Uncovering latent singularities from multifractal scaling laws in mixed asymptotic regime. Application to turbulence

J.-F. Muzy 1(a), E. Bacry 2, R. Baile 1,3 and P. Poggi 3

1 SPE UMR 6134, CNRS, Université de Corse - 20250 Corte, France, EU
2 CMAP, Ecole Polytechnique - 91128 Palaiseau, France, EU
3 SPE UMR 6134, CNRS, Université de Corse - Route des Sanguinaires, Vignola, 20200 Ajaccio, France, EU

received 21 March 2008; accepted in final form 5 May 2008
published online 10 June 2008

PACS 05.45.Df – Fractals
PACS 02.50.–r – Probability theory, stochastic processes, and statistics
PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion

Abstract – In this paper we revisit an idea originally proposed by Mandelbrot about the possibility to observe “negative dimensions” in random multifractals. For that purpose, we define a new way to study scaling where the observation scale \( \ell \) and the total sample length \( L \) are, respectively, going to zero and to infinity. This “mixed” asymptotic regime is parametrized by an exponent \( \chi \) that corresponds to Mandelbrot “supersampling exponent”. In order to study the scaling exponents in the mixed regime, we use a formalism introduced in the context of the physics of disordered systems relying upon traveling wave solutions of some non-linear iteration equation. Within our approach, we show that for random multiplicative cascade models, the parameter \( \chi \) can be interpreted as a negative dimension and, as anticipated by Mandelbrot, allows one to uncover the “hidden” negative part of the singularity spectrum, corresponding to “latent” singularities. We illustrate our purpose on synthetic cascade models. When applied to turbulence data, this formalism allows us to distinguish two popular phenomenological models of dissipation intermittency: We show that the mixed scaling exponents agree with a log-normal model and not with log-Poisson statistics.

Copyright © EPLA, 2008

Multifractal processes are random functions (or measures) that possess non-trivial scaling properties. They are widely used models in many areas of applied and fundamental fields. Well-known examples are turbulence, internet traffic, rainfall distributions or finance. For the sake of simplicity we will consider only non-decreasing multifractal processes (which increments define a multifractal measure) denoted hereafter \( M(x) \). In the sequel \( M(I) \) will stand for the measure of the interval \( I \), \( M(I) = \int_I dM \) and \( M(x, \ell) = M([x, x+\ell]) \). Multifractals are characterized by the scaling of the partition functions: If one covers the overall sample interval of length \( L \) with \( L/\ell \) disjoint intervals of size \( \ell \), \( \{I_i(i)\}_{i=1,...,L/\ell} \), one usually defines the order \( q \) partition function whose scaling behavior defines the exponent \( \tau_0(q) \):

\[
Z(q, \ell) = \sum_{i=1}^{L/\ell} M[I_i(i)]^q \sim \ell^{\tau_0(q)}, \quad (1)
\]

where the limit \( \ell \to 0 \) simply means that \( \ell/L \to 0 \), \( L \) being the large correlation scale usually referred to as the integral scale. When \( \tau_0(q) \) is a (concave) non-linear function of \( q \), the measure \( M(x) \) is said to be multifractal or intermittent. Within the multifractal formalism introduced by Parisi and Frisch (see, e.g., [1]), the non-linearity of \( \tau_0(q) \) is interpreted in terms of fluctuations of pointwise singularity exponents of the measure. Indeed, according to this formalism, \( \tau_0(q) \) is obtained as the Legendre transform of the singularity spectrum \( f_\alpha(q) \) that gives the (Hausdorff) dimension of the sets of points \( x \) of singularity \( \alpha \) \( (M(x, \ell) \sim \ell^\alpha) \). Therefore \( q \) can be interpreted as a value of the derivative of \( f_\alpha(q) \) and conversely \( \alpha \) is a value of the derivative of \( \tau_0(q) \). Multifractality is also closely related to the notion of stochastic self-similarity: The measure \( M(x) \) is self-similar in a stochastic sense if, for all \( s < 1 \), \( M(sx) \overset{\text{law}}{=} W_s M(x) \) where \( W_s \) is a positive random weight independent of \( M \). It can be easily shown that this stochastic equality implies that the expected value of \( Z(q, \ell) \) (i.e., the order \( q \) moment of the measure) behaves

\(\text{(a)}\) E-mail: muzy@univ-corse.fr
as a power law, i.e., $\mathbb{E}[Z(q, \ell)] = C_q \ell^{\tau(q)}$, where $\tau(q)$ is nothing but the cumulant generating function of $\ln W_n$. If one uses the terminology introduced in physics of disordered systems, the exponent $\tau(q)$ is defined from an “annealed” averaging while $\tau_0(q) \sim \mathbb{E}[\ln Z(q, \ell)]$ is the analog of a free energy computed as a “quenched” average. As will be discussed below, these two functions can be different for large values of $q$.

