ON A SMOOTH QUARTIC SURFACE CONTAINING 56 LINES
WHICH IS ISOMORPHIC AS A K3 SURFACE
TO THE FERMAT QUARTIC

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ABSTRACT. We give a defining equation of a complex smooth quartic surface containing 56 lines, and investigate its reductions to positive characteristics. This surface is isomorphic to the complex Fermat quartic surface, which contains only 48 lines. We give the isomorphism explicitly.

1. Introduction

The complex Fermat quartic surface
\[ x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0 \]
is a very interesting surface, because it lies at the intersection point of the two important classes of algebraic varieties: K3 surfaces and Fermat varieties. In this paper, we show that the complex K3 surface underlying the Fermat quartic surface has another smooth quartic surface \( X_{56} \subset \mathbb{P}^3 \) as a projective model. While the Fermat quartic surface contains only 48 lines, our new quartic surface \( X_{56} \) contains 56 lines, and hence these two surfaces are not projectively isomorphic. We present an explicit defining equation of \( X_{56} \), and describe the isomorphism between these two surfaces. It turns out that the isomorphism \( X_{48} \cong X_{56} \), the lines on \( X_{48} \) and \( X_{56} \), and the automorphisms of \( X_{48} \) and \( X_{56} \) are all defined over the 8th cyclotomic field \( \mathbb{Q}(\zeta) \), where
\[ \zeta := \exp(2\pi\sqrt{-1}/8). \]

We then study the reductions of \( X_{56} \) at primes of \( \mathbb{Z}[\zeta] \).

In the following, we denote the complex Fermat quartic surface by \( X_{48} \), and the complex K3 surface underlying \( X_{48} \) by \( X \).

The existence of a complex smooth quartic surface containing 56 lines that is isomorphic to \( X_{48} \) has been implicitly shown in the paper [1] by Degtyarev, Itenberg and Sertöz. This paper is one of the several works [1, 16, 17] on the number of lines lying on a smooth quartic surface that have been done after the seminal work [15], in which Rams and Schütt revised and corrected B. Segre’s classical work [21]. Degtyarev, Itenberg and Sertöz [1] proved the following theorem.

**Theorem 1.1.** The number of lines lying on a complex smooth quartic surface is either in \( \{64, 60, 56, 54\} \) or \( \leq 52 \).

The maximum number 64 of lines lying on a complex smooth quartic surface is attained by the Schur quartic. Schütt [20] has discovered the defining equations of complex smooth quartic surfaces containing 60 lines. On the other hand, in [1],

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the transcendental lattices of complex smooth quartic surfaces containing 56 lines are calculated. One of these surfaces, which we denote by \( X_{56} \), has the oriented transcendental lattice isomorphic to that of \( X_{48} \), and hence they are isomorphic over the complex number field by the result of [27] on the classification of complex K3 surfaces with Picard number 20. However, the defining equation of \( X_{56} \) and the description of the isomorphism \( X_{48} \cong X_{56} \) have been unknown.

Our main results are as follows. Let \( \mathbb{P}^3 \) and \( \mathbb{P}^3 \) be the projective spaces with homogeneous coordinates \((x_1 : x_2 : x_3 : x_4)\) and \((y_1 : y_2 : y_3 : y_4)\), respectively. Let \( S_X \) denote the Néron-Severi lattice of \( X_{48} \), and let \( h_{48} \in S_X \) be the class of a hyperplane section of \( X_{48} \subset \mathbb{P}^3 \).

**Theorem 1.2.** If \( h \in S_X \) is a very ample class such that \( \langle h, h \rangle = 4 \) and \( \langle h, h_{48} \rangle = 6 \), then the quartic surface \( X_h \) of \( X \) corresponding to \( h \) contains exactly 56 lines. The number of very ample classes \( h \in S_X \) satisfying \( \langle h, h \rangle = 4 \) and \( \langle h, h_{48} \rangle = 6 \) is 384.

Our quartic surface \( X_{56} \subset \mathbb{P}^3 \) corresponds to one of the 384 very ample classes in Theorem 1.2. For a prime \( \mathcal{P} \) of \( \mathbb{Z}[\zeta] \), let \( \kappa_\mathcal{P} \) denote the residue field at \( \mathcal{P} \), and \( \bar{\kappa}_\mathcal{P} \) an algebraic closure of \( \kappa_\mathcal{P} \).

**Theorem 1.3.** (1) We put \( A := -1 - 2\zeta - 2\zeta^3 \), \( B := 3 + A \), and
\[
\Psi := y_1^3y_2 + y_1y_2^3 + y_3y_4 + y_3y_4^3 + (y_1y_4 + y_2y_3)(A(y_1y_3 + y_2y_4) + B(y_1y_2 - y_3y_4)).
\]
Let \( X_{56} \) denote the surface in \( \mathbb{P}^3 \) defined over \( \mathbb{Q}(\zeta) \) by the equation \( \Psi = 0 \). Then the complex surface \( X_{56} \otimes \mathbb{C} \) is smooth, and contains exactly 56 lines, each of which is defined over \( \mathbb{Q}(\zeta) \). The projective automorphism group of \( X_{56} \otimes \mathbb{C} \) is of order 64. Moreover, if \( \mathcal{P} \) is a prime of \( \mathbb{Z}[\zeta] \) over a prime integer \( p > 3 \), then the surface \( X_{56} \otimes \bar{\kappa}_\mathcal{P} \) is also smooth, contains exactly 56 lines, each of which is defined over the finite field \( \kappa_\mathcal{P} \).

(2) There exists an isomorphism \( X_{48} \cong X_{56} \) defined over \( \mathbb{Q}(\zeta) \) such that the class \( h_{56} \in S_X \) of the pull-back of a hyperplane section of \( X_{56} \) satisfies \( \langle h_{48}, h_{56} \rangle = 6 \). This isomorphism induces an isomorphism from \( X_{48} \otimes \bar{\kappa}_\mathcal{P} \) to \( X_{56} \otimes \bar{\kappa}_\mathcal{P} \), if \( \mathcal{P} \) is a prime of \( \mathbb{Z}[\zeta] \) over a prime integer \( p > 3 \).

**Remark 1.4.** Since \( A = -1 - 2\sqrt{-2} \), the surface \( X_{56} \) is in fact defined over \( \mathbb{Q}(\sqrt{-2}) \).

The precise description and a geometric characterization of the class \( h_{56} \in S_X \) of the pull-back of a hyperplane section of \( X_{56} \) are given in Section 4. An explicit description of the isomorphism \( X_{48} \cong X_{56} \) is given in Table 4.1 where the pull-back of the rational functions \( y_1/y_2 \) on \( X_{56} \) are the rational functions \( f_1/f_2 \) on \( X_{48} \).

In the study of reductions of \( X_{56} \), we have to calculate Gröbner bases of ideals in polynomial rings with coefficients in \( \mathbb{Z}[\zeta] \) over the residue field \( \kappa_\mathcal{P} \) at infinitely many primes \( \mathcal{P} \) of \( \mathbb{Z}[\zeta] \). A simple computational trick for this task will be given in Section 5. For the actual computation, we used GAP [5]. Computational data is available from the author's webpage [26].

The Néron-Severi lattices of the Fermat quartic surfaces in characteristic 0 and in characteristic 3 were studied by Mizukami and Inose in 1970's. In particular, they proved that these Néron-Severi lattices are generated by the classes of lines. This fact is crucial for our construction of \( X_{56} \). See [10] and Section 6.1 of [19].

Note that we have the following classical theorem due to Matsumura and Mon- sky [9] Theorem 2].
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Theorem 1.5. If two smooth hypersurfaces of degree \( d \geq 3 \) in \( \mathbb{P}^n \) with \( n \geq 3 \) are isomorphic as abstract varieties but not projectively equivalent, then we have \((d,n) = (4,3)\).

Recently, Oguiso [13] informed us of his method of constructing pairs of complex smooth quartic surfaces that are isomorphic as K3 surfaces but are not projectively isomorphic. His result shows in particular that the graph of the isomorphism \( X_{48} \to X_{56} \) is a complete intersection of 4 hypersurfaces of bi-degree \( (1,1) \) in \( \mathbb{P}^3 \times \mathbb{P}^3 \).

