TOPOLOGY OF THE STANDARD MODEL I: FERMIONS

STEVE GERSTEN

Abstract. The Harari-Shupe model for fermions is extended to a topological model which contains an explanation for the observed fact that there are only three generations of fermions. Topological explanations are given for β-decay and for proton decay predicted in supersymmetry and string theories. An explanation is given for the observed fact that the three generations of fermions have such similar properties. The concept of “color” is incorporated into the model in a topologically meaningful way.

Preface to help non-grouptheorists get a start in reading this paper

Since some readers may not be familiar with the term van Kampen diagrams, I can refer them to an excellent article in Wikipedia; I’ll give a brief explanation shortly. The model I propose for particles of SM (standard model of particle physics) makes essential use of them, but the idea can be understood at a non-technical level as follows.

The model for particles is like Buckminster Fuller’s geodesic dome. The particles are the skin spanning the skeletal frame and the preons [Li] are the segments of that frame. So particles are 2-dimensional and preons are 1-dimensional. It makes no sense to say a particle (like a quark or electron) is composed of preons, in the way a sack of gum drops is composed of pieces of candy. A particle is the surface spanned by the 3 segments of Fuller’s frame that bound it. These segments

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I wish to thank Domingo Toledo for discussions. I am very grateful to Yong-Shi Wu for explaining the physics underlying “color”. Wikipedia articles on the standard model and related topics were useful and my thanks go to their anonymous authors .

1There is something new for grouptheorists here as well. In the word problem all that matters is the minimum number of 2-cells in a van Kampen diagram filling a relation R of a presentation, called the area of R. The diagram itself is irrelevant for the word problem. What is new in these articles is that the van Kampen diagrams are given an independent physical meaning. Thus both the proton and the positron are fillings of the word $aaa$ in the presentation $P$ to be defined shortly. But they have very different topological and physical properties.

2http://en.wikipedia.org/wiki/Van_Kampen_diagram
have names, $a$ or $b$, and orientations, that is, arrows. It will turn out that, for all particles in our theory, the arrows on one spanning triangle all point in the same direction; this is a deep fact and the explanation will be delayed to the sequel paper on bosons, but it is connected with the intrinsic spin of a particle. Particles may be detected, albeit indirectly, as in the case of quarks, which are not directly observable (each quark has a color and only “colorless” combinations of them (§4 below) are observable), but the segments of the skeleton may never be seen, but only inferred by the characteristics of surfaces spanning them.

I want to state from the beginning that I have only changed slightly the interpretation of the basic idea of [Ha] and [Sh], but not the idea itself. By interpreting their preons as the framework (or skeleton) rather than the content of particles, I can solve the problems that these papers pose concerning the standard model. It’s the paper I would have expected either of these authors to write had they been aware of the (admittedly arcane) field of combinatorial group theory. Thus my goals in this article are only to go as far as these authors could have gone in 1979 with the reinterpretation of “preons”. In the sequel article, I shall take the idea a step further by incorporating spin.

Now for a quick definition of van Kampen diagrams. It is the primary tool of combinatorial group theory (also known as geometric group theory). This branch of group theory is the study by geometry and topology of group presentations. A group presentation consists of generators and defining relations. A van Kampen diagram is a geometric way of representing relations. For more detail and for some superb drawings of van Kampen diagrams, consult the Wikipedia article.

An example of a presentation is that whose generators are all the edges of Fuller’s geodesic dome and whose defining relations are those determined by the set of triangles of the dome. We could call the group presented Buckminster Fuller’s group; it has not been studied to my knowledge. One van Kampen diagram is the dome itself.

The only presentation we are concerned with in these articles is $P = \langle a, b; aaa, bbb, aab, abb \rangle$. The group presented by $P$ is the cyclic group of order 3. Two van Kampen diagrams in $P$ may be seen in Figures 1 and 2 of this article. The second represents a proton and the three curvilinear triangles are, reading from left to right, the up, down, and up quarks. These two diagrams are of particular interest because they may be considered “spherical diagrams” by appending a face at infinity. So, with the exterior of the diagram compactified by adding a point at infinity, it becomes a sphere divided into four curvilinear triangles. These two

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3Figure 2 of this article is an example of a van Kampen diagram that represents a particle, the proton in this case, with the labels $a, b$ on the edges.

4The usual interpretation of the Harari-Shupe model is like the skeleton of a fish, whereas mine is like that of a lobster. Theirs has the skeleton on the inside, whereas I’ve moved it to the outside.

5The terminology “preon” is used in [Li] but does not occur in [Ha] nor in [Sh], which use the terms “rishon” and “quip” respectively.

6http://carlacapeto.files.wordpress.com/2010/11/buckminster-fuller-dome.jpg
spherical diagrams play a central role in characterizing SM in §5 below.\footnote{I have adopted Domingo Toledo’s suggestions to revise an earlier attempt at this preface in order to make it less technical.}

I cannot emphasize strongly enough that what is \textbf{not} being attempted is to calculate masses of particles nor to compute the fundamental constants of nature. The topological approach cannot and does not attempt to do this.

All that topology can do is to establish limits of what is possible, but it cannot fill in the details. For example my theory tells us that there is no fourth generation of fermions, but it cannot fill in the details about the masses of the three generations of fermions that exist. Only a metric theory can attempt to do this, and the mere beginnings of a metric theory are sketched at the end of the second paper on bosons. Do not expect me to calculate the value of “$g$” anytime soon.

Professor Wu suggested that I should emphasize also that the complex $K$ and its universal cover $\tilde{K}$, in which all my constructions take place, do not lie in 3-space (or even 4-space), but should be considered in a larger dimensional space, possibly in the fibre of a fibre bundle over Minkowski space. He also pointed out that the discreteness of the complex $K$ (as opposed to a continuous geometry) was a good feature that might sweeten the pill of higher dimensions. Those physicists who accept higher dimensional spaces as the venue of physics will appreciate this while those who reject them will reject this article whatever I do to try to reach them.

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Disclaimer.

§0. Introduction

In this article we propose a topological model for SM, the standard model of particle physics. A particle will turn out to be represented by a class of colored
van Kampen diagrams in the universal cover \( \tilde{K} \) of a complex \( K \) we construct in §3 (diagrams related by diamond moves represent the same particle; the concept of color is introduced in §4 below). The boundary label of a diagram is a conjugacy class \( \mathcal{c} \) of a single element in \( F = F(a, b) \), the free group with free basis \( \{a, b\} \), and is thus determined by the particle. Its antiparticle is represented by that diagram with the opposite orientation, and hence its boundary label is \( \overline{\mathcal{c}} \), the conjugacy class of inverses of elements of \( \mathcal{c} \). If a particle is its own antiparticle (as is the case for the photon, gluons, and the \( Z \)-boson), then \( \mathcal{c} = \overline{\mathcal{c}} \), which implies that \( \mathcal{c} \) consists only of the neutral element 1 of \( F \). Not all diagrams represent observable particles; the observability criterion is given in §4 below. At the end of §3 we illustrate the notions with van Kampen diagrams representing the proton, neutron, and their antiparticles without indicating the colors that must be attached to the faces.

The notions are unified in §5 to give a topological formulation for SM, including predictions for all particles, observable or not, which can exist. The emphasis in this paper is on fermions, which are the easiest to characterize, but the model applies to bosons as well. The deduction of the various species of bosons will be left to a future paper, but we have included a diagram for a Higgs boson, since people I have spoken to about this article have requested to see it and it is in the news because of its discovery by the LHC at CERN last year.

Finally, in §6, a purely topological explanation is given based on our model for the neutrino deficit, the fact that the neutrino flux from the sun observed in the laboratory is one third of the rate predicted by theory.

The correspondence between particles and van Kampen diagrams is the basic idea of this article, which is an expansion of ideas in papers of Harari [Ha] and Shupe [Sh] and which have been recently revived in an expository article [Li]. Viewing particles in this way leads to the complete model in §5, including answers to several of the questions raised in these papers. Among these are why there are only three generations of fermion, why the generations have such similar properties, and where the prediction of new fundamental particles stops.

A general reference for the algebraic topology used is [AH] and a reference for notions of combinatorial group theory is [BRS]. Other references to web articles will be provided as needed in footnotes.

§1. Review of Harari-Shupe model

Harari [Ha] and Shupe [Sh] have proposed a model wherein quarks and electrons are composite particles, composed each of three fundamental particles called “preons” by Lincoln in a recent article [Li]. In my notation the fundamental particles are \( a \) and \( b \) and their antiparticles are \( \overline{a} \) and \( \overline{b} \), respectively.\(^8\) The first generation of fermions consists of the electron \( e \), up quark \( u \), down quark \( d \), and electron neutrino \( \nu_e \); these are \( \overline{aa\overline{a}}, aab, \overline{ab\overline{b}} \) and \( bbb \), and their antiparticles are \( aaa, \overline{a\overline{a}b}, abb \), and \( \overline{b\overline{b}} \), respectively. Then the proton \( p \) is consists of two up and one down quark,

\(^8\)Both Shupe and Lincoln call \( a, \overline{a}, b, \overline{b} \) by +, −, 0, 0 respectively, whereas Harari calls them \( T, \overline{T}, V, \overline{V} \).
or \( p = aab + aab + \bar{ab} \), whereas the neutron \( n \) is two down and one up quark, or \( n = \bar{abb} + \bar{abb} + aab \). When one assigns an electric charge of \( +\frac{1}{3} \) to \( a \) and 0 to \( b \) and negatives of these numbers to the antiparticles, then the charges add up to the correct numbers 1 for the proton and 0 for the neutron.

There are two distinguished decay processes identified by Harari, \( d \rightarrow u + e + \bar{\nu}_e \) and \( u + u \rightarrow \bar{d} + e^+ \), where \( e^+ \) is the positron, antiparticle to the electron, and \( \bar{\nu}_e \) is the electron antineutrino. In my notation these are \( \bar{abb} \rightarrow aab + \bar{aaa} + \bar{bbb} \) and \( aab + aab \rightarrow abb + aab \). The first of these processes underlies \( \beta \)-decay or \( n \rightarrow p + e + \nu_e \) and the second underlies the decay of a proton which is predicted by supersymmetry and string theories. A hypothesis made is that the net number of \( a \)'s and of \( b \)'s (that is, the number of \( a \)'s minus the number of \( \bar{a} \)'s and similarly for the \( b \)'s) on one side of the reaction must equal the net number individually on the other side. So there is conservation of the net preon number of \( a \)'s and of \( b \)'s.

It should be emphasized that [Ha] and [Sh] were published in 1979 and there has been no confirmation of the prediction that these fermions are composite particles to date. In addition, there are two further generations of fermions identified in the laboratory, where the second generation is the muon, charm and strange quarks, and muon neutrino, and the third generation is the tau particle, top and bottom quarks, and tau neutrino. These appear to have similar properties to the first generation except for significantly higher masses. One of the most important open problems in particle physics to date has been to explain why there appear to be only three generations of fermions and why the three generations so closely resemble each other.

The reason this thirty year old puzzle is of interest today, and the reason [Li] was written, is that the large hadron collider LHC has come into operation and has discovered the existence of what appears to be a Higgs boson, the particle predicted over 40 years ago which is necessary to complete the “standard model” SM of particle physics. When the LHC is upgraded next year, it is hoped that other puzzles will be similarly unraveled, among which is the composite nature of fermions.

