Characteristic Classes and Integrable Systems. General Construction.

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Abstract

We consider topologically non-trivial Higgs bundles over elliptic curves with marked points and construct corresponding integrable systems. In the case of one marked point we call them the modified Calogero-Moser systems (MCM systems). Their phase space has the same dimension as the phase space of the standard CM systems with spin, but less number of particles and greater number of spin variables. Topology of the holomorphic bundles are defined by their characteristic classes. Such bundles occur if G has a non-trivial center, i.e. classical simply-connected groups, \( E_6 \) and \( E_7 \). We define the conformal version CG of G - an analog of GL(N) for SL(N), and relate the characteristic classes with degrees of CG-bundles. Starting with these bundles we construct Lax operators, quadratic Hamiltonians, define the phase spaces and the Poisson structure using dynamical r-matrices. To describe the systems we use a special basis in the Lie algebras that generalizes the basis of t’Hooft matrices for sl(N). We find that the MCM systems contain the standard CM systems related to some (unbroken) subalgebras. The configuration space of the CM particles is the moduli space of the holomorphic bundles with non-trivial characteristic classes.

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1 Introduction

The paper conventionally speaking contains two types of results. First, we construct topologically nontrivial holomorphic $G$-bundles over elliptic curves, where $G$ is a complex Lie group and describe their moduli space. Second, on the base of these results, we construct a new family of classical integrable systems related to simple Lie groups. We define the corresponding Lax operators, quadratic Hamiltonians and the classical dynamical elliptic $r$-matrices. The latter completes the classification list of classical elliptic dynamical $r$-matrices [16], where the underlying bundles are topologically trivial.

1. Non-trivial holomorphic bundles over elliptic curves.

Let $E_G$ be a principle $G$-bundle over an elliptic curve $\Sigma_\tau = \mathbb{C}/(\tau \mathbb{Z} + \mathbb{Z})$ and $\pi$ is a representation of $G$ in $V$. Following [42] we define a $G$-bundle $E = E_G \times_G V$ by the transition operators $Q$ and $\Lambda$ acting on sections of $E$ as

$$s(z + 1) = \pi(Q)s(z), \quad s(z + \tau) = \pi(\Lambda)s(z), \quad Q, \Lambda \in G,$$

where $Q$ and $\Lambda$ take values in $G$. The compatibility of this system dictates the following equation for the transition operators

$$Q(z + \tau)\Lambda(z)Q(z)^{-1}\Lambda^{-1}(z + 1) = Id.$$

Let $\zeta$ be an element of the center $Z(G)$ of $G$. Assume that $Q$ and $\Lambda$ satisfy the equation

$$Q(z + \tau)\Lambda(z)Q(z)^{-1}\Lambda^{-1}(z + 1) = \zeta.$$

Then $Q$ and $\Lambda$ can serve as transition operators only for a $G^{ad} = G/Z(G)$-bundle, but not for a $G$-bundle and $\zeta$ is an obstruction to lift a $G^{ad}$-bundle to a $G$-bundle.

More generally, consider a $G$-bundle over a Riemann surface $\Sigma$ and assume that $G$ has a nontrivial center $Z(G)$. It means that $G$ is a classical simply-connected group, or some of its subgroups, or a simply-connected group of type $E_6$ or $E_7$. The topologically non-trivial $G$-bundles are characterized by elements of $H^2(\Sigma, Z(G))$. We call them the characteristic classes, since for $G = Spin_n$ they coincide with the Shaferevich-Whitney classes.

It follows from [42] that it is possible to choose the constant transition operators. Then we come to the equation on $G$

$$Q\Lambda Q^{-1}\Lambda^{-1} = \zeta. \quad (1.1)$$

We describe the set $M(G) =$ (solutions of (1.1))/(conjugations), when $G = \tilde{G}$ is a simply-connected group. Assume that $Q$ is a semisimple element and $Q \in H_G$, where $H_G$ is a Cartan subgroup. Then $M(G) = (Q, \Lambda)$ is defined as

$$Q = \exp \left(2\pi i \frac{Q}{\hbar}\right)U, \quad \Lambda = \Lambda^0 V,$$

\footnote{The case $G \subset G$ and $Z(G)$ is nontrivial is also analyzed.}
\(\rho^\vee\) is a half-sum of positive coroots, \(h\) is the Coxeter number, \(\Lambda^0\) is an element of the Weyl group defined by \(\zeta\). It is a symmetry of the extended Dynkin diagram of \(\mathfrak{g} = \text{Lie}(\bar{G})\). \(V\) and \(U\) are arbitrary elements of the Cartan subgroup \(\mathcal{H}_0 \subset \mathcal{H}_G\) commuting with \(\Lambda^0\) and \(\tilde{\mathcal{H}}_0 = \text{Lie} \mathcal{H}_0\) is a Cartan subalgebra corresponding to a simple Lie subgroup \(\tilde{G}_0 \subset G\).

Since \((\Lambda^0)^l = 1\) for some \(l\), the adjoint action of \(\Lambda^0\) on \(g\) is an automorphism of order \(l\). All such automorphisms described in \([28]\). \(\text{Ad}(\Lambda^0)\) induces a \(\mu_l = \mathbb{Z}/l\mathbb{Z}\) gradation in \(g\)

\[g = \bigoplus_{k=0}^{l-1} g_k,\]

where \(g_0\) is a reductive subalgebra. The Lie algebra \(\tilde{g}_0 = \text{Lie}(\tilde{G}_0)\) in its turn is a subalgebra of \(g_0\). The concrete forms of invariant subalgebras are presented in Table 1. They will be calculated in \([33]\).

| Group          | \(\text{ord}(\Lambda^0)\) | \(\tilde{g}_0\)           | \(g_0\)             |
|----------------|---------------------------|---------------------------|---------------------|
| \(\text{SL}(N, \mathbb{C})\) \((N = pl)\) | \(N/p\)                   | \(\mathfrak{sl}_p\)     | \(\mathfrak{sl}_p \oplus \sum_{j=1}^{l-1} \mathfrak{g}_l\) |
| \(\text{SO}(2n + 1)\)          | 2             | \(\mathfrak{so}(2n - 1)\) | \(\mathfrak{so}(2n)\)  |
| \(\text{Sp}(2l)\)            | 2             | \(\mathfrak{so}(2l)\)    | \(\mathfrak{gl}_{2l}\) |
| \(\text{Sp}(2l + 1)\)         | 2             | \(\mathfrak{so}(2l + 1)\) | \(\mathfrak{gl}_{2l+1}\) |
| \(\text{SO}(4l + 2)\)         | 4             | \(\mathfrak{so}(2l - 1)\) | \(\mathfrak{so}(2l) \oplus \mathfrak{so}(2l) \oplus \mathbb{1}\) |
| \(\text{SO}(4l + 2)\)         | 2             | \(\mathfrak{so}(4l - 1)\) | \(\mathfrak{so}(4l) \oplus \mathbb{1}\) |
| \(\text{SO}(4l)\)            | 2             | \(\mathfrak{so}(2l)\)    | \(\mathfrak{so}(2l) \oplus \mathfrak{so}(2l)\) |
| \(\text{SO}(4l)\)            | 2             | \(\mathfrak{so}(4l - 3)\) | \(\mathfrak{so}(4l - 2) \oplus \mathbb{1}\) |
| \(E_6\)                     | 3             | \(\mathfrak{g}_2\)      | \(\mathfrak{so}(8) \oplus 2 \cdot \mathbb{1}\) |
| \(E_7\)                     | 2             | \(\mathfrak{f}_4\)      | \(\mathfrak{e}_6 \oplus \mathbb{1}\) |

**Table 1.** \(\Lambda^0\)-invariant subgroups and subalgebras

Since \(Z(\text{SO}(4l)) = \mu_2 \oplus \mu_2\) we take two different \(\Lambda^0_a\), \((a = 1, 2)\).

A big cell in the moduli space of trivial holomorphic bundles over an elliptic curve is a quotient of the Cartan subalgebra \(\mathfrak{h}_0\) of \(G\) under the action of some discrete group. For \(G = GL_N\) the moduli space was described by M. Atiyah \([1]\). For trivial \(G\)-bundles, where \(G\) is a complex simple group, it was done in \([6, 37]\). Nontrivial \(G\)-bundles and their moduli space was considered in \([20, 48]\).

It is important for applications to consider holomorphic bundles with quasi-parabolic structures at marked points at \(\Sigma_\tau\). It means that the automorphisms of the bundles (the gauge transformations) preserve flags \(Fl_\alpha\) located at \(n\) marked points \([49]\). The structure of a big cell \(\mathcal{M}^0_{g,n}\) (\(g = 1\)) in the moduli space of these bundles can be extracted from the moduli space \(\mathcal{M}(G)\) of solutions of (1.1). In the simplest case \(n = 1\)

\[\tilde{\mathcal{M}}^0_{1,1} = (\tilde{\mathcal{H}}_0/\tilde{W}_{BS}) \times (Fl/\mathcal{H}_0),\]

where \(\tilde{W}_{BS}\) are the Bernstein-Schwarzman generalizations \([6]\) of the affine Weyl groups \(W^{aff}(\tilde{G}_0)\), corresponding to different sublattices of the coweight lattice. Note that for the trivial bundles \(\Lambda^0\) can be chosen as \(Id\). In this case

\[\mathcal{M}^0_{1,1} = (\mathcal{H}_0/W_{BS}) \times (Fl/\mathcal{H}_G),\]

\[^{2}\text{Sp}(n)\) is a group preserving antisymmetric form in \(\mathbb{C}^{2n}\).
where \( \mathfrak{g} = \text{Lie}(\mathcal{H}_G) \). Thus, the big cell \( \tilde{M}_{1,1}^0 \) for the nontrivial \( G \) bundles is the same as the big cell \( M_{1,1}^0 \) for trivial \( \tilde{G}_0 \)-bundles. A detail description is given in Section 3.2.

By product, we obtained some additional results related to this subject. We describe an interrelation between the characteristic classes and degrees of some bundles. In the \( A_{N-1} \) case this relation is simple. The center of \( G = \text{SL}(N, \mathbb{C}) \) is the cyclic group \( \mu_N = \mathbb{Z}/N\mathbb{Z} \). The cohomology group \( H^2(\Sigma, \mathbb{Z}(\text{SL}(N, \mathbb{C}))) \) is isomorphic to \( \mu_N \). Represent elements of \( \mu_N \) as \( \exp \frac{2\pi i}{N} j \), \( j = 1, \ldots, N-1 \). Let \( \zeta \) be a generator of \( \mu_N \). Consider a principle \( \text{PGL}(N, \mathbb{C}) \) bundle with the characteristic classes \( \zeta \). It cannot be lifted to a \( \text{SL}(N, \mathbb{C}) \)-bundle, but can be lifted to a \( \text{GL}(N, \mathbb{C}) \) bundle. The degree of its determinant bundle \( \text{deg} E \) is \(-1\) and \( \zeta = \exp (-\frac{2\pi i}{N}) = \exp (\frac{2\pi i \text{deg} E}{\text{rank} E}) \).

We generalize this construction to other simple groups. To this end for a simple group \( G \) we define its conformal version \( CG \) (Definition 3.2). In particular, for the symplectic and orthogonal groups their conformal versions are groups preserving (anti)symmetric forms up to dilatations. It allows us to relate the characteristic classes of \( G \)-bundles to degrees of the determinant bundles of \( CG \) (Theorem 3.1).

We introduce a special basis in \( \mathfrak{g} = \text{Lie} G \). In the \( A_{N-1} \) case it is the basis of the finite-dimensional sin-algebra [19], generated by the t’Hooft matrices \( \mathcal{Q}, \Lambda \) \( (\mathcal{Q} \Lambda \mathcal{Q}^{-1} \Lambda^{-1} = \exp (\frac{2\pi i}{N}) ) \). We call it the generalized sinus (GS) basis and use it in the context of integrable systems.

2. Integrable systems.

Using the above construction we describe a new class of the finite-dimensional classical completely integrable systems related to simple Lie groups with nontrivial centers. They are generalizations of the elliptic Calogero-Moser systems, in general with spin degrees of freedom. Calogero-Moser systems (CM) were originally defined in quantum case by Francesco Calogero [11] and in classical case by Jurgen Moser [40], as an integrable model of one-dimensional nuclei. Now they play an essential role both in mathematics and in theoretical physics. \(^3\)

Their generalizations as integrable systems related to simple Lie groups has a long history. It was started more than thirty years ago [43], but the classical integrability was proved there only for the classical groups. It was done later in [8, 25]. They are the so-called spinless CM systems. The case of the \( A_{n-1} \) type (\( \text{SL}(n) \)) systems is very special. The integrability of these systems for rational and trigonometric potentials has a natural explanation in terms of Hamiltonian reduction [31, 43]. Later this approach was generalized for a wide class of classical integrable systems - the so-called Hitchin systems [24]. It was realized in [13, 23, 32, 39] that the \( A_{n-1} \) type CM systems with elliptic potential are particular examples of the Hitchin systems.

From the point of view of the Hitchin construction it is more natural to consider CM systems with spin, introduced in the \( A_{n-1} \) case in [21, 52]. \(^4\) Their description for all simple Lie algebras can be found in [8, 25]. Generically, the Hitchin systems come up as a result of the Hamiltonian reduction of the Higgs bundles [24]. Upon the reduction we obtain an integrable systems in the Lax form, where the Lax operator depends on a spectral parameter belonging to the base of the bundle. The reduced phase is the cotangent bundle to the moduli space of holomorphic bundles with the quasi-parabolic structure. The CM systems with spin appear as a result of the Hamiltonian reduction of the quasi-parabolic Higgs bundles over an elliptic curve with one marked point.

It turns out that the standard classification of the CM systems is based on topologically trivial bundles. The primary goal of this paper is a classification of MCM systems related to topologically non-trivial bundles. A particular examples related to \( \text{SL}(N, \mathbb{C}) \) are known. If the characteristic class of the bundle \( \zeta = \exp (-\frac{2\pi i}{N}) \), instead of interacting CM particles we come to

\(^3\)The mathematical aspects of the systems are discussed in [13].

\(^4\)The spinless CM systems considered in [8, 25] was described as Hitchin systems in [20].
the Euler-Arnold (EA) top \[2\] related to SL(N, \mathbb{C}) \[30, 34, 46\]. This top describes the classical degrees of freedom on a vertex in the vertex spin chain. The corresponding classical r matrix is non-dynamical \[5\]. But if \(N = pl\) there exists an intermediate situation \[36\] described in column 2:

| \(\zeta\) | 1 | \(\exp\left(-\frac{2\pi i p}{N}\right), \ N=pl\) | \(\exp\left(-\frac{2\pi i}{N}\right)\) |
|---|---|---|---|
| System | SL\(_N\)-CM system | SL\(_p\)-CM-system +l interacting EA-tops | SL\(_N\)-EA-top |

Table 2.

Integrable systems corresponding to different characteristic classes of SL(N) bundles.

In this paper we construct Lax operators, quadratic Hamiltonians and corresponding classical dynamical r-matrices for any simple complex Lie group G with a non-trivial center and arbitrary characteristic classes \(\zeta \in H^2(\Sigma, Z(G))\). The obtained elliptic r-matrices are completion of the list \[16, 38\], because the dynamical parameters belongs to the Cartan subalgebra \(\tilde{H}_0 \subset H_G\). This type of r-matrices in the trigonometric case were constructed in \[15, 47\], using an algebraic approach.

In fact, \(\tilde{H}_0\) is the same Cartan subalgebra that participates in the definition of the moduli space \(\Sigma\). Let us explain this phenomena. The phase space of the Hitchin systems is the moduli space \(\mathcal{M}^H_{\Sigma_n}\) of the Higgs bundles over a curve \(\Sigma_n\) with the quasi-parabolic structure at \(n\) marked points. It is a bundle over the moduli space \(\mathcal{M}^H_{\Sigma}\) of the Higgs bundles over the compact curve \(\Sigma\). The base \(\mathcal{M}^H_{\Sigma}\) can be interpreted as the phase space of interacting particles. It is the cotangent bundle to the moduli space \(\mathcal{M}_\Sigma\) of holomorphic bundles over \(\Sigma\). The fibers \(\mathcal{M}^H_{\Sigma_n} \to \mathcal{M}^H_{\Sigma}\) are coadjoint \(G\)-orbits located at the marked points. The coordinates on the orbits are called the spin variables. \(^5\) If the number of the marked points \(n = 1\) and the \(G\)-bundle over the elliptic curve has a trivial characteristic class, then the spin variables can be identify with angular velocities of the EA top related to \(G\). The inertia tensor of the top depends on coordinates of CM particles related to the same group \(G\). The configuration spaces of particles are the quotient of the Cartan algebra as in \(\Sigma\). It the space of dynamical parameters of \(r\).

For non-trivial bundles the configuration space of particles is quotient of the Cartan subalgebra \(\tilde{H}_0 \subset \tilde{H}\) and the dynamical r-matrices depends on variables belonging \(\tilde{H}_0\). The integrable system looks as interacting EA tops with parameters depending on coordinates of CM system related to \(\tilde{G}_0\). For this reason we call \(\tilde{G}_0\) the unbroken subgroup (see Table 1).

Solutions of \(\Sigma\) allows us to define the Lax operators for non-trivial bundles. We describe the Poisson brackets for the Lax operators in terms of classical dynamical r-matrix following the papers \[3, 7, 10, 17, 38\]. This form contains an anomalous term preventing the integrability of the system upon the Hamiltonian reduction with respect to action the Cartan subgroup \(\tilde{H}_0\). We prove the classical dynamical Yang-Baxter equation for the r-matrix defined in \[15, 50\].

It is worthwhile to emphasize that for the standard CM systems we deal in fact with a few different systems. More exactly, we have as many configuration spaces as a number of non-isomorphic moduli spaces. It amounts to existence of different sublattices in the coweight lattice containing the coroot lattice. A naive explanation of this fact is as follows. The potential of the system has the form \(\wp(\langle u, \alpha \rangle)\), where \(u\) is a coordinate vector, \(\alpha\) is a root and \(\wp\) is the

\(^5\)For elliptic curves the phase space of the spin variables is a result of a Hamiltonian reduction of the coadjoint orbits with respect to action of the Cartan subgroup.
Weierstrass function. Adding to $u$ any combination $\gamma_1 + \gamma_2 \tau$, where $\gamma_2 \in Q^\vee$-coroot lattice, does not change the potential, because $\wp((u, \alpha))$ is a double-periodic on the lattice $\tau \mathbb{Z} \oplus \mathbb{Z}$ and $(\gamma, \alpha)$ is an integer. Thus, the configuration space is the quotient $\mathcal{H}/(\tau Q^\vee + Q^\vee)$. It is the most big configuration space. But we can harmlessly shift as well by the coweight lattice $\tau P^\vee \oplus P^\vee$.

Then we come to a different configuration space (the smallest one). For $A_{N-1}$ root systems we describe in this way the $\text{SL}(N, \mathbb{C})$ and $\text{PSL}(N, \mathbb{C})$ CM systems. Their configuration spaces are different, while the Hamiltonians are the same. Evidently, this fact becomes important for the quantum systems. The same is valid for the systems with non-trivial characteristic class. But now one should consider the lattices related to the unbroken subgroups.

Finally, we should mention that in spite of apparent dissimilarity of Hamiltonians with different characteristic classes, the corresponding integrable systems are symplectomorphic. In particular, the MCM systems are symplectomorphic to the standard spin CM systems. The symplectomorphisms are provided by the so-called Symplectic Hecke Correspondence [34]. Following [29] the Symplectic Hecke Correspondence can be explained in terms of monopole solutions of the Bogomolny equation. Details can be found in [35].

Acknowledgments.
The work was supported by grants RFBR-09-02-00393, RFBR-09-01-92437-K Ea, NSh-3036.2008.2, RFBR-09-01-93166-NCNILa (A.Z. and A.S.), RFBR-09-02-93105-NCNILa (M.O.) and to the Federal Agency for Science and Innovations of Russian Federation under contract 14.740.11.0347. The work of A.Z. was also supported by the Dynasty fund and the President fund MK-1646.2011.1. A.L and M.O. are grateful for hospitality to the Max Planck Institute of Mathematics, Bonn, where the part of this work was done.

2 Holomorphic bundle.

Global description of holomorphic bundles

Here we define holomorphic bundles over a Riemann surface $\Sigma_g$ of genus $g$ following the approach developed in [42].

Let $\pi_1(\Sigma_g)$ be a fundamental group of $\Sigma_g$. It has $2g$ generators $\{a_\alpha, b_\alpha\}$, corresponding to the fundamental cycles of $\Sigma_g$ with the relation

$$\prod_{\alpha=1}^{g} [b_\alpha, a_\alpha] = 1,$$

(2.1)

where $[b_\alpha, a_\alpha] = b_\alpha a_\alpha b_\alpha^{-1} a_\alpha^{-1}$ is the group commutator.

Consider a finite-dimensional representation $\pi$ of a simple complex Lie group $G$ in a space $V$. Let $\mathcal{E}_G$ is a principle $G$-bundle over $\Sigma_g$. We define a holomorphic $G$-bundle $E = \mathcal{E}_G \times_G V$ (or in more detail $E_G$ or $E_G(V)$) over $\Sigma_g$ using $\pi_1(\Sigma_g)$. The bundle $E_G$ has the space of sections $\Gamma(E_G) = \{s\}$, where $s$ takes values in $V$. Let $\rho$ be a representation of $\pi_1$ in $V$ such that $\rho(\pi_1) \subset \pi(G)$. The bundle $E_G$ is defined by transition matrices of its sections around the fundamental cycles. Let $z \in \Sigma_g$ be a fixed point. Then

$$s(a_\alpha z) = \rho(a_\alpha) s(z), \quad s(b_\beta z) = \rho(b_\beta) s(z).$$

(2.2)

In what follows we use the second Eisenstein function $E_2(z)$. It differs from $\wp$ on a constant.
Thus, the sections are defined by their quasi-periodicities on the fundamental cycles. Due to (2.1) we have
\[\prod_{\alpha=1}^{g} [\rho(b_\alpha), \rho(a_\alpha)] = \text{Id}.\] (2.3)

The $G$-bundles described in this way are topologically trivial. To consider less trivial situation assume that $G$ has a non-trivial center $\mathcal{Z}(G)$. Let $\zeta \in \mathcal{Z}(G)$. Replace (2.3) by
\[\prod_{\alpha=1}^{g} [\rho(b_\alpha), \rho(a_\alpha)] = \zeta.\] (2.4)

Then the pairs $\hat{(\rho(a_\alpha), \rho(b_\beta))}$, satisfying (2.4), cannot describe transition matrices of $G$-bundle, but can serve as transition matrices of $G_{ad} = G/\mathcal{Z}(G)$-bundle. The bundle $E_G$ in this case is topologically non-trivial and $\zeta$ represents the characteristic class of $E_G$. It is an obstruction to lift $G_{ad}$ bundle to $G$ bundle. We will give a formal definition in Section 44.

The transition matrices can be deformed without breaking (2.3) or (2.4). Among these deformations are the gauge transformations
\[\rho(a_\alpha) \rightarrow f^{-1} \rho(a_\alpha)f, \quad \rho(b_\beta) \rightarrow f^{-1} \rho(b_\beta)f.\] (2.5)

The moduli space of holomorphic bundles $M_g$ is the space of transition matrices defined up to the gauge transformations. Its dimension is independent on the characteristic class and is equal to
\[\dim(M_g) = (g - 1) \dim(G).\] (2.6)

It means that the nonempty moduli spaces arise for the holomorphic bundles over surfaces of genus $g > 1$.

To include into the construction the surfaces with $g = 0, 1$ consider a Riemann surface with $n$ marked points and attribute $E$ with what is called the quasi-parabolic structure at the marked points. Let $B$ be a Borel subgroup of $G$. We assume that the gauge transformation $f$ preserves the flag variety $Fl = G/B$. It means that $f \in B$ at the marked points. It follows from (A.27)
\[\dim(M_{g,n}) = (g - 1) \dim(G) + n \dim(Fl) = (g - 1) \dim(G) + n \sum_{j=1}^{\text{rank } G} (d_j - 1).\] (2.7)

In the important for applications case $g = 1, n = 1 \dim(M_{g,1}) = \dim(Fl)$.

**Local description of holomorphic bundles and modification**

There exists another description of holomorphic bundles over $\Sigma_g$. Let $w_0$ be a fixed point on $\Sigma_g$ and $D_{w_0}$ ($D_{w_0}^\times$) be a disc (punctured disc) with a center $w_0$ with a local coordinate $z$. Consider a $G$-bundle $E_G = E_G \times_G V$ over $\Sigma_g$. It can be trivialized over $D$ and over $\Sigma_g \setminus w_0$. These two trivializations are related by a $G$ transformation $\pi(g)$ holomorphic in $D_{w_0}^\times$, where $D_{w_0}$ and $\Sigma_g \setminus w_0$ overlap. If we consider another trivialization over $D$ then $g$ is multiplied from the right by $h \in G$. Likewise, a trivialization over $\Sigma_g \setminus w_0$ is determined up to the multiplication on the left $g \rightarrow hg$, where $h \in G$ is holomorphic on $\Sigma_g \setminus w_0$. Thus, the set of isomorphism classes of $G$-bundles are described as a double-coset
\[G(\Sigma_g \setminus w_0) \setminus G(D_{w_0}^\times)/G(D_{w_0}),\] (2.8)
where $G(U)$ denotes the group of $G$-valued holomorphic functions on $U$.

To define a $G$-bundle over $\Sigma_g$ the transition matrix $g$ should have a trivial monodromy around $w_0$ $g(z e^{2\pi i}) = g(z)$ on the punctured disc $D_{w_0}^\times$. But if the monodromy is nontrivial

$$g(z e^{2\pi i}) = \zeta g(z), \quad \zeta \in Z(G),$$

then $g(z)$ is not a transition matrix. But it can be considered as a transition matrix for the $G^{ad}$-bundle, since $G^{ad} = G/Z(G)$. This relation is similar to (2.7).

