Chaotic Scattering on Individual Quantum Graphs

Z. Pluhařa and H. A. Weidenmüller

aFaculty of Mathematics and Physics, Charles University, 180 00 Praha 8, Czech Republic
bMax-Planck-Institut für Kernphysik, 69029 Heidelberg, Germany

For chaotic scattering on quantum graphs, the semiclassical approximation is exact. We use this fact and employ supersymmetry, the colour-flavour transformation, and the saddle-point approximation to calculate the exact expression for the lowest and asymptotic expressions in the Ericson regime for all higher correlation functions of the scattering matrix. Our results agree with those available from the random-matrix approach to chaotic scattering. We conjecture that our results hold universally for quantum-chaotic scattering.

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I. INTRODUCTION

We investigate the scattering matrix (S matrix) for scattering on quantum chaotic graphs. We derive an exact expression for the lowest S-matrix correlation function and asymptotic expressions for all higher S-matrix correlation functions in the Ericson regime. (All these functions are defined as wave-number averages of products of elements of S.) Our motivation for this work is the following.

The celebrated conjecture by Bohigas, Giannoni and Schmit (BGS, see Refs. [1]) postulates that the spectral fluctuation properties of closed dynamical quantum systems that are chaotic in the classical limit, coincide with those of one of Dyson’s three canonical random-matrix ensembles [2]. For the level-level correlator of general chaotic systems with unitary or orthogonal symmetry, the BGS conjecture has been demonstrated in Refs. [3–6]. For chaotic quantum graphs (which in some sense may be considered non-Hamiltonian systems) the analogous statement was proved in Refs. [7,8].

For open systems, the lowest S-matrix correlation function plays the same role as does the level-level correlator in closed systems. Generalizing the BGS conjecture one may, therefore, ask whether (depending on symmetry) that function assumes the same values for the random-matrix approach to scattering, for scattering on general dynamical chaotic systems, and for scattering on chaotic quantum graphs. For the random-matrix approach, the lowest S-matrix correlation function has been calculated both for the orthogonal [10] and for the unitary [11] cases. Some higher-order correlators have also been worked out, either exactly [12,13] or as leading-order terms in an asymptotic expansion [14,15]. However, we are not aware of any analytical results for the S-matrix correlation function either for general dynamical chaotic systems, or for chaotic quantum graphs. The present paper fills that gap in regard to scattering on graphs. We show that for the case of orthogonal symmetry, the lowest S-matrix correlation function for scattering on chaotic quantum graphs coincides with the one obtained from random-matrix theory (RMT). We also calculate asymptotic expressions for all higher S-matrix correlators in the Ericson regime and show that these coincide with RMT results if available. In the original papers on the subject [16–18] it was surmised that in the Ericson regime, the elements of S are random variables with a Gaussian probability distribution. We show that the surmise is correct only if the averages of all S-matrix elements vanish.

Chaotic scattering on quantum graphs was introduced in Refs. [19,20] and extensively investigated in these papers and in Refs. [21,22] where many of its properties were displayed with the help of numerical simulations. We benefit from the developments in these papers and from the recent discovery of “topological resonances” in the chaotic scattering on graphs [24]. We take advantage of the fact that chaotic quantum graphs are easier to handle than general dynamical chaotic systems because the propagator amplitudes are plane waves, and the semiclassical approximation is exact in that case. A brief account of this work was published in Ref. [25].

II. SCATTERING ON QUANTUM GRAPHS

A graph is a system of V vertices labeled α, β, . . . that are linked by B bonds. We consider graphs where every vertex α is linked by a single bond (αβ) to every other vertex β (“completely connected graph”). Then the number of bonds is B = V(V − 1)/2. Our results remain valid, however, if some bonds are missing, see Section IX below. The bonds (αβ) are often simply labeled with a running index b that ranges from 1 to B. The length of bond b is denoted by L_b. We assume that the lengths of all bonds are incommensurate (for every pair (b,b′) of different bonds (b ̸= b′) the ratio L_b/L_V is irrational). That assumption is necessary for the graph to be chaotic. We eventually consider the limit V → ∞. We assume that in that limit, the lengths of all bonds remain similar (i.e., obey L_{min} ≤ L_b ≤ L_{max} for all b).

Scattering occurs when a number Λ ≥ 1 of vertices is
linked by a single bond each (a “lead”) to infinity. The lead is labeled by the single vertex to which it is attached. The number \( \Lambda \) of leads defines the number of scattering channels and, thus, the dimension of the scattering matrix (\( S \) matrix). In the limit \( V \to \infty \) we keep the number \( \Lambda \) of channels fixed so that eventually \( \Lambda \ll V \). This is analogous to the RMT approach where the dimension \( N \) of the Hamiltonian matrix is taken to infinity while the number of channels is kept fixed. Without loss of generality we denote the \( \Lambda \) vertices coupled to a lead by \( \alpha = 1, \ldots, \Lambda \) while the indices \( \alpha > \Lambda \) denote vertices not coupled to a lead. We confine ourselves to scattering that is time-reversal invariant.

On each bond or lead of a quantum graph, the wave function is a linear combination of amplitudes \( \exp\{ikx\} \) and \( \exp\{-ikx\} \), with the same wave number \( k \) on all bonds or leads. The variable \( x \) measures the distance to one of the vertices attached to the bond (to the single vertex attached to the bond, respectively). The coefficients of the linear combination are determined by boundary conditions specified for each vertex \( \alpha \) in terms of a matrix \( \Gamma^{(\alpha)} \), \( \alpha = 1, \ldots, V \). The matrix \( \Gamma^{(\alpha)} \) relates outgoing amplitudes \( \mathcal{O} \) and incoming amplitudes \( \mathcal{I} \) on all bonds or on the lead connected to vertex \( \alpha \) by

\[
\mathcal{O} = \Gamma^{(\alpha)} \mathcal{I}.
\]

To ensure time-reversal invariance and flux conservation, \( \Gamma^{(\alpha)} \) is taken to be symmetric and unitary. With \( \beta, \gamma \neq \alpha \) the matrix \( \Gamma^{(\alpha)} \) has the form

\[
\begin{align*}
\Gamma^{(\alpha)} &= \begin{pmatrix}
\rho^{(\alpha)} & \tau^{(\alpha)}_\beta \\
\tau^{(\alpha)}_\beta & \sigma^{(\alpha)}_{\beta \gamma}
\end{pmatrix} \quad \text{for } \alpha \leq \Lambda, \\
\Gamma^{(\alpha)} &= \begin{pmatrix}
\sigma^{(\alpha)}_{\beta \gamma}
\end{pmatrix} \quad \text{for } \alpha > \Lambda.
\end{align*}
\]

Here \( \rho^{(\alpha)} \) describes backscattering on lead \( \alpha \), \( \tau^{(\alpha)}_\beta \) describes scattering from bond \( (\alpha \beta) \) to lead \( \alpha \) or vice versa, and the \((V-1)\)-dimensional matrix \( \sigma^{(\alpha)}_{\beta \gamma} \) describes scattering from bond \( (\alpha \beta) \) to bond \( (\alpha \gamma) \) or vice versa. For \( \alpha \leq \Lambda \) the symmetric matrix \( \sigma^{(\alpha)} \) is subunitary while for \( \alpha > \Lambda \), it is unitary.

### III. SCATTERING MATRIX

Given an incident wave in a single channel \( \alpha \leq \Lambda \) only, the boundary conditions \(^{[1]}\) completely define the total wave function. The amplitude of the outgoing wave in channel \( \beta \leq \Lambda \) is the element \( S_{\alpha \beta}(k) \) of the symmetric and unitary scattering matrix.

Since a full derivation of the \( S \) matrix is given in Ref. \(^{[20]}\), we confine ourselves to a somewhat heuristic argument. The element \( S_{\alpha \beta}(k) \) of the scattering matrix describes propagation of the wave amplitude from lead \( \alpha \) to lead \( \beta \) via multiple scattering within the graph. Such multiple scattering is governed by three elements: (i) scattering from one bond to another by one of the matrices \( \sigma^{(\alpha)} \) in Eqs. \(^{[2]}\), (ii) propagation along one of the bonds \( b \) described by the propagator \( \exp\{ikL_b\} \), (iii) scattering from a lead to one of the bonds or vice versa by an element \( \tau^{(\alpha)}_{\beta} \) in Eqs. \(^{[2]}\). With \( \rho^{(\alpha)} \) the amplitude for elastic backscattering on lead \( \alpha \), we write \( S_{\alpha \beta}(k) \) as the sum over all ways of propagating from lead \( \alpha \) to lead \( \beta \),

\[
S_{\alpha \beta}(k) = \rho^{(\alpha)} \delta_{\alpha \beta} + \tau^{(\alpha)}_{\beta} \exp\{ikL_{\alpha \beta}\} + \sum_\gamma \tau^{(\beta)}_{\gamma} \exp\{ikL_{\alpha \gamma}\} \sigma^{(\gamma)}_{\alpha \beta} \exp\{ikL_{\gamma \beta}\} \tau^{(\beta)}_{\gamma} + \ldots.
\]

The dots indicate terms of higher order in the matrices \( \sigma^{(\gamma)} \). The term containing the nth power of the \( \sigma^{(\gamma)} \)'s is the sum of all semiclassical trajectories that connect the vertices \( \alpha \) and \( \beta \) via passage through \((n+1)\) bonds. Each of the traversed bonds \( b \) yields the factor \( \exp\{ikL_b\} \). Since \( L_{\alpha \beta} = L_{\beta \alpha} \) and since \( \sigma^{(\gamma)} \) is symmetric, we conclude from Eq. \(^{[3]}\) that \( S_{\alpha \beta}(k) \) is symmetric.

