DIRECT DETECTION OF COLD DARK MATTER SUBSTRUCTURE

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ABSTRACT

We devise a method to measure the abundance of satellite halos in gravitational lens galaxies and apply our method to a sample of seven lens systems. After using Monte Carlo simulations to verify the method, we find that substructure comprises \(f_{\text{sat}} = 0.02\) (median, \(0.006 < f_{\text{sat}} < 0.07\) at 90% confidence) of the mass of typical lens galaxies, in excellent agreement with predictions of cold dark matter (CDM) simulations. We estimate a characteristic critical radius for the satellites of \(0.0001 < b < 0.0006\) (90% confidence). For a \(dn/dM \propto M^{-1.5}\) (\(M_{\text{low}} < M < M_{\text{high}}\)) satellite mass function, the critical radius provides an estimate for the upper mass limit of \(10^6 \, M_\odot \lesssim M_{\text{high}} \lesssim 10^9 \, M_\odot\). Our measurement confirms a generic prediction of CDM models and may obviate the need to invoke alternatives to CDM such as warm dark matter or self-interacting dark matter.

Subject headings: cosmology: theory — dark matter — galaxies: formation — gravitational lensing — large-scale structure of universe

1. INTRODUCTION

A discrepancy between the number of satellite halos expected from cold dark matter (CDM) simulations and the observed numbers of Galactic satellite galaxies is part of the prosecution’s case for a crisis in the CDM scenario for structure formation (e.g., Kauffmann, White, & Guiderdoni 1993; Moore et al. 1999; Klypin et al. 1999). Suggested solutions range from the mundane, such as the inhibition of star formation in the satellites by photoionization (e.g., Klypin et al. 1999; Bullock, Kravtsov, & Weinberg 2000), to the exotic, such as the disruption of the satellites by self-interacting dark matter (e.g., Spergel & Steinhardt 2000) or changes in the power spectrum (e.g., Bode, Ostriker, & Turok 2001; Colin, Avila-Reese, & Valenzuela 2000). The satellite crisis must also be closely related to the more general problem that the number of low-luminosity galaxies diverges only as \(1/L \sim 1/M\), while the number of CDM halos diverges as \(\sim 1/M^2\), implying that the probability of forming a visible galaxy in a low-mass halo must diminish as \(\sim M\) (e.g., Scoccimarro et al. 2001; Kochanek 2001; Chiu, Gnedin, & Ostriker 2001). In principle, the measured abundances of satellite halos should provide a strong test of the CDM scenario, but because the satellites used as evidence for a problem have low luminosities and (in many cases) low surface brightnesses, it is difficult to apply this test to any galaxy besides our own. Moreover, the test only considers the numbers of satellites with detectable optical emission, which is at best a lower bound on the number of CDM halos.

Gravitational lensing is the only probe that avoids both of these limitations, as was already noted by Moore et al. (1999). First, the test can be applied to many lens systems spanning a range of redshifts and physical properties. Second, because lensing phenomena couple directly to mass, lenses are sensitive to both luminous and dark substructures in CDM halos. Mao & Schneider (1998) pointed out that the anomalous image flux ratios observed in several lenses, particularly B1422+231, could be explained by substructures such as low-mass satellites in the primary-lens galaxy. The primary lens magnifies the perturbations from the substructure, making the brightest images particularly susceptible to the effects of substructure. Recently, Metcalf & Madau (2001) quantified the effects of CDM satellites using simulations and found that the effects should be readily detected, and Chiba (2002) demonstrated that plausible CDM satellite distributions could explain the anomalous flux ratios in B1422+231 and PG1115+080. Detailed studies of B1422+231 (Keeton 2002; Bradac et al. 2002) find that the observed perturbations require substructure with mass scales comparable to CDM substructure (\(\gtrsim 10^6 \, M_\odot\)) rather than stellar microlensing, and Metcalf & Zhao (2002) have shown that the anomalous flux ratios cannot be reproduced in a large family of smooth potentials for the primary lens.

The missing link is an approach for analyzing the gravitational lens data to estimate the properties of the satellite population. In this paper we develop such an analysis method and apply it to a sample of seven lenses to estimate the surface density and characteristic mass of the perturbing satellites. We focus on analyzing four-image radio lenses because using the radio lenses eliminates the problem of dust extinction and minimizes the problems from stellar microlensing due to the relatively large source size (see Koopmans & de Bruyn 2000). We analyzed the lenses MG 0414+0534 (Hewitt et al. 1992), B0712+472 (Jackson et al. 1998), PG 1115+080 (Weymann et al. 1980), B1422+231 (Patnaik et al. 1992), B1608+656 (Fassnacht et al. 1996), B1933+503 (Sykes et al. 1998), and B2045+265 (Fassnacht et al. 1999). Of these seven four-image lenses, six show anomalous flux ratios that might be due to the effects of substructure. We develop our formalism, characterize our model for the satellite distribution, and test our analysis methods in § 2. We apply it to the lens sample in § 3. In § 4 we review our conclusions and their limitations and then outline the observations needed to improve them.

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2. ANALYZING THE EFFECTS OF SUBSTRUCTURE ON GRAVITATIONAL LENSES

In this section we outline our mathematical approach to analyzing the lenses to determine the properties of the substructure (§ 2.1) and the physical model we use for the satellites composing the substructure (§ 2.2). In § 2.3 we discuss the relationship between our model and the physical properties of the substructure, such as its fractional surface density, mass and velocity scales, and linear sizes. In § 2.4 we outline our Monte Carlo models and test the analysis method.

2.1. A Linearized Approach to Analyzing Substructure

Unlike the primary-lens galaxy, which we can observe directly to determine its position and optical properties (e.g., Lehár et al. 2000; Kochanek et al. 2000), we can detect substructure only through its effects on the positions and fluxes of the lensed images. This means that estimates for the properties of the substructure are difficult to separate from the properties of the primary lens (the “macro model”) to adopt the language of the quasar microlensing literature because many perturbations are degenerate with changes in the macro model. For this reason, Mao & Schneider (1998) focused on merging image pairs for which macro models generically predict similar image fluxes but the observations sometimes find very different fluxes. We instead allow the macro model to compensate for the effects of substructure as part of our analysis. If we confine our analysis to typical four-image lenses, we have 14 constraints for determining the 10 parameters of a realistic macro lens model. If we confine our analysis to two-image lenses, we have eight constraints.

