Fast OBDD Reordering using Neural Message Passing on Hypergraph

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Abstract

Ordered binary decision diagrams (OBDDs) are an efficient data structure for representing and manipulating Boolean formulas. With respect to different variable orders, the OBDDs’ sizes may vary from linear to exponential in the number of the Boolean variables. Finding the optimal variable order has been proved a NP-complete problem. Many heuristics have been proposed to find a near-optimal solution of this problem. In this paper, we propose a neural network-based method to predict near-optimal variable orders for unknown formulas. Viewing these formulas as hypergraphs, and lifting the message passing neural network into 3-hypergraph (MPNN\textsubscript{3}), we are able to learn the patterns of Boolean formula. Compared to the traditional methods, our method can find a near-the-best solution with an extremely shorter time, even for some hard examples. To the best of our knowledge, this is the first work on applying neural network to OBDD reordering.

1 Introduction

Boolean functions are functions that take Boolean variables as arguments and return Boolean values. They were widely accepted as the modeling formalism for design, verification and synthesis of digital computers (Crama and Hammer 2011). More real-world problems, including cryptography, social choice theory, etc., can be formulated using Boolean functions.

Ordered binary decision diagrams (OBDDs) (Bryant 1992) are a standard data structure for representing and manipulating Boolean formulas. They are compact to store and efficient to operate. More importantly, they provide a canonical representation of Boolean functions. Given any two logically equivalent Boolean functions, their OBDDs are isomorphic.

A central problem in the application of OBDDs is to find a proper decision order of Boolean variables. With respect to different decision order, the OBDDs’ sizes may vary from linear to exponential in the number of the variables (Bryant 1986). Obviously, we prefer the decision order that minimizes the OBDD’s size.

However, the problem of finding the optimal order of an OBDD is NP-complete (Bollig and Wegener 1996). Many heuristics have been proposed to find a near-optimal solution of this problem. However, all existing techniques do not achieve a good balance between efficiency and effectiveness. Methods that can significantly reduce the OBDD’s size always take a long time. Methods that take advantages in speed always can’t achieve significant results. Although the problem is NP-complete, the Boolean formulas generated from real world (e.g. circuits, programs, etc.) do have some patterns. If we can utilize these patterns, it is possible to develop a technique that is both efficient and effective.

NNs have been applied to many areas, including computer vision, natural language processing, recommendation systems, etc. Surprisingly, there is no work for applying deep learning to the OBDD reordering problem. One possible reason is most NN frameworks are not suitable for learning on boolean formulas. For example, Recurrent Neural Networks (RNNs) (Medsker and Jain 1999) family can handle sequences learning problem. To apply RNNs, we must serialize Boolean formulas (usually in 3-conjunctive normal form (3-CNF)) into sequences. However, Boolean formula have rich invariances that such a sequential model would ignore, such as the permutation invariance of clauses (Selsam et al. 2018). For example, \((x_1 \land (x_2 \lor x_3))\) and \((x_2 \lor x_3) \land x_1\) are syntactically different but semantically equivalent. The sequential model will ignore the permutation invariance, take them as different input.

CNFs can be viewed as hypergraphs (Kolany 1993). If we can directly apply deep learning on hypergraph, the semantics of Boolean formula will not be wrecked. The message passing neural network (MPNN) framework (Gilmer et al. 2017) is a powerful deep learning technique for graphs. However, it cannot be utilized directly on hypergraph since the message function is defined on ordinary edges. In this paper, we lift the MPNN framework to 3-hypergraphs (MPNN\textsubscript{3}), define a message function on hyperedges. We implement the framework in gated graph neural network (GGNN). In ordinary graphs, GGNN models edges by square matrices, then uses matrix multiplication to generate message. We use non-square matrices to model hyperedges so that message can be generated and passed on hyperedges.

Compared with the existing techniques, MPNN\textsubscript{3} can give a near-optimal solution in an extremely short time, even for some hard examples that are unsolvable with the existing techniques. The main technical contributions of this paper are summarized as follows:

- We view OBDD reordering as deep learning on 3-hypergraph. To the best of our knowledge, this is the first...
neural network-based approach for OBDD reordering.

- Following the main idea of message passing, we lift the MPNN to 3-hypergraph.
- Experimental results show that our approach can find a near-optimal order in an extremely short time.