In order to write each term in Eq. \(^{[3]}\) as a matrix product, we arrange the quantities mentioned under points (i) to (iii) above in matrix form. (i) Scattering from one bond to another anywhere on the graph is described by the matrix \( \Sigma \). The non-zero elements of \( \Sigma \) are the elements of the matrices \( \sigma^{(\alpha)} \). We use two representations for \( \Sigma \). In “vertex representation”, \( \Sigma^{(V)} \) is block diagonal, each diagonal block carrying one of the matrices \( \sigma^{(\alpha)} \), \( \alpha = 1, \ldots, V \). By construction, \( \Sigma^{(V)} \) has dimension \( V(V-1) = 2B \) and is symmetric. Because of the factor two, it is necessary in “bond representation” to double the bond label and to use directed bonds (\( bd \)). This is done by arranging the \( B \) bonds (\( \alpha \beta \)) in lexicographical order (so that always \( \alpha < \beta \)), and by mapping the resulting sequence onto the sequence of integers \( b = 1, \ldots, B \). To double the bond label we refer to the bonds just constructed by a double label (\( b' \)). To every such “directed bond” (\( \alpha \beta \)) with \( \alpha < \beta \) we consider the bond (\( \beta \alpha \)) and label it as (\( b' \)). The number of directed bonds (\( bd \)) with \( d = \pm \) constructed in that way is \( 2B \). In the representation of directed bonds the matrix \( \sigma^{(\alpha)} \) with elements \( \sigma^{(\alpha)}_{\beta \gamma} \) is mapped onto the matrix \( \sigma_{\alpha \beta, \alpha \gamma} = \sigma_{bd, b'd'} \), with the bond labels \( b (b') \) determined by (\( \alpha, \beta \)) (by (\( \alpha, \gamma \)), respectively), with \( d = + (d = -) \) for \( \alpha < \beta \) (for \( \alpha > \beta \), respectively), and correspondingly for \( d' \). In bond representation, the matrix \( \Sigma \) is written as \( \Sigma^{(B)} \) and its elements are written as \( \Sigma_{bd,b'd'} \). The map \( \Sigma^{(V)} \to \Sigma^{(B)} \) involves an identical rearrangement of rows and columns. Therefore, \( \Sigma^{(B)} \) is also symmetric. As for point (iii), we arrange the elements \( \tau^{(\alpha)}_{\beta} \) in Eq. \(^{[2]}\) that describe scattering from lead \( \alpha \) to bond (\( \alpha \beta \)) in the form of a rectangular matrix \( \mathcal{T} \) with elements \( \mathcal{T}_{\alpha, bd} \). The index \( \alpha \) runs over all \( \Lambda \) leads, so that \( \mathcal{T} \) has \( \Lambda \) rows and \( 2B \) columns. The element \( \mathcal{T}_{\alpha, bd} \) is non-zero for every directed bond (\( bd \)) coupled to lead \( \alpha \) by the element \( \tau^{(\alpha)}_{\beta} \), with \( b \) determined by (\( \alpha, \beta \)) and \( d = + (d = -) \) for \( \alpha < \beta \) (for \( \alpha > \beta \), respectively). The element \( \tau^{(\alpha)}_{\beta} \) at the right end of
each term in Eq. (3) is correspondingly written as $T_{\gamma,\beta}^T$ with $T$ denoting the transpose. (iii) Amplitude propagation on the directed bonds ($bd$) is described by the matrix $\sigma^T_{\alpha} \exp\{i k L\}$. In bond representation the matrix $\exp\{i k L\}$ is diagonal with elements $\delta_{bd} \delta_{bd'} \exp\{i k L_{bd}\}$. For fixed $b$ the diagonal elements are the same for $d = +$ and for $d = -$. The matrix $\sigma^d_{\alpha}$ is the direct product of the first Pauli spin matrix in the two-dimensional space of directions $d$ and the unit matrix in bond space. The matrix $\sigma^d_{\alpha}$ reverses the direction of all bonds. Introduction of the factor $\sigma^d_{\alpha}$ is necessary in order for the terms in Eq. (4) to attain the form of a matrix product. With these definitions, Eq. (3) takes the form

$$S_{\alpha\beta}(k) = \rho^{(a)} \delta_{\alpha\beta} + \sum_{bd'} T_{\alpha,\beta} \exp\{i k L_{bd}\} (\sigma^T_{\alpha})_{bd,\beta} \sigma_{\alpha} \exp\{i k L_{bd'}\} \times \delta_{bd',\beta} + \ldots . \tag{4}$$

Carrying out the summation in Eq. (4) we obtain

$$S_{\alpha\beta}(k) = \rho^{(a)} \delta_{\alpha\beta} + \sum_{bd',\beta} T_{\alpha,\beta} (W^{-1})_{bd',\beta} \sigma_{\alpha} \exp\{i k L_{bd'}\} \times \delta_{bd',\beta} + \ldots . \tag{5}$$

or in matrix notation

$$S_{\alpha\beta}(k) = \delta_{\alpha\beta} \rho^{(a)} + (T W^{-1} T^T)_{\alpha\beta} \tag{6}$$

where

$$W = \exp\{\mathcal{P} L\} \sigma^T_{\alpha} - \Sigma^{(B)} . \tag{7}$$

Since $\exp\{-i k L\} \sigma^T_{\alpha}$ is symmetric, so is the matrix $W$. Without loss of generality we may assume that $\rho^{(a)}$ is real for all $a = 1, \ldots, \Lambda$. Indeed, for non-real $\rho^{(a)}$ we write $\rho^{(a)} = \exp\{2i \delta_{\alpha}\} \rho^{(a)}$ with both $\delta_{\alpha}$ and $\rho^{(a)}$ real. The transformation $S_{\alpha\beta} \rightarrow \exp\{-i \delta_{\alpha}\} S_{\alpha\beta} \exp\{-i \delta_{\beta}\}$ then removes all elastic scattering phase shifts $\delta_{\alpha}$.

The $S$ matrix must be unitary and, for a time-reversal invariant system, symmetric. Both properties follow from Eqs. (6) and (7) and from the fact that $W$ is symmetric. We mention in passing that our definitions differ from the ones used in Refs. [25, 26] where the factor $\sigma^T_{\alpha}$ is part of the matrix $\Sigma^{(B)}$. In Ref. [25] the factor $\sigma^T_{\alpha}$ in Eq. (7) was erroneously omitted. That factor eventually drops out of the calculation, however. Therefore, all results in Ref. [25] remain unchanged.

\section{Averages Over the Wave Number $k$ and Ergodicity}

While in Hamiltonian systems averages of products of $S$-matrix elements are taken over energy, in graph theory such averages are taken over the wave number $k$.

These are indicated by angular brackets. In Hamiltonian systems, averages over energy of a product containing $P \geq 1$ elements of $S$ and $Q \geq 1$ elements of $S^*$ cannot, in general, be carried out in closed form because the poles of $S$ and those of $S^*$ lie on opposite sides of the real energy axis, precluding the evaluation of the averages by contour integration. For quantum graphs the exact evaluation of the average over $k$ is possible provided the averaging interval is large compared to the minimum difference between any two $L_b$'s. Because of the incommensurability of the lengths $L_b$, the average over $k$ is then equivalent [1, 2] to a phase average so that for any function $F[\exp\{i k L_{b1}\}, \exp\{i k L_{b2}\}, \ldots]$ we have

$$\langle F[\exp\{i k L_{b1}\}, \exp\{i k L_{b2}\}, \ldots]\rangle_k = \frac{1}{(2\pi)^B} \prod_{i=1}^B \int_0^{2\pi} d\phi_i F[\exp\{i\phi_{b1}\}, \exp\{i\phi_{b2}\}, \ldots]. \tag{8}$$

The last line defines the phase average. For graphs the $B$ independent integrations over the angles $\phi_i$, $b = 1, \ldots, B$, can be done using supersymmetry. The remarkable identity [3, 4] follows [5] from an ergodicity argument. If the $L_b$ are incommensurate, the flow in $k$ (viewed as a flow in time) on the $B$-dimensional torus ($\exp\{i k L_{b1}\}, \exp\{i k L_{b2}\}, \ldots$) covers the torus densely. As the length of the averaging interval tends to infinity, the average over “time” (i.e., over $k$) can be replaced by an average over phase space (i.e., over the phases $\phi_b$).

We compare Eq. (8) with the corresponding result in RMT. In RMT, the average of an observable $\mathcal{O}(E)$ that depends on a random-matrix Hamiltonian with fixed symmetry is calculated as an ensemble average $\langle \mathcal{O}(E) \rangle_{\text{RM}}$. The quantity of physical interest is the average $\langle \mathcal{O}(E) \rangle_E$ of $\mathcal{O}(E)$ over energy $E$ for a given realization of the ensemble. The equality of both averages does not hold automatically for all observables or for all realizations of the ensemble and is controlled by the ergodicity criterion [3]

$$\langle \left( \langle \mathcal{O}(E) \rangle_{\text{RM}} - \langle \mathcal{O}(E) \rangle_E \right)^2 \rangle_{\text{RM}} = 0 . \tag{9}$$

