KRONECKER-WEBER PLUS EPSILON

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Abstract. We say that a group is almost abelian if every commutator is central and squares to the identity. Let $G$ be the Galois group of the algebraic closure of the field $\mathbb{Q}$ of rational numbers in the field of complex numbers. Let $G^{ab+\epsilon}$ be the quotient of $G$ universal for continuous homomorphisms to almost abelian profinite groups and let $\mathbb{Q}^{ab+\epsilon}/\mathbb{Q}$ be the corresponding Galois extension. We prove that $\mathbb{Q}^{ab+\epsilon}$ is generated by the roots of unity, the fourth roots of the rational primes and the square roots of certain algebraic sine-monomials. The inspiration for the paper came from recent studies of algebraic $\Gamma$-monomials by P. Das and by S. Seo.

1. Introduction

We say that a group is almost abelian if every commutator is central and squares to the identity. Let $G$ be the Galois group of the algebraic closure of the field of rational numbers $\mathbb{Q}$ in the field $\mathbb{C}$ of complex numbers. Let $G^{ab+\epsilon}$ be the quotient of $G$ universal for continuous homomorphisms to almost abelian profinite groups. Let $G^\epsilon$ be the kernel of the natural map of $G^{ab+\epsilon}$ to the abelianization $G^{ab}$ of $G$. By construction the group $G^\epsilon$ is central in $G^{ab+\epsilon}$ and killed by $2$. Let $\mathbb{Q}^{ab}$ (resp. $\mathbb{Q}^{ab+\epsilon}$) be the Galois extension of $\mathbb{Q}$ in $\mathbb{C}$ with Galois group $G^{ab}$ (resp. $G^{ab+\epsilon}$). The Kronecker-Weber theorem determines the structure of the group $G^{ab}$ and provides an explicit description of the field $\mathbb{Q}^{ab}$. The theory of Fröhlich in principle determines the structure of the group $G^{ab+\epsilon}$ but does not provide an explicit description of the field $\mathbb{Q}^{ab+\epsilon}$. Kummer theory identifies the Pontryagin dual of $G^\epsilon$ with $H^0(G^{ab}, \mathbb{Q}^{ab}\times/\mathbb{Q}^{ab}\times2)$. Our purpose in this paper is to exhibit for the latter group an explicit $\mathbb{Z}/2\mathbb{Z}$-basis, thereby obtaining a description of the field $\mathbb{Q}^{ab+\epsilon}$ as explicit as that provided for the field $\mathbb{Q}^{ab}$ by the Kronecker-Weber theorem. Our method is more or less elementary and independent of the theory of Fröhlich. The inspiration for our work came from the recent studies of algebraic $\Gamma$-monomials.

Our main results are as follows. Let $\mathcal{A}$ be the free abelian group on symbols of the form

$$[a] \quad (a \in \mathbb{Q}),$$

modulo the identifications

$$[a] = [b] \Leftrightarrow a - b \in \mathbb{Z}.$$
For all prime numbers $p < q$, if $2 < p$ put
\[
a_{pq} := \sum_{i=1}^{p-1} \left( \left\lfloor \frac{i}{p} \right\rfloor - \sum_{k=0}^{i-1} \left\lfloor \frac{i}{pq} + \frac{k}{q} \right\rfloor \right) - \sum_{j=1}^{q-1} \left( \left\lfloor \frac{j}{q} \right\rfloor - \sum_{t=0}^{j-1} \left\lfloor \frac{j}{pq} + \frac{t}{p} \right\rfloor \right),
\]
e. g.,
\[
a_{3,5} = \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{15} \right\rfloor - \left\lfloor \frac{4}{15} \right\rfloor - \left\lfloor \frac{1}{5} \right\rfloor.
\]
and if $2 = p$ put
\[
a_{pq} := \left( \left\lfloor \frac{1}{4} \right\rfloor - \sum_{k=0}^{p-1} \left\lfloor \frac{1}{4q} + \frac{k}{q} \right\rfloor \right) - \sum_{j=1}^{q-1} \left( \left\lfloor \frac{j}{q} \right\rfloor - \sum_{\ell=0}^{j-1} \left\lfloor \frac{j}{pq} + \frac{\ell}{p} \right\rfloor \right),
\]
e. g.,
\[
a_{2,3} := \left\lfloor \frac{1}{4} \right\rfloor - \left\lfloor \frac{5}{12} \right\rfloor - \left\lfloor \frac{1}{3} \right\rfloor.
\]
Let
\[\sin : A \rightarrow \mathbb{Q}^{ab}\times\]
be the unique homomorphism such that
\[\sin[a] = \begin{cases} 
2 \sin \pi a \left( = \left\lfloor 1 - e^{2\pi i a} \right\rfloor \right) & \text{if } 0 < a < 1 \\
1 & \text{if } a = 0 \end{cases} \quad (a \in \mathbb{Q} \cap [0, 1)).
\]
We prove that the family of real numbers
\[\left\{ \sqrt{\ell} \right\}_{\ell, \text{prime}} \bigcup \left\{ \sin a_{pq} \right\}_{p, q, \text{prime}, p < q}
\]
projects to a $\mathbb{Z}/2\mathbb{Z}$-basis of the group $H^0(G^{ab}, \mathbb{Q}^{ab} \times / \mathbb{Q}^{ab} \times 2)$ and hence that
\[\mathbb{Q}^{ab} + \varepsilon = \mathbb{Q}^{ab} \left( \left\{ \sqrt{\ell} \right\}_{\ell, \text{prime}} \bigcup \left\{ \sqrt{\sin a_{pq}} \right\}_{p, q, \text{prime}, p < q} \right).
\]
We actually prove more. We define a canonical injective homomorphism
\[D : H^0(G^{ab}, \mathbb{Q}^{ab} \times / \mathbb{Q}^{ab} \times 2) \rightarrow \bigwedge^2 H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \left( = \bigwedge^2 \mathbb{Q} \times / \mathbb{Q} \times 2 \right)
\]
and exhibit a preimage for each element of a natural $\mathbb{Z}/2\mathbb{Z}$-basis of the target. (In particular, it turns out that $D$ is an isomorphism.) Our main results are the “Auxiliary Formula” (§3.4) and the “Main Formula” (§4.3.4). Our main technical tools are the “Log Wedge Formula” (§3.3) and a family of combinatorial identities of “Das type” (§1.3).

Now the fact that the homomorphism $D$ is an isomorphism is not really new: one can easily deduce it from [Fröhlich 1983, Theorem 4.10, p. 56]. (See Remark 4.3.5 below for further discussion.) Rather, it is the explicit procedure developed here for inverting the map $D$ that is really new.

A key role is played in this paper by the universal odd ordinary distribution $U^-$, namely the quotient of $A$ by the subgroup generated by all expressions of the form
\[\left\lfloor a \right\rfloor - \sum_{i=0}^{N-1} \left\lfloor \frac{a+i}{N} \right\rfloor, \quad \left\lfloor a \right\rfloor - \left\lfloor 1 - a \right\rfloor \quad (a \in \mathbb{Q}, \ N: \text{positive integer}).
\]
A study of $U^-$ and related objects was made in [Kubert 1978a] and [Kubert 1978b], building on [Sinnott 1978]. In particular, it was proved that the torsion subgroup
of $U^-$ is killed by 2. Kubert’s results combined with the idea behind the algebraic-
ity criterion of [Koblitz-Ogus 1979] yield a more or less mechanical procedure for
determining whether a given element of $A$ represents a torsion element of $U^-$. The
double complex method of [Anderson 1999] yields a canonical $\mathbb{Z}/2\mathbb{Z}$-basis for the
torsion subgroup of $U^-$ indexed by finite sets of prime numbers of even cardinality.
In [Das 2000, Sec. 3 and Sec. 9] it is proved that the family $\{a_{pq}\}_{2<q<p}$ represents
the “two-odd-prime” part of the canonical basis. The method of Das can easily be
modified to prove that the family $\{a_{pq}\}_{p<q}$ represents the “two-prime” part of the
canonical basis.

The torsion subgroup $U^-$ plays an important role in the theory of algebraic
$\Gamma$-monomials. Let

$$\Gamma : A \to \mathbb{R}^\times$$

be the unique homomorphism such that

$$\Gamma([a]) = \begin{cases} \sqrt{2\pi}/\Gamma(a) & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0 \end{cases} \quad (a \in \mathbb{Q} \cap [0, 1)).$$

Now fix $a \in A$ representing a torsion element of $U^-$. By straightforward manipula-
tion of standard functional equations satisfied by $\Gamma(s)$ one verifies that $\Gamma(a)$ is an
algebraic number. Numbers of the form $\Gamma(a)$ are the so called algebraic $\Gamma$-
monomials. Under mild hypotheses, a reciprocity law [Deligne-Milne-Ogus-Shih 1982, Thm. 7.15,
p. 91] due to Deligne links $\Gamma(a)$ to an analogously defined Jacobi sum Hecke charac-
ter. Deligne reciprocity is the main motivation for studying algebraic $\Gamma$-monomials.

In [Das 2000] the algebraic $\Gamma$-monomials were related to algebraic sine-
monomials and properties of the latter were investigated. Again fix $a \in A$ representing a tor-
sion element of $U^-$. By [Das 2000, Thm. 6], under mild hypotheses, $\Gamma(a)$ factors as
the root of a rational number times $\sqrt{\sin a}$, and the factorization can in principle
be worked out explicitly. For example, one has

$$\Gamma(a_{3, 5}) = \frac{\Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{1}{3}\right)} \sqrt{\frac{\sin \frac{\pi}{5} \cdot \sin \frac{2\pi}{5}}{\sin \frac{\pi}{3} \cdot \sin \frac{\pi}{5}}} = 3^{-\frac{1}{2}} \sqrt{\sin a_{3, 5}}$$

and

$$\Gamma(a_{2, 3}) = \frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt{2\pi}} \sqrt{2 \cdot \sin \left(\frac{\pi}{12}\right)} \cdot \sin \left(\frac{\pi}{4}\right) = 2^{-\frac{1}{4}} \sqrt{\sin a_{2, 3}}.$$

Indeed, for all primes $p < q$ one has

$$\frac{\Gamma(a_{pq})}{\sqrt{\sin a_{pq}}} = \begin{cases} p^{-\frac{(p-1)^2}{16q} - \frac{(q-1)^2}{16p}} & \text{if } 2 < p, \\ 2^{-\frac{4}{p-1} - \frac{2}{q-1}} & \text{if } 2 = p, \end{cases}$$

as is explained below in Remark 4.3.3. By [Das 2000, Thm. 11], under mild hypo-
theses, the extension $\mathbb{Q}^{ab}\left(\sqrt{\sin a}\right)/\mathbb{Q}$ is Galois, and in fact the latter is true in
general; see Proposition 1.1.8 below. By [Das 2000, Thm. 22], if $a$ represents an
element of the canonical basis for the torsion subgroup of $U^-$ indexed by a finite
set of odd primes of cardinality at least four, then $\sqrt{\sin a} \in \mathbb{Q}^{ab}$.