1.1 Related Work

Many OBDD reordering algorithms have been proposed in the literatures. (Fujita, Matsunaga, and Kakuda 1991) and (Ishia, Sawada, and Yajima 1991) propose the window permutation algorithm by exchanging a variable with its neighbor in the ordering. (Rudell 1993) proposes the sifting algorithm which finds the optimum position of a variable by repeatedly move it forward or backward. (Günther and Drechsler 1998) applies linear transformations to minimize OBDD’s size. (Bollig, Löbbing, and Wegener 1995) applies simulated annealing to find a near-optimal order. Instead of using swap or exchange operation, this method defines a jump operation. (Drechsler, Becker, and Gröckel 1996) uses the genetic algorithm to optimize the OBDD’s size. Among all these heuristics, the genetic algorithm and the simulated annealing algorithm attain the best results but are also the most time consuming. (Grumberg, Livne, and Markovitch 2003) uses decision tree to learn variable pair permutation which is more likely to lead to a good order. In contrast, our approach can directly produce the total order of all variable, not pairwise.

There exist some works on applying neural networks to the OBDD-related topics. (Beg, Prasad, and Beg 2008) applies feed-forward and recurrent neural network to predict the OBDD’s size for a given Boolean function. (Barlett and Andrews 2002) studies the problem of converting fault trees to OBDDs. They propose a neural network approach for selecting one among several existing heuristics to construct the OBDD. Their approach is essentially a heuristic selection mechanism, and heavily depends on the available heuristics. In contrast, our NN approach can directly produce the total order of all variable, not pairwise.

Figure 1 shows two BDDs, where the edges to low(v) and high(v) are marked as dotted and solid lines, respectively. A 3-CNF is a CNF where all clauses have three or less literals. Any Boolean formula can be transformed into an equisatisfiable 3-CNF formula (Tseitin 1983). In the remainder of this paper, we assume all Boolean formulas are in 3-CNF.

2.2 Binary Decision Diagrams

A binary decision diagram (BDD) is a rooted, directed acyclic graph $G = (V, E)$ with a node set $V$ and an edge set $E$. Two types of nodes are contained in $V$, i.e., the terminal nodes and the nonterminal nodes. A terminal node has no outgoing edge, and is labelled with either 0 or 1. A nonterminal node $v$ is labelled with a variable $x$ (called the decision variable at this node), and has two successors, low($v$) and high($v$), where low and high indicate the decided values of $x$ being 0 and 1, respectively. Figure 1 shows two BDDs, where the edges to low($v$) and high($v$) are marked as dotted and solid lines, respectively.

Let $G_f$ be a BDD of $f$. Let $\varphi$ be a truth assignment to $X$. We can easily decide if $\varphi$ satisfies $f$ by traversing $G_f$ from its root to one of its terminal nodes. Let $v$ be the current node. If the variable $x$ labelled by $v$ is assigned 0 in $\varphi$, the next node on the path is low($v$); otherwise, if the variable $x$ is assigned 1, the next node is high($v$). The value that labels the final reached terminal node gives the value of the function. Taking the left BDD in Figure 1 as an example, with a truth assignment of $x_0 = 0, x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ and $x_5 = 0$, the value of $f$ can be quickly decided to be 0 by traversing the graph.

Let $\prec$ be a total order on $X$. An ordered BDD (OBDD) with respect to $\prec$ is a BDD such that the decision order of variables on all paths of this OBDD follow $\prec$. A reduce algorithm (Bryant 1986) can be repeatedly applied, to eliminate the possible redundancies in an OBDD. The resulting structure is called a reduced OBDD. Note that both BDDs in Figure 2 are reduced OBDDs.

Reduced OBDD is a canonical representation for Boolean functions. Given any two logically equivalent Boolean functions, their reduced OBDDs with respect to a variable order are isomorphic (Bryant 1986). In this paper, we assume all OBDDs are reduced OBDDs.
2.3 Variable Reordering Problem

Given an OBDD \( G \), we denote the size of \( G \) by \( |G| \), i.e., the number of nodes in \( G \). The size of an OBDD is highly sensitive to its variable order. Consider a Boolean function \( f = x_0 x_1 \lor x_2 x_3 \lor x_4 x_5 \). If we choose a variable order of \( x_0 < x_1 < x_2 < x_3 < x_4 < x_5 \), the OBDD’s size is 8 (the left OBDD in Figure 1). In contrast, if we choose another variable order of \( x_0 < x_2 < x_4 < x_1 < x_3 < x_5 \), the corresponding OBDD’s size is 16 (the right OBDD in Figure 1).

