ON THE SINGULARITIES OF THE
PLURICOMPLEX GREEN’S FUNCTION

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Abstract

It is shown that, on a compact Kähler manifold with boundary, the singularities of the pluricomplex Green’s function with multiple poles can be prescribed to be of the form \( \log \sum_{j=1}^{n} |f_j(z)|^2 \) at each pole, where \( f_j(z) \) are arbitrary local holomorphic functions with the pole as their only common zero. The proof is a combination of blow-ups and recent a priori estimates for the degenerate complex Monge-Ampère equation, and particularly the \( C^1 \) estimates away from a divisor.

1 Introduction

The Green’s function plays a central role in the study of functions of one complex variable or of two real variables. But while its natural generalization to functions of more real variables is as the fundamental solution of the Laplacian, its natural generalization to functions of several complex variables is rather as a fundamental solution of the complex Monge-Ampère equation. For our purposes, we shall consider the following broad definition. Let \( M \) be an \( n \)-dimensional compact Kähler manifold with smooth boundary \( \partial M \), and let \( \omega \) be a smooth non-negative closed \((1,1)\)-form on \( M \). Let \( PSH(M,\omega) \) the space of plurisubharmonic functions with respect to \( \omega \), i.e., \( f \in PSH(M,\omega) \) if and only if \( f \) is upper semi-continuous, and \( \omega + \frac{i}{2} \partial \bar{\partial} f \geq 0 \) on \( X \) in the sense of currents. Let \( \{p_1, \ldots, p_N\} \) be \( N \) distinct points on \( M \). Then a pluricomplex Green’s function \( G(z; p_1, \ldots, p_N) \) with poles at the points \( p_j \) is a function in \( PSH(M,\omega) \), bounded from above on \( M \), bounded on any compact subset of \( M \setminus \{p_1, \ldots, p_N\} \), which satisfies the equation

\[
(\omega + \frac{i}{2} \partial \bar{\partial} G)^n = 0 \quad \text{on} \quad M \setminus \{p_1, \ldots, p_N\} \tag{1.1}
\]

in the sense of pluripotential theory. In this paper, we mostly restrict ourselves to the case \( \partial M \neq \emptyset \), in which case we also impose the Dirichlet condition

\[
\lim_{z \to \partial M} G(z; p_1, \ldots, p_N) = 0. \tag{1.2}
\]

We note that the above conditions imply that \( (\omega + \frac{i}{2} \partial \bar{\partial} G)^n \) is a linear combination of Dirac measures supported at the poles \( p_j \). Indeed, the non-linear expression \( (\omega + \frac{i}{2} \partial \bar{\partial} G)^n \)

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is well-defined by the Bedford-Taylor construction [BT] as an \((n,n)\) non-negative current away from the unbounded locus of \(G\) (i.e., the complement in \(M\) of the largest open set where \(G\) is locally bounded). Since the unbounded locus consists only of isolated points, and hence is included in compact subsets of a finite union of Stein neighborhoods, the expression \((\omega + \frac{i}{2} \partial \bar{\partial} G)^n\) can again be defined as a non-negative \((n,n)\) current near these points by the constructions of Demailly [D] and Sibony [Si].

It has been known for a long-time that the pluricomplex Green’s function is not unique, even in the case of a simple pole \(p\) [BT], and that its singularities near \(p\) are not unique either. This is in marked contrast with the real Monge-Ampère equation, where the convex solution in the sense of Alexandrov on a convex domain \(\Omega\) of the equation \(\det D^2 u = c \delta_0, \ u|_{\partial \Omega} = 0\) with \(0 \in \Omega \subset \mathbb{R}^n, c > 0\), is unique. The graph of \(u\) is in this case just an inverted cone, with boundary given by the boundary of \(\Omega\), and vertex a point on the \(u\) axis, determined by the constant \(c\). In particular, the singularities of \(u\) at \(0\) are determined by \(\partial \Omega\) and \(c\). This difference between the real and the complex case can be partly attributed to the fact that the condition of convexity in the real case is much more stringent than the condition of plurisubharmonicity.

The pluricomplex Green’s function has been extensively studied over the years, using many different methods. It is not possible for us to provide a full list of references, but we shall try and indicate along some of the works closest in spirit to the present paper. One notable such example is the work of B. Guan [Gb], using PDE methods. There, building on the estimates of Yau [Y1] and Caffarelli, Kohn, Nirenberg, and Spruck [CKNS], he established the existence and \(C^{1,1}\) regularity of the pluricomplex Green’s function for strongly pseudoconvex domains in \(\mathbb{C}^n\), with prescribed singularity \(\log |z - p|^2\) at the pole. The \(C^{1,1}\) regularity was also obtained by Blocki [B1]. The methods of Guan extend to a singularity of the form \(\log \sum_{j=1}^n |f_j(z)|^2\), as long as the \(f_j(z)\) are holomorphic functions with only \(p\) as their common zero, and are globally defined on the domain.

The present paper has two goals. One goal is to establish the existence of pluricomplex Green’s functions with singularities at multi poles \(p_j, 1 \leq j \leq N\), given by arbitrary local analytic functions. The other is to begin developing a geometric/analytic approach to Monge-Ampère equations with measures on the right-hand side, where the singularities of the solution arise from blow-up constructions. Since blow-ups typically lead to degenerate Kähler forms, an essential tool in our approach is the recent existence theorems for the Dirichlet problem for complex Monge-Ampère equations with degenerate background form established in [PS4].

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2 The Main Results

Henceforth, $(M, \omega)$ will be a compact complex manifold of dimension $n$, with non-empty smooth boundary $\partial M$. Let $N$ be an arbitrary positive integer, and let $p_1, \cdots, p_N$ be $N$ distinct (interior) points in $M$. Our main result is the following:

**Theorem 1** Let $\omega$ be a Kähler form on $\tilde{M}$, and for each $1 \leq m \leq N$, let $f_{jm}(z)$, $1 \leq j \leq n$, be $n$ holomorphic functions defined in a neighborhood of $p_m$, with $p_m$ as their only common zero in this neighborhood. Then there exists a constant $\partial M$ smooth boundary $1 \leq H \leq n$, and any constant $0 < \varepsilon_0 < \varepsilon_0$, $1 \leq m \leq N$, there exists a unique function $G(z; p_1, \cdots, p_N) \in PSH(M, \omega) \cap C^\alpha(M \setminus \{p_1, \cdots, p_N\})$, which satisfies the equation (1.1), the boundary condition (1.2), and the following asymptotics near each pole $p_m$,

$$G(z; p_1, \cdots, p_N) = \varepsilon_m \log \left( \sum_{j=1}^{n} |f_{jm}(z)|^2 \right) + O(1). \quad (2.1)$$

Here $\alpha$ is any constant satisfying $0 < \alpha < 1$.

