Finite index operators on surfaces

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Abstract

We consider differential operators \( L \) acting on functions on a Riemannian surface, \( \Sigma \), of the form

\[
L = \Delta + V - aK,
\]

where \( \Delta \) is the Laplacian of \( \Sigma \), \( K \) is the Gaussian curvature, \( a \) is a positive constant and \( V \in C^\infty(\Sigma) \). Such operators \( L \) arise as the stability operator of \( \Sigma \) immersed in a Riemannian three-manifold with constant mean curvature (for particular choices of \( V \) and \( a \)).

We assume \( L \) is nonpositive acting on functions compactly supported on \( \Sigma \). If the potential, \( V := c + P \) with \( c \) a nonnegative constant, verifies either an integrability condition, i.e. \( P \in L^1(\Sigma) \) and \( P \) is non positive, or a decay condition with respect to a point \( p_0 \in \Sigma \), i.e. \( |P(q)| \leq M/d(p_0, q) \) (where \( d \) is the distance function in \( \Sigma \)), we control the topology and conformal type of \( \Sigma \). Moreover, we establish a Distance Lemma.

We apply such results to complete oriented stable \( H \)–surfaces immersed in a Killing submersion. In particular, for stable \( H \)–surfaces in a simply-connected homogeneous space with 4–dimensional isometry group, we obtain:

- There are no complete stable \( H \)–surfaces \( \Sigma \subset H^2 \times \mathbb{R}, H > 1/2 \), so that either \( K^+_e := \max \{0, K_e\} \in L^1(\Sigma) \) or there exist a point \( p_0 \in \Sigma \) and a constant \( M \) so that \( |K_e(q)| \leq M/d(p_0, q) \), here \( K_e \) denotes the extrinsic curvature of \( \Sigma \).

- Let \( \Sigma \subset E(\kappa, \tau), \tau \neq 0 \), be an oriented complete stable \( H \)–surface so that either \( \nu^2 \in L^1(\Sigma) \) and \( 4H^2 + \kappa \geq 0 \), or there exist a point \( p_0 \in \Sigma \) and a constant \( M \) so that \( |\nu(q)|^2 \leq M/d(p_0, q) \) and \( 4H^2 + \kappa > 0 \). Then:
  - In \( S^3_{\text{Berger}} \), there are no such a stable \( H \)–surface.
  - In \( \text{Nil}_3 \), \( H = 0 \) and \( \Sigma \) is either a vertical plane (i.e. a vertical cylinder over a straight line in \( \mathbb{R}^2 \)) or an entire vertical graph.
  - In \( \text{PSL}(2,\mathbb{R}) \), \( H = \sqrt{-\kappa}/2 \) and \( \Sigma \) is either a vertical horocylinder (i.e. a vertical cylinder over a horocycle in \( \mathbb{H}^2(\kappa) \)) or an entire graph.

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## 1 Introduction

Stable oriented domains $\Sigma$ on a constant mean curvature surface in a Riemannian three-manifold $\mathcal{M}^3$, are characterized by the *stability inequality* for normal variations $\psi N$ (see [1])

$$
\int_{\Sigma} \psi^2 |A|^2 + \int_{\Sigma} \psi^2 \text{Ric}_{\mathcal{M}^3}(N, N) \leq \int_{\Sigma} |\nabla \psi|^2,
$$

for all compactly supported functions $\psi \in H^{1,2}(\Sigma)$. Here $|A|^2$ denotes the square of the length of the second fundamental form of $\Sigma$, $\text{Ric}_{\mathcal{M}^3}(N, N)$ is the Ricci curvature of $\mathcal{M}^3$ in the direction of the normal $N$ to $\Sigma$ and $\nabla$ is the gradient w.r.t. the induced metric.

One writes the stability inequality in the form

$$
\frac{d^2}{dt^2} \left| \left( \text{Area}(\Sigma(t)) - 2H \text{Vol}(\Sigma(t)) \right) \right| = -\int_{\Sigma} \psi L \psi \geq 0,
$$

where $L$ is the linearized operator of the mean curvature

$$
L = \Delta + |A|^2 + \text{Ric}_{\mathcal{M}^3}.
$$

In terms of $L$, stability means that $-L$ is nonnegative, i.e., all its eigenvalues are non-negative. $\Sigma$ is said to have finite index if $-L$ has only finitely many negative eigenvalues.

The study of stable surfaces by considering Schrödinger-type differential operators on a surface $\Sigma$ with a metric $g$ of the form

$$
L := \Delta + V - aK,
$$

where $\Delta$ and $K$ are the Laplacian and Gaussian curvature associated to $g$ respectively, $a$ is a positive constant and $V$ is a nonnegative function, has been received an important number of contributions (see [9, 13, 12, 14, 15, 17, 23]), and even now it is a topic of interest. T. Colding and W. Minicozzi [5] introduced a new technique to study this type of operator based on the first variation formula for length and the Gauss-Bonnet formula. Using this technique they obtained an inequality which, when $a > 1/2$, gives quadratic area growth of the geodesic disks on the surface and the integrability of the potential $V$ at the same time (note that the stability operator can be realized with the right choice of $V$ and $a$). P. Castillon [2] used the ideas of Colding-Minicozzi to improve their result to $a > 1/4$. This technique allows them to control the topology and the conformal type of the surface. In the above works, the potential, $V$, is assumed to be nonnegative. Moreover, an important result for this kind of operators is the *Distance Lemma*,

A technique developed by Fischer-Colbrie [12], which bounds the intrinsic distance of any point in the surface to the boundary. This result have been done for $V \geq c$, $c$ a positive constant, and $a > 1/4$ (see [21] for a survey). In [10], the authors extended the Distance Lemma for nonnegative Schrödinger-type differential operators satisfying $V \geq c$, $c$ a positive constant, $0 < a \leq 1/4$, and assuming some control on the area growth of the geodesic disks by different methods. Also, they were able to control the topology of the surface. As we have mentioned, all these results depend on conditions on the potential $V$ and the constant $a$. Recently, Manzano-Pérez-Rodríguez [20] have imposed no condition on the potential, $Q := V - aK \in C^\infty(\Sigma)$, but $\Sigma$ is a complete parabolic\(^2\) surface with no boundary, and they obtained that, if the there exists a nonidentically zero bounded solution of $Lf = 0$, $-L$ nonnegative on $\Sigma$, then $f$ vanishes nowhere and the linear subspace of such functions is one dimensional.

In this paper, we drop the condition $V \geq 0$ for either the integrability of the potential or some decay at infinity. We will make them explicit. Those conditions allow us to obtain parabolicity of the surface or even a Distance Lemma.

The above achievements for Schrödinger-type operators have been used for proving results for stable $H$–surface in three-manifolds (see [10, 18, 21, 20, 24, 25] and references therein). The study of stable $H$–surface in a simply-connected homogeneous three-manifold with a four dimensional isometry group is a topic of increasing interest. These homogeneous spaces are denoted by $\mathbb{E}(\kappa, \tau)$, where $\kappa$ and $\tau$ are constant and $\kappa - 4\tau^2 \neq 0$. They can be classified as $\mathbb{M}^2(\kappa) \times \mathbb{R}$ if $\tau = 0$, with $\mathbb{M}^2(\kappa) = S^2(\kappa)$ if $\kappa > 0$ ($S^2(\kappa)$ the sphere of curvature $\kappa$), and $\mathbb{M}^2(\kappa) = H^2(\kappa)$ if $\kappa < 0$ ($H^2(\kappa)$ the hyperbolic plane of curvature $\kappa$). If $\tau$ is not equal to zero, $\mathbb{E}(\kappa, \tau)$ is a Berger sphere if $\kappa > 0$, a Heisenberg space if $\kappa = 0$ (of bundle curvature $\tau$), and the universal cover of $\text{PSL}(2, \mathbb{R})$ if $\kappa < 0$.

We apply our results to stable $H$–surfaces immersed in a Riemannian three-manifold which fiber over a Riemannian surface and whose fibers are the trajectories of a unit Killing vector field (see [11, 19, 26] and references therein). In particular, they include the simply-connected homogeneous spaces $\mathbb{E}(\kappa, \tau)$.

The paper is organized as follows. Section 2 is devoted to establish the notation and basic concepts. In Section 3 we study nonnegative differential operators with integrable potential, this means that $L_{a,c} := \Delta + V - aK$ satisfies that $V := c + P$, where $c$ is a nonnegative constant and $P$ is a nonpositive and integrable function on $\Sigma$, i.e., $P \in L^1(\Sigma)$ (see Definition 3.1).