We model the lens by combining a macrolensing potential \( \phi(x, p) \) defined by a set of parameters \( p \), with a localized perturbing potential for each image \( \delta \phi_i(x) \). For later notational simplicity, the source position and flux are considered part of the parameter vector \( p \). The time-delay surface near image \( i \) is

\[
\tau = \frac{1}{2}(u - x)^2 - \phi(x, p) - \delta \phi_i(x) = \tau_0(x, p) - \delta \phi_i(x),
\]

and we find images at solutions of \( V_\tau = 0 \) with an inverse magnification tensor of \( M^{-1} = VV^t \) and magnification \( M = |\mathbf{M}| \) (see Schneider, Ehlers, & Falco 1992).

We assume that the substructure produces small perturbations, so we can simplify its effects by expanding the lens equations as linear perturbations to a macro model for each image \( i \). We would like to linearize the equations so that the process of adjusting the macro models to compensate for the effects of substructure can be done rapidly. For macroparameters \( p_0 \), we find images at positions \( x_i^{(0)} \), with magnification tensors \( M_i^{(0)} \) at the solutions to \( V\tau_0(x_i^{(0)}, p_0) = 0 \). Expanding the lens equations about these solutions, the perturbed image positions are

\[
x_i^{(1)} = x_i^{(0)} + M_i^{(0)} \cdot (\delta x_i - \Delta p \cdot C_i),
\]

where \( \delta x_i = \nabla \delta \phi_i \) is the deflection produced by the substructure and

\[
C_i = \frac{d\nabla \delta \phi_i}{dp},
\]

evaluated at \( x = x_i^{(0)} \) and \( p = p_0 \) is the change in the macro model deflections produced by a small change \( \Delta p = p - p_0 \) in the macro model parameters. The perturbed image magnification is

\[
M_i^{(1)} = M_i^{(0)} + \frac{dM_i^{(0)}}{dp} \cdot \delta M + \frac{dM_i^{(0)}}{dx} \cdot \Delta x,
\]

which becomes

\[
M_i^{(1)} = M_i^{(0)} (1 + \delta m_i + \Delta p \cdot D_i + \delta x_i \cdot E_i),
\]

where

\[
\delta m_i = -\text{tr}(\mathbf{M} \delta \mathbf{M}_i^{-1}) \quad \text{for} \quad \delta \mathbf{M}_i^{-1} = -\nabla \nabla \delta \phi_i
\]
is the perturbation in the magnification due to the effects of the substructure on the magnification tensor,

\[
D_i = \frac{1}{M} \left( \frac{dM_i^{(0)}}{dp} - \frac{dM_i^{(0)}}{dx} \cdot \mathbf{M} \cdot \mathbf{C} \right)
\]

is the perturbation due to changing the lens parameters, and

\[
E_i = \frac{1}{M} \frac{dM}{dx} \cdot \mathbf{M}
\]
is the perturbation due to the effects of substructure on the deflections. The functions \( \delta m_i, D_i, \) and \( E_i \) are all evaluated at \( x = x_i^{(0)} \) and \( p = p_0 \). The flux of image \( i \) is \( f_i = f_i^{(0)} M_i = f_i^{(0)} (1 + \Delta f_i) M_i \), which we can linearize as

\[
f_i^{(1)} = f_i^{(0)} (1 + \delta m_i + \Delta p \cdot D_i + \delta x_i \cdot E_i),
\]

where \( \Delta p \cdot D_i + \Delta p \cdot J \cdot \Delta p \) and the fractional change in the source flux \( \Delta f \) is considered one of the model parameters.

We detect substructure as residuals in fits to the lens parameters that cannot be modeled by the macropotential. If we use a \( \chi^2 \) statistic, the fit statistic for the image positions is

\[
\chi^2 = \sum_{i=1}^N \left( \frac{x_i^{\text{obs}} - x_i^{(0)} - M_i^{(0)} (\delta x_i - \Delta p \cdot C_i)}{\sigma_x} \right)^2,
\]

where \( \sigma_x \approx 0.003 \) is the uncertainty in the observed image positions \( x_i^{\text{obs}} \). The fit statistic for the image fluxes is

\[
\chi^2_f = \sum_{i=1}^N \left( \frac{f_i^{\text{obs}} - f_i^{(0)} (1 + \delta m_i + \Delta p \cdot D_i + \delta x_i \cdot E_i)}{\sigma_{f_i}} \right)^2
\]

for observed fluxes \( f_i^{\text{obs}} \) and flux uncertainties \( \sigma_{f_i} \approx 0.05 f_i^{\text{obs}} \). The lens position, if measured, is constrained by

\[
\chi^2_{\Delta p} = \left( \frac{x_i^{\text{obs}} - x_i^{(0)} - \Delta p_i}{\sigma_x} \right)^2,
\]

where \( \Delta p \) represents the perturbations to the lens position and \( \sigma_x \approx 0.003 \) is the uncertainty in the observed position \( x_i^{\text{obs}} \). Because all the terms entering the fit statistic depend only linearly on \( \Delta p \), the fit statistic is a quadratic of the form

\[
\chi^2 = \chi^2_x + 2 \Delta p \cdot J + \Delta p \cdot J \cdot \Delta p,
\]

where \( \chi^2_{\Delta p} = \chi^2_x - \chi^2_{\Delta p} \) is minimized for \( \Delta p = -J^{-1} \cdot J \). With value \( \chi_{\min} \approx \chi^2_x - \chi^2_{\Delta p} \), if the
macro model were held fixed, the goodness of fit would be 
\( \chi^2_0 \), combining the effects of the substructure with the differ-
ences between the observations and the initial macro model. After adjus-
ting the macro model, the correction 
\( \Delta \chi^2 = 1 \cdot J \cdot J^{-1} \cdot 1 \) represents the ability of the macro model to 
mimic the effects of substructure.

