OPTIMAL CONDITION FOR BLOW-UP OF THE CRITICAL $L^q$ NORM FOR THE SEMILINEAR HEAT EQUATION

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Abstract. We shed light on a long-standing open question for the semilinear heat equation $u_t = \Delta u + |u|^{p-1}u$. Namely, without any restriction on the exponent $p > 1$ nor on the smooth domain $\Omega$, we prove that the critical $L^q$ norm blows up whenever the solution undergoes type I blow-up. A similar property is also obtained for the local critical $L^q$ norm near any blow-up point.

In view of recent results of existence of type II blow-up solutions with bounded critical $L^q$ norm, which are counter-examples to the open question, our result seems to be essentially the best possible result in general setting. This close connection between type I blow-up and critical $L^q$ norm blow-up appears to be a completely new observation.

Our proof is rather involved and requires the combination of various ingredients. It is based on analysis in similarity variables and suitable rescaling arguments, combined with backward uniqueness and unique continuation properties for parabolic equations.

As a by-product, we obtain the nonexistence of self-similar profiles in the critical $L^q$ space. Such properties were up to now only known for $p \leq p_S$ and in radially symmetric case for $p > p_S$, where $p_S$ is the Sobolev exponent.

1. Introduction and main results

1.1. Background. We consider the semilinear heat equation

$$
\begin{cases}
    u_t &= \Delta u + |u|^{p-1}u, & t > 0, \ x \in \Omega, \\
    u &= 0, & t > 0, \ x \in \partial \Omega, \\
    u &= u_0(x), & t > 0, \ x \in \Omega,
\end{cases}
$$

where $p > 1$ and $\Omega$ is a smooth (possibly unbounded) domain of $\mathbb{R}^N$ with $N \geq 1$. For any $u_0 \in L^\infty(\Omega)$, problem (1.1) has a unique, maximal classical solution $u$. We denote by $T = T(u_0) \in (0, \infty]$ its maximal existence time. If $T < \infty$, then blow-up occurs, in the sense that

$$
\lim_{t \to T} \|u(t)\|_\infty = \infty.
$$

Here and hereafter, we denote by $\| \cdot \|_q$ the norm of $L^q(\Omega)$ for $1 \leq q \leq \infty$. The notation is also extended to $q \in (0, 1)$ (although this is not a norm). In all this article, we use the notation

$$
p_S := (N+2)/(N-2)_+, \quad \beta = 1/(p-1)
$$

and

$$
q^* = N(p-1)/2
$$

(note that we allow values of $q^* < 1$).

The study of the well-posedness of problem (1.1) in Lebesgue spaces goes back to the late 1970’s-early 1980’s. As is well known, for nonlinear PDE’s in general, such properties are not only interesting in themselves, but also play an important role in the qualitative analysis of solutions (global existence, blow-up behavior); see, e.g. the monograph [44] and the references therein. The space $L^q$ is invariant under the natural scaling of the equation and it is known that the exponent

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$q^*$ is critical for local well-posedness in $L^q$ spaces and for the blow-up of $L^q$ norms. Namely, for $q \geq 1$, we have:

- If $q > q^*$, then problem (1.1) with $u_0 \in L^q(\Omega)$ is well posed, locally in time, and the existence time is uniformly positive for $u_0$ in bounded sets of $L^q(\Omega)$ (see [29, 3]). As a consequence, we have

$$T < \infty \quad \Rightarrow \quad \lim_{t \to T} \|u(t)\|_q = \infty \quad \text{for all } q > q^*.$$

- If $1 \leq q < q^*$, then problem (1.1) with $u_0 \in L^q(\Omega)$ is ill posed (both local existence and local uniqueness may fail; see [49, 27, 5]) and moreover (see [19]), there exist solutions such that $T < \infty$ and

$$\sup_{t \in (0,T)} \|u(t)\|_q < \infty.$$

- If $q = q^* > 1$, then problem (1.1) with $u_0 \in L^q(\Omega)$ is still well posed (see [49, 3]), but the existence time is no longer uniformly positive for $u_0$ in bounded sets of $L^q(\Omega)$ (see [3]), so that blow-up of the critical $L^q$ norm when $T < \infty$, even in the sense

$$\sup_{t \in (0,T)} \|u(t)\|_{q^*} = \infty,$$

cannot be deduced from the local existence theory.

The critical $L^q$ norm blow-up problem, i.e. the question whether (1.2) is true when $T < \infty$, has attracted a lot of attention in the past decades, but is still not fully understood. The following sufficient conditions for (1.2) when $T < \infty$ are known:

(a) $p = 1 + \frac{4}{N}$ (cf. [2] and see also [44 Proposition 16.3]);
(b) $p > 1 + \frac{2}{N}$ and $u_t \geq 0$ (see [51, 37]);
(c) $p = 1 + \frac{2}{N}$, $u \geq 0$, and $\Omega = \mathbb{R}^N$ or $\Omega$ bounded and $u$ bounded near $\partial \Omega$ (see [53]);
(d) $1 < p < p_S$ and $\Omega$ convex (see [22]);
(e) $p = p_S$, $\Omega = B_R$ or $\mathbb{R}^N$, and $u \geq 0$ radially symmetric (this follows from [18, 34]);
(f) $p > p_S$, $\Omega = B_R$ or $\mathbb{R}^N$, and $u$ radially symmetric (see [37, 35]).

Here and hereafter we denote by $B_r(a)$ the ball with radius $r$ centered at $a$ in $\mathbb{R}^N$, and simply by $B_r$ in the case $a = 0$. Up to recently, no counter-examples to property (1.2) were known, and it was somehow conjectured that (1.2) should be true for any blow-up solution of problem (1.1). However, it turns out that this is not the case: some special blow-up solutions constructed in [46, 12] do satisfy

$$\sup_{t \in (0,T)} \|u(t)\|_{q^*} < \infty.$$  \hfill (1.3)

Since the $L^{q^*}$ behavior of the solutions was not considered there, we give the proof in Appendix. This concerns the case $p = p_S$ with $N = 4$ or 5 (see [46, 12], respectively) and includes radial as well as nonradial situations. A key feature of these special solutions is that they undergo type II blow-up. Let us recall that blow-up is said to be of type I if

$$\limsup_{t \to T} (T - t)^{\frac{1}{3}} \|u(t)\|_{\infty} < \infty,$$  \hfill (1.4)

and type II otherwise. Type I blow-up is consistent with the natural scaling of the equation. In particular, if (1.4) holds, then the blow-up rate is actually comparable to that of the corresponding ODE, namely

$$C_1 \leq (T - t)^{\frac{1}{3}} \|u(t)\|_{\infty} \leq C_2, \quad t \in (0, T),$$

for some constants $C_1, C_2 > 0$. Sufficient conditions ensuring type I blow-up are summarized in Remark (1.11) below.

In view of the above results, a natural question is then: \hfill (1.5)

Does the critical $L^q$ norm blow up whenever blow-up is type I?
We will show that (1.5) is indeed true in full generality, for any \( p > 1 \) and without any restriction on \( \Omega \) or on the sign of \( u \). Moreover, we will show that the same property is true for the local critical \( L^q \) norm near any blow-up point. Recall that \( a \in \overline{\Omega} \) is said to be a blow-up point of \( u \) if \( \limsup_{t \to T, x \to a} |u(t, x)| = \infty \).

