WRONSKIANS OF FOURIER AND LAPLACE TRANSFORMS

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Abstract. Associated with a given suitable function, or a measure, on $\mathbb{R}$, we introduce a correlation function, so that the Wronskian of the Fourier transform of the function is the Fourier transform of the corresponding correlation function, and the same holds for the Laplace transform. We obtain two types of results. First, we show that Wronskians of the Fourier transform of a nonnegative function on $\mathbb{R}$ are positive definite functions and the Wronskians of the Laplace transform of a nonnegative function on $\mathbb{R}^+$ are completely monotone functions. Then we establish necessary and sufficient conditions in order that a real entire function, defined as a Fourier transform of a positive kernel $K$, belongs to the Laguerre-Pólya class, which answers an old question of Pólya. The characterization is given in terms of a density property of the correlation kernel related to $K$, via classical results of Laguerre and Jensen and employing Wiener’s $L^1$ Tauberian theorem. As a consequence we provide a necessary and sufficient condition for the Riemann hypothesis in terms of a density of the translations of the correlation function related to the Riemann $\xi$-function.

1. Introduction

A real entire function $\varphi$ is in the Laguerre-Pólya class, written $\varphi \in \mathcal{LP}$, if

$$\varphi(z) = cz^m e^{-\alpha z^2 + \beta z} \prod_{k=1}^{\infty} (1 + z/x_k) e^{-z/x_k}$$

for some $c, \beta \in \mathbb{R}$, $\alpha > 0$, $m \in \mathbb{N}_0$ and $x_k \in \mathbb{R} \setminus \{0\}$, such that $\sum_k x_k^{-2} < \infty$. The class $\mathcal{LP}$ consists of entire functions that are uniform limits on the compact sets of the complex plane of polynomials with only real zeros. This class of functions was studied first by Laguerre in the ninetieth century and then more extensively by Jensen, Pólya, Schur, Obrechkoff and others in the beginning of the twentieth century because of the efforts towards the Riemann hypothesis. The latter connection is straightforward and we recall it very briefly. The Riemann $\xi$-function is defined in terms of the $\zeta$-function by (see [19])

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

Define also $\Xi(z) = \xi(1/2 + iz)$. The Riemann hypothesis states that $\Xi$, represented also as

$$\Xi(z) = \int_{-\infty}^{\infty} \Phi(u) e^{-izu} du,$$

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with
\[ \Phi(t) = 2 \sum_{n=1}^{\infty} \left( 2n^4 \pi^2 e^{9t/2} - 3n^2 \pi e^{5t/2} \right) \exp(-n^2 \pi e^{2t}), \]
has only real zeros. Since \( \Xi(z) \) is an entire function of order one, the Riemann hypothesis is equivalent to the fact that it belongs to \( \mathcal{LP} \). Attempts to provide general tractable necessary and sufficient conditions for an entire function to be in \( \mathcal{LP} \) had failed, so that in 1926 Pólya \[14\] raised the question of characterizing the kernels \( K \) whose Fourier transforms
\[ \int_{-\infty}^{\infty} K(u)e^{-izu}du \]
belong to \( \mathcal{LP} \). We establish necessary and sufficient condition for Pólya’s problem for a subclass of Fourier transforms that contains \( \Xi(z) \), which is given in terms of a density of a family of functions in \( L^1(\mathbb{R}) \). In order to formulate it, for a given real function \( f \), we denote by \( \mathcal{T}(f) \) the span of its translations (or translates), that is,
\[ \mathcal{T}(f) := \left\{ \sum_{k=1}^{n} c_k f(x + a_k), \ a_k \in \mathbb{R}, \ n \in \mathbb{N} \right\}. \]

Our result implies the following necessary and sufficient condition for the Riemann hypothesis:

**Theorem 1.1.** The Riemann hypothesis is true if and only if, for each \( y \in (-1/2, 1/2) \setminus \{0\} \), the translates \( \mathcal{T}(\Phi_{2,y}) \) of the kernel
\[ \Phi_{2,y}(t) := \cosh(ty) \int_{-\infty}^{\infty} (t - 2s)^2 \Phi(t - s) \Phi(s) ds \]
are dense in \( L^1(\mathbb{R}) \).

Furthermore, the translates \( \mathcal{T}(\Phi_{2,y}) \) of \( \Phi_{2,y}(t) \) are dense in \( L^1(\mathbb{R}) \) for every fixed \( y \in (-1/2, 1/2) \) if and only if the zeros of \( \Xi(z) \) are real and simple.

It is worth mentioning that there are other density criteria for the Riemann hypothesis. We mention the classical Nyeman-Beurling criterion \[3, 13\] and its various generalizations and refinements due to Báez-Duarte and his collaborators \[1, 2\]. Other equivalent sufficient conditions for the Riemann hypothesis in terms of properties of the correlation kernel \( \Phi_{2,y}(t) \) will be stated in the end of Section 3.

The basic ingredients in the proof of Theorem \[1, 11\] are classical results of Jensen and Laguerre about entire functions in the Laguerre-Pólya class, Wiener’s \( L^1 \) Tauberian theorem, known also as The Wiener Approximation Theorem, and a tool that we develop in this paper, called the correlation function associated to a given function or a measure, which we now describe.

Let \( \mathcal{M}(E) \) be the set of Borel measures on \( E \in \mathbb{R} \). For \( \mu \in \mathcal{M}(E) \) and \( m = 0, 1, \ldots \), let \( \mu_m := \int_E t^m d\mu \) be the moment of \( \mu \). Let \( \mathcal{M}_N(E) \) denote the set that consists of \( \mu \in \mathcal{M}(E) \) for which \( \mu_n \) is finite for \( n = 0, 1, \ldots N \). Furthermore, we denote by \( \mathcal{M}^+(E) \) and \( \mathcal{M}_N^+ \) the subset of non-negative Borel measures, respectively.

**Definition 1.2.** Let \( \mu \in \mathcal{M}_{2n-2}(\mathbb{R}) \) be an absolutely continuous measure. For \( n = 2, 3, \ldots \) we define a correlation function
\[ \nu_n(t) := \nu_n(d\mu; t) = \int_{T^n(t)} \prod_{1 \leq i < j \leq n} (s_i - s_j)^2 \mu'(s_1) \ldots \mu'(s_n) ds, \]

where \( T^n(t) \) is the simplex in \( \mathbb{R}^n \) defined by
\[
T^n(t) := \{(s_1, \ldots, s_n) \in \mathbb{R}^n : s_1 + \cdots + s_n = t\}
\]
and \( ds \) is the Lebesgue measure on \( T^n(t) \). When \( d\mu = w(t)dt \), we also write \( \nu_n(w; t) \) and define \( \nu_1(t) := w(t) \).

The correlation function is well-defined and is closely related to the Wronskian determinants of integral transforms. We study this function in view of the Fourier and Laplace transforms below.

For \( \mu \in \mathcal{M}(\mathbb{R}) \), let \( F \) be the Fourier transform of \( \mu \) defined by, with \( i = \sqrt{-1} \),
\[
F \mu(x) := \tilde{\mu}(x) = \int_{\mathbb{R}} e^{-ixt}d\mu(t), \quad x \in \mathbb{R}.
\]
A measure \( \mu \in \mathcal{M}(\mathbb{R}) \) is called even if \( d\mu(t) = d\mu(-t) \). If \( \mu \) is even, then \( F \mu \) is a real valued function. Let \( \mathbb{R}_+ = [0, \infty) \). For \( \mu \in \mathcal{M}(\mathbb{R}_+) \), let \( L \mu \) be the Laplace transform of \( \mu \) defined by
\[
L \mu(x) := \int_{\mathbb{R}} e^{-xt}d\mu(t), \quad x \in \mathbb{R}.
\]

Let \( f \) be a function in \( C^{2n-2}(\mathbb{R}) \), the class of functions that have continuous derivatives of order \( 2n-2 \). The \( n \)-th Wronskian determinant of the function \( f \) is defined by
\[
W_n(f; x) := \det \begin{bmatrix}
 f(x) & f'(x) & \cdots & f^{(n-1)}(x) \\
 f'(x) & f''(x) & \cdots & f^{(n)}(x) \\
 \vdots & \vdots & \ddots & \vdots \\
 f^{(n-1)}(x) & f^{(n)}(x) & \cdots & f^{(2n-2)}(x)
\end{bmatrix}.
\]

