Perfect Reconstruction of Sparse Signals via Greedy Monte-Carlo Search

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We propose a Monte-Carlo-based method for reconstructing sparse signals in the formulation of sparse linear regression in a high-dimensional setting. The basic idea of this algorithm is to explicitly select variables or covariates to represent a given data vector or responses and accept randomly generated updates of that selection if and only if the energy or cost function decreases. This algorithm is called the greedy Monte-Carlo (GMC) search algorithm. Its performance is examined via numerical experiments, which suggests that in the noiseless case, GMC can achieve perfect reconstruction in undersampling situations of a reasonable level: it can outperform the $\ell_1$ relaxation but does not reach the algorithmic limit of MC-based methods theoretically clarified by an earlier analysis.¹ Additionally, an experiment on a real-world dataset supports the practicality of GMC.

1. Introduction

Variable selection is an important problem in statistics, machine learning and signal processing. One of the recent attractive topics regarding this study is compressed sensing,²⁻⁵ in which a sparse signal is reconstructed from (linear) measurements with smaller measurement number than the overall signal dimension. If we denote the measurement result as $y \in \mathbb{R}^M$ and the measurement matrix as $A \in \mathbb{R}^{M \times N}$, a naive formulation of compressed sensing can be...
written as
\[
\hat{x}(K) = \arg \min_x \left\{ \frac{1}{2} ||y - Ax||^2 \right\} \text{ subj. to } ||x||_0 \leq K,
\]
where \( || \cdot ||_0 \) denotes the \( \ell_0 \) norm giving the number of non-zero components and the \( K \)-sparse vector \( \hat{x}(K) (\in \mathbb{R}^N) \) becomes our estimator of the true signal \( x^0 \) embedded in the data generation process. This formulation requires solving discrete optimization problems; therefore, it involves serious computational difficulty when \( N \) and \( K \) are large.\(^6\) Due to this computational difficulty, certain approximations are needed to treat high-dimensional datasets.

Some representative approximations, such as orthogonal matching pursuit (OMP)\(^7,8\) and the iteratively reweighted least squares method (IRLS),\(^9\) are available. OMP is an approximate algorithm to directly solve eq. (1) by incrementing the set of used columns of \( A \) to approximate \( y \) in a greedy manner, whereas IRLS first relaxes the \( \ell_0 \) constraint to an \( \ell_p \) constraint, where \( p > 0 \), and then recursively solves the least squares problem with coefficients reweighted by the previous least squares solution; IRLS effectively solves eq. (1) with a small enough \( p \). Another more common method to find an approximate solution of eq. (1) is to relax \( ||x||_0 \) to \( ||x||_1 = \sum_{i=1}^N |x_i| \).\(^{10-13}\) This relaxation converts the problem into a convex optimization problem; hence, a unique solution can be efficiently obtained using versatile solvers. Even with this relaxation, it has been shown that a perfect reconstruction of the signal in the noiseless limit is possible in undersampling situations of a reasonable level.\(^{14,15}\) These findings have motivated using the \( \ell_1 \) relaxation (LASSO) for a wide range of problems and inspired a search for better inference methods enabling a perfect reconstruction with fewer measurements than the \( \ell_1 \) case with a reasonable computational cost. A successful example of this search is based on the Bayesian framework:\(^{16}\) it showed that a perfect reconstruction is possible with fewer measurements than the \( \ell_1 \) case with a sufficiently low computational cost when employing the so-called approximate message passing (AMP) algorithm. However, this impressive result is not perfect because prior knowledge about the generation of \( x^0 \) and \( y \) is required in the Bayesian framework but is not always available in realistic settings, and also its performance is guaranteed only for a certain class of measurement matrices. Solvers have their own advantages and disadvantages, and the appropriate choice depends on the available resources and purpose of the signal processing. Therefore, it is useful to increase the options of such solvers.