In order to test the CDM predictions for substructure, we need to es-
imate the statistical properties of the substructure rather than discuss the evidence for perturbations from sub-
structure in individual lenses. For a statistical model of the 
substructure characterized by the parameters \( s \), the probability of the \( \alpha \) th substructure realization for lens \( j \), \( \delta_{\alpha j} \), is 
\( P(\delta_{\alpha j}|s) \). Given a concrete model for the substructure \( \delta_{\alpha j} \), we can com-
pute the likelihood for fitting the data as the likelihood of obtaining the \( \chi^2 \) statistic we find after reoptimizing the 
macro model given the substructure realization, 
\( P(D_j|\delta_{\alpha j}) \). The Bayesian probability of the model parameters 
given the data for \( j = 1, \ldots, N \) lenses is

\[
P(s, \delta_{\alpha j}|D_j) \propto P(s)\Pi_{j=1}^N P(D_j|\delta_{\alpha j})P(\delta_{\alpha j}|s),
\]

where \( P(s) \) is the prior probability distribution of the parameters \( s \). As in all Bayesian probabili-
ties, the expression summed over all variables is normalized to unity. In 
estimating the statistical properties of the substructure, we are not interested in the likelihoods of the individual realiza-
tions, but in the marginalized distribution

\[
P(s|D_j) \propto P(s)\Pi_{j=1}^N P(D_j|\delta_{\alpha j})P(\delta_{\alpha j}|s),
\]

where we sum over the \( \alpha = 1, \ldots, N_M \) Monte Carlo real-
izations of the substructure for each lens. Typically, we use 
\( N_M = 10^5 \) realizations. As we vary the statistical properties of the substructure \( s \), the fraction of the realizations \( \delta_{\alpha j} \) that 
significantly improve the goodness of fit varies. These changes in the fraction of realizations that improve the fit 
relative to the macro model alone allow us to estimate the 
parameters \( s \) describing the substructure.

In our final analysis, we used random realizations of per-
turbing satellites to estimate the perturbations. It is worth 
mentioning, however, that the problem can be fully linear-
ized if we use a Gaussian model for the perturbations. If 
\( k = \{\delta \eta, \delta \chi \} \) is a d-dimensional vector of the perturbation 
variables and they have a covariance matrix \( \mathbf{V}^{-1} = (k^T k) \), 
then the Gaussian model for the probability distribution of the perturbations is

\[
P(k) = |\mathbf{V}|^{1/2}(2\pi)^{-d/2} \exp(-\frac{1}{2} k \cdot \mathbf{V} \cdot k).
\]

The matrix \( \mathbf{V} \) is proportional to the inverse of the satel-
lite surface density, so by combining equations (10), (11), (13), and (14), the marginalizing integrals over the shifts 
in the lens model and the distribution of satellite realizations 
can be done analytically using standard methods for 
linear algebra and Gaussian integrals to leave an 
expression depending only on the statistical properties of the 
substructure. We did not use this for our actual anal-
ysis because it was relatively easy to perform the neces-
sary Monte Carlo realizations and integrals needed to 
reproduce the true probability distribution for the 
substructure and its correlations, but we did use it as an 
internal check on our results. We mention it here because 
other studies may find it to be of similar utility.

### 2.2. Substructure Models

Before applying the above formalism to observed lens 
systems, it is necessary to calculate the expected amplitude of perturbations to lensed images from CDM substructure. 
This involves calculating the expected levels of astrometric 
shifts in the image positions and the rms fluctuations in the 
local convergence and shear. These quantities are then 
magnified by the local magnification tensor of the smooth 
macro model.

We model CDM substructure by randomly laying down 
subclumps of surface density. The substructures seen in 
CDM simulations appear to have mass profiles consistent 
with the “universal” Navarro-Frenk-White (NFW) profile; 
however, for simplicity we treat them as pseudo-Jaffe mod-
els [density \( \rho \propto r^{-2}(r^2 + a^2)^{-1} \); see Munoz, Kochanek, & 
Keeton 2001], with convergence, the surface density in units 
of the critical surface density,

\[
\kappa(r) = \frac{\Sigma_r}{\Sigma_c} = \frac{b_1}{r} - \frac{1}{(r^2 + a^2)^{1/2}},
\]

where the critical surface density for lensing is 
\( \Sigma_c = c^2 D_{OS}/4\pi G D_{LS} D_{LS} \). Here \( b \) is a length scale similar to 
the Einstein radius of the subclump and \( a \) is a tidal or break 
radius. Note that \( b, r, \) and \( a \) are angular lengths, which are 
related to physical sizes by multiplication by the distance to 
the lens \( D_{LS} \). The total mass of a clump is \( M = \pi ba \Sigma_c \), 
where \( \Sigma_c \) is the critical density in angular units. If the surface 
mass density of the perturbers is \( \Sigma \), the number of pertur-
bers per unit area is \( N = (\Sigma/\Sigma_c)/b_a \). To leading order, the 
variance in the image deflection, convergence, and shear are

\[
\langle |\delta \chi|^2 \rangle \approx \frac{3}{2} \Sigma \pi b_a, \quad \langle \kappa^2 \rangle \approx \frac{1}{2} \Sigma \ln \frac{a}{s},
\]

where we must introduce a core radius \( s \) to make the variance 
in the convergence and shear finite. These perturbations 
are then magnified by the macro model to produce the 
observed perturbations in the image positions and fluxes. If 
the scale \( b \) is determined by the satellite’s tidal radius, we 
have \( a = (b_0b_a)^{1/2} \), where \( b_0 \) is the critical radius of the 
primary lens (see, e.g., Metcalf & Madau 2001). Thus, for a 
fixed satellite surface density, the variance of the astrometry 
perturbation in units of the critical radius of the macrolens 
is roughly

\[
\frac{\langle |\delta \chi|^2 \rangle^{1/2}}{b_0} \approx 10^{-3} \left( \frac{10 \Sigma}{\Sigma_c} \right)^{1/2} \left( \frac{10^3 b_a}{b_0} \right)^{3/4},
\]

while the variance in the shear and convergence is roughly

\[
\langle \kappa^2 \rangle^{1/2} \approx \left( \frac{\langle |\delta \chi|^2 \rangle^{1/2}}{b_0} \right)^{1/4} \left( \frac{10 \Sigma}{\Sigma_c} \right)^{1/2} \left( \frac{10^3 b_a}{b_0} \right)^{1/4} \left( \ln \Lambda \right)^{1/2},
\]

where \( \ln \Lambda = \ln \left[ (\Sigma b_0/\Sigma_r)^{1/2} / s \right] \approx 10 \).

