Relative entropy is an exact measure of non-Gaussianity

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We prove that the closest Gaussian state to an arbitrary $N$-mode field state through the relative entropy is built with the covariance matrix and the average displacement of the given state. Consequently, the relative entropy of an $N$-mode state to its associate Gaussian one is an exact distance-type measure of non-Gaussianity. In order to illustrate this finding, we discuss the general properties of the $N$-mode Fock-diagonal states and evaluate their exact entropic amount of non-Gaussianity.

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I. INTRODUCTION

Distance-type measures for evaluating properties of the states involved in various protocols are at the heart of quantum information from its early days. The distance from a given state having a specific property to a reference set of states not having it has been recognized as a measure of that property [1, 5]. Two main problems are usually faced when defining a distance-type measure: choosing a convenient metric and identifying the reference set of states. Indeed, on the one hand, the metric used should be able to distinguish properly between quantum states. Accordingly, strong candidates observing the distinguishability criterion are the relative entropy [4, 5] and the Bures distance [6], although it is not always easy to calculate them. On the other hand, a reference set of states as large and relevant as possible is needed in order to get accurate results. Nevertheless, a few-parameter characterization of the reference states is highly desirable.

Had we chosen a reliable metric and identified an appropriate reference set of states, quantifying the given property could be an overwhelming extremization problem. The originally proposed properties to be quantified by distance-type measures were the nonclassicality of continuous-variable states [1] and the entanglement in both discrete- and continuous-variable settings [2, 3]. More recently, various distance-type degrees of polarization were also investigated [7]. We also mention a recent interesting proposal for a unified distance-type treatment of quantum correlations by means of the relative entropy [8]. Note that, in spite of not being a a true metric, the relative entropy is acceptable from the standpoint of state distinguishability due to the quantum Sanov theorem [4]. The probability of not distinguishing the states $\hat{\sigma}'$ and $\hat{\sigma}''$ after $\mathcal{N}$ measurements performed on $\hat{\sigma}'$ is

$$P_{\mathcal{N}}(\hat{\sigma}' \to \hat{\sigma}'') = \exp[-\mathcal{N}S(\hat{\sigma}'|\hat{\sigma}'')],$$

(1.1)

where the relative entropy between the states $\hat{\sigma}'$ and $\hat{\sigma}''$ is defined as the difference

$$S(\hat{\sigma}'|\hat{\sigma}'') := \text{Tr}[\hat{\sigma}' \ln(\hat{\sigma}') - \text{Tr}[\hat{\sigma}' \ln(\hat{\sigma}'')].$$

(1.2)

The relative entropy was successfully used as a measure of entanglement for pure bipartite states providing one of the few exact and general evaluations [2, 3]. In the mixed-state case, the tableau of exact results using relative entropy is much poorer.

In continuous-variable quantum information, the Gaussian states (GSs) and Gaussian transformations play an outstanding role from both the theoretical and practical viewpoints [2, 10]. On the theoretical side, the GSs have convenient phase-space representations in terms of Gaussian functions [11]. Experimentally, they are quite accessible and were effectively obtained with light, Bose-Einstein condensates, trapped ions, and Josephson junctions. Although the GSs proved to be important resources in quantum information processing [3, 11], it was quite recently realized that non-Gaussian states and operations could be more efficient in some quantum protocols such as teleportation, cloning, and entanglement distillation. To quantify the non-Gaussianity as a resource in such cases, some distance-type measures of this property were proposed [12, 13]. However, in order to avoid the expected complications of the necessary extremization procedures, a more pragmatic view prevailed in defining the degrees of non-Gaussianity [12, 16]. Let us designate by $\hat{\tau}_G$ the GS having the same mean displacement and covariance matrix as the given state $\hat{\rho}$. Instead of considering the whole reference set of the GSs, it has been convenient to choose $\hat{\tau}_G$ as a unique reference state for all the distances employed. A measure of non-Gaussianity was then evaluated as the distance from the given state $\hat{\rho}$ to the associate GS $\hat{\tau}_G$. In Refs. [12–14], Genoni et al. introduced the Hilbert-Schmidt and relative-entropy measures of non-Gaussianity. Then, in Ref. [15], a non-Gaussianity measure expressed as the difference between the Wehrl entropies of the states $\hat{\tau}_G$.
and " was proposed, while in Refs. the non-
Gaussianity based on the Bures metric was similarly intro-
duced.

