REAL FIBERED MORPHISMS AND ULRICH SHEAVES

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ABSTRACT. In this paper we define and study real fibered morphisms. Such morphisms arise in the study of real hyperbolic hypersurfaces in $\mathbb{P}^d$ and other hyperbolic varieties. We show that real fibered morphisms are intimately connected to Ulrich sheaves admitting positive definite symmetric bilinear forms.

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1. INTRODUCTION

1.1. Background. A homogeneous polynomial $f \in \mathbb{R}[x_0, \ldots, x_d]$ is called hyperbolic with respect to a point $e \in \mathbb{R}^{d+1}$ if $f(e) \neq 0$ and for every $x \in \mathbb{R}^{d+1}$ the roots of the univariate polynomial $f(e + tx)$ are all real. Hyperbolic polynomials were first studied in the context of partial differential equations, since they arise as symbols of hyperbolic (and hence the name) partial differential equations with constant coefficients. Such PDEs are of interest since the Cauchy problem is well-defined in this case, see for example [19] and [25]. The first to study geometric properties of hyperbolic polynomials was L. Gårding in [20]. L. Gårding showed that hyperbolic polynomials possess remarkable convexity properties and those results were extended by H. Bauschke, O. Güler, A. Lewis and H. Sendov in [5]. In the last years there has also been ample interest in hyperbolic polynomials from the areas of combinatorics [13] and optimization [22,40]. In that context the so-called generalized Lax conjecture is an important open question asking whether the feasible sets of hyperbolic programming are the same as the feasible sets of semidefinite programming. More recently, properties of stable polynomials, a special kind of hyperbolic polynomials, and certain determinantal representations of them were crucially used in the proof of the Kadison-Singer conjecture by Marcus, Spielman and Srivastava [35]. Petter Brändén reproved and slightly strengthened their results using convexity properties of hyperbolic polynomials, see [10]. Very recently,
hyperbolic polynomials also appeared in the context of exponential families in statistics [36].

One can reformulate the hyperbolicity property in a more geometric way, namely consider the hypersurface $X \subseteq \mathbb{P}^d$ cut out by $f$ and consider a point $e \in \mathbb{P}^d$ off $X$. Then $f$ is hyperbolic with respect to $e$ if and only if for every real line $L$ through $e$ we have that $L \cap X \subseteq X(\mathbb{R})$. This idea was generalized by V. Vinnikov and the second author in [43] to define hyperbolicity of a general real subvariety of $\mathbb{P}^d$ with respect to a real linear subspace of correct dimension. A well studied example of hyperbolic varieties that are not hypersurfaces are reciprocal linear spaces, i.e. the Zariski closure of the Cremona transform of a linear subspace in projective space. These varieties have been studied for example in the context of interior points methods for linear programming [15] and entropy maximization for log-linear models [41]. Hyperbolicity (though not called so) of these varieties was shown by Varchenko [44] and is used (not just) in the cited works at various points.

The classical example of a hyperbolic polynomial is the determinant of a generic symmetric $n \times n$ matrix. It can be shown that this polynomial is hyperbolic with respect to the identity matrix. This led P. D. Lax in 1958 [32] to ask whether every hyperbolic homogeneous polynomial in three variables admits a determinantal representation. More precisely, assume $f \in \mathbb{R}[x_0, x_1, x_2]$ is a homogeneous polynomial of degree $m$ hyperbolic with respect to $(1, 0, 0)$. P. D. Lax asked whether there exist symmetric matrices $A_0, A_1, A_2 \in M_m(\mathbb{R})$, with $A_0$ positive definite and $f = \text{det}(x_0A_0 + x_1A_1 + x_2A_2)$. It was observed by A. Lewis, P. Parrilo and M. Ramana in [33] that this follows from a result of V. Vinnikov and W. Helton in [23]. The conjecture fails for $d > 2$ even in a weakened form, see [6], [9] and [26] for more details. The generalized Lax conjecture described above can be formulated as follows. Given a homogeneous polynomial $f \in \mathbb{R}[x_0, \ldots, x_d]$ hyperbolic with respect to $e$, can we find another homogeneous polynomial $g \in \mathbb{R}[x_0, \ldots, x_d]$ hyperbolic with respect to $e$, such that the product $fg$ is hyperbolic with respect to every point in the connected component of $e$ in $\mathbb{R}^{d+1} \setminus Z(f)$ and $fg$ admits a symmetric determinantal representation definite at $e$. The best result known today regarding this conjecture is due to the first author in [31]. The reader is referred to [45] for an extensive overview of classical notions of hyperbolicity and determinantal representations.

The goal of this paper is to study hyperbolicity and determinantal representations in an invariant way. Let $X \subseteq \mathbb{P}^d$ be a real subvariety hyperbolic with respect to a linear subspace $V \subseteq \mathbb{P}^d$, then the linear projection with center $V$ defines a morphism over the reals from $X$ to $\mathbb{P}^k$ (here $k = \dim X$) that sends only real points to real points. Following the idea of A. Grothendieck, we isolate this property of a morphism between real (projective) varieties. In Section 2 we define real fibered morphisms as (finite, flat and surjective) morphisms that map only real points to real points. While in the context of hyperbolicity real fibered morphisms appear as linear projections there are also applications where this is not the case. For example in the study of Amoebas of algebraic varieties the question arises for which real hypersurfaces the logarithmic Gauss map is real fibered [37, 38]. We prove that real fibered morphisms are always unramified at smooth real points. We conclude that the Veronese embedding of $\mathbb{P}^k$ is not hyperbolic whenever $k \geq 2$. We say that $X$ is weakly hyperbolic if $X$ admits a real fibered morphism to $\mathbb{P}^k$. In the case of smooth projective curves this is equivalent to the corresponding Riemann
surface being of dividing type (sometimes also called type one). We then show that every such curve admits a hyperbolic embedding into $\mathbb{P}^3$. Therefore, in case of curves weak hyperbolicity is equivalent to hyperbolicity. This, however, fails if $\dim X > 1$ and we provide an example of a weakly hyperbolic variety, that admits no hyperbolic embeddings.

In Section 3 we recall the definition of determinantal representations from [43] and the notion of Ulrich sheaves from [16] and [17]. Ulrich modules were first studied by B. Ulrich and his collaborators, see for example [11] and [24]. D. Eisenbud and F.-O. Schreyer used Ulrich sheaves to construct determinantal and Pfaffian representations of Chow forms of subvarieties of $\mathbb{P}^d$. We show that in fact Ulrich sheaves can be identified with determinantal representations in the sense of [43]. This shows that every determinantal representation of a projective curve can be obtained using the algorithm described in [43].

In Section 4 we show that definite symmetric determinantal representations can be defined in terms of positive definite bilinear forms on Ulrich sheaves. Following [30] we define the relative notion of $f$-Ulrich sheaves with respect to a finite flat morphism $f : X \to Y$. We give a characterization of real fibered morphisms in terms of positive semidefinite bilinear forms on coherent sheaves which generalize the classic methods of checking real rootedness of univariate polynomials like the Hermite matrix or the Bézout matrix. In particular, we show that if $f : X \to Y$ is a finite flat morphism with a positive $f$-Ulrich sheaf on $X$, then $f$ is real fibered. We then proceed to formulate a question, which can be considered as a relative version of the generalized Lax conjecture.

1.2. Notations and Convention. In this paper, $\mathbb{K}$ will always denote the ground field, it may be either the field of real or complex numbers. By a $\mathbb{K}$-variety we mean a reduced, separated scheme of finite type over $\text{Spec} \mathbb{K}$, not necessarily irreducible. A morphism of $\mathbb{K}$-varieties is always meant to be a morphism over $\text{Spec} \mathbb{K}$. A curve over $\mathbb{K}$ is a $\mathbb{K}$-variety of pure dimension one. Now let $X$ be an $\mathbb{R}$-variety. We will write $X(\mathbb{K}) = \text{Hom}_{\text{Spec} \mathbb{K}}(\text{Spec} \mathbb{K}, X)$ for the set of $\mathbb{K}$-points. We can identify $X(\mathbb{R})$ with the set of points $x \in X$, such that the residue field $\kappa(x)$ of $X$ at $x$ is $\mathbb{R}$. We will write $X_{\mathbb{C}} = X \times_{\text{Spec} \mathbb{R}} \text{Spec} \mathbb{C}$ for the complexification.

For every $\mathbb{R}$-variety $X$ the complexification $X_{\mathbb{C}}$ of $X$ comes equipped with an involution $\tau$, i.e. an isomorphism of $\mathbb{R}$-varieties $\tau : X_{\mathbb{C}} \to X_{\mathbb{C}}$, such that $X(\mathbb{R})$ can be identified with $X_{\mathbb{C}}(\mathbb{C})^\tau = X(\mathbb{C})^\tau$, namely the fixed points of $\tau$. Given two $\mathbb{R}$-varieties $X$ and $Y$ and a morphism $f : X \to Y$, we have a corresponding morphism $f_C : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ and if we denote the involution of $X$ resp. $Y$ by $\tau$ resp. $\sigma$ we get that $f_C \circ \tau = \sigma \circ f_C$. Conversely, every morphism $g_C : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ that intertwines the involution descends to a morphism $g : X \to Y$.