### 2.3. Physical Scales and Interpretations

For this first attempt at modeling substructure, we con-
sider satellites with constant surface density \( \Sigma_r/\Sigma_c \), and critical 

radius \( b \). We scale the satellite break radius like a tidal 
radius with \( a = (b_0b_a)^{1/2} \) for \( b_0 \equiv 10 \). Near the critical 

radius of a moderately elliptical isothermal lens, the surface

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density is $\Sigma_c/2$, so the projected satellite mass fraction is $f_{sat} = 2\Sigma/\Sigma_c$. In the cylinder defined by the Einstein radius, roughly 50% (or 10% for a sphere) of the mass is dark matter (see, e.g., Keeton 2001a). The Einstein ring, where we make the measurement, is typically 1.0–1.5 effective radii from the lens center, and the dark matter fraction will be significantly higher than these average values at the edge of the cylinder where we see the images. Hence, we can interpret $f_{sat}$ as the fraction of the dark matter near the Einstein ring in substructure with only modest baryonic corrections.

We use only observational parameters in our models, which means that the physical parameters have small shifts between the lenses we consider because of the changing lens and source distances. To provide a sense of the physical scales, consider the parameters for PG1115+080 with source and lens redshifts of $z_s = 1.72$ and $z_l = 0.31$, respectively. The inner circular velocity of a satellite is $v_{circ} = 9.7(b/0.001)^{1/2}$ km s$^{-1}$, and its mass is $M = 3.4(b/0.001)^{1/2}10^6 h^{-1} M_\odot$. The angles $b$ and $a$ correspond to proper distances of $3(b/0.001) h^{-1}$ pc and $100(b/0.001)^{1/2} h^{-1}$ pc respectively. For the other lenses, the distances scale linearly with $D_{OL}$, the circular velocity scales as $(D_{LS}/D_{OS})^{-1/2}$, and the mass scales as $D_{OS}/D_{OL}D_{LS}$. These changes between lenses are sufficiently small compared to our logarithmic uncertainties to ignore.

We use satellites with fixed properties in our models, so our estimate of the mass scale is a weighted average of the satellite masses. Given our statistical uncertainties, models with a mass spectrum are unwarranted, and we should be able to estimate the effects of using a mass spectrum simply by matching the variance in the shear and astrometry perturbations (eq. [16]). The mass function of the satellite halos is $dN/dM \propto M^{-\alpha}$, with $1.7 \leq \alpha \leq 1.8$ (e.g., Moore et al. 1999; Klypin et al. 1999; Metcalf & Madau 2001; Springel et al. 2001; Helmi, White, & Springel 2002), which we limit to a finite range $M_{low} < M < M_{high}$ to avoid divergences in the total mass. With $\alpha < 2$, only the upper mass limit is important in estimating the perturbations. Our effective substructure mass $M$ is related to the upper mass limit by $M = M_{high}(2-\alpha)/(3-\alpha) = M_{high}/6$ if we match the amplitude of the astrometry perturbations and by $M = M_{high}(2-\alpha)/(7-3\alpha) = M_{high}/20$ if we match the shear perturbations. Given the precision with which we can currently estimate the characteristic mass scale, we choose not to include a satellite mass function. Crudely, we can estimate that $M_{high} \sim 10 M_\odot$–$20 M_\odot$.

2.4. Analysis Procedures and Monte Carlo Tests

In Figure 1 we outline our procedures for analyzing the lens data and for testing our method to ensure that it can recover the properties of the substructure accurately by following the treatment we use for each lens in the sample. We started (step 1) with the available data on the lens. The first processing step (step 2) was to model the lens with the LENS-SMODEL package (Keeton 2001b) using singular isothetical ellipsoids (SIE) in external shear fields for the mass distribution of the primary-lens galaxy. The SIE model is the only standard lens model that is consistent with general properties of the lens sample (see, e.g., Cohn et al. 2001, and references therein). Where needed, we added additional SIE lens components so as to reproduce the best standard models for each system. We used the observed astrometric uncertainties but broadened the uncertainties in the flux ratios to 20% to compensate for systematic errors and the contaminating effects of the substructure on the flux ratios. Effectively, we followed the procedures suggested by Mao & Schneider (1998) for modeling lenses in the presence of substructure.

For analyzing the real data (steps B1 and B2 in Fig. 1), we applied our linearized analysis method from § 2.1 to each lens, using the best-fit macro model from step 2 as the reference model ($p_0$), supplying the reference image positions and magnifications ($x_0^i$, $M_0^i$). Given the mass scale $b$ and surface number density $N$ of the substructure, we determined the angular radius $r_n = (n/N)^{1/2} = (n b a \Sigma_c/\Sigma)^{1/2}$, inside which we expected to find $n$ perturbing satellites. For each realization of the substructure (the $\delta_{ij}$ in eqs. [12] and [13]), we added $n = 10$ perturbing satellites inside radius $r_{10}$ from each image with corrections to avoid overcounting in models in which the $r_{10}$ regions of the individual images overlap. Each satellite perturbed all images, an effect that becomes important for mass scales $b \geq 0.01$. The model was in some senses still a “local” approximation because we assumed a constant surface density near all images and we did not generate a full global realization of the substructure distribution. The more distant satellites produced perturbations that were difficult to distinguish from changes in the macro model. We varied only the mass scale $b$ and surface density $\Sigma_c/\Sigma$ of the substructure using logarithmic priors for the two variables [$P(b) \propto 1/b$ and $P(\Sigma) \propto 1/\Sigma$]. The tidal radius was always set to $a = (b b_0)^{1/2}$ with $b_0 \equiv 1.0$. For each value of the mass scale and the surface density, we generated $10^3$ random realizations of the substructure. All parameters of the macro model were reopti-
mized for every substructure realization by minimizing the fit statistics in equations (10) and (11) combined with any ancillary constraints such as the position of the primary-lens galaxy. The Bayesian likelihood distribution (eq. [13]) was constructed by combining the likelihoods of fitting the data for lens $j$, $P(D_j | \theta_{ij})$ for each of the $\alpha = 1, \ldots, 10^5$ substructure realizations made for each of the set of substructure parameters $\theta$.

We tested the algorithm using Monte Carlo models following steps A1–A6 in Figure 1. The objective of the Monte Carlo sequence is to start from the best-fit macro model of each lens found in step 2 and then, by adding substructure and noise, generate a synthetic set of lens data that should be analogous to the real data. We started by taking the best-fit model for the lens (from step 2) and using its parameters and source properties as the true properties of a new Monte Carlo model. In step A1 we randomly placed $n = 5$ perturbing satellites inside the radius $r_5$ of each image based on the desired physical properties (mass scale $b$, surface density $\Sigma$, tidal radius $a$) of the substructure.

