An SDP Primal-Dual Approximation Algorithm for Directed Hypergraph Expansion and Sparsest Cut with Product Demands

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Abstract

We give approximation algorithms for the edge expansion and sparsest cut with product demands problems on directed hypergraphs, which subsume previous graph models such as undirected hypergraphs and directed normal graphs.

Using an SDP formulation adapted to directed hypergraphs, we apply the SDP primal-dual framework by Arora and Kale (JACM 2016) to design polynomial-time algorithms whose approximation ratios match those of algorithms previously designed for more restricted graph models. Moreover, we have deconstructed their framework and simplified the notation to give a much cleaner presentation of the algorithms.
1 Introduction

The edge expansion of an edge-weighted graph gives a lower bound on the ratio of the weight of edges leaving any subset $S$ of vertices to the sum of the weighted degrees of $S$. Therefore, this notion has applications in graph partitioning or clustering [KV04, MMV15, PSZ15], in which a graph is partitioned into clusters such that, loosely speaking, the general goal is to minimize the number of edges crossing different clusters with respect to some notion of cluster weights.

The edge expansion and the sparsest cut problems [LR99] can be viewed as a special case when the graph is partitioned into two clusters. Even though the involved problems are NP-hard, approximation algorithms have been developed for them in various graph models and settings, such as undirected [ARV09, ALN08] or directed graphs [ACMM05, AAC07], and uniform [ARV09, ACMM05] or general demands [AAC07, ALN08] in the case of sparsest cut. Recently, approximation algorithms have been extended to the case of undirected hypergraphs [LM14]. In this paper, we consider these problems for the even more general class of directed hypergraphs.

Directed Hypergraphs. We consider an edge-weighted directed hypergraph $H = (V, E, w)$, where $V$ is the vertex set of size $n$ and $E \subseteq 2^V \times 2^V$ is the set of $m$ directed hyperedges; Each directed hyperedge $e \in E$ is denoted by $(T_e, H_e)$, where $T_e \subseteq V$ is the tail and $H_e \subseteq V$ is the head; we assume that both the tail and the head are non-empty, and we follow the convention that the direction is from tail to head. We denote $r := \max_{e \in E}(|T_e| + |H_e|)$.

The function $w : E \to \mathbb{R}_+$ assigns a non-negative weight to each edge. Note that $T_e$ and $H_e$ do not have to be disjoint. This notion of directed hypergraph was first introduced by Galllo et al. [GLPN93], who considered applications in propositional logic, analyzing dependency in relational database, and traffic analysis.

Observe that this model captures previous graph models: (i) an undirected hyperedge $e$ is the special case when $T_e = H_e$, and (ii) a directed normal edge $e$ is the special case when $|T_e| = |H_e| = 1$.

Directed Hyperedge Expansion. In addition to edge weights, each vertex $u \in V$ has weight $\omega_u := \sum_{e \in E : u \in T_e \cup H_e} w_e$ that is also known as its weighted degree. Given a subset $S \subseteq V$, define $\overline{S} := V \setminus S$ and $\omega(S) := \sum_{u \in S} \omega_u$. Define the out-going cut $\partial^+(S) := \{e \in E : T_e \cap S \neq \emptyset \land H_e \cap \overline{S} \neq \emptyset\}$, and the in-coming cut $\partial^-(S) := \{e \in E : T_e \cap \overline{S} \neq \emptyset \land H_e \cap S \neq \emptyset\}$. The out-going edge expansion of $S$ is $\phi^+(S) := \frac{\sum_{e \in \partial^+(S)} w_e}{\omega(S)}$, and the in-coming edge expansion is $\phi^-(S) := \frac{\sum_{e \in \partial^-(S)} w_e}{\omega(S)}$. The edge expansion of $S$ is $\phi(S) := \min\{\phi^+(S), \phi^-(S)\}$. The edge expansion of $H$ is

$$\phi_H := \min_{\emptyset \neq S \subseteq V, \omega(S) \leq \omega(V)} \phi(S).$$

Directed Sparsest Cut with Product Demands. As observed in previous works such as [ARV09], we relate the expansion problem to the sparsest cut problem with product demands. For vertices $i \neq j \in V$, we assume that the demand between $i$ and $j$ is symmetric and given by the product $\omega_i \cdot \omega_j$. For $\emptyset \neq S \subseteq V$, its directed sparsity is $\vartheta(S) := \frac{\sum_{e \in \partial^+(S)} w_e}{\omega(S) \cdot \omega(S)}$. The goal is to find a subset $S$ to minimize $\vartheta(S)$.

Observe that $\omega(V) \cdot \vartheta(S)$ and $\frac{\sum_{e \in \partial^+(S)} w_e}{\min\{\omega(S), \omega(S)\}}$ are within a factor of 2 from each other. Therefore, the directed edge expansion problem on directed hypergraphs can be reduced (up to a constant factor) to the sparsest cut problem with product demands. Hence, for the rest of the paper, we just focus on the sparsest cut problem with product demands.

Vertex Weight Distribution. For the sparsest cut problem, the vertex weights $\omega : V \to \mathbb{R}_+$ actually do not have to be related to the edge weights. However, we do place restrictions on the skewness of the weight distribution. Without loss of generality, we can assume that each vertex has integer weight. For $\kappa \geq 1$, the weights $\omega$ are $\kappa$-skewed, if for all $i \in V$, $1 \leq \omega_i \leq \kappa$. In this paper, we assume $\kappa \leq n$.

Balanced Cut. For $0 < c < \frac{1}{2}$, a subset $S \subseteq V$ is $c$-balanced if both $\omega(S)$ and $\omega(V \setminus S)$ are at least $c \cdot \omega(V)$. 


1.1 Our Contributions and Results

Our first observation is a surprisingly simple reduction of the problem from the more general directed hypergraphs to the case of directed normal graphs.

**Fact 1.1 (Reduction to Directed Normal Graphs)** Suppose $H = (V, E)$ is a directed hypergraph with edge weights $w$ and vertex weights $\omega$. Then, transformation to a directed normal graph $\tilde{H} = (\tilde{V}, \tilde{E})$, where $|\tilde{V}| = n + 2m$ and $|\tilde{E}| = m + \sum_{e \in E} (|T_e| + |H_e|)$, is defined as follows.

The new vertex set is $\tilde{V} := V \cup \{v^e_T, v^H_e : e \in E\}$, i.e., for each edge $e \in E$, we add two new vertices; the old vertices retain their original weights, and the new vertices have zero weight.

The new edge set is $\tilde{E} := \{(v^e_T, v^H_e) : e \in E\} \cup \{(u, v^e_T) : e \in E, u \in T_e\} \cup \{(v^H_e, v) : e \in E, v \in H_e\}$. An edge of the form $(v^e_T, v^H_e)$ has its weight $w_e$ derived from $e \in E$, while all other edges have large weight $\tilde{\Omega} := n \sum_{e \in E} w_e$.

We overload the symbols for edge $w$ and vertex $\omega$ weights. However, we use $\tilde{\partial}^+(\cdot)$ for out-going cut in $\tilde{H}$.

Given a subset $S \subseteq V$, we define the transformed subset $\tilde{S} := S \cup \{v^e_T : e \cap T_e \neq \emptyset\} \cup \{v^H_e : H_e \subseteq S\}$.

Then, we have the following properties.

- For any $S \subseteq V$, $\omega(S) = \omega(\tilde{S})$ and $w(\tilde{\partial}^+(S)) = w(\tilde{\partial}^+(\tilde{S}))$.
- For any $T \subseteq \tilde{V}$, $\omega(T \cap V) = \omega(T)$; moreover, if $w(\tilde{\partial}^+(T)) < \tilde{\Omega}$, then $w(\tilde{\partial}^+(T \cap V)) = w(\tilde{\partial}^+(T))$.

Fact 1.1 implies that for problems such as directed sparsest cut (with product demands), max-flow and min-cut, it suffices to consider directed normal graphs.

**Semidefinite Program (SDP) Formulation.** Arora et al. [ARV09] formulated an SDP for the sparsest cut problem with uniform demands for undirected normal graphs. The SDP was later refined by Agarwal et al. [ACMM05] for directed normal graphs to give a rounding-based approximation algorithm. Since the method can be easily generalized to product demands with $\kappa$-skewed vertex weights by duplicating copies, we have the following corollary.