Our aim is to construct from $E$ a new bundle $\tilde{E}$ with a non-trivial characteristic class. This procedure is called a modification of bundle $E$. Smooth gauge transformations cannot change a topological type of bundles. The modification is defined by a singular gauge transformation at some point, say $w_0$. Since it is a local transformation we replace $\Sigma_g$ by a sphere $\Sigma_0 = \mathbb{C}P^1$, where $w_0$ corresponds to the point $z = 0$ on $\mathbb{C}P^1$. Since $z$ is local coordinate, we can replace $G(\Sigma_g \setminus w_0)$ in (2.8) by the group $G(\mathbb{C}(z))$. It is the group of Laurent series with $G$ valued coefficients. Similarly, $G(D_{w_0})$ is replaced by the power series $G(\mathbb{C}[[z]])$. It is clear from this description of the moduli space of bundles over $\mathbb{C}P^1$ that it is a finite dimensional space.

Transform $g(z)$ by multiplication from the right on $g(z) \rightarrow g(z)h(z)$ where $h(z)$ singular at $z = 0$. It is the singular gauge transformation mentioned above. Due to definition of $g(z)$, $h(z)$ is defined up to the multiplication from the right by $f(z) \in G(\mathbb{C}[[z]])$. On the other hand, since $g(z)$ is defined up to the multiplication from the right by an element from $G(\mathbb{C}[[z]])$, $h(z)$ is element of the double coset

$$G(\mathbb{C}[[z]]) \setminus G(\mathbb{C}(z))/G(\mathbb{C}[[z]]).$$

In particular, $h(z)$ is defined up to a conjugation. It means that as a representative of this double coset one can take a co-character (A.31) $h(z) \in t(G)$.

$$g(z) \rightarrow g(z)z^\gamma, \quad (z^\gamma = e(\ln(z\gamma))), \quad (2.9)$$

where $\gamma$ belongs to the coweight lattice $(\gamma = (m_1,m_2,\ldots,m_l) \in P^\vee)$. The monodromy of $z^\gamma$ is $\exp(-2\pi i\gamma)$. Since $\langle \alpha, \gamma \rangle \in \mathbb{Z}$ for any $\alpha \in g$ $A_{\alpha,\exp(-2\pi i\gamma)} x = x$. Then exp $-2\pi i\gamma$ an element of $Z(G)$ (A.43). If the transition matrix $g(z)$ defining $\tilde{E}$ has a trivial monodromy, the new transition matrix $\tilde{g}(z)$ acquires a nontrivial monodromy. In this way we come to a new bundle $\tilde{E}$ with a non-trivial characteristic class. The bundle $\tilde{E}$ is called the modified bundle. It is defined by the new transition matrix (2.9). If $\gamma \in Q^\vee$ then $\zeta = 1$ and the modified bundle $\tilde{E}$ has the same type as $E$.

This transformation of the bundle $E$ corresponds to transformations of its sections $\tilde{E}$

$$\Gamma(E) \xrightarrow{\Xi(\gamma)} \Gamma(\tilde{E}), \quad (\Xi(\gamma) \sim \pi(z^{m_1}, z^{m_2} \ldots z^{m_l})). \quad (2.10)$$

We say that this modification has a type $\gamma = (m_1,m_2,\ldots,m_l)$. Another name of the modification is the Hecke transformation. It acts on the characteristic classes of bundles as follows

$$\Xi(\gamma) : \ln \zeta(E) \rightarrow \ln \zeta(\tilde{E}) = \ln \zeta(E) + 2\pi i\gamma, \quad \gamma \in P^\vee/Q^\vee. \quad (2.11)$$

Consider the action of modification on sections (2.10) in more details. Let $V$ be a space of a finite-dimensional representation $\pi$ of $G$ with a highest weight $\nu$ and $\nu_j$ ($j = 1, \ldots, N$) is a set of its weights

$$\nu_j = \nu - \sum_{\alpha_m \in H} c^m_j \alpha_m, \quad c^m_j \in \mathbb{Z}, \quad c^m_j \geq 0. \quad (2.12)$$
It means that for $x \in \mathfrak{h}$, $\pi(x)\nu_j = \langle x, \nu_j \rangle \nu_j$. The weights belong to the weight diagram defined by the highest weight $\nu \in P$ of $\pi$. The space $V$ has the weight basis $(\nu_1^{s_1}, \ldots, \nu_N^{s_N})$ in $V$, where $s_1 = 1, \ldots, m_1, \ldots, s_N = 1, \ldots, m_N$ and $m_1, \ldots, m_N$ are multiplicities of weights. Thus, $M = \dim V = \sum m_j$.

Let us choose a trivialization of $E$ over $D$ by fixing this basis. Thereby, the bundle $E$ over $D$ is represented by a sum of $M$ line bundles $L_1 \oplus L_2 \oplus \ldots \oplus L_M$. Cartan subgroup $H$ acts in this basis in a diagonal way: for $s = (\nu_1^{s_1}, \ldots, \nu_N^{s_N})$

$$\pi(h) : \nu_j^{s_j} \rightarrow e(x, \nu_j)\nu_j^{s_j}, \quad h = e(x), \quad x \in \mathfrak{h}, \quad (e(x) = \exp(2\pi i x)).$$

Assume for simplicity that in (2.9) $g(z) = 1$. Then the modification transformation (2.10) of the sections assumes the form

$$\Xi(\gamma) : \nu_j^{s_j} \rightarrow z^{\langle \gamma, \nu_j \rangle} \nu_j^{s_j}, \quad j = 1, \ldots, M.$$  \hspace{1cm} (2.13)

It means that away from the point $z = 0$, where the transformations are singular, the sections of $E$ are the same as of $E$. But near $z = 0$ they are singular with the leading terms $|\nu_j^{s_j}| \sim z^{-\langle \gamma, \nu_j \rangle}$.

It is sufficient to consider the case when $\gamma = \varkappa^{\gamma}$ is a fundamental coweight and $\pi$ is a fundamental representation $\nu = \varkappa_k$. Then from (2.12) we have

$$z^{\langle \gamma, \nu_j \rangle} = z^{\langle \varkappa^{\gamma}, \varkappa_k - \sum_{\alpha_m \in \Pi} c_{\gamma}^m \alpha_m \rangle}.$$  \hspace{1cm} (2.13)

The weight $\varkappa_k$ can be expanded in the basis of simple roots $\varkappa_k = \sum_k A_{km} \alpha_m$, where $A_{jk}$ is the inverse Cartan matrix $(A_{jk} A_{ki} = \delta_{ji})$. Its matrix elements are rational numbers with the denominator $N = \ord(\mathcal{Z})$. Then from (2.12)

$$z^{\langle \gamma, \nu_j \rangle} \sim z^{\frac{1}{N} + m}, \quad l, m \in \mathbb{Z}.$$  \hspace{1cm} (2.13)

Note, that the branching does not happen for $G^{ad}$-bundles, because the corresponding weights $\nu_j$ belong to the root lattice $Q$ and thereby $\langle \gamma, \nu_j \rangle \in \mathbb{Z}$.

It is possible to go around the branching by multiplying the sections on a scalar matrix of the form $\text{diag}(z^{-A_{kl}}, \ldots, z^{-A_{Nl}})$. This matrix no longer belongs to the representation of $G$, because it has the determinant $z^{-\sum A_{kl}} (M = \dim V)$. It can be checked that $MA_{jk}$ is an integer number.

If $G = \text{SL}(N, \mathbb{C})$ the scalar matrix belongs to $\text{GL}(N, \mathbb{C})$. Thereby, after this transformation we come to a $\text{GL}(N, \mathbb{C})$-bundle. But this bundle is topologically non-trivial, because it has a non-trivial degree. In this way the characteristic classes for the $\text{SL}(N, \mathbb{C})$-bundles are related to another topological characteristic, namely to degrees of the $\text{GL}(N, \mathbb{C})$-bundles. We describe below the similar construction for other simple groups.

3 Holomorphic bundles over elliptic curves

Hereinafter we consider the bundles over an elliptic curve, described as the quotient $\Sigma_\tau \sim \mathbb{C}/(\tau \mathbb{C} \oplus \mathbb{C})$, $(\Im \tau > 0)$. There are two fundamental cycles corresponding to shifts $z \rightarrow z + 1$ and $z \rightarrow z + \tau$. Let $G$ be a complex simple Lie group. Sections of a $G$-bundle $E_G(V)$ over $\Sigma_\tau$ satisfy the quasi-periodicity conditions (2.2)

$$s(z + 1) = \pi(\mathcal{Q}) s(z), \quad s(z + \tau) = \pi(\Lambda) s(z), \quad (3.1)$$
where $Q, \Lambda$ take values in $G$. A bundle $\tilde{E}$ is equivalent to $E$ if its sections $\tilde{s}$ are related to $s$ as $\tilde{s}(z) = f(z)s(z)$, where $f(z)$ is invertible operator in $V$. It follows from (3.1) that the transition operators, have the form

$$\tilde{Q} = f(z + 1)Qf^{-1}(z), \quad \tilde{\Lambda} = f(z + \tau)\Lambda f^{-1}(z).$$

As we have mentioned, the moduli space $M_{1,n}$ is the quotient space of pairs $(Q, \Lambda)$ with respect to this action. In what follows we consider the simplest case $n = 1$, though our construction is applicable for arbitrary $n$.

The transition operators define a trivial bundle if $[Q, \Lambda] = \text{Id}$. Let $\zeta$ be an element of $\bar{G}$. To come to a nontrivial bundle we should find solutions $\Lambda, Q \in \bar{G}$ of the equation

$$\Lambda Q \Lambda^{-1} Q^{-1} = \zeta.$$  

It follows from (A.44) that the r.h.s. can be represented as $\zeta = e(-\varpi^\vee)$, where $\varpi^\vee \in P^\vee$. Then (3.3) takes the form

$$\Lambda Q \Lambda^{-1} Q^{-1} = e(-\varpi^\vee), \quad (e(x) = \exp(2\pi ix)).$$  

It follows from [42] that the transition operators can be chosen as constants. Therefore, to describe the moduli space of holomorphic bundles we should find a pair $Q, \Lambda \in G$ satisfying (3.3) and defined up to the conjugation

$$\Lambda \rightarrow f\Lambda f^{-1}, \quad Q \rightarrow fQf^{-1}. \quad (3.5)$$

Let $g = \bar{G}$ be a simply-connected group. Let us fix a Cartan subgroup $H \subset \bar{G}$. Assume that $Q$ is semisimple, and therefore is conjugated to an element from $H$. We will see that by neglecting non-semisimple transition operators we still define a big cell in the moduli space. Our goal is to find solutions of (3.3), where $Q$ is a generic element of a fixed Cartan subgroup $H \subset G$.

**Algebraic equation**

**Proposition 3.1** Solutions of (3.4) up to the conjugations have the following description.

- The element $\Lambda$ has the form $\Lambda = \Lambda^0 V$, where $\Lambda^0$ is defined uniquely by the coweight $\varpi^\vee$ ($\Lambda^0 = \Lambda^0_\varpi$). It is an element from the Weyl group $W$ preserving the extended coroot system $\Pi^\vee_{\text{ext}} = \Pi^\vee \cup \alpha_\varpi^\vee$, and in this way is a symmetry of the extended Dynkin diagram. $V \in H \bar{G}$ commutes with $\Lambda^0$.

- The element $Q$ has the form $Q = Q^0 U$, where

$$Q^0 = \exp 2\pi i\kappa, \quad \kappa = \frac{\rho^\vee}{h} \in \mathfrak{h},$$

where $h$ is the Coxeter number, $\rho^\vee = \frac{1}{2} \sum_{\alpha^\vee \in (R^\vee)^+} \alpha^\vee$ and $U$ commutes with $\Lambda^0$. \[7\]

**Proof**

In (3.6) $\kappa$ can be chosen from a fixed Weyl chamber $\bar{A}$. From (A.40) and (A.42) we find that if $\kappa_0 \in Q^\vee$ then $e(\kappa_0) = \text{Id}$. Therefore, by shifting $\kappa \rightarrow \kappa + \gamma, \gamma \in Q^\vee$, $\kappa$ can be put in $C_{\text{alc}}$ (A.16). Rewrite (3.3) as

$$\Lambda Q \Lambda^{-1} = \zeta Q, \quad \zeta = e(-\xi).$$  

Here $\Lambda$ is defined up to multiplication from $H \bar{G}$ and we write it in the form $\Lambda^0 V, V \in H \bar{G}$.
Lemma 3.1 There exists a conjugation $f$ such $Q \rightarrow Q$ and $\Lambda^0 V \rightarrow \Lambda^0 V_\lambda$, and $\Lambda^0 V_\lambda = V_\lambda \Lambda^0$.

Proof. Let us take $f \in H_C$. Then $f$ preserves $Q$. It acts on the second transition operator as

$$\Lambda^0 V \rightarrow f\Lambda^0 f^{-1} V = \Lambda^0 (\Lambda^0)^{-1} f\Lambda^0 f^{-1} V.$$ 

Define $V_\lambda$ as $V_\lambda = (\Lambda^0)^{-1} f\Lambda^0 f^{-1} V$. Our goal is to prove that there exists such $f$ that $V_\lambda$ commutes with $\Lambda^0$. In other words, $f\Lambda^0 f^{-1} V = \Lambda^0)^{-1} f\Lambda^0 f^{-1} V_\lambda$. Let $V = e(x)$, $f = e(y)$, $x$, $y \in H$, $\lambda = Ad_{\Lambda^0}$. Then the commutativity condition takes the form $(\lambda - 1) x = (\lambda^{-1} - 1) y + (\lambda - 1) y$.

Let $l$ be an order of $\Lambda^0$, $((\Lambda^0)^l = 1)$. Then a solution of this equation is given by a sum

$$y = \frac{1}{l} \sum_{i=1}^{l} i\lambda^i(x).$$

Thus $\Lambda^0$ and $V$ defines $V_\lambda = e(p)$ commuting with $\Lambda^0$, where $p$ is the average along the $\lambda$-orbit

$$p = \frac{1}{l} \sum_{i=0}^{l-1} \lambda^i(x).$$

On the next step we find $\Lambda^0$. Rewrite (3.4) in the form

$$\lambda(k) = \kappa - \xi, \quad \xi = w_j^\gamma, \quad \lambda = Ad_{\lambda},$$

where $\kappa \in C_{alc}$. Define a subgroup $\Gamma_{alc}$ of the affine Weyl group $W_a$ that preserves $C_{alc}$. It acts by permutations on its vertices. Equivalently, $\Gamma_{alc}$ acts by permutations of nodes of the extended Dynkin graph. The face of $C_{alc}$ belonging to the hyperplane $\langle \alpha_i, x \rangle = 0$ contains all vertices except $w_j^\gamma/n_j$. Similarly, the face belonging to the hyperplane $\langle \alpha_0, x \rangle = 1$ contains all vertices except 0. By this duality the permutations of vertices by $g = (\lambda, \xi) \in \Gamma_{alc}$ correspond to permutations of the faces, and in this way to permutations of the cooroots $\Pi^\text{ext}$. Instead of (3.3) consider $\lambda(C_{alc}) + \xi = C_{alc}$. The left hand side of this equation is a transformation $g = (\lambda, \xi) \in \Gamma_{calc}$. Let us take $\xi = w_j^\gamma$, where $w_j^\gamma$ is a fundamental coweight that is a vertex of $C_{alc}$ ($n_j = 1$ in (A.17)). Remember, that only these $w_j^\gamma$ define nontrivial elements of the quotient $P^\gamma/Q^\gamma$. Then we have

$$\lambda_j(C_{alc}) = C_{alc} - w_j^\gamma \equiv C_{alc}'.$$ 

The node 0 of $C_{alc}'$ is an image of the node $w_j^\gamma$ of $C_{alc}$ after the shift. Let us define $\lambda_j$. The Weyl group $W$ action on the Weyl alcoves that contains 0 is simple transitive. Therefore, there exists a unique $\lambda_j \in W$ such that $\lambda_j(C_{alc}) = C_{alc}'$. Then $(\lambda_j, w_j^\gamma) \in \Gamma_{calc}$ defines a transformation of $C_{alc}$, which is a permutation of its vertices (A.17) such that $w_j^\gamma \rightarrow 0$. Taking into account the action of $\Gamma_{alc}$ on the extended Dynkin graph we find

$$\lambda_j^\gamma(\alpha_k) = \begin{cases} \alpha_m & k \neq j \\ -\alpha_0 & k = j \end{cases}, \alpha_m \in \Pi.$$ 

Thus, taking $\xi = w_j^\gamma$ we find $\lambda_j$. 

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Fixed points of the $\Gamma_{ad,\lambda}$-action are solutions of (3.9). It will give us $\kappa$ and in this way $Q$. Let us prove that a particular solution of (3.9) is

$$\kappa = \frac{\rho^\vee}{h},$$

(3.12) where $h$ is the Coxeter number (A.7). The equation (3.9) is equivalent to

$$\langle \kappa, \lambda^\vee_j(\alpha_k) \rangle = \langle \kappa, \alpha_k \rangle - \delta_{jk}, \quad \alpha_k \in \Pi, k = 1,\ldots,l.$$

(3.13)

Since $\rho^\vee = \sum_{m=1}^{l} \varpi^\vee_m$ (see (A.13)) for $k \neq j$ (3.13) becomes a trivial identity. For $k = j$ using (A.6) we obtain $-\frac{i}{h} \sum_{m=1}^{l} n_m - \frac{i}{h} = -1$. It follows from (A.7) that it is again identity.

An arbitrary solution of (3.9) takes the form

$$\kappa = \frac{\rho^\vee}{h} + q, \quad q \in \text{Ker}(\lambda_j - 1).$$

In other words, the Weyl transformation $\lambda_j$ should preserve $q$.

Thus, taking in (3.3) $\zeta = \exp(-2\pi i \varpi^\vee_j)$ we find solutions ($\Lambda_j = \Lambda^0_j V_\lambda, Q$), where $\Lambda^0_j$ is a symmetry of the extended Dynkin graph corresponding to $\varpi^\vee_j$ and

$$Q = \exp 2\pi i \left(\frac{\rho^\vee}{h} + q \right).$$

(3.14)

The pair $(p, q)$ (3.8), belonging to the Cartan subalgebra $\mathfrak{h}$, plays the role of the moduli parameters of solutions to (3.4).

□

Remark 3.1 For $\text{Spin}(4n)$ there are two generators $\zeta_1$ and $\zeta_2$ of $Z(\text{Spin}(4n)) \sim \mu_2 \oplus \mu_2$ corresponding to the fundamental weights $\varpi_a, \varpi_b$ of the left and the right spinor representations. Arguing as above we will find two solutions $\Lambda_a$ and $\Lambda_b$ of (3.4), while $Q$ is the same in the both cases.

Consider a group $G$, $(G \supset G \supset G_{ad})$ and let $\Lambda, Q \in G$. Let us choose $\xi = \varpi$ such that it generates the group $t(G)$ of co-characters $t(G) = P^\vee (A.39), (A.40)$ $t(G) = \varpi + Q^\vee, \; l\varpi \in Q^\vee$. Then $\zeta = \exp(-\varpi)$ is a generator of center $Z(G) \sim P^\vee/t(G) = \mu_1$ (see (A.44)). Arguing as above we come to

**Proposition 3.2** The element $\Lambda$ is defined by the coweight $\varpi^\vee \in W$. It is a symmetry of the extended Dynkin diagram. $\Lambda$ is defined up to invariant elements from $\mathcal{H}_G$.

**•** Let

$$(\lambda(G) - 1)q = 0, \quad q \in \mathfrak{h}, \; \lambda(G) = A \text{d}_\Lambda(G).$$

(3.15)

A general solution of (3.4) is

$$\kappa = \frac{\rho^\vee}{h} + q,$$

(3.16)

Therefore, the group of cocharacters $t(G)$ defines a Weyl symmetry $\Lambda^0(G)$ of the extended Dynkin diagram $\Pi^{\text{ext}}$ such that $\langle \Lambda^0 \rangle \equiv (G) = Id$.

$\Lambda(G)$ and $Q$ play the role of transition operators of $G$-bundles over $\Sigma_r$. A generator $\zeta = \exp(\varpi)$ defines a characteristic class of the bundles. It is an obstruction to lift $G$-bundle to $G$-bundle.

**Remark 3.2** If $\xi \in Q^\vee$ then $\zeta = Id$. It means that we can take $\xi = 0$ as a representative of $P^\vee/Q^\vee$. Then $\lambda = 1$ (see (3.10)) and $\text{Ker}(\lambda - 1) = \mathfrak{h}$. In this case the bundle has a trivial characteristic class, but has holomorphic moduli defined by the vector $q \in \mathfrak{h}$. The corresponding Higgs bundle over $\Sigma_r/(z = 0)$ defines elliptic spin Calogero-Moser system.
The moduli space

We have described a $G$-bundle $E_G(V)$ by the transition operators $(\Lambda = \Lambda^0 e(\mathbf{p}), Q = e(\frac{\rho^\vee}{t} + \mathbf{q})$, where $\Lambda^0$ corresponds to the coweight $\pi^\vee \in P^\vee$. The topological type of $E$ is defined by an element of the quotient $P^\vee/t(G)$. Let us transform $(\Lambda, Q)$ taking in (3.12) $f = e(-qz)$. Since $f$ commutes with $\Lambda^0$ we come to new transition operators $Q = e(\kappa + \mathbf{q}) \rightarrow Q = e(\kappa)$, $\Lambda \rightarrow \Lambda^0 e(\mathbf{p} - q\tau)$. Denote $\mathbf{p} - q\tau = \mathbf{u}$. Then sections of $E_G(V)$ assume the quasi-periodicities

$$s(z + 1) = \pi(e(\kappa)) s(z), \quad s(z + \tau) = \pi(e(\mathbf{u})\Lambda^0) s(z). \tag{3.17}$$

Thus, we come to the transition operators

$$Q = e(\kappa), \quad \Lambda = e(\mathbf{u})\Lambda^0. \tag{3.18}$$

Here $\mathbf{u}$ plays the role of a parameter in the moduli space. In this subsection we describe it in details.

Trivial bundles

Consider first the simplest case $\Lambda = Id$ and $\mathbf{u} \in \mathfrak{h}$ (see Remark 3.2). It means, that $E$ has a trivial characteristic class. The transition transformations $\pi(e(\kappa)), \pi(e(\mathbf{u}))$ lie in of the Cartan subgroup $H_G$ of $G$.

Consider first a bundle $E_{\bar{G}}$ for a simply-connected group $\bar{G}$. Since $t(\bar{G}) \sim Q^\vee$, $\Lambda^0$ and due to (A.40), $e(\mathbf{u} + \gamma) = e(\mathbf{u})$ for $\gamma \in Q^\vee$. Taking into account that $\mathbf{u}$ lies in a Weyl chamber we conclude that in fact $\mathbf{u} \in C_{alc}$ as it was already established. Now apply the transformation $e(\gamma z)$

$$s(z) \rightarrow \pi(e(\gamma z)) s(z), \quad \gamma \in Q^\vee. \tag{3.19}$$

The sections are transformed as

$$s(z + 1) = \pi(e(\kappa)) s(z), \quad s(z + \tau) = \pi(e(\mathbf{u} + \gamma\tau)) s(z). \tag{3.20}$$

Thus, transition operators, defined by parameters $\mathbf{u}$ and $\mathbf{u} + \gamma_1 + \gamma_2 (\gamma_1,2 \in Q^\vee)$, describe equivalent bundles. The semidirect product of the Weyl group $W$ and the lattice $\tau Q^\vee \oplus Q^\vee$ is called the Bernstein-Schwarzman group $\mathfrak{g}$

$$W_{BS} = W \ltimes (\tau Q^\vee \oplus Q^\vee).$$

Thereby, $\mathbf{u}$ can be taken from the fundamental domain $C^{(sc)}$ of $W_{BS}$. Thus,

$$C^{(sc)} = \mathfrak{h}/W_{BS} \text{ is the moduli space of trivial } \bar{G} \text{-- bundles}. \tag{3.21}$$

Consider $G^{ad}$ bundle and let $e(\mathbf{u}) \in G^{ad}$. In this case $e(\gamma) = 1$ if $\gamma \in P^\vee$. Define the group

$$W_{BS}^{ad} = W \ltimes (\tau P^\vee \oplus P^\vee).$$

As above, we come to the similar conclusion:

$$C^{(ad)} = \mathfrak{h}/W_{BS}^{ad} \text{ is the moduli space of trivial } G^{ad} \text{-- bundles}. \tag{3.22}$$

*We will write $\mathbf{u}$ for nontrivial bundles reserving $\mathbf{u}$ for trivial bundles.
Consider a coweight $\varpi^\vee \in P^\vee$ such that $l \varpi^\vee \in Q^\vee$, the coweight lattice $P_l^\vee = \mathbb{Z} \varpi^\vee \oplus Q^\vee$ (A.43). Thus, $P_l^\vee / P_l^\vee \sim \mu_l$. Consider a group $G_l$ (A.33) and generated by a coweight 

$$P_l^\vee, \ l \varpi^\vee \in Q^\vee.$$  

The coweight sublattice $P_l^\vee$ is the group of its cocharacters $t(G_l)$ (A.39). Representations of $G_l$ are defined by the dual to $t(G_l)$ groups of characters $\Gamma(G_l)$ (A.36). The dual to $P_l^\vee$ lattice $P_p \subset P$ has the form 

$$P_p = \mathbb{Z} \varpi + Q, \ p \varpi \in Q.$$

By means of $P_l^\vee$ define the affine group of the Bernstein-Schwarzman type 

$$W_{BS}^{(l)} = W \ltimes (\tau P_l^\vee \oplus P_l^\vee).$$

Making use of the gauge transform $e(\gamma z) \in G_l, \ (\gamma \in P_l^\vee)$ we find that 

$$C^{(l)} = \mathcal{S}_l / W_{BS}^{(l)}$$

is the moduli space of trivial $G_l$-bundles. (3.23)

Consider the dual picture and the lattice $P_p^\vee$. It is formed by $Q^\vee$ and a coweight $\varpi^\vee$ 

$$P_p^\vee = \mathbb{Z} \varpi^\vee + Q^\vee, \ p \varpi^\vee \in Q^\vee.$$ 

The lattice $P_p^\vee$ plays the role of the group of cocharacters for the dual group $^L G_l = G_P = \bar{G} / \mu_p$, (A.33), while $P_l$ of defines characters of $G_P$. Again by means of the group 

$$W_{BS}^{(p)} = W \ltimes (P_p^\vee \oplus P_p^\vee \tau),$$

we find that 

$$C^{(p)} = \mathcal{S}_l / W_{BS}^{(p)}$$

is the moduli space of trivial $G_P$-bundles. (3.24)

Thus, for the $G, G_l, G_P, G_{ad}$ trivial bundles we have the following interrelations between their moduli space 

$$\begin{array}{ccc}
C^{(sc)} & \searrow & C^{(p)} \\
| & & | \\
\searrow & C^{(ad)} & \nearrow \\
C^{(l)} & & \\
\end{array}$$

(3.25)

Here arrows mean coverings. Note that $C^{(sc)}, C^{(ad)}$ and as well $C^{(l)}, C^{(p)}$ are dual to each other in the sense that the defining them lattices are dual.

