INVARIANT PROJECTIONS FOR OPERATORS THAT ARE FREE OVER THE DIAGONAL

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Abstract. Motivated by recent work of Au, Cébron, Dahlqvist, Gabriel, and Male, we study regularity properties of the distribution of a sum of two self-adjoint random variables in a tracial noncommutative probability space which are free over a commutative algebra. We give a characterization of the invariant projections of such a sum in terms of the associated subordination functions.

1. Introduction

Voiculescu’s analytic theory of operator-valued free probability [22, 23, 24] proved numerous times its essential role in the study of operator-valued distributions and freeness with amalgamation, and in their applications to random matrix theory (see, for instance, [17, 10, 15, 5, 11]). Recently, a new application of freeness with amalgamation to random matrix theory has been found by Au, Cébron, Dahlqvist, Gabriel, and Male: they show in [1] that independent permutation-invariant matrices are asymptotically free with amalgamation over the diagonal [1, Theorems 1.2, 2.2]. Motivated mainly by this result, we investigate in this short note the free additive convolution of operator-valued distributions with values in a commutative von Neumann algebra.

More specifically, we consider a tracial von Neumann algebra $(A, \tau)$ containing an Abelian von Neumann subalgebra $L$, and the unique trace-preserving conditional expectation $E: A \to L$. We assume that $X = X^*, Y = Y^* \in A$ are free with amalgamation over $L$. We assume that $X + Y$ has a nonzero invariant projection: there exists $a \in \mathbb{R}$ and $p = p^* = p^2 \in A \setminus \{0, 1\}$ such that $(X + Y)p = p(X + Y) = ap$. We ask whether this hypothesis imposes the existence of an invariant projection of $X$ and/or $Y$. This question was first answered in the case of scalar-valued distributions (i.e. when $L = \mathbb{C} \cdot 1$) by Bercovici and Voiculescu in [6]: the existence of $p$ requires the existence of an invariant projection $q$ for $X$ ($Xq = qX = a_1q$) and $r$ for $Y$ ($Yr = rY = a_2r$) such that $\tau(p) + 1 = \tau(q) + \tau(r)$ and $a = a_1 + a_2$ (see [6, Theorem 7.4]). The proof uses the analytic subordination functions of Voiculescu and Biane [21, 7].

In this note, we provide a characterization in terms of boundary properties of Voiculescu’s operator-valued subordination functions [23, 24] of elements $X, Y$ for which the above hypothesis is satisfied (see Theorem 3.3 below). Our result is nowhere near as satisfying as [6, Theorem 7.4], but one could not reasonably expect it to be: the reader is invited to consider the case when $L$ is isomorphic to the von Neumann algebra $L^\infty([0, 1])$ and recall that any two real-valued elements in $L$ are tautologically free, in order to construct a simple example of elements $X, Y \in L$ which are not constant on any Borel set of positive measure, but whose sum is constant on any desired Borel set of positive measure.
In recent years there were numerous results on the lack of invariant projections \[18, 9, 13, 2\], as well as the occurrence of “trivial” (in the above sense) invariant projections \[18, 14\]. As of now, we are not aware of results that indicate the existence and properties of invariant projections for \(X, Y\).

2. Analytic tools

Consider a tracial von Neumann algebra \((\mathcal{A}, \tau)\), and assume that \(\mathcal{L}\) is an Abelian von Neumann subalgebra of \(\mathcal{A}\). We shall assume throughout the paper that \(\mathcal{A}\) acts on the Hilbert space \(\mathcal{H} := L^2(\mathcal{A}, \tau)\), which is the completion of \(\mathcal{A}\) with respect to the inner product \(\langle \xi, \eta \rangle = \tau(\eta^* \xi)\). It is known (see, for instance, [20]) that there exists a unique trace-preserving conditional expectation \(E: \mathcal{A} \to \mathcal{L}\) which is the restriction to \(\mathcal{A}\) of the orthogonal projection from \(L^2(\mathcal{A}, \tau)\) onto \(L^2(\mathcal{L}, \tau)\). If \(T \in \mathcal{A}\), we write \(T \geq 0\) if \(T = T^*\) and the spectrum \(\sigma(T) \subseteq [0, +\infty)\), and we write \(T > 0\) to signify that \(T \geq 0\) and \(\sigma(T) \subseteq (0, +\infty)\). For any \(T \in \mathcal{A}\), there exists a decomposition in real and imaginary parts: \(T = \Re T + i \Im T\), where \(\Re T = \frac{T + T^*}{2}\) and \(\Im T = \frac{T - T^*}{2i}\). We define \(H^+(\mathcal{A}) = \{T \in \mathcal{A}: \Im T > 0\}\), and similar for \(L\) and any other von Neumann subalgebra of \(\mathcal{A}\).

Assume that \(X = X^*, Y = Y^* \in \mathcal{A}\) are free over \(\mathcal{L}\) with respect to \(E\) (see [22]). Define the analytic map
\[
G_X: H^+(\mathcal{L}) \to H^-(\mathcal{L}), \quad G_X(b) = E[(b - X)^{-1}].
\]
As shown in [23], \(G_X\) is a free noncommutative map in the sense of [12], whose matricial extension fully encodes the distribution of \(X\) with respect to \(E\). It is also known that \(G_X\) extends to a “neighborhood of infinity:” if \(\|b^{-1}\| < \|X\|^{-1}\), then \(G_X(b) = \sum_{n=0}^{\infty} E[b^{-1}(Xb^{-1})^n]\) converges in norm, so \(w \mapsto G_X(w^{-1})\) extends as an analytic map to the ball of center zero and radius \(1/\|X\|\).