The theory of [Das 2000] begs the question as to the nature of the Galois ex-
tensions of the form $\mathbb{Q}^{ab}\left(\sqrt{\sin a_{pq}}\right)/\mathbb{Q}$. In order to begin answering that question,
Das calculated (see [Das 2000, end of Sec. 16]) the structure of the central extension

\[ 1 \to \{ \pm 1 \} \to \text{Gal} \left( \mathbb{Q} \left( e^{\frac{2\pi i}{60}}, \sqrt{\sin a_{3.5}} \right) / \mathbb{Q} \right) \to (\mathbb{Z}/60\mathbb{Z})^\times \to 1 \]

explicitly and in particular proved that the middle group is nonabelian. That calculation of Das was the primary inspiration for this paper. Another important inspiration was the striking result [Seo 2001, Prop. 2.3] according to which, for all odd primes \( p < q \), one has

\[ (-1)^{v_p(\sin a_{pq})} = \left( \frac{p}{q} \right), \quad (-1)^{v_q(\sin a_{pq})} = \left( \frac{q}{p} \right) \]

where \( \left( \cdot \bigg| \frac{\cdot}{\cdot} \right) \) is the Legendre symbol, \( v_p \) is any additive valuation of \( \mathbb{Q}^{ab} \) above \( p \) such that \( v_p \left( 1 - e^{\frac{2\pi i}{p}} \right) = 1 \), and \( v_q \) is analogously chosen above \( q \). It follows that if \( \left( \frac{p}{q} \right) = -1 \) or \( \left( \frac{q}{p} \right) = -1 \), then \( \sqrt{\sin a_{pq}} \) cannot belong to \( \mathbb{Q}^{ab} \). Seo’s result greatly encouraged the author to attempt the calculations detailed in this paper.

2. Group-theoretical background

2.1. The category of \( \mathbb{N} \)-profinite groups. We say that a topological group \( G \) is \( \mathbb{N} \)-profinite if \( G \) is compact and there exists a chain \( U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots \) of open normal subgroups of \( G \) forming a neighborhood base at the origin. A morphism of \( \mathbb{N} \)-profinite groups is by definition a continuous homomorphism of topological groups; thus the \( \mathbb{N} \)-profinite groups form a category. The Galois group of any Galois extension of countable fields is \( \mathbb{N} \)-profinite. (But the Galois group of an algebraic closure of the field of rational functions in one variable over the field of complex numbers is not \( \mathbb{N} \)-profinite.) A compact topological group \( G \) is \( \mathbb{N} \)-profinite if and only if \( G \) has a countable neighborhood base consisting of open compact sets.

Proposition 2.2. Let \( G \) be an \( \mathbb{N} \)-profinite group. Let \( H \) be a closed normal subgroup of \( G \). There exists a function \( \vartheta : G \to G \) with the following properties:

\begin{itemize}
  \item \( \vartheta \) is continuous.
  \item \( \vartheta \) is constant on cosets of \( H \) in \( G \).
  \item \( \vartheta(\sigma) \in \sigma H \) for all \( \sigma \in G \).
\end{itemize}

Proof. Fix a descending chain \( U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots \) of open normal subgroups of \( G \) forming a neighborhood base at the origin. Choose a set \( S_i \) of representatives for the cosets of \( HU_i \) in \( G \). For each index \( i > 1 \) choose a set \( S_i \subseteq U_{i-1} \) of representatives for the cosets of \( HU_i \) in \( HU_{i-1} \). For each index \( i \) let \( \vartheta_i : G \to G \) be the unique map collapsing each coset of \( HU_i \) in \( G \) to its unique representative in the finite set \( S_1 \cdots S_i \). The uniform limit \( \vartheta \) of the maps \( \vartheta_i \) exists and has all the desired properties. \( \Box \)

2.3. Cochains, cocycles, coboundaries and cohomology classes. Let \( G \) be a topological group. Let \( A \) be a topological abelian group equipped with a continuous left \( G \)-action. For each nonnegative integer \( n \), let \( C^n(G, A) \) denote the group of continuous functions

\[ a = ((\sigma_1, \ldots, \sigma_n) \mapsto a_{\sigma_1, \ldots, \sigma_n}) : G^n \to A, \]
and equip the graded group
\[ C^*(G, A) := \bigoplus C^n(G, A) \]
with a differential \( \delta \) of degree 1 by the standard rule
\[
(\delta a)_{\sigma_1, \ldots, \sigma_{n+1}} = a_{\sigma_1, \ldots, \sigma_n} \cdot \sigma_{n+1} + \cdots + (-1)^{n+1} a_{\sigma_1, \ldots, \sigma_n+1} + \cdots + (-1)^n a_{\sigma_1, \ldots, \sigma_n}.
\]
Elements of \( C^n(G, A) \) will be called cochains of degree \( n \). In low degree the differential \( \delta \) takes the form
\[
\begin{align*}
(\delta a)_{\sigma_1} &= \sigma_1 a - a & \text{if } n = 0, \\
(\delta a)_{\sigma_1, \sigma_2} &= \sigma_1 a_{\sigma_2} - a_{\sigma_1, \sigma_2} + a_{\sigma_1} & \text{if } n = 1, \\
(\delta a)_{\sigma_1, \sigma_2, \sigma_3} &= \sigma_1 a_{\sigma_2, \sigma_3} - a_{\sigma_1, \sigma_2, \sigma_3} + a_{\sigma_1, \sigma_2, \sigma_3} - a_{\sigma_1, \sigma_2} & \text{if } n = 2.
\end{align*}
\]
Elements of the groups
\[
\begin{align*}
Z^n(G, A) &:= \ker (\delta : C^n(G, A) \to C^{n+1}(G, A)), \\
B^n(G, A) &:= \text{image } (\delta : C^{n-1}(G, A) \to C^n(G, A)), \\
H^n(G, A) &:= Z^n(G, A)/B^n(G, A)
\end{align*}
\]
will be called \( n \)-cocycles, \( n \)-coboundaries and cohomology classes of degree \( n \), respectively. If \( G \) is profinite and \( A \) is discretely topologized, then \( H^n(G, A) \) is the usual Galois cohomology group of degree \( n \).

2.4. Cup product. Let \( G \) be a topological group and let \( A \) be a commutative topological ring equipped with a continuous \( G \)-action. Given \( a \in C^p(G, A) \) and \( b \in C^q(G, A) \), put
\[
(a \cup b)_{\sigma_1, \ldots, \sigma_{p+q}} = a_{\sigma_1, \ldots, \sigma_p} \cdot \sigma_1 \cdots \sigma_p b_{\sigma_{p+1}, \ldots, \sigma_{p+q}},
\]
thereby defining the cup product \( a \cup b \in C^{p+q}(G, A) \) of the cochains \( a \) and \( b \). For \( p = q = 1 \) one has
\[
(a \cup b)_{\sigma_1, \sigma_2} = a_{\sigma_1} \cdot \sigma_1 b_{\sigma_2}.
\]
In general one has
\[
\delta(a \cup b) = (\delta a) \cup b + (-1)^p a \cup (\delta b)
\]
and thus the cup product construction induces a product in \( H^*(G, A) \).

2.5. Extensions in the category of \( \mathbb{N} \)-profinite groups. Let \( G \) be an \( \mathbb{N} \)-profinite group and let \( A \) be an \( \mathbb{N} \)-profinite abelian group equipped with a continuous left \( G \)-action. An exact sequence
\[
\Sigma : \quad 1 \to A \xrightarrow{i} U \xrightarrow{p} G \to 1
\]
in the category of \( \mathbb{N} \)-profinite groups such that
\[
u i(a)u^{-1} = i(p(u)a) \quad (u \in U, \ a \in A)
\]
is called an extension of \( G \) by \( A \) in the category of \( \mathbb{N} \)-profinite groups. If \( A \) is a trivial \( G \)-module, the homomorphism \( i : A \to U \) takes values in the center of \( U \); in such a case one says that \( \Sigma \) is a central extension. Extensions \( \Sigma \) and \( \Sigma' \) of \( G \) by \( A \) are said to be isomorphic if there exists an isomorphism \( \psi : U \to U' \) of \( \mathbb{N} \)-profinite groups such that \( \psi \circ i = i' \) and \( p' \circ \psi = p \). A set-theoretic splitting \( \vartheta : G \to U \) of \( \Sigma \) is a continuous map such that \( p \circ \vartheta = \id_G \). Set-theoretic splittings exist by Proposition 2.2. We say that \( \Sigma \) is trivial if there exists a set-theoretic splitting that is also a group homomorphism. Any two trivial extensions of \( G \) by \( A \) are
isomorphic. Each set-theoretic splitting $\vartheta$ of $\Sigma$ gives rise to a cochain $a \in C^2(G, A)$ by the rule
$$\vartheta(\sigma) \vartheta(\tau) = i(a_{\sigma, \tau}) \vartheta(\sigma \tau) \quad (\sigma, \tau \in G).$$

Associativity of the group law in $U$ forces $a$ to be a 2-cocycle. The cohomology class of the 2-cocycle $a$ depends only the isomorphism class of the extension $\Sigma$ and accordingly is denoted $[\Sigma]$. Every 2-cocycle representing the cohomology class $[\Sigma]$ arises from some set-theoretic splitting of $\Sigma$. The construction $\Sigma \mapsto [\Sigma]$ puts isomorphism classes of extensions of $G$ by $A$ in the category of $\mathbb{N}$-profinite groups into bijective correspondence with $H^2(G, A)$ and sends every trivial extension to 0.

