TAU FUNCTION AND HIROTA BILINEAR EQUATIONS FOR THE EXTENDED BIGRADED TODA HIERARCHY

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Abstract. In this paper we generalize the Sato theory to the extended bigraded Toda hierarchy (EBTH). We revise the definition of the Lax equations, give the Sato equations, wave operators, Hirota bilinear identities (HBI) and show the existence of \( \tau \) function \( \tau(t) \). Meanwhile we prove the validity of its Fay-like identities and Hirota bilinear equations (HBEs) in terms of vertex operators whose coefficients take values in the algebra of differential operators. In contrast with HBEs of the usual integrable system, the current HBEs are equations of product of operators involving \( e^\partial \) and \( \tau(t) \).

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1. Introduction

The Toda lattice equations is a set of nonlinear evolutionary differential-difference equations introduced by Toda ([1], [2]) describing an infinite system of masses on a line that interact through an exponential force which is used to explain nonergodic character in the well-known Fermi-Pasta-Ulam paradox. It was soon realized that this lattice equations is a completely integrable system, i.e. admits infinite conserved quantities and exact analytic solutions. It has important applications in many different fields such as classical and quantum fields theory. For our best knowledge, there are at least three important extensions of Toda lattice equation. The first one is the Toda hierarchy [3], which is in fact a two-dimensional extended hierarchy through infinite-dimensional matrix inspired by the Sato theory[4]. Recently, considering application to 2D topological fields theory and the theory of Gromov-Witten invariants ([5], [6], [7],[8]) of Toda lattice hierarchy, one replaced the discrete variables with continuous one. After continuous “interpolation” [9] to the whole Toda lattice hierarchy, it was found the flow of spatial translations was missing. In order to get a complete family of flows [10], the interpolated Toda lattice hierarchy was extended into the so-called extended Toda hierarchy(ETH) [9], which is the second extension of the Toda lattice equations. It was firstly conjectured and then shown ([5], [10], [11]) that the extended Toda hierarchy is the hierarchy describing the Gromov-Witten invariants of \( CP^1 \) by matrix models [12] which describe in the large \( N \) limit of the \( CP^1 \) topological sigma model. The HBEs of the ETH are given by Milanov ’s work [13]. The third extension of Toda lattice equations is the extended bigraded Toda hierarchy(EBTH), which are discovered independently two times from different concerns. The dispersionless version of extended bigraded Toda hierarchy was firstly introduced by S. Aoyama, Y. Kodama in [14].

In the dispersionless limit, the EBTH can be obtained from the dispersionless KP hierarchy.

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More recently, the extended bigraded Toda hierarchy was re-introduced by Gudio Carlet [15] who hoped that EBTH might also be relevant for some applications in 2D topological fields theory and in the theory of Gromov-Witten invariants. Specifically, Carlet [15] generalized the Toda lattice hierarchy by considering $N + M$ dependent variables and used them to provide a Lax pair definition of the extended bigraded Toda hierarchy. On the base of [15], Todor. E. Milanov and Hsian-Hua Tseng [16] described conjecturally one kind of Hirota bilinear equations (HBEs) which was proved to govern the Gromov-Witten theory of orbiford $c_{km}$. This naturally inspires us to consider the Sato theory of EBTH because the HBEs are the core knowledge of the integrable systems.

Sato and Sato proved one kind of algebraic identity in theorem of [4] about tau function of KP hierarchy which is now called the Fay-identity[17]. There are some important extensions, such as differential Fay-identity and its implication [18]. Furthermore, Takasaki and Takebe derived the differential Fay identity from the Hirota bilinear identity (HBI) of KP hierarchy and showed that the differential Fay identity is equivalent to KP hierarchy in appendix of [19]. In [20], it shows that differential (or difference, for the Toda hierarchy) Fay identities are generating functional expression of the full set of auxiliary linear equations and equivalent to the integrable hierarchies themselves. Lee-Peng Teo derived the Fay-like identities of tau function for the Toda lattice hierarchy from the HBI and prove that the Fay-like identities are equivalent to the hierarchy [21].

So the purpose of this paper is to establish Sato formulation of EBTH including its Lax equations, Sato equations, wave operators, HBIs, tau-functions, Fay-like identities and HBEs. An important feature of the current HBEs is that they are not differential equations of functions as the case of usual integrable systems. Actually, HBEs of the EBTH are equations of product of operators involving $e^{\partial_x}$ and $\tau(t)$. Here $\tau$ function is regarded as a zero order operator. In other words, the coefficients of vertex operators with a form of $e^{\partial_x + \sum \alpha, n \theta_{\alpha, n} \partial_{t\alpha, n} + \sum \beta, m \theta_{\beta, m} \partial_{t\beta, m}}$ in the HBEs, are not scalar-valued but take values in the algebra of differential operators, i.e. $\{\partial_x, x\}$. This subtle point of HBEs can be found in Milanov’s work [13]. Our work is an highly nontrivial extension of the results in [13] which is about ETH. We would like to stress that the current HBEs in section 5 are different from the one in [16].

The paper is organized as follows. In Section 2, we redefine the Lax equations using the roots and the logarithms of the Lax operator $L$ and give Zakharov-Shabat equation and Sato equations for the EBTH. By using the wave operators and their symbols, some bilinear identities are given in Section 3. In Section 4 we define the tau-function of EBTH and prove its existence, moreover we give some Fay-like identities from HBIs under some special cases. In Section 5 we give the HBEs of EBTH in the form of tau function and vertex operators, meanwhile we prove its validity with the help of HBI. In Section 6 we give the HBEs of bigraded Toda hierarchy (BTH) as a corollary. Section 7 is devoted to conclusions and discussions.

2. The EBTH

We describe the Lax form of the EBTH following [15]. Introduce firstly the lax operator

$$L = \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M}$$

which can be expressed in the following two different ways

$$L = P_L\Lambda^N P_L^{-1} = P_R\Lambda^{-M} P_R^{-1}.$$  (2.2)

Here, $N, M \geq 1$ are two fixed positive integers and $u_{-M}$ is a non-vanishing function. The variables $u_j$ are functions of the spatial variable $x$ and the shift operator $\Lambda$ acts on a function
Moreover, by using the second identity of eq.(2.2) and the non-vanishing character of \( \tilde{w}_0 \) we can also easily get the relation of \( u \) with the relations \( a \) by \( \Lambda a(x) = a(x + \epsilon) \), i.e. \( \Lambda \) is equivalent to \( e^{\epsilon \partial_x} \) where the spacing unit \( \epsilon \) is called string coupling constant. The operators \( P_L \) and \( P_R \) have the following forms

\[
\begin{align*}
P_L &= 1 + w_1 \Lambda^{-1} + w_2 \Lambda^{-2} + \ldots, \quad (2.3) \\
P_R &= \tilde{w}_0 + \tilde{w}_1 \Lambda + \tilde{w}_2 \Lambda^2 + \ldots, \quad (2.4)
\end{align*}
\]

where \( \tilde{w}_0 \) is not zero. The inverse operators of \( P_L \) and \( P_R \) are given by

\[
\begin{align*}
P_L^{-1} &= 1 - \Lambda^{-1} w_1' + \Lambda^{-2} w_2' + \ldots, \quad (2.5) \\
P_R^{-1} &= \tilde{w}_0' + \Lambda \tilde{w}_1' + \Lambda^2 \tilde{w}_2' + \ldots. \quad (2.6)
\end{align*}
\]

Note that the operator \( \Lambda^i \) are fixed at the left side of coefficients in inverse operators. The uniqueness is up to multiplying \( P_L \) and \( P_R \) from the right by operators in the form \( 1 + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \ldots \) and \( \tilde{a}_0 + \tilde{a}_1 \Lambda + \tilde{a}_2 \Lambda^2 + \ldots \) respectively whose coefficients are independent of \( x \). From the first identity of eq. (2.2), we can easily get the relation of \( u_i \) and \( w_j \) as following

\[
\begin{align*}
u_{N-1} &= w_1(x) - w_1(x + N\epsilon), \quad (2.7) \\
u_{N-2} &= w_2(x) - w_2(x + N\epsilon) - (w_1(x) - w_1(x + N\epsilon))w_1(x + (N - 1)\epsilon), \quad (2.8) \\
u_{N-3} &= w_3(x) - w_3(x + N\epsilon) - [w_2(x) - w_2(x + N\epsilon) - (w_1(x) - w_1(x + N\epsilon))w_1(x + (N - 1)\epsilon)] \\
&\quad w_1(x + (N - 2)\epsilon) - (w_1(x) - w_1(x + N\epsilon))w_2(x + (N - 1)\epsilon), \quad (2.9) \\
&\quad \ldots \quad \ldots \quad \ldots
\end{align*}
\]

Moreover, by using the second identity of eq. (2.2) and the non-vanishing character of \( \tilde{w}_0 \), we can also easily get the relation of \( u_i \) and \( \tilde{w}_j \) formally as following

\[
\begin{align*}
u_{-M} &= \frac{\tilde{w}_0(x)}{\tilde{w}_0(x - M\epsilon)}, \\
u_{-M+1} &= \frac{\tilde{w}_1(x) - \frac{\tilde{w}_0(x)}{\tilde{w}_0(x - M\epsilon)} \tilde{w}_1(x - M\epsilon)}{\tilde{w}_0(x - (M - 1)\epsilon)}, \\
u_{-M+2} &= \frac{\tilde{w}_2(x) - \frac{\tilde{w}_0(x)}{\tilde{w}_0(x - M\epsilon)} \tilde{w}_2(x - M\epsilon) - \frac{\tilde{w}_1(x) - \frac{\tilde{w}_0(x)}{\tilde{w}_0(x - M\epsilon)} \tilde{w}_1(x - M\epsilon)}{\tilde{w}_0(x - (M - 1)\epsilon)} \tilde{w}_1(x - (M - 1)\epsilon)}{\tilde{w}_0(x - (M - 2)\epsilon)}, \\
&\quad \ldots \quad \ldots \quad \ldots \\
u_{N-1} &= \frac{\tilde{w}_{M+N-1} - u_{-M} \tilde{w}_{M+N-1}(x - M\epsilon) - \cdots - u_{N-2} \tilde{w}_1(x + (N - 2)\epsilon)}{\tilde{w}_0(x + (N - 1)\epsilon)}, \\
u_N &= 1 = \frac{\tilde{w}_{M+N} - u_{-M} \tilde{w}_{M+N}(x - M\epsilon) - \cdots - u_{N-1} \tilde{w}_1(x + (N - 1)\epsilon)}{\tilde{w}_0(x + N\epsilon)}.
\end{align*}
\]

To write out explicitly the Lax equations of EBTH, fractional powers \( \mathcal{L}^{\frac{1}{\alpha}} \) and \( \mathcal{L}^{\frac{1}{\beta}} \) was defined by

\[
\mathcal{L}^{\frac{1}{\alpha}} = \Lambda + \sum_{k \leq 0} a_k \Lambda^k, \quad \mathcal{L}^{\frac{1}{\beta}} = \sum_{k \geq -1} b_k \Lambda^k,
\]

with the relations

\[
(\mathcal{L}^{\frac{1}{\alpha}})^N = (\mathcal{L}^{\frac{1}{\beta}})^M = \mathcal{L}.
\]
It was stressed that $\mathcal{L}_N^\pm$ and $\mathcal{L}_M^\pm$ are two different operators even if $N = M(N, M \geq 2)$ in [15] due to two different dressing operators. They can also be expressed as following

$$\mathcal{L}_N^\pm = \mathcal{P}_L \Lambda \mathcal{P}_L^{-1}, \quad \mathcal{L}_M^\pm = \mathcal{P}_R \Lambda^{-1} \mathcal{P}_R^{-1}.$$ 

Moreover, as [15] it is necessary to define the following two logarithms

$$\log_+ \mathcal{L} = \mathcal{P}_L N \epsilon \partial \mathcal{P}_L^{-1} = N \epsilon \partial - N \epsilon \mathcal{P}_L \mathcal{P}_L^{-1} = N \epsilon \partial + 2N \sum_{k>0} W_{-k}(x) \Lambda^{-k},$$

$$\log_- \mathcal{L} = -\mathcal{P}_R M \epsilon \partial \mathcal{P}_R^{-1} = -M \epsilon \partial + M \epsilon \mathcal{P}_R \mathcal{P}_R^{-1} = -M \epsilon \partial + 2M \sum_{k \geq 0} W_k(x) \Lambda^k,$$

where $\partial = \frac{d}{dx}$ and $\mathcal{P}_Lx, \mathcal{P}_Rx$ are differentiating $\mathcal{P}_L, \mathcal{P}_R$ respectively with respect to $x$. Now define

$$\log \mathcal{L} = \frac{1}{2N} \log_+ \mathcal{L} + \frac{1}{2M} \log_- \mathcal{L} = \sum_{k \in \mathbb{Z}} W_k \Lambda^k.$$ 

Given any difference operator $A = \sum_k A_k \Lambda^k$, the positive and negative projections are given by $A_+ = \sum_{k \geq 0} A_k \Lambda^k$ and $A_- = \sum_{k < 0} A_k \Lambda^k$. Similar to [15], we give the following definition.

