1. Introduction

Determining the space of free discrete two generator groups of Möbius transformations is an old and difficult problem. In this paper we show how to construct large balls of full dimension in this space. To do this, we begin with a marked discrete group of non-separating disjoint circle type, nsdc group (see section 2.2 for the definition). Such a group determines three disjoint or tangent planes. We prove that there is a whole family of discrete groups, which we term the nsdc-planar family, that share these planes.

From specific information about the nsdc planes which depends on the marking data, we show how to construct all the discrete groups in the nsdc planar family. In particular, we find a set of six real numbers that serve as parameters for this family. We then construct an embedding of our parameters into a classical representation of the full space of free discrete groups as a subset of $\hat{\mathbb{C}}^3$. We see that each planar family fills out a ball of full dimension in the classical embedding.

We remark that our construction does not use the usual theory of quasi-conformal deformations of a given group nor does it depend simply on the use of coverings and quotients that give commensurable groups.

The paper is organized as follows. In section 2 we give the basic notation and definitions and in section 3 we define the nsdc planar family. Sections 4 and 5 develop computational techniques and in section 6 we construct our new parameters. In section 7 we show how to our parameters are related to the classical parameters. Finally, in section 8 we give a necessary and sufficient condition for a marked group to belong to a given planar family.

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2. Notation and definitions

An element $X$ of $PSL(2, \mathbb{C})$ acts as a Möbius transformation on $\hat{\mathbb{C}}$, the complex sphere. The action extends in a natural way to hyperbolic three-space $\mathbb{H}^3$. When considered as the boundary of hyperbolic three-space, $\hat{\mathbb{C}}$ is referred to as the sphere at infinity.

Elements of $PSL(2, \mathbb{C})$ are classified as loxodromic, elliptic, parabolic according to the square of their traces. The classification of transformations can also be described by the action on $\hat{\mathbb{C}}$ or on $\hat{\mathbb{C}} \cup \mathbb{H}^3$.

We follow the notation of [1]. For $x$ and $y \in (\hat{\mathbb{C}} \cup \mathbb{H}^3)$ with $x \neq y$ we let $[x, y]$ denote the oriented hyperbolic geodesic passing through $x$ and $y$. The ends of the geodesic $[x, y]$ are by definition the points $v$ and $w$ with $\{v, w\} = [x, y] \cap \hat{\mathbb{C}}$. If $v$ and $w$ are the ends of $[x, y]$, then $[x, y] = [v, w]$. The notation $[x, y]$ also indicates a direction so that $[y, x]$ is the geodesic with the opposite orientation.

Following Fenchel [1], we include improper lines, in our considerations. A proper line is one whose ends are distinct. An improper line is one whose ends coincide.

A Möbius transformation fixes one or two points on $\hat{\mathbb{C}}$. A loxodromic transformation $A$ has two distinct fixed points as does an elliptic transformation. These transformations fix the geodesic $X$ in $\mathbb{H}^3$ whose ends are the fixed points of $A$ on $\hat{\mathbb{C}}$. This geodesic is called the axis of $A$ and is denoted by $Ax_A$.

A parabolic transformation has one fixed point. Using the terminology of improper lines, parabolic transformations also have axes. If $A$ is a parabolic transformation with fixed point $u \in \hat{\mathbb{C}}$, we consider $[u, u]$ to be its axis.

Every proper pair of lines in $\mathbb{H}^3$ has a unique common perpendicular. If $u, v, w, z$ are points in $\hat{\mathbb{C}}$, then two distinct improper lines $[u, u]$ and $[v, v]$ have the unique common perpendicular $[u, v]$. The common perpendicular to the improper line $[u, u]$ and the proper line $[v, w]$ is the geodesic through $u$ perpendicular to $[v, w]$.

With this terminology, every pair of elements $A, B \in PSL(2, \mathbb{C})$ determines a unique hyperbolic line in $\mathbb{H}^3$, the common perpendicular of their axes; we denote this common perpendicular by $L$ and denote the ends of $L$ by $n$ and $n'$.

