Circles in self dual symmetric $R$-spaces

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Abstract

Self dual symmetric $R$-spaces have special curves, called circles, introduced by Burstall, Donaldson, Pedit and Pinkall in 2011, whose definition does not involve the choice of any Riemannian metric. We characterize the elements of the big transformation group $G$ of a self dual symmetric $R$-space $M$ as those diffeomorphisms of $M$ sending circles in circles. Besides, although these curves belong to the realm of the invariants by $G$, we manage to describe them in Riemannian geometric terms: Given a circle $c$ in $M$, there is a maximal compact subgroup $K$ of $G$ such that $c$, except for a projective reparametrization, is a diametrical geodesic in $M$ (or equivalently, a diagonal geodesic in a maximal totally geodesic flat torus of $M$), provided that $M$ carries the canonical symmetric $K$-invariant metric. We include examples for the complex quadric and the split standard or isotropic Grassmannians.

Keywords: symmetric $R$-space, big transformation group, circle, diametrical geodesic, complex quadric, isotropic Grassmannian

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1 Introduction

Symmetric $R$-spaces are compact Riemannian symmetric spaces admitting a Lie group of diffeomorphisms, called the big transformation group, properly containing the isometry group. It is a celebrated theorem of Nagano [14] that, up to covers, these are essentially the only compact Riemannian symmetric spaces admitting such a group.

If the action of a group $G$ on a manifold $M$ is given, it is natural to look for a structure on $M$ whose automorphism group is $G$. For instance, the sphere $S^n$ admits two different big transformation groups, those of conformal and projective maps, respectively. The big transformation group of a compact Hermitian symmetric space $M$ is the group of its (anti-)holomorphic maps. For a general symmetric $R$-space $M$, the big transformation group has been identified as the group of diffeomorphisms

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preserving a geometrically defined quantity on M called the arithmetic distance \[17\], or preserving a so called generalized conformal structure (the smooth assignment of a cone in each tangent space of M, satisfying certain properties) \[6\] (see also \[2\]).

Self dual symmetric R-spaces have special curves, called circles, introduced in Burstall et al \[1\] in order to extend some aspects of the theory of Darboux transforms of isothermic surfaces in \(\mathbb{R}^3\) to significantly more general ambient spaces. Their definition does not involve the choice of any Riemannian metric. In the case of the sphere \(S^n\), their trajectories are the usual circles (intersections of \(S^n\) with planes at distance smaller than one from the origin). See the list of all self dual symmetric spaces at the end of \[1\] and detailed examples in Subsection 2.2 of that article and Section 4 below.

In Theorem 1 we characterize the elements of the big transformation group of a self dual symmetric R-space M as those diffeomorphisms of M sending circles in circles. Theorem 2 gives some geometrical properties of circles. Although circles in a self dual symmetric R-space M belong to the realm of the invariants of the big transformation group of M, we manage to describe them in Riemannian geometric terms: Given a circle \(c\) in M, there is a maximal compact subgroup \(K\) of \(G\) such that, except for a projective reparametrization, \(c\) is a diametrical geodesic in M (or equivalently, a diagonal geodesic in a maximal torus of M), provided that M carries the canonical symmetric \(K\)-invariant metric.

In \[12\], Langevin and O’Hara presented a new approach to the classical subject of conformal length of curves in the sphere, in terms of the canonical pseudo Riemannian structure on the space \(C\) of oriented circles: The circles osculating a curve \(\alpha\) in the sphere define a null curve in \(C\), whose \(\frac{1}{2}\)-dimensional length provides, generically, a conformally invariant parametrization of \(\alpha\). The curves we deal with in this article seem to be adequate to generalize that result to self dual symmetric R-spaces. We are currently working on the subject.

We adopt the approach of \[1\], only taking for granted the action of the big transformation group of M. We emphasize that no Riemannian metric on M is fixed from the beginning, in contrast to \[17\, 6\]. Since our proofs are based on their results, compatible symmetric Riemannian metrics are necessarily auxiliary constituents, even if they are not involved in some statements.

Next we recall from \[1\] (Sections 1 and 2 and Subsection 4.1) the definitions and basic properties of symmetric R-spaces, self duality and circles.

### 1.1 Self dual symmetric R-spaces

Let \(\mathfrak{g}\) be a real simple Lie algebra. Let \(\text{Inn}(\mathfrak{g})\) be the unique connected Lie subgroup of \(GL(\mathfrak{g})\) whose Lie algebra is \(\text{ad}(\mathfrak{g}) = \{\text{ad}_x \mid x \in \mathfrak{g}\} \subset gl(\mathfrak{g})\). For \(g \in \text{Inn}(\mathfrak{g})\), let \(I_g\) be the automorphism of \(\text{Inn}(\mathfrak{g})\) defined as usual by \(I_g(h) = ghg^{-1}\), satisfying \(\text{Ad}(g) = (dI_g)_e : \text{ad}(\mathfrak{g}) \to \text{ad}(\mathfrak{g})\). We may think of \(\text{Inn}(\mathfrak{g})\) as the adjoint group of \(\mathfrak{g}\), since \(\text{ad} : \mathfrak{g} \to \text{ad}(\mathfrak{g})\) is an isomorphism (\(\mathfrak{g}\) is simple).

**Definitions.** Let \(\mathfrak{g}\) be a simple real Lie algebra. A subalgebra \(\mathfrak{p}\) of \(\mathfrak{g}\) is said to be **height one parabolic** if the polar \(\mathfrak{p}^+ = \{x \in \mathfrak{g} \mid B(x,y) = 0 \text{ for all } y \in \mathfrak{p}\}\) with respect to the Killing form \(B\) of \(\mathfrak{g}\) is an abelian subalgebra of \(\mathfrak{g}\). Notice that \(\mathfrak{p}^+ \subset \mathfrak{p}\).

A **symmetric R-space** is a conjugacy class of height one parabolic subalgebras.
of $\mathfrak{g}$, that is, the orbit of a height one parabolic subalgebra of $\mathfrak{g}$ under the action of $\text{Inn}(\mathfrak{g})$.

Two height one parabolic subalgebras $p, q$ of $\mathfrak{g}$ are opposite if $p^\perp \cap q^\perp = \{0\}$, or equivalently, if $\mathfrak{g} = p \oplus q^\perp$. We adopt the nomenclature of [4]; in [1] they are said to be complementary.

The dual $M^*$ of a symmetric $R$-space $M$ is defined to be the set of parabolic subalgebras $q$ of $\mathfrak{g}$ for which $q^\perp$ is opposite to some $p \in M$. It turns out to be a symmetric $R$-space as well. One says that $M$ is self-dual if $M^* = M$, that is, if for any $p \in M$, any height one parabolic subalgebra $q$ of $\mathfrak{g}$ opposite to $p$ is in $M$. Equivalently, if for some $p \in M$, some height one parabolic subalgebra $q$ of $\mathfrak{g}$ opposite to $p$ is conjugate to $p$.

1.2 Circles in self dual symmetric $R$-spaces

Let $q$ be a height one parabolic subalgebra of a simple real Lie algebra $\mathfrak{g}$. Then $\exp|_{q^\perp} : q^\perp \to \exp(q^\perp)$ is a global diffeomorphism onto the abelian group $\exp(q^\perp)$. Let $M$ be a symmetric $R$-space and let $q \in M$. The set

$$\Omega_q = \{p \in M \mid p \text{ is opposite to } q\}$$

is open and dense in $M$. Moreover, $\exp(q^\perp)$ acts simply transitively on it. Hence, given $p \in \Omega_q$, the map

$$q^\perp \to \Omega_q, \quad x \mapsto (\exp x) \cdot p$$

is a diffeomorphism. It is called inverse-stereopjection with respect to $(p, q)$. This induces a linear isomorphism

$$\pi^p_q : q^\perp \to T_p M, \quad \pi^p_q(x) = \frac{d}{dt} \bigg|_0 \exp(tx) \cdot p$$

for any $p \in \Omega_q$. This identification remained mostly implicit in [6]. We make it explicit for the sake of clarity of the exposition.

**Definition.** Let $M$ be a self dual symmetric $R$-space. Let $p, p_1$ and $q$ be three points in $M$ which are pairwise opposite. Since the map in (1) is a diffeomorphism, then there exists a unique $y \in q^\perp$ such that $p_1 = \exp(y) \cdot p$. The curve $c : \mathbb{R} \cup \{\infty\} \to M$

$$c(t) = \exp(ty) \cdot p, \quad c(\infty) = q$$

is called the circle in $M$ through $p, p_1, q$.