**Corollary 1.2 (Approximation Algorithm for Directed Sparsest Cut with Product Demands)** For the directed sparsest cut problem with product demands (with $\kappa$-skewed vertex weights) on directed hypergraphs, there are randomized polynomial-time $O(\sqrt{\log \kappa n})$-approximate algorithms.

Are we done yet? Unfortunately, solving an SDP poses a major bottleneck in running time. Alternatively, Arora and Kale [AK16] proposed an SDP primal-dual framework that iteratively updates the primal and the dual solutions.

**Outline of SDP Primal-Dual Approach.** The framework essentially performs binary search on the optimal SDP value. Each binary search step requires iterative calls to some ORACLE. Loosely speaking, given a (not necessarily feasible) primal solution candidate of the minimization SDP, each call of the ORACLE returns either (i) a subset $S \subseteq V$ of small enough sparsity $\partial(S)$, or (ii) a (not necessarily feasible) dual solution with large enough objective value to update the primal candidate in the next iteration. At the end of the last iteration, if a suitable subset $S$ has not been returned yet, then the dual solutions returned in all the iterations can be used to establish that the optimal SDP value is large.

**Disadvantage of Direct Reduction.** For directed sparsest cut problem with uniform demands, the primal-dual framework gives an $O(\sqrt{\log n})$-approximate algorithm, which has running time $\tilde{O}(mn^{1.5} + n^{2+o(1)})$. If we apply the reduction in Fact 1.1 directly, the resulting running time for directed hypergraphs becomes $\tilde{O}(mrn^{1.5} + n^{2+o(1)} + m^{2+o(1)})$. The term $(mn)^{1.5}$ is due to a max-flow computation, which is not obvious how to improve. However, the extra $m^{2+o(1)}$ term is introduced, because the dimension of the primal domain

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1 After checking the calculation in [Kal07], we should actually be using $O(n^2)$ in the running time. Through personal communication with Kale, we are told that it might be possible reduce a factor of $O(n)$, using the “one-sided width” technique in [Kal07].
is increased. Therefore, we think it is worthwhile to adapt the framework in [Kal07] to directed hypergraphs to avoid the extra $n^{2+o(1)}$ term.

**Other Motivations.** We deconstruct the algorithm for directed normal graphs with uniform vertex weight in Kale’s PhD thesis [Kal07], and simplify the notation. The result is a much cleaner description of the algorithm, even though we consider more general directed hypergraphs and non-uniform vertex weights. As a by-product, we discover that since the subset returned by sparsest cut needs not be balanced, there should be an extra factor of $O(n^2)$ in the running time of their algorithm. We elaborate the details further as follows.

1. In their framework, they assume that in the SDP, there is some constraint on the trace $\text{Tr}(X) = I \cdot X$, which can be viewed as some dot-product with the identity matrix $I$. The important property is that every non-zero vector is an eigenvector of $I$ with eigenvalue 1. Therefore, if the smallest eigenvalue of $A$ is at least $-\epsilon$ for some small $\epsilon > 0$, then the sum $A + \epsilon I \succeq 0$ has non-negative eigenvalues. This is used crucially to establish a lower bound on the optimal value of the SDP.

However, for the SDP formulation of directed sparsest cut, the constraint loosely translates to $I \cdot X \leq O(\omega(S) \omega(S))$, where $S \subset V$ is some candidate subset. To achieve the claimed running time, one needs a good enough upper bound, which is achieved if the subset is balanced. However, for general $S$ that is not balanced, there can be an extra factor of $n$ in the upper bound, which translates to a factor of $O(n^2)$ in the final running time.

Instead, as we shall see, there is already a constraint $K \cdot X = 1$, where $K$ is the Laplacian matrix of the complete graph. Since $K$ is actually a scaled version of the identity operator on the space orthogonal to the all-ones vector $1$, a more careful analysis can use this constraint involving $K$ instead.

2. In capturing directed distance in an SDP [ACMM05], typically, one extra vector $v_0$ is added. However, in the SDP of [Kal07], a different vector $w_i$ is added for each $i \in V$, and constraints saying that all these $w_i$’s are the same are added. At first glance, these extra vectors $w_i$’s and constraints seem extraneous, and create a lot of dual variables in the description of the ORACLE. The subtle reason is that by increasing the dimension of the primal domain, the width of the ORACLE, which is measured by the spectral norm of some matrix, can be reduced.

Observe that the matrix $K$ does not involve any extra added vectors. If we do not use the trace bound on $\text{Tr}(X)$ in the analysis, then we cannot add any extra vectors in the SDP. This can be easily rectified, because we can just label any vertex in $V$ as 0 and consider two cases. In the first case, we formulate an SDP for the solution $S$ to include 0; in the second case, we formulate a similar SDP to exclude 0 from the solution. The drawback is that now the width of the ORACLE increases by a factor of $O(n)$, which leads to a factor of $O(n^2)$ in the number of iterations.

Therefore, in the end, we give a simpler presentation than [Kal07], but the asymptotic running time is the same, although an improvement as mentioned in Footnote 1 might be possible.

3. For each simple path, they add a generalized $\ell_2^2$-triangle inequality. This causes an exponential number of dual variables (even though most of them are zero). However, only triangle inequalities for triples are needed, because each triangle inequality for a long path is just a linear combination of inequalities involving only triples.

We summarize the performance of our modified primal-dual approach as follows.

**Theorem 1.3 (SDP Primal-Dual Approximation Algorithm for Directed Sparsest Cut)** Suppose the vertex weights are $\kappa$-skewed. Each binary search step of the primal-dual framework takes $T := \tilde{O}(\kappa^2 n^2)$ iterations. The running time of each iteration is $\tilde{O}(rtm^{1.5} + (\kappa n)^2)$.

The resulting approximation ratio is $O(\sqrt{\log \kappa n})$. 

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1.2 Related Work

As mentioned above, the most related work is the SDP primal-dual framework by Arora and Kale [AK16] used for solving various variants of the sparsest cut problems. The details for directed sparsest cut are given in Kale’s PhD thesis [Kal07]. We briefly describe the background of related problems as follows.

*Edge Expansion and Sparsest Cut.* Leighton and Rao [LR99] achieved the first $O(\log n)$-approximation algorithms for the edge expansion problem and the sparsest cut problem with general demands for undirected normal graphs. An SDP approach utilizing $\ell_2^2$-representation was used by Arora et al. [ARV09] to achieve $O(\sqrt{\log n})$-approximation for the special case of uniform demands; subsequently, $O(\sqrt{\log n} \cdot \log \log n)$-approximation has been achieved for general demands [ALN08] via embeddings of $n$-point $\ell_2^2$ metric spaces into Euclidean space with distortion $O(\sqrt{\log n} \cdot \log \log n)$. This embedding was also used to achieve $O(\sqrt{\log n} \cdot \log r \cdot \log \log n)$-approximation for the general demands case in undirected hypergraphs [CLTZ18], where $r$ is the maximum cardinality of an hyperedge.

For directed graphs, Agarwal et al. [ACMM05] generalized the separator theorem [ARV09] for $\ell_2^2$-representation vectors to the directed case and achieved an $O(\sqrt{\log n})$-approximation for the directed sparsest cut problem with uniform demands. The general demands variant for directed graphs seems to be much harder, as the best currently known polynomial-time approximation ratio is $O(n^{1/4})$ by [AAC07].

An $O(\sqrt{\log n})$-approximation for undirected hyperedge expansion has been achieved by Louis and Makarychev [LM14], who used hypergraph orthogonal separator as the main tool in rounding their SDP formulation. However, their orthogonal separator technique is more suitable for dealing with undirected hypergraphs. It is not immediately clear how to generalize their orthogonal separator to directed hypergraphs. Instead, we follow the approach in [ACMM05] and still can achieve the same approximation ratio of $O(\sqrt{\log n})$ for directed hyperedge expansion.

2 SDP Relaxation for Directed Sparsest Cut

We follow some common notation concerning sparsest cut (with uniform demands) in undirected [ARV09] and directed [ACMM05] normal graphs.

