Voltage fluctuations on a superconductor grain attached to a quantum wire

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(October 15, 1998)

When a finite superconductor is in contact with a 1D normal conductor, superconducting phase fluctuations lead to power-law response of the normal subsystem. As a result, the charge fluctuations on the superconductor at zero temperature have logarithmic correlator and a \(1/\omega^2\) power spectrum \((1/f\text{-noise})\). At higher temperatures \(1/\omega\) is pushed to the high frequency region, and \(1/\omega^2\) behavior prevails.

Correlated character of transport in mesoscopic systems strongly influences character of current and charge fluctuations there. Examples are non-Poissonian shot noise in quantum point contacts and NS junctions (and references therein) and giant current noise in superconducting quantum point contacts. Noise measurements may provide information about quantum correlations unattainable by any other means.

In this paper we will show that equilibrium fluctuations in a mesoscopic NS system reveal the many-body reaction of a conducting system to a sudden external perturbation, thus linking the noise to such phenomena as Fermi edge singularity (FES) and Anderson orthogonality catastrophe. Unlike the standard situation when the perturbation is due to creation of a scatterer, here the normal subsystem reacts to the change in the off-diagonal pairing potential in the superconductor, to which it is coupled via Andreev reflection processes on the NS boundary.

More specifically, consider the system shown in Fig.1. It consists of a mesoscopic superconducting grain \(S\) in contact with a 1D normal wire \(N\). We assume that the grain contains a large number of Cooper pairs, \(\langle n \rangle \gg 1\), and its capacitance \(C\) is large enough (so we are in the opposite limit to the Coulomb blockade regime), but the contact region is smaller than the Josephson screening length \(\lambda_J\), so that both the modulus and phase of the order parameter \(\Delta = |\Delta|e^{i\phi}\) can be taken constant there. Temperature is low enough, \(T \ll |\Delta|\), to neglect quasiparticle excitations in the superconductor. Finally, we assume that the 1D conductor is scattering free, and its length \(L \gg \xi_0\), where \(\xi_0\) is the superconducting coherence length.

If isolated, the superconducting phase in the grain would fluctuate, the system being in the state with fixed particle number (see e.g. Ch.7). To the 1D conductor brought in contact with the grain, these phase fluctuations will translate into fluctuating Andreev boundary condition.

First let us present a heuristic picture of the phenomenon at zero temperature. The effect of a sudden change in a boundary condition is a slow, power-law relaxation of the normal system to its new ground state, that is, Fermi edge singularity. It is conveniently described in terms of boundary conformal field theory, which allows a unified description of FES and Anderson orthogonality catastrophe. Both are characterized by scaling dimension \(x\) of the boundary condition changing operator \(O\), so that the correlation function \(\langle O(\tau)O(0)\rangle \sim \tau^{-2x}\), and the dimension \(x\) itself is related to the finite-size \(O(1/L)\) correction to the ground state energy shift due to the action of \(O\):

\[
x = \frac{L}{\pi} \Delta E_i / L.
\]

Hereafter we set \(\hbar = 1\) and the velocity of the excitations (Fermi velocity in the non-interacting case) \(v = 1\) unless stated otherwise; \(L\) is the characteristic size of the system (wire length in our case). This relation holds both for usual potential scattering and Andreev scattering.

The boundary condition changing operator in our case changes the phase of superconductor by \(\Phi\): \(O_\Phi = \exp[\Phi \frac{\partial}{\partial \Phi}] = \exp [i\Phi \hat{n}]\). Here we used the relation between \(\phi\) and the Cooper pair number operator on the grain, \(\hat{n} = \frac{1}{\sqrt{2}} \hat{\delta} \hat{n}\), which holds if \(\langle n \rangle \gg 1\). In agreement with general CFT argument, the Green’s function of \(O_\Phi\) is

\[
G_\Phi(\tau) = \langle \hat{O}_\Phi^\dagger(\tau)O_\Phi(0) \rangle \equiv \langle e^{-i\Phi \hat{n}(\tau)} e^{i\Phi \hat{n}(0)} \rangle = (\tau/\tau_0)^{-2x},
\]

with some cutoff parameter \(\tau_0\). Though by itself \(G_\Phi\) has no direct physical meaning, it is related to the correlator of charge fluctuations on the grain: \(K_Q(\tau) = \langle (\delta \hat{n}(\tau), \delta \hat{n}(0))_+ \rangle / 2\), where \(\delta \hat{n} = \hat{n} - \langle \hat{n} \rangle\). Obviously,

\[
\frac{K_Q(\tau) - K_Q(0)}{4e^2} = \frac{1}{2} \langle [\delta \hat{n}(\tau), \delta \hat{n}(0)]_+ \rangle - \langle \delta \hat{n}^2 \rangle = \frac{1}{2} \langle [\delta \hat{n}(\tau), \delta \hat{n}(0)]_+ \rangle - \langle \hat{n}^2 \rangle = \lim_{\Phi \to 0} \frac{1}{\Phi^2} \langle G_\Phi(\tau) + G_\Phi(\tau)^* - 1 \rangle.
\]

