A NOTE ON TRANS-SASAKIAN MANIFOLDS

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Abstract. In this paper, we obtain some sufficient conditions for a 3-dimensional compact trans-Sasakian manifold of type $(\alpha, \beta)$ to be homothetic to a Sasakian manifold. A characterization of a 3-dimensional cosymplectic manifold is also obtained.

1. Introduction

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold (cf. [2]). Then the product $\overline{M} = M \times R$ has a natural almost complex structure $J$ with the product metric $G$ being Hermitian metric. The geometry of the almost Hermitian manifold $(\overline{M}, J, G)$ dictates the geometry of the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ and gives different structures on $M$ like Sasakian structure, quasi-Sasakian structure, Kenmotsu structure and others (cf. [2], [3], [8]). It is known that there are sixteen different types of structures on the almost Hermitian manifold $(\overline{M}, J, G)$ (cf. [6]) and using the structure in the class $W_4$ on $(\overline{M}, J, G)$, a structure $(\varphi, \xi, \eta, g, \alpha, \beta)$ on $M$ called trans-Sasakian structure, was introduced (cf. [13]) that generalizes Sasakian and Kenmotsu structures on a contact metric manifold (cf. [3], [8]), where $\alpha, \beta$ are smooth functions defined on $M$. Since the introduction of trans-Sasakian manifolds, very important contributions of Blair and Oubiña [3] and Marrero [11] have appeared, studying the geometry of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$, Trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called cosymplectic, $\alpha$-Sasakian, and $\beta$-Kenmotsu manifolds respectively. Marrero [11] has shown that a trans-Sasakian manifold of dimension $\geq 5$ is either cosymplectic, or $\alpha$-Sasakian, or $\beta$-Kenmotsu. Since then, there is a concentration on studying geometry of 3-dimensional trans-Sasakian manifolds only (cf. [11], [1], [3], [2], [10]),

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putting some restrictions on the smooth functions $\alpha, \beta$ appearing in the definition of trans-Sasakian manifolds. There are several examples of trans-Sasakian manifolds constructed mostly on 3-dimensional Riemannian manifolds (cf. [3], [11], [13]). Moreover, as the geometry of Sasakian manifolds is very rich, and is derived from contact geometry, the question of finding conditions under which a 3-dimensional trans-Sasakian manifold is homothetic to a Sasakian manifold becomes more interesting. In this paper we consider this question and obtain two different sufficient conditions for a trans-Sasakian manifold to be homothetic to a Sasakian manifold. One of them is expressed in terms of the smooth functions $\alpha, \beta$ and a bound on certain Ricci curvature, and the other requires that the Reeb vector should be an eigenvector of the Ricci operator (cf. Theorems 3.1, 3.2). We also find a characterization of cosymplectic manifolds (cf. Theorem 4.1).

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**2. Preliminaries**

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold, where $\varphi$ is a $(1, 1)$-tensor field, $\xi$ a unit vector field and $\eta$ a smooth 1-form dual to $\xi$ with respect to the Riemannian metric $g$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(2.1)

$X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on $M$ (cf. [2]). If there are smooth functions $\alpha, \beta$ on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ satisfying

$$\nabla\varphi(X, Y) = \alpha (g(X, Y)\xi - \eta(Y)X) + \beta (g(\varphi X, Y)\xi - \eta(Y)\varphi X),$$

(2.2)

then this is said to be a trans-Sasakian manifold, where $(\nabla\varphi)(X, Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y), \quad X, Y \in \mathfrak{X}(M)$ and $\nabla$ is the Levi-Civita connection with respect to the metric $g$ (cf. [3], [11], [13]). We shall denote this trans-Sasakian manifold by $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ and it is called trans-Sasakian manifold of type $(\alpha, \beta)$. From equations (2.1) and (2.2), it follows that

$$\nabla_X \xi = -\alpha \varphi(X) + \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M).$$

(2.3)

It is clear that a trans-Sasakian manifold of type $(1, 0)$ is a Sasakian manifold (cf. [2]) and a trans-Sasakian manifold of type $(0, 1)$ is a Kenmotsu manifold (cf. [8]). A trans-Sasakian manifold of type $(0, 0)$ is called a cosymplectic manifold (cf. [7]).