All terms in Eq. (9) are ensemble averages or products thereof. Hence, Eq. (9) can, in principle, be tested in the framework of RMT for every observable. If fulfilled, Eq. (9) guarantees the equality of both averages for almost all members of the ensemble, i.e., with the exception of a set of measure zero. The excluded set contains integrable and other Hamiltonians that do not generate RMT fluctuations. These are so sparse that they do not contribute to the ensemble average in Eq. (9). In the case of graphs, Eq. (8) unconditionally guarantees the equality of wave-number average and phase average for all graphs with incommensurate bond lengths. We display the close analogy between Eq. (8) and Eq. (9) by grouping graphs into classes. A class $C$ is defined by the set of all graphs with the same number $V$ of vertices and
with the same $V$ matrices $\Gamma^{(\alpha)}$ in Eqs. (2) that define
the boundary conditions at each vertex. Graphs in $\mathcal{C}$
differ only in the lengths $L_b$ of the bonds. A given class
$\mathcal{C}$ contains both, graphs with incommensurate and with
commensurate bond lengths. With these definitions, the
phase average on the right-hand side of Eq. (8) can be
read as an ensemble average over all graphs in $\mathcal{C}$. Inde-
deed, with all $L_b$ obeying $L_{\text{min}} \leq L_b \leq L_{\text{max}}$, we
can decrease $L_{\text{min}}$ and/or increase $L_{\text{max}}$ such that for fixed
$k$, the quantity $k(L_{\text{max}} - L_{\text{min}}) = k\Delta L$ is a multiple of
$2\pi$. Then for any function $F(\exp\{i k L_b\})$ we have
\[
\frac{1}{\Delta L} \int_{L_{\text{min}}}^{L_{\text{max}}} \ dL_b \ F(\exp\{i k L_b\}) = \frac{1}{2\pi} \int_0^{2\pi} \ d\phi_b \ F(\exp\{i \phi_b\}) .
\]
(10)
This statement is restricted, of course, to the $L_b$-
dependence due to the propagator amplitudes $\exp\{i k L_b\}$. It
does not apply to any additional dependence on bond
lengths $L_b$ that arises, for instance, from the dependence of the
matrices $W$ in Eq. (22) on $\kappa_p$ and $\kappa_q$. Eqs. (8)
and (10) show that for every graph in $\mathcal{C}$ with incommen-
surate bond lengths, the $k$ average $\langle \ldots \rangle_k$ agrees with the
ensemble average $\langle \ldots \rangle_{(L_b)}$. That statement can be cast
into a form similar to Eq. (11).
\[
\left\langle \left( \langle F[\exp\{i k L_{b_1}\}, \exp\{i k L_{b_2}\}, \ldots \rangle_{k} \right)^2 \right\rangle_{\{L_b\}} \equiv 0 .
\]
(11)
Eq. (11) holds because among the real numbers, the ratio-
nal numbers form a subset of measure zero, and the same
is true of graphs in $\mathcal{C}$ with commensurate bond lengths
in relation to the totality of all graphs in $\mathcal{C}$. Eqs. (8)
and (11) display the close similarity of the ergodicity argu-
ment for RMT and for graphs. The difference is that in
the case of graphs, we know analytically which graphs
belong to the excluded subset of measure zero, we know
that Eq (8) holds strictly for graphs with incommensurate
bond lengths, and that a test of Eq. (11) is, therefore,
redundant.

V. AVERAGE S MATRIX

To calculate $\langle S_{\alpha\beta} \rangle$ we use Eq. (8) and phase average
every term of the series in Eq. (4). All terms containing
factors $\exp\{i k L_b\}$ vanish and we obtain
\[
\langle S_{\alpha\beta}(k) \rangle = \delta_{\alpha\beta} \rho^{(\alpha)} .
\]
(12)
As usual, we decompose $S$ into an average part and a
fluctuating part,
\[
S(k) = \langle S \rangle + S^\#(k)
\]
(13)
and have from Eqs. (12) and (6)
\[
S^\#(k) = T W^{-1} T^T .
\]
(14)
It also follows from Eq. (11) that the average of the product
of any number of $S$-matrix elements (not containing any element
$S^*$) is equal to the product of the averages. Thus for arbitrary positive integer $K$ and for any choice of the set $\{\alpha_i, \beta_i, k_i\}$ with $i = 1, \ldots, K$ we have
\[
\langle \prod_{i=1}^{K} S_{\alpha_i\beta_i}(k_i) \rangle = \prod_{i=1}^{K} \langle S_{\alpha_i\beta_i} \rangle .
\]
(15)
That same relation holds in RMT.

It is of interest to compare Eqs. (12) to (14) with the
corresponding results in a Hamiltonian theory of resonance
scattering. After removal of all elastic scattering phase shifts and with $E$ denoting the energy one writes there the matrix $S$ in the form \[26,27\]
\[
S_{\alpha\beta}(E) = \delta_{\alpha\beta} - 2i\pi \sum_{\mu \nu} W_{\alpha\mu}(D^{-1}(E))_{\mu\nu} W_{\nu\beta}
\]
(16)
where
\[
D_{\mu\nu}(E) = \delta_{\mu\nu} - H_{\mu\nu} + i\pi \sum_{\gamma=1}^{\Lambda} W_{\mu\gamma} W_{\gamma\nu} .
\]
(17)
By virtue of the coupling matrix elements $W_{\mu\nu} = W_{\alpha\mu}$, the
eigenvalues of the $N$-dimensional Hamiltonian matrix $H$
give rise to $N$ scattering resonances. Without reso-
nances, i.e., for $W_{\alpha\mu} = 0$ for all $\alpha$, the $S$ matrix equals
the unit matrix. The energy average of the $S$ matrix (10)
is \[27\]
\[
\langle S_{\alpha\beta} \rangle_E = \delta_{\alpha\beta} \frac{1 - x_\alpha}{1 + x_\alpha}
\]
(18)
with
\[
x_\alpha = \frac{\pi^2}{d} \frac{1}{N} \sum_{\mu=1}^{N} W_{\alpha\mu}^2 .
\]
(19)
Here $d$ is the mean spacing of the resonances (of the eigen-
values of $H$).

The standard interpretation (see Ref. \[27\]) of $\langle S \rangle_E$
identifies the average over energy with the fast part of
the reaction. The unitarity deficit of $\langle S \rangle_E$, expressed in terms of the “transmissions coefficients”
\[
T_\alpha = 1 - |\langle S_{\alpha\alpha} \rangle_E|^2 ,
\]
(20)
measures the flux that populates the long-lived reso-
nances scattering. After removal of all elastic scattering
phase shifts and with $E$ denoting the energy one writes there the matrix $S$ in the form \[26,27\]
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S_{\alpha\beta}(E) = \delta_{\alpha\beta} - 2i\pi \sum_{\mu \nu} W_{\alpha\mu}(D^{-1}(E))_{\mu\nu} W_{\nu\beta}
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the reaction. The unitarity deficit of $\langle S \rangle_E$, expressed in terms of the “transmissions coefficients”
\[
T_\alpha = 1 - |\langle S_{\alpha\alpha} \rangle_E|^2 ,
\]
(20)
measures the flux that populates the long-lived reso-
nances of the system. That interpretation also applies
to graphs. Indeed, Eqs. (12) and (20) and the unitarity
of $\Gamma^{(\alpha)}$ give
\[
T_\alpha = \sum_\beta |t_\beta^{(\alpha)}|^2
\]
(21)
so that \( T_\alpha \) is indeed the total coupling strength connecting lead \( \alpha \) with the graph. But here the analogy between scattering on graphs and scattering by Hamiltonian systems ends. While for fixed average coupling strength \((1/N)\sum_{\mu} W_{\alpha\mu}^2\) the coefficient \( x_\alpha \) in Eq. (12) changes with the density \((1/d)\) of the resonances (caused, for instance, by an increase of the dimension \( N \) of the Hamiltonian matrix), the analogous coefficient \( \rho(\alpha) \) in Eq. (12) is totally independent of the number \( V \) of vertices on the graph and, thus, of the density of resonances. The dependence of \( x_\alpha \) on \((1/d)\) is intuitively understood as due to the fact that the sum of the eigenphases of \( S \) increases by \( \pi \) over the width of a resonance. Averaging over the ensuing motion of the elements of \( S \) in the complex plane yields a subunitary average matrix. The complete lack of any dependence of \( \langle S_{\alpha\alpha} \rangle \) in Eq. (12) on \((1/d)\) is formally due to the fact that whenever there is scattering from vertex \( \alpha \) onto any bond in the graph, propagation along that bond causes the resulting contribution to \( \langle S \rangle \) to vanish. In physical terms that lack implies that every resonance on the graph gives on average a vanishing contribution. In view of these distinct differences it is not obvious that the fluctuation properties of the \( S \) matrix for chaotic scattering on graphs and for a random-matrix model of the Hamiltonian \( H \) in Eq. (17) should coincide.

VI. GENERATING FUNCTION

With all moments of \( S \) determined by Eq. (15) we turn to the calculation of moments involving both \( S \) and \( S^* \). In view of the decomposition (16) all such moments can be expressed in terms of the \((P,Q)\) correlation functions of the fluctuating part of the \( S \) matrix, defined as the average of a product of \( P \) elements of \( S^\text{fl} \) with arguments \( k + \kappa_p, p = 1, \ldots, P \) and \( Q \) elements of \( S^\text{th} \) with arguments \( k - \kappa_q, q = 1, \ldots, Q \). Without loss of generality we assume \( P \geq Q \geq 1 \). In view of Eq. (14) it suffices to work out the \((P,Q)\) correlation function of \( W^{-1} \) defined as

\[
\langle \prod_{p=1}^P W_{b_p d_p, b'_p d'_p}^{-1} (k + \kappa_p) \prod_{q=1}^Q (W_{b_q d_q, b'_q d'_q}^{-1} (k - \kappa_q))^* \rangle .
\]  

(22)

Let \( A \) denote a symmetric matrix in directed bond space that has non-zero elements only in the positions \((b_p d_p, b'_p d'_p)\),

\[
A_{b_d,v_d'} = \delta_{b b'} \delta_{d d'} \delta_{v v'} + \delta_{b' b} \delta_{d d'} \delta_{v v'}. 
\]  

(23)

We use the identity

\[
W_{b_d d_p, b'_d d'_p}^{-1} = \frac{1}{4} \frac{\partial}{\partial j} \text{det}(W + jA) \bigg|_{j=0}
\]  

(24)

and write the \((P,Q)\)-correlation function (22) as

\[
(P,Q) = \frac{1}{4^{P+Q}} \prod_{p=1}^P \prod_{q=1}^Q \frac{\partial}{\partial j_p} \frac{\partial}{\partial j_q} G \bigg|_{j_1 = \ldots = j_Q = 0} .
\]  

(25)

The generating function \( G \) is defined as

\[
G = \prod_{p=1}^P \frac{\text{det}(W + j_p A^{(p)})}{\text{det}(W - j_p A^{(p)})} \prod_{q=1}^Q \frac{\text{det}(W^* + j_q \tilde{A}^{(q)})}{\text{det}(W^* - j_q \tilde{A}^{(q)})} .
\]  

(26)

with \( A^{(p)} \) and \( \tilde{A}^{(q)} \) defined analogously to \( A \) in Eq. (23).

VII. AVERAGING THE GENERATING FUNCTION

To average \( G \) we use the supersymmetry approach of Refs. (10, 28) in the version of Refs. (5, 8). We aim at a representation which, although closely patterned after Ref. (8), is self-contained. That seems advisable because our treatment extends that of Ref. (8) to the general \((P,Q)\) correlation function.