Let $Fl$ be a flag variety located at the marked point. In this way we have defined a space $\tilde{\mathcal{M}}_{1,1} = (C^a, Fl) \ (a = (sc), (l), (ad))$ related to the moduli space of trivial bundles over $\Sigma_r$ with one marked point. But we still have a freedom to act on $Fl$ by constant conjugations from the Cartan subgroup $\mathcal{H}^a$. Thus, eventually we come to $\mathcal{M}_0^{0,1} = (C^a, Fl/\mathcal{H}^a)$. It has dimension of $\mathcal{M}_{1,1}$ (27). It is a big cell in $\mathcal{M}_{1,1}$. In our construction we have excluded non-semisimple elements $Q$. 

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Nontrivial bundles

Consider a general case $\Lambda^0 \neq Id$. It was explained above that $\Lambda^0$ corresponds to some characteristic class related to $\varpi^\vee \in P^\vee$, and $\varpi^\vee \notin Q^\vee$. In this case $\tilde{u} \in \text{Ker}(\lambda - 1)$, and in fact $\tilde{u} \in C_{alc} \cap \tilde{H}_0$, where $\tilde{H}_0$ is the invariant subalgebra $\lambda(\tilde{H}_0) = \tilde{H}_0$. There is a basis in $\tilde{H}_0$ defined by a system of simple coroots $\tilde{\Pi}^\vee$ (see Section 5.4). Moreover, the corresponding root system defines a simple Lie algebra $\tilde{g}_0$.

Let $\tilde{W}$ be the Weyl group $W$ of the root system $\tilde{R} = \tilde{R}(\tilde{\Pi})$

$$\tilde{W} = \{ w \in \tilde{W} | w(\tilde{R}) = \tilde{R} \}, \quad (3.26)$$

and

$$\tilde{Q}^\vee = \{ \gamma = \sum_{j=1}^{p} m_j \tilde{\alpha}_j^\vee, \ m_j \in \mathbb{Z} \} \quad (3.27)$$

is the coroot lattice generated by $\tilde{\Pi}^\vee$ [5.28]. Consider first $E_{\tilde{G}}$ bundles. As above, $e(\tilde{u} + \gamma) = e(\tilde{u})$, $\gamma \in \tilde{Q}^\vee$. The automorphism (3.19) for $\gamma \in \tilde{Q}^\vee$ commutes with $\Lambda$. Thus, $\tilde{u}$ and $\tilde{u} + \tau \gamma_1 + \gamma_2$ $\gamma_1,2 \in \tilde{Q}^\vee$ define equivalent $\tilde{G}$-bundles. Consider the semidirect products

$$\tilde{W}_{BS} = \tilde{W} \ltimes (\tau \tilde{Q}^\vee \oplus \tilde{Q}^\vee). \quad (3.28)$$

The fundamental domain in $\tilde{H}$ under the $\tilde{W}_{BS}$ action is the moduli space of $\tilde{G}$-bundles with characteristic classes defined by $\varpi^\vee$.

$$\tilde{C}_sc = \tilde{H}/\tilde{W}_{BS} \text{ is the moduli space of nontrivial } \tilde{G} - \text{bundles}, \quad (3.29)$$

Consider $E_{G_{ad}}$-bundles. Let $\tilde{\varpi}^\vee$ be fundamental coweights ($\langle \tilde{\varpi}^\vee_j, \tilde{\alpha}_k \rangle = \delta_{jk}$) and

$$\tilde{P}^\vee = \{ \gamma = \sum_{j=1}^{p} m_j \tilde{\varpi}^\vee_j, \ m_j \in \mathbb{Z} \} \quad (3.30)$$

is the coweight lattice in $\tilde{H}_0$. Define the semidirect product

$$\tilde{W}_{BS}^{ad} = \tilde{W} \ltimes (\tau \tilde{P}^\vee \oplus \tilde{P}^\vee). \quad (3.31)$$

A fundamental domain under its action

$$\tilde{C}_ad = \tilde{H}_0/\tilde{W}_{BS}^{ad} \text{ is the moduli space of nontrivial } G_{ad} - \text{bundles}, \quad (3.32)$$

is a moduli space of a $G_{ad}$-bundle with characteristic class defined by $\varpi^\vee$. If $\text{ord}(\mathbb{Z}(\tilde{G}))$ is not a primitive number then we again come to the hierarchy of the moduli spaces similar to (3.25).

As above the space $\mathcal{M}_{11}^{0} = (C^a, Fl/\tilde{H}_0)$ is a big cell in the moduli space of non-trivial bundles.

4 Characteristic classes and conformal groups

Characteristic classes

Let $E_{\tilde{G}}$ be is a principle $G$-bundle over $\Sigma$. Consider a finite-dimensional representation of complex group $G$ in a space $V$ and let $E_{\tilde{G}}(V)$ be the vector bundle $E_{\tilde{G}}(V) = E_{\tilde{G}} \times_G V$ induced by $V$. 

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The first cohomology $H^1(\Sigma_g, G(\mathcal{O}_\Sigma))$ of $\Sigma$ with coefficients in analytic sheaves define the moduli space $\mathcal{M}(G, \Sigma)$ of holomorphic $G$-bundles. Let $\tilde{G}$ be a simply-connected group and $G^{ad}$ be an adjoint group. Using (A.32) and (A.33) we write three exact sequences

\begin{align*}
1 \to Z(\tilde{G}) \to \tilde{G}(\mathcal{O}_\Sigma) \to G^{ad}(\mathcal{O}_\Sigma) & \to 1, \\
1 \to \mathbb{Z}_l \to G(\mathcal{O}_\Sigma) \to G_l(\mathcal{O}_\Sigma) & \to 1, \\
1 \to Z(G_l) \to G_l(\mathcal{O}_\Sigma) \to G^{ad}(\mathcal{O}_\Sigma) & \to 1,
\end{align*}

where $G_l = \tilde{G}/\mathbb{Z}_l$. Then we come to the long exact sequences

\begin{align*}
& H^1(\Sigma_g, \tilde{G}(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G^{ad}(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, Z(\tilde{G})) \sim Z(\tilde{G})) \to 0, \quad (4.1) \\
& H^1(\Sigma_g, \tilde{G}(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G_l(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, \mathbb{Z}_l) \sim \mu_l \to 0, \quad (4.2) \\
& H^1(\Sigma_g, G_l(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G^{ad}(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, Z(G_l)) \sim \mu_p \to 0. \quad (4.3)
\end{align*}

The elements from $H^2$ are obstructions to lift bundles, namely

\[
\zeta(E_{G^{ad}}) \in H^2(\Sigma_g, Z(\tilde{G})) \text{ -- obstructions to lift } E_{G^{ad}} \text{ -- bundle to } E_G \text{ -- bundle}, \\
\zeta(E_{G_l}) \in H^2(\Sigma_g, \mathbb{Z}_l) \text{ -- obstructions to lift } E_{G_l} \text{ -- bundle to } E_{\tilde{G}} \text{ -- bundle}, \\
\zeta(E_{G^{ad}}) \in H^2(\Sigma_g, Z(G_l)) \text{ -- obstructions to lift } E_{G^{ad}} \text{ -- bundle to } E_{\tilde{G}_l} \text{ -- bundle}.
\]

**Definition 4.1** Images of $H^1(\Sigma_g, G(\mathcal{O}_\Sigma))$ in $H^2(\Sigma_g, Z)$ are called the characteristic classes $\zeta(E_G)$ of $G$-bundles.

Since $\mathbb{Z}_l \to Z(\tilde{G}) \to Z(G_l)$, we have the following relations between these characteristic classes $\zeta(E_{G^{ad}}) = \zeta(E_G) \mod \mathbb{Z}_l$, and the characteristic class $\zeta(E_{G_l})$ coincides with $\zeta(E_{G^{ad}})$ as an obstruction to lift a $E_{G_l}$-bundle, treated as a $E_{G^{ad}}$-bundle to a $E_{\tilde{G}_l}$-bundle.

Consider a particular case $\tilde{G} = SL(N, \mathbb{C})$, $G^{ad} = PSL(N, \mathbb{C})$. Then the elements $\zeta \in Z(SL(N, \mathbb{C})) \sim \mu_N$ are obstructions to lift $PSL(N, \mathbb{C})$-bundles to $SL(N, \mathbb{C})$-bundles. They represent the characteristic classes of $PSL(N, \mathbb{C})$-bundles. On the other hand, the exact sequence

\[
1 \to \mathcal{O}^* \to GL(N, \mathbb{C}) \to PGL(N, \mathbb{C}) \to Id
\]

gives rise to the exact sequence of cohomology

\[
H^1(\Sigma_g, GL(N, \mathbb{C})) \to H^1(\Sigma_g, PGL(N, \mathbb{C})) \to H^2(\Sigma_g, \mathcal{O}^*). \quad (4.5)
\]

The Brauer group $H^2(\Sigma, \mathcal{O}^*)$ vanishes and, therefore, there are no obstructions to lift $PGL(N, \mathbb{C}) \sim PSL(N, \mathbb{C})$-bundles to $GL(N, \mathbb{C})$-bundles. A topological characteristic of a $GL(N, \mathbb{C})$-bundle is the degree of its determinant bundle. In following subsections we will construct an analog of $GL(N, \mathbb{C})$ for other simple groups. We call them the conformal groups. The main goal is to relate the characteristic classes to the degrees of some line bundles connected to the conformal groups.

**Conformal groups**

Here we introduce an analog of the group $GL(N, \mathbb{C})$ for other simple groups apart from $SL(N, \mathbb{C})$. Let

\[
\phi : Z(\tilde{G}) \to (\mathbb{C}^*)^r \quad (4.6)
\]
be an embedding of the center $Z(\bar{G})$ into algebraic torus $(\mathbb{C}^*)^r$ of minimal dimension ($r = 1$ for a cyclic center and $r = 2$ for $\mu_2 \times \mu_2$). Note that any two embeddings are conjugate from the left: $\phi_1 = A\phi_2$ for some automorphism $A$ of the torus $(\mathbb{C}^*)^r$. It is not true for $Z(\text{SL}(N, \mathbb{C})) = \mu_N$. But for other groups we deal with $\mu_2, \mu_3, \mu_4$ or $\mu_2 \times \mu_2$. In these cases nontrivial roots of unity coincide or they are inverse to each other. In the latter case $A : x \to x^{-1}$.

Consider the "anti-diagonal" embedding $Z(\bar{G}) \to \bar{G} \times (\mathbb{C}^*)^r$, $\zeta \mapsto (\zeta, \phi(\zeta)^{-1})$, $\zeta \in Z(\bar{G})$. The image of this map is a normal subgroup since $Z$ is the center of $\bar{G}$.

**Definition 4.2** The quotient

$$CG = (\bar{G} \times (\mathbb{C}^*)^r) / Z(\bar{G})$$

is called the conformal version of $\bar{G}$.

In the similar way the conformal version can be defined for any $G$ with a non-trivial center. If the center of $G$ is trivial as for $G^{ad}$ then $CG = G \times \mathbb{C}^*$.

The group $CG$ does not depend on embedding in $\mathbb{C}^*$ due to above remark about conjugacy of $\phi$'s. We have a natural inclusion $\bar{G} \subset CG$.

Consider the quotient torus $Z' = (\mathbb{C}^*)^r / Z(\bar{G}) \sim (\mathbb{C}^*)^r$. The last isomorphism is defined by $\lambda \sim \lambda^N$ for cyclic center and $(\lambda_1, \lambda_2) \sim (\lambda_1^2, \lambda_2^2)$ for $D_{even}$. The sequence

$$1 \to \bar{G} \to CG \to Z' \to 1 \quad (4.7)$$

is the analogue of

$$1 \to \text{SL}(N, \mathbb{C}) \to \text{GL}(N, \mathbb{C}) \to \mathbb{C}^* \to 1.$$ 

On the other hand, we have embedding $(\mathbb{C}^*)^r \to CG$ with the quotient $CG / (\mathbb{C}^*)^r = G^{ad}$. Then the sequence

$$1 \to (\mathbb{C}^*)^r \to CG \to G^{ad} \to 1 \quad (4.8)$$

is similar to the sequence

$$1 \to \mathbb{C}^* \to \text{GL}(N, \mathbb{C}) \to \text{PGL}(N, \mathbb{C}) \to 1.$$ 

Let $\pi$ be an irreducible representation of $\bar{G}$ and $\chi$ is a character of the torus $(\mathbb{C}^*)^r$. It follows from (4.7) that an irreducible representation $\bar{\pi}$ of $CG$ is defined as

$$\bar{\pi} = \pi \boxtimes \chi((\mathbb{C}^*)^r), \text{ such that } \bar{\pi}|_{Z(\bar{G})} = \chi \phi, \quad (\phi \in \mathfrak{g}). \quad (4.9)$$

Assume for the simplicity that $\pi$ is a fundamental representation. It means that the highest weight $\nu$ of $\pi$ is a fundamental weight. Let $\varpi^\vee$ be a fundamental coweight generating $Z(\bar{G})$ for $r = 1$. In other words, $\zeta = e(\varpi^\vee)$ is a generator of $Z(\bar{G})$ ($\zeta^N = 1$, $N = \text{ord}(Z(\bar{G}))$). Then $\bar{\pi}|_{Z(\bar{G})}$ acts as a scalar $e(\varpi^\vee, \nu)$. The highest weight can be expanded in the basis of simple roots $\nu = \sum_{\alpha \in \Pi} c^\nu_\alpha \alpha$. Then the coefficients $c^\nu_\alpha$ are rows of the inverse Cartan matrix. They have the form $k/N$, where $k$ is an integer. Therefore the scalar

$$e(\varpi^\vee, \nu) = e \left( \sum_{\alpha \in \Pi} c^\nu_\alpha \delta_{(\varpi^\vee, \alpha)} \right) \quad (4.10)$$

is a root of unity. On the other hand, let $\chi_m(\mathbb{C}^*) = w^m$ ($w \in \mathbb{C}^*$) be a character of $\mathbb{C}^*$, and $\phi(\zeta) = e(l/N)$. In terms of weights the definition of $\bar{\pi}$ (4.9) takes the form $e(\varpi^\vee, \nu) = e \left( \frac{ml}{N} \right)$. It follows from this construction that characters of $CG$ are defined by the weight lattice $P$ and the integer lattice $\mathcal{Z}$ with an additional restriction

$$\chi_{(\gamma, m)}(x, w) = \exp 2\pi i (\gamma, x) w^m, \quad (\gamma, \varpi^\vee) = \frac{ml}{N} + j, \quad \gamma \in P, \quad m, j \in \mathbb{Z}, \quad x \in \mathfrak{h}.$$ 

The case $D_{even}$ ($r = 2$) can be considered in the similar way.
Remark 4.1 Simple groups can be defined as subgroups of \(GL(V)\) preserving some multi-linear forms in \(V\). For examples, in the fundamental representations these forms are symmetric forms for \(SO\), antisymmetric forms for \(Sp\), a trilinear form for \(E_6\) and a form of fourth order for \(E_7\). In a generic situation \(G\) is defined as a subgroup of \(GL(V)\) preserving a three tensor in \(V^* \otimes V^* \otimes V\) \([22]\). The conformal versions of these groups can be alternatively defined as transformations preserving the forms up to dilatations. We prefer to use here the algebraic construction, but this approach justifies the name "conformal version".

The conformal versions can be also defined in terms of exact representations of \(\bar{G}\). Let \(V\) be such a representation and assume that \(Z(\bar{G})\) is a cyclic group. Then \(C\bar{G}\) is a subgroup of \(GL(V)\) generated by \(G\) and dilatations \(C\bar{G}^*\). The character \(\text{det} V\) is equal to \(\chi^{\dim(V)}\), where \(\chi\) is equal to \(4.10\) for fundamental representations.

For \(D_{\text{even}}\) we use two representations, i.e. the left and right spinors \(Spin^{L,R}\). The conformal group \(CSpin_{4k}\) is a subgroup of \(GL(Spin^L \oplus Spin^R)\) generated by \(Spin_{4k}\) and \(C^* \times C^*\), where the first factor \(C^*\) acts by dilatations on \(Spin^L\) and the second factor acts on \(Spin^R\). The character \(\text{det} Spin^L\) (det \(Spin^R\)) is equal to \(\lambda_1^{\dim(Spin_{4k}^L)}\) (\(\lambda_2^{\dim(Spin_{4k}^R)}\)), \((\dim(Spin_{4k}^{L,R}) = 2^{2k-1})\).

Characteristic classes and degrees of vector bundles

From the exact sequence \((4.8)\) and vanishing of the second cohomology of a curve \(H^2(\Sigma, O^*) = 0\) with coefficients in analytic sheaf we get that any \(G_{\text{ad}}(O)\)-bundle (even topological non-trivial with \(\zeta(G_{\text{ad}}(O)) \neq 0\)) can be lifted to a \(C\bar{G}(O)\)-bundle.

Let \(V\) be an exact representation either irreducible or the sum \(Spin^L \oplus Spin^R\) for \(D_{\text{even}}\). Then from \((4.6)\) one has an embedding of \(Z(\bar{G})\) to the automorphisms of \(V\)

\[
\phi_V : Z(\bar{G}) \hookrightarrow (C^*)^r = \text{Aut}_\bar{G}(V).
\]  

(4.11)

In particular case, when \(V\) is a space of a fundamental representation the center acts on \(V\) by multiplication on \((4.10)\).

Let \(E_{\bar{G}}\) be a principal \(C\bar{G}(O)\)-bundle. Denote by \(E(V) = E \otimes_{C\bar{G}} V\) (or \(E(Spin^{L,R})\)) a vector bundle induced by a representation \(V\) (\(Spin^{L,R}\) for \(D_{\text{even}}\)).

Theorem 4.1 Let \(E_{\text{ad}} = E(Ad)\) be the adjoint bundle with the characteristic class \(\zeta(E_{\text{ad}})\). The image of \(\zeta(E_{\text{ad}})\) under \(\phi_V\) \((4.11)\) is

\[
\phi_V(\zeta(E_{\text{ad}})) = \begin{cases} 
\exp(-2\pi i \deg(E(V))/\dim V), \\
\exp(-2\pi i \deg(E(Spin_{4k}^{L,R}))/2^{2k-1})
\end{cases}.
\]

\[9\] For \(G = GL(N, \mathbb{C})\) this theorem was proved in \([12]\).
Proof. Consider the commutative diagram

\[
\begin{array}{c}
1 & 1 \\
\downarrow & \downarrow \\
1 & \longrightarrow \ Z^1(\Omega_S) & \overset{\sim}{\longrightarrow} & \ Z^1(\Omega_S) & \longrightarrow & 1 \\
\downarrow_{[N]} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \longrightarrow (\Omega_S^*)^r & \longrightarrow & C\tilde{G}(\Omega_S) & \longrightarrow & G^{\text{ad}}(\Omega_S) & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \longrightarrow Z(\tilde{G}) & \longrightarrow & \tilde{G}(\Omega_S) & \longrightarrow & G^{\text{ad}}(\Omega_S) & \longrightarrow & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 \\
\end{array}
\]

and corresponding diagram of Čech cochains. Let \(\psi\) be a 1-cocycle with values in \(G^{\text{ad}}(\Omega_S)\). Consider its preimage as a cocycle with values in \(C\tilde{G}(\Omega_S)\). Due to definition of \(C\tilde{G}\) this cocycle is a pair of cochains \((\Psi, \nu)\) with values in \(\tilde{G}(\Omega_S)\) and \((\Omega_S^*)^r\) such that \(\phi_V(d\Psi)dv = 1 \in (\Omega^*)^r\), where \(d\) is the Čech coboundary operator. The cohomology class of \(d\Psi\) by definition is the characteristic class \(c\), so \(\phi_V\) of it is opposite to the class of \(dv\): \(\phi_V(\zeta(E_{ad})) = (dv)^{-1}\). Since \(\nu\) acts in \(V\) as a scalar \(\nu^{\text{dim} V}\), it is a one-cocycle as a determinant of this action. It represents the determinant of the bundle \(E(V)\). In this way \(\nu\) is a preimage of the cocycle \(\nu^{\text{dim} V}\) under the taking \(N = \dim (V)\) power \(\Omega^*[N] \to \Omega^*\), \(\nu \to \nu^N\), \(N = \dim (V)\).

Consider the long exact sequence

\[1 \to \mu_N \to (\Omega_S^*)^r \overset{[N]}{\to} \Omega_S^* \to 1, \quad (\mu_N = \mathbb{Z}/NZ).\]

It induces the map \(H^1(\Sigma, \Omega_S^*) \to H^2(\Sigma, \mu_N)\). The cocycle \(dv\) lies in the cohomology class which is an image of the class of \(\det E(V) = \nu^N\) under the coboundary map \(H^1(\Sigma, \Omega_S^*) \to H^2(\Sigma, \mu_N)\). Denote it by \(\text{Inv}_N = \text{Image}(\det E(V))\). Thus, by the definition, the class of \(dv\) equals to \(\text{Inv}_N(\det E(V)) = \text{Inv}_N(\zeta_1(E(V)))\).

The statement of the theorem follows from the following proposition

**Proposition 4.1** Let \(\gamma\) be a 1-cocycle with values in \(\Omega^*\). Then \(\text{Inv}_N(\gamma) = \exp (\frac{1}{N} 2\pi i \deg(\gamma))\).

**Proof**
Consider the diagram

\[
\begin{array}{c}
0 & \longrightarrow & \mu_N & \longrightarrow & (\Omega_S^*)^r & \overset{[N]}{\longrightarrow} & \Omega_S^* & \longrightarrow & 0 \\
\downarrow_{\exp} & \downarrow & \downarrow & \downarrow & \exp & \downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & \Omega_S & \overset{x_N}{\longrightarrow} & \Omega_S & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2\pi i \mathbb{Z} & \overset{x_N}{\longrightarrow} & 2\pi i \mathbb{Z} \\
\end{array}
\]

Let \(\gamma\) be a 1-cocycle of \(\Omega_S^*\). By definition its image in \(H^2(X, \mu_N)\) is equal to the coboundary of 1-cochain \(\gamma^{1/N}\) of \(\Omega_S^*\), \((\gamma^{1/N})^N = \gamma\). Let \(\log(\gamma)\) be a preimage of the cycle \(\gamma\) under
exponential map: \( \log(\gamma) \) is a 1-cochain of \( O_\Sigma \) and its coboundary equals to degree of \( \gamma \) times \( 2\pi i \). As the multiplication by \( N \) is invertible on \( O_\Sigma \), the cochain \( \frac{1}{N} \log(\gamma) \) is well-defined, due to commutativity of the diagram we can choose \( \exp\left(\frac{1}{N} \log(\gamma)\right) \) as \( \gamma^{1/N} \). Hence, the image of \( \gamma \) in \( H^2(X, \mu_N) \) equals to exponential of \( \frac{1}{N} \log(\gamma) \) equals exponential of degree of \( \gamma \) times \( \frac{2\pi i}{N} \). The case \( r = 2 \), can be analyzed in the same way. The theorem is proved. \( \square \)

Let as above \( \varpi^\vee \) be a fundamental coweight generating a center \( Z(\bar{G}) \) and \( \nu \) is weight of a representation of \( \bar{G} \) in \( V \). Then it follows from Theorem 4.1 and (4.10) that

\[
\deg(E(V)) = \dim(V)(\langle \varpi^\vee, \nu \rangle + k), \quad k \in \mathbb{Z}.
\] (4.12)

Then for the fundamental representations of \( \bar{G} \) we have the following realization of this formula.

| \( G \)          | \( \nu, \varpi^\vee \) | \( V \) | \( \deg(E(V)) \) |
|------------------|------------------------|------|--------------------|
| \( SL(n, \mathbb{C}) \) | \( \varpi_i^\vee \)     | \( n \) | \( -1 + kn \)     |
| \( \text{Spin}_{2n+1}(\mathbb{C}) \) | \( \varpi_n^\vee \)       | \( 2^n \) | \( 2^{n-1}(1 + 2k) \) |
| \( \text{Sp}_n(\mathbb{C}) \)       | \( \varpi_n^\vee \)       | \( 2^n \) | \( n(1 + 2k) \)   |
| \( \text{Spin}^{l,R}_{4n}(\mathbb{C}) \) | \( \varpi_{n,n-1}^\vee \) | \( 2^{n-1} \) | \( 2^{n-2}(1 + 2k) \) |
| \( \text{Spin}_{4n+2}(\mathbb{C}) \) | \( \varpi_n^\vee \)         | \( 2^n \) | \( 2^{n-2}(1 + 4k) \) |
| \( E_6(\mathbb{C}) \)          | \( \varpi_1^\vee \)      | \( 27 \) | \( 9(1 + 3k) \) |
| \( E_7(\mathbb{C}) \)          | \( \varpi_1^\vee \)      | \( 56 \) | \( 28(1 + 2k) \) |

\( (k \in \mathbb{Z}) \)

Table 3. Degrees of bundles for conformal groups.