Let $Ric$ be the Ricci tensor of a Riemannian manifold $(M, g)$. Then the Ricci operator $Q$ is a symmetric tensor field of type $(1, 1)$ defined by $Ric(X, Y) = g(QX, Y), \quad X, Y \in \mathfrak{X}(M)$. We prepare some tools for trans-Sasakian manifolds.
Lemma 2.1. Let \((M, \varphi, \xi, \eta, g, \alpha, \beta)\) be a 3-dimensional trans-Sasakian manifold. Then \(\xi(\alpha) = -2\alpha\beta\).

Proof. Using (2.3), we get that
\[
d\eta(X, Y) = -2\alpha g(\varphi X, Y), \quad X, Y \in \mathfrak{X}(M)
\]
and as a consequence, the 2-form \(\Omega\) defined by \(\Omega(X, Y) = \alpha g(\varphi X, Y)\) is closed. Using (2.1), (2.2), and (2.3) in \(d\Omega = 0\) after some trivial calculations, we arrive at
\[
\varphi \{X(\alpha)Y - Y(\alpha)X - 2\alpha\beta \eta(Y)X + 2\alpha\beta \eta(X)Y\} + g(\varphi X, Y)(\nabla \alpha + 2\alpha\beta \xi) = 0
\]
for all \(X, Y \in \mathfrak{X}(M)\). Operating \(\varphi\) on the equation above, we get
\[
Y(\alpha)X - X(\alpha)Y + 2\alpha\beta \eta(Y)X - 2\alpha\beta \eta(X)Y + X(\alpha)\eta(Y)\xi - Y(\alpha)\eta(X)\xi + g(\varphi X, Y)\varphi(\nabla \alpha) = 0.
\]
For a local orthonormal frame \(\{e_1, e_2, e_3\}\) on \(M\), taking \(X = e_i\) in the equation above, taking the inner product with \(e_i\) and adding the resulting equations, we get
\[
(2\alpha\beta + \xi(\alpha)) \eta(Y) = 0, \quad Y \in \mathfrak{X}(M)
\]
which gives
\[
(2\alpha\beta + \xi(\alpha)) \xi = 0
\]
and we obtain the result. \(\square\)

Lemma 2.2. Let \((M, \varphi, \xi, \eta, g, \alpha, \beta)\) be a 3-dimensional trans-Sasakian manifold. Then its Ricci operator satisfies
\[
Q(\xi) = \varphi(\nabla \alpha) - \nabla \beta + 2(\alpha^2 - \beta^2)\xi - g(\nabla \beta, \xi)\xi
\]
where \(\nabla \alpha, \nabla \beta\) are gradients of the smooth functions \(\alpha, \beta\).

Proof. We use (2.1), (2.2), and (2.3) to calculate
\[
R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi
\]
and after some easy computations we arrive at
\[
R(X, Y)\xi = Y(\alpha)\varphi X - X(\alpha)\varphi Y + X(\beta)(Y - \eta(Y))\xi - Y(\beta)(X - \eta(X))\xi + (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta \eta(Y)\varphi X - \eta(X)\varphi Y).
\]
The above equation gives
\[
\text{Ric}(Y, \xi) = g(\varphi(\nabla \alpha), Y) - g(\nabla \beta, Y) - g(\nabla \beta, \xi)\eta(Y) + 2(\alpha^2 - \beta^2)\eta(Y),
\]
which proves the result. \(\square\)

Next, we state the following result of [12], which we shall use in the sequel.
Theorem 2.1. Let \((M, g)\) be a Riemannian manifold. If \(M\) admits a Killing vector field \(\xi\) of constant length satisfying
\[
k^2 (\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi) = g(Y, \xi) X - g(X, Y) \xi
\]
for a nonzero constant \(k\) and any vector fields \(X\) and \(Y\), then \(M\) is homothetic to a Sasakian manifold.

3. Trans-Sasakian manifolds homothetic to Sasakian manifolds

In this section we study compact and connected 3-dimensional trans-Sasakian manifolds and obtain conditions under which they are homothetic to Sasakian manifolds. Our first result uses a bound on the Ricci curvature of the trans-Sasakian manifold in the direction of the vector field \(\xi\).

