CONSECUTIVE MINORS FOR DYSON’S BROWNIAN MOTIONS

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Abstract. In 1962, Dyson [Dys62] introduced dynamics in random matrix models, in particular into GUE (also for \( \beta = 1 \) and 4), by letting the entries evolve according to independent Ornstein-Uhlenbeck processes. Dyson shows the spectral points of the matrix evolve according to non-intersecting Brownian motions. The present paper shows that the interlacing spectra of two consecutive principal minors form a Markov process (diffusion) as well. This diffusion consists of two sets of Dyson non-intersecting Brownian motions, with a specific interaction respecting the interlacing. This is revealed in the form of the generator, the transition probability and the invariant measure, which are provided here; this is done in all cases: \( \beta = 1, 2, 4 \). It is also shown that the spectra of three consecutive minors ceases to be Markovian for \( \beta = 2, 4 \).

1. Introduction

In 1962, Dyson [Dys62] introduced dynamics in random matrix models, in particular into GUE, by letting the entries evolve according to independent Ornstein-Uhlenbeck processes. According to Dyson, the spectral points of the matrix evolve according to non-intersecting Brownian motions. The present paper addresses the question whether taking two consecutive principal minors leads to a diffusion on the two interlacing spectra of the minors, taken together. This is so! The diffusion is given by the Dyson diffusion for each of the spectra, augmented with a strong coupling term, which is responsible for a very specific interaction between the two sets of spectral points, to be explained in this paper. However the motion induced on the spectra of three consecutive minors is non-Markovian, for generic initial conditions. A further question: is the motion of two interlacing spectra a

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determinantal process? We believe this is not the case; but determinantal processes appear upon looking at a different space-time directions. These issues are addressed in another paper by the authors.

During the last few years, the question of interlacing spectra for GUE-minors have come up in many different contexts. In a recent paper, Johansson and Nordenstam [JN06], based on domino tilings results of Johansson [Joh05a], show that domino tilings of aztec diamonds provide a good discrete model for the consecutive eigenvalues of GUE-minors. In an effort to put some dynamics in the domino tiling model, Nordenstam [Nor10] then shows that the shuffling algorithm for domino tilings is a discrete version of an interlacing of two Dyson Brownian motions, introduced and investigated by Jon Warren [War07]. One might have suspected that the Warren process would coincide with the diffusion on the spectra of two consecutive principal minors. They are different!

Non-intersecting paths and interlaced processes (random walks and continuous processes) have been investigated by several authors in many different interesting directions; see e.g. [NF98], [Joh02], [Joh05a], [Joh05b], [TW04], [KT04], [O’C03], [MOW09], [Def08a], [KT04], [KS09], [AvM05], just to name a few. In particular, in [TW04, AvM05], partial differential equations were derived for the Dyson process and related processes.

The plan of this paper is the following. We state precisely all the results in Section 2. Some useful matrix equalities are derived in Section 3 which are used in Section 4 to derive transition densities for the various processes considered. Stochastic differential equations are derived in Sections 5 and 6. The fact that the the spectra of three consecutive minors are not Markovian for generic initial conditions is demonstrated the last Section.

There is a companion paper by the same authors aiming at determining the kernel for the point process related to the Dyson Brownian minor process along space-like paths [ANvM10].

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2. THE ORNSTEIN-UHLENBECK PROCESS AND DYSON’S BROWNIAN MOTION

Consider the space $\mathcal{H}_n^{(\beta)}$ of $n \times n$ matrices $B$, with entries $B_{k\ell} \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ ($\beta = 1, 2, 4$) satisfying the symmetry conditions

\begin{equation}
B_{k\ell} = B_{\ell k}^*.
\end{equation}

Any element $z \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ admits a decomposition $z = z^{(0)} + \sum_{r=1}^{\beta-1} z^{(r)} e_r$, with $e_i$’s satisfying

$e_1^2 = e_2^2 = e_3^2 = -1, \ e_1 e_2 = -e_2 e_1 = e_3, \ e_1 e_3 = -e_3 e_1 = -e_2, \ e_2 e_3 = -e_2 e_3 = e_1.$
The conjugate $^*$ of an element $z \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ and its norm are given by

$$z^* = z^{(0)} - \sum_{r=1}^{\beta-1} z^{(r)} e_r, \quad |z|^2 = zz^* = \sum_{r=0}^{\beta-1} z^{(r)2},$$

with $z$ admitting a polar decomposition $z = |z|u$, with $|u|^2 = \sum_{r=0}^{\beta-1} u^{(r)2} = 1$. The matrices $B \in \mathcal{H}_n^{(\beta)}$, as in (2.1), correspond to:

$$\mathcal{H}_n^{(\beta)} = \begin{cases} \text{real symmetric } n \times n \text{ matrices, for } \beta = 1 \\ \text{complex Hermitian } n \times n \text{ matrices, for } \beta = 2 \\ \text{self-dual Hermitian } n \times n \text{ “quaternionic” matrices, for } \beta = 4 \end{cases}$$

acting on it by conjugation.

For $\beta = 4$, it is well known that the quaternionic entries $z$ can be represented as follows

$$z = z^{(0)} + \sum_{r=1}^{\beta-1} z^{(r)} e_r \mapsto \hat{z} = \begin{pmatrix} z^{(0)} + iz^{(1)} & z^{(2)} + iz^{(3)} \\ -z^{(2)} + iz^{(3)} & z^{(0)} - iz^{(1)} \end{pmatrix}.$$ 

So, the $n \times n$ quaternionic matrices $B$ can be turned into $2n \times 2n$ self-dual Hermitian matrices $\hat{B}$, of which the real spectrum is doubly degenerate. Here, we shall define the $n$ distinct eigenvalues as the spectrum of $B$. Unless stated otherwise we shall be working with the $n \times n$ quaternionic matrices, rather than the $2n \times 2n$ Hermitian matrices. Also, when working with matrices having quaternionic entries, the trace will be defined in the usual way, that is as the sum of the diagonal entries of the $n \times n$-matrix.

The determinant of a matrix $B \in \mathcal{H}_n^{(4)}$ is given in terms of the $2n \times 2n$ matrix $\hat{B}$ (as defined in (2.3)), by the following procedure: first define the skew-symmetric $2n \times 2n$ matrix $\mathbb{B}$ by the following product:

$$\mathbb{B} := \hat{B} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I_n,$$

and then “det $B$” is defined as

$$\det B := \text{Pfaff}(\mathbb{B}) = (\det(\mathbb{B}))^{1/2} = \sum_p (-1)^{n-p} \prod_{1}^{\ell} B_{\alpha\beta} B_{\beta\gamma} \ldots B_{\delta\alpha}.$$
where \( p \) is any permutation of the indices \((1, 2, \ldots , n)\) consisting of \( \ell \) exclusive cycles of the form \((\alpha \rightarrow \beta \rightarrow \gamma \rightarrow \cdots \rightarrow \alpha)\); see Mehta [Meh88]. In particular, this means that

\[
\text{det}(\lambda I - B) = \prod_{i=1}^{n} (\lambda - \lambda_i), \quad \text{spec } B = \{\lambda_1, \ldots , \lambda_p\},
\]

with the \( \lambda_i \) being the double eigenvalues of \( B \).

The following normalization constant \( Z_{n,\beta} \) will come back over and over again:

\[
Z_{n,\beta}^{-1} := 2^{-\frac{n}{2}} \left( \frac{\beta}{\pi} \right)^{N_{n,\beta}}, \quad \text{with } N := N_{n,\beta} := \frac{n}{2} + \frac{\beta}{4} n(n-1).
\]

Dyson’s idea was to let the free parameters of the matrix evolve according to the SDE (Dyson process)

\[
dB_{ii} = -B_{ii} dt + \sqrt{\frac{2}{\beta}} dB_{ii}, \quad i = 1, \ldots , n
\]

\[
dB_{ij}^{(l)} = -B_{ij}^{(l)} dt + \frac{1}{\sqrt{\beta}} dB_{ij}^{(l)}, \quad i, j = 1, \ldots , n \text{ and } l = 0, \ldots , \beta - 1,
\]

where \( dB_{ii} \) and \( dB_{ij}^{(l)} \), for \( 1 \leq i, j \leq n \) and \( l = 0, \ldots , \beta - 1 \), are independent, standard Brownian motions. Since the Ornstein-Uhlenbeck diffusions are independent, the Dyson process on the matrix \( B \) has a generator, which is just the sum of the OU-processes above:

\[
A_{\text{Dys}} := \sum_{i=1}^{n} \left( \frac{1}{\beta} \frac{\partial^2}{\partial B_{ii}^2} - B_{ii} \frac{\partial}{\partial B_{ii}} \right) + \sum_{1 \leq i < j \leq n} \sum_{0 \leq \ell \leq \beta - 1} \left( \frac{1}{2\beta} \frac{\partial^2}{\partial B_{ij}^{(\ell)} \partial B_{ij}^{(\ell)}} - B_{ij}^{(\ell)} \frac{\partial}{\partial B_{ij}^{(\ell)}} \right),
\]

with transition probability, setting \( c := e^{-t} \) and using the constant (2.7),

\[
\mathbb{P}[B_t \in dB \mid B_0 = \bar{B}] =: p(t, \bar{B}, B) dB
\]

where \( dB \) is the product measure over all the independent parameters \( B_{ii}, \ B_{ij}^{(l)} \).

The transition probability (2.10) satisfies the Fokker-Planck equation

\[
\frac{\partial p}{\partial t} = A_{\text{Dys}}^T p,
\]

with

\[
A_{\text{Dys}} = \frac{2}{\beta} \left( \sum_{i=1}^{n} \frac{\partial}{\partial B_{ii}^\beta} h^\beta \frac{\partial}{\partial B_{ii}^\beta} \frac{1}{4} \sum_{1 \leq i < j \leq n} \sum_{0 \leq \ell \leq \beta - 1} \frac{\partial}{\partial B_{ij}^{(\ell)}^\beta} h^\beta \frac{\partial}{\partial B_{ij}^{(\ell)}^\beta} \frac{1}{4} \sum_{1 \leq i < j \leq n} \sum_{0 \leq \ell \leq \beta - 1} \frac{\partial}{\partial B_{ij}^{(\ell)}^\beta} h^\beta \frac{\partial}{\partial B_{ij}^{(\ell)}^\beta} \right).
\]
with a delta-function initial condition, \(p(t, \bar{B}, B)\big|_{t=0} = \delta(\bar{B}, B)\), and with invariant measure (density)

\[
\text{Inv1} \quad (2.13) \quad \lim_{t \to \infty} p(t, \bar{B}, B) = Z_{n, \beta}^{-1} h(B)^\beta, \quad \text{with } h := h(B) := e^{-\frac{1}{2} \text{Tr} \bar{B}^2}.
\]

Dyson discovered in [Dys62] the surprising fact that the process restricted to \(\text{spec}(B) := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) is Markovian as well. This is the content of Dyson’s celebrated Theorem (Theorem 2.1).

Before stating the main Theorem, we define diagonal matrices \(X = \text{diag}(x_1, \ldots, x_n)\) and \(Y = \text{diag}(y_1, \ldots, y_n)\), vectors \(w, v \in \mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n\) and the inner-product \(\langle w, v \rangle = \sum_1^n w_i v_i^*\). Then consider the integral

\[
\text{1.21} \quad (2.14) \quad G_n^\beta(X, Y; w, v) := \int_{U_n^{(\beta)}} dU e^{(\text{Tr} XUYU^{-1} + 2 \text{Re}(w, Uv))},
\]

and its integrand

\[
\text{Gint} \quad (2.15) \quad G_n^\beta(U; X, Y; w, v) := e^{(\text{Tr} XUYU^{-1} + 2 \text{Re}(w, Uv))}.
\]

For \(w = v = 0\), this is the more familiar integral

\[
(2.16) \quad F_n^{(\beta)}(X, Y) := G_n^{(\beta)}(X, Y; 0, 0) = \int_{U_n^{(\beta)}} dU e^{\text{Tr} XUYU^{-1}},
\]

which for \(\beta = 2\) gives the Harris-Chandra-Itzykson-Zuber formula:

\[
\text{HCIZ} \quad (2.17) \quad F_n^{(2)}(X, Y) = \frac{\det[e^{x_i y_j}]_{1 \leq i, j \leq n}}{\Delta_n(x) \Delta_n(y)} \prod_{r=1}^n r!.
\]

Does the integral (2.14) admit such a representation? This is an open problem.