2.6. Almost abelian $\mathbb{N}$-profinite groups. Let $G$ be an $\mathbb{N}$-profinite group and put
$$[\sigma, \tau] := \sigma \tau \sigma^{-1} \tau^{-1} \quad (\sigma, \tau \in G).$$

The abelianization $G^{ab}$ of $G$ is the abelian quotient of $G$ universal for morphisms to abelian $\mathbb{N}$-profinite groups. One has
$$G^{ab} = G/[G, G],$$
where $[G, G]$ is the closed subgroup of $G$ topologically generated by the set
$$\{[\sigma, \tau] \mid \sigma, \tau \in G\}.$$

We say that $G$ is almost abelian if $[\sigma, \tau]$ is central and 2-torsion for all $\sigma, \tau \in G$. We define $G^{ab+\varepsilon}$ to be the almost abelian quotient of $G$ universal for morphisms to almost abelian $\mathbb{N}$-profinite groups. One has
$$G^{ab+\varepsilon} = G/[G, G]^\varepsilon,$$
where $[G, G]^\varepsilon$ is the closed subgroup of $[G, G]$ topologically generated by the set
$$\{[\sigma, \tau]^2 \mid \sigma, \tau \in G\} \cup \{[\sigma, [\tau, \eta]] \mid \sigma, \tau, \eta \in G\}.$$

Put
$$G^\varepsilon := [G, G]/[G, G]^\varepsilon.$$

The preceding constructions fit together to form a canonical central extension
$$\Sigma_G : 1 \to G^\varepsilon \to G^{ab+\varepsilon} \to G^{ab} \to 1$$
of $\mathbb{N}$-profinite groups. We remark that the Pontryagin dual of $G^\varepsilon$ is canonically isomorphic to $H^1(G^\varepsilon, \mathbb{Z}/2\mathbb{Z})$ and that the latter is a vector space over $\mathbb{Z}/2\mathbb{Z}$ of at most countably infinite dimension.

**Proposition 2.7.** Let $G$ be an $\mathbb{N}$-profinite group. Let $H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^-$ be the quotient of $H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})$ by the subgroup consisting of all cohomology classes of the form $[\Sigma]$ for some extension $\Sigma$ of $G^{ab}$ by $\mathbb{Z}/2\mathbb{Z}$ the middle group of which is abelian. For each $c \in H^1(G^\varepsilon, \mathbb{Z}/2\mathbb{Z})$, view $c$ as a continuous homomorphism $G^\varepsilon \to \mathbb{Z}/2\mathbb{Z}$, and let
$$(a \mapsto c \circ a) : H^2(G^{ab}, G^\varepsilon) \to H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})$$
be the map induced by $c$ in degree 2 cohomology. The homomorphism
$$(c \mapsto c \circ [\Sigma_G]) : H^1(G^\varepsilon, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})$$
followed by the projection
$$H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^-$$
is injective.
Proof. Fix $0 \neq c \in H^1(G', \mathbb{Z}/2\mathbb{Z})$ and consider the extension
\[ \Sigma : 0 \to \mathbb{Z}/2\mathbb{Z} \to G_{\text{ab}}^+ / \ker c \to G_{\text{ab}} \to 1. \]
Then
\[ [\Sigma] = c \circ [\Sigma'] \]
and the middle group of $\Sigma$ is nonabelian by definition of $G_{\text{ab}}$. \hfill \Box

**Lemma 2.8.** Let $G = \prod_{i=1}^{\infty} G_i$ be an abelian $\mathbb{N}$-profinite group decomposed as a countable product of cyclic profinite groups $G_i$. Let an extension
\[ \Sigma : 0 \to \mathbb{Z}/2\mathbb{Z} \to U \to G \to 1 \]
of $\mathbb{N}$-profinite groups be given. Let
\[ a \in Z^2(G, \mathbb{Z}/2\mathbb{Z}) \]
be any 2-cocycle representing the cohomology class
\[ [\Sigma] \in H^2(G, \mathbb{Z}/2\mathbb{Z}). \]
For each index $i$ fix $\sigma_i \in G$ with the following properties:
\begin{itemize}
  \item $\sigma_i$ projects to a topological generator of $G_i$.
  \item $\sigma_i$ projects to 1 in $G_j$ for all indices $j \neq i$.
\end{itemize}
Put
\[ \alpha_{ij} := a_{\sigma_i,\sigma_j} + a_{\sigma_j,\sigma_i} \mod 2 \]
for all indices $i < j$. Then the following hold:
\begin{enumerate}
  \item $\alpha_{ij}$ vanishes for all but finitely many pairs of indices.
  \item $\alpha_{ij}$ vanishes if and only if $\sigma_i$ and $\sigma_j$ have liftings to $U$ that commute.
  \item $\alpha_{ij}$ depends only on the cohomology class $[\Sigma]$.
  \item If $\sigma_i$ or $\sigma_j$ is the square of an element of $G$, then $\alpha_{ij}$ vanishes.
  \item The middle group $U$ of $\Sigma$ is abelian if and only if all the $\alpha_{ij}$ vanish.
\end{enumerate}

Proof. Fix a set-theoretic splitting $\vartheta$ of $\Sigma$ giving rise to the 2-cocycle $a$. 

1. By hypothesis one has
\[ (\delta a)_{1,1} - (\delta a)_{1,1} \equiv a_{1,1} + a_{1,1} \equiv 0 \mod 2 \quad (\sigma \in G) \]
and there exists an open normal subgroup $V \subseteq G$ such that $a_{\sigma,\tau}$ depends only on the pair of cosets $(\sigma V, \tau V)$. It follows that if either $\sigma_i \in V$ or $\sigma_j \in V$, then $\alpha_{ij}$ vanishes. But one has $\sigma_i \in V$ for all but finitely many indices $i$.

2. The following statements are equivalent:
\begin{itemize}
  \item The coefficient $\alpha_{ij}$ vanishes.
  \item $\vartheta(\sigma_i)$ and $\vartheta(\sigma_j)$ commute.
  \item $\sigma_i$ and $\sigma_j$ have liftings to $U$ that commute.
  \item Any liftings of $\sigma_i$ and $\sigma_j$ to $U$ commute.
\end{itemize}

3. By what we have already proved, the coefficients $\alpha_{ij}$ depend only on the isomorphism class of the extension $\Sigma$ and hence only on the cohomology class $[\Sigma]$.

4. Suppose, say, that $\sigma_i = \tau^2$ for some $\tau \in G$. Then
\[ \vartheta(\tau)^2 c = \vartheta(\sigma_i), \quad \vartheta(\tau) \vartheta(\tau) \vartheta(\sigma_j) = \vartheta(\sigma_j) \vartheta(\tau) c' \]
where $c$ and $c'$ belong to the center of $U$, and hence
\[ \vartheta(\sigma_i) \vartheta(\sigma_j) = \vartheta(\tau) \vartheta(\tau) \vartheta(\tau) c' c = \vartheta(\sigma_j) \vartheta(\tau) c' \vartheta(\tau) c c = \vartheta(\sigma_j) \vartheta(\sigma_i), \]
i.e., $\vartheta(\sigma_i)$ and $\vartheta(\sigma_j)$ commute, and hence the coefficient $\alpha_{ij}$ vanishes.
5. All the coefficients \( \alpha_{ij} \) vanish if and only if the elements of the family \( \{ \vartheta(\sigma_i) \} \) commute among themselves. But the union of the latter family with \( \ker(U \to G) \) generates \( U \) topologically.

**Proposition 2.9.** Let \( G \) be an abelian \( \mathbb{N} \)-profinite group admitting decomposition as a countably infinite product of cyclic profinite groups. The cup product induces an isomorphism

\[
\bigwedge^2 H^1(G, \mathbb{Z}/2\mathbb{Z}) \cong H^2(G, \mathbb{Z}/2\mathbb{Z})^{\sim}
\]

of vector spaces over \( \mathbb{Z}/2\mathbb{Z} \), where \( H^2(G, \mathbb{Z}/2\mathbb{Z})^{\sim} \) is as defined in Proposition 2.7.

**Proof.** For each \( e \in Z^1(G, \mathbb{Z}/2\mathbb{Z}) = H^1(G, \mathbb{Z}/2\mathbb{Z}) \), the cohomology class of the 2-cocycle \( e \cup e \) vanishes in the quotient \( H^2(G, \mathbb{Z}/2\mathbb{Z})^{\sim} \) by Lemma 2.8. Therefore the cup product induces a well defined map

\[
\bigwedge^2 H^1(G, \mathbb{Z}/2\mathbb{Z}) \to H^2(G, \mathbb{Z}/2\mathbb{Z})^{\sim}.
\]

Fix a decomposition \( G = \prod_{i=1}^{\infty} G_i \) of \( G \) as a countably infinite product of cyclic profinite groups \( G_i \) and for each index \( i \) fix \( \sigma_i \in G \) with the properties specified in Lemma 2.8. Let \( I \) be the set of indices \( i \) such that \( G_i \not\subseteq G^2 \). For each index \( i \in I \) there exists a unique 1-cocycle \( e_i \in Z^1(G, \mathbb{Z}/2\mathbb{Z}) = H^1(G, \mathbb{Z}/2\mathbb{Z}) \) such that \( (e_i)_{\sigma_j} \equiv \delta_{ij} \mod 2 \) for all indices \( j \). Then the family

\[
\{e_i\}_{i \in I}
\]

is a \( \mathbb{Z}/2\mathbb{Z} \)-basis for \( H^1(G, \mathbb{Z}/2\mathbb{Z}) \) and hence the family

\[
\{e_i \wedge e_j\}_{i,j \in I, i < j}
\]

is a \( \mathbb{Z}/2\mathbb{Z} \)-basis for \( \bigwedge^2 H^1(G, \mathbb{Z}/2\mathbb{Z}) \). By Lemma 2.8 the cohomology classes of the 2-cocycles belonging to the family

\[
\{e_i \cup e_j\}_{i,j \in I, i < j}
\]

remain independent over \( \mathbb{Z}/2\mathbb{Z} \) in \( H^2(G, \mathbb{Z}/2\mathbb{Z})^{\sim} \). Therefore the map in question is injective. For any \( a \in Z^2(G, \mathbb{Z}/2\mathbb{Z}) \) the cohomology class of the two-cocycle

\[
a + \sum_{\substack{i,j \in I \atop i < j}} (a_{\sigma_i, \sigma_j} + a_{\sigma_j, \sigma_i}) e_i \cup e_j
\]

vanishes in the quotient \( H^2(G, \mathbb{Z}/2\mathbb{Z})^{\sim} \) by Lemma 2.8. Therefore the map in question is surjective.

