Co-existence of states in quantum systems

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Abstract. Co-existence of different states is a profound concept, which possibly underlies
the phase transition and the symmetry breaking. Because of a property inherent to quantum
mechanics (cf. uncertainty), the co-existence is expected to appear more naturally in quantum-
microscopic systems than in macroscopic systems. In this paper a mathematical theory
describing co-existence of states in quantum systems is presented, and the co-existence is
classified into 9 types.

1. Introduction
The boundary-value problem of nonlinear partial differential equation of elliptic-type:
\begin{align}
-\nabla^2 u - mu - V(u)u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}
\tag{1}
is studied, where \(m\) is a real number, and \(\Omega \in \mathbb{R}^3\) is a closed domain with a sufficiently smooth
boundary. The unknown complex function \(u\) consists of the unknown state \(\psi\) and the reference
state \(\bar{\psi}\):
\[u = \psi - \bar{\psi},\]
where \(\bar{\psi}\) (corresponding to a generalized concept of the vacuum) is not necessarily a solution of
Eq. (1), although the most simplest case \(\bar{\psi} = 0\) (the simplest vacuum) satisfies Eq. (1). Let a
part of the inhomogeneous term \(V(u)\), whose spectral set is assumed to be included in a real
axis, satisfy
\[
\partial_u(V(u)u)|_{u=0} = V_L.
\tag{2}
\]
For the simplicity \(V_L\), which corresponds to the signed strength of linearized interaction being
independent of \(u\), is assumed to be a real number. As is readily seen, the function \(u = \psi - \bar{\psi} = 0\)
is always a solution of this problem (refer to the trivial solution). In this sense let us imagine a
simple case when \(\bar{\psi} = 0\), and then the emergence of a solution \(\psi\) from another solution \(\tilde{\psi}\) is
true if \(u \neq 0\) is the solution of Eq. (1). Here we seek the non-trivial solution \(u \neq 0\ \ (\psi \neq \tilde{\psi})\) to
Eq. (1). The corresponding situation is nothing but the co-existence of different states \(\psi\) and \(\bar{\psi}\).

Equation (1) is associated with the stationary problem of nonlinear Schrödinger equations
as well as nonlinear Klein-Gordon equations. In the context of Klein-Gordon equations, it
is possible to associate \(\sqrt{-m}\) with the mass (if \(m < 0\)). Note that the statistical property
inherent to many-body system, which might bring about rather interesting physical properties,
is not taken into account in order to see the most fundamental properties associated with the
co-existence in both nonlinear Schrödinger equations and nonlinear Klein-Gordon equations.
2. Theory describing the co-existence

2.1. Mathematical settings

Let $X$ and $Y$ be functional spaces

$$X = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega),$$
$$Y = L^2(\Omega)$$

respectively (for mathematical notation, see [1]). An inclusion relation $X \subset Y$ is true. For $u \in X$, a mapping $f : R^1 \times X \to Y$ is defined by

$$f(\lambda, u) := -\nabla^2 u - mu - V(u)u.$$ 

The original master equation is written by

$$f(\lambda, u) = 0.$$ 

Since the trivial solution $u = 0$ always exists, $f(\lambda, 0) = 0$ is satisfied. According to the Sobolev embedding theorem $-\nabla^2$ is a $C^2$-mapping from $X$ to $Y$, where the detail setting of $V(u)$ is necessary to know the regularity of the mapping $f$. The space $W_0^{1,2}(\Omega)$ denotes all the functions included in $W^{1,2}(\Omega)$ satisfying $u|_{\partial\Omega} = 0$.

2.2. Linearized analysis

Linearized problem is derived. The Fréchet derivative of $f(\lambda, u)$ is calculated as

$$f_u(\lambda, 0)[u] = -\nabla^2 u - \lambda u = 0,$$ 

where $\lambda = m + V_L$. This corresponds to the master equation for the linearized eigen-value problem. It is well known that the linearized problem (with the Dirichlet boundary condition) is solvable. Furthermore it is known that a infinite set of eigen-values $\{\lambda_i\}_{i=0}^{\infty}$ of $-\nabla^2$ satisfy

- $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$;
- $\lambda_0$ is a simple eigen-value.