The key ingredient, and our starting point, in this study is the observation that the Wronskian of the Fourier (or Laplace) transform of a function is the Fourier (respectively, Laplace) transform of the corresponding correlation function. More precisely, we have the following theorem:

**Theorem 1.3.** For \( n = 2, 3, \ldots \) and \( f \in L^1(E) \) such that \( \int_E |f^{2n-2} f(t)|dt < \infty \),
\begin{enumerate}
  \item \( W_n(\mathcal{F} f; \cdot) = (-1)^n(n+1)/2 \mathcal{F}(\nu_n(f)) \) if \( E = \mathbb{R} \);
  \item \( W_n(\mathcal{L} f; \cdot) = \mathcal{L}(\nu_n(f)) \) if \( E = \mathbb{R}_+ \).
\end{enumerate}

This result allows us to prove, for example, that if \( f \) is a nonnegative even function, then \( (-1)^n(n-1)/2 W_n(\mathcal{F} f; x) \) is a strictly positive definite function on the real line, and \( W_n(\mathcal{L} f; x) \) is a completely monotone function on \( \mathbb{R}_+ \).

The paper is organised as follows. The next section is devoted to the study of the correlation functions, where we establish the latter results. Examples that illustrate our results are also given in the section. The functions in the Laguerre-Pólya class, represented as a Fourier transform, are studied in Section 3, where we give the proof of Theorem 1.1. In Section 4 we provide some comments and results concerning Wiener’s Tauberian theorem related to the main result established in Section 3.
2.1. Correlation functions and Wronskians. The integral in the definition \[1.2\] has \(n - 1\) folds and it can be written explicitly so as an integral over the simplex \(T^n(t) := \{ s \in \mathbb{R}^{n-1} : \alpha(s) := s_1 + \cdots + s_{n-1} \leq t \}\) of \(\mathbb{R}^{n-1}\). Indeed, we can write the correlation function \(\nu_n\) as

\[
\nu_n(w; t) = \int_{T^n(t)} \prod_{1 \leq i < j \leq n-1} (s_i - s_j)^2 \prod_{i=1}^{n-1} (t - \alpha(s) - s_i)^2 \times \mu'(s_1) \cdots \mu'(s_{n-1}) \mu'(t - \alpha(s)) ds_1 \cdots ds_{n-1}
\]

by setting \(s_n = t - \alpha(s) = t - s_1 - \cdots - s_{n-1}\) in \[1.2\].

We need a classical identity that is the cornerstone of our results. The identity can be found in [16, p. 62].

**Lemma 2.1.** Let \(f_i, g_i, 1 \leq i \leq n\), be integrable functions defined on \(\mathbb{R}\) such that \(f_i g_j \in L^1(\mathbb{R})\) for \(1 \leq i \neq j \leq n\). Then

\[
\det \left[ \int_{\mathbb{R}} f_i(t) g_j(t) d\mu(t) \right]_{i,j=1}^{n} = \int_{\mathbb{R}^n} \det \left[ f_j(t_i) \right]_{i,j=1}^{n} \det \left[ g_j(t_i) \right]_{i,j=1}^{n} \prod_{i=1}^{n} d\mu(t_i).
\]

For \(\mu \in \mathcal{M}(E)\) and \(n = 0, 1, \ldots\), let \(M_n(d\mu)\) be the moment matrix defined by

\[
M_n(d\mu) := \det \left[ \mu_{i+j} \right]_{i,j=0}^{n}.
\]

If \(d\mu(t) = w(t) dt\), we write this determinant as \(M_n(w)\). It is known that, if \(\mu \in \mathcal{M}_{2n}(E)\), then \(M_n(d\mu)\) is positive definite and, in particular, \(\det M_n(d\mu) > 0\).

**Lemma 2.2.** Let \(\mu \in \mathcal{M}_{2n-2}(\mathbb{R})\) be absolutely continuous. Then the correlation function \(\nu_n\) is integrable and

\[
\int_{\mathbb{R}} \nu_n(d\mu; x) dx = \det M_{n-1}(d\mu).
\]

Furthermore, if \(d\mu(t) = w(t) dt\) and \(w \in L^1(\mathbb{R})\), then \(\nu_n \in L^1(\mathbb{R})\).

**Proof.** Applying the previous lemma and using the Vandermonde determinant

\[
V(t_1, \ldots, t_n) := \det \left[ t_i^{j-1} \right]_{i,j=0}^{n-1} = \prod_{1 \leq i < j \leq n} (t_j - t_i),
\]

we obtain

\[
\int_{\mathbb{R}} \nu_n(t) dt = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} (s_i - s_j)^2 \mu'(s_1) \cdots \mu'(s_n) ds dt
\]

\[
= \int_{\mathbb{R}^n} V(s_1, \ldots, s_n)^2 d\mu(s_1) \cdots d\mu(s_n)
\]

\[
= \det \left[ \int_{\mathbb{R}} t_i^{j-1} d\mu(t) \right]_{i,j=0}^{n-1} = \det M_{n-1}(d\mu),
\]

where we have used \[2.2\] with \(f_j(t) = g_j(t) = t^j\).

If \(d\mu(t) = w(t) dt\) and \(w \in L^1(\mathbb{R})\), then the same proof shows that

\[
\int_{\mathbb{R}} |\nu_n(t)| dt \leq \int_{\mathbb{R}^n} V(s_1, \ldots, s_n)^2 w(s_1) \cdots w(s_n) ds_1 \cdots ds_n = \det M_{n-1}(|w|),
\]

which is finite and positive, so that \(\nu_n \in L^1(\mathbb{R})\).
Proof. Changing variables \( s_i \to -s_i \) in the definition of \( \nu_n \) shows that \( \nu_n(t) = \nu_n(-t) \) if \( d\mu \) is symmetric with respect to the origin. \( \square \)

**Lemma 2.4.** If \( d\mu(t) = f(t)dt \) and \( f \) is supported on the interval \((a, b)\), then \( \nu_n(f) \) is supported on \((na, nb)\).

Proof. If \( f \) is supported on \((a, b)\), then \( a \leq s_i \leq b \) in \((\mathbb{R}, \mathbb{R}_+^*)\). As a result, if \( t \leq na \), then \( t - s_1 - \cdots - s_{n-1} \leq a \), so that \( f(t - s \alpha(s)) = 0 \). Similarly, if \( t \geq nb \), then \( f(t - s \alpha(s)) = 0 \). \( \square \)

We are now ready to prove Theorem 1.3, which we restate below:

**Theorem 2.5.** For each \( n \geq 2 \), let \( \mu \in L^1(\mathbb{R}) \) such that \( t^{2n-2} \mu \in L^1(\mathbb{R}) \),

1. \( W_n(\mathcal{F} \mu; x) = (-1)^{n(n+1)/2} \mathcal{F}(\nu_n(f)) \) if \( E = \mathbb{R} \);
2. \( W_n(\mathcal{L} \mu; x) = \mathcal{L}(\nu_n(f)) \) if \( E = \mathbb{R}_+^* \).

Proof. We prove (2) first. Let \( d\mu = f(t)dt \). The condition \( t^{2n-2} \mu \in L^1(\mathbb{R}_+^*) \) implies that the derivatives of the Laplace transform \( \mathcal{L} \mu \), up to \( 2n - 2 \) order, are well defined continuous functions, so is \( W_n(\mathcal{L} \mu; x) \). Applying (2.2) with \( f_j(t) = g_j(t) = t^j \), we obtain

\[
W_n(\mathcal{L} \mu; x) = \det \left[ \int_{\mathbb{R}_+^*} (-t)^{i+j} e^{-tx} d\mu(t) \right]_{i,j=0}^{n-1}
\]

\[
= \int_{\mathbb{R}_+^*} \prod_{1 \leq i < j \leq n} (s_i - s_j)^2 e^{-x(s_1 + \cdots + s_n)} d\mu(s_1) \cdots d\mu(s_n)
\]

\[
= \int_{\mathbb{R}_+^*} \nu_n(f; t) e^{-tx} dt = \mathcal{L} \nu_n(f; x),
\]

which proves (2). The proof of (1) is similar. Taking the derivatives of \( \mathcal{F} \mu \) introduces powers of \( i \). Using

\[
\prod_{1 \leq i < j \leq n} (-1s_i + 1s_j)^2 = 1^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (s_i - s_j)^2 = (-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (s_i - s_j)^2,
\]

the proof follows as in the case (2). \( \square \)

Although these are simple relations, we give two nontrivial applications below to show that they are not as obvious as they appear to be.

