Fluctuations of the Fermi condensate in ideal gases

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Abstract. We calculate numerically and analytically the fluctuations of the fermionic condensate and of the number of particles above the condensate for systems of constant density of states. We compare the canonical fluctuations, obtained from the equivalent Bose condensate fluctuation, with the grandcanonical fermionic calculation. The fluctuations of the condensate are almost the same in the two ensembles, with a small correction comming from the total particle number fluctuation in the grandcanonical ensemble. On the other hand the number of particles above the condensate and its fluctuation is insensitive to the choice of ensemble.

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1. Introduction

Starting quite a long time ago, Auluck and Kothari [1], then May [2] and finally Viefers, Ravndal, and Haugset [3], discovered independently that the specific heat of nonrelativistic ideal gases in two-dimensional (2D) boxes does not depend on the exclusion statistics. This interesting result eventually did not receive the attention it deserved until 1995, when Lee [4] rederived it by introducing an unified way of writing the thermodynamic properties of ideal gases in terms of polylogarithmic functions [5].

This formulation represented also an important extension of the Auluck and Kothari result and triggered further investigations (see for example Refs. [6, 7, 9]).

Since under canonical conditions a Bose and its corresponding Fermi gas are similar at the thermodynamic level, they have been called thermodynamically equivalent. If we denote by $C_V(T, V, N)$ the heat capacity of a system at temperature $T$, volume $V$ and particle number $N$, then the heat capacities $C_{V,1}$ and $C_{V,2}$ of two thermodynamically equivalent systems are identical functions of $T$, $V$ and $N$. Using this property, all the thermodynamic systems may be divided into equivalence classes [9] and by doing this one may observe that all the systems of ideal particles of the same constant density of states (DOS), but obeying Bose, Fermi, or even fractional exclusion statistics [10], belong to the same equivalence class [9].

The equivalence between Bose and Fermi gases was critically examined by Pathria [7]. He showed that the Lee’s unified formulation of 2D ideal gases does not hold anymore below the Bose-Einstein condensation temperature of the Bose gas. Apparently, the 2D (or, more exactly, constant DOS) thermodynamic equivalence holds only above the Bose-Einstein condensation temperature.

On the other hand, Crescimanno and Landsberg [11] and one of us [9] showed that there is a one-to-one mapping between microscopic configurations of bosons, fermions or haldons, in systems with the same, constant DOS, which preserves the total excitation energy (i.e. the energy of the particles in the given configuration minus the energy of the system at zero kelvins is the same). Based on this theorem, the thermodynamic equivalence of systems with equally spaced spectra should hold at any temperature in any detail, so Pathria’s conclusion must be wrong. But what was overlooked there?

The method of mapping microscopic configurations between systems of different exclusion statistics introduced in Ref. [9] for systems with constant DOS was extended in Ref. [12] for systems with any DOS and was called exclusion statistics transformation (EST). Systems connected by EST are thermodynamically equivalent by construction. If we take for example a Fermi system, transform it by EST into a Bose system, and then calculate independently the thermodynamical properties of these two systems by maximizing the entropy in each of them, for constant $U$ (total energy) and $N$ – i.e. assuming grandcanonical distribution on the single particle energy levels – we loose again the thermodynamic equivalence that we started with, at least is most of the cases [12]! The obvious conclusion is that one or both of these grandcanonical distributions lead to results in dezacord with the canonical ensemble. The question which of these grandcanonical distribution is closest to the canonical distribution is very difficult to answer, since ab initio canonical calculations are not easy to perform on general, large systems.
1.1. The Fermi condensate

The concept of Fermi condensate was introduced in Refs. [12, 13]. For a general system, of spectrum consisting of single particle energy levels \( \epsilon_i \) (we enumerate them such that \( \epsilon_0 \leq \epsilon_1 \leq \ldots \)), one can calculate the grandcanonical probability, \( w_{N_0} \), of having the lowest \( N_0 \) energy levels occupied, the level \( N_0 + 1 \) free and all the other energy levels with any occupation number [13]:

\[
w_{N_0} = Z^{-1} \cdot \exp \left[ -\beta \cdot \sum_{i=0}^{N_0-1} \epsilon_i + \beta \mu N_0 \right] \cdot Z_{\text{ex}}(N_0, \beta, \beta \mu) \equiv \frac{Z_{N_0}}{Z}. \tag{1}
\]