In the present paper we choose the relative entropy to
measure the closeness of the quantum states of light.
Specifically, the non-Gaussianity of an arbitrary N-mode
state is reconsidered. Let G be the set of all the N-
mode GSs. We define here the entropic amount of non-
Gaussianity of a given N-mode state \( \hat{\rho} \) as its minimal
relative entropy to any N-mode GS:

\[
\delta_G[\hat{\rho}] := \min_{\hat{\rho}_G \in G} S(\hat{\rho} | \hat{\rho}_G). \tag{1.3}
\]

The paper is organized as follows. Section II recalls the
general structure of a GS. In Sec. III we prove that the
closest N-mode GS to an arbitrary N-mode state is pre-
cisely the associate GS of the latter. Section IV is devoted
to the special case of the Fock-diagonal states. In Sec. V
we present our concluding remarks.

II. GAUSSIAN DENSITY OPERATORS

In the following we deal with a state space that is the
tensor product of N one-mode Hilbert spaces:

\[
\mathcal{H} = \bigotimes_{j=1}^{N} \mathcal{H}_j. \tag{2.1}
\]

We start from two fundamental mathematical results: (i)
the algebraic theorem of Williamson and (ii) the existence of the metaplectic representation of the symplec-
tic group \( \text{Sp}(2N, \mathbb{R}) \) on the N-mode Hilbert space.
As a consequence, any N-mode GS \( \hat{\rho}_G \) can be
obtained by performing a suitable unitary transformation
of a well-defined N-mode thermal state (TS) \( \hat{\rho}_T \):

\[
\hat{\rho}_G = \hat{D}(\{\alpha\})\hat{U}(S)\hat{\rho}_T\hat{U}^\dagger(S)\hat{D}(\{\alpha\}). \tag{2.2}
\]

Recall that an N-mode TS is a product state

\[
\hat{\rho}_T = \bigotimes_{j=1}^{N} (\hat{\rho}_T)_j(\bar{n}_j) \tag{2.3}
\]

whose one-mode factors have the exponential form

\[
(\hat{\rho}_T)_j(\bar{n}_j) := \frac{1}{\bar{n}_j + 1} \exp(-\eta_j \hat{a}^\dagger_j \hat{a}_j), \tag{2.4}
\]

with \( \hat{a}_j := \frac{1}{\sqrt{2}}(\hat{q}_j + i\hat{p}_j) \) being the annihilation operator
of the mode \( j \). In Eq. (2.4), \( \bar{n}_j \) is the Bose-Einstein mean photon occupancy

\[
\bar{n}_j = [\exp(\eta_j) - 1]^{-1} \tag{2.5}
\]

and \( \eta_j \) is the positive dimensionless parameter

\[
\eta_j := \frac{\hbar \omega_j}{k_B T_j} = \ln \left( \frac{\bar{n}_j + 1}{\bar{n}_j} \right). \tag{2.6}
\]

Further, in Eq. (2.2), \( \hat{D}(\{\alpha\}) := \otimes_{j=1}^{N} \hat{D}_j(\alpha_j) \) stands
for an N-mode Weyl displacement operator, where \( \hat{D}_j(\alpha_j) := \exp(\alpha_j \hat{a}^\dagger_j - \alpha_j^* \hat{a}_j) \). Then \( S \) is the symplectic
\( 2N \times 2N \) matrix, \( S \in \text{Sp}(2N, \mathbb{R}) \), stated by the above-
cited Williamson theorem, and \( \hat{U}(S) \) is the corresponding unitary operator of the metaplectic representation on the
N-mode Hilbert space \( \mathcal{H} \). We designate the canonical
quadrature operators of the modes as follows:

\[
\hat{R}_{2j-1} := \hat{q}_j, \quad \hat{R}_{2j} := \hat{p}_j \quad (j = 1, 2, \ldots, N). \tag{2.7}
\]

We employ here Einstein’s summation convention and recall two unitary transformations of the quadratures:

\[
\hat{U}(S)\hat{R}_m\hat{U}^\dagger(S) = S_{im}\hat{R}_i, \tag{2.8}
\]

\[
\hat{D}(\{\alpha\})\hat{R}_m\hat{D}(\{\alpha\})^\dagger = \hat{R}_m - \bar{R}_m(\{\alpha\})\hat{1}. \tag{2.9}
\]