**Proposition 2.10.** For any \( \mathbb{N} \)-profinite group \( G \), the group \( [G,G]^\epsilon \) is the intersection of all relatively open subgroups of \( [G,G] \) that are of index 2 in \( [G,G] \) and normal in \( G \).

**Proof.** Let \( H \) be the intersection of all relatively open subgroups of \( [G,G] \) that are of index 2 in \( [G,G] \) and normal in \( G \). Since the quotient \( G/H \) is almost abelian, one has \( [G,G]^\epsilon \subseteq H \). Since the continuous linear functionals \( G^\epsilon \to \mathbb{Z}/2\mathbb{Z} \) separate points, one has \( H \subseteq [G,G]^\epsilon \).
3. The homomorphism $D$

3.1. The setting for the rest of the paper.

3.1.1. Let $\overline{\mathbb{Q}}$ be the algebraic closure of the field $\mathbb{Q}$ of rational numbers in the field $\mathbb{C}$ of complex numbers. Let $\mathbb{Q}^{ab}$ be the compositum of all subfields $K \subseteq \overline{\mathbb{Q}}$ abelian over $\mathbb{Q}$. Let $\mathbb{Q}^{ab+\epsilon}$ be the compositum of all subfields $K \subseteq \mathbb{Q}$ that are quadratic over $\mathbb{Q}^{ab}$ and Galois over $\mathbb{Q}$. (In the terminology of [Das 2000] such extensions $K/\mathbb{Q}$ would be called double coverings of $\mathbb{Q}^{ab}$.) Let $G$ denote the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$. One has

$$G^{ab} = \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}), \quad G^{ab+\epsilon} = \text{Gal}(\mathbb{Q}^{ab+\epsilon}/\mathbb{Q}), \quad G^\epsilon = \text{Gal}(\mathbb{Q}^{ab+\epsilon}/\mathbb{Q})$$

by Proposition 2.10 and the definitions.

3.1.2. Put $S := \{-1\} \cup \{\text{rational prime numbers}\}$.

The family

$$\{p \mod \mathbb{Q}^\times^2\}_{p \in S}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $\mathbb{Q}^\times/\mathbb{Q}^\times^2$. For each $p \in S$ we define a 1-cocycle

$$e_p \in Z^1(G^{ab}, \mathbb{Z}/2\mathbb{Z}) = H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$$

by the rule

$$\sigma \sqrt{p} = (-1)^{(e_p)\sigma} \sqrt{p} \quad (\sigma \in G^{ab}).$$

By Kummer theory the family

$$\{e_p\}_{p \in S}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$.

3.1.3. Let $G_{-1} \subset G^{ab}$ be the subgroup generated by the restriction of complex conjugation to $\mathbb{Q}^{ab}$. For each odd prime $p$, let $G_p \subset G^{ab}$ be the inertia subgroup at $p$. Let $G_2 \subset G^{ab}$ be the subgroup of the inertia subgroup at 2 fixing $\sqrt{-1}$. For all $p \in S$ the profinite group $G_p$ is cyclic and has nonzero 2-rank. By the Kronecker-Weber theorem the family $\{G_p\}_{p \in S}$ is the family of “coordinate axes” for a decomposition of $G^{ab}$ into a product of cyclic profinite groups indexed by $S$. It follows by Proposition 2.9 that we can identify $H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})$ with $\bigwedge^2 H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$. Note that the family

$$\{e_p \wedge e_q\}_{p,q \in S \atop p < q}$$

is a $\mathbb{Z}/2\mathbb{Z}$-basis for $\bigwedge^2 H^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$.

3.1.4. For each $p \in S$ we arbitrarily fix a topological generator $\sigma_p$ of $G_p$. Note that $\sigma_{-1}$ is the restriction to $\mathbb{Q}^{ab}$ of complex conjugation. Note that

$$(e_p)_{\sigma_q} \equiv \delta_{pq} \mod 2 \quad (p, q \in S),$$

i.e., the families $\{\sigma_p\}_{p \in S}$ and $\{e_p\}_{p \in S}$ are in a convenient way dual.
3.2. **The definition of D.** We define
\[ D : H^0(G^ab, \mathbb{Q}^{ab} \times /\mathbb{Q}^{ab} \times 2) \to 2 \bigwedge H^1(G^ab, \mathbb{Z}/2\mathbb{Z}) \]
to be the injective homomorphism obtained as the composite of the following three homomorphisms:

- The canonical isomorphism
  \[ H^0(G^ab, \mathbb{Q}^{ab} \times /\mathbb{Q}^{ab} \times 2) \cong H^1(G^e, \mathbb{Z}/2\mathbb{Z}) \]
  provided by Kummer theory combined with Proposition 2.10.

- The canonical injective homomorphism
  \[ H^1(G^e, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^ab, \mathbb{Z}/2\mathbb{Z})^- \]
  provided by Proposition 2.7.

- The inverse of the canonical isomorphism
  \[ 2 \bigwedge H^1(G^ab, \mathbb{Z}/2\mathbb{Z}) \cong H^2(G^ab, \mathbb{Z}/2\mathbb{Z})^- \]
  provided by Proposition 2.9.

The map \( D \) is the focus of our investigation.

3.3. **The Log Wedge Formula.** We carry out the diagram-chase necessary to make the map \( D \) usably explicit.

3.3.1. Fix \( u \in \mathbb{Q}^{ab} \times \) such that
\[ (u \mod \mathbb{Q}^{ab} \times 2) \in H^0(G^ab, \mathbb{Q}^{ab} \times /\mathbb{Q}^{ab} \times 2). \]

The 1-cocycle
\[ c \in Z^1(G^e, \mathbb{Z}/2\mathbb{Z}) = H^1(G^e, \mathbb{Z}/2\mathbb{Z}) \]
defined by the rule
\[ \sigma \sqrt{u} = (-1)^c \sigma \sqrt{u} \quad (\sigma \in G^e) \]
is the one corresponding via Kummer theory to the class \( u \mod \mathbb{Q}^{ab} \times 2 \).

3.3.2. Let
\[ \vartheta : G^{ab} \to G^{ab + e} \]
be any set-theoretic splitting of the canonical central extension \( \Sigma_G \). The 2-cocycle
\[ a \in Z^2(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \]
defined by the formula
\[ a_{\sigma, \tau} = c_{\vartheta(\sigma) \vartheta(\tau) \vartheta(\sigma \tau)^{-1}} \quad (\sigma, \tau \in G^{ab}) \]
represents the cohomology class \( c \circ [\Sigma_G] \in H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \).
3.3.3. Let 
\[ \{v_\sigma\}_{\sigma \in G^{ab}} \]
be a locally constant family of elements of \( \mathbb{Q}_{ab}^\times \) such that 
\[ \sigma u = v_\sigma^2 u \quad (\sigma \in G^{ab}). \]
Define a 1-cochain 
\[ f \in C^1(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \]
by the rule 
\[ \vartheta(\sigma)v_\sigma = (-1)^{f_\sigma}v_\sigma \sqrt{u} \quad (\sigma \in G^{ab}). \]
A straightforward if tedious calculation gives the formula 
\[ (-1)^{a_{\sigma,\tau}}v_\sigma = \vartheta(\sigma)\vartheta(\tau)\vartheta(\sigma\tau)^{-1}v_\sigma = (-1)^{(df)_{\sigma,\tau}}(dv)_{\sigma,\tau}\sqrt{u} \quad (\sigma, \tau \in G^{ab}). \]
The latter says that the 2-cocycle 
\[ \delta \log_{-1} v \in Z^2(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \]
also represents the cohomology class \( c \circ [\Sigma_G] \), where 
\[ \log_{-1} : \{\pm 1\} \overset{\sim}{\to} \mathbb{Z}/2\mathbb{Z} \]
is the evident (and only) isomorphism.

3.3.4. Put 
\[ \alpha_{p,q} := \frac{(dv)_{\sigma_p,\sigma_q}}{(dv)_{\sigma_\sigma,\sigma_p}} = \frac{\sigma_p v_{\sigma_p}/v_{\sigma_p}}{\sigma_q v_{\sigma_q}/v_{\sigma_p}} \in \{\pm 1\} \quad \begin{cases} p, q \in S & p < q \end{cases}. \]
By Lemma 2.8 one has \( \alpha_{p,q} = 1 \) for all but finitely many pairs \( \{p < q\} \) and the image in \( H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \) of the cohomology class of the 2-cocycle 
\[ \delta \log_{-1} v + \sum_{p,q \in S \atop p < q} \log_{-1} \alpha_{p,q} \cdot e_p \cup e_q \]
vanishes. We arrive finally at the result 
\[ D\left( u \mod \mathbb{Q}_{ab}^{\times 2} \right) = \sum_{p,q \in S \atop p < q} \log_{-1} \alpha_{p,q} \cdot e_p \cup e_q \]
to which in the sequel we refer as the Log Wedge Formula.

3.4. The Auxiliary Formula. Here is an easy first application of the Log Wedge Formula. Let \( \ell \) be any prime number. One has \( \sqrt{\ell} \in \mathbb{Q}(\zeta_{4\ell})^\times \) and hence there exists a unique function 
\[ (\sigma \mapsto v_\sigma) : G^{ab} \to \{1, i\} \]
factoring through \( \text{Gal}(\mathbb{Q}(\zeta_{4\ell})/\mathbb{Q}) \) such that 
\[ \sigma \sqrt{\ell} = v_\sigma^2 \sqrt{\ell} \quad \text{(equivalently: } v_\sigma^2 = (-1)^{(e_{\ell})\sigma}) \quad (\sigma \in G^{ab}). \]
For all \( p, q \in S \) one has 
\[ \sigma_p v_{\sigma_q}/v_{\sigma_p} = \begin{cases} -1 & \text{if } p = -1 \text{ and } q = \ell, \\ 1 & \text{otherwise}, \end{cases} \]
and hence 
\[ D\left( \sqrt{\ell} \mod \mathbb{Q}_{ab}^{\times 2} \right) = e_{-1} \cup e_\ell \]
by the Log Wedge Formula. We refer to the result above as the Auxiliary Formula.
4. The Main Formula

4.1. The universal ordinary distribution and related apparatus.

4.1.1. We denote by $\mathcal{A}$ the free abelian group on symbols of the form $[a]$ ($a \in \mathbb{Q}$) modulo the identifications $[a] = [b] \iff a - b \in \mathbb{Z}$.