Let the eigen-function corresponding to the eigen-value $\lambda_0$ be $u_0$ (i.e., $-\nabla^2 u_0 = \lambda_0 u_0$). First, according to the simple property of the eigen-value $\lambda_0$, it is clear that

$$\text{Ker}(f_u(\lambda_0, 0)) = \{ tu_0 : t \in R^1 \},$$

so that the dimension of $\text{Ker}(f_u(\lambda_0, 0))$ is equal to 1. Second, if there exists a solution $v \in X$ for $\nabla^2 v - \lambda_0 v = h$ with $h \in Y$, then

$$\text{R}(f_u(\lambda_0, 0)) = \left\{ h \in Y : \int_{\Omega} h(x)u(x)dx = 0 \right\},$$

so that $\text{R}(f_u(\lambda_0, 0))$ is a closed subset of $Y$ with its co-dimension 1 (cf. the Riesz-Schauder theory [1]). Third, it is valid that

$$f_{u\lambda}(\lambda_0, 0)[u] = -\lambda_0 u \notin \text{R}(f_u(\lambda_0, 0)).$$ 

Consequently, according to the bifurcation theory [2, 3], $\lambda = \lambda_0$ has been clarified to be a bifurcation point (corresponding to $(\lambda_0, 0)$ in Fig. 1). Note that only sufficient conditions for the existence of the bifurcation point is presented in the bifurcation theory.
Figure 1. 9 types of co-existence based on Eqs. (5) and (6): cases (i), (ii), and (iii) appear if $\mu_s(0) = 0$, cases (iv), (v), and (vi) appear if $\mu_s(0) > 0$, cases (vii), (viii), and (ix) appear if $\mu_s(0) < 0$; cases (i), (iv), and (vii) appear if $\mu_{ss}(0) > 0$, cases (ii), (v), and (viii) appear if $\mu_{ss}(0) = 0$, cases (iii), (vi), and (ix) appear if $\mu_{ss} < 0$.

2.3. Nonlinear analysis
Co-existence of different states (i.e., existence of non-trivial solution $u \neq 0$) is shown. We set a closed interval $[-\epsilon_0, \epsilon_0]$ and a $C^1$-function $\lambda(s)$ satisfying $\lambda(0) = \lambda_0$, where $s$ parametrizes the functional space $X$. Under the three conditions confirmed in Sec. 2.2, let the corresponding solution $u$ be represented by

$$u(\lambda, s, x) = su_0(\lambda, x) + sz(\lambda, s, x),$$

where $s$ is defined on the interval, and $z(\lambda, s, x)$ is a sufficiently smooth function of $s$ defined on $R^1 \times R^1 \times X$. The function $z(\lambda, s, x)$ satisfies $z(\lambda, 0, x) = 0$ and

$$\int_{\Omega} z(x)u_0(x)dx = 0.$$ 

The function $u(\lambda, s, x)$ satisfies the condition $u(\lambda, 0, x) = 0$, which means the existence of the trivial solution. It is useful to define a linear operator

$$A := -\nabla^2 - \lambda_0,$$

with its domain $X$, and then it is readily seen that $A$ is a self-adjoint operator in $Y$. The original equation is written by $Au = \mu(s)u + (V(u) - V_L)u$ with $\mu(s) = \lambda(s) - \lambda_0$, and the linearized problem is written by $Au_0 = 0$. By differentiating the original equation with respect to $s$, step
by step

\[(Au)_s = \mu_s u + \mu u_s + \partial_s(V(u)u) - V_L u_s\]
\[= \mu_s u + \mu u_s + (\partial_s V(u))u + V(u)u_s - V_L u_s\]

\[(Au)_{ss} = \mu_{ss} u + 2\mu_s u_s + \mu u_{ss} + \partial_s^2 V(u) u - V_L u_{ss}\]
\[= \mu_{ss} u + 2\mu_s u_s + \mu u_{ss} + (\partial_s^2 V(u))u + 2(\partial_s V(u))u_s + V(u)u_{ss} - V_L u_{ss}\]

\[(Au)_{sss} = \mu_{sss} u + 3\mu_{ss} u_s + 3\mu u_{sss} + \mu u_{sss} + \partial_s^3 V(u) u - V_L u_{sss}\]
\[= \mu_{sss} u + 3\mu_{ss} u_s + 3\mu u_{sss} + (\partial_s^3 V(u))u + 3(\partial_s^2 V(u))u_s + 3(\partial_s V(u))u_{ss} + V(u)u_{ss} - V_L u_{ss}\]

