Abstract—This paper studies scale-free protocol design for $H_2$ almost state synchronization of homogeneous networks of non-introspective agents in presence of external disturbances. The necessary and sufficient conditions are provided by designing collaborative linear dynamic protocols. The design is based on localized information exchange over the same communication network, which does not need any knowledge of the directed network topology and the spectrum of the associated Laplacian matrix. Moreover, the proposed protocol is scalable and achieves $H_2$ almost synchronization with a given arbitrary degree of accuracy for any arbitrary number of agents.

I. INTRODUCTION

In recent decades, the synchronization problem for multi-agent systems (MAS) has attracted substantial attention due to the wide potential for applications in several areas such as automotive vehicle control, satellites/robots formation, sensor networks, and so on. See for instance the books [19] and [34] or the survey paper [13].

State synchronization inherently requires homogeneous networks (i.e. agents which have identical models). Therefore, in this paper we focus on homogeneous networks. So far, most work has focused on state synchronization based on diffusive full-state coupling, where the agent dynamics progress from single- and double-integrator dynamics (e.g. [14], [17], [18]) to more general dynamics (e.g. [21], [30], [32]). State synchronization based on diffusive partial-state coupling has also been considered, including static design ([8], [10], [11]), dynamic design ([3], [22], [23], [25], [29], [31]), and the design with additional communication ([15], [6], and [21]). Recently, scale-free collaborative protocol designs are developed for homogeneous and heterogeneous MAS [1], [12] and for MAS subject to actuator saturation [7].

Meanwhile, if the agents have absolute measurements of their own dynamics in addition to relative information from the network, they are said to be introspective, otherwise, they are called non-introspective. There exist some results about these two types of agents, for example, introspective agents ([4], [35], etc), and non-introspective agents ([12], [33], etc).

Synchronization and almost synchronization in presence of external disturbances are studied in the literature, where three classes of disturbances have been considered namely:

1) Disturbances and measurement noise with known frequencies.
2) Deterministic disturbances with finite power.
3) Stochastic disturbances with bounded variance.

For disturbances and measurement noise with known frequencies, it is shown in [37] and [38] that actually exact synchronization is achievable. This is shown in [37] for heterogeneous MAS with minimum-phase and non-introspective agents and networks with time-varying directed communication graphs. Then, [38] extended this results for non-minimum phase agents utilizing localized information exchange.

For deterministic disturbances with finite power, the notion of $H_\infty$ almost synchronization is introduced by Peymani et.al for homogeneous MAS with non-introspective agents utilizing additional communication exchange [15]. The goal of $H_\infty$ almost synchronization is to reduce the impact of disturbances on the synchronization error to an arbitrarily degree of accuracy (expressed in the $H_\infty$ norm). This work was extended later in [16], [36], [39] to heterogeneous MAS with non-introspective agents and without the additional communication and for network with time-varying graphs. $H_\infty$ almost synchronization via static protocols is studied in [26] and [24] for MAS with passive and passifiable agents. Recently, in [28] necessary and sufficient conditions are provided for solvability $H_\infty$ almost synchronization of homogeneous networks with non-introspective agents and without additional communication exchange. Finally, we developed a scale-free framework for $H_\infty$ almost state synchronization for homogeneous network [9] utilizing suitably designed localized information exchange.

In the case of stochastic disturbances with bounded variance, the concept of stochastic almost synchronization is introduced by [40] and [41] which in the latter both stochastic disturbance and disturbance with known frequency are present. The idea of stochastic almost synchronization is to reduce the stochastic RMS norm of synchronization error arbitrary small in the presence of colored stochastic disturbances that can be modeled as the output of linear time invariant systems driven by white noise with unit power spectral intensities. By augmenting this model with agent...
model one can essentially assume that stochastic disturbance is white noise with unit power spectral intensities. In this case under linear protocols the stochastic RMS norm of synchronization error is the $H_2$ norm of the transfer function from disturbance to the synchronization error. As such one can formulate the stochastic almost synchronization equivalently in a deterministic framework requiring to reduce the $H_2$ norm of the transfer function from disturbance to synchronization error arbitrary small. This deterministic approach is referred to as almost $H_2$ synchronization problem which is equivalent to stochastic almost synchronization problem. Recent work on $H_2$ almost synchronization problem are [27], and [28] which provided necessary and sufficient conditions for solvability of $H_\infty$ almost synchronization for homogeneous networks with non-introspective agents and without additional communication exchange. Finally, $H_2$ almost synchronization via static protocols is also studied in [24] for MAS with passive and passifiable agents.

In this paper, we consider stochastic disturbances with bounded variance and we develop scale-free framework to solve $H_2$ almost state synchronization problem for homogeneous MAS. We design a class of linear parameterized dynamic protocol utilizing localized information exchange for both networks with full- and partial-state coupling. The linear dynamic protocol achieves $H_2$ almost state synchronization for any communication network with any number of agents which contains a spanning tree. The main contribution of this work is that the protocol design does not require any information of the communication network such as a lower bound of non-zero eigenvalue of the associated Laplacian matrix and the number of agents. It is worth to note that, so far in all the works of the literature on $H_2$ almost synchronization, the protocol design requires at least some information about the communication network such as bounds on the spectrum of associated Laplacian matrix and the number of agents.