When $a > 1/4$, for this kind of operators we get:

**Theorem 3.1.** Let $\Sigma$ be a Riemannian surface possibly with boundary. Suppose that $L_{a,c} := \Delta + V - aK$ is nonpositive acting on $f \in C^\infty_0(\Sigma)$ and has integrable potential with $c \geq 0$ and $a > 1/4$. Then, $\Sigma$ has Quadratic Area Bound (the area bound depending only on $a$, $c$ and $\|P\|_1$).

Moreover, if we assume $\Sigma$ is complete (without boundary), $\Sigma$ is conformally equivalent to a compact Riemann surface with a finite number of points removed.

\(^2\)A Riemannian manifold $\Sigma$ is parabolic if every positive subharmonic function on $\Sigma$ must be constant.
And, in the case \( c > 0 \), we can go further. First, we shall introduce a concept to understand correctly the next theorem. Let \( \Sigma \) be a Riemannian surface with boundary \( \partial \Sigma \), we say that the area of the geodesic disks goes to infinity as its radius goes to infinity if for any point \( p \in \Sigma \) and any \( s > 0 \) so that \( D(p, s) \cap \partial \Sigma = \emptyset \), where \( D(p, s) \) is the geodesic disk in \( \Sigma \) centered at \( p \) and radius \( s \), the function

\[
a(p, s) := \text{Area}(D(p, s)),
\]

goesto infinity if \( s \) goes to infinity. Now, we are ready for establishing:

**Theorem 3.2.** Let \( \Sigma \) be a Riemannian surface possibly with boundary. Suppose that \( L_{a, c} = \Delta + V - aK \) is nonpositive, has integrable potential with \( c > 0 \) and \( a > 1/4 \). Then, if the area of the geodesic disks goes to infinity as its radius goes to infinity, there exists a positive constant \( C \) such that

\[
\text{dist}_\Sigma(p, \partial \Sigma) \leq C, \quad \forall p \in \Sigma.
\]

In particular, if \( \Sigma \) is complete with \( \partial \Sigma = \emptyset \), then it must be either compact or parabolic with finite area. Moreover, when \( \Sigma \) is compact, it holds

\[
c A(\Sigma) - \|P\|_1 \leq 2a\pi \chi(\Sigma),
\]

where \( A(\Sigma) \) and \( \chi(\Sigma) \) denote the area and Euler characteristic of \( \Sigma \) respectively.

When \( 0 < a \leq 1/4 \), we obtain:

**Theorem 3.3.** Let \( \Sigma \) be a Riemannian surface with \( k - AAB \) (see Definition 2?) and possibly with boundary. Suppose that \( L_{a, c} = \Delta + V - aK \) is nonpositive, has integrable potential with \( c > 0 \) and \( 0 < a \leq 1/4 \). Then, there exists a positive constant \( C \) such that

\[
\text{dist}_\Sigma(p, \partial \Sigma) \leq C, \quad \forall p \in \Sigma.
\]

In particular, if \( \Sigma \) is complete with \( \partial \Sigma = \emptyset \) then it must be compact. Moreover, it holds

\[
c A(\Sigma) - \|P\|_1 \leq 2a\pi \chi(\Sigma),
\]

where \( A(\Sigma) \) and \( \chi(\Sigma) \) denote the area and Euler characteristic of \( \Sigma \) respectively.

In Section 4, we impose that the potential has linear decay with respect some point, specifically, \( L_{a, c} := \Delta + V - aK \) has **linear decay** if \( V := c + P \), where \( c \) is a nonnegative constant and \( P \) satisfies

\[
|P(q)| \leq M/d(p_0, q),
\]

for some point \( p_0 \in \Sigma \), where \( M \) is a nonnegative constant (see Definition 4.1). We prove
Theorem 4.1. Let $\Sigma$ be a complete Riemannian surface without boundary. Suppose that $L_{a,c} = \Delta + V - aK$ is nonpositive acting on $f \in C^\infty_0(\Sigma)$, has linear decay with $c > 0$ and $a > 1/4$. Then, $\Sigma$ is compact.

Next, in Section 5, we apply these abstract results to stable $H$-surfaces immersed in a Riemannian three-manifold. In particular, for stable $H$-surfaces in a Killing submersion

Theorem 5.3. Let $\Sigma$ be a complete oriented $H$-surface with finite index immersed in $\mathcal{M}(\kappa, \tau)$, $\mathcal{M}(\kappa, \tau)$ an orientable Killing submersion of bounded geometry so that $4H^2 + c(\Sigma) \geq 0$, where

$$c(\Sigma) := \inf \{\kappa(\pi(p)) : p \in \Sigma\}.$$ 

Set

$$P^- := \min \{0, -(Ke + \tau^2)\},$$

$$P^+ := \max \{0, -(Ke + \tau^2)\}.$$ 

Assume $P^- \in L^1(\Sigma)$. Then, one of the following statements hold:

- $\Sigma$ is a minimal graph with $\pi(\Sigma) = \mathbb{M}^2$ and $c(\Sigma) > 0$,
- $4H^2 + c(\Sigma) = 0$ and $\Sigma$ is either a vertical multigraph or a vertical cylinder of geodesic curvature $2H$ in $\mathbb{M}^2$.

And

Theorem 5.4. Let $\Sigma$ be a complete oriented stable $H$-surface in $\mathcal{M}(\kappa, \tau)$, $\mathcal{M}(\kappa, \tau)$ an orientable Killing submersion of bounded geometry so that $4H^2 + c(\Sigma) \geq 0$, where

$$c(\Sigma) := \inf \{\kappa(\pi(p)) : p \in \Sigma\}.$$ 

Set $P := Ke + \tau^2$ and assume there exist a point $p_0 \in \Sigma$ and a constant $M > 0$ so that

$$|P(q)| \leq M/d(p_0, q).$$ 

Then, $\Sigma$ is a minimal graph with $\pi(\Sigma) = \mathbb{M}^2$ and $c(\Sigma) > 0$.

But, if we restrict the above results to the three-dimensional simply-connected homogeneous spaces, we obtain:

Corollary 5.1. Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be an oriented complete stable $H$-surface satisfying one of the following conditions:
• \( H \geq 1/2 \) and \( \max \{0, K_e\} \in L^1(\Sigma) \),

• \( H > 1/2 \) and there exists a point \( p_0 \in \Sigma \) and a constant \( M > 0 \) so that

\[
|K_e(q)| \leq M/d(p_0, q).
\]

Then, \( H = 1/2 \) and \( \Sigma \) is either a vertical horocylinder (i.e. a vertical cylinder over a horocycle in \( \mathbb{H}^2 \)) or an entire vertical graph.

**Corollary 5.2.** Let \( \Sigma \subset \mathbb{E}(\kappa, \tau) \), \( \tau \neq 0 \), be an oriented complete stable \( H \)-surface satisfying one of the following conditions:

• \( 4H^2 + \kappa \geq 0 \) and \( \nu^2 \in L^1(\Sigma) \),

• \( 4H^2 + \kappa > 0 \) and there exist a point \( p_0 \in \Sigma \) and a constant \( M > 0 \) so that

\[
|\nu(p)|^2 \leq M/d(p_0, q).
\]

Then:

• In \( S^3_{\text{Berger}} \), there are no such a stable \( H \)-surface.

• In \( \text{Nil}_3 \), \( H = 0 \) and \( \Sigma \) is either a vertical plane (i.e. a vertical cylinder over a straight line in \( \mathbb{R}^2 \)) or an entire vertical graph.

• In \( \widetilde{\text{PSL}}(2, \mathbb{R}) \), \( H = \sqrt{-\kappa}/2 \) and \( \Sigma \) is either a vertical horocylinder (i.e. a vertical cylinder over a horocycle in \( \mathbb{H}^2(\kappa) \)) or an entire graph.

Finally, in the Appendix, we have compiled a sort of results we will make use along this paper for the sake of completeness.

## 2 Notation and preliminary results

Throughout this work, we denote by \( \Sigma \) a connected Riemannian surface, with Riemannian metric \( g \), and possibly with boundary \( \partial \Sigma \). Let \( p \in \Sigma \) be a point of the surface and \( D(p,s) \), for \( s > 0 \), denote the geodesic disk centered at \( p \) of radius \( s \). We assume that \( \overline{D(p,s)} \cap \partial \Sigma = \emptyset \).

Moreover, let \( r \) be the radial distance of a point \( q \) in \( D(p,s) \) to \( p \). We write \( D(s) = D(p,s) \) if we can omit the dependence on \( p \). We also denote

\[
\begin{align*}
l(s) & := \text{Length}(\partial D(s)) \\
a(s) & := \text{Area}(D(s)) \\
K(s) & := \int_{D(s)} K \\
\chi(s) & := \text{Euler characteristic of } D(s),
\end{align*}
\]
where $K$ is the Gaussian curvature associated to the metric $g$. In the case we cannot drop the dependence on $p$, we write $l(p, s), a(p, s), K(p, s)$ and $\chi(p, s)$ respectively.