**Definition 2.1 ($\ell_2^2$-Representation)** An $\ell_2^2$-representation for a set of vertices $V$ is an assignment of a vector $v_i$ to each vertex $i \in V$ such that the $\ell_2^2$-triangle inequality holds:

$$||v_i - v_j||^2 \leq ||v_i - v_k||^2 + ||v_k - v_j||^2, \quad \forall i, j, k \in V.$$ 

**Directed Distance [ACMM05].** We arbitrarily pick some vertex in $V$, and call it 0.

We first consider the case when 0 is always included in the feasible solution. Given an $\ell_2^2$-representation \(\{v_i\}_{i \in V}\), define the directed distance \(d : V \times V \to \mathbb{R}_+\) by

\[d(i, j) := ||v_i - v_j||^2 - ||v_i - v_0||^2 + ||v_j - v_0||^2.\]

It is easy to verify the directed triangle inequality: for all $i, j, k \in V$, $d(i, k) + d(k, j) \geq d(i, j)$.

For subsets $S \subseteq V, T \subseteq V$, we also denote $d(S, T) := \min_{i \in S, j \in T} d(i, j)$, $d(i, S) = d(\{i\}, S)$ and $d(S, i) = d(S, \{i\})$.

**Interpretation.** In an SDP-relaxation for directed sparsest cut, vertex 0 is always chosen in the solution $S \subseteq V$. For $i \in S$, $v_i$ is set to $v_0$; for $i \in \overline{S} = V \setminus S$, $v_i$ is set to $-v_0$. Then, it can be checked that $d(i, j)$ is non-zero iff $i \in S$ and $j \in \overline{S}$, in which case $d(i, j) = 8||v_i||^2$.

**The other case.** For the other case when 0 is definitely excluded from the solution $S$, it suffices to change the definition $d(i, j) := ||v_i - v_j||^2 - ||v_i + v_0||^2 + ||v_j + v_0||^2$. For the rest of the paper, we just concentrate on the case that 0 is in the solution $S$. 

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We consider the following SDP relaxation (where \( \{v_i : i \in V\} \) are vectors) for the directed sparsest cut problem with product demands on an edge-weighted hypergraph \( H = (V, E, w) \) with vertex weights \( \omega : V \to \{1, 2, \ldots, \kappa\} \). We denote \( \mathcal{W} := \sum_{i \in V} \omega_i \).

\[
\text{SDP} \quad \min \frac{1}{2} \sum_{e \in E} w_e \cdot d_e \quad (2.1)
\]

\[
\text{s.t.} \quad d_e \geq d(i, j), \quad \forall e \in E, \forall (i, j) \in T_e \times H_e \\
\|v_i - v_j\|^2 \leq \|v_i - v_k\|^2 + \|v_k - v_j\|^2, \quad \forall i, j, k \in V \quad (2.2)
\]

\[
\sum_{(i,j) \in T_e} \omega_i \omega_j \|v_i - v_j\|^2 = 1, \quad \{i,j\} \in \binom{V}{2} \quad (2.3)
\]

\[
d_e \geq 0, \quad \forall e \in E. \quad (2.4)
\]

**SDP Relaxation.** To see that SDP is a relaxation of the directed sparsest cut problem, it suffices to show that any subset \( S \subseteq V \) induces a feasible solution with objective function \( \vartheta(S) \). We set \( v_0 \) to be a vector with \( \|v_0\|^2 = \frac{1}{4\omega(S) \cdot \omega(S)} \). For each \( i \in V \), we set \( v_i := v_0 \) if \( i \in S \), and \( v_i := -v_0 \) if \( i \notin S \). Then, the value of the corresponding objective is

\[
\frac{1}{2} \sum_{e \in E} w_e \cdot d_e = \frac{1}{2} \sum_{e \in E} w_e \cdot \max_{(i,j) \in T_e \times H_e} \{d(i, j)\} = \frac{1}{2} \sum_{e \in \vartheta^+(S)} w_e \cdot (\|v_0 + v_0\|^2 - \|v_0 - v_0\|^2 + \|-v_0 - v_0\|^2) = \frac{w(\vartheta^+(S))}{\omega(S) \cdot \omega(S)} = \vartheta(S).
\]

**Trace Bound.** We have \( \sum_{i \in V} \|v_i\|^2 \leq \frac{n}{4\omega(S) \cdot \omega(S)} \leq O\left(\frac{n^2}{\log^2 n}\right) \). Note that if \( S \) is balanced, then the upper bound can be improved to \( O\left(\frac{n}{\log n}\right) \).

**SDP Primal-Dual Approach [AK16].** Instead of solving the SDP directly, the SDP is used as a tool for finding an approximate solution. Given a candidate value \( \alpha \), the primal-dual approach either (i) finds a subset \( S \) such that \( \vartheta(S) \leq O\left(\sqrt{\log n}\right) \cdot \alpha \), or (ii) concludes that the optimal value of the SDP is at least \( \frac{\alpha}{2} \). Hence, binary search can be used to find an \( O\left(\sqrt{\log n}\right) \)-approximate solution. This approach is described in Section 3.

### 3 SDP Primal-Dual Approximation Framework

We use the primal-dual framework by [AK16]. However, instead of using it just as a blackbox, we tailor it specifically for our problem to have a cleaner description.

**Notation.** We use a bold capital letter \( A \in \mathbb{R}^{V \times V} \) to denote a symmetric matrix whose rows and columns are indexed by \( V \).

The sum of the diagonal entries of a square matrix \( A \) is denoted by the trace \( \text{Tr}(A) \). Given two matrices \( A \) and \( B \), let \( A \bullet B := \text{Tr}(A^T B) \), where \( A^T \) is the transpose of \( A \). We use \( 1 \in \mathbb{R}^V \) to denote the all-ones vector.

**Primal Solution.** We use \( X \succeq 0 \) to denote a positive semi-definite matrix that is associated with the vectors \( \{v_i\}_{i \in V} \) such that \( X(i, j) = \langle v_i, v_j \rangle \).

We rewrite SDP (2.1) to an equivalent form as follows.
In the original notation [Kal07, p.59], the claimed constraint is
\[ X \geq 0; \quad d_e \geq 0, \quad \forall e \in E. \] (3.5)

We define the notation used in the above formulation as follows:

- For \((i, j) \in V \times V\), \(A_{ij}\) is the unique symmetric matrix such that
  \[ A_{ij} \cdot X = d(i, j) = \|v_i - v_j\|^2 - \|v_i - v_0\|^2 + \|v_j - v_0\|^2. \]

Since we consider a minimization problem, we just use \(X \succeq 0\) to represent a primal solution, and automatically set \(d_e := \max\{0, \max_{(i, j) \in T_e \times H_e} A_{ij} \cdot X\}\) for all \(e \in E\). As we shall see, this implies that corresponding dual variable \(y_{ij}^e \in \mathbb{R}\) can be set to 0.

Moreover, we do not need the constraint \(A_{ij} \cdot X \geq 0\), because we already have \(d_e \geq 0\).

- The set \(T\) contains elements of the form \(\{i, k\} \in (V/2) \times V\), where \(i, j, k\) are distinct elements in \(V\).

They are used to specify the \(\ell_2^2\)-triangle inequality.

For \(p = \begin{bmatrix} i & k \\ j & \end{bmatrix}\), \(T_p\) is defined such that
\[ T_p \cdot X = \|v_i - v_j\|^2 + \|v_j - v_k\|^2 - \|v_i - v_k\|^2. \]

Observe that in [Kal07], a constraint is added for every path in the complete graph on \(V\). However, these extra constraints are simply linear combinations of the triangle inequalities, and so, are actually unnecessary.

- As above, \(K\) is defined such that
  \[ K \cdot X = \sum_{(i, j) \in (V/2)} \omega_i \omega_j \|v_i - v_j\|^2. \]

Observe that any \(X \succeq 0\) can be re-scaled such that \(K \cdot X = 1\).

- **Optional constraint.** In [Kal07], an additional constraint is added, which in our notation becomes:
  \[ -I \cdot X \succeq -\Theta\left(\frac{n}{\sqrt{m}}\right). \]

However, this holds only if the solution \(S\) is balanced. For general cut \(S\), we only have the weaker bound: \(-I \cdot X \succeq -\Theta\left(\frac{n^2}{m}\right)\). As we shall see in the proof of Lemma 3.3, adding this weaker bound is less useful than the above constraint \(K \cdot X = 1\).

