SOME UNBOUNDED FUNCTIONS OF INTERMITTENT MAPS FOR WHICH THE CENTRAL LIMIT THEOREM HOLDS

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Abstract. We compute some dependence coefficients for the stationary Markov chain whose transition kernel is the Perron-Frobenius operator of an expanding map \( T \) of \([0, 1]\) with a neutral fixed point. We use these coefficients to prove a central limit theorem for the partial sums of \( f \circ T^i \), when \( f \) belongs to a large class of unbounded functions from \([0, 1]\) to \(\mathbb{R}\). We also prove other limit theorems and moment inequalities.

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1. Introduction

For \( \gamma \) in \([0, 1]\), we consider the intermittent map \( T_\gamma \) from \([0, 1]\) to \([0, 1]\), studied for instance by Liverani, Saussol and Vaienti (1999), which is a modification of the Pomeau-Manneville map (1980):

\[
T_\gamma(x) = \begin{cases} 
  x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[ \\
  2x - 1 & \text{if } x \in [1/2, 1] 
\end{cases}
\]

We denote by \( \nu_\gamma \) the unique \( T_\gamma \)-probability measure on \([0, 1]\). We denote by \( K_\gamma \) the Perron-Frobenius operator of \( T_\gamma \) with respect to \( \nu_\gamma \): for any bounded measurable functions \( f, g \),

\[
\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(K_\gamma(f)g).
\]

Let \((X_i)_{i \geq 0}\) be a stationary Markov chain with invariant measure \( \nu_\gamma \) and transition Kernel \( K_\gamma \). It is well known (see for instance Lemma XI.3 in Hennion and Hervé (2001)) that on the probability space \(([0, 1], \nu_\gamma)\), the random variable \((T_\gamma, T_\gamma^2, \ldots, T_\gamma^n)\) is distributed as \((X_n, X_{n-1}, \ldots, X_1)\). Hence any information on the law of

\[
S_n(f) = \sum_{i=1}^{n} f \circ T_\gamma^i
\]

can be obtained by studying the law of \(\sum_{i=1}^{n} f(X_i)\).

In 1999, Young proved that such systems (among many others) may be described by a Young tower with polynomial decay of the return time. From this construction, she was able to control the covariances \( \nu_\gamma(f \circ T^n \cdot (g - \nu_\gamma(g)) \) for any bounded function \( f \) and any \( \alpha \)-Hölder function \( g \), and then to prove that \( n^{-1/2}(S_n(f) - \nu_\gamma(f)) \) converges in distribution to a normal law as soon as \( \gamma < 1/2 \) and \( f \) is any \( \alpha \)-Hölder function. For \( \gamma = 1/2 \), Gouëzel (2004) proved that the central limit theorem remains true with the same normalization \( \sqrt{n} \) if \( f(0) = \nu_\gamma(f) \), and with the normalization

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\[ \sqrt{n \ln(n)} \text{ if } f(0) \neq \nu_{\gamma}(f). \] When \( 1/2 < \gamma < 1 \), he proved that if \( f \) is \( \alpha \)-Hölder and \( f(0) \neq \nu_{\gamma}(f), \)

\[ n^{-\gamma}(S_n(f) - \nu_{\gamma}(f)) \text{ converges to a stable law.} \]

At this point, two questions (at least) arise: 1) what happens if \( f \) is no longer continuous? 2) what happens if \( f \) is no longer bounded? For instance, for the uniformly expanding map \( T_0(x) = 2x - \lfloor 2x \rfloor \), the central limit theorem holds with the normalization \( \sqrt{n} \) as soon as \( f \) is monotonic and square integrable on \([0,1]\), that is not necessarily continuous nor bounded.

For the slightly different map \( \theta_{\gamma}(x) = x(1 - x^{\gamma})^{-1/\gamma} - [x(1 - x^{\gamma})^{-1/\gamma}] \), with the same behavior around the indifferent fixed point, Raugi (2004) (following a work by Conze and Raugi (2003)) has given a precise criterion for the central limit theorem with the normalization \( \sqrt{n} \) in the case where \( 0 < \gamma < 1/2 \) (see his Corollary 1.7). In particular, his result applies to a large class of non-continuous functions, which gives a quite complete answer to our first question for the map \( \theta_{\gamma} \). The result also applies to the unbounded function \( f(x) = x^{-a} \) with \( 0 < a < 1/2 - \gamma \). However, the function \( f \) is allowed to blow up near 0 only (if \( f \) tends to infinity when \( x \) tends to \( x_0 \in ]0,1[ \)), then the variation coefficient \( v(fh_{\gamma}, k) \), where \( h_{\gamma} \) is the density of the \( \theta_{\gamma} \)-invariant probability, is always infinite).

We now go back to the map \( T_{\gamma} \). In a short discussion after the proof of his Theorem 1.3, Gouëzel (2004) considers the case where \( f(x) = x^{-a} \), with \( 0 < a < 1 - \gamma \). He shows that, if \( 0 < a < 1/2 - \gamma \) then the central limit theorem holds with the normalization \( \sqrt{n} \), if \( a = 1/2 - \gamma \) then the central limit theorem holds with the normalization \( \sqrt{n \ln(n)} \), and if \( 0 < a < 1 - \gamma \) and \( \gamma \geq 1/2 \) then there is convergence to a stable law. Again, as for Raugi’s result (2004) concerning the map \( \theta_{\gamma} \), the function \( f \) is allowed to blow up only near 0.

On another hand, we know that for stationary Harris recurrent Markov chains with invariant measure \( \mu \) and \( \beta \)-mixing coefficients of order \( n^{-b}, b > 1 \), the central limit theorem holds with the normalization \( \sqrt{n} \) as soon as the moment condition \( \mu(|f|^p) < \infty \) holds for \( p > 2b/(b - 1) \). For \( T_{\gamma} \), the covariances decay is of order \( n^{(\gamma - 1)/\gamma} \), so that one can expect the moment condition \( \nu_{\gamma}(|f|^p) < \infty \) for \( p > (2 - 2\gamma)/(1 - 2\gamma) \). For instance, if \( f(x) = x^{-a} \), since the density of \( \nu_{\gamma} \) is of order \( x^{-\gamma} \) near 0, the moment condition is satisfied if \( 0 < a < 1/2 - \gamma \), which is coherent with Gouëzel’s result (2004). However, since the chain \( (K_{\gamma}, \nu_{\gamma}) \) is not \( \beta \)-mixing, the condition \( \nu_{\gamma}(|f|^p) < \infty \) for \( p > (2 - 2\gamma)/(1 - 2\gamma) \) alone is not sufficient to imply the central limit theorem, and one still needs some regularity on \( f \).

Let us now define the class of functions of interest. For any probability measure \( \mu \) on \( \mathbb{R} \), any \( M > 0 \) and any \( p \in ]1, \infty[ \), let \( \text{Mon}(M, p, \mu) \) be the class of functions \( g \) which are monotonic on some open interval of \( \mathbb{R} \) and null elsewhere, and such that \( \mu(|g| > t) \leq Mpt^{-p} \) for \( p < \infty \) and \( \mu(|g| > M) = 0 \) for \( p = \infty \). Let \( \mathcal{C}(M, p, \mu) \) be the closure in \( L^1(\mu) \) of the set of functions which can be written as \( \sum_{i=1}^n a_i g_i \), where \( \sum_{i=1}^n |a_i| \leq 1 \) and \( g_i \) belongs to \( \text{Mon}(M, p, \mu) \). Note that a function belonging to \( \mathcal{C}(M, p, \mu) \) is allowed to blow up at an infinite number of points.

In Corollary 6.1 of the present paper, we prove that if \( f \) belongs to the class \( \mathcal{C}(M, p, \nu_{\gamma}) \) for \( p > (2 - 2\gamma)/(1 - 2\gamma) \), then \( n^{-1/2}(S_n(f - \nu_{\gamma}(f)) \) converges in distribution to a normal law. We also give some conditions on \( p \) to obtain rates of convergence in the central limit theorem (Corollary 5.1), as well as moment inequalities for \( S_n(f - \nu_{\gamma}(f)) \) (Corollary 6.1). Finally, a central limit theorem for the empirical distribution function of \( (T_{\gamma}^i)_{1 \leq i \leq n} \) is given in the last section (Corollary 7.1).

To prove these results, we compute the \( \beta \)-dependence coefficients (cf Dedecker and Prieur (2005, 2007)) of the Markov chain \( (K_{\gamma}, \nu_{\gamma}) \). The main tool is a precise estimate of the Perron-Frobenius operator of the map \( F \) associated to \( T_{\gamma} \) on the Young tower, due to Maume-Deschamps (2001). Next, we apply some general results for \( \beta \)-dependent Markov chains. For the sake of simplicity, we give all
the computations in the case of the maps $T_\gamma$, but our arguments remain valid for many other systems modelled by Young towers.

