Gutkin billiard tables in higher dimensions and rigidity

Misha Bialy

School of Mathematical Sciences, Tel Aviv University, Israel
E-mail: bialy@post.tau.ac.il

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Abstract

Gutkin found a remarkable class of convex billiard tables in a plane that has a constant angle invariant curve. In this paper we prove that in dimension 3 only a round sphere has such a property. For dimensions greater than 3, a hypersurface with a Gutkin property different from a round sphere, if it exists, must be of constant width and, moreover, it must have very special geometric properties. In the 2D case we prove a rigidity result for Gutkin billiard tables. This is done with the help of a new generating function introduced recently for billiards in our joint paper with Mironov. In the present paper a formula for the generating function in higher dimensions is found.

Keywords: Birkhoff billiards, geodesics, bodies of constant width

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(Some figures may appear in colour only in the online journal)

1. Introduction and main results

Consider a convex compact domain in Euclidean space $\mathbb{R}^d$ bounded by a smooth hypersurface $S$ with positive principal curvatures everywhere. We shall call $S$ a Gutkin billiard table if there exists $\delta \in (0, \pi/2)$ such that for any pair of points $p, q \in S$ the following condition is satisfied: if the angle between the vector $\overrightarrow{pq}$ with the tangent hyperplane to $S$ at $p$ equals $\delta$, then the angle between $\overrightarrow{pq}$ and the tangent hyperplane at $q$ also equals $\delta$.

Notice that the case $\delta = \pi/2$ is classical and corresponds to bodies of constant width.

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Planar billiard tables with this property were found and studied in detail by Gutkin [8, 9] (see also [12]). He discovered that planar domains with this property, which are different from round discs, can exist only for those values of $\delta$ that satisfy for some integer $n > 3$ the equation
\[
\tan(n\delta) = n\tan(\delta).
\]
(1)

It is a conjecture of Gutkin that for any $\delta \in (0; \pi/2)$ at most one integer $n > 3$ can satisfy (1). Moreover, the shape of these domains is also very special. Namely, let $\rho(\phi)$ be the curvature radius as a function of the tangent angle $\phi$. Then the Fourier coefficients $c_k$ of $\rho$ all vanish for $k$ different from $n$ in (1). For example, if $\delta$ satisfies (1) for $n = 5$ then function $\rho(\phi) = a_0 + a_5 \cos 5\phi$ is an example of the radius of curvature function of a Gutkin billiard table. Notice that in this example the domain is also of constant width, so for a billiard ball map there are two constant angle invariant curves: one for the angle $\delta$ and another for the angle $\pi/2$.

It turns out that the property of equal angles becomes very rigid in higher dimensions.

**Theorem 1.1.** The only Gutkin billiard tables in $\mathbb{R}^3$ are round spheres.

Let us formulate now the result for the case $d > 3$. We have the following alternative.

**Theorem 1.2.** Let $S$ be a Gutkin billiard table in $\mathbb{R}^d$, $d > 3$, corresponding to an angle $\delta \in (0; \pi/2)$.

1. If $\delta$ is not a solution of equation (1) for an odd $n$, then $S$ is a round sphere.
2. If $\delta$ is a solution of equation (1) for some odd $n$, then $S$ is necessarily of constant width.

Moreover, every geodesic curve on $S$ that is tangent to a principal direction at some point of $S$ lies in a 2-plane and defines on this plane a 2D Gutkin billiard table.

**Remark 1.** It is plausible that equation (1) is in fact irrelevant in higher dimensions, i.e. $S$ must be a sphere also in case 2 of theorem 1.2. However we were not able to prove this.

The following discussion is important. It was proved in [5, 7] that for higher dimensional billiards only ellipsoids have convex caustics. A less restrictive object of dynamical importance would be an invariant hypersurface in the phase space. Obviously, if there is a convex caustic for a convex billiard table then the set of oriented lines tangent to the caustic form an invariant hypersurface in the phase space of all oriented lines. There are no examples, however, of invariant hypersurfaces besides ellipsoids. Gutkin billiards would provide an example of such a hypersurface, but theorems 1.1 and 1.2 tell us that they are very rare. It would be interesting to have an example and study further properties of invariant hypersurfaces for higher dimensional Birkhoff billiards.
Our next result on Gutkin billiard tables deals with the question of their integrability in the planar case.