In general, it is not possible to choose $\varepsilon_0$ to be arbitrary. In fact, if $\delta > 0$ is given, then it is easy to construct $(M, \omega)$ and $p \in M$, and local holomorphic functions $f_1, \cdots, f_n$, for which the maximal $\varepsilon_0$ is less than $\delta$. For example, let $M = X \times D$ where $(X, \omega)$ is a compact Kähler manifold with unit volume, and $D \subseteq \mathbb{C}$ is the unit disk. Let $p = (x, 0) \in X \times \{0\}$ and choose local coordinates $z_1, \cdots, z_n$ on $X$ centered at $x$. Suppose $G$ is a Greens function on $M$ with singularity $\varepsilon \log (|w|^2 + |z_1|^{2k} + \cdots + |z_n|^{2k})$, where $w$ is a coordinate on $\mathbb{C}$, centered at $0 \in \mathbb{C}$. Then $G(0, z_1, \cdots, z_n) \in PSH(X, \omega)$ has Lelong number $\varepsilon k$ so, by a result of Demailly [D], its Monge-Ampère mass is at least $\varepsilon k$. Thus $\varepsilon \leq \frac{1}{k}$.

However, the restriction on $\varepsilon_0$ can be removed for strongly pseudoconvex manifolds, i.e., manifolds $M$ admitting a $C^2$ function $\rho$ with $\partial M = \{\rho = 0\}$, and $i\partial\bar{\partial}\rho > 0$. In this case, we have the following solution of the Dirichlet problem:

**Theorem 2** Let $\omega$ be a non-negative smooth $(1, 1)$-form on $\tilde{M}$. Let $f_{jm}$, $1 \leq j \leq n$, $1 \leq m \leq N$ be as in the previous theorem. Assume that $M$ is strongly pseudoconvex. Then for any function $\varphi_b \in C^2(\partial M)$, and any constant $\varepsilon_0 > 0$, $1 \leq m \leq N$, there exists a unique function $G(z; p_1, \cdots, p_N) \in PSH(M, \omega) \cap C^\alpha(\tilde{M} \setminus \{p_1, \cdots, p_N\})$, which satisfies the equation (1.1), the asymptotics (2.1) near each pole $p_m$, and the Dirichlet boundary condition

$$\lim_{z \to \partial M} G(z; p_1, \cdots, p_N) = \varphi_b. \quad (2.2)$$

Again $\alpha$ is any constant satisfying $0 < \alpha < 1$. 

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Theorem 2 is closest to earlier results of B. Guan [Gb], Blocki [B1], and Lempert [L]. The regularity properties of the pluricomplex Green’s function are much more precise in these earlier works [Gb], [B1], and [L], but we gain here in generality, including the property that local holomorphic singularities can be assigned arbitrarily. We shall make use of this latter fact in the construction of new geodesic rays in the space of Kähler potentials (see section §5).

In general, the case of Kähler manifolds with boundary is quite different from the case of Kähler manifolds without boundary. Nevertheless, the idea used in the proof of Theorem 1 may also be used to yield results for singular Monge-Ampère equations on compact manifolds without boundary. The simplest example is the following:

**Theorem 3** Let \((X, \omega)\) be a compact Kähler manifold with unit volume. Let \(p \in X\), let \(f \in C^\infty(X)\) be positive, and assume \(\int_X f \omega^n = 1\). Let \(\delta_p\) be the Dirac measure concentrated at \(p\). Then for \(\varepsilon > 0\) sufficiently small, there exists a unique \(\varphi \in PSH(X, \omega) \cap C^{1,\alpha}(X \setminus \{p\})\) satisfying \(\varphi = \varepsilon \log |z|^2 + C^{1,\alpha}(X)\) near \(z = p\) and

\[
(\omega + \frac{i}{2} \partial \bar{\partial} \varphi)^n = (1 - \varepsilon) f \omega^n + \varepsilon \delta_p \quad \tag{2.3}
\]

As Coman-Guedj [CG] have shown, there are examples of Kähler manifolds for which \(\varepsilon\) must be strictly smaller than one.

We observe that the equation \((\omega + \frac{i}{2} \partial \bar{\partial} \varphi)^n = \mu\) on a compact Kähler manifold without boundary has been solved by Berman, Boucksom, Guedj, and Zeriahi [BBGZ], when \(\mu\) is a measure which does not charge pluripolar sets. The important case of measures \(\mu\) which charge pluripolar sets remains open. A general result is that of Ahag et al. [ACCH], who show that, on a hyperconvex domain in \(\mathbb{C}^n\), a non-negative measure is a complex Monge-Ampère measure if it is dominated by a Monge-Ampère measure. The regularity and precise singularities of the solutions are however still obscure at this moment. Theorem 3 provides another example of the solvability of a Monge-Ampère equation with measures charging pluripolar sets. In fact, it can be seen from its proof in section §4 that more general formulations are possible, with the singularity \(p\) replaced by the complex variety \(Z = \{s_1(z) = \cdots = s_k(z) = 0\}, 1 \leq k < n\), where the \(s_\alpha(z)\) are sections of a holomorphic vector bundles satisfying some non-degeneracy conditions.

A related and important problem is to provide flexible characterizations of the unbounded functions for which the Monge-Ampère measure is well-defined (see e.g. Cegrell [C] and Blocki [B2]).

### 3 Proof of Theorems 1 and 2

In this section we give the proof of Theorems 1 and 2. It will be seen that the argument does not depend essentially on \(N\), so we set \(N = 1\) and drop the index \(m\) to lighten the
notation. The bulk of the work is the proof of Theorem 1, and we can just indicate at the end the easy modifications for for Theorem 2. The key lemmas are the following:

**Lemma 1** Denote by \((M,p,f)\) the data consisting of the Kähler manifold \((M,\omega)\), the point \(p\), and the given local holomorphic functions \(f_j(z)\), \(1 \leq \alpha \leq n\) with \(p\) as their only common zero. Then there exists a complex manifold \(X' = X'(M,p,f)\) and a holomorphic map \(\pi' : X' \to M\) sending \(\partial X'\) to \(\partial M\), with the following properties:

(i) There is a closed, non-negative \((1,1)\)-form \(\Omega'\) on \(X'\) an effective divisor \(E'\), and an \(\varepsilon > 0\) with

\[
\Omega' - \varepsilon \frac{i}{2} \partial \bar{\partial} \log h_{E'} > 0
\]  

for some smooth metric \(h_{E'}\) on \(O(-E')\).