2. The main inequality

For any Markov kernel $K$ with invariant measure $\mu$, any non-negative integers $n_1, n_2, \ldots, n_k$, and any bounded measurable functions $f_1, f_2, \ldots, f_k$, define

$$K^{(n_1,n_2,\ldots,n_k)}(f_1, f_2, \ldots, f_k) = K^{n_1}(f_1)K^{n_2}(f_2)K^{n_3}(f_3) \cdots K^{n_k-1}(f_{k-1})K^{n_k}(f_k),$$

and

$$K^{(0)}(f_1, f_2, \ldots, f_k) = K^{(n_1,n_2,\ldots,n_k)}(f_1, f_2, \ldots, f_k) - \mu(K^{(n_1,n_2,\ldots,n_k)}(f_1, f_2, \ldots, f_k)).$$

For $\alpha \in ]0, 1]$ and $c > 0$, let $H_{\alpha,c}$ be the set of functions $f$ such that $|f(x) - f(y)| \leq c|x - y|^\alpha$.

**Theorem 2.1.** Let $\gamma \in ]0, 1]$, and let $f^{(0)} = f - \nu_\gamma(f)$. For any $\alpha \in ]0, 1]$, the following inequality holds:

$$\nu_\gamma\left( \sup_{f_1, \ldots, f_k \in H_{\alpha,1}} |K^{(0)}(f_1, f_2, \ldots, f_k)| \right) \leq C(\alpha, k)(\ln(n + 1))^2 / (n + 1)^{(1-\gamma)/\gamma}.$$  

In particular,

$$\nu_\gamma\left( \sup_{f \in H_{\alpha,1}} |K^n f - \nu_\gamma(f)| \right) \leq C(\alpha, 1)(\ln(n + 1))^2 / (n + 1)^{(1-\gamma)/\gamma}.$$  

**Proof of Theorem 2.1.** We refer to the paper by Young (1999) for the construction of the tower $\Delta$ associated to $T_\gamma$ (with floors $\Lambda_k$), and for the mappings $\pi$ from $\Delta$ to $[0, 1]$ and $F$ from $\Delta$ to $\Delta$ such that $T_\gamma \circ \pi = \pi \circ F$. On $\Delta$ there is a probability measure $\mu_0$ and an unique $F$-invariant probability measure $\nu$ with density $h_0$ with respect to $\mu_0$, and $\nu(\Lambda_k) = O(\ell^{-1/\gamma})$. The unique $T_\gamma$-invariant probability measure $\nu_\gamma$ is then given by $\nu_\gamma = \nu^{\pi}$. There exists a distance $\delta$ on $\Delta$ such that $\delta(x, y) \leq 1$ and $|\pi(x) - \pi(y)| \leq \kappa \delta(x, y)$. For $\alpha \in ]0, 1]$, let $\delta_\alpha = \delta^\alpha$, let $L_\alpha$ be the space of Lipschitz functions with respect to $\delta_\alpha$, and let $L_\alpha(f) = \sup_{x, y \in \Delta} |f(x) - f(y)| / \delta_\alpha(x, y)$. Let $L_{\alpha,c}$ be the set of functions such that $L_\alpha(f) \leq c$. For $\varphi$ in $H_{\alpha,c}$, the function $\varphi \circ \pi$ belongs to $L_{\alpha,c,\infty}$. Any function $f$ in $L_\alpha$ is bounded and the space $L_\alpha$ is a Banach space with respect to the norm $\|f\|_\alpha = L_\alpha(f) + \|f\|_\infty$. The density $h_0$ belongs to any $L_\alpha$ and $1/h_0$ is bounded. As in Maume-Deschamps (2001), we denote by $L_0$ the Perron-Frobenius operator of $F$ with respect to $\mu_0$, and by $P$ the Perron-Frobenius operator of $F$ with respect to $\nu$: for any bounded measurable functions $\varphi, \psi$,

$$m_0(\varphi \cdot \psi \circ F) = m_0(L_0(\varphi)\psi) \quad \text{and} \quad \nu(\varphi \cdot \psi \circ F) = \nu(P(\varphi)\psi).$$

We first state a useful lemma

**Lemma 2.1.** For any positive $n_1, n_2, \ldots, n_k$ and any bounded measurable functions $f_1, f_2, \ldots, f_k$ from $[0, 1]$ to $\mathbb{R}$, one has

$$K^{(n_1,n_2,\ldots,n_k)}(f_1, f_2, \ldots, f_k) \circ \pi = \mathbb{E}_\nu\left( P^{(n_1,n_2,\ldots,n_k)}(f_1 \circ \pi, f_2 \circ \pi, \ldots, f_k \circ \pi) | \pi \right).$$

We now complete the proof of Theorem 2.1 for $k = 2$, the general case being similar. Applying Lemma 2.1 it follows that

$$\sup_{f, g \in H_{\alpha,1}} |K^n(f^{(0)}K^n g^{(0)})(x) - \nu_\gamma(f^{(0)}K^n g^{(0)})|$$

$$\leq \mathbb{E}_\nu\left( \sup_{\phi, \psi \in L_{\alpha,\kappa}} |P^n(\phi^{(0)}P^n \psi^{(0)}) - \nu(\phi^{(0)}P^n \psi^{(0))}| | \pi = x \right).$$
Proof of Lemma 2.2. There exists $M_\alpha > 0$ such that, for any $\psi \in L_\alpha$
\[ |P^m\psi(x) - P^m\psi(y)| \leq M_\alpha \delta(x, y) \|\psi\|_\alpha \leq 2M_\alpha \delta(x, y) L_\alpha(\psi). \]

Hence, if $\psi \in L_{\alpha, k}$, then $P^m(\psi^{(0)})$ belongs to $L_{\alpha, 2M_\alpha k}$ and is centered, so that $\phi^{(0)} P^m \psi^{(0)}$ belongs to $L_{\alpha, 4M_\alpha k^2}$. It follows that
\[ \sup_{f,g \in H_{\alpha,1}} |K^n_\gamma(f^{(0)} K^n_\gamma g^{(0)})(x) - \nu(f^{(0)} K^n_\gamma g^{(0)})| \leq 4M_\alpha k^2 \|\phi\|_\alpha \left( \sup_{\varphi \in L_{\alpha,1}} |P^n(f^{(0)} P^n(g^{(0)})) - \bar{\nu}(\varphi)| \right) \pi = x. \]

Next, we apply the following Lemma, which is derived from Corollary 3.14 in Maume-Deschamps (2001).

Lemma 2.3. Let $v_\ell = (\ell + 1)^{(1-\gamma)/\gamma} (\ln(\ell + 1))^{-2}$. There exists $C_\alpha > 0$ such that
\[ E_\phi \left( \sup_{\varphi \in L_{\alpha,1}} |P^n(f^{(0)} P^n(g^{(0)})) - \bar{\nu}(\varphi)| \right) \pi = x \leq C_\alpha (\ln(n + 1))^2 (n + 1)^{(\gamma - 1)/\gamma} \sum_{\ell \geq 0} v_\ell |\bar{\nu}(\Lambda_{\ell})| \pi = x. \]

Hence
\[ \nu_\gamma \left( \sup_{f,g \in H_{\alpha,1}} |K^n_\gamma(f^{(0)} K^n_\gamma g^{(0)})(x) - \nu(f^{(0)} K^n_\gamma g^{(0)})| \right) \leq 4M_\alpha k^2 \sum_{\ell \geq 0} v_\ell |\bar{\nu}(\Lambda_{\ell})| \pi = x. \]

Since $\bar{\nu}(\Lambda_{\ell}) = O(\ell^{-1/\gamma})$, the result follows.

Proof of Lemma 2.1. We write the proof for $k = 2$ only, the general case being similar. Let $\varphi, f$ and $g$ be three bounded measurable functions. One has
\[ \nu_\gamma(\varphi K^n_\gamma(f K^n_\gamma g)) = \nu_\gamma(\varphi \circ T^{n+m}_\gamma \circ f \circ T^{m}_\gamma \circ g) = \bar{\nu}(\varphi \circ \pi \circ F^{n+m} \circ f \circ \pi \circ F^{m} \circ g \circ \pi) = \bar{\nu}(\varphi \circ \pi P^n(f \circ \pi P^n(g \circ \pi))) = \int \varphi(x) \bar{\nu}(P^n(f \circ \pi P^n(g \circ \pi)) \pi = x) \nu_\gamma(dx), \]
which proves Lemma 2.1 for $k = 2$.