More precisely, we examine the so-called total integrability in a strip between two neighboring invariant curves, which we now turn to explain. The term total integrability was suggested in [10] for geodesic flows. Consider a Gutkin billiard table $S$ corresponding to the angle $\delta$. Then $\delta$ is one of the solutions of equation (1), while (1) has as many as $\lfloor \frac{2}{\pi} \rfloor$ solutions in $[0, \frac{\pi}{2})$. Every solution $\delta_i$ corresponds to a constant angle invariant curve on the phase cylinder. Let us denote by $\Omega_{\delta_1, \delta_2}$ the strip between the two neighboring constant angle invariant curves on the cylinder. We shall say that the billiard is totally integrable in the strip $\Omega_{\delta_1, \delta_2}$ if the whole strip is foliated by rotational invariant curves (see figure 1).

**Theorem 1.3.** If Gutkin billiard table $S$ is totally integrable in the strip $\Omega_{\delta_1, \delta_2}$ between two neighboring invariant curves then $S$ is a circle.

Some discussion is in order. Total integrability, and more generally Hopf type rigidity for billiards, was found in [1] and by another method in [13] where the assumption was that rotational invariant curves occupy all (or almost all) phase space. In [2] a qualitative version of Hopf rigidity is obtained. One would like to relax the conditions of Hopf rigidity in some way. Theorem 1.3 gives a rigidity result on total integrability for Gutkin billiards on a strip between two invariant curves.

For the proof of theorem 1.3 we use a new generating function for convex billiards invented in [4]. We also discuss the formula for this function for the higher dimensional case. It turns out that for ellipsoids it coincides with the one found by Suris in [11].

In section 2 we use the symplectic nature of the problem and show a link between a billiard ball map and geodesics on the surface. Then we prove their planarity. In section 3 we prove theorems 1.1 and 1.2. Section 4 contains the proof of theorem 1.3. In section 5 we discuss the generating function in a higher dimensional case.

2. Higher dimensional Gutkin billiard tables

2.1. Symplectic properties

The proof of theorem 1.1 requires symplectic properties of billiards. Consider a Birkhoff billiard inside hypersurface $S$. The phase space $\Omega$ of the billiard consists of the set of oriented lines intersecting $S$. The space of oriented lines in $\mathbb{R}^d$ is isomorphic to $T^*S^{d-1}$ and hence carries natural symplectic structure. The Birkhoff billiard map acts on the space of oriented lines and preserves this structure. Another way to describe the same symplectic structure is the following. Every oriented line $l$ intersecting $S$ at $p$ corresponds to a unit vector with foot point $p$ on $S$. Orthogonal projection onto the tangent space $T_pS$ maps in a $1$–$1$ way the hemisphere of unit vectors with foot point $p$ on $S$ onto the unit ball of the tangent space $T_pS$. Thus the phase space of oriented lines intersecting $S$ is isomorphic to a unit (co-)ball bundle of $S$. The canonical symplectic form of this bundle coincides with that defined above. Here and later we identify co-vectors with vectors by means of the scalar product induced from $\mathbb{R}^d$.

Using these preliminaries, the main observation of this subsection is the following. The fact that $S$ corresponds to a Gutkin billiard table with the angle $\delta$ is equivalent to the fact that the hypersurface $\Sigma_\delta$ of the phase space $\Omega$ defined by

$$\Sigma_\delta = \{ (p, v) \in \Omega : p \in S, v \in T_pS, |v| = \cos \delta \}$$

is invariant under the billiard ball map. As a corollary we get the following.
Theorem 2.1. Given a Gutkin billiard table in $\mathbb{R}^d$, the billiard ball map transforms every characteristic of $\Sigma_\delta$ to a characteristic.

