Asymptotic Abelianness and Braided Tensor $C^*$–Categories

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Abstract

By introducing the concepts of asymptopia and bi–asymptopia, we show how braided tensor $C^*$–categories arise in a natural way. This generalizes constructions in algebraic quantum field theory by replacing local commutativity by suitable forms of asymptotic Abelianness.

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1 Introduction

The occurrence of superselection sectors in quantum theories was first recognized by Wick, Wightman and Wigner [20] who gave two important examples of situations leading to sectors, the univalence rule and the electric charge. For a while it looked as if little would be touched by their fundamental discovery. The unrestricted superposition principle had to be abandoned, but it at least remained valid within certain subspaces, the coherent subspaces. However the point of view persisted that pure states of the theory were described by the projective space associated to a preassigned Hilbert space, or rather a subset of that space to account for the new phenomenon.

But a drastic change in the basic picture allowed the new phenomenon to spawn new ideas and results: as was stressed very early by Haag, in quantum field theory the local observables are fundamental and generate an algebra. The different sectors provide inequivalent irreducible realizations of that algebra [13]. Furthermore, starting just from the vacuum sector and analyzing the structure of the algebra, we can in principle determine all sectors.

For superselection sectors describing localizable charges, this is possible by a selection criterion [6] singling out those irreducible representations. The mathematical object that emerges and describes the structure of the sectors is a symmetric tensor $C^*$–category with conjugates and irreducible unit. It was later shown that any such category is isomorphic to a category of unitary representations of some compact group, unique up to isomorphism [8]. Furthermore there is a canonical field net with ordinary Bose–Fermi commutation relations at spacelike separations [9] where this group, the gauge group, acts as automorphisms with the original net as fixed–point net.

The above selection criterion does not select all the relevant representations in every case. Although weaker physically significant conditions covering a large class of theories with short range forces have been analyzed with success [5], there is no known, or even plausible proposed criterion for singling out the relevant representations in all cases. In fact, it suffices to take the case of the electric charge, one of the key examples of [20], to realize that there are still unresolved problems. Essentially one would like in this case to arrive at a simple picture where the sectors are labelled by the electric charge corresponding to a gauge group $U(1)$. However for each value of the
electric charge there are myriads of representations differing by their infrared clouds, cf. [10, 2] and references therein. To find such a simple picture one would either have to take equivalence classes or choose a suitable subset of representations. Previous work on the problem of sectors in quantum electrodynamics include [1, 3, 4, 11, 12].

This paper has been a result of our attempts to describe the sector structure of quantum electrodynamics and of similar models exhibiting long range effects. In view of the central role played by the symmetric tensor $C^*$–category, we here propose a method of constructing such categories which might prove to be applicable in these cases but the scheme seems of interest in its own right.

In the simplest case of strictly localizable charges [6], one passes from the selected representations and their intertwiners to endomorphisms and their intertwiners using Haag duality. The endomorphisms of a $C^*$–algebra and their intertwiners form a tensor $C^*$–category, the endomorphisms being the objects and the intertwiners the arrows. The symmetry properties of the category can be deduced from analyzing the commutation properties of intertwiners.

The proposal described here allows for intertwiners not lying in the algebra where the endomorphisms act. In addition, our endomorphisms are not required to be locally inner but only asymptotically inner. We will show in Section 2 how an appropriate form of asymptotic Abelianness allows one to construct a tensor $C^*$–category from this structure without the additional input of Haag duality. Similarly, the symmetries on our derived tensor $C^*$–categories, discussed in Section 3, will reflect asymptotic rather than purely local commutation properties of intertwiners; yet they are exact symmetries although one might have anticipated that asymptotic Abelianness would just lead to an asymptotic notion of symmetry.

The general mechanism presented here might still be too limited to be directly applicable to quantum electrodynamics, but might elucidate some important aspects of that theory. As a matter of fact, it covers examples of theories exhibiting long range effects which go beyond the limits of the approaches previously studied.

One such example is provided by the model of charges of electromagnetic type discussed in [3], which actually stimulated the study of the more general structure discussed in the present paper. We briefly discuss at the end of Section 4 how that model fits into the scheme presented here.
Another interesting example is concerned with the superselection structure of Bose sectors of the gauge invariant part of the algebra of a free massive Fermi field on the two-dimensional space-time. Those sectors are usually described by a tensor category of localized automorphisms with a trivial Bose symmetry. The breakdown of Haag duality in this model, however, allows for different descriptions: for each real parameter $\lambda$, there is a tensor category describing the same sectors, where the intertwiners do not belong to the algebra, but fulfil the asymptotic Abelianness condition of Section 2. That category is equipped with a braiding which is not a symmetry, unless $\lambda = 0$, and arises from a bi-asymptopia as described in Section 3. For $\lambda = 0$, one gets back the usual symmetric tensor category of localized automorphisms [18].

2 Asymptotically Abelian Intertwiners

One of the very first steps in the theory of superselection sectors is to show how a tensor $C^*$-category may be obtained by passing from a $C^*$-category of representations to a $C^*$-category of endomorphisms. In this step duality plays a fundamental role. The aim here is to describe an alternative mechanism, in suitable mathematical generality and abstraction.

Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of unital $C^*$-algebras and $\Delta$ a semigroup of endomorphisms of $\mathcal{A}$. Given $\rho, \sigma \in \Delta$, we consider the corresponding space $(\rho, \sigma)$ of intertwiners

$$(\rho, \sigma) = \{ R \in \mathcal{B} : R\rho(A) = \sigma(A)R, A \in \mathcal{A} \}$$

and obtain in this way a $C^*$-category $\mathcal{T}$, where the composition of arrows (intertwiners) is denoted by $\circ$.

