Two almost-circles, and two real ones

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Abstract Implicit locus equations in GeoGebra allow the user to do experiments with generalization of the concept of ellipses, namely with \( n \)-ellipses. By experimenting we obtain a geometric object that is very similar to a set of two circles. We show that they are not circles and highlight the importance of symbolic verification.

Keywords \( n \)-ellipse · lemniscate of Booth · Viviani’s theorem · van Schooten’s theorem · GeoGebra · computer algebra · computer aided mathematics education · automated theorem proving · elimination · absolute factorization · true on parts

1 Introduction

The aim of this paper is to highlight the importance of using symbolic computations in educational contexts to verify conjectures. Today’s technology and the available tools make it possible to very easily sketch up conjectures and too quickly accept them as true statements—without double checking their truth.

Elementary planar Euclidean geometry deals with points, lines and circles and their subsets. At a higher level parabolas, ellipses and hyperbolas are introduced. But most learners never face the fact that it is very easy to obtain much more complicated geometric objects—sometimes they look very similar to the simple ones, and it can be difficult to tell the difference. The nature we are living in seems to be simple for the first impression: planets seem to have a spherical form; free falling objects follow a linear motion; or presence of floral symmetry is considered evident. But having a deeper look, these concepts are just partially true, and nature is far more complicated and complex than one usually expects.

This paper focuses on a geometric locus defined by a very simple equation that can be easily studied by secondary school students—actually a first time user of the dynamic geometry tool GeoGebra can obtain such loci in seconds. Therefore, to find a false conjecture is very simple. This paper highlights how important it is to discuss some very details in a classroom to avoid getting an incorrect conclusion, and some possible ways are listed how the situation could be resolved.
2 GeoGebra: a symbolic tool to generalize concepts

GeoGebra [8] is a well known dynamic geometry software package with millions of users worldwide. One of its main purposes is to visualize geometric invariants. Recently GeoGebra has been supporting investigation of geometric constructions also symbolically by exploiting the strength of the embedded computer algebra system (CAS) Giac [12]. A possible use of the embedded CAS is automated reasoning [14]. In this paper a particular use of the implicit locus derivation feature [1] is shown, by using the command LocusEquation with two inputs: a Boolean expression and the sought mover point. For example, given an arbitrary triangle ABC with sides \( a, b \) and \( c \), entering LocusEquation\((a == b, C)\) results in the perpendicular bisector \( d \) of \( AB \), that is, if \( C \) is chosen to be an element of \( d \), then the condition \( a = b \) is satisfied.

Obtaining implicit loci is a new method in GeoGebra to get interesting facts on classic theorems. These facts have deep connections to algebraic curves which usually describe generalization of the classic results. Sometimes it is computationally difficult to obtain the curves quickly enough, but some recent improvements in Giac’s elimination algorithm opened the road to very effectively investigate a large number of geometric constructions [10,11] including Holfeld’s 35th problem [7,10], a generalization of the Steiner-Lehmus theorem [10,18] or the right triangle altitude theorem [1], among many others.

We need to admit that the possibility to generalize well known theorems is a consequence of using unordered geometry [2, p. 97] in the applied tools and theories. In unordered geometry one cannot designate only the expression of sums of given quantities like the lengths of a segment, so the signed quantities will be considered at the same time. (See [2, p. 59] for an example on irreducible problems and indistinguishable cases.) Therefore we obtain a larger set of points (that is, an extended locus) for the resulting algebraic curve as expected. The obtained set may be inconvenient in some cases (since the output differs from the expected result), but it can be still fruitful to get some interesting generalizations.

3 A generalization of the definition of ellipse

Let us consider fixed points \( A_1, A_2, \ldots, A_n \) and a fixed segment with length \( s \) in the plane. We are searching for all points \( P \) such that

\[
\sum_{i=1}^{n} |A_i P| = s. \tag{3.1}
\]

In case \( n = 1 \) the obtained set is a circle with center \( A_1 \) and radius \( s \), in case \( n = 2 \) we obtain an ellipse with foci \( A_1 \) and \( A_2 \) and major axis \( s \). In a recent publication [3] by Árpád Fekete the geometry of the case \( n = 3 \) is observed by coloring the various curves for fixed points \( A_1, A_2, A_3 \) and various segments with length \( s \) (see Fig. 1).

This idea is, however, already studied by many others. The first appearance is at J.C. Maxwell’s work [19], but it is called also \( n \)-ellipse by Sekino, and by Nie, Parrilo and Sturmfels [21], multifocal ellipse by Erdős and Vincze, polyellipse by Melzak and Forsyth, and egglipse by Sahadevan, among others.