Here \( Z \) is the partition function of the system and \( Z_{\text{ex}}(N_0, \beta, \beta \mu) = \prod_{i=N_0+1}^{\infty} \{ 1 + \exp[-\beta(\epsilon_i - \mu)] \} \) is the partition function of the levels \( N_0 + 1, N_0 + 2, \ldots \). Obviously,

\[
Z = \sum_{N_0} Z_{N_0}. \tag{2}
\]

The probability distribution (1) may have a maximum at, say \( N_{0,\text{max}} \). The statistical interpretation of such a maximum is that in a physical system in contact with a particle reservoir, the lowest \( N_0 \approx N_{0,\text{max}} \) energy levels are always occupied. These \( N_0 \) particles form the Fermi condensate. At any finite temperature, \( N_0 \) is subject to fluctuations, denoted by \( \delta N_0 \).

The configurations of fermions may be transformed by EST into configurations of bosons [12, 9]. By this transformation the \( N_0 \) degenerate fermions will be transformed into the Bose-Einstein condensate of the corresponding Bose system, and hence the name of Fermi condensate. For this reason degenerate fermions will also be called Fermi condensate. For canonical Bose systems of constant DOS, the probability distribution of having \( N_0 \) particles in the condensate, \( w^c_{N_0} \), have been studied in detail before (see for example Ref.[15] for a review). By construction, the canonical probability distribution of having \( N_0 \) fermions in the condensate is also \( w^c_{N_0} \).

In Ref. [14] it was shown that in a system of constant density of states, \( N_{0,\text{max}} \) – which corresponds to the maximum of \( w_{N_0} \) – is close to the average canonical occupation of the ground state, but the two numbers are not the same. The distribution \( w_{N_0} \) is not symmetric around the maximum, so \( \langle N_0 \rangle \neq N_{0,\text{max}} \) (where by \( \langle \cdot \rangle \) and \( \langle \cdot \rangle^c \) we shall denote grandcanonical and canonical averages, respectively). Since if the Bose and Fermi condensates are separated from the rest of the particles, the grandcanonical Bose and Fermi distributions map onto each-other (see Section 5, Ref. [12]), the average number of particles in the Fermi condensate should be equal to the average ocupation of the bosonic ground state and in the Fermi system below the condensation temperature the usual Fermi distribution applies only to the particles above the condensate. For simplicity, to the fermions above the condensate we shall refer to as the particles in the thermally active layer.

It is well known that Fermi canonical and grandcanonical ensembles are equivalent. In this paper we shall investigate the equivalence between the grandcanonical description of the Fermi gas and its correspondent canonical description in terms of the Bose gas. This is a new type of equivalence, first mentioned in Ref. [12] and which seems not to hold for general systems. Here we shall discuss the simplest systems, namely the ones with constant density of states, \( \sigma \), and we shall compare \( w_{N_0} \) and \( w^c_{N_0} \). An important parameter of the system is the quantity \( \sigma k_B T \). For large \( \sigma k_B T \) we can do some analytical calculations, assuming that \( w_{N_0} \)
Figure 1. Probability distribution of $w_{N_0}$ as a function of $N_0$. The maximum of the probability distribution is located at $N_{0,\text{max}}$, which is given by the equation $(\mu - N_{0,\text{max}}/\sigma)/k_B T = \log(\sigma k_B T)$. For this particular plot $\sigma k_B T = 100$.

has a gaussian shape, but, as we shall see in section 2, this approximation is not too good for the evaluation of $\langle \delta^2 N_0 \rangle$, the mean square fluctuation of $N_0$. To correct this and to extend our calculations to lower values of $\sigma k_B T$, in Section 3 we do numerical calculations of $w_{N_0}$, $w_{N_0}^c$, $\langle \delta^2 N_0 \rangle$ and $\langle \delta^2 N_0 \rangle^c$ instead of the more rigorous $\langle (\delta N_0)^2 \rangle$ and $\langle (\delta N_0)^2 \rangle^c$, respectively.) We reobtain the known result: $\langle \delta^2 N_0 \rangle^c \rightarrow \zeta(2)(\sigma k_B T)^2$, i.e. the mean square fluctuation of $N_0$ is of the same order as the number of particles in the thermally active layer or the number of particles on the excited states in the Bose gas. We also obtain that $\langle \delta^2 N_0 \rangle - \langle \delta^2 N_0 \rangle^c$ converge to a constant value, 0.39, as $\sigma k_B T$ increases. This asymptotic behavior is proved analytically by the end of Section 3.