At some points we will make use of the real spectrum of a ring. Given a ring $A$ the real spectrum $\text{Sper} A$ is the set of all pairs $\alpha = (p, P)$ where $p$ is a prime ideal of $A$ and $P$ is an ordering of the residue field $\kappa(p)$. Let $\alpha = (p, P) \in \text{Sper} A$ and let $\rho_{A, p} : A \to \kappa(p)$ be the canonical homomorphism. We denote by $\text{Supp}(\alpha) = p$ the support of $\alpha$, i.e. the prime ideal corresponding to $\alpha$. For any element $f \in A$ we say that $f(\alpha) \geq 0$, i.e. $f$ is nonnegative in $\alpha$, if $\rho_{A, p}(f) \in P$. We write $f(\alpha) > 0$ if $f(\alpha) \geq 0$ and $f \notin \text{Supp}(\alpha)$. The closed points of $\text{Sper} A$ are those whose support is a maximal ideal. Now let $A$ be a finitely generated reduced $\mathbb{R}$-algebra and let $X = \text{Spec} A$ be the corresponding affine $\mathbb{R}$-variety. Since $\mathbb{R}$ has exactly one ordering
the closed points of \( \operatorname{Sper} A \) can be identified with the set \( X(\mathbb{R}) \). On \( \operatorname{Sper} A \) we consider the topology generated by the subbasis of open sets \( \{ \alpha \in \operatorname{Sper} A : f(\alpha) > 0 \} \) for \( f \in A \). In the case where \( A \) is a finitely generated \( \mathbb{R} \)-algebra the set of closed points is dense in \( \operatorname{Sper} A \) with respect to that topology. For \( \alpha, \beta \in \operatorname{Sper} A \) we say that \( \alpha \) specializes to \( \beta \) if \( \beta \in \overline{\{ \alpha \}} \). Every ring homomorphism \( A \to B \) induces a continuous map \( \operatorname{Sper} B \to \operatorname{Sper} A \). We refer to the book of Bochnak, Coste, and Roy [8] for proofs and more, basic results about the real spectrum.

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2. Real Fibered Morphisms

In this section, we work over the ground field \( \mathbb{R} \). We will write \( \mathbb{P}^d = \mathbb{P}^d_{\mathbb{R}} \) for the projective \( d \)-space over \( \mathbb{R} \). The goal of this section is to study real fibered morphisms of \( \mathbb{R} \)-varieties. We start with a definition.

**Definition 2.1.** Let \( f : X \to Y \) be a flat finite surjective morphism of \( \mathbb{R} \)-varieties. We say that \( f \) is real fibered if for every \( x \in X \), we have that \( f(x) \in Y(\mathbb{R}) \) if and only if \( x \in X(\mathbb{R}) \).

**Remark 2.2.** It is clear that every \( \mathbb{R} \)-point is sent to an \( \mathbb{R} \)-point. The real fibered property implies the converse as well. Now let \( g_C : X_C \to Y_C \) be a morphism that intertwines the involutions on \( X_C \) and \( Y_C \) which is flat, finite and surjective. Then by descent theory the morphism of \( \mathbb{R} \)-varieties \( g : X \to Y \) that we get is real fibered if and only if \( g_C \) maps only fixed points (of the involution) to fixed points.

**Remark 2.3.** The property of being real fibered is stable under base-change and composition.

To motivate the definition we recall the following definition of hyperbolic varieties from [43].

**Definition 2.4.** Let \( X \) be a projective \( \mathbb{R} \)-variety of dimension \( k \) and let \( \iota : X \to \mathbb{P}^d \) be an embedding. Let \( V \subseteq \mathbb{P}^d \) be a real \( d-k-1 \) dimensional linear subspace, such that \( \iota(X) \cap V = \emptyset \). We say that \( \iota \) is a hyperbolic embedding, or that \( X \) is hyperbolic with respect to \( V \) if for every real \( d-k \)-dimensional subspace \( U \) containing \( V \), we have \( U \cap \iota(X) \subseteq \iota(X(\mathbb{R})) \).

**Example 2.5.** Let \( f, g \in \mathbb{R}[s, t] \) be two homogeneous polynomials of the same degree that do not have a common (projective) zero. Assume that their zeros interlace, i.e. between each pair of consecutive (projective) zeros of \( f \) there is exactly one zero of \( g \) and vice versa. Then for all \( \lambda, \mu \in \mathbb{R} \), not both zero, the polynomial \( \lambda f + \mu g \) has only real roots [39, Thm. 6.3.8]. Thus the morphism \( \varphi : \mathbb{P}^1 \to \mathbb{P}^1, (s : t) \mapsto (f(s, t) : g(s, t)) \) is real fibered. In particular this shows that the rational normal curve of any degree is hyperbolic.

This can also be seen without using results about interlacing polynomials. The Möbius transformation \( \Phi(z) = \frac{z-1}{z+1} \) sends the real line (including infinity) to the unit circle. Conversely, every point that is sent to the unit circle lies on the real line. The map \( \psi_k(z) = z^k \) has the property that \( z \) is on the unit circle if and only if \( \psi_k(z) \) is on the unit circle for \( k \geq 1 \). This shows that the morphism \( \mathbb{P}^1 \to \mathbb{P}^1 \) that we get from \( \Phi^{-1} \circ \psi_k \circ \Phi \) is a real fibered morphism of degree \( k \) for all \( k \geq 1 \).
Assume that $X$ is equidimensional, Cohen–Macaulay and hyperbolic with respect to $V$. Then the linear projection from $V$ induces a finite surjective morphism $f : X \to \mathbb{P}^k$. It is flat since $X$ is equidimensional, Cohen–Macaulay and $\mathbb{P}^k$ is smooth (see for example [21, Prop. 15.4.2]). Furthermore, since $X$ is hyperbolic with respect to $V$ we have that $f$ is real fibered. Therefore, we make the following definition:

**Definition 2.6.** We say that an $\mathbb{R}$-variety $X$ is weakly hyperbolic if there exists a real fibered morphism $f : X \to \mathbb{P}^k$.

**Remark 2.7.** In the situation of the above definition let us denote by $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^k}(1)$. This is a line bundle on $X$ and since $f$ is finite this line bundle is ample.

The following proposition gives a condition when a weakly hyperbolic variety admits a hyperbolic embedding.

**Proposition 2.8.** Assume that $X$ is weakly hyperbolic, with $f : X \to \mathbb{P}^k$ being the real fibered morphism. If $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^k}(1)$ is very ample then $X$ admits a hyperbolic embedding.

**Proof.** Let $\iota : X \to \mathbb{P}(H^0(X, \mathcal{L})^*)$ be the embedding obtained from $\mathcal{L}$. Write $f = (\mu_0, \ldots, \mu_k)$, for $\mu_0, \ldots, \mu_k \in H^0(X, \mathcal{L})$. Then it is immediate that $\iota(X)$ is hyperbolic with respect to the real subspace $\mu_0 = \ldots = \mu_k = 0$. \qed

For any $\mathbb{R}$-variety $X$ we can equip $X(\mathbb{C})$ with the classical topology and $X(\mathbb{R})$ can be identified with a closed subset of $X(\mathbb{C})$ with respect to the classical topology. In the case where $X$ is a smooth, geometrically irreducible projective curve we have that $M = X(\mathbb{C})$ is a compact, real Riemann surface with real points $M(\mathbb{R}) = X(\mathbb{R})$. Recall that $M$ is called of dividing type if $M \setminus M(\mathbb{R})$ has two connected components. Then we have the following:

**Theorem 2.9.** Let $X$ be a smooth, geometrically irreducible projective curve over $\mathbb{R}$ and $M = X(\mathbb{C})$ the corresponding Riemann surface. The following are equivalent:

(i) $M$ is of dividing type.
(ii) $X$ is weakly hyperbolic.
(iii) $X$ admits a hyperbolic embedding into some $\mathbb{P}^d$.

**Proof.** It is immediate that (iii) implies (ii). Now let $M$ be a Riemann surface of dividing type, then by a theorem of L. Ahlfors [1, §4.2] there is a dividing function on $M$, namely a real meromorphic function, such that $f(p)$ is real (including infinity) if and only if $p \in M(\mathbb{R})$. In particular $X$ is weakly hyperbolic, hence (i) implies (ii) (cf. [42, 19 Prop. 15]). On the other hand if $X$ is weakly hyperbolic and $f$ is the corresponding real meromorphic function, then the inverse images under $f$ of the upper and lower half planes are precisely the two connected components of $X(\mathbb{C}) \setminus X(\mathbb{R})$ and hence (i) is equivalent to (ii).

Now let $f : X \to \mathbb{P}^1$ be real fibered and $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. In Example 2.5 we have seen that for any $n \geq 1$ there is a real fibered morphism $g : \mathbb{P}^1 \to \mathbb{P}^1$, such that $\mathcal{O}_{\mathbb{P}^1}(n) = g^*\mathcal{O}_{\mathbb{P}^1}(1)$ and for sufficiently large $n$ the line bundle $\mathcal{L}^n = (g \circ f)^*\mathcal{O}_{\mathbb{P}^1}(1)$ is very ample. Thus by Proposition 2.8 (ii) implies (iii).

**Remark 2.10.** Theorem 2.9 says that every weakly hyperbolic curve admits a hyperbolic embedding into some projective space. We will see in Example 2.24 that this is not true for higher dimensional varieties.
Corollary 2.11. Let $X$ be a smooth projective curve, such that $X(\mathbb{C})$ is a Riemann surface of dividing type. Then $X$ admits a hyperbolic embedding into $\mathbb{P}^3$ and a birational embedding into $\mathbb{P}^2$, such that the image has only node type singularities.

