or choice-based sampling.
Todd, 2009, Manski and Lerman, 1977], stratified sampling [Imbens and Lancaster, 1996], and exposure-based sampling such as the matched cohort design [Kennedy et al., 2015].

Under outcome-dependent sampling, common effect measures such as the average risk difference and the average risk ratio are not identified, but the conditional odds ratio is identified. The conditional odds ratio is often estimated under constant effect and/or parametric assumptions. Common in practice, logistic regression estimates the conditional odds ratio under the strong assumption that the parametric form is correctly specified [Prentice and Pyke, 1979]. A semiparametric approach relaxes these assumptions by placing restrictions directly on the odds ratio function, leaving other parts of the data-generating process unspecified [Chen, 2007, Tchetgen Tchetgen et al., 2010]. To avoid parametric assumptions altogether, a targeted maximum likelihood estimator (TMLE) with double robustness properties has been proposed for the population odds ratio, though only when the outcome rate is known [van der Laan, 2008].

The odds ratio can approximate or bound other For example, the odds ratio approximates the risk ratio under a rare disease assumption [Cornfield et al., 1951, Greenland and Thomas, 1982]. A recent work uses the odds ratio to upper bound the risk ratio under monotonicity assumptions on the treatment response and treatment selection [Jun and Lee, 2020]. To account for biased sampling, they target estimation of the outcome-conditional log odds ratio, and they propose a retrospective sieve logistic estimator to do so. The odds ratio can also be used to define a measure of interaction effect when there is more than one exposure of interest [VanderWeele and Vansteelandt, 2011].

Our work considers odds ratios as the effect of interest, in a setting where the outcome rate is unknown. We compare several odds ratio-based measures of effect, discussing their usefulness (Section 2), identifiability (Section 3), and collapsibility (Section 4). Of note is our novel observation that the geometric mean odds ratio is collapsible, unlike the standard (arithmetic aggregated) odds ratio. We offer a new definition of collapsibility that makes explicit the choice of aggregation. We provide the partial identification of the geometric mean odds ratio under outcome-dependent sampling. We propose a doubly robust-style estimation approach for the partially identified geometric odds ratio and describe conditions under which the estimator is \( \sqrt{n} \)-consistent and asymptotically normal in Section 6. Lastly we describe how to do inference for the geometric odds ratio over a range of possible values for the unknown outcome rate in Section 7.

2 Problem setting

Define \( Z = (X, A, Y) \) where \( X \in \mathbb{R}^d \) are covariates, \( A \in \{0, 1\} \) is a binary treatment or exposure, and \( Y \in \{0, 1\} \) a binary outcome. Let \( \mathcal{P} \) denote the space of probability distributions.

Our population of interest has distribution \( P \). We observe data \( (Z_1, Z_2, \ldots, Z_n) \) from a possibly biased distribution \( Q \), where the bias results from outcome-dependent sampling. Let \( \rho = P(Y = 1) \) and \( \omega = Q(Y = 1) \) denote the probability that \( Y = 1 \) under \( P \) and \( Q \) respectively. The quantity \( \omega \) is fixed and known—or if not known, can be easily estimated from the data. The quantity \( \rho \) is fixed but generally unknown. We make virtually no assumptions about knowledge of \( \rho \), requiring only that it lie in a user-specified range \( [\rho, \rho'] \). We anticipate in many settings the user to generally have some knowledge about \( \rho \) to inform the range, but if not, the user can specify \( \rho = \epsilon, \rho = 1 - \epsilon \).

While case-control data consists of samples from two different distributions, it can equivalently be viewed as independent and identically distributed draws from a modified distribution,
known as Bernoulli sampling [Breslow et al., 2000]. Under Bernoulli sampling, we first sample \( Y \sim \text{Bernoulli}(\omega) \), and then we sample \((X, A) \sim P(X, A \mid Y)\). We can relate \( P \) and \( Q \) as

\[
Q(z) = P(x, a \mid y)\omega = P(z)\frac{\omega}{\rho} \tag{1}
\]

We use potential (counterfactual) outcomes to define several measures of effect. Let \( Y^1 \) denote the potential outcome that would have been observed under treatment and \( Y^0 \) the potential outcome that would have been observed under no treatment. Expectations are taken with respect to the target distribution \( P \) unless otherwise noted via subscript.

**Definition 2.1 (Population odds ratio).** The population odds ratio describes the population odds of the outcome occurring under treatment relative to the odds under no treatment.

\[
\text{OR} := \frac{P(Y^1 = 1)}{P(Y^0 = 1)} \bigg/ \frac{P(Y^1 = 0)}{P(Y^0 = 0)} = \frac{\text{odds}(Y^1 = 1)}{\text{odds}(Y^0 = 1)}
\]

The population odds ratio describes the ratio of the odds of the outcome in a world where everyone is treated to the odds of the outcome in a world where no one is treated. In some settings, such as when the outcome is very rare, we may be interested in interventions targeting certain strata of the population. Then our target of interest is the conditional odds ratio, as given in the following definition.

**Definition 2.2 (Conditional odds ratio).** The conditional odds ratio, also known as the adjusted odds ratio, describes the odds of the outcome occurring under treatment relative to the odds under no treatment for a particular stratum of the population with measured covariates \( x \).

\[
\text{OR}(x) := \frac{P(Y^1 = 1 \mid X = x)}{P(Y^0 = 1 \mid X = x)} \bigg/ \frac{P(Y^1 = 0 \mid X = x)}{P(Y^0 = 0 \mid X = x)} = \frac{\text{odds}(Y^1 = 1 \mid X = x)}{\text{odds}(Y^0 = 1 \mid X = x)}
\]

Like many nonparametric function estimation problems, the conditional odds ratio is generally hard to estimate flexibly at fast rates. Logistic regression (without interactions) is probably the most commonly used method to estimate the conditional odds ratio in practice, but this approach will be biased when the conditional odds ratio varies across strata or when the linearity or logistic link function assumptions do not hold. On the other hand, nonparametric plugin methods suffer the curse of dimensionality. We may hope to avoid the curse of dimensionality by targeting the marginal effect since aggregated quantities can typically be estimated at faster rates even in nonparametric models. When we have randomly sampled iid observations, we can compare the importance of characterizing conditional effects against the reasonableness of the assumptions required, and we can then choose to target the marginal or conditional effect accordingly. When our data is from an outcome-dependent sample, however, this usual trade-off no longer applies because the marginal effect is not identified, as we show in the next section.

When the covariates are continuous or high-dimensional, we may desire an aggregation of the conditional odds ratio to describe a summary measure. The most common aggregation uses the arithmetic mean.

**Definition 2.3 (Arithmetic odds ratio).** The arithmetic aggregation of the conditional odds ratio is

\[
\alpha := E[\text{OR}(X)] \tag{2}
\]
A complicating issue is the population odds ratio could be smaller (or larger) than all conditional odds ratios. In other words, the odds ratio is not collapsible [Greenland and Robins, 1986, Miettinen and Cook, 1981, Robinson and Jewell, 1991].

**Definition 2.4 (Geometric odds ratio).** The geometric odds ratio describes the geometric mean of the conditional odds ratios,

\[
\gamma := \prod_{x} \left( OR(x) \right)^{dP(x)}
\]

where \( \prod \) denotes the product integral if \( X \) is continuous or the product operator if \( X \) is discrete. We can equivalently write the geometric odds ratio as

\[
\gamma = \exp \left( E[ \log(OR(X)) ] \right).
\]

**Miscellaneous notation** \( L \lesssim R \) indicates that \( L \leq C \cdot R \) for some universal constant \( C \). Define the squared \( L_2(P) \) norm of a function \( f \) as \( \|f\|^2 := \int (f(x))^2 dP(x) \). \( \mathbb{I}\{\} \) denotes the indicator function. For samples \( Z_1, Z_2, ..., Z_n \sim Q \), we denote sample averages \( \frac{1}{n} \sum_{i=1}^{n} f(Z_i) \) as \( Q_n(f(Z)) \).

### 3 Identifiability

To identify our targets in terms of observable data, we make the following assumptions:

**Assumption 3.1 (Consistency).** A unit that receives treatment \( a \) has outcome \( Y = Y^a \).

**Assumption 3.2 (Ignorability).** \( (Y^0, Y^1) \perp \! \! \! \perp A \mid X \).

**Assumption 3.3 (Overlap).** \( P(0 < P(A = 1 \mid X) < 1) = 1 \).

Assumption 3.1 requires that there is no interference between units; e.g., the potential outcome for a unit does not depend on the treatment assignment of other units. Assumption 3.2 requires that the treatment assignment is as good as random conditional on measured covariates. Assumption 3.3 requires that all covariate strata have some non-zero probability of receiving both treatment decisions.\(^2\) In settings where it is unreasonable to make these assumptions, sensitivity analysis can be conducted in order to assess whether the results are sensitive to violations of the assumptions [Luedtke et al., 2015, Robins et al., 2000, Rosenbaum et al., 2010].

### 3.1 Point identification under random sampling

If our data consists of random samples from the target distribution, then we can identify the population odds ratio, conditional odds ratio, arithmetic odds ratio, and geometric odds ratio respectively as

\(^2\)For estimation we will require a stronger boundedness assumption on overlap.
When our samples

The next section will consider outcome-dependent sampling, where the conditional effect

as a function of the unknown parameter

measures as a function of the unknown parameter

dependent sampling because we sample from

the conditional odds ratio due to the symmetry of the odds ratio \[\text{Cornfield et al., 1951}\]:

3.2 Outcome-dependent sampling

When our samples \((Z_1, Z_2, \ldots, Z_n)\) are drawn from the biased distribution \(Q\), we can still identify the conditional odds ratio due to the symmetry of the odds ratio [Cornfield et al., 1951]:

\[
OR = \frac{E[P(Y = 1 \mid A = 1, X)]}{E[P(Y = 0 \mid A = 1, X)]} / \frac{E[P(Y = 1 \mid A = 0, X)]}{E[P(Y = 0 \mid A = 0, X)]} \tag{5}
\]

\[
OR(x) = \frac{\text{odds}(Y = 1 \mid A = 1, X = x)}{\text{odds}(Y = 1 \mid A = 0, X = x)} \tag{6}
\]

\[
\alpha = E[OR(X)] \tag{7}
\]

\[
\gamma = \prod \{OR(x)\}^{dP(x)} \tag{8}
\]

The conditional, marginal, and aggregated effects are all identified under random sampling. The next section will consider outcome-dependent sampling, where the conditional effect remains identified, but the marginal and aggregated effects are partially identified.

We first define additional notation for regression functions:

\[
\mu_a(x) = Q(Y = 1 \mid X = x, A = a) \\
\nu_a(x) = P(Y = 1 \mid X = x, A = a).
\]

\(\mu_a(x)\) is point identified under outcome-dependent sampling. We can partially identify \(\nu_a(x)\) as a function of the unknown parameter \(\rho\).

Without prior knowledge about the outcome rate \(\rho\), we cannot estimate \(P(X)\). We can partially identify the aggregation measures as a function of the unknown parameter \(\rho\).

