FUNCTIONAL CALCULUS FOR A BOUNDED $C_0$-SEMIGROUP ON HILBERT SPACE

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Abstract. We introduce a new Banach algebra $A(C_+)$ of bounded analytic functions on $C_+ = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \}$ which is an analytic version of the Figa-Talamanca-Herz algebras on $\mathbb{R}$. Then we prove that the negative generator $A$ of any bounded $C_0$-semigroup on Hilbert space $H$ admits a bounded (natural) functional calculus $\rho_A : A(C_+) \to B(H)$. We prove that this is an improvement of the bounded functional calculus $B_0(C_+) \to B(H)$ recently devised by Batty-Gomilko-Tomilov on a certain Besov algebra $B_0(C_+)$ of analytic functions on $C_+$, by showing that $B_0(C_+) \subset A(C_+)$ and $B_0(C_+) \neq A(C_+)$. In the Banach space setting, we give similar results for negative generators of $\gamma$-bounded $C_0$-semigroups. The study of $A(C_+)$ involves dealing with Fourier multipliers on the Hardy space $H^1(\mathbb{R}) \subset L^1(\mathbb{R})$ of analytic functions.

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1. Introduction

Let $H$ be a Hilbert space and let $-A$ be the infinitesimal generator of a bounded $C_0$-semigroup $(T_t)_{t \geq 0}$ on $H$. To any $b \in L^1(\mathbb{R}_+)$, one may associate the operator $\Gamma(A, b) \in B(H)$ defined by

$$[\Gamma(A, b)](x) = \int_0^\infty b(t)T_t(x) \, dt, \quad x \in H.$$ 

The mapping $b \mapsto \Gamma(A, b)$ is the so-called Hille-Phillips functional calculus ([19], see also [16, Section 3.3]) and we obviously have

$$\|\Gamma(A, b)\| \leq C\|b\|_1, \quad b \in L^1(\mathbb{R}_+),$$

where $C = \sup_{t \geq 0} \|T_t\|$. This holds true as well for any bounded $C_0$-semigroup on Banach space. However we focus here on semigroups acting on Hilbert space.

If $(T_t)_{t \geq 0}$ is a contractive semigroup (i.e. $\|T_t\| \leq 1$ for all $t \geq 0$) on $H$, then we have the much stronger estimate $\|\Gamma(A, b)\| \leq \|\hat{b}\|_\infty$ for all $b \in L^1(\mathbb{R}_+)$, where $\hat{b}$ denotes the Fourier transform of $b$. This is a semigroup version of von Neumann’s inequality, see [16, Section 7.1.3] for a proof. Hence more generally, if $(T_t)_{t \geq 0}$ is similar to a contractive semigroup, then
there exists a constant $C \geq 1$ such that
\begin{equation}
\|\Gamma(A, b)\| \leq C\|\hat{b}\|_{\infty}, \quad b \in L^1(\mathbb{R}_+).
\end{equation}
However not all negative generators of bounded $C_0$-semigroups satisfy such an estimate. Indeed if $A$ is sectorial of type $\frac{\pi}{2}$, it follows from \cite{16} Section 3.3 that $A$ satisfies an estimate of the form (1.1) exactly when $A$ has a bounded $H^{\infty}$-functional calculus, see Subsection 4.4 for more on this.

The motivation for this paper is the search for sharp estimates of $\|\Gamma(A, b)\|$, and of the norms of other functions of $A$, valid for all negative generators of bounded $C_0$-semigroups. A major breakthrough was achieved by Haase \cite[Corollary 5.5]{17} who proved an estimate
\begin{equation}
\|\Gamma(A, b)\| \leq C\|L_b\|_{\mathcal{B}_0},
\end{equation}
where $\|\cdot\|_{\mathcal{B}_0}$ denotes the norm with respect to a suitable Besov algebra $\mathcal{B}_0(\mathbb{C}_+)$ of analytic functions, and

$$L_b : \mathbb{C}_+ = \{\text{Re}(\cdot) > 0\} \to \mathbb{C}, \quad L_b(z) = \int_0^\infty b(t)e^{-tz} \, dt,$$

is the Laplace transform of $b$. More recently, Batty-Gomilko-Tomilov \cite{5} (see also \cite{6}) extended Haase’s result by providing an explicit construction of a bounded functional calculus $\mathcal{B}_0(\mathbb{C}_+) \to B(H)$ associated with $A$, extending the Hille-Phillips functional calculus. It is worth mentioning the related works by Vitse \cite{35} and White \cite{37} (see also Subsection 4.5). We refer to \cite[Appendix]{5} for more information on $\mathcal{B}_0(\mathbb{C}_+)$ and on variants of this Besov algebra.

In this paper we introduce the space $\mathcal{A}(\mathbb{R}) \subset H^{\infty}(\mathbb{R})$ defined by

$$\mathcal{A}(\mathbb{R}) = \left\{ F = \sum_{k=1}^\infty f_k \ast h_k : (f_k)_{k \in \mathbb{N}} \subset BUC(\mathbb{R}), (h_k)_{k \in \mathbb{N}} \subset H^1(\mathbb{R}), \sum_{k=1}^\infty \|f_k\|_{\infty} \|h_k\|_1 < \infty \right\},$$

equipped with the norm $\|F\|_\mathcal{A} = \inf\{\sum_{k=1}^\infty \|f_k\|_{\infty} \|h_k\|_1\}$, where the infimum runs over all sequences $(f_k)_{k \in \mathbb{N}} \subset BUC(\mathbb{R})$ and $(h_k)_{k \in \mathbb{N}} \subset H^1(\mathbb{R})$ such that $\sum_{k=1}^\infty \|f_k\|_{\infty} \|h_k\|_1 < \infty$ and $F = \sum_{k=1}^\infty f_k \ast h_k$. The definition of this space is inspired by Peller’s paper \cite{28}, where a discrete analogue of $\mathcal{A}(\mathbb{R})$ was introduced to study functions of power bounded operators on Hilbert space. We also refer to \cite{36} for earlier results on this theme. Furthermore $\mathcal{A}(\mathbb{R})$ can be regarded as an $H^1(\mathbb{R})$-version of the Figa-Talamanca-Herz algebras $A_p(\mathbb{R})$, $1 < p < \infty$, for which we refer e.g. to \cite[Chapter 3]{9}.

We prove in Section 3 that $\mathcal{A}(\mathbb{R})$ is indeed a Banach algebra for pointwise multiplication. Next in Section 4 we introduce the natural half-plane version $\mathcal{A}(\mathbb{C}_+) \subset H^{\infty}(\mathbb{C}_+)$ of $\mathcal{A}(\mathbb{R})$, we prove that it contains $L_b$ for all $b \in L^1(\mathbb{R}_+)$, and we show (Corollary 4.6) that whenever $A$ is the negative generator of a bounded $C_0$-semigroup $(T_t)_{t \geq 0}$ on Hilbert space, there is a unique bounded homomorphism $\rho_A : \mathcal{A}(\mathbb{C}_+) \to B(H)$, such that
\begin{equation}
\rho_A(L_b) = \Gamma(A, b), \quad b \in L^1(\mathbb{R}_+).
\end{equation}

More precisely we show that
\begin{equation*}
\|\Gamma(A, b)\| \leq \left( \sup_{t \geq 0} \|T_t\| \right)^2 \|L_b\|_\mathcal{A}, \quad b \in L^1(\mathbb{R}_+).
\end{equation*}
This improves Haase’s estimate (1.2) mentioned above. Our work also improves [5] Theorem 4.4] in the Hilbert space case. Indeed we show in Section 5 that the Besov algebra considered in [16, 5] is included in \( \mathcal{A}(C_+) \), with an estimate \( | \cdot |_{\mathcal{A}} \lesssim | \cdot |_{B_0} \), and we also show that the converse is not true.

In general, our main result (Corollary 4.6) does not hold true on non-Hilbertian Banach spaces (see the start of Section 6). In Section 6, following ideas from [2, 1, 26], we give a Banach space version of Corollary 4.6 using the notion of \( \gamma \)-boundedness. Namely we show that if \( A \) is the negative generator of a bounded \( C_0 \)-semigroup \( (T_t)_{t \geq 0} \) on a Banach space \( X \), then the set \( \{ T_t : t \geq 0 \} \subset B(X) \) is \( \gamma \)-bounded if and only if there exists a \( \gamma \)-bounded homomorphism \( \rho_A : \mathcal{A}(C) \to B(X) \) satisfying (1.3). This should be regarded as a semigroup version of [26, Theorem 4.4], where a characterization of \( \gamma \)-bounded continuous representations of amenable groups was established.

Our results make crucial use of Fourier multipliers on the Hardy space \( H^1(\mathbb{R}) \). Section 2 is devoted to this topic. In particular we establish the following result of independent interest: if a bounded operator \( T : H^1(\mathbb{R}) \to H^1(\mathbb{R}) \) commutes with translations, then there exists a bounded continuous function \( m : \mathbb{R}^*_+ \to \mathbb{C} \) such that \( |m|_\infty \leq ||T|| \) and for any \( h \in H^1(\mathbb{R}) \), \( \widehat{T(h)} = mh \).

**Notation and convention.** We use the notations \( \mathbb{R}_+ = [0, \infty) \), \( \mathbb{R}^*_+ = (0, \infty) \) and \( \mathbb{R}_- = (-\infty, 0] \) on the real line.

We will use the following open half-planes of \( \mathbb{C} \),
\[
\mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Re}(z) > 0 \}, \quad P_+ := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}, \quad P_- := \{ z \in \mathbb{C} : \text{Im}(z) < 0 \}.
\]
Also for any real \( \alpha \in \mathbb{R} \), we set
\[
(1.4) \quad \mathcal{H}_\alpha = \{ z \in \mathbb{C} : \text{Re}(z) > \alpha \}.
\]
In particular, \( \mathcal{H}_0 = \mathbb{C}_+ \).

For any \( s \in \mathbb{R} \), we let \( \tau_s : L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \to L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \) denote the translation operator defined by
\[
\tau_s f(t) = f(t - s), \quad t \in \mathbb{R},
\]
for any \( f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \).

The Fourier transform of any \( f \in L^1(\mathbb{R}) \) is defined by
\[
\hat{f}(u) = \int_{-\infty}^{\infty} f(t) e^{-itu} \, dt, \quad u \in \mathbb{R}.
\]
Sometimes we write \( \mathcal{F}(f) \) instead of \( \hat{f} \). We will also let \( \mathcal{F}(f) \) or \( \hat{f} \) denote the Fourier transform of any \( f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \). Wherever it makes sense, we will use \( \mathcal{F}^{-1} \) to denote the inverse Fourier transform.

We will use several times the following elementary result (which follows from Fubini’s theorem and the Fourier inversion theorem).

**Lemma 1.1.** Let \( f_1, f_2 \in L^1(\mathbb{R}) \) such that either \( \hat{f}_1 \) or \( \hat{f}_2 \) belongs to \( L^1(\mathbb{R}) \). Then
\[
\int_{-\infty}^{\infty} f_1(t) f_2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_1(u) \hat{f}_2(-u) \, du.
\]
The norm on $L^p(\mathbb{R})$ will be denoted by $\| \cdot \|_p$. We let $C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$, resp. $C_b(\mathbb{R})$) denote the Banach algebra of continuous functions on $\mathbb{R}$ which vanish at infinity (resp. of bounded and uniformly continuous functions on $\mathbb{R}$, resp. of bounded continuous functions on $\mathbb{R}$), equipped with the sup-norm $\| \cdot \|_\infty$. We set

$$C_{00}(\mathbb{R}) = \{ f \in L^1(\mathbb{R}) : \hat{f} \in L^1(\mathbb{R}) \}. \tag{1.5}$$

This is a dense subspace of $C_0(\mathbb{R})$. Further we let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space on $\mathbb{R}$ and we let $M(\mathbb{R})$ denote the Banach algebra of all bounded Borel measures on $\mathbb{R}$.

We will use the identification $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$ (Riesz’s theorem) provided by the duality pairing

$$\langle \mu, f \rangle = \int_\mathbb{R} f(-t) \, d\mu(t), \quad \mu \in M(\mathbb{R}), \ f \in C_0(\mathbb{R}). \tag{1.6}$$

The use of a minus sign in this duality pairing will make the study of $\mathcal{A}(\mathbb{R})$ easier.

For any non empty open set $\mathcal{O} \subset \mathbb{C}$, we let $H^\infty(\mathcal{O})$ denote the Banach algebra of all bounded analytic functions on $\mathcal{O}$, equipped with the sup-norm $\| \cdot \|_\infty$.

Let $X,Y$ be (complex) Banach spaces. We let $B(X,Y)$ denote the Banach space of all bounded operators $X \to Y$. We simply write $B(X)$ instead of $B(X,X)$, when $Y = X$. We let $I_X$ denote the identity operator on $X$.

The domain of an operator $A$ on some Banach space $X$ is denoted by Dom($A$). Its kernel and range are denoted by Ker($A$) and Ran($A$), respectively. If $z \in \mathbb{C}$ belongs to the resolvent set of $A$, we let $R(z,A) = (zI_X - A)^{-1}$ denote the corresponding resolvent operator.

### 2. Fourier multipliers on $H^1(\mathbb{R})$

We denote by $H^1(\mathbb{R})$ the classical Hardy space, defined as the closed subspace of $L^1(\mathbb{R})$ of all functions $h$ such that $\hat{h}(u) = 0$ for all $u \leq 0$. For any $1 < p < \infty$, we denote by $H^p(\mathbb{R})$ the closure of $H^1(\mathbb{R}) \cap L^p(\mathbb{R})$ in $L^p(\mathbb{R})$. Also we let $H^\infty(\mathbb{R})$ denote the $w^*$-closure of $H^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ in $L^\infty(\mathbb{R})$. We recall (see e.g. [12], [20] or [23]) that for any $1 \leq p \leq \infty$, $H^p(\mathbb{R})$ coincides with the subspace of all functions $f \in L^p(\mathbb{R})$ whose Poisson integral $\mathcal{P}[f] : P_+ \to \mathbb{C}$ is analytic. The following is classical as well.

**Lemma 2.1.** Let $g \in L^\infty(\mathbb{R})$. Then $g \in H^\infty(\mathbb{R})$ if and only if $\int_{-\infty}^\infty g(t)h(t) \, dt = 0$ for all $h \in H^1(\mathbb{R})$.

**Proof.** This statement means that $H^\infty(\mathbb{R})$ is the annihilator of $H^1(\mathbb{R})$ in the usual $L^1/L^\infty$-duality. To prove it, it suffices to check the equivalent property that $H^1(\mathbb{R})$ is the pre-annihilator of $H^\infty(\mathbb{R})$.

If $g \in H^\infty(\mathbb{R})$ and $h \in H^1(\mathbb{R})$, then $gh \in H^1(\mathbb{R})$ hence $\int_{-\infty}^\infty g(t)h(t) \, dt = \hat{g}(0) = 0$, which proves one inclusion. Now define $c_u(t) = e^{-itu}$ for all $u \leq 0$ and all $t \in \mathbb{R}$. Then $c_u \in H^\infty(\mathbb{R})$, with $\mathcal{P}[c_u](z) = e^{-ituz}$ for all $z \in P_+$. If $h \in L^1(\mathbb{R})$ belongs to the pre-annihilator of $H^\infty(\mathbb{R})$, then $\hat{h}(u) = \int_{-\infty}^\infty c_u(t)h(t) \, dt = 0$ for all $u \leq 0$, that is, $h \in H^1(\mathbb{R})$. This proves the reverse inclusion. \[
\]

It is well-known that $H^p(\mathbb{R})$ also coincides with the subspace of all functions in $L^p(\mathbb{R})$ whose (distributional) Fourier transform has support in $\mathbb{R}_+$. In particular, $H^2(\mathbb{R})$ is the subspace
of all functions in $L^2(\mathbb{R})$ whose Fourier transform (regarded as an element of $L^2(\mathbb{R})$) vanishes almost everywhere on $\mathbb{R}_-$. This can be expressed by the identification
\[ \mathcal{F}(H^2(\mathbb{R})) = L^2(\mathbb{R}_+). \]
Let $m \in L^\infty(\mathbb{R}_+)$. Using (2.1), we may associate
\[ T_m : H^2(\mathbb{R}) \to H^2(\mathbb{R}) \]
\[ h \mapsto \mathcal{F}^{-1}(m\hat{h}), \]
and we have $\|T_m\| = \|m\|_\infty$. The function $m$ is called the symbol of $T_m$. Let $1 \leq p < \infty$. Assume that $T_m$ is bounded with respect to the $H^p(\mathbb{R})$-norm, that is, there exists a constant $C > 0$ such that
\[ \|\mathcal{F}^{-1}(m\hat{h})\|_p \leq C \|h\|_p, \quad h \in H^p(\mathbb{R}) \cap H^2(\mathbb{R}). \]

Then since $H^p(\mathbb{R}) \cap H^2(\mathbb{R})$ is dense in $H^p(\mathbb{R})$, $T_m$ uniquely extends to a bounded operator on $H^p(\mathbb{R})$ whose norm is the least possible constant $C$ satisfying (2.2). In this case we keep the same notation $T_m : H^p(\mathbb{R}) \to H^p(\mathbb{R})$ for this extension. Operators of this form are called bounded Fourier multipliers on $H^p(\mathbb{R})$. They form a subspace of $B(H^p(\mathbb{R}))$, that we denote by $\mathcal{M}(H^p(\mathbb{R}))$. It is plain that $\mathcal{M}(H^2(\mathbb{R})) \simeq L^\infty(\mathbb{R}_+)$ isometrically. In the sequel we will be mostly interested by $\mathcal{M}(H^1(\mathbb{R}))$.

The above definitions parallel the classical definitions of bounded Fourier multipliers on $L^p(\mathbb{R})$, that we will use without any further reference.

**Example 2.2.** Let $s \in \mathbb{R}$. For all $h \in L^1(\mathbb{R})$, one has $\hat{\tau_s h}(u) = e^{-isu}\hat{h}(u)$ for all $u \in \mathbb{R}$. Hence $\tau_s$ maps $H^1(\mathbb{R})$ into itself. Further $\tau_s$ is a bounded Fourier multiplier on $H^1(\mathbb{R})$, with symbol $m(u) = e^{-isu}$.