2.2. Differential geometric interpretation

Now notice that, since $\Sigma_\delta$ is a bundle of tangent vectors of constant length, then characteristics of $\Sigma_\delta$ are geodesics on $S$ equipped with their tangent vectors of the length $\cos \delta$. Thus the following differential geometric interpretation can be concluded from theorem 2.1.

Given a Gutkin billiard table $S \subset \mathbb{R}^d$, we shall denote by $n(p)$ the unit inner normal vector to $S$ at $p$. Let $\gamma$ be a geodesic curve on $S$. Denote by $s$ the arc length parameter on $\gamma$. Let $z(s)$ be the unit vector

$$z(s) = \cos \delta \; \dot{\gamma}(s) + \sin \delta \; n(\gamma(s)).$$

Consider the straight line segment $[\gamma(s); \Gamma(s)]$ such that $\Gamma(s)$ belongs to $S$ and

$$\Gamma(s) - \gamma(s) = l(s) z(s),$$

where $l(s)$ is the length of the segment $[\gamma(s); \Gamma(s)]$ (see figure 2).

Theorem 2.2. In the above notations, the curve $\Gamma$ is a geodesic on $S$ with regular parameter $s$ (not necessarily proportional to arc-length). Moreover the following two properties are valid:

1. The vectors $\{z(s), \dot{\Gamma}(s), \ddot{\Gamma}(s)\}$ belong to a 2-plane.
2. The angle between $\dot{\Gamma}(s)$ and $z(s)$ equals precisely $\delta$.

2.3. Deviation from the osculating 2-plane

Notice that since all principal curvatures of $S$ are assumed to be strictly positive, for any geodesic $\gamma$ on $S$ the curvature $k$ of $\gamma$ in $\mathbb{R}^d$ is strictly positive. Therefore we can write the first three Frenet equations for $\gamma$ as follows. Denote $v(s) = \dot{\gamma}(s)$, so

$$\dot{v}(s) = k(s) n(s).$$

(2)
Next, \( \dot{n} \) must be orthogonal to \( n \) and so belongs to the tangent space to \( S \). Hence it can be written as
\[
\dot{n}(s) = x \cdot v(s) + \tau(s)w(s),
\]
where \( w \) is a unit vector in \( \mathbb{R}^d \) orthogonal to \( \text{Span}\{v, n\} \). Differentiating \( \langle v, n \rangle \) along \( \gamma \) we get \( x \equiv -k \). So
\[
\dot{n}(s) = -k(s)v(s) + \tau(s)w(s).
\]
In a similar way
\[
\frac{d}{ds} \langle w, v \rangle = 0 = \langle \dot{w}, v \rangle + k \langle w, n \rangle = \langle \dot{w}, v \rangle,
\]
and hence \( \dot{w} \) is orthogonal to \( v \) and also to \( w \). Therefore we can write:
\[
\dot{w}(s) = -\tau(s)n(s) + \hat{w},
\]
where \( \hat{w} \) is orthogonal to \( \text{Span}\{v, n, w\} \).

Notice, that if \( d = 3 \) then \( w \) is just a bi-normal vector of \( \gamma \), \( \hat{w} \equiv 0 \) and (2), (4) and (5) are usual Frenet equations, where \( \tau \) is torsion of \( \gamma \). It is important that also in higher dimensions one concludes from (4) that the function \( \tau \) vanishes iff the curve \( \gamma \) lies in a 2-plane.

