UNIFORM APPROXIMATION OF POISSON INTEGRALS OF FUNCTIONS FROM THE CLASS $H_\omega$ BY DE LA VALLEE POUSSIN SUMS

A.S. SERDYUK and Ie.Yu. OVSII

Abstract

We obtain asymptotic equalities for least upper bounds of deviations in the uniform metric of de la Vallee Poussin sums on the sets $C^q_\beta H_\omega$ of Poisson integrals of functions from the class $H_\omega$ generated by convex upwards moduli of continuity $\omega(t)$ which satisfy the condition $\omega(t)/t \to \infty$ as $t \to 0$. As an implication, a solution of the Kolmogorov–Nikol’skii problem for de la Vallee Poussin sums on the sets of Poisson integrals of functions belonging to Lipschitz classes $H^\alpha$, $0 < \alpha < 1$, is obtained.

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1 Introduction

Let $L$ be the space of $2\pi$-periodic summable functions $f(t)$ with the norm $\|f\|_L = \int_{-\pi}^{\pi} |f(t)| dt$, let $L_\infty$ be the space of $2\pi$-periodic functions $f(t)$ with the norm $\|f\|_{L_\infty} = \text{ess sup}_t |f(t)|$, and let $C$ be the space of continuous $2\pi$-periodic functions $f(t)$ in which the norm is defined by the formula $\|f\|_C = \max_t |f(t)|$.

Further, let $C^q_\beta \mathcal{R}$ be the set of the Poisson integrals of functions $\varphi$ from $\mathcal{R} \subset L$, i.e., functions $f$ of the form

$$f(x) = A_0 + \frac{1}{\pi} \int_0^{2\pi} \varphi(x + t)P_{q,\beta}(t) dt, \quad A_0 \in \mathbb{R}, \quad x \in \mathbb{R}, \quad \varphi \in \mathcal{R},$$

(1)

where

$$P_{q,\beta}(t) = \sum_{k=1}^{\infty} q^k \cos \left( kt + \frac{\beta \pi}{2} \right), \quad q \in (0, 1), \quad \beta \in \mathbb{R},$$

is the Poisson kernel with parameters $q$ and $\beta$. If $\mathcal{R} = U_\infty$, where

$$U_\infty = \{ \varphi \in L_\infty : \|\varphi\|_{L_\infty} \leq 1 \},$$

then the classes $C^q_\beta \mathcal{R}$ will be denoted by $C^q_\beta U_\infty$, and if $\mathcal{R} = H_\omega$, where

$$H_\omega = \{ \varphi \in C : |\varphi(t') - \varphi(t'')| \leq \omega(|t' - t''|) \quad \forall t', t'' \in \mathbb{R},$$

and $\omega(t)$ is an arbitrary modulus of continuity, then $C^q_\beta \mathcal{R}$ will be denoted by $C^q_\beta H_\omega$. 

Denoting by $S_k(f; x)$ the $k$th partial sum of the Fourier series of the summable function $f$, we associate each function $f \in C_\beta^q H_\omega$ with the trigonometric polynomial of the form

$$V_{n,p}(f; x) = \frac{1}{p} \sum_{k=n-p}^{n-1} S_k(f; x), \quad p, n \in \mathbb{N}, \quad p \leq n. \quad (2)$$

The sums $V_{n,p}(f; x)$ appeared in [17] and are called the de la Vallée Poussin sums with parameters $n$ and $p$.

The purpose of the present work is to solve the Kolmogorov–Nikol’skii problem for de la Vallée Poussin sums, which consists of obtaining an asymptotic equality as $n - p \to \infty$ of the quantity

$$\mathcal{E}(\mathfrak{R}; V_{n,p}) = \sup_{f \in \mathfrak{R}} \|f(\cdot) - V_{n,p}(f; \cdot)\|_C, \quad (3)$$

where $\mathfrak{R} = C_\beta^q H_\omega$, $q \in (0, 1)$, $\beta \in \mathbb{R}$ and $\omega(t)$ is a given convex upwards modulus of continuity. Since for $p = 1$ $V_{n,p}(f; x) = V_{n,1}(f; x) = S_{n-1}(f; x)$, the quantity

$$\mathcal{E}(\mathfrak{R}; S_{n-1}) = \sup_{f \in \mathfrak{R}} \|f(\cdot) - S_{n-1}(f; \cdot)\|_C$$

is the special case of (3).

The problem of obtaining asymptotic equalities for the quantities of the form

$$\sup_{f \in \mathfrak{R}} \|f(\cdot) - V_{n,p}(f; \cdot)\|_X,$$

has a rich history in various function classes $\mathfrak{R}$ and metrics $X \subset L$ and connected with the names of A.N. Kolmogorov [2], S.M. Nikol’skii [3], A.F. Timan [16], S.B. Stechkin [14], A.V. Efimov [1], S.A. Telyakovskii [15], A.I. Stepanets [10], V.I. Rukasov [5] and many others. See [4], [7], [8], [9], [11] and [13] for more details on the history of this problem.

S.M. Nikol’skii [3] proved that, if $n \to \infty$ then

$$\mathcal{E}(C_\beta^q; S_{n-1}) = q^n \left( \frac{8}{\pi^2} K(q) + O(1)n^{-1} \right), \quad \beta \in \mathbb{R}, \quad (4)$$

where

$$K(q) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - q^2 \sin^2 t}}, \quad q \in (0, 1)$$

is the complete elliptic integral of the 1st kind. In [14] S.B. Stechkin improved the estimate of the remainder term in (4) by showing that

$$\mathcal{E}(C_\beta^q; S_{n-1}) = q^n \left( \frac{8}{\pi^2} K(q) + O(1) \frac{q}{(1 - q)n} \right), \quad \beta \in \mathbb{R}, \quad (5)$$

where $O(1)$ is a quantity uniformly bounded in $n$, $q$ and $\beta$.

In studying approximate properties of the Fourier sums $S_{n-1}(f; x)$ on the classes $C_\beta^q H_\omega$, A.I. Stepanets [10] proved that for any $q \in (0, 1)$, $\beta \in \mathbb{R}$ and arbitrary modulus of continuity $\omega(t)$ the equality

$$\mathcal{E}(C_\beta^q H_\omega; S_{n-1}) = q^n \left( \frac{4}{\pi^2} K(q) e_n(\omega) + O(1) \frac{\omega(1/n)}{(1 - q)^2n} \right), \quad (6)$$
holds as $n \to \infty$, where
\[
e_n(\omega) := \theta_\omega \int_0^{\pi/2} \omega \left( \frac{2t}{n} \right) \sin t \, dt,
\]
(7)
\[
\theta_\omega \in [1/2, 1], \quad (\theta_\omega = 1 \text{ if } \omega(t) \text{ is a convex upwards modulus of continuity}) \text{ and } O(1) \text{ is the same as in (5)}.
\]
Reasoning as in [10], V.I. Rukasov and S.O. Chaichenko [4] showed that if $q \in (0, 1)$, $\beta \in \mathbb{R}$, $n, p \in \mathbb{N}$, $p \leq n$ and $\omega(t)$ is an arbitrary modulus of continuity, then as $n \to \infty$,
\[
E(C_q^{\beta}H; V_{n,p}) = \frac{2q^{n-p+1}}{\pi(1-q^2)p} e_{n-p+1}(\omega) + 
+ O(1) \frac{q^{n-p+1}}{p} \omega \left( \frac{1}{n-p+1} \right) \left( \frac{q^p}{1-q^2} + \frac{1}{(1-q)^{\beta(n-p+1)}} \right),
\]
(8)
where $O(1)$ is a quantity uniformly bounded in $n, p, q$ and $\beta$.