2.1. Half-turns. For any hyperbolic line $[x, y]$ we let $H_{[x,y]}$ be the half-turn about the line with ends $x$ and $y$. We note that $H_{[x,y]}$ fixes every hyperbolic plane $\mathbb{P}$ whose horizon $C_P$ passes through $x$ and $y$ ([2]). The half-turn will interchange the exterior and the interior of $\mathbb{P} \in \mathbb{H}^3$. 

2.2. NSDC Groups. There is a natural way to attach a three generator group and six complex numbers to any a two generator group (see [2]).

Let \( G = \langle A, B \rangle \) and let \( L \) be the common perpendicular to \( A \) and \( B \). There are unique hyperbolic lines \( L_A \) and \( L_B \) such that \( A = H_{L_A} \cdot H_L \) and \( B = H_{L_B} \cdot H_L \). We let \( a \) and \( a' \) be the ends of \( L_A \) so that \( L_A = [a, a'] \) and let \( b \) and \( b' \) be those of \( L_B \) so that \( L_B = [b, b'] \). We define

\[ \text{Definition 2.1.} \quad \text{The marked three generator group determined by} \ G = \langle A, B \rangle \text{is denoted by} \ T_G \text{and defined by} \ T_G = \langle H_{L_A}, H_L, H_{L_B} \rangle. \]

By construction \( G \) is a normal subgroup of \( T_G \) of index at most two which immediately implies

\[ \text{Proposition 2.2.} \quad G \text{is discrete if and only if} \ T_G \text{is discrete.} \]

We also define

\[ \text{Definition 2.3.} \quad \text{The ortho-end of} \ G \text{is the six-tuple of complex numbers} \ (a, a', n, n', b, b'). \]

\[ \text{Definition 2.4.} \quad \text{Six points in} \ (a, a', n, n', b, b') \in \hat{\mathbb{C}}^6 \text{have the non-separating disjoint circle property if there are pairwise disjoint or tangent circles on} \ \hat{\mathbb{C}}, \ C_A, \ C_D \text{and} \ C_B \text{respectively passing through} \ a \text{and} \ a', \ n \text{and} \ n', \ b \text{and} \ b' \text{respectively such that no one circle separates the other two.} \]

\[ \text{Definition 2.5.} \quad \text{A marked group} \ G = \langle A, B \rangle \text{is a marked non-separating disjoint circle group or nsdc group if the ortho-end of} \ A \text{and} \ B \text{has the non-separating disjoint circle property.} \ G \text{is a non-separating disjoint circle group if some pair of generators for} \ G \text{has the non-separating disjoint circle property. The corresponding group} \ T_G \text{is also called an nsdc group.} \]

It was shown in [2] that

\[ \text{Proposition 2.6.} \quad \text{Let} \ G \text{be a two-generator subgroup of} \ PSL(2, \mathbb{C}). \ \text{If some ortho-end of} \ G \text{has the non-separating disjoint circle property, then} \ G \text{is discrete.} \]

3. The Planar Family of NSDC Groups

Let \( G \) be an nsdc group with ortho-end \( (a, a', n, n', b, b') \). Let the three circles \( C_A, C_D, C_B \) be a set of non-separating disjoint circles for this ortho-end. Let \( \mathbb{P}_A, \mathbb{P}_D \) and \( \mathbb{P}_B \) be planes in \( \mathbb{H}^3 \) whose respective horizons are \( C_A, C_D, \text{and} \ C_B \). Let \( L'_A \) be any hyperbolic line lying on \( \mathbb{P}_A \), let \( L'_B \) be any hyperbolic line lying on \( \mathbb{P}_B \), and let \( L' \) be any
hyperbolic line lying on \( \mathbb{P}_D \). Set \( T \mathcal{G}' = \langle H_{L_A'}, H_{L'}, H_{L_B'} \rangle \) and \( G' = \langle H_{L_A'} \cdot H_{L'}, H_{L_B'} \cdot H_{L'} \rangle \).