In step A2 we used the LENSMODEL package (Keeton 2001b) to find the nonlinear solutions for the new image positions and fluxes including all the substructure but keeping the macro model and source properties fixed. We used $n = 5$ perturbing satellites per image because of limitations on the maximum number of lenses in LENSMODEL. Tests varying the number of perturbers used both to generate and analyze the data suggested that the choices had no effects on the results in the sense that any biases were small compared to the statistical uncertainties. A modest fraction of realizations for B2045+265 produced extra images. We discarded these realizations. The existence of these solutions suggests that the substructure profile shape can be constrained by the production of extra images, but an exploration of these additional parameters is beyond the scope of our current study. After adding measurement errors to the image positions, lens positions, and image fluxes in step A3, we had a set of synthetic lens data that should be a realistic Monte Carlo model of the real data we started with in step 1.

The remainder of the analysis was identical to that for the real data. Step A4 matched step 2, in which we fitted the synthetic data using only a smooth macro model to provide the reference models and images for performing the substructure analysis. Step A5 matched step B1 for the real data, in which we applied our linearized substructure analysis to the noisy synthetic data and the reference model, and step A6 matched step B2 in which we combined the results to estimate the Bayesian likelihood distributions for the substructure parameters.

We illustrate the ability of our linearized analysis method to correctly extract the properties of substructure in three steps. First, we examine the sensitivity of our surface density estimate $f_{\text{sat}}$ to measurement errors. Next, we test our ability to estimate $f_{\text{sat}}$ when the satellite masses and structures are fixed to their true values. Finally, we test our ability to estimate simultaneously the surface density $f_{\text{sat}}$ and the mass scale $b$. In each case, we generate a Monte Carlo data set consisting of a perturbed realization for each of the seven lenses in our sample and then analyze it using the same procedures we apply to the real data.

In our first test, we examine whether measurement errors can be mistaken for substructure by adding random astrometry and flux errors to the models, fitting new macro models, and then analyzing the synthetic data using our method. This is unlikely to be a serious problem for the real data because the best-fit models typically have $\chi^2 \gg N_{\text{dof}}$ when we use realistic uncertainties for the image fluxes. However, this determines the range of $f_{\text{sat}}$ to which we are sensitive, in which the relevant scales are $10^{-4} \lesssim f_{\text{sat}} \lesssim 10^{-3}$ for normal satellite populations and $0.02 \lesssim f_{\text{sat}} \lesssim 0.15$ for CDM substructure. The results of two such simulations are shown in Figure 2. The formal, one-sided 90% confidence upper bounds are $f_{\text{sat}} \lesssim 0.004$, although this is very conservative because we only calculate the probability over the range $10^{-3} \lesssim f_{\text{sat}} \lesssim 1$. The peak probability and most of the integrated probability comes from still lower satellite fractions. Lenses with highly magnified images are more sensitive to substructure and constrain $\Sigma/\Sigma_c$ more strongly, with the upper limit varying as the inverse square of the maximum image magnification. Our lens sample has two “low-magnification” lenses ($M_{\text{max}} \lesssim 5$: B1608+656 and B1933+503) and five “high-magnification” lenses ($M_{\text{max}} \gtrsim 5$: MG 0414+0534, B0712+472, PG 1115+080, B1422+231, and B2045+265). Individual lenses can even show probability peaks at larger surface densities, but without the contrast between the peak and the probability at lower $f_{\text{sat}}$ needed to produce a signal at higher surface density in the full sample. If, however, we underestimate the flux errors, we can make a spurious detection of the substructure. When we examine a range of satellite mass scales $b$ as well as the surface density, we find that measurement errors produce no preferred mass scale for the substructure.

In the second set of simulations, we add perturbing satellites near each image with a surface density of $f_{\text{sat}} = 2\Sigma/\Sigma_c = 0.05$ and a mass profile defined by $b = 0.001$ and $a = 0.032$. The radius encompassing an average of five satellites, $r_5 = 0.080$, is much smaller than
The vertical line marks the true value of $f_{\text{sat}}$ of 56 lenses. The points on the solid curves mark the median probability combined probability of all eight realizations. This latter case mimics a sample of 56 lenses. The solid curves show the final Bayesian probability distributions for all eight simulations shown in Figure 3, the surface density cor-
responding to the median probability ranges from

$$2\Sigma/\Sigma_c \sim f_{\text{sat}}$$

The solid curves mark the median probability (triangles) and the regions encompassing

- 68.3% (1 $\sigma$, squares)
- 95.4% (2 $\sigma$, pentagons) of the likelihood in the Bayesian probability distribution.

The vertical line marks the true value of $f_{\text{sat}} = 0.05$.

distances between the images. We experimented with other models, but the results we present are typical. As we show in Figure 3, the estimated surface density is consistent with the surface density used to generate the data. In the eight simulations shown in Figure 3, the surface density corresponding to the median probability ranges from

$$f_{\text{sat}} = 0.01 \text{ to } 0.08$$

with uncertainties of a factor of 2.5 at 1 $\sigma$ and 3.0 at 90% confidence. The true surface density is within the 68.3% (1 $\sigma$) confidence region in four of the eight simulations and within the 90% confidence region for six of the eight simulations. If we combine all eight simulations to mimic a sample of 56 lenses, the surface density estimated by the median of the Bayesian likelihood distribution is

$$f_{\text{sat}} = 0.034,$$

with a 90% confidence range of

$$0.023 \leq f_{\text{sat}} \leq 0.048,$$

which marginally excludes the true value. The slightly low value for $f_{\text{sat}}$ could be due to chance, discarding the cases producing additional images, linearizing the problem, or the local approximation for the substructure.

We also examine the likelihood distribution in the two-dimensional space of the surface density $f_{\text{sat}}$ and the mass scale $b$, holding the internal structure of the satellites fixed with

$$a = (bb_0)^{1/2}$$

for $b_0 = 1''$. Figure 4 shows that the method can recover the mass scale, but less robustly than the surface density of the satellites. In these two-dimensional models, we find median estimates for the surface density ranging from $f_{\text{sat}} = 0.014$ to 0.074, with a factor of 3.8 uncertainty at 90% confidence. The ability to adjust the mass scale significantly increases the uncertainty in the surface density. The median estimates for the mass scale range from $b = 0''00016$ to $0''0027$. The uncertainty in the mass scale is usually an order of magnitude at 90% confidence. As we discuss in §2.2, we expect to be more sensitive to the surface mass density than to the mass scale. For constant astrometry perturbations, we expect $b \propto f_{\text{sat}}^{-2/3}$, and for constant shear or convergence perturbations, we expect $b \propto f_{\text{sat}}^{2}$ (see eqs. [17] and [18]). Neither slope is clearly reflected in the likelihood contours of Figure 4, suggesting that both types of perturbations contribute. If we combine all eight realizations to mimic a sample of 56 lenses, we recover the input model with modest uncertainties.