We cannot point identify the arithmetic odds ratio and geometric odds ratio under outcome-dependent sampling because we sample from \(P(X \mid Y)\), not \(P(X)\). Without prior knowledge about the outcome rate \(\rho\), we cannot estimate \(P(X)\). We can partially identify the aggregation measures as a function of the unknown parameter \(\rho\).

We first define additional notation for regression functions:

\[
\mu_a(x) = Q(Y = 1 \mid X = x, A = a) \\
\nu_a(x) = P(Y = 1 \mid X = x, A = a).
\]

\(\mu_a(x)\) is point identified under outcome-dependent sampling. We can partially identify \(\nu_a(x)\) as a function of the unknown \(\rho\) by applying Bayes’ Rules to obtain

\[
\nu_a(x; \rho) = \frac{P(A = a \mid X = x, Y = 1)P(X = x \mid Y = 1)\rho}{P(A = a, X = x \mid Y = 1)\rho + P(A = a, X = x \mid Y = 0)(1 - \rho)} \tag{9}
\]

The population odds ratio is partially identified as

\[
\frac{\rho E[\nu_1(X) \mid Y = 1] + (1 - \rho) E[\nu_1(X) \mid Y = 0]}{\rho E[\nu_0(X) \mid Y = 1] + (1 - \rho) E[\nu_0(X) \mid Y = 0]} \tag{10}
\]

The partial identification of the arithmetic odds ratio is

\[
\alpha(\rho) = \rho E[OR(X) \mid Y = 1] + (1 - \rho) E[OR(X) \mid Y = 0]. \tag{11}
\]

The geometric odds ratio is partially identified as

\[
\gamma(\rho) = \left(\prod_X OR(X)^{dP(X \mid Y = 1)}\right)^\rho \left(\prod_X OR(X)^{dP(X \mid Y = 0)}\right)^{1 - \rho} \tag{12}
\]
Or alternatively,

\[
\gamma(\rho) = \exp \left( \rho \mathbb{E}[\log (OR(X)) \mid Y = 1] + (1 - \rho)\mathbb{E}[\log (OR(X)) \mid Y = 0] \right) \\
= \exp \left( \rho \mathbb{E}[\log(\mu_1(X)) - \log(\mu_0(X)) \mid Y = 1] + (1 - \rho)\mathbb{E}[\log(\mu_1(X)) - \log(\mu_0(X)) \mid Y = 0] \right)
\]

(13)

This identification indicates that \(\mathbb{E}[\log(\mu_a(X)) \mid Y = y]\) is a key object for the geometric odds ratio, and we will see later that this function plays a central role in the efficiency theory and estimation. We use the following to denote this function:

\[
\psi_{a,y} := \mathbb{E}[\logit(\mu_{a}(X)) \mid Y = y]
\]

(14)

\(\psi_{a,y}\) is identified under outcome-dependent sampling. To recap, Eqs. 10-13 identify the marginal and aggregated odds ratios up to the unknown constant \(\rho\).

4 On collapsibility

A desirable quality for a measure of effect is that the marginal effect describes the effect for a representative unit. The property of collapsibility (Def. 4.1) formalizes this quality [Greenland et al., 1999, Whittemore, 1978]. In this section, we take a detour from the biased sampling design to discuss collapsibility in detail. For simplicity our example will use random sampling, but the ideas are generally applicable.

Remark 4.1. In a departure from standard usage, we use the term “marginal” to generically refer to a summary measure via an aggregation method that must be explicitly specified. As an example, the “marginal odd” of outcome \(Y\) in standard usage unambiguously refers to \(\frac{P(Y=1)}{P(Y=0)}\). However, we denote this quantity as the marginal odds with respect to arithmetic aggregation, differentiating it from other marginal measures such as, for example, the marginal odds with respect to geometric aggregation \(\prod \frac{P(Y=1|X=x)}{P(Y=0|X=x)}dP(x)\).

Collapsibility is often discussed with respect to the arithmetic mean, under which collapsibility requires that we can specify weights for conditional effects such that the marginal effects equals their weighted average [Hernán and Robins, 2021]. For instance, the risk ratio, \(RR = \mathbb{E}[Y|X]/\mathbb{E}[Y^0]\), is collapsible with respect to the arithmetic mean with weights \(\frac{P(X)}{\mathbb{E}[Y^0]}\). The odds ratio, however, is not collapsible with respect to the arithmetic mean [Greenland et al., 1999]. The population odds ratio is generally not equal to the conditional odds ratio, even if the conditional odds ratio is a constant.

Example: \(X = I\{\text{Female}\}\) with \(P(X = 1) = 0.5\). The risks under treatment and no treatment for women and men are given in the below table along with the corresponding conditional risk ratios, \(RR(X)\), and conditional odds ratios, \(OR(X)\). The marginal risk ratio with respect to arithmetic aggregation is \(\frac{10}{150} = 0.66\). Averaging the conditional risk ratios with weights \(\frac{P(X)}{\mathbb{E}[Y^0]}\) yields the marginal risk ratio under arithmetic aggregation.

| \(X\)     | \(P(Y = 1 \mid X, A = 1)\) | \(P(Y = 1 \mid X, A = 0)\) | \(RR(X)\) | \(OR(X)\) |
|-----------|----------------------------|----------------------------|------------|-----------|
| Female    | \(\frac{1}{6}\)           | \(\frac{9}{10}\)           | \(\frac{5}{27}\) | \(\frac{1}{35}\)    |
| Male      | \(\frac{1}{20}\)          | \(\frac{9}{14}\)           | \(\frac{7}{17}\) | \(\frac{1}{45}\)    |

The conditional odds ratio for women equals the conditional odds ratio for men. However
the marginal odds ratio under arithmetic aggregation is \( \approx \frac{3}{2} \times \frac{1}{45} \). It is not possible to find a weighted average of the conditionals that equals the marginal odds ratio.

While the odds ratio is not collapsible under the arithmetic mean, it is collapsible under the geometric mean. To demonstrate this, we introduce additional notation. Let \( f(a, b) : \mathbb{R}^2 \to \mathbb{R} \) denote an effect contrast and let \( g_w(x)(P) : \mathcal{P} \to \mathbb{R} \) denote a statistical functional that aggregates \( X \sim P \) with weighting function \( w(x) \). For example, letting \( p(x) \) denote the density of random variable \( X \sim P \), we describe the average risk difference (commonly referred to as average treatment effect) by specifying \( g_p(x)(P) = \int x p(x) dx \) and \( f(a, b) = a - b \). We consider aggregations that can be written as a Fréchet mean—that is, there is an associated distance function \( d \) such that

\[
g_w(x)(P) = \arg\min_{z \in \mathcal{X}} \int_{\mathcal{X}} w(x) d^2(z, x) dx.
\]

For ease of notation, we will write \( g(X) \) to indicate \( g(P) \) for the distribution \( P \) over \( X \).

**Definition 4.1.** A contrast \( f \) is collapsible with respect to aggregation method \( g \) if

\[
f\left( g_p(x)(\mu_1(X)), g_p(x)(\mu_0(X)) \right) = g_w(x)\left( f(\mu_1(X), \mu_0(X)) \right)
\]

for weights \( w(x) \) in the probability simplex and where \( p(x) \) denotes the density or pmf of \( X \sim P \).

The left hand side describes the marginal effect—that is, the contrast of the aggregations of \( \mu_a(x) \) for \( a \in \{0, 1\} \). The right hand side describes a weighted aggregation of the conditional contrasts.

Returning to our example, the average risk difference is collapsible with respect to the arithmetic mean using as weights the density of \( x \). We briefly remark on the weights \( p(x) \) and \( w(x) \). While \( p(x) = w(x) \) for the average risk difference, this need not be the case. The risk ratio has contrast \( f(a, b) = \frac{a}{b} \) and is collapsible under aggregation \( g_w(x)(P) = \int x w(x) dx \) with \( w(x) = p(x) \frac{E[Y | X = x]}{E[Y | Y = x]} \).

As far as we are aware, this expanded definition of collapsibility that explicitly incorporates the aggregation method has not appeared in the literature before. With this machinery in place, we make the novel observation that the odds ratio is collapsible under geometric aggregation.

**Proposition 4.1.** The odds ratio is collapsible with respect to the geometric mean with weights \( p(x) \).

If the conditional OR is a constant \( c \), then the geometric odds ratio also equals \( c \). Recall that this was not necessarily the case for the arithmetic odds ratio, which could take on a value other than \( c \). Since the geometric mean exhibits the desirable property of collapsibility, the remainder of this paper will consider estimation of the geometric odds ratio. We focus on the outcome-dependent sampling design, deferring results for the random sampling design to the Appendix. To motivate our estimation approach, we start by studying the efficiency theory.


5 Efficiency

5.1 Preliminaries

First, we restate and define additional notation for the nuisance functions:

\[ \mu_a(x) := Q(Y = 1 \mid X = x, A = a) \quad \text{for} \quad a \in \{0, 1\} \]
\[ \pi_a(x) := Q(A = a \mid X = x) \quad \text{for} \quad a \in \{0, 1\} \]
\[ \eta(x) := Q(Y = 1 \mid X = x) = \sum_{a=0}^{1} \pi_a(x) \mu_a(x). \]

Recall from Section 3.2 (Eqs. 13-14) that we can write our target geometric OR as a function of \( \rho \) and \( \psi_{a,y} \) for \((a, y) \in \{0, 1\}^2\):

\[ \gamma(\rho) = \exp \left( \rho (\psi_{1,1} - \psi_{0,1}) + (1 - \rho) (\psi_{1,0} - \psi_{0,0}) \right) \quad (16) \]

We will first provide a von Mises-type expansion for \( \psi_{a,y} \) using as an example \( a = 0 \) and \( y = 1 \) (see Appendix 9.3 for the general result) and subsequently provide the efficiency theory for \( \gamma \). Functioning as a distributional analog to the Taylor expansion for real-valued functions, the von Mises-type expansion of a target parameter describes two key elements for efficiency theory: the influence function and a remainder term. Influence functions enable us to construct estimators with desirable properties, such as second-order bias, which can achieve fast convergence rates even in nonparametric settings. The remainder term plays an important role in characterizing the error of such estimators (see Section 6). In a fully nonparametric model, the singular influence function is called the efficient influence function because it characterizes the efficiency bound in a local asymptotic minimax sense. The efficient influence function is therefore instructive for constructing optimal estimators. We direct the interested reader to Bickel et al. [1993], Hines et al. [2022], Kennedy [2022], Tsiatis [2006] for more information on influence functions. We first define notation to refer to the nuisance functions on the distribution \( Q \): let \( \bar{\eta}(x) := Q(Y = 1 \mid X = x) \) and similarly for \( \bar{\pi}(x) \) and \( \bar{\mu}(x) \).