**Definition 2.1.** The Lax equations of extended bigraded Toda hierarchy is given by

$$\frac{\partial \mathcal{L}}{\partial t_{\alpha,n}} = [A_{\alpha,n}, \mathcal{L}] \quad (2.11)$$

for $\alpha = N, N - 1, N - 2, \ldots, -M$ and $n \geq 0$. Here operators $A_{\alpha,n}$ are defined by

$$A_{\alpha,n} = \frac{\Gamma(2 - \frac{\alpha+1}{N})}{\epsilon \Gamma(n + 2 - \frac{\alpha+1}{N})_+} (\mathcal{L}^{n+1-\frac{\alpha+1}{N}})_+ \quad \text{for} \quad \alpha = N, N - 1, \ldots, 1, \quad (2.12a)$$

$$A_{\alpha,n} = -\frac{\Gamma(2 + \frac{\alpha}{M})}{\epsilon \Gamma(n + 2 + \frac{\alpha}{M})_-} (\mathcal{L}^{n+1+\frac{\alpha}{M}})_- \quad \text{for} \quad \alpha = 0, -1, \ldots, -M + 1, \quad (2.12b)$$

$$A_{-M,n} = \frac{2}{\epsilon n!} [\mathcal{L}^n (\log \mathcal{L} - \frac{1}{2} \frac{1}{M} + \frac{1}{N} C_n)]_+, \quad (2.12c)$$

and the constants $C_n$ are defined by

$$C_n = \sum_{k=1}^n \frac{1}{k} C_0 = 0. \quad (2.13)$$

The only difference of this definition from [15] is that we add the hierarchy when $\alpha = 1$ to the hierarchies in the definition of [15]. That hierarchy is in fact the Toda hierarchy which is also the hierarchy when $\alpha = 0$. We do this because it is necessary to introduce such an additional group of equations for proving the existence of tau function.

Particularly for $N = M = 1$ this hierarchy coincides with the extended Toda hierarchy introduced in [9]. If we consider $\mathcal{L}_N^\pm$ and $\mathcal{L}_M^\pm$ are two completely independent operators, the EBTH will imply well-known 2-dimensional Toda hierarchy. We can consider the EBTH as a kind of extended constrained 2-dimensional Toda hierarchy with constraint $(\mathcal{L}_N^\pm)^N = (\mathcal{L}_M^\pm)^M$. 


For the convenience to lead to the Sato equation, we define the following operators which are also similar to [15]:

\[
B_{\alpha,n} := \begin{cases} \frac{\Gamma(2 - \frac{\alpha}{n} - 1)}{\Gamma(n + 2 - \frac{\alpha}{n})} L^{n+1 - \frac{\alpha}{n}}, & \alpha = N \ldots 1, \\ \frac{\Gamma(2 + \frac{\alpha}{M} + 1)}{\Gamma(n + 2 + \frac{\alpha}{M})} L^{n+1 + \frac{\alpha}{M}}, & \alpha = 0 \cdots - M + 1, \\ \frac{2}{\Gamma(\alpha)} [L^n (\log L - \frac{1}{2} (\frac{1}{M} + \frac{1}{N}) c_n)], & \alpha = -M. \end{cases}
\] (2.14)

Then the following lemma can be got [15].

**Lemma 2.2.** The following equations hold

\[
\frac{\partial}{\partial \alpha,p} L^n = [A_{\alpha,p}, L^n],
\] (2.15)

\[
(L^+ \frac{1}{\alpha})_{\alpha,p} = [-(B_{\alpha,p})_-, L^+ \frac{1}{\alpha}],
\] (2.16)

\[
(L^- \frac{1}{\alpha})_{\alpha,p} = [(B_{\alpha,p})_+, L^- \frac{1}{\alpha}],
\] (2.17)

\[
(\log_+ L)_{\alpha,p} = [-(B_{\alpha,p})_-, \log_+ L],
\] (2.18)

\[
(\log_- L)_{\alpha,p} = [(B_{\alpha,p})_+, \log_- L],
\] (2.19)

and combine the last two equations into

\[
(\log L)_{\alpha,p} = [-(B_{\alpha,p})_-, \frac{1}{2N} \log_+ L] + [(B_{\alpha,p})_+, \frac{1}{2M} \log_- L].
\] (2.20)

**Proof.** See [15].

From the lemma above, noticing that \([\log_+ L, L^{k/N}] = 0\) and \([\log_- L, L^{k/M}] = 0\), we can easily get

\[
(\log L)_{\alpha,p} = [A_{\alpha,n}, \log L] = \begin{cases} [(B_{\alpha,n})_+, \log L], & \text{when } \alpha > 0, \\ [-(B_{\alpha,n})_-, \log L], & \text{when } \alpha \leq 0. \end{cases}
\] (2.21)

Using the lemma above, Carlet proved the following proposition.

**Proposition 2.3.** If \(L\) satisfies the Lax equations (2.11), then the following Zakharov-Shabat equations hold [15]

\[
(A_{\alpha,m})_{\beta,n} - (A_{\beta,n})_{\alpha,m} + [A_{\alpha,m}, A_{\beta,n}] = 0
\] (2.22)

for \(-M \leq \alpha, \beta \leq N\), \(m, n \geq 0\).

Using the Zakharov-Shabat eqs. (2.22) we can prove the following corollary.

**Corollary 2.4.** The following relation holds

\[
[\partial_{\beta,n}, \partial_{\alpha,m}] L = 0
\] (2.23)

for \(-M \leq \alpha, \beta \leq N\), \(m, n \geq 0\).

After the corollary above, we can prove the following lemma using the method in [3].

**Lemma 2.5.** The following two equations hold

\[
\frac{\partial}{\partial \beta,n} (B_{\alpha,m})_- - \frac{\partial}{\partial \alpha,m} (B_{\beta,n})_- - [(B_{\alpha,m})_-, (B_{\beta,n})_-] = 0,
\] (2.24)

\[
- \frac{\partial}{\partial \beta,n} (B_{\alpha,m})_+ + \frac{\partial}{\partial \alpha,m} (B_{\beta,n})_+ - [(B_{\alpha,m})_+, (B_{\beta,n})_+] = 0
\] (2.25)

here, \(-M \leq \alpha, \beta \leq N\), \(m, n \geq 0\).
Proof: We now only give the proof of a case of eqs. (2.24) which should be taken special care of because of the logarithm. As eqs. (2.22),

\[ \partial_{-M,n}(A_{\beta,m}) - \partial_{\beta,m}(A_{-M,n}) + [A_{\beta,m}, A_{-M,n}] = 0 \]

where \(-M + 1 \leq \beta \leq 0\), i.e.

\[ -\partial_{-M,n}(B_{\beta,m}) - \partial_{\beta,m}(B_{-M,n}) + [-(B_{\beta,m}), (B_{-M,n})] = 0. \]

Eqs. (2.15) lead to

\[ \partial_{\beta,m}L = \left[-(B_{\beta,m}), L\right] \quad (2.26) \]

Considering to eqs. (2.21) and using eqs. (2.26), we get

\[ \partial_{\beta,m}(B_{-M,n}) = \partial_{\beta,m}(\frac{2}{\epsilon n!}[L^n(\log L - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_n)]) \]

\[ = \left[-(B_{\beta,m}), \frac{2}{\epsilon n!}L^n[\log L - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_n]\right] \]

Then eqs. (2.22) imply

\[ 0 = \left[\partial_{-M,n} - (B_{-M,n})_+, \partial_{\beta,m} + (B_{\beta,m})_-\right] \]

\[ = \left[\partial_{-M,n} + (B_{-M,n})_-, B_{-M,n}, \partial_{\beta,m} + (B_{\beta,m})_-\right] \]

\[ = \left[\partial_{-M,n} + (B_{-M,n})_-, \partial_{\beta,m} + (B_{\beta,m})_-\right] + \left[\partial_{\beta,m} + (B_{\beta,m})_-, (B_{-M,n})_-\right] \]

This is just

\[ \partial_{-M,n}(B_{\beta,m}) - \partial_{\beta,m}(B_{-M,n}) + [(B_{-M,n})_-, (B_{\beta,m})_-] = 0. \]

One can further verify other identities easily by the same way. □

Considering the lemma above we can prove the following theorem.

**Theorem 2.6.** \( L \) is a solution to the EBTH if and only if there is a pair of dressing operators \( \mathcal{P}_L \) and \( \mathcal{P}_R \), which satisfies the following Sato equations

\[ \partial_{\alpha,n}\mathcal{P}_L = -(B_{\alpha,n})_-\mathcal{P}_L, \quad (2.27) \]

\[ \partial_{\alpha,n}\mathcal{P}_R = (B_{\alpha,n})_+\mathcal{P}_R, \quad (2.28) \]

where, \(-M \leq \alpha \leq N, n \geq 0\).

Proof: Using lemma 2.5 and a standard procedure given by [3] and [13], we can prove the theorem.

Sato equations can be regarded as the definitions of the wave operators, i.e. \( \mathcal{P}_L \) and \( \mathcal{P}_R \) in eq. (2.27) and eq. (2.28). It is unique up to multiplying \( \mathcal{P}_L \) and \( \mathcal{P}_R \) from the right by operators of the form \( 1 + a_1\Lambda^1 + a_2\Lambda^2 + \ldots \) and \( \tilde{a}_0 + \tilde{a}_1\Lambda + \tilde{a}_2\Lambda^2 + \ldots \) respectively, where \( a_i \) and \( \tilde{a}_j \) are independent of \( x \) and \( t_{\alpha,n} \) where \(-M \leq \alpha \leq N, n \geq 0\). We shall study identities related to the wave operators in next section. On the other hand, we shall show relations between tau function and \( w_i, \tilde{w}_i \) from Sato equations later.
3. Hirota Bilinear Identities of Wave Operators

We suppose the wave operators \( P_L, P_R \) and \( P_L^{-1}, P_R^{-1} \) given by eq.(2.3) to eq(2.6), then define the symbols \( P_L, P_R \) respectively, but \( \lambda \) does not change the symbols.

The left side of eq.(3.1)-eq.(3.4) means the operators \( P_L, P_R, P_L^{-1}, P_R^{-1} \) acting on the function \( \lambda^{\pm} \) in the bracket. We should note that the \( P_L^{-1} \) and \( P_R^{-1} \) are the inverse operators of \( P_L \) and \( P_R \) respectively, but \( P_L^{-1} \) and \( P_R^{-1} \) are not the inverse symbols of \( P_L \) and \( P_R \) respectively.

For simplicity of Hirota bilinear identities, we will introduce two series below.

\[
W_L(x,t,\Lambda) = P_L(x,t,\Lambda) \times \exp \left( \sum_{n \geq 0} \left[ \sum_{\alpha=1}^{N} \frac{\Gamma(2 - \frac{\alpha-1}{N})}{\Gamma(n+2 - \frac{\alpha-1}{N})} \frac{\lambda^{\alpha(n+1-\frac{\alpha-1}{N})}}{\epsilon} t_{\alpha,n} \right] + \sum_{n > 0} \frac{\Lambda^{\alpha n}}{n!} \left( \epsilon \partial_x - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-M,n}}{\epsilon} \right),
\]

\[
W_R(x,t,\Lambda) = P_R(x,t,\Lambda) \times \exp \left( -\sum_{n \geq 0} \left[ \sum_{\beta=-M+1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n+2 + \frac{\beta}{M})} \frac{\lambda^{-\beta(n+1-\frac{\beta}{N})}}{\epsilon} t_{\beta,n} \right] + \sum_{n > 0} \frac{\Lambda^{-\beta n}}{n!} \left( \epsilon \partial_x + \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-\beta,n}}{\epsilon} \right).
\]

If the series have forms

\[
W_L(x,t,\Lambda) = \sum_{i \in \mathbb{Z}} a_i(x,t,\partial_x) \Lambda^i \quad \text{and} \quad W_R(x,t,\Lambda) = \sum_{i \in \mathbb{Z}} b_i(x,t,\partial_x) \Lambda^i,
\]

\[
W_L^{-1}(x,t,\Lambda) = \sum_{i \in \mathbb{Z}} \Lambda^i a_i'(x,t,\partial_x) \quad \text{and} \quad W_R^{-1}(x,t,\Lambda) = \sum_{i \in \mathbb{Z}} \Lambda^i b_i'(x,t,\partial_x),
\]

then we denote their left symbols \( W_L, W_R \) and right symbols \( W_L^{-1}, W_R^{-1} \) as following

\[
W_L(x,t,\lambda) = \sum_{i \in \mathbb{Z}} a_i(x,t,\partial_x) \lambda^i = P_L(x,t,\lambda) \times \exp \left( \sum_{n \geq 0} \left[ \sum_{\alpha=1}^{N} \frac{\Gamma(2 - \frac{\alpha-1}{N})}{\Gamma(n+2 - \frac{\alpha-1}{N})} \frac{\lambda^{\alpha(n+1-\frac{\alpha-1}{N})}}{\epsilon} t_{\alpha,n} \right] + \sum_{n > 0} \frac{\Lambda^{\alpha n}}{n!} \left( \epsilon \partial_x - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-M,n}}{\epsilon} \right),
\]

\[
W_L^{-1}(x,t,\lambda) = \sum_{i \in \mathbb{Z}} a_i'(x,t,\partial_x) \lambda^i \times P_L^{-1}(x,t,\lambda),
\]

\[
W_R(x,t,\lambda) = \sum_{i \in \mathbb{Z}} b_i(x,t,\partial_x) \lambda^i = P_R(x,t,\lambda) \times \exp \left( -\sum_{n \geq 0} \left[ \sum_{\beta=-M+1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n+2 + \frac{\beta}{M})} \frac{\lambda^{-\beta(n+1-\frac{\beta}{N})}}{\epsilon} t_{\beta,n} \right] + \sum_{n > 0} \frac{\Lambda^{-\beta n}}{n!} \left( \epsilon \partial_x + \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-\beta,n}}{\epsilon} \right).
\]
After defining residue as $\text{Res}$ these operator-valued symbols are quite different from common symbols because $\lambda$ is not replaced by its corresponding symbol log $\lambda$. 