In general, for a Boolean function of the form \( f = x_0 x_1 \lor x_2 x_3 \lor \ldots \lor x_{n-2} x_{n-1} \), with the variable order \( x_0 < x_1 < \ldots < x_{n-1} \), its OBDD’s size is \( n + 2 \); while with the variable order \( x_0 < x_2 < \ldots < x_{2k-2} < x_1 < x_3 < \ldots < x_{2k-1} \) (assume \( n = 2k \)), its OBDD’s size becomes \( 2^{\frac{n}{8} + 1} \).

In other words, with respect to different variable orders, the OBDD’s size of a Boolean function may vary from linear to exponential in the number of variables (Bryant 1986).

The OBDD reordering problem is to find an optimal variable order for a given Boolean function, such that its OBDD’s size is minimal. This problem has been proved NP-complete (Bollig and Wegener 1996). Many heuristics (Fujita, Matsumaga, and Kakuda 1991; Ishiura, Sawada, and Tajima 1991; Rudell 1993; Drechsler, Becker, and Göckel 1996; Bollig, Löbbing, and Wegener 1995) have been proposed to find a near-optimal solution for this problem. Among all these heuristics, the genetic algorithm and simulated annealing algorithm often attain the best results (Drechsler, Becker, and Göckel 1996; Bollig, Löbbing, and Wegener 1995). However, both these two algorithms are quite time consuming. We seek for a reordering algorithm that can not only find a near-optimal solution, but also be time efficient.

2.4 Graph and Hypergraph

A graph \( G = (V, E) \) is defined by a set of vertices (also called nodes) \( V = \{1, 2, \ldots, |V|\} \) and a set of edge \( E \subset V \times V \) which defines the relation between nodes. A hypergraph \( H = (V, \tilde{E}) \) is a generalization of a graph in which an edge can connect more than two vertices, and thus \( \tilde{E} \subset \mathcal{P}(V) \), where \( \mathcal{P} \) means power set. The k-uniform hypergraph is a hypergraph such that all its hyperedges have exactly \( k \) nodes. We use k-hypergraph to represent the set of all k-uniform hypergraph. In this paper, we consider only 3-hyperedges i.e. \( \tilde{E} \subset V^3 \).

Let \( v \) be a node in a graph, the neighbors of \( v \) is the set of nodes that points to (or, passes messages to) \( v \), formally:

\[
NBR(v) = \{ i | (i, v) \in E \}
\]

In the next subsection, we will discuss how \( NBR(v) \) passes message to \( v \). But now, we need to lift the definition of \( NBR \) to hypergraph. We lift the idea of neighbors into left/right neighbors, which means each node can get message from both side in a hyperedge. Formally:

\[
NBR_H(v) = \{(l, r) | (l, v, r) \in \tilde{E} \} \cup \\
\{(l, r) | (l, r, v) \in \tilde{E} \} \\
\{(l, r) | (v, l, r) \in \tilde{E} \}
\]

The task of machine learning on graph domain can be either graph-level or node-level. In graph-level, a graph \( G \) is mapped to a vector of reals \( \tau(G) \in \mathbb{R}^m \). In node-level, \( \tau \) depends on a node \( n \) of \( G \), i.e. \( \tau(G, n) \in \mathbb{R}^m \). For example, compute the size of graph is a graph-level task, compute the degree of vertex is a node-level task.

2.5 The Message Passing Framework

The Message Passing Neural Network (MPNN) (Gilmer et al. 2017) is a general framework for supervised learning on graphs. It is originally a graph-level prediction framework for chemical compound, we slightly modify it into node-level prediction. The main idea of message passing is to embed each node into vector space, then iteratively refine the embedding. In an iteration, each node receives messages from its neighbors and updates its embedding accordingly.

In this paper, we also call embedding of node as state.