After the first version of this paper is submitted, Degtyarev [2] has proved that, up to projective equivalence, the K3 surface underlying \( X_{48} \) has exactly three smooth quartic surface models; \( X_{48}, X_{56}, \) and its complex conjugate \( \overline{X_{56}} \).

The plan of this paper is as follows. In Section 2, we review the theory of lattices, fix notation, and present two algorithms that are used throughout this paper. In Section 3, we describe the Néron-Severi lattice \( S_X \) of \( X_{48} \) by means of the 48 lines on it. In Section 4 we study very ample line bundles on \( X_{48} \) that give rise to an isomorphism to \( X_{56} \), and show how to obtain the defining equation of \( X_{56} \). We also compute the projective automorphism group of \( X_{56} \). In Section 5 we investigate the reductions of \( X_{56} \) at primes of \( \mathbb{Z}[\zeta] \).

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2. PRELIMINARIES ON LATTICES

A lattice \( L \) is a free \( \mathbb{Z} \)-module of finite rank with a non-degenerate symmetric bilinear form \( \langle , \rangle : L \times L \to \mathbb{Z} \). The orthogonal group \( O(L) \) of \( L \) acts on \( L \) from the right. The dual lattice \( L^\vee \) of \( L \) is a submodule of \( L \otimes \mathbb{Q} \) consisting of vectors \( v \in L \otimes \mathbb{Q} \) such that \( \langle v,x \rangle \in \mathbb{Z} \) holds for any \( x \in L \). The discriminant group \( \text{disc}(L) \) of \( L \) is defined to be \( L^\vee / L \). A lattice \( L \) is unimodular if \( \text{disc}(L) \) is trivial. A lattice \( L \) is even if \( \langle v,v \rangle \in 2\mathbb{Z} \) holds for any \( v \in L \). Let \( L \) be an even lattice. The \( \mathbb{Q} \)-valued symmetric bilinear form on \( L^\vee \) that extends \( \langle , \rangle : L \times L \to \mathbb{Z} \) defines a finite quadratic form \( q_L : \text{disc}(L) \to \mathbb{Q}/2\mathbb{Z} \), which is called the discriminant form of \( L \). Let \( O(q_L) \) denote the automorphism group of the finite quadratic form \( q_L \), which acts on \( \text{disc}(L) \) from the right. Then we have a natural homomorphism \( \eta_L : O(L) \to O(q_L) \).

See [11] for applications of the theory of discriminant forms.

A lattice \( L \) of rank \( n > 1 \) is hyperbolic if the signature of the real quadratic space \( L \otimes \mathbb{R} \) is \((1,n-1)\). We have the following algorithms. See [25] for details.

Algorithm 2.1. Let \( M \) be a free \( \mathbb{Z} \)-module of finite rank \( n > 1 \) with a \( \mathbb{Q} \)-valued non-degenerate symmetric bilinear form \( \langle , \rangle : M \times M \to \mathbb{Q} \) such that \( M \otimes \mathbb{R} \) is of signature \((1,n-1)\). (For example, \( M \) is a hyperbolic lattice or the dual lattice of a hyperbolic lattice.) Let \( h \in M \) be a vector such that \( \langle h,h \rangle > 0 \). Then, for given rational numbers \( a \) and \( b \), we can make the list of all vectors \( x \) of \( M \) that satisfy \( \langle h,x \rangle = a \) and \( \langle x,x \rangle = b \).

Algorithm 2.2. Let \( L \) be a hyperbolic lattice, and let \( h, h' \) be vectors of \( L \) that satisfy \( \langle h,h \rangle > 0 \), \( \langle h',h' \rangle > 0 \) and \( \langle h,h' \rangle > 0 \). Then, for a negative integer \( d \), we can make the list of all vectors \( x \) of \( L \) that satisfy \( \langle h,x \rangle > 0 \), \( \langle h',x \rangle < 0 \) and \( \langle x,x \rangle = d \).
Remark 2.3. These algorithms are based on an algorithm of positive-definite quadratic forms described in Section 3.1 of [25]. This algorithm can be made much faster by means of the lattice reduction basis [8]. See Section 2.7 of [3].

Let $L$ be an even hyperbolic lattice, and let $\mathcal{P}(L)$ be one of the two connected components of $\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}$, which we call a positive cone of $L$. A vector $r \in L$ is called a $(-2)$-vector if $\langle r, r \rangle = -2$ holds. For a $(-2)$-vector $r$, we have a reflection $s_r : x \mapsto x + \langle x, r \rangle r$ in the hyperplane

$$(r)^\perp := \{ x \in L \otimes \mathbb{R} \mid \langle r, x \rangle = 0 \}.$$ 

Each $s_r \in O(L)$ acts on $\mathcal{P}(L)$. The reflections $s_r$ with respect to all $(-2)$-vectors $r$ generate a subgroup $W(L)$ of $O(L)$. The closure in $\mathcal{P}(L)$ of a connected component of

$$\mathcal{P}(L) \setminus \bigcup (r)^\perp$$ 

is called a standard fundamental domain of the action of $W(L)$ on $\mathcal{P}(L)$. Let $\mathcal{N}$ be a standard fundamental domain, and let $h$ be an element of $\mathcal{N} \cap L$ such that $h \notin (r)^\perp$ for any $(-2)$-vectors $r$. Then Algorithm 2.2 applied to $d = -2$ provides us with a method to determine whether a given vector $h' \in \mathcal{P}(L) \cap L$ is contained in $\mathcal{N}$ or not.

The Néron-Severi lattice $S_Y$ of an algebraic $K3$ surface $Y$ is the $\mathbb{Z}$-module of numerical equivalence classes of divisors on $Y$ with the intersection pairing. The lattice $S_Y$ is even, and if its rank is $> 1$, it is hyperbolic. The class of a curve $C$ on $Y$ is denoted by $[C] \in S_Y$. Suppose that $\text{rank} S_X > 1$. It is well-known that the nef cone

$$\{ x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } Y \}$$

is a standard fundamental domain of the action of $W(S_X)$. When $Y$ is defined over $\mathbb{C}$, the second cohomology group $H^2(Y, \mathbb{Z})$ of a complex $K3$ surface $Y$ with the cup product is an even, unimodular lattice of signature $(3,19)$ containing $S_Y$ as a primitive sublattice.

3. The Fermat quartic surface $X_{48}$

The complex Fermat quartic surface is denoted by $X_{48} \subset \mathbb{P}^3$. The complex surface underlying $X_{48}$ is simply denoted by $X$. We describe the Néron-Severi lattice $S_X$ of $X$ in terms of the lines on $X_{48}$. In particular, we study the condition for an isometry $g \in O(S_X)$ of $S_X$ to extend to a Hodge isometry of $H^2(X, \mathbb{Z})$.

The transcendental lattice $T_X$ of $X$ is defined to be the orthogonal complement of $S_X$ in $H^2(X, \mathbb{Z})$. It is known [10], [27] that $S_X$ is of rank 20, and $T_X$ is isomorphic to the positive-definite lattice

$$T := \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

of rank 2. By [11], we have an anti-isometry $q_{S_X} \simeq -q_{T_X}$ of discriminant forms, and hence $|\text{disc}(S_X)| = 64$.

The surface $X_{48}$ contains exactly 48 lines. These lines are labelled by the tags $[i, [\mu, \nu]]$ in the following way, where $i$ is an integer satisfying $1 < i \leq 4$, and $\mu$ and $\nu$ are positive odd integers $\leq 7$. Let $j$ and $k$ be integers such that $j < k$ and $\{1, i, j, k\} = \{1, 2, 3, 4\}$. Then the line on $X_{48}$ labelled by $[i, [\mu, \nu]]$ is defined by

$$x_1 + \zeta^\mu x_i = 0, \quad x_j + \zeta^\nu x_k = 0.$$
have a common point and each line on $X$. All lines on $X$ are coplanar. The converse is also true; if three distinct lines on $X$ have a common point, which is a matrix $S$, their intersection matrix $G$ consists of four lines passing through $Q$, and the matrix $G$ at $\tau$ is written as a row vector with respect to this basis. Since the four lines $l_1, \ldots, l_4$ are on the plane $x_1 + \zeta x_2 = 0$, the class $h_{48}$ of the hyperplane section of $X_{48} \hookrightarrow \mathbb{P}^3$ is given by

$$h_{48} = [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].$$

All lines on $X_{48}$ are obtained in this way. Following [22], we call a point $Q \in X_{48}$ a $\tau$-point if the intersection $X_{48} \cap T_Q(X_{48})$ of $X_{48}$ and the tangent plane $T_Q(X_{48}) \subset \mathbb{P}^3$ to $X_{48}$ at $Q$ consists of four lines passing through $Q$. There exist exactly 24 $\tau$-points, and each line on $X_{48}$ contains exactly two $\tau$-points. If three distinct lines on $X_{48}$ have a common point $Q$, then $Q$ is a $\tau$-point, and hence these three lines are coplanar. The converse is also true; if three distinct lines on $X_{48}$ are coplanar, then they have a common point, which is a $\tau$-point.

