ON THE EIGENVALUE ESTIMATES FOR THE WEIGHTED LAPLACIAN ON METRIC GRAPHS

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Dedicated to Olga Aleksandrovna Ladyzhenskaya

to whom I am indebted for my formation as a mathematician

ABSTRACT. It is shown that the eigenvalues of the equation \( -\lambda \Delta u = V u \) on a graph \( G \) of final total length \( |G| \), with non-negative \( V \in L^1(G) \) and under appropriate boundary conditions, satisfy the inequality \( n^2\lambda_n \leq |G| \int_G V \, dx \), independently of geometry of a given graph. Applications and generalizations of this result are also discussed.

1. Introduction

Differential operator on a metric graph \( G \) is a family of differential expressions on its edges, complemented by appropriate matching conditions at the most of vertices and by some boundary conditions at the remaining vertices. In particular, for the Laplacian \( \Delta \) the differential expression is \( \Delta u = u'' \) and the matching conditions at vertices are the Kirchhoff conditions coming from the theory of electric networks. Our main goal in this paper is to investigate the behaviour of eigenvalues for the problem

\[
-\lambda \Delta u = V u; \quad u(x_0) = 0, \quad u'(v) = 0 \text{ if } v \in \partial G \setminus \{x_0\}
\]

on graphs of finite total length \( |G| \). In (1.1) \( V \) is a given real-valued, measurable weight function on \( G \) and \( x_0 \in G \) is a selected vertex. Note that the set \( \partial G \setminus \{x_0\} \) can be empty.

It is possible also to consider the similar problem with zero boundary conditions at several vertices, \( u(x_1) = \ldots = u(x_r) = 0 \), however this gives nothing new: by the variational principle, any estimate for the problem (1.1) implies the same estimate for the \( r \)-point problem. The accurate setting of the problem uses quadratic forms, see Section 3 for details. The way to insert the spectral parameter in (1.1) is motivated by technical reasons.

Here is our main result on the problem (1.1). Let \( \pm \lambda_n \) denote the positive and the negative eigenvalues of this problem.

Date: 28 December 2001.
1991 Mathematics Subject Classification. [.]
Theorem 1.1. Let $G$ be a connected graph of finite total length, $x_0 \in G$ be its arbitrary vertex, and let $V = \nabla \in L^1(G)$. Then the eigenvalues of the problem (1.1) satisfy the inequality
\[
\lambda_n^\pm \leq \frac{|G| \int_G V_\pm dx}{n^2}, \quad \forall n \in \mathbb{N}
\]
where $2V_\pm = |V| \pm V$. Along with the estimate (1.2), the Weyl-type asymptotics holds:
\[
n \sqrt{\lambda_n^\pm} \to \frac{\int_G \sqrt{V_\pm}(x)dx}{n}, \quad n \to \infty.
\]

The simplest example of metric graph is a single segment $[0, L] \subset \mathbb{R}$. The basic results for this case were obtained by M.Sh. Birman and the author [2] as far back as in 1971, as a particular case of the general result for many dimensions; see also an exposition in [3]. Namely, it was shown that eigenvalues of the equation $-\lambda u'' = Vu$ on the interval $(0, L)$, under the Dirichlet boundary conditions at its ends, satisfy the asymptotics (1.3) and the inequality $n^2 \lambda_n^\pm \leq C|G| \int_G V_\pm dx$, with some absolute constant $C$. The problem of the best possible constant in this inequality was not discussed in [2] and in [3], though the sharp estimate with $C = 1/4$ could be easily derived from Theorem 2.2 of the paper [1]. For the zero boundary condition only at one point, as in (1.1), the constant $C = 1$ in (1.2) is unimprovable even for the case of segment.

The most important feature of the estimate (1.2) for $G = [0, L]$ is its uniformity with respect to $V \in L^1(0, L)$. This proved quite useful for various applications. Theorem 1.1 shows that this estimate extends to arbitrary graphs of finite total length, with the same constant as for the single interval. The very possibility of such extension might look problematic, since the eigenvalues depend on the combinatorial structure of the graph, which can be quite diverse.

One can give a qualitative explanation of this effect. If $V \geq 0$, the eigenvalues of the problem (1.1) can be defined in terms of approximative characteristics of the unit ball of the Sobolev space $H^1(G, x_0)$ as a compact in the weighted space $L^2(G, V)$; indication of the point $x_0$ in the first notation reflects the boundary condition $u(x_0) = 0$. The points of a graph $G$ lie “closer to each other” than the points of the segment $[0, |G|]$. Therefore, one should expect that the dispersion of the values of a given function $u \in H^1(G, x_0)$ is smaller than that of a function $v \in H^1(0, |G|)$ having the same $L^2$-norm of the derivative. As a consequence, it is easier to approximate the class $H^1(G, x_0)$ than $H^1(0, |G|)$. Eventually this leads to the estimate (1.3).

This, a little bit naive argument gives no clue to the proof. Our proof is based upon a result (Theorem 2.1) of a rather combinatorial nature. This theorem can be considered as a far going generalization of Theorem 2.2 from [1].

Our results should be compared with the ones of the recent paper [6] by W.D. Evans, D.J. Harris, and J. Lang. Its subject is the behaviour of approximation numbers of the Hardy-type integral operators in the spaces $L^p$ on trees. If $p = 2$, the approximation numbers coincide with the singular numbers, and this is the link between our corresponding
results. Some of our results, including the asymptotics (1.3), could be derived from this paper. The techniques of [3], see especially its Section 3, is also based upon a certain combinatorial construction. It is substantially different from our one. The applications discussed in our paper are not touched upon by the authors of [3].

Let us describe the structure of the paper. In the next Section 2 we give the necessary information about graphs and trees and prove Theorem 2.1 which is our main technical tool. In Section 3 we define the Sobolev space $H^1(G, x_0)$ and give the variational formulation of the problem (1.1). We also state two auxiliary results, Theorems 3.1 and 3.2, on approximation on graphs. They are useful for the proof of Theorem 1.1. The proofs of all three theorems are given in Section 4. In Section 5 we discuss the result and give its application to the estimates of singular numbers for integral operators on graphs. In the last Section 6 we present a higher order analog of Theorem 1.1.

The Hilbert space structure is unnecessary for applications of Theorem 2.1. It can be also applied to piecewise-polynomial approximation of Sobolev spaces $W^{l,p}$ on graphs and trees. This leads to estimates of approximation numbers of embedding of $W^{l,p}$ in the weighted spaces $L^p(G, V)$ on graphs of finite total length. The results can be also applied to graphs and trees of infinite total length, in the spirit of the papers [8] and [6]. This material will be presented elsewhere.

2. GRAPHS AND TREES. PARTITIONS OF A TREE

Let $G$ be a graph with the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. Each edge $e$ of a metric graph is viewed as a non-degenerate line segment of finite length $|e|$. The quantity $|G| = \sum_{e \in E(G)} |e|$ is called the total length of the graph $G$. In this paper we always assume $|G| < \infty$. Even though, the number of edges of $G$ can be infinite. The distance $\rho(x, y)$ between any two points $x, y \in G$, and thus the metric topology on $G$, is introduced in a natural way. The natural measure $dx$ on $G$ is induced by the Lebesgue measure on its edges. For a measurable set $E \subset G$, its measure is denoted by $|E|$. This is consistent with the above notations $|e|, |G|.$

We always consider connected graphs, unless otherwise stipulated. We do not exclude graphs with multiple joins. In order to avoid unnecessary complications, we exclude graphs with loops. Recall that a loop is an edge whose endpoints coincide with each other.