Recall that a function \( \psi : \mathbb{R} \to \mathbb{R} \) is called positive definite if

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} c_i c_j \psi(x_i - x_j) \geq 0
\]

for all \( x_1, \ldots, x_N \in \mathbb{R} \) and \( N = 1, 2, \ldots \), and it is called strictly positive definite if \( \geq 0 \) is replaced by \( > 0 \) whenever \((c_1, \ldots, c_N)\) is not identically zero. The positive definite functions are characterized by Bochner’s theorem: a continuous function \( \psi \) is positive definite if and only if it is the Fourier transform of a finite non-negative Borel measure. Together with Theorem 2.5, we can then state the following theorem:

**Theorem 2.6.** For each \( n \in \mathbb{N} \), let \( \mu \) be an even measure in \( \mathcal{M}_{2n-1}^+(\mathbb{R}) \). Then the multiple of the Wronskian determinant

\[
W_n^{\mathcal{F}}(\mu; x) := (-1)^{n(n-1)/2} W_n(\mathcal{F} \mu; x)
\]

is a strictly positive definite function on the real line.
We only have to comment on the strictlypositiveness of the statement, which is not covered by Bochner’s theorem on the positive definite functions. However, it is known that if \( \psi \) is the Fourier transform of a finite non-negative Borel measure and the measure is not discrete, then \( \psi \) is strictly positive.

A function \( \phi : \mathbb{R}_+ \to \mathbb{R} \) in \( C(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_+) \) is called a completely monotone function if it satisfies
\[
(-1)^k \phi^{(k)}(x) \geq 0, \quad x > 0, \quad k = 0, 1, 2, \ldots.
\]
The completely monotone functions are characterized by Bernstein’s theorem: a function is completely monotone if and only if it is the Laplace transform of a finite non-negative Borel measure \( \mu \) on \( \mathbb{R}_+ \). Together with Theorem 2.7, we can then state the following theorem, which appeared first in [8, Corollary 4.6] and motivated our study in the present paper:

**Theorem 2.7.** Let \( n \in \mathbb{N} \) and \( \mu \in \mathcal{M}_{k+n-2}^+(\mathbb{R}_+) \). For each \( n \in \mathbb{N} \), \( W_n(\mathcal{L}_\mu; \cdot) \) is a completely monotone function on \( \mathbb{R}_+ \).

We state a corollary of Theorem 2.6 and Theorem 2.7. Let \( (\mathcal{F}_\mu)^{(k)} \) and \( (\mathcal{L}_\mu)^{(k)} \) denote the \( k \)-th order derivative of \( \mathcal{F}_\mu \) and \( \mathcal{L}_\mu \), respectively.

**Theorem 2.8.** Let \( k = 1, 2, \ldots \) and \( n = 2, 3, \ldots \). Then
\begin{enumerate}
\item If \( \mu \in \mathcal{M}_{k+n-2}^+(\mathbb{R}_+) \) is even, then \( (-1)^{nk/2}W_n((\mathcal{F}_\mu)^{(k)}; \cdot) \) is a strictly positive definite function on the real line provided either \( k \) or \( n \) is even.
\item If \( \mu \in \mathcal{M}_{k+n-2}^+(\mathbb{R}_+) \), then \( (-1)^{nk}W_n((\mathcal{L}_\mu)^{(k)}; \cdot) \) is a completely monotone function on the real line.
\end{enumerate}

**Proof.** For \( k = 1, 2, \ldots \), let \( \{\cdot\}^k \mu \) be the measure defined by \( d\{\cdot\}^k \mu(x) := x^k d\mu(x) \). Since \( (\mathcal{F}_\mu)^{(k)}(x) = (-1)^k \mathcal{F}(\{\cdot\}^k \mu; x) \), it is easy to see that
\[
W_n((\mathcal{F}_\mu)^{(k)}; x) = (-1)^{n(n+k+1)}W_n((\mathcal{F}(\{\cdot\}^k \mu); x),
\]
which is real valued and \( (-1)^{n(n+k+1)} = (-1)^{n(n+k+1)/2} \) if \( nk \) is even. Since \( \{\cdot\}^k \mu \in \mathcal{M}_{n-2}^+(\mathbb{R}) \), (1) follows from Theorem 2.6. Similarly, (2) follows as a corollary of Theorem 2.7.

In particular, in the case of \( n = 2 \), this shows that
\[
W_2((\mathcal{F}_\mu)^{(k)}; x) = (-1)^k \left( (\mathcal{F}_\mu)^{(k+1)}(x)^2 - (\mathcal{F}_\mu)^{(k)}(x)(\mathcal{F}_\mu)^{(k+2)}(x) \right)
\]
is a strictly positive definite function. For a nontrivial example of such results, we refer to Corollary 2.13 in the next subsection.

**2.2. Examples.** We give several examples to illustrate our results. First we point out that, by Theorem 2.7 and Theorem 2.6
\[
(4.4) \quad \nu_n(d\mu; x) = \mathcal{L}^{-1}[W_n(\mathcal{L}(d\mu); \cdot)](x) \quad \text{and} \quad \nu_n(d\mu; x) = \mathcal{F}^{-1}[W_n^F(d\mu; \cdot)](x),
\]
where we assume that \( d\mu \) is supported on \( \mathbb{R}_+ \) in the first identity. In general, taking the Fourier or Laplace transform of \( W_n^F(d\mu; \cdot) \) or \( W_n(\mathcal{L}(d\mu); \cdot) \), respectively, is difficult and the above identity may not be useful for determining the explicit formula of \( \nu_n \).

In some cases, however, it can be used as shown in our first two examples.

**Proposition 2.9.** If \( d\mu(t) = e^{-t^2/2} dt \), then
\[
\nu_n(d\mu; t) = a_n e^{-t^2/2n} \quad \text{with} \quad a_n = \frac{1}{\sqrt{2\pi n}} (2n)^{(n-1)/2} \prod_{k=1}^{n-1} k!.
\]
In particular, the span of \( \{e^{-(t-a)^2/2n} : a \in \mathbb{R}\} \) is dense in \( L^1(\mathbb{R}) \).

**Proof.** Let \( h(t) = e^{-t^2/2} \). It is well-known that \( \hat{h}(t) = \sqrt{2\pi}e^{-t^2} \). Furthermore, by the Rodrigues formula of the Hermite polynomials, it is easy to see that

\[
\frac{d^n}{dx^n}h(x) = \sqrt{2\pi}2^{-n/2}(-1)^n H_n(x/2).
\]

The closed formula of the Wronskian of the Hermite polynomials is known; see, for example, the identity \([11, (33)]\), from which we deduce that

\[
W_n(\mathcal{F}h; x) = \left((-1)^{(n-1)/2}(2\pi)^{n/2} \prod_{k=1}^{n-1} k!\right) e^{-nx^2/2}.
\]

Taking the inverse Fourier transform of this identity, the formula for \( \nu_n(t) \) follows from (1) in Theorem 2.10. \(\square\)

**Proposition 2.10.** Let \( d\mu_\alpha(t) = t^\alpha e^{-t}dt, \alpha > -1, \) be supported on \( \mathbb{R}_+ \). Then

\[
\nu_n(d\mu_\alpha; t) = a_n(t^{n(n+\alpha)-1}e^{-t} \quad \text{with} \quad a_n = \frac{\Gamma(\alpha + 1)n}{\Gamma(n(n+\alpha))} \prod_{k=1}^{n-1} k!(\alpha + 1)_k.
\]

**Proof.** Let \( h_\alpha(t) = t^\alpha e^{-t} \) on \( \mathbb{R}_+ \). The Laplace transform of \( h_\alpha \) is given by \( \mathcal{L}h_\alpha(t) = \Gamma(\alpha + 1)(1+t)^{\alpha-1} \), so that \( \mathcal{L}(h_\alpha(t))^{(k)} = (-)^k \Gamma(\alpha + k)(1+t)^{\alpha-k} \). It follows that

\[
W_n(\mathcal{L}h_\alpha; x) = \left(\prod_{k=1}^{n-1} k!(\alpha + 1)_k\right) \frac{\Gamma(\alpha + 1)^n}{(1+x)^{n(n+\alpha)}}.
\]

Indeed, the power of \( 1+x \) can be completely factored out from the determinant, so that \( W_n(\mathcal{L}h_\alpha; x) = c(1+x)^{-n(n+\alpha)} \) and the constant determinant can be evaluated to the value given. Taking the inverse Laplace transform of this identity, the formula of \( \nu_n(t) \) follows from (2) in Theorem 2.10. \(\square\)

Our next example uses the definition of the correlation function, or rather \([2.1]\), to derive an explicit formula for \( \nu_2 \) for a family of functions. Let \( {}_2F_1 \) be the standard Gauss hypergeometric function.