We choose the 20 lines $l_1, \ldots, l_{20}$ labelled by the following tags respectively:

$$[2, [1, 1]], [2, [1, 3]], [2, [1, 5]], [2, [1, 7]], [2, [3, 1]], [2, [3, 3]], [2, [3, 5]], [2, [5, 1]], [2, [5, 3]], [2, [5, 5]], [3, [1, 1]], [3, [1, 3]], [3, [1, 5]], [3, [3, 1]], [3, [3, 3]], [3, [3, 5]], [4, [1, 1]], [4, [1, 3]], [4, [1, 5]], [4, [3, 1]].$$

Their intersection matrix $G_{S_X}$ is given in Table 3.1. Since the determinant of this matrix is $-64 = -|\text{disc}(S_X)|$, we see that the classes of these 20 lines form a basis of $S_X$, and the matrix $G_{S_X}$ is the Gram matrix of $S_X$ with respect to this basis. From now on, every vector of $S_X$ is written as a row vector with respect to this basis. Since the four lines $l_1, \ldots, l_4$ are on the plane $x_1 + \zeta x_2 = 0$, the class $h_{48}$ of the hyperplane section of $X_{48} \hookrightarrow \mathbb{P}^3$ is given by

$$h_{48} = [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].$$
By Riemann-Roch theorem, the set of classes \([\ell]\) of lines \(\ell\) on \(X_{48}\) is equal to
\[
\mathcal{F}_{48} := \{ r \in S_X \mid \langle r, r \rangle = -2, \ \langle r, h_{48} \rangle = 1 \}.
\]
This set can be calculated by Algorithm 2.1. The class of each line is also computed from the intersection numbers with \(l_1, \ldots, l_{20}\).

We have a basis of \(S_X^\vee\) dual to the fixed basis \([l_1], \ldots, [l_{20}]\) of \(S_X\). To distinguish the vector representation with respect to the non-dual basis \([l_1], \ldots, [l_{20}]\) of \(S_X\) and that with respect to the dual basis of \(S_X^\vee\), we put a superscript \(\vee\) on the dual representation. Thus we have a relation
\[
x^\vee = x G_{S_X}
\]
between the non-dual vector representation \(x \in \mathbb{Q}^{20}\) of an element \(v \in S_X \otimes \mathbb{Q}\) and the dual representation \(x^\vee\) of \(v\).

Consider the following vectors of \(S_X^\vee\):
\[
s_1 := [3, 1, 2, 1, 3, 2, 2, 2, 3, 1, 2, 1, 3, 1]^\vee, \\
s_2 := [1, 3, 1, 1, 1, 3, 2, 1, 0, 1, 1, 2, 3, -1, 1, 2, 0, 2]^\vee.
\]
Then the elements
\[
\sigma_1 := s_1 \mod S_X, \quad \sigma_2 := s_2 \mod S_X
\]
of \(\text{disc}(S_X)\) form a basis of \(\text{disc}(S_X) \cong \mathbb{Z}/8\mathbb{Z}^2\), under which the discriminant form \(q_{S_X}\) is given by the matrix
\[
\frac{1}{8} \begin{bmatrix} 11 & 5 \\ 5 & 14 \end{bmatrix},
\]
where the diagonal components are in \(\mathbb{Q}/2\mathbb{Z}\) and the off-diagonal components are in \(\mathbb{Q}/\mathbb{Z}\). Let \(P\) be the \(20 \times 2\) matrix
\[
\begin{bmatrix} 7 & 2 & 5 & 6 & 0 & 6 & 6 & 7 & 2 & 7 & 6 & 4 & 6 & 2 & 4 & 2 & 0 & 4 & 0 \\ 0 & 5 & 3 & 2 & 7 & 6 & 3 & 1 & 7 & 6 & 0 & 6 & 2 & 0 & 2 & 6 & 4 & 4 & 4 \end{bmatrix},
\]
with components in \(\mathbb{Z}/8\mathbb{Z}\). Then the quotient homomorphism \(S_X^\vee \to \text{disc}(S_X)\) is given by \(x^\vee \mapsto x^\vee P\) with respect to the basis \(\sigma_1, \sigma_2\) of \(\text{disc}(S_X)\). When we are given an element \(g\) of \(O(S_X)\) as a \(20 \times 20\) matrix \(R_g\) with respect to the basis \([l_1], \ldots, [l_{20}]\) of \(S_X\), the automorphism \(\eta_{S_X}(g)\) of \(q_{S_X}\) induced by \(g\) is represented with respect to the basis \(\sigma_1, \sigma_2\) by the \(2 \times 2\) matrix
\[
\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} G_{S_X}^{-1} R_g G_{S_X} P
\]
with components in \(\mathbb{Z}/8\mathbb{Z}\).

Let \(t_1, t_2\) be a basis of the lattice \(T\) under which the Gram matrix is given in (3.1); and let \(\mathbb{C} \omega\) be a totally isotropic subspace of \(T \otimes \mathbb{C}\). We have
\[
T \otimes \mathbb{C} = \mathbb{C} \omega \oplus \mathbb{C} \bar{\omega}.
\]
Let \(t_1^\vee, t_2^\vee\) be the basis of \(T^\vee\) dual to the basis \(t_1, t_2\) of \(T\). Then the elements \(\tau_1 := t_1^\vee \mod T\) and \(\tau_2 := t_2^\vee \mod T\) form a basis of \(\text{disc}(T) \cong (\mathbb{Z}/8\mathbb{Z})^2\), under which \(q_T\) is given by the matrix
\[
\frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
As above, we can calculate the natural homomorphism \( \eta_T : O(T) \rightarrow O(q_T) \) explicitly. It is easy to see that \( O(T) \) is of order 8, \( O(q_T) \) is of order 16, and \( \eta_T \) is injective. Moreover, we see that the group 
\[
\tilde{\Gamma}_T := \{ \, g \in O(T) \mid \mathbb{C} \omega^g = \mathbb{C} \omega \, \}
\]
is of order 4. Since \( T \) is isomorphic to the transcendental lattice \( T_X \), there exists an isomorphism \( q_{S_X} \cong -q_T \) by \cite{11}. In fact, since \( O(q_T) \) is of order 16, there exist exactly 16 isomorphisms from \( q_{S_X} \) to \( -q_T \). For an isomorphism \( \varphi : q_{S_X} \cong -q_T \), let \( \varphi_* : O(q_{S_X}) \cong O(q_T) \) be the induced isomorphism. It turns out that the subgroup 
\[
\Gamma_{S_X} := \{ \, \gamma \in O(q_{S_X}) \mid \varphi_*(\gamma) \in \eta_T(\tilde{\Gamma}_T) \, \}
\]
of \( O(q_{S_X}) \) does not depend on the choice of \( \varphi \); we have
\[
(3.3) \quad \Gamma_{S_X} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right\} \subset \text{GL}_2(\mathbb{Z}/8\mathbb{Z}).
\]
Note that an isometry \( \tilde{g} \) of the lattice \( H^2(X, \mathbb{Z}) \) is a Hodge isometry if and only if \( \tilde{g} \) preserves \( T_X \) and its orientation. If \( \tilde{g} \) preserves \( T_X \), then the orientation of \( T_X \) is preserved if and only if \( \tilde{g}|_{T_X} \) belongs to \( \Gamma_T \) under an (and hence any) isometry \( T \cong T_X \). Hence, by \cite{11}, we see that an isometry \( g \in O(S_X) \) extends to a Hodge isometry of \( H^2(X, \mathbb{Z}) \) if and only if
\[
(3.4) \quad \eta_{S_X}(g) \in \Gamma_{S_X}.
\]
This condition can be checked computationally using \cite{12} and \cite{33}.