Let

$$L := \Delta + V - aK$$

be a differential operator on $\Sigma$ acting on piecewise smooth functions with compact support, i.e. $f \in C^\infty_0(\Sigma)$, where $a > 0$ is a constant, $V \in C^\infty(\Sigma)$ and $\Delta$ is the Laplacian operator associated to the metric $g$.

The index form of these kind of operators is

$$I(f) = \int_\Sigma \{ \| \nabla f \|^2 - V f^2 + aK f^2 \} , \quad (1)$$

where $\nabla$ and $\| \cdot \|$ are the gradient and norm associated to the metric $g$. One has

$$\int_\Sigma fLf = -I(f).$$

Thus, the nonpositivity of $L$ implies that the quadratic form, $I(f)$, associated to $L$ is nonnegative on compactly supported functions, i.e., $I(f) \geq 0$. In this case, we will say that $-L$ is stable.

$-L$ is stable if all the eigenvalues of $-L$ are nonnegative. $-L$ has finite index if has only finitely many negative eigenvalues. In this case, it is well known that there exists a compact set $K \subset \Sigma$ so that $-L$ is nonnegative acting on $f \in C^\infty_0(\Sigma \setminus K)$ (see [12]).

We recall now some topological concepts. For a compact surface $\Sigma$, its Euler characteristic is given by $\chi(\Sigma) := 2(1 - g_\Sigma) - n_\Sigma$, where $g_\Sigma$ and $n_\Sigma$ denote its genus and the number of connected components of its boundary respectively.

A noncompact surface $\Sigma$ is said to be of finite topology if there exists a compact surface $\Sigma$ without boundary and a finite number of pairwise disjoint closed disks $D_i \subset \Sigma, i = 1, \ldots, n$, so that $\Sigma$ is homeomorphic to $\Sigma \setminus \bigcup_{i=1}^n D_i$. In this case, the Euler characteristic of $\Sigma$ is $\chi(\Sigma) = 2(1 - g_\Sigma) - n$.

Moreover, we will need the following topological result (see [2, Lemma 1.4])

**Lemma 2.1.** Let $\Sigma$ be a complete Riemannian surface.

- If $\Sigma$ is of finite topology, then there exists $s_0$ such that for all $s \geq s_0$ we have $\chi(s) \leq \chi(\Sigma)$.
- If $\Sigma$ is not of finite topology then $\lim_{s \to +\infty} \chi(s) = -\infty$.

### 3 Nonpositive operators with integrable potential

In this Section we study differential operators with integrable potential on a Riemannian surface. First, we make explicit the kind of differential operators we are interested on:
Definition 3.1. Let \( \Sigma \) be a Riemannian surface. We say that \( L_{a,c} = \Delta + V - aK \) has integrable potential if \( L_{a,c} \) is a differential operator on \( \Sigma \) acting on piecewise smooth functions with compact support, i.e. \( f \in C_0^\infty(\Sigma) \), where \( a > 0 \) is constant, \( \Delta \) and \( K \) are the Laplacian and Gauss curvature associated to the metric \( g \) respectively. Moreover, we will assume that \( V := c + P \), where \( c \) is a nonnegative constant and \( P \) is a nonpositive and integrable function on \( \Sigma \), i.e., \( P \in L^1(\Sigma) \) or, equivalently, \( \|P\|_1 < +\infty \), where \( \|\cdot\|_1 \) denotes the \( L^1 \)-norm.

We will use the following condition on the area growth of \( \Sigma \).

Definition 3.2. Let \( \Sigma \) be a Riemannian surface possibly with boundary. We say that \( \Sigma \) has Asymptotic Area Bound of degree \( k \) \((k-AAB)\) if there exist positive constants \( k, C \in \mathbb{R}^+ \) such that for any point \( p \in \Sigma \) and any \( s > 0 \) so that \( D(p,s) \cap \partial \Sigma = \emptyset \), the function

\[
a(p, s) := \frac{\text{Area}(D(p, s))}{s^k} \to C \text{ if } s \to +\infty.
\]

If \( \Sigma \) is complete without boundary and for some point \( p \in \Sigma \) verifies

\[
\lim_{s \to \infty} \frac{\text{Area}(D(p, s))}{s^k} = C,
\]

we say that \( \Sigma \) has Asymptotic Area Growth of degree \( k \) \((k-AAG)\). Note that, by the Triangle Inequality, this condition does not depend on the point \( p \).

When \(-L\) is nonnegative, \( a > 1/4 \) and \( V \geq 0 \), it is known that \( \Sigma \) has quadratic area growth and the integrability of the potential (see [2], [21] for \( a > 1/4 \) or [5], [24] for \( a > 1/2 \)). When \( a \leq 1/4, V \geq 0 \) and we assume that \( \Sigma \) has asymptotic area growth of degree \( k \), we can obtain similar results (see [10]).

One of the most interesting consequence under the above conditions is that one can bound the distance of any point to the boundary, this is known as the Distance Lemma (see [21] or [10]). the Distance Lemma allows us to conclude that, if \( \Sigma \) is complete, it is compact.

We distinguish two cases depending on the value of \( a \). We start when \( a > 1/4 \), and first we will prove that the surface has Quadratic Area Bound.

Definition 3.3. Let \( \Sigma \) be a Riemannian surface possibly with boundary. We say that \( \Sigma \) has Quadratic Area Bound if there exists a positive constant \( C \) so that for any point \( p \in \Sigma \) and any \( s > 0 \) so that \( D(p, s) \cap \partial \Sigma = \emptyset \), the function

\[
a(p, s/2) \leq Cs^2.
\]

If \( \Sigma \) is complete without boundary and for some point \( p \in \Sigma \) verifies

\[
a(p, s) \leq Cs^2, \text{ for all } s > 0,
\]

we say that \( \Sigma \) has Quadratic Area Growth. Note that, by the Triangle Inequality, this condition does not depend on the point \( p \).
Actually, this will give us more information about the topology and conformal type of the surface.

**Theorem 3.1.** Let $\Sigma$ be a Riemannian surface possibly with boundary. Suppose that $L_{a,c} = \Delta + V - aK$ is nonpositive acting on $f \in C^\infty_0(\Sigma)$, has integrable potential with $c \geq 0$ and $a > 1/4$. Then, $\Sigma$ has Quadratic Area Bound (the area bound depending only on $a$, $c$ and $\|P\|_1$).

Moreover, if we assume $\Sigma$ is complete (without boundary), $\Sigma$ is conformally equivalent to a compact Riemann surface with a finite number of points removed.

**Proof.** Since $a > 1/4$, take $b \geq 1$ in (15) so that

$$-\alpha := b(1 - 4a) + 2a < 0.$$

Thus, applying Corollary 6.1, we obtain

$$\frac{\alpha}{s^2} \int_0^s (1 - r/s)^{2b-2} l(r) \leq 2a\pi - \int_{D(s)} (1 - r/s)^{2b} P,$$

where $P$ is a nonpositive integrable function, and so

$$-\int_{D(s)} (1 - r/s)^{2b} P \leq C \text{ for all } s > 0,$$

where $C$ is some positive constant depending only on $\|P\|_1$. Hence, since

$$\frac{\alpha}{s^2} \int_0^s (1 - r/s)^{2b-2} l(r) \geq \frac{\alpha}{2^{2b-2}s^2} a(s/2),$$

we obtain, inserting the above inequalities in (2),

$$\frac{\alpha}{2^{2b-2}s^2} a(s/2) \leq 2a\pi + C,$$

then

$$a(s/2) \leq \tilde{C}s^2,$$

where $\tilde{C}$ is some positive constant depending only on $a$, $c$ and $\|P\|_1$. This holds for any $p \in \Sigma \setminus \partial\Sigma$ and hence, $\Sigma$ has Quadratic Area Bound.

Now, we assume $\Sigma$ is complete. Take $b \geq 1$ so that $-\alpha := b(1 - 4a) + 2a < 0$. Then, applying Corollary 6.1, we have

$$0 \leq \frac{\alpha}{s^2} \int_0^s (1 - r/s)^{2b-2} l(r) \leq 2a\pi G(s) - \int_{D(s)} (1 - r/s)^{2b} P \leq 2a\pi G(s) + C,$$

(3)
where $C$ is some nonnegative constant depending only on $\|P\|_1$.