Under these circumstances, some of the authors proposed a solver based on Monte-Carlo (MC) sampling methods. The solver employs simulated annealing (SA),\(^{17}\) which is a versatile metaheuristic for optimization, and its performance was examined numerically and
analytically.\textsuperscript{1,18–20} The analytical result shows that a fairly wide parameter region exists in which the phase space structure is rather simple and a perfect reconstruction is possible; the existence limit of this region defines the algorithmic limit to achieve a perfect reconstruction and this limit is shown to be comparable with that of the Bayesian approach.\textsuperscript{11} Therefore, the SA-based algorithm, which is applicable for any measurement matrices, can exhibit comparable performance with a Bayesian method even without prior knowledge of the generative processes of \(x^0\) and \(y\), although its computational cost is higher than that of AMP.

The above result, including the larger computational cost of the SA-based algorithm and the simple structure of the phase space clarified by the analytical computation, inspires a further simplification of the MC-based algorithm, which is the goal of this study. In particular, we propose a greedy algorithm to solve eq. (1) based on MC sampling, and call it the greedy Monte-Carlo search (GMC) algorithm. GMC can also be regarded as a version of the SA-based algorithm quenched to zero temperature. Below we examine the performance of GMC via numerical experiments. The result suggests that the GMC performance does not reach that of the SA-based method but is better than LASSO.

2. Formulation and algorithm

Let us suppose a data vector \(y \in \mathbb{R}^M\) is generated from the measurement matrix \(A \in \mathbb{R}^{M \times N}\) and a \(K_0\)–sparse signal \(x^0 \in \mathbb{R}^N, \|x^0\|_0 = K_0\) via

\[
y = Ax^0.
\] (2)

We aim to infer the signal \(x^0\) from a given \(A\) and \(y\) for the underdetermined situation \(M < N\). We denote the estimated signal as \(\hat{x}\). To solve this underdetermined problem, we assume that the number of non-zero components \(K_0 = \|x^0\|_0\) is smaller than or equal to \(M\), where \(K_0 \leq M\).

The starting point of our inference framework is the least squares method. To apply this method to the underdetermined situation, we explicitly specify the variables or columns of \(A\) used to represent \(y\) by a binary vector \(c \in \mathbb{R}^N\): if the \(i\)th variable or column of \(A\) is used, then \(c_i\) is one; otherwise, it is zero. We call \(c\) the sparse weight. For any \(N\)–dimensional vector \(z \in \mathbb{R}^N\), the components whose corresponding sparse weights are unity (zero) are called active (inactive) components; we denote the vector of active components as \(z_c\), and that of inactive ones as \(z_{\bar{c}}\) (\(\bar{c} = 1 - c\)). Similarly, the submatrix of \(A\) consisting only of the active components is denoted as \(A_c\). If \(K = \sum_i c_i\) is smaller than or equal to \(M\), the following least squares estimator is unambiguously defined:

\[
\hat{x}_c(c) = \arg \min_{x_c \in \mathbb{R}^K} \left\{ \frac{1}{2} \|y - A_c x_c\|_2^2 \right\} = (A_c^T A_c)^{-1} A_c^T y, \tag{3a}
\]
\( \hat{x}_c(c) = 0. \) (3b)

The full dimensional form of this solution is denoted as \( \hat{x}(c) \). Using this least squares estimator, the original problem (1) can be represented as

\[
\hat{x}(K) = \hat{x}(\hat{c}),
\]

\[
\hat{c} = \arg \min_{c: \sum_i c_i = K} \left\{ \frac{1}{2} \| y - A \hat{x}_c(c) \|^2 \right\}.
\]

Therefore, we can solve the problem (1) by solving the combinatorial optimization problem (5) with respect to \( c \). To solve this problem, the SA-based algorithm was proposed in.\(^{19} \)

Although the SA-based algorithm has been shown to be effective.\(^{1,19,20} \) herein we examine a further simplified approach. In particular, we propose a simpler algorithm that solves eq. (5) in a greedy manner using MC sampling. We call this algorithm GMC, and the details are given in the following subsection. For convenience, we introduce a mean-square error (MSE) when representing \( y \) such that

\[
\epsilon_y(c|y, A) = \frac{1}{2M} \| y - A \hat{x}_c(c) \|^2.
\]

and call this the output MSE. This plays the role of ‘energy’ in the MC update as shown below.