Theorem 3.1. Let \((M, \phi, \xi, \eta, g, \alpha, \beta)\) be a 3-dimensional compact and connected trans-Sasakian manifold. If the Ricci curvature \(\text{Ric}(\xi, \xi)\) satisfies
\[
0 < \text{Ric}(\xi, \xi) \leq 2 (\alpha^2 + \beta^2),
\]
then \(M\) is homothetic to a Sasakian manifold.

Proof. Using (2.3) we immediately compute
\[
\delta \eta = \text{div} \xi = 2 \beta. \tag{3.1}
\]
Also, since \(d\eta(X, Y) = -2\alpha g(\phi X, Y)\), we obtain
\[
\|d\eta\|^2 = 8 \alpha^2. \tag{3.2}
\]
Now using (2.3), after some obvious calculations, we get
\[
\|\nabla \xi\|^2 = 2(\alpha^2 + \beta^2). \tag{3.3}
\]
Now, using (3.1)-(3.3) in the integral formula (cf. [14])
\[
\int_M \left( \text{Ric}(\xi, \xi) - \frac{1}{2} \|d\eta\|^2 + \|\nabla \xi\|^2 - (\delta \eta)^2 \right) = 0
\]
and the hypothesis of the theorem, we deduce that
\[
\text{Ric}(\xi, \xi) = 2 (\alpha^2 + \beta^2). \tag{3.4}
\]
Using Lemma [12], we have
\[
\text{Ric}(\xi, \xi) = -2\xi(\beta) + 2(\alpha^2 - \beta^2)
\]
which together with (3.4) gives
\[
\xi(\beta) = -2\beta^2. \tag{3.5}
\]
We claim that \(\beta\) must be a constant. If \(\beta\) is not a constant, then on the compact \(M\) it has a local maximum at some \(p \in M\). We have \((\nabla \beta)(p) = 0\) and the Hessian \(H_\beta\) is negative definite at this point \(p\). However, using the equation (3.5), we have \(\xi(\beta)(p) = -2 (\beta(p))^2 = 0\) and \(H_\beta(\xi, \xi)(p) = \xi(\beta)(p) = 0\).
A NOTE ON TRANS-SASAKIAN MANIFOLDS

4(β(p))³ = 0, (where we used ∇ξξ = 0), which yields a contradiction (as the Hessian is negative definite at p). Hence, β is a constant and this, combined with Stokes’ theorem applied to div(ξ) = 2β, proves that β = 0.

Since β = 0, the Lemma 2.1 gives ξ(α) = 0. We claim that α is a constant. If not, on compact M the smooth function α attains a local maximum at some point p ∈ M. At this point, the Hessian Hα is negative definite. However, for the unit vector field ξ, we have Hα(ξ, ξ) = 0, which fails to be negative definite at point p, which is a contradiction. Now, that α is a non-zero constant follows from the condition in the hypothesis. Thus, using (2.3), we compute

\[ α^{-2}(∇X∇Yξ - ∇∇XYξ) = g(Y, ξ)X - g(X, Y)ξ, \]

and this implies by Theorem 2.1 that M is homothetic to a Sasakian manifold.

□

As a direct consequence of the above theorem we have the following result, which has motivation from the fact that on a (2n + 1)-dimensional Sasakian manifold (M, ϕ, ξ, η, g) the Ricci operator satisfies Q(ξ) = 2nξ.

Corollary 3.1. Let (M, ϕ, ξ, η, g, α, β) be a 3-dimensional compact and connected trans-Sasakian manifold. If the vector field ξ satisfies Q(ξ) = 2α²ξ ≠ 0, then M is homothetic to a Sasakian manifold.

As pointed out earlier, on a (2n+1)-dimensional Sasakian manifold (M, ϕ, ξ, η, g), the Ricci operator satisfies Q(ξ) = 2nξ, that is, the Reeb vector field ξ is an eigenvector of the Ricci operator. This motivates the question of whether a 3-dimensional trans-Sasakian manifold (M, ϕ, ξ, η, g, α, β) satisfying Q(ξ) = λξ for a non-zero constant λ, is necessarily homothetic to a Sasakian manifold. We answer this question for compact connected 3-dimensional trans-Sasakian manifolds and show that they are homothetic to Sasakian manifolds.

Theorem 3.2. Let (M, ϕ, ξ, η, g, α, β) be a 3-dimensional compact and connected trans-Sasakian manifold. Then M is homothetic to a Sasakian manifold if and only if the vector field ξ satisfies Q(ξ) = λξ for a non-zero constant λ.