Let \(\mathcal{L}(X)\) denote the von Neumann algebra generated by \(\mathcal{L}\) and \(X\). Denote by \(E_X: \mathcal{A} \to \mathcal{L}(X)\) the unique trace-preserving conditional expectation from \(\mathcal{A}\) to \(\mathcal{L}(X)\). It is shown in [22] that there exists a free noncommutative analytic map \(\omega_1: H^+(\mathcal{L}) \to H^+(\mathcal{L})\) such that
\[
(1) \quad E_X[(b - X - Y)^{-1}] = (\omega_1(b) - X)^{-1}, \quad b \in H^+(\mathcal{L}) \text{ or } \|b^{-1}\| < \|X + Y\|^{-1}.
\]
A similar statement holds for a map \(\omega_2\), with \(X\) and \(Y\) interchanged. By applying \(E\) to (1) and using Voiculescu’s \(R\)-transform [22, 23], it is shown in [5] that
\[
(2) \quad G_{X + Y}(b)^{-1} = G_X(\omega_1(b))^{-1} = G_Y(\omega_2(b))^{-1} = \omega_1(b) + \omega_2(b) - b, \quad b \in H^+(\mathcal{L}).
\]
(See [4] for the scalar version of this relation.) Obviously, the above relation extends to \(b\) such that \(\|b^{-1}\| < \|X + Y\|^{-1}\). It is also shown in [23, 4] that
\[
(3) \quad 3\omega_j(b) \geq 3b, \quad \omega_j(b^*) = \omega_j(b)^*, \quad b \in H^+(\mathcal{L}), j = 1, 2.
\]
Given that \(\mathcal{L}\) is a commutative von Neumann algebra, hence isomorphic to an algebra of functions, we shall often write in the following \(1/b\) or \(\frac{1}{b}\) instead of \(b^{-1}\) for multiplicative inverses of elements of \(\mathcal{L}\).

As mentioned in the introduction, we shall be concerned with invariant projections for \(X + Y\). In the following, we characterize these objects in terms of resolvents. Thus, assume \(T = T^* \in \mathcal{A}\). Denote by \(\lim_{z \to a^+} -\frac{1}{z-a}\) the limit as \(z\) approaches \(a \in \mathbb{R}\) from the complex upper half-plane nontangentially to \(\mathbb{R}\).
Lemma 2.1. If there exists a \( p = p^* = p^2 \in A \setminus \{0\} \) and \( a \in \mathbb{R} \) such that
\[
\lim_{z \rightarrow a} (z - a)(z - T)^{-1} = p
\]
in the strong operator (so) topology, then \( Tp = pT = ap \). Conversely, if \( Tp = pT = ap \), then
\[
\text{so-} \lim_{z \rightarrow a} (z - a)(z - T)^{-1} = p.
\]

Proof. The essential part of the proof can be found for instance in [6]. We sketch it here for convenience. For any vector \( \xi \in \mathcal{H} \) of \( L^2 \)-norm equal to one, we write
\[
\| (z - a)(z - T)^{-1} \xi \|^2 = \langle (z - a)(z - T)^{-1} \xi, (z - a)(z - T)^{-1} \xi \rangle
\]
\[
= \left\langle (x - a)^2 + y^2 \left( (x - T)^2 + y^2 \right)^{-1}, \xi \right\rangle
\]
\[
= \int_{\mathbb{R}} \frac{(x - a)^2 + y^2}{(x - t)^2 + y^2} d\mu_{\xi, T}(t),
\]
where \( z = x + iy \) is the decomposition in real and imaginary parts of \( z \) and \( \mu_{\xi, T} \) is the distribution of the selfadjoint random variable \( T \) with respect to the expectation (state) \( \cdot \mapsto \langle \cdot, \xi \rangle \). The dominated convergence theorem guarantees that
\[
\lim_{z \rightarrow a} \int_{\mathbb{R}} \frac{(x - a)^2 + y^2}{(x - t)^2 + y^2} d\mu_{\xi, T}(t) = \mu_{\xi, T}(\{0\}),
\]
allowing us to conclude. \( \square \)

Remark 2.2. The above lemma together with the weak operator continuity of \( E, E_X \) allows us to conclude that
\[
\lim_{z \rightarrow a} (z - a)E \left[ (z - T)^{-1} \right] = E[p], \quad \lim_{z \rightarrow a} (z - a)E_X \left[ (z - T)^{-1} \right] = E_X[p],
\]
in the so topology. Similarly, we have
\[
\text{so-} \lim_{z \rightarrow a} \Re (z - a)(z - T)^{-1} = p, \quad \text{so-} \lim_{z \rightarrow a} \Im (z - a)(z - T)^{-1} = 0.
\]
In particular,
\[
\text{so-} \lim_{y \rightarrow 0} y(a - T) \left( (a - T)^2 + y^2 \right)^{-1} = 0, \quad \text{so-} \lim_{y \rightarrow 0} y^2 \left( (a - T)^2 + y^2 \right)^{-1} = p.
\]

We need one more (very simple) fact about the functions that behave like \( \omega \).

Lemma 2.3. Assume that \( f: H^+(\mathbb{C}) \rightarrow H^+(\mathcal{L}) \) is a free noncommutative function in the sense of [12]. For any \( a \in \mathbb{R} \), the so limit
\[
\lim_{y \rightarrow 0} y \Im f(a + iy)
\]
exists and is finite.