**Lemma 5.1.** We have the following von Mises expansion for \( \psi_{0,1} \):

\[ \psi_{0,1}(Q) = \psi_{0,1}(\bar{Q}) + \int \varphi_{0,1}(\bar{Q}) d(Q - \bar{Q}) + R_2(\bar{Q}, Q) \]

for \( R_2(\bar{Q}, Q) = \frac{\bar{\omega} - \omega}{\bar{\omega}} (\psi_{0,1}(\bar{Q}) - \psi_{0,1}(Q)) + \frac{1}{\omega} \int \left( \frac{\mu_0(x) - \bar{\mu}_0(x)}{\bar{\mu}_0(x)(1 - \bar{\mu}_0(x))} (\eta(x) - \bar{\eta}(x)) + \bar{\eta}(x) \frac{\mu_0(x) - \bar{\mu}_0(x)}{\bar{\mu}_0(x)(1 - \bar{\mu}_0(x))} \frac{\bar{\pi}(x) - \bar{\pi}(x)}{\bar{\pi}(x)} \right. \]
\[ + \left. \eta(x) \frac{\mu_0(x) - \bar{\mu}_0(x)}{\mu_0(x)(1 - \bar{\mu}_0(x))} \right) dQ \]

where \( \mu_0^*(x) \) lies between \( \bar{\mu}_0(x) \) and \( \mu_0(x) \) and

\[ \varphi_{0,1}(Z) = \frac{\eta(X) \logit(\mu_0(X))}{\omega} - \psi_{0,1} \]
\[ + \frac{\eta(X)}{\omega} \frac{(1 - A)(Y - \mu_0(X))}{\pi_0(X) \mu_0(X)(1 - \mu_0(X))} + \frac{\logit(\mu_0(X))}{\omega}(Y - \eta(X)) \]
\[ + \psi_{0,1} \frac{Y - \bar{\psi}_{0,1}}{\omega}. \]
Theorem 5.1. Theorem yields with influence function $\phi$ where each term in the $\phi$ where $\gamma$

Proof. Immediate from Lemma 9.2 for $a = 0$ and $y = 1$. □

5.2 Efficiency theory for $\gamma$

**Theorem 5.1.** We have the following von Mises-type expansion of our target $\gamma$:

$$\gamma(Q) = \gamma(\bar{Q}) + \gamma(\bar{Q}) \int \phi(\bar{Q})d(Q - \bar{Q}) + R_2(\bar{Q}, Q) \quad \text{where}$$

$$\phi(Q) = \rho(\varphi_{1,1}(Q) - \varphi_{0,1}(Q)) + (1 - \rho)(\varphi_{1,0}(Q) - \varphi_{0,0}(Q)).$$

Then, by Lemma 2 of [Kennedy et al., 2021], our target $\gamma(Q)$ is pathwise differentiable with influence function $\gamma(Q)\phi(z; Q)$.

Proof. For a $\gamma^*$ such that $\log(\gamma^*)$ lies between $\log(\gamma(Q))$ and $\log(\gamma(\bar{Q}))$, applying Taylor’s Theorem yields

$$\gamma(Q) = \gamma(\bar{Q}) + \gamma(\bar{Q}) \left( \log(\gamma(Q)) - \log(\gamma(\bar{Q})) \right) + \frac{1}{2} \left( \log(\gamma(Q)) - \log(\gamma(\bar{Q})) \right)^2 \gamma^*.$$  (17)

For the first order expression $\log(\gamma(Q)) - \log(\gamma(\bar{Q}))$ we apply Lemma 9.2 to obtain

$$\log(\gamma(Q)) - \log(\gamma(\bar{Q})) = \rho \left( \int \varphi_{1,1}(\bar{Q})d(Q - \bar{Q}) - \int \varphi_{0,1}(\bar{Q})d(Q - \bar{Q}) \right)$$

$$+ (1 - \rho) \left( \int \varphi_{1,0}(\bar{Q})d(Q - \bar{Q}) - \int \varphi_{0,0}(\bar{Q})d(Q - \bar{Q}) \right) + R_2(\bar{Q}, Q)$$

where each term in the $R_2(\bar{Q}, Q)$ is a second-order nuisance function error.

Substituting back into Eq. 17 yields

$$\gamma(Q) = \gamma(\bar{Q}) + \gamma(\bar{Q}) \int \phi(\bar{Q})d(Q - \bar{Q}) + R_2(\bar{Q}, Q) \quad \text{where}$$

$$\phi(Q) = \rho(\varphi_{1,1}(Q) - \varphi_{0,1}(Q)) + (1 - \rho)(\varphi_{1,0}(Q) - \varphi_{0,0}(Q))$$

□

For quick reference we will restate the influence function of $\gamma$ using notation that defines $\eta_y(x) := Q(Y = y \mid x)$ and $\omega_y := Q(Y = y)$:

$$\text{IF}(\gamma) = \left( \rho(\varphi_{1,1}(Z) - \varphi_{0,1}(Z)) + (1 - \rho)(\varphi_{1,0}(Z) - \varphi_{0,0}(Z)) \right) \gamma$$  (18)

where $\varphi_{a,y}(z; Q) = \text{IF}(\psi_{a,y}(z; Q))$

$$= \frac{\text{logit}(\mu_a(X))}{\omega_y} \mathbb{I}\{Y = y\} - \psi_{a,y} + \frac{\eta_y(X)}{\omega_y} \mathbb{I}\{A = a\} \frac{Y - \mu_a(X)}{\pi_a(X)\mu_a(X)(1 - \mu_a(X))}$$

$$+ \frac{\psi_{a,y} \mathbb{I}\{Y = y\}}{\omega_y}$$  (19)
The influence function for $\gamma$ indicates that in addition to requiring that the propensity scores be bounded away from zero and one, we will additionally require that the conditional variances $(1 - \mu_1(x))\mu_1(x)$ and $(1 - \mu_0(x))\mu_0(x)$ be bounded away from zero. This is notably different from the usual risk difference (ATE) setting, where we want the conditional variances to be small to improve efficiency.

**Efficiency bound**

The efficiency bound describes the local asymptotic minimax lower bound on the mean squared error for any estimator of the target parameter, analogous to the Cramer-Rao bound for parametric settings. This bound provides a benchmark against which we can compare estimators. Additionally, the efficiency bound illuminates which components affect the difficulty of the estimation problem. For additional details we refer the reader to Bickel et al. [1993], Kennedy [2022], Tsiatis [2006], Van der Laan et al. [2003]. Before stating our efficiency bound, we introduce notation that will simplify the result. We define the distribution-corrected log conditional odds ratio:

$$
\Psi(X,Y) = \left( Y\frac{\rho}{\omega} + (1-Y)\frac{1-\rho}{1-\omega} \right) \left( \text{logit}(\mu_1(X)) - \text{logit}(\mu_0(X)) \right)
$$

(20)

and the distribution-corrected aggregate log odds ratio:

$$
\Psi^*(Y) = Y\frac{\rho}{\omega}(\psi_{1,1} - \psi_{0,1}) + (1-Y)\frac{1-\rho}{1-\omega}(\psi_{1,0} - \psi_{0,0})
$$

(21)

**Theorem 5.2.** The nonparametric efficiency bound for estimating $\gamma$ is given by $\sigma^2 := \text{var}(\text{IF}(\gamma)) = \gamma^2\text{var}(\text{IF}(\text{log}(\gamma)))$, where $\text{var}(\text{IF}(\text{log}(\gamma)))$ equals

$$
\frac{1}{\pi_0(X)\mu_0(X)(1-\mu_0(X))} + \frac{1}{\pi_1(X)\mu_1(X)(1-\mu_1(X))} \left( \frac{\rho - \omega}{\omega(1-\omega)} \right)^2 
$$

The proof is given in Appendix 9.4.

The two terms that involve $\Psi^*(Y)$ result from having to estimate $\omega$: if $\omega$ is known by sampling design, then these terms drop from the bound. The coefficient $\left( \frac{\rho - \omega}{\omega(1-\omega)} \right) \eta(X) + \frac{1-\rho}{1-\omega}$ results from sampling bias. When $\rho = \omega$ this coefficient equals 1.3

Theorem 5.2 indicates that the difficulty of our estimation problem depends on the following factors:

1. Heterogeneity in the distribution-corrected log conditional odds ratio $\Psi(X,Y)$;
2. Heterogeneity in the distribution-corrected aggregate log odds ratio $\Psi^*(Y)$;
3. Covariance in the distribution-corrected conditional and aggregated log odds ratios;
4. Propensity scores $\pi(x)$;
5. Regression function variances $\mu_1(x)(1-\mu_1(x))$ and $\mu_0(x)(1-\mu_0(x))$;

---

3Compare to the bound under random sampling given in Appendix 9.5
6. Contrasts between the outcome rate in the target and sampled distributions, including the difference $\rho - \omega$ and the ratio $\frac{1-\rho}{1-\omega}$;

7. Variance in the outcome in the sampled distribution, $\omega(1-\omega)$;

8. Conditional outcome rates in the sampling distribution, $\eta(x)$.

The second line of the efficiency bound shows the efficiency bound decreases with the regression function variances, which is notably different from the average risk difference efficiency bound $\sigma_{ARD}^2$ [Hahn, 1998]:

$$\sigma_{ARD}^2 = E \left[ \frac{\mu_1(X)(1-\mu_1(X))}{\pi(X)} + \frac{\mu_0(X)(1-\mu_0(X))}{1-\pi(X)} + (\mu_1(X) - \mu_0(X) - E[\mu_1(X) - \mu_0(X)])^2 \right]$$

6 Estimation

We propose a doubly robust style estimator for $\gamma$ that relies on estimation of $\log(\gamma)$. Before providing the error analysis of our proposed estimator, we briefly remark on the role of sample splitting. Estimating our nuisance functions on a separate sample, which we denote by $\hat{Q}$, that is independent of the sample denoted by $Q_n$, enables us to avoid overfitting without having to rely on empirical process conditions. With iid data, we can obtain these independent samples simply by randomly partitioning the data into two or more folds. More generally, one can use cross-fitting, a procedure which swaps the samples and averages the results, to regain sample efficiency [Chernozhukov et al., 2018, Robins et al., 2008, Zheng and van der Laan, 2010]. For simplicity we present our analysis under single sample splitting. We note that the outcome rate $\omega$ can be estimated on the full data sample.

**Theorem 6.1.** Define the estimator for $\psi_{a,y}$ as

$$\hat{\psi}_{a,y} := Q_n(\phi_{a,y}(Z; \hat{\mu}_a, \hat{\eta}, \hat{\pi}_a))$$

where

$$\phi_{a,y}(Z; \hat{\mu}_a, \hat{\eta}, \hat{\pi}_a) = \logit(\hat{\mu}_a(X)) \mathbb{I}\{Y = y\}$$

$$+ \left( \frac{\hat{\eta}(X)}{\hat{\omega}} + (1-y) \frac{1-\hat{\eta}(X)}{1-\hat{\omega}} \right) \mathbb{I}\{A = a\} (Y - \hat{\mu}_a(X))$$

Assume the identification assumptions (3.1-3.3) hold and additionally assume the following five conditions hold.

1. **Convergence in probability in $L_2(P)$ norm:** $\left\| \phi_{a,y} - \hat{\phi}_{a,y} \right\| = o_P(1)$.