The dual to SDP is as follows:

\[
\begin{align*}
\text{Dual} & \quad \max \quad z \\
\text{s.t.} & \quad -\sum_{e \in E} \sum_{(i, j) \in T_e \times H_e} y_{ij}^e A_{ij} + \sum_{p \in T} f_p T_p + zK \leq 0 & \quad (3.6) \\
& \quad \sum_{(i, j) \in T_e \times H_e} y_{ij}^e \leq \frac{w_e}{2}, \quad \forall e \in E, & \quad (3.7) \\
& \quad f_p \geq 0, \quad \forall p \in T, & \quad (3.8) \\
& \quad y_{ij}^e \geq 0, \quad \forall e \in E, \forall (i, j) \in T_e \times H_e. & \quad (3.9)
\end{align*}
\]
Observe that, if we add the optional constraint \(-I \bullet X \geq -b\) in the primal, then this will create a dual variable \(x \geq 0\), which causes an extra term \(-bx\) in the objective function and an extra term \(-xI\) on the left hand side of the constraint.

To use the primal-dual framework \([AK16]\), we give a tailor-made version of the ORACLE for our problem.

**Definition 3.1 (ORACLE for SDP)** Given \(\alpha > 0\), ORACLE(\(\alpha\)) has width \(\rho\) (which can depend on \(\alpha\)) if the following holds. Given a primal candidate solution \(X \geq 0\) (associated with vectors \(\{v_i\}_{i \in V}\)) such that \(K \bullet X = 1\), it outputs either

1. a subset \(S \subseteq V\) such that its sparsity \(\vartheta(S) \leq O(\sqrt{\log kn}) \cdot \alpha\), or
2. some dual variables \((z, (f_p \geq 0 : p \in \mathcal{T}))\), where all \(y_{ij}\)'s are implicitly 0, and a symmetric flow matrix \(F \in \mathbb{R}^{V \times V}\) such that all the following hold:

   - \(z \geq \alpha\)
   - \((\sum_{p \in \mathcal{T}} f_p T_p + zK) \bullet X \leq F \bullet X\)
   - For all feasible primal solution \(X^*, F \bullet X^* \leq \frac{1}{2} \sum_{e \in E} w_e d_e^*,\) where \(d_e^* := \max\{0, \max_{(i,j) \in E} A_{ij} \bullet X^*\}\).
   - For all \(x \in \text{span}\{1\}\), \(Fx = 0\).
   - The spectral norm \(\|\sum_{p \in \mathcal{T}} f_p T_p + zK - F\|\) is at most \(\rho\).

Using ORACLE in Definition 3.1, we give the primal-dual framework for one step of the binary search in Algorithm 1. As in \([AK16]\), the running for each iteration is dominated by the call to the ORACLE.

**Algorithm 1: Primal-Dual Approximation Algorithm for SDP**

```
Input: Candidate value \(\alpha > 0\); ORACLE(\(\alpha\)) with width \(\rho\)
1 \(T\) is chosen as in Lemma 3.3, \(\eta \leftarrow \sqrt{\frac{\ln n}{T}}\);
2 \(W^{(1)} \leftarrow I \in \mathbb{R}^{V \times V}\);
3 for \(t = 1, 2, \ldots, T\) do
4    \(X^{(t)} \leftarrow \frac{W^{(t)} \bullet X}{\text{tr}(W^{(t)})}\);
5    \(X^{(t)} \leftarrow \frac{\text{ORACLE}(\alpha)\text{ with } X^{(t)}}{\text{ORACLE}(\alpha)}\);
6    if ORACLE returns some \(S \subseteq V\) then
7        return \(S\) and terminate.
8    end
9    Otherwise, the ORACLE returns some dual solution \((z^{(t)}, (f^{(t)}_p : p \in \mathcal{T}))\) and matrix \(F^{(t)}\) as promised in Definition 3.1
10   \(M^{(t)} \leftarrow -\frac{1}{\rho} \left(\sum_{p \in \mathcal{T}} f^{(t)}_p T_p + z^{(t)} K - F^{(t)}\right)\);
11   \(W^{(t+1)} \leftarrow \exp\left(\eta \sum_{\tau=1}^{t} M^{(\tau)}\right)\);
12 end
13 if no subset \(S\) is returned yet then
14    report the optimal value is at least \(\frac{\alpha}{2}\).
15 end
```

The following result is proved in \([AK16]\) Corollary 3.2

**Fact 3.2 (Multiplicative Update)** Given any sequence of matrices \(M^{(1)}, M^{(2)}, \ldots, M^{(T)} \in \mathbb{R}^{n \times n}\) that all have spectral norm at most 1 and \(\eta \in (0, 1]\), let \(W^{(1)} = I\), \(W^{(t)} = \exp\left(\eta \sum_{\tau=1}^{t-1} M^{(\tau)}\right)\), for \(t = 2, \ldots, T\); let \(F^{(t)} = \frac{W^{(t)} \bullet X}{\text{tr}(W^{(t)})}\), for \(t = 1, 2, \ldots, T\). Then, we have
\[
\sum_{t=1}^{T} M^{(t)} \cdot P^{(t)} \leq \lambda_{\min} \left( \sum_{t=1}^{T} M^{(t)} \right) + \eta T + \frac{\ln n}{\eta},
\]

where \( \lambda_{\min}(\cdot) \) gives the minimum eigenvalue of a symmetric matrix.

**Lemma 3.3 (Correctness)** Set \( T := \lceil \frac{\ln n^2}{\alpha^2} \rceil \). Suppose that in Algorithm 7 the ORACLE never returns any subset \( S \) in any of the \( T \) iterations. Then, the optimal value of SDP is at least \( \frac{\Theta}{T} \).

**Proof:** The proof follows the same outline as [AK16 Theorem 4.6], but we need to be more careful, depending on whether we use the constraint on \( I \cdot X \).

For \( t = 1, \ldots, T \), we use \( M^{(t)} \) as in Algorithm 1 and apply Fact 3.2. Definition 3.1 guarantees that \( M^{(t)} \cdot P^{(t)} \geq 0 \), because \( X^{(t)} \) is positively scaled from \( P^{(t)} \).

Hence, by Fact 3.2 we have \( \lambda_{\min} \left( \sum_{t=1}^{T} M^{(t)} \right) + \eta T + \frac{\ln n}{\eta} \geq 0 \).

By setting \( \eta := \sqrt{\frac{\ln n}{T}} \) and \( Z := \frac{1}{T} \sum_{t=1}^{T} M^{(t)} = \frac{1}{T} \sum_{t=1}^{T} (P^{(t)} - \sum_{p \in T} f^{(t)}_p T_p - z^{(t)} K) \), this is equivalent to \( \lambda_{\min}(Z) \geq -2\rho \cdot \sqrt{\frac{\ln n}{T}} \).

As in [AK16], we would like to add some matrix from the primal constraint to \( Z \) to make the resulting matrix positive semi-definite.

A possible candidate is \( K \), whose eigenvalues are analyzed as follows.

First, observe that for all \( x \in \text{span}\{1\} \), it can be checked that \( K x = T_p x = 0 \), for all \( p \in T \). Furthermore, Definition 3.1 guarantees that \( P^{(t)} x = 0 \), for all \( t \). Hence, it follows that \( Z x = 0 \), which implies that any negative eigenvalue of \( Z \) must be due to the space orthogonal to \( \text{span}\{1\} \).

We next analyze the eigenvectors of \( K \) in this orthogonal space. Consider a unit vector \( u \perp \text{span}\{1\} \), i.e., \( \sum_{i \in V} u_i = 0 \) and \( \sum_{i \in V} u_i^2 = 1 \).

Then, \( u^\top K u = \frac{1}{2} \sum_{i \in V} \sum_{j \in V} \omega_i \omega_j (u_i - u_j)^2 = \frac{2n^2}{T} \left[ \left( \sum_{i \in V} \delta_i u_i^2 \right) - \left( \sum_{i \in V} \delta_i u_i \right)^2 \right] \), where \( \delta_i := \frac{\omega_i}{\sqrt{n}} \) can be interpreted as some probability mass function. Hence, this term can be interpreted as some variance.

