Concept of semi-velocity and related wave equation

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Abstract

The semi-velocity is defined as an exponential function of the rapidity. Physically the semi-velocity is interpreted as the relativistic analogue of the phase velocity, a geometrical interpretation done within the framework of Beltrami-Klein and Poincaré models. A kinematical wave equation related with the Fermat principle and the concept of the semi-velocity is derived.

1 Introduction

The relativistic kinematics is a part of the relativistic mechanics independent of the inertial mass, $m$. The main notion of this theory, evidently, is the velocity, $V$. In the relativistic mechanics the dynamical notions like energy $cp_0$ and momentum $p$ are definite functions of the mass and the velocity and, usually, they are proportional to the proper mass $m$. The principal relationship of the relativistic mechanics is the mass-shell equation

$$p_0^2 - p^2 = m^2 c^2. \quad (1.1)$$
Factorize this equation with respect to the square of the momentum
\[ p^2 = (p_0 - mc)(p_0 + mc). \] (1.2)

In the kinematics instead of the product of the quantities the quotients of these quantities are used, e.g., the role of the momentum \( p^2 \) defined by the product of the quantities \( (p_0 \pm mc) \), in the kinematics we define the analogy of the square of the momentum as the following quotient
\[ k^2 = \frac{p_0 - mc}{p_0 + mc}. \] (1.3)

In the relativistic kinematics the velocity is defined by the quotient of the energy-momentum
\[ V = \frac{p}{p_0}. \]

Parametrization of the energy-momentum by the velocity leads us to the formulae proportional to the proper mass:
\[ p_0 = \frac{mc}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad p = \frac{mV}{\sqrt{1 - \frac{V^2}{c^2}}}. \] (1.4)

In terms of the velocity the quotient in (1.3) is written as
\[ k^2 = \frac{1 - \sqrt{1 - \frac{V^2}{c^2}}}{1 + \sqrt{1 - \frac{V^2}{c^2}}}. \] (1.5)

Evidently, this formula is the squared form of the equation
\[ k = \frac{c^2}{V}(1 \pm \sqrt{1 - \frac{V^2}{c^2}}). \] (1.6)

In the period of construction a bridge between wave mechanics and the classical mechanics the concept analogue of the phase velocity in the classical mechanics had played a central role.

On the basis of Hamilton-Jacobi equations E. Schrodinger has defined this quantity as follows [1]
\[ v = \frac{\mathcal{E}}{p}, \] (1.7)
where $p$ is the momentum and $\mathcal{E}$ is the total energy of a particle with mass $m$,

$$\mathcal{E} = \frac{p^2}{2m} + U(r). \quad (1.8)$$

Evidently, the analogue of the phase velocity differs of definition of the velocity in the classical mechanics. In absence of the external fields the phase velocity $v$ differs of the velocity $V$ by the factor one-half:

$$v = \frac{p^2/2m}{p} = \frac{1}{2} V. \quad (1.9)$$

By the de Broglie’s hypothesis the phase velocity is defined as [2]

$$v_{phase} = \frac{c^2}{V}, \quad (1.10)$$

where $V$ is the velocity of the particle regardless of wave behavior. In the case of free motion the formula of de Broglie (1.1) can be considered as a relativistic generalization of the Schrodinger definition (1.7). In fact, if $c\rho_0$ is relativistic expression for the kinetic energy, then,

$$c\rho_0 = \frac{c^2}{V}. \quad (1.11)$$

In the relativistic kinematics the velocity admits a trigonometrical representation via an hyperbolic angle

$$V = c \tanh(\psi). \quad (1.12)$$

In this representation the phase velocity corresponds one half of the hyperbolic angle

$$v = c \tanh\left(\frac{1}{2} \psi\right). \quad (1.13)$$

Furthermore, denoting the relativistic summation formula for velocities by the symbol

$$V = (v_1 \oplus v_2), \quad (1.14)$$

the formula (1.4) will be re-written as

$$V = (v \oplus v) = \frac{v + v}{1 + v^2/c^2} = \frac{2v}{1 + v^2/c^2}. \quad (1.15)$$
This formula we take as a basic formula of the wave kinematics. In the relativistic dynamics we propose to denominate this velocity as *semi-velocity*. The concept of the semi-velocity has its remarkable interpretation via the cross-ratio [6] and distance between two points in the hyperbolic space. This concept leads to new kind of wave equation which can be interpreted as *wave equation of the relativistic kinematics*.