For \( p = 1, \ldots, P \) and \( q = 1, \ldots, Q \) we define the matrices

\[
B_{p \pm 1}^{-1} = \Sigma(B) \mp j_p A^{(p)} , \quad B_{q \mp 1}^{-1} = \Sigma(B) \mp j_q \tilde{A}^{(q)} .
\]  

(27)

In directed bond space we define

\[
T_p = \exp\{i(k + \kappa_p) L / 2\} , \quad T_q = \exp\{i(k - \kappa_q) L / 2\} .
\]  

(28)

From Eq. (7) we have

\[
W(k + \kappa_p) \pm j_p A^{(p)} = T_p^* \sigma_1^p (1 - T_q^* \sigma_1^q B_{p \mp 1}^{-1} T_p) T_p^* .
\]  

(29)

Inserting that and the corresponding expression for \( W^* \) into Eq. (20) we see that the factors \( T_p^* \), \( T_q^* \) and the factors \( \sigma_1^p \) cancel in numerator and denominator. We obtain

\[
G = \prod_{p=1}^P \text{det}(1 - T_q^* \sigma_1^q B_{p \mp 1}^{-1} T_p) \prod_{q=1}^Q \text{det}(1 - T_q^* \sigma_1^q B_{q \mp 1}^{-1} T_q) .
\]  

(30)

To write \( G \) as a superintegral, we define the \( 8PB \)-dimensional supervector \( \psi_+ \) (the \( 8BQ \)-dimensional supervector \( \psi_- \), respectively), both with complex commuting (\( s = 1 \)) or anticommuting (\( s = 2 \)) elements. The factors \( 2BP \) and \( 2BQ \) account for the dimension of directed bond space and for the occurrence of the \((P,Q)\) factors in Eq. (30). A factor 2 is due to supersymmetry with \( s = 1, 2 \). Another factor 2 is due to the additional index \( x = 1, 2 \). That index is introduced as a preparatory step for the treatment of time-reversal invariance. We combine \( \psi_+ \) and \( \psi_- \) into a single supervector \( \psi \) and define \( \psi = \psi^\dagger \). We write the generating function as

\[
G = \prod_{p=1}^P \text{SDet} B_p \prod_{q=1}^Q \text{SDet} B_q^* \int \mathcal{D}(\bar{\psi}, \psi) \exp\{-A(\bar{\psi}, \psi)\} .
\]  

(31)
where $\text{SDet}$ denotes the superdeterminant and where
\begin{equation}
A(\tilde{\psi}, \psi) = \tilde{\psi}^T A \psi + \tilde{\psi} \tilde{A} \psi \tilde{\psi}.
\end{equation}

The matrices $A_+$ ($A_-$) have dimensions $8BP\ (8BQ$ respectively) and carry the indices $\{pbdsx\}$ ($\{qbdksz\}$ respectively). Moreover, the matrices $A_+$ and $A_-$ are block diagonal and contain the $P$ matrices $A_p$, $p = 1, \ldots, P$ (the $Q$ matrices $A_q$, $q = 1, \ldots, Q$, respectively) as diagonal blocks. These are defined by
\begin{equation}
A_p = \left( \begin{array}{cc}
1/T_p & T_p \\
B_p \sigma_1^d & B_p \sigma_1^d
\end{array} \right), \quad A_q = \left( \begin{array}{cc}
1/T_q & T_q \\
B_q \sigma_1^d & B_q \sigma_1^d
\end{array} \right).
\end{equation}

With $\sigma_3^s$, the third Pauli spin matrix in two-dimensional superspace labeled $s = 1, 2$ and in analogy to Eqs. (27) the $4B$ dimensional supersmatrices $B_p$ and $\bar{B}_q^*$ are defined by
\begin{equation}
B_p^{-1} = \Sigma(B) - \sigma_3^s j_p A(p), \quad \bar{B}_q^{-1} = \Sigma(B) - \sigma_3^q \bar{A}(q).
\end{equation}

All the matrices in Eqs. (33) and (34) are diagonal in superspace.

To account for time-reversal invariance we define
\begin{equation}
\Psi = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\psi \\
\sigma_1^d \psi^T
\end{array} \right), \quad \bar{\Psi} = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\bar{\psi} \sigma_1^d \\
\bar{\psi}^T
\end{array} \right).
\end{equation}

and, for all the supermatrices (jointly denoted by $\omega$) introduced above,
\begin{equation}
\Omega = \left( \begin{array}{cc}
\omega & 0 \\
0 & \sigma_1^d \omega^T \sigma_1^d
\end{array} \right).
\end{equation}

The matrix $\sigma_1^d$ accounts for time reversal in directed bond space $d = \pm$. The two new dimensions introduced by Eqs. (35) and (36) are denoted by the index $t = 1, 2$. We note that the dimensions of the supervectors $\psi$ and $\bar{\psi}$ remain unchanged. In the associated two-dimensional space we define
\begin{equation}
\tau = \sigma_1^d \left( \begin{array}{cc}
0 & \sigma_3^q \\
1 & 0
\end{array} \right).
\end{equation}

Then, $\Psi = \tau \bar{\Psi}^T$.

The generating function is now given by
\begin{equation}
G = \prod_{p=1}^P \text{SDet} B_p \prod_{q=1}^Q \text{SDet} \bar{B}_q \int d(\tilde{\psi}, \psi) \exp\{-A(\tilde{\Psi}, \Psi)\}
\end{equation}

where
\begin{equation}
A(\tilde{\Psi}, \Psi) = \tilde{\Psi}^T A_+ \tilde{\Psi} + \tilde{\Psi}^T A_- \Psi - \tilde{\Psi} \tilde{A}_- \Psi \tilde{\Psi}.
\end{equation}

The function $A(\tilde{\Psi}, \Psi)$ differs from the function $A(\tilde{\psi}, \psi)$ defined in Eq. (32). It also differs from the function $A(\tilde{\Psi}, \Psi, \tilde{Z}, \bar{Z})$ defined in Eq. (41) below. In the sense of Eq. (36), the matrices $A_{\pm}$ in Eq. (39) are extensions of the corresponding matrices in Eqs. (32), and analogously for the matrices in Eq. (41).

We use Eq. (8) and write the $k$ average of $G$ as a phase average over all $\phi_b = k L_b$. The latter average is calculated with the help of the colour-flavour transformation of Ref. 29. In general we have $P > Q$, and we must use that transformation in its most general form. The integrals over products of bond propagation amplitudes are mapped onto integrals over supermatrices $Z$ and $\bar{Z}$. We obtain
\begin{equation}
\langle \langle G \rangle \rangle_\phi = \prod_{p=1}^P \text{SDet} B_p \prod_{q=1}^Q \text{SDet} \bar{B}_q \int d(\tilde{\psi}, \psi) \int \text{d}(\tilde{Z}, \bar{Z}) \exp\{-A(\tilde{\Psi}, \Psi, \tilde{Z}, \bar{Z})\},
\end{equation}

where the action now has the form
\begin{equation}
A(\tilde{\Psi}, \Psi, \tilde{Z}, \bar{Z}) = \tilde{\Psi}_1 A_{11} \Psi_1 + \tilde{\Psi}_2 A_{22} \Psi_2.
\end{equation}

The indices are those of the auxiliary label $x = 1, 2$. The matrices $A_{11}$ and $A_{22}$ each have dimension $8BP\ (P + Q)$ and, in retarded/advanced space, are given by
\begin{equation}
A_{11} = \left( \begin{array}{cc}
1 & Z z_- \\
Z^\tau z_+ & 1
\end{array} \right), \quad A_{22} = \left( \begin{array}{cc}
B_+ \sigma_1^d & \bar{B}_+ \sigma_1^d \\
\bar{Z}_z^+ & \bar{B}_-^* \sigma_1^d
\end{array} \right).
\end{equation}

Here $B_+$ ($\bar{B}_-^*$) has dimension $8BP \ (8BQ)$, is block diagonal, and carries the $P$ matrices $B_p$ (the $Q$ matrices $\bar{B}_q^*$, respectively) in the diagonal blocks. Without change of notation $B_p$ and $\bar{B}_q^*$ now denote the supermatrices obtained from the supermatrices in Eqs. (34) by the doubling of matrix dimensions in Eq. (36). The matrices $z_{\pm}$ are block diagonal and in each block $p = 1, \ldots, P$ or $q = 1, \ldots, Q$ given by
\begin{equation}
(z_+)_p = \exp\{i \kappa_p \mathcal{L}\}, \quad (z_-)_q = \exp\{i \kappa_q \mathcal{L}\}.
\end{equation}

In the general case $P > Q$ the supermatrices $Z$ and $\bar{Z}$ are rectangular, $Z$ having $8BP$ rows and $8BQ$ columns, and conversely for $\bar{Z}$. Both $Z$ and $\bar{Z}$ are diagonal in bond space. For each bond index $b$, the bond-diagonal submatrices $Z_b$ are normalized according to
\begin{equation}
\int \text{d}(Z_b, \bar{Z}_b) \text{SDet}(1 - Z_b \bar{Z}_b) = 1.
\end{equation}

The integration measure is the flat Berezinian. In Boson-Fermion block notation, $Z$ and $\bar{Z}$ have the form
\begin{equation}
Z = \left( \begin{array}{cc}
Z_{BB} & Z_{BF} \\
Z_{FB} & Z_{FF}
\end{array} \right), \quad \bar{Z} = \left( \begin{array}{cc}
\bar{Z}_{BB} & \bar{Z}_{BF} \\
\bar{Z}_{FB} & \bar{Z}_{FF}
\end{array} \right),
\end{equation}

with
\begin{equation}
\bar{Z}_{BB} = Z_{BB}^\dagger \text{ and } \bar{Z}_{FF} = -Z_{FF}^\dagger.
\end{equation}

The eigenvalues of the positive definite Hermitian matrix $Z_{BB}^\dagger Z_{BB}$ are smaller than unity. The matrices
\begin{equation}
Z^\tau = \tau Z^T \tau^{-1} \text{ and } \bar{Z}^\tau = \tau \bar{Z}^T \tau^{-1}
\end{equation}
are simple transforms of $Z$ and $\tilde{Z}$, respectively, with $\tau$ defined in Eq. (37). In the retarded/advanced block notation of Eq. (42), the matrices $Z_b$ ($\tilde{Z}_b$) occur only in the retarded/advanced (in the advanced-retarded) non-diagonal blocks. This directly reflects the fact that $Z_b$ and $\tilde{Z}_b$ arise from averaging the product of two propagation amplitudes, one from the retarded and one from the advanced diagonal blocks, respectively.