We let the automorphism group \( \text{Aut}(X) \) act on \( X \) from the left, and act on \( S_X \) from the right by the pull-back. The following facts can be checked by direct computation by means of the data we have prepared so far. We consider the subgroup
\[
\text{Aut}(X_{48}) := \{ \, \gamma \in \text{PGL}_4(\mathbb{C}) \mid \gamma(X_{48}) = X_{48} \, \}
\]
on \( \text{Aut}(X) \), which is known to be of order 1536 and generated by the permutations of coordinates \( x_1, \ldots, x_4 \) and the scalar-multiplications by \( \zeta^2 \) of coordinates. We denote by
\[
G_{48} := \text{Im}(\text{Aut}(X_{48}) \rightarrow O(S_X))
\]
the image of \( \text{Aut}(X_{48}) \) in \( O(S_X) \) by the natural representation, which is injective. Since the set \( \mathcal{F}_{48} \) of classes of lines on \( X_{48} \) spans \( S_X \), the stabilizer subgroup
\[
\tilde{G}_{48} := \{ \, g \in O(S_X) \mid h^2_{48} = h_{48} \, \}
\]
of \( h_{48} \) is isomorphic to the group of permutations of \( \mathcal{F}_{48} \) that preserve the intersection numbers. The mapping
\[
g \mapsto (\langle l_1 \rangle^9, \ldots, \langle l_{20} \rangle^9)
\]
gives a bijection from \( \tilde{G}_{48} \) to the set of ordered lists \( (\langle l'_1 \rangle, \ldots, \langle l'_{20} \rangle) \) of elements of \( \mathcal{F}_{48} \) that satisfy
\[
(\langle l'_i \rangle, \langle l'_j \rangle) = (\langle l_i \rangle, \langle l_j \rangle) \text{ for all } i, j = 1, \ldots, 20.
\]
We calculate the set of all these \( (\langle l'_1 \rangle, \ldots, \langle l'_{20} \rangle) \) by the standard backtrack program (see \cite{16} for the meaning of the backtrack program), and calculate \( G_{48} \) as a list of elements of \( O(S_X) \). Since each line on \( X_{48} \) is defined over \( \mathbb{Q}(\zeta) \), the Galois group \( \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \) also acts on \( \mathcal{F}_{48} \) preserving the intersection numbers, and hence acts on \( S_X \). It turns out that \( \tilde{G}_{48} \) is of order 6144 and is generated by \( G_{48} \).
$A'(o_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A'(o_2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A'(o_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

\[A'(o_4) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad A'(o_5) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad A'(o_6) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\]

\[A(o_7) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad A'(o_8) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix}.\]

Table 3.2. Matrices $A'(o_i)$

and $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. By Torelli theorem [14], the subgroup $G_{48}$ of $\tilde{G}_{48}$ consists of elements $g \in G_{48}$ that satisfy the period-preserving condition [6].

We review the result of B. Segre [22] on the set of pairs of lines on $X_{48}$. The group $\text{Aut}(X_{48})$ acts on the 48 lines transitively. Let $P_i$ be the set of pairs of intersecting lines on $X_{48}$, and let $P_d$ be the set of pairs of disjoint lines on $X_{48}$. The orbit decompositions $P_i = o_1 \sqcup o_2 \sqcup o_3$ and $P_d = o_4 \sqcup \cdots \sqcup o_8$ of these sets by the action of $\text{Aut}(X_{48})$ are as follows:

\[
\begin{align*}
\{ [2, [1, 1]], [2, [1, 5]] \} & \in o_1 \subset P_i, \quad |o_1| = 48, \\
\{ [2, [1, 1]], [2, [1, 3]] \} & \in o_2 \subset P_i, \quad |o_2| = 96, \\
\{ [2, [1, 1]], [3, [1, 1]] \} & \in o_3 \subset P_i, \quad |o_3| = 192, \\
\{ [2, [1, 1]], [2, [5, 5]] \} & \in o_4 \subset P_d, \quad |o_4| = 24, \\
\{ [2, [1, 1]], [2, [3, 3]] \} & \in o_5 \subset P_d, \quad |o_5| = 96, \\
\{ [2, [1, 1]], [2, [3, 5]] \} & \in o_6 \subset P_d, \quad |o_6| = 96, \\
\{ [2, [1, 1]], [3, [1, 5]] \} & \in o_7 \subset P_d, \quad |o_7| = 192, \\
\{ [2, [1, 1]], [3, [1, 3]] \} & \in o_8 \subset P_d, \quad |o_8| = 384.
\end{align*}
\]

For each orbit $o_i$, we define an $8 \times 8$ matrix $A(o_i) = (a_{jk})$ as follows. Let $\{\ell, \ell'\}$ be a pair in $o_i$. We put

$$a_{jk} := \text{the number of lines } \ell'' \text{ such that } \{\ell, \ell''\} \in o_j \text{ and } \{\ell', \ell''\} \in o_k.$$

The $3 \times 3$ upper-left part $A'(o_i) = (a_{jk})_{1 \leq j, k \leq 3}$ of each of these matrices is given in Table 3.2.

Remark 3.1. Let $\{\ell, \ell'\}$ be a pair of intersecting lines. Then $\ell$ and $\ell'$ intersect at a $\tau$-point if and only if $\{\ell, \ell'\}$ belongs to $o_1$ or to $o_2$.

4. The Quartic Surface $X_{56}$

For $v \in S_X$, let $L_v \to X$ be a line bundle whose class is $v$. We say that $h \in S_X$ is a polarization of degree 4 if $\langle h, h \rangle = 4$ and the complete linear system $|L_h|$ is fixed-component free. By [18], if $h$ is a polarization of degree 4, then $|L_h|$ is base-point free and defines a morphism $\Phi_h: X \to \mathbb{P}^3$. We say that a polarization $h$ of degree 4 is very ample if $\Phi_h$ is an embedding.

Theorem 4.1. A class $h \in S_X$ with $\langle h, h \rangle = 4$ is a very ample polarization of degree 4 if and only if the following hold:
(a) \( \langle h, h_{48} \rangle > 0 \),
(b) \( \{ r \in S_X \mid \langle r, r \rangle = -2, \langle r, h_{48} \rangle > 0, \langle r, h \rangle < 0 \} \) is empty,
(c) \( \{ e \in S_X \mid \langle e, e \rangle = 0, \langle e, h \rangle = 1 \} \) is empty,
(d) \( \{ e \in S_X \mid \langle e, e \rangle = 0, \langle e, h \rangle = 2 \} \) is empty, and
(e) \( \{ r \in S_X \mid \langle r, r \rangle = -2, \langle r, h \rangle = 0 \} \) is empty.

If \( h \in S_X \) is a very ample polarization of degree 4, then the set \( \mathcal{F}_h \) of classes of lines contained in the image \( X_h \) of \( \Phi_h : X \to \mathbb{P}^3 \) is equal to

\[
\{ r \in S_X \mid \langle r, r \rangle = -2, \langle r, h \rangle = 1 \}.
\]

Proof. The condition (a) is equivalent to the condition that \( h \) is in the positive cone \( \mathcal{P}(S_X) \) of \( S_X \otimes \mathbb{R} \) containing \( h_{48} \). Suppose that (a) holds. Since the nef-cone of \( X \) is a standard fundamental domain of the action of \( W(S_X) \) on \( \mathcal{P}(S_X) \), the condition (b) is equivalent to the condition that \( h \) is nef. Suppose that (a) and (b) hold. By Proposition 0.1 of [12], the condition (c) is equivalent to the condition that \( |\mathcal{L}_h| \) is fixed-component free, and hence defines a morphism \( \Phi_h : X \to \mathbb{P}^3 \). Suppose that (a)–(c) hold. By [13], the condition (d) is equivalent to the condition that \( \Phi_h \) is not hyperelliptic, that is, \( \Phi_h \) is generically injective. Suppose that (a)–(d) hold. The condition (e) is equivalent to the condition that \( \Phi_h \) does not contract any \((-2)\)-curves, that is, the image \( X_h \) of \( \Phi_h \) is smooth. The second assertion is obvious. \( \square \)

Note that the conditions (a)–(e) can be checked by means of Algorithms 2.1 and 2.2 and that the set \( \mathcal{F}_h \) can be calculated by Algorithm 2.1.