In summary, the lenses are sensitive to surface densities of substructure exceeding $f_{\text{sat}} \gtrsim 0.004$ and samples of seven lenses can be used to determine the surface density and mass scale with reasonable accuracy. Our method shows no signs of biases in the recovered parameters given the statistical uncertainties expected for seven lenses. In much larger lens samples, or samples with very accurately measured image fluxes, we may underestimate the surface density slightly as a consequence of the simplifications made to allow rapid calculation (linearizing the problem and the “local” approximation for the substructure) and the elimination of realizations generating extra images.

3. THE PROPERTIES OF HALO SUBSTRUCTURE

Given these limitations, we now apply the analysis to the sample of seven real lenses. We assume an average uncertainty in the flux ratio measurements of 10%, but report results for uncertainties of 5% and 20% as well. For the few cases in which the flux measurement errors are larger, we use the measurement errors instead, but in most cases, the flux measurement errors are dominated by systematic uncertainties rather than measurement errors. Among the...
Fig. 5.—Results for the observed lens sample with $b = 0\arcsec 001$. The solid curves show the probability distributions assuming errors in the flux ratios of 5%, 10%, and 20%. The points on the curves mark the median surface density ($\Sigma_0$) and the regions encompassing 68.3% (1 $\sigma$: squares) and 95.4% (2 $\sigma$: pentagons) of the probability. The region between the vertical lines is the range of substructure mass fractions found in the Klypin et al. (1999) simulations. Normal satellite populations, with $10^{-4} \lesssim f_{\text{sat}} \lesssim 10^{-3}$, correspond to a region off the left edge of the figure.

systematic issues are variability and time delays, wavelength dependencies on the flux ratios, and any contribution from stellar microlensing. A detailed examination of these problems is beyond the scope of our present study. Given the current data for most lenses, the image flux uncertainties are certainly lower than 20%, probably lower than 10%, and unlikely to be lower than 5%. The errors in the image and lens positions are dominated by measurement errors rather than systematic errors.

We first analyze the data assuming a fixed mass scale of $b = 0\arcsec 001$ and a tidal radius of $a = 0\arcsec 032$. As shown in Figure 5, the results for the real lens sample have qualitative properties that are very similar to the results of the Monte Carlo simulations shown in Figure 3. The median estimate for the surface density depends on the assumed level of systematic uncertainties in the image flux ratios, with $f_{\text{sat}} = 2\Sigma / \Sigma_0 = 0.051$, 0.024, and 0.0097 for flux ratio uncertainties of 5%, 10%, and 20%, respectively. The 90% confidence ranges for the three cases are $0.027 < f_{\text{sat}} < 0.096$, $0.0098 < f_{\text{sat}} < 0.058$, and $0.0014 < f_{\text{sat}} < 0.037$, respectively. In all three cases, the distributions are broadly consistent with the $0.02 < f_{\text{sat}} < 0.15$ range found in the Klypin et al. (1999) simulations, and well above the $10^{-4} \lesssim f_{\text{sat}} \lesssim 10^{-3}$ range found in visible satellites (see Mao & Schneider 1998; Chiba 2002).

We also calculate the probabilities as a function of both $f_{\text{sat}}$ and the mass scale $b$ as shown in Figure 6. With 10% flux errors, the median estimates for the surface density and mass scale are $f_{\text{sat}} = 0.020$ and $b = 0\arcsec 0013$, with 90% confidence regions of $0.0058 < f_{\text{sat}} < 0.068$ and $0\arcsec 0001 < b < 0\arcsec 007$. For a $dn/dM \propto 1/M^2$ ($M_{\text{low}} < M < M_{\text{high}}$) satellite mass function, this implies that the upper mass scale is in a range $10^6 M_{\odot} \lesssim M_{\text{high}} \lesssim 10^9 M_{\odot}$ that is consistent with the expectations for satellites. There is a relatively strong covariance between the parameters $b$ and $f_{\text{sat}}$, with low surface densities requiring higher mass scales. The slope of the likelihood contours is very close to the $b \propto f_{\text{sat}}^{-3/2}$ slope corresponding to constant shear or convergence perturbations (see eq. [18]) rather than the flatter $b \propto f_{\text{sat}}^{-1}$ slope corresponding to constant astrometry perturbations (see eq. [17]). If we assume 5% flux errors, then the surface density and mass scales are restricted to larger values, with $0.013 < f_{\text{sat}} < 0.078$ and $0\arcsec 00036 < b < 0\arcsec 013$. If we assume 20% flux errors, a broader range is permitted, with $0.0016 < f_{\text{sat}} < 0.051$ and $0\arcsec 000015 < b < 0\arcsec 00023$. These calculations neglect the smoothing effects of the finite source size ($\Delta \theta \sim 0.01–1.0$ mas), which will wash out the perturbations from smaller satellites if $\Delta \theta \gtrsim b$. With a finite source size, we would require a larger satellite fraction to produce the same perturbations to the images. On a final if qualitative note, the general properties of the likelihood distributions for the real data are remarkably similar to those of the Monte Carlo simulations.

4. DISCUSSION

CDM simulations generically produce halos in which $\sim 2\%-15\%$ of the mass is comprised by substructure, which is 50–100 times more mass than is observed in the satellites.
of the Local Group (e.g., Moore et al. 1999; Klypin et al. 1999). This substructure problem and possible conflicts between rotation curves and density cusps and the observed and predicted angular momentum distributions in spiral galaxies have been interpreted as requiring significant modifications to the CDM paradigm (e.g., Spergel & Steinhardt 2000; Bode et al. 2001; Colin et al. 2000).