The supervectors $\psi$, $\psi$ can be integrated out. The prefactor $\prod_{p=1}^{P} \text{SDet} B_p \prod_{q=1}^{Q} \text{SDet} \tilde{B}_q$ cancels out, and the averaged generating function is (here and in what follows, we suppress the index $\phi$ on the averages)

$$\langle G \rangle = \int d(\tilde{Z}, Z) e^{-A(\tilde{Z}, Z)} ,$$

where $A(\tilde{Z}, Z)$ denotes the action

$$A(\tilde{Z}, Z) = -\text{STr} \ln(1 - Z\tilde{Z})$$

$$+ \frac{1}{2} \text{STr} \ln(1 - Z\ddot{Z})$$

$$+ \frac{1}{2} \text{STr} \ln(1 - z_+\sigma d B^{-1} z_+ \tilde{Z}^* \sigma d z_+ \tilde{Z})$$

and $\text{STr}$ the supertrace. Up to this point our results are exact.

VIII. SADDLE-POINT APPROXIMATION

We calculate $\langle G \rangle$ using the saddle-point approximation. We neglect small terms by putting $z_\pm = 1$ and $A^{(j)} = 0, \tilde{A}^{(j)} = 0$ for all $j$. Variation of the resulting action $A_0$ with respect to any element of $Z$ and the relations (47) yield (8) the saddle-point equation

$$\tilde{Z} \frac{1}{1 - ZZ} = Z^* \frac{1}{1 - ZZ^*} .$$

Variation of $A_0$ with respect to any element of $\tilde{Z}$ yields correspondingly

$$\frac{1}{1 - ZZ} Z = \frac{1}{1 - ZZ^*} \Sigma \tilde{Z}^* \Sigma .$$

Here $\Sigma$ is block diagonal in retarded-advanced space and is a multiple of the unit matrix in superspace. In all retarded (advanced) blocks $\Sigma$ carries the matrix $\sigma_0^d (\Sigma^{(B)})^*$, respectively. As in Eq. (7), the factors $\sigma_0^d$ are needed to ensure proper matrix multiplication in directed bond space. We have used that $\sigma_0^d (\Sigma^{(B)})^* = \sigma_0^d (\Sigma^{(B)})$, see Eq. (47). Eq. (50) implies $\tilde{Z} = Z^*$. The saddle-point equations (51) hold if (i) $Z \Sigma = \Sigma Z$ and if (ii) $\sigma_0^d (\Sigma^{(B)}) \sigma_0^d (\Sigma^{(B)})^* = 1$. Condition (ii) is fulfilled if and only if $\Sigma^{(V)} (\Sigma^{(V)})^* = 1$. We recall that $\Sigma^{(V)}$ is block-diagonal, each block carrying one of the matrices $\sigma^{(\alpha)}$ defined in Eqs. (2). These obey $\sigma^{(\alpha)} (\sigma^{(\alpha)})^* = 1$ for all $\alpha \leq \Lambda$. We put $\Sigma^{(V)} (\Sigma^{(V)})^* = 1$ throughout and account for the deviation due to the blocks $\alpha \leq \Lambda$ presently. As for condition (i), we proceed as in Refs. [7, 8] and write the universal saddle-point solutions $Z^{sp}$ and $\tilde{Z}^{sp}$ as

$$Z^{sp}_{pd_{bst}, qbd_{st}} = \delta_{bd} \delta_{dd'} Y_{bst}, q_{st}, t'$$

$$\tilde{Z}^{sp}_{qbd_{st}, pbd_{st}} = \delta_{bd} \delta_{dd'} Y_{bst}, q_{st}, t' ,$$

where the former label $x$ is replaced by $t$. Eqs. (52) guarantee that condition (i) is fulfilled. In retarded-advanced block representation, the matrix $Y (Y)$ has non-vanishing elements only in the retarded-advanced block (in the advanced-retarded block, respectively). As for $Z$, the eigenvalues of the positive definite Hermitian matrix $Y^{BB} Y^{BB}$ must be smaller than unity. The matrices $Y$ and $\tilde{Y}$ are linked by the symmetry properties (45) and (46). The independent variables in the matrices $Y$ and $\tilde{Y}$ span the saddle-point manifold. Using $\Sigma^{(V)} (\Sigma^{(V)})^* = 1$ and Eqs. (52) in Eq. (49) we find that the saddle-point action $A_0$ vanishes.

Corrections to $A_0 = 0$ are due to deviations from $\Sigma^{(B)} (\Sigma^{(B)})^* = 1$, and from $z_\pm = 1$. As for the former, we consider the last term of the action (49) (taken at the saddle point) for $z_\pm = 1$,

$$\frac{1}{2} \text{STr} \ln(1 - \Sigma \Sigma^* Y \tilde{Y}) .$$

We use the fact that in the space of directed bonds, both $Y$ and $\tilde{Y}$ are multiples of the unit matrix. We write the term (53) for every block $b$ in vertex representation. For each diagonal block $\alpha \leq \Lambda$ in the matrix $\Sigma^{(V)}$ we use the first of Eqs. (2), suppressing the index $\alpha$. With $\rho$ real, the unitary and symmetric matrix $\Gamma$ can be unitarily transformed into

$$
\begin{pmatrix}
\exp\{i\phi_1\} T^{1/2} & 0 \\
0 & 0 \\
\delta_{\mu \nu} \exp\{i\phi_{\mu} \}
\end{pmatrix}
$$

Here $\mu, \nu = 3, \ldots, V - 1$. The transmission coefficient $T$ is defined in Eq. (21), the phases $\phi_1$ and $\phi_2$ are real and arbitrary. Eq. (54) shows that $\sigma^* \sigma$ differs from the unit matrix only in the first diagonal element which is $1 - T$. Using that fact for all $\alpha \leq \Lambda$ in the term (53) and the resulting expression in the action (49), we obtain in the exponent of Eq. (48) the “channel-coupling term”

$$CC = -\frac{1}{2} \sum_{\alpha=1}^\Lambda \sum_{\beta=1}^\Lambda \text{STr}_{pst} \ln \left( 1 + T^{(\alpha)} Y Y^* \frac{1}{1 - Y Y^*} \right) .$$

The trace extends only over the indices indicated. Concerning the deviations from $z_\pm = 1$, we expand $Z = \gamma_{\pm}$ and the action $A$ in Eq. (49) around the saddle-point value $A_0 = 0$ up to first order in $\kappa_\rho$ and $\kappa_q$, putting $\Sigma \Sigma^* = 1$. With

$$\langle d_R \rangle = \frac{1}{\pi} \sum_b L_b$$

(56)
the average level density \[20\), we obtain in the exponent of Eq. \[48\] the “symmetry-breaking term”

\[
SB = i\pi \langle d_R \rangle \left( \text{STr}_{p=1} \kappa \frac{1}{1 - YY} + \text{STr}_{q=1} \kappa \frac{1}{1 - YY} \right).
\]

(57)

The matrix \(\kappa\) is \(\delta_{j_q, j_q'} \delta_{p, p'} \kappa_{p, p'}\), and correspondingly for \(\tilde{k}\). Collecting results we obtain

\[
\langle G \rangle = \int d(Y, \tilde{Y}) \left( \ldots \right) \exp \{ CC + SB \}.
\]

(58)

The term in big round brackets denotes the “source terms” (terms in \(\mathcal{A}\) that are proportional to one of the variables \(j_p, j_q\) in Eq. \(27\)). According to Eq. \(25\) these terms are needed to first order in every \(j_p, j_q\) only. However, expanding the action \(19\) and subsequently the exponential in Eq. \(18\) in powers of all the \(j\)’s creates a multitude of terms even if we keep only terms to first order. That is why these are not given here explicitly. For specific applications they are worked out in Sections \(X\) and \(XI\). The integration measure in Eq. \(58\) is again the flat Berezinian.

**IX. MASSIVE MODES**

In the derivation of Eq. \(48\) we have taken into account only the universal form \[22\) of the saddle-point solution. We have neglected all other parts of the matrices \(Z\) and \(\tilde{Z}\). In Sections \(X\) and \(XI\) we show that the resulting generating function \(\langle G \rangle\) in Eq. \(58\) gives rise to universal results for the \(S\) matrix of chaotic scattering. This fact motivates our neglect: In this paper we use graph theory as a tool to generate universal results without resorting to random-matrix theory.

Nevertheless we must address the question whether the neglect leading to Eq. \(48\) is justified. Technically the approximation leading to Eq. \(48\) is referred to as the “neglect of massive modes”. (While the saddle-point solution corresponds to the zero mode of the problem, other parts of the matrices \(Z\) and \(\tilde{Z}\) give rise to Gaussian superintegrals. The factors in the exponents play the role of masses.) Are there special graphs for which the universal saddle-point solution \[22\) is actually correct? Or is the neglect of massive modes perhaps even justified for all chaotic graphs? Ref. \[24\) demonstrates the existence of non-statistical resonance scattering on chaotic quantum graphs, refuting such hopes and underlining the need to establish the conditions under which the massive modes can be neglected. While a full treatment of that problem is beyond the scope of the present paper, we offer a conjecture that is based upon the following considerations.

For closed quantum graphs, the neglect of massive modes has been investigated \[30, 31\) with the help of the supersymmetry approach. The issue was quantum ergodicity. A graph is said to be quantum ergodic if in the semiclassical limit the moduli of the eigenfunctions are spread uniformly over the graph. Naively one might expect chaotic quantum graphs to be quantum ergodic. However, the existence of scars on graphs \[22\) refutes that expectation. Therefore, quantum ergodicity cannot be expected to hold universally for chaotic graphs. On the other hand, quantum ergodicity emerges in the supersymmetry approach \[30, 31\) as a universal property of graphs if the massive modes are neglected. Therefore, such neglect can apply only under special conditions.

