Scattering theory without large-distance asymptotics in arbitrary dimensions

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Received 16 December 2015, revised 20 June 2016
Accepted for publication 5 October 2016
Published 27 October 2016

Abstract
In conventional scattering theory, by large-distance asymptotics, at the cost of losing the information of the distance between target and observer, one imposes large-distance asymptotics to achieve a scattering wave function which can be represented explicitly by a scattering phase shift. In this paper, without large-distance asymptotics, we establish an arbitrary-dimensional scattering theory. Arbitrary-dimensional scattering wave functions, scattering boundary conditions, and phase shifts are given without large-distance asymptotics. We give a discussion of one- and two-dimensional scatterings. Moreover, we also suggest a dimensional renormalization scheme to remove the divergence encountered in singular-potential scattering.

Keywords: scattering wave function, scattering phase shift, large-distance asymptotics, arbitrary dimensions, dimensional renormalization

1. Introduction
In conventional scattering theory, the information of the distance between target and observer is lost due to large-distance asymptotics. By large-distance asymptotics, in conventional three-dimensional scattering theory, the solution of the radial Schrödinger equation in the asymptotic region and the scattering boundary condition are approximately represented as [1]

\begin{equation}
R_i(r) \approx \infty A_i \frac{\sin(kr - l\pi/2 + \delta_i)}{kr},
\end{equation}

\begin{equation}
\psi(r, \theta) \approx \sum_{l=0}^{\infty} (2l + 1)i^l \frac{\sin(kr - l\pi/2)}{kr} P_l(\cos \theta) + f(\theta) \frac{\psi_{kr}}{r},
\end{equation}

respectively, where $\delta_i$ is the scattering phase shift and $f(\theta)$ is the scattering amplitude.
Without large-distance asymptotics, in [2], the asymptotic solution (1.1) and the asymptotic scattering boundary condition (1.2) are replaced by the following exact solutions:

\[
R_i(r) = M_i \left( -\frac{1}{ikr} \right) \frac{d}{dr} \sin \left[ kr - \frac{l\pi}{2} + \delta_i + \Delta_i \left( -\frac{1}{ikr} \right) \right], \quad (1.3)
\]

\[
\psi(r, \theta) = \sum_{l=0}^{\infty} (2l + 1)^i M_i \left( -\frac{1}{ikr} \right) \frac{1}{kr} \sin \left[ kr - \frac{l\pi}{2} + \Delta_i \left( -\frac{1}{ikr} \right) \right] P_l(\cos \theta) + f(r, \theta) \frac{e^{ikr}}{r}, \quad (1.4)
\]

where \( M_i(x) = |y_i(x)| \) and \( \Delta_i(x) = \text{arg } y_i(x) \) are the modulus and argument of the Bessel polynomial \( y_i(x) \) [2], respectively, and \( P_l(x) \) is the Legendre polynomial.

In this paper, we establish a rigorous scattering theory without large-distance asymptotics in arbitrary dimensions.

In the following, we first rewrite the three-dimensional scattering theory established in [2] in a new form which is convenient to be generalized to arbitrary dimensions, and from which one can directly see what happens after a scattering. We will show that, like that in three-dimensional cases, the scattering phase shift is the only effect in arbitrary-dimensional elastic scatterings.

Moreover, it will be shown that the scattering theory is different in odd and even dimensions.

An arbitrary-dimensional scattering theory is important in quantum field theory. For example, in scattering spectral method, to perform a dimensional regularization procedure requires us to be able to carry out scattering theory calculations in arbitrary dimensions [3–5]. Moreover, two special cases, one- and two-dimensional scatterings, are important both in theories and experiments.

In singular-potential scattering, one encounters divergences. In an arbitrary-dimensional scattering theory, the value of spatial dimension appears as a parameter in the theory. This allows us to implement dimensional renormalization to remove divergences. Inspired by the renormalization technique in quantum field theory [3, 4], we suggest a dimensional renormalization treatment to remove the divergence encountered in scattering problems.

In section 2, we rewrite the three-dimensional scattering theory established in [2] in a new form. In section 3, we give an exact \( n \)-dimensional scattering wave function without large-distance asymptotics. In section 4, we construct a \( n \)-dimensional scattering boundary condition without large-distance asymptotics. In section 5, we rewrite the results given in sections 3 and 4 by sine functions, which is the form in conventional scattering theory. In section 6, we demonstrate how the scattering phase shift appears. In section 7, we give the \( n \)-dimensional scattering cross section. In section 8, as examples, we discuss one-, two-, and three-dimensional scatterings. In section 9, we demonstrate how to take large-distance asymptotics of a \( n \)-dimensional scattering theory. In section 10, we suggest a dimensional renormalization scheme in singular-potential scattering. The conclusion is given in section 11.

2. An alternative expression of three-dimensional scattering theory without large-distance asymptotics

In order to establish an arbitrary-dimensional scattering theory without large-distance asymptotics, in this section, we rewrite the three-dimensional scattering theory without large-
distance asymptotics given in [2] in a new form which is convenient to be generalized to arbitrary dimensions.

For a three-dimensional scattering, without large-distance asymptotics, the incident plane wave is

\[ \psi^{\text{in}}(r, \theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} [h^{(2)}_l(k r) + h^{(1)}_l(k r)] P_l(\cos \theta); \]  

(2.1)

After an elastic scattering, the wave function becomes

\[ \psi(r, \theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} [h^{(2)}_l(k r) + e^{2i\delta} h^{(1)}_l(k r)] P_l(\cos \theta), \]

(2.2)

where \( h^{(1)}_l(z) \) and \( h^{(2)}_l(z) \) are the first and second kind spherical Hankel functions.

The scattering boundary condition is

\[ \psi(r, \theta) = e^{ikr \cos \theta} + \sum_{l=0}^{\infty} a_l(\theta) h^{(1)}_l(k r), \]

(2.3)

where

\[ a_l(\theta) = (2l + 1) \frac{1}{2} (e^{2i\delta} - 1) P_l(\cos \theta). \]

(2.4)

**Proof.** The incident plane wave is

\[ \psi^{\text{in}}(r, \theta) = e^{ikr \cos \theta}. \]

(2.5)

Substituting the plane wave expansion \( e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} j_l(k r) P_l(\cos \theta) \) and \( j_l(z) = \frac{1}{2} [h^{(1)}_l(z) + h^{(2)}_l(z)] \) [2] into equation (2.5) proves equation (2.1) directly, where \( j_l(z) \) is the spherical Bessel function [6].

Next we prove equations (2.2) and (2.3).