(ii) The restriction \(\pi = \pi'|_{\bar{X}' \setminus E'}\) is a biholomorphism \(\pi : \bar{X}' \setminus E' \to \bar{M} \setminus p\) with

\[
\pi_* \Omega' = \omega + \varepsilon \frac{i}{2} \partial \bar{\partial} (\psi(z) \log \sum_{j=1}^{n} |f_j(z)|^2 + 1 - \psi(z))
\]  

where \(\psi(z)\) is a function which is 1 in a neighborhood of \(p\), and which is compactly supported in another such neighborhood.

**Lemma 2** Let \(X'\) be a complex manifold with smooth boundary of dimension \(n\), equipped with a non-negative closed form \(\Omega'\), with \(\Omega' - \varepsilon \frac{i}{2} \partial \bar{\partial} \log h_{E'} > 0\) for some effective divisor \(E'\) supported away from \(\partial M\) some smooth metric \(h_{E'}\) on \(O(-E')\), and some \(\varepsilon > 0\). Then there exists a unique function \(\Phi \in \text{PSH}(X', \Omega') \cap L^\infty(X') \cap C^\alpha(\bar{X}' \setminus E')\) which solves the Dirichlet problem

\[
(\Omega' + \frac{i}{2} \partial \bar{\partial} \Phi)^n = 0 \text{ on } X', \quad \Phi_{\partial X'} = 0.
\]  

Here \(\alpha\) is any constant satisfying \(0 < \alpha < 1\).

The theorem follows readily from the two lemmas. Let \(\Phi\) be the function given by Lemma 2 applied to the complex manifold \(X' = X'(M,p,f)\) and the non-negative form \(\Omega'\) of (3.2). Then \((\Omega' + \frac{i}{2} \partial \bar{\partial} \Phi)^n = 0\) on \(X'\). Set \(\varphi = \Phi \circ \pi\). Since \(\pi\) is a biholomorphism between \(X' \setminus E'\) and \(M \setminus p\), this implies \((\pi_* \Omega' + \frac{i}{2} \partial \bar{\partial} \phi)^n\) on \(M \setminus p\), i.e.

\[
0 = (\omega + \frac{i}{2} \partial \bar{\partial} (\varepsilon [\psi(z) \log \sum_{j=1}^{n} |f_j(z)|^2 + (1 - \psi(z))] + \varphi(z)))^n.
\]  

We can now set

\[
G(z; p) = \varepsilon [\psi(z) \log \sum_{j=1}^{n} |f_j(z)|^2 + (1 - \psi(z))] + \varphi(z) - \varepsilon.
\]
Clearly it satisfies the equation \((\omega + \frac{i}{2} \partial \bar{\partial} G)^n = 0\) on \(M \setminus p\), vanishes on \(\partial M\), and has the desired asymptotics near \(p\). It is \(\omega\)-plurisubharmonic on \(M \setminus p\), and bounded from above. Thus it extends to an \(\omega\)-plurisubharmonic function on \(M\). This shows that \(G(z; p)\) satisfies all the desired properties, and the existence part of the theorem is proved.

To prove the uniqueness, let \(\tilde{G} \in PSH(M, \omega)\) satisfy (1.1), (1.2) and (2.1). Let

\[
\tilde{\Phi} = \tilde{G} - \varepsilon [\psi(z) \log \sum_{j=1}^{n} |f_j(z)|^2 + (1 - \psi(z))] + \varepsilon
\]  

(3.6)

Then \(\tilde{\Phi} \in PSH(\Omega', X' \setminus E') \cap C^\alpha(X' \setminus E')\). Since it is bounded, it extends to a function, which by abuse of notation, will be denoted \(\Phi \in PSH(\Omega', X') \cap C^\alpha(X' \setminus E') \cap L^\infty(X)\). Moreover, \((\Omega' + \frac{i}{2} \partial \bar{\partial} \Phi')^n = 0\). This is certainly true on \(X' \setminus E'\) and, since \(\Phi'\) is bounded, is true on all of \(X'\). By Lemma 2, we have \(\Phi = \tilde{\Phi}\) and hence, \(\tilde{G} = G\).

Thus it suffices to establish Lemma 1 and Lemma 2. We begin with the proof of Lemma 1, which we break into several steps. Some of these are well-known, but we did not find the version that we needed in the literature, so we have provided a complete derivation.

### 3.1 Blow-ups of complex manifolds

We start by recalling the construction of the blow-up of a complex manifold along a smooth submanifold and some of its basic properties.

Let \(W\) be a complex manifold of dimension \(n\), and let \(Z \subset W\) be a submanifold of dimension \(d < n\). Let \(N = TW|_Z / T Z\) be the normal bundle of \(Z\) and let \(E = P(N)\). Let

\[
W' = (W \setminus Z) \cup E
\]

(3.7)

and define \(\pi : W' \to W\) by extending the map \(\pi : E \to Z\) to be the identity map on \(W \setminus Z\). The set \(W'\) has a natural complex structure for which \(\pi : W' \to W\) is a holomorphic map: The complex structure on \(W'\) is defined as follows:

First, we require that \(W \setminus Z \subseteq W'\) is an open set and the inclusion \(W \setminus Z \to W'\) is holomorphic.

Next, we let \(U_\alpha\) be a collection of coordinate balls in \(W\) such that \(Z \subseteq \cup_\alpha U_\alpha\) and such that \(U_\alpha \cap Z = \{(x^\alpha, y^\alpha) \in U_\alpha : y^\alpha = 0\}\). On \(U_\alpha \cap U_\beta\) we have \(z^\beta = \phi_\alpha^\beta(z^\alpha)\) with \(\phi_\alpha^\beta : U_{\alpha \beta} \to U_{\beta \alpha}\) a biholomorphic function between open subsets of \(C^n\). The derivative

\[
D\phi_\alpha^\beta = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}
\]

is an invertible \(n \times n\) matrix, where \(D_{11}\) has size \(d \times d\), and \(D_{22}\) has size \((n - d) \times (n - d)\).

Note that \(D_{22}(p)\) is invertible if \(p \in U_\alpha \cap Z = \{z^\alpha \in U_\alpha : y^\alpha = 0\}\). In fact, \(D_{22}\) is the isomorphism of the normal bundle of \(\{y^\alpha = 0\} \subseteq U_{\alpha \beta}\) to the normal bundle of
\[ \{ y^\beta = 0 \} \subseteq U_{\beta \alpha} \] induced by the biholomorphic map \( \phi^\beta_\alpha \). Let us spell out this point in more detail. We can write

\[ y^\beta_i = A^\beta_i y^\alpha_j, \quad 1 \leq i, j \leq n - d, \quad (3.8) \]

for certain (non-unique) holomorphic functions \( A^\beta_i \) on \( U_{\alpha \beta} \). This follows from the fact that \( \phi^\beta_\alpha \) takes the \( x^\alpha \)-axis (i.e. the set \( \{ y^\alpha = 0 \} \)) to the \( x^\beta \)-axis (i.e. the set \( \{ y^\beta = 0 \} \)). Then \( D_{22}(p) = (A^\beta_i(p)) \) is an invertible matrix.