Let us prove first that $\Sigma$ has finite topology. Assume $\Sigma$ has infinite topology, then

$$\liminf_{s \to +\infty} \chi(s) = -\infty,$$

and hence, there exists $s_0$ so that for all $s \geq s_0$, we have $\chi(s) \leq -M$, where $M := \frac{C+1}{2a\pi}$. Therefore

$$G(s) = -\int_0^s (f(r)^2)'\chi(r) = -\int_0^{s_0} (f(r)^2)'\chi(r) - \int_{s_0}^s (f(r)^2)'\chi(r) \leq -\int_0^{s_0} (f(r)^2)' + M \int_{s_0}^s (f(r)^2)' = - (f(s_0)^2 - f(0)^2) + M (f(s)^2 - f(s_0)^2) = -(M + 1)f(s_0)^2 + 1 = -(M + 1)(1 - s_0/s)^{2b} + 1,$$

and, letting $s \to +\infty$ in the above expression, we can see that

$$\lim_{s \to +\infty} G(s) \leq -M,$$

and so

$$\lim_{s \to +\infty} 2a\pi G(s) + C \leq -1,$$

which contradicts (3). Therefore, $\Sigma$ has finite topology.

Thus, since $\Sigma$ has finite topology and QAG, $\Sigma$ is conformally equivalent to a compact Riemann surface with a finite number of points removed. \qed

For the next result recall that, for a Riemannian surface $\Sigma$ with boundary $\partial \Sigma$, we say that the area of the geodesic disks goes to infinity as its radius goes to infinity if for any point $p \in \Sigma$ and any $s > 0$ so that $D(p, s) \cap \partial \Sigma = \emptyset$, where $D(p, s)$ is the geodesic disk in $\Sigma$ centered at $p$ and radius $s$, the function

$$a(p, s) := \text{Area}(D(p, s)),$$

goes to infinity if $s$ goes to infinity. Now, we can prove

**Theorem 3.2.** Let $\Sigma$ be a Riemannian surface possibly with boundary. Suppose that $L_{a,c} = \Delta + V - aK$ is nonpositive, has integrable potential, $V := P + c$, with $c > 0$ and $a > 1/4$. Then, if the area of the geodesic disks goes to infinity as its radius goes to infinity, there exists a positive constant $C$ (depending only on $a$, $c$ and $\|P\|_1$) such that

$$\text{dist}_{\Sigma}(p, \partial \Sigma) \leq C, \quad \forall p \in \Sigma.$$

In particular, if $\Sigma$ is complete with $\partial \Sigma = \emptyset$, then it must be either compact or parabolic with finite area. Moreover, when $\Sigma$ is compact, it holds

$$c \ A(\Sigma) - \|P\|_1 \leq 2a\pi \chi(\Sigma),$$

where $A(\Sigma)$ and $\chi(\Sigma)$ denote the area and Euler characteristic of $\Sigma$ respectively.
Proof. Let us suppose that the distance to the boundary were not bounded. Then there exists a sequence of points \( \{p_i\} \in \Sigma \) such that \( \text{dist}^\Sigma(p_i, \partial \Sigma) \rightarrow +\infty \). So, for each \( p_i \) we can choose a real number \( s_i \) such that \( s_i \rightarrow +\infty \) and \( \overline{D(p_i, s_i)} \cap \partial \Sigma = \emptyset \).

We argue as in Theorem 3.1. Take \( b \geq 1 \) so that \( -\alpha := b(b(1 - 4a) + 2a) < 0 \). Then, applying (15) to each disk \( D(p_i, s_i) \), we have

\[
ca(p_i, s_i/2) \leq C,
\]

where \( C \) is constant independing only on \( a, c \) and \( \|P\|_1 \).

Now, bearing in mind that the left hand side of the above inequality goes to infinity and the right hand side remains bounded, we obtain a contradiction.

Also, if \( \Sigma \) is complete and has not finite area, the above estimate and the Hopf-Rinow Theorem imply that \( \Sigma \) must be compact.

When \( \Sigma \) is compact, the last formula follows taking the constant function \( f \equiv 1 \) on \( \Sigma \).

Now, we focus on the case \( 0 < a \leq 1/4 \).

**Theorem 3.3.** Let \( \Sigma \) be a Riemannian surface with \( k - AAB \) and possibly with boundary. Suppose that \( L_{a,c} = \Delta + V - aK \) is nonpositive, has integrable potential with \( c > 0 \) and \( 0 < a \leq 1/4 \). Then, there exists a positive constant \( C \) such that

\[
\text{dist}_\Sigma(p, \partial \Sigma) \leq C, \quad \forall p \in \Sigma.
\]

In particular, if \( \Sigma \) is complete with \( \partial \Sigma = \emptyset \) then it must be compact. Moreover, it holds

\[
cA(\Sigma) - \|P\|_1 \leq 2a\pi \chi(\Sigma),
\]

where \( A(\Sigma) \) and \( \chi(\Sigma) \) denote the area and Euler characteristic of \( \Sigma \) respectively.

**Proof.** Let us suppose that the distance to the boundary were not bounded. Then there exists a sequence of points \( \{p_i\} \in \Sigma \) such that \( \text{dist}^\Sigma(p_i, \partial \Sigma) \rightarrow +\infty \). So, for each \( p_i \) we can choose a real number \( s_i \) such that \( s_i \rightarrow +\infty \) and \( \overline{D(p_i, s_i)} \cap \partial \Sigma = \emptyset \).

Take \( p \in \Sigma \) and \( s > 0 \) so that \( \overline{D(p, s)} \cap \partial \Sigma = \emptyset \). Let \( f(r) \) be the radial function given by (17). Set

\[
\omega := \int_{D(se^{-r})} V + \int_{D(s) \setminus D(se^{-r})} \left( \frac{\ln(s/r)}{s} \right)^{2b} V,
\]

then

\[
\omega = c \int_{D(s)} f(s)^2 + \int_{D(s)} Pf(s)^2 \geq c \int_{D(s)} f(s)^2 + \int_{D(s)} P
\]

\[
\geq c \int_{D(s)} f(s)^2 - \tilde{C},
\]

where \( \tilde{C} \) is constant depending only on \( a, c \) and \( P \).
where $\tilde{C}$ is a nonnegative constant depending only on $\|P\|_1$.

Now, let $\beta \in \mathbb{R}$ be a real number greater than one, then

$$
\int_{D(s)} f(s)^2 = \int_{D(se^{-s})} 1 + \int_{D(s) \setminus D(se^{-s})} \left( \frac{\ln(s/r)}{s} \right)^{2b} \geq \frac{\beta^{2b}}{s^{2b}} a(se^{-\beta}).
$$

Thus, joining the above inequalities we get

$$
\int_{D(se^{-s})} V + \int_{D(s) \setminus D(se^{-s})} \left( \frac{\ln(s/r)}{s} \right)^{2b} V \geq c\frac{\beta^{2b}}{s^{2b}} a(se^{-\beta}) - \tilde{C}.
$$

Now, choose $b > 1$ such that $2(b + 1) \geq k > 2b > 2$. Thus, by Corollary 6.2 and the above inequality

$$
c\frac{\beta^{2b}}{s^{2b}} a(se^{-\beta}) \leq C + \rho^+_{a,b}(\delta_0, s),
$$

where $C$ is a positive constant depending on $a$ and $\|P\|_1$.

Now, since $\Sigma$ has $k$–AAB and $k > 2b$ then for $s$ large enough we have that the left hand side goes to infinity

$$
c\frac{\beta^{2b}}{s^{2b}} a(se^{-\beta}) \sim s^{k-2b} \longrightarrow +\infty,
$$

but the right hand side remains bounded (see the asymptotic properties of $\rho^+$ in Corollary 6.2).

Thus, applying (4) to each disk $D(p_i, s_i)$, and bearing in mind that, from (5), the left hand side of (4) goes to infinity, and the right hand side remains bounded, we obtain a contradiction.

We still have to consider the case $k \leq 2$. Here, we use Corollary 6.1. Thus, for $b = 1$ and the $k$–AAB, $k \leq 2$, of $\Sigma$, the right hand side of (15) remains bounded by some positive constant $C'$ (depending on the $k$–AAB) as $s$ goes to infinity. But,

$$
\int_{D(s)} (1 - r/s)^{2b} V \geq c\frac{a(s/2)^2}{4} - \tilde{C},
$$

for a nonnegative constant $\tilde{C}$ depending only on $a$ and $\|P\|_1$. Thus, we obtain

$$
ca(s/2) \leq C \quad \text{for all} \quad s > 0,
$$

where $C$ is a constant depending on $a$, $\|P\|$ and the $k$–AAB.

Applying (6) to each disk $D(p_i, s_i)$, and bearing in mind that the left hand side of (6) goes to infinity and the right hand side remains bounded, we obtain a contradiction.