The paper is presented by the following sections.

Section 2 we introduce a concept of the semi-velocity and give its geometrical interpretation. In Section 3 define an analogue of the phase velocity in the relativistic mechanics and suggest a new kind of wave equation for the massive particle.

2 Geometrical interpretations of the semi-velocity

Let $V$ be a velocity of the point-particle defined with respect to coordinate time expressed via the hyperbolic angle $\psi$, the rapidity,

$$V = c \tanh(\psi). \quad (2.1)$$

Define the velocity $v$ at half-rapidity $\psi/2$ by the same trigonometric function

$$v = c \tanh(\psi/2). \quad (2.2)$$

Then, the velocity $V$ is the function of $(v)$ of the form

$$V = (v \boxplus v) = \frac{2v}{1 + v^2/c^2}. \quad (2.4)$$

The quantity $v$ we will denominate as the *semi-velocity*. In general, we introduce two types of the semi-velocities by

$$v_- v_+ = c^2, \quad v_- = v. \quad (2.5)$$

Formula (2.4) is valid for the both velocities:

$$V = (v_\pm \boxplus v_\pm) = \frac{2v_\pm}{1 + v_\pm^2/c^2}. \quad (2.6)$$
Equation (2.6) is converted into a quadratic equation with respect to $v_\pm$:

$$v^2 - \frac{2c^2}{V}v + c^2 = 0,$$

(2.7)

with roots

$$v_\pm = \frac{c}{V}(1 \pm \sqrt{1 - \frac{V^2}{c^2}}).$$

(2.8)

In terms of the dynamical variables the semi-velocity is defined by the formulae

$$\frac{v_\pm}{c} = \frac{p}{p_0 \mp mc} = \frac{p_0 \pm mc}{p},$$

(2.9)

correspondingly, the square of this equation is

$$\frac{v_\pm^2}{c^2} = \frac{p_0 \pm mc}{p_0 \mp mc}.$$

(2.10)

In addition, let us observe a connection of the semi-velocities with the Pythagoras spinors ($q_1, q_2$). The trio of dynamic variables, $p_0, p, mc$ are bilinear functions of the Pythagoras spinors

$$p^2 = q_1^2 q_2^2, \quad p_0 = \frac{1}{2}(q_1^2 + q_2^2), \quad mc = \frac{1}{2}(q_1^2 - q_2^2).$$

(2.11)

The semi-velocities are defined simply as fractions of the Pythagoras spinors

$$\frac{v_\pm^2}{c^2} = \frac{q_1^2}{q_2^2}, \quad \frac{v_-^2}{c^2} = \frac{q_2^2}{q_1^2}.$$

(2.12)

It is remarkable that the formula for semi-velocity admits exponential representation because [7]

$$\exp(2mc\phi) = \frac{p_0 + mc}{p_0 - mc}.$$

(2.13)

Then,

$$\exp(mc\phi) = \coth(\psi/2) = v_+, \quad v_- = \exp(-mc\phi).$$

(2.14)

In order to give a geometrical interpretation of the semi-velocity let us recall some elements of the Beltrami-Klein and the Poincaré models of the hyperbolic geometry.
The Klein disk model (also known as the Beltrami-Klein model) and the Poincaré disk model are both models that project the whole hyperbolic plane in a disk. The two models are related through a projection on or from the hemisphere model. The Klein disk model is an orthographic projection to the hemisphere model while the Poincaré disk model is a stereographic projection.

