An Extended Abelian Chern-Simons Model and the Symplectic Projector Method

L.R.U. Manssur*, †, A.L.M.A. Nogueira*, † and M.A. Santos‡

*Centro Brasileiro de Pesquisas Físicas, CBPF - DCP
Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil

†Universidade Católica de Petrópolis - Grupo de Física Teórica, GFT - UCP
Rua Barão do Amazonas 124, 25685-070 Petrópolis, Brazil

‡Laboratório Nacional de Computação Científica, LNCC-DMA
Av. Getúlio Vargas 333, 25651-070 Petrópolis, Brazil

§Dep. de Física, Univ. Fed. Rural do Rio de Janeiro (UFRRJ)
23851-180, Seropédica, RJ-Brazil

Abstract
The Symplectic Projector Method is applied to discuss quantisation aspects of an extended Abelian model with a pair of gauge potentials coupled by means of a mixed Chern-Simons term. We focus on a field content that spans an N=2-D=3 supersymmetric theory whenever scalar and fermionic matter is suitably coupled to the family of gauge potentials.

1 Introduction

Nearly ten years ago, a method was developed that treats the fundamental question of canonically quantising field theories based on gauge symmetries. In this method, a crucial point is to identify, among the original constrained coordinates, those quantities related to the true degrees of freedom, which we refer to as the physical variables. We shall call this procedure the Symplectic Projector Method (SPM).

Along this line of investigation, the SPM has been tested through a number of relevant situations, such as Classical Electrodynamics, the 2-Dimensional Bosonised Schwinger Model, the Christ-Lee Model and the Chern-Simons-Maxwell Theory.
In this work, we reassess the efficacy of the method in picking up the true (physical) field coordinates for a sort of extended Abelian gauge model with a Chern-Simons term coupling a pair of gauge potentials. Such a model is the 3D descent of a 4D gauge theory with a topological mass term involving the Kalb-Ramond 2-form gauge potential. Also, the model discussed here is the core of the (gauge) bosonic sector (we leave behind an additive bilinear for a massive scalar field) of an \( N = 2 \)-supersymmetric gauge theory that leads to (after suitable identifications of fields) an \( N = 2 \) model endowed with a rich structure of topological magnetic vortices \([5, 6]\).

Since our model displays a vector potential with a peculiar gauge transformation and without any dynamics, if taken on-shell, we believe it could also provide an interesting working example to test the consistency and the efficacy of the SPM.

Our paper is presented according to the following outline: in Section 2, we show explicitly the 4D origin of our model and perform its dimensional reduction towards the 3D mixed Chern-Simons theory; in Section 3, we establish the set of constraints and apply the SPM to pick up the physical variables. Finally, we present our General Conclusions.

2 The 4-Dimensional Model and Its Reduction

The 4-dimensional- \( U(1) \times U(1) \) model we start off is based on the presence of a vector potential, \( A_\mu \), together with a rank-2 gauge potential, \( B_{\hat{\mu}\hat{\nu}} = -B_{\hat{\nu}\hat{\mu}} \), the latter playing the role of a Kalb-Ramond field \([7]\). We use hatted indices to denote components with respect to 4-dimensional space, while the bare ones refer to \( D=3 \) (\( \hat{\mu} = 0, 1, 2, 3 \) and \( \mu = 0, 1, 2 \)). The corresponding field strengths are given as below:

\[
F_{\hat{\mu}\hat{\nu}} \equiv \partial_{\hat{\nu}} A_{\hat{\mu}} - \partial_{\hat{\mu}} A_{\hat{\nu}},
\]

\[
G_{\hat{\mu}\hat{\nu}\hat{\kappa}} \equiv \partial_{\hat{\kappa}} B_{\hat{\nu}\hat{\mu}} + \partial_{\hat{\nu}} B_{\hat{\kappa}\hat{\mu}} + \partial_{\hat{\mu}} B_{\hat{\nu}\hat{\kappa}},
\]

and the coupling with a general matter field is carried out by means of the extended gauge-covariant derivative as below:

\[
D_\mu \Phi \equiv (\partial_\mu + ieA_\mu + igG_\mu)\Phi,
\]

where \( G_\mu \) is the dual of the field strength 3-form:

\[
G_\mu = \frac{1}{3!} \epsilon_{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\rho}} G^{\hat{\nu}\hat{\kappa}\hat{\rho}}.
\]
This means that charged matter couples minimally to $A_\mu$ (coupling constant $e$) and non-minimally to $B_{\mu\nu}$ ($g$ is the coupling parameter governing the non-minimal interaction).

