Linear Phase Transition in Random Linear Constraint Satisfaction Problems

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Abstract

Our model is a generalized linear programming relaxation of a much studied random K-SAT problem. Specifically, a set of linear constraints $C$ on $K$ variables is fixed. From a pool of $n$ variables, $K$ variables are chosen uniformly at random and a constraint is chosen from $C$ also uniformly at random. This procedure is repeated $m$ times independently. We are interested in whether the resulting linear programming problem is feasible. We prove that the feasibility property experiences a linear phase transition, when $n \to \infty$ and $m = cn$ for a constant $c$. Namely, there exists a critical value $c^*$ such that, when $c < c^*$, the problem is feasible or is asymptotically almost feasible, as $n \to \infty$, but, when $c > c^*$, the "distance" to feasibility is at least a positive constant independent of $n$. Our result is obtained using the combination of a powerful local weak convergence method developed in Aldous [Ald92], [Ald01], Aldous and Steele [AS03], Steele [Ste02] and martingale techniques.

By exploiting a linear programming duality, our theorem implies the following result in the context of sparse random graphs $G(n,cn)$ on $n$ nodes with $cn$ edges, where edges are equipped with randomly generated weights. Let $M(n,c)$ denote maximum weight matching in $G(n,cn)$. We prove that when $c$ is a constant and $n \to \infty$, the limit $\lim_{n \to \infty} M(n,c)/n$, exists, with high probability. We further extend this result to maximum weight $b$-matchings also in $G(n,cn)$.

1 Introduction

The primary objective of the present paper is studying randomly generated linear programming problems. We are interested in scaling behavior of the corresponding objective value and some phase transition properties, as the size of the problem diverges to infinity. Our random linear programming problems are generated in a specific way. In particular, our linear programs have a fixed number of variables per constraint and the number of variables and constraints diverges to infinity in such a way that their ratio stays a constant.

Our motivation to consider this specific class of random linear programs has several sources. The main motivation is recent explosion of interest in random instances of boolean satisfiability (K-SAT) problems and ensuing phase transition phenomenon. The main outstanding conjecture in this field states that the satisfiability property of random K-SAT problem experiences a linear phase transition as the function of the ratio of the number of clauses to the number of variables. Our linear programming problem can be viewed as a generalized linear programming relaxation of the integer programming formulation of such random K-SAT problem.

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Tightly related to the K-SAT problem are problems of maximal cardinality cuts, independent sets, matchings and other objects, in sparse random graphs $G(n, \lfloor cn \rfloor)$, which are graphs on $n$ nodes and $\lfloor cn \rfloor$ edges selected uniformly at random from all the possible edges, and where $c > 0$ is some fixed constant. For future we drop the annoying notation $\lfloor \cdot \rfloor$, assuming that $cn$ is always an integer. It is easy to show that all of these objects scale linearly in $n$. It is conjectured that the size of each such object divided by $n$ converges to a constant, independent of $n$. This convergence is established only for the case of maximal matchings using direct methods [KS81], where the limit can be computed explicitly, but is open in other cases.

The main result of this paper states that the objective value of the random linear programming problem we consider, when divided by the number of variables converges with high probability (w.h.p.) to a certain limit. As a corollary we prove that, suitably defined, distance to feasibility in the same random linear programming problem experiences a linear phase transition, just as conjectured for random K-SAT problem. Furthermore, we show that, in a special case, the dual of this random linear programming problem is a linear programming relaxation of the maximum cardinality matching and more generally $b$-matching (defined later) problems in $G(n, cn)$. We show that these relaxations are asymptotically tight as the number of nodes $n$ diverges to infinity. As a corollary of our main result, we prove that maximum cardinality $b$-matching when divided by $n$ converge to a constant. These results hold even in the weighted version, where edges are equipped with randomly independently generated non-negative weights.

Our proof technique is a combination of a very powerful local weak convergence method and martingale techniques. The local weak convergence method was developed in Aldous [Ald92, Ald01], Aldous and Steele [AS03], Steele [Ste02]. The method was specifically used by Aldous for proving the $\zeta(2)$ conjecture for the random assignment problem. It was used in [Ald92] to prove that the expected minimum weight matching in a complete bipartite graph converges to a certain constant. Later Aldous proved [Ald01] that this limit is indeed $\zeta(2)$, as conjectured earlier by Mezard and Parisi [MP87]. Since then the local weak convergence method was used for other problems (see [AS03] for a survey), and seems to be a very useful method for proving existence of limits in problems like the ones we described, and in some instances also leads to actually computing the limits of interest. By an analogy with the percolation literature, we call these problems existence and computation of scaling limits in large random combinatorial structures. Such questions, many of them open, abound in percolation literature. For example the existence of limits of crossing probabilities in critical percolation have been established in several percolation models like triangular percolation using conformal invariance techniques [Scl01, SW01], but are still open in the case of other lattices, like rectangular bond and site critical percolation, see Langlands [LPSA94]. Whether a local weak convergence is a useful technique for addressing these questions seems worth investigation.

To the extend that we know, our result is the first application of the local weak convergence method to establishing phase transitions in random combinatorial structures. In the following section we describe in details randomly generated combinatorial problems we mentioned above, describe the existing results in the literature and list some outstanding conjectures. In Section 2 we describe our model and state our main results. We also give a short summary of the proof steps. Sections 3, 6, 7 are devoted to the proof of our main result. Section 8, is devoted to the applications to the maximum weight matching and $b$-matching in sparse random graphs. Section 9 is devoted to conclusions and some open problems.
2 Background: random K-SAT, sparse random graphs and scaling limits

2.1 Random K-SAT problem

A satisfiability or K-SAT problem is a boolean constraint satisfaction problem with a special form. A collection of \( n \) variables \( x_1, x_2, \ldots, x_n \) with values in \( \{0, 1\} \) is fixed. A boolean formula of the form \( C_1 \land C_2 \land \cdots \land C_m \) is constructed, where each \( C_i \) is a disjunctive clause of the form \( x_{i1} \lor \bar{x}_{i1} \lor x_{i2} \lor \cdots \lor x_{iK} \), where exactly \( K \) variables are taken from the pool \( x_1, \ldots, x_n \), some with negation, some without. The formulae is defined to be satisfiable if an assignment of variables \( x_i, i = 1, \ldots, n \) to 0 or 1 can be constructed such that all the clauses take value 1. The K-SAT problem is one of the most famous combinatorial optimization problem, see [PS98]. It is well known that the satisfiability problem is solvable in polynomial time for \( K = 2 \), and is NP-complete for \( K \geq 3 \).

Recently we have witnessed an explosion of interest in random instances of the K-SAT problem. This was motivated by computer science, artificial intelligence and statistical physics investigations, with phase transition phenomena becoming the focus of a particular attention. A random instance of a K-SAT problem with \( m \) clauses and \( n \) variables is obtained by selecting each clause uniformly at random from the entire collection of \( \binom{2n}{K} \) possible clauses where repetition of variables is (not) allowed. In particular, for each \( j = 1, 2, \ldots, m \), \( K \) variables \( x_{i1}, x_{i2}, \ldots, x_{iK} \) or their negations are selected uniformly at random from the pool \( x_1, \ldots, x_n \) to form a clause \( C_j = y_{i1} \lor \cdots \lor y_{ik} \), where each \( y_{ir} = x_{ir} \) or \( \bar{x}_{ir} \), equiprobably. This is done for all \( j = 1, \ldots, m \) independently. Whether the resulting formulae has a satisfying assignment \( \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}^n \) becomes a random event with respect to this random construction. The main outstanding conjecture for the random K-SAT problem is as follows.

**Conjecture 1** For every \( K \geq 2 \) there exists a constant \( c^*_K \) such that a random K-SAT formulae with \( n \) variables and \( m = cn \) clauses is satisfiable when \( c < c^*_K \) and is not satisfiable when \( c > c^*_K \), w.h.p. as \( n \to \infty \). In other words, the satisfiability experiences a linear sharp phase transition at \( m = c^*_Kn \).

That the problem experiences a sharp phase transition is proven by Friedgut [Fr95] in a much more general context. It is the linearity which is the main outstanding feature of this conjecture. The conjecture can be rephrased as follows: there does not exist \( c_1 > c_2 \) and two infinite sequences \( n_t^{(1)}, n_t^{(2)}, t = 1, 2, \ldots \), such that instances of K-SAT problem with \( n_t^{(1)} \) variables and \( c_1n_t^{(1)} \) clauses are satisfiable w.h.p., but instances with \( n_t^{(2)} \) variables and \( c_2n_t^{(1)} \) clauses are not satisfiable w.h.p., as \( t \to \infty \). One of the goals of our paper is to establish an analogue of this conjecture for generalized linear programming relaxations of the integer programming formulation (to be described below) of the random K-SAT problem.

Conjecture 1 is proven for the case \( K = 2 \). Specifically, \( c^*_2 = 1 \) was established by Goerdt [Goe92], [Goe96], Chvatal and Reed [CR92], Fernandez de la Vega [JV92]. For higher values of \( K \) many progressively sharper bounds on \( c^*_K \) (assuming it exists) are established by now. For \( K = 3 \) the best known upper and lower bounds are 4.506 and 3.42, obtained by Dubois, Boufkhad and Mandler [DBM00], and Kaporis, Kirovskis and Lalas [KKL02], respectively. It is known that \( c^*_K \), if exists, approaches asymptotically \( 2^K(\log 2 + o(1)) \) when \( K \) is large. [APa]. See also [AM02a], [FW02] for the related results. Talagrand [Tan01] approached the random K-SAT problem using the methods of statistical physics.

The interest in random K-SAT problem does not stop at the threshold value \( c^*_K \). For \( c > c^*_K \) (assuming Conjecture 1 holds), a natural question is what is the maximal number of clauses \( N(n, m) \leq m \) that can be satisfied by a single assignment of the \( n \) variables? It is shown in Coppersmith et al [CGHS03].
that for $K = 2$ and every $c > 1$, there exists a constant $\alpha(c) > 0$ such that $N(n, cn) \leq (c - \alpha(c))n$, w.h.p. The following conjecture from [CGHS03] then naturally extends Conjecture II.

**Conjecture 2** Assuming Conjecture [CGHS03] holds, $\lim_{n \to \infty} \frac{N(n, cn)}{cn}$ exists and is smaller than one, for all $c > c_K^*$.

In Section [CGHS03] we introduce a conjecture similar to the one above with respect to random linear programs.

### 2.2 Matching and $b$-matching in $G(n, cn)$

Let $G$ be a simple undirected graph on $n$ nodes $\{1, 2, \ldots, n\} \equiv [n]$ with the edge set $E$. A set of nodes $V \subset [n]$ in this graph is an independent set if no two nodes in $V$ are connected by an edge. A partition of nodes $[n]$ into two $k$ groups $V_1, V_2, \ldots, V_k$ such that $\cup_{1 \leq i \leq k} V_i = [n], V_i \cap V_i = \emptyset$ for all $i \neq i_2$, is defined to be a $k$-cut. The size of the $k$-cut is the total number of edges whose end points belong to different sets $V_i$. When $k = 2$, the $k$-cuts are simply referred to as cuts.

A matching is a collection of edges such that no two edges are incident to the same node. The size of the matching is the number of edges in it. A path is a collection of distinct nodes $C = \{i_1, \ldots, i_k\}$ such that the edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ belong to the edge set $E$. A cycle is a collection of distinct nodes $C = \{i_1, \ldots, i_k\}$ such that the edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$ belong to the edge set $E$.

Let $b \geq 1$ be a positive integer. A $b$-matching is a collection of edges $A \subset E$ such that every node is incident to at most $b$ edges from $A$. Naturally, $1$-matching is simply a matching. Note that $2$-matching is collection of node disjoint paths and cycles. We will also call it path/cycle packing.

Fix a constant $c > 0$. Let $G(n, cn)$ denote a simple undirected sparse random graph on $n$ nodes with $cn$ edges selected uniformly at random from all the possible $n(n-1)/2$ edges. This is a standard model of a sparse random graph. Denote by $\text{IND}(n, c), \text{CUT}(n, c, k) \ M(n, c, b)$ the size (cardinality) of the maximum independent set, $k$-cut and $b$-matching, respectively, in $G(n, cn)$. Suppose, in addition, the nodes and the edges of $G(n, cn)$ are equipped with random non-negative weights $W_{i,j}^{\text{node}}, W_{i,j}^{\text{edge}}$ drawn independently according to some common probability distributions $\mathbb{P}\{W_{i,j}^{\text{node}} \leq t\} \equiv w_{i,j}^{\text{node}}(t), \mathbb{P}\{W_{i,j}^{\text{edge}} \leq t\} \equiv w_{i,j}^{\text{edge}}(t)$. We assume throughout the paper that both $W^{\text{node}}$ and $W^{\text{edge}}$ have a bounded support $[0, B_w]$ (assumed the same for simplicity). Let $\text{IND}_w(n, c), \text{CUT}_w(n, c, k) \ M_w(n, c, b)$ denote the maximum weight independent set, $k$-cut and $b$-matching, respectively, where the weight of an independent set is the sum of weights of its nodes, and the weights of a cut and $b$-matching are defined as the sums of weights of edges in them.