where the functions are represented by \(u_s = u_0 + sz_s + z, u_{ss} = s^2z_{ss} + 2sz_s, \) and \(u_{sss} = s^3z_{sss} + 3sz_{sss} \) respectively. The derivatives of the inhomogeneous terms become

\[\partial_s V(u) = (\partial_s V(u)) u_s\]
\[\partial_s^2 V(u) = (\partial_s^2 V(u)) u_s^2 + (\partial_s V(u)) u_{ss}\]
\[\partial_s^3 V(u) = (\partial_s^3 V(u)) u_s^3 + 3(\partial_s^2 V(u)) u_s u_{ss} + (\partial_s V(u)) u_{sss}.

By taking \(s = 0\), the bi-linear forms become

\[(Au)_s|_{s=0} = \mu_s(0) u|_{s=0} + \mu(0) u_s|_{s=0} + \partial_u(V(u)u) u_s|_{s=0} - V_L u_s|_{s=0}
\[= V_L u_s|_{s=0} - V_L u|_{s=0} = 0,
\[(Au)|_{s=0} = 0,

\[(Au)_{ss}|_{s=0} = \mu_{ss}(0) u|_{s=0} + 2\mu_s(0) u_0 + z|_{s=0} + 2\mu(0) z_s|_{s=0} + \partial_s^2(V(u)u)|_{s=0} - 2V_L z_s|_{s=0}
\[= 2\mu_s(0) u_0 + \partial_s^2(V(u)u)|_{s=0} - 2V_L z_s|_{s=0},
\[(Au)|_{s=0} = 2\mu_s(0) u_0 + \partial_s^2(V(u)u)|_{s=0} - 2V_L z_s|_{s=0}.

\[(Au)_{sss}|_{s=0} = 3\mu_{ss}(0) u|_{s=0} + 3\mu_s(0) u_0 + z|_{s=0} + 6\mu_s(0) z_s|_{s=0} + 3\mu(0) z_{ss}|_{s=0}
\[+ \partial_s^3(V(u)u)|_{s=0} - 3V_L z_{ss}|_{s=0}
\[= 3\mu_{ss}(0) u_0 + 6\mu_s(0) z_s|_{s=0} + \partial_s^3(V(u)u)|_{s=0} - 3V_L z_{ss}|_{s=0},
\[(Au)|_{s=0} = 3\mu_{ss}(0) u_0 + 6\mu_s(0) z_s|_{s=0} + (\partial_s^3(V(u)u)|_{s=0} - (2V_L z_{ss}|_{s=0} u_0),

where \(u|_{s=0} = 0, z|_{s=0} = 0\), and \(\mu(0) = 0 \) are utilized, as well as Eq. (2). \((Au)|_{s=0} = (u_{ss}|_{s=0} = 0) \) due to \(Au_0 = 0\). Consequently

\[2\mu_s(0) = - (\partial_s^2(V(u)u)|_{s=0} u_0) + 2(V_L z_{ss}|_{s=0} u_0),
\[(5)

and the sign of \(\lambda_s(0) = \mu_s(0)\) is determined by \(- (\partial_s^2(V(u)u)|_{s=0} u_0) + 2(V_L z_{ss}|_{s=0} u_0)\). In the same manner \((Au)_{ss}|_{s=0} = (u_{ss}|_{s=0} = 0, Au_0) = 0\). It leads to

\[3\mu_{ss}(0) = - (\partial_s^3(V(u)u)|_{s=0} u_0) - (6\mu_s(0) z_s|_{s=0} u_0) + 3(V_L z_{ss}|_{s=0} u_0),
\[(6)

and the sign of \(\lambda_{ss}(0) = \mu_{ss}(0)\) is determined by \(- (\partial_s^3(V(u)u)|_{s=0} u_0) - (6\mu_s(0) z_s|_{s=0} u_0) + 3(V_L z_{ss}|_{s=0} u_0). \) In particular, if \(\lambda_s(0) = \mu_s(0) = 0\) is true, the sign of \(\lambda_{ss}(0)\) is determined by \(- (\partial_s^3(V(u)u)|_{s=0} u_0) + 3(V_L z_{ss}|_{s=0} u_0). \) According to Eqs. (5) and (6), the co-existence of states is classified into 9 types (Fig. 1). In Figure 1, around the neighbour of the bifurcation point \((\lambda_0, 0)\), two solutions co-exist in types (iv) to (ix), while the transition from single-existence to co-existence is described in types (i) and (iii).
Table 1. Systematic analysis for $\psi^k$-interaction theory. Possible classification of co-existence is shown in the column “Type”, where $\sigma = 4\eta(u_0 z_s|_{s=0}, u_0) - 2\eta(u_0^2, u_0)(z_s|_{s=0}, u_0)$.