**Notations and Background**

Given a matrix $A \in \mathbb{R}^{m \times n}$, $A^T$ and $A^*$ denote transpose and conjugate transpose of $A$ respectively while $\|A\|_2$ denotes the induced 2-norm (which has submultiplicative property). The $\text{im}(\cdot)$ denote the image of matrix (vector). A square matrix $A$ is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. $A \otimes B$ depicts the Kronecker product between $A$ and $B$. $I_n$ denotes the $n$-dimensional identity matrix and $0_n$ denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context. For a deterministic continuous-time signal $v(t)$, we denote the $L_2$ norm by $\|v\|_2$ and its Root Mean Square (RMS) value is defined by

$$\|v\|_{RMS} = \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T v(t)^2 dt \right)^{\frac{1}{2}},$$

and for a stochastic signal $v(t)$ which is modeled as wide-sense stationary stochastic process, the $\|v(t)\|_{RMS}$ is given by

$$\|v(t)\|_{RMS} = \left( \mathbb{E}[v^2(t)v(t)] \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2)

where $\mathbb{E}[\cdot]$ stands for the expectation operation. For stochastic signals that approach wide-sense stationarity as time $t$ goes on to infinity (i.e. for asymptotically wide-sense stationary signals) (2) is rewritten as

$$\|v(t)\|_{RMS} = \left( \lim_{t \to \infty} \mathbb{E}[v^2(t)v(t)] \right)^{\frac{1}{2}}.$$  \hspace{1cm} (3)

For a continuous-time system having a $q \times l$ stable transfer function $G(s)$, the $H_2$ norm of $G(s)$ is defined as

$$\|G\|_{H_2} = \left( \frac{1}{2\pi} \text{tr} \left( \int_{-\infty}^{+\infty} G(j\omega) G^*(j\omega) d\omega \right) \right)^{\frac{1}{2}}.$$  \hspace{1cm} (4)

By Parseval’s theorem, $\|G\|_{H_2}$ can be equivalently be defined as

$$\|G\|_{H_2} = \left( \text{tr} \left( \int_0^{+\infty} g(t)^2 dt \right) \right)^{\frac{1}{2}},$$  \hspace{1cm} (5)

where $g(t)$ is the weighting function or unit impulse (Dirac distribution) response matrix of $G(s)$, as such for single-input single-output system $\|G\|_{H_2} = \|g\|_{\text{RMS}}$. The $H_2$ norm of $G(s)$, can be interpreted as the RMS value of the output when the given system is driven by independent zero mean white noise with unit power spectral densities. Note that the $H_2$ norm of a stable transfer function $G(s)$ is finite if and only if it is strictly proper. The $H_\infty$ norm of $G(s)$ is defined as

$$\|G\|_{H_\infty} := \sup_{\omega} \sigma_{\text{max}}(G(j\omega))$$

where $\sigma_{\text{max}}$ is the largest singular value of $G(j\omega)$. Let $\omega(t)$ and $z(t)$ be energy signals which are respectively the input and the corresponding output of the given system. Then, The $H_\infty$ norm of $G(s)$ turns out to coincide with its RMS gain, namely

$$\|G\|_{H_\infty} = \|G\|_{\text{RMS gain}} \sup_{\|\omega\|_0 \neq 0} \frac{\|z\|_{\text{RMS}}}{\|\omega\|_{\text{RMS}}}$$

An important property of the $H_\infty$ norm is that it is submultiplicative. That is for transfer functions $G_1$ and $G_2$, we have

$$\|G_1G_2\|_{H_\infty} \leq \|G_1\|_{H_\infty}\|G_2\|_{H_\infty}.$$  \hspace{1cm} (6)

A weighted graph $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{W})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{W} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighting matrix. Each pair in $\mathcal{E}$ is called an edge, where $a_{ij} > 0$ denotes an edge $(i, j) \in \mathcal{E}$ from node $j$ to node $i$ with weight $a_{ij}$. Moreover, $a_{ii} = 0$ if there is no edge from node $j$ to node $i$. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A path from node $i_1$ to $i_k$ is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A directed tree with root $r$ is a subgraph of the graph $\mathcal{G}$ in which there exists a unique path from node $r$ to each node in this subgraph. A directed spanning tree is a directed tree containing all the nodes of the graph.

For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

which is a lower bound of non-zero eigenvalue of the associated graph. The main contribution of this work is that the protocol design does not require any information of the communication network such as a lower bound of non-zero eigenvalue of the associated Laplacian matrix and the number of agents.
is called the Laplacian matrix associated with the graph $G$. The Laplacian matrix $L$ has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $1$, i.e. a vector with all entries equal to 1. When graph contains a spanning tree, then it follows from [18, Lemma 3.3] that the Laplacian matrix $L$ has a simple eigenvalue at the origin, with the corresponding right eigenvector $1$, and all the other eigenvalues are in the open right-half complex plane.

II. PROBLEM FORMULATION

Consider a MAS composed of $N$ identical linear time-invariant agents of the form,

$$
\dot{x}_i = Ax_i + Bu_i + E\omega_i, \quad y_i = Cx_i, \quad (i = 1, \ldots, N)
$$

(4)

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are respectively the state, input, and output vectors of agent $i$, and $\omega_i \in \mathbb{R}^w$ is the external disturbance.

