A NORMAL GRAPH ALGEBRA

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Abstract. We define a normal graph algebra modeled on algebras used in genetics. Although the algebra does not always determine its graph, it often highlights special features. After developing basic properties of the algebra, we examine those of certain minimal graphs. We then apply the results to the Petersen graph, finding connections between some of its many aspects. For example, the outer automorphisms of Sym(6) emerge naturally. The normal algebra of the Petersen graph is unique among normal graph algebras.

1. Introduction

Algebraic methods play a prominent role in the investigation of graph properties. Many of these stem from the creation of an algebra from the graph, such as the algebra of polynomials in the adjacency matrix. In [6] and [11], the authors present a commutative nonassociative algebra defined by the incidence properties of a graph. It is inspired by Bernstein algebras introduced in genetic studies (for a survey of that topic, see [15]). This Bernstein graph algebra determines the graph itself [11, 18]. In the present paper, we define another nonassociative algebra from a graph that is a good bit simpler. It does not always determine the graph; for example, all trees of a given order have isomorphic algebras. But it seems to encapsulate certain graph properties. It leads to aspects of the graph in a natural way, sometimes by means of the automorphism group of the algebra.

1.1. Normal algebras. Following the lead in [10], we define a normal algebra:

Definition 1. A normal algebra consists of a finite dimensional vector space \( N \) over a field \( \mathbb{F} \) of characteristic not 2, endowed with a product \( (a, b) \rightarrow ab \). (We may write \( a \times b \) if juxtaposition is confusing.) The product has the following properties:

1. Bilinearity: for \( a, b, a', b' \) in \( N \) and \( \alpha, \beta, \alpha', \beta' \) in \( \mathbb{F} \),
\[
(\alpha a + \beta b)(\alpha' a' + \beta' b') = \alpha\alpha'(aa') + \alpha\beta'(ab') + \beta\alpha'(ba') + \beta\beta'(bb').
\]

2. Commutativity: \( ab = ba \).

3. Grading: \( N \) is the direct sum of two subspace \( U \) and \( \mathfrak{Z} \) for which
\[
\begin{align*}
u \delta &= 0 \text{ for } u \in U, \delta \in \mathfrak{Z}; \\
\delta \delta' &= 0 \text{ for } \delta, \delta' \in \mathfrak{Z}; \\
uu' &\in \mathfrak{Z} \text{ for } u, u' \in U.
\end{align*}
\]

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This is the framework presented in [10]. Bernstein algebras are not necessarily associative: \(a(bc)\) might not equal \((ab)c\). (Such an algebra is traditionally called a nonassociative algebra, even though it might accidentally be associative.) However, normal algebras are trivially associative, since any three-fold product is automatically 0. They are also special Jordan algebras, the algebra itself providing the associative algebra for which \(ab = \frac{1}{2}(ab + ba)\). (Chapter IV of the standard reference on nonassociative algebra [10] presents Jordan algebras.) None of these extras is used in this paper.

Needed technical aspects of normal algebras will be given on the spot. For example, a homomorphism of one algebra \(A\) to another \(A'\) is a linear transformation \(\varphi : A \rightarrow A'\) for which \(\varphi(ab) = \varphi(a)\varphi(b)\). If \(A\) and \(A'\) are normal, \(\varphi\) is also required to map the \(U\)- and \(Z\)-subspaces of \(A\) into the respective subspaces of \(A'\).

If \(X\) is a subset of an \(\mathbb{F}\)-vector space, the span \(\langle X \rangle\) of \(X\) is the set of \(\mathbb{F}\)-linear combinations of the members of \(X\), shortened to \(\langle x \rangle\) when \(X = \{x\}\). Two members \(u\) and \(v\) of a vector space are called proportional if \(\langle u \rangle = \langle v \rangle\), and we often indicate this by \(u \approx v\). Finally, for a vector space \(V\), \(\overline{V}\) is its dual space of \(\mathbb{F}\)-linear functionals on \(V\).

2. Normal graph algebras

Let \(G\) be a finite simple graph, one with no loops or multiple edges. The set of vertices of \(G\) is \(VG\) and \(EG\) is the set of edges. The order of \(G\) is \(p = |VG|\) and the size is \(q = |EG|\); these notations will hold throughout. (For both basic graph-theoretic concepts and topics in algebraic graph theory, see [9].) Edges are two-element subsets of \(VG\). Instead of the conventional notations \(\{x, y\}\) or \(xy\) for the edge with vertices \(x\) and \(y\), we write \([x, y]\), especially in the algebra framework where we see \(xy = [x, y]\). The vertices of an edge \([x, y]\) are called adjacent, denoted \(x \sim y\). Different edges are also called adjacent when they share a vertex.

The normal graph algebra \(NG\) of \(G\) has for its \(U\)-subspace, \(U_G\), the space of formal \(\mathbb{F}\)-linear combinations of the vertices, and for its \(Z\)-subspace, \(Z_G\), the formal \(\mathbb{F}\)-linear combinations of the edges. Thus \(U_G = \langle VG \rangle\) and \(Z_G = \langle EG \rangle\). The product in \(NG\) is defined for the basis elements, the vertices and edges, and extended through bilinearity. (The construction is similar to that in [6, Section 2]. It also resembles the definition of a group ring; the exposition in [5, p. 172] may be helpful.) Specifically, if \(x\) and \(y\) are vertices,

\[
xy = \begin{cases} 
[x, y] & \text{if } x \neq y \text{ and } x \sim y, \\
0 & \text{if } x \neq y \text{ and } x \not\sim y.
\end{cases}
\]

\[x^2 = \sum [x, y], \text{ summed over the } y \text{ adjacent to } x.
\]

All other basis products, such as \(x \times \epsilon, x \in VG, \epsilon \in EG\), are 0, in line with the definition [4]. Let two members of \(U_G\) be \(u = \sum_{x \in VG} \theta_x x\) and \(v = \sum_{x \in VG} \eta_x x\). Then

\[uv = \left(\sum_{x \in VG} \theta_x x\right) \left(\sum_{x \in VG} \eta_x x\right) = \sum_{x, y \in VG} \theta_x \eta_y xy.
\]

An edge \([a, b]\) appears in the last sum four times: in \(a^2, ab, ba,\) and \(b^2\). The contributions from these terms are

\[a^2 \theta_a \eta_a, \quad ab \theta_a \eta_b, \quad ba \eta_b \theta_a, \quad b^2 \theta_b \eta_b.
\]
Their sum is \( \theta_a \eta_a + \theta_a \eta_b + \eta_b \theta_a + \theta_b \eta_b = (\theta_a + \theta_b)(\eta_a + \eta_b) \). Thus

\[
(2.2) \quad uv = \sum_{[x,y] \in EG} (\theta_x + \theta_y)(\eta_x + \eta_y)[x, y] \text{ and } u^2 = \sum_{[x,y] \in EG} (\theta_x + \theta_y)^2[x, y].
\]

**Example 1.** Here are two trees: let \( T_1 \) be the path of order 4 with vertices \( a, b, c, d \) and edges 
\[ e = [a, b], f = [b, c], g = [c, d]. \]

Let \( T_2 \) be the claw with the same vertices, but edges 
\[ e = [a, b], f = [a, c], g = [a, d]. \]

First define elements \( u_0, u_1, u_2, u_3 \) in \( T_1 \) and \( T_2 \) by 
\[
T_1 \quad u_0 = a - b + c - d \quad u_1 = a \quad u_2 = \frac{1}{2}(a - b - c + d) \quad u_3 = d.
T_2 \quad a - b - c - d \quad b \quad c \quad d
\]

The \( u_i \) form bases of the two \( U \)-spaces. As a computational example, expanding out \( u_0 u_2 \) in \( T_1 \) directly from the definition (2.1) gives 
\[
\begin{align*}
\frac{1}{2} & (a - b + c - d)(a - b - c + d) \\
& = \frac{1}{2}(a^2 - ab - ac + ad - ba + b^2 + bc - bd) \\
& \quad + ca - cb - c^2 + cd - da + db + dc - d^2 \\
& = \frac{1}{2}(a - b + c - d - e + f - 0) \\
& \quad + 0 - f - e + g - 0 + 0 + 0 - 0.
\end{align*}
\]

The whole-scale cancelling shows that \( u_0 u_2 = 0 \). The computation of \( u_0 u_2 \) is much easier when done by the product formulas (2.2), because for each edge \([a, b], [b, c], [c, d]\) of \( T_1 \), the sums \( \theta_x + \theta_y \) for \( u_0 \) are all 0.