For all integers $r$, we shall prove that

$$W^{-1}_{L}(x, t, \lambda) = \sum_{j \in \mathbb{Z}} b_j(x, t, \partial_x)\lambda^j$$

for all integers $r \geq 0$. Just the same as the method used in [13], by induction on $\alpha$, we shall prove that

$$W_{L}(x, t, \Lambda)\Lambda^{Nr}W^{-1}_{L}(x, t', \Lambda) = W_{R}(x, t, \Lambda)\Lambda^{-Mr}W^{-1}_{R}(x, t', \Lambda)$$

for all integers $r \geq 0$. Just the same as the method used in [13], by induction on $\alpha$, we shall prove that

$$W_{L}(x, t, \Lambda)\Lambda^{Nr}(\partial^n W^{-1}_{L}(x, t, \Lambda)) = W_{R}(x, t, \Lambda)\Lambda^{-Mr}(\partial^n W^{-1}_{R}(x, t, \Lambda)).$$

When $\alpha = 0$, eq. (3.7) becomes

$$\mathcal{P}_{L}(x, t, \Lambda)\Lambda^{Nr}\mathcal{P}^{-1}_{L}(x, t, \Lambda) = \mathcal{P}_{R}(x, t, \Lambda)\Lambda^{-Mr}\mathcal{P}^{-1}_{R}(x, t, \Lambda).$$

which is obviously true according to the definition of wave operators. Suppose eq. (3.7) is true in the case of $\alpha \neq 0$. Note that

$$\partial_{\alpha, n}W := \begin{cases} 
\left[(\partial_{\alpha, n}\mathcal{P}_{L})\mathcal{P}^{-1}_{L} + \mathcal{P}_{L}\Gamma(\frac{\alpha - 1}{\Lambda^{N(n+1-\frac{\alpha}{N})}})\Lambda^{N(n+1-\frac{\alpha}{N})}\mathcal{P}^{-1}_{L}\right]W_{L}, & \alpha = N, N - 1, \ldots, 1, \\
(\partial_{\alpha, n}\mathcal{P}_{L})\mathcal{P}^{-1}_{L}W_{L}, & \alpha = 0 \cdots - M + 1, \\
\left[(\partial_{\alpha, n}\mathcal{P}_{L})\mathcal{P}^{-1}_{L} + \mathcal{P}_{L}\frac{\Lambda^{N1}}{\epsilon} \frac{1}{\frac{M}{N}}(\epsilon\partial_x + \frac{1}{M} + \frac{1}{N})\mathcal{P}^{-1}_{L}\right]W_{L}, & \alpha = -M,
\end{cases}$$

These operator-valued symbols are quite different from common symbols because $\epsilon\partial_x$ is not replaced by its corresponding symbol log $\lambda$.

After defining residue as $\text{Res}_{\lambda} \sum_{n \in \mathbb{Z}} \alpha_n \lambda^n = \alpha_{-1}$, we get the following proposition using the similar proof as [3] and [13].

**Proposition 3.1.** Let $t$ and $t'$ be time sequences such that $t_{-M,0} = t'_{-M,0}$. $\mathcal{P}_{L}$ and $\mathcal{P}_{R}$ are wave operators of the EBTH if and only if for all $m \in \mathbb{Z}$, $r \in \mathbb{N}(\text{including } 0)$, the following Hirota bilinear identity hold

$$\text{Res}_{\lambda}\left\{\lambda^{N_{r+m-1}} W_{L}(x, t, \epsilon\partial_x, \lambda)W^{-1}_{L}(x - m\epsilon, t', \epsilon\partial_x, \lambda)\right\} =$$

$$\text{Res}_{\lambda}\left\{\lambda^{-M_{r+m-1}} W_{R}(x, t, \epsilon\partial_x, \lambda)W^{-1}_{R}(x - m\epsilon, t', \epsilon\partial_x, \lambda)\right\}. \quad (3.5)$$

**Proof.**

$(\Rightarrow)$: Set $\alpha = (\alpha_{0,0}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \ldots; \alpha_{N-1,0}, \alpha_{N-1,1}, \alpha_{N-1,2}, \ldots; \alpha_{-M,1}, \alpha_{-M,2}, \ldots)$ be a multi index and

$$\partial^\alpha := \partial_{\alpha_{0,0}}\partial_{\alpha_{1,0}}\partial_{\alpha_{1,1}}\partial_{\alpha_{1,2}}\ldots; \partial_{\alpha_{N-1,0}}\partial_{\alpha_{N-1,1}}\partial_{\alpha_{N-1,2}}\ldots; \partial_{\alpha_{-M,1}}\partial_{\alpha_{-M,2}}\ldots,$$

where $\partial_{\alpha} = \partial / \partial t_{\alpha} \epsilon$ (we stress that $\partial / \partial t_{-M,0}$ is not involved). Firstly we shall prove the left statement leads to

$$W_{L}(x, t, \Lambda)\Lambda^{Nr}W^{-1}_{L}(x, t', \Lambda) = W_{R}(x, t, \Lambda)\Lambda^{-Mr}W^{-1}_{R}(x, t', \Lambda) \quad (3.6)$$

for all integers $r \geq 0$. Just the same as the method used in [13], by induction on $\alpha$, we shall prove that

$$W_{L}(x, t, \Lambda)\Lambda^{Nr}(\partial^\alpha W^{-1}_{L}(x, t, \Lambda)) = W_{R}(x, t, \Lambda)\Lambda^{-Mr}(\partial^\alpha W^{-1}_{R}(x, t, \Lambda)). \quad (3.7)$$

When $\alpha = 0$, eq. (3.7) becomes

$$\mathcal{P}_{L}(x, t, \Lambda)\Lambda^{Nr}\mathcal{P}^{-1}_{L}(x, t, \Lambda) = \mathcal{P}_{R}(x, t, \Lambda)\Lambda^{-Mr}\mathcal{P}^{-1}_{R}(x, t, \Lambda). \quad (3.8)$$

which is obviously true according to the definition of wave operators.
\[
\partial_{\alpha,n}W_R := \begin{cases} 
(B_{\alpha,n})_+ W_L, & \alpha = N \ldots 1, \\
-(B_{\alpha,n})_- W_L, & \alpha = 0 \cdots -M + 1, \\
[-(B_{\alpha,n})_+ - \frac{1}{c_m} [\mathcal{L}^n (\frac{1}{N} \log L - \frac{1}{2} (\frac{1}{M} + \frac{1}{N}) \mathcal{C}_n)]] W_L, & \alpha = -M,
\end{cases}
\]

By computation we get

\[
\partial_{\alpha,n}W_L := \begin{cases} 
(B_{\alpha,n})_+ W_L, & \alpha = N \ldots 1, \\
-(B_{\alpha,n})_- W_L, & \alpha = 0 \cdots -M + 1, \\
[-(B_{\alpha,n})_+ - \frac{1}{c_m} [\mathcal{L}^n (\frac{1}{M} \log L - \frac{1}{2} (\frac{1}{M} + \frac{1}{N}) \mathcal{C}_n)]] W_L, & \alpha = -M,
\end{cases}
\]

which implies

\[
(\partial_{\alpha,n}W_L) \Lambda'^{N_r}(\partial^{N_r}W_L^{-1}) = (\partial_{\alpha,n}W_R) \Lambda^{-M_r}(\partial^{N_r}W_R^{-1})
\]

by considering (3.7). Furthermore we get

\[
W_L \Lambda'^{N_r}(\partial_{\alpha,n}^{\partial^{N_r}}W_L^{-1}) = W_R \Lambda^{-M_r}(\partial_{\alpha,n}^{\partial^{N_r}}W_R^{-1}).
\]

Thus if we increase the power of \(\partial_{\alpha,n}\) by 1 then eq.(3.7) still holds. The induction is completed. Using the Taylor’s formula and eq.(3.7), expanding Both sides of eq.(3.6) about \(t = t'\), we can finish the proof of eq.(3.6).

Then we shall prove the right-side statement of the proposition is equivalent to identity eq.(3.5).

Let \(m \in \mathbb{Z}, r \in \mathbb{N}\) and \(t_{-M,0} = t'_{-M,0}\). Put

\[
W_L(x, t, \Lambda) = \sum_{i \in \mathbb{Z}} a_i(x, t, \partial_x) \Lambda^i \text{ and } W_R(x, t, \Lambda) = \sum_{i \in \mathbb{Z}} b_i(x, t, \partial_x) \Lambda^i,
\]

\[
W_L^{-1}(x, t, \Lambda) = \sum_{i \in \mathbb{Z}} \Lambda^i a'_i(x, t, \partial_x) \text{ and } W_R^{-1}(x, t, \Lambda) = \sum_{j \in \mathbb{Z}} \Lambda^j b'_j(x, t, \partial_x)
\]

and compare the coefficients in front of \(\Lambda^{-m}\) in eq.(3.6):

\[
\sum_{i+j=-m-Nr} a_i(x, t, \partial_x) a'_j(x - m \epsilon, t', \partial_x) = \sum_{i+j=-m+Mr} b_i(x, t, \partial_x) b'_j(x - m \epsilon, t', \partial_x).
\]

This equality can be written also as

\[
\text{Res}_\lambda \left\{ \lambda^{N_r+m-1} W_L(x, t, \epsilon \partial_x, \lambda) W_L^{-1}(x - m \epsilon, t', \epsilon \partial_x, \lambda) \right\} = \text{Res}_\lambda \left\{ \lambda^{-M_r+m-1} W_R(x, t, \epsilon \partial_x, \lambda) W_R^{-1}(x - m \epsilon, t', \epsilon \partial_x, \lambda) \right\}.
\]

\[(\Leftarrow\Rightarrow):\] We have proved that eq.(3.5) is equivalent to eq.(3.6). Now we will prove eq.(3.6) implies that operators \(P_L\) and \(P_R\) are wave operators of the EBTH.

Differentiate eq.(3.6) with respect to \(t_{\alpha,n}\) and then put \(t = t'\), we can get

\[
(\partial_{\alpha,n}P_L) P_L^{-1} + P_L C_{\alpha,n} P_L^{-1} = (\partial_{\alpha,n}P_R) P_R^{-1} - P_R C'_{\alpha,n} P_R^{-1}
\]
where
\[
C_{\alpha,n} := \begin{cases} 
\frac{\Gamma(2 - \frac{\alpha}{M})}{\alpha!(n + 2 - \frac{\alpha}{M})} \Lambda^{N(n+1-\frac{\alpha}{M})}, & \alpha = N \ldots 1, \\
0, & \alpha = 0 \ldots -M + 1, \\
\frac{1}{e^n!}[\Lambda^n(\epsilon \partial_x - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_n)], & \alpha = -M,
\end{cases}
\]
\[
C'_{\alpha,n} := \begin{cases} 
0, & \alpha = N \ldots 1, \\
\frac{\Gamma(2 + \frac{\alpha}{M})}{\alpha!(n + 2 + \frac{\alpha}{M})} \Lambda^{-N(n+1+\frac{\alpha}{M})}, & \alpha = 0 \ldots -M + 1, \\
\frac{1}{e^n!}[\Lambda^{-nM}(\epsilon \partial_x - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_n)], & \alpha = -M.
\end{cases}
\]

Since \((\partial_{\alpha,n}P_L)P_L^{-1}\) contains only negative powers of \(\Lambda\) and \((\partial_{\alpha,n}P_R)P_R^{-1}\) contains non-negative powers, we get eq. (2.27), eq. (2.28) by separating the negative and the positive part of the equation. Thus \(P_L, P_R\) is a pair of wave operators. This is the end the proof.

Although in the HBI eq. (3.5), the symbols are not scaled-valued, we can also think about the scalar-valued form of the HBI.