Now we are in position to give the interpretation of conditions 1 and 2 of theorem 2.2. The second condition is easy. Using Frenet equations (2) and (4) we compute:
\[
\dot{\Gamma} = \dot{\gamma} + l(\cos \delta v + \sin \delta n) + l(\cos \delta kn + \sin \delta (kv + \tau w))
\]
\[
= (1 + \dot{l}\cos \delta - kl\sin \delta)v + (\dot{l}\sin \delta + kl\cos \delta)n + (\tau l\sin \delta)w.
\]
Using (6) we compute
\[
|\dot{\Gamma}|^2 = (\dot{l} + \cos \delta)^2 + (kl - \sin \delta)^2 + \tau^2l^2\sin^2 \delta,
\]
and also
\[
\langle \dot{\Gamma}, z \rangle = \cos \delta(1 + \dot{l}\cos \delta - kl\sin \delta) + \sin \delta(\dot{l}\sin \delta + kl\cos \delta) = \dot{l} + \cos \delta.
\]
Condition 2 of theorem 2.2 reads:
\[
\langle \dot{\Gamma}, z \rangle = |\dot{\Gamma}| \cos \delta.
\]
It takes the form:
\[
\dot{l} + \cos \delta = \cos \delta \sqrt{(\dot{l} + \cos \delta)^2 + (kl - \sin \delta)^2 + \tau^2l^2\sin^2 \delta}.
\]
Simplifying (10) we have:
\[
\dot{l} + \cos \delta = \frac{\cos \delta}{\sin \delta} \sqrt{(kl - \sin \delta)^2 + \tau^2l^2\sin^2 \delta}.
\]
Since \( \dot{\Gamma} \) cannot vanish (\( s \) is a regular parameter on \( \Gamma \)) we can conclude from (11) the following.

**Proposition 2.3.** Terms \( (kl - \sin \delta) \) and \( \tau \) do not vanish simultaneously.

**Proof.** Indeed, if both \( (kl - \sin \delta) \) and \( \tau \) vanish at some point, then by (11) at this point \( \dot{l} + \cos \delta = 0 \) and hence by (8) and (9) \( \dot{\Gamma} = 0 \) contradicting the regularity of \( \Gamma \).
2.4. Differential equation on \( \tau \)

Condition 1 of theorem 2.2 reads that the vectors \( \{z, \dot{\Gamma}, \ddot{\Gamma}\} \) are linearly dependent. Denote by \( \pi \) the orthogonal projection onto the 3D space \( W = \text{Span}\{v, n, w\} \). Then, the three vectors \( \{\pi(z), \pi(\dot{\Gamma}), \pi(\ddot{\Gamma})\} \) must be linearly dependent as well. We need to compute \( \ddot{\Gamma} \). Differentiating (6) and using Frenet formulas we can write:

\[
\pi(\ddot{\Gamma}) = a_1 v + a_2 n + a_3 w,
\]

where

\[
\begin{align*}
a_1 &= (\dot{l} \cos \delta - 2k\dot{l}) \sin \delta - \dot{k} l \sin \delta - k^2 l \cos \delta, \\
a_2 &= 1 + \dot{l} + 2k\dot{l} \cos \delta - k^2 l \sin \delta + \ddot{l} \sin \delta + \dot{k} l \cos \delta - \tau^2 \dot{l} \sin \delta, \\
a_3 &= 2\tau l \sin \delta + \tau k\dot{l} \cos \delta + \ddot{l} \sin \delta.
\end{align*}
\]

(12)

Finally we write the determinant:

\[
D = \det \begin{vmatrix} 
\cos \delta & \sin \delta & 0 \\
1 + \dot{l} \cos \delta - k\dot{l} \sin \delta & \dot{l} \sin \delta + k\dot{l} \cos \delta & \tau l \sin \delta \\
a_1 & a_2 & a_3 
\end{vmatrix} = 0.
\]

Fortunately, there is no need to compute this determinant exactly, but to notice the following. Every term in the last column (also terms coming from \( a_3 \)) contains \( \tau \) or \( \dot{\tau} \) as a multiplier. Thus \( D \) can be written as

\[
D = A(s) \dot{\tau} + B(s) \tau. \tag{13}
\]

We do not care about \( B(s) \) but need to find \( A(s) \) explicitly. However this is easy because \( \dot{\tau} \) is present only in \( a_3 \). Thus

\[
A(s) = l \sin \delta (\cos \delta (l \sin \delta + k\dot{l} \cos \delta) - \sin \delta (1 + \dot{l} \cos \delta - k\dot{l} \sin \delta))
= l \sin \delta (kl - \sin \delta).
\]

So we have the following.

