Σ-pure-injective complexes for gentle algebras

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Abstract

We prove, for a finite-dimensional gentle algebra, that indecomposable Σ-pure-injective objects in the homotopy category of complexes of projectives must be shifts of string or band complexes. The key step in our proof uses a splitting result for linear relations, together with an analysis of the canonical multi-sorted language.

1. Introduction.

A gentle algebra has the form Λ = kQ/J where k is a field, Q is a quiver, kQ is the path algebra, and J is the admissible ideal in kQ generated by a set ρ of length 2 paths such that:

1) if v is a vertex then |A(v→)| ≤ 2 and |A(→v)| ≤ 2;
2) if y ∈ A then |{x ∈ A(h(y)→) | xy ∈ J}| ≤ 1 and |{z ∈ A(t(y)→) | yz ∈ J}| ≤ 1;
3) if y ∈ A then |{x ∈ A(h(y)→) | xy ∉ J}| ≤ 1 and |{z ∈ A(t(y)→) | yz ∉ J}| ≤ 1.

where |X| denotes the cardinality of any set X, and for any vertex u the set A(u→) (respectively A(→u)) consists of all arrows a whose tail t(a) (respectively head h(a)) is u.

A fundamental idea in model theory is to study structures via the formulas they satisfy. By the famous quantifier elimination result of Baur [3], any such formula in the language of modules (over a fixed ring) is a Boolean combination of positive-primitive (pp) formulas. A monomorphism of modules is called pure if it remains an embedding under any tensor product functor, and a module is called pure-injective if it is injective with respect to pure monomorphisms. A module is called Σ-pure-injective if any small direct sum of copies of it is pure-injective. The direct sum over a singleton shows Σ-pure-injectives are pure-injective.

The introduction of the Ziegler spectrum (a topological space whose points are indecomposable pure-injectives) motivated the provision of examples of these modules. Various examples were given by Ringel [25], where he conjectured a classification of pure-injective indecomposables for domestic string algebras. This conjecture was verified by Prest and Puninski [23] by studying coherent functors from the category of pp-pairs, defined by evaluation. More recently, in joint work [8] with Crawley-Boevey, we adapted the focus of [10] to classify Σ-pure-injective modules over (possibly non-domestic) string algebras, using a classification method from representation theory known as the functorial filtrations method. As was done in [23], the combinatorial properties defining string algebras in [8] were translated into properties of pp-formulas in order to classify modules up to isomorphism.

The Ziegler spectrum of a compactly generated triangulated category was defined by Krause [18], and Garkusha and Prest [13] subsequently gave a relation between the Ziegler spectrum of a (right hereditary or von Neumann regular) ring and the Ziegler spectrum of its derived category. Furthermore these authors gave a correspondence between the pp-formulas (in the canonical multi-sorted language) and coherent functors into the category of abelian groups. In this article, the combinatorial properties defining gentle algebras will be translated into properties of pp-formulas in order to classify complexes of projectives up to homotopy equivalence. We use results from [8] and a summary [7] of the authors PhD thesis.

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In [7] the author considered the homotopy category $K(\Lambda\text{-Proj})$ of unbounded complexes of projective modules. Here objects were classified in the full subcategory $K(\Lambda\text{-proj})$ of complexes with finitely generated homogeneous components. Our main result in this article (Theorem 1.1) provides a similar classification, in which we broaden the class of complexes we consider to $\Sigma$-pure-injectives. It is worth noting that the class of rings considered in [7] strictly contains the class of gentle algebras, and examples of these rings include infinite-dimensional algebras and rings of mixed characteristic. The aforementioned broadening comes with the restriction to the smaller class of (finite-dimensional) gentle algebras.

**Theorem 1.1.** Let $\Lambda$ be a gentle algebra. Every $\Sigma$-pure-injective in $K(\Lambda\text{-Proj})$ is a direct sum of string complexes and band complexes indexed by a $\Sigma$-pure-injective $k[T,T^{-1}]$-module.

Note that any indecomposable $\Sigma$-pure-injective $k[T,T^{-1}]$-module is isomorphic to: an indecomposable finite-dimensional module; a Prüfer module (an injective envelope of a simple); or the function field $k(T)$. String complexes have the form $P(C)$, and are indexed by an aperiodic homotopy word $C$. Band complexes are indexed by a periodic homotopy word together with an indecomposable $k[T,T^{-1}]$-module. These words are essentially combinatorial data: see Definitions 3.2 and 3.3 for details. Theorem 3.5 characterises when two shifts of string or band complexes are isomorphic.

The article is organised as follows. In §2 we recall the canonical multi-sorted language for compactly generated triangulated categories: in §2.1 we recall the notion of pure monomorphisms and pure-injective objects in these categories; and in §2.2 we restrict our focus to the study of $\Sigma$-pure-injective objects. In §3 we recall and study string and band complexes in the context of compactly generated homotopy categories: in §3.1 we recall homotopy words and define associated complexes of projective modules over gentle algebras; in §3.2 we recall results from [6] to classify left-bounded string complexes with bounded cohomology; and in §3.3 we explain why the homotopy category we are considering is a compactly generated (triangulated) category. In §4 we study the category of linear relations; in §4.1 we recall results from [8]; and in §4.2 we consider the notion of a homotopically minimal complex, explaining how homotopy words induce linear relations on such complexes. In §5 we outline the setting and definitions required to employ the functorial filtrations method; in §5.1 we construct various functors involved in the proof; in §5.2 a covering property is verified for these functors evaluated on $\Sigma$-pure-injective objects; and this property is exploited in §5.3 to check compatibility conditions between (string and band complexes) and (linear relations given by homotopy words). In §6 we provide a proof of Theorem 1.1.

2. **The canonical multi-sorted language for compactly generated triangulated categories.**

There are various characterisations for the purity of a module in terms of pp-formulas. Similarly, purity in compactly generated triangulated categories may be discussed in terms of formulas in a multi-sorted language.

**Notation 2.1.** Suppose $\mathcal{A}$ is an additive category. Denote the hom-sets $\mathcal{A}(X,Y)$. For any set $I$ and any collection $B = \{B_i \mid i \in I\}$ of objects in $\mathcal{A}$: if the categorical product $\prod_i B_i$ exists in $\mathcal{A}$, we write $p_{j,B} : \prod_i B_i \to B_j$ for the natural morphisms equipping it; in which case the universal property gives unique morphisms $v_{j,B} : B_j \to \prod_i B_i$ such that $p_{j,B}v_{j,B}$ is the identity on $B_j$ (for each $j$). Similarly $u_{j,B} : B_j \to \bigoplus_i B_i$ will denote the morphisms equipping the coproduct $\bigoplus_i B_i$ (if it exists); in which case there are unique morphisms $q_{j,B} : \bigoplus_i B_i \to B_j$ such that $q_{j,B}u_{j,B}$ is the identity.
Now fix an object $A$ in $\mathcal{A}$ and consider the covariant functor $\mathcal{A}(A, -)$. Note that both the product and coproduct of the collection $\mathcal{A}(A, B) = \{\mathcal{A}(A, B_i) \mid i \in I\}$ exist in the category of abelian groups. We identify $\bigoplus_{i \in I} \mathcal{A}(A, B_i)$ with the subgroup of $\prod_{i \in I} \mathcal{A}(A, B_i)$ consisting of tuples $(g_i \mid i \in I)$ such that $g_i = 0$ for all but finitely many $i \in I$. This means the canonical morphism $\iota_{I, \mathcal{A}(A, B)}$ is the inclusion of sets. If $\prod_{i \in I} B_i$ exists in $\mathcal{A}$ then map $\lambda_{A,B} : \mathcal{A}(A, \prod_{i \in I} B_i) \to \prod_{i \in I} \mathcal{A}(A, B_i)$ from the universal property is given by $f \mapsto (g_i \mid i \in I)$ for each $f \in \mathcal{A}(A, \prod_{i \in I} B_i)$. Similarly if $\bigoplus_{i \in I} B_i$ exists in $\mathcal{A}$ then map $\gamma_{A,B} : \bigoplus_{i \in I} \mathcal{A}(A, B_i) \to \mathcal{A}(A, \bigoplus_{i \in I} B_i)$ from the universal property is given by $\gamma_{A,B}(g_i \mid i \in I) = \sum_i u_i g_i$. Since the functor $\mathcal{A}(A, -)$ commutes with arbitrary limits, each of the morphisms $\lambda_{A,B}$ are isomorphisms.

**Assumption 2.2.** Throughout $[2]$ fix a triangulated category $\mathcal{T}$ with suspension functor $\Sigma$. We assume that $\mathcal{T}$ is skeletal small and that $\mathcal{T}$ has arbitrary coproducts.

An object $X$ is said to be compact if, for any set $I$ and collection $Y = \{Y_i \mid i \in I\}$ of objects in $\mathcal{T}$, the morphism $\gamma_{X,Y}$ is an isomorphism. The category $\mathcal{T}$ is said to be compactly generated if there exists a set $\mathcal{G}$ of compact objects in $\mathcal{T}$, such that there are no non-zero objects $Z$ in $\mathcal{T}$ satisfying $\mathcal{T}(G, Z) = 0$ for all $n \in \mathbb{Z}$ and all $G \in \mathcal{G}$. The set $\mathcal{G}$ is said to be a generating set if $\Sigma \mathcal{G} \subseteq \mathcal{G}$ for all $G \in \mathcal{G}$.

**Assumption 2.3.** Throughout $[2]$ we assume (in addition to Assumption 1) that $\mathcal{T}$ is compactly generated by a generating set $\mathcal{G}$. Note that, as a consequence of the Brown representability theorem (see for example $[17]$ §1.3, Lemma 1.5)), under the above assumptions $\mathcal{T}$ has arbitrary products.

**Definition 2.4.** (See for example $[11]$ Definition 34)). For a non-empty set $\mathcal{S}$ an $\mathcal{S}$-sorted predicate language $\mathcal{L}$ is a tuple $(\text{pred}_\mathcal{S}, \text{func}_\mathcal{S}, \text{arg}_\mathcal{S}, \text{sort}_\mathcal{S})$ where: each $s \in \mathcal{S}$ is called a sort; $\text{pred}_\mathcal{S}$ is a non-empty set of sorted predicate symbols; $\text{func}_\mathcal{S}$ is a set of sorted function symbols (considered disjoint with $\text{pred}_\mathcal{S}$); the arity function $\text{arg}_\mathcal{S}$ maps a natural number to each sorted predicate symbol and to each sorted function symbol; and the function $\text{sort}_\mathcal{S}$ maps any $n$-ary sorted predicate (respectively function) symbol to a sequence of $n$ (respectively $n + 1$) sorts.

For each sort $s$ we introduce a countable set $V_s$ of variables of sort $s$. The terms of $\mathcal{L}$ each have their own sort, and are defined inductively by stipulating: any variable $x$ of sort $s$ will be considered a term of sort $s$; and for any $F \in \text{func}_\mathcal{S}$ with $\text{sort}_\mathcal{S}(F) = (s_1, \ldots, s_n, s)$ and any terms $t_1, \ldots, t_n$ of sort $s_1, \ldots, s_n$ (respectively) we will consider $F(t_1, \ldots, t_n)$ as a term of sort $s$. Note that constant symbols, which are given by nullary sorted function symbols, are (therefore) also terms. The atomic formulas with which $\mathcal{L}$ is equipped are built from the equality $t =_a t'$ between terms $t, t'$ of common sort $s$ together with the formulas $R(t_1, \ldots, t_n)$ where $R \in \text{pred}_\mathcal{S}$, $\text{sort}_\mathcal{S}(R) = (s_1, \ldots, s_n)$ and where each $t_i$ is a term of sort $s_i$. First-order formulas $\varphi$ in $\mathcal{L}$ are built from: the variables of each sort; the atomic formulas; Boolean connectives $\land, \lor, \implies$, and $\neg$; and the symbols $\forall$ and $\exists$.

By a $\mathcal{L}$-structure we mean a tuple $Z = \langle Z, (R(Z) \mid R \in \text{pred}_\mathcal{S}), (F(Z) \mid F \in \text{func}_\mathcal{S}) \rangle$ where: $Z$ is a family of sets $s(Z)$ for each $s \in \mathcal{S}$; $R(Z)$ is a subset of $s_1(Z) \times \cdots \times s_n(Z)$ for any $R \in \text{pred}_\mathcal{S}$ with $\text{sort}_\mathcal{S}(R) = (s_1, \ldots, s_n)$; and $F(Z)$ is a map $s_1(Z) \times \cdots \times s_n(Z) \to s(Z)$ for any $F \in \text{func}_\mathcal{S}$ with $\text{sort}_\mathcal{S}(F) = (s_1, \ldots, s_n, s)$. If $\mathcal{Z}$ and $\mathcal{Y}$ are $\mathcal{L}$-structures then by a $\mathcal{L}$-homomorphism we mean a function $g_\mathcal{L} : s(\mathcal{Z}) \to s(\mathcal{Y})$ for each sort $s$ such that: for each $F \in \text{func}_\mathcal{S}$ with $\text{sort}_\mathcal{S}(F) = (s_1, \ldots, s_n, s)$ we have $g_\mathcal{L}(F(Z)(a_1, \ldots, a_n)) = F(Y)(g_\mathcal{L}(a_1), \ldots, g_\mathcal{L}(a_n))$ for all $(a_1, \ldots, a_n) \in s_1(Z) \times \cdots \times s_n(Z)$; and for each $R \in \text{pred}_\mathcal{S}$ with $\text{sort}_\mathcal{S}(R) = (s_1, \ldots, s_n)$ and each formula $\varphi(x_1, \ldots, x_n)$ where $(x_i)$ is a variable of sort $s_i$ and each $(a_1, \ldots, a_n) \in s_1(Z) \times \cdots \times s_n(Z)$; if $Z \models \varphi(a_1, \ldots, a_n)$ then $Y \models g_\mathcal{L}(\varphi(a_1, \ldots, a_n)).$
If \( Z \) and \( Y \) are \( \mathcal{L} \)-structures then we say that \( Z \) is a substructure of \( Y \) provided: \( s(Z) \subseteq s(Y) \) for each sort \( s \); for each \( F \in \text{func}_s \) with \( \text{sort}_s(F) = (s_1, \ldots, s_n, s) \) we have \( F(Z)(a_1, \ldots, a_n) = F(Y)(a_1, \ldots, a_n) \) for all \( (a_1, \ldots, a_n) \in s_1(Z) \times \cdots \times s_n(Z) \); and for each \( R \in \text{pred}_s \) with \( \text{sort}_s(R) = (s_1, \ldots, s_n) \) we have \( R(Z) = R(Y) \cap s_1(Z) \times \cdots \times s_n(Z) \). If \( Z \) is a substructure of \( Y \), then we say that \( Z \) is an elementary substructure of \( Y \) if, for each formula \( \varphi(x_1, \ldots, x_n) \) where \( x_i \) is a variable of sort \( s_i \) and each \( (a_1, \ldots, a_n) \in s_1(Z) \times \cdots \times s_n(Z) \) we have \( (Z, \varphi(a_1, \ldots, a_n)) \in Y \).

For a set \( S \), an \( S \)-sorted predicate language \( \mathcal{L} \) and a \( \mathcal{L} \)-structure \( Z \) we write: \( |\mathcal{L}| \) for the largest of the cardinalities \( |N| \) and \( |\text{pred}_s \cup \text{func}_s| \); and \( |Z| \) for the sum of the cardinalities \( |s(Z)| \) as \( s \) runs through the sorts.

We now recall the downward Löwenheim-Skolem theorem for many-sorted structures.

**Theorem 2.5.** (See for example [11] Theorem 37). Let \( S \) be a set and let \( \mathcal{L} \) be an \( S \)-sorted predicate language. Let \( Y \) be a \( \mathcal{L} \)-structure. Fix a subset \( s(R) \subseteq s(Y) \) for each \( s \in S \). Suppose that there is a cardinal \( \kappa \) such that \( \max\{|N|, |\mathcal{L}|, |s(R)|\} \leq \kappa \leq |Y| \) for each \( s \in S \). Then there is an elementary substructure \( Z \) of \( Y \) such that \( |Z| = \kappa \) and for each \( s \in S \) we have \( s(R) \subseteq s(Z) \).

As in Assumptions 2.3 and 2.3 let \( T \) be a triangulated category which: has arbitrary coproducts; is skeletally small; is compactly generated by a generating set \( G \); and (hence) has arbitrary products. Fix a non-empty full subcategory \( S \) of \( T \).