**Proposition 3.2.** Let \(1 \leq \alpha \leq N, -M + 1 \leq \beta \leq 0, m \in \mathbb{Z}, r \in \mathbb{N}\); HBI eq. (3.3) leads to the following scaled-valued Hirota bilinear identities

\[
\text{Res}_\lambda \left\{ \lambda^{Nr+m-1}[(\partial_{\alpha,n}P_L(x,t,\lambda))P_L^{-1}(x-m\epsilon, t, \lambda) + \frac{\lambda^nN}{n!}P_L(x,t,\lambda)P_L^{-1}(x-m\epsilon, t, \lambda) \right. \\
\left. - \frac{\lambda^nN}{n!}\frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_nP_L(x,t,\lambda)P_L^{-1}(x-m\epsilon, t, \lambda) \right\} = \\
\text{Res}_\lambda \left\{ \lambda^{-Mr+m-1}[(\partial_{\alpha,n}P_R(x,t,\lambda))P_R^{-1}(x-m\epsilon, t, \lambda) + \frac{\lambda^nM}{n!}P_R(x,t,\lambda)P_R^{-1}(x-m\epsilon, t, \lambda) \right. \\
\left. + \frac{\lambda^{-nM}}{n!}\frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_nP_R(x,t,\lambda)P_R^{-1}(x-m\epsilon, t, \lambda) \right\},
\]

\[
\text{Res}_\lambda \left\{ \lambda^{Nr+m-1}P_L(x,t,\lambda)P_L^{-1}(x-m\epsilon, t, \lambda) \right\} = \\
\text{Res}_\lambda \left\{ \lambda^{-M}(x,t,\lambda) P_R^{-1}(x-m\epsilon, t, \lambda) \right\}.
\]

**Proof.** Let operators in both sides of eq. (3.5) act on “1”, because

\[
\exp \left( \frac{\lambda^nN}{n!}(t-M_n-t'_{-M_n})\partial_x \right)P_L^{-1}(x-m\epsilon, t', \lambda)1 = P_L^{-1}(1 + \sum_{n>0} \frac{\lambda^nN}{n!}(t-M_n-t'_{-M_n}) - m\epsilon, t', \lambda),
\]
\[
\exp \left( \frac{\lambda^{-nM}}{n!}(t-M_n-t'_{-M_n})\partial_x \right)P_R^{-1}(x-m\epsilon, t', \lambda)1 = P_R^{-1}(1 + \sum_{n>0} \frac{\lambda^{-nM}}{n!}(t-M_n-t'_{-M_n}) - m\epsilon, t', \lambda),
\]
therefore the HBI eq. (3.5) becomes

\[
\text{Res}_\lambda \left\{ \lambda^{N_r+m-1} P_L(x, t, \lambda) \exp(\sum_{n \geq 0} \sum_{\alpha=1}^N \frac{\Gamma(2 - \frac{\alpha-1}{N})}{\Gamma(n + 2 - \frac{\alpha-1}{N})} \frac{\lambda^{N(n+1-\frac{\alpha}{N})}}{\epsilon} (t, a, n - t', a, n) - \sum_{n > 0} \lambda^{nN} \frac{1}{n!} \frac{1}{M} + \frac{1}{N} C_n (t_{-M,n} - t'_{-M,n}) \right\} \]

\[
\text{Res}_\lambda \left\{ \lambda^{N_r+m-1} P_L(x, t, \lambda) \exp(-\sum_{n \geq 0} \sum_{\beta=-M+1}^0 \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{N})}}{\epsilon} (t, a, n - t', a, n) + \sum_{n > 0} \lambda^{-nM} \frac{1}{n!} \frac{1}{M} + \frac{1}{N} C_n (t_{-M,n} - t'_{-M,n}) \right\} .
\]

(3.13)

To get eq. (3.9), we differentiate both sides of eq. (3.13) by \( t_{a,n} \) and let \( t = t' \). To get eq. (3.10), we differentiate both sides of eq. (3.13) by \( t_{\beta,n} \) and let \( t = t' \). To get eq. (3.11), we differentiate both sides of eq. (3.13) by \( t_{-M,n} \) and let \( t = t' \). To get eq. (3.12), we just let \( t = t' \) in eq. (3.13). \( \square \)

Moreover, HBI (3.5) can imply other interesting identities.

**Proposition 3.3.** Let \( 1 \leq \alpha \leq N, -M + 1 \leq \beta \leq 0, r \in \mathbb{N} \) and \( x - x' = me, m \in \mathbb{Z} \), HBI (3.5) leads to the following scalar-valued Hirota bilinear identities

\[
\text{Res}_\lambda \left\{ \lambda^{N_r-1} \left[ \left( \partial_{a,n} P_L(x, t, \lambda) \right) P_{L^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right] \right\} = \text{Res}_\lambda \left\{ \lambda^{-M_r-1} \left( \partial_{a,n} P_R(x, t, \lambda) \right) P_{R^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right\} ,
\]

(3.14)

\[
\text{Res}_\lambda \left\{ \lambda^{N_r-1} \left( \partial_{\beta,n} P_L(x, t, \lambda) \right) P_{L^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right\} = \text{Res}_\lambda \left\{ \lambda^{-M_r-1} \left( \partial_{\beta,n} P_R(x, t, \lambda) \right) P_{R^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right\} ,
\]

(3.15)

\[
\text{Res}_\lambda \left\{ \lambda^{N_r-1} \left( \partial_{-M,n} P_L(x, t, \lambda) \right) P_{L^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right\} = \text{Res}_\lambda \left\{ \lambda^{-M_r-1} \left( \partial_{-M,n} P_R(x, t, \lambda) \right) P_{R^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right\} .
\]

(3.16)

\[
\text{Res}_\lambda \left\{ \lambda^{N_r-1} P_L(x, t, \lambda) P_{L^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right\} = \text{Res}_\lambda \left\{ \lambda^{-M_r-1} P_R(x, t, \lambda) P_{R^{-1}}(x', t, \lambda) \lambda^{\frac{x-x'}{\epsilon}} \right\} .
\]

(3.17)
4. THE EXISTENCE OF TAU-FUNCTIONS

For shortness, denote by \([\lambda^{-1}]_N^M\), \([\lambda]^M\) the following sequences:

\[
[\lambda^{-1}]_{\alpha,n}^N := \begin{cases} \\
\frac{\Gamma(n+\frac{M}{2})}{\Gamma(n+1)} \epsilon \lambda^{-N(n+1)}, & \alpha = N, N - 1, \ldots, 1, \\
0, & \alpha = 0, -1, \ldots, (M - 1), \\
\frac{\Gamma(n+1)}{\Gamma(n+1+\beta)} \epsilon \lambda^{M(n+1)}, & \alpha = N, N - 1, \ldots, 1, \\
0, & \alpha = -M.
\end{cases}
\]

A function \(\tau\) depending only on the dynamical variables \(t\) and \(\epsilon\) is called the **tau-function of the EBTH** if it provides symbols related to wave operators as following,

\[
P_L := 1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \ldots := \frac{\tau(t-M, 0 + x - \frac{\epsilon}{2}, t - [\lambda^{-1}]_N^M; \epsilon)}{\tau(t-M, 0 + x - \frac{\epsilon}{2}, t; \epsilon)}, \quad (4.1)
\]

\[
P_L^{-1} := 1 + \frac{w'_1}{\lambda} + \frac{w'_2}{\lambda^2} + \ldots := \frac{\tau(t-M, 0 + x + \frac{\epsilon}{2}, t + [\lambda]^M; \epsilon)}{\tau(t-M, 0 + x + \frac{\epsilon}{2}, t; \epsilon)}, \quad (4.2)
\]

\[
P_R := \bar{w}_0 + \bar{w}_1 \lambda + \bar{w}_2 \lambda^2 + \ldots := \frac{\tau(t-M, 0 + x - \frac{\epsilon}{2}, t + [\lambda]^M; \epsilon)}{\tau(t-M, 0 + x - \frac{\epsilon}{2}, t; \epsilon)}, \quad (4.3)
\]

\[
P_R^{-1} := \bar{w}'_0 + \bar{w}'_1 \lambda + \bar{w}'_2 \lambda^2 + \ldots := \frac{\tau(t-M, 0 + x + \frac{\epsilon}{2}, t - [\lambda]^M; \epsilon)}{\tau(t-M, 0 + x + \frac{\epsilon}{2}, t; \epsilon)}. \quad (4.4)
\]

For a given pair of wave operators the tau-function is unique up to a non-vanishing function factor which is independent of \(x\), \(t-M, 0\) and \(t_{\alpha,n}\) with all \(n \geq 0\) and \(-M + 1 \leq \alpha \leq N\).

In this section we shall give a transparent and detailed proof of the existence of tau function for the EBTH according to the Sato theory ([1], [22]). Let \(t\) and \(t'\) be two different sequences of time variables with \(t_{\alpha,n} = t_{\alpha,n}'\), \(n \geq 0, r = 0\), then HBI eq. (3.5) becomes

\[
\text{Res}_\lambda \left\{ \lambda^{m-1} P_L(x, t, \lambda) e^{-\sum_{n \geq 0} \sum_{\alpha=1}^{N} \frac{\Gamma(2, \frac{\alpha-1}{2})}{\Gamma(\alpha + 2)} \lambda^{N(n+1-\frac{\alpha-1}{2})} (t_{\alpha,n}-t'_{\alpha,n})} P^{-1}_L(x - m \epsilon, t', \lambda) \right\} = \\
\text{Res}_\lambda \left\{ \lambda^{m-1} P_R(x, t, \lambda) e^{-\sum_{n \geq 0} \sum_{\beta=-M+1}^{M-1} \frac{\Gamma(2, \frac{\beta}{2})}{\Gamma(\beta + 2)} \lambda^{-M(n+1+\frac{\beta}{2})} (t_{\beta,n}-t'_{\beta,n})} P^{-1}_R(x - m \epsilon, t', \lambda) \right\}.
\]

(4.5)

By a straightforward computation, we can infer following lemma from eq. (4.5), which are necessary for our main theorem on tau function.

**Lemma 4.1.** The following three identities hold

\[
\log P_L(x, t, \lambda_1) - \log P_L(x, t - [\lambda_2^{-1}]^N, \lambda_1) = \log P_L(x, t, \lambda_2) - \log P_L(x, t - [\lambda_1^{-1}]^N, \lambda_2).
\]

(4.6)
Proof. For the proof of identity (4.6), we shall set $m = 1, t' = t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N$ in eq. (4.5). Using the identity

$$P_L(x, t, \lambda) = \log P_R(x, t, \lambda) - \log P_L(x + \epsilon, t + [\lambda_2]^M, \lambda_1) - \log P_R(x, t, \lambda_2) - \log P_R(x, t - [\lambda_1^{-1}]^N, \lambda_2).$$

(4.7)

$$\log P_R(x, t, \lambda_1) - \log P_R(x + \epsilon, t + [\lambda_2]^M, \lambda_1) = \log P_R(x, t, \lambda_2) - \log P_R(x + \epsilon, t + [\lambda_1]^M, \lambda_2).$$

(4.8)

Using the bilinear identity eq. (4.5) gives

$$\exp \left( \sum_{n \geq 0} \sum_{a=0}^{N-1} \frac{(\lambda_1^{-1} \lambda)^{N(n+1-a)}}{N(n + 1 - \frac{a}{N})} \lambda \right) (1 - \lambda_1^{-1} \lambda)^{-1},$$

the bilinear identity eq. (4.5) gives

$$\res_{\lambda} \left\{ P_L(x, t, \lambda) P_L^{-1}(x - \epsilon, t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N, \lambda) \right\} \frac{1}{(1 - \lambda^{-1}) - \frac{\lambda}{\lambda_2}} =$$

$$\res_{\lambda} \left\{ P_R(x, t, \lambda) P_R^{-1}(x - \epsilon, t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N, \lambda) \right\}.$$  

(4.9)

Using

$$(1 - \lambda_1^{-1} \lambda)^{-1} (1 - \lambda_2^{-1} \lambda)^{-1} = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \{(1 - \lambda_1^{-1} \lambda)^{-1} - (1 - \lambda_2^{-1} \lambda)^{-1}\} \lambda^{-1},$$

$$\res_{\lambda} \left\{ f(\lambda) \frac{1}{\lambda(1 - \frac{\lambda}{\lambda_1})} \right\} = f(\lambda_1),$$

where $f(\lambda) = 1 + \sum_{i=1}^{\infty} a_i \lambda^{-i}$ is a formal series of $\lambda$, then eq. (4.9) infers

$$P_L(x, t, \lambda_1) P_L^{-1}(x - \epsilon, t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N, \lambda_1) = P_L(x, t, \lambda_2) P_L^{-1}(x - \epsilon, t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N, \lambda_2).$$

(4.10)

Setting $\lambda_1 = \lambda$ and $\lambda_2 = \infty$, we obtain

$$P_L(x, t, \lambda) P_L^{-1}(x - \epsilon, t - [\lambda^{-1}]^N, \lambda) = 1,$$

(4.11)

which is equivalent to

$$P_L^{-1}(x - \epsilon, t - [\lambda^{-1}]^N, \lambda) = \frac{1}{P_L(x, t, \lambda)},$$

(4.12)

Using this identity, eq. (4.10) gives

$$\frac{P_L(x, t, \lambda_1)}{P_L(x, t - [\lambda_2^{-1}]^N, \lambda_1)} = \frac{P_L(x, t, \lambda_2)}{P_L(x, t - [\lambda_1^{-1}]^N, \lambda_2)},$$

(4.13)

or equivalently to eq. (4.6).