When projecting the same lines in both models on one disk both lines go through the same two ideal points (the ideal points remain on the same spot), also the pole of the chord in the Klein disk model is the center of the circle that contains the arc in the Poincaré disk model.

A point \((x, y)\) in the Poincaré disk model maps to
\[
\left( \frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2} \right) \tag{2.15}
\]
in the Klein model.

A point \((x, y)\) in the Klein model maps to
\[
\left( \frac{x}{1 + \sqrt{1 - x^2 - y^2}}, \frac{y}{1 + \sqrt{1 - x^2 - y^2}} \right) \tag{2.16}
\]
in the Poincaré disk model.

If \(u\) is a vector of norm less than one representing a point of the Poincaré disk model, then the corresponding point of the Klein disk model is given by
\[
s = \frac{2u}{1 + u \cdot u}. \tag{2.17}
\]
Conversely, from a vector \(s\) of norm less than one representing a point of the Beltrami-Klein model, the corresponding point of the Poincaré disk model is given by:
\[
u = \frac{s}{1 + \sqrt{1 - s \cdot s}} = \frac{1 - \sqrt{1 - s \cdot s}}{s \cdot s} \frac{s}{s \cdot s}. \tag{2.18}
\]

Thus, if \(P\) is a point a distance \(V/c\) from the center of the unit circle in the Beltrami-Klein model, then corresponding point in the Poincaré disk model a distance of \(v/c\) of the same radius:
\[
\frac{v}{c} = \frac{c}{V} \left(1 - \sqrt{1 - \frac{V^2}{c^2}}\right). \tag{2.19}
\]
Conversely, if \( P \) is a point a distance \( v/c \) from the center of the unit circle in the Poincaré disk model, then the corresponding point of the Beltrami-Klein model is a distance of \( V/c \) on the same radius:

\[
\frac{V}{c} = \frac{2v/c}{1 + v^2/c^2}.
\] (2.20)

By taking into account (2.9) we get

\[
\frac{v_-}{v_+} = \exp(2\psi) = \frac{1 - \sqrt{1 - V^2/c^2}}{1 + \sqrt{1 - V^2/c^2}}.
\] (2.21)

This formula coincides with the formula for distance between two points on the hyperbolic plane (see, for instance, [8], p.142). In this geometry distance between two point is defined by logarithmic formula

\[
\rho = \frac{k}{2} \log \frac{1 + \sqrt{1 - J^2}}{1 - \sqrt{1 - J^2}},
\] (2.22)

where \( J \) in coordinates Beltrami is defined by the formula

\[
J^2 = \frac{(z_1^2 - x_1^2 - y_1^2)(z_2^2 - x_2^2 - y_2^2)}{(z_1z_2 - x_1x_2 - y_1y_2)^2}.
\] (2.23)

Formula (2.22) written in the exponential form coincides with (2.20), hence

\[
V^2/c^2 = J^2.
\] (2.24)

If the equation of the fundamental conic is

\[
\Omega = \sum_{i,j=1}^{3} a_{ij}x_i y_j = 0,
\] (2.25)

then the distance between the two points \( x \) and \( y \), in homogeneous coordinates \((x_1, x_2, x_3)\) and \((y_1, y_2, y_3)\)

\[
\rho = \frac{k}{2} \log \frac{\Omega_{xy} + \sqrt{\Omega_{xy}^2 - \Omega_{xx}\Omega_{yy}}}{\Omega_{xy} - \sqrt{\Omega_{xy}^2 - \Omega_{xx}\Omega_{yy}}},
\] (2.26)