In the following Theorem, formulae (2.18), (2.20) and (2.22) are due to Dyson [Dys62].

**Theorem 2.1.** The Dyson process restricted to its spectrum \(\text{spec}(B) = \lambda := \{\lambda_1, \ldots, \lambda_n\}\) is Markovian with SDE given by:

\[
\text{1.4} \quad (2.18) \quad d\lambda_i = \left(-\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}\right) dt + \sqrt{\frac{2}{\beta}} db_{ii}, \quad i = 1, \ldots, n.
\]
Its transition probability, with $c := e^{-t}$,

$$P[\lambda_t \in d\lambda|\lambda_0 = \bar{\lambda}] = p_{\lambda}(t, \bar{\lambda}, \lambda) d\lambda_1 \cdots d\lambda_n = \frac{C^{-1}_{n,\beta}}{(1 - c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \sum_{i=1}^{n}(\lambda_i^2 + c^2 \bar{\lambda}_i^2)} F_n^{(\beta)} \left( \frac{\beta c}{1 - c^2} \lambda, \bar{\lambda} \right) |\Delta_n(\lambda)|^{\beta} \prod_{i=1}^{n} d\lambda_i,$$

satisfies the Dyson diffusion equation, with delta-function initial condition $(p_{\lambda}|_{t=0} = \delta(\lambda, \bar{\lambda}))$ (forward equation)

$$\frac{\partial p_{\lambda}}{\partial t} = A^{\top}_{\lambda} p_{\lambda}, \text{ with } A^{\top}_{\lambda} := \frac{1}{\beta} \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} (\Phi_n(\lambda))^{\beta} \frac{\partial}{\partial \lambda_i} (\Phi_n(\lambda))^{\beta}.$$

The generator is

$$A_{\lambda} = \sum_{i=1}^{n} \left( \frac{1}{\beta} \frac{\partial^2}{\partial \lambda_i^2} + \left( -\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \frac{\partial}{\partial \lambda_i} \right),$$

and the invariant measure of the Dyson process on $B$, projected onto $\text{spec}(B)$, is given by the GOE($n$), GUE($n$), GSE($n$) measure for $\beta = 1, 2, 4$ respectively:

$$C^{-1}_{n,\beta} (\Phi_n(\lambda))^{\beta} d\lambda_1 \cdots d\lambda_n, \text{ with } \Phi_n(\lambda) = e^{-\frac{1}{\beta} \sum_{i=1}^{n} \lambda_i^2} |\Delta_n(\lambda)|.$$

For completeness we shall prove (2.19), (2.22) in Section 4 and (2.18), (2.20) in Section 5.

It is remarkable that the Dyson process is not only Markovian upon restriction to the spectrum of any single principal minor $B$, $B^{(n-1)}$, $B^{(n-2)}$, ..., of sizes $n$, $n-1$, $n-2$, ..., but also upon restriction to any two consecutive principal minors, in particular,

$$(\text{spec } B, \text{spec } B^{(n-1)}) := (\lambda, \mu) := ((\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_{n-1})),$$

with intertwining property

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$ We denote by $A_{\lambda}$ and $A_{\mu}$ the generators of the consecutive spectra $B$ and $\text{spec } B^{(n-1)}$, as defined in (2.20). Define the characteristic polynomials of the two

\footnote{\textit{footC} \(C^{-1}_{n,\beta}\) is the norming constant for the Gaussian ensemble for general \(\beta\), as obtained from the Selberg formula (see Mehta [Meh04], formula (3.3.10)), (see (2.7) for \(N\))

\(C^{-1}_{n,\beta} = (2\pi)^{-\frac{n}{2}} \beta^n \prod_{j=1}^{n} \left( \frac{\Gamma \left( 1 + \frac{j}{\beta} \right)}{\Gamma \left( 1 + \frac{j}{2} \right)} \right).\)
consecutive minors $B$ and $B^{(n-1)}$, 

$$P_n(x) = \prod_{\alpha=1}^{n} (x - \lambda_\alpha), \quad P_{n-1}(x) = \prod_{\beta=1}^{n-1} (x - \mu_\beta),$$

and the Vandermonde determinants

$$\Delta_n(\lambda) := \prod_{j>i} (\lambda_j - \lambda_i) \geq 0,$$

$$\Delta_n(\lambda, \mu) := \prod_{i=1}^{n} \prod_{j=1}^{n-1} (\lambda_i - \mu_j) = \prod_{i=1}^{n} P_n(\lambda_i) = \prod_{i=1}^{n-1} P_{n-1}(\lambda_i),$$

with $\Delta_n(\lambda, \mu)(-1)^{n(n-1)} \geq 0$ because of the intertwining.

In order to state Theorem 2.2, we need the following property of an $n \times n$ matrix $B \in H_1^{(\beta)}$; not only can $B$ be conjugated by a matrix $U^{(n)} \in U_n^{(\beta)}$ (see (2.2)), in the standard way, such that

$$conj1 \quad (U^{(n)})^{-1} BU^{(n)} = \text{diag}(\lambda_1, \ldots, \lambda_n),$$

but also by a matrix of the form 

$$conj2 \quad \begin{pmatrix} U^{(n-1)} & 0 \\ 0 & 1 \end{pmatrix}$$

with $|u_i| = 1$ (angular variables) and with $r_i \geq 0$ for $1 \leq i \leq n - 1$ and $r_n$, given by

$$conj3 \quad r^2 := \frac{P_n(\mu_k)}{P_n^{(n-1)}(\mu_k)} \geq 0, \quad 1 \leq k \leq n - 1, \quad r_n := \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i.$$ 

The conjugation in (2.27) transforms the last column $v$ of $B$ in the last column of the bordered matrix $B_{bord}$ (except for the last entry); i.e.,

$$conj4 \quad U^{(n-1)} v = (r_1 u_1, \ldots, r_{n-1} u_{n-1})^\top, \quad B_{nn} = r_n, \quad v := (B_{1,n}, \ldots, B_{n-1,n})^\top.$$ 

These facts, (2.27), (2.28) and (2.29), will be discussed and shown in Section 3.

The next statement is the analogue of Theorem 2.1 for the case of the spectra of two consecutive minors.
Theorem 2.2. The Dyson process on $B$ restricted to
$(\text{spec } B, \text{spec } B(n-1)) = (\lambda, \mu) := ((\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_{n-1}))$
is a diffusion $(\lambda(t), \mu(t))$ as well, with the following SDE:

$$d\lambda_\alpha = \left(-\lambda_\alpha + \sum_{\varepsilon \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\varepsilon}\right) dt + \sqrt{\frac{2}{\beta}} \frac{P_n-1(\lambda_\alpha)}{P_n(\lambda_\alpha)} dt + \sum_{1 \leq i < j \leq n-1} \frac{\sqrt{2} r_i r_j}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)} dB_{ij} + \sum_{i=1}^{n-1} \frac{\sqrt{2} r_i}{\lambda_\alpha - \mu_i} dB_{ni} + dB_{nn},$$

$$d\mu_\gamma = \left(-\mu_\gamma + \sum_{\varepsilon \neq \gamma} \frac{1}{\mu_\gamma - \mu_\varepsilon}\right) dt + \sqrt{\frac{2}{\beta}} dB_{\gamma\gamma},$$
in terms of independent standard Brownian motions $\{dB_{ij}, dB_{n}\}_{1 \leq i < j \leq n}$. Its transition probability is given by:

$$p_{\lambda \mu}(t, (\tilde{\lambda}, \tilde{\mu}), (\lambda, \mu)) = \mathbb{P}[(\lambda_t, \mu_t) \in (d\lambda, d\mu) \mid (\lambda_0, \mu_0) = (\tilde{\lambda}, \tilde{\mu})]$$

$$= \frac{\hat{Z}_{n, \beta}^{-1}}{(1 - c^2)^N} e^{-\frac{\beta}{2(1 - c^2)} \sum_{i=1}^{n} (\lambda_i^2 + c^2 \lambda_i^2)} \int_{(S^{\beta-1})_{2(n-1)}} \prod_{i=1}^{n-1} d\Omega(\beta - 1)(u_i) d\Omega(\beta - 1)(\tilde{u}_i)$$

$$\times G_n^{(\beta)} \left( \frac{\beta c}{1 - c^2}, \frac{\beta c}{1 - c^2} \right) \left( r_i u_i, r_i \tilde{u}_i \right)^{n-1} \prod_{i<j} d\lambda_i d\lambda_j d\mu_j,$$

where the $r_i$’s are given by (2.28). It is also a solution of the following forward diffusion equation, with delta-function initial condition

$$\frac{\partial p_{\lambda \mu}}{\partial t} = A_T p_{\lambda \mu},$$

where

$$A^{T}_{\lambda \mu} := -\frac{2}{\beta} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \mu_j} \left( \frac{1}{(\lambda_i - \mu_j)^2} \frac{P_{n-1}(\lambda_i)}{P_{n}(\lambda_i)} - \frac{P_{n-1}(\mu_j)}{P_{n}(\mu_j)} \right)$$

The constant reads

$$\hat{Z}_{n, \beta}^{-1} = \frac{\beta^{n+1}(\Gamma(1 + \frac{\beta}{2}))^{n-1}}{(2\pi)^{\frac{n}{2}}(\pi)^{n-1}} \prod_{j=1}^{n-1} \Gamma(1 + \frac{\beta}{2})(\text{vol}(S^{\beta-1}))^{n-1},$$

and $\text{vol}(S^k) = 2\pi \prod_{i=1}^{k-1} \left( \frac{\beta^{\pi/2}}{\pi}(\cos \theta) d\theta \right)$ for $k \geq 2$, $\text{vol}(S^0) = 1$ and $\text{vol}(S^1) = 2\pi$, which is proved by induction on $k$; so $\text{vol}(S^2) = 4\pi$ and $\text{vol}(S^3) = 2\pi^2$. 

footZhat
and where $A_{\lambda}^T$ and $A_{\mu}^T$ are defined by (2.20). The Dyson process restricted to $(\lambda, \mu)$ has invariant measure, (see (2.23)),

\[
Z_{n,\beta}^{-1} \left( \text{vol}(S^{\beta-1}) \right)^{2(n-1)} e^{-\frac{\beta}{2} \sum_{i=1}^{n} \lambda_i^2} |\Delta_n(\lambda)\Delta_{n-1}(\mu)| |\Delta_n(\lambda, \mu)|^{\frac{\beta}{2} - 1} \prod_{i=1}^{n} d\lambda_i \prod_{i=1}^{n-1} d\mu_i.
\]

The SDE (2.30) and generator (2.32) are computed in Section 6 while the expressions for transition density (2.31) and invariant measure (2.34) are proved in Section 4.

Note that it is an immediate consequence of Theorem 2.1 that the generator $A_{\text{Dys}}$, defined in (2.9), acting on the $\lambda_i$ and $\mu_i$, has the form

\[
A_{\text{Dys}}(\lambda_i) = A_\lambda(\lambda_i) \quad \text{and} \quad A_{\text{Dys}}(\mu_i) = A_\mu(\mu_i),
\]

where $A_\lambda$ and $A_\mu$ are defined by (2.21).

Whereas all statements in this paper hold for $\beta = 1, 2, 4$, a part of it can be extended to general $\beta > 0$, as will be shown in section 6, after the proof of Theorem 2.2.

**Corollary 2.3.** For general $\beta > 0$, the SDE (2.30), in terms of the independent standard Brownian motions $\{db_{ij}, \tilde{b}_{ij}\}_{1 \leq i < j \leq n}$, defines a diffusion, whose generator is given by the same equations (2.32), and whose invariant measure is given by (2.34). Moreover, this diffusion restricted to the $\lambda_i$'s (or to the $\mu_i$'s) is the standard Dyson Brownian motion (2.18).

The following corollary shows that the $\mu_i$'s in $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ are repelled by the boundary and fluctuate in unison with the boundary points, when they get close.