Proof. Using Q(ξ) = λξ in Lemma 2.1 we have

\[ ϕ(∇α) - ∇β = (λ + ξ(β) - 2(α² - β²))ξ. \]  (3.7)

Taking the inner product with ξ in the above equation, we obtain

\[ ξ(β) = -\frac{λ}{2} + (α² - β²). \]  (3.8)

Inserting this value in (3.7), we have

\[ ϕ(∇α) - ∇β = \left(\frac{λ}{2} - (α² - β²)\right)ξ \]  (3.9)
and applying $\varphi$ to the above equation, we obtain
\[ \nabla \alpha = -2\alpha \beta \xi - \varphi(\nabla \beta). \tag{3.10} \]

If $A$ is a symmetric operator on the trans-Sasakian manifold $M$, we can choose a local orthonormal frame that diagonalizes $A$ and consequently, we have
\[ \sum g(\varphi(A e_i), e_i) = 0. \tag{3.11} \]

Now for $X \in \mathfrak{X}(M)$, we compute
\[ \nabla_X (\varphi(\nabla \beta) + 2\alpha \beta \xi) = (\nabla_X \varphi)(\nabla \beta) + \varphi(A \beta X) + 2X(\alpha \beta) \xi + 2\alpha \beta \nabla_X \xi \]
where $A \beta X = \nabla_X \nabla \beta$ is a symmetric operator $A \beta : \mathfrak{X}(M) \to \mathfrak{X}(M)$. Taking the inner product with $X$ in above equation and using equations (2.2) and (2.3), after some easy calculations we arrive at
\[ g(\nabla_X (\varphi(\nabla \beta) + 2\alpha \beta \xi), X) = \alpha X(\beta) \eta(X) - \alpha \xi(\beta) g(X, X) + \beta g(\varphi(A \beta X), X) + 2X(\alpha \beta) \eta(X) + 2\alpha \beta^2 g(X, X) \]

Taking trace in the equation above, in view of the equation (3.11), we get
\[ \text{div} (\varphi(\nabla \beta) + 2\alpha \beta \xi) = -2\alpha \xi(\beta) + 2\xi(\alpha \beta) + 4\alpha \beta^2 = 0, \tag{3.12} \]
where we used the fact that $\xi(\alpha) = -2\alpha \beta$. Thus using (3.12) in the equation (3.10), we conclude that $\Delta \alpha = \text{div}(\nabla \alpha) = 0$ on compact $M$, which proves that $\alpha$ is a constant. Using the constant $\alpha$ in the equation (3.9), we get
\[ -\nabla \beta = \left( \frac{\lambda}{2} - (\alpha^2 - \beta^2) \right) \xi \]
which together with the equation (3.8) gives
\[ \Delta \beta = -2\beta \xi(\beta) - \left( \frac{\lambda}{2} - (\alpha^2 - \beta^2) \right) \text{div} \xi \]
\[ = -2\beta \left( \frac{\lambda}{2} + (\alpha^2 - \beta^2) \right) - 2\beta \left( \frac{\lambda}{2} - (\alpha^2 - \beta^2) \right) \]
\[ = 0. \]

Here we used the fact that $\text{div} \xi = 2\beta$. Thus $\beta$ is a constant, which together with Stokes’ theorem and $\text{div} \xi = 2\beta$ proves that $\beta = 0$. If $\alpha = 0$, then (3.7) would imply $\lambda = 0$, which is a contradiction. Consequently, $\alpha$ is a non-zero constant which by the equation (3.1) satisfies
\[ \alpha^{-2} (\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi) = g(Y, \xi) - g(X, Y) \xi. \]
This proves that $M$ is homothetic to a Sasakian manifold. The converse is obvious. \[ \square \]
4. A CHARACTERIZATION OF COSYMPLECTIC MANIFOLDS

