The Kondo effect and weak localization

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I. INTRODUCTION

The word Kondo means battle in Swahili. This coincidence is fortuitous because in the Kondo effect, a battle inevitably ensues anytime a magnetic impurity is placed in a non-magnetic metal. Below some energy scale, the Kondo temperature ($T_k$) a lone magnetic impurity is robbed of its spin. Above the Kondo temperature, rapid spin-flip scattering produces a temperature-dependent correction to the resistivity of the form, $B_k \ln T$. Until recently, both the Kondo resistivity and $T_k$ were thought to be determined solely by the host metal and the magnetic impurity. However, numerous presentations in this volume attest, there is now overwhelming evidence that both are affected by the size of the sample as well as non-magnetic random scattering. In this paper, I will focus on the theoretical work we have performed on the experiments revealing that non-magnetic scattering suppresses the Kondo resistivity in thin Kondo alloys.

In Kondo alloys of the form Cu(Mn), Cu(Fe) and Au(Fe), Giordano and colleagues observed that introducing non-magnetic impurities suppressed the coefficient of the Kondo logarithm. The Kondo slope, $B_k$, is a monotonically decreasing function as the mean-free path is decreased. This result is surprising for two reasons. First, disorder gives rise to diffusive motion. Hence, relative to a clean sample, conduction electrons spend more time around a given magnetic impurity in the presence of disorder. Naively, this effect would result in an enhancement of the Kondo resistivity. Second, at the time of these experiments, the leading theoretical view was that disorder eliminates the Kondo logarithm and leads to a stronger algebraic divergence of the form $T^{d/2-2}$ in the resistivity. Everts and Keller were the first to argue for the emergence of a $1/\sqrt{T}$ in the Kondo self-energy for a $d=2$ system in the presence of random non-magnetic scattering. A few years later, Bohnen and Fisher argued, however, that such a term would not survive in the conductivity. More recently, Ohkawa and Fukuyama and Vladar and Zimanyi have developed an extensive diagrammatic scheme to re-investigate this problem and also concluded that the algebraic singularity dominates the Kondo $\ln T$. As a result, these groups conclude that static disorder can mask the Kondo resistivity as $T \to 0$. The experiments show no singularity of this sort, however. This complete lack of agreement between theory and experiment led us to re-evaluate the interplay between disorder and Kondo spin-flip scattering.

As our work is based heavily on the previous diagrammatic expansion of Fukuyama and
colleagues, it is first important to understand how the algebraic divergence emerges from their analysis. When non-magnetic impurities are present, the diffusive propagator that describes the resultant motion

\[ D(Q, \omega) \propto \frac{1}{(DQ^2 - i\omega)} \]  

(1)

has a diffusion pole. Here, \( Q \) and \( \omega \) are the net momentum and energy transfer and \( D = \frac{2\hbar}{eF\tau/dm} \). When such diffusive propagators are used to decorate the spin-flip vertices in the Kondo self-energy, the singular dependence found by Ohkawa and Fukuyama obtains as can be seen from the following argument. The most divergent contribution to the Kondo self-energy arises from the two-diffuson decoration of the Kondo spin-flip vertices. Diagrams of this form involve an integration over the internal momentum line:

\[
\sum_Q \frac{1}{(DQ^2 + |\omega|)^2} \propto \int \frac{Q^{d-1}dQ}{(DQ^2 + |\omega|)^2} \propto |\omega|^{d/2-2}.
\]  

(2)

The absolute value of the frequency appears here because we work in the finite-temperature Matsubara formalism. The Matsubara frequency \( \omega \) is proportional to temperature \( T \). Therefore, the temperature dependence due to diffusons and Cooperons is indeed \( T^{d/2-2} \), as can be also verified by a more careful calculation, and is a direct consequence of the diffusion poles.

The argument leading to the new algebraic dependence is certainly clear. However, it is well-known that spin-flip scattering can cut off the diffusion pole. Should this occur then the algebraic dependence will only be valid above a certain temperature, not as \( T \to 0 \). Of course, this requires that the feedback effect of spin-flip scattering on localization physics be included. It is this effect that has been absent from all previous treatments of the disorder/Kondo problem. Inclusion of the feedback effect of spin-flip scattering on localization has been the primary focus of our work. A key difference that the feedback effect introduces is a nontrivial density dependence into the Kondo problem. This difference arises because diffusive propagators which include the spin-flip scattering rate decorate the bare spin-flip vertices in the Kondo self-energy. The spin scattering rate is proportional to the concentration of magnetic impurities. Consequently, a non-zero spin-flip scattering rate arises only if all the magnetic impurities are averaged over. Hence, the feedback effect represents a departure from the single-impurity physics typically associated with the Kondo problem. That this state of affairs obtains naturally when disorder is present can be seen
from considering the standard weak-localization correction to the conductivity in a thin film, with \( \tau_\phi \) the dephasing time and \( \tau_o \) the elastic scattering time. Whenever localization physics is relevant, one has to decide which is the dominant dephasing process. Experiments show that the dephasing time is weakly dependent on temperature. This is consistent with a dephasing rate that is determined solely by spin-flip scattering. Hence, \( \hbar/\tau_s \propto n_s \hbar J^2 > \hbar/\tau_T \), where \( \tau_T \) is the dephasing time due to all other processes in the system. Consequently, if spin-flip scattering is the dominant dephasing process, the number of impurities has a lower bound. Our treatment does not include impurity-impurity effects, however. What is crucial here is that the contribution from each impurity must be averaged over to describe the dominant dephasing process. Our central result that is used to compare with the experiments can be derived simply from Eq. (3). In the presence of the Kondo logarithm, the spin-flip scattering rate is given by