Define a complex manifold \( U'_\alpha \) as follows:

\[ U'_\alpha = \{ (z, t) : z \in U_\alpha, t \in \mathbb{P}^{n-d-1} : y_i t_j = t_i y_j, 1 \leq i, j \leq n - d \} \quad (3.9) \]

Let \( f_\alpha : U'_\alpha \rightarrow \pi^{-1}(U_\alpha) \subseteq W' \) be the bijective map

\[ (z, t) \mapsto \begin{cases} z & \text{if } y \neq 0 \\ t_1 \frac{\partial}{\partial y_1} + \cdots + t_n - d \frac{\partial}{\partial y_{n-d}} & \text{if } y = 0 \end{cases} \quad (3.10) \]

We wish to use the maps \( f_\alpha \) to give \( W' \) a complex structure. To do this, we must show that the change of coordinate maps \( f^\beta_\alpha = f^{-1}_\beta \circ f_\alpha : U'_\alpha \rightarrow U'_\beta \) are holomorphic.

We proceed as follows: \( f^\beta_\alpha(z^\alpha, t^\alpha) = (z^\beta, t^\beta) \) where \( z^\beta = \phi^\beta_\alpha(z^\alpha) \) and

\[ \frac{t^\beta_i}{t^\beta_j} = \frac{y^\beta_i}{y^\beta_j} = \frac{A^\beta_i y^\alpha_k}{A^\beta_j y^\alpha_k} = \frac{A^\beta_i t^\alpha_k}{A^\beta_j t^\alpha_k} \quad (3.11) \]

If \( y^\beta_i \neq 0 \) for some \( i \), then the first equality implies that \( t^\beta_i \neq 0 \) so we can take \( t^\beta_i = 1 \) and \( t^\beta_j = \frac{y^\beta_j}{y^\beta_i} \), which is holomorphic.

If \( y^\beta_i = 0 \) for all \( i \) then we make use of the fact that \( D_{22} \) is invertible on \( Z \) so there exists \( i \) such that \( A^\beta_i t^\alpha_k \neq 0 \). We take \( t^\beta_i = 1 \) and \( t^\beta_j = \frac{A^\beta_i y^\alpha_k}{A^\beta_j y^\alpha_k} \), which is holomorphic. Thus we see that \( W' \) is a complex manifold.

Now let \( \pi : W' \rightarrow W \) be as above. Let \( p \in E \) and \( q = \pi(p) \). Then the discussion above shows that there exists a coordinate neighborhood \( \Omega \) of \( p \in W' \) and coordinates \((\zeta_0, \ldots, \zeta_d, \theta_1, \ldots, \theta_{n-d-1})\) centered at \( p \) with the following properties:

1) \( \zeta_i = z_i \circ \pi \) for some coordinate functions \( z_j \) on \( W \)
2) \( E \cap \Omega = \{ \zeta_0 = 0 \} \).
3) \( (\theta_1|_E, \ldots, \theta_{n-d-1}|_E) \) is a set of local coordinates of \( \pi^{-1}(p) \) centered at \( p \in E \).
4) If \( p \in U'_\alpha \) then \( \zeta_0|_j y^\alpha_j \) for all \( j \) and \( y^\alpha_j / \zeta_0 \) is nowhere vanishing for some \( j_0 \).

The last condition says that \( E \cap \Omega = \{ y^\alpha_j = 0 \} \).
This blow-up process can be iterated: If $Z' \subseteq W'$ is a smooth subvariety then we can construct $W'' = BL(Z', W')$ and we have maps $W'' \rightarrow W' \rightarrow W$. We say that $W''$ is an iterated blow-up.

If $\pi : W' \rightarrow W$ is an iterated blow-up, the exceptional divisor is by definition the smallest effective divisor $E \subseteq W'$ such that $W'\setminus E \rightarrow W\setminus \pi(E)$ is an isomorphism.

### 3.2 Analytic spaces

Let $X$ be a set. We say that $X$ is an analytic space if there is a complex manifold $W$ (called an ambient manifold) with $X \subset W$ and satisfying the following property: for every $p \in X$, there is an open set $p \in U \subset W$ and functions $f_1, \ldots, f_r : U \rightarrow \mathbb{C}$ such that

$$U \cap X = \{w \in W; f_1(w) = \cdots = f_r(w) = 0\}.$$ \hfill (3.12)

We denote by $X_{\text{reg}}$ the subset of $X$ where $X$ is locally a complex manifold, and by $X_{\text{sing}}$ its complement in $X$. A Kähler metric on $X$ is by definition a Kähler metric on $X_{\text{reg}}$ which is the restriction of a Kähler metric on an ambient manifold $W$.

**Lemma 3** Let $(M, p, f)$ be a data consisting of a complex manifold $M$, a point $p \in M$, and local holomorphic functions $f_1(z), \ldots, f_n(z)$ defined in a neighborhood of $p$, and with $p$ as their only common zero. Then we can associate to this data an analytic space $X = X(M, p, f)$ with the following properties:

(i) There is a biholomorphism between $X \setminus X_0$ and $M \setminus p$, for some subset $X_0$ of $X$ which is biholomorphically equivalent to $\mathbb{CP}^{n-1}$.

(ii) Let $\psi(z)$ be a cut-off function which is $1$ in a coordinate chart around $p$ in $M$, and $0$ outside another such chart. Then for all $\delta$ small enough, the pull-back to $X \setminus X_0$ of the form $\omega_\delta$ defined as

$$\omega_\delta = \omega + \frac{\delta i}{2} \partial \bar{\partial} (\psi(z) \log \sum_{j=1}^n |f_j(z)|^2 + 1 - \psi(z))$$ \hfill (3.13)

defined on $M \setminus p$ extends to a Kähler form on $X$.

