Darboux-Egoroff Metrics, Rational Landau-Ginzburg
Potentials and the Painlevé VI Equation

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Abstract

We present a class of three-dimensional integrable structures associated with the
Darboux-Egoroff metric and classical Euler equations of free rotations of a rigid
body. They are obtained as canonical structures of rational Landau-Ginzburg po-
tentials and provide solutions to the Painlevé VI equation.

1 Introduction

The flat coordinates and prepotentials play a prominent role in the study and clas-
sification of solutions to the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations
\cite{1,2,3,4}. Our interest is in finding the formalism which would provide a universal
approach to canonical integrable structures behind the WDVV equations. Here,
we understand canonical integrable structures as hierarchies formulated in terms of
canonical coordinates and with evolution flows obtained in a conventional way from
the Riemann-Hilbert factorization problem with twisting condition imposed \cite{5,6}.
This approach coincides with the framework of Darboux-Egoroff metric constructed
in terms of the Lamé coefficients and their symmetric rotation coefficients \cite{3}.
Additional reduction (scaling invariance) effectively turns the evolution flows into the
isomonodromic deformations (up to the Laplace transformation) and allows the flow
evolution equations to be rewritten as Schlesinger equations \cite{3}. One can associate
a tau function to the factorization problem which, upon imposition of the confor-
mal condition, coincides with the corresponding isomonodromic tau function
\cite{5,6}.
The tau function appears to be a key object in this construction due to the fact
that all the other objects of canonical formalism (e.g. rotation coefficients) can be
obtained from it. The conformal condition introduces a notion of scaling dimension
or homogeneity of the tau function which is convenient to use in order to classify
integrable models. For the integer scaling dimensions of the tau function the inte-
grable hierarchies can be reproduced by the (C)KP-like formalism \cite{7,8}. The tau
functions can be calculated in the Grassmannian approach to the multi-component
KP hierarchy \cite{7,10}. The structures with the tau-functions possessing fractional
scaling dimension lack such a clear underlying foundation. An exception to this rule

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is the trivial two-dimensional case where the tau function can be explicitly found for all models.

This work studies three-dimensional models with the tau functions having the scaling dimension equal to the fractional number of a 1/4 in the hope of providing data for its future general formalism. The integrable structures are obtained from Frobenius manifolds associated with a class of rational Lax functions \[1\]. The construction of canonical coordinates for these polynomials generalizes the well-known construction of canonical coordinates associated with the monic polynomials \[3, 12\]. It is worth noting that the space of monic polynomials endowed with a natural metric provides one of the most standard examples of the Frobenius manifolds.

We find a closed expression for the tau function for one of the considered models. We also associate the canonical structure to algebraic solutions for the Painlevé VI equations considered by Hitchin \[13, 14, 15\] and Segert \[16, 17\].

The paper is organized as follows. In Section 2, we define the canonical integrable structure behind the WDVV equations emphasizing its connection to the Darboux-Egoroff metric. The tau function plays a key role in this formalism. In this section we also introduce homogeneity as a scaling dimension. In addition, we discuss the relation of canonical integrable structures to the flat coordinates, structure constants and associativity equations.

The special case of three dimensions is presented in Section 3 where the Darboux-Egoroff equations are shown to take the form of the classical Euler equations of free rotations of a rigid body. Also, the scaling dimension of the tau function is found to be related to the integral of the Euler equations.

The rational potentials and its metric along with the associated canonical coordinates related to the Darboux-Egoroff metric are described in Section 4. Examples of the three-dimensional canonical integrable models derived from the rational potentials are given in Section 5 and its relation to the Painlevé VI equation is established. The Darboux-Egoroff metric is described in detail for the discussed three-dimensional models. The explicit form of the tau function is found for one of the models. Section 6 presents concluding remarks.

### 2 From factorization problem to Darboux-Egoroff metric

Let the “bare” wave (matrix) function be defined in terms of canonical coordinates \( \mathbf{u} = (u_1, \ldots, u_N) \) as

\[
\Psi_0(\mathbf{u}, z) = e^{z U}
\]

where \( U = \text{diag}(u_1, \ldots, u_N) \) and the unit matrix \( E_{jj} \) has the matrix elements \((E_{jj})_{kl} = \delta_{kj} \delta_{lj}\).

Define the “dressed” wave matrix \( \Psi \):

\[
\Psi(\mathbf{u}, z) = \Theta(\mathbf{u}, z) \Psi_0(\mathbf{u}, z)
\]
obtained by acting on the “bare” wave (matrix) function $\Psi_0(u, z)$ with the dressing matrix:

$$\Theta(u, z) = 1 + \theta^{(-1)}(u)z^{-1} + \theta^{(-2)}z^{-2} + \cdots.$$  \hspace{1cm} (2.3)

The “dressed” wave matrix $\Psi$ enters a factorization problem:

$$\Psi(u, z)g(z) = \Theta(u, z)\Psi_0(u, z)g(z) = M(u, z)$$  \hspace{1cm} (2.4)

where $g$ is a map from $z \in S^1$ to the Lie group $G$ and $M(u, z)$ is a positive power series in $z$:

$$M(u, z) = M_0(u) + M_1(u)z + M_2(u)z^2 + \cdots.$$  \hspace{1cm} (2.5)

If the map $g(z)$ is such that it satisfies the twisting condition $g^{-1}(z) = g^T(-z)$ then it follows that the matrices $\Theta(u, z)$ and $M(u, z)$ from equation (2.4) will also satisfy the twisting conditions $\Theta^T(u, -z) = \Theta^{-1}(u, z)$, $M^T(u, -z) = M^{-1}(u, z)$ [5, 6]. In this case, the matrix $\theta^{(-1)}(u)$ from expansion (2.3) must be symmetric, $\theta^{(-1)}(u) = (\theta^{(-1)})^T(u)$ and the matrix $M_0(u)$ from expression (2.5) must be orthogonal, $M_0^{-1}(u) = M_0^T(u)$.

The factorization problem (2.4) leads to the flow equations:

$$\frac{\partial}{\partial u_j} \Theta(u, z) = -\left(\Theta z E_{jj} \Theta^{-1}\right)_- \Theta(u, z)$$  \hspace{1cm} (2.6)

$$\frac{\partial}{\partial u_j} M(u, z) = \left(\Theta z E_{jj} \Theta^{-1}\right)_+ M(u, z)$$  \hspace{1cm} (2.7)

where $(\ldots)_\pm$ denote the projections on positive/negative powers of $z$. Due to equation (2.6) the one-form $\text{Res}_z (\text{tr} (\Theta^{-1}z(d\Theta/dz)dU))$ is closed. Conventionally, one associates to this one-form a $\tau$-function through

$$d \log \tau = \text{Res}_z \left(\text{tr} \left(\Theta^{-1}_z \frac{d\Theta}{dz} dU\right)\right)$$  \hspace{1cm} (2.8)

where $d = \sum_{j=1}^N \partial_j du_j$ with $\partial_j = \partial/\partial u_j$. One can verify that this definition of the tau function implies relation $\theta^{(-1)}_{ii} = -\partial_i \log \tau$ for the diagonal elements of the $\theta^{(-1)}$-matrix. We will parametrize the $\theta^{(-1)}$-matrix as follows

$$\theta^{(-1)}_{ij} = \begin{cases} 
\beta_{ij} & i \neq j \\
-\partial_i \log \tau & i = j 
\end{cases}$$  \hspace{1cm} (2.9)