This framework is supposed to model the situation of a set of representations and their intertwiners. Restricting attention to representations described by endomorphisms does not seem restrictive on the mathematical side. In the case of separable simple $C^*$-algebras $\mathcal{A}$ acting irreducibly on some Hilbert space $\mathcal{H}$, any other irreducible representation can be obtained (up to equivalence) by the action of some automorphism. One can choose that automorphism to be asymptotically inner in the sense that it is the limit of inner automorphisms induced by a continuous one parameter family of unitaries of the same
algebra (cf. [16], and references in there for previous results). If the algebra is also nuclear and not type I, any cyclic representation is obtained in a similar way by asymptotically inner endomorphisms [15]. In the non separable case, customary in quantum field theory, the physically motivated split property [7] implies that $\mathcal{A}$ is generated by a type $I_{\infty}$ funnel. Then all locally normal irreducible representations can also be described (up to equivalence) in one such representation by automorphisms [19].

The intertwiners between the representations considered, however, do not belong to the given $C^*$–algebra in general, so $\text{End}\mathcal{A}$ does not model the category of representations, in contrast to the case of localizable charges in quantum field theory fulfilling duality. If duality fails and $\mathcal{B}$ denotes the $C^*$-algebra generated by the dual net, our category $\mathcal{T}$ models again a category of representations with localizable charges. More generally, whenever the above mentioned theorems apply and $\mathcal{A}$ acts irreducibly on $\mathcal{H}$, we may always take $\mathcal{B} = \mathcal{B}(\mathcal{H})$. We take these facts as motivation for studying our more general structure.

The descriptions of representations given by the above mentioned general mathematical results, however, either ignore or only partially into account the local structure of $\mathcal{A}$ in quantum field theory. Yet certain aspects of locality are needed in order to turn $\mathcal{T}$ into a tensor $C^*$–category. Note that $\mathcal{T}$ does not yet have a tensor structure, for the arrows lie in $\mathcal{B}$ and each $\rho \in \Delta$ is defined only on $\mathcal{A}$.\footnote{\text{$\mathcal{T}$ acquires a tensor structure if the tensor product of endomorphisms is defined by composition and if the tensor product of intertwiners $R \in (\rho, \rho')$ and $S \in (\sigma, \sigma')$ given by $R \times S = R \rho(S)$ is meaningful as an element of $\langle \rho \sigma, \rho' \sigma' \rangle$.}} So what is required is an extension of $\Delta$ to a semigroup of endomorphisms of the $C^*$–algebra $\mathcal{A}_{\Delta} \subset \mathcal{B}$ generated by $\mathcal{A}$ and all the intertwiner spaces $(\rho, \sigma)$ in $\mathcal{B}$.

Two basic ingredients will help us to accomplish this task. First, we assume that any $\rho \in \Delta$ is asymptotically inner in $\mathcal{B}$, i.e. there are unitaries $U_m \in \mathcal{B}$ such that

$$U_m^* A U_m \to \rho(A)$$

(1)

in norm as $m \to \infty$ for each $A \in \mathcal{A}$. Secondly, letting $V_n$ denote the analogues of the $U_m$ for $\sigma$ instead of $\rho$, equation (1) implies that for all $R \in (\rho, \sigma)$, $A \in \mathcal{A}$

$$[V_n R U_m^*, A] \to 0$$
in norm as \( m, n \to \infty \). As we want to extend our morphisms to \( \mathcal{A}_\Delta \), we require that this asymptotic Abelianness holds for intertwiners too, i.e. given \( R \in (\rho, \sigma) \) and \( R' \in (\rho', \sigma') \)

\[
[V_nRU_m^*, R'] \to 0
\]

(2)

in norm as \( m, n \to \infty \). Indeed, condition (2) will allow us to use (1) as a definition of the desired extension of each \( \rho \in \Delta \) to \( \mathcal{A}_\Delta \).

Rather than spell out our assumptions in detail, we will first introduce some standard formalism from the theory of \( C^* \)-algebras enabling us to replace a sequence of unitaries in \( \mathcal{B} \) by a unitary in a larger \( C^* \)-algebra \( \mathcal{B} \). This simplifies what are already simple proofs by eliminating the indices.

Now the bounded sequences in \( \mathcal{B} \) with pointwise operations and norm defined as the supremum of the norms of elements of the sequence form a \( C^* \)-algebra and the subset of sequences that tend to zero in norm is a two-sided ideal in that algebra. \( \mathcal{B} \) is then the quotient \( C^* \)-algebra and, passing from an element of \( \mathcal{B} \) to the corresponding constant sequences, \( \mathcal{B} \) can and will be canonically identified with a \( C^* \)-subalgebra of \( \mathcal{B} \), denoted again by \( \mathcal{B} \).

We will use the following notation: generic elements of \( \mathcal{B} \) will be denoted by bold face letters \( \mathbf{B} \), whereas elements of \( \mathcal{A}, \mathcal{B} \) and of their canonical images in \( \mathcal{B} \) are denoted by \( A, B \). As indicated, the elements \( \mathbf{B} \in \mathcal{B} \) are equivalence classes of bounded sequences modulo sequences that tend to zero. If \( \{B_n\}_{n \in \mathbb{N}} \) represents \( \mathbf{B} \) then \( \| \mathbf{B} \| = \lim\sup_n \|B_n\| \). Obviously any subsequence of a bounded sequence is again bounded and we say that a subset \( S \subset \mathcal{B} \) is stable if it is closed under taking subsequences.

**Lemma 1** A subset of \( \mathcal{B} \) consisting of a single element \( \mathbf{B} \) is stable if and only if \( \mathbf{B} = B \in \mathcal{B} \).

**Proof.** Let \( \{B_n \in \mathcal{B}\}_{n \in \mathbb{N}} \) be a sequence representing \( B \in \mathcal{B} \), then \( B_n - B \to 0 \). Given any subsequence \( \{B_{n(i)}\}_{i \in \mathbb{N}} \), then \( B_{n(i)} - B \to 0 \) and \( \{B_{n(i)}\}_{i \in \mathbb{N}} \) again represents \( B \), so the subset consisting just of \( \mathbf{B} \) is stable. Conversely, if \( \{B_n\}_{n \in \mathbb{N}} \) does not represent an element of \( \mathcal{B} \), then \( \{B_n\}_{n \in \mathbb{N}} \) is not a Cauchy sequence. But then there is an \( \varepsilon > 0 \) and for each \( i \in \mathbb{N} \) an \( n(i), n'(i) \geq i \) with \( \|B_{n(i)} - B_{n'(i)}\| \geq \varepsilon \). Thus the subset consisting just of \( \mathbf{B} \) is not stable. \( \square \)