**Proof of Lemma 3:** Given the data $(M, p, f)$, let $U$ be a coordinate neighborhood centered at $p$ in $M$, define a space $V$ by

$$V = \{(z_1, \ldots, z_n), (y_1, \ldots, y_n) \in U \times \mathbb{CP}^{n-1}: y_i f_j(z) = y_j f_i(z)\}$$ \hfill (3.14)

and let $\pi : V \rightarrow U$ be the projection. We then define $X = X(M, p, f)$ by

$$X = (M \setminus \{p\} \cup V) / \sim$$ \hfill (3.15)
where, for \( m \in M \) and \( v \in V \) we say \( m \sim v \) if \( \pi(v) = m \). Note that the fiber \( X_0 \) of \( V \) above the point \( p \) is the entire projective space \( \mathbb{CP}^{n-1} \). We claim that \( X \) is an analytic space.

To show that \( X \) is an analytic space, we must find a complex manifold \( W \) such that \( X \subseteq W \) and such that \( X \) is locally defined by the simultaneous vanishing of a finite collection of holomorphic functions.

Let \( B \subseteq \mathbb{C}^n \) be a small open ball centered at the origin and let \( Z \subseteq U \times \mathbb{C}^n \) be the smooth manifold
\[
Z = \{(z, \xi) : z \in U, \xi \in B : f_1(z) - \xi_1 = \cdots = f_n(z) - \xi_n = 0\} \tag{3.16}
\]
If \( B \) is sufficiently small, then the image of the map \( Z \to U \) is compactly supported in \( U \). Thus \( Z \subseteq M \times B \) is a smooth submanifold whose image, when projected to \( M \), lies in a relatively compact subset of \( U \). Finally, let \( W = BL(Z, M \times B) \). Thus \( W \) is locally defined by
\[
W = \{(z, \xi, y) \in U \times B \times \mathbb{CP}^{n-1} : y_i(f_j(z) - \xi_j) = y_j(f_i(z) - \xi_i)\} \tag{3.17}
\]
Then \( W \) is a smooth manifold and \( X \subseteq W \) is defined by \( \xi_1 = \cdots = \xi_n = 0 \). This shows \( X \) is an analytic space.

We can define a Kähler metric on \( X \) as follows. Let \( \omega \) be a Kähler metric on \( M \). Extend \( \omega \) to a Kähler metric on \( M \times B \). Choose \( \psi \in \mathcal{C}^\infty(U) \) with the property that \( \psi \) equals one in a neighborhood \( p \) and \( \text{support}(\psi) \subseteq U \) is compact. Then the composition of the map \( U \times B \to U \) with \( \psi \) is a smooth function on \( U \times B \) which, by abuse of notation, is again denoted \( \psi \). By \( \text{support}(\psi) \subseteq U \) to be a sufficiently large compact set, we may assume \( \psi = 1 \) on \( Z \). Let
\[
\omega_\delta = \omega + i \delta \frac{1}{2} \partial \bar{\partial} (1 - \psi) + \psi \log \left( \sum |f_j(z) - \xi_j|^2 \right) \tag{3.18}
\]
on \( W \setminus E = (M \times B) \setminus Z \) and let
\[
\omega_\delta = \omega + \delta \frac{i}{2} \partial \bar{\partial} \log |y|^2 = \omega + \delta \omega_{FS} \tag{3.19}
\]
on the open neighborhood \( \{\psi = 1\}^o \) of \( E \) (i.e., the interior of the closed set \( \{\psi = 1\} \)). Here \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{CP}^{n-1} \). The two definitions are consistent, and define \( \omega_\delta \), a smooth \((1,1)\) form on \( W \). Since \( \omega + \delta \omega_{FS} > 0 \) in a fixed (independent of \( \delta \)) neighborhood of \( E \), we find that \( \omega_\delta > 0 \) on all of \( W \) for \( \delta \) sufficiently small. Thus \( \omega_\delta \) is a Kähler metric on \( W \) and its restriction to \( X \) is a Kähler metric on \( X \). The proof of Lemma 3 is complete.

For general functions \( f_1(z), \cdots, f_n(z) \), the space \( X(M, p, f) \) is not smooth. In order to obtain a smooth manifold, we use Hironaka’s theorem on resolution of singularities. One version of this theorem is the following:
Theorem 4 Let \( W \) be a complex manifold and let \( X \subset W \) be a complex analytic space. Then there exists an iterated blow up \( W' \) of \( W \), with corresponding holomorphic map \( \pi: W' \to W \) with the following property: let \( E \) be the exceptional divisor, and set

\[ X' = \pi^{-1}(X) \setminus E. \] (3.20)

Then \( X' \) is a smooth manifold and the map \( \pi: X' \to X \) is surjective. Moreover,

\[ E' = E \cap X' = \pi^{-1}(X_{\text{sing}}) \] (3.21)

is a divisor with normal crossings, and \( \pi: X' \setminus E' \to X_{\text{reg}} \) is an isomorphism.

3.3 Metrics on blow-ups

The following is the key property of blow-ups that we need:

Lemma 4 Let \( W \) be a smooth complex manifold and \( Z \subset W \) a smooth submanifold. Let \( \pi: W' \to W \) be the blow-up of \( W \) with center \( Z \), and let \( E \subset W \) be the exceptional divisor. If \( \omega \) is any Kähler metric on \( W \), then there exists a hermitian metric \( h_E \) on \( \mathcal{O}(-E) \) so that, for any compact subset \( K \) of \( W' \), there exists \( \varepsilon_K > 0 \) with

\[ \pi^* \omega - \varepsilon i \frac{1}{2} \partial \bar{\partial} \log h_E > 0 \] (3.22)

for all \( 0 < \varepsilon < \varepsilon_K \).

Proof of Lemma 4: Let \( \{ U_\alpha \} \subset W' \) be a locally finite collection of coordinate neighborhoods which cover \( Z \), and let \( z^\alpha = (x^\alpha, y^\alpha) \) be coordinates on \( U_\alpha \) with the property:

\[ U_\alpha \cap Z = \{ y^\alpha = 0 \} \]

Choose \( \psi_\alpha \in C^\infty_c(U_\alpha) \) such that \( 0 \leq \psi_\alpha \leq 1 \) and \( \psi = \sum_{\alpha=1}^r \psi_\alpha = 1 \) on \( Z \).