The algebras \( NT_1 \) and \( NT_2 \) have the same product table in terms of the \( u_i \) and the edges:
\[
\begin{array}{cccc}
\ u_0 & u_1 & u_2 & u_3 \\
0 & 0 & 0 & 0 \\
0 & e & 0 & 0 \\
0 & 0 & f & 0 \\
0 & 0 & 0 & g \\
\end{array}
\]

Hence \( NT_1 \) and \( NT_2 \) are isomorphic, an illustration of the introductory comment about trees.

The normal graph algebra of a single vertex with no edges is the one-dimensional \( \mathbb{F} \)-algebra \( \mathbb{F}^0 \) with all products 0. \( \mathbb{F}^0 \) is a direct summand of the normal graph algebra of any bipartite graph. In general, \( \mathcal{N}(G \cup H) \) is isomorphic to the direct sum of \( \mathcal{N}G \) and \( \mathcal{N}H \). For a connected bipartite graph of order at least 2, that \( \mathbb{F}^0 \) of a bipartite graph is not the algebra of a subgraph.
3. Short homomorphisms

The short algebra is the graph algebra of the spline graph, $S$, that has one vertex, $r$, and one edge, $s$, with $r$ the sole endpoint of $s$. This spline algebra $S$ has the single defining relation $r^2 = s$. For $S$, $US = \langle r \rangle$, the one-dimensional subspace spanned by $r$, and $3S = \langle s \rangle$. A short homomorphism $\varphi$ from $\text{alg}G$ (or any normal algebra) to $S$ is a normal homomorphism, meaning that in addition to the standard algebra homomorphism rule that $\varphi$ is a linear transformation for which $\varphi(mn) = \varphi(m)\varphi(n)$ when $m, n \in \text{alg}G$, we also have that $\varphi$ sends $U_G$ into $US$ and $3G$ into $3S$. This implies that $\varphi$ is described by two linear functionals, $\lambda$ and $\mu$, for which $\varphi(u) = \lambda(u)r$ and $\varphi(3) = \mu(3)s$, where $u \in U_G$ and $3 \in 3_G$. The homomorphism rule for $\varphi$ that $\varphi(u)\varphi(v) = \varphi(uv)$ for all $u, v \in U_G$ is equivalent to the demand that $\mu(uv) = \lambda(u)\lambda(v)$. Given a short homomorphism $\varphi$, we get $\lambda$ and $\mu$ satisfying the last equation. If we were just presented with a $\lambda$ and a $\mu$ and define $\varphi$ by $\varphi(u) = \lambda(u)r$ and $\varphi(3) = \mu(3)s$, then to verify that $\varphi$ is a short homomorphism, we would need to show that $\mu(uv) = \lambda(u)\lambda(v)$ for all $u, v \in U_G$.

However, by the linearity of the functionals, this last requirement for producing a short homomorphism from $\lambda$ and $\mu$ just has to be checked when $u$ and $v$ are taken to be vertices $x$ and $y$. If $x \neq y$, then $xy = 0$ if $x$ and $y$ are not adjacent. So in that case, $\mu(xy) = 0$, and then $\lambda(x)\lambda(y) = 0$. That is, one or both of $\lambda(x), \lambda(y)$ must be 0. If $x$ and $y$ are adjacent, then $xy = [x, y]$ and $\mu([x, y]) = \lambda(x)\lambda(y)$. Furthermore, since $x^2 = \sum [x, t]$ summed over the vertices $t$ adjacent to $x$, $\mu(x^2) = \sum \mu([x, t])$. Substituting from the previous equations gives $\lambda(x)^2 = \sum \lambda(x)\lambda(t)$. If $\lambda(x) = 0$, this is automatically true. But if $\lambda(x) \neq 0$, we can divide through and get $\lambda(x) = \sum \lambda(t)$. Thus the requirements on $\lambda$ and $\mu$ are:

\begin{align*}
(3.1) \\
1. & \text{If } x \text{ and } y \text{ are nonadjacent vertices, at least one of } \lambda(x) \text{ and } \lambda(y) \text{ is } 0. \\
2. & \text{If } \lambda(x) \neq 0, \text{ then } \lambda(x) = \sum \lambda(t), \text{ summed over the vertices } t \text{ adjacent to } x. \\
3. & \text{If } x \text{ and } y \text{ are adjacent vertices, then } \mu([x, y]) = \lambda(x)\lambda(y).
\end{align*}

Call such a $\lambda$ a short functional (always assumed nonzero). Given $\lambda \neq 0$ satisfying these conditions and then $\mu$ from the third one (extended linearly to all of $3_G$), the map $\varphi$ defined by $\varphi(u + z) = \lambda(u)r + \mu(z)s$ will be a short homomorphism on $\text{alg}G$. Scaling such a $\lambda$ by $\alpha \neq 0$ will produce a short homomorphism also; it is not a scalar multiple of the $\varphi$ for $\lambda$ because $\mu$ is scaled by $\alpha^2$. Call it a scaled version of $\varphi$ nevertheless, and indicate the collection of these “scalings” of $\varphi$ by $\langle \varphi \rangle$. We also collect the scalar multiples of $\lambda$ (including 0) into the set $\langle \lambda \rangle$. This is a one-dimensional subspace of $U_G$, the dual space of $U_G$, and so a point of the projective space $PU_G$, the set of one-dimensional subspaces of $U_G$.

4. Edges and short homomorphisms

There is a useful relation between short homomorphisms and edges. Begin with an edge $e = [a, b]$ of $G$. Let $\lambda_{e} \in U_G$ be defined as follows: $\lambda_{e}(a) = \lambda_{e}(b) = 1$ and $\lambda_{e}(c) = 0$ for all vertices other than $a$ and $b$. Then extend $\lambda$ linearly to all of $U_G$ (so $\lambda_{e}(aa + \beta b + \ldots) = a + \beta + \ldots$). Next, let $\mu_{e}([a, b]) = 1$ and let $\mu_{e}([x, y]) = 0$ for all other edges $[x, y]$ of $G$; again, extend this to all of $3_G$ by linearity. It is easy to verify that $\lambda_{e}$ satisfies the requirements (3.1) for producing a short homomorphism.
For example, the second requirement just comes down to \( \lambda_x(a) = \lambda_x(b) \), the only case needing examination.