Observe that the \( \kappa \)-skewness of the weights \( \omega \) implies that for all \( i \in V \), \( \delta_i \geq \frac{1}{\kappa n} \). Therefore, Lemma 3.4 below implies that \( u^\top K u \geq \frac{2n^2}{\kappa n} \).

Hence, by enforcing \( \epsilon \cdot \frac{2n^2}{\kappa n} \geq 2\epsilon \cdot \sqrt{\frac{\ln n}{T}} \), we have \( \lambda_{\min}(Z + \epsilon K) \geq 0 \).

Next, suppose \( X^* \) (with induced \( d^* \)) is an optimal primal solution to SDP. Then, Definition 3.1 implies that \( \frac{1}{T} \sum_{t=1}^{T} w_e d^*_e \geq \frac{1}{T} \sum_{t=1}^{T} P^{(t)} \cdot X^* \).

Since \( (Z + \epsilon K) \cdot X^* \geq 0 \), the optimal value is at least

\[
\frac{1}{T} \sum_{t=1}^{T} \left( \sum_{p \in T} f^{(t)}_p T_p \cdot X^* \right) + \frac{1}{T} \sum_{t=1}^{T} z^{(t)} K \cdot X^* - \epsilon K \cdot X^* \geq 0 + \frac{1}{T} \sum_{t=1}^{T} z^{(t)} \cdot 1 - \epsilon \cdot 1 \geq \alpha - \epsilon,
\]

where the last two inequalities come from the properties of primal feasible \( X^* \) and ORACLE, respectively.

Setting \( \epsilon = \frac{\alpha}{T} \) gives the result.

**Remark.** One can see that in the proof of Lemma 3.3 if one uses the weaker bound \( -I \cdot X \geq -\Theta(\alpha n^2) \). Then, the proof continues by choosing \( \nu = 2\rho \cdot \sqrt{\frac{\ln n}{T}} \), we have \( \lambda_{\min}(Z + \nu I) \geq 0 \).

Using the same argument, we conclude that the optimal value is at least
$$\alpha - \nu I \cdot X^* \geq \alpha - \nu \cdot \Theta(\frac{m^2}{n^2})$$.

Setting \( \frac{\alpha}{\nu} = \nu \cdot \Theta(\frac{m^2}{n^2}) \) gives \( T := \Theta\left(\frac{\nu^2 n^2 \ln n}{\alpha m^2}\right) \) in this case, which has an extra factor of \( O(n^2) \).

However, since we do not add any extra vectors in our primal domain, the width in our ORACLE in Theorem 5.1 has an extra \( O(n) \) factor compared to that in [Kal07], which brings back the \( O(n^2) \) factor we have saved earlier.

**Lemma 3.4 (Bounding the Variance)** For real numbers \( u_1, u_2, \ldots, u_n \) and \( \delta_0, \delta_1, \delta_2, \ldots, \delta_n \) such that \( \sum_{i=1}^{n} u_i = 0, \sum_{i=1}^{n} u_i^2 = 1, \sum_{i=1}^{n} \delta_i = 1 \) and \( \delta_i \geq \delta_0 > 0, \forall i \), we have \( \sum_{i=1}^{n} \delta_i u_i^2 - (\sum_{i=1}^{n} \delta_i u_i)^2 \geq \delta_0 \).

Moreover, we have \( \sum_{i=1}^{n} \delta_i u_i^2 - (\sum_{i=1}^{n} \delta_i u_i)^2 \leq \max \delta_i \).

**Proof:** Let \( u_1, \ldots, u_n \) be fixed and consider the function

$$g(\delta_1, \ldots, \delta_n) = \sum_{i=1}^{n} \delta_i u_i^2 - \left(\sum_{i=1}^{n} \delta_i u_i\right)^2$$

with domain \( \{(\delta_1, \ldots, \delta_n) | \sum_{i=1}^{n} \delta_i = 1, \delta_i \geq \delta_0, \forall i\} \).

We claim that the minimum can be obtained at some point where at most one \( \delta_i \) has value strictly greater than \( \delta_0 \). Indeed, suppose there are two variables, say \( \delta_1 \) and \( \delta_2 \), whose value is strictly greater than \( \delta_0 \). Consider \( h(x) = g(x, s - x, \delta_3, \ldots, \delta_n) \) where \( s = \delta_1 + \delta_2 \). Simplifying it, we know that the coefficient associated to \( x^2 \) in \( h \) is \(-((u_1 - u_2)^2) \leq 0 \), which means that we can shift either \( \delta_1 \) or \( \delta_2 \) to \( \delta_0 \) (and the other variable to \( s - \delta_0 \)) without increasing the value of \( g \).

Therefore, we only need to consider the case where there is at most one \( \delta_i > \delta_0 \). Without loss of generality, suppose \( \delta_2 = \delta_3 = \ldots = \delta_n = \delta_0 \) and thus \( \delta_1 = (1 - n \delta_0) + \delta_0 \). Then,

$$g(\delta_1, \ldots, \delta_n) = \delta_0 + (1 - n \delta_0) u_1^2 - ((1 - n \delta_0) u_1)^2 = \delta_0 + n \delta_0 (1 - n \delta_0) u_1^2 \geq \delta_0,$$

since \( \delta_0 \) cannot be greater than \( \frac{1}{n} \).  

**Corollary 3.5 (Non-zero Eigenvalues of \( K \))** All eigenvectors of \( K \) that are orthogonal to \( 1 \) have eigenvalues in the range \( [\frac{m^2}{n^2}, \frac{m n^2}{n}] \).

### 4 Hypergraph Flows and Demands

In order to facilitate the description of the ORACLE, we define some notation for flows and demands in hypergraphs.

**Definition 4.1 (Hypergraph Flow)** Given a directed hypergraph \( H = (V, E) \), a flow is defined as

\[ \bar{f} := (f_{ij}^e \geq 0 : e \in E, (i, j) \in T_e \times H_e) \].

The corresponding flow matrix is defined as

\[ F := \sum_{e \in E} \sum_{(i,j) \in T_e \times H_e} \bar{f}_{ij}^e \bar{A}_{ij}, \text{ where } \bar{A}_{ij} \text{ is defined in Section 3} \]

For \( i \in V \), the net amount of flow entering \( i \) is

\[ \sum_{e \in E} \sum_{(j',i) \in T_e \times H_e} \bar{f}_{ij'}^e - \sum_{(i,j) \in T_e \times H_e} \bar{f}_{ij}^e \].

A flow \( \bar{f} \) satisfies edge capacities \( c_e : E \to \mathbb{R}_+ \) if for all \( e \in E \),

\[ \sum_{(i,j) \in T_e \times H_e} \bar{f}_{ij} \leq c_e. \]

**Fact 4.2 (Constrained Flow)** Suppose a flow \( \bar{f} \) satisfies edge capacities \( c \). Then, for any primal solution \( X \succeq 0 \) (with induced \( \{d_e \geq 0 : e \in E\} \) as defined is Section 3), we have \( F \cdot X \leq \sum_{e \in E} c_e d_e \).

Next, we define the notion of demand, which is used to express the sources and the sinks of a flow later.
**Definition 4.3 (Demand)** Given a vertex set \( V \), a demand between ordered pairs in \( V \) is defined as
\[
\mathfrak{d} := (d_{ij} \geq 0 : (i, j) \in V \times V).
\]
The corresponding demand matrix is
\[
D := \sum_{(i,j) \in V \times V} d_{ij} A_{ij}.
\]

**Fact 4.4 (Spectral Norm of Demand Matrix \([AK16]\))** The matrix for demand \( \mathfrak{d} \) has spectral norm
\[
\|D\| \leq O(\sum_{ij} d_{ij}).
\]

**Definition 4.5 (Flow Decomposition)** Suppose \( p = (i_0, i_1, \ldots, i_k) \) is a directed path of length \( k \geq 2 \). Consider a flow of magnitude \( f \geq 0 \) along \( p \), which has the flow matrix
\[
F_p = \sum_{j=1}^{k-1} f_{p,j} T_{p,j} + f_{p} A_{i_0,i_k}.
\]
where \( p_j := \left[ \{i_0, i_{j+1}\}, i_j \right] \in T \), and the notation is defined in Section 3.