2. **Sample-splitting:** Nuisance functions $\hat{\pi}_1$, $\hat{\eta}$, $\hat{\mu}_1$, and $\hat{\mu}_0$ are estimated on $\hat{Q}$.

And for some $\epsilon \in (0,1)$,

3. **Strong overlap:** $Q(\epsilon < \pi_a(X)) = 1$ and $Q(\epsilon < \hat{\pi}_a(X)) = 1$.

---

---

4One could avoid the nuisance function estimation for $\eta$ since $\eta(x) = \pi_1(x)\mu_1(x) + \pi_0(x)\mu_0(x)$. However, in some cases it may be easier to estimate $\eta(x)$ directly than it is to estimate $\mu_a(x)$ or $\pi_a(x)$.
4. **Outcome variance:** $Q(\mu_a(X)(1 - \mu_a(X)) > \epsilon) = 1$ and $Q(\tilde{\mu}_a(X)(1 - \tilde{\mu}_a(X)) > \epsilon) = 1$ for $a \in \{0, 1\}$.

5. **Outcome base rate:** $0 < \epsilon < \omega < 1 - \epsilon$ and $0 < \epsilon < \hat{\omega} < 1 - \epsilon$.

Then the proposed estimator satisfies

$$
\hat{\psi}_{a,y} - \psi_{a,y} = O_P\left(\left|\hat{\omega} - \omega\right|^2 + \left|\hat{\eta} - \eta\right| \left|\hat{\mu}_a - \mu_a\right| + \left|\hat{\pi}_a - \pi_a\right| \left|\hat{\mu}_a - \mu_a\right| + \left|\hat{\mu}_a - \mu_a\right|^2\right)
+ (Q_n - Q)\left(\phi_{a,y}(Z; Q) - \frac{\mathbb{I}\{Y = y\}\psi_{a,y}}{y\omega + (1 - y)(1 - \omega)}\right) + o_P\left(\frac{1}{\sqrt{n}}\right)
$$

Theorem 6.1 demonstrates that our proposed estimator has second-order errors in the nuisance estimation errors, yielding “doubly-fast” rates. That is, we obtain a faster rate for our estimator even when estimating the nuisance function at slower rates. For example, to obtain $n^{-1/2}$ rates for our estimator, it is sufficient to estimate the nuisance functions at $n^{-1/4}$, allowing us to use flexible machine learning methods to nonparametrically estimate the nuisance functions under smoothness or sparsity assumptions. Since our error involves squared terms, this is not the usual double-robustness property that guarantees fast rates when either of the propensity or regression function is estimated at fast rates.

**Proof.** Apply Theorem 9.1 with $f(\mu_a(x)) = \text{logit}(\mu_a(x))$. □

**Corollary 6.1.** The estimator $\hat{\psi}_{a,y}$ is $\sqrt{n}$-consistent and asymptotically normal under the assumptions in Theorem 6.1 and the following conditions:

1. $\left|\hat{\pi}_a - \pi_a\right| = O_P(n^{-1/4})$
2. $\left|\tilde{\mu}_a - \mu_a\right| = o_P(n^{-1/4})$ for $a \in \{0, 1\}$
3. $\left|\hat{\eta} - \eta\right| = O_P(n^{-1/4})$
4. $\left|\hat{\omega} - \omega\right| = o_P(n^{-1/4})$

The limiting distribution is $\sqrt{n}(\hat{\psi}_{a,y} - \psi_{a,y}) \rightsquigarrow \mathcal{N}(0, \text{var}(\text{IF}(\psi_{a,y})))$ where $\text{var}(\text{IF}(\psi_{a,y})) = \text{var}\left(\phi_{a,y}(Z) - \frac{\mathbb{I}\{Y = y\}\psi_{a,y}}{\omega y + (1 - \omega)(1 - y)}\right)$.

**6.1 Estimation of $\gamma$**

Our proposed estimator for $\gamma(\rho)$ is

$$
\hat{\gamma}(\rho) = \exp\left(\rho \left(\hat{\psi}_{1,1} - \hat{\psi}_{0,1}\right) + (1 - \rho)\left(\hat{\psi}_{1,0} - \hat{\psi}_{0,0}\right)\right)
$$

(23)

where $\hat{\psi}_{a,y}$ is defined in Eq. 22.

**Corollary 6.2.** The estimator $\hat{\gamma}(\rho)$ is $\sqrt{n}$-consistent and asymptotically normal under the assumptions in Theorem 6.1 and in Corollary 6.1 for all $(a, y) \in \{0, 1\}^2$.

The limiting distribution is $\sqrt{n}(\hat{\gamma}(\rho) - \gamma(\rho)) \rightsquigarrow \mathcal{N}(0, \sigma^2)$ where $\sigma^2$ is given in Theorem 5.2.
7 Inference

This section discusses how to do inference when estimating $\gamma(\rho)$ over a user-specified range of values $[\rho, \overline{\rho}]$ for the unknown outcome rate $\rho$. We note that $\gamma(\rho)$ is monotonic in $\rho$, so our bound on $\gamma(\rho)$ has as endpoints $\gamma(\rho)$ and $\gamma(\overline{\rho})$. First we discuss how to obtain a confidence interval on the endpoints, using $\gamma(\overline{\rho})$ as an example. Then we show how these imply a confidence interval on the bound for $\gamma(\rho)$. Guided by our theoretical results in the previous sections, our proposed approach is influence-function based. Alternatively, one could use the corrected confidence interval approach in Imbens and Manski [2004] to give a confidence interval for $\gamma(\rho)$.

Based on the efficient influence function of $\gamma(\rho)$ (Eq. 18), we create a pseudo-outcome $\zeta(Z_i; \hat{\gamma}(\overline{\rho}))$ for each observation $Z_i$ as

$$\zeta(Z_i; \hat{\gamma}(\overline{\rho})) = \hat{\gamma}(\overline{\rho}) \left( \overline{\rho}(\hat{\phi}_{1,1}(Z_i) - \hat{\phi}_{0,1}(Z_i)) + (1 - \overline{\rho})(\hat{\phi}_{1,0}(Z_i) - \hat{\phi}_{0,0}(Z_i)) \right)$$  \hspace{1cm} (24)

where $\hat{\gamma}(\overline{\rho})$ and $\hat{\psi}_{a,y}$ for $(a, y) \in \{0, 1\}^2$ are estimates using the doubly robust approach in Sec. 6 and where, for $a \in \{0, 1\}$,

$$\hat{\phi}_{a,0}(Z) = \frac{\logit(\hat{\mu}_a(X))}{1 - \hat{\omega}} (1 - Y) + \frac{1 - \hat{\eta}(X)}{1 - \hat{\omega}} \frac{\hat{\pi}_a(X) \hat{\mu}_a(X)}{\hat{\mu}_a(X)(1 - \hat{\mu}_a(X))} - \frac{\hat{\psi}_{a,y}(1 - Y)}{1 - \hat{\omega}}$$

$$\hat{\phi}_{a,1}(Z) = \frac{\logit(\hat{\mu}_a(X))}{\hat{\omega}} Y + \frac{\hat{\eta}(X)}{\hat{\omega}} \frac{\hat{\pi}_a(X) \hat{\mu}_a(X)}{\hat{\mu}_a(X)(1 - \hat{\mu}_a(X))} - \frac{\hat{\psi}_{a,y}Y}{\hat{\omega}}.$$

We can then make use of the asymptotic normality results from the previous section. If the conditions in Corollary 6.2 are met, then Corollary 6.2 and Slutsky’s theorem give that a $100(1 - \alpha)\%$ asymptotic confidence interval for $\gamma(\overline{\rho})$ is

$$\hat{\gamma}(\overline{\rho}) \pm z_{1-\alpha/2} \sqrt{\frac{\text{var}(\zeta(Z_i; \hat{\gamma}(\overline{\rho})))}{n}}$$

where $z_\beta$ is the standard normal quantile of $\beta$ and the empirical variance is over $Z$, holding as fixed the estimated nuisance functions. We can repeat this process on the same sample to obtain the confidence interval for $\gamma(\rho)$.

To obtain the confidence interval for our bound on $\rho$, we define

$$\hat{\gamma}_{\text{min}}(\rho, \overline{\rho}) := \min(\hat{\gamma}(\rho), \hat{\gamma}(\overline{\rho}))$$

$$\hat{\gamma}_{\text{max}}(\rho, \overline{\rho}) := \max(\hat{\gamma}(\rho), \hat{\gamma}(\overline{\rho})).$$

**Proposition 7.1.** Under the conditions in Corollary 6.2, the interval $[l_\alpha, u_\alpha]$ where

$$l_\alpha := \hat{\gamma}_{\text{min}}(\rho, \overline{\rho}) - z_{1-\alpha/2} \sqrt{\frac{\text{var}(\zeta(Z_i; \hat{\gamma}_{\text{min}}(\rho, \overline{\rho})))}{n}}$$

$$u_\alpha := \hat{\gamma}_{\text{max}}(\rho, \overline{\rho}) + z_{1-\alpha/2} \sqrt{\frac{\text{var}(\zeta(Z_i; \hat{\gamma}_{\text{max}}(\rho, \overline{\rho})))}{n}}$$

gives a $100(1 - \alpha)\%$ asymptotic confidence interval for the bound on $\gamma(\rho)$ for $\rho \in [\rho, \overline{\rho}]$.

**Proof.** $\hat{\gamma}_{\text{min}}$ and $\hat{\gamma}_{\text{max}}$ are asymptotically normal under Corollary 6.2. Applying Slutsky’s Theorem, their variances are $\text{var}(\zeta(Z_i; \hat{\gamma}_{\text{min}}(\rho, \overline{\rho})))$ and $\text{var}(\zeta(Z_i; \hat{\gamma}_{\text{max}}(\rho, \overline{\rho})))$, respectively. Applying Slutsky’s Theorem once more, then $P(\rho > u_\alpha) = \frac{\alpha}{2}$ and $P(\rho < l_\alpha) = \frac{\alpha}{2}$. By the union bound, $P(\rho \notin [l_\alpha, u_\alpha]) \leq \alpha$. 

\[\Box\]
8 Conclusion

The geometric mean odds ratio has the desirable property of collapsibility, unlike the more commonly used arithmetic mean. Under outcome-dependent sampling, the geometric odds ratio is not point identified, but we can estimate the geometric odds ratio as a function of the unknown outcome rate. We detail the efficiency theory for the geometric odds ratio, describe a doubly robust estimation procedure that is $\sqrt{n}$-consistent and asymptotically normal under mild conditions, and propose an inference procedure to construct confidence intervals for the geometric odds ratio over a range of possible values for the unknown outcome rate $\rho$.

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9 Appendix

The appendix provides the derivations of identification results, proofs omitted from the main paper, and additional theoretical results.