1.2. Main results. Our main theorem is the following.

**Theorem 1.** Let \( \Omega \) be any smooth domain of \( \mathbb{R}^N \), \( p > 1 \) and \( u_0 \in L^\infty(\Omega) \). Let \( u \) be a type I blow-up solution of (1.1). Then the following holds.

(i) We have
\[
\lim_{t \to T} \|u(t)\|_{L^{q^*}(\Omega)} = \infty.
\]

(ii) For any blow-up point \( a \in \overline{\Omega} \) and any \( r > 0 \), we have
\[
\lim_{t \to T} \int_{\Omega \cap B_r(a)} |u(t, x)|^{q^*} \, dx = \infty.
\]

In view of the above mentioned counter-examples, Theorem 1 seems to be essentially the best possible result in general setting, and this close connection between type I blow-up and critical \( L^{q^*} \) norm blow-up appears to be a completely new observation. On the other hand, an important special class of blow-up solutions of (1.1) consists of (classical) backward self-similar solutions. These are of the form
\[
u(t, x) = (T - t)^{-\beta} \varphi\left(\frac{x}{\sqrt{T - t}}\right), \quad t \in [0, T), \ x \in \mathbb{R}^N,
\]
where \( \varphi \in C^2(\mathbb{R}^N) \) is a nontrivial bounded function, called the (backward self-similar) profile. A direct computation shows that \( u \) in (1.6) is a solution of (1.1) if and only if \( \varphi \) solves the elliptic equation
\[
\Delta \varphi - \frac{N}{2} \cdot \nabla \varphi + |\varphi|^{p - 1} \varphi - \beta \varphi = 0, \quad y \in \mathbb{R}^N.
\]
Moreover, for any backward self-similar solution \( u \), we have \( \|u(t)\|_{L^{q^*}(\mathbb{R}^N)} = \|\varphi\|_{L^{q^*}(\mathbb{R}^N)} (\leq \infty) \) for all \( t \in [0, T) \). As a by-product of Theorem 1, we thus have the following consequence for all possible backward self-similar profiles.

**Corollary 2.** Let \( N \geq 1 \) and \( p > 1 \). Let \( \varphi \in C^2(\mathbb{R}^N) \) be a nontrivial, bounded classical solution of (1.7). Then
\[
\varphi \notin L^{q^*}(\mathbb{R}^N).
\]

Let us recall (cf. [21]) that for \( p \leq p_S \), the only bounded backward self-similar profiles are the constants \( \varphi = \pm \kappa \) with \( \kappa = \beta^\beta \) (which implies conclusion (1.8) in that case). For \( N \geq 3 \) and
\[
p_S < p < p_L := 1 + \frac{6}{(N - 10)_+},
\]
there exist nonconstant, radial bounded profiles (see [32], [4], [33]), and they are known to behave like \( r^{-2/(p - 1)} \) as \( r \to \infty \), so that (1.8) is in particular true. As for nonradial profiles, they seem quite difficult to study for \( p > p_S \). The conclusion (1.8) is therefore significant.

**Remark 1.1.** (i) **Type I blow-up.** Blow-up is known to be type I whenever \( 1 < p < p_S \) and \( \Omega \) is convex (see [22], [24], [25]), or \( p > 1 \) and \( u_t \geq 0 \) (cf. [19], [45]). For positive solutions, this remains true for any \( \Omega \) (possibly nonconvex) if either \( N = 2 \) or \( p < N(N + 2)/(N - 1)^2 \) \((< p_S)\); see [42], [43]. However it is still unclear if this is true in the full range \((1, p_S)\) for nonconvex domains.
Concerning the supercritical range in the Sobolev sense, it is known (see [34], [40]) that for radial solutions, blow-up is always type I in a ball and under some conditions in \( \mathbb{R}^N \) for \( p < p_{JL} \), where

\[
p_{JL} := 1 + 4 \frac{N - 4 + 2\sqrt{N - 1}}{(N - 2)(N - 10)}.
\]

In the supercritical and critical range in the Sobolev sense, type I blow-up is also known to occur for some solutions which are neither radial nor increasing in time: see [10] for \( N = 3, p > p_s \), [38] for \( N = 4, p > 5, 7 \) for \( N \geq 7 \) and \( p = p_s \) and [8] for \( p = ps \).

(ii) **Type II blow-up.** The existence of type II solutions is up to now only known to occur for certain values in the range \( p \geq p_s \):

- \( p = p_s \) (see [46], [12]);
- \( p = (N + 1)/(N - 3) \) with \( N \geq 7 \) (see [11]);
- \( p \geq p_{JL} \); see [30], [29], [39], [47] in radial case and [6], [9] in nonradial case.

We note that one cannot expect general conclusions regarding the blow-up of the critical \( L^q \) norm for type II blow-up solutions, since there are examples of boundedness of the critical \( L^q \) norm as stated above (cf. [46], [12]) as well as of unboundedness.

In fact, for \( p > p_s \), radial type II blow-up solutions converge to the singular stationary solution \( c_*|x|^{-2/(p-1)} \) as \( t \) tends to \( T \) (see [35]), and this implies unboundedness of \( L^q \) norm. As for the nonradial solutions in [6], [9], [11], the unboundedness of \( L^q \) norm can be directly seen by simple computations, using the asymptotic form of the solution obtained in those works.

On the other hand, it seems a difficult open problem whether or not boundedness of the critical \( L^q \) norm may happen for type II blow-up solutions with \( p \neq p_s \). In fact, the construction of the known nonradial type II blow-up solutions is somewhat related to radial type II blow-up profiles in lower dimensions and we have no information on nonradial type II blow-up solutions completely independent of radial type II profiles.

**Remark 1.2.** (i) The results (b)-(e) mentioned in section 1.1 above become special cases of Theorem [1] since blow-up is known to be type I under the corresponding assumptions. On the other hand, for any \( p \leq p_s \), nothing seems to be known on blow-up type for nonmonotone sign-changing solutions when \( \Omega \) is nonconvex. Therefore, the result in (a) in the very special case \( p = 1 + 4/N \) is not a consequence of Theorem [1].

(ii) In Theorem [1], the function \( \|u(t)\|_{L^{q^*}(\Omega)} \) is allowed to take the value \( \infty \) (which arises, when \( \Omega \) is unbounded, if \( u(t) \notin L^{q^*}(\Omega) \) at some \( t \)). Of course, in the case \( q^* \geq 1 \) (i.e. \( p \geq 1 + 2/N \)), if we assume \( u_0 \in L^{q^*}(\Omega) \), then \( u(t) \in L^{q^*}(\Omega) \) for all \( t \in (0, T) \).

(iii) As a consequence of the proof of Theorem [1] for any \( p > 1 \) and any type I blow-up solution, we obtain an \( L^{q^*} \)-concentration property in backward space-time parabolas near every blow-up point (see Lemma [3.1]), which may be of independent interest.

1.3. **Outline of proof.** The proof of Theorem [1] is rather involved and requires the combination of various ingredients. Let us sketch the proof. We assume for contradiction that there exists a solution \( u \) undergoing type I blow-up at \( t = T \), whose \( L^{q^*} \) norm stays bounded along a sequence of times \( t_n \to T \).