The communication network provides each agent with a linear combination of its own outputs relative to that of other neighboring agents. In particular, each agent $i \in \{1, \ldots, N\}$ has access to the quantity,

$$
\bar{x}_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j),
$$

(5)

where $a_{ij} \geq 0$ and $a_{ii} = 0$ indicate the communication among agents while $\bar{e}_{ij}$ denote the coefficients of the associated Laplacian matrix $L$. This communication topology of the network can be described by a weighted and directed graph $G$ with nodes corresponding to the agents in the network and the weight of edges given by the coefficient $a_{ij}$.

The MAS (4) and (5) is referred to as MAS with full-state coupling.

Let $N$ be any positive number and define $\bar{x} = x - x_N$, while

$$
\bar{x} = \begin{pmatrix}
\bar{x}_1 \\
\bar{x}_2 \\
\vdots \\
\bar{x}_{N-1}
\end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_N
\end{pmatrix}.
$$

We denote by $T_{dist}$ the transfer function from $\omega$ to $\bar{x}$.

In this paper, we introduce an localized exchange of information among protocols. In particular, each agent $i \in \{1, \ldots, N\}$ has access to localized information, denoted by $\zeta_i$, of the form

$$
\zeta_i = \sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j)
$$

(6)

where $\xi_j \in \mathbb{R}^n$ is a variable produced internally by agent $j$ which will be appropriately chosen in the coming sections.

In this paper, we focus on scale-free stochastic almost state synchronization for MAS subject to external stochastic disturbances. The scale-free design framework does not require information of the communication topology and the size of the network.

We adopt a deterministic framework for stochastic almost state synchronization as explained in the Introduction that equivalently we focus on scale-free $H_2$ almost state synchronization problem for MAS subject to external stochastic disturbances. More specifically reducing the stochastic RMS norm of synchronization error to any arbitrary degree of accuracy is equivalent to reducing the $H_2$ norm of the transfer function of synchronization error to the disturbance with desired arbitrary degree of accuracy.

We formulate the scale-free $H_2$ almost state synchronization problem of a MAS with localized information exchange.

Problem 1 The scale-free $H_2$ almost state synchronization problem with localized information exchange (scale-free $H_2$-ASSWLIE) for MAS (4) and (5) is to find, if possible, a fixed linear protocol parameterized in terms of a scalar parameter $\rho$ of the form:

$$
\begin{alignat}{2}
\dot{x}_{i,c} &= A_c(\rho)x_{i,c} + B_c(\rho)\bar{z}_i + C_c(\rho)\bar{\xi}_i \\
u_i &= F_c(\rho)x_{i,c}
\end{alignat}
$$

(7)

where $\bar{\xi}_i$ is defined by (6), with $\bar{\xi}_i = H(x,x_{i,c})$ with $x_{i,c} \in \mathbb{R}^n$ such that for any number of agents $N$, and any communication graph $G$ we have:

- in the absence of the disturbance $\omega$, for all initial conditions the state synchronization

$$
\lim_{t \to \infty} (x_i - x_j) = 0 \quad \text{for all } i, j \in \{1, \ldots, N\}
$$

(8)

is achieved for any $\rho \geq 1$.

- in the presence of the disturbance $\omega$, for any $\gamma > 0$, one can render the $H_2$ norm from $\omega$ to $x_i - x_j$ less than $\gamma$ by choosing $\rho$ sufficiently large.

The architecture of the protocol (7) is shown in Figure 1.

Remark 1 We would like to emphasize that in our formulation of Problem 1, the protocol (7), i.e. $(A_c(\rho),B_c(\rho),C_c(\rho),F_c(\rho))$ to be solely designed based on agent model $(A,B,C,E)$ and independent of communication graph and the number of agents.
III. $H_2$ Almost State Synchronization: Solvability Conditions and Protocol Design

In this section, we will consider the $H_2$ almost state synchronization problem of a MAS for both cases of full-and partial-state coupling.

A. Full-state coupling

We use the following dynamic protocol with localized information exchanges.

**Protocol 1:** Full-state coupling

We design collaborative protocols for agent $i \in \{1, \ldots, N\}$ as

$$
\begin{align*}
\dot{x}_i &= A x_i + B u_i + \rho \zeta_i - \rho \dot{x}_i, \\
u_i &= - \rho B^T P x_i,
\end{align*}
$$

where $\rho$ is a parameter satisfying $\rho \geq 1$ while $P$ is the unique solution of algebraic Riccati equation

$$
A^T P + PA - PBB^T P + I = 0
$$

and $\zeta_i$ is defined by (5). The agents communicate $\bar{x}_i = x_i$, therefore each agent has access to local information

$$
\bar{x}_i = \sum_{j=1}^{N} a_{ij} (x_i - x_j).
$$

Then, we have the following theorem for scale-free $H_2$-ASSWILIE case.

**Theorem 1** Consider a MAS described by (4) and (5), where $C = I$.

1) The scale-free $H_2$-ASSWILIE problem as stated in Problem 1 is solvable if and only if

- (A, B) is stabilizable.
- All eigenvalues of $A$ are in the closed left half plane.
- The graph $\mathcal{G}$, describing the communication topology of the network, contains a directed spanning tree.
- $\text{im} E \subseteq \text{im} B$.

