Computing the differential Galois group of a parameterized second-order linear differential equation

Carlos E. Arreche

Abstract

We develop algorithms to compute the differential Galois group $G$ associated to a parameterized second-order homogeneous linear differential equation of the form

$$\frac{d^2}{dx^2}Y + r_1 \frac{d}{dx}Y + r_0 Y = 0,$$

where the coefficients $r_1, r_0 \in F(x)$ are rational functions in $x$ with coefficients in a partial differential field $F$ of characteristic zero. Our work relies on the procedure developed by Dreyfus to compute $G$ under the assumption that $r_1 = 0$. We show how to complete this procedure to cover the cases where $r_1 \neq 0$, by reinterpreting a classical change of variables procedure in Galois-theoretic terms.

1. Introduction

Consider a linear differential equation of the form

$$\delta_i^r Y + \sum_{i=0}^{n-1} r_i \delta_i^j Y = 0,$$

where $r_i \in K := F(x)$, the field of rational functions in $x$ with coefficients in a $\Pi$-field $F$, $\delta_j$ denotes the derivative with respect to $x$, and $\Pi := \{\delta_1, \ldots, \delta_m\}$ is a set of commuting derivations. Letting $\Delta := \{\delta_i\} \cup \Pi$, consider $K$ as a $\Delta$-field by setting $\delta_j x = 0$ for each $j$. The parameterized Picard-Vessiot theory of [5] associates a parameterized Picard-Vessiot (PPV) group to such an equation. In analogy with the Picard-Vessiot theory developed by Kolchin [12], the PPV-group measures the $\Pi$-algebraic relations amongst the solutions to (1.1). The differential Galois groups that arise in this theory are linear differential algebraic groups: subgroups of $GL_n$ that are defined by the vanishing of systems of polynomial differential equations in the matrix entries. The study of linear differential algebraic groups was pioneered in [4]. The parameterized Picard-Vessiot theory of [5] is a special case of an earlier generalization of Kolchin’s theory, presented in [17], as well as the differential Galois theory for difference-differential equations with parameters [10].

This work addresses the explicit computation of the PPV-group $G$ corresponding to a second-order parameterized linear differential equation

$$\delta_i^r Y + r_1 \delta_i^r Y + r_0 Y = 0,$$

where $r_1, r_0 \in F(x) =: K,$ and $F$ is a $\Pi$-field. In [7], Dreyfus applies results from [6] to develop algorithms to compute $G$, under the assumption that $r_1 = 0$ (see also [1] for a detailed discussion of Dreyfus’ results in the setting of one parametric derivation, and [3] for the computation of the unipotent radical). We complete these algorithms to compute $G$ when $r_1$ is not necessarily zero. Algorithms for higher-order equations are developed in [18, 19].

After performing a change of variables on (1.2), we obtain an associated equation (3.1) of the form $\delta_i^2 Y - qY = 0$, whose PPV-group $H$ is already known [3, 7]. In §3, we reinterpret this change-of-variables procedure in terms of a lattice (3.6) of PPV-fields. We recover the original PPV-group $G$ from this lattice in Proposition 3.3, which is a formal consequence of the parameterized Galois correspondence [5, Thm. 3.5]. This reinterpretation, whose non-parameterized analogue is probably well-known to the experts in classical Picard-Vessiot theory, comprises a theoretical procedure to recover the PPV-group of (1.2) from these data.

2010 Mathematics Subject Classification Primary 34M15; Secondary 12H20, 34M03, 20H20, 13N10, 37K20

Keywords: Parameterized differential equation, parameterized Picard-Vessiot theory, linear differential algebraic group.

This material is based upon work partially supported by a National Science Foundation (NSF) Graduate Research Fellowship (grant 40017-04-05) and by NSF grant CCF-0952591.
In §4, the main tools leading to the explicit computation of \( G \) obtained in Theorem 4.1 are Proposition 3.3 and the Kolchin-Ostrowski Theorem [13]. This strategy for computing \( G \) was already sketched in [1, §3.4], in the setting of one parametric derivation. The results here are sharper than those of [1], and the proofs are conceptually simpler.

In §5, we apply Theorem 4.1 and the results of [3, 7] to compute the PPV-group corresponding to a concrete parameterized second-order linear differential equation (5.1).

2. Preliminaries

We refer to [14, 20] for more details concerning the following definitions. Every field considered in this work is assumed to be of characteristic zero. A ring \( R \) equipped with a finite set \( \Delta := \{ \delta_1, \ldots, \delta_m \} \) of pairwise commuting derivations (i.e., \( \delta_i(ab) = a\delta_i(b) + \delta_i(a)b \) and \( \delta_i\delta_j = \delta_j\delta_i \) for each \( a, b \in K \) and \( 1 \leq i, j \leq m \)) is called a \( \Delta \)-ring. If \( R = K \) happens to be a field, we say that \((K, \Delta)\) is a \( \Delta \)-field. We often omit the parentheses, and simply write \( \delta a \) for \( \delta(a) \). For \( \Pi \subseteq \Delta \), we denote the subring of \( \Pi \)-constants of \( R \) by \( R^\Pi := \{ a \in K \mid \delta a = 0, \delta \in \Pi \} \). When \( \Pi = \{ \delta \} \) is a singleton, we write \( R^\delta \) instead of \( R^\Pi \).

The ring of differential polynomials over \( K \) (in \( m \) differential indeterminates) is denoted by \( K\{Y_1, \ldots, Y_m\}_\Delta \).

As a ring, it is the free \( K \)-algebra in the countably infinite set of variables

\[
\{ \theta Y_i \mid 1 \leq i \leq m, \theta \in \Theta \}; \quad \text{where} \quad \Theta := \{ \delta_1^{\eta_1} \cdots \delta_n^{\eta_n} \mid \eta_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq n \}
\]

is the free commutative monoid on the set \( \Delta \). The ring \( K\{Y_1, \ldots, Y_m\}_\Delta \) carries a natural structure of \( \Delta \)-ring, given by \( \delta_i(\theta Y_j) := (\delta_i \cdot \theta)Y_j \). We say \( p \in K\{Y_1, \ldots, Y_m\}_\Delta \) is a linear differential polynomial if it belongs to the \( K \)-vector space spanned by the \( \theta Y_j \), for \( \theta \in \Theta \) and \( 1 \leq j \leq m \). The \( K \)-vector space of linear differential polynomials will be denoted by \( K\{Y_1, \ldots, Y_m\}_\Delta^1 \). The ring of linear differential operators \( K[\Delta] \) is the \( K \)-span of \( \Theta \), and its (non-commutative) ring structure is defined by composition of additive endomorphisms of \( K \).

If \( M \) is a \( \Delta \)-field and \( K \) is a subfield such that \( \delta(K) \subseteq K \) for each \( \delta \in \Delta \), we say \( K \) is a \( \Delta \)-subfield of \( M \) and \( M \) is a \( \Delta \)-field extension of \( K \). If \( y_1, \ldots, y_n \in M \), we denote the \( \Delta \)-subfield of \( M \) generated over \( K \) by all the derivatives of the \( y_i \) by \( K\{y_1, \ldots, y_n\}_\Delta \).

We say that a \( \Delta \)-field \( K \) is \( \Delta \)-closed if every system of polynomial differential equations defined over \( K \) that admits a solution in some \( \Delta \)-field extension of \( K \) already has a solution in \( K \). This last notion is discussed at length in [11]. See also [5, 23].