Proof. The proof is based on the representation of free noncommutative maps of noncommutative half-planes provided by [10, 26]: there exists a completely positive map \( \rho: \mathcal{C}(\mathcal{X}) \rightarrow \mathcal{L} \), an element \( A = A^* \) and \( B \geq 0 \) in \( \mathcal{L} \) such that
\[
f(z) = A + zB + \rho \left[ (\mathcal{X} - z)^{-1} \right], \quad z \in H^+(\mathbb{C}).
\]
Then $3f(z) = 3zB + \rho[(\mathcal{X} - z)^{-1}3z(\mathcal{X} - \bar{z})^{-1}] = 3zB + \rho\left[3z/(\mathcal{X} - \bar{z})^2 + (3z)^2\right]$. Here $\mathcal{X}$ is a selfadjoint operator. Thus,

$$y(\Re f(z + iy))^{-1} = (B + \rho[(\mathcal{X} - a - iy)^{-1}(\mathcal{X} - a + iy)^{-1}])^{-1}.$$  

Trivially the map $y \mapsto (\mathcal{X} - a - iy)^{-1}(\mathcal{X} - a + iy)^{-1}$ is decreasing. This concludes the proof. □

Observe that commutativity of $\mathcal{L}$ plays no role in the proof of the previous lemma.

In this paper we shall make use also of the estimate

$$\left\| (\Re z)^{-\frac{1}{2}} (f(z) - f(w)) \left( (\Re f(z))^{-1} (f(z) - f(w))^* (\Re f(z))^{-\frac{1}{2}} \right) \right\| \leq \left\| (\Re z)^{-\frac{1}{2}} (z - w) (\Re w)^{-\frac{1}{2}} \right\|^2,$$

proven in [3, Proposition 3.1] for an arbitrary free noncommutative map $f$ between two noncommutative upper half-planes of two $C^*$-algebras. Since $\mathcal{L}$ is commutative, we sometimes write the above as

$$\frac{(f(z) - f(w))(f(z) - f(w))^*}{\Re f(z)^2} \leq \left\| (\Re z)^{-\frac{1}{2}} (z - w) (\Re w)^{-\frac{1}{2}} \right\|^2.$$

3. Invariant projections

Let us re-state our hypotheses: $(\mathcal{A}, \tau)$ is a tracial von Neumann algebra (with normal faithful $\tau$), $\mathcal{L} \subset \mathcal{A}$ is an Abelian von Neumann subalgebra of $\mathcal{A}$, $E: \mathcal{A} \to \mathcal{L}$ is the unique trace-preserving conditional expectation from $\mathcal{A}$ to $\mathcal{L}$, and $X = X^*, Y = Y^* \in \mathcal{A}$ are two bounded selfadjoint random variables which are free with respect to $E$ over $\mathcal{L}$. Also, $\mathcal{L}(X)$ (respectively $\mathcal{L}(Y)$) is the von Neumann algebra generated by $\mathcal{L}$ and $X$ (respectively $\mathcal{L}$ and $Y$), and $E_X: \mathcal{A} \to \mathcal{L}(X)$ (resp. $E_Y: \mathcal{A} \to \mathcal{L}(Y)$) is the unique trace-preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{L}(X)$ (resp. $\mathcal{L}(Y)$). Finally, $\mathcal{A}$ acts (faithfully) on the Hilbert space $\mathcal{H} := L^2(\mathcal{A}, \tau)$, which is the completion of $\mathcal{A}$ with respect to the inner product $(\xi, \eta) = \tau(\eta^* \xi)$.

We assume that there exists $a \in \mathbb{R}$ and $p = p^* = p^2 \in \mathcal{A} \setminus \{0\}$ such that

$$(X + Y)p = p(X + Y) = ap.$$  

As seen in Lemma 2.1 we have $\lim_{z \to a} (z - a)(z - X - Y)^{-1} = p$, and, by Remark 2.2

$$\lim_{z \to a}(z - a)E\left[(z - X - Y)^{-1}\right] = E[p], \quad \lim_{z \to a}(z - a)E_X\left[(z - X - Y)^{-1}\right] = E_X[p].$$
Remark 3.1. Ideally (as it will become clear from our proof below), we would wish that \( \ker \lim \lim_{a \to a} \{ (z-a)E_X [(z-X-Y)^{-1}] \} \). Using (1) and the above,

\[
E_X[p] = \lim_{t \to a} (z-a)E_X [(z-X-Y)^{-1}]
= \lim_{t \to a} (z-a)(\omega_1(z) - X)^{-1}
\]

\[
= \lim_{t \to a} \sqrt[3]{\omega_1(z)} \left( i - \frac{1}{\sqrt[3]{\omega_1(z)}} (X - \Re \omega_1(z)) \frac{1}{\sqrt[3]{\omega_1(z)}} \right)^{-1} \frac{1}{\sqrt[3]{\omega_1(z)}}
\]

Applying (8) to the above yields

\[
E[p] = \lim_{y \to 0} \frac{y}{3 \omega_1(a + iy)}
\]

\[
\times iy \left( iy - \sqrt[3]{\omega_1(a + iy)} (X - \Re \omega_1(a + iy)) \sqrt[3]{\omega_1(a + iy)} \right)^{-1}
\]

All limits take place in the so topology.

