THE ADDITIONAL SYMMETRIES FOR THE BTL AND CTL HIERARCHIES

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Abstract. In this paper, we construct the additional symmetries for the Toda lattice (TL) hierarchies of B type and C type (the BTL and CTL hierarchies), and show their algebraic structures are \(w_\infty^B \times w_\infty^B\) and \(w_\infty^C \times w_\infty^C\) respectively. And also we discuss the generating functions of the additional symmetries.

Keywords: the BTL and CTL hierarchies, additional symmetries

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1. Introduction

The Toda lattice (TL) hierarchy was first introduced by K. Ueno and K. Takasaki in [1] to generalize the Toda lattice equations [2]. Along the work of E. Date, M. Jimbo, M. Kashiwara and T. Miwa [3] on the KP hierarchy, K. Ueno and K. Takasaki in [1] develop the theory for the TL hierarchy: its algebraic structure, the linearization, the bilinear identity, \(\tau\) function and so on. Also the analogues of the B and C types for the TL hierarchy, i.e. the BTL and CTL hierarchies, are considered in [1], which are corresponding to infinite dimensional Lie algebras \(o(\infty)\) and \(sp(\infty)\) respectively. In this paper, we will focus on the study of the additional symmetries for the BTL and CTL hierarchies.

Symmetries have been playing vital roles in the study of the integrable system. The additional symmetry [4–11] is one of the most important symmetries, which has two different expressions. One of its expressions can be traced back to the master symmetry [12–18]. As an interesting generalization of usual symmetries of partial differential equations, master symmetries are introduced in references [12] and further developed in references [13–18]. The master symmetries are usually for the given explicit soliton equations, whose remarkable feature is its depending explicitly on the space \(x\) and time \(t\) variables for 1+1-dimensional case. When considering the integrable hierarchies, the master symmetries are usually called additional symmetries [4]. For the KP hierarchy, the corresponding additional symmetries are studied in references [5–9], which can be used to form \(w_\infty\) algebra when acting on the linear problem, while the additional symmetry for the TL hierarchy is investigated in references [3][10][11], which forms \(w_\infty \times w_\infty\) algebra when acting on the linear problem. The other expression for the additional symmetry is in the form of Sato Bäcklund transformation [3] defined by the vertex operator \(X(\lambda, \mu)\) acting on the \(\tau\) function. These two different expressions can be
linked by so-called Adler-Shiota-van Moerbeke (ASvM) formulas \([6,10]\) for the continuous integrable systems (the KP hierarchy) and also the discrete ones (the TL hierarchy). By the ASvM formulas, the associated algebra of additional symmetries can be lifted to its central extension, the algebra of Bäcklund symmetries. The additional symmetries are involved in so-called string equation and the generalized Virasoro constraints in matrix models of the 2d quantum gravity (see \([5,7,11]\) and references therein).

There are several interesting results about the additional symmetries for the BKP and CKP hierarchies from the views of the possible applications related to the string equations. For the BKP hierarchy, the corresponding additional symmetry are constructed by Takasaki \([19]\) in the operator form, and Virasoro constraints and the ASvM formula have been studied by Johan van de Leur \([20,21]\) using an algebraic formalism. Recently, Tu \([22]\) gave an alternative proof of the ASvM formula of the BKP hierarchy by using Dickeys method \([9]\). And the corresponding string equation was also constructed by Tu in \([23]\). As for the CKP hierarchy, the additional symmetry and string equation were well constructed by He in \([24]\). Inspired by these works and the relation between the TL hierarchy and the KP hierarchy, we shall in this paper establish the additional symmetries for the BTL and CTL hierarchies, and then investigate their corresponding algebraic structures, and study some interesting properties of them.

This paper is organized in the following way. In Section 2, we recall some basic knowledge about the BTL and CTL hierarchies. Then, we construct the additional symmetry for the BTL hierarchy and give their algebraic structure in Section 3. Next, in Section 4, the additional symmetry for the CTL hierarchy is also investigated and the corresponding algebraic structure is shown. At last, we devote Section 5 to some conclusions and discussions.

2. the BTL and CTL hierarchies

In this section, we will review some basic facts about the BTL and CTL hierarchies in the style of Adler & van Moerbeke \([8,10]\). One can refer to \([1]\) for more details about the BTL and CTL hierarchies.

First, consider the algebra

\[
\mathcal{D} = \{(P_1, P_2) \in \text{gl}(\mathbb{R}) \times \text{gl}(\mathbb{R}) \mid (P_1)_{ij} = 0 \text{ for } j - i \gg 0, \ (P_2)_{ij} = 0 \text{ for } i - j \gg 0\},
\]

which has the following splitting:

\[
\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_-,
\]

\[
\mathcal{D}_+ = \{(P,P) \in \mathcal{D} \mid (P)_{ij} = 0 \text{ for } |i - j| \gg 0\} = \{(P_1, P_2) \in \mathcal{D} \mid P_1 = P_2\},
\]

\[
\mathcal{D}_- = \{(P_1, P_2) \in \mathcal{D} \mid (P_1)_{ij} = 0 \text{ for } j \geq i, \ (P_2)_{ij} = 0 \text{ for } i > j\},
\]
with \((P_1, P_2) = (P_1, P_2)_+ + (P_1, P_2)_-\) given by

\[
(P_1, P_2)_+ = (P_{1u} + P_{2l}, P_{iu} + P_{2l}), \quad (P_1, P_2)_- = (P_{1l} - P_{2l}, P_{2u} - P_{1u}),
\]

where for a matrix \(P\), \(P_u\) and \(P_l\) denote the upper (including diagonal) and strictly lower triangular parts of \(P\), respectively. For \((P_1, P_2), (Q_1, Q_2) \in \mathcal{D}\), we define

\[
(P_1, P_2)(Q_1, Q_2) = (P_1Q_1, P_2Q_2), \quad (P_1, P_2)^{-1} = (P_1^{-1}, P_2^{-1}).
\]

