SHORT-RANGE ENTANGLEMENT AND INVERTIBLE FIELD THEORIES

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ABSTRACT. Quantum field theories with an energy gap can be approximated at long-range by topological quantum field theories. The same should be true for suitable condensed matter systems. For those with short range entanglement (SRE) the effective topological theory is invertible, and so amenable to study via stable homotopy theory. This leads to concrete topological invariants of gapped SRE phases which are finer than existing invariants. Computations in examples demonstrate their effectiveness.

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1. Introduction

The long-range behavior of gapped systems in condensed matter physics is accessible via topology. For noninteracting fermionic systems there is a classification of topological phases using ideas related to $K$-theory [K1]. Over the past few years the interacting case has been vigorously studied, for both fermionic systems and bosonic systems, with an emphasis on short-range entanglement (SRE); a small sampling of papers is [CGW1, CGW2, CGLW, GW, LV, VS, We, PMN, CFV, WPS, HW, WS, Ka1, Ka2, WGW, KTTW]. Particular symmetry protected topological (SPT) phases are captured by group cohomology [CGLW, GW], but other investigations (e.g. [VS]) reveal the existence of additional SRE phases and raise the question of a complete classification. In this paper we propose an invariant of bosonic and fermionic SRE topological phases constructed from effective field theory. Computations and examples demonstrate that it effectively detects known SRE phases.

Our proposal applies to gapped systems which at low energy (long time) can be approximated by topological field theories. Such a system must be sufficiently local that it can be formulated on arbitrary manifolds, and it must have a continuum limit which is a field theory, at least at low energy. We do not investigate microscopic behavior at all in this paper, but rather simply assume the existence of a long-range topological theory.\textsuperscript{1} Short range entanglement, or the absence of topological order, is a microscopic assumption. Kitaev [K2, K3, K4] has been studying SRE phases from first principles microscopically,\textsuperscript{2} and has suggested the macroscopic consequence that the long-range topological theory has a unique vacuum on any background manifold. We go further and assume that the long-range topological field theory describing an SRE phase is fully extended and invertible, concepts that we explain below. From mathematical investigations it has been known for a long time that fully extended, invertible, topological field theories are equivalent to maps between spectra in the sense of algebraic topology. This link with stable homotopy theory is the basis of our proposal.

\textsuperscript{1}Rather than a single long-range approximation, we envision a connected space of long-range theories.

\textsuperscript{2}We remark that there is an alternative microscopic definition of SRE proposed by Chen-Gu-Wen [CGW1].
While our proposal in §5.2 is specific and precise, we neither formulate nor prove a mathematical theorem which justifies it. Also, while we enumerate groups which should house invariants of SRE phases, we do not argue either that the effective field theory is a complete invariant or that every possible effective field theory is realized by a microscopic system. In place of proof the paper marshals evidence in two stages. Pre-§5.2 is a long conceptual march leading to the proposal. Post-§5.2 is a series of experimental checks, including the relationship to group cohomology, boundary terminations, and specific computations. We include a long discussion in §6.3 about detecting Kitaev’s $E_8$ phase using invertible topological field theories. There are additional possible effective field theories which are “4th roots”, perhaps an indication that not all possible effective theories are realized by microscopic systems. Another example, the “3d bosonic $E_8$ phase with half-quantized surface thermal Hall effect” [VS], [BCFV], is also treated in detail.

It may be useful to broadly characterize our proposal in field-theoretic language: whereas the group cohomology captures pure gauge theories, the additional SRE phases contain couplings to gravity or are purely gravitational. The SPT phases have no purely gravitational component. More fundamentally, the long-range field theory is envisioned as the low-energy behavior of the coupling to gravity of the original system. (And, if there are global symmetries, we gauge them and so couple to gauge theory too.)

Bordism as a tool to classify SPT phases appears differently in the recent papers of Kapustin [Ka1], [Ka2], [KTTW]. In unpublished work Kitaev [K2, K3, K4] develops a classification of SRE phases based on microscopic considerations. Their results and approach differ from ours, and it will be very interesting to reconcile them. It may be that there is more microscopic information in the physics which leads to different or additional input into the effective field theories. In particular, there are a few ingredients in our proposal (choice of tangential structure, choice of target spectrum) which involve leaps of faith and can easily be adjusted if further microscopic implications are discovered.

We begin in §2 with an exposition of several formal points in field theory: extended field theory, invertible field theory, relative field theory, anomalies, global symmetries and gauging, topological field theory, unitarity. While many of these concepts are familiar to physicists, the mathematical language may be unfamiliar and we hope to provide some bridge here. We touch on the cobordism hypothesis, which classifies fully extended topological theories, in §2.6; the precise nature of this classification is illuminated in §2.8 with a toy example in preparation for a later discussion of the “Kitaev $E_8$ phase” in §6.3. But the cobordism hypothesis is overkill for invertible theories. In §2.7 we describe the link between fully extended, invertible, topological field theories and stable homotopy theory in general terms. That involves particular Madsen-Tillmann spectra, which are unstable analogs of Thom’s bordism spectra; we describe them in §4. There we also discuss the crucial theorem of Galatius-Madsen-Tillmann-Weiss [GMTW] which identifies these spectra as geometric realizations of bordism categories. That these unstable bordism spectra are appropriate to field theories is natural since field theories are dimension-specific. The material in §2 and §4 is general background not particular to condensed matter systems.

In §3 we give much of the general argument about effective field theories for SRE topological phases. We begin in §3.1 with elementary thoughts indicating why topology may sufficiently describe the low energy behavior of gapped systems. Most of our assumptions are stated explicitly in §3.2. There is still conceptual work to translate those assumptions into concrete mathematical statements. Specifically, once we know that SRE gapped phases give rise to invertible topological
there are still parameters to choose: the tangential structure on manifolds representing space and the target category for the field theory. The latter is discussed in §5.1 separately for bosonic and fermionic theories; the bosonic case is a bit surprising and we settle on a kludge for the target spectrum. For the tangential structures we assume without much justification that in bosonic theories the space manifolds are oriented and in fermionic theories they are spin. Along the way we encounter a few tricky issues, for example the gauging of antilinear symmetries (§§2.4.3, 5.1.4) and implementation of unitarity (§4.2.5).

We state our proposal in §5.2. We divide theories into bosonic and fermionic. Also, symmetries may be anomalous and we propose a classification of anomaly theories and anomalous gauged theories as well. Possible effective invertible topological theories for SRE phases with fixed symmetry form an abelian group; for SPT phases they form a subgroup which we also delineate.

Our first deduction in §5.3 from the proposal is that the phases previously identified using group cohomology are included. In §6.1 we show that for bosonic theories in $d = 1$ space dimensions group cohomology provides a complete classification, which agrees with known results [CLW]. Already for fermionic theories in $d = 1$ the situation is more interesting, as described in §6.2: we detect the Majorana chain [K6] in our classification. In §6.3 we identify bosonic $d = 2$ SRE phases. These were introduced by Kitaev [K5], [K2] and are related to 2-spacetime\(^3\) dimensional chiral conformal field theories whose chiral central charge is an integer divisible by 8. These central charges do not show up in the usual account of the associated 3-spacetime dimensional topological field theory; there only the reduction mod 8 is used. Our explanation of how they fit in here, and so the role of the chiral central charge as a real number not taken mod 8, is based on the easier examples discussed in §2.8. In §6.4 we illustrate a constraint imposed by unitarizability. In §6.5 we compute that the abelian group of $d = 3$ bosonic time-reversal symmetric effective SRE field theories is isomorphic to $\mathbb{Z}/2\mathbb{Z})^2$. One generator is accounted for by group cohomology, and we claim the other is the 3d bosonic $E_8$ phase with half-quantized surface thermal Hall effect mentioned earlier. Finally, in §7 we give a general discussion of boundary conditions/terminations/excitations and use it to justify the aforementioned claim. An appendix includes topological computations which are needed in the text.

The notion of a non-extended invertible field theory arose in joint work with Greg Moore [FM1, §5.5]. Fully extended invertible topological theories, and the relation to stable homotopy theory, has been a longstanding discussion topic with Mike Hopkins and Constantin Teleman, as have many other general ideas described in §2. In particular, we used these ideas in [FHT] to construct a topological field theory based on the Verlinde ring. The specific application to SRE phases described here crystallized during the Symmetry in Topological Phases workshop in Princeton, and I thank the organizers for inviting me. I had long conversations with Alexei Kitaev after a first draft of this paper was complete, and those inspired a significant modification of §5.1.2 and §6.3. I thank him for sharing his perspectives. I also thank Zheng-Cheng Gu, Mike Hopkins, Anton Kapustin, Constantin Teleman, Ashvin Vishwanath, Kevin Walker, Oscar Randal-Williams, and Xiao-Gang Wen for very helpful conversations and correspondence.

\(^3\)The number $d$ above is the dimension of space; spacetime has dimension $d + 1$. 
2. Field theories from a bordism point of view

We begin with a formal viewpoint on the structure of a field theory, which is the lens through which we analyze the long-range effective topological theory in §3. In the mathematics literature this approach was abstracted in Segal’s axioms for two-dimensional conformal field theory [S1] and in Atiyah’s axioms for topological field theories [A1]. The lectures [S2] treat general quantum field theories from this perspective. Our focus in this paper is on invertible field theories, which we define in §2.2. Other general topics we quickly review include extended field theories, relative field theories, anomalies, gauging symmetries, and unitarity. We then focus on fully extended topological theories, for which the powerful cobordism hypothesis [BD, L, F1] provides a classification result. Invertible topological theories can be analyzed using homotopy theory, and in this section we explain why that is true but defer a more precise description to §4. We conclude with a few toy examples which illuminate subtleties we will encounter in the condensed matter systems of §6. The subtleties discussed there may have broader interest. There are many expositions of this material, in addition to the ones referenced earlier in this paragraph, and here we offer another. A somewhat different point of view on topological field theories may be found in [MW]. The reader may wish to use this section for reference and skip on first reading to later parts of the paper.

We remark that this formal viewpoint does not distinguish “classical” from “quantum”, and indeed we will give examples of both types. Another remark is that the field theories which arise in condensed matter physics are usually defined on spacetimes which are products of space and time, whereas the discussion in this section models theories defined on more general spacetimes. We discuss the necessary modification in §4.2 and account for it in the proposals of §5.2.

2.1. Field theories

\[
\begin{array}{ccc}
\bullet & \rightarrow & \mathcal{H} \\
\downarrow & t & \\
\downarrow & & \mathcal{H} \\
\downarrow & & U_t: \mathcal{H} \rightarrow \mathcal{H} \\
\downarrow & t_1 & \\
\downarrow & t_2 & \\
\downarrow & t_{1} + t_2 & \\
\hline
\end{array}
\]  

\[
U_{t_1 + t_2} = U_{t_1} \circ U_{t_2}
\]

Figure 1. Quantum mechanical evolution

Let \( n \) be the \textit{spacetime} dimension of a field theory \( F \), which we simply call the dimension of \( F \). We write \( n = d + 1 \) where \( d \) is the space dimension.\(^4\) The case \( n = 1 \) (\( d = 0 \)) is mechanics; there is only time. A quantum mechanical system assigns a complex vector space \( \mathcal{H} \) (the ‘quantum Hilbert space’) to a point and the time evolution \( U_t: \mathcal{H} \rightarrow \mathcal{H} \) to a closed interval of length \( t \). The group law \( U_{t_1 + t_2} = U_{t_1} \circ U_{t_2} \) is encoded by gluing intervals, as illustrated in Figure 1. An \( n \)-dimensional

\(^4\)In the condensed matter literature \( d \) is often called the dimension of the theory, whereas in quantum field theory and string theory literature it is \( n \) which is the dimension. We use the terms ‘spacetime dimension’ and ‘space dimension’ to avoid confusion.
Euclidean field theory assigns a *partition function* \( F(X) \in \mathbb{C} \) to a compact \( n \)-dimensional manifold \( X \) with no boundary. There is a complex vector space \( F(Y) \) for each compact \((n-1)\)-manifold, thought of as a spatial slice, and now this ‘quantum Hilbert space’ may depend on \( Y \). Roughly speaking, the vector space \( F(S^{n-1}) \) attached to a small sphere \( S^{n-1} \) is the space of local operators, and to a closed manifold \( X \) with \( k \) small open balls removed we attach the *correlation functions*

\[
(2.1) \quad F(X \setminus \bigcup_{i=1}^{k} B^n_\alpha) : F(S^{n-1}) \otimes \cdots \otimes F(S^{n-1}) \to \mathbb{C},
\]

as illustrated in Figure 2. Here all boundary components are “incoming”, whereas each interval in Figure 1 has an incoming boundary component and an outgoing boundary component, each a single point. There are analogous quantum evolution operators in any dimension for manifolds with both incoming and outgoing components, and the group law of quantum mechanics has a generalization.

![Figure 2. Correlation functions](image)

The mathematical expression of this formal structure is the assertion that

\[
(2.2) \quad F : \text{Bord}_{(n-1,n)} \to \text{Vect}
\]

is a homomorphism, or *functor*, between *symmetric monoidal categories*. The bordism category \( \text{Bord}_{(n-1,n)} \) consists of closed\(^5\) \((n-1)\)-manifolds and bordisms between them. A bordism \( X : Y_0 \to Y_1 \) is a compact \( n \)-manifold with boundary, the boundary has a continuous partition \( p : \partial X \to \{0, 1\} \) which divides it into incoming and outgoing components, and there are diffeomorphisms \( Y_0 \xrightarrow{\cong} (\partial X)_0 \) and \( Y_1 \xrightarrow{\cong} (\partial X)_1 \); see Figure 3 in which one should view time as flowing from left to right.\(^6\)

The target category has complex vector spaces as objects and linear maps as morphisms.\(^7\)

There are two kinds of composition. The internal composition glues morphisms (Figure 4) and the external composition is disjoint union. Similarly \( \text{Vect} \) has two composition laws: the internal composition is the usual composition of linear maps and the external composition is tensor product. The homomorphism \( (2.2) \) is required to preserve both composition laws. This formulation is rather compact and one must unpack it to see the usual structures in field theory.

---

\(^5\)A manifold is *closed* if it is compact without boundary.

\(^6\)In that figure the diffeomorphisms map open collar neighborhoods. This guarantees that the composition, or gluing, operation on bordisms yields smooth manifolds.

\(^7\)One should use *topological* vector spaces and *continuous* linear maps. The topology on the vector spaces is not relevant for topological field theories since the vector spaces are finite dimensional so have a unique linear topology.
Typically one does not have bare $n$-manifolds, but rather each $n$-manifold $X$ is endowed with a space\footnote{Some fields, such as gauge fields, have internal symmetries, and they form a stack rather than a space.} $\mathcal{F}(X)$ of fields. For example, in quantum mechanics (Figure 1) the 1-manifolds have a Riemannian metric: the total length represents time. Higher dimensional field theories are often formulated on Riemannian manifolds, though conformal field theories only require a conformal structure. Other possible fields include orientations, spin structures, scalar fields, spinor fields, etc.

There is a bordism category $\text{Bord}_{n,n}$ of manifolds equipped with a specified collection of fields, and a functor

\begin{equation}
F : \text{Bord}_{n,n} \to \text{Vect}
\end{equation}

represents a field theory with $\mathcal{F}$ as the set of (background) fields.

**Example 2.4.** To illustrate the notation, consider $n = 3$ spacetime dimensional Chern-Simons theory with gauge group $\mathbb{T} = U(1)$. There is a classical and a quantum theory. The fields $\mathcal{F}(X)$ in the classical theory on a 3-manifold $X$ consist of an orientation $o$ and a principal $\mathbb{T}$-bundle with connection $A$. A closed 3-manifold $X$ appears in the bordism category $\text{Bord}_{(2,3)}$ as a morphism $X : \varnothing^2 \to \varnothing^2$ from the empty 2-manifold to itself. We have $F(\varnothing^2) = \mathbb{C}$ and so $F(X) : \mathbb{C} \to \mathbb{C}$ is
multiplication by a complex number, and if we make the fields explicit we denote it as \( F(X; o, A) \).

It is given by the formula

\[
F(X; o, A) = \exp \left( -\frac{i}{2\pi} \int_{X,o} A \wedge dA \right)
\]

where implicitly, since \( A \) is a 1-form on the total space of a \( T \)-bundle, we have used a global section to pull it down to the base \( X \).

The orientation is used to define integration of differential forms. An oriented 2-manifold \( Y \) with \( T \)-connection has an attached Chern-Simons line \( F(Y; o, A) \), and a bordism with fields has a relative Chern-Simons invariant mapping between the Chern-Simons lines of the boundaries. This theory is invertible in the sense described in §2.2.

In the quantum theory \([W1]\) one integrates over the gauge field \( A \), and to get a well-defined 3-dimensional theory one needs in addition to the orientation \( o \) a field \( f \) which is a certain sort of “framing” called a \( p_1 \)-structure. The quantum invariant \( F(X; o, f) \in \mathbb{C} \) of a closed 3-manifold is an arbitrary complex number—for example, it vanishes for some manifolds—and the quantum vector spaces \( F(Y; o, f) \) do not necessarily have dimension one. So the quantum theory is not generally invertible.

Remark 2.6. This description of a field theory has an important deficiency: it does not encode smooth dependence on parameters. For example, the partition function on a closed \( n \)-manifold \( X \) must depend smoothly on the background fields in \( \mathcal{F}(X) \). (Example: the smooth dependence of the partition function of 2-dimensional Yang-Mills theory on the area of a surface.) Formally, the definitions are enlarged to include fiber bundles \( X \to S \) of \( n \)-manifolds with fields, and these must map to smoothly varying linear maps of smooth vector bundles over \( S \). For topological theories, which are our main concern, instead of families one usually postulates instead that the morphism sets in the categories \( \text{Bord}_{(n-1,n)}(\mathcal{F}) \) and \( \text{Vect} \) have a topology and all maps are continuous. There are two obvious topologies on the set of linear maps \( \text{Hom}(V_0, V_1) \) between two finite dimensional vector spaces, equivalently on the set of \( n \times m \) matrices. We can use the usual topology induced from the usual topology on the real numbers, or we can use the discrete topology. Both are used in the classification scheme of §5.2.

Remark 2.7. One should also allow smooth families of field theories \( F \). For topological field theories it is more natural to study continuous families, i.e., to form a topological space of topological field theories. This is crucial for the classification of gapped topological phases: if two gapped systems are connected by a continuous path, we expect their effective topological field theories can also be joined by a continuous path.

Finally, in this paper we will always consider fully extended field theories, usually topological. An \( n \)-dimensional fully extended theory assigns invariants to manifolds of all dimensions \( \leq n \). These compact manifolds with fields, which are now allowed corners, are organized into an algebraic structure called a symmetric monoidal \( (\infty, n) \)-category, denoted \( \text{Bord}_n(\mathcal{F}) \). The target for an extended field theory is a symmetric monoidal \( (\infty, n) \)-category \( \mathcal{C} \), which typically has its \( (n-1)^{st} \) loop space isomorphic to \( \text{Vect} \). If so, then an extended field theory

\[
F: \text{Bord}_n(\mathcal{F}) \to \mathcal{C}
\]

\(^9\)If a section does not exist, there is a more complicated definition.
restricts on \((n-1)\)- and \(n\)-manifolds to a usual field theory \((2.3)\). As already indicated, the
invariants attached to manifolds of dimension \(\leq n-2\) tend to be categorical in nature. A theory
which extends in this way is fully local, and it is natural to make this strong locality hypothesis for
the effective topological theory which comes from a gapped physical theory. See \([L]\) for a modern
description of fully extended topological field theories.

2.2. Invertible field theories

‘Invertibility’ refers to the tensor product operation on vector spaces and linear maps. First, \(\mathbb{C}\)
is a “unit element” for tensor product in the sense that for any vector space \(V\) we have an
isomorphism \(\mathbb{C} \otimes V \cong V\) which is naturally defined. Thus we call \(\mathbb{C}\) a tensor unit. A complex
vector space \(V\) is invertible if there exists a vector space \(V’\) and an isomorphism \(V \otimes V’ \cong \mathbb{C}\).
Since \(\dim(V \otimes V’) = (\dim V)(\dim V’),\) it follows immediately that if \(V\) is invertible, then \(\dim V = 1\).
Conversely, if \(V\) is 1-dimensional then \(V \otimes V^*\) is isomorphic to \(\mathbb{C}\). Thus the invertible vector spaces
are precisely the lines.

Invertibility of a linear map \(T: V_0 \rightarrow V_1\) under tensor product is equivalent to the usual definition
of invertibility, namely that there exist \(S: V_1 \rightarrow V_0\) such that the compositions \(S \circ T\) and \(T \circ S\)
are identity maps. If \(L\) is a line, then a linear map \(\lambda: L \rightarrow L\) is multiplication by a complex
number \(\lambda \in \mathbb{C}\) and it is invertible if and only if \(\lambda \neq 0\). We denote the nonzero complex numbers
as \(\mathbb{C}^\times\).

Invertible complex vector spaces and invertible linear maps between them form a subcategory
Line \(\subset\) Vect which by definition has the property that all morphisms are invertible. Such a category
is called a groupoid.

An invertible field theory \(F: \text{Bord}_n(\mathcal{F}) \rightarrow\) Vect is one for which all vector spaces \(F(Y)\) and
linear maps \(F(X)\) are invertible.\(^{10}\) We can express that by saying that \(F\) factors through a functor
\(\text{Bord}_n(\mathcal{F}) \rightarrow\) Line. Thus all quantum Hilbert spaces are one-dimensional and all propagations
are invertible. The tensor product of invertible theories is invertible, so (isomorphism classes of or
deformation classes of) invertible theories form an abelian group.