**Proposition 3.1.** \( T \mathcal{G}' \) and \( G' \) are both discrete.

**Proof.** Let \([\alpha, \alpha'], [\eta, \eta'], [\beta, \beta']\) be the ends of \( L_A', L' \) and \( L_B' \) respectively. The six-tuple \((\alpha, \alpha', \eta, \eta', \beta, \beta')\) is an ortho-end of \( G' \) and \( T \mathcal{G}' \) with nsdc circles \( C_A, C_D \) and \( C_B \). By proposition 2.6 \( G' \) is discrete and by proposition 2.2 \( T \mathcal{G}' \) is also.

The family of groups \( T \mathcal{G}' \) depends upon \( \mathbb{P}_A, \mathbb{P}_D \) and \( \mathbb{P}_B \). We write \( \mathbb{P} \) for \( \mathbb{P}_D \).

**Definition 3.2.** We call the triple of planes \((\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)\) a non-separating disjoint planar triple or an nsdc-triple if the horizons are a set of nsdc circles. We call the set of all nsdc groups with a fixed planar triple, a planar family of nsdc groups.

We want to explore all discrete groups corresponding to a given nsdc planar triple.

4. Ortho-ends and pull back circles

In this section we present notation needed to describe a triple of nsdc planes. We begin with \( G = \langle A, B \rangle \) and assume that the ortho-end of \( G \) is the six-tuple \((a, a', n, n', b, b')\) and that the nsdc-planes are \( \mathbb{P}_A, \mathbb{P}, \mathbb{P}_B \) respectively. The planes each are determined by their pull-back angles, which can be described as follows:

Let \( k \) and \( k' \) be any two points in \( \hat{\mathbb{C}} \). Any circle through \( k \) and \( k' \) has center lying on the perpendicular bisector of the line connecting \( k \) and \( k' \). Thus its center is at \( c_t = \frac{k + k'}{2} + it \frac{k - k'}{2} \) for some real number \( t \). We think of the center as having been pulled back \( t \) units from \( \frac{k + k'}{2} \).

We let \( \theta_t \) denote the angle from the radius connecting \( c_t \) to \( k \) to the Euclidean segment joining \( k \) and \( k' \) so that \( t = \frac{|k - k'|}{2} \cdot \tan \theta_t \). We call \( \theta_t \) the pull-back angle of the circle. Each circle passing through \( k \) and \( k' \) corresponds to a unique pull-back angle \( \theta_t \) with \( -\pi/2 \leq \theta_t < \pi/2 \).

If \( r(t) \) denotes the radius of the circle with pull back angle \( \theta_t \), we have \( r(t) = \frac{|k - k'|}{2 \cos \theta_t} \).

When we are discussing a fixed plane whose horizon passes through \( k \) and \( k' \), we write \( \theta_K \) for the pull-back angle. We write \( r_{\theta_K} \) for the radius of the horizon of the plane and \( c_{\theta_K} \) for its center.

In particular, if we have a family of nsdc-planes for \( G = \langle A, B \rangle \), we assume that \( \mathbb{P}_A \) has pull-back angle \( \theta_A \), \( \mathbb{P}_B \) has pull-back angle \( \theta_B \), and \( \mathbb{P} \) has pull-back angle \( \theta \).
5. Computations and perpendicul"urs

In what follows we will be interested in being able to compute quantities such as points and planes directly from the ortho-end \((a, a', n, n', b, b')\) and the three pull-back angles \((\theta_A, \theta, \theta_B)\).

We define three points \(v \in P, v_A \in P_A\) and \(v_B \in P_B\) by \(v = L \cap Ax_A, v_A = L_A \cap Ax_A, v_B = L_B \cap Ax_B\). These are all computable from the ortho-end and the pull-back angles.