The paradigm of self-similar measures are random multiplicative cascades (for the sake of simplicity we will exclusively focus, in this paper, on discrete cascades but all our results can be easily extended to recent continuous cascade constructions [2–4]) originally introduced by the Russian school for modelling the energy cascade in fully developed turbulence and to which a lot of mathematical studies have been devoted [5–8]. Let us summarize the main properties of these constructions. The integral scale from which the cascading process “starts” is denoted as $L$. A dyadic discrete cascade is built as follows: A measure is uniformly spread over the starting interval $[0, L]$ and one splits this interval in two equal parts: On each part, the density is multiplied by (positive) i.i.d. random factors $W$, such that $\mathbb{E}[W] = 1$. Each of the two sub-intervals is again cut in two equal parts and the process is repeated infinitely. It is convenient to introduce the random variable $\omega = \ln(W)$ whose probability density function will be denoted as $g(x)$. Thus $g(x)$ is Gaussian or Poisson for, respectively, log-normal and log-Poisson cascades. The stochastic self-similarity property of the limit measure $M$ associated with previous iterative construction can be directly proven and it is easy to show that $\tau(q)$ is related to the cumulant generating function of $\omega$:

$$\tau(q) = q - \ln(\mathbb{E}[e^{\omega q}]) - 1.$$  

(2)

Moreover, as first established by Molchan [7,9], the multifractal formalism holds for random cascades and one has the following relationship between $\tau_0(q)$ and $\tau(q)$, for $q > 0$ (note that a similar relationship holds for negative values of $q$ [7]):

$$\tau_0(q) = \tau(q) \text{ if } q \leq q_0 \text{ and } \tau_0(q) = \alpha_0 q \text{ if } q > q_0,$$  

(3)

where $q_0$ is the value of $q$ corresponding to the minimum value of $f_0(\alpha)$ (in general $f_0(\alpha_0) = 0$). This discrepancy between the annealed and quenched spectra has been extensively studied on a numerical ground in ref. [10] and was referred to as the “linearization effect”. It has notably been observed to be independent of the nature of the cascade and of the overall length $L$ of the sample. As will be emphasized below this effect has not been properly taken into account in the literature of turbulence (see, e.g., [11]).

One of the goals of this paper is to recover the linearization effect and to establish how it can be somehow controlled. For that purpose, we will consider the scaling of partition function (1) in some “mixed” asymptotic regime where, as the resolution becomes smaller, the total length of the sample is increased. Let us introduce some useful notations: We call $\ell$ the scale of observations (it can be the sampling scale or a multiple of it), $L$ the integral scale and $\mathcal{L}$ the total sample length. $N_{\ell}$ will refer to the number of samples per integral scale and $N_L$ the number of integral scales. We have obviously $N_L = L^{-1}$ and $N_{\ell} = \mathcal{L} L^{-1}$ and $N = N_{\ell} N_L = L\ell$ is the total number of samples. If $L$ is fixed, the limit $N \rightarrow +\infty$ can be conveniently controlled using an additional exponent $\gamma > 0$ as $N_L \sim N_L^\gamma$. Let us mention that such an exponent has been already introduced by Mandelbrot as an “embedding dimension” [12,13] in order to discuss the concept of negative dimension and latent singularities (see below). Within this framework, the definition (1) of the partition function depends on $\gamma$ and allows us to define a new exponent $\tau_\gamma(q)$ as follows:

$$Z(q, \ell, \gamma) = \sum_{i=1}^{\mathcal{L} - 1} M[I_{\ell}(i)]^q \sim \ell^{\tau_\gamma(q) - \gamma}.$$  

(4)