For vertices $v, w$ the notation $w \sim v$ means that there exists an edge $e \in E$ whose endpoints are $v$ and $w$. Connectedness of the graph means that for any two different vertices $v, w \in V$ there exists a finite sequence $\{v_k\}_{0 \leq k \leq m}$ of vertices, such that $v_0 = v, v_m = w$ and $v_k \sim v_{k-1}$ for each $k = 1, \ldots, m$. The combinatorial distance $\rho_{comb}(v, w)$ is defined as the minimal possible $m$ in this construction. We define $\rho_{comb}(v, v) = 0$ for any $v \in V$. The degree $d(v)$ of a vertex $v$ is defined as the total number of edges incident to $v$. The vertices $v$ with $d(v) = 1$ constitute the boundary $\partial G$ of the graph $G$. We suppose that $d(v) < \infty$ for each $v \in V$. It is often convenient to treat an arbitrary point
$x \in G$ as a vertex. We set $d(x) = 2$ for any $x \notin V(G)$ and write $v \sim x$ if $v \in V(G)$ is one of the endpoints of the vertex containing $x$. This remark concerns also the point $x_0$ appearing in (1.1) which actually can be an arbitrary point in $G$.

Given a subgraph $G \subset G$, we denote by $d_G(v)$ the degree of a vertex $v$ with respect to $G$.

A metric graph $G$ is compact if and only if $\#E(G) < \infty$. Any metric graph can be represented as the union of an expanding family of its compact subgraphs,

$$G = \bigcup_{m=1}^{\infty} G^{(m)}, \quad G^{(1)} \subset G^{(2)} \subset \ldots; \quad \#E(G^{(m)}) < \infty, \ \forall m \in \mathbb{N}.$$  

Indeed, choose an arbitrary vertex $x_0 \in G$ and define the subgraph $G^{(m)}$ as follows. A vertex $v \in V(G)$ belongs to $V(G^{(m)})$ if and only if $\rho_{\text{comb}}(x_0, v) \leq m$, and an edge $e \in E(G)$ belongs to $E(G^{(m)})$ if and only if its both ends lie in $V(G^{(m)})$. By the construction, the graph $G^{(m)}$ is connected and compact. It is evident that $\{G^{(m)}\}$ is an expanding family which covers the whole of $G$.

Let $G$ be a compact graph and $G, G_1, \ldots, G_n$ be its (connected) subgraphs. We say that the subgraphs $G_1, \ldots, G_n$ constitute a partition, or a splitting of $G$ if $G_1 \cup \ldots \cup G_n = G$ and $|G_i \cap G_j| = 0$ for any $i, j \in \{1, \ldots, n\}, \ i \neq j$. If $G, G_1$ are subgraphs of $G$ and $G_1 \subset G$, then connected subgraphs $G_2, \ldots, G_n$ can be always found which together with $G_1$ constitute a partition of $G$ (the property of complementability).

A pair $\{G, x\}$ where $G \subset G$ is a subgraph and $x \in G$ is a selected point, is called a punctured subgraph. If $\{G_j, x_j\}, \ j = 1, \ldots, n$ are punctured subgraphs such that $G = G_1 \cup \ldots \cup G_n$ is a partition of a subgraph $G$, then we say that $G$ is split into the union of punctured subgraphs.

Our auxiliary result, Theorem 2.1, which serves us as a basis for the proof of Theorem [11], concerns trees rather than arbitrary graphs. Recall that tree is a connected graph without cycles, loops and multiple joins. So, let $G = T$ be a tree. For any two points $x, y \in T$ there exists a unique simple polygonal path in $T$ connecting $x$ with $y$, we denote it by $\langle x, y \rangle$.

What was said above about partitions of graphs, applies to trees. Note that if $T = T_1 \cup T_2$ is a partition, then the intersection $T_1 \cap T_2$ consists of exactly one point.

Let $\Phi$ be a non-negative function defined on the set of all subtrees $T$ of a given tree $T$. We call it continuous if $\Phi(T) \to \Phi(T_0)$ as soon as $|T \triangle T_0| \to 0$. Recall that $T \triangle T_0 = (T \setminus T_0) \cup (T_0 \setminus T)$ is the symmetric difference of the sets $T, T_0$.

We call a continuous function $\Phi$ superadditive and write $\Phi \in \mathcal{S}(T)$ if for any subtree $T \subset T$ and any partition $T = T_1 \cup \ldots \cup T_n$ we have

$$\Phi(T_1) + \ldots + \Phi(T_n) \leq \Phi(T).$$  

Due to the complementability, (2.2) implies that any superadditive function is monotone:

$$T_1 \subset T \implies \Phi(T_1) \leq \Phi(T).$$
Any finite Borel measure $\mu$ on $T$ without atoms (i.e. points of positive measure) generates the continuous superadditive function $\Phi(T) = \mu(T)$. A more general example follows from Hölder’s inequality:

\[(2.3)\quad \Phi_1, \Phi_2 \in S(T), \quad \alpha_1, \alpha_2 > 0, \quad \alpha_1 + \alpha_2 = 1 \implies \Phi_1^{\alpha_1} \Phi_2^{\alpha_2} \in S(T).\]

Let $\{T, x\}$ be a punctured subtree of $T$. The tree $T$ splits in a unique way into the union of subtrees $\Theta_j \subset T$, $j = 1, \ldots, d_T(x)$, rooted at $x$ and such that $d_{\Theta_j}(x) = 1$ for each $j$. We call it the canonical partition of the punctured subtree $\{T, x\}$. Given a function $\Phi \in S(T)$, we define the function $\tilde{\Phi}(T, x)$ of punctured subtrees,

\[(2.4)\quad \tilde{\Phi}(T, x) = \max_{1 \leq j \leq d_T(x)} \Phi(\Theta_j).\]

Evidently, $\tilde{\Phi}(T, x) \leq \Phi(T)$.

Let in particular $T = T$. Each subtree $\Theta_j$ appearing in the canonical partition of $\{T, x\}$ is fully determined by indication of its initial edge $(x, v)$, $v \sim x$ and we denote this subtree by $\Theta_{(x, v)}$. For $T = T$ the definition (2.4) takes the form

\[(2.5)\quad \tilde{\Phi}(T, x) = \max_{v \sim x} \Phi(\Theta_{(x, v)}).\]

Our eigenvalue estimates will be derived from the following result on superadditive functions of subtrees.

**Theorem 2.1.** Let $T$ be a compact metric tree and $\Phi \in S(T)$. Then for any $n \in \mathbb{N}$ the tree $T$ can be split into the union of punctured subtrees $\{T_j, x_j\}$, $j = 1, \ldots, k$ in such a way that $k \leq n$ and

\[(2.5)\quad \max_{j=1,\ldots,k} \tilde{\Phi}(T_j, x_j) \leq (n + 1)^{-1} \Phi(T).\]

First, we prove a lemma.