**Corollary 2.4.** The nonnegative gaps $\mu_i - \lambda_i$ and $\lambda_{i+1} - \mu_i$ for $1 \leq i \leq n - 1$ satisfy, in the notation of (2.30),

\[
d(\mu_i - \lambda_i) = F_i(\lambda, \mu) dt + \sqrt{\mu_i - \lambda_i} \sum_{1 \leq k \leq n} \alpha_{k\ell} \tilde{b}_{k\ell}
\]

\[
d(\lambda_{i+1} - \mu_i) = \hat{F}_i(\lambda, \mu) dt + \sqrt{\lambda_{i+1} - \mu_i} \sum_{1 \leq k \leq n} \hat{\alpha}_{k\ell} \tilde{b}_{k\ell}
\]

with

\[
\begin{cases}
\text{some } \alpha_{k\ell} = O(1) \text{ for } \mu_i \simeq \lambda_i \text{ and some } \hat{\alpha}_{ij} = O(1) \text{ for } \mu_i \simeq \lambda_{i+1}. \\
F_i(\lambda, \mu) \big|_{\mu_i = \lambda_i} > 0, \quad \hat{F}_i(\lambda, \mu) \big|_{\mu_i = \lambda_{i+1}} > 0
\end{cases}
\]

This is to be compared with the Warren process [War07], which also describes two intertwined Dyson processes $\lambda$ and $\mu$, but with an entirely different interaction: namely the $\mu_i$'s near the boundaries of the intervals $[\lambda_i, \lambda_{i+1}]$ behave like the absolute value of one-dimensional Brownian motion near the origin.
As we saw, the Dyson process on $B$, restricted to the spectrum of one principal minor or the spectra of two consecutive minors leads to two Markov processes; opposed to that, we have the following statement, which will be proved in Section 7.

**Theorem 2.5.** The restriction of the Dyson process restricted to the following data

$$(\text{spec } B, \text{spec } B^{(n-1)}, \text{spec } B^{(n-2)}) := (\lambda, \mu, \nu)$$

is not Markovian for generic initial conditions on $B$, i.e., the joint spectra of any three neighbouring set of minors of $B$ are not Markovian, for $\beta = 2$ and 4.

### 3. Some Matrix Identities

In this section, we prove formula (2.27) for $r_k$ as in (2.28). In the course of doing that we will also prove the following formulas:

\[ \sum_{i=1}^{n-1} r_i^2 + \frac{r_n^2}{2} = \frac{1}{2} \left( \sum_{i=1}^{n} \lambda_i^2 - \sum_{i=1}^{n-1} \mu_i^2 \right) \quad \text{and} \quad \prod_{i=1}^{n-1} r_i^2 = \frac{\Delta_n(\lambda, \mu)}{\Delta_{n-1}(\mu)}. \]

One also has the (often used) identities

\[ \sum_{i=1}^{n-1} \frac{r_i^2}{\lambda_i - \mu_i} + r_n - \lambda = 0 \quad \text{and} \quad \frac{P_n'(\lambda_i)}{P_{n-1}(\lambda)} = \sum_{i=1}^{n-1} \left( \frac{r_i}{\lambda_i - \mu_i} \right)^2 + 1. \]

Finally, one has, for fixed $(\mu_1, \ldots, \mu_{n-1})$ and fixed $(u_1, \ldots, u_{n-1}),$

\[ \prod_{i=1}^{n-1} d\lambda_i d\sigma = (-1)^{n-1} \frac{\Delta_n(\lambda)}{\Delta_{n-1}(\mu)} \prod_{i=1}^{n} d\lambda_i. \]

**Proof.** From the form of the matrix $B_{\text{bord}}$ as in (2.27), one checks (see (2.25) and also the formula (2.5) for the determinant in the quaternionic case)

\[ \prod_{i=1}^{n-1} (\lambda_i - \lambda) = \det(B_{\text{bord}} - \lambda I) = \prod_{i=1}^{n-1} (\mu_i - \lambda) \left( \sum_{i=1}^{n-1} \frac{r_i^2}{\lambda - \mu_i} + r_n - \lambda \right), \]

from which it follows that

\[ \left( -\sum_{i=1}^{n-1} \frac{r_i^2}{\lambda - \mu_i} - r_n + \lambda \right) = \frac{P_n(\lambda)}{P_{n-1}(\lambda)} \]

\[ = \lambda - (\sigma_1(\lambda) - \sigma_1(\mu)) - \frac{1}{\lambda} (\sigma_1(\lambda)\sigma_1(\mu) + \sigma_2(\mu) - \sigma_2(\lambda) - \sigma_2(\mu)) + O\left( \frac{1}{\lambda^2} \right). \]

Then taking residues in formula (3.5) yields the first formulae (2.28) and thus the formula for $\prod_{i=1}^{n-1} r_i^2$ in (3.1). Comparing the coefficients of $\lambda^0$ and the $\lambda^{-1}$ on both sides of (3.5) yields the first formula of (3.1). Setting $\lambda = \lambda_i$ in the expression

\[ \sigma_k(\lambda) \quad \text{are symmetric polynomials:} \quad \sigma_1(\lambda) = \sum \lambda_i, \quad \sigma_2(\lambda) = \sum_{i<j} \lambda_i \lambda_j, \text{etc.} \]

\[ \text{The same for } \sigma_k(\mu) \]
and its derivative with regard to \( \lambda \) implies the two sets of \( n \) identities \( \text{(3.2)} \), in view of the definition of \( P_n \). Formula \( \text{(3.3)} \) amounts to computing the Jacobian determinant of the transformation from \( \lambda_1, \ldots, \lambda_n \) to \( r_1, \ldots, r_n \); to do this, take the differential of the first of the \( n \) expressions appearing in \( \text{(3.2)} \) (as functions of \( \lambda_1, \ldots, \lambda_n \) and \( r_1, \ldots, r_n \)), keeping the \( \mu_i \)'s fixed and use the second of the expressions \( \text{(3.2)} \):

\[
0 = \sum_{i=1}^{n-1} \frac{dr_i^2}{\lambda_i - \mu_i} + dr_n - \left( 1 + \sum_{i=1}^{n-1} \frac{r_i^2}{(\lambda_i - \mu_i)^2} \right) d\lambda_i
\]

\[
= \sum_{i=1}^{n-1} \frac{dr_i^2}{\lambda_i - \mu_i} + dr_n - \frac{P'_n(\lambda)}{P_{n-1}(\lambda)} d\lambda_i,
\]

which in matrix form reads

\[
\begin{pmatrix}
\frac{dr_1^2}{d\lambda_1} \\
\frac{dr_2^2}{d\lambda_2} \\
\vdots \\
\frac{dr_{n-1}^2}{d\lambda_{n-1}} \\
\frac{dr_n}{d\lambda_n}
\end{pmatrix} = \text{diag} \begin{pmatrix}
P'_n(\lambda_1) & \cdots & P'_n(\lambda_n)
\end{pmatrix} \begin{pmatrix}
d\lambda_1 \\
d\lambda_2 \\
\vdots \\
d\lambda_{n-1} \\
d\lambda_n
\end{pmatrix},
\]

where (by Cauchy’s determinantal formula)

\[
\Gamma := \begin{pmatrix}
\frac{1}{\lambda_1 - \mu_1} & \frac{1}{\lambda_1 - \mu_2} & \cdots & \frac{1}{\lambda_1 - \mu_{n-1}} & 1 \\
\frac{1}{\lambda_2 - \mu_1} & \frac{1}{\lambda_2 - \mu_2} & \cdots & \frac{1}{\lambda_2 - \mu_{n-1}} & 1 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\frac{1}{\lambda_{n-1} - \mu_1} & \frac{1}{\lambda_{n-1} - \mu_2} & \cdots & \frac{1}{\lambda_{n-1} - \mu_{n-1}} & 1 \\
\frac{1}{\lambda_n - \mu_1} & \frac{1}{\lambda_n - \mu_2} & \cdots & \frac{1}{\lambda_n - \mu_{n-1}} & 1
\end{pmatrix}, \quad \text{with } \det \Gamma = (-1)^{(n-1)(\frac{n}{2}+1)} \frac{\Delta_n(\lambda) \Delta_{n-1}(\mu)}{\Delta_n(\lambda, \mu)}.
\]

The formula \( \text{(3.7)} \) for the determinant follows from the observation that \( \det \Gamma \) has homogeneous degree \( 1 - n \) and vanishes when \( \Delta_n(\lambda) \Delta_{n-1}(\mu) \) does and blows up (simply) when and only when \( \Delta_n(\lambda, \mu) \) vanishes. Thus we have

\[
\frac{\partial(r_1^2, \ldots, r_n^2; r_n)}{\partial(\lambda_1, \ldots, \lambda_n)} = \prod_{i=1}^{n} \frac{P'_n(\lambda_i)}{P_{n-1}(\lambda_i)} (\det \Gamma)^{-1} = (-1)^{n-1} \frac{\Delta_n(\lambda)}{\Delta_{n-1}(\mu)}.
\]

This concludes the proof of formulas stated in the beginning of this section. \( \square \)

4. Transition Probabilities

A quick review of the Ornstein-Uhlenbeck process (see Feller [Fel71]): it is a diffusion on \( \mathbb{R} \), given by the one-dimensional SDE,

\[
dx = -\rho x \, dt + \frac{1}{\sqrt{\beta}} \, db,
\]

\( \text{T1} \) (4.1)
and it has transition probability \((c := e^{-\rho t})\)

\[
P[x_t \in dx \mid x_0 = \bar{x}] =: p_{\text{OU}}(t; \bar{x}, x) \, dx
\]

\[
= \left(\frac{\rho \beta}{\pi (1 - c^2)}\right)^{1/2} \exp \left(-\frac{\rho \beta (x - c\bar{x})^2}{1 - c^2}\right) \, dx.
\]

The transition probability is a solution of the forward (diffusion) equation, with \(\delta\)-function initial condition\(^4\)

\[
\frac{\partial p_{\text{OU}}}{\partial t} = \left(\frac{1}{2\beta} \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} (-\rho x)\right) p_{\text{OU}} = \frac{1}{2\beta} \left(\frac{\partial}{\partial x} \phi_{\beta}(x) \frac{\partial}{\partial x} \phi_{\beta}(x)\right) p_{\text{OU}},
\]

and invariant measure (density)\(^5\)

\[
\phi_{\beta}(x) = \sqrt{\frac{\rho \beta}{\pi}} e^{-\rho \beta x^2} = \lim_{t \to \infty} p_{\text{OU}}(t; \bar{x}, x).
\]

**Proof of transition probabilities** \((2.10)\), \((2.19)\) and \((2.31)\).

(i) The Fokker-Planck equation for the transition probability of the Dyson process. The Dyson process consists of running the free parameters of the matrix \(B \in \mathcal{H}_n^{(\beta)}\), as in \((2.1)\), according to independent Ornstein-Uhlenbeck processes, with \(\rho = 1\), the diagonal with \(\beta \to \beta/2\) and the off-diagonal parameters with \(\beta \to \beta\). Remembering the definition \((2.7)\) of \(N = N_{n,\beta}\) and the definition of the trace (after \((2.3)\)), one has, setting \(c = e^{-t}\), and using \((2.8)\), \((4.1)\) and \((4.2)\), the transition probability for the Dyson process is given by\(^3\)

\[
p(t, B, B) = \prod_{i=1}^n p_{\text{OU}}(t; B_{ii}, B_{ii}) \prod_{1 \leq i < j \leq n} \prod_{\ell=0}^{\beta-1} p_{\text{OU}}(t; B_{ij}(\ell), B_{ij}(\ell))
\]

\[
= \prod_{i=1}^n \left(\frac{e^{-\beta (B_{ii} - cB_{ii})^2}}{(2\pi(1 - c^2)/\beta)^2}\right) \prod_{1 \leq i < j \leq n} \prod_{\ell=0}^{\beta-1} \left(\frac{e^{-\beta (B_{ij}(\ell) - cB_{ij}(\ell))^2}}{(\pi(1 - c^2)/\beta)^2}\right)
\]

\[
= \frac{1}{2^{n/2} (\beta(1 - c^2))^{N_{n,\beta}}} e^{-\beta 2(1 - c^2) \text{Tr}(B - c\bar{B})^2} = \frac{Z_{n,\beta}^{-1}}{(1 - c^2)^{N_{n,\beta}}} e^{-\beta - \frac{2(1 - c^2) \text{Tr}(B - c\bar{B})^2}{2\rho}},
\]

\(^4\)The backward equation becomes the heat equation with \((x, t) \mapsto (xe^{\rho t}, \frac{1-e^{2\rho t}}{2\rho})\)

\(^5\)with constant \(Z_{n,\beta}^{-1} = 2^{-n/2}(\beta)^{-N_{n,\beta}}\).
yielding \( (2.10) \), while \( \lim_{t \rightarrow \infty} p(t, \bar{B}, B) = Z_{n,\beta}^{-1}(h(B))^\beta \) is immediate, showing \( (2.13) \). Moreover, from \( (4.2) \), one computes for \( p(t; \bar{B}, B) \),

\[
\frac{\partial}{\partial t} p(t; \bar{B}, B) = \frac{2}{\beta} \left[ \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial B_{ii}^\beta/2}(B_{ii}) \frac{\partial}{\partial B_{ii}^\beta/2}(B_{ii}) \right] p(t; \bar{B}, B)
\]

\[
+ \frac{1}{4} \sum_{1 \leq i < j \leq n} \left[ \frac{\partial}{\partial B_{ij}^\beta}(B_{ij}) \frac{\partial}{\partial B_{ij}^\beta}(B_{ij}) \right] p(t; \bar{B}, B)
\]

\[
= \frac{2}{\beta} \left[ \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial B_{ii}} h(B) \frac{\partial}{\partial B_{ii}} h(B) \right]
\]

\[
+ \frac{1}{4} \sum_{1 \leq i < j \leq n} \left[ \frac{\partial}{\partial B_{ij}} h(B) \frac{\partial}{\partial B_{ij}} h(B) \right] p(t; \bar{B}, B)
\]

with

\[
h(B) = \text{constant} \times \prod_{i=1}^{n} \phi_{\beta/2}(B_{ii}) \prod_{1 \leq i < j \leq n} \phi_{\beta}(B_{ij}^{(ij)}),
\]

proving \( (2.12) \).