We are focusing on finding a symbolic equation to describe these curves algebraically. We begin with the case \( n = 1 \) and start GeoGebra to create a segment with an arbitrary length \( s \). Then a point \( A_1 \) is added to the construction, freely chosen. As next steps, a free point \( P \) and the segment \( a = A_1P \) is created. Finally we use GeoGebra’s LocusEquation\((a == s, P)\) command to obtain the sought locus that produces a circle with center \( A_1 \) and radius \( s \). One can easily verify (at least, numerically) that the obtained symbolic equation (which is produced

\footnote{Recently a CindyJS [6,20] applet was written by the author that can produce the same output with just a couple of lines of code, based on a simple statement like colorplot(hue(re(|x-A|+|x-B|+|x-C|))). See the examples 66_ellipses.html, 66_3-ellipses.html and 66_4-ellipses.html in the folder examples/cindygl at https://github.com/CindyJS/CindyJS.}
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Fig. 1 A set of curves with fixed points $A_1, A_2, A_3$ and various segments $s$

Fig. 2 GeoGebra’s output on $\text{LocusEquation}(a==s, P)$

by GeoGebra’s built-in symbolic computation system) is indeed a circle (see Fig. 2). We note that $P$ is not a member of the circle in the GeoGebra construction—by issuing the $\text{LocusEquation}$ command we just want to know where to put $P$ to fulfill the requirement $a = s$.

As a further experiment one can go on with $n = 2$ and try to obtain an ellipse in a similar way. Surprisingly, the output will be sometimes not an ellipse but a hyperbola. The reason behind this is that the algebraic counterpart of the geometric setup cannot distinguish between “signed” and “unsigned” distances, that is, the formula given in (3.1) is internally realized as

$$\sum_{i=1}^{n} \pm |A_i P| = s. \quad (3.2)$$

As an immediate consequence, in certain cases the formulas $|A_1 P| - |A_2 P| = s$ and $|A_2 P| - |A_1 P| = s$ jump in: they actually represent the algebraic counterpart of the two branches of a hyperbola (see Fig. 3).

For $n = 3$ one can easily produce a somewhat more difficult curve (see Fig. 4), under the assumption $A_1 = (0, 2), A_2 = (1, 0), A_3 = (2, 0)$ and $s = 4$. In this case GeoGebra can also compute an algebraic equation of the curve, namely $E(x, y) = 9 x^8 + 9 y^8 + 36 x^2 y^6 + 54 x^4 y^4 + 36 x^6 y^2 - 72 x^7 - 48 y^7 - 72 x y^6 - 144 x^2 y^5 - 216 x^3 y^4 - 144 x^4 y^3 - 216 x^5 y^2 - 48 x^6 y - 220 x^6 - 372 y^6 + 480 x y^5 - 964 x^2 y^4 + 960 x^3 y^3 - 812 x^4 y^2 + 480 x^5 y + 2136 x^3 + 1712 y^5 + 1656 x y^4 + 2080 x^2 y^3 + 3792 x^3 y^2 + 368 x^4 y + 446 x^4 + 2846 y^4 - 8256 x y^3 + 5452 x^2 y^2 - 7104 x^3 y - 14424 x^3 - 6928 y^3 - 22008 x y^2 + 688 x^2 y + 4980 x^2 + 3132 y^2 + 17376 x y + 27720 x + 3600 y - 14175 = 0.$
Fig. 3 GeoGebra’s output on \texttt{LocusEquation}(a + b == s, P)—the same output can be obtained also with the command \texttt{LocusEquation}(a - b == s, P), among other combinations of signed sums.

Fig. 4 GeoGebra’s output on \texttt{LocusEquation}(a + b + c == s, P)
By using Maple’s evala(AFactor(…)) command (or by using Singular’s [4] absolute factorization library [5] which is freely available for all readers) we can learn that the polynomial \( E(x, y) \) is irreducible over \( \mathbb{C} \). In fact, as [21, Lemma 2.1] proves, in all non-degenerate cases the obtained polynomial is of degree 2^3 and irreducible over \( \mathbb{Q} \).

By considering the introductory comments and the paper [13], it seems clear that the internal loop belongs to the expression \( a + b + c = s \), while the others to some similar but signed expressions like \( a + b - c = s \), \( a - b + c = s \) and \( -a + b + c = s \). Theoretically also the expressions \( a - b - c = s \), \( -a - b - c = s \) and \( -a - b + c = s \) could occur, but some geometrical observations disallow those cases. (In fact, in other setups the latter three cases can also occur, see below in Fig. 9.) The union of all these curves is our extended locus.