Example 2.9. Here is a simple example with \(n = 1\). Fix a smooth manifold \(M\) of any dimension
and a smooth complex line bundle \(L \rightarrow M\) with connection. We define a 1-dimensional field theory
whose set of fields \(\mathcal{F}(X)\) on a 1-manifold \(X\) consists of a pair \((o, \phi)\) of an orientation \(o\) and a smooth
map \(\phi: X \rightarrow M\). Then to \(Y = pt_+\) a point with the positive orientation and \(\phi(pt) = m \in M\) we set
\(F(Y) = L_m\) to be the fiber of the line bundle \(L \rightarrow M\) at \(m\). To \(X = [0,1]\) with the usual
orientation and a map \(\phi: [0,1] \rightarrow M\) we assign the parallel transport \(F(X): L_{\phi(0)} \rightarrow L_{\phi(1)}\) along
the path \(\phi\). The reader can easily work out the values of \(F\) on other manifolds. We can make
a family of such field theories by varying the line bundle \(L \rightarrow M\) and its connection. The path
components of this family of field theories are parametrized by the topological equivalence classes

\(^{10}\)For a fully extended invertible field theory the value of \(F\) on any manifold of dimension \(\leq n\) is invertible under
the symmetric monoidal product of the target. A theorem of the author and Constantin Teleman asserts that for
oriented theories if the number \(F(S^n)\) is nonzero and the vector spaces \(F(S^p \times S^{n-1-p})\) are one-dimensional, then
\(F\) is invertible.
of line bundles $L 	o M$ (without connection.) In §4 and §5 we will learn that the set of path components can be computed by stable homotopy theory.\footnote{The computation: $[\Sigma^1 MTSO_1 \wedge M +, \Sigma^2 HZ] \equiv H^2(M; \mathbb{Z}).$ In all computations $[X, Y]$ denotes pointed homotopy classes of maps between the pointed spaces $X, Y.$}

**Example 2.10.** Continuing with $n = 1$ we now take the line bundle to be one of the fields, rather than being pulled back from an external manifold. Thus let $\mathcal{F}(X)$ consist of an orientation $o$ and a complex line bundle $L \to X$ with connection. The definition of the theory is similar to that in Example 2.9. (The two theories are related: choose the universal line bundle $L \to \mathbb{CP}^\infty$ in Example 2.9.) We continue this example in §2.8.

**Example 2.11.** Let $n = 2$ and suppose $\mathcal{F}$ includes just an orientation. The line $F(Y)$ attached to any closed 1-manifold $Y$ is the trivial line $\mathbb{C}$ and the number attached to any closed 2-manifold $X$ is

$$\lambda^{\text{Euler}(X)},$$

the exponential of the Euler number with base some $\lambda \in \mathbb{C}^\times.$ This is a connected\footnote{As agrees with the computation $[\Sigma^2 MTSO_2, \Sigma^3 HZ] \equiv H^3(BSO_2; \mathbb{Z}) = 0.$} family of theories with parameter space\footnote{The computation of the parameter space: $[\Sigma^2 MTSO_2, \Sigma^3 HZ] \equiv H^3(BSO_2; \mathbb{C}/\mathbb{Z}) \cong \mathbb{H}/\mathbb{Z}.$ Aficionados may relish the following. If we compose with $\Sigma^2 \mathbb{H}/\mathbb{Z} \to \Sigma^2 \mathbb{IC}/\mathbb{Z}$, then the theories with parameter $\lambda \in \mathbb{C}^\times$ and $-\lambda \in \mathbb{C}^\times$ become isomorphic. Here $\mathbb{IC}/\mathbb{Z}$ is the Brown-Comenetz dual of the sphere spectrum (§5.1.2). Note the numerical invariants of 2-manifolds only depend on $\lambda^2.$} $\mathbb{C}^\times.$ If we drop the orientation, then there is another theory not connected to this family;\footnote{We compute: $[\Sigma^2 MTSO_2, \Sigma^3 HZ] \equiv H^3(BO_2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$ Here $\tilde{\mathbb{Z}}$ is the nontrivial local system on $BO_2.$}

the invariant of a closed surface $X$ is $(-1)^{w_1^2(X)}$, where $w_1^2(X)$ is the characteristic number associated to the square of the first Stiefel-Whitney class of the tangent bundle. (It is nontrivial for the real projective plane, for example.)

Classical lagrangian field theories are invertible field theories: the invariant of a closed $n$-manifold is the exponentiated action $e^{iS(X)}.$ Here ‘$X$’ includes a choice of background fields.

**Example 2.13.** Finite gauge theories provide a typical example of a classical topological theory in any dimension $n$ [DW, FQ]. Let $G$ be a finite group and fix a cocycle which represents a cohomology class $\lambda \in H^n(BG; \mathbb{R}/\mathbb{Z}).$ The fields $\mathcal{F}(X)$ are an orientation and a principal $G$-bundle $P \to X.$ (As principal $G$-bundles have automorphisms—deck transformations—the fields in this case form a groupoid, or stack, rather than a space; see [FH] for one mathematical treatment.) Let $\lambda(P) \in \mathbb{R}/\mathbb{Z}$ be the pairing of the characteristic class $\lambda$ of $P$ in $H^n(X; \mathbb{R}/\mathbb{Z})$ with the fundamental class of the orientation. The invariant of the invertible field theory is $e^{2\pi i \lambda(P)}.$ Note that no orientation is required if the cocycle vanishes. This classical Dijkgraaf-Witten theory is invertible; the quantum theory, obtained by a finite sum over bundles $P$, is typically not invertible. The case $n = 3$ is a special case of Chern-Simons theory (briefly described in Example 2.4 with gauge group $\mathbb{T}$).

### 2.3. Relative field theories and anomalies

A field theory $F$ as described in 2.1 might be termed *absolute*. Suppose $\alpha$ is an (absolute) $(n+1)$-dimensional field theory. Then we can have an $n$-dimensional theory $F$ which is defined *relative*...
to $\alpha$. In fact, it is more precise and important to realize that we need only the truncation $\tau_{\leq n}\alpha$ of $\alpha$ which remembers the values on manifolds of dimension $\leq n$. Thus $\alpha$ need only be defined on such manifolds in the first place. To a closed $n$-manifold $X$ (with fields) the theory $\alpha$ assigns a vector space $\alpha(X)$. A relative theory $F$ then assigns either a linear map

$$
F(X) : \mathbb{C} \longrightarrow \alpha(X)
$$

or a linear map

$$
F(X) : \alpha(X) \longrightarrow \mathbb{C}.
$$

In the first case we evaluate the map on $1 \in \mathbb{C}$ to obtain a vector in $\alpha(X)$; in the second case $F(X)$ is a covector, an element of the dual vector space. There are similar statements for lower dimensional manifolds. In the first case we write

$$
F: 1 \longrightarrow \tau_{\leq n}\alpha
$$

and in the second

$$
F: \tau_{\leq n}\alpha \longrightarrow 1,
$$

where $1$ is the trivial theory.

If $\alpha$ is invertible, then $F$ is termed an anomalous field theory with anomaly theory $\alpha$.

We refer to [FT] for more explanations and for nontrivial examples. Here is an easy one, which illustrates the relationship with boundary conditions$^{15}$. Relative theories can be viewed as boundary conditions in any dimension, an idea we take up in §7.

**Example 2.18.** Let $n = 0$ and suppose $\alpha$ is a quantum mechanics theory (Figure 1) with Hilbert space $\mathcal{H}$ attached to a point. Then a relative theory $F: 1 \rightarrow \tau_{\leq 0}\alpha$ is determined by evaluating on 0-manifolds, and it is enough to evaluate on a point with each orientation. Since $\alpha(pt_{-}) = \mathcal{H}$ and $\alpha(pt_{+}) = \mathcal{H}^{*}$ we obtain a vector $\Omega_{F} \in \mathcal{H}$ and a dual vector $\theta_{F} \in \mathcal{H}^{*}$. We can use the relative theory $F$ as a boundary condition for $\alpha$, as illustrated in Figure 5.

### 2.4. Global symmetries and equivariant extensions

#### 2.4.1. General discussion

Let $G$ be a Lie group. For simplicity we discuss a non-extended field theory (2.3). A global symmetry group may have an action on fields, and it also may have an action on the category Vect. In the simplest cases those actions are trivial, and then $G$ is a global symmetry if the functor (2.3) lifts to a functor

$$
F: \text{Bord}_{(n-1,n)}(\mathcal{F}) \longrightarrow \text{Rep}_{G}
$$

$^{15}$Rather than a boundary ‘condition’, a relative theory can be viewed as a boundary ‘theory’.
into the category of representations of \( G \). More plainly, the group \( G \) acts on the vector space \( F(Y) \) attached to each \((n-1)\)-manifold and the linear maps assigned to bordisms are \( G \)-invariant.

In this situation we might “gauge the symmetry” or, in less ambiguous terms, construct a \( G \)-equivariant extension of the theory. This means that there is a new field which is a \( G \)-connection; if \( G \) is finite, then a \( G \)-connection is simply the underlying principal \( G \)-bundle. Whence the terminology: this is the gauge field. There is a new set of fields \( \tilde{F} \) which maps to the single field\(^{16} \) \( B_G \) of a \( G \)-connection, and the fiber over the trivial \( G \)-connection is the old set \( F \) of fields. The \( G \)-equivariant extension is a functor

\[
\tilde{F} : \text{Bord}_{(n-1,n)}(\tilde{F}) \to \text{Vect}
\]

whose restriction to the trivial \( G \)-connection is the original theory (2.19). This makes sense since the trivial \( G \)-connection has the group \( G \) as its automorphism group.

**Example 2.21.** A typical example in field theory is a \( \sigma \)-model into a Riemannian manifold \( M \) with a group \( G \) of isometries. The classical model makes sense in any spacetime dimension \( n \). The fields \( F(X) \) on an \( n \)-manifold consist of a metric \( g \) and a map \( \phi : X \to M \). In the \( G \)-equivariant extension an element of \( \tilde{F}(X) \) is a triple \((g, \Theta, \phi)\) where \( \Theta \) is a connection on a principal \( G \)-bundle \( P \to X \) and now \( \phi \) is a \( G \)-equivariant map \( P \to M \). The map to \( B_G \) sends \((g, \Theta, \phi)\) to \( \Theta \). If \( \Theta \) is the trivial \( G \)-connection then \( \phi \) is equivalent to a map \( X \to M \), and the deck transformations of the trivial bundle \( X \times G \to X \) become the original \( G \)-action on the fields.

This example does not fit our simplified description, since the global symmetry group \( G \) does act on the fields \( F \), but there is a modification which covers this situation. In the quantum \( \sigma \)-model we integrate over the field \( \phi \), and as there is no \( G \)-action on the remaining fields our description applies as is.

**2.4.2. Anomalies.** There may be obstructions to constructing this \( G \)-equivariant extension: see [KT] for a recent discussion and examples. One well-known example is the gauged WZW model [W5]. In good cases there is only a single obstruction which can be interpreted as an anomaly. In these cases the extension (2.20) does not exist but rather there is an invertible \((n+1)\)-spacetime dimensional theory \( \alpha \) and an extension \( \tilde{F} \) which is a theory relative to \( \alpha \) in the sense of §2.3.

**Example 2.22.** A standard example in \( n = 4 \) is quantum chromodynamics. This theory has a global \( SU_N \times SU_N \) symmetry (for \( N \) the number of flavors) which is anomalous.

We account for such anomalies in our proposals (§5.2).

\(^{16}\text{\(B_G\) is most naturally a simplicial sheaf on the category of smooth manifolds [FH]}\).
2.4.3. *Gauging antilinear symmetries.* In quantum mechanics, due to Wigner’s theorem, the global symmetry group $G$ is equipped with a homomorphism\(^{17}\)

\[
(2.23) \quad \phi: G \to \mu_2 = \{\pm 1\}
\]

which tracks whether a given symmetry acts linearly or antilinearly. Because states are lines in a Hilbert space, rather than vectors, the group $G$ acts projectively and there is an extension by the group $\mathbb{T}$ of unit norm scalars. For theories tied to spacetime, as opposed to abstract theories, one can also track whether or not symmetries reverse the orientation of time by another homomorphism

\[
(2.24) \quad t: G \to \mu_2.
\]

In many situations $t = \phi$, but that needn’t be so in general. See [FM2, §§1,3] for a general discussion.

We handle the extension by simply replacing $G$ with the extended symmetry group. In doing so we must take care that the group of scalars acts by scalar multiplication on all vector spaces in the theory. We discuss time-reversal in §4.2.3. Here we explain how to gauge antilinear symmetries.

Consider a 1-spacetime dimensional theory $F$, so a quantum mechanical model as in Figure 1. Let $F(\text{pt}_+) = \mathcal{H}$ and suppose $G$ is a group of global symmetries as above. For simplicity, assume $G$ is a discrete group. As explained in §2.4.1 a $G$-equivariant extension $\hat{F}$ is a theory on manifolds equipped with a principal $G$-bundle. Now $\mathcal{H}$ is the value of $\hat{F}$ on $\text{pt}_+$ equipped with the trivial (which means trivialized) $G$-bundle. Consider the 1-manifold $X = [0,1]$ with a (necessarily trivial, but not trivialized) $G$-bundle, and suppose there are trivializations over the endpoints $\{0,1\}$. Let $g \in G$ be the parallel transport. Then $\hat{F}(X,g): \mathcal{H} \to \mathcal{H}$ is the action of the global symmetry $g$. However, if $\phi(g) = -1$—i.e., if $g$ acts antilinearly—then this does not fit into (2.20) since the category $\text{Vect}$ has only linear maps. The way out is that $\phi$ defines a 2-dimensional invertible anomaly theory $\alpha$, and $\hat{F}$ is an anomalous theory with anomaly $\alpha$, as in (2.17). The anomaly theory assigns $\alpha(X,g) = \overline{\mathcal{C}}$, the complex conjugate to the trivial line of complex numbers, and then the relative theory gives a linear map

\[
(2.25) \quad \hat{F}(X,g): \overline{\mathcal{C}} \otimes \mathcal{H} \longrightarrow \mathcal{H}.
\]

(A linear map $\overline{\mathcal{C}} \otimes V \to W$ is equivalent to an antilinear map $V \to W$. See [FT, (2.7)] for the analog of (2.25) in an arbitrary relative theory.)

This discussion extends to higher dimensions. For extended field theories with values in a higher category $\mathcal{C}$ we would need to explain how complex conjugation acts on $\mathcal{C}$. In this paper we focus on invertible field theories, and we will implement this “antilinearity anomaly” using twisted cohomology; see §5.1.4.

\(^{17}\)Notation: $\mu_k = \{\lambda \in \mathbb{C} : \lambda^k = 1\}$ is the group of $k^{th}$ roots of unity.
2.5. Unitarity

The formal setup of topological field theory described here is based on the Euclidean version of quantum field theory. For Euclidean QFTs unitarity is expressed by both a reality condition and a reflection-positivity condition. (A standard reference is [GJ]; see [Detal, p. 690] for a heuristic explanation.) The unitarity condition for a fully extended topological theory (2.8) implements only the reality condition. Namely, assuming the fields $F$ include an orientation there is an involution of $\text{Bord}_n(F)$ which reverses the orientation. Also, assuming that the target $C$ is based on complex numbers, then it has an involution of complex conjugation. Unitarity is the statement that $F: \text{Bord}_n(F) \to C$ is equivariant for these involutions. A formal justification from the path integral stems from a basic fact: orientation reversal conjugates the Euclidean action.

In this paper we indicate how to implement unitarity for invertible topological field theories, which are maps of spectra. The involutions are quite explicit, and unitarity amounts to a twisted extension of the field theory to unoriented manifolds. One subtlety, which also occurs in non-invertible theories, is that for spin theories there are two notions of unitarity—the two different Euclidean pin groups lead to two orientation-reversing involutions on spin manifolds.\(^{18}\)

Because of the gaps in our understanding of unitarity, related to positivity and the choice of pin group, we do not implement unitarity fully in our proposal in §5.2. We discuss unitarity further in §§4.2.4, 4.2.5.

2.6. Topological field theories and the cobordism hypothesis

One can debate which theories deserve the moniker ‘topological’.\(^{19}\) We will say that a fully extended theory (2.8) is topological if the fields $F$ are topological, and the fields are topological if they satisfy homotopy invariance: if $f_1: X' \to X$ is a homotopy of local diffeomorphisms of $n$-manifolds, then the pullbacks by $f_0$ and $f_1$ on fields are equal.\(^{20}\) Thus orientations, spin structures, and $G$-bundles for discrete groups $G$ are all examples of topological fields. Metrics, conformal structures, and connections for positive dimensional Lie groups are all examples of non-topological fields. On the other hand, flat $G$-connections are topological fields for any Lie group $G$.

Fully extended topological field theories are a topic of current interest in topology and other parts of mathematics. The cobordism hypothesis, conjectured by Baez-Dolan [BD] and proved by Hopkins-Lurie in dimensions $\leq 2$ and in general by Lurie [L], is a powerful result which determines the space of fully extended topological theories of a fixed type. The ‘type’ refers to both the fields $F$ and the target $C$. Thus one speaks of “oriented” theories or “framed” theories, which tells about the topological fields in the theory. The theorem very roughly states that a theory $F$ is determined by its value $F(\text{pt})$ on the 0-manifold consisting of a single point. One can intuitively think of this as the value on an $n$-dimensional ball, and the idea is that any $n$-manifold is glued together from balls, so that if the theory is fully local then its values can be reconstructed from those on a point.

\(^{18}\)I thank Kevin Walker for emphasizing this point. I do not know a physical argument which distinguishes one of them as the preferred choice, though in specific examples often one is preferred over the other.

\(^{19}\)For example, with our definition classical Chern-Simons theory (Example 2.4) is not topological if the gauge group is not discrete.

\(^{20}\)Some fields, such as gauge fields, have internal symmetries and then the pullbacks are not strictly ‘equal’ but rather are ‘equivalent’ or ‘homotopic’.
Furthermore, the value on a point is constrained to satisfy strong finiteness conditions. We refer the reader to [L] and the expository account [F1].

Remark 2.26. If $n = 1$ and we let the field $\mathcal{F}$ be an orientation, then a theory

\[(2.27)\quad F : \text{Bord}_{(0,1)}(\mathcal{F}) \rightarrow \text{Vect}\]

is determined by the vector space $F(\text{pt}_+).$ The finiteness condition is that $F(\text{pt}_+)$ is finite dimensional. Of course, in usual quantum mechanics the quantum Hilbert space is typically infinite dimensional; not so in this topological version.

The cobordism hypothesis tells not just about individual theories, but rather about the collection of theories with fixed $\mathcal{F}$ and $\mathcal{C}.$ The first (easy) theorem is that this collection is an ordinary space rather than a more abstract category. One should think of this space as parametrizing families of theories, as in Example 2.9 and Example 2.11. Yet in the homotopical setting for field theories, it is only the homotopy type of the space of theories which is well-defined; see §2.8 for more discussion. The theorem in particular computes the set of path components of this space. Two theories lie in the same path component if and only if they can be continuously connected. This matches well the notion of a topological phase, and indeed the cobordism hypothesis is a powerful tool for distinguishing topological phases of gapped theories. In this paper we focus on invertible theories, which describe SRE phases, and the cobordism hypothesis reduces to a much easier statement, as we explain in §2.7. We emphasize that the cobordism hypothesis—for invertible and non-invertible theories—determines the complete homotopy type of the space of theories, not just the set of path components.

2.7. Invertible topological theories and maps of spectra

We begin with an analogy. Let $\mathbb{C}[x]$ be the ring of polynomials in a variable $x$ with complex coefficients and let $\mathbb{C}$ be the ring of complex numbers. Define the ring homomorphism $F : \mathbb{C}[x] \rightarrow \mathbb{C}$ which sends a polynomial $f(x)$ to its value $f(0)$ at $x = 0.$ Let $S \subset \mathbb{C}[x]$ denote the subset of polynomials with nonzero constant term. Note that $S$ is closed under multiplication: if $f_1, f_2 \in S,$ then $f_1 f_2 \in S.$ Extend the homomorphism $F$ to ratios of polynomials $f/g$ where $g \in S.$ This is for the simple reason that $g(0) \neq 0$ if $g \in S,$ so $f(0)/g(0)$ makes sense. We write $S^{-1}\mathbb{C}[x]$ for the ring of such ratios: we have inverted elements in $S.$ This inversion construction is easy in this case since $\mathbb{C}[x]$ is a commutative ring; it is trickier in the noncommutative case and in the categorical context to which we now turn.

Suppose

\[(2.28)\quad F : \text{Bord}_n(\mathcal{F}) \rightarrow \mathcal{C}\]

is an invertible topological field theory. By definition $F$ takes values in the subset $\mathcal{C}^\times \subset \mathcal{C}$ consisting of invertibles: invertible objects, invertible 1-morphisms, and invertible morphisms at all levels. Now, as in the analogy, the fact that all values are invertible\(^{21}\) means that the theory factors

\(^{21}\)In our analogy, only some values are invertible; here all are.
through the symmetric monoidal \((\infty, n)\)-category \(|\text{Bord}_n(\mathcal{F})|\) obtained by adjoining inverses of all morphisms:

\[
\begin{align*}
\text{Bord}_n(\mathcal{F}) & \xrightarrow{F} \mathcal{C} \\
|\text{Bord}_n(\mathcal{F})| & \xrightarrow{\tilde{F}} \mathcal{C}^\otimes
\end{align*}
\]

The map \(\tilde{F}\) encodes all information about the theory \(F\).