9.1 Identifications

9.1.1 Identifications under random sampling

\[
OR = \frac{E[Y^1]}{E[1-Y^1]} \frac{E[Y^0]}{E[1-Y^0]} \\
= \frac{E[E[Y^1|X]]}{E[E[1-Y^1|X]]} \frac{E[Y^0]}{E[1-Y^0]} \\
= \frac{E[E[Y^1|X,A=1]]}{E[E[1-Y^1|X,A=1]]} \frac{E[Y^0]}{E[1-Y^0]} \\
= \frac{E[Y^0]}{E[1-Y^0]} \\
\]

The first line holds by definition, the second by iterated expectation, the third from ignorability, and the fourth from consistency.

\[
\gamma = \prod \left\{ \begin{array}{c} P(Y=1 | X=x) \\ P(Y=0 | X=x) \\ P(Y^0=1 | X=x) \\ P(Y^0=0 | X=x) \end{array} \right\} dP(x) \\
= \prod \left\{ \begin{array}{c} P(Y^1=1 | A=1,X=x) \\ P(Y^0=0 | A=1,X=x) \\ P(Y^1=0 | A=0,X=x) \\ P(Y^0=1 | A=0,X=x) \end{array} \right\} dP(x) \\
= \prod \left\{ \begin{array}{c} P(Y=1 | A=1,X=x) \\ P(Y=0 | A=1,X=x) \\ P(Y=1 | A=0,X=x) \\ P(Y=0 | A=0,X=x) \end{array} \right\} dP(x) \\
\]

The first line holds by definition, the second from ignorability, and the third from consistency.

9.1.2 Partial identifications under outcome-dependent sampling

We first provide the derivation for the partial identification of \( \nu_a(x) := P(Y = 1 \mid X = x, A = a) \)
Then the law of total probability partially identifies the population odds ratio as

\[ OR = \frac{\mathbb{E}[\nu_2(x)]}{\mathbb{E}[1-\nu_2(x)]} \]

Then the law of total probability partially identifies the population odds ratio as

\[ OR(\rho) = \frac{\rho\mathbb{E}[\nu_2(X)|Y=1]+(1-\rho)\mathbb{E}[\nu_1(X)|Y=0]}{\rho\mathbb{E}[1-\nu_2(X)|Y=1]+(1-\rho)\mathbb{E}[\nu_0(X)|Y=0]} \]

### 9.2 Proofs

Some proofs will make use of the notation:

\[
\begin{align*}
\rho_y &:= P(Y = y) \quad \text{for } y \in \{0, 1\} \\
\omega_y &:= Q(Y = y) \quad \text{for } y \in \{0, 1\} \\
\eta_y(x) &:= Q(Y = y | X = x) \quad \text{for } y \in \{0, 1\}
\end{align*}
\]

The subscript will be omitted when context clearly indicates \( y = 1 \) (following the notation used in the main paper). Additionally we may overload some notation used in the main paper.

We first give a generic result for the von Mises expansion of smooth functions of the outcome regression functions that will be useful for deriving our main theoretical results.

#### 9.2.1 Second-Order Result for Functions of Regression Functions

**Lemma 9.1.** For \( a \in \{0, 1\}, y \in \{0, 1\}, \) and any twice differentiable function \( f \) of the regression function \( \mu_a(x) \), define \( \psi_{a,y}(Q) := \mathbb{E}[f(\mu_a(x)) | Y = y] = \int \frac{f(\mu_a(x))\eta_y(x)}{\omega_y} dQ \).

Then we can expand \( \psi_{a,y}(Q) \) as

\[
\begin{align*}
\psi_{a,y}(Q) &= \psi_{a,y}(Q) + \int \varphi_{a,y}(Q)d(Q - \bar{Q}) + R_2(Q, \bar{Q}) \\
R_2(Q, \bar{Q}) &= \frac{\ddot{\omega}_y - \omega_y}{\omega_y} (\psi_{a,y}(Q) - \psi_{a,y}(\bar{Q})) + \frac{1}{\omega_y} \int \left( f'(\mu_a(x))(\eta_y(x) - \bar{\eta}_y(x))(\mu_a(x) - \bar{\mu}_a(x)) \\
&+ \bar{\eta}_y(x)(f'(\bar{\mu}_a(x))(\mu_a(x) - \bar{\mu}_a(x))\frac{\bar{\pi}_{a}(x) - \bar{\pi}_a(x)}{\bar{\pi}_a(x)} + \eta(x)f''((\mu_a^*)(x))(\mu_a(x) - \bar{\mu}_a(x))^2 \right) dQ
\end{align*}
\]
where \( \mu_a^*(x) \) is between \( \bar{\mu}_a(x) \) and \( \mu_a(x) \) and where \( \varphi_{a,y} \) is

\[
\varphi_{a,y}(z; Q) = \frac{f(\mu_a(X))}{\omega_y} (\mathbb{I}(Y = y) - \eta_y(X)) + \frac{f'(\mu_a(X))\eta_y(X)}{\omega_y} \mathbb{I}\{A = a\}(Y - \mu_a(X)) + \frac{\eta_y(X)f(\mu_a(X))}{\omega_y} - \psi_{a,y}\mathbb{I}\{Y = y\}
\]

(25)

Since the remainder term \( R_2(\bar{Q}, Q) \) is a product of the nuisance function errors, we can apply Lemma 2 of Kennedy et al. [2021] to conclude that \( \psi_{a,y}(Q) \) is pathwise differentiable with efficient influence function \( \varphi_{a,y}(z; Q) \).

**Proof of Lemma 9.1**

Proof. We provide the proof for \( a = 1 \) and \( y = 1 \). Similar steps yield the result for other values of \((a, y) \in \{0, 1\}^2\).

The posited influence function of \( \psi_{1,1} \)

\[
\varphi_{1,1}(Z) = \frac{\eta(X)f(\mu_1(x))}{\omega_1} - \frac{\psi_{1,1}Y}{\omega_1} + \frac{f'(\mu_1(X))\eta(X)}{\omega_1} \frac{A(Y - \mu_1(X))}{\pi_1(X)} + \frac{f(\mu_1(X))}{\omega_1}(Y - \eta(X))
\]

gives \( \psi_{1,1}(Q) - \psi_{1,1}(\bar{Q}) - \int \varphi_{1,1}(Q)d(Q - \bar{Q}) \)

\[
= \psi_{1,1}(Q) - \psi_{1,1}(\bar{Q})
\]

\[
- \int \left( \frac{\eta(X)f(\mu_1(x))}{\omega_1} - \frac{\psi_{1,1}(Q)Y}{\omega_1} + \frac{f'(\mu_1(X))\eta(X)}{\omega_1} \frac{A(Y - \mu_1(X))}{\pi_1(X)} + \frac{f(\mu_1(X))}{\omega_1}(Y - \eta(X)) \right)dQ
\]

\[
= \frac{\bar{\omega}_1 - \omega_1}{\omega_1}(\psi_{1,1}(Q) - \psi_{1,1}(\bar{Q}))
\]

\[
- \frac{1}{\omega_1} \int \left( \frac{\eta(x)f(\mu_1(x))}{\omega_1} - \eta(x)f(\mu_1(x)) + f'(\mu_1(x))\eta(x) \frac{\pi(x)(\mu_1(x) - \bar{\mu}_1(x))}{\pi_1(x)} + f(\mu_1(x))(\eta(x) - \bar{\eta}(x)) \right)dQ
\]

\[
= \frac{\bar{\omega}_1 - \omega_1}{\omega_1}(\psi_{1,1}(Q) - \psi_{1,1}(\bar{Q}))
\]

\[
+ \frac{1}{\omega_1} \int \left( \eta(x)f'(\mu_1(x))(\mu_1(x) - \bar{\mu}_1(x)) + \eta(x)f''(\mu_1^*(x)) \frac{(\mu_1(x) - \bar{\mu}_1(x))^2}{2} - f'(\mu_1(x))\eta(x) \frac{\pi(x)(\mu_1(x) - \bar{\mu}_1(x))}{\pi_1(x)} \right)dQ
\]

\[
= \frac{\bar{\omega}_1 - \omega_1}{\omega_1}(\psi_{1,1}(Q) - \psi_{1,1}(\bar{Q}))
\]

\[
+ \frac{1}{\omega_1} \int \left( f'(\mu_1(x))(\eta(x) - \bar{\eta}(x))(\mu_1(x) - \bar{\mu}_1(x)) + \eta(x)f''(\mu_1^*(x)) \frac{(\mu_1(x) - \bar{\mu}_1(x))^2}{2} \right)dQ
\]

where the first equality makes use of the fact that \( \int \varphi_{1,1}(Q)dQ = 0 \), the second equality subtracts and adds term \( \psi_{1,1}(Q)\bar{\omega}_1/\omega_1 \), and the third equality applies iterated expectation. The
fourth equality applies a Taylor expansion for \( f(\mu_1(x)) \) around \( f(\bar{\mu}_1(x)) \) with the mean-value form remainder where \( \mu^*_1(x) \) is between \( \bar{\mu}_1(x) \) and \( \mu_1(x) \). The final equality rearranges the first and third terms of the integrand by adding and subtracting \( \bar{\eta}(x)\big(f'(\bar{\mu}_1(x))(\mu_1(x) - \bar{\mu}_1(x))\big) \).

\[ \square \]

As in the main paper, for our error analysis below we assume that our nuisance functions are estimated on a separate sample, which we denote by \( \hat{Q} \), that is independent and of the same size as the sample denoted by \( Q_n \). With iid data, we can simply random split the data into two samples. To regain sample efficiency, we can use cross-fitting, a procedure which swaps the samples and averages the results [Chernozhukov et al., 2018, Györfi et al., 2006, Robins et al., 2008, Van der Laan et al., 2003, Zheng and van der Laan, 2010]. Sample splitting (or cross-fitting) enables us to avoid overfitting without having to rely on empirical process conditions.

