Relativistic weak lensing from a fully non-linear cosmological density field

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Abstract. In this paper we examine cosmological weak lensing on non-linear scales and show that there are Newtonian and relativistic contributions and that the latter can also be extracted from standard Newtonian simulations. We use the post-Friedmann formalism, a post-Newtonian type framework for cosmology, to derive the full weak-lensing deflection angle valid on non-linear scales for any metric theory of gravity. We show that the only contributing term that is quadratic in the first order deflection is the expected Born correction and lens-lens coupling term. We use this deflection angle to analyse the vector and tensor contributions to the E- and B- mode cosmic shear power spectra. In our approach, once the gravitational theory has been specified, the metric components are related to the matter content in a well-defined manner. Specifying General Relativity, we write down a complete set of equations for a GR+\Lambda CDM universe for computing all of the possible lensing terms from Newtonian N-body simulations. We illustrate this with the vector potential and show that, in a GR+\Lambda CDM universe, its contribution to the E-mode is negligible with respect to that of the conventional Newtonian scalar potential, even on non-linear scales. Thus, under the standard assumption that Newtonian N-body simulations give a good approximation of the matter dynamics, we show that the standard ray tracing approach gives a good description for a \Lambda CDM cosmology.

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# Introduction

It has been understood since the dawn of General Relativity that one of the consequences of a gravitational theory incorporating the equivalence principle is the bending of light by gravitating masses \[23\]. An important consequence of this light bending is that galaxy images are distorted by gravitational influences along the line of sight. A statistical analysis of the shapes of galaxy images allows this weak gravitational lensing effect, known as cosmic shear, to be observed. Since the first measurements of cosmic shear \[6, 19, 31, 34\], increasingly precise measurements have been made of the cosmic shear power spectra, leading up to the latest CHTLens results \[20\]. Now, weak lensing is becoming an important tool for cosmology, with claims that it can place constraints on the equation of state of dark energy \[2\] and modified gravity models \[3\]. Current and future surveys include the Dark Energy Survey \[1\], LSST \[18\], SKA \[13\] and the ESA satellite mission Euclid \[28\]. These missions will yield a wealth of precise data, including tomographic weak lensing power spectra. However, these power spectra will largely come from non-linear scales in the universe, i.e. scales smaller than around 8-10 h⁻¹ Mpc, where the matter density contrast has gone non-linear.

Broadly speaking, cosmologists study linear and non-linear scales in different ways. On larger, linear scales, fully relativistic perturbative schemes are used where the matter and metric perturbations in the universe are assumed to be small. On smaller scales in the universe, gravity is assumed to be Newtonian and N-body simulations are run to examine the evolution of the density field. Weak-lensing calculations have been carried out both in
the Newtonian regime [7] and with GR perturbation theory (see e.g. [14, 15]). However, there are few studies that both apply on smaller, non-linear scales in cosmology and are fully relativistic, thus justifying the standard ray tracing approach to extracting lensing observables from N-body simulations.

The post-Friedmann formalism, introduced in [25, 26] and applied in [10], is a non-linear approximation scheme designed to study relativistic structure formation in the universe on all scales, including scales where the density contrast is large. It is based upon a post-Newtonian type expansion in inverse powers of the speed of light $c$, building the relativistic corrections on top of the Newtonian equations. The resulting equations are a non-linear approximation to the full Einstein equations that are valid on all scales. At leading order, the post-Friedmann formalism reproduces Newtonian cosmology [25, 26].

In this paper, we use the post-Friedmann metric to compute the complete weak-lensing deflection angle up to order $c^{-4}$, including terms that are quadratic in the first order deflection, the first post-Newtonian corrections to the two scalar gravitational potentials and the effects of the vector and tensor contributions to the metric. This is the first time that the complete weak-lensing deflection angle on non-linear scales has been computed in a fully relativistic manner, including scalar, vector and tensor contributions to the metric. We then examine how the different terms in the deflection angle contribute to the E- and B-mode power spectra. The calculation of the deflection angle here is purely geometrical, i.e. does not assume the Einstein equations, thus it should be valid for any metric theory of gravity.

The strength of this approach to weak lensing is that the metric components that contribute to the lensing can be consistently related to the matter inhomogeneities. We illustrate this with a set of equations demonstrating how each of the quantities that contributes to the deflection angle can be computed from N-body simulations for a $\Lambda$CDM cosmology and demonstrate that the contribution to the E-mode from the vector potential is negligible for such a cosmology. The equation for the deflection angle, equation (3.10), coupled with equations (5.5)–(5.12) demonstrating how to relate the metric perturbations to the matter in a $\Lambda$CDM cosmology, comprise the main results of this paper.

This paper is laid out as follows. In section 2 we will briefly introduce the relevant parts of the post-Friedmann formalism, which will then be used to calculate the weak-lensing deflection angle in section 3. In section 4, we will examine how the vector and tensor parts of the deflection angle contribute to the E- and B-mode power spectra of cosmic shear. Section 5 will examine how to relate the matter inhomogeneities in the universe to the metric potentials, including the example of the post-Friedmann vector potential. We conclude in section 6.

## 2 Post-Friedmann formalism

The post-Friedmann formalism is a new method that has been developed in [26] for the purpose of studying relativistic structure formation on all scales. It consists of performing an expansion in inverse powers of the speed of light $c$ in an analogous fashion to post-Newtonian theory in the Solar System [33]. However, the expansion is performed differently in a cosmological setting, see [26] for the full details. Note that one of the differences compared to the Solar System treatment is that the two scalar potentials are both included at leading order and are not assumed to be equal a priori. The post-Friedmann approach considers a dust (pressureless matter) cosmology with a cosmological constant and the perturbed FLRW metric, in Poisson gauge, is explicitly written in terms ordered in powers of $c^{-1}$. Including
terms up to $c^{-5}$, the metric thus looks as follows,

\[
\begin{align*}
g_{00} &= -\left[1 - \frac{2U_N}{c^2} + \frac{1}{c^4} (2U_N^2 - 4U_P)\right] \\
g_{0i} &= -\frac{aB^N_i}{c^3} - \frac{aB^P_i}{c^5} \\
g_{ij} &= a^2 \left[\left(1 + \frac{2V_N}{c^2} + \frac{1}{c^4} (2V_N^2 + 4V_P)\right) \delta_{ij} + \frac{h_{ij}}{c^4}\right].
\end{align*}
\]

(2.1)

Since this metric is in the Poisson gauge, the three-vectors $B^N_i$ and $B^P_i$ are divergenceless, $B^N_{i,i} = 0$ and $B^P_{i,i} = 0$. In addition, $h_{ij}$ is transverse and tracefree, $h^i_i = h^i_{ij} = 0$. Note that at this order, $h_{ij}$ is not dynamical, so it does not represent gravitational waves. Rather, $h_{ij}$ represents the transverse traceless part of the metric necessarily arising from non-linear effects. The $g_{00}$ and $g_{ij}$ scalar potentials have been split into their leading order, Newtonian, components ($U_N, V_N$) and their post-Friedmann components ($U_P, V_P$). Similarly, the vector potential has been split up into $B^N_i$ and $B^P_i$. From a post-Friedmann viewpoint, there are two different levels of perturbations in the theory, corresponding to terms of order $c^{-2}$ and $c^{-3}$, or of order $c^{-4}$ and $c^{-5}$ respectively. Inserting the above metric in Einstein’s equations and keeping only the leading order terms shows that $U_N = V_N$ and leads to the standard Newtonian cosmological equations, with a bonus, the vector potential $B^N_i$ is necessarily non-zero [10, 25, 26].