The domain and codomain of \(\tilde{F}\) are each a higher category, in fact an \(\infty\)-category, in which all arrows are invertible. Such categories are called \(\infty\)-groupoids. The basic idea is that an \(\infty\)-groupoid is equivalent to a space. This is easiest to see in the opposite direction: from a space \(S\) we can extract an \(\infty\)-groupoid \(\pi_{\leq \infty}S\). Putting aside the ‘\(\infty\)’ for a moment, we extract an ordinary groupoid called the fundamental groupoid \(\pi_{\leq 1}S\). Its objects are the points of \(S\) and a morphism \(x_0 \to x_1\) between points \(x_0, x_1 \in S\) is a continuous path \(x: [0, 1] \to S\) from \(x_0\) to \(x_1\) up to homotopy. This is a groupoid because paths are invertible: reverse time. The higher groupoids track homotopies of paths, homotopies of homotopies, etc. The conclusion is that \(\tilde{F}\) may be considered as a continuous map of spaces. This already brings us into the realm of topology. But more is true. The domain and codomain of \(\tilde{F}\) are symmetric monoidal \(\infty\)-groupoids, which induces more structure on the corresponding spaces. (Recall that the monoidal product on the bordism category is disjoint union and on the category \(\mathcal{C}\) it is some sort of tensor product.) Namely, those spaces are infinite loop spaces. So for each of the spaces \(S\) there exist a sequence of pointed spaces \(S_0 = S, S_1, S_2, \ldots\) such that the loop space of \(S_n\) is\(^{22}\) \(S_{n-1}\). Furthermore, the fact that \(F\) preserves the symmetric monoidal structure—a field theory takes compositions to compositions and disjoint unions to tensor products—implies that \(\tilde{F}\) is an infinite loop map. Such sequences of spaces are called spectra and an infinite loop map gives rise to a map of spectra.

The bottom line is that the space of invertible field theories (with specified \(\mathcal{F}, \mathcal{C}\)) is\(^{23}\) a space of maps in homotopy theory. We are interested in the abelian group of path components, which houses an invariant of gapped topological phases, and that is the group of homotopy classes of maps between spectra. Therefore, the computation of invertible field theories starts by recognizing the spectra in the domain and codomain, which in turn depend on the choice of \(\mathcal{F}\) and \(\mathcal{C}\). The domain spectra, obtained from bordism multicategories, will be discussed in §4. We remark that to study invertible field theories one does not need the cobordism hypothesis; the power of the latter is for more general non-invertible topological field theories.

Remark 2.30. The set of path components here has a natural abelian group structure. In fact, a spectrum \(\mathcal{X}\) determines a collection \(\{\pi_n\mathcal{X}\}_{n \in \mathbb{Z}}\) of abelian groups, its homotopy groups. There is additional information in the spectrum which binds these groups together, but a first heuristic is that a spectrum is some topological version of a \(\mathbb{Z}\)-graded abelian group.

Example 2.31. We continue with Example 2.11. In this case \(n = 2\) and the field \(\mathcal{F}\) is an orientation. Now the target \(\mathcal{C}\) is a 2-category, and we need to determine the spectrum corresponding to

---

\(^{22}\)The Clinton question: here best to take ‘is’='is homeomorphic to'. See §4.1 for further discussion.

\(^{23}\)The space is only determined up to homotopy equivalence, i.e., there is only a well-defined homotopy type.
the invertibles $\mathbb{C}^\times$. A typical choice is to take $\mathcal{C}$ to be the 2-category of complex linear categories. This conforms to the usual picture that a 2-dimensional field theory assigns a complex number to a closed 2-manifold, a complex vector space to a closed 1-manifold, and a complex linear category to a 0-manifold. In the invertible sub 2-groupoid $\mathbb{C}^\times$ all of these are “1-dimensional”. This means that the category is equivalent to Vect, the category of vector spaces; the vector spaces are isomorphic to $\mathbb{C}$; and the numbers we encounter are nonzero, so elements of $\mathbb{C}^\times$. When we make the corresponding spectrum $\mathcal{X}$, the fact that invertible categories are all equivalent implies $\pi_0 \mathcal{X} = 0$. The fact that all invertible 1-morphisms are equivalent implies $\pi_1 \mathcal{X} = 0$. Finally we come to $\pi_2 \mathcal{X}$, which captures the 2-morphisms $\mathbb{C}^\times$. Here we get two different answers, depending on whether we consider $\mathbb{C}^\times$ to have the discrete topology or the continuous topology. In the discrete case we have $\pi_2 \mathcal{X}_{\text{discrete}} \cong \mathbb{C}^\times$. For the ordinary topology we use $\pi_0 \mathbb{C}^\times = 0$, $\pi_1 \mathbb{C}^\times \cong \mathbb{Z}$ to deduce $\pi_2 \mathcal{X}_{\text{continuous}} = 0$, $\pi_3 \mathcal{X}_{\text{continuous}} \cong \mathbb{Z}$. Higher homotopy groups of $\mathcal{X}$ vanish. Hence each of $\mathcal{X}_{\text{discrete}}$ and $\mathcal{X}_{\text{continuous}}$ has only a single nonzero homotopy group. Such spectra are called Eilenberg-MacLane spectra and are basic building blocks. The notation is

\begin{equation}
\mathcal{X}_{\text{discrete}} \cong \Sigma^2 H\mathbb{C}^\times,
\mathcal{X}_{\text{continuous}} \cong \Sigma^3 H\mathbb{Z}.
\end{equation}

The domain spectrum, to be explained in §4, is denoted $\Sigma^2 MT SO_2$. Thus the two sets of path components are:

\begin{equation}
[\Sigma^2 MT SO_2, \Sigma^2 H\mathbb{C}^\times] \cong \mathbb{C}^\times,
[\Sigma^2 MT SO_2, \Sigma^3 H\mathbb{Z}] = 0.
\end{equation}

Resuming Example 2.11 we see that the first of the computations in (2.33) distinguishes the Euler theories, parametrized by the base $\lambda \in \mathbb{C}^\times$ of the exponential in (2.12). The discreteness in $\mathcal{X}_{\text{discrete}}$ means that theories for distinct $\lambda$ cannot be connected by a smooth path. In the usual topology they can, and this explains the second computation in (2.33): the space of theories is connected.

**Remark 2.34.** For a general invertible bosonic theory in higher than 2-spacetime dimensions we need more nonzero homotopy groups in the target spectrum. This surprise is discussed in §5.1.

**Remark 2.35.** The distinction between $\mathcal{X}_{\text{discrete}}$ and $\mathcal{X}_{\text{continuous}}$ is important and carries through to more elaborate target spectra. The discrete target, based on $\mathbb{C}^\times$ with the discrete topology, is where we detect individual theories. The other target, based on $\mathbb{C}^\times$ with its usual topology and manifested as $\mathbb{Z}$ shifted up one degree, is where we detect deformation classes of theories. In the next section we elaborate on the meaning of spaces of maps into targets such as $\mathcal{X}_{\text{continuous}}$.

### 2.8. An illuminating example

We have alluded several times to the homotopical setting of the cobordism hypothesis and so too the computations of deformation classes of invertible field theories. It often happens that the geometric interpretation of a homotopical computation is subtle. There are many examples in

\footnote{We reprise the following argument in §5.1.2.}
topology, enumerative geometry, etc. We illustrate in our present context with a simple example. In §6 we encounter a more sophisticated example of the same type in the classification of gapped topological phases.

In Example 2.10 we discussed an invertible field theory in $n = 1$ spacetime dimensions with fields

$$(2.36) \quad F_{\text{geometric}} = \{(o, L, \nabla)\}$$

a triple consisting of an orientation, complex line bundle, and connection. Then there is an obvious theory

$$(2.37) \quad F_{\text{geometric}} : \text{Bord}_1(F_{\text{geometric}}) \rightarrow \text{Line}$$

with values in the category of complex lines. Namely, to a point $pt_+$ with positive orientation and a line bundle $L \rightarrow pt_+$ we attach the line $L$. (A line bundle over a point is a single line.) If we reverse the orientation of $pt_+$, so consider $pt_-$, we take the dual line; if there are several points we form the tensor product. If $(L, \nabla) \rightarrow [0, 1]$ is a line bundle with covariant derivative over the interval with its usual orientation, the field theory assigns to it the parallel transport $F_{\text{geometric}} : L_0 \rightarrow L_1$ from the fiber over the initial endpoint to the fiber at the terminal endpoint. If $(L, \nabla) \rightarrow S^1$ is a line bundle with connection over an oriented circle, then $F_{\text{geometric}}$ assigns to it the holonomy, which is a number in $\mathbb{C}^\times$. This is a well-defined theory, and it is not “topological” according to our definition, since the connection $\nabla$ is not a homotopy-invariant field.

We can instead take the set of topological fields

$$(2.38) \quad F_{\text{topological}} = \{(o, L)\}$$

consisting of an orientation and a complex line bundle but no connection. Now we ask to classify topological field theories

$$(2.39) \quad F_{\text{topological}} : \text{Bord}_1(F_{\text{topological}}) \rightarrow \text{Line}$$

As emphasized in Example 2.31 it is important to specify which topology we use on linear isomorphisms in the category $\text{Line}$: the discrete or continuous topology. Here we use the continuous topology. The computation of equivalence classes of theories is

$$(2.40) \quad [\Sigma^1\text{MTSO}_1 \wedge BC_+^\times, \Sigma^2\mathbb{Z}] = [S^0 \wedge BC_+^\times, \Sigma^2\mathbb{Z}] = H^2(BC^\times; \mathbb{Z}) \cong \mathbb{Z}.$$ 

According to the discussions in §2.6 we conclude that the space of theories (2.39) is not connected, but rather there is an integer invariant which distinguishes deformation classes of theories. There is always a trivial theory—it sends every 0-manifold with fields to the trivial line $\mathbb{C}$ and every closed 1-manifold with fields to the number $1 \in \mathbb{C}^\times$—and it is in the deformation class of theories labeled by the integer 0 in (2.40). So we are led to ask:
**Question 2.41.** What theory (2.39) is labeled by the integer $k$ in (2.40)? Can we construct a single example of such a theory?

It is clear what to do on 0-manifolds. For example,

$$F_{\text{topological}}(L \to \text{pt}+) = L^\otimes k,$$

the $k^{\text{th}}$ tensor power of the line $L$. In other words, we observe that the truncation $\tau_{\leq 0} F_{\text{geometric}}$ of the geometric theory above does not use the covariant derivative, and so we take $F_{\text{topological}}$ on 0-manifolds to be that theory to the $k^{\text{th}}$ power.

What do we do on 1-manifolds? The theory $F_{\text{geometric}}$ uses the connection on $L \to [0, 1]$ to define a definite linear map—parallel transport—and the connection on $L \to S^1$ to define a definite number—the holonomy. But the now the fields (2.38) do not include a connection and we have no apparent way to determine these linear maps and numbers.

There are several ways out, and they illuminate what is is being computed in (2.40) and in later computations. Similar remarks apply to all topological theories and to the cobordism hypothesis.

The first comment, already made in Remark 2.6, is that a field theory gives invariants for families of manifolds with fields parametrized by a smooth manifold $S$, not just for single manifolds. So, for example, we can consider a family of points $\text{pt}_+$ parametrized by the 2-sphere $S = S^2$ endowed with a complex line bundle $L \to S^2$ on the total space. Let $d$ be the degree of this line bundle. A field theory $F_{\text{topological}}$ returns another line bundle over $S$, and according to (2.42) $F_{\text{topological}}$ of this family is the $k^{\text{th}}$ tensor power $L^\otimes k \to S$, which has degree $kd$. The ratio $kd/d = k$ of the degrees is the integer in (2.40), and it is detected by this 2-parameter family of points. Similarly, we can consider a 1-parameter family of circles $S \times S^1 \to S$ parametrized by $S = S^1$. The fields are an orientation along the fibers of this map and a line bundle $L \to S \times S^1$ on the total space, say of degree $d$. A field theory $F_{\text{topological}}$ returns a map $S \to \mathbb{C}^\times$ and what the computation (2.40) tells is that the winding number of this map around the origin in $\mathbb{C}$ is $kd$. In other words, the computation (2.40) determines homotopical information in any particular field theory in the deformation class, and that information may need to be measured in families.

This still does not construct a particular theory (2.39); it only constrains any such. To construct a theory we need to make some choices—which in itself is not surprising—but what may be surprising is that we can choose to change the domain $\text{Bord}_1(F_{\text{topological}})$ or the codomain Line in order to construct a particular theory. What’s more, we may go outside the realm of topological field theories and use more general field theories.

For instance we can replace $F_{\text{topological}}$ with $F_{\text{geometric}}$. There is an obvious map $F_{\text{geometric}} \to F_{\text{topological}}$ which forgets the connection $\nabla$, and the important point is that the fibers of this map are contractible. That is, the space of covariant derivatives $\nabla$ on a fixed line bundle $L \to X$ is a contractible space. (In fact, it is an infinite dimensional affine space.) If we make this substitution, then we know how to construct a theory. Namely, we take $F^{\otimes k}_{\text{geometric}}$ where $F_{\text{geometric}}$ is defined after (2.37). Another choice would be to fix a smooth model of the classifying space for line bundles, say by fixing a complex Hilbert space $\mathcal{H}$ and taking the classifying space to be the infinite dimensional projective space $\mathbb{P}(\mathcal{H})$. Furthermore, we fix a smooth line bundle $H \to \mathbb{P}(\mathcal{H})$ with covariant derivative. Augment the topological fields (2.38) by a contractible choice: a classifying...
map $\gamma: X \to \mathbb{P}(H)$ for each line bundle $L \to X$. Again there is a map from triples $(o, L, \gamma)$ to $(o, L)$ and the fibers are contractible. With these choices and replacements it is easy to construct a particular theory using parallel transport in the bundle $H^\otimes k \to \mathbb{P}(H)$, pulled back via the classifying map $\gamma$.

**Remark 2.43.** A different possibility is to keep the domain fields as is (2.38) but change the target category Line. We can replace it by the 2-category of $\mathbb{Z}$-gerbes; the automorphisms of any object comprise the category of $\mathbb{Z}$-torsors, and the automorphism group of any automorphism is $\mathbb{Z}$. This is the target of a 2-dimensional field theory with fields (2.38). This theory assigns to a line bundle $L \to X$ over a closed oriented surface its degree, which is an integer. The information on lower dimensional manifolds can be used to compute this degree locally; see [F2].

Another insight can be gleaned from (2.29), which we unwrap:

\[(2.44) \quad \text{Bord}_n(\mathcal{F}) \longrightarrow |\text{Bord}_n(\mathcal{F})| \xrightarrow{\bar{F}} C^\times\]

Let double vertical bars around a higher category denote its geometric realization, which is a topological space. The process of geometric realization inverts all arrows in the category, so factors through the single bar construction. In these terms what we calculate in (2.40) is the group of homotopy classes of maps of spaces

\[(2.45) \quad \|\text{Bord}_n(\mathcal{F})\| \longrightarrow \|C^\times\|\]

whereas a particular field theory $F_{\text{topological}}$ is a map of categories

\[(2.46) \quad \text{Bord}_n(\mathcal{F}) \longrightarrow C^\times\]

The functor ‘geometric realization’ takes a map of categories (2.46) to a map of spaces (2.45). Furthermore, Grothendieck’s homotopy hypothesis asserts that the functor of geometric realization is an equivalence between higher groupoids and spaces, so we expect to be able to invert

\[(2.47) \quad C^\times \longrightarrow \|C^\times\|\]

since $C^\times$ is a higher groupoid. Such an inversion would let us pass from a map of spaces (2.45) to a field theory (2.46), but in practice we cannot explicitly construct an inverse without making choices. In our example, $C^\times = \text{Line}$ is the groupoid of complex lines and its geometric realization as a space is $|C^\times| \simeq \mathbb{C}\mathbb{P}^\infty$. We need to identify that geometric realization with an explicit model of $\mathbb{C}\mathbb{P}^\infty$ to get back to one-dimensional vector spaces.

In summary, the most relevant interpretation we can offer of the computation (2.40) for this paper is our first: it computes homotopy classes of a class of theories obtained by augmenting the topological fields with geometric fields which constitute a contractible choice. This resonates with the construction of quantum Chern-Simons theory [W1], for example, in which a Riemannian metric is introduced to evaluate the path integral.

**Remark 2.48.** The complication here arises since we seek to interpret nontorsion elements of the group of deformation classes, computed in (2.40). Torsion elements, such as in footnote 14, lift to maps into $C^\times_{\text{discrete}}$ shifted down a degree, and so are realized by definite topological theories.
3. The long-range effective topological field theory

In this section we begin to apply the generalities about invertible topological field theories to short-range entangled phases. We start in §3.1 with some general remarks about scaling, energy gaps, and effective theories. In §3.2 we give general arguments about short-range entangled (SRE) phases in gapped condensed matter systems. In the last subsection §3.3 we bring in local symmetries, including time-reversal, and in particular define symmetry protected topological (SPT) phases in terms of invertible topological field theories.

3.1. Low energy, long time

In classical nonrelativistic physics there are three basic dimensions: length ($L$), time ($T$), and mass ($M$). (For mathematical discussions of dimensions and units, see [Ta, §2.1], [DF, §2.1].) The dimensions of other physical quantities can be expressed in terms of these. For example, energy has dimension

\begin{equation}
[E] = \frac{ML^2}{T^2}.
\end{equation}

Universal physical constants also have dimensions, for example Planck’s constant $\hbar$ and the speed of light $c$:

\begin{equation}
[h] = \frac{ML^2}{T}, \quad [c] = \frac{L}{T}.
\end{equation}

The constant $\hbar$ is present in any quantum system, the constant $c$ in any relativistic system, and both in a relativistic quantum system. These constants allow us to convert between dimensions, often silently by assuming units in which $\hbar = 1$ and $c = 1$. Thus in a quantum system we have

\begin{equation}
[E] = \frac{1}{T} \quad \text{(assuming } \hbar).\end{equation}

This dimensional analysis suggests that the low energy behavior of a quantum system is reflected in its long time behavior. In a relativistic quantum system we have $L = T$ using $c$, and so

\begin{equation}
[E] = M = \frac{1}{T} = \frac{1}{L} \quad \text{(assuming } \hbar, c),
\end{equation}

thereby relating low energy to low mass and long time to large distance.

Consider, then, a Riemannian manifold $M$ with Riemannian metric $g$. Let $\Delta$ be the Laplace operator on differential forms, which we take as the Hamiltonian of a nonrelativistic quantum system.\footnote{There are others, such as temperature and electric current, but they do not play a role in this discussion.} The eigenvalues of $\Delta$ are energies, so the low energy behavior involves the low lying eigenvalues. Equivalently, we can consider the long time evolution, which is $e^{it\Delta}$ for $t$ large. Nonzero eigenvalues lead to long time behavior, which is reframed in terms of a Hamiltonian eigenvalue.

\footnote{The dimensionally correct expression is $H = (\hbar^2/m)\Delta$, where $m$ = mass; instead, we set $\hbar = 1$ and $m = 1$.}
eigenvalues lead to oscillations, which tend to cancel out if $t$ is large, and once more we are led to the low lying spectrum.\footnote{In Riemannian geometry the heat operator $e^{-t\Delta}$ is more familiar and leads to the same conclusion.} Now we encounter a fundamental dichotomy. If $0$ is an isolated point of the spectrum, then for $t$ large the kernel of $\Delta$ dominates. However, if $0$ is not an isolated point of the spectrum, then the low lying continuous spectrum mixes inextricably with the kernel. Therefore, we isolate the kernel at long time only if there is a gap in the spectrum above $0$. This always happens if $M$ is compact. In that case the Hodge and de Rham theorems combine to prove that the kernel of the Laplace operator $\Delta$ on differential forms has topological significance: the dimension of the kernel of $\Delta$ on $\Omega^q(M)$ is the $q^{th}$ Betti number of $M$.

A gapped quantum system is one in which $0$ is an isolated point of the spectrum of the Hamiltonian $H$. By the dimensional analysis above, this energy gap is equivalent to a mass gap in a relativistic quantum system. One is interested in the long time (and large distance) behavior of a system since that is what we can observe, and often that behavior is described by an effective system. We use the general term ‘long-range’ for either ‘long time’ or ‘large distance’. For example, quantum field theories at large distance are often well-approximated by another quantum field theory, and the approximating theory contains more than the vacuum state if there is no mass gap; see [W2, §2.5] for a discussion. It is sometimes said that if there is a mass gap then the low energy theory is trivial, but there is a more nuanced truism:\footnote{I do not know where this idea originated; one older reference is [W3, p. 405].}

\((3.5)\) The low energy behavior of a gapped system is approximated by a topological field theory.

For this to make sense, we need to study the theory on arbitrary spacetimes, so couple to background gravity. That can be done for many field theories. For example, 4-dimensional Yang-Mills theory makes sense on arbitrary Riemannian manifolds, and conjecturally there is a mass gap, so one should obtain a 4-dimensional topological field in the low energy limit. (Here $n = 4$ is the spacetime dimension.)

\textit{Remark} 3.6. The $N = 1$ supersymmetric Yang-Mills theory also has a conjectural mass gap, and presumably the low energy effective topological theory is more interesting in that case.