We say that \( h \in S_X \) is an \( X_{56} \)-polarization if \( h \) is a very ample polarization of degree 4 such that \( X_h \subset \mathbb{P}^3 \) contains exactly 56 lines. The relative degree of a very ample polarization \( h \) of degree 4 is defined to be \( \langle h, h_{48} \rangle \).

Using Algorithm 2.1 we calculate the set

\[
\mathcal{H}_d := \{ v \in S_X \mid \langle v, h_{48} \rangle = d, \langle v, v \rangle = 4 \}
\]

for \( d = 1, \ldots, 6 \). Note that \( G_{48} \) acts on each \( \mathcal{H}_d \). We have \( \mathcal{H}_d = \emptyset \) for \( d < 4 \), and

\[
\mathcal{H}_4 = \{ h_{48} \}, \quad |\mathcal{H}_5| = 48, \quad |\mathcal{H}_6| = 48264.
\]

The action of \( G_{48} \) on \( \mathcal{H}_5 \) is transitive, and no vectors in \( \mathcal{H}_5 \) are nef. The action of \( G_{48} \) decomposes \( \mathcal{H}_6 \) into 60 orbits. Among the vectors in \( \mathcal{H}_6 \),

- 792 vectors in 5 orbits are not nef,
- 792 vectors in 5 orbits are nef, fixed-component free, but define hyperelliptic morphism,
- 46296 vectors in 48 orbits are nef, fixed-component free, define non-hyperelliptic morphism, but the images are singular, and
- the remaining 384 vectors in 2 orbits are very ample, and the images contain exactly 56 lines. The larger group \( \tilde{G}_{48} \) acts on these 384 vectors transitively.

Thus we obtain the following theorem.

**Theorem 4.2.** (1) If \( h \in S_X \) is a very ample polarization of degree 4 with relative degree 6, then \( h \) is an \( X_{56} \)-polarization. (2) There exist exactly 384 \( X_{56} \)-polarizations of relative degree 6. Up to the action of \( \text{Aut}(X_{48}) \) and \( \text{Gal}(\mathbb{Q} (\zeta)/\mathbb{Q}) \), there exists only one \( X_{56} \)-polarization of relative degree 6.

We describe \( X_{56} \)-polarizations of relative degree 6 geometrically.
Definition 4.3. An ordered list \((\ell_1, \ell_2, m_1, m_2, m_3, m_4, n)\) of seven lines on \(X_{48}\) is called an \(X_{56}\)-configuration if the following conditions are satisfied:

- \(\{\ell_1, \ell_2\} \in o_4\),
- \(\{\ell_1, m_1\} \in o_1, \{\ell_2, m_1\} \in o_1, \{\ell_1, m_i\} \in o_3, \{\ell_2, m_i\} \in o_3\) for \(i = 2, 3, 4\),
- \(\{m_1, m_k\} \in o_7\) for \(k = 2, 3, 4\),
- \(\{m_2, m_3\} \in o_5, \{m_2, m_4\} \in o_8, \{m_3, m_4\} \in o_8, \{\ell_1, n\} \in o_8, \{m_2, n\} \in o_8, \{m_3, n\} \in o_8, \{m_4, n\} \in o_7\).

We make the list of \(X_{56}\)-configurations. It turns out that the number of \(X_{56}\)-configurations is 6144. Comparing this list with the list of \(X_{56}\)-polarizations of relative degree 6, we obtain the following theorem.

Theorem 4.4. If \((\ell_1, \ell_2, m_1, \ldots, m_4, n)\) is an \(X_{56}\)-configuration, then the vector

\[
h := 3h_{48} - (\ell_1 + \ell_2 + m_1 + \cdots + m_4)
\]

is an \(X_{56}\)-polarization of relative degree 6.

Consider the seven lines \(\ell_1, \ell_2, m_1, \ldots, m_4, n\) labelled by the tags

\[(4.2) [2, [1, 1]], [2, [5, 5]], [2, [1, 5]], [3, [1, 1]], [3, [3, 3]], [4, [1, 7]], [3, [1, 3]],
\]

respectively. These lines form an \(X_{56}\)-configuration, and the corresponding \(X_{56}\)-polarization \(h_{56}\) is given by

\[
h_{56} = [1, 2, 1, 2, 0, 0, 0, 0, -1, -1, 0, 0, 0, -1, 0, 1, 1, 1, 0].
\]

Theorem 4.5. Let \(\Phi_{56}: X_{48} \to \mathbb{P}^3\) be the embedding induced by \(h_{56}\), and \(X_{56}\) the image of \(\Phi_{56}\). With a suitable choice of the homogeneous coordinates of \(\mathbb{P}^3\), the surface \(X_{56}\) is defined by the equation \(\Psi = 0\), where \(\Psi\) is given in Theorem 1.3.

Proof. Let \(\Gamma_d\) be the space of all homogeneous polynomials of degree \(d\) in the variables \(x_1, x_2, x_3, x_4\). We have a natural identification

\[
\Gamma_3 = H^0(X_{48}, \mathcal{L}_{3h_{48}}).
\]

Since \(\mathcal{L}_{h_{56}}\) is isomorphic to \(\mathcal{L}_{3h_{48}}(-\ell_1 - \ell_2 - m_1 - \cdots - m_4)\) as an invertible sheaf, the space \(H^0(X_{48}, \mathcal{L}_{h_{56}})\) is identified with the space of homogeneous polynomials of degree 3 in \(x_1, x_2, x_3, x_4\) that vanish along each of the lines \(\ell_1, \ell_2, m_1, \ldots, m_4\). Since we have explicit defining equations \(4.2\) of these lines, we can confirm that \(H^0(X_{48}, \mathcal{L}_{h_{56}})\) is of dimension 4, and calculate a basis \(f_1, f_2, f_3, f_4\) of \(H^0(X_{48}, \mathcal{L}_{h_{56}})\) by elementary linear algebra. We fix a basis of \(H^0(X_{48}, \mathcal{L}_{h_{56}})\) as in Table 4.1. Let \(\bar{\Gamma}_{12}\) denote the space of all homogeneous polynomials of degree 12 in \(x_1, x_2, x_3, x_4\) such that the degree with respect to \(x_1\) is \(\leq 3\). For \(g \in \bar{\Gamma}_{12}\), let \(\rho(g)\) denote the remainder on the division by \(x_1^4 + x_2^4 + x_3^4 + x_4^4\) under the lex monomial ordering \(x_1 > x_2 > x_3 > x_4\). Then we obtain a surjective homomorphism

\[
\rho: \Gamma_{12} \to \bar{\Gamma}_{12}.
\]

Therefore we have a natural identification

\[
\bar{\Gamma}_{12} = H^0(X_{48}, \mathcal{L}_{12h_{48}}).
\]

Let \(\Sigma_4\) be the linear space of all homogeneous polynomials of degree 4 in the variables \(y_1, y_2, y_3, y_4\). The substitution \(y_i \mapsto f_i\) for \(i = 1, \ldots, 4\) gives rise to a linear homomorphism \(\sigma: \Sigma_4 \to \Gamma_{12}\). The linear homomorphism \(\rho \circ \sigma\) is represented by a \(290 \times 35\) matrix. The kernel of \(\rho \circ \sigma: \Sigma_4 \to \bar{\Gamma}_{12}\) is of dimension 1 and is generated by the polynomial \(\Psi \in \Sigma_4\). □
We study the projective geometry of the surface $X_{56}$ more closely. The set

$$\mathcal{F}_{56} := \{ r \in S_X \mid \langle r, r \rangle = -2, \langle r, h_{56} \rangle = 1 \}$$

of classes of lines on $X_{56}$ can be easily calculated by Algorithm 2. It turns out that $\mathcal{F}_{56}$ spans $S_X$, and hence the stabilizer subgroup

$$\tilde{G}_{56} := \{ g \in O(S_X) \mid h_{56}^g = h_{56} \}$$

of $h_{56}$ in $O(S_X)$ is naturally isomorphic to the group of permutations of $\mathcal{F}_{56}$ that preserve the intersection numbers. We fix a list of vectors $[\lambda_1], \ldots, [\lambda_{30}]$ of $\mathcal{F}_{56}$ that form a basis of $S_X$, and calculate $\tilde{G}_{56}$ by the standard backtrack program in the same way as the calculation of $G_{48}$. It turns out that $\tilde{G}_{56}$ is of order 128. We put