Put $e(a) := e^{2\pi ia}$ ($a \in \mathbb{Q}$).

We equip $\mathcal{A}$ with the unique action of $G^{ab}$ such that

$$\sigma[a] = [b] \iff \sigma e(a) = e(b) \quad (a,b \in \mathbb{Q}, \sigma \in G^{ab}).$$

4.1.2. For each prime number $p$ we define $Y_p : \mathcal{A} \to \mathcal{A}$ to be the unique endomorphism such that

$$Y_p[a] := [a] - \sum_{i=0}^{p-1} \left[ \frac{a + i}{p} \right] \quad (a \in \mathbb{Q}).$$

The operator $Y_p$ is $G^{ab}$-equivariant. The operators $Y_p$ commute among themselves.

Put

$$U := \frac{\mathcal{A}}{\sum_p Y_p \mathcal{A}}, \quad U^- := \frac{\mathcal{A}}{(1 + \sigma_{-1}) \mathcal{A} + \sum_p Y_p \mathcal{A}},$$

thereby defining the universal ordinary distribution and the universal odd ordinary distribution, respectively. The action of $G^{ab}$ on $\mathcal{A}$ descends to $U$ and $U^-$. By [Kubert 1978a], the group $U$ is free abelian. It follows that the torsion subgroup of $U^-$ can be identified with the cohomology group $H^2(\langle \sigma_{-1} \rangle, U)$, and in particular that the torsion subgroup of $U^-$ is killed by 2. By [Kubert 1978b], the group $G^{ab}$ operates trivially on the torsion subgroup of $U^-$. 

4.1.3. We denote by $\mathcal{A}'$ the subgroup of $\mathcal{A}$ generated by all symbols of the form $[a]$ ($a \in \mathbb{Q} \setminus \frac{1}{2} \mathbb{Z}$).

Notice that $\mathcal{A}'$ is stable under the action of $G^{ab}$ and stable also under the action of all the operators $Y_p$. We call a homomorphism

$$H : \mathcal{A}' \to \mathcal{A}$$

a lifting operator if the identities

$$H^2 = H, \quad H + \sigma_{-1} H \sigma_{-1} = 1$$

hold, in which case

$$\mathcal{A}' = H \mathcal{A}' \oplus \sigma_{-1} H \mathcal{A}'$$

and one has implications

$$(1 + \sigma_{-1})x = 0 \quad \Rightarrow \quad x = (1 - \sigma_{-1}) H x \quad \quad \quad \quad (x \in \mathcal{A}').$$

$$(1 - \sigma_{-1})x = 0 \quad \Rightarrow \quad x = (1 + \sigma_{-1}) H x \quad \quad \quad \quad (x \in \mathcal{A}').$$
4.1.4. To each partition
\[(0, 1) \cap \left( \mathbb{Q} \setminus \frac{1}{2}\mathbb{Z} \right) = T \coprod \{1 - a \mid a \in T\}\]
is attached a unique lifting operator
\[H_T : \mathcal{A}' \to \mathcal{A}'\]
such that
\[H_T[a] = \begin{cases} [a] & \text{if } a \in T \\ 0 & \text{if } a \not\in T \end{cases} \quad (a \in (0, 1) \cap \left( \mathbb{Q} \setminus \frac{1}{2}\mathbb{Z} \right)).\]
Following [Das 2000], we call
\[H := H_{(0, 1/2) \cap \mathbb{Q}}\]
the canonical lifting operator.

4.1.5. We define
\[\xi : \mathcal{A} \to \mathbb{Q}^{ab} \times\]
to be the unique homomorphism such that
\[\xi[a] = \begin{cases} 1 - e(a) & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases} \quad (a \in [0, 1) \cap \mathbb{Q}).\]
The homomorphism \(\xi\) is Galois equivariant. One has
\[\xi(Y_p[a]) = \begin{cases} p^{-1} = (p^{-1/2})^2 & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases} \quad (p: \text{ prime, } a \in [0, 1) \cap \mathbb{Q})\]
and
\[\xi((1 + \sigma^-)[a]) = \begin{cases} 1 - e(a) \quad & \text{if } a = 0 \\ \frac{1}{4} \sin^2 \pi a & \text{if } a \neq 0 \end{cases} \quad (a \in [0, 1) \cap \mathbb{Q}).\]
It follows that \(\xi\) induces a \(G^{ab}\)-equivariant homomorphism
\[U^- \to \mathbb{Q}^{ab} \times / \mathbb{Q}^{ab} \times \]
and hence a homomorphism
\[(\text{torsion subgroup of } U^-) \to H^0 \left( G^{ab}, \mathbb{Q}^{ab} \times / \mathbb{Q}^{ab} \times \right).\]
Note also that \(\xi\) kills \(\sum_p Y_p \mathcal{A}'\).

4.1.6. We define a homomorphism
\[\sin : \mathcal{A} \to \mathbb{Q}^{ab} \times\]
by the rule
\[\sin[a] := \begin{cases} 2\sin \pi a & \text{if } a \in \mathbb{Q} \cap (0, 1), \\ 1 & \text{if } a = 0. \end{cases}\]
One then has
\[\sin a = |\xi(a)| \quad (a \in \mathcal{A}).\]
Let \(\mathcal{A}''\) denote the subgroup of \(\mathcal{A}'\) generated by elements of the form
\[[a] \quad (a \in (0, 1) \cap \mathbb{Q} \cap \mathbb{Z}_2),\]
where \(\mathbb{Z}_2\) denotes the 2-adic completion of \(\mathbb{Z}\). Let
\[P : \mathcal{A}'' \to \mathcal{A}''\]
be the unique idempotent endomorphism such that
\[ P[a] = \begin{cases} [a] & \text{if } a \text{ is a 2-adic unit} \\ 0 & \text{otherwise} \end{cases} \quad (a \in (0, 1) \cap \mathbb{Q} \cap \mathbb{Z}_2). \]
Now \(a \in (0, 1) \cap \mathbb{Q} \cap \mathbb{Z}_2\) is a 2-adic unit if and only if \(1 - a\) is not a 2-adic unit; it follows that one has an identity
\[ Pa + \sigma^{-1}P\sigma^{-1}a = a \quad (a \in A''). \]
Let
\[ \deg : A \to \mathbb{Z} \]
be the unique homomorphism such that
\[ \deg[a] = 1 \quad (a \in \mathbb{Q}). \]

**Proposition 4.1.7.** One has
\[ \text{sign } \sin a = \frac{\sigma \sin a}{\sin \sigma a} = (-1)^{\deg P(1-\sigma)a} \left( \frac{\sigma \sqrt{-1}}{\sqrt{-1}} \right)^{\deg a} \quad (a \in A'', \sigma \in G^{ab}). \]

**Proof.** For each \(a \in \mathbb{Q} \cap \mathbb{Z}_2\) there exists a solution \(\tilde{a} \in \mathbb{Q} \cap \mathbb{Z}_2\) of the congruence
\[ 2\tilde{a} \equiv a \mod \mathbb{Z} \]
unique modulo \(\mathbb{Z}\) because \((\mathbb{Q} \cap \mathbb{Z}_2)/\mathbb{Z}\) is the subgroup of \(\mathbb{Q}/\mathbb{Z}\) consisting of elements of odd order. For all \(a \in \mathbb{Q} \cap (0, 1) \cap \mathbb{Z}_2\) one has
\[ \frac{a}{2} \equiv \tilde{a} + \frac{1}{2} \mod \mathbb{Z} \iff a \text{ is a 2-adic unit}. \]
It follows that one has an identity
\[ |1 - e(a)| = 2 \sin \pi a = (-1)^{\deg P[a]} \frac{e(\tilde{a}) - e(-\tilde{a})}{e(1/4)} \quad (a \in (0, 1) \cap \mathbb{Q} \cap \mathbb{Z}_2). \]
Let
\[ \tilde{\xi} : A'' \to \mathbb{Q}^{ab \times} \]
be the unique homomorphism such that
\[ \tilde{\xi}[a] = e(\tilde{a}) - e(-\tilde{a}) \quad (a \in (0, 1) \cap \mathbb{Q} \cap \mathbb{Z}_2). \]
Then \(\tilde{\xi}\) is \(G^{ab}\)-equivariant and one has an identity
\[ \sin a = (-1)^{\deg P} \frac{\tilde{\xi}(a)}{e(1/4)^{\deg a}} \quad (a \in A''), \]
whence the result.

**Proposition 4.1.8.** For every \(a \in A\) representing a torsion element of \(U^-(\mathbb{Q})\), the extension \(\mathbb{Q}^{ab}(\sqrt{\sin a})/\mathbb{Q}\) is Galois. (Cf. [Das 2000, Thm. 11].)

**Proof.** For every \(a \in A\) the ratio \(\xi(a)/\sin a\) is a root of unity. Therefore the homomorphism
\[ (\text{torsion subgroup of } U^-) \to H^0(G^{ab}, \mathbb{Q}^{ab \times}/\mathbb{Q}^{ab \times 2}) \]
induced by \(\xi : A \to \mathbb{Q}^{ab \times}\) is also induced by \(\sin : A \to \mathbb{Q}^{ab \times}\).

### 4.2. Identities of Das type

We carry out by ad hoc methods tailored to our needs what are in effect diagram-chases through the double complex employed by [Das 2000].
4.2.1. Fix a lifting operator

\[ H : A' \to A', \]

distinct prime numbers \( p \) and \( q \), and \( x \in A \) such that

\[ Y_p x, Y_q x \in A', \quad (1 - \sigma_{-1}) x = 0. \]

One has

\[ (1 - \sigma_{-1}) Y_p x = 0, \quad (1 - \sigma_{-1}) Y_q x = 0 \]

and hence

\[ Y_p x = (1 + \sigma_{-1}) H Y_p x, \quad Y_q x = (1 + \sigma_{-1}) H Y_q x. \]

One has

\[ (1 + \sigma_{-1})(Y_q H Y_p - Y_p H Y_q) x = (Y_q (1 + \sigma_{-1}) H Y_p - Y_p (1 + \sigma_{-1}) H Y_q) x = 0, \]

hence

\[ (1 - \sigma_{-1})(H Y_p H Y_q - H Y_q H Y_p) x = (Y_p H Y_q - Y_q H Y_p) x, \]

and hence

\[ 2(H Y_p H Y_q - H Y_q H Y_p) x = (Y_p H Y_q - Y_q H Y_p) x + (1 + \sigma_{-1})(H Y_p H Y_q - H Y_q H Y_p) x. \]

We refer to the latter relation as the first Das identity.