The two “extreme” cases are: i) the $\gamma = 0$ case which corresponds to a fixed number of integral scales $N_L$ while $\ell \rightarrow 0$ and ii) the $\gamma = +\infty$ case which corresponds to a fixed observation scale $\ell$ while $N_L \rightarrow +\infty$. It results that for $\gamma = 0$ one recovers former definition (1) of $\tau_0(q)$ while $\tau_\gamma(q) = \tau(q)$ as defined in (2). In that respect $\gamma$ allows us to interpolate between quenched and annealed situations. In order to compute $\tau_\gamma(q)$, one needs to study the behavior of the probability law of $Z(q, \ell, \gamma)$ [14]. For that purpose, along the same line as in refs. [15,16], we will define $G(s, \ell) = \mathbb{E}[e^{-s q} Z(q, \ell, \gamma)]$. $G(s, \ell)$ is simply the Laplace transform of the law of $Z(q, \ell, \gamma)$ evaluated at $t = e^{-s q}$. Let $p, r$ be two integers and let us define the iteration $m \rightarrow m + 1$, $\ell \rightarrow 2^{-p} \ell$ and $\mathcal{L} \rightarrow 2 \mathcal{L}$. In other words, at each iteration step, the resolution is divided by $2^p$ while the number of independent integral scales is multiplied by $2^p$. One thus has $N_L = 2^m$ and $N_{\ell} = 2^m$ that corresponds to $\gamma = r/p$. Within this parametrization, $G(s, m, p, r)$ and $G(s, m, 1, 0)$ will be denoted as $H(s, m)$. If $M$ is a random cascade as defined previously, its self-similarity allows one to prove that $G(s, m, p, r)$ can be written as [17]

$$G(s, m, p, r) = (H(s, pm))^{2^m},$$  

(5)

where $H(s, m)$ satisfies the following recursion:

$$H(s, m + 1) = [H(s, m) * g(s + \ln 2)]^2,$$  

(6)

where $g(x)$ is the pdf of $\omega$ the logarithm of cascade weights and $\ast$ stands for the convolution product.

It is easy to see that eq. (6) has two uniform “stationary” solutions $H(s, n) = 0$ and $H(s, n) = 1$, the first one being stable while the latter is (linearly) unstable. The initial condition connects the stable state $H(-\infty, 0) = 0$ to the unstable one $H(\infty, 0) = 1$ and, as shown in [15,18,19], this kind of non-linear equation admits traveling wave
Multifractal scaling in mixed asymptotic regime

In order to illustrate our results, we have performed several numerical simulations on both continuous and discrete cascades whose statistics are log-normal (LN) and log-Poisson (LP). In the log-normal case one has \( \tau(q) = q(1 + \lambda^2/2) - \lambda^2 q^2/2 - 1 \) and therefore by solving eq. (7) one gets \( \alpha_q = 1 + \lambda^2/2 - \lambda \sqrt{2(1+\chi)} \) and \( q^*_\chi = \lambda^{-1} \sqrt{2(1+\chi)} \). For each value of \( \chi \), we have chosen \( L = 4096, \lambda^2 = 0.2 \) and \( N_L \) depends on \( \chi \), such that \( N_L N_q \) is fixed. One clearly sees that, as the value of \( \chi \) increases, the value of \( q^*_\chi \) increases, the slope \( \alpha_q \) of \( \tau(q) \) decreases and \( \tau(q) \) becomes closer to the annealed spectrum \( \tau(q) \) (solid line). The dashed lines represent the analytical expressions of \( \tau(q) \) derived from eqs. (8) and (9). In the inset the same estimates have been performed for a log-Poisson cascade. As expected, in that case \( \tau(q) \) weakly depends on \( \chi \) and rapidly converges towards \( \tau(q) \), represented by the solid line (see text).