For the classes $C_q^{\beta}H$ there are well-known estimates of the best uniform approximations by trigonometric polynomials of order not more than $\leq n-1$ (see, for example, [12, p. 509]):
\[
E_n(C_q^{\beta}H) = \sup_{f \in C_q^{\beta}H} \inf_{t_{n-1} \in t_{n-1}} \| f(\cdot) - t_{n-1}(\cdot) \|_C \asymp q^n \omega(1/n)
\]
(the notation $\alpha(n) \asymp \beta(n)$ as $n \to \infty$ means that there exist $K_1, K_2 > 0$ such that $K_1 \alpha(n) \leq \beta(n) \leq K_2 \alpha(n)$). It’s easy to see from (8) that if $n \to \infty$ and the values of the parameter $p$ are bounded, then the de la Vallée Poussin sums realize the order of the best uniform approximation on the classes $C_q^{\beta}H$. But in that case estimate (8) isn’t an asymptotic equality, taking the form
\[
E(C_q^{\beta}H; V_{n,p}) = O(1) \frac{q^{n-p+1}}{p(1-q^2)} \omega \left( \frac{1}{n-p+1} \right).
\]
Thus for $n-p \to \infty$ and bounded values of the parameter $p$ the question of the asymptotic behavior of a principal term of the quantities $E(C_q^{\beta}H; V_{n,p})$ was still unknown.

The proofs of (6) and (8) are based on the well-known Korneichuk–Stechkin lemma (see, e.g., [10]). In the present work we use a somewhat different way which, as will be seen later, proves itself in some important cases.

## 2 Main result

The main result of the paper is the next theorem.

**Theorem 1.** Let $q \in (0, 1)$, $\beta \in \mathbb{R}$, $n, p \in \mathbb{N}$, $p < n$ and let $\omega(t)$ be a convex upwards modulus of continuity. Then as $n-p \to \infty$:
\[
E(C_q^{\beta}H; V_{n,p}) = \frac{q^{n-p+1}}{p} \left( \frac{K_{p,q}}{\pi^2} e_{n-p+1}(\omega) + O(1) \frac{\omega(\pi)}{(1-q)^{\beta(p)}(n-p+1)} \right),
\]
(9)
where
\[
K_{p,q} := \int_0^{2\pi} \frac{\sqrt{1 - 2q^p \cos pt + q^{2p}}}{1 - 2q \cos t + q^2} \, dt, \tag{10}
\]

\[
e_{n-p+1}(\omega) = \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t \, dt,
\]

\[
\delta(p) := \begin{cases} 
2, & p = 1, \\
3, & p = 2, 3, \ldots,
\end{cases} \tag{11}
\]

and \(O(1)\) is a quantity uniformly bounded in \(n, p, q, \omega\) and \(\beta\).

Since
\[
\frac{2}{\pi} \omega\left(\frac{\pi}{k}\right) \leq e_k(\omega) \leq \omega\left(\frac{\pi}{k}\right), \quad k \in \mathbb{N},
\]

formula (9) is an asymptotic equality if and only if
\[
\lim_{t \to 0} \frac{\omega(t)}{t} = \infty. \tag{12}
\]

An example of moduli of continuity \(\omega(t)\) which satisfy (12), are the functions:
\[
\omega(t) = t^\alpha, \quad \alpha \in (0, 1),
\]
\[
\omega(t) = \ln^\alpha(t+1), \quad \alpha \in (0, 1),
\]

\[
\omega(t) = \begin{cases} 
0, & t = 0, \\
t^\alpha \ln\left(\frac{1}{t}\right), & t \in (0, e^{-1/\alpha}], \quad \alpha \in (0, 1], \\
\frac{1}{ae^\alpha}, & t \in [e^{-1/\alpha}, \infty),
\end{cases}
\]

\[
\omega(t) = \begin{cases} 
0, & t = 0, \\
\ln^{-\alpha}\left(\frac{1}{t}\right), & t \in (0, e^{-(1+\alpha)}], \quad \alpha \in (0, 1], \\
\frac{1}{(1+\alpha)^\alpha}, & t \in [e^{-(1+\alpha)}, \infty).
\end{cases}
\]

Putting \(\omega(t) = t^\alpha, \quad \alpha \in (0, 1)\), in the hypothesis of Theorem 1 and taking into account that in this case the class \(H_{\omega}\) becomes the well-known Hölder class \(H^\alpha\), we obtain the next statement.

**Theorem 2.** Let \(q \in (0, 1)\), \(\beta \in \mathbb{R}\), \(n, p \in \mathbb{N}\), \(p < n\) and \(\alpha \in (0, 1)\). Then the following asymptotic equality
\[
\mathcal{E}(C^q_{p}\alpha; V_{n,p}) =
\]

\[
= \frac{q^{n-p+1}}{p(n-p+1)\alpha} \left(2\alpha \frac{K_{p,q}}{\pi^2} \int_0^{\pi/2} t^\alpha \sin t \, dt + \frac{O(1)}{1-\delta(p)(n-p+1)^{1-\alpha}}\right), \tag{13}
\]

is true as \(n-p \to \infty\), where \(K_{p,q}\) and \(\delta(p)\) are defined by (10) and (11) respectively, and \(O(1)\) is a quantity uniformly bounded in \(n, p, q, \alpha\) and \(\beta\).

We note that asymptotic equality (13) for de la Vallée Poussin sums \(V_{n,p}\) with bounded \(p \in \mathbb{N}\setminus\{1\}\) is obtained for the first time.
Asymptotic behavior of the constant $K_{p,q}$ as $p \to \infty$ can be judged from the estimate
\[
K_{p,q} = \frac{2\pi}{1 - q^2} \left(1 + O(1)q^p\right),
\] (14)
uniform in $p$ and $q$ (see [7, p. 130]). The substitution of (14) into equality (9) enables us to obtain the asymptotic estimate of the quantity $\mathcal{E}(C_\beta^q H_\omega; V_{n,p})$ as $n - p \to \infty$ in which the principal term coincides with the first summand in the right-hand side of (8).