\[
1/\tau_s = 2/3\sqrt{3}\tau_0^0 \left( 1 - 4J_0N(0) \ln \frac{T_F}{T} + \cdots \right). \tag{4}
\]

Substitution of this result into Eq. (3) and expansion of the logarithm for \( T > T_k \) yields the contribution of spin-flip scattering to the conductivity

\[
\delta \sigma \approx \sigma_0 \left( \ln \frac{\tau_s^0}{\tau_o} - N(0)J_0 \ln \frac{T_F}{T} \right). \tag{5}
\]

for a \( d = 2 \) sample, with \( \sigma_0 \) the Drude conductivity. Because \( J_0 < 0 \), the Kondo logarithmic term enhances the spin-scattering time and in turn reduces the magnitude of the weak-localization correction. That is, spin-flip scattering produces an ‘antiloclization’ effect. Further, this correction is opposite in sign to the zeroth-order Kondo logarithm. Consequently, disorder leads to a suppression of the Kondo resistivity. The suppression of the Kondo resistivity follows immediately from three principles: 1) spin-flip scattering feeds back into the Kondo self-energy to regularize the algebraic divergence, 2) weak localization appears as a negative correction to the conductivity and 3) spin-flip scattering weakens the weak-localization effect. Hence, the net effect is a positive correction logarithmic correction to the conductivity which when added to the negative bare Kondo logarithm leads to a diminished logarithmic conductivity.
II. FORMULATION OF PROBLEM

The starting point for our analysis is a model Hamiltonian $H = H_o + H_{sd}$ that contains both normal impurities
\begin{equation}
H_o = \sum_{k\sigma} (\varepsilon_k - \varepsilon_F) a_{k\sigma}^\dagger a_{k\sigma} + \frac{v}{\Omega} \sum_{k,k',i} e^{i(k-k') \cdot R_i} a_{k\sigma}^\dagger a_{k'\sigma}
\end{equation}
as well as magnetic scatterers
\begin{equation}
H_{sd} = -\frac{J}{\Omega} \sum_{R_n, k,k',\sigma,\sigma'} e^{i(k-k') \cdot R_n} \sigma_{\sigma,\sigma'} \cdot S_n a_{k\sigma}^\dagger a_{k'\sigma}'.
\end{equation}
where $v$ measures the strength of the scattering with the non-magnetic disorder, $R_n$ denotes the position of the impurities, magnetic or otherwise, $S_n$ is the spin operator for the magnetic impurity at site $n$, and $\Omega$ is the volume. The two natural timescales in this problem are, $\tau_s^o$ and $\tau_o$, the bare magnetic and non-magnetic scattering times. In terms of the density of states of the host metal, $\rho_o$ and the concentrations of magnetic and non-magnetic scatterers, $n_s$ and $n_o$, respectively, we have that \( \bar{\hbar}/2\tau_s^o = 3\pi n_s \rho_o |J|^2/4 \) and \( \bar{\hbar}/2\tau_o = \pi n_o \rho_o |v|^2 \). The total scattering rate is \( 1/\tau = 1/\tau_s^o + 1/\tau_o \). To measure the strength of the non-magnetic disorder, we define $\lambda = \bar{\hbar}/(2\pi \varepsilon_F \tau_o)$. We assume that the concentration of localized spins is dilute so that long-range spin glass effects are irrelevant. Also, we work in the regime in which normal impurity scattering dominates, \( 1/\tau_o \gg 1/\tau_s^o \).