In Refs. \[30, 31\) such conditions have been formulated in terms of the analogue of our matrix \(\Sigma^{(B)}\). In contrast to the present case, for closed graphs that matrix contains exclusively the elements of the unitary matrices \(\sigma^{(u)}\) appearing in the second of Eqs. \(2\). For sequences of graphs with monotonically increasing vertex number \(V\) the neglect of massive modes is asymptotically \((V \to \infty)\) justified \[30, 31\) if the spectrum of eigenvalues of the matrix \(|(\sigma^{(B)} \Sigma^{(B)})_{bd, bd'}|^2\) possesses a gap separating it from zero.

In chaotic scattering the statistics of wave functions is important, too. That fact is known from the RMT approach to chaotic scattering \[27\). The eigenvalues and eigenfunctions of the random Hamiltonian \(H\) in Eqs. \(16\) and \(17\) are uncorrelated random variables. The fluctuations of the \(S\) matrix are dominated \[27\) by the fluctuations of the eigenfunctions both in the limit of very weakly overlapping resonances (where the fluctuations of the eigenvalues are totally unimportant) and in the limit of strongly overlapping resonances (where a picket-fence model for the eigenvalues \[16\) gives the same result as a full RMT treatment). Conversely, deviations of the eigenfunctions from RMT statistics may give rise to non-universal scattering. For chaotic quantum graphs, that has been shown for “topological resonances” \[24\).

These correspond to poles of the \(S\) matrix that can be moved to the real \(k\) axis by continuously changing some bond lengths, giving rise to a bound state embedded in the continuum. Physically, a topological resonance is related to a “cycle” (a closed loop on the graph). When the lengths of the bonds forming the cycle become rationally dependent, the resulting bound-state eigenfunction is localized on the cycle and, thus, very far from being uniformly distributed over the graph.

In view of these facts we conjecture that a prerequisite for universal chaotic scattering on an open chaotic quantum graph is quantum ergodicity of the corresponding closed graph (obtained by letting all coefficients \(\tau^{(s)}_{\beta}\) in Eqs. \(2\) tend to zero). Quantitatively we conjecture that the criterion establishing quantum ergodicity for closed graphs \[30, 31\) applies equally in the present case: The spectrum of our matrix \(|(\sigma^{(B)} \Sigma^{(B)})_{bd, bd'}|^2\) (which contains \(\Lambda\) subunitary matrices \(\sigma^{(u)}\)) should asymptotically \((V \to \infty \text{ with } \Lambda \text{ fixed})\) have a gap separating it from zero. The proof would require a special investigation.

The completely connected graphs defined in Section \(II\) contain many loops and may, thus, not obey our criterion. However, with a slight change of notation that allows for missing bonds, our derivation also holds for other types of connected graphs.
X. TWO-POINT FUNCTION

In the framework of the Hamiltonian approach of Eqs. (10) and (17) and with $H$ replaced by an ensemble of random matrices with orthogonal symmetry, the $S$-matrix two-point function was worked out in Ref. [10] as an ensemble average. Using the universal saddle-point solutions (52), we now calculate the $S$-matrix two-point correlation function for a single chaotic time-reversal invariant quantum graph and show that it coincides with the RMT result.

We consider the average generating function $G$ in Eqs. (55), (56) and (57) for the case $(P, Q) = (1, 1)$ where the matrices $Y$ and $\tilde{Y}$ are both square matrices of dimension four carrying indices $(st, s't')$. The summing over $p$ in Eq. (55) is redundant. In the symmetry-breaking term we put $\kappa_1 = \kappa = \kappa_1$. Then

$$SB(1, 1) = 2i\pi\kappa \langle d\kappa \rangle \text{STr}_{st} \left( \frac{1}{1 - YY} \right). \quad (59)$$

The part of the action (19) at the saddle point that is relevant for the source terms is for $z_+ = 1 = z_-$ given by

$$\frac{1}{2} \text{STr} \ln (1 - \sigma_1^0\tilde{B}_+^{-1}Y\sigma_1^0\tilde{B}_-^{-1}\tilde{Y}) \quad (60)$$

Differentiation of with respect to $j_p$ with $p = 1$ and to $\tilde{j}_q$ with $q = 1$ yields

$$\text{STr} \left( Y_{\sigma_1^0}(\Sigma^{(B)})^*\tilde{Y} 
\times \frac{1}{1 - \sigma_1^0(\Sigma^{(B)})^*Y\sigma_1^0(\Sigma^{(B)})^*Y} \alpha_3 \right)_{b\bar{d}p, b\bar{d}p}$$

$$\times \text{STr} \left( \tilde{Y}_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y \right)_{b\bar{d}q, b\bar{d}q}$$

$$+ \text{STr} \left( \left[ \frac{1}{1 - \sigma_1^0(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y} \alpha_3 \right]_{b\bar{d}p, b\bar{d}p} 
\times \left[ Y_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y \right]_{b\bar{d}q, b\bar{d}q} \right)$$

$$+ \text{STr} \left( \left[ \frac{1}{1 - \sigma_1^0(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y} \alpha_3 \right]_{b\bar{d}p, b\bar{d}p} 
\times \left[ Y_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y_{\sigma_1^0}(\Sigma^{(B)})^* Y \right]_{b\bar{d}q, b\bar{d}q} \right) \quad (61)$$

To calculate $(S_{\alpha,\beta}^a S_{\gamma,\delta}^{\alpha*})$ we use Eq. (5) and expression (61) with $T_{\alpha,\beta, b\bar{d}p}$, with $T_{\gamma, b\bar{d}q}$, and with $T^{*}_{\delta, b\bar{d}p, b\bar{d}q}$ and sum over the indices $\{b\bar{d}p, b\bar{d}p, b\bar{d}q, b\bar{d}q\}$. We rewrite the resulting expression in vertex space and apply the unitary transformation that brings all matrices $\sigma^{(a)}$ into the form [24]. That same transformation applied to the vector $T_{\alpha, b\bar{d}p}$ (fixed $\alpha$) yields a vector the components of which are all zero except for the first one that has the value $\exp \{i\delta^{(a)}_{\alpha, \beta} (T^{(a)})^{1/2} \}$. Combining the result with Eqs. (25), (55) and (59) we obtain

$$\langle S_{\alpha, \beta}^a S_{\gamma, \delta}^{\alpha*} \rangle = \frac{1}{16} \int d(Y, \tilde{Y}) F_{\alpha, \beta} \gamma \delta$$

$$\times \exp \left\{ - \frac{1}{2} \sum_{\tau=1}^\Lambda \text{STr} \ln \left( 1 + T^{(\tau)} Y_{\gamma, \delta} \right) \right\}$$

$$\times \exp \left\{ 2i\pi \kappa \langle d\kappa \rangle \text{STr}_{st} \left( \frac{1}{1 - YY} \right) \right\} \quad (62)$$

where

$$F_{\alpha, \beta} \gamma \delta = \delta_{\alpha, \beta} \gamma \delta (S_{\alpha, \gamma} (T^{(\alpha)}) (S_{\gamma, \delta}^{\alpha*} T^{(\gamma)})$$

$$\times \text{STr} \left( \frac{1}{1 - YY + T^{(\gamma)} YY} \right) \text{STr} \left( \frac{1}{1 - YY + T^{(\gamma)} YY} \right)$$

$$\times \frac{1}{2} \left( \delta_{\alpha, \gamma} \delta_{\beta, \delta} + \delta_{\alpha, \delta} \delta_{\beta, \gamma} \right) T^{(\alpha)} T^{(\beta)}$$

$$\times \left\{ \text{STr} \left( \frac{1}{1 - YY + T^{(\gamma)} YY} \right) \text{STr} \left( \frac{1}{1 - YY + T^{(\gamma)} YY} \right) + (\alpha \leftrightarrow \beta) \right\}. \quad (63)$$

We note that the phases appearing in expression (64) have cancelled. Eqs. (62) and (63) give the two-point function for chaotic scattering on graphs in terms of a superintegral.

It is not necessary to work out the remaining integrations because we now show that the result in Eqs. (62) and (63) coincides with the one obtained in RMT [10] where these remaining steps have been carried out. Equations in Ref. [10] are denoted by a prefixed letter $V$. In the comparison allowance must be made for the different definitions of the symbols STr and trg used in the two approaches.

We recall that in retarded-advanced representation the matrices $Y$ and $\tilde{Y}$ occupy the non-diagonal blocks. We denote the corresponding blocks of the solution of the saddle-point equation of Ref. [10] by $\sigma_{12}$ and $\sigma_{21}$. In Ref. [10] the saddle-point manifold is parametrized in the form $T_c^{-1} \sigma_D T_c$ where the matrix $\sigma_D$ is block diagonal and proportional to the unit matrix (minus the unit matrix) in the retarded (the advanced block, respectively). We use Eq. (V.D.19) for $T_c$ and obtain

$$\sigma_{12} = t_{12} \sqrt{1 + t_{21} t_{12}} \cdot \sigma_{21} = t_{21} \sqrt{1 + t_{21} t_{12}}. \quad (64)$$

(A common proportionality constant has been removed by scaling). The non-linear transformation (64) from the variables in $\sigma_{12}$ and $\sigma_{21}$ with a flat integration measure to the variables in $t_{12}$ and $t_{21}$ gives rise to the non-flat integration measure $d\mu(t)$ appearing in Eq. (V.7.23) and...
given in Eq. (V.8.4). We use the analogous substitutions for the matrices $Y$ and $\tilde{Y}$,
\begin{equation}
Y = t_1^0 \sqrt{1 + t_{12}^0 t_{12}^0} \quad \tilde{Y} = t_2^0 \sqrt{1 + t_{12}^0 t_{21}^0}.
\end{equation}
Since the integration measure $d(Y, \tilde{Y})$ is flat and since the transformations (64) and (65) are identical in form, the integration measure $d\mu(t_b)$ for the variables in $t_{12}^0$ and $t_{21}^0$ has the same form as $d\mu(t)$ provided that the symmetry properties of $t_{12}^0$ and $t_{21}^0$ and of $t_{12}$ and $t_{21}$ are the same. The matrices $t_{12}^0$ and $t_{21}^0$ share the symmetry properties of $Y$ and $\tilde{Y}$ in Eqs. (14) and (16). Table D.3 of Ref. [10] shows that these are also the symmetry properties of $t_{12}$ and $t_{21}$. Therefore, a direct comparison of the two-point function given in Eq. (V.7.23) with that for the quantum graph (the latter expressed in terms of $t_{12}^0$ and $t_{21}^0$) is meaningful.