We now briefly recall the main facts that we will need from the parameterized Picard-Vessiot theory [5] and the theory of linear differential algebraic groups [4, 15]. Let \( F \) be a \( \Pi \)-field, where \( \Pi := \{ \partial_1, \ldots, \partial_m \} \), and let \( K := F(x) \) be the field of rational functions in \( x \) with coefficients in \( F \), equipped with the structure of \((\{ \delta_i \} \cup \Pi \)-field determined by setting \( \delta_i x = 1, K^{\delta_i} = F, \) and \( \partial_i x = 0 \) for each \( i \). We will sometimes refer to \( \delta_i \) as the main derivation, and to \( \Pi \) as the set of parametric derivations. From now on, we will let \( \Delta := \{ \delta_i \} \cup \Pi \). Consider the following linear differential equation with respect to the main derivation, where \( r_i \in K \) for each \( 0 \leq i \leq n - 1 \):

\[
\delta_i^\circ Y + \sum_{i=0}^{n-1} r_i \delta_i^\circ Y = 0. \tag{2.1}
\]

**Definition 2.1.** We say that a \( \Delta \)-field extension \( M \supseteq K \) is a parameterized Picard-Vessiot extension (or PPV-extension) of \( K \) for (2.1) if:

(i) There exist \( n \) distinct, \( F \)-linearly independent elements \( y_1, \ldots, y_n \in M \) such that \( \delta_i^\circ y_j + \sum r_i \delta_i^\circ y_j = 0 \) for each \( 1 \leq j \leq n \).

(ii) \( M = K\{y_1, \ldots, y_n\}_\Delta \).

(iii) \( M^{\delta_i} = K^{\delta_i} \).
The parameterized Picard-Vessiot group (or PPV-group) is the group of $\Delta$-automorphisms of $M$ over $K$, and we denote it by $\Gamma(M/K)$. The $F$-linear span of all the $y_j$ is the solution space $S$.

If $F$ is $\Pi$-closed, it is shown in [5] that a PPV-extension of $K$ for (2.1) exists and is unique up to $K$-$\Delta$-isomorphism. Although this assumption allows for a simpler exposition of the theory, several authors [8, 25] have shown that, in many cases of practical interest, the parameterized Picard-Vessiot theory can be developed without assuming that $F$ is $\Pi$-closed. In any case, we may always embed $F$ in a $\Pi$-closed field [11, 23]. The action of $\Gamma(M/K)$ is determined by its restriction to $S$, which defines an embedding $\Gamma(M/K) \hookrightarrow GL_n(F)$ after choosing an $F$-basis for $S$. It is shown in [5] that this embedding identifies the PPV-group with a linear differential algebraic group (Definition 2.2), and from now on we will make this identification implicitly.

**Definition 2.2.** Let $F$ be a $\Pi$-closed field. We say that a subgroup $G \subseteq GL_n(F)$ is a linear differential algebraic group if $G$ is defined as a subset of $GL_n(F)$ by the vanishing of a system of polynomial differential equations in the matrix entries, with coefficients in $F$. We say that $G$ is $\Pi$-constant if $G \subseteq GL_n(F^\Pi)$.

There is a parameterized Galois correspondence [5, Thm. 3.5] between the linear differential algebraic subgroups $\Gamma$ of $\Gamma(M/K)$ and the intermediate $\Delta$-fields $K \subseteq L \subseteq M$, given by $\Gamma \mapsto M^\Gamma$ and $L \mapsto \Gamma(M/L)$. Under this correspondence, an intermediate $\Delta$-field $L$ is a PPV-extension of $K$ (for some linear differential equation with respect to $\delta_\Delta$) if and only if $\Gamma(M/L)$ is normal in $\Gamma(M/K)$. The restriction homomorphism $\Gamma(M/K) \to \Gamma(M/L)$ defined by $\sigma \mapsto \sigma|_L$ is surjective, with kernel $\Gamma(M/L)$.

The differential algebraic subgroups of the additive and multiplicative groups of $F$, which we denote respectively by $G_a(F)$ and $G_m(F)$, were classified by Cassidy in [4, Prop. 11, Prop. 31 and its Corollary]:

**Proposition 2.3 (Cassidy).** If $B \subseteq G_a(F)$ is a differential algebraic subgroup, then there exist finitely many linear differential polynomials $p_1, \ldots, p_s \in F\{Y\}_{\Pi}$ such that

$$B = \{b \in G_a(F) \mid p_i(b) = 0 \text{ for each } 1 \leq i \leq s\}.$$

If $A \subseteq G_m(F)$ is a differential algebraic subgroup, then either $A = \mu_\ell$, the group of $\ell$th roots of unity, or else $G_m(F^\Pi) \subseteq A$, and there exist finitely many linear differential polynomials $q_1, \ldots, q_s \in F\{Y_1, \ldots, Y_m\}_{\Pi}$ such that

$$A = \{a \in G_m(F) \mid q_i(\frac{\partial a}{\partial x}, \ldots, \frac{\partial a}{\partial x}) = 0 \text{ for } 1 \leq i \leq s\}.$$

### 3. Recovering the original group

Recall that $K := F(x)$ is the $\Delta$-field defined by: $F = K^{\delta_\Delta}$ is $\Pi$-closed field, $A := \{\delta_x\} \cup \Pi$, $\delta_x x = 1$, and $\partial x = 0$ for each $\partial \in \Pi$. Consider a second-order parameterized linear differential equation

$$\delta_\xi^2 Y - q Y = 0,$$

where $q \in K$. In [7], Dreyfus develops the following procedure to compute the PPV-group $H$ corresponding to (3.1) (see also [1, 3]). As in Kovacic’s algorithm [16], one first decides whether there exists $u \in \hat{K}$ such that

$$(\delta_x + u)(\delta_x - u) = \delta_\xi^2 - q,$$

where $\hat{K}$ is an algebraic closure of $K$. Expanding the left-hand side of (3.2) shows that such a factorization exists precisely when one can find a solution in $\hat{K}$ to the Riccati equation $P_q(u) = \delta_x u + u^2 - q = 0$. One can deduce structural properties of $H$ from the algebraic degree of such a $u$ over $K$ [16]. By [7, Thm. 2.10], precisely one of the following possibilities occurs.

I. If there exists $u \in K$ such that $P_q(u) = 0$, then there exist differential algebraic subgroups $A \subseteq G_m(F)$ and $B \subseteq G_a(F)$ such that $H$ is conjugate to

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in A, b \in B \right\}.$$
II. If there exists \( u \in \bar{K}, \) of degree 2 over \( K, \) such that \( P_q(u) = 0, \) then there exists a differential algebraic subgroup \( A \leq \mathbb{G}_m(F) \) such that \( H \) is conjugate to
\[
\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right| a \in A \} \cup \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \right| a \in A \}.
\]

III. If there exists \( u \in \bar{K}, \) of degree 4, 6, or 12 over \( K, \) such that \( P_q(u) = 0, \) then \( H \) is one of the finite primitive groups \( A_4^{SL_2}, S_4^{SL_2}, \) or \( A_5^{SL_2}, \) respectively (see [22, §4]).