Using again Remark 2.2 and the fact that

\[
\Im ((\omega_1(z) - X)^{-1}) = -(3 \omega_1(z) + (X - \Re \omega_1(z))(3 \omega_1(z))^{-1}(X - \Re \omega_1(z)))^{-1},
\]

we obtain

\[
E_X[p] = \lim_{y \to 0} yE_X \left[ \frac{y}{(a-X-Y)^2 + y^2} \right] = -\lim_{y \to 0} y\Im E_X \left[ (a+iy-X-Y)^{-1} \right]
= -\lim_{y \to 0} y\Im (\omega_1(a+iy) - X)^{-1}
\]

\[
= \lim_{y \to 0} \sqrt[3]{\omega_1(a + iy)}
\]

\[
\times \left( 1 + \left( \frac{y}{3 \omega_1(a + iy)} (X - \Re \omega_1(a + iy)) \sqrt[3]{\omega_1(a + iy)} \right)^2 \right)^{-1}
\]

\[
\times \sqrt[3]{\omega_1(a + iy)}
\]

(8) \leq \lim_{y \to 0} \frac{y}{3 \omega_1(a + iy)}.

Applying (8) to the above yields

\[
E[p] \leq \lim_{y \to 0} \frac{y}{3 \omega_1(a + iy)}.
\]

Remark 3.1. Ideally (as it will become clear from our proof below), we would wish that \( \ker \lim_{y \to 0} \frac{y}{3 \omega_1(a + iy)} = \{0\} \). That is obviously implied by \( \ker E[p] = \{0\} \).

Observe that if \( 0 \neq q = q^* = q^2 = \ker E[p] \), then \( E[qqq] = qE[p]q = 0 \), which implies
\[\tau(qpq) = \tau(E[qpq]) = \tau(0) = 0, \text{ so that } qpq = 0. \text{ Since } p \text{ is also a projection, we conclude from the faithfulness of } \tau \text{ that } pq = qp = 0, \text{ so that } p \perp q, \text{ or, equivalently, } p \le q^+. \text{ This means that there exists a nontrivial algebraic relation between an element from } L \setminus C \cdot 1, \text{ namely } q, \text{ and an element from } C\langle X + Y \rangle \setminus C \cdot 1, \text{ namely } p: pq = qp = 0.

Conversely, let us assume that \( o_1 = \ker \lim_{y \to 0} \frac{y}{\omega_1(a + iy)} \neq \{0\} \). Then \( \ker E_X[p] \ge \ker \lim_{y \to 0} \frac{y}{\omega_1(a + iy)} \), so that there exists an element \( o_1 = o_1^* = o_1^2 \in L \setminus C \cdot 1 \) such that \( o_1 \) and the element \( E_X[p] \in C\langle X \rangle \setminus C \cdot 1 \) satisfy a nontrivial algebraic relation: \( o_1 E_X[p] = E_X[p] o_1 = 0 \).

We study next the nontangential limit of the real part of \( \omega_1 \) (and thus also of \( \omega_2 \)) at \( a \). A few steps in this proof will not depend on the commutativity of \( L \).

Fix \( c \in \mathbb{R}, c \ge 2\|X + Y\| \) and \( y' \in (0, +\infty) \). We use inequality \( 4 \), applied to \( f = \omega_1, z = c + iy', w = a + iy \) in order to write

\[
\frac{1}{\Im(\omega_1(c + iy'))} \left( \omega_1(c + iy') - \omega_1(a + iy) \right) \left[ \frac{1}{\Im(\omega_1(c + iy'))} \right]^* \le \left\| \frac{1}{\sqrt{y'}} (c - a + iy - iy') \frac{1}{\sqrt{y'}} \right\|^2.
\]

This implies

\[
(\omega_1(c + iy') - \omega_1(a + iy)) \left[ \frac{y}{\Im(\omega_1(a + iy))} \right] (\omega_1(c + iy') - \omega_1(a + iy))^* \le \|c - a + iy\|^2 \omega_1'(c)(1).
\]

Expanding in real and imaginary parts, we obtain

\[
(\omega_1(c) - \omega_1(a + iy)) \left[ \frac{y}{\Im(\omega_1(a + iy))} \right] (\omega_1(c) - \omega_1(a + iy))^* \le \|c - a + iy\|^2 \omega_1'(c)(1).
\]

We conclude that

\[
\left\| \omega_1(c) - \omega_1(a + iy) \right\| \left[ \frac{y}{\Im(\omega_1(a + iy))} \right]^\frac{1}{2} \le \|c - a + iy\| \sqrt{\omega_1'(c)},
\]

so that, by elementary properties of the norm, and recalling that \( \frac{y}{\Im(\omega_1(a + iy))} \le 1 \),

\[
\left\| \Re\omega_1(a + iy) \left[ \frac{y}{\Im(\omega_1(a + iy))} \right]^\frac{1}{2} \le \|c - a + iy\| \sqrt{\omega_1'(c)} + \|\omega_1(c)\|,
\]

independently of \( y > 0 \). The bound from the above relation, while necessary, is not sufficient for our purposes. We need to show that \( \lim_{y \to 0} \frac{y}{\Im(\omega_1(a + iy))} \Re\omega_1(a +
Clearly the existence of \(\lim_{y \to 0} \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \) exists in the so topology and is finite. Clearly, this is implied by the existence of

\[
\text{so - lim}_{y \to 0} \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}.
\]

We write:

\[
\begin{align*}
\left[ \frac{y'}{3\omega_1(a + iy')} \right]^\frac{1}{2} \omega_1(a + iy') & \left[ \frac{y'}{3\omega_1(a + iy')} \right]^\frac{1}{2} \\
- \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \omega_1(a + iy) & \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \\
+ \left[ \frac{y'}{3\omega_1(a + iy')} \right]^\frac{1}{2} (\omega_1(a + iy') - \omega_1(a + iy) & \left[ \frac{y'}{3\omega_1(a + iy')} \right]^\frac{1}{2} - \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}) \\
+ \left( \left[ \frac{y'}{3\omega_1(a + iy')} \right]^\frac{1}{2} - \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \right) & \omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}.
\end{align*}
\]