In the sequel we say that a bounded operator $T : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ commutes with translations if $T\tau_s = \tau_s T$ for each $s \in \mathbb{R}$. By Example 2.2 any bounded Fourier multiplier on $H^1(\mathbb{R})$ commutes with translations. The next result implies that the converse is true and provides a sharp estimate on the symbol of an element of $\mathcal{M}(H^1(\mathbb{R}))$.

**Theorem 2.3.** Let $T \in B(H^1(\mathbb{R}))$ and assume that $T$ commutes with translations. Then there exists a bounded continuous function $m : \mathbb{R}_+^+ \to \mathbb{C}$ such that $\hat{T}h = m\hat{h}$ for all $h \in H^1(\mathbb{R})$ (and hence $T = T_m$). In this case, we have
\[ \|m\|_\infty \leq \|T\|. \]

**Proof.** Except the estimate (2.3), this statement can be deduced from [8, pp. 131-132], and from the fact that $\{h_1 + h_2(\cdot) : h_1, h_2 \in H^1(\mathbb{R})\} \subset L^1(\mathbb{R})$ coincides with the so-called real $H^1$-space (see also [11, Theorem 7.31]). We briefly prove the first part of our statement for completeness and then focus on the proof of (2.3).

We use Bochner spaces and Bochner integrals, for which we refer to [10]. Let $T \in B(H^1(\mathbb{R}))$ and assume that $T$ commutes with translations.

Let $h, g \in H^1(\mathbb{R})$. The identification $L^1(\mathbb{R}^2) = L^1(\mathbb{R}; L^1(\mathbb{R}))$ and the fact that
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(t-s)||g(s)| \, dt \, ds = \|h\|_1 \|g\|_1 < \infty \]
imply that \( s \mapsto g(s)\tau_s h \) is an almost everywhere defined function belonging to the Bochner space \( L^1(\mathbb{R}; L^1(\mathbb{R})) \). Since \( \tau_s h \) belongs to \( H^1(\mathbb{R}) \) for all \( s \in \mathbb{R} \), the latter is actually an element of \( L^1(\mathbb{R}; H^1(\mathbb{R})) \). Further its integral (which is an element of \( H^1(\mathbb{R}) \)) is equal to the convolution of \( h \) and \( g \), that is,

\[
(2.4) \quad h \ast g = \int_{-\infty}^{\infty} \tau_s h \cdot g(s) \, ds.
\]

It follows, by the assumption, that

\[
T(h \ast g) = \int_{-\infty}^{\infty} T(\tau_s h)g(s) \, ds = \int_{-\infty}^{\infty} \tau_s(Th)g(s) \, ds = T(h \ast g).
\]

Likewise \( T(h \ast g) = h \ast Th \), whence \( Th \ast g = h \ast Tg \). Applying the Fourier transform to the latter equality, one obtains

\[
(2.5) \quad \widehat{Th \ast g} = \widehat{h} \ast \widehat{Tg}.
\]

Now let \( f \in S(\mathbb{R}) \) satisfying \( f = 0 \) on \( \mathbb{R}_- \) and \( f > 0 \) on \( \mathbb{R}_+ \). The function \( g = F^{-1}(f) \) belongs to \( H^1(\mathbb{R}) \) and we may define \( m: \mathbb{R}^+_1 \to \mathbb{C} \) by

\[
(2.6) \quad m(u) = \frac{\widehat{Tg(u)}}{\widehat{g}(u)}, \quad u > 0.
\]

Obviously \( m \) is continuous. Furthermore it follows from (2.5) that for any \( h \in H^1(\mathbb{R}) \), we have \( \widehat{Th} = m\widehat{h} \) on \( \mathbb{R}^+_1 \). It therefore suffices to show that \( m \) is bounded and that (2.3) holds true.

For \( h \in H^1(\mathbb{R}) \) and \( g \in S(\mathbb{R}) \), one has

\[
\int_{-\infty}^{\infty} m(u)\widehat{h}(u)\widehat{g}(u) \, du = \int_{-\infty}^{\infty} \widehat{T(h)}(u)\widehat{g}(u) \, du = 2\pi \int_{-\infty}^{\infty} \widehat{T(h)}(-t)g(t) \, dt,
\]

by Lemma 1.1. This implies that

\[
\left| \int_{-\infty}^{\infty} m(u)\widehat{h}(u)\widehat{g}(u) \, du \right| \leq 2\pi \|T\| \|h\|_1 \|g\|_\infty.
\]

Replacing \( g \) by \( g_1 \ast g_2 \) for \( g_1, g_2 \in S(\mathbb{R}) \) and using \( \|g_1 \ast g_2\|_\infty \leq \|g_1\|_2 \|g_2\|_2 \), we deduce that

\[
\left| \int_{-\infty}^{\infty} m(u)\widehat{h}(u)\widehat{g_1}(u)\widehat{g_2}(u) \, du \right| \leq 2\pi \|T\| \|h\|_1 \|g_1\|_2 \|g_2\|_2 = \|T\| \|h\|_1 \|\widehat{g_1}\|_2 \|\widehat{g_2}\|_2.
\]

This implies \( \|m\widehat{h}\|_2 \leq \|T\| \|h\|_1 \|\widehat{g}\|_2 \) for all \( g_1 \in S(\mathbb{R}) \) and hence

\[
(2.7) \quad \|m\widehat{h}\|_\infty \leq \|T\| \|h\|_1,
\]

for all \( h \in H^1(\mathbb{R}) \).

Now fix a function \( \varphi \in S(\mathbb{R}) \) such that \( \varphi \geq 0 \), \( \text{Supp}(\widehat{\varphi}) \subset [-1, 1] \) and \( \varphi(0) = 1 \). Let \( a > 0 \) and define \( \varphi_a, h_a: \mathbb{R} \to \mathbb{C} \) by \( \varphi_a(t) = a\varphi(at) \) and \( h_a(t) = e^{ita}\varphi_a(t) \). Then

\[
\varphi_a(u) = \varphi\left(\frac{u}{a}\right) \quad \text{and} \quad \widehat{h}_a(u) = \widehat{\varphi}\left(\frac{u}{a}\right),
\]
for all \( u \in \mathbb{R} \). Hence \( h_a \in H^1(\mathbb{R}) \) and \( \hat{h}_a(a) = 1 \). Furthermore, \( \varphi_a \geq 0 \), hence \( \| \varphi_a \|_1 = \| \hat{\varphi}_a \|_1 = \| \hat{\varphi}_a(0) \| = \| \hat{\varphi}(0) \| = 1 \). Since \( \| h_a \|_1 = \| \varphi_a \|_1 \), we obtain that \( \| h_a \|_1 = 1 \). Applying (2.7) with \( h = h_a \) then yields
\[
|m(a)| = |m(a)\hat{h}_a(a)| \leq \| m\hat{h}_a \|_\infty \leq \| T \|.
\]
This proves the boundedness of \( m \) and (2.3). \( \square \)

**Remark 2.4.**

1. Let \( 1 < p < \infty \). Let \( S : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}) \) be a bounded Fourier multiplier. Then \( S \) maps \( H^p(\mathbb{R}) \) into itself and the restriction \( S_{H^p} : H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R}) \) is a bounded Fourier multiplier.

Let \( Q : L^p(\mathbb{R}) \rightarrow H^p(\mathbb{R}) \) be the Riesz projection and let \( J : H^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}) \) be the canonical embedding. Then conversely, for any bounded Fourier multiplier \( T : H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R}) \), \( S = JTQ \) is a bounded Fourier multiplier on \( L^p(\mathbb{R}) \), whose restriction to \( H^p(\mathbb{R}) \) coincides with \( T \). Thus \( \mathcal{M}(H^p(\mathbb{R})) \) can be simply regarded as a subspace of \( \mathcal{M}(L^p(\mathbb{R})) \), the space of bounded Fourier multipliers on \( L^p(\mathbb{R}) \).

It is well-known that a bounded operator \( L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}) \) is a Fourier multiplier if and only if it commutes with translations. Using the above reasoning, we deduce that a bounded operator \( H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R}) \) belongs to \( \mathcal{M}(H^p(\mathbb{R})) \) if and only if it commutes with translations.

2. Let \( p' = \frac{p}{p-1} \) be the conjugate exponent of \( 1 < p < \infty \). Using \( Q \) again, we see that given any \( m \in L^\infty(\mathbb{R}_+) \), the operator \( T_m : H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}) \) extends to a bounded Fourier multiplier on \( H^p(\mathbb{R}) \) if and only if it extends to a bounded Fourier multiplier on \( H^{p'}(\mathbb{R}) \). Thus,
\[
\mathcal{M}(H^p(\mathbb{R})) \simeq \mathcal{M}(H^{p'}(\mathbb{R}))
\]

isomorphically.

3. Recall that the bounded Fourier multipliers on \( L^1(\mathbb{R}) \) are the operators of the form \( h \mapsto \mu * h \), with \( \mu \in M(\mathbb{R}) \), and that the norm of the latter operator is equal to \( \| \mu \|_{M(\mathbb{R})} \) (see e.g. [33, Chapter I, Theorem 3.19]).

For any \( \mu \in M(\mathbb{R}) \), let \( R_\mu : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \) be the restriction of \( h \mapsto \mu * h \) to \( H^1(\mathbb{R}) \). This is a bounded Fourier multiplier whose symbol is equal to the restriction of \( \hat{\mu} \) to \( \mathbb{R}_+^\ast \). We set
\[
\mathcal{R} = \{ R_\mu : \mu \in M(\mathbb{R}) \} \subset \mathcal{M}(H^1(\mathbb{R})).
\]
In contrast with the result in part (1) of this remark, we have
\[
\mathcal{R} \neq \mathcal{M}(H^1(\mathbb{R})).
\]

Indeed this follows from [13, Remark (ii)].

The following lemma will play a crucial role.

**Lemma 2.5.** For any \( 1 \leq p < \infty \), we have \( \mathcal{M}(H^1(\mathbb{R})) \subset \mathcal{M}(H^p(\mathbb{R})) \).

**Proof.** By definition we have \( \mathcal{M}(H^1(\mathbb{R})) \subset \mathcal{M}(H^2(\mathbb{R})) \). By (2.8) we may assume that \( p \in (1, 2) \). Let \( \theta = 2(1 - \frac{1}{p}) \). Then in the complex interpolation method, we have
\[
[H^1(\mathbb{R}), H^2(\mathbb{R})]_\theta \simeq H^p(\mathbb{R}),
\]
by [29, Theorem 4.3]. The result follows at once. \( \square \)
3. Algebras $A_0$ and $\mathcal{A}$

We introduce and study new algebras of functions which will be used in Section 4 to establish a functional calculus for negative generators of bounded $C_0$-semigroups on Hilbert space. The next definitions are inspired by [28], see also the “Notes and Remarks on Chapter 6" in [30].

3.1. Definitions and properties.

**Definition 3.1.** We let $A_0(\mathbb{R})$ (resp. $\mathcal{A}(\mathbb{R})$) be the set of all functions $F: \mathbb{R} \to \mathbb{C}$ such that there exist two sequences $(f_k)_{k \in \mathbb{N}}$ in $C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$) and $(h_k)_{k \in \mathbb{N}}$ in $H^1(\mathbb{R})$ satisfying

$$\sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 < \infty \quad (3.1)$$

and

$$\forall s \in \mathbb{R}, \quad F(s) = \sum_{k=1}^{\infty} (f_k \ast h_k)(s). \quad (3.2)$$

For any $f \in L^\infty(\mathbb{R})$ and any $h \in H^1(\mathbb{R})$, $f \ast h$ belongs to $H^\infty(\mathbb{R})$. To prove this, let $k \in H^1(\mathbb{R}) \cap H^\infty(\mathbb{R})$. By Fubini’s theorem, and Lemma 2.1,

$$\int_{-\infty}^{\infty} (f \ast h)(t)k(t) \, dt = \int_{-\infty}^{\infty} f(u) \left( \int_{-\infty}^{\infty} h(t-u)k(t) \, dt \right) \, du = 0.$$

Indeed for all $u \in \mathbb{R}$, $h(\cdot - u) \in H^1(\mathbb{R})$ hence the integral of $h(\cdot - u)k$ is equal to 0. Since $H^1(\mathbb{R}) \cap H^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, this implies that $\int_{-\infty}^{\infty} (f \ast h)(t)k(t) \, dt = 0$ for all $k \in H^1(\mathbb{R})$. By Lemma 2.1 this proves that $f \ast h \in H^\infty(\mathbb{R})$.

We deduce that for any $f \in C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$) and any $h \in H^1(\mathbb{R})$, $f \ast h$ belongs to $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$ (resp. $BUC(\mathbb{R}) \cap H^\infty(\mathbb{R})$). Further for any $(f_k)_{k \in \mathbb{N}}$ and $(h_k)_{k \in \mathbb{N}}$ as in Definition 3.1 we have $\|f_k \ast h_k\|_{\infty} \leq \|f_k\|_{\infty} \|h_k\|_1$, and hence $\sum_{k=1}^{\infty} \|f_k \ast h_k\|_{\infty} < \infty$, by (3.1). This ensures the convergence of the series in (3.2) and implies that

$$A_0(\mathbb{R}) \subset C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{A}(\mathbb{R}) \subset BUC(\mathbb{R}) \cap H^\infty(\mathbb{R}).$$

**Definition 3.2.** For all $F \in A_0(\mathbb{R})$ (resp. $F \in \mathcal{A}(\mathbb{R})$), we set

$$\|F\|_{A_0} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \right\}, \quad \text{(resp.} \|F\|_{\mathcal{A}} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \right\} \text{),}$$

where the infimum runs over all sequences $(f_k)_{k \in \mathbb{N}}$ in $C_0(\mathbb{R})$ (resp. $BUC(\mathbb{R})$) and $(h_k)_{k \in \mathbb{N}}$ in $H^1(\mathbb{R})$ satisfying (3.1) and (3.2).

It is clear that

$$\|F\|_{\infty} \leq \|F\|_{A_0}, \quad F \in \mathcal{A}(\mathbb{R}).$$

To show that $\|\cdot\|_{A_0}$ and $\|\cdot\|_{\mathcal{A}}$ are complete norms, we make a connection with projective tensor products, which will be useful throughout this section.
If $X$ and $Y$ are any Banach spaces, the projective norm of $\zeta \in X \otimes Y$ is defined by
\[
\|\zeta\|_{\wedge} = \inf \left\{ \sum_k \|x_k\| \|y_k\| \right\},
\]
where the infimum runs over all finite families $(x_k)_k$ in $X$ and $(y_k)_k$ in $Y$ satisfying $\zeta = \sum_k x_k \otimes y_k$. The completion of $(X \otimes Y, \| \cdot \|_{\wedge})$, denoted by $X^\wedge \otimes Y$, is called the projective tensor product of $X$ and $Y$.

Let $Z$ be a third Banach space. To any $\ell \in B_2(X \times Y, Z)$, the space of bounded bilinear maps from $X \times Y$ into $Z$, one can associate a linear map $\circ \ell: X \otimes Y \to Z$ by the formula
\[
\circ \ell(x \otimes y) = \ell(x, y), \quad x \in X, y \in Y.
\]
Then $\circ \ell$ extends to a bounded operator (still denoted by) $\circ \ell: X^\wedge \otimes Y \to Z$, with $\|\circ \ell\| = \|\ell\|$. Further the mapping $\ell \mapsto \circ \ell$ yields an isometric identification
\[
(3.3) \quad B_2(X \times Y, Z) \simeq B(X^\wedge \otimes Y, Z).
\]
Let us apply the above property in the case when $Z = \mathbb{C}$. Using the standard identification $B_2(X \times Y, \mathbb{C}) = B(Y, X^*)$, we obtain an isometric identification
\[
(3.4) \quad (X^\wedge Y)^* \simeq B(Y, X^*).
\]
We refer to either [10, Chapter 8, Theorem 1 & Corollary 2] or [31, Theorem 2.9] for these classical facts.

Consider the bilinear map $\sigma: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \to C_0(\mathbb{R})$ defined by
\[
(3.5) \quad \sigma(f, h) = f \ast h, \quad f \in C_0(\mathbb{R}), \ h \in H^1(\mathbb{R}).
\]
Applying (3.3), let
\[
\circ \sigma: C_0(\mathbb{R})^\wedge \otimes H^1(\mathbb{R}) \to C_0(\mathbb{R})
\]
be associated with $\sigma$. Then $A_0(\mathbb{R}) = \text{Ran}(\circ \sigma)$. Through the resulting linear isomorphism between $A_0(\mathbb{R})$ and $(C_0(\mathbb{R})^\wedge \otimes H^1(\mathbb{R}))/\text{Ker}(\circ \sigma)$, $\| \cdot \|_{A_0}$ corresponds to the quotient norm on the latter space (this follows from either [10, Chapter 8, Proposition 9 (b)] or [31, Proposition 2.8]). Thus $(A_0(\mathbb{R}), \| \cdot \|_{A_0})$ is a Banach space and $\circ \sigma$ induces an isometric identification
\[
(3.6) \quad A_0(\mathbb{R}) \simeq \frac{C_0(\mathbb{R})^\wedge \otimes H^1(\mathbb{R})}{\text{Ker}(\circ \sigma)}.
\]
Similarly, $\| \cdot \|_A$ is a norm on $A(\mathbb{R})$ and $(A(\mathbb{R}), \| \cdot \|_A)$ is a Banach space.

**Remark 3.3.** It is clear from Definition 3.2 that $A_0(\mathbb{R}) \subset A(\mathbb{R})$ and that for any $F \in A_0(\mathbb{R})$, we have
\[
(3.7) \quad \|F\|_A \leq \|F\|_{A_0}.
\]
We will show in Proposition 3.12 below that this inequality is actually an equality.
3.2. $\mathcal{A}$ and $\mathcal{A}_0$ are Banach algebras.