Proof of Lemma 2.2. Applying Lemma 3.4 in Maume-Deschamps (2001) with $v_k = 1$, we see that there exists $D_\alpha > 0$ such that, for any $\psi$ in $L_\alpha$,
\[ |L_0^m\psi(x) - L_0^m\psi(y)| \leq D_\alpha \delta(x, y) \|\psi\|_\alpha. \]

Now $P^m(\psi) = L_0^m(\psi h_0)/h_0$. Since $1/h_0$ is bounded by $B(h_0)$, and since $h_0$ belongs to $L_\alpha$, it follows that
\[ |P^m\psi(x) - P^m\psi(y)| \leq D_\alpha B(h_0) \|h_0\|_\alpha \delta(x, y) \|\psi\|_\alpha. \]

Let $M_\alpha = D_\alpha B(h_0) \|h_0\|_\alpha$. Since $|P^m\psi(x) - P^m\psi(y)| = |P^m\psi^{(0)}(x) - P^m\psi^{(0)}(y)|$ and since $\|\psi^{(0)}\|_\infty \leq L_\alpha(\psi)$, it follows that
\[ |P^m\psi(x) - P^m\psi(y)| \leq M_\alpha \delta(x, y) \|\psi^{(0)}\|_\alpha \leq 2M_\alpha \delta(x, y) L_\alpha(\psi). \]
Proof of Lemma 2.3. Applying Corollary 3.14 in Maume-Deschamps (2001), there exists $B_\alpha > 0$ such that
\[ |c^\alpha_0 f - h_0 m_0(f)| \leq B_\alpha \|f\|_\alpha (\ln(n+1))^2 (n+1)^{(\gamma-1)/\gamma} \sum_{\ell \geq 0} v_\ell 1_{\Delta_\ell}. \]

It follows that, with the notations of the proof of Lemma 2.2,
\[ |P^n(f) - \bar{\nu}(f)| \leq B_\alpha B(h_0)\|h_0\|_\alpha \|f\|_\alpha (\ln(n+1))^2 (n+1)^{(\gamma-1)/\gamma} \sum_{\ell \geq 0} v_\ell 1_{\Delta_\ell}. \]

Since $|P^n(f) - \bar{\nu}(f)| = |P^n(f^{(0)}) - \bar{\nu}(f^{(0)})|$ and since $\|f^{(0)}\|_\infty \leq L_\alpha(f)$, it follows that
\[ |P^n(f) - \bar{\nu}(f)| \leq 2B_\alpha B(h_0)\|h_0\|_\alpha L_\alpha(f)(\ln(n+1))^2 (n+1)^{(\gamma-1)/\gamma} \sum_{\ell \geq 0} v_\ell 1_{\Delta_\ell}, \]
and the result follows.

3. The dependence coefficients

Let $X = (X_i)_{i \geq 0}$ be a stationary Markov chain with invariant measure $\mu$ and transition kernel $K$. Let $f_t(x) = 1_{x \leq t}$. As in Dedecker and Prieur (2005, 2007), define the coefficients $\alpha_k(n)$ of the stationary Markov chain $(X_i)_{i \geq 0}$ by
\[ \alpha_1(n) = \sup_{t \in \mathbb{R}} \mu(|K^n(f_t) - \mu(f_t)|), \quad \text{and for } k \geq 2,
\[ \alpha_k(n) = \alpha_1(n) \vee \sup_{2 \leq l \leq k} \sup_{n_2 \geq 1} \sup_{n_1 \geq 1} \sup_{t_1, \ldots, t_l \in \mathbb{R}} \mu(|K^{(0)}(n,n_2,\ldots,n_l)(f_{t_1}, f_{t_2}, \ldots, f_{t_l})|). \]

In the same way, define the coefficients $\beta_k(n)$ by
\[ \beta_1(n) = \mu\left(\sup_{t \in \mathbb{R}} |K^n(f_t) - \mu(f_t)|\right), \quad \text{and for } k \geq 2,
\[ \beta_k(n) = \beta_1(n) \vee \sup_{2 \leq l \leq k} \sup_{n_2 \geq 1} \sup_{n_1 \geq 1} \mu\left(\sup_{t_1, \ldots, t_l \in \mathbb{R}} |K^{(0)}(n,n_2,\ldots,n_l)(f_{t_1}, f_{t_2}, \ldots, f_{t_l})|\right). \]

Theorem 3.1. Let $0 < \gamma < 1$. Let $X = (X_i)_{i \geq 0}$ be a stationary Markov chain with invariant measure $\nu_\gamma$ and transition kernel $K_\gamma$. There exist two positive constants $C_1(\gamma)$ and $C_2(\delta, \gamma, k)$ such that, for any $\delta$ in $]0, (1 - \gamma)/\gamma[$ and any positive integer $k$,
\[ C_1(\gamma)(n+1)^{\frac{-1}{\gamma}} \leq \alpha_k(n) \leq \beta_k(n) \leq C_2(\delta, \gamma, k)(n+1)^{\frac{-1}{\gamma} + \delta}. \]

Proof of Theorem 3.1. Applying Proposition 2, Item 2, in Dedecker and Prieur (2005), we know that
\[ \nu_\gamma\left(\sup_{f \in H_{1,1}} |K^n_\gamma f - \nu_\gamma(f)|\right) \leq 2\alpha_1(n). \]

Hence, for any $\varphi$ such that $|\varphi| \leq 1$ and any $f$ in $H_{1,1}$,
\[ \nu_\gamma(\varphi \cdot (K^n_\gamma f - \nu_\gamma(f))) = \nu_\gamma(\varphi \circ T^n \cdot (f - \nu_\gamma(f))) \leq 2\alpha_1(n) \]

The lower bound for $\alpha_k(n)$ follows from the lower bound for $\nu_\gamma(\varphi \circ T^n \cdot (f - \nu_\gamma(f)))$ given by Sarig (2002), Corollary 1.
It remains to prove the upper bound. The point is to approximate the indicator \( f_t(x) = 1_{x \leq t} \) by some \( \alpha \)-Hölder function. Let
\[
   f_{t,\epsilon,\alpha}(x) = f_t(x) + \left( 1 - \left( \frac{x - t}{\epsilon} \right)^\alpha \right) 1_{t < x \leq t + \epsilon}.
\]
This function is \( \alpha \)-Hölder with Hölder constant \( \epsilon^{-\alpha} \). We now prove the upper bounds for \( k = 1 \) and \( k = 2 \) only, the general case being similar. For \( k = 1 \), one has
\[
   K^n_{\gamma}(f_{t-\epsilon,\epsilon,\alpha}) - \nu_\gamma(f_{t-\epsilon,\epsilon,\alpha}) - \nu_\gamma([t - \epsilon, t]) \leq K^n_{\gamma}(f_t) - \nu_\gamma(f_t) \leq K^n_{\gamma}(f_{t,\epsilon,\alpha}) - \nu_\gamma(f_{t,\epsilon,\alpha}) + \nu_\gamma([t, t + \epsilon]).
\]
Since the density \( g_{\nu_\gamma} \) of \( \nu_\gamma \) is such that \( g_{\nu_\gamma}(x) \leq V(\gamma)x^{-\gamma} \), we infer that for any real \( a, \nu_\gamma([a, a + \epsilon]) \leq V(\gamma)\epsilon^{1-\gamma}(1 - \gamma)^{-1} \). Consequently,
\[
   |K^n_{\gamma}(f_t) - \nu_\gamma(f_t)| \leq \epsilon^{-\alpha} \sup_{f \in H_{\alpha,1}} |K^n_{\gamma}(f) - \nu_\gamma(f)| + \frac{V(\gamma)}{1 - \gamma} \epsilon^{1-\gamma}.
\]
Applying Theorem 2.1 with \( \nu_\gamma \), we obtain that
\[
   \nu_\gamma \left( \sup_{t \in [0,1]} |K^n_{\gamma}(f_t) - \nu_\gamma(f_t)| \right) \leq C(\alpha, 1)\epsilon^{-\alpha}(\ln(n + 1))^2(n + 1)^{\frac{1}{\gamma - 1}} + \frac{V(\gamma)}{1 - \gamma} \epsilon^{1-\gamma}.
\]

The optimal \( \epsilon \) is equal to
\[
   \epsilon = \left( \frac{\alpha C(\alpha, 1)(\ln(n + 1))^2(n + 1)^{\frac{1}{\gamma - 1}}}{V(\gamma)} \right)^{\frac{1}{\gamma + 1 - \gamma}}.
\]
Consequently, for some positive constant \( D(\gamma, \alpha) \), one has
\[
   \nu_\gamma \left( \sup_{t \in [0,1]} |K^n_{\gamma}(f_t) - \nu_\gamma(f_t)| \right) \leq D(\gamma, \alpha) \left( (\ln(n + 1))^2(n + 1)^{\frac{1}{\gamma - 1}} \right) \epsilon^{\frac{1}{\alpha + 1 - \gamma}}.
\]
Choosing \( \alpha < \delta\gamma(1 - \gamma)/(1 - \gamma(1 + \delta)) \), the result follows for \( k = 1 \).