Describing scattering in the presence of a weakly disordered potential requires Cooperon and diffuson propagators. The traditional form of such propagators, $C(Q, \omega) = D(Q, \omega) \propto (DQ^2 - i\omega)^{-1}$, was used extensively in the early treatments of the disordered Kondo problem. However, as remarked in the introduction, such a procedure assumes that diffusive motion with a diffusion pole remains intact even in the presence of oscillating fields created by spin-flip scattering. It is this assumption that leads to the divergence found earlier by Fukuyama and co-workers. To alleviate this problem, we include the all-important feedback effect spin-flip scattering has on such diffusive processes. If all scattering processes are treated in the first Born approximation, the Dyson-like integral equation,
\begin{equation}
D_{\alpha\beta\delta\gamma} = \delta_{\alpha\beta} \delta_{\delta\gamma} + \overline{U_{\alpha\mu} U_{\nu\gamma}} \sum G_{\mu}^R G_{\nu}^A D_{\mu\delta\nu}
\end{equation}
describes all ladder diagrams with the spin-dependent potential
\begin{equation}
\overline{U_{\alpha\mu} U_{\nu\gamma}} = \frac{1}{\tau_o} \delta_{\alpha\mu} \delta_{\nu\gamma} + \frac{1}{3\tau_s} \sigma_{\alpha\mu} \cdot \sigma_{\nu\gamma}.
\end{equation}
The Greek letters denote the spin indices on the upper and lower electron lines in the diffusion ladder and repeated indices are summed over. The advanced and retarded Green functions are given by \((G^A_\mu)^{-1} = \epsilon_F - p_\mu^2/2m - i/2\tau + \nu h\) and \((G^R_\mu)^{-1} = \epsilon_F + \omega - p_\mu^2/2m + i/2\tau + \mu h\) respectively. Noting that
\[
\sum G^R_\mu G^A_\nu = -\sum \frac{G^R_\mu - G^A_\nu}{(G^R_\mu)^{-1} - (G^A_\mu)^{-1}}
\]
we arrive at the solution for the diffusion,
\[
D_{\alpha\beta\gamma\delta} = \frac{\hbar}{4\tau(DQ^2 - i\omega)} (\delta_{\alpha\beta}\delta_{\gamma\delta} + \sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta})
\]
\[
+ \frac{\hbar}{4\tau(DQ^2 - i\omega + 4/3\tau^0)} (3\delta_{\alpha\beta}\delta_{\gamma\delta} - \sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta}).
\]
The analogous integral equation for the Cooperon
\[
C_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + U_{\alpha\mu} U_{\gamma\nu} \sum G^R_\mu G^A_\nu C_{\mu\beta\nu\delta}
\]
can be solved analogously to yield,
\[
C_{\alpha\beta\gamma\delta} = \frac{\hbar}{4\tau(DQ^2 - i\omega + 2/\tau^0)} (\delta_{\alpha\beta}\delta_{\gamma\delta} - \sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta})
\]
\[
+ \frac{\hbar}{4\tau(DQ^2 - i\omega + 2/3\tau^0)} (3\delta_{\alpha\beta}\delta_{\gamma\delta} + \sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta}).
\]
where \(\alpha\beta\) and \(\gamma\delta\) are spin indices. The dot-product
\[
\sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta} = \sigma_{\alpha\beta x} \sigma_{\gamma\delta x} + \sigma_{\alpha\beta y} \sigma_{\gamma\delta y} + \sigma_{\alpha\beta z} \sigma_{\gamma\delta z}.
\]

III. SELF ENERGY

As is evident, even in the presence of spin-flip scattering, the diffusion still retains its diffusion pole in the \(S = 0\) channel. Hence, we will be back to where we started if the \(S = 0\) diffusion still contributes to the self-energy. We now show that this contribution vanishes identically to all orders of perturbation theory. Consider the self-energy diagrams shown in Fig. [I].

To illustrate how the self-energy diagrams in Fig. [I] are evaluated let’s focus on the first two diagrams with the diffusion vertex decorations. The sum of the two self-energy diagrams is
\[
\Sigma^D_{\alpha\beta\gamma\delta}(i\epsilon_n) = n_\uparrow f^3 T \sum_{\omega_\ell,\omega_m, Q, q} \Theta(-\epsilon_n(\epsilon_n + \omega_\ell)) V_{\alpha\beta\mu\nu}(i\omega_\ell, i\omega_m)
\]
\[
\times G(i\epsilon_n + i\omega_m, q) G(i\epsilon_n + i\omega_\ell, k + Q)
\]
\[
\times D_{\sigma\alpha\beta\gamma}(i\omega_\ell, Q) D_{\sigma\nu\eta\sigma}(i\omega_\ell, Q)
\]

(14)
FIG. 1: Feynman diagrams contributing to the Kondo self-energy. The dashed lines correspond to Abrikosov pseudofermions and the double solid lines to diffusons and double dashed lines to the Cooperons. The Greek letters indicate the spin. The X indicates a single non-magnetic impurity scattering event. Such diagrams are known as the rainbow diagrams.