Proposition 3.4. The spaces $\mathcal{A}_0(\mathbb{R})$ and $\mathcal{A}(\mathbb{R})$ are Banach algebras for the pointwise multiplication. Furthermore, $\mathcal{A}_0(\mathbb{R})$ is an ideal of $\mathcal{A}(\mathbb{R})$ and for any $F \in \mathcal{A}(\mathbb{R})$ and $G \in \mathcal{A}_0(\mathbb{R})$, we have

$$\|FG\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}} \|G\|_{\mathcal{A}_0}. \tag{3.8}$$

Proof. We will adapt an idea from [28]. Let $f_1, f_2 \in C_0(\mathbb{R})$ and $h_1, h_2 \in H^1(\mathbb{R})$. We note that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h_1(t)||h_2(t+s)| dt ds = \|h_1\|_1 \|h_2\|_1. \tag{3.9}$$

We define, for $s \in \mathbb{R}$, $\varphi_s : \mathbb{R} \to \mathbb{C}$ and $\psi_s : \mathbb{R} \to \mathbb{C}$ by

$$\varphi_s(t) = f_1(t)f_2(t-s) \quad \text{and} \quad \psi_s(t) = h_1(t)h_2(t+s).$$

Since $f_2$ is uniformly continuous, the function $s \mapsto \varphi_s$ is continuous from $\mathbb{R}$ into $C_0(\mathbb{R})$. Thus $s \mapsto \varphi_s$ belongs to the Bochner space $L^\infty(\mathbb{R}; C_0(\mathbb{R}))$. Using (3.9) and arguing as at the beginning of the proof of Theorem 2.3, we see that $s \mapsto \psi_s$ belongs to $L^1(\mathbb{R}; H^1(\mathbb{R}))$. It follows that $s \mapsto \varphi_s \ast \psi_s$ is defined almost everywhere and belongs to $L^1(\mathbb{R}; \mathcal{A}_0(\mathbb{R}))$. Moreover,

$$\int_{-\infty}^{\infty} \|\varphi_s \ast \psi_s\|_{\mathcal{A}_0} ds \leq \|f_1\|_\infty \|f_2\|_\infty \|h_1\|_1 \|h_2\|_1. \tag{3.10}$$

Indeed, using (3.9) and Fubini’s theorem,

$$\int_{-\infty}^{\infty} \|\varphi_s \ast \psi_s\|_{\mathcal{A}_0} ds \leq \int_{-\infty}^{\infty} \|\varphi_s\|_\infty \|\psi_s\|_{1} ds \leq \|f_1\|_\infty \|f_2\|_\infty \int_{-\infty}^{\infty} \|\psi_s\|_{1} ds,$$

which is equal to the right-hand side of (3.10).

The integral of $s \mapsto \varphi_s \ast \psi_s$ is an element of $\mathcal{A}_0(\mathbb{R})$. We claim that we actually have

$$\int_{-\infty}^{\infty} \varphi_s \ast \psi_s ds = (f_1 \ast h_1)(f_2 \ast h_2). \tag{3.11}$$

Indeed, using again (3.9) and Fubini’s theorem, we have for all $u \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} (\varphi_s \ast \psi_s)(u) ds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t)f_2(t-s)h_1(u-t)h_2(u-t+s) dt ds = \int_{-\infty}^{\infty} f_1(t)h_1(u-t) \int_{-\infty}^{\infty} f_2(t-s)h_2(u-(t-s)) ds dt = (f_1 \ast h_1)(u)(f_2 \ast h_2)(u).$$

Combining (3.11) and (3.10), we obtain that $(f_1 \ast h_1)(f_2 \ast h_2) \in \mathcal{A}_0(\mathbb{R})$, with

$$\|(f_1 \ast h_1)(f_2 \ast h_2)\|_{\mathcal{A}_0} \leq \|f_1\|_\infty \|f_2\|_\infty \|h_1\|_1 \|h_2\|_1.$$
Now let \( F, G \in A_0(\mathbb{R}) \) and let \( \varepsilon > 0 \). Consider sequences \((f_k^1)_{k \in \mathbb{N}}, (f_k^2)_{k \in \mathbb{N}} \) in \( C_0(\mathbb{R}) \) and \((h_k^1)_{k \in \mathbb{N}}, (h_k^2)_{k \in \mathbb{N}} \) in \( H^1(\mathbb{R}) \) such that

\[
F = \sum_{k=1}^{\infty} f_k^1 \ast h_k^1 \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k^1\|_\infty \|h_k^1\|_1 \leq \|F\|_{A_0} + \varepsilon,
\]
as well as

\[
G = \sum_{k=1}^{\infty} f_k^2 \ast h_k^2 \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k^2\|_\infty \|h_k^2\|_1 \leq \|G\|_{A_0} + \varepsilon.
\]

Then, using summation in \( C_0(\mathbb{R}) \), we have

\[
FG = \sum_{k,l=1}^{\infty} (f_k^1 \ast h_k^1)(f_l^2 \ast h_l^2).
\]

Further \((f_k^1 \ast h_k^1)(f_l^2 \ast h_l^2) \in A_0(\mathbb{R})\) for all \( k, l \geq 1 \), and

\[
\sum_{k,l=1}^{\infty} \|(f_k^1 \ast h_k^1)(f_l^2 \ast h_l^2)\|_{A_0} \leq \sum_{k,l=1}^{\infty} \|f_k^1\|_\infty \|f_l^2\|_\infty \|h_k^1\|_1 \|h_l^2\|_1
\]

\[
= \left( \sum_{k=1}^{\infty} \|f_k^1\|_\infty \|h_k^1\|_1 \right) \left( \sum_{l=1}^{\infty} \|f_l^2\|_\infty \|h_l^2\|_1 \right)
\]

\[
\leq (\|F\|_{A_0} + \varepsilon)(\|G\|_{A_0} + \varepsilon).
\]

Since \( A_0(\mathbb{R}) \) is complete, this shows that \( FG \in A_0(\mathbb{R}) \). Since \( \varepsilon > 0 \) is arbitrary, we obtain that \( \|FG\|_{A_0} \leq \|F\|_{A_0} \|G\|_{A_0} \). This shows that \( A_0(\mathbb{R}) \) is a Banach algebra.

Analogously, \( A(\mathbb{R}) \) is a Banach algebra. Moreover if \( f_1 \in BUC(\mathbb{R}) \) and \( f_2 \in C_0(\mathbb{R}) \), then for each \( s \in \mathbb{R} \), \( \varphi_s : t \mapsto f_1(t)f_2(t-s) \) belongs to \( C_0(\mathbb{R}) \). Hence the computations above show that \( A_0(\mathbb{R}) \) is an ideal of \( A(\mathbb{R}) \), as well as \( (3.8) \). \( \square \)

We note that the Banach algebra \( A_0(\mathbb{R}) \) can be naturally regarded as an \( H^1(\mathbb{R}) \)-version of the Figa-Talamanca-Herz algebras \( A_p(\mathbb{R}) \), \( 1 < p < \infty \) (see e.g. \[3\] Chapter 3).

### 3.3. Duality results and consequences.

The main aim of this subsection is to identify \( A_0(\mathbb{R})^* \) with a subspace of \( \mathcal{M}(H^1(\mathbb{R})) \), the space of bounded Fourier multipliers on \( H^1(\mathbb{R}) \). This requires the use of duality tools.

We let \( H^1(\mathbb{R}) \) (resp. \( H^\infty(\mathbb{R}) \)) be the subspace of all \( f \) in \( L^1(\mathbb{R}) \) (resp. \( L^\infty(\mathbb{R}) \)) such that \( f(-\cdot) \) belongs to \( H^1(\mathbb{R}) \) (resp. \( H^\infty(\mathbb{R}) \)). Recall the identification \( \mathcal{M}(\mathbb{R}) \simeq C_0(\mathbb{R})^* \) provided by \((1.6)\) and regard \( H^1(\mathbb{R}) \subset L^1(\mathbb{R}) \subset M(\mathbb{R}) \) in the usual way. We have

\[
(C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}))^\perp = H^1(\mathbb{R}).
\]

Indeed the inclusion \( \subset \) follows from \[12\] Chapter II, Theorem 3.8] whereas the reverse inclusion follows from Lemma \[2.1\] .

Set \( Z_0 := C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}) \) for convenience, then the above result yields an isometric identification

\[
(\frac{C_0(\mathbb{R})}{Z_0})^* \simeq H^1(\mathbb{R}).
\]
In the sequel, we let \( \dot{f} \in C_0(\mathbb{R}) \) denote the class of any \( f \in C_0(\mathbb{R}) \).

We note that for any \( f \in C_0(\mathbb{R}) \), \( h \in H^1(\mathbb{R}) \) and \( s \in \mathbb{R} \), we have
\[
(3.14) \quad (f \star h)(s) = \langle \tau_{-s} h, f \rangle.
\]

Thus \( f \star h = 0 \) for any \( f \in Z_0 \) and \( h \in H^1(\mathbb{R}) \). The bilinear map \( \sigma: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \to C_0(\mathbb{R}) \) defined by \((3.13)\) therefore induces a bilinear map \( \delta: \frac{C_0(\mathbb{R})}{Z_0} \times H^1(\mathbb{R}) \to C_0(\mathbb{R}) \) given by
\[
\delta(\dot{f}, h) = f \star h, \quad f \in C_0(\mathbb{R}), \ h \in H^1(\mathbb{R}).
\]

Let
\[
(3.15) \quad \delta^\circ: \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R}) \to C_0(\mathbb{R})
\]
be the bounded map induced by \( \delta \). Then the argument leading to \((3.6)\) shows as well that \( \mathcal{A}_0(\mathbb{R}) \) is equal to the range of \( \delta^\circ \) and that
\[
(3.16) \quad \mathcal{A}_0(\mathbb{R}) \simeq \left( \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R}) \right) / \text{Ker}(\delta^\circ).
\]

Since \( H^1(\mathbb{R}) \) is a dual space, by \((3.13)\), \( B(H^1(\mathbb{R})) \) is a dual space as well. Indeed applying \((3.4)\), we have an isometric identification
\[
(3.17) \quad B(H^1(\mathbb{R})) \simeq \left( \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R}) \right)^*.
\]

If we unravel the identifications leading to \((3.17)\), we obtain that the latter is given by
\[
(3.18) \quad \langle T, \dot{f} \otimes h \rangle = \int_{-\infty}^{\infty} (Th)(t)f(-t)\,dt, \quad T \in B(H^1(\mathbb{R})), \ f \in C_0(\mathbb{R}), \ h \in H^1(\mathbb{R}).
\]

**Lemma 3.5.** The space \( \mathcal{M}(H^1(\mathbb{R})) \) is \( w^* \)-closed in \( B(H^1(\mathbb{R})) \).

**Proof.** According to Theorem 2.3, an operator \( T \in B(H^1(\mathbb{R})) \) belongs to \( \mathcal{M}(H^1(\mathbb{R})) \) if and only if \( \tau_s T = T \tau_s \) for all \( s \in \mathbb{R} \). Hence it suffices to show that the maps \( T \mapsto \tau_s T \) and \( T \mapsto T \tau_s \) are \( w^* \)-continuous on \( B(H^1(\mathbb{R})) \), for all \( s \in \mathbb{R} \).

Fix some \( s \in \mathbb{R} \) and note that the mapping \( C_0(\mathbb{R}) \to C_0(\mathbb{R}) \) taking \( \dot{f} \) to \( \tau_s \dot{f} \) for any \( f \) in \( C_0(\mathbb{R}) \) is a well-defined isometric isomorphism. This implies the existence of an isometric isomorphism
\[
w_s: \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R}) \to \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R})
\]
such that \( w_s(\dot{f} \otimes h) = \tau_s \dot{f} \otimes h \) for all \( f \in C_0(\mathbb{R}) \) and all \( h \in H^1(\mathbb{R}) \). It readily follows from \((3.18)\) that for any \( T \in B(H^1(\mathbb{R})) \), and any \( f, h \) as above, we have
\[
\langle \tau_s T, \dot{f} \otimes h \rangle = \langle T, \tau_s \dot{f} \otimes h \rangle.
\]

Thus \( w_s^*: B(H^1(\mathbb{R})) \to B(H^1(\mathbb{R})) \) coincides with \( T \mapsto \tau_s T \). The latter is therefore \( w^* \)-continuous. The proof that \( T \mapsto T \tau_s \) is \( w^* \)-continuous is similar. \( \square \)
We introduce
\[ PM = \overline{\text{Span}}^w \{ \tau_s : s \in \mathbb{R} \} \subset B(H^1(\mathbb{R})). \]
A direct consequence of Lemma 3.5 is that
\[ PM \subset M(H^1(\mathbb{R})). \]
(3.19)

Lemma 3.6. Recall the mapping \( \overset{\circ}{\delta} \) from (3.15). Then
\[ \left[ \text{Ker}(\overset{\circ}{\delta}) \right]^\perp = PM. \]

Proof. Let \( s \in \mathbb{R} \) and let \( \Phi \in C_0(\mathbb{R}) \hat{\otimes} H^1(\mathbb{R}) \). According to [31, Proposition 2.8], we may choose two sequences \((f_k)_{k \in \mathbb{N}} \) in \( C_0(\mathbb{R}) \) and \((h_k)_{k \in \mathbb{N}} \) in \( H^1(\mathbb{R}) \) such that
\[ \sum_{k=1}^{\infty} \| f_k \|_\infty \| h_k \|_1 < \infty \quad \text{and} \quad \Phi = \sum_{k=1}^{\infty} f_k \otimes h_k. \]
Then by (3.14),
\[ \left[ \overset{\circ}{\delta}(\Phi) \right](s) = \sum_{k=1}^{\infty} (f_k \ast h_k)(s) = \sum_{k=1}^{\infty} \langle \tau_{-s} h_k, f_k \rangle = \langle \tau_{-s}, \Phi \rangle. \]
This shows that
\[ \text{Span}\{ \tau_s : s \in \mathbb{R} \}_\perp = \text{Ker}(\overset{\circ}{\delta}). \]
The result follows at once. \( \square \)

By standard duality and (3.16), the dual space \( A_0(\mathbb{R})^\ast \) may be identified with \( [\text{Ker}(\overset{\circ}{\delta})]^\perp \). Applying Lemma 3.6 and (3.18), we therefore obtain the following.

Theorem 3.7.

(1) For any \( T \in PM \), there exists a unique \( \eta_T \in A_0(\mathbb{R})^\ast \) such that
\[ \langle \eta_T, f \ast h \rangle = \int_{-\infty}^{\infty} (Th)(t)f(-t) \, dt \]
for any \( f \in C_0(\mathbb{R}) \) and any \( h \in H^1(\mathbb{R}) \).
(2) The mapping \( T \mapsto \eta_T \) induces a \( w^\ast \)-homeomorphic and isometric identification
\[ A_0(\mathbb{R})^\ast \simeq PM. \]

Remark 3.8. Recall \( \mathcal{R} \) from (2.9). It turns out that
\[ PM = \mathcal{R}^w. \]
Indeed for any \( s \in \mathbb{R} \), \( \tau_s : H^1(\mathbb{R}) \to H^1(\mathbb{R}) \) is equal to the convolution by the Dirac mass at \( s \), which yields \( \subset \). To show the converse inclusion, we observe that for any \( \mu \in M(\mathbb{R}) \) and any \( \Phi \in C_0(\mathbb{R}) \hat{\otimes} H^1(\mathbb{R}) \), we have
\[ \langle \mu, \overset{\circ}{\delta}(\Phi) \rangle = \langle R_\mu, \Phi \rangle. \]
(3.20)
Here we use the identification $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$ given by (1.6) on the left-hand side and we use (3.17) on the right-hand side. To prove this identity, let $f \in C_0(\mathbb{R})$ and $h \in H^1(\mathbb{R})$. Then

$$\langle \mu, \delta(f \otimes h) \rangle = \langle \mu, f \ast h \rangle = \int_{\mathbb{R}} \int_{-\infty}^{\infty} f(-t)h(t-s) \, dt \, d\mu(s) = \langle \mu, f \ast h, f \rangle.$$ 

This shows (3.20) when $\Phi = \hat{\delta} \otimes 0$. By linearity, this implies (3.20) for $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R})$. Then by density, we deduce (3.20) for all $\Phi$.

It clearly follows from (3.20) that $\mathcal{R} \subset [\text{Ker}(\delta)]^\perp$. By Lemma 3.6, this yields $\supset$.

We now give a few consequences of the above duality results.

**Proposition 3.9.** For any $b$ in $L^1(\mathbb{R}_+)$, the function $\hat{b}(-\cdot) : \mathbb{R} \to \mathbb{C}$ belongs to $A_0(\mathbb{R})$ and

$$||\hat{b}(-\cdot)||_{A_0} \leq ||b||_1.$$ 

Moreover the mapping $L^1(\mathbb{R}_+) \to A_0(\mathbb{R})$ taking $b$ to $\hat{b}(-\cdot)$ is a Banach algebra homomorphism with dense range.