§2. Conjugacy classes in a free group

There is a product among conjugacy classes \( c, d \) in a group whose result \( cd \) is the set of products of elements of the two sets. The product is associative and commutative. For each conjugacy class \( c \) there is the conjugacy class \( \bar{c} \) which consists of inverses of elements of \( c \).

We shall consider conjugacy classes of single elements in a free group \( F \) with free basis \( \{x_1, x_2, \ldots, x_n\} \). Each such conjugacy class has a representative which is a cyclically reduced word in the free basis. Denote the conjugacy class of \( w \in F \) by \([w]\). Given a set of conjugacy classes of words \([w_1], [w_2], \ldots, [w_m]\), we can form the CW complex \( K \) whose 1-skeleton is a bouquet of circles corresponding to the generators \( x_i \) of \( F \) and whose 2-cells \( e_{[w_j]} \) have attaching maps the words \( w_j \) in the 1-skeleton. We make the convention that we attach only one 2-cell for the pair
of conjugacy classes \([w], [\bar{w}]\) and that the 2-chain they determine satisfies thus \(e_{[w]} = -e_{[\bar{w}]}\).\(^9\) Note that 2-cycles \(Z_2(K)\) (with integer coefficients) are integer linear combinations of the \(e_{[w_j]}\) such that the net algebraic sum of each of the \(x_i\) in the attaching maps is zero (occurrences of \(x_i\) count as \(+1\) and occurrences of \(\bar{x}_i = x_i^{-1}\) count as \(-1\)). Since each conjugacy class \([c]\) determines a homology class in \(H_1(K)\), this condition is equivalent to the assertion that the sum of the homology classes determined by the \([w_i]\) is zero.

\section{2-cycles in the 2-complex \(K\)}

A klepton is defined as a conjugacy class of one of the four words \(aaa, aab, abb, bbb\) and their inverses in the free group \(F = F(a, b)\) with free basis \(\{a, b\}\). From now on let \(K\) be the 2-complex constructed in the preceding paragraph with 2-cells \(e_\kappa\) corresponding to the four conjugacy classes \(\kappa = [aaa], [aab], [abb], [bbb]\) in \(F\). Note that \(e_\kappa = -e_\kappa\) in the chain group \(C_2(K)\) for each klepton \(\kappa\).

We require a reaction among kleptons \(\kappa_1, \kappa_2, \ldots, \kappa_m\) and \(\lambda_1, \ldots, \lambda_m\), denoted by \(\sum \kappa_i \to \sum \lambda_j\), to conserve algebraic sum of the number of \(a\)'s and of \(b\)'s. It follows that \(\sum_i e_{\kappa_i} + \sum_j e_{\lambda_j}\) is a 2-cycle in \(K\). From now on we identify the 2-cell \(e_\kappa\) with the fermion corresponding to \(\kappa\).

A key observation is the fact, proved by an easy computation, that \(K\) has fundamental group \(\mathbb{Z}/3\mathbb{Z}\), the cyclic group of order 3. It follows that each of the 2-cells \(e_\kappa\) of \(K\) has 3 lifts to the universal cover \(\tilde{K}\) of \(K\).\(^10\) This gives the 12 fermions of the standard model. The antiparticles arise from the observation that each 2-cell \(e_\kappa\) is a map of the disc into \(K\). So the mapping with the opposite orientation, corresponding to the chain \(e_\kappa = -e_\kappa\) corresponds to the antiparticle \(e_\kappa\) of the fermion \(e_\kappa\).

\textbf{Scholium.} There are exactly 3 generations of fermions in the model \(\tilde{K}\) for the standard model, corresponding to the lifts of the 2-cells of \(K\). These are the 24 basic fermions of the standard model. \(\Box\)

\textbf{Proposition.} Every 2-cycle of \(K\) is spherical.

This means that the Hurewicz map \(\pi_2(K) \to H_2(K) = Z_2(K)\) is surjective.

\textit{Proof.}

Let us assume that \(\alpha e_{[abb]} + \beta e_{[aab]} + \gamma e_{[aaa]} + \delta e_{[bbb]}\) is a 2-cycle, where \(\alpha, \beta, \gamma, \delta\) are integers. It follows that we have linear equations \(\alpha + 2\beta + 3\gamma = 0, 2\alpha + \beta + 3\delta = 0\).\(^9\) From Gaussian elimination it follows that \(\alpha = \gamma - 2\delta, \beta = -2\gamma + \delta\), so the general solution is a linear combination \(\gamma v_1 + \delta v_2\) with integer coefficients \(\gamma, \delta\) of the integral vectors \(v_1 = [1, -2, 1, 0]\) and \(v_2 = [-2, 1, 0, 1]\). A second integral basis for the solutions is \(v_1 + v_2 = [-1, -1, 1, 1]\) and \(-v_1 = [-1, 2, -1, 0]\). It follows that to prove

\(^9\)This convention will play an important role in the sequel paper on bosons in the algorithm for determining spin of composite particles.

\(^{10}\)Note that \(\tilde{K}^{(1)}\) is the regular covering space of \(K^{(1)}\) associated to the kernel of the homomorphism \(F \to \mathbb{Z}/3\mathbb{Z}\) given by \(a \to 1, b \to 1 \pmod{3}\).
the theorem it suffices to show that the 2-cycles $-e_{[abb]} - e_{[aab]} + e_{[aaa]} + e_{[bbb]}$ and $-e_{[abb]} + 2e_{[aab]} - e_{[aaa]}$ are spherical (observe that these 2-cycles correspond to the reactions of $\beta$-decay $\bar{a}\bar{b} \rightarrow aab + \bar{a}\bar{a} + \bar{b}\bar{b}$ and $aab + aab \rightarrow abb + aaa$ identified in [Ha]). This is accomplished by explicit construction with the following spherical diagrams.

In each of these figures there are four 2-cells, where the fourth 2-cell is at infinity. One reads the boundary labels for the finite cells counterclockwise and the boundary label for the cells at infinity in the clockwise manner. Since each 2-cycle is represented by a map of 2-sphere into $K$, it follows that the generators of $Z_2(K)$ are spherical and hence all 2-cycles of $K$ are spherical. This completes the proof.\footnote{This result is a special case of the theorem that every integral 2-cycle on a 2-complex with finite cyclic fundamental group is spherical. The proof makes use of the edge-term exact sequence}
Remark 1. We shall calculate the kernel of the homomorphism $\pi_2(\tilde{K}) \to H_2(K)$ in §6. It turns out to have important physical significance.

The significance of the proposition is the following. A particle, like a proton or neutron, is represented by a van Kampen diagram $D$ (that is, a combinatorial map of an oriented singular disc $D$ into $K$) [BRS]. Given a spherical diagram, one can cut it along one edge to obtain a van Kampen diagram with freely trivial boundary label. This can then be combined with $D$ along an edge with the same boundary label in the same orientation to obtain a new disc diagram $D'$. Then one can do “diamond moves” ([BRS] page 115) on $D'$ and repeat the process any number of times. The result of such a sequence of moves is a reaction among fermions, and we can read off the result at the chain level $C_2(K)$ to get the reaction in terms of the Harari-Shupe model. The proposition says that all reactions among fermions, that is, beginning with one collection of fermions and ending with another, after projecting from $\tilde{K}$ to $K$, are obtained in this way.

Remark 2. The fact that processes affecting fermions are represented by spherical diagrams means they lift to the universal cover $\tilde{K}$ of $K$. Thus the same processes that affect the fermions of the first generation affect those of the second and third generations. This explains why these particles resemble so closely their first generation counterparts.

To finish this section, we observe that Fig. 2 can be interpreted as a disc diagram by ignoring the face at infinity. As such it is a van Kampen diagram for the proton. A van Kampen diagram for the antineutron is obtained from it by interchanging $a$ and $b$. Those for the antiparticles are obtained by reversing all arrows.\textsuperscript{12}

§4. Color

In this section we take the first steps toward incorporating color, the source of the strong nuclear force, into our model. To begin, let $[0], [1], [2]$ denote the residue classes of $0, 1, 2 \pmod{3}$. Let $N = [0] + [1] + [2]$ in $\mathbb{Z}G$, where $G = \mathbb{Z}/3\mathbb{Z}$ and let $M = \mathbb{Z}G/N\mathbb{Z}G$, where $M$ is defined to be the color group. In the literature, $[0], [1], [2]$ correspond to the colors red ($r$), green ($g$), and blue ($b$). Their negatives $-r, -g, -b$ in $M$ are called cyan, yellow, and magenta, respectively; they play no role in the discussion below, but enter in the discussion of antiparticles and gluons, which we shall not undertake here.\textsuperscript{13} The designation of colors is arbitrary and has nothing to do with perceived vision. It is important to make the distiction of the Serre spectral sequence along with the fact that $H_2(G, \mathbb{Z}) = 0$ for a finite cyclic group $G$. Since we need the explicit form of the generators shown in Figures 1 and 2, we have given our direct geometric argument.

\textsuperscript{12}The other van Kampen diagram for the proton is obtained by doing a diamond move on the two edges labelled “$a$” with origin the bottom vertex.

\textsuperscript{13}In Appendix 2 below I relate my color group $M$ to the root system $A_2$. When I wrote this section I was ignorant of the existing $SU(3)$ theory and formulated the notions out of thin air. After Professor Wu kindly explained the physicists’ theory, I saw that we were saying the same thing, and the appendix gives the relation.
between the color red $[0]$ and the neutral element $0$ in $M$. In the literature, $0$ is called both “white” and “colorless”. The discussion below is given in detail for the proton Fig. 2, but it can be extended to all the fermions (see the last paragraph of §3). The gluons will be discussed in a future paper along with the other bosons of the theory.

We lift the diagram Fig. 2 for the proton to $\tilde{K}$.\footnote{The lift here is a technical device to facilitate the definition of color. Technically this means that color, for example on the quarks constituting a proton, is defined by the local coefficient system $M$, or, what amounts to the same thing, a $\pi_1(K)$-module $M$ [AH]. This is the same thing as ordinary constant coefficients on the universal cover $\tilde{K}$. To define color on the other generations of quarks, one pulls back the coefficient system to $\tilde{K}$ via the covering map $\tilde{K} \to K$.}

This requires a choice of base point, which we take to be a lift $P_0$ of the bottom vertex. This determines the other vertices $P_1, P_2$ and the lifts of the edges $a_i, b_i$, $i = 0, 1, 2$, where an edge is labelled by its initial vertex. The situation is shown below in Fig. 3.\footnote{We calculate the chain in $C_2(K)$ associated to the lift of the disc diagram shown in Fig. 3 as follows. In the free group on edges of $K$ we have, reading around the diagram once counterclockwise from the base point $P_0$, $(a_0a_1b_2)(b_2b_1\tilde{a}_0)(a_0b_1a_2)$. The first term may be identified with a lift $\tilde{u}$ of the up quark and we shall identify the second with a lift $\tilde{d}$ of the down quark. For the third term, $a_0b_1a_2 = \tilde{a}_2(a_2a_0b_1)a_2$, so as conjugacy classes in the free group we have $[a_0b_1a_2] = g^2[a_0a_1b_2]$, where $g$ is the generator of $G = \mathbb{Z}/3\mathbb{Z}$ given by $a \to [1], b \to [1]$. Thus the 2-cell corresponding to $[a_0b_1a_2]$ maps onto the up quark, but this is different from the conjugacy class $[a_0a_1b_2]$; so the 2-cells are different. In $C_2(K)$ we have the 2-chain determined by the lift of the diagram of Fig. 3 is $\tilde{u} + \tilde{d} + g^2\tilde{u}$. It would appear that there is a mixing of generations of the up quark in the lift. If we take into account the face at infinity of Fig. 3, then we obtain the 2-cycle on $K$ given by $\tilde{u} + \tilde{d} + g^2\tilde{u} + \tilde{e}$, where $\tilde{e}$ is a lift of the electron.}

![Figure 3](image-url)

The rules for coloring a diagram are the following, where a coloring is an assignment of elements of $M$ to the faces, so that the face with the opposite orientation is assigned the negative of the color of that face.
1. A lepton is always white, so has 0 for color.
2. The color of a quark is always one of $r, g, b$ and the color of an antiquark is the negative of the color of the corresponding quark.
3. For a particle to be observable, the sum of the colorings of the faces must be 0.