In Eq. (2.9), \( \bar{R}_m(\{\alpha\}) \) denotes the expectation value
of the quadrature \( \bar{R}_m \) in the N-mode coherent state \( |\{\alpha\}\rangle\langle\{\alpha\}| \). We find it convenient to introduce the abbre-
viation

\[
\delta\bar{R}_m := \bar{R}_m - \bar{R}_m(\{\alpha\})\hat{1} \quad (m = 1, 2, \ldots, 2N), \tag{2.10}
\]

as well as the \( 2N \)-dimensional column vectors \( \bar{R} \) and \( \delta\bar{R} \),
whose components are the quadrature operators and , respectively. By using the diagonal matrix

\[
\eta := \sum_{j=1}^{N} \eta_j \sigma_0 \in \mathbb{M}_{2N}(\mathbb{R}), \tag{2.11}
\]

with \( \sigma_0 \) the \( 2 \times 2 \) identity matrix, a TS \( \hat{\rho}_T \) [Eqs. (2.3)
and (2.4)] reads

\[
\hat{\rho}_T = \exp \left[ -\sum_{j=1}^{N} \ln (\bar{n}_j + 1) + \frac{1}{4}\text{tr}(\eta) \right] \times \exp \left( -\frac{1}{2}\delta\bar{R}^T \eta \delta\bar{R} \right). \tag{2.12}
\]

Note that we have employed the symbol \( \text{tr} \) to designate
the trace of a matrix, while the symbol \( \text{Tr} \) denotes the
trace of an operator on the state space \( \mathcal{H} \). Substitu-
ton of Eq. (2.12) into Eq. (2.2) leads, via the transformation rules (2.8) and (2.9), to the exponential form of an
arbitrary Gaussian density operator:

\[
\hat{\rho}_G = \exp \left[ -\sum_{j=1}^{N} \ln (\bar{n}_j + 1) + \frac{1}{4}\text{tr}(\eta) \right] \times \exp \left[ -\frac{1}{2}(\delta\bar{R})^T (S\eta S^T) \delta\bar{R} \right]. \tag{2.13}
\]

Accordingly, the logarithm of any Gaussian density op-
erator is a quadratic polynomial in the canonical quadra-
ture operators [Eq. (2.7)]:

\[
\ln (\hat{\rho}_G) = -\frac{1}{2}(\delta\bar{R})^T (S\eta S^T) \delta\bar{R} \quad + \quad \sum_{j=1}^{N} \ln (\bar{n}_j + 1) + \frac{1}{4}\text{tr}(\eta) \hat{1}. \tag{2.14}
\]
Similar considerations are made in Ref. [21].

It is worth stressing that there is no need to specify the explicit form of the unitaries $U(S)$ in order to write Eq. (2.14). However, for one-mode GSs $(N = 1)$, Eq. (2.2) reduces to their familiar parametrization as displaced squeezed thermal states [22].

III. CLOSEST GAUSSIAN STATE

The main result of this paper is the following theorem.

Theorem 1. The nearest GS to a given $N$-mode state $\hat{\rho}$, as measured by the relative entropy, is precisely its associate GS $\hat{\tau}_{G}$.

Proof. We exploit the two equations that define the associate GS $\hat{\tau}_{G}$ of a given $N$-mode state $\hat{\rho}$:

$$\text{Tr} \left[ (\hat{\rho} - \hat{\tau}_{G}) \hat{R}_{l} \right] = 0,$$

$$\text{Tr} \left[ (\hat{\rho} - \hat{\tau}_{G}) \frac{1}{2} \left( \hat{R}_{l} \hat{R}_{m} + \hat{R}_{m} \hat{R}_{l} \right) \right] = 0.$$  \hfill (3.1) \hfill (3.2)