**Proof.** We will use the space $C_{00}(\mathbb{R})$ defined by (1.5). Let $C_{00} \ast H^1 \subset A_0(\mathbb{R})$ be the linear span of the functions $f \ast h$, for $f \in C_{00}(\mathbb{R})$ and $h \in H^1(\mathbb{R})$. By definition of $C_{00}(\mathbb{R})$, the Fourier transform maps $C_{00} \ast H^1$ into $L^1(\mathbb{R}_+)$. Then we consider

$$\mathcal{C}_{0,1} = \{ \hat{F} : F \in C_{00} \ast H^1 \} \subset L^1(\mathbb{R}_+).$$

Let $b \in C^\infty(\mathbb{R}_+^*)$ with compact support. Let $c \in C^\infty(\mathbb{R}_+^*)$ with compact support such that $c \equiv 1$ on the support of $b$, so that $\hat{b} = bc$. Then $\mathcal{F}^{-1}(b) \in C_{00}(\mathbb{R})$, $\mathcal{F}^{-1}(c) \in H^1(\mathbb{R})$ and the Fourier transform of $\mathcal{F}^{-1}(b) \ast \mathcal{F}^{-1}(c)$ is equal to $b$. Thus $b \in \mathcal{C}_{0,1}$. Consequently, $\mathcal{C}_{0,1}$ is dense in $L^1(\mathbb{R}_+)$. Let $b \in \mathcal{C}_{0,1}$ and let $F = \hat{b}(-\cdot)$, so that

$$\hat{F} = (2\pi)b.$$

Take finite families $(f_k)_k$ in $C_{00}(\mathbb{R})$ and $(h_k)_k$ in $H^1(\mathbb{R})$ such that $F = \sum_k f_k \ast h_k$. Pick $\eta \in A_0(\mathbb{R})^*$ such that $||\eta|| = 1$ and $||F||_{A_0} = \langle \eta, F \rangle$. By Theorem 3.7 there exists $T \in \mathcal{M}(H^1(\mathbb{R}))$ such that $||T||_{B(H^1)} = 1$ and for any $k$,

$$\langle \eta, f_k \ast h_k \rangle = \int_{-\infty}^{\infty} (Th_k)(u)f_k(-u) \, du.$$ 

By Theorem 2.3 the symbol $m$ of the multiplier $T$ satisfies $||m||_{\infty} \leq 1$. Moreover by Lemma 1.1 we have

$$\int_{-\infty}^{\infty} (Th_k)(u)f_k(-u) \, du = \frac{1}{2\pi} \int_0^{\infty} \hat{T}h_k(t)\hat{f}_k(t) \, dt = \frac{1}{2\pi} \int_0^{\infty} m(t)\hat{h}_k(t)\hat{f}_k(t) \, dt,$$
for all $k$. Summing over $k$, we deduce that
\[
\langle \eta, F \rangle = \frac{1}{2\pi} \sum_k \int_0^\infty m(t) \hat{h}_k(t) \hat{f}_k(t) \, dt = \frac{1}{2\pi} \int_0^\infty m(t) \hat{F}(t) \, dt,
\]
and hence
\[
\langle \eta, F \rangle = \int_0^\infty m(t) b(t) \, dt.
\]
We deduce that
\[
\|F\|_{A_0} \leq \|b\|_1.
\]
Since $C_{0,1}$ is dense in $L^1(\mathbb{R}_+)$ and $A_0(\mathbb{R})$ is complete, this estimate implies that $\hat{b}(\cdot) : \mathbb{R} \to \mathbb{C}$ belongs to $A_0(\mathbb{R})$ for any $b \in L^1(\mathbb{R}_+)$, with $\|\hat{b}(\cdot)\|_{A_0} \leq \|b\|_1$.

It is plain that $b \mapsto \hat{b}(\cdot)$ is a Banach algebra homomorphism. Its range contains $C_{00} \ast H^1$ and the latter is dense in $A_0(\mathbb{R})$, because $C_{00}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$. □

**Remark 3.10.** Let $\eta \in A_0(\mathbb{R})^*$, let $T \in \mathcal{M}(H^1(\mathbb{R}))$ be the multiplier associated with $\eta$, according to Theorem 3.7, and let $m \in C_b(\mathbb{R}_+) \ast C_0(\mathbb{R}_+)$ be the symbol of $T$. Then it follows from the previous result and its proof that for all $b \in L^1(\mathbb{R}_+)$,
\[
\langle \eta, \hat{b}(\cdot) \rangle = \int_0^\infty m(t) b(t) \, dt.
\]

**Remark 3.11.** For any $\lambda \in P_-$, we let $b_\lambda(t) = ie^{\lambda t}$, $t > 0$. Then $b_\lambda \in L^1(\mathbb{R}_+)$ and we have
\[
\hat{b}_\lambda(u) = \frac{1}{\lambda - u}, \quad u \in \mathbb{R}.
\]
Applying Proposition 3.9, we obtain that $(\lambda - \cdot)^{-1}$ belongs to $A_0(\mathbb{R})$ for any $\lambda \in P_-$. Since $A_0(\mathbb{R})$ is a Banach algebra, this implies that any rational function $F : \mathbb{R} \to \mathbb{C}$ with degree $\text{deg}(F) \leq -1$ and poles in $P_-$ belongs to $A_0(\mathbb{R})$.

We can now strengthen Remark 3.3 as follows.

**Proposition 3.12.** For any $F \in A_0(\mathbb{R})$, we have
\[
\|F\|_{A_0} = \|F\|_A.
\]

**Proof.** For any $N \in \mathbb{N}$, let $G_N : \mathbb{R} \to \mathbb{C}$ be defined by
\[
G_N(u) = \frac{N}{N - iu}, \quad u \in \mathbb{R}.
\]
Then $G_N = \overline{N + iN} \cdot (\cdot)$. We note that the sequence $(Ne^{\cdot})_{N \in \mathbb{N}}$ is a contractive approximate unit of $L^1(\mathbb{R}_+)$. It therefore follows from Proposition 3.9 that $(G_N)_{N \in \mathbb{N}}$ is a contractive approximate unit of $A_0(\mathbb{R})$.

Let $F \in A_0(\mathbb{R})$. By Proposition 3.4, we have
\[
(3.21) \quad \|FG_N\|_{A_0} \leq \|F\|_A \|G_N\|_{A_0} \leq \|F\|_A.
\]
Moreover $FG_N \to F$ in $A_0(\mathbb{R})$ hence $\|FG_N\|_{A_0} \to \|F\|_{A_0}$ when $N \to \infty$. We deduce that
\[ \|F\|_{A_0} \leq \|F\|_A. \]
Combining with (3.7), we obtain $\|F\|_{A_0} = \|F\|_A$. \[\square\]

**Remark 3.13.** According to [12, Chapter II, Theorem 3.8], $H^1(\mathbb{R}) \subset M(\mathbb{R})$ is the annihilator of $\{ (\lambda - \cdot)^{-1} : \lambda \in P_- \} \subset C_0(\mathbb{R})$. Hence we deduce from Remark 3.11 that $H^1(\mathbb{R})$ contains $A_0(\mathbb{R})^\perp$. By (3.12), we have $(C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}))^\perp = H^1(\mathbb{R})$. Hence $A_0(\mathbb{R})^\perp = (C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}))^\perp$. This implies that $A_0(\mathbb{R})$ is dense in $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$.

Using Proposition 3.9 or repeating the above argument, we also obtain that the space \( \{ b(-\cdot) : b \in L^1(\mathbb{R}^+) \} \) is dense in $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$.

### 3.4. Half-plane versions

For the purpose of studying functional calculus in the next three sections, we now introduce half-plane versions $A_0(\mathbb{C}_+)$ and $A(\mathbb{C}_+)$ of $A_0(\mathbb{R})$ and $A(\mathbb{R})$, respectively.

Let $F \in H^\infty(\mathbb{R})$. We may consider its Poisson integral $P[F] : P_+ \to \mathbb{C}$ and the latter is a bounded holomorphic function (see e.g. [12, Sect. I.3]). Then we define
\[ \tilde{F} : \mathbb{C}_+ \longrightarrow \mathbb{C} \]
by setting
\[ \tilde{F}(z) = P[F](iz), \quad z \in \mathbb{C}_+. \]
Note that the mapping $F \mapsto \tilde{F}$ is an isometric algebra isomorphism of $H^\infty(\mathbb{R})$ onto $H^\infty(\mathbb{C}_+)$. We set
\[ A_0(\mathbb{C}_+) = \{ \tilde{F} : F \in A_0(\mathbb{R}) \} \quad \text{and} \quad A(\mathbb{C}_+) = \{ \tilde{F} : F \in A(\mathbb{R}) \}. \]
We equip these spaces with the norms induced by $A_0(\mathbb{R})$ and $A(\mathbb{R})$, respectively. That is, we set $\|\tilde{F}\|_{A_0} = \|F\|_{A_0}$ (resp. $\|\tilde{F}\|_A = \|F\|_A$) for any $F \in A_0(\mathbb{R})$ (resp. $F \in A(\mathbb{R})$). By construction, we have
\begin{equation}
A_0(\mathbb{C}_+) \subset A(\mathbb{C}_+) \quad \text{and} \quad A(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+). \tag{3.22}
\end{equation}
It clearly follows from Proposition 3.3 and from the multiplicativity of the mapping $F \mapsto \tilde{F}$ that $A(\mathbb{C}_+)$ is a Banach algebra for pointwise multiplication and that $A_0(\mathbb{C}_+)$ is an ideal of $A(\mathbb{C}_+)$. Further the second inclusion in (3.22) is contractive and it follows from Proposition 3.12 that the first inclusion in (3.22) is an isometry.

Let $b \in L^1(\mathbb{R}^+)$. and consider $F = \hat{b}(-\cdot) : \mathbb{R} \to \mathbb{C}$. Then $\tilde{F}$ coincides with the Laplace transform $L_b : \mathbb{C}_+ \to \mathbb{C}$ of $b$ defined by
\begin{equation}
L_b(z) = \int_0^\infty e^{-tz}b(t) \, dt, \quad z \in \mathbb{C}_+. \tag{3.23}
\end{equation}
As a consequence of Proposition 3.9, we therefore obtain the following.

**Lemma 3.14.** For any $b \in L^1(\mathbb{R}^+)$, $L_b$ belongs to $A_0(\mathbb{C}_+)$ and $\|L_b\|_{A_0} \leq \|b\|_1$. Moreover the mapping $b \mapsto L_b$ is a Banach algebra homomorphism from $L^1(\mathbb{R}^+)$ into $A_0(\mathbb{C}_+)$, and the space $\{ L_b : b \in L^1(\mathbb{R}^+) \}$ is dense in $A_0(\mathbb{C}_+)$. 
4. Functional calculus on $A_0$ and $A$

The goal of this section is to construct a bounded functional calculus for the negative generator of a bounded $C_0$-semigroup on Hilbert space, defined on $A(\mathbb{C}_+)$. 

4.1. Half-plane holomorphic functional calculus. We need some background on the half-plane holomorphic functional calculus introduced by Batty-Haase-Mubeen in [7], to which we refer for details.

Let $X$ be an arbitrary Banach space. Let $\omega \in \mathbb{R}$ and recall the definition of $H_\omega$ from (1.4). Let $A$ be a closed and densely defined operator on $X$ such that the spectrum of $A$ is included in the closed half-plane $\overline{H_\omega}$ and

$$\forall \alpha < \omega, \quad \sup\{\|R(z, A)\| : \text{Re}(z) \leq \alpha\} < \infty. \tag{4.1}$$

Consider the auxiliary algebra

$$E(H_\alpha) := \{\varphi \in H^\infty(H_\alpha) : \exists s > 0, |\varphi(z)| = O(|z|^{-(1+s)}) \text{ as } |z| \to \infty\},$$

for any $\alpha < \omega$. For any $\varphi \in E(H_\alpha)$ and for any $\beta \in (\alpha, \omega)$, the assumption (4.1) insures that the integral

$$\varphi(A) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\beta + is)R(\beta + is, A) \, ds \tag{4.2}$$

is absolutely convergent in $B(X)$. Further its value is independent of the choice of $\beta$ (this is due to Cauchy’s Theorem for vector-valued holomorphic functions) and the mapping $\varphi \mapsto \varphi(A)$ in an algebra homomorphism from $E(H_\alpha)$ into $B(X)$. This definition is compatible with the usual rational functional calculus; indeed for any $\mu \in \mathbb{C}$ with $\text{Re}(\mu) < \alpha$ and any integer $m \geq 2$, the function

$$e_{\mu,m} : z \mapsto (\mu - z)^{-m},$$

which belongs to $E(H_\alpha)$, satisfies $e_{\mu,m}(A) = R(\mu, A)^m$.

Let $\varphi \in H^\infty(H_\alpha)$. We can define a closed, densely defined, operator $\varphi(A)$ by regularisation, as follows (see [7] and [16] for more on such constructions). Take $\mu \in \mathbb{C}$ with $\text{Re}(\mu) < \alpha$, and set $e = e_{\mu,2}$. Then $e\varphi \in E(H_\alpha)$ and $e(A) = R(\mu, A)^2$ is injective. Then $\varphi(A)$ is defined by

$$\varphi(A) = e(A)^{-1}(e\varphi)(A) \tag{4.3}$$

with domain $\text{Dom}(\varphi(A))$ equal to the space of all $x \in X$ such that $[(e\varphi)(A)](x)$ belongs to the range of $e(A)$ ($= \text{Dom}(A^2)$). It turns out that this definition does not depend on the choice of $\mu$.

The half-plane holomorphic functional calculus $\varphi \mapsto \varphi(A)$ satisfies the following “Convergence Lemma”, provided by [7, Theorem 3.1].

**Lemma 4.1.** Assume that $A$ satisfies (4.1) for some $\omega \in \mathbb{R}$ and let $\alpha < \omega$. Let $(\varphi_i)_i$ be a net of $H^\infty(H_\alpha)$ such that $\varphi_i(A) \in B(X)$ for all $i$ and let $\varphi \in H^\infty(H_\alpha)$ such that $\varphi_i \to \varphi$ pointwise on $R_\alpha$, when $i \to \infty$. If

$$\sup_i \|\varphi_i\|_{H^\infty(R_\alpha)} < \infty \quad \text{and} \quad \sup_i \|\varphi_i(A)\|_{B(X)} < \infty,$$

then $\varphi(A) \in B(X)$ and for any $x \in X$, $[\varphi_i(A)](x) \to [\varphi(A)](x)$ when $i \to \infty$. 


Let \((T_t)_{t \geq 0}\) be a bounded \(C_0\)-semigroup on \(X\) and let \(-A\) denote its infinitesimal generator. For any \(b \in L^1(\mathbb{R}_+)\), we set
\[
\Gamma(A, b) := \int_0^\infty b(t) \, T_t \, dt,
\]
where the latter integral is defined in the strong sense. The mapping \(b \mapsto \Gamma(A, b)\) is the so-called Hille-Phillips functional calculus. We refer to [16, Section 3.3] for information and background. We recall that this functional calculus is a Banach algebra homomorphism
\[
L^1(\mathbb{R}_+) \rightarrow B(X).
\]

We now provide a compatibility result between the half-plane holomorphic functional calculus and the Hille-Phillips functional calculus. This kind of compatibility properties is very common in functional calculus (see e.g. [16, Section 3.3]) and we follow a classical approach. Note that any \(A\) as above satisfies \((4.1)\) for \(\omega = 0\). Thus for any \(\varepsilon > 0\), the operator \(A + \varepsilon\) satisfies \((4.1)\) for \(\omega = \varepsilon\). For any \(b \in L^1(\mathbb{R}_+)\), this allows us to define \(L_b(A + \varepsilon)\), where \(L_b\) is the Laplace transform defined by \((3.23)\).

**Lemma 4.2.** Let \(b \in L^1(\mathbb{R}_+)\) and let \(-A\) be the generator of a bounded \(C_0\)-semigroup on \(X\). Then for any \(\varepsilon > 0\), we have
\[
L_b(A + \varepsilon) = \Gamma(A + \varepsilon, b).
\]

**Proof.** Let \(\varepsilon > 0\), let \(\beta \in (0, \varepsilon)\), and let \(b \in L^1(\mathbb{R}_+)\).

First suppose that \(L_b \in \mathcal{E}(\mathcal{H}_0)\). Then by the Laplace formula,
\[
L_b(A + \varepsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} L_b(\beta + is) \left( \int_0^\infty e^{ist} e^{-\varepsilon t} T_t \, dt \right) ds.
\]
Since \(L_b \in \mathcal{E}(\mathcal{H}_0)\), the function \(s \mapsto L_b(\beta + is)\) belongs to \(L^1(\mathbb{R})\). Hence by Fubini’s theorem,
\[
L_b(A + \varepsilon) = \frac{1}{2\pi} \int_0^\infty e^{\beta t} \left( \int_{-\infty}^{\infty} L_b(\beta + is) e^{ist} ds \right) e^{-\varepsilon t} T_t \, dt.
\]
By the Fourier inversion Theorem, we have
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} L_b(\delta + is) e^{ist} ds = e^{-\beta t} b(t)
\]
for each \(t > 0\). We deduce that
\[
L_b(A + \varepsilon) = \int_0^\infty b(t) e^{-\varepsilon t} T_t \, dt = \Gamma(A + \varepsilon, b).
\]

For the general case, let us consider \(c \in \mathcal{E}(\mathcal{H}_0)\) defined by \(c(z) = (1 + z)^{-2}\). We note that \(c\) is the Laplace transform of the function \(c \in L^1(\mathbb{R}_+)\) defined by \(c(t) = 2e^{-t}\). The product \(eL_b\), which belongs to \(\mathcal{E}(\mathcal{H}_0)\), is therefore the Laplace transform of \(b \star c\). Hence by the first part of this proof,
\[
(eL_b)(A + \varepsilon) = \Gamma(A + \varepsilon, b \star c).
\]

The multiplicity of the Hille-Phillips functional calculus yields \(\Gamma(A + \varepsilon, b \star c) = \Gamma(A + \varepsilon, c) \Gamma(A + \varepsilon, b)\). Further \(e(A + \varepsilon) = \Gamma(A + \varepsilon, c)\) by the Laplace formula. Thus we have
\[
e(A + \varepsilon) \Gamma(A + \varepsilon, b) = (eL_b)(A + \varepsilon).
\]
Applying \((4.3)\), we obtain that \(L_b(A + \varepsilon) = \Gamma(A + \varepsilon, b)\) as required. \(\square\)
4.2. Functional calculus on $\mathcal{A}_0(\mathbb{C}_+)$. Throughout this subsection, we fix a Hilbert space $H$, we let $(T_t)_{t \geq 0}$ be a bounded $C_0$-semigroup on $H$ and we let $-A$ denote its infinitesimal generator. We set

$$C := \sup \{ \|T_t\| : t \geq 0 \}.$$  

For any $f \in C_{00}(\mathbb{R})$ and any $h \in H^1(\mathbb{R})$, the function

$$b = (2\pi)^{-1} \hat{f}$$

belongs to $L^1(\mathbb{R}^+)$ and we have $\hat{b}(-\cdot) = f \ast h$. Consequently,

$$(f \ast h)^\sim = L_b.$$  

Further we have the following key estimate, which is inspired by [30, Proposition 4.16].

