Variational Inequalities For The Differences Of Averages Over Lacunary Sequences

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Abstract

Let \( f \) be a locally integrable function defined on \( \mathbb{R} \), and let \( (n_k) \) be a lacunary sequence. Define the operator \( A(n_k) \) by

\[
A(n_k)f(x) = \frac{1}{n_k} \int_0^{n_k} f(x - t) \, dt.
\]

We prove various types of new inequalities for the variation operator

\[
\mathcal{V}_s f(x) = \left( \sum_{k=1}^{\infty} |A(n_k)f(x) - A(n_{k-1})f(x)|^s \right)^{1/s}
\]

when \( 2 \leq s < \infty \).

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An increasing sequence \((n_k)\) of real numbers is called lacunary if there exists a constant \(\beta > 1\) such that
\[
\frac{n_{k+1}}{n_k} \geq \beta
\]
for all \(k = 0, 1, 2, \ldots\).

Let \(f\) be a locally integrable function defined on \(\mathbb{R}\). Let \((n_k)\) be a lacunary sequence and define the operator \(A_{n_k}\) by
\[
A_{n_k}f(x) = \frac{1}{n_k} \int_0^{n_k} f(x - t) \, dt.
\]

It is clear that
\[
A_{n_k}f(x) = \frac{1}{n_k} \chi_{(0,n_k)} * f(x)
\]
where * stands for convolution.

Consider the variation operator
\[
V_s f(x) = \left( \sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s \right)^{1/s}
\]
for \(2 \leq s < \infty\).

Analyzing the boundlessness of the variation operator \(V_s f\) is a method of measuring the speed of convergence of the sequence \(\{A_{n_k}f\}\).

Various types of inequalities for the two-sided variation operator
\[
V'_s f(x) = \left( \sum_{n=-\infty}^{\infty} \frac{1}{2^n} \int_x^{x+2^n} f(t) \, dt - \frac{1}{2^{n-1}} \int_x^{x+2^{n-1}} f(t) \, dt \right)^{1/s}
\]
when \(2 \leq s < \infty\) have been proven by the author in S. Demir [1], in this research we prove that same types of inequalities are also true for any lacunary sequence \((n_k)\) for the one-sided variation operator \(V_s f(x)\) for \(2 \leq s < \infty\).

**Lemma 1.** Let \((n_k)\) be a lacunary sequence with the lacunarity constant \(\beta\), i.e.,
\[
\frac{n_{k+1}}{n_k} \geq \beta > 1
\]
for all \( k = 0, 1, 2, \ldots \), and \( 1 \leq s < \infty \). Then there exists a sequence \((m_j)\)

such that

\[
\beta^2 \geq \frac{m_{j+1}}{m_j} \geq \beta > 1
\]

for all \( j \) and

\[
\left( \sum_{k=1}^{\infty} |A_n f(x) - A_{n-1} f(x)|^s \right)^{1/s} \leq \left( \sum_{j=1}^{\infty} |A_{n_j} f(x) - A_{n_{j-1}} f(x)|^s \right)^{1/s}.
\]

**Proof.** Let us start our construction by first choosing \( m_0 = n_0 \). If

\[
\beta^2 \geq \frac{n_1}{n_0} \geq \beta,
\]

define \( m_1 = n_1 \). If

\[
\frac{n_1}{n_0} > \beta^2,
\]

let \( m_1 = \beta n_0 \), then we have

\[
\beta^2 \geq \frac{m_1}{m_0} = \frac{\beta n_0}{n_0} = \beta \geq \beta.
\]

Also,

\[
\frac{n_1}{m_1} \geq \frac{\beta^2 n_0}{\beta n_0} = \beta.
\]

Again, if

\[
\frac{n_1}{m_1} \leq \beta^2,
\]

then choose \( m_2 = n_1 \). If this is not the case, choose \( m_2 = \beta^2 n_0 \leq n_1 \).