IV. If there is no \( u \) in \( \bar{K} \) such that \( P_q(u) = 0, \) then there exists a subset \( \Pi' \subset F \cdot \Pi \) consisting of \( F \)-linearly independent, pairwise commuting derivations \( \partial' \) such that \( H \) is conjugate to \( SL_2(F^{\Pi'}). \)

The computation of \( A \) in cases I and II is explained in [7]. When \( A \leq \mathbb{G}(F^{\Pi}) \) is \( \Pi \)-constant, an effective procedure to compute \( B \) in case I is given in [7, §2, p.7]. This procedure is extended to the case when \( A \) is not necessarily \( \Pi \)-constant in [3]. The computation of \( G \) in case III is reduced to Picard-Vessiot theory [5, Prop. 3.6(2)], and the algorithms of [22] can be used to compute \( G \) in this case. In case IV, for any \( \partial \in \mathcal{D} := F \cdot \Pi \) it is shown in [7, Prop. 2.8] that \( H \) is conjugate to a subgroup of \( SL_2(F^{\partial}) \) if and only if a certain 3rd-order inhomogeneous linear differential equations with respect to \( \delta, \) admits a solution in \( K, \) which can be decided effectively (see [20, Prop. 2.24] and [21, §3]). The results of [9] reduce the problem for general \( \Pi \) to the one-parameter situation.

We now apply these results to compute the PPV-group \( G \) corresponding to
\[
\delta^2_\lambda Y - 2r_1 \delta_\lambda Y + r_0 Y = 0,
\]
where \( r_1, r_2 \in K, \) and \( r_1 \) is not necessarily zero. The harmless normalization \( -2r_1 \) instead of \( r_1 \) will spare us the eyesore of a ubiquitous factor of \( 1/2 \) in what follows.

The solutions for (3.4) are related to the solutions of an associated unimodular equation by a classical change of variables. Letting \( q := r_1^2 - \delta, r_1 - r_0, \) let \( M \) denote a PPV-extension of \( K \) for (3.1), so \( H = \text{Gal}_\Lambda(M/K) \) is the corresponding PPV-group, which we assume has already been computed by [3, 7].

Let \( \{ \eta, \zeta \} \) denote a basis for the solution space of (3.1), and let \( U \) denote a PPV-extension of \( M \) for
\[
\delta_\lambda Y - r_1 Y = 0,
\]
and choose \( 0 \neq \zeta \in U \) such that \( \delta_\lambda \zeta = r_1 \eta. \) A computation shows that \( \{ \zeta \eta, \zeta \zeta \} \) is an \( F \)-basis for the solution space of (3.4), whence \( E := K \langle \zeta \eta, \zeta \zeta \rangle_\Delta \subseteq U \) is a PPV-extension of \( K \) for (3.4), and its PPV-group is \( G = \text{Gal}_\Delta(E/K). \)

Letting \( N := K \langle \zeta \rangle_\Delta \subseteq U, \) we see that \( N \) is a PPV-extension of \( K \) for (3.5), and we denote its PPV-group by \( D \) (the mnemonic is “determinant”). The computation of \( D \) is analogous to that of \( A \) in case I (see [7]). Finally, let \( R := M \cap N \subseteq U. \) Since \( D \leq \mathbb{G}_m(F) \) is abelian, \( \text{Gal}_\Delta(N/R) \leq D \) is normal, and therefore \( R \) is a PPV-extension of \( K, \) with PPV-group denoted by \( \Delta. \) We obtain the following lattice of PPV-extensions and PPV-groups.

![Diagram](image.png)
Remark 3.1. In fact, $U = K(\zeta \eta, \zeta \xi, \zeta \zeta_0)$ is a PPV-extension of $K$ for
\[
\delta_3^3 Y - (3r_1 + \delta_0 \frac{\delta_1}{\delta_2}) \delta_2^2 Y + (2r_1^2 - 2\delta_1 r_1 + r_0 + 2r_1 \delta_0 \frac{\delta_1}{\delta_2}) \delta_4 Y + \left(\delta_4 r_0 - r_1 r_0 - r_0 \frac{\delta_4}{\delta_2}\right) Y = 0. \tag{3.7}
\]
To verify that each of $\zeta \eta, \zeta \xi, \zeta \zeta_0$ and $\zeta$ satisfies (3.7), note that
\[
\delta_0^2 \xi = 2r_1 \delta_1 \xi + r_0 \xi = -q^2 \xi,
\]
and expand the following product in $K[\delta_i]$ to obtain the operator in (3.7):
\[
(\delta_1 - r_1 - \delta_0 \frac{\delta_1}{\delta_2}) \circ (\delta_1^2 - 2r_1 \delta_1 + r_0).
\]
Let $\Gamma := \text{Gal}_\Delta(U/K)$ be the PPV-group of $U$ over $K$. The choice of $\Delta$-field generators $\{\eta, \xi, \zeta\}$ for $U$ over $K$ produces the following embedding of $\Gamma$ in $\text{GL}_3(F)$:
\[
\gamma \mapsto \begin{pmatrix} a_\gamma & b_\gamma & 0 \\ c_\gamma & d_\gamma & 0 \\ 0 & 0 & e_\gamma \end{pmatrix}, \quad \gamma(\eta) = a_\gamma \eta + c_\gamma \xi; \\
\gamma(\xi) = b_\gamma \eta + d_\gamma \xi; \quad \gamma(\zeta) = c_\gamma \xi. \tag{3.8}
\]
Embedding $G$ in $\text{GL}_2(F)$ by means of the basis $\{\zeta \eta, \zeta \xi\}$, the surjection $\Gamma \twoheadrightarrow G$ is then given by $\gamma \mapsto e_{\gamma}^{-1} \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$, and therefore
\[
G \simeq \left\{ e_{\gamma}^{-1} a_{\gamma} e_{\gamma} c_{\gamma} d_{\gamma} \mid \gamma \in \Gamma \right\}. \tag{3.9}
\]
Our next task is to apply the parameterized Galois correspondence [5, Thm. 3.5] to the lattice (3.6) to compute $\Gamma$, and therefore $G$, in terms of $H$ and $D$. The arguments are familiar from classical Galois theory.

**Lemma 3.2.** The restriction homomorphisms
\[
\text{Gal}_\Delta(U/M) \to \text{Gal}_\Delta(N/R); \quad \text{and} \quad \text{Gal}_\Delta(U/N) \to \text{Gal}_\Delta(M/R),
\]
defined respectively by $\gamma \mapsto \gamma|_N$ and $\gamma \mapsto \gamma|_M$, are isomorphisms.

**Proof.** Since $U = M \cdot N$, these homomorphisms are injective. Since the image of $\text{Gal}_\Delta(U/M)$ in $\text{Gal}_\Delta(N/R)$ is Kolchin-closed, its fixed field $R'$ is an intermediate $\Delta$-field extension $R \subseteq R' \subseteq N$. Since every $f \in R'$ is fixed by every $\gamma \in \text{Gal}_\Delta(U/M)$, it follows that $f \in M$, whence $f \in R$. By [5, Thm. 3.5], the image of $\text{Gal}_\Delta(U/M)$ must be all of $\text{Gal}_\Delta(N/R)$. The surjectivity of $\text{Gal}_\Delta(U/N) \to \text{Gal}_\Delta(M/R)$ is proved analogously. \hfill \Box

**Proposition 3.3.** The canonical homomorphism
\[
\Gamma \to H \times_D D := \{ (\sigma, \tau) \in H \times D \mid \sigma|_R = \tau|_R \},
\]
given by $\gamma \mapsto (\gamma|_M, \gamma|_N)$, is an isomorphism.

