BILIPSCHITZ INVARIANTS FOR GERMS OF HOLOMORPHIC FOLIATIONS

RUDY ROSAS

ABSTRACT. In this paper we study bilipschitz equivalences of germs of holomorphic foliations in \((\mathbb{C}^2,0)\). We prove that the algebraic multiplicity of a singularity is invariant by such equivalences. Moreover, for a large class of singularities, we show that the projective holonomy representation is also a bilipschitz invariant.

1. INTRODUCTION

Given a reduced holomorphic curve \(f : (\mathbb{C}^2,0) \to (\mathbb{C},0)\), singular at \(0 \in \mathbb{C}^2\), we define its algebraic multiplicity as the degree of the first nonzero jet of \(f\), that is, \(\nu(f) = \nu\) where \(f = f_\nu + f_{\nu+1} + \cdots\) is the Taylor development of \(f\) and \(f_\nu \neq 0\). A well known result by Burau [2] and Zariski [18] states that \(\nu\) is a topological invariant, that is, given \(\tilde{f} : (\mathbb{C}^2,0) \to (\mathbb{C},0)\) reduced and a homeomorphism \(h : U \to \tilde{U}\) between neighborhoods of \(0 \in \mathbb{C}^2\) such that \(h(f^{-1}(0) \cap U) = \tilde{f}^{-1}(0) \cap \tilde{U}\) then \(\nu(f) = \nu(\tilde{f})\). This question in dimension greater than 2 is the celebrated Zariski’s Multiplicity Problem, posed in [19] (see [6] for a survey). It is well known that the topological equivalence between the curves \(f\) and \(\tilde{f}\) implies the topological equivalence between the foliations induced by \(df\) and \(d\tilde{f}\) on a neighborhood of \(0 \in \mathbb{C}^2\) (see [7]). Thus it is natural to extend the Zariski’s Multiplicity Problem for holomorphic foliations in \((\mathbb{C}^2,0)\), as was posed by J.F. Mattei. Consider a holomorphic vector field \(Z\) in \(\mathbb{C}^2\) with a singularity at \(0 \in \mathbb{C}^2\). If

\[ Z = Z_\nu + Z_{\nu+1} + \cdots, \quad Z_\nu \neq 0 \]

we define \(\nu = \nu(Z)\) as the algebraic multiplicity of \(Z\) at \(0 \in \mathbb{C}^2\). The vector field \(Z\) defines a holomorphic foliation by curves \(\mathcal{F}\) with isolated singularity in a neighborhood of \(0 \in \mathbb{C}^2\) and the algebraic multiplicity \(\nu(Z)\) depends only on the foliation \(\mathcal{F}\). Then: Is \(\nu(\mathcal{F})\) a topological invariant of \(\mathcal{F}\)? In [8], the authors give a positive answer if \(\mathcal{F}\) is a generalized curve, that is, if the desingularization of \(\mathcal{F}\) does not contain complex saddle-nodes. If \(\mathcal{F}\) is dicritical, that is, after a blow up the exceptional divisor is not invariant by the strict transform of \(\mathcal{F}\), the conjecture is also true: in this case, it is not difficult to show that the algebraic multiplicity of \(\mathcal{F}\) is equal to the index of \(\mathcal{F}\) (as defined in [8]) along a generic separatrix. Then the topological invariance of the algebraic multiplicity of a dicritical singularity is a consequence of the topological invariance of the index along a curve, which is proved in [8]. Recently has been considered the problem for some classes of...
equivalences. In [16] is proved the invariance of the algebraic multiplicity under equivalences that are differentiable at the singular point and in [15] is solved the problem for 1-foliations in any dimension but only for $C^1$ equivalences.

The following theorem is the first result of this work.

**Theorem 1.** Suppose that $\mathfrak{h}$ is biLipschitz, that is, there are positive constants $m, M$ such that

$$m|z_1 - z_2| \leq |h(z_1) - h(z_2)| \leq M|z_1 - z_2|$$

for all $z_1, z_2 \in \mathfrak{U}$. Then the algebraic multiplicity of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are the same.

It is worth mentioning that the biLipschitz hypothesis is also used in [14] to give a positive answer for the Zariski’s Multiplicity Problem.

Another object associated to a non-dicritical foliation is its projective holonomy representation. Cerveau and Sad in [4] pose the following problem: Assuming $\mathcal{F}$ is a non-dicritical generalized curve, it is true that the projective holonomy groups of $\mathcal{F}$ and $\mathcal{F}'$ are topologically conjugated? Also in [4] the authors give a positive answer for a generic class of foliations $\mathcal{F}$ and assuming that $h$ is a topologically trivial deformation. Stronger results in relation to this subject are obtained in [5], [10], [9], and [17]. We must remark the work of Marín and Mattei ([9]), who prove the topological invariance of the projective holonomy for a generic class of generalized curves, although the problem is still unsettled if we allow saddle node singularities after resolution. Thus, as a second result of this work we prove that the projective holonomy representation is a biLipschitz invariant for a large class of foliations $\mathcal{F}$.

**Theorem 2.** Let $\mathcal{F} \in \mathfrak{G}$ and let $\tilde{\mathcal{F}}$ be a foliation topologically equivalent to $\mathcal{F}$ by a biLipschitz homeomorphism. Then $\tilde{\mathcal{F}} \in \mathfrak{G}$ and the projective holonomy representations of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are topologically conjugated.

2. **Itinerary and some remarks**

Given foliations $\mathcal{F}$ and $\tilde{\mathcal{F}}$ with isolated singularities at $0 \in \mathbb{C}^2$, we say that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are topologically equivalent (at $0 \in \mathbb{C}^2$) if there is an orientation preserving homeomorphism $h : \mathfrak{U} \to \tilde{\mathfrak{U}}$, $h(0) = 0$ between neighborhoods of $0 \in \mathbb{C}^2$, taking leaves of $\mathcal{F}$ to leaves of $\tilde{\mathcal{F}}$. Such a homeomorphism is a topological equivalence between $\mathcal{F}$ and $\tilde{\mathcal{F}}$. Let $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ be the quadratic blow up at $0 \in \mathbb{C}^2$, let $E = \pi^{-1}(0)$ be the exceptional divisor and let $\mathcal{F}_s$ and $\tilde{\mathcal{F}}_s$ be the strict transforms of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ by $\pi$. We will always assume that $\mathcal{F}$ (and therefore $\tilde{\mathcal{F}}$) is non-dicritical, that is, the exceptional divisor $E$ is invariant by $\mathcal{F}_s$. We know that $h$ lifts to a homeomorphism

$$h = \pi^{-1}h \pi : \pi^{-1}(\mathfrak{U})\setminus E \to \pi^{-1}(\tilde{\mathfrak{U}})\setminus \tilde{E}$$

which takes leaves of $\mathcal{F}_s$ to leaves of $\tilde{\mathcal{F}}_s$ and such that $h(w) \to E$ as $w \to E$. Conversely, if $W$ and $\tilde{W}$ are neighborhoods of $E$ and $\tilde{h} : W \setminus E \to \tilde{W} \setminus E$ is a homeomorphism taking leaves of $\mathcal{F}_s$ to leaves of $\tilde{\mathcal{F}}_s$ and such that $\tilde{h}(w) \to E$ as $w \to E$, then $\tilde{h}$ induces a topological equivalence between $\mathcal{F}$ and $\tilde{\mathcal{F}}$. Thus, by simplicity, we will say that any such $\tilde{h}$ is a topological equivalence between $\mathcal{F}$ and
Moreover, when no confusion arises we will often denote \( \tilde{\mathcal{F}} \) and \( \mathcal{F} \), respectively.

The starting point in the proof of Theorem 3 is the following result proved in [16]. We say that a complex line \( \mathcal{P} \) passing through \( 0 \in \mathbb{C}^2 \) is \( \mathcal{F} \)-generic if the strict transform of \( \mathcal{P} \) by \( \pi \) meets no singular point of \( \mathcal{F} \) in the exceptional divisor.

**Theorem 3.** Let \( \mathcal{P} \) be a \( \mathcal{F} \)-generic complex line passing through \( 0 \in \mathbb{C}^2 \) and suppose that \( \mathcal{H} \) maps \( \mathcal{P} \cap \mathcal{U} \) into a \( \tilde{\mathcal{F}} \)-generic complex line. Then the algebraic multiplicity of \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) are equal.

The strict transform \( \tilde{P} \) of \( \mathcal{P} \) is a Hopf fiber passing through a point \( p \in E \). The hypothesis in Theorem 3 basically means that, for some neighborhood \( D \) of \( p \) in \( \tilde{P} \), we have that \( \mathcal{H} \) maps \( \tilde{D} = D \setminus \{p\} \) into a Hopf fiber. In particular this means that \( \mathcal{H}|_{\tilde{D}} \) extends continuously to \( p \), which is the essence of the Hypothesis in Theorem 3. In general there is no extension of \( \mathcal{H}|_{\tilde{D}} \) to \( p \), even with the bilipschitz hypothesis. However, the strategy in the proof of Theorem 3 is to use \( \mathcal{H} \) and the bilipschitz hypothesis to construct another topological equivalence satisfying the hypothesis in Theorem 3.

Assume from now on that \( \mathcal{H} \) is bilipschitz. By the Separatrix Theorem there exists an irreducible \( \mathcal{F} \)-invariant curve \( \mathcal{S} \) passing through \( 0 \in \mathbb{C}^2 \). Then \( \tilde{\mathcal{S}} = \mathcal{H}(\mathcal{S}) \) is an irreducible \( \tilde{\mathcal{F}} \)-invariant curve. We can assume that \( \mathcal{S} \) and \( \tilde{\mathcal{S}} \) are tangent to \( \{(x, y) \in \mathbb{C}^2 : y = 0\} \) at \( 0 \in \mathbb{C}^2 \). Let \( (t, x) \) be coordinates in \( \mathbb{C}^2 \) such that \( \pi(t, x) = (x, tx) \in \mathbb{C}^2 \). In this chart the exceptional divisor \( E \) is represented by \( \{x = 0\} \). Let \( \mathcal{S} \) and \( \tilde{\mathcal{S}} \) be the strict transform of \( \mathcal{S} \) and \( \tilde{\mathcal{S}} \) by \( \pi \). Then \( \mathcal{S} \) and \( \tilde{\mathcal{S}} \) pass through the point \((0, 0) \in E \), so this point is singular for \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \). The first step in order to prove Theorems 1 and 2 is the following proposition, proved in Section 3.

**Proposition 4.** Denote \( h(t, x) = (\tilde{t}, \tilde{x}) \). There exist positive constants \( \rho, L_1 < 1 < L_2 \) and a continuous positive function \( \delta : (0, \rho] \to \mathbb{R} \) such that, if \( |t| = r \in (0, \rho] \) and \( 0 < |x| \leq \delta(r) \), then

\[ L_1 |t| \leq |\tilde{t}| \leq L_2 |t|. \]

**Remark 5.** Clearly we can assume \( L_1 = \frac{1}{2} \) and \( L_2 = L \) for some constant \( L > 1 \).

Fix \( r_1, r_2 \in (0, \rho] \) with \( r_1 < r_2 < Lr_2 \leq 1/2 \) and such that:

1. \( \text{Sing}(\mathcal{F}) \cap \{|t| \leq r_2\} = \{(0, 0)\} \)
2. \( \text{Sing}(\tilde{\mathcal{F}}) \cap \{|t| \leq Lr_2\} = \{(0, 0)\} \).

From Proposition 4 there exist constants \( \delta_0, \tilde{\delta}_0 > 0 \) such that for any \( (t, x) \) in the set

\[ V := \{r_1 \leq |t| \leq r_2, 0 < |x| \leq \delta_0\} \]

the point \((\tilde{t}, \tilde{x}) = h(t, x) \) satisfies

\[ \frac{1}{L}|t| \leq |\tilde{t}| \leq L|t| \text{ and } |\tilde{x}| < \tilde{\delta}_0. \]

In particular \( V \) is mapped by \( h \) into the set

\[ \tilde{V} := \left\{ \frac{1}{L}r_1 \leq |t| \leq Lr_2, 0 < |x| \leq \tilde{\delta}_0 \right\}. \]
Remark 6. Observe that we can take \( r_1, r_2 \) and \( r_1/r_2 \) arbitrarily small. Moreover, fixed \( r_1 \) and \( r_2 \) we can take \( \delta_0 \) and \( \tilde{\delta}_0 \) arbitrarily small. These facts will be used later.

The set \( V \) is the local where we will modify \( h \) in order to satisfy the hypothesis in Theorem 3. Suppose that the punctured Hopf fiber \( \mathcal{D}^* \) is taken in \( V \). In general the set \( h(\mathcal{D}^*) \) can accumulates to a large set in the exceptional divisor, for example a point \( w \in h(\mathcal{D}^*) \) could oscillate infinitely many times around \( \{ t = 0 \} \) as \( w \to E \). Observe that, is \( \tilde{\delta}_0 \) is small enough, the dynamic of \( \mathcal{F}|_V \) is basically the suspension of the holonomy map of the leaf \( E \cap V \). We can see this dynamic as the 1-foliation induced by \( \mathcal{F} \) on a set

\[
T = \{(t, x) : |t| = r_{12}, |x| \leq \delta_1\}
\]

with \( r_1 < r_{12} < r_2 \) and \( 0 < \delta_1 \leq \delta_0 \). The main step to prove Theorem 1 will be locate the problem in the set \( T \). Thus we will construct (Proposition 20) another topological equivalence \( \tilde{h} \) which (for suitable \( r_{12} \) and \( \delta_1 \)) maps \( T \) into the set \( \tilde{T} = \{(t, x) \in \tilde{V} : |t| = \sqrt{r_1 r_2}\} \).

Clearly the 1-foliation induced by \( \tilde{\mathcal{F}} \) on \( \tilde{T} \) can be defined by a flow. The local punctured Hopf fibers in \( T \) are mapped by \( \tilde{h} \) into sets which in general are not Hopf fibers in \( \tilde{T} \), but the idea is to arrive to this situation by using the flow to move each point of \( \tilde{T} \) the correct amount. This idea is realized in Section 9 and we construct another topological equivalence \( \bar{h} \) mapping Hopf fibers in \( \tilde{T} \) into Hopf fibers in \( \tilde{T} \), which is the final step in order to prove Theorem 1. All the constructions summarized above can be performed in a neighborhood of each singularity of \( \mathcal{F} \) in \( E \) having a separatrix other than \( E \). This is the starting point in the proof of Theorem 2 in Section 13.

3. PROOF OF PROPOSITION 4

We use the following Lemma.

**Lemma 7.** If \( z_1, z_2 \in \text{dom}(h) \) and \( z_2 \neq 0 \), then

\[
\frac{m}{M} \frac{|z_1 - z_2|}{|z_2|} \leq \frac{|h(z_1) - h(z_2)|}{|h(z_2)|} \leq \frac{M}{m} \frac{|z_1 - z_2|}{|z_2|}.
\]

**Proof.** We have

\[
|h(z_2)| = |h(z_2) - h(0)| \geq m|z_2 - 0| = m|z_2|,
\]

hence

\[
|z_2| \leq (1/m)|h(z_2)|.
\]

Then

\[
|h(z_1) - h(z_2)| \leq M|z_1 - z_2| = M \frac{|z_1 - z_2|}{|z_2|} |z_2| \leq (M/m) \frac{|z_1 - z_2|}{|z_2|} |h(z_2)|
\]

and we obtain the right hand inequality. The other inequality is similar.
3.1. Proof of Proposition 4. Since $S$ and $\tilde{S}$ are tangent to $\{y = 0\}$ at $0 \in \mathbb{C}^2$, there exist neighborhoods $U$ and $\tilde{U}$ of $0 \in \mathbb{C}^2$ and a constant $l > 0$ such that:

(i) If $(x, y) \in S \cap U$, then $|y| \leq |x|^2$.
(ii) If $(x, y) \in \tilde{S} \cap \tilde{U}$, then $|y| \leq |x|^2$.
(iii) Given $(x, y) \in U$, there exists $y' \in \mathbb{C}$ such that $(x, y') \in S \cap U$.
(iv) $h(U) \subset \tilde{U}$.