where $G(i\epsilon, q)$ is the electron Green function

$$G(i\epsilon, q) = \frac{1}{i\epsilon + \epsilon_F - \hbar^2 q^2/2m + i(h/2\tau)\text{sign}(\epsilon)},$$

and the electron energies are given by the Matsubara frequencies, $\epsilon_n = (2n + 1)\pi T$. The pseudofermion energies are defined in terms of $z_k = (2k + 1)\pi T$ and $\omega_\ell = 2l\pi T$. The range of summation over the momentum $Q$ and energy $\omega_\ell$ transfers is limited by the range
of validity of the diffusion approximation, \(DQ^2 < \hbar/\tau\) and \(\omega_\ell < \hbar/\tau\). The step function \(\Theta(x)\) appears in the expression because the diffusion propagators are only non-zero if the impurity ladders connect electrons on different sides of the Fermi surface. The summation over momenta \(k'\) and \(k''\) in the Green functions adjacent to the spin vertices is already included in the definition of the diffuson. The Green functions in Eq. (14) can be simplified using

\[
\sum_q G(i\epsilon_n + i\omega_m, q) \approx -i\pi\rho_0(\varepsilon_F) \text{sign}(\epsilon_n + \omega_m),
\]

\[
G(i\epsilon_n + i\omega_\ell, k + Q) \approx -i\frac{2\tau}{\hbar} \text{sign}(\epsilon_n + \omega_\ell) = i\frac{2\tau}{\hbar} \text{sign}(\epsilon_n).
\] (16)

The first approximation can be obtained by integration around the Fermi surface. The second approximation makes use of the fact that the momentum \(k\) and energy \(\epsilon_n\) are close to the Fermi surface (within the energy shell of width \(T\)), and the momentum and energy transfers allowed by the diffusion propagator are less than \(\hbar/\tau\).

The pseudofermion part,

\[
V_{\alpha\beta\nu\eta}(\omega_\ell, \omega_m) = -\frac{1}{16} \left[ \frac{\delta_{m0}}{i\omega_\ell}(1 - \delta_{\ell0}) + \frac{\delta_{\ell0}}{i\omega_m}(1 - \delta_{m0}) - \frac{\delta_{\ell m}}{i\omega_\ell}(1 - \delta_{\ell0}) \right] (\sigma_{\alpha\beta} \cdot \sigma_{\nu\eta}).
\] (17)

involves a trace over the impurity spin states. The internal spin indices are not summed over because the electron spin can be flipped by the diffusons. After substituting all the ingredients into Eq. (14) and performing the summation over the spin indices, the self-energy becomes

\[
\Sigma_{3Q}^D(k, i\epsilon_n) = -6n_s\pi\rho_0J^3\tau T \sum_{Q,\omega_\ell} \Theta(-\epsilon_n(\epsilon_n + \omega_\ell)) \left[ \frac{\hbar/\tau}{DQ^2 + |\omega_\ell| + 4\hbar/3\tau_0^0} \right]^2 \text{sign}(\epsilon_n).
\]

Remarkably, the divergent \(S = 0\) part of the diffuson drops out, and as a result the singular temperature dependence in the resistivity disappears.

But what is the source of this cancellation and is it exact? By careful examination of the pseudofermion contribution Eq. (17), we see that the sum over the spin indices in the self-energy [Eq. (14)] separates into two identical sums of the form, \(\Sigma_{\alpha\beta} D_{\sigma_{\alpha\beta}\gamma} \sigma_{\alpha\beta}^a\). If we use the identity \(\Sigma_{\alpha\beta} (\sigma_{\nu\alpha} \cdot \sigma_{\beta\gamma}) \cdot \sigma_{\alpha\beta}^a = -\sigma_{\nu\gamma}^a\), we find immediately that the cancellation of the \(S = 0\) diffuson

\[
\sum_{\alpha\beta} D_{\nu\alpha\beta\gamma}^S = \sum_{\alpha\beta} (\delta_{\nu\alpha}\delta_{\beta\gamma} + \sigma_{\nu\alpha} \cdot \sigma_{\beta\gamma})\sigma_{\alpha\beta}^a = 0
\] (18)
from the 3\textsuperscript{rd} order Kondo self-energy is exact. To any order in $J$ in the most divergent approximation, the cancellation of the $S = 0$ diffuson can be seen as follows. Within this scheme, each diffuson encircles a vertex that is exactly equal to the Abrikosov vertex function $\Gamma = J_{\text{eff}} \sigma \cdot S$, with $J_{\text{eff}}$ defined within the parquet summation. When this function is now multiplied by $D^{S=0}$ and summed over the spin indices, the cancellation to all orders follows immediately from Eq. (18). Note that this cancellation relies on the spin algebra and hence is not tied to the approximations used to obtain $D^{S=0}$. Consequently, the cancellation of the $S = 0$ component of the diffuson is fundamentally tied to the fact that the Kondo interaction does not conserve the electron’s spin. This cancellation theorem which signifies that the Kondo logarithm remains in tact is in the spirit of Anderson’s theorem that non-magnetic impurities do not affect $T_c$ for s-wave superconductors. Recently, Chakravarty and Nayak\textsuperscript{17} have shown that in the very weak-disorder limit, a true Anderson theorem exists in which disorder does scale out of the Kondo problem.