Proof. Assume that we can embed $X$ in $\mathbb{P}^d$, such that the image is hyperbolic with respect to some $d-2$-dimensional real subspace $V \subseteq \mathbb{P}^d$. The tangent variety to $X$ is of dimension at most two and the secant variety is of dimension at most three. Thus if $d > 3$ we can find a real point in $V$ disjoint from the secant variety and project from it, we can perturb $V$ slightly if needed using [43, Thm. 3.10]. We obtain an embedding of $X$ into $\mathbb{P}^{d-1}$ hyperbolic with respect to the image of $V$. Hence we can embed $X$ hyperbolically into $\mathbb{P}^3$.

Since we can repeat this argument in case $d = 3$ and avoid only the tangent variety and secant varieties of lines that intersect our curve at $3$ or more points, we can project again to get a finite map from $X$ into $\mathbb{P}^2$, birational onto its image and the image will have only node type singularities. □

Remark 2.12. Let $X$ be a projective, geometrically irreducible, smooth, real curve. Let $g$ be its genus and let $s$ be the number of the connected components of $X(\mathbb{R})$. If $s = g+1$, then $X(\mathbb{C})$ will be of dividing type. If $X(\mathbb{C})$ is of dividing type, then $g+1-s$ will be even [27, §21].

Example 2.13. The TV-Screen is the plane quartic curve defined by $x^4 + y^4 - z^4$. Its genus is three and $X(\mathbb{R})$ has only one connected component. Thus $g+1-s = 3$ is odd and therefore the TV-Screen admits no hyperbolic embedding.

Example 2.14. The Edge quartic is the plane quartic curve defined by $25 \cdot (x^4 + y^4 + z^4) - 34 \cdot (x^2y^2 + x^2z^2 + y^2z^2)$. The set of its real points has four connected components. Thus it is not hyperbolic with respect to any point in the plane. But it can be embedded hyperbolically into some $\mathbb{P}^n$ since the corresponding Riemann surface is of dividing type. But we can also give such an embedding explicitly: Fix a point in each of the four ovals and consider the pencil of quadrics that pass through all these four points. Each such quadric will intersect the curve in eight real points. Thus we get a real fibered morphism to $\mathbb{P}^1$. This corresponds to a hyperbolic embedding of the edge quartic into $\mathbb{P}^5$ via the second Veronese embedding of $\mathbb{P}^2$. Every choice of four points in general position in $\mathbb{P}^2$ corresponds to a three dimensional subspace of $\mathbb{P}^5$. The curve will be hyperbolic with respect to such a subspace whenever the four points are chosen in such a way that each oval contains exactly one of the points.

Recall from [34, §20, Thm. 9] that if $A$ is an $\mathbb{R}$-algebra which is a finite dimensional $\mathbb{R}$-vector space, then the bilinear form $A \times A \to \mathbb{R}, (f,g) \mapsto \text{tr}(fg)$ is positive semidefinite if and only if $\text{Spec}(A)$ consist only of $\mathbb{R}$-points. Similarly, if $K \subseteq L$ is a field extension of degree $m$ and $P$ an ordering of $K$, then the signature (with respect to $P$) of the trace bilinear form is the number of different extensions of $P$ to $L$. To characterize real fibered morphisms we have the following theorem:

Theorem 2.15. Let $X$ and $Y$ be irreducible $\mathbb{R}$-varieties. Assume, furthermore, that $Y$ is smooth. Let $f : X \to Y$ be a finite, flat and surjective morphism, then the following are equivalent:

(i) The morphism $f$ is real fibered.
(ii) Every ordering of the function field $K$ of $Y$ has exactly $m$ extensions to the function field $L$ of $X$, where $m = [L : K]$ the degree of $f$.

Proof. Consider the $K$-bilinear form $b : L \times L \to K, (f,g) \mapsto \text{tr}_{L/K}(f : g)$. By the above remark we have that (ii) holds if and only if $b$ is positive definite with respect to every ordering of $K$. For every point $y \in Y(\mathbb{R})$ there exists an open affine neighborhood $U \subseteq Y$ of $y$, such that $f^{-1}(U) \subseteq X$ is affine and $O_X(f^{-1}(U))$ is a finite free module over $O_Y(U)$. Let $A = O_Y(U)$ and $B = O_X(f^{-1}(U))$. Therefore, the trace map $\text{tr}_{L/K} : L \to K$ satisfies $\text{tr}_{L/K}B \subseteq A$. Now (i) together with the above remark implies that the $A$-bilinear form

$$b : B \times B \to A, (a,b) \mapsto \text{tr}_{L/K}(ab)$$

is positive semidefinite in every closed point of the real spectrum $\text{Spec}(A)$ of $A$, thus it is positive semidefinite on $\text{Spec}(A)$ (the closed points are dense in $\text{Spec}(A)$ and therefore the closed set defined by the principal minors of the matrix associated to the bilinear form is everything). In particular, the $K$-bilinear form

$$b : L \times L \to K, (a,b) \mapsto \text{tr}_{L/K}(ab)$$

is positive definite on $S\text{Spec}(K) \subseteq \text{Spec}(A)$ which implies (ii). In order to prove $(ii) \Rightarrow (i)$, assume that $f^{-1}({\{y\}}) \not\subseteq X(\mathbb{R})$. This means that the bilinear form $b$ is not positive semidefinite in $y$. Since $y$ is a smooth point of $Y$, the Artin-Lang Theorem [7, Thm. 1.3] implies that $b$ is not positive semidefinite on $\text{Spec}(K)$. □

For the following recall that we assume all curves to be equidimensional.

**Corollary 2.16.** Let $f : X \to Y$ be a finite surjective morphism of curves over $\mathbb{R}$. Assume that $Y$ is smooth, then $f$ is real fibered if and only if $f \circ \pi$ is real fibered, where $\pi : \tilde{X} \to X$ is the normalization map.

Proof. Without loss of generality, we can assume that $X$ and $Y$ are irreducible. Since all $\mathbb{R}$-varieties of dimension one are Cohen-Macaulay we have that both $f$ and $f \circ \pi$ are flat. Now since the function fields of $X$ and of $\tilde{X}$ are the same, the claim follows immediately from the above Theorem. □

**Proposition 2.17.** Let $f : X \to Y$ be a finite surjective morphism of curves over $\mathbb{R}$. Let $p \in X(\mathbb{R})$ be such that $Y$ is smooth at $f(p)$. If the differential $d_p f : T_pX \to T_{f(p)}Y$ at $p$ is the zero map, then $f$ is not real fibered.

Proof. Without loss of generality we can assume that $Y$ is smooth and that $X$ and $Y$ are irreducible and affine. Let $\pi : \tilde{X} \to X$ be the normalization map and let $q \in \tilde{X}(\mathbb{R})$ be any point, such that $\pi(q) = p$ (we can assume that such a point $q$ exists, since otherwise $f \circ \pi$ and thus $f$ would fail to be a real fibered morphism). We have $d_q(f \circ \pi) = d_{q,f} \circ d_q \pi = 0$. Thus by the preceding corollary we can further restrict to the case where $X$ is smooth.

Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ and let $\tilde{f} : \text{Spec}(A) \to \text{Spec}(B)$ be the induced map between the real spectra. Let $d$ be the degree of $f$. By the Baer-Krull Theorem [18, Thm. 2.2.5] there are two distinct points $\alpha_1, \alpha_2 \in \text{Spec}(B)$ with support zero that specialize to $f(p)$. If $f$ is real fibered the theorem above implies that there are $2d$ distinct points in the preimage $\tilde{f}^{-1}(\{\alpha_1, \alpha_2\})$. By real going-up [3, Thm. 4.3] these specialize to points in the preimage of $f(p)$. But since $f$ is ramified at $p$, there are at most $d - 1$ points in the preimage of $f(p)$. By the pigeonhole principle we thus have at least one point in $\text{Spec}(A)$ to which at
least three distinct points with support zero specialize. But since $X$ is smooth this
contradicts the Baer-Krull Theorem.

Remark 2.18. The fact that a real fibered morphism $f : X \to \mathbb{P}^1$, where $X$ is a smooth
curve over $\mathbb{R}$, is unramified at real points can be easily seen using complex analysis.
Consider $f$ as a meromorphic function on $X(\mathbb{C})$ and consider its Laurent expansion in
some real local coordinate around a zero to see that it has to be a simple zero.

The following theorem has been proved in several special cases like for hyperbolic hypersurfaces [23] or reciprocal linear spaces [41]. However their methods
do not generalize to the case of arbitrary real fibered morphisms.

Theorem 2.19. Let $f : X \to Y$ be a real fibered morphism between two $\mathbb{R}$-varieties.
Let $p \in X(\mathbb{R})$ and $q = f(x) \in Y(\mathbb{R})$ be smooth points. Then the differential
$d_p f : T_p X \to T_q Y$ at $p$ is an isomorphism.

Proof. Assume that the differential $d_p f$ of $f$ at $p$ is not surjective. Let $C \subseteq Y$ be a
curve over $\mathbb{R}$ which is smooth at $q$ and whose tangent space intersects the image of
$d_p f$ trivially. Let $C' = X \times_Y C$ be the fiber of $f$ over $C$ which is again a curve. The
induced map $C' \to C$ is real fibered and its differential at $p$ is zero. This contradicts
the preceding lemma.