It follows from our considerations that replacing the transition matrix

\[
\Lambda \rightarrow \tilde{\Lambda} = e^{(\langle \varpi^\vee, \nu \rangle (z + \frac{\tau}{2}))} \Lambda
\]

defines the bundle of conformal group \( CG \) of degree (4.12).

5 GS-basis in simple Lie algebras

We pass from the Chevalley basis (A.22) to a new basis that is more convenient to define bundles corresponding to nontrivial characteristic classes. We call it the *generalized sin basis* (GS-basis), because for \( A_n \) case and degree one bundles it coincides with the sin-algebra basis (see, for example, [19]).

Let us take an element \( \zeta \in Z(\bar{G}) \) of order \( l \) and the corresponding \( \Lambda^0 \in W \) from (3.3). Then \( \Lambda^0 \) generates a cyclic group \( \mu_l = (\Lambda^0, (\Lambda^0)^2, \ldots, (\Lambda^0)^l = 1) \) isomorphic to a subgroup of \( Z(\bar{G}) \). Note that \( l \) is a divisor of \( \text{ord}(Z(\bar{G})) \). Consider the action of \( \Lambda^0 \) on \( \mathfrak{g} \). Since \( (\Lambda^0)^l = Id \) we have a \( l \)-periodic gradation

\[
\mathfrak{g} = \oplus_{\alpha=0}^{l-1} \mathfrak{g}_\alpha, \quad \lambda(\mathfrak{g}_\alpha) = \omega^\alpha \mathfrak{g}_\alpha, \quad \omega = \exp \frac{2\pi i}{l}, \quad \lambda = \text{Ad}_{\Lambda^0}, \quad (5.1)
\]

\[
[\mathfrak{g}_a, \mathfrak{g}_b] = \mathfrak{g}_{a+b}, \quad (\text{mod} l), \quad (5.2)
\]
where \( g_0 \) is a subalgebra \( g_0 \subset g \) and the subspaces \( g_\alpha \) are its representations.

Since \( \Lambda^0 \in W \) it preserves the root system \( R \). Define the quotient set \( T_l = R/\mu_l \). Then \( R \) is represented as a union of \( \mu_l \)-orbits \( R = \cup_{l}^l \). We denote by \( O(\beta) \) an orbit starting from the root \( \beta \)

\[
O(\beta) = \{ \beta, \lambda(\beta), \ldots, \lambda^{-1}(\beta) \}, \quad \beta \in T_l.
\]

The number of elements in an orbit \( O \) (the length of \( O \)) is \( l/p_\alpha = l_\alpha \), where \( p_\alpha \) is a divisor of \( l \). Let \( \nu_\alpha \) be a number of orbits \( O_\alpha \) of the length \( l_\alpha \). Then \( \sum \nu_\alpha l_\alpha \). Note, that if \( O(\beta) \) has length \( l_\beta \) \((l_\beta \neq 1)\), then the elements \( \lambda^k \beta \) and \( \lambda^{k+l_\beta} \beta \) coincide.

**Basis in \( \mathfrak{L} \)**

Transform first the root basis \( \mathcal{E} = \{ E_\beta, \beta \in R \} \) in \( \mathfrak{L} \). Define an orbit in \( \mathcal{E} \)

\[
E_\beta = \{ E_\beta, E_\lambda(\beta), \ldots, E_\lambda^{-1}(\beta) \}
\]

corresponding to \( O(\bar{\beta}) \). Again \( \mathcal{E} = \cup_{\beta \in T_l}^l E_\beta \).

For \( O(\bar{\beta}) \) define the set of integers

\[
J_{p_\alpha} = \{ m = mp_\alpha \mid m \in \mathbb{Z}, \quad a \text{ is defined } \pmod{l}, \quad \mu_\alpha = l/l_\alpha \}.
\]

"The Fourier transform" of the root basis on the orbit \( O(\bar{\beta}) \) is defined as

\[
t_\beta^a = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{ma} E_\lambda^{m}(\beta), \quad \omega = \exp \left( \frac{2\pi i}{l} \right), \quad a \in J_\beta.
\]

This transformation is invertible \( E_\lambda^{k}(\beta) = \frac{1}{\sqrt{l}} \sum_{a \in J_l} \omega^{-ka} t_\beta^a \), and therefore there is the one-to-one map \( \mathcal{E}_\beta \leftrightarrow \{ t_\beta^a, \quad a \in J_\beta \} \). In this way we have defined the new basis

\[
\{ t_\beta^a, \quad (a \in J_l, \quad \bar{\beta} \in T_l) \}.
\]

Since \( \lambda(E_\alpha) = E_\lambda(\alpha) \) we have for \( \Lambda e(\tilde{\mu}) \) \( (\tilde{\mu} \in \tilde{\mathfrak{h}}_0) \)

\[
Ad_{\Lambda}(t_\beta^a) = e(\langle \tilde{\mu}, \beta \rangle - \frac{a}{l} t_\beta^a), \quad e(x) = \exp(2\pi ix).
\]

It means that \( t_\beta^a \) \((\bar{\beta} \in T_l)\) is a part of basis in \( g_{l-a} \). Moreover,

\[
Ad_{Q}(t_\beta^a) = e(\langle \kappa, \beta \rangle) t_\beta^a.
\]

This relations follows from \( \mathfrak{g} \)(3.7) and \( \mathfrak{g}(3.14) \). We also take into account that \( Q \) and \( \Lambda \) commute in the adjoint representation and \( e(x)E_\alpha e(-x) = e(\langle x, \alpha \rangle)E_\alpha \) for \( x \in \tilde{\mathfrak{h}}_0 \).

Picking another element \( \Lambda' \) generating a subgroup \( \mathcal{Z}_{l'} \) \((l' \neq l)\) we come to another set of orbits and to another basis. We have as many types of bases as many of non-isomorphic subgroups in \( \mathcal{Z}(\tilde{G}) \).

21
The Killing form

Consider two orbits $O(\hat{\alpha})$ and $O(\hat{\beta})$, passing through $E_\alpha$ and $E_\beta$. Assume that there exists such integer $r$ that $\alpha = -\lambda^r(\beta)$. It implies that elements of two orbits are related as $\lambda^n(\alpha) = -\lambda^m(\beta)$ if $m - n = r$. In other words, $-\beta \in O(\hat{\alpha})$. In particular, it means that orbits have the same length. It follows from (5.4) and (A.25) that

$$ (\xi_1^\alpha, \xi_2^\beta) = \delta_{\alpha,-\lambda^r(\beta)} \delta^{(c_1+c_2,0 \ (mod \ l))} \omega^{-rc_1} \frac{2p_\alpha}{(\alpha, \alpha)}. \quad (5.8) $$

where $p_\alpha = l/l_\alpha$ and $l_\alpha$ is the length of $O(\hat{\alpha})$. In particular, $(\xi_1^\alpha, \xi_{-\alpha}^\alpha) = \frac{2p_\alpha}{(\alpha, \alpha)}$.

In what follows we need a dual basis $\xi_\alpha^B$

$$ (\xi_\alpha^B, \xi_{-\alpha}^B) = \delta^{(b_1+b_2,0 \ (mod \ l))} \delta^{(-\alpha_1,-\alpha_2)} \omega^{-ra_1} \frac{2p_\alpha}{(\alpha_1, \alpha_1)}. \quad (5.9) $$

The Killing form in this basis is inverse to (5.8)

$$ (\xi_\alpha^B, \xi_{-\alpha}^B) = \frac{(\alpha_1, \alpha_1)}{2p_\alpha}. \quad (5.10) $$

A basis in the Cartan subalgebra

Almost the same construction exists in $\mathfrak{h}$. Again let $\Lambda^0$ generates the group $\mu_l$. Since $\Lambda^0$ preserves the extended Dynkin diagram, its action preserves the extended coroot system $\Pi^{ext} = \Pi^\vee \cup \alpha_\mu^\vee$ in $\mathfrak{h}$. Consider the quotient $K_l = \Pi^{ext}/\mu_l$. Define an orbit $\mathcal{H}(\hat{\alpha})$ of length $l_\alpha = l/p_\alpha$ in $\Pi^{ext}$ passing through $H_\alpha \in \Pi^{ext}$

$$ \mathcal{H}(\hat{\alpha}) = \{H_{\alpha}, H_{\lambda(\alpha)}, \ldots, H_{\lambda^{l-1}(\alpha)}\}, \quad \hat{\alpha} \in K_l = \Pi^{ext}/\mu_l. $$

The set $\Pi^{ext}$ is a union of $\mathcal{H}(\hat{\alpha})$

$$ (\Pi^{ext})^{ext} = \bigcup_{\hat{\alpha} \in K_l} \mathcal{H}(\hat{\alpha}). $$

Define "the Fourier transform"

$$ \mathfrak{h}^c_\alpha = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{mc} H_{\lambda^{m}(\alpha)}, \quad \omega = \exp \frac{2\pi i}{l}, \quad c \in J_\alpha. \quad (5.11) $$

The basis $\mathfrak{h}^c_\alpha, \ (c \in J_\alpha, \ \hat{\alpha} \in K_l)$ is over-complete in $\mathfrak{h}$. Namely, let $\mathcal{H}(\hat{\alpha}_0)$ be an orbit passing through the minimal coroot $\{H_{\alpha_0}, H_{\lambda(\alpha_0)}, \ldots, H_{\lambda^{l-1}(\alpha_0)}\}$. Then the element $\mathfrak{h}^0_{\alpha_0}$ is a linear combination of elements $\mathfrak{h}^0_{\alpha}, \ (\alpha \in \Pi)$ and we should exclude it from the basis. We replace the basis $\Pi^\vee$ in $\mathfrak{h}$ by

$$ \mathfrak{h}^0_{\alpha}, \ (c \in J_\alpha), \quad \left\{ \begin{array}{ll} \hat{\alpha} \in \hat{K}_l \setminus \mathcal{H}(\hat{\alpha}_0), & \text{c = 0} \\ \hat{\alpha} \in K_l, & \text{c \neq 0}. \end{array} \right. \quad (5.12) $$

As before there is a one-to-one map $\Pi^\vee \leftrightarrow \{\mathfrak{h}^c_\alpha\}$.

The elements $(\mathfrak{h}^c_\alpha, \xi_\alpha^B)$ form GS basis in $\mathfrak{g}_{l-\alpha}$ (5.1).
The Killing form

The Killing form in the basis (5.12) can be found from (A.24)

\[(h_{a\bar{\alpha}}, h_{b\bar{\beta}}) = \delta(a+b,0) A_{a,\beta}^{\alpha}, \quad A_{a,\beta}^{\alpha} = 2 (\beta,\beta) l^{-1} \sum_{s=0}^{l-1} \omega^{-s} a_{\beta,\lambda'(\alpha)}, \quad (5.13)\]

where \(a_{\alpha,\beta}\) is the Cartan matrix (A.4).

The dual basis is generated by elements

\[(h_{a\bar{\alpha}}, h_{b\bar{\beta}}) = \delta(a+b,0) \delta_{\alpha,\beta}, \quad H_{a\bar{\alpha}} = \sum_{\beta \in \Pi} (A_{a,\beta}^{\alpha})^{-1} h_{-\alpha,\beta}, \quad H_{a\bar{\alpha}} = \sum_{\alpha \in \Pi} (A_{-\alpha,\beta}^{\alpha}) \delta_{\beta}^{-\alpha} \quad (5.14)\]

The Killing form in the dual basis takes the form

\[(h_{a\bar{\alpha}}, h_{b\bar{\beta}}) = \delta(a+b,0) (A_{a,\beta}^{\alpha})^{-1}. \quad (5.15)\]

In summary, we have defined the GS-basis in \(g\)

\[\{h_{a\bar{\alpha}}, h_{b\bar{\beta}}, (a, \beta, c, \bar{\alpha})\} \text{ are defined in (5.5), (5.12)}\],

and the dual basis

\[\{T_{a\bar{\alpha}}, T_{b\bar{\beta}}, (a, \beta, c, \bar{\alpha})\} \text{ are defined in (5.9), (5.14)}\],

along with the Killing forms.

Commutation relations

The commutation relations in the GS basis can be found from the commutation relations in the Chevalley basis (A.23). Taking into account the invariance of the structure constants with respect to the Weyl group action \(C_{\lambda\alpha,\lambda\beta} = C_{\alpha,\beta}\) it is not difficult to derive the commutation relations in the GS basis using its definition in the Chevalley basis (5.4), (5.11). In the case of root-root commutators we come to the following relations

\[[t_{a\alpha}, t_{b\beta}] = \begin{cases} \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{s} C_{\alpha,\lambda'\beta} t_{a+b,\alpha+\lambda'\beta}, & \alpha \neq -\lambda'\beta \\ \frac{1}{\sqrt{l}} \omega^{s} h_{a+b}^{\alpha+b} & \alpha = -\lambda'\beta \end{cases} \quad (5.18)\]

The Cartan-root commutators are:

\[[h_{a\alpha}, t_{m\beta}] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-s} a_{\alpha,\lambda'\beta} \frac{2}{(\alpha,\alpha)} t_{m,\beta}^{k+m} \]

\[[\delta_{a\alpha}, t_{m\beta}] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-s} (\alpha,\lambda'\beta) \frac{2}{(\alpha,\alpha)} (\hat{\alpha},\lambda'\beta) t_{m,\beta}^{k+m} \quad (5.19)\]

Here we denote by \(\hat{\alpha}\) the dual to the simple roots elements in the Cartan subalgebra:

\[(\hat{\alpha}_{i}, \hat{\beta}_{j}) = \delta_{ij} \quad (5.20)\]

In Section for explicit computations with Lax operators and r-matrices, it will be much more convenient to use the following normalized basis for Cartan subalgebra:

\[h_{a\alpha}^{k} = \frac{(\alpha,\alpha)}{2} h_{a\alpha}^{k}, \quad \delta_{a\alpha}^{k} = \frac{2}{(\alpha,\alpha)} \delta_{a\alpha}^{k} \quad (5.21)\]
This reparametrization leads to the following commutation relations:

\[
\begin{align*}
\left[ \bar{h}_k^\alpha, t^m_\beta \right] &= \frac{1}{\sqrt{l - 1}} \sum_{s=0}^{l-1} \omega^{-ks}(\alpha, \lambda^s_\beta) t^{k+m}_\beta \\
\left[ \bar{H}_k^\alpha, t^m_\beta \right] &= \frac{1}{\sqrt{l - 1}} \sum_{s=0}^{l-1} \omega^{-ks}(\bar{\alpha}, \lambda^s_\beta) t^{k+m}_\beta 
\end{align*}
\] (5.22)

The following simple formula expresses the decomposition of Cartan element in the basis of simple roots:

\[
\bar{h}_\beta^k = \sum_{\alpha \in \Pi} (\hat{\alpha}, \beta) \bar{h}_\alpha^k, \quad \beta \in R 
\] (5.23)

the connection of dual bases is clear from the following expression:

\[
\sum_{\beta \in \Pi} (\hat{\alpha}, \beta) \bar{h}_\beta^k = \sum_{\beta \in \Pi} (\alpha, \beta) \bar{H}_\beta^k 
\] (5.24)

The Cartan elements have the following ”sign”-property:

\[
\begin{align*}
\bar{h}_\lambda^\alpha &= -\bar{h}_\alpha^\lambda, \quad \bar{H}_\lambda^\alpha &= -\bar{H}_\alpha^\lambda, \\
\bar{S}^{\lambda^k}_\alpha &= -\bar{S}^{\lambda^k}_\alpha, \quad \bar{S}^{\delta^k}_\alpha &= -\bar{S}^{\delta^k}_\alpha 
\end{align*}
\] (5.25)

From the definition of GS-basis we simply find:

\[
\begin{align*}
t^k_{\lambda\alpha} &= \omega^{-ks} t^k_{\alpha}, \quad \bar{h}_\lambda^\alpha = \omega^{-ks} \bar{h}_\alpha^k, \quad \bar{H}_\lambda^\alpha = \omega^{-ks} \bar{H}_\alpha^k, \\
S^{\lambda^k}_{\lambda\alpha} &= \omega^{-ks} S^{\lambda^k}_{\alpha}, \quad S^{h^k}_{\lambda\alpha} = \omega^{-ks} S^{h^k}_{\alpha}, \quad S^{\delta^k}_{\lambda\alpha} = \omega^{-ks} S^{\delta^k}_{\alpha} 
\end{align*}
\] (5.26)

The same identities we suppose for classical variables (see Section 7):

\[
\begin{align*}
S^{\lambda^k}_{\lambda\alpha} &= \omega^{-ks} S^{\lambda^k}_{\alpha}, \quad S^{h^k}_{\lambda\alpha} = \omega^{-ks} S^{h^k}_{\alpha}, \quad S^{\delta^k}_{\lambda\alpha} = \omega^{-ks} S^{\delta^k}_{\alpha} 
\end{align*}
\] (5.27)

Invariant subalgebra

Consider the invariant subalgebra \( g_0 \). It is generated by the basis \( \{ t^0_\beta, \bar{h}^0_\alpha \} \). In particular, \( \{ h^0_\beta \} \), \( \{ h^1_\beta \} \), \( \{ h^2_\beta \} \) form a basis in the Cartan subalgebra \( \tilde{\mathfrak{h}}_0 \subset \mathfrak{h} \) (dim \( \tilde{\mathfrak{h}}_0 = p < n \)).

We pass from \( \{ h^0_\beta \} \) to a special basis in \( \tilde{\mathfrak{h}}_0 \):

\[
\tilde{\Pi}^\vee = \{ \tilde{\alpha}_k^\vee \mid k = 1, \ldots, p \}. 
\] (5.28)

It is constructed in the following way. Consider a subsystem of simple coroots

\[
\Pi^\vee_I = \Pi^{ext \vee} \setminus \mathcal{O}(\tilde{\alpha}_0^\vee) 
\] (5.29)

(see (5.12)). In other words, \( \Pi_I^\vee \) is a subset of simple coroots that does not contain simple coroots from the orbit passing through \( \alpha_0 \). For \( A_{n-1}, B_n, E_6 \) and \( E_7 \) the coroot basis \( \tilde{\Pi}^\vee \) is a result of an averaging along the \( \lambda \) orbits in \( \Pi^\vee_I \):

\[
\hat{\alpha}^\vee = \sum_{m=1}^{l-1} H^{m(\alpha)} \lambda, \quad H_\alpha \in \Pi^\vee_I 
\] (5.30)

In the \( C_n \) and \( D_n \) cases this construction is valid for almost all coroots except the last on the Dynkin diagram (see Remark 10.1 below). Consider the dual vectors \( \tilde{\Pi} = \{ \tilde{\alpha}_k^\vee \mid k = 1, \ldots, p, \langle \tilde{\alpha}_k^\vee, \tilde{\alpha}_k^\vee \rangle = 2 \} \) in \( \tilde{\mathfrak{h}}_0^* \).
Proposition 5.1  The set of vectors in $\tilde{\mathfrak{h}}_0$

$$\tilde{\Pi} = \{ \tilde{\alpha}_k | k = 1, \ldots, p \},$$  

is a system of simple roots of a simple Lie subalgebra $\tilde{\mathfrak{g}}_0 \subset \mathfrak{g}_0$ defined by the root system $\tilde{R} = \tilde{R}(\tilde{\Pi})$ and the Cartan matrix $(\tilde{\alpha}_k, \tilde{\alpha}_\ell^\vee)$.

We will check this statement case by case.

Let $R_1 = R_1(\Pi_1)$ be a subset of roots generated by simple roots $\Pi_1 = \Pi^{\text{ext}} \setminus \mathcal{O}(a_0)$. It is invariant under $\lambda$ action. The root system $\tilde{R}$ of $\tilde{\mathfrak{g}}_0$ corresponds to the $\lambda$-invariant set of $R_1$. Consider the complementary set of roots $R \setminus R_1$ and the set of orbits

$$\mathcal{T}_1' = (R \setminus R_1)/\mu_1.$$  

It is a subset of all orbits $\mathcal{T}_1 = R/\mu_1$. Therefore, $\mathcal{T}_1 = \tilde{R} \cup \mathcal{T}_1'$. The $\lambda$-invariant subalgebra $\mathfrak{g}_0$ contains the subspace

$$V = \{ \sum_{\tilde{\beta} \in \mathcal{T}_1'} a_{\tilde{\beta}} \tilde{\beta}^\vee, \ a_{\tilde{\beta}} \in \mathbb{C} \}.$$  

Then $\mathfrak{g}_0$ is a sum of $\tilde{\mathfrak{g}}_0$ and $V$

$$\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \oplus V.$$  

The components of this decomposition are orthogonal with respect to the Killing form and $V$ is a representation of $\tilde{\mathfrak{g}}_0$. We find below the explicit forms of $\mathfrak{g}_0$ for all simple algebras from our list.

Let $\mathfrak{h}'$ be a subalgebra of $\mathfrak{h}$ with the basis $\mathfrak{h}_c'$ and $\mathfrak{h}'$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}_0$. Then

$$\mathfrak{h}' = \tilde{\mathfrak{g}}_0 \oplus \mathfrak{h}'.$$  

We summarize the information about invariant subalgebras in Table 3.

| $\Pi$ | $\mathcal{Z}(\mathbb{G})$ | $\varphi^\vee_j$ | $\Pi_1$ | $l = \text{ord}(\Lambda)$ | $\tilde{\mathfrak{g}}_0$ | $\mathfrak{g}_0$ |
|---|---|---|---|---|---|---|
| $A_{N-1}$, $(N = p!)$ | $\mu_N$ | $\varphi^\vee_{N-1}$ | $\mathcal{U}_1 A_{p-1}$ | $N/p$ | $\text{sl}_p$ | $\text{sl}_p \oplus \varphi^\vee_{j=1} \mathfrak{g}_p$ |
| $B_n$ | $\mu_2$ | $\varphi^\vee_{n}$ | $\mathcal{O}_{2n-1}$ | 2 | $\text{so}(2n-1)$ | $\text{so}(2n)$ |
| $C_{2l}$, $(l > 1)$ | $\mu_2$ | $\varphi^\vee_{2l}$ | $A_{2l-1}$ | 2 | $\text{so}(2l)$ | $\text{gl}_{2l}$ |
| $D_{2l+1}$, $(l > 1)$ | $\mu_4$ | $\varphi^\vee_{2l+1}$ | $A_{2l}$ | 2 | $\text{so}(2l + 1)$ | $\text{gl}_{2l+1}$ |
| $D_{2l+1}$, $(l > 1)$ | $\mu_4$ | $\varphi^\vee_{2l+1}$ | $A_{2l-2}$ | 4 | $\text{so}(2l - 1)$ | $\text{so}(2l) \oplus \text{so}(2l) \oplus \mathbb{1}$ |
| $D_{2l}$, $(l > 2)$ | $\mu_2 \oplus \mu_2$ | $\varphi^\vee_{2l}$ | $D_{2l}$ | 2 | $\text{so}(4l - 1)$ | $\text{so}(4l - 2) \oplus \mathbb{1}$ |
| $E_6$ | $\mu_3$ | $\varphi^\vee_6$ | $D_4$ | 3 | $\mathfrak{g}_2$ | $\text{so}(8) \oplus 2 \cdot \mathbb{1}$ |
| $E_7$ | $\mu_2$ | $\varphi^\vee_7$ | $\mathfrak{e}_6$ | 2 | $\mathfrak{f}_4$ | $\mathfrak{e}_8$ |

**Table 4**

Invariant subalgebras $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{\Pi}$ and $\mathfrak{g}_0$ of simple Lie algebras.

The coweights generating central elements are displaced in column 3.

---
In the invariant simple algebra $\tilde{g}_0$ instead of the basis $(h^0_\alpha, t^0_\beta)$ we use the Chevalley basis and incorporate it in the GS-basis

$$\{h^0_\alpha, t^0_\beta\} \rightarrow \{\tilde{g}_0 = (H_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{\Pi}, E_{\tilde{\beta}}, \tilde{\beta} \in \tilde{R}), V = (t^0_\beta, \tilde{\beta} \in T')\}.$$ (5.36)

**Remark 5.1** For any $\xi \in Q^\vee$ a solution of (3.10) is $\Lambda = \text{Id}$. In this case $\tilde{g}_0 = g$ and GS-basis is the Chevalley basis.

**The GS basis from a canonical basis in $\mathfrak{h}$**

Let $(e_1, e_2, \ldots, e_n)$ be a canonical basis in $\mathfrak{h}$, where $((e_j, e_k) = \delta_{jk})$. Since $\Lambda$ preserves $\mathfrak{h}$ we can consider the action of $\mu_l$ on the canonical basis. Define an orbit of length $l_s = l/p_s$ passing through $e_s O(s) = \{e_s, \lambda(e_s), \ldots, \lambda^{l-1}(e_s)\}$.

The Fourier transform along $O(s)$ takes the form

$$h^c_s = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{mc} \lambda^m(e_s), \ c \in J_p, \ \omega = \exp \left(\frac{2\pi i}{l}\right),$$ (5.37)

where $J_p = \{c = mp \mod(l) | m \in \mathbb{Z}\}$. Consider the quotient $C_l = (e_1, e_2, \ldots, e_n)/\mu_l$. Then we can pass from the canonical basis to the GS basis

$$(e_1, e_2, \ldots, e_n) \leftrightarrow \{h^c_s, s \in C_l\}.$$  

The Killing form is read of from (5.37)

$$[h^c_{s1}, h^c_{s2}] = \delta_{(s1, s2)} \delta^{(c_1, c_2)}.$$ (5.38)

Then the dual generators are

$$\mathfrak{h}^c_s = h^{-c}_s.$$ (5.39)

The commutation relations in $\mathfrak{g}$ in this form of GS basis take the form

$$[h^k_{s1}, t^k_{\beta}] = \frac{1}{\sqrt{l}} \sum_{r=0}^{l-1} \omega^{-rk_1} \lambda^r(\beta), e_s t^k_{\beta} = \frac{1}{\sqrt{l}} \sum_{s} (\alpha^\vee, e_s) h^{k_1+k_2}_s, \ if \ \alpha = -\lambda^r(\beta) \ for \ some \ r.$$ (5.40)

We obtain the last relation from (5.4) and from the expansion $h^k_\alpha = \sum_s (\alpha^\vee, e_s) h^k_s$.