$$\text{Aut}(X_{56}) := \{ \gamma \in \text{PGL}_4(\mathbb{C}) \mid \gamma(X_{56}) = X_{56} \},$$

and consider its image

$$G_{56} := \text{Im}(\text{Aut}(X_{56}) \to O(S_X))$$

by the natural representation. Then $G_{56}$ is a subgroup of $\tilde{G}_{56}$ consisting of elements that satisfy the period-preserving condition. It turns out that $G_{56}$ is isomorphic to the group

$$(((\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) : \mathbb{Z}/2\mathbb{Z}) : \mathbb{Z}/2\mathbb{Z})$$

of order 64. The action of $\text{Aut}(X_{56})$ decomposes $\mathcal{F}_{56}$ into three orbits $O_8$, $O_{16}$, $O_{32}$ of size 8, 16, and 32, respectively. We have

$$h_{56} = \frac{1}{8} \sum_{r \in O_{32}} r = \frac{1}{8} \left( 2 \sum_{r \in O_8} r + \sum_{r \in O_{16}} r \right).$$

The intersection $\mathcal{F}_{48} \cap \mathcal{F}_{56}$ consists of 30 classes. (In fact, in searching for a basis of $H^0(X_{48}, \mathcal{L}_{h_{56}})$ that gives a simple defining equation of $X_{56}$, we have used these
and the morphism $\Phi_{56}$, we can easily compute the defining equations of these 30 lines.

We say that a finite set $\{\mu_1, \ldots, \mu_N\}$ of lines in $\mathbb{P}^3$ has a unique common intersecting line if there exists a unique line in $\mathbb{P}^3$ that intersects all of $\mu_1, \ldots, \mu_N$. If we know the defining equations of $\mu_1, \ldots, \mu_N$, then we can determine whether $\{\mu_1, \ldots, \mu_N\}$ has a unique common intersecting line, and if it has, we can calculate the defining equation of the common intersecting line. Suppose that we know the defining equations of the 48 lines on $X_{48}$, but we do not know the defining equation of a line $\lambda_{N+1}$ on $X_{56}$. Since we have the set $\mathcal{F}_{56}$ of classes of lines on $X_{56}$, we can make the subset $\{\lambda_1', \ldots, \lambda_N'\}$ of $\{\lambda_1, \ldots, \lambda_N\}$ consisting of lines that intersect $\lambda$. If $\{\lambda_1', \ldots, \lambda_N'\}$ has a unique common intersecting line, then we can calculate the defining equation of $\lambda_{N+1}$. Starting from the 30 lines, we can calculate the defining equations of the remaining 26 lines on $X_{56}$. Since we know the permutation action of $G_{56}$ on $\mathcal{F}_{56}$, we can calculate each element of $\text{Aut}(X_{56})$. By these calculations, we obtain the following theorem.

**Theorem 4.6.** The subgroup $\text{Aut}(X_{56})$ of $\text{PGL}_4(\mathbb{C})$ is generated by the following elements of order 4.

$$
\gamma_1 = \begin{bmatrix}
1 & \zeta^2 & -\zeta + \zeta^2 - \zeta^3 & 1 - \zeta + \zeta^3 \\
-1 + \zeta - \zeta^3 & -\zeta + \zeta^2 - \zeta^3 & -\zeta^2 & 1 \\
-1 & \zeta^2 & -\zeta + \zeta^2 - \zeta^3 & -1 + \zeta - \zeta^3 \\
-1 + \zeta - \zeta^3 & \zeta - \zeta^2 + \zeta^3 & \zeta^2 & 1 \\
\end{bmatrix},
$$

$$
\gamma_2 = \begin{bmatrix}
1 & \zeta^2 & \zeta + \zeta^2 - \zeta^3 & 1 - \zeta - \zeta^3 \\
\zeta^2 & 1 & 1 - \zeta - \zeta^3 & \zeta + \zeta^2 - \zeta^3 \\
\zeta + \zeta^2 - \zeta^3 & 1 - \zeta - \zeta^3 & -1 & -\zeta^2 \\
1 - \zeta - \zeta^3 & \zeta + \zeta^2 - \zeta^3 & -\zeta^2 & -1 \\
\end{bmatrix}.
$$

The lines on $X_{56}$ are obtained from the following lines $\lambda^{(8)}$, $\lambda^{(16)}$, $\lambda^{(32)}$ by the action of $\text{Aut}(X_{56})$, where the size of the orbit of $\lambda^{(n)}$ are $n$.

- $\lambda^{(8)} : y_1 + \zeta^2 y_3 = y_2 - \zeta^2 y_3 = 0$,
- $\lambda^{(16)} : y_1 + (-\zeta + \zeta^2 - \zeta^3) y_3 = y_2 + (-\zeta + \zeta^2 - \zeta^3) y_3 = 0$,
- $\lambda^{(32)} : y_1 = 3y_2 + (-1 - \zeta - \zeta^3) y_3 + (-\zeta + \zeta^2 + \zeta^3) y_4 = 0$.

**Corollary 4.7.** Every line $\lambda$ on $X_{56}$ is defined by an equation $M_\lambda y = 0$, where $M_\lambda$ is a $2 \times 4$ matrix in the row-reduced echelon form with components in $\mathbb{Z}[\zeta, 1/3]$, and $y = [y_1, y_2, y_3, y_4]^T$.

5. **Reductions of $X_{56}$ at Primes of $\mathbb{Z}[\zeta]$**

5.1. **Buchberger algorithm and the reduction.** We need a slight enhancement of the Buchberger algorithm to calculate Gröbner bases over $\mathbb{K}_P$ at all but finitely many primes $P$ of $\mathbb{Z}[\zeta]$ simultaneously. This method must have been used by many people without fanfare, but we cannot find any appropriate references.

We fix a monomial ordering on the set of monomials of variables $z_1, \ldots, z_n$. Let $F$ be a field. We use the notation in Chapter 2 of [4]. In particular, for a non-zero polynomial $f \in F[z_1, \ldots, z_n]$, let $\text{LC}(f) \in F$ denote the leading coefficient.
for \( f, g \in F[z_1, \ldots, z_n] \) and a finite subset \( H \subset F[z_1, \ldots, z_n] \), let \( S(f, g) \) be the \( S \)-polynomial of \( f \) and \( g \), and \( \overline{f}^H \) the remainder on the division of \( f \) by \( H \).

Suppose that \( F \) is a number field, and let \( R \) be the integer ring of \( F \). For a prime \( P \) of \( R \), let \( R_P \) denote the localization of \( R \) at \( P \), \( R^*_P \) the group of units of \( R_P \), and \( \kappa_P \) the residue field of \( R \) at \( P \). For a polynomial \( f \in R_P[z_1, \ldots, z_n] \), let \( f \mod P \in \kappa_P[z_1, \ldots, z_n] \) denote the reduction of \( f \) at \( P \), and for a subset \( H \) of \( R_P[z_1, \ldots, z_n] \), let \( H \mod P \) denote the set of reductions at \( P \) of polynomials in \( H \). The following lemma follows immediately from the definition of the \( S \)-polynomial and the division algorithm.

**Lemma 5.1.** Let \( f \) and \( g \) be polynomials in \( R_P[z_1, \ldots, z_n] \), and \( H \) a finite subset of \( R_P[z_1, \ldots, z_n] \). We have \( S(f, g) \in F[z_1, \ldots, z_n] \) and \( \overline{f}^H \in F[z_1, \ldots, z_n] \).

1. Suppose that both of \( \text{LC}(f) \) and \( \text{LC}(g) \) belong to \( R^*_P \). Then \( S(f, g) \) belongs to \( R_P[z_1, \ldots, z_n] \), and \( S(f, g) \mod P \) is equal to the \( S \)-polynomial of the polynomials \( f \mod P \) and \( g \mod P \) in \( \kappa_P[z_1, \ldots, z_n] \).

2. Suppose that \( \text{LC}(h) \in R^*_P \) for any \( h \in H \). Then \( \overline{f}^H \) belongs to \( R_P[z_1, \ldots, z_n] \), and \( \overline{f}^H \mod P \) is equal to the remainder on the division of \( f \mod P \in \kappa_P[z_1, \ldots, z_n] \) by the subset \( H \mod P \) of \( \kappa_P[z_1, \ldots, z_n] \).