Due to equation (2.6) the symmetric off-diagonal components $\beta_{ij}$ satisfy

$$\partial_j \beta_{ik} = \beta_{ij} \beta_{jk}, \quad i, j, k \text{ distinct}$$  \hspace{1cm} (2.10)

It follows from the definition (2.2) that the “dressed” wave matrix $\Psi$ will satisfy the flow equations identical to those in (2.7):

$$\frac{\partial}{\partial u_j} \Psi(u, z) = \left(\Theta z E_{jj} \Theta^{-1}\right)_+ \Psi(u, z)$$  \hspace{1cm} (2.11)
The projection in (2.11) contains only two terms: $(\Theta z E_{jj} \Theta^{-1})_{+} = zE_{jj} + V_{j}$ where $V_{j} \equiv [\theta^{(-1)}, E_{jj}]$. Accordingly,
\[
\frac{\partial \Psi}{\partial u_{j}} = (zE_{jj} + V_{j})\Psi \tag{2.12}
\]
and $\partial \Theta / \partial u_{j} = z[E_{jj}, \Theta] + V_{j} \Theta$. Projecting on diagonal directions yields $\partial_{i} \partial_{j} \log \tau = -\beta_{ij}^{2}$ for $i \neq j$. Hence the dressing matrix can be expressed in terms of one object only, the $\tau$-function.

We now define Euler and identity vectorfields $E$ and $I$ as:
\[
E = \sum_{i=1}^{N} u_{i} \frac{\partial}{\partial u_{i}} , \quad I = \sum_{i=1}^{N} \frac{\partial}{\partial u_{i}} \tag{2.13}
\]
Their action on $\Psi$ can easily be found from (2.12) by summing over $j$:
\[
E(\Psi) = (zU + V) \Psi , \quad I(\Psi) = z\Psi \tag{2.14}
\]
where we introduced matrix $V = [\theta^{(-1)}, U]$ with components
\[
V_{ij} = (u_{j} - u_{i})\theta_{ij}^{(-1)} = (u_{j} - u_{i})\beta_{ij} , \quad i, j = 1, \ldots, N . \tag{2.15}
\]
Similarly, $E(\Theta) = z[U, \Theta] + V\Theta$.

Notice, that $E(\Psi_{0}) = zd\Psi_{0}/dz$. We will require that the same equality holds for the action of the grading operator $zd/dz$ and of the Euler operator $E$ on $\Psi$:
\[
z \frac{d}{dz} \Psi = E(\Psi) = (zU + V) \Psi \tag{2.16}
\]
The relation (2.16) is called the conformal condition [3]. It holds provided $E(\Theta) = zd\Theta/dz$. Also, from equation (2.6) we get by summing over all indices $j$ that $I(\Theta) = 0$. In particular, for $\theta^{(-1)}$ we get
\[
E(\beta_{ij}) = -\beta_{ij} , \quad I(\beta_{ij}) = 0 \tag{2.17}
\]
\[
E(\partial_{i} \log \tau) = -\partial_{i} \log \tau , \quad I(\partial_{i} \log \tau) = 0 \tag{2.18}
\]
Since, $E\partial_{i} = \partial_{i} E - \partial_{i}$ we see that $E \log \tau$ = constant or $E(\tau)$ = constant $\tau$ is compatible with the conformal condition (2.18). This constant defines the scaling dimension (or homogeneity) of the tau function.

Plugging relation $E\Theta = zd\Theta/dz$ into the formula (2.8) for the $\tau$-function one obtains [3]:
\[
\partial_{j} \log \tau = \text{Res}_{z} \left( \text{tr} \left( \Theta^{-1} E \Theta E_{jj} \right) \right) = \frac{1}{2} \text{tr} (V_{j} V) \tag{2.19}
\]
Comparing with expression for the isomonodromy tau-function $\tau_{I}$ [3] one concludes that $\tau_{I} = 1/\sqrt{\tau}$ [3, 8]. The isomonodromy tau-function $\tau_{I}$ gives the Hamiltonian formulation $\partial_{j} V = \{ H_{j}, V \}$ for equation
\[
\partial_{j} V = [V_{j}, V] \tag{2.20}
\]
via the formula: $\partial_j \log \tau_i = H_j$. Equation (2.20) follows from the compatibility of equations (2.12), (2.14) and (2.16). One clearly sees from (2.19) and (2.20) that

$$I(\log \tau) = 0, \quad I(V) = 0$$

(2.21)
since $\sum_{j=1}^{N} V_j = 0$.

The similarity transformation $V \to \mathcal{V} = M_0^{-1} V M_0$ transforms $V$ to the constant matrix $\mathcal{V}$ ($\partial_j \mathcal{V} = 0$) due to the flow equations $\partial_j M_0 = V_j M_0$, which follow from (2.7). Assume, now that there exists an invertible matrix $S$ which diagonalizes $\mathcal{V}$:

$$S^{-1} \mathcal{V} S = \mu = \sum_{j=1}^{N} \mu_j E_{jj}$$

(2.22)

where $\mu$ is the constant diagonal matrix. Next, define a matrix

$$M(u) = M_0(u) S = (m_{ij}(u))_{1 \leq i,j \leq N}$$

(2.23)

which governs transformation from the canonical coordinates $u_j, j = 1, \ldots, N$ to the flat coordinates $x^\alpha, \alpha = 1, \ldots, N$. Let, constant non-degenerate metric be given by the matrix :

$$\eta = (\eta_{\alpha\beta})_{1 \leq \alpha,\beta \leq N} = M^T M = S^T S, \quad \text{and denote} \quad \eta^{-1} = (\eta^{\alpha\beta})_{1 \leq \alpha,\beta \leq N},$$

(2.24)
hence $\eta_{\alpha\beta} = \sum_{i=1}^{N} m_{i\alpha} m_{i\beta}$. Then the derivatives with respect to the flat coordinates $x^\alpha, \alpha = 1,\ldots, N$ are given by

$$\frac{\partial}{\partial x^\alpha} = \sum_{i=1}^{N} \frac{m_{i\alpha}}{m_{i1}} \frac{\partial}{\partial u_i}$$

(2.25)

with the reversed relation being

$$\frac{\partial}{\partial u_i} = \sum_{\alpha,\beta=1}^{N} \eta^{\alpha\beta} m_{i1} m_{i\beta} \frac{\partial}{\partial x^\alpha}.$$  

(2.26)

The structure constants

$$c_{\alpha\beta\gamma} = \sum_{i=1}^{N} \frac{m_{i\alpha} m_{i\beta} m_{i\gamma}}{m_{i1}}.$$  

(2.27)

satisfy the associativity equation

$$\sum_{\delta\gamma=1}^{N} c_{\alpha\beta\delta} \eta^{\delta\gamma} c_{\gamma\sigma\rho} = \sum_{\delta\gamma=1}^{N} c_{\alpha\sigma\delta} \eta^{\delta\gamma} c_{\gamma\beta\rho}, \quad \alpha, \beta, \sigma = 1, \ldots, N$$