**Theorem 9.1.** Let \( f \) be any twice differentiable function of the regression function \( \mu_a \) with bounded second derivative. Define the estimator \( \hat{\psi}_{a,y} \) for the target \( \psi_{a,y} := \mathbb{E}[f(\mu_a(X)) \mid Y = y] \) as

\[
\hat{\psi}_{a,y} := Q_n(\phi_{a,y}(Z; \hat{\mu}_a, \hat{\eta}_y, \hat{\pi}_a)) \quad \text{where} \\
\phi_{a,y}(Z) = f(\mu_a(X)) \frac{\eta_y(X)}{\omega_y} + f'(\mu_a(X)) \frac{\eta_y(X) \mathbb{I}\{A = a\}(Y - \mu_a(X))}{\pi_a(X)} + \frac{f(\mu_a(X))}{\omega_y} (\mathbb{I}\{Y = y\} - \eta_y(X))
\]

(26)

Under the following conditions,

1. \( \|\hat{\phi}_{a,y} - \phi_{a,y}\| = o_P(1) \)
2. Sample-splitting: \( \hat{\pi}_a, \hat{\eta}_y, \) and \( \hat{\mu}_a \) are estimated on samples from \( \hat{Q} \).
3. \( Q(\pi_a(X) > \epsilon) = 1 \) and \( Q(\hat{\pi}_a(X) > \epsilon) = 1 \) for some \( \epsilon > 0 \)
4. \( \omega_y > \epsilon > 0 \) and \( \hat{\omega}_y > \epsilon > 0 \) for \( y \in \{0,1\} \) for some \( \epsilon > 0 \)
5. The target \( \psi_{a,y} \) is bounded and well-defined,

then the estimator \( \hat{\psi}_{a,y} := Q_n(\phi_{a,y}(Z; \hat{\mu}_a, \hat{\eta}_y, \hat{\pi}_a)) \) satisfies

\[
\hat{\psi}_{a,y} - \psi_{a,y} = O_P\left( \|\hat{\omega}_y - \omega_y\|^2 + \|\hat{\eta}_y - \eta_y\| \|\hat{\mu}_a - \mu_a\| + \|\hat{\pi}_a - \pi_a\| \|\hat{\mu}_a - \mu_a\| + \|\hat{\mu}_a - \mu_a\|^2 \right) \\
+ (Q - Q_n) \left( \phi_{a,y}(Z; Q) - \frac{\mathbb{I}\{Y = y\} \psi_{a,y}(Q)}{\omega_y} \right) + o_P\left( \frac{1}{\sqrt{n}} \right)
\]

**Proof of Theorem 9.1**

*Proof.*

\[
\psi_{a,y}(Q) - Q_n(\phi_{a,y}(Z; \hat{\mu}_a, \hat{\eta}_y, \hat{\pi}_a)) = \underbrace{\psi_{a,y}(Q) - Q(\phi_a(Z; Q)) + (Q - Q_n)(\phi_{a}(Z; Q) - \phi_a(Z; Q))}_{A} \\
+ (Q - Q_n)(\phi_a(Z; Q))
\]
where the first equality holds by definition (Eq. 26 and 25) and the second equality uses Lemma 9.1. We can write the first term as

\[
\hat{\omega}_y - \omega_y \psi_{a,y}(Q) = \left( \hat{\omega}_y - \omega_y \right) \psi_{a,y}(Q)
\]

where the last line applies the conditions given in Theorem 9.1.

For term A, we have that

\[
\psi_{a,y}(Q) = \psi_{a,y}(Q) + \hat{\psi}_{a,y}(Q) - \int \varphi_{a,z}(Z; \hat{Q}) dQ
\]

(27)

where the second equality uses Cauchy-Schwarz inequality.

For term B, since \( Q_n \) is the empirical measure on an independent sample from \( \hat{Q} \), we can apply Lemma 2 of Kennedy et al. [2020] with our assumption that \( \| \phi_{a,y}(Z; \hat{Q}) - \phi_{a,y}(Z; Q) \| = o_P(1) \):

\[
(Q - Q_n)(\phi_{a,y}(Z; \hat{Q}) - \phi_{a,y}(Z; Q)) = o_P\left( \frac{1}{\sqrt{n}} \right)
\]

We combine the results for term A and term B with term C to obtain:

\[
\psi_{a,y}(Q) - Q_n(\phi_{a,y}(Z; \mu_a, \hat{\eta}_y, \hat{\pi}_a)) = (Q - Q_n) \left( \phi_a(Z; Q) - \frac{\| Y = y \| \psi_{a,y}(Q)}{\omega_y} \right) + o_P \left( \frac{1}{\sqrt{n}} \right)
\]

\[
+ \| \hat{\eta}_y - \eta_y \| \| \mu_a - \mu_a \| + \| \hat{\pi}_a - \pi_a \| \| \hat{\pi}_a - \pi_a \| + \| \hat{\pi}_a - \pi_a \|^2 + o_P \left( \frac{1}{\sqrt{n}} \right)
\]

\[
\square
\]
9.3 General efficiency theory for \( \psi_{a,y} \)

We provide the generalization of Lemma 5.1 for any \((a, y) \in \{0, 1\}^2\).

**Lemma 9.2.** We have the following von Mises expansion for \( \psi_{a,y} \)

\[
\psi_{a,y}(Q) = \psi_{a,y}(Q) + \int \varphi_{a,y}(Q)d(Q - \bar{Q}) + R_2(Q, Q)
\]

\[
R_2(Q, Q) = \frac{\tilde{\omega}_y - \omega_y}{\omega_y} (\psi_{a,y}(Q) - \psi_{a,y}(\bar{Q})) + \frac{1}{\omega_y} \int \left( \frac{\mu_a(x) - \bar{\mu}_a(x)}{\bar{\mu}_a(x)(1 - \bar{\mu}_a(x))} (\eta_y(x) - \bar{\eta}_y(x)) \right.
\]

\[
+ \bar{\eta}_y(x) \frac{\mu_a(x) - \bar{\mu}_a(x)}{\bar{\mu}_a(x)(1 - \bar{\mu}_a(x))} \pi_a(x)
\]

\[
+ \eta(x) \frac{\mu^*_a(x) - 1/2}{\mu^*_a(x)^2(1 - \mu^*_a(x))^2} \left( \mu_a(x) - \bar{\mu}_a(x) \right)^2 \right) dQ.
\]

where \( \mu^*_a(x) \) lies between \( \bar{\mu}_a(x) \) and \( \mu_a(x) \) and

\[
\varphi_{a,y}(Z) = \frac{\eta_y(X) \logit(\mu_a(X))}{\omega_y} - \psi_{a,y}
\]

\[
+ \frac{\eta_y(X)}{\omega_y} \frac{1}{\pi_a(X) \mu_a(X)(1 - \mu_a(X))} + \frac{\logit(\mu_a(X))}{\omega_y} (\mathbb{I}\{Y = y\} - \eta_y(X))
\]

\[
+ \psi_{a,y} - \frac{\psi_{a,y} \mathbb{I}\{Y = y\}}{\omega_y}.
\]

Since the remainder term \( R_2(Q, Q) \) is a product of the nuisance function errors, we can apply Lemma 2 of Kennedy et al. [2021] to conclude that \( \psi_{a,y}(Q) \) is pathwise differentiable with efficient influence function \( \varphi_{a,y}(z; Q) \).

**Proof of Lemma 9.2**

Proof. Let \( f(x) = \logit(x) \). Then \( f'(x) = \frac{1}{x(1-x)} \) and \( f''(x) = \frac{2x-1}{x^2(1-x)^2} \). We apply Lemma 9.1 for \( \psi_{a,y} \) to get

\[
\psi_{a,y}(Q) = \psi_{a,y}(Q) + \int \varphi_{a,y}(Q)d(Q - \bar{Q}) + R_2(Q, Q)
\]

\[
R_2(Q, Q) = \frac{\tilde{\omega}_y - \omega_y}{\omega_y} (\psi_{a,y}(Q) - \psi_{a,y}(\bar{Q})) + \frac{1}{\omega_y} \int \left( \frac{\mu_a(x) - \bar{\mu}_a(x)}{\bar{\mu}_a(x)(1 - \bar{\mu}_a(x))} (\eta_y(x) - \bar{\eta}_y(x)) \right.
\]

\[
+ \bar{\eta}_y(x) \frac{\mu_a(x) - \bar{\mu}_a(x)}{\bar{\mu}_a(x)(1 - \bar{\mu}_a(x))} \pi_a(x)
\]

\[
+ \eta(x) \frac{\mu^*_a(x) - 1/2}{\mu^*_a(x)^2(1 - \mu^*_a(x))^2} \left( \mu_a(x) - \bar{\mu}_a(x) \right)^2 \right) dQ.
\]

where \( \mu^*_a(x) \) is between \( \bar{\mu}_a(x) \) and \( \mu_a(x) \) and \( \varphi_{a,y} \) is

\[
\varphi_{a,y}(z; Q) = \frac{\logit(\mu_a(X))}{\omega_y} \mathbb{I}\{Y = y\} + \frac{\eta_y(X)}{\omega_y} \frac{1}{\pi_a(X) \mu_a(X)(1 - \mu_a(X))} - \frac{\psi_{a,y} \mathbb{I}\{Y = y\}}{\omega_y}
\]

\( \square \)
9.4 Proof of Theorem 5.2

Proof. We first state some preliminaries that will be useful in proving the result. First, since \( A \) is a binary random variable, then for any function \( f(x, y) \),

\[
E[I\{A = 1\}I\{A = 0\}f(X, Y)] = 0
\]  

(30)

Second,

\[
E[I\{A = a\}(Y - \mu_a(X))^2 | X] = \mu_a(X)(1 - \mu_a(X))\pi_a(X)
\]  

(31)

\[
E[YI\{A = a\}(Y - \mu_a(X)) | X] = \mu_a(X)(1 - \mu_a(X))\pi_a(X)
\]  

(32)

\[
E[(1 - Y)I\{A = a\}(Y - \mu_a(X)) | X] = -\mu_a(X)(1 - \mu_a(X))\pi_a(X)
\]  

(33)

Using the influence function \( \text{IF}(\log(\gamma)) = \sum_{y=0}^{1} \rho_y(\varphi_{1,y}(Z) - \varphi_{0,y}(Z)) \), we can write the variance as

\[
\text{var}\left( \sum_{y=0}^{1} \rho_y(\varphi_{1,y}(Z) - \varphi_{0,y}(Z)) \right) =
\]

\[
\text{var}\left( \sum_{y=0}^{1} \rho_y\left( \frac{I\{Y = y\}}{\omega_y} \right) \logit(\mu_1(X)) - \frac{I\{Y = y\}}{\omega_y} \logit(\mu_0(X)) \right) - \sum_{y=0}^{1} \rho_y\left( \psi_{1,y} - \psi_{0,y} \right)
\]

\[
+ \sum_{y=0}^{1} \rho_y\left( \frac{\eta_y(X)}{\omega_y} \frac{I\{A = 1\}}{\pi_1(X)\mu_1(X)(1 - \mu_1(X))} - \frac{\eta_y(X)}{\omega_y} \frac{I\{A = 0\}}{\pi_0(X)\mu_0(X)(1 - \mu_0(X))} \right)
\]

\[
+ \sum_{y=0}^{1} \rho_y\left( \psi_{1,y} - \psi_{0,y} \right) - \sum_{y=0}^{1} \rho_y\left( \frac{\psi_{1,y}I\{Y = y\}}{\omega_y} - \frac{\psi_{0,y}I\{Y = y\}}{\omega_y} \right)
\]

\[
\text{var}\left( \frac{\Psi(X, Y) - \log(\gamma) + \log(\gamma) - \Psi^*(Y) \right)
\]

\[
+ \left( \rho_0 \frac{\eta_0(X)}{\omega_0} + \rho_1 \frac{\eta_1(X)}{\omega_1} \right) \left( \frac{I\{A = 1\}Y - \mu_1(X)}{\pi_1(X)\mu_1(X)(1 - \mu_1(X))} - \frac{I\{A = 0\}Y - \mu_0(X)}{\pi_0(X)\mu_0(X)(1 - \mu_0(X))} \right)
\]

where the first equality applies Eq. 19, and the second applies Eq. 20 and 21.