**Lemma 2.2.** Let $T$ be a compact metric tree and $\Phi \in S(T)$. Then for any $\varepsilon$, $0 < \varepsilon < \Phi(T)$, there exists a partition $T = T' \cup T''$, such that

\[(2.6)\quad \Phi(T') \leq \Phi(T) - \varepsilon\]

and for the only point $x \in T \cap T''$ the inequality holds:

\[(2.7)\quad \tilde{\Phi}(T, x) \leq \varepsilon.\]

**Proof.** Without loss of generality, we can assume $\Phi(T) = 1$. Take any vertex $v_0 \in \partial T$, then $\tilde{\Phi}(T, v_0) = \Phi(T) = 1$. There is a unique vertex $v_1 \sim v_0$. Now we choose the vertices $v_2 \sim v_1, \ldots, v_{k+1} \sim v_k, \ldots$ as follows. If $v_k$ is already chosen, we define $v_{k+1}$ as the vertex different from $v_{k-1}$ and such that

\[(2.8)\quad \Phi(\Theta_{(v_k, v_{k+1})}) = \max_{w \sim v_k, w \neq v_{k-1}} \Phi(\Theta_{(v_k, w)}) = \tilde{\Phi}(\Theta_{(v_k, v_{k+1})}, v_k).\]
If there are several vertices \( w \sim v_k \) at which the maximum in the middle term of (2.8) is attained, then any of them can be chosen as \( v_{k+1} \). The described procedure is always finite, it terminates when we arrive at a vertex \( v_m \in \partial T \). On the path \( \mathcal{P} = (v_0, v_m) \) we introduce the natural ordering, i.e., \( y \succeq x \) means that \( x \in (v_0, y) \). We write \( y \succ x \) if \( y \geq x \) and \( y \neq x \).

Let \( x \in \mathcal{P} \) be not a vertex of \( T \), then \( v_{k-1} \prec x \prec v_k \) for some \( k = 1, \ldots, m \). Denote
\[
T^+_x = \Theta(x, v_k), \quad T^-_x = \Theta(x, v_{k-1}).
\]
We also define the subtrees \( T^\pm_x \) for \( x = v_0, \ldots, v_m \). Namely,
\[
T^+_v = T(v_k, v_{k-1}), \quad k = 1, \ldots, m;
T^-_v = \bigcap_{v_{k-1} \prec x \prec v_k} T^+_x, \quad v \sim v_{k-1}.
\]
Finally, \( T^-_0 = \{v_0\}, T^-_m = \{v_m\} \) are degenerate subtrees. Clearly, for any \( x \in \mathcal{P} \) \( T = T^+_x \cup T^-_x \) is a partition of the punctured tree \( \{T, x\} \), and \( T^+_x \cap T^-_x = \{x\} \).

The function \( F(x) = \Phi(T^+_x) \) is well defined on \( \mathcal{P} \) and is continuous everywhere except possibly for the vertices \( v_0, \ldots, v_m \). By the construction,
\[
\Phi(T^+_x, x) = F(x), \quad x \neq v_0, \ldots, v_m,
\]
and
\[
\lim_{x \prec v_k, x \rightarrow v_k} F(x) = F(v_k);
\]
\[
\lim_{x \succ v_k, x \rightarrow v_k} F(x) = \Phi(T(v_k, v_{k+1})) = \Phi(T^+_x, v_k) \leq F(v_k).
\]
Here the second equality follows from (2.8). It is clear that \( F(x) \) is non-increasing along the path \( \mathcal{P} \). Besides,
\[
0 = F(v_m) < \varepsilon < F(v_0) = 1.
\]
Therefore, there exists a point \( x \in \mathcal{P} \) such that
\[
\Phi(T_x, x) \leq \varepsilon \leq F(x).
\]
We take \( T = T^+_x \) and \( T' = T^-_x \). The inequality (2.7) is satisfied and (2.6) is implied by superadditivity:
\[
\Phi(T') \leq 1 - \Phi(T) = 1 - F(x) \leq 1 - \varepsilon.
\]

Proof of Theorem 2.4. 1. Let \( n = 1 \). Then we apply the result of Lemma 2.2 with \( \varepsilon = \Phi(T)/2 \). Let \( T = T \cup T' \) be the corresponding partition, then \( \Phi(T', x) \leq \Phi(T') \leq \Phi(T)/2 \). Consider the canonical partition of the punctured tree \( \{T, x\} \). Each subtree of this partition is contained either in \( T \) or in \( T' \), therefore
\[
\Phi(T, x) \leq \max(\Phi(T, x), \Phi(T', x)) \leq \Phi(T)/2.
\]
Thus, (2.5) with \( k = n = 1 \) is satisfied if we take \( T_1 = T \) and \( x_1 = x \).

2. We proceed by induction. Suppose that the result is already proved for \( n = n_0 - 1 \). Let \( T = T \cup T' \) be the partition constructed according to Lemma 2.2 for \( \varepsilon = (n_0 + 1)^{-1} \Phi(T) \). Then

\[
\Phi(T') \leq n_0(n_0 + 1)^{-1} \Phi(T).
\]

By the inductive hypothesis, there exists a splitting of \( T' \) into the union of the family of punctured subtrees \( \{T_j, x_j\}, j = 1, \ldots, k \) such that \( k \leq n_0 - 1 \) and for each \( j \)

\[
\tilde{\Phi}(T_j, x_j) \leq n_0^{-1} \Phi(T') \leq (n_0 + 1)^{-1} \Phi(T).
\]

Adding to this family the punctured subtree \( \{T_{k+1}, x_{k+1}\} = \{T, x\} \), we obtain the desired partition of \( T \) for \( n = n_0 \).

\[\Box\]

3. Variational setting of the problem. Reduction to the case of trees

3.1. Sobolev spaces on a graph. Below \( \| \cdot \|_p, 1 \leq p \leq \infty \) stands for the norm in the space \( L^p(G) \) and \( L_+(G) \) stands for the cone of all non-negative elements in \( L^1(G) \).

We say that a function \( u \) on \( G \) belongs to the Sobolev space \( L^{1,2}(G) \) if \( u \) is continuous on \( G \), the restriction of \( u \) to each edge \( e \) lies in \( H^1(e) \), and \( u' \in L^2(G) \). The functional \( \|u'\|_2 \) defines on \( L^{1,2}(G) \) a semi-norm which vanishes on the one-dimensional subspace of constant functions.

Let \( G \) be a graph of finite total length and let \( \xi, x \) be its two arbitrary points. Choose a simple polygonal path \( \mathcal{L} \) in \( G \) connecting \( \xi \) with \( x \), let its length be \( t_0 \). Parametrizing \( \mathcal{L} \) by the path length, we can regard the restriction \( u|\mathcal{L} \) as a function on the line segment \([0, t_0]\). It follows from the equality \( u(x) - u(\xi) = \int_0^{t_0} u'(t)dt \) that

\[
|u(x) - u(\xi)|^2 \leq t_0 \int_0^{t_0} |u'(t)|^2 dt \leq |G| \int_G |u'(x)|^2 dx.
\]

This shows that any function \( u \in L^{1,2}(G) \) lies in the Hölder class of order \( 1/2 \).

A step function \( v \) on \( G \) is a function which takes only a finite number of different values, each on a connected subset of \( G \). We denote by \( \text{Step}(G) \) the linear space (non-closed linear subspace of \( L^\infty(G) \)) of all step functions on \( G \).

The following result on the approximation of functions \( u \in L^{1,2}(G) \) by step-functions will be used when proving Theorem 1.1. Let punctured subgraphs \( \{G_j, x_j\}, j = 1, \ldots, k \) constitute a partition of the graph \( G \). We associate with this partition the linear operator

\[
P : u \mapsto v = \sum_{j=1}^k u(x_j) \chi_j
\]

where \( \chi_j \) stands for the characteristic function of the set \( G_j \). It is clear that the operator \( P \) acts from \( L^{1,2}(G) \) into \( \text{Step}(G) \) and its rank is less or equal to \( k \).
**Theorem 3.1.** Let $G$ be a compact graph and $V \in L_+(G)$. Then for any $n \in \mathbb{N}$ there exists a partition of $G$ into punctured subgraphs $\{G_j, x_j\}$, $j = 1, \ldots, k$, such that $k \leq n$ and for the corresponding operator $P$ given by (3.2) we have

$$
\int_G \left| u - Pu \right|^2Vdx \leq \frac{|G| \int_G Vdx}{(n+1)^2} \|u'\|_2^2, \quad \forall u \in L^{1,2}(G).
$$

The assumption that the graph $G$ is compact is important for the proof. To exhaust the general case, we need one more statement.