Charge fluctuations are measured by e.g. monitoring the electrostatic potential of the grain.

The dimension \(x\) is easily found if notice that the energy correction entering is simply the Josephson energy of a ballistic 1D SNS junction of length \(L\), with
phase difference \( \Phi \) at zero temperature: \( E_J(\Phi) = \Phi^2/(4\pi L) \), so that \( x = LE_J(\Phi)/\pi = [\Phi/(2\pi)]^2 \). Substituting this and (2) into (3), we find for the charge fluctuations on the grain

\[
\frac{K_Q(\tau)}{4e^2} = \langle \delta n^2 \rangle - \frac{1}{2\tau^2} \ln \frac{\tau}{\tau_0}.
\]

For the spectral density of charge fluctuations, \( S_Q(\omega) = 2 \int_0^\infty K_Q(\tau) \cos(\omega \tau) d\tau \), such logarithmic behavior of correlator translates into \( S_Q(\omega) \propto 1/\omega (1/f\text{-noise}) \) in the corresponding frequency interval; it is often observed in spin glasses, magnetic polycrystals and other systems with wide distribution of relaxation times [4-13]. In our case the source of slow relaxation times is the normal system with its power-law approach to the new ground state. The idea of invoking infrared singularity to explain scaling that underlies current 1/f-noise observed in normal conductors was exploited in e.g. model of “quantum 1/f-noise” due to Bremsstrahlung, but it encountered fatal difficulties (see [13] and references therein). In contrast, we see no fundamental difficulty in the present mechanism, which will be shown to be consistent with fluctuation-dissipation theorem.

Expression (3) is obviously only an intermediate asymptotics for \( 1 \ll \tau/\tau_0 \ll \exp[2\pi^2 \langle \delta n^2 \rangle] \). Therefore we need a more consistent approach, taking into account effects of charging energy and finite temperature. We will see that the region of 1/f-fluctuations is pushed by thermal effects to high frequencies, and 1/\( \omega^2 \)-type spectrum prevails except at very low temperature.

We will describe the wire in terms of Tomonaga-Luttinger liquid (TLL), which allows us to treat simultaneously non-interacting (1D Fermi-liquid) and interacting case of such S-TLL system [16]. Though as we will see interactions do not change qualitative picture of the phenomenon, they may lead to interesting side effects.

Spin and charge degrees of freedom in TLL separate. Since spin is totally reflected by NS interface both in the process of ideal Andreev reflection and ideal normal reflection, it is irrelevant and will be dropped. For the charge, the Lagrangian density is \( \mathcal{L} = 1/(2\pi K)|\partial_x \theta|^2 - (\partial_x \theta)^2 \), where \( K \) is the coupling constant, which is unity for non-interacting electrons.

Physically, the boson field \( \theta(x, t) \) corresponds to the phase of Charge Density Wave (CDW), while its dual \( \phi(x, t) \) to the superconducting phase of the Cooper pair \( \Theta \). For a perfect NS interface, the boundary value of the superconducting phase of the TLL should be equal to that of the superconductor, which in the absence of phase fluctuations gives a Neumann boundary condition for the field \( \theta(x, t) \) in the wire [14], equivalent to a Dirichlet boundary condition on \( \phi(0, t) = \Phi = \text{const.} \).

The charge \( Q \) on the grain can be related to the boundary value of the “CDW” field \( \theta \). Indeed, the current in TLL can be written as

\[
J(x, t) = -\frac{\sqrt{2e}}{\pi} \frac{\partial \theta(x, t)}{\partial t}.
\]

Therefore the electric charge of the superconductor at time \( t \) is given by

\[
Q(t) = Q(-\infty) - \int_{-\infty}^t J(0, t) dt = -\frac{\sqrt{2e}}{\pi} \theta(0, t),
\]

where we have defined \( Q \) and \( \theta \) so that \( Q = \theta(0, t) \) corresponds to the neutral superconductor grain. Similar identification was used in studies of Coulomb Blockade [15], although our application is to quite different regime.

