A Short Note on Zero-error Computation for Algebraic Numbers by IPSLQ *

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1 Introduction

Being exact, symbolic computation is usually inefficient due to the well-known problem of intermediate swell. Being efficient, numerical computation only gives approximate results. For both efficiency and reliability, an idea that obtains exact results by using approximate computing has been of interest. We call such methods **zero-error computation**.

The output of zero-error algorithms are exact expressions, but the intermediate process (partially) uses appropriate numerical methods, so that they are symbolic-numeric. At the end of these algorithms the errors become zero (i.e., exact results) by certain gap theorem from background knowledge, though errors appear at (almost) every step in these algorithms. For instance, applying the algorithm presented in [ZF07], one can recover the exact value of a rational number from its approximation; more generally, the algorithm in [KLL88] is to reconstruct algebraic numbers. In many zero-error computation algorithms, such as reconstruction algebraic number [KLL88], polynomial factorization [H02, WCF14], etc., the problem to be solved is finally converted to finding an integer relation.

The PSLQ algorithm is one of the most popular algorithm for finding non-trivial integer relations for several real numbers. Although it has been theoretically proved that the PSLQ algorithm [FBA99] is to some extent equivalent to the HJLS algorithm [HJLS89], under the exact real arithmetic computational model (see, e.g., [Mei01]), the PSLQ algorithm seems more practical. The problem of finding the minimal polynomial from an approximation \( \alpha \) of a degree algebraic number \( \alpha \), equivalent to finding an integer relation for the vector \( (1, \alpha, \ldots, \alpha^d) \), was first solved in [KLL88] by using the celebrated LLL algorithm [LLL82]. This routine has been recently improved in [HN12]. Naturally, the PSLQ algorithm is applicable to the algebraic number reconstruction problem as well [QFCZ12].

*Key words: zero-error computation, algebraic number, integer relation. This is a modified version of [FCW13].
Given an approximation to $\alpha$, a degree bound $d$ and an upper bound $M$ on its height, if we do not know the exact degree of the algebraic number in advance, then no matter whether one uses PSLQ or LLL, one has to search an integer relation for the vector $(1, \alpha, \ldots, \alpha^i)$ from $i = 2, 3, \ldots$ until the degree bound $d$. Hence, if the complexity of an algorithm for finding an integer relation is $O(P(n, M))$ for an $n$-dimensional vector, then the complexity of the minimal polynomial algorithm, based on the integer relation finding algorithm, is $O(d \cdot P(d, M))$. Our main contribution in the present work is to give the incremental PSLQ algorithm (IPSLQ), based on which, the corresponding algebraic number reconstruction algorithm has the complexity only $O(P(d, M))$, even though we do not know the exact degree of the algebraic number.

2 The Incremental PSLQ Algorithm

The main difference between IPSLQ (Algorithm 1) and PSLQ is the following: PSLQ considers $x_1, \ldots, x_n$; IPSLQ considers $x_i, \ldots, x_n$, if the vector $(x_i, \ldots, x_n)$ has no relation with 2-norm less than $M$ (see Step 2(a)v) then add $x_{i-1}$ to the left.

When reconstructing the minimal polynomial, we apply IPSLQ with input as $(x_1, \ldots, x_n) = (\alpha^{n-1}, \ldots, \alpha, 1)$. Now, if the vector $(x_i, \ldots, x_n)$ has no relation with 2-norm less than $M$, the results of previous iterations can be reused in the next iteration. However, the traditional methods can not use the information produced by the previous iterations. Therefore, the complexity of IPSLQ for minimal polynomial without knowing the degree is only $O(P(d, M))$, which is the same as PSLQ for minimal polynomial with knowing the degree.

Algorithm 1 (IPSLQ).

Input: A vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ and a positive number $M$.
Output: Either return an integer relation for $x$, or return "$\lambda_1(x) > M$".

1. Compute $H_x \in \mathbb{R}^{n \times (n-1)}$. Set $H := H_x$, $A := I_n$ and $B := I_n$. Size-reduce $H$ and update $A$ and $B$.
2. For $k$ from $n-1$ to 1 do
   (a) While $h_{n-1, n-1} \neq 0$ do
      i. Choose $r$ such that $\gamma^r |h_{r, r}| = \max_{j \in \{k, \ldots, n-1\}} \{\gamma^j |h_{j, j}|\}$.
      ii. Swap the $r$-th and the $(r + 1)$-th rows of $H$ and update $A$ and $B$.
      iii. If $r < n - 1$ then update $H$ to L-factor of $H$.
      iv. Size-reduce $H$ and update $A$ and $B$.
      v. If $\max_{j \in \{k, \ldots, n-1\}} |h_{j, j}| < 1/M$ then do the following: If $k > 1$ then go to Step 2; Else return "$\lambda_1(x) > M$".
   (b) Return the last column of $B$. 

2
3 Experiments

The following experiments are preliminary and to compare the performance between traditional PSLQ and IPSLQ for minimal polynomial reconstruction.

Consider approximations of \( \alpha = 3^{1/s} + 2^{1/t} \) with 500 decimal digits. Running these experiments in Maple 15 with \texttt{Digits :=500} gives a preliminary experimental results in Table 1. Note that here \texttt{Digits :=500} may not be necessary for many examples (see [QFCZ12] for the a detailed error control). In Table 1, the input degree bound and height bound in these tests are \( d \) and \( M + 1 \); the exact degree and height of \( \alpha \) are \( d - 1 \) and \( M \), respectively. All these experimental results are obtained by using a Windows 7 (32 bits mode) PC with AMD Athlon II X4 645 processor (3.10 GHz) and 4 GB memory.

| No. | s | t | d | M     | \( T_{\text{IPSLQ}} \) | \( T_{\text{PSLQ}} \) | \( \frac{T_{\text{PSLQ}}}{T_{\text{IPSLQ}}} \) |
|-----|---|---|---|-------|-----------------|-----------------|----------------|
| 1   | 2 | 2 | 5 | 10    | 0.08            | 0.16            | 2.00           |
| 2   | 2 | 3 | 7 | 36    | 0.16            | 0.64            | 4.00           |
| 3   | 3 | 3 | 10| 125   | 0.89            | 5.34            | 6.00           |
| 4   | 3 | 4 | 13| 540   | 3.14            | 21.34           | 6.79           |
| 5   | 2 | 7 | 15| 5103  | 6.91            | 45.91           | 6.64           |
| 6   | 3 | 6 | 19| 10278 | 23.37           | 144.11          | 6.17           |
| 7   | 4 | 5 | 21| 11160 | 32.73           | 249.54          | 7.62           |
| 8   | 5 | 5 | 26| 57500 | 78.95           | 838.99          | 10.63          |
| 9   | 5 | 6 | 31| 538380| 186.28          | 2089.87         | 11.22          |
| 10  | 6 | 6 | 37| 4281690| 421.94          | 4313.99         | 10.22          |

Table 1: The comparison of IPSLQ and PSLQ for minimal polynomial

Note that there exists a built-in function \texttt{IntegerRelations:-PSLQ} in Maple 15, but for the comparison in Table 1, we implement the PSLQ algorithm by ourselves. The reasons we do not use the built-in function is that there does not exist a height parameter in the built-in function. This may cause that the built-in function will go on the iterations even if the height has been greater than \( M \).

In our implementations of PSLQ and IPSLQ, the same function uses the same technique for fairness. According to Table 1, the IPSLQ algorithm is obviously faster than the PSLQ algorithm. Meanwhile the ratio between \( T_{\text{PSLQ}} \) and \( T_{\text{IPSLQ}} \) seems to get larger and larger with increasing \( d \).

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