2.1 GMC

The outline of GMC is as follows. Starting from a given initial configuration or state of \( c \), we update the state in an MC manner. We randomly generate a new state \( c' \) and update it as \( c \rightarrow c' \) if and only if the output MSE or energy decreases. If the energy cannot be decreased by any single ‘local flip’, then the algorithm stops and returns the current state as the output.

Comprehensively, during the update, we require the number of non-zero components \( K = \sum_i c_i \) to remain constant. To efficiently generate a new state satisfying this requirement, we employ the ‘pair flipping’ of two sparse weights: one equals 0 and the other equals 1. In particular, we randomly choose an index \( i \) from ONES \( \equiv \{ k | c_k = 1 \} \) and another index \( j \) from ZEROS \( \equiv \{ k | c_k = 0 \} \) and we set \( (c'_i, c'_j) = (0, 1) \) while keeping the other components unchanged. The pseudocode for pair flipping is given in Alg. 1. We define one MC step (MCS) as \( N \) pair-flipping trials.

Pair flipping gives a concrete meaning to the ‘local flip’ mentioned above. Accordingly, we can introduce the stopping condition for GMC as follows. If the configuration of \( c \) is invariant during \( t_{\text{wait}} \) MCSs, we examine all states accessible via a single pair flip from the current state \( c \) and compute the associated energy values; if there is no lower energy state
Algorithm 1 MC update with pair flipping

1: \textbf{procedure} MCpf\((c, y, A)\) \hspace{1cm} \triangleright \text{MC routine with pair flipping}

2: \hspace{1cm} \text{ONES} \leftarrow \{k|c_k = 1\}, \text{ZEROS} \leftarrow \{k|c_k = 0\}

3: \hspace{1cm} \text{randomly choose } i \text{ from ONES and } j \text{ from ZEROS}

4: \hspace{1cm} c' \leftarrow c

5: \hspace{1cm} (c'_i, c'_j) \leftarrow (0, 1)

6: \hspace{1cm} (\epsilon_y, \epsilon'_y) \leftarrow (\epsilon_y(c|y, A), \epsilon_y(c'|y, A)) \hspace{1cm} \triangleright \text{Calculate energy}

7: \hspace{1cm} \text{if } \epsilon'_y < \epsilon_y \text{ then}

8: \hspace{2cm} c \leftarrow c'

9: \hspace{2cm} \epsilon_y \leftarrow \epsilon'_y

10: \hspace{1cm} \text{end if}

11: \hspace{1cm} \text{return } c, \epsilon_y

12: \hspace{1cm} \text{end procedure}

than the current one, then the algorithm stops; otherwise, the state is updated to the lowest energy state of the locally accessible ones and the usual update is continued. The pseudocode for the entire GMC procedure is given in Alg. 2. Typically, we set \(t_{\text{wait}} = 10\) in the following experiments.

If we denote the necessary MCSs until convergence as \(N_{\text{conv}}\), the scaling of the total computational cost of GMC is \(O(K(N - K)N_{\text{energy}} + N_{\text{conv}}NN_{\text{energy}})\), where the first term is for the search of locally accessible states and the last term is for the MC update; \(N_{\text{energy}}\) is the computational cost of the energy, which is estimated as \(O(K^3 + MK^2)\) when a naive method of matrix inversion is used but can be reduced to \(N_{\text{energy}} = O(K^2 + MK)\) by using the technique mentioned in.\(^{19}\) Certainly, for the scaling, the computational cost of GMC is in the same order as that of the SA-based algorithm; however, GMC is actually faster because it does not need the annealing procedure required in the SA-based algorithm, reducing the computational cost by a factor of \(O(1)\). In the next section, we examine the actual behaviour of GMC using numerical experiments. The scaling of \(N_{\text{conv}}\) will also be examined.