Recalling that

\[
\left\| \left[ \frac{y'}{3\omega_1(a + iy')} \right]^\frac{1}{2} (\omega_1(a + iy') - \omega_1(a + iy)) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \right\| \leq |y - y'|
\]

assures us that the middle term on the right hand side of the equality above converges in norm to zero as \(y, y' \to 0\). As shown in Lemma 2.3 above, so-\(\lim_{y \to 0} \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}\) exists and is strictly between 0 and 1. Thus,

\[
\text{so - lim}_{y, y' \to 0} \left( \left[ \frac{y'}{3\omega_1(a + iy')} \right]^\frac{1}{2} - \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \right) = 0.
\]

Clearly \(\omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} = [y3\omega_1(a + iy)]^\frac{1}{2} + \Re \omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}\). Since \(\Re \omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}\) has been shown to be bounded in [1], and \(y3\omega_1(a + iy)\) is known to be uniformly bounded as \(y \to 0\) by a universal constant depending only on the first two moments of \(X\) and \(Y\), it follows that \(\omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}\) is uniformly bounded as \(y \to 0\). Generally, if \(a_t \to \infty\) in the so topology and \(b_t\) is uniformly bounded in norm, then \(b_t a_t\) converges to zero in the so topology. Indeed, for any \(\xi \in \mathcal{H}\), \(|b_t a_t \xi|_2 \leq \|b_t\| \|a_t \xi|_2 \leq (\sup_{\xi} \|b_t\|) \|a_t \xi|_2 \to 0\). This guarantees that the first term on the right hand side of the equality above converges in the so topology to zero as \(y, y' \to 0\). Finally, under the above assumptions, \(a_t, b_t \to 0\) in the wo topology: \(|(a_t b_t \xi, \eta)| \leq \|b_t \xi|_2 \|a_t \xi|_2 \leq (\sup_{\xi} \|b_t||a_t \xi|_2 \to 0\). This guarantees that the last term on the right hand side of the equality above converges in the wo topology to zero as \(y, y' \to 0\). This shows that the family

\[
\left\{ \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \right\}_{y > 0}
\]

is Cauchy, hence convergent in the wo topology. Up to this point, we did not need the fact that \(\mathcal{L}\) is an Abelian von
Neumann algebra. However, since \( \omega_1 \) takes values in a commutative algebra, it follows trivially that the third (last) term on the right hand side of the above relation converges also in the so topology to zero, which proves the existence and finiteness of the so limit \((10)\).

Let us denote
\[
\omega_1^w(a) := \lim_{y \to 0} \frac{y}{\Im\omega_1(a + iy)} \left( \frac{y}{\Im\omega_1(a + iy)} \right) \]
and
\[
\omega_1^s(a) := \lim_{y \to 0} \frac{y}{3\omega_1(a + iy)},
\]
where the limits are in the so topology. We need one more lemma in order to be able to state and complete the proof of our main result.

**Lemma 3.2.** Consider a family \( \{Y_n\}_{n \in \mathbb{N}} \subset \mathcal{A} \) of selfadjoint elements uniformly bounded in norm. Assume that so-limit \( Y_n \to Y \) and that there exists a sequence \( \{y_n\}_{n \in \mathbb{N}} \subset (0, 1) \) converging to zero and an element \( r \in \mathcal{A} \setminus \{0\} \) such that
\[
\text{wo- lim } n \to \infty iy_n(iy_n - Y_n)^{-1} = r.
\]
Then \( \ker Y \neq 0 \). Moreover, \( Yr = 0 = rY \).

**Proof.** We claim that \( Yr = 0 \). Indeed,
\[
Yr = Y \lim_{n \to \infty} iy_n(iy_n - Y_n)^{-1} = \lim_{n \to \infty} iy_nY(iy_n - Y_n)^{-1}
\]
\[
= \lim_{n \to \infty} iy_nY(iy_n - Y_n)^{-1} + iy_n(Y - Y_n)(iy_n - Y_n)^{-1}.
\]
Since \( Y_n = Y_n^* \) and there is an \( I > 0 \) such that \( \|Y_n\| < I \) for all \( n \), by continuous functional calculus the first term is bounded in norm by
\[
\max_{t \in [-I, I]} \left| \frac{iyn t}{yn - t} \right| = \max_{t \in [-I, I]} \left| \frac{ynt}{\sqrt{y_n^2 + t^2}} \right| = \frac{yn I}{\sqrt{y_n^2 + I^2}} \to 0 \text{ as } yn \searrow 0.
\]
If \( \xi, \eta \in \mathcal{H} \), then
\[
\left| \langle iy_n(Y - Y_n)(iy_n - Y_n)^{-1}\xi, \eta \rangle \right| = \left| \langle iy_n(iy_n - Y_n)^{-1}\xi, (Y - Y_n)\eta \rangle \right|
\]
\[
\leq \left\| \frac{iyn}{yn - Y_n} \xi \right\|_2 \left\| (Y - Y_n)\eta \right\|_2 \to 0
\]
as \( yn \searrow 0 \), according to our hypothesis that \( Y_n \to Y \) in the so topology. We conclude that \( \langle Yr\xi, \eta \rangle = 0 \) for all \( \xi, \eta \in \mathcal{H} \), so that \( Yr = 0 \) in \( \mathcal{L} \), as claimed. Since \( r \neq 0 \), any element \( \xi \neq 0 \) which is in the range of \( r \) must belong to the kernel of \( Y \).