A graph $G$ of finite total length is not necessarily a compact metric space. Let $\overline{G}$ be its compactification. Any function $u \in L^{1,2}(G)$ is uniformly continuous on $G$ and hence admits the unique continuous extension to $\overline{G}$. We keep the same symbol $u$ for the extended function.

**Theorem 3.2.** Let $G$ be a graph of finite total length and $V \in L_+(G)$. Then for any $n \in \mathbb{N}$ there exist points $x_1, \ldots, x_k \in G$ such that $k \leq n$ and the inequality

$$
\int_G \left| u \right|^2Vdx \leq \frac{|G| \int_G Vdx}{(n+1)^2} \|u'\|_2^2
$$

holds for any function $u \in L^{1,2}(G)$ satisfying the conditions $u(x_1) = \ldots = u(x_k) = 0$.

Proofs of Theorems 3.1 and 3.2 are given in the next section, before proving Theorem 1.1.

3.2. **Space $H^1(G, x_0)$ and operators $B_V$.** Let a point $x_0 \in G$ be given. It is convenient to assume that $x_0$ is a vertex. Consider the Hilbert space

$$
H^1(G, x_0) = \{u \in L^{1,2}(G) : u(x_0) = 0\},
$$

equipped with the scalar product

$$(u, v)_{H^1(G, x_0)} = (u', v')_{L^2(G)}.$$

Inequality (3.1) (with $\xi = x_0$) shows that this scalar product is non-degenerate.

Let $V$ be a function from the space $L^1(G)$. Consider the quadratic form

$$
b_V[u] = \int_G |u|^2Vdx.
$$

It follows from the inequality (3.1) (again, with $\xi = x_0$) that $b_V$ is bounded in the space $H^1(G, x_0)$, i.e.

$$
|b_V[u]| \leq |G| \|u'\|_2^2 \int_G |V(x)|dx, \quad \forall u \in H^1(G, x_0).
$$

Therefore, the quadratic form $b_V[u]$ generates a bounded linear operator, say $B_V$, in the space $H^1(G, x_0)$. It is easy to see that the operator $B_V$ is compact; actually, compactness will automatically follow from the estimates we obtain in the next section. This operator is self-adjoint provided the function $V$ is real-valued and it is non-negative provided
\( V \geq 0 \) a.e. As usual, it is natural to identify the spectrum of the problem (1.1) with the spectrum of the operator \( B_V \). Still, we recall the corresponding argument.

The Laplacian \(-\Delta\) on \( G \), with the boundary conditions as in (1.1), is defined as the self-adjoint operator in \( L^2(G) \), associated with the quadratic form \( \int_G |u'|^2 \, dx \) considered on the domain \( H^1(G, x_0) \). Given an element \( f \in L^2(G) \), the equality \(-\Delta u = f\) under these boundary conditions means that \( u \) is the unique function in \( H^1(G, x_0) \) such that

\[
\int_G u \varphi' \, dx = \int_G f \varphi \, dx, \quad \forall \varphi \in H^1(G, x_0).
\]

(3.7)

The Euler–Lagrange equation reduces to \(-u'' = f\) on each edge. The continuity of \( u \) on the whole of \( G \) and the boundary condition \( u(x_0) = 0 \) are implied by the inclusion \( u \in H^1(G, x_0) \). At each vertex \( v \neq x_0 \) the solution \( u \) meets the natural condition in the sense of Calculus of Variations. Namely, let \( e_1, \ldots, e_{d(v)} \) be the edges adjacent to a given vertex \( v \), oriented in such direction that \( v \) is their initial point. Then the condition at \( v \), traditionally called Kirchhoff’s condition, is

\[
(u \upharpoonright e_1)'(v) + \ldots + (u \upharpoonright e_{d(v)})'(v) = 0.
\]

If, in particular, \( d(v) = 1 \), this turns into \( u'(v) = 0 \) which is the boundary condition required by (1.1).

The requirement \( f \in L^2(G) \) is unnecessary for the existence of a solution \( u \in H^1(G, x_0) \) of the equation (or “integral identity”) (3.7). The solution does exist if and only if \( f \) is such that the expression in the right-hand side generates a continuous anti-linear functional on the space \( H^1(G, x_0) \). One has to take into account that the solution of (3.7) for \( f \not\in L^2(G) \) does not belong to the domain of the Laplacian considered as the operator in \( L^2(G) \). Still, it is conventional to interpret (3.7) as the weak form of the equation \(-\Delta u = f\). In particular, this is the case if \( f \in L^1(G) \), due to the embedding \( H^1(G, x_0) \subset C(\overline{G}) \).

According to the above interpretation, the equation (1.1) means that

\[
\lambda \int_G u \varphi' \, dx = \int_G V u \varphi \, dx, \quad \forall \varphi \in H^1(G, x_0).
\]

(3.8)

For \( V \in L^1(G) \) we have also \( V u \in L^1(G) \), so that the general scheme applies.

On the other hand, the equation \( B_V u = \lambda u \) means that

\[
b_V [u, \varphi] := \int_G V u \varphi' \, dx = \lambda \int_G u \varphi' \, dx, \quad \forall \varphi \in H^1(G; x_0).
\]

Comparing this with (3.8), we see that the eigenpairs for both equations are the same.

Below we use for the eigenvalues of (1.1) the notations \( \pm \lambda_n \) (B\( V \)). If \( V \geq 0 \), we write \( \lambda_n \) instead of \( \lambda_n^+ \).
3.3. **Reduction to the case of trees.** In order to have the possibility to use Theorem 2.1, it is necessary to reduce the problem to the case of trees. The procedure of such reduction (cutting cycles) is rather standard. Nevertheless, we describe it in detail.

**Lemma 3.3.** Let $G$ be a compact metric graph. Then there exist a compact metric tree $T$ and a continuous mapping $\tau : T \to G$ such that the operator $\tau^* : u(x) \mapsto u(\tau(x))$ defines an isometry of the space $L^{1,2}(G)$ onto a subspace of finite codimension in $L^{1,2}(T)$.

**Proof.** Let $G$ be a compact connected graph which is not a tree. Take any edge $e_0 = \langle v, w \rangle \in E(G)$ which is part of a cycle in $G$, then the subgraph $G_{e_0} = G \setminus \text{Int}(e_0)$, with $\mathcal{V}(G_{e_0}) = \mathcal{V}(G)$ and $\mathcal{E}(G_{e_0}) = \mathcal{E}(G) \setminus \{e_0\}$, is connected.

Choose a point $x \in \text{Int} e_0$ and replace it by a pair $\{x_1, x_2\}$ of new vertices. This gives rise to a new graph $G_1$ whose rigorous definition is as follows. Its sets of vertices and edges are defined by

$$
\mathcal{V}(G_1) = \mathcal{V}(G) \cup \{x_1, x_2\}, \quad \mathcal{E}(G_1) = \mathcal{E}(G_{e_0}) \cup \{(v, x_1), (w, x_2)\}.
$$

For any edge $e \in \mathcal{E}(G_{e_0})$ its endpoints and its length in $G_1$ are defined to be the same as in $G_{e_0}$. We set

$$
|\langle v, x_1 \rangle|_{G_1} = |\langle v, x \rangle|_G, \quad |\langle w, x_2 \rangle|_{G_1} = |\langle w, x \rangle|_G.
$$

Evidently, $|G_1| = |G|$.