3. Numerical experiments

Herein, the performance of the GMC algorithm is numerically examined. Both simulated and real-world datasets are used.
Algorithm 2 Greedy Monte-Carlo Search (GMC)

1: procedure GMC(c\text{ini}, y, A, t\text{wait})
2: \hspace{1em} c ← c\text{ini}, (M, N) ← size(A), K ← \sum_i c_i, t ← 0 \hspace{1em} \triangleright \text{Initialization}
3: while do
4: \hspace{2em} c\text{pre} ← c
5: \hspace{2em} for i = 1 : N do \hspace{1em} \triangleright \text{One MCS}
6: \hspace{3em} (c, \epsilon_y) ← MC_{PF}(c, y, A)
7: \hspace{2em} end for
8: \hspace{2em} if c = c\text{pre} then \hspace{1em} \triangleright \text{State update check}
9: \hspace{3em} t ← t + 1
10: \hspace{2em} else
11: \hspace{3em} t ← 0
12: \hspace{2em} end if
13: \hspace{2em} if t ≥ t\text{wait} then \hspace{1em} \triangleright \text{Exhaustive search of locally accessible states}
14: \hspace{3em} Set \mathcal{E} as a (K(N − K))-dim vector, and C as a \text{N} \times (K(N − K))-dim matrix
15: \hspace{3em} a ← 1 \hspace{1em} \triangleright \text{Index for locally accessible states}
16: \hspace{3em} for i = 1 : K do
17: \hspace{4em} k ← ONES(i) \hspace{1em} \triangleright \text{i-th component of ONES}
18: \hspace{4em} for j = 1 : (N − K) do
19: \hspace{5em} c' ← c \hspace{2em} \triangleright j-th component of ZEROS
20: \hspace{5em} l ← ZEROS(j) \hspace{1em} \triangleright \text{Pair flipping}
21: \hspace{5em} (c'_k, c'_l) ← (0, 1)
22: \hspace{5em} C(:, a) ← c', \mathcal{E}(a) ← \epsilon_y(c'|y, A),
23: \hspace{5em} a ← a + 1
24: \hspace{4em} end for
25: \hspace{4em} end for
26: \hspace{3em} if No component of \mathcal{E} is lower than \epsilon_y then \hspace{1em} \triangleright \text{Break the while loop}
27: \hspace{4em} Break
28: \hspace{3em} else
29: \hspace{4em} \hspace{1em} a^* = \arg\min_y \mathcal{E}(a) \hspace{1em} \triangleright \text{Choose one randomly if multiple minimums exist}
30: \hspace{4em} \epsilon_y ← \mathcal{E}(a^*), c ← C(:, a^*)
31: \hspace{4em} t ← 0
32: \hspace{3em} end if
33: \hspace{2em} end if
34: \hspace{2em} end while
35: \hspace{1em} return c
36: end procedure
3.1 Simulated dataset

In this subsection, we examine the performance of GMC on simulated datasets, particularly focusing on whether a perfect reconstruction of $x^0$ is achieved. To more directly quantify the reconstruction accuracy of $x^0$, we introduce another MSE in addition to $\epsilon_y$:

$$\epsilon_x(c) = \frac{1}{2N} ||x^0 - \hat{x}(c)||_2^2,$$

which is referred to as the input MSE hereafter.

Our simulated datasets are generated as follows. Each component of the design matrix $A \in \mathbb{R}^{M \times N}$ is independent and identically distributed (i.i.d.) from $\mathcal{N}(0, N^{-1})$; the non-zero components of the $K_0$-sparse signal $x^0 \in \mathbb{R}^N$ are also i.i.d. from $\mathcal{N}(0, 1/\rho_0)$, where $\rho_0 = K_0/N$ is the density of the non-zero components, setting the power of the signal to unity. The data vector is then obtained by eq. (2) given $A$ and $x^0$. This setup is identical to that of the theoretical computation in; thus, we can directly compare the results. We follow the limit of theoretical computation, where the thermodynamic limit $N \to \infty$ is considered while maintaining the ratios $\alpha = M/N, \rho = K/N, \rho_0 = K_0/N$ in $O(1)$.