As advertised, the cancellation of the diffusion pole suppresses the algebraic divergence of the self-energy. To see how this emerges, we continue with our analysis of the first two diagrams in Fig.1). We limit ourselves to 2D case, and hence

$$
\Sigma_{3q}^D(k, i\epsilon_n) = 6ln_s^2\rho_0^2J^3\lambda\tau T \sum_{\omega_n} \frac{\Theta(-\epsilon_n(\epsilon_n + \omega_\ell))}{\omega_\ell(\omega_\ell + 4\hbar/3\tau_0)} \text{sign}(\epsilon_n)
$$

$$
= \frac{3}{2}ln_s^2\rho_0^2J^3\lambda\hbar\frac{\hbar}{\tau T} \sum_{m=0}^{\infty} \left( m + \frac{\epsilon_n}{2\pi T} + \frac{1}{2} \right) \left( m + \frac{\epsilon_n}{2\pi T} + \frac{1}{2} + \frac{2\hbar}{3\pi T \tau_0} \right) \text{sign}(\epsilon_n).
$$

where $\lambda$ is the dimensionless disorder defined earlier. From this self energy, we define the scattering rate as follows:

$$
\frac{\hbar}{2\tau_D^q} = \int \left( \frac{\partial f}{\partial \epsilon} \right) \left( -\text{Im} \Sigma_{3q}^D(k, \epsilon + i0) \right) d\epsilon
$$

$$
= -\int f(\epsilon) \frac{\partial \text{Im} \Sigma_{3q}^D(k, \epsilon + i0)}{\partial \epsilon} d\epsilon.
$$

Clearly, one cannot evaluate this expression just by setting $\epsilon = 0$ because of the singular temperature dependence in the self-energy. We will compute this expression by contour integration in the complex $\epsilon$-plane. The self-energy has two poles in the upper half-plane. Hence if we close the contour in the lower half-plane, then the integral will be equal to the sum of the residues in the points where the Fermi function $f(\epsilon)$ has poles, $\epsilon_k = -i(2k+1)\pi T$:

$$
\frac{\hbar}{2\tau_D^q} = 2\pi i T \sum_k \left. \frac{\partial \text{Im} \Sigma_{3q}^D(k, \epsilon + i0)}{\partial \epsilon} \right|_{\epsilon = \epsilon_k}
$$
\[ \sum_{k,m=0}^{\infty} \frac{1}{(m+k+1)(m+k+1+\eta)} \left[ \frac{1}{(m+k+1)} + \frac{1}{(m+k+1+\eta)} \right] \]

where \( A \) is the coefficient of the self-energy and \( \eta = \frac{2\hbar}{3\pi T \tau_s^0} \). In the double sum over \( m \) and \( k \) there are \( (m+k+1) \equiv N \) identical elements. Therefore, the sum can be transformed to a sum over \( N \) times \( N \). Now if we use the series expansion for the digamma, \( \psi(x) = d\ln \Gamma(x)/dx \), and trigamma, \( \psi'(x) = d^2\ln \Gamma(x)/dx^2 \) functions,

\[ \psi(1+x) = -\gamma + x \sum_{N=1}^{\infty} \frac{1}{N(N+x)} \]

\[ \psi'(1+x) = \sum_{N=1}^{\infty} \frac{1}{(N+x)^2}, \]

and define a new function

\[ F(x) \equiv \frac{\psi(1+x) + \gamma}{x} + \psi'(1+x) \]

then the result for the contribution to the scattering rate becomes

\[ \frac{\hbar}{2\tau_D} = -\frac{3}{2} n_s \rho_0^2 J^3 \lambda \frac{\hbar}{\tau T} F \left( \frac{2\hbar}{3\pi T \tau_s^0} \right). \quad (19) \]

For small argument, \( x \ll 1 \), \( F(x) \approx \zeta(2) = \pi^2/6 \); for large argument \( x \gg 1 \), \( F(x) \approx \ln(x)/x \). Hence we conclude that there are two regimes, corresponding to "high" \( (\hbar/\tau_s^0 > T) \) and "low" \( (\hbar/\tau_s^0 < T) \) impurity concentrations, in which the diffuson corrections behave logarithmically in temperature and as \( 1/T \), respectively. In other words, the \( 1/T \) behavior is cut-off at the temperatures below \( \hbar/\tau_s^0 \). Now let us consider the rest of the diagrams, namely the diagrams that involve Cooperon propagators, and the diagrams with the external single impurity line.