4.2.2. Fix lifting operators

\[ H, \bar{H} : A' \to A', \]

distinct prime numbers \( p \) and \( q \), and \( x \in A \) such that

\[ Y_p x, Y_q x \in A', \quad (1 - \sigma_{-1}) x = 0. \]

Put

\[ a := (H Y_p H Y_q - H Y_q H Y_p) x, \quad \bar{a} := (\bar{H} Y_p H Y_q - \bar{H} Y_q H Y_p) x, \]

and

\[ b := (Y_p H (H - \bar{H}) Y_q - Y_q H (H - \bar{H}) Y_p) x, \quad c := H(a - \bar{a}) - H b. \]

One has

\[ (1 + \sigma_{-1})(H - \bar{H}) Y_q x = 0, \quad (1 + \sigma_{-1})(H - \bar{H}) Y_p x = 0, \]

hence

\[ (H - \bar{H}) Y_q x = (1 - \sigma_{-1}) H (H - \bar{H}) Y_q x, \]
\[ (H - \bar{H}) Y_p x = (1 - \sigma_{-1}) H (H - \bar{H}) Y_p x, \]

hence

\[ (1 - \sigma_{-1}) (a - \bar{a}) = (Y_p (H - \bar{H}) Y_q - Y_q (H - \bar{H}) Y_p) x = (1 - \sigma_{-1}) b \]

and hence

\[ a - \bar{a} = b + (1 + \sigma_{-1}) c. \]

We refer to the latter relation as the second Das identity.
4.2.3. Fix a lifting operator

\[ H : A' \rightarrow A' \]

distinct primes \( p \) and \( q \), and \( x \in A \) such that

\[ Y_p x, Y_q x \in A', \quad (1 - \sigma)x = 0 \quad (\sigma \in G^{ab}) . \]

Put

\[ a := (HY_p HY_q - HY_q HY_p)x . \]

For each \( \sigma \in G^{ab} \) put

\[ b_\sigma := Y_p H(1 - \sigma)HY_q x - Y_q H(1 - \sigma)HY_p x \in Y_p A' + Y_q A' \]

\[ c_\sigma := H(1 - \sigma)a - Hb_\sigma \in A' . \]

After making the substitution \( \tilde{H} = \sigma H \sigma^{-1} \) in the second Das identity and simplifying, one obtains the family of identities

\[ (1 - \sigma)a = b_\sigma + (1 + \sigma^{-1})c_\sigma \quad (\sigma \in G^{ab}) \]

to which we refer in the sequel as the Das conjugation formula. Note that all the expressions \( b_\sigma \) belong to the kernel of the homomorphism \( \xi \). It follows that the Das conjugation formula specializes under the homomorphism \( \xi \) to a family of numerical identities

\[ \sigma \xi(a) = \xi(a)/\sin^2 c_\sigma \quad (\sigma \in G^{ab}) \]

in which the expressions \( b_\sigma \) figure not at all.

Remark 4.2.4. We got the idea for the technical tool that we are calling the Das conjugation formula from the intriguing example [Das 2000, end of Sec. 16].

4.3. Das classes and the statement of the Main Formula.

4.3.1. Let a lifting operator

\[ H : A' \rightarrow A' \]

and primes \( p < q \) be given. Put

\[ a := (HY_p HY_q - HY_q HY_p) \cdot \begin{cases} [0] & \text{if } 2 < p < q, \\ [0] + [1/2] & \text{otherwise.} \end{cases} \]

In the present situation the identities of Das type have the following implications:

- \( a \) represents a 2-torsion \( G^{ab} \)-invariant element of \( U^- \) independent of \( H \).
- \( \xi(a) \) is a real number and hence \( \xi(a) = \pm \sin a \).
- \( \sin a \) represents a class in \( H^0 \left( G^{ab}, \mathbb{Q}^{ab}/\mathbb{Q}^{ab} \times 2 \right) \) independent of \( H \).
- One has a family of numerical identities

\[ \sigma \sin a = \sin a / \sin^2 c_\sigma \quad (\sigma \in G^{ab}) \]

where \( \{c_\sigma\} \) is the cochain figuring in the Das conjugation formula.

We call the torsion element of \( U^- \) represented by \( a \) the Das class associated to the pair \( \{p < q\} \) of prime numbers.
4.3.2. For all primes \( p < q \), put
\[
\mathbf{a}_{pq} := (\mathbf{H}Y_p \mathbf{H}Y_q - \mathbf{H}Y_q \mathbf{H}Y_p) : \begin{cases} 
[0] & \text{if } 2 < p < q, \\
[0] + \lfloor 1/2 \rfloor & \text{otherwise}, 
\end{cases}
\]
thereby defining the canonical representative of the Das class associated to the pair \( \{p < q\} \) of primes. For \( 2 < p < q \) one has
\[
\mathbf{a}_{pq} = \left( \sum_{i=1}^{q-1} \left( \frac{i}{p} \right) - \sum_{k=0}^{q-1} \left( \frac{i}{pq} + \frac{k}{q} \right) \right) - \left( \sum_{j=1}^{q-1} \left( \frac{j}{q} \right) - \sum_{\ell=0}^{p-1} \left( \frac{j}{pq} + \frac{\ell}{p} \right) \right).
\]
For \( 2 = p < q \) one has
\[
\mathbf{a}_{pq} = \left( \left[ \frac{1}{4} \right] - \sum_{k=0}^{\frac{q-1}{2}} \left( \frac{1}{4q} + \frac{k}{q} \right) \right) - \sum_{j=1}^{\frac{q-1}{2}} \left( \frac{j}{q} \right) + \left( \frac{1}{2q} + \frac{j}{q} \right) - \left( \frac{j}{2q} \right) - \left( -\frac{1}{4q} + \frac{j}{q} \right).
\]
The discovery that \( \mathbf{a}_{pq} \) for \( 2 < p < q \) represents a torsion element of \( U^- \) is due to Das; see [Das 2000, Sec. 9].

**Remark 4.3.3.** The double complex method of [Anderson 1999] gives rise to a canonical \( \mathbb{Z}/2\mathbb{Z} \)-basis for the torsion subgroup of \( U^- \) indexed by finite sets of prime numbers of even cardinality. In [Das 2000, §3 and §9] it is proved that the family \( \{\mathbf{a}_{pq}\}_{2 < p < q} \) represents the “two-odd-prime” part of the canonical basis. The method of Das can easily be modified to show that the family \( \{\mathbf{a}_{pq}\}_{p < q} \) represents “two-prime” part of the canonical basis.

4.3.4. The main result of this paper is the relation
\[
\mathbf{D}(\sin \mathbf{a}_{pq} \mod \mathbb{Q}^{ab} \times 2) = e_p \wedge e_q \quad (p < q \text{ prime})
\]
to which we refer in the sequel as the Main Formula. The Main and Auxiliary Formulas together immediately imply that
\[
\mathbb{Q}^{ab+\varepsilon} = \mathbb{Q}^{ab} \left( \left\{ \sqrt{7} \right\}_{\ell \text{ prime}} \cup \left\{ \sqrt{\sin \mathbf{a}_{pq}} \right\}_{p, q \text{ prime}} \right).
\]
Thus we get a quite explicit description of the field \( \mathbb{Q}^{ab+\varepsilon} \).

**Remark 4.3.5.** The following is a corollary to the Main and Auxiliary Formulas:

(1) For \( G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), the canonical injective map
\[
(c \mapsto c \circ [\Sigma_G^')] : H^1(G^\varepsilon, \mathbb{Z}/2\mathbb{Z}) \to H^2(G^{ab}, \mathbb{Z}/2\mathbb{Z})^{-}
\]
defined in Proposition 2.7 is an isomorphism.

But (1) is not really new. One can get it very easily from the theory of [Fröhlich 1983], as we now explain. Let \( S \) be a finite set of primes of \( \mathbb{Q} \) including 2 and the infinite prime. Let \( \Gamma \) be the maximal pro-2 quotient of \( G \) unramified outside the primes in \( S \). To prove (1) it is enough to prove that the canonical injective map
\[
(e \mapsto e \circ [\Sigma_G^\varepsilon]) : H^1(\Gamma^\varepsilon, \mathbb{Z}/2\mathbb{Z}) \to H^2(\Gamma^{ab}, \mathbb{Z}/2\mathbb{Z})^{-}
\]
is an isomorphism. Let \( (\tau \mapsto \tau^{ab}) : \Gamma \to \Gamma^{ab} \) be the canonical projection. For each odd finite prime \( p \in S \), choose any \( \tau_p \in \Gamma \) such that \( \tau_p^{ab} \) topologically generates the inertia subgroup of \( \Gamma^{ab} \) at \( p \). Choose any \( \tau_\infty \in \Gamma \) such that \( \tau_\infty^{ab} \) topologically generates the subgroup of the inertia subgroup of \( \Gamma^{ab} \) at 2 fixing \( \sqrt{-1} \). Choose any \( \tau_\infty \in \Gamma \) so that \( \tau_\infty^{ab} \in \Gamma^{ab} \) is the automorphism induced by complex conjugation.
The family \( \{\tau_p\}_{p \in S} \) is then a minimal set of generators for \( \Gamma \); in particular, the 2-rank of \( \Gamma^{ab} \) is \( N \), where \( N \) is the cardinality of \( S \). By \cite[Theorem 4.10, p. 56]{Froehlich1983} a complete set of defining relations for \( \Gamma \) as a pro-2 group is lifted from the family of relations

\[
(\tau_p^{ab})^{p-1} = 1 \quad (p: \text{odd finite prime in } S); \quad (\tau_\infty^{ab})^2 = 1.
\]