Without large-distance asymptotics, in [2], we show that the scattering boundary condition can be expressed as

\[ \psi(r, \theta) = e^{ikr \cos \theta} + f(r, \theta) \frac{e^{ikr}}{r} \]

(2.6)

with

\[ f(r, \theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) (e^{2i\delta} - 1) P_l(\cos \theta) j_l\left( -\frac{1}{ikr} \right). \]

(2.7)

By equation (2.4), we can rewrite \( f(r, \theta) \) as

\[ f(r, \theta) = \frac{1}{k} \sum_{l=0}^{\infty} a_l(\theta)(-i)^{l+1} j_l\left( -\frac{1}{ikr} \right). \]

(2.8)

Substituting equation (2.8) into equation (2.6) and using \( h^{(1)}_l(i) = (-i)^{l+1} (e^{i\zeta}/\zeta) j_l(i/\zeta) \) [2], we arrive at equation (2.3).

Finally, substituting equation (2.1) into (2.3) and using (2.4) prove equation (2.2). ■

By large-distance asymptotics, in conventional scattering theory, it is proved that the phase shift is the only effect after an elastic scattering and all information of an elastic scattering is embedded in a scattering phase shift [1].
Without large-distance asymptotics, it is proved by comparing equations (2.1) and (2.2) that, the only effect after an elastic scattering is still a phase shift on the outgoing wave function: the incoming part, represented by \( h^{(2)}_l(kr) \), does not change anymore; the outgoing part, represented by \( h^{(1)}_l(kr) \), changes a phase factor \( e^{2i\delta} \).

Naturally, when taking large-distance asymptotics, the above result will reduce to conventional scattering theory: \( \sum_{i=0}^{\infty} a_i(\theta)h^{(1)}_l(kr) \stackrel{r \to \infty}{\sim} \infty f(\theta)e^{ikr}/r \) with the scattering amplitude \( f(\theta) = \sum_{i=0}^{\infty} a_i(\theta)/(i^{l+1}k) \).

### 3. \( n \)-dimensional scattering wave function

For a \( n \)-dimensional scattering, the radial wave equation with a spherical potential reads [3]

\[
\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} + k^2 - \frac{l(l+n-2)}{r^2} - V(r) \right] R_l(r) = 0.
\]  

(3.1)

The solution of the asymptotic equation of the radial equation (3.1), i.e., equation (3.1) with \( V(r) = 0 \), can be solved exactly:

\[
R_l(r) = C_l h^{(2)}_{l+(n-3)/2}(kr) r^{-(n-3)/2} + D_l h^{(1)}_{l+(n-3)/2}(kr) r^{-(n-3)/2}.
\]  

(3.2)

It should be noted that in conventional scattering theory the exact solution of the asymptotic equation (3.1) is approximated by an asymptotic solution of the asymptotic equation, like equation (1.1).

The \( n \)-dimensional wave function can be expressed as

\[
\psi(r, \theta) = \sum_{l=0}^{\infty} R_l(r) C_l^{n/2-1}(\cos \theta),
\]

where \( C_l^\lambda(z) \) is the Gegenbauer polynomial, a generalization of the Legendre polynomial [6]. Then, by equation (3.2), we arrive at

\[
\psi(r, \theta) = \sum_{l=0}^{\infty} C_l \left[ h^{(2)}_{l+(n-3)/2}(kr) r^{-(n-3)/2} + e^{2i\delta} h^{(1)}_{l+(n-3)/2}(kr) r^{-(n-3)/2} \right] C_l^{n/2-1}(\cos \theta),
\]  

(3.3)

where \( e^{2i\delta} = D_l/C_l \) defines the phase shift [2].

### 4. \( n \)-dimensional scattering boundary condition

A scattering is determined by the Schrödinger equation with a scattering boundary condition. In conventional scattering theory, the scattering boundary condition is the Sommerfeld radiation condition which is constructed under large-distance asymptotics. In our preceding work [2], without large-distance asymptotics, instead of the Sommerfeld radiation condition, we construct a scattering boundary condition, equation (1.4) or, equivalently, equation (2.3), which preserves the information of the distance between the target and observer.

#### 4.1. Scattering boundary condition

In the following, without large-distance asymptotics, we construct the \( n \)-dimensional scattering boundary condition.
Generally speaking, a scattering boundary condition is a wave function at an asymptotic distance, consisting of two parts: the incident wave $\psi^i$ and the scattering wave $\psi^s$, i.e., $\psi = \psi^i + \psi^s$. In three-dimensional conventional scattering theory, $\psi^s$ is chosen as being in proportion to $e^{ikr}/r$, since the asymptotics of the scattering wave function is $R^r - \infty e^{ikr}/r$ and only the outgoing wave $R^r - \infty e^{ikr}/r$ remains in the scattering wave function when $r \to \infty$ [7]. Without the asymptotic approximation, as shown in equation (3.2), the solution is $h_{1+(n-3)/2}^1(kr)/r^{(n-3)/2}$ and only the outgoing wave $h_{1+(n-3)/2}^1(kr)/r^{(n-3)/2}$ remains in the scattering wave function. To retrieve the information of the distance, we construct the scattering boundary condition by $h_{1+(n-3)/2}^1(kr)/r^{(n-3)/2}$ rather than its asymptotics $e^{ikr}/r$.

To generalize the three-dimensional scattering boundary condition to $n$ dimensions, we replace the three-dimensional outgoing wave $h_{1+(n-3)/2}^1(kr)$ in equation (2.3) with $n$-dimensional outgoing wave $h_{1+(n-3)/2}^1(kr)/r^{(n-3)/2}$;

$$\psi(r, \theta) = e^{ikr} cos \theta + \sum_{l=0}^{\infty} a_l(\theta) \frac{h_{1+(n-3)/2}^1(kr)}{r^{(n-3)/2}}. \quad (4.1)$$

The expression of $a_l(\theta)$ will be given in the following.

4.2. $a_l(\theta)$

In a scattering theory without large-distance asymptotics, $a_l(\theta)$ plays the role of the partial wave scattering amplitude in conventional scattering theory, and the information of the scattering is embedded in $a_l(\theta)$. In this section, we calculate $a_l(\theta)$ in $n$ dimensions.

By using the $n$-dimensional plane wave expansion [6]

$$e^{ikr} cos \theta = \frac{\Gamma(n/2 - 1)}{\sqrt{\pi} (k/2)^{(n-3)/2}} \sum_{l=0}^{\infty} (2l + n - 2)! \frac{\hat{h}_{1+(n-3)/2}^l(kr)}{r^{(n-3)/2}} C_l^{n/2-1}(cos \theta) \quad (4.2)$$

and $h(z) = \frac{1}{2}[h_z^1(z) + h_z^2(z)]$, we can rewrite the scattering boundary condition (4.1) as

$$\psi(r, \theta) = \sum_{l=0}^{\infty} \frac{\Gamma(n/2 - 1)}{\sqrt{\pi} (k/2)^{(n-3)/2}} (2l + n - 2)! \frac{C_l^{n/2-1}(cos \theta) + a_l(\theta) \hat{h}_{1+(n-3)/2}^l(kr)}{r^{(n-3)/2}}$$

$$+ \frac{\Gamma(n/2 - 1)}{\sqrt{\pi} (k/2)^{(n-3)/2}} \sum_{l=0}^{\infty} (2l + n - 2)! \frac{\hat{h}_{1+(n-3)/2}^l(kr)}{r^{(n-3)/2}} C_l^{n/2-1}(cos \theta). \quad (4.3)$$