Including the charging energy associated with \( Q \), \( E_Q = \frac{2e^2}{\pi c^2} \theta(0, t)^2 \), we arrive at the effective action of the problem:

\[
S = \int \int dx \frac{1}{2} (\partial_x \theta)^2 - \int dt \frac{u}{2} \Theta(0, t)^2.
\]

Here \( \Theta = \theta/\sqrt{\pi K} \) and \( u = 2 e^2 k/(\pi c) \) (\( u = 2 e^2 k/(\pi c) \) in conventional units). This is a free boson action with a mass term only on the boundary, and this effective theory is exactly solvable. The calculation of the charge fluctuation in the superconductor is reduced to the calculation of the correlation function of the boundary field \( \Theta(0, t) \). The physics of the problem is contained in a proper choice of the boundary condition at \( x = 0 \) (on the NS boundary) [14].

The Matsubara Green’s function of \( \Theta \) (for \( \tau = i\tau \)) \( \mathcal{G}(x, \tau) = -(T_r \Theta(x, \tau) \Theta(0, 0)) \), is the Green’s function of the Laplacian on the plane, with the boundary conditions \( \partial_x \Theta(x = 0, \tau) = u \Theta(x = 0, \tau) \) and \( \Theta(x, \tau + \beta) = \Theta(x, \tau) \), where \( \beta = 1/T \) is the inverse temperature. As a consequence, the Green’s function \( \mathcal{G} \) for the source at \( (x_0, 0) \) satisfies

\[
(\partial_x^2 + c^2) \mathcal{G}(x, \tau) = \delta(x - x_0) \delta(\tau) \quad \text{(8)}
\]

\[
\partial_x \mathcal{G}(x = 0, \tau) = u \mathcal{G}(x = 0, \tau). \quad \text{(9)}
\]

Because we are interested in the correlation function of the boundary, we take the limit \( x_0 \to 0 \). Integrating (8) over \( x \) from \( 0 \) to \( x_0 + 0 \), we see that the boundary condition (8) must be now modified:

\[
\partial_x \mathcal{G}(x = 0, \tau) = u \mathcal{G}(x = 0, \tau) - \delta(\tau). \quad \text{(10)}
\]

For Fourier components we obtain an ordinary differential equation:

\[
\partial^2 \mathcal{G}(x, \omega_n) = \omega_n^2 \mathcal{G} \quad \text{with the boundary condition} \quad \partial_x \mathcal{G}(x = 0, \omega_n) = u \mathcal{G}(x = 0, \omega_n) - 1.
\]

Imposing the asymptotic condition \( \lim_{x \to \infty} \mathcal{G}(x, \tau) \to 0 \), we find

\[
\mathcal{G}(x, \omega_n) = \frac{1}{u + |\omega_n|} e^{-|\omega_n| x}. \quad \text{(11)}
\]

The Matsubara Green’s function for the boundary field is \( \mathcal{G}(x = 0, \omega_n) = -1/(u + |\omega_n|) \).
The correlation functions in the real time are obtained by an analytic continuation from the Matsubara Green’s function. In a standard way, we obtain the retarded and advanced Green’s functions, $G^{R(A)}(\omega) = -1/(u \mp i\omega)$. Their mismatch at the real axis, which is pure imaginary, gives the spectral density of the system: $\rho(\omega) = -2\text{Im}G^R(\omega) = 2\omega/(u^2 + \omega^2)$. The spectrum of electrostatic potential fluctuations on the grain is given by:

$$S_V(\omega) = \frac{S_Q(\omega)}{C^2} = \int_{-\infty}^{\infty} \frac{1}{2C^2} (Q(t)Q(0) + Q(0)Q(t)) e^{i\omega t} dt = \frac{2ke^2}{\pi e^2 \rho(\omega)} \frac{1}{2} \coth \left( \frac{\beta \omega}{2} \right) = \frac{2k^2 e^2}{\pi C^2} \frac{\omega}{u^2 + \omega^2} \coth \left( \frac{\beta \omega}{2} \right). \tag{12}$$

This result has the form expected from fluctuation-dissipation theorem (FDT).