It follows that the family of commutators

\[
\{[\tau_p, \tau_q] \}_{p, q \in S \atop p \neq q}
\]

projects to a \( \mathbb{Z}/2\mathbb{Z} \)-basis for \( \Gamma^\epsilon \); in particular, the 2-rank of \( \Gamma^\epsilon \) is \( \frac{N(N-1)}{2} \) where \( N \) is the cardinality of \( S \). By \cite[Theorem 4.10, p. 56]{Froehlich1983} a complete set of defining relations for \( \Gamma \) as a pro-2 group is lifted from the family of relations

\[
(\tau_p^{ab})^{p-1} = 1 \quad (p: \text{odd finite prime in } S); \quad (\tau_\infty^{ab})^2 = 1.
\]

Remark 4.3.6. Let \( \Gamma : A \to \mathbb{R}^\times \) be the unique homomorphism such that

\[
\Gamma([a]) = \begin{cases} 
\sqrt{2\pi}/\Gamma(a) & \text{if } 0 < a < 1 \\
1 & \text{if } a = 0 
\end{cases} \quad (a \in \mathbb{Q} \cap [0, 1)).
\]

The well known functional equations satisfied by the \( \Gamma \)-function become in the present context the identities

\[
\Gamma([a] + [-a]) = 2 \sin \pi a, \quad \Gamma(Y_p[a]) = p^{1/2-a} \quad (a \in \mathbb{Q} \cap (0, 1)).
\]

For all primes \( p < q \), we then have

\[
\frac{\Gamma(a_{pq})}{\sqrt{\sin a_{pq}}} = \sqrt{\Gamma \left( (Y_p H Y_q - Y_q H Y_p) \cdot \begin{cases} 
0 & \text{if } 2 < p \\
0 + \lfloor 1/2 \rfloor & \text{if } 2 = p
\end{cases} \right)} = \begin{cases} 
\frac{p^{-\frac{\pi}{2}(a-1/2)^2} q^{\frac{\pi}{2}(a+1/2)^2}}{\sin a_{pq}} & \text{if } 2 < p \\
2^{-\frac{\pi}{2}(a-1/2)} q^{\frac{1}{2}} & \text{if } 2 = p
\end{cases}
\]

by the first Das identity. The formula above (in the case \( 2 < p \)) was discovered by Das (see \cite[Sec. 9]{Das2000}) and the proof of that formula is where we got the idea for the technical tool that we are calling the first Das identity.

Remark 4.3.7. The relations standing between the Main Formula, the index formulas of \cite{Simott1978}, Deligne reciprocity \cite[Thm. 7.15, p. 91]{Deligne-Milne-Ogus-Shih1982}, the theory of \cite{Froehlich1983}, the theory of \cite{Das2000}, the theory of the group cohomology of the universal ordinary distribution (see \cite{Ouyang2001} and references therein) and Stark’s conjecture and its variants (see \cite{Tate1984}) deserve to be thoroughly investigated. We have only scratched the surface here. Stark’s conjecture is relevant in view of the well known expansion

\[
\sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)
\]

of the Hurwitz zeta function at \( s = 0 \).
Remark 4.3.8. The papers [Thakur 1991], [Sinha 1997], and [Bae-Gekeler-Yin 2000] (just to mention the first several coming to mind) suggest that there ought to be an analogue of the Main Formula over a global field of positive characteristic equipped with a distinguished place.

Remark 4.3.9. Perhaps there is an analogue of the Main Formula over an imaginary quadratic field involving elliptic units. This possibility seems especially intriguing.

5. Proof of the Main Formula

Fix prime numbers $p < q$.

5.1. Reductions.

5.1.1. Put

$$x := \begin{cases} [0] & \text{if } 2 < p, \\ [0] + [1/2] & \text{if } 2 = p. \end{cases}$$

Fix a partition

$$(0, 1) \cap \left( \mathbb{Q} \setminus \frac{1}{2} \mathbb{Z} \right) = T \coprod \{1 - a \mid a \in T\}$$

and put

$$H := H_T.$$ 

Presently we are going to make an advantageous choice for the set $T$. Put

$$a := (HY_p HY_q - HY_q HY_p) x.$$ 

Then $a$ represents the Das class associated to the pair $\{p < q\}$ of primes and

$$\sin a_{pq} \equiv \sin a \mod \mathbb{Q}^{ab \times 2}.$$ 

5.1.2. In the present situation the Das conjugation formula reads

$$(1 - \sigma)a = b_\sigma + (1 + \sigma_{-1})c_\sigma \quad (\sigma \in G^{ab})$$

where

$$b_\sigma := (Y_p H(1 - \sigma)HY_q - Y_q H(1 - \sigma)HY_p) x,$$

$$c_\sigma := H(1 - \sigma)a - Hb_\sigma,$$

and one has a family of numerical identities

$$\sigma \sin a = \sin a / \sin^2 c_\sigma \quad (\sigma \in G^{ab}).$$

By construction the function

$$\sigma \mapsto \sin c_\sigma : G^{ab} \to \mathbb{Q}^{ab \times}$$

factors through $\text{Gal}\left( \mathbb{Q} \left( e\left( \frac{1}{pq}\right) \right) / \mathbb{Q} \right)$ and takes values in $\mathbb{Q} \left( e\left( \frac{1}{pq}\right) \right)$.
5.1.3. The Log Wedge Formula says that
\[ D(\sin a_{pq} \mod \mathbb{Q}_{ab}^\times) = D(\sin a \mod \mathbb{Q}_{ab}^\times) = \sum_{r,s \in S \atop r < s} \log_{-1} \alpha_{rs} \cdot e_r \wedge e_s \]
where
\[ \alpha_{rs} := \frac{\sigma_s \sin c_{\sigma_r}}{\sigma_r \sin c_{\sigma_s}} \in \{\pm 1\}. \]
(We employ \( r \) and \( s \) as dummy variables here since \( p \) and \( q \) have already been fixed.)

Since all numbers of the form \( \sin a \) are positive real, we can calculate \( \alpha_{rs} \) just by keeping track of signs, i.e., we have
\[ \alpha_{rs} = \frac{\text{sign } \sigma_s \sin c_{\sigma_r}}{\text{sign } \sigma_r \sin c_{\sigma_s}}. \]

One has \( \sin c_{\sigma_{r-1}} = 1 \) and hence \( \alpha_{-1, \ell} = 1 \) for all primes \( \ell \). For all primes \( \ell \) distinct from \( p \) and \( q \), one has \( \sin c_{\sigma_{\ell}} = \sin c_1 = 1 \) and moreover \( \sigma_{\ell} \) operates trivially on the field \( \mathbb{Q}(\sqrt{-1}) \). (Recall that \( \sqrt{-1} = -1 \).) It follows that \( \alpha_{rs} = 1 \) for all primes \( r < s \) such that \( \{r, s\} \neq \{p, q\} \). It remains now only to prove that \( \alpha_{pq} = -1 \).

5.2. Calculation of \( \alpha_{pq} \) in the case \( 2 < p \). Assume that \( 2 < p \), i.e., that both \( p \) and \( q \) are odd.

5.2.1. We have
\[ \alpha_{pq} = (-1)^{\deg (P(1-\sigma_q)c_{\sigma_p} - P(1-\sigma_p)c_{\sigma_q})} \]
by Proposition \[4.1.7\]. In this way we reduce the calculation of \( \alpha_{pq} \) to an essentially combinatorial problem.

5.2.2. Put
\[ \Theta_p := \sigma_p^0 + \cdots + \sigma_p^{\frac{p-1}{2}}, \quad \Theta_q := \sigma_q^0 + \cdots + \sigma_q^{\frac{q-1}{2}}, \]
thereby defining elements of the integral group ring of \( G_{ab} \). The relations
\[ (1 - \sigma_p) \Theta_p[a + b] = \left(1 - \sigma_p^{a+b} \right)[a + b] = [a + b] - [-a + b] \]
\[ (1 - \sigma_q) \Theta_q[a + b] = \left(1 - \sigma_q^{a+b} \right)[a + b] = [a + b] - [a - b] \]
figure crucially in our calculations.

5.2.3. We make an advantageous choice for the set \( T \) now. We assume that \( T \) has been chosen in such a way that the associated lifting operator \( H = H_T \) has the following properties:
\[ H \sigma_i^p \left[ \frac{q}{p} \right] = \begin{cases} \sigma_p^i \left[ \frac{p}{q} \right] & \text{for } 0 \leq i < (p-1)/2, \\ 0 & \text{for } (p-1)/2 \leq i < p-1, \end{cases} \]

\[ H \sigma_i^q \left[ \frac{p}{q} \right] = \begin{cases} \sigma_q^i \left[ \frac{q}{p} \right] & \text{for } 0 \leq i < (q-1)/2, \\ 0 & \text{for } (q-1)/2 \leq i < q-1, \end{cases} \]

\[ H \sigma_i^p \sigma_j^q \left[ -\frac{1}{p} + \frac{1}{q} \right] = \begin{cases} \sigma_p^i \sigma_q^j & \text{for } 0 \leq i < (p-1)/2 \text{ and } 0 \leq j < q-1, \\ 0 & \text{for } (p-1)/2 \leq i < p-1 \text{ and } 0 \leq j < q-1. \end{cases} \]

The reason for making this choice is to force a lot of cancellation. It was in order to have the freedom to make this choice that we took the trouble to formulate the definition of the Das class in terms of an arbitrarily chosen lifting operator.