With the definitions $\alpha_1 = 2t_{12} t_{21}$ and $\alpha_2 = 2t_{12} t_{12}$ in Eq. (V.7.20), the channel coupling term in Eq. (V.7.23) coincides with the one in Eq. (62). In the symmetry-breaking term in Eq. (V.7.23) we must replace energies by wave numbers. We identify the inverse $1/d$ of the mean level spacing $d$ with the average level density $\langle d_R \rangle$.

The definition (V.3.12a) of the energy difference $\varepsilon$ implies the substitution $\varepsilon \rightarrow -2\kappa$. Then the symmetry-breaking term in Eq. (V.7.23) becomes equal to the one in Eq. (62). For the source terms we use the text below Eqs. (V.3.12c) and (V.7.12) and identify both $I(1)$ and $I(2)$ with $\sigma_1^{\alpha_1} \delta_{\nu_1}$.

The source terms in Eq. (V.7.23) become equal to the terms in Eq. (59), including the numerical factors. According to Eq. (V.8.6b) the factor $c$ in Eq. (V.7.23) is equal to unity. Therefore, the entire expression (V.7.23) coincides with our result in Eqs. (62) and (63).

We have shown that the two-point function of the $S$ matrix for a chaotic quantum graph coincides with the RMT result. The explicit form of that function is given in Eq. (V.8.10) and need not be repeated here.

\section{XI. Ericson Regime}

Progress beyond the two-point function derived in the previous Section is possible in the Ericson regime, defined by the condition $\sum_\alpha T^{(\alpha)} \gg 1$. The terms of leading order in an asymptotic expansion in inverse powers of $\sum_\alpha T^{(\alpha)}$ can be worked out for all $(P, Q)$ correlation functions.

To set the stage we first derive the asymptotic form of the two-point function, following Ref. [32]. In Eq. (62) we expand both the channel-coupling term and the symmetry-breaking term in powers of $Y \tilde{Y}$, keeping only the lowest-order terms. That gives
\begin{equation}
\exp \left\{-\frac{1}{2} \sum_{\tau=1}^\Lambda T^{(\tau)} + i\pi(\kappa + \tilde{\kappa})(d_R)\right\} \text{STr}_{st} \left(Y \tilde{Y}\right).
\end{equation}
Because of later applications we have not put $\kappa = \tilde{\kappa}$. The product $Y \tilde{Y}$ carries the factor $\sum_\tau T^{(\tau)} \gg 1$. Terms of higher order in $Y \tilde{Y}$ produce higher-order terms in $(\sum_\tau T^{(\tau)})^{-1}$ and are, therefore, neglected. We proceed likewise in the source terms, keeping only terms bilinear in $Y$ and $\tilde{Y}$. We obtain
\begin{equation}
F_{\alpha \beta \gamma \delta} \approx \left(\delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma}\right) T^{(\alpha)} T^{(\beta)} \text{STr}_{st} \left(\sigma_4^\alpha Y \sigma_4^\beta \tilde{Y}\right).
\end{equation}
The resulting Gaussian integrals are easily evaluated and give
\begin{equation}
\langle S^{\alpha \beta}(k + \kappa) S^{\gamma \delta}(k - \tilde{\kappa}) \rangle = \left(\delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma}\right) T^{(\alpha)} T^{(\beta)} \sum_{\tau=1}^{\Lambda} T^{(\tau)} - 2i\pi(\kappa + \tilde{\kappa})(d_R) \text{STr}_{st} \left(Y \tilde{Y}\right).
\end{equation}
Replacing wave numbers by energies as in Section XI gives exactly the expression obtained for RMT in Refs. [14, 15, 32].

Starting from $\langle G \rangle$ in Eq. (58), we use the same approximation scheme for the general $(P, Q)$ correlation function. The rectangular matrices $Y$ and $\tilde{Y}$ consist of blocks of dimension four each, denoted by $Y_{pq}$ and $\tilde{Y}_{pq}$, with elements $Y_{pq}$, $\tilde{Y}_{pq}$, with elements $(Y_{pq})_{st,s't'} = Y_{pst,s't'}$ and $(\tilde{Y}_{pq})_{st,s't'} = \tilde{Y}_{qst,pq}$. The integration measure $d(Y, \tilde{Y})$ being flat we have
\begin{equation}
d(Y, \tilde{Y}) = \prod_{p=1}^P \prod_{q=1}^Q d(Y_{pq}, \tilde{Y}_{qp}).
\end{equation}
The exponent in Eq. (58) is approximated by
\begin{equation}
\sum_{p=1}^P \sum_{q=1}^Q \left[-\frac{1}{2} \sum_{\tau=1}^\Lambda T^{(\tau)} + i\pi(d_R)(\kappa_p + \tilde{\kappa}_q)\right] \text{STr}_{st} \left(Y_{pq} \tilde{Y}_{qp}\right).
\end{equation}
We expand the source terms (last term of the action (49) with $Z$ replaced by $Y$, $\tilde{Z}$ by $\tilde{Y}$, $z_\pm$ by 1) in powers of $Y$ and $\tilde{Y}$, retaining only terms linear in both $Y$ and $\tilde{Y}$. Only these combine to expressions of the form $\sum_{pq} f_{pq} Y_{pq} \tilde{Y}_{qp}$ (with some matrices $f_{pq}$) that according to Eq. (70) give the leading-order contribution in the expansion in inverse powers of $\sum_\tau T^{(\tau)}$. We obtain
\begin{equation}
\sum_{p=1}^P \sum_{q=1}^Q \text{STr}_{bdst} \ln(1 - \sigma_4^d B_{-1}^p Y \sigma_4^d \tilde{B}_{-1}^q \tilde{Y}) \approx -\frac{1}{2} \sum_{p=1}^P \sum_{q=1}^Q \text{STr}_{bdst} \left(\sigma_4^d (B_{-1}^p Y \sigma_4^d \tilde{B}_{-1}^q \tilde{Y})\right).
\end{equation}
Since $P \geq Q$ we first focus on the fact that we need $P$ source terms, each one deriving from one of the factors $(B_{-1}^p)_{pq}$ and carrying a different element $f_{pq}$, $p = 1, \ldots, P$, see Eq. (24). Expanding the exponential of the term (71) in a Taylor series we accordingly keep the term
\begin{equation}
\frac{(-)^P}{2^P P!} \sum_{p=1}^P \sum_{q=1}^Q \text{STr}_{bdst} \left(-\sigma_4^d f_{pq} A(p) Y_{pq} (\sigma_4^d \tilde{B}_{-1}^q \tilde{Y})\right)^P.
\end{equation}
Differential with respect to \( j_p \) at \( j_p = 0 \) for \( p = 1, \ldots, P \) gives
\[
\left( \frac{-\rho}{2} \right)^P \prod_{p=1}^{P} \left\{ \sum_{q=1}^{Q} \text{STr}_{bdst} \left( -\sigma^d_1 \sigma^d_2 A^{(p)} Y_{pq} \sigma^d_1 (\hat{B}^{-1*})_q \hat{Y}_{qp} \right) \right\} .
\] (73)

We define
\[
X_{pq} = \text{STr}_{bdst} \left( -\sigma^d_1 \sigma^d_2 A^{(p)} Y_{pq} \sigma^d_1 (\hat{B}^{-1*})_q \hat{Y}_{qp} \right)
\] (74)
and write expression (73) in the form
\[
\left( \frac{-\rho}{2} \right)^P \sum_{q_1=1}^{Q} X_{1q_1} \times \cdots \times \sum_{q_P=1}^{Q} X_{Pq_P} .
\] (75)

Differential with respect to \( \hat{j}_q \) at \( \hat{j}_q = 0 \) for \( q = 1, \ldots, Q \) forces \( Q \) of the \( P \) summand variables \( q_1, \ldots, q_P \) to take the values \( 1, 2, \ldots, Q \). These are denoted by \( q_1, q_2, \ldots, q_Q \). The set \( \{q_1, q_2, \ldots, q_Q\} \) is a permutation of the set \( \{1, 2, \ldots, Q\} \). In the corresponding factors \( X_{pq} \), the matrices \( (\hat{B}^{-1*})_q \), are replaced by \( -\sigma^d_1 \sigma^d_2 \hat{A}^{(q)} \), yielding \( X_{pq} \rightarrow \hat{X}_{pq} = \text{STr}(\sigma^d_1 \sigma^d_2 A^{(p)} Y^d_1 \sigma^d_2 A^{(q)} Y) \).

For \( P > Q \), \((P-Q)\) factors of the form \( \sum_q X_{pq} \) remain unaffected by the differentiation. Each of these gives \( \sum_q (\hat{X}_{pq})_{j_q=0} \). Changing notation we write \( r_1, \ldots, r_Q \) for those indices \( p \) that appear in one of the factors \( X_{pq} \). The set \( \{r_1, r_2, \ldots, r_Q\} \) is a subset of \( \{1, 2, \ldots, P\} \). The remaining indices in \( \{1, 2, \ldots, P\} \) are denoted by \( s_i, i = 1, \ldots, (P-Q) \). Expression (73) takes the form
\[
\sum_{\text{selections}} \prod_{i=1}^{P-Q} \left( \sum_q (\hat{X}_{pq})_{j_q=0} \right) \prod_{\text{permutations}} \hat{X}_{r_i,i} .
\] (76)
The sum with index “selections” runs over all possibilities of selecting the \( \binom{P}{P-Q} \) indices \( s_1 < s_2 < \ldots < s_{P-Q} \) from the set \( \{1, 2, \ldots, P\} \). The sum with index “permutations” runs over all permutations of the remaining indices \( r_1, r_2, \ldots, r_Q \).