Let \( h^i_v \) be the embedding of node \( v \) at time \( t \), \( E_{iv} \) be the embedding of edge \( (i, v) \), and \( a_v \) be a handcrafted feature of \( v \). The \( h^0_v \) is initialized by the zero-padding of \( a_v \). Formally, message passing is defined by message function \( M_i \) and vertex update function \( U_i \).

\[
m^{t+1}_v = \lambda * \sum_{i \in NBR(v)} M_i(h^t_i, E_{iv}, h^t_v) \\
h^{t+1}_v = U_i(h^t_v, m^{t+1}_v)
\]

where \( M_i(h^t_i, E_{iv}, h^t_v) \) is the message \( i \) sent to \( v \), The \( \lambda \) can be \( 1 \) or \( 1/|NBR(v)| \) for different message aggregation strategies. All the incoming messages of \( v \) will be aggregated by average if \( \lambda = 1/|NBR(v)| \). For \( \lambda = 1 \), the messages is aggregated by suming up. After the message passing, we read out each prediction \( y_v \) of node \( v \), from its final refined embedding \( h^T_v \) and handcrafted feature \( a_v \).

\[
y_v = O(h^T_v, a_v)
\]

We collect the prediction of all node \( y = (y_1, y_2, \ldots, y|V|) \) as the output of neural network.

Notice that the \( M, U, O \) are all undefined by now. While the MPNN is a framework, each design of \( M, U, O \) defines a concrete Neural Network. For example, Gated Graph Neural Networks (GGNN) (Li et al. 2016), Deep Tensor Neural Networks (DTNN) (Schütt et al. 2017) are all instance of MPNN, which define two different \( M, U, O \). In fact, The MPNN originally came from the abstraction of at least eight notable NNs that operate on graphs. Our work of lifting message passing is on the message function \( M_i \). For implementation, we will use GGNN as the instance of MNPP in this paper.
In GGNN, the embedding of node is in vector space $h^t_v \in \mathbb{R}^h$, the embedding of edge is in matrix space $E_{ij}^t \in \mathbb{R}^{h \times h}$. We use matrix $A_k$ to model the $k^{th}$ type of edges. The parameter of $A_k$ is learned from training of neural network. Let $\gamma$ be the type of edge $E_{ij}$, the embedding of edge of determined by its type $A_{\gamma}$. 

The message function is designed as matrix multiplication $M_{ij}(h^t_v, E_{ij}, h^t_{\gamma}) = E_{ij}A_{\gamma}h^t_v$. The update function is $U_{ij}(h^t_v, m_{ij}^{t+1}) = \text{GRU}(h^t_v, m_{ij}^{t+1})$, where GRU is the Gated Recurrent Unit introduced in (Cho et al. 2014). The same update function is used at each time step $t$. Finally $O(h^t_v, x_v) = g_{O}(h^t_v, x_v)$, where $g_{O}$ is a fully connected neural network, and $[\cdot \! , \cdot]$ means concatenation of vectors.

Several GGNN can be composed successively as several layers (Li et al. 2016) in a way that the output (i.e. final state) of the current message passing process is used as the initial state of the next message passing process. In each layer (i.e. message passing process), the message passing is repeated for several times, with the same parameters of NN. But different layers have different parameters. We denote $h^t_v(k,l)$ as the state of node $v$ on the $t^{th}$ timestep in the $k^{th}$ layer, and $T_k$ as the number of timesteps in the $k^{th}$ layer. The layered GGNN can be formalized as $h^t_v(k,l+1) = h^t_v(k,T_k)$.

The idea of residual connection (i.e. skipping over layers) (He et al. 2016) can also be introduced into the connection of GGNN layers. The incoming message of each node can be concatenated to the final state of several previous layer before that is fed into $U$. For example, the message of each node $v$ in the $4^{th}$ GGNN layer, can be concatenated to the final state of $0^{th}$ and $2^{nd}$ layers.

$h^t_v(4,l+1) = U_{4,l}(h^t_v(4,l), [m^t_v(4,l+1), h^t_v(0,T_0), h^t_v(2,T_2)])$

The residual connection is used to reduce the problem of vanishing gradients in backpropagation.

## 3 OBDD Reordering as DL on Hypergraph

Neural network (NN) has been proven a powerful machine learning technique for nonlinear data-fitting problem (Hornik, Stinchcombe, and White 1989). In this section, we show how an OBDD problem can be reduced to a deep learning problem on 3-uniform hypergraph. We utilize NN to learn the patterns of “good” OBDD variable orders from real-world example. After the training phase, NN can predict a good variable order for a given 3-CNF formula in a short time.