2) The linear dynamic Protocol 1 solves scale-free $H_2$-ASSWILIE. In other words, for any number of agents $N$ and any graph $\mathcal{G}$ in the absence of the disturbance $\omega$, for any $\rho \geq 1$, the state synchronization (8) is achieved for any initial conditions while in the presence of the disturbance $\omega$, for any $\gamma > 0$, the $H_2$ norm from $\omega$ to $x_i - x_j$ is less than $\gamma$ by choosing $\rho$ sufficiently large.

To prove this theorem, we need the following lemma.

**Lemma 1** Let a Laplacian matrix $L \in \mathbb{R}^{N \times N}$ be given associated with a graph that contains a directed spanning tree. We define $\bar{L} \in \mathbb{R}^{(N-1) \times (N-1)}$ as the matrix $L = [\bar{L}_{ij}]$ with $\bar{L}_{ij} = \ell_{ij} = \ell_{ij} - \ell_{Nj}$. Then the eigenvalues of $\bar{L}$ are equal to the nonzero eigenvalues of $L$.

**Proof:** We have

$$
\bar{L} = (I - L) (I - 0)^T
$$

Assume that $\lambda$ is a nonzero eigenvalue of $L$ with eigenvector $x$, then

$$
\bar{x} = (I - L) x
$$

satisfies

$$
(I - L) \bar{x} = (I - L) (I - 0)^T (I - L) x = (I - L) \bar{x}
$$

for $Lx = \lambda x$ and since $L1 = 0$ we have $L(I - 0)^T (I - L) = L$, then we find that

$$
\bar{L} \bar{x} = (I - L) (I - L)^T (I - L) x = (I - L) \bar{x}
$$

This shows that $\bar{x}$ is an eigenvector of $\bar{L}$ if $\lambda \neq 0$. It is easily seen that $\bar{x} = 0$ if and only if $\lambda = 0$. Conversely if $\bar{x}$ is an eigenvector of $\bar{L}$ with eigenvalue $\lambda$ then it is easily verified that $\bar{x} = L (I - 0)^T \bar{x}$ is an eigenvector of $L$ with eigenvalue $\lambda$.

**Proof of Theorem 1:** Firstly, let $\bar{x}_i = x_i - x_N$ and $\bar{x}_i = x_i - x_N$. We find:

$$
\dot{\bar{x}}_i = A \bar{x}_i + B (u_i - u_N) + E (\omega_i - \omega_N),
$$

$$
\dot{\bar{x}}_i = A \bar{x}_i + B (u_i - u_N) + \rho \sum_{j=1}^{N-1} \bar{\ell}_{ij} (\bar{x}_i - \bar{x}_j),
$$

$$
u_i - u_N = - \rho B^T P \bar{x}_i.
$$

Next, we define

$$
\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix}, \quad \bar{\zeta} = \begin{pmatrix} \bar{\zeta}_1 \\ \vdots \\ \bar{\zeta}_{N-1} \end{pmatrix}
$$

and we obtain the following closed-loop system

$$
\dot{\bar{x}} = (I \otimes A) \bar{x} - \rho (I \otimes B B^T P) \bar{\zeta} + (I \otimes C) \omega,
$$

$$
\dot{\bar{\zeta}} = (I \otimes A) \bar{\zeta} - \rho (I \otimes B B^T P) \bar{x} + \rho (I \otimes C) \omega
$$

where $L$ as defined in Lemma 1 and $\Pi = (I - L)$. Let $e = \bar{x} - \bar{\zeta}$, we can obtain

$$
\dot{e} = [I \otimes (A - \rho B B^T P)] e + (I \otimes C) \omega
$$

According to Lemma 1, we have that the real part of the eigenvalues of $\bar{L}$ are positive. Therefore, there exists a nonsingular transformation matrix $T$ such that

$$
(T \otimes I)(I \otimes A - \rho L \otimes I) (T^{-1} \otimes I) = I \otimes A - \rho J \otimes I,
$$

where

$$
J = \begin{pmatrix} \lambda_2 & 0 & \cdots & 0 \\ J_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ J_{N1} & \cdots & J_{N,N-1} & \lambda_N \end{pmatrix}
$$

with $\text{Re}(\lambda_i) > 0$ ($i = 2, \cdots, N$), where $\lambda_i$ are the nonzero eigenvalues of $L$.

Then, for the stability of (15), we just need to prove the stability of $A - \rho \lambda_i I$ for $i = 1, \cdots, N - 1$. Since the eigenvalues of $A$ are in the closed left half plane and $\rho \geq 1$,
$A - \rho \lambda I$ is asymptotically stable, i.e. the real part of its eigenvalues are all negative. Since (15) is asymptotically stable, we find that $I \otimes A - \rho \hat{L} \otimes I$ is asymptotically stable. Further, since $A - \rho BB^T P$ is asymptotically stable too, we have the closed-loop system consisting of (13) and (14) is asymptotically stable without disturbance $\omega$. Namely the protocol (9) can achieve the state synchronization. And we just need to prove the $H_2$ norm from $\omega$ to $\bar{x}$ can be made arbitrary small.