**Proof.** Injectivity follows from the fact that $U = M \cdot N$. To establish surjectivity, let $\sigma \in H$ and $\tau \in D$ be such that $\sigma|_R = \tau|_R =: \lambda \in \Lambda$. Now choose $\tilde{\lambda} \in \Gamma$ such that $\tilde{\lambda}|_R = \lambda$, and define elements
\[
\sigma' := \sigma \circ \tilde{\lambda}|_M^{-1} \in \text{Gal}_\Delta(M/R); \quad \text{and} \quad \tau' := \tau \circ \tilde{\lambda}|_N^{-1} \in \text{Gal}_\Delta(N/R).
\]
By Lemma 3.2, there exist $\tilde{\sigma} \in \text{Gal}_\Delta(U/N)$ and $\tilde{\tau} \in \text{Gal}_\Delta(U/M)$ such that $\tilde{\sigma}|_M = \sigma'$ and $\tilde{\tau}|_N = \tau'$. A computation shows that $\gamma := \tilde{\sigma} \circ \tilde{\tau} \circ \tilde{\lambda} \in \Gamma$ satisfies $\gamma|_M = \sigma$ and $\gamma|_N = \tau$. \hfill \Box

**Corollary 3.4.** The PPV-group $\Lambda = \{1\}$ if and only if $\Gamma \simeq H \times D$. In this case,
\[
G \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (a, b, c, d) \in H, \, e \in D \right\}.
\]

We will now apply Proposition 3.3 to compute $G$ in cases I, II, III, and IV.
4. Explicit computations

In case 1, there exists a solution \( u \in K \) for the Riccati equation \( P_y(u) = 0 \). We may choose the basis \( \{\eta, \xi\} \) for the solution space of (3.1) such that \( \delta_x \eta = u \eta \) and \( \delta_x (\xi) = \eta^{-2} \). The embedding \( H \hookrightarrow \mathop{SL}_2(F) \) is then given by the formulae \( \sigma(\eta) = a_{\sigma} \eta \) and \( \sigma(\xi) = a_{\sigma}^{-1} \xi + b_{\sigma} \eta \) (cf. Remark 3.1), and there are differential algebraic subgroups \( A \leq G_m(F) \) and \( B \leq G_a(F) \) such that \( H \) is defined by (3.3) (see [1, 7] for more details). We let \( \Theta := \Pi^{\Pi} \) denote the free commutative monoid on the set \( \Pi \) (see §2). The \( \Delta \)-field \( L := K[\eta]_\Delta \) is a PPV-extension of \( K \) for \( \delta_x Y - u Y = 0 \), and \( A \simeq \text{Gal}_{\Delta}(L/K) \). We are ready to state the main result of this section.

**Theorem 4.1.** In case 1, with notation as above, exactly one of the following possibilities holds:

(i) There exist integers \( k_1, k_2 \in \mathbb{Z} \), with \( \text{gcd}(k_1, k_2) \) as small as possible, such that the image group

\[
\{ a^{k_1} \mid a \in A \} \neq \{1\},
\]

and such that there exists an element \( f \in K \) satisfying

\[
k_1 u - k_2 r_1 = \delta_x f.
\]

(ii) Case (i) doesn’t hold, and there exist linear differential polynomials \( p, q \in F\{Y_1, \ldots, Y_m\}_\Pi \) such that the image group

\[
\{ p \left( \frac{\partial^{a_1}}{a_1}, \ldots, \frac{\partial^{a_s}}{a_s} \right) \mid a \in A \} \neq \{0\},
\]

and such that there exists an element \( f \in K \) satisfying

\[
p(\partial_1 u, \ldots, \partial_m u) - q(\partial_1 r_1, \ldots, \partial_m r_1) = \delta_x f.
\]

The \( F \)-vector space generated by such pairs \( (p, q) \) admits a finite \( F \)-basis \( \{(p_1, q_1), \ldots, (p_s, q_s)\} \).

(iii) There exist linear differential polynomials \( p \in F\{Y\}_\Pi \) and \( q \in F\{Y_1, \ldots, Y_m\}_\Pi \) such that the image group

\[
\{ p(b) \mid b \in B \} \neq \{0\},
\]

and such that there exists an element \( f \in K \) satisfying

\[
p(\eta^{-2}) - q(\partial_1 r_1, \ldots, \partial_m r_1) = \delta_x f.
\]

The \( F \)-vector space generated by such pairs \( (p, q) \) admits a finite \( F \)-basis \( \{(p_1, q_1), \ldots, (p_s, q_s)\} \).

(iv) \( \Lambda = \{1\} \).

Consequently, in each of these cases \( G \) coincides with the subset of matrices in

\[
\left\{ \begin{pmatrix} e^a & e^b \\ 0 & e^{-1} \end{pmatrix} \right\} \quad (4.1)
\]

that satisfy the corresponding set of conditions below:

1. In case (i), \( e^{k_1} = e^{k_2} \).
2. In case (ii), \( p_i \left( \frac{\partial^{a_1}}{a_1}, \ldots, \frac{\partial^{a_s}}{a_s} \right) = q_i \left( \frac{\partial^{b_1}}{b_1}, \ldots, \frac{\partial^{b_s}}{b_s} \right) \) for \( 1 \leq i \leq s \).
3. In case (iii), \( p_i(b) = q_i \left( \frac{\partial^{e_1}}{e_1}, \ldots, \frac{\partial^{e_s}}{e_s} \right) \) for \( 1 \leq i \leq s \).
4. In case (iv), there are no further conditions.

We will show slightly more in the course of the proof. We collect these facts in the form of criteria that eliminate from consideration certain possibilities from Theorem 4.1 without testing them directly, based on the data of \( H \) and \( D \) only.
COROLLARY 4.2. The following cases refer to the list of possibilities in the first part of Theorem 4.1.

(1) In case (i), either $A$ and $D$ are both finite, or else they coincide as subgroups of $\mathbb{G}_m(F)$.

(2) In case (ii), neither $A$ nor $D$ is $\Pi$-constant.

(3) In case (iii), $A \subseteq \{\pm 1\}$, $B \neq \{0\}$, and $D$ is not $\Pi$-constant. It follows that $\gamma^2 \in K$, and therefore the test comprised in (iii) concerns elements of $K$ only.

Proof of Theorem 4.1. That the possibilities (i)–(iv) are exhaustive and mutually exclusive will be proved in Propositions 4.4 and 4.6 below. Let us prove that each of these possibilities implies that $G$ is defined as a subset of (4.1) by the corresponding equations contained in (1)–(4), and that no more equations are required. In case (iv), the fact that $G$ coincides with (4.1) is Corollary 3.4.

In case (i), a computation shows that $\gamma^{k_1} \zeta^{-k_2} = cf$ for some $c \in F$. Letting $a_\gamma := \frac{\gamma}{\eta}$ and $b_\gamma := \frac{\gamma}{\zeta}$ for $\gamma \in \Gamma$, we see that $\gamma(cf) = c\eta$ implies that $a_\gamma^{k_1} = e_\gamma^{k_2}$, and that $A$ is finite if and only if $D$ is finite. If $A$ and $D$ are infinite, Theorem 2.3 shows that $A$ and $D$ are defined by the same linear differential polynomials (see Proposition 2.3), because

$$k_1 b_\gamma \left( \frac{\partial a_\gamma}{\partial a_\gamma}, \ldots, \frac{\partial m a_\gamma}{\partial a_\gamma} \right) = k_2 q \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m a_\gamma}{\partial \gamma} \right)$$

(4.2)

for every $\gamma \in \Gamma$ and every $q \in F \{Y_1, \ldots, Y_m\}_{\Pi}$, and the integers $k_1$ and $k_2$ are both different from zero. It follows from (4.2) that there are no more differential-algebraic relations defining $G$ as a subset of (4.1) which involve only $A$ and $D$. To see that there are no more relations at all, it suffices to show that $B$ is in the kernel of $H \to \Lambda$ (by Proposition 3.3), or equivalently that $\Lambda \subseteq L$, which follows from Propositions 4.4 and 4.6. This concludes the proof of (1).