Then the BTL (or CTL) hierarchy is defined in the Lax forms as

\[
\partial_{x_{2n+1}} L = [(L_1^{2n+1}, 0)_{\cdot}, L] \quad \text{and} \quad \partial_{y_{2n+1}} L = [(0, L_2^{2n+1})_{\cdot}, L], \quad n = 0, 1, 2, \ldots
\]

where the Lax operator \(L\) is given by a pair of infinite matrices

\[
L = (L_1, L_2) = \left( \sum_{-\infty < i \leq 1} \text{diag}[a_i^{(1)}(s)]A_i, \sum_{1 \leq i < \infty} \text{diag}[a_i^{(2)}(s)]A_i \right) \in \mathcal{D}
\]

with \(\Lambda = (\delta_{j-i,1})_{i, j \in \mathbb{Z}}\), and \(a_1^{(k)}(s)\) and \(a_2^{(k)}(s)\) depending on \(x = (x_1, x_3, x_5, \ldots)\) and \(y = (y_1, y_3, y_5, \ldots)\), such that

\[
a_1^{(1)}(s) = 1 \quad \text{and} \quad a_2^{(1)}(s) \neq 0 \quad \forall s
\]

and satisfies the BTL (or CTL) constraint \([1]\)

\[
L^T = -(J, J) L(J^{-1}, J^{-1}) \quad \text{(or} \quad L^T = -(K, K) L(K^{-1}, K^{-1}) \text{)},
\]

where \(J = ((-1)^i\delta_{i+j,0})_{i, j \in \mathbb{Z}}\), \(K = \Lambda J\) and \(T\) refers to the matrix transpose.

The Lax operator of the BTL (or CTL) hierarchy \([1]\) has the representation

\[
L := W(\Lambda, \Lambda^{-1})W^{-1} = S(\Lambda, \Lambda^{-1})S^{-1}
\]

in terms of two pairs of wave operators \(W = (W_1, W_2)\) and \(S = (S_1, S_2)\), where

\[
S_1(x, y) = \sum_{i \geq 0} \text{diag}[c_i(s; x, y)]\Lambda^{-i}, \quad S_2(x, y) = \sum_{i \geq 0} \text{diag}[c_i^*(s; x, y)]\Lambda^i
\]

and

\[
W_1(x, y) = S_1(x, y)e^{\xi(x, \Lambda)}, \quad W_2(x, y) = S_2(x, y)e^{\xi(y, \Lambda^{-1})}
\]

with \(c_0(s; x, y) = 1\) and \(c_i^*(s; x, y) \neq 0\) for any \(s\), and \(\xi(x, \Lambda^{\pm 1}) = \sum_{n \geq 0} x_{2n+1} \Lambda^{\pm 2n+1}\). Obviously, \(W = (W_1, W_2)\) are not uniquely determined, but have the arbitrariness

\[
W_1(x, y) \mapsto W_1(x, y)f_1(\Lambda), \quad W_2(x, y) \mapsto W_2(x, y)f_2(\Lambda).
\]

Here \(f_1(\Lambda) = \sum_{i \geq 0} f_1^i \Lambda^{-i}\) and \(f_2(\Lambda) = \sum_{i \geq 0} f_2^i \Lambda^i\) \((f_0^1 = 1, \ f_0^2 \neq 0)\) are formal Laurent series with constant scalar coefficients. Under an appropriate choice of \(f_i(\Lambda)\), \(W = (W_1, W_2)\) satisfies

\[
J^{-1}W_i^T J = W_i^{-1} \quad \text{for BTL (or} \quad K^{-1}W_i^T K = W_i^{-1} \text{for CTL),} \quad i = 1, 2.
\]
The wave operators evolve according to

\[ \partial_{x_{2n+1}} S = -(L^{2n+1}_1, 0)_-, \quad \partial_{y_{2n+1}} S = -(0, L^{2n+1}_2)_-, \]
\[ \partial_{x_{2n+1}} W = (L^{2n+1}_1, 0)_+, \quad \partial_{y_{2n+1}} W = (0, L^{2n+1}_2)_+. \]