Here we show that the anomalous flux ratios observed in a sample of seven gravitational lenses can be interpreted as requiring a mass fraction of $0.006 < f_{\text{sat}} < 0.07$ (90% confidence) in satellite halos, which is remarkably consistent with CDM predictions. This estimate assumes 10% errors (measurement + systematic) in the estimates of image fluxes, but the predicted surface density remains consistent with the expectations for CDM over the plausible 5%–20% range for these uncertainties. The estimates are always well above the $10^{-4} \leq f_{\text{sat}} \leq 10^{-3}$ range predicted for known satellite populations (see Mao & Schneider 1998; Chiba 2002). This can be consistent with CDM and the lower density of Galactic satellites if star formation is suppressed in most such satellites, as already discussed by Klypin et al. (1999) and Bullock et al. (2000). For the $dn/dM \propto M^{-1.8}$ ($M_{\text{low}} < M < M_{\text{high}}$) mass function expected for satellites (e.g., Moore et al. 1999; Klypin et al. 1999), our test provides a rough estimate of the upper mass scale $M_{\text{high}} \approx 10^{6} - 10^{9} M_{\odot}$. While this is uncomfortably close to the masses capable of disrupting stellar disks and globular clusters (e.g., Moore et al. 1999), Font et al. (2001) find that the expected CDM substructure is consistent with the survival of thin galactic disks. Thus, our result confirms a surprising if generic prediction of CDM models and can be regarded as a major success of the CDM model. By the same token, alternatives to CDM that aim to suppress small-scale power (warm dark matter) or to destroy small satellites (self-interacting dark matter) are accordingly disfavored.

We believe that three other explanations, systematic errors in the data, unmodeled coherent structures in the lens, and stellar microlensing, are unlikely. While there are systematic errors in the lens data, the anomalous flux ratios that drive the detection of substructure are present at levels far above the measurement errors and appear in multiple observations at differing wavelengths over periods of years. They may be misinterpreted but cannot be eliminated. They are also unlikely to be due to coherent structures in the lens galaxy. While we analyzed the lenses using singular isothermal ellipsoids in an external shear for the macro model, Metcalf & Zhao (2002) have shown that the flux ratios cannot be explained by a broad range of macro models. The typical lens galaxy, including all seven discussed here, is an early-type galaxy whose surface brightness profile is well modeled by a smooth, elliptical de Vaucouleurs profile (e.g., Lehar et al. 2000; Kochanek et al. 2000), with no obvious photometric residuals. Coherent features in the lenses such as spiral arms would be trivially detected in most cases.

Moreover, if we need $f_{\text{sat}} \sim 0.01$ in compact components such as satellites to perturb the images, we would require a far bigger mass fraction in large-scale coherent structures that cannot produce perturbations isolated to a single image.

The most problematic alternative explanation is stellar microlensing, which is the same physical phenomenon but produced by the stellar populations we know to be present in the lens galaxy. The basic argument against microlensing is that it has too small a characteristic angular scale (micro-arseconds) to produce large, long-lived flux ratio anomalies, given the sizes of typical radio sources. The Compton limit and direct VLBI observations of the lenses mean that typical sources are resolved on scales of 10–1000 $\mu$as that are large enough to suppress the effects of stellar microlensing. The one apparent case of microlensing of a radio source, B1600+434, is probably due to a supraluminal subcomponent of the radio source, where Doppler boosting gives the source a smaller effective size and a rapid modulation timescale (see Koopmans & de Bruyn 2000). Even in B1600+434, microlensing provides only a ~5% rms variation in the fluxes. Moreover, many of the radio lenses also have constant flux ratios on long timescales (years), which are difficult to reconcile with producing flux ratio anomalies using the stars. Finally, our method provides an estimate for a characteristic mass scale that is grossly inconsistent with stellar microlensing. This is reinforced by detailed analyses of B1422+231 (Keeton 2002; Bradac et al. 2002), which find mass scales compatible with CDM substructure but not stellar microlensing. In summary, satellites are the most natural explanation, and the required densities are comparable to that expected in CDM and higher than that observed in normal satellite populations. Whether the substructure is dark or luminous cannot be addressed directly because of the enormous distances.

Our examination of the problem is a preliminary one, and our estimates can be extended and improved if the following points are addressed. First, the entire question of the image fluxes and their uncertainties needs to be carefully reconsidered. We use a fixed measurement error of 10% for the image fluxes, but the estimated surface density and its uncertainties are affected by differences between the true errors and the errors used in the analysis. Until now, there has been little motivation for determining image flux ratios with high precision (say 1% accuracy), but improved analyses will need such high precision. Lens monitoring and time delay measurements, already important for using the lenses to determine the Hubble constant without the systematic problems of the local distance scale (e.g., Schechter 2002), are needed to eliminate the effects of source variability on the flux ratios. In optical lenses, observations over a broad range of wavelengths are needed to provide accurate corrections for extinction (see Falco et al. 1999).

Second, improved observations of the lenses are needed. The lensed images of the host galaxies of the radio sources, which are relatively easy to observe using deep infrared imaging with the Hubble Space Telescope (HST), can be used to constrain the macro model (e.g., Kochanek, Keeton, & McLeod 2001). Unlike the unresolved images of quasars or the marginally resolved images of radio cores, large lensed structures such as the host galaxies ($D_{\text{H}} \geq 0.7$) constrain the macro model without being affected by substructure. Combining the large-scale constraints with the compact images allows us to probe the substructure while limiting the ability of the macro model to mask its effects. Simultaneously, very high resolution, high dynamic range VLBI observations to map thin, extended radio structures can be used to extend the search for substructure over larger regions in each lens (e.g., Wambsganss & Paczyński 1992; Metcalf & Madau 2001). If the VLBI observations can show that the anomalous flux ratios are consistent with the geometric structure of the image, then we can completely rule out microlensing as an alternative explanation. Finally, careful searches for additional but faint VLBI images pro-
duced by the substructure may be a powerful means to constrain the density profiles of the satellites. We have already found that our assumed density satellite profile occasionally produces additional detectable images, suggesting that a shallower density profile would be preferred.

Third, the analysis can be expanded to include complete treatments of the mass spectrum, the density profiles of the substructure, and the effects of finite-sized sources. These additional complications were unwarranted in this first calculation because with only seven lenses, all we can realistically say we have measured are the average properties of the substructure. Any model producing the same average shear and astrometry perturbations should be consistent with the data. It is clear from our Monte Carlo simulations, in which our model would occasionally generate additional images, that the density distribution of the more massive substructures can be constrained by limits on the production of extra images. Given our estimated angular scales for the substructure perturbations and the dominance of the mass spectrum by the higher mass halos, our results should be little affected by finite source sizes. If the typical radio source is 1 mas, then we are modestly understimating the surface density.