3.1.1 Result for the simulated dataset

Herein, we examine the perfect reconstruction ratio obtained by GMC when the sparsity is correctly specified $\rho = \rho_0$. In particular, we prepare $N_{\text{init}}$ initial conditions with the correct sparsity $\rho = \rho_0$ for each sample of $(x^0, A)$, run GMC from the initial conditions and compute the ratio achieving a perfect reconstruction of the true signal. We call this ratio the success rate ($P_{\text{suc}}$). The ensemble of $P_{\text{suc}}$ for $N_{\text{samp}}$ samples quantifies the performance of GMC and is therefore investigated below. For simplicity, we fix $N_{\text{init}} = 100$ in the following experiments.

The average value of $P_{\text{suc}}$ over $N_{\text{samp}} = 100$ is listed against the system size in Table I. The examined system sizes are $N = 100, 200, 400, 800, 1000$ and the other parameters are

| $N$  | 100  | 200  | 400  | 800  | 1000 |
|------|------|------|------|------|------|
| $P_{\text{suc}}$ | 0.56±0.02 | 0.54±0.02 | 0.52±0.02 | 0.67±0.07 | 0.75±0.04 |

**Table I.** The system size dependence of the average value of $P_{\text{suc}}$. The dependence seems to be absent.

fixed to $\alpha = 0.5, \rho_0 = \rho = 0.2$. The error bar is the standard error over the $N_{\text{samp}}$ samples. The result implies that a system size dependence is very weak or absent and GMC can stably find the true solution.

Further, we investigate the ratio of samples exhibiting $P_{\text{suc}} > 0, P_{\text{samp}}$, for a wide range
of parameters \((\alpha, \rho_0)\). This allows us to capture the practical algorithmic limit of the perfect reconstruction by GMC. A heat map of \(P_{\text{samp}}\) for the overall region of \((\alpha, \rho_0)\) at \(N = 100\) is plotted in the left panel of Fig. 1. Herein, \(P_{\text{samp}}\) is computed for \(N_{\text{samp}} = 100\) samples. The theoretically derived algorithmic limit is depicted by a white line, and a sharp change of the colour occurs around the line. This implies that GMC achieves the limit. To examine this point, we further investigate the system size dependence of \(P_{\text{samp}}\). The result is indicated in the plot of \(P_{\text{samp}}\) against \(\alpha\) at \(\rho = 0.2\) for different system sizes. The black dotted line \((\ell_0)\) denotes the limit derived in\(^1\) while the black dashed line \((\ell_1)\) is the theoretical limit of the \(\ell_1\) relaxation.\(^{14, 15}\)

![Phase diagram](image_url)

**Fig. 1.** (Left) A heat map of \(P_{\text{samp}}\) at \(N = 100\) for the overall range of the parameters \((\alpha, \rho_0)\). The white line is the theoretically derived limit obtained in\(^1\) (Right) Plots of \(P_{\text{samp}}\) against \(\alpha\) at \(\rho = 0.2\) for different system sizes. The black dotted line \((\ell_0)\) denotes the limit derived in\(^1\) while the black dashed line \((\ell_1)\) is the theoretical limit of the \(\ell_1\) relaxation.\(^{14, 15}\)

Finally, we list the necessary MCSs until convergence in Table 2. Herein, the average value over \(N_{\text{samp}} = 100\) is shown with the standard error. The result implies a linear increase in \(N_{\text{conv}}\) with respect to \(N\). Therefore, the total computational cost of GMC is scaled as \(O(K(N - K)N_{\text{energy}} + N^2 N_{\text{energy}})\). This is certainly not cheap; however, GMC can achieve a perfect reconstruction even for rather large values of \(\rho_0\), which is not possible using other greedy algorithms such as OMP. This again suggests that GMC is a reasonable choice for...
Fig. 2. The system size dependence of $N_{\text{conv}}$. A linear increasing tendency with respect to $N$ is observed.

sparse linear regression.

3.2 Real-world dataset

Herein, we apply GMC to a dataset of Type Ia supernovae. Our dataset is a part of the data from,\textsuperscript{21,22} which was screened using a certain criterion.\textsuperscript{23} This dataset was recently treated using a number of sparse estimation techniques, and a set of important variables known to be empirically significant has been reproduced.\textsuperscript{20,23–25} In these studies, LASSO and $\ell_0$ cases were treated and cross-validation (CV) was employed for the hyperparameter estimation. We reanalyse this dataset using GMC. The parameters of the screened data are $M = 78$ and $N = 276$, and a standardization, i.e. centering $y$ and columns of $A$ and normalizing the columns of $A$ to be unit norm, was employed in the pre-processing.