In the general case $(p = 1, 2, \ldots, n)$, as follows from [6, p. 215], the values of the constant $K_{p,q}$ can be expressed through the values of the complete elliptic integral of the 1st kind $K(q^p)$ by means of the following equation:
\[
K_{p,q} = 4 \frac{1 - q^{2p}}{1 - q^2} K(q^p), \quad p \in \mathbb{N}, \quad q \in (0, 1).
\] (15)

Using (15), formulas (9) and (13) have the next representations
\[
\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \frac{q^{n-p+1}}{p} \left(\frac{4}{\pi^2} \frac{1 - q^{2p}}{1 - q^2} K(q^p) e_{n-p+1}(\omega) + \frac{O(1)\omega(\pi)}{(1 - q^{\delta(p)}) (n - p + 1)}\right),
\]
(9')
\[
\mathcal{E}(C_\beta^q H^\alpha; V_{n,p}) = \frac{q^{n-p+1}}{p(n - p + 1)^\alpha} \left(\frac{\alpha + 2}{\pi^2} \frac{1 - q^{2p}}{1 - q^2} K(q^p) \int_0^{\pi/2} t^\alpha \sin t \, dt + \frac{O(1)}{(1 - q^{\delta(p)}) (n - p + 1)^{1-\alpha}}\right).
\]
(13')

In addition to Theorem 1 we present the sequent result.

**Theorem 3.** Let $q \in (0, 1)$, $\beta \in \mathbb{R}$, $n, p \in \mathbb{N}$, $p < n$ and let $\omega(t)$ be a convex upwards modulus of continuity. Then as $n - p \to \infty$:
\[
\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \frac{q^{n-p+1}}{p} \left(\frac{4J_{q,p}}{\pi^2} \frac{1 - q^p}{1 - q} e_{n-p+1}(\omega) + \frac{O(1)}{(1 - q^{\delta(p)}) (n - p + 1)} \omega \left(\frac{1}{n - p + 1}\right)\right),
\]
(16)
where the two-sided estimate
\[
\frac{1 + q^p}{1 + q} K(q^p) \leq J_{q,p} \leq K(q),
\] (17)
holds for $J_{q,p} = J_{q,p}(n, \omega)$, and $e_{n-p+1}(\omega)$, $\delta(p)$ and $O(1)$ have the same meaning as in Theorem 1.

For $p = 1$, as appears from (17), $J_{q,1} = J_{q,1}(n, \omega) = K(q)$. In this case relation (16) becomes the asymptotic equality which is the special case of (6).
3 Proof of main result

The proof of Theorem 1 consists of three steps.

Step 1. We single out a principal value of the quantity $E(C_q^\beta H_\omega; V_{n,p})$.

To this end we consider first the deviation

$$\rho_{n,p}(f; x) := f(x) - V_{n,p}(f; x), \quad f \in C_q^\beta H_\omega$$

(18)

and on the basis of equalities (1) and (2) we write the representation

$$\rho_{n,p}(f; x) = \frac{1}{\pi p} \int_0^{2\pi} \varphi(x + t) \sum_{k=n-p}^{n-1} P_{q,\beta,k+1}(t) \, dt,$$

(19)

in which $\varphi \in H_\omega$, and

$$P_{q,\beta,m}(t) = \sum_{j=m}^{\infty} q^j \cos \left( j t + \frac{\beta \pi}{2} \right), \quad m \in \mathbb{N}, \; q \in (0, 1), \; \beta \in \mathbb{R}.$$  

Since

$$\int_0^{2\pi} \sum_{k=n-p}^{n-1} P_{q,\beta,k+1}(t) \, dt = 0$$

and, according to [10, p. 118],

$$P_{q,\beta,m}(t) = q^m Z_q(t) \cos \left( m t + \theta_q(t) + \frac{\beta \pi}{2} \right),$$

where

$$Z_q(t) := \frac{1}{\sqrt{1 - 2q \cos t + q^2}},$$

$$\theta_q(t) := \arctg \frac{q \sin t}{1 - q \cos t},$$

it follows from (19) that

$$\rho_{n,p}(f; x) = \frac{1}{\pi p} \int_0^{2\pi} (\varphi(x + t) - \varphi(x)) Z_q(t) \sum_{k=n-p+1}^{n} q^k \cos \left( k t + \theta_q(t) + \frac{\beta \pi}{2} \right) \, dt.$$  

(20)

By virtue of formula (17) in [7, p. 126] the equality

$$\sum_{k=n-p+1}^{n} q^k \cos \left( k t + \theta_q(t) + \frac{\beta \pi}{2} \right) =$$

$$= Z_q(t) q^{n-p+1} \left[ \cos \left( (n - p + 1) t + \frac{\beta \pi}{2} \right) G_{p,q}(t) - \sin \left( (n - p + 1) t + \frac{\beta \pi}{2} \right) H_{p,q}(t) \right]$$

(21)
holds, where
\[ G_{p,q}(t) = \cos 2\theta_q(t) - q^p \cos(pt + 2\theta_q(t)), \]
\[ H_{p,q}(t) = \sin 2\theta_q(t) - q^p \sin(pt + 2\theta_q(t)). \]

Representing the functions \( G_{p,q}(t) \) and \( H_{p,q}(t) \) in the form
\[ G_{p,q}(t) = \frac{\cos(2\theta_q(t) - \theta_{qp}(pt))}{Z_{qp}(pt)}, \quad H_{p,q}(t) = \frac{\sin(2\theta_q(t) - \theta_{qp}(pt))}{Z_{qp}(pt)}, \]
where
\[ Z_{qp}(t) := \frac{1}{\sqrt{1 - 2q^p \cos t + q^{2p}}}, \quad \theta_{qp}(t) := \arctg \frac{q^p \sin t}{1 - q^p \cos t}, \]
from (21) we get
\[
\sum_{k=n-p+1}^{n} q^k \cos \left( kt + \theta_q(t) + \frac{\beta \pi}{2} \right) =
\]
\[
= q^{n-p+1} \frac{Z_q(t)}{Z_{qp}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{qp}(pt) + \frac{\beta \pi}{2} \right). \tag{22}
\]

Relations (20) and (22) imply the following equality
\[
\rho_{n,p}(f; x) =
\]
\[
= \frac{q^{n-p+1}}{\pi p} \int_0^{2\pi} (\varphi(x+t) - \varphi(x)) \frac{Z_q^2(t)}{Z_{qp}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{qp}(pt) + \frac{\beta \pi}{2} \right) dt. \tag{23}
\]

The right-hand side of (23) is 4-periodic in \( \beta \). Therefore, it will be assumed below that \( \beta \in [0,4) \).