Recall \( S \) is a triangulated subcategory if: for any object \( X \) of \( S \) and any \( n \in \mathbb{Z} \) the object \( \Sigma^n X \) lies in \( S \); and for any distinguished triangle \( X \rightarrow Y \rightarrow Z \rightarrow \Sigma X \), if two of the objects \( X \), \( Y \), or \( Z \) lies in \( S \), then so does the third. Note that the subcategory \( S \) of \( T \) is a triangulated subcategory if and only if: any object in \( T \) which is isomorphic to an object in \( S \) is an object in \( S \), and; \( S \) together with the restriction of \( \Sigma \) defines a triangulated category, where any distinguished triangle in \( S \) is a distinguished triangle in \( T \).

**Notation 2.6.** We write \( T^c \) for the triangulated subcategory of \( T \) consisting on compact objects. We write \( \text{Mod-}T^c \) for the category of additive contravariant functors \( T^c \rightarrow \text{Ab} \) where \( \text{Ab} \) is the category of abelian groups.

**Definition 2.7.** [13] §3 The canonical language \( \mathcal{L}^T \) of \( T \) is given by a \( G \)-sorted predicate language \( \langle \text{pred}_G, \text{func}_G, \arg, \text{sort}_G \rangle \) defined as follows. The set \( \text{pred}_G \) consists of a symbol \( 0_G \) with \( \text{sort}_G(0_G) = G \) for each \( G \in G \). The set \( \text{func}_G \) consists of: a binary operation \( +_G \) with \( \text{sort}_G(+_G) = (G, G, G) \) for each \( G \in G \); and a unary operation \( - \circ \alpha \) with \( \text{sort}_G(- \circ \alpha) = (H, G) \) for each \( G, H \in G \) and each map \( \alpha \in T(G, H) \). The variables of sort \( G \in G \) will be denoted \( v_G \). Let \( \text{Ax}(T) \) be the set of axioms expressing the positive atomic diagram of \( T^c \), including the specification that all functions are additive. Consequently the category \( \text{Mod}(\text{Ax}(T)) \) of models for \( \text{Ax}(T) \) is just the category \( \text{Mod-}T^c \). The objects of \( T \) are regarded as structures \( M \) for this language via the functor which takes such an object \( M \) to the functor \( T(-, M) \).

2.1. **Purity in compactly generated triangulated categories.**

In what follows we discuss the notion of purity in the context of triangulated categories. Terminology about triangulated categories may be found in the book of Neeman [22].
Definition 2.8. [18 Definition 1.1] Let \( \mathcal{T} \) be compactly generated. A morphism \( L \to M \) is pure monomorphism if the induced map \( \mathcal{T}(X, L) \to \mathcal{T}(X, M) \) is a monomorphism for each compact object \( X \), and an object \( M \) of \( \mathcal{T} \) is pure-injective provided every pure monomorphism \( M \to N \) is a section. We say that an object \( M \) of \( \mathcal{T} \) is \( \Sigma \)-pure-injective if, for any set \( I \), the coproduct \( \bigoplus_i M \) is pure-injective.

Lemma 2.9 is analogous to the equivalence of (i) and (ii) in [15 Theorem 6.4].

Lemma 2.9. Let \( L, M \) be objects in \( \mathcal{T} \). Then the natural transformations \( \mathcal{T}(-, L) \to \mathcal{T}(-, M) \) are precisely the \( \Sigma^T \)-homomorphisms \( L \to M \). Furthermore, there is a pure monomorphism \( L \to M \) if and only if \( L \) is a pure substructure of \( M \).

Proof. Recall that \( L = \mathcal{T}(-, L) \) and \( M = \mathcal{T}(-, M) \) define \( \Sigma^T \)-structures by setting \( G(L) = \mathcal{T}(G, L) \) and \( G(M) = \mathcal{T}(G, M) \) for each sort \( G \in \mathcal{G} \). Since \( L \) and \( M \) are the contravariant hom-functors, any morphism \( \gamma : L \to M \) defines a natural transformation \( \mathcal{T}(-, L) \to \mathcal{T}(-, M) \) by postcomposition with \( \gamma \). Note that any such natural transformation is, by Definition 2.7, the same thing as a \( \Sigma^T \)-homomorphism \( L \to M \), since formulas and predicate and function symbols will be preserved by construction.

By [13 Proposition 3.1] any pp-formula \( \varphi(v_G) \) is equivalent to a divisibility formula \( \exists u_H : v_G = u_H \alpha \) where \( \alpha : G \to H \) is morphism and \( G, H \in \mathcal{G} \). By definition \( L \) is a pure substructure of \( M \) if and only if (for any \( (f, g) \in G(L) \times H(L) \) such that \( \gamma g = \gamma f \alpha \) we must have \( g = f \alpha \)). This is equivalent to the condition that the morphism \( \mathcal{T}(X, L) \to \mathcal{T}(X, M) \) given by \( g \mapsto \gamma g \) is a monomorphism for each compact object \( X \), which by definition is the same as saying \( \gamma \) is a pure monomorphism. The result follows.

Definition 2.10. Let \( I \) be a set and let \( M \) be an object of \( \mathcal{T} \). By the universal properties of the product and coproduct of the collection \( M = \{ M \mid i \in I \} \) there exists a unique summation morphism \( \sigma_{I, M} : \bigoplus_i M \to M \) and a unique canonical morphism \( \iota_{I, M} : \bigoplus_i M \to \prod_i M \) satisfying \( \sigma_{I, M} u_{i,M} = 1_M \) and \( \iota_{I, M} u_{i,M} = v_{i,M} \) for each \( i \).

There are various ways to characterise both pure-injective and \( \Sigma \)-pure-injective objects in a module category, see for example [15 Theorem 7.1] and [15 Theorem 8.1] respectively. It what follows we discuss analogous statements for compactly generated triangulated categories.

Theorem 2.11. [18 Theorem 1.8, (1.5)] An object \( M \) of \( \mathcal{T} \) is pure-injective if and only if for each set \( I \) the summation morphism \( \sigma_{I, M} \) factors through the canonical morphism \( \iota_{I, M} \).

Theorem 2.11 is analogous to the equivalence of (ii) and (vi) in [15 Theorem 7.1]. Similarly Proposition 2.12 is analogous to the equivalence of (i) and (ii) in [15 Theorem 8.1].

Proposition 2.12. An object \( M \) of \( \mathcal{T} \) is \( \Sigma \)-pure-injective if and only if for each set \( I \) the canonical morphism \( \iota_{I, M} \) is a section.

Proof. Suppose that \( \iota_{I, M} \) is a section for each set \( I \). We now show \( M \) is \( \Sigma \)-pure-injective. Choose a set \( T \). Let \( N \) be the coproduct \( \bigoplus_i M \) of the collection \( M = \{ M \mid t \in T \} \).
Let $S$ be any set and consider the collection $N = \{ N \mid s \in S \}$. By Theorem 2.11 it suffices to show find a map $\theta_{S,N} : \prod_s N \to N$ such that $\sigma_{S,N} = \theta_{S,N} \circ \eta_{S,N}$. For each $(s,t) \in S \times T$ the morphisms $u_{s,N} \circ u_{t,M}$ satisfy the universal property of the coproduct $\bigoplus_{s,t} M$, and so we assume $u_{s,t,M} = u_{s,N} \circ u_{t,M}$ without loss of generality. Consider the morphisms $\varphi_{s,t,M} = q_t \circ \eta_{s,N}$ for each $(s,t) \in S \times T$. Since $u_{s,t,M} = u_{s,N} \circ u_{t,M}$ we have $q_t \circ \eta_{s,N} = q_t \circ \eta_{s,N} \circ u_{t,M}$ by uniqueness. Consequently $\varphi_{s,t,M} \circ u_{s,N} = u_{s,N} \circ \varphi_{s,t,M}$ is the identity on $M$. By the universal property of the product, there is a morphism $\omega : \prod_s N \to \prod_{s,t} M$ such that $p_{s,t,\omega} = \varphi_{s,t,M}$ for each $(s,t)$. It suffices to let $\theta_{S,N} = \sigma_{S,N} \circ \pi_{S \times T}$, $\omega$. The proof that $\sigma_{S,N} = \theta_{S,N} \circ \eta_{S,N}$ is a straightforward application of the uniqueness of the involved morphisms.

For the converse, suppose $M$ is $\Sigma$-pure-injective, and let $I$ be a set. Let $X$ be a compact object in $T$. In general: the morphism $\lambda_{X,M}$ is an isomorphism; the canonical morphism $\iota_{I,T(X,M)}$ is injective; and $\lambda_{X,M} \circ \iota_{I,T(X,M)} = \iota_{I,T(X,M)} \circ (\lambda_{X,M} \circ \iota_{I,T(X,M)})$. Since $X$ is compact the morphism $\gamma_{X,M}$ is an isomorphism, and so the morphism $T(X,\iota_{I,M})$ is injective. This means that the morphism $\iota_{I,M}$ is a pure monomorphism. Since the domain of $\iota_{I,M}$ is pure-injective by assumption, the morphism $\iota_{I,M}$ is a section. \qed

**Definition 2.13.** Recall the canonical language $\mathcal{L}^T$ from Definition 2.7. Any formula in $\mathcal{L}^T$ lying in the closure of the set of equations under conjunction and existential quantification is called a 
"pp-formula", and a "pp-definable subgroup" is the solution set of a pp-formula. For any morphism $\alpha : G \to H$ and any object $M$ in $T$ such that $G$ and $H$ are compact, let $T(\alpha,M) : T(H,M) \to T(G,M)$ be the induced map $u \mapsto u \circ \alpha$. Following Proposition 6.6 and Proposition 6.7(ii) respectively.

**Lemma 2.14.** Let $\varphi(v_G)$ be a pp-formula (in one free variable of sort $G \in \mathcal{G}$) in $\mathcal{L}^T$. (i) Let $L,M$ be objects in $T$ with $\mathcal{L}^T$-structures $L = T(-,L)$ and $M = T(-,M)$. If there is a pure monomorphism $L \to M$ then we have $\varphi(L) = T(G,L) \cap \varphi(M)$. (ii) For any set $I$ and any collection $M = \{ M_i \mid i \in I \}$ of objects in $T$ with $\mathcal{L}^T$-structures $M_i = T(-,M_i)$ we have $\bigoplus_i \varphi(M_i) \simeq \varphi(\bigoplus_i M_i)$ and $\varphi(\prod_i M_i) \simeq \prod_i \varphi(M_i)$.

Every subgroup $M\alpha$ of an object $M$ in $T$ is the set $\varphi(M)$ of solutions $v$ to the pp-formula $\varphi(v) = \exists u : v = u \alpha$. We abuse notation by writing $\varphi(M)$ for $\varphi(M)$.

**Proof of Lemma 2.14** (i) Any pp-formula is equivalent to a divisibility formula by Proposition 3.1. Hence $\varphi(M) = M\alpha$ and $\varphi(L) = L\alpha$ for some morphism $\alpha : G \to H$.

(ii) If $t : M \to N$ and $\alpha : G \to H$ are morphisms in $T$ such that $G$ and $H$ are compact, then $tv \in N\alpha$ for any $v \in M\alpha$. Hence, for any pp-formula $\varphi$ in $\mathcal{L}^T$, the assignment of objects $M \mapsto \varphi(M)$ from $T$ to the category of abelian groups defines a functor $\varphi$.

By the existence of products and coproducts in $T$ and the functoriality of $\varphi$, the universal properties give morphisms $\delta : \bigoplus_i \varphi(M_i) \to \varphi(\bigoplus_i M_i)$ and $\mu : \varphi(\prod_i M_i) \to \prod_i \varphi(M_i)$. By Lemma 4.3 the functor $\varphi$ is coherent, so by the equivalence of statements (1) and (3) from Theorem A the morphisms $\delta$ and $\mu$ are isomorphisms. Note that $\varphi(M)$ is a subgroup of $T(G,M)$, and $\delta$ and $\mu$ are the restrictions of $\gamma_{G,M}$ and $\lambda_{G,M}$ where $M = \{ M \mid t \in T \}$. \qed

By Proposition 3.2 one has quantifier elimination in $\mathcal{L}^T$. This is because triangulated categories have weak kernels and weak cokernels. Later we use that compactly generated categories have weak limits and weak colimits (see Remark 2.17).
2.2. $\Sigma$-purity in compactly generated triangulated categories.

Lemma 2.15, together with its subsequent proof below, is analogous to the equivalence of (ii) and (iii) in [15 Theorem 8.1].

**Lemma 2.15.** Let $M$ be an object $M$ in $\mathcal{T}$. Then $M$ is $\Sigma$-pure-injective if and only if, for any compact object $G$ in $\mathcal{T}$, every descending chain of (pp-definable subgroups of $M$ of sort $G$) must stabilise.

**Proof.** Fix a descending chain

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

of (pp-definable subgroups of $M$ of sort $G$). Hence there is a collection of compact objects $H_n$ in $\mathcal{T}$ such that $\alpha_n \in \mathcal{T}(G, H_n)$ for each $n \in \mathbb{N}$. For a contradiction we assume this chain does not stabilise. After relabelling, we can assume that $M_\alpha \neq M_{\alpha+1}$ for each $n \in \mathbb{N}$, and so we choose $f_n \in \mathcal{T}(H_n, M)$ such that $f_n \alpha_n \notin M_{\alpha+1}$. Consider the collection $M = \{M \mid n \in \mathbb{N}\}$.

Since $M$ is $\Sigma$-pure-injective, by Proposition 2.12 the canonical morphism $\lambda_{M,\mathbb{N}} : \bigoplus_{\mathbb{N}} M \to \prod_{\mathbb{N}} M$ is a section, and so there is some morphism $\pi_{M,\mathbb{N}} : \prod_{\mathbb{N}} M \to \bigoplus_{\mathbb{N}} M$ such that $\pi_{M,\mathbb{N}} \circ \lambda_{M,\mathbb{N}}$ is the identity on $\bigoplus_{\mathbb{N}} M$. Let $f_\alpha = (f_n \alpha_n \mid n \in \mathbb{N})$, considered as an element of $\prod_{\mathbb{N}} \mathcal{T}(G, M)$. Fix $n \in \mathbb{N}$ and let $\varphi_n(u_G)$ be the formula $\exists u_{H_n} : u_G = u_{H_n} \alpha_n$. Let $M = \{M \mid n \in \mathbb{N}\}$. Recall that the morphism $\lambda_{G,M}$ is always an isomorphism, and since $G$ is compact, $\gamma_{G,M}$ is also an isomorphism. Let

$$\omega = (\gamma_{G,M})^{-1} \mathcal{T}(G, \pi_{M,\mathbb{N}})(\lambda_{G,M})^{-1}$$

and $\omega(f_\alpha) = (w_n \mid n \in \mathbb{N})$. The contradiction we will find is that $w_\ell \neq 0$ for all $\ell \in \mathbb{N}$ (which contradicts that $\omega$ has codomain $\bigoplus_{\mathbb{N}} \mathcal{T}(G, M)$). Fix $l \in \mathbb{N}$. Let $x_n = f_n \alpha_n$ and $y_n = 0$ for all $n \in \mathbb{N}$ with $n \leq l$, and otherwise let $x_n = 0$ and $y_n = f_n \alpha_n$. This gives $f_\alpha = f_{\alpha \leq l} + f_{\alpha > l}$ where $f_{\alpha \leq l} = (x_n \mid n \in \mathbb{N})$ and $f_{\alpha > l} = (y_n \mid n \in \mathbb{N})$ and so

$$f_{\alpha \leq l} = (f_0 \alpha_0, \ldots, f_l \alpha_l, 0, 0, \ldots), \quad f_{\alpha > l} = (0, \ldots, 0, f_{l+1} \alpha_{l+1}, f_{l+2} \alpha_{l+2}, \ldots).$$

Note that $f_{\alpha \leq l} \in \bigoplus_{n \in \mathbb{N}} \mathcal{T}(G, M)$. Furthermore, since the chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ is descending, we have $f_n \alpha_n \in \varphi_1(M)$ for all $n > l$ and so $f_{\alpha > l} \in \prod_{n \in \mathbb{N}} \varphi_1(M)$. As in the proof of Lemma 2.14(ii), the restrictions of $(\lambda_{G,M})^{-1}$ and $(\gamma_{G,M})^{-1}$ respectively define isomorphisms $\prod_{n \in \mathbb{N}} \varphi_{l+1}(M) \to \varphi_{l+1}(\bigoplus_{n \in \mathbb{N}} M)$ and $\varphi_{l+1}(\bigoplus_{n \in \mathbb{N}} M) \to \prod_{n \in \mathbb{N}} \varphi_{l+1}(M)$. Similarly, $\mathcal{T}(G, \pi_{M,\mathbb{N}})$ restricts to define a morphism $\varphi_{l+1}(\bigoplus_{n \in \mathbb{N}} M) \to \varphi_{l+1}(\bigoplus_{n \in \mathbb{N}} M)$. Altogether we have that $\omega$ restricts to a morphism $\prod_{n \in \mathbb{N}} \varphi_{l+1}(M) \to \bigoplus_{n \in \mathbb{N}} \varphi_{l+1}(M)$. Let $\omega(f_{\alpha > l}) = (z_n \mid n \in \mathbb{N})$, and so $z_n \in \varphi_{l+1}(M)$ for all $n$.