**Lemma 4.3.** For any $f \in C_{00}(\mathbb{R})$ and any $h \in H^1(\mathbb{R})$,

$$\|\Gamma(A, (2\pi)^{-1} \hat{h})\| \leq C^2 \|f\|_\infty \|h\|_1.$$  

**Proof.** We fix $f \in C_{00}(\mathbb{R})$. Let $w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ and let $h = wv$. By definition,

$$\Gamma(A, (2\pi)^{-1} \hat{h}) = \frac{1}{2\pi} \int_0^\infty \hat{f}(t) \hat{wv}(t) T_t \, dt.$$  

By assumption, $\hat{w}$ and $\hat{v}$ belong to $L^1(\mathbb{R}^+)$ hence $\hat{wv} = (2\pi)^{-1} \hat{w} \ast \hat{v}$ belongs to $L^1(\mathbb{R}^+)$. Further $f$ belongs to $L^1(\mathbb{R})$. We can therefore apply Fubini’s Theorem and we obtain that

$$\frac{1}{2\pi} \int_0^\infty \hat{f}(t) \hat{wv}(t) T_t \, dt = \frac{1}{2\pi} \int_0^\infty f(s) \left( \int_0^\infty \hat{wv}(t) e^{-its} T_t \, dt \right) ds.$$  

Note that for any $s \in \mathbb{R}$,

$$\int_0^\infty \hat{wv}(t) e^{-its} T_t \, dt = \Gamma(A + is, \hat{wv}).$$  

According to the multiplicativity of the Hille-Phillips functional calculus, we have

$$\int_0^\infty \hat{wv}(t) e^{-its} T_t \, dt = \frac{1}{2\pi} \left( \int_0^\infty \hat{w}(r) e^{-irs} T_r \, dr \right) \left( \int_0^\infty \hat{v}(t) e^{-its} T_t \, dt \right).$$  

Let $W, V : \mathbb{R} \to B(H)$ be defined by

$$W(s) = \int_0^\infty \hat{w}(r) e^{-irs} T_r \, dr \quad \text{and} \quad V(s) = \int_0^\infty \hat{v}(t) e^{-its} T_t \, dt, \quad s \in \mathbb{R}.$$  

It follows from above that for any $x, x^* \in H$, we have

$$\langle \Gamma(A, (2\pi)^{-1} \hat{h}) x, x^* \rangle = \frac{1}{4\pi^2} \int_{-\infty}^\infty f(s) \langle W(s)x, V(s)^* x^* \rangle \, ds.$$  

Applying the Cauchy-Schwarz inequality, we deduce

$$|\langle \Gamma(A, (2\pi)^{-1} \hat{h}) x, x^* \rangle| \leq \frac{1}{4\pi^2} \|f\|_\infty \left( \int_{-\infty}^\infty \|W(s)x\|^2 \, ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^\infty \|V(s)^* x^*\|^2 \, ds \right)^{\frac{1}{2}}.$$  

According to the Fourier-Plancherel equality on $L^2(\mathbb{R}; H)$, we have

$$\int_{-\infty}^\infty \|W(s)x\|^2 \, ds = 2\pi \int_0^\infty |\hat{w}(r)|^2 \|T_r x\|^2 \, dr.$$
This implies
\[ \int_{-\infty}^{\infty} \|W(s)x\|^2 ds \leq 2\pi C^2 \int_{0}^{\infty} |\widehat{w}(r)|^2 \|x\|^2 dr = 4\pi^2 C^2 \|w\|^2 \|x\|^2. \]
Similarly, we have
\[ \int_{-\infty}^{\infty} \|V(s)x^*\|^2 ds \leq 4\pi^2 C^2 \|v\|^2 \|x^*\|^2. \]
Hence,
\[ |\langle \Gamma(A, (2\pi)^{-1}\widehat{f_h})x, x^* \rangle| \leq C^2 \|f\|_\infty \|w\|_2 \|v\|_2 \|x\| \|x^*\|. \]
Since this is true for all \( x, x^* \), we have proved that
\[ \|\Gamma(A, (2\pi)^{-1}\widehat{f_h})\| \leq C^2 \|f\|_\infty \|w\|_2 \|v\|_2. \]

Now let \( h \) be an arbitrary element of \( H^1(\mathbb{R}) \). As is well-known (see e.g. [12, Exercise 1, p. 84]), there exist \( w, v \in H^2(\mathbb{R}) \) such that \( h = wv \) and \( \|w\|^2_2 = \|v\|^2_2 = \|h\|_1 \).

Since \( \mathcal{F}(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R}) \), it follows from (2.1) that \( \mathcal{F}(H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})) = L^2(\mathbb{R}_+) \cap \mathcal{S}(\mathbb{R}) \). Since \( L^2(\mathbb{R}_+) \cap \mathcal{S}(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}_+) \), we readily deduce that \( H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}) \) is dense in \( H^2(\mathbb{R}) \).

Thus, there exist sequences \( (w_k)_{k \in \mathbb{N}}, (v_k)_{k \in \mathbb{N}} \) in \( H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}) \) such that \( w_k \to w \) and \( v_k \to v \) in \( H^2(\mathbb{R}) \), when \( k \to \infty \). This implies that
\[ \|w_k\|_2 \|v_k\|_2 \to \|h\|_1, \]
when \( k \to \infty \) and \( w_k v_k \to wv = h \) in \( H^1(\mathbb{R}) \), when \( k \to \infty \). Consequently,
\[ \|\Gamma(A, (2\pi)^{-1}\widehat{f_{w_kv_k}}) - \Gamma(A, (2\pi)^{-1}\widehat{f_h})\| \to 0 \]
when \( k \to \infty \). Indeed,
\[ \|\Gamma(A, (2\pi)^{-1}\widehat{f_{w_kv_k}}) - \Gamma(A, (2\pi)^{-1}\widehat{f_h})\| = \left\| \int_{0}^{\infty} \widehat{f(t)}(w_kv_k - wv)(t)T_t dt \right\| \]
\[ \leq C\|f\|_1 \|(w_kv_k - wv)\|_\infty \]
\[ \leq C\|f\|_1 \|w_kv_k - wv\|_1. \]

For all \( k \in \mathbb{N} \), we have
\[ \|\Gamma(A, (2\pi)^{-1}\widehat{f_{w_kv_k}})\| \leq C^2 \|f\|_\infty \|u_k\|_2 \|v_k\|_2, \]
by the first part of the proof. Passing to the limit, we obtain (4.3).

We now arrive at the main result of this subsection.

**Theorem 4.4.** There exists a unique bounded homomorphism \( \rho_{0,A} : \mathcal{A}_0(\mathbb{C}_+) \to B(H) \) such that
\[ \rho_{0,A}(L_b) = \int_{0}^{\infty} b(t)T_t dt \]
for all \( b \in L^1(\mathbb{R}_+) \). Moreover \( \|\rho_{0,A}\| \leq C^2. \)
Proof. By Lemma 4.3 and the density of $C_0(\mathbb{R})$ in $C_0(\mathbb{R})$, there exists a unique bounded bilinear map

$$u_A : C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow B(H)$$

such that $u_A(f, h) = \Gamma(A, (2\pi)^{-1}\hat{h})$ for each $(f, h) \in C_0(\mathbb{R}) \times H^1(\mathbb{R})$. Moreover $\|u_A\| \leq C^2$.

For each $\varepsilon > 0$, $(A + \varepsilon)$ is the negative generator of the semigroup $(e^{-\varepsilon t}T_t)_{t \geq 0}$. Therefore, in the same manner as above, one can define $u_{A+\varepsilon} : C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow B(H)$ and we have the uniform estimate

$$\forall \varepsilon > 0, \quad \|u_{A+\varepsilon} : C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow B(H)\| \leq C^2. \quad (4.7)$$

We claim that for each $\varepsilon > 0$, we have

$$u_{A+\varepsilon}(f, h) = (f \ast h)^\sim(A + \varepsilon), \quad f \in C_0(\mathbb{R}), \ h \in H^1(\mathbb{R}). \quad (4.8)$$

(We recall that the operator on the right-hand side is defined by the half-plane holomorphic functional calculus. In particular the above formula shows that $(f \ast h)^\sim(A + \varepsilon)$ is bounded.)

Recall that if $f \in C_0(\mathbb{R})$, then $b = (2\pi)^{-1}\hat{f} \in L^1(\mathbb{R}_+)$ and $(f \ast h)^\sim = L_b$. Hence (4.8) is given by Lemma 4.2 in this case. In the general case, let $(f_n)_{n \in \mathbb{N}}$ be a sequence of $C_0(\mathbb{R})$ such that $f_n \rightarrow f$ in $C_0(\mathbb{R})$, when $n \rightarrow \infty$. Then $u_{A+\varepsilon}(f_n, h) \rightarrow u_{A+\varepsilon}(f, h)$, hence $(f_n \ast h)^\sim(A + \varepsilon) \rightarrow u_{A+\varepsilon}(f, h)$. Moreover $(f_n \ast h)^\sim \rightarrow (f \ast h)^\sim$ in $H^\infty(C_+)$. Therefore by the Convergence Lemma (4.1) $(f \ast h)^\sim(A + \varepsilon)$ is bounded and (4.8) holds true.

Next we show that in $B(H)$ we have

$$u_{A+\varepsilon}(f, h) \xrightarrow{\varepsilon \rightarrow 0} u_A(f, h), \quad f \in C_0(\mathbb{R}), \ h \in H^1(\mathbb{R}). \quad (4.9)$$

In the case when $f \in C_0(\mathbb{R})$,

$$u_{A+\varepsilon}(f, h) = \frac{1}{2\pi} \int_0^\infty \hat{f}(t)\hat{h}(t)e^{-\varepsilon t}T_t \ dt$$

for all $\varepsilon \geq 0$. Hence

$$\|u_A(f, h) - u_{A+\varepsilon}(f, h)\| \leq \frac{C}{2\pi} \int_0^\infty |\hat{f}(t)\hat{h}(t)|(1 - e^{-\varepsilon t}) \ dt.$$

This integral tends to 0 when $\varepsilon \rightarrow 0$, by Lebesgue’s dominated convergence theorem. This yields the result in this case. The general case follows from the density of $C_0(\mathbb{R})$ in $C_0(\mathbb{R})$ and the uniform estimate (4.7).

We now construct $\rho_{0.A}$. Let $F \in \mathcal{A}_0(\mathbb{R})$ and consider two sequences $(f_k)_{k \in \mathbb{N}}$ of $C_0(\mathbb{R})$ and $(h_k)_{k \in \mathbb{N}}$ of $H^1(\mathbb{R})$ satisfying (3.1) and (3.2). We let

$$F_N = \sum_{k=1}^N f_k \ast h_k, \quad N \geq 1.$$ 

For any fixed $\varepsilon > 0$, it follows from (4.8) that for any $N \geq 1$,

$$\hat{F}_N(A + \varepsilon) = \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k).$$
We have both that \( \widetilde{F}_N \to \widetilde{F} \) in \( H^\infty(\mathbb{C}_+) \) and that \( \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) \to \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) \) in \( B(H) \). Appealing again to Lemma \ref{lem:4.11}, we deduce that \( \widetilde{F}(A + \varepsilon) \in B(H) \) and that

\[
\tag{4.10}
\widetilde{F}(A + \varepsilon) = \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k).
\]

We observe that

\[
\tag{4.11}
\sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) \xrightarrow{\varepsilon \to 0} \sum_{k=1}^\infty u_A(f_k, h_k)
\]

in \( B(H) \). To check this, let \( a > 0 \) and choose \( N \geq 1 \) such that \( \sum_{k=N+1}^\infty \| f_k \| \| h_k \|_1 \leq a \). We have

\[
\left\| \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\| \leq \left\| \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\|
\]

\[
+ \sum_{k=N+1}^\infty \| u_{A+\varepsilon}(f_k, h_k) \| + \sum_{k=N+1}^\infty \| u_A(f_k, h_k) \|.
\]

By the uniform estimate \( \ref{lem:4.11} \), this implies that

\[
\left\| \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\| \leq \left\| \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\| + 2Ca.
\]

Applying \( \ref{lem:4.9} \), we deduce that

\[
\left\| \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^\infty u_A(f_k, h_k) \right\| \leq 3Ca
\]

for \( \varepsilon > 0 \) small enough, which shows the result.

Combining \( \ref{lem:4.10} \) and \( \ref{lem:4.11} \) we obtain that \( \widetilde{F}(A + \varepsilon) \) has a limit in \( B(H) \), when \( \varepsilon \to 0 \). We set

\[
\rho_{0,A}(\widetilde{F}) := \lim_{\varepsilon \to 0} \widetilde{F}(A + \varepsilon).
\]

It is plain that \( \rho_{0,A} : \mathcal{A}_0(\mathbb{C}_+) \to B(H) \) is a linear map. It follows from the construction that

\[
\| \rho_{0,A}(\widetilde{F}) \|_{\mathcal{A}_0} \leq C^2 \| \widetilde{F} \|_{\mathcal{A}_0}
\]

for any \( F \in \mathcal{A}_0(\mathbb{R}) \), hence \( \rho_{0,A} \) is bounded with \( \| \rho_{0,A} \| \leq C^2 \).

Let \( b \in L^1(\mathbb{R}_+) \). By the compatibility Lemma \ref{lem:4.2}, we have

\[
L_b(A + \varepsilon) = \int_0^\infty b(t)e^{-\varepsilon t}T_t \, dt
\]

for all \( \varepsilon > 0 \). Passing to the limit and using Lebesgue’s dominated convergence theorem, we obtain \( \ref{lem:4.6} \).

It follows from the density of \( \{ L_b : b \in L^1(\mathbb{R}_+) \} \) in \( \mathcal{A}_0(\mathbb{C}_+) \), given by Lemma \ref{lem:3.14}, that \( \rho_{0,A} \) is unique. Moreover the multiplicativity of the Hille-Phillips functional calculus ensures that \( \rho_{0,A} \) is a Banach algebra homomorphism. \( \square \)
Remark 4.5. Let $F \in \mathcal{A}_0(\mathbb{R})$ and let $(f_k)_{k \in \mathbb{N}}$ and $(h_k)_{k \in \mathbb{N}}$ be sequences of $C_0(\mathbb{R})$ and $H^1(\mathbb{R})$, respectively, satisfying (3.1) and (3.2). It follows from the proof of Theorem 4.4 that

$$
(4.12) \quad \rho_{0,A}(F) = \sum_{k=1}^{\infty} u_A(f_k, h_k).
$$

This equality shows that the right-hand side of (4.12) does not depend on the choice of $(f_k)_{k \in \mathbb{N}}$ and $(h_k)_{k \in \mathbb{N}}$. The reason why we did not take (4.12) as a definition of $\rho_{0,A}$ is precisely that we did not know a priori that $\sum_{k=1}^{\infty} u_A(f_k, h_k)$ was independent of the representation of $F$.

4.3. Functional calculus on $\mathcal{A}(\mathbb{C}_+)$. We keep the notation from the previous subsection. We can extend Theorem 4.4 as follows.

Corollary 4.6. There exists a unique bounded homomorphism $\rho_A: \mathcal{A}(\mathbb{C}_+) \to B(H)$ extending $\rho_{0,A}$. Moreover $\|\rho_A\| \leq C^2$.

Proof. We follow an idea from [5], using regularization. Consider the sequence $(G_N)_{N \in \mathbb{N}}$ defined in the proof of Proposition 3.12. Then

$$
\widetilde{G}_N(z) = \frac{N}{N + z}, \quad z \in \mathbb{C}_+, \quad N \geq 1.
$$

For any $\varphi \in \mathcal{A}(\mathbb{C}_+)$, we let $S_\varphi$ be the operator defined by

$$
S_\varphi = (1 + A)\rho_{0,A}(\varphi \widetilde{G}_1),
$$

with domain $\text{Dom}(S_\varphi) = \{x \in H : [\rho_{0,A}(\varphi \widetilde{G}_1)](x) \in \text{Dom}(A)\}$. In this definition, we use the fact that $\varphi \widetilde{G}_1$ belongs to $\mathcal{A}_0(\mathbb{C}_+)$, which follows from Proposition 3.3. It is clear that $S_\varphi$ is closed. Further $\text{Dom}(A) \subset \text{Dom}(S_\varphi)$, hence $S_\varphi$ is densely defined. More precisely, if $x \in \text{Dom}(A)$, then $x = \rho_{0,A}(\varphi \widetilde{G}_1)(1 + A)(x)$ hence

$$
[\rho_{0,A}(\varphi \widetilde{G}_1)](x) = [\rho_{0,A}(\varphi \widetilde{G}_1^2)](1 + A)(x) = (1 + A)^{-1}[\rho_{0,A}(\varphi \widetilde{G}_1)](1 + A)(x)
$$

belongs to $\text{Dom}(A)$ and we have

$$
(4.13) \quad S_\varphi(x) = \rho_{0,A}(\varphi \widetilde{G}_1)(1 + A)(x).
$$

Since $\rho_{0,A}$ is multiplicative, we have $\rho_{0,A}(\varphi \widetilde{G}_N \varphi \widetilde{G}_1) = \rho_{0,A}(\varphi \widetilde{G}_N)(1 + A)^{-1}$ for any $N \geq 1$. Moreover as noticed in the proof of Proposition 3.12 $(G_N)_{N \in \mathbb{N}}$ is an approximate unit of $\mathcal{A}_0(\mathbb{R})$, hence $\varphi \widetilde{G}_N \varphi \widetilde{G}_1 \to \varphi \widetilde{G}_1$ in $\mathcal{A}_0(\mathbb{C}_+)$, when $N \to \infty$. We deduce, using (4.13), that for any $x \in \text{Dom}(A)$,

$$
S_\varphi(x) = \lim_{N} \rho_{0,A}(\varphi \widetilde{G}_N \varphi \widetilde{G}_1)(1 + A)(x) = \lim_{N} \rho_{0,A}(\varphi \widetilde{G}_N)(x).
$$

For any $N \geq 1$,

$$
\|\rho_{0,A}(\varphi \widetilde{G}_N)\| \leq C^2 \|\varphi \widetilde{G}_N\|_{\mathcal{A}_0} \leq C^2 \|\varphi\|_{\mathcal{A}},
$$

by (3.8). Consequently, $\|S_\varphi(x)\| \leq C^2 \|\varphi\|_{\mathcal{A}} \|x\|$ for any $x \in \text{Dom}(A)$. This shows that $\text{Dom}(S_\varphi) = H$ and $S_\varphi \in B(H)$.