From (13) and (14), we have

$$T_{\text{ax}}(s) = (I \ 0) \left( \begin{matrix} T_3 \\ 0 \end{matrix} \right) (P \otimes \omega) \left( \begin{matrix} 0 \\ (P \otimes \epsilon) \end{matrix} \right)$$

$$= T_3^{-1} \left( I + \rho I \otimes (BB^T P) \right) T_2^{-1} (P \otimes \omega) \left( B \otimes X \right)$$

$$= T_3^{-1} (P \otimes \omega) \left( B \otimes X \right)$$

since we have (12) such that $E = BX$ for an $X$, where

$$T_3 = s I - (I \otimes A - \rho \hat{L} \otimes I)$$

$$T_3 = s I - (I \otimes A - \rho BB^T P)$$

Then, we have

$$\|T_{\text{ax}}(s)\|_{H_2} \leq \|T_3^{-1} (P \otimes \omega) \left( B \otimes X \right)\|_{H_2}$$

$$+ \|\rho T_3^{-1} (I \otimes B) (I \otimes B^T P) T_2^{-1} (P \otimes \omega) \left( B \otimes X \right)\|_{H_2}$$

We know that $T_3^{-1} (P \otimes \omega)$ is the transfer matrix of the following system:

$$\dot{\rho} = [I \otimes (A - \rho BB^T P)] \rho + (P \otimes \omega) \left( B \otimes X \right) w.$$ 

Thus let

$$\Theta_0 = \frac{\|\Pi \|_2^2 \|X\|_2^2}{\rho} P^{-1}$$

with $P$ satisfying (10), then we have

$$\Theta_0 (A - \rho BB^T P)^T + (A - \rho BB^T P) \Theta_0 + \Pi \Pi^T \otimes BX X^T B^T$$

$$= \frac{\|\Pi \|_2^2 \|X\|_2^2}{\rho} \left( P^{-1} A^T + A P^{-1} \right) - \|\Pi \|_2^2 \|X\|_2^2 BB^T$$

$$= - \frac{1}{\rho} \|\Pi \|_2^2 \|X\|_2^2 P^{-2} - (1 - \frac{1}{\rho}) \|\Pi \|_2^2 \|X\|_2^2 BB^T < 0$$

since $\rho > 1$ and $P > 0$. According to (28, Lemma 3), the $H_2$ norm from $w$ to $\rho$ is less than $\rho^{-1} \|\Pi \|_2^2 \|X\|_2^2 P^{-1} \|_2$, i.e.

$$\|T_3^{-1} (P \otimes \omega) \left( B \otimes X \right)\|_{H_2} < \|\Pi \|_2^2 \|X\|_2^2 P^{-1} \|_2.$$

Meanwhile, because $T_3^{-1} (I \otimes B)$ is a strictly proper transfer matrix, we have

$$\|\rho T_3^{-1} (I \otimes B) (I \otimes B^T P) T_2^{-1} (P \otimes \omega) \left( B \otimes X \right)\|_{H_2}$$

$$\leq \|T_3^{-1} (I \otimes B)\|_{H_2} \|\rho (I \otimes B^T P) T_2^{-1} (P \otimes \omega) \left( B \otimes X \right)\|_{H_2}.$$ 

We know that $T_3^{-1} (I \otimes B)$ is the transfer matrix of the following system:

$$\dot{\bar{x}} = (A - \rho BB^T P) \bar{x} + B w.$$ 

Let

$$\Theta = \rho^{-1} P^{-1},$$

with $P$ satisfying (10), then we have

$$\Theta (A - \rho BB^T P)^T + (A - \rho BB^T P) \Theta + BB^T$$

$$= \rho^{-1} (P^{-1} A^T + A P^{-1}) - BB^T$$

$$= \rho^{-1} P^{-1} - (1 - \rho^{-1}) BB^T < 0$$

since $\rho > 1$ and $P > 0$. Correspondingly, the $H_2$ norm from $w$ to $q$ is less than $\frac{1}{\rho} \|P^{-1} \|_2$, i.e.

$$\|T_3^{-1} (I \otimes B)\|_{H_2} < \rho^{-1} \|P^{-1} \|_2.$$ 

Meanwhile, since $\hat{L}$ is invertible, there exists a constant $W_2 > 0$ such that $\|T_2^{-1} \|_{H_2} \leq W_2$, and thus obtain

$$\|T_{\text{ax}}\|_{H_2} \leq \|T_3^{-1} (P \otimes \omega) \left( B \otimes X \right)\|_{H_2}$$

$$+ \|T_3^{-1} (I \otimes B) \|_{H_2} \|\rho (I \otimes B^T P) T_2^{-1} (P \otimes \omega) \left( B \otimes X \right)\|_{H_2}$$

$$\leq \rho^{-1} \left( \|\Pi \|_2^2 \|X\|_2^2 \|P^{-1} \|_2 + W_2 \|P^{-1} \|_2 \|B^T P \|_2 \|B \|_2 \|X \|_2 \|2 \right)$$

i.e.