6 **General description of systems with non-trivial characteristic classes**

**The Lax operators and Symplectic Hecke Correspondence**

Consider a meromorphic section $\Phi$ of the adjoint bundle $End E_G \otimes K$, where $K$ is a canonical class. The pair $(\Phi, E_G)$ is called the Higgs bundle over $\Sigma$. The set of these pairs is a cotangent bundle to the space of holomorphic bundles equipped with a canonical symplectic structure.

\[\text{For A_n and E_6 root systems it is convenient to choose canonical bases in } \mathfrak{h} \oplus \mathbb{C}.\]
Evidently, the gauge transformations \((2.3)\) can be lifted to the Higgs bundle as canonical transformations with respect to the symplectic form. The Hamiltonian reduction of the Higgs bundle under this action leads to integrable systems \((24)\) and the Higgs field become the Lax operator \(L\). The moduli space of a Higgs bundle becomes a phase space of an integrable system. This construction is valid for curves with marked points. In this case we deal with the Higgs bundle with quasi-parabolic structures at the marked points. It implies that the Lax operators have first order poles at the marked poles with residues belong to generic coadjoint orbits \(O\). The coadjoint orbits are affine spaces over the flag varieties mentioned at Section 2. The dimension of the moduli space of the Higgs bundle is twice of \(\dim M_{g,n} \) \((2.7)\)

\[
\dim (T^*M_{g,n}) = 2(g - 1) \dim (G) + n \dim (O).
\]

The commuting integrals are generated by the quantities \((L_d^j)\), where \(d_j\) are degrees of invariant polynomials. The integrals are coefficients of expansion \((L_d^i)\) in the basis of functions on \(\Sigma\) with prescribed singularities at the marked points.

For \(g = 1, n = 1\) the phase space has dimension of a coadjoint orbit \(2\sum_{j=1}^{\text{rank}G} (d_j - 1)\). In this case \(L\) satisfies the conditions

\[
L(z + 1) = QL(z)Q^{-1}, \quad L(z + \tau) = \Lambda L(z)\Lambda^{-1},
\]

where \(Q\) and \(\Lambda\) are solutions of \((3.3)\), and

\[
\bar{\partial}L(z) = S\delta(z, \bar{z}).
\]

In other words \(Res|_{z=0} L(z) = S\). These conditions fix \(L\).

To make dependence on the characteristic class \(\zeta(E_G)\) explicit we will write \(L(z)^{\gamma^\vee}\), if the Lax matrix satisfies the quasi-periodicity conditions with \(\Lambda = \Lambda_{\gamma^\vee}, Q_{\gamma^\vee}\) where \(\Lambda_{\gamma^\vee}, Q_{\gamma^\vee}\) are solutions of \((3.3)\) with \(\zeta = e(-\gamma^\vee), \gamma^\vee \in P^\vee\).

The modification \(\Xi(\gamma)\) of \(E_G\) changes the characteristic class \((2.1)\). It acts on \(L^{\gamma^\vee}\) as follows

\[
L^{\gamma^\vee}\Xi(\gamma) = \Xi(\gamma)L^{\gamma^\vee + \gamma}.
\]

In this form it was introduced in \((31)\). It is called the Symplectic Hecke Correspondence, because it acts as a symplectomorphism on phase spaces of Hitchin integrable systems. In implies that the Higgs systems corresponding to the bundles with different characteristic classes are in fact symplectomorphic. In particular, they are symplectomorphic to the standard CM systems.

The action \((6.4)\) allows one to write down condition on \(\Xi(\gamma)\). Since \(L^{\gamma^\vee}\) has a simple pole at \(z = 0\) the modified Lax matrix \(L^{\gamma^\vee + \gamma}\) should have also a simple pole at \(z = 0\). Decompose \(L^{\gamma^\vee}\) and \(L^{\gamma^\vee + \gamma}\) in the Chevalley basis \((A.21), (A.22)\)

\[
L^{\gamma^\vee} = L_\delta(z) + \sum_{\alpha \in R} L_\alpha(z)E_\alpha, \quad L^{\gamma^\vee + \gamma} = \tilde{L}_\delta(z) + \sum_{\alpha \in R} \tilde{L}_\alpha(z)E_\alpha.
\]

Expand \(\alpha\) in the basis of simple roots \((A.22)\) \(\alpha = \sum_{j=1}^{\ell} f^j j_\alpha j\) and \(\gamma\) in the basis of fundamental coweights \(\gamma = \sum_{j=1}^{\ell} m_j \gamma^\vee_j\). Assume that \(\langle \gamma, \alpha_j \rangle \geq 0\) for simple \(\alpha_j\). In other words \(\gamma\) is a dominant coweight. Then \(\langle \gamma, \alpha \rangle = \sum_{j=1}^{\ell} m_j n^j\) is an integer number, positive for \(\alpha \in R^+\) and negative for \(\alpha \in R^-\). From \((2.10)\) and \((6.3)\) we find

\[
L_{\delta_j}^{\gamma^\vee + \gamma}(z) = L_{\delta_j}^{\gamma^\vee}(z), \quad L_{\alpha_j}^{\gamma^\vee + \gamma}(z) = z^{\langle \gamma, \alpha \rangle} L_{\alpha_j}^{\gamma^\vee}(z).
\]
In a neighborhood of \( z = 0 \) \( L_\alpha(z) \) should have the form

\[
L_\alpha^{\omega_\gamma}(z) = a_{(\gamma,\alpha)} z^{- (\gamma,\alpha)} + a_{(\gamma,\alpha)+1} z^{- (\gamma,\alpha)+1} + \ldots, \quad (\alpha \in R^-),
\]

otherwise the transformed Lax operator becomes singular. It means that the type of the modification \( \gamma \) is not arbitrary, but depends on the local behavior of the Lax operator. It allows one to find the dimension of the space of the Hecke transformation. We do not need it here.

Now consider on a global behavior of \( L(z) \) (6.3). Then we find that \( \Xi(\gamma) \) should intertwine the quasi-periodicity conditions

\[
\Xi(\gamma, z + 1) Q_{\omega_\gamma} = Q_{\omega_\gamma + \gamma} \Xi(\gamma, z), \quad \Xi(\gamma, z + \tau) \Lambda_{\omega_\gamma} = \Lambda_{\omega_\gamma + \gamma} \Xi(\gamma, z).
\]

Solutions of these equations do not belong to the fixed Cartan subgroup \( \mathcal{H} \subset G \) as in (2.10). So it is not easy to find a consistent behavior of \( \Xi(\gamma) \) and \( L^{\omega_\gamma} \) at \( z = 0 \) as in (6.5). For \( G = SL(N,\mathbb{C}) \), \( \gamma = \omega_\gamma \) and special residue of \( L \) it was done in [34].

**The Lax matrix. Explicit form**

Assume that \( L \) has a residue at \( z = 0 \) taking values in a coadjoint orbit \( \mathcal{O} \subset g^* \).

\[
\text{Res } L|_{z=0} = S = \sum_{\alpha \in \Pi} \frac{1}{2}(\alpha,\alpha) S^0_\alpha \sum_{\beta \in \Pi} a_{\alpha,\beta}^{-1} H_\beta + \sum_{\beta \in R} S^g_\beta (\beta,\beta) E_{-\beta} \tag{6.6}
\]

\[
= \sum_{j=1}^n S_j e_j + \sum_{\beta \in R} S^g_\beta (\beta,\beta) E_{-\beta}.
\]

We identify \( g^* \) and \( g \) by means the Killing form \([A.24], [A.25]\). Then the coordinates are linear functionals on \( g \) \( S^0_\alpha = (S, H_\alpha) \), or \( S_j = (S, e_j) \), \( S^g_\beta = (S, E_\beta) \).

The Poisson brackets for \( S^0_\alpha \), \( (S_j, S^g_\beta) \) have the same structure constants as \( g \) \([A.23]\). To define a generic orbit \( \mathcal{O} \) we pick the Casimir functions \( C_j(S) \).

 Rewrite (6.6) in the dual GS-basis. We use the gradation (5.1) to define the Lax matrix

\[
L(z) = \sum_{a=0}^{l-1} L_a(z), \tag{6.7}
\]

where the zero component is decomposed according with (5.34) \( L_0(z) = \tilde{L}_0(z) + L'_0(z) \). Then

\[
S = \text{Res } L|_{z=0} = \text{Res } \tilde{L}_0|_{z=0} + \text{Res } L'_0|_{z=0} + \sum_{a=1}^{l-1} \text{Res } L_a|_{z=0} = \tilde{S}_0 + S'_0 + \sum_{a=1}^{l-1} S_a,
\]

where

\[
\tilde{S}_0 = \sum_{\alpha,\beta \in \Pi} \tilde{S}^0_\alpha (\alpha,\alpha)(\beta,\beta) 4(\tilde{\alpha},\tilde{\beta}) H_\beta + \sum_{\beta \in R} \tilde{S}^g_\beta (\tilde{\beta},\tilde{\beta}) E_{-\beta},
\]

\[
S'_0 = \sum_{\beta \in T'_1} S^g_\beta \tau_0^\beta, \quad S_a = \sum_{\alpha \in K_t} S^0_\alpha a_{\alpha}^{l-a} + \sum_{\beta \in T'_1} S^c_\beta \tau_{l-a}^{-\beta}, \tag{6.8}
\]

(see (5.32), (5.33)). Again, the coordinates

\[
S^0_\alpha = (S_\alpha, h^a_\alpha), \quad S'_\beta = (S_\beta, a^0_\beta), \quad S^c_\beta = (S_\beta, c_\beta), \quad \tilde{S}^0_\alpha = (S, H_\alpha), \quad \tilde{S}^g_\beta = (S, E_\beta)
\]

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have the structure constants of the Poisson brackets as in (5.18), (5.19), (5.40). We can pass from one data to another by the Fourier transform introduced above. We rewrite \( \tilde{S}_0 \) in terms of a canonical basis \((e_1, \ldots, e_n)\) in the invariant Cartan algebra \( \hat{h}_0 \)

\[
\tilde{S}_0 = \sum_{j=1}^{n} \tilde{S}_j e_j + \sum_{\beta \in R} \tilde{S}_\beta \frac{\langle \beta, \beta \rangle}{2} E_{-\beta}.
\]
(6.9)

It follows from (5.6), (5.7) and from definition of the dual basis (5.9), (5.14) that

\[
Ad_{\Lambda}(\Sigma^\alpha_\beta) = e\left(\frac{c_i}{l} - \langle \tilde{u}, \beta \rangle\right) \Sigma^\alpha_\beta, \quad Ad_{\Lambda}(\delta^\alpha_\beta) = e\left(\frac{c_i}{l}\right) \delta^\alpha_\beta, \quad (e(x) = \exp(2\pi i x)).
\]

In addition, we have

\[
Ad_{\Lambda}(\delta^\alpha_\beta) = \delta^\alpha_\beta, \quad Ad_{\Lambda}(H_{\tilde{a}}) = H_{\tilde{a}},
\]
(6.10)

\[
Ad_{\Lambda}(\Sigma^\alpha_\beta) = e\langle \kappa, \beta \rangle \Sigma^\alpha_\beta, \quad Ad_{\Lambda}(E_{\tilde{a}}) = e\langle \kappa, \tilde{a} \rangle E_{\tilde{a}}.
\]
(6.11)

Using (A.14) we obtain \( \langle \kappa, \alpha \rangle = f_\alpha / h \). Then the last relation assumes the form

\[
Ad_{\Lambda}(\Sigma^\alpha_\beta) = e\left(-f_\beta / h\right) \Sigma^\alpha_\beta, \quad Ad_{\Lambda}(E_{\tilde{a}}) = e\left(f_\alpha / h\right) E_{\tilde{a}}.
\]
(6.12)

There are also the evident relations:

\[
Ad_{\Lambda}(E_{\tilde{a}}) = e\langle \langle \tilde{u}, \bar{\alpha} \rangle \rangle E_{\tilde{a}}, \quad Ad_{\Lambda}(H_{\tilde{a}}) = H_{\tilde{a}}, \quad \tilde{u} \in \tilde{H}.
\]

The quasi-periodicity conditions and the existence of pole at \( z = 0 \) dictate the form of the components for \( a \neq 0 \). We define matrix element of Lax operator using \( \phi(u, z) \) (B.3). Let

\[
\varphi^\alpha_\beta(x, z) = e\langle \langle \kappa, \beta \rangle \rangle \phi(x + \kappa \tau, \beta) + \frac{a}{l} z = e(\kappa f_\beta / h) \phi(x, \beta) + \tau f_\beta / h + \frac{a}{l}\zeta, \quad \kappa, \beta \in \tilde{H}. \]
(6.13)

The last equality follows from the identity \( \langle \kappa, \alpha \rangle = \frac{1}{l} \langle \rho^\gamma, \alpha \rangle = f_\alpha / h \) (see (A.14)). It follows from (B.6) that \( \varphi^\alpha_\beta(x, z + 1) = e\langle \langle \kappa, \beta \rangle \rangle \varphi^\alpha_\beta(x, z), \varphi^\alpha_\beta(x, z + \tau) = e\langle -\langle \kappa, -\beta \rangle - \frac{a}{l} \rangle \varphi^\alpha_\beta(x, z). \)

Then from (6.1) we find

\[
L_a(z) = \sum_{\beta \in \tilde{H}_l} S^\alpha_\beta \varphi^\alpha_\beta \phi^\alpha_\beta(z) + \sum_{\beta \in \tilde{H}_l} S^\alpha_\beta \varphi^\alpha_\beta(-\tilde{u}, z) \Sigma^\beta_\gamma,
\]
(6.14)

and \( L'_a(z) = \sum_{\beta \in \tilde{H}_l} S^\alpha_\beta \varphi^\alpha_\beta(-\tilde{u}, z) \Sigma^\beta_\gamma. \)

In the canonical basis in \( \tilde{H} \) (B.37) \( L_a(z) \) takes the form

\[
L_a(z) = \sum_{s \in \tilde{H}_l} S^\alpha_\beta \phi^\alpha_\beta(z) + \sum_{\beta \in \tilde{H}_l} S^\alpha_\beta \varphi^\alpha_\beta(-\tilde{u}, z) \Sigma^\beta_\gamma.
\]

It follows from (6.13), (B.4) and (B.5), and that \( L_a(z) \) has the required quasi-periodicities and the residues.

We replace the basis on the dual basis using (5.9) and (5.39) and finally obtain

\[
L_a(z) = \sum_{s \in \tilde{H}_l} S^\alpha_\beta \phi^\alpha_\beta(z) + \sum_{\beta \in \tilde{H}_l} S^\alpha_\beta \varphi^\alpha_\beta(-\tilde{u}, z) \Sigma^\beta_\gamma \frac{\langle \beta, \beta \rangle}{2p_\beta},
\]
(6.15)

\[
L'_a(z) = \sum_{\beta \in \tilde{H}_l} S^\alpha_\beta \varphi^\alpha_\beta(-\tilde{u}, z) \Sigma^\beta_\gamma \frac{\langle \alpha, \alpha \rangle}{2p_\alpha}.
\]
(6.16)
Consider the invariant subalgebra $\tilde{g}_0$. For $\tilde{g}_0$ we write down the Lax matrix in the Chevalley basis. Let $p \leq n$ be a rank of $\tilde{g}_0$, $(e_1, \ldots, e_p)$ is a canonical basis in $\tilde{g}_0$ and $E_\alpha$ are generators of the root subspaces. The matrix elements of $L_0$ are constructed by means of $\varphi_\beta^0$ and the Eisenstein functions [B.9]:

$$\tilde{L}_0(z) = \sum_{j=1}^{p} (v_j + \tilde{S}_j^0 E_1(z)) e_j + \sum_{\beta \in R} \tilde{S}_\beta^0 \varphi_\beta^0(-\tilde{u}, z) E_\beta.$$  

(6.17)

Here $\tilde{v} = (v_1, \ldots, v_p)$ are momenta vector, dual to $\tilde{u} = (u_1, \ldots, u_p)$. The Lax operator (6.17) differs from the standard Lax operator related to $\tilde{g}_0$

$$\tilde{L}_0(z) = \sum_{j=1}^{p} (v_j + S_j^0 E_1(z)) e_j + \sum_{\alpha \in R} S_\alpha^0 \varphi(-\tilde{u}, \tilde{u}, z) E_\alpha.$$  

It is gauge equivalent to the previous one after $\tilde{u} \rightarrow \tilde{u} + \kappa$. For this reason we call $\tilde{g}_0$ the unbroken subalgebra. The operator $\tilde{L}_0(z)$ has the needed residue (see (B.13) and (B.14)). However, the Cartan term containing $E_1(z)$ breaks the quasi-periodicities (see (B.13)), because there are no double-periodic functions with one pole on $\Sigma_T$. To go around this problem we use the Poisson reduction procedure.

The Lax matrix. Poisson reduction

The Lax element we have defined depends on the spin variables representing an orbit $O$, on the moduli vector $\tilde{u}$ in the moduli space described in Section 5 and the dual covector $v$. It is a Poisson manifold $\mathbb{P}$ with the canonical brackets for $\tilde{v}$, $\tilde{u}$ and the Poisson-Lie brackets for $S$.

$$\mathbb{P} = T^\ast C \times O = \{\tilde{v}, \tilde{u}, S\}, \quad \tilde{u} \in C, \quad S \in O.$$  

(6.18)

It has dimension $\dim (O) + 2 \dim (\tilde{g}_0)$.

Consider the Poisson algebra $A = C^\infty(\mathbb{P})$ of smooth function on $\mathbb{P}$. Let $\epsilon \in \tilde{g}_0$ and $\gamma$ is a small contour around $z = 0$. Consider the following function $\mu_\epsilon = \{\epsilon, L(\tilde{v}, \tilde{u}, S)\} = (\epsilon, S_\alpha^0), S_\alpha^0 = \sum_{j=1}^{p} S_j^0 e_j. \quad$ It generates the vector field on $\mathbb{P} \quad V_\epsilon : L(\tilde{v}, \tilde{u}, S) \rightarrow \{\mu_\epsilon, L(\tilde{v}, \tilde{u}, S)\} = [\epsilon, L(\tilde{v}, \tilde{u}, S)].$

Let $A^{inv}$ be an invariant subalgebra of $A$ under $V_\epsilon$ action. Then $I = \{\mu_\epsilon F(\tilde{v}, \tilde{u}, S) \mid F(\tilde{v}, \tilde{u}, S) \in A\}$ is the Poisson ideal in $A^{inv}$. The reduced Poisson algebra is the factor-algebra

$$A^{red} = A^{inv} / I = A / \tilde{\mathcal{H}}_0, \quad (\tilde{\mathcal{H}}_0 = \exp \tilde{g}_0).$$

The reduced Poisson manifold $\mathbb{P}^{red}$ is defined by the moment constraint $S_0^0 = 0$ (or $S_{0,s}^0 = 0$) and dim $\tilde{\mathcal{H}}$ gauge fixing constraints on the spin variables that we do not specify

$$\mathbb{P}^{red} = \mathbb{P} / \tilde{\mathcal{H}}_0 = \mathbb{P}(S_{0,s}^0 = 0) / \tilde{\mathcal{H}}_0, \quad \dim (\mathbb{P}^{red}) = \dim (\mathbb{P}) - 2 \dim (\tilde{\mathcal{H}}_0) = \dim (O).$$

Thus, after the reduction we come to the Poisson manifold that has dimension of the coadjoint orbit $O$, but the Poisson structure on $\mathbb{P}^{red}$ is not the Lie-Poisson structure. The Poisson brackets on $\mathbb{P}^{red}$ are the Dirac brackets [12].

Due to the moment constraints we come to the Lax operator that has the correct periodicity. It depends on variables \{\tilde{v}, \tilde{u}, S\} $\in \mathbb{P}^{red}$

$$\tilde{L}_0(z) = \sum_{j=1}^{n} v_j e_j + \sum_{\beta \in R} S_{\beta}^0 \varphi_\beta^0(\tilde{u}, z) E_\beta.$$  

(6.19)

Here $S_{\beta}^0$ are not free due to the gauge fixing.

30
Hamiltonians

Let us define commuting integrals. For this purpose consider the ring of invariant polynomials on \( \mathfrak{g} \). It is generated \( n \) homogeneous polynomials \( P_1, P_2, \ldots, P_n \) of degrees \( d_1 = 2, \ldots, d_n = h \). It follows from the general approach \([21]\) that \( P_j(L(z)) \) generate commuting integrals. They are double periodic meromorphic functions on \( \Sigma_\tau \) and thereby can be expanded in the basis of elliptic functions

\[
\frac{1}{m_j} P_j(L(z)) = I_{j,0} + I_{j,2}E_2(z) + \ldots + I_{j,j}E_j(z).
\]

The coefficients \( I_{j,k} \) \((0 \leq k \leq m_j, \ k \neq 1)\) become commuting independent integrals. The highest order coefficients \( I_{jj} \) are Casimir functions fixing the orbits. The coefficient \( I_{j,1} \) vanishes, because there is no double periodic functions with one simple pole. The number of rest coefficients is equal to \( \frac{1}{2} \dim(\mathcal{O}) \).

Consider the second order in the spin variables Hamiltonian \( H = I_{1,0} \) coming from the expansion \( \frac{1}{2} P_1(L(z)) = H + I_{2,2}E_2(z). \) We represent it in the form

\[
H = \tilde{H}_0 + H' + \sum_{a=1}^{M} H_a, \quad M = \left\lceil \frac{1}{2} \right\rceil.
\]

Due to the orthogonality of \( L_a \) and \( L_b \) \((a \neq -b \mod l)\) with respect to the Killing form the Hamiltonians \( \tilde{H}, H' \) and \( H_b \) are determined by pairing of the corresponding Lax operators

\[
\tilde{H}_0 = \frac{1}{2}(\tilde{L}_0(z), \tilde{L}_0(z))|_{\text{const}}, \quad H' = \frac{1}{2}(L_0'(z), L_0'(z))|_{\text{const}}, \quad H_a = \frac{1}{2}(L_a(z), L_{l-a}(z))|_{\text{const}}
\]

To calculate the Hamiltonians we use \([5.10], [5.38], (B.15)\). Then we come to the following expressions \(^{11}\)

\[
\tilde{H}_0 = \frac{1}{\sqrt{2}} \sum_{s=1}^{n} v_s^2 - \sum_{\beta \in R} \frac{1}{(\beta, \beta)} \tilde{S}^\gamma_{\beta} \tilde{S}^\gamma_{-\beta} E_2((\tilde{u} - \kappa \tau, \tilde{\beta})).
\]

As it was noted above \( \tilde{H}_0 \) is the elliptic CM Hamiltonian related to \( \tilde{\mathfrak{g}}_0 \). Using \([5.8]\) we find

\[
H' = - \sum_{\alpha, \beta \in T_l} \delta_{\beta, -\lambda'(\alpha)} \frac{(\beta, \beta)}{p_\beta} S^\alpha_{\beta} S^{\alpha'}_{\beta} E_2((\tilde{u} - \kappa \tau, \beta)).
\]

Similarly, from \([5.15], (5.10) \) and \([5.13]\)

\[
H_a = - \frac{1}{\sqrt{2}} \sum_{s \in C_l} E_2(\frac{a}{r}) S^\alpha_{\beta} S^\alpha_{\beta} - \sum_{\alpha, \beta \in T_l} \delta_{\alpha, -\lambda'(\beta)} \omega^{-ar} \frac{(\alpha, \alpha)}{p_\alpha} S^\beta_{\alpha} S^{\beta}_{\alpha} E_2((\tilde{u} - \kappa \tau, \beta)).
\]

The Hamiltonians \( H', H_a \) are Hamiltonians of EA tops with the inertia tensors depending on \( \tilde{u} \).