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Here,
\[ \Omega_{kj} = z_k z_j - x_k x_j - y_k y_j \]  
then, the following identity holds true
\[ \Omega_{11} \Omega_{22} - \Omega_{12}^2 = (x_1 y_2 - x_2 y_1)^2 - (y_1 z_2 - y_2 z_1)^2 - (x_1 z_2 - x_2 z_1)^2. \]  
In this model, the velocity \( V \) is presented by formula
\[ V^2/c^2 = \frac{\Omega_{11} \Omega_{22}}{\Omega_{12}^2}, \]  
and the rapidity \( \xi \) with the distance \( \rho \) and speed of light with curvature \( k \). According to identity (2.28), the energy \( p_0 \), momentum \( p \) and the mass \( mc \) can be identified as follows
\[ p^2 = (z_1^2 - x_1^2 - y_1^2)(z_2^2 - x_2^2 - y_2^2), \quad p_0 = z_1 z_2 - x_1 x_2 - y_1 y_2, \]
\[ m^2 c^2 = (x_1 y_2 - x_2 y_1)^2 - (y_1 z_2 - y_2 z_1)^2 - (x_1 z_2 - x_2 z_1)^2. \]  

2.2 Semi-velocity and Chebyshev polynomials.

The Chebyshev polynomials of the first kind are defined by the recurrence relations \[ T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x). \]
The corresponding difference equation can be associated with unit defined by the equation \[ H^2 = 2x H - 1, \quad H_\pm = x \pm \sqrt{x^2 - 1}, \]  
and
\[ H_\pm^n = A_n + H_\pm B_n, \]  
where \( A_n \) and \( B_n \) are functions of the variable \( x \) satisfying recurrences that can be presented in the matrix form
\[ \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2x \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}. \]
\[ \frac{c}{V} = x. \]
Then the formula (2.8) for semi-velocity coincides with $H_{\pm}$:

$$\frac{v_{\pm}}{c} = (x \pm \sqrt{x^2 - 1}) = H_{\pm}.$$  \hspace{1cm} (2.35)

This fact prompts us to construct the Chebyshev sequences from the components of the semi-velocity. The Chebyshev polynomials of first kind are given by

$$T_n(x) = \frac{1}{2c}(v_n^+ + v_n^-).$$ \hspace{1cm} (2.36)

The semi-velocity as the function of the velocity satisfy the equation

$$\frac{d}{dx}v_{\pm} = \pm \frac{v_{\pm}}{\sqrt{x^2 - 1}},$$ \hspace{1cm} (2.37)

The Chebyshev polynomials of the first kind in the study of differential equations arises as solutions of the equation

$$[(1 - x^2)\partial_x^2 - x\partial_x + n^2]T_n(x) = 0.$$ \hspace{1cm} (2.38)

And the Chebyshev equations of the second kind

$$U_n(x) = \frac{H_{n+1}^n - H_{n+1}^n}{2\sqrt{x^2 - 1}}$$ \hspace{1cm} (2.39)

verify the following differential equation

$$[(1 - x^2)\partial_x^2 - 3x\partial_x + n(n + 2)]U_n(x) = 0.$$ \hspace{1cm} (2.40)

The exponential generating function is

$$\sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = \frac{1}{2}(e^{v-t/c} + e^{v+t/c}).$$ \hspace{1cm} (2.41)

### 3 Derivation of the wave equation from Fermat’s principle

**The Fermat law.**

The wave equation of the quantum mechanics, the Schrodinger equation, had been obtained on the basis of Hamilton principle and the Hamilton-Jacobi equation [5]. Our goal is to pass the same way, but by using as an
starting platform the Fermat principle. In refs. [11] and [4] it has been established that for the electromagnetic waves radiating in a medium the velocities $v_\pm$ are related with refractive index $n$ according to the following formulae

$$\frac{v^2}{c^2} = \frac{p_0 - mc}{p_0 + mc} = \frac{1}{n^2},$$

(3.1, a)

for $n > 1$, and

$$\frac{v^2}{c^2} = \frac{p_0 + mc}{p_0 - mc} = \frac{1}{n^2},$$

(3.1, b)

for $n < 1$.

Consequently, the velocities $v_\pm$ have to be interpreted as the phase velocities related in ordinary way with refractive index $n$.