Combining Eqs. (69) and (70) with expression (76) we see that the superintegral factorizes into a product of \( PQ \) terms, each factor having the form
\[
\int d(Y_{pq}, \hat{Y}_{qp}) \left( \cdots \right) \exp \left\{ \left[ \cdots \right] \text{STr}_{bdst} \left( Y_{pq} \hat{Y}_{qp} \right) \right\} .
\] (77)
The content of the big straight brackets is the same as in Eq. (70). The content of the big round brackets depends upon whether the index pair \((pq)\) does or does not occur in expression (76). If not, the big round brackets contain the factor unity, and the superintegral gives unity. If it does, the big round brackets contain the factor \( X_{pq} \) or the factor \( (X_{pq})_{j_q=0} \), as the case may be. In the first case the Gaussian superintegral (including the normalization factor \( 1/16 \) in Eq. (23)) is the same as occurs in the derivation of Eq. (68) and gives the same result, with a proper replacement of \( \kappa, \tilde{\kappa}, \alpha, \beta, \gamma, \delta \). The Gaussian superintegral containing in the integrand the factor \( (X_{pq})_{j_q=0} \) is easily worked out. The matrix \( (\Sigma^{(B)})^* \) in \( (X_{pq})_{j_q=0} \) does not couple to any channel. Therefore, the expression analogous to the result (68) involves only two channels and vanishes unless both channel indices are equal. The result is
\[
\mathcal{F}_{\alpha_p}(\kappa_p) = -\sum_{q=1}^{Q} \frac{T^{(\alpha_p)}(\delta_{\alpha_p,\alpha_{q^*}})}{\sum_{\gamma} T^{(\gamma)} - 2i\pi (\kappa_p + \tilde{\kappa}_q)(d_H)}
\] (78)
where again the phase \( \phi_1 \) in Eq. (54) cancels out. The sum over \( q \) arises because in the advanced block, the matrix \( \sigma^d_1 \hat{B}^{-1} \) carries the same entry \( \sigma^d_1(\Sigma^{(B)})^* \) in every subblock labeled \( q \).

Collecting results we obtain
\[
\left( \prod_{p=1}^{P} \sum_{q=1}^{Q} \sigma^d_{\alpha_p, \beta_p}(k + \kappa_p) \sigma^d_{\alpha_{q^*}, \beta_{q^*}}(k - \tilde{\kappa}_q) \right) = \sum_{\text{selections}} \prod_{j=1}^{P-Q} \mathcal{F}_{\alpha_j}(\kappa_j')
\] \times \sum_{\text{permutations}} \prod_{q=1}^{Q} \left( \sigma^d_{\alpha, \beta}(k + \tilde{\kappa}_q) \sigma^d_{\alpha^*, \beta^*}(k - \tilde{\kappa}_q) \right) .
\] (79)
The sum labeled “selections” goes over all \( \binom{P}{P-Q} \) possibilities to select \( (P-Q) \) matrix elements \( \sigma^d \) from the first factor on the left-hand side. These give rise to the first product which vanishes unless every selected element is diagonal. The remaining \( Q \) elements \( \sigma^d \), symbolically written as \( \sigma^d_{\alpha, \beta} \), appear as first factors in the angular brackets on the right-hand side. The sum labeled “permutations” extends over all permutations of these elements. Each of the terms in angular brackets on the right-hand side is equal to the asymptotic form (18) of the two-point function.

For \( P \geq Q \) Eq. (71) gives the universal part of all \( (P,Q) \) matrix correlation functions in the Ericson regime \( \sum \gamma T^{(\gamma)} \gg 1 \). The corresponding expressions for \( P < Q \) are obtained by complex conjugation. Combined with Eqs. (68) and (78), the universal part of the distribution of the \( k \)-dependent \( \sigma^d \) matrix is thus, completely known for graphs in the Ericson regime.

We compare Eqs. (71), (68) and (78) with previous results. All of these were derived in the framework of random-matrix theory, most approaches using Eqs. (16) and (17) and a symmetric and real random-matrix Hamiltonian \( H \). In the comparison we use the substitutions and replacements listed in Section X. We have mentioned already that the asymptotic form of the \((1,1)\) correlation function was calculated in Refs. [14, 15, 32]. In Ref. [14] that was done with the help of a picket-fence model for the eigenvalues of the GOE matrix \( H \) in Eq. (17), using the Gaussian distribution of the
eigenfunctions as the only stochastic element. The full random-matrix approach was utilized in Ref. [15], results were obtained with the help of the replica trick. The supersymmetry approach was used in Ref. [32]. The results agree with each other and with our result (68). Our asymptotic result for the (2, 2) correlation function agrees with that of Ref. [14] and with that of Ref. [12]. For the RMT approach the complete (2, 1) correlation function was worked out in Refs. [12, 13] with the help of supersymmetry. Our result in Eqs. (79) and (68) agrees with the asymptotic form of the expression given there. To the best of our knowledge, the papers cited contain the entire available analytical information on $S$ matrix correlation functions for systems with orthogonal symmetry. (Either of the approaches used in Refs. [14] and [12] could, in principle, be used to calculate higher correlation functions $(P, Q)$ in the Ericson regime. In both cases the effort becomes prohibitive, however, with increasing values of $P$ and/or $Q$.) We conclude that within that body of information, there is complete agreement between universality of the scattering matrix requires that all elements of $S$ obey $|S_{\alpha\beta}| \leq 1$. That constraint also restricts the distribution of $S$. In the Ericson regime, the constraint is easily fulfilled for the non-diagonal elements of $S$. All these vanish on average, and Eq. (68) shows that their variances are small compared to unity for $\sum T^{(\tau)} \gg 1$. The tails of a Gaussian centered at zero with width as given by Eq. (68) are very close to zero at the unit circle in the complex plane, and the Gaussian distribution is, therefore, asymptotically consistent with the constraint. For every diagonal element $S_{\alpha\alpha}$ with $(S_{\alpha\alpha}) \neq 0$ the situation differs. The distribution is now centered at $(S_{\alpha\alpha})$, and it is squeezed between that center and the unit circle. That is why it differs from a Gaussian. The squeezing becomes stronger as $|\langle S_{\alpha\alpha}\rangle|$ increases, and one would expect that the deviations from the Gaussian are biggest for $|\langle S_{\alpha\alpha}\rangle| \rightarrow 1$ whereas Eq. (79) shows that $F^\alpha_\alpha(k)$ is biggest for $|\langle S_{\alpha\alpha}\rangle| = 1/\sqrt{2}$. The reason is that according to Eq. (13), unitarity eventually forces $|S_{\alpha\alpha}(fl)|$ to decrease as $|\langle S_{\alpha\alpha}\rangle|$ increases so that $|\langle S_{\alpha\alpha}\rangle| \rightarrow 0$ for $|\langle S_{\alpha\alpha}\rangle| \rightarrow 1$.

The implications of deviations of the $S$-matrix distribution from the Gaussian form for cross-section fluctuations have been discussed in Ref. [33].

**XII. SUMMARY AND CONCLUSIONS**

Following up on our earlier work [25] we have in this paper studied universal aspects of scattering on chaotic quantum graphs. Starting point is the exact semiclassical expansion for the $S$ matrix. Using an ergodicity argument we have calculated averages of products of $S$-matrix elements over the wave number as phase averages. We have compared ergodicity for graphs and for RMT.

The average $S$ matrix is easy to calculate. It does not depend on the density of resonances and, thus, differs characteristicly from its counterpart in RMT. Formal expressions for all higher $(P, Q)$ $S$-matrix correlation functions were obtained with the help of supersymmetry, the colour-flavour transformation, and the saddle-point approximation to the resulting superintegrals. We conjecture that our neglect of massive modes is asymptotically justified for sequences of graphs that when closed are quantum ergodic. These formal results were used to calculate the $(1, 1)$ correlation function exactly. That function is equal to its counterpart in RMT for all values of the number $\Lambda$ of channels. We also calculated all $(P, Q)$ correlation functions in the Ericson regime, thus determining the complete $S$-matrix distribution function in that regime. That was done by calculating the leading-order terms in an asymptotic expansion in inverse powers of the sum $\sum T^{(\tau)}$ of transmission coefficients. These terms agree with the corresponding results of RMT inasmuch as the latter are known. In the Ericson regime the $S$ matrix has a Gaussian distribution only if all average $S$-matrix elements vanish. The deviations which arise
otherwise are due to the unitarity constraints on $S$.

Our results suggest several lines of future research. First, an investigation along the lines of Refs. [30, 31] is called for to see whether our conjecture regarding the neglect of massive modes is valid. Second, if the conjecture holds it would be interesting to know which types of graphs obey the criterion of asymptotic quantum ergodicity. Third, the saddle-point manifold being known, it may be possible to obtain exact closed expressions for the $(2, 1)$ and $(2, 2)$ correlation functions, similar in form to Eq. (8.10) of Ref. [10]. If universally valid, these would be useful for the analysis of intensity fluctuations of chaotic scattering in several areas of physics.

By confining ourselves to the saddle-point solution, we have focused attention on universal aspects of chaotic scattering on graphs. To discuss the significance of our results for scattering on chaotic quantum systems, we first recall the situation in closed systems. Here, the proven equality of the level-level correlation functions for RMT, for dynamical (i.e., Hamiltonian) chaotic quantum systems [3–6], and for an arbitrary chaotic quantum graph [7, 8] strongly suggests that the spectral fluctuations in all three types of systems are identical (including all higher correlation functions), in line with the original BGS conjecture [1]. Concerning scattering systems, we have proved the equivalence of chaotic scattering on a single graph and of the random-matrix model of Eqs. [16, 17]. Our result goes beyond that for the case of closed systems because it encompasses not only the universal $(1, 1)$ $S$-matrix correlation function but also the universal $(P, Q)$ correlation functions in the Ericson regime (inasmuch as the latter are available for the random-matrix model). As in the case of closed systems, these facts strongly suggest that all $S$-matrix correlation functions for both scattering systems coincide. To complete the analogy to closed systems it would be necessary to calculate the $(1, 1)$ $S$-matrix correlation function for scattering on dynamical chaotic quantum systems. It would be highly surprising if that would differ from our and the RMT result, and similarly for higher correlation functions. Therefore, we conjecture that scattering on graphs, scattering on Hamiltonian chaotic quantum systems, and the RMT approach to chaotic scattering, are completely equivalent.

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