We now prove the result for \( k = 2 \). Clearly, the four following inequalities hold:
\[
   \begin{align*}
   K^n_{\gamma}(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) &\leq K^n_{\gamma}(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) + \nu_\gamma([t, t + \epsilon]) + \nu_\gamma([s, s + \epsilon]), \\
   K^n_{\gamma}(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) &\geq K^n_{\gamma}(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) - \nu_\gamma([t - \epsilon, t]) - \nu_\gamma([s - \epsilon, s]), \\
   \nu_\gamma(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) &\geq \nu_\gamma(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) - 2\nu_\gamma([t, t + \epsilon]) - \nu_\gamma([s, s + \epsilon]), \\
   \nu_\gamma(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) &\leq \nu_\gamma(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) + 2\nu_\gamma([t - \epsilon, t]) + \nu_\gamma([s - \epsilon, s]).
   \end{align*}
\]
Consequently,
\[
   |K^n_{\gamma}(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) - \nu_\gamma(f_t^{(0)}K^n_{\gamma}f_s^{(0)})| \leq \epsilon^{-\alpha} \sup_{f, g \in H_{\alpha,1}} |K^n_{\gamma}(f^{(0)}K^n_{\gamma}g^{(0)}) - \nu_\gamma(f^{(0)}K^n_{\gamma}g^{(0)})| + \frac{5V(\gamma)}{1 - \gamma} \epsilon^{1-\gamma}.
\]
Applying Theorem 2.1, we obtain that
\[
   \nu_\gamma \left( \sup_{t \in [0,1]} |K^n_{\gamma}(f_t^{(0)}K^n_{\gamma}f_s^{(0)}) - \nu_\gamma(f_t^{(0)}K^n_{\gamma}f_s^{(0)})| \right) \leq C(\alpha, 2)\epsilon^{-\alpha}(\ln(n + 1))^2(n + 1)^{\frac{1}{\gamma - 1}} + \frac{5V(\gamma)}{1 - \gamma} \epsilon^{1-\gamma},
\]
and the proof can be completed as for \( k = 1 \).
4. Central limit theorems

In this section we give a central limit theorem for \( S_n(f - \nu_\gamma(f)) \) when \( f \) belongs to the class \( \mathcal{C}(M, p, \mu) \) defined in the introduction. Note that any function \( f \) with bounded variation (BV) such that \( |f| \leq M_1 \) and \( ||df|| \leq M_2 \) belongs to the class \( \mathcal{C}(M_1 + 2M_2, \infty, \mu) \). Hence, any BV function \( f \) belongs to \( \mathcal{C}(M, \infty, \mu) \) for some \( M \) large enough. If \( g \) is monotonic on some open interval of \( \mathbb{R} \) and null elsewhere, and if \( \mu(|g|^{p}) \leq M^{p} \), then \( g \) belongs to \( \text{Mon}(M, p, \mu) \). Conversely, any function in \( \mathcal{C}(M, p, \mu) \) belongs to \( \mathbb{L}^{q}(\mu) \) for \( 1 \leq q < p \).

**Theorem 4.1.** Let \( X = (X_i)_{i \geq 0} \) be a stationary and ergodic (in the ergodic theoretic sense) Markov chain with invariant measure \( \mu \) and transition kernel \( K \). Assume that \( f \) belongs to \( \mathcal{C}(M, p, \mu) \) for some \( M > 0 \) and some \( p \in ]2, \infty[ \), and that

\[
\sum_{k>0} (\alpha_1(k))^{1/p} < \infty.
\]

The following results hold:

1. The series

\[
\sigma^2(\mu, K, f) = \mu((f - \mu(f))^2) + \sum_{k>0} \mu((f - \mu(f))K^k(f))
\]

converges to some non negative constant, and \( n^{-1}\text{Var}(\sum_{i=1}^{n} f(X_i)) \) converges to \( \sigma^2(\mu, K, f) \).

2. Let \( (D([0, 1], d) \) be the space of cadlag functions from \([0, 1] \) to \( \mathbb{R} \) equipped with the Skorohod metric \( d \). The process \( \{n^{-1/2} \sum_{i=1}^{[nt]} (f(X_i) - \mu(f)), t \in [0, 1]\} \) converges in distribution in \( D([0, 1], d) \) to \( \sigma(\mu, K, f)W \), where \( W \) is a standard Wiener process.

3. One has the representation

\[
f(X_1) - \mu(f) = m(X_1, X_0) + g(X_1) - g(X_0)
\]

with \( \mu(|f|^{p/(p-1)}) < \infty \), \( \mathbb{E}(m(X_1, X_0)|X_0) = 0 \) and \( \mathbb{E}(m^2(X_1, X_0)) = \sigma^2(\mu, K, f) \).

**Corollary 4.1.** Let \( \gamma \in ]0, 1/2[ \). If \( f \) belongs to the class \( \mathcal{C}(M, p, \nu) \) for some \( M > 0 \) and some \( p > (2 - 2\gamma)/(1 - 2\gamma) \), then \( n^{-1/2} S_n(f - \nu_\gamma(f)) \) converges in distribution to \( \mathcal{N}(0, \sigma^2(\nu_\gamma, K, f)) \).

**Remark 4.1.** We infer from Corollary 4.1 that the central limit theorem holds for any BV function provided \( \gamma < 1/2 \). Under the same condition on \( \gamma \), Young (1999) has proved that the central limit theorem holds for any \( \alpha \)-Hölder function. For the map \( \theta_\gamma(x) = x(1 - x^\gamma)^{-1/\gamma} - [x(1 - x^\gamma)^{-1/\gamma}] \) and \( \gamma < 1/2 \), the central limit theorem for BV functions is a consequence of Corollary 1.7(i) in Raugi (2004).

Two simple examples.

1. Assume that \( f \) is positive and non increasing on \([0, 1]\), with \( f(x) \leq Cx^{-a} \) for some \( a \geq 0 \). Since the density \( g_{\nu_\gamma} \) of \( \nu_\gamma \) is such that \( g_{\nu_\gamma}(x) \leq V(\gamma)x^{-\gamma} \), we infer that

\[
\nu_\gamma(f > t) \leq \frac{C_{a}\gamma V(\gamma)}{1 - \gamma}t^{-1/a}.
\]

Hence the CLT holds as soon as \( a < \frac{1}{2} - \gamma \).
(2) Assume now that $f$ is positive and non decreasing on $]0, 1]$ with $f(x) \leq C(1 - x)^{-a}$ for some $a \geq 0$. Here
\[\nu_{\gamma}(f > t) \leq \frac{V(\gamma)}{1 - \gamma} \left(1 - \left(\frac{C}{t}\right)^{1/\gamma}\right)^{1 - \gamma}.\]
Hence the CLT holds as soon as $a < \frac{1}{2} - \frac{\gamma}{2(1 - \gamma)}$.

**Proof of Theorem 4.1** Let $f \in C(M, p, \mu)$. From Dedecker and Rio (2000), Items (1) and (2) of Theorem 4.1 hold as soon as
\[\sum_{n>0} \|f(X_0) - \mu(f)\|_1 < \infty.\]
Assume first that $f = \sum_{i=1}^{k} a_i g_i$, where $\sum_{i=1}^{k} |a_i| \leq 1$, and $g_i$ belongs to $\text{Mon}(M, p, \mu)$. Clearly, the series on left side is bounded by
\[\sum_{i=1}^{k} \sum_{j=1}^{k} |a_i a_j| \sum_{n>0} \|g_i(X_0) - \mu(g_i)(\mathbb{E}(g_j(X_n)|X_0) - \mu(g_j))\|_1.\]
Here, we use the following lemma

**Lemma 4.1.** Let $g_i$ and $g_j$ be two functions in $\text{Mon}(M, p, \mu)$ for some $p \in [2, \infty]$. For any $1 \leq q \leq p$ one has
\[\|\mathbb{E}(g_j(X_n)|X_0) - \mu(g_j)\|_q \leq 2M^{\frac{p}{p - q}} (2\alpha_1(n))^{\frac{p-q}{p}}.\]
For any $1 \leq q < p/2$, one has
\[\|g_i(X_0) - \mu(g_i)(\mathbb{E}(g_j(X_n)|X_0) - \mu(g_j))\|_q \leq 4M^2 \left(\frac{p}{p - 2q}\right)^{\frac{1}{q}} (2\alpha_1(n))^{\frac{p - 2q}{pq}}.\]

> From Lemma 4.1 with $q = 1$, we conclude that
\[\sum_{n>0} \|f(X_0) - \mu(f)(\mathbb{E}(f(X_n)|X_0) - \mu(f))\|_1 \leq \frac{4pM^2}{p - 2} \sum_{n>0} (2\alpha_1(n))^{\frac{p - 2}{p}}.\]
Since the bound (4.1) is true for any function $f = \sum_{i=1}^{k} a_i g_i$, it is true also for any $f$ in $C(M, p, \mu)$, and Items (1) and (2) follow.