Defining “resummed” variables, such as $\Phi = 2U_N + c^{-2} (2U_N^2 - 4U_P)$, then calculating the Einstein equations and linearising them, reproduces linear GR perturbation theory in Poisson gauge. Thus, this approach is capable of describing structure formation on the largest scales. The energy momentum tensor is constructed from the dust four-velocity and is also expanded in powers of $c^{-1}$, allowing the Einstein equations to be derived at different orders in the expansion. Crucially, whilst performing calculations using this $c^{-1}$ expansion, no assumptions are made regarding the amplitude of perturbations, notably the density contrast, so this expansion and the corresponding equations are valid even on scales where the density contrast is greater than unity. This will allow us to use the metric and Einstein equations from this formalism to calculate weak-lensing spectra on fully non-linear scales. The leading order gravitational equations in the $c^{-1}$ expansion and some of the higher order gravitational equations are introduced in section 5 for relating the metric potentials to N-body simulations. The derivation of these equations in a ΛCDM cosmology is in [26].

3 The deflection angle

In this section we will calculate the full weak-lensing deflection angle in the post-Friedmann formalism, up to order $c^{-4}$. By truncating at this order, we include the leading order contributions from each of the possible post-Newtonian corrections, $U_P, V_P, B^N_i$ and $h_{ij}$. We will follow the method outlined in [15], based on the second order geodesic equation from [27], but we will adapt the method for an expansion in powers of $c^{-1}$. For this work, when examining the Christoffel symbols, we found that we can treat post-Friedmann $c^0$ and $c^{-1}$ quantities analogously to zeroth order in perturbation theory, $c^{-2}$ quantities analogously to first order in perturbation theory and $c^{-3}$ or $c^{-4}$ quantities analogously to second order in perturbation theory. This will ensure that we include all of the terms that are relevant for our calcula-
A second order treatment of the geodesic equation is sufficient for a post-Friedmann calculation up to order \(c^{-4}\), since every metric potential is at least of order \(c^{-2}\), so any third order terms will be at least of order \(c^{-6}\).

We will then compare our result to related calculations in the literature. In practice, what we will see is that the deflection angle from the post-Friedmann expansion up to order \(c^{-4}\) is similar to the second order calculation from perturbation theory. The key differences are: the leading order term is composed of the Newtonian scalar potential arising from the fully non-linear density field, as opposed to the linear potential from perturbation theory; there is a post-Friedmann correction to the scalar potential and, a priori, the vector and tensor terms cannot be neglected; and the equations relating the metric perturbations in the deflection angle formula to the matter inhomogeneities on non-linear scales will also be different compared to perturbation theory.

3.1 Post-Friedmann deflection angle calculation

The starting points for the deflection angle calculation are the metric above 2.1, truncated at order \(c^{-4}\),

\[
\begin{align*}
g_{00} &= -1 + \frac{2U_N}{c^2} - \frac{1}{c^4} \left(2U_N^2 - 4U_P\right) \\
g_{0i} &= -\frac{aB_i}{c^3} \\
g_{ij} &= a^2 \left[ \left(1 + \frac{2V_N}{c^2} + \frac{1}{c^4} \left(2V_N^2 + 4V_P\right) \right) \delta_{ij} + \frac{h_{ij}}{c^4} \right],
\end{align*}
\]

and the spatial parts of the geodesic equation

\[
\frac{d^2x^i}{d\chi^2} = -\Gamma^i_{\mu\nu} \frac{dx^\mu}{d\chi} \frac{dx^\nu}{d\chi}.
\]

The Christoffel symbols for this metric, ordered as described above, are in appendix A. We will be solving the geodesic equation following [15]. Integrating the geodesic equation shows the transverse deflection of the light ray to be given by

\[
\begin{align*}
x^{(1)}(\chi, \vec{\theta}) &= \int_0^{\chi} d\chi \left(\chi - \chi_s\right) f^{(1)}(\chi, \vec{\theta}) \\
x^{(2)}(\chi, \vec{\theta}) &= \int_0^{\chi} d\chi \left(\chi - \chi_s\right) f^{(2)}(\chi, \vec{\theta}),
\end{align*}
\]

where the number in brackets represents the order, in a perturbative sense, of the deflection. The co-ordinates are such that \(\chi\) is the comoving distance and is the \(z\) co-ordinate, \(\vec{\theta}\) is the transverse displacement from the \(z\) axis and the \(x\) and \(y\) co-ordinates can be written as \(\chi\theta_1\) and \(\chi\theta_2\) respectively. The upper limit of the integral, \(\chi_s\), is the comoving distance that the photon originated from. From here onwards, a lower case latin letter will be used to denote 3 dimensional spatial indices (i=1,2,3) and an upper case latin letter will be used to denote 3 dimensional spatial indices (i=1,2,3).
denote indices that are one of the transverse spatial directions only \( (1, 2) \). The terms in the integral are given by

\[
\begin{align*}
  f^{(1)I}(x, \theta) &= -\Gamma^{(1)I}_{\alpha\beta} p^{(0)\alpha} p^{(0)\beta} \\
  f^{(2)I}(x, \theta) &= -\Gamma^{(0)I}_{\alpha\beta} p^{(1)\alpha} p^{(1)\beta} - 2\Gamma^{(1)I}_{\alpha\beta} p^{(0)\alpha} p^{(0)\beta} \\
  & \quad - 2\partial_{\sigma}\Gamma^{(0)I}_{\alpha\beta} x^{(1)\sigma} p^{(0)\alpha} p^{(0)\beta} - \partial_{\sigma}\Gamma^{(1)I}_{\alpha\beta} x^{(1)\sigma} p^{(0)\alpha} p^{(0)\beta} \\
  & \quad - \frac{1}{2} \partial_{\sigma}\partial_{\tau}\Gamma^{(0)I}_{\alpha\beta} x^{(1)\sigma} x^{(1)\tau} p^{(0)\alpha} p^{(0)\beta} - \Gamma^{(2)I}_{\alpha\beta} p^{(0)\alpha} p^{(0)\beta},
\end{align*}
\]  

(3.5)

where \( p^{(i)\alpha} \) is the photon direction vector and at zeroth order is normalised to \( p^{(0)\alpha} = (-a, 0, 0, 1) \). This is because we are working in the small angle approximation and means that we can ignore terms involving \( p^{(0)I} \). The first order photon direction vector \( (p^{(1)\alpha}) \), deflection \( (x^{(1)\alpha}) \) and geodesic terms \( (f^{(1)\alpha}) \) are all in appendix A, these will be needed for calculating the six terms contributing to \( f^{(2)I} \). We will examine the six terms individually, discarding any terms of order \( c^{-5} \) or above, before combining them into the final deflection angle.

Noting that each \( p^{(1)\alpha} \) or \( x^{(1)\alpha} \) term includes a factor of \( c^{-2} \) and that all zeroth order Christoffel symbols include a factor of \( c^{-1} \) due to the time derivative, a quick examination shows that the first, third and fifth terms are all at least of order \( c^{-5} \), so can be neglected. These terms serve to illustrate how, whilst the post-Newtonian expansion up to order \( c^{-4} \) is similar to second order perturbation theory, some of the terms don’t survive. All three of these terms are at least of order \( c^{-5} \) because the zeroth order (background) universe is spatially homogeneous, meaning that the only derivative of the zeroth order universe that survives is the time derivative. This is considered to be higher order than a spatial derivative in the post-Friedmann formalism, so has a factor \( c^{-1} \) associated with it.