**Proposition 2.11.** For \( \alpha, \beta > -1, \) let \( w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta \chi_{[-1,1]}(t) \). Then \( \nu_2(w_{\alpha,\beta}; t) = 0 \) if \( |t| > 2 \) and

\[
\nu_2(w_{\alpha,\beta}; t) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(\alpha + 1)}{2^{2\alpha + 1}\Gamma(\alpha + \frac{3}{2})} \left(2 - t\right)^{2\alpha + 3} \left(2t\right)^\beta {}_2F_1\left(-\beta, \alpha + 1, \alpha + \frac{3}{2}; \frac{(2-t)^2}{8t}\right)
\]

if \( 0 \leq t \leq 2 \), and \( \nu_2(w_{\alpha,\beta}; -t) = \nu_2(w_{\beta,\alpha}; t) \).

**Proof.** Directly from the expression of \( \nu_2 \) at \([2.1]\), we obtain that

\[
\nu_2(t) = \int_{-\infty}^{\infty} (2s - t)^2 w(s) w(t-s) ds,
\]

from which it follows immediately that \( \nu_2(w_{\alpha,\beta}; t) = 0 \) if \( |t| > 2 \) and \( \nu_2(w_{\alpha,\beta}; -t) = \nu_2(w_{\beta,\alpha}; t) \). Furthermore, for \( 0 \leq t \leq 2 \), changing variable \( u = -1 + 2(1-s)/(2-t) \)
Corollary 2.12. For $\lambda > -1/2$, let $w_\lambda(t) := (1 - t^2)^{\lambda-1/2} \chi_{[-1,1]}(t)$. Then

\begin{equation}
\nu_2(w_{\alpha,\beta}; t) = \int_{-1}^{1} (2s - t)^2 w_{\alpha,\beta}(s) w_{\alpha,\beta}(t - s) ds
\end{equation}

\begin{align*}
&= \frac{(2 - t)^{2\alpha+3}}{2^{2\alpha+1}} \int_{-1}^{1} \left( 2t + \frac{(2 - t)^2}{4}(1 - u^2) \right)^\beta u^2(1 - u^2)^\alpha du \\
&= \frac{(2 - t)^{2\alpha+3}}{2^{2\alpha+1}} (2t)^\beta \int_{-1}^{1} \left( 1 + \frac{(2 - t)^2}{8t}(1 - u^2) \right)^\beta u^2(1 - u^2)^\alpha du.
\end{align*}

Let $s = -(2 - t^2)/(8t)$. Expanding $(1 + s(1 - u^2))^\beta$ in infinite series, we obtain

\begin{align*}
\int_{-1}^{1} (1 - s(1 - u^2))^\beta u^2(1 - u^2)^\alpha du &= \sum_{n=0}^{\infty} \frac{(-\beta)_n}{n!} \int_{-1}^{1} u^2(1 - u^2)^{\alpha+n} du s^n \\
&= \frac{\Gamma(3/2)\Gamma(\alpha + 1)}{\Gamma(\alpha + 3/2)} \sum_{n=0}^{\infty} \frac{(-\beta_n(\alpha + 1))}{(\alpha + 3/2)n!} s^n,
\end{align*}

the last summation is the $2F_1$ function. This completes the proof. \qed

In particular, the above $2F_1$ function is nonnegative on the interval $[-2, 2]$.

The Fourier transform of $w_{\alpha,\beta}$ is given by

\[ \mathcal{F}w_{\alpha,\beta}(x) = \frac{\Gamma(\frac{3}{2})\Gamma(\lambda + \frac{1}{2})}{2^{\lambda+1}\Gamma(\lambda + 2)} (2 - |t|)^{2\lambda+2}|t|^{\lambda+\frac{1}{2}} 2F_1 \left( \frac{-\lambda + \frac{1}{2}, \lambda + \frac{1}{2}}{\lambda + 2}; \frac{(2 - |t|)^2}{8|t|} \right) \chi_{[-2,2]}(t). \]

In particular, the above $2F_1$ function is nonnegative on the interval $[-2, 2]$.

The Fourier transform of $w_\lambda$ can be expressed in terms of the Bessel function $J_\lambda(x)$. Indeed, by the integral representation of the Bessel function,

\[ \mathcal{F}w_\lambda(x) = \int_{-1}^{1} e^{-itx} (1 - t^2)^{\lambda-1/2} dt = \sqrt{\pi} \Gamma(\lambda + 1/2) \left( \frac{2}{x} \right)^\lambda J_\lambda(x). \]

Using the identity $J_\lambda'(x) = (J_{\lambda-1}(x) - J_{\lambda+1}(x))/2$, it follows that

\[ -W_2(\mathcal{F}w_\lambda; x) = 4^\lambda \pi \Gamma(\lambda + \frac{1}{2})^2 x^{-2(1+\lambda)} \times \left[ x^2 J_{\lambda-1}(x)^2 - (2\lambda - 1)x J_{\lambda+1}(x)J_\lambda(x) + (x^2 - 2\lambda) J_\lambda(x)^2 \right]. \]

By Theorem 2.5, this function is equal to the Fourier transform of $\nu_2(w_\lambda; t)$, which is non-trivial since a direct verification is not immediate and, in fact, looks formidable for a generic parameter $\lambda$. In the case of $\lambda = 1/2$, $w_{1/2}$ is the characteristic function $w_{1/2}(t) = \chi_{(-1,1)}(t)$. In this case,

\[ \mathcal{F}w_{1/2}(t) = \frac{2\sin t}{t} \quad \text{and} \quad \nu_2(w_{1/2}; t) = \frac{1}{3} (2 - |t|)^3. \]

It is known that if $\phi$ is a positive definite function, then

\[ \phi(0) > 0 \quad \text{and} \quad |\phi(x)| \leq \phi(0), \quad \forall x \in \mathbb{R}. \]

However, a positive definite function is not necessarily positive everywhere.
Corollary 2.13. For $\lambda > -1/2$, let $\mathcal{J}_\lambda(x) := (\frac{1}{t})^\lambda J_\lambda(t)$. For $n = 2, 3, \ldots$, the Wronskian determinant $W_n^{(\lambda)}(x) := (-1)^{n(n-1)/2} W_n(\mathcal{J}_\lambda; x)$ is a strictly positive definite function on $\mathbb{R}$ and $W_n^{(\lambda)}(x) \leq W_n^{(\lambda)}(0)$ for $x \in \mathbb{R}$.

Even in the case of $w_0(t) = \chi_{[-1,1]}(t)$, that the Wronskian $W_n^F(x)$ leads to a positive definite function appears to be nontrivial. For example, the case $n = 2$ shows that

$$W_2^F(x) = 2 \frac{-1 + 2x^2 + \cos(2x)}{x^4}$$

is a strictly positive function on the real line and $W_2^F(x) \leq 4/3$ for all $x \in \mathbb{R}$.