The image of $c$ is in fact a circle. Moreover, if the circles determined by two triples of (appropriately ordered) pairwise opposite points have the same trajectories, then their parametrizations differ in a fractional linear transformation. See the precise assertion in Proposition 4.4 of [1].
1.3 The big transformation group

Now we recall from [6] the definition of a Lie group $G$ acting on $M$ which is in general bigger than Inn $(\mathfrak{g})$. For instance, for the conformal structure of $S^n$, $G$ will be the group $O_+(1, n+1)$ of all conformal maps, not only the group Inn $(o(1, n+1)) = O_0(1, n+1)$ consisting of those conformal maps preserving the orientation.

Let $M$ be a self dual symmetric $R$-space. Given two opposite elements $p, q$ in $M$, we consider the group

$$G_0(p, q) = \{g \in \text{Aut } (\mathfrak{g}) | g(p) = p \text{ and } g(q) = q\}$$

and call

$$G = G_0(p, q) \text{ Inn } (\mathfrak{g}),$$

which is a subgroup of Aut $(\mathfrak{g})$ with the same Lie algebra ad $(\mathfrak{g})$ as Inn $(\mathfrak{g})$. This follows from the fact that if $h \in G_0(p, q)$ and $g \in \text{Inn } (\mathfrak{g})$, then $hgh^{-1} \in \text{Inn } (\mathfrak{g})$.

By Proposition 2.5 in [1], the group Inn $(\mathfrak{g})$ acts transitively on the pairs of opposite elements of $M$. Hence, $G_0(p, q)$ and $G_0(p', q')$ are conjugate by an element of Inn $(\mathfrak{g})$. Therefore, the group $G$ depends only of the symmetric $R$-space $M$, and not on the choice of opposite $p, q$ in $M$. It is called the big transformation group of $M$.

The following theorem characterizes the big transformation group of a self dual symmetric $R$-space.

**Theorem 1** Let $M$ be a self dual symmetric $R$-space and let $G$ be its big transformation group. Then a diffeomorphism $g$ of $M$ sends circles into circles if and only if $g \in G$. More precisely, given a diffeomorphism $g$ of $M$, the following assertions are equivalent.

a) The diffeomorphism $g$ belongs to $G$.

b) For any circle $c$ in $M$, $g \circ c$ is a circle in $M$.

c) For any circle $c$ in $M$, $g \circ c$ is a reparametrization of a circle in $M$.

1.4 Diametrical geodesics of symmetric Riemannian metrics on $R$-spaces

It is well-known that any maximal compact subgroup $K$ of $G$ (they are all conjugate with each other) acts transitively on $M$ and there is a unique (up to homotheties) Riemannian metric $g$ on $M$ such the action of $K$ is by isometries of $g$. By abuse of nomenclature we refer to it as the $K$-invariant Riemannian metric on $M$. Moreover, $(M, g)$ is a compact, irreducible Riemannian symmetric space with cubic maximal tori (see [13], and also [3] for a geometrical proof of the rectangularity of maximal tori).

Let $T$ be an $r$-dimensional cubic flat torus of volume $\lambda^r$, that is, $T = \mathbb{R}^r/\lambda \mathbb{Z}^r$ endowed with the Riemannian metric such that the canonical projection $p : \mathbb{R}^r \to T$ is a local isometry. A geodesic $\gamma : \mathbb{R} \to T$ is said to be diagonal if there is an isometry $F$ of $T$ such that $\gamma (ct) = F(p(t, \ldots, t))$ for some $c \neq 0$ and all $t$.

Let $M$ be a compact Riemannian manifold with associated distance $d$. The diameter of $M$, denoted by $\text{diam } M$, is the maximum of the numbers $d(p, q)$ among all pairs of points $p, q$ in $M$. Two points $p, q \in M$ are said to be diametrical if $d(p, q) =$
diam $M$. If $M$ is symmetric, a unit speed geodesic $\gamma : \mathbb{R} \to M$ is said to be diametrical if $\gamma (t), \gamma (t + \text{diam} M)$ are diametrical points in $M$ for all $t$. A nonconstant geodesic in $M$ is said to be diametrical if some (any) unit speed reparametrization is so.

As a corollary of the main result in [16], we have that a geodesic in a symmetric $R$-space is diametrical if and only if it is a diagonal geodesic in a (cubic) maximal torus of $N$, in particular periodic with period 2 diam $M$ if it is parametrized by arc length.

From now on, we set $\tan \left( \frac{\pi}{2} + k\pi \right) = \infty$ for any $k \in \mathbb{Z}$.

**Theorem 2** Let $c : \mathbb{R} \cup \{ \infty \} \to M$ be a circle in a self dual symmetric $R$-space $M$ with big transformation group $G$. Then there exists a maximal compact subgroup $K$ of $G$ such that the curve $\gamma : \mathbb{R} \to M$ defined by

$$\gamma (t) = c (\tan (\pi t))$$

(3)

is a diametrical geodesic of $M$, provided that $M$ is endowed with the symmetric $K$-invariant Riemannian metric.

Conversely, given a diametrical geodesic $\gamma$ with period 1 of a Riemannian self dual symmetric $R$-space $M$, then the curve $c : \mathbb{R} \cup \{ \infty \} \to M$ uniquely determined by (3) is a circle in $M$.

The geodesic and the circle coincide up to a projective reparametrization in the following sense: Let $f, \varphi$ be the functions

$$\mathbb{R} \cup \{ \infty \} \xleftarrow{f} \mathbb{R}/\mathbb{Z} \xrightarrow{\varphi} S^1 \subset \mathbb{C}$$

defined by $f (t + \mathbb{Z}) = \tan (\pi t)$ and $\varphi (t) = e^{2\pi ti}$. Then $\varphi \circ f^{-1}$ is the projective map $s \mapsto (1 - s^2 + 2si) / (1 + s^2)$.

I would like to express my gratitude to Cristián U. Sánchez for introducing and promoting in Córdoba the beautiful and central concept of symmetric $R$-space.

**2 Preliminaries**

**2.1 Prevalent vectors**

We fix a self dual symmetric $R$-space $M$, the orbit of height one parabolic subalgebra of the simple Lie algebra $\mathfrak{g}$ under the group $\text{Inn} (g)$. The following definition, taken from [4], is motivated by Lemma 4.1 in [1] and Proposition 3 below is a restatement of the latter. It provides a Lie algebraic condition on a tangent vector to $M$ to be the initial velocity of a circle in $M$.

**Definition.** Let $M$ be a self dual symmetric $R$-space with big transformation group $G$. A vector $y \in \mathfrak{g}$ is said to be prevalent if

$$y \in p^1 \quad \text{and} \quad \text{Ker} \ (\text{ad}_y)^2 = p$$

(4)

for some $p \in M$. 


Remarks.  a) In the second equation the inclusion $\supset$ is always valid.

b) By Proposition 7.6 in [4], the notion of prevalence makes sense only for self dual symmetric $R$-spaces. Also, in that Ph.D. thesis, prevalent vectors are called regular vectors, but we prefer a more specific term. The importance of the concept was apparent in [1], but they did not give it a name. It was also present in [6] (the nondegeneracy of $P(X)$), but involving a Cartan involution $\tau$, which is an alien structure in our setting (cf. the proof of Proposition 6 below).

Proposition 3 Let $M$ be a self dual symmetric $R$-space with big transformation group $G$. Let $p$ and $q$ be two opposite elements of $M$ and let $y \in q^\perp$. The following assertions are equivalent:

a) The vector $y$ is prevalent.

b) The map

$$\left(\text{ad}_y\right)^2|_p : p^\perp \rightarrow q^\perp$$

is a linear isomorphism.

c) For any $t \neq 0$, $\exp (ty) \cdot p$ is an element of $M$ opposite to $p$ and $q$.

d) The curve $c : \mathbb{R} \cup \{\infty\} \rightarrow M$, $c(t) = \exp (ty) \cdot p$, $c(\infty) = q$, is the circle through $p$, $\exp (y) \cdot p$ and $q$.

Proposition 4 Given $y \in g$, there is at most one $p \in M$ satisfying condition (4).