It is well known and simple to prove that $\text{IND}(n, c), \text{CUT}(n, c, k) \ M(n, c, 1)$ are all $\Theta(n)$ w.h.p. as $n$ diverges to infinity. For example since a fixed node $i$ is isolated with a positive constant probability, then $\mathbb{E}[\text{IND}(n, c)] = \Theta(n)$. Since any matching is also a $b$-matching for $b \geq 1$, then $\mathbb{M}(n, c, b) = \Theta(n)$. Also the length of the longest path in $G(n, cn)$ is also $\Theta(n)$, thanks to the result of Frieze [Fri89].

It is natural to suspect then that the expected values of these objects divided by $n$ converge to a constant, both in the unweighted and weighted cases. In other words, the scaling limits exist for these objects. In fact, it is conjectured in [Ald] and [CGHS03], respectively, that the scaling limits

$$\lim_{n \to \infty} \frac{\mathbb{E}[\text{IND}(n, c)]}{n}, \quad \lim_{n \to \infty} \frac{\mathbb{E}[\text{CUT}(n, c)]}{n}$$

exist. The existence of these limits for expectation would also imply almost sure limits, by application of Azuma’s inequality.
The scaling limit of the form (1) is in fact proven for maximum cardinality matchings $M(n, c)$ by Karp and Sipser [KS81]. The result was strengthened later by Aronson, Frieze and Pittel [APF98]. Karp and Sipser proved that, almost surely,

$$
\lim_{n \to \infty} \frac{M(n, c)}{n} = 1 - \frac{\gamma_*(c) + \gamma^*(c) + \gamma_*(c)\gamma^*(c)}{2},
$$

where $\gamma_*(c)$ is the smallest root of the equation $x = c \exp(-c \exp(-x))$ and $\gamma^*(c) = c \exp(-\gamma_*(c))$. Their algorithmic method of proof is quite remarkable in its simplicity. We briefly describe the argument below and explain why, however, it does not apply to the case of weighted matchings. Suppose we are given a (non-random) graph $G$. Then the following algorithm finds a maximum matching (clearly there could be many maximum matchings): while the graph contains any leaves, pick any leaf of a tree and the corresponding edge. Mark the edge and delete the two nodes incident to the edge and all the other leaves that share the (deleted) parent with the selected leaf. Delete all the edges incident to these leaves and delete the edge between the parent and its own parent. Repeat while there are still leaves in the graph. When the graph has no more leaves select a maximum size matching in the remaining graph. It is a fairly easy exercise to prove that this remaining matching plus the set of marked edges is a maximum matching. This fact is used to prove (2).

One notes, however, that when edges of the graph are equipped with some weights, the Karp-Sipser algorithm does not necessarily work anymore. Occasionally it might be better to include an edge between a parent of a leaf and and a parent of a parent of a leaf and, as a result, not include the edge incident to the leaf. Therefore, the Karp-Sipser algorithm may produce a strictly suboptimal matching and the results (2) do not hold for the weighted case. Moreover, it is not clear how to extend the Karp-Sipser heuristic to $b$-matchings. In this paper we prove the convergence (2) for the maximum weight $b$-matchings. The proof uses the main result of the paper and the linear programming duality, though we are not able to compute the limits. Naturally, our result applies to the non-weighted case – maximum cardinality $b$-matching. To the best of our knowledge this is a new result.

The case of maximum weight matching with random weights is treated by Aldous and Steele [AS03] for the case of a randomly generated tree on $n$ nodes. That is, consider a tree selected uniformly at random from the set of all possible $n^{-2}$ labelled trees. The limit of the sort (2) is proven and computed using the local weak convergence method, when the edges of this tree are equipped with exponentially distributed random weights. The tree structure of the underlying graph helps very much the analysis. In our case, however, the random graph $G(n, cn)$ contains a linear size non-tree ”giant” component, [JLR00], when $c > 1/2$, and the results of [AS03] are not applicable.

Yet another scaling limit question is the existence of the limits for probability of $k$-colorability in $G(n, cn)$. A graph is defined to be $k \ge 2$ colorable if there exists a function mapping vertices of $G$ to colors $1, \ldots, k$ such that no two nodes connected by an edge have the same color. The following conjecture proposed by Erdos is found in Alon and Spencer [AS92].

**Conjecture 3** For every positive integer $k \ge 2$ there exists a critical value $c_k^*$ such that the graph $G(n, cn)$ is w.h.p. $k$-colorable for $c < c_k^*$ and w.h.p. not $k$-colorable for $c > c_k^*$.

This conjecture is very similar in spirit to Conjecture 1. For a survey of existing results see a recent Molloy’s survey [Mol01]. For related results see also [AM02a], [COMS03].

### 3 Model and the main results

There is a natural way to describe a K-SAT problem as an integer programming problem. The variables are $x_i$, $i = 1, 2, \ldots, n$ which take values in $\{0, 1\}$. Each clause $C_j$ is replaced by a linear constraint of the
form \( x_1 + (1-x_2) + x_3 + \ldots \geq 1 \), where term \((1-x)\) replaces \(\bar{x}\). For example a clause \(C = x_3 \lor x_7 \lor \bar{x}_2 \lor \bar{x}_4\) in a 4-SAT problem is replaced by a constraint \(x_3 + x_7 + (1-x_2) + (1-x_4) \geq 1\). It is easy to check that an assignment of \(x_2, x_3, x_4, x_7\) to 0 and 1 gives \(C\) value 1 if and only if the corresponding constraint is satisfied. Clearly, these constraints can be created for all the possible clauses. In the present paper we study the linear programming (LP) relaxation of this integer programming problem, where the restriction \(x_i \in \{0,1\}\) is replaced by a weaker restriction \(x_i \in [0,1]\). Note, that this relaxation by itself is not interesting, as the assignment \(x_i = 1/2\) for all \(i=1,2,\ldots,n\) makes all of the linear constraints feasible. However, the problem becomes non-trivial when we generalize the types of constraints that can be generated on the variables \(x_i\), and this is described in the following subsection.

### 3.1 Random K-LSAT problem

Our setting is as follows. Consider a fixed collection of \(K\) variables \(y_1, y_2, \ldots, y_K\) which take values in some bounded interval \([B_1, B_2]\) and a fixed collection \(C\) of linear constraints on these variables: \(\sum_{k=1}^{K} a_{rk}y_k \leq b_r, r=1,2,\ldots,|C|\), where the values \(a_{rk}, b_r\) are arbitrary fixed reals. The \(r\)-th constraint can also be written in a vector form \(a_r y \leq b_r\), where \(a_r = (a_{r1},\ldots,a_{rK})\) and \(y = (y_1,\ldots,y_K)\). We fix \(c > 0\) and let \(m = cn\), where \(n\) is a large integer. A random instance of a linear constraint satisfaction problem with \(n+m\) variables \(x_1, \ldots, x_n, \psi_1, \ldots, \psi_m\) and \(m\) constraints is constructed as follows. For each \(j = 1,2,\ldots,m\) we perform the following operation independently. We first select \(K\) variables \(x_{i_1}, x_{i_2}, \ldots, x_{i_K}\) uniformly at random from \(x_i, i = 1,2,\ldots,n\). Whether the variables are selected with or without replacement turns out to be irrelevant to the results of this paper, as it is the case for random K-SAT problem. However, the order with which the variables are selected is relevant, since the constraints are not necessarily symmetric. Then we select \(1 \leq r \leq |C|\) also uniformly at random. We then generate a constraint

\[
C_j : \sum_{k=1}^{K} a_{rk}x_{ik} \leq b_r + \psi_j.
\]

Here is an example of an instance with \(K = 3, n = 10, m = 4, |C| = 2\). Say the first constraint \(C_1\) is \(2y_1 + 3y_2 - y_3 \leq 5\), and the second constraint \(C_2\) is \(-y_1 + y_2 + 4y_3 \leq 2\). An example of an instance where first three constraints are type \(C_1\) and the fourth is type \(C_2\) is

\[
\begin{align*}
(2x_5 + 3x_4 - x_9 \leq 5 + \psi_1) \land (2x_1 + 3x_3 - x_4 \leq 5 + \psi_2) \land \\
(2x_2 + 3x_1 - x_{10} \leq 5 + \psi_3) \land (-x_5 + x_8 + 4x_7 \leq 2 + \psi_4).
\end{align*}
\]

The central question is what are the optimal values of \([B_1, B_2], \psi_j \geq 0\), which minimize the sum \(\sum \psi_j\) subject to the constraints \(C_j\). That is, we consider the following linear programming problem:

\[
\text{Minimize } \sum_{1 \leq j \leq m} \psi_j, \text{ subject to } : C_1, C_2, \ldots, C_m, \ x_i \in [B_1, B_2], \psi_j \geq 0.
\]

In words, we are seeking a solution \(x_j\) which is as close to satisfying the constraints \(\sum_{k=1}^{K} a_{rk}x_{ik} \leq b_r\) as possible. If the optimal value of this linear programming problem is zero, that is \(\psi_j = 0\) for all \(j\), then all of these constraints can be satisfied. Naturally, the objective value of the linear program (4) is a random variable. We denote this random variable by \(\mathcal{LP}(n, c)\). Note, that the linear program (4) is always feasible, by making \(\psi_j\) sufficiently large. In fact, clearly, in the optimal solution we must
have \( \psi_j = \max(0, \sum_{k=1}^K a_{rk} x_{ik} - b_r) \). We refer to the linear program (4) as a random linear constraint satisfaction (LSAT) problem, or random K-LSAT problem.

The following conditions on the set of constraints \( C \) will be used below.

- **Condition A.** For any constraint \( a_r y \leq b_r, 1 \leq r \leq |C| \) for any \( k \leq K \) and any value \( z \in [B^1_k, B^2_k] \) there exist values \( y_1, \ldots, y_K \in [B^1_k, B^2_k] \) such that \( y_k = z \) and the constraint is satisfied.

- **Condition B.** There exist a positive integer \( l \) and a constant \( \nu > 0 \) such that for any \( K \)-dimensional cube \( I \) of the form \( \prod_{1 \leq k \leq K} [\frac{z_k - k}{l}, \frac{z_k + k}{l}], B^1_k \leq \frac{i_k}{l} < B^2_k, i_k \) integer, there exists at least one constraint \( \sum a_{rk} y_k \leq b_r \) from \( C \) such that for every \( y \in I \), \( \sum a_{rk} y_k - b_r \geq \nu \). That is, every point of the cube \( I \) deviates from satisfying this constraint by at least \( \nu \).

The analogue of the Condition A clearly holds for random K-SAT problem. Given any clause \( y_1 \lor y_2 \lor \cdots \lor y_K \) and \( k \leq K \), if \( y_k \) is set to be 0 or 1, we still can satisfy the clause, by satisfying any other variable. The following is an example of an LSAT problem where Conditions A and B are satisfied. Fix \( K = 3 \). Let \( B^1_k = 0, B^2_k = 1 \), and let \( C \) be a collection of all eight constraints of the type \(-y_1 - y_2 - y_3 \leq -7/4, -(1 - y_1) - y_2 - y_3 \leq -7/4, \ldots, -(1 - y_1) - (1 - y_2) - (1 - y_3) \leq -7/4 \). Condition A is checked trivially. We claim that Condition B holds for \( l = 2 \) and \( \nu = 1/4 \). Select any cube \( I \) with side-length \( 1/l = 1/2 \). For example \( I = [0, 1/2] \times [1/2, 1] \times [1/2, 1] \). Consider constraint \(-y_1 - (1 - y_2) - (1 - y_3) \leq -7/4 \). For any \( y \in I \) we have \(-y_1 - (1 - y_2) - (1 - y_3) \geq -7/4 + 1/4 = -7/4 + \nu \). Other cases are analyzed similarly.

Consider now the following generalization of the linear program (4). For each \( j = 1, 2, \ldots, m \) generate a random variable \( W_j \), independently from some common distribution \( \mathbb{P}\{W_j \leq t\} \) with a bounded support \([-B_w, B_w]\). Let \( w_x \geq 0 \) and \( w_\psi > 0 \) be fixed non-negative constants. Our random linear program in variables \( x_i, \psi_j \) is constructed exactly as above except each constraint \( C_j : \sum_{1 \leq r \leq K} a_{rk} x_{ik} \leq b_r + \psi_j \) is replaced by

\[
C_j : \sum_{1 \leq r \leq K} a_{rk} x_{ik} \leq b_r + W_j + \psi_j, \tag{5}
\]

and the objective function is replaced by

\[
\text{Minimize } w_x \sum_{1 \leq i \leq n} x_i + w_\psi \sum_{1 \leq j \leq m} \psi_j, \tag{6}
\]

subject to:

\[
C_1, C_2, \ldots, C_m, \quad x_i, \psi_j \in [B^1_k, B^2_k], \quad \psi_j \geq 0.
\]

This particular form of the linear program might look unnatural at first. But note that setting \( B_w = w_x = 0, w_\psi = 1 \), turns this into exactly linear program (4). We will show later that this general format is useful when we study \( b \)-matchings in sparse random graphs \( G(n, cn) \). We denote the optimal value of the linear program (4) by \( G\mathcal{L}\mathcal{P}(n, c) \). As before, this linear program is always feasible, by making \( \psi_j \) sufficiently large. Since we assumed \( w_\psi > 0 \), then in the optimal solution

\[
\psi_j = \max(0, a_{rk} x_{ik} - b_r - W_j). \tag{7}
\]

We now state the main result of this paper. In words, our result asserts that the scaling limit of \( G\mathcal{L}\mathcal{P}(n, c)/n \) exists.
Theorem 1 For every \( c \geq 0 \), the limit
\[
\lim_{n \to \infty} \frac{GLP(n,c)}{n} = f(c)
\]
exists w.h.p. That is, there exists \( f(c) \geq 0 \) such that for every \( \epsilon > 0 \),
\[
P\{|\frac{GLP(n,c)}{n} - f(c)| > \epsilon\} \to 0
\]
as \( n \to \infty \).