Showing that \( rY = 0 \) is similar. We have:
\[
rY = \lim_{n \to \infty} iy_n(iy_n - Y_n)^{-1}Y
\]
\[
= \lim_{n \to \infty} iy_nY(iy_n - Y_n)^{-1} + iy_n(iy_n - Y_n)^{-1}(Y - Y_n).
\]
The first term tends to zero in norm, while the second term, when applied to \( \langle \xi, \eta \rangle \), is dominated by \( \|Y - Y_n\| \xi_2 \|iy_n(iy_n + Y_n)^{-1}\eta_2 \), which tends to zero. \( \square \)

Let us state now our main result.
Theorem 3.3. Let \( X, Y \) be selfadjoint, free over the commutative von Neumann algebra \( \mathcal{L} \). Assume that there exists a nonzero projection \( p \) and \( a \in \mathbb{R} \) such that \( (X + Y)p = p(X + Y) = ap \). Denote by \( \omega_1, \omega_2 \) Voiculescu’s analytic subordination functions associated to \( X \) and \( Y \), respectively. Then:

1. \( \ker \left( \sqrt{\omega_1^3(a)}X \sqrt{\omega_1^3(a)} - \omega_1^R(a) \right) \cap \ker \omega_1^3(a) \neq \{0\}; 
2. \( \ker \left( \sqrt{\omega_2^3(a)}Y \sqrt{\omega_2^3(a)} - \omega_2^R(a) \right) \cap \ker \omega_2^3(a) \neq \{0\}; 
3. \( E \left[ \ker \left( \sqrt{\omega_1^3(a)}X \sqrt{\omega_1^3(a)} - \omega_1^R(a) \right) \right] + E \left[ \ker \left( \sqrt{\omega_2^3(a)}Y \sqrt{\omega_2^3(a)} - \omega_2^R(a) \right) \right] \geq E[p] + \Xi, \)

where \( \Xi = \lim_{y \to 0} \frac{(3G_{X+Y}(a + iy))}{(4E[p]^2 + (3G_{X+Y}(a + iy))^2)} \) is an operator between \( \mathcal{L} \) and \( 1 \). We have \( \Xi = 1 \) and equality in the above whenever \( E[p] > 0 \).

Proof. Let us return to equality (5): we have

\[
E[p] = \text{so-} \lim_{y \to 0} \left[ \sqrt{\frac{y}{3\omega_1(a + iy)}} \times iy \left( \frac{y}{3\omega_1(a + iy)}(X - \Re \omega_1(a + iy)) \frac{y}{3\omega_1(a + iy)} \right)^{-1} \times \sqrt{\frac{y}{3\omega_1(a + iy)}} \right].
\]

As shown above,

\[
\lim_{y \to 0} \sqrt{\frac{y}{3\omega_1(a + iy)}}(X - \Re \omega_1(a + iy)) \sqrt{\frac{y}{3\omega_1(a + iy)}} = \omega_1^3(a) \frac{1}{2} X \omega_1^3(a) \frac{1}{2} - \omega_1^R(a),
\]

so-convergence to a bounded selfadjoint element. We have also seen that the family \( \sqrt{\frac{y}{3\omega_1(a + iy)}}(X - \Re \omega_1(a + iy)) \sqrt{\frac{y}{3\omega_1(a + iy)}} \) is uniformly bounded in norm as \( y \in (0, 1) \). Since in the above relation (5), \( 0 \leq E[p] \neq 0 \) and \( \sqrt{\frac{y}{3\omega_1(a + iy)}} \), \( y > 0 \), is bounded from below by the positive nonzero element \( E[p] \), it follows that the middle factor in the right hand side cannot converge to zero. Also, if \( \ker \omega_1^3(a) \neq 0 \), then \( \ker \left( \sqrt{\omega_1^3(a)}X \sqrt{\omega_1^3(a)} - \omega_1^R(a) \right) \geq \ker \omega_1^3(a) \). Indeed, we may write

\[
\sqrt{\frac{y}{3\omega_1(a + iy)}}(X - \Re \omega_1(a + iy)) \sqrt{\frac{y}{3\omega_1(a + iy)}} = \sqrt{\frac{y}{3\omega_1(a + iy)}} X \sqrt{\frac{y}{3\omega_1(a + iy)}} - \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \Re \omega_1(a + iy) \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2} \left[ \frac{y}{3\omega_1(a + iy)} \right]^\frac{1}{2}.
\]
We recall from (9) that \( \| \Re \omega_1(a + iy) \left[ \frac{y}{\Im \omega_1(a + iy)} \right] ^{\frac{1}{2}} \| \) is uniformly bounded as \( y \to 0 \). Elementary operator theory informs us that the norm of an operator on a Hilbert space dominates its spectral radius, with equality for normal elements. Since 
\[ \sigma \left( \Re \omega_1(a + iy) \left[ \frac{y}{\Im \omega_1(a + iy)} \right] ^{\frac{1}{2}} \right) \cup \{0\} = \sigma \left( \left[ \frac{y}{\Im \omega_1(a + iy)} \right] ^{\frac{1}{2}} \Re \omega_1(a + iy) \left[ \frac{y}{\Im \omega_1(a + iy)} \right] ^{\frac{1}{2}} \right) \cup \{0\}, \]

it follows that the spectral radius, and hence the norm, of the right-hand side, selfadjoint, operator is uniformly bounded as \( y \to 0 \). Since the kernel of a positive operator equals the kernel of any of its positive powers, we conclude that if \( \omega_1(a) \xi = 0 \), then \( (\sqrt{\omega_1(a)} \mathcal{X} \sqrt{\omega_1(a)} - \omega_1^R(a)) \xi = 0 \). Since