**Theorem 2.4.** Condition 2 of theorem 2.2 implies the following differential equation on \( \tau \):

\[ l \sin \delta (kl - \sin \delta) \dot{\tau} + B(s) \tau = 0. \]

Moreover, if \( \tau \) vanishes at one point it must vanish identically.

**Proof.** Indeed consider the subset \( Z \) of \( \mathbb{R} \) defined by:

\[
Z = \{s \in \mathbb{R} : \tau(s) = 0\}.
\]

By definition, \( Z \) is obviously a closed set. On the other hand, it follows from proposition 2.3 that if \( \tau(s_0) = 0 \) then \( (k(s_0))(l(s_0) - \sin \delta) \) does not vanish and hence by the uniqueness for the differential equation, \( \tau(s) \) vanishes in a neighborhood of \( s_0 \). So \( Z \) is an open set. Thus \( Z \) coincides with the whole real line. \( \square \)
2.5. Planarity of geodesics

As a consequence of theorem 2.4 we get planarity of some geodesic curves of $S$.

**Theorem 2.5.** Every geodesic curve on $S$ which at some point $p$ passes in a principal direction lies necessarily in a 2-plane spanned by this direction and the normal line at $p$. Moreover, this geodesic curve has a principal direction at every point where it passes.

**Proof.** Principal directions correspond to the eigenvectors of a shape operator. If $\gamma$ has principal direction at $p = \gamma(0)$ then

$$\lim_{s \to 0} n(\gamma(s)) = -k\dot{\gamma}(0).$$

Comparing this formula with Frenet formula (4) we get

$$\tau(0) = 0.$$ 

Hence by theorem 2.4 function $\tau(s)$ vanishes identically, so the geodesic curve lies in the 2-plane. Since at the points of the geodesic curve normal to $S$ equals that of the geodesic, then the derivative of the normal satisfies (4) at every point with $\tau = 0$, which means that this geodesic has principal direction everywhere on its way. □

3. Proof of theorem 1.1

In this theorem we have $d = 3$, so $S$ is 2D and for every point $p \in S$ either $p$ is an umbilical or there are precisely two orthogonal principal directions. Let $p$ be a non-umbilic point, so in a neighborhood of $p$ there are two orthogonal unit vector fields $v_1$ and $v_2$ going in principal directions. Moreover it follows from theorem 2.5 that these vector fields are orthogonal geodesic vector fields, i.e. integral curves are geodesics. In such a case, passing to curvature coordinates on $S$, it is easy to see that the Riemannian metric of $S$ must be flat in the neighborhood of $p$. Indeed in the curvature coordinates $(x, y)$ the metric takes the form

$$ds^2 = E(x, y)dx^2 + G(x, y)dy^2.$$ 

Since $\{x = \text{constant}\}$ and $\{y = \text{constant}\}$ are geodesics we get $E = E(x)$, $G = G(y)$, but then the metric is flat. The flatness of the metric yields a contradiction, since $S$ is assumed to have positive principal curvatures. This argument implies that all points of $S$ are umbilical and hence $S$ is a round sphere. This completes the proof in the 3D case.

4. Proof of theorem 1.2

Now we are in the case $d > 3$. Suppose $\gamma$ is a geodesic curve of $S$ lying in a 2-plane $\sigma$. Then, as a section of the convex hypersurface, $\gamma$ is a convex closed curve in the plane $\sigma$. In addition, since the normal to $S$ and the normal to $\gamma$ are the same, then the planar billiard inside $\gamma$ is a 2D Gutkin billiard, and therefore is very special, as explained in the introduction.
4.1. Proof of constant width