By the same calculation as before, \( m_0, m_1, m_2 \) are part of a lacunary sequence satisfying

\[
\beta^2 \geq \frac{m_{k+1}}{m_k} \geq \beta > 1.
\]

To continue the sequence, either \( m_3 = n_1 \) if

\[
\frac{n_1}{m_2} \leq \beta^2
\]

or \( m_3 = \beta^3 n_0 \) if

\[
\frac{n_1}{m_2} > \beta^2.
\]
Since $\beta > 1$, this process will end at some $k_0$ such that $m_{k_0} = n_1$. The remaining elements $m_k$ are constructed in the same manner as the original $n_k$, with necessary terms added between two consecutive $n_k$ to obtain the inequality

$$\beta^2 \geq \frac{m_{k+1}}{m_k} \geq \beta > 1.$$ 

Let now

$$J(k) = \{ j : n_{k-1} < m_j \leq n_k \}$$

then we have

$$A_{n_k} f(x) - A_{n_{k-1}} f(x) = \sum_{j \in J(k)} (A_{m_j} f(x) - A_{m_{j-1}} f(x))$$

and thus we get

$$|A_{n_k} f(x) - A_{n_{k-1}} f(x)| = \left| \sum_{j \in J(k)} (A_{m_j} f(x) - A_{m_{j-1}} f(x)) \right|$$

$$\leq \sum_{j \in J(k)} |A_{m_j} f(x) - A_{m_{j-1}} f(x)|.$$ 

This implies that

$$\sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)| \leq \sum_{k=1}^{\infty} \sum_{j \in J(k)} |A_{m_j} f(x) - A_{m_{j-1}} f(x)|.$$ 

$$= \sum_{j=1}^{\infty} |A_{m_j} f(x) - A_{m_{j-1}} f(x)|.$$ 

Thus we have

$$\left( \sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)|^s \right)^{1/s} \leq \left( \sum_{j=1}^{\infty} |A_{m_j} f(x) - A_{m_{j-1}} f(x)|^s \right)^{1/s}.$$ 

and this completes the proof. \qed

**Remark 1.** We know from Lemma 1 that

$$\left( \sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)|^s \right)^{1/s} \leq \left( \sum_{j=1}^{\infty} |A_{m_j} f(x) - A_{m_{j-1}} f(x)|^s \right)^{1/s}.$$ 

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and the new sequence \((m_j)\) satisfies

\[
\beta^2 \geq \frac{m_{j+1}}{m_j} \geq \beta > 1
\]

for all \(j \in \mathbb{Z}^+\). Therefore, we can assume without loss of generality,

\[
\beta^2 \geq \frac{n_{k+1}}{n_k} \geq \beta > 1
\]

for all \(k \in \mathbb{Z}^+\) when we are proving any result for \(V_s(x)\).

Since \(\frac{1}{n_k} = \frac{n_1}{n_2} \cdot \frac{n_2}{n_3} \cdots \frac{n_{k-1}}{n_k}\)

we can also assume that

\[
\frac{1}{n_k} \leq \frac{1}{\beta^{2(k-1)}},
\]

for all \(k = 0, 1, 2, \ldots\).

**Lemma 2.** Let \((n_k)\) be a lacunary sequence, and let \(\gamma\) denote the smallest positive integer satisfying

\[
\frac{1}{\beta} + \frac{1}{\beta^\gamma} \leq 1.
\]

If \(i \geq j + \gamma, 0 < y \leq n_j \text{ and } n_j < x < n_{i+1}\), then

\[
\chi_{(y,n_k)}(x) - \chi_{(0,n_k)}(x) = 0
\]

unless \(k = i\) in which case

\[
\chi_{(y,n_k)}(x) - \chi_{(0,n_k)}(x) = \chi_{(n_i,y,n_i)}.\]

**Proof.** Since \((n_k)\) is a lacunary sequence there exists a constant \(\beta > 1\) such that

\[
\frac{n_{k+1}}{n_k} \geq \beta
\]

for all \(k\).

We can assume that

\[
\beta^2 \geq \frac{n_{k+1}}{n_k} \geq \beta
\]

(1)
for all $k$ by Remark 1. Since we have

$$\frac{n_l}{n_k} = \frac{n_l}{n_{l+1}} \cdot \frac{n_{l+1}}{n_{l+2}} \cdot \ldots \cdot \frac{n_{k-1}}{n_k}$$

and

$$\frac{1}{\beta} \leq \frac{n_k}{n_{k+1}} \leq \frac{1}{\beta^{k-l}}$$

for all $k$, we see that

$$\frac{1}{\beta^{2(k-l)}} \leq \frac{n_l}{n_k} \leq \frac{1}{\beta^{k-l}}$$

for all $k > l$.