Fourth, larger samples of lenses can reduce the considerable Poisson uncertainties. At least two additional radio four-image lenses have been discovered (B0128+437: Phillips et al. 2000; B1555+375: Marlow et al. 1999) in the Cosmic Lens All-Sky Survey we used as the basis for our analysis, but lack the HST imaging data needed to accurately determine the position of the lens galaxy. Two-image lenses, while less optimal because of their lower average magnifications, can be included in the analysis when additional lensed structures such as the images of the quasar host galaxy or VLBI subcomponents provide the constraints needed to break the degeneracies between the macro model and the substructure we expect for a simple two-image lens.

Finally, we note that other probes of substructure may be possible in Local Group galaxies. Very recently, Ibata, Lewis, & Irwin (2001a) and Ibata et al. (2001b) have suggested that the paucity of tidal streamers in the Milky Way halo may betray the presence of halo substructure. Johnston, Spergel, & Haydn (2002) similarly analyze tidal debris from the disrupted Sagittarius dwarf, and find that stars in these tidal tails appear to be more scattered than expected for debris orbiting in a smooth halo. Thus, there are tantalizing hints of evidence for substructure within our own halo, and further work along this avenue may lead to more definite conclusions than is currently possible. Whatever the outcome of these local studies, however, only gravitational lenses can directly detect CDM satellites in which star formation has been suppressed.

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REFERENCES

Bode, P., Ostriker, J. P., & Turok, N. 2001, ApJ, 556, 93
Bradač, M., Schneider, P., Steinmetz, M., Lombardi, M., & King, L. J. 2002, A&A, submitted (astro-ph/0112038)
Bullock, J. S., Kravtsov, A. V., & Weinberg, D. H. 2000, ApJ, 539, 517
Chiba, M. 2002, ApJ, 565, 17
Chu, W. A., Gnedin, N. Y., & Ostriker, J. P. 2001, ApJ, 563, 21
Cohn, J. D., Kochanek, C. S., McLeod, B. A., & Keeton, C. R. 2001, ApJ, 554, 1216
Colin, P., Avila-Reese, V., & Valenzuela, O. 2000, ApJ, 542, 622
Falco, E. E., Impey, C. D., Kochanek, C. S., Lehár, J., McLeod, B. A., Rix, H.-W., Keeton, C. R., Munoz, J. A., & Peng, C. Y. 1999, ApJ, 523, 617
Fassnacht, C. D., Womble, D. S., Neugebauer, G., Browne, I. W. A., Readhead, A. C. S., Matthews, K., & Pearson, T. J. 1996, ApJ, 460, L103
Fassnacht, C. D., et al. 1999, AJ, 117, 658
Font, A. S., Navarro, J. F., Stadel, J., & Quinn, T. 2001, ApJ, 563, L1
Helmi, A., White, S. D. M., & Springel, V. 2002, Phys. Rev. D, submitted (astro-ph/0201289)
Hewitt, J. N., Turner, E. L., Lawrence, C. R., Schneider, D. P., & Brody, J. P. 1992, AJ, 104, 968
Ibata, R. A., Lewis, G. F., & Irwin, M. J. 2001a, MNRAS, submitted (astro-ph/0110890)
Ibata, R. A., Lewis, G. F., Irwin, M. J., & Cambresy, L. 2001b, MNRAS, submitted (astro-ph/0110691)
Jackson, N., et al. 1998, MNRAS, 296, 483
Johnston, K. V., Spergel, D. N., & Haydn, C. 2002, ApJ, 570, 656
Kauffmann, G., White, S. D. M., & Guiderdoni, B. 1993, MNRAS, 264, 201
Keeton, C. 2001a, ApJ, 561, 46
___. 2001b, ApJ, submitted (astro-ph/0102340)
___. 2002, ApJ, submitted (astro-ph/0111595)
Klypin, A., Kravtsov, A. V., Valenzuela, O., & Prada, F. 1999, ApJ, 522, 82
Kochanek, C. S. 2001, in Proc. The Dark Universe, ed. M. Livio (Cambridge: Cambridge Univ. Press), in press (astro-ph/0108160)
Kochanek, C. S., Falco, E. E., Impey, C. D., Lehár, J., McLeod, B. A., Rix, H.-W., Keeton, C. R., Munoz, J. A., & Peng, C. Y. 2000, ApJ, 543, 131
Kochanek, C. S., Keeton, C. R., & McLeod, B. A. 2001, ApJ, 547, 50
Koopmans, L. V. E., & de Bruyn, A. G. 2000, A&A, 358, 793
Lehar, J., Falco, E. E., Kochanek, C. S., McLeod, B. A., Munoz, J. A., Impey, C. D., Rix, H.-W., Keeton, C. R., & Peng, C. Y. 2000, ApJ, 536, 584
Mao, S., & Schneider, P. 1998, MNRAS, 295, 587
Marlow, D. R., et al. 1999, AJ, 118, 654
Metcalfe, R. B., & Madau, P. 2001, ApJ, 563, 9
Metcalfe, R. B., & Zhao, H. 2002, ApJ, 567, L5
Moore, B., Ghigna, S., Governato, F., Lake, G., Quinn, T., Stadel, J., & Tozzi, P. 1999, ApJ, 524, L19
Munoz, J. A., Kochanek, C. S., & Keeton, C. R. 2001, ApJ, 558, 657
Patnaik, A. R., Browne, I. W. A., Walsh, D., Chaffee, R. H., & Holtz, C. B. 1992, MNRAS, 254, 655
Phillips, P. M., et al. 2000, MNRAS, 319, L7
Schechter, P. 2002, in IAU Symp. 201, New Cosmological Data and the Values of the Fundamental Parameters, ed. A. N. Lasenby, A. Wilkinson, & A. W. Jones (San Francisco: ASP), in press (astro-ph/0009048)
Schneider, P., Ehlers, J., & Falco, E. E. 1992, Gravitational Lenses (Berlin: Springer)
Scoccimarro, R., Sheth, R. K., Hui, L., & Jain, B. 2001, ApJ, 546, 20
Spergel, D. N., & Steinhardt, P. J. 2000, Phys. Rev. Lett., 84, 3760
Springel, V., White, S. D. M., Tormen, G., & Kauffmann, G. 2001, MNRAS, 328, 726
Sykes, C. M., et al. 1998, MNRAS, 301, 310
Wambsganss, J., & Paczynski, B. 1992, ApJ, 397, L1
Weinmann, R. J., Latham, D., Roger, J., Angel, P., Green, R. F., Liebert, J., Turner, D. A., Turner, D. E., & Tyson, J. A. 1980, Nature, 285, 641