Now we define a mapping $\tau_1 : G_1 \to G$. Namely, $\tau_1$ is identical on $G_{e_0}$ and isometrically sends the edge $\langle v, x_1 \rangle$ onto $\langle v, x \rangle$ and the edge $\langle w, x_2 \rangle$ onto $\langle w, x \rangle$. It is clear that the mapping $\tau_1$ is continuous and the corresponding mapping $\tau_1^* : u(x) \mapsto u(\tau_1(x))$ is an isometry of the space $L^{1,2}(G)$ into $L^{1,2}(G_1)$. Its image consists of all functions $u \in L^{1,2}(G_1)$ such that $u(x_1) = u(x_2)$. Therefore, the image is a subspace of codimension 1.

The number of simple cycles in $G_1$ is smaller than that for the initial graph $G_0 := G$. Therefore, repeating this construction, we obtain a finite sequence of graphs $G_0, \ldots, G_m$ such that the graph $G_m$ has no cycles and loops (i.e. is a tree) and a sequence of corresponding mappings $\tau_j : G_j \to G_{j-1}$, $j = 1, \ldots, m$. The tree $T = G_m$ and the mapping $\tau = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_m$ satisfy all the requirements of Lemma.

It is useful to note also that

$$
\int_G |u|^2 V dx = \int_T |\tau^* u|^2 \tau^* V dx
$$

for any $V \in L^1(G)$ and any $u \in C(G)$.

4. **Proof of the main results**

First of all we prove Theorem 3.1 and then derive Theorem 3.2 from it. Theorem 1.1 follows from the latter almost immediately.
4.1. **Proof of Theorem 3.1.**

1. Let first $G = T$ be a compact tree, $\{T, \xi\}$ be its punctured subtree and $T = \Theta_1 \cup \ldots \cup \Theta_{d_T(\xi)}$ be the canonical partition of $\{T, \xi\}$. Applying the inequality (3.1) to each subtree $\Theta_j$, we get for any $u \in L^1(T)$:

$$
\int_{\Theta_j} |u(x) - u(\xi)|^2 V dx \leq |\Theta_j| \int_{\Theta_j} V dx \int_{\Theta_j} |u'|^2 dx, \quad j = 1, \ldots, d_T(\xi)
$$

and hence,

$$
\int_T |u(x) - u(\xi)|^2 V dx \leq \int_T |u'|^2 dx \max_{j=1, \ldots, d_T(\xi)} \left( |\Theta_j| \int_{\Theta_j} V dx \right).
$$

(4.1)

Consider a function of subtrees $T \subset T$,

$$
\Phi(T) = \Phi(T; V) = |T|^{1/2} \left( \int_T V dx \right)^{1/2}.
$$

(4.2)

By (2.3), $\Phi \in S(T)$. The inequality (4.1) can be written as

$$
\int_T |u(x) - u(\xi)|^2 V dx \leq \Phi^2(T, \xi) \int_T |u'|^2 dx.
$$

(4.3)

Here $\Phi$ is the function of punctured subtrees associated with the function (1.2), cf. (2.4).

Suppose now that the tree $T$ is split into the union of punctured subtrees $\{T_1, x_1\}, \ldots, \{T_k, x_k\}$. Let $P$ be the corresponding operator (3.2). It follows from (4.3) that

$$
\int_T |u - Pu|^2 V dx = \sum_{j=1}^k \int_{T_j} |u(x) - u(x_j)|^2 V dx \leq \sum_{j=1}^k \Phi^2(T_j, x_j) \int_{T_j} |u'|^2 dx
$$

$$
\leq \|u\|_2^2 \left( \max_j \Phi(T_j, x_j) \right)^2, \quad \forall u \in L^1(T).
$$

Now we apply Theorem 2.1 to find that for any $n \in \mathbb{N}$ there exists a partition of $T$ into $k$ punctured subtrees, such that $k \leq n$ and

$$
\int_T |u - Pu|^2 V dx \leq (n + 1)^{-2} |T| \int_T V dx \|u\|_2^2, \quad \forall u \in L^1(T).
$$

For the case of trees the proof is complete.

2. Let $G$ be an arbitrary compact graph. Let $T$ and $\tau : T \to G$ be a compact tree and a mapping constructed according to Lemma 3.3. Without loss of generality, we assume that the pre-image $\tau^{-1}(x_0)$ consists of a single point.

Let $\{T_j, x_j\}$, $j = 1, \ldots, k$ be the partition of $T$, such that (3.3) (with $T$ instead of $G$) is satisfied. Take $G_j = \tau(T_j)$ and $x_j = \tau(x_j)$. Since the mapping $\tau$ is continuous, each set $G_j$ is closed and connected, and hence is a subgraph of $G$. The punctured subgraphs $\{G_j, x_j\}$ constitute a partition of $G$. Now an elementary computation shows that (3.3) is fulfilled for the graph $G$ if we take as $P$ the operator (3.2) corresponding to this partition.
4.2. **Proof of Theorem 3.2.** By the standard limiting argument, the inequality (3.1) extends from the graph \( G \) to its compactification \( \overline{G} \).

Let \( \{G^{(m)}\}_{1 \leq m < \infty} \) be the family of compact subgraphs of \( G \), such that (2.1) is fulfilled. Fix a number \( n \in \mathbb{N} \). For any \( m \), let

\[
P_m : u \mapsto \sum_{j=1}^{k_m} u(x_j^m) \chi_j^m
\]

be the operators (3.2) for the subgraphs \( G^{(m)} \) and the weight functions \( V \restriction G^{(m)} \). These operators depend also on \( n \) but we do not reflect this dependence in the notations.

According to the inequality (3.3), we have for any \( m \) and for any function \( u \in L^{1,2}(G) \) normalized by the condition \( \int_G |u'|^2 dx = 1 \):

\[
(4.4) \quad \int_{G^{(m)}} |u - \sum_{j=1}^{k_m} u(x_j^m) \chi_j^m|^2 V dx \leq (n + 1)^{-2}|G^{(m)}| \int_{G^{(m)}} V dx \int_{G^{(m)}} |u'|^2 dx \leq (n + 1)^{-2}|G| \int_G V dx.
\]

For different values of \( m \), the numbers \( k_m \) may be different. Denote by \( k = k(V) \) the minimal number such that \( \# \{m : k_m = k\} = \infty \). Thinning out the sequence \( \{G^{(m)}\} \), we find points \( \overline{x}_1, \ldots, \overline{x}_k \in \overline{G} \) (not necessarily different), such that \( x_j^m \to \overline{x}_j \) as \( m \to \infty \) for all \( j = 1, \ldots, k \).