\(^{11}\)In what follows we shall use the standard CM Hamiltonians, where the coordinate vector is shifted \( \tilde{u} \rightarrow \tilde{u} + \kappa \tau. \)

7 The proof of the classical RLL-relation

Proposition 7.1 • The r-matrix and the Lax operator described above define the Poisson brackets on the reduced phase space \( P^{\text{red}} \) via RLL-equation:

\[
\{L(z) \otimes 1, 1 \otimes L(w)\} = [L(z) \otimes 1 + 1 \otimes L(w), r(z, w)] - \frac{\sqrt{t}}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |a|^2 \partial_1 \phi_a^k(z - w) S_{\alpha}^{a,0} t_a^k \otimes t_{-a}^{-k} \tag{7.24}
\]

These brackets have the following explicit form:

\[
\left\{ S_{\alpha}^{a,0}, S_{\beta}^{b,0} \right\} = \begin{cases} \frac{1}{\sqrt{t}} \sum_{s=0}^{l-1} \omega^s C_{\alpha,\lambda}^{\gamma} S_{\alpha+\lambda,\beta}^{\epsilon+b}, & \alpha \neq -\lambda, \beta \\ \frac{2\alpha}{\sqrt{t}} \omega^s b S_{\alpha}^{a+b}, & \alpha = -\lambda, \beta \end{cases}
\]

\[
\left\{ S_{\alpha}^{0,0}, S_{\beta}^{m,0} \right\} = \frac{1}{\sqrt{t}} \sum_{s=0}^{l-1} \omega^{-k s} (\alpha, \lambda, \beta) S_{\beta}^{0,k+m}
\]

\[
\left\{ S_{\alpha}^{0,0}, S_{\beta}^{0,m} \right\} = \frac{1}{\sqrt{t}} \sum_{s=0}^{l-1} \omega^{-k s} (\alpha, \lambda, \beta) S_{\beta}^{0,k+m}
\]

\[
\left\{ v_{\alpha}, S_{\beta}^{a,0} \right\} = \left\{ v_{\alpha}, S_{\beta}^{b,0} \right\} = \left\{ v_{\alpha}, S_{\beta}^{a,b} \right\} = 0
\]

\[
\left\{ u_{\alpha}, S_{\beta}^{a,0} \right\} = \left\{ u_{\alpha}, S_{\beta}^{b,0} \right\} = \left\{ u_{\alpha}, S_{\beta}^{a,b} \right\} = 0
\]

It will be convenient to subdivide the Lax operator and the classical r-matrix into Cartan and root parts:

\[
L(z) = L_R(z) + L_\beta(z) + L_{\beta 0}^0(z) \tag{7.26}
\]

\[
r(z, w) = r_R(z, w) + r_\beta(z, w)
\]

where

\[
L_R(z) = \frac{1}{2} \sum_{a=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \varphi_a^\beta(z) S_{\beta}^{0,-a} t_a^\beta
\]

\[
L_\beta(z) = \sum_{a=1}^{l-1} \sum_{\alpha \in \Pi} \varphi_a^\alpha(z) S_{\alpha}^{0,-a} h_a^\alpha + L_{\beta 0}^0(z) = \sum_{ \alpha \in \Pi} \left( \psi_0^\alpha + E_1(z) S_{\alpha}^{0,0} \right) h_a^\alpha
\]

and:

\[
r_R(z, w) = \frac{1}{2} \sum_{a=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi_a^\alpha(z - w) t_a^\alpha \otimes t_{-a}^{-\alpha}, \quad r_\beta(z, w) = \sum_{a=0}^{l-1} \sum_{\alpha \in \Pi} \varphi_a^\alpha(z - w) S_{\alpha}^{0,-a} \otimes h_{-a}^\alpha
\]

Note that in these formulae the summation spreads over all roots but not over orbits. Obviously these definitions of Lax operators and r-matrices are equivalent if one takes into account
identities \((5.23), (5.27)\) and \((13.16)\). The proof of the proposition of this section is purely technical calculation and to simplify intermediate expressions we will use normalized Cartan generators defined in Section \(5\). In terms of these generators the Cartan part of Lax operator and \(r\)-matrix take the form:

\[
\begin{align*}
    r_S(z, w) &= \sum_{a=0}^{l-1} \sum_{\alpha \in \Pi} \varphi^a_0(z - w) \delta^a_\alpha \otimes \bar{h}^a_\alpha = \sum_{a=0}^{l-1} \sum_{\alpha \in \Pi} \varphi^a_0(z - w) \delta^-a_\alpha \otimes \bar{h}^a_\alpha \\
    L_S(z) &= \sum_{a=1}^{l-1} \sum_{\alpha \in \Pi} \varphi^a_0(z) \bar{S}^a_\alpha \cdot \bar{h}^a_\alpha. \quad L^0_S(z) = \sum_{\alpha \in \Pi} \left( \varphi^0_\alpha + E_1(z) \bar{S}^{0,0}_\alpha \right) \bar{h}^0_\alpha
\end{align*}
\]

(7.29)

**proof**

Non-vanishing terms in the left hand side of \((7.24)\) are:

\[
\begin{align*}
    \{ L(z) \otimes 1, 1 \otimes L(w) \} &= \{ L_R(z) \otimes 1, 1 \otimes L_R(w) \} + \{ L_R(z) \otimes 1, 1 \otimes L_B(w) \} + \{ L_R(z) \otimes 1, 1 \otimes L_B(w) \} + \{ L_B(z) \otimes 1, 1 \otimes L_R(w) \} + \{ L_B(z) \otimes 1, 1 \otimes L_B(w) \}
\end{align*}
\]

where we denoted by \(t^k_{\alpha} \otimes t^m_{\beta}, \ h^0_\alpha \otimes t^m_{\beta}\) etc. the corresponding tensor structure of the brackets. Our strategy is to check the equation \((7.24)\) for all possible tensor structures using the commutation relations \((5.18)\) and Poisson brackets \((7.25)\). From the previous expression we see that we should consider only three structures: \(t^k_{\alpha} \otimes t^m_{\beta}, \ h^0_\alpha \otimes t^m_{\beta}\) and \(h^0_\alpha \otimes t^m_{\beta}\). The computations for the rest terms \(t^k_{\alpha} \otimes h^0_\alpha\) and \(t^k_{\alpha} \otimes h^0_\alpha\) are symmetric to \(h^0_\alpha \otimes t^m_{\beta}\) and \(h^0_\alpha \otimes t^m_{\beta}\) cases respectively.

The commutator \([t^k_{\alpha}, t^m_{\beta}]\) depends on whether \(\alpha \in \mathcal{O}(-\beta)\) or not, therefore, it will be convenient to consider the case of tensor structure \(t^k_{\alpha} \otimes t^{-k}_{\alpha}\) separately. The table below describes the terms in the right hand side of the equation \((7.24)\) getting contributions to the corresponding tensor structures:

| \(t^k_{\alpha} \otimes t^m_{\beta}\) | \([L_R(z) \otimes 1, r_R(z, w)], \ [1 \otimes L_R(w), r_R(z, w)]\) |
|-----------------|-----------------|
| \(h^0_\alpha \otimes t^m_{\beta}\) | \([L_R(z) \otimes 1, r_R(z, w)], \ [1 \otimes L_R(w), r_R(z, w)]\) |
| \(h^0_\alpha \otimes t^m_{\beta}\) | \([L_R(z) \otimes 1, r_R(z, w)], \ [1 \otimes L_R(w), r_R^0(z, w)]\) |
| \(t^k_{\alpha} \otimes t^{-k}_{\alpha}\) | \([L^0_B(z) \otimes 1, r_R(z, w)], \ [1 \otimes L^0_B(w), r_R(z, w)]\) |

(7.30)

Therefore we prove \((7.24)\) by checking four equations for every tensor structure:

\[
\begin{align*}
    \{ L_R(z) \otimes 1, 1 \otimes L_R(w) \} \bigg|_{t^k_{\alpha} \otimes t^m_{\beta}} - \bigg[ L_R(z) \otimes 1, r_R(z, w) \bigg] \bigg|_{t^k_{\alpha} \otimes t^m_{\beta}} - \bigg[ 1 \otimes L_R(w), r(w, z) \bigg] \bigg|_{t^k_{\alpha} \otimes t^m_{\beta}} &= 0 \quad (7.31)
\end{align*}
\]

for for generic \(m, k, \alpha\) and \(\beta\),

\[
\begin{align*}
    \{ L_S(z) \otimes 1, 1 \otimes L_R(w) \} \bigg|_{h^0_\alpha \otimes t^m_{\beta}} - \bigg[ L_R(z) \otimes 1, r_S(z, w) \bigg] \bigg|_{h^0_\alpha \otimes t^m_{\beta}} - \bigg[ 1 \otimes L_R(w), r_S(z, w) \bigg] \bigg|_{h^0_\alpha \otimes t^m_{\beta}} &= 0 \quad (7.32)
\end{align*}
\]

for \(k \neq 0\),

\[
\begin{align*}
    \{ L^0_B(z) \otimes 1, 1 \otimes L_R(w) \} \bigg|_{h^0_\alpha \otimes t^m_{\beta}} - \bigg[ L_R(z) \otimes 1, r_R(z, w) \bigg] \bigg|_{h^0_\alpha \otimes t^m_{\beta}} - \bigg[ 1 \otimes L_R(w), r_R(z, w) \bigg] \bigg|_{h^0_\alpha \otimes t^m_{\beta}} &= 0 \quad (7.33)
\end{align*}
\]
and
\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\}\bigg|_{t_\alpha^k \otimes t_\beta^m} - \left[ L_R^0(z) \otimes 1, r_R(z, w) \right] - \left[ 1 \otimes L_R^0(w), r_R(z, w) \right] =
\]
\[= - \frac{\sqrt{7}}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \partial_\alpha \varphi^k_\alpha(z - w) S_{\alpha - \lambda}^k \otimes t_{-\alpha}^{-k} \quad (7.34)\]

In the next subsections we check these identities using only commutation relations (7.25) and (7.26).

**t_\alpha^k \otimes t_\beta^m -terms for generic \( \alpha \) and \( \beta \)**

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\} = \left\{ \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi^k_\alpha(z) S_{-\alpha}^k \otimes 1, \frac{1}{2} \sum_{m=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \varphi^m_\beta (w) S_{-\beta}^{-m} \otimes 1 \right\} =
\]
\[= \frac{1}{4} \sum_{k,m=0}^{l-1} \sum_{\alpha, \beta \in R} |\alpha|^2 |\beta|^2 |\omega^{-m} \varphi^k_\alpha(z) \varphi^m_\beta (w) C_{-\alpha, -\lambda} S_{-\alpha - \lambda}^{-k - m} t_\alpha^k \otimes t_\beta^m \quad (7.25)\]

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\}\bigg|_{t_\alpha^k \otimes t_\beta^m} = \frac{1}{4\sqrt{7}} \sum_{k,m,s=0}^{l-1} \sum_{\alpha, \beta \in R} |\alpha|^2 |\beta|^2 |\omega^{-m} \varphi^k_\alpha(z) \varphi^m_\beta (w) C_{-\alpha, -\lambda} S_{-\alpha - \lambda}^{-k - m} t_\alpha^k \otimes t_\beta^m \quad (7.35)\]

\[
\left[ L_R(z) \otimes 1, r_R(z, w) \right] = \left[ \frac{1}{2} \sum_{b=0}^{l-1} \sum_{\gamma \in R} |\gamma|^2 \varphi^b_\gamma(z) S_{-\gamma}^{-b} \otimes 1, \frac{1}{2} \sum_{m=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \varphi^{-m}_\beta(z - w) t_{-m}^{-m} \otimes t_\beta^m \right] =
\]
\[= \frac{1}{4} \sum_{b,m=0}^{l-1} \sum_{\beta, \gamma \in R} |\gamma|^2 |\beta|^2 |\omega^{-m} \varphi^b_\gamma(z) \varphi^{-m}_\beta(z - w) S_{-\gamma}^{-b} \otimes t_\gamma^b \otimes t_\beta^m \quad (5.13)\]

\[= \frac{1}{4\sqrt{7}} \sum_{b,m,s=0}^{l-1} \sum_{\beta, \gamma \in R} |\gamma|^2 |\beta|^2 |\omega^{-m} \varphi^b_\gamma(z) \varphi^{-m}_\beta(z - w) S_{-\gamma}^{-b} \otimes t_\gamma^b \otimes t_\beta^m \quad (B.16)\]

\[= \frac{1}{4\sqrt{7}} \sum_{k,m,s=0}^{l-1} \sum_{\alpha, \beta \in R} |\alpha|^2 |\beta|^2 |\omega^{-m} \varphi^k_\alpha(z) \varphi^{-m}_\beta(z - w) S_{-\alpha - \lambda}^{-k - m} \otimes t_\alpha^k \otimes t_\beta^m \quad (B.16)\]

Therefore we get:

\[
\left[ L_R(z) \otimes 1, r_R(z, w) \right]\bigg|_{t_\alpha^k \otimes t_\beta^m} =
\]
\[= \frac{1}{4\sqrt{7}} \sum_{k,m,s=0}^{l-1} \sum_{\alpha, \beta \in R} |\alpha|^2 |\beta|^2 |\omega^{-m} \varphi^k_\alpha(z) \varphi^{-m}_\beta(z - w) \otimes t_\alpha^k \otimes t_\beta^m \quad (7.36)\]
The last term:

\[
\left[ 1 \otimes L_R(w), r(w, z) \right] = \left[ \frac{1}{2} \sum_{\substack{p=0 \gamma \in R}} \frac{1}{2} \sum_{k=0 \alpha \in R} \frac{1}{2} \sum_{p=0 \gamma \in R} \frac{1}{2} \sum_{k=0 \alpha \in R} \left| \alpha \right|^2 \varphi_{\alpha}^p (z-w) \varphi_{\alpha}^k (z-w) S_{-\gamma}^{\alpha,-p} t_{\alpha}^k \otimes t_{-\alpha}^{-k} \right] =
\]

\[
= \frac{1}{4} \sum_{k,p=0 \alpha,\gamma \in R} \left| \alpha \right|^2 \varphi_{\alpha}^p (z-w) S_{-\gamma}^{\alpha,-p} t_{\alpha}^k \otimes t_{-\alpha}^{-k} \quad (7.18)
\]

\[
= \frac{1}{4} \sum_{k,p=0 \alpha,\gamma \in R} \left| \alpha \right|^2 \varphi_{\alpha}^p (z-w) S_{-\gamma}^{\alpha,-p} w^{-ks} C_{\gamma,-\lambda^s} t_{\alpha}^k \otimes t_{-\alpha}^{-k} \quad (7.27)
\]

\[
= \frac{1}{4} \sum_{k,m,s=0 \alpha,\beta \in R} \left| \alpha \right|^2 \varphi_{\alpha}^k (z-w) S_{-\alpha,-\lambda^s}^\beta C_{-\alpha,-\lambda^s} t_{\alpha}^k \otimes t_{\beta}^{m} \quad (7.28)
\]

therefore:

\[
\left[ 1 \otimes L_R(w), r(w, z) \right] \bigg|_{h_{\alpha}^k \otimes t_{\beta}^m} =
\]

\[
= \frac{1}{4} \sum_{k,m,s=0 \alpha,\beta \in R} \left| \alpha \right|^2 \varphi_{\alpha}^k (z-w) S_{-\alpha,-\lambda^s}^\beta C_{-\alpha,-\lambda^s} t_{\alpha}^k \otimes t_{\beta}^{m} \quad (7.37)
\]

Summing up \((7.35), (7.36)\) and \((7.37)\) and using Fay identity \((17.7)\) we obtain:

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\} \bigg|_{h_{\alpha}^k \otimes t_{\beta}^m} - \left[ L_R(z) \otimes 1, r_R(z, w) \right] \bigg|_{h_{\alpha}^k \otimes t_{\beta}^m} - \left[ 1 \otimes L_R(w), r(w, z) \right] \bigg|_{h_{\alpha}^k \otimes t_{\beta}^m} = 0
\]

\[\hat{h}_{\alpha}^k \otimes t_{\beta}^m\text{-terms for } k \neq 0\]

Let us consider the following Poisson bracket:

\[
\left\{ L_\delta(z) \otimes 1, 1 \otimes L_R(w) \right\} = \left\{ \sum_{\alpha,\beta \in \Pi} \varphi_{\alpha}^k (z) S_{\alpha}^{\beta,-k} \hat{h}_{\alpha}^k \otimes 1, 1 \right\}
\]

\[
= \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\beta \in \Pi} \sum_{\alpha \in \Pi} \left| \beta \right|^2 \varphi_{\alpha}^m (w) S_{-\beta}^{\alpha,-m} \right. \bigg|_{h_{\alpha}^k \otimes t_{\beta}^{m}} ^{7.25}
\]

\[
= \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\beta \in \Pi} \sum_{\alpha \in \Pi} \left| \beta \right|^2 \varphi_{\alpha}^m (z) \varphi_{\beta}^m (w) \omega^{ks} (\hat{\alpha}, \lambda^s) \right. \bigg|_{h_{\alpha}^k \otimes t_{\beta}^{m}} ^{7.25}
\]

and therefore:

\[
\left\{ L_\delta(z) \otimes 1, 1 \otimes L_R(w) \right\} \bigg|_{\hat{h}_{\alpha}^k \otimes t_{\beta}^{m}} = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\beta \in \Pi} \sum_{\alpha \in \Pi} \left| \beta \right|^2 \varphi_{\alpha}^m (z) \varphi_{\beta}^m (w) \omega^{ks} (\hat{\alpha}, \lambda^s) \right. \bigg|_{h_{\alpha}^k \otimes t_{\beta}^{m}} ^{7.38}
\]
In the second part we need to compute the Cartan part of the commutator:

\[
\left[ L_R(z) \otimes 1, r_R(z, w) \right] \bigg|_{h^0 \otimes t^m_{\beta}} = \frac{1}{2} \left( \frac{1}{2} \sum_{p=0}^{l-1} \sum_{\gamma, \beta \in R} |\gamma|^2 \varphi_\gamma^p(z) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} \right) = \left( \frac{1}{2} \sum_{p=0}^{l-1} \sum_{\gamma, \beta \in R} |\gamma|^2 \varphi_\gamma^p(z) \varphi_\beta^m(z-w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} \right) \bigg|_{h^0 \otimes t^m_{\beta}}
\]

To separate the needed part from the expression in the bracket we must take into account only those terms for which \(\gamma \in \mathcal{O}(\beta)\). For this purpose it convenient to make use of the "delta"-symbol of the orbit:

\[
\delta \left( \alpha \in \mathcal{O}(\beta) \right) = \begin{cases} 
1 & \text{if } \alpha \in \mathcal{O}(\beta) \\
0 & \text{if } \alpha \notin \mathcal{O}(\beta)
\end{cases}
\]

Using this notation we rewrite the last expression in the form:

\[
\frac{1}{2} \sum_{p=0}^{l-1} \sum_{\gamma, \beta \in R} \delta \left( \gamma \in \mathcal{O}(\beta) \right) |\gamma|^2 |\beta|^2 \varphi_\gamma^p(z) \varphi_\beta^m(z-w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\beta \in R} (\alpha, \lambda^\beta) \delta \left( \gamma \in \mathcal{O}(\beta) \right) \left| \beta \varphi_\gamma^p(z) \varphi_\beta^m(z-w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} \right|
\]

\[
= \frac{1}{2} \left( \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\beta \in R} (\alpha, \lambda^\beta) \omega^k |\beta|^2 \varphi_\gamma^p(z) \varphi_\beta^m(z-w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} \right)
\]

here we use the denotation \(\theta(\alpha, \beta) = s\) if \(\alpha = -\lambda^\beta\). Finally we obtain:

\[
\left[ L_R(z) \otimes 1, r_R(z, w) \right] \bigg|_{h^0 \otimes t^m_{\beta}} = \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\beta \in R} (\alpha, \lambda^\beta) \omega^k |\beta|^2 \varphi_\gamma^p(z) \varphi_\beta^m(z-w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta}
\]

For the last, third term we have:

\[
\left[ 1 \otimes L_R(w), r_R(z, w) \right] \bigg|_{h^0 \otimes t^m_{\beta}} = \frac{1}{2} \left( \frac{1}{2} \sum_{a=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \varphi_\beta^a(w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{2} \sum_{a=0}^{l-1} \sum_{\beta \in R} |\beta|^2 \varphi_\beta^a(w) \varphi_\beta^k(z-w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{2} \sum_{a=0}^{l-1} \sum_{\beta \in R} \omega^k (\alpha, \lambda^\beta) |\beta|^2 \varphi_\beta^a(w) \varphi_\beta^k(z-w) S^\gamma_{-\gamma} \varphi_\beta^m(z-w) t^m_{-\beta} \otimes t^p_{\beta} \right)
\]
\[
\frac{1}{2 \sqrt{t}} \sum_{m,k,s=0}^{l-1} \frac{1}{R} \sum_{R} \sum_{\alpha \in \Pi} \omega^{sk} (\alpha, \lambda^s \beta) |\beta|^2 \varphi_{k+m}^m (w) \varphi_{k} (z-w) S^{n-k-m}_{\alpha} S^{m}_{\beta} k_t \otimes t^m_{\beta} \]
\]

and we get:

\[
\left[ 1 \otimes L_R(w), r_R(z, w) \right] |_{h^0_{\alpha} \otimes t^m_{\beta}} = -\frac{1}{2 \sqrt{t}} \sum_{m,k,s=0}^{l-1} \frac{1}{R} \sum_{R} \sum_{\alpha \in \Pi} \omega^{sk} (\alpha, \lambda^s \beta) |\beta|^2 \varphi_{k+m}^m (w) \varphi_{k} (z-w) S^{n-k-m}_{\alpha} S^{m}_{\beta} k_t \otimes t^m_{\beta} \] 

Summing up (7.38), (7.39), (7.40) and using Fay identity (7.7) we obtain:

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\} |_{h^0_{\alpha} \otimes t^m_{\beta}} - \left[ L_R(z) \otimes 1, r_R(z, w) \right] |_{h^0_{\alpha} \otimes t^m_{\beta}} - \left[ 1 \otimes L_R(w), r_R(z, w) \right] |_{h^0_{\alpha} \otimes t^m_{\beta}} = 0
\]

\[h^0_{\alpha} \otimes t^m_{\beta}\text{-terms}\]

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\} = \left\{ \sum_{\alpha \in \Pi} \left( \hat{w}^0_{\alpha} + E_1(z) S^0_{\alpha} \right) h^0_{\alpha} \otimes 1, \frac{1}{2} \sum_{m=0}^{l-1} \sum_{R} |\beta|^2 \varphi^m (w) S^{n-m}_{\alpha} 1 \otimes t^m_{\beta} \right\} = \frac{1}{2} \sum_{\alpha \in \Pi} \sum_{m=0}^{l-1} \sum_{R} |\beta|^2 \left( \left\{ \hat{w}^0_{\alpha} \varphi^m (w) \right\} S^{n-m}_{\alpha} + E_1(z) \varphi^m (w) \left\{ S^0_{\alpha}, S^{n-m}_{\alpha} \right\} \right) h^0_{\alpha} \otimes t^m_{\beta}
\]

therefore we have:

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\} |_{h^0_{\alpha} \otimes t^m_{\beta}} = \frac{1}{2 \sqrt{t}} \sum_{\alpha \in \Pi} \sum_{m=0}^{l-1} \sum_{R} |\beta|^2 S^{n-m}_{\alpha} (\hat{w}^0_{\alpha} \varphi^m (w)) \left( \left( \hat{w}^0_{\alpha} \varphi^m (w) \right) S^{n-m}_{\alpha} + E_1(z) \varphi^m (w) \right) h^0_{\alpha} \otimes t^m_{\beta}
\]

The calculation of the second term \[L_R(z) \otimes 1, r_R(z, w) \] \[h^0_{\alpha} \otimes t^m_{\beta}\] is analogous to the computation of (7.39), and to get the answer we need to put \(k = 0\) in the expression (7.39):

\[
\left[ L_R(z) \otimes 1, r_R(z, w) \right] |_{h^0_{\alpha} \otimes t^m_{\beta}} = \frac{1}{2 \sqrt{t}} \sum_{m,k,s=0}^{l-1} \frac{1}{R} \sum_{R} \sum_{\alpha \in \Pi} (\hat{w}^0_{\alpha} \varphi^m (w)) \varphi^m_{-\beta} (z-w) S^{n-m}_{\alpha} S^{m}_{\beta} k_t \otimes t^m_{\beta}
\]

Analogously, we obtain the answer for \[1 \otimes L_R(w), r_R(z, w) \] \[h^0_{\alpha} \otimes t^m_{\beta}\] substituting into (7.40) \(k = 0\), and replacing \(\varphi^0 (z-w) \rightarrow E_1(z-w)\):

\[
\left[ 1 \otimes L_R(w), r_R(z, w) \right] |_{h^0_{\alpha} \otimes t^m_{\beta}} = -\frac{1}{2 \sqrt{t}} \sum_{m,k,s=0}^{l-1} \frac{1}{R} \sum_{R} \sum_{\alpha \in \Pi} (\hat{w}^0_{\alpha} \varphi^m (w)) \varphi^m_{-\beta} (z-w) E_1(z-w) S^{n-m}_{\alpha} S^{m}_{\beta} k_t \otimes t^m_{\beta}
\]
Therefore we have:

\[
\left\{ L_R^0(z) \otimes 1, 1 \otimes L_R(w) \right\}\bigg|_{t_a^k \otimes t_{-a}^k} - \left[ L_R(z) \otimes 1, r_R(z, w) \right]\bigg|_{t_a^k \otimes t_{-a}^k} - \left[ 1 \otimes L_R(w), r_D(z, w) \right]\bigg|_{t_a^k \otimes t_{-a}^k} = 0
\]

\( t_a^k \otimes t_{-a}^k\)-terms

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\}\bigg|_{t_a^k \otimes t_{-a}^k} =
\left\{ \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \left| \alpha \right|^2 \varphi_\alpha^k(z) \varphi_{-\lambda^\alpha}^{k} \sum_{\lambda^\alpha} \left( S_{\alpha}^k \right) \otimes 1, \frac{1}{2} \sum_{m=0}^{l-1} \sum_{\beta \in R} \left| \beta \right|^2 \varphi_\beta^m(z) \varphi_{-\lambda^\beta}^{m} \sum_{\lambda^\beta} \left( 1 \otimes t_{-a}^m \right) \right\}\bigg|_{t_a^k \otimes t_{-a}^k} =
\left( \frac{1}{4} \sum_{k,m=0}^{l-1} \sum_{\alpha,\beta \in R} \left| \alpha \right|^2 \left| \beta \right|^2 \varphi_\alpha^k(z) \varphi_\beta^m(z) \left( S_{\alpha}^k \otimes S_{-\lambda^\beta}^m \right) \sum_{\lambda^\beta} \left( t_a^k \otimes t_{-a}^m \right) \right)\bigg|_{t_a^k \otimes t_{-a}^k}
\]

To separate the needed part we must take into account only those terms for which \( m = -k \), and \( \beta \in O (-\alpha) \), therefore we put \( m = -k \), \( \beta = -\lambda^\alpha \), and take the sum over \( s \in \{0, \ldots, l_\alpha - 1\} \):

\[
\frac{1}{4} \sum_{k=0}^{l-1} \sum_{s=0}^{l_\alpha - 1} \sum_{\alpha \in R} \left| \alpha \right|^4 \left| \varphi_\alpha^k(z) \varphi_{-\lambda^\alpha}^{k} \right| \left( S_{\alpha}^k \right) \sum_{\lambda^\alpha} \left( t_a^k \otimes t_{-a}^k \right) = \frac{1}{4} \sqrt{\frac{1}{l-1} \sum_{k=0}^{l-1} \sum_{s=0}^{l_\alpha - 1} \sum_{\alpha \in R} \left| \alpha \right|^4 \left| \varphi_\alpha^k(z) \varphi_{-\lambda^\alpha}^{k} \right| \left( S_{\alpha}^k \right) \sum_{\lambda^\alpha} \left( t_a^k \otimes t_{-a}^k \right)} = \frac{1}{4} \sqrt{\frac{1}{l_\alpha - 1} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \left| \alpha \right|^4 \left| \varphi_\alpha^k(z) \varphi_{-\lambda^\alpha}^{k} \right| \left( S_{\alpha}^k \right) \sum_{\lambda^\alpha} \left( t_a^k \otimes t_{-a}^k \right)}
\]

Therefore we have:

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\}\bigg|_{t_a^k \otimes t_{-a}^k} = -\frac{1}{4} l_\alpha \sum_{k=0}^{l-1} \sum_{\alpha \in R} \left| \alpha \right|^2 \left| \varphi_\alpha^k(z) \varphi_{-\lambda^\alpha}^{k} \right| \left( S_{\alpha}^k \right) \sum_{\lambda^\alpha} \left( t_a^k \otimes t_{-a}^k \right)
\]

The second term:

\[
\left[ L_R^0(z) \otimes 1, r_R(z, w) \right] = \sum_{\gamma \in \Pi} \left( \tilde{v}_\gamma^0 + E_1(z) S_{\gamma}^0 \right) h_\gamma^0 \otimes 1, \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \left| \varphi_\alpha^k(z - w) \right| t_a^k \otimes t_{-a}^k
\]

\[
= \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \left( \tilde{v}_\gamma^0 + E_1(z) S_{\gamma}^0 \right) \left| \varphi_\alpha^k(z - w) \right| h_\gamma^0 t_a^k \otimes t_{-a}^k
\]

\[
= \frac{1}{2} \sqrt{\frac{1}{l_\alpha} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \sum_{\gamma \in \Pi} \left( \tilde{v}_\gamma^0 + E_1(z) S_{\gamma}^0 \right) \left| \varphi_\alpha^k(z - w) \right| t_a^k \otimes t_{-a}^k}
\]

and we obtain:

\[
\left[ L_R^0(z) \otimes 1, r_R(z, w) \right] = \frac{1}{2 \sqrt{l_\alpha}} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \sum_{\gamma \in \Pi} \left( \tilde{v}_\gamma^0 + E_1(z) S_{\gamma}^0 \right) \left| \varphi_\alpha^k(z - w) \right| t_a^k \otimes t_{-a}^k
\]
the last term:

$$
\left[ 1 \otimes L^0_B(w), r_R(z, w) \right] = \left[ \sum_{\gamma \in \Pi} \left( t^0_{\gamma} + E_1(w) S^0_{\gamma \alpha} \right) 1 \otimes t^0_{\gamma}, \frac{1}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi_{\alpha}^k (z - w) t^k_{\alpha} \otimes t^{-k}_{-\alpha} \right]
$$

Obviously the result differs from the previous expression \((7.46)\) only by sign and by substitution \(E_1(z) \to E_1(w)\), therefore:

$$
\left[ 1 \otimes L^0_B(w), r_R(z, w) \right] = -\frac{1}{2\sqrt{l}} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \sum_{\gamma \in \Pi} \left( t^0_{\gamma} + E_1(w) S^0_{\gamma \alpha} \right) |\alpha|^2 \varphi_{\alpha}^k (z - w) (\gamma, \lambda^s \alpha) t^k_{\alpha} \otimes t^{-k}_{-\alpha}
$$

The sum of the last two expressions \((7.46), (7.46)\):

\[
\left[ L^0_B(z) \otimes 1, r_R(z, w) \right] + \left[ 1 \otimes L^0_B(w), r_R(z, w) \right] = \left[ L^0_B(z) \otimes 1, r_R(z, w) \right] - \left[ L^0_B(w) \otimes 1, r_R(z, w) \right] - \left[ 1 \otimes L^0_B(w), r_R(z, w) \right] = -\frac{\sqrt{l}}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi_{\alpha}^k (z - w) t^k_{\alpha} \otimes t^{-k}_{-\alpha}
\]

and finally:

\[
\left\{ L_R(z) \otimes 1, 1 \otimes L_R(w) \right\} \bigg|_{t^k_{\alpha} \otimes t^{-k}_{-\alpha}} - \left[ L^0_B(z) \otimes 1, r_R(z, w) \right] - \left[ 1 \otimes L^0_B(w), r_R(z, w) \right] = -\frac{\sqrt{l}}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2 \varphi_{\alpha}^k (z - w) t^k_{\alpha} \otimes t^{-k}_{-\alpha}
\]

The last step here is by the second Fay identity \((B.8)\).