We first show a concentration property for the \( L^{q^*} \) norm near blow-up points, by means of an analysis in backward similarity variables. It is classical to analyze global solution of the new equation generated by the change of variables, instead of the original solution blowing up in finite time (cf. [21], [22]). The fact that the energy is nonincreasing in time in convex domains plays a crucial role there. We here deal with all domains not necessarily convex. One of the main points in our method is to use a criterion to exclude blow-up at a given point involving a certain weighted \( L^1 \) norm, based on suitable delayed smoothing effects in weighted \( L^q \) spaces.
Next supposing without loss of generality that the origin is a blow-up point, we introduce the sequence of rescaled solutions
\[ v_n(s, y) = (T - t_n)^\beta u(T + (T - t_n)s, \sqrt{T - t_n}y). \]

Using boundedness of the (local) \( L^q \) norm, type I estimates and the above delayed smoothing effects in similarity variables, we can show that, up to a subsequence:
\[ v(s, y) := \lim_{n \to \infty} v_n(s, y) \]
exist and solves (1.1) in \((-2, 0) \times D\) (where \( D \) is the whole space or a half-space depending on the location of the blow-up point),
\[ |v_n(s, y)| \leq C, \quad \text{for } s \in [-1, 0) \text{ and } |y| \geq R, \quad \text{with some } C, R > 0, \]
and
\[ \int_{B_1(y_0)} |v_n(0, y)|^q dy \to 0 \quad \text{as } n \to \infty, \quad \text{for } |y_0| \geq R. \]

From this it can be deduced that \( v \) is bounded in \([-1, 0) \times \{|y| \geq R\} \cap D\) and that
\[ v(0, y) = 0 \quad \text{in } \{|y| \geq R\} \cap D. \]

Applying backward uniqueness in a half-space (cf. [17]) and then unique continuation, we infer that \( v \equiv 0 \) in \([-1, 0) \times D\). This leads to a contradiction with the concentration property given in the first part.

**Remark 1.3.** (i) Let us mention that a method based on backward uniqueness and unique continuation was used in [17] (see also references therein) to show that if a solution to the Navier-Stokes equations in \( \mathbb{R}^3 \) has a singularity, then the \( L^3 \) norm diverges to \( \infty \). The \( L^3 \) norm is critical there, however the situation is quite different from ours. In particular, while type I estimate is not needed in [17], it is crucially needed here since there exist counter-examples to the conclusion of Theorem 1 without type I estimate.

On the other hand, a related method was recently applied in [41] to prove that only type II blow-up is possible in the parabolic-parabolic Keller-Segel system in \( \mathbb{R}^2 \) (unlike in higher dimensions \( N \geq 3 \), where type I blow-up does occur). The possibility of type I blow-up was ruled out there by a contradiction argument. The system consists of two parabolic equations for two components, in which one equation is linear and the other has nonlinear drift term, and the linear equation is helpful in a priori estimates.

More generally, it is common in the case of the Keller-Segel system and nonlinear heat equation to need type I estimate (see, e.g., the monograph [44]). The methods to obtain the so called \( \varepsilon \)-regularity and to show concentration of the local critical \( L^q \) norm at a blow-up point are essentially different in Navier-Stokes, Keller-Segel and nonlinear heat equations, due to distinctive structural features.

(ii) We point out that for problem (1.1) in the special case \( p < p_S \) and \( \Omega \) convex, blow-up of the critical \( L^q \) norm was obtained in [22] as a consequence of the analysis of the local self-similar blow-up profile. The local self-similar profile analysis was extended in [31] to the case \( p \leq p_S, \Omega \) bounded and \( u_0 \geq 0 \) under the assumption of type I blow-up, and one can thereby infer blow-up of the critical \( L^q \) norm in that case. This approach is restricted to the range \( p \leq p_S \) and is very different from our proof, which does not require any information on the local self-similar blow-up profile, and on the contrary provides new information on this profile as a by-product when \( p > p_S \) (cf. Corollary 2).

The outline of the paper is as follows. Section 2 provides some important preliminaries to our proofs and consists of two subsections. In subsection 2.1 we recall the classical framework of similarity variables from [21] and we give a suitable version of a criterion from [1, 48] to exclude blow-up at a given point. In subsection 2.2 we recall some known results on backward uniqueness.
for parabolic equations. Section 3 is then devoted to the proof of Theorem 1. Finally, in Appendix, we verify that the special solutions constructed in [46], [12] remain bounded in \( L^{q^*} \) (these papers did not address the \( L^{q^*} \) behavior of the solutions).

2. Preliminaries

2.1. Similarity variables and local criterion for excluding blow-up. Denote by \( \tilde{u}(t, \cdot) \) the extension of \( u(t, \cdot) \) by 0 to the whole of \( \mathbb{R}^N \) and let \( a \in \mathbb{R}^N \). Following [21], we define the (backward) similarity variables around \((T, a)\) by

\[
s = -\log(T - t), \quad y = \frac{x - a}{\sqrt{T - t}} = e^{s/2}(x - a),
\]

set \( s_0 := -\log T \), and define the rescaled function by

\[
w(s, y) = w_a(s, y) := (T - t)^{\beta} \tilde{u}(t, x - a) = (T - t)^{\beta} \tilde{u}(t, a + y\sqrt{T - t}), \quad s_0 \leq s < \infty, \quad y \in \mathbb{R}^N.
\]

We note that the rescaling point \( a \) is usually taken in \( \Omega \). However, the above is well defined for any \( a \in \mathbb{R}^N \) and this will be convenient in what follows.

Denoting the transformed domains by

\[
\Omega_s := e^{s/2}(\Omega - a), \quad s > s_0,
\]

the PDE in (1.1) becomes, in similarity variables:

\[
w_s - \mathcal{L}w = w^p - \beta w, \quad s_0 < s < \infty, \quad y \in \Omega_s,
\]

where

\[
\mathcal{L} = \Delta - \frac{y}{2} \cdot \nabla = \rho^{-1} \nabla \cdot (\rho \nabla), \quad \rho(y) = e^{-|y|^2/4}.
\]

We stress that the PDE in (2.1) is of course only satisfied in \( \Omega_s \) (not in the whole \( \mathbb{R}^N \)), and that \( w(s, \cdot) = 0 \) outside \( \Omega_s \). For any \( \phi \in L^\infty(\mathbb{R}^N) \), we put

\[
\|\phi\|_{L^p} = \left( \int_{\mathbb{R}^N} |\phi(y)|^q \rho(y) \, dy \right)^{1/q}, \quad 1 \leq q < \infty,
\]

and note that

\[
\| \cdot \|_{L^q} \leq C(N) \| \cdot \|_{L^r}, \quad 1 \leq q < r < \infty.
\]

The following Proposition 2.1 is twofold:

- Assertion (i) gives a local criterion to exclude blow-up at a given point, which is essentially due to [1], [18]. We here state and prove a version adapted to our needs. We note that, although the criterion concerns possible blow-up points \( a \in \overline{\Omega} \), the corresponding estimate (2.5) remains true for any \( a \in \mathbb{R}^N \), which will be convenient in the proof of Theorem 1. As a key difference with the earlier local criterion in [23] and its proof (see also [31]), the method from [1], [18], that we use here, relies on delayed smoothing effects in weighted \( L^q \) spaces (cf. Lemma 2.2) instead of rescaled energy, hence allowing to cover possibly nonconvex domains and/or \( p > p_S \).

- Assertion (ii) rules out blow-up at space infinity in case \( \Omega \) is unbounded and \( u_0 \) belongs to \( L^m \) for some finite \( m \). When \( \Omega \) is convex the conclusion was first proved in [23] under the assumption \( u_0 \in H^1(\Omega) \). The general case was obtained in [20]. We shall give a different proof, allowing a common treatment to both assertions (i) and (ii).