Now we show that the desired result is satisfied for the points found. Indeed, let \( u \in L^{1,2}(G) \), \( \int_G |u'|^2 dx = 1 \) and \( u(\overline{x}_1) = \ldots = u(\overline{x}_k) = 0 \). Then we have for each \( m \):

\[
(4.5) \quad \left( \int_{G^{(m)}} |u|^2 V dx \right)^{1/2} \leq \left( \int_{G^{(m)}} |u - P_m u|^2 V dx \right)^{1/2} + \left( \int_{G^{(m)}} \sum_{j=1}^{k} u(x_j^m) \chi_j^m|^2 V dx \right)^{1/2}.
\]

If \( m \) is large enough, then \( |u(x_j^m)| = |u(x_j^m) - u(\overline{x}_j)| < \varepsilon \) for all \( j = 1, \ldots, k \) where \( \varepsilon > 0 \) is arbitrarily small. Then

\[
(4.6) \quad \int_{G^{(m)}} \sum_{j=1}^{k} u(x_j^m) \chi_j^m|^2 V dx \leq \varepsilon^2 \int_{G^{(m)}} V dx \leq \varepsilon^2 \int_G V dx.
\]

Letting \( m \to \infty \) in the inequality (4.5) and taking (4.4) and (4.6) into account, we arrive at the desired inequality (3.4).

4.3. **Proof of Theorem 1.1.** Let first \( V \in L_+(G) \). Fix a number \( n \in \mathbb{N} \) and find points \( \overline{x}_1, \ldots, \overline{x}_k \) according to Theorem 3.2. The subspace

\[
\{u \in H^1(G, x_0) : u(\overline{x}_1) = \ldots = u(\overline{x}_k) = 0\} \subset H^1(G, x_0)
\]
is of codimension $k \leq n$, and for $n > 1$ the inequality
\begin{equation}
\lambda_n(B_V) \leq |G|n^{-2} \int_G Vdx, \quad V \geq 0
\end{equation}
follows from (3.4) by the variational principle. The same inequality for $n = 1$ is implied by the estimate (3.6). This completes the proof of (1.2) for $V \geq 0$. The result for sign-indefinite $V$ follows from (4.7) by the variational principle, due to the inequalities $\pm b_V[u] \leq b_{V\pm}[u]$.

The asymptotics (1.3) is an almost immediate consequence of the estimate (1.2). Indeed, suppose first that the weight function $V$ has compact support, i.e. it vanishes outside a compact subgraph $G \subset \mathcal{G}$. Then we insert additional conditions $u(v) = 0$ at all the vertices $v \in G$, $v \neq x_0$. Since the number of these conditions is finite, they do not affect the spectral asymptotics. For the new problem the result follows from the well known asymptotic formula for a single interval, and we are done. In the general case we fix $\varepsilon > 0$ and find a compactly supported function $V_\varepsilon$ such that $\|V - V_\varepsilon\|_1 < \varepsilon$. For the operator $B_{V_\varepsilon}$ the asymptotics (1.3) is satisfied, and for the operator $B_V - B_{V_\varepsilon} = B_{V-V_\varepsilon}$ we have the estimate
\begin{equation}
\lambda_n(|B_V - B_{V_\varepsilon}|) \leq |G|\varepsilon n^{-2}, \quad \forall n \in \mathbb{N}.
\end{equation}
Now the asymptotics (1.3) for the operator $B_V$ is implied by Lemma on continuity of the asymptotic coefficients, see [4], Lemma 1.18.

5. Complementary remarks. Applications to integral operators on graphs

5.1. On sharpness of the estimate (1.2). Consider the simplest case when $G$ is a single segment $[a, b] \subset \mathbb{R}$ and $V \geq 0$. The analog of (1.2) for the eigenvalue problem $-\Lambda u'' = Vu, \quad u(a) = u(b) = 0$ is the inequality
\begin{equation}
4n^2\Lambda \leq (b - a) \int_a^b Vdx.
\end{equation}
Its sharpness was discussed in [4], Section 3.2. It was shown there that for any $\varepsilon > 0$ and any fixed $n_0 \in \mathbb{N}$ a function $V = V_{\varepsilon, n_0}$ can be found, such that the corresponding eigenvalue $\Lambda_{n_0}$ satisfies the inequality
\begin{equation}
4n_0^2\Lambda_{n_0} \geq (1 - \varepsilon)(b - a) \int_a^b Vdx.
\end{equation}
So, the constant in (5.1) is sharp.

Turn now to the eigenvalue problem $-\lambda u'' = Vu, \quad u'(0) = u(L) = 0$, which is just our problem (1.1) for the graph $G = [0, L]$, with $x_0 = L$. Each eigenvalue $\lambda_n$ of this problem coincides with the eigenvalue $\Lambda_{2n-1}$ for the equation $-\lambda u'' = V(|x|)u$ on the interval
$(-L, L)$, with the zero boundary conditions at both ends. Using the above result, we find a function $V \geq 0$ such that

$$(2n_0 - 1)^2 \lambda_{n_0} = (2n_0 - 1)^2 \Lambda_{2n_0 - 1} \geq (1 - \varepsilon)L \int_0^L V dx.$$ 

Taking here $n_0 = 1$, we see that constant factor 1 in (1.2) is the best possible. However, for each particular $n$ the factor 1 is probably not sharp.

Note also that the inequality (3.1) implies the estimate

$$\|B_V\| \leq \text{diam}(G) \int_G |V| dx,$$

where $\text{diam}(G) := \sup\{\rho(x, y) : x, y \in G\}$. It follows that the inequality (1.2) can be replaced by

$$\lambda_{\pm n} \leq \min\left(\frac{|G|}{n^2}, \text{diam}(G)\right) \int_G V_{\pm} dx.$$ 

This can be useful when dealing with graphs of small diameter but large total length.

5.2. Singular numbers of the operator $a(x)(-\Delta)^{-1/2}$ in $L^2(G)$. Recall that the singular numbers ($s$-numbers) of a compact operator $B$ acting between two Hilbert spaces, are defined as the non-zero eigenvalues of any of two self-adjoint compact operators $|B| := (B^*B)^{1/2}$ and $|B^*| := (BB^*)^{1/2}$.

According to the Hilbert space theory, for each $u \in H^1(G, x_0)$ we have $\|u\|_2 = \|(\Delta)^{1/2}u\|_2$. Here $\Delta$ is the operator whose rigorous description was given in Subsection 3.2. For any $V \in L_+(G)$ and for any non-zero element $u \in H^1(G, x_0)$ the equality holds

$$\frac{\int_G |u| \sqrt{V} dx}{\int_G |u'||2 dx} = \left(\frac{\|V^{1/2}(-\Delta)^{-1/2}w\|_2}{\|w\|_2}\right)^2, \quad w = (-\Delta)^{1/2}u.$$ 

Taking into account the variational description of the eigenvalues and the $s$-numbers, we conclude from (5.2) that

$$\lambda_n(B_V) = s_n^2(V^{1/2}(-\Delta)^{-1/2}), \quad \forall n \in \mathbb{N}.$$ 

Let now $a(x)$ be an arbitrary function from $L^2(G)$ and $V(x) = |a(x)|^2$, then $a(x) = \psi(x)V^{1/2}(x)$ where $|\psi(x)| = 1$ a.e. Multiplication by $\psi$ is a unitary operator. Therefore, the operators $V^{1/2}(x)(-\Delta)^{-1/2}$ and $a(x)(-\Delta)^{-1/2}$ are compact simultaneously, and have the same $s$-numbers. This leads to a useful consequence of the estimate (1.2),

$$s_n(a(x)(-\Delta)^{-1/2}) \leq \frac{|G|^{1/2}\|a\|_2}{n}, \quad \forall n \in \mathbb{N}; \quad a \in L^2(G).$$
5.3. **The case $V \equiv 1$.** In this case $B_V = (-\Delta)^{-1}$, thus $\lambda_n(B_V) = \lambda_{n-1}^V(-\Delta)$ and the relations (1.2) and (1.3) turn into

$$|G|^2 \lambda_n(-\Delta) \geq n^2, \quad \forall n \in \mathbb{N};$$

(5.4)

$$\frac{\sqrt{\lambda_n(-\Delta)}}{n} \to \frac{\pi}{|G|}.$$  

Even for this simplest case, the estimate (5.4) is informative, since it is uniform with respect to all graphs of a given length, independently of their combinatorial structure. By means of standard perturbation arguments, the asymptotics (5.5) extends to the operators $-\Delta + q$ of Sturm – Liouville type with the real-valued, bounded potential $q$. This considerably improves the result of Theorem 5.4 of the paper [5]. For such operators on the so-called regular trees, another proof of the asymptotics (5.5) was recently given in [9].