The last assertion is rather standard. From the first inequality of Lemma 4.1 with $q = p/(p - 1)$, we infer that if $\sum_{n>0} (\alpha_1(n))^{(p-2)/p} < \infty$, then $\sum_{n>0} \|f(X_n)|X_0) - \mu(f\|_{p/(p-1)} < \infty$ for any $f$ in $C(M, p, \mu)$. It follows that $g(x) = \sum_{k=1}^{\infty} \mathbb{E}(f(X_k) - \mu(f)|X_0 = x)$ belongs to $\mathbb{L}^{p/(p-1)}(\mu)$ and that $m(X_1, X_0) = \sum_{k\geq 1} (\mathbb{E}(f(X_k)|X_0) - \mathbb{E}(f(X_k)|X_1))$ belongs to $\mathbb{L}^{p/(p-1)}(\mu)$. Clearly
\[f(X_1) - \mu(f) = m(X_1, X_0) + g(X_0) - g(X_1),\]
with $\mathbb{E}(m(X_1, X_0)|X_0) = 0$. Moreover, it follows from the preceding result that
\[\lim_{n \to \infty} \frac{1}{\sqrt{n}} \left\| \sum_{k=1}^{n} m(X_k, X_{k-1}) \right\|_1 = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left\| \sum_{k=1}^{n} (f(X_k) - \mu(f)) \right\|_1 \leq \sigma(\mu, K, f).\]
By Theorem 1 in Esseen and Janson (1985), it follows that $\mathbb{E}(m^2(X_1, X_0)) = \sigma^2(\mu, K, f)$. 
Proof of Lemma 4.1. We only prove the second inequality (the proof of the first one is easier). Let 
\( r = q/(q-1) \) and let \( B_r (\sigma(X_0)) \) be the set of \( \sigma(X_0) \)-measurable random variables such that \( \| Y \|_r \leq 1 \). By duality,
\[
\|(g_i(X_0) - \mu(g_i))(\mathbb{E}(g_j(X_n)|X_0) - \mu(g_j))\|_q = \sup_{Y \in B_r(\sigma(X_0))} \mathbb{E}(Y (g_i(X_0) - \mu(g_i))(g_j(X_n) - \mu(g_j))) = \sup_{Y \in B_r(\sigma(X_0))} \text{Cov}(Y (g_i(X_0) - \mu(g_i), g_j(X_n))).
\]
Define the coefficients \( \alpha_{k,g}(n) \) of the sequence \( (g(X_i))_{i \geq 0} \) as in Section 3 with \( g \circ f_t \) instead of \( f_t \). If \( g \) is monotonic on some open interval of \( \mathbb{R} \) and null elsewhere, the set \( \{ x : g(x) \leq t \} \) is either some interval or the complement of some interval, so that \( \alpha_{k,g}(n) \leq 2^k \alpha_k(n) \). Let \( Q_Y \) be the generalized inverse of the tail function \( t \rightarrow \mathbb{P}(|Y| > t) \). From Theorem 1.1 and Lemma 2.1 in Rio (2000), one has that
\[
\text{Cov}(Y g_i(X_0), g_j(X_n)) \leq 2 \int_{0}^{\alpha_{1,g_i}(n)} Q_Y(u) Q_{g_i(X_0)}(u) Q_{g_j(X_0)}(u) du.
\]
In the same way, applying first Theorem 1.1 in Rio (2000) and next Fréchet’s inequality (1957) (see also Inequality (1.11b) in Rio (2000)),
\[
\text{Cov}(Y \mu(g_i), g_j(X_n)) \leq 2 \mu(|g_i|) \int_{0}^{2\alpha_{1,g_i}(n)} Q_Y(u) Q_{g_i(X_0)}(u) du \leq 2 \int_{0}^{2\alpha_{1,g_i}(n)} Q_Y(u) Q_{g_i(X_0)}(u) Q_{g_j(X_0)}(u) du.
\]
Since \( \int_{0}^{1} Q_Y(u) du \leq 1 \), it follows that
\[
\|(g_i(X_0) - \mu(g_i))(\mathbb{E}(g_j(X_n)|X_0) - \mu(g_j))\|_q \leq 4 \left( \int_{0}^{2\alpha_{1,g_i}(n)} Q_{g_i(X_0)}(u) Q_{g_j(X_0)}(u) du \right)^{1/q}.
\]
Since \( g_i \) and \( g_j \) belong to \( \text{Mon}(M,p,\mu) \) for some \( p > 2q \), we have that \( Q_{g_i(X_0)}(u) \) and \( Q_{g_j(X_0)}(u) \) are smaller than \( Mu^{-1/p} \), and the result follows.

Proof of Corollary 4.1. We have seen that \( (T_{\gamma_1}^{1}, \ldots, T_{\gamma_n}^{n}) \) is distributed as \( (X_n, \ldots, X_1) \) where \( (X_i)_{i \geq 0} \) is the stationary Markov chain with invariant measure \( \nu_\gamma \) and transition kernel \( K_\gamma \). Consequently, on the probability space \( ([0,1], \nu_\gamma) \), the sum \( S_n(f - \nu_\gamma(f)) \) is distributed as \( \sum_{i=1}^{n} (f(X_i) - \nu_\gamma(f)) \), so that \( n^{-1/2} S_n(f - \nu_\gamma(f)) \) satisfies the central limit theorem if and only if \( n^{-1/2} \sum_{i=1}^{n} (f(X_i) - \nu_\gamma(f)) \) does. Moreover, we infer from Theorem 3.1 that
\[
\alpha_1(n) = O(n^{-\frac{\epsilon}{\gamma}})
\]
for any \( \epsilon > 0 \). Consequently, if \( p > (2 - 2\gamma)/(1 - 2\gamma) \), one has that \( \sum_{k>0}^{\alpha_1(n)} \frac{p-2}{p} < \infty \) so that Theorem 4.1 applies: the central limit theorem holds provided that \( f \) belongs to \( \mathcal{C}(M,p,\nu_\gamma) \).
5. Rates of convergence in the CLT

Let \( c \) be some concave function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \), with \( c(0) = 0 \). Denote by \( \text{Lip}_c \) the set of functions \( g \) such that

\[
|g(x) - g(y)| \leq c(|x - y|).
\]

When \( c(x) = x^\alpha \) for \( \alpha \in [0, 1] \), we have \( \text{Lip}_c = H_{\alpha,1} \). For two probability measures \( P, Q \) with finite first moment, let

\[
d_c(P, Q) = \sup_{g \in \text{Lip}_c} |P(f) - Q(f)|.
\]

When \( c = \text{Id} \), we write \( d_c = d_1 \). Note that \( d_1(P, Q) \) is the so-called Kantorovič distance between \( P \) and \( Q \).

**Theorem 5.1.** Let \( X = (X_i)_{i \geq 0} \) be a stationary Markov chain with invariant measure \( \mu \) and transition kernel \( K \). Let \( \sigma^2(f) = \sigma^2(\mu, K, f) \) be the non-negative number defined in Theorem 4.1, and let \( G_{\sigma^2(f)} \) be the Gaussian distribution with mean 0 and variance \( \sigma^2(f) \). Let \( P_n(f) \) be the distribution of the normalized sum \( n^{-1/2} \sum_{i=1}^{n} (f(X_i) - \mu(f)) \).

1. Assume that \( f \) belongs to \( \mathcal{C}(M, p, \mu) \) for some \( M > 0 \) and some \( p \in ]2, \infty[ \), and that

\[
\sum_{k>0} (\alpha_1(k))^{p-2} < \infty.
\]

If \( \sigma^2(f) = 0 \), then \( d_c(P_n(f), \delta_{\{0\}}) = O(c(n^{-1/2})) \).

2. If \( f \) belongs to \( \mathcal{C}(M, p, \mu) \) for some \( M > 0 \) and some \( p \in ]3, \infty[ \), and if

\[
\sum_{k>0} k^{p-3} < \infty,
\]

then \( d_c(P_n(f), G_{\sigma^2(f)}) = O(c(n^{-1/2})) \).

3. If \( f \) belongs to \( \mathcal{C}(M, p, \mu) \) for some \( M > 0 \) and some \( p \in ]3, \infty[ \), and if

\[
\alpha_2(k) = O(k^{-(1+\delta)p/(p-3)}) \text{ for some } \delta \in ]0, 1[,
\]

then \( d_c(P_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2})) \).

**Corollary 5.1.** Let \( \delta \in [0, 1] \) and \( \gamma < 1/(2+\delta) \), and let \( \mu_n(f) \) be the distribution of \( n^{-1/2} S_n(f - \nu_{\gamma}(f)) \). If \( f \) belongs to the class \( \mathcal{C}(M, p, \nu_{\gamma}) \) for some \( M > 0 \) and some \( p > (3 - 3\gamma)/(1 - (2+\delta)\gamma) \), then \( d_c(\mu_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2})) \), where \( \sigma^2(f) = \sigma^2(\nu_{\gamma}, K, f) \).