Let $(t, x)$ be such that $x \neq 0$ and $(x, y) := \pi(t, x) = (x, tx)$ is contained in $U$. By (iii) there exists $y'$ such that $(x, y') \in S \cap U$. Then

$$|(x, y) - (x, y')| = |y - y'| \leq |y| + |y'| \leq |t||x| + l|x|^2$$

$$= (|t| + l|x|)|x| \leq (|t| + l|x|)(x, y)|,$$

hence

$$\frac{|(x, y) - (x, y')|}{|(x, y')|} \leq |t| + l|x|.$$

and by Lemma 7

$$\frac{|h(x, y) - h(x, y')|}{|h(x, y')|} \leq \frac{M}{m}(|t| + l|x|).$$

Since $h(S) = \tilde{S}$ and $(x, y') \in S \cap U$, by (iv) we have $(\tilde{x}, \tilde{y}) := h(x, y') \in \tilde{S} \cap \tilde{U}$, hence $|\tilde{y}| \leq l|\tilde{x}|^2$ and therefore

$$|h(x, y) - h(x, y')| \leq \frac{M}{m}(|t| + l|x|)(\tilde{x}, \tilde{y}) \leq \frac{M}{m}(|t| + l|x|)(|\tilde{x}| + |\tilde{y}|)$$

$$\leq \frac{M}{m}(|t| + l|x|)(|\tilde{x}| + l|\tilde{x}|^2),$$

that is:

$$|h(x, y) - h(x, y')| \leq \frac{M}{m}(|t| + l|x|)(1 + l|\tilde{x}|)|\tilde{x}|.\quad (3.1)$$

If $h(x, y) = (\tilde{x}, \tilde{y})$, then

$$|h(x, y) - h(x, y')| = |(\tilde{x}, \tilde{y}) - (\tilde{x}, \tilde{y})| = |(\tilde{x} - \tilde{x}, \tilde{y} - \tilde{y})| \geq |\tilde{x} - \tilde{x}| \geq |\tilde{x} - |\tilde{x}|$$

and by equation (3.1)

$$|\tilde{x} - |\tilde{x}| \leq \frac{M}{m}(|t| + l|x|)(1 + l|\tilde{x}|)|\tilde{x}|,$$

and so

$$|\tilde{x}| \leq \frac{1}{1 - \frac{M}{m}(|t| + l|x|)(1 + l|\tilde{x}|)}|\tilde{x}|.\quad (3.2)$$

On the other hand:

$$|h(x, y') - (\tilde{x}, 0)| = |(\tilde{x}, \tilde{y}) - (\tilde{x}, 0)| = |\tilde{y}| \leq l|\tilde{x}|^2,$$

then

$$|\tilde{y}| \leq |(\tilde{x} - \tilde{x}, \tilde{y})| = |(\tilde{x}, \tilde{y}) - (\tilde{x}, 0)| = |h(x, y) - (\tilde{x}, 0)|$$

$$\leq |h(x, y) - h(x, y')| + |h(x, y') - (\tilde{x}, 0)|,$$

by equations (3.1) and (3.3)

$$|\tilde{y}| \leq \frac{M}{m}(|t| + l|x|)(1 + l|\tilde{x}|)|\tilde{x}| + l|\tilde{x}|^2 = \left(\frac{M}{m}(|t| + l|x|)(1 + l|\tilde{x}|) + l|\tilde{x}|\right)|\tilde{x}|.$$
and by equation [3.2]

\[ |\ddot{y}| \leq \left( \frac{M(|t| + l|x|)(1 + l|x|)}{1 - \frac{M}{m}(|t| + l|x|)(1 + l|x|)} \right) |\tilde{x}|, \]

hence

\[ |\ddot{t}| \leq \frac{M(|t| + l|x|)(1 + l)f(|x|)) + lf(|x|)}{1 - \frac{M}{m}(|t| + l|x|)(1 + l|f(|x|))}. \]

By Puiseux Theorem there is a local parametrization of \((S,0)\) of the form \(\psi(\zeta) = (\zeta^n, \psi_2(\zeta))\) defined on a small disc \(|\zeta| \leq \varrho\). Denote \(\mathfrak{h} = (h_1, h_2)\). Given \(s \in [0, \varrho^n]\) define

\[ f(s) = \sup\{|h_1(\psi(\zeta))| : |\zeta| \leq s^{1/n}\}. \]

It is easy to see that \(f\) is a strictly increasing continuous function. Observe that

\[ |\tilde{x}| = |h_1(x, y')| = |h_1(\psi(x^{1/n}))| \leq f(|x|). \]

Using this fact in equation [3.5] we obtain

\[ |\ddot{t}| \leq \frac{M(|t| + l|x|)(1 + l)f(|x|)) + lf(|x|)}{1 - \frac{M}{m}(|t| + l|x|)(1 + l|f(|x|))}. \]

Let \(f^{-1}\) be the inverse function of \(f\) and consider the strictly increasing function \(\eta(r) = \min\{f^{-1}(r), r/l\}\) defined on an interval \([0, \rho]\). Suppose now \(|t| = r \in (0, \rho]\) and \(|x| \leq \eta(r)\). Using the inequalities \(|x| \leq r/l\) and \(|x| \leq f^{-1}(r)\) in equation [3.6] we obtain

\[ |\ddot{t}| \leq \frac{M(|t| + l(r/l))(1 + l|f^{-1}(r)|)) + lf^{-1}(r)|}{1 - \frac{M}{m}(|t| + l(r/l))(1 + l|f^{-1}(r)|))} = \frac{2M}{2M}r(1 + lr) + lr \]

Clearly we can take \(\rho\) small enough such that \((1 + lr) \leq 2\) and \(2M/r(1 + lr) \leq 1/2\) and therefore we obtain

\[ |\ddot{t}| \leq \frac{2M}{2M}r(2) + lr = Lr, \]

were \(L = (8M/m + 2l)\).

If we apply the above arguments to \(\mathfrak{h}^{-1}\) we find positive constants \(\tilde{\rho}, \tilde{L}\) and a strictly increasing continuous function \(\tilde{\eta} : [0, \tilde{\rho}] \to [0, +\infty)\) such that \(|t| \leq \tilde{L}|\tilde{t}|\), whenever \(|\tilde{t}| = \tilde{r} \in [0, \tilde{\rho}]\) and \(|\tilde{x}| \leq \tilde{\eta}(\tilde{r})\). Define the functions \(f' : [0, \rho] \to [0, +\infty]\) and \(f_r : [0, \eta(r)] \to [0, +\infty] |r \in (0, \rho)|\) by \(f'(0) = 0, f_r(0) = 0\) and

\[ f'(s) = \inf\{|\ddot{t}| : |t| = s, 0 < |x| \leq \eta(s)\}, \]

\[ f_r(s) = \sup\{|\ddot{x}| : |t| = r, 0 < |x| \leq s| \}

if \(s > 0\). Clearly \(f'(s) \leq Ls \leq L\rho\) and by reducing \(\rho\) if necessary we can assume \(f'(s) \leq \tilde{\rho}\), hence \(\tilde{\eta} \circ f'\) is well defined. For \(r \in (0, \rho]\) define \(\delta(r) > 0\) as follows. If \(f_r(\eta(r)) \leq \tilde{\eta} \circ f'(r)\) define \(\delta(r) = \eta(r)\), otherwise we choose \(\delta(r)\) such that \(f_r(\delta(r)) = \tilde{\eta} \circ f'(r)\); this is possible because the function \(f_r\) is continuous and \(f_r(0) = 0\). In any case we have \(f_r(\delta(r)) \leq \tilde{\eta} \circ f'(r)\). Moreover, since \(f_r\) is increasing we have \(\delta \leq \eta\). Consider now \((t, x)\) such that \(|t| = r \in (0, \rho]\) and \(|x| \leq \delta(r)\). Then

\[ |\tilde{x}| \leq \sup\{|\ddot{x}| : |t| = r, 0 < |x| \leq \delta(r)| = f_r(\delta(r)) \leq \tilde{\eta} \circ f'(r). \]
But \( \tilde{\eta} \circ f'(r) \leq \tilde{\eta}(\tilde{t}) \) because \( \tilde{\eta} \) is increasing, hence \( |\tilde{x}| \leq \tilde{\eta}(\tilde{t}) \) and by definition of \( \tilde{\eta} \) we have \( |t| \leq \tilde{L}|\tilde{t}| \). Since \( \delta \leq \eta \) we have \( |x| \leq \eta(r) \), hence \( |t| \leq L|t| \). Therefore

\[
L_1|t| \leq |\tilde{t}| \leq L_2|t|,
\]

where \( L_1 = 1/\tilde{L} \) and \( L_2 = L \). Now it is sufficient to find a continuous function \( \delta' \) such that \( 0 < \delta' \leq \delta \). By using a partition of unity it is easy to see that it suffices to show that \( \liminf \delta'(r') > 0 \) for all \( r' \in (0, \rho] \). Suppose by contradiction that there is a sequence \( (r_n) \) with \( r_n \to r \) and \( \delta(r_n) \to 0 \). By definition we have \( \delta(r_n) = \eta(r_n) \) or \( f_n(\delta(r_n)) = \tilde{\eta} \circ f'(r_n) \). But \( \delta(r_n) = \eta(r_n) \) does not occur infinitely many times because \( \tilde{\eta} \) is continuous and \( \eta(r) > 0 \). Then assume that

\[
(3.7) \quad f_n(\delta(r_n)) = \tilde{\eta} \circ f'(r_n)
\]

for all \( n \in \mathbb{N} \) and suppose moreover that \( f'(r_n) \to \epsilon \geq 0 \). As we will show at the end of the proof, \( f_n(s) \) depends continuously on \( (r, s) \). Then, if \( n \to \infty \) from equation 3.7 we obtain \( f_n(0) = \tilde{\eta}(\epsilon) \). But \( f_n(0) = 0 \) and therefore \( \epsilon = 0 \), that is, \( f'(r_n) \to 0 \).

By definition of \( f' \) we may take \( w_n = (t_n, x_n) \) with \( |t_n| = r_n, 0 < |x_n| \leq \eta(r_n) \) and such that \( |t_n| \to 0 \) as \( n \to \infty \), where \( (t_n, x_n) = h(t_n, x_n) \). We can assume \( t_n \to \tilde{t}, x_n \to \bar{x} \) with \( |\tilde{t}| = r, 0 \leq |\bar{x}| \leq \eta(r) \). But \( \bar{x} \neq 0 \) implies by continuity of \( h \) that \( |t_n| \to |\tilde{t}| > 0 \), where \( (\tilde{t}, \bar{x}) = h(\tilde{t}, \bar{x}) \), contradiction. Then we have \( x_n \to 0 \). Take \( \tilde{r} > 0 \) such that \( \tilde{L}\tilde{r} < r \) and set

\[
\tilde{T} = \{ (t, x) : |t| = \tilde{r}, 0 < |x| \leq \tilde{\eta}(\tilde{r}) \}.
\]

As we have seen \( h^{-1}(\tilde{T}) \) is contained in \( \{ |t| \leq \tilde{L}\tilde{r} \} \). Then, since \( \tilde{L}\tilde{r} < r \), provided \( n \) is large enough the point \( w_n \) can be connected with the set

\[
T = \{ (t, x) : |t| = r, 0 < |x| \leq \eta(r) \}
\]

by a curve \( \alpha_n \) contained in the set

\[
\{ \tilde{L}\tilde{r} < |t| \leq r, 0 < |x| \}.
\]

In particular \( \alpha_n \) does not meet the set \( h^{-1}(\tilde{T}) \) and consequently the curve \( h(\alpha_n) \) does not meet \( \tilde{T} \). Moreover, since \( \alpha_n \) meets \( T \) and \( h(T) \) is contained in

\[
\{ L_1r \leq |t| \} = \{ \frac{1}{L}r \leq |t| \} \subset \{ \tilde{r} < |t| \},
\]

then \( h(\alpha_n) \) meets the set \( \{ \tilde{r} < |t| \} \). Observe that \( \alpha_n \) may be taken arbitrarily close to the exceptional divisor for \( n \) large enough. Then by connectedness we see that, since \( h(\alpha_n) \) does not meet \( \tilde{T} \), necessarily \( h(\alpha_n) \) is contained in the set \( \{ \tilde{r} < |t| \} \).

In particular \( h(w_n) \in \{ \tilde{r} < |t| \} \) and therefore \( |t_n| > \tilde{r} > 0 \) for \( n \in \mathbb{N} \) large enough, which is a contradiction. It remains to prove that \( f_r(s) \) depends continuously on \( (r, s) \). Fix \( r_0 > 0, s_0 \in (0, \eta(r_0)) \). Since \( h(w) \) tends to the exceptional divisor as \( w \) tends to the exceptional divisor we may find \( \epsilon > 0 \) such that

\[
f_r(s) = \sup \{ |\bar{x}| : |t| = r, \epsilon < |x| \leq s \}
\]

for all \( (r, s) \) close enough to \( (r_0, s_0) \). From here the continuity at \( (r_0, s_0) \) is easy to establish. If \( s_0 = 0 \) clearly we have \( f_{r_0}(s_0) = 0 \). If \( \epsilon > 0 \) is small enough the set

\[
V = \{ r - \epsilon \leq |t| \leq r + \epsilon, 0 < |x| \leq \epsilon \}
\]

is mapped by \( h \) arbitrarily close to the exceptional divisor. Then \( |\bar{x}| \) is uniformly small for \( (t, x) \in V \) and the continuity at \( (r_0, s_0) \) follows.
Corollary 8. Let $L_1$ and $L_2$ be as in Proposition 19. Then there exist a constant $\hat{\rho} > 0$ and a continuous positive function $\hat{\delta} : (0, \hat{\rho}) \to \mathbb{R}$ such that, if $|\hat{t}| = \tau \in (0, \hat{\rho}]$ and $0 < |\hat{x}| \leq \hat{\delta}(\tau)$, then

$$L_1|\hat{t}| \leq |	au| \leq L_2|\hat{t}|.$$  

Proof. This is a corollary of the proof of Proposition 19. □

Remark 9. As in remark 5 we will assume $L_1 = 1/L, L_2 = L$.

4. Complex time distortion

In this Section we fix complex flows associated to the foliations on $V$ and $\tilde{V}$ and study the distortion of the complex time induced by $h$. Thus we establish Equation 4.1 and state Proposition 10 which give us some estimations for the complex time distortion.

Since the foliations $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are tangent to the exceptional divisor, we can assume that $\mathcal{F}|_V$ and $\tilde{\mathcal{F}}|_{\tilde{V}}$ are generated by holomorphic vector fields

$$Z = t \frac{\partial}{\partial t} + xQ \frac{\partial}{\partial x}$$

and

$$\tilde{Z} = t \frac{\partial}{\partial t} + x\tilde{Q} \frac{\partial}{\partial x},$$

where $Q$ and $\tilde{Q}$ are holomorphic functions on $V$ and $\tilde{V}$ respectively. Consider the complex flows $\phi^T = (\phi_1^T, \phi_2^T)$ and $\tilde{\phi}^T = (\tilde{\phi}_1^T, \tilde{\phi}_2^T)$ associated to $Z$ and $\tilde{Z}$ respectively. Clearly we have $\phi_1^T(t, x) = \phi_1^T(t, x) = te^\tau$. Consider $\varphi = (t, x) \in V$ and

$$T \in \{ \tau + \theta i \in \mathbb{C} : \ln \frac{r_1}{|\tau|} \leq \tau \leq \ln \frac{r_2}{|\tau|} \}$$

such that the path $\gamma^T_w(s) = \phi^T(\varphi), s \in [0, 1]$ is contained in $V$. Since $h \circ \gamma^T_w$ is a path starting at $h(\varphi)$ and contained in a leaf of $\tilde{\mathcal{F}}|_{\tilde{V}}$, we can write $h \circ \gamma^T_w(s) = \tilde{\phi}^T(s)(h(\varphi))$, where $\sigma : [0, 1] \to \mathbb{C}$ is a continuous path with $\sigma(0) = 0$. Then define

$$\tilde{T}(w, T) = \sigma(1),$$

that is: “the complex time between $h(\varphi)$ and $h \phi^T(\varphi)$”.

Put $\tilde{T} = \tilde{\tau} + i\tilde{\theta}$. Since $h \circ \gamma^T_w(1) = \tilde{\phi}^T(1)(h(\varphi))$, then

$$h(\phi_1^T(w), \phi_2^T(w)) = (\phi_1^T h(w), \phi_2^T h(w)).$$

and by Equation 2.1 for the point $\phi^T(w)$:

$$\frac{1}{L} |\phi_1^T(w)| \leq |\tilde{\phi}_1^T h(w)| \leq L |\phi_1^T(w)|.$$  

Then, if $h(w) = (\tilde{\varphi}, \tilde{x})$ we have

$$\frac{1}{L} |\tau e^\tau| \leq |\tilde{\tau} e^\tau| \leq L |\tau e^\tau|$$

and together with equation 2.1 we obtain

$$\tau - 2 \ln L \leq \tilde{\tau}(w, T) \leq \tau + 2 \ln L,$$

which give us a control on the real part of the complex time $\tilde{T}$ only in terms of the real part of the complex time $T$. Sections 6 and 7 are devoted to prove the
Proposition 10. There exist increasing homeomorphisms \( \varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R} \) such that one of the following situations holds:

1. \( \varphi_1(\theta) \leq \hat{\theta}(w, T) \leq \varphi_2(\theta) \) for any \( (w, T) \) such that \( \hat{\theta} \) is defined.
2. \( \varphi_1(\theta) \leq -\hat{\theta}(w, T) \leq \varphi_2(\theta) \) for any \( (w, T) \) such that \( \hat{\theta} \) is defined.

Remark 11. Case 1 and case 2 in Proposition 10 happen according to \( h \) preserves or reverses the natural orientations of the leaves.

5. Homological Compatibility

In this section we describe the map \( h : V \to \tilde{V} \) at homology level. Thus, Proposition 12 and Lemma 13 discard any homological obstruction to perform the constructions in the rest of the paper. Proposition 12 is a special version of a kind of results previously obtained in [9] (Theorem 6.2.1) and [17] (section 5).

Proposition 12. Let \( (a, b) \in V, (\tilde{a}, \tilde{b}) \in \tilde{V} \) and consider the loops \( \alpha, \beta : [0, 1] \to V, \tilde{\alpha}, \tilde{\beta} : [0, 1] \to \tilde{V} \) given by \( \alpha(s) = (ae^{s2\pi i}, b), \beta = (a, be^{s2\pi i}), \tilde{\alpha}(s) = (\tilde{a}e^{s2\pi i}, \tilde{b}) \) and \( \tilde{\beta}(s) = (\tilde{a}, be^{s2\pi i}) \). Put \( \xi = 1 \) or \(-1\) according to \( h \) preserves or reverses the natural orientation of the leaves. Then, in the first homology group of \( \tilde{V} \) we have

\[ [h(\alpha)] = \xi\tilde{\alpha} \quad \text{and} \quad [h(\beta)] = \xi\tilde{\beta}. \]

Let \( w \in V \) and consider the map \( \phi_w \) defined by \( \phi_w(T) = \phi^T(w_1) \) and whose domain is the connected component of \( 0 \in \mathbb{C} \) of the set where \( \phi^T(w_1) \) is defined. Clearly we have to cases:

1. The map \( \phi_w \) is injective, or
2. There exists \( k > 0 \) such that \( \phi_w \) has period \( 2k\pi i \). In this case we say that \( w \) has period \( 2k\pi i \).

In the same way we define the map \( \tilde{\phi}_w \) for any \( w \in \tilde{V} \). Observe that \( \tilde{T}(w, T) \) is defined if \( T \) is in the domain of \( \phi_w \).

Lemma 13. Let \( \xi \) as in Proposition 12. Then \( w \) has period \( 2k\pi i \) if and only if \( h(w) \) has period \( \xi 2k\pi i \) and

\[ \tilde{T}(w, T + 2k\pi i) = \tilde{T}(w, T) + \xi 2k\pi i \]

for any \( T \) in the domain of \( \phi_w \).

Proof. This is an easy application of Proposition 12. \( \square \)

5.1. Proof of Proposition 12. For some \( m_1, n_1, m_2, n_2 \in \mathbb{Z} \) we have

\[ h(\alpha) = m_1\tilde{\alpha} + n_1\tilde{\beta} \quad \text{and} \quad h(\beta) = m_2\tilde{\alpha} + n_2\tilde{\beta} \]

in \( H_1(\tilde{V}) \).