### 3.1 Inputs

The input of neural network is a 3-Hypergraph. The labels on hyperedges are finite and discrete, we call it types just like what we did in normal graph. Each 3-CNF is converted to a 3-Hypergraph. Let $X = \{x_1, x_2, \ldots, x_N\}$ be the variable set of a given CNF. The vertex set $V$ of the converted hypergraph $H = (V, E)$ is $V = X \cup \{x_{\perp}\}$, where the $x_{\perp}$ is a special node that represents $False$. Each clause in 3-CNF is converted to a hyperedge directly from it’s variables. The type of each hyperedge is decided by the type of each literal (i.e. $+,-$). Especially, the type of literal $False$ is 0. For example: the clause $x_{325} \lor x_{174} \lor x_{299}$ is converted to the hyperedge $(x_{325}, x_{174}, x_{299})$ with the type $+,-,$ and the clause $x_{5} \lor x_{7}$ is converted to the hyperedge $(x_{5}, x_{7}, x_{\perp})$ with the type $-+0$. For simplicity, we use $(a,b,c)$ to represent the hyperedge $(x_{a}, x_{b}, x_{c})$. 

To start the message passing, each node $v$ needs a handcrafted feature $a_v$, to initialize $h^0_v$. We sort those variables primarily by the frequencies of occurrence, secondarily by the frequency of positive literal if variables appear same times. Lastly we use lexicographic order of variable name if they are still same. We use the position of $v$ in the sorting order to construct an one-hot vector as $a_v$. If $v$ is the $i^{th}$ variable, the $i^{th}$ element of $a_v$ is 1, other elements are 0s. For $a_{\perp}$, we use zero vector to initialize. Let us take $(x_1 \lor x_2)(x_1 \lor x_2 \lor x_3)$ as an example, we use $a_{\perp} = (0,0,0)$, $a_1 = (0,1,0)$, $a_2 = (1,0,0)$, $a_3 = (0,0,1)$ as handcrafted features, use zero-padding $h^0_v = (a_v,0,\cdots) \in \mathbb{R}^h$ to initialize $h^0_v$. This encoding method ensures $h_v$ almost independent of the name of Boolean variable $v$.

It should be noted that, the hypergraph is only converted from real-world example. After the training phase, NN can predict a good variable order for a given 3-CNF formula in a short time.

### 3.2 Outputs

Outputs of the OBDD reordering problem are variable orders. We want the neural network to find a near-optimal order in a short time.

A variable order can be specified as a permutation of variables. For example, the variable orders of the two OBDDs in Figure 1 are $x_0 x_1 x_2 x_3 x_4 x_5$ and $x_0 x_2 x_4 x_1 x_3 x_5$, respectively. However, the variable permutation is not a proper format of the neural network’s output. Generally, a neural network requires its output to be a differentiable structure such that the gradient descent algorithm can work on (Rumelhart, Hinton, and Williams 1986).

To this end, we let the output of the OBDD reordering problem to be a vector of real numbers, called the depth vector. Formally, given a variable $x \in X$, denote $\text{Depth}(x) \in \mathbb{R}$. 

![](image.png)
For example, a depth vector of the right OBDD in Figure 1 is (1.7, 6.7, 3.3, 8.3, 5.0, 1.0). The less the depth value is, the more front the corresponding variable in the order. With the above depth vector, we can quickly decide the variable order: $x_0 \prec x_2 \prec x_4 \prec x_1 \prec x_3 \prec x_5$.

### 3.3 Loss Function

After the definition of input and output, we also need a suitable loss function for our task. Since the final order computed from the final depth vector is only related to the order of the values, but not the detailed values in the vector. We use the angle $\theta$ of the predicted vector to the expected vector to measure the error, i.e.,

$$
\text{loss}(y, y^*) = \theta(y, y^*) = \arccos\left( \frac{y \cdot y^*}{\|y\| \|y^*\|} \right) \cdot \frac{180^\circ}{\pi}
$$

where $y$ is the prediction of NN, $y^*$ is the target vector, and $\|\cdot\|$ means 2-norm. Notice that, each element $y^*$ is the near-optimal OBDD depth of corresponding variable (i.e. node in hypergraph). We have already convert the OBDD reordering task into node-level learning task on hypergraph. We don’t care the state of whole hypergraph but do focus on the prediction of each node (i.e. depth of variable).