\[ \Sigma_{\eta q}(k, i\epsilon_n) = n_s J^3 T \sum_{\omega, \omega_m, Q, q} \Theta(-\epsilon_n + \omega) V_{\alpha\beta\nu\eta}(i\omega, i\omega_m) \times G(i\epsilon_n + i\omega_m, q) G(i\epsilon_n + i\omega, k + Q) \times C_{\sigma\alpha\gamma\nu}(i\omega, Q) C_{\beta\gamma\eta\sigma}(i\omega, Q). \quad (20) \]

The only difference compared to the diffuson self-energy contribution is the different spin indexing of the Cooperons compared to the diffusons. This is because the Cooperon propagators need to be "crossed" in order to have the same momentum transfers without phase space restrictions. Summing over the spin indices reduces the problem to one in which the product of the Cooperons is spin independent and equal to:

\[ \tilde{C}^2 = \frac{\hbar^2}{2\tau^2} \left[ \frac{1}{(DQ^2 - i\omega + 2/\tau_s^0)^2} + \frac{1}{(DQ^2 - i\omega + 2/3\tau_s^0)^2} \right]. \quad (21) \]
Now we can continue in exactly the same way as in the diffusion case to obtain for the scattering rate correction
\[ \frac{\hbar}{2 \tau_C} = -\frac{3}{2} n_s \rho_0^2 J^3 \lambda \frac{\hbar}{\tau T} \left[ \frac{1}{2} F\left( \frac{\hbar}{\pi T \tau_0^0} \right) + \frac{1}{2} F\left( \frac{\hbar}{3 \pi T \tau_0^0} \right) \right]. \] (22)

At high temperatures, \( T > \hbar/\tau_0^0 \) we again recover the \( 1/T \) behavior, and for low temperatures, \( T < \hbar/\tau_0^0 \), the logarithmic behavior obtains.

Finally, we need to consider the set of diagrams that contain one external impurity line (the rainbow diagrams). We will prove now that such diagrams are equal to the corresponding diagrams without the impurity line times the factor of \(-1/2\). Unlike the diagrams that we considered before, the internal Green function \( G(i\epsilon_n + i\omega, k + Q) \) can no longer be replaced by its value at the Fermi surface, \( 2i\tau/\hbar \). Instead, a sum over the intermediate momentum of a product of three Green functions needs to be computed:
\[
|v|^2 \sum_{k'} \frac{1}{(i\epsilon_n + \epsilon_F - \epsilon_{k'} + i\hbar/2\tau)^2 (i\epsilon_n + i\omega - \epsilon_F - \epsilon_{k'-Q} + i\hbar/2\tau)}.
\]
\[
\approx |v|^2 \int_{-\infty}^{\infty} \rho_0 d(-x) \frac{1}{(x + i\hbar/2\tau)^2 (x - i\hbar/2\tau)} = 2\pi i \rho_0 |v|^2 \frac{1}{(2i\hbar/2\tau)^2} = -i\tau/\hbar. \] (23)

In deriving this relation, we set \( Q \) and \( \omega \) to zero, since they are small. Therefore the sum of the diagrams with and without the external impurity lines is two times smaller that sum of the diffusion and Cooperon diagrams that we derived before. However, there is a factor of 2 that comes from two possible internal electron lines to which the diffusion propagators can be attached. As a result, the sum of all diagrams happens to be exactly equal to the sum of two contributions that we already computed. Hence the total quantum correction to the scattering time due to the Kondo diagrams in Fig. 4 is
\[ \frac{1}{\tau_q} = \frac{1}{\tau_D^q} + \frac{1}{\tau_C^q}, \]
with \( 1/\tau_D^q \) and \( 1/\tau_C^q \) given by Eq. (19) and Eq. (22), respectively.

**IV. CONDUCTIVITY**

The total conductivity is a sum of the Drude contribution with the transport scattering time, and the weak localization correction. The transport scattering rate is composed of elastic scattering, the second order in \( J \) spin scattering, the third order in \( J \) (Kondo)
scattering and the quantum corrections computed above.

\[
\frac{1}{\tau_{tr}} = \frac{1}{\tau_0} + \frac{1}{\tau_0^0} + \frac{1}{\tau_s^K} + \frac{1}{\tau_q^D} + \frac{1}{\tau_C^q}.
\]  
(24)

Of these contributions, only the last three have non-trivial temperature dependence. The total conductivity is

\[
\sigma(T) = \frac{e^2 n \tau_{tr}}{m} + \delta \sigma_{WL} = \sigma_0 \left( 1 - \frac{\tau}{\tau_s^K} - \frac{\tau}{\tau_q^D} - \frac{\tau}{\tau_C^q} + \frac{\delta \sigma_{WL}}{\sigma_0} \right),
\]  
(25)

where \( \sigma_0 \) is the temperature-independent part of the conductivity. There are also more complex conductivity diagrams that involve both the spin-dependent pseudofermion part and the diffusion propagators, but they can be shown to cancel out.