Since for any \( \varphi \in H_\omega \) the function \( \varphi_1(u) = \varphi(u+h), \ h \in \mathbb{R}, \) also belongs to \( H_\omega \), then from (23) we have
\[
\mathcal{E}(C^p_\beta H_\omega; V_{n,p}) =
\]
\[
= \frac{q^{n-p+1}}{\pi p} \sup_{\varphi \in H_\omega} \left| \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^2(t)}{Z_{qp}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{qp}(pt) + \frac{\beta \pi}{2} \right) dt \right|, \tag{24}
\]
where
\[ \Delta(\varphi, t) := \varphi(t) - \varphi(0). \]

Denote by \( \mathcal{J}_{n,p,q,\beta}(\varphi) \) the integral in the right-hand side of (24), that is
\[
\mathcal{J}_{n,p,q,\beta}(\varphi) = \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^2(t)}{Z_{qp}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{qp}(pt) + \frac{\beta \pi}{2} \right) dt. \tag{25}
\]

Our further goal is to find an asymptotic estimation of the integral \( \mathcal{J}_{n,p,q,\beta}(\varphi) \) as \( n-p \to \infty \). For this reason, without loss of generality, we shall assume that the numbers \( n \) and \( p \) have been chosen such that
\[
n - p \geq \frac{6}{1-q}. \tag{26}
\]
First we show that

\[ J_{n,p,q,\beta}(\varphi) = \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^4(t)}{Z_q^p(pt)Z_{q,n,p}^2(t)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{qp}(pt) + \frac{\beta\pi}{2} \right) dt + O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)}, \]

(27)

where

\[ Z_{q,n,p}(t) := \frac{Z_q(t)}{\sqrt{n-p+1+2q(\cos t - q)Z_q^2(t)-pq^p(\cos pt - q^p)Z_{qp}(pt)}} \]

\[ \alpha_q := \left\lfloor \frac{3q}{1-q} \right\rfloor + 2, \]

(29)

and \( [m] \) denotes the integral part of the number \( m \). By virtue of (26) and the obvious inequality

\[ \left| pq^p \cos pt - q^p Z_{qp}(pt) \right| \leq \left| \frac{pq^p}{1-q} \right| \leq \frac{q}{1-q}, \quad q \in (0, 1), \quad p \in \mathbb{N}, \quad t \in \mathbb{R}, \]

(30)

the quantity under the radical sign in (28) is always positive. Set

\[ R_{n,p,q,\beta}(\varphi) := J_{n,p,q,\beta}(\varphi) - \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^4(t)}{Z_q^p(pt)Z_{q,n,p}^2(t)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{qp}(pt) + \frac{\beta\pi}{2} \right) dt = \]

(31)

To prove equality (27) it suffices to show that the estimate

\[ R_{n,p,q,\beta}(\varphi) = O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)} \]

(32)

holds, where the quantity \( \delta(p) \) is defined by (11). Indeed, considering the inequality \( |\Delta(\varphi, t)| \leq \omega(|t|) \), from (31) we get

\[ |R_{n,p,q,\beta}(\varphi)| \leq \omega(2\pi) \int_0^{2\pi} \frac{Z_q^2(t)}{Z_{qp}(pt)} \left| 1 - \frac{Z_q^2(t)}{Z_{q,n,p}(t)} \right| dt. \]

(33)

After performing elementary transformations and taking into account (30), we find

\[ \left| 1 - \frac{Z_q^2(t)}{Z_{q,n,p}^2(t)} \right| = \frac{2q(\cos t - q)Z_q^2(t) - pq^p(\cos pt - q^p)Z_{qp}^2(pt) + 1 - \alpha_q}{n-p+\alpha_q} \]

\[ < \frac{1}{n-p+\alpha_q} \left( 2q|\cos t - q|Z_q^2(t) + pq^p|\cos pt - q^p|Z_{qp}^2(pt) + \frac{3q}{1-q} + 1 \right) \]

\[ \leq \frac{1}{n-p+\alpha_q} \left( \frac{2q}{1-q} + \frac{q}{1-q} + \frac{3q}{1-q} + 1 \right) = \frac{O(1)}{(1-q)(n-p+1)}. \]

(34)
From the estimate
\[
\frac{Z_q^2(t)}{Z_{qp}(pt)} = \frac{1}{\sqrt{1 - 2q \cos t + q^2}} \leq \frac{1}{1 - q}, \quad t \in \mathbb{R}, \quad p = 1,
\]
and (14) it follows that
\[
\frac{Z_q^2(t)}{Z_{qp}(pt)} = \frac{\sqrt{1 - 2q^p \cos pt + q^{2p}}}{1 - 2q \cos t + q^2} = \frac{O(1)}{(1 - q)^{\delta(p) - 1}}, \quad t \in \mathbb{R}, \quad p \in \mathbb{N}. \quad (35)
\]
Comparing (33)–(35), we obtain (32), and with it estimate (27).