$$\|T_{\text{ax}}\|_{H_2} < \rho^{-1} M$$

with

$$M = \|P^{-1} \|_2 \|\Pi \|_2 \left( \|\Pi \|_2 \|X\|_2^2 + W_2 \|B^T P \|_2 \|B \|_2 \|X \|_2 \right)$$

for $\rho > 1$. It means that we have

$$\|T_{\text{ax}}(s)\|_{H_2} \leq \rho^{-1} M.$$ 

Now, we will prove the necessity. Assume we have a protocol of the form (7) that achieves synchronization for any possible graph in the absence of disturbances. It is easily seen that this requires that condition (a) is satisfied. On the other hand, we have that

$$\left\{ \begin{array}{l}
\dot{\bar{x}} = (I \otimes A) \bar{x} + (\hat{L} \otimes B \hat{F}(\rho)) \bar{x} \\
\dot{\bar{y}} = (I \otimes \hat{A}_r(\rho)) \bar{y} + (\hat{L} \otimes B_r(\rho) \hat{C}) \bar{y} + (\hat{L} \otimes \hat{C}(\rho) H_r) \bar{y}
\end{array} \right.$$ 

must be asymptotically stable for all possible Laplacian matrices. By letting $\hat{L} \to 0$ we see that in the limit the system must have all eigenvalues in the closed left half plane which yields that condition (b) must be satisfied. It is well-known that state synchronization is impossible to achieve if the network does not have a directed spanning tree. Finally, from the result on $H_2$ almost disturbance decoupling in [20, Corollary 2.4], we find that (d) is also a necessary condition.

### B. Partial-state coupling

In this subsection, we will consider $H_2$ almost state synchronization via partial-state coupling.
Protocol 2: Partial-state coupling

We design collaborative protocols for agent \(i \in \{1, \ldots, N\}\) as

\[
\begin{align*}
\dot{x}_i &= A\dot{x}_i - \rho BB^TP\hat{\chi}_i + \delta^{-2}Q_2\chi_i - C\dot{x}_i \\
\dot{\chi}_i &= A\chi_i + Bu_i + \rho \dot{x}_i - \rho \hat{\chi}_i \\
u_i &= -\rho BB^P\chi_i,
\end{align*}
\]  
(16)

where \(P > 0\) is the unique solution of (10). Since \((A, E, C, 0)\) is minimum-phase and left invertible, then for any \(\rho \geq 1\), there exists \(\delta > 0\) small enough such that \(Q_2 > 0\) is the unique solution of

\[
Q_2A^T + AQ_2 + EE^T - \delta^{-2}Q_2C'CQ_2 + \rho^2Q_2^2 = 0. 
\]  
(17)

In this protocol, agents communicate \(\hat{\chi}_i = \chi_i\), i.e., each agent has access to localized information (11), while \(\hat{\chi}_i\) is defined by (5).

Then, we have the following theorem for MAS via partial-state coupling.

**Theorem 2** Consider a MAS described by (4) and (5).

1. The scale-free \(H_2\)-ASSWLIE problem stated in Problem 1 is solvable if and only if
   a) \((A, B)\) are stabilizable and \((C, A)\) are detectable.
   b) All eigenvalues of \(A\) are in the closed left half plane.
   c) \((A, E, C, 0)\) is minimum phase and left invertible.
   d) The graph \(\mathcal{G}\), describing the communication topology of the network, contains a directed spanning tree.
   e) \(\text{im}E \subseteq \text{im}B\) (i.e. (12)).

2. The linear dynamic Protocol 2 solves Scale-Free \(H_2\)-ASSWLIE, for any number of agents \(N\) and any graph \(\mathcal{G}\) such that in the absence of disturbance \(\omega\), for any \(\rho \geq 1\), the state synchronization (8) is achieved for any initial conditions and in the presence of disturbance \(\omega\), for any \(\gamma > 0\), the \(H_2\) norm from \(\omega\) to \(x_i - x_j\) is less than \(\gamma\) by choosing \(\rho\) sufficiently large.

Proof of Theorem 2: Similar to Theorem 1 and by defining

\[
\begin{align*}
\dot{x}_i &= \dot{x}_i - \dot{x}_N, \\
\hat{x}_i &= A\dot{x}_i + B(u_i - u_N) + E(\omega - \omega_N) \\
\hat{\chi}_i &= A\chi_i - \rho BB^TP\sum_{j=1}^{N-1} \tilde{\epsilon}_{ij}\hat{x}_j + \delta^{-2}Q_2\chi_i - \chi_i \\
\hat{\chi}_i &= A\hat{\chi}_i + B(u_i - u_N) + \rho \hat{x}_i - \rho \sum_{j=1}^{N-1} \tilde{\epsilon}_{ij}\hat{x}_j
\end{align*}
\]

We define

\[
\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{N-1} \end{pmatrix}, \quad \bar{\chi} = \begin{pmatrix} \bar{\chi}_1 \\ \vdots \\ \bar{\chi}_{N-1} \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}
\]

then we have the following closed-loop system

\[
\begin{align*}
\dot{x} &= (I \otimes A)x - \rho (I \otimes BB^P)\bar{x} + (\Pi \otimes E)\omega \\
\hat{x} &= (I \otimes (A - \frac{1}{\delta^2}Q_2C'C))\bar{x} - \rho (\hat{\chi} \otimes BB^P) + \frac{1}{\delta^2}(\hat{\chi} \otimes Q_2C'C)\bar{x} \\
\hat{\chi} &= (I \otimes (A - \rho L \otimes I))\bar{\chi} - \rho (I \otimes BB^P)\bar{\chi} + \rho \bar{\chi}
\end{align*}
\]