The derived expressions for \( \tau_q^D \) and \( \tau_C^q \) have simple asymptotic behavior. For \( d = 2 \) in the limit \( T \gg \hbar/\tau_0^s \), we recover the inverse temperature dependence

\[
\frac{\hbar}{2\tau_C^q} = \frac{\hbar}{2\tau_D^q} \approx -\frac{\pi \hbar \rho_0 \lambda J}{3\tau} \frac{\hbar}{T \tau_0^s} \ll -\rho_0 \lambda J \hbar T
\]  
(26)

of Refs. (13,14). Without the diffusion pole cancellation, the lower bound in temperature for the \( 1/T \) behavior would be set by \( \max[\hbar/\tau_0^s, T_K] \), where \( \tau_0^s \) is the inelastic scattering time. We find here that by explicitly including spin-scattering in the diffusion propagators, the algebraic behavior occurs when \( \hbar/(\tau_0^s T) \ll 1 \). We will see later that as a result of this restriction, the contribution of the \( 1/T \) term to the conductivity is negligible. In the opposite regime, \( T \ll \hbar/\tau_0^s \), the scattering rates

\[
\frac{\hbar}{2\tau_D^q} = -\frac{3}{2} \rho_0 \lambda J \frac{\hbar}{T \tau_0^s} \ln \frac{\hbar}{T \tau_0^s}
\]

are both logarithmic functions of temperature.

The weak-localization contribution is given in Eq. (3). We collect all the contributions discussed above to determine the temperature-dependent conductivity. In the temperature range \( T_K < T < \hbar/\tau_0^s \), Cooperon, diffusion, and weak-localization corrections are all logarithmic in temperature. Combining the results from Eq. (27) with the weak-localization correction, we find that the magnitude of the logarithmic part of the conductivity

\[
\sigma^T = \sigma_0 4 \frac{T \rho_0 J}{\tau_0^s} \left( 1 + 0.75 \frac{\tau_0^s}{\tau} \right) \ln \frac{\epsilon_F}{T}.
\]  
(28)
The first term in this expression arises from the unperturbed Kondo effect and the latter from the interplay with disorder. Inclusion of disorder in the self-energy, even after inclusion of the negative WL correction, enhances the Kondo resistivity relative to a clean system result.

For temperatures $T \gg \bar{\hbar}/\tau_s^0$, the self-energy contribution to the relaxation time scales as $1/T$, whereas the weak-localization correction is proportional to $\ln T$. However, comparison of the magnitude of these corrections reveals that the weak-localization term dominates, and the magnitude of the resultant temperature-dependent conductivity

$$
\sigma_T = \sigma_0 \frac{4\pi \rho_0 J}{\tau_s^0} \left( 1 - \frac{\lambda \tau_s^0}{\tau} \right) \ln \frac{\varepsilon_F}{T}
$$

(29)
is suppressed by the disorder. Let us now apply our results to thin films with a thickness, $L$. We are interested in thin films, such that $\ell < L \ll L_\phi$. Because $\ell < L$, the electron gas is characterized by a 3-dimensional density of states $\rho_0 = 1/(2\pi)^2 (2m/\hbar^2)^{3/2} \epsilon_F^{1/2}$ and diffusion constant given by $D = 2\hbar \epsilon_F \tau / 3m$. Since the dephasing length $L_\phi$ exceeds the film thickness, such a film should be treated as quasi-2D with respect to weak localization. That means that the momentum-transfer summation in the diffusion propagators must be restricted to the plane, or $\sum_Q \rightarrow (1/L) \sum_Q(2D)$. The density of states that arises from converting this sum into an integral is the two-dimensional density of states, $\rho_{0D}^2 = \pi \rho_0 / k_F$. Hence, the self-energy diagrams with the diffusion propagators will generate a size-dependence to the conductivity of the form $1/(k_F L)$. The explicit finite-size weak-localization correction is

$$
\delta \sigma_{WL} = -\frac{e^2}{2\pi^2 \hbar L} \ln \left( \frac{3\sqrt{3} \tau_s}{2\tau} \sinh \left( \frac{L}{\ell} \right) \frac{\ell}{L} \right).
$$

The size-dependence under the logarithm yields an effective size dependence in the spin-relaxation time. This size dependence should be observable in the standard WL magnetoresistance measurements in the weak magnetic fields. However, it will not affect the temperature dependence of the conductivity. The only size dependence that is coupled to the temperature is the $1/L$ prefactor of the weak-localization correction.