**Proposition 2.1.** Let \( \Omega \) be any smooth domain of \( \mathbb{R}^N \), \( p > 1 \) and \( u_0 \in L^\infty(\Omega) \). Assume that \( u \) is a blow-up solution of (1.1) satisfying the type I estimate

\[
\|u(t)\|_\infty \leq M(T - t)^{-\beta}, \quad 0 \leq t < T,
\]

for some constant \( M > 0 \).
Let $\alpha \in \mathbb{R}^N$ and let $w_\alpha$ be the rescaled solution by similarity variables around $(T, \alpha)$. For each $q \geq p$ with $q > Np/2$, there exist $s^*, \epsilon_0, C_0 > 0$ depending only on $N, p, q, C$ such that, if

$$\|w_\alpha(\sigma)\|_{L_\rho^q} < \epsilon_0 \quad \text{for some } \sigma > s_0,$$

then

$$\|w_\alpha(\sigma + s)\|_{L_\rho^q} < C_0 e^{-\beta s} \|w_\alpha(\sigma)\|_{L_\rho^q} \quad \text{for all } s \geq s^*.$$  \hspace{1cm} (2.4)

Moreover, if $\alpha \in \Omega$ and \cite{24} is satisfied, then $\alpha$ is not a blow-up point of $u$.

(ii) Assume that $\Omega$ is unbounded and that $u_0 \in L^m(\Omega)$ for some $m \in [1, \infty)$. Then $u$ does not blow up at infinity, namely there exist $R, C > 0$ such that

$$|u(t, x)| \leq C, \quad 0 < t < T, \ x \in \Omega \cap \{x \geq R\}.$$

In view of the proof of Proposition 2.1, we introduce the semigroup $(T(s))_{s \geq 0}$ associated with $\mathcal{L}$. More precisely, for each $\phi \in L^\infty(\mathbb{R}^N)$, we set $T(s)\phi := z(s, \cdot)$, where $z$ is the unique classical solution of

$$\left\{ \begin{array}{l}
  z_s = \mathcal{L}z, \quad s > 0, \ y \in \mathbb{R}^N, \\
  z(0, y) = \phi(y), \quad y \in \mathbb{R}^N.
\end{array} \right.$$

An essential property of $(T(s))_{s \geq 0}$ is given by the following Lemma (see \cite{28} and cf. also \cite{27}).

**Lemma 2.2.**

(i) For all $1 \leq q \leq \infty$,

$$\|T(s)\phi\|_{L_q^\infty} \leq \|\phi\|_{L_q^\infty}, \quad s \geq 0, \ \phi \in L^\infty(\mathbb{R}^N).$$  \hspace{1cm} (2.6)

(ii) (Delayed regularizing effect) For all $1 \leq m < q < \infty$, there exist $c_0, s^* > 0$ such that

$$\|T(s)\phi\|_{L_q^m} \leq c_0 \|\phi\|_{L_q^m}, \quad s \geq s^*, \ \phi \in L^\infty(\mathbb{R}^N).$$  \hspace{1cm} (2.7)

**Proof of Proposition 2.1** (i) Set $w := w_\alpha$. For given $s_1 \geq s_0$, denote by $W$ the solution of

$$\left\{ \begin{array}{l}
  W_s - \mathcal{L}W = |w|^p - \beta W, \quad s > s_1, \ y \in \mathbb{R}^N, \\
  W(s_1, y) = |w(s_1, y)|, \quad y \in \mathbb{R}^N.
\end{array} \right.$$

Clearly $W$ exists globally and we claim that

$$|w| \leq W \quad \text{in } (s_1, \infty) \times \mathbb{R}^N.$$  \hspace{1cm} (2.8)

To check this, in order to avoid any difficulty related to the application of the maximum principle in a time dependent domain, we just note that, going back to the original variables by setting

$$V(t, x) := (T - t)^{-\beta} W \left( -\log(T - t), (x - a)/\sqrt{T - t} \right), \quad t_1 < t < T, \ x \in \mathbb{R}^N,$$

with $t_1 = T - e^{-s_1}$, we have

$$\left\{ \begin{array}{l}
  V_t - \Delta V = |w|^p, \quad t_1 < t < T, \ x \in \Omega, \\
  V \geq 0, \quad t_1 < t < T, \ x \in \partial \Omega, \\
  V(t_1, x) = |w(t_1, x)|, \quad x \in \Omega.
\end{array} \right.$$

Therefore, $u \leq V$ and $-u \leq V$ in $(t_1, T) \times \Omega$ by the maximum principle. Consequently, $|\bar{u}| \leq V$ in $(t_1, T) \times \mathbb{R}^N$, and claim (2.8) follows.

Next, by the variation of constants formula, we deduce that

$$e^{\beta s} |w(s_1 + s)| \leq e^{\beta s} W(s_1 + s) \leq T(s) |w(s_1)| + \int_0^s e^{\beta \tau} T(s - \tau) |w(s_1 + \tau)|^p d\tau, \quad s_1 \geq s_0, \ s > 0.$$  \hspace{1cm} (2.9)

On the other hand, since

$$\|w(s)\|_{\infty} \leq M, \quad s \geq s_0,$$  \hspace{1cm} (2.10)
due to (2.3), W satisfies
\[
\begin{cases}
W_s - \mathcal{L} W \leq C_1 W, & s > s_1, \ y \in \mathbb{R}^N, \\
W(s_1, y) = |w(s_1, y)|, & y \in \mathbb{R}^N,
\end{cases}
\]
where \(C_1 := M^{p-1}\). Therefore
\[
|w(s_1 + s)| \leq e^{C_1 s^* T(s)} |w(s_1)|, \quad s_1 \geq s_0, \ s > 0.
\]
Let \(s^*\) be given by Lemma 2.2(i) with \(m = 1\) and set \(\varepsilon := \|w(\cdot\|L^1_0,_{\varepsilon}\). It follows from (2.4), (2.11), and Lemma 2.2(i) that
\[
\|w(\cdot\|L^1_0,_{\varepsilon} \leq C_2 \varepsilon, \quad 0 < s \leq s^*, \quad \text{with } C_2 := e^{C_1 s^*}.
\]
Let now
\[
T_0 = \sup \left\{s > 0; \ e^{\beta s} \|w(\cdot + s^* + \tau)\|_{L^1_0} \leq 2C_2 \varepsilon, \ \tau \in [0, s]\right\},
\]
Note that \(T_0 > 0\) and suppose for contradiction that \(T_0 < \infty\). We have, by (2.12) and (2.13),
\[
\|w(\cdot + s^* + \tau)\|_{L^1_0} \leq 2C_2 \varepsilon e^{-\beta s}, \quad -s^* \leq s \leq T_0.
\]
For \(0 \leq \tau \leq T_0\), (2.7), (2.11) and (2.14) imply
\[
\|w(\cdot + s^* + \tau)\|_{L^1_0} \leq e^{C_1 s^*} T(s^*) w(\cdot + \tau)\|_{L^1_0} \leq \|w(\cdot + \tau)\|_{L^1_0}
\]
\[
\leq 2C_2 \varepsilon e^{C_1 s^*} \|w(\cdot + \tau)\|_{L^1_0} e^{-\beta(\tau - s^*)} = C_3 \varepsilon e^{-\beta \tau},
\]
with \(C_3 = 2C_2 \varepsilon e^{C_1 s^*}\). Using (2.2), (2.6), (2.9) with \(s_1 = \sigma + s^*, \ 2.12\) and (2.15), we deduce that, for all \(s \in [0, T_0]\),
\[
e^{\beta s} \|w(\cdot + s^* + s)\|_{L^1_0} \leq \|w(\cdot + s^*)\|_{L^1_0} + \int_0^s e^{\beta \tau} \|w(\cdot + s^* + \tau)\|_{L^1_0}^p d\tau
\]
\[
\leq C_2 \varepsilon + C(N, p)(C_3 \varepsilon)^p \int_0^s e^{-\beta(p-1) \tau} d\tau
\]
\[
\leq C_2 \varepsilon + C(N, p)p(C_3 \varepsilon)^p(\beta(p-1))^{-1} e^p.
\]
Applying this with \(s = T_0\), it follows that \(2C_2 \varepsilon \leq C_2 \varepsilon + C(N, p)\|p\|_{L^1_{\varepsilon}}(\beta(p-1))^{-1} e^p\), hence \(C_2 \leq C(N, p, q, M) e^p\). Assuming \(\varepsilon \leq \varepsilon_0\) with \(\varepsilon_0 = \varepsilon_0(N, p, q, M) > 0\) sufficiently small, we thus necessarily have \(T_0 = \infty\). It then follows from (2.15) that
\[
\|w(\cdot + s)\|_{L^1_0} \leq C_0 \varepsilon e^{-\beta s}, \quad s \geq s^*,
\]
with \(C_0 = C_0(N, p, q, M) := C_3 \varepsilon^p = 2C_0 \exp[2(M^{p-1} + \beta)s^*]\) that is, (2.5) holds. (For the proof that \(a\) is not a blow-up point, see at the end of the proof of assertion (ii).)