Thus one possible coloring of the diagram Fig. 3 is to assign the faces from left to right the colors $r, g, b$, whose sum is 0 (they lift to different faces in $\tilde{K}$), and assign colors arbitrarily to the other 2-cells of $\tilde{K}$, subject to rule 1. Call such a coloring $f$. Note that this van Kampen diagram in $\tilde{K}$ can be viewed as a spherical diagram where the fourth face is at infinity and corresponds to the lift of an electron. As such the coloring extends to one of the spherical diagram where again the sum of the colors assigned to the faces is 0.

So we can reformulate the definition of a coloring of the diagram $D$ for a fermion in $\tilde{K}$ as the pull-back to $D$ of a 2-cochain $f \in C_2(\tilde{K}, M)$ with values in $M$, where all leptons are assigned 0, where all lifts of quarks in the diagram are assigned colors from $r, g, b$, and where the sum of the values assigned to the faces of $D$ is 0. The rules for a spherical diagram are the same. So the cochain $f$ in the previous paragraph can be considered by pull-back as either defined on the disc or as a cochain $\tilde{f}$ on the sphere containing the disc obtained, in this example, by extension by 0.

The main point here is how to change colorings, that is, to change cochains in such a way as to preserve rules 1–3 and in a topologically meaningful way. This is done by means of coboundaries of 1-cochains. For example, in the coloring $f$ indicated above, we can interchange the colors of the first two faces by $\delta h$, where $h$ is the 1-cochain which is zero on all edges except on $b_1$, and where $h(b_1) = -r + g$. To interchange the colors of the second and third faces, we use $\delta k$, where $k$ is zero on all edges except on $b_2$, where $k(b_2) = g - b$. That rule 3 is preserved is checked directly or, more fundamentally, follows from the fact that $< \delta c, z > = 0$ for all 1-cochains $c$ and all 2-cycles $z$ (applied to the spherical diagram). In this way all colorings consistent with the rules are obtained.

These 1-cochains correspond physically to gluons and are the source of the strong nuclear force. We shall offer another interpretation of gluons in a subsequent article on bosons, more in keeping with the ideas of §3 of processes applied to a fermion.

§5. Standard Model

In this section we propose our version of the standard model SM. There are alternative versions of SM, some of which were proposed in order to cope with the possibility that the Higgs boson might not exist. Now that the data confirm its existence with better than 99% probability, we shall eschew treating these alternative theories from our point of view.

We define a fundamental domain to be one of the two spherical diagrams shown in Figures 1 and 2 or one obtained from it by replacements $a, b \to b, a$, $a, b \to \bar{a}, \bar{b}$, or $a, b \to \bar{b}, \bar{a}$. Here replacing $a, b$ by $\bar{a}, \bar{b}$, etc., in a diagram means to reverse the arrows.
We also admit as fundamental domains completely reducible\textsuperscript{16} two-faced and four-faced spherical diagrams consisting of cells $e_\kappa$ and their oppositely oriented cells $-e_\kappa = e_{\bar{\kappa}}$ (these degenerate\textsuperscript{17} examples have corresponding 2-chains 0 in $C_2(K)$). So in a four-faced completely reducible fundamental domain one can order the four 2-cells $\alpha, \beta, \gamma, \delta$ so that $\alpha$ and $\beta$ form a reducible pair across a common edge, and $\gamma$ and $\delta$ form a reducible pair across a common edge.\textsuperscript{18}

It should be remarked that not all 2-faced spherical diagrams are considered fundamental domains. For example, one can take the electron neutrino $e_\kappa$ with $\kappa = [\text{bb}b]$, rotate it a third of a revolution, and glue the two together. This is a perfectly acceptable spherical diagram, and indeed it plays a key role in §6 below. However it is not considered a fundamental domain because it is not reducible. The rotation prevents any edge from acting as a mirror.

A fundamental process for particle physics is the lift $\tilde{f} : S \rightarrow \tilde{K}$ of a spherical diagram $f : S \rightarrow K$, where $S$ is a fundamental domain. A fundamental particle is a triple $(D, \tilde{f}, h)$, where $\tilde{f} : S \rightarrow \tilde{K}$ is a fundamental process lifting $f : S \rightarrow K$, where $D \subset S$ is a van Kampen diagram, and where $h \in C^2(\tilde{K}, M)$ is a coloring in the color group $M$ satisfying
1. the pull back $\tilde{f}^*(h) \in C^2(S, M)$ vanishes on leptons, and
2. the values of $\tilde{f}^*(h)$ on quarks are in $\{r, g, b\}$ and the values on antiquarks are in $\{-r, -g, -b\}$.

The fundamental particle is observable if in addition
3. the sum of the values of $\tilde{f}^*(h)$ on the faces of $S$ is 0; the invariant formulation\textsuperscript{19} of this is that $< [S], \tilde{f}^*(h) > = 0$, where $[S]$ here denotes the fundamental 2-cycle of the sphere\textsuperscript{20}.

Furthermore, we demand that any diagram $D'$ obtained from $D$ in the triple $(D, \tilde{f}, h)$ by diamond moves be observable if the original triple is observable (the 2-cells of $D'$ are mapped in the same way as those of $D$, so condition 3 is preserved).

Another way of formulating the last condition is to give a van Kampen diagram $\tilde{f} : D' \rightarrow \tilde{K}$ and 2-cochain $h \in C^2(\tilde{K}, M)$ such that after a sequence of diamond moves on $D'$ one obtains either a fundamental particle $D$ or a spherical diagram $S \rightarrow \tilde{K}$ satisfying 1–3. In the latter case, if $D = S$, then the boundary label of $D'$

\textsuperscript{16}A diagram is called reducible if it contains a pair of faces with an edge in common so that the faces are mapped mirror-wise across that edge. The faces themselves are called a reducible pair. In a reducible diagram one can remove the reducible pair and the edge between them and sew up to boundary of the hole created to obtain a new diagram with two fewer faces. A diagram is called completely reducible if its faces can be paired off into reducible pairs.

\textsuperscript{17}These degenerate diagrams play an important role in the sequel article on bosons.

\textsuperscript{18}It is also possible for both $\alpha$ and $\beta$ to be a reducible pair and for $\gamma$ and $\delta$ to be reducible, all at the same time, as happens in the case of the hypothetical graviton, to be considered in the sequel article on bosons. In Appendix 5 below I consider completely reducible 6-faced diagrams in order to treat the $\Delta^{++}$ baryon, as a result of a challenge from Professor Wu. Presumably this could continue with completely reducible $2n$-faced diagrams in order to handle higher resonances.

\textsuperscript{19}The notation $< x, y >$ means to result of evaluating a cochain $y$ on the chain $x$.

\textsuperscript{20}$[S] \in C_2(S, \mathbb{Z})$ is the sum of the oriented 2-cells in cell decomposition of the 2-sphere $S$. It is readily checked that $[S]$ is a 2-cycle.
must have been freely trivial, so that \( S \) was obtained by sewing up the boundary completely. In all cases, \( D' \) has at most four 2-cells, where the case of four 2-cells is the case of the freely trivial boundary label (this is also the case where the particle is its own antiparticle).

Note that the invariant formulation of condition 3 above is automatically satisfied for a coboundary \( h = \delta c \), with \( c \in C^1(\tilde{K}, M) \); that is, \( < \tilde{f}_* [S], \delta c > = 0 \), because \( [S] \) is a cycle and \( < [S], \tilde{f}_* (h) > = < \tilde{f}_* [S], \delta c > = < \partial \tilde{f}_* [S], c > = < \tilde{f}_* (\partial [S]), c > = 0 \) by adjointness. The main result of this section is a converse.

**Theorem.** If we let \( k = \tilde{f}_* (h) \), with \( \tilde{f} : S \to \tilde{K} \) and \( h \in C^2(\tilde{K}, M) \) as above, and if 1–3 are satisfied, then \( k = \delta \tilde{f}_* (c) \) for some cochain \( c \in C^1(\tilde{K}, M) \).

**Sketch of proof.** We sketch the argument for the proton, namely, Figure 3. The cases of the neutron and the antiparticles follow by applying the symmetries of interchanging \( a \) and \( b \) and of reversing all arrows, as in §3.

Since we have to consider spherical diagrams, we consider Figure 3 together with the face at infinity, which is a lepton. We pick a coloring of the finite faces, say \( x, y, z \) where these are colors chosen without repetition from \([0], [1] \) and \([2]\), read from left to right. The color of the face at infinity is white, namely 0. Then the conditions on \( c \) restricted to \( \tilde{f}(S) \), the image of \( S \) under \( \tilde{f} \), are four linear equations in five unknowns \( c(a_0), c(a_1), c(a_2), c(b_1), c(b_2) \) on the left side (referring to the notation of Figure 3), with variables \( x, y, z \) on the right side. For example, the first of these equations, corresponding to the face at infinity, is \( c(a_0) + c(a_1) + c(a_2) = 0 \). When we row reduce the system, the last row of the row reduction corresponds to the equation equation \( 0 = x + y + z \). But this compatibility condition is satisfied because \( x + y + z = [0] + [1] + [2] = 0 \), by the definition of \( M \). So the equations are solvable.

This determines \( c \) on \( \tilde{f}(S)^{(1)} \). Then we extend \( c \) in any way to all of \( \tilde{K}^{(1)} \), thereby completing the argument.

**Remark.** Electric charge of a composite particle is given by the coboundary of a 1-cochain \( h \) on \( K \). Namely define \( h \) to be \( \frac{1}{3} \) on an edge labeled \( a \), \( -\frac{1}{3} \) on the oppositely oriented edge, and \( h \) of an edge labeled \( b \) with any orientation is defined to be 0. Then \( \delta h \in C^2(K, \mathbb{R}) \) calculates the net electric charge of any van Kampen diagram. In observable particles the electric charge must be an integer. Note that to calculate color, we must use cochains on \( \tilde{K} \) (the two faces in Figure 2 that represent up quarks have different colors in the proton because they lift to different 2-cells in \( \tilde{K} \)), whereas electric charge can be defined from the projection into \( K \).

**Remark.** The weak isospin \( T_3 \) of particle in SM\textsuperscript{21} governs how it interacts in the weak interaction. It is analogous to electric charge and is given by the coboundary of the 1-cochain \( w \) determined by \( w(a) = \frac{1}{6}, w(\tilde{a}) = -\frac{1}{6}, w(b) = \frac{1}{6}, w(\tilde{b}) = -\frac{1}{6} \). In the higher generations, one projects first from \( \tilde{K} \) to \( K \) and then applies \( w \) to determine the weak isospin.