In case (ii), a computation shows that

$$p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) - q \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) \in K.$$

Since, for each $\gamma \in \Gamma$,

$$\gamma (p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right)) = p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) + p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right);$$

and

$$\gamma (l \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right)) = l \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) + l \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right),$$

the parameterized Galois correspondence implies that

$$p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) = q \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right)$$

for each $\gamma \in \Gamma$. If $A$ (resp. $D$) were $\Pi$-constant, then $\frac{\partial 1}{\partial \gamma}$ (resp. $\frac{\partial m}{\partial \gamma}$) would belong to $K$ for every $\partial j \in \Pi$, which is impossible. In particular, $A$ is infinite, so Lemma 4.5 says that $B$ is in the kernel of $H \to \Lambda$. By Proposition 4.6, in case (ii)

$$K \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) \cap K \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) = R.$$

By Proposition 3.3, there are no more equations defining $G$ in (4.1). This concludes the proof of (2).

In case (iii), a computation shows that

$$p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right) \in K.$$

Let $b_\gamma := \frac{\gamma}{\eta} - \frac{\zeta}{\eta} = \frac{\zeta}{\eta} - \frac{\zeta}{\eta}$ for each $\gamma \in \Gamma$. Since

$$\gamma (p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right)) = p \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right),$$

and

$$\gamma (q \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right)) = q \left( \frac{\partial 1}{\partial \gamma}, \ldots, \frac{\partial m}{\partial \gamma} \right),$$

letting
we see that \( p(\frac{\delta}{\eta}) \notin L \), and \( p(b) = q(\frac{\delta_1}{e_1}, \ldots, \frac{\delta_m}{e_m}) \). Since the element \( h := p(\frac{\delta}{\eta}) \in R \) does not belong to \( L \), \( B \) is not in the kernel of \( H \rightarrow \Lambda \). By Lemma 4.5, this implies that \( \Lambda = \{ \pm 1 \} \). By Proposition 4.4, in this case \( L \cap R = K \). Therefore, the equations defining \( G \) in (4.1) do not involve \( A \). This concludes the proof of (3). \( \square \)

In proving Theorem 4.1, it is convenient to treat separately the cases where \( \Lambda = \{ \pm 1 \} \) and \( \Lambda \not\subset \{ \pm 1 \} \). This is done in Proposition 4.4 and Proposition 4.6, respectively. These results are obtained as consequences of the Kolchin-Ostrowski Theorem [13].

**Theorem 4.3 (Kolchin-Ostrowski).** Let \( K \subseteq V \) be a \( \delta_\ast \)-field extension such that \( V^{\delta_i} = K^{\delta_i} \), and suppose that \( e_1, \ldots, e_m, f_1, \ldots, f_n \in V \) are elements such that \( \frac{\delta_i e_i}{e_i} \in K \) for each \( 1 \leq i \leq m \), and \( \delta_i f_j \in K \) for each \( 1 \leq j \leq n \).

Then, these elements are algebraically dependent over \( K \) if and only if at least one of the following holds:

(i) There exist integers \( k_i \in \mathbb{Z} \), not all zero, such that \( \prod_{i=1}^{m} e_i^{k_i} \in K \).

(ii) There exist elements \( c_j \in K^{\delta_j} \), not all zero, such that \( \sum_{j=1}^{n} c_j f_j \in K \).

The following result implies Theorem 4.1 in case \( \Lambda = \{ \pm 1 \} \).

**Proposition 4.4.** If \( \Lambda = \{ \pm 1 \} \), then \( \eta^2 \in K \) and exactly one of the following possibilities holds:

(i) \( \Lambda = \{ \pm 1 \} \), \( D \) is finite of even order \( 2k \), and \( u - kr_1 = \frac{\delta_x f}{f} \)

for some \( f \in K \). Moreover, in this case \( R = L \).

(ii) There exist linear differential polynomials \( p \in F\{Y\}_1 \Pi \) and \( q \in F\{Y_1, \ldots, Y_m\}_1 \Pi \) such that the image group

\[ \{ p(b) \mid b \in B \} \neq \{ 0 \}, \]

and such that there exists an element \( f \in K \) satisfying

\[ p(\eta^{-2}) - q(\delta_1 r_1, \ldots, \delta_m r_1) = \delta_i f. \]

The \( F \)-vector space generated by such pairs \((p, q)\) admits a finite \( F \)-basis \( \{(p_1, q_1), \ldots, (p_s, q_s)\} \). Moreover, in this case \( R \cap L = K \).

(iv) \( \Lambda = \{ 1 \} \).

**Proof.** If \( \Lambda = \{ \pm 1 \} \), then \( \sigma \eta = \pm \eta \) for every \( \sigma \in H \), whence \( \eta^2 \in K \) and \( L = K(\eta) \). If \( \Lambda \neq \{ 1 \} \), there exists an element \( f \in R \) such that \( f \notin K \), and there exist non-constant rational functions \( Q_1(Y, Z_\theta, \hat{\theta}) \) and \( Q_2(Y, Z_\hat{\theta}) \) with coefficients in \( K \), where the variables are indexed by \( \theta \in \Pi \) and \( \hat{\theta} \in \Theta \), such that

\[ Q_1(\zeta, \theta \frac{\delta_x}{\zeta}) = f = Q_2(\eta, \theta \frac{\delta_x}{\eta}). \]

(4.3)

We may assume that the powers of \( \zeta \) (resp. \( \eta \)) appearing in the numerator and denominator of \( Q_1 \) (resp. \( Q_2 \)) are algebraically independent over \( K \), and that the \( \frac{\delta_x}{\zeta} \) (resp. \( \frac{\delta_x}{\eta} \)) appearing in \( Q_1 \) (resp. \( Q_2 \)) are \( F \)-linearly independent. Since \( \delta_i \frac{\delta_x}{\zeta} \in K \) and \( \delta_i \frac{\delta_x}{\eta} \in K \) for each \( \theta \in \Theta \) and \( \hat{\theta} \in \Pi \), Theorem 4.3 implies that these \( \delta_i \frac{\delta_x}{\eta} \) (resp. \( \theta \frac{\delta_x}{\eta} \)) are algebraically independent over \( K(\zeta) \) (resp. \( L \)). Clearing denominators in (4.3) shows that the elements \( \zeta, \eta, \frac{\delta_x}{\zeta}, \frac{\delta_x}{\eta} \) are algebraically dependent over \( K \). Since \( \frac{\delta_x}{\zeta}, \frac{\delta_x}{\eta} \in K \) for each \( \theta \in \Pi \) and \( \hat{\theta} \in \Theta \), Theorem 4.3 says that there exist integers \( k_1, k_2 \in \mathbb{Z} \), none of them zero, such that \( \eta^{k_1} \zeta^{k_2} \in K \), or else there exist \( c_\theta \in F \), almost all zero but not all zero, and \( d_\theta \in F \), almost all zero but not all zero, where \( 1 \leq j \leq m \), such that

\[ \sum_{\theta} c_\theta \frac{\delta_x}{\eta} - \sum_{j, \theta} d_{j, \theta} \theta \frac{\delta_x}{\zeta} \in K. \]  