Let \( f \) be a section of \( \mathcal{O}(-E) \) over an open set \( \Omega \). This means that \( f \) is a holomorphic function on \( \Omega \) which vanishes on \( E \cap \Omega \). Let

\[ |f|_{h_E}^2 = \frac{|f|^2}{(1 - \psi) + \sum_\alpha \psi_\alpha (|y^\alpha_1|^2 + \cdots + |y^\alpha_{n-d}|^2)} \] (3.23)

Observe that this makes sense: We may assume that \( E \cap \Omega \neq \emptyset \) and that \( \Omega \) is a small open set such that \( \psi \circ \pi = 1 \) on \( \Omega \). Choose coordinates \( (\zeta, \theta) \) as in section 13.1. Thus \( \zeta = (\zeta_0, ..., \zeta_d) \) and \( \theta = (\theta_1, ..., \theta_{n-d-1}) \). Since \( f \) vanishes on \( E \) we have \( f = g\zeta_0 \) for some holomorphic function \( g \) and

\[ |f|_{h_E}^2 = \frac{|f|^2}{\sum_\alpha \psi_\alpha (|y^\alpha_1|^2 + \cdots + |y^\alpha_{n-d}|^2)} = \frac{|g|^2}{\sum_\alpha \psi_\alpha (|y^\alpha_1/\zeta_0|^2 + \cdots + |y^\alpha_{n-d}/\zeta_0|^2)} \] (3.24)
This shows that $h_E$ is a well defined smooth metric on $O(-E)$.

Next we claim that $(\pi^*\omega - \frac{i}{2}\varepsilon\partial\bar{\partial}\log h_E)(p) > 0$ for all $p \in E$ and sufficiently small $\varepsilon > 0$. To see this, fix $p \in E$. If $F(\zeta, \theta)$ is smooth in a neighborhood of $\{p\}$ then

$$\partial_i \partial_j F = \begin{pmatrix} A & B \\ tB & D \end{pmatrix}$$

(3.25)

where $A$ has size $(d+1) \times (d+1)$, $B$ has size $(d+1) \times (n-d-1)$ and $D$ has size $(n-d-1) \times (n-d-1)$. We see that in these coordinates,

$$\omega(p) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

(3.26)

with $A > 0$. Now we write

$$-i\partial\bar{\partial}\log h_E(p) = \begin{pmatrix} X & Y \\ tY & D \end{pmatrix}$$

(3.27)

Since $\psi$ is independent of $\theta_j$ we have

$$D_{ij} = \frac{\partial^2}{\partial \theta_j \partial \theta_k} \log \sum_\alpha \psi_\alpha(p)(|y_1^\alpha/\zeta_0|^2 + \cdots + |y_{n-d}^\alpha/\zeta_0|^2)$$

(3.28)

Finally we observe that $\sum_{i,j=1}^{n-d-1} D_{ij} d\theta_i \wedge d\theta_j$ is the pullback of a Fubini-Study metric with respect to a holomorphic map whose derivative has maximal rank. Thus $D > 0$.

The claim now follows from the following linear algebra fact:

**Lemma 5** Let $A, X$ be $(d+1) \times (d+1)$ hermitian matrices and $D$ an $(n-d-1) \times (n-d-1)$ hermitian matrix. Let $Y$ be a $(d+1) \times (n-d-1)$ matrix. Assume $A > 0$ and $D > 0$. Then for $\lambda > 0$ sufficiently large we have

$$M(\lambda) = \lambda \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} X & Y \\ tY & D \end{pmatrix} > 0$$

(3.29)

**Proof.** We may assume $A$ to be diagonal with positive diagonal entries $a_0, \ldots, a_d$. Then $\det(M(\lambda))$ is a polynomial of degree $d$, with real coefficients in $\lambda$ whose leading coefficient is $a_0 \cdots a_d > 0$. Thus $\det(M(\lambda)) > 0$ for $\lambda > 0$ sufficiently large.

The same argument shows that the determinants of all the square submatrices of $M(\lambda)$ which are situated in the lower right corner of $M(\lambda)$, also have positive determinants for $\lambda$ sufficiently large. This proves Lemma 5.

Since $(\pi^*\omega - \frac{i}{2}\varepsilon\partial\bar{\partial}\log h_E) > 0$ on $Z$, we see that there is an open neighborhood $U$ of $Z \cap K$ on which $(\pi^*\omega - \frac{i}{2}\varepsilon\partial\bar{\partial}\log h_E) > 0$ for all sufficiently small $\varepsilon > 0$. Choosing $\varepsilon > 0$ sufficiently small, we have $(\pi^*\omega - \frac{i}{2}\varepsilon\partial\bar{\partial}\log h_E) > 0$ on all of $K$. This proves Lemma 4.

The resolution of singularities in Hironaka’s theorem will usually require an iteration of blow-ups. Thus we need the following generalization of Lemma 4:
Lemma 6 Let $W$ be a compact complex manifold and let $\pi : W' \to W$ be an iterated blow-up of $W$. Let $E \subset W'$ be the exceptional divisor. If $\omega$ is any Kähler metric on $W$, then there exists an effective divisor $E'$ on $W'$ supported on $E$, and a hermitian metric $h_{E'}$ on $O(-E')$ so that there exists $\varepsilon > 0$ with

$$
\pi^*\omega - \varepsilon \frac{i}{2} \partial \bar{\partial} \log h_{E'} > 0.
$$

(3.30)

Proof of Lemma 6: Let $\pi : W' \to W$ be the composition of two blow-ups, $\pi = \pi_2 \circ \pi_1$, with $\pi_2 : W' = W_2 \to W_1$, $\pi_1 = W_1 \to W$. Apply Lemma 4 to $\pi_1$ and the Kähler metric $\omega$ on $W$. If $E_1$ is the exceptional divisor of $\pi_1$, we obtain a metric $h_{E_1}$ in on the line bundle $O(-E_1)$ on $W_1$ with $(\pi_1)^*\omega - \varepsilon_1 \frac{i}{2} \partial \bar{\partial} \log h_{E_1} > 0$ on $W_1$. Apply next Lemma 4 to $\pi_2$ and the Kähler metric $(\pi_1)^*\omega - \varepsilon_1 \frac{i}{2} \partial \bar{\partial} \log h_{E_1}$ on $W_1$. We obtain then a metric $h_{E_2}$ on $O(-E_2)$, where $E_2$ is the exceptional divisor of $\pi_2$, with $\pi_2^*(\pi_1^*\omega - \varepsilon_1 \frac{i}{2} \partial \bar{\partial} \log h_{E_1}) - \varepsilon_2 \frac{i}{2} \partial \bar{\partial} \log h_{E_2} > 0$ on $W_2$. We can take $\varepsilon_1 = \frac{1}{n_1}$ and $\varepsilon_2 = \frac{1}{n_1 n_2}$ for $n_1$ and $n_2$ large enough integers. We can then write

$$
\pi_2^*(\pi_1^*\omega - \varepsilon_1 \frac{i}{2} \partial \bar{\partial} \log h_{E_1}) - \varepsilon_2 \frac{i}{2} \partial \bar{\partial} \log h_{E_2} = \pi^*\omega - \frac{1}{n_1 n_2} \frac{i}{2} \partial \bar{\partial} \log h_{E_2} (3.31)
$$

so the lemma holds in this case with the line bundle given by $O(-E_2) \otimes \pi_2^*O(-n_2E_1)$. Clearly the argument extends to any finite number of blow-ups, and the lemma is proved.