As an application of Proposition 2.11 in Section 4, $W_2^A(x) > 0$ on $\mathbb{R}$. In this case, the inequality in Corollary 2.13 gives:

Corollary 2.14. For $\lambda > -1/2$ and $x \in \mathbb{R}$,

$$0 < (\mathcal{J}_\lambda'(x))^2 - \mathcal{J}_\lambda(x)\mathcal{J}_\lambda''(x) \leq \frac{1}{2} \left( \frac{1}{\Gamma(\lambda + 1)^2} - \frac{1}{\Gamma(\lambda)\Gamma(\lambda + 2)} \right).$$

In particular, the right hand side becomes $1/2$ when $\lambda = 0$.

We give another example, where the “tent function” $w_A$ is such that its Fourier transform is the function $(\sin t/t)^2$.

Proposition 2.15. Let $w_A(t) = \frac{1}{4}(2 - |t|)_+$, where $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ otherwise. Then the correlation function $\nu_2(w_A; \cdot)$ is given by

$$(2.6) \quad \nu_2(w_A; t) = \frac{1}{480} \chi_{[-4,4]}(t) \left\{ \begin{array}{ll} (4 - |t|)^5 - 2(2 - |t|)^5 - 40(2 - |t|)^3, & 0 \leq t \leq 2, \\ (4 - |t|)^5, & t \geq 2. \end{array} \right.$$

Proof. As in the proof of Proposition 2.11 it is easy to see that $\nu_2(w_A)$ is supported on $[-4,4]$ and it is an even function. Assume $0 \leq t \leq 4$. It follows then that

$$\nu_2(w_A; t) = \frac{1}{16} \int_{-2+t}^2 (2s - t)^2(2 - s)(2 - |t - s|)ds.$$

Evaluating the integral gives the stated result.  

For the Laplace transform, we need $d\mu$ supported on $\mathbb{R}_+$. We give one example.

Proposition 2.16. For $\alpha, \beta > -1$, let $u_{\alpha,\beta}(t) = t^\alpha(1-t)^\beta \chi_{[0,1]}(t)$. Then $\nu_2(u_{\alpha,\beta}; t) = 0$ if $t > 2$,

$$\nu_2(u_{\alpha,\beta}; t) = \frac{\Gamma(\frac{3}{2})\Gamma(\alpha + 1)}{2^{2\alpha + 1}\Gamma(\alpha + \frac{3}{2})} t^{2\alpha + 3}(1-t)^{\beta} F_1(-\beta, \alpha + 1; \alpha + \frac{5}{2}; \frac{t^2}{4(1-t)})$$

if $0 \leq t \leq 1$, and

$$\nu_2(u_{\alpha,\beta}; t) = \frac{\Gamma(\frac{3}{2})\Gamma(\beta + 1)}{2^{2\beta + 1}\Gamma(\beta + \frac{3}{2})} (2-t)^{2\beta + 3}(t-1)^{\alpha} F_1(-\alpha, \beta + 1; \beta + \frac{5}{2}; \frac{(2-t)^2}{4(1-t)})$$

if $1 \leq t \leq 2$.

Proof. Since $u_{\alpha,\beta} \left( \frac{1-t}{2} \right) = w_{\alpha,\beta}(t)/2^{\alpha + \beta}$, a simple change variable shows that

$$\nu_2(u_{\alpha,\beta}; t) = 2^{-2\alpha - 2\beta - 1} \int_{u_1 + u_2 = 2 - 2t} u_{\alpha,\beta}(u_1)w_{\alpha,\beta}(u_2)du_1du_2$$

$$= 2^{-2\alpha - 2\beta - 1} \nu_2(w_{\alpha,\beta}; 2(1-t)),$$

from which the stated formula follows from the previous proposition.  

For $\alpha, \beta > -1$, the Laplace transform of $u_{\alpha, \beta}$ is given by
\[
\mathcal{L} u_{\alpha, \beta}(x) = \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \mathcal{F}_1 \left( \frac{\alpha + 1}{\alpha + \beta + 2}; -t \right).
\]
According to Theorem 2.7, the Wronskian of this function is a completely monotone function. In the case of $\alpha = \beta = \lambda - 1/2$, we write $u_{\lambda}(t) = (t(1-t))^{\lambda-1/2}$ and the Laplace transform is given in terms of the modified Bessel function $I_n(t)$ of the first kind,
\[
\mathcal{L} u_{\lambda}(t) = \sqrt{\pi} \Gamma\left(\frac{\lambda-1}{2}\right) t^{-\lambda} e^{-t} I_n\left(\frac{t}{2}\right) e^{-t},
\]
where $I_n(t) = i^n J_n(it)$ is real valued. In particular, in the case of $\alpha = \beta = 0$, we have
\[
\mathcal{L} \chi_{[0,1]}(t) = 1 - e^{-t} t,
\]
which is a completely monotone function. The Wronskian of this function is also completely monotone. The simplest case $n = 2$ shows that
\[
W_{\mathcal{L}}(t) = \frac{2 \cosh(t) - t^2 - 2}{t^4} e^{-t}
\]
is completely monotone.

3. Fourier transforms in the Laguerre-Pólya class

In this section we establish the necessary and sufficient conditions for an entire function, represented as a Fourier transform of an absolutely continuous Borel measure, to belong to the Laguerre-Pólya class in terms of density of the corresponding correlation functions. In fact, we consider an additional assumption on the entire function that is satisfied by the Riemann $\Xi$-function. As it has been mentioned, our result provides an answer of the problem of Pólya [14] for this specific subclass.

Recall that a real entire function $\varphi$ is in the Laguerre-Pólya class, $\varphi \in \mathcal{LP}$, if
\[
\varphi(z) = cz^m e^{-\alpha z^2 + \beta} \prod_{k=1}^{\infty} \left( 1 + z/x_k \right) e^{-z/x_k}
\]
for some $c, \beta \in \mathbb{R}$, $\alpha > 0$, $m \in \mathbb{N}_0$ and $x_k \in \mathbb{R} \setminus \{0\}$ such that $\sum_k x_k^{-2} < \infty$.

Already Laguerre observed that if $\varphi \in \mathcal{LP}$, then
\[
[\varphi^{(j)}(x)]^2 - \varphi^{(j-1)}(x)\varphi^{(j+1)}(x) \geq 0, \quad x \in \mathbb{R}, \quad j \in \mathbb{N}.
\]
Although the Laguerre inequalities (3.1) consist of an infinite set of conditions, they are only necessary for an entire function $\varphi$ to belong to the Laguerre-Pólya class. In fact, (3.1) follow from the most simple Laguerre inequalities
\[
|\varphi''(x)|^2 - \varphi'(x)\varphi''(x) \geq 0, \quad x \in \mathbb{R}
\]
and the fact that $\mathcal{LP}$ is closed under differentiation. In 1913 Jensen [9] made the ingenious observation that $\varphi$ should have only real zeros if $|\varphi(z)|^2$ is either an increasing function along all rays perpendicular to the real line or convex along all such lines. Let $\varphi$ be a real entire function
\[
\varphi(z) = \phi(x, y) + i\psi(x, y), \quad z = x + iy,
\]
whose real and imaginary parts are
\[
\phi(x, y) := \Re \varphi(z) = \frac{1}{2} (\varphi(z) + \varphi(\bar{z})) \quad \text{and} \quad \psi(x, y) := \Im \varphi(z) = \frac{1}{2i} (\varphi(z) - \varphi(\bar{z})).
\]
Jensen’s criteria state that, under a mild additional restriction, the zeros of \( \varphi \) should be real if and only if
\[
|\varphi(z)|^2 = [\phi(x, y)]^2 + [\psi(x, y)]^2
\]
is either an increasing function of \( y \in [0, \infty) \) or it is a convex function of \( y \in (-\infty, \infty) \). He observed that
\[
\frac{1}{2} \frac{\partial^2}{\partial y^2} |\varphi(z)|^2 = |\varphi'(z)|^2 - \Re(\varphi(z)\varphi''(z)).
\]
Jensen’s convexity criterion states:

**Theorem A.** Let \( \varphi(z) = e^{-az^2} \varphi_1(z) \), \( a \geq 0 \), \( \varphi \not\equiv 0 \), where \( \varphi_1 \) is a real entire function of genus 0 or 1. Then \( \varphi \in \mathcal{LP} \) if and only if
\[
|\varphi'(z)|^2 - \Re(\varphi(z)\varphi''(z)) \geq 0 \quad \text{for all } z \in \mathbb{C}.
\]

Jensen’s result was refined recently by Csordas and Vishnyakova [6] who proved that if \( \varphi(z) \) is a real entire function, \( \varphi \not\equiv 0 \), and Jensen’s inequalities (3.5) hold then \( \varphi \in \mathcal{LP} \).