By defining \(e = \bar{x} - \hat{\chi}\) and \(\bar{e} = (\hat{L} \otimes I)\bar{x} - \bar{\chi}\), we can obtain

\[
\begin{align*}
\dot{\bar{x}} &= (I \otimes (A - \rho BB^P))\bar{x} + (\Pi \otimes E)\omega \\
\bar{e} &= (I \otimes (A - \delta^{-2}Q_2C'C))\bar{e} + (\Pi \otimes E)\omega \\
\dot{\bar{e}} &= (I \otimes (A - \rho L \otimes I))\bar{e} + \rho \bar{e} + (\Pi \otimes E)\omega
\end{align*}
\]  
(18)

From Theorem 1 and (17), we have \(A - \rho BB^P, I - A - \rho L \otimes I\) and \(A - \delta^{-2}Q_2C'C\) are stable for \(\rho \geq 1\) and some \(\delta > 0\). It means that system (18) is asymptotically stable without disturbance \(\omega\). So, we have the protocol (16) can achieve the state synchronization. Meanwhile, we can obtain the following result by choosing the Lyapunov function

\[
V_0 = \bar{e}^2(I \otimes Q_2^{-1})\bar{e}
\]

for \(\bar{e}\) with \(Q_2\) satisfying (17). Then we have

\[
\begin{align*}
V_0 &= \bar{e}^2[I \otimes (Q_2^{-1}A + A'Q_2^{-1} - 2\delta^{-2}C'C)]\bar{e} \\
&\quad + 2\bar{e}(\bar{L} \Pi \otimes Q_2^{-1})\omega \\
&\leq -\bar{e}^2[I \otimes (\rho^2I + Q_2^{-1}EE^TQ_2^{-1})]\bar{e} \\
&\quad + \bar{e}^2(I \otimes Q_2^{-1}EE^TQ_2^{-1})\bar{e} + \omega^2(\Pi \otimes L^2\Pi \otimes I)\omega \\
&\leq -\rho^2\|\bar{e}\|_2^2 + \|L\|_2^2\|\Pi\|_2^2\|\omega\|_2^2
\end{align*}
\]

with \(\rho \geq 1\). By integrating the above inequality, we have

\[
\int_0^\infty -\rho^2\|\bar{e}(t)\|_2^2 + \|L\|_2^2\|\Pi\|_2^2\|\omega(t)\|_2^2 \, dt \geq 0
\]

for zero initial conditions and hence

\[
\rho^2\|\bar{e}\|_2^2 \leq \|L\|_2^2\|\Pi\|_2^2\|\omega\|_2^2
\]

i.e.

\[
\|T_{o\bar{e}}\|_{H_2} \leq \|L\|_2\|\Pi\|_2\|\omega\|_2 \rho^{-1}
\]  
(19)

for \(\rho \geq 1\).

Then we need to prove the \(H_2\) norm from \(\omega\) to \(\bar{x}\) (or \(x_i - x_j\)) can be made arbitrary small. From (18), we have

\[
T_{o\bar{e}}(s) = (I \otimes 0) \begin{pmatrix} T_3 & 0 & -\rho T_4 \\ 0 & T_1 & 0 \\ 0 & -\rho T_2 & T_2 \end{pmatrix}^{-1} \begin{pmatrix} (\Pi \otimes E) \\ (\Pi \Pi \otimes E) \\ (\Pi \otimes E) \end{pmatrix}
\]

\[
= T_3^{-1}(I \otimes B)T_0
\]

where \(T_1 = sI - I \otimes (A - \delta^{-2}Q_2C'C), T_2, T_3\) are as same as the definition in Theorem 1, \(T_4 = I \otimes BB^P\),

\[
T_0 = (\Pi \otimes X) + \rho^2(I \otimes B^TP)T_2^{-1}T_{o\bar{e}} + \rho(I \otimes B^P)T_2^{-1}(\Pi \otimes BX)
\]

and \(T_{o\bar{e}} = T_1^{-1}[(\hat{L} \Pi) \otimes E]\).

Similar to the proof of Theorem 1, we have

\[
\|T_{o\bar{e}}\|_{H_2} \leq \|T_3^{-1}[\Pi \otimes (BX)]\|_{H_2} \\
+ \|T_3^{-1}(I \otimes B)\rho^2(I \otimes B^P)T_2^{-1}T_{o\bar{e}}\|_{H_2} \\
+ \|T_3^{-1}(I \otimes B)\rho(I \otimes (B^P))T_2^{-1}(\Pi \otimes BX)\|_{H_2} \\
\leq \|T_3^{-1}[\Pi \otimes (BX)]\|_{H_2} \\
+ \|T_3^{-1}(I \otimes B)\|_{H_2} \|\hat{L}\|_{H_2}^2 \|\rho(I \otimes (B^P))T_2^{-1}T_{o\bar{e}}\|_{H_2} \\
+ \|T_3^{-1}(I \otimes B)\|_{H_2} \|\Pi \otimes (B^P)\|_{H_2} \|T_2^{-1}(\Pi \otimes BX)\|_{H_2}
\]
because \( T_3^{-1}(I \otimes B) \) is a strictly proper transfer matrix. Then, we have
\[
\| T_{oo} \|_{H_2} \leq \| T_3^{-1}(I \otimes (BX)) \|_{H_2} \\
+ p^2 \| T_3^{-1}(I \otimes B) \|_{H_2} \| B^2P \|_2 \| T_2^{-1}(H_H \| T_{oo} \|_{H_2}} \\
+ p \| T_3^{-1}(I \otimes B) \|_{H_2} \| B^2P \|_2 \| T_2^{-1}(H_H \| T_{oo} \|_{H_2} \| BX \|_2). 
\]