We now define $\rho_A: \mathcal{A}(\mathbb{C}_+) \to B(H)$ by $\rho_A(\varphi) = S_\varphi$. It is clear from above that $\rho_A$ is linear and bounded, with $\|\rho_A\| \leq C^2$. It extends $\rho_{0,A}$ because if $F \in \mathcal{A}_0(\mathbb{C}_+)$, then we have $\rho_{0,A}(\varphi \widetilde{G}_1) = \rho_{0,A}(\widetilde{G}_1)\rho_{0,A}(\varphi) = (1 + A)^{-1}\rho_{0,A}(\varphi)$, hence $S_\varphi = \rho_{0,A}(\varphi)$.
Let \( \varphi_1, \varphi_2 \in A(\mathbb{C}_+) \). For any integers \( N_1, N_2 \geq 1 \), we have
\[
\rho_{0,A}(\varphi_1\varphi_2\widetilde{G}_{N_1}\widetilde{G}_{N_2}) = \rho_{0,A}(\varphi_1\widetilde{G}_{N_1})\rho_{0,A}(\varphi_2\widetilde{G}_{N_2}),
\]
because \( \rho_{0,A} \) is multiplicative. We deduce that \( \rho_{0,A}(\varphi_1\varphi_2\widetilde{G}_{N_1}) = \rho_{0,A}(\varphi_1\widetilde{G}_{N_1})\rho_A(\varphi_2) \) for all \( N_1 \geq 1 \), by letting \( N_2 \to \infty \). Next we obtain \( \rho_A(\varphi_1\varphi_2) = \rho_A(\varphi_1)\rho_A(\varphi_2) \) by letting \( N_1 \to \infty \). Thus \( \rho_A \) is multiplicative.

The uniqueness property is clear. \( \square \)

We will show in Remark 5.3 below that the functional calculus \( \rho_A \) from Corollary 4.6 is compatible with the Hille-Phillips functional calculus on \( M(\mathbb{R}_+) \).

4.4. Operators with a bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus. The goal of this subsection is to explain the connections between our main results (Theorem 4.4, Corollary 4.6) and \( H^\infty \)-functional calculus.

We will assume that the reader is familiar with sectorial operators and their \( H^\infty \)-functional calculus, for which we refer to [16] or [21, Chapter 10]. Using standard notation, for any \( \theta \in (0, \pi) \) we let \( \Sigma_\theta = \{ z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta \} \) and
\[
H^\infty(\Sigma_\theta) = \{ \varphi \in H^\infty(\Sigma_\theta) : \exists s > 0, |\varphi(z)| \leq \min \{|z|^s, |z|^{-s}| \} \text{ on } \Sigma_\theta \}.
\]

Let \( (T_t)_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on some Banach space \( X \), with generator \( -A \). Recall that \( A \) is a sectorial operator of type \( \frac{\pi}{2} \).

The following lemma is probably known to specialists, we include a proof for the sake of completeness. In part (i), the operator \( \varphi(A) \) is defined by \( \text{(4.2)} \) whereas in part (ii), the operator \( \varphi(A) \) is defined by \( \text{(4.3)} \). It is worth noting that if \( \varphi \in \mathcal{E}(\mathcal{H}_0) \cap H^\infty_0(\Sigma_\theta) \), then these two definitions coincide.

**Lemma 4.7.** The following assertions are equivalent.

(i) There exists a constant \( C > 0 \) such that for all \( \alpha < 0 \) and for all \( \varphi \in \mathcal{E}(\mathcal{H}_0) \),
\[
(4.14) \quad \| \varphi(A) \| \leq C \| \varphi \|_{H^\infty(\mathbb{C}_+)}.\]

(ii) There exists a constant \( C > 0 \) such that for all \( \theta \in (\frac{\pi}{2}, \pi) \) and for all \( \varphi \in H^\infty_0(\Sigma_\theta) \),
\[
(4.15) \quad \| \varphi(A) \| \leq C \| \varphi \|_{H^\infty(\mathbb{C}_+)}.\]

(iii) There exists a constant \( C > 0 \) such that for all \( b \in L^1(\mathbb{R}_+) \),
\[
\left\| \int_0^\infty b(t)T_t \, dt \right\| \leq C \| \widehat{b} \|_{\infty}.\]

**Proof.** Assume (i). By the approximation argument at the beginning of [7, Section 5], \( (4.14) \) holds as well for any \( \varphi \in H^\infty(\mathcal{H}_0) \). Let \( b \in L^1(\mathbb{R}_+) \). The function \( L_\varepsilon(\cdot + \varepsilon) \) belongs to \( H^\infty(\mathcal{H}_{-\varepsilon}) \) for any \( \varepsilon > 0 \), hence we have
\[
\| L_\varepsilon(A + \varepsilon) \| \leq C \| L_\varepsilon(\cdot + \varepsilon) \|_{H^\infty(\mathbb{C}_+)} \leq C \| L_\varepsilon \|_{H^\infty(\mathbb{C}_+)} = C \| \widehat{b} \|_{\infty}.
\]
Applying Lemma 4.2 and letting \( \varepsilon \to 0 \), we obtain \( (4.15) \), which proves (iii).

The fact that (iii) implies (ii) follows from [16, Lemma 3.3.1 & Proposition 3.3.2], see also [25, Lemma 2.12].
Assume (ii) and let us prove (i). For any \( \varepsilon \in (0, 1) \), consider the rational function \( q_\varepsilon \) defined by
\[
q_\varepsilon(z) = \frac{\varepsilon + z}{1 + \varepsilon z}, \quad z \neq \frac{-1}{\varepsilon}.
\]
We may and do assume that \( \alpha \in (-1, 0) \) when proving (i). Fix some \( \varepsilon \in (0, 1) \). It is easy to check (left to the reader) that \( q_\varepsilon \) maps \( H_\alpha \) into itself. Moreover there exists \( \theta \in \left( \frac{\pi}{2}, \pi \right) \) such that \( q_\varepsilon \) maps \( \Sigma_\theta \) into \( \mathbb{C}_+ \).
Let \( \varphi \in \mathcal{E}(H_\alpha) \), then
\[
\varphi_\varepsilon := \varphi \circ q_\varepsilon : H_\alpha \cup \Sigma_\theta \to \mathbb{C}
\]
is a well-defined bounded holomorphic function. Moreover we have
\[
(4.16) \quad \|\varphi_\varepsilon\|_{H^\infty(H_\alpha)} \leq \|\varphi\|_{H^\infty(H_\alpha)}.
\]
By \([16] \) Lemma 2.2.3, \( \varphi_\varepsilon \) belongs to \( H^\infty(\Sigma_\theta) \oplus \text{Span}\{1, (1+\cdot)^{-1}\} \). Further the definition of \( \varphi_\varepsilon(A) \) provided by the functional calculus of sectorial operators coincides with the definition of \( \varphi_\varepsilon(A) \) provided by the half-plane functional calculus. Hence for some constant \( C' > 0 \) not depending on \( \varepsilon \), we have
\[
\|\varphi_\varepsilon(A)\| \leq C'\|\varphi_\varepsilon\|_{H^\infty(\mathbb{C}_+)} \leq C'\|\varphi\|_{H^\infty(\mathbb{C}_+)},
\]
by (ii). Since \( \varphi_\varepsilon \to \varphi \) pointwise on \( H_\alpha \), it now follows from \((4.16)\) and the Convergence Lemma \([4] \) that \( \|\varphi(A)\| \leq C'\|\varphi\|_{H^\infty(\mathbb{C}_+)} \), which proves (i). \( \square \)

We say that \( A \) admits a bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus if one of (equivalently, all of) the properties of Lemma \([4,7] \) hold true. If \( A \) is sectorial of type \( < \frac{\pi}{2} \), the latter is equivalent to \( A \) having a bounded \( H^\infty \)-functional calculus of angle \( \frac{\pi}{2} \) is the usual sense. The main feature of the “bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus” property considered here is that it may apply to the case when the sectorial type of \( A \) is not \( < \frac{\pi}{2} \).

We now come back to the specific case when \( X = H \) is a Hilbert space. Here are a few known facts in this setting:

(f1) If \( (T_t)_{t \geq 0} \) is a contractive semigroup (that is, \( \|T_t\| \leq 1 \) for all \( t \geq 0 \)), then \( A \) admits a bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus. See \([16] \) Section 7.1.3 for a proof and more on this theme.

(f2) We say that \( (T_t)_{t \geq 0} \) is similar to a contractive semigroup if there exists an invertible operator \( S \in B(H) \) such that \( (ST_tS^{-1})_{t \geq 0} \) is a contractive semigroup. A straightforward application of the previous result is that in this case, \( A \) admits a bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus.

(f3) If \( A \) is sectorial of type \( < \frac{\pi}{2} \), then \( A \) admits a bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus (if and only if \( (T_t)_{t \geq 0} \) is similar to a contractive semigroup. This goes back to \([24] \) Section 4).

(f4) There exist sectorial operators of type \( < \frac{\pi}{2} \) which do not admit a bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus, by \([27,4] \) (see also \([16] \) Section 7.3.4)).

(f5) There exists a bounded \( C_0 \)-semigroup \( (T_t)_{t \geq 0} \) such that \( A \) admits a bounded \( H^\infty(\mathbb{C}_+) \)-functional calculus but \( (T_t)_{t \geq 0} \) is not similar to a contractive semigroup. This follows from \([24] \) Proposition 4.8] and its proof.
We now establish analogues of Theorem 4.4 and Corollary 4.6 in the case when $A$ admits a bounded $H^\infty(\mathbb{C}_+)$-functional calculus. Just as we did in Subsection 3.4 we set
\[
C_0(\mathbb{C}_+) = \{\widehat{F} : F \in C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})\} \quad \text{and} \quad \mathcal{C}(\mathbb{C}_+) = \{\widehat{F} : F \in C_b(\mathbb{R}) \cap H^\infty(\mathbb{R})\}.
\] Since $\{\widehat{b}(- \cdot) : b \in L^1(\mathbb{R}_+)\}$ is dense in $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$, by Remark 3.13 the following is straightforward.

**Proposition 4.8.** Assume that $A$ admits a bounded $H^\infty(\mathbb{C}_+)$-functional calculus on $H$. Then there exists a unique bounded homomorphism $\nu_{0,A} : C_0(\mathbb{C}_+) \to B(H)$ such that
\[
(4.17) \quad \nu_{0,A}(L_b) = \int_0^\infty b(t)T_t \, dt
\]
for all $b \in L^1(\mathbb{R}_+)$. Now arguing as in the proof of Corollary 4.6 we deduce the following.

**Corollary 4.9.** Assume that $A$ admits a bounded $H^\infty(\mathbb{C}_+)$-functional calculus on $H$. Then there exists a unique bounded homomorphism $\nu_A : \mathcal{C}(\mathbb{C}_+) \to B(H)$ such that (4.17) holds true for all $b \in L^1(\mathbb{R}_+)$. Of course when the above corollary applies, $\nu_A$ is an extension of the mapping $\rho_A$ from Corollary 4.6. Thus our main results (Theorem 4.4, Corollary 4.6) should be regarded as a way to obtain a “good” functional calculus for negative generators of bounded $C_0$-semigroups which do not admit a bounded $H^\infty(\mathbb{C}_+)$-functional calculus.

4.5. **Note added in May 2022.** A first version of this paper has circulated since the beginning of 2021. A few months later, together with Safoura Zadeh, we proved in [3, Section 4] that the inclusion $k(3.19)$ is actually an equality. Equivalently (see Theorem 3.7 above), we have
\[
A_0(\mathbb{R})^* \simeq \mathcal{M}(H^1(\mathbb{R})).
\] The paper [3] also contains a new proof of Theorem 4.4 based on a description of the so-called $S^1$-bounded Fourier multipliers on $H^1(\mathbb{R})$ and on a tensor product estimate of independent interest, inspired by an old result of White [37, Section 5].

5. **Comparison with the Besov functional calculus**

In this section we compare the functional calculus constructed in Section 4 (Theorem 4.4 and Corollary 4.6) with the Besov functional calculus from [17, Subsection 5.5] and [5]. We start with some background on the analytic homogeneous Besov space used in the latter paper. We refer to [5, Section 6] for further details.

Let $\psi \in \mathcal{S}(\mathbb{R})$ such that $\text{Supp}(\psi) \subset [\frac{1}{2}, 2]$, $\psi(t) \geq 0$ for all $t \in \mathbb{R}$, and $\psi(t) + \psi(\frac{t}{2}) = 1$ for all $t \in [1, 2]$. For any $k \in \mathbb{Z}$, we let $\psi_k \in \mathcal{S}(\mathbb{R})$ be defined by $\psi_k(t) = \psi(2^{-k}t)$, $t \in \mathbb{R}$. A key property of the sequence $(\psi_k)_{k \in \mathbb{Z}}$ is that for any $k_0 \in \mathbb{Z}$, we have
\[
(5.1) \quad \forall t \in [2^{k_0}, 2^{k_0+1}) : \quad \psi_{k_0}(t) + \psi_{k_0+1}(t) = 1 \quad \text{and} \quad \psi_k(t) = 0 \text{ if } k \notin \{k_0, k_0 + 1\}.
\]
Next define $\phi_k = F^{-1}(\psi_k)$. It is plain that for any $k \in \mathbb{Z}$,
\[
(5.2) \quad \phi_k \in H^1(\mathbb{R}) \quad \text{and} \quad \|\phi_k\|_1 = \|\phi_0\|_1.
\]
It follows that for any \( F \in BUC(\mathbb{R}) \) and any \( k \in \mathbb{Z} \), \( F \ast \phi_k \) belongs to \( BUC(\mathbb{R}) \cap H^\infty(\mathbb{R}) \). We define a Besov space \( \mathcal{B}_0(\mathbb{R}) \) by
\[
\mathcal{B}_0(\mathbb{R}) = \left\{ F \in BUC(\mathbb{R}) : \sum_{k \in \mathbb{Z}} \| F \ast \phi_k \|_\infty < \infty \text{ and } F = \sum_{k \in \mathbb{Z}} F \ast \phi_k \right\}.
\]
This is a Banach space for the norm \( \| F \|_{\mathcal{B}_0} = \sum_{k \in \mathbb{Z}} \| F \ast \phi_k \|_\infty \).

This space is denoted by \( \mathcal{B}_{dyad} \) in [5, Section 6].

Next we set \( \mathcal{B}_{00}(\mathbb{R}) = \mathcal{B}(\mathbb{R}) \cap C_0(\mathbb{R}) \), equipped with the norm of \( \mathcal{B}_0(\mathbb{R}) \). Then \( \mathcal{B}_{00}(\mathbb{R}) \) is a closed subspace of \( \mathcal{B}(\mathbb{R}) \) and we clearly have \( \mathcal{B}_{00}(\mathbb{R}) \subset C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}) \) and \( \mathcal{B}_0(\mathbb{R}) \subset BUC(\mathbb{R}) \cap H^\infty(\mathbb{R}) \).

We wish to underline that the above definitions of \( \mathcal{B}_0(\mathbb{R}) \) and \( \mathcal{B}_{00}(\mathbb{R}) \) do not depend on the choice of the function \( \psi \). More precisely if \( \psi^{(1)}, \psi^{(2)} \) are two functions as above and if we let \( \mathcal{B}_0^{\psi^{(1)}}(\mathbb{R}) \) and \( \mathcal{B}_0^{\psi^{(2)}}(\mathbb{R}) \) denote the associated spaces, then \( \mathcal{B}_0^{\psi^{(1)}}(\mathbb{R}) \) and \( \mathcal{B}_0^{\psi^{(2)}}(\mathbb{R}) \) coincide as vector spaces and the norms \( \| \cdot \|_{\mathcal{B}_0^{\psi^{(1)}}} \) and \( \| \cdot \|_{\mathcal{B}_0^{\psi^{(2)}}} \) are equivalent. We refer to [5, Section 6] and the references therein for these properties.

Similarly to Subsection 3.4, we introduce half-plane versions of \( \mathcal{B}_0(\mathbb{R}) \) and \( \mathcal{B}_{00}(\mathbb{R}) \), by setting
\[
\mathcal{B}_{00}(\mathbb{C}_+) = \{ \tilde{F} : F \in \mathcal{B}_{00}(\mathbb{R}) \} \quad \text{and} \quad \mathcal{B}_0(\mathbb{C}_+) = \{ \tilde{F} : F \in \mathcal{B}(\mathbb{R}) \}.
\]
According to [5, Proposition 6.2], the space \( \mathcal{B}_0(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+) \) coincides with the space \( \mathcal{B}_0 \) considered by Batty-Gomilko-Tomilov in [5, Subsection 2.2]. By [5, Subsection 2.4], we have
\[
\left\{ L_b : b \in L^1(\mathbb{R}_+) \right\} \subset \mathcal{B}_{00}(\mathbb{C}_+).
\]

Moreover Batty-Gomilko-Tomilov established the following remarkable functional calculus result.

**Theorem 5.1.** ([5, Theorem 4.4], [6, Theorem 6.1]) Let \( X \) be a Banach space, let \( (T_t)_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on \( X \) and let \( -A \) denote its generator. The following are equivalent.

(i) There exists a constant \( K > 0 \) such that
\[
\int_{-\infty}^{\infty} \langle R(\beta + it, A)^2(x), x^* \rangle \, dt \leq \frac{-K}{\beta} \| x \| \| x^* \|
\]
for all \( \beta < 0 \), all \( x \in X \) and all \( x^* \in X^* \).

(ii) There exists a bounded homomorphism \( \gamma_A : \mathcal{B}_0(\mathbb{C}_+) \to B(X) \) such that
\[
\gamma_A(L_b) = \int_0^\infty b(t)T_t \, dt, \quad b \in L^1(\mathbb{R}_+).
\]

In this case, \( \gamma_A \) is unique.
Condition (i) in Theorem 5.1 goes back at least to [14] and [32]. In fact, condition (i) can be defined for any closed and densely defined operator $A$ satisfying (4.1) for $\omega = 0$. Then it follows from [14, 32] that (i) actually implies that $-A$ generates a bounded $C_0$-semigroup on $X$. (See also [7, Theorem 6.4].) Conversely, if $X = H$ is a Hilbert space, it is proved in [14, 32] that if $-A$ generates a bounded $C_0$-semigroup, then $A$ satisfies (i). (The assumption that $X = H$ is a Hilbert space is crucial here, see the beginning of Section 6 for more on this.)