**Classical Yang-Baxter equation**

**Proposition 7.2** The \(r\)-matrix \((7.28)\) satisfies the classical dynamical Yang-Baxter equation:

\[
[r_{12}(z, w), r_{13}(z, x)] + [r_{12}(z, w), r_{23}(w, x)] + [r_{13}(z, x), r_{23}(w, x)] - [r_{12}(z, w), r_{23}(w, x)] = \frac{\sqrt{l}}{2} \sum_{k=0}^{l-1} \sum_{\alpha \in R} \frac{|\alpha|^2}{2} t^k_{\alpha} \otimes t^{-k}_{-\alpha} \partial_1 \varphi_{\alpha}^k (z - w) - \frac{|\alpha|^2}{2} t^k_{\alpha} \otimes t^{-k}_{-\alpha} \partial_1 \varphi_{\alpha}^k (z - x) + \frac{|\alpha|^2}{2} t^k_{\alpha} \otimes t^{-k}_{-\alpha} \partial_1 \varphi_{\alpha}^k (w - x) = 0
\]

**Proof**

Let us examine the "non-dynamical" part (the upper line) of \((7.47)\):

off Cartan part:
\[
\sum_{k,n=0}^{l-1} \sum_{\alpha,\beta \in R} \varphi^k_\alpha(z-w)\varphi^n_\beta(z-x) [t^k_\alpha, t^n_\beta] \otimes c^k_\alpha \otimes c^n_\beta + \]
\[
\varphi^k_\alpha(z-w)\varphi^n_\beta(w-x) t^k_\alpha \otimes [t^{-k}_\alpha, t^n_\beta] \otimes t^{-n}_\beta + \]
\[
\varphi^k_\alpha(z-x)\varphi^n_\beta(w-x) t^k_\alpha \otimes t^n_\beta \otimes [t^{-k}_\alpha, t^{-n}_\beta] = \]
\[
\sum_{k,n,s=0}^{l-1} \sum_{\alpha,\beta \in R} \frac{|\alpha|^2|\beta|^2}{4\sqrt{I}} \varphi^k_\alpha(z-w)\varphi^n_\beta(z-x) \omega^{n_s} C_{\alpha,\lambda^s_\beta} t^{k+n}_{\alpha+\lambda^s_\beta} \otimes c^k_\alpha \otimes c^n_\beta + \]
\[
\frac{|\alpha|^2|\beta|^2}{4\sqrt{I}} \varphi^k_\alpha(z-w)\varphi^n_\beta(w-x) \omega^{n_s} C_{-\alpha-\lambda^s_\beta,\lambda^s_\beta} t^{k+n}_{-\alpha-\lambda^s_\beta} \otimes t^{-n}_\beta + \]
\[
\frac{|\alpha|^2|\beta|^2}{4\sqrt{I}} \varphi^k_\alpha(z-x)\varphi^n_\beta(w-x) \omega^{-n_s} C_{-\alpha,-\lambda^s_\beta} t^k_\alpha \otimes t^n_\beta \otimes t^{-k-n}_{-\alpha+\lambda^s_-\beta} \]

Making shifts \( k \rightarrow k+n \) and \( \alpha \rightarrow \alpha + \lambda^s(\beta) \) in the second line of (7.49) we have:
\[
\frac{|\alpha + \lambda^s_\beta|^2|\beta|^2}{4\sqrt{I}} \varphi^{k+n}_{\alpha+\lambda^s_\beta}(z-w)\varphi^n_\beta(w-x) \omega^{n_s} C_{-\alpha-\lambda^s_\beta,\lambda^s_\beta} t^{k+n}_{\alpha+\lambda^s_\beta} \otimes t^{-n}_\beta + \]

Similarly, one should make the following substitutions in the third line: \( \beta \rightarrow -\beta, n \rightarrow -n, k \leftrightarrow n, \alpha \rightarrow \alpha + \lambda^s_\beta, n \rightarrow n+k, \alpha \leftrightarrow \beta, \alpha \rightarrow \lambda^s_\alpha, \beta \rightarrow \lambda^s_\beta. \) Finally:
\[
\frac{|\alpha + \lambda^s_\beta|^2|\alpha|^2}{4\sqrt{I}} \varphi^{k+n}_{\alpha+\lambda^s_\beta}(z-x)\varphi^{-k}_{-\lambda^s_-\alpha}(w-x) \omega^{k+s} C_{-\alpha-\lambda^s_\beta,\lambda^s_\alpha} t^{k+n}_{\alpha+\lambda^s_\beta} \otimes t^{-k-\alpha}_\beta \otimes t^{-\lambda^s_\beta}_\alpha \]

Using the property \( t^k_{\lambda^s_\alpha} = \omega^{-k \cdot \lambda^s}\delta^k_\alpha \) we have:
\[
\frac{|\alpha + \lambda^s_\beta|^2|\alpha|^2}{4\sqrt{I}} \varphi^{k+n}_{\alpha+\lambda^s_\beta}(z-x)\varphi^{-k}_{-\lambda^s_-\alpha}(w-x) \omega^{n_s} C_{-\alpha-\lambda^s_\beta,\lambda^s_\alpha} t^{k+n}_{\alpha+\lambda^s_\beta} \otimes t^{-\alpha}_\beta \otimes t^{-\lambda^s_\beta}_\alpha \]

By the definition of the structure constants \( C_{\alpha,\beta} \) it is easy to show that \( C_{-\alpha-\lambda^s_\beta,\lambda^s_\alpha} = -\frac{|\beta|^2}{|\alpha + \lambda^s_\beta|^2} C_{\alpha,\beta} \) and \( C_{-\alpha,-\lambda^s_\beta,\lambda^s_\beta} = \frac{|\beta|^2}{|\alpha + \lambda^s_\beta|^2} C_{\alpha,\beta}. \)

Now we can combine all three lines and get a common multiple for
\[
\frac{|\alpha|^2|\beta|^2}{4\sqrt{I}} \omega^{n_s} C_{\alpha,\lambda^s_\beta} t^{k+n}_{\alpha+\lambda^s_\beta} \otimes c^k_\alpha \otimes c^n_\beta \]

The multiple is
\[
\varphi^k_\alpha(z-w)\varphi^n_\beta(z-x) - \varphi^{k+n}_{\alpha+\lambda^s_\beta}(z-w)\varphi^n_\beta(w-x) + \varphi^{k+n}_{\alpha+\lambda^s_\beta}(z-x)\varphi^{-k}_{-\lambda^s_-\alpha}(w-x) \]

It vanishes due to Fay identity. The proof of this fact is direct.

Cartan part:

Consider terms with Cartan elements in the third component of the tensor product (the other components can be obtained by the cyclic permutations). There are two origins for this type terms: 1. a direct appearance from \([r_{12}(z,w), r_{13}(z,x)]\) and \([r_{12}(z,w), r_{23}(w,x)]\) 2. appearance from \([r_{13}(z,x), r_{23}(w,x)]\) due to Cartan part of the commutator \([t^{-k}_\alpha, c^n_\beta]:\)
\[
\sum_{k,n=0}^{l-1} \sum_{\alpha \in R, \beta \in \Pi} \frac{|\alpha|^2}{2\sqrt{I}} \varphi^k_\alpha(z-w)\varphi^n_\beta(z-x) [t^k_\alpha, f^n_\beta] \otimes t^{-k}_\alpha \otimes h^{-n}_\beta + \]
\[
\sum_{k,n=0}^{l-1} \sum_{\alpha \in R, \beta \in \Pi} \frac{|\alpha|^2}{2\sqrt{I}} \varphi^k_\alpha(z-w)\varphi^n_\beta(z-x) [t^k_\alpha, f^n_\beta] \otimes t^{-k}_\alpha \otimes h^{-n}_\beta + \]
\[
\sum_{k,n=0}^{l-1} \sum_{\alpha \in R, \beta \in \Pi} \frac{|\alpha|^2}{2\sqrt{I}} \varphi^k_\alpha(z-w)\varphi^n_\beta(z-x) [t^k_\alpha, f^n_\beta] \otimes t^{-k}_\alpha \otimes h^{-n}_\beta + \]
\[
40
\[
\sum_{k,n=0}^{l-1} \sum_{\alpha \in R, \beta \in \Pi} \frac{|\alpha|^2}{2\sqrt{l}} \varphi^k_\alpha(z-w) \varphi^n_\alpha(w-x) t^k_\alpha \otimes [t^{-k}_\alpha, \delta^n_\beta] \otimes h^{-n}_\beta + \\
\sum_{k,n=0}^{l-1} \sum_{\alpha, \beta, -\lambda' \alpha \in R} \frac{|\alpha|^2|\beta|^2}{4\sqrt{l}} \varphi^k_\alpha(z-x) \varphi^n_\beta(w-x) t^k_\alpha \otimes t^n_\beta \otimes [t^{-k}_\alpha, \lambda^n_\beta] = \\
- \sum_{k,n,s=0}^{l-1} \sum_{\alpha \in R, \beta \in \Pi} \frac{|\alpha|^2|\beta|^2}{4\sqrt{l}} \varphi^{k-n}_\alpha(z-w) \varphi^n_\alpha(z-x) (\hat{\beta}, \lambda^{-s}_\alpha) \omega^{ns} t^k_\alpha \otimes t^{n-k}_\alpha \otimes h^{-n}_\beta - \\
\sum_{k,n=0}^{l-1} \sum_{\alpha, \beta, -\lambda' \alpha \in R} \frac{|\alpha|^2|\beta|^2}{4\sqrt{l}} \varphi^k_\alpha(z-x) \varphi^n_\beta(w-x) (\hat{\beta}, -\lambda^{-s}_\alpha) \omega^{ns} t^k_\alpha \otimes t^{-n-k}_\alpha \otimes h^{-n-k}_\beta
\]

The first lines contain a sum. Let us analyze the first line. We have the following sum over \(\beta\):
\[
\sum_{\beta \in \Pi} \frac{|\beta|^2}{2} (\hat{\beta}, \lambda^{-s}_\alpha) \omega^{ns} h^{-n}_\beta.
\]
For arbitrary root \(\gamma\):
\[
\sum_{\beta \in \Pi} \frac{|\beta|^2}{2} (\hat{\beta}, \gamma) h^{-n}_\beta = \frac{|\gamma|^2}{2} h_n.
\]
Now, using the property \(h^k_{\lambda^s(\alpha)} = \omega^{-k}\gamma^k h_n^k\) and \(|\lambda| = |\alpha|\) we have:
\[
\sum_{\beta \in \Pi} \frac{|\beta|^2}{2} (\hat{\beta}, \lambda^{-s}_\alpha) \omega^{ns} h^{-n}_\beta = \frac{|\alpha|^2}{2} \omega^{ns} h^{-n}_{\lambda^{-s}(\alpha)} = \frac{|\alpha|^2}{2} h^{-n}_\alpha.
\]
Note that the result does not depend on \(s\). Thus the sum over \(s\) makes the common multiple \(l\) and the first line equals:
\[
-l \sum_{k,n=0}^{l-1} \sum_{\alpha \in R} \frac{|\alpha|^2}{2\sqrt{l}} \varphi^{k-n}_\alpha(z-w) \varphi^n_\alpha(z-x) t^k_\alpha \otimes t^{-n-k}_\alpha \otimes h^{-n}_\alpha
\]
In the same way for the second line of (7.51) we have:
\[
l \sum_{k,n=0}^{l-1} \sum_{\alpha \in R} \frac{|\alpha|^2}{2\sqrt{l}} \varphi^k_\alpha(z-w) \varphi^n_\alpha(w-x) t^k_\alpha \otimes t^{-n-k}_\alpha \otimes h^{-n}_\alpha
\]
In the third line of (7.51) we collect all terms with tensor structure of type \(t^k_\alpha \otimes t^{-n-k}_\alpha \otimes h^{-n}_\alpha\). In this line \(\beta = -\lambda'\alpha\) for some \(r\). Then \(\omega^{nr} t^k_\alpha \otimes t^{-l}_\alpha \otimes h^{-n}_\alpha = -t^k_\alpha \otimes t^{-l}_\alpha \otimes h^{-n}_\alpha\) From the definition it follows that \(p_{-\alpha} = p_\alpha = \frac{1}{l_\alpha}\), where \(l_\alpha\) describes the minimal orbit, i.e. \(l_\alpha\) is a minimal nonzero number with the property \(\lambda^{l_\alpha} \alpha = \alpha\). The number of desired terms equals \(l_\alpha\).

Finally we have:
\[
\sqrt{l} \sum_{k,n=0}^{l-1} \sum_{\alpha \in R} -\frac{|\alpha|^2}{2} \varphi^{k-n}_\alpha(z-w) \varphi^n_\alpha(z-x) t^k_\alpha \otimes t^{-n-k}_\alpha \otimes h^{-n}_\alpha + \\
- \frac{|\alpha|^2}{2} \varphi^k_\alpha(z-w) \varphi^n_\alpha(w-x) t^k_\alpha \otimes t^{-n-k}_\alpha \otimes h^{-n}_\alpha - \\
\frac{|\alpha|^2}{2} \varphi^k_\alpha(z-x) \varphi^{-n-k}_\alpha(w-x) t^k_\alpha \otimes t^{-n-k}_\alpha \otimes h^{-n}_\alpha
\]
(7.52)
For $n \neq 0$ the sum vanishes due to Fay identity:

$$\varphi_k^α(z-w)\varphi_0^n(z-x) - \varphi_k^α(z-w)\varphi_0^0(z-x) + \varphi_k^α(z-x)\varphi_0^{-n}(w-x) = 0$$

For $n = 0$ the common multiple equals:

$$-\varphi_k^α(z-w)E_1(z-x) + \varphi_k^α(z-w)E_1(w-x) - \varphi_k^α(z-x)\varphi_0^{k-α}(w-x) = \partial_1\varphi_k^α(z-w)$$

The later is exactly compensated by "dynamical" part of YB:

$$-\sqrt{l\frac{l-1}{2}} \sum_{k=0}^{l-1} \sum_{α \in R} \frac{|α|^2}{2} \varphi_0^{-k} \otimes \varphi_0^{k} \otimes \bar{h}_0 \partial_1\varphi_k^α(z-w)$$

□

8 Appendix A. Simple Lie groups. Facts and notations, [9, 45]

Roots and weights.

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$, $\dim V = n$ and $V^*$ is its dual and $\langle \cdot, \cdot \rangle$ is a pairing between $V$ and $V^*$. A finite system of vectors $R = \{α\}$ in $V^*$ is called a root system, if
1. $R$ generates $V^*$;
2. For any $α \in R$ there exists a coroot $α^\vee \in V$ such that $\langle α, α^\vee \rangle = 2$ and the reflection in $V^*$

$$s_α : x \mapsto x - \langle x, α^\vee \rangle α$$

(A.1)

preserving $R$;
3. $\langle β, α^\vee \rangle \in \mathbb{Z}$ for any $β \in R$;
4. For $α \in R$ $nα \in R$ iff $n = \pm 1$.

The dual system $R^\vee = \{α^\vee\}$ is the root system in $V$. If $V$ and $V^*$ are identified by a scalar product $(\cdot, \cdot)$, then $α^\vee = \frac{2α}{(α,α)}$. The group of automorphisms of $V^*$ generated by reflections (A.1) is the Weyl group $W(R)$. The groups $W(R)$ and $W(R^\vee)$ are isomorphic and $W(R^\vee)$ acts on $V$ as

$$s_α : x \mapsto x - \langle x, α^\vee \rangle α, \quad x \in V^*.$$

A basis $Π = (α_1, \ldots, α_l)$ of simple roots in $R$ is defined in such a way that any $α \in R$ is decomposed as

$$α = \sum_{j=1}^{n} f_j^α α_j, \quad f_j^α \in \mathbb{Z}, \quad (A.2)$$

and all $f_j^α$ are positive (in this case $α$ is a positive root), or negative ($α$ is a negative root). In other words, the root system is an union of positive and negative roots $R = R^+ \cup R^-$. The sum

$$f_α = \sum_{α_j \in Π} f_j \quad (A.3)$$

is called the level of $α$. 

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The matrix of order \( n \)
\[
a_{jk} = \langle \alpha_j, \alpha_k \rangle, \quad \alpha_j \in \Pi, \quad \alpha_k \in \Pi^\vee
\]
(A.4)
is called the Cartan matrix. The Dynkin diagram is constructed by means of \( a_{jk} \).

Let \( S^W \) be an algebra of polynomials on \( V \) invariant with respect to \( W \)-action. There exists a basis in \( S^W \) of \( l \) homogeneous polynomials of degrees \( d_1 = 2, d_2, \ldots, d_l \). The degrees are uniquely defined by the root system \( R \). The number of roots can be read off from the degrees
\[
\# R = 2 \sum_{i=1}^{l} (d_i - 1).
\]
(A.5)

The simple roots generate the root lattice in \( V^* \)
\[
Q = \sum_{j=1}^{n} n_j \alpha_j, \quad (n_j \in \mathbb{Z}, \alpha_j \in \Pi).
\]

There exists a unique a maximal root in \(-\alpha_0 \in R^+\)
\[- \alpha_0 = \sum_{\alpha_j \in \Pi} n_j \alpha_j.
\]
(A.6)

Its level is equal to \( h - 1 \), where
\[
h = 1 + \sum_{\alpha_j \in \Pi} n_j
\]
(A.7)
is the Coxeter number. The minimal root \( \alpha_0 \) is defined as a minimal element in \(-R^+ = R^-\). The extended system \( \Pi^\text{ext} = \Pi \cup (-\alpha_0) \) generates the affine Cartan matrix \( a^{(1)}_{jk} \) of order \( n + 1 \) and extended Dynkin diagram.

Let \( X \) be an union of hyperplane \( \langle x, \alpha \rangle = 0, \alpha \in R, x \in V \). The connected components of \( V \setminus X \) are called the Weyl chambers. One of them
\[
C^+ = \{ x \in V | \langle x, \alpha \rangle > 0, \alpha \in R^+ \}\]
(A.8)
is the positive Weyl chamber. The Weyl group acts simply-transitively on the set of the Weyl chambers. The simple coroots \( \Pi^\vee = (\alpha_1^\vee, \ldots, \alpha_l^\vee) \) form a basis in \( V \) and generate the coroot lattice
\[
Q^\vee = \sum_{j=1}^{n} n_j \alpha_j^\vee \subset V, \quad n_j \in \mathbb{Z}.
\]
(A.9)

The fundamental weights \( \varpi_j \in V^* \), \( (j = 1, \ldots, n) \) are defined by condition \( \langle \varpi_j, \alpha_k^\vee \rangle = \delta_{jk} \). In this way the weight lattice \( P = \sum_{j=1}^{l} m_j \varpi_j \subset V^* \) is dual to the coroot lattice \( Q^\vee \).

The simple roots are related to fundamental weights by the Cartan matrix
\[
\alpha_k = a_{kj} \varpi_j.
\]
(A.10)

Similarly, the fundamental coweights are defined as
\[
\langle \alpha_k, \varpi_j^\vee \rangle = \delta_{kj}.
\]
(A.11)

The coweight lattice
\[
P^\vee = \sum_{j=1}^{l} m_j \varpi_j^\vee, \quad m_j \in \mathbb{Z}, \quad \langle \varpi_j^\vee, \alpha_k \rangle = \delta_{jk}
\]
(A.12)
is dual to the root lattice $Q$.

The half-sum of positive roots $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ is equal to sum of fundamental weights $\rho = \frac{1}{2} \sum_{j=1}^{n} \varpi_j$. We define the dual vector in $V$

$$\rho^\vee = \frac{1}{2} \sum_{\alpha \in R^\vee} \alpha^\vee = \sum_{j=1}^{n} \varpi_j^\vee. \quad (A.13)$$

Then from (A.2) and (A.3) the level of $\alpha$ is equal

$$f_\alpha = \langle \rho^\vee, \alpha \rangle. \quad (A.14)$$

**Affine Weyl group.**

The affine Weyl group $W_a$ is a semidirect product $Q^\vee \rtimes W$ of the Weyl group $W$ and the group $Q^\vee$. It acts on $V^*$ as

$$x \to x - \langle \alpha, x \rangle \alpha^\vee + k\beta^\vee, \quad \alpha^\vee, \beta^\vee \in R^\vee \ k \in \mathbb{Z}. \quad (A.15)$$

This transformation is an affine reflection and the hyperplanes $\{\langle \alpha, x \rangle \in \mathbb{Z}\}$ are invariant with respect to this action. Connected components of the set $V \setminus \{\langle \alpha, x \rangle \in \mathbb{Z}\}$ are called the Weyl alcoves. Their closure are fundamental domains of the $W_a$-action. Let us choose an alcove belonging to $C^+$ (A.8)

$$C_{alc} = \{x \in V | \langle \alpha, x \rangle > 0, \alpha \in \Pi, (a_0, x) > -1\}. \quad (A.16)$$

It has nodes

$$C_{alc} = \{0, \varpi_1^\vee/n_1, \ldots, \varpi_n^\vee/n_j\}. \quad (A.17)$$

Here $n_j$ are the coefficients of expansion of the maximal root (A.6).

Consider a semidirect product

$$W'_a = P^\vee \rtimes W. \quad (A.18)$$

In particular, the shift operator

$$x \to x + \gamma, \gamma \in P^\vee \quad (A.19)$$

is an element from $W'_a$. It follows from this construction that the factor group

$$W'_a/W_a \sim P^\vee/Q^\vee \sim \mathbb{Z}(\bar{G}). \quad (A.20)$$

**Chevalley basis in $g$.**

Let $g$ be a simple Lie algebra over $\mathbb{C}$ of rank $n$ and $\mathfrak{h}$ is a Cartan subalgebra. Let $\mathfrak{h} = V + iV$, where $V$ is the vector space defined above with the root system $R$. The algebra $g$ has the root decomposition

$$g = \mathfrak{h} + \mathfrak{g}, \quad \mathfrak{g} = \sum_{i \in R} \mathfrak{R}_i, \quad \dim_{\mathbb{C}} \mathfrak{R}_i = 1. \quad (A.21)$$

The Chevalley basis in $g$ is generated by

$$\{E_{\beta_j} \in \mathfrak{R}_{\beta_j} , \beta_j \in R; H_{\alpha_k} \in \mathfrak{h}, \alpha_k \in \Pi\}, \quad (A.22)$$
where $H_{\alpha_k}$ are defined by the commutation relations

$$
\begin{align*}
[E_{\alpha_k}, E_{-\alpha_k}] &= H_{\alpha_k}, \\
[H_{\alpha_k}, E_{\pm \alpha_j}] &= a_{kj} E_{\pm \alpha_k}, \quad \alpha_k, \alpha_j \in \Pi.
\end{align*}
$$

where $C_{\alpha,\beta}$ are structure constants of $\mathfrak{g}$.