5.4. **Estimates of singular numbers for integral operators on graphs.** The above results admit immediate applications to the estimates of singular numbers of the integral operators on graphs. We restrict ourselves with the simplest case when the operator acts in $L^2(G)$ according to the formula

$$ (Ku)(x) = \int_G K(x, y)u(y)dy. $$

(5.6)

We suppose that the kernel $K(x, y)$ belongs to the space $L^2_y(G, W^1_x(G))$. This means that $K$ is measurable on $G \times G$, for almost all $y \in G$ the function $K(\cdot, y)$ lies in the space $L^{1,2}(G)$ and, moreover,

$$ \mathcal{M}(K, G) := \int_G \int_G (|K(x, y)|^2 + |G|^2 |K'_x(x, y)|^2) dx dy < \infty. $$

The expression for $\mathcal{M}(K, G)$ is homogeneous with respect to the similitudes of the graph $G$. This explains why the factor $|G|^2$ is included.

**Theorem 5.1.** Let $G$ be a graph of finite total length and $K \in L^2_y(G, W^1_x(G))$. Then the following inequality is satisfied for the $s$-numbers of the operator $K$ given by (5.6):

$$ \sum_{n=1}^{\infty} n^2 s^2_n(K) \leq 32 |G|^2 \mathcal{M}(K, G). $$

(5.7)

If in addition there exists a point $x_0 \in G$ such that $K(x_0, y) = 0$ for almost all $y \in G$, then the inequality (5.7) can be refined:

$$ \sum_{n=1}^{\infty} n^2 s^2_n(K) \leq 8 |G|^2 \int_G \int_G |K'_x(x, y)|^2 dx dy. $$

(5.8)
Proof. We follow the approach of [7], Section II.10.4.

We start with the proof of the second statement of Theorem. Its assumption means that the function $K(\cdot, y)$ lies in $H^1(G, x_0)$ for almost all $y \in G$. This allows one to write

$$K(\cdot, y) = (-\Delta_x)^{-1/2}L(\cdot, y)$$

where $L(\cdot, y) = (-\Delta_x)^{1/2}K(\cdot, y)$ and $-\Delta_x$ is the operator $-\Delta$ (with the boundary conditions as in ([13], cf. Subsection 3.2), acting in the variable $x$. This representation of the kernel yields factorization of the corresponding operator, $K = (-\Delta_x)^{-1/2}L$.

From the estimate (5.3) for $a \equiv 1$ we derive $s_n((-\Delta)^{-1/2}) \leq |G|n^{-1}$. The operator $L$ is of the Hilbert-Schmidt class and moreover,

$$\sum_{n=1}^{\infty} s_n^2(L) = \int_G \int_G |L(x, y)|^2 dx dy = \int_G \int_G |K_x(x, y)|^2 dx dy. \quad (5.9)$$

Due to an inequality by Ky Fan, see [7], Corollary II.2.2,

$$s_{2n}(K) \leq s_{2n-1}(K) \leq s_n((-\Delta)^{-1/2}) s_n(L) \leq |G|n^{-1}s_n(L).$$

Together with (5.9) this implies (5.8).

Let us turn to the proof of (5.1). The function $K(\cdot, y)$ is continuous on $G$ for almost all $y \in G$. Choose a point $x_0 \in G$ and split the kernel $K$ into the sum $K = K^0 + \tilde{K}$ where $K^0(x, y) = K(x_0, y)$. The operator $K^0$ has rank one, and its only non-zero singular number is given by

$$s_1^2(K^0) = |G| \int_G |K(x_0, y)|^2 dy \leq 2M(K, G). \quad (5.10)$$

Let us explain the latter inequality. Any function $w \in L^{1,2}(G)$ assumes its mean value $\hat{w} = |G|^{-1}\int_G w dx$ at some point $\xi \in \overline{G}$. Using the inequality (5.1) (which extends from $G$ to any $\xi \in \overline{G}$ by the continuity), we obtain $|w(x_0) - \hat{w}|^2 \leq |G| \int_G |w'|^2 dx$. Also, $|G||\hat{w}|^2 \leq \int_G |w|^2 dx$. Therefore, $|G||w(x_0)|^2 \leq 2 \int_G (|w|^2 + |G|^2|w'|^2) dx$, whence (5.10).

Since $K^0$ is a rank one operator, one has $s_{n+1}(K) \leq s_n(\tilde{K}), \forall n \in \mathbb{N}$. For $\tilde{K}$ the inequality (5.8) is satisfied, because $\tilde{K}(x_0, y) = 0$ a.e. Therefore,

$$\sum_{n=2}^{\infty} n^2 s_n^2(\tilde{K}) \leq \sum_{n=1}^{\infty} (n + 1)^2 s_n^2(\tilde{K}) \leq 32|G|^2 \int_G \int_G |K'_x(x, y)|^2 dx dy. \quad (5.11)$$

Further, (5.8) implies $s_1(\tilde{K}) \leq \sqrt{8}M(K, G)$. The inequality (5.7) follows from here, (5.11) and (5.10), due to the triangle inequality $s_1(K) \leq s_1(\tilde{K}) + s_1(K^0) \leq 3\sqrt{2}M(K, G)$. \hfill \qed
Corollary 5.2. Under the assumptions of Theorem 5.1 the singular numbers $s_n(K)$ of the operator (5.6) satisfy the estimate

$$s_n(K) \leq \frac{4\sqrt{6}}{n^{3/2}} M^{1/2}(K, G), \quad \forall n \in \mathbb{N}$$

and besides, $s_n(K) = o(n^{-3/2})$.

Proof. Denote $s_n = s_n(K)$ and $C^2 = 32M(K, G)$. The estimate (5.12) is implied by the inequality

$$\frac{n^3}{3} s_n^2 \leq s_n^2 \sum_{k=1}^{n} k^2 \leq \sum_{k=1}^{n} k^2 s_k^2 \leq C^2, \quad \forall n \in \mathbb{N}.$$

The relation $s_n(K) = o(n^{-3/2})$ follows from the inequality

$$cn^3 s_n^2 \leq \sum_{k=[n/2]}^{n} k^2 s_k^2, \quad n > 1, \ c > 0$$

in which the right-hand side tends to zero as $n \to \infty$, due to the convergence of the series in (5.7). \qed

We would like to emphasize that the estimates (5.7), (5.8) and (5.12) are uniform with respect to all graphs of a given length. Specific values of the constants in these estimates are not so important.

In the same way, it is possible to study similar operators (with $dy$ in (5.6) replaced by $d\mu(y)$) acting between the spaces $L^2(G, \mu)$ and $L^2(G, \nu)$, where $\mu$ and $\nu$ are finite Borelian measures. Such operators appear in various applications, see the paper [4]. Note also that for the case of trees an analog of Theorem 5.1 for $l$ times differentiable kernels can be derived from Theorem 6.1 of the next section.