Then $a_l(\theta)$ can be achieved immediately by equating the coefficients in equations (4.3) and (3.3):

$$\frac{\Gamma(n/2 - 1)}{\sqrt{\pi} (k/2)^{(n-3)/2}} (2l + n - 2)! \frac{1}{2} C_l^{n/2-1}(cos \theta) + a_l(\theta) = C_l e^{2i\delta} C_l^{n/2-1}(cos \theta), \quad (4.4)$$

$$\frac{\Gamma(n/2 - 1)}{\sqrt{\pi} (k/2)^{(n-3)/2}} (2l + n - 2)! \frac{1}{2} = C_l. \quad (4.5)$$

Substituting equation (4.5) into equation (4.4) gives

$$a_l(\theta) = \frac{\Gamma(n/2 - 1)}{\sqrt{\pi} (k/2)^{(n-3)/2}} (2l + n - 2)! \frac{1}{2} (e^{2i\delta} - 1) C_l^{n/2-1}(cos \theta). \quad (4.6)$$

It should be emphasized that $n = 2$ is a removable singularity of $a_l(\theta)$, which will be discussed in section 8.2.
5. Representing scattering wave function by sine function

In conventional scattering theory, the scattering wave function is approximately expressed by a sine function. In [2], we show that the scattering wave function, in fact, can be exactly expressed by a sine function. In this section, we represent the $n$-dimensional scattering wave function by a sine function exactly.

5.1. Radial wave function

In order to represent the scattering wave function by a sine function, we first rewrite the spherical Hankel function as

$$h^{(1)}_{\nu}(z) = e^{i(z-\nu \pi/2)/2} \frac{1}{iz} J_{\nu} \left( -\frac{1}{iz} \right), \quad (5.1)$$

$$h^{(2)}_{\nu}(z) = -e^{-i(z-\nu \pi/2)/2} \frac{1}{iz} J_{\nu} \left( \frac{1}{iz} \right). \quad (5.2)$$

Here we introduce

$$\mathcal{Y}_{\nu}(z) = \left( \frac{2}{z} \right)^{\nu+1/2} U \left( \nu + 1, 2(\nu + 1), \frac{2}{z} \right), \quad (5.3)$$

where $U(a, b; z)$ is the Tricomi confluent hypergeometric function [6]. Notice that $\mathcal{Y}_{\nu}(z)$ recovers the Bessel polynomial in odd dimensions.

The radial wave function (3.2) then can be expressed as

$$R_{\nu}(r) = C_{\nu} \left\{-e^{i(kr-(l+(n-3)/2)(\pi/2))} \frac{1}{r^{(n-3)/2}} \mathcal{Y}_{l+(n-3)/2} \left( \frac{1}{ikr} \right) \right.$$

$$+ e^{2i\theta} e^{i(kr-(l+(n-3)/2)(\pi/2))} \frac{1}{r^{(n-3)/2}} \mathcal{Y}_{l+(n-3)/2} \left( -\frac{1}{ikr} \right) \right\}; \quad (5.4)$$

notice that here $e^{2i\theta} = D_l/C_l$. Then the radial wave function (5.4) can be represented by a sine function:

$$R_{\nu}(r) = M_{\nu} \left\{-\frac{1}{ikr} \frac{A_{l}}{kr r^{(n-3)/2}} \sin \left[ kr - \left( l + \frac{n - 3}{2} \right) \frac{\pi}{2} + \Delta_{l} \left( -\frac{1}{ikr} \right) \right] \right\}; \quad (5.5)$$

where $A_{l} = 2\sqrt{C_{l}D_{l}}$ and $M_{\nu} = \left| \mathcal{Y}_{l+(n-3)/2} \left( \frac{1}{ikr} \right) \right|$ and $\Delta_{l} = \arg \mathcal{Y}_{l+(n-3)/2} \left( -\frac{1}{ikr} \right)$ are the modulus and argument of $\mathcal{Y}_{l+(n-3)/2} \left( -\frac{1}{ikr} \right)$, respectively.

When employing large-distance asymptotics, the radial wave function becomes

$$R_{\nu}(r) \sim \frac{A_{l}}{kr r^{(n-3)/2}} \sin \left[ kr - \left( l + \frac{n - 3}{2} \right) \frac{\pi}{2} + \Delta_{l} \right], \quad (5.6)$$

where asymptotics $U(a, b; z) \sim 1/e^a$ [6] and equation (9.4) which will be proved in section 9 are used.

5.2. $\mathcal{Y}_{\nu}(z)$ in odd and even dimensions

As will be shown in the following, the function $\mathcal{Y}_{\nu}(z)$ defined by equation (5.3) is different in odd and even dimensions: in odd dimensions $\mathcal{Y}_{\nu}(z)$ is a polynomial and in even dimensions $\mathcal{Y}_{\nu}(z)$ is an infinite series.
The Tricomi confluent hypergeometric function \( U(a, b; z) \), when \( b = m + 1 \) \((m = 0, 1, 2, \ldots)\), can be expanded as \([6]\)

\[
U(a, m + 1, z) = \frac{(-1)^{m+1}}{m! \Gamma(a - m)} \sum_{k=0}^{\infty} \frac{(a)_k}{(m+1)_k k!} z^k \\
\times [\ln z + \psi(a + k) - \psi(1 + k) - \psi(m + k - 1)] \\
+ \frac{1}{\Gamma(a)} \sum_{k=1}^{m} \frac{(k-1)! (1-a+k)_{m-k}}{(m-k)!} \frac{1}{z^k},
\]

where \((\alpha)_n = \alpha (\alpha + 1) \ldots (\alpha + n - 1) = \Gamma(\alpha + n)/\Gamma(\alpha)\) is the Pochhammer symbol and \(\psi(z) = \Gamma'(z)/\Gamma(z)\) is the digamma function \([6]\).