Thus if $(T_t)_{t \geq 0}$ is a bounded $C_0$-semigroup with generator $-A$ on Hilbert space, then the property (ii) in Theorem 5.1 holds true. It is therefore natural to compare Corollary 4.6 with that property. This is the aim of the rest of this section.

**Proposition 5.2.** We have

$$B_0(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad B_{00}(\mathbb{C}_+) \subset \mathcal{A}_0(\mathbb{C}_+).$$

Moreover there exists a constant $K > 0$ such that $\| \varphi \|_A \leq K \| \varphi \|_{B_0}$ for any $\varphi \in B_0(\mathbb{C}_+)$.  

**Proof.** It follows from (5.1) that

$$\psi_k = \psi_k(\psi_{k-1} + \psi_k + \psi_{k+1}), \quad k \in \mathbb{Z}.$$ 

Consequently,

$$\phi_k = \phi_k \ast (\phi_{k-1} + \phi_k + \phi_{k+1}), \quad k \in \mathbb{Z}.$$ 

Let $F \in B(\mathbb{R})$. Applying the above identity, we have

$$F = \sum_{k \in \mathbb{Z}} F \ast \phi_k \ast (\phi_{k-1} + \phi_k + \phi_{k+1}).$$

Appealing to (5.2), we observe that $F \ast \phi_k \in BUC(\mathbb{R})$ and $\phi_{k-1} + \phi_k + \phi_{k+1} \in H^1(\mathbb{R})$ for each $k \in \mathbb{Z}$, and that

$$\sum_{k \in \mathbb{Z}} \|F \ast \phi_k\|_\infty \|\phi_{k-1} + \phi_k + \phi_{k+1}\|_1 \leq 3 \|\phi_0\|_1 \|F\|_{B_0}.$$ 

This shows that $F \in \mathcal{A}(\mathbb{R})$, with

$$\|F\|_A \leq 3 \|\phi_0\|_1 \|F\|_{B_0}.$$ 

This yields $B(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+)$. The above argument also shows that $B_0(\mathbb{C}_+) \subset \mathcal{A}_0(\mathbb{C}_+)$. \qed

Let $H$ be a Hilbert space and let $A$ be the negative generator of a bounded $C_0$-semigroup on $H$. We already noticed that $A$ satisfies property (ii) in Theorem 5.1. According to Proposition 5.2 and (4.6), the functional calculus $\rho_A: \mathcal{A}(\mathbb{C}_+) \to B(H)$ from Corollary 4.6 extends the functional calculus $\gamma_A: B(\mathbb{C}_+) \to B(H)$.

It turns out that the extension from $\gamma_A$ to $\rho_A$ is an actual improvement, because of the following result.

**Theorem 5.3.** We have

$$B_0(\mathbb{C}_+) \neq \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad B_{00}(\mathbb{C}_+) \neq \mathcal{A}_0(\mathbb{C}_+).$$
We need some preparation before coming to the proof. We use an idea from [34, Paragraph 2.6.4]. First for the definition of the Besov space $B_0(\mathbb{R})$, we make the additional assumption that $\psi(t) = 1$ for any $t \in \left[\frac{1}{2}, 1\right]$. This is allowed by the aforementioned fact that the definition of $B_0(\mathbb{R})$ does not depend on $\psi$. This implies that $\text{Supp}(\psi) \subset \left[\frac{1}{2}, \frac{3}{2}\right]$. Second we fix a non-zero function $f_0 \in \mathcal{S}(\mathbb{R})$ such that $\text{Supp}(f_0) \subset \left[\frac{3}{4}, 1\right]$. Next for any integer $n \geq 0$, we set $N_n = 2^n - 1$ and $f_n = \tau_{N_n} f_0 = f_0(\cdot - N_n)$. By construction, $\text{Supp}(\psi_k) \subset [2^{k-1}, \frac{3}{2} 2^k]$ for all $k \in \mathbb{Z}$ and $\text{Supp}(f_n) \subset [2^n - \frac{1}{4}, 2^n]$ for all $n \geq 0$. We derive that

\begin{equation}
\forall k \geq 0 : \quad f_k \psi_k = f_k \quad \text{and} \quad f_n \psi_k = 0 \quad \text{if} \quad n \neq k,
\end{equation}

as well as

\begin{equation}
\forall n, n' \geq 0 : \quad f_n f_{n'} = 0 \quad \text{if} \quad n \neq n'.
\end{equation}

**Lemma 5.4.** There exists a bounded continuous function $m : \mathbb{R}_+^* \to \mathbb{C}$ such that

\begin{equation}
\sup_{k \in \mathbb{Z}} \| \hat{m} \psi_k \|_1 < \infty
\end{equation}

and the mapping $T_m : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ does not belong to $\mathcal{M}(H^1(\mathbb{R}))$.

**Proof.** Using the definitions preceding the lemma, we set

$$m(t) = \sum_{n=0}^{\infty} e^{iN_n t} f_n(t), \quad t > 0.$$ 

At most one term is non zero in this sum, hence this is well-defined and $m \in C_b(\mathbb{R}_+^*)$. Let $k \geq 0$. According to (5.5), we have $m \psi_k = e^{iN_k} \cdot f_k$ hence

$$\| \hat{m \psi_k} \|_1 = \| \hat{f}_k(\cdot - N_k) \|_1 = \| \hat{f}_k \|_1 = \| f_0 \|_1.$$ 

Since $m \psi_k = 0$ if $k < 0$, this shows (5.7).

Define

$$g_N = \mathcal{F}^{-1} \left( \sum_{n=0}^{N} e^{-iN_n \cdot f_n} \right)$$

for all $N \geq 0$. Then $g_N \in \mathcal{S}(\mathbb{R}) \cap H^1(\mathbb{R})$ hence $g_N \in H^p(\mathbb{R})$ for any $1 \leq p \leq \infty$. Let us estimate its $L^p$-norm. On the one hand, we have

$$\| g_N \|_1 \leq \sum_{n=0}^{N} \| \mathcal{F}^{-1}(e^{-iN_n \cdot f_n}) \|_1 = \sum_{n=0}^{N} \| [\mathcal{F}^{-1}(f_n)](\cdot - N_n) \|_1 = \sum_{n=0}^{N} \| \mathcal{F}^{-1}(f_n) \|_1,$$

hence

$$\| g_N \|_1 \leq (N + 1) \| \mathcal{F}^{-1}(f_0) \|_1.$$ 

On the other hand, for any $t \in \mathbb{R}$, we have

$$g_N(t) = \sum_{n=0}^{N} \mathcal{F}^{-1}(f_n)(t - N_n),$$
hence
\[ |g_N(t)| \leq \sum_{n=0}^{N} |g_0(t - N_n)|. \]

Since \( g_0 = \mathcal{F}^{-1}(f_0) \in \mathcal{S}(\mathbb{R}) \) we infer that
\[ \sup_{N \geq 0} \|g_N\|_{\infty} < \infty. \]

For any \( 1 < p < \infty \), we have \( \|g_N\|_p \leq \|g_N\|_1^{\frac{1}{p}} \|g_N\|_{\infty}^{1 - \frac{1}{p}} \), hence the above estimates imply the existence of a constant \( K > 0 \) such that
\[ (5.8) \quad \|g_N\|_p \leq KN^\frac{1}{p}, \quad N \geq 1. \]

By (5.6), we have
\[ m\hat{g}_N = \left( \sum_{n=0}^{\infty} e^{iN_n \cdot f_n} \right) \left( \sum_{n=0}^{N} e^{-iN_n \cdot f_n} \right) = \sum_{n=0}^{N} f_n^2. \]

For any \( n \geq 0 \), \( \text{Supp}(f_n^2) \subset [2^{n-1}, 2^n] \) hence by \[15\], Theorem 5.1.5., we have an estimate
\[ \|\mathcal{F}^{-1}(m\hat{g}_N)\|_p \approx \left( \sum_{n=0}^{N} |\mathcal{F}^{-1}(f_n^2)|^2 \right)^{\frac{1}{2}}. \]

Further, \( f_n^2 = f_0^2(\cdot - N_n) \) hence \( |\mathcal{F}^{-1}(f_n^2)| = |\mathcal{F}^{-1}(f_0^2)| \) for any \( n \geq 0 \). Consequently,
\[ \left( \sum_{n=0}^{N} |\mathcal{F}^{-1}(f_n^2)|^2 \right)^{\frac{1}{2}} = (N + 1)^{\frac{1}{2}} |\mathcal{F}^{-1}(f_0^2)|. \]

Thus we have
\[ \|\mathcal{F}^{-1}(m\hat{g}_N)\|_p \approx N^{\frac{1}{2}}. \]

Comparing with (5.8) we deduce that if \( 2 < p < \infty \), then \( T_m : H^2(\mathbb{R}) \to H^2(\mathbb{R}) \) is not a bounded Fourier multiplier on \( H^p(\mathbb{R}) \). By Lemma 2.5 we deduce that \( T_m \notin \mathcal{M}(H^1(\mathbb{R})) \). \( \square \)

**Proof of Theorem 5.3** If \( \mathcal{A}(\mathbb{C}_+) \) were equal to \( \mathcal{B}_0(\mathbb{C}_+) \), we would have \( \mathcal{A}_0(\mathbb{C}_+) = \mathcal{B}_{00}(\mathbb{C}_+) \) which in turn is equivalent to \( \mathcal{A}_0(\mathbb{R}) = \mathcal{B}_{00}(\mathbb{R}) \). So it suffices to show that this equality fails. Let us assume, by contradiction, that \( \mathcal{A}_0(\mathbb{R}) = \mathcal{B}_{00}(\mathbb{R}) \).

Let \( m \) be given by Lemma 5.4. Let \( b \in L^1(\mathbb{R}_+) \). For any \( k \in \mathbb{Z} \), we have
\[ \psi_k b = \frac{1}{2\pi} \mathcal{F}(\phi_k \ast \hat{b}(\cdot)). \]
Hence using (5.1), (5.4) and Lemma 1.1, we have
\[ \int_{-\infty}^{\infty} m(t)b(t) \, dt = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} m(t)\psi_k(t)b(t) \, dt \]
\[ = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} (\psi_{k-1}(t) + \psi_k(t) + \psi_{k+1}(t))m(t)\psi_k(t)b(t) \, dt \]
\[ = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathcal{F}(\psi_{k-1} + \psi_k + \psi_{k+1})m(u)\mathcal{F}(\psi_kb)(-u) \, du \]
\[ = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathcal{F}(\psi_{k-1} + \psi_k + \psi_{k+1})m(u)[\phi_k \ast \hat{b}(-\cdot)](u) \, du \]
Therefore,
\[ \left| \int_{-\infty}^{\infty} m(t)b(t) \, dt \right| \leq \sum_{k \in \mathbb{Z}} \left\| \mathcal{F}(\psi_{k-1} + \psi_k + \psi_{k+1})m \right\|_1 \left\| \phi_k \ast \hat{b}(-\cdot) \right\|_\infty. \]
Applying (5.7), we deduce the existence of a constant \( K > 0 \) such that
\[ \left| \int_{-\infty}^{\infty} m(t)b(t) \, dt \right| \leq K \sum_{k \in \mathbb{Z}} \left\| \phi_k \ast \hat{b}(-\cdot) \right\|_\infty = K \left\| \hat{b}(-\cdot) \right\|_{\mathcal{B}_0}. \]
Therefore there exists \( \eta \in \mathcal{B}_{00}(\mathbb{R})^* \) such that
\[ \langle \eta, \hat{b}(-\cdot) \rangle = \int_{-\infty}^{\infty} m(t)b(t) \, dt, \quad b \in L^1(\mathbb{R}_+). \]
By assumption, \( \eta \in \mathcal{A}_0(\mathbb{R})^* \). Applying Theorem 3.7, let \( T \in \mathcal{M}(H^1(\mathbb{R})) \) be associated to \( \eta \) and let \( m_0 \in C_b(\mathbb{R}_+^*) \) be the symbol of \( T \). Then by Remark 3.10 we have
\[ \langle \eta, \hat{b}(-\cdot) \rangle = \int_{-\infty}^{\infty} m_0(t)b(t) \, dt, \quad b \in L^1(\mathbb{R}_+). \]
We deduce that \( m_0 = m \), and this contradicts the fact that \( T_m \notin \mathcal{M}(H^1(\mathbb{R})) \). \( \square \)

We conclude this section with a series of remarks.

Remark 5.5. Let \( A \) be as in Subsections 1.2 and 1.3. Let \( \mu \in M(\mathbb{R}_+), \) with \( \mu(\{0\}) = 0. \) According to [5, Subsection 2.2 & Proposition 6.2], its Laplace transform \( L_\mu : \mathbb{C}_+ \to \mathbb{C} \) belongs to \( \mathcal{B}_0(\mathbb{C}_+) \). Hence \( L_\mu \) belongs to \( \mathcal{A}(\mathbb{C}_+) \), by Proposition 5.2. The argument in the proof of Corollary 1.6 shows that \( \rho_A(L_\mu) \) is the strong limit of \( \rho_{0,A}(L_\mu \hat{G}_N) \), when \( N \to \infty \). Define \( c_N(t) = Ne^{-Nt} \) for any \( t > 0 \) and recall that \( \hat{G}_N = L_{c_N} \). Then \( L_\mu \hat{G}_N = L_{\mu \ast c_N} \) for any \( N \geq 1 \). Further \( \mu \ast c_N \to \mu \) narrowly, when \( N \to \infty \). It therefore follows from (1.6) that
\[ [\rho_A(L_\mu)](x) = \int_{\mathbb{R}_+} T_t(x) \, d\mu(t), \quad x \in H. \]

Remark 5.6. Let \( \mathcal{D} \subset H^1(\mathbb{R}) \) be the space of all \( h \in H^1(\mathbb{R}) \) such that \( \text{Supp}(\hat{h}) \) is a compact subset of \( \mathbb{R}_+^* \). It is well-known that \( \mathcal{D} \) is dense in \( H^1(\mathbb{R}) \). To check this, take any \( h \in H^1(\mathbb{R}) \) and recall that there exist \( v, w \in H^2(\mathbb{R}) \) such that \( h = vw \). Let \( (d_n)_{n \in \mathbb{N}} \) and \( (c_n)_{n \in \mathbb{N}} \) be
sequences of $C_b(\mathbb{R}_+^*)$ with compact supports such that $d_n \to \hat{w}$ and $c_n \to \hat{v}$ in $L^2(\mathbb{R}_+).$ Then $\mathcal{F}^{-1}(d_n) \to w$ and $\mathcal{F}^{-1}(c_n) \to v$ in $L^2(\mathbb{R}),$ hence $\mathcal{F}^{-1}(d_n)\mathcal{F}^{-1}(c_n) \to h$ in $H^1(\mathbb{R}).$ Now it is easy to see that $\mathcal{F}^{-1}(d_n)\mathcal{F}^{-1}(c_n)$ belongs to $\mathcal{D}$ for any $n \in \mathbb{N}.$

Let $BUC \ast \mathcal{D} \subset \mathcal{A}(\mathbb{R})$ be the linear span of the functions $f \ast h,$ for $f \in BUC(\mathbb{R})$ and $h \in \mathcal{D}.$ It follows from above that this is a dense subspace of $\mathcal{A}(\mathbb{R}).$

Let $\mathcal{G} \subset H^\infty(\mathbb{R})$ be the space of all $F \in H^\infty(\mathbb{R})$ such that $\text{Supp}(\hat{F})$ is a compact subset of $\mathbb{R}_+^*.$ Then we have

$$BUC \ast \mathcal{D} \subset \mathcal{G} \subset B_0(\mathbb{R}).$$

The first inclusion is obvious and the second one is given by [5, Lemma 2.4].

It follows that $B_0(\mathbb{R})$ is dense in $\mathcal{A}(\mathbb{R}),$ or equivalently that $B_0(\mathbb{C}^+)$ is dense in $\mathcal{A}(\mathbb{C}^+).$

Also $B_{00}(\mathbb{C}^+)$ is dense in $\mathcal{A}_{00}(\mathbb{C}^+),$ by (5.3) and Lemma 3.14.

Remark 5.7. It follows from [5, Subsection 2.2] that for any $\varphi \in B_0(\mathbb{C}^+),$ $\lim_{y \to \infty} \varphi(y) = 0,$ where the limit is taken for $y$ going to $\infty$ along the real axis. We noticed in Remark 5.6 that $B_0(\mathbb{C}^+)$ is dense in $\mathcal{A}(\mathbb{C}^+).$ Since $\| \cdot \|_{H^\infty(\mathbb{C}^\ast)} \leq \| \cdot \|_{\mathcal{A}(\mathbb{C}^\ast)},$ this implies that any element of $\mathcal{A}(\mathbb{C}^\ast)$ is the uniform limit of a sequence of $B_0(\mathbb{C}^\ast).$ Consequently, $\lim_{y \to \infty} \varphi(y) = 0$ for any $\varphi \in \mathcal{A}(\mathbb{C}^\ast).$

Thus the algebra $\mathcal{A}(\mathbb{C}^\ast)$ (equivalently, the algebra $\mathcal{A}(\mathbb{R}))$ does not contain any non-zero constant function and hence is not unital.

Remark 5.8. The following problem is open: Let $-A$ be the generator of a bounded $C_0$-semigroup on a Hilbert space. Is the Cayley transform $V = (A - I_H)(A + I_H)^{-1}$ power bounded? This problem is discussed in [5, Section 5.5], to which we refer for information.

Let $v \in H^\infty(\mathbb{C}^\ast)$ be defined by $v(z) = (z - 1)(z + 1)^{-1}.$ It follows from Remark 3.13 that for any integer $n \geq 1,$ the function $\varphi_n : \mathbb{C}^\ast \to \mathbb{C}$ defined by $\varphi_n(z) = v(z)^n - (-1)^n$ belongs to $\mathcal{A}_0(\mathbb{C}^\ast)$ and

$$V^n = (-1)^n I_H + \rho_{\mathcal{A}}, \varphi_n).$$

Hence it would be interesting to determine the behaviour of $\| \varphi_n \|_{\mathcal{A}_0}.$ It is shown in [6, Section 5.1] that $\| \varphi_n \|_{B_0} \approx \log(n).$ We do not know if the asymptotic behaviour of $\| \varphi_n \|_{\mathcal{A}_0}$ differs from the one of $\| \varphi_n \|_{B_0}.$

6. $\gamma$-Bounded semigroups on Banach spaces

In general, Theorem 4.4 and Corollary 4.6 do not hold true if $H$ is replaced by an arbitrary Banach space. Indeed it is shown in [6, Corollary 6.7] that the translation semigroup $(T_t)_{t \geq 0}$ on $L^p(\mathbb{R}),$ for $1 \leq p \neq 2 < \infty,$ does not satisfy condition (i) in Theorem 5.1. Hence by the latter theorem and Proposition 5.2, the mapping

$$L_b \mapsto \int_0^\infty b(t)T_t dt, \quad b \in L^1(\mathbb{R}_+),$$

is not bounded with respect to the $\mathcal{A}_0(\mathbb{C}^\ast)$-norm.