**Lemma 2** If \( U \) is a unitary from \( \mathcal{B} \) then there is a representing sequence consisting of unitaries.
Proof. If \( \{B_n(i)\}_{i \in \mathbb{N}} \) is a representing sequence for \( U \), then \( B_n^* B_n \) represents \( U^* U = I \) so that \( B_n^* B_n \to 1 \). Similarly \( B_n^* B_n \to 1 \). Thus \( |B_n| \) is invertible for all sufficiently large \( n \) and if we then set \( U_n = B_n |B_n|^{-1}, U_n^* U_n = 1 \) whilst \( U_n^* U_n = B_n |B_n|^{-2} B_n^* \) is a projection tending in norm to 1. Consequently, \( U_n \) is unitary for all sufficiently large \( n \) and, setting \( U_n = 1 \) for other values of \( n \) to make it unitary, the result follows. □

Now if \( \mathcal{U}(\mathcal{B}) \) denotes the set of unitaries in \( \mathcal{B} \), and \( U \in \mathcal{U}(\mathcal{B}) \), then
\[
\tau(A) \triangleq U^* A U, \quad A \in \mathcal{A}
\]
is a morphism of \( \mathcal{A} \) into \( \mathcal{B} \) and we set
\[
\mathcal{U}(\mathcal{B})_\tau \triangleq \{ U \in \mathcal{U}(\mathcal{B}) : \text{Ad} U^* \upharpoonright A = \tau \}.
\]

Lemma 3 \( \tau(A) \subset \mathcal{B} \) if and only if \( \mathcal{U}(\mathcal{B})_\tau \) is stable.

Proof. If \( U \in \mathcal{U}(\mathcal{B})_\tau \) and \( \mathcal{U}(\mathcal{B})_\tau \) is stable then \( U^* A U \) is stable and hence, by Lemma 1, an element of \( \mathcal{B} \) for each \( A \in \mathcal{A} \). Conversely if \( \tau(A) \subset \mathcal{B} \) then \( \mathcal{U}(\mathcal{B})_\tau \) is stable from the definition. □

Note that if we consider the \( C^* \)-category \( \mathcal{C} \) whose objects are the morphisms of \( \mathcal{A} \) into \( \mathcal{B} \) inner in \( \mathcal{B} \) and arrows their intertwiners then all objects are obviously unitarily equivalent since \( U \in \mathcal{U}(\mathcal{B})_\tau \) is a unitary in \( \mathcal{C} \) from \( \tau \) to the identity automorphism \( \iota \). Now \( \mathcal{A}' \cap \mathcal{B} \) is the set of arrows from \( \iota \) to \( \iota \) in \( \mathcal{C} \) so if \( U \in \mathcal{U}_\tau, V \in \mathcal{U}_\upsilon \) then \( R \in (\tau, \upsilon) \) if and only if \( V RU^* \in \mathcal{A}' \cap \mathcal{B} \).

Obviously if \( \rho(\mathcal{A}) \subset \mathcal{A} \) so that we are dealing with endomorphisms of \( \mathcal{A} \) into \( \mathcal{B} \) and arrows their intertwiners then all objects are obviously unitarily equivalent since \( U \in \mathcal{U}(\mathcal{B})_\tau \) is a unitary in \( \mathcal{C} \) from \( \tau \) to the identity automorphism \( \iota \). Now \( \mathcal{A}' \cap \mathcal{B} \) is the set of arrows from \( \iota \) to \( \iota \) in \( \mathcal{C} \) so if \( U \in \mathcal{U}_\tau, V \in \mathcal{U}_\upsilon \) then \( R \in (\tau, \upsilon) \) if and only if \( V RU^* \in \mathcal{A}' \cap \mathcal{B} \).

Now by Lemma 2, saying that an endomorphism of \( \mathcal{A} \) is unitarily implemented in \( \mathcal{B} \) is the same as saying that it is asymptotically inner in \( \mathcal{B} \). So the set of such endomorphisms of \( \mathcal{A} \) obviously forms a semigroup. We begin here by studying a subsemigroup \( \Delta \) and the associated \( C^* \)-category \( \mathcal{T} \) of intertwiners in \( \mathcal{B} \). Just as we defined \( \mathcal{B} \) from \( \mathcal{B} \) so we can define a \( C^* \)-category \( \mathcal{T} \) by taking bounded sequences of arrows from \( \mathcal{T} \) with pointwise composition \( \circ \) and the supremum norm and quotient by the ideal of sequences of arrows that tend to zero in norm. The proof of Lemma 2 shows that a unitary arrow of \( \mathcal{T} \) has a representing sequence consisting of unitaries from \( \mathcal{T} \). \( \mathcal{T} \) can again be considered as a \( C^* \)-subcategory of \( \mathcal{T} \) in a natural way.

We now want to extend our semigroup of endomorphisms to the \( C^* \)-algebra \( \mathcal{A}_\Delta \subset \mathcal{B} \) generated by \( \mathcal{A} \) and the intertwiners \( \mathcal{I} \subset \mathcal{B} \) so
that the extended endomorphisms remain asymptotically inner in $\mathcal{B}$. The above discussion indicates what hypotheses will be necessary. We assume that for each $\rho \in \Delta$, we are given a stable subset $U_\rho \subset U(\mathcal{B})_\rho$ with $U_\rho \cap \mathcal{I} \neq \emptyset$, where $\mathcal{I}$ here refers to the union of its Hom-sets. Thus in each $U_\rho$ there is at least one sequence of intertwiners between morphisms belonging to $\Delta$ and

$$U_\rho U_\sigma \cap U_{\sigma \rho} \neq \emptyset.$$ 

Furthermore, we suppose that asymptotic Abelianness holds in the sense of Equation (2), i.e. given $R \in (\rho, \sigma) \subset \mathcal{B}$ and $R' \in (\rho', \sigma') \subset \mathcal{B}$ and $U \in U_\rho$, $V \in U_\sigma$,

$$[VRU^*, R'] = 0.$$ 

Such a coherent assignment $U : \rho \mapsto U_\rho$ for $\rho \in \Delta$ will be called an asymptopia for $\Delta$. Note that there are in general several such asymptopias.