(2.28)

and are given by derivations of the prepotential $F$:

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial x^\alpha \partial x^\beta \partial x^\gamma}.$$ 

(2.29)
The metric \( g = \sum_{\alpha, \beta=1}^{N} \eta_{\alpha \beta} dx^{\alpha} dx^{\beta} \) equals in terms of the canonical coordinates to:

\[
g = \sum_{i=1}^{N} h_{i}^{2} (du_{i})^{2} \tag{2.30}
\]

with Lamé coefficients \( h_{i} = m_{i1} \) being such that the corresponding rotation coefficients:

\[
\beta_{ij} = \frac{1}{h_{j}} \frac{\partial h_{i}}{\partial u_{j}} \tag{2.31}
\]

satisfy conditions of the Darboux-Egoroff metric, namely \( \beta_{ij} = \beta_{ji} \) together with the relations (2.10) and \( I(\beta_{ij}) = 0 \). Note, that the Darboux-Egoroff condition is equivalent to square of the Lamé coefficient being a gradient of some potential \( \phi : h_{i}^{2} = \partial \phi / \partial u_{i} \).

For the Lamé coefficients \( h_{i} \) with \( \sum_{i=1}^{N} h_{i}^{2} = \sum_{i=1}^{N} m_{i1}^{2} = \eta_{11} \) the homogeneity must be zero for the constant metric tensor with \( \eta_{11} \neq 0 \) since

\[
0 = E(\sum_{i=1}^{N} h_{i}^{2}) = \sum_{i=1}^{N} 2h_{i}^{2} = 2\eta_{11}c \tag{2.34}
\]

From relation \( E(\eta_{\alpha \beta}) = (d_{F} - d_{1} - d_{\alpha} - d_{\beta}) \eta_{\alpha \beta} \), obtained by acting with the Euler vectorfield on both sides of (2.29), it follows that \( d_{F} = 3d_{1} \) for \( \eta_{11} \neq 0 \) and if \( d_{F} \neq 3d_{1} \) then \( \eta_{11} = 0 \). Hence the value of the homogeneity of the prepotential indicates when the homogeneity of the Lamé coefficients vanish.

For the class of models we consider here the Lamé coefficients \( h_{i} \) are given by the formula:

\[
h_{i}^{2} = \frac{\partial x_{1}}{\partial u_{i}} \tag{2.35}
\]
Note, also that $E(h_i) = ch_i$ is consistent with the conformal condition in (2.17) for the arbitrary constant homogeneity $c$.

For the Darboux-Egoroff metric the identity vectorfield vanishes when acting on the Lamé coefficient as follows from

$$I(h_i) = \sum_{j=1}^{N} \beta_{ij} h_j = \sum_{j=1}^{N} \frac{1}{h_i} \frac{\partial h_j}{\partial u_i} h_j = \frac{1}{2h_i} \frac{\partial \eta_{11}}{\partial u_i} = 0$$

(2.37)

3 The Three-dimensional Case

Let us now consider the three-dimensional manifolds. In this case, we can rewrite the antisymmetric matrix $V$ as:

$$V = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-\omega_3 & 0 & \omega_1 \\
\omega_2 & -\omega_1 & 0
\end{pmatrix}$$

(3.1)

or $(V)_{ij} = (u_j - u_i)\beta_{ij} = \epsilon_{ijk}\omega_k$. From (2.17) and (2.21) we see that $\omega_k$ vanishes when acted on by the vectorfields $E$ and $I$. That makes $\omega_k$ effectively a function of one variable $s$ such that $E(s) = I(s) = 0$. Let us choose

$$s = \frac{u_2 - u_1}{u_3 - u_1}$$

(3.2)

Then equation (2.20) takes a form equivalent to the Euler top equations:

$$\frac{d\omega_1}{ds} = \frac{\omega_2\omega_3}{s}$$

(3.3)

$$\frac{d\omega_2}{ds} = \frac{\omega_1\omega_3}{s(s - 1)}$$

(3.4)

$$\frac{d\omega_3}{ds} = \frac{\omega_1\omega_2}{1 - s}$$

(3.5)

One verifies that $d(\sum_{k=1}^{3} \omega_k^2)/ds = 0$. Consequently,

$$\sum_{k=1}^{3} \omega_k^2 = -R^2$$

(3.6)

with $R$ being a constant is the integral of equations (3.3)-(3.5). The same constant $R$ characterizes the homogeneity of the tau function, as we will show now. Starting from expression (2.19), one finds for the scaling dimension

$$E(\log \tau) = \frac{1}{2} \sum_{j=1}^{3} u_j \text{ tr } (V_j V) = \frac{1}{2} \text{ tr } (V^2) = \frac{1}{2} \text{ tr } (\mu^2) = \frac{1}{2} \sum_{\alpha=1}^{3} \mu_{\alpha}^2$$

(3.7)

Recalling that $(V)_{ij} = \epsilon_{ijk}\omega_k$ we can rewrite the above as:

$$E(\log \tau) = \frac{1}{2} \sum_{j=1}^{3} \sum_{i=1}^{3} (\epsilon_{ijk}\omega_k)^2 = -\sum_{k=1}^{3} \omega_k^2 = R^2$$

(3.8)
and since \( \mu_\alpha = 1 - d_\alpha + d/2 \) with \( d = d_F - 3 \)

\[
R^2 = \sum_{\alpha=1}^{3} \left( \frac{1}{2} - \frac{d_F}{2} + d_\alpha \right)^2
\]  

(3.9)

We have seen above, that for \( \eta_{11} \) different from zero the homogeneity of the Lamé coefficients \( h_i \) must vanish. In such case, the Lamé coefficients \( h_i \) depend only on one variable \( s \) due to the fact that \( I(h_i) = E(h_i) = 0 \). The relations \( \partial_i h_i^2 = \partial_j h_j^2 \) translate for the function \( h^2_i(s) \) to

\[
s \frac{dh_i^2}{ds} = (s-1) s \frac{dh_i^2}{ds} = (1-s) \frac{dh_i^2}{ds}
\]  

(3.10)

Also, since

\[
\omega_k = \frac{u_j - u_i \partial h_i^2}{2h_i h_j \partial u_j}, \quad i, j, k \text{ cyclic}
\]  

(3.11)

we find e.g.

\[
\omega_3 = \frac{s}{2h_1 h_2} \frac{dh_1^2}{ds}, \quad \omega_2 = \frac{s}{2h_1 h_3} \frac{dh_1^2}{ds}
\]  

(3.12)

and so \( h_3 \omega_2 = h_2 \omega_3 \) and similarly \( h_1 \omega_2 = h_2 \omega_1 \). We conclude that

\[
\omega_i^2 = - \frac{R^2}{\eta_{11}} h_i^2, \quad i = 1, 2, 3
\]  

(3.13)

and comparing equations (3.3)-(3.5) with equation (3.10) we obtain like in [17]:

\[
s \frac{dh_i^2}{ds} = (s-1) s \frac{dh_i^2}{ds} = (1-s) \frac{dh_i^2}{ds} = -2i R \sqrt{\eta_{11}} h_i h_2 h_3
\]  

(3.14)

4 **Rational Landau - Ginsburg Models**

Following Aoyama and Kodama [11] we study a rational potential:

\[
W(z) = \frac{1}{n+1} z^{n+1} + a_{n-1} z^{n-1} + \ldots + a_0 + \frac{v_1}{z - v_{m+1}} + \frac{v_2}{2(z - v_{m+1})^2} + \ldots \\
+ \frac{v_m}{m(z - v_{m+1})^m}
\]  

(4.1)

which is known to characterize the topological Landau-Ginzburg (LG) theory. The rational potential in this form can be regarded as the Lax operator of a particular reduction of the dispersionless KP hierarchy [11, 19, 20].