The preceding lemma can be extended to the case of complex analytic sets:

Lemma 7 Let $X$ be a complex analytic set, and $\omega$ a Kähler metric on $X$. Let $\pi : X' \to X$ be a resolution of singularities and $E \subset X'$ the exceptional divisor. Then there is a divisor $E'$ on $X'$, whose support is contained in $E$, a hermitian metric $h_{E'}$ on the line bundle $O(-E')$, and an $\varepsilon > 0$ such that $\pi^*\omega - \varepsilon \frac{i}{2} \partial \bar{\partial} \log h_{E'} > 0$.

Proof of Lemma 7: By definition of a Kähler metric on $X$, the metric $\omega$ extends to a Kähler metric on an ambient space $W$ of $X$. By definition of resolution of singularities, the map $\pi$ extends to a map $\pi : W' \to W$, with $X' \subset W'$, and $W'$ an iterated blow-up. The metric on the line bundle $O(-E')$ on $W'$ obtained from Lemma 6 applied to $W$ and the Kähler form $\omega$ restricts to a metric on the line bundle $O(-E')$ on $X'$ with the desired property.

We can now give the proof of Lemma 1: We apply Lemma 3, to obtain the analytic space $X = X(M, p, f)$ and the Kähler form $\omega_{\delta}$ with the properties stated there. The space $X$ is only an analytic set, but we can apply Theorem 4 to obtain a resolution of singularities $\pi : X' \to X$. We can then apply Lemma 7 to obtain a line bundle $O(-E')$ on $X'$ with $\pi^*(\omega_{\delta}) - \varepsilon \frac{i}{2} \partial \bar{\partial} \log h_{E'} > 0$ on $X'$. Set

$$
\Omega' = \pi^*\omega_{\delta}.
$$

(3.32)
Since the resolution of singularities is a biholomorphism of $X' \setminus E'$ to $X_{reg}$, and $X_{reg}$ contains (a biholomorphic image of) $M \setminus p$, the Kähler form $\Omega' = \pi^*(\omega_\delta)$ retains the same expression (3.13) on $M \setminus p$, and is hence given by the expression (3.2). The proof of Lemma 1 is complete.

It remains only to establish Lemma 2. In the special case when the background form $\Omega'$ is actually strictly positive, this has been proved by X.X. Chen [Ch] and Blocki [B4]. But for our purposes, it is essential to allow degeneracies in $\Omega'$, as such degeneracies arise due to blow-ups. The full Lemma 2, allowing for degeneracies, is actually the main result of [PS4], stated there as Theorem 2. The desired solution of the homogeneous complex Monge-Ampère equation (3.3) is obtained as a $C^\alpha$ limit on compact subsets of $X' \setminus E$ of solutions of elliptic equations where the right hand side tends to 0. The key estimate is the following pointwise $C^1$ estimate ([PS4], Theorem 1)

$$|\nabla \varphi(z)| \leq C_1 \exp(C_2 \varphi(z))$$

(3.33)

for the solutions of the Dirichlet problem for the equation

$$(\omega + \frac{i}{2} \partial \bar{\partial} \varphi)^n = F(z, \varphi)\omega^n$$

(3.34)

where $\omega$ is a Kähler form on $X'$. Here $C_1$ and $C_2$ are strictly positive constants which depend only on upper bounds for $\inf_{X'} \varphi$, $\sup_{X' \times [\inf_{X'} \varphi, \infty]} F$, $\sup_{X' \times [\inf_{X'} \varphi, \infty]} (|\nabla F_{\frac{1}{n}}| + |\partial \varphi F_{\frac{1}{n}}|)$, $\|\varphi\|_{C^1(\partial X')}$ and a lower bound for the holomorphic bisectional curvature of $\omega$. The point of the estimate (3.33) is that it does not require an upper bound for $\sup_{X'} \varphi$. It is used to obtain the existence of convergent subsequences in $C^\alpha(X' \setminus E')$ for any $0 < \alpha < 1$ of solutions $\varphi$ of the equation (3.34) which may tend to $+\infty$ along a divisor $E'$. The proof of (3.33) makes essential use of a differential inequality for solutions of complex Monge-Ampère equations due to Blocki [B4]. We refer to [PS4] for the complete details. The proof of the main theorem is complete.

3.4 The case of strongly pseudoconvex manifolds

We give now the proof of Theorem 2.

First we observe that the theorem can be reduced to the case of boundary value 0 and $\omega$ strictly positive. Indeed, for any boundary value $\varphi$, we can pick an extension $\hat{\varphi}$ of $\varphi$ to $M$ with the property that $i\partial \bar{\partial} \hat{\varphi} > 0$. This can be done by choosing any extension, and adding a large positive multiple of $i\partial \bar{\partial} \rho$. Next, set $\hat{\omega} = \omega + \frac{i}{2} \partial \bar{\partial} \hat{\varphi}$, and $G = G - \hat{\varphi}$. The equation (1.1) can be rewritten as

$$(\omega + \frac{i}{2} \partial \bar{\partial} G)^n = (\hat{\omega} + \frac{i}{2} \partial \bar{\partial} \hat{G})^n = 0 \text{ on } M \setminus \{p_1, \ldots, p_N\}.$$ 

(3.35)
So if we can solve this equation for $\hat{G} \in PSH(M, \omega)$ with boundary value 0, then the function $G = \hat{G} + \hat{\varphi} \in PSH(M, \hat{\omega})$ is a solution of the original problem.

Next, assume that the boundary value is 0 and the form $\omega$ is strictly positive. We note that the restriction to $\varepsilon_j$ small in the proof of Theorem 1 is just due to the requirement that the form $\omega_\delta$ of (3.13) be strictly positive. But in the present case, for any $\delta$, it suffices to replace the form $\omega_\delta$ by the form $\omega_\delta + A(\delta)i\partial \bar{\partial} \rho$ with $A$ large enough, in order to obtain a form which is strictly positive. The rest of the proof applies verbatim. The proof of Theorem 2 is complete.