What is noteworthy is that, up to scalars, these are the only short functionals. To see this, let \( \lambda \) be a short functional and let \( K \) be the set of vertices \( x \) for which \( \lambda(x) \neq 0 \). If \( x, y \in K \) and \( x \neq y \), then as \( \mu(xy) = \lambda(x)\lambda(y) \) and neither factor is 0, \( \mu(xy) \neq 0 \). That means \([x, y]\) is an edge. Thus any two members of \( K \) are adjacent, so that \( K \) is a clique in \( G \). By the summation condition 2 in (3.1), when \( x \in K \), then \( \lambda(x) = \sum \lambda(t) \), summed over the \( t \) adjacent to \( x \). The only \( t \) adjacent to \( x \) that contribute are the members in \( K \), since the ones outside \( K \) have \( \lambda(t) = 0 \). So \( \lambda(x) = \sum_{t \in K \setminus \{x\}} \lambda(t) \). Add \( \lambda(x) \) to both sides to get \( 2\lambda(x) = \sum_{t \in K} \lambda(t) \). The right side doesn’t depend on \( x \); call it \( \kappa \): \( \lambda(x) = \kappa/2 \) for all \( x \in K \). Then all the \( \lambda(t) \) in \( \sum_{t \in K} \lambda(t) \) are \( k/2 \), and we get \( \kappa = |K| \times \kappa/2 \). This has to be interpreted as an equation in \( \mathbb{F} \); simplified, it reads \( |K| = 2 \). But we can’t conclude that \( |K| \) is numerically 2 unless either \( |K| < 2 + \text{char} \mathbb{F} \) or \( \text{char} \mathbb{F} = 0 \). That will certainly be the case when \( \text{char} \mathbb{F} \neq 0 \) if the number of vertices \( p \) of \( G \) has \( p < 2 + \text{char} \mathbb{F} \). If \( G \) has no complete subgraphs of order at least \( 2 + \text{char} \mathbb{F} \) when \( \text{char} \mathbb{F} \neq 0 \), then we can still conclude that \( |K| = 2 \). So we’ll restrict \( \mathbb{F} \) either to have characteristic 0 or to have \( p < 2 + \text{char} \mathbb{F} \).

That assumption being made, we conclude that \( |K| \) is really 2. This means \( K \) is the pair of vertices \( a, b \) of some edge, \([a, b]\). As we can scale \( \lambda \), we may take \( \lambda(a) = \lambda(b) = 1 \) and all other vertex values \( \lambda(c) = 0 \). Then the only edge having nonzero \( \mu \)-value is \([a, b]\). Thus we have exactly the recipe for the short homomorphism corresponding to \([a, b]\). In summary:

**Theorem 1.** Let \( \mathbb{F} \) be restricted by having either \( \text{char} \mathbb{F} = 0 \) or \( p < 2 + \text{char} \mathbb{F} \), \( p \) the order of graph \( G \). Then up to scalars, the short functionals \( \lambda \) are exactly those produced from each edge \([a, b]\) by the assignment \( \lambda(a) = \lambda(b) = 1 \), with all other vertex values \( \lambda(c) = 0 \).

With \( u = \sum_{x \in V_G} \theta_x x \), \( \lambda_x(u) = \sum_{x \in V_G} \lambda_x(x) \). Then if \( \epsilon = [x, y] \), \( \lambda_x(u) = \theta_x + \theta_y \). Thus in the product equation (2.2),

\[
(4.1) \quad uv = \sum_{\epsilon} \lambda_x(u)\lambda_x(v) \quad \text{and} \quad u^2 = \sum_{\epsilon} \lambda_x(u)^2.
\]

5. **Weights**

In this section we assign weights to the members of \( \mathfrak{z} \) by using short homomorphisms. For future use, let \( \Lambda \) be a collection of nonzero scalar multiples of the \( \lambda_x \), one for each edge \( \epsilon \), and \( M \) a similar set for the \( \mu_x \).

**Definition 2.** Let \( \mathfrak{z} \in \mathfrak{Z}_G \). The support \( \text{supp} \ (\mathfrak{z}) \) is the set of edges appearing with a nonzero coefficient when \( \mathfrak{z} \) is written as a linear combination of edges. The weight \( \text{wt} \ (\mathfrak{z}) \) of \( \mathfrak{z} \) is the size \(|\text{supp} \ (\mathfrak{z})|\).

Since \( \mu_x(f) = 0 \) for any edge \( f \) other than \( \epsilon \), \( \text{wt} \ (\mathfrak{z}) \) is the number of \( \mu_x \) for which \( \mu_x(\mathfrak{z}) \neq 0 \). That count would be the same if we used functionals proportional to the \( \mu_x \). Hence

**Lemma 1.** For \( \mathfrak{z} \in \mathfrak{Z}_G \), \( \text{wt} \ (\mathfrak{z}) \) is the number of members \( \mu \) of \( M \) for which \( \mu(\mathfrak{z}) \neq 0 \).

The edges themselves are proportional to the \( \mathfrak{z} \in \mathfrak{Z}_G \) for which \( \text{wt} \ (\mathfrak{z}) = 1 \). Consequently we can determine the edges, to scalars, entirely from the structure of \( NG \).
By equation (4.1), for \( u, v \in U_G \),

\[
\text{supp}(u^2) = \{ \epsilon \in EG | \lambda(\epsilon) u \neq 0 \} \quad \text{and} \quad \text{supp}(v^2) = \{ \epsilon \in EG | \lambda(\epsilon) v \neq 0 \}.
\]

Then it also follows from (4.1) that

\[
\text{supp}(uv) = \text{supp}(u^2) \cap \text{supp}(v^2).
\]

5.1. **The annihilator.** The definition of \( NG \) implies that \( Z_G \) annihilates the algebra. Modifying the annihilator concept a bit, we let \( \text{ann} G \), the *annihilator* of \( NG \), mean just the set of \( u \in U_G \) for which \( uU_G = 0 \).

**Proposition 1.** The annihilator \( \text{ann} G \) is the set of \( u \in U_G \) for which \( u^2 = 0 \). It is also the intersection \( \cap_{\lambda \in \Lambda} \ker \lambda \) of the kernels of the \( \lambda \in \Lambda \).

**Proof.** If \( u \in \text{ann} G \), then certainly \( u^2 = 0 \). Conversely, if \( u^2 = 0 \), then \( \text{supp}(u^2) = \emptyset \), making \( \text{supp}(uv) = \emptyset \) by (5.1). So \( uv = 0 \) for any \( v \in U_G \). We have \( u^2 = 0 \) just when \( \lambda(u) = 0 \) for each edge \( \epsilon \), by (4.1). As \( \lambda(u) \approx \lambda(v) \) for that \( \lambda \in \Lambda \) corresponding to \( \epsilon \), \( u^2 = 0 \) exactly when \( \epsilon \in \cap_{\lambda \in \Lambda} \ker \lambda \).

**Corollary 1.** If \( G \) is a connected bipartite graph, then

\[
\text{ann} G = \left( \sum_{x \in X} x - \sum_{y \in Y} y \right),
\]

where \( X \) and \( Y \) are the two parts of \( G \).

**Proof.** Let \( u = \sum_{x \in V_G} \theta_x x \). The requirement \( \lambda(\epsilon) u = 0 \) for an edge \( \epsilon = [x, y] \) reads \( \theta_x + \theta_y = 0 \), that is, \( \theta_y = -\theta_x \). Then the connectedness of \( G \) forces the coefficients for \( x \in X \) to have one value, say \( \theta \), and those for \( y \in Y \) to have the opposite, \( -\theta \).

5.2. **The incidence matrix.** The *incidence matrix* \( I(G) \) is the matrix with rows indexed by \( V(G) \) and columns by \( E(G) \), with 1 in row \( x \) and column \( \epsilon \) if \( x \) is a vertex of \( \epsilon \), and 0 if not [9, p. 165]. Matrix \( I(G) \) can also be viewed as displaying in column \( \epsilon \) the values of \( \lambda(\epsilon) \) on the vertices. The matrix obtained by replacing \( \lambda(\epsilon) \) by its representative in \( \Lambda \) has the same rank as \( I(G) \), since it is a scaling of the columns. A row dependence of \( I(G) \) corresponds to a member \( u = \sum_x \theta_x x \) for which \( \lambda(\epsilon) u = 0 \) for all edges \( \epsilon \); that is, a \( u \) in \( \cap_{\lambda \in \Lambda} \ker \lambda \). Thus by [1] \( \text{rank} I(G) = p - \text{dim} \text{ann} G \), \( p \) the order of \( G \). As each bipartite component of \( G \), including isolated vertices, contributes 1 to \( \text{dim} \text{ann} G \), the number of such components is \( \text{dim} \text{ann} G \). Hence:

**Proposition 2.** The rank of the incidence matrix \( I(G) \) of a graph \( G \) of order \( p \) is given by

\[
\text{rank} I(G) = p - k_b(G),
\]

where \( k_b(G) \) is the number of its bipartite connected components [9, Theorem 8.2.1]. Thus this rank is determined by the algebra \( NG \).