(ii) The transition probability \( (2.19) \) and the invariant measure for the \( \lambda_t \) process. Set

\[\lambda = (\lambda_1, \ldots, \lambda_n) \quad \text{and} \quad \bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n).\]

By the Weyl integration formula, given \( B = U\lambda U^{-1} \) and initial condition \( \bar{B} = \bar{U}\bar{\lambda}\bar{U}^{-1} \), express \( dB = d(U\lambda U^{-1}) \) in formula \( (2.10) \) in terms of spectral and angular variables \( dB = Z_{n,\beta}^{-1}|\Delta_n(\lambda)|^\beta dU \prod d\lambda_i \), with Haar measure \( dU \) on \( U_n^{(\beta)} \) normalized such that \( \text{Vol}(U_n^{(\beta)}) = 1 \), with \( C_{n,\beta}^{-1} \) defined in footnote \( \mathbb{1} \). This yields, using the transition probability \( (4.3) \),

\[
(4.8) \quad \mathbb{P}[B_t \in dB \mid B_0 = \bar{B}] = p(t; \bar{B}, B) dB
\]

\[
= \frac{C_{n,\beta}^{-1}}{(1 - c^2)^N} e^{-\frac{a}{2(1 - c^2)} \sum_i (\lambda_i^2 + c^2 \bar{\lambda}_i^2)}
\]

\[
\times e^{\frac{d a}{c} \text{Tr} U \lambda U^{-1} \bar{U} \bar{\lambda} \bar{U}^{-1}} |\Delta_n(\lambda)|^\beta dU \prod d\lambda_i.
\]

Note that the constant \( C_{n,\beta}^{-1} \) is compatible with the fact that for \( t \rightarrow \infty \) this transition probability tends to the GUE-probability; see below.

We now compute the transition probability \( p_\lambda(t; \bar{\lambda}, \lambda) d\lambda \) for the spectrum of the Dyson process; this will be a model to compute the transition probability for the
(\lambda_t, \mu_t)$-process. So, consider
\begin{equation}
\tag{4.9}
p_\lambda(t; \bar{\lambda}, \lambda)d\lambda = \mathbb{P}(\lambda_t \in d\lambda | \lambda_0 = \bar{\lambda})
\end{equation}
Next we compute the two probabilities in the integrand of the integral (4.11):
\begin{equation}
\tag{4.10}
\mathbb{P} \left[ B_t \in dB \mid B_0 = \bar{B} \right] \mathbb{P} \left[ B_0 \in d\bar{B} \mid \text{spec}(B_0) = \lambda \right]
\end{equation}
using (4.8) above, using the following conditional probability formula:
\begin{equation}
\tag{4.11}
\mathbb{P} \left[ B_0 \in d\bar{B} \mid \text{spec}(B_0) = \lambda \right] = \frac{\mathbb{P} \left[ B_0 \in d\bar{B}, \text{spec}(B_0) \in d\lambda \right]}{\mathbb{P} \left[ \text{spec}(B_0) \in d\lambda \right]} = d\bar{U}
\end{equation}
and finally using the integration (2.14),
\begin{equation}
\int_{U_n^{(\beta)}} e^{\frac{\beta c}{1-c^2} \text{Tr} U \lambda U^{-1} U \lambda U^{-1}} dU = \int_{U_n^{(\beta)}} e^{\frac{\beta c}{1-c^2} \text{Tr} U \lambda U^{-1}} dU = F_n^{(\beta)} \left( \frac{\beta c}{1-c^2} \lambda, \bar{\lambda} \right),
\end{equation}
thus yielding (2.19).
Letting $t \to \infty$, (equivalently $c \to 0$) in (4.8) proves formula (2.22) for the invariant measure, taking into account that $F_n^{(\beta)}(0, Y) = \text{vol}(U_n^{(\beta)}) = 1$.

\textbf{(ii) Proof of the transition probability (2.31) and the invariant measure for the $(\lambda_t, \mu_t)$ process.} The proof of (2.31) in Theorem 2.2 proceed along similar lines. First observe the identity
\begin{equation}
\tag{4.11}
\mathbb{P} \left[ B_t \in dB \mid (\text{spec}(B_0^{(n)}), \text{spec}(B_0^{(n-1)})) = (\bar{\lambda}, \bar{\mu}) \right]
\end{equation}
Next we compute the two probabilities in the integrand of the integral (4.11):
\begin{enumerate}
\item The first integral equals $p(t; \bar{B}, B)dB$, as in (4.4). Since the Haar measure $dB$ is the product measure over all the free parameters, one will express $dB$ as the product of Haar measure $dB^{(n-1)}$ on the $(n-1) \times (n-1)$ minor and the measure $\prod_{0 \leq i \leq \beta-1} dB_{in}^{(\ell)} dB_{nn}$ on the last row and column, remembering the expression (2.7)
for $N_{n,\beta}$, thus giving:

$$
\mathbb{P}[B_t \in dB | B_0 = \bar{B}] = \frac{Z_{n,\beta}^{-1}}{(1-c^2)^{N_{n,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \text{Tr}(B-c\bar{B})^2} dB
$$

= \frac{Z_{n,\beta}^{-1}}{(1-c^2)^{N_{n-1,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \text{Tr}(B^{(n-1)}-c\bar{B}^{(n-1)})^2} dB^{(n-1)}

\times \frac{Z_{n,\beta}^{-1} Z_{n-1,\beta}^{-1}}{(1-c^2)^{N_{n-1,\beta}}} e^{-\frac{\beta}{2(1-c^2)} \sum_{1 \leq i \leq n-1} (B_{in}^{(t)}-c\bar{B}_{in}^{(t)})^2 + \frac{1}{2}(B_{nn}^{(n)}-c\bar{B}_{nn})^2}

\times \prod_{i=1}^{n-1} \prod_{\ell=0}^{\beta-1} dB_{in}^{(t)} dB_{nn}^{(n)}.

As mentioned prior to (1.8), one can set in (4.12),

$$
\text{B}_0 \quad (4.13) \quad dB^{(n-1)} = Z_{n-1,\beta} C_{n-1,\beta}^{-1} |\Delta_{n-1}(\mu)|^{\bar{\beta}} dB^{(n-1)} \prod_{i=1}^{n-1} d\mu_i.
$$

In (2.27), it was shown that upon conjugation by an appropriate matrix $U^{(n-1)} \in \mathcal{U}_{n-1}^{(\beta)}$, the matrix $B$ could be transformed in the bordered matrix $B_{\text{bord}}$, as in (2.27) and (2.29), with $(r_1 u_1, \ldots, r_{n-1} u_{n-1})^{\top} = U^{(n-1)} v$ and $|u_i| = 1$. Using the same inner-product as in the formula just preceding (2.14), but for $n-1$-vectors, and using the associated norm $\| \|$, one finds, using the above,

$$
\text{B}_1 \quad (4.14) \quad \sum_{1 \leq i \leq n-1, 0 \leq \ell \leq \beta-1} (B_{in}^{(t)}-c\bar{B}_{in}^{(t)})^2

= \| v - c\bar{v} \|^2

= \| U^{(n-1)} (v - c\bar{v}) \|^2

= \| U^{(n-1)} v \|^2 + c^2 \| U^{(n-1)} \bar{v} \|^2 - 2c \text{Re}\langle U^{(n-1)} v, U^{(n-1)} \bar{v} \rangle

= \| U^{(n-1)} v \|^2 + c^2 \| U^{(n-1)} \bar{v} \|^2 - 2c \text{Re}\langle U^{(n-1)} v, (U^{(n-1)} (U^{(n-1)})^{-1} (U^{(n-1)} \bar{v}) \rangle

= \sum_{1}^{n-1} r_i^2 + c^2 \sum_{1}^{n-1} \bar{r}_i^2

- 2c \text{Re}\langle (r_1 u_1, \ldots, r_{n-1} u_{n-1})^{\top}, (U^{(n-1)} (U^{(n-1)})^{-1})(\bar{r}_1 \bar{u}_1, \ldots, \bar{r}_{n-1} \bar{u}_{n-1})^{\top} \rangle
Given that \( U^{(n-1)} \) is fixed and that \( \det(U^{(n-1)}) = 1 \), and since the expressions \( u_i \) in (2.27) have \( |u_i| = 1 \), the differential below can be written in terms of a product of differentials \( d(r_i u_i) \) along \( S^{\beta-1} \subset \mathbb{R}^\beta \), expressed in polar coordinates, thus yielding differentials involving the \( r_i \)'s and volume elements on the unit sphere \( S^{\beta-1} \):

\[
\prod_{i=1}^{n-1} \prod_{\ell=0}^{\beta-1} dB_{i\ell}^{(\ell)} = \prod_{i=1}^{n-1} dv_i = \prod_{i=1}^{n-1} d((U^{(n-1)})^{-1}(r_1 u_1, \ldots, r_{n-1} u_{n-1})^\top)_i
\]

\[
(4.15)
\]

Thus all together, setting (4.13), (4.14) and (4.15) in (4.12), we have shown that (4.16)

\[
\mathbb{P}[B_i^{(n)} \in dB | B_0^{(n)} = B] = \frac{C_{n-1, \beta}}{(1 - e^2)^{N_{n-1, \beta}}} e^{-\frac{\beta}{(1 - e^2)^2} \sum_{i=1}^{n-1} (\mu_i^2 + c^2 \bar{\mu}_i^2)} \\
\times |\Delta_{n-1}(\mu)|^{\beta} e^{\frac{\beta c}{1 - e^2} Tr(U^{(n-1)}(\bar{U}^{(n-1)})^{-1})^{-1} \mu U^{(n-1)}(\bar{U}^{(n-1)})^{-1} \bar{\mu} dU^{(n-1)} \prod_{i=1}^{n-1} d\mu_i} \\
\times \frac{Z_{n, \beta} Z_{n-1, \beta}}{(1 - e^2)^{N_{n, \beta} - N_{n-1, \beta}}} e^{-\frac{\beta}{(1 - e^2)^2} \left( \sum_{i=1}^{n-1} r_i^2 + \frac{1}{2} r_{1}^2 + c^2 \left( \sum_{i=1}^{n-1} r_i^2 + \frac{1}{2} r_{1}^2 \right) \right)} \\
\times e^{\frac{2\beta e}{1 - e^2} Re((r_1 u_1)^{n-1} U^{(n-1)}(\bar{U}^{(n-1)})^{-1} (\bar{r}_1 \bar{u}_1)^{n-1}) \frac{\beta c}{1 - e^2} Tr_r r_n \sum_{i=1}^{n-1} d\mu_i} \\
= \frac{Z_{n, \beta} Z_{n-1, \beta}}{(1 - e^2)^{N_{n, \beta} - C_{n-1, \beta}}} e^{-\frac{\beta}{(1 - e^2)^2} \sum_{i=1}^{n} (\lambda_i^2 + c^2 \bar{\lambda}_i^2)} e^{\frac{\beta c}{1 - e^2} r_n} |\Delta_{n}(\lambda)| |\Delta_{n-1}(\mu)| \frac{\beta}{\vec{n}^{\beta-1}} \\
\times G_{n-1}^{(\beta)} U^{(n-1)}(\bar{U}^{(n-1)})^{-1} \frac{\beta c}{1 - e^2} \mu, \bar{\mu}; \frac{\beta c}{1 - e^2} (r_1 u_1)^{n-1}, (\bar{r}_1 \bar{u}_1)^{n-1} \\
\times dU^{(n-1)} \prod_{i=1}^{n-1} d\Omega_i^{(\beta-1)}(u_i) \prod_{i=1}^{n} d\mu_i \prod_{i=1}^{n} d\lambda_i.
\]

In the last equality we have used identities (3.1) and (3.3) and the definition (2.15) of \( G_{n-1}^{(\beta)} \).