To prove identity (4.7), we shall set $m = 0, t' = t - [\lambda_1^{-1}]^N + [\lambda_2]^M$ in eq. (4.5). In this case, using the identities

$$\exp \left( \sum_{n \geq 0} \sum_{a=0}^{N-1} \frac{(\lambda_1^{-1} \lambda)^{N(n+1-a)}}{N(n + 1 - \frac{a}{N})} \lambda \right) (1 - \lambda_1^{-1} \lambda)^{-1},$$

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Consider another residue formula

\[
\text{Res}_\lambda \left\{ P_L(x, t, \lambda) P_L^{-1}(x, t - [\lambda_1^{-1}]^N + [\lambda_2]^M, \lambda) \lambda^{-1} \frac{1}{1 - \frac{\lambda}{\lambda_1}} \right\} = \text{Res}_\lambda \left\{ P_R(x, t, \lambda) P_R^{-1}(x, t - [\lambda_1^{-1}]^N + [\lambda_2]^M, \lambda) \lambda^{-1} \frac{1}{1 - \frac{\lambda}{\lambda_2}} \right\}. \tag{4.14}
\]

Using formula

\[
f_{\lambda}(x, t, \lambda) = \frac{1}{\lambda - \lambda_1},
\]

where \( f(\lambda) = a_0 + \sum_{i=1}^{\infty} a_i \lambda^i \) is a formal series of \( \lambda \), eq. (4.14) further leads to

\[
P_L(x, t, \lambda_1) P_L^{-1}(x, t - [\lambda_1^{-1}]^N + [\lambda_2]^M, \lambda_1) = P_R(x, t, \lambda_2) P_R^{-1}(x, t - [\lambda_1^{-1}]^N + [\lambda_2]^M, \lambda_2). \tag{4.16}
\]

Setting \( \lambda_1 = \infty \) and \( \lambda_2 = \lambda \) in above equation, then

\[
P_R(x, t, \lambda) P_R^{-1}(x, t + [\lambda]^M, \lambda) = 1, \tag{4.17}
\]

\[
P_R^{-1}(x, t + [\lambda]^M, \lambda) = \frac{1}{P_R(x, t, \lambda)}. \tag{4.18}
\]

Using identity eq.(4.12), eq.(4.16) and eq.(4.18), we get

\[
\frac{P_L(x, t, \lambda_1)}{P_L(x + \epsilon, t + [\lambda_2]^M, \lambda_1)} = \frac{P_R(x, t, \lambda_2)}{P_R(x, t - [\lambda_1^{-1}]^N, \lambda_2)}, \tag{4.19}
\]

which is equivalent to eq.(4.7).

For proving identity(4.8), we set \( m = -1, t' = t + [\lambda_1]^M + [\lambda_2]^M \) in eq.(4.15). The bilinear identity eq.(4.15) gives

\[
\text{Res}_\lambda \left\{ P_L(x, t, \lambda) P_L^{-1}(x + \epsilon, t + [\lambda_1]^M + [\lambda_2]^M, \lambda) \lambda^{-2} \right\} = \text{Res}_\lambda \left\{ P_R(x, t, \lambda) P_R^{-1}(x + \epsilon, t + [\lambda_1]^M + [\lambda_2]^M, \lambda) \frac{\lambda^{-2}}{(1 - \frac{\lambda}{\lambda_1})(1 - \frac{\lambda}{\lambda_2})} \right\}. \tag{4.20}
\]

Using formula

\[
(1 - \lambda_1 \lambda^{-1})^{-1}(1 - \lambda_2 \lambda^{-1})^{-1} = \frac{1}{\lambda_1 - \lambda_2} \{(1 - \lambda_1 \lambda^{-1})^{-1} - (1 - \lambda_2 \lambda^{-1})^{-1}\} \lambda
\]

and residue formula eq.(4.15), eq.(4.20) further gives

\[
P_R(x, t, \lambda_1) P_R^{-1}(x + \epsilon, t + [\lambda_1]^M + [\lambda_2]^M, \lambda_1) = P_R(x, t, \lambda_2) P_R^{-1}(x + \epsilon, t + [\lambda_1]^M + [\lambda_2]^M, \lambda_2). \tag{4.21}
\]
Using identity (4.17), eq.(4.21) leads
\[
\frac{P_R(x, t, \lambda_1)}{P_R(x + \epsilon, t + [\lambda_2]^M, \lambda_1)} = \frac{P_R(x, t, \lambda_2)}{P_R(x + \epsilon, t + [\lambda_1]^M, \lambda_2)},
\] (4.22)
which is equivalent to eq.(4.8). So the proof of the lemma is completed now.

By lemma 4.1, we get the following theorem.

**Theorem 4.2.** Given a pair of wave operators \( \mathcal{P}_L \) and \( \mathcal{P}_R \) of the EBTH there exists a unique corresponding tau-function up to a non-vanishing function factor which is independent of \( t_{-M,0} \) and \( t_{a,n}, \ n \geq 0, -M + 1 \leq a \leq N - 1 \).

**Proof.** As [22], the proof is a little complicated and the process can be divided into three steps. For the first step, we shall define a 1-form \( \omega \), and then give the translational invariance of \( d\omega \). Then we will prove the 1-form is closed in the second step which leads to the existence of tau function \( \tau(t) \). The third step is devoted to give the certain value of integration constants such that we can get the symbols of dressing operators by \( \tau(t) \). To this end, define

\[
\omega_L(\epsilon, x, t) := -\sum_{\alpha=1}^{N} \sum_{n \geq 0} dt_{\alpha,n} \operatorname{Res}_\lambda \left\{ \frac{\Gamma(2 - \frac{\alpha-1}{N})}{\Gamma(n + 2 - \frac{\alpha-1}{N})} \frac{\lambda^{N(n+1-\frac{\alpha-1}{N})}}{\epsilon} (\partial_{\lambda} + \sum_{n' \geq 0} \sum_{\alpha'=1}^{N} \frac{\Gamma(n' + 2 - \frac{\alpha'-1}{N})}{\Gamma(2 - \frac{\alpha'-1}{N})} \epsilon^{N(n' + 1 - \frac{\alpha'-2}{N})} \partial_{\lambda t_{\alpha',n'}}) \log P_L(x, t, \lambda) \right\}, \tag{4.23}
\]
\[
\omega_R(\epsilon, x, t) := \sum_{\beta=-M+1}^{0} \sum_{n \geq 0} dt_{\beta,n} \operatorname{Res}_\lambda \left\{ \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} (\partial_{\lambda} - \sum_{n' \geq 0} \sum_{\beta'=-M+1}^{0} \frac{\Gamma(n' + 2 + \frac{\beta'}{M})}{\Gamma(2 + \frac{\beta'}{M})} \epsilon^{M(n' + 1 + \frac{\beta'-1}{M})} \partial_{\lambda t_{\beta',n'}}) \log P_R(x, t, \lambda) \right\}. \tag{4.24}
\]

Using the three identity eq.(4.6), eq.(4.7) and eq.(4.8) in lemma 4.1 we get
\[
\omega_L(x, t) - \omega_L(x, t - [\lambda^{-1}]^N) = -d_L \log P_L(x, t, \lambda), \tag{4.25}
\]
\[
\omega_R(x, t) - \omega_R(x, t - [\lambda^{-1}]^N) = -d_R \log P_L(x, t, \lambda), \tag{4.26}
\]
\[
\omega_L(x, t) - \omega_L(x + \epsilon, t + [\lambda]^M) = -d_L \log P_R(x, t, \lambda), \tag{4.27}
\]
\[
\omega_R(x, t) - \omega_R(x + \epsilon, t + [\lambda]^M) = -d_R \log P_R(x, t, \lambda), \tag{4.28}
\]
where
\[
d_L = \sum_{\alpha=1}^{N} \sum_{n \geq 0} dt_{\alpha,n} \frac{\partial}{\partial t_{\alpha,n}}, \tag{4.29}
\]
\[
d_R = \sum_{\beta=-M+1}^{0} \sum_{n \geq 0} dt_{\beta,n} \frac{\partial}{\partial t_{\beta,n}}.
\]
Here we only give the proof of eq.(4.25) using identity eq.(4.6) in the following, the other there equations can be got in the same way.

\[
\omega_L(x, t) - \omega_L(x, t - [\lambda^{-1}]^N)
\]
we define

In fact equations (4.25)-(4.28) can be seen as a generalization of eqs.(3.16) in [21]. Moreover Eq.(4.25) and Eq.(4.26) lead to

Eq.(4.27) and Eq.(4.28) lead to

When \( \lambda = 0 \), eq.(4.31) lead to

Differentiate both sides of equations in eq.(4.30), eq.(4.31) and eq.(4.32), we get

\[
\begin{align*}
  d\omega(x, t) &= d\omega(x, t - [\lambda^{-1}]^N), \quad (4.33) \\
  d\omega(x, t) &= d\omega(x + \epsilon, t + [\lambda]^M), \quad (4.34) \\
  d\omega(x, t) &= d\omega(x + \epsilon, t), \quad (4.35)
\end{align*}
\]
which shows \( d\omega(x,t) \) is independent of \( x,t,\alpha,n, -M + 1 \leq \alpha \leq N, n \geq 0 \). Without loss of generality, we can assume

\[
d\omega(x,t) = \sum_{\alpha,\beta = -M + 1}^{N} \sum_{n,m \geq 0} a(\epsilon)_{\alpha,n,\beta,m} dt_{\alpha,n} \wedge dt_{\beta,m}
\]  

(4.36)

where \( a(\epsilon)_{\alpha,n,\beta,m} \) are independent of \( x,t,\alpha,n, -M + 1 \leq \alpha \leq N, n \geq 0 \) and \( a(\epsilon)_{\alpha,n,\beta,m} = -a(\epsilon)_{\beta,m,\alpha,n} \). So

\[
\omega(x,t) = \sum_{\beta = -M + 1}^{N} \sum_{m \geq 0} \left( \sum_{\alpha = -M + 1}^{N} \sum_{n \geq 0} a(\epsilon)_{\alpha,n,\beta,m} t_{\alpha,n} \right) dt_{\beta,m} + dF(\epsilon, x, t)
\]

(4.37)

for arbitrary function \( F(\epsilon, x, t) \). Taking \( \omega(x,t) \) in eq.(4.37) back into the equation (4.30) and (4.31), then

\[
-d \log P_L(x, t, \lambda) = dF(x,t) - dF(x,t - [\lambda^{-1}]N) + \sum_{\beta = -M + 1}^{N} \sum_{m \geq 0} \left( \sum_{\alpha = -M + 1}^{N} \sum_{n \geq 0} a(\epsilon)_{\alpha,n,\beta,m} \right) dt_{\beta,m},
\]

(4.38)

\[
-d \log P_R(x, t, \lambda) = dF(x,t) - dF(x + \epsilon,t + [\lambda]^N) - \sum_{\beta = -M + 1}^{N} \sum_{m \geq 0} \left( \sum_{\alpha = -M + 1}^{N} \sum_{n \geq 0} a(\epsilon)_{\alpha,n,\beta,m} \right) dt_{\beta,m}.
\]

(4.39)

Furthermore, two identities above lead to

\[
\log P_L(x, t, \lambda) = F(x,t - [\lambda^{-1}]N) - F(x,t) - \sum_{\beta = -M + 1}^{N} \sum_{m \geq 0} \left( \sum_{\alpha = -M + 1}^{N} \sum_{n \geq 0} a(\epsilon)_{\alpha,n,\beta,m} \right) \frac{\Gamma(n + 1 - \frac{\alpha - 1}{N})}{\eta^{(2 - \frac{\alpha - 1}{N})}} \epsilon \lambda^{-N(n + 1 - \frac{\alpha - 1}{N})} t_{\beta,m} + H_L(\epsilon, x, t-M,n, \lambda),
\]

(4.40)

\[
\log P_R(x, t, \lambda) = F(x + \epsilon,t + [\lambda]^N) - F(x,t) + \sum_{\beta = -M + 1}^{N} \sum_{m \geq 0} \left( \sum_{\alpha = -M + 1}^{N} \sum_{n \geq 0} a(\epsilon)_{\alpha,n,\beta,m} \right) \frac{\Gamma(n + 1 + \frac{\alpha}{M})}{\eta^{(2 + \frac{\alpha}{M})}} \epsilon \lambda^{N(n + 1 + \frac{\alpha}{M})} t_{\beta,m} + H_R(\epsilon, x, t-M,n, \lambda),
\]

(4.41)

where the functions \( H_L(\epsilon, x, t-M,n, \lambda) = \sum_{i=1}^{\infty} H_{L,i}(\epsilon, x, t-M,n) \lambda^{-i} \) and \( H_R(\epsilon, x, t-M,n, \lambda) = \sum_{i=1}^{\infty} H_{R,i}(\epsilon, x, t-M,n) \lambda^{i} \) are independent on \( t, \alpha, (M + 1 \leq \alpha \leq N) \). Taking these results back into eq.(4.30), eq.(4.37) and eq.(4.38), then

\[
\sum_{\beta = 1}^{N} \sum_{m \geq 0} \left( \sum_{\alpha = 1}^{N} \sum_{n \geq 0} a(\epsilon)_{\alpha,n,\beta,m} \right) \frac{\Gamma(n + 1 - \frac{\alpha - 1}{N})}{\eta^{(2 - \frac{\alpha - 1}{N})}} \epsilon \lambda^{-N(n + 1 - \frac{\alpha - 1}{N})} \frac{\Gamma(m + 1 - \frac{\beta - 1}{N})}{\eta^{(2 - \frac{\beta - 1}{N})}} \epsilon \lambda^{N(m + 1 - \frac{\beta - 1}{N})} =
\]

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conclude that there exists a non-vanishing function is denoted as all the other time variables except \( t \) for all \( -M < \beta \leq N \); \( m, n \geq 0 \). So from identity (4.36), we have \( d\omega(x, t) = 0 \). We thus conclude that there exists a non-vanishing function \( \tau(\epsilon, x, t) \) such that

\[
\omega(\epsilon, x, t) = d\log \tau(x - \frac{\epsilon}{2}, t).
\]

