Bifurcation in a bubble translating and expanding in a sound field

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Abstract. We study the interaction of the motion of an expanding and translating bubble with a sound field, using the method of effective potential adopted by Landau and Lifshitz. We work under the assumption that the amplitude of the sound is small, while the frequency of the sound is large compared to the characteristic oscillation frequency of the bubble. We find that the equilibrium value of the mean radius $R$ of the bubble exhibits a bifurcation as the amplitude of the high frequency sound field increases in a neighbourhood of the value zero.

1. Introduction

The importance of bubble dynamics in cavitation [2], underwater explosion [6], ultrasonic cleaning devices and medical application of ultrasonics [12], sonoluminescence [7] and other applications like processing of material [1] in a space laboratory is well known.

Rayleigh [14] studied the adiabatic expansion of a spherical gas bubble in an inviscid fluid. He derived the differential equation for growth of the radius $R(t)$ of the bubble. Plesset [13] modified this equation to include the viscosity of the outside liquid. The motion of a gas bubble, which translates with velocity $U$ as it expands, has been considered by Chakraborty [4] and Chakraborty and Tuteja [5], assuming a spherical bubble shape. Detailed numerical results have been given in Ref. [5] when the pressure in the liquid far from the bubble is constant.

In the absence of the translational motion of the expanding bubble, its motion in a sound field has been discussed by Lauterborn [11] and Keller and Miksis [9]. In the present paper, we investigate the interaction of the motion of an expanding and translating bubble with a sound field. In our study we shall use a method applied by Landau and Lifshitz [10] in their study of stability of an inverted pendulum when the pivot oscillates vertically.

This method has been discussed by Blackburn, Smith and Grønbeck-Jensen [3]. In this method the concept of an “effective potential” is used. The oscillating amplitude of the pivot is taken as small and the period of oscillations is assumed to be small compared to a characteristic period of the pendulum. Using the same method, in our discussion of the bubble motion in a sound field, we work under the assumption that the amplitude of the sound field is small, while the frequency of this field is large compared to a characteristic frequency of the oscillation of the bubble. We find that the equilibrium value for the mean radius $R$ of the bubble exhibits a bifurcation [8] as the sound field amplitude increases from a critical value which is zero or nearly so.
2. Formulation

We consider a spherical gas bubble of radius $R$ at time $t$ which is translating with velocity $U$ in an inviscid liquid of density $\rho$ and surface tension coefficient $\sigma$. If $p_e$ is the pressure in the liquid at a large distance from the bubble, $p_{g0}$, $U_0$ and $R_0$ are the gas pressure, velocity of translation and radius initially, respectively, then the equations governing the motion of the bubble [4] are

$$R\frac{d^2 R}{dt^2} + \frac{3}{2} \left( \frac{dR}{dt} \right)^2 - \frac{U^2}{4} + \frac{1}{\rho} \left\{ p_e - p_{g0} \left( \frac{R_0}{R} \right)^3 + \frac{2\sigma}{R} \right\} = 0$$  \hspace{1cm} (1)

and

$$UR^3 = U_0 R_0^3 = k,$$  \hspace{1cm} (2)

where $k$ is constant. Using eq. (2), we eliminate $U$ from (1) and obtain the equation

$$\frac{d^2}{dt^2}(R^{5/2}) - \frac{5}{8} k^2 R^{-11/2} + \frac{5}{2} R^{1/2} \rho \left\{ p_e - p_{g0} \left( \frac{R_0}{R} \right)^3 + \frac{2\sigma}{R} \right\} = 0$$  \hspace{1cm} (3)

Defining

$$r = R^{5/2},$$  \hspace{1cm} (4)

and taking $\gamma = 4/3$ for simplicity, we find that eq. (3) becomes

$$\frac{d^2r}{dt^2} - \frac{5}{8} k^2 r^{-11/5} + \frac{5}{2} r^{1/5} \rho \left\{ p_e - p_{g0} \left( \frac{r_0}{r} \right)^{8/5} + 2\sigma r^{-2/5} \right\} = 0$$  \hspace{1cm} (5)

where $r_0 = R_0^{5/2}$.