(ii) The second probability in (4.11) takes on the following value:

\[
(4.17)
\]

\[
\mathbb{P} \left[ B_0^{(n)} \in dB \left| \text{spec}(B_0^{(n)}), \text{spec}(B_0^{(n-1)}) = \left( \bar{\lambda}, \bar{\mu} \right) \right. \right] = d\bar{U}^{(n-1)} \frac{\prod_{i=1}^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i)}{(\text{vol}(S^{\beta-1}))^{n-1}}.
\]
Indeed, the probability (4.16), when \( t \to \infty \) (which amounts to letting \( c \to 0 \)), tends to the invariant measure for the Dyson Brownian motion; instead of the usual representation in the variables \((\lambda_i, U^{(n)})\), this gives the expression of the GUE-probability in the variables \((\lambda_i, \mu_j, u_k, U^{(n-1)})\):

\[
\mathbb{P}[B^{(n)} \in dB] = Z_{n, \beta}^{-1}e^{-\frac{\beta}{2} \text{Tr} B^2} dB \\
= \lim_{t \to \infty} \mathbb{P}[B^{(n)}_t \in dB|B^{(n)}_0 = \bar{B}] \\
= f_{n, \beta}(\lambda, \mu) dU^{(n-1)} \prod_{i=1}^{n-1} d\Omega_i^{(\beta-1)}(u_i) \prod_{i=1}^{n-1} d\mu_i \prod_{i=1}^{n-1} d\lambda_i
\]

with (using \( G_{n-1}^{(\beta)}(U; 0, \bar{\mu}; 0, (\bar{r}_i \bar{u}_i))^{n-1} = 1 \))

\[
f_{n, \beta}(\lambda, \mu) := \frac{Z_{n, \beta}^{-1}Z_{n-1, \beta}^{-1}}{C_{n-1, \beta}} e^{-\frac{\beta}{2} \sum_{i=1}^{n} \lambda_i^2} |\Delta_n(\lambda)\Delta_{n-1}(\mu)| |\Delta_{n-1}(\lambda, \mu)|^{\frac{\beta}{2} - 1}.
\]

This also shows that \( H_n^{(\beta)}(\lambda, \mu) \simeq U_{n-1}^{(\beta)} \times (S^{(\beta-1)})^{n-1} \). Using (4.18), one checks that the conditional probability equals

\[
P \left[ B_0^{(n)} \in d\bar{B} \mid (\text{spec}(B_0^{(n)}), \text{spec}(B_0^{(n-1)})) = (\bar{\lambda}, \bar{\mu}) \right] \\
= \frac{P \left[ B_0^{(n)} \in d\bar{B}, (\text{spec}(B_0^{(n)}), \text{spec}(B_0^{(n-1)})) \in (d\bar{\lambda}, d\bar{\mu}) \right]}{P \left[ (\text{spec}(B_0^{(n)}), \text{spec}(B_0^{(n-1)})) \in (d\bar{\lambda}, d\bar{\mu}) \right]} \\
= \frac{f_{n, \beta}(\bar{\lambda}, \bar{\mu}) d\bar{U}^{(n-1)} \prod_{i=1}^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i) \prod_{i=1}^{n-1} d\bar{\mu}_i \prod_{i=1}^{n} d\bar{\lambda}_i}{f_{n, \beta}(\bar{\lambda}, \bar{\mu}) \prod_{i=1}^{n} d\bar{\mu}_i \prod_{i=1}^{n} d\bar{\lambda}_i \int_{U_n^{(\beta)} \times (S^{(\beta-1)})^{n-1}} \bar{U}^{(n-1)} \prod_{i=1}^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i)} \\
= \frac{d\bar{U}^{(n-1)} \prod_{i=1}^{n-1} d\Omega_i^{(\beta-1)}(\bar{u}_i)}{(\text{vol}(S^{(\beta-1)})^{n-1})}. 
\]
confirming expression (4.17). Setting (4.16) and (4.17) in (4.11) and using the integral (2.14) and the identification just after (4.19), one computes:

\[ p_{\lambda, \mu}(t; (\bar{\lambda}, \bar{\mu}), (\lambda, \mu))d\lambda d\mu \]

\[ = \int_{B_0 \in \mathcal{H}_0^{(\beta)}(\lambda, \mu)} P \left[ B_0^{(n)} \in dB \mid B_0^{(0)} = B \right] \]

\[ = \int_{\bar{B} \in \mathcal{H}_0^{(\beta)}(\bar{\lambda}, \bar{\mu})} P \left[ B_0^{(n)} \in dB \right] \left( \text{spec}(B_0^{(n)}), \text{spec}(B_0^{(n-1)}) = (\bar{\lambda}, \bar{\mu}) \right) \]

\[ = \int \left( \frac{Z_{n, \beta} Z_{n-1, \beta} C_{n-1, \beta}}{(1-c^2)^{N_{n, \beta}}} \right)^2 e^{-\frac{\bar{\alpha}}{2(1-c^2)} \sum_{i=1}^{n-1} (\bar{\lambda}_i^2 + c^2 \bar{\lambda}_i^2)} e^{\frac{\bar{\alpha}}{1-c^2} r_n r_{n-1}} \times G_{n-1}^{(\beta)} \int \left( \frac{U^{(n-1)}(U^{(n-1)})^{-1}}{1-c^2} \mu; \frac{\beta c}{1-c^2} (r_i u_i)^{n-1}, (\bar{r}_i \bar{u}_i)^{n-1} \right) dU^{(n-1)} dU^{(n-1)} \times |\Delta_n(\lambda) \Delta_{n-1}(\mu) | \Delta_{n-1}(\lambda, \mu) \right]^{\frac{1}{2}-1} d\lambda d\mu \]

\[ = \frac{\hat{Z}_{n, \beta}^{-1}}{(1-c^2)^{N_{n, \beta}}} e^{-\frac{\bar{\alpha}}{2(1-c^2)} \sum_{i=1}^{n-1} (\bar{\lambda}_i^2 + c^2 \bar{\lambda}_i^2)} e^{\frac{\bar{\alpha}}{1-c^2} r_n r_{n-1}} \times \int \left( \frac{\lambda_{n, \beta}}{1-c^2} \mu; \frac{\beta c}{1-c^2} (r_i u_i)^{n-1}, (\bar{r}_i \bar{u}_i)^{n-1} \right) \prod_{i=1}^{n-1} d\Omega_i^{(\beta-1)}(u_i) d\Omega_i^{(\beta-1)}(\bar{u}_i) \]

were we used the translation invariance of \( dU^{(n-1)} \) and \( \text{vol}(U^{(n-1)}) = 1 \); also one checks the value of the constant \( \hat{Z}_{n, \beta}^{-1} \) to be the one given in footnote 2. This establishes formula (2.31) for the transition probability of the \((\lambda_t, \mu_t)\)-process.

The statements concerning the invariant measures, (2.22) and (2.34) follow immediately from (2.19), (2.41), (2.31), by letting \( t \to \infty \) in the transition probability. This concludes the proof of the formulae for the transition probabilities (2.10), (2.19) and invariant measure (2.34), appearing in Theorems 2.1 and 2.2. \( \square \)

**Remark 4.1.** The diffusion equation (2.32), which will be established in section 5, can also be used to confirm the form of the invariant measure, at least for \( \beta = 2 \). On general grounds, the density of the invariant measure, namely

\[ I_{\lambda, \mu}(\lambda, \mu) := C e^{-\frac{\beta}{2} \sum \lambda_i^2} |\Delta_n(\lambda) \Delta_{n-1}(\mu) | \Delta_{n-1}(\lambda, \mu) \right]^{\frac{1}{2}-1}, \]

is a null vector of the forward equation, i.e.

\[ A^\top I_{\lambda, \mu}(\lambda, \mu) = \left( A_{\lambda}^\top + A_{\mu}^\top + A_{\lambda, \mu}^\top \right) I_{\lambda, \mu}(\lambda, \mu) = 0, \]

with \( A \) defined in (2.32). For \( \beta = 2 \), more is true; namely

\[ A_{\lambda}^\top(\lambda) I_{\lambda, \mu} = \frac{n(n-1)}{2} I_{\lambda, \mu}, \quad A_{\mu}^\top(\mu) I_{\lambda, \mu} = \frac{n(n-1)}{2} I_{\lambda, \mu}. \]
Once this is shown, it follows that

\[(4.25) \quad \mathcal{A}_{\lambda \mu}^T I_{\lambda \mu}(\lambda, \mu) = -n(n-1)I_{\lambda \mu}(\lambda, \mu).\]

So it suffices to prove \[(4.24)\]. First observe that \(\Delta_n(\lambda)\) and \(\Delta_{n-1}(\mu)\) are harmonic functions, i.e.

\[
\sum_{i=1}^{n} \left( \frac{\partial}{\partial \lambda_i} \right)^2 \Delta_n(\lambda) = 0, \quad \sum_{i=1}^{n-1} \left( \frac{\partial}{\partial \mu_i} \right)^2 \Delta_{n-1}(\mu) = 0,
\]

and also homogeneous functions so that acted upon by the Euler operators,

\[
\sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial \lambda_i} \Delta_n(\lambda) = \frac{n(n-1)}{2} \Delta_n(\lambda), \quad \sum_{i=1}^{n-1} \mu_i \frac{\partial}{\partial \mu_i} \Delta_{n-1}(\mu) = \frac{(n-1)(n-2)}{2} \Delta_{n-1}(\mu).
\]

Now compute from \[(2.20)\] and \[(4.22)\] that (remember \(\Phi_n(\lambda) := e^{-\frac{1}{2} \sum_i^\lambda \lambda_i^2} |\Delta_n(\lambda)|\))

\[
\mathcal{A}_{\lambda \mu}^T I_{\lambda \mu} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} (\Phi_n(\lambda))^2 \frac{\partial}{\partial \lambda_i} \Delta_n(\lambda) \Delta_{n-1}(\mu) e^{-\sum_i^\lambda \lambda_i^2} \frac{\partial}{\partial \lambda_i} \Delta_n(\lambda)
\]

\[
= -\frac{\Delta_{n-1}(\mu)}{2} \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} e^{-\sum_i^\lambda \lambda_i^2} \frac{\partial}{\partial \lambda_i} \Delta_n(\lambda)
\]

\[
= -\frac{1}{2} e^{-\sum_i^\lambda \lambda_i^2} \Delta_{n-1}(\mu) \sum_{i=1}^{n} \left( \frac{\partial^2}{\partial \lambda_i^2} - 2\lambda_i \frac{\partial}{\partial \lambda_i} \right) \Delta_n(\lambda)
\]

\[
= \frac{n(n-1)}{2} I_{\lambda \mu}
\]
and also that

\[
\mathcal{A}^\top_\mu I_{\lambda\mu} = \frac{1}{2} \sum_{i=1}^{n-1} \frac{\partial}{\partial \mu_i} (\Phi_{n-1}(\mu))^2 \frac{\partial}{\partial \mu_i} \frac{\Delta_n(\lambda) \Delta_{n-1}(\mu) e^{-\sum_i^\mu \lambda_i^2}}{(\Phi_{n-1}(\mu))^2}
\]

\[
= -\frac{1}{2} \Delta_n(\lambda) e^{-\sum_i^\mu \lambda_i^2} \left\{ \sum_{i=1}^{n-1} \frac{\partial}{\partial \mu_i} (\Phi_{n-1}(\mu)) e^{-\sum_i^\mu \lambda_i^2} \right\} \frac{\partial}{\partial \mu_i} \left( \Delta_{n-1}(\mu) e^{-\sum_i^{n-1} \mu_i^2} \right)
\]

\[
= -\frac{1}{2} \Delta_n(\lambda) e^{-\sum_i^\mu \lambda_i^2} \left( \sum_{i=1}^{n-1} \left( \frac{\partial^2}{\partial \mu_i^2} - 2 \mu_i \frac{\partial}{\partial \mu_i} \right) \Delta_{n-1}(\mu) - 2(n-1) \Delta_{n-1}(\mu) \right)
\]

\[
= \frac{1}{2} \Delta_n(\lambda) e^{-\sum_i^\mu \lambda_i^2} \left( (n-1)(n-2) + 2(n-1) \right) \Delta_{n-1}(\mu)
\]

\[
= \frac{n(n-1)}{2} I_{\lambda\mu}.
\]

This ends the proof of identities (4.23).