2. Analytical evaluation of the fluctuations

First we analyse Eq. (1) analytically in the limit $\sigma k_B T \gg 1$. To do this we take

$$\log Z_{N_0} = \left[-\beta \left(\frac{\sigma \epsilon_0^2}{2} - \epsilon_0\right) + \beta \mu (\sigma \epsilon_0 - 1)\right] + \sigma \int_{\epsilon_0}^{\infty} d\epsilon \log \left[1 + e^{-\beta(\epsilon - \mu)}\right],$$  \hspace{1cm} (3)

where $\epsilon_0$ is the energy of the $N_0^{th}$ single-particle level, the integral represents $\log Z_{\epsilon_0}(N_0, \beta, \beta \mu)$. The shape of the probability distribution is depicted in figure 4. Since $\sigma \epsilon_0 = N_0$ and $\partial \log Z_{N_0}/\partial \epsilon_0 = \sigma^{-1}(\partial \log Z_{N_0}/\partial \epsilon_0)$, the value of $\epsilon_0$ corresponding to the maximum of probability is given by the equation

$$\frac{\partial \log Z_{N_0}}{\partial \epsilon_0} \bigg|_{\epsilon_{0,\text{max}}} = - \sigma \left\{ \log \left[1 + e^{\beta(\epsilon_{0,\text{max}} - \mu)}\right] - (\sigma k_B T)^{-1}\right\} = 0,$$  \hspace{1cm} (4)
\textit{Fluctuations of the Fermi condensate}

and for \( \sigma k_B T \gg 1 \)

\[ \epsilon_{0,\text{max}} = \mu - k_B T \log[\sigma k_B T]. \tag{5} \]

We observe here that \( \beta(\mu - \epsilon_{0,\text{max}}) \) depends only on \( \sigma k_B T \) and not on \( \mu \), i.e., on the total particle number, as long as \( \mu > k_B T \log[\sigma k_B T] \). Therefore, as \( T \) decreases and \( \mu \) becomes bigger than \( k_B T \log[\sigma k_B T] \) the probability distribution (11) forms a maximum at \( N_0 > 0 \). We say that at this temperature the condensate starts to form and the equation \( \mu = k_B T \log[\sigma k_B T] \) defines the condensation temperature.

At low temperatures the maximum of \( w_{N_0} \) becomes sharp and \( \epsilon_0 \) approaches \( \mu \). In this temperature range we shall approximate \( w_{N_0} \) around the maximum by a gaussian distribution:

\[ w_{N_0} \approx w(N_{0,\text{max}}) \cdot \exp \left[ -\frac{(N_0 - N_{0,\text{max}})^2}{2\Delta^2} \right] \tag{6} \]

The width of the gaussian is

\[ \Delta^{-2} = -\left. \frac{\partial^2 \log Z_{N_0}}{\partial N_0^2} \right|_{N_0,\text{max}} = (\sigma k_B T)^{-1} - (1 + \sigma k_B T)^{-1} \approx (\sigma k_B T)^{-2}. \tag{7} \]

Equation (6) amounts to the use of the approximative function \( Z_{N_0}^{(a)} \) instead of \( Z_{N_0} \):

\[ Z_{N_0}^{(a)} = \exp \left\{ \log Z_{N_0} + \frac{1}{2} \frac{\partial^2 \log Z_{N_0}}{\partial N_0^2} \delta^2 N_0 \right\} = Z_{N_{0,\text{max}}} \exp \left\{ -\frac{1}{2} \frac{\delta^2 N_0}{(\sigma k_B T)^2} \right\}. \tag{8} \]

To check the approximation we calculate first the total partition function (2) as

\[ Z^{(a)} = \int_{-\infty}^{\infty} d(\delta N_0) Z_{N_0}^{(a)} \approx Z_{N_{0,\text{max}}} \cdot \int_{-\infty}^{\infty} d(\delta N_0) \exp \left[ -\frac{1}{2} \frac{\delta^2 N_0}{(\sigma k_B T)^2} \right] \]

\[ = Z_{N_{0,\text{max}}} \sqrt{2\pi \sigma k_B T}, \tag{9} \]