Let us show now that $S$ has a constant width. Take a point $p \in S$ and any two orthogonal principal directions $v_1, v_2$ at $p$. Then geodesics $\gamma_1, \gamma_2$ in these directions are simple closed convex curves contained in the 2-planes $\sigma_1, \sigma_2$. Intersection of these planes is precisely the normal line $l_p$ through $p$. Therefore the curves $\gamma_1$ and $\gamma_2$ must intersect at a unique point in addition to $p$, which lies on this normal line $l_p$. Indeed, in addition to $p$, $\gamma_1$ must intersect $l_p$ in some other point $p'$, also $\gamma_2$ must intersect $l_p$ in some point $p''$. Then there are three points $p, p', p''$ of the intersection of the line $l_p$ with the convex hypersurface. Therefore, $p' = p''$. Moreover, $l_p$ is orthogonal to $S$ at $p'$. Indeed, the normal to $S$ at $p'$ coincides with the normal to $\gamma_1$ and to $\gamma_2$ because they are geodesic curves on $S$; thus this normal is parallel to the intersection of the 2-planes $\sigma_1, \sigma_2$, which is $l_p$. So we have proved the so-called double normal property, which is known to be equivalent to the constant width condition [6] (see figure 3). In addition the plane curves $\gamma_1$ and $\gamma_2$ are of course also of constant width.

4.2. Finishing proof of Theorem 1.2

Let us show now that every point of $S$ is umbilical in the case when $\delta$ is not a solution (1) for odd $n$. In other words, we claim that all principal curvatures of any point $p$ are all equal to $\frac{1}{R}$, where $R$ is the half width of $S$. This implies immediately that $S$ must be a sphere. In order to prove the claim, take any two principal directions at $p$ and the geodesics $\gamma_1$ and $\gamma_2$ as above. Recall that $\gamma_1$ and $\gamma_2$ are planar Gutkin billiards for the same angle $\delta$. Under the assumptions of theorem 1.2 there are two possibilities.

In the first case $\delta$ is not a solution of (1) for any $n$. In this case by the Gutkin result $\gamma_1$ and $\gamma_2$ are circles with the same diameter.

In the second case $\delta$ is a solution of (1) for an even $n$. In such a case [8] the Fourier expansion of $\rho_1(\phi), \rho_2(\phi)$ contains only harmonics of even multiples of $\phi$. On the other hand we
know from the previous argument that \( \gamma_1 \) and \( \gamma_2 \) are of constant width. Therefore, we have for their curvature radii:

\[
\rho_{1,2}(\phi + \pi) + \rho_{1,2}(\phi) = \text{constant}.
\]

But this is not possible for functions having even harmonics only.

So we conclude that in both cases the curves \( \gamma_1 \) and \( \gamma_2 \) are circles. Moreover since they have the same diameter, then they also have the same curvature \( \frac{1}{R} \). This proves the claim and theorem 1.2.

5. Rigidity of planar Gutkin billiard tables

5.1. Proof of theorem 1.3

In order to prove theorem 1.3 we use a new generating function for Birkhoff billiards found in [4]. We fix a point inside \( S \) and use the coordinates \((p, \phi)\) on the space of oriented lines intersecting \( S \). Here \( \phi \) is the angle between the \( x \) axis and the positive normal to the line, and \( p \) is a signed distance to the line. It is proved in [4] that in these coordinates billiard map is a twist map and can be given with the help of generating function

\[
S = 2h \left( \frac{\phi_1 + \phi_2}{2} \right) \sin \left( \frac{\phi_2 - \phi_1}{2} \right),
\]

(14)

where \( h(\phi) \) denotes the support function of the curve \( S \).

Remark 2. In [4] we used this function near the boundary showing its advantage in the proof of KAM type results for billiards. In fact in the coordinates \((p, \phi)\) on the billiard ball map satisfy twist conditions globally with

\[
S = 2h \left( \frac{\phi_1 + \phi_2}{2} \right) \left| \sin \left( \frac{\phi_2 - \phi_1}{2} \right) \right|.
\]

(15)

It turns out that in higher dimensions the billiard ball map is still a twist map in these symplectic coordinates and has a very simple generating function, which we shall derive in the next section.