Then by equation \((5.3)\), for \(n \geq 2\), we have

\[
\mathcal{J}_{l+(n-3)/2}(z) = \left(\frac{2}{z}\right)^{(l+(n-1)/2)l+(n-3)/2} \sum_{j=0}^{\infty} \frac{\Gamma(j+1)}{\Gamma(-l-n/2+5/2+j)\Gamma(2l+n-2-j)} \left(\frac{z}{2}\right)^{j+1} \\
+ \left(\frac{2}{z}\right)^{(l+(n-1)/2)l+(n-3)/2} \frac{(-1)^{l+n-1}}{(2l+n-2)! \Gamma(-l-(n-3)/2)} \\
\times \sum_{j=0}^{\infty} \frac{(l+(n-1)/2)_{j!}}{(2l+n-1)_{j!}} \left(\frac{z}{2}\right)^{j} \\
\times \left[\ln \left(\frac{2}{z}\right) + \psi \left(l+n-1/2+j\right) - \psi \left(1+j\right) - \psi \left(2l+n-3+j\right)\right].
\]

For odd-dimensional cases \((n \neq 1)\), i.e., \(n = 3, 5, \ldots\), the second term in equation \((5.8)\) equals zero because \(1/\Gamma(-l-(n-3)/2) = 0\). Moreover, because \(1/\Gamma(-l-n/2+5/2+j) = 0\) when \(-l-n/2+5/2+j = 0, -1, -2, \ldots\), the summation of \(j\) in fact begins with \(j = l+(n-3)/2\) rather than \(j = 0\). Therefore, in odd dimensions \((n \neq 1)\), \(\mathcal{J}_{l+(n-3)/2}(z)\) is in fact a polynomial,

\[
\mathcal{J}_{l+(n-3)/2}(z) = \left(\frac{2}{z}\right)^{(l+(n-1)/2)l+(n-3)/2} \sum_{j=l+(n-3)/2}^{2l+n-3} \frac{\Gamma(j+1)}{\Gamma(-l-n/2+5/2+j)\Gamma(2l+n-2-j)} \left(\frac{z}{2}\right)^{j+1}.
\]

This result can be rewritten as

\[
\mathcal{J}_{\nu}(z) = \sum_{j=0}^{\nu} \frac{(\nu+j)!}{j! \Gamma(\nu-j)!} \left(\frac{z}{2}\right)^{j} = \mathcal{J}_{\nu}(z),
\]

where \(\mathcal{J}_{\nu}(z)\) is just the Bessel polynomial. That is to say, \(\mathcal{J}_{\nu}(z)\) recovers the Bessel polynomial \(y_{\nu}(z)\) in odd dimensions.

On the contrary, for even-dimensional cases, \(n\) is an even number and \(\nu = l+(n-3)/2\) is a half-integer; as a result, \(\mathcal{J}_{\nu}(z)\) is an infinite series rather than a polynomial.

6. Wave functions before and after a scattering: phase shifts

In this section, similar to equations \((2.1)\) and \((2.2)\), we write out the \(n\)-dimensional wave functions before and after a scattering. The phase shift is the only effect in an elastic scattering
process, i.e., all information of an elastic scattering process is embedded in a scattering phase shift \[1\]. The result given below will show how a scattering phase shift appears.

The \(n\)-dimensional incident plane wave, by equation (4.2), can be expressed as

\[
\psi_{\text{in}}(r, \theta) = \frac{\Gamma(n/2 - 1)}{\sqrt{\pi} \Gamma(k/2)n^{n-3}/2} \sum_{l=0}^{\infty} (2l + n - 2)^l
\times \frac{1}{2} \left[ \frac{h_{l+1/2}^{(2)}(kr)}{r^{n-3/2}} + \frac{h_{l+1}^{(1)}(kr)}{r^{n-3/2}} \right] C_l^{n/2-1}(\cos \theta). \tag{6.1}
\]

After an elastic scattering, the wave function, by equations (4.1) and (4.6), becomes

\[
\psi(r, \theta) = \frac{\Gamma(n/2 - 1)}{\sqrt{\pi} \Gamma(k/2)n^{n-3}/2} \sum_{l=0}^{\infty} (2l + n - 2)^l
\times \frac{1}{2} \left[ \frac{h_{l+1/2}^{(2)}(kr)}{r^{n-3/2}} + \frac{\exp(\mathrm{i} \theta) h_{l+1}^{(1)}(kr)}{r^{n-3/2}} \right] C_l^{n/2-1}(\cos \theta). \tag{6.2}
\]

Comparing the wave functions before and after a scattering process, equations (6.1) and (6.2), we can see that after a scattering, the incoming part which is represented by \(h_{l+1/2}^{(2)}(kr)/r^{(n-3)/2}\), does not change anymore, while a phase factor \(\exp(\mathrm{i} \theta)\) appears in the outgoing part which is represented by \(\frac{\exp(\mathrm{i} \theta) h_{l+1}^{(1)}(kr)}{r^{n-3/2}}\). This reveals that the only effect after an elastic scattering is a phase shift on the outgoing wave function.

7. \(n\)-dimensional differential scattering cross section

Without large-distance asymptotics, the \(n\)-dimensional differential scattering cross section is

\[
\frac{d\sigma}{d\Omega} = \frac{j^s \cdot dS}{j^{\text{in}}} = \frac{|j^s|}{j^{\text{in}}} \frac{\pi^{n-1}}{\cos \gamma} = \frac{\sqrt{(j^s_0)^2 + (j^s_\gamma)^2}}{j^{\text{in}}}, \tag{7.1}
\]

where \(dS = \hat{n} r^{n-1} d\Omega\) with the \(n\)-dimensional solid angle \(d\Omega = \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots \sin \theta_{n-2} d\theta_1 d\theta_2 \ldots d\theta_{n-2} d\phi\) and \(\gamma\) is the angle between \(j^s\) and \(\hat{r}\),

\[
\tan \gamma = \frac{j^s_\gamma}{j^s_0}. \tag{7.2}
\]

The \(n\)-dimensional differential scattering cross section (7.1) can be rewritten as

\[
\frac{d\sigma}{d\Omega} = \frac{j^s_\gamma}{j^{\text{in}}} \left[ 1 + \left( \frac{j^s_0}{j^s_\gamma} \right)^2 \right] r^{n-1}. \tag{7.3}
\]

The leading contribution of the \(n\)-dimensional differential scattering cross section is

\[
\frac{d\sigma}{d\Omega} = \frac{j^s_\gamma}{j^{\text{in}}} r^{n-1} = \frac{1}{k} \operatorname{Im} \left( \psi^s \frac{\partial}{\partial r} \psi^s \right) r^{n-1}. \tag{7.4}
\]

Substituting \(\psi^s = \psi - \psi_{\text{in}} = \sum_{l=0}^{\infty} a_l(\theta) h_{l+1/2}^{(1)}(kr)/r^{(n-3)/2}\) into equation (7.4), we have
\[
\frac{d\sigma}{d\Omega} = r^{n-1} \frac{1}{2ik} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} a_l^*(\theta) a_{l'}(\theta) W_l \left[ \frac{h_{l'}^{(2)}(r) \rho^{(n-3)/2}}{\rho^{(n-3)/2}}, \frac{h_l^{(1)}(r)}{\rho^{(n-3)/2}} \right], \tag{7.5}
\]

where \( W_l[f(r), g(r)] = f(r) \frac{\partial}{\partial r} g(r) - g(r) \frac{\partial}{\partial r} f(r) \) is the Wronskian determinant.

### 8. One-, two-, and three-dimensional scatterings

In this section, we discuss one-, two-, and three-dimensional scatterings, respectively. These three kinds of scattering can occur in real physical systems.