Suppose that a finite set \( \{ f_1, \ldots, f_s \} \) of non-zero polynomials in \( R[z_1, \ldots, z_n] \) is given. Let \( I_P \) be the ideal of \( F[z_1, \ldots, z_n] \) generated by \( \{ f_1, \ldots, f_s \} \). For a prime \( P \) of \( R \), let \( I_P \) be the ideal of \( \kappa_P[z_1, \ldots, z_n] \) generated by \( \{ f_1, \ldots, f_s \} \mod P \). A Gröbner basis \( G \) of \( I_P \) is calculated by the Buchberger algorithm. We initialize \( G := \{ f_1, \ldots, f_s \} \). If \( \overline{S(f_i, f_j)}^G \) is non-zero for a pair of \( f_i, f_j \in G \), we add \( f_t := \overline{S(f_i, f_j)}^G \) to \( G \). We continue this process until no new non-zero polynomials \( S(f_i, f_j)^G \) are obtained.

We introduce a variable set \( C \) in the Buchberger algorithm. We initialize

\[
C := \{ \text{LC}(f_i) \mid i = 1, \ldots, s \},
\]

and, whenever a new non-zero polynomial \( f_t := \overline{S(f_i, f_j)}^G \) is added to \( G \), we add \( \text{LC}(f_t) \) to \( C \). From Lemma 5.1, we obtain the following proposition.

**Proposition 5.2.** Let \( P \) be a prime of \( R \). Suppose that, when the algorithm terminates, we have \( C \subset R^*_P \). Then we have \( G \subset R_P[z_1, \ldots, z_n] \), and \( G \mod P \) is a Gröbner basis of the ideal \( I_P \) of \( \kappa_P[z_1, \ldots, z_n] \).

Since \( C \) is a finite set, we can calculate a finite set \( S \) of prime integers such that \( G \mod P \) is a Gröbner basis of \( I_P \) for any prime \( P \) over \( p \notin S \). More precisely, for \( \alpha \in F \), let \( d(\alpha) \) denote the least positive integer such that \( d(\alpha)\alpha \in R \), and let \( n(\alpha) \in \mathbb{Z} \) be the norm of \( d(\alpha)\alpha \in R \) over \( \mathbb{Z} \). Let \( \tilde{C} \) denote the subset

\[
\{ d(\alpha) \mid \alpha \in C \} \cup \{ n(\alpha) \mid \alpha \in C \}
\]

of \( \mathbb{Z} \setminus \{0\} \). For a finite set \( T \) of non-zero integers, let \( \mathcal{P}(T) \) denote the set of prime integers that divide at least one element of \( T \). Then \( C \subset R^*_P \) holds for any prime \( P \) over \( p \notin \mathcal{P}(\tilde{C}) \).

In fact, this naive method often fails to work in practice, because some elements of \( \tilde{C} \) can be so large that we cannot calculate their prime factors. (For example, in the proof of Theorem 5.3 below, this method led us to a factorization of a composite integer \( > 10^{80} \), which was impossible.) To overcome this difficulty, we
use the following trick. Let $T_1, \ldots, T_N$ be finite sets of non-zero integers. Then we have
\[ \mathcal{P}(T_1) \cap \cdots \cap \mathcal{P}(T_N) = \mathcal{P}(\gcd(T_1, \ldots, T_N)), \]
where
\[ \gcd(T_1, \ldots, T_N) := \{ \gcd(t_1, \ldots, t_N) \mid t_1 \in T_1, \ldots, t_N \in T_N \}. \]
Since the calculation of the greatest common divisor of large integers is much easier than the calculation of prime factors of these integers, we often manage to calculate $\mathcal{P}(T_1) \cap \cdots \cap \mathcal{P}(T_N)$ even when the calculation of $\mathcal{P}(T_i)$ is intractable.

For example, suppose that $I_F$ contains 1, and let us calculate a finite set $S$ of prime integers such that $1 \in I_P$ holds for any prime $P$ of $R$ over $p \notin S$. We carry out the Buchberger algorithm several times under various choices of monomial ordering, and obtain the sets $C_1, \ldots, C_N$ of non-zero integers for these choices. Note that, if $p \notin \mathcal{P}(C_\nu)$ for at least one $\nu$, then $i \in I_P$ for any prime $P$ of $R$ over $p$. Hence the intersection $S$ of these $\mathcal{P}(C_i)$ has the desired property.

By means of this method, we write the following algorithms.

**Algorithm 5.3.** Let $V$ be a subscheme of $\mathbb{P}^3$ defined by a homogeneous polynomial $\psi \in R[z_1, \ldots, z_4]$ such that $V \otimes F$ is a smooth surface. Then we can make a finite set $S$ of prime integers such that $V \otimes \kappa_P$ is a smooth surface for any prime $P$ of $R$ over $p \notin S$. Executing the Buchberger algorithm in the field $\kappa_P$ for the primes $P$ over $p \in S$, we can make the complete set of primes $P$ such that $V \otimes \kappa_P$ is not a smooth surface.

**Algorithm 5.4.** We say that a finite set $\{\mu_1, \ldots, \mu_N\}$ of lines in $\mathbb{P}^3$ defined over a field has no common intersecting lines if there exist no lines in $\mathbb{P}^3$ that intersect all of $\mu_1, \ldots, \mu_N$. Let $\{\mu_1, \ldots, \mu_N\}$ be a set of subschemes of $\mathbb{P}^3$ defined over $R$ such that $\{\mu_1 \otimes F, \ldots, \mu_N \otimes F\}$ is a set of distinct $N$ lines with no common intersecting lines. We can make a complete set $B$ of primes of $R$ such that $\{\mu_1 \otimes \kappa_P, \ldots, \mu_N \otimes \kappa_P\}$ is a set of distinct $N$ lines that has no common intersecting lines for any $P \notin B$.

### 5.2. Reductions of $X_{56}$

Let $X_{56}$ be the projective scheme over $\mathbb{Z}[\zeta]$ defined by the homogeneous equation $\Psi = 0$ in $\mathbb{P}^3$. The generic fiber $X_{56} \otimes \mathbb{Q}(\zeta)$ is the surface $X_{56}$. For a prime $P$ of $\mathbb{Z}[\zeta]$, let $\overline{\kappa}_P$ denote an algebraic closure of $\kappa_P$. We define $X_{56}(P)$ to be the pullback of $X_{56}$ by $\mathbb{Z}[\zeta] \to \overline{\kappa}_P$.

Recall that $A = -1 - 2\zeta - 2\zeta^3$ and $B = 3 + A$. There exists only one prime $P_2$ of $\mathbb{Z}[\zeta]$ over 2, and we have $A \mod P_2 = 1$. There exist exactly two primes $P_3$ and $P_3'$ of $\mathbb{Z}[\zeta]$ over 3, for which we have $A \mod P_3 = 0$ and $A \mod P_3' = 1$. It is easy to see that $X_{56}(P_2)$ and $X_{56}(P_3')$ are singular at the point $\overline{(1 : 0 : \sqrt{-1} : 0)}$.

**Proposition 5.5.** The surface $X_{56}(P_3)$ is projectively isomorphic over $\kappa_{P_3} \cong \mathbb{F}_9$ to the Fermat quartic surface in characteristic 3. In particular, $X_{56}(P_3)$ is smooth, and contains 112 lines, each of which is defined over $\kappa_{P_3}$.

**Proof.** The surface $X_{56}(P_3)$ is defined by $y_1^3y_2 + y_1y_2^3 + y_3^3y_4 + y_3y_4^3 = 0$, which is a non-degenerate Hermitian form in 4 variables over $\mathbb{F}_9$. Hence $X_{56}(P_3)$ is projectively isomorphic to the Fermat quartic surface in characteristic 3 over $\mathbb{F}_9$ by [23]. For the number of lines on this surface, see [23] or [7].

By Corollary 4.7, the lines on $X_{56}$ reduce to lines on $X_{56}(P)$ for any prime $P$ over $p > 3$. 