We turn to condensed matter systems, which we treat very heuristically. We assume the quantum Hilbert space $\mathcal{H}$ is the tensor product over a set $S$ of sites of finite dimensional Hilbert spaces

\begin{equation}
\mathcal{H} = \bigotimes_{s \in S} \mathcal{H}_s
\end{equation}

and we may assume $\mathcal{H}_s = V$ is a fixed Hilbert space independent of the site $s$; the sites are initially arranged in a regular pattern, such as a lattice; and the Hamiltonian $H$ is local in that it is a sum

\begin{equation}
H = \sum_{s \in S} H_s,
\end{equation}

where $H_s = T \otimes \text{id}$ relative to a decomposition $\mathcal{H} = (\bigotimes_{s'} \mathcal{H}_{s'}) \otimes (\bigotimes_{s''} \mathcal{H}_{s''})$ in which $s'$ runs over sites in a small vicinity of $s$ and the operator $T$ is independent of $s$. The sites $S$ are located in a
background space $Y$ of dimension $d$. (We emphasize that $d$ is the dimension of space and $d+1$ the dimension of spacetime.) This is a nonrelativistic system, and Galilean boosts have been broken by the sites, which don’t move in time. Thus time is completely separate from the geometry of space. Initially $Y$ typically lies in flat Euclidean space, and coupling to gravity in this context means that the system can be studied on any curved $d$-dimensional manifold $Y$. Fix a compact $Y$ and imagine that the finite set of sites $S$ becomes more and more dense. With no pretense of precision we assume that the system is gapped in that 0 is the lowest eigenvalue of $H$ and the spectrum of $H$ has a fixed size gap above 0 which persists in the limit that $S$ becomes dense. Furthermore, we assume the kernel of $H$ stabilizes to a finite dimensional vector space $F_{pYq}$. In this situation we would like to assert (3.5): there is an effective $d$-space dimensional topological field theory $F$ which approximates the low energy/long time behavior of the system. The vector spaces $F_{pYq}$ are part of that theory. The deformation class of the theory $F$ is then a topological invariant of the original system, much the same way that the Betti numbers are topological invariants of the Laplace operator on a compact Riemannian manifold.

Remark 3.9. Some parts of the gluing laws of a topological field theory are clearly going to hold. For example, if $Y = Y' \| Y''$ is a disjoint union, and we envision a system as described after Remark 3.6, then the quantum Hilbert space (3.7) of $Y$ is the tensor product of those on $Y'$ and $Y''$ and the Hamiltonian (3.8) decomposes accordingly. Therefore, so too does the kernel $F(Y)$. One expects a more subtle decomposition emerges from a continuum limit process.

Remark 3.10. This procedure does not obviously give numerical invariants on all compact $(d+1)$-dimensional manifolds. We expect invariants corresponding to time evolution, and since the field theory is defined for compact manifolds we must take circular time $S^1$. When we return to the initial time the manifold and its fields can undergo a symmetry, so the manifold we obtain is a fiber bundle $X^{d+1} \to S^1$. We do expect invariants for these mapping cylinders. This kind of impoverished $(d+1)$-dimensional field theory, perhaps with variations, goes under many names: a $(d+\epsilon)$-dimensional theory, a theory of $H$-type, . . . . Many, in fact most, of the effective theories of $H$-type that we encounter do extend to full $(d+1)$-dimensional theories, and it may be that additional considerations in the microscopic theory can imply that property of the long-range topological theory. Even more is possible. A 1-parameter family of $(d+1)$-manifolds parametrized by $S$ is the total space of a fiber bundle $M^{d+2} \to S^1$, and the theory gives a map $S^1 \to \mathbb{C}^\times$. Its winding number is an integer invariant associated to $M$. It may happen that these integer invariants are defined for arbitrary closed $(d+2)$-manifolds, not just mapping cylinders. This would impose a more severe constraint on the low energy effective theory, as we illustrate in §6.3. Geometrically, an $H$-type theory gives invariants for manifolds equipped with a rank $d$ bundle stably equivalent to the tangent bundle.

Remark 3.11. If one starts with a gapped quantum field theory, at first glance one can imagine its low energy behavior giving rise to a specific topological field theory, or at least a contractible space of theories depending on some mild choices in the approximation. For a condensed matter system there seem to be more choices as one must, in addition to any cutoffs, take a continuum limit. This leads to the expectation that we obtain a space of theories—again presumably contractible, but hopefully at least connected—and makes it more plausible that one could encounter the phenomenon highlighted in §2.8. Indeed, we will in §6.3.
3.2. Short-range entanglement hypothesis and its consequences

We make explicit the assumptions beyond (3.5) which underlie our proposal in §5.2. The most drastic of these is invertibility, which is an expression of short-range entanglement; see (6) below.

(1) We assume that the low energy effective topological theory $F$ is fully extended, in the sense discussed after Remark 2.7. This is a strong form of locality, and it seems reasonable since the continuum limit theory is obtained from local discrete systems, as in (3.7) and (3.8). To define the extended theory we must specify what sorts of (higher categorical) invariants $F$ computes on low dimensional manifolds. In other words, we must specify the target $\mathcal{C}$ for the field theory; see (2.8). Since we restrict to invertible theories, we need only specify a spectrum. That is one of the key choices to be made; see §5.1 for a full discussion.

(2) We assume that $F$ is unitary, since microscopic condensed matter systems are typically unitary.

(3) One important consideration which affects the choice of $\mathcal{C}$ is the topology on the space of theories, as already mentioned and illustrated in Remark 2.7, Example 2.11, and Example 2.31. To the extent that the theory $F$ has numerical invariants of closed $(d+1)$-manifolds, they lie in $\mathbb{C}$; the invariants of closed $d$-manifolds are $\mathbb{C}$-vector spaces. In both cases we use the usual topology on $\mathbb{C}$ to allow the theory $F$ to deform by continuously varying the numerical invariants, the linear maps between vector spaces, etc. Two microscopic gapped systems related by a continuous deformation are considered to define the same topological phase, and they should give rise to effective topological field theories which are deformation equivalent. On the other hand, we also consider anomaly theories $\alpha$, which can arise when gauging an anomalous global symmetry. (See §2.3 and the text before Example 2.22.) In that case we do not allow continuous deformations of the anomaly theory: anomalous theories relative to distinct anomaly theories should not be viewed as the same topological phase. Thus, for the classification of anomalies we take the discrete topology on $\mathbb{C}$.

(4) Another important choice is of the background fields $\mathcal{F}$ in the low energy effective topological theory (2.8). We expect only topological fields. Our choice here is based on limited experience, and is more or less a guess. Naively one may think a lattice system on a manifold induces a framing, but we hope there is rotational invariance which allows us to formulate the theory on more general manifolds. As orientations are typically used to define local Hamiltonians (3.8), we require invariance only under rotations connected to the identity. Therefore, for purely bosonic systems we choose $\mathcal{F}$ to include an orientation, and if the theory includes fermions then we augment this to a spin structure. In the fermionic case one can think that the coupling of such a system to background gravity involves spacetime spinor fields, whence the necessity of a spin structure. There is an additional field—a background principal bundle—if we gauge a global symmetry, as we elaborate below in §3.3.

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29 I thank Xiao-Gang Wen for explaining this point to me.
(5) As already discussed in Remark 3.10 we take $F$ to be a theory defined in $d$ space dimensions which defines complex numbers only on special closed $(d+1)$-manifolds. In Remark 3.10 we explain then that integers are obtained for special closed $(d+2)$-manifolds. ‘Special’ in both cases means the manifold is equipped with a rank $d$ vector bundle and a stable isomorphism with the tangent bundle.

(6) Finally, we assume $F$ is invertible, which is the hypothesis of no long-range entanglement, referred to as short-range entanglement. The macroscopic definition given in some literature (for example [VS]) is that $\dim F(Y) = 1$ for all closed $d$-manifolds. This is precisely invertibility at this level—dimension $d$—of the field theory. It is not too much of a stretch to extrapolate that to invertibility at all levels. The theorem mentioned in footnote 10 supports this extrapolation.

3.3. Symmetries and SPT phases

Suppose that a Lie group $G$ acts as global symmetries of a condensed matter system. This can include internal symmetries which act on the local Hilbert space $\mathcal{H}_s = V$ in (3.7). It can also include time-reversal symmetry, since we consider time as external to space and so time-reversal symmetries fix the points of space. We do not, however, consider symmetries which move points of space since we want to consider the theory on an arbitrary $d$-manifold $Y$. For example, this rules out rotation and reflection symmetries of theories on Euclidean space.

Just as we study the condensed matter system on arbitrary space manifolds $Y$ to explore its low energy behavior (coupling to gravity), to explore the effect of the global symmetry we attempt to construct an equivariant extension, as in §2.4. Recall that this means that we augment the set of fields to include a $G$-connection on a principal $G$-bundle. As mentioned after Example 2.21 there may be obstructions to constructing a $G$-invariant extension. We assume that any obstruction, if it exists, can be expressed as an anomaly theory in one higher dimension. Passing to the low energy approximation, we stipulate that there is a topological anomaly theory $\alpha$ and the original long-range effective theory $F$ is extended to be an anomalous theory $\tilde{F}$ with anomaly $\alpha$. The topological theories $\alpha$ and $\tilde{F}$ have a principal $G$-bundle as an additional background field—the choice of connection does not appear in the classification of effective topological theories. (See the discussion in §2.8.) Also, both theories are truncated only on manifolds of dimension $\leq d$, as in §3.2(4). As for any anomaly theory, $\alpha$ is invertible. We make the hypothesis that for short-range entanglement the $G$-equivariant long-range topological theory $\tilde{F}$ is also invertible. When we come to classify anomalies in §5.2 we use the considerations of §3.2(2) to guide the choice of topology on the space of anomaly theories.

In summary, then, we will assume one of two cases. Either the original long-range effective theory $F$ extends to a $G$-equivariant theory $\tilde{F}$, or there is an anomaly $\alpha$ and there is an anomalous $G$-equivariant extension $\tilde{F}$. In both cases the original set of fields $\mathcal{F}$ in $F$ is augmented to

\begin{equation}
\tilde{\mathcal{F}} = \mathcal{F} \cup \{G\text{-bundle}\}.
\end{equation}

There is an embedding $\mathcal{F} \rightarrow \tilde{\mathcal{F}}$ which chooses the trivial $G$-bundle. Restriction along this map allows us to study the effective theory $\tilde{F}$ ignoring the symmetry $G$. This restriction is simply the
theory $F$. A topological phase is said to be *symmetry protected* if this restriction $F$ is the trivial theory. (Notice that the anomaly theory $\alpha$ is trivialized under this restriction since the original long-range effective theory $F$ is an absolute theory—it has no anomaly.)

4. Bordism and homotopy theory

In this section we recall the *Madsen-Tillmann* spectra which appear in the study of invertible topological field theories. In the notation of §2.7 the simplest of these is the spectrum

\[(4.1) \quad \Sigma^n \text{MTO}_n \cong |\text{Bord}_n| \]

which is the geometric realization of the bordism category, obtained by inverting all morphisms. The Madsen-Tillmann spectra were introduced in [MT] and versions of (4.1)—including nontrivial fields $\mathcal{F}$—are proved in [MaWe], [GMTW], [Ay]. Those theorems treat the geometric realization of a topological 1-category, whereas the right hand side of (4.1) is the geometric realization of an $(\infty, n)$-category. The $(\infty, n)$ version is stated in [L, §2.5] and the techniques to prove it are most likely contained in the cited references. We proceed to use the $(\infty, n)$ statement as the basis for our proposal in §5.2. We give a brief introduction in §4.1; the class notes [F3] contain much more detail. In §4.2 we note the modifications to include symmetry and the modifications for theories of $H$-type discussed in Remark 3.10. We also explain orientation-reversal and unitarity in this context.

4.1. Madsen-Tillmann spectra

Pontrjagin studied smooth maps

\[(4.2) \quad f: S^{n+q} \to S^q \]

by choosing a regular value $p \in S^q$ and focusing on the inverse image $M = f^{-1}(p) \subset S^{n+q}$, an $n$-dimensional closed submanifold. It carries an additional topological structure. Namely, for all $m \in M$ the differential

\[(4.3) \quad df_m: T_mS^{n+q}/T_mM \to T_pS^q \]

is an isomorphism from the normal space to $M$ at $m$ to a fixed vector space, the tangent space of $S^q$ at $p$. This is a *framing* of the normal bundle. We emphasize that it is a *normal*, rather than tangential, framing. The submanifolds for different regular values and homotopic maps are all *framed bordant* in the sense that for any two $M_0, M_1$ there exists a compact $(n+1)$-dimensional manifold $N \subset [0, 1] \times S^{n+q}$ with boundary $\{0\} \times M_0 \sqcup \{1\} \times M_1$; the manifold $N$ carries a normal

---

30The ‘$\Sigma^n$’ denotes a shift, or suspension, and is present because of an unfortunate indexing convention. What should appear on the left hand side of (4.1) is the 0-space of the Madsen-Tillmann spectrum, which is standardly denoted ‘$\Omega^\infty \Sigma^n \text{MTO}_n$’, but we blur the notation.
framing which agrees with that of \( M_0 \) and \( M_1 \) on the boundary. This is the fundamental connection between bordism and homotopy theory [Mi, §7], which was elaborated and given computational punch by Thom in his PhD thesis [Th]. If we seek to understand not \( n \)-dimensional closed submanifolds of a fixed sphere \( S^{n+q} \) but rather \( n \)-dimensional abstract closed framed manifolds, then a basic theorem of Whitney tells that every abstract manifold embeds in some sphere, so it suffices to take \( q \) large. We increase \( q \) by suspending (4.2); large \( q \) is realized by iterated suspension. The suspension of (4.2) is a map between the suspension of the domain and codomain spheres, and the suspension of a sphere is a sphere of dimension one greater. A sequence of spaces related by suspension is a spectrum, and Thom introduced special spectra to compute bordism groups. The manifolds classified by these bordism groups may carry geometric structures (framings, orientations, spin structures, ...) on their stable normal bundle.

The bordism question for invertible topological field theory differs in a fundamental way. A field theory has a definite spacetime dimension \( n \), and the geometric structures live on the \( n \)-dimensional tangent bundle. So whereas Thom’s theory is stable normal, the Madsen-Tillmann spectra which arise encode unstable\( ^{31} \) tangential bordism. We sketch the basic example \( MTO_n \) and indicate the modifications \( MTSO_n \) for oriented \( n \)-manifolds and \( MTS\text{Spin}_n \) for spin \( n \)-manifolds.

Formally, a prespectrum \( T \) is a sequence \( \{ T_q \}_{q \in \mathbb{Z}} \) of pointed spaces and pointed maps \( s_q: \Sigma T_q \rightarrow T_{q+1} \), where ‘\( \Sigma \)’ denotes suspension. It is a spectrum if the induced maps \( t_q: T_q \rightarrow \Omega T_{q+1} \) are homeomorphisms, where ‘\( \Omega \)’ denotes based loops. Any prespectrum has an associated spectrum, and furthermore it suffices to define \( T_q \) for \( q \geq q_0 \) for some \( q_0 \in \mathbb{Z} \). The simplest example, indicated above, is the sphere prespectrum with \( T_q = S^q \). We now construct a prespectrum whose associated spectrum is \( MTO_n \) for a fixed \( n \in \mathbb{Z}^\geq 0 \).

Let \( Gr_n(W) \) be the Grassmannian of \( n \)-dimensional subspaces of the real vector space \( W \). A point of \([V] \in Gr_n(W)\) is an \( n \)-dimensional subspace \( V \subset W \). The Grassmannian is a smooth manifold, and there is a tautological rank \( n \) universal subbundle

\[
(4.4) \quad S \rightarrow Gr_n(W)
\]

whose fiber at \([V]\) is \( V \). Even better, there is a tautological exact sequence

\[
(4.5) \quad 0 \rightarrow S \rightarrow W \rightarrow Q \rightarrow 0
\]

of vector bundles; the fiber of the universal quotient bundle \( Q \rightarrow Gr_n(W) \) at \( V \in Gr_n(W) \) is the quotient vector space \( W/V \), and the vector bundle \( W \rightarrow Gr_n(W) \) has constant fiber \( W \). For any integer \( q > 0 \) we define the space \( T_{n+q} \) to be the Thom space of the universal quotient bundle

\[
(4.6) \quad Q(q) \rightarrow Gr_n(\mathbb{R}^{n+q}).
\]

The Thom space is obtained from the manifold \( Q(q) \) by introducing an inner product on the vector bundle (4.6) and collapsing the subspace of all vectors of norm \( \geq R \) to a point, where \( R > 0 \) is any

\( ^{31} \)MT spectra are still part of stable homotopy theory—they are spectra—but the dimension is fixed, not stabilized.
real number, as indicated in Figure 6. The structure map \( s_{n+q} \) is obtained by applying the Thom space construction to the map

\[
\begin{array}{ccc}
\mathbb{R} \oplus Q(q) & \longrightarrow & Q(q+1) \\
\downarrow & & \downarrow \\
Gr_n(\mathbb{R}^{n+q}) & \longrightarrow & Gr_n(\mathbb{R}^{n+q+1})
\end{array}
\]

(4.7)

The bottom arrow takes a subspace \( V \subset \mathbb{R}^{n+q} \) and regards it as a subspace of \( \mathbb{R}^{n+q+1} \) all of whose vectors have first coordinate zero.

![Figure 6. The Thom space of a vector bundle \( V \to M \)](image)

More useful to us is the \( n \)th suspension \( \Sigma^n MTO_n \), which is represented by the shift of this prespectrum with \( (\Sigma^n T)_q = T_{n+q} \). A map \( S^m \to \Sigma^n MTO_n \) is represented by a pointed map \( S^{m+q} \to T_{n+q} \) for some \( q \) large, so a map \( f: S^{m+q} \to Q(q) \) which sends the basepoint of \( S^{m+q} \) to a vector of norm \( \geq R \). Suppose the map is transverse to the zero section \( Z \subset Q(q) \). Then the pullback \( M := f^{-1}(Z) \subset S^{m+q} \) is a submanifold of dimension \( m \). Its normal bundle is identified via \( df \) with the normal bundle to \( Z \) in \( Q(q) \), which in turn is identified with the vector bundle (4.6).

But because of the degrees it is more natural to consider the virtual bundle \( Q(q) - \mathbb{R}^{n+q} \approx -S(n) \) whose pullback is then identified with the negative of the tangent bundle to \( M \), stabilized to have rank \( n \). (Recall that \( S(n) \to Gr_n(\mathbb{R}^{n+q}) \) is the universal subbundle (4.4).) The choice of embedding into a sphere \( S^{m+q} \) disappears in the limit \( q \to \infty \) and, as in Thom’s bordism theory, we obtain abstract manifolds rather than embedded ones.

**Remark 4.8.** The pullback of \( S(n) \to Gr_n(\mathbb{R}^{n+q}) \) to \( M \) is a rank \( n \) bundle which is equipped with a stable isomorphism to the tangent bundle \( TM \).

More generally, for any smooth manifold \( S \), a map \( S \times S^m \to \Sigma^n MTO_n \) leads to a smooth proper map \( \pi: N \to S \) with \( \dim N - \dim S = m \) and with a rank \( n \) vector bundle \( V \to N \) equipped with a stable isomorphism with \( TN - \pi^*TS \). The Galatius-Madsen-Tillmann-Weiss theorem [GMTW] asserts that the spectrum \( \Sigma^n MTO_n \) classifies a bordism theory of proper fiber bundles, rather than arbitrary proper maps \( \pi \).

**Remark 4.9.** There is a sequence of maps

\[
\Sigma^1 MTO_1 \longrightarrow \Sigma^2 MTO_2 \longrightarrow \Sigma^3 MTO_3 \longrightarrow \cdots
\]

(4.10)
whose “limit” is the Thom spectrum $MO$ which classifies unoriented manifolds. In this precise sense the Madsen-Tillmann spectra approximate Thom spectra.

To construct the Madsen-Tillmann spectra $MTSO_n$ ($MTSpin_n$) use the Grassmannian of oriented (spin) subspaces.

Quite generally, a map $S \rightarrow T$ of spectra represents a $T$-cohomology class on $S$. The $T$-cohomology of a Madsen-Tillmann spectrum $S = \Sigma^n MTSO_n$ is isomorphic to the $T$-cohomology of the space $BSO_n$ by the Thom isomorphism. For example, if $T = \Sigma^q H\mathbb{Z}$ then there is a Thom isomorphism

$$\Sigma^n MTSO_n, \Sigma^q H\mathbb{Z} \cong H^q(BSO_n; \mathbb{Z}).$$

There is an orientation condition which is satisfied here because we consider the Madsen-Tillmann spectrum with group $SO_n$. If instead we use $O_n$, then we obtain cohomology with twisted coefficients:

$$\Sigma^n MTO_n, \Sigma^q H\mathbb{Z} \cong H^q(BO_n; \mathbb{Z}^w),$$

where $\mathbb{Z}^w \rightarrow BO_n$ is the nontrivial local system with holonomy $-1$ around the nontrivial loop in $BO_n$. The general form of the twisted Thom isomorphism is

$$\Sigma^n MTO_n, \Sigma^q T \cong T^{\tau+q}(BO_n),$$

where $\tau$ is the twisting of $T$-cohomology defined by the virtual vector bundle $\mathbb{R}^n - S(n) \rightarrow BO_n$. There are similar statements for the groups $SO_n$ and $Spin_n$. We refer to [ABGHR] and references therein for a modern treatment of twisted cohomology, orientations, and the Thom isomorphism.

