Anomalous Diffusion in Velocity Space

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The problem of anomalous diffusion in the momentum space is considered on the basis of the appropriate probability transition function (PTF). New general equation for description of the diffusion of heavy particles in the gas of the light particles is formulated on basis of the new approach similar to one in coordinate space [1]. The obtained results permit to describe the various situations when the probability transition function (PTF) has a long tail in the momentum space. The effective friction and diffusion coefficients are found.

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I. INTRODUCTION

Interest in anomalous diffusion is conditioned by a large variety of applications: semiconductors, polymers, some granular systems, plasmas in specific conditions, various objects in biological systems, physical-chemical systems, et cetera.

The deviation from the linear in time \( < r^2(t) > \propto t \) dependence of the mean square displacement have been experimentally observed, in particular, under essentially nonequilibrium conditions or for some disordered systems. The average square separation of a pair of particles passively moving in a turbulent flow grows, according to Richardson’s law, with the third power of time [2]. For diffusion typical for glasses and related complex systems [3] the observed time dependence is slower than linear. These two types of anomalous diffusion obviously are characterized as superdiffusion \( < r^2(t) > \propto t^\alpha (\alpha > 1) \) and subdiffusion \( (\alpha < 1) \) [4]. For a description of these two diffusion regimes a number
of effective models and methods have been suggested. The continuous time random walk (CTRW) model of Scher and Montroll [5], leading to strongly subdiffusion behavior, provides a basis for understanding photoconductivity in strongly disordered and glassy semiconductors. The Levy-flight model [6], leading to superdiffusion, describes various phenomena as self-diffusion in micelle systems [7], reaction and transport in polymer systems [8] and is applicable even to the stochastic description of financial market indices [9]. For both cases the so-called fractional differential equations in coordinate and time spaces are applied as an effective approach [10].

However, recently a more general approach has been suggested in [1], [11], which avoid the fractional differentiation, reproduce the results of the standard fractional differentiation method, when the last one is applicable, and permit to describe the more complicated cases of anomalous diffusion processes. In [12] these approach has been applied also to the diffusion in the time-dependent external field.

In this paper the problem of anomalous diffusion in the momentum (velocity) space will be considered. In spite of formal similarity, diffusion in the momentum space is very different physically from the coordinate space diffusion. It is clear already because the momentum conservation, which take place in the momentum space has no analogy in the coordinate space.

The anomalous diffusion in the velocity space is weakly investigated. Some attempts to investigate the influence of the long tails of correlation functions in velocity space have been done recently by W. Ebeling and M.Yu. Romanovsky (Contr. Plasma Physics, in print, private communication). The consequent way to describe the anomalous diffusion in the velocity space is, according to our knowledge, still absent.

In Section II the diffusion equation in coordinate space for a homogeneous system is shortly reviewed. The diffusion in velocity space for the cases of normal and anomalous behavior of the probability transition function PTF is presented in Sections III, IV.

II. DIFFUSION IN THE COORDINATE SPACE ON THE BASIS OF A MASTER-TYPE EQUATION

Let us consider diffusion in coordinate space on the basis of the master equation, which describes the balance of grains coming in and out the point $r$ at the moment $t$. The structure
of this equation is formally similar to the master equation in the momentum space (see, e.g., [1],[11]). Of course, for coordinate space there is no conservation law, similar to that in momentum space:

$$\frac{df_g(r, t)}{dt} = \int d\mathbf{r}' \{ W(\mathbf{r}, \mathbf{r}') f_g(\mathbf{r}', t) - W(\mathbf{r}', \mathbf{r}) f_g(\mathbf{r}, t) \}.$$ (1)

The probability transition $W(\mathbf{r}, \mathbf{r}')$ describes the probability for a grain to transfer from the point $\mathbf{r}'$ to the point $\mathbf{r}$ per unit time. We can rewrite this equation in the coordinates $\rho = \mathbf{r}' - \mathbf{r}$ and $\mathbf{r}$ as:

$$\frac{df_g(r, t)}{dt} = \int d\rho \{ W(\rho, \mathbf{r} + \rho) f_g(\mathbf{r} + \rho, t) - W(\rho, \mathbf{r}) f_g(\mathbf{r}, t) \}.$$ (2)