In this section, we study 3-dimensional compact trans-Sasakian manifolds, and obtain a characterization of cosymplectic manifolds. Let \((M, \varphi, \xi, \eta, g, \alpha, \beta)\) be a 3-dimensional trans-Sasakian manifold. Then for each point \(p \in M\) there is a neighbourhood \(U\) of \(p\), where we have a local orthonormal frame \(\{e, \varphi e, \xi\}\) for a unit vector field \(e\) on \(U\) called an adapted frame. Using the equations (2.1), (2.2) and (2.3), we obtain the following local structure equations defined on \(U\):

\[
\nabla_e \xi = \beta e - \alpha \varphi e, \quad \nabla_{\varphi e} \xi = \alpha e + \beta \varphi e, \quad \nabla_\xi \xi = 0, \quad (4.1)
\]

\[
\nabla_e e = \gamma \varphi e - \beta \xi, \quad \nabla_{\varphi e} e = -\delta \varphi e - \alpha \xi, \quad \nabla_\xi e = \lambda \varphi e, \quad (4.2)
\]

\[
\nabla_e \varphi e = -\gamma e + \alpha \xi, \quad \nabla_{\varphi e} \varphi e = \delta e - \beta \xi, \quad \nabla_\xi \varphi e = -\lambda e, \quad (4.3)
\]

where \(\gamma, \delta, \lambda\) are smooth functions defined on \(U\). Using the above equations, we compute

\[
R(e, \varphi e) \xi = (e(\alpha) - \varphi e(\beta)) e + (e(\beta) + \varphi e(\alpha)) \varphi e
\]

\[
R(\varphi e, \xi) e = (\varphi e(\lambda) + \xi(\delta) + \beta \delta - \gamma \alpha - \gamma \lambda) \varphi e + (\xi(\alpha) + 2\alpha \beta) \xi
\]

\[
R(\xi, e) \varphi e = (e(\lambda) - \xi(\gamma) - \beta \gamma - \delta \alpha - \delta \lambda) e + (\xi(\alpha) + 2\alpha \beta) \xi.
\]

Adding these three equations, we conclude that

\[
e(\alpha) - \varphi e(\beta) + e(\lambda) - \xi(\gamma) = \beta \gamma + \delta \alpha + \delta \lambda, \quad (4.4)
\]

\[
e(\beta) + \varphi e(\alpha) + \varphi e(\lambda) + \xi(\delta) = \gamma \alpha + \gamma \lambda - \beta \delta, \quad (4.5)
\]

and the third component gives the result in the Lemma [2.1]. Also, we have

\[
R(\xi, \varphi e) e = (\xi(\gamma) - e(\lambda) + \beta \gamma + \alpha \delta + \lambda \delta) \varphi e + (-\xi(\beta) + \alpha^2 - \beta^2) \xi
\]

and

\[
R(\xi, \varphi e) \varphi e = (\xi(\delta) + \varphi e(\lambda) + \beta \delta - \alpha \gamma - \lambda \gamma) e + (-\xi(\beta) + \alpha^2 - \beta^2) \xi
\]

Using the two equations above in \(Q(\xi) = R(\xi, e)e + R(\xi, \varphi e)\varphi e\), we obtain

\[
Q(\xi) = (\xi(\delta) + \varphi e(\lambda) + \beta \delta - \alpha \gamma - \lambda \gamma) e
\]

\[
+ (\xi(\gamma) - e(\lambda) + \beta \gamma + \alpha \delta + \lambda \delta) \varphi e
\]

\[
+ 2 (-\xi(\beta) + \alpha^2 - \beta^2) \xi.
\]

This together with the equations (4.4) and (4.5) gives

\[
Q(\xi) = - (e(\beta) + \varphi e(\alpha)) e + (e(\alpha) - \varphi e(\beta)) \varphi e + 2 (-\xi(\beta) + \alpha^2 - \beta^2) \xi. \quad (4.6)
\]

Recall that in Theorem [3.2] the vector field \(\xi\) being an eigenvector of the Ricci operator corresponding to a non-zero eigenvalue makes the trans-Sasakian manifold homothetic to a Sasakian manifold. Similarly, we have the following characterization of cosymplectic manifolds.
Theorem 4.1. Let \((M, \varphi, \xi, \eta, g, \alpha, \beta)\) be a 3-dimensional compact and connected trans-Sasakian manifold. Then \(M\) is a cosymplectic manifold if and only if the Ricci operator \(Q\) annihilates the vector field \(\xi\).