#### 8.1. One-dimensional scattering

For a one-dimensional scattering, \( n = 1 \), by equation (4.6), we have

\[
a_l(\theta) = -k(2l - 1) \frac{1}{2} (e^{2i\delta_l} - 1) C_l^{-1/2}(\cos \theta). \tag{8.1}
\]

In the one-dimensional case, \( \theta \) can only take two possible values, 0 and \( \pi \). Therefore,

\[
C_l^{-1/2}(1) = C_l^{-1/2}(-1) = C_l^{-1/2}(-1) = -C_l^{-1/2}(1) = 1,
\]

\[
C_l^{-1/2}(\cos \theta) = 0, \quad l = 0, 1. \tag{8.2}
\]

Thus we have

\[
a_0(0) = a_0(\pi) = -\frac{1}{2} k (e^{2i\delta_0} - 1),
\]

\[
a_1(0) = -a_1(\pi) = -\frac{1}{2} ik (e^{2i\delta_1} - 1),
\]

\[
a_l(0) = a_l(\pi) = 0, \quad l = 0, 1. \tag{8.3}
\]

From equation (7.5), we obtain the differential scattering cross section at \( \theta = 0 \) and \( \theta = \pi \):

\[
\sigma(0) = \sin^2 \delta_0 + \sin^2 \delta_1 + 2 \cos(\delta_0 - \delta_1) \sin \delta_0 \sin \delta_1, \tag{8.4}
\]

\[
\sigma(\pi) = \sin^2 \delta_0 + \sin^2 \delta_1 - 2 \cos(\delta_0 - \delta_1) \sin \delta_0 \sin \delta_1. \tag{8.5}
\]

In a one-dimensional scattering, we are interested in transmissivity \( T \) and reflectivity \( R \):

\[
T = \frac{\sigma(0)}{\sigma(0) + \sigma(\pi)} = \frac{1}{2} + \frac{\cos(\delta_0 - \delta_1) \sin \delta_0 \sin \delta_1}{\sin^2 \delta_0 + \sin^2 \delta_1}, \tag{8.6}
\]

\[
R = \frac{\sigma(\pi)}{\sigma(0) + \sigma(\pi)} = \frac{1}{2} - \frac{\cos(\delta_0 - \delta_1) \sin \delta_0 \sin \delta_1}{\sin^2 \delta_0 + \sin^2 \delta_1}. \tag{8.7}
\]

It can be seen that in one dimension the scattering result is independent of the distance \( r \).

#### 8.2. Two-dimensional scattering

In a two-dimensional scattering, we encounter a singularity in \( a_l(\theta) \) given by equation (4.6). We will show that, however, \( n = 2 \) is a removable singularity.

We can see from equation (4.6) that \( n = 2 \) is a singularity of the gamma function \( \Gamma(n/2 - 1) \), but, meanwhile, \( n = 2 \) is also a zero of the Gegenbauer polynomial \( C_n^{(n/2 - 1)}(\cos \theta) \). This makes \( n = 2 \) a removable singularity.
When $n = 2$, $a_i(\theta)$ given by equation (4.6) reduces to

$$a_i(\theta) = \text{Deg}(l) \frac{1}{2} \frac{\sqrt{2k}}{\pi} (e^{2ik\gamma} - 1) \cos (l\theta),$$

where \(\text{Deg}(l)\) is the degeneracy,

\[
\text{Deg}(l) = 1, \quad l = 0,
\]

\[
\text{Deg}(l) = 2, \quad l \neq 0.
\]

The differential scattering cross section can be obtained by equation (7.5) with $n = 2$:

$$\frac{d\sigma}{d\Omega} = \frac{r}{2\pi k} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} a_i^*(\theta) a_i'(\theta) W_{l'} \left[ \frac{\tilde{h}_{l'-1/2}(kr)}{r^{-1/2}}, \frac{\tilde{h}_{l'-1/2}(kr)}{r^{-1/2}} \right]$$

$$= \frac{kr}{2\pi} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \text{Deg}(l) \text{Deg}(l') (-i)^l l' (e^{-2ik\gamma} - 1)(e^{2ik\gamma} - 1) \cos (l\theta) \cos (l'\theta)$$

$$\times \frac{1}{2i k} W_{l'} [\tilde{F} \tilde{h}_{l'-1/2}(kr), \tilde{F} \tilde{h}_{l'-1/2}(kr)].$$

Performing large-distance asymptotics gives

$$\frac{d\sigma}{d\Omega} \sim \frac{1}{2\pi k} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \text{Deg}(l) \text{Deg}(l') (e^{-2ik\gamma} - 1)(e^{2ik\gamma} - 1) \cos (l\theta) \cos (l'\theta)$$

$$= |f(\theta)|^2.$$  

8.3. Three-dimensional scattering

The three-dimensional result can be obtained directly by setting the dimension $n = 3$ in the above result.

Equation (4.6) with $n = 3$, by the relation $C_l^{1/2}(\cos \theta) = P_l(\cos \theta)$ [6], gives equation (2.4). The differential scattering cross section can then be obtained by equation (7.5):

$$\frac{d\sigma}{d\Omega} = r^2 \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} a_i^*(\theta) a_i'(\theta) \frac{W_{l'} [\tilde{h}_{l'-1/2}(kr), \tilde{h}_{l'-1/2}(kr)]}{2ik}.$$  

This result agrees with the result given by [2].

9. $n$-dimensional scattering with large-distance asymptotics

In the above, we obtain a $n$-dimensional scattering theory without large-distance asymptotics, which contains the information of the distance between target and observer. In this section, we demonstrate how this result reduces to the conventional scattering result when taking $r \rightarrow \infty$ asymptotics.

Using [6]

$$h_{\nu}^{(1/2)}(z) = \sqrt{\frac{\pi}{2z}} H_{\nu}^{(1/2)}(z)$$

(9.1)
and

\[ H^{(1,2)}_\nu(z) = \mp \frac{2}{\sqrt{\pi}} \text{e}^{\pm i\pi \nu} (2z)^\nu \text{e}^{\pm iz} U \left( \nu + \frac{1}{2}, 2\nu + 1, \mp 2iz \right), \]  

(9.2)

where \( H^{(1,2)}_\nu(z) \) are the Hankel functions of the first kind and the second kind, we have

\[ h^{(1)}_{l+(n-3)/2}(kr) = \frac{\text{e}^{ikr}}{kr} (-i)^{l+(n-1)/2} (-2ikr)^{l+(n-1)/2} U \left( l + \frac{n-1}{2}, 2l + n - 1, -2ikr \right). \]

(9.3)

Using large-distance asymptotics of the Tricomi confluent hypergeometric function, \( U(a, b, z) \to \infty z^{-a} \), we arrive at

\[ h^{(1)}_{l+(n-3)/2}(kr) \to \infty \frac{\text{e}^{ikr}}{kr} (-i)^{l+(n-1)/2}. \]

(9.4)

The \( n \)-dimensional scattering condition (4.1) becomes

\[ \psi(r, \theta) \to \infty \text{e}^{ikr} \cos \theta + f(\theta) \frac{\text{e}^{ikr}}{r^{(n-1)/2}}, \]