Owing to the quadratic structure of the operator (2.14), Eqs. (3.1) and (3.2) imply the identity

$$\text{Tr}[\hat{\rho} \ln (\hat{\rho}_{G})] = \text{Tr}[\hat{\tau}_{G} \ln (\hat{\rho}_{G})],$$

which holds for any $N$-mode GS $\hat{\rho}_{G}$. In particular,

$$\text{Tr}[\hat{\rho} \ln (\hat{\tau}_{G})] = \text{Tr}[\hat{\tau}_{G} \ln (\hat{\tau}_{G})] = - S(\hat{\tau}_{G}),$$

where $S(\hat{\rho})$ denotes the von Neumann entropy of the state $\hat{\rho}$. Let us write the relative entropies of the given state $\hat{\rho}$ and its associate GS $\hat{\tau}_{G}$ to an arbitrary GS $\hat{\rho}_{G} \in \mathcal{G}$:

$$S(\hat{\rho} | \hat{\rho}_{G}) = - \text{Tr}[\hat{\rho} \ln (\hat{\rho}_{G})] - S(\hat{\rho}),$$

$$S(\hat{\tau}_{G} | \hat{\rho}_{G}) = - \text{Tr}[\hat{\tau}_{G} \ln (\hat{\rho}_{G})] - S(\hat{\tau}_{G}).$$  \hfill (3.5) \hfill (3.6)

On account of Eqs. (3.3) and (3.4), Eq. (3.3) reads

$$S(\hat{\tau}_{G} | \hat{\rho}_{G}) = - \text{Tr}[\hat{\rho} \ln (\hat{\rho}_{G})] + \text{Tr}[\hat{\tau}_{G} \ln (\hat{\tau}_{G})].$$  \hfill (3.7)

By subtracting Eq. (3.7) from Eq. (3.5) we get the formula

$$S(\hat{\rho} | \hat{\rho}_{G}) - S(\hat{\tau}_{G} | \hat{\rho}_{G}) = - \text{Tr}[\hat{\rho} \ln (\hat{\tau}_{G})] - S(\hat{\rho})$$

$$= S(\hat{\rho} | \hat{\tau}_{G}).$$  \hfill (3.8)

An equivalent form of Eq. (3.8) is

$$S(\hat{\rho} | \hat{\rho}_{G}) - S(\hat{\rho} | \hat{\tau}_{G}) = S(\hat{\tau}_{G} | \hat{\rho}_{G}) \geq 0.$$  \hfill (3.9)

According to Eq. (3.9), the $N$-mode GS $\hat{\tau}_{G}$ is the closest GS, $\hat{\rho}_{G} \in \mathcal{G}$, to the given $N$-mode state $\hat{\rho}$, via the relative entropy:

$$S(\hat{\rho} | \hat{\tau}_{G}) = \min_{\hat{\rho}_{G} \in \mathcal{G}} S(\hat{\rho} | \hat{\rho}_{G}).$$  \hfill (3.10)

This concludes the proof of Theorem 1.

Corollary 1. The entropic non-Gaussianity, defined by Eq. (1.3), coincides with the original entropic measure introduced in Ref. [13]:

$$\delta_{S}[\hat{\rho}] = S(\hat{\rho} | \hat{\tau}_{G}).$$  \hfill (3.11)

In fact, we proved here that the original entropic non-Gaussianity has the significance of an exact entropic amount of non-Gaussianity. If the state $\hat{\rho}$ is Gaussian, then the minimal value (1.3) is reached for $\hat{\tau}_{G} = \hat{\rho}$, the unique state for which the relative entropy of non-Gaussianity $\delta_{S}[\hat{\rho}]$ [Eq. (1.3)] vanishes.

We point out that insertion of Eq. (3.4) into Eq. (3.8) allows one to express the minimal relative entropy $S(\hat{\rho} | \hat{\tau}_{G})$ as a difference of von Neumann entropies:

$$S(\hat{\rho} | \hat{\tau}_{G}) = S(\hat{\tau}_{G}) - S(\hat{\rho}).$$  \hfill (3.12)

Consequently, among all the $N$-mode states with given first- and second-order moments of the canonical quadrature variables [27], the state with maximal von Neumann entropy is the unique Gaussian one, namely, the $N$-mode state $\hat{\tau}_{G}$ [28].