We consider the case when the pressure $p_e$ in the liquid at a large distance from the bubble is modulated harmonically, due to a sound field, with frequency $\omega$ and a relative amplitude $\varepsilon$ ($\ll 1$) around its static value $p_{e0}$, so that $p_e$ is given by

$$p_e = p_{e0}(1 + \varepsilon \cos \omega t).$$  \hspace{1cm} (6)

We use $p_{g0}$ and $U_0$ as characteristic pressure and speed. We also use $r_0$ and $T_0$ as scales of $r$ and $t$, respectively, where $T_0 = R_0/U_0$. We define the dimensionless quantities $r'$, $t'$, $\omega'$ and $p_e'$ as

$$r' = r/r_0, \quad t' = t/T_0, \quad \omega' = \omega T_0, \quad p_e' = p_e/p_{g0}.$$

We may then write the equations (5) and (6) as

$$\frac{d^2r'}{dt'^2} = \frac{5}{8} r'^{-11/5} \left\{ 1 - \frac{4p_{g0}}{\rho U_0^2} \left( \frac{p_e'}{p_{g0} R_0} r'^{1/5} + \frac{2\sigma}{p_{g0} R_0} r'^{2} - r'^{4/5} \right) \right\}$$  \hspace{1cm} (7)

and

$$p_e' = p_{e0}(1 + \varepsilon \cos(\omega' t')).$$  \hspace{1cm} (8)

Finally, equations (7) and (8) take the forms

$$\frac{d^2r}{dt^2} = \frac{5}{8} r^{-11/5} \left\{ 1 - \frac{4p_{g0}}{\rho U_0^2} \left( p_e r^{12/5} + \frac{2\sigma}{p_{g0} R_0} r^2 - r^{4/5} \right) \right\},$$  \hspace{1cm} (9)

and

$$p_e = \frac{p_{e0}}{p_{g0}}(1 + \varepsilon \cos \omega t),$$  \hspace{1cm} (10)

where the dashes have been omitted from $r'$, $t'$, $\omega'$ and $p_e'$. Following Landau and Lifshitz [10] we suppose that $r$ in equation (9) can be split up into a fast component $\xi$ and a slow component $\eta$ so that

$$r = \xi + \eta.$$  \hspace{1cm} (11)
We shall assume that $|\xi|$ is small, as $\varepsilon$ is taken as small. In writing (11) we have assumed that 
\[ \omega \gg 1 \] (12)

Equations (9) and (10), in view of (11), then yield
\[
\frac{d^2 \xi}{dt^2} + \left( \frac{1}{2} \frac{p_{g0}}{\rho U_0^2} \left\{ \frac{p_{e0}}{p_{g0}} \eta^{-4/5} - \frac{2\sigma}{p_{g0}R_0} \eta^{-6/5} + 7\eta^{-12/5} \right\} \right) \xi + \frac{11}{8} \eta^{-16/5} = 0
\]
\[ = -5\varepsilon \frac{p_{g0} p_{e0}}{2 \rho U_0^2 p_{g0}} \eta^{1/5} \cos(\omega t) \]
(13)

and
\[
\frac{d^2 \eta}{dt^2} = \frac{5}{8} \eta^{-11/5} \left\{ 1 - \frac{4p_{g0}}{\rho U_0^2} \left[ \frac{p_{e0}}{p_{g0}} \eta^{12/5} + \frac{2\sigma}{p_{g0}R_0} \eta^2 - \eta^{4/5} \right] \right\} - \frac{4p_{g0} p_{e0}}{5p U_0^2 p_{g0}} \eta^{7/5} \langle \varepsilon \xi \cos(\omega t) \rangle
\]
\[ + \left[ \frac{360}{25} \frac{p_{g0}^2}{\rho U_0^2} \frac{p_{e0}}{p_{g0}} \eta^{2/5} - \frac{24}{25} \frac{p_{g0}^2}{\rho U_0^2} \frac{\sigma}{p_{g0}R_0} + \frac{168}{25} \frac{p_{g0}^2}{\rho U_0^2} \eta^{-6/5} + \frac{88}{25} \eta^{-2} \right] \langle \xi^2 \rangle \]
(14)
where \( \langle \cdot \rangle \) denotes the average over a period $\frac{2\pi}{\omega}$. Terms of order $\varepsilon^3$ and smaller have been neglected in writing (13) and (14). Equation (13) can be integrated to yield the fast component as
\[
\xi = \frac{5\varepsilon \frac{p_{g0} p_{e0}}{2 \rho U_0^2 p_{g0}} \cos(\omega t) \eta^{17/5}}{\omega^2 \eta^{16/5} - 1 - \frac{p_{g0}}{2 \rho U_0^2} \left\{ \frac{p_{e0}}{p_{g0}} \eta^{12/5} - \frac{2\sigma}{p_{g0}R_0} \eta^2 + 7\eta^{4/5} \right\} - \frac{11}{8}}.
\]
(15)