The remaining second, fourth and sixth terms in equation (3.5) will all contribute to the deflection angle at \( c^{-4} \). In all three of these terms, we will make use of \( p^{(0)I} = 0 \).

For the second term, \( -2\Gamma^{(1)I}_{\alpha\beta} p^{(0)\alpha} p^{(1)\beta} \), the \( \alpha = 0, \beta = j \) and \( \alpha = 3, \beta = 0 \) terms are both zero for our purposes: for the first of these, the first order Christoffel symbol \( \Gamma^{(1)i}_{0j} \) is of order \( c^{-5} \) and the \( p^{(1)j} \) is of order \( c^{-2} \) making the term of order \( c^{-5} \). For the second of these, the 1st order Christofel symbol \( \Gamma^{1I}_{03} = 0 \). Then the remainder of this second term is

\[
-2\Gamma^{(1)I}_{\alpha\beta} p^{(0)\alpha} p^{(1)\beta} = \frac{4}{c^4} U^J_N \int_0^\chi d\chi' U_{N,3}^J - \frac{1}{c^2} V_N^3 \int_0^\chi d\chi' \left( U_{N}^J + V_{N}^j \right) + \frac{2}{c^4} V_N^j \int_0^\chi d\chi' \left( U_{N,3}^J - V_{N,3}^3 \right).
\]  

(3.6)

The fourth term contains the first order Christoffel symbol, which is non-zero if the lower indices are both spatial or both temporal. In addition, the \( \partial_0 \) term can also be ignored because of the extra factor of \( c^{-1} \), which makes the term of order \( c^{-5} \).

\[
-\partial_{\sigma}\Gamma^{(1)I}_{\alpha\beta} x^{(1)\sigma} p^{(0)\alpha} p^{(0)\beta} = \frac{(U_{N}^J + V_N^J)_{,j}}{c^4} \int_0^\chi (\chi - \chi') \left[ U_{N}^J + V_{N}^j \right] d\chi' \\
+ \frac{(U_{N}^J + V_N^J)_{,3}}{c^4} \int_0^\chi (\chi - \chi') \left[ U_{N}^3 - V_{N}^3 \right] d\chi'.
\]  

(3.7)

Note that the \( p^{(0)0} \) component here is different to [15] due to the use of cosmic time in our metric (3.1).
The 6th term contains the “true” 2nd order quantities due to the second order Christoffel symbol, including the vector and tensor contributions.

\[-\Gamma_{\alpha\beta}^{(2)}p_{\mu}^{(0)}p_{\nu}^{(0)} = \frac{B_{3}^{I} - B_{3}^{I} \beta}{c^{3}} + \frac{a}{c^{4}} B^{I} + \frac{1}{2c^{2}} h_{33}^{I} - \frac{1}{c^{2}} h_{33}^{I} + \frac{1}{c^{4}} \left( 2(U_{P} + V_{P})^{I} - 2U_{N}^{I} (U_{N} + V_{N}) \right). \tag{3.8}\]

We can now combine these three terms together to give the full second order contribution to the deflection angle.

\[x^{(2)}I(\theta) = \int_{0}^{\chi_{s}} d\chi (\chi_{s} - \chi) f^{(2)}I(\chi, \theta) = \int_{0}^{\chi_{s}} d\chi (\chi_{s} - \chi) \left[ \frac{1}{c^{3}} \left( B_{3}^{I} - B_{3}^{I} \beta \right) - \frac{1}{c^{4}} V_{N,3} \int_{0}^{\chi} d\chi' \left( U_{N}^{I} + V_{N}^{I} \right) \right. \]

\[\left. + \frac{4}{c^{2}} U_{N}^{I} f_{0}^{x} d\chi' U_{N,3} + \frac{2}{c^{3}} V_{N}^{I} f_{0}^{x} d\chi' (U_{N,3} - V_{N,3}) \right] \]

\[\left. + \frac{(U_{N} + V_{N})_{I}}{c^{2}} \int_{0}^{\chi} (\chi - \chi') \left[ U_{N}^{3} - V_{N}^{3} \right] d\chi' + \frac{2}{c^{2}} \left( U_{P} + V_{P} \right) \right] \]

\[\left. + \frac{2}{c^{2}} \left( U_{P} + V_{P} \right) \right] - \frac{1}{c^{2}} h_{33}^{I} + \frac{2}{c^{2}} \left( U_{N} + V_{N} \right) \right]. \tag{3.9}\]

It is known, see e.g. [14], that perturbations varying rapidly along the line of sight contribute little to the deflection angle since regions of positive and negative overdensity cancel out. Thus, only modes with small \(k_{3}\) contribute. This was considered more quantitatively by [15] in the context of higher order lensing, where they argue that contributing modes will have \(k_{3}\) of order of the Hubble radius, \(3000 h^{-1} \text{Mpc}\). For a typical mode contributing to the deflection angle, \(k_{3}\) will thus be at most of order 1% of \(k_{1}\) or \(k_{2}\). Since derivatives with respect to \(x_{i}\) become a factor of \(k_{i}\) in fourier space, we can neglect the \(A_{3,I}\) derivatives with respect to the \(A_{I,I}\) derivatives for all of the potentials \(A = \{U_{N}, V_{N}, B_{i}, U_{P}, V_{P}, h_{ij}\}\) in the expression above, these terms will contribute negligibly to the power spectrum and can safely be ignored. In addition, we can also ignore the last term in this expression: considering a simple order of magnitude estimate, the scalar potential is of order \(10^{-5}\), so this term is of order \(10^{5}\) times smaller than the first order term and we neglect it. For comparison, the Born correction term, \(\frac{(U_{N} + V_{N})_{I}}{c^{2}} \int_{0}^{\chi} (\chi - \chi') \left[ U_{N}^{3} + V_{N}^{3} \right] d\chi'\), is of order 10% of the leading order term [32].

Combining the first and second order terms we arrive at the complete deflection angle

\[x^{I}(\theta) = \int_{0}^{\chi_{s}} d\chi (\chi_{s} - \chi) \left[ \frac{(U_{N} + V_{N})_{I}}{c^{2}} + \frac{2}{c^{2}} \left( U_{P} + V_{P} \right) \right] \]

\[\left. + \frac{(U_{N} + V_{N})_{I}}{c^{2}} \int_{0}^{\chi} (\chi - \chi') \left[ U_{N}^{I} + V_{N}^{I} \right] d\chi' \right]. \tag{3.10}\]

This equation is the first main result of this paper. It contains all of the terms that are required for a complete description of the weak-lensing deflection angle up to order \(c^{-4}\) on non-linear scales. Since the derivation of this equation is purely geometrical, it should hold for all metric theories of gravity. For cosmological observables to be computed, it should
be supplemented by a set of equations that relates the metric perturbations to the matter inhomogeneities for the chosen gravity theory. In section 5, we present the equations (5.5)–(5.12) that relate the metric perturbations to the matter inhomogeneities for a GR ΛCDM cosmology.