Theorem 5.6. Suppose that $P$ is a prime of $\mathbb{Z}[\zeta]$ over a prime integer $p > 3$. Then $X_{56}(P)$ is smooth, and contains exactly 56 lines, each of which is obtained by the reduction of a line on $X_{56}$ at $P$. Moreover, the isomorphism $X_{48} \cong X_{56}$ given in Table 4.2 reduces at $P$ to an isomorphism $X_{48} \otimes \overline{\kappa}_P \cong X_{56}(P)$.

Proof. The smoothness of $X_{56}(P)$ can be proved by Algorithm 5.3. We show that $X_{56}(P)$ contains exactly 56 lines, and that they are obtained by the reduction of lines on $X_{56}$. The fact that the 56 lines on $X_{56}$ reduce to distinct 56 lines on $X_{56}(P)$ keeping the intersection numbers can be easily proved. We will show that there exist no other lines on $X_{56}(P)$. Let $S_{X_{56}}(P)$ denote the Néron-Severi lattice of $X_{56}(P)$. Recall that the set $\mathcal{F}_{56}$ of classes of the 56 lines on $X_{56}$ spans $S_X$. Hence the reduction of lines on $X_{56}$ induces a natural embedding

$$S_X \hookrightarrow S_{X_{56}(P)}$$

of lattices. From now on, we regard $S_X$ as a sublattice of $S_{X_{56}(P)}$ by this embedding. In particular, $\mathcal{F}_{56}$ is a subset of the set of classes of lines on $X_{56}(P)$. It is enough to show that, if $\lambda$ is a line on $X_{56}(P)$, then its class $[\lambda] \in S_{X_{56}(P)}$ is in $\mathcal{F}_{56}$. Let $Q_P$ denote the orthogonal complement of $S_X$ in $S_{X_{56}(P)}$. Then $Q_P$ is either of rank 0 or negative-definite of rank 2. (See [23] for the problem when $Q_P$ is of rank 2.) We have

$$S_X \oplus Q_P \subset S_{X_{56}(P)} \subset S_X \oplus Q_P^\vee.$$

We denote the projections by

$$\text{pr}_S : S_{X_{56}(P)} \to S_X^\vee, \quad \text{pr}_Q : S_{X_{56}(P)} \to Q_P^\vee.$$

Let $h_{56}(P) \in S_{X_{56}(P)}$ denote the class of a hyperplane section of $X_{56}(P) \subset \mathbb{P}^3$. Since $\mathcal{F}_{56}$ is perpendicular to $Q_P^\vee$, we have $\langle \text{pr}_S(h_{56}(P)), r \rangle = \langle h_{56}(P), r \rangle = 1$ for any $r \in \mathcal{F}_{56}$. The class $h_{56} \in S_X$ is characterized in $S_X \otimes \overline{\kappa}$ by the property that $\langle h_{56}, r \rangle = 1$ holds for any $r \in \mathcal{F}_{56}$. Hence we have $\text{pr}_S(h_{56}(P)) = h_{56}$. Since $\langle h_{56}(P), h_{56}(P) \rangle = \langle h_{56}, h_{56} \rangle = 4$ and $Q_P$ is either 0 or negative-definite, we obtain $\text{pr}_Q(h_{56}(P)) = 0$, and therefore we have

$$h_{56}(P) = h_{56}.$$

Suppose that there exists a line $\nu$ on $X_{56}(P)$ such that $[\nu] \notin \mathcal{F}_{56}$. Since $h_{56}(P) = h_{56}$ and $h_{56} \perp Q_P$, we have $\langle \text{pr}_S([\nu]), h_{56} \rangle = 1$. Since $\langle \text{pr}_Q([\nu]), \text{pr}_Q([\nu]) \rangle \leq 0$, we have $\langle \text{pr}_S([\nu]), \text{pr}_S([\nu]) \rangle \geq -2$. For any $\lambda \in \mathcal{F}_{56}$, we have $\langle [\nu], [\lambda] \rangle \in \{0, 1\}$. Since $Q_P \perp \mathcal{F}_{56}$, we have $\langle \text{pr}_S([\nu]), [\lambda] \rangle \in \{0, 1\}$ for any $[\lambda] \in \mathcal{F}_{56}$. Therefore $\text{pr}_S([\nu])$ belongs to

$$\mathcal{F}_{56}^\prime := \{ r' \in S_X^\vee \mid \langle r', h_{56} \rangle = 1, \langle r', r' \rangle \geq -2, \langle r', [\lambda] \rangle \in \{0, 1\} \text{ for any } [\lambda] \in \mathcal{F}_{56} \}.$$ 

We calculate $\mathcal{F}_{56}^\prime$ by Algorithm 2.1. It turns out that $\mathcal{F}_{56}^\prime$ consists of 56 vectors. For each $r' \in \mathcal{F}_{56}^\prime$, we calculate the set $\Lambda(r') = \{ \lambda_1', \ldots, \lambda_k' \}$ of lines on $X_{56}$ such that

$$\{ [\lambda_1], \ldots, [\lambda_k] \} = \{ [\lambda] \in \mathcal{F}_{56} \mid \langle [\lambda], r' \rangle = 1 \}.$$

Then $\nu$ is a common intersecting line of the set

$$\Lambda(\text{pr}_S([\nu])) \otimes \kappa_P = \{ \lambda_1 \otimes \kappa_P, \ldots, \lambda_k \otimes \kappa_P \}$$

of lines over $\kappa_P$. On the other hand, since we know the defining equations over $\mathbb{Z}[\zeta, 1/3]$ of lines $\lambda_1', \ldots, \lambda_k'$, we can see by Algorithm 5.4 that $\Lambda(r') \otimes \kappa_P$ has no common intersecting lines for any $r' \in \mathcal{F}_{56}^\prime$ if $P$ is over $p > 3$. Thus we obtain a contradiction. (See Remark 5.7 for what happens when $p = 3$.)
Next, we investigate the reduction at $P$ of the isomorphism $\Phi_{56}: X_{48} \cong X_{56}$ given in Table 11. The equality (4.3) implies that, as an invertible sheaf, the line bundle $L_{h_{56}}^\otimes 8$ is isomorphic to $O(D)$, where $D$ is a linear combination of lines on $X_{56}$. Hence the embedding $S_X \hookrightarrow S_{X_{56}(P)}$ induced by the reduction of lines on $X_{56}$ maps the class $h_{56}$ of $L_{h_{56}}$ to the class of the line bundle $L_{h_{56}} \otimes \overline{k}_P$. Since $h_{56} = h_{56}(P)$, the line bundle $L_{h_{56}} \otimes \overline{k}_P$ is the very ample line bundle associated with the embedding $X_{56}(P) \hookrightarrow \mathbb{P}^3$. We confirm that $\lambda \mod P := (f_1 \mod P, \ldots, f_4 \mod P)$ are linearly independent over $\kappa_P$. Hence $\lambda \mod P$ form a basis of the space of the global sections of $L_{h_{56}} \otimes \overline{k}_P$.

**Remark 5.7.** We investigate the lines on $X_{56}(P_3)$, where $P_3$ is the prime of $\mathbb{Z}[\zeta]$ in Proposition 5.5. For each line $\lambda$ on $X_{56}$, we have an invertible $2 \times 2$ matrix $U_\lambda$ with components in $\mathbb{Q}(\zeta)$ such that $U_\lambda M_\lambda$ has components in the localization $\mathbb{Z}[\zeta][P_3]$, and that $M_\lambda := U_\lambda M_\lambda \mod P_3$ is a matrix of row-reduced echelon form of rank 2. Hence the equation $M_\lambda y = 0$ defines a line on $X_{56}(P_3)$ defined over $\kappa_{P_3}$. Using these equations, we can make the reduction $\lambda \mapsto \lambda \otimes \kappa_{P_3}$ of lines on $X_{56}$ to lines on $X_{56}(P_3)$. Since this reduction keeps the intersection numbers, it induces an embedding $S_X \hookrightarrow S_{X_{56}(P_3)}$. For each $r' \in \mathcal{F}'_{56}$, the set $\Lambda(r') \otimes \kappa_{P_3}$ of lines over $\kappa_{P_3}$ has a unique common intersecting line, and the common intersecting line is contained in $X_{56}(P_3)$. Thus we obtain the $112 = |\mathcal{F}_{56}| + |\mathcal{F}'_{56}|$ lines on $X_{56}(P_3)$.

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