Remark 2.20. Looking at the proof of Theorem 2.19 it is not hard to see that the
statement is still true if we drop the assumptions on $f : X \to Y$ being finite, flat and
surjective (which is part of our definition of real fibered morphisms). Namely, if we
require the morphism to be dominant and to satisfy $x \in X(\mathbb{R}) \iff f(x) \in Y(\mathbb{R})$ for all
$x \in X$, then the morphism will automatically have finite fibers over real points and
from the proof it follows that it is unramified at smooth real points.

The following corollary is a partial generalization of [23, Thm. 5.2].

Corollary 2.21. Assume that $X$ is a smooth weakly hyperbolic $\mathbb{R}$-variety of dimension
$k \geq 2$ with $f : X \to \mathbb{P}^k$ being a real fibered morphism. Then $X(\mathbb{R})$ is a disjoint union
of $s$ manifolds homeomorphic to $S^k$ and $r$ manifolds homeomorphic to $\mathbb{P}^k(\mathbb{R})$ where
$2s + r = \deg f$. In particular, every real fibered morphism $X \to \mathbb{P}^k$ has to be of degree
$\deg f$.

Proof. By the above Theorem $X(\mathbb{R})$ is a covering space of $\mathbb{P}^k(\mathbb{R})$ with $\deg f$ many sheets. Since $k \geq 2$ we have that $\pi_1(\mathbb{P}^k(\mathbb{R})) \cong \mathbb{Z}/2$. Hence every connected
component of $X(\mathbb{R})$ is either homeomorphic to $S^k$ or to $\mathbb{P}^k(\mathbb{R})$ and the formula
$2s + r = \deg f$ follows from counting the sheets.

We have seen in Example 2.5 that the rational normal curve is always hyperbolic.
This not true anymore for Veronese varieties of higher dimension.

Corollary 2.22. Let $X = V_m(\mathbb{P}^k)$ be the Veronese embedding of $\mathbb{P}^k$ ($k, m \geq 2$) into
$\mathbb{P}^N$, where $N = \binom{k+m}{m} - 1$. Then $X$ is not hyperbolic, i.e. there is no real linear
subspace $V \subseteq \mathbb{P}^N$ of dimension $N - k - 1$, such that $V \cap X = \emptyset$ and $X$ is hyperbolic
with respect to $V$.

Proof. Suppose towards a contradiction that $X$ is hyperbolic with respect to some
real linear subspace $V \subseteq \mathbb{P}^d$ of dimension $N - 1 - k$. Then the projection from $V$
is a real fibered morphism $f : X \to \mathbb{P}^k$ of degree $m^k$ which is a contradiction to the
preceding corollary.
Remark 2.23. Two bivariate homogeneous forms with real coefficients of the same
degree that interlace have the property that every nonzero polynomial in their span
over \( \mathbb{R} \) has only real zeros. One can ask whether such a phenomenon exists for forms in
more than two variables. More precisely, is it possible to find \( d+1 \) homogeneous forms
\( f_0, \ldots, f_d \in \mathbb{R}[x_0, \ldots, x_d] \) in \( d+1 \) variables of degree \( m \) for \( d, m > 1 \) without common
zeros such that every \( d \) linearly independent forms in their span over \( \mathbb{R} \) have just real
common zeros? The preceding corollary gives a negative answer to this question.

Example 2.24. In this example we will consider a double cover of the projective
plane branched along a smooth curve of degree \( 2m \) without real points. We will
show that if \( m \geq 2 \) this is a weakly hyperbolic variety that does not admit a hyper-
monic embedding into some projective space.

Let \( p \in \mathbb{R}[x_0, x_1, x_2] \) be a positive definite, homogeneous polynomial of degree
\( 2m \), such that the curve in \( \mathbb{P}^2 \) defined by \( p \) is smooth. Let \( X \) be the hypersurface
defined by \( y^2 = p(x_0, x_1, x_2) \) in the weighted projective space \( \mathbb{P}(1, 1, m) \) where
\( x_0, x_1, x_2 \) are homogeneous coordinates of weight 1 and \( y \) is a homogeneous coordi-
nate of weight \( m \). We have that \( X \) is a smooth projective variety and the projection
morphism \( f : X \to \mathbb{P}^2 \) is finite of degree two. Moreover, it is real fibered since \( p \)
is positive definite. If there was any embedding \( \iota \) of \( X \) into some projective space
as a hyperbolic variety, there would be a real fibered linear projection from \( \iota(X) \) to
\( \mathbb{P}^2 \). By Corollary 2.21 this would also have to be of degree two. This means that \( X \)
would be isomorphic to a smooth quadric surface \( Q \) in \( \mathbb{P}^3 \). For example, comparing
Hodge numbers shows that this can not be: We have \( h^{1,1}(Q) = 1 \) but \( h^{1,1}(X) \neq 1 \)
for \( m \geq 2 \) (see for example [4, Chapter 17]).

The previous example shows the existence of a weakly hyperbolic surface which
can not be embedded hyperbolically to some \( \mathbb{P}^n \) but rather to some weighted pro-
jective space. In the following we show that this can always be done.

Lemma 2.25. Let \( Y = \mathbb{P}(1, \ldots, 1, n, \ldots, n) \) be the weighted projective space. Let us
write \( y = (y_0, y_1, \ldots, y_d) \) and set \( V = \{ y \in Y \mid y_0 = y_1 = \cdots = y_k = 0 \} \). Consider
the projection from \( V \), namely \( f : Y \setminus V \to \mathbb{P}^k \). This map realizes \( Y \setminus V \) as the total
space of \( \mathcal{O}_{\mathbb{P}^k}(n)(d-k) \).

Proof. Clearly, we have that \( Y \setminus V \) is a total space of a vector bundle over \( \mathbb{P}^k \) since
over each point we perform coordinate-wise addition in the coordinates of weight
\( n \). Consider the distinguished affine open sets in \( \mathbb{P}^k \), for \( j = 0, \ldots, k \) \( U_j = \{ x_j \neq 0 \} \).
Note that \( f^{-1}(U_j) = \{ x_j \neq 0 \} \) with coordinates
\[
(x_0/x_j, \ldots, x_{j-1}/x_j, x_{j+1}/x_j, \ldots, x_{k+1}/x_j^n, \ldots, x_d/x_j^n).
\]
Now on the intersection of \( U_j \) with \( U_i \) we get that the transition maps are diagonal
with the coordinate to power \( n \) on the diagonal and this corresponds precisely to
\( \mathcal{O}_{\mathbb{P}^k}(n)(d-k) \). \qed

Theorem 2.26. Let \( X \) be a real projective \( k \)-dimensional weakly hyperbolic variety.
Let \( f : X \to \mathbb{P}^k \) be the real fibered morphism, then we can embed \( X \) into a projective
space $Y = \mathbb{P}(1, \ldots, 1, n, \ldots, n)$, such that the following diagram commutes:

$$
\begin{array}{c}
X \\
\downarrow f \\
\pi \\
\downarrow \\
\mathbb{P}^k
\end{array}
\quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
Y \\
\rightarrow
\end{array}
$$

(2.1)

Here $\pi$ is the projection on the first $k + 1$ coordinates.

**Proof.** Since the morphism $f$ is finite and flat we know that $f_* \mathcal{O}_X$ is a vector bundle on $\mathbb{P}^k$. Furthermore, we have the trace morphism $\mathcal{O}_X \to \mathcal{O}_{d-k}$ and we obtain a decomposition $f_* \mathcal{O}_X \cong \mathcal{O}_{d-k} \oplus \mathcal{E}$, where $\mathcal{E}$ is some vector bundle. Let $n$ be a positive integer, such that $\mathcal{E}(n)$ is generated by global sections, i.e., there exits an epimorphism $\mathcal{O}_{d-k}^{\oplus m} \to \mathcal{E}(n)$. Now we get from applying the relative spec construction to the sheaf of algebras $\mathcal{O}_{d-k}^{\oplus m}$ that we shall denote by $Y$. The epimorphism $\iota^#$ induces the closed embedding $\iota: X \to Y$ and by construction we have the commutative diagram (2.1). Now it remains to apply the previous lemma to obtain that $Y$ is the complement of a linear subspace in a weighted projective space and $\pi$ is the associated projection. \qed

3. Admissible Determinantal Representations and Ulrich Sheaves

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^d = \mathbb{P}_K^d$ (where $d \geq 2$) and let $X = \text{Supp}(\mathcal{F})$. We assume that $\dim X = k < d$ and that $X$ is pure-dimensional. Recall from [17, Prop. 2.1] that $\mathcal{F}$ is called Ulrich, if the module of twisted global sections $M = \oplus_{j \in \mathbb{N}} \mathbb{H}^0(\mathbb{P}^d, \mathcal{F}(j))$ is a maximally generated maximal Cohen-Macaulay module in the sense of [11]. Namely, $M$ is a Cohen-Macaulay module of dimension $k + 1$ over the ring $S = \oplus_{j \in \mathbb{N}} \mathbb{H}^0(\mathbb{P}^d, \mathcal{O}_{d-k}(j)) \cong K[z_0, \ldots, z_d]$ and it admits a linear minimal resolution:

$$
0 \to F_{d-k} \to \cdots \to F_1 \to F_0 \to M \to 0 .
$$

Where each $F_j$ is a free graded $S$-module generated in degree $j$. Furthermore, it was shown in [17] that each $F_i$ is of rank $\binom{d-k}{j}$, where $n = \deg \mathcal{F}$. In particular $\text{rank } F_{d-k} = \text{rank } F_0 = n$. Hence, passing to the associated sheaves we get the following resolution by vector bundles:

$$
0 \to \mathcal{O}_{d-k}(-d + k)^n \to \cdots \to \mathcal{O}_{d-k}(-1)^{n(d-k)} \to \mathcal{O}_{d-k} \to \mathcal{F} \to 0 .
$$

Conclude that for every point $x \in \mathbb{P}^d$, the module $\mathcal{F}_x$ is a Cohen-Macaulay module over $\mathcal{O}_{d-k}$, and thus applying $\mathcal{H}om(-, \mathcal{F}_x)$ to the complex will yield a free resolution of $\mathcal{E}$ over $\mathcal{F}$. Twisting this resolution by $\mathcal{O}_{d-k}(k-d)$ we get that the functor $\mathcal{E} \mathcal{O}_{d-k}(-, \mathcal{F}_x)$ induces a duality on the category of Ulrich sheaves.