In what follows we shall need expressions for second derivatives of \( S \). We introduce the notations (see figure 4):
\[ \phi = \frac{\phi_1 + \phi_2}{2}; \quad \alpha = \frac{\phi_2 - \phi_1}{2}. \]

Then we have following formulas for second partial derivatives of \( S \):

\[ S_{11}(\phi_1, \phi_2) = \frac{1}{2}(h''(\phi) - h(\phi)) \sin \alpha - h'(\phi) \cos \alpha; \]

\[ S_{22}(\phi_1, \phi_2) = \frac{1}{2}(h''(\phi) - h(\phi)) \sin \alpha + h'(\phi) \cos \alpha; \]

\[ S_{12}(\phi_1, \phi_2) = \frac{1}{2}(h''(\phi) + h(\phi)) \sin \alpha. \]

From the last formula the twist condition \( S_{12} > 0 \) holds true since

\[ h''(\phi) + h(\phi) = \rho(\phi) \]

is the curvature radius.

The following statement follows in a standard way ([1, 2]) from the assumption of total integrability.

**Proposition 5.1.** For any Gutkin billiard table such that \( \Omega_{\delta_1, \delta_2} \) is foliated by rotational invariant curves the following inequality holds:

\[ \int_{\Omega_{\delta_1, \delta_2}} (S_{11} + 2S_{12} + S_{22}) d\mu \leq 0, \tag{16} \]

where \( d\mu \) is the invariant measure.

Next we compute the integral (16).

First notice that the invariant measure can be written in the form:

\[ d\mu = S_{12} \, d\phi_1 \, d\phi_2 = \frac{1}{2} \rho(\phi) \sin \alpha \, d\phi_1 \, d\phi_2. \]

Using this and passing from \((\phi_1, \phi_2)\) to \((\phi, \alpha)\) we get the integral

\[ I = \int_{\delta_1}^{\delta_2} \int_0^{2\pi} (S_{11} + 2S_{12} + S_{22}) S_{12} \, d\phi \, d\alpha \leq 0. \]

Substituting the exact expressions for the derivatives and applying the Fubini theorem and integration by parts we compute:

\[ I = 2 \int_{\delta_1}^{\delta_2} \int_0^{2\pi} h''(\phi)(h''(\phi) + h(\phi)) \sin^2 \alpha \, d\phi \, d\alpha \]

\[ = 2 \int_{\delta_1}^{\delta_2} \sin^2 \alpha \, d\alpha \cdot \int_0^{2\pi} h''(\phi)(h''(\phi) + h(\phi)) \, d\phi \]

\[ = 2 \int_{\delta_1}^{\delta_2} \sin^2 \alpha \, d\alpha \cdot \int_0^{2\pi} [(h''(\phi))^2 - (h'(\phi))^2] \, d\phi. \]

Notice that the last integral is non-negative by the Wirtinger inequality applied to the function \( h' \). Comparing with (16) we get \( I = 0 \) and hence the equality in the Wirtinger inequality.
However, the equality in the Wirtinger inequality is possible only for 

\[ h' = a \cos \phi + b \sin \phi \]

that is 

\[ h = h_0 + a \sin \phi - b \cos \phi, \]

which means that the curve \( S \) is a circle. This completes the proof of theorem 1.3.

5.2. Formula for generating functions in higher dimensions

Consider a Birkhoff billiard inside a convex hypersurface \( S \) in \( \mathbb{R}^d \). The billiard ball map \( T \) acts on the subset \( \Omega \) of the space of oriented lines intersecting \( S \). The latter is isomorphic to \( T^*S^{d-1} \); every oriented line can be represented uniquely in the form:

\[ l = \{ m + nt \mid |n| = 1, m \perp n \}; \]

\[ (m,n) \in T^*S^{d-1}, n \in S^{d-1}, m \in T_n^*S^{d-1} \]

(as before we identify tangent and cotangent spaces).