Proof. Suppose that $q \in M$ also satisfies conditions (4). Since $q$ and $p$ have the same dimension, it suffices to check that $p \subset q$. Let $z \in p$. Since $y \in p^\perp$, $\text{ad}_y z \in [p^\perp, p] \subset p^\perp$. Hence, $\text{ad}_y \text{ad}_y z \in [p^\perp, p^\perp] = \{0\}$ ($p^\perp$ is abelian). Thus $(\text{ad}_y)^2 z = 0$ and so $z \in \text{Ker} (\text{ad}_y)^2 = q$. □

Remark. We wonder whether prevalent vectors in $g$ can be characterized as those elements of $g$ belonging to exactly one $p^\perp$ with $p \in M$.

The notion of prevalence of a vector in $g$ defines the concept of prevalent tangent vector of $M$:

Definition. Let $M$ be a self dual symmetric $R$-space. A tangent vector $X \in T_qM$ is said to be prevalent if $X = \left| \frac{d}{dt} \right|_0 e^{ty} \cdot q$ for some prevalent vector $y \in p^\perp$ with $p$ opposite to $q$.

This notion is independent of the choice of $p \in \Omega_q$. Indeed, suppose that $p' \in M$ is opposite to $q$. Then there exists a unique $z \in q^\perp$ such that $p' = e^z \cdot p$. We have that

$$e^z y = y + [z, y] + \frac{1}{2} [z, [z, y]]$$

(see the proof of Lemma 4.1 in [1]), where the second and third terms of the right hand side are in $[q^\perp, p^\perp] \subset q \cap p$ and $[q^\perp, q]$ respectively, and hence both are contained in $q$. Let $\pi : G \rightarrow M$ be the projection given by $\pi (g) = g \cdot q$, which satisfies that $\text{Ker} (d\pi_e) = q$. Hence,

$$X = d\pi_e (y) = d\pi_e \left( y + [z, y] + \frac{1}{2} [z, [z, y]] \right) = \left| \frac{d}{dt} \right|_0 \exp (te^z y) \cdot q,$$
where $e^2 y \in p'$ is prevalent, since it is conjugate to $y$.

For further reference we recall that given opposite elements $p$ and $q$ in a symmetric $R$-space $M$, there exists $z \in g$, called the characteristic element of the pair $p, q$, satisfying that $p^\perp, p \cap q$ and $q^\perp$ are the eigenspaces of the operator $\text{ad}_z$ with eigenvalues $1, 0$ and $-1$, respectively.

### 2.2 Generalized conformal structures

We collect some facts from [6] and [10] Part II on characterizations of the big transformation group of a symmetric $R$-space.

Let $V$ be a finite dimensional vector space. An algebraic subset $C$ of $V$ is called a prehomogeneous generalized cone if it is stable by homotheties and the subgroup of $GL(V)$ preserving $C$ has an open orbit. Let $M$ be a smooth manifold with the same dimension as $V$. A locally flat generalized conformal structure on $M$ with typical cone $C$ assigns to each point $q \in M$ a subset $C_q$ of $T_q M$ in such a way that for any $p \in M$ there is an open neighborhood $U$ of $p$ and a smooth map $\varphi : U \times V \to TM$ such that for each $q \in U$, $\varphi(q, \cdot)$ is a linear isomorphism from $V$ to $T_q M$ sending $C$ to $C_q$.

Given a symmetric $R$-space $M$, Gindikin and Kaneyuki define in Lemmas 3.1 and 3.2 of [6] a locally flat generalized conformal structure $K$ on $M$, where $C = \partial V$ for a certain subset $V$ of $q^\perp$ for some $q \in M$ (where $\partial$ denotes the frontier), and prove the following theorem (Theorem 3.3).

**Theorem 5** [6] The group of automorphisms of $K$ is exactly $G$, provided that the symmetric $R$-space $M$ has rank larger than one.

In the case of a symmetric $R$-space $M$ of type $C$ (or equivalently, if $M$ is self dual, see the proof of Proposition 6 below), essentially the same result has been obtained by Bertram in [2].

**Proposition 6** Let $M$ be a self dual symmetric $R$-space. Then, for all $p \in M$,

$$K_p = \partial \{ X \in T_p M \mid X \text{ is prevalent} \} .$$

**Proof.** First, we recall the definition of the generalized conformal structure $K$ on $M$: Given $p \in M$, take any $q \in M$ opposite to $p$ and define $K_p = \pi^*_q (\partial V_r)$, where $V_r$ turns out to be equal to $\{ x \in q^\perp \mid x \text{ is prevalent} \}$, provided that $M$ is self dual. More precisely, let $z$ be the characteristic element of the pair $p, q$ and define $P : q^\perp \to \text{End} (q^\perp)$ as in 2.7 of [6] by

$$P (x) (y) = \frac{1}{2} \left[ [\tau y, x], x \right] = \frac{1}{2} (\text{ad}_x)^2 (\tau y),$$

where $\tau$ is a Cartan involution of $g$ satisfying $\tau (z) = -z$.

Now, comparing the table of self dual symmetric $R$-spaces at the end of [11] with Table 2 (page 601) in [9] (or Table 5 on page 135 of [10] Part II), we see that a symmetric $R$-space is self dual if and only if certain root system $\Delta (g, c)$ is of type $C$ (there should be a better reason for this to be true). In this case, one has by (2.24) in [6] that

$$V_r = \{ x \in q^\perp \mid \det P (x) \neq 0 \} .$$

(6)
By the definition of prevalent tangent vectors in $T_pM$, it suffices to check that $V_x$ consists exactly of the prevalent vectors in $q^\perp$. Indeed, suppose that $x \in V_x$, that is, $(\text{ad}_x)^2 \circ \tau : q^\perp \to q^\perp$ is a linear isomorphism. But since $\tau_{q^\perp} : q^\perp \to p^\perp$ is an isomorphism, then $(\text{ad}_x)^2 : p^\perp \to q^\perp$ also is an isomorphism. Thus, $x$ is prevalent. The other inclusion holds by similar arguments.

3 Proofs of the theorems

Proof of Theorem 1. First we check that $a) \Rightarrow b)$. Let $p, p_1$ and $q$ be three points in $M$ which are pairwise opposite and let $c : \mathbb{R} \cup \{\infty\} \to M$ be the circle through $p, p_1, q$, that is, $c(t) = e^{ty} \cdot p$ and $c(\infty) = q$ for a unique $y \in q^\perp$. Let $g \in G$. Since $g$ is an automorphism of $g$, then $g \cdot p, g \cdot p_1, g \cdot q$ are pairwise opposite in $M$ and $g$ also preserves the Killing form; in particular, $g \cdot y \in g \cdot q^\perp = (g \cdot q)^\perp$. Hence,

$$g(c(t)) = ge^{ty} \cdot p = e^{ty} \cdot gp, \quad g(\infty) = g \cdot q$$

is the circle through $g \cdot p, g \cdot p_1, g \cdot q$.

Since the implication $b) \Rightarrow c)$ is obvious, it remains only to prove $c) \Rightarrow a)$. Suppose that $g$ is a diffeomorphism of $M$ satisfying $c)$. For the sphere $S^n$ we know from [8] that the assertion is true (see also Theorem 3 in [5], a much more general result). The projective spaces $P^n(\mathbb{F})$, where $\mathbb{F} = \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ ($n = 2$ in the last case), are not self dual (see the table at the end of [11]). Thus, we may suppose that $\text{rank}(M) > 1$. By Theorem 5 it suffices to prove that $g$ is an automorphism of the generalized conformal structure $\mathcal{K}$.