5. **Itô’s Lemma and Dyson’s Theorem**

To fix notation we repeat some well known facts from stochastic calculus in a way that will be useful later. Given a diffusion \( X_t \in \mathbb{R}^n \), given by the SDE

\[
dX_t = \sigma(X_t)dB_t + a(X_t)dt,
\]

where \( dB_t \) is a vector of independent standard Brownian motions, where \( x, a(x) \in \mathbb{R}^n \) and \( \sigma(x) \) an \( n \times n \) matrix. Then the generator of this diffusion is given by

\[
\mathcal{A} = \frac{1}{2} \sum_{i,j} (\sigma \sigma^\top)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i a_i(x) \frac{\partial}{\partial x_i},
\]

and, by straightforward verification,

\[
(\sigma \sigma^\top)_{ij}(x) = \mathcal{A}(x_i x_j) - x_i \mathcal{A}(x_j) - x_j \mathcal{A}(x_i) = \left( \frac{dX_i dX_j}{dt} \right)(x)
\]

\[a_i(x) = \mathcal{A}x_i.
\]

The transition density \( p(t, \bar{x}, x) \) is a solution of the forward equation (in \( x \))

\[
\frac{\partial p}{\partial t} = \mathcal{A}^\top p.
\]

Moreover for a function \( g : \mathbb{R}^n \to \mathbb{R}^p \) with \( g \in \mathcal{C}^2 \), the SDE for \( Y_t = g(X_t) \) has the form

\[
dY_k = \sum_i \frac{\partial g_k}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g}{\partial x_i x_j} dX_i dX_j = \sum_j \left( \sum_i \frac{\partial g_k}{\partial x_i} \sigma_{ij} \right) dB_j + h_k dt,
\]

6The subscript \( t \) in \( X_t \) and \( B_t \) will often be omitted.
for \( k = 1, \ldots, p \) and for some function \( h_k \); i.e., the local martingale part only depends on first derivatives of \( g \). This follows from the standard multiplication rules of stochastic calculus (\( dt dt = 0, dt db = 0 \) and \( db db = \delta_{ij} dt \)):

\[
dX_i dX_j = (a_i dt + \sum_{\ell=1}^n \sigma_{i\ell} db_\ell)(a_j dt + \sum_{k=1}^n \sigma_{jk} db_k)
\]

\[
= (\sum_{\ell=1}^n \sigma_{i\ell} db_\ell)(\sum_{k=1}^n \sigma_{jk} db_k) = (\sigma \sigma^\top)_{ij} dt.
\]

More details can be found in any book on stochastic calculus, for example [McK05] or [Øks03]. As a warm-up exercise, we first prove Dyson’s original result, namely the formulae for the SDE and for the generator of Theorem 2.1, including some consequences.

**Proof of (2.18) and (2.20) in Theorem 2.1.** The Dyson process is invariant under conjugation by \( U \in U_n(\beta) \); to be precise from (2.9),

\[
p(t; \overline{BU}^{-1}, UB^{-1}) = p(t; \overline{B}, B).
\]

Therefore, we are free, at any fixed choice of \( t \), to reset

\[
B(t) \mapsto UB(t)U^{-1}, \text{ for any } U \in U_n(\beta).
\]

At any given time \( t \), diagonalize the matrix \( B \) to yield \( \text{diag}(\lambda_1, \ldots, \lambda_n) \) and consider the perturbation

\[
diag(\lambda_1, \ldots, \lambda_n) + [dB_{ij}],
\]

where one defines the \( n \times n \) matrix, for \( 1 \leq i < j \leq n \),

\[
\text{perturb} [dB_{ij}] := \begin{pmatrix}
\cdots & 0 & \cdots & dB_{ij}^{(0)} + \sum_{\ell=1}^{\beta-1} dB_{ij}^{(\ell)} e_\ell \\
\cdots & \ddots & \ddots & \ddots \\
\cdots & \cdots & \cdots & 0 \\
0 & \cdots & dB_{ij}^{(0)} - \sum_{\ell=1}^{\beta-1} dB_{ij}^{(\ell)} e_\ell & \cdots \\
\end{pmatrix}
\]

and, for \( i = 1, \ldots, n \),

\[
[dB_{ii}] := \text{diag}(0, \ldots, dB_{ii}, \ldots, 0),
\]

with, by (2.8),

\[
(5.9) \quad dB_{ij}^{(\ell)} dB_{ij}^{(\ell')} = \delta_{ii'} \delta_{jj'} \delta_{\ell \ell'} \frac{dt}{\beta} \quad dB_{ii} dB_{jj} = 2 \delta_{ij} \frac{dt}{\beta} \quad dB_{ij}^{(\ell)} dB_{kk} = 0.
\]

Remember by Ito’s formula (5.5), one only needs to keep track of at most second order changes of the arguments.
Thus for non-diagonal perturbations \((i \neq j)\), one checks\(^7\)

\[
0 = \det \left( \text{diag}(\lambda_1, \ldots, \lambda_n) + [dB_{ij} - \lambda I] \right)_{\lambda \to \lambda_0 + d\lambda_0}
= \prod_{\substack{\ell = 1 \\ \ell \neq i, j}}^n (\lambda_\ell - \lambda) \left( (\lambda - \lambda_i)(\lambda - \lambda_j) - \sum_{\ell=0}^{\beta-1} (dB_{ij}^{(\ell)})^2 \right)
\tag{3.10}
\]

\[
\text{(non-zero function)} \times d\lambda_\alpha, \quad \text{for } \alpha \neq i, j,
\]

\[
\text{(non-zero function)} \times ((\lambda_i - \lambda_j) d\lambda_i - \sum_{\ell=0}^{\beta-1} (dB_{ij}^{(\ell)})^2) \quad \text{for } \alpha = i,
\]

\[
\text{(non-zero function)} \times ((\lambda_j - \lambda_i) d\lambda_j - \sum_{\ell=0}^{\beta-1} (dB_{ij}^{(\ell)})^2) \quad \text{for } \alpha = j,
\]

showing that an off-diagonal perturbation of the diagonal matrix \(B(t) = \text{diag}(\lambda_1(t), \ldots, \lambda_n(t))\) yields

\[
d\lambda_i = \sum_{\ell=0}^{\beta-1} (dB_{ij}^{(\ell)})^2, \quad d\lambda_j = \frac{\sum_{\ell=0}^{\beta-1} (dB_{ij}^{(\ell)})^2}{\lambda_j - \lambda_i}, \quad \text{and } d\lambda_\alpha = 0 \quad \text{for } \alpha \neq i, j.
\]

For diagonal perturbations \((i = j)\), one finds

\[
\det \left( \text{diag}(\lambda_1, \ldots, \lambda_n) + [dB_{ii} - \lambda I] \right)_{\lambda \to \lambda_0 + d\lambda_0} = \prod_{\ell=1}^n (\lambda_\ell - \lambda) \left( \lambda_i + dB_{ii} - \lambda \right)
\]

\[
\text{(non-zero function)} \times d\lambda_\alpha, \quad \text{for } \alpha \neq i
\]

\[
\text{(non-zero function)} \times (dB_{aa} - d\lambda_\alpha), \quad \text{for } \alpha = i.
\]

and thus

\[
d\lambda_\alpha = 0 \quad \text{for } \alpha \neq i \quad \text{and } d\lambda_\alpha = dB_{aa} \quad \text{for } \alpha = i.
\]

Then summing up all the perturbations, one finds the SDE \((2.18)\) announced in Theorem 2.1, using \(dB_{ii} = -B_{ii} dt + \sqrt{\frac{\beta}{2}} dB_{ii} = -\lambda_i dt + \sqrt{\frac{\beta}{2}} dB_{ii}\) and formula (5.9),

\[
d\lambda_i = \left( dB_{ii} + \sum_{j \neq i} \frac{\sum_{\ell=0}^{\beta-1} (dB_{ij}^{(\ell)})^2}{\lambda_i - \lambda_j} \right)
\]

\[
= \left( -\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) dt + \sqrt{\frac{\beta}{2}} dB_{ii}, \quad \text{for } i = 1, \ldots, n.
\]

Then translating the SDE into the generator of the diffusion on \((\lambda_1, \ldots, \lambda_n)\), one finds, by (5.3), that

\[
A_i = \sum_{\ell=1}^n \left( \frac{1}{\beta} \frac{\partial^2}{\partial \lambda_i^2} + \left( -\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \frac{\partial}{\partial \lambda_i} \right),
\tag{5.11}
\]

\(^7\)Remember, for \(\beta = 4\), the determinant is defined in (2.4).
and thus

\[ A_\lambda^\top = \sum_1^n \left( \frac{1}{\beta} \frac{\partial^2}{\partial \lambda_i^2} - \frac{\partial}{\partial \lambda_i} \left( -\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \right) \]

\[ = \frac{1}{\beta} \sum_{i=1}^n \frac{\partial}{\partial \lambda_i} (\Phi_n(\lambda))^\beta \frac{\partial}{\partial \lambda_i} \left( \frac{1}{\Phi_n(\lambda)} \right)^\beta, \]

with \( \Phi_n(\lambda) \) as in (2.22), confirming formula (2.20) in Theorem 2.1. Finally, \( A_{\text{Dys}} \lambda_i = A_\lambda \lambda_i \), mentioned in (2.35), follows from the fact that the generator \( A_{\text{Dys}} \) restricted to the functions \((\lambda_1, \ldots, \lambda_n)\) equals \( A_\lambda \), as a consequence of (5.1) to (5.4); of course, this holds for the spectrum of every principal minor of the matrix \( B \). □

6. SDE for the Dyson Process on the Spectra of Two Consecutive Minors

In this section we prove the formulas (2.30) for the \( \lambda \)- and \( \mu \)-SDE’s, together with the generator (2.32).

**Proof of SDE (2.30) and generator (2.32) in Theorem 2.2.** Using the same idea as in the proof of (2.18) and (2.20) in the Section 5, we choose, at time \( t \), to conjugate the matrix \( B \) so as to have the form \( B_{\text{bord}} \) of (2.27) and let the matrix \( B_{\text{bord}} \) evolve according to the Dyson process. We will consider only the first order effects on the \( \lambda \)’s and ignore second order effects.