### 4.2. Variations

#### 4.2.1. Global symmetry groups.

Let $G$ be a Lie group. As explained in §3.3 the long-range effective topological theory approximating a condensed matter system with global symmetry group $G$ includes a $G$-bundle among its background fields; see (3.12). Such a bundle can be obtained from a universal $G$-bundle $EG \rightarrow BG$ by pullback. To incorporate the bundle into the Madsen-Tillmann spectrum $MTO_n$ replace (4.6) with the pullback bundle

$$Q(q) \rightarrow Gr_n(\mathbb{R}^{n+q}) \times BG$$

over the Cartesian product. A map $f : S^{m+q} \rightarrow Q(q)$ which is transverse to the zero section of (4.14) gives a manifold $M \subset S^{m+q}$ and a $G$-bundle $P \rightarrow S^{m+q}$. The structure maps defined from (4.7) extend to incorporate the $BG$ factor. The spectrum so obtained is denoted

$$MTO_n \wedge BG_+$$

The wedge, pronounced “smash”, is the appropriate product for pointed spaces; the ‘$+$’ denotes a disjoint basepoint.\(^{32}\) Of course, there are similar spectra to (4.15) for oriented and spin manifolds.

---

\(^{32}\)The classifying space $BG$ does have a basepoint which represents the trivial $G$-bundle, and we use it at the end of §5.2 to implement the embedding described after (3.12) in the discussion of SPT phases. The basepoint of $BG$ is ignored in the construction of $BG_+ = BG \amalg pt$, the disjoint union of $BG$ and a point.
4.2.2. Theories of $H$-type. In a $d$-space dimensional theory we obtain numerical invariants only for $(d + 1)$-manifolds which are essentially products: time is not mixed with space; see Remark 3.10. In terms of (2.8) an “$H$-type” theory is defined on the subcategory of $\text{Bord}_{d+1}(\mathcal{F})$ which only contains top dimensional bordisms which are fibered over 1-manifolds:

$$F: \text{Bord}_d(\mathcal{F}) \rightarrow \mathcal{C}$$

An invertible theory of that form, if $\mathcal{F}$ is empty, is a map out of the spectrum $\Sigma^d MTO_d$. In terms of the explicit prespectrum described in §4.1, the $q^{th}$-space of $\Sigma^d MTO_d$ is the Thom space of

$$Q(q) \rightarrow \text{Gr}_d(\mathbb{R}^{d+q}).$$

We explained in the paragraph after (4.7) that manifolds in the usual $(d + 1)$-spacetime dimensional theory have tangent bundles stabilized to rank $d + 1$. Here in the space theory they are stabilized to rank $d$. So to get a bundle of rank $d + 1$ we simply add an extra rank one trivial bundle $\mathbb{R}$; the fiber represent time, which is visibly a product and does not mix with space.

4.2.3. Time-reversal symmetries. The general picture of symmetries in quantum mechanics (§2.4.3) distinguishes time-preserving vs. time-reversing (2.24) from linear vs. antilinear (2.23). Often, of course, they are equal dichotomies. In that sense an accounting of antilinearity suffices to account for time-reversal. But more to the point, as we consider theories of $H$-type (§4.2.2) in which there is no explicit time, there is no need to track (2.24) other than through antilinearity.

4.2.4. Orientation-reversal and unitarity. This is to implement the unitarity (§2.5). The geometric realization of the oriented bordism category $\text{Bord}_n(\mathcal{F})$ is $\Sigma^n MTSO_n$, if $\mathcal{F}$ consists of an orientation. Thus orientation-reversal induces an involution on $\Sigma^n MTSO_n$. Introducing the notation $MV$ for the Thom spectrum of the virtual vector bundle $V \to M$, we can summarize (4.7) as

$$\Sigma^n MTSO_n = BSO_n \mathbb{R}^{n-S(n)},$$

where $S(n) \to BSO_n$ is the universal rank $n$ bundle. Orientation-reversal on rank $n$ bundles is represented as the deck transformation of the double cover

$$BSO_n \rightarrow BO_n.$$ 

There is an induced involution on the Thom spectrum (4.18).

In a unitary theory orientation-reversal maps to complex conjugation. Complex conjugation on $\mathbb{C}^x$ corresponds to the involution

$$z \mapsto -\bar{z}$$

on $\mathbb{C}/\mathbb{Z}$ via exponentiation. Let $H\mathbb{C}^x$ be the target of an invertible unitary theory $\Sigma^n MTSO_n \to \Sigma^n H\mathbb{C}^x$. This represents a twisted cohomology class of $\Sigma^n MTO_n$. The twisted Thom isomorphism theorem (4.12) identifies this twisted cohomology group with

$$H^n(BO_n; \mathbb{C}/\mathbb{Z}),$$
where the coefficients are twisted by the complex conjugation action (no sign). Note the short exact sequence

\[
\begin{align*}
0 & \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0,
\end{align*}
\]

which is equivariant for the \((-1)\)-action on \(\mathbb{Z}\) and the action \((4.20)\) on the other two groups. So when we map to \(H\mathbb{Z}\) or \(I\mathbb{Z}\) we use the \((-1)\)-action in place of the action \((4.20)\) on \(HC/\mathbb{Z}\) and \(IC/\mathbb{Z}\). We remark that the twisted Thom isomorphism involves more complicated twistings for \(IC/\mathbb{Z}\).

An analogous construction works if we replace \(SO_n\) with \(\text{Spin}_n\) and \(O_n\) with \(\text{Pin}_n\). However, there are two choices for \(\text{Pin}_n\). If we view \(\text{Pin}_n\) as embedded in the real Clifford algebra \(\text{Cliff}_n\), then the choices depend on the sign in the relation \(\gamma_2^2 = \pm 1\) which defines \(\text{Cliff}_n\); see [ABS]. So it seems there are two distinct notions of unitarity for spin theories. We will simply write ‘Pin’ and not investigate the distinction further in this paper.

We now sketch precisely how we implement the \((-1)\)-involution on spectra.

**Construction 4.23** \((-1)\)-action on spectra. Let \(H \to \mathbb{R}P^\infty\) denote the real Hopf line bundle. The Thom spectrum \((\mathbb{R}P^\infty)^H - \mathbb{R}\) of the reduced Hopf bundle is a bundle, or sheaf, of spectra over \(\mathbb{R}P^\infty\) whose typical fiber is the sphere spectrum. The holonomy around the nontrivial loop acts on the homotopy groups \(\pi_* S^0\) of the sphere as multiplication by \(-1\). In other words, for each \(q\) the homotopy groups make a local system over \(\mathbb{R}P^\infty\) with fiber \(\pi_q S^0\) and holonomy \(-1\). Any other spectrum \(T\) is a module over \(S^0\) and there is an induced bundle of spectra over \(\mathbb{R}P^\infty\) with fiber \(T\); the holonomy acts on \(\pi_* T\) as multiplication by \(-1\).

See [ABGHR] for a precise definitions, statements, and proofs of these assertions about bundles of spectra and the twisted Thom isomorphism.

### 4.2.5. Interlude on unitarity.

See §2.5 for a general discussion of unitarity, and §4.2.4 for its implementation in invertible theories. Here we compute and interpret groups of low-dimensional oriented unitary theories, using \((4.21)\).

The group of \(n = 1\) spacetime dimensional oriented theories with Eilenberg-MacLane target is

\[
\begin{align*}
[S^1 MTSO_1, S^1 HC/\mathbb{Z}] & \cong H^1(BSO_1; \mathbb{C}/\mathbb{Z}) = 0;
\end{align*}
\]

there is only the trivial theory \(F\) which assigns the trivial line \(L = \mathbb{C}\) to the oriented point \(pt_+\) and the number 1 to the oriented circle. This theory is clearly unitarizable. A unitary structure is data, and according to \((4.21)\) the group of isomorphism classes of unitarity data is

\[
\begin{align*}
H^1(BO_1; \mathbb{C}/\mathbb{Z}) & \cong H^1(\mathbb{R}P^\infty; \mathbb{C}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},
\end{align*}
\]

where the local system \(\mathbb{C}/\mathbb{Z} \to BO_1\) has holonomy \(-1\) around the nontrivial loop.\(^{33}\) The two equivalence classes of unitarity data on \(F\) are easily explained. Writing \(F(pt_+) = L\) the unitarity data

\[^{33}\text{Here are two methods to compute (4.25): (1) use a cell structure on \(\mathbb{R}P^\infty\) with a single cell in each dimension, trivialize the local system over each cell, and derive the cochain complex}
\]

\[
\begin{align*}
\mathbb{C}/\mathbb{Z} & \xrightarrow{1-c} \mathbb{C}/\mathbb{Z} \xrightarrow{1+c} \mathbb{C}/\mathbb{Z} \xrightarrow{1-c} \cdots
\end{align*}
\]

in which \(c\) is complex conjugation; (2) use the short exact coefficient sequence \((4.22)\) where \(\mathbb{C}, \mathbb{C}/\mathbb{Z}\) are local systems with holonomy \(-1\) and \(\mathbb{Z}\) is untwisted.
provides an isomorphism \( F(\text{pt}_-) \xrightarrow{\cong} \mathcal{T} \). The oriented interval with both endpoints incoming is a bordism \( \text{pt}_- \amalg \text{pt}_+ \to \mathcal{D}_0 \) and applying \( F \) and the unitarity isomorphism we obtain a nondegenerate hermitian form \( h: \mathcal{T} \otimes L \to \mathbb{C} \). Such a form is either positive or negative, which accounts for (4.25).

**Remark 4.27.** The **positivity** condition in a unitary theory means we should exclude the negative form, so only allow a unique isomorphism class of unitarity data. The group (4.21) only implements the correct action of orientation-reversal, not the positivity, but I do not know how to pick out the subgroup corresponding to positive unitarity data in higher dimensions.

As a further illustration we consider \( n = 2 \) spacetime dimensional oriented theories with Eilenberg-MacLane target. These theories are discussed in Example 2.11 and Example 2.31. The group of oriented theories is

\[
[\Sigma^2 MTSO_2, \Sigma^2 H \mathbb{C}/\mathbb{Z}] \cong H^2(BSO_2; \mathbb{C}/\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z},
\]

and the theory \( F_z \) corresponding to \( z \in \mathbb{C}/\mathbb{Z} \) has

\[
F_z(X) = \lambda^\text{Euler}(X)
\]

for a closed oriented 2-manifold \( X \), where \( \lambda = e^{2\pi iz} \in \mathbb{C}^\times \). Since \( X \) has an orientation-reversing involution, only those theories with \( F_z(X) \) real can possibly be unitarizable, which forces \( z \in i\mathbb{R} \). That is consistent with the computation of (4.21) for \( n = 2 \):

\[
H^2(BO_2; \mathbb{C}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus i\mathbb{R},
\]

together with the computation of the map

\[
H^2(BO_2; \mathbb{C}/\mathbb{Z}) \longrightarrow H^2(BSO_2; \mathbb{C}/\mathbb{Z}),
\]

which includes \( \frac{1}{2}\mathbb{Z}/\mathbb{Z} \oplus i\mathbb{R} \hookrightarrow \mathbb{C}/\mathbb{Z} \). For \( z = 1/2 - ix \) we have (4.29) with \( \lambda = -e^{2\pi x} \), but since the Euler number is even the numerical invariants are positive. The line \( F_z(S^1) = L \) attached to the oriented circle has a real structure, since \( S^1 \) has an orientation-reversing involution (reflection), and the cylinder with both boundaries incoming gives a nondegenerate real symmetric bilinear form \( h: L \otimes L \to \mathbb{C} \). Glue 2-disks to each incoming boundary component to deduce that \( h(\ell, \ell) = \lambda^2 \) is positive. The conclusion is that all theories with parameter \( \lambda \in \mathbb{R}^\times \subset \mathbb{C}^\times \) are uniquely unitarizable.

5. SRE phases

There is one more preliminary before we can state our proposed topological invariant of short-range entangled (SRE) phases: we must specify our choice of target spectrum for classifying long-range effective topological theories. Recall that in general to define a topological field theory (2.8)
we need to specify a target higher category $\mathcal{C}$, but for an invertible theory we need the much weaker information of the sub-groupoid $C^\times$ of invertibles; see (2.29). While concrete arguments give information about the highest few homotopy groups of $C^\times$, we can only guess at the structure lower down, which is increasingly relevant as the dimension of the theory increases. In the general fermionic case we take a universal choice, the Brown-Comenetz dual to the sphere spectrum. In the bosonic case we have solid arguments to determine the “top part” of the spectrum, but after that only sparse data points. For that reason we take the same target spectrum as in the fermionic case—after all, theories with only bosons are special cases of general theories—though in space dimension $d = 1$ the “top part”, an Eilenberg-MacLane spectrum, is all that is relevant and so we use it instead. We explain these choices in §5.1. Our main proposal is stated precisely in §5.2. In §5.3 we work out the relationship to the group cohomology classifications in the literature $[CGLW]$, $[GW]$.

5.1. Target spectra

The discussions in Example 2.31 and especially §3.2(2) are relevant here.

5.1.1. Preliminary: duals to the sphere spectrum. Let $A$ be an abelian group, which might be discrete or have a topology. (In the latter case there is an additional hypothesis: $A$ is locally compact.) Examples: $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{R}/\mathbb{Z}$. There are two notions of dual group we might consider. The first is the Pontrjagin dual, which is the group of continuous homomorphisms $A \to \mathbb{R}/\mathbb{Z}$. The Pontrjagin dual of $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, the Pontrjagin dual of $\mathbb{Z}$ is isomorphic to $\mathbb{R}/\mathbb{Z}$, and the Pontrjagin dual of $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{Z}$. On the other hand, we can consider an integral dual $\text{Hom}(A, \mathbb{Z})$. However, the naive interpretation is not so well behaved. For example, if $A = \mathbb{Z}/n\mathbb{Z}$ there are no nonzero homomorphisms $A \to \mathbb{Z}$. The resolution is to consider $\text{Hom}(A, \mathbb{Z})$ in the derived sense, which means that we replace $A$ by a free chain complex whose homology in degree 0 is $A$ and then compute $\text{Hom}$. The result is called $\text{Ext}^\bullet(A, \mathbb{Z})$ and is the “correct” integral dual.

Both notions of duality exist in the world of spectra; see $[An]$, $[HS$, Appendix B], $[FMS$, Appendix B], $[HeSt]$ for precise definitions and discussion. Recall that a spectrum $T_*$ has an associated $\mathbb{Z}$-graded abelian group $\pi_* T$. For the sphere spectrum $S^0$ the first several homotopy groups are

$$\pi_{\{0,1,2,\ldots\}} S^0 \cong \{ \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/24\mathbb{Z}, 0, 0, \ldots \}$$

The analog of the Pontrjagin dual is the Brown-Comenetz dual. For the sphere we denote it as $IC/\mathbb{Z}$ (We replace $\mathbb{R}/\mathbb{Z}$ with $\mathbb{C}/\mathbb{Z}$ in topology it is often $\mathbb{Q}/\mathbb{Z}$ instead.) Its homotopy groups are the Pontrjagin dual groups to (5.1):

$$\pi_{\{0,-1,-2,\ldots\}} IC/\mathbb{Z} \cong \{ \mathbb{C}/\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/24\mathbb{Z}, 0, 0, \ldots \}$$

Note carefully the indexing difference on the left hand side between (5.2) and (5.1). The analog of the integral dual is the Anderson dual. For the sphere we denote it as $IZ$. Its homotopy groups
are the integral dual to (5.1); the torsion groups are degree shifted:

\[(5.3) \quad \pi_{\{0, -1, -2, \ldots\}} \mathbb{Z} \cong \{\mathbb{Z}, 0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/24\mathbb{Z}, 0, 0, \ldots\}\]

There is a fiber sequence \(I \mathbb{Z} \to I \mathbb{C} \to I \mathbb{C}^n\) and a corresponding long exact sequence of homotopy groups. Note \(I \mathbb{C} \to H \mathbb{C}\) is an Eilenberg-MacLane spectrum with a single nonzero homotopy group.

These spectra enjoy universal properties which make them universal targets for invertible field theories. Recall that the notation \([X, X']\) is used for the abelian group of homotopy classes of maps \(X \to X'\) between spectra. If \(X\) is any spectrum, then

\[(5.4) \quad [X, \Sigma^n I \mathbb{C}/\mathbb{Z}] = I \mathbb{C}/\mathbb{Z}^n(X) \cong \text{Hom}(\pi_n X, \mathbb{C}/\mathbb{Z}).\]

(To compute the \(I \mathbb{C}/\mathbb{Z}\) cohomology of a space \(Y\), set \(X = \Sigma^n Y\) the suspension spectrum, so \(\pi_n X\) is the \(n^{th}\) stable homotopy group of \(Y\).) Thus an invertible \(n\)-spacetime dimensional theory whose values on closed \(n\)-manifolds are numbers in \(\mathbb{C}/\mathbb{Z}\) pushes uniquely to a field theory with values in \(I \mathbb{C}/\mathbb{Z}\), and any theory with target \(I \mathbb{C}/\mathbb{Z}\) is determined by its numerical values on \(n\)-manifolds. For maps from any spectrum \(X\) into \(I \mathbb{Z}\) there is a short exact sequence

\[(5.5) \quad 0 \to \text{Ext}^1(\pi_{n-1} X, \mathbb{Z}) \to I \mathbb{Z}^n(X) \to \text{Hom}(\pi_n X, \mathbb{Z}) \to 0\]

which is split, but not canonically.

5.1.2. **Bosonic theories.** We continue with theories \(F\) defined in \(d\)-space dimensions and assume first that \(F\) is a bosonic theory. The target is a \((d + 1)\)-category \(\mathcal{C}^x_{\text{bose}}\), but since \(F\) is invertible we need only the groupoid \(\mathcal{C}^x_{\text{base}}\). Standard quantum mechanics dictates that \(F(Y)\) is a complex vector space for any closed \(d\)-manifold \(Y\), and since \(F\) is invertible it must be 1-dimensional. Furthermore, a diffeomorphism of \(Y\) acts as an invertible map \(F(Y) \to F(Y')\), which is multiplication by a scalar in \(\mathbb{C}^x\). Thus the truncation \(\Omega^d\mathcal{C}^x_{\text{base}}\) of the target groupoid to the top two levels is the groupoid Line of complex lines. A crucial point, discussed in §3.2(2), is that we use the continuous topology on the morphism spaces, in particular on \(\mathbb{C}^x\). This gives information about some homotopy groups of \(\mathcal{C}^x_{\text{base}}\):

\[(5.6) \quad \pi_d \mathcal{C}^x_{\text{base}} = 0, \quad \pi_{d+1} \mathcal{C}^x_{\text{base}} = 0, \quad \pi_{d+2} \mathcal{C}^x_{\text{base}} \cong \mathbb{Z}.\]

The first equality expresses the fact that any two lines are isomorphic; the latter two that \(\mathbb{C}^x\) is connected with infinite cyclic fundamental group. Furthermore, \(\pi_k \mathcal{C}^x_{\text{base}} = 0\) for \(k > d + 2\) since there are no non-identity \(k\)-morphisms for \(k > d + 1\).

It remains to determine the lower homotopy groups. We have quite solid information about the next homotopy group down, as we know the nature of the higher categorical invariant usually attached to manifolds of dimension \(d - 1\) by the theory \(F\). For example, a standard choice is to

---

\(^{34}\)In a unitary theory this scalar has unit norm, but the partition function of a general \((d + 1)\)-manifold need not.
assign a linear category to a closed \((d - 1)\)-manifold, and since all invertible linear categories are isomorphic deduce

\[(5.7) \quad \pi_{d-1}O^\times_{\text{bose}} = 0.\]

An alternative choice is to assign an invertible complex algebra to a closed \((d - 1)\)-manifold, and the triviality of the Brauer group of \(\mathbb{C}\) leads to the same conclusion \((5.7)\). At this stage we have reproduced the first part of Example 2.31, in which the target is a 2-category and there are no further homotopy groups. Note that \((5.6)\) and \((5.7)\) amount to an Eilenberg MacLane spectrum \(\Sigma^{d+2}HZ\): a single nonzero homotopy group. This is the full story for \(d \leq 1\).

What is perhaps surprising is that we postulate a nonzero homotopy group at the next level down:

\[(5.8) \quad \pi_{d-2}O^\times_{\text{bose}} \neq 0.\]

One rationale for \((5.8)\) is the 4-dimensional integral invertible oriented extended topological field theory

\[(5.9) \quad \sigma : \text{Bord}_4(\text{orientation}) \longrightarrow C^\times_{\text{bose}}(d = 2)\]

for which the integer invariant \(\sigma(W)\) of a closed oriented 4-manifold \(W\) is its signature \(\text{Sign}(W)\). In its analytic incarnation this invariant involves only differential forms on \(W\), not spinor fields, so in that sense is bosonic. The theory factors through a map

\[(5.10) \quad \Sigma^4 MTSO_4 \longrightarrow \Sigma^4 KO\]

which can be viewed as the universal symbol \([\text{FHT, } \S 3]\) of the signature operator in dimension 4. The relevant stretch of homotopy groups of the target spectrum in \((5.10)\) is

\[(5.11) \quad \pi_{\{0,1,2,3,4\}} \cong \{\mathbb{Z}, 0, 0, 0, \mathbb{Z}\}.\]

This theory does not factor through \(\Sigma^4HZ\), so the bottom \(\mathbb{Z}\) in \((5.11)\) cannot be replaced by 0. Another piece of evidence for at least a nonzero homotopy group in this spot \((\pi_{d-2}C^\times_{\text{bose}})\) comes from conformal nets. These are a possible target for 3-dimensional theories with numerical invariants in \(\mathbb{C}/\mathbb{Z}\), and conjecturally not all invertible conformal nets are isomorphic. (See \([\text{DH}]\) for a discussion of conformal nets in this context.)