Consider the function
\[
y_1(t) := t + \frac{1}{n - p + \alpha_q}
\left(2\theta_q(t) - \theta_{qp}(pt) + (1 - \alpha_q)t + \frac{\beta \pi}{2}\right). \quad (36)
\]
On the strength of the fact that \((\theta_{qp}(pt))' = pq^p(\cos pt - q^p)Z_{qp}^2(pt)\), the equality
\[
y_1'(t) = 1 + \frac{1}{n - p + \alpha_q}
\left(2q(\cos t - q)Z_q^2(t) - pq^p(\cos pt - q^p)Z_{qp}^2(pt) + 1 - \alpha_q\right) =
\]
\[
= \frac{Z_q^2(t)}{Z_{q,n,p}(t)}, \quad (37)
\]
holds, where \(Z_{q,n,p}(t)\) is defined by (28). In view of (30)
\[
- \frac{5q + 1}{1 - q} \leq 2q(\cos t - q)Z_q^2(t) - pq^p(\cos pt - q^p)Z_{qp}^2(pt) + 1 - \alpha_q < 0. \quad (38)
\]
Thus, we see from (26) and (37) that the next two-sided estimate
\[
\frac{1}{3} < y_1'(t) < 1 \quad (39)
\]
holds. So \(y_1\) has the inverse function \(y(t) = y_1^{-1}(t)\), whose derivative \(y'\) by (37) satisfies the relation
\[
y'(t) = \frac{1}{y_1'(y(t))} = \frac{Z_{q,n,p}(y(t))}{Z_q^2(y(t))}. \quad (40)
\]
Making the change of variables \(t = y(\tau)\) in (27), we obtain by (37) the relation
\[
\mathcal{J}_{n,p,q,\beta}(\varphi) =
\]
\[
= \int_{y_1(0)}^{y_1(2\pi)} \Delta(\varphi, y(\tau)) \frac{Z_q^4(y(\tau))}{Z_{qp}(py(\tau))Z_{q,n,p}(y(\tau))} \frac{Z_{q,n,p}^2(y(\tau))}{Z_q^2(y(\tau))} \cos \left((n - p + \alpha_q)\tau\right) d\tau +
\]
\[
+ O(1) \frac{\omega(\pi)}{(1 - q)^{\delta(p)}(n - p + 1)} =
\]
\[
= \int_{y_1(0)}^{y_1(2\pi)} \Delta(\varphi, y(\tau)) \frac{Z_{q,n,p}^2(y(\tau))}{Z_{qp}(py(\tau))} \cos \left((n - p + \alpha_q)\tau\right) d\tau +
\]
\[
+ O(1) \frac{\omega(\pi)}{(1 - q)^{\delta(p)}(n - p + 1)}. \quad (41)
\]
We set
\[ x_k := \frac{k\pi}{n - p + \alpha_q}, \quad \tau_k := x_k + \frac{\pi}{2(n - p + \alpha_q)}, \quad k \in \mathbb{N} \]  
and
\[ l_n(\tau) = \begin{cases} Z_q(y(\tau_k)) - Z_q(y(\tau_{k+1})), & \tau \in [x_k, x_{k+1}], \quad k = 2, 3, \ldots, k_0 - 2, k_0 - 1, \\ 0, & \tau \in [y_1(0), x_2) \cup (x_{k_0}, y_1(2\pi)], \end{cases} \]
where \( k_0 \) is an index such that \( \tau_{k_0} \) is the nearest to the left of \( y_1(2\pi) \) root of the function \( \cos((n - p + \alpha_q)\tau) \). Using this notations, from (41) we get
\[ J_{n,p,q,\beta}(\varphi) = \int_{x_2}^{x_{k_0}} \Delta(\varphi, y(\tau))l_n(\tau)\cos((n - p + \alpha_q)\tau)\,d\tau + R_{n,p,q}^{(1)}(\varphi) + R_{n,p,q}^{(2)}(\varphi) + O(1)\frac{\omega(\pi)}{(1 - q)\delta(p)(n - p + 1)}, \]  
where
\[ R_{n,p,q}^{(1)}(\varphi) := \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau))\left(\frac{Z_q^2(y(\tau))}{Z_{qp}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{qp}(py(\tau_k))}\right)\cos((n - p + \alpha_q)\tau)\,d\tau, \]
\[ R_{n,p,q}^{(2)}(\varphi) := \int_{y_1(0)}^{y_1(2\pi)} \Delta(\varphi, y(\tau))\frac{Z_q^2(y(\tau))}{Z_{qp}(py(\tau))}\cos((n - p + \alpha_q)\tau)\,d\tau. \]
Since \( y(t) \) is an increasing function (see (39), (40)), then
\[ y(x_{k+1}) \leq y(x_{k_0}) < y(y_1(2\pi)) = 2\pi, \quad k = 2, k_0 - 1 \]
and so for \( R_{n,p,q}^{(1)}(\varphi) \) the following trivial estimate
\[ \left|R_{n,p,q}^{(1)}(\varphi)\right| \leq \sum_{k=2}^{k_0-1} \omega(y(x_{k+1})) \int_{x_k}^{x_{k+1}} \left|\frac{Z_q^2(y(\tau))}{Z_{qp}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{qp}(py(\tau_k))}\right|\,d\tau < \omega(2\pi) \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left|\frac{Z_q^2(y(\tau))}{Z_{qp}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{qp}(py(\tau_k))}\right|\,d\tau \]
holds.
We will show that
\[ \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left|\frac{Z_q^2(y(\tau))}{Z_{qp}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{qp}(py(\tau_k))}\right|\,d\tau = \frac{O(1)q}{(1 - q)\delta(p)(n - p + 1)}. \]  
For this purpose, we consider the derivative
\[ \left(\frac{Z_q^2(t)}{Z_{qp}(pt)}\right)' = -2q \sin t \frac{Z_q^2(t)}{Z_{qp}(pt)} + q^p \sin pt Z_{qp}(pt)Z_q^2(t) =: J_{q,p}^{(1)}(t) + J_{q,p}^{(2)}(t) \]  
(46)
and estimate separately the summands of the right-hand side of (46). Taking into account (35) and the estimate
\[ q|\sin t|Z_q^2(t) = \frac{q|\sin t|}{1 - 2q \cos t + q^2} < \sum_{k=1}^{\infty} q^k = \frac{q}{1 - q}, \]  
we obtain for \( J_{q,p}^{(1)}(t) \)
\[ |J_{q,p}^{(1)}(t)| \leq 2q|\sin t|Z_q^2(t)\frac{Z_q^2(t)}{Z_q^p(pt)} = O(1)\frac{q}{(1 - q)^{\delta(p)}}. \]  
If \( p = 1 \), then in view of (47) we have for \( J_{q,p}^{(2)}(t) \)
\[ |J_{q,p}^{(2)}(t)| < \frac{q}{(1 - q)^2}. \]  
For any \( p = 2, 3, \ldots \), one easily shows that
\[ |J_{q,p}^{(2)}(t)| < \frac{q^{p+1}}{(1 - q)^2(1 - q^p)} < \frac{q}{(1 - q)^3}. \]  
Therefore we finally obtain
\[ |J_{q,p}^{(2)}(t)| < \frac{q}{(1 - q)^{\delta(p)}}, \ p \in \mathbb{N}. \]  
Combining (46), (48) and (49), we arrive at the estimate
\[ \left| \left( \frac{Z_q^2(t)}{Z_q^p(pt)} \right)' \right| = O(1)\frac{q}{(1 - q)^{\delta(p)}}, \ t \in [0, 2\pi]. \]  
Since by (39) and (40) \(|y'(t)| < 3\), applying Lagrange’s theorem on finite increments, we find
\[ \left| \frac{Z_q^2(y(\tau)) - Z_q^2(y(\tau_k))}{Z_q^p(py(\tau)) - Z_q^p(py(\tau_k))} \right| = \]
\[ = O(1)\frac{q}{(1 - q)^{\delta(p)}}|y(\tau) - y(\tau_k)| = O(1)\frac{q}{(1 - q)^{\delta(p)}}|\tau - \tau_k| = \]
\[ = O(1)\frac{q}{(1 - q)^{\delta(p)}(n - p + \alpha_q)} = \]
\[ = O(1)\frac{q}{(1 - q)^{\delta(p)}(n - p + 1)}, \ \tau \in [x_k, x_{k+1}], \ k = 2, k_0 - 1. \]  
It results from (50) that
\[ \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left| \frac{Z_q^2(y(\tau)) - Z_q^2(y(\tau_k))}{Z_q^p(py(\tau)) - Z_q^p(py(\tau_k))} \right| d\tau = O(1)\frac{qx_{k_0}}{(1 - q)^{\delta(p)}(n - p + 1)}. \]  
But since
\[ x_{k_0} < y_1(2\pi) = \int_0^{2\pi} y_1'(t) dt + y_1(0) = \]