(ii) We may assume without loss of generality that \(m > 1\). Fix \(p \geq p\) with \(q > Np/2\). Pick \(A > 0\) large enough so that \(T^{\beta-N/2} \|u_0\|_{L^\infty} \int_{|z| > A} e^{-|z|^2/4T} dz \leq \varepsilon_0/2\), where \(\varepsilon_0 = \varepsilon_0(N, p, q, M)\) is given by assertion (i). Since \(u_0 \in L^m(\Omega)\) with \(m > 1\), choosing \(R_0 > 2A\) sufficiently large, for any \(a \in \Omega\) such that \(|a| > R_0\), we get
\[
\|u_0(s_0)\|_{L^1_0} = T^{\beta-N/2} \int_{|z| \leq A} |\tilde{u}_0(a + z)| e^{-|z|^2/4T} dz
\]
\[
\leq T^{\beta-N/2} \int_{|z| \leq A} |\tilde{u}_0(a + z)| e^{-|z|^2/4T} dz + T^{\beta-N/2} \|u_0\|_{L^\infty} \int_{|z| > A} e^{-|z|^2/4T} dz
\]
\[
\leq T^{\beta-N/2} \left( \int_{|z| \leq A} |\tilde{u}_0(a + z)|^m dz \right)^{1/m} \left( \int_{|z| > A} e^{-m|z|^2/4(m-1)T} dz \right)^{(m-1)/m} + \frac{\varepsilon_0}{2}
\]
\[
\leq T^{\beta-N/2} \left( \int_{|y| \geq R_0/2} |\tilde{u}_0(y)|^m dy \right)^{1/m} \left( \int_{\mathbb{R}^N} e^{-m|z|^2/4(m-1)T} dz \right)^{(m-1)/m} + \frac{\varepsilon_0}{2} < \varepsilon_0.
It then follows from (2.4) that
\[
\int_{\mathbb{R}^N} |w_a(s_0 + s, y)|^q \rho(y) \, dy \leq C_0^q e_0^q e^{-\beta q s}, \quad s \geq s^*. \tag{2.18}
\]

Restated in terms of \( u \), (2.18) means that
\[
\int_{\mathbb{R}^N} |\tilde{u}(t, a + y\sqrt{T-t})|^q e^{-|y|^2/4} \, dy \leq C := (C_0 e_0 T^{-\beta})^q, \quad \tau < t < T, \quad |a| \geq R_0,
\]
with \( \tau := (1 - e^{-s^*})T \). Set \( \tau_1 := \max(\tau, T - 1) \). For each \( a \in \Omega \) with \( |a| \geq R := R_0 + 2 \), integrating with respect to \( \xi \in B_2(a) \) and using Fubini’s theorem and the change of variable \( z = \xi + y\sqrt{T-t} \), it follows that, for all \( t \in [\tau_1, T) \),
\[
Ce^{1/4}|B_2(0)| \geq \int_{|y|<1} \int_{|\xi-a|<2} |\tilde{u}(t, \xi + y\sqrt{T-t})|^q \, d\xi \, dy \\
\geq \int_{|y|<1} \int_{|z-y\sqrt{T-t}|<2} |\tilde{u}(t, z)|^q \, dz \, dy \geq |B_1(0)| \int_{|z-a|<1} |\tilde{u}(t, z)|^q \, dz.
\]

Since \( q > Np/2 \), we deduce from standard parabolic regularity properties that
\[
|u(t, x)| \leq C_1, \quad \tau_2 < t < T, \quad x \in \Omega \cap \{|x| \geq R\},
\]
for some constant \( C_1 > 0 \) and \( \tau_2 = (\tau_1 + T)/2 \), which implies assertion (ii).

Finally, to show that \( a \in \Omega \) is not a blow-up point under assumption (2.4), we observe that, by continuity, we have \( \|w_b(\sigma)\|_{L_\sigma^q} < \varepsilon_0 \) for all \( b \in \mathbb{R}^N \) such that \( |b-a| \leq r \), with \( r > 0 \) sufficiently small. The first part of assertion (i) then guarantees that for \( q > Np/2 \)
\[
\|w_b(\sigma + s)\|_{L_\sigma^q} < C_0 \varepsilon_0 e^{-\beta s}, \quad s \geq s^*,
\]
for all such \( b \). The conclusion then follows from a similar argument as in the preceding paragraph. \( \square \)

2.2. Backward uniqueness. One of the key ingredients in our proof of Theorem [III] is a backward uniqueness theorem in a half-space, which was given by [17] (also see [14], [15], [16]) to treat a problem on regularity of solutions to the Navier-Stokes equations. Since we are concerned with blow-up problem, the initial data in backward uniqueness theorem has singularities. However we will deal with an auxiliary problem whose singularities remain confined in a ball, so that this backward uniqueness theorem applies conveniently to our purpose (working in half-spaces which do not intersect this ball). Although the backward uniqueness theorem was given in more general form for example in [17] Theorem 5.1], we describe it in the following form as needed here.

**Proposition 2.3.** Let \( \mathbb{R}^N_+ = \{ y = (y_1, y_2, \cdots, y_N) \in \mathbb{R}^N : y_1 > 0 \} \) and \( S > 0 \). Let \( w \in C^{1,2}([0, S] \times \mathbb{R}^N_+; \mathbb{R}) \) satisfy
\[
|w(s, y)| \leq e^{C_1|y|^2} \quad \text{in } (0, S) \times \mathbb{R}^N_+
\]
for some \( C_1 > 0 \). If \( w \) fulfills
\[
|w_s + \Delta w| \leq C_2 |w| \quad \text{in } (0, S) \times \mathbb{R}^N_+
\]
for some \( C_2 > 0 \) and
\[
w(0, y) = 0 \quad \text{in } \mathbb{R}^N_+, \]
then \( w \equiv 0 \) in \([0, S] \times \mathbb{R}^N_+\).
As a first consequence of Proposition 2.1(i), we have the following lemma, which shows that the $L^{q^*}$ norm of any type I blow-up solution has to concentrate in space-time parabolas near every blow-up point.