We propose to begin with the 4D action as follows:

$$\mathcal{L}_{4D} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{12} G_{\mu\nu\kappa}^2 + \frac{1}{2} m e^{\hat{\mu}\hat{\nu}\hat{\kappa}} A_\mu \partial_\nu B_{\kappa\lambda} + \text{(matter-gauge terms)}; \quad (5)$$

we do not specify the matter-gauge coupling terms as we wish to discuss the quantisation of the gauge potential sector exclusively. For a discussion of the complete model, we refer the reader to the works of refs. [5, 6]. $m$ is the mass parameter associated to the presence of a massive spin-1 gauge boson in the spectrum [8].

The idea is now to dimensionally reduce the model to (1+2) dimensions, adopting the Scherk ansatz [9], by simply assuming that all fields do not depend on $x^3$:

$$\partial_3 \text{(fields)} = 0. \quad (6)$$

$A_\mu$ yields two independent fields in 3D, namely, a gauge potential, $A_\mu$, and a scalar, $\varphi \equiv A_3$; on the other hand, two vector potentials stem from $B_{\mu\nu}$:

$$B_\mu \equiv B_{\mu3} \quad (7)$$

and

$$Z_\mu \equiv \frac{1}{2} \epsilon_{\mu\kappa\lambda} B^{\kappa\lambda}. \quad (8)$$

The gauge transformations read now as given below:

$$A'_\mu = A_\mu + \partial_\mu \alpha, \quad (9)$$

$$B'_\mu = B_\mu + \partial_\mu \beta, \quad (10)$$

$$Z'_\mu = Z_\mu + \epsilon_{\mu\kappa\lambda} \partial_\nu \xi^\lambda, \quad (11)$$

where $\alpha$, $\beta$ and $\xi^\lambda$ are arbitrary functions: $\alpha$ is the gauge parameter associated to the gauge symmetry of $A_\mu$, whereas $\beta$ and $\xi^\lambda$ are the 3D descents ($\beta \equiv \xi_3$) of the (vector) gauge parameter associated to $B_{\mu\nu}$. The reduced model exhibits a $[U(1)]^3$-symmetry; the extra Abelian factor comes about by virtue of the 4D gauge symmetry of the Kalb-Ramond field [10]. Moreover, eq.(11) displays an unusual gauge transformation for the vector potential $Z_\mu$, as we have anticipated. Such an exchange of rôles between the longitudinal and transverse sectors of the vector potential $Z_\mu$ (the longitudinal part is now gauge-invariant) has as a counterpart odd expressions for some of the constraints, as we shall see in Section 3.
The relationships between the 4D and 3D field strengths are readily worked out and
the following expressions can be shown to hold:

\[ F_{\mu\nu}^2 \hat{\mu} \hat{\nu} = F_{\mu\nu}^2 - 2(\partial_\mu \varphi)^2, \] (12)

\[ G_{\mu\nu\kappa}^2 \hat{\mu} \hat{\nu} \hat{\kappa} = -3G_{\mu\nu}^2 + 6(\partial_\mu Z^\mu)^2, \] (13)

\[ \epsilon^{\mu\nu\kappa\lambda} A_\mu \partial_\nu B_\kappa = 2\epsilon^{\mu
u\kappa} A_\mu \partial_\nu B_\kappa - 2\varphi(\partial_\mu Z^\mu), \] (14)

so that the 3D Lagrangian for the gauge fields takes over the form:

\[ L_{3D}^{\text{gauge}} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{4} G_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu Z^\mu)^2 + m\epsilon^{\mu\nu\kappa} A_\mu \partial_\nu B_\kappa - m\varphi(\partial_\mu Z^\mu), \] (15)

where \( F_{\mu\nu} \) and \( G_{\mu\nu} \) are the field strengths corresponding to \( A_\mu \) and \( B_\mu \), respectively. One should also notice the presence of a non-diagonal Chern-Simons term along with a partner that mixes \( \varphi \) and \( Z^\mu \). The 3D action of eq. (15) is invariant under the \( U(1) \times U(1) \times U(1) \)-symmetry quoted in eqs. (9), (10) and (11).