\[
\tau(\chi)(q) = \begin{cases} \tau(q) & \text{if } q < q^*_\chi, \\
q = \alpha_q & \text{otherwise,} \\
\end{cases}
\]

where \( \alpha_q \) is defined in (7). Let us note that a rigourous proof of eqs. (8) and (9) will be provided in [14]. In the case \( \chi = 0 \) one recovers standard linearization effect (eq. (3)) which has been generalized to any value of \( \chi \) in the mixed asymptotic regime. As \( \chi \) increases so does \( q^*_\chi \) and \( \tau(q) \) continuously converges towards \( \tau(q) \). Singularities \( \alpha_q > \alpha_0 \) have been qualified by Mandelbrot as “latent” because they are only observable for large enough values of the “supersampling” exponent \( \chi \).
than a critical value function have different behavior for values of equation, that quenched and annealed averaged partitions from the log-Poisson prediction (dashed line). The log-normal expression (solid line) which is very different one clearly see that the data perfectly match the linear asymptotics slope of $\lambda$. Since both models are traditionally used to describe the spatial fluctuations of energy dissipation in fully developed turbulence. [1,22]. We naturally reproduced the spatial longitudinal velocity signal is such that $\sqrt{\partial v/\partial x}$, as a function of $q$. In fig. 3, we have plotted the values of the asymptotic slope of $\tau(q)$, $\alpha$, as a function of $\sqrt{1+\chi}$. One clearly see that the data perfectly match the linear log-normal expression (solid line) which is very different from the log-Poisson prediction (dashed line).

To summarize, we have shown in this paper, using traveling wave solutions of some cascade non-linear iteration equation, that quenched and annealed averaged partitions function have different behavior for values of $q$ larger than a critical value $q_c$, analog of the glass transition temperature in spin glass systems. This difference can be controlled using some “supersampling” exponent $\chi$ which defines an asymptotic limit that mixes small scales and large number of samples regimes. By analyzing the value of the multifractal scaling exponents for various values of $\chi$, one can distinguish between different cascade models. In the context of the modelling of energy dissipation intermittency in fully developed turbulence, we have provided evidences supporting log-normal statistics against log-Poisson statistics.

***

We thank B. CHABAUD and B. CASTAING for the permission to use their turbulence experimental data.

REFERENCES

[1] FRIECH U., Turbulence: The Legacy of A. N. Kolmogorov (Cambridge University Press, Cambridge) 1995.
[2] MUZY J.-F., DELOUR J. and BACRY E., Eur. Phys. J. B, 17 (2000) 537.
[3] BARRAL J. and MANDELBROT B. B., Probab. Theory Relat. Fields, 124 (2002) 409.
[4] BACRY E. and MUZY J.-F., Commun. Math. Phys., 236 (2003) 449.
[5] MANDELBROT B. B., J. Fluid Mech., 62 (1974) 1057.
[6] KAHANE J. P. and PEYRIÈRE J., Adv. Math., 22 (1976) 131.
[7] MOLCHAN G. M., Commun. Math. Phys., 179 (1996) 681.
[8] LIU Q. S., Asian J. Math., 6 (2002) 145.
[9] MOLCHAN G. M., Phys. Fluids, 9 (1997) 2387.
[10] LAHERMES B., ABRY P. and CHAINSIS P., Int. J. Wavelets, Multiresolut. Inf. Proc., 2 (2004) 497.
[11] VAN DE WATER W. and HERWEIJER J. A., J. Fluid Mech., 387 (1999) 3.
[12] MANDELBROT J.-F. and MANDLBROT B. B., Physica A, 163 (1990) 306.
Multifractal scaling in mixed asymptotic regime

[13] Chhabra A. B. and Sreenivasan K., Phys. Rev. A, 43 (1991) 114.

[14] Bacry E., Gloter A., Hoffman M. and Muzy J.-F., preprint arXiv:0805.0194 (2008).

[15] Derrida B. and Spohn H., J. Stat. Phys., 51 (1988) 817.

[16] Carpentier D. and LeDoussal P., Phys. Rev. E, 63 (2001) 026110.

[17] Muzy J.-F., Bacry E. and Baile R., in preparation (2008).

[18] Brunet E., Influence des effets de taille finie sur la propagation d’un front. Distribution de l’énergie libre d’un polymère dirigé en milieu aléatoire, PhD Thesis, Université de Paris VII (2000).

[19] Dean D. S. and Majumdar S. N., Phys. Rev. E, 64 (2001) 045101.

[20] Brunet E. and Derrida B., Phys. Rev. E, 62 (1997) 2597.

[21] Majumdar S. N. and Krapivsky, Phys. Rev. E, 65 (2001) 036127.

[22] She Z. S. and Lévéque E., Phys. Rev. Lett., 72 (1994) 336.

[23] Chanal O., Chabaud B., Castaing B. and Hebral B., Eur. Phys. J. B, 17 (2000) 309.