**Lemma 3.1.** Let $\Omega$ be any smooth domain of $\mathbb{R}^N$, $p > 1$ and $u_0 \in L^\infty(\Omega)$. Assume that $u$ is a blow-up solution of (1.1) satisfying the type I estimate (2.3) with $M > 0$. Then there exist $\eta_1, k_1 > 0$, depending only on $M, p, N$, such that for any blow-up point $a \in \Omega$, we have

$$\int_{\Omega \cap \{|x-a| \leq k_1 \sqrt{T-t}\}} |u(t,x)|^{q^*} \, dx \geq \eta_1, \quad 0 \leq t < T.$$

**Proof of Lemma 3.1.** Let $a \in \Omega$. Recalling the notation in Section 2.1, we set $w(s,y) = w_a(s,y) = (T-t)^{3/2} \tilde{u}(t,a+y\sqrt{T-t})$, $s_0 < s < \infty$, $y \in \mathbb{R}^N$, where $\tilde{u}$ is the extension of $u$ by 0 to the whole of $\mathbb{R}^N$ and $s_0 = -\log T$. By Proposition 2.1(i), there exists $\varepsilon_0 = \varepsilon_0(M, p, N) > 0$ such that:

- if $\|w(s)\|_{L^p_\rho} \leq \varepsilon_0$ for some $s \in [-\log T, \infty)$, then $a$ is not a blow-up point.

Now choose $k_1 > 0$ such that $\int_{|y|>k_1} e^{-|y|^{2/4}} \, dy \leq \varepsilon_0/2M$. If $a$ is a blow-up point then, using (2.10), we have for all $s \geq s_0$,

$$\varepsilon_0 \leq \int_{\mathbb{R}^N} |w(s,y)| e^{-|y|^{2/4}} \, dy \leq \int_{|y| \leq k_1} |w(s,y)| e^{-|y|^{2/4}} \, dy + M \int_{|y|>k_1} e^{-|y|^{2/4}} \, dy \leq \int_{|y| \leq k_1} |w(s,y)| \, dy + \varepsilon_0/2,$$

hence

$$\int_{|y| \leq k_1} |w(s,y)| \, dy \geq \varepsilon_0/2. \quad (3.1)$$

Moreover, since $q^* = N(p - 1)/2$, we have

$$\int_{|x-a| \leq k_1 \sqrt{T-t}} |\tilde{u}(t,x)|^{q^*} \, dx = (T-t)^{N/2} \int_{|y| \leq k_1} |\tilde{u}(t,a+y\sqrt{T-t})|^{q^*} \, dy = \int_{|y| \leq k_1} |w(s,y)|^{q^*} \, dy.$$

If $p \geq 1 + 2/N$, hence $q^* \geq 1$, the conclusion then directly follows from (3.1) and Hölder’s inequality. If $p \in (1, 1 + 2/N)$, hence $q^* \in (0, 1)$, the conclusion follows from (2.10) and (3.1), by writing

$$\int_{|y| \leq k_1} |w(s,y)|^{q^*} \, dy \geq M q^{q^*-1} \int_{|y| \leq k_1} |w(s,y)| \, dy \geq M q^{q^*-1} \varepsilon_0/2.$$

**Proof of Theorem 1**. The proof is divided into five steps.

**Step 1. Preliminaries.** We shall first prove the local assertion (ii). Let $a$ be a blow-up point of $u$. We may take $T = 1$ and $a = 0$ without loss of generality. As before, $u$ is extended by 0 to the whole of $\mathbb{R}^N$ (we shall just write $u$ instead of $\tilde{u}$ without risk of confusion). Also we shall denote $B_r = B_r(0)$.

Assume for contradiction that

$$\liminf_{t \to 1} \int_{\Omega \cap B_{r_0}} |u(t,x)|^{q^*} \, dx < \infty$$

for some $r_0 > 0$. Thus there exist $M_1 > 0$ and a sequence $t_n \to 1$ such that

$$\|u(t_n)\|_{L^{q^*}(B_{r_0})} \leq M_1. \quad (3.2)$$
Moreover, as a consequence of Lemma 3.1, $u$ has finitely many blow-up points in the ball $B_{r_0}$. Therefore, there exists $r \in (0, r_0)$ such that 0 is the only blow-up point in $B_r$ and, by standard parabolic estimates, there exists $U \in L^{q*}(B_r) \cap L_\text{loc}^\infty(\overline{B_r} \setminus \{0\})$ such that
\[
\lim_{t \to 1} u(t, x) = U(x), \quad \text{for } 0 < |x| \leq r,
\]
where the convergence is uniform on compact subsets of $\overline{B_r} \setminus \{0\}$.

**Step 2.** Rescaling and lower $L^{q*}$ estimate of the rescaled functions in some ball centered at 0. We rescale $u$ as
\[
v_n(s, y) = (1 - t_n)^{\beta} u(1 + (1 - t_n)s, y\sqrt{1 - t_n}), \quad s \in (-2, 0), \; y \in \mathbb{R}^N.
\]
The function $v_n$ satisfies
\[
\partial_t v_n - \Delta v_n = |v_n|^{p-1} v_n, \quad s \in (-2, 0), \; y \in \Omega_n := (1 - t_n)^{-1/2}(\Omega \cap B_r).
\]
By the type I estimate (2.3), we have
\[
|v_n(s, y)| \leq M|s|^{-\beta}, \quad s \in (-2, 0), \; y \in \mathbb{R}^N.
\]
As $n \to \infty$, the domain $\Omega_n$ converges to $D := \mathbb{R}^N$ if $0 \in \Omega$ and to $D := \mathbb{R}^N_+$ if $0 \in \partial\Omega$ after suitable rotation if necessary. By parabolic estimates, it follows that, up to a subsequence,
\[
v_n \text{ converges locally uniformly in } (-2, 0) \times \overline{D} \text{ to } v,
\]
where $v$ is a classical solution of
\[
v_t - \Delta v = |v|^{p-1} v, \quad (s, y) \in (-2, 0) \times D,
\]
with $v = 0$ on $(-2, 0) \times \partial D$ in case $a = 0 \in \partial\Omega$. In this second case, we extend $v$ by 0 outside $\overline{D}$.

Also, by Lemma 3.1 since 0 is a blow-up point of $u$, there exist $k_1, \eta_1 > 0$ such that
\[
\int_{|y| \leq k_1} |v_n(s, y)|^{q^*} \, dy = (1 - t_n)^{N/2} \int_{|y| \leq k_1} |u(1 + (1 - t_n)s, y\sqrt{1 - t_n})|^{q^*} \, dy
\]
\[
= \int_{|x| \leq k_1 \sqrt{1 - t_n}^{-1}} |u(1 + (1 - t_n)s, x)|^{q^*} \, dx \geq \eta_1, \quad -2 < s < 0,
\]
where we used $q^* = N(p - 1)/2$. In view of (3.7), we deduce that
\[
\int_{|y| \leq k_1} |v(s, y)|^{q^*} \, dy \geq \eta_1, \quad -2 < s < 0.
\]

**Step 3.** Uniform a priori estimates of the rescaled functions $v_n$ for large $|y|$. We shall show that there exist $s_1 \in (0, 1)$ and $C_1, K_1 > 0$ such that, for all $K_2 > K_1$ and all $n \geq n_1(K_2),$
\[
|v_n(s, y)| \leq C_1, \quad -s_1 \leq s < 0, \; K_1 \leq |y| \leq K_2.
\]
This step is more technical and requires some interplay between the different versions of the solution: $v_n$, $v$, $u$, and $w_b$, with a certain range of rescaling points $b$. In what follows, $C$ denotes a generic constant varying from line to line.