However, \( NG \) does not in general determine the entire number \( k(G) \) of connected components of \( G \), as will be illustrated at the end of the next section. A variant \( I'(G) \) of the incidence matrix of \( G \) is presented in [2, p. 24]: orient the edges of \( G \) by assigning a direction to each one. Then if \( \epsilon = [a, b] \) with the direction of \( \epsilon \) being \( a \rightarrow b \), change the 1 in the \( (a, \epsilon) \) position of \( I(G) \) to \(-1 \). Proposition 4.3 of [2] now shows that \( \text{rank} I'(G) = p - k(G) \).
6. Squares of weight 1

As we noted, a member $u$ of $U_G$ has $\text{wt}(u^2) = 1$ exactly when $u^2 \approx \epsilon$ for some edge $\epsilon$. The product formula \[1\] shows that $u^2 \approx \epsilon$ requires $\lambda_\epsilon(u) \neq 0$ and $\lambda_\epsilon(u) = 0$ for all edges $\epsilon$ other than $\epsilon$. If such an $u$ exists, we can scale it to have $\lambda_\epsilon(u) = 1$, making $u^2 = \epsilon$. The focus of this section will be connected edge-square graphs, those for which every edge is a square.

**Proposition 3.** Let $G$ be connected and let $\epsilon = [a, b]$ be an edge. Then $u^2 = \epsilon$ for some $u \in U_G$ if and only if one of the following holds:

1. $\epsilon$ is in an odd cycle and $G - \epsilon$ is bipartite;
2. $\epsilon$ is a bridge and at least one component of $G - \epsilon$ is bipartite.

**Proof.** Suppose that $u = \sum_{x \in V_G} \theta_x x$ and $u^2 = \epsilon$. Then $\lambda_\epsilon(u) = \theta_a + \theta_b \neq 0$; so $\theta_a \neq 0$, say. Follow a path starting at $a$ not beginning with $\epsilon$. Then to have $\theta_x + \theta_y = 0$ for all other edges $[x, y]$ along that path, the coefficients $\theta_x$ will have to alternate between $\theta_a$ and $-\theta_a$. If the path reaches $b$, it must be that $\theta_b = \theta_a$ to avoid $\theta_a + \theta_b = 0$. Thus if $\epsilon$ is in a cycle, the cycle must be odd, so that $G$ is not bipartite. But $G - \epsilon$ is bipartite, with $\{x | \theta_x = \theta_a\}$ and $\{y | \theta_y = -\theta_a\}$ giving the needed parts. If $\epsilon$ is not in a cycle, then deleting $\epsilon$ separates $G$ into two connected components. The one containing $a$ is bipartite, shown again by the two parts described but just for that component. Hence one of the two conditions holds.

Conversely, if condition 1 holds, assign $\theta_a$ and $\theta_b$ both to be $1/2$. Then define $\theta_x = 1/2$ for $x$ in the part of $G - \epsilon$ containing $a$ and $b$, and $\theta_y = -1/2$ for $y$ in the other part. As any edge $[x, y]$ other than $\epsilon$ connects vertices in different parts, $\theta_x + \theta_y = 0$ and $u^2 = \epsilon$. If condition 2 holds, suppose that the component containing $a$ is bipartite, with parts $X$ and $Y$, and $a \in X$. Put $\theta_x = 1$ for $x \in X$ and $\theta_y = -1$ for $y \in Y$. For any $z$ in the other component containing $b$, put $\theta_z = 0$. Again $u^2 = \epsilon$. 

For characterizing edge-square graphs, the relevant graphs are unicyclic graphs–connected graphs having exactly one cycle \[1\]. They in turn are characterized among connected graphs as those for which order and size are equal. Such a graph comes from a tree by adding an edge to complete a cycle.

**Proposition 4.** If $G$ is a tree or a unicyclic graph whose cycle is odd, then $G$ is edge-square.

**Proof.** Both statements follow from Proposition \[6\] \qed

Let $G$ be a connected edge-square graph. With $EG = \{\epsilon_1, \ldots, \epsilon_q\}$, let $u_i \in U_G$ give $u_i^2 = \epsilon_i$, $1 \leq i \leq q$. By (??), $\text{supp}(u_i u_j) = \emptyset$ for $i \neq j$, forcing $u_i u_j = 0$. This implies that the $u_i$ are linearly independent, so that $q \leq p$, as $G$ is connected, $q = p - 1$ or $p$. Define a normal algebra $O_p$ over $\mathbb{F}$ ($O$ for “one”) with basis $w_1, \ldots, w_p, j_1, \ldots, j_p$, by the relations

\begin{align}
O_p : \quad \begin{cases}
  w_i^2 = j_i, & 1 \leq i \leq p; \\
  w_i w_j = 0, & 1 \leq i, j \leq p, \ i \neq j; \\
  \text{all other basis products} = 0.
\end{cases}
\end{align}

(The spline algebra is $O_1$.) Its $U$- and $3$-spaces are $\langle w_1, \ldots, w_p \rangle$ and $\langle j_1, \ldots, j_p \rangle$. It is easy to check that the short homomorphisms of $O_p$ are proportional to those given by $w_i \rightarrow r$, $j_i \rightarrow s$, and $w_j \rightarrow 0$, $j_j \rightarrow 0$ for $i \neq j$, where $1 \leq i, j \leq p$. 

A NORMAL GRAPH ALGEBRA 7
Theorem 2. For a connected edge-square graph $G$, there are the two parameter possibilities $q = p - 1$ or $q = p$ with corresponding realizations:

1. $q = p - 1$: then $\mathcal{N}G$ is isomorphic to $F^0 \oplus \mathcal{O}_{p-1}$. In this case, $\dim \text{ann}G = 1$ and $G$ is a tree.
2. $q = p$: then $\mathcal{N}G$ is isomorphic to $\mathcal{O}_p$. Here $\dim \text{ann}G = 0$, and $G$ is unicyclic with an odd cycle.

Proof. If $q = p - 1$, $G$ is a tree and bipartite. As earlier, if the parts are $X$ and $Y$, $u_0 = \sum_{x \in X} x - \sum_{y \in Y} y \in \text{ann}G$. The isomorphism comes from the remarks leading to $\mathcal{O}_p$, with $u_i \longleftrightarrow w_i$ and $e_i \longleftrightarrow \delta_i$, $1 \leq i \leq p - 1$. That $\dim \text{ann}G = 1$ follows from the fact that if $w \in \langle w_1, \ldots, w_p \rangle$ and $w^2 = 0$, then $w = 0$. So $\text{ann}G = \langle u_0 \rangle$, giving the $\mathbb{F}^0$ term.

When $q = p$, the matching is the same, but with $1 \leq i \leq p$. Since a unicyclic graph $G$ whose cycle is even is bipartite and $\text{ann}G \neq 0$, the cycle in item 2 must be odd.

The direct sum $\mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \oplus \ldots \oplus \mathcal{O}_{p_n}$ is isomorphic to $\mathcal{O}_p$, with $p = \sum p_i$. Thus if $G$ is the union of $n$ graphs of the type in item 2 of the theorem, $\mathcal{N}G$ depends only on $p$ and not the number of components.

7. Squares of weight 2

Now suppose that $G$ is a graph for which every pair of edges is the support of a square of a member of $\mathcal{U}_G$: a pair-square graph. Assume that $G$ has no isolated vertices. We can also assume that $G$ is connected. For if not, any pair of edges with one edge in one component and the other in another would separately have each edge supporting a square from its component. That means all edges would support squares and the graph would be edge-square.