Now, if $\Sigma$ is complete, then the above estimate and the Hopf-Rinow Theorem imply that $\Sigma$ must be compact. The last formula follows by taking the constant function $f \equiv 1$ as above. \qed
We have worked with nonnegative differential operators $-L_{a,c}$. This means that all the eigenvalues of $-L_{a,c}$ are nonnegative. $-L_{a,c}$ has finite index if has only finitely many negative eigenvalues or, equivalently, if there exists a compact set $K \subset \Sigma$ so that $-L_{a,c}$ is nonnegative acting on $f \in C^\infty_0(\Sigma \setminus K)$ (see [12]). As a consequence of these proofs we have the following

**Corollary 3.1.** Under the hypothesis of Theorem 3.1, if $-L_{a,c}$, $c = 0$, has finite index and $\Sigma$ is complete with $\partial \Sigma = \emptyset$, then $\Sigma$ is conformally equivalent to a compact Riemann surface with a finite number of points removed.

And

**Corollary 3.2.** Under the hypothesis of Theorem 3.2 or Theorem 3.3, if $-L_{a,c}$ has finite index and $\Sigma$ is complete with $\partial \Sigma = \emptyset$, then it must be either compact or parabolic with finite area.

### 4 Non positive operators with linear decay

We follow the notation of the previous section. First, let us make explicit the operators we will work with.

**Definition 4.1.** Let $\Sigma$ be a Riemannian surface. We say that $L_{a,c} = \Delta + V - aK$ has **Linear Decay** if $L_{a,c}$ is a differential operator on $\Sigma$ acting on piecewise smooth functions with compact support, i.e. $f \in C^\infty_0(\Sigma)$, where $a > 0$ is constant, $\Delta$ and $K$ are the Laplacian and Gauss curvature associated to the metric $g$ respectively. Moreover, we will assume that $V := c + P$, where $c$ is a nonnegative constant and $P$ satisfies

$$|P(q)| \leq M/d(p_0, q),$$

for some point $p_0 \in \Sigma$, here $M$ is a nonnegative constant

We continue assuming that $L_{a,c}$ has linear decay. In this case, we obtain a stronger result.

**Theorem 4.1.** Let $\Sigma$ be a complete Riemannian surface with $\partial \Sigma = \emptyset$. Suppose that $L_{a,c} = \Delta + V - aK$ is nonpositive acting on $f \in C^\infty_0(\Sigma)$, has linear decay, with $c > 0$ and $a > 1/4$. Then, $\Sigma$ is compact.

**Proof.** Since $P$ has linear decay and $\Sigma$ is complete, there exists $s_0 > 0$ so that $|P(q)| \leq c/2$ for all $q \in \Sigma \setminus D(p_0, s_0)$ (take $s_0 := 2M/c$). Set $\Sigma := \Sigma \setminus D(p_0, s_0)$, where $\partial \Sigma = \partial D(p_0, s_0)$.

On the one hand, from [21, Theorem 2.8], the distance from every point $q \in \Sigma$ to the boundary of $\Sigma$ satisfies:

$$d(q, \partial \Sigma) \leq \pi \sqrt{\left(1 + \frac{1}{4a} - 1\right) \frac{a}{c}}.$$

On the other hand, the distance from every point $p \in D(p_0, s_0)$ to $\partial \Sigma$ is bounded by $2M/c$. That is, the distance from any two points in $\Sigma$ is uniformly bounded, depending only on $M$, $a$ and $c$. Therefore, since $\Sigma$ is complete, the Hopf-Rinow Theorem implies that $\Sigma$ is compact. □
5 Stable surfaces in three-manifolds

Let $\Sigma$ be a two-sided surface with constant mean curvature $H$ (in short, $H$-surface) in a Riemannian three-manifold $M$. Throughout the rest of the paper, for the sake of simplicity, we will assume the ambient manifold $M$ is orientable without mention it. We will assume $M$ has bounded geometry, that is, $M$ has bounded sectional curvatures and injectivity radius bounded from below. $\Sigma$ is stable if (see [27] for the minimal case or [1] for the constant mean curvature case)

$$\int_{\Sigma} \psi^2 |A|^2 + \int_{\Sigma} \psi^2 \text{Ric}_M(N, N) \leq \int_{\Sigma} |\nabla \psi|^2$$

for all compactly supported functions $\psi \in H^1_c(\Sigma)$. Here $|A|^2$ denotes the the square of the length of the second fundamental form of $\Sigma$, $\text{Ric}_M(N, N)$ is the Ricci curvature of $M$ in the direction of the normal $N$ to $\Sigma$ and $\nabla$ is the gradient w.r.t. the induced metric.

One writes the stability inequality in the form

$$\frac{d^2}{dt^2} \left( \text{Area}(\Sigma(t)) - 2H \text{Volume}(\Sigma(t)) \right) = -\int_{\Sigma} \psi L \psi \geq 0,$$

where $L$ is the linearized operator of the mean curvature

$$L = \Delta + |A|^2 + \text{Ric}_M.$$

In terms of $L$, stability means that $-L$ is nonnegative, i.e., all its eigenvalues are nonnegative. $\Sigma$ is said to have finite index if $-L$ has only finitely many negative eigenvalues. It is well known that the stability operator $L$ can be written as

$$L = \Delta - K + (4H^2 - K_e + S),$$

where $K$ and $K_e$ are the Gaussian curvature and extrinsic curvature (i.e., the product of the principal curvatures) of $\Sigma$, and $S$ is the scalar curvature of $M$. Hence, as a direct application of the previous results, we have

**Theorem 5.1.** Assume $M$ has bounded geometry. Let $\Sigma \subset M$ be a complete oriented $H$-surface with finite index. Set $\varepsilon \geq 0$ and

$$P^-_{\varepsilon} := \min \left\{ 0, 4H^2 - K_e + S - \varepsilon \right\},$$

$$P^+_{\varepsilon} := \max \left\{ 0, 4H^2 - K_e + S - \varepsilon \right\}$$

- If $P^-_0 \in L^1(\Sigma)$, $\Sigma$ is conformally equivalent to a compact Riemann surface with a finite number of points removed. Moreover, $P^+_0 \in L^1(\Sigma)$.
- If $P^-_{\varepsilon} \in L^1(\Sigma)$ for some $\varepsilon > 0$, then $\Sigma$ is compact.
Proof. Since $-L$ has finite index and

\[
L = \Delta - K + (4H^2 - K_e + S)
= \Delta - K + \varepsilon + P_\varepsilon^+ + P_\varepsilon^-
\geq \Delta - K + \varepsilon + P_\varepsilon^-,
\]

then $-L\varepsilon$, where $L\varepsilon := \Delta - K + \varepsilon + P_\varepsilon^-$, has finite index. Thus, applying either Corollary 3.1 if $\varepsilon = 0$ or Corollary 3.2 if $\varepsilon > 0$, we obtain the result.

When $\varepsilon > 0$, we still have to remove the case when $\Sigma$ is parabolic with finite volume. Since $\mathcal{M}$ has bounded geometry and $\Sigma$ is complete and has constant mean curvature, from [4, Proposition 2.1], we get that each end of $\Sigma$ has infinite area, a contradiction. So, $\Sigma$ must be compact.

Let us see that $P_0^+ \in L^1(\Sigma)$. Apply equation (15) for $b = 1$, then

\[
\int_{D(s)} V(1 - r/s)^2 \leq 2a\pi - \frac{a(s)}{s^2},
\]

where $V := 4H^2 - K_e + S$. Set $V := P_0^+ + P_0^-$. Now, the above inequality can be written as

\[
1/2 \int_{D(s/2)} P_0^+ \leq \int_{D(s)} P_0^+(1 - r/s)^2 \leq 2a\pi - \frac{a(s)}{s^2} - \int_{D(s)} (1 - r/s)^2 P_0^-,
\]

and since the right hand side of the above inequality is bounded as $s$ goes to infinity, we get that $P_0^+ \in L^1(\Sigma)$. ■

Also, we can drop the assumption about the bounded geometry of the ambient space in the next result:

**Theorem 5.2.** Let $\Sigma \subset \mathcal{M}$ be a complete oriented stable $H$–surface. Set $\varepsilon > 0$ and

\[
P_{\varepsilon} := 4H^2 - K_e + S - \varepsilon.
\]

Assume there exist $p_0 \in \Sigma$ and a constant $M > 0$ so that

\[
|P_{\varepsilon}(q)| \leq M/d(p_0, q). \tag{7}
\]

Then, $\Sigma$ is compact

Proof. Note that the stability operator $L$ can be written as

\[
L = \Delta - K + \varepsilon + P_{\varepsilon}.
\]

Then, under the assumption (7), $L$ has linear decay (see Definition 4.1). Hence, applying Theorem 4.1, we obtain the result. ■
We focus now on surfaces immersed on a Killing submersion $\mathcal{M}$, that is, $\mathcal{M}$ is a Riemannian submersion over a Riemannian surface $\mathbb{M}^2$ whose fibers are the trajectories of an unit Killing field. In [11], the geometry of this kind of submersion is studied. In some sense, these spaces behaves like a simply-connected homogeneous space $E(\kappa, \tau)$.