In view of (15), the averages can be evaluated as
\[
\langle \varepsilon \xi \cos(\omega t) \rangle = \frac{5}{4} \varepsilon^2 \eta^{17/5} \frac{p_{g0} p_{e0}}{\rho U_0^2 p_{g0}} \omega^2 \eta^{16/5} - 1 - \frac{p_{g0}}{2 \rho U_0^2} \left\{ \frac{p_{e0}}{p_{g0}} \eta^{12/5} - \frac{2\sigma}{p_{g0}R_0} \eta^2 + 7\eta^{4/5} \right\} - \frac{11}{8}
\]
(16)
\[
\langle \xi^2 \rangle = \frac{25}{8} \varepsilon^2 \left( \frac{p_{g0}^2}{\rho U_0^2} \frac{p_{e0}}{p_{g0}} \right)^2 \eta^{34/5} \omega^2 \eta^{16/5} - 1 - \frac{p_{g0}}{2 \rho U_0^2} \left\{ \frac{p_{e0}}{p_{g0}} \eta^{12/5} - \frac{2\sigma}{p_{g0}R_0} \eta^2 + 7\eta^{4/5} \right\} - \frac{11}{8}
\]
(17)

We define the dimensionless quantities
\[
U^* = \left( \frac{\rho U_0^2}{p_{g0}} \right)^{1/2}, P = \frac{p_{e0}}{p_{g0}}, S = \frac{2\sigma}{p_{g0}R_0}.
\]
(18)

We note that $U^*, P$ and $S$ are proportional to translational velocity of the bubble, steady part of the pressure due to the sound field, surface tension coefficient of the liquid outside the bubble, respectively. Using (18) we can write, in view of (16) and (17), the equation (14) as
\[
\frac{d^2 \eta}{dt^2} = \frac{5}{8} \eta^{-11/5} \left( \Phi + E^2 / \Omega + E^2 \bar{P} / \Omega^2 \right)
\]
(19)
where
\[
E = \varepsilon P U^{* -2}
\]
(20)
is the control parameter, and we have introduced the abbreviations
\[ \Phi = 1 - 4U^{*2}(P \eta^{12/5} + S \eta^2 - \eta^{1/5}), \tag{21} \]
\[ \Omega = -O \eta^{16/5} + \frac{U^{*2}}{2}(P \eta^{12/5} - S \eta^2 + 7\eta^{4/5}) + \frac{11}{8}, \tag{22} \]
\[ \bar{P} = 45PU^{*2} \eta^{36/5} - \frac{3}{2}SU^{*2} \eta^{34/5} + 21U^{*2}\eta^{28/5} + 11\eta^{24/5} \tag{23} \]
and
\[ O = \omega^2 \tag{24} \]

Following Landau and Lifshitz ([10]), we introduce the effective potential \( V \) and write equation (19) as
\[ \frac{d^2 \eta}{dt^2} = -\frac{dV}{d\eta}, \tag{25} \]
where
\[ \frac{dV}{d\eta} = -\frac{5}{8}\eta^{-11/5}[\Phi + E^2/\Omega + E^2\bar{P}/\Omega^2]. \tag{26} \]

3. Bifurcation of the mean bubble radius

To discuss possible bifurcations of the bubble radius, we find the equilibrium points \((E, \eta)\), in the \( E - \eta \) plane, characterized by \( dV/d\eta = 0 \) (cf. (25)). From (26) we have, at an equilibrium point,
\[ \Phi + E^2/\Omega + E^2\bar{P}/\Omega^2 = 0. \tag{27} \]

By back transformation from \( \eta \) to \( R \), using (4), we identify the mean radius by
\[ R = \eta^{2/5}. \tag{28} \]

The abbreviations (21)-(23) now become
\[ \Phi = 1 - 4U^{*2}(PR^6 + SR^5 - R^2), \tag{29} \]
\[ \Omega = -OR^8 + U^{*2}(PR^6 - SR^5 + 7R^2) + \frac{11}{8}, \tag{30} \]
\[ \bar{P} = 45PU^{*2}R^{18} - \frac{3}{2}SU^{*2}R^{17} + 21U^{*2}R^{14} + 11R^{12} \tag{31} \]