If $\text{supp}(u^2) = \{e, f\}$, then by scaling $u$, we can take $\lambda_e(u) = 1$ and have $u^2 = e + \alpha^2 f$ for some $\alpha \neq 0$. If $u^2 = e + \beta^2 g$ for some third edge $g$, with $\beta \neq 0$ and $\lambda_e(v) = 1$, then $(u - v)^2 = \alpha^2 f + \beta^2 g$. It follows that if for some fixed edge $e$, all the edge pairs containing $e$ are square supports, then all edge pairs are square supports. However, if $e$ is a square, then all edges support squares (since now $\alpha = 0$), and the graph is again edge-square. So we assume that $G$ is connected with no isolated vertices, and that no edge alone supports a square.

Fix one edge $e_0$ and on numbering the remaining edges $e_1, \ldots, e_{q-1}$, let $u_i^2 = e_0 + \alpha_i^2 e_i, 1 \leq i \leq q - 1$, with $\alpha_i = \lambda_{e_i}(u_i)$. Then $u_i u_j = e_0, 1 \leq i, j \leq q - 1, i \neq j$. The $u_i$ are linearly independent: if $\sum \beta_i u_i = 0$, the product with $u_j$ gives $(\sum \beta_i) e_0 + \beta_j \alpha_j^2 e_j = 0$, making $\beta_j = 0$ for all $j$. In particular, $p \geq q - 1$. Now set up a normal $\mathbb{F}$-algebra $\mathcal{T}_p$ ($\mathcal{T}$ for “two”) with dimension $2p + 1$ and basis $w_1, \ldots, w_p, \delta_0, \delta_1, \ldots, \delta_p$. Again, the $U$-space is $\langle w_1, \ldots, w_p \rangle$ and the $3$-space $\langle \delta_0, \delta_1, \ldots, \delta_p \rangle$. The defining relations are

$$\mathcal{T}_p : \begin{cases} w_i w_j = \delta_0 + \delta_{ij} \delta_i, 1 \leq i, j \leq p; \\
\text{all other basis products 0.} \end{cases} \tag{7.1}$$

(Here $\delta_{ij}$ is the Kronecker delta.) It will be useful to have a description of the short functionals of $\mathcal{T}_p$:
Lemma 2. Let \( \varphi \) be a short homomorphism of the algebra \( T_p \), \( p \geq 3 \), with associated functionals \( \lambda \) and \( \mu \). Then \( \varphi \) is proportional to one of the following:

- \( \varphi_0 : \lambda_0(w_i) = 1, \mu_0(3_0) = 1, \mu_0(3_i) = 0, 1 \leq i \leq p; \)
- \( \varphi_i, 1 \leq i \leq p : \lambda_i(w_j) = \delta_{ij}, \mu_i(3_0) = 0, \mu_i(3_j) = \delta_{ij}, 1 \leq i, j \leq p. \)

Proof. Let \( \lambda \) and \( \mu \) be the functionals of a short homomorphism \( \varphi \) of \( T_p \). First suppose that \( \mu(3_0) \neq 0 \). Then as \( \lambda(w_i)\lambda(w_j) = \mu(3_0) \) for \( i \neq j \), all \( \lambda(w_i) \) must all have the same nonzero value. Scaling it to be 1 makes \( \mu(3_0) = 1 \) and then all \( \mu(3_j) = 0 \), from \( \lambda(w_i)^2 = \mu(3_0) + \mu(3_i) \). This is the recipe for \( \varphi_0 \).

Now assume that \( \mu(3_0) = 0 \). Then for some \( i, \mu(3_i) \neq 0 \). Since \( 3_i = w_i^2 - 3_0 \), \( \mu(w_i) \neq 0 \). Then \( \lambda(w_i)\lambda(w_j) = \mu(3_0) \) for \( i \neq j \) reads \( \lambda(w_i)\lambda(w_j) = 0 \) and gives \( \lambda(w_j) = 0 \). This in turn shows that \( \mu(3_j) = 0 \). Scaling produces \( \varphi_i \).

That the descriptions of the \( \varphi_i \) do give short homomorphisms is straight-forward.

\( \square \)

Theorem 3. Let \( G \) be a connected graph such that every pair of edges is the support of a square of a member of \( U_G \). Suppose also that \( G \) is not an edge-square graph, so that \( p \geq 4 \). Then \( q = p \) or \( q = p + 1 \), and the following hold:

1. \( q = p : G \) is an even cycle and \( NG \) is isomorphic to \( F^0 \oplus T_{p-1} \).
2. \( q = p + 1 : NG \) is isomorphic to \( T_p \).

Proof. We saw above that \( p \geq q - 1 \), so that \( p - 1 \leq q \leq p + 1 \), since \( G \) is connected. The case \( q = p - 1 \) is excluded, as then \( G \) is a tree and edge-square. If \( q = p \), \( G \) is unicyclic. A terminal edge supports the square of its degree 1 vertex, making \( G \) edge-square again. So \( G \) is a cycle. If \( p \) is odd, Proposition 8 implies that \( G \) is still edge-square. Thus \( p \) must be even. In that case, \( \dim \text{ann} G = 1 \), by Corollary 1. Then \( U_G = \text{ann} G \oplus \langle u_1, \ldots, u_{p-1} \rangle \), the \( u_i \) as preceding the definition of \( T_p \). The subspace \( \langle u_1, \ldots, u_{p-1} \rangle + \delta G \) is a subalgebra isomorphic to \( T_p \) by the correspondence \( u_i \rightarrow w_i, e_i \rightarrow 3_0 \), and \( e_i \rightarrow \alpha_i^{-2}3_i, 1 \leq i \leq p - 1 \). That shows \( NG \) to be isomorphic to \( F^0 \oplus T_q \).

For \( q = p + 1 \), \( \text{ann} G = 0 \), since now \( U_G = \langle u_1, \ldots, u_p \rangle \). The same correspondence shows that \( NG \) is isomorphic to \( T_p \).}

To see that for an even cycle \( C \), \( NC \) really is a pair-square graph, index the vertices around \( C \) as \( x_0, \ldots, x_{q-1} \). Reading indices modulo \( q \), we first have \( x_i^2 = [x_{i-1}, x_i] + [x_i, x_{i+1}] \). For \( [x_{i-1}, x_i] \) and \( [x_j, x_{j+1}] \) disjoint,

\( (x_i - x_{i+1} + x_{i+2} - \ldots + (-1)^{j-1}x_j)^2 = [x_{i-1}, x_i] + [x_j, x_{j+1}] \).

Thus all pairs of edges do support squares.

What about pair-square graphs \( G \) with \( q = p + 1 \)?

Theorem 4. Let \( G \) be a pair-square connected graph for which \( q = p + 1 \). (Such a graph is not edge-square.) Then \( G \) is a doubly-odd paddle graph; that is, it has two edge-disjoint odd cycles either sharing one vertex or connected by a path. Moreover, any such graph is pair-square.

Proof. Graph \( G \) has no terminal edges (they are squares), so all vertex degrees are at least 2. Those degrees being \( \delta_1, \ldots, \delta_p \), \( \sum_{i=1}^{p} \delta_i = 2q \) gives \( \sum_{i=1}^{p} (\delta_i - 2) = 2 \). Then on renumbering, either \( \delta_1 \) and \( \delta_2 \) are both 3 and \( \delta_i = 2 \) for \( i \geq 3 \), or else \( \delta_1 = 4 \) and \( \delta_i = 2, i \geq 2 \). In the first case, vertices \( x_1 \) and \( x_2 \) might be connected by three
paths. But then two of the path lengths would have the same parity. Removing an edge from the third path leaves a bipartite graph, making that edge a square by Proposition 3. From the comments above, that would show \( G \) to be edge-square, which is excluded. So \( x_1 \) and \( x_2 \) are on single cycles joined by a path between \( x_1 \) and \( x_2 \). Both cycles are odd, by the same squared-edge argument. In the second case, if \( x_1 \) has degree 4, it is on two cycles meeting just at \( x_1 \). Again, both cycles must be odd.