IV. NON-GAUSSIANITY OF A FOCK-DIAGONAL STATE

We designate by $\{n\} := \{n_{1}, n_{2}, \ldots, n_{N}\}$ the occupancy label of a standard $N$-mode Fock state,

$$|\{n\}\rangle := \bigotimes_{j=1}^{N} |n_{j}\rangle_{j} |n_{j}\rangle_{j}.$$

An $N$-mode Fock-diagonal state $\hat{\rho}_{F}(\lambda)$ is a mixture of Fock states [41]:

$$\hat{\rho}_{F}(\lambda) = \sum_{\{n\}} \lambda_{\{n\}} |\{n\}\rangle \langle \{n\}|.$$

As a matter of fact, Eq. (4.1) is the spectral resolution of the density operator $\hat{\rho}_{F}(\lambda)$, whose eigenprojections are the Fock states (4.1) and whose eigenvalues $\lambda_{\{n\}}$, besides satisfying the general requirements

$$\lambda_{\{n\}} \geq 0, \quad \sum_{\{n\}} \lambda_{\{n\}} = 1,$$

are otherwise arbitrary. By the same token, Eq. (4.2) displays the feature of $\hat{\rho}_{F}(\lambda)$ of being a classically correlated state [8, 24].

The one-mode reductions of the Fock-diagonal state (FDS) (4.2), are the states $\hat{\rho}_{j}(\lambda) := \text{Tr}_{H_{j}}[\hat{\rho}_{F}(\lambda)]$, where the Hilbert space $H_{j}$ is a tensor product (2.1) with the factor $H_{j}$ omitted. They have the spectral decompositions

$$\hat{\rho}_{j}(\lambda) = \sum_{n=0}^{\infty} (\lambda_{j})_{n} |n\rangle_{j} \langle n| \quad (j = 1, 2, \ldots, N),$$

where $|n\rangle_{j}$ denotes the $n$-mode Fock state, and $\lambda_{\{n\}}$ is a vector and $\lambda_{j}$ is a scalar.
with the eigenvalues
\[
(\lambda_j)_{n_j} := \sum_{\{n\}^{n_j}} \lambda_{\{n\}} \quad (j = 1, 2, \ldots, N).
\]

In view of a general result obtained by Modi et al. [8], the nearest product state to the FDS \([4.2]\) is the tensor product of its one-mode marginals \([4.3]\):
\[
\hat{\rho}_\mu(\lambda) := \bigotimes_{j=1}^{N} \hat{\rho}_j(\lambda).
\] (4.4)

The product state \([4.4]\) is a special FDS, with factorized photon-number probabilities:
\[
|\Pi(\lambda)|_{\{n\}} := \prod_{j=1}^{N} (\lambda_j)_{n_j}.
\] (4.5)

The relative entropy
\[
S(\hat{\rho}_F(\lambda) | \hat{\rho}_\mu(\lambda)) = S(\hat{\rho}_\mu(\lambda)) - S(\hat{\rho}_F(\lambda))
\] (4.6)
is called the total mutual information of the state \(\hat{\rho}_F(\lambda)\) [8]. We are ready to state the following theorem.

**Theorem 2.** The nearest GS to a given \(N\)-mode Fock-diagonal state \(\hat{\rho}_F(\lambda)\), as measured by the relative entropy, is the \(N\)-mode TS with the same mean photon occupancies of the modes.

**Proof.** It is easy to see that the mean displacements of any FDS are all equal to zero. For instance,
\[
\langle q_j \rangle = \sum_{n=0}^{\infty} (\lambda_j)_n n \langle q_j \rangle |n\rangle_j = 0.
\]

In this manner, we evaluate both the expectation values and the covariances of the quadrature observables and find
\[
\langle \hat{R}_j \rangle = \text{Tr} \left[ \hat{R}_j \hat{\rho}_F(\lambda) \right] = 0,
\] (4.7)
\[
\sigma(\hat{R}_l, \hat{R}_m) := \text{Tr} \left[ \frac{1}{2} \left( \hat{R}_l \hat{R}_m + \hat{R}_m \hat{R}_l \right) \hat{\rho}_F(\lambda) \right] - \langle \hat{R}_l \rangle \langle \hat{R}_m \rangle = 0 \quad (l \neq m).
\] (4.8)