(4.45)

In fact the function \( \tau(x - \frac{\epsilon}{2}, t) \) can be written in another form as \( \tau(t_{-M,0} + x - \frac{\epsilon}{2}, \tilde{t}) \), where \( \tilde{t} \) is denoted as all the other time variables except \( t_{-M,0} \). Therefore eq.(4.32) can be rewritten as

\[
\tilde{\omega}(\epsilon, x, t) = \frac{\tau(x + \frac{\epsilon}{2}, t)}{\tau(x - \frac{\epsilon}{2}, t)}.
\]

(4.46)

From eq.(4.37), we can take \( F(\epsilon, x, t) = \log \tau(x - \frac{\epsilon}{2}, t) \). So eq.(4.40) and eq.(4.41) give us

\[
\log P_L(x, t, \lambda) = \log \tau(x - \frac{\epsilon}{2}, t - [\lambda^{-1}]^N) - \log \tau(x - \frac{\epsilon}{2}, t) + H_L(\epsilon, x, t_{-M,0}, \lambda),
\]

(4.47)

\[
\log P_R(x, t, \lambda) = \log \tau(x + \frac{\epsilon}{2}, t + [\lambda]^M) - \log \tau(x - \frac{\epsilon}{2}, t) + H_R(\epsilon, x, t_{-M,0}, \lambda).
\]

(4.48)

Substituting these into the definition of \( \omega \) and using eq.(4.45) we will see that \( H_L(\epsilon, x, \lambda), H_R(\epsilon, x, \lambda) \) are all zero. eq.(4.47), eq.(4.12), eq.(4.48) and eq.(4.18) will give the equations eq.(4.1), eq.(4.12), eq.(4.3) and eq.(4.4) in the definition of \( \tau \). So the proof of existence of tau function is finished. \( \square \)
Next we shall consider the Fay-like identities on the tau functions. To this end, by taking the definition of tau function in (4.1), (4.2), (4.3) and (4.4) into eq. (4.5) and replacing \( x - \frac{1}{2} \) by \( x \) in the tau function, we get the following Hirota bilinear identity

\[
\text{Res}_\lambda \left\{ \lambda^{m-1} \tau(x, t - [\lambda^{-1}]^N) \times \tau(x - (m - 1)\epsilon, t') + [\lambda^{-1}]^N \epsilon \xi_L(t-t') \right\} \\
= \text{Res}_\lambda \left\{ \lambda^{m-1} \tau(x + \epsilon, t + [\lambda]^M) \times \tau(x - m\epsilon, t' - [\lambda]^M) \epsilon \xi_R(t-t') \right\}, \tag{4.49}
\]

where

\[
\xi_L(t-t') = \sum_{n \geq 0} \sum_{\alpha=1}^{N} \frac{\Gamma(2 - \frac{\alpha-1}{\lambda})}{\Gamma(n + 2 - \frac{\alpha-1}{\lambda})} \frac{\lambda^{n+1-\frac{\alpha-1}{\lambda}}}{\epsilon} (t_{\alpha,n} - t'_{\alpha,n}),
\]

\[
\xi_R(t-t') = -\sum_{n \geq 0} \sum_{\beta=-M+1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} (t_{\beta,n} - t'_{\beta,n}).
\]

To better understand these identities, following special cases are given explicitly.

Similar to [21], we can choose other cases in different values of \( m, t, t' \) which lead to the following Fay-like identities:

**I.** \( m = 0, t' = t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N \). In this case the Hirota bilinear identity (4.49) will lead to

\[
\text{Res}_\lambda \left\{ \tau(x, t - [\lambda^{-1}]^N) \times \tau(x + \epsilon, t' + [\lambda^{-1}]^N) \frac{1}{(1 - \lambda \lambda_1^{-1})(1 - \lambda \lambda_2^{-1}) \lambda} \right\} \\
= \text{Res}_\lambda \left\{ \tau(x + \epsilon, t + [\lambda]^M) \times \tau(x, t' - [\lambda]^M) \frac{1}{\lambda} \right\}.
\]

Using

\[
(1 - \lambda_1^{-1})^{-1}(1 - \lambda_2^{-1})^{-1} = \frac{\lambda_1^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}}(1 - \lambda_1^{-1})^{-1} = \frac{\lambda_2^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}}(1 - \lambda_2^{-1})^{-1},
\]

we get

\[
\frac{\lambda_1^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_1^{-1}]^N) \tau(x + \epsilon, t' + [\lambda_1^{-1}]^N) - \frac{\lambda_2^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_2^{-1}]^N) \tau(x + \epsilon, t' + [\lambda_2^{-1}]^N) = \tau(x + \epsilon, t) \tau(x, t').
\]

It further leads to

\[
\frac{\lambda_1^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_1^{-1}]^N) \tau(x + \epsilon, t - [\lambda_2^{-1}]^N) - \frac{\lambda_2^{-1}}{\lambda_1^{-1} - \lambda_2^{-1}} \tau(x, t - [\lambda_2^{-1}]^N) \tau(x + \epsilon, t - [\lambda_1^{-1}]^N) = \tau(x + \epsilon, t) \tau(x, t - [\lambda_1^{-1}]^N - [\lambda_2^{-1}]^N).
\] \tag{4.50}

**II.** \( m = 0, t' = t + [\lambda]^M + [\lambda_2]^M \). In this case the Hirota bilinear identity (4.49) will lead to

\[
\text{Res}_\lambda \left\{ \tau(x, t - [\lambda^{-1}]^N) \times \tau(x + \epsilon, t' + [\lambda^{-1}]^N) \lambda^{-1} \right\} \\
= \text{Res}_\lambda \left\{ \tau(x + \epsilon, t + [\lambda]^M) \times \tau(x, t' - [\lambda]^M) \lambda^{-1} \frac{1}{(1 - \lambda^{-1}\lambda_1)(1 - \lambda^{-1}\lambda_2)} \right\}.
\]

Using

\[
(1 - \lambda^{-1}\lambda_1)^{-1}(1 - \lambda_2^{-1})^{-1} = \frac{\lambda_1}{\lambda_1 - \lambda_2}(1 - \lambda_1^{-1})^{-1} - \frac{\lambda_2}{\lambda_1 - \lambda_2}(1 - \lambda_2^{-1})^{-1},
\]
we get
\[ \tau(x, t) \tau(x + \epsilon, t') \]
\[ = \frac{\lambda_1}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_1]^M) \tau(x, t' - [\lambda_1]^M) - \frac{\lambda_2}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_2]^M) \tau(x, t' - [\lambda_2]^M). \]

It further leads to
\[ \tau(x, t) \tau(x + \epsilon, t + [\lambda_1]^M + [\lambda_2]^M) \]
\[ = \frac{\lambda_1}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_1]^M) \tau(x, t + [\lambda_2]^M) - \frac{\lambda_2}{\lambda_1 - \lambda_2} \tau(x + \epsilon, t + [\lambda_2]^M) \tau(x, t + [\lambda_1]^M). \] (4.51)

III. \( m = 1, t' = t - [\lambda_1^{-1}]^N + [\lambda_2]^M. \) In this case the Hirota bilinear identity (4.49) will lead to
\[ \text{Res}_\lambda \left\{ \tau(x, t - [\lambda^{-1}]^N) \times \tau(x, t' + [\lambda^{-1}]^N) \frac{1}{1 - \lambda \lambda_1^{-1}} \right\} \]
\[ = \text{Res}_\lambda \left\{ \tau(x + \epsilon, t + [\lambda]^M) \times \tau(x - \epsilon, t' - [\lambda]^M) \frac{1}{1 - \lambda^{-1} \lambda_2} \right\}, \]

which is equivalent to
\[ \lambda_1 \left( \tau(x, t - [\lambda_1^{-1}]^N) \tau(x, t' + [\lambda_1^{-1}]^N) - \tau(x, t) \tau(x, t') \right) = \lambda_2 \tau(x + \epsilon, t + [\lambda_2]^M) \tau(x - \epsilon, t' - [\lambda_2]^M). \]

It further implies
\[ \lambda_1 \left( \tau(x, t - [\lambda_1^{-1}]^N) \tau(x, t + [\lambda_2]^M) - \tau(x, t) \tau(x, t - [\lambda_1^{-1}]^N + [\lambda_2]^M) \right) \]
\[ = \lambda_2 \tau(x + \epsilon, t + [\lambda_2]^M) \tau(x - \epsilon, t - [\lambda_1^{-1}]^N). \] (4.52)

These identities can be used to prove the Adler-Shiota-van Moerbeke (ASvM) formula. We tried that but got stuck by some difficulty. We will omit it because the center of our consideration in this paper is the HBEs of the EBTH which will appear in the next section.

As the end of this section, we would like to show the close relations between tau function and dynamical functions \( w_i \) and \( \tilde{w}_i \) from Sato equation. Calculate the residue of eq.(2.27), it implies
\[ \partial_{a,n} w_1 = -\text{Res}_\Lambda B_{a,n}, \] (4.53)

where the residue is the coefficient of term \( \Lambda^{-1} \). According to eq.(4.2), we have
\[ P_L = 1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \cdots = 1 - \frac{\epsilon \partial_{N,0} \tau(x, t)}{\lambda \tau(x, t)} + \cdots, \] (4.54)

which implies \( w_1 = -\epsilon \partial_{N,0} \log \tau(x, t) \). Taking \( w_1 \) into eq.(4.53), we have
\[ \epsilon \partial_{a,n} \partial_{N,0} \log \tau(x, t) = \text{Res}_\Lambda B_{a,n}, \] (4.55)

Additionally, comparing the coefficient of \( \Lambda^0 \) on both sides of eq.(2.28), we can get
\[ \partial_{a,n} \tilde{w}_0 = \tilde{w}_0 \text{Res}_\Lambda [(B_{a,n}) \Lambda^{-1}]. \]

Further considering eq.(4.46) and replacing \( x - \frac{t}{2} \) by \( x \), we can get
\[ \partial_{a,n} \log \tilde{w}_0 = \partial_{a,n} \log \frac{\tau(x + \epsilon)}{\tau(x)} = \text{Res}_\Lambda [(B_{a,n}) \Lambda^{-1}]. \] (4.56)

The relations between tau function and other dynamical functions also can be given by tedious calculation from Sato equations.
5. The HBEs of the EBTH

In this section we continue to discuss the fundamental properties of the tau function, i.e., the Hirota bilinear equations. So we introduce the following vertex operators

\[
\Gamma^{\pm a} := \exp \left( \pm \sum_{n \geq 0} \frac{\Gamma(n+1 - \frac{\alpha}{N})}{\Gamma(n+2 - \frac{\alpha}{N})} \exp \left( \lambda \sum_{n \geq 0} \frac{\Gamma(n + 1 - \frac{\alpha}{N})}{\Gamma(n+2 - \frac{\alpha}{N})} \frac{\partial}{\partial t_{a,n}} \right) n! t_{-M,n} \right)
\]

\[
\times \exp \left( \mp \frac{\epsilon}{2} \partial_{-M,0} \mp [\lambda^{-1}]_{\beta} \right)
\]

\[
\Gamma^{\pm b} := \exp \left( \pm \sum_{n \geq 0} \frac{1}{\beta} \sum_{\beta = -M+1}^{0} \exp \left( \lambda \sum_{n \geq 0} \frac{1}{\beta} \sum_{\beta = -M+1}^{0} \frac{\Gamma(n + 1 + \frac{\beta}{M})}{\Gamma(n+2 + \frac{\beta}{M})} \frac{\partial}{\partial t_{\beta,n}} \right) n! t_{-M,n} \right)
\]

\[
\times \exp \left( \mp \frac{\epsilon}{2} \partial_{-M,0} \mp [\lambda]_{\beta} \right)
\]

where

\[
[\lambda^{-1}]_{\beta} = \sum_{n \geq 0} \sum_{\alpha = 1}^{N} \frac{\Gamma(n + 1 - \frac{\alpha}{N})}{\Gamma(n+2 - \frac{\alpha}{N})} \epsilon^{\lambda^{-N(n+1 - \frac{\alpha}{N})}} \frac{\partial}{\partial t_{a,n}},
\]

\[
[\lambda]_{\beta} = \sum_{n \geq 0} \sum_{\beta = -M+1}^{0} \frac{\Gamma(n + 1 + \frac{\beta}{M})}{\Gamma(n+2 + \frac{\beta}{M})} \epsilon^{\lambda^{M(n+1 + \frac{\beta}{M})}} \frac{\partial}{\partial t_{\beta,n}}.
\]

We can see that the coefficients of the vertex operators \( \Gamma^{\pm a} \otimes \Gamma^{\pm a} \) and \( \Gamma^{\pm b} \otimes \Gamma^{\pm b} \) are multi-valued function because of the logarithmic terms \( \log \lambda \). There are monodromy factors \( M^a \) and \( M^b \) respectively as following between two different ones in adjacent branches around \( \lambda = \infty \)

\[
M^a = \exp \left\{ \frac{2\pi i}{\epsilon} \sum_{n \geq 0} \frac{\lambda^{nN}}{n!} (t_{-M,n} \otimes 1 - 1 \otimes t_{-M,n}) \right\}, \tag{5.1}
\]

\[
M^b = \exp \left\{ \frac{2\pi i}{\epsilon} \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} (t_{-M,n} \otimes 1 - 1 \otimes t_{-M,n}) \right\}. \tag{5.2}
\]