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\[
\int_0^{2\pi} y_1'(t) \, dt + \frac{\beta \pi}{2(n-p+\alpha_q)} < 2\pi + \frac{\beta \pi}{2} < 4\pi,
\]

(45) follows from (51). Estimates (44) and (45) imply

\[
| R_{n,p,q}^{(1)}(\varphi) | = O(1) \frac{\omega(\pi)q}{(1-q)^{\delta(p)(n-p+1)}}. \tag{52}
\]

Further, considering that

\[
x_2 - y_1(0) \leq \frac{2\pi}{n-p+\alpha_q} < \frac{2\pi}{n-p+1},
\]

\[
y_1(2\pi) - x_{k_0} \leq \tau_{k_0+1} - x_{k_0} = \frac{3\pi}{2(n-p+\alpha_q)} < \frac{3\pi}{2(n-p+1)}
\]

and using (35), we find

\[
| R_{n,p,q}^{(2)}(\varphi) | = O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p) - 1}(n-p+1)}. \tag{55}
\]

From (43), (52) and (55) we obtain

\[
J_{n,p,q,\beta}(\varphi) = \int_{x_{k_0}}^{x_{k_0+1}} \Delta(\varphi, y(\tau)) l_n(\tau) \cos \left( (n-p+\alpha_q)\tau \right) d\tau +
\]

\[
+ O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)(n-p+1)}}, \quad \varphi \in H_\omega, \quad n-p \to \infty, \tag{56}
\]

where \( O(1) \) is quantity uniformly bounded relative to all parameters under consideration.

Comparing (24), (25) and (56) we conclude that

\[
\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \frac{g^{n-p+1}}{\pi p} \left( \sup_{\varphi \in H_\omega} | J_{n,p,q,\beta}(\varphi) | + \frac{O(1)\omega(\pi)}{(1-q)^{\delta(p)(n-p+1)}} \right), \tag{57}
\]

in which

\[
J_{n,p,q,\beta}(\varphi) := \int_{x_{k_0}}^{x_{k_0+1}} \Delta(\varphi, y(\tau)) l_n(\tau) \cos \left( (n-p+\alpha_q)\tau \right) d\tau =
\]

\[
= \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{qp}(py(\tau_k))} \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos \left( (n-p+\alpha_q)\tau \right) d\tau. \tag{58}
\]

Step 2. Using formula (57) we find an upper bound for \( \mathcal{E}(C_\beta^q H_\omega; V_{n,p}) \).

With this goal, dividing each integral

\[
\int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos \left( (n-p+\alpha_q)\tau \right) d\tau, \quad k = 2, k_0 - 1
\]

into two integrals over \((x_k, \tau_k)\) and \((\tau_k, x_{k+1})\), and setting \( z = 2\tau_k - \tau \) in the last integral, we obtain

\[
\left| \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos \left( (n-p+\alpha_q)\tau \right) d\tau \right| =
\]
and consequently

\[
\begin{aligned}
\mid \varphi(y(\tau)) - \varphi(y(2\tau_k - \tau))\mid \\
\leq \omega(y(2\tau_k - \tau) - y(\tau)) \leq \omega(y'(c_k)(2\tau_k - 2\tau)), \\
k = 2, k_0 - 1.
\end{aligned}
\]

We choose \( c_k \in [x_k, x_{k+1}] \) such that

\[
y'(c_k) = \max_{\tau \in [x_k, x_{k+1}]} y'(\tau).
\]

Then for any \( \tau \in [x_k, \tau_k] \)

\[
y(2\tau_k - \tau) - y(\tau) = \int_{\tau}^{2\tau_k - \tau} y'(x) dx \leq y'(c_k)(2\tau_k - 2\tau)
\]
and consequently

\[
\mid \varphi(y(\tau)) - \varphi(y(2\tau_k - \tau))\mid \\
\leq \omega(y(2\tau_k - \tau) - y(\tau)) \leq \omega(y'(c_k)(\tau_k - \tau)), \\
k = 2, k_0 - 1.
\]

From (59), in view of (60), we find that

\[
\begin{aligned}
\left| \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos \left( (n - p + \alpha_q)\tau \right) d\tau \right| \\
\leq \int_{x_k}^{\tau_k} \omega\left(2y'(c_k)(\tau_k - \tau)\right) |\cos \left((n - p + \alpha_q)\tau\right)| d\tau \\
\leq \frac{1}{n - p + \alpha_q} \int_0^{\pi/2} \omega\left(\frac{2y'(c_k)t}{n - p + \alpha_q}\right) \sin t dt < \\
= \frac{1}{n - p + \alpha_q} \int_0^{\pi/2} \omega\left(\frac{2t}{n - p + 1}\right) \sin t dt + \\
+ \frac{O(1)}{n - p + \alpha_q} \max_{t \in (0, \pi/2]} \left| \omega\left(\frac{2y'(c_k)t}{n - p + 1}\right) - \omega\left(\frac{2t}{n - p + 1}\right) \right|, \\
k = 2, k_0 - 1.
\end{aligned}
\]

Because for any convex upwards modulus of continuity \( \omega \)

\[
\omega(b) - \omega(a) \leq \omega(a) \frac{b - a}{a}, \\
0 < a < b,
\]
then taking into consideration that by (39) and (40) \( y'(c_k) > 1 \), we have

\[
\begin{aligned}
\max_{t \in (0, \pi/2]} \left| \omega\left(\frac{2y'(c_k)t}{n - p + 1}\right) - \omega\left(\frac{2t}{n - p + 1}\right) \right| \\
\leq \omega\left(\frac{\pi}{n - p + 1}\right) (y'(c_k) - 1), \\
k = 2, k_0 - 1.
\end{aligned}
\]

By virtue of (40),

\[
y'(c_k) - 1 = \frac{Z_{q,n,p}^2(y(c_k))}{Z_{q,n,p}^2(y(c_k))} - 1 = \frac{a_q - 1 - \lambda_{p,q}(c_k)}{n - p + 1 + \lambda_{p,q}(c_k)},
\]