### 4 Neural Network for 3-Hypergraph

In this section, we discuss how to generate messages on hyperedges. Firstly we define a message function $\mathcal{M}$ on hyperedges, then introduce non-square matrices to model each edge type from massive data. Following this idea, the first thing we need to do, is to design a form of message function which can generate message on 3-hyperedges. To achieve this goal, message function $\mathcal{M}$ should be defined on hyperedges:

$$
\hat{m}^{t+1}_v = \hat{\lambda} \sum_{(i, j) \in \text{NBR}_H(v)} \mathcal{M}_t(h^t_i, E_{i \leftarrow j}, h^t_j, h^t_v)
$$

where $\hat{\lambda}$ can be either 1 or $1/|\text{NBR}_H(v)|$. What needs to be emphasized is that, the modification of message function is the only modification of the MPNN framework. The update function $U$ and the readout function $\mathcal{O}$ all remain the same.

Our motivation is to keep the framework of message pass unchanged, but lift the message generation on hyperedges.

### 4.2 Hyperedge Message Functions

We have already extended the framework of MPNN into hypergraph, now we discuss how to implement the message functions in GGNN. The key idea of GGNN is to use square matrices to model ordinary edges. Each edge type is modeled by a matrix $A_k \in \mathbb{R}^{h \times h}$. Finally GGNN uses a matrix multiplication $E_{i \leftarrow j} h^t_i$ to implement the message function and generate messages. The $A_k$ can be seen as a mapping from node state to message: $A_k : \mathbb{R}^h \to \mathbb{R}^h$. We need to lift the mapping into $\hat{A}_k : \mathbb{R}^{2h^2} \to \mathbb{R}^h$ since one node gets two neighbors in a hyperedge. So we lift the square matrix $A_k \in \mathbb{R}^{h \times h}$ into non-square matrix $\hat{A}_k \in \mathbb{R}^{h \times 2h^2}$, and use $\hat{E}_{i \leftarrow j} h^t_i$ to implement message functions:

$$
\mathcal{M}_t(h^t_i, E_{i \leftarrow j}, h^t_j, h^t_v) = \hat{E}_{i \leftarrow j} \left[ h^t_i \right] \in \mathbb{R}^h
$$

Now we have a method to generate pass messages on hypergraphs. We also find a way to implement the MPNN on the top of existing MPNN.

### 4.3 Implementation on The Top of MPNN

Since the matrix can be partitioned into blocks. The hyperedge message function $\mathcal{M}$ can also be rewritten as block matrix multiplication. Following this idea, we surprisingly found that the $\hat{M}$ can be reduced to two ordinary message passings on ordinary graph. This makes it possible to implement MPNN on the top of existing MPNN. The key is to decompose the hyper message passing. Firstly we divide the matrix into 2 blocks, i.e. $\hat{E}_{i \leftarrow j} = [E_{i \leftarrow v}, E_{i \leftarrow v}]$, then:

$$
\hat{m}^{t+1}_v = \hat{\lambda} \sum_{(i, j) \in \text{NBR}_H(v)} \mathcal{M}_t(h^t_i, E_{i \leftarrow j}, h^t_j, h^t_v)
$$

We call a graph $G_H = (V, E)$ a derived graph of $H = (V, \hat{E})$ when

$$
E = \{(i, j) | (i, j, k) \in \hat{E}\} \cup \{(j, k) | (i, j, k) \in \hat{E}\} \cup \{(k, i) | (i, j, k) \in \hat{E}\}
$$

and denote $G^{-1}_H = (V, E^{-1})$ as the reverse graph of $G_H$ where $E^{-1} = \{(j, i) | (i, j) \in E\}$. We get that

$$
\hat{\lambda} \sum_{i \in \text{NBR}_L(v)} E_{i \leftarrow v} h^t_i = \overline{\hat{m}}^{t+1}_v
$$

$$
\hat{\lambda} \sum_{j \in \text{NBR}_R(v)} E_{j \leftarrow v} h^t_j = \overline{\hat{m}}^{t+1}_v
$$

where $\overline{\hat{m}}^{t+1}_v$ is the message of $v$ on $G_H$, $\overline{\hat{m}}^{t+1}_v$ is the message of $v$ on $G^{-1}_H$. Finally we get

$$
\hat{m}^{t+1}_v = \overline{\hat{m}}^{t+1}_v + \overline{\hat{m}}^{t+1}_v
$$

which means that the MPNN can be used to implement the message passing of hypergraph by decomposing a hypergraph into the derived graph and it’s reverse.
5 Implementation and Evaluation

We implement our algorithm on the top of TensorFlow [Abadi et al. 2016], and used ADAM [Kingma and Ba 2015] for the learning rate control. All experiments were performed on GeForce GTX 1080 Ti GPU and an Intel Xeon E5 CPU.