Given $p \in M$, we must show that $dg_p(\mathcal{K}_p) = \mathcal{K}_{g(p)}$. Since $dg_p$ is a linear isomorphism, by Proposition 6 it is equivalent to show that $dg_p(X)$ is prevalent for any prevalent tangent vector $X \in T_pM$. Suppose $X = \pi_q^p(x)$ for some prevalent vector $x \in q^\perp$. By Proposition 3 the curve $c : \mathbb{R} \cup \{\infty\} \to M$, $c(t) = \exp(tx) \cdot p$, $c(\infty) = q$ is a circle in $M$. Now, by the hypothesis, $g \circ c = \bar{c} \circ \phi$ for some circle $\bar{c}$ in $M$ and some diffeomorphism $\phi$ of $\mathbb{R} \cup \{\infty\}$. Suppose that $\bar{c}$ is the circle through $\bar{p}, \bar{p}_1$ and $\bar{q}$, that is, $\bar{c}(t) = \exp((\bar{\pi}) \cdot \bar{p}$ for some $\bar{\pi} \in q^\perp$ with $\bar{q}$ opposite to $\bar{p}$ and $\bar{p}_1$. In particular, $\bar{\pi}$ is prevalent by Proposition 3. We call $t_o = \phi(0)$ and assuming that $t_o \neq \infty$, we compute

$$dg_p(X) = (g \circ c)'(0) = (\bar{c} \circ \phi)'(0) = \bar{c}'(t_o) \phi'(0),$$

with $\phi'(0) \neq 0$. Now, calling $h = \exp(t_o \bar{\pi})$, we have that

$$\bar{c}'(t_o) = \frac{d}{ds} \bigg|_{s=0} \bar{c}(t_o + s) = \frac{d}{ds} \bigg|_{s=0} \exp(t_o \bar{\pi}) \exp(s \bar{\pi}) \cdot \bar{p} = \frac{d}{ds} \bigg|_{s=0} \exp(sh \bar{\pi}) \cdot \bar{p} = \pi_{h\bar{q}}^{\bar{h}\bar{p}}(h \bar{\pi}).$$

Since $h$ is an automorphism and $\bar{\pi}$ is prevalent, we have that $dg_p(X)$ is prevalent. Therefore, $g$ is an automorphism of $\mathcal{K}$, as desired. For the case $t_o = \infty$ use a reparametrization as in the last assertion of Subsection 1.2.

Proof of Theorem 2. The general setting of the proof is from [17] and it uses results from [6, 9]. Only we should keep in mind that in contrast to those articles, no maximal compact subgroup of $G$ (or equivalently, no symmetric Riemannian metric on $M$) is fixed. The notation is a mixture of that of the different sources. Suppose
that the circle $c$ is given by $c(0) = p$, $c(\infty) = q$ and $c(t) = \exp (tx) \cdot p$ for all $t$, with $x \in q^\perp$; in particular, $x$ is prevalent.

We consider the decomposition $g = p^+ \oplus g_0 \oplus q^\perp$, with $g_0 = p \cap q$, which turns out to be the Lie algebra of $G_0 (p, q)$. Let $z \in g_0$ be the associated characteristic element. By Theorem I.2.3 in [10] Part II there exists a Cartan involution $\tau$ on $g$ such that $\tau (p^+) = q^\perp$ and $\tau (g_0) = g_0$; in particular, $\tau (z) = -z$ (see (1.4) in [17]). Let $g = \mathfrak{k} \oplus \mathfrak{r}$ be the decomposition associated with $\tau$, that is, $\mathfrak{k}$ and $\mathfrak{r}$ are the eigenspaces of $\tau$ with eigenvalues $1$ and $-1$. Define

$$\overline{K} = \{ g \in G \mid g\tau = \tau g \},$$

which is a maximal compact subgroup of $G$, with Lie algebra $\mathfrak{k}$, acting transitively on $M$, with isotropy subgroup $K_0 = \overline{K} \cap G_0 (p, q)$ at $p$. Thus, one can identify $M = \overline{K}/K_0$.

Let $a$ be a maximal abelian subspace of $\mathfrak{r}$ containing $z$. Let $\Sigma = \Delta (g, a)$ be the irreducible root system of $g$ relative to $a$ and $\Sigma_i = \{ \gamma \in \Sigma \mid B (\gamma, z) = i \}$ for $i = 0, \pm 1$, where $B$ denotes the Killing form of $g$.

Choose a linear order in $\Sigma$ such that the set of positive roots $\Sigma^+ \subset \Sigma_0 \cup \Sigma_1$ and a maximal system of strongly orthogonal roots $\{ \beta_1, \ldots , \beta_r \} \subset \Sigma_1$ such that $\beta_1$ is the highest root in $\Sigma$. Let $g^a$ be the root space for the root $\alpha \in \Sigma$. For each $i = 1, \ldots , r$ choose a vector $X_i \in g^\beta_i \subset p^+$ with $|X_i|^2 = 2/B (\beta_i, \beta_i)$ and set $X_i = \tau (X_i) \in g_{-\beta} \subset q^\perp$. Then one has $[X_{\beta_i}, X_{\beta_j}] = -2\beta_i/B (\beta_i, \beta_i)$.

Now, $\mathfrak{k}$ decomposes as $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{m}_t$, where $\mathfrak{k}_0 = g_0 \cap \mathfrak{k}$ and $\mathfrak{m}_t = (p^+ \oplus q^\perp) \cap \mathfrak{k}$. Via the differential of the canonical projection $\overline{K} \rightarrow \overline{K}/K_0$ one identifies $T_0 M = \mathfrak{m}_t$. Let $A_i = \pi (X_i + X_{-i}) \in \mathfrak{m}_t$. By (1.21) in [17], $A_1, \ldots , A_r$ is a basis of a maximal abelian subspace of $\mathfrak{m}_t$ and moreover its $\mathbb{Z}$-span is the lattice of a cubic maximal torus $T$ in $M$. Let $A = \sum_{i=1}^r A_i$. Then $\gamma (t) = \exp (tA) \overline{K}_0$ is a diagonal geodesic in $T$, and hence diametrical in $M$.

From the proof of Lemma 2.3 in [17] we have that

$$\exp \left( \sum_{i=1}^r x_i X_{-i} \right) \cdot p = \exp \left( \frac{1}{\pi} \sum_{i=1}^r \arctan (x_i) A_i \right) \cdot p$$

for any $x_1, \ldots , x_n \in \mathbb{R}$. In particular, taking $x_i = \tan (\pi t)$ for all $i$ and calling $\overline{X} = \sum_{i=1}^r X_{-i} \in q^\perp$, we have

$$\gamma (t) = \tilde{c} (\tan (\pi t)),$$

where $\tilde{c} (s) = \exp (s \overline{X}) \cdot p$. Now, $\overline{X} = \pm O_{r,0}$ in the notation of (1.17) in [6] and by Theorem 2.9 there, it belongs to $V_r$. Hence, by the proof of Proposition [6] $\overline{X}$ is prevalent and so $\tilde{c}$ is a circle in $M$ by Proposition [3].

On the other hand, by [6], the $G_0 (p, q)$-orbit of any element in $V_r$ is the whole $V_r$ (actually, Kanayuki studies the finer $G_0^a (p, q)$-orbit structure, where the superscript $0$ refers to the identity component, but we do not need that). Hence, there is $h \in G_0 (p, q)$ such that $h \overline{X} = X$. Let us show that $K = \overline{\mathfrak{h}} \overline{K} h$ is a maximal compact subgroup of $G$ satisfying the required property. Indeed, since $h \mathfrak{p} = \mathfrak{p}$, we have that

$$h \tilde{c} (s) = h \exp (s \overline{X}) \cdot p = h \exp (s \overline{X}) h^{-1} h \cdot p = \exp (s h \overline{X}) \cdot p = \exp (s \overline{X}) \cdot p = c (s)$$
for all $s$. Hence
\[ \gamma(t) = c(\tan(\pi t)) = h\bar{c}(\tan(\pi t)) = h\bar{\gamma}(t) \]
holds for all $t$. Call $\bar{g}$ the $\mathbf{K}$-invariant Riemannian metric on $M$. Hence $g = h^*\bar{g}$ is a metric in $M$ which is $K$-invariant. Since $h : (M, \bar{g}) \to (M, g)$ is an isometry and $\bar{\gamma}$ is a diametrical geodesic of $(M, \bar{g})$, then the same is true for $\gamma = h \circ \bar{\gamma}$ in $(M, g)$, as desired.

Now we deal with the converse. The fact that $M$ has a fixed Riemannian structure of that type is equivalent to fixing a subgroup $\mathbf{K}$ as in (7). One follows the arguments above until the conclusion that $\bar{c}$ is a circle (since the maximal torus is cubic, one can consider any diagonal).

\[ \square \]

4 Examples

The examples below complement those in Subsection 2.2 of [1] or approach them from a more concrete (but also quite general) perspective; for instance, the seven families of split isotropic Grassmannians and their circles are treated in a uniform way. We include the circles of the Grassmannian of oriented planes (aka complex quadric), which is a self dual symmetric $\mathbb{R}$-space not described in detail in [1] and is very different from the split standard or isotropic Grassmannians (see Subsection 4.5 below).