In the work of ref. [5], the potentials \( A_\mu \) and \( B_\mu \) were suitably identified with each other in a consistent way in connection with an N=2-D=3 supersymmetric version of the Maxwell-Chern-Simons model with anomalous magnetic couplings of matter to the gauge fields.

Before going ahead to discuss the symplectic quantisation of the model under consideration, we should perhaps mention that the 3 massive degrees of freedom that propagate in 4D [8] are now accommodated in \( A_\mu \) (1 d.f.), \( B_\mu \) (1 d.f.) and \( \varphi \) (1 d.f.); the vector potential \( Z_\mu \), featured with the unusual gauge transformation given in eq. (11) and with the potentially dangerous longitudinal kinetic term \((\partial_\mu Z^\mu)^2\), does not propagate any on-shell degree of freedom (in fact, as displayed in eq.(8), \( Z_\mu \) is just the dual of the well-known non-propagating 3D Kalb-Ramond field). This means that one of the \( U(1) \)-factors has no dynamical significance. The application of the Symplectic Projector Method to reassess the quantisation of this peculiar 3D gauge model in the sequel shall illustrate this procedure in a more evident way.

### 3 Constraints and relevant degrees of freedom

In view of what we have set previously, we define our model by means of the Lagrangian density:

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} G^{\mu\nu} G_{\mu\nu} + \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(\partial_\mu Z^\mu)^2 - m\varphi(\partial_\mu Z^\mu) + m\epsilon^{\mu\nu\rho} B_\mu \partial_\nu A_\rho, \] (16)
with
\[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (17) \]
\[ G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu, \quad (18) \]
and the metric signature \( \eta^{\mu\nu} = (+, -, -) \). Picking up the canonically conjugate momenta, we have:
\[ \pi^\mu \equiv \frac{\delta L}{\delta (\partial_0 A_\mu)} = - F^{0\mu} + m \varepsilon^{0\mu\nu} B_\nu, \quad (19) \]
which yields
\[ \pi^0 = 0 \quad (20) \]
\[ \pi^i = - F^{0i} + m \varepsilon^{0ik} B_k. \quad (21) \]
Also,
\[ P^\mu \equiv \frac{\delta L}{\delta (\partial_0 B_\mu)} = - G^{0\mu}, \quad (22) \]
or
\[ P^0 = 0 \quad (23) \]
\[ P^i = - G^{0i}. \quad (24) \]
For the scalar field,
\[ \pi_\varphi \equiv \frac{\delta L}{\delta (\partial_0 \varphi)} = \partial_0 \varphi. \quad (25) \]
Finally,
\[ \pi'^\mu \equiv \frac{\delta L}{\delta (\partial_0 Z_\mu)} = \left( -m \varphi + \partial^\beta Z_\beta \right) \eta^{\mu\nu} \quad (26) \]
or
\[ \pi'^0 = \left( -m \varphi + \partial^\beta Z_\beta \right) \quad (27) \]
\[ \pi'^i = 0. \quad (28) \]