Assuming that the characteristic displacements are small one may expand Eq. (2) and arrive at the Fokker-Planck form of the equation for the density distribution $f_g(\mathbf{r}, t)$

$$\frac{df_g(r, t)}{dt} = \frac{\partial}{\partial r_\alpha} \left[ A_\alpha(r) f_g(r, t) + \frac{\partial}{\partial r_\beta} (B_{\alpha\beta}(r) f_g(r, t)) \right].$$ (3)

The coefficients $A_\alpha$ and $B_{\alpha\beta}$, describing the acting force and diffusion, respectively, can be written as functionals of the PTF in the coordinate space $W$ (with the dimension $s$) in the form:

$$A_\alpha(\mathbf{r}) = \int d^s \rho \rho_\alpha W(\rho, \mathbf{r})$$ (4)

and

$$B_{\alpha\beta}(\mathbf{r}) = \frac{1}{2} \int d^s \rho \rho_\alpha \rho_\beta W(\rho, \mathbf{r}).$$ (5)

For the isotropic case the probability function depends on $\mathbf{r}$ and the modulus of $\rho$. For a homogeneous medium, when $r$-dependence of the PT is absent, the coefficients $A_\alpha = 0$ while the diffusion coefficient is constant with $B_{\alpha\beta} = \delta_{\alpha\beta} B$, where B is the integral

$$B = \frac{1}{2s} \int d^s \rho \rho^2 W(\rho).$$ (6)

This consideration cannot be applied to specific situations in which the integral in Eq. (6) is infinite. In that case we have to examine the general transport equation (1). We will now consider the problem for the homogeneous and isotropic case, when the PT function depends only on $|\rho|$. By Fourier-transformation we arrive at the following form [11] of Eq. (1):

$$\frac{df_g(k, t)}{dt} = \int d^s \rho [\exp(i\rho k) - 1] W(|\rho|) f_g(k, t) \equiv X(k) f_g(k, t),$$ (7)
where \(X(k) \equiv X(k)\). Let us assume a simple form of the PT function with a power dependence on the distance \(W(\rho) = C/|\rho|^\alpha\), where \(C\) is a constant and \(\alpha > 0\). Such type singular dependence is typical for jump diffusion probability in heteropolymers in solution (see, e.g., [13], where the different applications of anomalous diffusion are considered on the basis of the fractional differentiation method). For the one-dimensional case we find:

\[
X(k) \equiv -4\int_0^\infty du \sin^2 \left(\frac{k u}{2}\right) W(u) = -2^{3-\alpha}C|k|^{\alpha-1} \int_0^\infty d\zeta \frac{\sin^2 \zeta}{\zeta^\alpha}. \tag{8}
\]

For the values \(1 < \alpha < 3\) this function is finite and equal to

\[
X(k) = -\frac{C \Gamma((3-\alpha)/2)|k|^{\alpha-1}}{2^\alpha \sqrt{\pi} \Gamma(\alpha/2)(\alpha-1)}, \tag{9}
\]

where \(\Gamma\) is the Gamma-function. At the same time the integral in Eq. (6) for such a type of PT functions is infinite, because usual diffusion is absent.

At the same time the integral (6) for such a type of PT functions is infinite, because usual diffusion is absent. The procedure considered for the simplest cases of power dependence of the PT function is equivalent to the equation with fractional space differentiation [10],[13]:

\[
\frac{df_g(x, t)}{dt} = C \Delta^{\mu/2} f_g(x, t), \tag{10}
\]

where \(\Delta^{\mu/2}\) is a fractional Laplacian, a linear operator, whose action on the function \(f(x)\) in Fourier space is described by \(\Delta^{\mu/2} f(x) = -(k^2)^{\mu/2} f(k) = -|k|^{\mu} f(k)\). In the case considered above \(\mu \equiv (\alpha - 1)\), where \(0 < \mu < 2\). For more general PT functions, which (for arbitrary values \(\rho\)) are not proportional to the \(\alpha\) power of \(\rho\), the method described above is also applicable, although the fractional derivative does not exist.