\textsuperscript{21}http://en.wikipedia.org/wiki/Weak_interaction
Consequently, three of the fundamental forces of nature, the strong and weak nuclear forces and the electric charge, are determined by coboundaries of 1-cochains with different value groups. This is a remarkable unifying property and serves to strengthen our case that SM is the algebraic topology of the covering map \( \tilde{K} \rightarrow K \).\(^{22}\) There is no evidence at present that gravity fits into this scheme, and indeed it would be most remarkable (and stretch the imagination nearly to breaking point) if it too were determined by the coboundary of a 1-cochain.

As a last tidbit for this section in response to requests we draw an avatar of Higgs boson below in Fig. 4. The argument that this is correct is deferred to the sequel article on bosons.

\[\text{Figure 4}\]

\section*{6. Neutrino oscillations}

This section gives a physical interpretation for the processes in \( Z_2(\tilde{K}) = \pi_2(K) \) which are not lifts of processes from \( Z_2(K) \). For many years it was a mystery that the observed neutrinos coming from the sun were a third of the predicted value. The explanation finally provided was that a neutrino in free space oscillates between the three types, electron neutrino \( \nu_e \), muon neutrino \( \nu_\mu \), and tau neutrino \( \nu_\tau \). Since only \( \nu_e \) was observed in the experiment, what we observe is only a third of the actual production from nuclear reactions (due to \( \beta \)-decay) in the sun.

We begin by noting that, by an Euler characteristic computation, the rank of \( Z_2(\tilde{K}) \) is 8. We can account for 6 free generators by lifts of the classes in \( \pi_2(K) \)

\(^{22}\)See the Appendix below for a historical perspective on this assertion.
represented by the spherical diagrams in Figures 1 and 2. We shall now account for the other two free generators.

Let \( \tilde{\nu} \) denote a lift of the 2-cell \( e_\kappa \) in \( K \), where \( \kappa = [bbb] \). Let \( g \) be a generator for the covering group \( G \cong \mathbb{Z}/3\mathbb{Z} \) of \( \tilde{K} \). Then an easy calculation shows that \( g\tilde{\nu} - \tilde{\nu} \in Z_2(\tilde{K}) = H_2(\tilde{K}) \), and it is undetectable in \( K \). We also have the class \( g(g\tilde{\nu} - \tilde{\nu}) = g^2\tilde{\nu} - g\tilde{\nu} \) in the kernel. A calculation shows that these two chains are linearly independent in \( C_2(\tilde{K}) \) and hence in \( Z_2(\tilde{K}) \). It follows that we have accounted for the extra two free generators of \( Z_2(\tilde{K}) \) that do not arise from lifts of cycles in \( K \).

Now recall our premise from §3 that processes involving fermions arise from 2-cycles. Thus \( g\tilde{\nu} - \tilde{\nu}, g^2\tilde{\nu} - g\tilde{\nu}, \) and \( \tilde{\nu} - g^2\tilde{\nu} \) are processes; these processes exchange the three neutrinos \( \nu_e, \nu_\mu \) and \( \nu_\tau \) in cyclic fashion, accounting for the oscillation of neutrinos. Of course we cannot deduce from the topology alone the rates of the processes, so this is all the information we can hope to gain from this type of argument about neutrino oscillation.

Now we can replace \( b \) by \( a \) in the argument above and deduce that there exist processes which exchange cyclically the electron, muon, and tau particle. Such oscillations have not yet been detected in the laboratory.

There is fortunately a check for the correctness of these ideas, as follows. The process by which the muon decays is \( \mu^- \rightarrow \nu_\mu + e^- + \nu_e \). Now \( \mu^- = g\tilde{e} \), where \( \tilde{e} \) is the lift of the electron to \( \tilde{K} \) and \( g \) is as above a generator for the covering group. Also \( \nu_\mu = g\tilde{\nu}_e \), where \( \tilde{\nu}_e \) is the lift of \( \nu_e \), the electron neutrino. Bringing all terms to the left side, the decay process above is compatible with our theory if \( g\tilde{e} - g\tilde{\nu}_e - \tilde{e} + \tilde{\nu}_e \) is a 2-cycle in \( Z_2(\tilde{K}) \). But this last expression is \( (g-1)\tilde{e} - (g-1)\tilde{\nu}_e \), which is indeed a 2-cycle.

§7. Questions and comments

1. Shupe’s ”no mixing rule” ([Sh] p. 88), which states, on our terminology, that \( aab, abb \), and their inverses do not occur in the construction of the complex \( K \), is at the heart of this paper. In the sequel paper on bosons, we offer a justification for this rule.

2. There is an uncanny symmetry in this model. Namely the Klein four-group, which appears as the subgroup of \( \text{Aut}(F) \) consisting of the identity, \( \{a, b\} \rightarrow \{b, a\} \), \( \{a, b\} \rightarrow \{\bar{a}, \bar{b}\} \), and \( \{a, b\} \rightarrow \{\bar{b}, \bar{a}\} \), acts on the theory. This symmetry demands further explanation.

3. The definition of the 2-complex \( K \) looks arbitrary although it arose naturally from a reinterpretation of the work of [Ha] and [Sh]. I can prove the following result which may serve to motivate it for mathematicians.

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\(^{23}\)Z\(_2(\tilde{K})\) maps onto \( Z_2(K) \), as follows from the Proposition of §3, and \( Z_2(K) \) is free abelian, so the kernel is a direct summand; it follows that it is of rank 2.

\(^{24}\)See http://en.wikipedia.org/wiki/Muon
Theorem. There exists an infinite dimensional CW-complex $L$ with finite $n$-skeleton for all $n \geq 0$ with the following properties.

1. $L$ is contractible; in particular it is simply connected.
2. $L$ is acted on freely and cellularly by the group $G = \mathbb{Z}/3\mathbb{Z}$; hence the projection $L \to L/G$ is the universal covering.
3. $L(2) = \tilde{K}$.
4. $L(2^{n+1})$ is homotopy equivalent to $S^{2n+1}$ for all $n \geq 1$.

The argument makes use of the periodic resolution for $\mathbb{Z}$ over $\mathbb{Z}[G]$, which is modified in low dimensions ($\leq 3$) to account for the unusual presentation $P$ for $G$.

It follows from the theorem that $L$ is homotopy equivalent to $BG$, the classifying space of principal $G = \mathbb{Z}/3\mathbb{Z}$-fibrations, which is a space of type $K(G, 1)$.

Question. Can $\tilde{K}$ be equivariantly imbedded in $S^5$ for the $G = \mathbb{Z}/3\mathbb{Z}$-actions? Here $G$ acts on $S^5$ by the diagonal action of third roots of unity on the unit vectors of $\mathbb{C}^3$. In the sequel paper we show that $\tilde{K}$ imbeds equivariantly in $S^{2n+1}$ for $n$ sufficiently large ($n \geq 6$).

Appendix 1. Historical remark

C-N Yang [Ya] relates a conversation he had with Andre Weil in which the latter suggested that the particles that were being discovered might be explained by “geometry and topology”. Yang writes that he did not understand this, and no one at the meeting thought to ask him what he meant. As far as Weil’s suggestion about geometry, that has proved to be correct in the development of Yang-Mills theory. But there is no precedent in the literature that what he had in mind may also have included pure topology.

Appendix 2. Color group $M$ and $A_2$

This Appendix will relate the color group $M$ to $A_2$, which is the root system for $SU(3)$; this lie group is known to physicists as the theory that describes the strong nuclear force.

The notation follows §4 for the definition of $M$. As usual $G$ denotes the cyclic group of order 3 and $g$ is a generator. We make the complex numbers $\mathbb{C}$ into a $G$-module by letting $g$ act by multiplication by $\zeta = e^{2\pi i/3}$, a primitive third root of unity. Geometrically this represents a rotation through 120 degrees.

Theorem. The map $g \to \zeta$ extends to an homomorphism of $G$-modules $\phi : \mathbb{Z}[G] \to \mathbb{C}$ with the properties that

1. the image of $\phi$ is $\mathbb{Z}[\zeta]$, the subring of $\mathbb{C}$ generated by $\zeta$ and the unit element,
2. the kernel of $\phi$ is $NZ[G] = (1 + g + g^2)\mathbb{Z}[G]$, so
3. $\phi$ induces an isomorphism $M \to \mathbb{Z}[\zeta]$, and
4. the images of the colors $\pm[j]$, where $j = 0, 1, 2$, form the root system $A_2$; thus $\phi(\pm[j]) = \pm \zeta^j$ comprise the set of sixth roots of unity in $\mathbb{C}$.

For example let us prove that the kernel of $\phi$ is the ideal generated by $N = 1 + g + g^2$ in $\mathbb{Z}[G]$ and that $\phi$ induces an isomorphism $M \to \mathbb{Z}[\zeta]$. The element $N$
is in the kernel since $\zeta$ is a primitive third root of unity, so satisfies the polynomial $1 + z + z^2 = \frac{z^3 - 1}{z - 1}$. The rank of $\mathbb{Z}[\zeta]$ is 2, since it is the set of algebraic integers in an algebraic number field $\mathbb{Q}[\zeta]$ of degree 2 over the rationals $\mathbb{Q}$. So a rank count shows that the induced map $M \to \mathbb{Z}[\zeta]$ is an isomorphism.

Appendix 3. Conservation laws in SM and quantum numbers

In §1 it was pointed out that the fundamental conservation law involving preons is that the net numbers of $a'$s and $b'$s are individually conserved in a reaction predicted by SM. From this followed the result that there are only three generations of fundamental fermions of SM, etc.

In this appendix we shall define conservation laws and quantum numbers for elementary particles and show that, if we consider lifts to the universal cover $\tilde{K}$, then the preon number for a lift $a_i$ or $b_i$ of a preon is a quantum number. It follows that there is no theoretical reason that one cannot detect the preon number directly.

**Definition.** A conservation law in $A$, where $A$ is a ring with unit, is a 2-cochain $c \in C^2(\tilde{K}, A) = \text{Hom}(C_2(\tilde{K}, \mathbb{Z}), A)$ such that $< z, c > = 0$ for all $z \in Z_2(\tilde{K}, \mathbb{Z})$. Here $< z, c >$ is the result of evaluating $c$ on the chain $z$ to yield an element of $A$. A quantum number for the conservation law $c$ is the result of evaluating $c$ on a particle (that is, on a van Kampen diagram in $\tilde{K}$).

The reason this is called a conservation law is that a reaction is given by $\kappa_1 + \kappa_2 + \cdots + \kappa_p \to \lambda_1 + \cdots + \lambda_q$, so it follows that $c(\kappa_1) + c(\kappa_2) + \cdots + c(\kappa_p) = c(\lambda_1) + \cdots + c(\lambda_q)$. By adjointness, it follows that, if $c = \delta h$ with $h \in C^1(\tilde{K}, A)$, then $c$ is a conservation law. Thus it follows from the remarks of §5 that electric charge and weak isospin $T_3$ are quantum numbers with values in $\mathbb{R}$. The next result says the converse is true.

**Proposition.** If $c \in C^2(\tilde{K}, A)$ is a conservation law, then $c = \delta h$ for some $h \in C^1(\tilde{K}, A)$. Thus the set of conservation laws in $A$ is $B^2(\tilde{K}, A)$.

**Proof.** We are given $c \in C^2(\tilde{K}, A) = \text{Hom}(C_2(\tilde{K}, \mathbb{Z}), A)$ such that $c$ vanishes on $Z_2(\tilde{K}, \mathbb{Z}) \subset C_2(\tilde{K}, \mathbb{Z})$.