The vector wave functions \( \Psi = (\Psi_1, \Psi_2) \) and the adjoint wave functions \( \Psi^* = (\Psi_1^*, \Psi_2^*) \), can also be introduced as

\[ \Psi_i(x, y; \lambda) = (\Psi_i(n; x, y; \lambda))_{n \in \mathbb{Z}} := W_i(x, y)\chi(\lambda), \]
\[ \Psi_i^*(x, y; \lambda) = (\Psi_i^*(n; x, y; \lambda))_{n \in \mathbb{Z}} := (W_i(x, y)^{-1})^T \chi^*(\lambda), \]

with \( \chi(\lambda) = (\lambda^i)_{i \in \mathbb{Z}} \), and \( \chi^*(\lambda) = \chi(\lambda^{-1}) \), which satisfy the following differential equations:

\[ \partial_{x_{2n+1}} \Psi = (L^{2n+1}_1, 0)_+ \Psi, \quad \partial_{y_{2n+1}} \Psi = (0, L^{2n+1}_2)_+ \Psi, \]
\[ \partial_{x_{2n+1}} \Psi^* = -((L^{2n+1}_1, 0)_+)^T \Psi^*, \quad \partial_{y_{2n+1}} \Psi^* = -((0, L^{2n+1}_2)_+)^T \Psi^*. \]

At last, in order to define the additional symmetries of the BTL and CTL hierarchies in next sections, it is necessary to introduce the Orlov-Shulman operator \[48\]

\[ M = (M_1, M_2) := (W_1\varepsilon W_1^{-1}, W_2\varepsilon^* W_2^{-1}), \]

where \( \varepsilon = \text{diag}(i)_{i \in \mathbb{Z}} \cdot \Lambda^{-1} \) and \( \varepsilon^* = -J\varepsilon J^{-1} \). These operators satisfy

\[ L\Psi = (\lambda, \lambda^{-1})\Psi, \quad M\Psi = (\partial_{\lambda}, \partial_{\lambda^{-1}})\Psi, \quad [L, M] = (1, 1), \]

and

\[ \partial_{x_{2n+1}} M = [(L^{2n+1}_1, 0)_+, M], \quad \partial_{y_{2n+1}} M = [(0, L^{2n+1}_2)_+, M]. \]

3. The Additional Symmetry for the BTL Hierarchies

In this section, we shall construct the additional symmetry for the BTL hierarchy. Just as the case of the TL hierarchy, the Orlov-Shulman operators given by \[14\] still play an important role.

In this case, we can still similarly introduce the additional independent variables \( x^*_{m, l} \) and \( y^*_{m, l} \), and define the additional flows for the BTL hierarchy as

\[ \partial_{x^*_{m, l}} W := -(A_{1ml}(M_1, L_1), 0)_- W, \quad \partial_{y^*_{m, l}} W := -(0, A_{2ml}(M_2, L_2))_+ W, \]

where \( A_{iml}(M_i, L_i) \) are polynomials in \( L_i \) and \( M_i \) and their explicit forms will be determined next.

The action on \( \Psi, L \) and \( M \) of the additional flows is given by the following proposition.

**Proposition 1.**

\[ \partial_{x^*_{m, l}} \Psi = -(A_{1ml}(M_1, L_1), 0)_- \Psi, \quad \partial_{y^*_{m, l}} \Psi = -(0, A_{2ml}(M_2, L_2))_+ \Psi, \]
\[ \partial_{x^*_{m, l}} L = [-(A_{1ml}(M_1, L_1), 0)_-, L], \quad \partial_{y^*_{m, l}} L = [-(0, A_{2ml}(M_2, L_2))_-, L], \]
\[ \partial_{x^*_{m, l}} M = [-(A_{1ml}(M_1, L_1), 0)_-, M], \quad \partial_{y^*_{m, l}} M = [-(0, A_{2ml}(M_2, L_2))_-, M]. \]
Proof. Firstly, (18) follows from (10) and (17). Then for the action on $L$, according to (1) and (17),

$$ \partial_{x_{m,l}} L = (\partial_{x_{m,l}} W) \Lambda W^{-1} - W \Lambda W^{-1} (\partial_{x_{m,l}} W) W^{-1} $$

$$ = -(A_{1ml}(M_1, L_1), 0)_- W \Lambda W^{-1} + W \Lambda W^{-1} (A_{1ml}(M_1, L_1), 0)_- W $$

$$ = -(A_{1ml}(M_1, L_1), 0)_- L + L (A_{1ml}(M_1, L_1), 0)_- $$

$$ = [-(A_{1ml}(M_1, L_1), 0)_-, L]. $$

And $\partial_{y_{m,l}} L = [- (0, A_{2ml}(M_2, L_2))_-, L]$ can be similarly derived.

At last, by (14), a similar proof leads to (20).

Next, we will show (17) is indeed the symmetry flow of the BTL hierarchy in the proposition below, and thus it is called the additional symmetry.

Proposition 2.

$$ [\partial_{x_{m,l}}, \partial_{x_{2n+1}}] = [\partial_{y_{m,l}}, \partial_{x_{2n+1}}] = [\partial_{x_{m,l}}, \partial_{y_{2n+1}}] = [\partial_{y_{m,l}}, \partial_{y_{2n+1}}] = 0. $$