From Theorem 1, we have
\[
\| T_3^{-1}(\Pi \otimes BX) \|_{H_2} < \rho^{-1} \| \Pi \|_2 \| X \|_2 \| P^{-1} \|_2 \quad \text{and} \quad \| T_3^{-1}(I \otimes B) \|_{H_2} < \rho^{-1} \| P^{-1} \|_2.
\]

for \( \rho \geq 1 \). Meanwhile, according to the results of Theorem 1, we have
\[
\| T_2^{-1}(H_H \| T_2^{-1} \| H_\infty \leq \rho^{-1}W_2
\]

for \( \rho \geq 1 \). Thus, for \( \rho \geq 1 \), we can obtain
\[
\| T_{oo(s_i-s_j)} \|_{H_2} < \rho^{-1}W
\]

with
\[
W = \| P^{-1} \|_2 \| \Pi \|_2 \| X \|_2 \| \Pi \|_2 \| X \|_2 \| \Pi \|_2 \| X \|_2 \| \Pi \|_2 \| X \|_2 \| \Pi \|_2 \| X \|_2 \\
+ W_2 \| B^2P \|_2 \| L \|_2 + W_2 \| B^2P \|_2 \| BX \|_2. 
\]

for \( \rho \geq 1 \), i.e.
\[
\| T_{oo(s_i-s_j)} \|_{H_2} < \rho^{-1}W. 
\]

As the next step, we will prove the necessity. Similar to the proof of Theorem 1, we have that (a), (b) and (d) are necessary conditions. From the result on \( H_2 \) almost disturbance decoupling in [20, Theorem 2.3], we find that (c) and (e) are also necessary conditions in case of partial-state coupling.

**Remark 2** We would like to emphasize that Protocol 1 and 2 do not need any information about the number of agents and Laplacian matrix \( L \) associated with the communication graph. On the other hand, the parameter \( \rho \) in our protocol design is used to reduce the impact of disturbance on synchronization error. However, when we need a better disturbance rejection level (smaller \( H_2 \) norm), then we need to increase \( \rho \). It worth to mention that in the absence of disturbance we can choose \( \rho \) arbitrarily.

**Remark 3** Note that Protocol 1 and 2 can generate a signal which is part of their state and communicate with their neighbors over the same communication graph that leads to achieving scalable protocols. Meanwhile, these protocols are universal since they can work for any communication graph with any number of agents as long as the graph contains a directed spanning tree. Meanwhile the design methodology is scalable since it is a one shot design based on an explicit linear structure.

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**IV. NUMERICAL EXAMPLE**

In this section we will illustrate the effectiveness of our protocol design with a numerical example for \( H_2 \) state synchronization with partial-state coupling. Consider agent models (4) with
\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad E = B. 
\]

For this agent model, we obtain Protocol 2, by solving algebraic Riccati equations (10) and (17) for three values of \( \rho = 4, \rho = 6 \) and \( \rho = 10 \) (The Riccati equation (17) is solved with \( \delta = 0.0004 \)).

We create two homogeneous MAS with different number of agents and different communication topologies to show the designed protocol is scale-free, i.e. it is independent of information of the communication network and the number of agents \( N \).

- **Case I:** In this case, we consider MAS with 3 agents and communication topology \( A_1 \), with \( a_{21} = a_{32} = 1 \). The result of exact state synchronization in the absence of disturbance and \( H_2 \) almost state synchronization in presence of white noises with unit power spectral densities for \( i = 1, \ldots, N \) are shown in Figure 2. The results show that by increasing \( \rho \), one can decrease the impact of disturbances on synchronization error.

- **Case II:** Next, we consider a MAS with 20 agents and associated adjacency matrix \( A_2 \), with \( a_{16} = a_{21} = a_{32} = a_{43} = a_{54} = a_{65} = a_{76} = a_{87} = a_{98} = a_{10,9} = a_{11,10} = a_{12,11} = a_{13,12} = a_{14,13} = a_{15,14} = a_{15,6} = a_{16,15} = a_{17,16} = a_{18,17} = a_{19,18} = a_{20,18} = 1. \)
The simulation results also show that by increasing one-shot protocol design, for any graph with any number of agents. The simulation results show that the protocol design is independent of the communication graph and is scale free so that we can achieve $H_2$ almost state synchronization with one-shot protocol design, for any graph with any number of agents. The simulation results also show that by increasing the value of $\rho$, almost state synchronization is achieved with higher degree of accuracy.

V. Conclusion

In this paper, we studied $H_2$ almost state synchronization of homogeneous networks of non-introspective agents. A parameterized scale-free linear dynamic protocol, parameterized in scalar $\rho$, was developed using localized information exchange over the same communication network and solely based on agent models. In particular, in the absence of disturbance, we achieved synchronization for any $\rho > 1$ and in the presence of disturbance we achieved almost state synchronization for a given arbitrary degree of accuracy by choosing $\rho$ sufficiently large. Despite all the existing results, our design methodology was scale-free so that we did not need any information about the communication network such as bounds on the associated Laplacian matrix and the number of agents. As our future work, we aim to extend the scale-free designs proposed in this paper to the broader classes of agent models.

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