In each of the following examples the smooth manifold $M$ is acted on transitively by a semisimple Lie group $H$. Each point $p \in M$ is identified with its infinitesimal stabilizer $\mathfrak{p}$, that is, the Lie algebra of the isotropy subgroup of $H$ at $p$. If $H$ is simple, then $M$ is a symmetric $\mathbb{R}$-space, $H$ coincides with the big transformation group $G$ of $M$ up to coverings and connected components and $\mathfrak{p}$ is a height one parabolic subalgebra of $\mathfrak{g} = \text{Lie}(G) = \text{Lie}(H)$.

4.1 The sphere

The group $H = O_o(1, n + 1)$ acts on the sphere $S^n$ by orientation preserving conformal diffeomorphisms. Let $q$ be the infinitesimal stabilizer of a point $q \in S^n$ and let $F_q : S^n \to q^\perp \cup \{\infty\}$ be the standard stereographic projection with $F_q(q) = \infty$. Then, after conjugation by $F_q$, exp $\mathfrak{q}^+$ consists of the diffeomorphisms of $q^\perp \cup \{\infty\}$ fixing $\infty$ and acting on $q^\perp$ by translations. Two points in $S^n$ are opposite if and only if they are different. The trajectories of the circles are the usual circles: intersections of $S^n$ with planes at distance smaller than one from the origin. The circle $c$ determined by three distinct points $p, p_1$ and $q$ is given by $F_q(c(t)) = (1 - t)F_q(p) + tF_q(p_1)$ and $c(\infty) = q$.

4.2 The split standard Grassmannians

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (where the skew field $\mathbb{H}$ of the quaternions acts on $\mathbb{H}^m$ on the right). The group $H = SL(m, \mathbb{F})$ acts in a canonical way on the Grassmannian $M$ of $k$-dimensional subspaces of $\mathbb{F}^m$, which is a symmetric $\mathbb{R}$-space. A subspace $P$ is identified with the subalgebra $\mathfrak{p}$ of $\mathfrak{sl}(m, \mathbb{F})$ consisting of all the $\mathbb{F}$-linear transformations of $\mathbb{F}^m$ preserving $P$ and $\mathfrak{p}^\perp$ consists of the $\mathbb{F}$-linear transformations vanishing on $P$ with image contained in $P$. Two subspaces $P, Q \in M$ are opposite if and only
if $P \cap Q = \{0\}$. Now, $M$ is self dual if and only if $m = 2n$. In this case, which is the only we consider henceforth, circles in $M$ have the form
\[
c(t) = \{x + tT(x) | x \in P\}, \quad c(\infty) = Q,
\]
for some complementary subspaces $P, Q$ of the same dimension in $M$ and some $\mathbb{F}$-linear isomorphism $T : P \to Q$. Let $\{u_1, \ldots, u_n\}$ be an $\mathbb{F}$-basis of $P$ and let $g$ be the $\mathbb{F}$-Hermitian symmetric (positive definite) inner product on $\mathbb{F}^{2n}$ such that $\{u_1, \ldots, u_n, Tu_1, \ldots, Tu_n\}$ is orthonormal. Then $K = \text{Aut}_0 (g)$ is a maximal compact subgroup of $H$ and $\gamma : \mathbb{R} \to M$ defined by
\[
\gamma (s) = \text{span} \ \{\cos (\pi s) u_i + \sin (\pi s) T(u_i) | i = 1, \ldots, n\}
\]
is a geodesic in $M$ provided that it carries the (unique up to homotheties) $K$-invariant Riemannian metric. Clearly, (3) is satisfied.

### 4.3 The split isotropic Grassmannians associated with the seven split inner products

We recall some basic facts about bilinear and sesquilinear forms from Thorbergsson [18] (see also [7, 15]). Let $V$ denote a right vector space over $\mathbb{F}$ where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\sigma : \mathbb{F} \to \mathbb{F}$ be either the identity or conjugation in $\mathbb{F}$, denoted as usual $\text{conj} (\alpha) = \bar{\alpha}$ (in particular, $\sigma$ is the identity if $\mathbb{F} = \mathbb{R}$). We recall that $\overline{\overline{w}} = \overline{w}$ holds in $\mathbb{H}$.

Let $f : V \times V \to \mathbb{F}$ be a map that is additive in both arguments and non-degenerate, and let $\varepsilon$ be either 1 or $-1$. The map $f$ is said to be an $\varepsilon$-symmetric $\sigma$-sesquilinear form on $V$, or briefly a $(\sigma, \varepsilon)$-form on $V$, if
\[
f(x\alpha, y\beta) = \sigma (\alpha) f(x, y)\beta \quad \text{and} \quad f(x, y) = \varepsilon \sigma (f(y, x))
\]
for all $x$ and $y$ in $V$ and all $\alpha$ and $\beta$ in $\mathbb{F}$. Since $\mathbb{H}$ is not commutative, $\sigma$ must be the conjugation if $\mathbb{F} = \mathbb{H}$. In the case $\mathbb{F} = \mathbb{C}$, $(\sigma, 1)$-forms differ inessentially from $(\sigma, -1)$-forms: If $f$ is a $(\sigma, -1)$-form, then $if$ is a $(\sigma, 1)$-form.

Let $f$ be a $(\sigma, \varepsilon)$-form on $V$. A subspace $W$ in $V$ is said to be totally isotropic if $f(x, y) = 0$ for all $x$ and $y$ in $W$. We will consider only split $(\sigma, \varepsilon)$-forms, that is, the dimension of any maximal totally isotropic subspace is half the dimension of $V$; in particular, the dimension of $V$ is even. There are seven non-isomorphic families of them. In the table below we recall the standard models $f : \mathbb{F}^{2n} \times \mathbb{F}^{2n} \to \mathbb{F}$. In all cases, $\mathbb{F}^n \times \{0\}$ and $\{0\} \times \mathbb{F}^n$ are examples of maximal totally isotropic subspaces.

| $\sigma$ | $\varepsilon$ | $\mathbb{F}$ | Type | $f ((x, y), (x', y'))$ | $\mathcal{U} (f)$ |
|---------|--------------|--------------|------|----------------------|------------------|
| id      | 1            | $\mathbb{C}$ | C-symmetric | $x^+y' + y^+x'$ | $O_{2n} (\mathbb{C})$ |
| id      | $-1$         | $\mathbb{R}, \mathbb{C}$ | $\mathbb{F}$-symplectic | $x^+y' - y^+x'$ | $Sp_{2n} (\mathbb{F})$ |
| conj    | 1            | $\mathbb{R}, \mathbb{C}, \mathbb{H}$ | split $\mathbb{F}$-Hermitian | $\bar{x}^+y' + \bar{y}^+x'$ | $O_{n,n}, U_{n,n}, Sp_{n,n}$ |
| conj    | $-1$         | $\mathbb{H}$ | $\mathbb{H}$-skew Hermitian | $\bar{x}^+y' - \bar{y}^+x'$ | $SO_{2n}^*$ |

If $f$ is a split $(\sigma, \varepsilon)$-form over $\mathbb{F}$, we call $M = \text{Gr}_0 (f)$ the set of all maximal isotropic subspaces of $V$ for $f$ and put $H = \mathcal{U} (f)$, the Lie group of automorphisms of $f$, which acts transitively on $\text{Gr}_0 (f)$. 

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11
All the split isotropic Grassmannians $\text{Gr}_0 (f)$ are self dual, except in the case of a split real Hermitian (o equivalent, split real symmetric) form on $\mathbb{R}^{2n}$ with $n$ odd (see the table at the end of [11]). A maximal isotropic subspace $P$ is identified with the Lie algebra $\mathfrak{p}$ of the parabolic subgroup of $\mathcal{U}(f)$ fixing $P$ as a subspace of $V$. Two parabolic subalgebras $\mathfrak{p}, \mathfrak{q}$ in $M$ are opposite if and only if the corresponding maximal isotropic subspaces $P, Q$ are complementary.

One says that an additive map $S : V \to V$ is id-linear if it is $\mathbb{F}$-linear and conj-linear (also called $\mathbb{F}$-antilinear) if $S(ua) = S(u) \bar{\alpha}$ for any $u \in V$ and $\alpha \in \mathbb{F}$. Antilinear maps on linear spaces over $\mathbb{H}$ do not exist.

**Definition.** Let $f : V \times V \to \mathbb{F}$ be a split $(\sigma, \varepsilon)$-form over $\mathbb{F}$. A $\bar{\sigma}$-linear map $J : V \to V$ is said to be compactly adapted to $f$ if

$$J^2 = \varepsilon \text{id} \quad \text{and} \quad f(Jx, Jy) = \sigma f(x, y)$$

for all $x, y \in V$ and the symmetric Hermitian form $D : V \times V \to \mathbb{F}$ given by $D(x, y) = f(Jx, y)$ is (positive or negative) definite.