Take neighborhoods \( W \) and \( \tilde{W} \) of the exceptional divisor \( E = \pi^{-1}(0) \) with the following properties:

1. \( W \) contains \( V \)
2. \( W \cap S \) is closed in \( W \) and it is homeomorphic to a disc
3. \( h(W \setminus E) = \tilde{W} \setminus E \)
4. \( \pi(W) \) and \( \pi(\tilde{W}) \) are homeomorphic to balls.
Let $S = \pi(W \cap S)$ and $\tilde{S} = h(S)$. Since $\pi(W)$ is homeomorphic to $\mathbb{C}^2$ and $S$ is closed in $\pi(W)$ and homeomorphic to $\mathbb{C}$, we have by Alexander’s duality that $H_1(\pi(W)\setminus S) \cong \mathbb{Z}$. Then, since $W^* = W \setminus (E \cup \tilde{S})$ is homeomorphic to $\pi(W)\setminus S$ we have $H_1(W^*) \cong \mathbb{Z}$. In the same way, if $\tilde{W}^* = \tilde{W} \setminus (E \cup \tilde{S})$ we have $H_1(\tilde{W}^*) \cong \mathbb{Z}$. Let $D \subset W$ and $\tilde{D} \subset \tilde{W}$ be small complex discs transverse to $S$ and $\tilde{S}$ respectively. If $D$ and $\tilde{D}$ are small enough we have that $\partial D$ and $\partial \tilde{D}$ are generators of $H_1(W^*)$ and $H_1(\tilde{W}^*)$ respectively. Since $h$ preserves orientation it follows from the topological invariance of the intersection number\footnote{See [5] p. 200.} that

$$h(\partial D) = \xi \partial \tilde{D} \text{ in } H_1(\tilde{W}^*).$$

Without loss of generality we can assume that $\tilde{\alpha}$ and $\tilde{\beta}$ are contained in a set $\tilde{V}_\varepsilon = \{(t, x) \in \tilde{V} : |x| \leq \varepsilon\}$ such that $\tilde{V}_\varepsilon \subset \tilde{W}^*$. Then we can write equations (5.1) in $H_1(\tilde{V}_\varepsilon)$ and consequently we have

$$h(\alpha) = m_1 \tilde{\alpha} + n_1 \tilde{\beta} \text{ and } h(\beta) = m_2 \tilde{\alpha} + n_2 \tilde{\beta} \text{ in } H_1(\tilde{W}^*).$$

Moreover, by equisingularity there exist $p, q \in \mathbb{Z}$ such that

$$\alpha = p \partial D \text{ and } \beta = q \partial D \text{ in } H_1(W^*),$$

$$\tilde{\alpha} = p \partial \tilde{D} \text{ and } \tilde{\beta} = q \partial \tilde{D} \text{ in } H_1(\tilde{W}^*).$$

Then, since $h(W^*) = \tilde{W}^*$ from equations (5.2), (5.3) and (5.4) we obtain

$$\xi p = m_1 p + n_1 q,$$
$$\xi q = m_2 p + n_2 q.$$

Put

$$\mathfrak{V}_\varepsilon = \{(t, x) : |t| \leq r_2, 0 < |x| \leq \varepsilon\}$$

for $\varepsilon > 0$ and

$$\tilde{\mathfrak{V}} = \{(t, x) : |t| \leq L r_2, 0 < |x| \leq \tilde{\delta}_0\}.$$

**Assertion.** If $\varepsilon > 0$ is taken small enough $h(\mathfrak{V}_\varepsilon)$ is contained in $\tilde{\mathfrak{V}}$.

Take $w \in \mathfrak{V}_\varepsilon$. If $w \in V$ clearly we have $h(w) \in \tilde{V} \subset \mathfrak{V}$, so we assume $w \notin V$ and in particular

$$w \in \mathfrak{Y}_\varepsilon = \{(t, x) \in \mathfrak{V}_\varepsilon : |t| < r_2\}.$$

From Proposition\footnote{Clearly we can assume $L r_2 \leq \tilde{\delta}$.} we see that $w$ can be connected by a path $\gamma$ in $\mathfrak{Y}_\varepsilon$ with a point $w' \in \mathfrak{Y}_\varepsilon$ such that $h(w') \in \mathfrak{V}$. It follows from Corollary\footnote{See [5] p. 200.} that the set

$$\Sigma = \{(t, x) : |t| = L r_2, 0 < |x| \leq \tilde{\delta}(L r_2)\}$$

is mapped by $h^{-1}$ outside $\mathfrak{Y}_\varepsilon$. In particular $h(\gamma)$ does not meet $\Sigma$. Then, since $\varepsilon$ small implies $h(\gamma)$ close to the exceptional divisor, we deduce that $h(\gamma)$ can not leave $\mathfrak{V}$ and therefore $h(w) \in \mathfrak{V}$ for $\varepsilon$ small enough.

Without loss of generality we can assume $\alpha, \beta \subset \mathfrak{V}_\varepsilon$ and since $\alpha = 0$ in $H_1(\mathfrak{V}_\varepsilon)$ the assertion above implies

$$h(\alpha) = 0 \text{ in } H_1(\mathfrak{V}).$$
Let $\partial S$ and $\partial \tilde{S}$ be positively oriented loops in $S$ and $\tilde{S}$ such that $\pi(\partial S)$ and $\pi(\partial \tilde{S})$ are generators of $H_1(S\setminus \{0\})$ and $H_1(\tilde{S}\setminus \{0\})$ respectively. Clearly we have
\begin{equation}
(5.7) \quad h(\partial S) = \xi \partial \tilde{S} \text{ in } H_1(\tilde{\mathcal{V}}).
\end{equation}

By equisingularity there exist $k \in \mathbb{Z}$ such that
\begin{align*}
\partial S &= k \beta \text{ in } H_1(\mathcal{V}_\epsilon), \\
\partial \tilde{S} &= k \tilde{\beta} \text{ in } H_1(\tilde{\mathcal{V}}).
\end{align*}

Then, since $h(\mathcal{V}_\epsilon) \subset \tilde{\mathcal{V}}$ we have
\begin{align*}
h(\partial S) &= kh(\beta) \text{ in } H_1(\tilde{\mathcal{V}}), \\
\partial \tilde{S} &= k \tilde{\beta} \text{ in } H_1(\tilde{\mathcal{V}}),
\end{align*}
and by (5.7) we obtain
\begin{equation}
(5.8) \quad h(\beta) = \xi \tilde{\beta} \text{ in } H_1(\tilde{\mathcal{V}}).
\end{equation}

Since $\tilde{V} \subset \tilde{\mathcal{V}}$ we can write equations (5.1) in $H_1(\tilde{\mathcal{V}})$ and using that $\tilde{\alpha} = 0$ in $H_1(\tilde{\mathcal{V}})$ we obtain:
\begin{equation}
(5.9) \quad h(\alpha) = n_1 \tilde{\beta} \text{ and } h(\beta) = n_2 \tilde{\beta} \text{ in } H_1(\tilde{\mathcal{V}}).
\end{equation}

Thus, from equations (5.6), (5.8) and (5.9) we obtain $n_1 = 0$ and $n_2 = \xi$ and together with equation (5.5) Proposition 12 follows.

6. Lipschitz condition along the leaves

In this section we study the consequences of the Lipschitz condition along the leaves and the following proposition is the main result of this section. We use this result in the proof of Proposition 11 in Section 7.

Proposition 14. There exists a constant $M_0 > 0$ with the following property. If $\gamma : [0,1] \to V$ is a continuous rectifiable path contained in a leaf of $\mathcal{F}$, then
\[ \frac{1}{M_0} l(\gamma) \leq l(h \circ \gamma) \leq M_0 l(\gamma). \]

We need some lemmas.

Lemma 15. For $j = 1,2$ put $w_j = (t_j, x_j)$, $z_j = \pi(w_j) = (x_j, t_j x_j)$ and suppose $|x_j|, |t_j| \leq \frac{1}{2}$.

1. If $\xi = \max\{|x_1|, |x_2|\} \neq 0$, then
\[ |w_1 - w_2| \leq \frac{1}{\xi} |z_1 - z_2|. \]

2. If $|x_1 - x_2| \leq \eta |x_1| |t_1 - t_2|$ for some $\eta > 0$, then
\[ |z_1 - z_2| \leq (2\eta + 1) |x_1| |w_1 - w_2|. \]

3. Here $l(\gamma)$ denote the length of $\gamma$ induced by the norm $|(t,x)| = |t| + |x|$. 
Proof. we have
\[ |z_1 - z_2| = |(x_1, t_1x_1) - (x_2, t_2x_2)| = |x_1 - x_2| + |t_1x_1 - t_2x_2| \]
\[ = |x_1 - x_2| + |(t_1 - t_2)x_1 - t_2(x_2 - x_1)| \]
\[ \geq |x_1 - x_2| + |x_1||t_1 - t_2| - |t_2||x_2 - x_1|, \]
but \(|t_1|, |x_1| \leq \frac{1}{2},\) then
\[ |z_1 - z_2| \geq \frac{1}{2} |x_1 - x_2| + |x_1||t_1 - t_2| \]
\[ \geq |x_1||x_1 - x_2| + |x_1||t_1 - t_2| \]
\[ = |x_1||w_1 - w_2| \]
and therefore \(|w_1 - w_2| \leq \frac{1}{|x_1|}|z_1 - z_2|.|\) In the same way \(|w_1 - w_2| \leq \frac{1}{|x_2|}|z_1 - z_2|.|\) and item (1) follows.

On the other hand we have
\[ |z_1 - z_2| = |x_1 - x_2| + |(t_1 - t_2)x_1 - t_2(x_2 - x_1)| \]
\[ \leq |x_1 - x_2| + |x_1||t_1 - t_2| + |t_2||x_2 - x_1| \]
\[ \leq |x_1 - x_2| + |x_1||t_1 - t_2| + |x_2 - x_1| = 2|x_1 - x_2| + |x_1||t_1 - t_2| \]
\[ \leq 2\eta |x_1||t_1 - t_2| + |x_1||t_1 - t_2| \leq (2\eta + 1)|x_1||t_1 - t_2| \]
\[ \leq (2\eta + 1)|x_1||w_1 - w_2|. \]

\[ \square \]

Remark 16. By remark 6 we may assume \(\delta_0, \tilde{\delta}_0 \leq \frac{1}{2}.\) Then Lemma 15 holds for points \(w_j\) in \(V\) or \(\tilde{V}\).

Lemma 17. For \(j = 1, 2\) let \(w_j = (t_j, x_j) \in V\) and denote \(h(w_j) = (\tilde{t}_j, \tilde{x}_j).\) Given \(\eta > 0,\) there is a constant \(M_0 = M_0(\eta, m, M)\) such that, if \(|x_1 - x_2| \leq \eta|x_1||t_1 - t_2|\) and \(|\tilde{x}_1 - \tilde{x}_2| \leq \eta|\tilde{x}_1||\tilde{t}_1 - \tilde{t}_2|,\) then
\[ 1/M_0|w_1 - w_2| \leq |h(w_1) - h(w_2)| \leq M_0|w_1 - w_2|. \]

Proof. Put \(z_j = \pi(w_j).\) Since
\[ |h(z_1)| = |\tilde{x}_1| + |\tilde{t}_1||\tilde{x}_1| \leq 2|\tilde{x}_1|, \]
then
\[ |\tilde{x}_1| \geq \frac{1}{2} |h(z_1)| \geq \frac{m}{2} |z_1| \geq \frac{m}{2} |x_1|, \]
\[ \text{hence } \tilde{r} := \text{Max} \{|\tilde{x}_1|, |\tilde{x}_2|\} \geq \frac{m}{2} |x_1|. \] In the same way \(\tilde{r} \geq \frac{m}{2} |x_2|\) and therefore
\[ \tilde{r} \geq \frac{m}{2} r, \]
where \(r = \text{Max} \{|x_1|, |x_2|\}.\) Then by Lemma 15 we have
\[ |h(w_1) - h(w_2)| \leq \frac{1}{\tilde{r}} |h(z_1) - h(z_2)| \leq \frac{2}{m\tilde{r}} |h(z_1) - h(z_2)| \]
\[ \leq \frac{2M}{m\tilde{r}} |z_1 - z_2| \leq \frac{2M}{m\tilde{r}} (2\eta + 1)|x_1||w_1 - w_2| \]
\[ \leq \frac{2M}{m}(2\eta + 1)|w_1 - w_2|, \]
so we obtain the right hand inequality with \(M_0 = \frac{2M}{m}(2\eta + 1).\) The other hand is similar. \(\square\)
Let \( Q \) and \( \tilde{Q} \) be as in Section 4. Since \(|Q|\) and \(|\tilde{Q}|\) are bounded it is easy to see the following fact:

**Fact 18.** There exist \( \eta > 0 \) with the following property: if \( \Gamma(s) = (t(s), x(s)) \) is a differentiable path contained in a leaf of \( \mathcal{F}|_V \) or \( \tilde{\mathcal{F}}|_{\tilde{V}} \), then

\[
|x'(s)| \leq \frac{\eta}{2}|x(s)||t'(s)|.
\]

6.1. **Proof of Proposition 14.** Let

\[ 0 = s_0 < \ldots < s_n = 1 \]

be a partition of \([0, 1]\). Denote \( \gamma(s_j) = (t_j, x_j) \). If the partition is fine enough we have that each segment \([t_{j-1}, t_j]\) is contained in

\[
\{ t \in \mathbb{C}, \frac{1}{L} \leq |t| \leq L r_2 \}.
\]

Fix \( j \) and let \( \Gamma = (\Gamma_1, \Gamma_2) : [0, 1] \to V \) be a path tangent to \( \mathcal{F} \) with \( \Gamma(0) = \gamma(s_{j-1}) \), \( \Gamma(1) = \gamma(s_j) \) and such that \( \Gamma_1(s) = (1 - s)t_{j-1} + st_j \). Then by the fact above we have

\[
|x_j - x_{j-1}| = |\Gamma_2(1) - \Gamma_2(0)| \leq \int_0^1 |\Gamma_2'(s)|ds \leq \int_0^1 \frac{\eta}{2}|\Gamma_2(s)||\Gamma_1'(s)|ds.
\]

We can assume the partition to be small enough such that \(|\Gamma_2(s)| \leq 2|x_j|\). Then

\[
|x_j - x_{j-1}| \leq \int_0^1 \frac{\eta}{2}|\Gamma_2(s)||\Gamma_1'(s)|ds \leq \eta|x_j|\int_0^1 |\Gamma_1'(s)|ds,
\]

that is:

\[
(6.1) \quad |x_j - x_{j-1}| \leq \eta|x_j||t_j - t_{j-1}|.
\]

Denote \( h(\gamma(s_j)) = (\tilde{t}_j, \tilde{x}_j) \). Provided the partition is fine enough and by working with the path \( h \circ \gamma \) we prove as above that

\[
(6.2) \quad |\tilde{x}_j - \tilde{x}_{j-1}| \leq \eta|\tilde{x}_j||\tilde{t}_j - \tilde{t}_{j-1}|.
\]

From equations (6.1) and (6.2) and Lemma 17 we obtain

\[
\frac{1}{M_0} |\gamma(t_j) - \gamma(t_{j-1})| \leq |h \circ \gamma(t_j) - h \circ \gamma(t_{j-1})| \leq M_0 |\gamma(t_j) - \gamma(t_{j-1})|
\]

and the proposition follows.

7. **Proof of Proposition 14**

As a direct consequence of Fact 18 if \((t(s), x(s))\) is a path in a leaf of \(\mathcal{F}|_V\) or \(\tilde{\mathcal{F}}|_{\tilde{V}}\), we have that \(|x'(s)| \leq \eta|t'(s)|\). We start with the following proposition.

**Proposition 19.** There exist constants \(a_1, a_2, b_1, b_2 > 0\) such that

\[
a_1|\theta| - b_1 \leq |\hat{\theta}(w, T)| \leq a_2|\theta| + b_2.
\]
Proof. Fix \( (w, T) \), \( T = \tau + \theta i \) such that \( \gamma(s) = \phi^{Ts}(w), \ s \in [0,1] \) is contained in \( V \). Put \( \gamma(s) = (t(s), x(s)) \) and observe that \( t(s) = r(s)e^{i\alpha(s)}, \) where \( r \) and \( \alpha \) are differentiable monotone functions. Then

\[
 l(\gamma) = \int_0^1 |t'(s)|ds + \int_0^1 |x'(s)|ds \\
\leq \int_0^1 |t'(s)|ds + \int_0^1 |\eta(t')|ds \\
= (1 + \eta) \int_0^1 |t'(s)e^{i\alpha(s)} + r(s)i\alpha'(s)e^{i\alpha(s)}|ds \\
\leq (1 + \eta) \int_0^1 |r'|ds + (1 + \eta)Lr_1 \int_0^1 |\alpha'|ds \\
\leq (1 + \eta)(2r_1 - r_1) + (1 + \eta)Lr_2|\theta|, 
\]

since \( r \) and \( \alpha \) are monotone. By Proposition 14 we have

\[
 l(h \circ \gamma) \leq M_0(1 + \eta)|r_2 - r_1| + M_0(1 + \eta)Lr_2|\theta|.
\]

On the other hand, take a differentiable path \( \tilde{\gamma}(s) = (\tilde{r}(s)e^{i\tilde{\alpha}(s)}, \tilde{x}(s)) \) homotopic with fixed endpoints to \( h \circ \gamma(s) \) in \( \tilde{V} \) and such that \( l(h \circ \gamma) \geq l(\tilde{\gamma}) - 1 \). Clearly \( \tilde{\theta}(w, T) = \tilde{\alpha}(1) - \tilde{\alpha}(0) \). Then

\[
 l(h \circ \gamma) \geq l(\tilde{\gamma}) - 1 = \int_0^1 |\tilde{r}'e^{i\tilde{\alpha}} + \tilde{r}i\tilde{\alpha}'e^{i\tilde{\alpha}}|ds + \int_0^1 |\tilde{x}'|ds - 1 \\
\geq \int_0^1 |\tilde{r}' + \tilde{r}i\tilde{\alpha}'|ds - 1 \geq \int_0^1 |\tilde{r}i\tilde{\alpha}' - 1| \geq \frac{1}{L}r_1 \int_0^1 |\tilde{\alpha}'|ds - 1 \\
\geq \frac{1}{L}r_1 |\tilde{\alpha}(1) - \tilde{\alpha}(0)| - 1 = \frac{1}{L}r_1 |\tilde{\theta}| - 1 
\]

and together with equation (7.1) we obtain

\[
|\tilde{\theta}| \leq L\frac{M_0(1 + \eta)r_2}{r_1}|\theta| + L\frac{M_0(1 + \eta)(r_2 - r_1) + 1}{r_1},
\]

which give us the right inequality of the proposition. We can perform the same arguments with \( h^{-1} \) to obtain the left inequality. \( \square \)

Proof of Proposition 16. Fix a constant \( c > 0 \) such that \( a_1c - b_1 > 0 \). From proposition 14 we have

\[
0 < a_1\theta - b_1 \leq |\tilde{\theta}(w, T)| \leq a_2\theta + b_2 
\]

whenever \( \theta \geq c \). By connectedness we have two cases:

1. \( 0 < a_1\theta - b_1 \leq \tilde{\theta}(w, T) \leq a_2\theta + b_2 \) whenever \( \theta \geq c \).
2. \( 0 < a_1\theta - b_1 \leq -\tilde{\theta}(w, T) \leq a_2\theta + b_2 \) whenever \( \theta \geq c \).

In the same way we have the following possibilities:

1. \( a_2\theta - b_2 \leq \tilde{\theta}(w, T) \leq a_1\theta + b_1 < 0 \) whenever \( \theta \leq -c \).
2. \( a_2\theta - b_2 \leq -\tilde{\theta}(w, T) \leq a_1\theta + b_1 < 0 \) whenever \( \theta \leq -c \).