6. Spaces $H^1(T, x_0)$ and operators $B_{l, V}$.

Let us discuss the higher order analogs of the space $L^{1,2}$. Here a serious obstacle arises, since the continuity of derivatives $u', \ldots, u^{(l-1)}$ at the vertices should be included in the definition. However, the derivatives of odd order change their sign depending on the orientation on edges, so that for $l > 1$ the space $L^{1,2}(G)$ can be well defined only for oriented graphs, and for different choices of orientation such spaces are substantially different. For this reason, we define the spaces $L^{1,2}$ only on trees, since for them a natural orientation does exist.

So, let $G = T$ be a tree of finite total length and let $x_0 \in T$ be a vertex selected (the root). The natural partial ordering on the rooted tree $\{T, x_0\}$ is introduced as follows:

$$x \leq y \iff x \in \langle x_0, y \rangle.$$

Recall that $\langle x_0, y \rangle$ is the unique simple polygonal path in $T$ connecting $x_0$ with $y$. We always parametrize the edges of $T$ in the direction, compatible with this partial ordering.
Now we are in a position to define the space $H^l(T, x_0)$ for arbitrary $l \in \mathbb{N}$. A function $u$ on $T$ belongs to $H^l(T, x_0)$ if $u$ is continuous on $T$, the restriction of $u$ to each edge $e$ lies in $H^l(e)$, the functions $u', \ldots, u^{(l-1)}$ extend from $T \setminus V(T)$ to the whole of $T$ as continuous functions, $u(x_0) = \ldots = u^{(l-1)}(x_0) = 0$ and $u^{(l)} \in L_2(T)$. We consider $H^l(T, x_0)$ as the Hilbert space with the scalar product $(u, v)_{H^l(T, x_0)} = (u^{(l)}, v^{(l)})_{L_2(T)}$ and the corresponding norm.

Let $\xi, x \in T$ and $\xi \preceq x$. Consider the Taylor polynomial

$$P_{l-1}(t; u, \xi) = \sum_{k=0}^{l-1} \frac{u^{(k)}(\xi) t^k}{k!},$$

then, due to our agreement about orientation,

$$u(x) - P_{l-1}(\rho(\xi, x); u, \xi) = \frac{1}{(l-1)!} \int_{[\xi, x]} u^{(l)}(y) \rho^{l-1}(x, y) dy.$$ 

By Cauchy’s inequality,

$$((l-1)!)^2 |u(x) - P_{l-1}(\rho(\xi, x); u, \xi)|^2 \leq \frac{\rho^{2l-2}(\xi, x)}{2l-1} \int_{[\xi, x]} |u^{(l)}(y)|^2 dy. \tag{6.1}$$

Given a function $V \in L^1(T)$, let $b_V$ is the corresponding quadratic form, cf. (3.3). It follows from the inequality (6.1) for $\xi = x_0$ that

$$|b_V[u]| \leq C'(l) \|T\|^{2l-1} \|u^{(l)}\|^2 \int_T |V(x)| dx, \quad \forall u \in H^l(T, x_0),$$

$$C'(l) = ((l-1)!)^{-2}(2l-1)^{-1}. \tag{6.2}$$

Therefore, the quadratic form $b_V[u]$ generates in $H^l(T, x_0)$ a bounded linear operator which we denote by $B_{b_V}$. Its eigenpairs $\{\lambda, u\}$ correspond to the problem

$$\lambda(-\Delta) u = V u, \quad u(x_0) = u'(x_0) = \ldots = u^{(l-1)}(x_0) = 0; \quad u^{(l)}(v) = \ldots = u^{(2l-1)}(v) = 0 \text{ if } v \in \partial T \setminus \{x_0\}. \tag{6.3}$$

For rooted trees, Theorem 6.1 is a particular case of the following result.

**Theorem 6.1.** Let $T$ be a rooted tree of finite total length, $x_0$ be its root, and let $V = V \in L^1(T)$. Then the eigenvalues of the problem (6.2) satisfy the inequality

$$\lambda_n^\pm \leq C(l) \frac{|T|^{2l-1} \int_T V_\pm dx}{n^{2l}}, \quad C(l) = \frac{l^{2l}}{(l-1)!^2 (2l-1)}, \quad \forall n \in \mathbb{N}. \tag{6.3}$$

Along with the estimate (6.3), the Weyl-type asymptotics holds,

$$n \left( \frac{\lambda_n^\pm}{\sqrt{n}} \right)^{\frac{1}{l}} \to \pi^{-1} \int_T (V_\pm(x))^{\frac{1}{l}} dx, \quad n \to \infty.$$
Outline of the proof. We discuss only the estimate (6.3) for the case of compact trees and \( V \in L_+(T) \), since the rest needs no serious changes compared with the proof of Theorems 3.2 and 1.1. We also suppose that \( d(x_0) = 1 \). Otherwise, the operator \( B_{l,V} \) splits into the orthogonal sum of similar operators for each of \( d(x_0) \) subtrees constituting the canonical partition of the punctured tree \( \{T, x_0\} \), cf. Section 2. The estimate (6.3) for \( B_{l,V} \) easily follows from the same estimate for its corresponding parts.

Consider a function of subtrees \( T \subset T \):

\[
\Phi(T) = |T|^{1-\frac{1}{2l}} \left( \int_T V \, dx \right)^{\frac{1}{2l}},
\]

(6.4)

and let \( \tilde{\Phi}(T, x) \) be the function of punctured subtrees associated with \( \Phi \), cf. (2.4). Suppose that the punctured subtrees \( \{T_j, x_j\}, j = 1, \ldots, k \) constitute a partition of \( T \) and let \( \chi_j \) be the characteristic function of \( T_j \). It can be easily derived from (6.1) that for \( u \in H^l(T, x_0) \) the inequality

\[
\int_T |u - \sum_{j=1}^k P_{l-1}(\rho(x, x_j); u, x_j)\chi_j|^2 V \, dx \leq C''(l)\|u^{(l)}\|_2^2 \left( \max_{j=1, \ldots, k} (\tilde{\Phi}(T_j, x_j)) \right)^{2l},
\]

holds, provided all the subtrees \( \{T_j, x_j\} \) are oriented coherently to the orientation on \( T \), that is if \( x_j \preceq x \) for any \( x \in T_j \).

Theorem 2.1 applies to the function \( \Phi \) but this does not lead automatically to the inequality (6.3). Indeed, we have to check that all the subtrees \( \{T_j, x_j\} \) are properly oriented. For this purpose, let us return to Lemma 2.2 whose consequence is Theorem 2.1. The proof of Lemma started with choosing a vertex \( v_0 \in \partial T \), then a path \( P \) and subtrees \( T^+_x \) for each \( x \in P \) were constructed. The assumption \( d(x_0) = 1 \) means that \( x_0 \in \partial T \). If we take \( v_0 = x_0 \), then all the subtrees \( T^+_x \) are oriented coherently to the orientation on \( T \), and the scheme goes through.

Applying inequality (2.5) to the function \( \Phi \) introduced by (6.4) and using the variational principle, we find that

\[
\lambda_{Nl+1}(B_{l,V}) \leq C'(l) \frac{|T|^{2l-1}}{N^{2l}} \int_T V \, dx, \quad \forall N \in \mathbb{N}; \quad V \in L_+(T).
\]

The estimate (6.3) for all \( n \in \mathbb{N} \), with \( C(l) = C'(l)l^{2l} \), follows from here due to the monotonicity in \( n \) of eigenvalues.

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