In fact, $D$ is a symmetric Hermitian form: We have that

$$D(x\alpha, y\beta) = f(J(x\alpha), y\beta) = f(J(x) \sigma(\bar{\alpha}), y\beta) = \sigma^2(\bar{\alpha}) f(Jx, y) \beta = \bar{\alpha}D(x, y) \beta$$

and similarly, $D(y, x) = \overline{D(x, y)}$ for all $x, y \in V$.

The following result of Ju. Neretin will be helpful to find an appropriate symmetric Riemannian metric on $\text{Gr}_0 (f)$ such that a given circle is a diametrical geodesic, up to projective reparametrizations.

**Lemma 7** [15] Let $f : V \times V \to \mathbb{F}$ be a split $(\sigma, \varepsilon)$-form over $\mathbb{F}$. If $J : V \to V$ is compactly adapted to $f$, then $\mathcal{U}^J(f) = \{ A \in \mathcal{U}(f) \mid A \circ J = J \circ A \}$ coincides with $\mathcal{U}(f) \cap \mathcal{U}(D)$ and is a maximal compact subgroup of $\mathcal{U}(f)$.

**Lemma 8** a) If $P$ and $Q$ are complementary maximal isotropic subspaces of $V$ and $\mathcal{B} = \{u_1, ..., u_n\}$ is a basis of $P$ as an $\mathbb{F}$-space, then there exists a basis $\mathcal{B}' = \{v_1, ..., v_n\}$ of $Q$, called a dual basis of $\mathcal{B}$ with respect to $f$, such that $f(u_i, v_j) = \delta_{ij}$ for all $i, j = 1, ..., n$.

b) Given bases $\mathcal{B}$ and $\mathcal{B}'$ of $V$ as above, the $\bar{\sigma}$-linear map $J$ such that $Ju_i = v_i$ and $Jv_i = \varepsilon u_i$ is compactly adapted to $f$.

**Proof.** The first assertion is well-known. Regarding the second statement, straightforward computations show that (8) holds for all $x, y \in V$. We check that the associated form $D$ is $\varepsilon$-definite. Indeed,

$$D(u_i, u_j) = f(Ju_i, u_j) = f(v_i, u_j) = \varepsilon \sigma(f(u_j, v_i)) = \varepsilon \delta_{ij},$$

and in the same manner, $D(v_i, v_j) = \varepsilon \delta_{ij}$. □

**Theorem 9** Let $f$ be a split $(\sigma, \varepsilon)$-form on a vector space $V$ over $\mathbb{F}$. Let $P$ and $Q$ be two complementary maximal isotropic subspaces of $V$ and let $T : P \to Q$ be an $\mathbb{F}$-linear isomorphism satisfying

$$f(Tx, y) + f(x, Ty) = 0$$

(9)
for all $x, y \in P$. Then $c : \mathbb{R} \cup \{\infty\} \to \text{Gr}_0 (f)$ defined by
\[
c(t) = \{x + tT(x) \mid x \in P\}, \quad c(\infty) = Q,
\]
is a circle in the split isotropic Grassmannian $\text{Gr}_0 (f)$, and all circles have this form.

Moreover, there exist a basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ of $P$ and a map $J : V \to V$ compactly adapted to $f$ such that
\[
\gamma(s) = \text{span} \ \{\cos s \ u_i + \sin s \ T(u_i) \mid i = 1, \ldots, n\}
\]
is a geodesic in $\text{Gr}_0 (f)$ satisfying $\gamma(s) = c(\tan s)$ for all $s \in \mathbb{R} \cup \{\infty\}$, provided that the metric on $\text{Gr}_0 (f)$ is invariant by the maximal compact subgroup $U^J (f)$ of $U (f)$. The relationship among $T, \mathcal{B}$ and $J$ is made explicit in the proof.

**Proof.** The fact that circles have the form (10) is analogous to the similar assertion for the split standard Grassmannians. Notice that $c(t)$ is isotropic if and only if $f(x + tT(x), y + tT(y)) = 0$ for all $x, y \in P$, and this is equivalent to (9), since $P$ and $Q$ are isotropic.

Let $h : P \times P \to \mathbb{F}$ be defined by $h(x, y) = f(x, Ty)$. We compute
\[
h(x, y) = f(x, Ty) = -f(Tx, y) = -\varepsilon \sigma f(y, Tx) = -\varepsilon \sigma h(y, x),
\]
which is nondegenerate since $T$ is nonsingular.

We consider first the case $\varepsilon = -1$. Then $h$ is bilinear symmetric or Hermitian symmetric. By the basis theorem [7], there exists a basis $\mathcal{B} = \{u_1, \ldots, u_n\}$ of $P$ such that the matrix of $h$ with respect to $\mathcal{B}$ is $\text{diag} (I_k, -I_{n-k})$ for some $k \in \{0, 1, \ldots, n\}$ ($I_\ell$ is the $\ell \times \ell$ identity matrix). We call $v_i = \rho_iTu_i$, where $\rho_i = 1$ for $i = 1, \ldots, k$ and $-1$ otherwise. We compute
\[
f(u_i, v_j) = f(u_i, \rho_jTu_j) = \rho_j h(u_i, u_j) = (\rho_j)^2 \delta_{ij} = \delta_{ij}.
\]
Hence $\{v_1, \ldots, v_n\}$ is a dual basis of $\{u_1, \ldots, u_n\}$ with respect to $f$.

Let $J : V \to V$ be the unique $\bar{\sigma}$-linear map such that $Ju_i = v_i$ and $Jv_i = \varepsilon u_i = -u_i$ for all $i = 1, \ldots, n$. By Lemma [8]b), $J$ is compactly adapted to $f$ and so $K = \text{def} \ U^J (f)$ is a maximal compact subgroup of $U (f)$ by Lemma [7]

Now, let $S : V \to V$ be the $\mathbb{F}$-linear map defined by
\[
S|_P = T \quad \text{and} \quad S|_Q = -T^{-1}.
\]
In particular, $Su_i = Tu_i = \rho_i v_i$ and $Sv_i = -T^{-1}v_i = -\rho_i u_i$. Let us see that $S \in \text{Lie} \ U^J (f)$. Indeed,
\[
JSu_i = J(\rho_i v_i) = -\rho_i u_i = Sv_i = SJu_i,
\]
and one checks similarly that $JSv_i = SJv_i$ and that $f$ is skew symmetric with respect to $f$.

Let $L$ be the isotropy subgroup of $K$ at $P$. Its Lie algebra $\mathfrak{l}$ consists of the linear maps preserving $P$ and $Q$. Since $S$ interchanges $P$ and $Q$, $\text{tr}(SU) = 0$ for all $U \in \mathfrak{l}$. This means that $S$ belongs to the orthogonal complement of $\mathfrak{l}$ with
respect to the Killing form. Since \((K, L)\) is a symmetric pair, it is well-known that then \(\gamma(s) = \exp\left(sS\right) L\) is a geodesic in \(K/L\), provided that the latter carries a \(K\)-invariant Riemannian metric. We have that

\[
\exp\left(sS\right) (u_i) = \cos s u_i + \sin s T(u_i),
\]

\[
\exp\left(sS\right) (T(u_i)) = -\sin s u_i + \cos s T(u_i).
\]

Therefore, after the canonical identification of \(K/L\) with \(\text{Gr}_0(f)\), we have that

\[\gamma(s) = \exp\left(sS\right) P = \text{span} \left\{\cos s u_i + \sin s T(u_i) \mid i = 1, \ldots, n\right\} = c(\tan s).\]

Next we consider the remaining case \(\varepsilon = 1\). Then \(h\) is bilinear skew symmetric or Hermitian skew symmetric. By the basis theorem, \(n\) is even, say \(n = 2k\), and there exists a basis \(B = \{u_1, \ldots, u_n\}\) of \(P\) such that the matrix of \(h\) with respect to that basis satisfies \(h(u_i, u_{k+i}) = -h(u_{k+i}, u_k) = 1\) for \(i = 1, \ldots, k\) and the other coefficients are zero. Calling

\[v_i = T(u_{k+i}) \quad \text{and} \quad v_{k+i} = -T(u_i)\]

for \(i = 1, \ldots, k\), a straightforward computation yields that \(\{v_1, \ldots, v_n\}\) is a dual basis of \(B\) with respect to \(f\). Define \(S\) as above in (11) and \(J : V \rightarrow V\) as the unique \(\bar{\sigma}\)-linear map such that \(Ju_i = v_i\) and \(Jv_i = u_i = \varepsilon u_i\) for all \(i = 1, \ldots, n\). Mutatis mutandis, all arguments for the case \(\varepsilon = -1\) apply. \(\square\)