Suppose items 1 and 2 hold. Then we may find \( w_0 \in V \) and \( c_1, c_2 \geq c \) such that

\[
\tilde{\theta}(w_0, -c_1i) = \tilde{\theta}(w_0, c_2i).
\]

Denote \( w_1 = \phi^{-c_1i}(w_0) \). Then, from equation (7.2) it is easy to see that

\[
\tilde{\theta}(w_1, (c_1 + c_2)i) = 0,
\]
which contradicts Proposition \[19\]. In the same way we see that items 2 and 1’ does not simultaneously hold. Suppose that we have 1 and 1’. The other case is similar. Define \( f_1(\theta) = a_1\theta - b_2, \) \( f_2(\theta) = a_2\theta + b_1 \) for \( \theta \leq -c, \) \( f_1(\theta) = a_1\theta - b_1, \) \( f_2(\theta) = a_2\theta + b_2 \) for \( \theta \geq c \) and linearly extend \( f_1 \) and \( f_2 \) to \([-c, c]\). Clearly \( f_1 \leq \vartheta \leq f_2 \) on \( |\vartheta| \geq c \). Since \( \vartheta \) is bounded on \( |\vartheta| \leq c \), we can put \( \vartheta_1 = f_1 - C, \vartheta_2 = f_2 + C \) for suitable \( C > 0 \) to obtain \( \vartheta_1 \leq \vartheta \leq \vartheta_2 \).

\[\square\]

8. Regularization

This section is devoted to prove Proposition \[20\] which basically asserts that we can find a topological equivalence \( \hat{h} \) mapping \( T \) into \( \tilde{T} \), as we have announced in Section \[2\]. This proposition is the main construction of this work and is based on two key facts (Lemma \[22\] and Proposition \[21\]) which are proved in Section \[12\].

**Proposition 20.** Close to the exceptional divisor there exists a topological equivalence \( h \) between \( F \) and \( \tilde{F} \) with the following properties:

1. \( h = h \) outside \( V \).
2. \( h \) maps \( V \) into \( \tilde{V} \).
3. On \( V \) we have \( \hat{h}(w) = \phi T(w)(h(w)) \), where \( T : V \to \mathbb{C} \) is a continuous function such that \( |\text{Im}(T)| \) is bounded by some constant \( \tilde{\theta}_0 > 0 \).
4. There exist \( r_{12} \in (r_1, r_2) \) and \( \delta_1 \in (0, \delta_0) \) such that \( \hat{h} \) maps the set

\[ T = \{(t, x) \in V : |t| = r_{12}, |x| \leq \delta_1 \} \]

into the set

\[ \tilde{T} = \{(t, x) \in \tilde{V} : |t| = \sqrt{r_1 r_2} \}. \]

From now on we assume the first case in Proposition \[10\] that is:

\[ (8.1) \quad \vartheta_1(\theta) \leq \tilde{\theta}(w, \tau + \theta i) \leq \vartheta_2(\theta) \] for all \((w, \tau + \theta i)\) such that \( \tilde{\theta} \) is defined.

For the second case the arguments works in the same way. Then, given \( \tilde{\theta}_0 > 0 \) we can take \( \theta_0 > 0 \) and \( \tilde{\theta}_1 > 0 \) such that

\[ (8.2) \quad \tilde{\theta}(w, -\theta_0) \leq -\tilde{\theta}_0 \leq \tilde{\theta}(w, \theta_0) \]

and

\[ (8.3) \quad \tilde{\theta}(w, [-\theta_0, \theta_0]) \subset [-\tilde{\theta}_1, \tilde{\theta}_1], \]

whenever \( \tilde{\theta}(w, -\theta_0) \) and \( \tilde{\theta}(w, \theta_0) \) are defined. We fix the constants \( \tilde{\theta}_0, \tilde{\theta}_1 \) and \( \theta_0 \) although we will specify later how \( \tilde{\theta}_0 \) is chosen.

Set

\[ R = \{\tau + \theta i \in \mathbb{C} : 0 \leq \tau \leq \ln \frac{r_2}{r_1}, |\theta| \leq \theta_0 \} \]

and

\[ \tilde{R} = \{\tau + \theta i \in \mathbb{C} : -2 \ln L \leq \tau \leq \ln \frac{r_2}{r_1} + 2 \ln L, |\theta| \leq \tilde{\theta}_1 \}. \]

Consider

\[ \partial V_1 = \{|t| = r_1, 0 < |x| \leq \delta_0' \} \]

with \( \delta_0' \) small enough such that for all \( w \in \partial V_1 \) we have that \( \tilde{T}(w, T) \) is defined for all \( T \in R \). In particular, the segment

\[ I_w = \{\phi s^{\ln \frac{r_2}{r_1}}(w), s \in [0, 1]\} \]
is contained in $V$. Set

$$V' = \bigcup_{\mathfrak{m} \in \partial V_1} I_{\mathfrak{m}}.$$  

We will redefine the function $h$ on each $I_{\mathfrak{m}}$. Naturally $\bar{h}(I_{\mathfrak{m}})$ must be a segment with endpoints $\{h(\mathfrak{m}), h(\phi^{\ln 2}L_{\mathfrak{m}})\}$. Clearly $\bar{T}(\mathfrak{m}, T) \in \bar{R}$ for all $T \in R$ and we can define the map $\bar{T}_{\mathfrak{m}} : R \to \bar{R}, \bar{T}_{\mathfrak{m}}(T) = \bar{T}(\mathfrak{m}, T)$. It is not difficult to see that we have the following properties:

1. $\bar{T}_{\mathfrak{m}}$ is a homeomorphism onto its image.
2. $\bar{T}_{\mathfrak{m}}(R)$ contains the rectangle

$$(8.4) \quad \bar{R}_0 = \{2 \ln L \leq \tau \leq \ln \frac{r_2}{r_1} - 2 \ln L, -\bar{\theta}_0 \leq \theta \leq \bar{\theta}_0\}.$$  

Here we use the equations (8.2) and (4.1). Clearly the points $\bar{T}_{\mathfrak{m}}(0) = 0$ and $\bar{T}_{\mathfrak{m}}(\ln \frac{r_2}{r_1}) := a_{\mathfrak{m}}$ are contained in $\partial \bar{T}_{\mathfrak{m}}(R)$. Consider the Poincaré metric in the interior of $\bar{T}_{\mathfrak{m}}(R)$ and let $\gamma$ be a geodesic in $\bar{T}_{\mathfrak{m}}(R)$ with $\lim_{s \to -\infty} \gamma(s) = 0$ and $\lim_{s \to +\infty} \gamma(s) = a_{\mathfrak{m}}$. Although the parameterized geodesic $\gamma$ is not uniquely determined the set $\Gamma_{\mathfrak{m}} = \gamma(\bar{R}) \cup \{0, a_{\mathfrak{m}}\}$ is well defined and depends continuously on $\mathfrak{w}$.

**Proposition 21.** If $r_2/r_1$ and $\bar{\theta}_0$ are large enough, then $\Gamma_{\mathfrak{m}}$ intersects once and transversely each line $\{\tau = c\}$ with $c$ in the interval

$$\left[\frac{1}{2} \ln \frac{r_2}{r_1} - \ln L, \frac{1}{2} \ln \frac{r_2}{r_1} + \ln L\right].$$  

**Idea of the proof.** For $r_2/r_1$ and $\bar{\theta}_0$ arbitrarily large we have that, after "normalization", the region $\bar{T}_{\mathfrak{m}}(R)$ is arbitrarily close to $\bar{R}$ and the line $\{\tau = c\}$ is arbitrarily close to the center line $\{\tau = \frac{1}{2} \ln \frac{r_2}{r_1}\}$ of $\bar{R}$. Then the geodesic $\Gamma_{\mathfrak{m}}$ should be arbitrarily close to the geodesic $\{\theta = 0\}$ in $\bar{R}$ in such way $\Gamma_{\mathfrak{m}}$ intersects once and transversely the line $\{\tau = c\}$. The complete proof is given in Section 12. \hfill $\Box$

8.1. **Proof of Proposition 20.** Naturally we assume $r_2/r_1$ and $\bar{\theta}_0$ large enough according to Proposition 21. If $h(\mathfrak{w}) = (h_1(\mathfrak{w}), h_2(\mathfrak{w}))$ it is easy to see that

$$c_{\mathfrak{m}} = \ln \frac{\sqrt{r_1 r_2}}{|h_1(\mathfrak{w})|}$$  

belongs to the interval in Proposition 21. Then $\Gamma_{\mathfrak{m}}$ intersects once and transversely the line $\{\tau = c_{\mathfrak{m}}\}$. Let $o_{\mathfrak{m}}$ be the intersection point between $\Gamma_{\mathfrak{m}}$ and $\{\tau = c_{\mathfrak{m}}\}$. Since this intersection is transversal the point $o_{\mathfrak{m}}$ depends continuously on $\mathfrak{w}$. Now there is a unique parameterized geodesic $\gamma_{\mathfrak{m}}$ such that

$$\lim_{s \to -\infty} \gamma_{\mathfrak{m}}(s) = 0, \quad \lim_{s \to +\infty} \gamma_{\mathfrak{m}}(s) = a_{\mathfrak{m}}, \quad \gamma_{\mathfrak{m}}(0) = o_{\mathfrak{m}}$$  

and $\gamma_{\mathfrak{m}}$ depends continuously on $\mathfrak{w}$. We are now in position to define $\bar{h}$ on $I_{\mathfrak{m}}$.

Put $w = \phi^{s \ln \frac{r_2}{r_1}}(\mathfrak{w}), s \in [0, 1]$, fix an increasing diffeomorphism $f : (0, 1) \to \mathbb{R}$ and define

$$\bar{h}(w) = \bar{\gamma}_{\mathfrak{m}}(f(s))(h(\mathfrak{w})), \text{ if } s \in (0, 1).$$

$$\bar{h}(w) = h(\mathfrak{w}), \text{ if } s = 0 \text{ or } -1.$$
Thus, $I_w$ is mapped onto $\hat{\phi}^f(h(w))$. Naturally we set $\tilde{h} = h$ outside $V' = \bigcup_{w \in \partial V_1} I_w$.

If
$$T(w) = \gamma_w(f(s)) - T(w, s \ln \frac{r_2}{r_1})$$
by the flow property we have
$$\tilde{h}(w) = \tilde{\phi}^f(h(w)).$$
Then, since $\gamma_w(f(s))$ and $T(w, s \ln \frac{r_2}{r_1})$ are contained in $\tilde{R}$, we obtain item 3 of Proposition 20. Put $r_{12} = r_1(\frac{r_2}{r_1})^{s_m}$, where $s_m = f^{-1}(0)$. Suppose $w \in I_m$ is contained in the set $\{(t, x) \in V : |t| = r_{12}\}$. Then $w = \phi^{s_m \ln \frac{r_2}{r_1}}(w)$, hence
$$\tilde{h}(w) = \tilde{\phi}^{s_m \ln \frac{r_2}{r_1}}(h(w)) = \tilde{\phi}^{s_m(0)}(h(w)) = \tilde{\phi}^{s_m}(h(w))$$
and
$$|\tilde{\phi}^{s_m}(h(w))| = |h_1(w)| e^{Re(s_m)} = |h_1(w)| e^{c_m} = \sqrt{r_1 r_2},$$
which proves item 4.

It is easy to see that, if restricted to a small enough neighborhood of the exceptional divisor, $\tilde{h}$ has the following properties:

1. $\tilde{h}$ is continuous.
2. $\tilde{h}$ maps leaves of $\mathcal{F}$ into leaves of $\tilde{\mathcal{F}}$.
3. $\tilde{h}$ maps points in $V'$ into $h(V')$.
4. $\tilde{h}(w)$ tends to the exceptional divisor as $w$ tends to the exceptional divisor.

Then, in order to prove that $\tilde{h}$ defines a topological equivalence between $\mathcal{F}$ and $\tilde{\mathcal{F}}$ it suffices to show that $\tilde{h}$ is injective, which reduces to show that the sets $\{h(I_w)\}_{w \in \partial V_1}$ are disjoint. We will use the following lemma, which is proved in Section 12.

**Lemma 22.** Let $U$ be a Jordan region in $\mathbb{C}$. Consider the Poincaré-Urbini metric in $U$ and let $\gamma$ be a geodesic with $\lim_{s \to -\infty} \gamma(s) = \zeta_1 \in \partial U$ and $\lim_{s \to \infty} \gamma(s) = \zeta_2 \in \partial U$ ($\zeta_1 \neq \zeta_2$). Let $C \subset \partial U$ be one of the two segments determined by $\{\zeta_1, \zeta_2\}$. Let $\tilde{U}$ be a Jordan region such that
$$U \subseteq \tilde{U}$$
and $C \subset \partial \tilde{U}$.

Consider the Poincaré metric on $\tilde{U}$ and let $\tilde{\gamma}$ be the geodesic in $\tilde{U}$ with $\lim_{s \to -\infty} \tilde{\gamma}(s) = \zeta_1$ and $\lim_{s \to \infty} \tilde{\gamma}(s) = \zeta_2$. Let $\Omega$ be the Jordan region bounded by $\gamma$ and $C$. Then $\tilde{\gamma}$ is disjoint of $\Omega$.

Let $w_1, w_2$ be different points in $\partial V_1$. In order to prove that $\tilde{h}(I_{w_1}) \cap \tilde{h}(I_{w_2}) = \emptyset$, since
$$\tilde{h}(I_{w_1}) \subset h(\phi^R(w_1)), \quad \tilde{h}(I_{w_2}) \subset h(\phi^R(w_2))$$
and $h$ is injective, it suffices to consider the case
$$\phi^R(w_1) \cap \phi^R(w_2) \neq \emptyset.$$ 
Reordering $w_1, w_2$ if necessary we have the following:

1. $w_2 = \phi^{\alpha_1}(w_1)$ for some $\alpha > 0$.
2. $\phi^T(w_1)$ is defined for all $T$ in
$$R = \{\tau + \theta i : 0 \leq \tau \leq \ln \frac{r_2}{r_1}, -\theta_0 \leq \theta \leq \alpha + \theta_0\}.$$
Let
\[ R_\alpha = \{ \tau + \theta i : 0 \leq \tau \leq \ln \frac{r_2}{r_1}, \alpha - \theta \leq \alpha + \theta_0 \} \]
and consider the Jordan regions \( \tilde{T}_{m_1}(R) \), \( \tilde{T}_{m_1}(R) \) and \( \tilde{T}_{m_1}(R_\alpha) \). By applying Lemma \[22\] to the regions \( \tilde{T}_{m_1}(R) \subseteq \tilde{T}_{m_1}(R) \) we see that the geodesic \( \gamma_1 \) in \( \tilde{T}_{m_1}(R) \) “joining” the points \( \tilde{T}_{m_1}(0) \) and \( \tilde{T}_{m_1}(T) \) is disjoint of \( \Omega_0 \), where \( \Omega_0 \) is the region bounded by the curves:

1. The geodesic \( \Gamma_{m_1} \),
2. \( C^0_1 : \tilde{T}_{m_1}(-s\theta_0), s \in [0, 1] \),
3. \( C^0_2 : \tilde{T}_{m_1}(s \ln \frac{r_2}{r_1} - \theta_0), s \in [0, 1] \),
4. \( C^0_3 : \tilde{T}_{m_1}(\ln \frac{r_2}{r_1} - \theta_0), s \in [0, 1] \).

Then, if \( \Omega_1 \) is the region bounded by \( \gamma_1, C^0_1, C^0_2 \) and \( C^0_3 \) we have \( \Gamma_{m_1} \subseteq \Omega_1 \). Since the geodesic \( \gamma_2 \) in \( \tilde{T}_{m_1}(R) \) "joining" the points \( \tilde{T}_{m_1}(\alpha i) \) and \( \tilde{T}_{m_1}(\ln \frac{r_2}{r_1} + \alpha i) \) does not meet \( \gamma_1 \), we have that \( \Omega_1 \) is contained in the region \( \Omega_2 \) bounded by the curves:

1. The geodesic \( \gamma_2 \),
2. \( C^1_1 : \tilde{T}_{m_1}((1 - s)\alpha i - s\theta_0), s \in [0, 1] \),
3. \( C^1_2 : \tilde{T}_{m_1}(s \ln \frac{r_2}{r_1} - \theta_0), s \in [0, 1] \),
4. \( C^1_3 : \tilde{T}_{m_1}(\ln \frac{r_2}{r_1} + (1 - s)\alpha i - s\theta_0), s \in [0, 1] \).

By applying Lemma \[22\] to the regions \( \tilde{T}_{m_1}(R_\alpha) \subseteq \tilde{T}_{m_1}(R) \) we have that the geodesic \( \gamma_3 \) in \( \tilde{T}_{m_1}(R_\alpha) \) "joining" the points \( \tilde{T}_{m_1}(\alpha i) \) and \( \tilde{T}_{m_1}(\ln \frac{r_2}{r_1} + \alpha i) \) is disjoint of \( \Omega_2 \), hence \( \gamma_3 \) is disjoint of \( \Gamma_{m_1} \subseteq \Omega_1 \subseteq \Omega_2 \). By the definition of \( \tilde{T}_w \) and the flow property we obtain the following relation:

\[ \tilde{T}_{m_1}(\alpha i + T) = \tilde{T}_{m_1}(\alpha i) + \tilde{T}_{m_2}(T) \]

Then \( \tilde{T}_{m_1}(R_\alpha) = \tilde{T}_{m_1}(\alpha i) + \tilde{T}_{m_2}(R) \) and, since the map \( z \mapsto z + \tilde{T}_{m_1}(\alpha i) \) is an isometry between \( \tilde{T}_{m_2}(R) \) and \( \tilde{T}_{m_2}(R_\alpha) \), we deduce that \( \gamma_3 = \tilde{T}_{m_1}(\alpha i) + \Gamma_{m_2} \). Then

\[
\begin{align*}
\tilde{\phi}^{\gamma_3}(h(w_1)) &= \tilde{\phi}^{\gamma_3}(h(w_1)) + \Gamma_{m_2}(h(w_1)) \\
&= \tilde{\phi}^{\Gamma_{m_2}}(\tilde{\phi}^{\gamma_3}(h(w_1))) \\
&= \tilde{\phi}^{\Gamma_{m_2}}(h(w_2)) \\
&= \tilde{h}(I_{m_2}).
\end{align*}
\]

At this point we consider two cases.

**Case 1.** The map \( \tilde{\phi}_h(w_1) : T \mapsto \tilde{\phi}^T(h(w_1)) \) is injective. Then, since \( \gamma_3 \) is disjoint of \( \Gamma_{m_1} \), we have that \( \tilde{\phi}_h(w_1)(\gamma_3) \) is disjoint of \( \tilde{\phi}_h(w_1)(\Gamma_{m_1}) \), that is, \( \tilde{h}(I_{m_2}) \) is disjoint of \( \tilde{h}(I_{m_1}) \).