For the case of purely power dependence of PT the non-stationary solution for the density distribution describes so-called super-diffusion (or Levy flights). The solution of Eq. (10) in Fourier space reads:

\[
f_g(k, t) = \exp(-C|k|^{\mu}t), \tag{11}\]

which in coordinate space corresponds to a so-called symmetric Levy stable distribution:

\[
f_g(x, t) = \frac{1}{(kt)^{1/\mu}} L \left[ \frac{x}{(kt)^{1/\mu}}; \mu, 0 \right]. \tag{12}\]

For the general case it follows from Eq. (7) that

\[
f_g(k, t) = C_1 \exp[X(k)t], \tag{13}\]
with some constant $C_1$.

The consideration on the basis of PTF function given above, permits to avoid the fractional differentiation method and to consider more general physical situations of the non-power probability transitions for arbitrary space dimension.

### III. DIFFUSION IN THE VELOCITY SPACE ON THE BASIS OF A MASTER-TYPE EQUATION

Let us consider now the main problem formulated in the introduction, namely, diffusion in velocity space ($V$-space) on the basis of the respective master equation, which describes the balance of grains coming in and out the point $p$ at the moment $t$. The structure of this equation is formally similar to the master equation in the coordinate space Eq. (2)

$$\frac{df_g(p, t)}{dt} = \int dq \left\{ W(q, p + q)f_g(p + q, t) - W(q, p)f_g(p, t) \right\}.$$  \hspace{1cm} (14)

Of course, for coordinate space there is no conservation law, similar to that in the momentum space. The probability transition $W(p, p')$ describes the probability for a grain with momentum $p'$ (point $p'$) to transfer from this point $p'$ to the point $p$ per unit time. The momentum transferring is equal $q = p' - p$. Assuming in the beginning that the characteristic displacements are small one may expand Eq. (2) and arrive at the Fokker-Planck form of the equation for the density distribution $f_g(p, t)$

$$\frac{df_g(p, t)}{dt} = \partial_{\alpha} A_\alpha(p, t) + \frac{1}{2} \partial_{\beta\gamma} B_{\alpha\beta}(p, t).$$ \hspace{1cm} (15)

The coefficients $A_\alpha$ and $B_{\alpha\beta}$ describing the friction force and diffusion, respectively.

Because the velocity of heavy particles is small, the $p$-dependence of the PTF can be neglected for calculation of the diffusion, which in this case is constant $B_{\alpha\beta} = \delta_{\alpha\beta} B$, where $B$ is the integral

$$B = \frac{1}{2s} \int d^s q q^2 W(q).$$ \hspace{1cm} (17)

If to neglect the $p$-dependence of the PTF at all we arrive to the coefficient $A_\alpha = 0$ (while the diffusion coefficient is constant). This neglecting, as well known is wrong, and the coefficient $A_\alpha$ for the Fokker-Planck equation can be determined by use the argument that
the stationary distribution function is Maxwellian. On this way we arrive to the standard form of the coefficient \( MT A_\alpha(p) = p_\alpha B \), which is one of the forms of Einstein relation. For the systems far from equilibrium this argument is not acceptable.