**Remark 5.1.** We infer from Corollary 5.1 that if \( f \) is BV, then \( d_1(\mu_n(f), G_{\sigma^2(f)}) = O(n^{-1/2}) \) if \( \gamma < 1/3 \), and \( d_1(\mu_n(f), G_{\sigma^2(f)}) = O(n^{-\delta/2}) \) if \( \gamma < 1/(2+\delta) \). Denote by \( d_{BV}(P, Q) \) the uniform distance between the distribution functions of \( P \) and \( Q \). If \( f \) is \( \alpha \)-Hölder, Guézel (2005, Theorem 1.5) has proved that \( d_{BV}(\mu_n(f), G_{\sigma^2(f)}) = O(n^{-1/2}) \) if \( \gamma < 1/3 \), and \( d_{BV}(\mu_n(f), G_{\sigma^2(f)}) = O(n^{-\delta/2}) \) if \( \gamma = 1/(2+\delta) \). In fact, from a general result of Bolthausen (1982) for Harris recurrent Markov chains, we conjecture that the results of Corollary 5.1 are true with \( d_{BV} \) instead of \( d_1 \).

**Two simple examples (continued).**

1. Assume that \( f \) is positive and non increasing on \( [0, 1] \), with \( f(x) \leq Cx^{-a} \) for some \( a \geq 0 \). Let \( \delta \in [0, 1] \) and \( \gamma < 1/(2+\delta) \). If \( a < \frac{1}{3} - \frac{(2+\delta)\gamma}{3} \), then \( d_c(\mu_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2})) \).
(2) Assume that \( f \) is positive and non-increasing on \([0,1]\), with \( f(x) \leq C(1-x)^{-a} \) for some \( a \geq 0 \). Let \( \delta \in [0,1] \) and \( \gamma < 1/(2+\delta) \). If \( a < \frac{1}{\delta} - \frac{1}{3(1-\gamma)} \), then \( d_c(\mu_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2})) \).

**Proof of Theorem 5.1.** From the Kantorovič-Rubinstein theorem (1957), there exists a probability measure \( \pi \) with margins \( P \) and \( Q \), such that \( d_1(P, Q) = \int |x-y| \pi(dx, dy) \). Since \( c \) is concave, we then have

\[
d_c(P, Q) = \sup_{f \in H_c} \left| \int (f(x) - f(y)) \pi(dx, dy) \right| \leq \int c(|x-y|) \pi(dx, dy) \leq c(d_1(P, Q)).
\]

Hence, it is enough to prove the theorem for \( d_1 \) only.

If \( \sum_{k>0}(\alpha_1(k))^{(p-2)/p} < \infty \), \( f \) belongs to \( C(M,\mu,p) \) for some \( M > 0 \) and some \( p \in ]2,\infty[ \), and \( \sigma^2(f) = 0 \), it follows from Theorem 4.1 that \( f(X_1) = g(X_0) - g(X_1) \) with \( \mu(|g|) < \infty \). Hence

\[
d_1(P_n(f), \delta(0)) \leq \frac{2\mu(|g|)}{\sqrt{n}},
\]

and Item (1) is proved.

> From now, we assume that \( \sigma^2(f) > 0 \) (otherwise, the result follows from Item (1)). If \( f = g_1 - g_2 \), where \( g_1, g_2 \) belong to \( \text{Mon}(M,\mu,p) \) for some \( M > 0 \) and some \( p \in ]3,\infty[ \), Item (2) of Theorem 5.1 follows from Theorem 3.1(b) in Dedecker and Rio (2007). In fact the proof remains unchanged if \( f \) belongs to \( C(M,\mu,p) \) for some \( M > 0 \) and some \( p \in ]3,\infty[ \).

It remains to prove Item (3). Let \( Y_k = f(X_k) - \mu(f) \), \( \sigma^2(f) = \sigma^2 \), and \( s_m = \sum_{i=1}^m Y_i \). Define

\[
W_m = A_m + B_m, \quad \text{with} \quad A_m = \mathbb{E}(s_m^2|X_0) - m\sigma^2 \quad \text{and} \quad B_m = 2\sum_{k=1}^m \mathbb{E}(Y_k \sum_{i>m} Y_i|X_0).
\]

> From Theorem 2.2 in Dedecker and Rio (2007), we have that, if \( \sum_{k>0} \|Y_0|E(Y_k|X_0)\|_1 < \infty \),

\[
(5.2) \quad \sqrt{n}d_1(P_n(f), G_{\sigma^2}) \leq C\ln(n) + \sum_{m=1}^{\lceil \sqrt{n} \rceil} \frac{\|Y_0|E(Y_i|X_0)\|_1}{m\sigma^2} + D_{1,n} + D_{2,n},
\]

where

\[
D_{1,n} = \sum_{m=1}^n \frac{1}{\sigma \sqrt{m}} \sum_{i \geq m} \|Y_0|E(Y_i|X_0)\|_1 \quad \text{and} \quad D_{2,n} = \sum_{m=1}^n \frac{1}{2\sigma^2 m} \sum_{k=1}^m \|\sigma^2 + Y_0^2|E(Y_k|X_0)\|_1.
\]

> From Lemma 4.1 with \( q = 1 \), the bound (4.1) holds for any \( f \) in \( C(M,\mu,p) \) for \( p > 2 \). Consequently, if \( \alpha_2(k) = O(k^{-(1+\delta)p/(p-3)}) \) for some \( \delta \in [0,1] \) and \( p > 3 \), then \( \sum_{k>0} \|Y_0|E(Y_k|X_0)\|_1 < \infty \), so that the bound (5.2) holds. Moreover \( n^{-1/2}D_{1,n} = O(n^{-1/2}\ln(n) \vee n^{-\delta}) \). Arguing as in Lemma 4.1 one can prove that

\[
\|Y_0^2|E(Y_k|X_0)\|_1 \leq C(M,p)(\alpha_1(k))^{p-3\over p},
\]

so that \( n^{-1/2}D_{2,n} = O(n^{-1/2}\ln(n)) \).

> Arguing as in Lemma 4.1 one can prove that, for \( 0 < k < i \),

\[
(5.3) \quad \|Y_0 + 2\sigma Y_k|E(Y_i|X_0)\|_1 \leq \|Y_0 + 2\sigma Y_k E(Y_i|X_0)\|_1 \leq C(M,p,\sigma)(\alpha_1(i-k))^{p-3\over p}.
\]
Consequently,

\[
\frac{1}{\sqrt{n}} \sum_{m=1}^{\lceil \sqrt{2n} \rceil} \left\|(Y_0 + 2\sigma)B_m \right\|_1 = O\left( \frac{1}{\sqrt{n}} \sum_{m=1}^{\lceil \sqrt{2n} \rceil} \frac{1}{m\sigma^2} \sum_{k=1}^{m} \sum_{j>i}^{m} \frac{1}{(i-k)^{1+\delta}} \right) = O(n^{-\delta/2}).
\]

Now,

\[
\frac{\left\|(Y_0 + 2\sigma)A_m \right\|_1}{m} \leq \frac{2}{m} \sum_{i=1}^{m} \sum_{j=i}^{m} \left\|(Y_0 + 2\sigma)(E(Y_i Y_j | X_0) - E(Y_i Y_j)) \right\|_1 + \left(\|Y_0\|_1 + 2\sigma\right) \frac{1}{m} E(s_m^2) - \sigma^2 \right|.
\]

For the second term on right hand, we have

\[
\left| \frac{1}{m} E(s_m^2) - \sigma^2 \right| \leq 2 \sum_{k=1}^{\infty} k \land m \frac{E(Y_0 Y_k)}{m} = O\left( \sum_{k>0} k \land m \frac{1}{m} (\alpha_1(k)) \frac{p-2}{p} \right) = O(m^{-\delta}),
\]

so that

\[
\frac{1}{\sqrt{n}} \sum_{m=1}^{\lceil \sqrt{2n} \rceil} \left| \frac{1}{m} E(s_m^2) - \sigma^2 \right| = O(n^{-\delta/2}).
\]

To complete the proof of the theorem, it remains to prove that

\[
\frac{1}{\sqrt{n}} \sum_{m=1}^{\lceil \sqrt{2n} \rceil} \left( \frac{2}{m} \sum_{i=1}^{m} \sum_{j=i}^{m} \left\|(Y_0 + 2\sigma)(E(Y_i Y_j | X_0) - E(Y_i Y_j)) \right\|_1 \right) = O(n^{-\delta/2}).
\]

Applying first (5.3), we have for \(j > i\),

\[
\left\|(Y_0 + 2\sigma)(E(Y_i Y_j | X_0) - E(Y_i Y_j)) \right\|_1 \leq 2(C(M, p, \sigma)(\alpha_1(j - i)) \frac{p-3}{p}).
\]