To evaluate the efficiency of our approach, up to 7 typical OBDD reordering algorithms are compared:

- **MPNN**: our neural network approach;
- **GA**: the genetic algorithm (Drechsler, Becker, and Göckel 1996) for OBDD reordering;
- **LINEAR**: a combination of sifting variable up and down and linear transformations of two adjacent variables (Günther and Drechsler 1998);
- **RAND**: randomly choose pairs of variables and swap them in the order (Somenzi 2015);
- **G-SIFT**: the group sifting method (Panda and Somenzi 1995);
- **WIN2**: the iterating window algorithm (Felt et al. 1993), with its window size been set to 2.
- **WIN3**: also the (Felt et al. 1993), but window size is 3.

In MPNN, we embed each node to 500-dimensional vector space. We have 5 GGNN layers, the each layer correspondingly propagate 2, 2, 1, 2, 1 times. The 2\textsuperscript{rd} layer has residual connections from 0\textsuperscript{th} layer, and the 4\textsuperscript{rd} layer has residual connections from both 0\textsuperscript{th} and 2\textsuperscript{th} layer. We use average function to do message aggregation.

5.1 Data Set and Benchmark

We choose the LGSynth91 benchmark (Yang 1991) as our data set. We collect the circuits in Berkeley Logic Interchange Format (blif) (Berkeley 1992) format, convert them to And-Inverter Graph (aig) (Biere 2007) format, extract the transition relation boolean formula into equisatisfiable 3-CNF. Note that the genetic algorithm (Drechsler, Becker, and Göckel 1996) attains the best result among all OBDD reordering algorithms. For each sample, we run the genetic algorithm to compute the near-optimal variable order, and use this order as the label. We set a time-out of 30 minutes, being all counted. There are 28 samples that can finish GA labeling in time limit, and call them the hard benchmark. The samples in the hard benchmark are all challenging enough for OBDD. We are very curious about the performance of MPNN\textsubscript{3} on the hard benchmark.

5.2 Results on Time

To evaluate the efficiency of those algorithm, we compare their computation time of giving a result of near-optimal order. We only consider the time of perform algorithms, the time of building initial OBDD is not included. The result of average computation time (seconds) is in Table 1.

| Algorithm | GA | LINEAR | MPNN\textsubscript{3} | G-SIFT | RAND | WIN3 | WIN2 |
|-----------|----|--------|----------------|--------|------|------|------|
| Time(sec) | 43.50 | 12.29 | 0.01 | 12.92 | 9.65 | 0.58 | 0.24 |

Table 1: Average Computation Time

The WIN2 and WIN3 are quite efficient among the traditional methods. The GA takes longest time to give a best result. The RAND makes balance between compression ratio and time. However, MPNN\textsubscript{3} is the fastest algorithm. We go further and fit a curve of time for each algorithm in Figure 3. The horizontal axis lists the sizes of the input OBDDs, the vertical axis shows the average computation time of different reordering algorithms. Note that the vertical axis is logarithmic.

5.3 Results on Compression Ratio

To evaluate the accuracy of reordering algorithms, we compare their compression ratios.

![Figure 3: Curve of OBDD Size and Computation Time](image)

We observe that GA slows down quickly with the increasing of OBDD’s size. In contrast, MPNN\textsubscript{3} is not sensitive to the size of the input OBDD. Recall that the inputs of the MPNN\textsubscript{3} are CNFs, instead of OBDDs. To conclude, our approach can get a near-optimal variable order in a short time. But will such a fast speed of MPNN\textsubscript{3} affect the quality of its solution?
Given a Boolean function $f$ and a variable order, we denote the original OBDD by $G$. All reordering algorithms are respectively applied to $f$ to produce a new variable order. The OBDD with respect to the new order is denoted by $G'$. In the experiments, we use CUDD (Somenzi 2015) to evaluate the corresponding OBDD’s size. If $|G'| < |G|$, we adopt the new order. Let $A$ be a reordering algorithm, we use $\eta_A = (|G| - |G'|)/|G|$ to measure the compression ratio. The average compression ratios of each algorithm on test set are shown in Figure 4. The horizontal axis indicates 7 algorithms and the vertical axis shows their compression ratios. From Figure 4, observe that GA always gets the best compression ratio. This conforms to the existing results in literatures (Drechsler, Becker, and Göckel 1996; Jindal and Bansal 2017). The MPNN$_3$ achieves $3^{rd}$ best result, and the results of the top-4 algorithms are close. To see more details of those samples, we fit a curve of compression ratio for each algorithm in Figure 5. The horizontal axis lists the sizes of the input OBDDs, and the vertical axis shows the average compression ratio of different reordering algorithms. Note that the horizontal axis is logarithmic. We find that smaller OBDDs are harder to be compressed. This is understandable, since the difference between linear and exponential OBDD size is smaller when the number of variables is smaller. The curves of top-4 algorithms are close and GA is always better than other algorithms. The WIN2 and WIN3 is not such effective but quick. How MPNN$_3$ performs in hard benchmark? we will talk it in next subsection.