where

\[ \log Z_{N_{0,\text{max}}} = -\beta \left( \frac{\epsilon_{0,\text{max}}^2}{\sigma} - \epsilon_{0,\text{max}} \right) + \beta \mu (\sigma \epsilon_{0,\text{max}} - 1) + \sigma \int_{\epsilon_{0,\text{max}}}^{\infty} d\epsilon \log \left[ 1 + e^{-\beta(\epsilon - \mu)} \right] \]

\[ = \frac{\sigma k_B T}{2} \left[ (\beta \mu)^2 - \log^2(\sigma k_B T) \right] - \log(\sigma k_B T) + \sigma k_B T L_i(1) \exp[\beta(\mu - \epsilon_{0,\text{max}})], \tag{10} \]

where \( L_i \) is Euler’s dilogarithm function [5]. Using Eq. (3) and the expansion

\[ L_i(z) \big|_{z \to -1} \approx \frac{(\log |z|)^2}{2} \left[ 1 + \frac{\pi^2}{6} \left( \frac{2}{(\log |z|)^2} \right) \right], \tag{11} \]

valid for any \( n > 1 \), we obtain the approximation

\[ \log Z^{(a)} \approx \frac{\sigma k_B T}{2} (\beta \mu)^2 + \frac{\pi^2}{6} \sigma k_B T + \log \sqrt{2\pi}. \tag{12} \]

On the other hand, the exact partition function is

\[ \log Z = \sigma \int_{0}^{\infty} d\epsilon \log \left[ 1 + e^{-\beta(\epsilon - \mu)} \right] = \sigma k_B T L_i(1) \exp(-\beta \mu), \tag{13} \]

which, if we apply again the approximation (11) we get

\[ \log Z = \frac{\sigma k_B T}{2} (\beta \mu)^2 + \frac{\pi^2}{6} \sigma k_B T. \tag{14} \]
The expansions of $\log Z$ and $\log Z^{(a)}$ are identical up to order $\sigma_B T / (\beta \mu)^2$, or $\sigma_B T / \log^2 (C_B T)$. The term $\log \sqrt{2\pi} \approx 0.92$ from equation (12) may be neglected, since is smaller than the order $\sigma_B T / \log^2 (\sigma_B T) \gg 1$, for $\sigma_B T \gg 1$.

The fluctuation of the number of particles in the condensate, in the approximation (9), is

$$\sqrt{\langle \delta^2 N_0 \rangle} = \sigma_B T.$$  \hspace{1cm} (15)

On the other hand, the corresponding canonical fluctuation may be calculated for example by saddle point method applied to the equivalent Bose system \[15, 16\] and gives

$$\sqrt{\langle \delta^2 N_0 \rangle^c} = \sqrt{\zeta(2)} \sigma_B T,$$  \hspace{1cm} (16)

where $\zeta(x)$ is the Riemann Zeta function. Obviously, the two analytical approximations, (15) and (16), do not coincide and the question that remains to be answered is whether these distributions are indeed different, or simply the gaussian approximation (3) is not good enough.

3. Numerical evaluation of the fluctuations

In this section we calculate the fluctuations numerically by introducing a recursion relation. From Eq. (11) we obtain

$$\frac{w_{N_0+1}}{w_{N_0}} = \frac{\exp[-\beta(\epsilon_{N_0} - \mu)]}{1 + \exp[-\beta(\epsilon_{N_0+1} - \mu)]}$$  \hspace{1cm} (17)

and the value of $N_{0,max}$ may be found by solving

$$\frac{\exp[-\beta(\epsilon_{N_{0,max}} - \mu)]}{1 + \exp[-\beta(\epsilon_{N_{0,max}+1} - \mu)]} = 1.$$  \hspace{1cm} (18)

If the density of states is constant and $\epsilon_{i+1} - \epsilon_i = \sigma^{-1}$ for any $i$, then Eq. (18) becomes

$$\frac{\exp[-\beta(\epsilon_{0,max} - \mu)]}{1 + \exp[-\beta(\epsilon_{0,max} + \sigma^{-1} - \mu)]} = 1.$$  \hspace{1cm} (19)