Our first application of Theorem 1 is the following result. It establishes a linear phase transition property for the random K-LSAT problem. Recall that \( LP(n,c) \) is the optimal value of the linear programming problem (4).

Theorem 2 There exists a constant \( c^*_K > 0 \) such that, w.h.p. as \( n \to \infty \),
\[
\lim_{n \to \infty} \frac{LP(n,c)}{n} = 0,
\]
for all \( c < c^*_K \), and
\[
\liminf_{n \to \infty} \frac{LP(n,c)}{n} > 0,
\]
for all \( c > c^*_K \). Moreover, if Condition A holds, then \( c^*_K > 0 \), and if Condition B holds, then \( c^*_K < +\infty \).

In what sense does the theorem above establish a linear phase transition? It is conceivable that for a collection of constraints \( C \), the following situation occurs: there exist two constants \( c_1 > c_2 \) and two sequences \( n_t^{(1)}, n_t^{(2)}, t = 0,1,2,\ldots \), such that for \( c = c_1 \) the corresponding optimal values of the random K-LSAT problem satisfy w.h.p. \( \lim_t LP(n_t^{(1)},c)/n_t^{(1)} = 0 \), but for \( c = c_2 \), \( \liminf_t LP(n_t^{(2)},c)/n_t^{(2)} \geq \delta(c) > 0 \). In other words, the critical density \( c \) oscillates between different values. This is precisely the behavior that Conjectures 1 and 2 rule out for random K-SAT problem. Our theorem states that such a thing is impossible for the random K-LSAT problem. There exists a linear function \( c^*_K.n \) such that, w.h.p., below this function the instance is very close to being feasible, but above this function the scaled ”distance” \( \min(1/n) \sum \psi_j \) to feasibility is at least a positive constant.

The statement of the theorem above does not fully match its analogue, Conjecture 1 as, using the auxiliary variables \( \psi_j \) we converted the feasibility problem to the optimality problem. Now consider the collection of constraints \( C_j \) where \( \psi_j \) are set to be zero, and we ask the question whether the collection of constraints in \( \mathcal{H} \) has a feasible solution. We suspect that this problem does experience a linear phase transition, but we do not have a proof at the present time.

Conjecture 4 Let \( c^*_K \) be the value introduced in Theorem 2. Then, w.h.p. as \( n \to \infty \), the random K-LSAT problem with \( cn \) constraints is satisfiable if \( c < c^*_K \) and is not satisfiable if \( c > c^*_K \).

In this paper we use local weak convergence method to prove Theorem 1. While our approach is very much similar to the one used in [Ald92], there are several distinctive features of our problem. In particular, we do not use an infinite tree construction and instead consider a sequence of finite depth trees with some corresponding sequence of probability measures. Then we use a Martingale Convergence Theorem for the ”projection” step. This simplifies the proofs significantly.
3.2 Maximum weighted $b$-matching

We return to the setting of Subsection 2.2. We have a sparse random graph $G(n, cn)$, where $c$ is a positive constant. The edges of these graph are equipped with random weights $W_{i,j}$ which are selected independently from a common distribution $\mathbb{P}\{W_{i,j} \leq t\} = \text{wedge}(t), 0 \leq t \leq B_w < \infty$, where $[0, B_w]$ is the support of this distribution. Again let $\mathcal{M}_w(n, c, b)$ denote the maximum weight $b$-matching in $G(n, cn)$, where $b \geq 1$ is an integer.

**Theorem 3** For every $c > 0$ the limit

$$\lim_{n \to \infty} \frac{\mathcal{M}_w(n, c, b)}{n} \equiv g(c)$$

exists w.h.p.

The probability in the statement of the theorem is both with respect to the randomness of $G(n, cn)$ and with respect to the random weights. This theorem is proven in Section 8. We use linear programming duality and certain linear programming formulation of the maximum weight $b$-matching problem in order to related it to our main result, Theorem 1.

3.3 Proof plan

Below we outline the main steps in proving our main result, Theorem 4. Let $\mathbb{E}[\cdot]$ denote the expectation operator. The general scheme of the proof follows the one from Aldous [Ald92].

1. We first observe that, as in the case of a random K-SAT problem, in the limit as $n \to \infty$, the (random) number of constraints containing a fixed variable $x$ from the pool $x_1, \ldots, x_n$ is distributed as a Poisson random variable with parameter $cK$, denoted henceforth as Pois($cK$).

2. For every $c > 0$ we introduce

$$\lambda(c) \equiv \lim_{n \to \infty} \frac{\mathbb{E}[\mathcal{LP}(n, c)]}{n}.$$  

Our goal is to show that in fact convergence $\lim_n \frac{\mathbb{E}[\mathcal{LP}(n, c)]}{n}$ holds, and therefore we can set $f(c) = \lambda(c)$. The convergence w.h.p. will be a simple consequence of Azuma’s inequality. Then, in order to prove Theorem 2, we prove that $c_K = \sup\{c : f(c) = 0\}$ satisfies the properties required by the theorem.

3. We consider a subsequence $n_1, n_2, \ldots, n_i, \ldots$ along which $\frac{\mathbb{E}[\mathcal{LP}(n_i, c)]}{n_i}$ converges to $\lambda(c)$. Let $X_1, \ldots, X_{n_i}, \Psi_1, \ldots, \Psi_{cn_i} \in [B^2_1, B^2_2]^{n_i} \times [0, \infty)^{cn_i}$ denote a (random) optimal assignment which achieves the optimal value $\mathcal{LP}(n_i)$. For each $n_i$ we pick a variable $x_1$ from the pool $x_1, \ldots, x_{n_i}$ (the actual index is irrelevant) and consider its depth $d$ neighborhood appropriately defined, where $d$ is some fixed constant. We then consider the optimal solution $(X(n_i, d), \Psi(n_i, d))$ restricted to this $d$-neighborhood. We consider the probability distribution $\mathcal{P}(d, n_i)$ which describes the joint probability distribution for the values of $(X_i, \Psi_j, W_j)$ for $X_i, \Psi_j$ in the $d$-neighborhood as well as the graph-theoretic structure of this neighborhood.

We show that for each fixed $d$, the sequence of probability measures $\mathcal{P}(d, n_i)$ is tight in its corresponding probability space. As a result, there exists subsequence of $n_i$ along which the probability distribution $\mathcal{P}(d, n_i)$ converges to a limiting probability distribution $\mathcal{P}(d)$ for every fixed $d$. Moreover, we show that the subsequence can be selected in such a way that the resulting probability
distributions are consistent. Namely, for every \( d < d' \), the marginal distribution of \( P(d') \) in \( d' \)-neighborhood is exactly \( P(d) \). We will show that, since the sequence \( n_i \) was selected to achieve the optimal value \( \mathbb{E}[GLP(n_i, c)] \approx \lambda(c)n \), then

\[
\mathbb{E}[w_x X_1 + \frac{w_x}{K} \sum_j \Psi_j] = \lambda(c),
\]

where the expectation is with respect to \( P(d) \) and the summation is over all the constraints \( C_j \) containing \( X_1 \).

The sequence of probability distributions \( P(d), d = 1, 2, \ldots \) was used by Aldous in [Ald92] to obtain an invariant (with respect to a certain pivot operator) probability distribution on some infinite tree. Our proof does not require the analysis of such a tree, although similar invariant measure can be constructed.

4. We consider a random sequence \( \mathbb{E}[X_1|\mathbb{S}_d], d = 1, 2, \ldots \) where \( X_1 \in [B_{x_1}^1, B_{x_1}^2] \) is, as above, the value that is assigned to the variable \( x_1 \) by an optimal solution, and \( \mathbb{S}_d \) is the filtration corresponding to the sequence of probability measures \( P(d), d = 1, 2, \ldots \). We prove that the sequence \( \mathbb{E}[X_1|\mathbb{S}_d], d = 1, 2, \ldots \) is a martingale.

5. This is the "projection" step in which for any \( \epsilon > 0 \) and an arbitrary large \( n \) we construct a feasible solution to the system of constraints \( \mathbb{S} \) which achieves the expected objective value at most \( (\lambda(c) + \epsilon)n \). Given any large \( n \) and an instance of a random linear program \( \mathbb{S} \) with variables \( x_1, x_2, \ldots, x_n, \psi_1, \ldots, \psi_{cn} \) and constraints \( C_j, 1 \leq j \leq cn \), for each variable \( x_i, 1 \leq i \leq n \) we consider its \( d \)-neighborhood, where \( d \) is a very large constant. We let the value of \( x_i \) be \( \mathbb{E}[X_i|\mathbb{S}_d] \) where the expectation is conditioned on the observed \( d \)-neighborhood of the variable \( x_i \) and this information is incorporated by filtration \( \mathbb{S}_d \). By construction, this value is in \( [B_{x_1}^1, B_{x_1}^2] \). Then we set \( \Psi_j \) to the minimal value which satisfies the \( j \)-th constraint for the selected values of \( x_i \), for all \( j = 1, 2, \ldots, cn \). Using a martingale convergence theorem and property \( \mathbb{E} \) we show that for a randomly chosen variable \( x_i \), the corresponding value of \( \mathbb{E}[w_x X_i + \frac{w_x}{K} \sum_j \Psi_j] \) is smaller than \( \lambda(c) + \epsilon \), when \( n \) and \( d \) are sufficiently large. We sum the expectation above over all \( x_i \) and observe that each constraint belongs in the sum \( \sum_j \) of exactly \( K \) variables \( x_i \). Then the sum of these expectations is \( \mathbb{E}[w_x \sum_{1 \leq i \leq n} X_i + w_\psi \sum_{1 \leq j \leq cn} \Psi_j] \) which is exactly the objective function. We use this to conclude that the expected value of the objective function is at most \( (\lambda(c) + \epsilon)n \).

4 Poisson trees and some preliminary results

We begin by showing that in order to prove Theorem 1 it suffices to prove the existence of a limit \( \mathbb{E}[GLP(n, c)] \) for the expected value of the optimal cost \( \mathbb{E}[GLP(n, c)] \). Indeed, note that given an instance of a linear program \( \mathbb{S} \), if we change one of the constraints \( C_j \) to any other constraint from the pool \( \mathbb{C} \) and change the value of \( W_j \) to any other value in \([-B_w, B_w]\), and leave all the other constraints intact, then the optimal value \( \mathbb{E}[GLP(n, c)] \) changes by at most a constant. Using a corollary of Azuma’s inequality (see Corollary 2.27 [JLR00] for the statement and a proof), we obtain that \( \mathbb{P}(|\mathbb{E}[GLP(n, c)]| / \mathbb{E}[GLP(n, c)] > \epsilon) \) converges to zero exponentially fast for any \( \epsilon > 0 \). Then the convergence \( \lim_{n \to \infty} \mathbb{E}[GLP(n, c)] / n \) implies that \( \mathbb{E}[GLP(n, c)] / n \) holds w.h.p. Thus, from now on we concentrate on proving the existence of the limit

\[
\lim_n \frac{\mathbb{E}[GLP(n, c)]}{n}.
\]
A random instance of a linear program naturally leads to a sparse weighted $K$-hypergraph structure on a node set $x_1, \ldots, x_n$. Specifically, for each constraint $C_j, 1 \leq j \leq cn$ we create a $K$-edge $(x_{i_1}, \ldots, x_{i_K}, r, w_j)$, if $C_j$ contains exactly variables $x_{i_1}, \ldots, x_{i_K}$ in this order, the constraint is type $r, 1 \leq r \leq |C|$ and the corresponding random variable $W_j$ has value $w_j$. This set of edges completely specifies the random instance. We first study the distribution of the number of edges containing a given fixed node $x = x_1, \ldots, x_n$.

**Lemma 4** Given node $x$ from the pool $x_1, \ldots, x_n$, the number of edges (constraints $C_j$) containing $x$ is distributed as Pois$(cK)$, in the limit as $n \to \infty$.

**Proof**: The probability that a given edge does not contain $x$ is $1 - K/n$ if variables are taken without replacement and $((n - 1)/n)^K = 1 - K/n + o(1/n)$ if taken with replacement. The probability that exactly $s$ edges contain $x$ is then asymptotically \(\binom{cn}{s}(K/n)^s(1-K/n)^{cn-s}\). When $s$ is a constant and $n \to \infty$, this converges to \(\frac{(cK)^s}{s!}\) \(\exp(-cK)\).