If \(P\) is any hyperbolic plane, \(L^P\) a hyperbolic line on \(P\) and \(x\) a point on \(L^P\), there is a line perpendicular to \(P\) passing through \(x\) which we denote by \(V_x^P\) and a unique line perpendicular to \(L^P\) and \(V_x^P\) lying on \(P\) and passing through \(x\) which we denote \(M_x^P\). If there is no confusion we will omit the superscript \(P\) from the notation for these lines. Let \(L_W\) be a line with ends \([w, w']\) and suppose it lies on a plane \(P_W\) with pull-back angle \(\theta_W\). Then we can compute that
\[
\zeta_{\theta_W} = \frac{w + w'}{2} + i \frac{(w - w')^2}{4} \tan \theta_W.
\]
Thus for any \(x\) on \(L_W\)
\[
P_{xW}^P = \left[\frac{w + w'}{2} + i \frac{(w - w')^2}{4} \tan \theta_W, x\right].
\]
Also since one end of \(M_{xW}^P\) must be \(\frac{w + w'}{2} + i (r_{\theta_W})\), we have
\[
M_{xW}^P = \left[\frac{w + w'}{2} + i (r_{\theta_W}), x\right].
\]
The point is that this explicit calculation only depends upon \(\theta_W\). We note that a line perpendicular to \(P_W\) passing through a point \(x\) in \(P_W\) is the line \([c_W, x]\). We are interested in these perpendicul"urs when \(x = v_A\).

6. Constructing the planar family

A loxodromic transformation that is a pure translation by distance \(d\) with zero rotation angle is called a hyperbolic transformation with translation length \(d\).

**Definition 6.1.** Let \(X\) be any hyperbolic line and \(\tau\) any angle. We let \(R_{X, \tau}\) be the elliptic transformation that is rotation through an angle \(\tau\) about the line \(X\). If \(d\) is any positive real number, we let \(T_{X, d}\) be the hyperbolic transformation whose axis is \(X\) and whose translation length is \(d\).

We let our rotation angles lie in the interval \((-\pi, \pi]\) in order to keep track of the orientation of a line under the rotation.
We note that given any nsdc planar family, these two types of moves, rotation about a line perpendicular to a plane of the family and parallel transport along a line lying in a plane of the family preserve the family. Specifically we prove

**Lemma 6.2.** Assume that \( G = \langle A, B \rangle \) is of nsdc type in the \((\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)\) planar family. Let \( L_A' = g(L_A) \) where \( g = g_1, ..., g_n \) and for each \( i \), either

- \( g_i = T_{X_i,d_i} \) where \( X_i \in \mathbb{P}_A \) and \( d_i \) is a real number or
- \( g_i = R_{X_i,\tau_i} \) where \( X_i \) is perpendicular to \( \mathbb{P}_A \) and \( \tau_i \) is any angle \(-\pi < \tau_i \leq \pi\).

Then \( \langle H_{L_A'}, H_L, H_{L_B} \rangle \) is a discrete group in the \((\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)\) planar family. If \( A' = H_{L_A'} H_L \), then \( G' = \langle A', B \rangle \) is a discrete group of nsdc type.

**Proof.** By construction each of the \( g_i \) maps lines on \( \mathbb{P}_A \) to lines on \( \mathbb{P}_A \); \( g_i H_{L_A} g_i^{-1} = H_{g_i(L_A)} \).

Thus these moves preserve the planar family. Next we see that every triple of geodesics \( (L_{A'}, L', L_{B'}) \) lying in the respective planes \( \mathbb{P}_A, \mathbb{P}, \mathbb{P}_B \) of the planar family can be so obtained. Precisely,

**Proposition 6.3.** Assume that \( G = \langle A, B \rangle \) is of nsdc type with ortho-end \((a, a', n, n', b, b')\) and planar family \((\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)\) with pull back-angles \((\theta_A, \theta, \theta_B)\). Let \( v_A = L_A \cap Ax_A \). Then \( G' = \langle A', B \rangle \) is in the nsdc planar family with \( A' = H_{L_A'} H_L \) where \( L_{A'} \) is obtained from \( L_A \) by the following (where the plane \( \mathbb{P}_A \) is implied).

1. \( X = R_{V_{v_A},\tau}(M_{v_A}) \) where \( \tau \) is an angle with \(-\pi \leq \tau < \pi\),
2. \( Y = R_{V_{v_A},\tau}(L_A) \), and
3. \( L_{A'} = T_{X,d}(Y) \) for some positive number \( d \).

Conversely, given any \( L_{A'} \in \mathbb{P}_A \) there exists an angle \( \tau \), \(-\pi \leq \tau < \pi\) and a positive number \( d \) so that \( L_{A'} \) can be obtained from \( L_A \) by the above construction.