### 4.4 The compact classical Lie groups \(SO_{2m}, U_n\) and \(Sp_n\)

It is well known that these Lie groups \(K\) are, only in another guise, some connected components of the split isotropic Grassmannians for a split \(\mathbb{F}\)-Hermitian form \(f\) over \(\mathbb{R}, \mathbb{C}\) and \(\mathbb{H}\), respectively. In fact, there is bijection between the group \(K\) and the corresponding Grassmannian:

\[\phi : K \rightarrow \text{Gr}_0(f), \quad \phi(A) = \text{graph}(A).\]

One can take \(H = U(f)\), that is, \(O(2m, 2m), U(n, n)\) and \(Sp(n, n)\), acting on \(K\) by birational transformations: If \(A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in H\), with \(a, b, c, d \in \mathbb{F}^{n \times n}\) \((n = 2m)\) if \(\mathbb{F} = \mathbb{R}\) and \(A \in K\), then \(A : A\) is the unique \(B \in K\) such that

\[
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ Ax \end{pmatrix} \in \text{graph}(B),
\]

for all \(x \in \mathbb{F}^n\), that is, \(B = (bI_n + dA)(aI_n + cA)^{-1}\).

It easy to check that two operators \(S, T \in K\) are opposite if and only if \(Su \neq Tu\) for any unit vector \(u \in \mathbb{F}^{2m}\). One may identify \(SO_{2m}\) with the set of positions of an extended (that is, not contained in a codimension two subspace) rigid body \(B\) in \(\mathbb{R}^{2n}\) with the center of mass at the origin. Three positions of \(B\) turn out to be pairwise opposite when passing from one to another no unit vector is fixed. In this case there is a distinguished circle of positions of \(B\) joining them.

We only recall the form of the diagonal geodesics in standard position. They are \(t \mapsto (R_t, \ldots, R_t)\) for \(SO_{2m}\) (where \(R_t\) is the rotation through angle \(t\)) and \(t \mapsto (e^{it}, \ldots, e^{it})\) for the remaining groups.
4.5 The complex quadric or Grassmannian of oriented planes

This Grassmannian is very different from the ones in the previous subsections. For instance, in contrast to them, two oriented planes which are opposite may intersect and even coincide as unoriented planes.

Let $M$ be the Grassmannian of oriented planes in $\mathbb{R}^{2+n}$. We set $N = 2 + n$ and consider on $\mathbb{C}^N$ the standard bilinear form given by $\langle (z_1, \ldots, z_N), (w_1, \ldots, w_N) \rangle = z_1^\top w_1 + \cdots + z_N^\top w_N$, which is isomorphic to the one presented in (4.3). Then $M$ can be identified with the (complex) projectivization $Q$ of the quadric $\{ w \in \mathbb{C}^N \mid \langle w, w \rangle = 0 \}$ as follows:

$$
\psi : M \to Q, \quad \psi (u \wedge v) = \mathbb{C} (u + iv),
$$

for any orthonormal set $\{u, v\}$ in $\mathbb{R}^N$ (see [11]). The group $SO(N, \mathbb{C})$ acts transitively on $Q$, in the obvious way. Let $P_X$ be the isotropy group of the complex null line $\mathbb{C}X$. Its Lie algebra is

$$
p_X = \{ T \in so(N, \mathbb{C}) \mid X \text{ is an eigenvector of } T \}
$$

and its polar is

$$
p_X^\perp = \{ T \in so(N, \mathbb{C}) \mid TX = 0 \text{ and } T (X^\perp) \subset \mathbb{C}X \},
$$

which is abelian, so $p_X$ has height one. By the list at the end of [1], $M$ is a self dual symmetric $R$-space.

For the sake of simplicity, in the following we suppose that $N > 3$. On the one hand, the arguments can be easily adapted to the case $N = 3$. On the other hand, the action of $SO(3, \mathbb{C})$ above on the Grassmannian of oriented planes in $\mathbb{R}^3$ is equivalent to the well-known action of the direct conformal group on $S^2$.

We recall from Lemma 7.124 in [7] the definition of the characteristic angles of a pair of oriented planes in $\mathbb{R}^N$.

**Proposition 10** [7]. Given $P, Q$ oriented planes in $\mathbb{R}^N$ with $N > 3$, there exist an orthonormal set $\{u_1, v_1, u_2, v_2\}$ in $\mathbb{R}^N$ and angles $\alpha, \beta$ satisfying

$$
0 \leq \alpha \leq \beta \quad \text{and} \quad \alpha + \beta \leq \pi
$$

such that $P = u_1 \wedge u_2$ and $Q = (\cos \alpha \ u_1 + \sin \alpha \ v_1) \wedge (\cos \beta \ u_2 + \sin \beta \ v_2)$.

The angles $\alpha, \beta$ are uniquely determined by $P, Q$ and are called the characteristic angles of the pair $P, Q$.

The following proposition characterizes the pairs of opposite oriented planes and implies that two opposite oriented planes need not be complementary; they may even coincide as unoriented subspaces: $u \wedge v$ is opposite to $v \wedge u$ for any orthonormal set $\{u, v\}$ in $\mathbb{R}^N$.

**Proposition 11** Given $P, Q \in M$ with $\psi (P) = \mathbb{C}X$ and $\psi (Q) = \mathbb{C}Y$ for some non-zero null vectors $X, Y \in \mathbb{C}^N$, the following assertions are equivalent:

a) $P$ and $Q$ are opposite.

b) $\langle X, Y \rangle \neq 0$.

c) The characteristic angles of the pair $P, Q$ are distinct.
Proof. a) ⇒ b) Suppose that $\langle X, Y \rangle = 0$. If $X \in \mathbb{C}Y$, then $P = Q$ and so clearly $P$ and $Q$ are not opposite. If $X$ and $Y$ are linearly independent, by Lemma 12 below one can choose an ordered basis $\{X, X', Y, Y', Z_1\}$ such that the associated Gram matrix is diag $(R, R, I_{N-4})$, where $R$ is as in (14). Let $T$ be the linear transformation on $\mathbb{C}^N$ defined by $T(X') = Y$, $T(Y) = -X$ and $T$ equal zero on the remaining elements of the basis. Then $T \neq 0$ and a straightforward computation shows that $T \in p_X^1 \cap p_Y^1$. Therefore $P$ and $Q$ are not opposite.

b) ⇒ a) Suppose that $\langle X, Y \rangle \neq 0$. Then $X, Y$ together with $X^\perp \cap Y^\perp$ span $\mathbb{C}^N$. If $T \in p_X^1 \cap p_Y^1$, by (13), $T(X) = 0$, $T(Y) = 0$ and $T(X^\perp \cap Y^\perp) \subset \mathbb{C}X \cap \mathbb{C}Y = \{0\}$. Hence $T = 0$ and so $P$ and $Q$ are opposite.

c) Suppose that the characteristic angles of $P$ and $Q$ are $\alpha, \beta$. Then $P$ and $Q$ may be written as in Proposition 10 and

$$
\langle u_1 + iu_2, \cos \alpha \ u_1 + \sin \alpha \ v_1 + i (\cos \beta \ u_2 + \sin \beta \ v_2) \rangle = \cos \alpha - \cos \beta,
$$

from which the assertion follows. \hfill \Box

Lemma 12 Let $X, Y$ be two linearly independent null vectors in $\mathbb{C}^N$ such that $\langle X, Y \rangle = 0$. Then $N > 3$ and there exist linearly independent null vectors $X', Y'$ such that Gram matrix with respect to the basis $\{X, X', Y, Y'\}$ is diag $(R, R)$ with

$$
R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Proof. Suppose first that $X^\perp \neq Y^\perp$. Since both subspaces have the same dimension $N - 1$, there exist vectors $U \in X^\perp$ and $V \in Y^\perp$ with $\langle U, Y \rangle = \langle V, X \rangle = 1$. In particular, $X, Y, U, V$ are linearly independent. Now, a straightforward computation shows that one can choose $a, b, c, d \in \mathbb{C}$ such that $X' = U + aX + bY$ and $Y' = V + cX + dY$ satisfy the required conditions.