**Theorem 4** Let $\mathcal{A}_\Delta$ denote the $C^*$–subalgebra of $\mathcal{B}$ generated by $\mathcal{A}$ and $\mathcal{I}$, and let $U$ be some asymptopia for $\Delta$. Then every $\rho \in \Delta$ has a unique extension $\rho_U$ to an endomorphism of $\mathcal{A}_\Delta$ such that

$$\rho_U(A) = U^*AU, \ A \in \mathcal{A}_\Delta \ U \in U_\rho.$$ 

Furthermore $(\rho \sigma)_U = \rho_U \sigma_U$ and $(\rho, \sigma) = (\rho_U, \sigma_U)$. Thus $\mathcal{I}$ inherits the structure of a tensor $C^*$–category from $\text{End}\mathcal{A}_\Delta$.

We first prove the following lemma.

**Lemma 5** If $\rho, \sigma, \tau \in \Delta$ and $S \in (\sigma, \tau)$ then $\rho_U(S) \equiv U^*SU$ is independent of the choice of $U \in U_\rho$ and is an element of $\mathcal{A}_\Delta$.

**Proof.** If $U, U' \in U_\rho$,

$$U^*SU - U'^*SU' = U^*(SUU'^* - UU'^*S)U' = U^*[S, U1_\rho U'^*]U' = 0$$

as required. In particular, the subset consisting of the single element $U^*SU$ is stable since $U_\rho$ is stable. Thus by Lemma 1, $\rho_U(S) \in \mathcal{B}$ and since we can choose $U \in \mathcal{I}$, $\rho_U(S) \in \mathcal{A}_\Delta$. \hfill $\Box$

**Proof of Theorem 4.** Lemma 5 shows that each $\rho \in \Delta$ has the required unique extension to an endomorphism $\rho_U$ of $\mathcal{A}_\Delta$. The condition of asymptotic Abelianness shows that we do not lose any intertwiners.
in this way, i.e. that $(\rho, \sigma) = (\rho_U, \sigma_U)$. Since $VU \in U_{\rho_\sigma}$ for some $U \in U_\rho$ and $V \in U_\sigma$, $(\rho_\sigma)_U = \rho_U\sigma_U$. Thus $\mathcal{T}$ can be identified with a full tensor $C^*$–subcategory of $\text{End}\,\mathcal{A}_\Delta$ completing the proof of the theorem. \hfill $\square$

For a given semigroup $\Delta$ of endomorphisms, the $C^*$–category $\mathcal{T}$ is determined by the inclusion $\mathcal{A} \subset \mathcal{B}$. Its tensor structure is determined by the choice of asymptopia $\mathcal{U}$. Different asymptopias can lead to different tensor structures, in other words to different extensions of $\Delta$ to $\mathcal{A}_\Delta$, unless they are included in a common asymptopia. $\rho \mapsto U_{\rho_\mathcal{U}}$, the set of all unitaries in $\mathcal{B}$ inducing $\rho_\mathcal{U}$ on $\mathcal{A}_\Delta$, is obviously a maximal asymptopia containing the given asymptopia. Thus two asymptopias lead to the same tensor structure if and only if there is an asymptopia containing both. If we consider the set of asymptopias ordered under inclusion, the different tensor structures correspond to the different path–components of this set.

**Theorem 6** Given an inclusion of unital $C^*$–algebras $\mathcal{A} \subset \mathcal{B}$ and a semigroup $\Delta$ of endomorphisms of $\mathcal{A}$, the path–components of the set of asymptopias for $\Delta$ are in natural 1–1 correspondence with the set of maximal asymptopias and with the set of extensions of $\Delta$ to a semigroup of asymptotically inner endomorphisms of the $C^*$–algebra $\mathcal{A}_\Delta$ generated by $\mathcal{A}$ and the intertwiners for $\Delta$ in $\mathcal{B}$.

It is of interest that the above framework for studying semigroups of endomorphisms can be extended so as to cover the case of representations and their intertwiners. One then deals with a set $\Delta$ of morphisms of $\mathcal{A}$ into $\mathcal{B}$ and the associated $C^*$–category of intertwiners. We suppose that for each $\rho \in \Delta$, we are given a set $U_\rho$ of unitary operators in $\mathcal{B}$ with $U_\rho \cap \mathcal{T} \neq \emptyset$ such that

$$\rho(A) = U^*AU$$

for each $A \in \mathcal{A}$, $U \in U_\rho$ and $\rho \in \Delta$. In particular, the inclusion mapping $\iota$ of $\mathcal{A}$ into $\mathcal{B}$ is in $\Delta$ with $U_\iota$ consisting just of the identity of $\mathcal{B}$. Furthermore, we require that for each $\rho, \sigma \in \Delta$, there is a unique $\tau \in \Delta$ such that given $U \in U_\rho$ there is a $V \in U_\sigma$ with $VU \in U_\tau$. As before, $R \in (\rho, \sigma)$ if and only if $VRU^* \in \mathcal{B} \cap \mathcal{A}'$. The intertwiners are again supposed to be asymptotically Abelian. Under these circumstances we have the following generalization of Theorem 4.

**Theorem 7** Let $\mathcal{A}_\Delta$ denote the $C^*$–subalgebra of $\mathcal{B}$ generated by $\mathcal{A}$ and $\mathcal{T}$, then every $\rho \in \Delta$ has a unique extension $\rho_\mathcal{U}$ to an endomor-
phism of \( \mathcal{A}_\Delta \) such that

\[
\rho_U(A) = U^*A U, \quad A \in \mathcal{A}_\Delta, \quad U \in \mathcal{U}_\rho.
\]

Furthermore \((\rho, \sigma) = (\rho_U, \sigma_U)\) and the set \(\Delta_U\) of the extended morphisms constitutes a unital semigroup of endomorphisms of \(\mathcal{A}_\Delta\). Thus \(\mathcal{T}\) inherits the structure of a tensor \(C^*\)-category from \(\text{End}\mathcal{A}_\Delta\).