Table II shows the GMC result for the supernovae dataset. Herein, we changed the value of $K$ from 1 to 5, and for each $K$ we examined ten different initial conditions and picked the lowest energy configuration among the tens as our final estimator $\hat{c}$. The result implies that the output MSE decreases as $K$ grows, and some variables such as “2” and “1” seem to be frequently selected.

| $K$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| $\epsilon_y$ | 0.0312 | 0.0204 | 0.0179 | 0.0158 | 0.0139 |
| variables | {2} | {1,2} | {1,2,233} | {1,2,94,233} | {2,6,170,221,225} |

Table II. The output MSE $\epsilon_y$ and selected variables’ indices for the Type Ia supernovae dataset from\textsuperscript{23} obtained by GMC. The final estimator of $c$ is the least $\epsilon_y$ configuration obtained by ten independent runs of GMC with different initial conditions.
Table III. LOO CV error obtained for $K = 1$–$5$ for the type Ia supernova data set.

| $K$ | 1  | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|----|
| $\epsilon_{CV}$ | 0.0328 | 0.0239 | 0.0261 | 0.0302 | 0.0342 |

To scrutinize this result, we also computed the leave-one-out (LOO) cross-validation (CV) error:

$$
\epsilon_{CV}(K = \sum_{i} \hat{c}^{\mu}_{i}, \forall \mu) = \frac{1}{2M} \sum_{\mu=1}^{M} \left( y_{\mu} - \sum_{i=1}^{N} A_{\mu i} x_{\mu i}(\hat{c}^{\mu}) \right)^{2},
$$

(8)

where $x^{\mu}(c)$ is the solution of eq. (3) given $c$ and “$\mu$th LOO system $\{A^{\mu}, y^{\mu}\}$”. $y^{\mu}$ is the data vector whose $\mu$th component is removed from the original one as $y^{\mu} = (y_{1}, y_{2}, \cdots, y_{\mu-1}, y_{\mu+1}, \cdots, y_{M})^{T}$, and $A^{\mu}$ is defined similarly. In our present case, the $\mu$th LOO estimator $\hat{c}^{\mu}$ is computed by running GMC for the $\mu$th LOO system, and thus can be different for each LOO system.

The result of a typical single run of GMC is shown in Table III. Here, the LOO CV error takes its minimum at $K = 2$, implying that the best model is obtained at this sparsity level. For further quantification of statistical correlations between variables, we count how many times each variable was selected in this LOO CV procedure, following the way of.

Table IV summarizes the results for five variables from the top for $K = 1$–$5$. This indicates that no variables other than “2” representing light width were chosen stably, whereas variable “1” representing color was selected with high frequencies for $K \leq 4$. Table IV shows that the frequency of “1” being selected is significantly reduced for $K = 5$. This is presumably due to the strong statistical correlations between “1” and the newly added variables. In addition, the results for $K \geq 4$ varied as we rerun GMC with different initial conditions. These observations mean that we could select at most only light width and color as the relevant variables. This conclusion is consistent with those of recent papers using a number of methods including the SA-based one. This provides additional evidence for the practicality of the GMC algorithm.

4. Summary

In this study, inspired by the theoretical result and the SA-based algorithm of, we proposed an MC-based greedy algorithm called GMC for sparse linear regression. GMC is simpler than the SA-based algorithm but still it can achieve the perfect reconstruction in undersampling situations of a reasonable level, as shown by the numerical experiments on synthetic datasets. These experiments also suggest that GMC can outperform the $\ell_{1}$ relaxation
Table IV. The top five variables selected by the $M = 78$ LOO CV for $K = 1–5$. The result is similar to that of\textsuperscript{20} using the SA-based method.