Now, we are ready to write down the canonical Hamiltonian of the theory:
\[ H_c = \pi^\mu \partial_0 A_\mu + P^\mu \partial_0 B_\mu + \pi'^\mu \partial_0 Z_\mu + \pi_\varphi \partial_0 \varphi - \mathcal{L} \]
\[ = \frac{1}{2} \pi_i^2 + \frac{1}{2} P_i^2 + \frac{1}{2} \pi_\varphi^2 + \frac{1}{2} \pi'^2 + A_0 (\partial_i \pi_i) + B_0 (\partial_i P_i - m \varepsilon_{0ij} \partial_i A_j) + \]
\[ + \frac{1}{4} F_{ij} F_{ij} + m \varepsilon_{0ik} \pi_i B_k + \frac{1}{2} m^2 B_k B_k + \frac{1}{4} G_{ij} G_{ij} + \frac{1}{2} (\partial_i \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \]
\[ + \pi'^0 (m \varphi + \partial_0 Z_0). \quad (29) \]
The primary Hamiltonian is just
\[ H_p = H_c + v_i \Omega_i, \]  
(30)
where the primary constraints are
\[ \begin{align*}
\Omega_1 &= \pi_0 \approx 0 \\
\Omega_2 &= P_0 \approx 0 \\
\Omega_3 &= \pi'_1 \approx 0 \\
\Omega_4 &= \pi'_2 \approx 0.
\end{align*} \]  
(31)
As one can immediately notice, \( \Omega_3 \) and \( \Omega_4 \) are the first counterparts of the \( Z_\mu \)'s unusual features to be brought about in the set of constraints. The consistency condition imposed on the latter yields:
\[ \begin{align*}
\Omega_5 &= \partial_i \pi_i \approx 0 \\
\Omega_6 &= \partial_i P_i - m \varepsilon_{0ij} \partial_i A_j \approx 0 \\
\Omega_7 &= \pi'_0 - f(t) \approx 0,
\end{align*} \]  
(32)
for some arbitrary \( f(t) \). They are all first class constraints; the gauge-fixing conditions will be so chosen that
\[ \begin{align*}
\Omega_8 &= A_0 \approx 0 \\
\Omega_9 &= B_0 \approx 0 \\
\Omega_{10} &= Z_1 \approx 0 \\
\Omega_{11} &= Z_2 \approx 0 \\
\Omega_{12} &= \partial_i A_i \approx 0 \\
\Omega_{13} &= \partial_i B_i \approx 0 \\
\Omega_{14} &= Z_0 \approx 0,
\end{align*} \]  
(33)
where we have imposed the "Coulomb" gauge-fixing on \( Z_\mu \) in straight analogy to the usual procedure as applied onto \( A_\mu \) (and \( B_\mu \)). As a net result, the collection of constraints related to \( Z_\mu \) already indicate that its phase space variables are to be excluded from the dynamical subset.

We have now to build the matrix \( g_{ij}(x, y) = \{ \Omega_i(x), \Omega_j(y) \} \). Adopting the notational convention \( \delta \equiv \delta^2 (x - y) \), we get:
\(g(x, y) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 \\
\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 \\
0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 \\
0 & 0 & -\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\partial^x_i \partial^y_j \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\partial^x_i \partial^y_j \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \tag{34}
\]

whose inverse reads as below:

\(g^{-1}(x, y) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +\nabla^{-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +\nabla^{-2} & 0 \\
-\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\nabla^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\nabla^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}. \tag{35}
\)
We shall label the fields and their corresponding momenta as follows:

\[
(A^0, A^1, A^2, \varphi, B^0, B^1, B^2, Z^0, Z^1, Z^2, \pi_0, \pi_1, \pi_2, \pi_\varphi, P_0, P_1, P_2, \pi'_0, \pi'_1, \pi'_2) \equiv \\
(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{10}, \xi_{11}, \xi_{12}, \xi_{13}, \xi_{14}, \xi_{15}, \xi_{16}, \xi_{17}, \xi_{18}, \xi_{19}, \xi_{20}).
\]

At this stage, we are able to calculate the symplectic projector defined by the expression [2]:

\[
\Lambda^i_j (x, y) = \delta^i_j \delta^2 (x - y) - \varepsilon^{ik} \int d^2z \, d^2w \left[ \delta_k(x) \Omega^m(z) \right] g_{mn}^{-1}(z, w) \left[ \delta_j(y) \Omega^n(w) \right],
\]

where

\[
\delta_k(x) \Omega^m(z) \equiv \frac{\delta \Omega^m(z)}{\delta \xi_k(x)},
\]

and \( \varepsilon \equiv \{ \varepsilon^{ik} \} \) is the symplectic matrix. The prescription to obtain the projected variables is

\[
\xi^{is}(x) = \int d^2y \, \Lambda^i_j (x, y) \xi^j (y),
\]

yielding the following expressions for those which are non-trivial:

\[
\begin{align*}
\xi^{2s}(x) &= A^{1\perp}(x) \quad (39) \\
\xi^{3s}(x) &= A^{2\perp}(x) \quad (40) \\
\xi^{4s}(x) &= \varphi(x) \quad (41) \\
\xi^{6s}(x) &= B^{1\perp}(x) \quad (42) \\
\xi^{7s}(x) &= B^{2\perp}(x) \quad (43) \\
\xi^{12s}(x) &= \pi_1^\perp(x) + m \int d^2y \partial_{x^2} \nabla^{-2}(x, y) \left[ \partial_{y_1} B_1(y) + \partial_{y_2} B_2(y) \right] \\
&= \pi_1^\perp(x) \quad (44) \\
\xi^{13s}(x) &= \pi_2^\perp(x) - m \int d^2y \partial_{x^2} \nabla^{-2}(x, y) \left[ \partial_{y_1} B_1(y) + \partial_{y_2} B_2(y) \right] \\
&= \pi_2^\perp(x) \quad (45) \\
\xi^{14s}(x) &= \pi_\varphi(x) \quad (46) \\
\xi^{16s}(x) &= P_{1\perp}(x) + m \partial_{x_1} \int d^2y \nabla^{-2}(x, y) \left[ \partial_{y_1} A_2(y) - \partial_{y_2} A_1(y) \right] \\
&= P_{1\perp}(x) + m \partial_{x_1} \int d^2y \nabla^{-2}(x, y) \left[ \nabla \times A_1^\perp \right]_y \quad (47) \\
\xi^{17s}(x) &= P_{2\perp}(x) + m \partial_{x_2} \int d^2y \nabla^{-2}(x, y) \left[ \partial_{y_1} A_2(y) - \partial_{y_2} A_1(y) \right] \\
&= P_{2\perp}(x) + m \partial_{x_2} \int d^2y \nabla^{-2}(x, y) \left[ \nabla \times A_2^\perp \right]_y . \quad (48)
\end{align*}
\]
The reduced canonical Hamiltonian is obtained from the previous primary Hamiltonian taking into account the constraints and the gauge conditions, now looked upon as strong equalities. We obtain:

\[ H'_{c} = \frac{1}{2} \pi_{i}^{2} + \frac{1}{2} p_{i}^{2} + \frac{1}{2} \pi_{i}^{2} + \frac{1}{2} \pi_{0}^{2} + \frac{1}{4} F_{ij} F_{ij} + m \varepsilon_{0k} \pi_{i} B_{k} + \frac{1}{2} m^{2} B_{k} B_{k} \]

\[ + \frac{1}{4} G_{ij} G_{ij} + \frac{1}{2} (\partial_{i} \varphi)^{2} + \frac{1}{2} m^{2} \varphi^{2} + \pi_{0} (m \varphi), \]

or, in symplectic notation,

\[ H'_{c} = \frac{1}{2} (\xi_{12}^{2} + \xi_{13}^{2}) + \frac{1}{2} (\xi_{16}^{2} + \xi_{17}^{2}) + \frac{1}{2} \xi_{14}^{2} + \frac{1}{2} (\partial_{i} \xi_{3})^{2} + \frac{1}{2} (\partial_{2} \xi_{2})^{2} + \]

\[- (\partial_{1} \xi_{3}) (\partial_{2} \xi_{2}) - m (\xi_{12} \xi_{7} - \xi_{13} \xi_{6}) + \frac{1}{2} m^{2} (\xi_{6}^{2} + \xi_{7}^{2}) + \frac{1}{2} (\partial_{1} \xi_{7})^{2} + \frac{1}{2} (\partial_{2} \xi_{6})^{2} + \]

\[- (\partial_{1} \xi_{7}) (\partial_{2} \xi_{6}) + \frac{1}{2} (\partial_{1} \xi_{4})^{2} + \frac{1}{2} (\partial_{2} \xi_{4})^{2} + \frac{1}{2} m^{2} \xi_{4}^{2} + \xi_{18} (m \xi_{4}). \]

The physical Hamiltonian density is obtained by rewriting the one given above in terms of the projected variables:

\[ H^{*} = \frac{1}{2} (\xi_{12}^{*2} + \xi_{13}^{*2}) + \frac{1}{2} (\xi_{16}^{*2} + \xi_{17}^{*2}) + \frac{1}{2} \xi_{14}^{*2} + \frac{1}{2} (\partial_{i} \xi_{3}^{*})^{2} + \frac{1}{2} (\partial_{2} \xi_{2}^{*})^{2} + \]

\[- (\partial_{1} \xi_{3}^{*}) (\partial_{2} \xi_{2}^{*}) - m (\xi_{12}^{*} \xi_{7}^{*} - \xi_{13}^{*} \xi_{6}^{*}) + \frac{1}{2} m^{2} (\xi_{6}^{*2} + \xi_{7}^{*2}) + \frac{1}{2} (\partial_{1} \xi_{7}^{*})^{2} + \]

\[+ \frac{1}{2} (\partial_{2} \xi_{6}^{*})^{2} - (\partial_{1} \xi_{7}^{*}) (\partial_{2} \xi_{6}^{*}) + \frac{1}{2} (\partial_{1} \xi_{4}^{*})^{2} + \frac{1}{2} (\partial_{2} \xi_{4}^{*})^{2} + \frac{1}{2} m^{2} \xi_{4}^{*2}. \]