**Case 2.** The map \( \tilde{\phi}_h(w_1) : T \mapsto \tilde{\phi}^T(h(w_1)) \) is not injective. In this case, provided \( h(w_1) \) is close enough to the exceptional divisor the leaf of \( \tilde{F}_1 \) through \( h(w_1) \) is compact and by Lemma \[13\] the maps \( T \mapsto \tilde{\phi}^T(w_1) \) and \( T \mapsto \tilde{\phi}^T(h(w_1)) \) have period \( 2\pi k i \) for some \( k \in \mathbb{N} \) and

\[ \tilde{T}_{m_1}(T + 2k\pi i) = \tilde{T}_{m_1}(T) + \xi 2k\pi i. \]
Therefore, for any \( n \in \mathbb{Z} \), in the arguments above we can change \( \alpha \) by \( \alpha + n2\pi \), we put \( \gamma_3^k \) instead \( \gamma_3 \) and thus we obtain that \( \Gamma_{\mathbb{Z}} \) is disjoint of

\[
\gamma_3^k = \bar{T}_{\mathbb{Z}}, ((\alpha + n\xi2\pi)i) + \Gamma_{\mathbb{Z}}
\]

\[
= \bar{T}_{\mathbb{Z}}, (\alpha i) + n\xi2\pi i + \Gamma_{\mathbb{Z}}
\]

\[
= \gamma_3 + n\xi2\pi i
\]

for any \( n \in \mathbb{Z} \). Then, since \( \bar{\phi}^{\gamma_3}(h(\mathbb{W}_1)) = \bar{h}(I_{\mathbb{W}_2}) \) we deduce that \( \Gamma_{\mathbb{Z}} \) is disjoint of the inverse image of \( \bar{h}(I_{\mathbb{W}_2}) \) by \( T \mapsto \bar{\phi}^T(h(\mathbb{W}_1)) \). Therefore \( \bar{h}(I_{\mathbb{W}_1}) \) is disjoint of \( \bar{h}(I_{\mathbb{W}_2}) \).

9. Proof of Theorem 11

Let \( \bar{h} \) as in Section 8 As we have seen in Proposition 20 for some \( \delta_1 > 0 \) we have that \( \bar{h} \) maps the set

\[
T = \{|t| = r_{12}, 0 < |x| \leq \delta_1\}
\]

into the set

\[
\bar{T} = \{|t| = \sqrt{r_1r_2}, 0 < |x| \leq \delta_0\}
\]

In this section we construct a topological equivalence \( \bar{h} \) between \( \mathcal{F} \) and \( \bar{\mathcal{F}} \) mapping the Hopf fibers in \( T \) into Hopf fibers in \( \bar{T} \). Clearly this finishes the proof of Theorem 11. The key of the proof is the following proposition which shows that there is a Hopf fiber in \( T \) whose image by \( \bar{h} \) in \( \bar{T} \) can be "redressing" to a Hopf fiber in \( \bar{T} \).

**Proposition 23.** By reducing \( \delta_1 \) if necessary, there exist:

1. A punctured Hopf fiber \( \mathcal{D}^s = \{(t_0, x) : 0 < |x| < \delta_1\} \) with \( |t_0| = r_{12} \)
2. A Hopf fiber \( \mathcal{D} = \{(t_0, x) : |x| < \delta_0\} \) with \( |t_0| = \sqrt{r_1r_2} \)
3. A real function \( \sigma \) defined on \( \mathcal{A} = \bar{h}(\mathcal{D}^s) \),

such that the following properties hold:

1. \( \bar{\phi}^{\sigma(\bar{w})i}(\bar{w}) \in \bar{T} \) and \( f(\bar{w}) := \bar{\phi}^{\sigma(\bar{w})i}(\bar{w}) \in \bar{\mathcal{D}} \) for all \( \bar{w} \in \mathcal{A} \) and all \( s \in [0, 1] \)
2. \( f : \mathcal{A} \to \bar{\mathcal{D}} \) is a homeomorphism onto its image
3. \( f(\mathcal{A}) = \Omega \setminus o \), where \( o = (t_0, 0) \in \bar{\mathcal{D}} \) and \( \Omega \) is a topological disc
4. \( f(\bar{w}) \) tends to \( o \) as \( \bar{w} \) tends to the exceptional divisor.

**Proof.** We give the proof of this proposition in Sections 10 and 11. □

We perform the proof of Theorem 11 in two steps.

**Step 1.** As a first step, for some \( \bar{T}_1 \subset \bar{T} \) we construct a homeomorphism \( f : \bar{T}_1 \to \bar{T} \) preserving \( \bar{\mathcal{F}} \) and redressing the image by \( \bar{h} \) of the Hopf fibers in \( T \). Set

\[
C = \{|t| = r_{12}, x = 0\},
\]

\[
\bar{C} = \{|t| = \sqrt{r_1r_2}, x = 0\},
\]

\[
\zeta = (t_0, 0) \text{ and } \bar{\zeta}_0 = (t_0, 0). \text{ Consider the natural orientation on } C. \text{ Let } \gamma \text{ be a oriented circle in } T \text{ homotopic to } C \text{ in } \bar{T} \text{ and take a diffeomorphism } g : C \to \bar{C},
\]

\[
g(\zeta_0) = \bar{\zeta}_0 \text{ such that } g(C) \text{ is homotopic to } \bar{h}(\gamma) \text{ in } \bar{T}. \text{ Put}
\]

\[
\bar{T}_1 = \{(t, x) \in \bar{T} : 0 < |x| \leq \delta_1\}
\]

and assume \( \delta_1 > 0 \) be such that

1. \( \bar{\phi}^{2\pi i}(w) \in \bar{T} \) for all \( w \in \bar{T}_1, \ s \in [-1, 1] \)
Given $\zeta \in C$, define $\vartheta(\zeta) \in [0,2\pi)$ by $\zeta = \phi^{\vartheta(\zeta)}(\zeta_0)$ and let $\hat{\vartheta}(\zeta) \in \mathbb{R}$ be such that $\phi^{\hat{\vartheta}(\zeta)}(\zeta), s \in [0,1]$ is a positive reparameterization of the path $g\phi^{\hat{\vartheta}(\zeta)}(\zeta_0), s \in [0,1]$. Clearly $\vartheta$ and $\hat{\vartheta}$ are continuous on $C \setminus \{\zeta\}$ and they have a simply discontinuity at $\zeta_0$. Let $\pi_1$ be the projection $(t,x) \to t$ in $T$. Given $\bar{w} \in \bar{T}_1$, put $\zeta(\bar{w}) = \pi_1 \circ \bar{h}^{-1}(\bar{w})$ and let $\theta(\bar{w}) \in \mathbb{R}$ be such that $\hat{\phi}^{-s\theta(\bar{w})}(\bar{w}), s \in [0,1]$ is a positive reparameterization of

$$\bar{h}\phi^{-s\theta(\zeta(\bar{w}))}(\bar{h}^{-1}(\bar{w})), s \in [0,1].$$

From [2] and the definition of $\theta$ it is easy to see that $\hat{\phi}^{-\theta(\bar{w})}(\bar{w}) \in \mathcal{A}$ for all $\bar{w} \in \bar{T}_1$. Let $\sigma$ be the function given by Proposition [2]. We extend $\sigma$ to $\bar{T}_1$ by putting:

$$(9.1) \quad \sigma(\bar{w}) = -\theta(\bar{w}) + \sigma(\hat{\phi}^{-\theta(\bar{w})}(\bar{w})) + \hat{\vartheta}(\zeta(\bar{w})).$$

**Assertion.** $\sigma$ is continuous and $\hat{\phi}^{s\sigma(\bar{w})}(\bar{w}) \in \bar{T}$ for all $\bar{w} \in \bar{T}_1, s \in [0,1]$.

Fix $\tilde{w}_1 \in \mathcal{A}$. It is sufficient to show that $\sigma(\bar{w}) \to \sigma(\tilde{w}_1)$ whenever $\bar{w} \to \tilde{w}_1 \in \mathcal{A}$ with $\pi \leq \vartheta(\zeta(\bar{w})) < 2\pi$. If $\vartheta(\zeta(\bar{w})) \to 2\pi$ we have that $\theta(\bar{w}) \to \theta_0$, where $\theta_0$ is such that $\hat{\phi}^{-s\theta_0}(\tilde{w}_1), s \in [0,1]$ is a positive reparameterization of

$$\bar{h}\phi^{-s\theta_0}(\bar{h}^{-1}(\tilde{w}_1)), s \in [0,1].$$

Then

$$\bar{w}_0 := \hat{\phi}^{-\theta_0}(\tilde{w}_1) = \bar{h}\phi^{-2\pi}(\bar{h}^{-1}(\tilde{w}_1)) \in \mathcal{A}.$$ 

Let $\gamma : [0,1] \to \mathcal{A}$ be any path such that $\gamma(0) = \bar{w}_0$ and $\gamma(1) = \tilde{w}_1$. For all $t \in [0,1]$ define the paths

$$\gamma_t(s) = \hat{\phi}^{t\sigma(\gamma(s))}(\gamma(s)), s \in [0,1]$$

and

$$\alpha_t(s) = \hat{\phi}^{[(1-s)\sigma(\tilde{w}_1) + s(\sigma(\bar{w}_0) - \theta_0)]}(\tilde{w}_1), s \in [0,1].$$

The paths $\alpha_1 \ast \gamma_t$ are closed and give a homotopy between $\alpha_0 \ast \gamma$ and $\alpha_1 \ast \gamma_1$. By the definition of $\theta_0$, the path $\alpha_0$ is homotopic in $\bar{T}$ to the path $\bar{h}\phi^{s2\pi}(\bar{h}^{-1}(\tilde{w}_1)), s \in [0,1]$. Then $\alpha_0 \ast \gamma$ is homotopic to the path $\bar{h}(\alpha_1 \ast \gamma)$, where

$$\bar{\alpha}(s) = \hat{\phi}^{-s2\pi}(\bar{h}^{-1}(\tilde{w}_1)), s \in [0,1]$$

and $\bar{\gamma} = \bar{h}^{-1} \circ \gamma$. But the path $\alpha_1 \ast \gamma$ is homotopic to $-C$ in $\bar{T}$. Then, it follows from the definition of $q$ that $\alpha_0 \ast \gamma$ is homotopic to $g(-C)$ in $\bar{T}$. Therefore $\alpha_1 \ast \gamma_1$ is homotopic to $g(-C)$ in $\bar{T}$. Observe that, since $\gamma_1 \subset \bar{D}$, the path $\alpha_1 \ast \gamma_1$ is homotopic in $\bar{T}$ to the closed path

$$\gamma(1-s)\sigma(\tilde{w}_1) + s(\sigma(\bar{w}_0) - \theta_0)]i(\tilde{\zeta_0}), s \in [0,1].$$

Then $g(-C)$ is homotopic to

$$\hat{\phi}^{s(\sigma(\bar{w}_0) - \sigma(\tilde{w}_1) - \theta_0)}(q), s \in [0,1],$$

where $q = \hat{\phi}^{s(\tilde{w}_1)}(\tilde{\zeta_0})$. On the other hand, since $\vartheta(\zeta(\bar{w})) \to 2\pi$ as $\bar{w} \to \tilde{w}_1$ with $\pi < \vartheta(\zeta(\bar{w})) < 2\pi$, it follows from the definition of $\vartheta$ that $\hat{\vartheta}(\zeta(\bar{w})) \to \xi 2\pi$, where $\xi \in \{1, -1\}$ is such that $\hat{\phi}^{-s\xi 2\pi}(\tilde{\zeta_0}), s \in [0,1]$ is a positive reparameterization of

$$g(-C) = \{g\hat{\phi}^{-s\xi 2\pi}(\tilde{\zeta_0}), s \in [0,1]\}.$$
Then \( g(-C) \) is homotopic to
\[
\{ \tilde{\phi}^{-s\xi 2\pi i}(\tilde{\tau}_0), s \in [0,1] \} = \{ \tilde{\phi}^{-s\xi 2\pi i}(q), s \in [0,1] \}.
\]
It follows that the paths \( \{ \tilde{\phi}^{-s(\sigma(\tilde{w}_0) - \sigma(\tilde{w}_1) - \theta_0)i}(q) \} \) and \( \{ \tilde{\phi}^{-s\xi 2\pi i}(q) \} \) are homotopic in \( \overline{\mathbf{T}} \) and this implies that
\[
\xi 2\pi = -\sigma(\tilde{w}_0) + \sigma(\tilde{w}_1) + \theta_0.
\]
Thus, if \( \tilde{w} \to \tilde{w}_1 \) with \( \pi < \vartheta(\xi(\tilde{w})) < 2\pi \), we have that \( \vartheta(\tilde{w}) \to \theta_0, \sigma(\tilde{w}) \to \sigma(\tilde{w}_0), \vartheta(\xi(\tilde{w})) \to \xi 2\pi = -\sigma(\tilde{w}_0) + \sigma(\tilde{w}_1) + \theta_0 \) and replacing in (9.1) we obtain that \( \vartheta(\tilde{w}) \to \sigma(\tilde{w}_1) \). Therefore \( \sigma \) is continuous. On the other hand it is easy to see that \( \tilde{\phi}^{x(\vartheta)}(\tilde{w}) \in \overline{\mathbf{T}} \) for all \( \tilde{w} \in \tilde{T}_1, s \in [0,1] \). Assertion is proved.

Define the map
\[
f : \tilde{T}_1 \to \tilde{T} \\
f(\tilde{w}) = \tilde{\phi}^{\sigma(\tilde{w})}(\tilde{w}).
\]
This map \( f \) is an extension of the map \( f : \mathcal{A} \to \tilde{D} \) given by Proposition 23. Given \( \xi = (t_\xi,0) \in C \), put \( g(\xi) = (\tilde{t}_\xi,0) \) and define the sets
\[
\mathcal{A}_\xi = \tilde{h}(\{(t_\xi, x) : 0 < |x| < \delta_1 \}) \\
\tilde{D}_\xi = \{(\tilde{t}_\xi, x) : |x| < \delta_0 \}.
\]
Observe that \( f(\tilde{w}) \in \tilde{D}_\xi \) for all \( \tilde{w} \in \mathcal{A}_\xi \cap \tilde{T}_1 \). Moreover, the map
\[
f_\xi = f|_{\mathcal{A}_\xi \cap \tilde{T}_1} : \mathcal{A}_\xi \cap \tilde{T}_1 \to \tilde{D}_\xi
\]
can be expressed as \( f_\xi = \tilde{\rho} f_0 \tilde{h} \rho h^{-1} \), where \( \rho(w) = \phi^{-\vartheta(t_\xi)}(w) \), \( \tilde{\rho}(w) = \tilde{\phi}^{\vartheta(t_\xi)}(w) \) and \( f_0 = f|_{\mathcal{A}_\xi \cap \tilde{T}_1} \). Clearly \( \rho \) and \( \tilde{\rho} \) are diffeomorphisms and by Proposition 23 the map \( f_0 \) is a homeomorphism. Then \( f_\xi \) is a homeomorphism onto its image and \( f_\xi(\tilde{w}) \) tends to the divisor as \( \tilde{w} \) tends to the divisor. Then we have the following:

1. \( f \) is a homeomorphism onto its image
2. \( f \) preserves the 1-foliation induced by \( \tilde{F} \) on \( \tilde{T} \)
3. \( f(\tilde{w}) \) tends to the divisor as \( \tilde{w} \) tends to the divisor
4. \( f \) maps \( \mathcal{A}_\xi \cap \tilde{T}_1 \) into the Hopf fiber \( \tilde{D}_\xi \). Thus, for some \( \delta_2 > 0 \) we have that \( f \circ h \) maps each Hopf fiber \( \{(t_\xi, x) : 0 < |x| < \delta_2 \} \) into the Hopf fiber \( \{(\tilde{t}_\xi, x) : 0 < |x| < \delta_0 \} \).

Step 2. Now, we will extend \( f \) as a topological equivalence of \( \tilde{F} \) with itself. Thus we will define \( \tilde{h} = f \circ h \), which clearly will finish the proof of Theorem 1. First, for some \( \varepsilon, \delta_2 > 0 \) we will extend \( f \) to the set
\[
\mathcal{T} = \{(t,x) : \sqrt{r_1 r_2} - \varepsilon \leq |t| \leq \sqrt{r_1 r_2} + \varepsilon, 0 < |x| \leq \delta_2 \}
\]
in such way \( f = id \) on the boundary
\[
\partial \mathcal{T} = \{(t,x) : |t| = \sqrt{r_1 r_2} \pm \varepsilon, 0 < |x| \leq \delta_2 \}
\]
Take first any \( \delta_2 \in (0, \delta_1) \). There is \( \varepsilon > 0 \) with the following property: If \( \tilde{w} \in \mathcal{T} \), there exists a unique \( \tau(\tilde{w}) \in \mathbb{R} \) such that \( \tilde{\phi}^{\tau(\tilde{w})}(\tilde{w}) \) is contained in \( \tilde{T}_1 \). Put \( \vartheta(\tilde{w}) = \tilde{\phi}^{\tau(\tilde{w})}(\tilde{w}) \). Take a continuous function
\[
\nu : [\sqrt{r_1 r_2} - \varepsilon, \sqrt{r_1 r_2} + \varepsilon] \to [0,1]
\]
such that \( \nu(\sqrt{r_1^2 + r_2^2} + z) = 0 \) and \( \nu(\sqrt{r_1^2 + r_2^2}) = 1 \). Then denote \( \tilde{w} = (\tilde{t}, \tilde{x}) \) and extend \( \sigma \) and \( f \) by the expressions:

\[
\sigma(\tilde{w}) = \nu(|\tilde{t}|)\sigma(\tilde{w}) \\
f(\tilde{w}) = \tilde{\sigma}(\tilde{w})(\tilde{w}).
\]

Naturally we extend \( f \) as the identity map outside \( \mathcal{T} \). Then it is not difficult to see that close to the exceptional divisor \( f \) is a topological equivalence of \( \mathcal{F} \) with itself.

10. Proof of Proposition 23 in the non-hyperbolic case

In this section we assume that the holonomy of \( \mathcal{F}|_{\mathcal{V}} \) at the leaf \( \{x = 0\} \) is
non-hyperbolic. Fix a punctured Hopf fiber \( D_0^* = \{(t_0, x) : 0 < |x| < \delta_0\} \) with \( |t_0| = r_{1,2} \). Clearly in general the set \( \mathcal{A}_0 = h(D_0^*) \) is not a Hopf fiber, but we can use the flow to move each point of \( \mathcal{A}_0 = h(D_0^*) \) the correct amount to obtain a Hopf fiber.