However, if one differentiates the right-hand side of (3.4) twice with respect to \( y \), uses the fact that both \( \phi(x, y) \) and \( \psi(x, y) \) are harmonic functions and applies the Cauchy-Riemann equations, obtains (see also [4, 6])
\[
|\varphi'(z)|^2 - \Re(\varphi(z)\varphi''(z)) = [\phi_x(x, y)]^2 - \phi(x, y)\phi_{xx}(x, y) + [\psi_y(x, y)]^2 - \psi(x, y)\psi_{yy}(x, y) = - \left[ W_2(\phi(\cdot, y); x) + W_2(\psi(\cdot, y); x) \right].
\]

Therefore, Jensen’s result can be rewritten in the following form:

**Corollary 3.1.** Let \( \varphi(z) \), defined by (3.5), be a real entire function that obeys the requirements in Theorem A. Then \( \varphi \in \mathcal{LP} \) if and only if
\[
W_2(\phi(\cdot, y); x) + W_2(\psi(\cdot, y); x) \leq 0 \quad \text{for all } z = x + iy \in \mathbb{C}.
\]

Let \( K \) be an even function that decreases rapidly enough at infinity, so that
\[
\varphi(z) := \int_{-\infty}^{\infty} K(u)e^{-iu^2} du
\]
is an entire function of the form described in Theorem A. Then its real and imaginary part are given by
\[
\phi(x, y) = \int_{\mathbb{R}} \cosh(sy)K(s)\cos(sx)ds, \quad \psi(x, y) = \int_{\mathbb{R}} \sinh(sy)K(s)\sin(sx)ds.
\]

The observations presented in this section up to now are classical and they are due to Jensen [9], Pólya [15] and developed further by Csordas and Varga [5]. Roughly speaking, they say that \( \varphi \in \mathcal{LP} \) if and only if the sum of the Laguerre quantities (3.2) for the real and imaginary part of \( \varphi \), considered as functions of \( x \), are nonnegative for any fixed \( y \).

In what follows we restrict our study to a narrower class of entire function which obey specific properties that are verified for the Riemann \( \Xi(z) \). The main features we employ are the facts that the zeros of \( \Xi \) lie in the horizontal strip \( S_{1/2} = \{ z : |3z| < 1/2 \} \) because the nontrivial zeros of \( \zeta \) lie in the critical strip and it possesses the Hadamard factorization
\[
\Xi(z) = \Xi(0) \prod_{k=1}^{\infty} \left( 1 - \frac{z}{z_k} \right), \quad z_k \in S_{1/2}.
\]
It is worth mentioning that Riemann stated the latter representation without proof in [17] and it was established rigorously by Hadamard.

Therefore, in what follows, we consider real entire functions \( \varphi \) with the following properties:

(i) \( \varphi(z) \) is represented as a Fourier transform \([57]\) of an even kernel \( K \) in the Schwartz space \( S(\mathbb{R}) \) (see [18]);

(ii) The zeros of \( \varphi \) belong to the horizontal strip

\[ S_\alpha = \{ z = x + iy : -\alpha < y < \alpha \} \]

with width \( 2\alpha > 0 \);

(iii) The function \( \varphi \) possesses the Hadamard factorization

\[ \varphi(z) = \varphi(0) \prod_{k=1}^{\infty} \left( 1 - \frac{z}{z_k} \right), \quad z_k \in S_\alpha. \]  

We shall prove that an entire function which obeys these requirements belongs to the Laguerre-Pólya class if and only if it satisfies the strict inequalities \((3.5)\), or equivalently the strict inequalities \((3.6)\), for every \( y \neq 0 \). We shall employ the results from the previous sections and the celebrated Wiener’s \( L^1 \) Tauberian Theorem [20, Theorem II]:

**Theorem B.** If \( K \in L^1(\mathbb{R}) \), a necessary and sufficient condition for the set of all its translations to be dense in \( L^1(\mathbb{R}) \) is that its Fourier transform \( \mathcal{F}(K) \) should have no real zeros.

It is worth mentioning that Wiener’s \( L^2 \) Tauberian Theorem [20, Theorem I] states that the translates of \( K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) are dense in \( L^2(\mathbb{R}) \) if and only if its Fourier transform, extended to \( L^2(\mathbb{R}) \) via Plancherel’s theorem, is such that its real zeros should form a set of zero measure.

Thus, the main result in this section reads as follows:

**Theorem 3.2.** Let the real entire function \( \varphi \) satisfy the above properties (i), (ii) and (iii). Then \( \varphi \in \mathcal{LP} \) if and only if, for every fixed \( y \in (-\alpha, \alpha) \setminus \{0\} \), the translates \( \mathcal{T}(K_{2,y}) \) of the kernel

\[ K_{2,y}(t) = \cosh(ty) \int_{-\infty}^{\infty} (t - 2s)^2 K(t - s)K(s) \, ds \]

are dense in \( L^1(\mathbb{R}) \).

Furthermore, the translates \( \mathcal{T}(K_{2,y}) \) of \( K_{2,y}(t) \) are dense in \( L^1(\mathbb{R}) \) for every fixed \( y \in (-\alpha, \alpha) \) if and only if the zeros of \( \varphi \) are real and simple.

We shall need the following simple technical result:

**Lemma 3.3.** Let \( \varphi \) be defined by \([3.7]\). For a fixed \( y \), let \( \mu_y^\varphi(s) := \cosh(sy)K(s) \) and \( \mu_y^\psi(s) := \sinh(sy)K(s) \). Then

\[ \phi(x,y) = (\mathcal{F}\mu_y^\varphi)(x) \quad \text{and} \quad \psi(x,y) = (\mathcal{F}\mu_y^\psi)(x). \]

**Proof.** Since \( K \) is even, changing variable \( s \mapsto -s \) shows that

\[ (\mathcal{F}\mu_y^\varphi)(x) = \int_\mathbb{R} \cosh(sy)K(s)e^{-isz} \, ds = \int_\mathbb{R} \cosh(sy)K(s)\cos(sx) \, ds = \phi(x,y) \]

The case of \( \mathcal{F}\mu_y^\psi \) follows similarly. \( \square \)
We shall need also some results concerning entire functions in the Hermite–Biehler class (see [12] Chapter 7). The entire function $\omega$ is said to be in this class, denoted $\omega \in HB$, if it has no zeros in the closed lower half-plane $\Im z \leq 0$ and $|\omega(z)/\bar{\omega}(z)| < 1$ for every $z$ with $\Im z > 0$. Here $\bar{\omega}(z) = \overline{\omega(z)}$. Let $\omega(z) = P(z) + iQ(z)$, where the real and imaginary parts $P$ and $Q$ are real entire functions. An important fact we shall need is the following (see [12] Theorem 3 on pp. 311-312):

**Theorem C.** If $\omega \in HB$ then the zeros of $P(z)$ and $Q(z)$ are real and strictly interlace.

Similarly, the Hermite-Biehler class $HB$ consists of entire functions $\omega$ with no zeros in the open lower half-plane $\Im z < 0$ and $|\omega(z)/\bar{\omega}(z)| \leq 1$ for every $z$ with $\Im z > 0$. An essential difference between $HB$ and $\overline{HB}$ is that the real and imaginary parts $P$ and $Q$ of a function from $\overline{HB}$ may have common real zeros. In general, if $R(z)$ is the canonical product corresponding to those common zeros of $\omega \in \overline{HB}$, then $\omega(z) = R(z)\omega_1(z)$ with $\omega_1 \in HB$.

Now we are in a position to prove Theorem 3.2.

### 3.1. Proof of the sufficiency

In this subsection we prove that the condition about the density of the correlation functions implies that $\varphi \in LP$ under milder conditions that do not require the Hadamard factorization (3.8).