(4.4)
First suppose that η^k_1ζ^k_2 ∈ K as above. Since Λ ≠ {1}, we may assume that k_1 = 1, η ≠ K, and therefore ζ^k_2 ∉ K. Now η^k_2ζ^k_2 ∈ K implies that ζ^2k_2 ∈ K, whence D is finite. Let k > 0 be the smallest integer k_2. Then K(ζ^k) = L = R, the order of D is 2k, and
\[ u - kr_1 = \frac{δ_1(ηζ^{-1})}{ηζ} = δ_1f. \]

Supposing instead that there are elements c_0, d_{0,θ} ∈ F as in (4.4), set
\[ p := \sum_θ c_0θY ∈ F\{Y\}_1; \quad \text{and} \quad q := \sum_{j,θ} d_{j,θ}θY_j ∈ F\{Y_1, \ldots, Y_m\}_1, \]
so that (4.4) now reads
\[ p(\frac{ζ}{η}) - q(\frac{δ_1ζ}{ζ}, \ldots, \frac{δ_mζ}{ζ}) = f ∈ K. \]
Since Λ ≠ {1}, we may assume that p(\frac{ζ}{η}) ∉ K, and that
\[ q(\frac{δ_1ζ}{ζ}, \ldots, \frac{δ_mζ}{ζ}) ∉ K. \]
This implies that D is infinite, and hence connected [4, 15], because whenever D is finite we have that \( \frac{2ζ}{ζ} ∈ K \) for each θ ∈ Π. Let \( b_γ := \frac{γζ}{η} - \frac{ζ}{η} \) and \( e_γ := \frac{ζ}{ζ} \) for each γ ∈ Γ, and note that
\[ γ(p(\frac{ζ}{η})) = p(\frac{ζ}{η}) + p(b_γ). \]
Hence, there exists \( b ∈ B \) such that \( p(b) ≠ 0 \). Observe that \( δ_0p(\frac{ζ}{η}) = p(η^{-2}); \) and
\[ δ_0(q(\frac{δ_1ζ}{ζ}, \ldots, \frac{δ_mζ}{ζ})) = q(δ_1r_1, \ldots, δ_mr_1). \]
The finite-dimensionality of the F-vector space generated by such pairs \( p,q \) follows from the fact that both N and M have finite algebraic transcendence degree over K (since \( H/R_u(H) ≃ Λ ⊆ \{±1\} \) is Π-constant [19]), so we only need to consider finitely many elements from \{θ \frac{ζ}{η}, θ \frac{2ζ}{ζ} | θ ∈ Π, θ ∈ Θ\}.

We now consider the case when A ⊆ \{±1\}. We begin with a preliminary result.

Lemma 4.5. If A ⊆ \{±1\}, then B is in the kernel of the restriction homomorphism H → Λ.

Proof. To show that A ⊆ L, we proceed by contradiction: suppose that f ∈ R and f ∉ L. There exist non-constant rational functions \( Q_1(Y, Z_{0,θ}) \) and \( Q_2(Z_θ) \) with coefficients in L, where the variables are indexed by θ ∈ Θ and \( \overline{θ} ∈ Π \), such that
\[ Q_1(ζ, θ \frac{2ζ}{ζ}) = f = Q_2(θ \frac{ζ}{η}). \]
We assume without loss of generality that the \( θ \frac{2ζ}{ζ} \) (resp. \( θ \frac{ζ}{η} \)) appearing in \( Q_1 \) (resp. \( Q_2 \)) are algebraically independent over L. Clearing denominators in (4.5) shows that the elements \( ζ, θ \frac{2ζ}{ζ}, θ \frac{ζ}{η} \) are algebraically dependent over L. Since
\[ \frac{δ_1ζ}{ζ}, δ_1θ \frac{2ζ}{ζ}, δ_1θ \frac{ζ}{η} ∈ L \]
for each θ ∈ Θ and \( \overline{θ} ∈ Π \), and since \( f ∉ L \), by Theorem 4.3 there exist \( c_0 ∈ F = L^{∞}, \) almost all zero but not all zero, and \( d_{0,θ} ∈ F \), almost all zero but not all zero, such that
\[ \sum_{θ ∈ Θ} c_0θ \frac{ζ}{η} + \sum_{θ ∈ Θ, θ ∈ Θ} d_{0,θ}θ \frac{2ζ}{ζ} =: g ∈ L. \]
Applying $\delta_\epsilon$ on both sides of (4.6), we obtain
\[ \sum_{\theta \in \Theta} c_\theta \theta(\eta^{-2}) + \sum_{\delta \in \Pi, \theta \in \Theta} d_{\delta, \theta} \theta \partial r_1 = \delta \epsilon. \] (4.7)

Now choose $\sigma \in \text{Gal}_\Delta(L/K)$ such that $a_\sigma := \frac{c_\eta}{\eta} \in F^\Pi$ and $a_\sigma^2 \neq 1$, and apply $(\sigma - 1)$ to both sides of (4.7), to obtain
\[ (a_\sigma^{-2} - 1) \sum_{\theta \in \Theta} c_\theta \theta(\eta^{-2}) = \delta \epsilon(\sigma g - g). \]

But this implies that $\sum c_\theta \theta(\frac{\eta}{\epsilon}) \in \theta$, a contradiction. This concludes the proof that $R \subseteq L$. \hfill \Box

**Proposition 4.6.** If $A \not\subseteq \{\pm 1\}$, then $R \subseteq L$, and exactly one of the following possibilities holds:

(i) There exist integers $k_1, k_2 \in \mathbb{Z}$, with $\gcd(k_1, k_2)$ as small as possible, such that the image group
\[ \{a^k \mid a \in A\} \neq \{0\}, \]
and such that there exists an element $f \in K$ satisfying
\[ k_1 u - k_2 r_1 = \frac{\partial f}{f}. \]

(ii) Case (i) doesn’t hold, and there exist linear differential polynomials $p, q \in F\{Y_1, \ldots, Y_m\}_1$ such that the image group
\[ \{p(\frac{\partial u}{u}, \ldots, \frac{\partial u}{u}) \mid a \in A\} \neq \{0\}, \]
and such that there exists an element $f \in K$ satisfying
\[ p(\partial_1 u, \ldots, \partial_m u) - q(\partial_1 r_1, \ldots, \partial_m r_1) = \delta \epsilon f. \]

The $F$-vector space generated by such pairs $(p, q)$ admits a finite $F$-basis $\{(p_1, q_1), \ldots, (p_s, q_s)\}$. Moreover, in this case
\[ K(\frac{\partial_1 \eta}{\eta}, \ldots, \frac{\partial_m \eta}{\eta})_\Delta \cap K(\frac{\partial_1 \zeta}{\zeta}, \ldots, \frac{\partial_m \zeta}{\zeta})_\Delta = R. \] (4.8)

(iv) $\Lambda = \{1\}$.

**Proof.** Since $A \not\subseteq \{\pm 1\}$, Lemma 4.5 says that $R \subseteq L$. Assume that $\Lambda \neq \{1\}$, and let $f \in R$ such that $f \not\in K$. Then there exist non-constant rational functions $Q_1(Y, Z, \theta)$ and $Q_2(Y, Z, \theta)$ with coefficients in $K$, where the variables are indexed by $\theta \in \Pi$ and $\theta \in \Theta$, such that
\[ Q_1(\eta, \theta\frac{\partial n}{n}) = f^2 = Q_2(\zeta, \theta\frac{\partial \xi}{\xi}). \] (4.9)

We assume without loss of generality that the $\theta\frac{\partial n}{n}$ (resp. $\theta\frac{\partial \xi}{\xi}$) appearing in $Q_1$ (resp. $Q_2$) are algebraically independent over $K$ (cf. the proof of Proposition 4.4). Clearing denominators in (4.9) shows that the elements $\eta, \zeta, \theta\frac{\partial n}{n}, \theta\frac{\partial \xi}{\xi}$ are algebraically dependent over $K$.