Given \( \tilde{w}_1, \tilde{w}_2 \in \mathcal{A}_0 \), take a continuous path \( \gamma = (\gamma_1, \gamma_2) \) in \( \mathcal{A}_0 \) with \( \gamma(0) = \tilde{w}_1 \) and \( \gamma(1) = \tilde{w}_2 \). We can write \( \gamma(s) = r(s)e^{i\alpha(s)} \) for some continuous functions \( r \) and \( \alpha \). Define \( \Theta(\tilde{w}_1, \tilde{w}_2) = \alpha(1) \). From Lemma 12 we see that \( \Theta(\tilde{w}_1, \tilde{w}_2) \) does not depend on the path \( \gamma \). The function \( \Theta \) measures the oscillation of \( \mathcal{A}_0 \) around \( \{t = 0\} \), that is, the deviation of \( \mathcal{A}_0 \) from being a Hopf fiber.

Remark 24. It is easy to see from the definition of \( \Theta \) that

\[
\Theta(\tilde{w}_1, \tilde{w}_2) + \Theta(\tilde{w}_2, \tilde{w}_3) = \Theta(\tilde{w}_1, \tilde{w}_3)
\]

for any \( \tilde{w}_1, \tilde{w}_2, \tilde{w}_3 \in \mathcal{A}_0 \). From this and the mean value theorem we deduce that, if \( \Theta(\tilde{w}_1, \tilde{w}_2) = \epsilon_1 + \ldots + \epsilon_k \) with \( \epsilon_j > 0 \) and \( C \) is a simple path connecting \( \tilde{w}_1 \) and \( \tilde{w}_2 \), then there are intermediary ordered points \( \tilde{w}_0, \ldots, \tilde{w}_k \) in \( C \) with \( \tilde{w}_0 = \tilde{w}_1 \), \( \tilde{w}_k = \tilde{w}_2 \) and such that \( |\Theta(\tilde{w}_{j-1}, \tilde{w}_j)| = \epsilon_j \).

Proposition 23 is based in the following proposition.

**Proposition 25.** There exists \( \delta_1 \in (0, \delta_0) \) and \( \mu_3 > 0 \) such that the set

\[
\mathcal{A}_1 = h(\{(t_0, x) : 0 < |x| < \delta_1\})
\]

has the following property: for any \( \tilde{w}_1, \tilde{w}_2 \in \mathcal{A}_1 \), \( \tilde{w}_1 = (\tilde{t}_1, \tilde{x}_1) \) we have that \( \tilde{\phi}_s(\tilde{w}_1) \) is well defined and \( |\tilde{\phi}_s(\tilde{w}_1)| \leq \mu_3 \sqrt{|\tilde{x}_1|} \) for \( |s| \leq |\Theta(\tilde{w}_1, \tilde{w}_2)| \).

This section is organized as follows. We start with a lemma and the intermediary propositions 27, 28, 29 and 30. In Subsection 10.1 we prove a preparatory proposition and give the proof of Proposition 25. Finally, in Subsection 10.2 we prove Proposition 23. It is worth mentioning that Propositions 29 and 30 are independent of the Lipschitz hypothesis.

**Lemma 26.** Let \( \tilde{w}_1 \) and \( \tilde{w}_2 \) be points in \( \mathcal{A}_0 \) such that

\[
|\Theta(\tilde{w}_1, \tilde{w}_2)| = \pi.
\]

Let \( z_1, z_2 \in \mathbb{C}^2 \) be such that \( h(z_j) = \pi(\tilde{w}_j), j = 1, 2 \). Then

\[
|z_1 - z_2| \geq \frac{r_1 m}{LM}|\tilde{z}_2|.
\]

**Proof.** Since \( |\Theta(\tilde{w}_1, \tilde{w}_2)| = \pi \), it is easy to see that \( |\tilde{w}_1 - \tilde{w}_2| \geq 2r_1 / L \). Then, if \( h(z_j) = (\tilde{x}_j, \tilde{y}_j) \), by Lemma 15 we have

\[
|z_1 - z_2| \geq \frac{1}{M}|h(z_1) - h(z_2)| \geq \frac{1}{M}|\tilde{x}_2||\tilde{w}_1 - \tilde{w}_2| \geq \frac{2r_1}{LM}|\tilde{x}_2|.
\]
But $|h(z_2)| = |\bar{x}_2| + |\bar{y}_2| \leq |\bar{x}_2| + Lr_2|\bar{x}_2| \leq 2|\bar{x}_2|$, hence

$$|z_1 - z_2| \geq \frac{2r_1}{LM}|\bar{x}_2| \geq \frac{2r_1|h(z_2)|}{LM} \geq \frac{r_1m}{LM}|\bar{x}_2|.$$  

□

**Proposition 27.** Let $(t_0, x_1), (t_0, x_2)$ be points in $D_0^*$ and denote $\bar{w}_1 = (\bar{t}_1, \bar{x}_1) = h(t_0, x_1)$ and $\bar{w}_2 = (\bar{t}_2, \bar{x}_2) = h(t_0, x_2)$. Then there exists a constant $\mu_1 > 0$ with the following property. If $x_2 = \lambda x_1$ with $0 < \lambda \leq 1$ and

$$|\Theta(\bar{w}_1, \bar{w}_2)| = 2n\pi \quad \text{for some } n \in \mathbb{N},$$

then

$$|\bar{x}_2| \leq \frac{2M|\bar{x}_1|}{(1 + \mu_1)^n}.$$  

**Proof.** Consider the segment

$$L = \{(sx_1, st_0x_1) : \lambda \leq s \leq 1\}$$

in $\mathbb{C}^2 \setminus \{0\}$. Clearly $z_1 = (x_1, t_0x_1)$ and $z_2 = (\lambda x_1, \lambda t_0x_1)$ are the endpoints of $L$ and $\bar{w}_1 = h \circ \pi^{-1}(z_1)$, $\bar{w}_2 = h \circ \pi^{-1}(z_2)$. Since the simple path $h \circ \pi^{-1}(L)$ connects $\bar{w}_1$ with $\bar{w}_2$ and $|\Theta(\bar{w}_1, \bar{w}_2)| = 2n\pi$, we can find ordered points

$$w_0 = h \circ \pi^{-1}(j_0), \ldots, w_{2n} = h \circ \pi^{-1}(j_{2n})$$

in $h \circ \pi^{-1}(L)$, with $j_0 = z_1, j_{2n} = z_2$ such that

$$|\Theta(w_j, w_{j+1})| = \pi$$

for all $j = 0, \ldots, 2n - 1$. Then by Lemma 20 we have that

$$|j_j - j_{j+1}| \geq \frac{r_1m}{LM}|j_j|.$$  

Observe that $|j_j - j_{j+1}| = |\bar{j}_j| - |\bar{j}_{j+1}|$, because $\bar{j}_j$ and $\bar{j}_{j+1}$ are contained in the segment $L$. Then $|\bar{j}_j| - |\bar{j}_{j+1}| \geq \frac{r_1m}{LM}|\bar{j}_{j+1}|$, so

$$|\bar{j}_j| \leq \frac{|\bar{j}_j|}{1 + \frac{r_1m}{LM}}$$

and it follows that

$$|\bar{j}_j| \leq \frac{|j_0|}{(1 + \frac{r_1m}{LM})^j}$$

for all $j = 1, \ldots, 2n$. In particular, we have that

$$|z_2| \leq \frac{|z_1|}{(1 + \frac{r_1m}{LM})^{2n}}.$$  

Recall that

$$M|z_2| \geq |h(z_2)| = |\pi(\bar{w}_2)| = |(\bar{x}_2, \bar{t}_2\bar{x}_2)| > |\bar{x}_2|,$$

hence

$$|z_2| \leq \frac{|z_1|}{(1 + \frac{r_1m}{LM})^{2n}}.$$  

On the other hand:

$$m|z_1| \leq |h(z_1)| = |\pi(\bar{w}_1)| = |(\bar{x}_1, \bar{t}_1\bar{x}_1)| = |\bar{x}_1| + |\bar{t}_1||\bar{x}_1| < 2|\bar{x}_1|$$

and we obtain

$$|z_1| \leq (2/m)|\bar{x}_1|.$$
From [10.2] and [10.3]

\[ \frac{1}{M} |\bar{x}_2| \leq |z_2| \leq \frac{|z_1|}{(1 + \frac{r_1 m}{LM})^{2n}} \leq \frac{(2/m)|\bar{x}_1|}{(1 + \frac{r_1 m}{LM})^{2n}}. \]

hence

\[ |\bar{x}_2| \leq \frac{2M|m|x_1|}{(1 + \frac{r_1 m}{LM})^{2n}}. \]

Finally, we take \( \mu_1 > 0 \) be such that \( 1 + \mu_1 = (1 + \frac{r_1 m}{LM})^2 \).

\[ \square \]

**Proposition 28.** There exists a constant \( K_1 > 0 \) with the following property. If \( \tilde{w}_1 = h(t_0, x_1) \) and \( \tilde{w}_2 = h(t_0, x_2) \) are points in \( A_0 \) such that \( |x_1| = |x_2| \), then

\[ |\Theta(\tilde{w}_1, \tilde{w}_2)| \leq 2K_1 \pi. \]

**Proof.** Let \( |x_1| = |x_2| = \rho \) and let \( S \subset \mathbb{C}^2 \) be the circle \( \{ (\zeta, t_0 \zeta) : \zeta \in \mathbb{C}, |\zeta| = \rho \} \). Let \( C \subset S \) be an arc of \( S \) joining \( z_1 = (x_1, t_0 x_1) \) and \( z_2 = (x_2, t_0 x_2) \). Suppose that for some \( n \in \mathbb{N} \) we have

\[ |\Theta(\tilde{w}_1 - \tilde{w}_2)| > 2n \pi. \]

Then as in the proof of Proposition 27 there are ordered points \( z_0 < \ldots < z_{2n} \) in \( C \) with \( z_0 = z_1, z_2 \leq z_2 \) and such that, if \( w_j = h \circ \pi^{-1}(\tilde{z}_j) \), then

\[ |\Theta(w_j, w_{j+1})| = \pi \]

for all \( j = 0, \ldots, 2n - 1 \). Then by Lemma 26 we have

\[ |\delta_j - \delta_{j+1}| \geq \frac{r_1 m}{LM} |\delta_{j+1}| \]

for all \( z = 0, \ldots, 2n - 1 \). Since \( \delta_j \in C \), then \( \delta_j = (\zeta_j, t_0 \zeta_j) \) with \( |\zeta_j| = \rho \) and therefore

\[ |\delta_j - \delta_{j+1}| = |(\zeta_j - \zeta_{j+1}, t_0(\zeta_j - \zeta_{j+1}))| = |(1, t_0)||\delta_j - \delta_{j+1}|, \]

\[ |\delta_{j+1}| = |(\zeta_{j+1}, t_0 \zeta_{j+1})| = |(1, t_0)||\delta_{j+1}| = |(1, t_0)| \rho. \]

Thus, from (10.4) we have

\[ |(1, t_0)||\delta_j - \delta_{j+1}| \geq \frac{r_1 m}{LM} |(1, t_0)| \rho, \]

\[ |\zeta_j - \zeta_{j+1}| \geq \frac{r_1 m}{LM} \rho, \]

hence

\[ \sum_{j=0}^{2n-1} |\zeta_j - \zeta_{j+1}| \geq 2n \frac{r_1 m}{LM} \rho. \]

Observe that the sum above is less or equal than the length of the circle \( |\zeta| = \rho \) in \( \mathbb{C} \), that is, \( 2\pi \rho \). Then

\[ 2n \frac{r_1 m}{LM} \rho \leq 2\pi \rho, \]

and so

\[ n \leq \frac{\pi LM}{r_1 m}. \]

Thus, it is sufficient to take an integer \( K_1 > \frac{\pi LM}{r_1 m} \).

\[ \square \]

**Proposition 29.** There exists a constant \( s_0 > 0 \) such that for any \( w = (t, x) \in \tilde{V} \) with \( |x| \leq \delta_0/2 \) we have that \( \phi^s(t, x) \) is defined for \( s \in [-s_0, s_0] \) and \( |\phi_2^s(w)| \leq 2|x| \) for all \( s \in [-s_0, s_0] \).
Proof. Denote \( \tilde{\phi}^x(w) = (r(s), \alpha(s)) \). If we set \( \eta_1 = \eta L r_2 / 2 \) from Fact \([18] \) we have that \( |\alpha'(s)| \leq \eta_1 |\alpha(s)| \). Define \( f(s) = |\alpha(s)|^2 \). Then
\[
|f'(s)| \leq 2 |\alpha'(s)| |\alpha(s)| \leq 2 \eta_1 |\alpha(s)|^2, \\
|f'(s)| \leq 2 \eta_1 f(s).
\]
Thus, if \( f(s) \leq f(0)e^{2n_1 |s|} \) for all \( s \in I \), hence \( |\alpha(s)| \leq |x| e^{n_1 |s|} \) for all \( s \in I \). If \( S > 0 \) is maximal such that \( \tilde{\phi}^x(w) \) is defined on \( (-S, S) \) and contained in \( V \), then, either \( S = \infty \), or there exists a sequence \( (s_k) \) with \( |s_k| \to S \) such that \( |\alpha(s_k)| \) tends to \( \tilde{\alpha}_0 \). But \( \lim |\alpha(s_k)| \leq \lim |x| e^{n_1 |s_k|} \) as \( s \to S \), hence \( \tilde{\alpha}_0 \leq |x| e^{n_1 S} \leq (\tilde{\alpha}_0/2)e^{n_1 S} \) and therefore \( S \geq \log_2 \frac{\tilde{\alpha}_0}{2} \). Finally it is sufficient to take \( s_0 < \frac{\log_2 \tilde{\alpha}_0}{2} \).

By successively applications of Proposition \([20] \) we obtain:

**Proposition 30.** If \( w = (t, x) \in \tilde{V} \) is such that \( |x| \leq \tilde{\alpha}_0/2^n \) with \( n \in \mathbb{N} \), then \( \tilde{\phi}^x(w) \) is defined for \( s \in [-ns_0, ns_0] \) and \( |\tilde{\phi}_2^x(w)| \leq 2^n |x| \) for all \( s \in [-ns_0, ns_0] \).

### 10.1. Proof of Proposition \([25] \)

We start with a preparatory proposition.

**Proposition 31.** There exist \( \mu_2, \mu_3 > 0 \) with the following property. If \( n \in \mathbb{N} \) and \( w = (t, x) \in A_0 \) is such that \( |x| \leq \frac{\mu_2}{(1+\mu_1)^n} \), then \( \tilde{\phi}^x(w) \) is defined and
\[
|\tilde{\phi}_2^x(w)| \leq \mu_3 \sqrt{|x|}
\]
for all \( s \in [-2n\pi, 2n\pi] \).

*Proof.* The holonomy of \( \tilde{\mathcal{F}}_{\tilde{V}} \) at the leaf \( \{x = 0\} \) is represented by the maps
\[
x \mapsto \tilde{\phi}^{2\pi i}(t, x),
\]
where \( L_1 r_1 \leq |t| \leq L_2 r_2 \). Let \( \lambda \) be the multiplier of those maps at \( x = 0 \). We can find \( \tilde{\delta}_3 \in (0, \tilde{\delta}_0) \) such that, if \( |x| \leq \tilde{\delta}_3 \) and \( L_1 r_1 \leq |t| \leq L_2 r_2 \), then
\[
|\tilde{\phi}_2^{2\pi i}(t, x) - \lambda x| \leq (\sqrt{1+\mu_1} - 1)|x|
\]
and
\[
|\tilde{\phi}_2^{-2\pi i}(t, x) - \lambda^{-1} x| \leq (\sqrt{1+\mu_1} - 1)|x|.
\]
In this case, since \( |\lambda| = 1 \) we have
\[
|\tilde{\phi}_2^{2\pi i}(t, x)| \leq \sqrt{1+\mu_1}|x|.
\]
Take a natural \( n_0 \geq \frac{2\pi}{\tilde{s}_0} \) and define \( \mu_2 = \frac{\delta_3}{2^{n_0}} \). We prove first that \( |x| \leq \frac{\mu_2}{(1+\mu_1)^n} \) implies that \( \tilde{\phi}^x(w) \) is defined for all \( s \in [-2n\pi, 2n\pi] \). We proceed by induction. Suppose \( |x| \leq \frac{\mu_2}{1+\mu_1} \). Then \( |x| \leq \frac{\delta_3}{2^{n_0}} \) and by Proposition \([30] \) we have that \( \tilde{\phi}^x(t, x) \) is defined for all \( s \in [-n_0 s_0, n_0 s_0] \supset [-2\pi, 2\pi] \), which proves case \( n = 1 \). Now, suppose that \( |x| \leq \frac{\mu_2}{(1+\mu_1)^n} \) implies that \( \tilde{\phi}^x(w) \) is defined for all \( s \in [-2n\pi, 2n\pi] \) and let \( w = (t, x) \) be such that \( |x| \leq \frac{\mu_2}{(1+\mu_1)^{n+1}} \). Then \( |x| \leq \frac{\mu_2}{(1+\mu_1)^{n+1}} \) and by induction hypothesis we have that \( \tilde{\phi}^x(w) \) is defined for all \( s \in [-2n\pi, 2n\pi] \). Since \( |x| \leq \tilde{\delta}_3 \) we have
\[
|\tilde{\phi}_2^{2\pi i}(w)| \leq \sqrt{1+\mu_1}|x| \leq \frac{\mu_2}{(1+\mu_1)^{n+1}} \leq \tilde{\delta}_3.
\]
Again, since $|\tilde{\phi}_2^{2\pi i}(w)| \leq \tilde{\delta}_3$ we have

$$|\tilde{\phi}_2^{2\pi i}(w)| = |\tilde{\phi}_2^{2\pi i}(\tilde{\phi}_2^{2\pi i}(w))| \leq \sqrt{1 + \mu_1}|\tilde{\phi}_2^{2\pi i}(t, x)| \leq \sqrt{1 + \mu_1^2} |x| \leq \frac{\mu_2}{(1 + \mu_1)^n} \leq \tilde{\delta}_3.$$  

If we proceed successively we obtain

$$|\tilde{\phi}_2^{2\pi i}(w)| \leq \sqrt{1 + \mu_1^n} |x| \leq \frac{\mu_2}{(1 + \mu_1)^{n+1}} \leq \tilde{\delta}_3.$$  

Then, if $w_1 = \tilde{\phi}_2^{2\pi i}(w)$, by case $n = 1$ we have that $\tilde{\phi}_2^{i\xi}(w_1)$ is defined for all $s \in [-2\pi, 2\pi]$. From this and the flow property we deduce that $\tilde{\phi}_2^{i\xi}(w)$ is defined for all $s \in [-2(n+1)\pi, 2n\pi]$. On the other hand, there is a constant $c > 0$ such that $|\tilde{\phi}_2^{i\xi}(t, x)| \leq c|x|$ for all $s \in [-2\pi, 2\pi]$, $|x| \leq \tilde{\delta}_3$. Suppose $|x| \leq \frac{\mu_2}{(1 + \mu_1)^n}$ and let $s \in [0, 2n\pi]$. The case $s \in [-2n\pi, 0]$ is similar. Put $s = 2\pi k + \xi$ with $k \in \mathbb{Z}$ and $\xi \in [0, 2\pi)$. Then

$$|\tilde{\phi}_2^{i\xi}(w)| \leq |\tilde{\phi}_2^{i\xi}(\tilde{\phi}_2^{2\pi k i}(w))| \leq c|\tilde{\phi}_2^{2\pi k i}(w)| \leq c \sqrt{1 + \mu_1^n} |x|.$$  

But $|x| \leq \frac{\mu_2}{(1 + \mu_1)^n}$ implies

$$\sqrt{1 + \mu_1^k} \leq \sqrt{1 + \mu_1^n} \leq \frac{\mu_2}{\sqrt{|x|}},$$

and therefore

$$|\tilde{\phi}_2^{i\xi}(w)| \leq c \sqrt{\mu_2 \sqrt{|x|}},$$

hence we set $\mu_3 = c \sqrt{\mu_2}$.  