**Proof.** Note that Lemma 5 retains its validity in this new context and that \(\rho(A) \subset \mathcal{A}_\Delta\) since \(\mathcal{U}_\rho \cap \mathcal{T} \neq \emptyset\). Thus \(\rho_U(A) = U^*A U \subset \mathcal{A}_\Delta\).

Now let \(\rho, \sigma \in \Delta\). Choose \(\tau \in \Delta\), \(U \in \mathcal{U}_\rho\) and \(V \in \mathcal{V}_\sigma\) such that \(V U \in \mathcal{U}_\tau\). Since \(\text{Ad}U^*V^* = \text{Ad}U^*\text{Ad}V^*\), we conclude that \(\rho_U \sigma_U = \tau_U\). Thus \(\Delta_U\) is a semigroup with unit. Given \(T \in (\rho, \sigma)\) then the set of \(B \in \mathcal{B}\) such that

\[
TU^*BU = V^*BV T,
\]

\[
TU^*B^*U = V^*B^*V T,
\]

is a \(C^*\)-subalgebra of \(\mathcal{B}\) containing \(\mathcal{A}\). It also contains every arrow of \(\mathcal{T}\) since intertwiners are asymptotically Abelian. This shows that \((\rho, \sigma) = (\rho_U, \sigma_U)\). \qed

We conclude with the remark that all results of this section can be generalized in the following sense. Instead of considering an inclusion of unital \(C^*\)-algebras, we can consider a fixed unital \(C^*\)-algebra \(\mathcal{L}\), say, and a set \(\Delta\) of morphisms between unital \(C^*\)-subalgebras, closed under composition. Given \(\rho, \sigma \in \Delta\), between two such subalgebras \(\mathcal{A}\) and \(\mathcal{B}\), we set

\[
(\rho, \sigma) = \{R \in \mathcal{L} : R \rho(A) = \sigma(A)R, \ A \in \mathcal{A}\},
\]

and obtain in this way a \(C^*\)-category \(\mathcal{T}\). In the presence of an asymptopia \(\mathcal{U}\) for \(\Delta\), the morphism \(\rho\) has a unique extension to a morphism \(\rho_U\) between \(\mathcal{A}_\Delta\) and \(\mathcal{B}_\Delta\), the \(C^*\)-subalgebras of \(\mathcal{L}\) generated by \(\mathcal{A}\) and \(\mathcal{T}\) and \(\mathcal{B}\) and \(\mathcal{T}\), respectively. In this way, \(\mathcal{T}\) inherits the structure of a \(2-C^*\)-category from the \(2-C^*\)-category of morphisms and intertwiners between unital \(C^*\)-subalgebras of \(\mathcal{L}\), cf. [14]. The reader should have no difficulty in enunciating and proving the analogue of the results of this section.

The above generalization may prove relevant to the theory of superselection sectors first because it may prove advantageous to restrict a representation to a subnet of the observable net and secondly because it may not be possible to get an endomorphism of the subnet by passing to an equivalent representation.
3 The Emergence of Braiding

In view of the results in the preceding section, we can now forget about the inclusion \( A \subset B \), work with a single unital \( C^* \)-algebra \( A \) (corresponding to the previous algebra \( A_\Delta \)) and suppose that we are given a tensor \( C^* \)-category \( \mathcal{T} \) with \( \Delta \) as its set of objects realized as a full tensor subcategory of \( \text{End} A \). How it was obtained, in particular, which asymptopia, if any, was used to construct it, is for the moment quite irrelevant.

We also note that the \( C^* \)-category \( \mathcal{T} \) now becomes a tensor \( C^* \)-category in a natural way. The tensor product of objects is just the pointwise product of the individual sequences of endomorphisms whilst if arrows \( R \) and \( S \) are represented by sequences \( R_n \in (\rho_n, \rho'_n) \) and \( S_n \in (\sigma_n, \sigma'_n) \), their tensor product \( R \times S \) is represented by \( R_n\rho_n(S_n) \in (\rho_n\sigma_n, \rho'_n\sigma'_n), n \in \mathbb{N} \). We further note that a sequence of endomorphisms \( \rho_n \) of \( A \) defines in a canonical way a unital morphism \( \rho \) of \( A \) into \( A \), two sequences \( \rho_n \) and \( \sigma_n \) defining the same morphism if and only if \( \rho_n(A) - \sigma_n(A) \to 0 \) for every \( A \in \mathcal{A} \). Adjoining intertwiners, we get a \( C^* \)-category \( \text{Mor}(A, A) \). There is now an obvious canonical \( * \)-functor \( F \) from \( \mathcal{T} \) to \( \text{Mor}(A, A) \) mapping a sequence of endomorphisms of \( A \) onto the induced morphism from \( A \) to \( A \) and acting as the identity on intertwiners. The effect of \( F \) is to identify sequences of endomorphisms with the same asymptotic behaviour.

The task in this section is to show how to get a braiding and to develop criteria for deciding whether the braiding is a symmetry. The idea here is to define the braiding using suitable norm convergent sequences of unitaries and we will again realize these sequences as unitary operators in \( \mathcal{A} \). However, the conditions we need are now somewhat different. In particular, as a braiding is a function of two objects, we assume that there are for each object \( \rho \in \Delta \) two stable sets of unitaries \( \mathcal{U}_\rho, \mathcal{V}_\rho \) which are contained in \( \mathcal{T} \), each unitary \( W \) implementing \( \rho \), i.e. \( \rho(A) = W^*AW, A \in \mathcal{A} \). This means that \( F(W) \) is an intertwiner in \( \mathcal{A} \) from \( \rho \) to the identity automorphism \( \iota \). Furthermore, we require the following notion of asymptotic Abelianness, where we work in the category \( \text{Mor}(\mathcal{A}, \mathcal{A}) \): given two intertwiners \( R \in (\rho, \rho') \) and \( S \in (\sigma, \sigma') \) of \( \mathcal{T} \) and \( U \in \mathcal{U}_\rho, U' \in \mathcal{U}_{\rho'}, V \in \mathcal{V}_\sigma \) and \( V' \in \mathcal{V}_{\sigma'} \) then
\[
F(U'R^* \times V'S^*) - F(V'SV^* \times U'R^*) = 0.
\]
This form of asymptotic Abelianness of the tensor product of arrows...
will allow us to define the universal rule for interchanging tensor products, in very much the same way as condition (2) allowed us to define the tensor product of arrows.