We now introduce a notion of a $d$-neighborhood of a variable $x$. A collection of constraints $C_{i_1}, C_{i_2}, \ldots, C_{i_r}, 1 \leq i_j \leq m$ from an instance of linear program is defined to be a chain of length $r$ if for all $j = 1, \ldots, r - 1$ the constraints $C_{i_j}$ and $C_{i_{j+1}}$ share at least one variable. Fix a variable $x$ from the pool $x_1, 1 \leq i \leq n$. We say that a variable $x' \in \{x_1, \ldots, x_n\}$ belongs to a $d$-neighborhood of $x$ if $x$ is connected to $x'$ by a chain of length at most $d$. We say that a constraint $C_j$ belongs to the $d$-neighborhood of $x$ if all of its variables belong to it. The variables $W_j$ and $\psi_j$ in these constraints are also assumed to be a part of this neighborhood. In particular, a 1-neighborhood of $x$ is the set of constraints $C_j$ which contain $x$ together with variables in these constraints. If no constraints contain $x$, the 1-neighborhood of $x$ is just $\{x\}$. For $d \geq 1$, we let $B(x, d, n)$ denote the $d$-neighborhood of $x$, and let $\partial B(x, d, n) = B(x, d, n) \setminus B(x, d - 1, n)$ denote the boundary of this neighborhood, where $\partial B(x, 0, n)$ is assumed to be $\emptyset$. Of course $B(x, d, n)$ and $\partial B(x, d, n)$ are random. In graph-theoretic terms, $B(x, d, n)$ is a sub-graph of the original $K$-hypergraph corresponding to nodes with distance at most $d$ from $x$.

A chain $C_{i_1}, C_{i_2}, \ldots, C_{i_r}$ is defined to be a cycle if $C_{i_1}, C_{i_2}, \ldots, C_{i_r-1}$ are distinct and $C_{i_r} = C_{i_1}$. The following observation is a standard result from the theory of random graph. A simple proof is provided for completeness.

**Lemma 5** Let $r, r'$ be fixed constants. The expected number of cycles of length $r$ is at most $(K^2c)^r$. Moreover, the expected number of variables with distance at most $r'$ from some size-$r$ cycle is at most $r^r(K^2+K^{r+1})$. In particular, for constants $r, r'$ the probability that a randomly and uniformly selected variable $x$ is at distance at least $r'$ from any size-$r$ cycle is at least $1 - \frac{r^r(K^2+K^{r+1})}{n}$.

**Proof**: Fix any $r$ variables, say $x_1, \ldots, x_r$ and let us compute the expected number of cycles $C_1, \ldots, C_r$ such that $C_j$ and $C_{j+1}$ share variable $x_j, j = 1, \ldots, r$, (where $C_{r+1}$ is identified with $C_1$). Constraint $C_j$ contains variables $x_{j-1}, x_j (x_0 = x_r)$. There are at most $n^{K^2}/(K - 2)!$ choices for other variables in $C_j$. For each such choice, the probability that a constraint consisting of exactly this selection of variables is present in the random instance is at most $\frac{(cn)/(n^K/K!)}{K!c/n^K-1}$. Finally, there are at most $n^r$ ways to select (with order) $r$ variables $x_1, \ldots, x_r$. Combining, we obtain that, asymptotically, the expected number of length-$r$ cycles is at most

$$n^r \left( \frac{n^{K-2}}{(K-2)!} K! \frac{c}{n^{K-1}} \right)^r < (K^2c)^r.$$ 

By Lemma 4 for any fixed variable $x$ the expected number of variables with distance at most $r'$ from $x$ is asymptotically at most $(cK)^{r'}$. Each size $r$ cycle contains at most $rK$ variables. Therefore, the total
expected number of variables with distance at most \( r' \) to some size \( r \) cycle is at most \((K^2c)^r (rK)(cK)^{r'} = r^{c+r'}K^{2r+r'}\).

Applying Lemmas 4 and 5 we obtain the following proposition, which is well known in the context of random graphs [JLR00].

**Proposition 1** Given a variable \( x \), the number of constraints in \( \mathcal{B}(x, 1, n) \) is distributed as \( \text{Pois}(cK) \), in the limit as \( n \to \infty \). Also \( \mathcal{B}(x, d, n) \) is distributed as a depth-\( d \) Poisson tree. That is, if \( \partial \mathcal{B}(x, d, n) \) contains \( r \) constraints and \( r(K-1) \) variables, then the number of constraints in \( \partial \mathcal{B}(x, d+1, n) \) is distributed as \( \text{Pois}(crK(K-1)) \). Moreover, these constraints do not share variables, other than variables in \( \partial \mathcal{B}(x, d, n) \). In short, the constraints in \( \mathcal{B}(x, d, n) \) are distributed as first \( d \) steps of a Galton-Watson (branching) process with outdegree distribution \( \text{Pois}(cK) \), in the limit as \( n \to \infty \).

We finish this section by showing how Theorem 1 implies Theorem 2. We noted before that the linear program (1) is a special case of the linear program (6), via setting \( W_j = 0 \) with probability one and by setting \( w_x = 0, w_\psi = 1 \). Assuming limit \( f(c) \) defined in Theorem 1 exists, let us first show that it is a non-decreasing function of \( c \). This is best seen by the following coupling arguments. For any \( c_1 < c_2 \) and large \( n \) consider two instances of random linear program (6) with \( m = c_1n \) and \( m = c_2n \), where the second is obtained by adding \((c_2-c_1)n\) additional constraints to the first instance (we couple two instances). For each realization of two linear programs, such an addition can only increase or leave the same the value of the objective function. Note that in both cases we divide the objective value by the same \( n \). We conclude \( f(c_1) \leq f(c_2) \).

Let \( c^*_K \equiv \sup\{ c \geq 0 : f(c) = 0 \} \). Clearly the set in this definition includes \( c = 0 \), and therefore is non-empty. The definition of \( c^*_K \) of course includes the possibilities \( c^*_K = 0, c^*_K = \infty \) and clearly satisfies 3 and 10.

The proof of the second part of Theorem 2 follows from the following proposition.

**Proposition 2** Under Condition \( \mathcal{A} \), for any \( c < 1/K^2 \), a random \( K \)-LSAT instance has the optimal value \( \mathbb{E}[\mathcal{LP}(n, c)] = O(1) \). Under Condition \( \mathcal{B} \), there exists \( \bar{c}_K > 0 \) such that for all \( c > \bar{c}_K \), there exists \( \delta(c) > 0 \) for which \( \mathbb{E}[\mathcal{LP}(n, c)] \geq \delta(c)n \).

**Proof:** Suppose Condition \( \mathcal{A} \) holds. The technique we use is essentially "set one variable at a time" algorithm, used in several papers on random K-SAT problem to establish lower bounds on critical values of \( c \), see, for example [AS00]. Set variable \( x_1 \) to any value in \([B^1_1, B^2_1]\) (actual value is irrelevant). For every constraint containing \( x_1 \) (if any exist), set other variables in this constraint so that this constraints are satisfied with corresponding values of \( \psi \) equal to zero. This is possible by Condition \( \mathcal{A} \). For every variable which is set in this step, take all the other constraint containing this variable (if any exist) and set its variables so that again these constraints are satisfied with \( \psi = 0 \). Continue in this fashion. If at any stage some newly considered constraint \( C_j \) contains some variable which was set in prior stages, set \( \psi \) to the minimum value which guarantees to satisfy this constraint. At the worst we put \( \psi = \max_x(|b_1| + \sum |a_{r,k}| \max(|B^1_2|, |B^2_2|)) \). Note, however, that this situation occurs only if \( C_j \) belongs to some cycle. Applying the first part of Lemma 5 for \( c < 1/K^2 \) the total expected number of constraints which belong to at least one cycle is at most \( \sum_{r=1}^{\infty} r(K^2c)^r = O(1) \). Therefore, the total expected number of constraints, for which we set \( \psi \) positive, is also \( O(1) \) and \( \mathbb{E}[\mathcal{LP}(n, c)] = O(1) \).

Suppose now Condition \( \mathcal{B} \) holds. The proof is very similar to proofs of upper bounds on critical thresholds for random K-SAT problems and uses the following moment argument. Consider an instance of random K-LSAT with \( n \) variables \( m = cn \) constraints, where \( c \) is a very large constant, to be specified later. Consider any \( n \)-dimensional cube \( I \) of the form \( \prod_{1 \leq k \leq n} [\frac{1}{P}, \frac{1}{P+1}], B^1_1 \leq \frac{1}{P}, B^2_2 \). Consider the optimal cost \( \mathcal{LP}(n, c) \) corresponding to solutions \( x, \psi \), such that \( x \) is restricted to belong to \( I \). Let,
w.l.g., $x_1^j, \ldots, x_K^j$ be the variables which belong to the $j$-th constraint, $1 \leq j \leq m$. By Condition $B$, with probability at least $1/|C|$, $C_j$ is such that the corresponding $v_j$ must be at least $\nu$. We obtain that the expected cost $\mathbb{E}[\mathcal{L}P(n, c)] \geq \nu(1/|C|)m$, when $x \in I$. Moreover, since the events $"C_j"$ are independent for all $j$, then, applying Chernoff bound, $\mathbb{P}\{\mathcal{L}P_1(n, c) \leq (1/2)\nu(1/|C|)m\} \leq e^{-\frac{m}{8|C|}}$ where the optimum $\mathcal{L}P_1(n, c)$ of the linear program is taken over the subset $x \in I$. Since there are at most $m$ different cubes $I$, and every solution $x$ belongs to at one of them (several for points $x$ on the boundary between two cubes) then $\mathbb{P}\{\mathcal{L}P(n, c) \leq (1/2)\nu(1/|C|)m\} \leq m e^{-\frac{m}{8|C|}}$. By taking $m = cn$ with $c$ sufficiently large, we obtain that the probability above is exponentially small in $n$. \hfill $\Box$

## 5 Limiting probability distributions

In this and the following two sections we prove the existence of the limit (14). Fix $c > 0$ and take $n$ to be large. We assume that the labelling (order) of the variables $x_1, \ldots, x_n$ is selected independently from all the other randomness of the instance. In graph-theoretic sense we have a random labelled hypergraph with labels of the nodes independent from the edges of the graph. We noted above that

$$\psi_j \leq \max_{1 \leq i \leq |C|} \left( \sum_k (|B_1^i| + |B_2^i|)K|a_{rk}| + |b_r| \right) \equiv B_{\psi}.$$  

Let, $X(n), \Psi(n)$ denote an optimal (random) assignment of the random linear programming problem (14), where $X(n) = (X_1, \ldots, X_n), \Psi(n) = (\Psi_1, \ldots, \Psi_m)$. That is $x_i = X_i, \psi_j = \Psi_j$ achieve the objective value $\mathcal{G}LP(n, c)$. If the set of optimal solutions contains more than one solution (in general it can be uncountable) select a solution $X(n), \Psi(n)$ uniformly at random from this set. Define

$$\lambda(c) = \liminf_n \frac{\mathbb{E}[\mathcal{G}LP(n, c)]}{n},$$

and find a subsequence $n_t, t = 1, 2, \ldots$ along which

$$\lim_t \frac{\mathbb{E}[\mathcal{G}LP(n_t, c)]}{n_t} = \lambda(c).$$

Fix a variable $x$ from the set of all $n_t$ $x$-variables. Since labelling is chosen independent from the instance, we can take $x = x_1$, w.l.g. Denote by $X(d, n_t), \Psi(d, n_t), W(d, n_t)$ the collection of $X, \Psi$ and $W$-variables which belong to the neighborhood $B(x_1, d, n_t)$. In particular $X_1 \in X(d, n_t)$ and the number of $\Psi$ variables is the same as the number of $W$ variables which is the number of constraints $C_j$ in $B(x_1, d, n_t)$. Denote by $\mathcal{P}(d, n_t)$ the joint probability distribution of $(B(x_1, d, n_t), X(d, n_t), \Psi(d, n_t), W(d, n_t))$. We omit $x = x_1$ in the notation $\mathcal{P}(d, n_t)$ since, by symmetry, this joint distribution is independent of the choice of $x$. The support of this probability distribution is $\mathcal{X}(d) = \cup(T, [B_1^2, B_2^2]|T| \times [0, B_{\psi}]^{E(T)} \times [-B_w, B_w]^{E(T)} \cup \mathcal{E}$, where the first union runs over depth-$d$ rooted trees $T$ with root $x_1$, $|T|$ is number of nodes in $T$, $E(T)$ is the number of constraints in $T$, and $\mathcal{E}$ is a singleton event which represents the event that $B(x_1, d, n_t)$ is not a tree and contains a cycle. In particular, $\mathcal{X}(d)$ is a countable union of compact sets. We have from Proposition 1 that $\lim_{t \to \infty} \mathcal{P}(d, n_t)(\mathcal{E}) = 0$. Observe that $\mathcal{X}(d) \subset \mathcal{X}(d+1)$ for all $d$. As we showed in Proposition 1 the marginal distribution of $T$ with respect to $\mathcal{P}(d, n_t)$ is depth-$d$ Poisson tree, in the limit as $t \to \infty$.

Recall, that a sequence of probability measures $\mathcal{P}_n$ defined on a joint space $\mathcal{X}$ is said to be weakly converging to a probability measure $\mathcal{P}$ if for any event $A \subset \mathcal{X}$, $\lim_{n \to \infty} \mathcal{P}_n(A) = \mathcal{P}(A)$. We also need the following definition and a theorem, both can be found in [Dur96].
Definition 1 A sequence of probability measures $\mathcal{P}_n$ on $\mathcal{X}$ is defined to be tight if for every $\epsilon > 0$ there exists a compact set $K \subset \mathcal{X}$ such that $\limsup_n \mathbb{P}\{X \setminus K\} < \epsilon$.

Theorem 6 Given a tight sequence of probability measures $\mathcal{P}_n$ on $\mathcal{X}$ there exists a probability measure $\mathcal{P}$ on $\mathcal{X}$ and a subsequence $\mathcal{P}_{n_t}$ that weakly converges to $\mathcal{P}$.