\[
\left\| iy \left( iy - \frac{y}{\Im \omega_1(a + iy)} (X - \Re \omega_1(a + iy)) \sqrt{\frac{y}{\Im \omega_1(a + iy)}} \right)^{-1} \right\| \leq 1, \quad y > 0,
\]

in a von Neumann algebra, there exists a sequence \( y_n \) converging to zero so that the above converges in the weak operator (wo) topology. Choose such a limit point and call it \( r \). (Note that, in this particular case, the adjoint of the above also converges, and necessarily to \( r^* \).) We have established above that \( \ker(\sqrt{\omega_1^3(a)} X \sqrt{\omega_1^3(a)} - \omega_1^{R}(a)) \supseteq \ker \omega_1^{3}(a) \). If this inequality were an equality, then the inequality \( \text{ran}(r) \leq \ker(\sqrt{\omega_1^3(a)} X \sqrt{\omega_1^3(a)} - \omega_1^{R}(a)) \) provided by Lemma 3.2 would imply \( \text{ran}(r) \leq \ker \omega_1^{3}(a) \). In particular, we would obtain that the right-hand side of (3) converges to zero \( \text{ran}(r) \), contradicting the fact that \( p \), and hence \( E_{X[p]} \), is non-zero. Thus, necessarily

\[
\ker \left( \sqrt{\omega_1^{3}(a)} X \sqrt{\omega_1^{3}(a)} - \omega_1^{R}(a) \right) \supseteq \ker \omega_1^{3}(a).
\]

This way, we conclude that

\[
\ker \left( \sqrt{\omega_1^{3}(a)} X \sqrt{\omega_1^{3}(a)} - \omega_1^{R}(a) \right) \supset \ker \omega_1^{3}(a) \neq \{0\}.
\]

The statement for \( \omega_2 \) and \( Y \) follows the same way.

Let us establish next the relation between the kernels from items (1) and (2). Let us take the imaginary part in (2) (we use commutativity of \( \mathcal{L} \) in an essential way):

\[
\Im \omega_1(a + iy) + \Im \omega_2(a + iy) = y + \frac{-yG_{X+Y}(a + iy)}{(yG_{X+Y}(a + iy))^2 + (yG_{X+Y}(a + iy))^2}.
\]

(Recall that \( yG_{X+Y}(a + iy) < 0 \).) We multiply with \( -yG_{X+Y}(a + iy) \) to obtain

\[
-y \Im \omega_1(a + iy) yG_{X+Y}(a + iy) + \Im \omega_2(a + iy) yG_{X+Y}(a + iy) = -y^2G_{X+Y}(a + iy) + \frac{(yG_{X+Y}(a + iy))^2}{(yG_{X+Y}(a + iy))^2 + (yG_{X+Y}(a + iy))^2}.
\]

\[\text{If the bounded sequence } r_n \text{ converges wo to } r \text{ and the positive sequence } x_n \text{ converges so to } x, \text{ then } \langle x_n r_n \xi, \eta \rangle = \langle r_n \xi, x_n \eta \rangle = \langle r_n \xi, (x_n - x) \eta \rangle + \langle r_n \xi, x \eta \rangle. \text{ By Cauchy-Schwartz, } \langle r_n \xi, (x_n - x) \eta \rangle \to 0, \text{ and by hypothesis } \langle r_n \xi, x \eta \rangle \to \langle x \xi, x \eta \rangle = \langle x \xi, \eta \rangle. \]
It is easy to verify that the right hand side converges when \( y \searrow 0 \), at least along a subsequence. Let us analyze each of the two terms on the left hand side separately:

\[
-3\omega_1(a + iy)^3 G_X + Y(a + iy) = -3\omega_1(a + iy)\bar{3} G_X(\omega_1(a + iy))
\]

\[
= 3\omega_1(a + iy) \times E \left[ \left(3\omega_1(a + iy) + (\Re \omega_1(a + iy) - X)(3\omega_1(a + iy) - X)\right)^{-1}\right]
\]

\[
= E \left[ \left(1 + \left(3\omega_1(a + iy) - X\right)^{-1}\right)^2\right]^{-1}
\]

\[
= y^2 E \left[ y^2 + \left(\sqrt{\frac{y}{3\omega_1(a + iy)}}(\Re \omega_1(a + iy) - X)\right)^2\right]^{-1}
\]

We recognize under the expectation the square of the selfadjoint shown to so-converge to \( \sqrt{\omega_1^2(a) X \sqrt{\omega_1^2(a) - \omega_1^2(a)}} \), and which has been shown to be uniformly norm-bounded in \( y \). Since the square of a bounded family of selfadjoints converges whenever the family converges, we obtain that

\[
\lim_{y \searrow 0} \left( \left(3\omega_1(a + iy) - X\right)^{-1}\right)^2 = \left(\sqrt{\omega_1^2(a) X \sqrt{\omega_1^2(a) - \omega_1^2(a)}}\right)^2
\]

The family

\[
Z_y := y^2 \left( y^2 + \left(\sqrt{\frac{y}{3\omega_1(a + iy)}}(\Re \omega_1(a + iy) - X)\right)^2\right)\]

is uniformly bounded from above by 1 and positive in the von Neumann algebra \( \mathcal{L}(X) \). Pick, as before, a subsequence \( y_n \searrow 0 \) such that \( Z_{y_n} \) converges to an element \( s_1 \geq 0 \) in \( \mathcal{L}(X) \). As in the proof of Lemma \( \ref{2} \), we have \( \left(\sqrt{\omega_1^2(a) X \sqrt{\omega_1^2(a) - \omega_1^2(a)}}\right)^2 = s_1 \left(\sqrt{\omega_1^2(a) X \sqrt{\omega_1^2(a) - \omega_1^2(a)}}\right)^2 = 0 \). It is quite clear that \( s_1 \neq 0 \).