Proof. By equations (1), (9), (16), (17) and (19),

$$ [\partial_{x_{m,l}}, \partial_{x_{2n+1}}] W = \partial_{x_{m,l}} ((L^{2n+1}_1, 0)_+ W) + \partial_{x_{2n+1}} ((A_{1ml}(M_1, L_1), 0)_- W) $$

$$ = -[(A_{1ml}(M_1, L_1), 0)_-, (L^{2n+1}_1, 0)_+ W] + [(L^{2n+1}_1, 0)_+, (A_{1ml}(M_1, L_1), 0)_- W] $$

$$ - (L^{2n+1}_1, 0)_+ (A_{1ml}(M_1, L_1), 0)_- W + (A_{1ml}(M_1, L_1), 0)_- (L^{2n+1}_1, 0)_+ W $$

$$ = [(L^{2n+1}_1, 0)_+, (A_{1ml}(M_1, L_1), 0)_- W] - [(L^{2n+1}_1, 0)_-, (A_{1ml}(M_1, L_1), 0)_+ W] $$

$$ + [(L^{2n+1}_1, 0)_+, (A_{1ml}(M_1, L_1), 0)_- W] - [(L^{2n+1}_1, 0)_-, (A_{1ml}(M_1, L_1), 0)_+ W] $$

$$ = 0. $$

Note that, the term $[(L^{2n+1}_1, 0)_-, (A_{1ml}(M_1, L_1), 0)_+ W]$ in the third equality and $[(L^{2n+1}_1, 0)_+, (A_{1ml}(M_1, L_1), 0)_- W]$ vanish, since $(P^-)_+ = 0$ and $(P^+)_- = 0$ for $P \in \mathcal{D}$.

The proofs for others are almost the same.

So now, the remaining work is to determine the explicit forms of $A_{1ml}(i = 1, 2)$. Before doing this, a useful lemma is introduced first.

Lemma 3.

$$ \Lambda^{-1} \varepsilon \Lambda = J^{-1} \varepsilon^T J, \quad \Lambda \varepsilon^T \Lambda^{-1} = J^{-1} \varepsilon^T J. $$

Proof. By recalling $\varepsilon := \text{diag}[s] \Lambda^{-1}$ and $J = ((-1)^i \delta_{i+j,0})_{i,j \in \mathbb{Z}} = J^{-1}$, we compare the $(i, j)$-entries on each side of the first relation in (21). For the left side,

$$ (\Lambda^{-1} \varepsilon \Lambda)_{i,j} = (\Lambda^{-1} \text{diag}[s] \Lambda^{-1} \Lambda)_{i,j} $$

$$ = (\Lambda^{-1} \varepsilon \Lambda)_{i,j} $$

$$ = J^{-1} \varepsilon^T J. $$

For the right side,

$$ (\Lambda \varepsilon^T \Lambda^{-1})_{i,j} = (\Lambda \varepsilon^T \Lambda^{-1})_{i,j} $$

$$ = (\Lambda \varepsilon^T \Lambda^{-1})_{i,j} $$

$$ = J^{-1} \varepsilon^T J. $$

Therefore, we have

$$ \Lambda^{-1} \varepsilon \Lambda = J^{-1} \varepsilon^T J, \quad \Lambda \varepsilon^T \Lambda^{-1} = J^{-1} \varepsilon^T J. $$
and on the right side

\[(J^{-1}\varepsilon^T J)_{i,j} = \sum_{\alpha, \beta} (J^{-1})_{i,\alpha}(\varepsilon^T)_{\alpha,\beta} J_{\beta,j}\]

\[= \sum_{\alpha, \beta} (-1)^i \delta_{i+\alpha,0} (\Delta \text{diag}[s])_{\alpha,\beta} (-1)^j \delta_{\beta,j,0}\]

\[= (-1)^{i+j} (\text{diag}[s + 1] \Lambda)_{-i,-j}\]

\[= (-1)^{i+j} (-i + 1) \delta_{-i+1,-j} = (i - 1) \delta_{i-1,j},\]

where the facts \(\Lambda^T = \Lambda^{-1}\) and \((-1)^{i+j} \delta_{-i+1,-j} = (-1)^{(j+1)+j}\delta_{i-1,j} = -\delta_{i-1,j}\) are used. Thus the first relation in (21) can be got.

As for the second relation, we start from transposing the first relation in (21), which leads to \(\Lambda^{-1}\varepsilon^T \Lambda = J \varepsilon J^{-1} = -\varepsilon^*\), with remembering \(\varepsilon^* := J \varepsilon J^{-1}\) and \(J^T = J^{-1} = J\). Therefore

\[J^{-1}\varepsilon^* T J = -J^{-1}(\Lambda^{-1}\varepsilon^T \Lambda)^T J\]

\[= -J^{-1}\Lambda^{-1}\varepsilon \Lambda J\]

\[= -J^{-1}(J^{-1}\varepsilon^T J)J\]

\[= -\varepsilon^T = -\Lambda \Lambda^{-1}\varepsilon^T \Lambda \Lambda^{-1} = \Lambda \varepsilon^* \Lambda^{-1}.\]

\[\square\]

Remark 1: for the BTL hierarchy, by (7) and (21),

\[M^T = (W(\varepsilon, \varepsilon^*) W^{-1})^T = (W^{-1})^T (\varepsilon^T, \varepsilon^* T) W^T\]

\[= (J, J)W(J^{-1}, J^{-1})(\varepsilon^T, \varepsilon^* T)(J, J)W^{-1}(J^{-1}, J^{-1})\]

\[= (J, J)W(J^{-1}\varepsilon^T J, J^{-1}\varepsilon^* T J)W^{-1}(J^{-1}, J^{-1})\]

\[= (J, J)W(\Lambda^{-1}\varepsilon \Lambda, \Lambda \varepsilon^* \Lambda^{-1})W^{-1}(J^{-1}, J^{-1})\]

\[= (J, J)W(\Lambda^{-1}, \Lambda)(\varepsilon, \varepsilon^*)(\Lambda, \Lambda^{-1})W^{-1}(J^{-1}, J^{-1})\]

\[= (J, J)W(\Lambda^{-1}, \Lambda)W^{-1}W(\varepsilon, \varepsilon^*) W^{-1}W(\Lambda, \Lambda^{-1})W^{-1}(J^{-1}, J^{-1})\]

\[= (J, J)(L^{-1}_1, L^{-1}_2)(M_1, M_2)(L_1, L_2)(J^{-1}, J^{-1})\]

\[= (J, J)L^{-1} ML(J^{-1}, J^{-1}).\] (22)