Let $\mathcal{M}$ be a three-dimensional Killing submersion, then $\pi : \mathcal{M} \to \mathbb{M}^2$ over a surface $(\mathbb{M}^2, g)$ with Gauss curvature $\kappa$, and the fibers, i.e. the inverse image of a point at $\mathbb{M}^2$ by $\pi$, are the trajectories of a unit Killing vector field $\xi$, and hence geodesics. Denote by $\langle \cdot, \cdot \rangle$, $\nabla$, $\wedge$, $\bar{R}$ and $[,]$ the metric, Levi-Civita connection, exterior product, Riemann curvature tensor and Lie bracket in $\mathcal{M}$, respectively. Moreover, associated to $\xi$, we consider the operator $J : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ given by

$$JX := X \wedge \xi, \quad X \in \mathfrak{X}(\mathcal{M}).$$

Given $X \in \mathfrak{X}(\mathcal{M})$, $X$ is vertical if it is always tangent to fibers, and horizontal if always orthogonal to fibers. Moreover, if $X \in \mathfrak{X}(\mathcal{M})$, we denote by $X^v$ and $X^h$ the projections onto the subspaces of vertical and horizontal vectors respectively. In particular (see [11, Proposition 2.6])

**Proposition 5.1.** Let $\mathcal{M}$ be as above. There exists a function $\tau : \mathcal{M} \to \mathbb{R}$ so that

$$\nabla_X \xi = \tau X \wedge \xi,$$

(8)

here $\nabla$ denotes the Levi-Civita connection on $\mathcal{M}$.

Actually, it can be shown that $\tau$ only depends on $\mathbb{M}^2$ (this is a personal communication of A. Jimenez, and it will appear in a forthcoming paper). This makes natural the following notation:

**Definition 5.1.** A Riemannian submersion over a surface $\mathbb{M}^2$ whose fibers are the trajectories of an unit Killing vector field $\xi$ will be called Killing submersion and denoted by $\mathcal{M}(\kappa, \tau)$, where $\kappa$ is the Gauss curvature of $\mathbb{M}^2$ and $\tau$ is given in Proposition 5.1.

Moreover, we remind here [11, Lemma 2.8]

**Lemma 5.1.** Let $\mathcal{M}(\kappa, \tau)$ be a Riemannian submersion with unit Killing vector field $\xi$. Let $\{X, Y\} \in T\mathcal{M}(\kappa, \tau)$ be an orthonormal basis of horizontal vector fields so that $\{X, Y, \xi\}$ is positively oriented. Then

$$\bar{K}(X \wedge Y) = \kappa - 3\tau^2,$$

(9)

$$\bar{K}(X \wedge \xi) = \tau^2.$$

(10)

Moreover, the scalar curvature $S$ of $\mathcal{M}(\kappa, \tau)$ at $p \in \mathcal{M}(\kappa, \tau)$ is given by

$$S(p) = \kappa - \tau^2.$$

(11)

Now, we can announce:
Theorem 5.3. Let $\Sigma$ be a complete oriented $H$–surface with finite index immersed in $\mathcal{M}(\kappa, \tau)$, $\mathcal{M}(\kappa, \tau)$ a Killing submersion of bounded geometry so that $4H^2 + c(\Sigma) \geq 0$, where
\[ c(\Sigma) := \inf \{ \kappa(p) : p \in \Sigma \}. \]

Set
\[ P^- := \min \{ 0, -(K_e + \tau^2) \}, \]
\[ P^+ := \max \{ 0, -(K_e + \tau^2) \}. \]

Assume $P^- \in L^1(\Sigma)$. Then, one of the following statements hold:

- $\Sigma$ is a minimal graph with $\pi(\Sigma) = \mathbb{M}^2$ and $c(\Sigma) > 0$,
- $4H^2 + c(\Sigma) = 0$ and $\Sigma$ is either a vertical multigraph or vertical cylinder of geodesic curvature $2H$ in $\mathbb{M}^2$.

Proof. The linearized operator for the mean curvature is given by
\[ L := \Delta - K + (4H^2 - K_e + S). \]

Now, from (11), we might rewrite the above inequality as
\[ L := \Delta - K + 4H^2 + \kappa - (K_e + \tau^2) \geq \Delta - K + c + P^-, \]
where $c := 4H^2 + c(\Sigma) \geq 0$. Then $-\bar{L}$, where $\bar{L} := \Delta - K + c + P^-$, is nonnegative acting on $f \in C^\infty_0(\Sigma)$ and has integrable potential.

If $4H^2 + c(\Sigma) > 0$, applying Theorem 3.2, $\Sigma$ is either compact or parabolic with finite area.

Let us prove now that $\Sigma$ cannot be parabolic with finite area. Since $\mathcal{M}(\kappa, \tau)$ has bounded geometry and $\Sigma$ is complete and has constant mean curvature, from [4, Proposition 2.1], we get that $\Sigma$ has infinite area. So, $\Sigma$ must be compact.

Set $\nu := \langle \xi, N \rangle$, where $N$ is the unit normal vector field along $\Sigma$, $\nu$ is a bounded Jacobi function, i.e., $\bar{L}\nu = 0$. Since $\Sigma$ is stable and compact, elementary elliptic theory asserts that either $\nu$ vanishes identically or $\nu > 0$.

If $\nu$ vanishes identically, $\Sigma := \pi^{-1}(\alpha)$, i.e., it is a vertical cylinder over a complete curve $\alpha \subset \mathbb{M}^2$ of geodesic curvature $2H$. Then, the stability operator of $\Sigma := \pi^{-1}(\alpha)$ is given by
\[ L = \Delta + (4H^2 + \kappa) \]
since $K_e = -\tau^2$ on a vertical cylinder (see [11, Proposition 2.10]). Then
\[ L \geq \Delta + c, \]
where $c := 4H^2 + c(\Sigma) > 0$. 

17
Let \( \lambda_1(L) \) denote the first eigenvalue of \(-L\). Thus,

\[
\lambda_1(L) \leq \lambda_1(\Delta + c) = \lambda_1(\Delta) - c = -c < 0,
\]

where \( \lambda_1(\Delta) = 0 \) since \( \Sigma \) is isometrically a plane. So, \( \Sigma \) is unstable, a contradiction.

Therefore, \( \nu \) never vanishes. Since \( \Sigma \) is compact, there exists \( \varepsilon > 0 \) so that \( \nu \geq \varepsilon > 0 \), and hence \( \pi(\Sigma) = \mathbb{M}^2 \). Let \( j(\mathbb{M}^2) \) denote the Cheeger constant, i.e.,

\[
j(\mathbb{M}^2) := \inf_{\Omega \subset \mathbb{M}^2} \left\{ \frac{A(\Omega)}{L(\partial \Omega)} \right\},
\]

where \( \Omega \) varies over open domains on \( \mathbb{M}^2 \) with compact closure and smooth boundary.

Let \( \Omega \subset \mathbb{M}^2 \) be a relatively compact domain with smooth boundary \( \partial \Omega \). Since \( \nu \geq \varepsilon > 0 \), there exists a compact set \( \Sigma_0 \subset \Sigma \) which is a \( H \)-graph over \( \Omega \). From the Divergence Theorem

\[
2HA(\Omega) = \int_{\Omega} \text{div}(N^h) = \int_{\partial \Omega} g(N^h, \eta) \leq L(\partial \Omega),
\]

where \( A(\Omega) \) and \( L(\partial \Omega) \) are the area and the length of \( \Omega \) and \( \partial \Omega \) (w.r.t. \( g \)) respectively. Moreover, \( \text{div} \) is the divergence operator on \((\mathbb{M}^2, g)\). Thus,

\[
2H \leq j(\mathbb{M}^2). \tag{12}
\]

Since \( \mathbb{M}^2 \) is compact (recall \( \pi(\Sigma) = \mathbb{M}^2 \) and \( \Sigma \) is compact), \( j(\mathbb{M}^2) = 0 \). So, from (12) and \( 4H^2 + c(\Sigma) > 0 \), we get \( H = 0 \) and \( c(\Sigma) > 0 \).