To find the coefficients in the kinetic equation, which are applicable also to slowly decreasing PT functions, let us use a more general way, based on the difference of the velocities of the light and heavy particles. For calculation of the function \( A_\alpha \) we have taken into account that the function \( W(q, p) \) is scalar and depends on \( q, q \cdot p, p \). Expanding \( W(q, p) \) on \( q \cdot p \) one arrive to the approximate representation of the functions \( W(q, p) \) and \( W(q, p + q) \):

\[
W(q, p) \simeq W(q) + \tilde{W}'(q)(q \cdot p) + \frac{1}{2} \tilde{W}''(q)(q \cdot p)^2.
\]

\[
W(q, p + q) \simeq W(q) + \tilde{W}'(q)(q \cdot p) + \frac{1}{2} \tilde{W}''(q)(q \cdot p)^2 + q^2 \tilde{W}'(q),
\]

where \( \tilde{W}'(q) \equiv \partial W(q, q \cdot p)/\partial (q p) |_{q \cdot p=0} \) and \( \tilde{W}''(q) \equiv \partial^2 W(q, q \cdot p)/\partial (q p)^2 |_{q \cdot p=0} \).

Then, with the necessary accuracy, \( A_\alpha \) equals

\[
A_\alpha(p) = \int d^4 q_\alpha q_\beta p_\beta \tilde{W}'(q) = p_\alpha \int d^4 q_\alpha q_\alpha \tilde{W}'(q) = \frac{p_\alpha}{s} \int d^4 q_\alpha q_\alpha \tilde{W}'(q)
\]

If for the function \( W(q, p) \) the equality \( \tilde{W}'(q) = W(q)/2MT \) is fulfilled, then we arrive to the usual Einstein relation

\[
MT A_\alpha(p) = p_\alpha B
\]

Let us check this relation for the Boltzmann collisions, which are described by the PT-function \( W(q, p) = w_B(q, p) \) [11]:

\[
w_B(q, p) = \frac{2\pi}{\mu^2 q} \int_{q/2\mu}^{\infty} du \frac{d\sigma}{do} \left[ \arccos \left( 1 - \frac{q^2}{2\mu^2 u^2} \right), u \right] f_b(u^2 + v^2 - q \cdot v / \mu),
\]

where \( (p = Mv) \) and \( d\sigma/do \) and \( f_b \) are respectively the differential cross-section for scattering and the distribution function for the light particles. For the equilibrium Maxwellian distribution \( f_b^0 \) the equality \( \tilde{W}'(q) = W(q)/2MT \) is evident and we arrive to the usual Fokker-Planck equation in velocity space with the constant diffusion \( D \equiv B/M^2 \) and friction \( \beta \equiv B/MT = DM/T \) coefficients, which satisfy the Einstein relation.

For some non-equilibrium situations the PTF can possess a long tail. In this case we have derive a generalization of the Fokker-Planck equation in spirit of the above consideration for the coordinate case, because the diffusion and friction coefficients in the form Eqs. (17), (20)
diverge for large \( q \) if the functions have the asymptotic behavior \( W(q) \sim 1/q^\alpha \) with \( \alpha \leq s + 2 \) and (or) \( \tilde{W}'(q) \sim 1/q^\beta \) with \( \beta \leq s + 2 \).

Let us insert in Eq. (14) the expansions for \( W \) (as an example we choose \( s = 3 \), the arbitrary \( s \) can be considered by the similar way). With necessary accuracy we find

\[
\frac{df_g(p,t)}{dt} = \int dq\{f_g(p + q,t)[1 + q_0\partial/\partial p_0][W(q) + \tilde{W}'(q)(q \cdot p) + \frac{1}{2}\tilde{W}''(q)(q \cdot p)^2] - f_g(p,t)[W(q) + \tilde{W}'(q)(q \cdot p) + \frac{1}{2}\tilde{W}''(q)(q \cdot p)^2]\}
\]

After the Fourier-transformation \( f(r) = \int \frac{dp}{(2\pi)^4} \exp(ipr)f(p,t) \) Eq. (23) reads:

\[
\frac{df_g(r,t)}{dt} = \int dq\{\exp(-iqr)[W(q) - i\tilde{W}'(q)(q \cdot \partial/\partial r) + \frac{1}{2}\tilde{W}''(q)(q \cdot \partial/\partial r)^2) - W(q) - i\tilde{W}'(q)(q \cdot \partial/\partial r) - \frac{1}{2}\tilde{W}''(q)(q \cdot \partial/\partial r)^2]\}f_g(r,t)
\]