The following proposition is a key result of this section.

Proposition 3 For each $d = 1, 2, \ldots$ there exists a probability measure $\mathcal{P}(d)$ on $\mathcal{X}(d)$ such that $\mathcal{P}(d, n_t)$ weakly converges to $\mathcal{P}(d)$. The sequence of probability measures $\mathcal{P}(d)$, $d = 1, 2, \ldots$ is consistent in the sense that for every $d < d'$, $\mathcal{P}(d)$ is the marginal distribution of $\mathcal{P}(d')$ on $\mathcal{X}(d)$. The probability of the event $E$ is equal to zero, with respect to $\mathcal{P}(d)$. Finally, with respect to $\mathcal{P}(d)$,

$$\mathbb{E}[w_x X_1 + \frac{w}{K} \sum_j \Psi_j] = \lambda(c),$$

where the summation is over all the constraints $C_j$ in $T(d)$ containing the root variable $x_1$.

Proof: The proof is similar to the one in [Ald92] for constructing a similar limiting probability distribution of optimal matchings in a complete bi-partite graphs with random weights, where a compactness argument plus Kolmogorov’s extension theorem is used to obtain the limiting measures on infinite tree. In our case, since we limit ourselves to trees with bounded depths, the proofs can be simplified by using the tightness argument. Fix $d \geq 1$. We claim that the sequence of measures $\mathcal{P}(d, n_t)$ is tight on $\mathcal{X}(d)$. By Proposition[1] according to the measure $\mathcal{P}(d, n_t)$ the neighborhood $B(x_1, d, n_t)$ approaches in distribution a depth-$d$ Poisson tree with parameter $cK$. In particular, the expected number of constraints in this neighborhood is smaller than $cK + c^2 K^3 + \ldots + c^d K^{2d-1} \equiv M_0$, in the limit $t \to \infty$. Fix $\epsilon > 0$. By Markov’s inequality the total number of constraints in $B(x_1, d, n_t)$ is at most $M$ with probability at least $1 - \epsilon$, for $M > M_0/\epsilon$ and $n_t$ sufficiently large. This implies that, moreover, each $x$-variable in $B(x_1, d, n_t)$ belongs to at most $M$ constraints (has degree at most $M$) with probability at least $1 - \epsilon$.

Let $\mathcal{X}(d, M) \subset \mathcal{X}(d)$ denote $\bigcup (T, [B_{x_1}^1, B_{x_2}^1]^{|T|} \times [0, B_\psi]^{|E(T)|} \times [-B_w, B_w]^{|E(T)|})$, where the trees $T$ are restricted to have degree bounded by $M$. That is, each variable in such a tree belongs to at most $M$ constraints – one towards the root and $M - 1$ in the opposite direction, and the root $x$ belongs to at most $M$ constraints. The number of trees $T$ in $\mathcal{X}(d, M)$ is finite and, as a result, the set $\mathcal{X}(d, M)$ is compact, as it is a finite union of sets of the form $\{T\} \times [B_{x_1}^1, B_{x_2}^1]^{|T|} \times [0, B_\psi]^{|E(T)|} \times [-B_w, B_w]^{|E(T)|}$. We showed above that according to $\mathcal{P}(d, n_t)$, the neighborhood $B(x_1, d, n_t)$ belongs to $\mathcal{X}(d, M)$ with probability at least $1 - \epsilon$, for all sufficiently large $t$. This proves that the sequence of measures $\mathcal{P}(d, n_t)$ is tight. Then, applying Theorem[2] there exists a weakly converging subsequence $\mathcal{P}(d, n_{t_i})$. We find such a sequence for $d = 1$ and denote it by $\mathcal{P}(1, n_{t_i}^{(1)})$, $t = 1, 2, \ldots$. Again using Theorem[3] for $d = 2$ there exists a subsequence of $\mathcal{P}(2, n_{t_i}^{(1)})$ which is weakly converging. We denote it by $\mathcal{P}(2, n_{t_i}^{(2)})$. We continue this for all $d$ obtaining a chain of sequences $n_{t_i}^{(1)} \supset n_{t_i}^{(2)} \supset \cdots \supset n_{t_i}^{(d)} \supset \cdots$. Select a diagonal subsequence $n_{t_i}^{*} \equiv n_{t_i}^{(*)}$ from these sequences. Then all the convergence above holds for this diagonal subsequence. In particular, for every $d$, $\mathcal{P}(d, n_{t_i}^{*})$ converges to some probability measure $\mathcal{P}(d)$. Moreover, these measures, by construction, are consistent. Meaning, for every $d < d'$, the marginal distribution of $\mathcal{P}(d')$ onto $\mathcal{X}(d)$ is simply $\mathcal{P}(d)$. Since, from Proposition[4] the probability of the event $E$ according to $\mathcal{P}(d, n_t)$ approaches zero, then the probability of the same event with respect to $\mathcal{P}(d)$ is just zero, for every $d$. 

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To complete the proof of the proposition we need to establish (18). Note that in random instances of linear program (16), with \( n_t \) \( x \)-variables, when we sum the expression \( w_x X_i + \frac{w_x}{K} \sum_j \Psi_j \) over all \( x_i, i = 1, 2, \ldots, n_t \) we obtain \( w_x \sum_{1 \leq i \leq n_t} X_i + \frac{w_x}{K} \sum_{1 \leq j \leq c_{n_t}} \Psi_j \), since each variable \( \Psi_j \) appeared in exactly \( K \) sums, corresponding to \( K \) variables in \( j \)-th constraint. From (17) we obtain that \( \lim_t \mathbb{E}[w_x X_i + \frac{w_x}{K} \sum_j \Psi_j] = \lambda(c) \), where the expectation is with respect to measure \( \mathcal{P}(d, n_t) \). Since all the random variables involved in \( w_x X_i + \frac{w_x}{K} \sum_j \Psi_j \) are bounded and since \( \mathcal{P}(d, n_t) \) converges weakly to \( \mathcal{P}(d) \), then the convergence carries on to the expectations. This implies that with respect to \( \mathcal{P}(d) \), \( \mathbb{E}[w_x X_i + \frac{w_x}{K} \sum_j \Psi_j] = \lambda(c) \). \( \square \)

6 Filtration and a martingale with respect to \( \mathcal{P}(d) \)

Let us summarize the results of the previous section. We considered a sequence of spaces \( \mathcal{X}(1) \subset \mathcal{X}(2) \subset \cdots \), where \( \mathcal{X}(d) = \bigcup (T, [B_1^d, B_2^d]^{|T|}) \times [0, \mathcal{B}_\Psi]^{|T|} \times [-B_w, B_w]|^{|T|}) \) and the union runs over all depth-\( d \) trees \( T \). Recall, that the event \( \mathcal{E} \) was used before to generically represent the event that \( \mathcal{B}(x, d, n) \) is not a tree. The probability of this event is zero with respect to the limiting probability distribution \( \mathcal{P}(d) \) we constructed, so we drop this event from the space \( \mathcal{X}(d) \). We have constructed a probability measure \( \mathcal{P}(d) \) on each \( \mathcal{X}(d) \) as a weak limit of probability measures \( \mathcal{P}(d, n_t) \), defined on \( d \)-neighborhoods of a variable \( x_1 \). Denote by \((T(d), X(d), \Psi(d), W(d))\) the random vector distributed according to \( \mathcal{P}(d) \). Note, that \( T(d) \) and \( W(d) \) are independent from each other, first distributed as a depth-\( d \) Poisson tree, second distributed as i.i.d. with the distribution function \( \mathbb{P}\{W \leq t\} \), yet \( X(d) \) and \( \Psi(d) \) depend on both \( T(d) \) and \( W(d) \).

Using Kolmogorov’s extension theorem, the probability measures \( \mathcal{P}(d) \) can be extended to a probability measure \( \mathcal{P} \) on space \( \bigcup (T, [B_1^d, B_2^d]^{|T|}) \times [0, \mathcal{B}_\Psi]^{|T|} \times [-B_w, B_w]|^{|T|}) \), where the union runs over all finite and infinite trees \( T \). This extension is used in Aldous [Al92] to construct a spatially invariant measure on infinite trees. For our purpose this extension, while possible, is not necessary, and instead we use a martingale convergence techniques.

Note, that, since the measures \( \mathcal{P}(d) \) are consistent, they correspond to a certain filtration on the increasing sequence of sets \( \mathcal{X}(d), d = 1, 2, \ldots \). Then we can look at \( \mathbb{E}[X_i|\mathcal{X}(d)](T(d), W(d)] \) as a stochastic process indexed by \( d = 1, 2, \ldots \). Another way of looking at this stochastic process is as follows. A root \( x = x_1 \) is fixed. At time \( d = 1 \) we sample constraints \( C \) containing \( x_1 \) with the corresponding values of \( W_j \). One way of looking at this stochastic process is to sample constraints \( C \) containing \( x_1 \) with the corresponding values of \( W_j \). Using the \( \mathcal{P}(1) \) distribution. Recall that the number of constraints is distributed as Pois\( (c_K) \), the type of each constraint is chosen uniformly at random from \( |C| \) types, and \( W_j \) values are selected i.i.d. using distribution function \( \mathbb{P}\{W \leq t\} \). For this sample we have a conditional probability distribution of \( (X(1), \Psi(1)) \), that is of \( x \) and \( \psi \) variables in 1-neighborhood of \( x_1 \). Then, for each variable in this depth-1 tree except for \( x_1 \) we again sample constraints and \( W_j \) values to obtain a depth-2 Poisson tree \( T(2) \) and \( W(2) \). We obtain a distribution of \( (X(2), \Psi(2)) \) conditioned on \( T(2), W(2) \), and so on. On every step we reveal a deeper layer of the tree and and the corresponding values of \( W_j \)-s to obtain a new conditional probability distribution for \( (X(d), \Psi(d)) \).

The technical lemma that follows is needed to prove Proposition 4 below. This lemma is sometimes defined as tower property of conditional expectations. The proof of it can be found in many books on probability and measure theory, see for example Theorem 1.2, Chapter 4 in [Dur96].

**Lemma 7** Let \( X \) and \( Y \) be dependent in general random variables, where \( X \in \mathcal{X} \) and \( Y \) takes values in some general set \( \mathcal{Y} \). Let \( f : \mathcal{Y} \to \mathcal{Y} \) be a measurable function. Then \( \mathbb{E}[\mathbb{E}[X|Y]|f(Y)] = \mathbb{E}[X|f(Y)] \).

The conditional expectations in this lemma are understood as follows. \( \mathbb{E}[X|Y] \) is an expectation of \( X \) conditioned on the \( \sigma_Y \)-algebra generated by the random variable \( Y \). Similarly, \( \mathbb{E}[X|f(Y)] \) is an
expectation of $X$ conditioned on the $\sigma_f(Y)$-algebra generated by the random variable $f(Y)$. Of course $\sigma_f(Y) \subset \sigma_Y$.

As above let $x_1$ be the root of our random tree $T(d)$ and let $X_1$ be the corresponding random variable from the vector $(T(d), X(d), \Psi(d), W(d))$. Conditioned on the event that $x_1$ belongs to at least one constraint $C_j$, select any such a constraint and let, w.l.g. $x_2, \ldots, x_K$ be the variables in this constraint. For every $k = 2, \ldots, K$ and every $d \geq 2$, consider $B(x_k, d) \equiv T(x_k, d)$, the $d$-neighborhood of the variable $x_k$. One way to introduce this neighborhood formally is to select any $d' > d + 1$, sample $T(d')$ from $\mathcal{P}(d')$ and then consider the $d$-neighborhood around $x_k$, as a subtree of the tree $T(d')$. But since, by consistency, the distribution of these neighborhood is the same for all $d' > d + 1$, then we can simply speak about selecting $T(x_k, d)$. Note, that $T(x_1, d) = T(d)$. Let $W(x_k, d)$ denote the collection of $W_j$ variables corresponding to constraints in $T(x_k, d)$. To simplify the notations, we will let $T_W(x_k, d)$ stand for the pair $(T(x_k, d), W(x_k, d))$.

*Proposition 4* For every $k = 1, 2, \ldots, K$ a random sequence $\mathbb{E}[X_k|T_W(x_k, d), d = 1, 2, \ldots]$ is a martingale with values in $[B^1_1, B^2_2]$. Moreover, the sequence $\mathbb{E}[X_k|T_W(x_d \text{ mod } (K), d), d = 1, 2, \ldots]$ is also a martingale.

**Remark**: To understand better the meaning of the first part of the proposition, imagine that we first sample depth-1 tree $T_W(1)$. In case this tree is not trivial ($T_W(1) \neq \{x_1\}$), we fix a constraint from this tree and variable $x_k$ from this constraint. Then we start revealing trees $T_W(x_k, 2), T_W(x_k, 3), \ldots$. This defines the stochastic process of interest.

The second part states that even if we reveal trees rooted in a round-robin fashion at variables $x_1, x_2, \ldots, x_K, x_1, x_2, \ldots$, then we still obtain a martingale. That is the sequence $\mathbb{E}[X_k|T_W(x_1, 1)], \mathbb{E}[X_k|T_W(x_2, 2)], \ldots, \mathbb{E}[X_k|T_W(x_K, K)], \mathbb{E}[X_k|T_W(x_1, K + 1)], \mathbb{E}[X_k|T_W(x_2, K + 2)], \ldots$, is a martingale.