Let $\gamma$ denote the smallest positive integer satisfying

$$\frac{1}{\beta} + \frac{1}{\beta^{\gamma}} \leq 1.$$

We see from (2) that

$$n_j + n_k \leq n_{k+1}$$

(3)

for all $k \geq j + \gamma - 1$.

It is easy to see that for $k > i$,

$$0 < y \leq n_j \leq n_i < x < n_{i+1} \leq n_k < y + n_k,$$

and this implies that

$$\left[ \chi(y,x+n_k)(x) - \chi(0,n_k)(x) \right] \cdot \chi(n_i,n_{i+1})(x) = 0.$$

For $k \leq i - 1$, we see by (3) that

$$n_k < y + n_k \leq n_j + n_{i-1} \leq n_i.$$

Then we have

$$\chi(y,x+n_k)(x) \cdot \chi(n_i,n_{i+1})(x) = \chi(0,n_k)(x) \cdot \chi(n_i,n_{i+1})(x) = 0.$$

Suppose now that $k = i$, by (3) we have

$$y < n_i < y + n_i \leq n_j + n_i \leq n_{i+1}$$

and this implies that

$$\chi(y,x+n_i)(x) - \chi(0,n_i)(x) = \chi(y,x+n_i) \cdot \chi(n_i,n_{i+1})(x) = \chi(n_i,y+n_i)(x).$$
Let 

\[ \phi_k(x) = \frac{1}{n_k} \chi_{(0,n_k)}(x) \]

and define the kernel operator \( K : \mathbb{R} \to \ell^s(\mathbb{Z}^+) \) as

\[ K(x) = \{ \phi_k(x) - \phi_{k-1}(x) \}_{k \in \mathbb{Z}^+}. \]

It is clear that

\[ V_s f(x) = \| K * f(x) \|_{\ell^s(\mathbb{Z}^+)} \]

\[ = \left( \sum_{k=1}^{\infty} |\phi_k * f(x) - \phi_{k-1} * f(x)|^s \right)^{1/s} \]

\[ = \left( \sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)|^s \right)^{1/s} \]

where \(*\) denotes convolution, i.e.,

\[ K * f(x) = \int K(x - y) \cdot f(y) \, dy. \]

Let \( B \) be a Banach space. We say that the \( B \)-valued kernel \( K \) satisfies \( D_r \) condition, for \( 1 \leq r < \infty \), and write \( K \in D_r \), if there exists a sequence \( \{ c_l \}_{l=1}^{\infty} \) of positive numbers such that \( \sum_l c_l < \infty \) and such that

\[ \left( \int_{S_l(|y|)} \| K(x - y) - K(x) \|_B^r \, dx \right)^{1/r} \leq c_l |S_l(|y|)|^{-1/r'}, \]

for all \( l \geq 1 \) and all \( y > 0 \), where \( S_l(|y|) \) denotes the spherical shell \( 2^l|y| < |x| < 2^{l+1}|y| \) and \( \frac{1}{r} + \frac{1}{r'} = 1. \)

When \( K \in D_1 \) we have the Hörmander condition:

\[ \int_{|x|>2|y|} \| K(x - y) - K(x) \|_B \, dx \leq C \]

where \( C \) is a positive constant which does not depend on \( y > 0 \).
Lemma 3. Let $\gamma$ denote the smallest positive integer satisfying $$\frac{1}{\beta} + \frac{1}{\beta^\gamma} \leq 1.$$ and let $1 \leq r, s < \infty$, $i \geq j + \gamma$, and $0 < y \leq n_j$. Then

$$\left( \int_{n_i}^{n_{i+1}} \|K(x - y) - K(x)\|_{\ell^r(\mathbb{Z}^+)}^r \, dx \right)^{1/r} \leq C_i n_i^{1/r - 1},$$

i.e., $K$ satisfies $D_r$ condition for $1 \leq r < \infty$.