Because of the constraints (3) on the Lax operators for the BTL hierarchy, we cannot just let \(A_{iml} = M_i^m L_i\), which will lead to contradiction. By considering the constraints (3), we have the following proposition for \(A_{iml}\). For convenience, denote \(A_{ml}(M, L) = (A_{iml}(M_1, L_1), A_{2ml}(M_2, L_2)).\)
Proposition 4. For the BTL hierarchy, it is sufficient to ask for
\[
A_{ml}(M, L)^T = -(J, J)A_{ml}(M, L)(J^{-1}, J^{-1}),
\]
thus we can take
\[
A_{ml}(M, L) = M^m L^l - (-1)^l L^{l-1} M^m L.
\]

Proof. Since the constraints (7) for the BTL hierarchy, the action of \( \partial_{x_{ml}}^* \) and \( \partial_{y_{ml}}^* \) on the transpose of the wave operator \( W \) can be obtained in two different way. The first is to transpose the relation (17), that is
\[
\partial_{x_{ml}}^* W^T = -W^T(A_{ml}(M_1, L_1), 0)_+^T, \quad \partial_{y_{ml}}^* W^T = -W^T(0, A_{2ml}(M_2, L_2))^T.
\]
The second is to use the constraints (7),
\[
\begin{align*}
\partial_{x_{ml}}^* W^T &= \partial_{x_{ml}}^* ((J, J)W^{-1}(J^{-1}, J^{-1})) = -(J, J)W^{-1}(\partial_{x_{ml}}^* W)W^{-1}(J^{-1}, J^{-1}) \\
&= (J, J)W^{-1}(A_{ml}(M_1, L_1), 0)_-(J^{-1}, J^{-1}) \\
&= W^T(J, J)(A_{ml}(M_1, L_1), 0)_+(J^{-1}, J^{-1}), \\
\partial_{y_{ml}}^* W^T &= \partial_{y_{ml}}^* ((J, J)W^{-1}(J^{-1}, J^{-1})) = -(J, J)W^{-1}(\partial_{y_{ml}}^* W)W^{-1}(J^{-1}, J^{-1}) \\
&= (J, J)W^{-1}(0, A_{2ml}(M_2, L_2))_-(J^{-1}, J^{-1}) \\
&= W^T(J, J)(0, A_{2ml}(M_2, L_2))_+(J^{-1}, J^{-1}).
\end{align*}
\]
The consistence of these two results lead to
\[
(A_{ml}(M_1, L_1), 0)_+^T = (J, J)(A_{ml}(M_1, L_1), 0)_-(J^{-1}, J^{-1}), \\
(0, A_{2ml}(M_2, L_2))^T = (J, J)(0, A_{2ml}(M_2, L_2))_-(J^{-1}, J^{-1}),
\]
thus from the fact that if \( A \in o(\infty) \), then \( (A)_+ \in o(\infty) \), it is sufficient to ask for
\[
(A_{ml}(M_1, L_1), 0)_+^T = (J, J)(A_{ml}(M_1, L_1), 0)_+(J^{-1}, J^{-1}), \\
(0, A_{2ml}(M_2, L_2))^T = (J, J)(0, A_{2ml}(M_2, L_2))_+(J^{-1}, J^{-1}),
\]
that is,
\[
A_{ml}(M, L)^T = -(J, J)A_{ml}(M, L)(J^{-1}, J^{-1}).
\]

From (23) and (22), we have
\[
(M^m L^l)^T = (L^l)^T(M^m)^T = (-1)^l(J, J)L^l(J^{-1}, J^{-1})(J, J)L^{-1} M^m L(J^{-1}, J^{-1})
\]
\[
= (-1)^l(J, J)L^{l-1} M^m L(J^{-1}, J^{-1}).
\]
Thus \( (L^{l-1} M^m L)^T = (-1)^l(J, J)M^m L^l(J^{-1}, J^{-1}) \), since \( J^T = J^{-1} = J \). Therefore \( A_{ml}(M, L) = M^m L^l - (-1)^l L^{l-1} M^m L \) satisfies the requirement of (24).

Next, we will discuss the algebraic structure of the additional symmetry of the BTL hierarchy.
**Proposition 5.** Acting on the wave matrices $W$, $\partial_{x_{ml}}^*$ and $\partial_{y_{ml}}^*$ form a algebra of $w_{\infty}^B \times w_{\infty}^B$ in the case of the BTL hierarchy.