Recall it suffices to show $w_\ell \neq 0$ where $\omega(f_\alpha) = (w_n \mid n \in \mathbb{N})$. From the above we have

$$(w_0, \ldots, w_l, w_{l+1}, \ldots) = \omega(f_\alpha) = \omega(f_{\alpha \leq l} + f_{\alpha > l}) = f_{\alpha \leq l} + \omega(f_{\alpha > l}) = (f_0 \alpha_0 + 2a, \ldots, f_l \alpha_l + 2l, f_{l+1} \alpha_{l+1} + 2l+1, \ldots),$$

and so $w_\ell \neq 0$ as otherwise $\varphi_{l+1}(M) \ni -z_l = f_l \alpha_l \notin \varphi_{l+1}(M)$.

We have shown that if $M$ is $\Sigma$-pure-injective then any descending chain of (pp-definable subgroups of $M$ of sort $G$) stabilises. This was done by given an analogous to that given in the proof that (ii) implies (iii) in [15 Theorem 8.1]. By providing a similar analogous proof that (iii') implies (ii') in [15 Theorem 8.1], one can show that if all of the aforementioned descending chains stabilise (for each compact object $G$), then $M$ is $\Sigma$-pure-injective.

Corollary 2.16, together with its subsequent proof below, is analogous to parts (i) and (ii) of [15 Corollary 8.2].
Corollary 2.16. Let $M$ be a $\Sigma$-pure-injective object in $T$.

(i) For any set $I$ the objects $M^{(I)}$ and $M^I$ are $\Sigma$-pure-injective.

(ii) If $\lambda : L \to M$ in $T$ is a pure monomorphism then $L$ is $\Sigma$-pure-injective and $\lambda$ is a section.

Proof. (i) By Lemma 2.14(ii) we have that $\varphi(M^{(I)}) \simeq \varphi(M^I)$ and $\varphi(M^I) \simeq \varphi(M)^I$ for any pp-formula $\varphi$. Since pp-definable subgroups are pp-definable subgroups (of a particular sort), any descending chain of such subgroups of $M^{(I)}$ (respectively $M^I$) gives rise to a descending chain of such subgroups in $M$. The result follows by Lemma 2.15.

(ii) By Lemma 2.14(i) we have that $\varphi(L) = T(G,L) \cap \varphi(M)$ for any pp-formula $\varphi$ of sort $G$. By Lemma 2.14 this means $L$ must be $\Sigma$-pure-injective. Since this means $L$ is pure-injective, and so $\lambda$ is a section.

Remark 2.17. We recall some ideas used by Beligiannis [5] and Garkusha and Prest [13]. Let $\mathcal{C}$ be a set considered as a small category with arrows denoted by $\tau_{a,b} : a \to b$ for $a,b \in \mathcal{C}$, and let $\mathcal{A}$ be an additive category. Let $H : \mathcal{C} \to \mathcal{A}$ be a functor. A weak colimit of $H$ is an object $N$ together with morphisms $n_c : H(c) \to N$ (all in $\mathcal{A}$) for each $c \in \mathcal{C}$ such that: $n_a = n_b H(\tau_{a,b})$ for any arrow $\tau_{a,b}$ in $\mathcal{C}$; and if $\{m_c : H(c) \to M \mid c \in \mathcal{G}\}$ is a set of arrows in $\mathcal{A}$ such that $m_a = m_b H(\tau_{a,b})$ for any arrow $\tau_{a,b}$ then there exists a morphism $\omega : N \to M$ (which in general is not unique) such that $\omega n(c) = m(c)$ for all $c \in \mathcal{C}$.

For example, the weak cokernel of a morphism $f : A \to B$ in $\mathcal{A}$ is a morphism $h : B \to C$ such that $hf = 0$, and for any morphism $g : B \to D$ such that $gf = 0$ we have $g = ah$ for some morphism $a : C \to D$. We let $\text{w.colim}(H)$ denote the weak colimit of $H$. Dually one can define the notions of a weak kernel of such a map $f$, and more generally the notion of a weak limit of such a functor $H$.

Let $\mathcal{A}$ be the compactly generated triangulated category $T$. We show that any functor $H : \mathcal{C} \to T$ has a weak colimit in $T$. The dual argument will show that any such $H$ has a weak limit in $T$. By [5] 2.2 it suffices to show that $T$ has all coproducts and all weak cokernels. We are assuming that $T$ has all products, and that $T$ is compactly generated. Recall that by the Brown representability theorem $T$ has arbitrary coproducts.

So it suffices to show $T$ has weak cokernels. Let $f : A \to B$ be an morphism in $T$. Consider the morphism $h : B \to C$ given by completing $f$ to a triangle $A \to B \to C \to A[1]$. Applying the covariant functor $T(A,-) : T \to \text{Ab}$ to this triangle yields a sequence of abelian groups. Since this sequence is exact, $h$ is the required weak cokernel of $f$.

The proof of Theorem 2.18 is analogous to the respective proof that (i) implies (v) in 15 Theorem 8.1]. We proceed in a similar spirit to the proof of Lemma 2.15.

Theorem 2.18. Let $M$ be a $\Sigma$-pure-injective object of $T$. Then for any set $I$ the product $M^I$ is a direct sum of indecomposable $\Sigma$-pure-injective objects of $T$.

Proof. Let $K = M^I$. Without loss of generality it suffices to assume $I$ is infinite. By Assumption 2.3 there are no non-zero objects $Z$ in $T$ satisfying $T(G,Z) = 0$ for all $n \in \mathbb{Z}$ and all $G \in \mathcal{G}$. We can assume $K \neq 0$, and so there is some $G \in \mathcal{G}$ for which $T(G,K) \neq 0$. So we choose some non-zero $f \in T(G,K)$. Let $K$ be the $\Sigma^T$-structure $T(\cdot,K)$ where $\Sigma^T$ is the canonical language for $T$. Consider the formula $\psi = \neg \varphi$ where $\varphi = \exists u_G : u_G = f$. Note that the collection of $\Sigma^T$-substructures $L$ of $K$ lies in the power set of the set of the functor $T(G,M)$ over all $G \in \mathcal{G}$. By Definition 2.7 there is an object $L$ of $T$ such that $L$ is the functor $T(\cdot,L)$. Consider the set $\Psi$ consisting of all pure substructures $L$ of $K$ such that $L \models \psi$. 

Note that $0_{G} = T(G,0) = 0$ which lies in $\Psi$. Let $\{\mathbb{N}_{c} \mid c \in \mathbb{C}\}$ be a totally ordered subset in $\Psi$ such that $\mathbb{N}_{c}$ is a substructure of $\mathbb{N}_{c'}$ whenever $c < c'$ in $\mathbb{C}$. By construction there is a pure monomorphism $\gamma_{c} : \mathbb{N}_{c} \to K$ for each $c \in \mathbb{C}$ and a morphism $t_{a,b} : \mathbb{N}_{a} \to \mathbb{N}_{b}$ whenever $a < b$ (in $\mathbb{C}$) such that $\gamma_{a} = \gamma_{b}t_{a,b}$. By Remark 2.17 there is a weak colimit $P$ of this directed system which comes equipped with morphisms $n_{c} : \mathbb{N}_{c} \to P$ such that $n_{a} = n_{b}t_{a,b}$ whenever $a < b$. By the defining property of weak colimits there is a morphism $\omega : P \to K$ such that $\omega n_{c} = \gamma_{c}$ for each $c$. Applying the functor $T(G, -)$ to these equations shows that $n_{c}$ is a pure monomorphism because $\gamma_{c}$ is a pure monomorphism. By Corollary 2.16(i) we have that $K$ is $\Sigma$-pure-injective. By Corollary 2.16(ii) this means that $n_{c}$ and $\gamma_{c}$ are sections. This shows that $\omega$ must be a pure monomorphism. So the $\Sigma^{T}$-structure $P = T(-, P)$ is an upper bound of $\{\mathbb{N}_{c} \mid c \in \mathbb{C}\}$ in $\Psi$. Thus $\Psi$ is a non-empty partially ordered set with respect to inclusion, and every chain in $\Psi$ has an upper bound in $\Psi$. By the axiom of choice $\Psi$ contains a maximal element $N$.

Since $N$ is a pure substructure of $K$, by Lemma 2.9 there is a pure monomorphism $\gamma : N \to K$. By Corollary 2.16(ii), $\gamma$ splits, and so $K \cong N \oplus U$ for some object $U$ of $T$. Note that $U \neq 0$, since otherwise $K \cong N$ which would give $K \cong N$ contradicting that $K \not\cong \varphi$ and $N \not\cong \psi$. Furthermore any split monic $U \to K$ must be a pure monomorphism, and so $U$ defines a pure substructure of $K$. Suppose that $U \cong V \oplus W$ in $T$ and so $T(G,U \cong T(G,V) \oplus T(G,W)$ in the category $\mathbb{Ab}$ of abelian groups. For a contradiction suppose $V \neq 0 \neq W$. Since $f \neq 0$ we cannot have $f \in T(G,V) \cap T(G,W)$, and so without loss of generality $f \notin T(G,V)$, which means $f \notin T(G,V \oplus N)$ (since otherwise $f$ would lie in either $T(G,V)$ or $T(G,N)$). The element $R$ in $\Psi$ given by $R = N \oplus V$ contradicts the maximality of $N$, and so $V = 0$. This shows $U$ is indecomposable. Let $\Theta$ be the set of collections $\{U_{i} \mid i \in I\}$ of pure substructures $U_{i} \cong T(-, U_{i})$ for indecomposable objects $U_{i}$ in $T$ which admit a pure monomorphism $\bigoplus U_{i} \to K$. As above any chain in $\Theta$ has an upper bound in $\Theta$, and the existence of the object $U$ means $\Theta$ is non-empty. Again the choice of a maximal element of $\Theta$ shows that $\bigoplus U_{i} \cong K$. \hfill \Box

3. String and band complexes.

Notation 3.1. Let $P$ be the set of non-trivial paths $p \notin J$ with head $h(p)$ and tail $t(p)$. For each $t > 0$ and each vertex $v$ let $P(t, v \to)$ (respectively $P(t, \to v)$) be the set of paths $p \in P$ of length $t$ with $t(p) = v$ (respectively $h(p) = v$). Let $A$ be the set of arrows in $Q$, $A(v \to) = P(1, v \to)$ and $A(\to v) = P(1, \to v)$. The composition of $a \in A(\to v)$ and $b \in A(u \to)$ is $ba$ if $u = v$, and $0$ otherwise.

3.1. Homotopy words.

We now recall the language of homotopy words developed in [7].

Definition 3.2. [7, Definition 3.1] A homotopy letter $q$ is one of $\gamma, \gamma^{-1}, d_{a}$, or $d_{a}^{-1}$ for $\gamma \in P$ and an arrow $a$. Those of the form $\gamma$ or $d_{a}$ will be called direct, and those of the form $\gamma^{-1}$ or $d_{a}^{-1}$ will be called inverse. The inverse $q^{-1}$ of a homotopy letter $q$ is defined by setting $(\gamma)^{-1} = \gamma^{-1}$, $(\gamma^{-1})^{-1} = \gamma$, $(d_{a})^{-1} = d_{a}^{-1}$ and $(d_{a}^{-1})^{-1} = d_{a}$.

Let $I$ be one of the sets $\{0, \ldots, m\}$ (for some $m \geq 0$), $\mathbb{N} = \{-n \mid n \in \mathbb{N}\}$, or $\mathbb{Z}$. For $I \neq \{0\}$ a homotopy I-word is a sequence of homotopy letters

\[
C = \begin{cases} 
  l_{1}^{-1}r_{1} \ldots l_{m}^{-1}r_{m} & \text{(if } I = \{0, \ldots, m\}) \\
  l_{1}^{-1}r_{1}l_{2}^{-1}r_{2} & \text{(if } I = \mathbb{N}) \\
  \ldots l_{1}^{-1}r_{1}l_{0}^{-1}r_{0} & \text{(if } I = \{-n \mid n \in \mathbb{N}\}) \\
  \ldots l_{1}^{-1}r_{1}l_{0}^{-1}r_{0}l_{1}^{-1}r_{1}l_{2}^{-1}r_{2} & \text{(if } I = \mathbb{Z})
\end{cases}
\]
(which will be written as $C = \ldots l_i^{-1}r_i \ldots$ to save space) such that:

(i) any homotopy letter in $C$ of the form $l_i^{-1}$ (respectively $r_i$) is inverse (respectively direct);

(ii) any sequence of 2 consecutive letters in $C$, which is of the form $l_i^{-1}r_i$, is one of $\gamma^{-1} d_{l(\gamma)}$ or $d_{l(\gamma)}^{-1} \gamma$ for some $\gamma \in P$; and

(iii) any sequence of 4 consecutive letters in $C$ of the form $l_i^{-1}r_i l_{i+1}^{-1}r_{i+1}$ is one of

(a) $\gamma^{-1} d_{l(\gamma)}^{-1} \gamma$ where $h(\gamma) = h(\lambda)$ and $l(\gamma) \neq l(\lambda)$;

(b) $d_{l(\gamma)}^{-1} \gamma^{-1} d_{l(\gamma)} \gamma$ where $t(\gamma) = h(\lambda)$ and $f(\gamma) l(\lambda) \in J$;

(c) $d_{l(\gamma)}^{-1} \gamma \lambda^{-1} d_{l(\lambda)}$ where $t(\gamma) = t(\lambda)$ and $f(\gamma) \neq f(\lambda)$;

(d) $d_{l(\gamma)}^{-1} \gamma^{-1} d_{l(\gamma)} \gamma$ where $h(\gamma) = t(\lambda)$ and $f(\lambda) l(\lambda) \in J$.

For $I = \{0\}$ there are trivial homotopy words $1_{v,1}$ and $1_{v,-1}$ for each vertex $v$.

The head and tail of any path $\gamma \in P$ are already defined and we extend this by setting $h(d_a^{-1}) = h(a)$ for any arrow $a$ and $h(q^{-1}) = t(q)$ for all homotopy letters $q$. For each $i \in I$ there is an associated vertex $v_C(i)$ defined by: $v_C(i) = t(i+1)$ for $i \leq 0$ and $v_C(i) = t(r_i)$ for $i > 0$ provided $C = \ldots l_i^{-1}r_i \ldots$ is non-trivial; and $v_{1_{v,-1}}(0) = v$ otherwise.

If $\gamma \in P$ and $a = l(\gamma)$ let $H(\gamma^{-1} d_a) = -1$ and $H(d_a^{-1} \gamma) = 1$. Let $\mu_C(0) = 0$ and

$$
\mu_C(i) = \begin{cases} 
H(l_1^{-1} r_1) + \cdots + H(l_i^{-1} r_i) & \text{(if } 0 < i \in I) \\
-H(l_0^{-1} r_0) + \cdots + H(l_{i+1}^{-1} r_{i+1}) & \text{(if } 0 > i \in I) 
\end{cases}
$$

[4] Definition 2. For $n \in \mathbb{Z}$ let $P^n(C)$ be the sum $\bigoplus AE_{v_C(i)}$ over $i \in \mu_C^{-1}(n)$. For each $i \in I$ let $b_{i,C}$ denote the coset of $v_C(i)$ in $P(C)$ (in degree $\mu_C(i)$). If the dependency on $C$ is irrelevant let $b_{i,C} = b_i$. We define the complex $P(C)$ by extending the assignment $d_{P(C)}(b_i) = b_i^+ + b_i^-$ linearly over $\Lambda$ for each $i \in I$, where

$$
b_i^+ = \begin{cases} 
\alpha b_{i+1} & \text{(if } i + 1 \in I, l_{i+1}^{-1} r_{i+1} = d_{l(\alpha)}^{-1} \alpha) \\
0 & \text{(otherwise)} 
\end{cases}
$$

and

$$
b_i^- = \begin{cases} 
\beta b_{i-1} & \text{(if } i - 1 \in I, l_i^{-1} r_i = d_{l(\beta)}^{-1} \beta) \\
0 & \text{(otherwise)} 
\end{cases}
$$

Let $|C|_i = [\gamma^{-1}]$ if $l_i^{-1} r_i = d_{l(\gamma)}^{-1} \gamma$ and $|C|_i = [\gamma]$ if $l_i^{-1} r_i = \gamma^{-1} d_{l(\gamma)}$. Then $[C] = \ldots [C]_i \ldots$ defines a generalised string or a generalised band as in Bekkert and Merklen [4 §4.1].