Finally, the equations of motion are obtained directly from the Hamilton-Jacobi equations by means of Poisson parentheses of the projected variables with the Hamiltonian \( \int d^{2}y \ H^{*}(y) \). In so doing, we arrive at:

\[ \dot{\xi}_{4}^{*}(x) = \int d^{2}y \ \{ \xi_{4}^{*}(x), H^{*}(y) \} = \xi_{14}^{*}(x) ; \]

this yields

\[ \ddot{\xi}_{4}^{*} = \dot{\xi}_{14}^{*} = \int d^{2}y \ \{ \xi_{14}^{*}, H^{*}(y) \} = -m^{2} \xi_{4}^{*} + \nabla^{2} \xi_{4}^{*} , \]

or

\[ (\Box + m^{2}) \xi_{4}^{*} = 0 . \]
Analogously, we obtain

\begin{align}
\dot{\xi}_2^* &= \partial_2 \partial_2 \xi_2^* - \partial_1 \partial_2 \xi_3^* - m \xi_{17}^* \\
\dot{\xi}_3^* &= \partial_1 \partial_1 \xi_3^* - \partial_1 \partial_2 \xi_2^* + m \xi_{16}^* \\
\dot{\xi}_6^* &= -m^2 \xi_6^* + \partial_2 \partial_2 \xi_6^* - \partial_1 \partial_2 \xi_7^* - m \xi_{13}^* \\
\dot{\xi}_7^* &= -m^2 \xi_7^* + \partial_1 \partial_1 \xi_7^* - \partial_1 \partial_2 \xi_6^* + m \xi_{12}^*.
\end{align}

(55)

Now, eqs.(55) can be rephrased in a much simpler form if one chooses, without loss of generality, the momentum to lay upon the x-axis \((\vec{k} = (k, 0))\), selecting the components \((A_2, B_2)\) to be the transverse ones. One can easily notice that such a choice would cancel the variables \(\xi_2^*, \xi_6^*\) and \(\xi_{12}^*\), and render the variables \(\xi_{16}^*\) and \(\xi_{17}^*\) equal to:

\begin{align}
\xi_{16}^* &= -m \xi_3^*, \\
\xi_{17}^* &= P_{2}^\perp.
\end{align}

The set of independent variables would correspondingly be specified by the pairs \((\xi_3^*, \xi_{13}^*)\), \((\xi_4^*, \xi_{14}^*)\) and \((\xi_7^*, \xi_{17}^*)\). The remaining equations of motion would then read:

\begin{align}
\Box \xi_4^* &= -m^2 \xi_4^*, \\
\Box \xi_3^* &= -m^2 \xi_3^*, \\
\Box \xi_7^* &= -m^2 \xi_7^*.
\end{align}

(56)

So, we have a massive scalar, \(\xi_4^*\), and two massive transverse vector fields, \(\xi_3^*\) and \(\xi_7^*\), according to what could be expected from our counting of degrees of freedom in the framework of the 3D Lagrangian given by eq.(16), if we were to keep the mapping \(B_{\mu\nu} \rightarrow Z_{\mu}\) in mind. Such an outcome also matches the natural allocation of physical degrees of freedom that could be inferred from the original 4D model. Moreover, the fact that the two vectors provide the physical sector with equivalent transverse massive contributions indicates the room for a consistent mapping into a new model, a procedure that can be implemented through the identification of the vector fields (and partners, in a supersymmetric context), as performed in Ref.[5].

4 General Conclusions

We have reassessed the efficacy of the SPM in selecting the true dynamical set of phase space variables for a gauge 3D model hosting a peculiar topological term, namely, a \textit{mixed}
Chern-Simons bilinear. As a consequence of the dimensional reduction procedure, which defines our model as a descent of the 4D Cremmer-Scherk-Kalb-Ramond model \cite{7,10}, a counterpart for the mixed Chern-Simons term shows up as a source of mass for the scalar field. The SPM has proven to be efficient in casting the physical variables and exhibiting their dynamics through the field equations \cite{57}, where the expected topological mass generation is explicitly displayed.