**Proof.** By Corollary 3.6 the entire function $\varphi$ certainly belongs to $LP$ provided that, for any fixed $y \in \mathbb{R}$, the inequality $W_2(\phi(\cdot,y); x) + W_2(\psi(\cdot,y); x) \leq 0$ holds for every $x \in \mathbb{R}$. By Lemma 3.3

$$\phi(x, y) = (\mathcal{F}\mu_y^\varphi)(x) \quad \text{and} \quad \psi(x, y) = (\mathcal{F}\mu_y^\psi)(x).$$

It follows immediately from Proposition 3.1 in [5] that when $y$, with $|y| \geq \alpha$, is fixed, $\phi(x, y)$ and $\psi(x, y)$ are entire functions of the variable $x$ and that they belong to $LP$. Hence, the Laguerre inequalities (3.2) hold for them with $j = 1$. In other words

$$[\phi_x(x, y)]^2 - \phi(x, y)\phi_{xx}(x, y) \geq 0,$$

$$[\psi_x(x, y)]^2 - \psi(x, y)\psi_{xx}(x, y) \geq 0,$$

for every $z = x + iy \not\in S_\alpha$.

It remains to establish (3.6) for $z \in S_\alpha$. By Theorem 2.3 the Wronskians of $\phi(x, y)$ and $\psi(x, y)$, considered as functions of $x$, are the Fourier transforms of the corresponding correlations functions,

$$W_2(\phi(\cdot, y); x) = W_2(\mathcal{F}\mu_y^\phi; x) = -\mathcal{F}(\nu_2(\mu_y^\phi))(x) = -\mathcal{F}(\nu_2(\cosh(ty)K(t)))(x),$$

and

$$W_2(\psi(\cdot, y); x) = W_2(\mathcal{F}\mu_y^\psi; x) = -\mathcal{F}(\nu_2(\mu_y^\psi))(x) = -\mathcal{F}(\nu_2(\sinh(ty)K(t)))(x).$$

By the definition of the correlation function, it is easy to see that

$$\nu_2(\cosh(\cdot)y)K(\cdot); t) + \nu_2(\sinh(\cdot)y)K(\cdot); t) = \cosh(ty)\nu_2(K; t).$$

Therefore, it follows that

$$W_2(\phi(\cdot, y); x) + W_2(\psi(\cdot, y); x) = -\mathcal{F}(\cosh(\cdot)y)\nu_2(K; \cdot))(x).$$

We shall prove now that for every $y \in (\alpha, \alpha) \setminus \{0\}$,

$$\mathcal{F}(\cosh(\cdot)y)\nu_2(K; \cdot))(x) \geq 0 \quad \text{for every} \quad x \in \mathbb{R}.$$
It is clear that the inequalities (3.9) hold for every such $y$ and for $x = 0$. Indeed,

$$[\phi_x(0, y)]^2 - \phi(0, y)\phi_{xx}(0, y) = \int_{\mathbb{R}} \cosh(sy)K(s)ds \int_{\mathbb{R}} s^2 \cosh(sy)K(s)ds > 0,$$

$$[\psi_x(0, y)]^2 - \psi(0, y)\psi_{xx}(0, y) = \left(\int_{\mathbb{R}} s \sinh(sy)K(s)ds\right)^2 > 0,$$

which shows that both $W_2(\phi(\cdot, y); 0)$ and $W_2(\psi(\cdot, y); 0)$ are negative. Since the translates of $\cosh(\cdot)y$ are dense in $L^1(\mathbb{R})$, then its Fourier transform does not change sign for $x \in \mathbb{R}$, so that the strict inequalities (3.9) hold for $y \in (-\alpha, \alpha) \setminus \{0\}$. It is clear then that (3.9) also hold for $y = 0$ by continuity.

Finally, observe that, if the $L^1$-density holds for $y = 0$ too, then by Wiener’s theorem, since $\varphi(x) = \phi(x, 0)$,

$$|\varphi'(x)|^2 - \varphi(x)\varphi''(x) > 0, \quad x \in \mathbb{R}.$$ 

Therefore, $\varphi$ does not have multiple zeros. □

3.2. Proof of the necessity. We shall prove that if $\varphi \in \mathcal{L}P$ is represented as a Fourier transform and has an Hadamard factorization \((3.8)\), then the translations of the correlation function are dense in $L^1(\mathbb{R})$ not only for every $y \in (-\alpha, \alpha) \setminus \{0\}$ but for every $y \in \mathbb{R} \setminus \{0\}$. Because of Wiener’s Tauberian Theorem, we need to show that the strict inequalities (3.9) hold. In fact, we shall see that $[\phi_x(x, y)]^2 - \phi(x, y)\phi_{xx}(x, y) > 0$ and $[\psi_x(x, y)]^2 - \psi(x, y)\psi_{xx}(x, y) > 0$ for all $z = x + iy$, $y \neq 0$, which, in Jensen’s terminology means that $\varphi$ is strictly convex with respect to $y$. However, we shall employ the Hadamard factorization (3.8).

It is worth mentioning that Levin [12, p. 307] pointed out that the requirement $|\omega(z)/\bar{\omega}(z)| < 1, \Im z > 0$ is secured by the fact that the zeros lie in upper half plane when $\omega$ is a polynomial and the same holds for entire functions of order zero because of the Phragmén-Lindelöf theorem. In fact, the same holds for functions with Hadamard factorization (3.8). Indeed, if $\varphi$ is such an entire function in the Laguerre-Pólya class, and $a > 0$, then both $\varphi(z - ia)$ and $\varphi(ia - z)$ belong to $HB$. We sketch the proof that the “horizontal translation” $\varphi_a(z) := \varphi(z - ia) \in HB$. Since $\varphi(z) = \varphi(0) \prod (1 - z/x_k)$, $x_k \in \mathbb{R}$, then

$$\varphi_a(z) = \varphi(0)\varphi(-ia) \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k + ia}\right).$$

Hence, the zeros of $\varphi_a$ belong to the upper half-plane. Moreover, if $\Im z > 0, z = x + iy$ with $y > 0$, then

$$\left|\frac{\varphi_a(z)}{\varphi_a'(z)}\right| = \left|\prod_{k=1}^{\infty} \frac{(1 - (x + iy)/(x_k + ia))}{(1 - (x - iy)/(x_k + ia))}\right|.$$ 

So the quotient of the corresponding terms is $|x_k - x + i(a - y)|/|x_k - x + i(a + y)| < 1$. This implies immediately that there is $q \in (0, 1)$, such that the quotients of the finite products above are limited from above by $q$. Thus, $|\varphi_a(z)/\varphi_a'(z)| < 1$ when $\Im z > 0$ and $\varphi(z - ia) \in HB$. Then, if $\varphi(z - ia) = P_a(z) + iQ_a(z)$, the real and imaginary parts $P_a(z)$ and $Q_a(z)$ are (see (3.4))

$$P_a(x) = \int_{\mathbb{R}} K(u) \cosh(au) \cos(xu)du = \phi(x, a),$$

$$Q_a(x) = \int_{\mathbb{R}} K(u) \sinh(au) \sin(xu)du = \psi(x, a).$$
Since, by Theorem C, the zeros of $P_a(x)$ and $Q_a(x)$ are real and strictly interlace, then each has only simple zeros and, consequently, satisfies the strict Laguerre inequalities

$$\left[ \phi_x(x,a) \right]^2 - \phi(x,a)\phi_xx(x,a) > 0,$$

$$\left[ \psi_x(x,a) \right]^2 - \psi(x,a)\psi_xx(x,a) > 0.$$  

Although the reasonings up to now concern the case $a > 0$, they hold analogously for $a < 0$, so that the latter inequalities hold for all $a \neq 0$. The above relations between the Laguerre inequalities and the Wronskians yield that, for every fixed $y \neq 0$,

$$\mathcal{F}(\cosh(\cdot y)\nu_2(K;\cdot))(x) > 0 \text{ for every } x \in \mathbb{R}.$$  

Finally, Theorem B implies that the translates of the correlation kernel $K_{2,y}(t)$ must be dense in $L^1(\mathbb{R})$ for every fixed $y \neq 0$.

It is worth mentioning that without the restriction (iii), one may prove only that $\varphi(z - ia)$ and $\varphi(ia - z)$ belong to $\mathcal{HB}$, as was done in [5]. In that case one guarantees that the Laguerre inequalities do hold but not the strict ones and in order to prove the necessity, the strict ones are fundamental.