At first, we need to compute the (first order) variation of the \( \lambda_\alpha \)’s as a function of the (first order) variation of the entries:

**Case 1:** Consider the perturbation of \( B_{\text{bord}} \), using the notation (5.8) for \([dB_{ij}]\), namely

\[ B_{\text{bord}} + [dB_{ij}], \text{ for } 1 \leq i < j \leq n - 1. \] 

(6.1)

Up to first order, one must compute the effect of the perturbation on each of the eigenvalues \( \lambda_\alpha \), by explicitly computing the characteristic polynomial of the bordered matrix (2.27) with the extra non-diagonal perturbation; then, by neglecting
the second order terms in $dB$, one finds

$$0 = \det(B_{\text{bord}} + [dB_{ij}] - \lambda I) \bigg|_{\lambda \to \lambda_\alpha + d\lambda_\alpha}$$

(6.2)

$$= \prod_{\ell=1}^{n-1} (\mu_\ell - \lambda)$$

(6.3)

$$\left( \sum_{\ell=1}^{n-1} \frac{r_\ell^2}{\lambda - \mu_\ell} + r_n - \lambda + \frac{2r_ir_j}{(\lambda - \mu_i)(\lambda - \mu_j)} \sum_{\ell=0}^{\beta-1} dB_{ij}^{(\ell)} (u_i u_j^*)_\ell \right) \bigg|_{\lambda \to \lambda_\alpha + d\lambda_\alpha}.$$  

(6.4)

Setting $\lambda \mapsto \lambda_\alpha + d\lambda_\alpha$ in this expression, shows that the product $\prod_{\ell=1}^{n-1} (\mu_\ell - \lambda_\alpha - d\lambda_\alpha)$ is of the form $(\text{non-zero-function}) + (\text{function}) \times d\lambda_\alpha$, whereas the second part gives, by Taylor expanding in $\lambda_\alpha$, keeping in the expression first order terms only, evaluated by (3.2), and noticing that the $0$th-order term vanishes (again using (3.2)), we find

$$- \frac{P'_n(\lambda_\alpha)}{P_{n-1}(\lambda_\alpha)} d\lambda_\alpha + \frac{2r_ir_j}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)} \sum_{\ell=0}^{\beta-1} dB_{ij}^{(\ell)} (u_i u_j^*)_\ell = 0.$$  

(6.5)

Finally, adding up the first order contributions from all the perturbations $dB_{ij}^{(0)} + \sum_{1 \leq i < j \leq n-1}^{\beta-1} dB_{ij}^{(\ell)}$, yields

$$d\lambda_\alpha = \frac{P_{n-1}(\lambda_\alpha)}{P'_n(\lambda_\alpha)} \sum_{1 \leq i < j \leq n-1} \frac{2r_ir_j}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)} \sum_{\ell=0}^{\beta-1} dB_{ij}^{(\ell)} (u_i u_j^*)_\ell.$$  

(3.19)

Case 2: For the perturbation $[dB_{ii}]$, with $i = 1, \ldots, n-1$, again neglecting the second order terms,

$$0 = \det(B_{\text{bord}} + [dB_{ii}] - \lambda I) \bigg|_{\lambda \to \lambda_\alpha + d\lambda_\alpha}$$

$$= \prod_{\ell=1}^{n-1} (\mu_\ell + \delta_{\ell i} dB_{ii} - \lambda_\alpha - d\lambda_\alpha)$$

$$\left( \sum_{\ell=1}^{n-1} \frac{r_\ell^2}{\lambda_\alpha + d\lambda_\alpha - \mu_\ell - \delta_{\ell i} dB_{ii}} + (r_n + \delta_{n i} dB_{ii}) - \lambda_\alpha - d\lambda_\alpha \right)$$

(6.6)

8Notice that $2 \text{Re} dB_{ij}^* u_i u_j^* = DB_{ij}^* u_i u_j^* + (DB_{ij}^* u_i u_j^*)_\ell = 2 \sum_{\ell=0}^{\beta-1} dB_{ij}^{(\ell)}(u_i u_j^*)_\ell$, using $(ab)^* = b^* a^*$. Remember, for $\beta = 4$, quaternion multiplication does not commute and for the determinant of a matrix, use formula (2.5).
Upon expanding this expression as a function of \( \lambda, \mu \) up to first order, noticing as before that the first part does not matter, and using again (3.2), this leads to

\[
- \frac{P'_n(\lambda)}{P_{n-1}(\lambda)} d\lambda_a + \frac{r_i^2}{(\lambda_a - \mu_i)^2} dB_{ii} = 0 \quad \text{for } i = 1, \ldots, n - 1, \\
- \frac{P'_n(\lambda)}{P_{n-1}(\lambda)} d\lambda_a + dB_{nn} = 0 \quad \text{for } i = n,
\]

and thus summing up all the contributions coming from the \( dB_{ii} \) for \( i = 1, \ldots, n \), one finds

\[
(6.8) \quad d\lambda_a = \frac{P_{n-1}(\lambda)}{P'_n(\lambda)} \left( \sum_{i=1}^{n-1} \frac{r_i^2}{(\lambda_a - \mu_i)^2} dB_{ii} + dB_{nn} \right)
\]

Case 3: For the perturbation \([dB_{in}]\), \( i = 1, \ldots, n - 1 \),

\[
0 = \det(B_{\text{bord}} + [dB_{in}] - \lambda I) \bigg|_{\lambda \to \lambda_a + d\lambda_a}
= \prod_{\ell=1}^{n-1} (\mu_\ell - \lambda)
\left( \sum_{k=1}^{n-1} \frac{r_k^2 + r_i(u_i dB_{in}^* + u_i^* dB_{in}) \delta_{ik} + r_n - \lambda}{\lambda - \mu_k} \right) \bigg|_{\lambda \to \lambda_a + d\lambda_a}.
\]

Then, using \( u_i dB_{in}^* + dB_{in} u_i^* = 2 \sum_{\ell=0}^{\beta-1} u_i^{(\ell)} dB_{in}^{(\ell)} \), using formula (3.2) and finally summing up over all perturbations of the last row and column (\( 1 \leq i \leq n - 1 \)) yields

\[
(6.9) \quad d\lambda_a = \frac{P_{n-1}(\lambda)}{P'_n(\lambda)} \sum_{i=1}^{n-1} 2r_i \sum_{\ell=0}^{\beta-1} u_i^{(\ell)} dB_{in}^{(\ell)} \frac{\lambda - \mu_i}{\lambda_a - \mu_i}.
\]

Then summing up the three contributions (6.5), (6.8) and (6.9) gives us the total first order contribution to \( d\lambda_a \):

\[
d\lambda_a = \frac{P_{n-1}(\lambda)}{P'_n(\lambda)} \left\{ \sum_{1 \leq i < j \leq n-1} \frac{2r_i r_j}{(\lambda_a - \mu_i)(\lambda_a - \mu_j)} \sum_{\ell=0}^{\beta-1} dB_{ij}^{(\ell)} (u_i u_j^*)^{(\ell)} \right\} + \sum_{i=1}^{n-1} \frac{r_i^2}{(\lambda_a - \mu_i)^2} dB_{ii} + dB_{nn} + \sum_{i=1}^{n-1} 2r_i \sum_{\ell=0}^{\beta-1} u_i^{(\ell)} dB_{in}^{(\ell)} \frac{\lambda - \mu_i}{\lambda_a - \mu_i} \right\}.
\]
We now set the SDE’s (2.28) for the $dB_{ii}$, $dB_{ij}^{(\ell)}$ into the equation obtained above, thus yielding, by (3.5),

\begin{equation}
(6.10) \quad d\lambda_\alpha = F_\alpha^{(n)}(\lambda) \, dt + \sqrt{2} \frac{P_{n-1}(\lambda_\alpha)}{\beta P_n(\lambda_\alpha)} \left\{ \sum_{1 \leq i < j \leq n-1} \frac{\sqrt{2} \, r_i \, r_j}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)} \sum_{\ell=0}^{\beta-1} (u_i u_j^*)^{(\ell)} dB_{ij}^{(\ell)} \right. \\
+ \sum_{i=1}^{n-1} \left( \frac{r_i}{\lambda_\alpha - \mu_i} \right)^2 dB_{ii} + dB_{nn} \\
+ \sqrt{2} \sum_{i=1}^{n-1} \frac{r_i}{\lambda_\alpha - \mu_i} \sum_{\ell=0}^{\beta-1} u_i^{(\ell)} dB_{ii}^{(\ell)}
\end{equation}

for some function $F_\alpha^{(n)}(\lambda)$ to be determined later. Notice that in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the norm $|v|$ satisfies $|vw| = |v||w|$ and $|v| = |v^*|$. Therefore, when $|u_i| = 1$, we also have $|u_i u_j^*| = 1$, implying that

\[ \tilde{d}b_{ii} := \sum_{\ell=0}^{\beta-1} u_i^{(\ell)} dB_{ii}^{(\ell)} \quad \text{and} \quad \tilde{d}b_{ij} := \sum_{\ell=0}^{\beta-1} (u_i u_j^*)^{(\ell)} dB_{ij}^{(\ell)} \]

are both standard Brownian motions on the sphere $S^{\beta-1}$; since they are different linear combinations, they are independent standard Brownian motions, and independent of $dB_{ii}$, $1 \leq i \leq n$. This is precisely formula (2.30) of Theorem 2.2 namely

\begin{equation}
(6.11) \quad d\lambda_\alpha = F_\alpha^{(n)}(\lambda) dt + \sqrt{2} \frac{P_{n-1}(\lambda_\alpha)}{\beta P_n(\lambda_\alpha)} \times \left( \sum_{1 \leq i < j \leq n-1} \frac{\sqrt{2} \, r_i \, r_j \, dB_{ij}}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)} + \sum_{i=1}^{n-1} \frac{r_i^2 dB_{ii}}{(\lambda_\alpha - \mu_i)^2} + \sum_{i=1}^{n-1} \sqrt{2} \, r_i \, dB_{nn} + dB_{nn} \right).
\end{equation}

The SDE for the Dyson process induced on the $(n-1) \times (n-1)$ upper-left minor is given by the first formula of Theorem 2.1 with $n \rightarrow n - 1$ and $\lambda \rightarrow \mu$, yielding the formula in (2.30). Therefore the product of the SDEs in (2.30), together with identity (2.25) yields

\begin{equation}
(6.12) \quad \frac{d\lambda_\alpha d\mu_j}{dt} = \frac{2}{\beta} \frac{P_{n-1}(\lambda_\alpha)}{P_n(\lambda_\alpha)} \left( \frac{r_j}{\lambda_\alpha - \mu_j} \right)^2 = -\frac{2}{\beta} \frac{P_{n-1}(\lambda_\alpha)P_n(\mu_j)}{(\lambda_\alpha - \mu_j)^2} \frac{P_{n-1}(\lambda_\alpha)P_n(\mu_j)}{P_n(\lambda_\alpha)P_{n-1}(\mu_j)}.
\end{equation}
Moreover,
\[
\frac{d \lambda_\alpha d \lambda_\gamma}{dt} = A_{\mathrm{Dys}}(\lambda_\alpha \lambda_\gamma) - \lambda_\alpha A_{\mathrm{Dys}}(\lambda_\gamma) - \lambda_\gamma A_{\mathrm{Dys}}(\lambda_\alpha) \\
= A_\lambda(\lambda_\alpha \lambda_\gamma) - \lambda_\alpha A_\lambda(\lambda_\gamma) - \lambda_\gamma A_\lambda(\lambda_\alpha) \\
= 2(\text{coefficient of } \frac{\partial^2}{\partial \lambda_\alpha \lambda_\beta} \text{ in } A_\lambda) \\
= \frac{2}{\beta} \delta_\alpha \gamma
\]

and similarly,

\[
\text{diag } (6.13) \quad \frac{d \mu_i d \mu_j}{dt} = \frac{2}{\beta} \delta_{ij}.
\]

These identities can also be computed from the expressions (6.11) of \(d \lambda_\alpha\) in terms of the \(\lambda_i, \mu_j\), as done in the remark below. From Ito’s formula (5.5) it then follows that

\[
(d \lambda_1, \ldots, d \lambda_n, d \mu_1, \ldots, d \mu_{n-1}) \\
= (A_{\mathrm{Dys}} \lambda_1, \ldots, A_{\mathrm{Dys}} \lambda_n, A_{\mathrm{Dys}} \mu_1, \ldots, A_{\mathrm{Dys}} \mu_{n-1}) dt + \sigma(\lambda, \mu)db_t,
\]

where, according to (2.35) and (2.21),

\[
A_{\mathrm{Dys}}(\lambda_i) = A_\lambda(\lambda_i) = -\lambda_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j},
\]

\[
A_{\mathrm{Dys}}(\mu_i) = A_\mu(\mu_i) = -\mu_i + \sum_{j \neq i} \frac{1}{\mu_i - \mu_j},
\]

establishing the form of \(F^{(n)}_\alpha(\lambda)\) in (6.11), thus yielding (2.30). Identities (2.30), (6.12) and (6.13), together with Ito’s formula (5.5), then establish the formula (2.33) for \(A_{\lambda}\). \(\square\)