| $K$ | variables | times selected |
|-----|-----------|----------------|
| 1   | 2         | 78 0 0 0 0     |
| 2   | 1 275     | 78 77 1 0 0   |
| 3   | 1 233 14 15 | 78 78 70 3 2 |
| 4   | 1 233 94 97  | 78 76 74 66 2|
| 5   | 233 1 268 36 | 78 37 32 29 26|

which is the most commonly used method for sparse estimation. An additional experiment on a real-world dataset of supernovae also supported the practicality of GMC. These results imply that the energy landscape of the sparse linear regression problem is simple and exhibits a funnel-like structure\textsuperscript{26} in the reconstructable region. We believe that this finding inspires further algorithms of sparse linear regression as well as new models for complex systems, such as glasses and proteins, based on sparse variable selection.

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References

1) T. Obuchi, Y. Nakanishi-Ohno, M. Okada, and Y. Kabashima: Journal of Statistical Mechanics: Theory and Experiment 2018 (2018) 103401.

2) E. J. Candès and T. Tao: IEEE transactions on information theory 51 (2005) 4203.

3) E. J. Candès, J. Romberg, and T. Tao: IEEE Transactions on information theory 52 (2006) 489.

4) E. J. Candès and T. Tao: IEEE transactions on information theory 52 (2006) 5406.

5) D. L. Donoho: IEEE Transactions on information theory 52 (2006) 1289.

6) B. K. Natarajan: SIAM journal on computing 24 (1995) 227.

7) Y. C. Pati, R. Rezaiifar, and P. S. Krishnaprasad: Signals, Systems and Computers, 1993. 1993 Conference Record of The Twenty-Seventh Asilomar Conference on, 1993, pp. 40–44.

8) G. M. Davis, S. G. Mallat, and Z. Zhang: Optical engineering 33 (1994) 2183.

9) R. Chartrand and W. Yin: Iterative Reweighted Algorithms for Compressive Sensing (2008).

10) R. Tibshirani: Journal of the Royal Statistical Society. Series B (Methodological) (1996) 267.

11) N. Meinshausen and P. Bühlmann: 2004.

12) O. Banerjee, L. E. Ghaoui, A. d’Aspremont, and G. Natsoulis: Proceedings of the 23rd international conference on Machine learning, 2006, pp. 89–96.

13) J. Friedman, T. Hastie, and R. Tibshirani: Biostatistics 9 (2008) 432.

14) D. Donoho and J. Tanner: Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 367 (2009) 4273.

15) Y. Kabashima, T. Wadayama, and T. Tanaka: Journal of Statistical Mechanics: Theory and Experiment 2009 (2009) L09003.

16) F. Krzakala, M. Mézard, F. Sausset, Y. Sun, and L. Zdeborová: Journal of Statistical Mechanics: Theory and Experiment 2012 (2012) P08009.

17) S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi: science 220 (1983) 671.

18) Y. Nakanishi-Ohno, T. Obuchi, M. Okada, and Y. Kabashima: Journal of Statistical Mechanics: Theory and Experiment 2016 (2016) 063302.
19) T. Obuchi and Y. Kabashima: Journal of Physics: Conference Series, Vol. 699, 2016, p. 012017.

20) T. Obuchi and Y. Kabashima: 2016 24th European Signal Processing Conference (EUSIPCO), 2016, pp. 1247–1251.

21) A. V. Filippenko, M. Ganeshalingam, W. Li, and J. M. Silverman: Monthly Notices of the Royal Astronomical Society 425 (2012) 1889.

22) The SNDB: http://heracles.astro.berkeley.edu/sndb/info.

23) M. Uemura, K. S. Kawabata, S. Ikeda, and K. Maeda: Publications of the Astronomical Society of Japan 67 (2015).

24) T. Obuchi and Y. Kabashima: Journal of Statistical Mechanics: Theory and Experiment 2016 (2016) 53304.

25) Y. Kabashima, T. Obuchi, and M. Uemura: 2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Sep. 2016, pp. 596–600.

26) P. E. Leopold, M. Montal, and J. N. Onuchic: Proceedings of the National Academy of Sciences 89 (1992) 8721.