**Proof**: Recall, that the optimal values $X(d)$ are a weak limit of optimal values $X(d, n^*_t), t = 1, 2, \ldots$. Then, every $X_t \in [B^1_1, B^2_2]$ almost surely. To prove the martingale property we use Lemma 7 where $X = X_k, Y = T_W(x_k, d + 1)$ and $f$ is a projection function which projects a depth-$d$ + 1 tree $T_W(x_k, d + 1)$ onto a depth-$d$ tree $T_W(x_k, d)$ by truncating the $d + 1$-st layer. Applying the lemma, we obtain $\mathbb{E}[\mathbb{E}[X_k|T_W(x_k, d + 1)]|T_W(d)] = \mathbb{E}[X_k|T_W(x_k, d)]$, which means precisely that the sequence is a martingale. The proof of the second part is exactly the same, we just observe that $T_W(x_1, 1) \subset T_W(x_2, 2) \subset \cdots \subset T_W(x_d \text{ mod } (K), d) \subset \cdots$, and we let $f$ to be a projection of $T_W(x_d \text{ mod } (K), d + 1)$ onto $T_W(x_d \text{ mod } (K), d)$.

The following is a classical result from the probability theory.

*Theorem 8* [Martingale Convergence Theorem]. Let $X_n, n = 1, 2, \ldots$ be a martingale which takes values in some finite interval $[-B, B]$. Then $X_n$ converges almost surely to some random variable $X$. As a consequence, $|X_{n+1} - X_n|$ converges to zero almost surely.

A proof of this fact can be found in [Dur96]. Here we provide, for completeness, a very simple proof of a weak version of MCT, stating that $|X_{n+1} - X_n|$ converges to zero in probability. This is the only version we need for this paper. We have

\begin{equation}
0 \leq \mathbb{E}[(X_{n+1} - X_n)^2] = \mathbb{E}[X_{n+1}^2] - 2\mathbb{E}[X_{n+1}X_n] + \mathbb{E}[X_n^2].
\end{equation}

But $\mathbb{E}[X_{n+1}X_n] = \mathbb{E}[(\mathbb{E}[X_{n+1}|X_n])X_n] = \mathbb{E}[X_n^2]$, where $\mathbb{E}[X_{n+1}|X_n] = X_n$ is used in the second equality. Combining with (19), we obtain $0 \leq \mathbb{E}[(X_{n+1} - X_n)^2] \leq \mathbb{E}[X_{n+1}^2] - \mathbb{E}[X_n^2]$ and therefore $\mathbb{E}[X_n^2]$ is an increasing subsequence. Since $\mathbb{E}[X_n^2] \leq B^2$, then this sequence is converging and, from (19), $\mathbb{E}[(X_{n+1} -
using also a martingale, and again applying Theorem 8, we obtain that 
for sufficiently large $d$

$$\lambda$$

values of $\lambda$

That is $\bar{\Psi}_j$ is the optimal value to be assigned to $\psi_j$ when the variables $x_k$ take values $\bar{X}_k$. Observe that, just like $\bar{X}_k$, $\bar{\Psi}_j$ is a random variable. Its value is determined by the random tree $T(x_k, d)$ and the values of $W_j$, and, importantly, it is different from $\Psi_j$. The following proposition is the main technical result of this section. Jumping ahead, the usefulness of this proposition is that if we assign the value $\bar{X}_k$ to each variable $x_k$ and take $\psi_j = \bar{\Psi}_j$ for each constraint containing $x_1$, then we obtain that the corresponding objective value of the linear program (3) "per variable" $x_1$ almost achieves value $\lambda(c)$, provided $d$ is sufficiently large. We will use this in the following section for the projection step, where for a random instance of linear program (3), we assign value $\bar{X}_i$ for every variable $x_i, 1 \leq i \leq n$ and obtain an objective value close to $\lambda(c)n$, for arbitrary large $n$.

**Proposition 5** For every $\epsilon > 0$ there exists sufficiently large $d = d(\epsilon)$ such that $\mathbb{E}[w_x \bar{X}_1 + \frac{w_x}{K} \sum_j \bar{\Psi}_j] < \lambda(c) + \epsilon$, where the summation is over all the constraints in $\mathcal{J}(x_1)$.

**Proof:** Fix $\delta > 0$ very small. Fix any constraint $C_j \in \mathcal{J}(x_1)$ and a variable $x_k$ in it. We first show that for sufficiently large $d$, $\mathbb{P}\{|\bar{X}_k - \mathbb{E}[X_k|T^W(x_k, d)]| > \delta\} < \delta$. In other words, the expected values of $X_k$ conditioned on $T^W(x_k, d)$ and $T^W(x_1, d)$ are sufficiently close to each other. For this purpose we use the martingale convergence theorem and Proposition 4. Take largest integer probability.

Let $\bar{X}_k \equiv \mathbb{E}[X_k|T^W(x_k, d)]$ and let

$$\Psi_j \equiv \max(0, \sum_{1 \leq k \leq K} a_{rk} \bar{X}_{\sigma(k)} - b_r - W_j).$$

That is $\Psi_j$ is the optimal value to be assigned to $\psi_j$ when the variables $x_k$ take values $\bar{X}_k$. Observe that, just like $\bar{X}_k$, $\Psi_j$ is a random variable. Its value is determined by the random tree $T(x_k, d)$ and the values of $W_j$, and, importantly, it is different from $\Psi_j$. The following proposition is the main technical result of this section. Jumping ahead, the usefulness of this proposition is that if we assign the value $\bar{X}_k$ to each variable $x_k$ and take $\psi_j = \Psi_j$ for each constraint containing $x_1$, then we obtain that the corresponding objective value of the linear program (3) "per variable" $x_1$ almost achieves value $\lambda(c)$, provided $d$ is sufficiently large. We will use this in the following section for the projection step, where for a random instance of linear program (3), we assign value $\bar{X}_i$ for every variable $x_i, 1 \leq i \leq n$ and obtain an objective value close to $\lambda(c)n$, for arbitrary large $n$.

**Proposition 5** For every $\epsilon > 0$ there exists sufficiently large $d = d(\epsilon)$ such that $\mathbb{E}[w_x \bar{X}_1 + \frac{w_x}{K} \sum_j \bar{\Psi}_j] < \lambda(c) + \epsilon$, where the summation is over all the constraints in $\mathcal{J}(x_1)$.

**Proof:** Fix $\delta > 0$ very small. Fix any constraint $C_j \in \mathcal{J}(x_1)$ and a variable $x_k$ in it. We first show that for sufficiently large $d$, $\mathbb{P}\{|\bar{X}_k - \mathbb{E}[X_k|T^W(x_k, d)]| > \delta\} < \delta$. In other words, the expected values of $X_k$ conditioned on $T^W(x_k, d)$ and $T^W(x_1, d)$ are sufficiently close to each other. For this purpose we use the martingale convergence theorem and Proposition 4. Take largest integer probability.

Let $\bar{X}_k \equiv \mathbb{E}[X_k|T^W(x_k, d)]$ and let

$$\Psi_j \equiv \max(0, \sum_{1 \leq k \leq K} a_{rk} \bar{X}_{\sigma(k)} - b_r - W_j).$$

That is $\Psi_j$ is the optimal value to be assigned to $\psi_j$ when the variables $x_k$ take values $\bar{X}_k$. Observe that, just like $\bar{X}_k$, $\Psi_j$ is a random variable. Its value is determined by the random tree $T(x_k, d)$ and the values of $W_j$, and, importantly, it is different from $\Psi_j$. The following proposition is the main technical result of this section. Jumping ahead, the usefulness of this proposition is that if we assign the value $\bar{X}_k$ to each variable $x_k$ and take $\psi_j = \Psi_j$ for each constraint containing $x_1$, then we obtain that the corresponding objective value of the linear program (3) "per variable" $x_1$ almost achieves value $\lambda(c)$, provided $d$ is sufficiently large. We will use this in the following section for the projection step, where for a random instance of linear program (3), we assign value $\bar{X}_i$ for every variable $x_i, 1 \leq i \leq n$ and obtain an objective value close to $\lambda(c)n$, for arbitrary large $n$.

**Proposition 5** For every $\epsilon > 0$ there exists sufficiently large $d = d(\epsilon)$ such that $\mathbb{E}[w_x \bar{X}_1 + \frac{w_x}{K} \sum_j \bar{\Psi}_j] < \lambda(c) + \epsilon$, where the summation is over all the constraints in $\mathcal{J}(x_1)$.

**Proof:** Fix $\delta > 0$ very small. Fix any constraint $C_j \in \mathcal{J}(x_1)$ and a variable $x_k$ in it. We first show that for sufficiently large $d$, $\mathbb{P}\{|\bar{X}_k - \mathbb{E}[X_k|T^W(x_k, d)]| > \delta\} < \delta$. In other words, the expected values of $X_k$ conditioned on $T^W(x_k, d)$ and $T^W(x_1, d)$ are sufficiently close to each other. For this purpose we use the martingale convergence theorem and Proposition 4. Take largest integer $t$ such that $tK + k \leq d$. Since, by the first part of the proposition, $\mathbb{E}[X_k|T^W(x_k, d)], d = 1, 2, \ldots$ is a martingale, and since $d - (tK + k) < K$ then, applying Theorem 4, $\mathbb{E}[X_k|T^W(x_k, d)] - \mathbb{E}[X_k|T^W(x_k, d)]$ becomes very small w.h.p. as $d$ becomes large. In particular,

$$\mathbb{P}\{|\mathbb{E}[X_k|T^W(x_k, d)] - \mathbb{E}[X_k|T^W(x_k, d)]| > \delta\} < \delta,$$

for sufficiently large $d$. By the second part of Proposition 4, $\mathbb{E}[X_k|T^W(x_k, d)]$, $d = 1, 2, \ldots$ is a also a martingale, and again applying Theorem 4, we obtain that

$$\mathbb{P}\{|\mathbb{E}[X_k|T^W(x_k, d)] - \mathbb{E}[X_k|T^W(x_k, d)]| > \delta\} < \delta,$$

Finally, again applying the first part of Proposition 4 to the sequence $\mathbb{E}[X_1|T^W(x_1, d)], d = 1, 2, \ldots$, and using $d - (tK + 1) < 2K$, we obtain

$$\mathbb{P}\{|\mathbb{E}[X_k|T^W(x_1, d)] - \mathbb{E}[X_k|T^W(x_1, d)]| > \delta\} < \delta,$$

for sufficiently large $d$. Combining and using $\bar{X}_k = \mathbb{E}[X_k|T^W(x_k, d)]$, we obtain

$$\mathbb{P}\{|\bar{X}_k - \mathbb{E}[X_k|T^W(x_1, d)]| > 3\delta\} < \delta,$$
as claimed.

For every constraint $C_j \in \mathcal{J}(x_1)$ introduce

\begin{equation}
\hat{\Psi}_j \equiv \max(0, \sum_k a_{rk} \mathbb{E}[X_{\sigma(k)}|T^W(x_1, d)] - b_r - W_j),
\end{equation}

where $\sum_k a_{rk} x_{\sigma(k)} \leq b_r + W_j + \psi_j$ is the expanded form of the constraint $C_j$. That is $\hat{\Psi}_j$ is defined just like $\bar{\Psi}_j$, except for conditioning is done on the tree $T^W(x_1, d)$ instead of $T^W(x_k, d)$. Applying (20), (21) and recalling $\hat{\Psi}_j, \bar{\Psi}_j \in [0, B_\psi]$, we obtain that for our constraint $C_j$

\begin{equation}
P\{|\hat{\Psi}_j - \bar{\Psi}_j| > 3\delta K \max_{r_k} |a_{rk}|\} < K\delta.
\end{equation}

Recall, that, according the measure $\mathcal{P}(d)$, $J(x_1)$ is distributed as Pois($cK$). Select a value $M(\delta) > 0$ sufficiently large, so that $P\{J(x_1) > M(\delta)\} < \delta$. Conditioning on the event $J(x_1) \leq M(\delta)$ we obtain

\begin{align*}
P\left\{ \sum_j |\hat{\Psi}_j - \bar{\Psi}_j| > 3\delta M(\delta) K \max_{r_k} |a_{rk}| \Big| J(x_1) \leq M(\delta) \right\} &
\leq P\left\{ \exists C_j \in \mathcal{J}(x_1) : |\hat{\Psi}_j - \bar{\Psi}_j| > 3\delta K \max_{r_k} |a_{rk}| \Big| J(x_1) \leq M(\delta) \right\} \\
&
\leq M(\delta) P\left\{ |\hat{\Psi}_j - \bar{\Psi}_j| > 3\delta K \max_{r_k} |a_{rk}| \Big| J(x_1) \leq M(\delta) \right\} \\
&
\leq \frac{M(\delta)K\delta}{1 - \delta} < M(\delta)(K + 1)\delta,
\end{align*}

where the summation is over constraints in $\mathcal{J}(x_1)$, the last inequality holds provided $\delta < 1/K$, and $j$ in (21) corresponds to any fixed constraint in $\mathcal{J}(x_1)$. Combining with the event $J(x_1) > M$, we obtain from above that, without any conditioning,

\begin{equation}
P\left\{ \sum_j |\hat{\Psi}_j - \bar{\Psi}_j| > 3\delta M(\delta) K \max_{r_k} |a_{rk}| \right\} < M(\delta)(K + 1)\delta + \delta < M(\delta)(K + 2)\delta.
\end{equation}