**Proof.** According to (23), we have

$$[A_{ml}(M, L), A_{nk}(M, L)]^T = [A_{nk}(M, L)^T, A_{ml}(M, L)^T]$$

$$= (J, J)[A_{nk}(M, L), A_{ml}(M, L)](J^{-1}, J^{-1}) = -(J, J)[A_{ml}(M, L), A_{nk}(M, L)](J^{-1}, J^{-1}),$$

thus $A_{ml}(M, L)$ constitute a closed algebra and we can set

$$[A_{ml}(M, L), A_{nk}(M, L)] := C_{ml, nk}^{pq} A_{pq}(M, L).$$  (28)

On the other hand, from (17)

$$[\partial_{x_{ml}}^*, \partial_{x_{nk}}^*]W = -\partial_{x_{ml}}^* ((A_{1nk}(M_1, L_1), 0)_{-} W) + \partial_{x_{nk}}^* ((A_{1ml}(M_1, L_1), 0)_{-} W)$$

$$= [(A_{1ml}(M_1, L_1), 0)_{-}, (A_{1nk}(M_1, L_1), 0)_{-}]_{-} W + (A_{2nk}(M_2, L_2), 0)_{-} (A_{1ml}(M_1, L_1), 0)_{-} W$$

$$- [(A_{1nk}(M_1, L_1), 0)_{-}, (A_{1ml}(M_1, L_1), 0)_{-}]_{-} W - (A_{1ml}(M_1, L_1), 0)_{-} (A_{1nk}(M_1, L_1), 0)_{-} W$$

$$= [(A_{1ml}(M_1, L_1), 0)_{-}, (A_{1nk}(M_1, L_1), 0)_{-}]_{-} W - [(A_{1ml}(M_1, L_1), 0)_{-}, (A_{1nk}(M_1, L_1), 0)_{-}]_{-} W$$

$$- [(A_{1nk}(M_1, L_1), 0)_{-}, (A_{1ml}(M_1, L_1), 0)_{-}]_{-} W$$

$$= [(A_{1ml}(M_1, L_1), 0)_{-}, (A_{1nk}(M_1, L_1), 0)_{-}]_{-} W + [(A_{1ml}(M_1, L_1), 0), (A_{1nk}(M_1, L_1), 0)_{-}]_{-} W$$

$$= [(A_{1ml}(M_1, L_1), 0), (A_{1nk}(M_1, L_1), 0)]_{-} W,$$

and similarly

$$[\partial_{y_{ml}}^*, \partial_{y_{nk}}^*]W = [(0, A_{2ml}(M_2, L_2)), (0, A_{2nk}(M_2, L_2))]_{-} W,$$

$$[\partial_{x_{ml}}^*, \partial_{y_{nk}}^*]W = [(A_{1ml}(M_1, L_1), 0), (0, A_{2nk}(M_2, L_2))]_{-} W.$$

Thus by (28), we get

$$[\partial_{x_{ml}}^*, \partial_{x_{nk}}^*]W = C_{nk, ml}^{pq} \partial_{x_{pq}}^* W, \quad [\partial_{y_{ml}}^*, \partial_{y_{nk}}^*]W = C_{nk, ml}^{pq} \partial_{y_{pq}}^* W, \quad [\partial_{x_{ml}}^*, \partial_{y_{nk}}^*]W = 0.$$

That is,

$$[\partial_{x_{ml}}^*, \partial_{x_{nk}}^*] = C_{nk, ml}^{pq} \partial_{x_{pq}}^*, \quad [\partial_{y_{ml}}^*, \partial_{y_{nk}}^*] = C_{nk, ml}^{pq} \partial_{y_{pq}}^*, \quad [\partial_{x_{ml}}^*, \partial_{y_{nk}}^*] = 0.$$  (29)

At last, let’s introduce the generating function of the additional symmetry for the BTL hierarchy

$$Y(\lambda, \mu) = (Y_1(\lambda, \mu), Y_2(\lambda, \mu)) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}(A_{m, m+l}(M, L))_{-},$$  (30)

and show its relation with the (adjoint) wave functions. For this, let’s see a lemma developed in [5].
Lemma 6. Given two operators \( U = (U_1, U_2), V = (V_1, V_2) \in \mathcal{D} \) depending on \( x \) and \( y \), we have:

\[
U_1 V_1 = \text{res}_z \frac{1}{z}(U_1 \chi(z)) \otimes (V_1^T \chi^*(z)), \\
U_2 V_2 = \text{res}_z \frac{1}{z}(U_2 \chi(z^{-1})) \otimes (V_2^T \chi^*(z^{-1}))
\]

where \( \text{res}_z \sum_j a_j z^j = a_{-1} \).

Proposition 7. For the BTL hierarchy,

\[
Y(\lambda, \mu) = \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} \left( \frac{\mu - \lambda}{m!} \right)^m \lambda^{-l-m-1} (M^m L^{m+l} - (-1)^{m+l} L^{m+l-1} M^m L)_-.
\]

We have

\[
Y_1(\lambda, \mu) = \lambda^{-1} (\Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda) - \Psi_1(x, y; -\lambda) \otimes \Psi_1^*(x, y; -\mu))_-, \\
Y_2(\lambda, \mu) = \lambda^{-1} (\Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1}) - \Psi_2(x, y; -\lambda^{-1}) \otimes \Psi_2^*(x, y; -\mu^{-1}))_-.
\]