**Proof.** If \( L_{A'} \) is any line on \( \mathbb{P}_A \), we can drop a perpendicular from \( v_A \) to \( L_{A'} \), whether or not the lines \( L_A \) and \( L_{A'} \) are disjoint. Let \( X \) be the geodesic on which this perpendicular lies, oriented towards \( v_A \), and let \( d \) be the distance from \( L_{A'} \) to \( v_A \) along \( X \). Then \( T^{-1}_{X,d}(L_{A'}) = Y \) passes through \( v_A \) and makes some angle \( \tau \) with \( L_A \). We can rotate \( Y \) to \( L_A \) by applying \( R_{V_{v_A},\tau} \). (Alternatively, rotating by \( \pi/2 + \tau \) sends \( Y \) to \( M_{v_A} \).) Combining these moves we have

\[
L_{A'} = (T_{R_{V_{v_A},\tau}(L_A),d} \circ R_{V_{v_A},\tau})(L_A) = (T_{R_{V_{v_A},\pi/2+\tau}(M_{v_A}),d} \circ R_{V_{v_A},\tau})(L_A)
\]

Thus any line \( L'_{A} \) on \( \mathbb{P}_A \) determines a positive number \( d = d_A \) and an angle \( \tau = \tau_A \) in \((-\pi, \pi)\). Conversely, given the initial data
(θ_A, a and a') and the pair (d_A, τ_A) we can use this construction to find L_A'.

**Definition 6.4.** For the fixed planar family \((P_A, P, P_B)\) with pull back-angles \((θ_A, θ, θ_B)\) we call the sequence of moves in the proposition the \((d_A, τ_A)\) moves determining L_A'.

**Corollary 6.5.** All of these moves can be calculated directly from \((a, a', n, n', b, b)\) and \((θ_A, θ, θ_B)\).

**Proof.** This is the content of section 5 on perpendiculars.

In summary we have

**Theorem 6.6.** Let \(G = \langle A, B \rangle\) be a fixed marked group of nsdc type in the planar family \((P_A, P, P_B)\) with pull back-angles \((θ_A, θ, θ_B)\). Then the set of triples

\[
D = \{(d_A, τ_A), (d, τ), (d_B, τ_B) ∈ ([0, ∞), (−π, π])^3\}
\]

form a set of moduli for the family. That is, given an ortho-end \((a, a', n, n', b, b')\) in \(\hat{C}^6\) and pull-back angles \((θ_A, θ, θ_B)\) there are six real numbers \((d_A, τ_A, d, τ, d_B, τ_B)\) such that every nsdc group \(G' = \langle A', B' \rangle\) in the planar family \((P_A, P, P_B)\) can be obtained through the moves \((d_A, τ_A), (d, τ)\) and \((d_B, τ_B)\) giving the lines \(L_A', L\) and \(L_B\) respectively and hence the marked group \(G'\).

**Proof.** Proposition 6.3 shows that given each pair \((d_A, τ_A)\), appropriate moves can be chosen to find \(L_A'\) and conversely each \(L_A'\) determines such a pair; the same is true for \(L'\) and \(L_B\). Thus all marked groups with these nsdc planes are determined. The point is that given the planar triple, or equivalently the three angles \((θ_A, θ_B, θ)\), every ortho-end that gives a group in this planar nsdc family can be obtained.

### 7. Relation to classical moduli

The marked nsdc groups are a subset of the space \(S\) of marked discrete free groups \(G = \langle A, B \rangle\). Classically, \(S\) can be embedded into a subset of \(C^3\) by choosing as moduli \((\text{tr}A, \text{tr}B, \text{tr}AB^{-1})\). Specifically, one picks a marked group \(G\) as base point and arbitrarily chooses elements in \(SL(2, \hat{C})\) (again called \(A\) and \(B\)) to represent the generators. The signs of the traces of these elements depends on this choice but once it is made, there is a unique matrix corresponding to every other group element. The space \(S\) is then obtained by quasiconformal deformation. The computation of the full boundary of \(S\) in this embedding is an open hard question.