Using Eqs. (17) and (19) we may now calculate numerically $N_{0,max}$, $\langle N_0 \rangle$, and $\langle \delta^2 N_0 \rangle$. If $\sigma_B T \gg 1$, by writing $\exp[-\beta(\epsilon_{0,max} + \sigma^{-1} - \mu)] \approx \exp[-\beta(\epsilon_{0,max} - \mu)](1-(\sigma_B T)^{-1})^{-1}$ Eq. (19) may be simplified to $\exp[\beta(\mu - \epsilon_{0,max})] = \sigma_B T$, which is the same as Eq. (5). Moreover, since around the maximum $\exp[\beta(\mu - \epsilon_{0,max})] \gg 1$, in the relevant energy interval we may transform Eq. (17) into

$$\frac{w_{N_0+1}}{w_{N_0}} = \{\exp[\beta(\epsilon_{N_0} - \mu)] + \exp[-(\sigma_B T)^{-1}]\}^{-1} \approx 1 - e^{\beta(\epsilon_{N_0} - \mu)} + (\sigma_B T)^{-1}.$$  \hspace{1cm} (20)

Let us now analyse the equivalent Bose gas. If again the system has a constant density of states we denote $q \equiv e^{-1/(\sigma_B T)}$. Then the canonical partition function for a system of $N_{ex}$ particles is \[16\]

$$Z^b_{N_{ex}} = \prod_{k=1}^{N_{ex}} \frac{1}{1 - q^k}.$$  \hspace{1cm} (21)

In a canonical system of $N$ particles, the probability $w^b_{N_{ex}}$ to have exactly $N_{ex}$ particles in the excited states (not on the ground state) is proportional to $Z^b_{N_{ex}} - Z^b_{N_{ex} - 1}$ \[16\], so we have

$$w^b_{N_{ex}} = \frac{q^{N_{ex}}}{Z} \prod_{k=1}^{N_{ex}} \frac{1}{1 - q^k}.$$  \hspace{1cm} (22)
Since $N_{\text{ex}} \equiv N - N_0$, the relative probability which corresponds to Eq. (17) for fermions is

$$\frac{w_{N_{\text{ex}}-1}^b}{w_{N_{\text{ex}}}^b} = \frac{1 - q^{N_{\text{ex}}}}{q} = \{1 - \exp[N_{\text{ex}}/(\sigma k_B T)]\} \exp[(\sigma k_B T)^{-1}]. \quad (23)$$

The most probable $N_{\text{ex}}$ is given by

$$[1 - \exp(-\beta N_{\text{ex}}/\sigma)] \exp[(\sigma k_B T)^{-1}] = 1. \quad (24)$$

We want now to compare Eqs. (17) and (23) in the limit $\sigma k_B T \gg 1$. For this we take a Fermi and a Bose system with the same number of particles, $N$. In the Fermi system we define $\epsilon_F = N/\sigma$ (Fermi energy). We shall assume that both systems are below the condensation temperature and the number of particles in the condensate is $N_0$. Above the condensate we have $N_{\text{ex}}$ particles. For a condensed gas $\epsilon_F - \mu < \sigma^{-1}$, so we can express $N_{\text{ex}}$ in Eq. (23) as $N_{\text{ex}} = \sigma(\epsilon_F - \epsilon_0) = \sigma(\mu - \epsilon_0)$. By doing so, Eq. (23) becomes

$$\frac{w_{N_{\text{ex}}+1}^b}{w_{N_{\text{ex}}}^b} = \{1 - \exp[\beta(\epsilon_{n_0} - \mu)]\} \exp[(\sigma k_B T)^{-1}]$$

$$\approx 1 - \exp[\beta(\epsilon_{n_0} - \mu)] + (\sigma k_B T)^{-1},$$

which is identical to Eq. (20). Therefore the two probability distributions $w_{N_0}^b$ and $w_{N_{\text{ex}}}^b$ approach each other in the limit of large systems, i.e., when $\sigma k_B T \gg 1$.

The numerical calculations, based on Eqs. (17) and (23) are plotted in figure 2. We can observe that already for $\sigma k_B T$ bigger than 1, the fluctuation of the particle number in the condensate is almost the same for the canonical Bose and grandcanonical Fermi systems. This justify the approach taken in Ref. [14], and for $\sigma k_B T \gg 1$, $N_0$ may be calculated directly as the average number of particles in the Bose condensate, rather than by Eq. (4).