Nie, Parrilo and Sturmfels give a very similar example as seen in Fig. 1 (see [21, Fig. 3]), by expressing that all curves are smooth, except those that contain either \( A_1, A_2 \) or \( A_3 \). Also, colored set of curves are shown (in [21, Fig. 4]), similarly to Fig. 4. In addition, they explain why the three extra curves appear in the extended locus. By using the same notions, we will say that GeoGebra displays the Zariski-closure of a 3-ellipse in Fig. 4. The length of segment \( s \) can also be called radius of an \( n \)-ellipse, and the points \( A_1, \ldots, A_n \) will be called its foci.

### 4 Two almost-circles

By selecting different positions for foci \( A_1, A_2, A_3 \) and different radii for \( s \) we can obtain a geometrically rich set of objects. The online GeoGebra applet https://www.geogebra.org/m/tuf3uzf9 can be used by the reader for own experiments. According to [21], the appearing curves are of 8th grade in all cases, except the degenerate ones, for example, the setup \( A_1 = (-1, 0), A_2 = (0, 0), A_3 = (1, 0), s = 0 \) produces a quartic curve that looks like a lemniscate (Fig. 5), having the equation \( L(x, y) = 3x^4 + 6x^2y^2 - 12x^2 + 3y^4 + 4y^2 = 0 \). By checking the web page https://en.wikipedia.org/wiki/Hippopede and learning that the equation \( L(x, y)/3 = 0 \) is equivalent to \((x^2 + y^2)^2 = 4x^2 - \frac{4}{3}y^2\), we identify the obtained curve as a lemniscate of Booth.

Some interesting outputs of the 8th grade curve can be found, among others, for setups \( A_1 = (-1, 0), A_2 = (1, 0), A_3 = (0, 1.73), s = 4 \) (see Fig. 6) and \( A_1 = (-1, 0), A_2 = (0, 0), A_3 = (1, 0), s = 1 \) (see Fig. 7).

For the former setup (in Fig. 6) we can also learn that, even if the extended curve is just one loop, different parts of the loop belong to different signed sums. By using a technique “dynamic coloring” described in [17] we can assign RGB components to the signs appearing in the sums, namely, “red” for the sign of \( |A_1P| \), “green” for the
Fig. 6 An interesting output for $A_1 = (-1, 0)$, $A_2 = (1, 0)$, $A_3 = (0, 1.73)$, $s = 4$

The sign of $|A_2 P|$ and “blue” for the sign of $|A_3 P|$ in the signed sum

$$\sum_{i=1}^{3} \pm |A_i P|.$$  

$A$ – sign adds an RGB component, while $a +$ sign removes it. For example, the signed sum $+|A_1 P| + |A_2 P| - |A_3 P|$ corresponds to $+\text{red} + \text{green} - \text{blue}$, that is, red and green are not used, only blue. On the other hand, the signed sum $-|A_1 P| - |A_2 P| + |A_3 P|$ corresponds to $-\text{red} - \text{green} + \text{blue}$, that is, red and green are used (but not blue) which means yellow in the RGB system. With this kind of coloring the black color corresponds to the unsigned 3-ellipse (which is typically the internal loop or the very internal part of the extended locus). See Fig. 8 and 9.

Surprisingly enough, the curve in Fig. 7 looks like a union of two circles. Factorization, however, does not suggest this conjecture, because the obtained polynomial, $C(x, y) = 9 x^8 + 9 y^8 + 36 x^2 y^6 + 54 x^4 y^4 + 36 x^6 y^2 - 100 x^6 - 4 y^6 - 108 x^2 y^4 - 204 x^4 y^2 + 182 x^4 - 10 y^4 - 84 x^2 y^2 - 100 x^2 - 4 y^2 + 9$, is irreducible over $\mathbb{C}$.