Consider the Gauss map

\[ G : S \to S^{d-1}, x \mapsto n(x), \]

where \( n(x) \) is the outer unit normal to \( S \) at \( x \). With the help of \( G \) it is easy to write a support function for \( S \):

\[ h(n) = \langle G^{-1} n, n \rangle, \quad n \in S^{d-1}. \]

**Theorem 5.2.** The billiard ball map \( T \) can be described with the help of generating function \( S \) as follows

\[ T : (m_1, n_1) \mapsto (m_2, n_2) \Leftrightarrow m_1 = D_1 S, \quad m_2 = -D_2 S, \]

where \( S : S^{d-1} \times S^{d-1} \setminus \Delta \to \mathbb{R} \), is the function defined by the formula:

\[ S(n_1, n_2) = \langle G^{-1} \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right), n_1 - n_2 \rangle \]

\[ = h \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right) |n_1 - n_2|. \tag{17} \]

Moreover the twist condition is satisfied in the following sense: the linear operators

\[ D_{12} = D_1 \circ D_2 : T_n S^{d-1} \to T^*_n S^{d-1}, \quad D_{21} = D_2 \circ D_1 : T_n S^{d-1} \to T^*_n S^{d-1} \]

are isomorphisms.

**Remark 3.**

1. One can check that the formula (17) for \( S \) coincides with (15) in the case \( d = 2 \).
2. In higher dimensions the derivatives of the generating function related to the usual Birkhoff coordinates were computed in [3] and are also very useful.

**Proof.** Take any two distinct unit vectors \( n_1, n_2 \). Since \( G \) is a diffeomorphism, there is a unique point \( P \) on the surface \( S \) with the normal vector equal to
\[ n(P) = \frac{n_1 - n_2}{|n_1 - n_2|}. \]

Then, by the construction, the straight line \( l_1 \) incoming to \( P \) in the direction \( n_1 \) is reflected to a line \( l_2 \) outgoing in the direction \( n_2 \). We need to compute the derivative of

\[ \left\langle G^{-1} \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right), n_1 - n_2 \right\rangle \]

along the tangent vector \( \xi \in T_n S^{d-1} \). By the Leibniz rule it has two terms. The first term, when we differentiate \( G^{-1} \), vanishes, since \( P = G^{-1} \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right) \) varies in the tangent space to \( S \) and \( n_1 - n_2 \) is orthogonal to it. Thus the differential \( D_1S \) acts on \( \xi \) by the formula

\[ D_1S : \xi \mapsto \left\langle G^{-1} \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right), \xi \right\rangle. \]

This functional can be identified with the orthogonal projection of the vector \( G^{-1} \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right) \) along \( n_1 \), i.e.

\[ D_1S = G^{-1} \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right) - \left\langle G^{-1} \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right), n_1 \right\rangle n_1, \]

However, the last one is precisely \( m_1 \), since the line \( l_1 \) goes through \( P \). Analogously one computes \( D_2S \) as well as the second differentials. \( \square \)

It turns out that the formula (17) has been worked out already for the case of ellipsoids [11]. Indeed, we shall compute \( S \) for the ellipsoids more explicitly.

**Example 1.** Let \( S \) be an ellipsoid in \( \mathbb{R}^d \),

\[ S = \{ \langle A^{-1}x, x \rangle = 1 \}, \]

given by a positive definite symmetric matrix \( A \).

We compute the Gauss map and the support function. Given \( n \), we need to find \( x \in S \) with \( n(x) = n \). Write:

\[ \mu n = A^{-1}x, \]

for some \( \mu \). Then we have:

\[ \langle A^{-1}x, x \rangle = \langle \mu n, \mu An \rangle = 1. \]

So we get:

\[ \mu = \langle An, n \rangle^{-\frac{1}{2}}. \]

Therefore

\[ G^{-1}(n) = \langle An, n \rangle^{-\frac{1}{2}} An, \]

and the support function:

\[ M \]
\[ h(n) = \langle G^{-1}n, n \rangle = \langle An, n \rangle^{\frac{1}{2}}. \]

Thus finally we get
\[
S(n_1, n_2) = \langle A \left( \frac{n_1 - n_2}{|n_1 - n_2|} \right), \frac{n_1 - n_2}{|n_1 - n_2|} \rangle^{\frac{1}{2}} |n_1 - n_2| \\
= \langle A (n_1 - n_2), n_1 - n_2 \rangle^{\frac{1}{2}}. \tag{18}
\]

The last formula coincides with one found in [11] for the ellipsoid.

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