We need a second bound for this quantity. Assume first that \(f = \sum_{i=1}^{k} a_i g_i\), where \(\sum_{i=1}^{k} |a_i| \leq 1\) and \(g_i\) belongs to Mon\((M, p, \sigma)\). Let \(g_i^{(0)} = g_i - \mu(g_i)\). We have that

\[
\left\|Y_0(E(Y_i Y_j | X_0) - E(Y_i Y_j)) \right\|_1 \leq \sum_{l=1}^{k} \sum_{q=1}^{k} \sum_{r=1}^{k} |a_l a_q a_r| \left\|g_i^{(0)}(X_0)(E(g_q^{(0)}(X_i)g_r^{(0)}(X_j)|X_0) - E(g_q^{(0)}(X_i)g_r^{(0)}(X_j))) \right\|_1.
\]

For three real-valued random variables \(A, B, C\), define the numbers \(\alpha(A, B)\) and \(\tilde{\alpha}(A, B, C)\) by

\[
\alpha(A, B) = \sup_{s, t \in \mathbb{R}} |\text{Cov}(1_{A \leq s}, 1_{B \leq t})|,
\]

\[
\tilde{\alpha}(A, B, C) = \sup_{s, t, u \in \mathbb{R}} |E((1_{A \leq s} - \mathbb{P}(A \leq s))(1_{B \leq t} - \mathbb{P}(B \leq t))(1_{C \leq u} - \mathbb{P}(C \leq u))|.
\]

(note that \(\tilde{\alpha}(A, B, B) \leq \tilde{\alpha}(A, B))\). Let

\[
A = |g_i^{(0)}(X_0)| \text{sign}\{E(g_q^{(0)}(X_i)g_r^{(0)}(X_j)|X_0) - E(g_q^{(0)}(X_i)g_r^{(0)}(X_j))\},
\]

\[
\text{sign}\{E(g_q^{(0)}(X_i)g_r^{(0)}(X_j)|X_0) - E(g_q^{(0)}(X_i)g_r^{(0)}(X_j))\}.
\]
and note that $Q_A = Q_{g_i^0}(X_0)$. From Proposition 6.1 and Lemma 6.1 in Dedecker and Rio (2007), we have that

$$\|g_{i}^{(0)}(X_0)(\mathbb{E}(g_{i}^{(0)}(X_i)g_{r}^{(0)}(X_j)|X_0) - \mathbb{E}(g_{i}^{(0)}(X_i)g_{r}^{(0)}(X_j)))\|_1 = \mathbb{E}((A - \mathbb{E}(A))g_{i}^{(0)}(X_i)g_{r}^{(0)}(X_j))$$

$$\leq 16 \int_{0}^{\bar{\alpha}(A,g_{r}(X_i),g_{r}(X_j))/2} Q_{g_{i}^{(0)}(X_0)}(u)Q_{g_{r}(X_0)}(u)Q_{g_{r}(X_0)}(u)\,du.$$ 

Note that $Q_{g_{i}^{(0)}(X_0)} \leq Q_{g_{r}(X_0)} + \|g_{l}(X_0)\|_1$. Hence, by Fréchet’s inequality (1957),

$$\int_{0}^{\bar{\alpha}(A,g_{r}(X_i),g_{r}(X_j))/2} Q_{g_{i}^{(0)}(X_0)}(u)Q_{g_{r}(X_0)}(u)Q_{g_{r}(X_0)}(u)\,du$$

$$\leq 2 \int_{0}^{\bar{\alpha}(A,g_{r}(X_i),g_{r}(X_j))/2} Q_{g_{r}(X_0)}(u)Q_{g_{r}(X_0)}(u)Q_{g_{r}(X_0)}(u)\,du.$$ 

Since $\{g_{l}(x) \leq t\}$ is some interval of $\mathbb{R}$, we have that for $j \geq 1$

$$\bar{\alpha}(A, g_{q}(X_i), g_{r}(X_j)) \leq 4\bar{\alpha}(A, X_i, X_j) \leq 4\alpha_2(i),$$

and for $i = j$,

$$\bar{\alpha}(A, g_{q}(X_i), g_{r}(X_i)) \leq 4\bar{\alpha}(A, X_i, X_i) \leq 4\alpha_2(i).$$

Since $Q_{g_{r}(X_0)}(u) \leq Mu^{-1/p}$, it follows that, for $1 \leq i \leq j$,

$$\|g_{l}(X_0)(\mathbb{E}(g_{q}(X_i)g_{r}(X_j)|X_0) - \mathbb{E}(g_{q}(X_i)g_{r}(X_j)))\|_1 \leq \frac{32M^3p}{p-3}(2\alpha_2(i))^{\frac{p-3}{p}}.$$ 

Consequently, for any $f$ in $C(M, p, \mu)$ with $p > 3$,

$$\|Y_0(\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j))\|_1 \leq \frac{32M^3p}{p-3}(2\alpha_2(i))^{\frac{p-3}{p}}.$$ 

In the same way,

$$2\sigma\|\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j)\|_1 \leq \frac{32\sigma M^2p}{p-2}(2\alpha_2(i))^{\frac{p-2}{p}}.$$ 

It follows that, for any $1 \leq i \leq j$,

$$\text{(5.6)} \quad \|(|Y_0| + 2\sigma)(\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j))\|_1 \leq D(M, p, \sigma)(\alpha_2(i))^{\frac{p-3}{p}}.$$ 

Combining (5.5) and (5.6), we infer that

$$\sum_{i=1}^{m} \sum_{j=i}^{m} \|(|Y_0| + 2\sigma)(\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j))\|_1 = O(m^{1-\delta}),$$

and (5.4) easily follows. This completes the proof.
6. Moment inequalities

**Theorem 6.1.** Let \( X = (X_i)_{i \geq 0} \) be a stationary Markov chain with invariant measure \( \mu \) and transition kernel \( K \). If \( f \) belong to \( C(M,p,\mu) \) for some \( M > 0 \) and some \( p > 2 \), then, for any \( 2 \leq q < p \)

\[
\left\| \sum_{i=1}^{n}(f(X_i) - \mu(f)) \right\|_q \leq \sqrt{2q} \left( n\|f(X_0) - \mu(f)\|_q^2 + 4M^2 \left( \frac{p}{p-q} \right)^2 \sum_{k=1}^{n-1}(n-k)(2\alpha_1(k)\frac{2(p-q)}{pq}) \right)^{\frac{1}{2}}.
\]

**Corollary 6.1.** Let \( 0 < \gamma < 1 \). Let \( f \) belong to \( C(M,p,\mu) \) for some \( M > 0 \) and some \( p > 2 \), and let \( 2 \leq q < p \).

1. \( f \) positive and non increasing on \([0,1]\), with \( f(x) \leq Cx^{-a} \) for some \( a > 0 \).

2. \( f \) positive and non increasing on \([0,1]\), with \( f(x) \leq C(1-x)^{-a} \) for some \( a > 0 \).

**Remark 6.1.** Assume that \( \gamma < (p-2)/(2p-2) \). By Chebichev inequality applied with \( 2 \leq q < 2p(1-\gamma)/(\gamma p + 2(1-\gamma)) \), we infer from Item (1) that for any \( \epsilon > 0 \),

\[
\nu_{\gamma} \left( \left\| S_n(f - \nu_{\gamma}(f)) \right\| > x \right) \leq \frac{C}{(n\sqrt{2})^{\gamma(1-\gamma)/(\gamma p + 2(1-\gamma))}}.
\]

Assume now that \( (p-2)/(2p-2) \leq \gamma < 1 \). By Chebichev inequality applied with \( q = 2 \), we infer from Item (2) that for any \( \epsilon > 0 \),

\[
\nu_{\gamma} \left( \left\| S_n(f - \nu_{\gamma}(f)) \right\| > x \right) \leq \frac{C}{x^{2n(1-\gamma)/(\gamma p - \epsilon)}}.
\]

When \( f \) is BV (case \( p = \infty \)) and \( \gamma < 1 \), we obtain that, for any \( \epsilon > 0 \) and any \( x > 0 \),

\[
\nu_{\gamma} \left( \left\| S_n(f - \nu_{\gamma}(f)) \right\| > x \right) \leq \frac{C}{n(1-\gamma)/\gamma - \epsilon}.
\]

Note that Melbourne and Nicol (2007) obtained the same bound when \( f \) is \( \alpha \)-Hölder and \( \gamma < 1/2 \).