We can actually prove the fact that Fig. 7 does not correspond to two circles. Let us assume the contrary, that is, the right curve is a circle. Clearly, points $D = (-1/3, 0)$ and $E = (3, 0)$ are on the extended locus, because $-|A_1 D| + |A_2 D| + |A_3 D| = -2/3 + 1/3 + 4/3 = 1$ and $-|A_1 E| + |A_2 E| + |A_3 E| = -4 + 3 + 2 = 1$. So we need to assume that the right circle has center $(3 - 1/3, 0) = (4/3, 0)$ and radius 5/3. (See Fig. 10.)
When considering point $F' = (0, 1)$ (this situation is not shown in Fig. 10) we can still be optimistic. Indeed, $-|A_1 F'| + |A_2 F'| + |A_3 F'| = -\sqrt{2} + 1 + \sqrt{2} = 1$, and $F'$ is lying on the circle since $\sqrt{1^2 + (4/3)^2} = 5/3$. On the other hand, the point $F = (1.8, 1.6) = (9/5, 8/5)$ lies on the circle but not on the extended locus: $\sqrt{(\frac{9}{5} - 4)^2 + (\frac{8}{5})^2} = 5/3$, $-\sqrt{(1.8 - (-1))^2 + 1.6^2 + \sqrt{1.8^2 + 1.6^2 + \sqrt{(1.8 - 1)^2 + 1.6^2}} = -\sqrt{10.4 + 5.8 + \sqrt{3.2}} \approx 0.972270 < 1$. Nevertheless, this is still below an error of 3%. In general, the error is always below $3.429\%$.

5 Two real circles

Viviani’s well known theorem for planar triangles states that the sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle’s altitude.

A minor modification of Viviani’s theorem can lead to a statement that involves indeed two circles. We refer here to [10, Fig. 7] that already reports this result by automated reasoning. Here we repeat a similar kind of proof by using GeoGebra 5.0.575.0 in three different ways. Namely,

**Proposition 5.1** Let $A_1 A_2 A_3$ be a regular triangle. The locus of points $P$ such that $|A_1 P| + |A_2 P| = |A_3 P|$ is a circular arc of the circumcircle of $A_1 A_2 A_3$.

This proposition is often called van Schooten’s theorem [23].

For a symbolic proof we refer to the GeoGebra construction in Fig. 11: a regular triangle is constructed via the Regular Polygon tool, and the command LocusEquation($a + b == c, P$) is issued after $P, a, b$ and $c$ are defined. Notably, the output is a union of two circles. The reason behind this is that the regular triangle is ambiguous: the point $A_3$ can actually be on the other side of segment $A_1 A_2$. The locus equation is $D(x, y) =$
Fig. 8 Dynamic coloring for $A_1 = (-1, 0), A_2 = (1, 0), A_3 = (0, 1.73), s = 4$

$$3x^4 + 3y^4 + 6x^2y^2 - 6x^2 - 10y^2 + 3 = 0,$$
and its factorized form is

$$D(x, y) = 3 \left( x^2 + y^2 + \frac{2}{3} \sqrt{3}y - 1 \right) \cdot \left( x^2 + y^2 - \frac{2}{3} \sqrt{3}y - 1 \right)$$

that clearly corresponds to a union of two circles. In fact, it is the Zariski-closure of the circular arc over $\mathbb{Q}$.

Another way to prove the same statement in GeoGebra is to draw the circumcircle of a regular triangle $A_1A_2A_3$ and put a point $P'$ on it, then, by connecting it with $A_1, A_2$ and $A_3$, respectively, we get segments with length $e, f$ and $g$. Now the command `ProveDetails(e + f == g)` gives the output `{true, {
"A_1 = A_2", "e = f + g", "f = e + g"}}` which can be interpreted in the following complex algebraic geometrical way:

**Proposition 5.2** Let $A_1A_2A_3$ be a regular triangle. The locus of points $P$ such that $|A_1P| + |A_2P| = |A_3P|$ is the circumcircle of $A_1A_2A_3$, except eventually those points on the circle such that $A_1 = A_2$, $|A_1P| = |A_2P| + |A_3P|$ or $|A_2P| = |A_1P| + |A_3P|$.

A third method is to enter the command `Relation(e + f, g)` and obtain the result as shown in Fig. 12, and then, by pressing the button “More...”, getting the information given in Fig. 13.

For the meaning of “true on parts, false on parts” we refer to [15], but roughly speaking, this means that the case $e + f = g$ covers the generally observed equation $e \pm f \pm g = 0$ just partly.
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Fig. 9  Dynamic coloring for $A_1 = (-4, 0), A_2 = (0, 0), A_3 = (4, 0), s = 1$

Fig. 10  A sketch that explains the situation for the two non-circles, a black semicircle is also drawn to express the difference.

At the end of this section we give references to two simple elementary proofs. One is based on using Ptolemy’s theorem and available at Wikipedia [23]. The other one is Viglione’s idea that was published as a “proof without words” in [22]. A GeoGebra applet explaining it in more detailed is available at https://www.geogebra.org/m/kwgp4abk.