Proof. Firstly, according to Lemma 6

\[
L_1^{m+l-1} M_1^m L_1 = L_1^{m+l-1} M_1^m W_1 A W_1^{-1},
\]

and similarly

\[
M_1^m L_1^{m+l} = \text{res}_z (z^{-1+m+l} \partial_z^m \Psi_1(x, y; z)) \otimes \Psi_1^*(x, y; z),
\]

then

\[
Y_1(\lambda, \mu) = \text{res}_z \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} \left( (z^{-1+m+l} \partial_z^m \Psi_1(x, y; z)) \otimes \Psi_1^*(x, y; z) \right)_-
\]

\[
-(-1)^{m+l} \partial_z^m \left( z^{m+l-1} \Psi_1(x, y; z) \right) \otimes \Psi_1^*(x, y; z)_-
\]

\[
= \text{res}_z \delta(\lambda, z) \left( z^{1-\lambda} e^{\mu-\lambda} \partial_z \Psi_1(x, y; z) \otimes \Psi_1^*(x, y; z) \right)_-
\]

\[
+ \text{res}_z e^{\mu-\lambda} \partial_z \left( z^{1-\lambda} \delta(-\lambda, z) \Psi_1(x, y; z) \otimes \Psi_1^*(x, y; z) \right)_-
\]

\[
= \left( \lambda^{-1} e^{\mu-\lambda} \partial_z \Psi_1(x, y; z) \otimes \Psi_1^*(x, y; z) \right)_-
\]

\[
+ \text{res}_z \left( (\mu - \lambda + z)^{-1} \delta(-\lambda, z + \mu - \lambda) \Psi_1(x, y; z + \mu - \lambda) \Psi_1^*(x, y; z) \right)_-
\]

\[
= \lambda^{-1} (\Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda) - \Psi_1(x, y; -\lambda) \otimes \Psi_1^*(x, y; -\mu))_-,
\]

where \( \delta(\lambda, z) = \sum_{n=-\infty}^{\infty} \frac{z^n}{\lambda^{n+1}} \) and \( \text{res}_z (\delta(\lambda, z) f(z)) = f(\lambda) \) is used.

\[1(A \otimes B)_{ij} = A_i B_j\]
Similarly, 
\[ M^m_2 L^{m+l}_2 - (-1)^{m+l} L^{m+l-1}_2 M^m_2 L_2 \]

\[ = \text{res}_z \left( (z^{-1+m+l}) \nabla^m_2 \Psi_2(x, y, z^{-1}) \right) \otimes \Psi^*_2(x, y, z^{-1}) \]

\[ - (-1)^{m+l} \nabla^m_2 \left( (z^{m+l-1}) \Psi_2(x, y, z^{-1}) \right) \otimes \Psi^*_2(x, y, z^{-1}) \]

and

\[ Y_2(\lambda, \mu) = \lambda^{-1}(\Psi_2(x, y; \mu^{-1}) \otimes \Psi^*_2(x, y; \lambda^{-1}) - \Psi_2(x, y; -\lambda^{-1}) \otimes \Psi^*_2(x, y; -\mu^{-1}))_. \]

\[ \square \]

4. The Additional Symmetry for the CTL Hierarchy

In this section, the additional symmetry for the CTL hierarchy will be given.

Similar to the case of the above section, introduce additional independent variables \( x^*_{m,l} \) and \( y^*_{m,l} \), and define the additional flows for the CTL hierarchy as follows

\[ \partial_{x^*_{m,l}} W := -(A_{1ml}(M_1, L_1), 0)_- W, \quad \partial_{y^*_{m,l}} W := -(0, A_{2ml}(M_2, L_2))_- W. \] (36)

The same computations lead to the following propositions.

Proposition 8.

\[ \partial_{x^*_{m,l}} \Psi = -(A_{1ml}(M_1, L_1), 0)_- \Psi, \quad \partial_{y^*_{m,l}} \Psi = -(0, A_{2ml}(M_2, L_2))_- \Psi, \] (37)

\[ \partial_{x^*_{m,l}} L = -(A_{1ml}(M_1, L_1), 0)_- L, \quad \partial_{y^*_{m,l}} L = -(0, A_{2ml}(M_2, L_2))_- L, \] (38)

\[ \partial_{x^*_{m,l}} M = -(A_{1ml}(M_1, L_1), 0)_- M, \quad \partial_{y^*_{m,l}} M = -(0, A_{2ml}(M_2, L_2))_- M. \] (39)

Proposition 9.

\[ [\partial_{x^*_{m,l}}, \partial_{x_{2n+1}}] = [\partial_{y^*_{m,l}}, \partial_{x_{2n+1}}] = [\partial_{x^*_{m,l}}, \partial_{y_{2n+1}}] = [\partial_{y^*_{m,l}}, \partial_{y_{2n+1}}] = 0. \]

Thus, (36) is indeed the symmetry of the CTL hierarchy. So now we will try to determine the explicit forms of \( A_{1ml} \). For this, let’s see an important lemma first.