### 3.3. Theorem [1.1] and further equivalent statements.

It is clear now that Theorem 1.1 follows from Theorem 3.2 by setting $K(t) = \Phi(t)$ and having in mind that $\Xi(z)$ obeys all the requirements imposed on $\varphi(z)$.

Wiener’s $L^1$ Tauberian theorem has several equivalent formulations, as given in [10, Theorem 8.1, p. 82], which can be used to derive other equivalent forms of Theorem 3.2. We state one such result given in terms of the convolution $f * g$.

**Theorem 3.4.** Under the assumption in Theorem 3.2, $\varphi \in \mathcal{LP}$ if and only if, for every fixed $y \in (-\alpha, \alpha) \setminus \{0\}$, the testing equation $K_{2,y} * g = 0$ for bounded $g$ implies $g = 0$.

Recall that $\Phi_{2,y}$ is defined in [1.1]. Applying the above theorem with $K = \Phi$ gives the following corollary.

**Corollary 3.5.** The Riemann hypothesis is true if and only if, for each $y \in (-1/2, 1/2) \setminus \{0\}$, the testing equation $\Phi_{2,y} * g = 0$ for bounded $g$ implies $g = 0$.

There are two additional equivalent statements of Wiener’s theorem in [10, Theorem 8.1, p. 82] that provide further equivalent forms of Theorems 3.2 and 1.1. We omit the details.

### 4. Further results concerning Wiener’s theorems

It is quite clear that Wiener proved his density theorems aiming at a proof of the Prime Number Theorem. At the first glance, these theorems are peculiar. They guarantee that the translates of the the Gauss kernels $\exp(-ax^2), a > 0$, are dense in both $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. However, according to Wiener’s results, the translates of the characteristic functions of a compact interval and of the “tent function” in Example 2.15 are dense in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$. Indeed, their Fourier transforms are, up to a normalization or scaling, the sinc function $\text{sinc}(t) = \sin t/t$ and its square. The same holds for the whole family of the Gegenbauer weights $\omega_{\lambda}$ in Corollary 2.12 whose Fourier transforms are $J_\lambda$ defined in Corollary 2.13. This is so because translations of the argument are permitted, but not scaling. However, Theorem 3.2 allows us to build a vast class of functions whose translates are dense in $L^1(\mathbb{R})$. This class is generated by the correlation functions that correspond to those $\mathcal{LP}$-functions that obey
the requirements in Theorem 3.2. Let us first state the following general consequence of Wiener’s theorems and Theorem 1.3.

**Theorem 4.1.** Let \( f \in L^1(\mathbb{R}) \) be a nonnegative function, and \( t^{2n-2}f \in L^1(\mathbb{R}) \) for \( n \in \mathbb{N}, n \geq 2 \).

1. \( W_n(\mathcal{F}f; x) \) has no real zeros if, and only if, the span of the translates of the functions \( \nu_n(x) := \nu_n(f;x+a), a \in \mathbb{R}, \) are dense in \( L^1(\mathbb{R}) \).
2. If \( \nu_n(f) \in L^2(\mathbb{R}) \), then the real zeros of \( W_n(\mathcal{F}f; x) \) form a set of measure zero set if, and only if, the span of the translates of the functions \( \nu_n(x) := \nu_n(f;x+a), a \in \mathbb{R}, \) are dense in \( L^2(\mathbb{R}) \).

**Proof.** By Theorem 2.5, \( W_n(\mathcal{F}f; x) \) is equal to, up to a sign, \( \mathcal{F}\nu_n(f) \). By Lemma 2.2, \( \nu_n(f) \in L^1(\mathbb{R}) \). Hence, (1) follows immediately from Wiener’s \( L^1 \) Tauberian theorem (see [20, Theorem II] or Theorem B in the next section). Similarly, (2) is a consequence of Wiener’s \( L^2 \) Tauberian theorem [20, Theorem I]. \( \square \)

More specifically, Theorem 3.2 immediately yields:

**Corollary 4.2.** Let \( \varphi \in \mathcal{LP} \) satisfy the properties (i), (ii) and (iii) in Section 3. Then the translates \( \mathcal{T}(K_{2,y}) \) of the kernel

\[
K_{2,y}(t) = \cosh (ty) \int_{-\infty}^{\infty} (t-2s)^2 K(t-s)K(s) \, ds
\]

are dense in \( L^1(\mathbb{R}) \) for every fixed \( y \in \mathbb{R} \setminus \{0\} \). Furthermore, if zeros of \( \varphi \) are real and simple, then the translates \( \mathcal{T}(K_{2,y}) \) of \( K_{2,y}(t) \) are dense in \( L^1(\mathbb{R}) \) for every fixed \( y \in \mathbb{R} \).

It is known that both sinc\(^2(t) \) and \( \mathcal{J}_\lambda(t) \) (see [7]) belong to the Laguerre-Pólya class and they obey the requirements of Theorem 3.2; moreover, the zeros of \( \mathcal{J}_\lambda(t) \) are simple while the zeros of sinc\(^2(t) \) are obviously double. Then the Corollaries 4.2 and 2.12 imply the following:

**Corollary 4.3.** The translates of the following functions are dense in \( L^1(\mathbb{R}) \):

a) \( \cosh(yt) \nu_2(w_\lambda;t), \quad y \in \mathbb{R}, \) where \( \nu_2(w_\lambda;t) \) is defined in (2.5);

b) \( \cosh(yt) (2-|t|)^3, \quad y \in \mathbb{R}, \) \( (\lambda = 1/2 \) of (a))

c) \( \cosh(yt) \nu_2(w_\lambda;t), \quad y \in \mathbb{R} \setminus \{0\}, \) where \( \nu_2(w_\lambda;t) \) is defined in (2.6).

Finally, we prove that the translates of the correlation functions \( \nu_2(w_{\alpha,\beta};x) \) in Proposition 2.11 are also dense in \( L^1(\mathbb{R}) \).

**Proposition 4.4.** For \( \alpha, \beta > 0 \), the family of functions \( \{\nu_2(w_{\alpha,\beta};x+a) : a \in \mathbb{R}\} \) is dense in \( L^1(\mathbb{R}) \). Equivalently, the Wronskian of the Fourier transform of \( w_{\alpha,\beta} \), \( W_n(\mathcal{F}w_{\alpha,\beta};x) \) does not change sign on \( \mathbb{R} \).

**Proof.** As shown in [10, Theorem 8.1, p. 82], one of the equivalent statements for the density of \( \{\nu_2(w_{\alpha,\beta};x+a) : a \in \mathbb{R}\} \) in \( L^1(\mathbb{R}) \) is that the test equation

\[
\int_{\mathbb{R}} \nu_2(w_{\alpha,\beta};t) \phi(x-t) \, dt = 0 \quad \forall x \in \mathbb{R},
\]

where \( \phi \) is a continuous and bounded function, has only trivial solution \( \phi(x) = 0 \). Since \( \nu_2(w_{\alpha,\beta}) \) is nonnegative and has compact support on \([-2,2]\), the measure \( dm(t) := \)
\[ \nu_2(w_{\alpha,\beta}; t) dt \] is a finite nonnegative measure. If \[ \int_{\mathbb{R}} \phi(x-t) d\mu(t) = 0 \] for all \( x \in \mathbb{R} \) then, by Fatou’s lemma,
\[
\int_{\mathbb{R}} \lim_{n \to \infty} \frac{\phi(t + n^{-1}) - \phi(t)}{n^{-1}} dm(t) \leq \liminf_{n \to \infty} \int_{\mathbb{R}} \frac{\phi(t + n^{-1}) - \phi(t)}{n^{-1}} dm(t) = 0
\]
and the same inequality holds if the left hand side contains a negative sign, from which it follows that \( \int_{\mathbb{R}} \phi'(t) dm(t) = 0 \) almost everywhere. Hence, \( \phi(t) = 0 \). This proves the density. By Theorem 2.5 and Wiener’s \( L^1 \) Tauberian theorem, the Wronskian of the Fourier transform of \( w_{\alpha,\beta} \) does not change sign. \( \square \)

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