Since $\hat{\Psi}_j, \bar{\Psi}_j \in [0, B_\psi]$, then the bound above implies

\begin{equation}
|\mathbb{E}[\sum_j \hat{\Psi}_j] - \mathbb{E}[\sum_j \bar{\Psi}_j]| \leq B_\psi M(\delta)(K + 2)\delta + 3\delta M(\delta) K \max_{r_k} |a_{rk}|.
\end{equation}

Our final step is to relate $\hat{\Psi}_j$ to the random variables $\Psi_j$, where $(X(d), \Psi(d))$ are drawn according to the probability distribution $\mathcal{P}(d)$. The distinction between $\hat{\Psi}_j$ and $\Psi_j$ is somewhat subtle and we expand on it here. As we observed above, the values of $\Psi(d)$ are determined almost surely by $\Psi_j = \max(0, \sum_k a_{rk} X_{\sigma(k)} - b_r - W_j)$, with respect to the measure $\mathcal{P}(d)$, when the corresponding constraint is $\sum_k a_{rk} x_{\sigma(k)} \leq b_r + W_j + \psi_j$. These values of $\Psi_j$ are different, however, from $\hat{\Psi}_j$ which are obtained by first taking the expectations of $X_k$ conditioned on trees $T^W(x_1, d)$ and then setting $\hat{\Psi}_j$ according to (22). Naturally, $\Psi_j$ and $\hat{\Psi}_j$ are related to each other. Note, that for every constraint $C_j$ almost surely

\begin{equation}
\Psi_j \geq \sum_k a_{rk} X_{\sigma(k)} - b_r - W_j, \quad \Psi_j \geq 0.
\end{equation}
By taking the conditional expectations and using the linearity of inequalities above, we obtain
\[
\mathbb{E}[\Psi_j | T^W(x_1, d)] \geq \sum_k a_{rk} \mathbb{E}[X_{\sigma(k)} | T^W(x_1, d)] - b_r - W_j, \quad \mathbb{E}[\Psi_j | T^W(x_1, d)] \geq 0,
\]
where we use the trivial equality $\mathbb{E}[W_j | T^W(x_1, d)] = W_j$. From the definition of $\Psi_j$ in (22), we obtain then $\mathbb{E}[\Psi_j | T^W(x_1, d)] \geq \Psi_j$ almost surely, with respect to the random variables $T^W(x_1, d)$. As a result
\[
\sum_j \mathbb{E}[\Psi_j | T^W(x_1, d)] \geq \sum_j \Psi_j,
\]
then
\[
\mathbb{E}[\sum_j \Psi_j] \geq \mathbb{E}[\sum_j \Psi_j].
\]
Recall from the last part of Proposition 3 that with respect to measure $\mathcal{P}(d)$,
\[
\mathbb{E}[w_x X_1 + \frac{w_{\psi}}{K} \sum_j \Psi_j] = \lambda(c).
\]
Combining this with (27), (28), and using a simple observation $\mathbb{E}[\hat{X}_1] = \mathbb{E}[\mathbb{E}[X_1 | T^W(x_1, d)]] = \mathbb{E}[X_1]$, we obtain
\[
\mathbb{E}[w_x \hat{X}_1 + \frac{w_{\psi}}{K} \sum_j \hat{\Psi}_j]
\leq \mathbb{E}[w_x X_1 + \frac{w_{\psi}}{K} \sum_j \Psi_j] + w_{\psi} B_{\psi} M(\delta)(K + 2)\delta + 3\delta M(\delta)K \max_{rk} |a_{rk}|
\]
\[
= \lambda(c) + w_{\psi} B_{\psi} M(\delta)(K + 2)\delta + 3\delta M(\delta)K \max_{rk} |a_{rk}|.
\]
Recall that $J(x_1)$ is distributed as $\text{Pois}(cK)$ and, in particular, has exponentially decaying tails. Therefore, for any $\epsilon > 0$ we can find sufficiently small $\delta$ and the corresponding $M(\delta)$ such that $\delta M(\delta) < \epsilon$ and $\mathbb{P}\{J(x_1) > M(\delta)\} < \delta$. All the other values in the right-hand side of the bound (29) are constants. Therefore, for any $\epsilon > 0$, we can find sufficiently small $\delta > 0$ such that the right-hand side is at most $\lambda(c) + \epsilon$. Choosing $\delta$ sufficiently large for this $\delta$, we obtain the result. \qed

7 Projection

In this section we complete the proof of Theorem 1 by proving the existence of the limit (14). We use the limiting distribution $\mathcal{P}(d)$ constructed in Section 4 and "project" it onto any random instance of linear program (6) with $n$ $x$-variables and $cn$ constraints $C_1, \ldots, C_{cn}$.

Fix $c > 0$ and $\epsilon > 0$ and take $n$ to be a large integer. We construct a feasible solution $x_i \in [B_{x_i}^1, B_{x_i}^2]$, $\psi_j \geq 0, 1 \leq i \leq n, 1 \leq j \leq cn$ as follows. For each variable $x_i$, consider its depth-$d$ neighborhood $\mathcal{B}(x_i, d, n)$, where $d = d(\epsilon)$ is taken as in Proposition 4. If the $\mathcal{B}(x_i, t, n)$ is a tree $T(x_i, d)$, then we set the value of $x_i$ equal to $X_i(n) \equiv \mathbb{E}[X_i | T^W(x_i, d)]$, where the expectation is with respect to the measure $\mathcal{P}(d)$. That is we observe the depth-$d$ tree $T(x_i, d)$ together with values $W_j$ corresponding to the constraints $C_j$ in this tree and set the values $x_i$ to be equal to the expectation of $X_i$ condition on this observation. If, on the other hand, $\mathcal{B}(x_i, t, n)$ is not a tree, then we assign any value to $x_i = X_i(n)$, for example $x_i = B_{x_i}^1$. Once we have assigned values to $x_i$ in the manner above, for every constraint $C_j : \sum_k a_{rk} x_{ik} \leq b_r + W_j + \psi_j$, we set its corresponding value of $\psi_j$ to $\Psi_j(n) \equiv \max\{0, \sum_k a_{rk} x_{ik} - b_r - W_j\}$ - the optimal choice for given values of $x_i$-s.
Proposition 6 For every $\epsilon > 0$, the solution constructed above has expected cost at most $(\lambda(c) + \epsilon)n$ for all sufficiently large $n$.

Since $\epsilon$ was an arbitrary constant and since $\lambda(c)$ was defined by \([10]\), the proposition shows that the assignment above satisfies $\lim_{n \to \infty} \frac{E[G\mathcal{L}\mathcal{P}(n, c)]}{n} = \lambda(c)$. Therefore the proposition implies Theorem \([4]\)

Proof: Select one of the $n$ variables $x_i$ uniformly and random. W.l.g. assume it is $x_1$. Fix $\epsilon_0 > 0$ very small. We claim that when $n$ is sufficiently large, we have

\[ \mathbb{E}[w_x X_1(n) + \frac{w_\psi}{K} \sum_j \Psi_j(n)] \leq \lambda(c) + 3\epsilon_0, \]

where the summation is over all constraints $C_j$ containing $x_1$. First let us show that this implies the statement of the proposition. Indeed, multiplying the inequality by $n$, and recalling that $x_1$ was uniformly selected from $x_i$, $1 \leq i \leq n$, we obtain that $\mathbb{E}[w_x \sum_{1 \leq i \leq n} X_i(n) + w_\psi \sum_j \Psi_j(n)] \leq (\lambda(c) + 3\epsilon_0)n$, where again we used the fact that each variable $\Psi_j(n)$ was counted exactly $K$ times. By taking $\epsilon_0 < \epsilon/3$, we obtain the required bound.

Consider the neighborhood $\mathcal{B}(x_1, d + 1, n)$ and suppose first that it is not a tree. Denote this event by $\mathcal{E}_n$. From the second part of Lemma \([8]\), $\mathbb{P}\{\mathcal{E}_n\} = O(1/n)$, where the notation $O(\cdot)$ involves constants $d, c$ and $K$. Recall, that the values of $\Psi_j(n)$ never exceed $B_\psi$. Then we have

\[
\mathbb{E}[w_x X_1(n) + \frac{w_\psi}{K} \sum_j \Psi_j(n) | \mathcal{E}_n] \\
\leq w_x \max \{|B_1^1|, |B_2^2|\} + w_\psi B_\psi \mathbb{E}[|J(x_1, n)| | \mathcal{E}_n] \\
= w_x \max \{|B_1^1|, |B_2^2|\} + w_\psi B_\psi \frac{\mathbb{E}[|J(x_1, n)|1\{\mathcal{E}_n\}]}{\mathbb{P}\{\mathcal{E}_n\}},
\]

where $1\{\mathcal{E}_n\}$ is the indicator of an event $\mathcal{E}_n$. Recall from the proof of Lemma \([4]\) that the probability $J(x_1, n) = s$ is asymptotically $\left( \frac{cn}{s} \right)(K/n)^s(1 - K/n)^{cn-s}$ for large $n$. It follows that the probability that $J(x_1, n)$ exceeds $\log^2 n$ is at most $1/n^3$ for sufficiently large $n$ (more accurate bounds can be obtained \([10]\), which are not required here). Then

\[
\mathbb{E}[J(x_1, n)1\{\mathcal{E}_n\}] \\
\leq \mathbb{E}[J(x_1, n)1\{\mathcal{E}_n\}1\{J(x_1, n) \leq \log^2 n\}] + \mathbb{E}[J(x_1, n)1\{\mathcal{E}_n\}1\{J(x_1, n) > \log^2 n\}] \\
\leq \log^2 n \mathbb{P}\{\mathcal{E}_n\} + \frac{cn}{n^3} \mathbb{P}\{\mathcal{E}_n\},
\]

where we used the fact that $J(x_1, n) \leq cn$ with probability one. But since $\mathbb{P}\{\mathcal{E}_n\} = O(1/n)$, we obtain that $\mathbb{E}[(w_x X_1(n) + \frac{w_\psi}{K} \sum_j \Psi_j(n))1\{\mathcal{E}_n\}] = o(1)$.

Suppose now that $\mathcal{B}(x_1, d + 1, n)$ is a tree $T(x_1, d)$, that is the event $\mathcal{E}_n$ occurs. Select any of the constraints containing $x_1$ (if any exist), and let $x_2, \ldots, x_K$ be the variables in this constraints. Note that then $\mathcal{B}(x_k, d, n)$ are also trees $T(x_k, d)$ and the corresponding values $X_k(n) = \mathbb{E}[X_k|T^W(x_k, d)]$, $k = 2, 3, \ldots, K$ for these variables are set based only on the observed trees $T(x_k, d)$ and values $W_j$, that is exactly as $X_k$ where defined in the previous section. Then the same correspondence holds between $\Psi_j(n)$ and $\bar{\Psi}_j$. Applying Proposition \([5]\) and the first part of Proposition \([3]\) that is the fact that $\mathcal{B}(x_1, n, d)$ converges to $T(x_1, d)$ distributed according to $\mathcal{P}(d)$, we obtain that, for sufficiently large $n,$
\[ \mathbb{E}[w_x X_1(n) + \frac{w_r}{K} \sum_j \Psi_j(n)] \leq \lambda(c) + 2\epsilon_0, \] where the second \( \epsilon_0 \) comes from approximating \( B(x_1, n, d) \) by \( T(x_1, d) \). Combining with the case of non-tree \( B(x_1, d, n) \) we obtain

\[ \mathbb{E}[w_x X_1(n) + \frac{w_r}{K} \sum_j \Psi_j(n)] \leq \lambda(c) + 2\epsilon_0 + o(1) < \lambda(c) + 3\epsilon_0. \]

for sufficiently large \( n \), just as required by (30). This concludes the proof of Theorem 1. \[\square\]

8 Applications to maximum weight \( b \)-matchings in sparse random graphs

The main goal of this section is proving Theorem 3. We begin with a linear programming formulation of the maximum weight matching problem. Suppose we have (a non-random) graph with \( n \) nodes and \( m \) undirected edges represented as pairs \((i, j)\) of nodes. Denote by \( E \) the edge set of the graph. The edges are equipped with (non-random) weights \( 0 \leq w_{i,j} \leq w_{\text{max}} \). Given \( V \subset [n] \), let \( \delta(V) \) denote the set of edges with exactly one end point in \( V \). A classical result from the theory of combinatorial optimization [Sch03, Theorem 32.2] states that the following linear programming problem provides an exact solution (namely, it is a tight relaxation) of the maximum weight matching problem:

\begin{align*}
\text{(32)} \quad & \text{Maximize} \quad \sum_{i,j} w_{i,j} x_{i,j} \\
\text{subject to:} & \sum_{j} x_{i,j} \leq b, \quad \forall i = 1, 2, \ldots, n,
\end{align*}

\begin{align*}
\text{(33)} \quad & \sum_{i} x_{i,j} \leq b, \\
\text{(34)} \quad & \sum_{i,j \in V} x_{i,j} + \sum_{(i,j) \in A} x_{i,j} \leq \frac{b|V| + |A| - 1}{2}, \quad \forall V \subset [n], A \subset \delta(V) \text{ such that } b|V| + |A| \text{ is odd},
\end{align*}

\begin{align*}
\text{(35)} & \quad 0 \leq x_{i,j} \leq 1.
\end{align*}

Specifically, there exists an optimal solution of this linear programming problem which is always integral and it corresponds to the maximum weight \( b \)-matching. We denote the optimal value of the linear program above by \( \mathcal{LPM}(G) \).