Proof. Let $\Phi_k(x, y) = \phi_k(x - y) - \phi_k(x)$. Then it is easy to check that

$$K(x - y) - K(x) = \{\Phi_k(x, y) - \Phi_{k-1}(x, y)\}_{k \in \mathbb{Z}^+}.$$

On the other hand, because of a property of the norm we have

$$\|K(x - y) - K(x)\|_{\ell^r(\mathbb{Z}^+)} = \|\Phi_k(x, y) - \Phi_{k-1}(x, y)\|_{\ell^r(\mathbb{Z}^+)} \leq \|\Phi_k(x, y)\|_{\ell^r(\mathbb{Z}^+)} + \|\Phi_{k-1}(x, y)\|_{\ell^r(\mathbb{Z}^+)} \leq 2\|\Phi_{k-1}(x, y)\|_{\ell^r(\mathbb{Z}^+)},$$

where $x$ and $y$ are fixed and $\|\Phi_{k-1}(x, y)\|_{\ell^r(\mathbb{Z}^+)}$ is the $\ell^r(\mathbb{Z}^+)$-norm of the sequence whose $k^{\text{th}}$-entry is $\Phi_k(x, y)$. 

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We now have

\[
\left( \int_{n_i}^{n_{i+1}} \| K(x-y) - K(x) \|_{\ell^r(Z^+)}^r \right)^{1/r} \leq 2 \left( \int_{n_i}^{n_{i+1}} \| \Phi_{k-1}(x,y) \|_{\ell^r(Z^+)}^r \right)^{1/r} \\
\leq 2 \left( \int_{n_i}^{n_{i+1}} \| \Phi_{k-1}(x,y) \|_{\ell^1(Z^+)}^r \right)^{1/r} \\
= 2 \left( \int_{n_i}^{n_{i+1}} \left( \sum_{n_i < n_k - 1} \frac{1}{n_k-1} \chi_{(n_i,y+n_i)}(x) \right)^r \right)^{1/r} \\
= 2 \left( \int_{n_i}^{n_{i+1}} \left( \sum_{n_i < n_k - 1} \frac{1}{\beta^2(k-2)} \chi_{(n_i,y+n_i)}(x) \right)^r \right)^{1/r} \\
\leq 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \cdot \frac{1}{n_i} \cdot \left( \int_{n_i}^{n_{i+1}} \chi_{(n_i,y+n_i)}(x) \right)^r \right)^{1/r} \\
= 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \cdot \frac{1}{n_i} \cdot y^{1/r} \\
\leq 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \frac{1}{\beta(i-j)/r} n_i^{1/r-1}
\]

where in the last inequality we used

\[ y \leq n_j \leq n_i \frac{i-j}{\beta i-j} \]

by (2), and this completes our proof with

\[ C_i = 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \frac{1}{\beta(i-j)/r} \]

\[ \Box \]

**Lemma 4.** Let \( \{n_k\} \) be a lacunary sequence then there exists a constant \( C > 0 \) such that

\[ \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| < C \]

for all \( x \in \mathbb{R} \), where \( \phi_k(x) = \frac{1}{n_k} \chi_{(0,n_k)}(x) \), and \( \hat{\phi}_k \) is its Fourier transform.
Proof. First note that we have
\[ I(x) = \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| = \sum_{k=1}^{\infty} \left| \frac{1-e^{-ixn_k}}{xn_k} - \frac{1-e^{-ixn_{k-1}}}{xn_{k-1}} \right|. \]

Let
\[ I(x) = \sum_{\{k:|x|n_k \geq 1\}} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| + \sum_{\{k:|x|n_k < 1\}} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| = I_1(x) + I_2(x). \]

Let us now fix \( x \in \mathbb{R} \) and let \( k_0 \) be the first \( k \) such that \( |x|n_k \geq 1 \). Since \( \hat{\phi}_k(x) \) is an even function we can assume without the loss of generality that \( x \geq 0 \).

We clearly have
\[ I_1(x) \leq \sum_{\{k:|x|n_k \geq 1\}} \frac{4}{|x|n_k}. \]

Since the sequence \( \{n_k\} \) is lacunary there exists a constant \( \beta > 1 \) such that
\[ \frac{n_{k+1}}{n_k} \geq \beta \]
for all \( k \in \mathbb{N} \). Also note that in summation, \( I_1 \), the term with index \( n_{k_0} \) is the term with smallest index since it is the first term satisfying condition \( |x|n_k \geq 1 \) and the sequence \( \{n_k\} \) is increasing. On the other hand, we have
\[ \frac{n_{k_0}}{n_k} = \frac{n_{k_0}}{n_{k+1}} \cdot \frac{n_{k_0+1}}{n_{k_0+2}} \cdot \frac{n_{k_0+2}}{n_{k_0+3}} \cdots \frac{n_{k-1}}{n_k} \leq \frac{1}{\beta^k}. \]