Concerning the results obtained through Dirac’s \cite{11} method for the canonical quantisation of gauge systems, as applied to our 3D model, the dynamics turns out to be the same. This convergence of outcomes can be seen as another check of consistency for the SPM. Moreover, in contrast to the possibility, \textit{in principle}, to reduce the phase space through Dirac’s approach, by using the SPM we always place ourselves in the right context for totally identifying, with the very explicit corresponding expressions, the true physical variables, an outcome lying on the very heart of the geometrical nature of the method.

The analysis we have pursued in this work happens to be an achievement in the sense of fully and rigorously specifying the dynamics generated in the gauge sector of an interesting N=2 off-shell supersymmetric model, one that can be mapped (through suitable identifications of fields) into another N=2 system in which topological self-dual vortex solutions can be found on-shell \cite{5,8}. The presence of two massive transverse vector excitations along with a massive scalar mode in the (bosonic) gauge sector is thus the relevant information confirmed as part of our knowledge about the above mentioned N=2-D=3 model. Also, applying the Symplectic Projector Method has proven to be a consistent choice for the complete clarification of the symplectic structure underlying the phase space spanned by this gauge model.

\textbf{Acknowledgements}

The authors are grateful to Prof. J.A. Helayël-Neto for useful suggestions, discussions and the always kindest support to our work. L.R.U. Manssur and M.A. Santos also express their gratitude to the GFT-UCP (Group of Theoretical Physics), at Universidade Católica de Petrópolis - UCP (Petrópolis, Brazil), where part of this work has been done, for the kind and warm hospitality.

\textbf{E-mail contact:}

†leon@cbpf.br, ⋆nogue@cbpf.br
References

[1] C. Marcio do Amaral, Nuovo Cim., B 25 (1975) 817; C. Marcio do Amaral and P. Pitanga, Rev. Bras. Fís., 12, no 3, (1982) 473; P. Pitanga and K.C. Mundim, Nuovo Cim., A 101 (1989) 345; P. Pitanga, Nuovo Cim., A 103 (1990) 1529; P. Pitanga, Projectors in Constrained Dynamics, in portuguese, PhD. thesis, CBPF, 1991; as a brief review we suggest: M.A. De Andrade, M.A. Santos and I.V. Vancea, Mod. Phys. Lett. A 16 (2001) 1907 (hep-th/0108197);

[2] M.A. Santos, J. C. Mello and P. Pitanga, Z. Phys. C55 (1992) 271;

[3] M.A. Santos, Braz. J. Phys., 23, no 2 (1993) 214;

[4] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. 140 (1982) 372, concerning the model; M.A. Santos and J. A. Helayël-Neto, hep-th/9905063 (submitted for publication), concerning the application of the SPM;

[5] H.R. Christiansen, M.S. Cunha, J.A. Helayël-Neto, L.R.U. Manssur and A.L.M.A. Nogueira, Int. J. Mod. Phys. A14 (1999) 147 (hep-th/9802096);

[6] H.R. Christiansen, M.S. Cunha, J.A. Helayël-Neto, L.R.U. Manssur and A.L.M.A. Nogueira, Int. J. Mod. Phys. A14 (1999) 1721 (hep-th/9805128);

[7] M. Kalb and P. Ramond, Phys. Rev. D9 (1974) 2273;

[8] Winder A. Moura-Melo, N. Panza and J.A. Helayël-Neto, Int. J. Mod. Phys. A14 (1999) 3949; Winder A. Moura-Melo, M.Sc. thesis, in portuguese, CBPF, 1998; A. Lahiri, Mod. Phys. Lett. A8 (1993) 2403;

[9] J. Scherk, Extended Supersymmetry and Extended Supergravity Theories, in Recent Developments in Gravitation, Cargèse 1978, Edited by M. Lévy and S. Deser, Plenum Press; L. Brink, J.H. Schwarz and J. Scherk, Nucl. Phys. B121 (1977) 77;

[10] E. Cremmer and J. Scherk, Nucl. Phys. B72 (1974) 117;

[11] P. A. M. Dirac, Can. J. Math, 2 (1950) 129; P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, Academic Press, New York) 1967.