Let $\mathcal{F}$ be an Ulrich sheaf on $\mathbb{P}^d$, then from the resolution we get that the Hilbert polynomial of $\mathcal{F}$ is $\chi(\mathcal{F}(e)) = h^0(\mathcal{F})(e^ck)$ and in particular the degree of $\mathcal{F}$ is $h^0(\mathcal{F})$. Now let $X$ be a projective variety of pure dimension $k$ and fix an embedding $i: X \to \mathbb{P}^d$ given by a line bundle $\mathcal{L} = i^* \mathcal{O}_{d-k}(1)$, we say that a sheaf $\mathcal{F}$ on $X$
is Ulrich with respect to \( L \) if \( i_* \mathcal{F} \) is Ulrich. In this case we have also that if we decompose into irreducible components \( X = X_1 \cup \cdots \cup X_r \), then we have that 
\[
\deg \mathcal{F} = \sum_{j=1}^r \text{rank}(\mathcal{F}|_{X_j}) \deg(X_j),
\]
where the degree of each component is with respect to the embedding \( i \), see [17] for more details. When the embedding is fixed and there exists such a sheaf on \( X \) of degree \( n \), then we will simply say that \( X \) admits an Ulrich sheaf of degree \( n \).

There is yet another equivalent way to define Ulrich sheaves, given a subscheme \( X \subseteq \mathbb{P}^d \) of pure dimension \( k \), we can realize \( X \) as a branched covering of \( \mathbb{P}^k \) by means of a linear projection from a linear subspace of \( \mathbb{P}^d \) of dimension \( d - k - 1 \).

One can define then a sheaf \( \mathcal{F} \) supported on \( X \) (scheme-theoretically) to be Ulrich if for a general linear projection \( \pi : X \rightarrow \mathbb{P}^k \), we have that there exists a positive integer \( m \), such that \( \pi_* \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^k}^m \), see [17, Prop. 2.1] for the equivalence of those definitions.

For the rest of this section we will work over the complex numbers \( \mathbb{C} \) unless explicitly otherwise stated. We now recall a definition of determinantal representations of subvarieties of \( \mathbb{P}^d = \mathbb{P}^d_\mathbb{C} \) introduced in [43].

**Definition 3.1.** We say that \( X \subseteq \mathbb{P}^d \) of dimension \( k \) admits a Livsic-type determinantal representation, if there exists a tensor \( \gamma \in \wedge^{k+1} \mathbb{C}^{d+1} \otimes M_n(\mathbb{C}) \), such that the set of closed points \( p \in \mathbb{P}^d \), satisfying \( \gamma \wedge p \) has non-trivial kernel considered as a linear map from \( \mathbb{C}^n \) to \( \wedge^{k+2} \mathbb{C}^{d+1} \otimes \mathbb{C}^n \), is precisely the set of closed point of \( X \).

Note that this implies that \( X \) supports the kernel sheaf of the vector bundle map \( \mathcal{O}_{\mathbb{P}^d}(-1)^n \rightarrow \mathcal{O}_{\mathbb{P}^d}^{(d+1)_n} \) associated to \( \gamma \). Let us denote this coherent sheaf \( \mathcal{K} \).

Consider the divisor associated to \( \mathcal{K} \), namely if we decompose \( X = X_1 \cup \cdots \cup X_r \) into irreducible components, and for each component we set \( n_j \) to be the dimension of the fiber of \( \mathcal{K} \) at the generic point of \( X_j \). We then define the cycle of \( \gamma \) to be:

\[
Z(\gamma) = \sum_{j=1}^r n_j [X_j].
\]

Let us denote (here \( [H] \) is the class of a hyperplane):

\[
\deg(\gamma) = \deg(Z(\gamma)) = \int_{\mathbb{P}^d} Z(\gamma) \cdot [H]^k = \sum_{\dim X_j = k} n_j \deg(X_j).
\]

**Definition 3.2.** We say that \( X \) admits an admissible (very reasonable in the parlance of [43]) determinantal representation if \( X \) admits a determinantal representation \( \gamma \in \wedge^{k+1} \mathbb{C}^{d+1} \otimes M_n(\mathbb{C}) \) and \( \deg(\gamma) = n \).

The next theorem shows that in fact \( X \) admits an admissible determinantal representation if and only if \( X \) has an Ulrich sheaf.

**Theorem 3.3.** The following conditions for a subvariety \( X \subseteq \mathbb{P}^d \) of dimension \( k \), are equivalent:

(i) \( X \) admits an Ulrich sheaf of degree \( n \).

(ii) \( X \) admits an admissible determinantal representation \( \gamma \), such that 
\[
\deg(\gamma) = n.
\]

**Proof.** Assume that there exists an Ulrich sheaf \( \mathcal{F} \) supported on \( X \). Denote the module of twisted global sections of \( \mathcal{F} \) by \( M = \oplus_{j \in \mathbb{H}} H^0(\mathbb{P}^d, \mathcal{F}(j)) \). Consider the
linear free resolution of $M$:

\[(3.1) \quad 0 \to F_{d-k} \xrightarrow{\psi_{d-k}} \cdots \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 \to M \to 0.\]

Similarly to [17], we compose the differentials $\gamma = \psi_1 \circ \cdots \circ \psi_{d-k}$ as if the entries were elements of the tensor algebra and it is immediate that we get a $n \times n$ matrix of linear forms, due to the ranks of $F_j$ and the fact that they are generated in degree $j$. Furthermore, since those are differentials of a complex we get that $\gamma \in \wedge^{d-k}(\mathbb{C}^{d+1})^\ast \otimes \text{Mat}_n(\mathbb{C}) \cong \wedge^{k+1} \mathbb{C}^{d+1} \otimes \text{Mat}_n(\mathbb{C})$. We want to show that the transpose $\gamma^T$ is an admissible determinantal representation of $X$. For $v \in \mathbb{C}^{d+1}$ the matrix $\gamma \wedge v$ is (up to a sign) the same as the one that we obtain by plugging in the $v_i$ for the $x_i$ in one of the $\psi_j$ before composing them. Thus, since $X$ is the support of $F$ we see that $\gamma \wedge v$ has a nontrivial left-kernel if and only if $v \in X$. The left-kernel of $\gamma \wedge v$ for a general point $v$ from an irreducible component $X_j$ of $X$ is exactly $\text{rank}(F|_{X_j})$. Thus it follows from

$$n = \deg F = \sum_{j=1}^r \text{rank}(F|_{X_j}) \deg(X_j)$$

that $\gamma^T$ is an admissible determinantal representation of Livsic-type.

To prove the converse note that if $X$ admits an admissible determinantal representation, then using the conditions described in [43, Cor. 2.20] and the discussion after it, we get that $\gamma$ can be reduced to $d-k$ commuting $n \times n$ matrices of linear forms, $T_1, \ldots, T_{d-k}$, write $T = (T_1 \cdots T_{d-k})$ for the long matrix obtained from concatenating the matrices. By changing variables we may assume that $T_j = z_j - T_{0,j}$, where $T_{0,j}$ is a matrix of linear forms in the variables $z_{d-k+1}, \ldots, z_d$, hence in particular the $T_j$ define an action of the polynomial ring $\mathbb{C}[y_1, \ldots, y_{d-k}]$ on $S^n$ and the $y_j$ form a homogeneous $S^n$-sequence. In this case $X$ is precisely the variety of points in $x \in \mathbb{P}^d$, such that $0 \in \text{Sp}(T_1(x), \ldots, T_{d-k}(x))$, where $\text{Sp}$ stands for the joint spectrum of commuting matrices. Therefore, we can write the Koszul complex associated to the matrices $T_1, \ldots, T_{d-k}$:

$$0 \to S(-d+k)^n \to \cdots \to S(-1)^{n(d-k)} \xrightarrow{T^\ast} S^n \to M \to 0.$$

This Koszul complex is exact and the sheaf associated to $M$ is thus the Ulrich sheaf on $X$. The fact that the degree is $n$ is immediate from the resolution. \qed

From the proof of the preceding Theorem we also get the following statement for Ulrich sheaves on $\mathbb{R}$-varieties.

**Corollary 3.4.** Let $X \subseteq \mathbb{P}^d_\mathbb{R}$ be a projective $\mathbb{R}$-variety. The following are equivalent:

(i) There exists an Ulrich sheaf $\mathcal{F}$ on $\mathbb{P}^d_\mathbb{R}$ supported on $X$,

(ii) There is a tensor $\gamma \in \wedge^{k+1} \mathbb{R}^{d+1} \otimes \text{Mat}_n(\mathbb{R})$ which is an admissible determinantal representation of $X_\mathbb{C}$.