Then, the matrix can be expressed as
\[
F_p = \sum_{p \in T} f_{p} T_{p} + D, \text{ for some appropriate } f_{p}'s \text{ and demand matrix } D.
\]
Observe that the flow decomposition is not unique.

## 5 Implementation of ORACLE(\( \alpha \))

We give the implementation of the ORACLE as in Definition 3.1. For \( \alpha > 0 \), the input to the ORACLE(\( \alpha \)) is some \( X \succeq 0 \) such that \( K \cdot X = 1 \). By the standard Cholesky factorization, we also have the associated vectors \( (v_i : i \in V) \). Then, we have:
\[
\sum_{\{i,j\} \in (V^2)} \omega_i \omega_j \|v_i - v_j\|^2 = 1.
\]
Below is the main result of this section.

**Theorem 5.1** Given a candidate value \( \alpha > 0 \) and primal \( X \succeq 0 \) such that \( K \cdot X = 1 \), the ORACLE returns one of the following:

1. A subset \( S \) with directed sparsity \( \vartheta(S) = \frac{w(\vartheta^+(S))}{\omega(S) \omega(S)} = O(\sqrt{\log \kappa n}) \cdot \alpha \).
2. Dual variables \( (z, (f_p : p \in T)) \) and flow matrix \( F \) satisfying Definition 3.7.

   Moreover, the spectral norm satisfies
\[
\|\sum_p f_p T_p + z K - F\| \leq O(\alpha m^2 \sqrt{\log \kappa n}).
\]

The running time is \( \tilde{O}(n^{1.5} + (\kappa n)^2) \), where \( \kappa \) is the skewness of vertex weights, \( m = |E| \) and \( r = \max_{e \in E} (|T_e| + |H_e|) \).

Following Lemma 8 and Lemma 6 in [Kal07], two cases are analyzed, based on whether the vectors are concentrated around some vector.

### 5.1 Case 1: Vectors Concentrated Case

This is similar to [Kal07] Lemma 6.

For vertex \( i \) and radius \( r \), define \( B(i, r) := \{j \in V : \|v_i - v_j\|^2 \leq r^2\} \).

We consider the case that there exists some vertex \( i_0 \in V \) such that \( \omega(B(i_0, \sqrt{2m})) \geq \frac{m}{4} \). This can be verified in time \( O(n^2) \).
Lemma 5.2 (Case 1 of ORACLE) Suppose there exists some vertex \( i_0 \in V \) such that \( \omega(B(i_0, \frac{1}{\sqrt{8M}})) \geq \frac{\alpha}{4} \).

Then, there is an algorithm with running time \( O((rm)^{1.5}) \), where \( m = |E| \) and \( r = \max_{e \in E}(|T_e| + |H_e|) \) that outputs one of the following:

1. A subset \( S \) with directed sparsity \( \vartheta(S) = \frac{w(\partial^+(S))}{\omega(S)\omega(S)} = O(\alpha) \).

2. Dual variables \((z, (f_p : p \in T))\) and flow matrix \( F \) satisfying Definition 3.7.

Moreover, the spectral norm satisfies \( \|\sum_p f_p T_p + zK - F\| \leq O(\alpha M^2) \).

Proof:

Let \( L := B(i_0, \frac{1}{\sqrt{8M}}) \) and \( R = V \setminus L \). Following step 1 of the proof of [Kal07, Lemma 5], we denote \( \Delta(j, L) := \min_{i \in L} \|v_i - v_j\|^2 \) and obtain

\[
1 = \sum_{i,j \in L} \omega_i \omega_j \|v_i - v_j\|^2 \\
\leq \sum_{i,j \in L} \omega_i \omega_j (2 \|v_i - v_{i_0}\|^2 + 2 \|v_{i_0} - v_j\|^2) = 2 \sum_{i \in V} \omega_i (M - \omega_i) \|v_i - v_{i_0}\|^2 \\
\leq 2 \sum_{i \in V} \omega_i M (2 \Delta(i, L) + 2 \cdot \frac{1}{8M^2}) = 4M \sum_{i \in V} \omega_i (\Delta(i, L) + \frac{1}{8M^2}) .
\]

Thus, \( \sum_{j \in R} \omega_j \Delta(j, L) = \sum_{i \in V} \omega_i \Delta(i, L) \geq \frac{1}{2M} - \sum_{i \in V} \omega_i \frac{1}{8M^2} = \frac{1}{8M^2} \). Let \( \gamma = \frac{\omega(R)}{\omega(L)} \); note that \( \gamma \leq \frac{32M/4}{32M/4} = 3 \).

Now consider the two quantities \( Q_L := \sum_{i \in L} \gamma \omega_i \|v_0 - v_i\|^2 \) and \( Q_R := \sum_{j \in R} \omega_j \|v_0 - v_j\|^2 \). We first consider the case that \( Q_L \leq Q_R \).

Max-Flow Instance in Directed Hypergraph. We consider the following max-flow instance \( G \). Each directed edge \( e \) in the original hypergraph \( H = (V, E) \) has capacity \( c_e = \frac{\omega_p}{2} \).

Source. We add an extra source vertex \( s \), and edges \( \{(s, i) : i \in L\} \), each of which has capacity \( 8\gamma M \omega_i \alpha \).

Sink. We add a sink vertex \( t \), and edges \( \{(j, t) : j \in R\} \), each of which has capacity \( 8\gamma M \omega_j \alpha \).

A max-flow can be computed in \( G \), for instance, by using the reduction to directed normal graph in Fact 1.1.

Case A. Suppose the flow does not saturate all source (and sink) edges, i.e., the flow is less than \( 8\gamma M \omega(R) \cdot \alpha \).

Let \( S \) be the set of vertices in \( V \) that are reachable from \( s \) in the residual graph. Denote \( V_s := L \setminus S \) and \( V_t := R \cap S \). Observe that \( w(\partial^+(S)) \leq 8\gamma M (\omega(R) - \gamma \omega(V_s) - \omega(V_t)) \). As

\[
\max\{\omega(S), \omega(S)\} \geq \frac{M}{2}
\]

and

\[
\min\{\omega(S), \omega(S)\} \cdot \gamma \geq \min\{\omega(L \setminus V_s), \omega(R \setminus V_t)\} \cdot \gamma \geq \omega(R) - \gamma \omega(V_s) - \omega(V_t),
\]

the algorithm returns \( S \) with \( \vartheta(S) = \frac{w(\partial^+(S))}{\omega(S)\omega(S)} \leq O(\alpha) \), as required.

Case B. Suppose the max flow saturates all edges from \( s \) (and thus also saturates all edges going into \( t \)). The max flow induces a flow \( f \) in the original graph, by ignoring the newly added edges. Let \( F \) be the resulting flow matrix as in Definition 4.1 and consider the corresponding flow decomposition \( F := \sum_{p \in T} f_p T_p + D \) as in Definition 4.5, where each non-zero demand \( \delta_{ij} \) in \( D \) must be from \((i, j) \in L \times R \).

The algorithm returns dual variable \((z = \alpha, (f_p : p \in T))\). It suffices to check that the conditions in Definition 3.1 are satisfied.
Obviously, $z \geq \alpha$. Moreover, since $F$ respects edge capacities, by Fact 4.2 $F \cdot X^* \leq \frac{1}{2} \sum_{e \in E} w_e d_e^*$, for any $X^* \succeq 0$ with associated directed distance $d^*$. From Definition 4.1 $F$ is a linear combination of $A_{ij}$, each of which has 1 as a 0-eigenvector.

We next verify the condition $(\sum_{p \in T} f_p T_p + zK) \cdot X \leq F \cdot X$. Since the candidate matrix $X$ satisfies $K \cdot X = 1$, this reduces to $D \cdot X = \sum_{i,j \in R} d_{ij} (i, j) \geq \alpha$.

The saturation condition implies that for each $i \in L$, $\sum_{j \in R} d_{ij} = 8 \gamma \omega_i \alpha$; similarly, for each $j \in R$, $\sum_{i \in L} d_{ij} = 8 \omega_j \alpha$.