We now have
\[ I_1(x) \leq \sum_{\{k:|x|n_k \geq 1\}} \frac{4}{|x|n_k} = \sum_{\{k:|x|n_k \geq 1\}} \left[ \frac{4n_{k_0}}{|x|n_{k_0}n_k} \right] \leq \frac{4}{|x|n_{k_0}} \sum_{\{k:|x|n_k \geq 1\}} \frac{n_{k_0}}{n_k} \leq \frac{4}{|x|n_{k_0}} \sum_{\{k:|x|n_k \geq 1\}} \frac{1}{\beta^k}. \]
since $\frac{1}{|x|n_{k_0}} \leq 1$ and $n_{k_0} = \frac{1}{\beta x}$. Also, since
\[
\sum_{k=1}^{\infty} \frac{1}{\beta^k} = \frac{1}{1 - \frac{1}{\beta}}
\]
we clearly see that
\[
I_1(x) \leq C_1
\]
for some constant $C_1 > 0$.

To control the summation $I_2$ let us first define the function $F$ as
\[
F(r) = \frac{1 - e^{-ir}}{r},
\]
then we have $\tilde{\phi}_k(x) = F(xn_k)$. Now by the Mean Value Theorem there exists a constant $\xi \in (xn_k, xn_{k+1})$ such that
\[
|F(xn_{k+1}) - F(xn_k)| = |F'(\xi)||xn_{k+1} - xn_k|.
\]
Also, it is easy to verify that
\[
|F'(x)| \leq \frac{x + 2}{x^2}
\]
for $x > 0$.

Now we have
\[
|F(xn_{k+1}) - F(xn_k)| = |F'(\xi)||xn_{k+1} - xn_k|
\leq \frac{\xi + 2}{\xi^2} |x|(n_{k+1} - n_k)
\leq \frac{xn_{k+1} + 2}{x^2 n_k^2} |x|(n_{k+1} - n_k)
= \frac{2n_{k+1}}{n_k^2} (n_{k+1} - n_k).
Thus we have

\[ I_2(x) = \sum_{\{k: |x| n_k < 1\}} |F(xn_{k+1}) - F(xn_k)| \]

\[ \leq \sum_{\{k: |x| n_k < 1\}} \frac{2}{|x|n_k} \cdot \frac{2n_{k+1}}{n_k^2} (n_{k+1} - n_k) \]

\[ \leq \sum_{\{k: |x| n_k < 1\}} \frac{4n_{k+1}^2}{n_k^2 |x|} \left( \frac{1}{n_k} - \frac{1}{n_{k+1}} \right) \]

\[ = \sum_{\{k: |x| n_k < 1\}} \frac{16}{|x|} \left( \frac{1}{n_{k+1}} - \frac{1}{n_k} \right) \]

\[ \leq \frac{16}{|x| n_{k+1}} \]

\[ \leq 16. \]

We thus conclude that

\[ I(x) = I_1(x) + I_2(x) \leq C_1 + 16 = C \]

for all \( x \in \mathbb{R} \) and this completes our proof. \( \square \)

**Lemma 5.** Let \( s \geq 2 \) and \((n_k)\) be a lacunary sequence. Then there exists a constant \( C > 0 \) such that

\[ \|V_s f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})} \]

for all \( f \in L^2(\mathbb{R}) \).

**Proof.** Since

\[ \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 \leq \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|, \]

it is clear from Lemma 4 that there exists a constant \( C > 0 \) such that

\[ \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 < C \]
for all $x \in \mathbb{R}$.