The following corollary is a strengthening of [43, Thm. 6.2].

**Example 3.5.** It was shown in [12] that the variety of $m \times n$ matrices of rank at most $r$ admits a rank one Ulrich sheaf for all $1 \leq r \leq \min\{m, n\}$. Thus, determinantal varieties have determinantal representations.
Remark 3.6. Given an Ulrich sheaf with support $X \subseteq \mathbb{P}^d$ and rank $n$, the above theorem shows that there exists $d-k$ commuting matrices of linear forms $T_1, \ldots, T_{d-k}$ that endow $S^n$ with a structure of a Cohen-Macaulay module over the polynomial ring $\mathbb{C}[t_1, \ldots, t_{d-k}]$. Let $I$ be the radical homogeneous ideal that is an invertible complex matrix. Hence we have that $j$ is the radical of the ideal of maximal minors of the long matrix $T = (T_1 \cdots T_{d-k})$, we get an induced action of $T_j$ on $(S/I)^n$.

The maximality assumption on the degree of the cycle $Z(\gamma)$ translates to the fact that for a generic point $x \in \mathbb{P}^d$, the induced matrices on the fiber are semi-simple. One can relax this condition slightly, by considering instead of the kernel of the matrices $T_1, \ldots, T_{d-k}$, the torsion submodule, namely if we write $m \subseteq \mathbb{C}[t_1, \ldots, t_{d-k}]$ for the irrelevant ideal $(t_1, \ldots, t_{d-k})$, then:

$$\Gamma_m((S/I)^n) = \{ \xi \in (S/I)^n \mid \exists k \in \mathbb{N}, m^k \xi = 0 \}.$$ 

This is in fact a graded $S$-submodule of $(S/I)^n$ and thus induces a sheaf supported on $X$. Let us denote this sheaf by $\mathcal{G}$. Now we decompose $X$ into irreducible components $X = X_1 \cup \cdots \cup X_r$ and define the following cycle:

$$Z(\mathcal{G}) = \sum_{j=1}^r n_j [X_j],$$

where $n_j$ is now the dimension of the generic fiber of $\mathcal{G}$. One can show that if $V \subseteq \mathbb{P}^d$ is a $d-k-1$-dimensional linear subspace, such that $V \cap X = \emptyset$, then for every $V \subseteq U \subseteq \mathbb{P}^d$ a $d-k$-dimensional subspace, the fibers of $\mathcal{G}$ at points $x \in U \cap X$ correspond to “generalized” joint eigenspaces of $T_1(x), \ldots, T_d(x)$. This generalizes [43, Lem. 2.12]. We will say a determinantal representation is Jordan-admissible if $\deg Z(\mathcal{G}) = n$. It is easy then to generalize [43, Thm. 2.18] and thus obtain a different version of the above theorem, namely that $X$ is the reduced support of an Ulrich sheaf if and only if $X$ admits a Jordan-admissible determinantal representation.

We will say that two admissible tensors $\gamma_1$ and $\gamma_2$ are similar, if there exist matrices $A, B \in \text{GL}_n(\mathbb{C})$, such that $\gamma_1 = A \gamma_2 B$. Then we have the following result:

**Proposition 3.7.** The association of isomorphism classes of Ulrich sheaves and similarity classes of determinantal representations described in Theorem 3.3 is a bijection.

**Proof.** Assume that we have an isomorphism $\varphi: \mathcal{F}_1 \to \mathcal{F}_2$. Let $M_j$ be the module of twisted global sections of $\mathcal{F}_j$. Then $\varphi$ induces an isomorphism $M_1 \cong M_2$, that can be lifted to an isomorphism of their resolutions:

$$0 \longrightarrow S(-d-k)^n \longrightarrow \cdots \longrightarrow S(-1)^{n(d-k)}T \longrightarrow S^n \longrightarrow M_1 \longrightarrow 0.$$ 

$$0 \longrightarrow S(-d-k)^n \longrightarrow \cdots \longrightarrow S(-1)^{n(d-k)}T \longrightarrow S^n \longrightarrow M_2 \longrightarrow 0.$$ 

Note that each $A_j$ is an invertible complex matrix. Hence we have that $\psi_{1j} = A_{j-1}^{-1} \psi_{2j} A_j$. Now note that the constructions of Theorem 3.3 are inverse to each other and that the admissible determinantal representations are determined by the commuting pencils from [43, Cor. 2.20] (see also the discussion following the corollary referenced). □

**Corollary 3.8.** Every projective curve $X \subseteq \mathbb{P}^d$ admits an admissible determinantal representation of size $\deg X$ and the algorithm of [43, Thm. 6.2] constructs them all.
Proof. It was shown in [17] that every projective curve \( X \subseteq \mathbb{P}^d \) admits an Ulrich sheaf, such sheaves correspond precisely to non-special line bundle of degree \( g - 1 \) on the normalization of \( X \).

\[ \square \]

4. Real Varieties and Bilinear Forms

In this section, we work again over the ground field \( \mathbb{R} \). Again we write \( \mathbb{P}^d = \mathbb{P}^d_{\mathbb{R}} \). Let \( X \) be an \( \mathbb{R} \)-variety and let \( \mathcal{E} \) be a coherent sheaf on \( X \). A non-degenerate \( \mathcal{E} \)-valued bilinear form on \( \mathcal{F} \) is a map \( \varphi: \mathcal{F} \otimes \mathcal{F} \to \mathcal{E} \), such that the adjoint morphism \( \kappa: \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}) \) is an isomorphism. We will call a form symmetric, if \( \varphi = \varphi \circ \epsilon \), where \( \epsilon: \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F} \) is natural "twist" morphism. If \( \mathcal{F} \) is reflexive and \( \mathcal{E} \) is a line bundle, then we have that \( \varphi \) is symmetric if and only if we have that \( \kappa^t = \kappa \), where \( \kappa \) is the morphism obtained by applying \( \mathcal{H}om_{\mathcal{O}_X}(\cdot, \mathcal{E}) \) to \( \kappa \).

**Definition 4.1.** Let \( \mathcal{F} \) be a coherent sheaf on an \( \mathbb{R} \)-variety \( X \) and assume that it admits a non-degenerate symmetric \( \mathcal{O}_X \)-valued bilinear form \( \varphi \). Then we say that \( \varphi \) is positive (resp. negative) definite if for every closed point \( x \in X(\mathbb{R}) \) we have that the induced form on the fiber of \( \mathcal{F} \) is positive (resp. negative) definite.

**Remark 4.2.** If \( \mathcal{F} \cong \mathcal{O}_X^\bullet \) then a \( \mathcal{O}_X \)-valued bilinear form is positive (resp. negative) definite if and only if the induced form on the global section is positive (resp. negative) definite.

**Remark 4.3.** More generally one can define definite \( \mathcal{L} \)-valued bilinear forms on \( \mathcal{F} \), for \( \mathcal{L} \) a line bundle on \( \mathcal{F} \), following [28, §II.7.2]. Namely, we will say that a non-degenerate bilinear \( \mathcal{L} \) valued form on \( \mathcal{F} \) is (semi-)definite if for every closed point \( x \in X(\mathbb{R}) \) we have that the induced form on the fiber of \( \mathcal{F} \) is (semi-)definite. In this case however, one can not speak of positive or negative definite forms, since this depends on the chosen trivialization.

**Definition 4.4.** Let \( X \subseteq \mathbb{P}^d \) be a projective \( \mathbb{R} \)-variety of dimension \( k \). Let \( \gamma \in \wedge^{k+1} \mathbb{R}_d^{d+1} \otimes M_d(\mathbb{R}) \) be a Livsic-type determinantal representation of \( X_\mathbb{C} \). If for some (and hence for every) basis \( e_0, \ldots, e_d \) of \( \mathbb{R}^{d+1} \) we have that

\[ \gamma = \sum_{I \subseteq \{0, \ldots, d\}, |I| = k+1} \gamma_I e_I \]

with every \( \gamma_I \) symmetric, we say that \( \gamma \) is a real symmetric Livsic-type determinantal representation.

We have seen in the preceding section that admissible determinantal representations of a variety \( X \subseteq \mathbb{P}^d \) correspond to Ulrich sheaves supported on \( X \). It was shown in [43, Prop. 3.12] that the existence of a real symmetric admissible Livsic-type determinantal representation for \( X_\mathbb{C} \) which is positive definite at a linear space \( \mathcal{V} \) (of correct dimension) implies that \( X \) is hyperbolic with respect to \( \mathcal{V} \). In the following we elaborate what the properties of being real symmetric and positive definite at a certain linear space mean for the corresponding Ulrich sheaf. Note that given a finite flat surjective linear projection \( f: X \to \mathbb{P}^k \), we have \( f^* \omega_{\mathbb{P}^k} \cong \omega_X \) (this is a coherent sheaf since \( X \) is Cohen-Macaulay, it is a line bundle if \( X \) is Gorenstein). Since \( f \) is finite we have for all coherent sheaves \( \mathcal{H} \) on \( \mathbb{P}^d \) that \( f^* \mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(f, \mathcal{H}) \) and in particular \( f^* \omega_{\mathbb{P}^k} \cong (f^* \omega_{\mathbb{P}^k})^\vee \otimes \omega_X \). Let us denote by \( \mathcal{G} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^* \mathcal{O}_{\mathbb{P}^k}) \), then by Grothendieck duality we have the
isomorphism:
\[ f_* \mathcal{G} = f_* \mathcal{H} \mathcal{om}_{\mathcal{O}_X}(\mathcal{F}, f^! \mathcal{O}_{p^k}) \cong \mathcal{H} \mathcal{om}_{\mathcal{O}_{p^k}}(f_* \mathcal{F}, \mathcal{O}_{p^k}). \]

Hence in particular if \( \mathcal{F} \) is an Ulrich sheaf supported on \( X \) and has a non-degenerate \( f^! \mathcal{O}_{p^k} \)-valued bilinear form on it and \( \kappa : \mathcal{F} \rightarrow \mathcal{G} \) is the adjoint isomorphism, then \( f_*(\kappa) \) corresponds to a non-degenerate \( \mathcal{O}_{p^k} \)-valued bilinear form on \( \mathcal{O}_{p^k} \).