5.2.4. We make some preliminary calculations. One has

\[ Y_q[0] = -\Theta_q \left( \left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor -\frac{p}{q} \right\rfloor \right), \quad Y_p[0] = -\Theta_p \left( \left\lfloor \frac{q}{p} \right\rfloor + \left\lfloor -\frac{q}{p} \right\rfloor \right), \]

hence

\[ HY_q[0] = -\Theta_q \left[ \frac{p}{q} \right], \quad HY_p[0] = -\Theta_p \left[ \frac{q}{p} \right], \]

and hence

\[ H(1 - \sigma_q)HY_q[0] = -\left[ \frac{p}{q} \right], \quad H(1 - \sigma_p)HY_p[0] = -\left[ \frac{q}{p} \right]. \]

One then has

\[ Y_pH(1 - \sigma_q)HY_q[0] = \left[ \frac{p}{q} \right] - \left[ \frac{1}{q} \right] - \Theta_p \left( \left\lfloor \frac{1}{p} + \frac{1}{q} \right\rfloor + \left\lfloor -\frac{1}{p} + \frac{1}{q} \right\rfloor \right), \]

\[ Y_qH(1 - \sigma_p)HY_p[0] = \left[ \frac{q}{p} \right] - \left[ \frac{1}{p} \right] - \Theta_q \left( \left\lfloor \frac{1}{p} + \frac{1}{q} \right\rfloor + \left\lfloor -\frac{1}{p} + \frac{1}{q} \right\rfloor \right), \]

\[ Y_pHY_q[0] = -\Theta_q \left( \left\lfloor \frac{q}{p} \right\rfloor - \left\lfloor \frac{1}{q} \right\rfloor - \Theta_p \left( \left\lfloor \frac{1}{p} + \frac{1}{q} \right\rfloor + \left\lfloor -\frac{1}{p} + \frac{1}{q} \right\rfloor \right) \right), \]

\[ Y_qHY_p[0] = -\Theta_p \left( \left\lfloor \frac{q}{p} \right\rfloor - \left\lfloor \frac{1}{p} \right\rfloor - \Theta_q \left( \left\lfloor \frac{1}{p} + \frac{1}{q} \right\rfloor + \left\lfloor -\frac{1}{p} + \frac{1}{q} \right\rfloor \right) \right). \]
5.2.5. It follows that

\[ a = (HY_p HY_q - HY_q HY_p)[0] = \Theta_p \Theta_q \left( \left[ \frac{1}{p} \right] - \left[ \frac{1}{q} \right] \right) - H \Theta_q \left( \left[ \frac{1}{q} \right] - \left[ \frac{1}{p} \right] \right), \]

\[ H(1 - \sigma_p)a = \Theta_q \left( \left[ \frac{1}{p} \right] + \left[ \frac{1}{q} \right] \right) + H(1 - \sigma_p)H \Theta_p \left( \left[ \frac{1}{q} \right] - \left[ \frac{1}{p} \right] \right), \]

\[ HY_q H(1 - \sigma_p)HY_p[0] = -H \left( \left[ \frac{1}{p} \right] - \left[ \frac{1}{q} \right] \right), \]

\[ c_{\sigma_p} = H(1 - \sigma_p)a - HY_p H(1 - \sigma_p)HY_q[0] + HY_q H(1 - \sigma_p)HY_p[0] = \Theta_q \left( \left[ \frac{1}{p} \right] + \left[ \frac{1}{q} \right] \right) + H(1 - \sigma_q)H \Theta_p \left( \left[ \frac{1}{q} \right] - \left[ \frac{1}{p} \right] \right), \]

\[ HY_p H(1 - \sigma_q)HY_q[0] = \Theta_p \left( \left[ \frac{1}{p} \right] + \left[ \frac{1}{q} \right] \right) - H \left( \left[ \frac{1}{q} \right] - \left[ \frac{1}{p} \right] \right), \]

\[ c_{\sigma_q} = H(1 - \sigma_q)a - HY_p H(1 - \sigma_q)HY_q[0] + HY_q H(1 - \sigma_q)HY_p[0] = -\Theta_p \left( \left[ \frac{1}{p} \right] - \left[ \frac{1}{q} \right] \right) - H(1 - \sigma_q)H \Theta_q \left( \left[ \frac{1}{q} \right] - \left[ \frac{1}{p} \right] \right) + H \left( \left[ \frac{1}{q} \right] - \left[ \frac{1}{p} \right] \right). \]

5.2.6. Finally, one has

\[ P(1 - \sigma_q)c_{\sigma_p} - P(1 - \sigma_p)c_{\sigma_q} = P \left( \left[ \frac{\frac{1}{p} + \frac{1}{q}}{2} \right] \right) - P \left( \left[ \frac{\frac{1}{p} - \frac{1}{q}}{2} \right] \right) \]

\[ = P \left[ \frac{\frac{1}{p} + \frac{1}{q}}{2} \right] - P \left[ \frac{\frac{1}{p} - \frac{1}{q}}{2} \right]. \]

Now consider the rational numbers \( \frac{1}{p} + \frac{1}{q} \) and \( \frac{1}{p} - \frac{1}{q} \). Both numbers are 2-adically integral and belong to the open unit interval of the real line, but one and only one of them (namely the former) is a 2-adic unit. Therefore one has \( \alpha_{pq} = -1 \).

5.3. Calculation of \( \alpha_{pq} \) in the case \( 2 = p \). Assume that \( 2 = p \).

5.3.1. One has \( c_{\sigma_2} = c_1 = 0 \) in this case and hence

\[ \alpha_{2q} = \text{sign} \sigma_2 \sin c_{\sigma_q}. \]
5.3.2. We may assume that the set $T$ has been chosen in such a way that the lifting operator $H = H_T$ has the following properties:

$$H\sigma_q^i \left[ \frac{4}{q} \right] = \begin{cases} 
\sigma_q^i \left[ \frac{4}{q} \right] & \text{for } 0 \leq i < (q-1)/2, \\
0 & \text{for } (q-1)/2 \leq i < q-1, 
\end{cases}$$

$$H \left[ \frac{\nu}{q} \right] = \begin{cases} 
\left[ \frac{\nu}{q} \right] & \text{for } \nu = q, \\
0 & \text{for } \nu = -q, 
\end{cases}$$

$$H\sigma_q^i \left[ \frac{1}{2} + \frac{2}{q} \right] = \begin{cases} 
\sigma_q^i \left[ \frac{1}{2} + \frac{2}{q} \right] & \text{for } 0 \leq i < (q-1)/2, \\
0 & \text{for } (q-1)/2 \leq i < q-1, 
\end{cases}$$

$$H\sigma_q^i \left[ \frac{\nu}{q} + \frac{1}{q} \right] = \begin{cases} 
0 & \text{for } 0 \leq i < q-1 \text{ and } \nu = 1, \\
\sigma_q^i \left[ \frac{\nu}{q} + \frac{1}{q} \right] & \text{for } 0 \leq i < q-1 \text{ and } \nu = -1. 
\end{cases}$$

5.3.3. Put

$$\Theta_q := \left( \sigma_q^0 + \cdots + \sigma_q^{(q-3)/2} \right).$$

The relations

$$(1 - \sigma_q)\Theta_q[a + b] = \left(1 - \sigma_q^{\frac{a-b}{2}}\right) [a + b] = [a + b] - [a - b] \quad \left( a \in \frac{1}{4} \mathbb{Z}, \ b \in \frac{1}{q} \mathbb{Z}/\mathbb{Z} \right)$$

are crucial to our calculations.

5.3.4. Recall that in the case $p = 2$ one has

$$x = [0] + \left[ \frac{1}{2} \right].$$
We have

\[ Y_q x = -\Theta_q \left( \left[ \frac{4}{q} \right] + \left[ -\frac{1}{4} \right] + \left[ \frac{1}{2} + \frac{1}{2} \right] + \left[ \frac{1}{2} - \frac{2}{q} \right] \right) \]

\[ H Y_q x = -\Theta_q \left( \left[ \frac{4}{q} \right] + \left[ \frac{1}{2} + \frac{2}{q} \right] \right) \]

\[ H(1 - \sigma_q) H Y_q x = -\left( \left[ \frac{4}{q} \right] + \left[ \frac{1}{2} + \frac{2}{q} \right] \right) \]

\[ Y_2 H(1 - \sigma_q) H Y_q x = -\left( \left[ \frac{4}{q} \right] - \left[ \frac{2}{q} \right] - \left[ \frac{1}{4} + \frac{1}{2} \right] - \left[ -\frac{1}{2} + \frac{1}{2} \right] \right) \]

\[ Y_2 x = -\left[ \frac{4}{q} \right] - \left[ \frac{2}{q} \right] \]

\[ H Y_2 x = -\left[ \frac{2}{q} \right] \]

\[ (Y_2 H Y_q - Y_q H Y_2) x = -\Theta_q \left( \left[ \frac{4}{q} \right] - \left[ \frac{2}{q} \right] - \left[ \frac{1}{4} + \frac{1}{2} \right] - \left[ -\frac{1}{4} + \frac{1}{2} \right] \right) \]

\[ + \left[ \frac{4}{q} \right] - \left[ \frac{1}{4} \right] - \Theta_q \left( \left[ \frac{1}{4} + \frac{1}{2} \right] + \left[ \frac{1}{2} - \frac{1}{2} \right] \right) \]

\[ a = \Theta_q \left[ -\frac{1}{4} + \frac{1}{2} \right] - H \Theta_q \left( \left[ \frac{4}{q} \right] - \left[ \frac{2}{q} \right] \right) + H \left( \left[ \frac{4}{q} \right] - \left[ \frac{1}{4} \right] \right) \]

\[ H(1 - \sigma_q) a = \left[ -\frac{1}{4} + \frac{1}{2} \right] - \left[ -\frac{1}{4} - \frac{1}{2} \right] - H(1 - \sigma_q) H \Theta_q \left( \left[ \frac{4}{q} \right] - \left[ \frac{2}{q} \right] \right) \]

\[ H Y_2 H(1 - \sigma_q) H Y_q x = \left[ -\frac{1}{4} + \frac{1}{2} \right] - H \left( \left[ \frac{4}{q} \right] - \left[ \frac{2}{q} \right] \right) \]

\[ c_{\sigma_q} = H(1 - \sigma_q) a - H Y_2 H(1 - \sigma_q) H Y_q x + H Y_q H(1 - \sigma_q) H Y_2 x \]

\[ = -\left[ -\frac{1}{4} + \frac{1}{2} \right] + H(1 - (1 - \sigma_q) H \Theta_q) \left( \left[ \frac{4}{q} \right] - \left[ \frac{2}{q} \right] \right) \]

5.3.5. Finally, one has

\[ \frac{1}{\sin c_{\sigma_q}} \equiv \sin \pi \left( \frac{1}{4} + \frac{1}{4q} \right) \equiv \frac{1}{\sqrt{2}} \mod \left( \mathbb{R} \cap \mathbb{Q} \left( e \left( \frac{1}{4q} \right) \right) \right) \]

because the sines and cosines of all integral multiples of the angle $\pi/q$ belong to the field $\mathbb{R} \cap \mathbb{Q} \left( e \left( \frac{1}{4q} \right) \right)$, and hence $\alpha_{2q} = \text{sign} \sigma_2 \sqrt{2} = -1$. The proof of the Main Formula is complete.

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