4 Proof of Theorem 3

Let $X' = BL(X, p)$, the blow up of the point $p$, and let $\pi : X' \to X$ be the projection map. Choosing, as before, a cut-off function $\psi$ which is supported in a neighborhood of the point $p$. Then for $\varepsilon$ sufficiently small,

$$\omega_\varepsilon = \omega + \frac{i}{2} \varepsilon \partial \bar{\partial} (\psi \log |z|^2 + (1 - \psi))$$

extends to a smooth Kähler metric $\omega'$ on the smooth manifold $X'$. Consider the equation on $X'$

$$(\omega' + \frac{i}{2} \partial \bar{\partial} \varphi')^n = cf(\pi^* \omega)^n$$

for a function $\varphi' \in PSH(X', \omega')$ with $c$ a normalization constant so that both sides have the same total volume. This equation can be rewritten as

$$(\omega' + \frac{i}{2} \partial \bar{\partial} \varphi')^n = cF(\omega')^n$$

where $F \equiv f(\pi^* \omega')^n$ is a smooth non-negative function. A careful examination of Yau’s treatment [Y1] of equations of the form (4.3) shows that a priori upper bounds for $\|\varphi'\|_{C^0(X')}$ and for $\|\Delta \varphi'\|_{C^0(X')}$ do not require a strictly positive lower bound for $F$. Thus the equation (4.3) admits a generalized solution $\varphi \in PSH(X', \omega') \cap C^{1, \alpha}(X')$ for any $0 < \alpha < 1$. Restricting to $X' \setminus E$ we get

$$(\omega + \frac{i}{2} \partial \bar{\partial} (\varepsilon (\psi \log |z|^2 + 1 - \psi) + \varphi'))^n = cf \omega^n \text{ on } X' \setminus \{p\}$$

Thus, if we let $\varphi = \varepsilon \psi \log |z|^2 + (1 - \psi) + \varphi'$ we get

$$(\omega + \frac{i}{2} \partial \bar{\partial} \varphi)^n = \varepsilon \delta_p + cf \omega^n$$

which implies $c = 1 - \varepsilon$. The proof of Theorem 3 is complete.

Clearly the proof extends to the cases of local singularities $f_j(z)$, when the analytic set $X = X(M, p, f)$ is a smooth manifold. It is easy to formulate conditions on the $f_j(z)$ which would insure this property, but we leave this to the interested reader.
5 Geodesics in the space of Kähler potentials

Let \((X, \omega_0)\) be a compact Kähler manifold without boundary. A well-known conjecture of Yau [Y2] is that the existence of a Kähler form in the class \([\omega_0]\) with constant scalar curvature should be equivalent to the stability of \((X, [\omega_0])\) in the sense of geometric invariant theory. Suitable notions of stability have been proposed by Tian [T] and Donaldson [D98, D02] (see also [PS2] [PSSW] for some other notions of stability, and [PS5] for a survey). In particular, in [D98], Donaldson introduces a notion of stability based on the behavior of the \(K\)-energy functional of Mabuchi near infinity along geodesic rays in the space of Kähler potentials. Such rays have been constructed from test configurations (see [PS2][PS3][CT][SZ][RWN], and also [AT] in the analytic category, using the Cauchy-Kowalevska theorem). Here we illustrate Theorem 1 by constructing certain new rays, exploiting the fact that local singularities can be prescribed near infinity.

More precisely, the space \(\mathcal{K}\) of Kähler potentials is defined by

\[
\mathcal{K} = \{ \varphi \in C^\infty(X); \omega_\varphi \equiv \omega_0 + \frac{i}{2} \partial \bar{\partial} \varphi > 0 \}. \tag{5.1}
\]

It carries a natural Riemannian structure defined by the \(L^2\) norm on \(T_\varphi(\mathcal{K})\) with respect to the volume form \(\omega_\varphi^n\). A path \((-T, 0] \ni t \mapsto \varphi(\cdot, t)\) is a geodesic if and only if it satisfies the equation

\[
\ddot{\varphi} - g_{\varphi\bar{\varphi}}^{jk} \partial_j \dot{\varphi} \partial_k \dot{\varphi} = 0. \tag{5.2}
\]

where \(g_{\varphi\bar{\varphi}}^{jk}\) is the metric corresponding to the Kähler form \(\omega_\varphi\). A key observation due to Donaldson [D98] and Semmes [Se] is that this equation is equivalent to the homogeneous complex Monge-Ampère equation

\[
(\pi^* \omega_0 + \frac{i}{2} \partial \bar{\partial} \Phi)^{n+1} = 0 \tag{5.3}
\]
on the manifold \(M = X \times \{ e^{-T} < |w| < 1 \}\), for the function \(\Phi\) defined by

\[
\Phi(z, w) = \varphi(z, \log |w|), \tag{5.4}
\]

and where \(\partial\) is now with respect to both \(z\) and \(w\). The end points of the geodesic paths in \(\mathcal{K}\) correspond to Dirichlet boundary conditions for the equation (5.3) on \(M\). Generalized geodesics will correspond to generalized solutions of the equation (5.3) in the sense of pluripotential theory, which are invariant under the rotation \(w \rightarrow e^{i\theta} w\).

For the purpose of stability, we are particularly interested in geodesic rays, which correspond to \(T = \infty\), and the manifold \(M\) is given by \(M = X \times D^\times\), with \(D^\times = \{ 0 < |w| < 1 \}\) being the pointed disk. We compactify \(M\) into \(\hat{M} = X \times D\), by adjoining the central fiber \(X_0 = X \times \{ 0 \}\). Then \(\omega + \frac{i}{2} \partial \bar{\partial} |w|^2\) is a Kähler metric on \(M\). Let \(p_1, \ldots, p_N \in M\).
be any $N$ distinct points in the central fiber, i.e., $\pi(p_{\alpha}) = 0 \in \mathbb{C}$, where $\pi : M \to D$ is the projection on the second factor. For each $\alpha$, let $U_{\alpha}$ be a neighborhood of $p_{\alpha}$ in $X$, and let $f_{1\alpha}(z, w), \ldots, f_{n+1,\alpha}(z, w)$ be any $n + 1$ holomorphic functions on $U_{\alpha} \times D$, with the property that their only common zero is at $(p_{\alpha}, 0)$ and with $\sum_{j=1}^{n} |f_{j\alpha}(z, w)|^2$ invariant under the rotation $w \to e^{i\theta} w$. Theorem 1 gives then a function $G(z, w; p_{1}, \ldots, p_{N})$ with the prescribed singularities (2.1) and satisfying the equation $(\omega + \frac{i}{2} \partial \bar{\partial}(|w|^2 + G))^{n+1} = 0$ on $X \times D^\times$. We may choose the blow ups in the proof of Theorem 1 to be equivariant under the above rotation. The function $G(z, w; p_{1}, \ldots, p_{N})$ must also be invariant under rotation, so $|w|^2 + G$ defines a generalized geodesic ray in the space of Kähler metrics. Because of the singularities at $(p_{\alpha}, 0)$, the rays are not trivial (i.e., $\varphi$ is not constant along the ray). They are different from those previously obtained in the literature.
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