3.2 Comparison to previous results

We can compare our final deflection angle (3.10) to the deflection angle in [15], where the deflection angle was calculated according to second order perturbation theory. The key difference is the regime of validity of the two approaches, in the sense that the deflection angle in [15] uses perturbation theory, and thus requires all of the metric and matter inhomogeneities to be small whereas, in our approach, the density field can be non-linear. This will be more evident in section 5 when we examine the equations relating the metric perturbations to the matter inhomogeneities for a ΛCDM cosmology. In particular, the metric perturbations in [15] cannot be related to matter inhomogeneities on non-linear scales in the universe, such as those in an N-body simulation. Nonetheless, there are similarities in the two deflection angles: as is well known, the linear perturbation theory scalar potential and leading order non-linear Newtonian potential contribute to the deflection angle in the same fashion. In addition, the only contributing “leading-order squared” term in both deflection angles is the term giving rise to the Born correction and lens-lens coupling. The form of the contributions of the vectors and tensors to the deflection angle are the same, although the second order vector and tensors are argued to be negligible in [15]. The same argument does not allow the vectors and tensors to be neglected here as we are dealing with scales where structure formation is non-linear.

For clarity we note that, in [15], the focus is on examining the second order perturbations to the metric and expanding the geodesic equation up to second order. We are interested in performing the same calculation here, except allowing for a fully non-linear density field. Additional perturbative corrections have been studied more recently, such as examining the luminosity distance to redshift relationship in a perturbed universe, and thus correctly interpreting the redshift of distant sources [16, 30]. Typically, these results are presented in terms of the magnification, and few papers have fully examined the effects of these additional corrections on cosmic-shear surveys. In [5], several of these corrections are examined in terms of their effect on the convergence and it is found that they are negligible compared to the standard first order lensing term.

The key difference between this work and [32] is the philosophy behind the metric. In our approach, there is an explicit connection between the metric and the matter inhomogeneities in a cosmological context. This will be important for section 5. There are differences in the details of the deflection angle too: there is an additional second order scalar term contributing to the deflection angle in [32]. This is the $U_N U_{NI}$ term that appeared in (3.9) but was argued to be negligible both here and in [15]. In [32], this term has been kept in case the PPN parameters are large enough to make the term observable. Furthermore, the deflection angle in this paper includes the contribution from tensor modes.

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3There are several other notable works on second order perturbation theory. In particular, [9] provides a comprehensive full-sky treatment. In the small-angle limit, the results in that paper agree with those of [15], to which we compare our results.

4See also [5] for a more recent examination of the contribution of vectors and tensors to the weak lensing convergence from the point of view of perturbation theory. The same result as in [15] is found.
A further difference between this work and [15, 32] is the sign of the Born correction term, although we note that the sign of our Born correction term agrees with other works, such as [21], once the difference in convention for the scalar potentials is taken into account.

4 Deflection angle to observable power spectra

Weak gravitational lensing has multiple effects on galaxy images, namely convergence ($\kappa$), rotation ($\rho$) and shear ($\gamma_1$ and $\gamma_2$). These effects are described by the distortion tensor, which is the Jacobian of the deflection angle (3.10).

$$\psi_{ij} \equiv \chi_s \frac{\partial x^i}{\partial \vartheta_j} \equiv \begin{pmatrix} -\kappa - \gamma_1 & -\gamma_2 + \rho \\ -\gamma_2 - \rho & -\kappa + \gamma_1 \end{pmatrix}.$$  \hfill (4.1)

Since a statistical analysis is required to detect weak lensing effects, the observables are the power spectra of the E- and B-modes of shear. These are defined to be particular combinations of the power spectra of the components of shear, $\gamma_1$ and $\gamma_2$. In the flat sky, small angle limit where $\psi_{ij}$ is defined as the Jacobian of the deflection angle as above, the E- and B-mode power spectra are equal to the power spectra of the convergence and rotation respectively. The power spectrum of the components of the distortion tensor is defined by

$$\langle \tilde{\psi}_{ij} (\vec{l}) \tilde{\psi}^*_{LM} (\vec{l}') \rangle = (2\pi)^2 \delta^2 (\vec{l} - \vec{l}') P_{ijLM} (l),$$  \hfill (4.2)

where $\vec{l}$ is the 2-d fourier wavenumber, conjugate to the two dimensional vector $\vec{\vartheta}$. The convergence and rotation power spectra are computed from the appropriate combinations of $P_{ijLM} (l)$, respectively

$$P_\kappa = \frac{1}{4} \left( P_{\psi 1111} + P_{\psi 2222} + 2 P_{\psi 1122} \right) = P_E$$

$$P_\rho = \frac{1}{4} \left( P_{\psi 1212} + P_{\psi 2121} - 2 P_{\psi 1221} \right) = P_B$$  \hfill (4.3)

The E- and B-mode power spectra from the scalar potentials, both the first order term and higher order Born correction and lens-lens coupling term, have been exhaustively studied in [21], where the scalar potentials in that analysis should be replaced by the fully non-linear Newtonian ($U_N$ and $V_N$) potentials for analysing non-linear scales. The post-Friedmann scalar potentials, $U_P$ and $V_P$, will contribute solely to the E-mode, in the same manner as the leading order scalar term, as the term involving $U_P$ and $V_P$ contributes to the components of the distortion tensor is the same fashion as the leading order scalar term. We will now examine how the vector and tensor modes contribute to the E- and B-mode power spectra, in order to fully catalogue which of the terms in equation (3.10) contribute to the E-mode and which can contribute to the B-mode. We will see that the vectors and tensors both produce an E-mode, but, to order $c^{-4}$, only the time derivative of the vector potential generates a B-mode.

4.1 Vectors

Starting from equation (3.10) and removing the scalar and tensor terms, we have

$$x^I (\vec{\vartheta}) = \chi_s \vartheta^s I = \int^\chi_s_0 d\chi (\chi_s - \chi) \left[ \frac{B^I_s}{c^3} + \frac{1}{c^4} a \dot{B}^I \right].$$  \hfill (4.4)
Following [14], we use this expression to get the distortion in terms of the vector,

\[ \psi_{\text{ij}} = \int_0^{\chi_{\infty}} d\chi g(\chi) \left[ \frac{B_{3,\text{ij}}}{c^3} + \frac{a\dot{B}_{3,\text{ij}}}{c^4} \right], \]

(4.5)

with \( g(\chi) \equiv \chi \int_0^\chi d\chi' (1 - \chi') W(\chi') \), where we are now dealing with a distribution of sources, rather than a single source, \( \chi_{\infty} \) is the distance to furthest source in the survey and \( W(\chi') \) is the normalised distribution of the sources. Since \( B_i \) is divergence-less, the vector power spectra can be defined as follows,

\[ \langle \hat{B}_i(\vec{k}) \hat{B}_j^*(\vec{k}') \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_{ij}(k), \]

\[ \langle \hat{B}_i(\vec{k}) \hat{B}_j^*(\vec{k}') \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_{ij}(k), \]

(4.6)

where \( P_{ij} = \left( \delta_{ij} - \vec{k}_i \vec{k}_j \right) \). This expression arises from noticing that, for a divergenceless vector, \( k^i B_i(\vec{k}) = 0 \). Thus, we can construct an orthonormal basis in fourier space, \( \{ \vec{k}^i, e^i, \bar{e}^i \} \), such that the basis vectors can be shown to satisfy \( e_i e_j + \bar{e}_i \bar{e}_j + \vec{k}_i \vec{k}_j = \delta_{ij} \) and the vector in fourier space can be expressed in terms of the two “polarisations” as \( \hat{B}_i(\vec{k}) = e_i B(\vec{k}) + \bar{e}_i \hat{B}(\vec{k}) \).