Noticeable relative differences between canonical and grandcanonical results appear only for $\sigma k_B T$ about 1 or below. For these values of $\sigma k_B T$ the fluctuations depend on the exact location of $\mu$, with respect to the single particle levels. For example let’s say that $\mu \in (\epsilon_{N-1}, \epsilon_N)$, where $\epsilon_{N-1}$ and $\epsilon_N$ are two consecutive energy levels. In the limit $\beta(\mu - \epsilon_{N-1}) \rightarrow \infty$ and for $N_0 = N$, equation (17) becomes

$$\frac{w_N}{w_{N-1}} \approx \exp[\beta(\mu - \epsilon_{N-1})]. \quad (26)$$

For $\beta(\mu - \epsilon_{N-1}) \rightarrow \infty$ we can calculate $\langle N_0 \rangle$ and $\langle \delta^2 N_0 \rangle$ by taking into account only the levels $\epsilon_{N-1}$ and $\epsilon_N$ and we obtain

$$\langle N_0 \rangle = \frac{\exp[\beta(\mu - \epsilon_{N-1})](N + 1) + N}{\exp[\beta(\mu - \epsilon_{N-1})] + 1} = N + 1 - \exp[\beta(\mu - \epsilon_{N-1})] \quad (27)$$

and

$$\langle \delta^2 N_0 \rangle = \frac{\exp[\beta(\mu - \epsilon_{N-1})] \exp[-2\beta(\mu - \epsilon_{N-1})] + (1 - \exp[-\beta(\mu - \epsilon_{N-1})])^2}{\exp[\beta(\mu - \epsilon_{N-1})] + 1}$$

$$= \frac{1 - \exp[-\beta(\mu - \epsilon_{N-1})]}{\exp[\beta(\mu - \epsilon_{N-1})] + 1} \approx \exp[-\beta(\mu - \epsilon_{N-1})] \quad (28)$$

Therefore, for any $\mu \in (\epsilon_N, \epsilon_{N+1})$,

$$\lim_{T \rightarrow 0} \left( \langle \delta^2 N_0 \rangle^{1/2} \left[ \frac{\zeta^{1/2}(2)}{(\sigma k_B T)^{-1}} \right]^0 \right) = 0. \quad (29)$$
The situation is different if for example $\mu = \epsilon_N$. Then, applying the same algorithm as above, we get $\langle N_0 \rangle = N + 0.5$ and $\langle \delta^2 N \rangle^{1/2} = 0.5$. In figure 2 we plotted $\langle \delta^2 N \rangle^{1/2}/[\xi^{1/2}(2) \sigma_{kT}]$ (a) and $\langle \delta^2 N \rangle^{1/2}$ (b) for $\mu = \epsilon_N + 0.1 \cdot i \sigma^{-1} (i = 0, 1, \ldots, 5)$. The fluctuations normalized to the asymptotic value, $\xi^{1/2}(2) \sigma_{kT}$, are quite different for $\sigma_{kT} \lesssim 1$, but the absolute values of the fluctuations are very close for any $\sigma_{kT}$ for both types of systems and any choice of $\mu$.

We notice also in figure 2(b) that although the difference $\sqrt{\langle \delta^2 N_0 \rangle} - \langle \delta^2 N_0 \rangle^c$ is very small for any $\sigma_{kT}$, it does not converge to zero as $\sigma_{kT} \to \infty$. Numerically we obtain

$$\sqrt{\langle \delta^2 N_0 \rangle} - \langle \delta^2 N_0 \rangle^c \approx 0.39 \quad \text{for } \sigma_{kT} \gg 1. \quad \text{(30)}$$

To explain this difference, let us note that the fluctuation of $N_0$ in the grandcanonical ensemble, $\delta N_0 \equiv N_0 - \langle N_0 \rangle$, may be viewed as the superposition of fluctuations coming from two sources: the canonical fluctuation of $N_0$ around its average value, corresponding to the total particle number $N$, denoted by $\delta^N N_0 \equiv N_0 - \langle N_0 \rangle_N$, and the fluctuation of $\langle N_0 \rangle_N$ due to the grandcanonical fluctuation of $N$. Assuming small fluctuations, the variation of $\langle N_0 \rangle_N$ due to the variation of $N$ may be written as

$$\delta \langle N_0 \rangle_N = \frac{\partial \langle N_0 \rangle}{\partial N} \cdot \delta N.$$ 

Collecting all these together we write

$$\delta N_0 = N_0 - \langle N_0 \rangle \equiv N_0 - \langle N_0 \rangle_N + \langle N_0 \rangle_N - \langle N_0 \rangle = \delta N_0^c + \frac{\partial \langle N_0 \rangle}{\partial N} \cdot \delta N. \quad \text{(31)}$$