According to Eq. \([4.3]\), the covariance matrix (CM) \(V_F(\lambda)\) of the FDS \(\hat{\rho}_F(\lambda)\) [Eq. \([4.2]\)] is the direct sum of the \(2 \times 2\) CMs \(V_j(\lambda)\) of the reduced single-mode states \(\hat{\rho}_j(\lambda)\) [Eq. \([4.3]\)]:
\[
V_F(\lambda) = \bigoplus_{j=1}^{N} V_j(\lambda).
\] (4.9)

We are left to find the CM \(V_j(\lambda)\) of a reduced one-mode state \(\hat{\rho}_j(\lambda)\) [Eq. \([4.3]\)]. In order to simplify the notation, we consider a Fock-diagonal single-mode state
\[
\hat{\rho}(\lambda) = \sum_{n=0}^{\infty} \lambda_n |n\rangle \langle n|,
\] (4.10)
with arbitrary photon-number probabilities \(\lambda_n\). This is an undisplaced state whose covariances are
\[
\sigma(q_j, q_j) = \sum_{n=0}^{\infty} \lambda_n \langle q_j^2 |n\rangle = \langle n\rangle + \frac{1}{2},
\] (4.11)
\[
\sigma(\hat{p}_j, \hat{p}_j) = \sum_{n=0}^{\infty} \lambda_n \langle \hat{p}_j^2 |n\rangle = \langle n\rangle + \frac{1}{2},
\] (4.12)
\[
\sigma(q_j, \hat{p}_j) = \sum_{n=0}^{\infty} \lambda_n \langle \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) |n\rangle = 0.
\] (4.13)

Accordingly, the CM \(V(\lambda)\) of the FDS \(\hat{\rho}(\lambda)\) [Eq. \([4.10]\)] depends solely on its mean number of photons \(\langle n\rangle\) and is proportional to the \(2 \times 2\) identity matrix \(\sigma_0\):
\[
V(\lambda) = \left( \langle n\rangle + \frac{1}{2} \right) \sigma_0, \quad \langle n\rangle = \sum_{n=0}^{\infty} n \lambda_n.
\] (4.14)

Recall that a one-mode TS is Fock-diagonal and is fully determined by its mean photon occupancy \(\bar{n}\) [Eq. \([2.5]\)] via the spectral representation
\[
\hat{\rho}_T(\bar{n}) = \frac{1}{\bar{n} + 1} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^n |n\rangle \langle n|.
\] (4.15)

Therefore, its CM
\[
V_T(\bar{n}) = \left( \bar{n} + \frac{1}{2} \right) \sigma_0
\] (4.16)
coincides with \(V(\lambda)\) [Eq. \([4.13]\)] provided the mean numbers of photons in the two FDSs are equal: \(\bar{n} = \langle n\rangle\). Hence the closest GS to the FDS \([4.10]\) is the one-mode TS \(\hat{\rho}_T(\langle n\rangle)\), with the same mean photon number.

The above conclusions apply to each reduced one-mode state \(\hat{\rho}_j(\lambda)\) [Eq. \([4.3]\)] whose CM is therefore of the form \([4.14]\):
\[
V_j(\lambda) = \left( \langle n\rangle_j + \frac{1}{2} \right) \sigma_0, \quad \langle n\rangle_j = \sum_{n=0}^{\infty} n \lambda_{\{n\}}.
\] (4.17)

This is also the CM of a single-mode TS \(\langle \hat{\rho}_T \rangle_j(\langle n\rangle_j)\) whose Bose-Einstein mean occupancy \([2.5]\) is \(\bar{n}_j = \langle n\rangle_j\),
\[
\langle \hat{\rho}_T \rangle_j(\langle n\rangle_j) = \frac{1}{\langle n\rangle_j + 1} \times \sum_{n=0}^{\infty} \left( \frac{\langle n\rangle_j}{\langle n\rangle_j + 1} \right)^n |n\rangle_j \langle n|.
\] (4.18)

Substitution of Eq. \([4.17]\) into Eq. \([4.9]\) shows that the CM of the FDS \(\hat{\rho}_F(\lambda)\) [Eq. \([4.2]\)] is diagonal:
\[
V_F(\lambda) = \bigoplus_{j=1}^{N} \left[ \left( \langle n\rangle_j + \frac{1}{2} \right) \sigma_0 \right].
\] (4.19)
In accordance with the remark preceding Eq. 4.18, this coincides with the CM of the $N$-mode TS
\[ \hat{\rho}_T(\lambda) := \bigotimes_{j=1}^{N} (\hat{\rho}_T)_j(n_j). \] (4.20)

The $N$-mode TS [4.20] is the associate GS of the FDS [4.2]. According to Theorem 1, this is the nearest GS to the given FDS $\hat{\rho}_F(\lambda)$, as measured by the relative entropy. The remark that both states have the same mean numbers of photons in all the $N$ modes concludes the proof of Theorem 2.