**Definition 3.3.** [7] Definitions 3.3 Let $C$ be a homotopy word. Write $I_C$ for the subset of $\mathbb{Z}$ where $C$ is a homotopy $I_C$-word. Let $t \in \mathbb{Z}$. If $I_C = \mathbb{Z}$ we let $C[t] = l_t^{-1} r_t | l_{t+1}^{-1} r_{t+1} \ldots$. That is, in the language of generalised strings and bands, if $I_C = \mathbb{Z}$ let $[C[t]]_i = [C]_{i+t}$. If instead $I_C \neq \mathbb{Z}$ we let $C = C[t]$.

The inverse $C^{-1}$ of $C$ is defined by $(1_{e,-})^{-1} = 1_{e,-}$ if $I = \{0\}$, and otherwise inverting the homotopy letters and reversing their order. Note the homotopy $\mathbb{Z}$-words are indexed so that

$$
(\ldots l_{i-1}^{-1} r_{i-1} l_i^{-1} r_i l_{i+1}^{-1} r_{i+1} \ldots)^{-1} = \ldots r_2^{-1} l_2^{-1} r_1^{-1} l_1 | r_0^{-1} l_0 r_1^{-1} l_1 \ldots
$$

[7] Definitions 3.5] We say $C$ is periodic if $I_C = \mathbb{Z}$, $C = C[p]$ and $\mu_C(p) = 0$ for some $p > 0$. In this case the minimal such $p$ is the period of $C$, and we say $C$ is $p$-periodic. We say $C$ is aperiodic if $C$ is not periodic.

If $C$ is periodic of period $p$ then by [7] Lemma 3.4] $P^n(C)$ is a $\Lambda$-$k[T,T^{-1}]$-bimodule where $T$ acts on the right by $b_i \mapsto b_{i-p}$. By translational symmetry the map $d_{P^n(C)} : P^n(C) \to P^{n+1}(C)$ is $\Lambda \otimes_k k[T,T^{-1}]$-linear. For a $k[T,T^{-1}]$-module $V$ we define $P(C,V)$ by $P^n(C,V) = P^n(C) \otimes_k k[T,T^{-1}]$ and $d^n_{P^n(C,V)} = d^n_{P^n(C)} \otimes 1_V$ for each $n \in \mathbb{Z}$.

[7] Definition 3.12] A string complex has the form $P(C)$ where $C$ is aperiodic. If $V$ is a $k[T,T^{-1}]$-module we call $P(C,V)$ a band complex provided $C$ is a periodic homotopy $\mathbb{Z}$-word and $V$ is an indecomposable $k[T,T^{-1}]$-module.
At this point it is worth stating in full two results from [7]. Theorem 3.4 classifies objects in the homotopy category of complexes with finitely generated homogeneous components. Theorem 3.5 characterises when two shifts of string or band complexes are isomorphic.

**Theorem 3.4.** [7 Theorem 1.1] Let $\Lambda$ be a gentle algebra.

(i) Every object in $\mathcal{K}(\Lambda\text{-proj})$ is isomorphic to a (possibly infinite) direct sum of shifts of string complexes $P(C)$ and shifts of band complexes $P(C,V)$.

(ii) Each shift of a string or band complex is an indecomposable object in $\mathcal{K}(\Lambda\text{-Proj})$.

**Theorem 3.5.** [7 Theorem 1.2] Let $\Lambda$ be a gentle algebra. Let $C$ and $E$ be homotopy words, let $V$ and $W$ be $k[T,T^{-1}]$-modules and let $n \in \mathbb{Z}$.

(i) If $C$ and $E$ are aperiodic, then $P(C)[n] \cong P(E)$ in $\mathcal{K}(\Lambda\text{-Proj})$ if and only if:

(a) we have $I_C = \{0, \ldots, m\}$ and $(I_E, E, n) = (I_C, C, 0)$ or $(I_C, C^{-1}, \mu_C(m))$; or

(b) we have $I_C = \pm \mathbb{N}$ and $(I_E, E, n) = (\pm \mathbb{N}, C, 0)$ or $(\mp \mathbb{N}, C^{-1}, 0)$; or

(c) we have $I_C = \mathbb{Z}$ and $(I_E, E, n) = (\mathbb{Z}, C^{\pm 1}[t], \mu_C(\pm t))$ for some $t \in \mathbb{Z}$.

(ii) If $C$ and $E$ are periodic, then $P(C,V)[n] \cong P(E,W)$ in $\mathcal{K}(\Lambda\text{-Proj})$ if and only if:

(a) we have $E = C[t]$, $V \cong W$ and $n = \mu_C(t)$ for some $t \in \mathbb{Z}$; or

(b) we have $E = C^{-1}[t]$, $V \cong \text{res}$, $W$ and $n = \mu_C(-t)$ for some $t \in \mathbb{Z}$.

(iii) If $C$ is aperiodic and $E$ is periodic, then $P(C)[n] \not\cong P(E, V)$ in $\mathcal{K}(\Lambda\text{-Proj})$.

**Definition 3.6.** [7 Definition 6.8] Choose a sign $s(q) \in \{\pm 1\}$ for each homotopy letter $q$ in the set $\mathbf{A}^\pm$ of homotopy letters of the form $\alpha \circ \alpha^{-1}$ such that if distinct letters $q$ and $q'$ from $\mathbf{A}^\pm$ have the same head, then they have the same sign if and only if $\{q, q'\} = \{\alpha^{-1}, \beta\}$ with $\alpha \beta \notin \mathbf{P}$. Now let $s(\gamma) = s(l(\gamma))$, $s(\gamma^{-1}) = s(l(\gamma)^{-1})$, and $s(d_{n+1}^{\alpha \beta}) = -s(\alpha)$ for each $\gamma \in \mathbf{P}$ and each arrow $\alpha$. For a (non-trivial) finite or $\mathbb{N}$-homotopy word $C$ we let $h(C)$ and $s(C)$ be the head and sign of the first letter of $C$. For any vertex $v$ let $s(1_{v, \pm 1}) = \pm 1$ and $h(1_{v, \pm 1}) = v$.

Let $D$ and $E$ be homotopy words where $I_{D^{-1}} \subseteq \mathbb{N}$ and $I_E \subseteq \mathbb{N}$. If $u = h(D^{-1})$ and $\epsilon = -s(D^{-1})$ let $D_{\epsilon, \epsilon} = D$. If $v = h(E)$ and $\delta = s(E)$ we let $1_{v, \delta}E = E$. The composition $DE$ is the concatenation of the homotopy letters in $D$ with those in $E$. The result is a homotopy word if and only if $h(D^{-1}) = h(E)$ and $s(D^{-1}) = -s(E)$ [6 Proposition 2.1.13]. If $D = \ldots \mathcal{l}_1^{-1} r_1 l_0^{-1} r_0$ is a $\mathbb{N}$-word and $E = l_1^{-1} r_1 l_2^{-1} r_2 \ldots$ is an $\mathbb{N}$-word, write $DE = \ldots l_0^{-1} r_0 | l_1^{-1} r_1 \ldots$.

**Example 3.7.** (See also [7 Example 3.2]). Consider the gentle algebra $\Lambda = kQ/\mathcal{J}$ given by $\mathcal{J} = \langle a^2, gf, hg, fh, sr, ts, rt, b^2 \rangle$ where $Q$ is the quiver

![Diagram](image-url)
3.2. Left bounded string complexes.

We now compute the kernel of the differential map for any string complex.

**Definition 3.8.** Let $C$ be a homotopy $I$-word. For each $i \in I$ we define the path $\kappa(i)$ by

(a) $\kappa(i) = e_{\nu(C)}(i)$ if $(i - 1) \notin I$ or $l_i^{-1}r_i = d_i^{-1}\gamma$ and $(i + 1) \notin I$ or $l_{i+1}^{-1}r_{i+1} = \gamma^{-1}d_y$.

(b) $\kappa(i) = f(\tau)$ if $(i - 1) \in I$ and $l_i^{-1}r_i = d_i^{-1}\gamma$ and $(i + 1) \in I$ and $l_{i+1}^{-1}r_{i+1} = \gamma^{-1}d_y$.

(c) $\kappa(i) = f(\gamma)$ if $(i - 1) \in I$ and $l_i^{-1}r_i = \tau^{-1}d_z$ and $(i + 1) \in I$ and $l_{i+1}^{-1}r_{i+1} = \gamma^{-1}d_y$.

(d) $\kappa(i) = \beta$ if $(i - 1) \notin I$ and $\beta y = 0$ and $(i + 1) \in I$ and $l_{i+1}^{-1}r_{i+1} = d_i^{-1}\gamma$.

(e) $\kappa(i) = \alpha$ if $(i - 1) \in I$ and $l_i^{-1}r_i = \tau^{-1}d_z$ and $(i + 1) \notin I$ and $\alpha z = 0$.

(f) $\kappa(i) = 0$ if $(i - 1) \in I$ and $l_i^{-1}r_i = \tau^{-1}d_z$ and $(i + 1) \in I$ and $l_{i+1}^{-1}r_{i+1} = d_i^{-1}\gamma$.

Note that for any $i \in I$ exactly one of the ((a), (b), (c), (d), (e) and (f)) is true. We say that the $i^{th}$ kernel part is: full in case (a); a left (resp. right) arm in case (b) (resp. (c)); a left (resp. right) peripheral arm in case (d) (resp. (e)); and 0 in case (f).

**Corollary 3.9.** [6 Corollary 2.7.8] Let $C$ be a homotopy $I$-word. For any $n \in \mathbb{Z}$ we have $\ker(d_{P(C)}^n) = \bigoplus_{t \in \eta^{-1}(n)} \Lambda^i b_t$.

To streamline proposition 3.13 we use the following notation.

**Definition 3.10.** For any vertex $v$ and $\delta = \pm 1$, let $W_{v, \delta}$ be the set of homotopy $I$-words with $I \subseteq \mathbb{N}$, head $v$ and sign $\delta$. Let $C$ be a finite homotopy word. Let $h(C) = u$, $h(C^{-1}) = v$, $s(C) = \delta$ and $s(C^{-1}) = \epsilon$. Note $1_{u, \delta} C = C$ and $C 1_{v, -\epsilon}$ are homotopy words. Let $W_{v}^+(C)$ be the union of $\{1_{u, \delta} C\}$ and the (potentially empty) set of homotopy $I$-words of the form

$$B_+ = \begin{cases} (\alpha(m))^{-1}d_\alpha(m) \cdots (\alpha(1))^{-1}d_\alpha(1) & \text{(if } I = \{0, \ldots, m\} \text{ for some } m > 0) \\ \cdots \alpha(3)^{-1}d_\alpha(3) \alpha(2)^{-1}d_\alpha(2) \alpha(1)^{-1}d_\alpha(1) & \text{(if } I = -\mathbb{N}) \end{cases}$$

where $(\alpha(i)) \in A$ for all $i > 0$ in $I$, and $s(\alpha(1)) = \delta$. 

\[
\nonumber
\]
Dually let \( W^-_n(C) \) be the union of \( \{1_{u,\varepsilon}\} \) and the set of homotopy I-words of the form
\[
B_- = \begin{cases} \alpha(m) \ldots \alpha(1) & (\text{if } I = \{0, \ldots, m\} \text{ for some } m > 0) \\ \ldots \alpha(3) \alpha(2) \alpha(1) & (\text{if } I = \{-\infty\}) \\ \end{cases}
\]
Similarly: let \( W^+_n(C) \) be the union of \( \{1_{v,-\varepsilon}\} \) and the set of homotopy I-words of the form
\[
D_+ = \begin{cases} \beta(1) \ldots \beta(m) & (\text{if } I = \{0, \ldots, m\} \text{ for some } m > 0) \\ \ldots \beta(2) \beta(3) & (\text{if } I = \mathbb{N}) \\ \end{cases}
\]
where \((\beta(i)) \in A\) for all \( i > 0 \) in \( I \), and \( s(\beta(1)) = \varepsilon \); and let \( W^-_n(C) \) be the union of \( \{1_{u,-\varepsilon}\} \) and the set of homotopy I-words of the form
\[
D_- = \begin{cases} (\beta(1))^{-1} \beta(2) \ldots (\beta(m))^{-1} \beta(1) & (\text{if } I = \{0, \ldots, m\} \text{ for some } m > 0) \\ (\beta(1))^{-1} \beta(2) & (\text{if } I = \mathbb{N}) \\ \end{cases}
\]
The remark above implies \( B_{\pm} \) and \( C \) are both homotopy words for each \( B_{\pm} \in W^\pm_n(C) \) and each \( C \in W^\pm_n(C) \). Note that, by \[ 8 \text{Lemma 2.1.14}, \] if \( B_{\pm}, B'_{\pm} \in W^\pm_n(C) \) and \( I_{B_{\pm}} = I_{B'_{\pm}} \), then \( B_{\pm} = B'_{\pm} \). This means there exists an unique homotopy I-word \( C_{\varepsilon} \in W^\pm_n(C) \) (respectively \( C_{\varepsilon} \in W^\pm_n(C) \) such that \( I \) is maximal with respect to inclusion. Similarly there is a unique homotopy I-word \( C_{\varepsilon} \in W^\pm_n(C) \) (respectively \( C_{\varepsilon} \in W^\pm_n(C) \)) such that \( I \) is maximal with respect to inclusion. For each \( i \in \{\varepsilon, K\} \) and each \( r \in \{\varepsilon, K\} \), let \( C(I) \) be the composition \( C_iC \) of \( C_i \) and \( C \); let \( C(r) \) be the composition \( C_iC_r \) of \( C_i \) and \( C_r \); and let \( C(1, r) \) be the composition \( C_iC_r \) of \( C_i \), \( C \), and \( C_r \).

Consider the full subcategory \( K^{-b}(\text{proj}-\Lambda) \) of \( K(\text{Proj}-\Lambda) \) (respectively \( K^{-b}(\Lambda-\text{proj}) \) of \( K(\Lambda-\text{proj}) \)) consisting of right-bounded complexes with bounded cohomology and finitely generated homogeneous components. We now describe the objects in the category \( K^{-b}(\Lambda-\text{proj}) \) by adapting the proof of \[ 8 \text{Lemma 2.7.5} \]. The proof is essentially the same, but for completeness we essentially repeat it.

**Proposition 3.11.** The following statements hold.

(i) A shift of a string complex \( P(A) \) lies in \( K^{+,b}(\Lambda-\text{proj}) \) if and only if there exists a finite homotopy word \( C \) such that \( A = C, A = C(\varepsilon), A = C(\varepsilon) \) or \( A = C(\varepsilon, \varepsilon) \).

(ii) A shift of a band complex \( P(D, V) \) lies in \( K^{+,b}(\Lambda-\text{proj}) \) if and only if the indecomposable \( k[T, T^{-1}] \)-module \( V \) is finitely-dimensional as a \( k \)-vector space.

**Proof.** (i) Let \( A \) be a homotopy I-word. Suppose firstly that there is a sequence \( \{i_n | n \in \mathbb{N}\} \in I^\mathbb{N} \) such that the \( i_n \)th kernel part is full for each \( n \). Since \( P(A) \) is bounded below \( \{\mu_A(i_n) | n \in \mathbb{N}\} \) does not have a lower bound.

This means there is a sequence \( \{i_{n(r)} | r \in \mathbb{N}\} \in I^\mathbb{N} \) such that \( \mu_A(i_{n(r)}) > \mu_A(i_{n(r+1)}) \) for all \( r \). By definition, for each \( r \) we have \( b_{i_{n(r)}} \notin \text{im}(d_{P(A)}) \), and the assumption on \( (i_n) \) gives \( b_{i_{n(r)}} \in \text{ker}(d_{P(A)}) \), which contradicts that \( P(A) \) has bounded cohomology.

Hence we have shown that there are no sequences \( \{i_n | n \in \mathbb{N}\} \in I^\mathbb{N} \) such that the \( i_n \)th kernel part is full for each \( n \). So we can choose \( l \in I \) such that \( A_{\geq l} = d_{i_{(1)}}^{i_{(2)}} \gamma_1 \ldots \gamma_2 \) for a sequence of paths \( \gamma_1 \in P \) such that \( f(\gamma_2) - f(\gamma_1) = 0 \) for each \( j \geq 1 \). Now choose \( q \in \mathbb{Z} \) such that \( H^p(P(A)) = 0 \) for all \( p < q \). Choose \( t > l \) such that \( \mu_A(i) < q \) for each \( i > t \).