Due to the assumed isotropy of the universe, these polarisations satisfy

\[ \langle B(\vec{k}) B^*(\vec{k}') \rangle = \langle \hat{B}(\vec{k}) \hat{B}^*(\vec{k}') \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_B(k) \]

(4.7)

Thus,

\[ \langle \hat{B}_i(\vec{k}) \hat{B}_j^*(\vec{k}') \rangle = \left\langle \left( e_i B(\vec{k}) + \bar{e}_i \hat{B}(\vec{k}) \right) \left( e_j B^*(\vec{k}') + \bar{e}_j \hat{B}^*(\vec{k}') \right) \right\rangle \]

\[ = \left( e_i e_j + \bar{e}_i \bar{e}_j \right) (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_B(k) \]

\[ = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_{ij} P_B(k). \]

(4.8)

Using the expressions (4.6), and taking \( k_3 \ll k_1, k_2 \), the power spectrum (4.2) arising from the vector potential is

\[ P_{I,\text{JJLM}}^\phi(l) = \int_0^{\chi_{\infty}} d\chi g^2(\chi) \frac{l_{IJLM}}{\chi^2} \left( \frac{l_{IL}}{\chi^2} P_B(l/\chi) + \frac{a^2}{c^5} \left( \delta_{IJ} - \hat{i}_I \hat{i}_L \right) P_B(l/\chi) \right). \]

(4.9)

Taking the appropriate combinations of \( P_{I,\text{JJLM}}^\phi \) yields the final expressions for the E- and B-mode power spectra, \( P_E \) and \( P_B \) respectively, as

\[ P_E(l) = \frac{1}{4} \int_0^{\chi_{\infty}} d\chi g^2(\chi) \frac{l^4}{\chi^6} \frac{P_B(l/\chi)}{c^5}, \]

(4.10)

\[ P_B(l) = \frac{1}{4} \int_0^{\chi_{\infty}} d\chi g^2(\chi) \frac{a^2 l^2}{\chi^4} \frac{P_B(l/\chi)}{c^5}. \]

(4.11)
4.2 Tensors

Considering just the tensor contribution to the deflection angle, equation (3.10) and following the same steps as for the vector, the tensor contribution to the distortion tensor is

$$\psi_{IJ}(\vec{\theta}) = \int_0^{\chi_{\text{max}}} d\chi g(\chi) \left( \frac{h_{33}^{33}}{2c^4} \right).$$

(4.12)

We can straightforwardly see that the rotation, and therefore the B-mode will be zero, as the contribution from tensor perturbations is symmetric in $I$ and $J$. The power spectrum of the convergence will be non-zero and we will compute that now. First note that, similarly to the vectors, the tensors can be split into the two polarisations in Fourier space as follows,

$$h_{ij}(\vec{x}) = \int_{-\infty}^{\infty} d^3k e^{i\vec{k} \cdot \vec{x}} \left( h(\vec{k}) q_{ij} + \bar{h}(\vec{k}) \bar{q}_{ij} \right),$$

(4.13)

where $q_{ij}$ and $ar{q}_{ij}$ can be expressed in terms of the vectors $e_i$ and $\bar{e}_i$, where again the vectors $e_i$ and $\bar{e}_i$ form an orthonormal basis with $\hat{k}$, such that they satisfy $e_i e_j + \bar{e}_i \bar{e}_j + \hat{k}_i \hat{k}_j = \delta_{ij}$. In terms of these vectors,

$$q_{ij} = \frac{1}{\sqrt{2}} (e_i e_j - \bar{e}_i \bar{e}_j),$$

$$\bar{q}_{ij} = \frac{1}{\sqrt{2}} (e_i \bar{e}_j + \bar{e}_i e_j),$$

(4.14)

and the tensor power spectra are defined by

$$\langle h(\vec{k}) h^*(\vec{k'}) \rangle = \langle \bar{h}(\vec{k}) \bar{h}^*(\vec{k'}) \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k'}) P_h(k),$$

$$\langle h(\vec{k}) \bar{h}^*(\vec{k'}) \rangle = \langle \bar{h}(\vec{k}) h^*(\vec{k'}) \rangle = 0.$$  

(4.15)

When computing the contribution to the convergence, we will need to calculate $q_{33}^2 + \bar{q}_{33}^2$ whilst neglecting the contribution of $k_3$. This works out to be $1/2$, so the convergence power spectrum from the tensors is simply

$$P_E(l) = \frac{1}{8} \int_0^{\chi_{\text{max}}} d\chi g^2(\chi) \frac{l^4}{\chi^6} \frac{P_h(l/\chi)}{c^8}.$$  

(4.16)

5 Relating to matter content and Newtonian simulations

The strength of using the post-Friedmann formalism for weak lensing is that it is possible to connect the metric perturbations with the matter inhomogeneities in the universe, even on scales where the density contrast is greater than unity. Here, we will show how the metric perturbations can be calculated from fully non-linear matter perturbations in a ΛCDM universe. This section will use the application of the post-Friedmann formalism to a ΛCDM cosmology in [26]. These equations will not necessarily apply for a dark energy or modified gravity cosmology, so the post-Friedmann approach would have to be applied to these cosmologies in order to fully examine lensing on non-linear scales. We leave this to future work.

We will also explain which metric quantities can be fully calculated directly from ΛCDM N-body simulations, and discuss the circumstances in which the other quantities can be extracted. This creates the opportunity for all of the contributions to the deflection angle (3.10) to be calculated from standard Newtonian simulations, and thus for the complete weak lensing predictions on fully non-linear scales to be calculated.
5.1 Newtonian regime

Considering the leading order Einstein equations in the post-Friedmann $c^{-1}$ expansion yields the gravitational equations in the Newtonian regime [26]

\[
\frac{1}{c^4 a^2} \nabla^2 V_N = -\frac{4\pi G}{c^2} \rho_b \delta \tag{5.1}
\]

\[
\frac{2}{c^4 a^2} \nabla^2 (V_N - U_N) = 0 \tag{5.2}
\]

\[
\frac{1}{c^3} \left[ \frac{2a}{a^2} U_{N,i} + \frac{2}{a} V_{N,ii} - \frac{1}{2a^2} \nabla^2 B_i^N \right] = \frac{8\pi G \rho_b}{c^3} (1 + \delta) \nu_i, \tag{5.3}
\]

where $\delta$ is the fully non-linear density contrast and $\nu$ is the velocity of the cold dark matter. As expected, we have the Poisson equation for $U_N$ and the equality of the $U_N$ and $V_N$ potentials. In addition, there is a vector potential that is sourced by the Newtonian density and velocity fields. These three equations dictate how to extract $U_N$, $V_N$ and $B_i^N$ from N-body simulations. Using the curl of equation (5.3), the power spectrum of the post-Friedmann vector potential was calculated from N-body simulations in [10, 29] and found to be of order $10^{-5}$ times the power spectrum of the non-linear Newtonian scalar potential over a range of scales and redshifts. This is up to two orders of magnitude larger than predicted by second order perturbation theory [10, 29], cf. [24]. Using this result in equation (4.10), we can calculate the contribution of the vector potential to the E-mode, which will clearly be negligible compared to the signal from the leading order Newtonian scalar potential. This is the first time that the contribution of the vector potential to the cosmological weak-lensing signal has been shown to be negligible on non-linear scales and illustrates the power of our approach.