**Corollary 2.** The entropic non-Gaussianity (3.11) of a FDS [4.2] and its nearest product state $\hat{\pi}_\rho(\lambda)$ have the same closest GS through the relative entropy.

Indeed, besides having zero mean quadratures of the field modes [4.3], the FDSs $\hat{\rho}_F(\lambda)$ [Eq. 4.2] and $\hat{\pi}_\rho(\lambda)$ [Eq. 4.24] possess the same diagonal CM $\gamma_F(\lambda)$ [Eq. 4.20]. Accordingly, they have a common associate GS. In view of Theorem 2, this is the TS with the same mean occupancies of the modes, $\hat{\rho}_T(\lambda)$ [Eq. 4.20]. By virtue of Theorem 1, their common associate GS [4.20] is also the nearest GS to both FDSs.

Moreover, the difference of the relative entropies of non-Gaussianity (3.12)
\[ S(\hat{\rho}_F(\lambda)|\hat{\rho}_T(\lambda)) = S(\hat{\rho}_F(\lambda)) - S(\hat{\rho}_T(\lambda)) \] (4.22)

and
\[ S(\hat{\pi}_\rho(\lambda)|\hat{\rho}_T(\lambda)) = S(\hat{\rho}_T(\lambda)) - S(\hat{\pi}_\rho(\lambda)) \] (4.23)
is equal to the total mutual information 4.6 of the FDS $\hat{\rho}_F(\lambda)$:
\[ S(\hat{\rho}_F(\lambda)|\hat{\rho}_T(\lambda)) - S(\hat{\rho}_F(\lambda)|\hat{\rho}_T(\lambda)) = S(\hat{\rho}_F(\lambda)|\hat{\rho}_F(\lambda)). \] (4.24)

It follows that the common associate GS $\hat{\rho}_T(\lambda)$ is closer to the product state $\hat{\pi}_\rho(\lambda)$ than to the given FDS $\hat{\rho}_F(\lambda)$:
\[ \delta_S[\hat{\rho}_F(\lambda)] \geq \delta_S[\hat{\pi}_\rho(\lambda)], \] (4.25)

with saturation of the inequality if and only if the chosen FDS is a product state: $\hat{\rho}_F(\lambda) = \hat{\pi}_\rho(\lambda)$. The entropic non-Gaussianity (3.11) of the product state $\hat{\pi}_\rho(\lambda)$ [Eq. 4.24] is a sum of $N$ single-mode terms:
\[ \delta_S[\hat{\pi}_\rho(\lambda)] = \sum_{j=1}^{N} \left[ (\langle n_j \rangle + 1) \ln (\langle n_j \rangle + 1) - \langle n_j \rangle \ln (\langle n_j \rangle) \right] + \sum_{j=1}^{N} \left\{ \sum_{n=0}^{\infty} \lambda_{(n)} \ln [\lambda_{(n)}] \right\}. \] (4.26)

**V. DISCUSSION AND CONCLUSIONS**

The importance of Theorem 1 lies in its accuracy and generality: It provides an *exact* result that holds for *any* state of the quantum radiation field. The identity between the associate GS and the nearest one, as measured by the relative entropy, gives a deep significance to the former and a quite simple recipe of evaluating the latter without any extremization hurdle. Concerning the proof, we mention again the central roles of Williamson’s theorem and of the metaplectic representation, leading to the general form (2.3) of a GS. In addition, we exploited just the basic properties of the relative entropy of being a non-negative quantity that vanishes if an only if the two states coincide. The fact that the entropic amount of non-Gaussianity is the difference of two von Neumann entropies allows one some simple evaluations and comparisons.

One may wonder about the relevance of our choice of the FDSs as an insightful example. However, there are several reasons for this preference.