As a first step let us recall as the definition of analogue of the phase velocity had been used in the procedure of modification of the Eikonal equation into the Hamilton-Jacobi equation. Let us start with the Eikonal equation in a geometrical wave theory of the form

$$\frac{dW}{dt} = n = \frac{c}{v},$$

(3.2)

admits direct integration. By taking into account $dl/dt = v$, $dl/v = dt$, integrate equation (3.2) to obtain

$$W/c = \int_1^2 \frac{dl}{v} = \int_1^2 dt = t_2 - t_1.$$

Thus, the value $W/c$ is an interval of time $T_{21} = t_2 - t_1$ during of which the wave front passes given distance $d_{12} = l_1 - l_2$.

The time of radiation of the light in the medium with refractive index $n$ is given by the integral

$$T = \int \frac{dl}{v(x,y,z)} = \int \frac{n}{c} dl.$$

(3.3)

According to Fermat’s law this time has to be minimal (extremal):

$$\delta T = 0.$$  

(3.4)
Now, instead of the phase velocity \( v \) let us use its expression via the velocity \( V \) defined with respect to the coordinate time. The function \( v = v(V) \) is given in the explicit form by
\[
\frac{v}{c} = \frac{c}{V}(1 \pm \sqrt{1 - \frac{V^2}{c^2}}).
\] (3.5)

Substitute this formula into the integral
\[
W = \int_1^2 \frac{V dl}{c(1 + \sqrt{1 - \frac{V^2}{c^2}})} = \int_1^2 \frac{V^2 dt}{c(1 + \sqrt{1 - \frac{V^2}{c^2}})} = \int_1^2 dt \left(1 - \sqrt{1 - \frac{V^2}{c^2}}\right).
\] (3.6)

Inside the integral we have the Lagrange function
\[
L = 1 - \sqrt{1 - \frac{V^2}{c^2}},
\] (3.7)

which differs of the relativistic Lagrangian only by the constant.

**3.2 The Hamilton-Jacobi equation.**

With aim to adopt the eikonal equation (3.2) to the dynamics of the particle, first of all, let us use relationship of refractive index of the medium express with the phase velocity \( v \),
\[
n^2 = \frac{c^2}{v^2}.
\] (3.8)

Perform replacement on making use of formula (1.1) in (2.1), this gives,
\[
(\nabla W)^2 = \frac{c^2}{v^2} \rightarrow (\nabla \frac{W}{c})^2 = \frac{p^2}{c^2} \rightarrow (\nabla (\mathcal{E} W/c))^2 = p^2.
\] (3.9)

For the simplest case of the inertial free motion define
\[
W = -v_0 t + r \frac{v_0}{v},
\] (3.10)

so that,
\[
\frac{\partial W}{\partial t} = -v_0, \quad \frac{\partial W}{\partial r} = \frac{v_0}{v}.
\] (3.11)

Define the function of action by
\[
S = \mathcal{E} W/c = -\mathcal{E} t + pr.
\] (3.12)
Differential form of this equation

\[ dS = -\mathcal{E} dt + p dr, \]  

(3.13)

offers formulas

\[ \frac{\partial S}{\partial t} = -\mathcal{E}, \quad \frac{\partial S}{\partial r} = p. \]  

(3.14)

Substituting these definitions into formula for the energy (1.2) we come to Hamilton-Jacobi equations

\[ |\nabla S|^{2} = 2m(\mathcal{E} - U), \quad \frac{\partial S}{\partial t} = -\mathcal{E}, \]  

(3.15)

From Hamilton-Jacobi theory it follows that the wave front of the function of action \( S \) is moving with the phase velocity

\[ v = \frac{\mathcal{E}}{\sqrt{2m(\mathcal{E} - U)}}, \]  

(3.16)

in fact,

\[ \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{dr}{dt} \frac{\partial S}{\partial r} = 0 \rightarrow \frac{dr}{dt} = -\frac{\partial S}{\partial t} / \frac{\partial S}{\partial r} = \frac{\mathcal{E}}{p}. \]  