That a doubly-odd paddle graph is pair-square is a matter of coefficient assignment verification, along the lines of the argument for even cycles above. □

The classical “butterfly” graph is the doubly-odd paddle graph of order 5. All doubly-odd paddle graphs of the same order \( p \) have isomorphic normal algebras \( T_p \). The number of such graphs is the number of pairs \( \{m, n\} \) of odd integers \( n, m \geq 3 \) with \( n + m \leq p + 1 \), \( n = m \) allowed. The sequence of counts, starting at \( p = 5 \), is presented in [17, Sequence A008642]. (The number of unrestricted paddle graphs is also there [17, Sequence A033638].)

8. Edge coherence

A set of distinct edges \( e_1, \ldots, e_n \) in a graph \( G \) is called coherent if there is a matching set of scalars \( \alpha_1, \ldots, \alpha_n \), not all 0, such that at each vertex of \( G \), the sum of the scalars for the edges incident with that vertex is 0. The coherence is proper if none of the \( \alpha_i \) is 0. Although this concept seems to require knowledge of the edge-vertex incidence relation, it actually depends only on the normal graph algebra \( N G \):

**Lemma 3.** Let \( e_1, \ldots, e_n \) be a set of distinct edges of the graph \( G \). Let \( \lambda_1, \ldots, \lambda_n \) be corresponding short functionals. Then the \( e_i \) are coherent if and only if the \( \lambda_i \) are linearly dependent.

**Proof.** Scaling the \( \lambda_i \) does not affect their dependence, so we may assume that \( \lambda_i = \lambda_{e_i} \). Then \( \sum \alpha_i \lambda_i = 0 \) is a dependence just when at each vertex \( x \), \( \sum \alpha_i \lambda_i(x) = 0 \). This in turn means that \( \sum \alpha_j = 0 \), summed over the edges \( e_j \) incident with \( x \). That is just the condition for coherence, with \( \alpha_i \) matching \( e_i \). □

**Definition 3.** A graph \( G \) is called minimally coherent if its edges are coherent, but no proper subset of them is.

For instance, the assignment of 1 and \( -1 \) alternately to the edges of an even cycle shows it to be coherent. But no proper subset of edges is coherent, since its edge-induced subgraph contains terminal edges that could not be assigned nonzero scalars. The smallest minimally coherent graph is a 4-cycle.

As the dimension of \( \hat{U}_G \) is \( p \), the order of \( G \), any set of \( p + 1 \) or more edges is coherent. So if \( G \) is minimally coherent, its size is at most \( p + 1 \).

**Theorem 5.** The minimally coherent graphs are the even cycles and the doubly-odd paddle graphs of Theorem 4.

**Proof.** The minimality of an even cycle was just noted. A doubly-odd paddle graph of order \( p \) has normal algebra isomorphic to \( T_p \) (Theorems 3 and 4). The short functionals of \( T_p \) in Lemma ?? satisfy \( \lambda_0 = \sum_{i=1}^{p} \lambda_i \), but no proper subset of them is linearly dependent. Consequently, doubly-odd paddle graphs are minimally coherent.
Conversely, let \( G \) be a minimally coherent graph of order \( p \) and size \( q \). Then \( G \) is connected, since a coherence requires a coherence in at least one component. So \( q = p \) or \( q = p + 1 \). At \( q = p \), \( G \) is unicyclic with no terminal edges and so a cycle. Up to scaling, the only possible coherence in a cycle is the alternating assignment of 1 and \(-1\) to its edges going around it, and that does not work if the cycle is odd. Thus \( G \) is an even cycle.

Now let \( q = p + 1 \). The proof runs as in Theorem \([2]\). Once again, \( G \) cannot contain any even cycles, because they would be graphs of lower size with coherent edges. Thus \( G \) is indeed a doubly-odd paddle graph. \( \square \)

Incidentally, this theorem implies a variant of a well-known one of Pósa [8] on the existence of pairs of edge-disjoint cycles:

**Corollary 2.** Let \( G \) be a graph of order \( p \) and size \( q \) with \( p \geq 4 \) and \( q \geq p + 1 \). Then \( G \) contains either an even cycle or a pair of edge-disjoint odd cycles.

**Proof.** As pointed out, the edges of \( G \) are coherent. A minimal coherent set of edges induces a minimally coherent subgraph of \( G \) providing the needed cycle set. (A direct proof makes a good exercise!) \( \square \)

9. **Automorphisms**

We plan to investigate automorphisms of normal graph algebras in a later paper. Here we present a few comments. The automorphism group \( \text{aut}\, G \), the group of permutations of \( VG \) whose induced actions on 2-element subsets of \( VG \) permute edges, induces automorphisms of \( \mathcal{N}G \). We call these **graphical automorphisms** and also refer to their set as \( \text{aut}\, G \). Other automorphisms are **nongraphical**. There are **scalar automorphisms**, maps scaling the members of \( \mathcal{U}_G \) by a nonzero scalar \( \alpha \) and members of \( \mathcal{Z}_G \) by \( \alpha^2 \). Their subgroup of the automorphism group \( \text{aut}\, \mathcal{N}G \) of \( \mathcal{N}G \) will be denoted \( \mathcal{P}_G \) (\( \mathcal{P}^\# \) is the set of nonzero members of \( \mathcal{P} \)).

For \( g \in \text{aut}\, \mathcal{N}G \) and a short homomorphism with functionals \( \lambda \) and \( \mu \), define \( g\varphi \) to be \( \varphi\circ g^{-1} \), \( g\lambda = \lambda\circ g^{-1} \), and \( g\mu = \mu\circ g^{-1} \). Then \( g\varphi \) is also a short homomorphism, with functionals \( g\lambda \) and \( g\mu \). Thus \( g \) permutes the spans of the members of \( M \) and so preserves weights: \( \text{wt}(g\varphi) = \text{wt}(\varphi) \), \( \varphi \in \mathcal{Z}_G \). The edges being proportional to the \( \varphi \in \mathcal{Z}_G \) of weight 1, \( g \) permutes the edge spans \( \langle \varepsilon \rangle \). (For \( g \in \text{aut}\, G \), this is automatic.)

If \( H \) is a spanning subgraph of \( G \), one with \( VH = VG \), \( \mathcal{N}H \) is not necessarily a subalgebra of \( \mathcal{N}G \). But it is a quotient, by the map \( x \to x \) for \( x \in VG \), \( \varepsilon \to \varepsilon \) for \( \varepsilon \in EH \), and \( \varepsilon \to 0 \) for \( \varepsilon \notin EH \). The kernel is the rather trivial ideal \( \langle EG - EH \rangle \), for which \( w\langle EG - EH \rangle = 0 \) for all \( w \in \mathcal{N}G \). An automorphism \( g \) of \( \mathcal{N}G \) that permutes the edge spans of \( EH \) (and so of \( EG - EH \)) induces an automorphism \( g\mid H \) of \( \mathcal{N}H \).

An **edge-scaling** automorphism is one that scales each edge. One might hope that such an automorphism is scalar, but that may not be so. For example, the automorphism group of \( \mathcal{O}_p \) \([6,4]\) is isomorphic to the monomial group on \( \langle w_1, \ldots, w_p \rangle \). If the monomial matrix is \( [a_{ij}] \), the matrix for the action on \( \langle z_1, \ldots, z_p \rangle \) is \( [a_{ij}^2] \). In particular, diagonal transformations can scale the edges by arbitrary nonzero squares.