(9.5)

where the \( n \)-dimensional scattering amplitude

\[ f(\theta) = \sum_{l=0}^{\infty} a_l(\theta) \frac{(-i)^{l+(n-1)/2}}{k}. \]

(9.6)

By equation (4.6), the large-distance asymptotic \( n \)-dimensional scattering amplitude can be expressed as

\[ f(\theta) = \frac{1}{2ik} \frac{(-i)^{n-3/2} \Gamma(n/2 - 1)}{\sqrt{\pi}} \sum_{l=0}^{\infty} (2l + n - 2)(e^{2\delta} - 1) C^{n/2-1}_l \cos \theta. \]

(9.7)

The large-distance asymptotic \( n \)-dimensional differential scattering cross section, by equations (9.4) and (7.5), reads

\[ \frac{d\sigma}{d\Omega} \to \infty \frac{1}{2k} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} a^*_l(\theta) a_{l'}(\theta) W_l \left( \frac{1}{r^{(n-3)/2}} \text{e}^{-ikr} r^{l+(n-1)/2}, \right) \]

\[ = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \frac{r^{l+(n-1)/2} a^*_l(\theta) (-i)^{l+(n-1)/2} a_{l'}(\theta)}{k} \]

\[ = |f(\theta)|^2. \]

(9.8)

Integrating the differential scattering cross section gives the large-distance asymptotic \( n \)-dimensional total cross section:

\[ \sigma = \frac{4\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \frac{1}{k^{n-1}} \sum_{l=0}^{\infty} (2l + n - 2)(l + 1) \delta l \sin^2 \delta. \]

(9.9)
10. Dimensional renormalization for singular-potential scattering

One purpose to establish an arbitrary dimensional theory is to implement dimensional renormalization [3, 4]. In quantum field theory, for instance, the scattering spectrum method [8–10] is based on an arbitrary dimensional scattering theory. In the following, as an example, we demonstrate through examples how to use a dimensional renormalization treatment to remove the divergence in scattering problems in quantum mechanics.

Once a potential is a singular potential, one encounters divergences. In order to remove the divergence, we need a renormalization procedure. In quantum field theory, a powerful renormalization technique is dimensional renormalization. To implement dimensional renormalization, we need an arbitrary dimensional theory in which the value of spatial dimension \( n \) appears as a renormalization parameter. In this section, we show that an arbitrary scattering theory can be applied to the problem of singular potentials to remove divergences.

In order to illustrate the validity of dimensional renormalization, for simplicity, we consider the Born approximation. Though in principle we can develop a Born approximation method without large-distance asymptotics, we will only use the Born approximation given by conventional scattering theory to illustrate this problem. The validity of the Born approximation without large-distance asymptotics is the same as that in conventional scattering theory but is more accurate.

To demonstrate the validity of renormalization, by taking the Lennard–Jones potential as an example, through three different approaches, we perform renormalization treatments. It will be shown that the results obtained by these three different approaches are the same.

In the following, we take the Lennard–Jones potential \( V(r) = \eta \left( \frac{\alpha}{r^{12}} - \frac{2 \beta}{r^6} \right) \) as an example to testify the validity of the dimensional renormalization through a comparison with other two renormalization treatments.

The Lennard–Jones potential with \( m = 12 \) reads

\[
V(r) = \eta \left( \frac{\alpha}{r^{12}} - \frac{2 \beta}{r^6} \right).
\]

A divergence appears in the first-order Born approximation of \( s \)-wave scattering phase shift [11]:

\[
\delta_0(k) = -\frac{\pi}{2} \int_0^\infty r dr \eta \left( \frac{\alpha}{r^{12}} - \frac{2 \beta}{r^6} \right) J_{6/2}^2(kr),
\]

where \( J_n(z) \) is the Bessel function of the first kind. In the following, we remove this divergence through three renormalization treatments, dimensional renormalization, the analytic continuation approach, and the minimal-subtraction scheme, to show the validity.

10.1. Dimensional renormalization

Using the \( n \)-dimensional Born approximation [3], we have the first-order scattering phase shift,

\[
\delta_1^{(n)}(k) = -\frac{\pi}{2} \int_0^\infty r dr V(r) J_{6/2+z-1}^2(kr).
\]

The \( n \)-dimensional \( s \)-wave phase shift of the Lennard–Jones potential (10.1) can be obtained by performing the integral directly,
\[\delta_0^{(n)}(k) = -\frac{\pi}{2} \int_0^\infty r dr \eta \left( \frac{\alpha}{r^12} - \frac{2\beta}{r^6} \right) J_{n/2-1}(kr) = -\frac{63\pi\alpha\eta k^{10} \Gamma(n/2 - 6)}{2^{10}\Gamma(n/2 + 5)} + \frac{3\pi\beta\eta k^{4}\Gamma(n/2 - 3)}{2^4\Gamma(n/2 + 2)}. \quad (10.4)\]

Then directly putting \(n = 3\) gives
\[\delta_0^{(3)}(k) = \frac{2}{155\ 925}\pi\alpha\eta k^{10} + \frac{2}{15}\pi\beta\eta k^{4}. \quad (10.5)\]

This is a finite result.

### 10.2. Analytic continuation approach

As comparison, we now use another renormalization treatment to remove the divergence, which is based on analytic continuation.

The integral \(I = \int_0^\infty f(x) dx\) will diverge, if the expansion of \(f(x)\) at \(x = 0\) has negative-power terms. In order to remove the divergence, we use the analytic continuation technique. Concretely, we rewrite the integral as \([12]\)

\[I = \int_0^\infty f(x) dx = \int_0^1 \left( f(x) - \sum_{n=2}^N a_n \frac{1}{x^n} \right) dx + \sum_{n=2}^N a_n \frac{x^{1-n}}{1-n} + \int_1^\infty f(x) dx, \quad (10.6)\]

where \(a_n\) is the expansion coefficient and \(N\) equals the highest negative power of the expansion of \(f(x)\). Here the integral is split into two parts: \(\int_0^\infty dx\) and \(\int_0^1 dx\). The integral \(\int_1^\infty dx\) is well defined. The divergence encountered in the integral \(\int_0^1 dx\) is removed by \(\sum_{n=2}^N a_n \frac{1}{x^n}\). The basis of equation \((10.6)\) is essentially analytic continuation.