In the low temperature/high frequency limit $\beta \omega \gg 1$, the spectrum is proportional to $|\omega|/(u^2 + \omega^2)$. At frequency much higher than the “infrared cutoff” $u$, the spectrum is indeed $1/\omega$ as expected from the heuristic argument \cite{1}. On the other hand, at smaller $\omega$ fluctuations are suppressed, because finite capacitance obviously excludes large fluctuation of the charge in the grain. The $1/\omega$ spectrum, which is independent of the interaction parameter $K$. It automatically arises from the free boson field theory once the boson field itself is identified with a physical observable, because the correlation function of the boson field is logarithmic. We have thus shown an example of “quantum $1/f$ noise” naturally derived within the framework of equilibrium statistical mechanics and field theory. Although the $1/\omega$ behavior is only expected at very low temperature and high frequency, we believe this presents certain interest both from the point of view of translation of infrared divergences into $1/f$-spectrum, and of its destruction by thermal fluctuations.

At higher temperature/lower frequency $\beta \omega \gg 1$, the spectrum is proportional to $T/(u^2 + \omega^2)$. The spectrum is (restoring the fundamental constants) $4kTke^2/\pi u^2$ for frequency larger than inverse capacitance $\omega \gg u$. In the opposite limit of small frequency $\omega \ll u$, (but still $\omega/\beta \gg 1$), $\omega$ fluctuation is simply a white noise with the magnitude $2RHkT/(4K)$ where $RH = h/e^2$, again in a complete agreement with FDT. The transition from low- to high- temperature behavior occurs at $\beta \omega \sim 1$, namely $kT \sim hf$ (see Fig.2). For $T = 10$K, the boundary is at $f \sim 200$MHz. This would be within the capabilities of modern experiment; we expect the crossover between $1/\omega^2$ and $1/\omega$ behavior could be directly observable.

The charge fluctuation in the grain means the current flowing through the wire also fluctuates. The correlation function of the current can be obtained by a similar method. For example, the Matsubara Green’s function of the current at $J(x_0)$ is given by $(2e^2\omega_n^2/\pi^2)G_{x_0}(x_0, \omega_n)$, where

$$G_{x_0}(x_0, \omega_n) = -\frac{1}{2|\omega_n|} \left[ 1 - \frac{u - |\omega_n|}{u + |\omega_n|} e^{-2|\omega_n| x_0} \right] \tag{13}$$

is the Green’s function of the Laplacian with the source at $x_0$. The $1/\omega$ fluctuation of the grain voltage translates to an $\omega$-linear fluctuation of the current, although a similar behavior persists even for the current far from the grain.

So far, we have considered a semi-infinite wire connected to the grain. In the remainder of this paper, we discuss how our results are affected by wire being finite.

Instead of manufacturing a very long wire, it would be easier to connect the other end of wire to a normal reservoir. We assume that the junction to the reservoir is adiabatic (no backscattering), and the phase information is immediately lost in the reservoir (perfect thermalization). Then the situation is not qualitatively different from the case of the semi-infinite wire, since the particles can leave to infinity (reservoir), and the usual reasoning of Landauer formalism applies \cite{21}; the transport of the charge from/to the grain controlled by the geometric restriction (narrow 1D channel) implies that our theory applies to the more realistic case of the finite wire connected to a normal reservoir, if the interaction effects in the wire can be neglected.

When the interaction in the wire is not negligible, the problem is more subtle, since a quasiparticle excitation inside the wire is a collective motion of many electrons and cannot be simply transferred to the reservoir. While a rigorous theory of junction between a TLL and normal reservoir is not yet established, it has been studied by modelling the reservoir with a (semi-infinite) 1D Fermi liquid. As a simplest model, we study a finite wire of length $L$, in which the coupling constant is $K$, attached to semi-infinite wire with coupling constant $K'$, $K \equiv \alpha K'$. To model the reservoir by a 1D Fermi liquid, $K'$ should be unity but we shall obtain the solution for general values of $K'$ because it can be applied also to other interesting cases. It has been found that normal and Andreev-type reflections occur at the boundary if $K \neq K'$. \cite{22}

By a similar line of arguments as before, we obtain the Matsubara Green’s function of the boundary field $\Theta$ as

$$G(0, \omega_n) = \frac{-[\cosh (\omega_n L) + \alpha \sinh (|\omega_n|L)]}{(u + \alpha|\omega_n|) \cosh (\omega_n L) + (|\omega_n| + \alpha u) \sinh (\omega_n L)} \tag{14}$$

Since the temperature effect is given by the same FDT factor $\coth (\beta \omega/2)$ as in the semi-infinite wire case, we concentrate on how the spectral density $\rho(\omega) = -2\text{Im}G^R(\omega)$ is affected.