**Theorem 4.5.** Let \( X \subseteq \mathbb{P}^d \) be a projective \( \mathbb{R} \)-variety of dimension \( k \). Let \( E \subseteq \mathbb{P}^d \) be a real linear subspace of dimension \( d - k - 1 \), such that \( E \cap X = \emptyset \) and let \( f : X \rightarrow \mathbb{P}^k \) be the linear projection from center \( E \). Then the following are equivalent:

(i) There exists an Ulrich sheaf \( \mathcal{F} \) supported on \( X \) together with a non-degenerate \( f^! \mathcal{O}_{p^k} \)-valued symmetric bilinear form \( \varphi \) on \( \mathcal{F} \), such that the corresponding \( \mathcal{O}_{p^k} \)-valued form on \( \mathcal{O}_{p^k} \) is positive definite.

(ii) The complexification \( X_{\mathbb{C}} \) admits an admissible real symmetric determinantal representation \( \gamma \in \wedge^{k+1} \mathbb{R}^{d+1} \otimes M_n(\mathbb{R}) \), such that \( \gamma(E) \) is a positive definite matrix.

**Proof.** Let \( R = H^0(\mathcal{O}_{p^k} \cong \mathbb{R}[z_{d-k+1}, \ldots, z_d] \) and let \( S = H^0(\mathcal{O}_E \cong \mathbb{R}[z_0, \ldots, z_d] \). We can assume that under these identifications the linear projection \( f \) corresponds to the inclusion \( \mathbb{R}[z_{d-k+1}, \ldots, z_d] \hookrightarrow \mathbb{R}[z_0, \ldots, z_d] \).

First assume that there is such an Ulrich sheaf. Furthermore, let \( M = H^0(\mathcal{F}). \) As an \( R \)-module we have \( M = R^N \) and \( z_1, \ldots, z_{d-k} \) act like matrices \( A_1, \ldots, A_{d-k} \) with homogeneous elements of degree one from \( R \) as entries. Since \( \varphi \) is a morphism of sheaves of modules on \( X \), these matrices are selfadjoint with respect to the corresponding symmetric bilinear form. Since it is positive definite, there is a basis of \( R^N \) with respect to which \( A_1, \ldots, A_{d-k} \) are symmetric. Let \( T_i = z_i - A_i \). The free resolution of \( M \) is given by the Koszul complex associated to \( T_1, \ldots, T_{d-k} \). If we compose the differentials as if their entries were elements of the exterior algebra, we get the matrix

\[ T = \sum_{\sigma \in \mathcal{S}_d} \text{sgn}(\sigma) \cdot T_{\sigma(1)} \wedge \cdots \wedge T_{\sigma(d-k)} \]

where the product is again to be taken as if the entries were elements from the exterior algebra. We have seen that this matrix gives us an admissible determinantal representation. We want to show that \( T \) is symmetric. An elementary calculation verifies that

\[ (T_{\sigma(1)} \wedge \cdots \wedge T_{\sigma(d-k)})^T = \text{sgn}(\tau) \cdot (T_{\tau(d-k)}^T \wedge \cdots \wedge T_{\tau(1)}^T) \]

where \( \tau \in \mathcal{G}_{d-k} \) is the permutation which maps \( \tau(j) = d - k - j + 1 \) for all \( j = 1, \ldots, d - k \). Therefore, since the \( T_i \) are symmetric, we have that \( T \) is symmetric. Moreover we see that \( T \) evaluated at \( E \) is the identity matrix, since \( T_i \) evaluated at \( \delta_i \) is the identity matrix for all \( i = 1, \ldots, d - k \).

Now we assume that there is such an admissible symmetric determinantal representation \( \gamma \) as in (ii). We can assume that \( \gamma(E) \) is the identity matrix. Let us define an Ulrich sheaf on \( X \), let \( A_1, \ldots, A_{d-k} \) be the matrices of linear forms of size \( n \times n \) obtained from \( \gamma \) as in the second part of the proof of Theorem 3.3. We note that the linear projection from \( X \) turns \( \mathbb{P}^d \setminus E \) into a vector bundle over \( \mathbb{P}^k \). This vector bundle is in fact \( \mathcal{O}_{p^k}(1)^{d-k} \). Thus the commuting matrices \( A_1, \ldots, A_{d-k} \) give \( \mathcal{O}_{p^k}^n \), the structure of a module over \( \text{Sym}(\mathcal{O}_{p^k}(-1)^{d-k}) \). This action factors through \( f_* \mathcal{O}_X \), since the matrices arise from a determinantal representation of \( X \). Since \( f \) is finite we get a coherent sheaf \( \mathcal{F} \) on \( X \), such that \( f_* \mathcal{F} \cong \mathcal{O}_{p^k}^n \), i.e. \( \mathcal{F} \) is Ulrich. We
consider the isomorphism $\mathcal{O}_{P^k}^m \cong \mathcal{H}om_{\mathcal{O}_{P^k}}(\mathcal{O}_{P^k}^m, \mathcal{O}_{P^k})$ which induces the symmetric $\mathbb{R}$-bilinear form on the global sections $\mathcal{R}^m = \Gamma(\mathbb{P}^k, \mathcal{O}_{P^k}^m)$ that has the standard basis of $\mathbb{R}^n$ as orthonormal basis (and thus is positive definite). Since the matrices are symmetric we have that this isomorphism is in fact a map of $f_*\mathcal{O}_X$-modules. Hence we get an isomorphism of $f_*\mathcal{O}_X$-modules:

$$f_*\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_{P^k}}(f_*\mathcal{F}, \mathcal{O}_{P^k}) \cong f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{O}_{P^k}).$$

Hence, we obtain an isomorphism of $\mathcal{O}_X$-modules $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{O}_{P^k})$, that corresponds to a non-degenerate $f^!\mathcal{O}_{P^k}$-valued bilinear form on $\mathcal{F}$.

\[\square\]

**Remark 4.6.** It was shown in [24] that if $X \subseteq \mathbb{P}^d$ is a complete intersection, then $X$ has an Ulrich sheaf. If a variety $X \subseteq \mathbb{P}^d$ admits an Ulrich sheaf that satisfies the positivity conditions of the preceding theorem, then $X$ must be hyperbolic. But not every hyperbolic complete intersection has such an Ulrich sheaf: Brändén [9] constructed a hyperbolic hypersurface, such that no power of its defining polynomial can be written as the determinant of a real symmetric matrix with linear entries that is positive definite at some point. On the other hand, it was shown in [31] that for every smooth hyperbolic hypersurface $X \subseteq \mathbb{P}^d$ there is a hypersurface $Y \subseteq \mathbb{P}^d$, such that $X \cup Y$ admits an Ulrich sheaf with the positivity conditions of the above theorem. This is very much related to the generalized Lax conjecture, see for example [45, Conjecture 3.3]. The following question is the natural extension of the above mentioned result to varieties of higher codimension.

**Question 4.7.** Let $X \subseteq \mathbb{P}^d$ be a smooth and irreducible variety of dimension $k$ which is hyperbolic with respect to the linear space $E$. Is there a variety $Y \subseteq \mathbb{P}^d$ of the same dimension as $X$ and also hyperbolic with respect to $E$ such that $X \cup Y$ admits an Ulrich sheaf $\mathcal{F}$ together with a nondegenerate symmetric $f^!\mathcal{O}_{P^k}$-valued bilinear form $\varphi$ on $\mathcal{F}$, such that the corresponding symmetric form on $\mathcal{O}_{P^k}^m$ is positive definite? Here $f : X \cup Y \to \mathbb{P}^k$ denotes the linear projection from center $E$.

Let $f : X \to Y$ be a finite flat morphism of degree $m$. Let $\mathcal{F}$ be a coherent sheaf on $X$. We say that $\mathcal{F}$ is $f$-positive (resp. $f$-nonnegative) if it admits a non-zero $f^!\mathcal{O}_Y$-valued bilinear form, such that the induced $\mathcal{O}_Y$-valued bilinear form on $f_*\mathcal{F}$ is symmetric and positive definite (resp. positive semidefinite). Following [30] one can define the notion of $f$-Ulrich sheaves, namely a sheaf $\mathcal{F}$ on $X$ is $f$-Ulrich if there exists a positive integer $r$, such that $f_*\mathcal{F} \cong \mathcal{O}_Y^r$. We will say that an $f$-Ulrich sheaf $\mathcal{F}$ is positive if it is $f$-positive. To show the connection between $f$-nonnegative sheaves and real fibered morphisms we need a lemma:

**Lemma 4.8.** Let $L/K$ be a finite extension of fields, let $V$ be a finite dimensional vector space over $L$. Assume that there is an $L$-linear, non-zero homomorphism $\varphi : V \to \text{Hom}_K(V, K)$, such that the corresponding $K$-bilinear form on $V$ is symmetric and positive semidefinite with respect to some ordering $P$ on $K$, then $P$ has exactly $[L : K]$ extensions to $L$.