**Two simple examples (continued).**

1. Assume that \( f \) is positive and non increasing on \([0,1]\), with \( f(x) \leq Cx^{-a} \) for some \( a > 0 \).

2. Assume that \( f \) is positive and non increasing on \([0,1]\), with \( f(x) \leq C(1-x)^{-a} \) for some \( a > 0 \).
Proof of Theorem 6.1} From Proposition 4 in Dedecker and Doukhan (2003) (see also Theorem 2.5 in Rio (2000)), we have that, for any \( q \geq 2 \),
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - \mu(f)) \right\|_q \leq \sqrt{2q} \left( \frac{n}{q} \left| f(X_0) - \mu(f) \right|^2 \right)^{\frac{1}{2}} + \sum_{k=1}^{n-1} (n-k) \left| (f(X_0) - \mu(f)) (\mathbb{E}(f(X_k)|X_0) - \mu(f)) \right|^{\frac{1}{2}}.
\]
Assume first that \( f = \sum_{i=1}^{k} a_i g_i \), where \( \sum_{i=1}^{k} \left| a_i \right| \leq 1 \), and \( g_i \) belongs to \( \text{Mon}(M, p, \mu) \). Clearly,
\[
\left\| (f(X_0) - \mu(f)) (\mathbb{E}(f(X_n)|X_0) - \mu(f)) \right\|_{q/2} \leq \sum_{i=1}^{k} \sum_{j=1}^{k} \left| a_i a_j \right| \left| (g_i(X_0) - \mu(g_i)) (\mathbb{E}(g_j(X_n)|X_0) - \mu(g_j)) \right|_{q/2}.
\]
Applying Lemma 4.1, we obtain that
\[
\left\| (f(X_0) - \mu(f)) (\mathbb{E}(f(X_n)|X_0) - \mu(f)) \right\|_{q/2} \leq 4M^2 \left( \frac{p}{p-q} \right)^{2/q/2} (2\alpha_1(n))^{2(p-q)/pq}.
\]
Clearly, this inequality remains valid for any \( f \in \mathcal{C}(M, p, \mu) \), and the result follows.

7. The empirical distribution function

Theorem 7.1. Let \( X = (X_i)_{i \geq 0} \) be a stationary Markov chain with invariant measure \( \mu \) and transition kernel \( K \). Let \( F_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{X_i \leq t} \) and \( F_\mu(t) = \mu(-\infty, t] \).

1. If \( X \) is ergodic (in the ergodic theoretic sense) and if \( \sum_{k=1}^{\infty} \beta_1(k) < \infty \), then, for any probability \( \pi \) on \( \mathbb{R} \), the process \( \{ \rho_n(F_n(t) - F_\mu(t)), t \in \mathbb{R} \} \) converges in distribution in \( L^2(\pi) \) to a tight Gaussian process \( G \) with covariance function

\[
\text{Cov}(G(s), G(t)) = C_{\mu, K}(s, t) = \mu(f_t^{(0)} f_s^{(0)}) + 2 \sum_{k=0}^{\infty} \mu(f_t^{(0)} K^k f_s^{(0)}).
\]

2. Let \( (D(\mathbb{R}), d) \) be the space of cahlq functions equipped with the Skorohod metric \( d \). If \( \beta_2(k) = O(k^{-2-\epsilon}) \) for some \( \epsilon > 0 \), then the process \( \{ \sqrt{n}(F_n(t) - F_\mu(t)), t \in \mathbb{R} \} \) converges in distribution in \( (D(\mathbb{R}), d) \) to a tight Gaussian process \( G \) with covariance function \( C_{\mu, K} \).

Corollary 7.1. Let \( F_{n, \gamma}(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{T_i \leq t} \).

1. If \( 0 < \gamma < 1/2 \), then, for any probability \( \pi \) on \([0, 1]\), the process \( \{ \sqrt{n}(F_{n, \gamma}(t) - F_{\nu_\gamma}(t)), t \in [0, 1]\} \) converges in distribution in \( L^2(\pi) \) to a tight Gaussian process \( G_\gamma \) with covariance function \( C_{\nu_\gamma, K_\gamma} \).

2. If \( 0 < \gamma < 1/3 \), the process \( \{ \sqrt{n}(F_{n, \gamma}(t) - F_{\nu_\gamma}(t)), t \in [0, 1]\} \) converges in distribution in \( (D([0,1]), d) \) to a tight Gaussian process \( G_\gamma \) with covariance function \( C_{\nu_\gamma, K_\gamma} \).

Remark 7.1. Denote by \( \| \cdot \|_{p, \pi} \) the \( L^p(\pi) \)-norm. If \( \gamma < 1/2 \), we have that, for any \( 1 \leq p \leq 2 \),
\[
\sqrt{n} \left\| F_{n, \gamma} - F_{\nu_\gamma} \right\|_{p, \pi} \text{ converges in distribution to } \|G_\gamma\|_{p, \pi}.
\]
In particular, if \( \pi = \lambda \) is the Lebesgue measure on \([0, 1]\) and \( q = p/(p-1) \), we obtain that
\[
\frac{1}{\sqrt{n}} \sup_{\| f \|_{L^1} \leq 1} |S_n(f - \nu_\gamma(f))| \text{ converges in distribution to } \|G_\gamma\|_{p, \lambda}.
\]
For $p = 1$ and $q = \infty$, we obtain the limit distribution of the Kantorović distance $d_1(F_{n,\gamma}, F_{\nu})$:

$$\sqrt{n}d_1(F_{n,\gamma}, F_{\nu}) = \frac{1}{\sqrt{n}} \sup_{f \in H_{1,1}} |S_n(f - \nu(f))| \text{ converges in distribution to } \int_0^1 |G_\gamma(t)| dt.$$  

Now if $\gamma < 1/3$, the limit in (7.7) holds for any $p \geq 1$.

Note that, for Harris recurrent Markov chains, Item (2) of Theorem 7.1 holds as soon as the sum of the $\beta$-mixing coefficients of the chain is finite. Hence, we conjecture that Item (2) of Corollary 7.1 remains true for $\gamma < 1/2$.

**Proof of Theorem 7.1.** Item (1) has been proved in Dedecker and Merlevède (2007, Theorem 2, Item 2) and Item (2) in Dedecker and Prieur (2007, Proposition 2).

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**References**

[1] E. Bolthausen (1982), The Berry-Esseen theorem for strongly mixing Harris recurrent Markov chains. *Z. Wahrsch. verw. Gebiete.* 60, 283-289.

[2] J.-P. Conze and A. Raugi (2003), Convergence of iterates of a transfer operator, application to dynamical systems and to Markov chains. *ESAIM Probab. Stat.* 7, 115-146.

[3] J. Dedecker and P. Doukhan (2003), A new covariance inequality and applications, *Stochastic Process. Appl.* 106, 63-80.

[4] J. Dedecker and F. Merlevède (2006), The empirical distribution function for dependent variables: asymptotic and nonasymptotic results in $L^p$. *ESAIM Probab. Stat.* 11, 102-114.

[5] J. Dedecker and C. Prieur (2005), New dependence coefficients. Examples and applications to statistics. *Probab. Theory Relat. Fields* 132, 203-236.

[6] J. Dedecker and C. Prieur (2007), An empirical central limit theorem for dependent sequences. *Stochastic Process. Appl.* 117, 121-142.

[7] J. Dedecker and E. Rio (2000), On the functional central limit theorem for stationary processes. *Ann. Inst. H. Poincaré Probab. Statist.* 36, 1-34.

[8] J. Dedecker and E. Rio (2007), On mean central limit theorems for stationary sequences. *Accepted for publication in Ann. Inst. H. Poincaré.*

[9] C-G. Esseen and S. Janson (1985), On moment conditions for normed sums of independent variables and martingale differences. *Stochastic Process. Appl.* 19, 173-182.

[10] M. Fréchet (1957), Sur la distance de deux lois de probabilités. *C. R. Acad. Sci. Paris.* 244, 689-692.

[11] S. Gouëzel (2004), Central limit theorem and stable laws for intermittent maps. *Probab. Theory Relat. Fields* 128, 82-122.

[12] S. Gouëzel (2005), Berry-Esseen theorem and local limit theorem for non uniformly expanding maps. *Ann. Inst. H. Poincaré Probab. Statist.* 41, 997-1024.

[13] H. Hennion and L. Hervé (2001), Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness. *Lecture Notes in Mathematics* 1766, Springer.

[14] L. V. Kantorovič and G. Š. Rubinštejn (1957), On a functional space and certain extremum problems. *Dokl. Akad. Nauk SSSR* 115, 1058-1061.

[15] C. Liverani, B. Saussol and S. Vaienti (1999), A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems.* 19, 671-685.
[16] V. Maume-Deschamps (2001), Projective metrics and mixing properties on towers. Trans. Amer. Math. Soc. 353, 3371-3389.
[17] I. Melbourne and M. Nicol (2007), Large deviations for nonuniformly hyperbolic systems. To appear in Trans. Amer. Math. Soc.
[18] A. Raugi (2004), Étude d’une transformation non uniformément hyperbolique de l’intervalle [0,1[. Bull. Soc. math. France 132, 81-103.
[19] Y. Pomeau and P. Manneville (1980), Intermittent transition to turbulence in dissipative dynamical systems. Commun. Math. Phys. 74, 189-197.
[20] E. Rio (2000), Théorie asymptotique des processus aléatoires faiblement dépendants. Mathématiques et applications de la SMAI. 31, Springer.
[21] O. Sarig (2002), Subexponential decay of correlations. Inv. Math. 150, 629-653.
[22] L-S. Young (1999), Recurrence times and rates of mixing. Israel J. Math. 110, 153-188.

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