**Proof of Proposition 23** Take $\delta_1 \in (0, \delta_0)$ such that $A_1$ is contained in the set

$$\{(t, x) \in \bar{V} : |x| \leq \frac{m\mu_2}{2M(1 + \mu_1)^{K_1+1}}\}.$$  

Let $\tilde{w}_1 = h(w_1), \tilde{w}_2 = h(w_2) \in A_1$, put $w_1 = (t_0, x_1), w_2 = (t_0, x_2), \tilde{w}_1 = (\tilde{t}_1, \tilde{x}_1), \tilde{w}_2 = (\tilde{t}_2, \tilde{x}_2)$ and assume $|x_1| \leq |x_2|$. Take $w' = (t_0, x')$ with $|x'| = |x_2|$ and such that $x_1 = \lambda x'$ for $\lambda \in (0, 1)$. Since

$$|\tilde{x}_1| \leq \frac{m\mu_2}{2M(1 + \mu_1)^{K_1+1}} \leq \frac{\mu_2}{(1 + \mu_1)^{K_1+1}},$$

by Proposition 21 we have that $\tilde{\phi}_2^{i\xi}(w_1)$ is defined for $|s| \leq 2(K_1 + 1)\pi$. Then we can assume $|\Theta(\tilde{w}_1, \tilde{w}_2)| > 2(K_1 + 1)\pi$. But $\Theta(\tilde{w}_1, \tilde{w}_2) = \Theta(\tilde{w}_1, \tilde{w}'') + \Theta(\tilde{w}'', \tilde{w}_2)$ and $|\Theta(\tilde{w}'', \tilde{w}_2)| \leq 2K_1\pi$ by Proposition 28 so we have $|\Theta(\tilde{w}_1, \tilde{w}'')| = 2\kappa\pi$ with $\kappa > 1$. Let $n \in \mathbb{N}$ be such that $n \leq \kappa < n + 1$. It follows from the mean value theorem that there exists $w'' = (t_0, x'')$ with $x''$ in the segment joining $x_1$ and $x'$ such that $|\Theta(\tilde{w}_1, \tilde{w}'')| = 2n\pi$, where $\tilde{w}'' = (\tilde{t}'', \tilde{x}'') = h(w'')$. Then, by Proposition 27 we have

$$|\tilde{x}_1| \leq \frac{(2M/m)|\tilde{x}|'}{(1 + \mu_1)^n},$$

Since $|x'| \in A_1$ we have that

$$|\tilde{x}|' \leq \frac{m\mu_2}{2M(1 + \mu_1)^{K_1+1}}.$$  

Then

$$|\tilde{x}_1| \leq \frac{(2M/m)}{(1 + \mu_1)^n} \frac{m\mu_2}{2M(1 + \mu_1)^{K_1+1}} = \frac{\mu_2}{(1 + \mu_1)^{n+K_1+1}}.$$
then, if we define $\tilde{\sigma}$ that

$$ |\Theta(\tilde{w}_1, \tilde{w}_2)| \leq |\Theta(\tilde{w}_1, \tilde{w}')| + |\Theta(\tilde{w}', \tilde{w}_2)| \leq 2\kappa \pi + 2K_1 \pi \leq 2(n + K_1 + 1)\pi. $$

10.2. Proof of Proposition 28

Set $D^* = \{(t_0, x) : 0 < |x| < \delta_1\}$, fix $\tilde{w}_2 \in A_1 = h(D^*)$ and define $\sigma_1(\tilde{w}_1) = \Theta(\tilde{w}_1, \tilde{w}_2)$ for all $\tilde{w}_1 \in A_1$. By Proposition 25 we have that $\tilde{\alpha}^s(\tilde{w}_1)$ is defined and $|\tilde{\alpha}^s(\tilde{w}_1)| \leq \mu_3 \sqrt{|x|}$ for $|s| \leq 2(n + K_1 + 1)\pi$. Clearly we can assume $\delta_1$ to be small enough such that, for each $\zeta \in S^1$, the radial segment

$$ \{(s, c) : \frac{1}{T}r_1 \leq s \leq Lr_2\} $$

can be lifted to the leaf of $F|\tilde{\gamma}$ through any $\tilde{\phi}^s(\tilde{w}_1)$, $|s| \leq |\sigma_1(\tilde{w}_1)|$. Then, if $\tilde{w}_1 = (\tilde{t}_1, \tilde{x}_1)$, we have that $\tilde{\phi}^T(\tilde{w}_1)$ is defined in all $A$ contained in $f(\tilde{w}_1)$. Denote $\tilde{f}(\tilde{w}) = (f_1(\tilde{w}), f_2(\tilde{w}))$ and observe that

$$ f(\tilde{w}) = \tilde{\phi}^T(\tilde{w}_1), $$

then, if we define $\sigma(\tilde{w}) = \sigma_1(\tilde{w}_1) - \text{Im} T(\tilde{w}_1)$ we see that $\tilde{\phi}^s(\tilde{w})$ is defined for all $s \in [0, 1]$. By the definition of $\Theta$ it is easy to see that $f(\tilde{w}) := \tilde{\phi}^s(\tilde{w})$ is contained in a Hopf fiber $\tilde{D} = \{(\tilde{t}_0, x) : |x| < \tilde{\delta}_0\}$ with $|\tilde{t}_0| = \sqrt{T_1 r_2}$. Denote $f(\tilde{w}) = (f_1(w), f_2(w))$ and observe that

$$ f(\tilde{w}) = \tilde{\phi}^T(\tilde{w}_1). $$

Since $|\text{Re} \tilde{T}|$ is bounded we find a constant $\mu_4 > 0$ such that

$$ |f_2(\tilde{w})| = |\tilde{\phi}^T(\tilde{w}_1)| \leq \mu_4|\tilde{\phi}^s(\tilde{w}_1)| \leq \mu_4 \mu_3 \sqrt{|x|}. $$

Then, since $|\tilde{x}_1| \to 0$ as $|x| \to 0$, we deduce that $f(\tilde{w}) \to o \in \tilde{D}$ as $\tilde{w}$ tends to the exceptional divisor. It remains to prove that $f$ is injective. Suppose that $f(\tilde{w}) = f(\tilde{w}')$. Let $\gamma : [0, 1] \to A$ be a curve joining $\tilde{w}$ and $\tilde{w}'$. Consider the paths

$$ \alpha(s) = \tilde{\phi}^{(1-s)}(\tilde{w}), s \in [0, 1] $$

and

$$ \beta(s) = \tilde{\phi}^{s}(\tilde{w'}), s \in [0, 1]. $$
Let \( \vartheta \) be the closed path \( \gamma \ast \beta \ast \alpha \). For \( t \in [0, 1] \) define the paths

\[
\gamma_t(s) = \hat{\vartheta}^{t \sigma \gamma_0}(\gamma(s)), \quad s \in [0, 1],
\]

\[
\alpha_t(s) = \hat{\vartheta}^{(1-s+ts)\sigma}(\tilde{w}), \quad s \in [0, 1]
\]

and

\[
\beta_t(s) = \hat{\vartheta}^{(s+(1-t)s)\sigma}(\tilde{w}), \quad s \in [0, 1].
\]

It is easy to see that \( \gamma_t \ast \beta_t \ast \alpha_t \) define a homotopy between \( \vartheta \) and a path contained in \( \tilde{D} \). Then \( \vartheta \) does not link the set \( \{ t = 0 \} \) and therefore, by Lemma \( \text{[12]} \) the path \( \tilde{h}^{-1}(\vartheta) \) does not link \( \{ t = 0 \} \). Observe that the path \( \tilde{h}^{-1}(\vartheta) \) has the part \( \tilde{h}^{-1}(\gamma) \) contained in \( D^* \). The other part \( \tilde{h}^{-1}(\beta \ast \alpha) \) is a path contained in a leaf of the foliation \( \mathcal{F}|_V \). Since \( \tilde{h}^{-1}(\beta \ast \alpha) \) joins \( \tilde{h}^{-1}(\tilde{w}) \) and \( \tilde{h}^{-1}(\tilde{w}') \) (points in \( D^* \)) we have that \( \tilde{h}^{-1}(\tilde{w}) \in g(\tilde{h}^{-1}(\tilde{w}')) \), where \( g \) is the holonomy map associated to the projection of \( \tilde{h}^{-1}(\vartheta) \) in \( \{ x = 0 \} \). Then, since \( \tilde{h}^{-1}(\vartheta) \) does not link \( \{ t = 0 \} \), we have that \( g = \text{id} \), hence \( \tilde{w} = \tilde{w}' \).

11. Proof of Proposition \( \text{[23]} \) in the hyperbolic case

In this section we assume that the holonomy of \( \tilde{\mathcal{F}}|_V \) at the leaf \( \{ x = 0 \} \) is hyperbolic. Let \( D_0^* \) as in section \( \text{[10]} \) and put \( A_0 = \tilde{h}(D_0^*) \). For \( \tilde{w}_1, \tilde{w}_2 \in A_0 \) define \( \Theta(\tilde{w}_1, \tilde{w}_2) \) as in Section \( \text{[10]} \) fix \( \tilde{w}_2 \in A_0 \) and define \( \sigma(\tilde{w}) = \Theta(\tilde{w}, \tilde{w}_2) \). Given \( z \in A_0 \) take a complex disc \( \Sigma_z \) passing through \( z \) and transverse to \( \mathcal{F} \). In a neighborhood \( U_z \) of \( z \) is well defined a leaf preserving projection \( \pi_z : U_z \rightarrow \Sigma_z \). It is not difficult to prove, since \( A_0 \) is a continuous transversal to \( \tilde{\mathcal{F}} \), that in a small neighborhood \( \Delta_z \) of \( z \) in \( A_0 \) the restriction \( \pi_z : \Delta_z \rightarrow \Sigma_z \) is a homeomorphism onto its image. The charts \( \{ \pi_z \}_{z \in A_0} \) define a natural complex structure on \( A_0 \). Then \( A_0 \) is analytically equivalent to an annulus

\[
\{ z \in \mathbb{C} : 0 \leq r < |z| \leq 1 \}
\]

for some \( r \geq 0 \). The holonomy of \( \tilde{\mathcal{F}}|_V \) at the leaf \( \{ x = 0 \} \) is represented by a contractive function \( g : D_0 \rightarrow D_0 \), where

\[
D_0 = \{(t_0, x) : |x| < \delta_0\}.
\]

Consider the map \( \hat{g} = \tilde{h} \circ g \circ \hat{h}^{-1} \). Clearly \( \hat{g} : A_0 \rightarrow A_0 \) is not trivial at homology level and it is holomorphic, because it is continuous and leaf preserving. Then, since \( g' \) is not an isomorphism, it follows from the annulus theorem (see \( \text{[13]}, \text{p. 211} \)) that \( r = 0 \) and \( A_0 \) is therefore analytically equivalent to a punctured disc.

By using linearizing coordinates we may assume that the foliation \( \tilde{\mathcal{F}}|_{\tilde{T}} \) extends to the set

\[
\bar{T}_\infty = \{(t, x) : |t| = \sqrt{r_1 r_2}, x \in \mathbb{C}\}
\]

and is the suspension of a hyperbolic automorphism of \( \mathbb{C} \). In this case \( \hat{\vartheta}^T(w) \) is defined for all \( w \in \bar{T}_\infty \) and all \( T \in \mathbb{R} \). Then \( f(\hat{w}) := \hat{\vartheta}^{\sigma}(\hat{w}) \) is well defined and it is contained in the Hopf fiber

\[
\bar{D}_\infty = \{ (t_0, x) : x \in \mathbb{C}\}
\]

through \( \hat{w}_2 \) for all \( \hat{w} \in A_0 \).

Observe that \( f : A_0 \rightarrow \bar{D}_\infty \) is holomorphic, because it is a continuous leaf preserving map. Identifying \( A_0 \) with \( \mathbb{D} \setminus \{0\} \) and \( \bar{D}_\infty \) with \( \mathbb{C} \) we have by Riemann Extension Theorem that \( f \) extends to a holomorphic map \( f : \mathbb{D} \rightarrow \mathbb{C}, f(0) = 0 \). Then \( f(\hat{w}) \) tends to \( o = (t_0, 0) \) as \( \hat{w} \) tends to the exceptional divisor. Since \( \tilde{\mathcal{F}}|_{\bar{T}_\infty} \)
is the suspension of a hyperbolic automorphism of $\mathbb{C}$, there exists a set $\tilde{T} \subset \tilde{T}$ such that

1. $\tilde{T}$ contains all segment of orbit with endpoints in $\tilde{T}$
2. $\tilde{T}$ contains the set $\{(t, x) : |t| = \sqrt{r_1 r_2}, |x| < \epsilon\}$ for some $\epsilon > 0$.

Since $f(0) = 0$, we can set $D^* = \{(t_0, x) : |x| < \delta_1\}$ with $\delta_1 > 0$ such that $A = \bar{h}(D^*)$ and $f(A)$ are contained in $\tilde{T}$. The proof of the injectivity of $f|_A$ given in 10.2 also works in this case and Proposition 23 follows.

12. Proof of Proposition 21 and Lemma 22

Given a proper subdomain $D \subset \mathbb{C}$ we denote by $\omega_D(z)$ the harmonic measure with pole at $z \in D$. Recall that $\omega_D(z)$ is a probability measure on $\partial D$ and, fixed a Borel subset $B$ of $\partial D$, the function $z \mapsto \omega_D(z, B)$ is harmonic on $D$. We will use the following subordination principle.

**Theorem 32.** Let $D_1$ and $D_2$ be domains in $\mathbb{C}$ with non-polar boundaries. Suppose that $D_1 \subset D_2$ and let $B$ be a Borel subset of $\partial D_1 \cap \partial D_2$. Then

$$\omega_{D_1}(z, B) \leq \omega_{D_2}(z, B)$$

for all $z \in D_1$. Moreover, if $\omega_{D_1}(z, B) > 0$ the equality holds only if $D_1 = D_2$.

For the proof of this result and general background on harmonic measures we refer to [12].

12.1. Proof of Lemma 22

It is known that the geodesic $\tilde{\gamma}$ is given by the level set

$$\omega_D(z, C) = \frac{1}{2}.$$ 

By Theorem 32 we have $\omega_U(z, C) < \omega_D(z, C)$ for all $z \in U$. Then for $z$ in the geodesic $\tilde{\gamma}$ we have $\omega_U(z, C) < \frac{1}{2}$. But it is easy to see that $\omega_U(z, C) \geq \frac{1}{2}$ for all $z \in \bar{U} \cap U$ and the lemma follows.

12.2. Proof of Proposition 21

Let $\tilde{D}$ be the interior domain of $\tilde{T}_m(R)$ and let $\tilde{C} = \tilde{T}_m(\partial R^+)$ where $\partial R^+ = \{\tau + \theta i : \tau \in \partial R : \theta \geq 0\}$. In $\tilde{D}$, the geodesic $\Gamma_m$ is defined by the level set

$$\omega_{\tilde{D}}(z, \tilde{C}) = \frac{1}{2}.$$

Provided $\theta_0$ is big enough there are real numbers $\alpha_1, \alpha_2$ with $|\alpha_1|, |\alpha_2| \leq \theta_0$ such that

1. $\tilde{T}_m(\alpha_1 i) = \tau_1 + \vartheta_1(0)i$, where $\tau_1$ is maximal with $-2 \ln L \leq \tau_1 \leq 2 \ln L$.
2. $\tilde{T}_m(\ln \frac{r_2}{r_1} + \alpha_2 i) = \tau_2 + \vartheta_1(0)i$, where $\tau_2$ is minimal with $\ln \frac{r_2}{r_1} - 2 \ln L \leq \tau_2 \leq \ln \frac{r_2}{r_1} + 2 \ln L$.