Let \( g \) be an edge-scaling automorphism. Then for each edge \( \varepsilon \), \( g\lambda = \varepsilon\varepsilon_\varepsilon \lambda \) for some \( \varepsilon_\varepsilon \in \mathcal{P}^\# \). Let \( \varepsilon_1, \ldots, \varepsilon_n \) be a minimal coherent set of edges, with corresponding short functionals \( \lambda_i \), and let \( \sum \varepsilon_i\lambda_i = 0 \) be a dependence showing coherence.
Applying $g$ gives $\sum \alpha_i \varepsilon_i \lambda_i = 0$. By the minimality, it must be that all $\varepsilon_i$ are the same, otherwise some combination of the two dependency sums would have fewer nonzero terms.

**Definition 4.** Let $\mathcal{MC}$ be the family of minimally coherent graphs. A graph $G$ is called $\mathcal{MC}$-edge connected if for any two edges $e$ and $f$ of $G$, there is a sequence $G_1, \ldots, G_m$ of subgraphs $G_i$ of $G$ belonging to $\mathcal{MC}$ such that $e \in E_{G_1}$, $f \in E_{G_m}$, and $E_{G_i} \cap E_{G_{i+1}} \neq \emptyset$ for $1 \leq i \leq m - 1$.

If such a graph has no isolated vertices, it is connected.

**Proposition 5.** Let $G$ be an $\mathcal{MC}$-edge connected graph for which $\det G = 0$ ($G$ thus has no isolated vertices and is not bipartite). If $g \in \text{aut} \mathcal{V}G$ is edge-scaling, then $g$ is a scalar automorphism.

**Proof.** If $H$ is a minimally coherent subgraph of $G$, then by the preceding discussion, there is an $\varepsilon_H \in \mathbb{F}^\#$ for which $g \lambda_e = \varepsilon_H \lambda_e$ for all $e \in E_H$. Then the edge overlap provided by Definition 4 implies the existence of an $\varepsilon \in \mathbb{F}^\#$ for which $g \lambda_e = \varepsilon \lambda_e$ for all edges $e$ of $G$. Consequently $\lambda_e(g^{-1}u) = \lambda_e(\varepsilon u)$ for $u \in U_G$. Since $\cap_{e \in E_G} \ker \lambda_e = 0$, by (1) $g^{-1}u = \varepsilon u$, that is, $gu = \varepsilon^{-1}u$. Hence $g$ is a scalar automorphism, as wished. \hfill $\square$

10. The Petersen Graph

In this section we apply some of the above results to the Petersen graph, $P$ \cite{12} (see \cite{3} for data). Its vertices will be taken as the 2-element subsets of $\{1, 2, 3, 4, 5\}$, with $\{i, j\}$ abbreviated to $ij$. Then $ij \sim kl$ just when the four indices are all different, and we denote the corresponding edge by $ijkl$. If $AB$ is an edge, where $A$ and $B$ are vertex pairs, then $BA$ is the same edge, and within $A$ and $B$ the two indices can be switched. The symmetric group $\text{Sym}(5)$ acts on $P$, and in fact, $\text{aut} P$ is isomorphic to it \cite{12} Theorem 4.6).

The Petersen graph contains ten hexagons, all equivalent under $\text{Sym}(5)$. A representative one has consecutive vertices $14, 35, 24, 15, 34, 25$. Its stabilizer is the subgroup $\langle (123), (12), (45) \rangle$. The three pairs of edges $\{1435, 2435\}, \{1435, 1524\}, \{1435, 1534\}$ represent the three orbits of edge pairs: adjacent, skew, and opposite in a hexagon. Armed with this, the proof of the following lemma is straightforward (but perhaps tedious), as are all the proofs from here on:

**Lemma 4.** Up to scalars, there are fifteen members $u$ of $U_P$ for which $\text{wt}(u^2) = 3$: the ten vertices $ij$ and the five sums indicated in this table (the $-1$’s in $2u_i$ are at the vertices showing an $i$). The five $u_i$ form a $\text{Sym}(5)$ orbit:

\[
\begin{array}{cccccccccccc}
12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \\
2u_1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
2u_2 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
2u_3 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
2u_4 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
2u_5 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
\end{array}
\]
Here are some representative products to which Sym(5) can be applied to produce other products:

\begin{equation}
\begin{aligned}
&u_2^2 = 2345 + 2435 + 2534; \\
&u_1 \times 12 = 0, u_1 \times 13 = 0, u_1 \times 14 = 0, u_1 \times 15 = 0; \\
&u_1 \times 23 = 2345, u_1 \times 45 = 2345; \\
&u_i u_j = 0, i \neq j.
\end{aligned}
\end{equation}

Notice that $12^2 = 1234 + 1235 + 1245$, the sum of the edges of the claw at $12$, while $u_1^2$ is a sum of three mutually opposite edges.

**Remark 1.** There is an enhanced notation that can be used here: Label $u_i$ by $i6$, $1 \leq i \leq 5$, and augment each edge symbol by the pair $k6$, where $k$ does not appear in the original four edge indices. The earlier switching rules extend to these symbols. Thus $123456 = 436512$. Then the products involving the $u_i$ are obtained just as for $NP$ itself. The products in (10.2) now read:

\begin{equation}
\begin{aligned}
&12^2 = 12345 + 12435 + 12534; \\
&16 \times 12 = 0, 16 \times 13 = 0, 16 \times 14 = 0, 16 \times 15 = 0; \\
&16 \times 23 = 162345, 16 \times 45 = 162345; \\
&i6 \times j6 = 0, i \neq j.
\end{aligned}
\end{equation}

We’ll continue to use this notation.

There are six 5-element sets of the spans of these fifteen members of $U_P$ with the property that the products from any two different spans of the set are 0. They are

\begin{equation}
\begin{array}{l}
\mathcal{F}_1 \langle 12 \rangle \langle 13 \rangle \langle 14 \rangle \langle 15 \rangle \langle 16 \rangle \\
\mathcal{F}_2 \langle 12 \rangle \langle 23 \rangle \langle 24 \rangle \langle 25 \rangle \langle 26 \rangle \\
\mathcal{F}_3 \langle 13 \rangle \langle 23 \rangle \langle 34 \rangle \langle 35 \rangle \langle 36 \rangle \\
\mathcal{F}_4 \langle 14 \rangle \langle 24 \rangle \langle 34 \rangle \langle 45 \rangle \langle 46 \rangle \\
\mathcal{F}_5 \langle 15 \rangle \langle 25 \rangle \langle 35 \rangle \langle 45 \rangle \langle 56 \rangle \\
\mathcal{F}_6 \langle 16 \rangle \langle 26 \rangle \langle 36 \rangle \langle 46 \rangle \langle 56 \rangle
\end{array}
\end{equation}

The group $\text{aut} NP$ permutes these six sets, with Sym(5) (as $\text{aut} P$) permuting the first five by the subscripts. Sym(5) stabilizes $\mathcal{F}_6$, in which it permutes the $(k6)$ by the $k$’s.

**Proposition 6.** Suppose that $g \in \text{aut} NP$ fixes each $\mathcal{F}_i$. Then $g$ is a scalar automorphism.

**Proof.** From $(ij) = \mathcal{F}_i \cap \mathcal{F}_j$ we infer that $g$ fixes each vertex, up to scalars. Then $g$ is edge-scaling. Since $P$ satisfies the hypotheses of Proposition 5, $g$ is scalar. (In fact, this conclusion just needs $P$ connected.) $\square$

As a result, we have a homomorphism of $\text{aut} NP$ into Sym(6), with kernel $\mathbb{F}_P^\#$.

**Theorem 6.** For the Petersen graph, $\text{aut} NP/\mathbb{F}_P^\#$ is isomorphic to Sym(6) acting on $\{\mathcal{F}_1, \ldots, \mathcal{F}_6\}$.