By the universal coefficient theorem, $H^2(\tilde{K}, A) = \text{Hom}(Z_2(\tilde{K}, \mathbb{Z}), A)$, where we have used the facts that $H_1$ vanishes (since $\tilde{K}$ is simply connected) and $H_2(\tilde{K}, \mathbb{Z}) = Z_2(\tilde{K}, \mathbb{Z})$ since there are no 3-cells. Also $Z^2(\tilde{K}, A) = C^2(\tilde{K}, A)$ since every 2-cochain is a 2-cocycle. Thus $H^2(\tilde{K}, A) = C^2(\tilde{K}, A)/B^2(\tilde{K}, A) = \text{Hom}(Z_2(\tilde{K}, \mathbb{Z}), A)$. This may be expressed as the short exact sequence

$$0 \to B^2(\tilde{K}, A) \to C^2(\tilde{K}, A) \to \text{Hom}(Z_2(\tilde{K}, \mathbb{Z}), A) \to 0.$$
Also we have the surjective map $C^1(\tilde{K}, A) \to B^2(\tilde{K}, A)$, which may be spliced with the first map in the short exact sequence to yield the exact sequence

$$C^1(\tilde{K}, A) \xrightarrow{\delta} C^2(\tilde{K}, A) \to \text{Hom}(Z_2(\tilde{K}, Z), A) \to 0.$$  

Now $c \in C^2(\tilde{K}, A) = \text{Hom}(C_2(\tilde{K}, Z), A)$ and the map on the right in the last exact sequence is obtained by restricting a homomorphism to $Z_2(\tilde{K}, Z)$ and then considering the image to lie in $A$ via the map $Z \to A$. It follows that the image of $c$ in $\text{Hom}(Z_2(\tilde{K}, Z), A)$ is zero, and, by exactness, $c = \delta h$ for some $h \in C^1(\tilde{K}, A)$. 

Let us specialize now to the case $A = \mathbb{R}$. Then $C^2(\tilde{K}, \mathbb{R})$ is of dimension 12 while the rank of $Z_2(\tilde{K}, Z)$ was calculated in §6 to be 8. It follows that the dimension of $B^2(\tilde{K}, \mathbb{R})$ is 4, and hence there are 4 linearly independent conservation laws in SM.28

Let us focus attention on one that has not been mentioned before. Let $f \in C^1(\tilde{K}, \mathbb{R})$ be given by $f(a_i) = \frac{1}{3}$, $f(b_i) = -\frac{1}{3}$, $f(\bar{a}_i) = -\frac{1}{3}$, $f(\bar{b}_i) = \frac{1}{3}$, for $i \pmod{3}$. If $h$ and $w$ are 1-cochains on $\tilde{K}$ determining charge and weak isospin $T_3$ respectively (in the notation of §5), then $f = 2(h - w)$. This identifies $\delta f$ with the weak hypercharge $Y_W$.29

We can calculate the vector space $B^2(\tilde{K}, \mathbb{R})$ of conservation laws over $\mathbb{R}$ as follows. We have $B^2 = C^1/Z^1$ where $Z^1 = B^1$. From these relations we see that $Z^1(\tilde{K}, \mathbb{R})$ is the set of functions $f$ on the 1-cells $a_i, b_i$ of $\tilde{K}$ so that $f(a_i) = f(b_i)$, $f(a_0) + f(a_1) + f(a_2) = 0$, for $i \pmod{3}$. This is a 2-dimensional subspace of the 6-dimensional space $C^1$, so the 4-dimensional quotient $B^2$ can be effectively determined. Here is the result of this calculation.

If we let $x$ be one of $a_i, b_i, i \pmod{3}$, we define $\Delta_x \in C^1(\tilde{K}, \mathbb{R})$ by $\Delta_x(x) = 1, \Delta_x(\bar{x}) = -1$, and $\Delta_x(y) = 0$ for all other $y$. Then for each lift $\bar{\kappa}$ of an elementary fermion $\kappa$ we have $\delta \Delta_x(\bar{\kappa}) = +1$ if the boundary label of $\bar{\kappa}$ contains $x$, $\Delta_x(\bar{\kappa}) = -1$ if the boundary label contains $\bar{x}$, and $\Delta_x(\bar{\kappa}) = 0$ otherwise.

**Proposition.** For each $x \in \{a_i, b_i\}, i \pmod{3}$, $\delta \Delta_x$ is a conservation law, and for each lift $\bar{\kappa}$ of an elementary fermion $\delta \Delta_x(\bar{\kappa})$ is a quantum number. For any collection of 4 of the 6 lifts $a_i, b_i$ the corresponding conservation laws are linearly independent. 

**Remark.** The proposition provides a test for the structure of the proton $p$ proposed in Figure 2. For any lift $\bar{p}$ of $p$ to $\tilde{K}$, we see that $\delta \Delta_{a_i}(\bar{p}) = 1$ for all $i \pmod{3}$. This structure also has the consequence that the two lifts of $u$ in $\bar{p}$ must be of different generations, as was pointed out in the footnote of §4. If we tried the structure $z = [a_0a_1b_2] + [b_2b_1a_0] + [a_0a_1b_2]$ instead (which projects to $2u + d$ but is not the continuous lift given by covering space theory), we would find $\delta \Delta_{a_1}(z) = 1, \delta \Delta_{a_2}(z) = 2, \delta \Delta_{a_3}(z) = 0$. So these new quantum numbers may be useful tools in determining the structure of composite particles as van Kampen diagrams.

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28 The same result holds for every field.

29 http://en.wikipedia.org/wiki/Hypercharge. In the form $Q = I_3 + \frac{1}{3} Y$ the relation is attributed to Gell-Mann and Nishijima.

30 So $\kappa$ is either an electron, neutrino, quark or an antiparticle of one of these.
Appendix 4. β-decay revisited

When I explained to a physicist how van Kampen diagrams gave a consistent interpretation to the ideas of [Ha] and [Sh], which solved many of the open problems these papers posed, he expressed interest in seeing in detail how β-decay works from the topological point of view (the point of view of Feynman diagrams is well-known\textsuperscript{31} $d \rightarrow u + e^- + \bar{\nu}_e$ but that leaves unexplained what is actually happening at the vertices of the Feynman diagram).

Examine Figure 5 below.

The left half of the diagram is the down quark $d$ whereas the right half is the Higgs boson. The latter is obtained from the spherical diagram Figure 1 by making two cuts along edges labeled $a$ and $b$ and spaying the result out as a van Kampen diagram in the plane; so this yields is a 2-cycle and acts on the van Kampen diagram for $d$ in the manner described in §3 following Remark 1.\textsuperscript{32}

The faces labeled $d$ and $\bar{d}$ are mapped mirror-wise across their common edge, so we may do a reduction, removing the pair and that edge, and sew up the hole to create the diagram Figure 6 below.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.png}
\caption{Figure 6}
\end{figure}

\textsuperscript{31}http://en.wikipedia.org/wiki/Beta\_decay

\textsuperscript{32}This description is purely topological and ignores entirely the question of magnitude of energies involved. So the Higgs in question is of the nature of a virtual particle, arising from the vacuum state only ephemerally and disappearing immediately after interacting with the down quark.
The two shaded faces are $e$ and $\bar{\nu}_e$ which are now shed to leave the up quark $u$. So the net result is the reaction $d \to u + e^- + \bar{\nu}_e$. As a parenthetical remark, we observe that the union of those two shaded faces constitute the $W^-$ boson, so the reaction can be rewritten as the composite of two reactions, $d \to W^- + u$ and $W^- \to e^- + \bar{\nu}_e$.

Appendix 5. $\Delta^{++}$ baryon

Professor Wu challenged me to describe the $\Delta^{++}$ baryon\textsuperscript{33} in my terms. The result is shown below in Figure 7.

It shows three $u$ quarks with a single vertex in common (like the radiation hazard sign). The spin of the resulting particle is $\frac{3}{2}$, while the remarks of §5 show how to calculate that the electric charge is 2 and the weak isospin $T_3$ is $\frac{3}{2}$. The second diagram on the right shows how to imbed $\Delta^{++}$ in a diagram with 6 faces and freely trivial boundary label, so the enlarged diagram folds up to a spherical diagram and represents a 2-cycle. That 2-cycle is 0, for the diagram is completely reducible.

\textsuperscript{33}en.wikipedia.org/wiki/Delta_baryon
All other spherical diagrams previously considered here have 2 or 4 faces, so this is the first where 6 faces are needed. That fact by itself makes Professor Wu’s question interesting and motivated me to include it here.

Appendix 6. Quantum numbers and the first Chern class

In §5 electric charge was interpreted as the coboundary of a 1-cochain on $\tilde{K}$ which is evaluated on particles (that is, on van Kampen diagrams). This is satisfactory from the point of view of algebraic topology but it is desirable to have a direct geometric interpretation. The answer given below is that charge, up to a factor of $\frac{1}{3}$, is the first Chern class of an explicitly constructed line bundle. This involves an interpretation of the preons $a$ and $b$ as sections of a complex line bundle, which is then pursued to deal with other quantum numbers.

Let $D$ be the van Kampen diagram for a fundamental fermion of SM, so $D$ represents a positron, an up quark, an anti-down quark, or an electron neutrino. We view $D$ as a map to $K$; as pointed out in §5, charge is defined on $K$ and pulls back to $\tilde{K}$ via the covering map $\tilde{K} \to K$. Thus the boundary label of $D$ is either $aaa$, $aab$, $abb$, or $bbb$. The cases of their antiparticles involve only a change of sign in the construction. Thus the boundary of $D$ is subdivided into 3 segments, each of which is mapped onto an edge $a$ or $b$ of $K$, and the end points of the segments are mapped to the unique vertex of $K$.

Consider the trivial $U(1)$-bundle $D \times S^1$ over $D$. Here $U(1) = S^1$ is considered as the unit interval $I = [0, 1]$ in $\mathbb{R}$ with end points identified, $S^1 = I/0 \sim 1$. Define a section $\sigma$ of the bundle over the boundary $\partial D$ of $D$ by the identity map of $S^1$ beginning at the base point 0 over each segment labeled $a$ and by the constant map 0 over the segment labeled $b$. Note that if we consider this section as a map $f_\sigma : S^1 \to S^1$, the degree of the map is 3, 2, 1 or 0 in the respective cases $aaa$, $aab$, $abb$, or $bbb$.

**Theorem 1.** The electric charge of the fermion is $\frac{1}{3}$ of the first Chern class in $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$ of the complex line bundle over $S^2$ with clutching function $f_\sigma$. \[34\]

**Remark.** Equivalently, the electric charge is the obstruction to extending $\sigma$ to a never-vanishing section of the trivial line bundle $D \times \mathbb{C}$ over $D$, up to a factor of $\frac{1}{3}$; here $S^1$ is considered as the image of the exponential map $t \mapsto e^{2\pi it}$ in $\mathbb{C}$, $0 \leq t \leq 1$.

**Proof of Theorem.** The first Chern class of the line bundle is the degree of the clutching function $f_\sigma : S^1 \to S^1$.