(i) The set $C_F$ of all the $N$-mode FDSs is the convex hull of the Fock basis in the Hilbert space $\mathcal{H}_N$.

(ii) Any FDS $\hat{\rho}_F(\lambda)$ is fully determined by its photon-number distribution $\{\lambda_{(n)}\}$.

(iii) The FDSs are commuting states. This property makes calculations easier.

(iv) Any FDS has zero mean quadratures of the modes.

(v) The FDSs are classically correlated states [8, 24]. Their projective quantum discord [8] vanishes and a *forteriori* their original quantum discords [25] are zero.

(vi) The nearest product state to a FDS [4.2] is the state [4.4] which also belongs to the convex set $C_F$.

(vii) As shown by Theorem 2, the closest GS to a FDS [4.2] is the TS [4.20] with the same mean numbers of photons. This state, like any $N$-mode TS, belongs to the set $C_F$.

(viii) The TSs are the only FDSs that are Gaussian.

There is one more feature of the FDSs that allows us to understand some aspects regarding the relationship between the set of all the $N$-mode states and its convex subset $C_F$. Specifically, a FDS [4.2] can be prepared starting from any $N$-mode state $\hat{\rho}_F$ with the same photon-number probabilities: $\langle n | \hat{\rho}_F | n \rangle = \lambda_{(n)}$. The required quantum operation is a nonselective ideal von Neumann-Lüders measurement [26] of the numbers of photons in all the $N$ field modes. By performing it, an
N-mode state $\hat{\rho}|\lambda$ is transformed to the FDS \eqref{eq:5.1}:

$$
\hat{\rho}_F(\lambda) = \sum_{\{n\}} |\{n\} \rangle \langle \{n\}| \hat{\rho}|\lambda|\{n\} \rangle \langle \{n\}|.
$$ \hspace{1cm} \text{(5.1)}

The nonselective projective measurement \eqref{eq:5.1} is a Lüders map \cite{27} denoted by $\Lambda_{\{n\}}$:

$$
\hat{\rho}_F(\lambda) = \Lambda_{\{n\}}(\hat{\rho}|\lambda).
$$ \hspace{1cm} \text{(5.2)}

Note the formula

$$
S(\hat{\rho}|\lambda|\hat{\rho}_F(\lambda)) = S(\hat{\rho}_F(\lambda)) - S(\hat{\rho}|\lambda|).
$$ \hspace{1cm} \text{(5.3)}

Equation \eqref{eq:5.3} implies the following extremum property.

\textit{Proposition 1.} From all the N-mode states with a given photon-number distribution, $\{\lambda_{\{n\}}\}$, the FDS $\hat{\rho}_F(\lambda)$ has the maximal von Neumann entropy:

$$
S(\hat{\rho}_F(\lambda)) = \max S(\hat{\rho}|\lambda|).
$$ \hspace{1cm} \text{(5.4)}

We specialize another general result obtained by Modi et al. \cite{28} in the form of the following statement.

\textit{Proposition 2.} Let us consider an arbitrary N-mode state $\hat{\rho}|\lambda$ whose photon-number probabilities are $\lambda_{\{n\}}$. Then the closest N-mode FDS to the given state $\hat{\rho}|\lambda$ by way of the relative entropy is the FDS $\hat{\rho}_F(\lambda)$ that has the same photon-number distribution $\{\lambda_{\{n\}}\}$:

$$
S(\hat{\rho}|\lambda|\hat{\rho}_F(\lambda)) \leq S(\hat{\rho}|\lambda|\hat{\rho}_F(\mu)),
$$ \hspace{1cm} \text{(5.5)}

or, equivalently,

$$
S(\hat{\rho}|\lambda|\hat{\rho}_F(\lambda)) = \min_{\{\mu_{\{n\}}\}} S(\hat{\rho}|\lambda|\hat{\rho}_F(\mu)).
$$ \hspace{1cm} \text{(5.6)}

This means that the relative entropy \eqref{eq:5.5} is the entropic distance from an arbitrary N-mode state $\hat{\rho}|\lambda$ to the convex set $C_F$ of all the N-mode FDSs.

To conclude, the FDSs have some conspicuous properties and at the same time they can be handled rather easily. This justifies why we chose them as an illustrative class of field states.

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