5.2 Gravitational equations up to order $1/c^4$

We now consider the gravitational equations beyond the leading order Newtonian regime. In particular, $U_P$, $V_P$ and $h_{ij}$ are given by [26]

\[
\frac{1}{c^4} \nabla^2 \nabla^2 V_P = \frac{1}{c^4} \left( \frac{1}{2} \nabla^2 \nabla^2 V_P^2 - \frac{5}{4} \nabla^2 \left( V_{N,i} V_{N,j}^i \right) - 2\pi G a^2 \bar{\rho} \left( \nabla^2 (v^2 (1 + \delta)) - 3 \dot{a} (v_i (1 + \delta)) \right) \right)
\]

\[
\frac{1}{c^4} \nabla^2 \nabla^2 (U_P - V_P) = \frac{1}{c^4} \left( A_{i,j}^{ij} + 8\pi G a^2 \bar{\rho} S_{i,j}^{ij} \right)
\]

\[
\frac{1}{c^4} \nabla^2 \nabla^2 h_i^j = \frac{1}{c^4} \left[ \begin{array}{c}
-A_{k,li}^{l,k} - \nabla^2 A_{k,li}^{l,k} \nu_i + 2\nabla^2 A_{k,li}^{l,k} \nu_l \\
+ 2\nabla^2 A_{k,li}^{l,k} \nu_i + 2\nabla^2 A_{k,li}^{l,k} \nu_l \\
+ 8\pi G a^2 \bar{\rho} \left( -S_{k,li}^{l,k} - \nabla^2 S_{k,li}^{l,k} + 2\nabla^2 S_{k,li}^{l,k} \right) \\
+ 8\pi G a^2 \bar{\rho} \left( + 2\nabla^2 S_{k,li}^{l,k} - 2\nabla^2 S_{k,li}^{l,k} \right)
\end{array} \right]
\]

\[
A_i^j = 2V_{N,i} V_{N}^j - \frac{2}{3} \delta_i^j V_{N,k} V_{N}^k
\]

\[
S_i^j = (1 + \delta) \left( \nu_i \nu^j - \frac{1}{3} \delta_i^j v^2 \right). \tag{5.4}
\]

These quantities cannot be extracted fully from standard N-body simulations. This is because, when solving the dynamical equations at order $c^{-4}$, the density and velocity fields will
no longer quite follow the same dynamics as in the Newtonian regime. However, it is expected that the non-linear matter dynamics in our universe is Newtonian to a good approximation and this is borne out by the small value of the vector potential in the N-body simulations. Thus, we propose to extract the leading order contribution to these post-Friedmann quantities by substituting the Newtonian density and velocity fields into these equations. Of course, if these quantities turn out to be large, then this suggests that Newtonian gravity is not as good an approximation to the matter dynamics as is currently believed and modified N-body simulations will need to be developed. Assuming this turns out to not be the case, we propose the following system of equations for extracting all of the metric potentials that contribute to the weak-lensing deflection angle from ΛCDM N-body simulations:

\[ \nabla^2 V_N = -4\pi G \rho_b \delta^N \]  
\[ V_N = U_N \]  
\[ \nabla \times \nabla^2 B_i^N = -16\pi G a^2 \rho_b \nabla \times \left[ (1 + \delta^N) u_i^N \right] \]  
\[ \nabla^2 \nabla^2 V_P = \left( \frac{1}{2} \nabla^2 \nabla^2 V_N - \frac{5}{4} \nabla^2 \left( V_{N,i} V_N^i \right) \right) - 2\pi G a^2 \bar{\rho} \left[ \nabla^2 \left( u^N_2 \left( 1 + \delta^N \right) \right) - 3\dot{a} \left( u_i^N \left( 1 + \delta^N \right) \right) \right] \]  
\[ 4 \frac{3}{3} \nabla^2 \nabla^2 \left( U_P - V_P \right) = \left( A_{i,j} + 8\pi G a^2 \bar{\rho} S_{i,j} \right) \]  
\[ \nabla^2 \nabla^2 \nabla^2 h^j_i = \left[ \begin{array}{c} -A_{i,li} - \nabla^2 A_{k,l,ki} + 2\nabla^2 A_{i,k} \delta_{ij} - 2\nabla^2 \nabla^2 A_i^j \\ +8\pi G a^2 \bar{\rho} \left( -S_{i,li} - \nabla^2 S_{k,l,ki} + 2\nabla^2 S_{i,k} \right) \\ +8\pi G a^2 \bar{\rho} \left( +2\nabla^2 S_{k,l,ki} \delta_{ij} - 2\nabla^2 \nabla^2 S_i^j \right) \end{array} \right] \]  
\[ A_i^j = 2V_{N,i} V_N^j - \frac{2}{3} \delta_i^j V_{N,k} V_N^k \]  
\[ S_i^j = (1 + \delta^N) \left( v_i^N v_N^j - \frac{1}{3} \delta_i^j v^N \right) \]

where \( \delta^N \) and \( v_i^N \) are the Newtonian density and velocity fields as found in a ΛCDM N-body simulation. This set of equations represents the second main result of this paper and allows all of the potentials that contribute to weak-lensing from non-linear scales to be computed from the Newtonian density and velocity fields in an N-body simulation. The equation for the difference between \( V_P \) and \( U_P \) is of particular topical interest, since the difference between the two scalar potentials has been suggested as a smoking gun for the existence of modified gravity (see e.g [4]). This assertion is only strictly correct if the quantity \( V_P - U_P \) is measured from standard N-body simulations using equation (5.9) and found to be small.

5.3 Order of magnitude upper bound for the B-mode power spectrum

The B-mode power spectrum is not generated at leading order by the Newtonian potential, so any effects generating this will be easier to detect than effects that generate an E-mode. As shown above, up to order \( c^{-4} \), other than the second order Born correction term that can be computed from the Newtonian potential, the only generator of the B-mode is the time derivative of the vector potential. We wish to get an estimate of the power spectrum of the time derivative of the vector potential, in order to determine whether the B-mode signal may be seen by future surveys. The power spectrum of the post-Friedmann vector
potential was calculated in \cite{10, 29}. Ideally, the power spectrum of the time derivative of the post-Friedmann vector potential would be extracted directly from simulations, as has been done for the scalar potential \cite{11}. The time derivative of the scalar potential is calculated by taking the time derivative of the Poisson equation and substituting for $\dot{\delta}$ using the continuity equation. An equivalent approach for the vector potential requires substituting for $\tilde{v}$ using the Euler equation, which introduces a term $U_N \delta$. Although we are able to extract these two fields individually, we have been unable to extract the combined term from the simulations. Investigations \cite{29} using the DTFE code \cite{12} code suggest that, in general, extracting two fields A and B from simulations and taking the product yields a different numerical answer to directly extracting the combined field AB from the simulations.