The space of rational potentials from (4.1) is naturally endowed with the metric:

\[
g(\partial_t W, \partial_t W) = \text{Res}_{z \in \text{Ker} W^*} \left( \frac{\partial_t W \partial_t W^*}{W^*} \right) dz
\]  

(4.2)

where \( \partial_t W = \partial_t a_{n-1} z^{n-1} + \ldots + \partial_t a_0 + \frac{\partial v_1}{z - v_{m+1}} + \ldots \) describes a tangent vector to the space of rational potentials obtained by taking derivative of all coefficients with
respect to their argument. $W'(z)$ is a derivative with respect to $z$ of the rational potential $W$:

$$W'(z) = z^n + (n-1)a_{n-1}z^{n-2} + \ldots - \frac{v_m}{(z-v_{m+1})^{m+1}} \quad (4.3)$$

Next, we find the flat coordinates $x_\alpha, \alpha = 1, \ldots, m + 1$ and $\bar{x}_\gamma, \gamma = 1, \ldots, n$ such that

$$g(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial x_\beta}) = \eta_{\alpha \beta}, \quad g(\frac{\partial W}{\partial \bar{x}_\alpha}, \frac{\partial W}{\partial \bar{x}_\beta}) = \bar{\eta}_{\alpha \beta}, \quad g(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial \bar{x}_\gamma}) = 0 \quad (4.4)$$

with constant and non-degenerate matrices $\eta_{\alpha \beta}$ and $\bar{\eta}_{\alpha \beta}$.

Consider first the function $w = w(W, z)$ such that $W(z) = w^{-m}/m$ and $z = x_{m+1} + x_m w + \ldots + x_1 w^m = \sum_{\alpha=1}^{m+1} x_\alpha w^{\alpha-1}$. We take $z \sim x_{m+1}$ or $w \ll 1$. It follows that

$$W' dz = -\frac{1}{w^{m+1}} dw, \quad \frac{\partial W}{\partial x_\alpha} = W' \frac{\partial z}{\partial x_\alpha} = W' w^\alpha \quad (4.5)$$

Consequently:

$$g(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial x_\beta}) = -\text{Res}_{z=\infty} \left( \frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'} \right) dz \quad (4.6)$$

$$= -\text{Res}_{z=\infty} \left( W' w^{\alpha-1} w^m \right) dz = \text{Res}_{w=\infty} \left( \frac{w^{m-\alpha} w^m}{w^{m+1}} \right) dw = \delta_{\alpha+\beta=m+2} \quad (4.7)$$

Hence $x_\alpha$ are flat coordinates with the metric $\eta_{\alpha \beta} = \delta_{\alpha+\beta=m+2}$. The coefficients $v_j, j = 1, \ldots, m + 1$ of $W(z)$ are given in terms of the flat coordinates as [11]:

$$v_k = \sum_{\alpha_1+\ldots+\alpha_k = (k-1)m+k} x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_k}, \quad k = 1, \ldots, m \quad (4.8)$$

Examples are:

$$v_m = (x_m)^m, \quad v_{m-1} = (m-1)x_{m-1}(x_m)^{m-1}, \ldots, \quad v_1 = x_1 \quad (4.9)$$

To represent the remaining coefficients of $a_i, i = 1, \ldots, n$ of $W$ in terms of the flat coordinates we consider a relation:

$$z = w + \frac{\bar{x}_1}{w} + \frac{\bar{x}_2}{w^2} + \ldots + \frac{\bar{x}_n}{w^n} \quad (4.10)$$

valid for large $z$ and $w \gg 1$. In this limit we impose a relation $W = w^{n+1}/(n+1)$ from which it follows that

$$W' dz = w^n dw, \quad \frac{\partial W}{\partial \bar{x}_\gamma} = W' \frac{\partial z}{\partial \bar{x}_\gamma} = W' w^{-\gamma} \quad (4.11)$$

We find

$$g(\frac{\partial W}{\partial \bar{x}_\gamma}, \frac{\partial W}{\partial \bar{x}_\delta}) = \text{Res}_{z=\text{Ker}W} \left( \frac{(\partial W/\partial \bar{x}_\gamma)(\partial W/\partial \bar{x}_\delta)}{W'} \right) dz \quad (4.12)$$

$$= \text{Res}_{w=\infty} \left( W' w^{-\gamma} w^{-\delta} \right) dw = \text{Res}_{w=0} w^{-\gamma-\delta} dw = \delta_{\gamma+\delta=n+1}$$
Hence $x_{\gamma}$ are flat coordinates with the metric $\tilde{\eta}_{\gamma\delta} = \delta_{\gamma+\delta=n+1}$. By similar considerations $\eta_{\alpha\gamma} = 0$ for $\alpha = 1, \ldots, m+1, \gamma = 1, \ldots, n$.

From expression (4.9) and $W(z) = w^{n+1}/(n+1)$ one can find relations between coefficients $a_{\gamma}$ and $\tilde{x}_{\gamma}$ starting with $a_{n-1} = -\tilde{x}_1$ and so on.

We will now show how to associate to the rational potentials $W$ canonical coordinates $u_i, i = 1, \ldots, n+m+1$ for which the metric (4.2) becomes a Darboux-Egoroff metric.

Let $\alpha_i, i = 1, \ldots, n+m+1$ be roots of the rational potential $W(z)$ in (4.3).

Equivalently, $W'(\alpha_i) = 0$ for all $i = 1, \ldots, n+m+1$. Thus $W'(z)$ can be rewritten as

$$W'(z) = \frac{\prod_{j=1}^{n+m+1} (z - \alpha_j)}{(z - v_{m+1})^{m+1}} \quad (4.12)$$

Next, define the canonical coordinates as

$$u_i = W(\alpha_i), \quad i = 1, \ldots, n+m+1 \quad (4.13)$$

The identity :

$$\delta_j^i = \frac{\partial u_i}{\partial u_j} = \frac{\partial W(\alpha_i)}{\partial u_j} = W'(\alpha_i) \frac{\partial \alpha_i}{\partial u_j} + \frac{\partial W}{\partial u_j}(\alpha_i) = \frac{\partial W}{\partial u_j}(\alpha_i) \quad (4.14)$$

implies that

$$\frac{\partial W}{\partial u_j}(z) = \frac{\partial a_{n-1}}{\partial u_j} z^{n-1} + \ldots + \frac{\partial a_0}{\partial u_j} + \frac{\partial v_1/\partial u_j}{z - v_{m+1}} + \ldots + \frac{v_m}{(z - v_{m+1})^{m+1}} \frac{\partial v_{m+1}}{\partial u_j} \quad (4.15)$$

can be rewritten as

$$\frac{\partial W}{\partial u_j}(z) = \frac{\prod_{k=1, j \neq k}^{n+m+1} (z - \alpha_k)}{(z - v_{m+1})^{m+1}} \frac{(\alpha_j - v_{m+1})^{m+1}}{\prod_{k=1, j \neq k}^{n+m+1} (\alpha_j - \alpha_k)} \prod_{j=1}^{n+m+1} (\alpha_j - \alpha_k) \quad (4.16)$$