We do not have any information about lower homotopy groups. So we could take the shifted Postnikov truncation \(\Sigma^{d-6}KO\langle 4, \ldots, 8 \rangle\) as a reasonable choice of target spectrum \(C^\times_{\text{bose}}\). But the \(\mathbb{Z}\) at the bottom of the spectrum is overkill—a smaller torsion group will do—and so instead for theories in dimension \(d \geq 2\) we use the universal choice, the Anderson dual of the sphere.

**Hypothesis 5.12.** The target spectrum for classifying long-range effective theories of a \(d\)-space dimensional bosonic system is \(\Sigma^3HZ\) for \(d = 1\) and \(\Sigma^{d+2}IZ\) for \(d \geq 2\).

**Remark** 5.13. The classification of anomalies has similar target spectra, but as indicated in \(\S 3.2(2)\) we use instead the discrete topology on \(\mathbb{C}^\times\). Thus for \(d \geq 2\) the target which classifies anomaly theories is \(\Sigma^{d+2}IC/\mathbb{Z}\); for \(d = 1\) it is \(\Sigma^3HC/\mathbb{Z}\).
5.1.3. Fermionic theories. The hypothesis for theories with fermions is different. Namely, the dichotomy between bosonic and fermionic states in quantum mechanical systems is encoded by stipulating that the quantum Hilbert space be \( \mathbb{Z}/2\mathbb{Z}\)-graded: states with even grading are bosonic and states with odd grading are fermionic. That persists in the long-range effective theory: vacua are either bosonic or fermionic. So the complex lines \( F_p \) in an invertible long-range theory are either even or odd. The existence of distinct isomorphism classes of \( \mathbb{Z}/2\mathbb{Z}\)-graded lines modifies (5.6):

\[
\pi_d \hat{C} \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_{d+1} C \cong 0, \quad \pi_{d+2} C \cong \mathbb{Z}.
\]

The \( k \)-invariant is nonzero; it is the composition \( \beta \circ Sq^2 \) of the integer Bockstein and the Steenrod square.

As in the bosonic case higher homotopy groups vanish. But now we expect many nontrivial lower homotopy groups. For example, we expect (5.7) to be replaced by

\[
\pi_{d-1} \hat{C} \cong \mathbb{Z}/2\mathbb{Z}
\]

since the super Brauer group of invertible \( \mathbb{Z}/2\mathbb{Z}\)-graded complex algebras has two elements (represented by even and odd complex Clifford algebras). The idea that modules over Clifford algebras may be used as the state space of a quantum system is familiar in condensed matter theory; see also [F5]. From the super Brauer category we compute how the three Eilenberg-MacLane spectra fit together, though we do not do so here. We can gather information about \( \pi_{d-2} \hat{C} \) by arguments analogous to those in §5.1.2. For example, there is an invertible theory

\[
\Sigma^4 MT \Spin_4 \longrightarrow KO
\]

which assigns the A-genus to a closed spin 4-manifold. (The universal symbol is quaternionic, which explains the absence of shift in \( KO \).) This suggests that \( \pi_{d-2} \hat{C} \neq 0 \). Invertible fermionic conformal nets are another clue, but the only knowledge is conjectural.

Therefore, by fiat really, we make a universal choice for the target spectrum, namely the Anderson spectrum \( \mathbb{I} \). Notice that its first four homotopy groups (5.3) agree with (5.14) and (5.15) and more precisely the Postnikov truncations are equivalent.

**Hypothesis 5.17.** The target spectrum for classifying long-range effective theories of a \( d \)-space dimensional fermionic system is \( \Sigma^{d+2} \mathbb{I} \).

**Remark 5.18.** We use the shift \( \Sigma^{d+2} \mathbb{I} \mathbb{C}/\mathbb{I} \) as a target to classify anomaly theories.

5.1.4. Antilinear symmetries. As explained in §2.4.3 antilinear symmetries lead to anomalous field theories after gauging. For invertible theories we account for the anomaly using twisted cohomology.

We implement complex conjugation on the target spectra by the universal \((-1\rangle\)-action defined in Construction 4.23. Consider, for example, the Eilenberg-MacLane spectrum \( \mathbb{I} \mathbb{C} \). In degree 1 a cohomology class is represented by a map to \( \mathbb{C} \), and the \((-1\rangle\)-action is complex conjugation on \( \mathbb{C} \). The classifying space for a degree 1 class is \( \mathbb{C} \), and Construction 4.23 gives a fiber bundle (not principal!) with fiber \( \mathbb{C} \) over \( \mathbb{R}^\mathbb{C} \) whose holonomy acts as complex conjugation on \( \mathbb{C} \). In
degree 2, as just explained, a cohomology class is represented by a complex line bundle and this action is complex conjugation on complex line bundles. In degree 3 there is a similar story with bundles of complex algebras, and it is reasonable to extend this picture to all degrees. The same story applies to $H_C/Z$, which one can view as flat elements in $H_Z$ (with a degree shift). For example, an element in degree 0 in $H_C/Z$ is a locally constant map to $\mathbb{C}^\times$. Similar considerations apply to other target spectra. The top nonzero homotopy group of $I_Z$ (respectively $I_C/Z$) is the same as that of $H_Z$ (respectively $H_C/Z$), and the $(-1)$-action is trivial on the $\mathbb{Z}/2\mathbb{Z}$ homotopy groups in (5.14) and (5.15) is trivial. Our tentative grasp on lower homotopy groups puts further justification beyond reach.

5.2. Topological invariants of short-range entangled phases

We propose a home for long-range effective topological field theories of gapped systems with short-range entanglement. We do not know if the map from the microscopic phases to the deformation classes of field theories is either injective or surjective. Regardless, the evidence presented in the remainder of the paper suggests that it is a very effective invariant.

Assume the theory is $d$-space dimensional and has a global symmetry group $G$, which is a Lie group equipped with a smooth homomorphism

\[(5.19) \quad \phi: G \longrightarrow \mu_2 = \{\pm 1\}\]

which encodes linearity vs. antilinearity: an element $g \in G$ with $\phi(g) = 1$ acts linearly and an element $g \in G$ with $\phi(g) = -1$ acts antilinearly. Recall our assumption, stated before (3.12), that there is a $G$-equivariant extension of the long-range effective theory (non-anomalous case) or that there is an anomaly and an anomalous extension (anomalous case).

The hypotheses underlying the proposal are stated in §3.2 and §5.1. We use the twisted Thom isomorphism (4.13) (and its variations for $SO$ and Spin replacing $O$).

5.2.1. Bosonic theories. According to Hypothesis 5.12 the target spectrum is $\Sigma^{d+2}H_Z$ for $d = 1$ and $\Sigma^{d+2}I_Z$ for $d \geq 2$. We use the notation ‘$\Sigma^{d+2}T_Z$’ for this target spectrum. Note that $T_Z$ is a ring spectrum. Let $\tau_{T_Z}$ denote the Thom twisting of the ring spectrum $T_Z$ associated to the virtual bundle (see (4.18))

\[(5.20) \quad \mathbb{R}^d - S(d) \longrightarrow BSO_d,\]

and $\bar{\tau}_{T_Z}$ the analogous Thom twisting for $O_d$ in place of $SO_d$. Let $w_{T_Z}$ denote the $(-1)$-twist of $T_Z$ (Construction 4.23) associated to the double cover $BSO_d \rightarrow BO_d$. The homomorphism (5.19) determines a $(-1)$-twist of $T_Z$ associated to the double cover $35 BG_0 \rightarrow BG$, where $G_0 = \ker \phi$; we denote it $\phi_{T_Z}$. Degree shifts are Thom twistings of trivial bundles, whence sums of twistings and degrees are defined.

\[35\text{If } \phi \text{ is identically } +1, \text{ then } G_0 = G \text{ and } \phi_{T_Z} = \phi \text{ is trivial.}\]
Proposal 5.21 (bosonic theories). Short-range entangled phases of a $d$-space dimensional bosonic theory with global symmetry group $G$ map to the abelian group

$$SRE_{\text{bose}}(d, G, \phi) = T\mathbb{Z}^{\tau_{IZ} + \phi_{IZ} + d + 2}(\text{BSO}_d \times B G)$$

in the non-anomalous case. The unitarizable theories lie in the image of the map\(^{36}\)

$$T\mathbb{Z}^{\tau_{IZ} + w_{IZ} + \phi_{IZ} + d + 2}(\text{BO}_d \times B G) \rightarrow T\mathbb{Z}^{\tau_{IZ} + \phi_{IZ} + d + 2}(\text{BSO}_d \times B G).$$

Anomalies are classified by the abelian group

$$\text{Anom}_{\text{bose}}(d, G, \phi) = T\mathbb{C}/\mathbb{Z}^{\tau_{IZ} + \phi_{IZ} + d + 2}(\text{BSO}_d \times B G)$$

and the anomalous theories with fixed anomaly map to a torsor for the abelian group (5.22).

An anomaly theory has spacetime dimension $d + 2$, but here is only defined on manifolds of dimension $\leq d$; this explains the degrees in (5.24). The last assertion is that any two choices of anomalous theories with the same anomaly are related by tensoring with a non-anomalous theory.

5.2.2. Fermionic theories. The proposal for fermionic theories is similar: we simply swap out the Eilenberg-MacLane spectra we used in the $d = 1$ bosonic case for the Anderson and Brown-Comenetz spectra (§5.1.3) and assume all manifolds are spin in addition to being oriented. The corresponding Thom twistings of (5.20) are denoted $\tau_{IZ}$ and $\bar{\tau}_{IZ}$; the $(-1)$-twisting of $IZ$ associated to the double cover $\text{BSO}_d \rightarrow \text{BO}_d$ is $w_{IZ}$; and the $(-1)$-twisting of $IZ$ associated to the double cover $BG_0 \rightarrow BG$ is $\phi_{IZ}$.

Proposal 5.25 (fermionic theories). Short-range entangled phases of a $d$-space dimensional fermionic theory with global symmetry group $G$ map to the abelian group

$$SRE_{\text{fermi}}(d, G, \phi) = IZ^{\tau_{IZ} + \phi_{IZ} + d + 2}(\text{BSpin}_d \times B G)$$

in the non-anomalous case. The unitarizable theories lie in the image of the map

$$IZ^{\tau_{IZ} + w_{IZ} + \phi_{IZ} + d + 2}(\text{BPin}_d \times B G) \rightarrow IZ^{\tau_{IZ} + \phi_{IZ} + d + 2}(\text{BSpin}_d \times B G).$$

Anomalies are classified by the abelian group

$$\text{Anom}_{\text{fermi}}(d, G, \phi) = I\mathbb{C}/IZ^{\tau_{IZ} + \phi_{IZ} + d + 2}(\text{BSpin}_d \times B G)$$

and the anomalous theories with fixed anomaly map to a torsor for the abelian group (5.26).

\(^{36}\)A more precise proposal would identify the subgroup of the domain of (5.23) representing unitary theories which satisfy positivity. As I do not know how to do this—see §4.2.5—we settle for a weaker formulation.
5.2.3. **Symmetry protected topological phases.** Now we address the question of symmetry protected topological (SPT) phases. The “symmetry protection” means that the effective topological field theory $F$ is trivial when the symmetry is ignored. As explained at the end of §3.3 this means that the $G$-extension $\tilde{F}$ is trivial when restricted to the trivial $G$-bundle. The basepoint of $BG$ determines an embedding $X \mapsto X \wedge BG_+$ for any spectrum $X$, and so a restriction map

$$[X \wedge BG_+, X'] \longrightarrow [X, X']$$

(5.29)

for any spectrum $X'$. When $X$ is a Madsen-Tillmann spectrum we can rewrite (5.29) using the twisted Thom isomorphism; then the map is pullback along the inclusion

$$BO_d \hookrightarrow BO_d \times BG$$

defined by the basepoint of $BG$.

**Proposal 5.31.**

(i) Symmetry protected topological phases of a $d$-space dimensional bosonic theory with global symmetry group $G$ map to the kernel

$$SPT_{bose}(d, G, \phi) = \ker \left( T\mathbb{Z}^{\tau_{T\mathbb{Z}}+d+2}(BSO_d \times BG) \longrightarrow T\mathbb{Z}^{\tau_{T\mathbb{Z}}+d+2}(BSO_d) \right)$$

(5.32)

of the indicated restriction map constructed from (5.30).

(ii) Symmetry protected topological phases of a $d$-space dimensional fermionic theory with global symmetry group $G$ map to the kernel

$$SPT_{fermi}(d, G, \phi) = \ker \left( I\mathbb{Z}^{\tau_{I\mathbb{Z}}+d+2}(BSpin_d \times BG) \longrightarrow I\mathbb{Z}^{\tau_{I\mathbb{Z}}+d+2}(BSpin_d) \right)$$

(5.33)

of the indicated restriction map.

### 5.3. Relation to group (super) cohomology

From the definition of $T\mathbb{Z}$ at the beginning of §5.2.1 we construct a map of spectra

$$\Sigma^{d+2}H\mathbb{Z} \longrightarrow \Sigma^{d+2}T\mathbb{Z}.$$  

(5.34)

It induces the second map in the composition

$$H^{d+2}(BG; \mathbb{Z}_\phi) \longrightarrow H^{d+2}(BSO_d \times BG; \mathbb{Z}_\phi) \longrightarrow T\mathbb{Z}^{\tau_{T\mathbb{Z}}+d+2}(BSO_d \times BG);$$

(5.35)

the first is induced from the projection $BSO_d \times BG \to BG$. Here $\mathbb{Z}_\phi \to BG$ is the local system defined by (5.19). Note that ordinary cohomology is oriented for oriented vector bundles, which explains why the twisting $\tau_{T\mathbb{Z}}$ is trivialized when restricted under (5.34). The homomorphism (5.35) maps the group cohomology phases discussed\(^{37}\) in [CGLW] to $SRE_{bose}(d, G, \phi)$. Furthermore, the image of (5.35) lies in the subgroup $SPT_{bose}(d, G, \phi)$ of symmetry protected phases; see (5.32).

---

\(^{37}\)To compare it helps to observe that $H^{d+2}(BG; \mathbb{Z}_\phi) \cong H^{d+1}(BG; U(1)_\phi)$, where $U(1)$ has its continuous topology.
Remark 5.36. If \( G \) is discrete, then the topological cohomology of \( BG \) is isomorphic to the group cohomology of the group \( G \). More generally, if \( G \) is a (finite dimensional) Lie group, then a theorem of D. Wigner [Wi] states that the topological cohomology of \( BG \) with coefficients in a discrete \( G \)-module is isomorphic to the Borel cohomology of \( G \); see [St] for more on group cohomology.

For theories with fermions Gu and Wen [GW] introduced a group “super” cohomology theory. In fact, it can be identified with a certain generalized cohomology theory of the classifying space \( BG \). This generalized cohomology theory, which we simply call \( E \), had already appeared in at least a few contexts in theoretical physics: (1) in spin Chern-Simons theories [J] and (2) in QCD, in the Wess-Zumino term of the long-range effective theory of pions [F4]. The spectrum \( E \) has two nonzero homotopy groups:

\[
\pi_0 E \cong \mathbb{Z}, \quad \pi_{-2} E \cong \mathbb{Z}/2\mathbb{Z}.
\]

The \( k \)-invariant which relates them is nonzero. We defer to [F4, §1] for generalities on this cohomology theory.

The two nontrivial homotopy groups in \( E \) occur in (5.14), shifted up by degree \( d + 2 \), and as the \( k \)-invariant match there is a map \( E \to I\mathbb{Z} \) of spectra. The theory \( E \) is oriented for spin bundles, as proved in [F4, Proposition 4.4]. Therefore, there is a homomorphism\(^{38}\)

\[
E^{\phi_E + d + 2}(B Spin_d \times BG) \longrightarrow SRE_{\text{fermi}}(d, G, \phi) = I\mathbb{Z}^{\tau_{I\mathbb{Z}} + \psi_{I\mathbb{Z}} + d + 2}(B Spin_d \times BG).
\]

The projection \( B Spin_d \times BG \to BG \) induces an inclusion

\[
E^{\phi_E + d + 2}(BG) \longrightarrow E^{\phi_E + d + 2}(B Spin_d \times BG)
\]

which, after composition with (5.38), induces a homomorphism of the group “super” cohomology into \( SRE_{\text{fermi}}(d, G, \phi) \), as expected. Note that the image of \( E^{\phi_E + d + 2}(BG) \) in \( SRE_{\text{fermi}}(d, G, \phi) \) lies in the subgroup \( SPT_{\text{fermi}}(d, G, \phi) \) of symmetry protected phases; see (5.33).

6. Computations and special cases

We illustrate how the proposed invariants of gapped short-range entangled (SRE) phases in §5.2 detect phases not covered by the group cohomology classification discussed in §5.3. We organize the discussion by space dimension \( d \) and by whether or not the theory includes fermions. The examples treated here are non-anomalous. Clearly there are many more computations and analyses which can be carried out.

\(^{38}\)The map (5.38) is part of a long exact sequence; the terms which come before and after are maps into the spectrum which is the cofiber of \( E \to I\mathbb{Z} \) so has vanishing homotopy group in degrees \( \geq -2 \). The spin orientation of \( E \) explains why \( \tau_{I\mathbb{Z}} \) does not appear in the domain of (5.38).
6.1. $d = 1$ bosonic theories: group cohomology

This is the one case in which there is nothing beyond the group cohomology classification. There are two reasons for this: (1) the group $SO_1$ which governs the spatial tangential structure is trivial, and (2) for $d = 1$ we have $TZ = HZ$. More formally, from (5.22) and (5.32) we deduce

$$SRE_{\text{bose}}(1, G, \phi) = SPT_{\text{bose}}(1, G, \phi)$$

(6.1)

$$= HZ^{\phi_{HZ}+3}(BSO_1 \times BG)$$

$$\cong H^3(BG; \mathbb{Z}_\phi).$$

6.2. $d = 1$ fermionic theories

Since the shifted Madsen-Tillmann spectra are connective (in this case the relevant spectrum is $\Sigma^1MT\text{Spin}_1 \wedge BG_+$), we can replace the codomain $\Sigma^3IZ$ of an invertible topological field theory by its connective cover. That connective cover is a module for $ko$-theory, which is connective real $K$-theory. (See [F5, §4], for example, where that connective cover is the theory called ‘$R^{-1}$‘.) In particular, the connective cover is Spin-oriented so the Thom twisting is trivial. Hence

$$SRE_{\text{fermi}}(1, G, \phi) \cong IZ^{\phi_{IZ}+3}(B\text{Spin}_1 \times BG).$$

(6.2)

The subgroup coming from the $BG$ factor is

$$IZ^{\phi_{IZ}+3}(BG).$$

(6.3)

Remark 6.4. This already goes beyond the group “super” cohomology theory $E$, since $E$ has two nonzero cohomology groups (5.37), whereas the truncation of $IZ$ we are using here has a third nonzero homotopy group $\pi_{-3} \cong \mathbb{Z}/2\mathbb{Z}$.

Consider the special case $G = \mu_2$ with $\phi$ nontrivial; this is the case of a time-reversal symmetry which squares to the identity. Then (6.3) is cyclic of order 8:

$$IZ^{\phi_{IZ}+3}(B\mu_2) \cong \mathbb{Z}/8\mathbb{Z}.$$  

(6.5)

One proof of (6.5) is [DFM, Theorem 3.13].

Remark 6.6. One interpretation of the left hand side of (6.5) is the group of degree shifts of $KO$-theory, which is the Brauer group of real $\mathbb{Z}/2\mathbb{Z}$-graded central simple algebras. This is surely very closely related to the classification in [FK, §V].

Because the group $\text{Spin}_1 \cong \mathbb{Z}/2\mathbb{Z}$ is nontrivial, there are additional SRE phases (6.2) not captured by (6.3). One example is the truncation to 1-space dimension of the 2-spacetime dimensional “Arf theory”. The invariant of a 2-dimensional closed spin manifold is ±1 according to the Arf invariant of the quadratic form defining the spin structure: even spin structures have invariant +1 and odd spin structures have invariant −1. The invariants on 1- and 0-dimensional manifolds are also
explicit. The invariant of $S^1$ is the trivial even line for the bounding spin structure and the odd line for the nonbounding spin structure. The invariant of $pt_+$ with the standard spin structure is the complex Clifford algebra $\text{Cliff}^\mathbb{C}_1$. (For simplicity we take the target 2-category $\mathcal{C}$ of the theory to be algebras-bimodules-intertwiners. Equivalently, we can assign to $pt_+$ the category of $\mathbb{Z}/2\mathbb{Z}$-graded $\text{Cliff}^\mathbb{C}_1$-modules.) For more on the Arf theory, and a beautiful geometric application, see [G].

It appears that the Arf theory is realized as the long-range effective topological theory of the Majorana chain [K6] in its nontrivial phase. For example, the description of the effective theory [K6, (15)] does assign the Clifford algebra $\text{Cliff}^\mathbb{C}_1$ to a point.