If \( 4H^2 + c(\Sigma) = 0 \), \( \Sigma \) is parabolic. Then, [20, Corollary 2.5] asserts that \( \nu \) vanishes identically or never vanishes. That is, \( \Sigma \) is either a vertical cylinder over a complete curve of geodesic curvature \( 2H \) in \( \mathbb{M}^2 \), or \( \Sigma \) is a complete multigraph.

This finishes the proof. \( \square \)

Also,

**Theorem 5.4.** Let \( \Sigma \) be a complete oriented stable \( H \)-surface in \( \mathcal{M}(\kappa, \tau) \) so that \( 4H^2 + c(\Sigma) > 0 \), where

\[
c(\Sigma) := \inf \{ \kappa(\pi(p)) : p \in \Sigma \}.
\]

Set \( P := K_e + \tau^2 \), assume there exist a point \( p_0 \in \Sigma \) and a constant \( M > 0 \) so that

\[
|P(q)| \leq M/d(p_0, q).
\]

Then, \( \Sigma \) is a minimal graph with \( \pi(\Sigma) = \mathbb{M}^2 \) and \( c(\Sigma) > 0 \).
Proof. We might write the stability operator $L$ as

$$L = \Delta - K + (4H^2 + \kappa) - (K_e + \tau^2),$$

then $L \geq \tilde{L}$, where

$$\tilde{L} := \Delta - K + c - P,$$

where $c := 4H^2 + c(\Sigma) > 0$. Thus, $\tilde{L}$ is a nonnegative operator with linear decay. So, from Theorem 4.1, $\Sigma$ is compact. Therefore, arguing as in Theorem 5.3, we obtain that $H = 0$, $c(\Sigma) > 0$ and $\pi(\Sigma) = \mathbb{M}^2$.

To finish, we give some consequences of the above results for stable $H-$surface in $\mathbb{E}(\kappa, \tau)$. Nelli-Rosenberg [22] proved that there are no stable $H-$surfaces, $H > 1/\sqrt{3}$, in $\mathbb{H}^2 \times \mathbb{R}$, by proving a Distance Lemma. In general, thanks to a Distance Lemma, i.e. an intrinsic estimate of the distance to the boundary, Rosenberg [25] proved that there are no complete stable $H-$surfaces in $\mathbb{E}(\kappa, \tau)$ provided $H^2 > \frac{\tau^2 - \kappa}{3}$, unless $\mathbb{S}^2(\kappa) \times \{0\}$ in $\mathbb{S}^2(\kappa) \times \mathbb{R}$. Moreover, we might assume that $\kappa < \tau^2$, since the case $\kappa \geq \tau^2$ is done (see [25]), a stable complete $H-$surface in $\mathbb{E}(\kappa, \tau)$, $\kappa \geq \tau^2$ is a slice in $\mathbb{S}^2(\kappa) \times \mathbb{R}$. When, $\kappa < \tau^2$, one can improves Rosenberg’s result using a compactness argument. In [20], the authors proved that there exists $\varepsilon > 0$ such that $H^2 < \frac{\tau^2 - \kappa}{3} - \varepsilon$ for any complete stable $H-$surface in $\mathbb{E}(\kappa, \tau)$ with $\kappa < \tau^2$.

Here, we prove that there are no complete $H-$surfaces, $H > 1/2$, in $\mathbb{H}^2 \times \mathbb{R}$ under some conditions on the extrinsic curvature.

**Corollary 5.1.** Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be an oriented complete stable $H-$surface satisfying one of the following conditions:

- $H \geq 1/2$ and $\max \{0, K_e\} \in L^1(\Sigma)$,
- $H > 1/2$ and there exist a point $p_0 \in \Sigma$ and a constant $M > 0$ so that

$$|K_e(q)| \leq M/d(p_0, q).$$

Then, $H = \frac{3}{2}$ and $\Sigma$ is either a vertical horocylinder (i.e. a vertical cylinder over a horocycle in $\mathbb{H}^2$) or an entire vertical graph.

**Proof.** We apply either Theorem 5.3 or Theorem 5.4 depending on the condition that $\Sigma$ verifies. In any case, $4H^2 + 1 = 0$ and $\Sigma$ is either a vertical cylinder over a complete curve of geodesic curvature $2H = 1$, that is, a horocycle in $\mathbb{H}^2$, or $\Sigma$ is a complete multigraph. In the latter case, Hauswirth-Rosenberg-Spruck [16] proved that $\Sigma$ is an entire graph. \(\Box\)

And, for stable $H-$surfaces in either the Heisenberg space or $\tilde{\text{PSL}}(2, \mathbb{R})$, we obtain

**Corollary 5.2.** Let $\Sigma \subset \mathbb{E}(\kappa, \tau)$, $\tau \neq 0$, be an oriented complete stable $H-$surface satisfying one of the following conditions:
- $4H^2 + \kappa \geq 0$ and $\nu^2 \in L^1(\Sigma)$,
- $4H^2 + \kappa > 0$ and there exist a point $p_0 \in \Sigma$ and a constant $M > 0$ so that
  $$|\nu(p)|^2 \leq M/d(p_0, q).$$

Then:
- In $\mathbb{S}^3_{\text{Berger}}$, there are no such a stable $H$-surface.
- In $\text{Nil}_3$, $H = 0$ and $\Sigma$ is either a vertical plane (i.e. a vertical cylinder over a straight line in $\mathbb{R}^2$) or an entire vertical graph.
- In $\tilde{\text{PSL}}(2, \mathbb{R})$, $H = \sqrt{-\kappa}/2$ and $\Sigma$ is either a vertical horocylinder (i.e. a vertical cylinder over a horocycle in $\mathbb{H}^2(\kappa)$) or an entire graph.

Proof. Note that, by the Gauss equation for a surface immersed in $\mathbb{E}(\kappa, \tau)$ (see [6]), i.e.,
$$K = K_e + \tau^2 + (\kappa - 4\tau^2)\nu^2,$$
we might write the stability operator as
$$L = \Delta - 2K + (4H^2 + \kappa) + (\kappa - 4\tau^2)\nu^2.$$

So, we apply Theorem 5.3 or Theorem 5.4 depending on the condition that $\Sigma$ verifies. We get:
- In $\mathbb{S}^3_{\text{Berger}}$, $4H^2 + \kappa > 0$ and so $\Sigma$ is compact, but there are no compact, oriented stable $H$-surfaces in $\mathbb{S}^3_{\text{Berger}}$ (see [21, Corollary 9.6]).
- In $\text{Nil}_3$, $H = 0$ and $\Sigma$ is either a vertical cylinder over a complete curve of geodesic curvature $H = 0$, that is, a straight line in $\mathbb{R}^2$, or $\Sigma$ is a complete multigraph. In the latter case, Daniel-Hauswirth [7] proved that $\Sigma$ is an entire graph.
- In $\tilde{\text{PSL}}(2, \mathbb{R})$, the proof is similar as above. Now, $\Sigma$ is an entire graph follows from [8].
6 Appendix

We recall here some results we have used along the paper for the sake of completeness. The first one is a general inequality for \( I(f) \) (see (1)) following the method developed by T. Colding and W. Minicozzi in [5]. We establish here the formula how it was stated in [10, Lemma 3.1], but the proof can be found in [2].

We denote

\[
\begin{align*}
  l(s) &= \text{Length}(\partial D(s)) \\
  a(s) &= \text{Area}(D(s)) \\
  K(s) &= \int_{D(s)} K \\
  \chi(s) &= \text{Euler characteristic of } D(s)
\end{align*}
\]

**Lemma 6.1** (Colding-Minicozzi stability inequality). Let \( \Sigma \) be a Riemannian surface possibly with boundary and \( K \neq 0 \). Let us fix a point \( p_0 \in \Sigma \) and positive numbers \( 0 \leq \varepsilon < s \) such that \( \overline{D(s)} \cap \partial \Sigma = \emptyset \). Let us consider the differential operator \( L_a = \Delta + V - aK \), where \( V \in C^\infty(\Sigma) \) and \( a \) is a positive constant, acting on \( f \in C_0^\infty(\Sigma) \). Let \( f : D(s) \longrightarrow \mathbb{R} \) a nonnegative radial function, i.e. \( f \equiv f(r) \), such that

\[
\begin{align*}
  f(r) &\equiv 1, \quad \text{for } r \leq \varepsilon \\
  f(r) &\equiv 0, \quad \text{for } r \geq s \\
  f'(r) &\leq 0, \quad \text{for } \varepsilon < r < s
\end{align*}
\]

Then, the following holds

\[
I(f) \leq 2a (\pi G(s) - f'(\varepsilon)l(\varepsilon)) - \int_{D(s)} V f(r)^2 + \int_{\varepsilon}^{s} \left\{ (1 - 2a)f'(r)^2 - 2af(r)f''(r) \right\} l(r),
\]

where

\[
G(s) := -\int_{\varepsilon}^{s} (f(r)^2)'\chi(r).
\]

Moreover, we should mention that we have relaxed here the hypothesis on \( V \), but from the proof, we can see that we do not really use the fact that \( V \geq 0 \).