Consider

$$g(\frac{\partial W}{\partial u_i}, \frac{\partial W}{\partial u_j}) = \text{Res}_{z \in \text{Ker} W'} \left( \frac{(\partial W/\partial u_i)(\partial W/\partial u_j)}{W'} \right) dz \quad (4.17)$$

Recalling (4.12) and (4.16) we find that $g(\partial W/\partial u_i, \partial W/\partial u_j) = 0$ for $i \neq j$. For $i = j$, we find

$$g(\frac{\partial W}{\partial u_i}, \frac{\partial W}{\partial u_i}) = \text{Res}_{z \in \text{Ker} W'} \left( \frac{(\partial W/\partial u_i)^2}{W'} \right) dz = \frac{(\alpha_i - v_{m+1})^{m+1}}{\prod_{j=1, j \neq i}^{n+m+1} (\alpha_i - \alpha_j)} = \frac{\partial a_{n-1}}{\partial u_i} \quad (4.18)$$

where the last identity was obtained by comparing coefficients of the $z^{n-1}$ term in (4.13) and (4.16).
Hence, in terms of the coordinates \( u_i \) the metric can be rewritten as \( g = \sum_{i=1}^{N} h_i^2(u) (du_i)^2 \) with the Lamé coefficients:

\[
h_i^2(u) = \frac{\partial a_{n-1}}{\partial u_i}
\]

The fact that \( h_i^2(u) \) is a gradient ensures that the rotation coefficients \( \beta_{ij} \) are symmetric and therefore the metric becomes the Darboux-Egoroff metric when expressed in terms of the orthogonal curvilinear coordinates \( u_i \).

## 5 N=3 Models, Examples of Rational Landau-Ginsburg models

### 5.1 n=m=1 model

Consider the model with \( n = m = 1 \):

\[
W(z) = \frac{1}{2} z^2 + x_1 + \frac{x_2}{z - x_3}
\]

where as coefficients we used the flat coordinates \( x_1 = -\bar{x}_1 \) and \( x_2, x_3 \) corresponding to \( x_1, x_2 \) of the previous section. The flat coordinates \( x_\alpha, \alpha = 1, 2, 3 \) are related to the flat metric:

\[
\eta^{\alpha\beta} = \eta_{\alpha\beta} = \text{Res}_{z \in \text{Ker} W'} \left( \frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'} \right) dz = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

The metric tensor can be derived from the more general expression involving the structure constants

\[
c^{\alpha\beta\gamma} = \text{Res}_{z \in \text{Ker} W'} \left( \frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)(\partial W/\partial x_\gamma)}{W'} \right) dz
\]

through relation \( \eta^{\alpha\beta} = c^{\alpha\beta 1} \). The non-zero values of the components of \( c^{\alpha\beta\gamma} \) are found from (5.3) to be:

\[
c^{111} = 1, \ c^{123} = 1, \ c^{222} = 1/x_2, \ c^{233} = x_3, \ c^{333} = x_2
\]

the other values can be derived using that \( c^{\alpha\beta\gamma} \) is symmetric in all three indices. These values can be reproduced from the formula (2.23) with the prepotential:

\[
F(x_1, x_2, x_3) = \frac{1}{6} x_2 (x_3)^3 + \frac{1}{6} (x_1)^3 + x_1 x_2 x_3 + \frac{1}{2} (x_2)^2 \left( \log x_2 - \frac{3}{2} \right)
\]

The prepotential satisfies the quasi-homogeneity relation (2.33) with \( d_F = 3 \) with respect to the Euler vectorfield:

\[
E = x_1 \frac{\partial}{\partial x_1} + \frac{3}{2} x_2 \frac{\partial}{\partial x_2} + \frac{1}{2} x_3 \frac{\partial}{\partial x_3} = x_1 \frac{\partial}{\partial x_1} + \frac{1}{2} x_2 \frac{\partial}{\partial x_2} + \frac{3}{2} x_3 \frac{\partial}{\partial x_3}
\]
We now adopt a general discussion of canonical coordinates to the case \( n = m = 1 \). Let \( \alpha_i, i = 1, 2, 3 \) be roots of the polynomial (4.3), which at present case is \( W'(z) = z-x_2/(z-x_3)^2 \). So, \( \alpha_i \) satisfy \( W'(\alpha_i) = 0 \) or \( \alpha_i(\alpha_i - x_3)^2 - x_2 = 0 \) for all \( i = 1, 2, 3 \).

Then, it follows by taking derivatives of \( \alpha_i(\alpha_i - x_3)^2 = x_2 \) with respect to \( x_2, x_3 \) that
\[
\frac{\partial \alpha_i}{\partial x_3} = \frac{2\alpha_i}{3\alpha_i - x_3}, \quad \frac{\partial \alpha_i}{\partial x_2} = \frac{1}{(\alpha_i - x_3)(3\alpha_i - x_3)} \tag{5.7}
\]
and further that
\[
\frac{\partial u_i}{\partial x_3} = \frac{x_2}{(\alpha_i - x_3)^2} = \alpha_i, \quad \frac{\partial u_i}{\partial x_2} = \frac{1}{\alpha_i - x_3} \tag{5.8}
\]
for the canonical coordinates \( u_i = W'(\alpha_i) = \frac{1}{2}\alpha_i^2 + x_1 + x_2/(\alpha_i - x_3) \). We now present a method of inverting the derivatives in (5.8) or alternatively to find the matrix elements \( m_{ij} \) of the matrix \( M \) from relation (2.23). The sum of the canonical coordinates is equal to \( \sum_{i=1}^3 u_i = 3x_1 + x_3^2 \) and therefore
\[
1 = 3 \frac{\partial x_1}{\partial u_i} + 2x_3 \frac{\partial x_3}{\partial u_i} = h_i^2 \left( 3 + 2x_3 \frac{\partial u_i}{\partial x_2} \right) \tag{5.9}
\]
where we used the fact that
\[
\frac{\partial x_3}{\partial u_i} = m_{i1} \frac{\partial u_i}{\partial x_2} \tag{5.10}
\]
because of
\[
\frac{\partial x_3}{\partial u_i} = m_{i1} m_{i\alpha}, \quad \frac{\partial u_i}{\partial x_\alpha} = \eta_{\alpha\beta} \frac{m_{i\beta}}{m_{i1}}, \quad h_i^2 = m_{i1}^2 \tag{5.11}
\]
Hence, from relation (5.9) it holds that \( h_i^2 = \left( 3 + 2x_3 \frac{\partial u_i}{\partial x_2} \right)^{-1} \) or by using equation (5.8) that
\[
\frac{\partial x_1}{\partial u_i} = h_i^2 = \frac{\alpha_i - x_3}{3\alpha_i - x_3} \tag{5.12}
\]
Plugging the last equation into equation (5.10) and using relation (5.8) we obtain
\[
\frac{\partial x_3}{\partial u_i} = \frac{1}{3\alpha_i - x_3} \tag{5.13}
\]
Similarly, from
\[
\frac{\partial x_2}{\partial u_i} = m_{i1} \frac{\partial u_i}{\partial x_3} \tag{5.14}
\]
we obtain
\[
\frac{\partial x_2}{\partial u_i} = \frac{x_2}{(\alpha_i - x_3)(3\alpha_i - x_3)} = \frac{\alpha_i(\alpha_i - x_3)}{(3\alpha_i - x_3)} \tag{5.15}
\]
Furthermore,
\[
\frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_i}{\partial x_2} \frac{\partial x_2}{\partial u_j} + \frac{\partial \alpha_i}{\partial x_3} \frac{\partial x_3}{\partial u_j} \tag{5.16}
\]
gives for \( i \neq j \):
\[
\frac{\partial \alpha_i}{\partial u_j} = \frac{1}{(3\alpha_i - x_3)(3\alpha_j - x_3)} \left( \frac{\alpha_j(\alpha_j - x_3)}{(3\alpha_i - x_3)} + 2\alpha_i \right) \tag{5.17}
\]
and for $i = j$:

$$
\frac{\partial \alpha_i}{\partial u_i} = \frac{3\alpha_i}{(3\alpha_i - x_3)^2}
$$

Using (5.12) and (5.17) we find the rotation coefficients defined in (2.31) to be

$$
\beta_{ij} = -\frac{(\alpha_k - x_3)(3\alpha_k - x_3)}{(3\alpha_i - x_3)(3\alpha_j - x_3)} \frac{1}{\sqrt{(\alpha_i - x_3)(3\alpha_i - x_3)(\alpha_j - x_3)(3\alpha_j - x_3)}}
$$

Its square is then

$$
\beta_{ij}^2 = -\frac{1}{4} \frac{1}{(\alpha_i - \alpha_j)^2 (4x_3 - 3\alpha_k)^2} \frac{\partial x_1}{\partial u_k},
$$

where $i, j, k$ are cyclic. Recall that in equation (3.1) we have introduced the functions $\omega_k = (u_j - u_i)\beta_{ij}$, where again we used the cyclic indices $i, j, k$. The difference of canonical coordinates can be written as: $u_j - u_i = (\alpha_i - \alpha_j)(3\alpha_k - 4x_3)/2$ which together with equation (5.19) yields:

$$
\omega_k^2 = -\frac{1}{4} h_k^2 = -\frac{1}{4} \frac{\partial x_1}{\partial u_k} = -\frac{1}{4} \frac{\alpha_k - x_3}{3\alpha_k - x_3}
$$

Since $I = \sum_{i=1}^{3} \partial / \partial u_i = \partial / \partial x_1$ then

$$
\sum_{k=1}^{3} \omega_k = -\frac{1}{4}, \quad E(\log \tau) = \frac{1}{4}
$$

The explicit form of the roots $\alpha_i$ is needed to find expressions for $\omega_k$ and its dependence on the parameter $s$. It is convenient to introduce $q = x_2/(x_3)^3$ and $a_i = \alpha_i/x_3$ which satisfy equation $a_i(a_i - 1)^2 = q$. Let us furthermore introduce a parameter $\omega$ such that $q = 4(\omega^2 - 1)^2/(\omega^2 + 3)^3$. This parametrization makes it possible to obtain the compact expressions for $\omega_k$. The three solutions to the algebraic equation

$$
a(a - 1)^2 = q = 4\frac{(\omega^2 - 1)^2}{(\omega^2 + 3)^3}
$$

are:

$$
a_1 = \frac{4}{\omega^2 + 3}, \quad a_2 = \frac{(\omega + 1)^2}{\omega^2 + 3}, \quad a_3 = \frac{(\omega - 1)^2}{\omega^2 + 3}
$$

Note, that $a_2 \leftrightarrow a_3$ under $\omega \leftrightarrow -\omega$ transformation, which shows that $\omega$ is a purely imaginary variable. First, we find that the variable $s$ from (3.2) can be expressed as:

$$
s = \frac{(a_2 - a_1)(3a_3 - 4)}{(a_3 - a_1)(3a_2 - 4)} = \frac{(\omega - 3)^3(\omega + 1)}{(\omega + 3)^3(\omega - 1)}
$$

Next, from relations $h_i^2 = (a_i - 1)/(3a_i - 1)$ and equation (5.21) we derive:

$$
\omega_1^2 = -\frac{1}{4} \frac{(\omega^2 - 1)}{(\omega^2 - 9)}, \quad \omega_2^2 = \frac{1}{4} \frac{(\omega + 1)}{\omega(\omega - 3)}, \quad \omega_3^2 = -\frac{1}{4} \frac{(\omega - 1)}{\omega(\omega + 3)}
$$
They provide solutions to the Euler top equations (3.3)-(3.5). The corresponding function \[ y(\omega) = \frac{(\omega - 3)^2(\omega + 1)}{(\omega + 3)(\omega^2 + 3)} \] (5.27)

connected with \( \omega_k \)'s through relations \[ \text{[13, 14, 15]} \]:

\[
\begin{align*}
\omega_1^2 &= -\frac{(y - s)y^2(y - 1)}{s} \left( v - \frac{1}{2(y - s)} \right) \left( v - \frac{1}{2y} \right), \\
\omega_2^2 &= \frac{(y - s)^2y(y - 1)}{s(1 - s)} \left( v - \frac{1}{2(y - 1)} \right) \left( v - \frac{1}{2y} \right), \\
\omega_3^2 &= -\frac{(y - s)y^2(y - 1)^2}{(1 - s)} \left( v - \frac{1}{2y} \right) \left( v - \frac{1}{2(y - s)} \right)
\end{align*}
\] (5.28)

with the auxiliary variable \( v \) defined by equation

\[
\frac{dy}{ds} = \frac{y(y - 1)(y - s)}{s(s - 1)} \left( 2v - \frac{1}{2y} - \frac{1}{2(y - 1)} + \frac{1}{2(y - s)} \right) \] (5.29)

is a solution of the Painlevé VI equation \[ \text{[13, 14, 15]} \]:

\[
\frac{d^2y}{ds^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - s} \right) \left( \frac{dy}{ds} \right)^2 - \left( \frac{1}{s} + \frac{1}{s - 1} + \frac{1}{y - s} \right) \frac{dy}{ds} + \frac{y(y - 1)(y - s)}{s^2(s - 1)^2} \left[ \frac{1}{8} - \frac{s}{8y^2} + \frac{s - 1}{8(y - 1)^2} + \frac{3s(s - 1)}{8(y - s)^2} \right].
\] (5.30)

Introducing parameter \( x = (\omega - 3)/(\omega + 3) \) one can rewrite expressions (5.27) and (5.28) as:

\[ y = \frac{x^2(x + 2)}{x^2 + x + 1}, \quad s = \frac{x^3(x + 2)}{2x + 1}, \] (5.31)

which reproduces the \( k = 3 \) Poncelet polygon solution of Hitchin \[ \text{[14, 15]} \]. Note, also that taking an inverse of solution (5.27) and letting \( \omega \to -\omega \) produces the \( k = 6 \) Poncelet polygon solution of Hitchin \[ \text{[14, 15]} \]:

\[ y^{-1}(-\omega) = \frac{(\omega - 3)(\omega^2 + 3)}{(\omega - 1)(\omega + 3)^2} = \frac{x(x^2 + x + 1)}{2x + 1}. \] (5.32)