(3.17)

3.3 The kinematic Wave equation.

Transformation of the Eikonal equation into relativistic Hamilton-Jacobi equation is worked out in a similar way. Write

\[ (\nabla W)^{2} = \frac{c^{2}}{v^{2}} = \frac{p^{2}}{\mathcal{E}^{2}}, \]  

(3.18)

where

\[ \mathcal{E} = cp_{0} + U(r), \]  

(3.19)

and

\[ p^{2} = (\mathcal{E} - U(r) - mc^{2})(\mathcal{E} - U(r) + mc^{2}). \]  

(3.20)

where

\[ \frac{\partial S}{\partial t} = -\mathcal{E}. \]  

(3.21)

Since

\[ n^{2} = \frac{v^{2}}{c^{2}} = \frac{p_{0} - mc}{p_{0} + mc} = \frac{\mathcal{E}/c + mc - U(r)/c}{\mathcal{E}/c - mc - U(r)/c}, \]  

(3.22)
The Eikonal equation we present of the form

\[ (\nabla W)^2 = n^2 = \frac{p_0 - mc}{p_0 + mc} = \frac{\mathcal{E}/c + mc - U(r)/c}{\mathcal{E}/c - mc - U(r)/c}. \tag{3.23} \]

This equation can be understood as an analogue of the relativistic Hamilton-Jacobi equation

\[ (\nabla S)^2 = p^2 = (p_0 - mc)(p_0 + mc) = (\mathcal{E}/c + mc - U(r)/c)(\mathcal{E}/c - mc - U(r)/c). \tag{3.24} \]

The wave equation in the medium is written as

\[ (\nabla^2 + n^2 \kappa_0^2)\Psi = 0, \tag{3.25} \]

\[ \kappa_0 = \frac{2\pi}{\lambda_0}. \tag{3.26} \]

Substitute here the definition of \( n^2 \) via the semi-velocity \( k^2 \) from (1.5). We get

\[ \nabla^2 \Psi + \kappa_0^2 \frac{1 - \sqrt{1 - \frac{V^2}{c^2}}}{1 + \sqrt{1 - \frac{V^2}{c^2}}} \Psi = 0. \tag{3.27} \]

In presence of the external potential field \( U(r) \) on making use of the formula \( \mathcal{E} = p_0c + U(r) \) we arrive at the following wave equation

\[ \nabla^2 \Psi + \kappa_0^2 \frac{\mathcal{E} - mc^2 - U(r)}{\mathcal{E} + mc^2 - U(r)} \Psi = 0. \tag{3.28} \]

In accordance with (3.15) we write

\[ \left( \frac{\partial W}{\partial r} \right)^2 = \left( \frac{c}{v} \right)^2 \left( \frac{\partial W}{\partial t} \right)^2. \tag{3.29} \]

Then, the corresponding wave equation is formulated as

\[ \nabla^2 \Psi + \frac{\mathcal{E} - mc^2 - U(r)}{\mathcal{E} + mc^2 - U(r)} \frac{\partial^2}{c^2 \partial t^2} \Psi = 0. \tag{3.30} \]
4 Concluding remarks

The formula of the semi-velocity had been used in various branches of the relativistic dynamics and models of the hyperbolic geometry. Furthermore, in these models the notion of the semi-velocity had played a central role. Physics of the accelerators, colliders, is one of the examples where the concept of the semi-velocity as the relative velocity of particles, it has been introduced. We have shown that the semi-velocity is related:

(1) with the refractive index of the medium;
(2) with the velocity of radiating electro-magnetic field;
(3) with the phase velocity of the massive elementary particle;
(4) with the cross-ratio;
(5) with the distance in the hyperbolic geometry.

Connection "phase velocity—semi-velocity" leads to a new kind of the wave equation, connection "semi-velocity—cross-ratio" leads to geometrical interpretation of the relativistic kinematics.

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