For that purpose this data should, in addition, be compatible with products in the sense that given \( \rho, \rho' \in \Delta \), we can find \( U \in U_{\rho} \) and \( U' \in U_{\rho'} \) such that \( U \times U' \in U_{\rho \rho'} \) and analogously for the second set of unitaries \( \{ V_{\rho} \} \). Any assignment \( (U, V) : \rho \mapsto U_{\rho}, V_{\rho} \) with these properties will be referred to as a bi-asymptopia for \( \Delta \).

**Theorem 8** Let \( (U, V) \) be a bi-asymptopia for \( \Delta \). Given \( \rho, \sigma \in \Delta \), then

\[
\varepsilon(\rho, \sigma) = F(V^* \times U^*) \circ F(U \times V)
\]

is independent of the choice of \( U \in U_{\rho} \) and \( V \in V_{\sigma} \) and is in \( (\rho \sigma, \sigma \rho) \). Furthermore, if \( R \in (\rho, \rho') \) and \( S \in (\sigma, \sigma') \) then

\[
\varepsilon(\rho', \sigma') \circ R \times S = S \times R \circ \varepsilon(\rho, \sigma)
\]

and if \( \tau \in \Delta \) then

\[
\varepsilon(\rho \sigma, \tau) = \varepsilon(\rho, \tau) \times 1_{\sigma} \circ 1_{\rho} \times \varepsilon(\sigma, \tau),
\]

\[
\varepsilon(\rho, \sigma \tau) = 1_{\sigma} \times \varepsilon(\rho, \tau) \circ \varepsilon(\rho, \sigma) \times 1_{\tau}.
\]

In other words, \( \varepsilon \) is a braiding for the full subcategory of \( \text{End} A \) generated by \( \Delta \).

**Proof.**

\[
F(V^* \times U^*) \circ F(U \times V) = F(V'^* \times U'^*) \circ F(U' \times V')
\]

\[
= F(V^* \times U^*) \circ (F((U \circ U'^*) \times (V \circ V'^*))
\]

\[
- F((V \circ V'^*) \times (U \circ U'^*))) \circ F(U' \times V') = 0
\]

proving independence of the choice of \( U \in U_{\rho} \) and \( V \in V_{\sigma} \). But then \( F(V^* \times U^*) \circ F(U \times V) \) is stable so that \( \varepsilon(\rho, \sigma) \in (\rho \sigma, \sigma \rho) \). Furthermore,

\[
F(V'^* \times U'^*) \circ F(U' \times V') \circ R \times S
\]

\[
- S \times R \circ F(V^* \times U^*) \circ F(U \times V)
\]

\[
= F(V'^* \times U'^*) \circ (F(U'RU^* \times V'SV^*)
\]

\[
- F(V'SV^* \times U'R \U^*)) \circ F(U \times V) = 0,
\]

and we deduce that

\[
\varepsilon(\rho', \sigma') \circ R \times S = S \times R \circ \varepsilon(\rho, \sigma).
\]
Now, pick $U \in \mathcal{U}_\rho$ and $V \in \mathcal{U}_\sigma$ such that $U \times V \in \mathcal{U}_{\rho\sigma}$. Then, if $W \in \mathcal{V}_\tau$,

$$F(W^* \times U^* \times V^*) \circ F(U \times V \times W) = F(W^* \times U^* \times 1_\sigma) \circ F(U \times W \times 1_\sigma) \circ F(1_\rho \times W^* \times V^*) \circ F(1_\rho \times U \times W),$$

and we conclude that

$$\varepsilon(\rho\sigma, \tau) = \varepsilon(\rho, \tau) \times 1_\sigma \circ 1_\rho \times \varepsilon(\sigma, \tau).$$

Now,

$$\varepsilon^{-1}(\rho, \sigma) = \varepsilon(\sigma, \rho)^{-1} = F(U^* \times V^*) \circ F(V \times U)$$

and as we have here just interchanged the roles of $\mathcal{U}$ and $\mathcal{V}$, we deduce that

$$\varepsilon(\rho, \sigma\tau) = 1_\sigma \times \varepsilon(\rho, \tau) \circ \varepsilon(\rho, \sigma) \times 1_\tau$$

as stated. $\square$

There still remains the question of whether $\varepsilon$ is in fact a symmetry, i.e. whether $\varepsilon = \varepsilon^{-1}$. But in the above proof, we have just seen that we pass from $\varepsilon$ to $\varepsilon^{-1}$ by interchanging the roles of $\mathcal{U}$ and $\mathcal{V}$. Hence one way of tackling the problem is to ask when two bi-asymptopias give rise to the same braiding and here we may follow our discussion in the case of asymptopias.

Two bi-asymptopias will obviously give rise to the same braiding if one is included in the other, or if their intersection is still a bi-asymptopia; thus if we order the set of bi-asymptopias under inclusion, we can divide that set into path-components, and the braiding associated to a bi-asymptopia will depend only on its path-component.

A difference from the case of asymptopias is worth noting here: whilst $\rho \mapsto \mathcal{U}_{\rho_{ui}}$ defines a maximal asymptopia, where in particular equation (2) is automatically satisfied, the notion of asymptotic Abelianness entering in the definition of bi-asymptopias requires a further mutual (asymptotic) commutativity, equation (3). Hence there is no a priori unique maximal bi-asymptopia containing a given one, just as in general there is no unique maximal Abelian subalgebra containing a given Abelian subalgebra. Thus in this case we cannot say a priori that different braiding correspond to different path components of asymptopias. But by the previous discussion we have

**Theorem 9** The bi–asymptopia $\{\mathcal{U}, \mathcal{V}\}$ gives rise to a symmetry if it lies in the same path–component as $\{\mathcal{V}, \mathcal{U}\}$. 