We now proceed to calculate the underlying \( \tau \)-function. Our knowledge of the \( \tau \)-function is based on equation (2.19) from which we derive that

\[
\partial_j \log \tau = \sum_{i=1}^{3} \beta_{ij}^2 (u_i - u_j).
\] (5.33)

The identity \( I(\log \tau) = 0 \), shows that \( \tau = \tau(x_2, x_3) \) is a function of two variables \( x_2, x_3 \). Furthermore, it satisfies:

\[
E(\log \tau) = \left( \frac{3}{2} x_2 \frac{\partial}{\partial x_2} + \frac{1}{2} x_3 \frac{\partial}{\partial x_3} \right) \log \tau = \frac{1}{4}
\] (5.34)
A solution to the above equation is

\[ \log \tau = \frac{1}{4} \left( \frac{1}{3} \log x_2 + \log x_3 \right) + f \left( \frac{1}{3} \log x_2 - \log x_3 \right) \]  

(5.35)

where \( f(\cdot) \) is an arbitrary function of its argument. In order to determine the function \( f \) we use equation (5.33) to calculate the derivative

\[ \frac{\partial \log \tau}{\partial x_3} = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_3} \log \tau = \sum_{i,j=1}^{3} \alpha_j \beta_{ij}^2 (u_i - u_j) \]  

(5.36)

A calculation based on equation (5.20) yields:

\[ x_3 \frac{\partial}{\partial x_3} \log \tau = \frac{1}{8} \frac{1}{1 - \frac{27}{4} q} = \frac{1}{4} - f' \left( \frac{1}{3} \log x_2 - \log x_3 \right) \]  

(5.37)

where the last equality was obtained by comparing with equation (5.35) (recall that \( q = x_2/(x_3)^3 \)). Integration gives (ignoring an inessential integration constant):

\[ f \left( \frac{1}{3} \log x_2 - \log x_3 \right) = \frac{1}{24} \left( \log q + \log(-4 + 27q) \right) \]  

(5.38)

Using that \( x_2 = qx_3^3 \) we can now rewrite \( \log \tau \) as

\[ \log \tau = \frac{1}{4} \log x_2^2 + \frac{1}{24} \log \left( q^3(-4 + 27q) \right) \]  

(5.39)

Inserting parametrization of \( q \) from (5.23) and using relation \( u_2 - u_3 = 8x_3^2\omega^3(\omega^2 + 3)^{-2} \) we obtain the following expression for \( \log \tau \):

\[ \log \tau = \log(u_2 - u_3)^{1/4} + \frac{1}{24} \log \left( (\omega - 1)^6(\omega + 1)^6(\omega - 3)^2(\omega + 3)^2\omega^{-16} \right) \]  

(5.40)

It is easy to confirm \( I(\log \tau) = 0 \) and \( E(\log \tau) = 1/4 \) based on this expression.

### 5.2 n=0,m=2 model

Consider the model with \( n = 0, m = 2 \) in (4.1):

\[ W(z) = z + \frac{x_1}{z - x_3} + \frac{x_2^2}{2(z - x_3)^2} \]  

(5.41)

The flat coordinates \( x_\alpha, \alpha = 1, 2, 3 \) are related to the flat anti-diagonal metric:

\[ \eta^{\alpha\beta} = \eta_{\alpha\beta} = \text{Res}_{z \in \text{Ker} W'} \left( \frac{\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'} \right) dz = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]  

(5.42)

The model is characterized by the prepotential

\[ F = \frac{1}{2} x_3^2 x_1 + \frac{1}{2} x_2^2 x_3 + \frac{1}{2} x_1^2 \log(x_2) \]  

(5.43)
which generates the structure constants according to relation (2.29) and possesses homogeneity $d_f = 4$ with respect to the Euler operator

$$E = 2x_1 \frac{\partial}{\partial x_1} + \frac{3}{2} x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} = x_1 \frac{\partial}{\partial x_1} + \frac{3}{2} x_2 \frac{\partial}{\partial x_2} + 2 x_3 \frac{\partial}{\partial x_3}$$

Plugging the values $d_1 = 1, d_2 = 3/2, d_3 = 2, d_F = 4$ into the relation (3.9) we find that the homogeneity of $\log \tau$ is again equal to $R^2 = 1/4$. The roots $\alpha_i, i = 1, 2, 3$ of $W''(\alpha_i) = 0$ with $W$ given in (5.44) satisfy equation:

$$(\alpha - x_3)^3 - (\alpha - x_3)x_1 - x_2^2 = 0$$

It is convenient to introduce instead variables $f_i = \alpha_i - x_3$ which satisfy

$$f^3 - f x_1 - x_2^2 = 0$$

Clearly, $f_i$ are functions of $x_1$ an $x_2$ only and by taking derivatives of (5.46) we obtain:

$$\frac{\partial f_i}{\partial x_1} = \frac{f_i}{3 f_i^2 - x_1}, \quad \frac{\partial f_i}{\partial x_2} = \frac{2 x_2}{3 f_i^2 - x_1}$$

In terms of variables $f_i$ the canonical coordinates become

$$u_i = W(\alpha_i) = x_3 + \frac{3}{2} f_i + \frac{1}{2} x_1$$

and satisfy $\sum_{i=1}^{3} u_i = 3 x_3 - x_1^2/2 x_2^2$. We find that the Lamé coefficients are given by:

$$h_i^2 = \frac{\partial x_1}{\partial u_i} = \frac{f_i^3}{3 f_i^2 - x_1}$$

The homogeneity of $h_i^2$ is found after applying a general formula $E \partial_i = \partial_i E - \partial_i$ to the above equality with the result

$$E(h_i^2) = E(\frac{\partial x_1}{\partial u_i}) = \partial_i E(x_1) - \partial_i x_1 = (d_3 - 1) \frac{\partial x_1}{\partial u_i} = (d_3 - 1) h_i^2 = h_i^2.$$  

Here we used that $d_3 = \sum_{\alpha=1}^{3} d_{\alpha} \eta^{\alpha_1} = 2$ for the model under consideration. Thus the current Lamé coefficients $h_i^2$ will not depend on only one variable $s$. Notice, that in the previous $n = m = 1$ model the value $d_1 = \sum_{\alpha=1}^{3} d_{\alpha} \eta^{\alpha_1} = 1$ was consistent with homogeneity of $h_i^2$ being zero.

The corresponding rotation coefficients

$$\beta_{ij} = \frac{1}{2 \sqrt{\frac{\partial^2 x_1}{\partial u_i \partial u_j}}}$$

are calculated straightforwardly from knowledge of (5.49) and following derivatives

$$\frac{\partial f_i}{\partial u_j} = \frac{2 f_i^3 f_j + f_i^3 f_i}{(3 f_i^2 - x_1)(3 f_j^2 - x_1)}, \quad i \neq j$$
They are explicitly given by

\[
\beta_{ij} = \frac{1}{\sqrt{f_if_j}} \frac{1}{\sqrt[3]{f_i^2 - 4x_1}} \frac{N_k}{(f_i - f_j)^2 (3f_k^2 - x_1)^{5/2}}
\]  

(5.53)

where \(N_k = 8x^4_2 f_k^2 - 4x_1 x^4_2 + f_i^4 f_j^4\) with indices \(i, j, k\) being cyclic.