If $(\ , \ )$ is a scalar product in $\mathfrak{g}$ then $H_{\alpha}$ can be identified with coroots as

$$
H_{\alpha} = \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}.
$$

Therefore,

$$
(H_{\alpha}, H_{\beta}) = \frac{4(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)} = \frac{2}{(\alpha,\alpha)} a_{\alpha,\beta}.
$$

The Killing form in the subspace $\mathfrak{L}$ is expressed in terms of $(\alpha,\alpha)$

$$
(E_{\alpha}, E_{\beta}) = \delta_{\alpha,-\beta} \frac{2}{(\alpha,\alpha)}.
$$

The structure constants $C_{\alpha,\beta}$ possess the obvious properties. Then

$$
\begin{align*}
C_{\alpha,\beta} &= -C_{\beta,\alpha} \\
C_{\lambda_{\alpha,\beta}} &= C_{\alpha,\lambda^{-1}_{\beta}}, \quad \lambda \in W, \\
C_{\alpha+\beta,-\alpha} &= \frac{|\beta|^2}{|\alpha+\beta|^2} C_{-\alpha,-\beta}
\end{align*}
$$

The first property is obvious from definition (A.23), the second one reflects the fact that $\lambda$ is automorphism of the algebra and the third is the consequence of the invariance of the Killing form. Indeed, using the Killing form, we can define the structure constants as the product:

$$
C_{\alpha,\beta} = \frac{|\alpha + \beta|^2}{2} \left( E_{-\alpha,-\beta}, [E_{\alpha}, E_{\beta}] \right).
$$

From invariance of the Killing form we get:

$$
C_{\alpha+\beta,-\alpha} = \frac{|\beta|^2}{2} \left( E_{-\beta}, [E_{\alpha+\beta}, E_{-\alpha}] \right) = \frac{|\beta|^2}{2} \left( E_{\alpha+\beta}, [E_{-\alpha}, E_{-\beta}] \right) = \frac{|\beta|^2}{|\alpha+\beta|^2} C_{-\alpha,-\beta}
$$

Consider the ring of invariant polynomials on $\mathfrak{g}$ It has $n = \text{rank} \mathfrak{g}$ generators, which we can take to be homogeneous of degrees $d_1, \ldots, d_r = h$. Let $B$ be a Borel subgroup of $G$. It is generated by Cartan subgroup of $G$ and by negative root subspaces $\exp(\sum_{\alpha \in R^-} E_{\alpha})$. The coset space $Fl = G/B$ is called the flag variety. It has dimension (see (A.5))

$$
\dim Fl = \sum_{j=1}^l (d_j - 1).
$$

The coadjoint orbits

$$
\mathcal{O} = \{ \text{Ad}_g^* S_0 \mid g \in G, \ S_0 \text{ is a fixed element of } \mathfrak{g}^* \}.
$$

(A.28)
is a generalization of a cotangent bundle to the flag varieties,\footnote{It is a cotangent bundle if $S_0$ is a Jordan element. If $S_0$ is semisimple, then $O$ is the torsor over $Fl$.} and for generic orbits

\[ \dim O = 2 \sum_{j=1}^{l} (d_j - 1). \] (A.29)

Consider a Cartan subgroup $\mathcal{H} \subset G$. Let $\mathcal{N}(\mathcal{H})$ be a normalizer of $\mathcal{H}$. Then

\[ W(R) \sim \mathcal{N}(\mathcal{H})/\mathcal{H}. \] (A.30)

**Centers of simple groups.**

Let $\tilde{G}$ be an universal covering of $G$. The group $\tilde{G}$ is simply-connected and in all cases apart $G_2$, $F_4$ and $E_8$ has a non-trivial center $Z(\tilde{G})$.

| $G$ | Lie ($\tilde{G}$) | $Z(\tilde{G})$ |
|-----|-----------------|----------------|
| $\text{SL}(n, \mathbb{C})$ | $A_{n-1}$ | $\mu_n$ |
| $\text{Spin}_{2n+1}(\mathbb{C})$ | $B_n$ | $\mu_2$ |
| $\text{Sp}_n(\mathbb{C})$ | $C_n$ | $\mu_2$ |
| $\text{Spin}_{4n}(\mathbb{C})$ | $D_{2n}$ | $\mu_2 \oplus \mu_2$ |
| $\text{Spin}_{4n+2}(\mathbb{C})$ | $D_{2n+1}$ | $\mu_4$ |
| $E_6(\mathbb{C})$ | $E_6$ | $\mu_3$ |
| $E_7(\mathbb{C})$ | $E_7$ | $\mu_2$ |

Table 7

Centers of universal covering groups

$(\mu_N = \mathbb{Z}/N\mathbb{Z})$

The factor-group $P^\vee/Q^\vee$ is isomorphic to the center $Z(\tilde{G})$ of simply-connected group $\tilde{G}$. It is a cyclic group except $g = D_{4l}$. The order of $Z(\tilde{G})$ is defined in terms of the Cartan matrix

\[ \text{ord} (Z(\tilde{G})) = \det (a_{kj}). \] (A.31)

The adjoint group $G^{ad}$ is the factor group

\[ G^{ad} = \tilde{G}/Z(\tilde{G}). \] (A.32)

In the cases $A_{n-1}$ ($n$ is non-prime), and $D_n$ the center $Z(\tilde{G})$ has non-trivial subgroups $Z_l \sim \mu_1 = \mathbb{Z}/l\mathbb{Z}$. Then there exists the factor groups

\[ G_l = \tilde{G}/Z_l, \quad G_p = G_l/Z(G_l), \] (A.33)

where $Z(G_l)$ is the center of $G_l$ and $Z(G_l) \sim \mu_p = Z(\tilde{G})/Z_l$.

Consider in detail the group $\tilde{G} = Spin_{4n}(\mathbb{C})$. It has a non-trivial center

\[ Z(Spin_{4n}) = (\mu_2^L \times \mu_2^R), \quad \mu_2 = \mathbb{Z}/2\mathbb{Z}, \] (A.34)

where three subgroups can be described in terms of their generators as

\[ \mu_2^L = \{(1, 1), (-1, 1)\}, \quad \mu_2^R = \{(1, 1), (1, -1)\}, \quad \mu_2^{\text{diag}} = \{(1, 1), (-1, -1)\}. \]
Therefore there are three intermediate subgroups between $\bar{G} = Spin_{4n}(\mathbb{C})$ and $G^{ad}$

\[
\begin{array}{cccc}
Spin_{4n}^R & \overset{Spin_{4n}^L}{\leftarrow} & SO(4n) = Spin_{4n}/\Gamma^{diag} & \overset{Spin_{4n}^R}{\rightarrow} \frac{G^{ad}}{Spin_{4n}/(\mu_2^L \times \mu_2^R)}
\end{array}
\] (A.35)

**Characters and cocharacters.**

Let $\mathcal{H}$ be a Cartan subgroup $\mathcal{H} \subset G$. Define the group of characters\(^{13}\)

\[
\Gamma(G) = \{ \chi : \mathcal{H} \rightarrow \mathbb{C}^* \}.
\] (A.36)

This group can be identified with a lattice group in $\mathfrak{h}^*$ as follows. Let $x = (x_1, x_2, \ldots, x_n)$ be an element of $\mathfrak{h}$, and $e^{2\pi i x} \in \mathcal{H}$. Define $\gamma \in V^*$ such that $\chi_{\gamma} = e^{2\pi i \langle \gamma, x \rangle} \in \Gamma(G)$. Then

\[
\Gamma(G) = P, \quad \Gamma(G^{ad}) = Q,
\] (A.37)

and $\Gamma(G^{ad}) \subseteq \Gamma(G_1) \subseteq \Gamma(\bar{G})$. The fundamental weights $\varpi_k$ ($k = 1, \ldots, n$) (simple roots $\alpha_k$) form a basis in $\Gamma(G)$ ($\Gamma(G^{ad})$). Let $Z(G)$ be a cyclic group and $p$ is a divisor of $\text{ord}(Z(G))$ such that $l = \text{ord}(Z(\bar{G}))/p$. Then the lattice $\Gamma(G_1)$ is defined as

\[
\Gamma(G_1) = Q + \varpi \mathbb{Z}, \quad p\varpi \in Q.
\] (A.38)

Define the dual groups of cocharacters $t(G) = \Gamma^*(G)$ as holomorphic maps

\[
t(G) = \{ \mathbb{C}^* \rightarrow \mathcal{H} \}.
\] (A.39)

In another way

\[
t(G) = \{ x \in \mathfrak{h} \mid \chi(e^{2\pi i x}) = 1 \}.
\] (A.40)

A generic element of $t(G)$ takes the form

\[
z^\gamma = \exp 2\pi i \gamma \ln z \in \mathcal{H}_G, \quad \gamma \in \Gamma(G), \quad z \in \mathbb{C}^*.
\] (A.41)

In particular, the groups $t(\bar{G})$ and $t(G^{ad})$ are identified with the coroot and the coweight lattices

\[
t(\bar{G}) = Q^\vee, \quad t(G^{ad}) = P^\vee,
\] (A.42)

and $t(\bar{G}) \subseteq t(G_1) \subseteq t(G^{ad})$. It follows from (A.38) that

\[
t(G_1) = Q^\vee + \varpi^\vee \mathbb{Z}, \quad l\varpi^\vee \in Q^\vee.
\] (A.43)

The center $Z(G)$ of $G$ is isomorphic to the quotient

\[
Z(G) \sim P^\vee/t(G),
\] (A.44)

while $\pi_1(G) \sim t(G)/Q^\vee$. In particular,

\[
Z(G) = P^\vee/t(\bar{G}) \sim P^\vee/Q^\vee.
\] (A.45)

Similarly, the fundamental group of $G^{ad}$ is $\pi_1(G^{ad}) \sim t(G^{ad})/Q^\vee \sim P^\vee/Q^\vee$.

The triple $(R, t(G), \Gamma(G))$ is called the root data. A Langlands dual to $G$ group $^L G$ is defined by the root data $(R^L, t^L(G), \Gamma^L(G))$, where

\[
t^L(G) \sim \Gamma^L(G), \quad \Gamma^L(G) \sim t(G).
\] (A.46)

In particular, in the simply-laced cases $^L G = G^{ad}$.

---

\(^{13}\) The holomorphic maps of $\mathcal{H}$ to $\mathbb{C}^*$ such that $\chi(xy) = \chi(x)\chi(y)$ for $x, y \in \mathcal{T}$.  

**Table 8**

Duality in simple groups.

| Root system | \(G\) | \(LG\) |
|-------------|------|-------|
| \(A_n\), \(N = n + 1 = pl\) | \(G_t = \text{SL}(N, \mathbb{C})/\mu_t\) | \(G_p = \text{SL}(N, \mathbb{C})/\mu_p\) |
| \(B_n\) | \(\text{Spin}(2n + 1)\) | \(\text{Sp}(n)/\mu_2\) |
| \(C_n\) | \(\text{Sp}(n)\) | \(\text{SO}(2l + 1)\) |
| \(D_n, n = 2l + 1\) | \(\text{Spin}(4l + 1)\) | \(\text{SO}(4l + 2)/\mu_2\) |
| \(D_n, n = 2l\) | \(\text{SO}(4l)\) | \(\text{SO}(4l)/\mu_2\) |
| \(l = 2m\) | \(\text{Spin}^L(8m)\) | \(\text{Spin}^L(8m)\) |
| \(l = 2m + 1\) | \(\text{Spin}^R(8m + 2)\) | \(\text{Spin}^R(8m + 2)\) |
| \(E_6\) | \(E_6\) | \(E_6/\mu_3\) |
| \(E_7\) | \(E_7\) | \(E_7/\mu_2\) |

9 Appendix B. Elliptic Functions, \([41, 51]\)

The basic function is the theta-function.

\[
\vartheta(z|\tau) = q^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n(n+1)\tau + 2nz)}.
\] (B.1)

It is a holomorphic function on \(\mathbb{C}\) with simple poles at the lattice \(\tau \mathbb{Z} + \mathbb{Z}\) and the quasi-periodicities

\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -q^{-\frac{1}{2}} e^{-2\pi iz} \vartheta(z),
\] (B.2)

Define the relation of the theta-functions

\[
\phi(u, z) = \frac{\vartheta(u + z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}.
\] (B.3)

It follows from (B.1) and (B.2) that it is a meromorphic function of \(z \in \mathbb{C}\) with simple poles at the lattice \(\tau \mathbb{Z} + \mathbb{Z}\) and

\[
\text{Res} \phi(u, z)|_{z \in (\tau \mathbb{Z} + \mathbb{Z})} = 1,
\] (B.4)

and the quasi-periodicities

\[
\phi(u, z + 1) = \phi(u, z), \quad \phi(u, z + \tau) = e^{-2\pi i u} \phi(u, z).
\] (B.5)

Since \(\phi(u, z) = \phi(z, u)\)

\[
\phi(u + 1, z) = \phi(u, z), \quad \phi(u + \tau, z) = e^{-2\pi iz} \phi(u, z).
\] (B.6)

We also need two Fay identities for \(\phi(z, w)\), the first one:

\[
\phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1 + u_2, z_1)\phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2)\phi(u_1, z_1 - z_2) = 0
\] (B.7)

and its degenerate form:

\[
\phi(u_1, z)\phi(u_2, z) - \phi(u_1 + u_2, z)(E_1(u_1) + E_1(u_2)) - \partial_z \phi(u_1 + u_2, z) = 0,
\] (B.8)
where \( E_1(z) \) is the first Eisenstein function
\[
E_1(z) = \partial_z \log \theta(z).
\] (B.9)

The second Eisenstein function is
\[
E_2(z) = \partial_z^2 \log \theta(z) = -\partial_z E_1(z).
\] (B.10)

They are related to the Weierstrass functions as follows
\[
\zeta(z|\tau) = E_1(z|\tau) + 2\eta_1(\tau)z,
\] (B.11)
and
\[
\wp(z|\tau) = E_2(z) - 2\eta_1(\tau).
\] (B.12)

Here
\[
\eta_1(\tau) = \frac{24}{\pi i} \frac{\eta'(\tau)}{\eta(\tau)} = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n),
\]
and \( \eta(\tau) \) is the Dedekind function.

\( E_1(z) \) is quasi-periodic
\[
E_1(z + 1|\tau) = E_1(z|\tau), \quad E_1(z + \tau|\tau) = E_1(z|\tau) - 2\pi i,
\] (B.13)
and has simple poles at at the lattice \( \tau\mathbb{Z} + \mathbb{Z} \)
\[
\text{Res} \zeta(z|\tau)_{z \in (\tau\mathbb{Z} + \mathbb{Z})} = 1.
\] (B.14)

\( E_2(z) \) is double-periodic with second order poles at the lattice. It is related to \( \phi(u, z) \) as
\[
\phi(u, z)\phi(-u, z) = E_2(z) - E_2(u).
\] (B.15)

\( E_2(z) \) and its derivatives \( \partial_z^k E_2(z) \) form a basis in a space of double periodic function on \( \Sigma_\tau = \mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z}) \).

The most important object for construction of Lax operators and \( r \)-matrices is the function defined as follows:
\[
\varphi_\alpha^k(z) = e^{2\pi i \langle \kappa, \alpha \rangle z} \phi \left( <u + \kappa \tau, \alpha > + \frac{k}{N}, z \right).
\]

Here \( u \) and \( \kappa \) are vectors defined in Proposition 3.1, \( \alpha \) is a root of the corresponding Lie algebra. Note, that to safe space we omit the \( u \)-dependence of the function in its definition.

An important property of function \( \varphi_\alpha^k(z) \) is its \( \lambda \)-invariance, more precisely, this function satisfies:
\[
\varphi^{k}_{\lambda \alpha}(z) = \varphi^{k}_\alpha(z) \quad \varphi^{k}_{\alpha + \lambda \beta}(z) = \varphi^{k}_{\alpha + \beta}(z).
\] (B.16)

Indeed:
\[
\varphi^{k}_{\lambda \alpha}(z) = e^{2\pi i < \kappa, \lambda \alpha > z} \phi \left( <u + \kappa \tau, \lambda \alpha > + \frac{k}{N}, z \right) = \\
e^{2\pi i < \lambda^{-1} \kappa, \alpha > z} \phi \left( \lambda^{-1}(u + \kappa \tau), \alpha > + \frac{k}{N}, z \right) = \\
e^{2\pi i < \kappa + \varpi', \alpha > z} \phi \left( <u + \kappa \tau + \varpi' \tau, \alpha > + \frac{k}{N}, z \right),
\]
where we use the invariance of vector \( u \): \( \lambda u = u \). By definition of the Cartan element \( \kappa \), it transforms under \( \lambda \)-action as
\[
\lambda^{-1} \kappa = \kappa + \omega^\vee
\]
where \( \omega^\vee \in Q^\vee \), therefore \( <\kappa, \alpha> = n \) - is an integer number and we have:
\[
e^{2\pi i <\kappa, \alpha> + n} z^\phi \left( \left( u + \omega^\vee \tau, \alpha > + n + \frac{k}{N}, z \right) \right) =
\]
\[
e^{2\pi i <\kappa, \alpha> + n + \frac{k}{N}} z^\phi \left( \left( u + \omega^\vee \tau, \alpha > + k, z \right) \right) = \phi^k(\alpha)(z).
\]

Here, in the last step we have used \( \phi(w + \tau, z) = \exp(-2\pi i z)\phi(w, z) \). The proof of the second identity in \([3,16]\) is equivalent.

References

[1] M. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. 7 (1957), 414-452.
[2] V. Arnold, *Mathematical Methods in Classical Mechanics*, Springer, 1978.
[3] J. Avan, M. Talon, *Classical R-matrix structure for the Calogero model*, Phys.Lett. B303 (1993) 33-37, arXiv:hep-th/9210128.
[4] O. Babelon, C-M. Viallet, *Hamiltonian structures and Lax equations*, Preprint Univer. Paris 6 N37 (1989).
[5] A. Belavin, V. Drinfeld, *Solutions of the classical Yang - Baxter equation for simple Lie algebras*, Funct. Analysis and Its Applic., 16, (N 3) (1982), 159-180.
[6] J. Bernstein, O. Schwarzman, *Chevalley’s theorem for complex crystallographic Coxeter groups*, (Russian) Funktsional. Anal. i Prilozhen. 12 (1978), no. 4, 79–80.
[7] E. Billey, J. Avan, O. Babelon, *The r-matrix structure of the Euler-Calogero-Moser model*, Phys. Lett. A, 186 (1994), 114-118.
[8] A. Bordner, E. Corrigan, R. Sasaki, *Calogero-Moser models: I. A new formulation*, Progr. Theoret. Phys. 100 (1998), 1107-1129.
[9] N. Bourbaki, *Lie Groups and Lie Algebras: Chapters 4-6*, Springer-Verlag, Berlin-Heidelberg-New York, (2002).
[10] H. Braden, T. Suzuki, *R-matrices for Elliptic Calogero-Moser Models*, Lett.Math.Phys. 30 (1994) 147-158, arXiv:hep-th/9309033.
[11] F. Calogero, *Solution of the one-dimensional n-body problem with quadratic and/or inversely quadratic pair potentials*, J.Math.Phys., 12, (1971), 419-436; *Exactly solvable one-dimensional many-body problem*, Lett.Nuovo Cim. 13 (1975), 411.
[12] P. Dirac, *Lectures on quantum mechanics*, (1967) Yeshiva Univ., NY, Academic Press.
[13] B. Enriquez and V. Rubtsov, *Hitchin systems, higher Gaudin operators and R-matrices*, Math. Res. Lett. 3 (1996), 343–357.
[14] P. Etingof, *Lectures on Calogero-Moser systems*, arXiv:math/0606233.
[15] P. Etingof, O. Schiffmann, *Lectures on the dynamical Yang-Baxter equations*, arXiv:math/9908064.
[16] P. Etingof, and A. Varchenko, *Geometry and classification of solutions of the classical dynamical Yang-Baxter equation*, Comm. Math. Phys., 192 (1998), 77-120.
[17] L. Feher *Poisson-Lie dynamical r-matrices from Dirac reduction*, Czech. J. Phys. 54 (2004) 1265-1274.

50
[18] G. Felder, *Conformal field theory and integrable systems associated with elliptic curves*, Proc. of the ICM 94, (1994) 1247-1255, Birkhäuser.

[19] D. Fairlie, P. Fletcher and C. Zachos, *Infinite Dimensional Algebras and a Trigonometric Basis for the Classical Lie Algebras*, Journal of Mathematical Physics, 31 (1990), 1088-1094.

[20] R. Friedman, J. Morgan, *Holomorphic principal bundles over elliptic curves*.

[21] J. Gibbons, and T. Hermsen, *A generalization of the Calogero-Moser systems*, Physica 11D (1984), 337-348.

[22] N. Gordeev, V. Popov, *Automorphism groups of finite dimensional simple algebras*, Ann. of Math., 158, (2003), 1041-1065.

[23] A. Gorsky, N. Nekrasov, *Elliptic Calogero-Moser system from two dimensional current algebra*, [hep-th/9401021](http://arxiv.org/abs/hep-th/9401021).

[24] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. Jour. 54 (1987), 91-114.

[25] E. D’Hoker, and D. H. Phong, *Calogero-Moser Lax pairs with spectral parameter for general Lie algebras*, Nuclear Phys. B 530 (1998), 537-610.

[26] J. Hurtubise, E. Markman, *Calogero-Moser systems and Hitchin systems*, Commun. Math. Phys. 223 (2001) 533-552, [arXiv:math/9912161](http://arxiv.org/abs/math/9912161).

[27] N. Jacobson, *Exceptional Lie algebras*, Lecture Notes in Pure and Applied Mathematics, (1971) NY.

[28] V. Kac, *Automorphisms of finite order of semisimple Lie algebras*, Funct. Anal. and Applic., 3, (1969), 94-96.

[29] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, [arXiv:hep-th/0604151](http://arxiv.org/abs/hep-th/0604151).

[30] B. Khesin, A. Levin, M. Olshanetsky, *Bihamiltonian structures and quadratic algebras in hydrodynamics and on non-commutative torus*, Comm. Math. Phys., 250 (2004) 581-612.

[31] D. Kazdan, B. Kostant, S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure and Appl. Math., 31 (1978) 481-507.

[32] A. Levin, M. Olshanetsky, *Isomonodromic deformations and Hitchin Systems*, Amer. Math. Soc. Transl. (2), 191 (1999) 223-262.

[33] A. Levin, M. Olshanetsky, A. Smirnov, A. Zotov, *Integrable systems and Characteristic Classes for Simple Lie Algebras*.

[34] A. Levin, M. Olshanetsky, A. Zotov, *Hitchin Systems - Symplectic Hecke Correspondence and Two-dimensional Version*, Comm. Math. Phys. 236 (2003) 93-133.

[35] A. Levin, M. Olshanetsky, A. Zotov, *Monopoles and modifications of bundles over elliptic curves*, SIGMA 5 (2009), 065, [arXiv:0811.3056](http://arxiv.org/abs/0811.3056).

[36] A. Levin, and A. Zotov, *Integrable systems of interacting elliptic tops*, Theor. Math. Phys., 146:1, (2006), 55-64.

[37] E. Looijenga, *Root systems and elliptic curves*, Invent. Math. 38 (1976), 17-32.

[38] Luen-Chau Li, Ping Xu *Integrable spin Calogero-Moser systems* Commun. Math. Phys. 231 (2002), 257-286.

[39] E. Markman, *Spectral curves and integrable systems*, Comp. Math. 93, (1994) 255-290.

[40] J. Moser, *Three integrable systems connected with isospectral deformations*, Adv. Math. 16, (1975), 1-23.

[41] D. Mumford, *Tata Lectures on Theta I, II*, Birkhäuser Boston, 1983, 1984.
[42] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. **82** (1965) 540-64.

[43] M. Olshanetsky, A. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Physics Reports, v.71 (1981), 313-400.

[44] M. Olshanetsky, A. Perelomov, *Explicit solution of the Calogero model in the classical case and geodesic flows on symmetric space of zero curvature*, Lett. Nuovo Cim., **16** (1976), 333-339.

[45] A. Onishchik, and E. Vinberg, *Seminar on Lie groups and algebraic groups*, Moscow (1988), (in Russian) English transl. Springer-Verlag, Berlin-Heidelberg-New York (1990).

[46] A. Reyman and M. Semenov-Tian-Schansky, *Lie algebras and Lax equations with spectral parameter on elliptic curve*, (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **150** (1986), Voprosy Kvant. Teor. Polya i Statist. Fiz. 6, 104–118, 221; translation in J. Soviet Math., **46**, no. 1, (1989), 1631–1640.

[47] O. Schiffmann, *On classification of dynamical r-matrices*, Math. Res. Letters, **5**, (1998), 13-30.

[48] C. Schweigert, *On moduli spaces of flat connections with non-simply connected structure group*, Nucl. Phys. B **492** (1997), 743-755.

[49] C. Simpson, *Harmonic bundles on Noncompact Curves*, Journ. Am. Math. Soc., **3** (1990) 713–770.

[50] E. Sklyanin, *Dynamical r-matrices for the Elliptic Calogero-Moser Model*, Alg.Anal. **6** (1994) 227-237; St.Petersburg Math.J. **6** (1995) 397-406 [arXiv:hep-th/9308060]

[51] A. Weyl, *Elliptic functions according to Eisenstein and Kronecker*, Springer-Verlag, Berlin-Heidelberg-New York, (1976).

[52] S. Wojciechowski, *An integrable marriage of the Euler equations with the Calogero-Moser systems*, Phys. Lett. A, **111** (1985), 101-103.