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We close with a few comments. First, when does a braiding arise from a bi-asymptopia? Note that if $\varepsilon$ is a braiding for $\mathcal{T}$, then this braiding extends in an obvious way to a braiding for $\mathcal{T}$. Writing $\rho$ to denote an object of $\mathcal{T}$, i.e. a sequence of elements of $\Delta$, $U \in (\rho, \rho)$ and $V \in (\sigma, \sigma)$, then

$$V \times U \circ \varepsilon(\rho, \sigma) = \varepsilon(\rho, \sigma) \circ U \times V.$$ 

So if we give ourselves a braiding $\varepsilon$ for $\mathcal{T}$ then

$$\varepsilon(\rho, \sigma) = F(V^* \times U^* \circ U \times V)$$

if and only if $F(\varepsilon(\rho, \sigma)) = 1_i$. Next,

$$\varepsilon(\rho', \sigma') \circ (U'RU^*) \times (V'SV^*) = (V'SV^*) \times (U'RU^*) \circ \varepsilon(\rho, \sigma).$$

Finally, $F(\rho) = F(\rho') = F(\sigma) = I$ and $F(\varepsilon(\rho, \sigma)) = F(\varepsilon(\rho', \sigma)) = 1_i$ implies $F(\rho \rho') = I$ and $F(\varepsilon(\rho \rho', \sigma)) = 1_i$. So we conclude

**Theorem 10** Let $\varepsilon$ be a braiding for $\mathcal{T}$ and suppose given two mappings $\mathcal{U}, \mathcal{V}$ from the morphisms $\rho$ into stable subsets of $\mathcal{U}(\mathcal{B})_{\rho}$ such that each $\mathcal{U}_\rho$ and each $\mathcal{V}_\sigma$ are non-empty. If, given any pair $U \in (\rho, \rho)$ from $\mathcal{U}_\rho$ and $V \in (\sigma, \sigma)$ from $\mathcal{V}_\sigma$, $F(\rho) = F(\sigma) = I$, $F(\varepsilon(\rho, \sigma)) = 1_i$, then $\mathcal{U}, \mathcal{V}$ can be extended to a bi-asymptopia $\{\mathcal{U}, \mathcal{V}\}$ giving $\varepsilon$. Furthermore, we can even take $\mathcal{U}$ and $\mathcal{V}$ to be closed under composition on the right by unitary intertwiners and stable under tensor products.

As a final comment, we note that our notion of asymptotic Abelian-ness of the $\times$–product implies the corresponding notion for the operator product. In fact, $F(U'R U^*) = F(U'R U^* \times 1_{\sigma'})$ and

$$F(V'SV^*) = F(V'SV^* \times 1_{\rho}) = F(V'SV^* \times U_{1_{\rho}}U^*).$$

Hence

$$F(U'R U^*) \circ F(V'SV^*) = F(U'R U^* \times 1_{\sigma'}) \circ F(U_{1_{\rho}}U^* \times V'SV^*)$$

when we have a bi-asymptopia. Thus

$$F(U'R U^*) \circ F(V'SV^*) = F(U'R U^* \times V'SV^*)$$

and

$$F(U'R U^*) \circ F(V'SV^*) = F(V'SV^*) \circ F(U'R U^*)$$

as stated.
4 Algebraic Quantum Field Theory

We briefly outline the relations of the preceding general analysis to algebraic quantum field theory, giving in particular some examples of asymptopias.

To begin with, we consider the semigroup of localized morphisms $\rho$ of an observable net $\mathcal{A}$ on Minkowski space which are transportable as representations on the vacuum Hilbert space of the theory. The $C^*$-algebra $\mathcal{A}$ of Section 2 can then be thought of as the $C^*$-inductive limit of the net $\mathcal{A}(O)$ of local algebras associated with double cones $O$, $\mathcal{A} = \bigcup_{O} \mathcal{A}(O)$. The intertwiners in the sense of representations belong to the dual net $\mathcal{A}^d$, $\mathcal{A}^d(O) = \bigcap_{O_1 \subset O} \mathcal{A}(O_1)'$, where as usual $O'$ denotes the spacelike complement of the double cone $O$. The role of $\mathcal{B}$ is played by the $C^*$-algebra generated by the net $\mathcal{A}^d$, $\mathcal{B} = \bigcup_{O} \mathcal{A}^d(O)$. We know that if $U_m \in (\rho, \rho_m)$ is unitary and $\rho_m$ is localized in $O_m$ then

$$\rho(A) = U_m^* A U_m, \quad A \in \mathcal{A}(O), \quad O_m \subset O'.$$

Hence our endomorphisms are asymptotically inner in $\mathcal{B}$. If we assume Haag duality we have $\mathcal{A} = \mathcal{B}$, so we do not need to extend our endomorphisms and $\rho \mapsto U(\mathcal{B})_\rho$ is the unique maximal asymptopia.

If we just assume essential duality, then in space–time dimension $d > 2$, we know that if $R_i \in (\rho_i, \sigma_i)$, $i = 1, 2$ then $R_1 R_2 = R_2 R_1$ if $\rho_1$ and $\sigma_1$ are localized spacelike to $R_2$. Thus we can define an asymptopia $U$ by taking $U_\rho$ to consist of all unitaries of $U(\mathcal{B})$ with representing sequences $U_m \in (\rho, \rho_m)$, the $\rho_m$ being localized in double cones $O_m$ tending spacelike to infinity. In this case, $\mathcal{A}_\Delta = \mathcal{B}$ and our Theorem 2 is just a variant of a known result cf. [17, §3.4.6].

The reader’s attention is also drawn to the case of charges localizable in spacelike cones [5] where the intertwiners likewise do not lie in the observable algebra and where the treatment in [9] has aspects in common with the present paper, cf. Lemma 5.5 of [9].

In space–time dimension $d = 2$, the spacelike complement of a double cone has two path–components, a spacelike left and a spacelike right. As far as the commutation properties of intertwiners $R_i \in (\rho_i, \sigma_i)$ go, we merely know then that $R_1 R_2 = R_2 R_1$ if $\rho_1$ and $\sigma_1$ are both localized left spacelike to $R_2$ or both localized right spacelike.
to $R_2$. Thus we can define two asymptopias $U^\ell$ and $U^r$ as above by letting $O_m$ tend spacelike to left infinity or right infinity, respectively. The restricted commutation properties of intertwiners show that these two asymptopias lead to different tensor structures, in general. The vacuum representation of the observable net then induces a solitonic representation of the corresponding field net.