Now suppose that $X^\perp = Y^\perp$. Let $X, Y, Z_3, \ldots, Z_N$ be a basis of $\mathbb{C}^N$, where the first $N - 1$ vectors generate $X^\perp$, and call $A$ the corresponding Gram matrix. Then the first two rows of $A$ have all their coefficients zero, except the last one, in each case. This implies that $A$ is singular. A contradiction, since the inner product is nondegenerate. \hfill \Box

Next we describe the circles $c$ in $M$ with $c(0) = e_1 \wedge e_2$ and $c(\infty) = e_2 \wedge e_1$. Notice that these oriented planes are opposite by Proposition 11 since $(e_1 + ie_2, e_2 + ie_1) = 2i \neq 0$; also, their characteristic angles are $0, \pi$.

Theorem 13 Let $P_0 = e_1 \wedge e_2$ and $Q = e_2 \wedge e_1$. An oriented plane $P_1$ is opposite to $P$ and $Q$ if and only if it has the form

$$
P_1 = (\cos \alpha \ e_1 + \sin \alpha \ u) \wedge (\cos \beta \ e_2 + \sin \beta \ v)
$$

for some orthonormal sets $\{e_1, e_2\}$ and $\{u, v\}$ with $e_1 \wedge e_2 = e_1 \wedge e_2$ and $u, v \in \{e_1, e_2\}^\perp$, and some $0 \leq \alpha < \beta$ with $\alpha + \beta < \pi$.

The circle $c$ through $P_0, P_1$ and $Q$ is given by $c(t) = (u_t \wedge v_t) / \|u_t\|^2$, with

$$
u_t = (1 + t^2C) \ e_1 + 2ta \ u \quad \text{and} \quad v_t = (1 - t^2C) \ e_2 + 2tb \ v,
$$

where $C = A_{12}$.
where
\[ a = \frac{\sin \alpha}{\cos \alpha + \cos \beta}, \quad b = \frac{\sin \beta}{\cos \alpha + \cos \beta} \quad \text{and} \quad C = \frac{\cos \alpha - \cos \beta}{\cos \alpha + \cos \beta}. \tag{16} \]

Proof. The first assertion follows from Proposition \[11\] and the fact that if \( \alpha, \beta \) are the characteristic angles of the pair \( P_0, P_1 \), then the characteristic angles of \( Q, P_1 \) are \( \alpha, \pi - \beta \). Without loss of generality we may suppose that \( e_1 = e_1 \) and \( e_2 = e_2 \).

All circles \( c \) with \( c(0) = P_0 \) and \( c(\infty) = Q \) have the form \( c(t) = \exp(tZ) \cdot P_0 \) for some \( Z \in \mathbb{R}_{e_2}^{1} \) with \( Z \) prevalent. The basis \( \mathcal{B} = \{ e_1 + ie_2, e_1 - ie_2, e_3, \ldots, e_N \} \) has Gram matrix \( \text{diag}(2, I_n) \), where \( R \) is as in \[\text{(14)}\]. By \[\text{(13)}\], the matrix of \( Z \) with respect to the basis \( \mathcal{B} \) has the form
\[ [Z]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -z^t \\ 2z & 0 & 0 \end{pmatrix} \tag{17} \]
for some \( z \in \mathbb{C}^n \). A straightforward computation shows that \( Z \) is prevalent if and only if \( \langle z, z \rangle \neq 0 \). We compute
\[ \exp(tZ)_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ -t^2z^t & 1 & -tz^t \\ 2tz & 0 & 1 \end{pmatrix}. \]
Setting \( z = xu + iyv \), with \( 0 < x < y \) (hence \( \langle z, z \rangle \neq 0 \) and so \( Z \) is prevalent), we have that
\[ \exp(tZ)(e_1 + ie_2) = (e_1 + ie_2) - t^2z^t(e_1 - ie_2) + t2z \]
\[ = ((1 + t^2)(y^2 - x^2))e_1 + t2xu + i((1 - t^2)(y^2 - x^2))e_2 + t2yv. \tag{18} \]

Suppose first that \( \alpha > 0 \). Then, by \[\text{(15)}\], for \( P_1 = \exp(Z) \cdot (e_1 + ie_2) \) it suffices that
\[ \cot \alpha = \frac{1 + (y^2 - x^2)}{2x} \quad \text{and} \quad \cot \beta = \frac{1 - (y^2 - x^2)}{2y}. \tag{19} \]
A straightforward computation shows that \( x = a, y = b \) is a solution of the system of equations (notice that \( C = b^2 - a^2 \)). The case \( \alpha = 0 \) follows from (simpler) similar arguments. \( \square \)

Proposition 14 The curve \( \gamma_o : \mathbb{R} \to M \) defined by
\[ \gamma_o(s) = e_1 \land (\cos(2\pi s) \cdot e_2 + \sin(2\pi s) \cdot e_4), \]
is a diametrical geodesic in \( M \) endowed with the standard Riemannian metric, that is, the one invariant by \( \text{SO}(N) \). It satisfies that \( \gamma_o(s) = c_o(\tan(\pi s)) \), where \( c_o : \mathbb{R} \cup \{ \infty \} \to M \) is the circle through \( e_1 \land e_2, e_1 \land e_4 \) and \( e_2 \land e_1 \), that is,
\[ c_o(t) = e_1 \land \left( \frac{1 - t^2}{1 + t^2} e_2 + \frac{2t}{1 + t^2} e_4 \right). \tag{20} \]
Proof. Consider the lattices $\Gamma = 2\pi \mathbb{Z}^2$ and $\Lambda = \pi (\mathbb{Z} (1, 1) + \mathbb{Z} (-1, 1))$, which contains $\Gamma$. Let $\phi : \mathbb{R}^2 \to M$ be defined by

$$\phi (s, t) = (\cos s \ e_1 + \sin s \ e_3) \land (\cos t \ e_2 + \sin t \ e_4),$$

which can be pushed down to the quotients as indicated in the following commutative diagram ($p_T$ and $p$ are the canonical projections). The map $\phi_T$ is a double covering of the maximal torus $T^2 = \text{def} \ \text{Image} (\phi_\Lambda)$ of $M$.

![Diagram]

Then $t \mapsto \phi (0, t) = e_1 \land (\cos t \ e_2 + \sin t \ e_4)$ is a diagonal geodesic in $T^2$, and hence also its reparametrization $\gamma_o (s) = \phi (0, 2\pi s)$. A straightforward computation shows that $\gamma_o (s) = c_o (\tan (\pi s))$. By (18), the curve $c_o$ is the circle through $e_1 \land e_2$, $e_1 \land e_4$ and $e_2 \land e_1$. □

Now, given any circle $c$ through $P_0$ and $Q$, we explicit the Riemannian metric on $M$ for which $c$ is a diametral geodesic up to a projective reparametrization.

**Proposition 15** Let $c$ be the circle through $P_0$, $P_1$ and $Q$ as in Theorem 13. Let $O \in SO(N, \mathbb{C})$ whose matrix with respect to the basis $\mathcal{B}$ is

$$\text{diag} \ (1/r, r, R_\sigma, I_{n-2}),$$

where $a + ib = r (\sinh \sigma + i \cosh \sigma)$ and

$$R_\sigma = \begin{pmatrix} \cosh \sigma & -i \sinh \sigma \\ i \sinh \sigma & \cosh \sigma \end{pmatrix}.$$

Then $\gamma (s) = c (\tan (\pi s))$ is a diagonal geodesic in a maximal torus of $M$, provided that $M$ is endowed with a Riemannian metric invariant by $O^{-1} \ SO_N \ O$.

Proof. By (18), the curve $c_o$ in (20) coincides up to the identification $\psi$ in (12) with $t \mapsto \mathbb{C} \exp (tZ) (e_1 + ie_2)$, where $Z$ is as in (17) with $z = ie_4$. On the other hand, the matrix of $W = \text{Ad} (O) (Z)$ with respect to $\mathcal{B}$ has the form (17) with $z = ae_3 + ibe_4$ and so, again by (18), $c (t)$ corresponds with $\mathbb{C} \exp (tW) (e_1 + ie_2)$ via $\psi$. Hence $c (t) = O \cdot c_o (t)$ holds for all $t$. Now, the statement follows from Proposition 14. □

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