In order to offset the complication we need to generalize the concept of vertex operators which leads it to be not scalar-valued any more. So we introduce the following vertex operators

\[
\Gamma_{a}^{\delta} = \exp \left( - \sum_{n \geq 0} \frac{\lambda^{nN}}{\epsilon n!} (\epsilon \partial_{x}) t_{-M,n} \right) \exp(x \partial_{-M,0}), \tag{5.3}
\]

\[
\Gamma_{b}^{\delta} = \exp \left( - \sum_{n \geq 0} \frac{\lambda^{-nM}}{\epsilon n!} (\epsilon \partial_{x}) t_{-M,n} \right) \exp(x \partial_{-M,0}). \tag{5.4}
\]

Then

\[
\Gamma_{a}^{\delta \#} \otimes \Gamma_{a}^{\delta} = \exp(x \partial_{-M,0}) \exp \left( \sum_{n \geq 0} \frac{\lambda^{nN}}{\epsilon n!} (\epsilon \partial_{x}) (t_{-M,n} - t'_{-M,n}) \right) \exp(x \partial'_{-M,0}), \tag{5.5}
\]

\[
\Gamma_{b}^{\delta \#} \otimes \Gamma_{b}^{\delta} = \exp(x \partial_{-M,0}) \exp \left( \sum_{n \geq 0} \frac{\lambda^{-nM}}{\epsilon n!} (\epsilon \partial_{x}) (t_{-M,n} - t'_{-M,n}) \right) \exp(x \partial'_{-M,0}). \tag{5.6}
\]
After computation we get

\[(\Gamma^\#_a \otimes \Gamma^\delta_a) M^a = \exp \left\{ \frac{2\pi i}{\epsilon} \sum_{n>0} \frac{\lambda^n N}{n!} (t_{-M,n} - t'_{-M,n}) \right\} \]

\[\exp \left( \frac{2\pi i}{\epsilon} ((t_{-M,0} + x) - (t'_{-M,0} + x + \sum_{n>0} \frac{\lambda^n N}{n!} (t_{-M,n} - t'_{-M,n})) \right) (\Gamma^\#_a \otimes \Gamma^\delta_a) \]

\[= \exp \left( \frac{2\pi i}{\epsilon} (t_{-M,0} - t'_{-M,0}) \right) (\Gamma^\#_a \otimes \Gamma^\delta_a) , \]

\[\left(\Gamma^\#_b \otimes \Gamma^\delta_b \right) M^b = \exp \left\{ \frac{2\pi i}{\epsilon} \sum_{n>0} \frac{\lambda^{-n M}}{n!} (t_{-M,n} - t'_{-M,n}) \right\} \]

\[\exp \left( \frac{2\pi i}{\epsilon} ((t_{-M,0} + x) - (t'_{-M,0} + x + \sum_{n>0} \frac{\lambda^{-n M}}{n!} (t_{-M,n} - t'_{-M,n})) \right) (\Gamma^\#_b \otimes \Gamma^\delta_b) \]

\[= \exp \left( \frac{2\pi i}{\epsilon} (t_{-M,0} - t'_{-M,0}) \right) (\Gamma^\#_b \otimes \Gamma^\delta_b) . \]

Thus when \(t_{-M,0} - t'_{-M,0} \in \mathbb{Z}\epsilon\), \((\Gamma^\#_a \otimes \Gamma^\delta_a) (\Gamma^a \otimes \Gamma^{-a})\) and \((\Gamma^\#_b \otimes \Gamma^\delta_b) (\Gamma^{-b} \otimes \Gamma^b)\) are all single-valued near \(\lambda = \infty\).

We will say that \(\tau\) satisfies the **HBEs of the EBTH** if

\[\text{Res}_\lambda \left( \lambda^{n r - 1} (\Gamma^\#_a \otimes \Gamma^\delta_a) (\Gamma^a \otimes \Gamma^{-a}) - \lambda^{-M r - 1} (\Gamma^\#_b \otimes \Gamma^\delta_b) (\Gamma^{-b} \otimes \Gamma^b) \right) (\tau \otimes \tau) = 0 \]  \hspace{1cm} (5.7)

computed at \(t_{-M,0} - t'_{-M,0} = m\epsilon\) for each \(m \in \mathbb{Z}, r \in \mathbb{N}\). Now we should note that the vertex operators take value in algebra \(A[[\epsilon]]\) whose element is like \(\sum_{i \geq 0} c_i(x, t, \epsilon) \partial^i\).

**Theorem 5.1.** Function \(\tau(t, \epsilon)\) is a tau-function of the extended bigraded Toda hierarchy at a certain spatial point if and only if it satisfies the Hirota bilinear equations \(\text{[5.7]}\).

**Proof.** Note that the \(\tau(t)\) now is independent of variable \(x\) because the \(x\) takes a fixed value, for example \(x = x_0\) (constant). However, in the following proof, \(x\) will appear in the \(\tau(t)\) due to the action of vertex operator on \(\tau(t)\). For example, \(e^{x \partial_{-M,0} \tau(t)} = \tau(t_{-M,0} + x, \bar{t})\) where \(\bar{t}\) is just as the definition in the proof of the existence of tau function.

We just need to prove that the HBEs are equivalent to the right side in Proposition \(\text{[3.1]}\). By a straightforward computation we can get the following four identities

\[\Gamma^\#_a \Gamma^\delta_a \tau = \tau(t_{-M,0} + x - \epsilon/2, \bar{t}) \lambda^{t_{-M,0}/\epsilon} W_L(t, t, \epsilon \partial_x, \lambda) \lambda^{x/\epsilon} , \]  \hspace{1cm} (5.8)

\[\Gamma^\#_a \Gamma^{-a} \tau = \lambda^{-t_{-M,0}/\epsilon} \lambda^{-x/\epsilon} W^{-1}_L(t, t, \epsilon \partial_x, \lambda) \tau(t + t_{-M,0} + \epsilon/2, \bar{t}) , \]  \hspace{1cm} (5.9)

\[\Gamma^\#_b \Gamma^{-b} \tau = \tau(t + t_{-M,0} - \epsilon/2, \bar{t}) \lambda^{t_{-M,0}/\epsilon} W^{-1}_R(t, t, \epsilon \partial_x, \lambda) \lambda^{x/\epsilon} , \]  \hspace{1cm} (5.10)

\[\Gamma^\#_b \Gamma^\delta_b \tau = \lambda^{-t_{-M,0}/\epsilon} \lambda^{-x/\epsilon} W^{-1}_R(t, t, \epsilon \partial_x, \lambda) \tau(t + t_{-M,0} + \epsilon/2, \bar{t}) . \]  \hspace{1cm} (5.11)

Here \(\bar{t}\) is denoted as all the other time variables except \(t_{-M,0}\). We should note that we take the left side of eq.(5.8)-eq.(5.11) not as functions but operators involving \(e^{\partial_x}\). We should pay more attention to the different operations of the operators \(\partial_x, \partial_{\alpha,n}\) and \(\partial_{\beta,m}\), for example,
The above formula shows the relationship between $e^{\theta_x}$ and $\tau(t)$ is a product of operators, but the relationship between $e^{\theta_{a,n}}$ (or $e^{\theta_{b,n}}$) and $\tau(t)$ is a action of the former on the latter.

For simplifying the proof, we first introduce following operators,

\begin{align}
D &= \sum_{n \geq 0} \sum_{\alpha = 1}^{N} \frac{\Gamma(2 - \frac{\alpha - 1}{N}) \lambda^{N(n+1 - \frac{\alpha - 1}{N})}}{\Gamma(n + 2 - \frac{\alpha - 1}{N})} \frac{\frac{1}{2}(1 + \frac{1}{N})C_n}{\epsilon} t_{\alpha,n}, 
\end{align}

\begin{align}
E &= \sum_{n > 0} \frac{\lambda^{nN}}{n!} (\log \lambda - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_n) \frac{t_{-M,n}}{\epsilon}, 
\end{align}

then, with the help of above identities eq. (5.12) and eq. (5.13), the left hand side of identity eq. (5.8) can be expressed by

\begin{align}
\Gamma^a \# \Gamma^n \tau &= \exp(x\partial_{-M,0}) \exp \left( \sum_{n > 0} \frac{\lambda^{nN}}{\epsilon n!} (\epsilon \partial_x) t_{-M,n} \right) \times \\
&\exp \left\{ \sum_{n \geq 0} \left[ \sum_{\alpha = 1}^{N} \frac{\Gamma(2 - \frac{\alpha - 1}{N}) \lambda^{N(n+1 - \frac{\alpha - 1}{N})}}{\Gamma(n + 2 - \frac{\alpha - 1}{N})} \frac{\frac{1}{2}(1 + \frac{1}{N})C_n}{\epsilon} t_{\alpha,n} + \frac{\lambda^{nN}}{n!} (\log \lambda - \frac{1}{2}(\frac{1}{M} + \frac{1}{N})C_n) \frac{t_{-M,n}}{\epsilon} \right] \right\} \\
&\times \exp \left\{ -\frac{\epsilon}{2} \partial_{-M,0} - [\lambda^{-1}]_0 \right\} \tau(t; \epsilon) \\
&= \exp\{D\} \exp\{E\} \exp\{x\partial_{-M,0}\} \exp\left( \sum_{n > 0} \frac{\lambda^{nN}}{\epsilon n!} (\epsilon \partial_x) t_{-M,n} \right) \\
&\exp\{ (\log \lambda) \frac{t_{-M,0}}{\epsilon} \} \tau(t_{-M,0} - \frac{\epsilon}{2}, \bar{t} - [\lambda^{-1}]_0) \\
&= \tau(t_{-M,0} + x - \frac{\epsilon}{2}, \bar{t} - [\lambda^{-1}]_0) \exp\{D\} \exp\{E\} \exp\{ (\log \lambda) \frac{t_{-M,0} + x}{\epsilon} \} \\
&\exp\left( \sum_{n > 0} \frac{\lambda^{nN}}{\epsilon n!} (\epsilon \partial_x) t_{-M,n} \right).
\end{align}

Taking eq. (5.14) into it, then substituting $D$ and $E$ by eq. (5.15) and eq. (5.16),
\[ \Gamma_\alpha^a \Gamma^a \tau = \tau(t_{-M,0} + x - \epsilon/2, \bar{\ell}) P_L(x, t, \lambda) \exp \left( \sum_{n>0} \frac{\lambda^{nN}}{en!} (\epsilon \partial_x) t_{-M,n} \right) \]

\[ \exp \left\{ \sum_{n \geq 0} \sum_{\alpha=1}^N \frac{\Gamma(2 - \frac{\alpha-1}{N})}{\Gamma(n + 2 - \frac{\alpha-1}{N})} \frac{\lambda^{n(n+1) - \frac{\alpha-1}{N}}}{\epsilon} t_{\alpha,n} + \sum_{n>0} \frac{\lambda^{nN}}{n!} \left( \log \lambda - \frac{1}{2} \frac{1}{M} + \frac{1}{N} C_n \right) \right\} \]

\[ \exp \left\{ \log \lambda \right\} \left[ t_{-M,0} + \left( \sum_{n>0} \frac{\lambda^{nN}}{n!} t_{-M,n} \right) + x \right] / \epsilon \}

\[ = \tau(t_{-M,0} + x - \epsilon/2, \bar{\ell}) P_L(x, t, \lambda) \times \]

\[ \exp \left\{ \sum_{n \geq 0} \sum_{\alpha=1}^N \frac{\Gamma(2 - \frac{\alpha-1}{N})}{\Gamma(n + 2 - \frac{\alpha-1}{N})} \frac{\lambda^{n(n+1) - \frac{\alpha-1}{N}}}{\epsilon} t_{\alpha,n} + \sum_{n>0} \frac{\lambda^{nN}}{n!} \left( \epsilon \partial_x - \frac{1}{2} \frac{1}{M} + \frac{1}{N} C_n \right) t_{-M,n} \right\} \]

\[ \exp \left\{ \log \lambda \right\} (t_{-M,0} + x) / \epsilon \}

\[ = \tau(t_{-M,0} + x - \epsilon/2, \bar{\ell}) \lambda^{t_{-M,0} - \epsilon} W_L(x, t, \epsilon \partial_x, \lambda) \lambda^{x/\epsilon}. \]

The other three identities are derived in similar way which will be shown in detail in the appendix.