Our plan for proving Theorem 3 is as follows. We first show that when the graph has very few small cycles (and this will turn out to be the case for \( G(n, cn) \)), the optimal value \( \mathcal{LPM}^0(G) \) of the modified linear program, obtained from (32)–(35) by dropping the constraints (34), is very close to \( \mathcal{LPM}(G) \). In the context of random graphs \( G(n, cn) \), this will imply that the difference \( |\mathcal{LPM}(G(n, cn)) - \mathcal{LPM}^0(G(n, cn))| = o(n) \), w.h.p. We then take the dual of the modified linear program (32), (33), (35), and show that it has the form (36). Applying Theorem 1, we will obtain that the limit \( \lim_n \mathcal{LPM}^0(G(n, cn))/n \) exists w.h.p. This will imply the existence of the limit \( \lim_n \mathcal{LPM}(G(n, cn))/n \) w.h.p. Since \( \mathcal{LPM}(G(n, cn)) \) is the maximum weight \( b \)-matching in \( G(n, cn) \), that is \( M_w(n, c, b) \), and Theorem 3 follows.

Naturally, \( \mathcal{LPM}^0(G) \geq \mathcal{LPM}(G) \).

**Proposition 7** Given a weighted graph \( G \) and \( d \geq 3 \) let \( L(d) \) denote the set of cycles of length \( < d \) in \( G \) and let \( M(d) \) denote the total number of edges in \( L(d) \). Then

\begin{align*}
\text{(36)} \quad & \frac{d - 1}{d} (\mathcal{LPM}^0(G) - M(d)w_{\text{max}}) \leq \mathcal{LPM}(G) \leq \mathcal{LPM}^0(G).
\end{align*}
Proof: We already observed that the right-hand side of (36) holds. We now concentrate on the left-hand side bound.

Let \( x^0 = (x^0)_{i,j} \) be an optimal solution of the linear program (32), (33), (35) with its the optimal value \( \mathcal{LPM}^0(G) \). In the graph \( G \) we delete all the \( M(d) \) edges which contribute to \( L(d) \) together with the corresponding values of \( x^0_{i,j} \). Consider the resulting solution \( x^1 = (x^1)_{i,j} \) to the linear program corresponding to the reduced weighted graph \( G^1 \), which now does not contain any cycles of length less than \( d \). The objective value of the linear program (32), (33), (35) corresponding to the solution \( x^1 \) is at least \( \mathcal{LPM}^0(G) - M(d)w_{\max} \), since, by constraint (35), \( x^0_{i,j} \leq 1 \) for all the edges \((i,j)\). We further modify the solution \( x^1 \) to \( x^2 \) by letting \( x^2_{i,j} = (1 - 1/d)x^1_{i,j} \) for every edge \((i,j)\) in the graph \( G^1 \). The objective value of the linear program (32), (33), (35) corresponding to \( x^2 \) is then at least \( \frac{d-1}{d}(\mathcal{LPM}^0(G) - M(d)w_{\max}) \). We claim that in fact \( x^2 \) is a feasible solution to the linear program (32), (33), (35), implying (36) and completing the proof of the proposition.

Clearly, constraints (33) and (35) still hold. We concentrate on (33). Consider any set \( V \subset [n] \) and \( A \subset \delta(V) \) such that \( |V| + |A| \) is odd. Assume first \( |V| + |A| < d \). Let \( \hat{V} \) denote the union of \( V \) and the end points of edges in \( A \). Then \( |\hat{V}| < d \). Since \( G^1 \) does not contain any cycles of size \( d \), then \( \hat{V} \) does not contain any cycles at all, and therefore is a forest. In particular it is a bipartite graph. Let \( \hat{x}^1 \) denote the sub-vector of the vector \( x^1 \) corresponding to edges with both ends \( \hat{V} \). Since (33) holds for \( x^1 \), then it also holds for the vector \( \hat{x}^1 \) for all nodes in \( \hat{V} \). A classical result from a combinatorial optimization theory states that for every bipartite graph, the polytope corresponding to the degree constraints (33) has only integral extreme points, which are \( b \)-matchings (in the reduced graph \( \hat{V} \)). This follows from the fact that this polytope when described in matrix form \( Bx \leq b \) corresponds to the case when \( B \) is totally unimodular. We refer the reader to [Sch03] for the details. But since any integral solution \( \hat{x}^1 \) corresponding to the \( b \)-matching must satisfy the constraints (34) by Theorem 32.2 in [Sch03], then these constraints are automatically satisfied by \( x^1 \). Moreover then they are satisfied by \( x^2 \). We proved that (36) holds whenever \(|V| + |A| < d \).

Suppose now \(|V| + |A| \geq d \). For the solution \( x^1 \), let us sum the constraints (33) corresponding to all the nodes in \( V \) and sum the right-hand side constraints (35) corresponding to all the edges in \( A \). Each value \( x_{i,j} \) for \( i,j \in V \) is counted twice, once for node \( i \) and once for node \( j \). Each value \( x_{i,j} \) for \((i,j) \in A \) is also counted twice, once for constraint (33) for the node \( i \) and once for constraint (35) for the edge \((i,j) \). Then we obtain \( 2(\sum_{i,j \in V} x_{i,j} + \sum_{(i,j) \in A} x_{i,j}) \leq b|V| + |A| \). This implies \( \sum_{i,j \in V} x^2_{i,j} + \sum_{(i,j) \in A} x^2_{i,j} \leq (1 - 1/d)\frac{b|V| + |A|}{2} \leq \frac{b|V| + |A|}{2} \), since by assumption, \( b|V| + |A| \geq |V| + |A| \geq d \).

Again we showed that the constraint (33) holds for the solution \( x^2 \). \( \square \)

We return to our main setting – sparse random graph \( G(n,cn) \), with edges equipped with randomly generated weights \( W_{i,j} \), drawn according to a distribution function \( w_{\text{edge}}(t), t \geq 0 \) with support \([0,B_w]\). Let \( E = E(G(n,cn)) \) denote the edge set of this graph. We denote the value of \( \mathcal{LPM}^0(G(n,cn)) \) by \( \mathcal{LPM}^0(n,c) \) for simplicity. That is, \( \mathcal{LPM}^0(n,c) \) the optimal (random) value of the linear program (32), (33), (35) on the graph \( G(n,cn) \).

**Proposition 8** \( \text{W.h.p. as } n \to \infty \)

\[
\mathcal{LPM}^0(n,c) - o(n) \leq M_{\text{w}}(n,c,b) \leq \mathcal{LPM}^0(n,c).
\]

**Proof:** Let \( d(n) \) be a very slowly growing function of \( n \). It is well known that in \( G(n,cn) \) w.h.p. the total number of edges which belong to at least one cycle with size \( < d(n) \) is \( o(n) \) (far more accurate bounds can be obtained). Thus \( M(d(n)) = o(n) \) w.h.p. Note also that from (35), we have \( \mathcal{LPM}^0(n,c) \leq w_{\max}bn = O(n) \). Applying Proposition 7 with \( d = d(n) \), we obtain the result. \( \square \)
Our final goal is proving the convergence w.h.p. of $\mathcal{LPM}^0(n, c)/n$. We use linear programming duality for this purpose. Consider the dual of the linear program (32), (33), (35), generated on the weighted graph $G(n, cn)$. It involves variables $y_1, \ldots, y_n$ and has the following form.

(38) \[ \text{Minimize } b \sum_{1 \leq i \leq n} y_i + \sum_{(i,j) \in E} \psi_{i,j} \]

subject to:

(39) \[ y_i + y_j + \psi_{i,j} \geq W_{i,j}, \quad \forall (i,j) \in E \]

(40) \[ y_i, \psi_{i,j} \geq 0. \]

The objective value of this linear program is also $\mathcal{LPM}^0(n, c)$, thanks to the strong duality of linear programming. The linear program above is almost of the form (6) that we need in order to apply Theorem 1. Let us rewrite the linear program above in the following equivalent form

(41) \[ \text{Minimize } b \sum_{1 \leq i \leq n} y_i + \sum_{(i,j) \in E} \psi_{i,j} \]

subject to:

(42) \[ (-1)y_i + (-1)y_j \leq -W_{i,j} + \psi_{i,j}, \quad \forall (i,j) \in E \]

(43) \[ y_i, \psi_{i,j} \geq 0, \]

Let the set of constraints $C$ of the linear program (40) contain only one element $(-1)y_1 + (-1)y_2 \leq 0$. We set $B_x^1 = 0, B_x^2 = B_w$, where, as we recall, $[0, B_w]$ is the support of the distribution of $W_{i,j}$. We also set $w_x = b$ and $w_\psi = 1$. Our linear program has now the form (40) except for we need to consider in addition the constraints $y_i \leq B_w$. We claim that in fact these constraints are redundant. In fact any value of $y_i$ which exceeds $B_w \geq W_{i,j}$ can be decreased to $B_w$, resulting in a smaller value of the objective function and still honoring all the constraints. Thus, we may replace $0 \leq y_i \leq B_w$ simply by $y_i \geq 0$. We conclude that the linear program (41), (42), (43) has form (6). Applying Theorem 1 there exists a function $g(c) \geq 0$ such that w.h.p. $\mathcal{LPM}^0(n, c)/n \to g(c)$ as $n \to \infty$. Finally, applying Proposition 8 we obtain (11). This concludes the proof of Theorem 3.

9 Discussion

The results of the present paper lead to several interesting open questions. In addition to Conjecture 4 stated in Section 3 it seems that the following analogue of Conjecture 2 is reasonable.

**Conjecture 5** Consider random $K$-LSAT problem (linear program (21)) with $n$ variables and $m$ constraints, where all $\psi$ variables are set to zero. Let $N(n,m)$ denote the maximum cardinality subset of constraints $C_j$ which is feasible. For any $c > 0$, the limit $\lim_{n \to \infty} (N(n, cn)/n)$ exists. Moreover, this limit is strictly smaller than unity, for all $c > c^*_K$.

An interesting group of questions relates to the behavior of the function $f(c) = \lim_n \mathbb{E}[\mathcal{LPM}(n)]/n$, which, by results of this paper is equal to zero for $c < c^*$, in the specific context of linear program (40). What can be said about $f(c)$ near $c^*_K$, when $c^*_K < \infty$? It is not hard to show that $f(c)$ is continuous and non-decreasing function of $c$. Is it differentiable in $c^*_K$? Is it convex, concave or neither? Similar questions arise in connection with percolation probability $\theta(p)$ which is a function of a bond or site probability $p$. It is known that for every dimension $d$ there exists a critical value $p^*_d$, for which the
probability $\theta(p)$ of existence of an infinite component containing the origin in $d$-dimensional bond/site percolation model, is equal to zero for $p < p^*_d$ and is positive for $p > p^*_d$ (whether $\theta(p) = 0$ for $p = p^*_d$ is a major outstanding problem in percolation theory), see [Gr99]. The behavior of $\theta(p)$ near $p^*_d$ is a well-known open problem. We also refer the reader to [APd] for the related discussion on scaling limits and universality of random combinatorial problems.

Computing the values of $c^*_K$ is another question which seems so far beyond the techniques applied in this paper. The cases where local weak convergence methods lead to computation of limiting values seem to be related to special structures of the corresponding problems. For example a very clever recursion was used by Aldous to prove $\zeta(2) = \pi^2/6$ limit for the expected value of the minimum weight assignment, [Ald01]. See also Bandyopadhyay [Ban02], who investigates some properties of the limiting measure arising in the proof of $\zeta(2)$ result.

Similarly, a special structure was used in [AS03] to prove $\zeta(3)$ limit for minimum spanning tree on a complete graph with exponentially distributed weights. (This problem was originally solved by Frieze [Fri85], using combinatorial methods). Uncovering such special structure for our problem for the purposes of computing $c^*_K$ is an interesting question.

A separate course of investigation is to formulate and study randomly generated integer programming problems as a unifying framework for studying random k-SAT, coloring, maximum k-cuts, maximum independent sets, and other problems. For example, under which conditions on the set of prototype constraints does the feasibility problem experience a sharp transition? These conditions should be generic enough to include the problems mentioned above. Also we suspect that other results from polyhedral combinatorics can be of use when studying these problems within an integer programming framework.

Finally, it should be clear what goes wrong when one tries to use local weak convergence approach for random K-SAT problem, for example along the lines of Theorem [1]. Our approach is built on using the values like $E[X_k|\cdot]$ to construct a feasible solution, but these expectations are not necessarily integers. Digging somewhat deeper into the issue, it seems that local weak convergence method in general is not very hopeful for resolving Conjecture [1] since it looks only into constant size neighborhoods of nodes. To elaborate somewhat this point consider maximal independent set problem in $r$-regular random graphs, discussed in [AS03]. For almost any node in such a graph, its constant size neighborhood is a $r$-regular tree, and, as such, the neighborhoods are indistinguishable. In such circumstances it seems hard to try to concoct a solution which is based only on neighborhoods of nodes. Some long-range structural properties of these graphs like structure of cycles have to be considered. We refer the reader to [AS03] for a further discussion of this issue.

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