The result for \(\omega_k = (u_j - u_i) \beta_{ij}\) is

\[
\omega_k = r \frac{3g_k^2 - 4}{4(g_k - 3r)(3g_k^2 - 1)^5} \left(8g_k^4 - 4g_k^2 + \frac{x^2}{g_k^2}\right)^2
\]

(5.54)

where \(g_k\) are three roots of the equation \(g^3 - g - r = 0\) with \(r = x^2_2 / x_1^{3/2}\).

One verifies that \(\omega_2^2\) from (5.54) do indeed satisfy \(\sum_{k=1}^3 \omega_k^2 = -1/4\) and the Euler top equations (3.3)-(3.5). This model provides an example of the Lamé coefficients with non-zero homogeneity and consequently \(\omega_k^2\) are not proportional to the Lamé coefficients.

Is it possible though to make another choice for solutions of the Euler top equation in this model. Consider, namely

\[
\tilde{h}_i^2 = \frac{\partial x_3}{\partial u_i} = \frac{f_i^2}{3f_i^2 - x_1}
\]

(5.55)

with the properties \(\sum_{i=1}^3 \tilde{h}_i^2 = 1\) and \(E(\tilde{h}_i^2) = (d_1 - 1)\tilde{h}_i^2 = 0\)

The corresponding rotation coefficients

\[
\tilde{\beta}_{ij} = \frac{1}{2} \frac{\partial^2 x_3}{\partial u_i \partial u_j}
\]

(5.56)

are found to be given by

\[
\tilde{\beta}_{ij} = \frac{x^2_2}{2} \frac{3f_k^2 - x_1}{(3f_i^2 - x_1)^{3/2}(3f_j^2 - x_1)^{3/2}}
\]

(5.57)

which produces \(\tilde{\omega}_k = (u_j - u_i) \tilde{\beta}_{ij}\) equal to

\[
\tilde{\omega}_k^2 = -\frac{1}{16} \tilde{h}_k^2 = -\frac{1}{16} \frac{f_k^2}{(3f_k^2 - x_1)}
\]

(5.58)

which satisfy \(\sum_{k=1}^3 \omega_k^2 = -1/16\) and the Euler top equations (3.3)-(3.5).

6 Discussion

This paper shows how to derive the canonical integrable structures for some class of rational Lax functions associated with a particular reduction of the dispersionless KP hierarchy. This derivation generalizes the well-known construction of the monic polynomials \([3, 12]\). The three-dimensional examples provide solutions to the
Painlevé VI equation. Given that the flows of the canonical integrable models can essentially be reformulated as isomonodromic deformations its connection to the sixth Painlevé equation is not surprising.

Much of the discussion is centered around the tau functions from which all the objects of the Darboux-Egoroff metric can be derived. The tau functions of the three-dimensional examples had a scaling dimension of $R^2 = 1/4$ and the corresponding prepotentials contained logarithmic terms. For the scaling dimensions, $R^2 = n^2$ such that $n$ is an integer, the multi-component KP hierarchy provides a framework for the construction of canonical integrable hierarchies. The longterm goal of this work is to search for a universal approach to the formulation of the canonical integrable models which would include models with fractional scaling dimensions as the ones encountered in examples based on the rational potentials of the LG type.

Further progress is needed for the classification of relevant rational reductions of the Toda and KP hierarchies and the canonical integrable models with associated Darboux-Egoroff metrics which can be derived from related rational Lax functions. This problem is currently under investigation.

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References
[1] E. Witten, On the structure of the topological phase of two-dimensional gravity, Nucl. Phys. B 340, 281 (1990)
[2] R. Dijkgraaf, H. Verlinde and E. Verlinde, Topological strings in $D < 1$, Nucl. Phys. B 352, 59 (1991)
[3] B. Dubrovin, Geometry of 2D topological field theories, Springer Lect. Notes Math. 1620, 120 (1995) [arXiv:hep-th/9407018]
[4] B. Dubrovin, Integrable systems in topological field theory, Nucl. Phys. B 379, 627 (1992)
[5] J. van de Leur, Twisted $GL_n$ loop group orbit and solutions of the WDVV equations, Intern. Math. Research Notices 11, 551 (2001) [arXiv:nlin.si/0004021]
[6] H. Aratyn, J.F. Gomes and A.H. Zimerman, Multidimensional Toda equations and topological-anti-topological fusion, to appear in J. Geom. Phys., arXiv:hep-th/0107056
[7] J.W. van de Leur and R. Martini, The construction of Frobenius manifolds from KP tau-functions, Commun. Math. Phys. 205, 587 (1999) [arXiv:solv-int/9808008]
[8] H. Aratyn and J. van de Leur, Integrable structure behind WDVV equations, contribution to NEEDS’01 Conference in Cambridge, to appear in Theor. Math. Phys., [arXiv:hep-th/0111243]

[9] H. Aratyn and J. van de Leur, Solutions of the WDVV equations and integrable hierarchies of KP type, [arXiv:hep-th/0104092]

[10] H. Aratyn and J. van de Leur, paper in preparation

[11] S. Aoyama and Y. Kodama, Topological Landau-Ginzburg theory with a rational potential and the dispersionless KP hierarchy, Commun. Math. Phys. 182, 185 (1996) [arXiv:hep-th/9505122]; Topological conformal field theory with a rational W potential and the dispersionless KP hierarchy, Mod. Phys. Lett. A 9, 2481 (1994) [arXiv:hep-th/9404011].

[12] N.J. Hitchin, Frobenius manifolds. With notes by David Calderbank. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 488, Gauge theory and symplectic geometry, Montreal, PQ (1995) 69–112

[13] N.J. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, J. Diff. Geom. 42, 30 (1995)

[14] N.J. Hitchin, Poncelet polygons and the Painlevé transcendents, Geometry and Analysis, Oxford University Press, Bombay, 1996, 151–185

[15] N.J. Hitchin, A new family of Einstein metrics, manifolds and geometry, Symposia Mathematica series v. 36, De Bartolomeis, Triceri, and Vesentini eds., Cambridge University Press.

[16] J. Segert, Painlevé solutions from equivariant holomorphic bundles, Preprint (1996), available from www.math.missouri.edu/~jan/papers/painpreprint.pdf

[17] J. Segert, Frobenius manifolds from Yang-Mills instantons, Math. Res. Lett. 5, no. 3, 327 (1998) [arXiv:dg-ga/9710003]

[18] B. Dubrovin and Y. Zhang, Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants, [arXiv:math.dg/0108160]

[19] J. H. Chang and M. H. Tu, Topological field theory approach to the generalized Benney hierarchy, [arXiv:hep-th/0108055].

[20] I.A.B. Strachan, Degenerate Frobenius manifolds and the bi-Hamiltonian structure of rational Lax equations, J. Math. Phys. 40, 5058 (1999) [arXiv:solv-int/9807004]