If there is some \( j > t - l \) where \( \gamma_j \) has length greater than \( 1 \) then \( d_{P(A)}(b_{i+j}) = \gamma_j b_i b_{i+j+1} \) and so \( f(\gamma_j) b_i b_{i+j+1} \notin \text{im}(d_{P(A)}) \). By Corollary 3.9 we have \( f(\gamma_j) b_1 b_{i+j+1} \in \text{ker}(d_{P(A)}) \), which contradicts that \( H^n(P(A)) = 0 \) where \( n = \mu_A(l + j + 1) \).
Hence $\gamma_j$ is an arrow for each $j > t - l$. Now let $\alpha_h = \gamma_{j+k}$ for each integer $h > 0$. Since the quiver $Q$ is finite there is some $h > 0$ such that $\alpha_h = \alpha_{h+n}$ for some $n > 0$, which means $\alpha_h = \alpha_{h+n}$ for each $h > 0$. Altogether we have $A_\lambda = (a^{-1}_n d_{\alpha_n}, \ldots, a^{-1}_1 d_{\alpha_1})^{-\infty}$, as required.

(ii) This follows from the fact that any band complex $P(C, V)$ is a bounded complex whose homogeneous component in degree $n$ is a direct sum of $|\mu^{-1}_C(n)| \times \dim_k(V)$ indecomposable projective modules of the form $\Lambda e_v$.  

\[3.3. \text{Compactness in homotopy categories of gentle algebras.}\]

Let us start by noting that derived categories of modules are compactly generated.

**Example 3.12.** If $A$ is a (unital) ring, then the derived category $\mathcal{D}(A\text{-Mod})$ of complexes of $A$-modules is compactly generated. Here $\{A[n] \mid n \in \mathbb{Z}\}$ is a generating set, where $A[n]$ denotes the complex consisting of the $A$-module $A$ concentrated in degree $n$. Furthermore, the compact objects of $\mathcal{D}(A\text{-Mod})$ are the bounded complexes of finitely generated projective $A$-modules.

We now use the above to identify which string complexes, and which band complexes, define compact objects in the triangulated category $\mathcal{K}(\Lambda\text{-Proj})$. Let $(--)^* = \text{Hom}_{\Lambda\text{-Mod}}(-, \Lambda)$, the contravariant functor from the category $\Lambda\text{-Mod}$ of left $\Lambda$-modules to the category $\text{Mod}\Lambda$ of right $\Lambda$-modules. Note that for any exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ in $\text{Mod}\Lambda$ the sequence $0 \rightarrow N^* \rightarrow M^* \rightarrow L^*$ in $\text{Mod}\Lambda$ is exact. Since $\Lambda$ is right noetherian, $(-)^*$ restricts to a functor $(-)^*| : \text{mod}\Lambda \rightarrow \text{mod}\Lambda$ between full subcategories of finitely generated modules. Similarly let $^*(-)|$ be the restriction of the functor $\text{Hom}_{\text{Mod}\Lambda}(-, \Lambda) : \text{Mod}\Lambda \rightarrow \Lambda\text{-Mod}$ to $\text{mod}\Lambda \rightarrow \Lambda\text{-mod}$.

We write $K^c(\Lambda\text{-Proj})$ for the full subcategory of the triangulated category $\mathcal{K}(\Lambda\text{-Proj})$ consisting of compact objects. Proposition 3.13 is essentially due to Jorgenson [16]. We sketch the proof, to verify our setup is a specialisation of [16] Setup 3.1. We then use these details from the proof of [1 Proposition 3.6].

**Proposition 3.13.** [16 Theorem 3.2] The following statements hold.

(i) The restriction $(-)^*$ defines a triangle equivalence $K^c(\Lambda\text{-Proj}) \rightarrow K_{--}^b(\text{proj}\Lambda)$;

(ii) The triangulated category $K(\Lambda\text{-Proj})$ is compactly generated;

(iii) Any indecomposable compact object of $K(\Lambda\text{-Proj})$ is isomorphic to:

(a) a shift of a string complex of the form $P(A)$ where, for some finite homotopy word $C$, we have $A = C$, $A = C(\gamma)$, $A = C(\gamma')$, $A = C(\gamma', \gamma)$;

(b) or a shift of a band complex $P(D, V)$ where the indecomposable $k[T, T^{-1}]$-module $V$ is finite-dimensional over $k$.

**Remark 3.14.** We recall the notion of thick-ness in a triangulated category. A non-empty full triangulated subcategory $S$ of $\mathcal{T}$ is called thick if, for any object $X$ of $\mathcal{S}$, if there is an isomorphism $X \cong X' \oplus X''$ in $\mathcal{T}$ then $X'$ and $X''$ are objects of $\mathcal{S}$. Given a set $\mathcal{X}$ of objects in $\mathcal{S}$, the thick subcategory $\text{thick}_{\mathcal{T}}(\mathcal{X})$ of $\mathcal{T}$ generated by $\mathcal{S}$, is defined as follows. We define, inductively, a subcategory $X_n$ of $\mathcal{T}$ for each $n \in \mathbb{Z}$ with $n > 0$. For the case $n = 1$ write $X_1$ for the full subcategory of $\mathcal{T}$ consisting of the objects in $\mathcal{X}$ together with the zero object 0. Suppose now, for some fixed arbitrary $n \in \mathbb{Z}$ with $n > 0$, subcategories $X_1, \ldots, X_n$ of $\mathcal{T}$ have been defined. Any morphism $f : X \rightarrow Y$ in $\mathcal{T}$ defines a mapping cone, an object $Z$ completing $f$ to a distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$. Any two mapping cones of $f$ are isomorphic, and for each morphism $f$ in $X_n$ we choose a representative $c(f)$ of the isoclass of $Z$. 

Let \( X_{n+1} \) be the full subcategory of \( T \) consisting of the objects \( X \) in \( T \) such that either: \( X \) is an object in \( X_n \); or \( X = c(f) \) for some morphism \( f \) in \( X_n \). We let thick\(_T(X) \) be the full subcategory of \( T \) consisting of the objects \( X \) in \( T \) such that \( X \) lies in some \( X_n \).

**Proof of Proposition 3.13.** Since \( \Lambda \) is a finite-dimensional \( k \)-algebra, the jacobson radical \( \text{rad}(\Lambda) \) is nilpotent, and the quotient ring \( \Lambda/\text{rad}(\Lambda) \) is semisimple. In particular, by a well-known equivalence due to Bass \cite{2} Theorem P, for all integers \( n \geq 0 \), the limit of modules of projective-dimension at most \( n \) must have projective-dimension at most \( n \).

By a well-known result of Lazard \cite{10} Théorème 1.2], over any unital ring, every flat module is a limit of free modules, which are projective. Altogether we have shown that any flat \( \Lambda \)-module is projective. Furthermore, \( \Lambda \) is coherent because it is noetherian. So the criterion from \cite{16} Setup 2.1] are met. We now follow \cite{16} Construction 2.3], and then apply \cite{16} Theorems 2.4 and 3.2]. Let \( M \) be an object in \( \Lambda\text{-mod} \), let \( N = (M)^* \) and let

\[
P_M = \cdots \longrightarrow P^2_M \longrightarrow P^1_M \longrightarrow P^0_M \longrightarrow 0 \longrightarrow \cdots
\]

be a projective resolution of \( N \), where both \( N \) and \( P_M \) are considered as objects in \( K^{−\text{b}}(\text{proj}-\Lambda) \) such that the modules \( N \) and \( P^0_M \) lie in degree 0. This means there is a quasi-isomorphism \( \phi : P_M \twoheadrightarrow N \) in \( K^{−\text{b}}(\text{proj}-\Lambda) \), which corresponds to a map \( \theta : P^0_M \to N \). Applying duality defines a homomorphism of left \( \Lambda \)-modules \( *(\theta) : * (P^0_M) \to * (N) \), and defines a morphism \( * (P_M) \to * (N) \) in the category \( K(\Lambda\text{-Proj}) \) which we label \( *(\phi) \), where \( *(N) \) is a complex concentrated in degree 0. Note that both \( P_M \) and \( * (P_M) \) depend functorially on \( M \).

(i), (ii) As in \cite{16} Construction 2.3] note that there is a set of isomorphism classes of objects \( M \) in \( \Lambda\text{-mod} \). Consequently there is only a set of isomorphism classes of objects in \( K(\text{proj}-\Lambda) \) of the form \( *(\text{proj})[n] \) with \( n \in \mathbb{Z} \), and we let \( \mathcal{G} \) consist of one object from each such isomorphism class. Let \( \mathcal{H} \) be the set of objects in \( K(\text{proj}-\Lambda) \) of the form \( *(G) \) where \( G \in \mathcal{G} \). Let \( \mathcal{C} = \text{thick}\_K(\text{proj}-\Lambda)(\mathcal{G}) \) and \( \mathcal{D} = \text{thick}\_K(\text{proj}-\Lambda)(\mathcal{H}) \). By \cite{16} Theorem 2.4] the sets \( \mathcal{G} \) and \( \mathcal{H} \) are generating sets of compact objects. By \cite{16} Theorem 3.2] the restriction \( (−)^* \) defines a triangle equivalence \( \mathcal{C} \to \mathcal{D} \). As in the proof of \cite{16} Theorem 3.2] we have \( \mathcal{D} = K^{−\text{b}}(\text{proj}-\Lambda) \).

(ii) Let \( M \) be an indecomposable object in \( K^c(\text{proj}-\Lambda) \). By part (i) we have that \( (−)^* \) is an object of \( K^{−\text{b}}(\text{proj}-\Lambda) \). This means \( M \) must have been an object of \( K^{+\text{b}}(\text{proj}-\Lambda) \). The result follows from Proposition 3.11.

Lemma 3.15 below is the analogue of the statement that (i) implies (v) in \cite{15} Theorem 8.1].

**Lemma 3.15.** There is a cardinal \( \kappa \) such that every indecomposable pure-injective object of \( K(\Lambda\text{-Proj}) \) has cardinality at most \( \kappa \).

**Proof.** Let \( T \) be the triangulated category \( K(\Lambda\text{-Proj}) \). Recall the canonical language \( \Sigma^T \) defined and discussed in \cite{2} By Proposition 3.11 the category \( T \) is compactly generated by a set \( \mathcal{G} \). Let \( \kappa = \max\{|\mathcal{G}|, |\Sigma^T|, |k|, |\mathcal{G}|\} \) (where \( k \) is the ground field of \( \Lambda = kQ/\mathcal{J} \)). Let \( U \) be a fixed indecomposable pure-injective object in \( K(\Lambda\text{-Proj}) \).

By Definition 2.4 we have that \( |\mathcal{U}| \) is the sum of the cardinals \( |T(G, U)| \) as \( G \) runs through \( \mathcal{G} \). This shows \( \kappa \leq |\mathcal{U}| \), and so by Theorem 2.25] there is an elementary substructure \( Z \) of \( U \) such that \( |Z| = \kappa \). Since \( Z \) is a pure substructure of \( U \) there is a pure monomorphism \( Z \to U \) in \( T \). By Corollary 2.16] (i) we must have that the morphism \( Z \to U \) splits, and since \( U \) is indecomposable, this means \( U \simeq Z \) which has cardinality \( \kappa \).
4. Linear relations.

The proof of [8 Theorem 1.1] uses the functorial filtrations method, going back to work of Gelfand and Ponomarev [14], which was written in the language of additive relations in the sense of Mac Lane [20]. The aforementioned method depends on a certain splitting result for finite-dimensional k-linear relations, see [14 Theorem 3.1], [24 §2] and [12 §7]. Given k-vector spaces V and W a linear relation from V to W (or on V if W = V) is a k-subspace C of the direct sum V ⊕ W. This notion generalises the graph of a k-linear map V → W.

4.1. Kronecker representations and relations.

The category k-Rel of linear relations has as objects the pairs (V, C) where C is a relation on M, and has morphisms (V, C) → (W, D) given by k-linear maps f : V → W with (f(u), f(v)) ∈ D for all (u, v) ∈ C. Let Γ be the Kronecker quiver, given by two arrows p and q with common tail u and common head v, and let kΓ be the path algebra.

Let a be the well-known equivalence from the category kΓ-Mod of left kΓ-modules to the category k-Rep(Γ) of k-representations (ϕp, ϕq : Lu → Lv) of Γ. Any relation C on V defines an object (πp, πq : C → V) of k-Rep(Γ) by choosing πp (respectively πq) to be the composition of the inclusion C ⊆ V ⊕ V with the first (respectively second) projection V ⊕ V → V.

In this way there is a fully-faithful additive functor k-Rel → k-Rep(Γ) whose essential image, denoted k-Rep(Γ)rel, is the full subcategory of representations (ϕp, ϕq : Lu → Lv) such that the induced map of the k-module product ϕp × ϕq : Lu → Lv ⊕ Lv is injective. Let kΓ-Modrel be the essential image of the restriction of a to k-Rep(Γ)rel. Denote the induced equivalences

\[ \iota : k-\text{Rel} \to k-\text{Rep}(\Gamma)_\text{rel}, \eta : k-\text{Rep}(\Gamma)_\text{rel} \to k\Gamma\text{-Mod}_\text{rel}, \]

and the corresponding (quasi-inverses of these) equivalences

\[ \lambda : k-\text{Rep}(\Gamma)_\text{rel} \to k-\text{Rel}, \mu : k\Gamma\text{-Mod}_\text{rel} \to k-\text{Rep}(\Gamma)_\text{rel}. \]

The compositions η and λμ equip k-Rel with various structural properties inherited from the category kΓ-Mod. We document some of the said properties below.

**Lemma 4.1.** The category k-Rel has all limits and all coproducts.

**Proof.** Note kΓ-Modrel consists of modules X where eux → eux ⊕ eux, x ↦ (px, qx) is injective. This property is closed under taking equalisers, products and coproducts. □

**Remark 4.2.** A sequence of relations

\[ 0 \to (U, B) \to (V, C) \to (W, D) \to 0 \]

is exact provided that the underlying sequences of k-vector spaces

\[ 0 \to U \to V \to W \to 0 \text{ and } 0 \to B \to C \to D \to 0 \]

are exact. For a set I and an object (Vi, Ci) of k-Rel for each i the set of pairs ((vi), (vi′)) with (vi, vi′) ∈ Ci for each i defines a the product \( \prod (Vi, Ci) \) of the objects (Vi, Ci). Similarly the coproduct \( \bigoplus (Vi, Ci) \) is given by the relation on \( \bigoplus Vi \) consisting of pairs ((vi), (vi′)) as above, but where additionally \( vi = vi′ = 0 \) for all but finitely many i.

If there exists an object (V, C) of k-Rel with (Vi, Ci) = (V, C) for each i, then the universal property of the coproduct defines a summation map \( \sigma_I : \bigoplus (V, C) \to (V, C) \). By the equivalence of (ii) and (vi) in [15 Theorem 7.1] the object (V, C) is pure-injective if, for any set I, \( \sigma_I \) extends to a map \( \prod (V, C) \to (V, C) \). Similarly by the equivalence of (i) and (ii) in [15 Theorem 8.1] the object (V, C) is Σ-pure-injective if, for any set I, \( \sigma_I \) is a section.
DEFINITION 4.3. For an object $(V,C)$ of $k$-$\text{Rel}$ let $Cv = \{w \in W: (v,w) \in C\}$ for any $v \in V$, and for a subset $U \subseteq M$ let $CU$ be the union $\bigcup Cu$ over $u \in U$. When $C$ is the graph of a map $f$ then $CU$ is the image of $U$ under $f$. Furthermore, let

$$C'' = \{m \in M : \exists (m_n) \in M^n \text{ with } (m_n, m_{n+1}) \in C \text{ and } m = m_0\},$$

$$C' = \{m \in M : \exists (m_n) \in M^n \text{ with } (m_n, m_{n+1}) \in C, m = m_0 \text{ and } m_n = 0 \text{ for } n \gg 0\}.$$

In the sense of Ringel [24, §2], $C'$ is equal to the stable kernel $\bigcup_{n>0} C^n 0$, and $C''$ is a subspace of the stable image $\bigcap_{n>0} C^n V$. Furthermore if $\dim_k(V) < \infty$ then the inclusion of $C''$ in the stable image is an equality [10, Lemma 4.2]. Define subspaces $C^2 \subseteq C^4 \subseteq V$ by

$$C^2 = C'' \cap (C^{-1})'' \text{ and } C^4 = C'' \cap (C^{-1})' + (C^{-1})'' \cap C'.$$

By [10, Lemma 4.5] the quotient $C^2/C^4$ is a $k[T,T^{-1}]$-module with the action of $T$ given by

$$T(v + C^4) = w + C^4 \text{ if and only if } w \in C^2 \cap (C^4 + Cv).$$

We say $(V,D)$ is automorphic if both projection maps $D \to V$ are isomorphisms, and that $(V,C)$ is split provided that there is a subspace $W$ of $V$ such that $C^2 = C^4 \oplus W$ and $(W,C|_W)$ is automorphic [10, §4].