In replacing $\mathfrak{A}$ by the algebra $\mathcal{A}$, we are, on the one hand, simplifying the mathematical setting by suppressing the net structure and avoiding all reference to spacetime. But we also have in mind applications where the endomorphisms are no longer strictly localized but only asymptotically inner.

For an example going beyond the standard setting of strictly local or cone–localized charges, we turn to the model expounded in [3] and based on the free massless scalar field. $\mathcal{A}$ is here the $C^*$–algebra generated by Weyl operators $W(f), f \in \mathcal{L},$

$$\mathcal{L} \doteq \omega^{-\frac{1}{2}}D(\mathbb{R}^3) + i\omega^{\frac{1}{2}}D(\mathbb{R}^3).$$

Here $D(\mathbb{R}^3)$ denotes the space of smooth real-valued functions with compact support and $\omega$ the energy operator. $\mathcal{L}$ is equipped with the scalar product

$$(f, f') \doteq \int d^3x \overline{f(x)} f'(x),$$

determining the symplectic form

$$\sigma(f, f') = -\Im(f, f')$$

and the usual vacuum state:

$$\omega(W(f)) = e^{-\frac{1}{4}(f,f)}.$$

The $C^*$–algebra $\mathcal{B}$ can be taken to be the algebra of all bounded operators on the vacuum Hilbert space. For $\Delta$, we take the group $\Gamma$ of automorphisms of $\mathcal{A}$ generated by the space

$$\mathcal{L}_\Gamma \doteq \omega^{-\frac{1}{2}}D(\mathbb{R}^3) + i\omega^{\frac{1}{2}}D(\mathbb{R}^3).$$

For convenience, we use the same symbol $\gamma$ to denote the element of $\mathcal{L}_\Gamma$, parametrized by smooth functions $g$ and $h$,

$$\gamma = i\omega^{-\frac{1}{2}}g + \omega^{-\frac{1}{2}}h, \quad g, h \in D(\mathbb{R}^3),$$
and the automorphism it generates so that
\[ \gamma(W(f)) = e^{i \sigma(\gamma,f)}W(f), \]
where the symplectic form \( \sigma \) is defined on \( \mathcal{L}_\Gamma \) so as to extend that on the subspace \( \mathcal{L} \) by
\[ \sigma(\gamma, \gamma') = \int d^3p \omega^{-2} \left( \tilde{g}(-p)\tilde{h}'(p) - \tilde{g}'(-p)\tilde{h}(p) \right). \]

The sectors are characterized by the charge
\[ \int d^3x g(x). \]
They are translation invariant and if \( \gamma_a \) denotes the translate of \( \gamma \) by \( a \), we have unitary intertwiners \( U_a \in (\gamma, \gamma_a) \) unique up to a phase. We define \( \mathcal{U}_\gamma \) to be the set of equivalence classes of sequences of unitaries \( U_a \in (\gamma, \gamma_a) \) for which \( a \) tends spacelike to infinity and \( a_0/|a| \to 0 \). Then
\[ U_a^* W(f) U_a = e^{i \sigma(\gamma-\gamma_a,f)}W(f) \to \gamma(W(f)), \]
as follows from the asymptotic behaviour of the symplectic form, Theorem 3 of [3]. Obviously, \( \mathcal{U}_\gamma \mathcal{U}_\delta = \mathcal{U}_{\delta\gamma} \). Hence to show that the assignment \( \gamma \mapsto \mathcal{U}_\gamma \) defines an asymptopia, it suffices to check that the intertwiners are asymptotically Abelian. Now if \( (\gamma, \delta) \neq 0 \), it consists of multiples of \( W(\delta - \gamma) \). Hence the intertwiners are asymptotically Abelian if
\[ \lim_{a,b} [W(\delta_b - \gamma_a), W(\delta'_b - \gamma'_a)] = 0, \]
whenever \( \gamma \) and \( \delta \) are equivalent and \( \gamma' \) and \( \delta' \) are equivalent. The norm of this commutator of Weyl operators is
\[ |e^{i \sigma(\delta_b - \gamma_a, \delta'_b - \gamma'_a)} - 1| \]
and asymptotic Abelianness follows from Theorem 3 of [3].

In order to define a bi-asymptopia, we may take \( \mathcal{U}_\gamma \) to consist of representing sequences \( U_a \in (\gamma, \gamma_a) \), where \( a \) tends to spacelike infinity inside some (open) spacelike cone \( S \), and define \( \mathcal{V}_\gamma \) similarly using the spacelike cone \( -S \). In view of our previous computations, we need only verify the condition of asymptotic Abelianness. Since every non-zero element of \( (\gamma, \gamma') \) is a multiple of \( W(\gamma' - \gamma) \), it suffices to check that
\[ W(\gamma'_{a'} - \gamma_a) \times W(\delta'_{b'} - \delta_b) - W(\delta'_{b'} - \delta_a) \times W(\gamma'_{a'} - \gamma_a) \]
tends to zero as \( a, a', -b \) and \(-b'\) go spacelike to infinity in \( S \). But the norm of this expression is

\[
|e^{i\sigma(\gamma', \delta')}e^{i\sigma(\delta, \gamma)} - 1|
\]

and tends to zero as required by Theorem 3 of [3].

It is easy to see by direct computation that the braiding determined by this bi-asymptopia is given by

\[
\varepsilon(\gamma, \delta) = e^{-i\sigma(\gamma, \delta)}
\]

cf. Sect. 5 of [3], and is therefore a symmetry, but it is instructive to derive this from Theorem 9. Obviously, if we replace the spacelike cone \( S \) in the definition of the bi-asymptopia by a smaller spacelike cone \( S_1 \) we remain in the same path–component. The same is therefore true if \( S \cap S_1 \neq \emptyset \). By a sequence of such moves, we may interchange \( S \) and \(-S\) so that by Theorem 9 our braiding is a symmetry.

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