Finally we note that van Schooten’s theorem is about a special case of a 3-ellipse, namely with points $A_1 = (-1, 0), A_2 = (1, 0), A_3 = (0, \sqrt{3})$, and radius $s = 0$, with the remark that the sum is a signed one.
6 Pedagogical implications

Today’s technology is ready to answer very difficult questions quickly if the problem is entered in a suitable way. GeoGebra’s recent capabilities allow the user—including the student—to find challenging curves with very little efforts. The output is sometimes surprisingly similar to well-known geometric objects and the difference cannot be told in a trivial way.

The same issue may occur on simpler challenges as well. Here we refer to string art parabolas (that look like a circle, see Fig. 14 and [9]), some concrete setups of Wittgenstein’s rod (that look like ellipses, but they are of the 6th degree, see Fig. 15, available at https://tinyurl.com/wittgensteins-rod as a GeoGebra activity) and several other sextic curves that are defined by 4-bar linkages (they partially look like straight lines, see Fig. 16 for an example of
both a seemingly straight line and a seemingly perfect circular arc). For this latter case we refer to a recent paper [16] that describes linkages that can be constructed manually with LEGO parts and also via a computer program to study their motions. In fact, all of these almost-curves can be created with just a couple of steps when using a dynamic geometry system like GeoGebra.

Today’s mathematics teachers should be warned about these similarities and the lack of matches. Luckily, technology is ready enough to help distinguishing between correct and erroneous conclusions.

7 A final remark: The mysterious 1.73 . . . or $\sqrt{3}$?

In Fig. 6 and 8 we used $A_3 = (0, 1.73)$ instead of $A_3 = (0, \sqrt{3})$. The reason behind this is that the latter case leads to a set of two curves since we cannot designate the point $A_3$ uniquely, but just two candidates at the same time,
Fig. 17 The mysterious case $A_3 = (0, \sqrt{3})$

namely $(0, \pm \sqrt{3})$. From the algebraic geometry point of view, however, we cannot avoid getting both curves at the same time. The union of them is the extended locus we are actually searching for.

The applet https://www.geogebra.org/m/w9wypbwq shows the extended locus for the latter case. It is a polynomial of degree 16, namely

$$M(x, y) = 81x^{16} + 81y^{16} + 648x^2y^4 + 2268x^4y^{12} + 4536x^6y^{10} + 5670x^8y^8 + 4536x^{10}y^6 + 2268x^{12}y^4 + 648x^{14}y^2 - 8712x^{14} - 9144y^{14} - 63576x^2y^{12} - 189432x^4y^{10} - 313560x^6y^8 - 311400x^8y^6 - 185544x^{10}y^4 - 61416x^{12}y^2 + 351580x^{12} + 387292x^2y^{10} + 5627364x^4y^8 + 7383536x^6y^6 + 5448804x^8y^4 + 2144328x^{10}y^2 - 6518584x^{10} - 7593736y^{10} - 37894104x^2y^8 - 74639248x^4y^6 - 72489712x^6y^4 - 34669416x^8y^2 + 53832870x^8 + 70725030y^8 + 307042904x^2y^6 + 457163108x^4y^4 + 274677848x^6y^2 - 156876856x^6 - 330575368y^6 - 1126682824x^2y^4 - 1131480568x^4y^2 + 207726172x^8 + 753521116y^4 + 1567247576x^2y^2 - 129609480x^2 - 67785900y^2 + 31102929 = 0,$

and it must be irreducible over the rationals. Fig. 17 shows the plot of $M(x, y) = 0$.

On the other hand, $M$ is reducible over $\mathbb{Q}[\sqrt{3}]$. Maple’s `evala(解除因子(...))` command can provide the two factors. One of them is as follows: $9x^8 - 24x^6y\sqrt{3} + 36x^6y^2 - 484x^6y^2 - 72x^4y^3\sqrt{3} + 968x^4y\sqrt{3} + 54x^4y^4 - 1380x^4y^2 + 6518x^4y^2 - 72x^2y^5\sqrt{3} + 1904x^2y^3\sqrt{3} - 14152x^2y\sqrt{3} + 36x^2y^6 - 1308x^2y^6 + 11100x^2y^2 - 11620x^2 - 24y^7\sqrt{3} + 936y^5\sqrt{3} - 6440y^3\sqrt{3} + 13208y\sqrt{3} + 9y^8 - 412y^6 + 4598y^4 - 13852y^2 + 5577.$
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