Now let’s consider the variance of term 2. Observe that term 2 has mean zero, so the variance of term 2 is:

\[
E\left[ \left( \rho_0 \frac{\eta_0(X)}{\omega_0} + \rho_1 \frac{\eta_1(X)}{\omega_1} \right)^2 \left( \frac{I\{A = 1\}Y - \mu_1(X)}{\pi_1(X)\mu_1(X)(1 - \mu_1(X))} - \frac{I\{A = 0\}Y - \mu_0(X)}{\pi_0(X)\mu_0(X)(1 - \mu_0(X))} \right) \right] =
\]

\[
E\left[ \left( \rho_0 \frac{\eta_0(X)}{\omega_0} + \rho_1 \frac{\eta_1(X)}{\omega_1} \right)^2 \left( \frac{1}{\pi_1(X)\mu_1(X)(1 - \mu_1(X))} + \frac{1}{\pi_0(X)\mu_0(X)(1 - \mu_0(X))} \right) \right]
\]

21
where we used iterated expectation, the fact that $\mathbb{I}\{A = a\}^2 = \mathbb{I}\{A = a\}$, Eq. 31, and Eq. 30. We now convert to the notation scheme used in the main paper where $p = \rho_1$ and $\eta(x) = \eta_1(x)$ to rewrite

$$
\left(\rho_0 \frac{\eta_0(X)}{\omega_0} + \rho_1 \frac{\eta_1(X)}{\omega_1}\right) = 1 - \rho + \left(\frac{\rho - \omega}{\omega(1 - \omega)}\right)\eta(X)
$$

Now let’s consider the covariance:

$$
\text{cov}\left(\Psi(X, Y) - \Psi^s(Y), \left(\rho_0 \frac{\eta_0(X)}{\omega_0} + \rho_1 \frac{\eta_1(X)}{\omega_1}\right)\right)
\begin{align*}
= & \mathbb{E}\left[\left(\Psi(X, Y) - \Psi^s(Y)\right)\left(\rho_0 \frac{\eta_0(X)}{\omega_0} + \rho_1 \frac{\eta_1(X)}{\omega_1}\right)\right] \mathbb{E}\left[\left(\Psi(X, Y) - \Psi^s(Y)\right)\left(\frac{\mathbb{I}\{A = 1\}(Y - \mu_1(X))}{\pi_1(X)\mu_1(X)(1 - \mu_1(X))} - \frac{\mathbb{I}\{A = 0\}(Y - \mu_0(X))}{\pi_0(X)\mu_0(X)(1 - \mu_0(X))}\right) | X\right]

= & 0
\end{align*}
$$

where the first equality made use of the fact that both terms are mean zero, the second applies iterated expectation, and the last line used Eqs. 32-33.

Putting this all together gives

$$
\text{var}(\text{IF}(\log(\gamma))) = \text{var}(\Psi(X, Y)) + \text{var}(\Psi^s(Y)) - 2\text{cov}(\Psi(X, Y), \Psi^s(Y))
$$

\begin{align*}
+ \mathbb{E}\left[\left(\frac{1 - \rho}{1 - \omega} + \left(\frac{\rho - \omega}{\omega(1 - \omega)}\right)\eta(X)\right) \mathbb{E}\left[\left(\frac{1}{\pi_1(X)\mu_1(X)(1 - \mu_1(X))} + \frac{1}{\pi_0(X)\mu_0(X)(1 - \mu_0(X))}\right) | X\right]\right]
\end{align*}

\[
\square
\]

### 9.5 Additional results: Random sampling

In this section we consider the simpler setting of estimating $\gamma$ under random sampling. Although causal estimands like the risk difference are more commonly used in random sampling, the odds ratio may nonetheless be of interest. Additionally, this analysis will differentiate which of the factors contributing to the difficulty of estimation (see Section 5) originate from the choice of estimand versus which result from the sampling scheme.

Under random sampling, we can point identify $\gamma$ (see Eq. 8).

**Lemma 9.3.** We have the following von Mises expansion for the target geometric mean:

$$
\gamma(P) = \gamma(\bar{P}) + \gamma(\bar{P}) \int \phi(\bar{P})d(P - \bar{P})) + R_2(\bar{P}, P)
$$

(34)

with

$$
R_2(\bar{P}, P) = \int \frac{(\bar{P}(x) - \pi(x))(\mu_1(x) - \bar{\mu}_1(x))}{\bar{\mu}_1(x)(1 - \bar{\mu}_1(x))\pi(x)} + \frac{(\bar{P}(x) - \pi(x))(\mu_0(x) - \bar{\mu}_0(x))}{\bar{\mu}_0(x)(1 - \bar{\mu}_0(x))(1 - \pi(x))}
$$

$$
+ \frac{(\mu_1^*(x) - 1/2)(\mu_1(x) - \bar{\mu}_1(x))^2}{\mu_1^*(x)^2(1 - \mu_1^*(x))^2} - \frac{(\mu_0^*(x) - 1/2)(\mu_0(x) - \bar{\mu}_0(x))^2}{\mu_0^*(x)^2(1 - \mu_0^*(x))^2}dP(x)
$$

\[
22
\]
Theorem 9.2. The efficiency bound with influence function.

Proof. For a \( \gamma^* \) such that \( \log(\gamma^*) \) lies between \( \log(\gamma(\bar{P})) \) and \( \log(\gamma(P)) \), applying Taylor’s Theorem

\[
\gamma(P) = \gamma(\bar{P}) + \gamma(\bar{P})(\log(\gamma(P)) - \log(\gamma(\bar{P}))) + \frac{1}{2}(\log(\gamma(P)) - \log(\gamma(\bar{P})))^2 \gamma^*
\]

where the second equality applies Lemma 9.4.

Then, by Lemma 2 of [Kennedy et al., 2021], our target \( \gamma(P) \) is pathwise differentiable with influence function \( \gamma(P)\phi(z; P) \).

\[ \tag{35} \]

\[ \tag{36} \]

\[ \tag{37} \]

\[ \tag{38} \]

\[ \tag{39} \]

\[ \tag{40} \]

Theorem 9.2 indicates that the difficulty of estimation problem under random sampling depends only on three factors:

1. Propensity scores \( \pi(x) \);
2. Regression function variances \( \mu_1(x)(1 - \mu_1(x)) \) and \( \mu_0(x)(1 - \mu_0(x)) \);
3. Heterogeneity in the log odds ratio \( \mathbb{E} \left[ \left( \log \left( \frac{\mu_1(x)}{\mu_0(x)} \right) - \log(\gamma) \right)^2 \right] \).

The efficiency bound decreases with the regression function variances, which is notably different from the average risk difference efficiency bound Hahn [1998].

Proof. First we will state some preliminaries that are useful in deriving the variance of the efficient influence function.

For any function \( g(X) \),

\[ \mathbb{E}[A(Y - \mu_1(X))g(X)] = 0 \]

\[ \mathbb{E}[(1 - A)(Y - \mu_0(X))g(X)] = 0 \]

This holds from iterated expectation:

\[ \mathbb{E}[A(Y - \mu_1(X))g(X)] = \mathbb{E}[\mathbb{E}[A(Y - \mu_1(X)) \mid X]g(X)] = \mathbb{E}[\mathbb{E}[(Y - \mu_1(X)) \mid X, A = 1]\pi(X)g(X)] = 0. \]

Additionally,

\[ \mathbb{E}[A(Y - \mu_1(X))^2 \mid X] = \mathbb{E}[(Y - \mu_1(X))^2 \mid X, A = 1]\pi(X) \]

\[ \mathbb{E}[(1 - A)(Y - \mu_0(X))^2 \mid X] = \mathbb{E}[(Y - \mu_0(X))^2 \mid X, A = 0](1 - \pi(X)). \]
We also restate the influence function for log(\(\gamma\)):

\[
\varphi = \log \left( \frac{\frac{\mu_1(x)}{1-\mu_1(x)}}{\frac{\mu_0(x)}{1-\mu_0(x)}} \right) + \frac{A}{\pi(X)} \frac{Y - \mu_1(x)}{1 - \mu_1(X)} - \frac{1 - A}{1 - \pi(X)} \frac{Y - \mu_0(x)}{1 - \mu_0(X)} - \log(\gamma). \tag{41}
\]

With these in hand, we have

\[
\text{var}(\varphi) = \text{var}(\varphi) = \mathbb{E}[\varphi^2] = \mathbb{E}\left[ \left( \log \left( \frac{\frac{\mu_1(x)}{1-\mu_1(x)}}{\frac{\mu_0(x)}{1-\mu_0(x)}} \right) - \log(\gamma) \right)^2 + \left( \frac{A}{\pi(X)} \frac{Y - \mu_1(x)}{1 - \mu_1(X)} - \frac{1 - A}{1 - \pi(X)} \frac{Y - \mu_0(x)}{1 - \mu_0(X)} \right)^2 \right] + 2 \mathbb{E}\left[ \left( \log \left( \frac{\frac{\mu_1(x)}{1-\mu_1(x)}}{\frac{\mu_0(x)}{1-\mu_0(x)}} \right) - \log(\gamma) \right) \left( \frac{A}{\pi(X)} \frac{Y - \mu_1(x)}{1 - \mu_1(X)} - \frac{1 - A}{1 - \pi(X)} \frac{Y - \mu_0(x)}{1 - \mu_0(X)} \right) \right]
\]

where the first line uses the fact that the centered influence function has mean-zero, the second line uses our definition of \(\varphi\) in Equation 41, and the third uses the properties 37-38. We can simplify the second term:

\[
\mathbb{E}\left[ \left( \frac{A}{\pi(X)} \frac{Y - \mu_1(x)}{1 - \mu_1(X)} - \frac{1 - A}{1 - \pi(X)} \frac{Y - \mu_0(x)}{1 - \mu_0(X)} \right)^2 \right] = \mathbb{E}\left[ \frac{A^2}{\pi(X)^2} \frac{(Y - \mu_1(x))^2}{(1 - \mu_1(X))^2 \mu_1(X)^2} - \frac{(1 - A)^2}{(1 - \pi(X))^2} \frac{(Y - \mu_0(x))^2}{(1 - \mu_0(X))^2 \mu_0(X)^2} \right] - 2 \mathbb{E}\left[ \frac{A}{\pi(X)} \frac{Y - \mu_1(x)}{1 - \mu_1(X)} - \frac{1 - A}{1 - \pi(X)} \frac{Y - \mu_0(x)}{1 - \mu_0(X)} \right]
\]

\[
= \mathbb{E}\left[ \frac{A}{\pi(X)^2} \frac{(Y - \mu_1(x))^2}{(1 - \mu_1(X))^2 \mu_1(X)^2} - \frac{1 - A}{(1 - \pi(X))^2} \frac{(Y - \mu_0(x))^2}{(1 - \mu_0(X))^2 \mu_0(X)^2} \right] + 2 \mathbb{E}\left[ \frac{1}{\pi(X)} \frac{(Y - \mu_1(x))^2}{(1 - \mu_1(X))^2 \mu_1(X)^2} - \frac{1}{(1 - \pi(X))} \frac{(Y - \mu_0(x))^2}{(1 - \mu_0(X))^2 \mu_0(X)^2} \right]
\]

The second equality uses the fact that \(A^2 = A\) which also implies that \(A(1 - A) = 0\). The third uses iterated expectation and properties 39-40. The fourth uses the fact that \(\mathbb{E}[(Y - \mu_1(X))^2 \mid X, A = 1] = (1 - \mu_1(X))\mu_1(X)\) and \(\mathbb{E}[(Y - \mu_0(X))^2 \mid X, A = 0] = (1 - \mu_0(X))\mu_0(X)\).