We now obtain

$$\|V_s f\|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left( \sum_{k=1}^{\infty} |\phi_k * f(x) - \phi_{k-1} * f(x)|^\rho \right)^{2/\rho} dx$$

$$\leq \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\phi_k * f(x) - \phi_{k-1} * f(x)|^2 dx$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\phi_k * f(x) - \phi_{k-1} * f(x)|^2 dx$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |(\phi_k - \phi_{k-1}) * f(x)|^2 dx$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\Delta_k * f(x)|^2 dx \quad (\Delta_k(x) = \phi_k(x) - \phi_{k-1}(x))$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\hat{\Delta_k}(x)|^2 \cdot |\hat{f}(x)|^2 dx$$

$$= \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\hat{\Delta_k}(x)|^2 \cdot |\hat{f}(x)|^2 dx$$

$$= \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\hat{\phi_k}(x) - \hat{\phi}_{k-1}(x)|^2 \cdot |\hat{f}(x)|^2 dx$$

$$\leq C \int_{\mathbb{R}} |\hat{f}(x)|^2 dx$$

$$= C \int_{\mathbb{R}} |f(x)|^2 dx \quad \text{(by Plancherel’s theorem)}$$

$$= C \|f\|_{L^2(\mathbb{R})}^2$$

as desired. \qed

**Remark 2.** Since for $s \geq 2$, we have proved in Lemma 3 that the kernel operator $K(x) = \{\phi_k(x) - \phi_{k-1}(x)\}_{k \in \mathbb{Z}}$ satisfies $D_r$ condition for $1 \leq r < \infty$, it specifically satisfies $D_1$ condition. We also have proved in Lemma 5 that
$Tf = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}^+}$ is a bounded operator from $L^2(\mathbb{R})$ to $L^2_{L^s(\mathbb{Z}^+)}(\mathbb{R})$ since $\|K * f(x)\|_{L^p(\mathbb{Z}^+)} = V_s f(x)$. Therefore, $Tf = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}^+}$ is an $\ell^s$-valued singular operator of convolution type for $s \geq 2$.

**Lemma 6.** A singular integral operator $T$ mapping $A$-valued functions into $B$-valued functions can be extended to an operator defined in all $L^p_A$, $1 \leq p < \infty$, and satisfying

(i) $\|Tf\|_{L^p_B} \leq C_p \|f\|_{L^p_A}$, \quad $1 < p < \infty$,
(ii) $\|Tf\|_{W^{1,p}_B} \leq C_1 \|f\|_{L^p_A}$,
(iii) $\|Tf\|_{L^1_B} \leq C_2 \|f\|_{H^{1}_A}$,
(iv) $\|Tf\|_{\text{BMO}(B)} \leq C_3 \|f\|_{L^\infty(A)}$, \quad $f \in L_c^\infty(A)$,

where $C_p, C_1, C_2, C_3 > 0$, and $L_c^\infty(A)$ is the space of bounded functions with compact support.

**Proof.** This is Theorem 1.3 of Part II in J. L. Rubio de Francia et al [5].

The following theorem is our first result:

**Theorem 1.** Let $2 \leq s < \infty$, and $(n_k)$ be a lacunary sequence. Then there exists a constant $C > 0$ such that

$$\|V_s f\|_{L^1(\mathbb{R})} \leq C \|f\|_{H^1(\mathbb{R})}$$

for all $f \in H^1(\mathbb{R})$.

**Proof.** This follows from Remark 2 and Lemma 6 (iii) since $\|K * f(x)\|_{L^p(\mathbb{Z}^+)} = V_s f(x)$.

**Remark 3.** We have proved that $Tf = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}^+}$ is an $\ell^s$-valued singular operator of convolution type for $s \geq 2$. By applying Lemma 6 to this observation we also provide a different proof for the following known facts for $s \geq 2$ (see [4]) since $\|K * f(x)\|_{L^p(\mathbb{Z}^+)} = V_s f(x)$.

(i) $\|V_s f\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}$, \quad $1 < p < \infty$,
(ii) $\|V_s f\|_{W^{1,p}(\mathbb{R})} \leq C_1 \|f\|_{L^1(\mathbb{R})}$,
(iii) $\|V_s f\|_{\text{BMO}(\mathbb{R})} \leq C_2 \|f\|_{L^\infty(\mathbb{R})}$, \quad $f \in L_c^\infty(\mathbb{R})$, \quad $f \in L_c^\infty(\mathbb{R})$, \quad $f \in L_c^\infty(\mathbb{R})$.
where \( C_p, C_1, C_2 > 0 \).

Let \( w \in L^1_{\text{loc}}(\mathbb{R}) \) be a positive function. We say that \( w \) is an \( A_p \) weight for some \( 1 < p < \infty \) if the following condition is satisfied:

\[
\sup_I \left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty,
\]

where the supremum is taken over all intervals \( I \) in \( \mathbb{R} \).