**Proof.** To prove this, we must find an additional automorphism outside of Sym(5) that produces a transposition. Define $t$ as follows; the members of $U_P$ whose images are shown form a basis of $U_P$:

$t : 1k \rightarrow k6, 2 \leq k \leq 5; ij \rightarrow ij, 2 \leq i < j \leq 5.$
Since $\sum_{k=2}^5 1k = \sum_{k=2}^5 k6$, $t$ fixes the common sum and so fixes $16$, which is $-\frac{1}{2}\sum_{k=2}^5 1k + \frac{1}{2}\sum_{2 \leq i < j \leq 5} ij$. Moreover, $t^2$ is the identity, so that $k6 \mapsto 1k$ for $2 \leq k \leq 5$. The induced map of the edges can be described this way: first, $t$ fixes any edge showing $16$ as one of its three pairs. For an edge $1hijk6$, $t1hijk6 = 1kijh6$; for instance, $123456 = 153426$. It is routine to verify that $t$, with these edge images, is an automorphism. (The fact that $t$ commutes with the members of $\text{Sym}(5)$ fixing $1$ simplifies the work.) The effect of $t$ on the $\mathcal{F}_i$ is that $t$ switches $\mathcal{F}_1$ and $\mathcal{F}_6$ and fixes the others. This $t$ is just the transposition we need.

In the notation of Remark 11, $t$ is the transposition $(16)$. Moreover, $(\text{aut} P, t)$ is isomorphic to $\text{Sym}(6)$, and $\text{aut} \mathcal{N}P$ is isomorphic to $\mathbb{F}^\# \times \text{Sym}(6)$.

The group $\text{Sym}(6)$ famously has outer automorphisms $\text{ Inn}(6)$, and they are hiding in all this. There are six pairs of disjoint pentagons in $P$, and each pair occurs in five minimally coherent subgraphs of size 11. Two pentagons share at most two edges, so that two such subgraphs involving different pairs of pentagons meet in at most nine edges. So different minimally coherent subgraphs with the same pentagon pair meet in that pair, with its ten edges. By this count, the edge span sets of the pentagon pairs can be identified from $\mathcal{N}P$, as can then the sets of five edge spans showing the 11th edges of the subgraphs. The edges of such a set form a 1-factor of $P$.

We obtain six sets $\mathcal{H}_1, \ldots, \mathcal{H}_6$ of five edge spans. They are permuted by $\text{aut} \mathcal{N}P$ (recorded on the subscripts). The spans given are cycled by $(12345)$ from $\text{aut} P$, with $\mathcal{H}_6$ fixed.

$$\begin{align*}
\mathcal{H}_1 &= \langle 123456 \rangle \langle 132546 \rangle \langle 143526 \rangle \langle 152436 \rangle \langle 162345 \rangle \\
\mathcal{H}_2 &= \langle 123546 \rangle \langle 132456 \rangle \langle 142536 \rangle \langle 153426 \rangle \langle 162435 \rangle \\
\mathcal{H}_3 &= \langle 124536 \rangle \langle 132546 \rangle \langle 142356 \rangle \langle 153426 \rangle \langle 162435 \rangle \\
\mathcal{H}_4 &= \langle 124536 \rangle \langle 132456 \rangle \langle 143526 \rangle \langle 152346 \rangle \langle 162345 \rangle \\
\mathcal{H}_5 &= \langle 123456 \rangle \langle 134526 \rangle \langle 142536 \rangle \langle 152346 \rangle \langle 162345 \rangle \\
\mathcal{H}_6 &= \langle 123546 \rangle \langle 134526 \rangle \langle 142356 \rangle \langle 152436 \rangle \langle 162354 \rangle 
\end{align*}$$

Once again, we have an isomorphism of $\text{aut} \mathcal{N}P/\mathbb{F}^\#$ with $\text{Sym}(6)$, but now acting on $\{\mathcal{H}_1, \ldots, \mathcal{H}_6\}$.

From the actions on the $\mathcal{F}_i$ and the $\mathcal{H}_i$, composition by way of $\text{aut} \mathcal{N}P/\mathbb{F}^\#$ produces an automorphism of $\text{Sym}(6)$. For instance, the transposition $(12)$ on the $\mathcal{F}_i$ effects $(15)(26)(34)$ on the $\mathcal{H}_i$, and (123) produces $(164)(253)$. The automorphism is indeed outer. The enhanced notation shows that a dual-syntheme correspondence, suggested by J. J. Sylvester and described in [7], is present in (10.5). Each edge appears in two rows, and matching the row index pair with the edge sets up such a correspondence. For example, $12 \leftrightarrow 162345$ and $36 \leftrightarrow 142356$. An outer automorphism also provides a correspondence, matching transpositions with triple transpositions. For more on this topic, see [5], Chapter 6.]

We end with a uniqueness theorem for the algebra $\mathcal{N}P$, the first such result in this paper. The sequel will contain other uniqueness theorems.

**Theorem 7.** Let $Q$ be a second graph of order 10 and size 15 for which $\mathcal{N}P$ and $\mathcal{N}Q$ are isomorphic normal algebras. Then the graphs $P$ and $Q$ are isomorphic.

**Proof.** Let $\varphi : \mathcal{N}Q \to \mathcal{N}P$ be an isomorphism. As in Section 9, the $\varphi$-images of edge spans and their associated form and functional spans are that sort of item of $\mathcal{N}P$. First observe that $P$ has no 4-cycles: its smallest minimally coherent
subgraphs are hexagons. Then $Q$ also has no 4-cycles, since $\varphi$ would match them with 4-cycles in $P$. Each edge of $P$ is in a hexagon, so that is true of $Q$. This implies that $Q$ is connected. Next, $NP$ contains no $u$ with $\text{wt}(u^2) = 1$ or 2. This is again a coefficient-assignment problem (automorphisms of $NP$ may be exploited to allow checking just the edge $123456$ and the pair $123456, 132456$ for potential supports). It follows that the minimum degree of $Q$ is at least 3, so that in fact $Q$ is regular of degree 3 (cubic). In addition, the $\varphi$-images of the vertices must be proportional to ten of the 15 members of the set $\{ij|1 \leq i < j \leq 6\}$ from Lemma 4.

Suppose that among the missing five there are two that are disjoint, such as $12$ and $36$. Their product $124536$ is an edge that must also appear as a product from the ten. But that would require two of $12, 36,$ and $45$ to be among the ten also, which can’t be. Therefore the excluded five form one of the sets described after Remark 1. Composing $\varphi$ with a member of $\text{aut}NP$, if needed, allows us to assume that the five are $16, 26, 36, 46,$ and $56$. Then $\varphi$ maps the vertices of $Q$ to scalar multiples of the vertices of $P$, inducing a map $\tilde{\varphi}$ of vertices. For $x, y \in VQ$, $x \sim y$ just when $xy \neq 0$. That in turn requires $((\varphi x)(\varphi y)) \neq 0$, which is equivalent to $\tilde{\varphi} x \sim \tilde{\varphi} y$. So $\tilde{\varphi}$ preserves adjacency and establishes the hoped-for isomorphism between $Q$ and $P$. □

There are 19 cubic graphs of order 10, tabulated with pertinent data in [4, pp. 13–14] (they are drawn in [14, p. 127] too, also with data). Their graph algebras are mutually nonisomorphic. For suppose that $G$ and $H$ are two of these graphs with $NG$ and $NH$ isomorphic. Then $G$ and $H$ have the same number of minimally coherent subgraphs of each size. Those of size 4 are the 4-cycles. There are none of size 5, and of the two possibilities of size 6, only the 6-cycle can appear, because the butterfly graph has a vertex of degree 4. Thus $G$ and $H$ must have the same numbers of 4-cycles and 6-cycles.

There are just two pairs of the graphs for which these numbers match. First, #14 and #18 (C10 and C19 in [14]) have two 4-cycles and eight 6-cycles. But the number of edges appearing in all the 4-cycles of #14 is seven, and in #18, eight. Similarly, #12 and #17 (C16 and C11 in [14]) have three 4-cycles and seven 6-cycles. Again the numbers of edges in the 4-cycles are different: nine for #12 and 11 for #17. So all the algebras are indeed nonisomorphic.

In further investigations, 4-cycles will play a prominent role.

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