First, split the integral in equation \((10.2)\) into two parts:
\[\delta_0(k) = [\delta_0(k)]_0^0 + [\delta_0(k)]_0^\infty, \quad (10.7)\]

where
\[\delta_0(k) = \frac{\pi}{2} \int_0^\epsilon r dr \eta \left( \frac{\alpha}{r^12} - \frac{2\beta}{r^6} \right) J_{1/2}(kr), \quad (10.8)\]
\[\delta_0(k) = \frac{\pi}{2} \int_\epsilon^\infty r dr \eta \left( \frac{\alpha}{r^12} - \frac{2\beta}{r^6} \right) J_{1/2}(kr), \quad (10.9)\]

where \(\epsilon\) is a finite number.
The integral in equation (10.9) can be performed directly,

\[
\left[ \delta_0(k) \right]_0^\infty = - \frac{\alpha \eta}{22ke^{11}} + \frac{\beta \eta}{5ke^5} - \text{Si}(2ke) \left( \frac{4\alpha \eta k^{10}}{155925} + \frac{4\beta \eta k^4}{15} + 2\pi \alpha \eta k^{10} + \frac{2\pi \beta \eta k^4}{15} \right)
\]

\[
+ \sin(2ke) \left[ - \frac{\alpha \eta}{110e^{10}} + \frac{\alpha \eta^2}{1980e^8} - \frac{\alpha \eta^4}{20790e^6} \right]
\]

\[
+ \left( \frac{\alpha \eta k^6}{103950} + \frac{\beta \eta}{10} \right) \frac{1}{e^4} - \left( \frac{\alpha \eta k^8}{155925} - \frac{\beta \eta k^2}{15} \right) \frac{1}{e^2}
\]

\[
+ \cos(2ke) \left[ - \frac{\alpha \eta k^6}{22ke^{11}} + \frac{\alpha \eta k^8}{495e^7} + \frac{\alpha \eta^3}{6930e^5} - \left( \frac{\alpha \eta^5}{51975} + \frac{\beta \eta}{5k} \right) \frac{1}{e^3} \right]
\]

\[
+ \left( \frac{\alpha \eta k^7}{155925} + \frac{\beta \eta k}{15} \right) \frac{1}{e^5} - \left( \frac{2\alpha \eta k^9}{155925} - \frac{2\beta \eta k^3}{15} \right) \frac{1}{e^7}
\]

\[
(10.10)
\]

where \( \text{Si}(z) \) is the sine integral [6].

Now we deal with \( \left[ \delta_0(k) \right]_0^r \).

Expanding the integrand in \( \left[ \delta_0(k) \right]_0^r \) around \( r = 0 \) gives

\[
r\eta \left[ \frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right] J_{1/2}^2 (kr) = D(r),
\]

where

\[
D(r) = \frac{2\alpha \eta k^{10}}{r^{10}} - \frac{2\alpha \eta k^2}{3} + \frac{4\alpha \eta k^4}{15} - \left( \frac{2\alpha \eta k^9}{315\pi} + \frac{4\beta \eta k^3}{3} \right) \frac{1}{r^4}
\]

\[
+ \left( \frac{4\alpha \eta k^8}{14175\pi} + \frac{4\beta \eta k^6}{3} \right) \frac{1}{r^2} + \ldots
\]

(10.12)

According to equation (10.6), subtracting \( D(r) \) from the integral in equation (10.8) gives

\[
- \frac{\pi}{2} \int_0^\infty dr \left[ \eta \left( \frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2 (kr) - D(r) \right]
\]

\[
= \frac{\alpha \eta}{22ke^{11}} - \frac{\alpha \eta k^2}{9ke^5} + \frac{\alpha \eta^3}{21e^7} - \left( \frac{\beta \eta}{5k} + \frac{\alpha \eta k^5}{225} \right) \frac{1}{e^5}
\]

\[
+ \left( \frac{\alpha \eta k^6}{945} + \frac{2\beta \eta k^2}{3} \right) \frac{1}{e^3} - \left( \frac{2\alpha \eta k^9}{14175} + \frac{2\beta \eta k^3}{3} \right) \frac{1}{e} + 4\text{Si}(2ke) \left( \frac{\alpha \eta k^{10}}{155925} + \frac{\beta \eta k^4}{15} \right)
\]

\[
+ \sin(2ke) \left[ - \frac{\alpha \eta}{110e^{10}} + \frac{\alpha \eta^2}{1980e^8} - \frac{\alpha \eta^4}{20790e^6} \right]
\]

\[
+ \left( \frac{\alpha \eta k^6}{103950} + \frac{\beta \eta k^2}{15} \right) \frac{1}{e^2}
\]

\[
- \left( \frac{\alpha \eta k^7}{155925} + \frac{\beta \eta k}{15} \right) \frac{1}{e^4} + \left( \frac{2\alpha \eta k^9}{155925} + \frac{2\beta \eta k^3}{15} \right) \frac{1}{e^6}
\]

(10.13)
Introducing

\[
D_\epsilon(r) = \frac{2\alpha \eta k}{\pi r^{10\epsilon}} - \frac{2\alpha \eta k^3}{3\pi r^{8\epsilon}} + \frac{4\alpha \eta k^5}{45\pi r^{6\epsilon}} - \left( \frac{2\alpha \eta k^7}{315\pi} + \frac{4\beta \eta k}{r^{4\epsilon}} \right) \frac{1}{r^{4\epsilon}} + \cdots \tag{10.14}
\]

with \(D_\epsilon(r)|_{\epsilon=1} = D(r)\).
Integrating \(D_\epsilon(r)\) and then putting \(s=1\) give

\[
-\frac{\pi}{2} \int_0^\infty \mathrm{d}r D_\epsilon(r) \bigg|_{\epsilon=1} = \frac{\eta k}{14175} \left[ \frac{14175\alpha}{(10s - 1)\epsilon^{10s-1}} - \frac{4725\alpha k^2}{(8s - 1)\epsilon^{8s-1}} + \frac{630\alpha k^4}{(6s - 1)\epsilon^{6s-1}} - \frac{45(630\beta + \alpha k^6)}{(4s - 1)\epsilon^{4s-1}} + \frac{2k^2(4725\beta + \alpha k^6)}{(2s - 1)\epsilon^{2s-1}} \right] \bigg|_{s=1} \tag{10.15}
\]

\[
= \frac{\alpha \eta k}{9\epsilon^3} - \frac{\alpha \eta k^3}{21\epsilon^3} + \frac{2\alpha \eta k^5}{225\epsilon^3} - \frac{\alpha \eta k^7}{945} + \frac{2\beta \eta k}{3\epsilon} \left( 1 + \frac{2\beta \eta k^3}{3} + \frac{2\alpha \eta k^9}{14175} \right) \frac{1}{\epsilon^3}.
\]

Then we arrive at

\[
\delta_0(k) = -\frac{\pi}{2} \int_0^\infty \mathrm{d}r \left[ \eta \left( \frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{5/2}^2(kr) - D(r) \right] + \left( -\frac{\pi}{2} \int_0^\infty \mathrm{d}r D_\epsilon(r) \bigg|_{\epsilon=1} \right) \\
- \frac{\pi}{2} \int_0^\infty \mathrm{d}r \eta \left( \frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{5/2}^2(kr) \\
= \frac{2}{155925} \pi \alpha \eta k^{10} + \frac{2}{15} \pi \beta \eta k^4. \tag{10.16}
\]

This result agrees with that given by dimensional renormalization.