In joint work with Crawley-Boevey [8] we considered $k$-linear relations $(V,C)$ as Kronecker modules, via the first and second projections of $D$ onto $V$, in order to prove the following.

COROLLARY 4.4. [8, Corollary 1.3] Let $(V,C)$ be $\Sigma$-pure-injective object of $k$-$\text{Rel}$. Then $(V,C)$ is split and $C^4/C^2$ is a $\Sigma$-pure-injective $k[T,T^{-1}]$-module.

4.2. Homotopic minimality and induced relations.

Let $C_{\min}(\Lambda$-$\text{Proj}$) and $K_{\min}(\Lambda$-$\text{Proj}$) be the full subcategories of $\mathcal{C}(\Lambda$-$\text{Proj}$) and $\mathcal{K}(\Lambda$-$\text{Proj}$) consisting of homotopically minimal complexes: that is, whose objects $M$ are complexes in $\mathcal{C}(\Lambda$-$\text{Proj}$) such that $\text{im}(d^n_M) \subseteq \text{rad}(M^{n+1})$ for all $n \in \mathbb{Z}$.

Since $\Lambda$ is a finite-dimensional $k$-algebra, the jacobson radical $\text{rad}(\Lambda)$ is nilpotent, and the quotient ring $\Lambda/\text{rad}(\Lambda)$ is semisimple. This means $\Lambda$ is a perfect ring, and consequently every object in $\Lambda$-$\text{Mod}$ has a projective cover.

COROLLARY 4.5. [7, Corollary 4.3]. The subcategory $K_{\min}(\Lambda$-$\text{Proj}$) of $\mathcal{K}(\Lambda$-$\text{Proj}$) is dense.

REMARK 4.6. Let $t: M \to N$ be an isomorphism in the category $K_{\min}(\Lambda$-$\text{Proj}$) with inverse $s: N \to M$. Write $\tau: M \to N$ and $\sigma: N \to M$ for the corresponding morphisms in the category $C_{\min}(\Lambda$-$\text{Proj}$). Consider the induced morphisms $\tilde{\tau}^n: M^n/\text{rad}(M^n) \to N^n/\text{rad}(N^n)$ and $\tilde{\sigma}^n: N^n/\text{rad}(N^n) \to M^n/\text{rad}(M^n)$ of $\Lambda$-modules (for each $n \in \mathbb{Z}$). By construction the morphisms $\sigma \tau - \text{id}_M$ and $\tau \sigma - \text{id}_N$ in $C_{\min}(\Lambda$-$\text{Proj}$) are null-homotopic.

Since $M$ and $N$ are homotopically minimal this means $\tilde{\tau}^n$ is an isomorphism with inverse $(\tilde{\tau}^n)^{-1} = \tilde{\sigma}^n$. Since $\Lambda$ is a perfect ring it must be a smeiperfect ring. By [7, Remark 3.11] this means that each of the morphisms $\tau^n$ is an isomorphism, and so $\tau$ is an isomorphism.

We have shown that the restriction $C_{\min}(\Lambda$-$\text{Proj}) \to K_{\min}(\Lambda$-$\text{Proj}$) of the quotient functor $\mathcal{C}(\Lambda$-$\text{Proj}) \to \mathcal{K}(\Lambda$-$\text{Proj})$ reflects isomorphisms.

ASSUMPTION 4.7. For the remainder of [12] we let $M$ be an object of $K_{\min}(\Lambda$-$\text{Proj}$).
**Definition 4.8.** Fix an arbitrary vertex $v$. Let $d_{v,M}$ be the $k$-linear endomorphism of $e_vM$ defined by the restriction of $d_M$. By [4] Lemma 5 we have $e_v\text{rad}(M) = \bigoplus bM$ where $b$ runs through $A(\to v)$. For any arrow $a \in A(\to v)$ let $\pi_a : \bigoplus bM \to aM$ (respectively $\iota_a : aM \to \bigoplus bM$) be the canonical retraction (respectively section) of $k$-vector spaces.

By [7] Lemma 6.3(i) there is a $k$-linear endomorphism $d_{a,M}$ of $e_{h(a)}M$ defined by $d_{a,M}(m) = \iota_a(\pi_a(d_{v,M}(m)))$. Furthermore, $d_{c,M} = \sum d_{c,M}$ where the sum runs over $c \in A(\to v)$.

**Remark 4.9.** It is worth noting some properties of the maps $d_{a,M}$ defined above, which together make up [7] Lemma 6.3(ii). For any $\tau \in P$ and any $x \in e_{\tau(M)}$:

(i) if there exists $\sigma \in A$ such that $\tau\sigma \in P$ then $d_{\tau(\tau),M}(\tau x) = \tau d_{\sigma,M}(x)$
(ii) if $\tau \sigma \not\in P$ for all $\sigma \in A$ then $d_{\tau(\tau),M}(\tau x) = 0$;
(iii) if $h(\theta) = h(\tau)$ for some arrow $\theta \not\equiv \iota(M)$ then $d_{\theta,M}(\tau x) = 0$;
(iv) if $h(\phi) = h(\tau)$ for some arrow $\phi$ then $d_{\phi,M}d_{\tau(\tau),M} = 0$; and
(v) if $\tau x \in \text{im}(d_{\tau(\tau),M})$ then $d_{\tau,M}(x) = 0$ for any $\zeta \in A$ where $\tau\zeta \in P$.

**Definition 4.10.** Let $q$ be a homotopy letter (that is, one of $\gamma$, $\gamma^{-1}$, $d_{\alpha}$ or $d_{\alpha}^{-1}$ for some path $\gamma \in P$ or some arrow $\alpha$). For any subset $U$ of $e_{t(q)}M$ define the subset $qU$ of $e_{h(q)}(M)$ by

$$
\gamma U = \{ym \in e_{h(\gamma)}M | m \in U\}, \\
\gamma^{-1}U = \{m \in e_{h(\gamma)}M | \gamma m \in U\}, \\
d_{\alpha}U = \{d_{\alpha,M}(m) \in e_{h(\alpha)}M | m \in U\}, \\
d_{\alpha}^{-1}U = \{m \in e_{h(\alpha)}M | d_{\alpha,M}(m) \in U\}.
$$

For any vertex $v$ and any subset $U$ of $e_{v,M}$ let $1_{v,\pm 1}U = U$. When $U = e_{t(q)}M$ we let $qM = qU$. When $U = e_{t(q)}\text{rad}(M)$ we let $\text{grad}(M) = qU$. When $U = \{u\}$ we let $qu = qU$.

**Remark 4.11.** By [7] Corollary 6.6 if $a$ is an arrow then $a^{-1}d_{\alpha}d_{\beta}\text{rad}(M) \subseteq e_{t(a)}\text{rad}(M)$, and furthermore given an arrow $b$ with $ab \in P$ we have $(ab)^{-1}ad_{b} = b^{-1}d_{b}$. By [7] Corollary 6.7 if $\alpha, \beta,\gamma,\sigma,\alpha\beta \in P$, $h(\gamma) = h(\sigma)$ and $l(\gamma) \not\equiv l(\sigma)$ then we have the following inclusions

$$
\beta^{-1}d_{l(\beta)}M \subseteq (\alpha\beta)^{-1}d_{l(\alpha)}M, \\
d_{l(\alpha)}^{-1}\alpha\beta M \subseteq d_{l(\alpha)}^{-1}\alpha M, \\
\alpha^{-1}d_{l(\alpha)}M \subseteq d_{l(\beta)}^{-1}\beta 0, \\
\gamma M \subseteq d_{l(\sigma)}^{-1}\sigma 0, \\
d_{l(\alpha)}M \subseteq d_{l(\sigma)}^{-1}\sigma 0.
$$

Let $a, b \in A$ and let $C, Ca^{-1}d_{a}$ and $Cd_{a}^{-1}b$ be homotopy words. By [7] Corollary 6.9 we have that: if $\gamma' \in P$ is longer than $\gamma \in P$ and $f(\gamma') = f(\gamma) = a$ then $C^{-1}d_{l(\gamma)}M \subseteq C\gamma'^{-1}d_{l(\gamma')}M$; and if $\tau' \in P$ is longer than $\tau \in P$ and $l(\tau') = l(\tau)$ then $Cd_{l(\tau')^{-1}}M \subseteq Cd_{l(\tau)^{-1}}M$.

We now define functors $C^\pm : C_{\min}(A-Proj) \to k\text{-Mod}$ (see [7] Corollary 6.13]). The inclusions above are used to determine compatibility properties between these functors.

**Definition 4.12.** [7] Definition 6.12 Let $C \in W_{v,\delta}$. Suppose $IC$ is finite. If $a$ is an arrow and $Cd_{a}^{-1}a$ is a homotopy word let $C^+(M) = C^+(M)$ be the intersection $\bigcap_{\beta} Cd_{a}^{-1}\beta \text{rad}(M)$ over $\beta \in P$ with $l(\beta) = a$. By [6] Lemma 2.1.19, if there are finitely many such $\beta$ then $C^+(M) = Cd_{a}^{-1}0$, and otherwise $C^+(M) = \bigcap_{\beta} Cd_{a}^{-1}\beta M$. If there is no such arrow $a$ we let $C^+(M) = CM$.

If there exists an arrow $b$ where $Cb^{-1}d_{b}$ is a homotopy word let $C^{-}(M) = C^{-}(M)$ be the union $\bigcup_{\alpha} Ca^{-1}d_{b}M$ over all $\alpha \in P$ with $l(\alpha) = b$. Otherwise let $C^{-}(M) = C(\sum d_{(+)}M + \sum c(-)M)$ where $c(\pm)$ runs through all arrows with head $h(C^{-1})$ and sign $\pm s(C^{-1})$.

Suppose instead $IC = \mathbb{N}$. In this case let $C^+(M)$ be the set of all $m \in e_vM$ with a sequence of elements $(m_i) \in \prod_{i \in \mathbb{N}} e_{v(c)}M$ satisfying $m_0 = m$ and $m_i = l_{i+1}^-m_{i+1}m_{i+1}$ for each $i \geq 0$, and let $C^{-}(M)$ be the subset of $C^+(M)$ where each sequence $(m_i)$ is eventually zero.
Remark 4.13. Let $C \in W_{\alpha, \delta}$. By [7 Corollary 6.13] we have that:
(i) the assignments $M \mapsto C^+(M)$, $M \mapsto C^-(M)$ and $(M \mapsto CM$ for when $C$ is finite) respectively define subfunctors $C^+$, $C^-$ and $(C$ for when $C$ is finite) of the forgetful functor $C_{\text{min}}$\text{(A-Proj)} \rightarrow k$-Mod such that $C^- \leq C^+$;
(ii) if $I_C$ is finite then the functor $C$ preserves small coproducts and products; and
(iii) the functors $C^\pm$ preserve small coproducts.

Definition 4.14. [7 Remark 7.7] If $B$ and $D$ are homotopy words such that $C = B^{-1}D$ is a $p$-periodic homotopy word then there is a homotopy word $E = l_1^{-1}r_1 \cdots l_p^{-1}r_p$ with

$$B = r_1^{-1} \cdots l_1^{-1}r_1 \cdots l_p^{-1}r_p \cdots$$

$$D = l_1^{-1} \cdots l_p^{-1}r_1 \cdots l_1^{-1}r_1 \cdots$$

In this case we write $D = E \infty$, $B = (E^{-1}) \infty$ and $C = \infty E \infty$. For each $n \in \mathbb{Z}$ let $E_M(n) = \{(m, m') \in e_v M^n \oplus e_v M^n | m' \in E_m\}$. By [10 Lemma 4.5] there is a $k$-vector space automorphism of $E_M(n)^2/E_M(n)^2$ defined by sending $m + E_M(n)^2$ to $m' + E_M(n)^2$ if and only if $m' \in E_M(n)^2 \cap (E_M(n)^2 + E_M(n)m)$.

Lemma 4.15. Suppose $M$ is a $\Sigma$-pure-injective object of $K_{\text{min}}$(A-Proj). Suppose $E = l_1^{-1}r_1 \cdots l_p^{-1}r_p$ is a homotopy word with head $v$. Let $B = (E^{-1}) \infty$ and $D = E \infty$, so that $C = B^{-1}D$ is the $p$-periodic homotopy word $E \infty \infty$. Then for any $n \in \mathbb{Z}$ the object $(e_v M^n, E_M(n))$ of $k$-Rel is $\Sigma$-pure-injective.

Proof. Let $I$ be a set and $K = M^I$. By Theorem 2.18 and Lemma 3.15 there is a cardinal $\kappa$ and a set $S$ such that $K \cong \bigoplus_{s \in S} U_s$ where each $U_s$ is an indecomposable $\Sigma$-pure-injective object of $K_{\text{min}}$(A-Proj) whose corresponding structure $U_s$ (in the category of models for the canonical language of $K$(A-Proj)) has cardinality at most $\kappa$. By Remark 4.6 this means $K \cong \bigoplus_{s \in S} U_s$ in the category $C_{\text{min}}$(A-Proj) of complexes. As in the proof of [7 Corollary 6.13(ii)], which is precisely the statement of Remark 4.13(ii), since $\Lambda$ is semilocal and noetherian we have that

$$(e_v \bigoplus U^n_s, E_{\bigoplus U_s}(n)) = \bigoplus ((e_v U^n_s, E_{U_s}(n)), (e_v K^n, E_{K}(n)) = (e_v M^n, E_M(n))^I$$

where the coproducts run over $s \in S$. Altogether we have shown that $(e_v M^n, E_M(n))^I$ is isomorphic to a direct sum of objects with cardinality at most $\kappa$. By considering objects in $k$-Rel as modules over the Kronecker algebra, and by the equivalence of (i) and (v) in [15 Theorem 8.1], this shows $(e_v M^n, E_M(n))$ is $\Sigma$-pure-injective.

5. Functorial filtrations.

Definition 5.1. [7 Definition 8.1] Let $\Sigma$ be the set of all triples $(B, D, n)$ where $B^{-1}D$ is a homotopy word (equivalently $(B, D) \in W_{\alpha, \pm 1} \times W_{\alpha, \mp 1}$) and $n$ is an integer.

Assumption 5.2. In [5] fix $(B, D, n), (B', D', n') \in \Sigma$ and let $C = B^{-1}D$ and $C' = B'^{-1}D'$.

Definition 5.3. [7 Definition 8.3] We write $C \sim C'$ if and only if $C' = C^{\pm 1}[t]$ for some $t \in \mathbb{Z}$. So either $(C' = C^{\pm 1}$ and $I_C \neq \mathbb{Z} \neq I_C$) or $(C' = C^{\pm 1}[t]$ and $I_C = I_C' = \mathbb{Z}$) [6 Lemma 2.2.17] (see also [10 Lemma 2.1]). Define the axis $a_{B, D} \in \mathbb{Z}$ of $(B, D)$ by $C_{\leq a_{B, D}} = B^{-1}$ and $C_{> a_{B, D}} = D$. If $I_C = \{0, \ldots, m\}$ then $a_{D, B} = m - a_{B, D}$; if $I_C = \pm \mathbb{N}$ then $a_{D, B} = -a_{B, D}$; and if $I_C = \mathbb{Z}$ then $a_{B, D} = 0$ [6 Lemma 2.2.15].
If \( C \sim C' \) let
\[
\begin{align*}
    r(B, D; B', D') = & \begin{cases} 
        \mu_C(a_{B', D'}) - \mu_C(a_{B, D}) & \text{(if } C' = C \text{ is not a homotopy } \mathbb{Z}\text{-word)} \\
        \mu_C(a_{D', B'}) - \mu_C(a_{B, D}) & \text{(if } C' = C^{-1} \text{ is not a homotopy } \mathbb{Z}\text{-word)} \\
        \mu_C(\pm t) & \text{(if } C' = C^{\pm 1}[t] \text{ is a homotopy } \mathbb{Z}\text{-word)} 
    \end{cases}
\end{align*}
\]
We write \((B, D, n) \sim (B', D', n')\) if and only if \( B^{-1} D \) and \( B'^{-1} D' \) are equivalent and \( n' - n = r(B, D; B', D') \).

By \cite[Lemma 2.2.19]{6} we have that \( r(B, D; B', D') = -r(B', D'; B, D) \) and \( r(B, D; B'', D'') = r(B, D; B', D') + r(B', D'; B'', D'') \) for all \((B, D, n') \in \Sigma\) with \( B'^{-1}D'' \sim B'^{-1}D', \) and so \( \sim \) is an equivalence relation. Let \( \Sigma(s) \) be the set of \((B, D, n) \in \Sigma\) where \( B^{-1} D \) is aperiodic, and \( \Sigma(b) \) the set of such \((B, D, n)\) where \( B^{-1} D \) is periodic. Note that the relation \( \sim \) on \( \Sigma \) restricts to an equivalence relation \( \sim_s \) (respectively \( \sim_k \)) on \( \Sigma(s) \) (respectively \( \Sigma(b) \)). Let \( I(s) \subseteq \Sigma(s) \) (respectively \( I(b) \subseteq \Sigma(b) \)) denote a chosen collection of representatives \((B, D, n)\), one for each equivalence class of \( \Sigma(s) \) (respectively \( \Sigma(b) \)). Let \( I = I(s) \sqcup I(b) \).