We can rewrite this equation as

\[
\frac{df_g(r,t)}{dt} = A(r)f(r) + B_\alpha(r)\frac{\partial}{\partial r_\alpha}f(r,t) + C_{\alpha\beta}(r)\frac{\partial^2}{\partial r_\alpha \partial r_\beta}f(r,t)
\]

where

\[
A(r) = \int dq[\exp(-iqr) - 1]W(q) = 4\pi \int_0^\infty dq q^2 \left[ \frac{\sin(qr)}{qr} - 1 \right] W(q)
\]

\[
B_\alpha \equiv r_\alpha B(r); \quad B(r) = -\frac{i}{r^2} \int dqqr[\exp(-iqr) - 1]\tilde{W}'(q) = \frac{4\pi}{r^2} \int_0^\infty dq q^2 \left[ \cos(qr) - \frac{\sin(qr)}{qr} \right] W'(q)
\]

\[
C_{\alpha\beta}(r) \equiv r_\alpha r_\beta C(r) = -\frac{1}{2} \int dq dq_0 q_0[\exp(-iqr) - 1]\tilde{W}''(q)
\]

\[
C(r) = -\frac{1}{2r^4} \int dq(qr)^2[\exp(-iqr) - 1]\tilde{W}''(q) = \frac{2\pi}{r^2} \int_0^\infty dq q^4 \left[ \frac{2\sin(qr)}{q^3r^3} - \frac{2\cos(qr)}{q^3r^2} - \frac{\sin(qr)}{qr} + \frac{1}{3} \right] W''(q)
\]

For the isotropic function \( f(r) = f(r) \) one can rewrite Eq. (25) in the form

\[
\frac{df_g(r,t)}{dt} = A(r)f(r) + B(r)r\frac{\partial}{\partial r}f(r) + C(r)r^2\frac{\partial^2}{\partial r^2}f(r)
\]
For the case of strongly decreasing PDF the exponent under the integrals for the functions $A(r)$, $B(r)$ and $C(r)$ can be expanded

$$A(r) \simeq - \frac{r^2}{6} \int dq q^2 W(q); \quad B(r) \simeq - \frac{1}{6} \int dq q^2 \tilde{W}'(q); \quad C(r) \simeq 0. \quad (31)$$

Then the simplified kinetic equation for the case of short-range on $q$-variable PTF (non-equilibrium, in general case) reads

$$\frac{df_g(r,t)}{dt} = A_0 r^2 f(r) + B_0 r \frac{\partial}{\partial r} f(r), \quad (32)$$

where $A_0 \equiv -1/6 \int dq q^2 W(q)$ and $B_0 \equiv -1/6 \int dq q^2 \tilde{W}'(q)$.

Stationary solution of Eq. (30) for $C(r) = 0$ reads

$$f_g(r,t) = C \exp \left[ - \int_0^r dr' \frac{A(r')}{r'B(r')} \right] = C \exp \left[ - \frac{A_0 r^2}{2B_0} \right] \quad (33)$$

The respective normalized stationary momentum distribution equals

$$f_g(p) = \frac{N_g B^{3/2}_0}{(2\pi A_0)^{3/2}} \exp \left[ - \frac{B_0 p^2}{2A_0} \right] \quad (34)$$

Therefore in Eq. (31) the constant $C = N_g$. Equation (32) and this distribution are the generalization of the Fokker-Planck case for normal diffusion on non-equilibrium situation, when the prescribed $W(q,p)$ is determined, e.g., by some non-Maxwellian distribution of the small particles $f_b$. To show this by other way let us make the Fourier transformation of (25) with $C = 0$ and the respective $A$ and $B_\alpha$:

$$\frac{df_g(p,t)}{dt} = -A_0 \frac{\partial^2}{\partial p^2} f_g(p,t) - B_0 \frac{\partial}{\partial p_\alpha} p_\alpha f_g(p,t), \quad (35)$$

Therefore we arrive to the Fokker-Planck type equation with the friction coefficient $\beta \equiv -B_0$ and diffusion coefficient $D = -A_0/M^2$. In general these coefficients (Eq. (31)) do not satisfy to the Einstein relation.