10.3. Minimal-subtraction scheme

The minimal-subtraction scheme is simply to remove the poles in divergent quantities [13].
First directly cut off the lower limit of the integral in equation (10.2):

\[
\delta_0(k, \epsilon) = -\frac{\pi}{2} \int_\epsilon^\infty \mathrm{d}r \eta \left( \frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{5/2}^2(kr). \tag{10.17}
\]

For \(\epsilon > 0\), the integral is convergent.
Performing the integral in equation (10.17) gives

\[ \delta_0(k, \epsilon) = -\frac{\alpha \eta}{22k^{11}} + \frac{\beta \eta}{5k^{5}} + \frac{2\pi \alpha \eta k^{10}}{155 \, 925} + \frac{2\pi \beta \eta k^{4}}{15} - \text{Si}(2k\epsilon) \left( \frac{4\alpha \eta k^{10}}{155 \, 925} + \frac{4\beta \eta k^{4}}{15} \right) \]

\[ + \cos(2k\epsilon) \left( \frac{\alpha \eta}{22k^{11}} - \frac{\alpha \eta k^{3}}{495 \epsilon^{3}} + \frac{\alpha \eta k^{5}}{6930 \epsilon^{5}} - \frac{\alpha \eta k}{51 \, 975 \epsilon^{5}} - \frac{\beta \eta}{5k^{5}} \right) \]

\[ + \left( \frac{\alpha \eta k^{7}}{155 \, 925} + \frac{\beta \eta k}{15} \right) \frac{1}{\epsilon} - \left( \frac{2\alpha \eta k^{9}}{155 \, 925} + \frac{2\beta \eta k^{3}}{15} \right) \frac{1}{\epsilon^{2}} \]

\[ + \sin(2k\epsilon) \left[ -\frac{\alpha \eta}{110 \epsilon^{10}} + \frac{\alpha \eta k^{2}}{1980 \epsilon^{8}} - \frac{\alpha \eta k^{4}}{20 \, 790 \epsilon^{6}} + \frac{\alpha \eta k^{6}}{103 \, 950} + \frac{\beta \eta}{10} \right] \frac{1}{\epsilon^{2}} \]

\[ \left( \frac{\alpha \eta k^{8}}{155 \, 925} + \frac{\beta \eta k^{2}}{15} \right) \frac{1}{\epsilon^{2}} \]  

(10.18)

Expanding \( \delta_0(k, \epsilon) \) around \( \epsilon = 0 \) gives

\[ \delta_0(k, \epsilon) = -\frac{\alpha \eta k^{9}}{141 \, 75} - \frac{\alpha \eta k^{3}}{211 \epsilon^{3}} + \frac{2\alpha \eta k^{5}}{225 \epsilon^{5}} + \frac{2\pi \alpha \eta k^{10}}{14 \, 175 \epsilon} + \frac{2\pi \beta \eta k^{4}}{155 \, 925} + O(\epsilon). \]

(10.19)

Take \( \epsilon \to 0 \) and, according to the minimal-subtraction scheme, dropping out the terms that diverge when \( \epsilon \to 0 \) give

\[ \delta_0(k) = \frac{2}{155 \, 925} \pi \alpha \eta k^{10} + \frac{2}{15} \pi \beta \eta k^{4}. \]

(10.20)

The fact that the above three renormalized results agree with each other demonstrates the validity of these three renormalization schemes.

### 11. Conclusions

In this paper, without large-distance asymptotics, we establish a \( n \)-dimensional scattering theory.

According to Euler’s scheme, functions can be classified by their asymptotics [14]. Therefore, for scatterings of short-range potentials, the scattering wave function is classified by the solution of the asymptotic equation of the Schrödinger equation, i.e., the Schrödinger equation with \( V(r) = 0 \). That is to say, the solution of the asymptotic equation is the asymptotics of the scattering wave function. In conventional scattering theory, however, the solution of the asymptotic equation is replaced by an approximate asymptotic solution of the asymptotic equation. What we do in the preceding work [2] (three-dimensions) and in the present paper (arbitrary dimensions) is to provide an accurate scattering theory in which the scattering wave function is restricted by the exact solution of the asymptotic equation rather than by an approximate solution of the asymptotic equation.

An arbitrary-dimensional theory has special importance in renormalization, e.g., the arbitrary-dimensional scattering theory in the scattering spectral method [3, 4]. In the present paper, we suggest a dimensional renormalization treatment to remove the divergence encountered in singular-potential scattering. Moreover, our result can also help us to develop a scattering spectrum method without large-distance asymptotics.
In particular, it is often useful to consider a physical problem in arbitrary dimensions. Arbitrary-dimensional scattering problems have been considered in many aspects: scatterings of the Fermi pseudopotential [15], scatterings of long-range potentials [16], and scatterings of black holes [17]. Besides scattering problems, there are also many arbitrary-dimensional theories. In quantum mechanics and statistic physics, one considers the $n + 4$-dimensional scalar spheroidal harmonics [18], the Landau problem in even-dimensional space $CP^k$ [19], and Fermi gases in arbitrary dimensions [20]; in field theory, one considers Yang–Mills and gravity theories in arbitrary dimensions [21]; in gravity and cosmology theories, one considers the thermodynamic curvature of the Kerr and Reissner–Nordström black holes in arbitrary dimensions [22], an arbitrary-dimensional gravitational theory with a negative cosmological constant, arbitrary-dimensional AdS Black Holes [23], and arbitrary-dimensional AdS black branes [24].

Starting from the result given by the present paper and the preceding work [2], one can reconsider many scattering related problems. The analytic property of scattering amplitudes, which used to be treated by conventional scattering theory [25–29], can be now discussed without large-distance asymptotics in arbitrary dimensions. An arbitrary-dimensional Lippmann–Schwinger equation without large-distance asymptotics can be constructed in the frame of the scattering theory given in the present paper. An arbitrary-dimensional vector and tensor scattering theories without large-distance asymptotics can be established, e.g., electromagnetic scatterings and gravitational wave scatterings, which are usually studied in the frame of conventional scattering theory [30–33]. The acoustic scattering is a scalar scattering. Two- and three-dimensional acoustic scattering theories without large-distance asymptotics can also be established. In conventional acoustic scattering theory, large-distance asymptotics is imposed [34–36] and, therefore, the information of the distance between target and observer is lost. A very important application is to consider inverse scattering problems without large-distance asymptotics in arbitrary dimensions; this is a fundamental problem and is studied under large-distance asymptotics [37, 38]. Based on the relation of two important quantum field theory methods [4, 39], the scattering spectrum method [8–10] and the heat-kernel method [40–43], we establish a heat-kernel method for calculating phase shifts [5]. This result can be generalized to arbitrary-dimensional cases. The result given by [2] and the present paper can be also applied to scatterings of long-range potentials, which is usually studied under large-distance asymptotics [44]. In particular, we will discuss the scattering of a wave on a black hole in arbitrary dimensions, while research in literature is usually under large-distance asymptotics [45–49]. String theory related scatterings [50] can be also considered without large-distance asymptotics.

Acknowledgments

We are very indebted to Dr G Zeitrauman for his encouragement. This work is supported in part by NSF of China under Grant No. 11575125 and No. 11375128.

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