By (3.2), (3.4) and recalling $q^* = N(p - 1)/2$, we have
\[
\int_{|y| \leq r \sqrt{1 - t_n}} |v_n(-1, y)|^{q^*} \, dy = (1 - t_n)^{N/2} \int_{|y| \leq r \sqrt{1 - t_n}} |u(t_n, y\sqrt{1 - t_n})|^{q^*} \, dy
\]
\[
= \int_{B_r} |u(t_n, x)|^{q^*} \, dx \leq M_1^{q^*},
\]
hence $v(-1, \cdot) \in L^{q^*}(D)$ by Fatou’s Lemma. By Proposition 2.1(ii), we deduce the existence of $R > 0$ such that
\[
\sup_{(-1,0) \times (\overline{D} \setminus \{|y| \geq R\})} |v| =: M_0 < \infty.
\]
Note that, in view of (3.8) and (3.11), by parabolic regularity, \( v \) extends to a function
\[
v \in C^{1,2}((-1,0] \times (\mathcal{D} \cap \{|y| > R\})).
\] (3.12)

On the other hand, by (3.7) and (3.11), for any \( \varepsilon \in (0,1) \) and \( K > R \), there exists \( n_0(\varepsilon, K) \) such that for all \( n \geq n_0(\varepsilon, K) \), we have
\[
|v_n(s, y)| \leq M_0 + 1, \quad -1 \leq s \leq -\varepsilon, \quad R \leq |y| \leq 3K,
\]
hence in particular
\[
(\varepsilon(1-t_n))^\beta |u(1-\varepsilon(1-t_n), y\sqrt{1-t_n})| = e^\beta |v_n(s, y)| \leq (M_0 + 1)e^\beta, \quad R \leq |y| \leq 3K.
\]
For any \( \xi \in \mathbb{R}^N \), this can be rewritten as
\[
(\varepsilon(1-t_n))^\beta |u(1-\varepsilon(1-t_n), \xi\sqrt{1-t_n} + (y - \xi)\varepsilon^{-1/2}\sqrt{\varepsilon(1-t_n)})| \leq (M_0 + 1)e^\beta, \quad R \leq |y| \leq 3K.
\]
In terms of solutions rescaled by similarity variables (cf. section 2), this amounts to
\[
\left| w_{\xi\sqrt{1-t_n}}(\log(\varepsilon(1-t_n)), (y - \xi)\varepsilon^{-1/2}) \right| \leq (M_0 + 1)e^\beta, \quad R \leq |y| \leq 3K.
\]
In particular, for any \( K \geq 2R \) and any \( \xi \) such that \( K \leq |\xi| \leq 2K \), we have, for all \( n \geq n_0(\varepsilon, K) \),
\[
\left| w_{\xi\sqrt{1-t_n}}(\log(\varepsilon(1-t_n)), z) \right| \leq (M_0 + 1)e^\beta, \quad |z| \leq K\varepsilon^{-1/2}/2.
\] (3.13)

Next take \( q \geq p \) with \( q > Np/2 \) and let \( \varepsilon_0 \) be as in Proposition 2.1(i). Recall that
\[
\|w_b(\tau)\|_\infty \leq M \quad \text{for any } \tau \in [s_0, \infty) \text{ and } b \in \mathbb{R}^N,
\]
owing to (2.3). From now on, we fix \( \varepsilon = \varepsilon_1 \in (0,1) \) and pick \( R_1 > 2 \), in such a way that
\[
(M_0 + 1)\varepsilon_1^\beta \int_{\mathbb{R}^N} e^{-|z|^2/4} \, dz \leq \frac{\varepsilon_0}{2} \quad \text{and} \quad M \int_{|z| > R_1} e^{-|z|^2/4} \, dz \leq \frac{\varepsilon_0}{2}.
\] (3.14)

Set \( K_1 := 2\max(R, \varepsilon_1^{1/2}R_1) > 4 \) and take any \( K \geq K_1 \). It follows from (3.13)-(3.15) that, for all \( n \geq n_1(K) := n_0(\varepsilon_1, K) \),
\[
\|w_{\xi\sqrt{1-t_n}}(\log(\varepsilon_1(1-t_n)), \cdot)\|_{L^p_{|\tau)}} \leq M \int_{|\tau| > R_1} e^{-|\tau|^2/4} \, d\tau \quad \text{and} \quad (M_0 + 1)\varepsilon_1^\beta \int_{|\tau| \leq R_1} e^{-|\tau|^2/4} \, d\tau \leq \varepsilon_0, \quad \tau \geq \tau_n + s^*, \quad K \leq |\xi| \leq 2K.
\]

By Proposition 2.1(i) applied with \( \sigma = \tau_n := -\log(\varepsilon_1(1-t_n)) \), we deduce that
\[
\|w_{\xi\sqrt{1-t_n}}(\tau, \cdot)\|_{L^p_{\tau_n}} \leq C\varepsilon_0 e^{-\beta(\tau - \tau_n)}, \quad \tau \geq \tau_n + s^*, \quad K \leq |\xi| \leq 2K, \quad n \geq n_1(K).
\] (3.16)

Now set \( t'_n := 1 - \varepsilon_1 e^{-s^*}(1-t_n) \). Restating (3.10) in terms of \( u \), using
\[
u(t, b + z\sqrt{1-t}) = e^{\beta\tau} w_b(\tau, z), \quad \text{with } \tau = -\log(1-t),
\]
we thus have, for all \( n \geq n_1(K) \),
\[
\int_{\mathbb{R}^N} \left| u(t, \xi\sqrt{1-t_n} + y\sqrt{1-t_n}) \right|^q \rho(y) \, dy \leq (C\varepsilon_0)^q e^{\beta q \tau_n} = C(1-t_n)^{-\beta q}, \quad t \in [t'_n, 1), \quad K \leq |\xi| \leq 2K.
\]

Going back to \( v_n \), noting that
\[
v_n(s, \xi + y\sqrt{|s|}) = (1-t_n)^\beta u(t, \xi\sqrt{1-t_n} + y\sqrt{1-t_n}), \quad \text{with } t = 1 + (1-t_n)s,
\]
and that
\[
1 + (1-t_n)s \geq t'_n = 1 - e^{-s^*} \varepsilon_1(1-t_n) \iff s \geq -e^{-s^*} \varepsilon_1,
\]
we obtain, for each \( K \leq |\xi| \leq 2K \),
\[
\int_{|y| \leq 1} |v_n(s, \xi + y\sqrt{|s|})|^q \, dy \leq e^{1/4} \int_{\mathbb{R}^N} |v_n(s + y\sqrt{|s|})|^q \rho(y) \, dy \leq C, \quad -\varepsilon_1 e^{-s^*} \leq s < 0.
\]
For any $K \geq K_1$ and each $\xi_0$ with $K + 2 \leq |\xi_0| \leq 2K - 2$ (recalling $K_1 > 4$), we integrate with respect to $\xi \in B_2(\xi_0)$ and use the change of variable $z = \xi + y\sqrt{|s|}$. We thus get, for all $n \geq n_1(K)$,

$$C \geq \int_{|\xi-\xi_0| \leq 2} \int_{|y| \leq 1} |v_n(s, \xi + y\sqrt{|s|})|^q dy d\xi$$

$$= \int_{|y| \leq 1} \int_{|z-y\sqrt{|s|}-\xi_0| \leq 2} |v_n(s, z)|^q dz dy \geq \int_{|y| \leq 1} \int_{|z-\xi_0| \leq 2} |v_n(s, z)|^q dz dy,$$

hence

$$\int_{|z-\xi_0| \leq 2} |v_n(s, z)|^q dz \leq C, \quad -\varepsilon_1 e^{-\varepsilon s} \leq s < 0.$$

By parabolic regularity applied to equation (3.5), we deduce that, for all $K \geq K_1$ and $n \geq n_1(K)$,

$$|v_n(s, y)| \leq C, \quad \text{for all } s \in [-s_1, 0) \text{ and } K + 2 \leq |y| \leq 2K - 2,$$

with $s_1 := (\varepsilon_1/2)e^{-\varepsilon s}$, which implies the claim (3.10).