Theorem 4.3 implies that either there are integers $k_1, k_2 \in \mathbb{Z}$, none of them zero, such that $\eta^{k_1} \zeta^{-k_2} \in K$, or else there exist $c_{j, \theta} \in F$, almost all zero but not all zero, and $d_{j, \theta} \in F$, almost all zero but not all zero, where $1 \leq j \leq m$, such that
\[ \sum_{j, \theta} c_{j, \theta} \theta\frac{\partial n}{n} - \sum_{j, \theta} d_{j, \theta} \theta\frac{\partial \xi}{\xi} \in K. \] (4.10)
If \( \eta^k \zeta^{-k} = f \in K \) for \( k_1, k_2 \in \mathbb{Z} \) as above, since \( \Lambda \neq \{1\} \) we may assume that \( \eta^k \notin K \). Hence, there exists \( a \in A \) such that \( d^k \neq 1 \), and
\[
k_1 u - k_2 r_1 = \frac{\delta_i(\eta^k \zeta^{-k})}{\eta^i \zeta^{-i}} = \frac{\delta_i f}{f}.
\]
If such integers \( k_1 \) and \( k_2 \) do not exist, then (4.8) is satisfied.

Now suppose there are elements \( c_j, d_j \in F \) as in (4.10), and set
\[
p := \sum_{j, \theta} c_j \theta Y_j \in F \{ Y_1, \ldots, Y_m \}^1; \quad \text{and} \quad q := \sum_{j, \theta} d_j \theta Y_j \in F \{ Y_1, \ldots, Y_m \}^1,
\]
so that (4.10) now reads
\[
p(\frac{\partial_1 n}{n}, \ldots, \frac{\partial_m n}{n}) - q(\frac{\partial_1 \zeta}{\zeta}, \ldots, \frac{\partial_m \zeta}{\zeta}) \in K.
\]
Since \( \Lambda \neq \{1\} \), we may assume that
\[
p(\frac{\partial_1 n}{n}, \ldots, \frac{\partial_m n}{n}) \notin K \quad \text{and} \quad q(\frac{\partial_1 \zeta}{\zeta}, \ldots, \frac{\partial_m \zeta}{\zeta}) \notin K.
\]
Since, for each \( \gamma \in \Gamma \),
\[
\gamma(p(\frac{\partial_1 n}{n}, \ldots, \frac{\partial_m n}{n})) = p(\frac{\partial_1 n}{n}, \ldots, \frac{\partial_m n}{n}) + p(\frac{\partial_1 a}{a}, \ldots, \frac{\partial_m a}{a}),
\]
there exists \( a \in A \) such that \( p(\frac{\partial_1 a}{a}, \ldots, \frac{\partial_m a}{a}) \neq 0 \). Note that
\[
\delta_i(p(\frac{\partial_1 n}{n}, \ldots, \frac{\partial_m n}{n})) = p(\partial_1 u, \ldots, \partial_m u) \quad \text{and} \quad \delta_i(q(\frac{\partial_1 \zeta}{\zeta}, \ldots, \frac{\partial_m \zeta}{\zeta})) = q(\partial_1 r_1, \ldots, \partial_m r_1).
\]
The finite-dimensionality of the \( F \)-vector space generated by such pairs \((p, q)\) follows from the fact that both \( L \) and \( N \) have finite algebraic transcendence degree over \( K \), so we only need to consider finitely many elements from the set \( \{ \theta \frac{\partial n}{n} \mid \theta \in \Pi ; \theta \in \Theta \} \). This concludes the proof of Proposition 4.6.

We will now apply Proposition 3.3 to compute \( G \) in case II. Recall there exists a solution \( u \) to the Riccati equation \( P_0(u) = 0 \) such that \( u \) is quadratic over \( K \). We denote by \( \bar{u} \) the unique Galois conjugate of \( u \), and set \( w := u - \bar{u} \). Then \( w^2 \in K \), so \( \frac{\delta w}{w} = v \in K \). There is a differential algebraic subgroup \( A \leq \mathbb{G}_m(F) \) such that \( H \simeq A \times \{ \pm 1 \} \).

**Proposition 4.7.** In case II, with notation as above, exactly one of the following possibilities holds:

(i) \( D \) is finite of even order \( 2k \), and \( v - kr_1 = \frac{\delta f}{f} \) for some \( f \in K \).

(ii) \( \Lambda = \{1\} \).

Consequently, in each of these cases \( G \) coincides with the subset of matrices in
\[
\left\{ \begin{pmatrix} e_1 a & 0 \\ 0 & e_1 a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -e_2 a \\ e_2 a^{-1} & 0 \end{pmatrix} \mid a \in A; e_1, e_2 \in D \right\}
\]
that satisfy the corresponding set of conditions below:

(1) In case (i), \( e_1^k = 1 \) and \( e_2^k = -1 \).

(2) In case (ii), there are no further conditions.

**Proof:** Since the commutator subgroup \( [H, H] \) of \( H \) coincides with \( \{ (a, 0) \mid a \in A \} \), and \( \Lambda \) is abelian, the surjection \( H \to \Lambda \) factors through \( H/[H, H] \simeq \{ \pm 1 \} \), the PPV-group of the quadratic subextension \( K(u) \subset M \).

Therefore, \( R \subseteq K(u) \) and \( \Lambda \) is finite of order at most 2.

If \( D \leq \mathbb{G}_m(F) \) is infinite, then it is also connected [4], so its only finite quotient is \( \Lambda = \{1\} \). If \( D \) is finite of odd order, then \( \Lambda = \{1\} \) is the only common quotient of \( \{ \pm 1 \} \) and \( D \). Hence, if \( \Lambda \neq \{1\} \), \( D \) must be finite of even
order 2k. Since \( D = \mu_{2k} \) is cyclic, the field \( K(\zeta^k) \) is the only quadratic subextension of \( K(\zeta) \). By Theorem 4.3 or classical Galois theory, \( K(w) = K(\zeta^k) \) if and only if \( w\zeta^{-k} \in K \). Otherwise, \( R = K(w) \cap K(\zeta^k) \) coincides with \( K \), which is impossible. If \( w\zeta^{-k} := f \in K \), we see that
\[
v - kr_1 = \frac{\delta(w\zeta^{-k})}{w\zeta^{-k}} = \delta f.
\]
Letting \( a_\gamma := \frac{\gamma w}{w} \) and \( e_\gamma := \frac{\gamma}{\zeta} \) for \( \gamma \in \Gamma \), we see that \( a_\gamma e_\gamma^{-k} = 1 \). In other words, \( \gamma w = w \) if and only if \( e_\gamma^k = 1 \), and \( \gamma w = -w \) if and only if \( e_\gamma^k = -1 \). To conclude the proof of (1), note that the elements of \( H \) that fix \( w \) are precisely those of the form \( \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \) for \( a \in A \) [16, 24].