In the case of equilibrium $W$-function (e.g., $f_b = f_b^0$, see above) the equality $\tilde{W}'(q) = W(q)/2MT_b$ is fulfilled. Then $A(r)/rB(r) = MT_br$ ($A_0 = MT_bB_0$). Only in this case the Einstein relation between the diffusion and friction coefficients exists and the standard Fokker-Planck equation is valid.

### IV. THE MODEL OF ANOMALOUS DIFFUSION IN $V$ - SPACE

Now we can calculate the coefficients for the models of anomalous diffusion.
In this paper we calculate only the simple model system of the hard spheres with the different masses \(m\) and \(M \gg m\), \(d\sigma/d\omega = a^2/4\). Let us suppose that in the model under consideration the small particles are described by the prescribed stationary distribution \(f_b = n_b\phi_b/u_0^3\) (where \(\phi_b\) is non-dimensional distribution, \(u_0\) is the characteristic velocity for the distribution of the small particles) and \(\xi \equiv (u^2 + v^2 - q \cdot v/\mu)/u_0^2\).

\[
W_s(q, p) = \frac{n_b a^2 \pi}{2u_0^2 q_4} \int_{(q^2/4\mu^2 + v^2 - q \cdot v/\mu)/u_0^2}^{\infty} d\xi \cdot \phi_b(\xi). \tag{36}
\]

If the distribution \(\phi_b(\xi) = 1/\xi^\gamma\) \((\gamma > 1)\) possess a long-tail we get

\[
W_s(q, p) = \frac{n_b a^2 \pi}{2u_0^2 q_4} \frac{\xi^{1-\gamma}}{(1-\gamma)\xi^{1-\gamma}} = \frac{n_b a^2 \pi}{2u_0^2 q_4} \frac{\xi_0^{1-\gamma}}{(1-\gamma)} \tag{37}
\]

where \(\xi_0 \equiv (q^2/4\mu^2 + v^2 - q \cdot v/\mu)/u_0^2\).

For the case \(p = 0\) the value \(\xi_0 \to \tilde{\xi}_0 \equiv q^2/4\mu^2 u_0^2\) and we arrive to the expression for anomalous \(W \equiv W_s\)

\[
W_s(q, p = 0) = \frac{n_b a^2 \pi}{2^{3-2\gamma}(\gamma - 1)\mu^4 2\gamma u_0^3 q^2\gamma - 1} \equiv \frac{C_a}{q^{2\gamma - 1}}. \tag{38}
\]

The function \(A(r)\), according to Eq. \(26\)

\[
A(r) \equiv 4\pi \int_0^\infty dq q^2 \left[ \frac{\sin(qr)}{qr^2} - 1 \right] W(q) = 4\pi C_a \int_0^\infty dq \frac{1}{q^{2\gamma - 3}} \left[ \frac{\sin(qr)}{qr} - 1 \right] \tag{39}
\]

Comparing the reduced equation (see below) in the velocity space with the diffusion in coordinate space \(2\gamma - 1 \leftrightarrow \alpha\) and \(W(q) = C/q^{2\gamma - 1}\) we can establish that the convergence of the integral in the right side of Eq. \(39\) \((3d\) case) is provided if \(3 < 2\gamma - 1 < 5\) or \(2 < \gamma < 3\). The inequality \(\gamma < 3\) provides the convergence for small \(q\) \((q \to 0)\) and the inequality \(\gamma > 2\) provides the convergence for \(q \to \infty\).

Now to determine the structure of the transport process and the kinetic equation in the velocity space we have find the functions \(\tilde{W}'(q)\) and \(\tilde{W}''(q)\).