In this section we show how these trace parameters are related to the nsdc moduli we found above.
7.1. **Skew Hexagons.** Let \( G = \langle A, B \rangle \) be a fixed group of nsdc type in the planar family \( (\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B) \) with pull back-angles \( (\theta_A, \theta, \theta_B) \) and let \( T G \) be the corresponding group. As we saw in section 2.2, we can write \( A = H_L \cdot H_L, B = H_L \cdot H_L, \) and \( AB^{-1} = H_L \cdot H_L. \) Here we assume the axes are oriented from their repelling fixed points to their attracting fixed points. We assume \( L \) is oriented from \( Ax_A \) to \( Ax_B, \) \( L_A \) is oriented from \( Ax_A \) to \( Ax_{AB}^{-1} \) and \( L_B \) is oriented from \( Ax_{AB}^{-1} \) to \( Ax_B. \)

Following Fenchel, [1], we can form a *skew right angled hexagon* associated to \( G, \) which we denote by \( H_G \) with sides:

\[
H_G \text{ has sides } L_A, Ax_A, L, Ax_B, L_B, Ax_{AB}^{-1}
\]

Here we adopt the convention that we label sides of a hexagon by the hyperbolic line upon which they lie taking the segment indicated by the order. That is, six ordered geodesics determine six vertices: if the geodesics are \( s_1, s_2, s_3, s_4, s_5, s_6, \) set \( v_i = s_i \cap s_{i+1} \) for \( i \leq i \leq 6 \) with indices taken modulo 6 and let the hexagon side denoted by \( s_i \) be the segment of \( s_i \) traversed from \( v_{i-1} \) to \( v_i \) where \( v_0 = v_6. \) With this convention it is clear which segment of \( s \) we are talking about, and, therefore, we do not distinguish notationally between the segment \( s \) and the geodesic \( s \) on which it lies. We write \( s \) for both.

The labeling gives an orientation to the hexagon and its sides. Note that the orientation of a side within the hexagon may be opposite to the orientation of the geodesic containing the side.

Relative to the orientation of the hexagon we define the complex length \( \delta_i = \delta(s_i) \) of the side \( s_i \) as follows: \( t_i = \Re \delta(s_i) \) is the hyperbolic length of \( s_i; \psi_i = \Im \delta(s_i) \) is the angle from \( T_{s_i,t_i}(s_{i-1}) \) to \( s_{i+1} \) with indices taken modulo 6. Here \( \Re \) and \( \Im \) denote the real and the imaginary parts of a complex number. The complex lengths satisfy the "cosine rule"

\[
\cosh(\delta_{i+4}) = \cosh(\delta_i) \cosh(\delta_{i+2}) + \cosh(\delta_{i+1}) \sinh(\delta_i) \sinh(\delta_{i+2}).
\]

It follows that the complex lengths of any three alternating sides or any three adjacent sides determine the other three lengths.

For example, in the hexagon \( H_A \) we have \( \delta(Ax_A) = t_A + i\psi_A \) where \( t_A \) is the hyperbolic distance along \( Ax_A \) between \( L_A \) and \( L \) and \( \psi_A \) is the angle from \( T_{Ax_A,t_A}(L_A) \) to \( L. \) (Define \( t_B, \psi_B \) and \( t_{AB^{-1}} = t, \psi_{AB^{-1}} = \psi \) similarly.) It is a standard computation that \( \mathrm{tr} A = 2 \cosh(t_A + i\psi_A) \) and similarly for \( \mathrm{tr} B \) and \( \mathrm{tr} AB^{-1}. \)

Using the cosine rule, these three traces determine the complex lengths of the other three sides, \( L_A, L, L_B, \) of the hexagon and the complex lengths of the \( L_A, L, L_B \) sides determine the traces.
7.2. Hexagons for the planar family. Let $G' = \langle A', B \rangle$ be a group in the planar family where $L_{A'}$ is obtained from $L_A$ by $(d_A, \tau_A)$ moves. We can form the pentagon $P_{A,A'}$ with ordered sides $Ax_A, L, Ax_{A'}, L_{A'}, X_{v_A}$ where $X_{v_A}$ is the perpendicular from $v_A$ to $L_{A'}$. We may consider the pentagon as a degenerate skew hexagon where the length of the (degenerate) fifth side has no real part as follows:

$s_1 = Ax_A; \quad \delta_1 = t_A + i\psi_A$
$s_2 = L; \quad \delta_2 = t_L + \psi_L$
$s_3 = -Ax_{A'}; \quad \delta_3 = t_{A'} + i(\psi_{A'} + \pi)$
$s_4 = L_{A'}; \quad \delta_4 = t_{L_{A'}} + i\psi_{L_{A'}}$
$s_5; \quad \delta_5 = -i\tau_A$
$s_6 = X_{v_A}; \quad \delta_6 = d_A$

**Theorem 7.1.** Given the fixed marked group $G$ and the parameters $(d_A, \tau_A), (d, \tau), (d_B, \tau_B)$ we can find the traces of the elements $A', B', A'B'^{-1}$ of any group in the planar family $(\mathbb{P}_A, \mathbb{P}, \mathbb{P}_B)$.

**Proof.** We use the cosine rule applied to the degenerate skew hexagon above to find $\delta_3$ and hence the trace of $A'$. We get $A'$ from its axis and its trace and thus obtain the group $G' = \langle A', B \rangle$. We can also find the complex distance $\delta_2$ along the side $L$. Since we know complex distance along $L$ from $Ax_A$ to $Ax_B$ in $\mathcal{H}_G$, we can compute the distance along $L$ from $Ax_{A'}$ to $Ax_B$ in $\mathcal{H}_{G'}$. Using the cosine rule again, we get all the complex lengths in the hexagon $\mathcal{H}_{G'}$ and hence the traces $(\text{tr}A', \text{tr}B, \text{tr}A'B)$.

Since we can get to any group in the family by a sequence of moves as above, we can find its trace moduli as well. □

As an immediate corollary we have

**Corollary 7.2.** The constructions above determine an embedding of $\mathcal{C}$ into $\mathcal{S}$. Thus each planar family fills out a ball in $\mathcal{S}$ of full dimension.

8. Ortho-ends in the planar family

Given a planar nsdc family with ortho-end $(a, a', n, n', b, b')$ and planes $(\mathbb{P}_a, \mathbb{P}, \mathbb{P}_B)$ with pull back angles $\theta_A, \theta, \theta_B$, we can find necessary and sufficient conditions for a six-tuple $(\alpha, \alpha', \eta, \eta', \beta, \beta')$ to be the ortho-end of a group in the family.

As usual, we work one plane at a time. That is, we look at the six-tuple $(\alpha, \alpha', n, n', b, b')$. We have
Proposition 8.1. \((\alpha, \alpha', n, n', b, b')\) lies in the NSDC-\(\theta_A\) planar family of \((a, a', n, n', b, b')\) if and only if

\[
\frac{(a + a') - (\alpha + \alpha') - i(a-a')(|a-a'|)}{i(\alpha - \alpha')} \in \mathbb{R}.
\]

Proof. We need to show that this condition is equivalent to \(a, a', \alpha, \) and \(\alpha'\) lying on the pull back circle \(\theta_A\). Recall that the pull-back angle, the pull-back distance, and the center of the pull-back circle are related by

\[
t_{\theta_A} = \frac{|a - a'|}{2} \cdot \tan \theta_A
\]

(1)

\[
c_{\theta_A} = c_{t_{\theta_A}} = \frac{a + a'}{2} + i\frac{(a - a')(|a - a'|)}{4} \cdot \tan \theta_A = \frac{a + a'}{2} + i\frac{(a - a')}{2} t_{\theta_A}
\]

(2)

Equations (1) and (2) must also hold for \(\alpha, \alpha', \theta_{\theta_A'}\) and \(t_{\theta_A'}\). The four points lie on the same circle precisely when \(c_{\theta_A} = c_{\theta_{A'}}\). This happens if and only if there is a real \(t_{\theta_{A'}}\) with

\[
\frac{a + a'}{2} + i\frac{(a - a')(|a - a'|)}{4} \cdot \tan \theta_A = \frac{\alpha + \alpha'}{2} + i\frac{(\alpha - \alpha')}{2} t_{\theta_{A'}}
\]

\[\square\]

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