Lemma 10.

\[ \varepsilon = K \varepsilon^T K^{-1}, \quad \varepsilon^* = K \varepsilon^T K^{-1}. \] (40)

Proof. By noticing \( K = \Lambda J \), (21) leads to (40). \[ \square \]

Remark 2: In the case of the CTL hierarchy, according to (7) (40) and also \( K^T = K^{-1} = -K \),

\[ M^T = (W(\varepsilon, \varepsilon^*) W^{-1})^T = (W^{-1})^T(\varepsilon, \varepsilon^*) W \]

\[ = (K^T, K^T) W((K^{-1})^T, (K^{-1})^T)(\varepsilon^T, \varepsilon^T) (K, K) W^{-1} (K^{-1}, K^{-1}) \]
\[ -(K, K) W(K, K)(\varepsilon^T, \varepsilon^*) (K, K) W^{-1}(K^{-1}, K^{-1}) \]
\[ = (K, K) W(K, K)(\varepsilon^T, \varepsilon^*) (K^{-1}, K^{-1}) W^{-1}(K^{-1}, K^{-1}) \]
\[ = (K, K) W(\varepsilon, \varepsilon^*) W^{-1}(K^{-1}, K^{-1}) \]
\[ = (K, K) M(K^{-1}, K^{-1}). \]  

(41)

Just as the case of the BTL hierarchy, we can not just let \( A_{ml} = M^m L^l_i \) in the CTL hierarchy because of the constraints [3] on the Lax operators. Similar investigation leads to the following proposition.

**Proposition 11.** For the CTL hierarchy, it is enough to require

\[ A_{ml}(M, L)^T = -(K, K) A_{ml}(M, L)(K^{-1}, K^{-1}), \]

so we can let

\[ A_{ml}(M, L) = M^m L^l - (-1)^l L^l M^m. \]

(42)

By the same way as the BTL hierarchy, we find

**Proposition 12.** Acting on the wave matrices \( W \), for the CTL hierarchy, \( \partial_{x^*_{ml}} \) and \( \partial_{y^*_{ml}} \) form a algebra of \( w^C_{\infty} \times w^C_{\infty} \).

At last, as for the generating function of the additional symmetry of the CTL hierarchy

\[ Y(\lambda, \mu) = (Y_1(\lambda, \mu), Y_2(\lambda, \mu)) = \sum_{m=0}^{\infty} \left( \frac{\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} (A_{m,m+l}(M, L))_\lambda \right), \]

(43)

we have

**Proposition 13.** For the CTL hierarchy,

\[ Y(\lambda, \mu) = \sum_{m=0}^{\infty} \left( \frac{\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1} (M^m L^{m+l} - (-1)^l L^l M^m) \right)_\lambda. \]

(44)

We have

\[ Y_1(\lambda, \mu) = (\lambda^{-1}\Psi_1(x, y; \mu) \otimes \Psi_1^*(x, y; \lambda) - \mu^{-1}\Psi_1(x, y; -\lambda) \otimes \Psi_1^*(x, y; -\mu))_\lambda, \]

(45)

\[ Y_2(\lambda, \mu) = (\lambda^{-1}\Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^*(x, y; \lambda^{-1}) - \mu^{-1}\Psi_2(x, y; -\lambda^{-1}) \otimes \Psi_2^*(x, y; -\mu^{-1}))_\lambda. \]

(46)

**Proof.** The same computations as the BTL hierarchy lead to

\[ M_1^m L_1^{m+l} - (-1)^m L_1^{m+l} M_1^m = \text{res}_z \left( \left( z^{-1+m+l} \partial_z^m \Psi_1(x, y; z) \right) \otimes \Psi_1^*(x, y; z) \right), \]

\[-(-1)^l \partial_z^m (z^{m+l} \Psi_1(x, y; z) \otimes \Psi_1^*(x, y; z)), \]

\[ M_2^m L_2^{m+l} - (-1)^l L_2^{m+l} M_2^m = \text{res}_z \left( \left( z^{-1+m+l} \partial_z^m \Psi_2(x, y; z) \right) \otimes \Psi_2^*(x, y; z) \right), \]

\[-(-1)^l \partial_z^m (z^{m+l} \Psi_2(x, y; z) \otimes \Psi_2^*(x, y; z)). \]
and
\[
Y_1(\lambda, \mu) = \left( \lambda^{-1}\Psi_1(x, y; \mu) \otimes \Psi_1^{\dagger}(x, y; \lambda) - \mu^{-1}\Psi_1(x, y; -\lambda) \otimes \Psi_1^{\dagger}(x, y; -\mu) \right)_-,
\]
\[
Y_2(\lambda, \mu) = \left( \lambda^{-1}\Psi_2(x, y; \mu^{-1}) \otimes \Psi_2^{\dagger}(x, y; \lambda^{-1}) - \mu^{-1}\Psi_2(x, y; -\lambda^{-1}) \otimes \Psi_2^{\dagger}(x, y; -\mu^{-1}) \right)_-.
\]

5. Conclusions and Discussions

To summarize, we have constructed the additional symmetries for the BTL and CTL hierarchies in (17)(24) and (36)(43) respectively. And the additional symmetry flows on \( \Psi, L \) and \( M \) for the BTL and CTL hierarchies are given in Proposition 1 and 8 respectively. When acting on the wave matrices \( W \), (a) \( \partial_{x_{ml}}^* \) and \( \partial_{y_{ml}}^* \) form a algebra of \( w^B_{\infty} \times w^B_{\infty} \) in the case of the BTL hierarchy; (b) For the CTL hierarchy, \( \partial_{x_{ml}}^* \) and \( \partial_{y_{ml}}^* \) form a algebra of \( w^C_{\infty} \times w^C_{\infty} \). The generating functions of the additional symmetries for the BTL and CTL hierarchies are discussed in Proposition 7 and 13, which may be helpful for the study of the ASvM formulas in the future. These results indicate again the essential difference between the BTL and CTL hierarchies from the point of view of the symmetry.

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