Below the condensation, $\partial \langle N_0 \rangle / \partial N = 1$ (temperature stays constant). Moreover, well below the condensation temperature $\delta N_0^c$ and $\delta N$ are independent, since the condensate may be viewed as a reservoir of particles of zero energy [17], and Eq. 31 leads to

$$\langle \delta^2 N_0 \rangle = \langle \delta^2 N_0 \rangle^c + \langle \delta^2 N \rangle. \quad \text{(32)}$$
For high enough $\sigma k_B T$ and $\beta \mu$ we have $\langle \delta^2 N \rangle \approx \sigma k_B T$, which, if plugged in Eq. (32) gives

$$\langle \delta^2 N_0 \rangle \approx \sqrt{\zeta(2)}(\sigma k_B T)^2 + \sigma k_B T \approx \sqrt{\zeta(2)}\sigma k_B T + \frac{1}{2\sqrt{\zeta(2)}}. \quad (33)$$

As we expect, $(2\sqrt{\zeta(2)})^{-1} \approx 0.39$.

Using the same method we calculate the fluctuation of the number of particles in the thermally active layer without doing any extra numerics. Again we denote by $\langle N_{ex} \rangle$ the average number of particles in the thermally active layer and the fluctuation $\delta N_{ex}$ can again be written as

$$\delta N_{ex} \equiv N_{ex} - \langle N_{ex} \rangle = N_{ex} - \langle N_{ex} \rangle_N + \langle N_{ex} \rangle_N - \langle N_{ex} \rangle = \delta^c N_{ex} + \frac{\partial \langle N_{ex} \rangle}{\partial N} \delta N. \quad (34)$$

By $\langle N_{ex} \rangle_N$ we denote the average number of particles in the thermally active layer at fixed $N$. Well below the condensation temperature $\langle N_{ex} \rangle_N$ does not depend on $N$, so from Eq. (34) we get $\delta N_{ex} = \delta^c N_{ex}$ and

$$\langle \delta^2 N_{ex} \rangle = \langle \delta^2 N_{ex} \rangle^c = \langle \delta^2 N_0 \rangle^c. \quad (35)$$

4. Conclusions

The thermodynamic equivalence between ideal bosons and fermions with the same constant density of states $\sigma$ apparently is lost below the Bose-Einstein condensation temperature $T_c$. On the other hand it was proven that if the Bose and the Fermi systems have the same spectrum consisting of nondegenerate, equidistant single particle states (like for example particles in a one-dimensional harmonic potential), then the canonical thermodynamic equivalence between the two systems is preserved down to zero temperature in the smallest details. This apparent contradiction is due to the fact that below the condensation temperature $T_c$, in the Fermi gas the values of intensive parameters like the chemical potential and also the population of single particle levels in the canonical ensemble are changed slightly from their corresponding values in the grandcanonical ensemble.

Below $T_c$, to the Bose-Einstein condensate in the Bose system it corresponds in the Fermi system a degenerate gas of the same number of particles, at the lowest part of the spectrum. Because of this correspondence the degenerate fermionic subsystem is called here the Fermi condensate. The Fermi condensate is manifested also in the probability distribution of the grandcanonical ensemble. One can calculate the probability to have $N_0$ degenerate particles (see Eq. 1). Below $T_c$ this probability distribution has a maximum for $N_0 > 0$ and we showed numerically and analytically in Section 3 that the grandcanonical average of $N_0$ is the same as the canonical average.

In this paper we did both, analytical and numerical calculations of the number of particles in the condensate and in the thermal active layer. We calculated also their fluctuations. The grandcanonical fluctuation of $N_0$ is almost the same as the fluctuation in the canonical ensemble. Although the average values $\langle N_0 \rangle$ and $\langle N_0 \rangle^c$ are identical, for large values of $\sigma k_B T$ the fluctuations $\langle \delta^2 N_0 \rangle^{1/2}$ and $\langle \delta^2 N_0 \rangle^c^{1/2}$ differ by a small, but constant value, 0.39. This is due to the extra contribution to $\langle \delta^2 N_0 \rangle^{1/2}$, given by the grandcanonical fluctuation of the total particle number.

The fermions in the thermally active layer correspond to the bosons on the excited energy levels. Canonical and grandcanonical averages of $N_{ex}$ are the same. Moreover, well below the condensation temperature, where Maxwell Deamon’s
ensemble is applicable [17], the fluctuation of $N_{ex}$ is the same in both, canonical and grandcanonical ensembles [35].

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