We now combine these results in the low and high-temperature limits discussed earlier. In the two limits, we obtain conductivities

$$
\sigma_T = \begin{cases} 
\sigma_0 \frac{4\pi \rho_0 J}{\tau_s^0} \left( 1 + \frac{0.25 \hbar \tau_s}{m k_F e L^2} \right) \ln \frac{\varepsilon_F}{T} & \text{if } T_K \ll T < \hbar / \tau_s^0 \\
\sigma_0 \frac{4\pi \rho_0 J}{\tau_s^0} \left( 1 - \frac{1.5 \hbar \tau_s}{m k_F e L T} \right) \ln \frac{\varepsilon_F}{T} & \text{if } T_K, \hbar / \tau_s^0 \ll T
\end{cases}
$$

(30)

that have an explicit size and disorder correction that scales as $1/(\ell^2 L)$. The fact that only the coefficient of $\ln T$, but not the form of the temperature dependence, is modified is
a direct consequence of the diffusion pole cancellation theorem. When magnetic impurity density is high, we find an enhancement of the Kondo logarithm. This is an intuitive result since, qualitatively, diffusive motion of electrons is expected to enhance the probability of repeated scattering that generates the Kondo effect. The surprising finding is that in the other regime, an overall suppression of the logarithmic correction in the conductivity is obtained. While the self-energy enhancement is always present, as it can be seen from the positive self-energy corrections to the transport scattering rate [Eqs. (19) and (22)], this effect is completely overwhelmed in the conductivity by the WL correction which also acquires $\ln T$ dependence due to the Kondo contribution to the dephasing rate.

V. EXPERIMENTAL APPLICATIONS

In the experiments of Blachly and Giordano, the impurity concentration was such that, $\hbar/\tau_s^0 \sim 0.1$ K, which is much less than the Kondo temperature for Cu(Fe), $T_K \sim 3$ K. Therefore, the second of Eqs. (30) should apply. Figure 2 shows a comparison between the experimental data of Blachly and Giordano and the theoretical predictions. Each black square corresponds to one sample. The best fit to the data was obtained with $\tau_s^0 = 1.3 \times 10^{-10}$ s, whereas experimentally the spin scattering time is on the order of $10 \times 10^{-11}$. This discrepancy also persists for the Cu(Mn) alloys for thicknesses of order 750 – 400Å. However, for the thinnest Cu(Mn) alloys Jacobs and Giordano have shown that excellent agreement exists between theory and experiment for $\tau_s^0 = 6 \times 10^{-11}$, which is well within the experimental uncertainty of the measured value, $\tau_s^0 = 6 \times 10^{-11}$.

While theory and experiment are in good agreement for thin samples, there is a key experimental ambiguity that surrounds these results, namely is there a well-defined Kondo temperature for the thinnest samples and for those with mean free paths of order 250Å. This question is most relevant in light of the experiments of Yanson and colleagues who have shown that in point contacts, huge fluctuations in the Kondo temperature occur for contact diameters of order 100Å. For an inhomogeneous system, the density of states becomes position dependent: $\rho_0(x) = \rho_0 + \delta \rho_0(x)$. Consequently, the position-dependent Kondo temperature will be

$$T_K(x) = \varepsilon_F \exp[1/2\rho_0(x)J] \approx \varepsilon_F \exp[1/2\rho_0(x)J] = T_K \exp[-\delta \rho_0(x)/2\rho_0^2 J].$$

(31)
Typically $\rho_0 J \sim 0.1$, which means that even a 10% change in the density of states can produce 100% change in the Kondo temperature. The effect is even stronger for alloys with lower Kondo temperatures. From elementary scattering theory, $\delta \rho_0(x) = \rho_0 1/\sqrt{k_F \lambda}$, where $\lambda$ is the smaller of the mean-free path and the sample thickness. For $\ell = 100\text{Å}$, $\delta \rho_0(x)/\rho_0 = .1$. Hence, we expect a 100% change in the Kondo temperature for such samples. Those impurities having high Kondo temperatures will not contribute to the Kondo resistivity. Consequently, fluctuations in the density of states can effectively decrease the concentration of active spin-flip scattering centers that could contribute to the Kondo logarithm. This will lead to an enhancement in the spin-scattering rate over the bulk value. Hence, the question as to how well-defined the Kondo temperature is in the thinnest and most disordered samples should be resolved before a complete experimental understanding of the Kondo effect in dirty alloys can be reached.

![Graph](image)

**FIG. 2:** Comparison of the theoretical Kondo resistivity predicted from the second of Eqs. (30) with the experimental data of Blachly and Giordano. The horizontal axis measures the strength of the static disorder through the mean-free path.
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