By substituting four equations eq.(5.8)-eq.(5.11) into the HBEs (5.7) we find:

\[ \text{Res}_\lambda \left\{ \lambda^{N^r-1} \Gamma_{\delta} \# \Gamma^a \tau \otimes \Gamma^a \Gamma_{-\delta} - \lambda^{-M^r-1} \Gamma_{\delta} \# \Gamma_{-\delta} \tau \otimes \Gamma^b \Gamma_{-\delta} \right\} \]

\[ = \text{Res}_\lambda \left\{ \tau(x - \epsilon/2, t) \lambda^{N^r-1} \lambda^{(t_{-M,0} - t_{-M,0} - \epsilon)/\epsilon} W_L(x, t, \epsilon \partial_x, \lambda) W_{L}^{-1}(x, \epsilon \partial_x, \lambda) \tau(x + \epsilon/2, t) \right\} \]

\[ - \tau(x - \epsilon/2, t) \lambda^{-M^r-1} \lambda^{(t_{-M,0} - t_{-M,0} - \epsilon)/\epsilon} W_R(x, t, \epsilon \partial_x, \lambda) W_{R}^{-1}(x, \epsilon \partial_x, \lambda) \tau(x + \epsilon/2, t) \right\} . \]

Note here \( \tau(x - \frac{\epsilon}{2}, t) = \tau(t_{-M,0} + x - \frac{\epsilon}{2}, \bar{\ell}) \) as eq.(5.45). Let \( t_{-M,0} - t_{-M,0} = m\epsilon \) and consider that \( W_L(x, t, \epsilon \partial_x, \lambda), W_{L}^{-1}(x, \epsilon \partial_x, \lambda) \) and \( W_R(x, t, \epsilon \partial_x, \lambda), W_{R}^{-1}(x, \epsilon \partial_x, \lambda) \) are all not scaled-valued but take values in the algebra of differential operator. Therefore the HBEs lead to

\[ \text{Res}_\lambda \left\{ \lambda^{m+N^r-1} W_L(x, t_{-M,0}, \epsilon \partial_x, \lambda) W_{L}^{-1}(x, t_{-M,0} - m\epsilon, \epsilon \partial_x, \lambda) - \lambda^{m-M^r-1} W_R(x, t_{-M,0}, \epsilon \partial_x, \lambda) W_{R}^{-1}(x, t_{-M,0} - m\epsilon, \epsilon \partial_x, \lambda) \right\} = 0, \]

which can also be written as

\[ \text{Res}_\lambda \left\{ \lambda^{m+N^r-1} W_L(x, t_{-M,0}, \epsilon \partial_x, \lambda) W_{L}^{-1}(x - m\epsilon, t_{-M,0}, \epsilon \partial_x, \lambda) - \lambda^{m-M^r-1} W_R(x, t_{-M,0}, \epsilon \partial_x, \lambda) W_{R}^{-1}(x - m\epsilon, t_{-M,0}, \epsilon \partial_x, \lambda) \right\} = 0. \]

This is just eq.(3.5). So the proof is finished. \[ \square \]

6. THE HBEs OF BTH

Excluding the variables of \( t_{-M,n}, n \geq 1 \), we obtain a Hirota bilinear equations for the bigraded Toda hierarchy (BTH). Similar to EBTH, we introduce the following vertex operators

\[ \Gamma^{\pm c} = \exp \left\{ \pm \left[ \sum_{\alpha=1}^N \frac{\Gamma(2 - \frac{\alpha-1}{N})}{\Gamma(n + 2 - \frac{\alpha-1}{N})} \frac{\lambda^{n(n+1) - \frac{\alpha-1}{N}}}{\epsilon} \right] \left[ t_{\alpha,n} \right] \right\} \times \exp \left\{ \frac{\epsilon}{2} \partial_{-M,0} + \sum_{n \geq 0} \left[ \sum_{\alpha=1}^N \frac{\Gamma(n + 1 - \frac{\alpha-1}{N})}{\Gamma(2 - \frac{\alpha-1}{N})} \lambda^{-N(n+1) - \frac{\alpha-1}{N}} \partial_{t_{\alpha,n}} \right] \right\} , \]
\[ \Gamma^{\pm d} = \exp \left\{ \pm \sum_{\beta = -M+1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} \right\} \times \exp \left\{ \mp \frac{\epsilon}{2} \partial_{-M,0} \mp \sum_{n \geq 0} \left[ \sum_{\beta = -M+1}^{0} \frac{\Gamma(n + 1 + \frac{\beta}{M})}{M\Gamma(2 + \frac{\beta}{M})} \lambda^{M(n+1+\frac{\beta}{M})} \frac{\partial}{\partial t_{\beta,n}} \right] \right\} \right. 

In this case, because there is no logarithmic term in the vertex operators, so we need not generalize the vertex operators. Just as a result of that, the vertex operator will take values in scaled function of \( \epsilon, t, \lambda \).

**Corollary 6.1.** A non-vanishing function \( \tau(t_{-M,0}; t_{\alpha,n}, \ldots; \epsilon) \) is a tau-function of the BTH \((t_{-M,n}, n \geq 1 \text{ excluded})\) if and only if for each \( m \in \mathbb{Z}, r \in \mathbb{N} \),

\[
\text{Res}_{\lambda} \left\{ \lambda^{Nr+m-1} \Gamma^c \otimes \Gamma^{-c} - \lambda^{-Mr+m-1} \Gamma^{-d} \otimes \Gamma^d \right\} (\tau \otimes \tau) = 0, \tag{6.1}
\]

when \( t_{-M,0} - t'_{-M,0} = m\epsilon \).

**7. Conclusions and Discussions**

In previous sections, we have succeeded in extending Sato theory to the EBTH. Starting from the revised definition of the Lax equations, we have given Sato equations, wave operators, Hirota bilinear identities related to the wave operators, the existence of the tau function and its important properties including Fay-like identities and Hirota bilinear equations. In particular, this hierarchy deserves further studying and exploring because of its potential applications in topological quantum fields and Gromov-Witten theory. Our main support of this statement currently is that ETH describes the Gromov-Witten invariants of \( CP^1 \). We would like to point out that our Lax equations are revised from Carlet’s result [15], but our proof on the existence of the tau functions is more transparent than it.

Our future work will contain the applications of this kind of HBEs in the topological fields theory and string theory, the virasoro constraint of EBTH from the point of string equation and ASvM formula.

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8. Appendix

Proof of the identities eq.(5.9)-eq.(5.11):
By a similar calculation of eq.(5.8), we have
\[
\Gamma_a^\delta a^{-a_T} = \exp\left( - \sum_{n>0} \frac{\lambda}{e^n!} (\epsilon \partial_x) t_{-M,n} \right) \times \exp \left( x \partial_{-M,0} \right)
\]
\[
\exp \left\{ - \sum_{n \geq 0} \sum_{\alpha = 1}^{N} \frac{\Gamma(2 - \frac{\alpha}{N})}{\Gamma(n + 2 - \frac{\alpha}{N})} \frac{\lambda^{N(n+1-\frac{\alpha}{N})}}{\epsilon} t_{\alpha,n} + \frac{\lambda^{n} N}{n!} \left( \log \lambda - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_{n} \right) \right\} \\
\times \exp \left\{ \frac{\epsilon}{2} \partial_{-M,0} + \left[ \lambda^{-1} \right]^{N} \right\} (t)
\]

\[
= \exp \left\{ \left[ - \sum_{n \geq 0} \sum_{\alpha = 1}^{N} \frac{\Gamma(2 - \frac{\alpha}{N})}{\Gamma(n + 2 - \frac{\alpha}{N})} \frac{\lambda^{N(n+1-\frac{\alpha}{N})}}{\epsilon} t_{\alpha,n} - \sum_{n > 0} \frac{\lambda^{n} N}{n!} \left( \epsilon \partial_{x} - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_{n} \right) \right] \right\} (t + t_{-M,0} + \epsilon/2, \bar{t} + [\lambda^{-1}])
\]

\[
= \lambda^{-\frac{t_{-M,0} + \epsilon}{\epsilon}} \exp \left\{ \left[ - \sum_{n \geq 0} \sum_{\alpha = 1}^{N} \frac{\Gamma(2 - \frac{\alpha}{N})}{\Gamma(n + 2 - \frac{\alpha}{N})} \frac{\lambda^{N(n+1-\frac{\alpha}{N})}}{\epsilon} t_{\alpha,n} - \sum_{n > 0} \frac{\lambda^{n} N}{n!} \left( \epsilon \partial_{x} - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_{n} \right) \right] \right\} (t + t_{-M,0} + \epsilon/2, \bar{t})
\]

So eq. (5.9) is proved.

For the convenience of the proof of eq. (5.10), we introduce an identity

\[
\exp \left\{ x \partial_{-M,0} \right\} \exp \left\{ \sum_{n > 0} \frac{\lambda^{-n \frac{M}{N}}}{\epsilon n!} (\epsilon \partial_{x}) t_{-M,n} \right\} \exp \left\{ (\log \lambda) \frac{t_{-M,0}}{\epsilon} \right\}
\]

\[
= \exp \left\{ \sum_{n > 0} \frac{\lambda^{-n \frac{M}{N}}}{\epsilon n!} (\epsilon \partial_{x}) t_{-M,n} \right\} \exp \left\{ (\log \lambda) \frac{t_{-M,0}}{\epsilon} + x - \sum_{n > 0} \frac{\lambda^{-n \frac{M}{N}}}{\epsilon n!} t_{-M,n} \right\},
\]

and define two operators

\[
F = \sum_{n \geq 0} \left[ \sum_{\beta = -M+1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} \right],
\]

\[
G = \sum_{n > 0} \frac{\lambda^{-n \frac{M}{N}}}{n!} \left( - \log \lambda - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_{n} \right) \frac{t_{-M,n}}{\epsilon},
\]

then

\[
\Gamma_{b}^{\delta \#} \Gamma_{b}^{-\#}
\]

\[
= \exp \left( x \partial_{-M,0} \right) \times \exp \left( \sum_{n > 0} \frac{\lambda^{-n \frac{M}{N}}}{\epsilon n!} (\epsilon \partial_{x}) t_{-M,n} \right)
\]

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\[
\exp \left\{ - \sum_{n \geq 0} \left[ \sum_{\beta = -M + 1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} + \frac{\lambda^{-nM}}{n!} \left( - \log \lambda - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-M,n}}{\epsilon} \right] \right\} \\
\times \exp \left\{ \frac{\epsilon}{2} \partial_{-M,0} + [\lambda^{M}] \right\} \tau(t; \epsilon)
\]

Using identity eq. (8.1), then substituting \( F \) and \( G \) given by eq. (8.2) and eq. (8.3), we have

\[
\Gamma^b_b \Gamma^{-b}_b \tau = \tau(x + t_{-M,0} + \epsilon/2, \bar{t} + [\lambda^{M}])
\]

\[
\exp \left\{ - \sum_{n \geq 0} \left[ \sum_{\beta = -M + 1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} - \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} \left( - \epsilon \partial_x - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-M,n}}{\epsilon} \right] \right\}
\]

\[
\exp \left\{ \sum_{n > 0} \frac{\lambda^{-nM}}{n!} t_{-M,n} \log \lambda \right\} \exp \left\{ (\log \lambda)(t_{-M,0} + x - \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} \epsilon t_{-M,n})/\epsilon \right\}
\]

\[
= \tau(x + t_{M,0} - \epsilon/2, \bar{t}) P_R(x, t, \lambda)
\]

\[
\exp \left\{ - \sum_{n \geq 0} \left[ \sum_{\beta = -M + 1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} + \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} \left( \epsilon \partial_x + \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-M,n}}{\epsilon} \right] \right\}
\]

\[
\lambda^{\frac{t_{-M,0} + x}{\epsilon}}
\]

which is eq. (5.11).

Similarly, to prove eq. (5.11), we have

\[
\Gamma^b_b \Gamma^{-b}_p \tau
\]

\[
= \exp \left\{ \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} \left( \epsilon \partial_x \right) t_{-M,n} \right\} \times \exp \left\{ \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} \left( \epsilon \partial_x \right) t_{-M,n} \right\}
\]

\[
\times \exp \left\{ - \frac{\epsilon}{2} \partial_{-M,0} - [\lambda^{M}] \right\} \tau(t)
\]

\[
= \exp \left\{ - \sum_{n \geq 0} \left[ \sum_{\beta = -M + 1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} + \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} \left( - \epsilon \partial_x - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-M,n}}{\epsilon} \right] \right\}
\]

\[
\exp \left\{ - \sum_{n > 0} \frac{\lambda^{-nM}}{n!} \epsilon \log \lambda \right\} \exp \left\{ \left( - \log \lambda \right)(t_{-M,0} + x)/\epsilon \right\} \tau(x + t_{-M,0} - \epsilon/2, \bar{t} - [\lambda^{M}])
\]

\[
= \lambda^{\frac{t_{-M,0} + x}{\epsilon}} \exp \left\{ - \sum_{n \geq 0} \left[ \sum_{\beta = -M + 1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} + \sum_{n \geq 0} \frac{\lambda^{-nM}}{n!} \left( - \epsilon \partial_x - \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) C_n \right) \frac{t_{-M,n}}{\epsilon} \right] \right\}
\]
\[
\begin{align*}
\frac{t_{-M,n}}{\epsilon} \left\{ \tau(x + t_{-M,0} - \epsilon/2, \bar{t}) - [\lambda^{-1}]^M \right\} \\
= \lambda^{-t_{-M,0} + \epsilon} \exp \left\{ \sum_{n \geq 0} \sum_{\beta = -M + 1}^{0} \frac{\Gamma(2 + \frac{\beta}{M})}{\Gamma(n + 2 + \frac{\beta}{M})} \frac{\lambda^{-M(n+1+\frac{\beta}{M})}}{\epsilon} t_{\beta,n} + \sum_{n > 0} \frac{\lambda^{-nM}}{n!} \right. \\
\left. \left( -\epsilon \partial_x + \frac{1}{2} \left( \frac{1}{M} + \frac{1}{N} \right) \right) \right\}
\end{align*}
\]