Next we generalize the construction just made to other quantum numbers. We do the construction in $\tilde{K}$ rather than in $K$, as we did in the case of electric charge, which is invariant under deck transformations of the covering map $\tilde{K} \to K$, since we want to allow the possibility of quantum numbers that are not invariant under deck transformations.

\[34\] For the notion of clutching function, see en.wikipedia.org/wiki/Clutching_construction or Hatcher’s book on vector bundles http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html, p. 22.
If $\alpha_i$ and $\beta_i$ are integers, $i \mod 3$, let $F$ be the section of the trivial $U(1)$ bundle over $\tilde{K}^{(1)}$ defined by maps of $S^1$ to itself of degree $\alpha_i$ over edges $a_i$ and $\beta_i$ over edges $b_i$. The obstruction to extending $F$ over all of $\tilde{K}$ (which depends only on the integers $\alpha_i, \beta_i$ and not on representative maps) can be calculated as follows. Let $e_{\kappa_i}$ be one of the 2-cells of $\tilde{K}$. Then the section $F$ is defined over the boundary $\partial e_{\kappa_i}$ of $e_{\kappa_i}$, which is a covering $\kappa_i$ of one of the fundamental fermions $\kappa$ (so $\kappa$ is one of $[aaa], [aab], [abb]$ or $[bbb]$ or their inverses). Let $S_{\kappa_i}$ be the 2-sphere which consists of two identical copies of $e_{\kappa_i}$ glued along their common boundary by the identity map, and let $L_{\kappa_i}$ be the complex line bundle over $S_{\kappa_i}$ given by the clutching function $F$. Then the obstruction to extending $F$ over $e_{\kappa_i}$ is the first Chern class $c_1(L_{\kappa_i}) \in H^2(S_{\kappa_i}, \mathbb{Z}) \cong \mathbb{Z}$. Thus the obstruction to extending $F$ over all of $\tilde{K}$ is the collection of first Chern classes $c_1(L_{\kappa_i}) \in \mathbb{Z}$. This amounts to 24 integers, counting lifts of antiparticles of the fundamental fermions. Call this 24-tuple of integers $q_F$.

**Theorem 2.** For every conservation law $c \in B^2(\tilde{K}, \mathbb{Q})$ there is a section $F$ of the trivial $U(1)$ bundle over $\tilde{K}^{(1)}$ and a positive integer $N$ so that $\frac{1}{N}q_F$ is the collection of quantum numbers associated to $c$. The collection $q_F$ is given by the first Chern classes of complex line bundles $L_{\kappa_i}$ constructed above.

For the proof one takes $h \in C^1(\tilde{K}, \mathbb{Q})$ so that $\delta h = c$, where $N$ is a number that clears all denominators. Then one applies the construction of the preceding paragraph to $Nh$.

**Example.** If $h(a_i) = 1, h(b_i) = 0$ for all $i \mod 3$ and $N = 3$, then we obtain the electric charge lifted to $\tilde{K}$.

Another way of stating the result is as follows. Let $X$ denote the double of $\tilde{K}$ along $\tilde{K}^{(1)}$. So $X$ consists of two disjoint copies of the 2-complex $\tilde{K}$ identified along their common subcomplex $\tilde{K}^{(1)}$. Take two copies of the trivial complex line bundle over $\tilde{K}$ and identify them over $\tilde{K}^{(1)}$ by the isomorphism given by $F$. This produces a line bundle $L$ over $X$. Then $c_1(L) \in H^2(X, \mathbb{Z})$ contains all the information in the collection $q_F$. Namely, if we take the double of the 2-cell $e_{\kappa_i}$ lying inside $X$, this is an imbedded copy of the 2-sphere $S_{\kappa_i}$. So we can restrict the bundle $L$ to $S_{\kappa_i}$ and take its first Chern class, thereby recovering the quantum number up to the factor of $N$.

There is a formal way to incorporate the multiplicative factor $N$ into the answer. One considers the group $\text{Pic}(X) \otimes \mathbb{Q}$ where $\text{Pic}(X)$ is the Picard group of isomorphism classes of complex line bundles under the operation of tensor product and $\mathbb{Q}$ is the rational numbers. The element $L \otimes \frac{1}{N}$ is then an invariant of the conservation law $c$ over $\mathbb{Q}$ and incorporates all the quantum numbers by the operation of taking the first Chern class.

\[\text{TOPOLOGY OF THE STANDARD MODEL I: FERMIONS}\]

\[\text{21}\]

\[\text{If } \alpha_i \text{ and } \beta_i \text{ are integers, } i \mod 3, \text{ let } F \text{ be the section of the trivial } U(1) \text{ bundle over } \tilde{K}^{(1)} \text{ defined by maps of } S^1 \text{ to itself of degree } \alpha_i \text{ over edges } a_i \text{ and } \beta_i \text{ over edges } b_i. \text{ The obstruction to extending } F \text{ over all of } \tilde{K} \text{ (which depends only on the integers } \alpha_i, \beta_i \text{ and not on representative maps) can be calculated as follows. Let } e_{\kappa_i} \text{ be one of the 2-cells of } \tilde{K}. \text{ Then the section } F \text{ is defined over the boundary } \partial e_{\kappa_i} \text{ of } e_{\kappa_i}, \text{ which is a covering } \kappa_i \text{ of one of the fundamental fermions } \kappa \text{ (so } \kappa \text{ is one of } [aaa], [aab], [abb] \text{ or } [bbb] \text{ or their inverses). Let } S_{\kappa_i} \text{ be the 2-sphere which consists of two identical copies of } e_{\kappa_i} \text{ glued along their common boundary by the identity map, and let } L_{\kappa_i} \text{ be the complex line bundle over } S_{\kappa_i} \text{ given by the clutching function } F. \text{ Then the obstruction to extending } F \text{ over } e_{\kappa_i} \text{ is the first Chern class } c_1(L_{\kappa_i}) \in H^2(S_{\kappa_i}, \mathbb{Z}) \cong \mathbb{Z}. \text{ Thus the obstruction to extending } F \text{ over all of } \tilde{K} \text{ is the collection of first Chern classes } c_1(L_{\kappa_i}) \in \mathbb{Z}. \text{ This amounts to 24 integers, counting lifts of antiparticles of the fundamental fermions. Call this 24-tuple of integers } q_F.\]

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\[\text{\textsuperscript{35}The construction of the bundle } L \text{ from the data } F \text{ mimics the clutching function construction for bundles over spheres. It works because } X \setminus \tilde{K}^{(1)} \text{ is a disjoint union of cells each of whose closures in } X \text{ is an imbedded disc. So the argument for characterizing bundles over spheres by clutching functions works for characterizing bundles over } X.\]
Example. Consider a lift \( \tilde{e} \) of the electron to \( \tilde{K} \), so \( \tilde{e} \) is a 2-cell with boundary label \([\tilde{a}_2\tilde{a}_1\tilde{a}_0]\). The 2-sphere \( S_{\tilde{e}} \) is the double of the closed cell \( \tilde{e} \) over its boundary. If we take the electric charge as the conservation law, then the map of the boundary \( \partial\tilde{e} \) to \( U(1) \) has degree \(-3\) and \( N = 3 \). The element \( L \otimes \frac{1}{3} \) of \( \text{Pic}(X) \) restricted to \( S_{\tilde{e}} \) is the complex line bundle over \( S_{\tilde{e}} \) with clutching function a map of degree \(-1\) of the circle to \( U(1) \). This is the tautological line bundle over \( \mathbb{C}P^1 \) whose associated principal bundle with fibre \( U(1) = S^1 \) is the Hopf fibration over \( S_{\tilde{e}} \). In terms of algebraic geometry, this is the bundle \( O(-1) \) over \( \mathbb{C}P^1 \).

If instead of the electron we take a lift of the positron, the result is the hyperplane bundle \( O(1) \) over \( \mathbb{C}P^1 \). In more colorful language, the charge of the electron is the Hopf bundle and the charge of the positron is the hyperplane bundle over \( \mathbb{C}P^1 \).

Remark. The 2-complex \( X \), constructed in this section as a formal device for relating conservation laws to complex line bundles, plays a fundamental role in the sequel paper on bosons. It will be shown there that the Higgs mechanism for generating masses of particles in SM can be understood in terms of the geometry of \( X \) and that the involution interchanging the two copies of \( \tilde{K} \) and fixing \( \tilde{K}(1) \) can be understood as changing handedness of particles (left for right and right for left).

As pointed out in the lead paragraph of this section, we have given a direct geometric interpretation for the preons \( a, b \) as maps of one circle to another (from a closed 1-cell of \( K \) to \( U(1) \)), of degree 1 for \( a \) and degree 0 for \( b \). Such an interpretation is lacking in [Ha] and [Sh], where the preons are merely labels.

An open problem is to explain why nature chose this strange mechanism \( K, \tilde{K}, X \) for particle physics and why the number 3 as in 3-fold covering space.

Appendix 7. Colorings as conservation laws

In this appendix we give examples of conservation laws where the coefficient ring \( A \) is the color group \( M \). We shall use the isomorphic ring \( \mathbb{Z}[\zeta] \) (see Appendix 3) where \( \zeta = e^{2\pi i/3} \).

Recall that in this model the color set \( A_2 \) is the subset \( \pm \zeta^j \) where \( j \) ranges over the integers modulo 3.

Definition. A coloring of \( \tilde{K} \) is an element \( c \in B^2(\tilde{K}, \mathbb{Z}[\zeta]) \) such that

1. \( c \) vanishes on every lepton, and
2. \( c \) assumes all the elements of \( \{1, \zeta, \zeta^2\} \) as values on the set of lifts of the up quark \( aab \) and also on all the lifts of the down quark \( \bar{a}\bar{b}\bar{b} \).

Condition (1) is means physically that leptons do not feel the strong force. It follows from (2) that \( c \) cannot assume the same value on two different 2-cells of \( \tilde{K} \) which lift the same quark of \( K \). Since \( c \) is a coboundary, \( c \) vanishes on all 2-cycles. Since \( c \) is a cochain, \( c(\kappa) = -c(\bar{\kappa}) \), where \( \bar{\kappa} \) is the oppositely oriented 2-cell to \( \kappa \).

It is not immediately clear that there are any colorings (the catch is to show that a candidate cochain vanishes on all 2-cycles in \( Z_2(\tilde{K}, \mathbb{Z}) \)). That is the content of the next result.
Theorem. There are six colorings of $\tilde{K}$, which are hence conservation laws with values in $\mathbb{Z}[\zeta]$.

Proof of theorem. The crux of the argument consists of constructing a family of elements of $B^2(\tilde{K}, \mathbb{Z}[\zeta])$. This will be done by explicit construction of 2-cochains which vanish on $\pi_2(\tilde{K}) = Z_2(\tilde{K}, \mathbb{Z})$. We recall from §6 that $Z_2(\tilde{K}) =: Z_2(\tilde{K}, \mathbb{Z})$ is a free abelian group of rank 8. Two of the free generators are leptonic in the sense that they are of the form $g\kappa - \kappa$, where $\kappa$ is a lift of a lepton and $g \in G \cong \mathbb{Z}/3\mathbb{Z}$, so any observable cochain will vanish on them. Thus there are six remaining linear conditions of vanishing that must be satisfied.

The six vanishing conditions arise from the lifts of the spherical diagrams represented by Figures 1 and 2. It is easier to visualize these in terms of van Kampen diagrams, so we make cuts along edges to obtain disc diagrams; for convenience we have shown one lift each of the two disc diagrams below in Figure 8 (one can check that these are obtained by cutting spherical diagrams along edges since the boundary labels are freely trivial).