We propose the estimator
\( \gamma_P = \exp \left( P_n(\phi(Z; \hat{\mu}_1, \hat{\mu}_0, \hat{\pi})) \right) \) where

\[
\phi(Z; \mu_1, \mu_0, \pi) = \log \left( \frac{\frac{\mu_1(X)}{1-\mu_1(X)}}{\frac{\mu_0(X)}{1-\mu_0(X)}} \right) + \frac{A}{\pi(X)} \frac{Y - \mu_1(x)}{(1 - \mu_1(x))\mu_1(x)} - \frac{1 - A}{1 - \pi(X)} \frac{Y - \mu_0(x)}{(1 - \mu_0(x))\mu_0(x)}.
\]

**Corollary 9.1.** The estimator \( \gamma_P \) is \( \sqrt{n} \)-consistent and asymptotically normal under the identification assumptions (3.1-3.3) and the following conditions:

1. **Strong overlap:** \( P(\epsilon < \pi(X) < 1 - \epsilon) = 1 \) and \( P(\epsilon < \hat{\pi}(X) < 1 - \epsilon) = 1 \) for some \( \epsilon \in (0, 1) \).
2. **Outcome variance:** \( P(\mu_a(X)(1 - \mu_a(X)) > \epsilon) = 1 \) and \( P(\hat{\mu}_a(X)(1 - \hat{\mu}_a(X)) > \epsilon) = 1 \) for \( a \in \{0, 1\} \) and for some \( \epsilon \in (0, 1) \).
3. **Convergence in probability in \( L_2(P) \) norm:** \( \| \phi - \hat{\phi} \| = o_P(1) \). This can be achieved by using plug-in estimates of the nuisance functions.
4. **Sample-splitting:** \( \hat{\pi}, \hat{\mu}_0 \) and \( \hat{\mu}_1 \) are estimated on samples from \( \tilde{P} \).
5. \( \| \hat{\pi} - \pi \| = O_P(n^{-1/4}) \).
6. \( \| \hat{\mu}_a - \mu_a \| = o_P(n^{-1/4}) \) for \( a \in \{0, 1\} \).

The limiting distribution is \( \sqrt{n}(\hat{\gamma}_P - \gamma) \rightsquigarrow N(0, \sigma^2) \) where \( \sigma^2 \) is given in the next theorem.

**Proof.** Lemma 9.5. \( \square \)

### 9.5.1 Useful lemmas for the random sampling setting

**Lemma 9.4.** We have the following von Mises expansion of \( \log(\gamma) \):

\[
\log(\gamma(P)) = \log(\gamma(\tilde{P})) + \int \varphi(\tilde{P})d(P - \tilde{P}) + R_2(\tilde{P}, P)
\]

for

\[
R_2(\tilde{P}, P) = \int \frac{(\hat{\pi}(x) - \pi(x))(\mu_1(x) - \bar{\mu}_1(x))}{(\mu_1(x))(1 - \mu_1(x))\pi(x)} + \frac{(\hat{\pi}(x) - \pi(x))(\mu_0(x) - \bar{\mu}_0(x))}{(\mu_0(x))(1 - \mu_0(x))(1 - \pi(x))} + \frac{\mu_*^1(x) - 1/2}{\mu_*^1(x)^2(1 - \mu_*^1(x)^2)} \frac{1}{\mu_*^0(x)^2(1 - \mu_*^0(x)^2)}dP(x)
\]

where \( \mu_*^a(x) \) lies between \( \hat{\mu}_a(x) \) and \( \bar{\mu}_a(x) \) for \( a \in \{0, 1\} \) and

\[
\varphi(z; P) = \log \left( \frac{\frac{\mu_1(X)}{1 - \mu_1(X)}}{\frac{\mu_0(X)}{1 - \mu_0(X)}} \right) - \log(\gamma) + \frac{A(Y - \mu_1(X))}{\mu_1(X)(1 - \mu_1(X))\pi(X)} - \frac{(1 - A)(Y - \mu_0(X))}{\mu_0(X)(1 - \mu_0(X))(1 - \pi(X))}.
\]

**Proof of Lemma 9.4**

**Proof.** We can write the log geometric mean as

\[
\log(\gamma(P)) = \psi_1(P) - \psi_0(P) \text{ where } \psi_1(P) = \int_X \log \left( \frac{\mu_1(x)}{1 - \mu_1(x)} \right) dP(x)
\]

\[
\psi_0(P) = \int_X \log \left( \frac{\mu_0(x)}{1 - \mu_0(x)} \right) dP(x)
\]

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Then, let $f(x) = \log \left( \frac{x}{1-x} \right)$ and apply Lemma 9.1 for $\psi_a$ to get

$$
\psi_a(P) = \psi_a(\bar{P}) + \int \varphi_a(\bar{P})d(P - \bar{P}) + R_2(\bar{P}, P)
$$

$$
R_2(\bar{P}, P) = \int \left( \frac{\tilde{\pi}_a(x) - \pi_n(x)}{\hat{\mu}_a(x)(1 - \hat{\mu}_a(x))} \right) dP(x).
$$

Then we subtract the von Mises expansion for $\psi_0(P)$ from the expansion for $\psi_1(P)$ to obtain

$$
\log(\gamma(P)) = \log(\gamma(\bar{P})) + \int \varphi_1(\bar{P}) - \varphi_0(\bar{P})d(P - \bar{P}) + R_2(\bar{P}, P)
$$

$$
R_2(\bar{P}, P) = \int \left( \frac{\tilde{\pi}(x) - \pi(x)}{\hat{\mu}_1(x)(1 - \hat{\mu}_1(x))} \right) dP(x).
$$

We propose the following to estimate $\log(\gamma)$: $\hat{\psi} := P_n(\phi(Z; \hat{\mu}_1, \hat{\mu}_0, \hat{\pi}))$ where

$$
\phi(Z; \mu_1, \mu_0, \pi) = \log \left( \frac{\mu_1(x)}{1 - \mu_1(x)} \right) + \frac{A(Y - \mu_1(X))}{\mu_1(X)(1 - \mu_1(X))\pi(X)} - \frac{(1 - A)(Y - \mu_0(X))}{\mu_0(X)(1 - \mu_0(X))}(1 - \pi(X)).
$$

**Lemma 9.5.** Assume the identification assumptions (3.1-3.3) hold and additionally assume the following conditions hold:

1. **Strong overlap:** $P(\epsilon < \pi(X) < 1 - \epsilon) = 1$ and $P(\epsilon < \hat{\pi}(X) < 1 - \epsilon) = 1$ for some $\epsilon \in (0, 1)$.
2. **Outcome variance:** $P(\mu_a(X)(1 - \mu_a(X)) > \epsilon) = 1$ and $P(\hat{\mu}_a(X)(1 - \hat{\mu}_a(X)) > \epsilon) = 1$ for $a \in \{0, 1\}$ and for some $\epsilon \in (0, 1)$.
3. **Convergence in probability in $L_2(P)$ norm:** $\| \phi - \hat{\phi} \| = o_P(1)$. This can be achieved by using plug-in estimates of the nuisance functions.
4. **Sample-splitting:** $\hat{\pi}$, $\hat{\mu}_0$ and $\hat{\mu}_1$ are estimated on samples from $\hat{P}$.

For the error of the proposed estimator, we have $P_n(\phi(Z; \hat{\mu}_1, \hat{\mu}_0, \hat{\pi})) - \log(\gamma) =

$$(P_n - P)(\varphi(Z; P)) + O_P \left( \sum_{a=0}^{1} \left( \| \tilde{\pi} - \pi \| \| \mu_a - \hat{\mu}_a \| + \| \mu_a - \hat{\mu}_a \|^2 \right) \right) + o_P \left( \frac{1}{\sqrt{n}} \right)
$$

**Proof.** We define the shorthand $\psi = \log(\gamma)$ and $\hat{\psi} = P_n(\phi(Z; \hat{\mu}_1, \hat{\mu}_0, \hat{\pi}))$.

$$
\hat{\psi} - \psi = P_n(\hat{\phi}) - \psi(P)
$$

$$
= P_n(\hat{\phi}) + \psi(\bar{P}) - \psi(P)
$$

$$
= (P_n - P)(\hat{\phi} - \varphi) + P(\hat{\phi} - \varphi) + P_n(\varphi) + \psi(\bar{P}) - \psi(P)
$$

$$
= (P_n - P)(\hat{\phi} - \varphi) + (P_n - P)(\varphi) + P(\hat{\phi}) + \psi(\bar{P}) - \psi(P)
$$

$$
= (P_n - P)(\hat{\phi} - \varphi) + (P_n - P)(\varphi) + R_2(\bar{P}, P)
$$
where \( R_2(\hat{P}, P) \) is given in Lemma 9.4.
Under the conditions of this lemma we have

\[
R_2(\hat{P}, P) = O_P \left( \int (\hat{\pi}(x) - \pi(x))(\hat{\mu}_1(x) - \mu_1(x))dP(x) + (\hat{\pi}(x) - \pi(x))(\hat{\mu}_0(x) - \mu_0(x)) + (\hat{\mu}_1(x) - \mu_1(x))^2 + (\hat{\mu}_0(x) - \mu_0(x))^2dP(x) \right).
\]

Applying the Cauchy-Schwarz inequality yields

\[
R_2(\hat{P}, P) = O_P \left( \|\hat{\pi}(x) - \pi(x)\|\|\hat{\mu}_1(x) - \mu_1(x)\| + \|\hat{\pi}(x) - \pi(x)\|\|\hat{\mu}_0(x) - \mu_0(x)\| + \|\hat{\mu}_1(x) - \mu_1(x)\|^2 + \|\hat{\mu}_0(x) - \mu_0(x)\|^2 \right).
\]

Because \( P_n \) is the empirical measure on an independent sample from \( \hat{P} \), we can apply Lemma 2 of Kennedy et al. [2020] along with Condition 3 of the Lemma statement to obtain

\[
(P_n - P)(\hat{\varphi} - \varphi) = o_p\left(\frac{1}{\sqrt{n}}\right).
\]

Putting this all together gives

\[
\hat{\psi} - \psi = (P - P_n)(\varphi(Z; P)) + O_P \left( \sum_{a=0}^{1} \left( \|\hat{\pi} - \pi\|\|\mu_a - \hat{\mu}_a\| + \|\mu_a - \hat{\mu}_a\|^2 \right) \right) + o_p\left(\frac{1}{\sqrt{n}}\right).
\]

\[\Box\]
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