*Proof.* After dividing out the kernel of $\varphi$ we can restrict to the case where $\varphi$ is injective. Let $\alpha \in L$ be a primitive generator, i.e. $L = K[\alpha]$. Let $p$ be the minimal polynomial of $\alpha$. It suffices to show that $p$ splits over the real closure of $K$ with respect to $P$ that we will denote by $R$. Note that $p$ is the minimal polynomial of the $K$-linear map $f_\alpha : V \to V$ defined by $v \mapsto \alpha v$. Since $\varphi$ is $L$-linear, we have that $f_\alpha$
is selfadjoint with respect to the $K$-bilinear form defined by $\varphi$. Since the bilinear form induced on $V \otimes_K R$ is positive definite it admits an orthogonal basis. The corresponding matrix of $f_\alpha$ with respect to this basis is symmetric. Thus all of the roots of its characteristic polynomial lie in $R$. We conclude that $p$ splits over $R$. □

**Theorem 4.9.** Let $X$ and $Y$ be irreducible $\mathbb{R}$-varieties and assume that $Y$ is smooth. Let $f : X \to Y$ be a finite flat morphism. Then $f$ is real fibered if and only if there is an $f$-non-negative coherent sheaf $\mathcal{F}$ with $\text{Supp} \mathcal{F} = X$.

**Proof.** By Theorem 2.15 it suffices to show that $f$ restricted to an open dense subset is real fibered. Thus by Grothendieck's Generic Freeness Lemma we can assume without loss of generality that $X$ and $Y$ are affine and $\mathcal{F}$ is a positive $f$-Ulrich sheaf. Denote by $L$ the field of functions on $X$ and by $K$ the field of functions on $Y$. Let us write $A = \mathcal{O}_Y(Y), B = \mathcal{O}_X(X)$ and $M = \mathcal{F}(X)$. Represent the positive definite symmetric bilinear form on $A^{mr}$ by a positive definite symmetric $mr \times mr$ matrix $T$. That means that the leading minors are positive at every closed point of $\text{Sper} \ A$ and thus at every point of $\text{Sper} \ A$. Now the bilinear form translates to an isomorphism $M \to \text{Hom}_A(M, \text{Hom}_A(B, A)) \equiv \text{Hom}_A(M, A)$. Localizing at the generic point gives us a bilinear form that satisfies the assumptions of Lemma 4.8. Thus $f$ is real fibered by Theorem 2.15.

Conversely, let $f$ be real fibered. We will show that $\mathcal{F} = \mathcal{O}_X$ is $f$-non-negative. Since $f$ is flat and finite we can define the trace morphism $\text{Tr}_{X/Y} : f_*\mathcal{O}_X \to \mathcal{O}_Y$. This gives us a map $f_*\mathcal{O}_X \to \mathcal{H}\text{om}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)$ defined as follows: For any open subset $U$ of $Y$ and any section $a \in \mathcal{O}_X(f^{-1}(U))$ we define the image of $a$ to be the map $b \mapsto (\text{Tr}_{X/Y})_U(ab)$, cf. [2, Ch. VI, §6]. The corresponding $\mathcal{O}_Y$-valued bilinear form is positive semidefinite since $f$ is real fibered (cf. the remarks and references before Theorem 2.15) and by Grothendieck duality it corresponds to an $f^!\mathcal{O}_Y$-valued bilinear form on $\mathcal{O}_X$. Therefore, $\mathcal{O}_X$ is $f$-non-negative. □

**Remark 4.10.** Theorem 4.9 includes the classic methods to check whether an univariate polynomial has only real roots or not (see for example [29]) as special cases. The so-called Hermite matrices correspond to $f^!\mathcal{O}_Y$-valued bilinear forms on the structure sheaf $\mathcal{O}_X$ and the so-called Bézout matrices correspond to $f^!\mathcal{O}_Y$-valued bilinear forms on the sheaf $f^!\mathcal{O}_Y$, cf. [31, Section 3].

Theorem 4.9 says in particular that the existence of a positive $f$-Ulrich sheaf implies that $f$ is real fibered. This motivates the following relative version of Question 4.7:

**Question 4.11.** Let $f : X \to Y$ be a real fibered morphism of irreducible $\mathbb{R}$-varieties. Does there exist a real fibered morphism $g : Z \to Y$ and a closed embedding of $X$ into $Z$ over $Y$, such that $Z$ has a positive $g$-Ulrich sheaf?

**Remark 4.12.** Note that this question differs from Question 4.7, since we do not require $X$ and $Z$ to be embedded into the same projective space and the maps to be linear projections. If we set $Y = \mathbb{P}^k$, we get a relaxed version of Question 4.7, which is also open.

The following Lemma shows that the duality on Ulrich sheaves can be described in terms of the Grothendieck duality.

**Lemma 4.13.** Assume $X$ is a smooth projective variety of dimension $k$ equipped with an embedding $i : X \to \mathbb{P}^d$ given by a very ample line bundle $\mathcal{L}$ on $X$. Let $f : X \to \mathbb{P}^k$
be a linear projection from a linear subspace $V \subseteq \mathbb{P}^d$ of dimension $d - k - 1$ that does not intersect $i(X)$. Assume that $\mathcal{F}$ is an Ulrich vector bundle on $X$ with respect to $\mathcal{L}$. Set $\mathcal{G} = \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{O}_{\mathbb{P}^d})$, then $i_*\mathcal{G} \cong \mathcal{E}\text{xt}^{d-k}_{\mathcal{O}_{\mathbb{P}^d}}(i_*\mathcal{F}, \omega_{\mathbb{P}^d})(k + 1)$.

**Proof.** Note that $\mathcal{F}$ is a linear projection we have that $f \mathcal{O}_{\mathbb{P}^d} \cong (f^*\omega_{\mathbb{P}^d})^\vee \otimes_{\mathcal{O}_X} \omega_X$. Now since $f$ is a general linear projection we have that $f^*\mathcal{O}_{\mathbb{P}^d}(1) \cong \mathcal{L}$ and thus $f^*\mathcal{O}_{\mathbb{P}^d} \cong \mathcal{L}^{k+1} \otimes_{\mathcal{O}_X} \omega_X$. Now on the other hand we have that $i^!\omega_{\mathbb{P}^d}(k + 1) \cong \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{k+1}$. We conclude that:

$$i_*\mathcal{G} \cong i_* \mathcal{H}\text{om}(\mathcal{F}, i^!\omega_{\mathbb{P}^d}(k + 1)) \cong \mathcal{E}\text{xt}^{d-k}_{\mathcal{O}_{\mathbb{P}^d}}(i_*\mathcal{F}, \omega_{\mathbb{P}^d})(k + 1).$$

The last equality follows from Grothendieck duality (the shift in the $\mathcal{E}\text{xt}$ is due to the difference in dimensions), see for example [14, Ch. 3.5].

Recall from [43, Thm. 2.18] that the fact that $X$ admits a positive definite admissible determinantal representation implies that the Chow form of $X$ admits a determinantal representation in the following form:

$$\det \left( \sum_{0 \leq i_0 < \ldots < i_d \leq d} p_{i_0 \ldots i_d} A_j \right).$$

Here the $A_j$ are constant Hermitian matrices, the $p_{i_0 \ldots i_d}$ are the Plücker coordinates and at some real point on the Grassmannian the above matrix is definite. From Corollary 2.22 we immediately get the following:

**Corollary 4.14.** Let $X = \mathcal{V}_m(\mathbb{P}^k)$, the Veronese embedding of $\mathbb{P}^k$ into $\mathbb{P}^N$, $k, m \geq 2$, where $N = \left( \binom{k+m}{k} - 1 \right)$, then the Chow form of $X$, i.e. the resultant $k + 1$ forms of degree $m$ in $k + 1$ variables, does not admit a representation as a determinant of a matrix of linear forms as above, where the $A_j$ are Hermitian and for some real point on the Grassmannian the above matrix is positive definite.

**Proof.** If the Chow form of $X$ would have had such a determinantal representation it would be hyperbolic in the sense of [43, Prop. 3.5] and therefore $X$ itself would be hyperbolic contradicting Corollary 2.22. □

**Example 4.15.** Recall from Example 2.5 that any two interlacing polynomials give a linear space with respect two which the rational normal curve is hyperbolic. Take for example $f = s^3 - 4st^2$ and $g = s^2t - t^3$. The morphism $\varphi$ then corresponds to the projection of the twisted cubic $C = \{(s^3 : s^2t : st^2 : t^3) : (s : t) \in \mathbb{P}^1\}$ from the linear space defined by $x_0 = 4x_2$ and $x_1 = x_3$. The matrix of the determinantal representation from [43, Exp. 6.4] for $C$ at this linear space is

$$\begin{pmatrix}
1 & 0 & -1 \\
0 & 3 & 0 \\
-1 & 0 & 4
\end{pmatrix}.$$

This matrix is positive definite as expected.

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