For $\alpha = 1$, the result reduces to the semi-infinite wire case, as it should be. For $\alpha \approx 1$, the overall shape of the spectral density remains the same. However, a variation with the period $1/L$ is superimposed on the dependence for semi-infinite wire (Fig.3). This is the resonant structure due to (normal and Andreev) reflections at the boundary between two wires. If the junction between the
wire and the reservoir is smooth and not abrupt, we expect that the resonant structure is smeared out; then the observed spectrum would be quite close to the prediction for the semi-infinite wire.

If $\alpha$ far from 1, the resonant structure dominates. In particular, the case $K' \to 0$ (or $K' \to \infty$) is equivalent to a finite wire with open end (or a finite wire attached to a very large superconductor reservoir), respectively. In these limits $\alpha \to \infty$, $(\alpha \to 0)$ the spectral density develops into a set of discrete $\delta$-functions with peak spacings of order $1/L$, rather than a continuous function of $\omega$. This reflects the finiteness of the system. In the $L \to \infty$ limit the $\delta$-functions become dense and converge to the smooth function we have obtained for the semi-infinite wire.

It is also instructive to take the opposite limit of short wire $L \to 0$. For the open end case ($K' \to 0$), the resonances are found at $\omega \sim (2n - 1)\pi/(2L)$ where $n$ is a natural number. For the wire connected to the superconductor reservoir, in the short-wire limit, the resonances are found at $\omega \sim n\pi/L, \ n = 1, 2, \ldots$. The resonance which would correspond to $n = 0$ is actually affected by the finite capacitance, and its location is given by $\omega = 1/\sqrt{LC}$. This can be simply understood from the following argument: from the conjugate relation between the phase and the charge, the effective Hamiltonian of the system is given by (see e.g. $[\text{Eq.1}]$) $H = -\frac{e^2}{2(L^2)} \left( \frac{\partial}{\partial \Phi} \right)^2 + \frac{1}{\sqrt{LC}} (\Phi - \Phi_0)^2$, where $\Phi$ is the superconducting phase of the grain, the first and second term represent the charging energy of the grain and the Josephson energy of the wire, respectively. This is a Hamiltonian of a harmonic oscillator with the eigenfrequency $1/\sqrt{LC}$, exactly where the resonance is found.

In conclusion, we have shown that quantum phase fluctuations in a superconducting grain in contact with a normal wire are translated into equilibrium charge fluctuations in the system, with specific spectrum demonstrating transition from $1/\omega^2$ to $1/\omega$ behavior at very low temperature. The mechanism of fluctuations is reaction of the normal subsystem to sudden change of the off-diagonal pairing potential on the NS boundary (Fermi edge singularity), leading in the low-temperature limit to quantum $1/f$ noise. Similar effect can be expected to exist also in systems with superconducting grain in contact with 2D conductors under appropriate conditions.

While the experimental observation of this effect presents a certain challenge, we hope it to be possible in near future. Concerning the experiments, impurity scatterings and physical effects of probing will require further theoretical study, which is however beyond scope of the present paper.

We are grateful to I. Affleck, N. Kokubo and S. Okuma for valuable discussions and to M. Büttiker and R. Fazio for helpful comments on the manuscript.

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[19] When there are more than one channels in the wire, the action is a sum of free boson ones, as far as the wire can be treated as collection of independent TLL’s. The voltage fluctuation is also given by the sum of single-channel contributions. In particular, if there are $N$ identical channels, the fluctuation is multiplied by $N$, and it suffices to consider the single-channel case. (If interchannel interactions cannot be neglected , this simple picture is no longer applicable for wires with $N > 1$ due to appearance of collective interchannel excitations [Ya.M. Blanter and M.Büttiker, Europhys. Lett. 42, 535 (1998).])
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FIG. 1. Superconducting grain $S$ in contact with a 1D conducting wire $N$. Tip $P$ probes the electrostatic potential of the grain.

FIG. 2. Reduced spectral density of voltage noise in a grain, $\bar{S}(\omega) = S_v(\omega)/[K e^2/\pi c^2]$ (Eq.(12)), as a function of $\bar{\omega} \equiv \omega/u$ for different values of $\bar{T} \equiv T/u$.

FIG. 3. Resonant structure of spectral density for $\alpha = 0.1$ (dots), $\alpha = 1$ (solid line), and $\alpha = 10$. Dimensionless length of the wire $L = 1$; $T = 1$. Dashed straight line illustrates $1/\bar{\omega}$ dependence.