(64)
where
\[
\lambda_{p,q}(c_k) = 2q(\cos y(c_k) - q)Z_q^2(y(c_k)) - pq^p(\cos py(c_k) - q^p)Z_q^2(py(c_k)).
\]
In view of (30),
\[
|\lambda_{p,q}(c_k)| \leq \frac{3q}{1-q}.
\]
Hence it follows from (64), taking into account (26), that
\[
y'(_k) - 1 \leq \frac{\alpha_q - 1 + \frac{3q}{1-q}}{n + p + 1 - \frac{3q}{1-q}} < \frac{6}{(1-q)(n+p+1)} \leq \frac{12}{(1-q)(n+p+1)},
\]
Comparing (61), (63) and (65), we obtain
\[
\left| \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos(n - p + \alpha_q) \tau \, d\tau \right| \leq \\
\leq \frac{1}{n + p + \alpha_q} \int_0^{\pi/2} \omega \left( \frac{2t}{n - p + 1} \right) \sin t \, dt + \\
+ \frac{O(1)}{(1-q)(n-p+1)} \omega \left( \frac{1}{n - p + 1} \right), \quad k = 2, k_0 - 1.
\]
Applying to each integral in (58) estimate (66), we have
\[
|I_{n,p,q,\beta}(\varphi)| \leq \frac{1}{n + p + \alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_q^p(py(\tau_k))} \int_0^{\pi/2} \omega \left( \frac{2t}{n - p + 1} \right) \sin t \, dt + \\
+ \frac{O(1)}{(1-q)^\delta(p)(n-p+1)} \omega \left( \frac{1}{n - p + 1} \right).
\]
Let us show that
\[
\frac{\pi}{n - p + \alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_q^p(py(\tau_k))} = \int_0^{2\pi} \frac{Z_q^2(t)}{Z_q^p(pt)} \, dt + \frac{O(1)}{(1-q)^\delta(p)(n-p+1)}.
\]
For this, we represent the left-hand side of (68) as
\[
\frac{\pi}{n - p + \alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_q^p(py(\tau_k))} = \int_{y_1(0)}^{y_1(2\pi)} \frac{Z_q^2(y(\tau))}{Z_q^p(py(\tau))} \, d\tau + R_{n,p,q}^{(1)} + R_{n,p,q}^{(2)},
\]
where
\[
R_{n,p,q}^{(1)} := - \left( \int_{y_1(0)}^{y_1(2\pi)} + \int_{x_k}^{y_1(2\pi)} \right) \frac{Z_q^2(y(\tau))}{Z_q^p(py(\tau))} \, d\tau,
\]
\[
R_{n,p,q}^{(2)} := \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left( \frac{Z_q^2(y(\tau_k))}{Z_q^p(py(\tau_k))} - \frac{Z_q^2(y(\tau))}{Z_q^p(py(\tau))} \right) \, d\tau.
\]
By virtue of (35), (53) and (54)

\[ R_{n,p,q}^{(1)} = \frac{O(1)}{(1-q)^{\delta(p)-1}(n-p+1)}; \]  

(70)

and by (45)

\[ R_{n,p,q}^{(2)} = \frac{O(1)q}{(1-q)^{\delta(p)}(n-p+1)}. \]  

(71)

Combining (69)–(71), we can write

\[
\frac{\pi}{n-p+\alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(t_k))}{Z_q(py(t_k))} = \int_{y_1(0)}^{y_1(2\pi)} \frac{Z_q^2(y(\tau))}{Z_q(py(\tau))} d\tau + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} =
\]

\[
= \int_0^{2\pi} \frac{Z_q^2(t)}{Z_q(p\tau)} dt + \int_0^{2\pi} \frac{Z_q^2(t)}{Z_q(p\tau)} (y'_1(t) - 1) dt + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)}. \]  

(72)

But in view of (37) and (38)

\[ |y'_1(t) - 1| < \frac{6}{(1-q)(n-p+1)}. \]

Thus, in consideration of (35), we obtain from (72) equality (68).

Estimates (57), (67) and (68) imply that as \( n-p \to \infty \)

\[
E(C_\beta H_\omega; V_{n,p}) \leq \frac{q^{n-p+1}}{\pi^2 p} K_{p,q} \int_0^{\pi/2} \omega \left( \frac{2t}{n-p+1} \right) \sin t dt +
\]

\[ + O(1) \frac{q^{n-p+1} \omega(\pi)}{(1-q)^{\delta(p)}(n-p+1),} \]  

(73)

where

\[ K_{p,q} = \int_0^{2\pi} \frac{Z_q^2(t)}{Z_q(p\tau)} dt, \]  

(74)

and \( O(1) \) is a quantity uniformly bounded in \( n, p, q, \omega \) and \( \beta \).

Step 3. We now show that (73) is an equality. For this, in view of (57), it is sufficient to show that there exists a function \( \varphi^* \in H_\omega \) such that

\[
I_{n,p,q,\beta}(\varphi^*) = \frac{K_{p,q}}{\pi} \int_0^{\pi/2} \omega \left( \frac{2t}{n-p+1} \right) \sin t dt + \frac{O(1)\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)}, \]  

(75)

where \( I_{n,p,q,\beta}(\varphi^*) \) is defined by (58). To this end, we set

\[
\varphi_i(t) := \begin{cases} 
\frac{1}{2} \omega(2y_1(t) - 2\tau_i), & t \in [y(\tau_i), y(x_{i+1})], \\
\frac{1}{2} \omega(2\tau_{i+1} - 2y_1(t)), & t \in [y(x_{i+1}), y(\tau_{i+1})], \end{cases} \quad i = s, k_0 - 1,
\]

\[
s = \begin{cases} 
2, & \text{if } k_0 \text{ is odd}, \\
3, & \text{if } k_0 \text{ is even},
\end{cases}
\]

\[ \begin{array}{c} 
\tilde{\varphi}(t) := (-1)^{i+1} \varphi_i(t), & t \in [y(\tau_i), y(\tau_{i+1})], \quad i = s, k_0 - 1.
\end{array} \]
Since $\tau_{k_0} \leq y_1(2\pi)$ and by (42), (36) and $\beta \in [0, 4)$, the inequality $\tau_s > y_1(0)$ holds, it follows that $y(\tau_{k_0}) \leq 2\pi$ and $y(\tau_s) > 0$. Consider the function

$$
\varphi^*(t) := \begin{cases} \bar{\varphi}(t), & t \in [y(\tau_s), y(\tau_{k_0})], \\ 0, & t \in [0, 2\pi] \setminus [y(\tau_s), y(\tau_{k_0})], \end{cases} \quad \varphi^*(t) = \varphi^*(t + 2\pi). \tag{76}
$$

We show that, if (26) holds, then $\varphi^* \in H_\omega$. The construction of $\varphi^*$ shows that it suffices to establish the inequality

$$
|\varphi^*(t') - \varphi^*(t'')| \leq \omega(t'' - t'),
$$

where $t' \in [y(x_i), y(\tau_i)]$ and $t'' \in [y(\tau_i), y(x_{i+1})], \ i = s + 1, k_0 - 1$.