If \(p \neq 0\) to find \(\tilde{W}'(q)\) and \(\tilde{W}''(q)\) we have use the full value \(\xi_0 \equiv (q^2/4\mu^2 + p^2/M^2 - q \cdot p/\mu)/u_0^2\) and it derivatives on \(q \cdot p\) at \(p = 0\), \(\xi_0' = -1/Mu_0^2\) and \(\xi_0'' = 0\). Then

\[
\tilde{W}'(q, p) = \frac{n_b a^2 \pi}{2M u_0^2 q} \xi_0^{-\gamma}; \quad \tilde{W}''(q, p) = \frac{n_b a^2 \pi \gamma}{2M^2 u_0^2 q} \xi_0^{-\gamma - 1} \tag{40}
\]

Therefore for \(p = 0\) \((\xi_0 \to \tilde{\xi}_0)\) we obtain the functions

\[
\tilde{W}'(q) = \frac{(4\mu^2 u_0^2)^\gamma n_b a^2 \pi}{2M u_0^2 q^{2\gamma + 1}}; \quad \tilde{W}''(q) = \frac{(4\mu^2 u_0^2)^{\gamma + 1} n_b a^2 \pi \gamma}{2M^2 u_0^2 q^{2\gamma + 3}} \tag{41}
\]
We have establish now the conditions of convergence the integrals for \( B(r) \) and \( C(r) \).

\[
B(r) = \frac{4\pi}{r^2} \int_0^\infty dq q^2 \left[ \cos(qr) - \frac{\sin(qr)}{qr} \right] W'(q)
\]

(42)

Convergence \( B(r) \) exists for small \( q \) if \( \gamma < 2 \) and for large \( q \to \infty \) for \( \gamma > 1/2 \).

Finally for \( C(r) \) convergence is determined by the equalities \( \gamma < 2 \) for small \( q \) and \( \gamma > 1 \) for large \( q \)

\[
C(r) = \frac{2\pi}{r^2} \int_0^\infty dq q^4 \left[ \frac{2\sin(qr)}{q^4 r^3} - \frac{2\cos(qr)}{q^2 r^2} - \frac{\sin(qr)}{qr} + \frac{1}{3} \right] W''(q)
\]

(43)

Therefore to provide convergence for \( A, B, C \) for large \( q \) we have provide convergence for \( A \), that means \( \gamma > 2 \). To provide convergence for small \( q \) enough to provide convergence for \( B \) and \( C \), that means \( \gamma < 2 \). Therefore for the purely power behavior of the function \( f_b(\xi) \) convergence is absent. However, for existence of the anomalous diffusion in the momentum space in reality the convergence for small \( q \) is always provided, e.g. by finite value of \( v \) or by change of the small \( q \)-behavior of \( W(q) \) (compare with the examples of anomalous diffusion in coordinate space [1]). Therefore the ”anomalous diffusion in velocity space” for the power behavior of \( W(q), W'(q) \) and \( W''(q) \) on large \( q \) exists if for large \( q \) the asymptotic behavior of \( W(q \to \infty) \sim 1/q^{2\gamma-1} \) with \( \gamma > 2 \). At the same time the expansion of the exponential function in Eqs. (20)-(29) under the integrals, which leads to the Fokker-Planck type kinetic equation is invalid for the power-type kernels \( W(q, p) \).

V. CONCLUSIONS

In the previous sections we shortly reviewed the anomalous diffusion in the coordinate space and firstly consequently considered the problem of anomalous diffusion in momentum (velocity) space. The new kinetic equation for anomalous diffusion in velocity space is established. For the normal diffusion the friction and diffusion coefficient are found for the non-equilibrium case. For equilibrium case the usual Fokker-Planck equation is reproduced as the particular case. The model of anomalous diffusion in velocity space is described on the basis of the respective expansion of the kernel in master equation and the conditions of the convergence for the coefficients of the kinetic equation are found.
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