**Step 4.** Backward uniqueness argument and completion of proof of assertion (ii). From (3.6) and (3.10), applying parabolic regularity again, we deduce that, for any $K_2 > K_1$, the limit

$$v_n(0, y) = \lim_{s \to 0} v_n(s, y) \quad \text{exists for } K_1 \leq |y| \leq K_2 \text{ and } n \geq n_1(K_2)$$

and that

$$v_n(0, y) \to v(0, y), \quad \text{locally uniformly for } |y| \geq K_1 \quad (3.17)$$

(recalling 3.12) and the fact that $v$ is extended by 0 outside $D = \mathbb{R}^N_+$ if $a = 0 \in \partial\Omega$. But, by (3.3) and 3.4, taking $n_1(K_2)$ larger if necessary, we also have

$$v_n(0, y) = (1 - t_n)^3 u(1, y\sqrt{1 - t_n}) = (1 - t_n)^3 U(y\sqrt{1 - t_n}), \quad K_1 \leq |y| \leq K_2, \quad n \geq n_1(K_2).$$

Since $U \in L^{\alpha^*}(B_r)$, for each $y_0$ with $|y_0| \geq K_1 + 1$, we deduce that, for all $n \geq n_1(|y_0| + 1)$,

$$\int_{B_1(y_0)} |v_n(0, y)|^q dy = \int_{B_1(y_0)} (1 - t_n)^{N/2} |U(y\sqrt{1 - t_n})|^q dy$$

$$\leq \int_{|z| \leq ((|y_0| + 1)\sqrt{1 - t_n})^2} |U(z)|^q dz \to 0, \quad n \to \infty.$$

Writing

$$\|v(0, \cdot)\|_{L^{\alpha^*}(B_1(y_0))} \leq \|v(0, \cdot) - v_n(0, \cdot)\|_{L^{\alpha^*}(B_1(y_0))} + \|v_n(0, \cdot)\|_{L^{\alpha^*}(B_1(y_0))}$$

and since the first term also converges to 0 by (3.17), we infer that

$$v(0, y) = 0, \quad |y| \geq K_1.$$

By backward uniqueness (cf. Proposition 2.3), in view of (3.3), (3.11), (3.12), we deduce that

$$v(s, y) = 0, \quad -1 < s < 0, \quad |y| \geq K_1.$$

Finally, by unique continuation (e.g. Theorem 4.1 of [17]), it follows that actually $v \equiv 0$ on $(-1, 0) \times D$. We refer also to [13] and references therein for unique continuation theorem. But this contradicts the lower bound in (3.3). Assertion (ii) follows.

**Step 5.** Proof of assertion (i). We may assume that $u(t_0) \in L^{\alpha^*}(\Omega)$ for some $t_0 \in (0, T)$ since otherwise the conclusion is immediate. Since $u(t_0) \in L^{\alpha^*}(\Omega) \cap L^\infty(\Omega)$, we have $u(t_0) \in L^m(\Omega)$ for some $m \in [1, \infty)$. It follows from Proposition 2.1(ii) that

$$|u(t, x)| \leq C, \quad t_0 < t < T, \quad x \in \Omega \cap \{|x| \geq R\}$$

for some $C, R > 0$. Therefore, there exists at least a blow-up point $a \in \Omega$. The conclusion then follows from assertion (ii).
4. Appendix: Type II blow-up solutions with bounded critical $L^q$ norm

Since the papers [46] and [12] did not explicitly address the $L^{q^*}$ behavior of the solutions that they constructed, we shall here check that they do satisfy property (1.3).

4.1. Case $p = p_S$ and $N = 5$ (cf. [12]). Set

$$U(x) = c(N)(1 + |x|^2)^{-(N-2)/2},$$

which is a regular positive solution of $\Delta u + u^p = 0$ in $\mathbb{R}^N$ (the standard Aubin-Talenti bubble), and denote its rescaling by $u_\lambda(x) = a^{-2/(p-1)}U(a^{-1}x)$, $a > 0$. The type II blow-up solution in [12] (in the case of a single blow-up point) is of the form

$$u(t, x) = U_{\lambda(t)}(x) + U^*(x) + \theta(t, x), \quad t \in (0, T), \ x \in \Omega,$$

where $\Omega$ is some smooth bounded domain of $\mathbb{R}^N$ with $0 \in \Omega$. Here $\lambda(t) \to 0$ as $t \to T$, $U^* \in L^\infty(\Omega)$ and $\theta(t, \cdot) \to 0$ in $L^\infty(\Omega)$. Since $q^* = N(p-1)/2 = 2N/(N-2)$ we have

$$\int_\Omega |U_{\lambda(t)}(x)|^{q^*} \, dx \leq \lambda^{-N}(t) \int_{\mathbb{R}^N} |U(x/\lambda(t))|^{q^*} \, dx = C \int_{\mathbb{R}^N} (1 + |y|^2)^{-N} \, dy = C,$$

where $C$ denotes generic constants. It follows that $u$ satisfies (1.3).

4.2. Case $p = p_S$ and $N = 4$ (cf. [46]). The solution is also of the form (4.1), this time with $\Omega = \mathbb{R}^N$, $\nabla U^* \in L^2(\mathbb{R}^N)$, $\nabla \theta(t, \cdot) \to 0$ in $L^2(\mathbb{R}^N)$, and $u_0 \in L^1(\mathbb{R}^N)$ (cf. [46] Theorem 1.1]). We note that $u(t, \cdot) \in H^1(\mathbb{R}^N)$ for each $t \in (0, T)$, due to the local well-posedness of (1.1) in that space for $p = p_S$ (see e.g. [44], Example 51.28)). It then follows from the Sobolev inequality that

$$\|u(t, \cdot)\|_{L^{q^*}(\mathbb{R}^N)} \leq \|U_{\lambda(t)}(\cdot)\|_{L^{q^*}(\mathbb{R}^N)} + \|u(t, \cdot) - U_{\lambda(t)}(\cdot)\|_{L^{q^*}(\mathbb{R}^N)} \leq C + C\|\nabla u(t, \cdot) - \nabla U_{\lambda(t)}(\cdot)\|_{L^2(\mathbb{R}^N)} \leq C\|\nabla U^* + \nabla \theta(t, \cdot)\|_{L^2(\mathbb{R}^N)} \leq C,$$

and we conclude that $u$ also satisfies (1.3).

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