### 5.4 Results on Stress Test

It is challenging for BDD-method in large circuits. We set the timeout for 12 hours, and give 110GB memory for each samples. Firstly, there are 46.2% of hard benchmark cannot even build an initial OBDD, we call them very-hard benchmark for simplicity. 50% of very-hard benchmark are out of time for 12 hours (OOT), others are out of memory for 110GB (OOM). The traditional methods are performed on the initial OBDD, so they are failed on those task. However, recall that the prediction of MPNN$_3$ doesn’t rely on the initial OBDD. We directly use the order of MPNN$_3$ to build OBDD. 41.7% samples in very-hard benchmark can build the OBDD using the order of MPNN$_3$, others are all OOT, not OOM, which means it still has some possibility for them to build OBDDs if we give more time.

For the rest of hard benchmark, which traditional method can be performed, we compare MPNN$_3$ with Win2, Win3, Rand. There are 2 samples OOT for Rand, we lists some results in Table 2.

| Name     | Vars | Nodes | MPNN$_3$ | WIN2 | WIN3 | RAND |
|----------|------|-------|----------|------|------|------|
| cordic   | 106  | 9M    | 99% 0.01 | 14    | 17   | 47%  1701 |
| s298     | 133  | 2M    | 80% 0.01 | 14    | 16   | 47%  122  |
| s344     | 144  | 41M   | 97% 0.01 | 24    | 71   | 34%  11194|
| s349     | 148  | 47M   | 90% 0.01 | 25    | 73   | 36%  6658 |
| mux      | 153  | 10M   | 85% 0.01 | 30    | 9    | 47%  537  |
| sct      | 159  | 3M    | 83% 0.01 | 17    | 2    | 53%  61   |
| lal      | 164  | 219M  | 99.7% 0.01| 14    | 444  | 47%  1807 |
| s382     | 185  | 12M   | 52% 0.01 | 14    | 13   | 64%  1232 |
| s386     | 185  | 0.5M  | 40% 0.01 | 12    | 0.4  | 20%  17   |
| s400     | 193  | 13M   | 62% 0.01 | 43    | 12   | 63%  2580 |
| s444     | 193  | 11M   | 89% 0.01 | 16    | 17   | 18%  1123 |
| s420     | 210  | 43M   | 93% 0.01 | 27    | 69   | 43%  5732 |
| s510     | 244  | 7M    | 99% 0.01 | 38    | 5    | 41%  438  |
| s526     | 248  | 594M  | 92% 0.01 | 10    | 1433 | 52%  5251 |

The first column is the name of samples, the second column is the number of variables. The third column is the size of initial OBDD, where the ‘M’ means million($10^6$). Others are result of each algorithm. The first column of each algorithm result is the compression ratio, the second column is time in seconds. The result shows that MPNN$_3$ achieves a very good result in the stress test, totally beats WIN2 and beats WIN3 in most case. The compression ratios of MPNN$_3$ is also competitive to RAND, with 2 case can not finish measure using RAND in 12 hours. The speed of MPNN$_3$ is extremely fast.

### 6 Conclusions

In this paper, we apply MPNN$_3$ to minimize OBDDs, lift the neural message passing on 3-hypergraph to recieve 3-CNF as input. We perform experiments to compare our approach to classical algorithms on variable reordering of OBDDs. Experimental results show that our approach can get competitive compression ratio in an extremely short time. There are many complex relationships in real world can be modeled by hypergraphs. In the future, we plan to apply MPNN$_3$ to more fields.
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