Proof. Suppose that \(Q(\xi) = 0\) holds. Then (4.6) gives
\[
e(\beta) = -\varphi e(\alpha), \quad e(\alpha) = \varphi e(\beta), \text{ and } \xi(\beta) = \alpha^2 - \beta^2.
\] (4.7)
Applying Lemma 2.1 and the equations (2.2), (2.3), (4.1)-(4.3) and (4.7), we obtain
\[
\Delta \alpha = ee(\alpha) + \varphi \varphi e(\alpha) + \xi \xi(\alpha) - \nabla e(\alpha) - \nabla \varphi \varphi e(\alpha) - \nabla \xi \xi(\alpha)
\]
\[
= [e, \varphi e](\beta) - 2(\alpha \beta) + \gamma e(\beta) - \delta \varphi \varphi e(\beta) - 4\alpha \beta^2
\]
\[
= -\beta(\gamma e + \alpha \xi)(\beta) - (\delta \varphi \varphi e)(\beta) - 2\xi(\alpha \beta) + \gamma e(\beta) - \delta \varphi \varphi e(\beta) - 4\alpha \beta^2
\]
\[
= 2\alpha \xi(\beta) - 2\xi(\alpha \beta) - 4\alpha \beta^2 = 0.
\]
Thus thanks to compactness of \(M\) we have proved that \(\alpha\) is a constant. If \(\alpha \neq 0\), then Lemma 2.1 implies that \(\beta = 0\) and consequently the equation (4.7) gives \(\alpha = 0\), which is a contradiction. Hence \(\alpha = 0\) and the equation (4.7) gives \(\xi(\beta) = -\beta^2\), that is, \(\text{div}(\beta \xi) = \beta^2\), where we used \(\text{div} \xi = 2\beta\), which follows from the equation (2.3). Using Stokes’ theorem in \(\text{div}(\beta \xi) = \beta^2\), we obtain \(\beta = 0\). That is, \(M\) is a cosymplectic manifold. Conversely, if \(M\) is a cosymplectic manifold, then the equation (4.6) gives that \(Q(\xi) = 0\). □

References
[1] AL-SOLAMY, F. R.—KIM, J.-S.—TRIPATHI, M. M.: On \(\eta\)-Einstein trans-Sasakian manifolds, An. Stiint. Univ. “Al.I.Cuza” din Iasi 57 (2011), no. 2, 417–440.
[2] BLAIR, D. E.: Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics 509, Springer (1976).
AARTS, J. M.—LUTZER, D. J.: Pseudo-completeness and the product of Baire spaces, Pacific J. Math. 48 (1973), 1–10.
[3] BLAIR, D. E.—OUBINA, J. A.: Conformal and related changes of metric on the product of two almost contact metric manifolds, Publ. Mat. 34 (1990), no. 1, 199–207.
[4] DE, U. C.—SARKAR, A.: On three-dimensional trans-Sasakian manifolds, Extracta Math. 23 (2008), no. 3, 265–277.
[5] DE, U. C.—TRIPATHI, M. M.: Ricci tensor in 3-dimensional trans-Sasakian manifolds, Kyungpook Math. J. 43 (2003), no. 2, 247–255.
[6] GRAY, A.—HERVELLA, L. M.: The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl. (4) 123 (1980), 35–58.
[7] FUJIMOTO, A.—MUTO, H.: On cosymplectic manifolds, Tensor 28 (1974), 43–52.
[8] KENMOTSU, K.: A class of almost contact Riemannian manifolds, Tohoku Math. J. 24 (1972), 93-103.
[9] KIM, J.-S.—PRASAD, R.—TRIPATHI, M. M.: On generalized Ricci-recurrent trans-Sasakian manifolds, J. Korean Math. Soc. 39 (2002), no. 6, 953–961.
[10] KIRICHENKO, V. F.: On the geometry of nearly trans-Sasakian manifolds, (Russian) Dokl. Akad. Nauk 397 (2004), no. 6, 733–736.
[11] MARRERO, J. C.: *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. (4) 162 (1992), 77–86.

[12] OKUMURA, M.: *Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures*, Tôhoku Math. J. (2) 16 (1964) 270–284.

[13] OUBINA, J. A.: *New classes of almost contact metric structures*, Publ. Math. Debrecen 32 (1985), no. 3-4, 187–193.

[14] YANO, K.: *Integral formulas in Riemannian Geometry*, Marcel Dekker (1970).

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