In the remainder of this section we illustrate some relevant computational techniques. More elaborate techniques are needed in higher dimensions, as we illustrate in the appendix. The Arf theory is predicted by

\begin{equation}
IZ^3(\Sigma^1 MT \text{Spin}_1) \cong IZ^3(B \text{Spin}_1) \cong \mathbb{Z}/2\mathbb{Z}.
\end{equation}

In fact, since the Arf theory extends to a 2-spacetime dimensional theory, we can detect it in the group $IZ^3(\Sigma^2 MT \text{Spin}_2)$. It is illuminating to first compute

\begin{equation}
IZ^2(\Sigma^2 MT \text{Spin}_2) \cong IZ^2(B \text{Spin}_2)
\cong IZ^2(\text{CP}^\mathbb{C})
\cong \text{Hom}(\pi^2_3\text{CP}^\mathbb{C}, \mathbb{C}/\mathbb{Z})
\cong \text{Hom}(\pi^2_3\text{CP}^\mathbb{C} \times \pi^2_3S^0, \mathbb{C}/\mathbb{Z})
\cong \mathbb{C}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\end{equation}

The first line is the Thom isomorphism; the third the defining property of $IZ/\mathbb{Z}$; the fourth the general fact $\Sigma^3(X_+) \cong \Sigma^3X \vee S^0$; and the last line the results $\pi^2_3(\text{CP}^\mathbb{C}) \cong \mathbb{Z}$ (computed in [Li], for example) and $\pi^2_3S^0 \cong \mathbb{Z}/2\mathbb{Z}$. This is the set of 2-dimensional invertible spin theories with target $IZ/\mathbb{Z}$. It includes the family of Euler theories (Example 2.11) parametrized by $\mathbb{C}/\mathbb{Z}$ and the Arf theory. The group of path components of (6.8) is

\begin{equation}
IZ^3(\Sigma^2 MT \text{Spin}_2) \cong IZ^3(\text{CP}^\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}.
\end{equation}

One computation of this group uses the split short exact sequence (5.5)

\begin{equation}
0 \longrightarrow \text{Ext}(\pi_2\Sigma^2 MT \text{Spin}_2) \longrightarrow IZ^3(\Sigma^2 MT \text{Spin}_2) \longrightarrow \text{Hom}(\pi_3\Sigma^2 MT \text{Spin}_2, \mathbb{Z}) \longrightarrow 0
\end{equation}

and the computations $\pi^2_3(\text{CP}^\mathbb{C}) = 0$ and $\pi^2_3(S^0) \cong \mathbb{Z}/24\mathbb{Z}$.

The Atiyah-Hirzebruch spectral sequence provides another means to compute the generalized cohomology groups (6.8) and (6.9). The relevant portion of the $E_2$ page of the spectral sequence

\begin{equation}
E^{pq}_2 = H^p(\text{CP}^\mathbb{C}; IZ^q(pt)) \longrightarrow IZ^{p+q}(\text{CP}^\mathbb{C})
\end{equation}
for computing (6.9) is shown in Figure 7. The differentials emanating from the initial column vanish, as can be seen from the splitting \( pt \to \mathbb{CP}^\infty \to pt \) of the projection to a point. That reasoning also applies to the spectral sequence

\[
E_2^{pq} = H^p(\mathbb{CP}^\infty; IC/\mathbb{Z}^q(pt)) \longrightarrow IC/\mathbb{Z}^{p+q}(\mathbb{CP}^\infty),
\]

a portion of which is shown in Figure 8, and it also applies to prove that the short exact sequence

\[
0 \longrightarrow \mathbb{C}/\mathbb{Z} \longrightarrow IC/\mathbb{Z}^2(\mathbb{CP}^\infty) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
\]

splits. (One reads off (6.13) from the \( E_\infty \) page.)

6.3. \( d = 2 \) bosonic theories: Kitaev \( E_8 \) phase and chiral central charge

To investigate SRE phases at the other extreme from those detected by group cohomology, we set \( G = \{1\} \) to be the trivial group. Then necessarily \( \phi: G \to \mu_2 \) is the trivial homomorphism. Unwinding Proposal 5.21 we have

\[
SRE_{\text{bose}}(2, \{1\}, 1) = [\Sigma^2 MTSO_2, \Sigma^4 IZ],
\]
the group of $H$-type oriented theories with target the Anderson dual of the sphere. The following computations are carried out in Appendix A. Recall that the group $\pi_4 \text{MSO}$ is Thom’s oriented bordism group of 4-manifolds, which is isomorphic to $\mathbb{Z}$ via homomorphism which attaches to each closed oriented 4-manifold $M$ its signature $\text{Sign}(M)$.

**Proposition 6.15.**

(i) $\pi_3 \Sigma^2 \text{MTSO}_2 = 0$.

(ii) $\pi_4 \Sigma^2 \text{MTSO}_2 \cong \mathbb{Z}$ and the composition

\[
\pi_4 \Sigma^3 \text{MTSO}_2 \to \pi_4 \text{MSO} \xrightarrow{\text{Sign}} \mathbb{Z}
\]

maps the generators to $\pm 4$.

(iii) $\pi_3 \Sigma^3 \text{MTSO}_3 = 0$.

(iv) $\pi_4 \Sigma^3 \text{MTSO}_3 \cong \mathbb{Z}$ and the composition

\[
\pi_4 \Sigma^3 \text{MTSO}_3 \to \pi_4 \text{MSO} \xrightarrow{\text{Sign}} \mathbb{Z}
\]

maps the generators to $\pm 2$.

We interpret these computations using (5.5). First, (i) and (ii) imply

\[
[\Sigma^2 \text{MTSO}_2, \Sigma^4 \mathbb{I}Z] \cong \mathbb{Z}.
\]

Furthermore, the generating field theory extends to a $\mathbb{Z}$-valued invertible 4-dimensional oriented theory whose numerical invariant is 4 times the signature. Assertions (iii) and (iv) together imply

\[
[\Sigma^3 \text{MTSO}_3, \Sigma^4 \mathbb{I}Z] \cong \mathbb{Z}
\]

and the generating field theory extends to a $\mathbb{Z}$-valued invertible 4-dimensional theory whose numerical invariant is 2 times the signature. A 2-dimensional theory which generates the group (6.18) does not extend to 3 dimensions, and a 3-dimensional theory which generates the group (6.19) does not extend to 4 dimensions. We relate these factors to known geometric facts about $\eta$-invariants and determinant lines.

The interpretation of the homotopical computations (6.14), (6.19) is not obvious at first glance; indeed that mystery was the motivation for §2.8. Furthermore, the example discussed there—in particular, the contractible choice replacing $F_{\text{topological}}$ with $F_{\text{geometric}}$—has a clear analog in our current situation. If we include a Riemannian metric as a field, then there is a 3-spacetime dimensional theory whose invariant of a closed oriented Riemannian 3-manifold is the exponentiated Atiyah-Patodi-Singer $\eta$-invariant [APS].\(^{39}\) As in Remark 2.43 there is a related 4-dimensional theory (called $\tilde{\alpha}_8$ below) whose value on a closed oriented 4-manifold $M$ is a multiple of the signature $\text{Sign}(M)$. If we use the $\eta$-invariant associated to the signature operator, then the multiple is 1.

---

\(^{39}\)More precisely, for a self-adjoint operator $B$ the invariant is $\exp(\pi i (\eta_B + h_B))$, where $h_B = \dim \ker B$. 
But we can use instead the \( \eta \)-invariant associated to the self-duality operator; the corresponding boundary operator \( B \) is \( 1/2 \) that of the signature operator. (This \( \eta \)-invariant appears in quantum Chern-Simons theory \cite{FG, (1.27)}. This theory represents a generator of \((6.19)\); the multiple of the signature is \( 1/2 \). The invariants of 2-manifolds are determinant lines, and the determinant line of the 2-dimensional signature operator on a closed oriented Riemannian manifold has a natural 4th root: the determinant line of the \( \hat{c} \)-operator. Determinant lines of \( \hat{c} \) provide a generator of \((6.18)\); the multiple of the signature is \( 1/4 \).

**Remark 6.20.** The discussion in §4.1, especially Remark 4.8, and also Remark 3.10 are relevant here. A theory classified by \((6.19)\) gives integer invariants of 4-manifolds whose stable tangent bundle can be represented by a rank 3 vector bundle. Such manifolds have even signature. (For example, the 4th Stiefel-Whitney number vanishes. It is the reduction modulo 2 of the Euler number which is equal to the signature modulo 2.) An example of such a 4-manifold is the mapping cylinder of a 3-manifold, which fibers over the circle, but then the signature vanishes. A nontrivial example is the connected sum \( M = \left( \mathbb{C}P^2 \right)^{\# 2} \# (S^1 \times S^3)^{\# 3} \), which has signature 2 and Euler number 0. The vanishing Euler number implies that \( M \) admits a nonzero vector field, which splits an oriented line bundle off of \( TM \). Similarly, 4 divides the signature of a compact oriented 4-manifold whose stable tangent bundle is 2-dimensional. In this case there are nontrivial examples which are fiber bundles; the base and fiber are both compact oriented 2-manifolds. The first examples are due to Atiyah \cite{A3}; an example with signature exactly 4 is constructed in \cite[Theorem 1]{EKKOS}.

We now argue that the SRE phase referred to in the literature as “Kitaev’s \( E_8 \) phase” or “Kitaev’s \( E_8 \) state” (see \cite{K5}, \cite{K2}, \cite{LV}) has the field theory whose invariant is exactly the signature as its low energy approximation.

To make a first connection to \( E_8 \) Chern-Simons, we recall that the gravitational Chern-Simons invariant enters into the quantization of classical Chern-Simons theory as a counterterm \cite[(2.20)]{W1}. Its appearance means that in general quantum Chern-Simons theory is anomalous as an oriented theory. The anomaly is an invertible 4-dimensional theory

\[
\alpha_{\tilde{c}} : \Sigma^4 MTSO_4 \longrightarrow \Sigma^4 IC^X
\]

whose invariant on a closed oriented 4-manifold \( M \) is

\[
e^{2\pi ic\text{Sign}(M)/8} = e^{2\pi icp_1(M)/24}.
\]

The anomaly depends only on the mod 8 reduction \( \tilde{c} \) of the *chiral central charge* \( c \in \mathbb{R} \) of the corresponding 2-dimensional chiral Wess-Zumino-Witten model.

**Remark 6.23.** Walker’s approach \cite{Wa} to quantum Chern-Simons theory uses bounding 4-manifolds to control the framing dependence. In joint work with Constantin Teleman (so far unpublished) we prove that a modular tensor category is invertible as an object in the 4-category of braided tensor categories, and we use it to define an invertible oriented 4-dimensional topological field theory which is precisely \( \alpha_{\tilde{c}} \). Note that the modular tensor category determines \( \tilde{c} = c \pmod{8} \)—see \cite[(172)]{K5}, for example—but it does not determine \( c \in \mathbb{R} \). The usual approach to quantum Chern-Simons
theory in the mathematics literature is to lift to a theory of manifolds with a \((w_1,p_1)\)-structure. A \(w_1\)-structure is a trivialization of \(w_1\): an orientation. A \(p_1\)-structure is similar \([BHMV]\), but its geometric avatars are not as simple as an orientation. For example, a \(p_1\)-structure on a 3-manifold can be given by a “2-framing” \([A2]\). In this way one obtains Chern-Simons as an extended theory of 1-, 2-, and 3-dimensional manifolds, but to do so one needs to lift \(c \mod 8\) to \(c \mod 24\). Of course, given \(c \in \mathbb{R}\) there is a preferred choice, but starting from a modular tensor category there are 3 choices.

We observe that given a chiral central charge \(c \in \mathbb{R}\) there is a 4-dimensional invertible theory

\begin{equation}
\tilde{\alpha}_c: \Sigma^4 MTSO_4 \rightarrow \Sigma^4 H\mathbb{R}
\end{equation}

whose invariant on a closed oriented 4-manifold \(M\) is the real number

\begin{equation}
c \text{ Sign}(M)/8.
\end{equation}

The anomaly theory \(\alpha_c\) in (6.21) is obtained by composing (6.24) with the map \(\Sigma^4 H\mathbb{R} \rightarrow \Sigma^4 I\mathbb{C}^\times\) induced by the exponential map \(e^{2\pi i(-)}: \mathbb{R} \rightarrow \mathbb{C}^\times\); see (5.4). If \(c = 8n\) for some \(n \in \mathbb{Z}\), then (6.24) factors through an integral theory

\begin{equation}
\tilde{\alpha}_{8n}: \Sigma^4 MTSO_4 \rightarrow \Sigma^4 I\mathbb{Z}
\end{equation}

These integral topological theories are not part of the usual quantum Chern-Simons theory: only the exponential (6.21) of (6.24) occurs (as the “framing anomaly” theory). For the theories in (6.26) the framing anomaly is trivial. What does occur in Chern-Simons is the invertible metric 3-dimensional theory whose partition function is the exponentiated \(\eta\)-invariant to a suitable power \([W1]\). \(E_8\) Chern-Simons at level 1, or Chern-Simons theory for the maximal torus of \(E_8\) (with its Cartan matrix specifying the level, or “K-matrix”), has chiral central charge \(c = 8\). The \(\eta\)-invariant which occurs is associated to the signature operator, and this is the class of theories we associate to Kitaev’s \(E_8\) phase.

**Remark 6.27.** Proposition 6.15 implies that there are additional possibilities for an \(H\)-type 2-dimensional theory: a “4th root” of the effective theory of Kitaev’s \(E_8\) phase. Such theories are associated with chiral central charge \(c = 2\), and in general the 2-dimensional \(H\)-type theories are associated with chiral central charge divisible by 2. This matches the conformal anomaly in 2-dimensional conformal field theory—see \([S1, (5.9)]\), for example: if the chiral central charge is not divisible by 2, then a \(p_1\)-structure is needed to define the theory. Even if we require the theory to extend to 3-manifolds, there is still the possibility of dividing by 2, so having chiral central charge divisible by 4. So it appears that our proposal allows for more effective topological theories than have been seen so far by SRE phases.
6.4. $d=2$ bosonic theories: mixed gravitational/gauge phases

Continuing with $d=2$ space dimensional theories, we give an example to illustrate the unitarizability restriction in (5.23). Now we allow a global symmetry group $G$.

We focus on the Kunneth component

\[(6.28) \quad H^2(BS\Omega_2; \mathbb{Z}) \otimes H^2(BG; \mathbb{Z}_\phi) \subset H^4(BS\Omega_2 \times BG; \mathbb{Z}_\phi) \to IZ^\tau_{I\mathbb{Z}} + \phi_{I\mathbb{Z}} + 4(BSO_2 \times BG) = SPT_{\text{bose}}(2, G, \phi).\]

The group $H^2(BS\Omega_2; \mathbb{Z})$ is infinite cyclic with generator the Euler class $e$. For simplicity let $G$ be finite and $\phi$ the trivial homomorphism. Then $H^2(BG; \mathbb{Z}) \cong H^1(BG; \mathbb{C}^\times)$ is isomorphic to the group of abelian characters $\chi: G \to \mathbb{C}^\times$. Fix one and let $\lambda_\chi \in H^2(BG; \mathbb{Z})$ be the corresponding class. Let $F$ be the theory which corresponds to $e \otimes \lambda_\chi$ in (6.28). We remark in passing that $F$ does not extend to a 3-spacetime dimensional theory: the Euler class $e$ is not the restriction of a class in $H^2(BSO_2; \mathbb{Z}) = 0$. The main point: this theory is not unitarizable. To see this it suffices to restrict along $HZ \to IZ$ in (5.23), since we are trying to hit $e \otimes \lambda_\chi$ which lies in ordinary cohomology. There is an isomorphism $\tau_{HZ} \cong w_{HZ}$ of twistings in the domain of (5.23), so the relevant group is $H^1(BO_2 \times BG; \mathbb{Z}_\phi)$ and the relevant Kunneth component is $H^2(BO_2; \mathbb{Z}) \otimes H^2(BG; \mathbb{Z}_\phi)$. The Euler class $e$ does not drop to a class in $H^2(BSO_2; \mathbb{Z})$.

Remark 6.29. It is instructive to compute something nontrivial in the theory $F$. Let $Y$ be a closed oriented 2-manifold and $P \to Y$ a principal $G$-bundle. Then $F(P \to Y)$ is a complex line. Suppose $\varphi$ is an automorphism of $P \to Y$, so a map

\[(6.30) \quad P \xrightarrow{\varphi} P \quad \text{and} \quad Y \xrightarrow{\varphi} Y\]

of principal $G$-bundles covering an orientation-preserving diffeomorphism of $Y$. Gluing the ends of $[0, 1] \times P \to [0, 1] \times Y$ using $\varphi$ we obtain a principal $G$-bundle $Q_\varphi \to X_\varphi$ over the mapping cylinder $X_\varphi$. Note that the mapping cylinder is a 3-manifold which is the total space of a fiber bundle $X_\varphi \to S^1$ with typical fiber $Y$. The rank 2 relative tangent bundle $T(X_\varphi/S^1) \to X_\varphi$ is oriented so has an Euler class $e(X_\varphi/S^1) \in H^2(X_\varphi; \mathbb{Z})$. The $G$-bundle $Q_\varphi \to X_\varphi$ has a characteristic class $\lambda_\chi(Q_\varphi) \in H^1(X_\varphi; \mathbb{C}^\times)$. Then the action of $\varphi$ on the line $F(P \to Y)$ is multiplication by

\[(6.31) \quad \langle e(X_\varphi/S^1) \sim \lambda_\chi(Q_\varphi), [X_\varphi] \rangle \in \mathbb{C}^\times.\]

In (6.31) we pair the cup product of the characteristic classes with the fundamental class of the oriented 3-manifold $X_\varphi$. A special case of note: $\varphi$ is the identity diffeomorphism, $P \to Y$ is the trivial bundle, and the gauge transformation $\varphi$ is given by an element $g \in G$ (assuming $Y$ is connected). Then (6.31) reduces to $\chi(g)^{\text{Euler}(Y)}$, where $\text{Euler}(Y) \in \mathbb{Z}$ is the Euler number.

\[\text{It does drop to a class on } BO_2 \text{ with twisted coefficients, but that is not relevant here.}\]
6.5. \( d = 3 \) bosonic theories: time-reversal symmetry

Set \( G = \mu_2 \) and \( \phi: \mu_2 \rightarrow \mu_2 \) the identity map. The group cohomology captures a subgroup of the group of SRE phases:

\[
H^5(\mu_2; \mathbb{Z}_\phi) = H^5(\mathbb{R}P^\infty; \mathbb{Z}_\phi) \cong \mathbb{Z}/2\mathbb{Z},
\]

as appears in [CGLW]. Another nontrivial SRE phase of order 2 was introduced in [VS] where it was emphasized that this goes beyond the group theory computation. This SRE phase is predicted by Proposal 5.21.

**Proposition 6.33.** We have

\[
SRE_{\text{bose}}(3, \mu_2, \text{id}) = \text{IZ}^{r_\mathbb{Z}+\phi r_\mathbb{Z}+5}(\text{BSO}_3 \times \mathbb{R}P^\infty) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

Furthermore, the map

\[
i : H^5(\text{BSO}_3 \times \mathbb{R}P^\infty; \mathbb{Z}_\phi) \rightarrow \text{IZ}^{r_\mathbb{Z}+\phi r_\mathbb{Z}+5}(\text{BSO}_3 \times \mathbb{R}P^\infty)
\]

is surjective and

\[
H^5(\text{BSO}_3 \times \mathbb{R}P^\infty; \mathbb{Z}_\phi) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

We defer most of the proof to the appendix; here we briefly sketch two ways to compute (6.36).

The first is a direct approach. Use the chain complex\(^{41}\)

\[
\begin{align*}
Z & \xleftarrow{0} 0 \xleftarrow{0} Z \xleftarrow{2} Z \xleftarrow{0} Z \xleftarrow{2} Z \cdots \\
& \text{for } \text{BSO}_3 \text{ and the chain complex}
\end{align*}
\]

\[
\begin{align*}
Z & \xleftarrow{2} Z \xleftarrow{0} Z \xleftarrow{2} Z \xleftarrow{0} Z \xleftarrow{2} Z \xleftarrow{0} Z \cdots \\
& \text{for } \mathbb{R}P^\infty \text{ with the nontrivial local system } \mathbb{Z}_\phi \rightarrow \mathbb{R}P^\infty.
\end{align*}
\]

Compute the cohomology of the cochain complex obtained by applying \( \text{Hom}(\_, \mathbb{Z}) \) to the tensor product of (6.37) and (6.38). An alternative approach is to apply the Kunneth formula for cohomology [Sp, §5.5], which in this case gives a split short exact sequence

\[
0 \rightarrow [H^\bullet(\text{BSO}_3; \mathbb{Z}) \otimes H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}_\phi)]^5 \rightarrow H^5(\text{BSO}_3 \times \mathbb{R}P^\infty; \mathbb{Z}_\phi) \rightarrow [H^\bullet(\text{BSO}_3; \mathbb{Z}) * H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}_\phi)]^6 \rightarrow 0
\]

Here ‘*’ denotes the torsion product of abelian groups. The tensor product in the kernel of (6.39) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) generated by the nonzero class in \( H^5(\mathbb{R}P^\infty; \mathbb{Z}_\phi) \cong \mathbb{Z}/2\mathbb{Z} \) and the tensor product of \( p_1 \in H^4(\text{BSO}_3; \mathbb{Z}) \) and the nonzero class \( a \in H^1(\mathbb{R}P^\infty; \mathbb{Z}_\phi) \). The quotient group in (6.39) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), which is the torsion product \( H^3(\text{BSO}_3; \mathbb{Z}) * H^3(\mathbb{R}P^\infty; \mathbb{Z}_\phi) \).

\(^{41}\)This is the minimal chain complex [Ha, Proposition 3E.3] derived from the homology of \( \text{BSO}_3 \).
Claim 6.40. The image \( i(p_1 \otimes a) \) of \( p_1 \otimes a \) under (6.35) is the long-range effective topological theory of the SRE phase identified in [VS].

We argue for this claim in §7.

Remark 6.41. Since \( i(p_1 \otimes a) \) is torsion we can lift it to the group \( IC/\mathbb{Z}^{\tau_{12} + \phi_{12} + 4}(BSO_3 \times \mathbb{R}P^2) \) where it is easier to identify a particular theory in this class; see Remark 2.48.