On physical grounds, we shall always restrict ourselves to the considerations of positive and real roots \( R \) of (27), where \( \Phi, \Omega, \bar{P} \) are now given by (29)-(31). We shall examine the change in the equilibrium value of \( R \) as the control parameter \( E \) increases from zero. A point of bifurcation or branch point \((E_{\text{critical}}, R^*)\) in \( E - R \) plane, where \( R^* \) is a root of the equation (27) at \( E = E_{\text{critical}} \), and where (cf.(20)) \( E_{\text{critical}} \) is small \((\varepsilon \ll 1)\), is identified by a change of the number of real and positive roots \( R \) of (27) in a small neighbourhood of \( R^* \), when \( E \) varies in a small neighbourhood of \( E_{\text{critical}} \). As \( E \to 0 \) we assume that \( E/\Omega \to 0 \) (so that \( \Phi \to 0 \) (cf. (27)), and the equation (27) reduces (cf. (29)) to the equation
\[ PR^6 + SR^5 - R^2 - \frac{1}{4U^{*2}} = 0. \tag{32} \]

A real positive root \( R = R_0 \) of (32) gives an equilibrium point of (25) when \( E = 0 \), i.e. in the absence of a sound field. Since there is one change of sign in (32), in the coefficients of powers of \( R \), there cannot be more than one positive real root of (32) by Descartes’ rule of signs [15]. Since the left hand side of (32) is positive for large \( R \) and negative for small \( R \), there is exactly one positive real root \( R = R_0 \) of (32), for a given set of values of \( P, S \) and \( U^* \). When \( E \neq 0 \), but
is small, the equation (27) shows that there is an equilibrium point \((E, R)\), where \(R\) is a real positive root of
\[
\Phi \Omega^2 + E^2 \Omega + E^2 \bar{P} = 0 \tag{33}
\]
Before treating the case \(E \neq 0\), we first examine if \(E = 0\) and \(R = R_0\) is a point of bifurcation, and for this purpose, we find out the positive real root \(R = R_0\) of (32) for a given set of values of \(P, S\) and \(U^*\). We then determine \(O\) such that for this value of \(O\) the equation
\[
\Omega = 0 \tag{34}
\]
is satisfied at \(R = R_0\). As \(E \to 0\), the equation (33) then reduces to
\[
\Phi \Omega^2 = 0 \tag{35}
\]
which has three repeated real positive roots \(R = R_0\) (one from \(\Phi = 0\), and two from \(\Omega^2 = 0\)), so that as \(E\) increases from the value 0, three branches of real solutions \(R\) of equation (33) may possibly emanate from the point \(E = 0, R = R_0\) in the \(E - R\) plane indicating that \(E = 0, R = R_0\) is a bifurcation point. Different sets of values of \(P, S\) and \(U^*\) are chosen and corresponding values of \(R_0\) and \(O\) are determined from (32) and (34). Only those sets of values of \(P, S\) and \(U^*\) are taken which give \(O\) large compared to 1, since (24) is true and (12) holds. Then \(E\) is increased gradually from the value \(E = 0\), and real positive roots of \(R\) of equation (33) are determined. Only in a few exceptional cases we have found three branches of real positive roots \(R\) of (33), which emanate from the point \(E = 0, R = R_0\) in the \(E R\) plane.

In such an exceptional case we have a branch point or bifurcation point at \(E = 0, R = R_0\). One such bifurcation point, and the branches of the real roots \(R\) of equation (33) emanating from this bifurcation point, constituting a bifurcation diagram [8], are shown in Fig. 1, where \(S = 0.03, P = 0.11, U^* = 0.277\). The stability of the bubble at equilibrium, at different values of the mean radius \(R\) for different \(E\), along different branches of the solution \(R\) are also indicated in the bifurcation diagram in Fig. 1. (There is stability of the bubble at an equilibrium point \((E, R)\) if \(-d^2 V/d\eta^2 < 0\), at the corresponding point \((E, \eta)\), where \(dV/d\eta\) is given by (26), otherwise there is instability at that point \((E, R)\).) In most cases, for a given combination of values of \(S, P\) and \(U^*\), the point \(E = 0, R = R_0\) is found not to be a branch point or bifurcation point. One such case is shown in Fig. 2.