Remark 4.8. In case III, \( H \) is a finite subgroup of \( \text{SL}_2(F) \). If \( D \subseteq \mathbb{G}_m(F) \) is infinite, then \( \Lambda = \{1\} \), since it is finite and connected. If \( D = \mu_4 \) is finite, then \( U \) is algebraic over \( K \), and therefore so is \( E \). By [5, Prop. 3.6(2)], \( G \) coincides with the (non-parameterized) PV-group of \( (3.4) \). We only sketch the computation of \( G \) in this case.

For each factor \( \ell \) of \( s \) and each character \( \chi : H \to \mu_4 \) of order \( \ell \), there is an element \( w_\chi \in M \) such that \( K(w_\chi) \subset M \) is cyclic of order \( \ell \) and \( \chi(\sigma) = w_\chi^\sigma \) for each \( \sigma \in H \). Thus \( K(w_\chi) \) is the fixed field of \( \ker(\chi) \). Such an element \( w_\chi \) can be computed effectively (cf. the semi-invariants discussed in [22, §4.3.1]). Let
\[
v_\chi := \frac{\delta w_\chi}{w_\chi} \in K.
\]
If there exist integers \( 0 < k_1 < \ell \) and \( 0 < k_2 < \frac{\ell}{4} \) such that
\[
k_1 v_\chi - k_2 r_1 = \frac{\delta f}{f}
\]
for some \( f \in K \), then
\[
G \simeq \{(\sigma, e) \in H \times D \mid \chi(\sigma)^{k_1} = e^{k_2}\}.
\]
If no such \( k_1 \) and \( k_2 \) can be found for any \( \chi \in H^s \), the character group of \( H \), then \( \Lambda = \{1\} \).

When \( H \) is finite in cases I and II (i.e., \( A \) is finite and \( B = 0 \)), the computation of \( G \) performed in Theorem 4.1 and Proposition 4.7 coincides with the one just described.

Remark 4.9. In case IV, there is a finite subset \( \Pi' \) of the \( F \)-span of \( \Pi \) consisting of \( F \)-linearly independent, pairwise commuting derivations such that \( H \) is isomorphic to the simple group \( \text{SL}_2(F^{1\Pi'}) \) [4]. Therefore, the only abelian quotient of \( H \) in this case is \( \Lambda = \{1\} \).

Example

We let \( K = F(x) \) denote the \( \Delta \)-field of the previous sections, where \( \Pi := \{\partial_1, \partial_2\} \), \( \partial_j := \frac{\partial}{\partial t_j} \) for \( j = 1, 2 \), and \( F \) denotes a \( \Pi \)-closed field containing \( \mathbb{Q}(t_1, t_2) \) [11, 23]. We now compute the PPV-group \( G \) corresponding to
\[
\delta_\zeta^2 Y - 2 \left( \frac{t_1 - t_2}{x} + \frac{t_1 - t_2}{x-1} \right) \delta_\zeta Y + \left( \frac{(t_1-2t_1)(t_1-1)+2(t_1-t_2)x}{x^2} + \frac{t_1(2t_2-t_1+1)-2(t_1-t_2)(x-1)}{(x-1)^2} \right) Y = 0.
\]
(5.1)

Note that \( r_1 = \frac{t_1 - t_2}{x} + \frac{t_1 - t_2}{x-1} \), and the coefficient \( q \) in the unimodular equation (3.1) associated to (5.1) in this case is
\[
q = \frac{t_1(t_1-1)(1-2x)}{x^2} + \frac{(t_1-t_2)(2t_1x-t_1-t_2-1)}{(x-1)^2}.
\]
The Riccati equation \( \delta_\zeta u + u^2 = q \) admits the solution \( u = \frac{t_1}{x} + \frac{t_1 - t_2}{x-1} \). Hence, we are in case I, and there are differential algebraic subgroups \( A \subseteq \mathbb{G}_m(F) \) and \( B \subseteq \mathbb{G}_a(F) \) such that the PPV-group \( H \) for (3.1) is defined by (3.3). By [7], for every \( a \in A \) and every linear differential polynomial \( q \in F\{Y_1, Y_2\}^1_{\Pi} \),
\[
q(\frac{\partial_1 u}{a}, \frac{\partial_2 u}{a}) = 0 \iff q(\partial_1 u, \partial_2 u) \in \delta_\zeta(K).
\]
(5.2)

Since \( \partial_1 u = \frac{1}{x} + \frac{1}{x-1} \) and \( \partial_2 u = -\frac{1}{x-1} \), we see that
\[
A = \{a \in \mathbb{G}_m(F) \mid \partial_j(\frac{\partial u}{a}) = 0 = \partial_j(\frac{\partial_2 u}{a}) \text{ for } j = 1, 2\}.
\]
A similar computation shows that the PPV-group $D$ for (3.5) is defined by the same linear differential polynomials as $A$.

To compute $B$, let $H'$ denote the (non-parameterized) PV-group of (3.1). The unipotent radical $B'$ of $H'$ is $G_a(F)$, by Kovacic’s algorithm [16] (see also [2, proof of Cor. 3.3] for a similar computation). Since the only derivation
\[ \partial \in F \cdot \partial_1 \oplus F \cdot \partial_2 \]
such that $\partial u \in \delta_x(K)$ is $\partial = 0$, the main result of [3] implies that $B = B' = G_a(F)$.

Having computed $H$ and $D$, we apply Theorem 4.1 to compute $G$. For any integers $k_1, k_2 \in \mathbb{Z}$,
\[ k_1 u - k_2 r_1 = \frac{t_1(k_1 - k_2) + t_2 k_2}{x} + \frac{t_1 k_1 - t_2 (k_1 - k_2)}{x - 1}. \] (5.3)
If $k_1 u - k_2 r_1 = \frac{f}{x}$ for some $f \in K$, the residues of (5.3) are integers, which is impossible unless $k_1 = 0 = k_2$, and therefore case (i) of Theorem 4.1 doesn’t hold. Now the relations
\[ \partial_1 u + \partial_2 u = \partial_1 r_1 \quad \text{and} \quad \partial_1 r_1 + \partial_2 r_1 = -\partial_2 u, \]
correspond to the linear differential polynomials
\[ p_1 := Y_1 + Y_2; \quad q_1 := Y_1; \quad p_2 := -Y_2; \quad q_2 := Y_1 + Y_2. \]
Letting $a \in A$ such that $\partial_1 a = 0$ and $\partial_2 a = a$, we see that $p_i(\frac{\partial_a u}{a}, \frac{\partial_a u}{a}) \neq 0$ for $i = 1, 2$, and we have verified the conditions of Theorem 4.1(ii).

Since $\theta \partial_j u = 0 = \theta \partial_j r_1$ for each $\theta \in \Theta$ and $1 \leq j \leq 2$, the set
\[ \{(p_1, q_1); (p_2, q_2)\} \]
forms a basis for the $F$-vector space of pairs $(p, q)$, with $p, q \in F\{Y_1, Y_2\}_1$, such that
\[ p(\partial_1 u, \partial_2 u) - q(\partial_1 r_1, \partial_2 r_1) \in \delta_x(K). \]
Therefore, the PPV-group for (5.1) is
\[ G \simeq \left\{ \begin{pmatrix} ea & eb \\ 0 & ea^{-1} \end{pmatrix} \bigg| a, e \in A; b \in B; \frac{\partial_a u}{a} + \frac{\partial_a u}{a} = \frac{\partial_e u}{e}, \frac{\partial_e u}{e} + \frac{\partial_e u}{e} = -\frac{\partial_u a}{a} \right\}. \]

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Carlos E. Arreche carreche@gc.cuny.edu
Mathematics Department, The Graduate Center of the City University of New York, New York, NY 10016