Here, we will consider a simple analytic approximation in order to estimate the order of magnitude of $B_i$, and hence whether the generated B-mode may be observable. From the simulations used in \cite{10, 29}, we can see that the vector potential is of order 3 times the power spectrum of the scalar potential, over a wide range of scales and redshifts. As this ratio only varies slowly over time, we will use an approximate form for the time derivative of the vector potential in order to obtain an order of magnitude estimate of the B-mode power spectrum:

$$P_B(k) = (3 \times 10^{-5}) P_\Phi(k) \approx \left( \frac{\dot{a}}{a} \right)^2 (f(a) - 1)^2 (3 \times 10^{-5}) P_\Phi(k),$$  \hspace{1cm} (5.13)

where the metric potentials are now taken to be dimensionless, so $\Phi_{\text{now}} = \Phi_{\text{previous}}/c^2$ and $B_{\text{now}} = B_{\text{previous}}/c^3$ and we have used the linear theory prediction to relate $P_\Phi(k)$ to $P_\Phi(k)$, with $f(a)$ being the logarithmic growth rate of density perturbations. Taking $(f(a) - 1)^2$ to be of order 1, the B-mode spectrum from $\dot{\delta}$ can now be expressed as

$$P_B(l) = \int_0^{\chi_{\text{max}}} d\chi g^2(\chi) \frac{\dot{a}}{a} P_\Phi(l/\chi) \left[ \frac{a^2 \chi^2}{4l^2 c^2} \left( \frac{\dot{a}}{a} \right)^2 (3 \times 10^{-5}) \right].$$ \hspace{1cm} (5.14)

Considering the term in square brackets, for the purpose of computing an approximate upper bound, we will set $a = 1$ and evaluate $\dot{a}$ for $a = 1$. Since $g(\chi) \to 0$ for large $\chi$, we will also set a nominal $\chi_0$ as the largest contributing scale. Setting $\chi^2 = \chi_0^2$, we can now remove the factor in square brackets from the integral to allow the B-mode to be written in terms of the standard E-mode power spectrum,

$$P_B(l) \leq P_\Phi^E(l) \chi_0^2 \frac{\dot{a}}{4l^2 c^2} \left( \frac{\dot{a}}{a} \right)^2 (3 \times 10^{-5}),$$ \hspace{1cm} (5.15)

where $P_\Phi^E(l) = \int_0^{\chi_{\text{max}}} d\chi g^2(\chi) \frac{\dot{a}}{a} P_\Phi(l/\chi)$ is the standard E-mode generated by the scalar potential. For the range of scales we are considering here, $l$ will range from 10 up to 1000 h$^{-1}$Mpc. To calculate the upper bound, we will set $\chi_0 = 1000$ h$^{-1}$Mpc and $l = 10$. Then

$$P_B(l) \leq P_\Phi^E(l) \times 10^{-8}. \hspace{1cm} (5.16)$$

The final step is to estimate how much the linear theory used in equation (5.13) underestimates the fully non-linear time derivative of the scalar potential. We can see from \cite{11} that this effect is scale dependent, but the linear theory prediction for the time derivative of the scalar potential typically underestimates its amplitude by 2-3 orders of magnitude on non-linear scales. Since the behaviour of the vector over time is similar to the behaviour of
the scalar, as shown by the almost constant ratio between the two power spectra, we will assume a similar relationship between the magnitude of the linear and non-linear amplitude of \( P_B \). Thus, we consider that \( P_B(l) \) could be up to three orders of magnitude larger than in equation (5.16). This still leaves our generous upper bound as \( P_B(l) \approx P_\Phi^s(l) \times 10^{-5} \), i.e. the power spectrum of the B-mode that a future survey might see has an upper limit of \( 10^{-5} \) times the power spectrum of the standard E-mode spectrum from the scalar potential. This is unlikely to be found with the next generation of surveys.

We can compare this result to the known sources of the B-mode in a \( \Lambda \)CDM universe. In particular, the Born-correction, lens-lens coupling and reduced shear contributions to the B-mode are calculated in [21] using perturbation theory. The cosmological parameters used for our calculation of the B-modes are slightly different, however, we can partially mitigate this by comparing the ratios of each B-mode to the respective standard first order scalar E-mode. In our simple estimate above, the B-modes from the vector potential goes as \( l^{-2} \) times the standard E-modes, whereas the B-mode from the Born correction and lens-lens coupling in [21] scales similarly to the standard E-mode. Comparing the orders of magnitude, it appears that the B-modes from the vector can only dominate over those from the Born correction and lens-lens coupling for small \( l \) (\( < 100 \)). Since our calculation is valid only for small angles (large \( l \)), there are likely to be few scales where the contribution from the vector is dominant compared to that from the Born correction and lens-lens coupling. Furthermore, the signal from the vector will always be subdominant to the signal from the reduced shear that is calculated in [21]. We note that this comparison is only an estimate, as the calculation in [21] is performed using perturbation theory. However, this should also be mitigated through the comparison of the ratios of the B-mode signals to the respective E-mode signals and any uncertainties are likely to be less than the approximations used in deriving our order of magnitude upper bound.

Thus far, we have considered only the power spectra of the lensing observables. One interesting possibility that we have not considered is whether the post-Friedmann vector potential could be observed through the bispectrum. The bispectrum of cosmic shear has attracted increasing attention recently and there have been several measurements, e.g. [17]. A full second order perturbative analysis was carried out in [8] (again using the full-sky formalism). In this work they showed that the second order vector contribution to the metric contributed negligibly to the E-mode of the bispectrum in comparison to the geometric corrections (such as the Born correction and lens-lens coupling terms), which are themselves smaller than the contribution of non-linear physics (i.e. the non-linear evolution of the scalar gravitational potential). As the ratio of the vector potential to the scalar potential is similar on linear and non-linear scales [10, 29], it is unlikely that the post-Friedmann vector potential will be relevant for the E-mode bispectrum. No full perturbative analysis has been done for the B-mode, however given that the geometric corrections are larger than the vector potential for the E-mode bispectrum and for the B-mode power spectrum, it seems unlikely that the vector potential will be a dominant contribution to the B-mode bispectrum. A full investigation of the effects of the metric generated by non-linear structure formation on the weak-lensing bispectra is beyond the scope of this paper, and we leave it to future work to address this interesting question.

6 Conclusion

Weak gravitational lensing is rapidly becoming an important tool in cosmology and promises to constrain both the parameters composing the concordance model of cosmology and physics
beyond the ΛCDM model. Weak-lensing calculations predominantly use relativistic perturbation theory, which works well for the largest scales. However, future surveys will yield the majority of their information on non-linear scales, requiring the use of ray tracing through N-body simulations.

In this paper we have calculated the full weak-lensing deflection angle up to order $c^{-4}$ using the post-Friedmann formalism. Crucially, this formalism doesn’t require the matter inhomogeneities to be small, so this calculation of the deflection angle can be used on fully non-linear scales in our universe. This deflection angle, equation (3.10), is the first main result of this paper and should hold for any metric theory of gravity.

The main advantage of using the post-Friedmann formalism for lensing is that the metric potentials can be related to the matter perturbations in the universe on non-linear scales. The second main result of this paper is the set of equations (5.5)–(5.12), derived from [25, 26], showing how the metric potentials contributing to the deflection angle can be computed from N-body simulations. These equations are valid for a ΛCDM cosmology. As an example, we have used the calculation of the post-Friedmann vector potential in [10, 29] to show that the vector contribution to the E-mode power spectrum is unlikely to be detected, even on fully non-linear scales.