7. Boundary conditions and long-range topological field theories

We begin in §7.1 with a general discussion of spatial boundary conditions for field theories and condensed matter systems. We specialize to the case at hand: the bulk theory is gapped and the long-range topological theory is invertible. We apply these general ideas in §7.2 to argue for Claim 6.40, which locates the 3d \( E_8 \) phase with half-quantized surface thermal Hall effect introduced in [VS] and further investigated in [BCFV]. We recover some key aspects of that theory from our topological viewpoint.

7.1. Boundary terminations

Suppose we are given a theory \( F \) in \( n \) spacetime dimensions. It may be a quantum field theory or a condensed matter theory. Then to a compact \((n - 1)\)-manifold \( Y \) with empty boundary we obtain a complex vector space of states. We would like to extend to allow compact \((n - 1)\)-dimensional manifolds \( Y \) which have nonempty boundary. These boundaries are spatial, not temporal. In this case we expect to impose boundary conditions \( \beta \) which essentially close off the boundary. In other words, the pair \((Y, \beta)\) behaves as a closed \((n - 1)\)-manifold for the pair of theories \((F, \beta)\), and the \((F, \beta)\) theory attaches to it a vector space of states. (Without the boundary condition \( \beta \) we expect instead a module for an algebra more complicated than \( C \), or an object in a category more complicated than \( \text{Vect} \).) Furthermore, we expect \( \beta \) to be local. In classical physics a spatial boundary condition is typically a local constraint on fields: a boundary condition for a system of classical partial differential equations. In quantum physics a spatial boundary condition is a relative field theory (§2.3)

\[
(7.1) \quad \beta: \tau_{n-1} F \rightarrow 1.
\]

The theory \( F \) evaluates on \( Y \) to a map \( F(Y): \text{Vect} \rightarrow F(\partial Y) \) and the boundary condition \( \beta \) evaluates on \( \partial Y \) to a map \( \beta(\partial Y): F(\partial Y) \rightarrow \text{Vect} \). The composition \( \beta(\partial Y) \circ F(Y): \text{Vect} \rightarrow \text{Vect} \) is tensor product with the vector space associated to \( Y \) in the theory \( (F, \beta) \). If \( F \) is an invertible field theory, then \( \beta \) is an anomalous theory with anomaly \( F \). We call \((F, \beta)\) a bulk-boundary pair.

Remark 7.2 (Vocabulary). A quantum boundary condition \( \beta \) in quantum field theory is sometimes called a D-brane, a term most appropriate in the context of 2-spacetime dimensional conformal field theories. In condensed matter physics \( \beta \) goes by a name like edge termination or surface
termination, depending on the dimension of the theory. Sometimes the word ‘excitation’ is used in place of ‘termination’.

Remark 7.3. If $F$ is a topological field theory with values in an $(\infty, n)$-category $C$, then a boundary condition is a 1-morphism $F(\text{pt}) \to 1$ in $C$. The dual map $1 \to F(\text{pt})$ may be considered as an “object in $F(\text{pt})$”. For example, if $n = 2$ and $C$ is a 2-category of categories, then $\beta$ is literally an object in the 1-category $F(\text{pt})$. Boundary conditions in topological theories are a special case of a much more general construction [L, Example 4.3.22].

Remark 7.4. If $F$ is a $d$-space dimensional theory of $H$-type, then (7.1) is replaced by

\[(7.5) \quad \beta : \tau_{\leq d-1} F \longrightarrow 1.\]

There is not a unique boundary condition for a given $F$, but rather $F$ determines a collection of boundary conditions. These formal considerations can lead to physical consequences. One important example in condensed matter physics is the integer quantum Hall effect; see [W4] for an account aimed at mathematicians.

Suppose the theory $F$ is the long-range topological approximation to a gapped $d$-space dimensional system of $H$-type. Then if a boundary condition produces a combined bulk-boundary pair which is still gapped, we expect that the long-range topological approximation is a bulk-boundary pair $(F, \beta)$ of topological theories. If $F$ describes a short-range entangled phase—that is, $F$ is invertible—then $\beta$ is a $(d-1)$-space dimensional anomalous theory with anomaly $F$. We implicitly assume that the truncation $\tau_{\leq d-1} F$ of $F$ is nontrivial. If, furthermore, $F$ describes an SPT phase, then we arrive at the following trichotomy.

A long-range effective boundary condition:

\[(7.6) \quad \begin{align*}
(i) & \text{ produces a gapless bulk-boundary pair which preserves the symmetry,} \\
(ii) & \text{ is non-anomalous and breaks the symmetry, or} \\
(iii) & \text{ is anomalous, symmetric, and exhibits long-range entanglement.}
\end{align*}\]

Possibility (ii) arises since by the definition of an SPT phase $F$ restricts to a trivial theory when the symmetry is broken, and a theory relative to the trivial theory is non-anomalous. For (iii) we observe that an invertible theory relative to an invertible theory $\tau_{\leq d-1} F$ is a trivialization of $\tau_{\leq d-1} F$, so if $\tau_{\leq d-1} F$ is not trivial then any relative theory must not be invertible. In the physics lingo it is not short-range entangled but rather is long-range entangled—it “exhibits topological order”. The trichotomy (7.6) is a restatement of an assertion in the introduction to [VS].

---

42The choice of direction of the arrow in (7.1) reflects our choice that $\partial Y$ is outgoing rather than incoming. There is an equivalent exposition with the other choice, and we would not need the dual here.
7.2. The invertible field theory of an exotic $d = 3$ bosonic phase

We turn now to the SPT phase identified as $i(p_1 \otimes a)$ at the end of §6.5.

The argument that $i(p_1 \otimes a)$ corresponds to the 3d $E_8$ phase with half-quantized surface thermal Hall effect is based on (iii) in the trichotomy (7.6). Consider a long-range effective boundary condition—surface termination—which is time-reversal symmetric and anomalous with anomaly $i(p_1 \otimes a)$. This boundary condition is a relative 3-spacetime dimensional topological theory. We claim that any Chern-Simons theory with chiral central charge

\[(7.7) \quad c \equiv 4 \pmod{8}\]

is such an effective boundary condition: it satisfies (iii) in the trichotomy. (See §6.3 for a topological discussion of chiral central charge.) One of the simplest examples is Chern-Simons theory for the maximal torus of $SO_8$, which was proposed for this role in [VS, §VII] and was realized as a boundary condition in the exactly soluble Hamiltonian constructed in [BCFV].

To justify the claim begin with the short exact coefficient sequence

\[(7.8) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \frac{1}{2}\mathbb{Z} \longrightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z} \longrightarrow 0\]

in which the first map is the inclusion. Identify $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$, which in particular is untwisted, and so write\(^{43}\) $p_1 \otimes a$ as the image of

\[(7.9) \quad \frac{1}{2}p_1 \pmod{\mathbb{Z}} \in H^4(BSO_3; \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \subset H^4(BSO_3 \times \mathbb{R}P^\infty; \frac{1}{2}\mathbb{Z}/\mathbb{Z})\]

under the connecting homomorphism in the long exact sequence deduced from (7.8). The image $i$ in the Brown-Comenetz dual of the sphere is computed via the sequence of maps

\[(7.10) \quad H^4(BSO_3; \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \cong H^4(\Sigma^4MTSO_3; \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \longrightarrow H^4(\Sigma^4MTSO_3; \mathbb{C}/\mathbb{Z}) \cong [\Sigma^4MTSO_3, \Sigma^4\mathbb{C}/\mathbb{Z}] \longrightarrow [\Sigma^4MTSO_3, \Sigma^4\mathbb{C}/\mathbb{Z}] \cong \mathbb{C}/\mathbb{Z}.\]

The last isomorphism follows from Proposition 6.15(iv). Since (7.9) has order 2, so does its image, and checking against (6.22) we identify it with the anomaly theory $\alpha_{\overline{c}}$ with $\overline{c} \equiv 4 \pmod{8}$. As explained in §6.3, $\alpha_{\overline{c}=4}$ is the (framing) anomaly of any quantum Chern-Simons theory whose chiral central charge satisfies (7.7).

\(^{43}\)This maneuver allows us to write the torsion class $p_1 \otimes a$ as a class in one lower degree, and so identify it with particular field theories. This circumvents the issues raised in §2.8 about nontorsion classes; see Remark 2.48.
Appendix A. Some homotopy groups of Madsen-Tillmann spectra

We prove Proposition 6.15 and Proposition 6.33. I thank Oscar Randal-Williams for sharing his expertise, for correcting a mistake in a previous version of Proposition 6.15, and for providing a few arguments in the proof.

First recall some facts about Madsen-Tillmann spectra. Set $X_n = \Sigma^n MTSO_n$. Then $X_1 \simeq S^0$. Let $\Sigma^\infty Y$ denote the suspension spectrum of a pointed space $Y$. The fibration

\begin{equation}
X_{n-1} \longrightarrow X_n \longrightarrow \Sigma^n \Sigma^\infty(BSO_n)_+
\end{equation}

is proved in [GMTW, Proposition 3.1] and [FHT, Lemma 3.8]. For any space $Y$ we have

\begin{equation}
\Sigma^\infty(Y_+) \simeq S^0 \vee \Sigma^\infty Y.
\end{equation}

The stable homotopy groups of a pointed space $Z$ are $\pi_j^s Z = \pi_j(\Sigma^\infty Z)$. The oriented version of (4.10) expresses the Thom spectrum $MSO$ as the colimit of a sequence of maps $X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \cdots$. The homotopy groups of $MSO$ are Thom’s oriented bordism groups. The long exact sequence of homotopy groups deduced from (A.1) implies that

\begin{equation}
\pi_j X_n \overset{\cong}{\longrightarrow} \pi_j MSO, \quad j < n,
\end{equation}

is an isomorphism and that there is an exact sequence

\begin{equation}
\pi_{n+1} X_{n+1} \overset{\chi}{\longrightarrow} Z \longrightarrow \pi_n X_n \longrightarrow \pi_n MSO \longrightarrow 0.
\end{equation}

An element of $\pi_{n+1} X_{n+1}$ is represented by a closed oriented $(n + 1)$-manifold $W$, and its image under $\chi$ is the Euler number of $W$. For $n$ even the map $\chi$ is zero. For $n = 3$ the map $\chi$ is surjective, since $\chi(\mathbb{CP}^2 \# S^1 \times S^3) = 1$.

**Proof of Proposition 6.15.** First apply (A.4) with $n = 3$ to derive the exact sequence

\begin{equation}
\pi_4 X_3 \longrightarrow \pi_4 X_4 \overset{\chi}{\longrightarrow} Z \longrightarrow \pi_3 X_3 \longrightarrow \pi_3 MSO
\end{equation}

As remarked above the Euler characteristic map $\chi$ is onto, and since $\pi_3 MSO = 0$ we deduce $\pi_3 X_3 = 0$, which is (iii) in the proposition. Next, apply (A.4) with $n = 4$ to deduce $\pi_4 X_4 \cong \mathbb{Z} \times \mathbb{Z}$ and the composition $\pi_4 X_4 \rightarrow \pi_4 MSO \overset{\text{Sign}}{\longrightarrow} \mathbb{Z}$ is surjective. Then a stretch of the long exact sequence of homotopy groups deduced from (A.1) with $n = 4$ is

\begin{equation}
\begin{array}{cccc}
\pi_1^s(BSO_4)_+ & \longrightarrow & \pi_4 X_3 & \longrightarrow & \pi_4 X_4 & \overset{\chi}{\longrightarrow} & \pi_0^s(BSO_4)_+ & \longrightarrow & \pi_3 X_3 \\
\mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}
\end{equation}
from which $\pi_4 X_3 \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. To see that it is the former, let $F$ be the fiber of the spectrum map $X_3 \to H\mathbb{Z}$ which represents the generator of $H^0(X_3; \mathbb{Z}) \cong \mathbb{Z}$. So there is a cofiber sequence

$$F \to X_3 \to H\mathbb{Z}. \tag{A.7}$$

Then (A.3) and the vanishing of $\pi_3 X_3$ imply that $\pi_j F = 0$, $j \leq 3$, whence the Hurewicz map $\pi_4 F \to H_4 F$ is an isomorphism. In addition, the map $\pi_4 F \to \pi_4 X_3$ is an isomorphism. Figure 9 is a schematic depiction of the long exact sequence of $\mathbb{F}_2$-cohomology groups induced by the cofiber (A.7), where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is the field of 2 elements. The $\mathbb{F}_2$-cohomology of a spectrum is a $\mathbb{Z}$-graded $\mathbb{F}_2$-vector space which is a module for the Steenrod algebra. The dots indicate basis elements and the vertical arrows the action of the Steenrod operations $Sq^1, Sq^2$. The degrees in the figure ascend from 0 to 5. The $\mathbb{F}_2$-cohomology of $H\mathbb{Z}$ is isomorphic to the group of cohomology operations $H\mathbb{Z} \to H\mathbb{Z}/2\mathbb{Z}$ and is computed by a theorem of Serre. The generators are the operations $Sq^2, Sq^3, Sq^4, Sq^5$ (preceded by reduction modulo 2), and the action of the Steenrod operations is given by the Adem relations. For $X_3$ the generators are $w_2 u, w_3 u, w_2^2 u, w_2 w_3 u$ where $u \in H^0(X_3; \mathbb{F}_2)$ is the mod 2 Thom class of the virtual bundle (5.20), which stably is minus the canonical rank 3 bundle $S(3) \to BSO_3$. Its total Stiefel-Whitney class is

$$\tilde{w} = \frac{1}{1 + w_2 + w_3} = 1 + w_2 + w_3 + w_2^2 + \cdots, \tag{A.8}$$

the inverse of the Stiefel-Whitney class of $S(3) \to BSO_3$. The action of the total Steenrod operation $Sq = 1 + Sq^1 + Sq^2 + \cdots$ is $Sq(u) = \tilde{w} u$. The horizontal arrow in degree 0 follows from the definition of $X_3 \to H\mathbb{Z}$, and the arrows in degrees 2,3,4 from the module structure, as does the lack of a horizontal arrow in degree 5. Exactness then implies the existence of a class in $H^4(F; \mathbb{F}_2)$ which maps to $Sq^5 \in H^5(H\mathbb{Z}; \mathbb{F}_2)$. Thus

$$\mathbb{F}_2 \cong H^4(F; \mathbb{F}_2) \cong \text{Hom}(H_4 F, \mathbb{F}_2) \cong \text{Hom}(\pi_4 F, \mathbb{F}_2) \cong \text{Hom}(\pi_4 X_3, \mathbb{F}_2) \tag{A.9}$$

and we conclude $\pi_4 X_3 \not\cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, whence $\pi_4 X_3 \cong \mathbb{Z}$. For the last claim in (iv) we revisit (A.5). A 4-manifold which represents a class in $\pi_4 X_3$ has vanishing $w_4$, since $w_4$ is a stable characteristic
class and vanishes for rank 3 bundles. Thus its Euler number is even, and since the Euler number and signature are congruent modulo 2 its signature is also even. Then observe that the 4-manifold \((\mathbb{C}P^2)^\#2 \# (S^1 \times S^3)^\#3\) represents an element of \(\pi_4X_3\) (since it has vanishing Euler characteristic, so a nonvanishing vector field which splits a line bundle off its tangent bundle) and has signature 2.

Similar techniques prove (i) and (ii). (An alternative is to use the Madsen-Weiss theorem \([\text{MaWe}]\) and known facts about the stable homology of mapping class groups of surfaces.) First

\[(A.10) \quad \pi_{\{0,1,2\}}X_2 \cong \{\mathbb{Z}, 0, \mathbb{Z}\}\]

from \((A.3)\) and \((A.4)\) with \(n = 2\). There is a nontrivial \(k\)-invariant connecting these homotopy groups; if not, then \(H^2(X_2;\mathbb{F}_2) \cong H^2(BSO_2;\mathbb{F}_2)\) would be 2-dimensional. Let \(C\) denote the spectrum with these two nonzero homotopy groups and nontrivial \(k\)-invariant. Its \(\mathbb{F}_2\)-cohomology is worked out in Figure 10 using the cofiber sequence \(\Sigma^2HZ \to C \to HZ\). All cohomology in degree 1 vanishes. Also, the nontrivial connecting map \(H^2(\Sigma^2HZ;\mathbb{F}_2) \to H^3(HZ;\mathbb{F}_2)\) is the \(k\)-invariant. Let \(F'\) be the fiber of the Postnikov map \(X_2 \to C\), and consider the cofiber sequence \(F' \to X_2 \to C\). The induced maps on \(\mathbb{F}_2\)-cohomology are worked out in Figure 11; the nonzero cohomology is in degrees 0,2,4,5. We deduce

\[(A.11) \quad H^4(F';\mathbb{F}_2) \cong \mathbb{F}_2,\]
and from Hurewicz $\pi_{\leq 3}F' = 0$ and $\pi_4F' \to H_4F'$ is an isomorphism. Now the long exact sequence of homotopy groups induced from (A.1) with $n = 2$ includes the stretch

\begin{equation}
\pi_4S^0 \longrightarrow \pi_4X_2 \longrightarrow \pi_2^+(BSO_2) \longrightarrow \pi_3S^0 \longrightarrow 0 \quad \text{Z} \times \text{Z}/2\text{Z} \quad \text{Z}/24\text{Z}
\end{equation}

(A.12)

This implies $\pi_4X_2 \cong \text{Z}$ or $\text{Z} \times \text{Z}/2\text{Z}$; (A.11) rules out the latter since $\pi_4F' \cong \pi_4X_2$ and $H^4(F'; \mathbb{F}_2) \cong \text{Hom}(H_4F', \mathbb{F}_2) \cong \text{Hom}(\pi_4F', \mathbb{F}_2)$. For the last statement in (ii) consider the long exact sequence of homotopy groups induced from (A.1) with $n = 3$:

\begin{equation}
\pi_4X_2 \longrightarrow \pi_4X_3 \longrightarrow \pi_2^+(BSO_3) \longrightarrow \pi_3X_2 \longrightarrow \text{Z} \times \text{Z} \times \text{Z}/2\text{Z} \longrightarrow 0
\end{equation}

(A.13)

and so the first homomorphism is multiplication by $\pm 2$ on generators.

\begin{proof}
Proof of Proposition 6.33. As a preliminary we prove that $\pi_5X_3$ is finite. Since this group is finitely generated, an equivalent assertion is $\pi_5X_3 \otimes \mathbb{Q} = 0$. To prove this observe that $X_3 \to MSO$ induces an isomorphism on rational homology in degrees $\leq 7$, whence also on rational homotopy groups in that range. Collating with (A.3) and facts in the previous proof we have

\begin{equation}
\pi_{\{0,1,2,3,4,5\}}X_3 \cong \{\text{Z}, 0, 0, \text{Z}, \text{finite}\}.
\end{equation}

(A.14)

Introduce the mapping spectrum

\begin{equation}
A = \text{Map}(X_3, \Sigma^5 I\text{Z}).
\end{equation}

(A.15)

From\(^{44}\) (5.5) and (A.14) we deduce

\begin{equation}
\pi_{\{0,1,2,3,4,5\}}A \cong \{0, \text{Z}, 0, 0, \text{Z}\},
\end{equation}

(A.16)

and $\pi_{\geq 6}A = 0$. The cohomology group in (6.34) is $A^\phi_\ast(\mathbb{R}P^\infty)$, which we compute using the Atiyah-Hirzebruch spectral sequence. The rows in the $E_2$ page, shown in Figure 12, are twisted cohomology groups of $\mathbb{R}P^\infty$. All differentials vanish in this range, for degree reasons, whence $A^\phi_\ast(\mathbb{R}P^\infty)$ is isomorphic to $\text{Z}/2\text{Z} \times \text{Z}/2\text{Z}$ or $\text{Z}/4\text{Z}$, depending on whether there is a group extension.

Define

\begin{equation}
B = \text{Map}(X_3, \Sigma^5 H\text{Z}).
\end{equation}

(A.17)

\(^{44}\)More simply, the $\text{Z}$-graded homotopy group of the Anderson dual to $X_3$ is the derived $R\text{Hom}(\pi_\ast X_3, \text{Z})$; there is a shift of 5 in (A.16)
Then

(A.18) \[ \pi_j B = [\Sigma^3 X_3, \Sigma^5 HZ] \cong H^{5-j}(X_3; \mathbb{Z}) \cong H^{5-j}(BSO_3; \mathbb{Z}), \]

where the last step is the Thom isomorphism. Hence

(A.19) \[ \pi_{(0,1,2,3,4,5)} B \cong \{0, \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0, 0, \mathbb{Z}\}. \]

The map \( HZ \to IZ \) induces a map \( B \to A \) and so a map of Atiyah-Hirzebruch spectral sequences. The \( E_2 \) page of the spectral sequence for \( B^{\phi_B}(\mathbb{RP}^\infty) \) is shown in Figure 13. The group \( B^{\phi_B}(\mathbb{RP}^\infty) \cong (\mathbb{Z}/2\mathbb{Z})^{x^3} \) was computed after (6.36), and it follows that \( d_2 : E^{3,-2}_2 \to E^{1,-1}_2 \) vanishes and there is no group extension passing from the degree 0 part of the \( E_\infty \) page to \( B^{\phi_B}(\mathbb{RP}^\infty) \). The map of spectral sequences now implies that there is no group extension in the \( A \)-spectral sequence either, that \( A^{\phi_A}(\mathbb{RP}^\infty) \cong (\mathbb{Z}/2\mathbb{Z})^{x^2} \), and that \( i : B^{\phi_B}(\mathbb{RP}^\infty) \to A^{\phi_A}(\mathbb{RP}^\infty) \) is surjective. \[ \square \]
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