| $k$ | $\partial_s V(u)$ | $\partial^2_s V(u)$ | $\partial^3_s V(u)|_{u=0}$ | $\partial^3_s V(u)|_{s=0}$ | $\mu_s(0)$ | $\mu_{ss}(0)$ | $\sigma$ | Type |
|-----|-----------------|-----------------|-----------------|-----------------|---------|---------|-----------------|-----|
| 3   | $-\eta u_s$    | $-\eta u_{ss}$ | $-2\eta u_0^2$ | $-12\eta u_0 z_s|_{s=0}$ | $\eta(u_0^2, u_0)$ | $\eta(u_0^2, u_0)$ | $\eta(u_0^2, u_0)$ | $\sigma$ | (i) |
| 4   | 0               | $-2\eta u_0^2$ | 0               | $-6\eta u_0^2$ | 0       | 0       | $2\eta(u_0^2, u_0)$ | (ii) |
| $\geq 5$ | 0               | 0               | 0               | 0               | 0       | 0       | 0               | (iii) |

3. Application to $\psi^k$-interaction theory

If the Lagrangian includes the $k$th-order nonlinearity in its interaction part (for example, see textbooks of particle physics), the inhomogeneous term of the master equation becomes

$$V(u)u = -\eta u^{k-1},$$

for integers $k \geq 1$, where $\eta$ is assumed to be a real number. Here $V_L = 0$ and $V(u)|_{s=0} = V(u)|_{u=0} = 0$ are true. The first derivative is

$$\partial_s V(u)|_{u=0} = -(k-2)\eta u^{k-3}|_{u=0}$$

for $k \geq 3$, so that it is equal to $-\eta$ for $k = 3$, and zero for $k \geq 4$. The second derivative is

$$\partial^2_s V(u)|_{u=0} = -(k-2)(k-3)\eta u^{k-4}|_{u=0}$$

for $k \geq 4$, so that it is equal to zero for $k = 3$, $-2\eta$ for $k = 4$, and zero for $k \geq 5$. The third derivative is

$$\partial^3_s V(u)|_{u=0} = -(k-2)(k-3)(k-4)\eta u^{k-5}|_{u=0}$$

for $k \geq 5$, so that it is equal to zero for $k \leq 4$, $-6\eta$ for $k = 5$, and zero for $k \geq 6$. Results are summarized in Table 1. In case of $k = 4$ ($\psi^4$-interaction theory), the non-trivial solution corresponds to type (i) of Fig. 1 if $\eta > 0$, to type (ii) if $\eta = 0$, and to type (iii) if $\eta < 0$. In particular when $\eta > 0$, the co-existence emerges only if $m > \lambda_0$ (cf. spontaneous symmetry breaking).

If there is no interaction (free particle condition: $\eta = 0$), $\mu_s(0) = \mu_{ss}(0) = 0$ is true, and the co-existence is classified into type (ii). If the interaction is linear ($V(u) = V_L \neq 0$; $\psi^2$-interaction theory), the derivatives are

$$\partial_s V(u)|_{u=0} = \partial^2_s V(u)|_{u=0} = \partial^2_s V(u)|_{u=0} = 0,$$

so that $\partial_s V(u) = \partial^2_s V(u) = \partial^3_s V(u) = 0$. It leads to $\partial^3_s V(u)|_{s=0} = V_L u_{ss}|_{s=0}$ and $\partial^3_s V(u)|_{s=0} = V_L u_{ss}|_{s=0} = 0$ so that $\mu_s(0) = -\mu_{ss}(0) = 0$ and $\mu_{ss}(0) = 0$. The co-existence is classified into type (ii). As a result the nonlinearity can be identified by the classification other than type (ii).

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