---

4Here $\vartheta_1$ is the function given by Proposition 10
From item (1) above we obtain
\[ \vartheta_1(0) = \hat{\vartheta}(w, \alpha_1 i). \]

Then by Proposition 10 we have
\[ \vartheta_1(0) = \hat{\vartheta}(w, \alpha_1 i) \geq \vartheta_1(\alpha_1), \]
so \( \alpha_1 \leq 0 \) because \( \vartheta_1 \) is increasing. In the same way we have \( \alpha_2 \leq 0 \). Then, if \( C_1 \) is the superior segment in \( \partial R \) defined by the points \( \alpha_1 i \) and \( (\ln \frac{\alpha_1}{\alpha_1} + \alpha_2 i) \), we have \( \partial R^+ \subset C_1 \) and therefore \( \bar{C}_1 := \overline{\tau_\alpha(C_1)} \) contains \( C \). Therefore
\[ (12.1) \quad \omega_{\bar{D}}(z, \bar{C}) \leq \omega_{\bar{D}}(z, \bar{C}_1) \text{ for all } z \in \bar{D}. \]

Let \( \bar{D}_1 \) be the region bounded by:

1. \( \partial \bar{D} \setminus \bar{C}_1 \)
2. \( \{ \tau + \vartheta_1(0)i : \tau_1 \leq \tau \leq 2 \ln L \} \)
3. \( \{ \tau + \vartheta_1(0)i : -2 \ln L \leq \tau \leq \tau_2 \} \)
4. \( \{ \tau + \vartheta_1(0)i : \theta \geq \vartheta_1(0) \} \)

Clearly \( \bar{D}_1 \subset \bar{D} \) and \( \partial \bar{D} \setminus \bar{C}_1 \subset \partial \bar{D}_1 \cap \partial \bar{D} \). Then by Theorem 32 we have
\[ \omega_{\bar{D}_1}(z, \partial \bar{D} \setminus \bar{C}_1) \leq \omega_{\bar{D}}(z, \partial \bar{D} \setminus \bar{C}_1) \text{ for all } z \in \bar{D}_1. \]

From this, if \( \bar{C}_2 \) is the subset of \( \partial \bar{D}_1 \) composed by the sets given in items 2, 3 and 4 above we have that
\[ (12.2) \quad \omega_{\bar{D}}(z, \bar{C}_1) \leq \omega_{\bar{D}_1}(z, \bar{C}_2) \text{ for all } z \in \bar{D}_1. \]

Let \( \bar{D}_2 \) be the connected component containing \( z \) of the complement of \( \bar{C}_2 \) in the interior of \( R \). Clearly we have \( \bar{D}_1 \subset \bar{D}_2 \) and \( \bar{C}_2 \subset \partial \bar{D}_1 \cap \partial \bar{D}_2 \). Then by Theorem 32 we have
\[ (12.3) \quad \omega_{\bar{D}_1}(z, \bar{C}_2) \leq \omega_{\bar{D}_2}(z, \bar{C}_2) \text{ for all } z \in \bar{D}_1. \]

Set
\[ \bar{C}_3 = \{ \tau + \vartheta_1(0)i : \theta \leq \vartheta_1(0) \} \]

and let \( \bar{D}_3 \) be the region bounded by

1. \( \bar{C}_2 \)
2. \( \bar{C}_3 \)
3. \( \{ \tau + \vartheta_1(0)i : -2 \ln L \leq \tau \leq \tau_1 \} \)
4. \( \{ \tau + \vartheta_1(0)i : \tau_2 \leq \tau \leq \ln \frac{\alpha_1}{\alpha_1} + 2 \ln L \} \).

Again by Theorem 32 we have
\[ \omega_{\bar{D}_3}(z, \bar{C}_3) \leq \omega_{\bar{D}_2}(z, \bar{C}_3) \text{ for all } z \in \bar{D}_3, \]

so
\[ \omega_{\bar{D}_2}(z, \partial \bar{D}_2 \setminus \bar{C}_3) \leq \omega_{\bar{D}_3}(z, \partial \bar{D}_3 \setminus \bar{C}_3) \text{ for all } z \in \bar{D}_3. \]

Then, since \( \bar{C}_2 \subset \partial \bar{D}_2 \setminus \bar{C}_3 \) we have
\[ (12.4) \quad \omega_{\bar{D}_3}(z, \bar{C}_2) \leq \omega_{\bar{D}_3}(z, \partial \bar{D}_3 \setminus \bar{C}_3) \text{ for all } z \in \bar{D}_3. \]

Therefore, if \( \bar{C}_4 = \partial \bar{D}_3 \setminus \bar{C}_3 \) from equations (12.1), (12.2), (12.3) and (12.4) we obtain:
\[ (12.5) \quad \omega_{\bar{D}}(z, \bar{C}) \leq \omega_{\bar{D}_3}(z, \bar{C}_4) \text{ for all } z \in \bar{D} \cap \bar{D}_3. \]

\(^5 \bar{R}_0 \) is given in 8.4.
Fixed \( z \in \bar{D} \cap \bar{D}_3 \), let \( f : \bar{R} \to \bar{D} \) be a homeomorphism such that:

1. \( f \) maps \( R = \text{int}(\bar{R}) \) biholomorphically onto \( D \)
2. \( f(z) = 0 \).

Set

\[
\partial \bar{R}^+ = \{ \tau + \theta i \in \partial \bar{R} : \theta \geq 0 \}.
\]

We know that

\[
(12.6) \quad \omega_{\bar{D}_4}(z, \bar{C}_4) = \omega_{f(\bar{D}_4)}(0, f(\bar{C}_4))
\]

and

\[
(12.7) \quad \omega_R(z, \partial \bar{R}^+) = \omega_D(0, f(\partial \bar{R}^+)).
\]

It is not difficult to see that, if \( \ln \frac{2}{r_1} \) and \( \tilde{\theta}_0 \) are large, the Jordan curve \( f(\partial \bar{D}_4) \) is uniformly close to \( \partial D \) and the segment \( f(\bar{C}_4) \) is uniformly close to \( f(\partial \bar{R}^+) \). Thus, given \( \epsilon > 0 \) and given a compact set \( K \subset \text{int}(\bar{R}_0) \), it can be proved that we can take \( \ln \frac{2}{r_1} \) and \( \tilde{\theta}_0 \) large enough such that

\[
(12.8) \quad \omega_{f(\bar{D}_4)}(0, f(\bar{C}_4)) \leq \omega_D(0, f(\partial \bar{R}^+)) + \epsilon
\]

for all \( z \in K \) and therefore by (12.5) (12.6) and (12.7)

\[
\omega_{\bar{D}_4}(z, \bar{C}_4) \leq \omega_R(z, \partial \bar{R}^+) + \epsilon
\]

for all \( z \in K \). By similar arguments we can obtain

\[
\omega_{\bar{D}_4}(z, \bar{C}_4) \geq \omega_R(z, \partial \bar{R}^+) - \epsilon
\]

for all \( z \in K \). Thus, when \( \ln \frac{2}{r_1} \) and \( \tilde{\theta}_0 \) tends to \( +\infty \) we have that the function \( z \mapsto \omega_{\bar{D}_4}(z, \bar{C}_4) \) converges to the function \( z \mapsto \omega_R(z, \partial \bar{R}^+) \) uniformly on the compacts subsets of \( \text{int}(\bar{R}_0) \) and uniformly on \( w \in \partial V_1 \). So, fixed a compact set \( K \subset \text{int}(\bar{R}_0) \), when \( \ln \frac{2}{r_1} \) and \( \tilde{\theta}_0 \) tends to \( +\infty \) we have that, uniformly on \( w \in \partial V_1 \), the level set

\[
\{ z \in K : \omega_{\bar{D}_4}(z, \bar{C}_4) = \frac{1}{2} \}
\]

tends in the \( C^\infty \) topology to the level set

\[
\{ z \in K : \omega_R(z, \partial \bar{R}^+) = \frac{1}{2} \} = \{ \tau + \theta i \in K : \theta = 0 \}
\]

and the proposition follows.

13. Projective holonomy representation

Given a non-dicritical\(^6\) foliation \( \mathcal{F} \), we define the projective holonomy representation of \( \mathcal{F} \) as follows. Let \( \mathcal{F}_s \) be the strict transform of \( \mathcal{F} \). Clearly \( E^* = E \setminus \text{Sing}(\mathcal{F}_s) \) is a leaf of \( \mathcal{F}_s \). Let \( q \) be a point in \( E^* \) and \( \Sigma \) a small complex disc passing through \( q \) and transverse to \( \mathcal{F}_s \). For any loop \( \gamma \) in \( E^* \) based on \( q \) there is an holonomy map \( H_F(\gamma) : (\Sigma, q) \mapsto (\Sigma, q) \) which only depends on the homotopy class of \( \gamma \) in the fundamental group \( \Gamma = \pi_1(E^*) \). The map \( H_F : \Gamma \to \text{Diff}(\Sigma, q) \) is known as the projective holonomy representation of \( \mathcal{F} \). Identifying \( (\Sigma, q) \simeq (\mathbb{C}, 0) \) the image

\(^6\)Here we use the good dependence on the boundary of the uniformization of a Jordan region (see \( [11] \) p.26) and the invariance of harmonic measures by conformal maps.

\(^7\)In the sense that after a single blow up we obtain an invariant exceptional divisor.
of $H_F$ defines up to conjugation a subgroup of $\text{Diff}(\mathbb{C}, 0)$ which is known as the holonomy group of $F$.

The representations $H : \Gamma \to \text{Diff}(\mathbb{C}, 0)$ and $H' : \Gamma' \to \text{Diff}(\mathbb{C}, 0)$ are topologically conjugated if there exist an isomorphism $\varphi : \Gamma \to \Gamma'$ and a germ of homeomorphism $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $H' \circ \varphi(\gamma) = h \circ H(\gamma) \circ h^{-1}$ for all $\gamma \in \Gamma$.

13.1. Proof of Theorem 2. Let $p_1, ..., p_k$ be the singularities of $F$ in $E$. Let $S_1, ..., S_k$ be irreducible separatrices through $p_1, ..., p_k$ respectively all different from $E$. Let $S_j = h(S_j)$ and let $\tilde{p}_j$ be the point where $S_j$ meets the exceptional divisor $E$. Let $(t, x)$ be coordinates in $\hat{\mathbb{C}}^2$ such that $\pi(t, x) = (x, tx) \in \mathbb{C}^2$. We can assume that the points $p_j$ and $\tilde{p}_j$ are contained in this coordinates, that is, $p_j = (t_j, 0)$ and $\tilde{p}_j = (\tilde{t}_j, 0)$ with $t_j, \tilde{t}_j \in \mathbb{C}$. We can perform the constructions of the proof of Theorem 1 for each $S_j$ to obtain:

1. Another topological equivalence $\tilde{h}$ between $F$ and $\tilde{F}$
2. Some constants $r_j, \tilde{r}_j > 0$

such that the following properties hold:

1. The sets $\{|t - t_j| \leq r_j\}$ are pairwise disjoint
2. The sets $\{|t - \tilde{t}_j| \leq \tilde{r}_j\}$ are pairwise disjoint and they does not contain singularities of $\tilde{F}$ other than the $\tilde{p}_j$
3. For each $j$ and any $w \in \text{dom}(\tilde{h})$ we have that:
   - $w \in \{|t - t_j| < r_j\}$ if and only if $\tilde{h}(w) \in \{|t - \tilde{t}_j| < \tilde{r}_j\}$
   - $w \in \{|t - t_j| > r_j\}$ if and only if $\tilde{h}(w) \in \{|t - \tilde{t}_j| > \tilde{r}_j\}$
   - $w \in \{|t - t_j| = r_j\}$ if and only if $\tilde{h}(w) \in \{|t - \tilde{t}_j| = \tilde{r}_j\}$
4. The Hopf fibers in $\{|t - t_j| = r_j\}$ are mapped by $\tilde{h}$ into Hopf fibers in $\{|t - \tilde{t}_j| = \tilde{r}_j\}$.

Let $W$ and $\tilde{W}$ be the complement in $\hat{\mathbb{C}}^2$ of the sets

$$\bigcup_{j=1}^{k}\{|t - t_j| \leq r_j\}$$

and

$$\bigcup_{j=1}^{k}\{|t - \tilde{t}_j| \leq \tilde{r}_j\}$$

respectively. Let $\Sigma$ be a local Hopf fiber in $\{|t - \tilde{t}_1| = \tilde{r}_1\}$ passing through a point $\tilde{q} \in E$ such that $\tilde{h}^{-1}(\Sigma)$ is contained in a local Hopf fiber $\Sigma$ in $\{|t - t_1| = r_1\}$. Let $\tilde{\gamma}$ be a loop in $E \cap \tilde{W} \setminus \text{Sing}(\tilde{F})$ based on $\tilde{q}$ and let $\tilde{f} : (\Sigma, 0) \to (\Sigma, 0)$ be the holonomy associated to $\tilde{\gamma}$. This holonomy map induces the map

$$f = \tilde{h}^{-1}\tilde{f}\tilde{h} : (\Sigma, 0) \to (\Sigma, 0).$$

Let $\tilde{\gamma}_1 \subset \tilde{W}$ be a loop based on a point $\tilde{q}_1 \in \tilde{\Sigma} \setminus \{\tilde{q}\}$ such that $\tilde{\gamma}_1$ is close to $\tilde{\gamma}$ and disjoint of the exceptional divisor $E$. Let $\varphi(\tilde{\gamma}) \subset E \cap W$ be the loop based at $q = \Sigma \cap E$ given by the projection of $\tilde{h}^{-1}(\tilde{\gamma}_1)$ by the Hopf fibration. The map $\tilde{\gamma} \mapsto \varphi(\tilde{\gamma})$ induces a homomorphism between $\pi_1(E \setminus \text{Sing}(F), \tilde{q})$ and $\pi_1(E \setminus \text{Sing}(F), q)$. Moreover, if $\tilde{F}$ does not have singularities other than the $\tilde{p}_j$ we have that $\varphi$ is an isomorphism.
Assertion. The holonomy map associated to $\varphi(\gamma)$ coincides with the map $f$.

Let $w \in \Sigma \setminus \{q\}$ and consider the path $\alpha = \tilde{h}^{-1}(\bar{\alpha})$, where $\bar{\alpha}$ is the lift to the leaf through $h(w)$ of the loop $\gamma$. Clearly the path $\alpha$ is contained in a leaf and connects $w$ with $f(w)$. We know that the set $E \cap W$ can be retracted into a subset $B$ which is a bouquet of $(k-1)$ circles based on $q$. By using the Hopf Fibration we can "lift" this retraction to the leaves close to the exceptional divisor. Thus, if $w$ is close enough to $q$, the path $\alpha$ is homotopic in the leaf through $w$ to a path $\beta$ contained in the Hopf fibrations restricted to $B$. Clearly the path $\beta$ connects $w$ with $f(w)$. Let $\beta_0$ be the projection of $\beta$ on $B$. Clearly $\beta_0 = \varphi(\gamma)$ in $\pi_1(E \setminus \text{Sing}(F), q)$. Let $\gamma$ be a geodesic representant of $\beta_0$ in $\pi_1(B, q)$. The homotopy between $\beta_0$ and $\gamma$ can be performed in the image of $\beta_0$. Let $\gamma_w$ be the lift of $\gamma$ to the leaf through $w$. Then it is not difficult to see that $\beta$ is homotopic to $\gamma_w$ by a homotopy contained in the image of $\beta$. Therefore $\beta$ is homotopic in the leaf through $w$ to the path $\gamma_w$ and in particular we have that $f(w)$ is the endpoint of $\gamma_w$. But the endpoint of $\gamma_w$ is just the image of $w$ by the holonomy map associated to $\varphi(\gamma)$.

It remains to prove that $\tilde{F}$ does not have singularities other than the $\tilde{p}_j$. Suppose that there exists a singularity $\tilde{p}$ of $\tilde{F}$ different from all $\tilde{p}_j$. Let $\tilde{\gamma}$ be a loop based on $\tilde{p}$ of type $\tilde{\gamma} = \alpha \sigma \alpha^{-1}$ where $\alpha$ is a loop surrounding closely the singularity $\tilde{p}$. It is easy to see in this case that $\varphi(\tilde{\gamma}) = 0$ in $\pi_1(E \setminus \text{Sing}(F), q)$. Therefore, it follows from the assertion above that the holonomy map associated to $\tilde{\gamma}$ is the identity. Let $D \subset E$ be a small disc such that $\overline{D \cap \text{Sing}(F)} = \{\tilde{p}\}$. Then the circle $\partial D$ lift to a circle $\delta$ contained in a leaf close to $\partial D$. The circle $\tilde{h}^{-1}(\delta)$ is contained in a leaf $L$ and its projection in $E \cap W$ is null-homotopic. As in the proof of the assertion, provided $\delta$ is close enough to $\partial D$ we have that $\tilde{h}^{-1}(\delta)$ is null homotopic in $L \cap W$. Then $\tilde{h}^{-1}(\delta)$ is the boundary of a simply connected domain in $L$. Therefore $\delta$ is the boundary of a simply connected domain $\Omega$ in the leaf containing $\delta$. Moreover we have $\Omega \subset \tilde{W}$, so we have that $\Omega$ is a disc close to $D$ as $\delta$ is close to $\partial D$, but this property is only possible if $\tilde{p}$ is a regular point.

References

[1] Bers L., Riemann Surfaces, New York University, Institute of Mathematical Sciences.
[2] Burau W., Kennzeichnung der schlauchknoten, Abh. Math. Sem. Han. Univ. 9(1932), 125-133.
[3] Camacho C., Lins A., Sad P., Topological invariants and equidesingularization for holomorphic vector fields, J. Differential Geometry 20(1984) 143-174.
[4] D. Cerveau and P. Sad, Problèmes de modules pour les formes différentielles singul ières dans le plan complexe, Comment. Math. Helv. 61 (1986), no. 2, 222–253.
[5] Dold A., Lecture notes on algebraic topology, Die Grunlehren der Mathematischen Wissenschaften, Band 200, Springer-Verlag, 1972.
[6] Eyral C., Zariski’s Multiplicity Question - A survey, New Zealand Journal of Mathematics Volume 36 (2007), 253-276.
[7] King H., Topological type of isolated critical points, Ann. Math. (2) 107 (1978), no. 2, 385-397.
[8] Marín, D., Moduli spaces of germs of holomorphic foliations in the plane, Ann. Math. (2) 170 (2009) 518-539.
[9] Marín D., Mattei J.-F., Monodromy and topological classification of germs of holomorphic foliations, Ann. Sci. Éc. Norm. Supér. sér. 4, 3 (2012), http://arxiv.org/abs/1004.1552.
[10] Ortíz-Bobadilla I., Rosales-González E., Voronin S. M., Extended holonomy and topological invariance of vanishing holonomy group, J. Dyn. Control Syst. 14 (2008), no. 3, 299–358.
[11] Pommerenke Ch.: *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften 299, A Series of Comprehensive Studies in Mathematics, Springer-Verlag, 1992, MR1217706, Zbl 0762.30001.

[12] Ransford T., *Potential Theory in the Complex Plane*, London Mathematical Society Student Texts 28, Cambridge University Press, 1995.

[13] Remmert R., *Classical Topics in Complex Function Theory*, Graduate Texts in Mathematics, Springer-Verlag, 1998.

[14] Risler J.J., Trotman D., *Bilipschitz invariance of the multiplicity*, Bull. London Math. Soc. 29 (1997) 200-204.

[15] Rosas R., *The C1 invariance of the algebraic multiplicity of a holomorphic vector field*, Journal of Differential Geometry 83 (2009) 337-376.

[16] Rosas R., *The differentiable invariance of the algebraic multiplicity of a holomorphic vector field*, Journal of Differential Geometry 83 (2009) 337-376.

[17] Rosas R., *Constructing equivalences with some extensions to the divisor and topological invariance of projective holonomy*, to appear at Comm. Math. Helv.

[18] Zariski O., *On the topology of algebraic singularities*, Amer. Journ. of Math., 54(1932), 453-465.

[19] Zariski O., *Some open questions in the theory of singularities*, Bull. Amer. Math. Soc. 77 (1971), 481-491.

E-mail address: rudy.rosas@pucp.pe
E-mail address: rudy@imca.edu.pe

Pontificia Universidad Católica del Perú, Av Universitaria 1801, San Miguel, Lima, Perú.

Instituto de Matemática y Ciencias Afines, calle Los Biólogos 245, Lima 12, Perú.