5.1. Constructive and refined functors.

**Definition 5.4.** \cite[Definition 8.5]{7} Let \( P = P(C)[\mu_C(a_{B, D}) - n] \) and let \( V \) and \( V' \) be vector spaces with bases \((v_{\lambda} | \lambda \in \Omega)\) and \((v'_{\lambda'} | \lambda' \in \Omega')\) and let \( f : V \to V' \) be \( k\)-linear. Define \( f_{x, \lambda} \in k \) by \( f(v_{\lambda}) = \sum_{\lambda'} f_{x, \lambda} v'_{\lambda'} \). If \((B, D, n) \in I(b)\) then \( a_{B, D} = 0, f \) is \( k[T, T^{-1}]\)-linear and \( T \) defines automorphisms \( \varphi_V \) of \( V \) and \( \varphi_{V'} \) of \( V' \) with \( f_{x} \varphi_V = \varphi_{V'} f \).

If \((B, D, n)\) lies in \( I(s) \) (respectively \( I(b) \)) we use \( S_{B, D, n} \) to denote a functor \( k\text{-Mod} \to C_{\text{min}}(\Lambda\text{-Proj}) \) (respectively \( k[T, T^{-1}]\text{-Mod} \to C_{\text{min}}(\Lambda\text{-Proj}) \)), defined as follows. On objects \( V \) define the homogeneous component \( S^m_{B, D, n}(V) \) of the complex \( S_{B, D, n}(V) \) in degree \( m \in \mathbb{Z} \) by \( P^m \otimes_k V \) (respectively \( P^m \otimes_k [T, T^{-1}] V \)). Define the corresponding differential \( d^m_{S_{B, D, n}} \) by \( d^m_{S_{B, D, n}} \) (respectively \( d^m_{S_{B, D, n}} \)) in degree \( m \in \mathbb{Z} \). Similarly we can define the map \( S^m_{B, D, n}(f) \) of the image \( S_{B, D, n}(f) \) of \( S_{B, D, n} \) on a morphism \( f \) by \( \text{id}_P^m \otimes f \).

**Corollary 5.5.** \cite[Corollary 8.6]{7} Suppose \((B, D, n) \sim (B', D', n') \) in \( \Sigma \).

(i) If \( C \) is aperiodic then \( S_{B, D, n} \simeq S_{B', D', n'} \).

(ii) If \( C \) is periodic and \( C' = C^t \) for some \( t \in \mathbb{Z} \) then \( S_{B, D, n} \simeq S_{B', D', n'} \).

(iii) If \( C \) is periodic and \( C = C^{-t} \) for some \( t \in \mathbb{Z} \) then \( S_{B, D, n} \simeq S_{B', D', n'} \) respectively.

**Assumption 5.6.** For the remainder of \cite[fix an object \( M \) of \( K_{\text{min}}(\Lambda\text{-Proj}) \).**

**Definition 5.7.** \cite[Definition 7.6]{7} For any \( n \in \mathbb{Z} \) consider the \( k\)-subspaces of \( e_v M^n \)
\[
F^+_{B, D, n}(M) = M^n \cap (B^+(M) \cap D^+(M)),
\]
\[
F^-_{B, D, n}(M) = M^n \cap (B^+(M) \cap D^-(M) + B^-(M) \cap D^+(M)),
\]
\[
G^+_{B, D, n}(M) = M^n \cap (B^-(M) + D^+(M) \cap B^+(M)).
\]
Define the quotients \( F_{B, D, n}(M) \) and \( G_{B, D, n}(M) \) by
\[
F_{B, D, n}(M) = F^+_{B, D, n}(M)/F^-_{B, D, n}(M), \quad G_{B, D, n}(M) = G^+_{B, D, n}(M)/G^-_{B, D, n}(M).
\]
Let
\[
\tilde{F}_{B, D, n}(M) = F^+_{B, D, n}(M)/F^-_{B, D, n}(M), \quad \tilde{G}_{B, D, n}(M) = G^+_{B, D, n}(M)/G^-_{B, D, n}(M)
\]
where
\[
\tilde{F}^\pm_{B, D, n}(M) = F^\pm_{B, D, n}(M) + e_v \text{rad}(M^n), \quad \tilde{G}^\pm_{B, D, n}(M) = G^\pm_{B, D, n}(M) + e_v \text{rad}(M^n).
\]
Let $\bar{A}^\pm(M) = A^\pm(M) + e_{h(C)} \text{rad}(M)$ for any homotopy word $A$ with $I_A \subseteq \mathbb{N}$.

**Remark 5.8.** By [7] Lemma 7.5 we have

\[
\begin{align*}
\bar{F}^+_B,D,n(M) &= e_v M^n \cap \bar{B}^+(M) \cap \bar{D}^+(M), \\
\bar{F}^-_B,D,n(M) &= e_v M^n \cap ((B^+(M) \cap \bar{D}^-(M)) + (B^-(M) \cap \bar{D}^+(M))), \text{ and} \\
\bar{G}^+_B,D,n(M) &= e_v M^n \cap (B^-(M) + \bar{D}^+(M) \cap \bar{B}^+(M)).
\end{align*}
\]

If $C$ is aperiodic $\bar{F}_B,D,n, \bar{F}_B,D,n, G^*_B,D,n,$ and $\bar{G}^*_B,D,n$ all define naturally isomorphic additive functors [6] Lemma 2.2.7]. Furthermore note that $\text{im}(q) \subseteq \text{rad}(N)$ for any null-homotopic morphism $q : M \to N$ between homotopically minimal complexes of projectives, and so $F_B,D,n, \bar{F}_B,D,n, G^*_B,D,n,$ and $\bar{G}^*_B,D,n$ all define functors $K_{\text{min}}(\Lambda\text{-Proj}) \to k\text{-Mod}$ [6] Corollary 2.2.8].

As above, and by [10] Lemma 4.5], if $C$ is periodic then $F_B,D,n, \bar{F}_B,D,n, G^*_B,D,n,$ and $\bar{G}^*_B,D,n$ all define naturally isomorphic functors $K_{\text{min}}(\Lambda\text{-Proj}) \to k[T,T^{-1}]\text{-Mod}.$

Recall the involution $\text{res}_s$ of $k[T,T^{-1}]\text{-Mod}$ which swaps the action of $T$ and $T^{-1}$.

**Lemma 5.9.** Fix $(B, D, n), \ (B', D', n'), \ C$ and $C'$ as in Assumption [5]2.

(i) [7] Lemma 8.4] (see also [10] Lemma 7.1]).

(a) If $C$ is aperiodic then $F_B,D,n \simeq F_B,D,n; \text{ and}$
(b) If $C$ is periodic then $F_B,D,n \simeq \text{res}_s, kF_B,D,n.$

(ii) [7] Corollary 8.6].

(a) If $C$ is aperiodic then $G_B,D,n \simeq G_B',D',n'.$
(b) If $C$ is periodic and $C' = C[i] \text{ for some } i \in \mathbb{Z}$ then $G_B,D,n \simeq G_B',D',n'$.
(c) If $C$ is periodic and $C' = C[-1][i] \text{ for some } i \in \mathbb{Z}$ then $G_{B,D,n} \simeq \text{res}_s, kG_B',D',n'$.

(iii) [7] Lemma 10.5] (see also [10] Lemma 8.2]. Let $C$ be aperiodic and let $P = P(C).$

(a) If $i \in I$ then $\bar{F}^\pm_{C(i,1),C(i,-1),n}(P[C(i) - n]) = \bar{F}^\pm_{C(i,1),C(i,-1),n}(P[C(i) - n]) + kb_i.$
(b) If $C' = C$ and $n - n' = \mu_C(a,b,D) - \mu_C(a',b',D')$ then id $\simeq \bar{F}_{B,D,n} \simeq \bar{S}_{B,D,n}.$
(c) If $(B, D, n)$ is not equivalent to $(B', D', n')$ then $\bar{F}^\pm_{B',D',n'}(P[C(a,b,D) - n]) = 0.$

(iv) [7] Lemma 10.6] (see also [7] Lemma 8.5]). Let $C$ be aperiodic, let $V$ be an indecomposable $k[T,T^{-1}]$-module, let $P = P(C, V)$ and let $i \in \{0, \ldots, p - 1\}$.

(a) We have $\bar{F}^\pm_{C(i,1),C(i,-1),n}(P[C(i) - n]) = \bar{F}^\pm_{C(i,1),C(i,-1),n}(P[C(i) - n]) + \sum_{\lambda} kb_i, \lambda.$
(b) If $C' = C[m]$ and $n - n' = \mu_C(m)$ then id $\simeq \bar{F}_{B',D',n'} \simeq \bar{S}_{B,D,n};$ and
(c) If $(B, D, n)$ is not equivalent to $(B', D', n')$ then $\bar{F}_{B',D',n'}(P[n]) = 0.$

5.2. Compactness and Covering.

**Lemma 5.10.** [6] Lemma 2.4.1. Suppose $M$ is $\Sigma$-pure-injective. Let $r \in \mathbb{Z}$ and $\delta = \pm 1.$ Let $U$ be an $r$-subspace of $e_v, M^\delta$ with $e_v, \text{rad}(M^\delta) \subseteq U.$

(i) (See also [10] Lemma 10.4]). If $H$ is a linear variety in $e_v, M^\delta$ and $m \in H \setminus U,$ then there is a homotopy word $C \in \mathcal{W}_{v,\delta}$ such that $H \cap (U + m)$ meets $C^+(M)$ but not $C^-(M).$

(ii) (See also [10] Lemma 10.5]). If $m \in e_v, M^\delta \setminus U$ then there are words $B \in \mathcal{W}_{v,\delta}$ and $D \in \mathcal{W}_{v,-\delta}$ such that $U + m$ meets $G_{B,D,r}(M)$ but not $G_{B,D,r}(M).$

The proof of Lemma [5.10] is given at the end of [5.2].

**Definition 5.11.** [6] Definition 2.1.17] If $A = \ldots l_i^{-1}r_i \ldots$ is a homotopy word and $i \in I_A,$ let $A_i = l_i^{-1}r_i$ and $A_{\leq i} = \ldots l_i^{-1}r_i$ given $i - 1 \in I_A,$ and otherwise $A_i = A_{\leq i} = 1_{h(A),s(A)}.$
Similarly let $A_{>i} = l_{i+1}^{-1} r_{i+1} \cdots$ given $i+1 \in I_A$ and otherwise $A_{>i} = 1_{h(A-1),-s(A-1)}$. In the same way we can define the homotopy words $A_{<i}$ and $A_{\geq i}$ with $A_{\leq i} = A_{<i} A_i$ and $A_i A_{>i} = A_{\geq i}$.

**Lemma 5.12.** [6 Lemma 2.4.8] (see also [10 Lemma 10.3]). Fix an integer $r$ and some $\delta \in \{\pm 1\}$. For any non-empty subset $S$ of $e_r M^r$ which does not meet $\text{rad}(M)$ there is a homotopy word $C \in W_{e,\delta}$ such that either:

(i) $C$ is finite and $S$ meets $C^+(M)$ but not $C^-(M)$; or

(ii) $C$ is a homotopy $\mathbb{N}$-word and $S$ meets $C_{\leq n} M$ but not $C_{\leq n} \text{rad}(M)$ for each $n \geq 0$.

In Lemma 5.12 we do not require that $M$ is an object of $\mathcal{K}_{\text{min}}(\Lambda-\text{proj})$. In Corollary 5.14 we do consider this setting, and show $S \cap C^+(M) \neq \emptyset = S \cap C^-(M)$ in case (ii) of Lemma 5.12.

Recall Definition 3.10.

**Lemma 5.13.** (Realisation) If $C \in W_{e,\delta}$ and $I_C = \{0, \ldots, t\}$ then there is some $n \in I_C(\Omega, \Omega)$ such that $(C(\Omega, \Omega))_{>n} \leq n+1 = C$ and

$$CM = \{f(b_{n,C(\Omega, \Omega)}) \mid f : P(C(\Omega, \Omega)) \to M \text{ in } \mathcal{C}_{\text{min}}(\Lambda-\text{proj})\}.$$

**Proof of Lemma 5.13** Without loss of generality assume $C$ is non-trivial, say $C = l_1^{-1} r_1 \cdots l_t^{-1} r_t$. Let $I = I_C(\Omega, \Omega)$, the subset of $\mathbb{Z}$ such that $C(\Omega, \Omega)$ is a homotopy $I$-word. Let $I_-$ and $I_+$ be the subsets of $\mathbb{N}$ for which $(C(\Omega, \Omega))^{-1}$ is a homotopy $I$-word and $C(\Omega, \Omega)$ is a homotopy $I_+$-word. If $I_-$ is finite then we let $I_+ = \{0, \ldots, n(\pm)\}$. If $I \subseteq \mathbb{N}$ then $I_-$ is finite, and we let $n = n(-)$. If $I = -\mathbb{N}$ then $I_+$ is finite, and we let $n = -t - n(+)$. If $I = \mathbb{Z}$ let $n = 0$. We firstly show $CM \subseteq X$ where we let

$$X = \{f(b_{n,C(\Omega, \Omega)}) \mid f : P(C(\Omega, \Omega)) \to M \text{ in } \mathcal{C}_{\text{min}}(\Lambda-\text{proj})\}.$$ 

Let $m \in CM$, and so there exists $m_0, \ldots, m_{n+t} \in M$ such that $m = m_n$ and $l_i m_i = r_i m_{i+1}$ for all $i$ with $n \leq i < t$. To show $m \in X$ it suffices to construct a sequence $(m_i \mid i \in I)$ of elements $m_i \in M$ such that $b_{i,C(\Omega, \Omega)} \to m_i$ defines a morphism of complexes $f : P(C(\Omega, \Omega)) \to M$. Note that $n - h, n + t + j \in I$ for all $h \in I_-$ and all $j \in I_+$. We begin by iteratively constructing $m_{n-h} \in M$ for all $h \in I_-$ and $m_{n+t+j} \in M$ for all $j \in I_+$, noting that $m_n$ and $m_{n+t}$ have already been defined. Suppose that $h \in I_-, m_{n-h}$ has been defined and that $h + 1 \in I_-$. By construction $(C(\Omega, \Omega))_{h+1} = d_{n+1}^{-1} \gamma$ for some arrow $\gamma$. Furthermore $\text{im}(d_{\gamma,M}) \subseteq \gamma M$, and we choose $m_{n-(h+1)} \in e((\gamma))M$ such that $d_{\gamma,M}(m_{n-h}) = \gamma m_{n-(h+1)}$.

Similarly if $(j \in I_+, m_{n+t+j}$ has been defined and $j + 1 \in I_+)$ then $(C(\Omega, \Omega))_{j+1} = d_{\beta}^{-1} \beta$ for some arrow $\beta$, and we choose $m_{n+t+(j+1)} \in e((\beta))M$ such that $d_{\beta,M}(m_{n+t+j}) = \beta m_{n+t+(j+1)}$. It is straightforward to check that $f(b_{i,C(\Omega, \Omega)}) = m_i$ $(i \in I)$ satisfies $f(b_{i,C(\Omega, \Omega)} + b_{i,C(\Omega, \Omega)}) = d_M(m_i)$ for all $i$. This is done by separating the cases $i < n, i = n, n < i < n + t, i = n + t$ and $i > n + t$. The cases $i < n$ and $i > n + t$ are similar. As are the cases $i = n$ and $i = n + t$. This shows $CM \subseteq X$. The proof that $X \subseteq CM$ is similar, easier, and omitted.