We say that the function \( w \) is an \( A_\infty \) weight if there exist \( \delta > 0 \) and \( \epsilon > 0 \) such that given an interval \( I \) in \( \mathbb{R} \), for any measurable \( E \subset I \),

\[
|E| < \delta \cdot |I| \implies w(E) < (1 - \epsilon) \cdot w(I).
\]

Here

\[
w(E) = \int_E w.
\]

It is well known and easy to see that \( w \in A_p \implies w \in A_\infty \) if \( 1 < p < \infty \).

We say that \( w \in A_1 \) if given an interval \( I \) in \( \mathbb{R} \) there is a positive constant \( C \) such that

\[
\frac{1}{|I|} \int_I w(y) \, dy \leq C w(x)
\]

for a.e. \( x \in I \).

**Lemma 7.** Let \( A \) and \( B \) be Banach spaces, and \( T \) be a singular integral operator mapping \( A \)-valued functions into \( B \)-valued functions with kernel \( K \in D_r \), where \( 1 < r < \infty \). Then, for all \( 1 < \rho < \infty \), the weighted inequalities

\[
\left\| \left( \sum_j \|Tf_j\|_B^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{p,\rho}(w) \left\| \left( \sum_j \|f_j\|_A^\rho \right)^{1/\rho} \right\|_{L^p(w)}
\]

hold if \( w \in A_{p/r} \) and \( r' \leq p < \infty \), or if \( w \in A_{p'} \) and \( 1 < p \leq r' \). Likewise, if \( w(x)^{r'} \in A_1 \), then the weak type inequality

\[
w \left( \left\{ x : \left( \sum_j \|Tf_j(x)\|_B^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_\rho(w) \frac{1}{\lambda} \int \left( \sum_j \|f_j(x)\|_A^\rho \right)^{1/\rho} w(x) \, dx
\]

holds.
Proof. This is Theorem 1.6 of Part II in J. L. Rubio de Francia et al [5].

Our next result is the following:

**Theorem 2.** Let \(2 \leq s < \infty\). Then, for all \(1 < \rho < \infty\), the weighted inequalities

\[
\left\| \left( \sum_j (V_s f_j)^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{p,\rho}(w) \left\| \left( \sum_j |f_j|^\rho \right)^{1/\rho} \right\|_{L^p(w)}
\]

hold if \(w \in A_{p/r'}\) and \(r' \leq p < \infty\), or if \(w \in A_p^r\) and \(1 < p \leq r'\). Likewise, if \(w(x)^{r'} \in A_1\), then the weak type inequality

\[
w \left( \left\{ x : \left( \sum_j (V_s f_j(x))^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_{\rho}(w) \frac{1}{\lambda} \int \left( \sum_j |f_j(x)|^\rho \right)^{1/\rho} w(x) \, dx
\]

holds.

Proof. We have proved for \(2 \leq s < \infty\) that \(Tf = \{ (\phi_k - \phi_{k-1}) * f \}_{k \in \mathbb{Z}^+}\) is an \(\ell^s\)-valued singular integral operator of convolution type and its kernel operator \(K(x) = \{ \phi_k(x) - \phi_{k-1}(x) \}_{k \in \mathbb{Z}^+}\) satisfies \(D_r\) condition for \(1 \leq r < \infty\). Thus the result follows from Lemma 7 and the fact that \(\|K * f(x)\|_{L^s(\mathbb{Z}^+)} = V_s f(x)\).

In particular we have the following corollary:

**Corollary 3.** Let \(2 \leq s < \infty\). Then the weighted inequalities

\[
\| V_s f \|_{L^p(w)} \leq C_{p,\rho}(w) \| f \|_{L^p(w)}
\]

hold if \(w \in A_{p/r'}\) and \(r' \leq p < \infty\), or if \(w \in A_p^r\) and \(1 < p \leq r'\). Likewise, if \(w(x)^{r'} \in A_1\), then the weak type inequality

\[
w \left( \{ x : V_s f(x) > \lambda \} \right) \leq C_{\rho}(w) \frac{1}{\lambda} \int |f(x)| w(x) \, dx
\]

holds.
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