---

9As an alternative way, (2.35) and (6.12) suffice to establish (2.33), with (6.12) needed to establish the coupling \(A_{\lambda \mu}\).
Remark 6.1. Note that the identities (6.13) can be computed as well from the SDE (6.11), using residue calculations:

\[
\frac{d\lambda}{\alpha} \frac{d\lambda}{\gamma} \frac{dt}{\beta} = 2 \left( \frac{P_{n-1}(\lambda_\alpha)}{P_n(\lambda_\alpha)} \right) \left( \frac{P_{n-1}(\lambda_\gamma)}{P_n(\lambda_\gamma)} \right) \left( \sum_{1 \leq i < j \leq n-1} \frac{2 r_i^2 r_j^2}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)(\lambda_\gamma - \mu_i)(\lambda_\gamma - \mu_j)} \right) + \sum_{i=1}^{n-1} \frac{r_i^4}{(\lambda_\alpha - \mu_i)^2(\lambda_\gamma - \mu_i)^2} + \sum_{i=1}^{n-1} \frac{2 r_i^2}{(\lambda_\alpha - \mu_i)(\lambda_\gamma - \mu_i)} + 1 \right) = \frac{2}{\beta} \delta_{\alpha \gamma}
\]

We now turn to the proof of Corollaries 2.3 and 2.4:

Proof of Corollary 2.3. Note, using logarithmic derivatives, that

\[
(\text{Invariant measure (2.34)}) \beta A^\top (\text{Invariant measure (2.34)})^{-1}
\]

is a quadratic polynomial in \( \beta \), which by Theorem 2.2 vanishes for \( \beta = 1, 2, 4 \) and thus it vanishes identically in \( \beta \). That the process restricted to \( \lambda \) or \( \mu \) is the standard Dyson process follows from the form of the generator \( A^\top \).

\[ \square \]

Proof of Corollary 2.4. In order to study the stochastic behavior of \( \mu_\alpha - \lambda_\alpha \) and \( \lambda_{\alpha+1} - \mu_\alpha \) when \( \mu_\alpha \) gets close to \( \lambda_\alpha \) or \( \lambda_{\alpha+1} \), one rewrites the Brownian part of \( d(\lambda_\alpha - \mu_\alpha) \) as follows:

\[
\sqrt{\frac{\beta}{2}} \text{Brownian part of } d(\lambda_\alpha - \mu_\alpha)
\]

\[
= \frac{P_{n-1}(\lambda_\alpha)}{P_n(\lambda_\alpha)} \left( \sum_{1 \leq i < j \leq n-1} \sqrt{2} r_i r_j \frac{r_i^2}{(\lambda_\alpha - \mu_i)(\lambda_\alpha - \mu_j)} + \sum_{i=1}^{n-1} \frac{r_i^2}{(\lambda_\alpha - \mu_i)^2} + \sum_{i=1}^{n-1} \frac{2 r_i^2}{(\lambda_\alpha - \mu_i)} + \frac{1}{\beta} \right) \text{db}_{\alpha \alpha}.
\]

At first notice that for \( \mu_\alpha \simeq \lambda_\alpha \), one has, using the expression (2.28) for \( r_k^2 \),

\[
\frac{P_{n-1}(\lambda_\alpha)}{P_n(\lambda_\alpha)} = \mathcal{O}(\mu_\alpha - \lambda_\alpha), \quad r_\alpha = \mathcal{O}(\sqrt{\mu_\alpha - \lambda_\alpha}) \quad \text{and} \quad r_i = \mathcal{O}(1)\quad \text{for } i \neq \alpha,
\]

from which one deduces that the first line on the right hand side of (6.16) has order \( \mathcal{O}(\sqrt{\mu_\alpha - \lambda_\alpha}) \). Using again (2.28), the second line of (6.16) multiplied with
$P_n'(\lambda_\alpha)P_n'(\mu_\alpha)$ reads:

$$
P_n'(\lambda_\alpha)P_n'(\mu_\alpha) = 
\left( \frac{P_n'(\lambda_\alpha) - P_{n-1}'(\lambda_\alpha)}{\lambda_\alpha - \mu_\alpha} \right) \left( \frac{P_n(\mu_\alpha) - P_n'(\lambda_\alpha)}{\mu_\alpha - \lambda_\alpha} \right) - P_{n-1}'(\mu_\alpha)P_n'(\lambda_\alpha) = \mathcal{O}(\mu_\alpha - \lambda_\alpha),
$$

Then

$$
dt\text{-part of } d(\mu_\alpha - \lambda_\alpha) \bigg|_{\mu_\alpha = \lambda_\alpha} = 
\left( \lambda_\alpha - \mu_\alpha + \sum_{1 \leq j < n} \frac{1}{\mu_\alpha - \mu_j} - \sum_{1 \leq j < n} \frac{1}{\lambda_\alpha - \lambda_j} \right)\bigg|_{\mu_\alpha = \lambda_\alpha} = 
\sum_{1 \leq j < n} \frac{\mu_j - \lambda_j}{(\lambda_\alpha - \mu_j)(\lambda_\alpha - \lambda_j)} + \frac{1}{\lambda_n - \lambda_\alpha} > 0,
$$

which follows from the inequalities (for $1 \leq j \leq n - 1$),

$$
\mu_j - \lambda_j \geq 0, \ (\lambda_\alpha - \mu_j)(\lambda_\alpha - \lambda_j) \geq 0 \text{ and } \lambda_n - \lambda_\alpha > 0.
$$

This proves the first relation (2.36), while the second one is done in a similar way.

\section{The Eigenvalues of Three Consecutive Minors}

In this section we shall prove Theorem 2.5, which affirms that for the Dyson process the joint spectra of any three consecutive minors is not Markovian, although the Markovianess of the spectra holds for any one or any two consecutive minors.

Note that given an Itô diffusion $X_t \in \mathbb{R}^n$, with stochastic differential equation

$$
dX_t = a(X_t)dt + \sigma(X_t)db_t, \text{ as in (5.1), and generator } A, \text{ the process restricted to } Y_i = \varphi_i(X), 1 \leq i \leq \ell \text{ is not Markovian (at least for generic initial conditions) if the generator fails to preserve the field of functions } F(Y) \text{ generated by the } (Y_1, \ldots, Y_\ell) := (\varphi_1(X), \ldots, \varphi_\ell(X)), \text{ i.e.}
$$

$$
A F(Y) \not\in F(Y), \tag{7.1}
$$

and provided the diffusion does not hit the $Y$-boundary of the domain.

\textbf{Proof of Theorem 2.5} In order to show the non-Markovianess of

$$
\Gamma := (\lambda, \mu, \nu) = (\text{spec } B, \text{spec } B^{(n-1)}, \text{spec } B^{(n-2)})
$$

it suffices to find a function, such that the function, obtained by applying the Dyson-generator to it, is not a function of $(\lambda, \mu, \nu)$. We pick a function of the
product form $xy = g(\Gamma)h(\Gamma)$, where

$$x := g(\Gamma) := \sum_{i=1}^{n-2} B_{ii} \quad \text{and} \quad y := h(\Gamma) := \det B$$

are two independent functions. Then, according to formula (5.3)

$$A_{\text{Dys}}xy = \frac{dx dy}{dt} + x A_{\text{Dys}}y + y A_{\text{Dys}}x.$$  

(7.2)

Since $x$ and $A_{\text{Dys}}x$ are functions of $\nu$ only and since $y$ and $A_{\text{Dys}}y$ are functions of $\lambda$ only, $x A_{\text{Dys}}y + y A_{\text{Dys}}x$ is a function of $(\lambda, \nu)$ only. Therefore, to establish non-Markovianess of $(\lambda, \mu, \nu)$, it suffices to show that $\frac{dx dy}{dt}$ is not only a function of $(\lambda, \mu, \nu)$. Since, by Itô’s formula (5.5),

$$dx \, dy = \sum_{i=1}^{n-2} dB_{ii} \left( \sum_{j=1}^{n} \frac{\partial \det B}{\partial B_{jj}} dB_{jj} + \sum_{1 \leq i < j \leq n} \sum_{\ell=0}^{\beta-1} \frac{\partial \det B}{\partial B_{ij}^{(\ell)}} dB_{ij}^{(\ell)} \right)$$

$$= \frac{2}{\beta} dt \sum_{i=1}^{n-2} \frac{\partial \det B}{\partial B_{ii}} = \frac{2}{\beta} \sum_{i=1}^{n-2} \det(\text{minor}_{ii}(B)) dt,$$

it suffices to show that the right hand side is not a function of $(\lambda, \mu, \nu)$ only. Here $\text{minor}_{ii}$ denotes removing row $i$ and column $i$ of the matrix.

For example in the case $\beta = 2, n = 3$, this amounts to showing that the determinant of the lower-right $2 \times 2$ principal minor of $B$ is not a function of $(\lambda, \mu, \nu)$ only; to do this, it is convenient to reparametrize the matrix as

$$B = \begin{pmatrix} B_{11} & \rho_3 e^{im_3} & \rho_2 e^{-im_2} \\ \rho_3 e^{-im_3} & B_{22} & \rho_1 e^{im} \\ \rho_2 e^{im_2} & \rho_1 e^{-im} & B_{33} \end{pmatrix}.$$  

Using the following formulae

$$B_{11} = \nu_1,$$

$$B_{22} = \mu_1 + \mu_2 - \nu_1,$$

$$B_{33} = \lambda_1 + \lambda_2 + \lambda_3 - \mu_1 - \mu_2,$$

$$\rho_3^2 = (\mu_2 - \nu_1)(\nu_1 - \mu_1),$$

the lower-right $2 \times 2$ principal minor of $B$ reads

$$\det(\text{minor}_{11}(B)) = B_{22} B_{33} - \rho_1^2$$

$$= (\mu_1 + \mu_2 - \nu_1)(\lambda_1 + \lambda_2 + \lambda_3 - \mu_1 - \mu_2) - \rho_1^2.$$
One observes that
\[
0 = \det B - \lambda_1 \lambda_2 \lambda_3 \\
= B_{11} B_{22} B_{33} - \lambda_1 \lambda_2 \lambda_3 - \sum_{i=1}^{3} \rho_i^2 B_{ii} + 2 \rho_1 \rho_2 \rho_3 \cos(\eta_1 + \eta_2 + \eta_3) \\
= F_1(\lambda, \mu, \nu) - \rho_1^2 \nu_1 - \rho_2^2 (\mu_1 + \mu_2 - \nu_1) \\
+ 2 \rho_1 \rho_2 \sqrt{(\mu_2 - \nu_1)(\nu_1 - \mu_1)} \cos(\eta_1 + \eta_2 + \eta_3)
\]

and
\[
0 = \text{Tr} B^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \sum_{i=1}^{3} B_{ii}^2 + 2 \sum_{i=1}^{3} \rho_i^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \\
= F_2(\lambda, \mu, \nu) + 2(\rho_1^2 + \rho_2^2),
\]

where \( F_i(\lambda, \mu, \nu) \) are functions of the spectral data \((\lambda, \mu, \nu)\). Upon solving these two equations in \(\rho_1\) and \(\rho_2\), one notices that, in particular, \(\rho_1\) is a function of \(\cos(\eta_1 + \eta_2 + \eta_3)\) and the spectral data \((\lambda, \mu, \nu)\), hence showing that \(\det(\text{minor}_{11}(B))\) is not a function of \((\lambda, \mu, \nu)\) only; thus the same is true for \(A_{\text{Dys}xy}\). This proves that \(A_{\text{Dys}xy}\) does not belong to the field of functions depending on \((\lambda, \mu, \nu)\).

More generally, by a perturbation argument about \(B^{(n-1)} = \text{diag}(\mu_1, \ldots, \mu_{n-1})\), one shows similarly that
\[
\sum_{i=1}^{n-2} \det(\text{minor}_{ii}(B)) \notin \mathcal{F}(\lambda, \mu, \nu),
\]
for \(\beta = 2\) and \(4\).

Finally, the boundary of the process \((\lambda, \mu, \nu)\) is given by the subvariety where some of the \(\mu_i\)’s hit the \(\lambda_j\)’s or the \(\nu_k\)’s; that is when \(P_n(\mu_i) = 0\) or \(P_{n-1}(\nu_k) = 0\) for some \(1 \leq i \leq n - 1\) or for some \(1 \leq k \leq n - 2\); \(\tau_{2j}(\mu, \nu) = 0\) for \(1 \leq j \leq n - 2\). From Corollary 2.4 one sees that the process never reaches that boundary. This ends the proof of Theorem 2.5. \(\square\)

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