**Figure 1.** The bifurcation diagram when \(S = 0.003, U^* = 0.277, P = 0.11, O = 2.296\), and critical \(E\) is 0.0. The variations of \(R\), the real roots of (33), with \(E\) are shown. The stable and unstable cases are shown by continuous and broken lines, respectively.

**Figure 2.** The bifurcation diagram when \(S = 0.5, U^* = 1.0, P = 1.1, O = 7.764\), and critical \(E\) is 0.1028. The variations of \(R\), the real roots of (33), with \(E\), are shown. The stable and unstable cases are shown by continuous and broken lines, respectively.
As $E$ increases from the value $E = 0$, only one branch of the real solution for $R$ is found to emanate from the point $E = 0, R = R_0$. Of the three repeated roots $R = R_0$, at $E = 0$ for the equilibrium equation (33), as $E$ increases from $E = 0$ value, two roots become complex conjugate and the variation of positive imaginary part of one such root is shown in Fig. 3 for the combination $S = 0.5, P = 1.1, U^* = 1.0$. This imaginary part gradually increases with $E$, reaches a maximum and then decreases to become zero at a value $E = E_{critical}$. At value $E = E_{critical}$, $R = R^*$, a real quantity. When $E = E_{critical}$ in (33), we have three branches of real solutions for $R$ of the equilibrium equation (33) of which one branch starts from $E = 0, R = R_0$ and other two branches originate from $E = E_{critical}$ and $R = R^*$ in $E - R$ plane. The point $E = E_{critical}$ and $R = R$ in $E - R$ plane is now the point of bifurcation. The equilibrium value of $R$ on one branch of the bifurcation diagram are stable, and those on other branch are unstable. Fig. 2 shows the bifurcation diagram for the combination of values $S = 0.5, P = 1.1, U^* = 1.0$ for which Fig. 3 is valid. Similarly Fig. 4 gives one more bifurcation diagram for a different set of values of $S, P$ and $U^*$.

![Figure 3](image1.png)  
**Figure 3.** The variation of $R_i$, the imaginary part of the two complex conjugate roots $R$ of (33), with $E$, when $S = 0.05, U^* = 1.0, P = 1.1, O = 7.764$ and critical $E$ is 0.1028

![Figure 4](image2.png)  
**Figure 4.** The bifurcation diagram when $S = 0.5, U^* = 0.5, P = 2.38, O = 79.69$ and critical $E$ is 0.0021. The variations with $E$, of the real roots $R$ of (33), are shown. The stable and unstable cases are shown by continuous and broken lines, respectively.

In Fig. 5 we give the variation of $O$, where $O$ is proportional to the square of frequency of sound field (cf. (24)) for which bifurcation is observed, with translating velocity $U^*$ of the bubble, when $S = 0.5, P = 1.125$. We note that $O$ decreases as velocity $U^*$ of the translating bubble is increased. Similar results are observed for other values of the parameters $S$ and $P$. In Fig. 6 we give the variation of $E_{critical}$ as $U^*$ changes but $S$ and $P$ kept fixed ($S = 0.5, P = 1.125$) Similar results are observed for other values of $P$ and $S$. 


Figure 5. The variation of $O$, with $U^*$, for which bifurcation is observed, when $S = -0.5, P = 1.125$

Figure 6. The variation of $E_c$, the critical value of $E$ at which bifurcation occurs, with $U^*$, when $S = 0.5, P = 1.125$

4. Concluding remarks

In this paper, we have discussed the bifurcations in an expanding and translating bubble in a sound field when the outside fluid is incompressible and inviscid. We have examined the roots of an algebraic equation (33) of degree 22 in $R$ for different values of $S, P, U^*$ and $E$. A set of values of the parameters $S, P$ and $U^*$ are so chosen that the value of $O$, determined from (34), for which bifurcation is possible, is large compared to 1, and the value of $E_{\text{critical}}$ is also small. Only in rare cases $E_{\text{critical}} = 0$ is found to be true.

For a given set of values of $S$ and $P$, the value of $O$, for which bifurcation is possible, is found to decrease as $U^*$ increases (Fig. 5).

For a given set of values of $S$ and $P$, as $U^*$ increases, $E_{\text{critical}}$ is found to decrease first, to come to a minimum value, and then to increase again (Fig. 6.).

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