Therefore, we have

$$\sum_{(i,j) \in L \times R} d_{ij} \|v_i - v_j\|^2 \geq \sum_{j \in R} 8 \omega_j \alpha \cdot \Delta(j, L) \geq \alpha,$$

which implies $\sum_{(i,j) \in L \times R} d_{ij} (d(i, j) + d(j, i)) \geq 2 \alpha$, since $d(i, j) + d(j, i) = 2 \|v_i - v_j\|^2 \forall i, j \in V$.

On the other hand,

$$\sum_{(i,j) \in L \times R} d_{ij} (d(i, j) - d(j, i)) = \sum_{(i,j) \in L \times R} 2d_{ij} \|v_0 - v_j\|^2 - \sum_{(i,j) \in L \times R} 2d_{ij} \|v_0 - v_i\|^2$$

$$= \sum_{j \in R} 2 \cdot 8 \omega_j \alpha \|v_0 - v_j\|^2 - \sum_{i \in L} 2 \cdot 8 \gamma \omega_i \alpha \|v_0 - v_i\|^2$$

$$= 16 \alpha \omega (Q_R - Q_L) \geq 0,$$

by the assumption $Q_L \leq Q_R$. Thus, we conclude that $\sum_{(i,j) \in L \times R} d_{ij} d(i, j) \geq \alpha$, as required.

**Bound on Spectral Norm.** Observe that

$$\left\| \sum_{p \in T} f_p T_p + zK - F \right\| = \|\alpha K - D\| \leq \|\alpha\| K + \|D\| \leq \frac{\alpha \omega \omega^2}{\eta} + O(\alpha \omega^2),$$

where the bound for $\|K\|$ comes from Corollary 3.5 and $\|D\| \leq O(\sum_{i,j} d_{ij})$ comes from Fact 4.4. Assuming $\kappa \leq n$, the spectral norm is at least $O(\alpha \omega^2)$, as required.

If $Q_L = \sum_{i \in L} \gamma \omega_i \|v_0 - v_i\|^2 > Q_R = \sum_{j \in R} \omega_j \|v_0 - v_j\|^2$, then we just reverse the directions of all edges touching $s$ or $t$ in $G$ and compute the max-flow from $t$ to $s$. The argument is analogous.

**Running time.** The most expensive step, a max-flow computation in a directed (normal) graph with $O(\alpha \omega^2)$ edges, which can be done in $O((\alpha \omega m)^{1.5})$ time using the algorithm of [GR98]. Hence, the running time is $O((\alpha \omega m)^{1.5})$.

### 5.2 Case 2: Vectors Well-Spread Case

This case is similar to [Kal07] Lemma 6. We will use [Kal07] Lemma 14 and [Kal07] Lemma 7, which we state below without proof.

**Lemma 5.3 (Lemma 14 of [Kal07]).** Suppose $|V| = n$ and each vertex $i \in V$ is associated with a vector $v_i \in \mathbb{R}^n$ such that $\|v_i\|^2 \leq 1$, $\forall i \in V$ and $\sum_{(i,j) \in E} \langle v_i - v_j, u \rangle \geq a \eta n^2$ for some $\alpha > 0$. Then for at least $\frac{\alpha \eta n^2}{2}$ fraction of directions $u$, there exist $S, T \subseteq V$, each of size at least $\frac{\alpha \eta n^2}{2}$, such that $\langle v_j - v_i, u \rangle \geq \frac{a \eta}{48 \sqrt{n}}$, $\forall (i, j) \in S \times T$.

Vertex pair $(i, j)$ is said to be a $(\eta, \sigma)$-stretched pair along a unit vector (also called a direction) $u$ if $\|v_i - v_j\|^2 \leq \frac{\eta}{\sqrt{\log n}}$ and $\langle v_j - v_i, u \rangle \geq \frac{\sigma}{\sqrt{n}}$.

**Lemma 5.4 (Lemma 7 of [Kal07]).** Let $v_1, v_2, \ldots, v_n$ be vectors of length at most 1 such that for a $\gamma$ fraction of directions $u$, there exists a matching of $(\eta, \sigma)$-stretched pairs along $u$ of size at least $\gamma n$. Let $\mu > 0$ be a given constant. Then there is a randomized algorithm which, in time $O(n^2 + \frac{1}{\mu} \log n^{1+\mu})$, finds $k$ vertex-disjoint
Lemma 5.5 (Case 2 of Oracle) Suppose after the pre-processing step, we have obtained subset $S \subseteq V$ and $i_0 \in S$ as described above.

Then, there is an $O((rm)^{1.5} + (\kappa n)^2)$-time algorithm that outputs one of the following:

1. A subset $S'$ with directed sparsity $\vartheta(S') = O(\sqrt{\log km}) \cdot \alpha$.

2. Dual variables $(z, (f_p : p \in T))$ and flow matrix $F$ satisfying Definition 8.7.

Moreover, the spectral norm satisfies $\left\| \sum_p f_p T_p + z K - F \right\| \leq O(\alpha m^2 \sqrt{\log km})$.

Proof: Define $\tilde{v}_i := \frac{m}{3}(v_i - v_{i_0})$ for each $i \in V$. Thus we have:

- $\omega(S) \geq \Omega(m)$,
- $\tilde{v}_{i_0} = 0$,
- $\|\tilde{v}_i\|^2 \leq 1, \forall i \in S$, and
- $\sum_{i,j \in V} \omega_{i,j} \|v_i - v_j\|^2 \geq \Omega(m^2)$.

We treat vertex $i$ with weight $\omega_i \in \{1, 2, \ldots, \kappa\}$ as $\omega_i$ identical copies, each of which has unit weight. They form a multiset $\tilde{V}$ with $|\tilde{V}| = m$. Applying Lemma 5.3 on $\tilde{V}$, we know that with constant possibility, for some appropriate constants $c, \sigma > 0$, we can obtain a unit vector $u$ and subsets $L_0, R_0 \subseteq S$ each of weight at least $c m$ such that $(\tilde{v}_j - \tilde{v}_i, u) \geq \frac{c}{\sqrt{m}} \forall i \in L_0, j \in R_0$. 

Pre-processing. In this case, for all $i \in V$, $\omega(B(i, \frac{1}{\sqrt{8m}})) < \frac{m}{4}$.

First, we claim that there is a vertex $i_0$ such that $\omega(B(i_0, 3/2m)) \geq m/2$, as otherwise for all vertices $i \in V$, there are vertices $j \in V$ with total weight greater than $2m/2$ such that $\|v_i - v_j\|^2 > 9/2m^2$, which implies that

$$\sum_{i,j \in V} \omega_{i,j} \|v_i - v_j\|^2 = \frac{1}{2} \sum_{i \in V} \sum_{j \neq i} \omega_{i,j} \|v_i - v_j\|^2 > \frac{1}{2} \cdot \frac{m}{2} \cdot \frac{m}{2} \cdot \frac{9}{2m^2} > 1,$$

a contradiction. Let $S := B(i_0, 3/2m)$. For every $i \in S$, since $\omega(B(i, \frac{1}{\sqrt{8m}})) < \frac{m}{4}$, we conclude that $\omega(S \setminus B(i, \frac{1}{\sqrt{8m}})) > \frac{m}{2} - \frac{m}{4} = \frac{m}{4}$ and thus

$$\sum_{i,j \in S} \omega_{i,j} \|v_i - v_j\|^2 > \frac{1}{2} \cdot \frac{m}{2} \cdot \frac{m}{4} \cdot \frac{1}{8m^2} = \Omega(1).$$

Therefore, we obtain a subset $S$ and a vertex $i_0 \in S$ such that

- $\omega(S) \geq \Omega(m)$;
- $\|v_i_0 - v_i\|^2 \leq \frac{9}{m}$, $\forall i \in S$;
- $\sum_{i,j \in \binom{S}{2}} \omega_{i,j} \|v_i - v_j\|^2 \geq \Omega(1)$. 


Let \( r \) be the median distance from \( \hat{v}_0 \) to the vectors \( \{ \hat{v}_i : i \in L_0 \} \), concerning the weight distribution \( \{ \omega_i \} \).