![Figure 8](image_url)

The left figure is a lift of the diagram for the Higgs boson that participates in $\beta$-decay (see Figure 5 in Appendix 4). We make the convention that the bottom triangle labeled $\tilde{d}$ represents the down quark. It follows that the top triangle labeled $\tilde{u}$ is the anti-particle to the up quark (since $\beta$-decay does not mix the generations of quarks). The two remaining triangles are leptons. So the vanishing condition on the cochain $c$ we are attempting to construct is $c(\tilde{d}) = c(\tilde{u})$. Similar conditions must hold in the other generations, since the process of $\beta$-decay lifts to these generations (see §3). Thus we obtain additional conditions $c(g\tilde{d}) = c(g\tilde{u})$ for all $g \in G$. This gives three of the six linear conditions that $c$ must satisfy to be an observable cochain.

The three colors $c(g\tilde{d}) \in \{1, \zeta, \zeta^2\}$, $g \in G$, may be chosen in any manner so as
to satisfy that they exhaust this set of colors. That means there are six choices of potential cochains if we consider their values restricted to the lifts of the down quark. But the conditions \( c(g\bar{d}) = c(g\bar{u}) \) of the previous paragraph determine completely the cochain \( c \), since the value of \( c \) on leptons is 0. It remains to establish that each of these six cochains vanishes on cycles \( Z_2(K) \).

We pass to the diagram on the right in Figure 8. The bottom triangle is seen to be \( \bar{d} \), and a calculation shows that the left and right triangles are \( g^2\bar{u} \) and \( g\bar{u} \) respectively.\(^{36}\) The top triangle is a lepton, so is assigned 0 by the cochain \( c \) under construction. Thus the values of \( c \) on the quark 2-cells in the diagram are \( \{c(g^2\bar{u}), c(\bar{d}), c(g\bar{u})\} = \{c(g^2\bar{d}), c(\bar{d}), c(g\bar{d})\} = \{1, \zeta, \zeta^2\} \) The same result holds for all translates of the diagram by \( g \in G \). But \( 1 + \zeta + \zeta^2 = 0 \) in \( \mathbb{Z}[\zeta] \), so it follows that the three remaining linear conditions are satisfied and that all six cochains \( c \) vanish on \( Z_2(K) \). Hence, by the first proposition of Appendix 3, \( c \in B^2(\bar{K}, \mathbb{Z}[\zeta]) \). □

Example. If we consider the coloring \( c \) given by \( d \rightarrow 1, g\bar{d} \rightarrow \zeta, g^2\bar{d} \rightarrow \zeta^2, u \rightarrow 1, g\bar{u} \rightarrow \zeta, g^2\bar{u} \rightarrow \zeta^2, \) then one can check that \( c = \delta L \), for \( L \in C^1(\bar{K}, \mathbb{Z}[\zeta]) \), where \( L \) is given by \( L(b_i) = 0 \) \( i \pmod{3} \), \( L(a_0) = 1 + \zeta^2, L(a_1) = 1 + \zeta, L(a_2) = \zeta + \zeta^2 \). Other choices of \( H' \) with \( \delta H' = c \) differ from \( H \) by a 1-coboundary.

If we apply a permutation \( \sigma \) of the set \( \{1, \zeta, \zeta^2\} \) to obtain the 1-cochain \( H_\sigma \), then their coboundaries \( \delta H_\sigma \) give all six of the colorings as \( \sigma \) ranges over the set of permutations of the set \( S \).

Question. What additional physically meaningful conservation laws are there in other rings \( A \)? A banal example in \( \mathbb{Z}/2\mathbb{Z} \) is the number of particles modulo 2; this is an invariant since, physically, creation and destruction of particles occurs in pairs, or, mathematically, since all spherical diagrams in \( \bar{K} \) have an even number of faces.

Appendix 8. Preons are determined by charge, color, and weak isospin

Here we tie together all notions of this paper and prove that the preon number (net number of \( a_i \)'s and \( b_i \)'s) in the composition of elementary fermions of SM are determined linearly by four of the conservation laws of SM, namely, electric charge, weak isospin, and a color conservation law of Appendix 7 and its complex conjugate.

Recall from §5 that electric charge is determined as \( \delta h \) where \( h \in C^1(\bar{K}, \mathbb{Q}) \) is given by \( h(a) = \frac{1}{3}, h(b) = 0 \). We lift \( h \) to \( \bar{K} \) and multiply by 3 to get \( H \in C^2(\bar{K}, \mathbb{Z}) \) given by \( H(a_i) = 1, H(b_i) = 0 \) for \( i \pmod{3} \). The conservation law \( \delta H \in B^2(\bar{K}, \mathbb{Z}) \subset B^2(\bar{K}, \mathbb{C}) \) will serve for our purposes the role of electric charge.

Recall also from §5 the weak isospin is determined by \( \delta w \), where \( w \in C^1(\bar{K}, \mathbb{Q}) \) is given by \( w(a) = w(b) = \frac{1}{3} \). We lift \( w \) to \( \bar{K} \) and multiply by 6 to get \( W \in C^2(\bar{K}, \mathbb{Z}) \) given by \( W(a_i) = W(b_i) = 1 \) for all \( i \pmod{3} \). The conservation law \( \delta W \in B^2(\bar{K}, \mathbb{Z}[\zeta]) \subset B^2(\bar{K}, \mathbb{C}) \) will serve for our purposes the role of weak isospin.

\(^{36}\)Recall \( g \) is the generator of the covering group \( G \cong \mathbb{Z}/3\mathbb{Z} \) where \( g \) is given by \( a \rightarrow 1, b \rightarrow 1 \) \( \pmod{3} \). The action of \( g \) on the lifts \( a_i, b_i \) is to raise the indices by 1 \( \pmod{3} \).
Let $L \in C^1(\hat{K}, \mathbb{Z}[\zeta])$ be the element of $C^1(\hat{K}, \mathbb{Z}[\zeta])$ determined in the Example of Appendix 7, so $L(b_1) = 0$, $L(a_0) = 1 + \zeta^2$, $L(a_1) = 1 + \zeta$, $L(a_2) = \zeta + \zeta^2$. We determined $L$ there so that $\delta L \in B^2(\hat{K}, \mathbb{Z}[\zeta]) \subset B^2(\hat{K}, \mathbb{C})$ is a color conservation law.

If $\bar{L}$ is the complex conjugate function, then, noting that $\bar{\zeta} = \zeta^2$ and $\bar{\zeta^2} = \zeta$, we have $\bar{L}(a_0) = 1 + \zeta, \bar{L}(a_1) = 1 + \zeta^2$, $\bar{L}(a_2) = \zeta + \zeta^2$, and $\bar{L}(b_1) = 0$. Note that $\bar{L}$ is itself one of the color conservation laws; in fact $\bar{L} = L_\sigma$, where $\sigma$ is the permutation of the set $\{1, \zeta, \zeta^2\}$ that interchanges $\zeta$ and $\zeta^2$ and leaves $1$ fixed. Note that $L + \bar{L} = \text{Re}(L)$, where $\text{Re}$ denotes the real part, so $(L + \bar{L})(a_0) = (L + \bar{L})(a_1) = 1$ and $(L + \bar{L})(a_2) = -2$.

Remark. If $\sigma$ is an even permutation of the set $\{1, \zeta, \zeta^2\}$, then $L_\sigma$ is of the form $\zeta^j L$ for some integer $j$, whereas $\bar{L}$ is of the form $L_\sigma$ for $\sigma$ a transposition.\footnote{The fact that the six color conservation laws are related in this way is a reflection of the fact that $S_3$, the Weyl group of the root system $A_2$, is generated by a 3-cycle (multiplication by $\zeta$) and a 2-cycle (complex conjugation).}

We can now state the main result.

**Theorem.** The four conservation laws $\delta H, \delta W, \delta L$ and $\delta(L + \bar{L})$ are linearly independent elements of $B^2(\hat{K}, \mathbb{C})$.

**Remark.** We calculated in Appendix 3 the dimension of the vector space $B^2(\hat{K}, \mathbb{R})$ to be 4 and pointed out in the footnote that the same result holds over every field; so it follows from the theorem that $B^2(\hat{K}, \mathbb{C})$ has basis over $\mathbb{C}$ given by $\delta H, \delta W, \delta L$ and $\delta(L + \bar{L})$. The next result follows immediately from the theorem; recall the definition of $\Delta_x$ for $x \in \{a_i, b_i\}$ from Appendix 3.

**Corollary.** For each $x \in \{a_i, b_i\}, \ i \ (\text{mod } 3)$, the conservation law $\delta \Delta_x$ is a linear combination over $\mathbb{C}$ of the four conservation laws $\delta H, \delta W, \delta L$ and $\delta(L + \bar{L})$.

**Remark.** The significance of the corollary is that the preon composition of all of the elementary fermions considered in $\hat{K}$ is completely determined by conservation laws of charge, weak isospin, and color. Consequently the preon composition of all composite particles (that is, the preons occurring in the boundary labels of their van Kampen diagrams in $\hat{K}$) is completely determined by these four conservation laws. So one is able to check a hypothetical structure of a composite particle by calculating the appropriate linear combination of conservation laws on the constituents.

**Proof of theorem.** Assume that a linear combination $\kappa \delta H + \lambda \delta W + \mu \delta L + \nu \delta(L + \bar{L}) = 0$, where $\kappa, \lambda, \mu, \nu \in \mathbb{C}$. Evaluate this relation on a lift of the positron $[a_0a_1a_2]$, noting that $\delta L$ and $\delta \bar{L}$ vanish on $[a_0a_1a_2]$, to get $3\kappa + 3\lambda = 0$. Next evaluate the relation on a lift of the electron neutrino $[b_0b_1b_2]$ to get $3\lambda = 0$. It follows that $\kappa = \lambda = 0$, so the linear combination simplifies to $\mu \delta L + \nu \delta(L + \bar{L}) = 0$.

Note that $\tilde{d} = [b_0b_1a_2]$ and $\tilde{g}d = [b_1b_2a_0]$. Evaluate the relation $\mu \delta L + \nu \delta(L + \bar{L}) = 0$ on $\tilde{d}$ and on $\tilde{g}d$ to get the equations $\mu L(a_i) + \nu \text{Re} L(a_i) = 0$ for $i = 2, 0$; that
is, $\mu(\zeta + \zeta^2) - 2\nu = 0$ and $\mu(1 + \zeta^2) + \nu = 0$. These equations have the unique solution $\mu = \nu = 0$, so the only linear relation among $\delta H, \delta W, \delta L$ and $\delta(L + \bar{L})$ is the trivial relation. It follows that they are linearly independent, and the theorem is established.

The theorem of color conservation suggests that restrictions on particles (van Kampen diagrams in $\tilde{K}$) can be directly related to $\tilde{K}$. With this in mind, we make two definitions.

**Definition.** An admissible particle is a van Kampen diagram $f : D \to \tilde{K}$ satisfying

1. for each pair $\alpha, \alpha'$ of distinct 2-cells in $D$ similarly oriented representing quarks with $f(\alpha) = f(\alpha')$ there is a coloring $c \in B^2(\tilde{K}, \mathbb{Z}[\zeta])$ so that $c(f(\alpha)) \neq c(f(\alpha'))$, and
2. at most two leptons appear in $D$, and their images under $f$ are not conjugate under the covering group $G$; that is, if $\alpha$ and $\alpha'$ distinct similarly oriented 2-cells of $D$ which represent leptons, then there is no $g \in G$ so that $g(f(\alpha)) = f(\alpha')$.