Indeed, that follows from \( \ref{7} \) the same way as above. In particular, we have ker\( \left(\sqrt{\omega_1^2(a) X \sqrt{\omega_1^2(a) - \omega_1^2(a)}}\right)^2 \geq s_1 \), and so ker\( \left(\sqrt{\omega_1^2(a) X \sqrt{\omega_1^2(a) - \omega_1^2(a)}}\right) \geq s_1 \). Since

\[
E[s_1] + E[s_2] = E[p] + \lim_{y \searrow 0} (y^2 G_X X + Y(a + iy))^2 + (y^2 G_X X + Y(a + iy))^2
\]

the inequality in item (3) of our theorem follows from the monotonicity of \( E \). The limit in the right hand side is easily seen to be between \( \frac{4E[p]}{4E[p]+1} \) and 1. Finally, if \( E[p] > 0 \), then \( \Re \omega_1(a + iy) \) converges as \( y \to 0 \) to a selfadjoint \( \omega_1(a) \) (see \( \ref{3} \) Theorem 2.2]), and, according to \( \ref{3} \) Relation (4.2)], \( \Re \omega_1(a + iy) - \omega_1(a)) \to 0 \) as \( y \to 0 \). Then \( \ref{7} \) yields \( \omega_1(a) + \omega_2(a) = a \) and thus \( \omega_1(a + iy) - \omega_1(a)) G_X (\omega_1(a + iy)) + (\omega_2(a + iy) - \omega_2(a)) G_Y (\omega_2(a + iy)) = iy G_X X + Y(a + iy) \). We write \( \omega_1(a + iy) - \omega_1(a)) G_X (\omega_1(a + iy)) = iy G_X X + Y(a + iy)) \) and so the first term tends to zero. We claim that the second converges

\[^2\text{If } Y_y = Y_y^* \to Y \text{ as } y \to 0, \text{ then } Y^2 - Y = (Y - Y_y) \bar{Y} + Y_y (Y - Y_y), \text{ and } \|Y_y (Y_y - Y)\|_2^2 = \langle (Y - Y) Y_y^* (Y - Y) \xi, \xi \rangle \leq \|Y_y\|^2 \|Y - Y\|_2^2 \to 0. \text{ The other term converges trivially to zero.}\]
to \( \ker(\omega'_1(a)^{-\frac{1}{2}}(X - \Re \omega_1(a))\omega'_1(a)^{-\frac{1}{2}}) \) (just here, we agree to denote \( \omega'_1(a)^{(1)} \) by \( \omega'_1(a) \)). Indeed,

\[
i3\omega_1(a + iy)G_X(\omega_1(a + iy)) - iyE \left[ \left( iy - \omega'_1(a)^{-\frac{1}{2}}(X - \Re \omega_1(a))\omega'_1(a)^{-\frac{1}{2}} \right)^{-1} \right]
\]

\[
= i3\omega_1(a + iy)G_X(\omega_1(a + iy)) - iy\omega'_1(a)E \left[ (iy\omega'_1(a) - (X - \Re \omega_1(a)))^{-1} \right]
\]

\[
= \left[ \frac{3\omega_1(a + iy)}{y} - \omega'_1(a) \right] iyG_{X + Y}(a + iy)
\]

\[
+ \omega'_1(a)iyE \left[ (\omega_1(a + iy) - X)^{-1} (iy\omega'_1(a) - X + \Re \omega_1(a)) \right.
\]

\[
- \Re \omega_1(a + iy) - i3\omega_1(a + iy) + X (iy\omega'_1(a) - (X - \Re \omega_1(a)))^{-1} \right].
\]

As shown in [3] Theorem 2.2, \( \frac{3\omega_1(a + iy)}{y} \) increases to \( \omega'_1(a) \) as \( y \searrow 0 \) (convergence in so topology) and \( iyG_{X + Y}(a + iy) \to E[p] \), so the first term goes to zero. Next,

\[
iy(\omega_1(a + iy) - X)^{-1} \left[ \frac{3\omega_1(a + iy)}{y} - \omega'_1(a) \right] iy(iy\omega'_1(a) - (X - \Re \omega_1(a)))^{-1}
\]

has the first and third factors bounded, while the middle one converges to zero in the so topology. Finally, precisely the same statement holds for the last product, namely

\[
iy(\omega_1(a + iy) - X)^{-1} \left[ \frac{3\omega_1(a + iy)}{y} - \omega'_1(a) \right] iy(iy\omega'_1(a) - (X - \Re \omega_1(a)))^{-1}
\]

Thus, the above tends to zero in the so topology, guaranteeing that

\[
\ker(\omega'_1(a)^{-\frac{1}{2}}(X - \Re \omega_1(a))\omega'_1(a)^{-\frac{1}{2}}) + \ker(\omega'_2(a)^{-\frac{1}{2}}(Y - \Re \omega_2(a))\omega'_2(a)^{-\frac{1}{2}}) = 1 + E[p].
\]

\[\square\]

Note that, under the very favourable hypothesis on \( E[p] \), discussed in Remark 3.1 the result above, and its proof, closely parallels the corresponding result and proof from [6]. This seems to justify the statement that the Julia-Carathéodory derivative is an important tool in the understanding of invariant projections for sums of random variables which are free over a von Neumann algebra.

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