In view of the convexity (upwards) of the modulus of continuity

$$
\frac{1}{2} (\omega(t_1) + \omega(t_2)) \leq \omega\left(\frac{t_1 + t_2}{2}\right).
$$

Then, considering that by (39) $y'_1 \in (\frac{1}{4}, 1)$, we get

$$
|\varphi^*(t') - \varphi^*(t'')| = |\bar{\varphi}(t') - \bar{\varphi}(t'')| = \varphi_{i-1}(t') + \varphi_i(t'') =
\frac{1}{2} (\omega(2\tau_i - 2y_1(t')) + \omega(2y_1(t'') - 2\tau_i)) \leq
\omega(y_1(t'') - y_1(t')), \ i = s + 1, k_0 - 1.
$$

Let us now verify that $\varphi^*(t)$ is the desired function, which means that $\varphi^*(t)$ satisfies (75). Since by (76)

$$
\varphi^*(y(\tau)) = \begin{cases} \frac{(-1)^{i+1}}{2} \omega(2\tau - 2\tau_i), & \tau \in [\tau_i, x_{i+1}], \ i = s, k_0 - 1, \\ \frac{(-1)^{i+1}}{2} \omega(2\tau_{i+1} - 2\tau), & \tau \in [x_{i+1}, \tau_{i+1}], \end{cases}
$$

it follows that

$$
\int_{x_k}^{x_{k+1}} \Delta(\varphi^*, y(\tau)) \cos \left( (n - p + \alpha_q) \tau \right) d\tau =
\frac{(-1)^k}{2} \left( \int_{x_k}^{\tau_k} \omega(2\tau - 2\tau_k) \cos \left( (n - p + \alpha_q) \tau \right) d\tau -
\int_{\tau_k}^{x_{k+1}} \omega(2\tau - 2\tau_k) \cos \left( (n - p + \alpha_q) \tau \right) d\tau \right) =
= \int_0^{\pi/2(n-p+\alpha_q)} \omega(2t) \sin \left( (n - p + \alpha_q) \tau \right) d\tau =
= \frac{1}{n - p + \alpha_q} \int_0^{\pi/2} \omega\left( \frac{2t}{n - p + \alpha_q} \right) \sin t dt, \ k = s + 1, k_0 - 1. \tag{77}
$$

By (77) and (35), we obtain

$$
I_{n,p,q,\beta}(\varphi^*) = \sum_{k=2}^{k_0-1} \frac{Z_k^2(y(\tau_k))}{Z_{q^2}(\beta y(\tau_k))} \int_{x_k}^{x_{k+1}} \Delta(\varphi^*, y(\tau)) \cos \left( (n - p + \alpha_q) \tau \right) d\tau =
$$

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$$= \frac{1}{n-p+\alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q,p}(p y(\tau_k))} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+\alpha_q}\right) \sin t \, dt +$$

$$\frac{O(1)}{(n-p+1)(1-q)^{\delta(p)-1}} \omega\left(\frac{1}{n-p+1}\right).$$

From (78), in view of (68) and (74), we find

$$I_{n,p,q,\beta}(\varphi^*) = \frac{K_{p,q}}{\pi} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+\alpha_q}\right) \sin t \, dt +$$

$$\frac{O(1)}{(n-p+1)(1-q)^{\delta(p)-1}} \omega\left(\frac{1}{n-p+1}\right).$$

Based on (62) and (29), we can write

$$\max_{t \in (0,\pi/2)} \left| \omega\left(\frac{2t}{n-p+\alpha_q}\right) - \omega\left(\frac{2t}{n-p+1}\right) \right| =$$

$$= \frac{O(1)}{n-p+1} \frac{\alpha_q-1}{\omega\left(\frac{1}{n-p+1}\right)} = \frac{O(1)}{(n-p+1)(1-q)^{\delta(p)}} \omega\left(\frac{1}{n-p+1}\right).$$

Comparing (79), (80) and taking into account that by (35),

$$K_{p,q} = \frac{O(1)}{(1-q)^{\delta(p)-1}},$$

we arrive at (75).

Combining (57) and (75), we obtain the estimate

$$E(C_{\beta}^q H_\omega; V_{n,p}) \geq \frac{q^{n-p+1}}{\pi^2 p} K_{p,q} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t \, dt +$$

$$+ \frac{q^{n-p+1} \omega(\pi)}{(1-q)^{\delta(p)} p(n-p+1)},$$

in which $O(1)$ is a quantity uniformly bounded in $n, p, q, \omega$ and $\beta$. From (73) and (82) we get asymptotic formula (9). Theorem 1 is proved.

Proof of Theorem 3. Since the sequence $e_k(\omega)$ is monotonically decreasing (see (7)), from (2) and (6) it follows that

$$E(C_{\beta}^q H_\omega; V_{n,p}) \leq \frac{1}{p} \sum_{k=n-p+1}^n E(C_{\beta}^q H_\omega; S_{k-1}) \leq$$

$$\leq \frac{q^{n-p+1}}{p} \left( \frac{4}{\pi^2} \frac{1-q^p}{1-q} K(q) e_{n-p+1}(\omega) + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} \omega\left(\frac{1}{n-p+1}\right) \right).$$

On the other hand, if we analyze the proof of (57), it is easy to see that for any function $\varphi \in H_\omega$ the estimate

$$E(C_{\beta}^q H_\omega; V_{n,p}) \geq \frac{q^{n-p+1}}{\pi p} \left( |I_{n,p,q,\beta}(\varphi)| + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} \|\Delta(\varphi, \cdot)\|_C \right)$$

(84)
holds. For the function $\varphi^*(t)$ defined by (76), we have from (79)–(81) that

\[
I_{n,p,q,\beta}(\varphi^*) = \frac{K_{p,q}}{\pi} \int_{0}^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t \, dt + \frac{O(1)}{(1-q)^{\delta(p)(n-p+1)}} \omega\left(\frac{1}{n-p+1}\right).
\]

Since $\|\Delta(\varphi^*, \cdot)\|_C = \frac{1}{2} \omega\left(\frac{\pi}{n-p+\alpha_q}\right)$, comparing (84) and (85), we obtain

\[
E(C_{\beta}^{q} H_{\omega}; V_{n,p}) \geq \frac{q^{n-p+1}}{p} \left(\frac{K_{p,q}}{\pi^2} e_{n-p+1}(\omega) + \frac{O(1)}{(1-q)^{\delta(p)(n-p+1)}} \omega\left(\frac{1}{n-p+1}\right)\right).
\]

From (83), (86) and (15), relation (16) follows. Theorem 3 is proved. ■

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Contact information: Department of the Theory of Functions, Institute of Mathematics of Ukrainian National Academy of Sciences, 3, Tereshenkivska st., 01601, Kyiv, Ukraine

E-mail: serdyuk@imath.kiev.ua, ievgen.ovsii@gmail.com