**Corollary 5.14.** If $M$ is $\Sigma$-pure-injective $I_C = \mathbb{N}$ then $C^+(M) = C_{\leq l} M$ for some $l > 0$.

**Proof.** Recall the canonical language $L = \Sigma^*$ of the compactly generated triangulated category $T = \mathcal{K}(\Lambda-\text{proj})$. Recall that for any compact object $G$ of $T$ a subgroup of finite definition of $M$ of sort $G$ has the form $M_G$, and is defined as the set of morphisms $fa \in T(G, M)$ where $f$ runs through $T(G, M), a \in T(G, M)$ is fixed and $H$ is compact.
Let $C = l_1^{-1} r_1 l_2^{-1} r_2 \ldots$. If $l_1^{-1} r_1$ has the form $\tau^{-1} d_i(\tau)$ then let $I = \{0\}$ and $B = 1_{n,\delta}$ and $D = C_{>1}$ so that $C = B \tau^{-1} d_i(\tau)$. Otherwise there is a non-trivial homotopy $I$-word $B$ such that $B_{\leq s}$ has the form $d_i^{-1}(\gamma^s(1)) \gamma(1) \ldots d_i^{-1}(\gamma^s(s)) \gamma(s)$ whenever $0 < s \in I$, and either $(I = \mathbb{N}$ and $B = C)$ or $(I = \{0, \ldots, p\}$ and $C_{\leq p+1} = B \tau^{-1} d_i(\tau_{i+1})$. Let $G = P(B(\omega))$ and $H_n = P((C_{\leq n})(\omega, \omega))$ for each $n$. We now define, for sufficiently large $s > 0$, a morphism $a_s : G \to H_s$.

Suppose firstly $I = \{0, \ldots, p\}$, so we have that $C_{\leq p} = B$ and that $l_1^{-1} r_1 \tau_{i+1}$ has the form $\tau^{-1} d_i(\tau)$. So for any $s > p$ there is a morphism $a_s : G \to H_n$ given by $b_{i,k}(\omega) \mapsto b_{i,k}(C_{\leq r}(\omega, \omega))$ for any $i \in I_B(\omega)$. Suppose instead $I = \mathbb{N}$, in which case we let $p = 0$. In this case $(C_{\leq s})(\omega, \omega)) = C(\omega) = B(\omega)$ and hence $G = H_s$ for all $s$, and we let $a_s$ be the identity $G \to G$. Consider the finite word $C_{\leq p}$. Applying Lemma 5.13 gives some $n \in I_B(\omega, \omega)$ such that, for any $s \geq p + 1$, $C_{\leq s} M$ is the set of images $f(a_s(b_1, B(\omega))))$ as $f$ runs through morphisms of complexes $H_s \to M$.

Note that $G$ and $H_n$ are compact objects of $T$ by Proposition 5.13. So the above may be rewritten as $C_{\leq s} M = \{g(b_0, C_{\leq s} ) | g \in M_{a_0}\}$. Let $\varphi_n(v_G)$ be the pp-formula in $G$ given by $\exists u_{H_n} : v_G = f u_{H_n}$. Since $C_{\leq n+1} M \subseteq C_{\leq n} M$ for each $n$, we have that $M_{a_{n+1}} = \varphi_{n+1}(M) \supseteq M_{a_{n+2}} = \varphi_{n+2}(M) \supseteq \ldots$ is a descending chain of pp-definable subgroups of $M$ of sort $G$. By Lemma 5.13 this chain stabilises. As above this must mean the chain $C_{\leq 1} M \supseteq C_{\leq 2} M \supseteq \ldots$ stabilises, as required.

**Proof of Lemma 5.14** (i) Let $S = H \cap (U + m)$. Note $S \cap \text{rad}(M) = \emptyset$ since $e_s \text{rad}(M') \subseteq U$ and $m \notin U$. So by Lemma 5.12 there is a homotopy word $C$ such that either $C$ is finite and $S \cap C^+(M) \neq \emptyset = S \cap C^-(M)$, or $C$ is a homotopy N-word and for all $n \geq 0$ we have $S \cap C_{\leq n} M \neq \emptyset = S \cap C_{\leq n} \text{rad}(M)$. We may assume $I_C = \mathbb{N}$, and so $S \cap C^+(M) = S \cap C_{\leq n} M \neq \emptyset$ for some $l > 0$ by Corollary 5.14 as required.

(ii) The argument in the proof of [10] Lemma 10.5] adapts with few complications.

5.3. **Local and global mapping properties.**

Recall: $\Sigma(s)$ (respectively $\Sigma(b)$) is the set of triples $(B, D, n) \in \Sigma$ where $B^{-1} D$ is aperiodic (respectively periodic); $\mathcal{I}(s)$ (respectively $\mathcal{I}(b)$) is a collection of representatives $(B, D, n)$; and $\mathcal{I} = \mathcal{I}(s) \cup \mathcal{I}(b)$ (see Definition 5.3).

**Lemma 5.15.** [7] Lemma 2.5.2 (see also [10] Lemma 8.3)]. If $(B, D, n) \in \mathcal{I}(s)$ and $B = (\bar{u}_\lambda | \lambda \in \Omega)$ is a $k$-basis of $F_{B, D, n}(M)$ then there is a morphism of complexes $\theta_{B, D, n, M} : \bigoplus P(C)[\mu_C(a, B, D) - n] \to M$ such that $F_{B, D, n}(\theta_{B, D, n, M})$ is an isomorphism.

Recall Definition 4.14 if $E$ is a homotopy $\{0, \ldots, p\}$ word such that $p > 1$ and $C = \Sigma E^\infty$ is a $p$-periodic then the relation $E(M)(n)$ on $e_s(M)$ is defined to be the set of pairs $(m, m')$ with $m' \in Em$. Lemma 5.16 below will be applied in the context of $\Sigma$-pure-injective complexes.

**Lemma 5.16.** [7] Lemma 12.4 (see also [10] Lemma 8.6)]. Suppose $(B, D, n) \in \mathcal{I}(b)$, say where $C = \Sigma E^\infty$ is periodic of period $p > 0$ and $E = l_1^{-1} r_1 \ldots l_p^{-1} r_p$. If the relation $E(M)(n)$ on $e_v M^n$ is split then there is a morphism $\theta_{B, D, n, M} : P(C, U) [-n] \to M$ of complexes such that $F_{B, D, n}(\theta_{B, D, n, M})$ is an isomorphism.

Note that the statements of [7] Lemma 12.4 and Lemma 5.16 are equivalent: any relation $(V, C)$ that admits a reduction which meets in 0 must have been split by [6] Corollary 1.4.33,

\[ \Sigma \text{-pure-injective complexes for gentle algebras} \]
Lemma 5.17. [7, Lemma 12.5] (see also [10, Lemma 10.5] and [9, p. 163]). Let $\theta : P \to M$ be a morphism in $\mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$ where $M$ is $\Sigma$-pure-injective. If $F_{B,D,n}(\theta)$ is surjective for each $(B, D, n) \in \Sigma$ then $\theta^i$ is surjective for each $i$.

Proof. For a contradiction suppose that $\theta^i$ is not surjective for some $i \in \mathbb{Z}$. Since $\Lambda$ is perfect $M^i$ is a projective cover of $M/rad(M^i)$, and so $rad(M^i)$ is a superfluous submodule of $M^i$. This means $e_vim(\theta^i) + e_vrad(M^i)$ is contained in a maximal $k$-subspace $U$ of $e_vM^i$. Since $e_vrad(M^i) \subseteq U$ and $U \neq e_vM^i$, by Lemma 5.10(ii) for some element $m \in e_vM^i \setminus U$ there are homotopy words $B \in W_{v,\delta}$ and $D \in W_{v,-\delta}$ for which $(B^{-1}D)$ is a homotopy word and $U + m$ meets $G_{B,D,i}^+(M)$ but not $G_{B,D,i}^-(M)$. From here one can show $F_{B,D,i}(\theta)$ is not surjective by adapting the argument from the proof of [10, Lemma 10.6].

Assumption 5.18. For Lemma 5.19 we fix a direct sum $N$ of shifts of string and band complexes as follows. Let $S$ and $B$ be sets, $\{t(\sigma), s(\beta) \mid \sigma \in S, \beta \in B\}$ be a set of integers, $\{V^\beta \mid \beta \in B\}$ be a set of objects from $k[T, T^{-1}]-\text{Mod}$ and $\{A(\sigma), E(\beta) \mid \sigma \in S, \beta \in B\}$ be a set of homotopy words, where each $A(\sigma)$ is aperiodic and each $E(\beta)$ is $p_\beta$-periodic. Let

$$N = \left( \bigoplus_{\sigma \in S} P(A(\sigma))[-t(\sigma)] \right) \oplus \left( \bigoplus_{\beta \in B} P(E(\beta), V^\beta)[-s(\beta)] \right)$$

Lemma 5.19. [6, Lemma 2.5.6] (see also [10, Lemma 9.4]). Let $N$ be the direct sum of string and band complexes from Assumption 5.18. Let $\theta : N \to M$ be a map in $\mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$ where $F_{B,D,n}(\theta)$ is injective for all $(B, D, n) \in \Sigma$. Then each $\theta^i$ is injective.

6. Completing the proof of the main theorem.

Assumption 6.1. In [6] we let $\mathcal{I}$ be a set, $\Xi : \mathcal{C}_{\text{min}}(\Lambda-\text{Proj}) \to \mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$ be the quotient and $S_i : \mathcal{A}_i \to \mathcal{C}_{\text{min}}(\Lambda-\text{Proj})$ and $F_i : \mathcal{K}_{\text{min}}(\Lambda-\text{Proj}) \to \mathcal{A}_i$ ($i \in \mathcal{I}$) be additive functors.

Definition 6.2. (See [7, Definition 4.4]). Let $Z$ be a full subcategory of $\mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$. We say that $\{(S_i, F_i) \mid i \in \mathcal{I}\}$ detects objects in $Z$ if the following statements hold.

(i) For any $i \in \mathcal{I}$:

(FFI) the functor $F_i \Xi S_i$ is dense and reflects isomorphisms;

(FFII) $F_i \Xi S_i \simeq 0$ for each $j \in \mathcal{I}$ with $j \neq i$;

(FFIII) $F_i$ preserves small coproducts; and

(FFIV) for each object $M$ in $Z$ there exists an object $A_{i,M}$ in $\mathcal{A}_i$ and a morphism $\gamma_{i,M} : \Xi(S_i(A_{i,M})) \to M$ in $\mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$ such that $F_i(\gamma_{i,M})$ is an isomorphism.

(ii) For all morphisms $\theta : N \to M$ in $\mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$:

(FFV) if $M$ lies in $Z$ and $F_i(\Xi(\theta))$ is epic for all $i \in \mathcal{I}$ then each $\theta^a$ is epic; and

(FFVI) if $N = \bigoplus_{i \in \mathcal{I}} S_i(A_i)$ and $F_i(\Xi(\theta))$ is monic for each $i \in \mathcal{I}$ then each $\theta^a$ is monic.

Lemma 6.3. (See [7, Lemma 4.5]). Let $Z$ be a full subcategory of $\mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$ where $\{(S_i, F_i) \mid i \in \mathcal{I}\}$ detects objects in $Z$. Any object $M$ of $Z$ is isomorphic to $\bigoplus_{i \in \mathcal{I}} \Xi(S_i(A_{i,M}))$.

Note that Definition 6.2 and Lemma 6.3 are essentially [7, Definition 4.4, Lemma 4.5], the difference being we have replaced the category $\mathcal{P}_\Lambda$ of $\Lambda-\text{Proj}$ from [7] with an arbitrary full subcategory $\mathcal{X}$ of $\mathcal{K}_{\text{min}}(\Lambda-\text{Proj})$. The proof of [7, Lemma 4.5] generalises to our setting with no further complications. We now verify the hypotheses of Lemma 6.3.
Recall, from Definition 6.2, the equivalence relation on the triples \((B, D, n)\) where \(B^{-1}D\) is a homotopy word and \(n \in \mathbb{Z}\). Recall that \(\mathcal{I} = \mathcal{I}(s) \sqcup \mathcal{I}(b)\) where \(\mathcal{I}(s)\) (respectively \(\mathcal{I}(b)\)) denotes a chosen set of representatives \((B, D, n)\) such that \(B^{-1}D\) is aperiodic (respectively periodic).

Recall that if \((B, D, n)\) lies in \(\mathcal{I}(s)\) (respectively \(\mathcal{I}(b)\)) then the functor \(S_{B, D, n}\) has the form \(k\text{-Mod} \rightarrow \mathcal{C}_{\text{min}}(\Lambda\text{-Proj})\) (respectively \(k[T, T^{-1}]\text{-Mod} \rightarrow \mathcal{C}_{\text{min}}(\Lambda\text{-Proj})\)), and the functor \(F_{B, D, n}\) has the form \(\mathcal{K}_{\text{min}}(\Lambda\text{-Proj}) \rightarrow k\text{-Mod}\) (respectively \(\mathcal{K}_{\text{min}}(\Lambda\text{-Proj}) \rightarrow k[T, T^{-1}]\text{-Mod}\)).

Recall Definition 6.2.

**Proposition 6.4.** [7 Proposition 13.1] (see also [9] p. 163, Proposition)]. Let \(\mathcal{M} = \Lambda\text{-Mod}\) and \(\mathcal{I} = \mathcal{I}(s) \sqcup \mathcal{I}(b)\); and for each \(i = (B, D, n) \in \mathcal{I}\) let

\[
\mathfrak{A}_i = \begin{cases} (k\text{-Mod}) & \text{(if } B^{-1}D \text{ is aperiodic),} \\ (k[T, T^{-1}]\text{-Mod}) & \text{(if } B^{-1}D \text{ is periodic).} \end{cases}
\]

The collection \(\{(S_{B, D, n}, F_{B, D, n}) \mid (B, D, n) \in \mathcal{I}\}\) detects the objects in the full subcategory \(\mathcal{Z}\) of \(\mathcal{K}_{\text{min}}(\Lambda\text{-Proj})\) consisting of \(\Sigma\)-pure-injective objects.

In the proof of Proposition 6.4(ii) we verify the conditions FFII, FFIII, FFIV, FFV, and FFVI from Definition 6.2. Later we use Proposition 6.4 in the context of Lemma 6.2.

**Proof.**

(i) Since \(\Lambda\) is a perfect ring every \(\Lambda\)-module has a projective cover.

(ii) FFII Let \((B, D, n) \in \Sigma\) and \(B^{-1}D = C\). If \(C\) is aperiodic (respectively periodic) then by Lemma 6.9(iii) (respectively Lemma 6.9(ivb)) we have \(F_{B, D, n} \simeq S_{B, D, n} \simeq \text{id.}\)

(ii) FFII Let \((B', D', n') \in \mathcal{I}\). If \((B', D', n') \neq (B, D, n) \in \mathcal{I}(s)\) then \(F_{B', D', n'}(P(C)[\mu_C(a_{B,D}) - n]) = 0\) by Lemma 6.9(iiic) where \(C = B^{-1}D\). This shows \(F_{B', D', n'} \simeq S_{B, D, n} = 0\) since \(F_{B', D', n'} \simeq F_{B, D, n'}\). If \((B, D, n) \in \mathcal{I}(b)\) then the proof is similar and uses Lemma 6.9(ivc).

(ii) FFIII By Remark 4.13 each of the subfunctors \(C^\pm\) of the forgetful functor \(\mathcal{C}_{\text{min}}(\Lambda\text{-Proj}) \rightarrow k\text{-Mod}\) commutes with arbitrary direct sums. It follows that \(F_{B, D, n}^\pm\) commutes with direct sums of objects in \(\mathcal{K}_{\text{min}}(\Lambda\text{-Proj})\) (see [6] Lemma 2.1.21 [for details]).

(ii) FFIV Let \(M\) be an object in \(\mathcal{Z}\). If \((B, D, n) \in \mathcal{I}(s)\), by Lemma 5.13 there is a vector space \(U\) and a morphism \(\theta_{B, D, n} : S_{B, D, n}(U) \rightarrow M\) for which \(F_{B, D, n}(\theta_{B, D, n})\) is an isomorphism. Now suppose instead \((B, D, n) \in \mathcal{I}(b)\), say where \(B^{-1}D = \infty E\) is periodic of period \(p > 0\), and where \(E = l_1^{-1}r_1 \ldots l_p^{-1}r_p\).

Note \(F_{B, D, n}^+(M) = E(n)^p\) and \(F_{B, D, n}^-(M) = E(n)^q\), so \(E(n)^2/E(n)^q = F_{B, D, n}(M)\). By Lemma 4.13 the object \((e, M^n, E_M(n))\) of \(k\text{-Rel}\) is \(\Sigma\)-pure-injective. By Corollary 4.4 the relation \((e, M^n, E_M(n))\) is split. The required morphism exists by Lemma 5.16.

(iii) FFV, FFVI Let \(\theta : N \rightarrow M\) be an morphism in the category \(\mathcal{K}_{\text{min}}(\Lambda\text{-Proj})\). If \(M\) is an object of \(\mathcal{Z}\) and \(F_{B, D, n}(\theta)\) is epic for all \((B, D, n) \in \mathcal{I}\) then \(\theta^n\) is epic for each \(n \in \mathbb{N}\) by Lemma 6.17. This shows FFV holds, and similarly FFVI holds by Lemma 6.19.

**Proof of Theorem 1.1.** The statement of Theorem 1.1 is precisely the statement of Lemma 6.3 after applying Proposition 6.4.

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