Condition (1) is independent of which conservation law $c$ is chosen to apply in the definition. This follows from the fact that all the 6 possible candidates for $c$ are related to each other by translation by $\zeta^j$ or by complex conjugation, as we remarked earlier.

The lepton condition (2) is more speculative; the justification is that it is difficult to see how, for example, an electron neutrino $\nu_e$ and a muon neutrino $\nu_\mu$ can bind together inside a particle. However $\nu_e$ and $\bar{\nu}_\mu$ together give one of the free generators of $\pi_2(\tilde{K})$ which was used in §6 to give a topological explanation for neutrino oscillations. On the other hand, the electron $e^-$ and electron anti-neutrino $\bar{\nu}_e$ are similarly oriented in the van Kampen diagram for the Higgs particle; but they have different preon representations which are not conjugate under the covering group $G$.

**Definition.** An admissible particle $f : D \to \tilde{K}$ is observable if the sum of the values of any cochain $c \in B^2(\tilde{K}, \mathbb{Z}[\zeta])$ on the faces $f(\alpha)$ is 0, where $\alpha$ ranges over the 2-cells of $D$.

By what was remarked, if the condition is satisfied for one color conservation law, it is satisfied for all of them.

**Remark.** The admissibility condition incorporates the Pauli exclusion principle, that two bound fermions cannot have the same quantum numbers. The observability condition strengthens the color conservation laws; it says no color can leak out of an observable particle. It is a way to formulate the experimental observation that no isolated quark has ever been observed.\(^{38}\)

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\(^{38}\)Color can however have an indirect second order effect on distinct particles by exchange of virtual particles, similar to the way van der Waal forces affect molecules even though the individual atoms are electrically neutral.
Ansatz\textsuperscript{39}. All particles of SM are admissible; thus even virtual particles that participate in reactions but are not observed are admissible. All particles that can be detected in an experiment satisfy the observability condition.

**Open problem.** Give a consistent geometric model that explains the algebraic conditions in the definitions of admissibility and observability of particles. That is, the appearance of the Eisenstein integers $\mathbb{Z}[\zeta]$ suggests that there is a discrete geometry underlying these conditions, but I do not yet know what this geometry is.

**Disclaimer**

I received the following in a letter from M. Gromov: “... in order to publish the article and not to be scorned at by the physicists community, Bogomolov, who as much as myself finds your article interesting and provocative, asks, if you agree, to do the following.

(1) To have author’s (yours) disclaimer on the relation of the article to “real physics” and an emphasis on the mathematical contents in it.

(2) A one page appendix by a professional physicist commenting on the original physical input of your article. The more critical it could be the better it would fare in the face of the physicists community.”

To comply with their request, I wrote the following disclaimer:

“I have been requested to include a disclaimer on the relation of this article to “real physics”. I am a mathematician, not a physicist, in my seventh year of retirement; prior to that I worked for 25 years in the field of combinatorial group theory, and earlier than that I published articles in fields involving algebra and topology, but not physics. Last November 2012 I read the Scientific American article [Li] and understood quickly that the Harari-Shupe theory suitably interpreted implied the solution to the generation problem, stated as an open problem in [Li]. I read all I could find in Wikipedia, having had no prior background in the physics involved, and asked questions of physicists, most of which were ignored and one of which was rebuffed with the statement that “there are no open problems; good luck with the math.” The accompanying letter from D. Singleton shows that I was encouraged to do real physics, to calculate masses of elementary particles and magnetic moments. One exception was Professor Wu, who listened to me in a meeting arranged by my colleague Domingo Toledo and made pertinent suggestions. He said I had solved the generation problem, but also told me that my paper would never be accepted by a physics journal.

As a further irony, I submitted the article to arXiv in the category Group Theory. The editorial board vetted the article and resubmitted it as General Physics.”

As for the point about criticism by a professional physicist, I received the following letter on 1 March 2013 from Douglas Singleton (unedited except for punctuation and TeX formatting):

\begin{footnote}{39}An Ansatz is an educated guess that is verified later by its results. Source: http://en.wikipedia.org/wiki/Ansatz\end{footnote}
From: Douglas Singleton dougs@csufresno.edu
To: Steve Gersten sg@math.utah.edu
Hi Professor Gersten,

Sorry for the delayed reply. I’m currently on sabbatical which in principle means I should have more time, but also in expectation of this I have taken on extra projects.

Anyway your general question and suggestion (that electrons, photons, gluons, etc.) should have some sub-structure is interesting and in fact I tried to think of such a model some time ago, but there are a lot of problems such models face.

First a non-important comment – you had said that you have had a hard time getting physicists to look at your work. I think this is a combination of the fact that composite models face many challenges (I’ll detail some below) but as well the formalism/notation you use is not in the standard tool kit of physicists so you would either need to “translate” your work for physicists or start a campaign to educate physicists to the formalism you use. For example there is a book on differential forms by H. Flanders from Purdue from the 1960s and in the foreword he specifically says his intent (he is a mathematician) was to educate physicists and engineers to the beauty of differential forms, wedge products, etc. In the 90s when I was in grad school differentials forms were still not part of the standard curriculum. I learned them (somewhat) on my own using Flanders book. In any case if you use unfamiliar formalism/notation this will explain why most physicist will not take a look at your work.

For example in the opening of your fermion paper you talk about “Kampen diagrams”. I looked at your pictures and these look similar to Dynkin diagrams/root diagrams/weight diagrams from group theory (have a look at Howard Georgi’s book on Lie Groups for physicists and your Kampen diagrams look similar to some of these other diagrams I mention, but then again not exactly the same). In turn I scanned both articles and did not find any obvious mention of SU(2), SU(3), U(1) or even string theory groups like E(8) x E(8) or any of the other exceptional Lie groups. It may be that these are discussed in a manner/formalism I do not know, but in any case this would explain why physicists have ignored your work. Also I checked the arXiv (both hep-th and math-ph) and there was no mention of “Kampen diagrams” – I did a “full text” search for “Kampen” and found only a guy named van Kampen and then three papers which referenced van Kampen’s work – but no reference to Kampen diagrams. A google search turned up Kampen diagrams but in looking through the first 2–3 pages I did not see any physicists working with them. Again in some sense this is not really important since one can attribute this to the “bad” math education of physicists, but as a practical matter it will mean your work will have an uphill battle getting attention from the physics community.

OK but let me move on to substantive comments. The first questions you should be able to calculate an answer for if your model is correct is

(i) What is the mass of the electron, muon, tau in terms of the more fundamental masses of your “preons” a, b? (By the way I very much liked Harari’s work and
this inspired me to try my hand at this type of model building but my attempt foundered on exactly this and the following questions). Or at least you need to calculate the mass ratio $e/\mu$ and $\mu/\tau$.

(ii) What are the values of the CKM (Cabibbo-Kobayashi-Maskawa) mixing elements between the three generations?

If your model is correct and useful then you should be able to calculate the above quantities in terms of the input parameters of your model (I guess the parameter should be the masses and mixings of the $a$ and $b$ “preons”). If you could do this then people would pay attention even if they did not at first understand your notation/formalism.

Next I have some general comments which seem problematic.

(a) Electrons, muons, quarks, etc. seem to be made of three “preons” and these $a, b$ “preons” I assume are spin 1/2 so that the results electron, quarks etc. will come out to be spin 1/2. However if electrons are composite in this way (rather than being fundamental Dirac spinors) then their g-factor is no longer guaranteed to be $g \sim 2$ (up to QED corrections) but should be calculated from the underlying theory. In other words you need to take your composite electron, calculate its g-factor and show that it is about 2. For example the proton and neutron are both known to be spin 1/2 but they are composite spin 1/2 particles being made up of (at least at the first level) quarks. Now if one assumes that the quarks are fundamental Dirac particles with $g \sim 2$ one can to a pretty good job at getting the g-factors for the proton and neutron ($\sim 5.59$ and $\sim -3.83$ respectively) in terms of the more fundamental g-factors of the quarks taken at their Dirac value. Note that the g-factors of the proton and neutron are *very* different from 2. Also the calculation of these values from QCD models is only at the 5% level as compared to the 0.000000001% level of QED calculations of the electron g-factor. This is said to be due to the very large non-perturbative quantum corrections coming from the strong interaction. If you could get your model to give the correct g-factor of the electron, muon, tau, etc. to the 5% level this would good enough since that is what is done in the case of protons and neutrons.

(b) Your photon seems to be composed of $\bar{a}a$ i.e. the photon is also a bound state. But the photon is experimentally known to be massless to a very high degree and theoretically this is said to come from gauge invariance (which can be broken via the Higgs mechanism but the Higgs in our Universe does not do this for the photon so gauge invariance means the photon is exactly massless and this agrees very well with experiment). Now if your photon is a bound state of more fundamental spin 1/2 particles it is highly unlikely that it would be massless to such a high degree. This would require an extremely unlikely cancellation of the binding energy of the system against the masses of the $a$ and $\bar{a}$. In any case you need to show that your model gives the mass of the bound state $\bar{a}a$ (i.e. photon) as zero to some fantastic accuracy. To do this you will need to specify the energy scale of the interaction which binds your “preons” together. I also did not see mention of this and it is important – what is the energy scale/interaction strength of the interaction which binds your preons together? For example in the strong interaction the dimensionless
coupling constant at low energies is \(\sim 1 - 5\) (the range is large since QCD is poorly understood in the low energy limit). For the E\& M interaction the coupling strength at low energies is \(\sim 1/137\) (which also explains why perturbation theory works at low energy for QED but fails from QCD).

(c) Also if your photon is composed of a fermion-anti-fermion both of which are charged then it will have in general some magnetic moment, electric dipole moment, etc. It is well known experimentally that the photon does not have these thus in your model you should explain why your composite photon has no moments to such a high degree. The same comment applies to the W and Z bosons which in your composite model would have – barring some cancellation – magnetic, electric dipole moments in general. In fact the W from the SM has a spin and magnetic dipole momentum that lead to a g-factor of 2 (same as for a fundamental Dirac particle). Because of this some scattering processes involving the W boson have “gauge amplitude zeros” in the differential scattering cross section which been confirmed to the 5% level. The original article on this is K.O. Mikaelian, M.A. Samuel, and D. S ahdev, Phys. Rev. Lett. 43, 746-749(1979). Thus your model would also need to how a composite W would get a g-factor so close to 2.

Anyway the first task - if you haven’t done it or if I missed it in the two articles - would be to calculate the electron, muon, tau masses (or at least their ratios) in terms of the parameters of your model. Then next would be to calculate the values of CKM elements. Another good place to test your model would be in predicting the mass structure of the neutrinos. The oscillation data for neutrinos give two possible mass structures – regular and inverted. If your model could predict which is the correct mass structure for neutrinos this would be a good *prediction* of the model (the other things I ask above are post-dictions).

Sorry for the delay and then long email. I was also going to explain a bit about my idea for composite leptons and quarks but as I said for my model I got stuck on exactly the above questions.

Best,

Doug

A final word from the author, your truly. Part of Singleton’s letter refers to part II of the article, which is under revision. The preface to the present article is an attempt to, at the very least, introduce the concept of van Kampen diagram, to give a readable reference in Wikipedia, and to give a down-to-earth analogy with Buckminster Fuller’s geodetic dome, so that the concept will not appear so strange.

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Mathematics Department, University of Utah, 155 S. 1400 E., Salt Lake City, UT 84112-0090, http://www.math.utah.edu/~sg

E-mail address: sg at math dot utah dot edu