Thus far, we have been unable to extract the time derivative of the vector potential from the simulations. However, we have calculated a simple order of magnitude estimate of the B-mode power spectrum that this time derivative would generate, showing it to be very small. Thus, there are no significant sources of the B-mode spectrum on non-linear scales in a ΛCDM cosmology, allowing its use as one control of systematics as is often done in analysis of weak lensing data. We also have not yet extracted the post-Friedmann scalar potentials $U_P$ and $V_P$ or the tensor modes from the N-body simulations, however these are higher order than the vector potential and so cannot be significantly larger than the vector potential if using Newtonian simulations for ΛCDM cosmologies has any validity. Thus, their contributions to the E-mode should be equally negligible. We leave it to future work to verify by direct extraction from simulations using equations (5.5)–(5.12) that these are indeed small. If this is the case then, for a ΛCDM cosmology, the use of ray tracing tracing through N-body simulations, taking into account only the Newtonian scalar gravitational potential, is valid on all scales. If these quantities are found to be not small then the use of Newtonian simulations for ΛCDM cosmologies requires serious scrutiny.

In order to investigate lensing on fully non-linear scales for dark energy or modified gravity cosmologies, equivalent equations to (5.5)–(5.12) would need to be derived. It is possible that the post-Friedmann vector potential, and/or its time derivative, are larger in modified gravity theories (such as f(R) gravity), due to both the modified behaviour of galaxies [22] and the intrinsically relativistic nature of a scalar field. We leave it to future work to develop the modified Einstein equations in dark energy and modified gravity cosmologies using the post-Friedmann formalism. Once the equations have been derived, they can be applied to modified N-body simulations in order to calculate the potentials contributing to the deflection angle (3.10).

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A Useful quantities

For completeness, we include here the inverse metric and Christoffel symbols associated with the metric (3.1), as well as the first order $x^{(1)\alpha}$, $p^{(1)\alpha}$ and $f^{(1)\alpha}$ quantities.

A.1 Inverse metric and Christoffel symbols

The inverse metric is given by

\[
g^{00} = -1 - \frac{2U_N}{c^2} + \frac{1}{c^4} (2U_N^2 - 4U_P)
\]

\[
g^{0i} = -aB^i
\]

\[
g^{ij} = a^{-2} \left[ \left( 1 - \frac{2V_N}{c^2} + \frac{1}{c^4} (2V_N^2 - 4V_P) \right) \delta^{ij} - \frac{h^{ij}}{c^4} \right].
\]

(A.1) (A.2)

The only non-zero Christoffel symbols at zeroth order are

\[
\Gamma^0_{ij} = \frac{a\dot{a}\delta_{ij}}{c},
\]

\[
\Gamma^i_{0j} = \frac{\dot{a}\delta^i_j}{ac}.
\]

(A.3)

The first order Christoffel symbols are given by

\[
\Gamma^0_{0i} = -\frac{U_{N,i}}{c^2},
\]

\[
\Gamma^i_{00} = -\frac{U^i_N}{a^2c^2}
\]

\[
\Gamma^i_{jk} = \frac{\delta^i_{jk}V_{N,k}}{c^2} + \frac{\delta^i_j V_{N,j}}{c^2} - \frac{\delta_{jk}V_i^j}{c^2},
\]

(A.4)

and the second order Christoffel symbols are given by

\[
\Gamma^0_{00} = -\frac{\dot{U}_N}{c^3}
\]

\[
\Gamma^0_{0i} = -\frac{2U_{P,i}}{c^4} - \frac{\dot{a}B^i}{c^4}
\]

\[
\Gamma^i_{ij} = \frac{aB_{i,j} + aB_{j,i}}{2c^2} + \frac{2a\dot{a}\delta_{ij}U_N}{c^3} + \frac{2a\dot{a}\delta_{ij}U_N}{c^3} + \frac{a^2\delta_{ij}\dot{V}_N}{c^3}
\]

\[
\Gamma^i_{00} = -\frac{\dot{a}B^i + \dot{B}_i - \frac{\dot{a}}{2ac^3}}{c^4} - \frac{2U_{P,i}}{a^2c^4} + \frac{2V_NU_i^j}{a^2c^4} + \frac{2U_NU_i^j}{a^2c^4}
\]

\[
\Gamma^i_{0j} = \frac{B^i_j - B^i_j}{2ac^3} + \frac{\delta^i_j\dot{V}_N}{c^3}
\]

\[
\Gamma^i_{jk} = \frac{\dot{a}\delta_{jk}B^i}{c^4} + \frac{h^i_{j,k} + h^i_{k,j} - h^i_{j,k}}{2c^4} + \frac{2\delta^i_j V_{P,k}}{c^4} + \frac{2\delta^i_k V_{P,j}}{c^4} - \frac{2\delta_{jk}V_i^j}{c^4}.
\]

(A.5)
A.2 First order $x^{(1)\alpha}$, $p^{(1)\alpha}$ and $f^{(1)\alpha}$

The first order $f^{(1)\alpha}$ functions are given by

\begin{align}
  f^{(1)0}(\chi, \vec{\theta}) &= -\frac{2aU_{N,3}}{c^2}, \\
  f^{(1)I}(\chi, \vec{\theta}) &= \frac{U^I_N}{c^2} + \frac{V^I_N}{c^2}, \\
  f^{(1)3}(\chi, \vec{\theta}) &= \frac{U^3_N}{c^2} - \frac{V^3_N}{c^2},
\end{align}

(A.6)

In addition, the first order deflections and p-vectors are defined as

\begin{align}
  x^{(1)\alpha}(\vec{\theta}) &= \int_0^{\chi_s} d\chi (\chi_s - \chi) f^{(1)\alpha}(\chi, \vec{\theta}) \\
  p^{(1)0}(\chi) &= p^{(1)0}(\chi = 0) - \int_0^{\chi} f^{(1)0}(\chi') d\chi' \\
  p^{(1)I}(\chi) &= -\int_0^{\chi} f^{(1)I}(\chi') d\chi' \\
  p^{(1)3}(\chi) &= -\int_0^{\chi} d\chi' (\chi - \chi') \left( \frac{2aU_{N,3}}{c^2} \right)
\end{align}

(A.7)

So, in our case, the first order $p^{(1)\alpha}$ and $x^{(1)\alpha}$ vectors are given by

\begin{align}
  p^{(1)0}(\chi) &= \int_0^{\chi} d\chi' \frac{2aU_{N,3}}{c^2} \\
  p^{(1)I}(\chi) &= -\int_0^{\chi} d\chi' \left( \frac{U^I_N}{c^2} + \frac{V^I_N}{c^2} \right) \\
  p^{(1)3}(\chi) &= -\int_0^{\chi} d\chi' \left( \frac{U^3_N}{c^2} - \frac{V^3_N}{c^2} \right) \\
  x^{(1)0}(\chi) &= -\int_0^{\chi} d\chi' (\chi - \chi') \frac{2aU_{N,3}}{c^2} \\
  x^{(1)I}(\chi) &= \int_0^{\chi} d\chi' (\chi - \chi') \left( \frac{U^I_N}{c^2} + \frac{V^I_N}{c^2} \right) \\
  x^{(1)3}(\chi) &= \int_0^{\chi} d\chi' (\chi - \chi') \left( \frac{U^3_N}{c^2} - \frac{V^3_N}{c^2} \right). \tag{A.8}
\end{align}

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