Define \( L_0^+ := \{ i \in L_0 : \| \hat{v}_i - \hat{v}_0 \| \geq r \} \), \( L_0^- := \{ i \in L_0 : \| \hat{v}_i - \hat{v}_0 \| \leq r \} \), \( R_0^+ := \{ i \in R_0 : \| \hat{v}_i - \hat{v}_0 \| \geq r \} \) and \( R_0^- := \{ i \in R_0 : \| \hat{v}_i - \hat{v}_0 \| \leq r \} \).

If \( \omega(R_0^+) \geq \omega(R_0^-) \), set \( L = L_0^- \) and \( R = R_0^+ \); else set \( L = R_0^- \) and \( R = L_0^+ \). This guarantees that both \( L \) and \( R \) have weight at least \( \frac{c_\beta}{2} \) and \( \forall (i, j) \in L \times R \),
\[
\tilde{d}(i, j) := \| \hat{v}_i - \hat{v}_j \|^2 - \| \hat{v}_i - \hat{v}_0 \|^2 + \| \hat{v}_j - \hat{v}_0 \|^2 \geq \| \hat{v}_i - \hat{v}_j \|^2.
\]

**Max-Flow in Directed Hypergraph.** Define \( \beta := \frac{39C_{\text{max}}}{4} \), where \( C = C(\gamma, \frac{1}{\sqrt{d}}, \sigma) \) and \( s = s(\gamma, \frac{1}{\sqrt{d}}, \sigma) \) are constants determined from Lemma 5.4. We add a source \( s \) and connect an edge from \( s \) to each \( i \in L \) with capacity \( \beta \| \log \| \sqrt{\omega_i} \| \alpha \) and similarly, add a sink \( t \) and connect an edge from each \( j \in R \) to \( t \) with capacity \( \beta \| \log \| \sqrt{\omega_j} \| \alpha \). Each edge \( e \) in the original hypergraph has capacity \( c_e = \frac{C_{\mu}}{2} \). Again, a max-flow can be computed in \( \tilde{O}((rn)^{1.5}) \) time.

We consider cases based on the total value flow out of the source. By removing the extra edges, we can form a flow matrix \( F \) in the original graph, and consider the flow decomposition \( F = \sum_{p \in T} f_p T_p + D \) as in Definition 4.8. Observe that the value of the flow is \( \sum_{i \in L, j \in R} \delta_{ij} \), where the demands \( \delta_{ij} \) are determined by the demand matrix \( D \).

**Case A.** Suppose the total flow is less than \( \frac{c_{\beta \omega}}{4} \sqrt{\| \log \( \| \sqrt{\omega_i} \| \alpha \) } \), and let \( S' \subseteq V \) be the corresponding induced min-cut connecting to the source.

The total weight of the saturated edges touching the source (or sink) is at most \( \frac{c_{\beta \omega}}{4} \) so there are at least \( \Omega(\| \omega \|) \) weight of vertices on both sides of the cut.

Thus, the algorithm returns \( S' \), where directed sparsity \( \hat{\delta}(S') \leq O(\sqrt{\| \log \| \sqrt{\omega_i} \|} \cdot \alpha \leq O(\sqrt{\log(\| \omega \|)}) \cdot \alpha \), as required.

**Case B.** Suppose \( D \bullet X \geq \alpha \). In this case, the algorithm returns the dual solution \( (z = \alpha, (f_p : p \in T)) \) and the flow matrix \( F \), which satisfies Definition 5.1, as in the proof of Lemma 5.2.

Again, the spectral norm is bounded by \( \| z \|_{K - D} \| \leq O(\alpha \| \omega \|^2 \sqrt{\log \| \omega \|}) \), because it is dominated by \( \| D \| \leq O(\sum_{i \in L, j \in R} \delta_{ij}) \leq O(\alpha \| \omega \|^2 \sqrt{\log \| \omega \|}) \).

**Case C.** In the remaining case, the total flow is at least \( \frac{c_{\beta \omega}}{4} \sqrt{\| \log \| \sqrt{\omega_i} \| \alpha \} \) but \( D \bullet X \leq \alpha \). We try to find a path along which the path inequality is drastically violated.

Observe that \( \sum_{i \in L, j \in R} \delta_{ij} d(i, j) = D \bullet X \leq \alpha \). On the other hand, the value of the flow is \( \sum_{i \in L, j \in R} \delta_{ij} \geq \frac{c_{\beta \omega}}{4} \sqrt{\| \log \| \sqrt{\omega_i} \| \alpha} \). Hence, by Markov’s Inequality, among the total flow \( \sum_{i \in L, j \in R} \delta_{ij} \), at least half of it is routed between pairs \( (i, j) \) such that \( \frac{8}{c_{\beta \omega}} \sqrt{\| \log \| \sqrt{\omega_i} \|} \geq d(i, j) \geq \| v_i - v_j \|^2 \), where the last inequality follows from the choice of \( L \) and \( R \). Equivalently, at least half of the total flow is routed between pairs \( (i, j) \) such that \( \| v_i - v_j \|^2 \leq \frac{8}{9c_{\beta \omega}} \sqrt{\| \log \| \sqrt{\omega_i} \|} \).

Following the arguments in [Kal07], Lemma 15], we can show that there is a matching in \( \tilde{V} \) of size at least \( \frac{c_{\omega}}{4} \), such that each matched pair is copied from some \( (i, j) \in L \times R \) with \( \| v_i - v_j \|^2 \leq \frac{8}{9c_{\beta \omega}} \sqrt{\| \log \| \sqrt{\omega_i} \|} \).

Let \( \gamma' \) be the fraction of directions \( u \) such that there is such a matching. Note that \( \gamma' \) is determined once all the vectors are given, while we do not need to compute its exact value. If \( \gamma' \geq \gamma/2 \), then after trying \( O(\log n) \) random directions we will, with high probability, end up in Case A or B, or successfully find a matching. If \( \gamma' < \gamma/2 \), then each trial will result in Case A or B with probability at least \( \gamma - \gamma' \geq \gamma/2 \), so trying \( O(\log n) \) random directions will make ORACLE end up in Case A or B with high probability.

Now assume that the matching mentioned above exists. The choice of \( \beta \) guarantees that \( \eta = \frac{8}{9c_{\beta \omega}} = \frac{\mu s}{4c} \). We apply Lemma 5.4 on \( \omega_i \) copies of \( \hat{v}_i \), \( \forall i \in V \), with parameter \( \mu = 1 \) and \( k = 1 \). So in \( \tilde{O}((\| \omega \|)^2) \) time,
we can find a path $q = (i_0, i_1, \ldots, i_k)$ whose $\ell_2$-path inequality is violated by at least $s$. Specifically, define $T_q := \{ \{i_0, i_j+1\} : 1 \leq j \leq k-1 \}$. Then, the violation condition is $\sum_{p \in T_q} T_p \cdot X \leq -\frac{9s}{W_2}$.

Next, we define the dual solution returned. We set $z = \alpha$ and $F = 0$. For each $p \in T_q$, we set $f_p := \frac{W_2}{9s} \alpha$; set other $f_p$’s to 0.

We next check that the conditions in Definition 3.1 are satisfied. Most of them are straightforward. In particular,

$$\left( \sum_{p \in T} f_p T_p + zK \right) \cdot X = \frac{W_2}{9s} \sum_{p \in T_q} T_p \cdot X + \alpha \leq \frac{W_2}{9s} \alpha \cdot (\frac{9s}{W_2}) + \alpha = 0.$$

**Bound on Spectral Norm.** Observe that for the path $q$, the degree of each vertex is at most 2 in the path. Hence, $\|\sum_{p \in T_q} T_p\| \leq O(1)$. Therefore, we have

$$\left\| \sum_{p \in T} f_p T_p + zK - F \right\| \leq \frac{W_2}{9s} \alpha \cdot O(1) + \alpha \cdot \frac{\kappa W_2}{n} \leq O(\alpha W_2),$$

assuming $\kappa \leq n$.

**Analysis of Running Time.** The max-flow computation takes $\tilde{O}(\sqrt{rm})$ time. Running the algorithm in Lemma 5.4 takes $\tilde{O}((\kappa n)^2)$ time, since we only need to find 1 violating path, instead of $\Theta(\sqrt{n \log n})$ as indicated in [Kal07]. Hence, the running time is $\tilde{O}(\sqrt{rm} + (\kappa n)^2)$.

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