It is well known that the kind of results we can obtain for nonnegative operator of the form

\[
L_a := \Delta + V - aK
\]

where \( V \geq 0 \) and \( a \) is a positive constant, depend strongly on the value of \( a \).

The most studied case is when \( a > 1/4 \) (see [2] or [21]). When \( a > 1/4 \), we use the following radial function

\[
f(r) = \begin{cases} 
(1 - \frac{r}{s})^b & 0 \leq r \leq s \\
0 & r \geq s
\end{cases}
\]

(14)
where \( s > 0, b \geq 1 \) and \( r \) is the radial distance of a point \( p \) in \( D(s) \) to \( p_0 \). So now, we establish a formula developed by Meeks-Pérez-Ros [21]. Such a formula follows from Lemma 6.1 with the test function given by (14).

**Corollary 6.1.** Let \( \Sigma \) be a Riemannian surface possibly with boundary and \( K \neq 0 \). Fix a point \( p_0 \in \Sigma \) and a positive number \( s > 0 \) such that \( D(s) \cap \partial \Sigma = \emptyset \). Suppose that the differential operator \( L_a = \Delta + V - aK \) is nonpositive on \( C_0^\infty(\Sigma) \), where \( V \in C^\infty(\Sigma) \) and \( a > 1/4 \) is a constant. For \( b \geq 1 \), we have

\[
\int_{D(s)} (1 - r/s)^{2b} V \leq 2aG(s)\pi + \frac{b(b(1 - 4a) + 2a)}{s^2} \int_0^s (1 - r/s)^{2b-2} l(r),
\]

where

\[
G(s) := \frac{2b}{s} \int_0^s (1 - r/s)^{2b-1} \chi(r) \, dr \leq 1.
\]

If \( a \leq 1/4 \) (see [10]), we will work with the special radial function given by

\[
f(r) = \begin{cases} 
1 & r \leq se^{-s} \\
\left(\frac{\ln(s/r)}{s}\right)^b & se^{-s} \leq r \leq s \\
0 & r \geq s
\end{cases}
\]

where \( s > 0, b \geq 1 \) and \( r \) is the radial distance of a point \( p \) in \( D(s) \) to \( p_0 \). Now, we use the above test function (17) in Lemma 6.1 (see [10, Corollary 6.1 and Theorem 6.1] for details).

**Corollary 6.2.** Let \( \Sigma \) be a Riemannian surface with \( k - \text{AAB} \), possibly with boundary and \( K \neq 0 \). Fix a point \( p_0 \in \Sigma \) and a positive number \( s > 0 \) such that \( D(s) \cap \partial \Sigma = \emptyset \). Suppose that the differential operator \( L_a = \Delta + V - aK \) is nonpositive on \( C_0^\infty(\Sigma) \), where \( V \in C^\infty(\Sigma) \) and \( 0 < a \leq 1/4 \) is a constant. Let \( b \geq 1 \) so that \( 2(b + 1) \geq k \), then

\[
\int_{D(se^{-s})} V + \int_{D(s) \setminus D(se^{-s})} \left(\frac{\ln(s/r)}{s}\right)^b V \leq 2a \left(\pi + \frac{2\pi - K(se^{-s})}{s}\right) + \rho_{a,b}(\delta_0, s),
\]

where we denote

\[
K(r_1) = \min_{[0,r_1]} \{ K(r) \},
\]

and \( \rho_{a,b}^+ \) is a function depending on \( s \) so that

\[
\rho_{a,b}^+(\delta_0, s) \longrightarrow \begin{cases} 
0 & \text{for } 2(b + 1) > k \text{ if } s \to +\infty, \\
C^+ & \text{for } 2(b + 1) = k
\end{cases}
\]

for a positive constant \( C^+ \). Moreover,

\[
\lim_{s \to \infty} \frac{K(se^{-s})}{s} = 0.
\]
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References

[1] L. Barbosa, M. doCarmo and J. Eschenburg, Stability of hypersurfaces with constant mean curvature in Riemannian manifolds, Math. Z., 197 (1988), 123-138.

[2] P. Castillon, An inverse spectral problem on surfaces, Comment. Math. Helv., 81 no 2 (2006), 271–286.

[3] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, INC., 1984.

[4] X. Cheng, L. Cheung and D. Zhou, The structure of stable constant mean curvature hypersurfaces, Tohoku Math. Jour., 60 (2008), 101–121.

[5] T. Colding and W. Minicozzi, Estimates for parametric elliptic integrands, Internat. Math. Res. Notices, 6 (2002), 291–297.

[6] B. Daniel, Isometric immersions into 3-dimensional homogenous manifolds, Comment. Math. Helv., 82 no 1 (2007), 87–131.

[7] B. Daniel and L. Hauswirth, Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg Group, Proc. Lond. Math. Soc. (3) 98 no 2 (2009), 445–470.

[8] B. Daniel, L. Hauswirth and P. Mira, Constant mean curvature surfaces in homogeneous manifolds. Preprint, 2009. Published preliminarily by the Korea Institute for Advanced Study.

[9] M. do Carmo and C. K. Peng, Stable minimal surfaces in $\mathbb{R}^3$ are planes, Bull. Amer. Math. Soc., 1 (1977), 903–906.

[10] J.M. Espinar and H. Rosenberg, A Colding-Minicozzi stability inequality and its applications, Trans. A.M.S., 363 (2011), 2447–2465.

[11] J. M. Espinar and Inés Silva, Locally convex surfaces immersed in a Killing submersion. Preprint.

[12] D. Fischer-Colbrie, On complete minimal surfaces with finite Morse index in three manifolds, Invent. Math., 82 (1985), 121–132.

[13] D. Fischer-Colbrie and R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm. Pure Applied Math., 33 (1980), 199–211.

[14] R. Gulliver, Index and total curvature of complete minimal surfaces, Geometric measure theory and the calculus of variations (Arcata, Calif. 1984), Proc. Sympos. Pure Math. 44, Amer. Math. Soc., Providence, RI. (1986), 207–211.
[15] R. Gulliver, *Minimal surfaces of finite index in manifolds of positive scalar curvature*, Lecture Notes in Mathematics: Calculus of variations and partial differential equations (Trento, 1986) **1340**, Springer, Berlin (1988), pp. 115–122.

[16] L. Hauswirth, H. Rosenberg and J. Spruck, *On complete mean curvature 1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Comm. Anal. Geom., **16** no 5 (2005), 989–1005.

[17] S. Kawai, *Operator $\Delta - aK$ on surfaces*, Hokkaido Math. Journal, **17** (1988), 147–150.

[18] L. Mazet, *Optimal length estimates for stable CMC surfaces in 3-space-forms*, Proc. A.M.S., **137** (2009), 2761–2765.

[19] C. Leandro, H. Rosenberg, *Renovable singularities for sections of Riemannian submersions of prescribed mean curvature*, Bull. Sci. Math., **133** (2009), 445-452.

[20] J. M. Manzano, J. Pérez and M. M. Rodríguez, *Parabolic stable surfaces with constant mean curvature*. To appear in Cal. Var. and PDEs.

[21] W. Meeks, J. Pérez and A. Ros, *Stable constant mean curvature hypersurfaces*, Handbook of Geometric Analysis, volume 1 (2008), pages 381–380. International Press, edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon, ISBN: 978-1-57146-130-8.

[22] B. Nelli and H. Rosenberg, *Global properties of constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Pacific J. Math., **226** no 1 (2006), 137–152.

[23] A. V. Pogorelov, *On stability of minimal surfaces*, Soviet Math. Dokl., **24** (1981), 274–276.

[24] H Rosenberg, *Some recent developments in the theory of minimal surfaces*, XXIV Coloquio Brasileiro de Matematica, Publicações Cceos Matematicas, IMPA, Rio de Janeiro (2003), 1–48.

[25] H Rosenberg, *Constant Mean Curvature Surfaces in Homogeneously Regular 3-Manifolds*, Bull. Aust. Math. Soc., **74** (2006), 227–238.

[26] H. Rosenberg, R. Souam and E. Toubiana, *General curvature estimates for stable $H$-surfaces in 3-manifolds and applications*, Journal of Differential Geometry, **84**(2010), 623–648.

[27] R. Schoen and S.T. Yau, *Harmonic maps and the topology of stable hypersurfaces and manifolds of nonnegative Ricci curvature*, Comm. Math. Helv. **39** (1976), 333–341.