In this section we will however establish Banach space versions of Theorem 4.4 and Corollary 4.6 on Banach spaces, involving $\gamma$-boundedness. We start with some background and basic facts on this topic and refer to [21, Chapter 9] for details and more information.
Let $X$ be a Banach space. Let $(\gamma_n)_{n \geq 1}$ be a sequence of independent complex valued standard Gaussian variables on some probability space $\Sigma$ and let $G_0 \subset L^2(\Sigma)$ be the linear span of the $\gamma_n$. We denote by $G(X)$ the closure of

$$G_0 \otimes X = \left\{ \sum_{k=1}^{N} \gamma_k \otimes x_k : x_k \in X, N \in \mathbb{N} \right\}$$

in the Bochner space $L^2(\Sigma; X)$, equipped with the induced norm. Next we let $G'(X^*)$ denote the closure of $G_0 \otimes X^*$ in the dual space $G(X)^*$.

A bounded set $T \subset B(X)$ is called $\gamma$-bounded if there exists a constant $C \geq 0$ such that for all finite sequences $(S_k)_{k=1}^{N} \subset T$ and $(x_k)_{k=1}^{N} \subset X$, we have:

$$(6.1) \quad \left\| \sum_{k=1}^{N} \gamma_k \otimes S_k(x_k) \right\|_{G(X)} \leq C \left\| \sum_{k=1}^{N} \gamma_k \otimes x_k \right\|_{G(X)}.$$ 

The least admissible constant $C$ in the above inequality is called the $\gamma$-bound of $T$ and is denoted by $\gamma(T)$.

Let $Z$ be any Banach space and let $\text{Ball}(Z)$ denote its closed unit ball. A bounded operator $\rho: Z \to B(X)$ is called $\gamma$-bounded if the set $\rho(\text{Ball}(Z)) \subset B(X)$ is $\gamma$-bounded. In this case we set $\gamma(\rho) = \gamma(\rho(\text{Ball}(Z)))$.

We now turn to the definition of $\gamma$-spaces, which goes back to the paper [22] (which began to circulate 20 years ago). Let $H$ be a Hilbert space. A bounded operator $T: H \to X$ is called $\gamma$-summing if

$$\|T\|_{\gamma} := \sup \left\{ \left\| \sum_{k=1}^{N} \gamma_k \otimes T(e_k) \right\|_{G(X)} \right\} < \infty,$$

where the supremum is taken over all finite orthonormal systems $(e_k)_{k=1}^{N}$ in $H$. We let $\gamma_{\infty}(H; X)$ denote the space of all $\gamma$-summing operators and we endow it with the norm $\| \cdot \|_{\gamma}$. Then $\gamma_{\infty}(H; X)$ is a Banach space. Any finite rank bounded operator is $\gamma$-summing. We let $\gamma(H; X) \subset \gamma_{\infty}(H; X)$ denote the closure of the space of finite rank bounded operators in $\gamma_{\infty}(H; X)$. In the sequel, finite rank bounded operators are represented by the algebraic tensor product $H^* \otimes X$ in the usual way.

Following [22] Section 5, we let $\gamma_+'(H^*; X^*)$ be the space of all bounded operators $S: H^* \to X^*$ such that

$$\|S\|_{\gamma'} := \sup \left\{ \|\text{tr}(T^*S)\| : T: H \to X, \text{rank}(T) < \infty, \|T\|_{\gamma} \leq 1 \right\} < \infty.$$ 

Then $\| \cdot \|_{\gamma'}$ is a norm on $\gamma_+'(H^*; X^*)$ and according to [22] Proposition 5.1, we have

$$(6.2) \quad \gamma_+'(H^*; X^*) = \gamma(H; X)^*$$

isometrically, through the duality pairing

$$(S, T) \mapsto \text{tr}(T^*S), \quad T \in \gamma(H; X), \ S \in \gamma_+'(H^*; X^*).$$

We will focus on the case when $H$ is an $L^2$-space. Let $(\Omega, \mu)$ be a $\sigma$-finite measure space. We identify $L^2(\Omega)^*$ and $L^2(\Omega)$ in the usual way. A function $\xi: \Omega \to X$ is called weakly-$L^2$ if
for each \( x^* \in X^* \), the function \( \langle x^*, \xi(\cdot) \rangle \) belongs to \( L^2(\Omega) \). Then the operator \( x^* \mapsto \langle x^*, \xi(\cdot) \rangle \) from \( X^* \) into \( L^2(\Omega) \) is bounded. If \( \xi \) is both measurable and weakly-\( L^2 \), then its adjoint takes values in \( X \) and we let \( \mathbb{I}_\xi : L^2(\Omega) \to X \) denote the resulting operator. More explicitly,

\[
\langle x^*, \mathbb{I}_\xi(g) \rangle = \int_\Omega g(t) \langle x^*, \xi(t) \rangle \, d\mu(t) , \quad g \in L^2(\Omega) , \, x^* \in X^* .
\]

We let \( \gamma(\Omega; X) \) be the space of all measurable and weakly-\( L^2 \) functions \( \xi : \Omega \to X \) such that \( \mathbb{I}_\xi \) belongs to \( \gamma(\Omega; X) \), and we write \( \| \xi \|_\gamma = \| \mathbb{I}_\xi \|_\gamma \) for any such function.

Likewise a function \( \zeta : \Omega \to X^* \) is called weakly*-\( L^2 \) if for each \( x \in X \), the function \( \langle \zeta(\cdot), x \rangle \) belongs to \( L^2(\Omega) \). In this case, the operator \( x \mapsto \langle \zeta(\cdot), x \rangle \) from \( X \) into \( L^2(\Omega) \) is bounded and we let \( \mathbb{I}_\zeta : L^2(\Omega) \to X^* \) denote its adjoint. We let \( \gamma_+(\Omega; X^*) \) be the space of all weakly*-\( L^2 \) functions \( \zeta : \Omega \to X \) such that \( \mathbb{I}_\zeta \) belongs to \( \gamma_+(\Omega; X^*) \), and we write \( \| \zeta \|_{\gamma_+} = \| \mathbb{I}_\zeta \|_{\gamma_+} \) for any such function.

Note that our space \( \gamma_+(\Omega; X^*) \) is a priori bigger than the one from [22, Definition 4.5], where only measurable functions \( \Omega \to X^* \) are considered.

**Lemma 6.1.** For any \( \xi \in \gamma(\Omega; X) \) and any \( \zeta \in \gamma_+(\Omega; X^*) \), the function \( t \mapsto \langle \zeta(t), \xi(t) \rangle \) belongs to \( L^1(\Omega) \) and in the duality \( (\mathcal{B}.2) \), we have

\[
\langle \mathbb{I}_\zeta, \mathbb{I}_\xi \rangle = \int_\Omega \langle \zeta(t), \xi(t) \rangle \, d\mu(t) .
\]

Moreover

\[
\int_\Omega |\langle \zeta(t), \xi(t) \rangle| \, d\mu(t) \leq \| \xi \|_{\gamma} \| \zeta \|_{\gamma_+} .
\]

If we consider measurable functions \( \zeta : \Omega \to X^* \) only, the above statement is provided by [22, Corollary 5.5]. The fact that this holds as well in the more general setting of the present paper follows from the proof of [21, Theorem 9.2.14].

The main result of this section is the following.

**Theorem 6.2.** Let \( (T_t)_{t \geq 0} \) be a bounded \( C_0 \)-semigroup on \( X \) and let \( A \) be its negative generator. The following assertions are equivalent.

(i) The semigroup \( (T_t)_{t \geq 0} \) is \( \gamma \)-bounded, that is, the set \( \mathcal{T}_A = \{ T_t : t \geq 0 \} \) is \( \gamma \)-bounded;

(ii) There exists a \( \gamma \)-bounded homomorphism \( \rho_{0,A} : \mathcal{A}_0(\mathbb{C}_+) \to B(X) \) such that \((4.6)\) holds true for all \( b \in L^1(\mathbb{R}_+) \).

In this case, \( \rho_{0,A} \) is unique and \( \gamma(\mathcal{T}_A) \leq \gamma(\rho_{0,A}) \leq \gamma(\mathcal{T}_A)^2 \).

Further there exists a unique bounded homomorphism \( \rho_A : \mathcal{A}(\mathbb{C}_+) \to B(X) \) extending \( \rho_{0,A} \), this homomorphism is \( \gamma \)-bounded and \( \gamma(\rho_A) = \gamma(\rho_{0,A}) \).

A thorough look at the proofs of Theorem 4.4 and Corollary 4.6 reveals that in Subsections 4.2 and 4.3, the Hilbertian structure was used only in Lemma 4.3. So without any surprise the main point in proving Theorem 6.2 is the following \( \gamma \)-bounded version of Lemma 4.3.

**Lemma 6.3.** Let \( (T_t)_{t \geq 0} \) be a \( \gamma \)-bounded \( C_0 \)-semigroup on \( X \) and let \( A \) be its negative generator. Let \( C = \gamma(\mathcal{T}_A) \). Then the set

\[
(6.3) \quad \{ \Gamma(A, (2\pi)^{-1} \hat{f} w) : f \in C_{00}(\mathbb{R}), w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}), \{ \| f \|_\infty, \| w \|_2, \| v \|_2 \} \leq 1 \}
\]
is $\gamma$-bounded, with $\gamma$-bound $\leq C^2$.

Proof. Let $N \in \mathbb{N}$ and let $f_1, \ldots, f_N \in C_{00}(\mathbb{R}), w_1, \ldots, w_N, v_1, \ldots, v_N \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ such that $\|f_k\|_\infty \leq 1, \|w_k\|_2 \leq 1$ and $\|v_k\|_2 \leq 1$ for any $k = 1, \ldots, N$. We set

$$S_k = \Gamma(A, (2\pi)^{-1}\hat{f}_k \hat{w}_k \hat{v}_k), \quad k = 1, \ldots, N.$$  

Let $x_1, \ldots, x_N \in X$ and $x_1^*, \ldots, x_N^* \in X^*$. Following the notation in the proof of Lemma 4.3, we define, for any $k = 1, \ldots, N$, two strongly continuous functions $W_k, V_k : \mathbb{R} \to B(X)$ by

$$W_k(s) = \int_0^\infty \hat{\omega}_k(r) e^{-irs}T_r \, dr \quad \text{and} \quad V_k(s) = \int_0^\infty \hat{\nu}_k(t) e^{-its}T_t \, dt, \quad s \in \mathbb{R}. $$

According to (4.5),

$$\sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle = \frac{1}{4\pi^2} \sum_{k=1}^N \int_{-\infty}^\infty f_k(s)\langle W_k(s)x_k, V_k(s)^*x_k^* \rangle \, ds,$$

hence

$$\left| \sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \right| \leq \frac{1}{4\pi^2} \sum_{k=1}^N \int_{-\infty}^\infty \left| \langle W_k(s)x_k, V_k(s)^*x_k^* \rangle \right| \, ds.$$

We let $\mathbb{N}_N = \{1, \ldots, N\}$ for convenience. We will use $\gamma$-spaces on either $\mathbb{R}$ or $\mathbb{N}_N \times \mathbb{R}$. For any $k = 1, \ldots, N$, the function

$$\alpha_k := W_k(\cdot)x_k : \mathbb{R} \to X$$

is measurable and weakly-$L^2$. Likewise,

$$\beta_k := V_k(\cdot)^*x_k^* : \mathbb{R} \to X^*$$

is weakly*-L$^2$. If we are able to show that $\alpha_k \in \gamma(\mathbb{R}; X)$ and $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$ for any $k = 1, \ldots, N$, then Lemma 6.1 ensures that

$$\sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \leq \frac{1}{4\pi^2} \left\| (k, s) \mapsto W_k(s)x_k \right\|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} \left\| (k, s) \mapsto V_k(s)^*x_k^* \right\|_{\gamma'_+(\mathbb{N}_N \times \mathbb{R}; X^*)}. $$

Our aim is now to check that $\alpha_k \in \gamma(\mathbb{R}; X)$ and $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$ for any $k$ and to estimate the right-hand side of (6.4).

By assumption, $T_A = \{T_t : t \geq 0\}$ is $\gamma$-bounded. According to the Multiplier Theorem stated as [18, Theorem 6.1], there exists a bounded operator

$$M : \gamma(L^2(\mathbb{R}); X) \to \gamma(L^2(\mathbb{R}); X)$$

with norm $\leq C = \gamma(T_A)$, mapping $\gamma(\mathbb{R}; X)$ into itself, and such that for any $\xi \in \gamma(\mathbb{R}; X)$, $[M(\xi)](t) = T_t(\xi(t))$ if $t \geq 0$, and $[M(\xi)](t) = 0$ if $t < 0$. Further by the Extension Theorem stated as [21, Theorem 9.6.1], $\mathcal{F} \otimes I_X : L^2(\mathbb{R}) \otimes X \to L^2(\mathbb{R}) \otimes X$ admits a (necessarily unique) bounded extension

$$\Psi : \gamma(L^2(\mathbb{R}); X) \to \gamma(L^2(\mathbb{R}); X),$$
with norm $\leq \sqrt{2\pi}$. According to Lemma 2.19, $\mathbb{I}_{\alpha_k} = (\Psi \circ M)(\hat{w}_k \otimes x_k)$ for any $k = 1, \ldots, N$. This shows that $\alpha_k \in \gamma(\mathbb{R}; X)$. Let $(e_k)_k^{N}$ be the canonical basis of $\ell^2_N$. It follows from above that

$$\| (k, s) \mapsto W_k(s)x_k \|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} = \left\| \sum_{k=1}^{N} e_k \otimes (\Psi \circ M)(\hat{w}_k \otimes x_k) \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)}$$

$$\leq \sqrt{2\pi} C \left\| \sum_{k=1}^{N} e_k \otimes \hat{w}_k \otimes x_k \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)}.$$

The finite sequence $(e_k \otimes \hat{w}_k)_k^{N}$ is an orthogonal family of $L^2(\mathbb{N}_N \times \mathbb{R})$. Consequently,

$$\left\| \sum_{k=1}^{N} e_k \otimes \hat{w}_k \otimes x_k \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)} = \left\| \sum_{k=1}^{N} \| \hat{w}_k \|_2 \gamma_k \otimes x_k \right\|_{G(X)}$$

$$\leq \max_k \| \hat{w}_k \|_2 \left\| \sum_{k=1}^{N} \gamma_k \otimes x_k \right\|_{G(X)}.$$

Since $\| \hat{w}_k \|_2 = \sqrt{2\pi} \| w_k \|_2 \leq \sqrt{2\pi}$ for any $k = 1, \ldots, N$, we finally obtain that

$$\| (k, s) \mapsto W_k(s)x_k \|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} \leq 2\pi C \left\| \sum_{k=1}^{N} \gamma_k \otimes x_k \right\|_{G(X)}.$$

We now analyse the $\beta_k$. Fix $k$ and consider $g \in L^2(\mathbb{R})$ and $x \in X$. Using Lemma 6.1 we have

$$\langle \mathbb{I}_{\beta_k}(g), x \rangle = \int_{-\infty}^{\infty} g(s) \langle x_k^*, V_k(s)x \rangle \, ds$$

$$= \int_{-\infty}^{\infty} g(s) \mathcal{F}(\hat{v}_k \langle x_k^*, T_\chi(x) \rangle) \, ds$$

$$= \int_{0}^{\infty} \hat{g}(t) \hat{v}_k(t) \langle x_k^*, T_\chi(x) \rangle \, dt$$

$$= \langle \hat{v}_k \otimes x_k^*, M(\hat{g} \otimes x) \rangle$$

$$= \langle \hat{v}_k \otimes x_k^*, (M \circ \Psi)(\hat{g} \otimes x) \rangle$$

$$= \langle (\Psi^* \circ M^*)(\hat{v}_k \otimes x_k^*), \hat{g} \otimes x \rangle.$$ This shows that $\beta_k \in \gamma^*_+(\mathbb{R}; X^*)$, with $\mathbb{I}_{\beta_k} = (\Psi^* \circ M^*)(\hat{v}_k \otimes x_k^*)$. Now arguing as in the $W_k(\cdot)x_k$ case, we obtain that

$$\| (k, s) \mapsto V_k^*(s)x_k^* \|_{\gamma'(\mathbb{N}_N \times \mathbb{R}; X^*)} \leq 2\pi C \left\| \sum_{k=1}^{N} \gamma_k \otimes x_k^* \right\|_{G'(X)}.$$

We now implement these estimates in (6.4) to obtain that

$$\left| \sum_{k=1}^{N} \langle S_k(x_k), x_k^* \rangle \right| \leq C^2 \left\| \sum_{k=1}^{N} \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^{N} \gamma_k \otimes x_k^* \right\|_{G'(X)}.$$
By the very definition of $G'(X)$, this means that
\[
\left\| \sum_{k=1}^{N} \gamma_k \otimes S_k(x_k) \right\|_{G'(X)} \leq C^2 \left\| \sum_{k=1}^{N} \gamma_k \otimes x_k \right\|_{G(X)},
\]
which completes the proof. \qed

**Proof of Theorem 6.2.** Assume (i). By Lemma 6.3 any element in the set (6.3) has norm $\leq C^2$. Hence the proof of Theorem 4.3 shows the existence of a unique bounded homomorphism $\rho_{0,A} : A_0(\mathbb{C}_+) \to B(X)$ such that (4.6) holds true for all $b \in L^1(\mathbb{R}_+)$. 

To prove $\gamma$-boundedness of $\rho_{0,A}$, we introduce the set
\[
\mathcal{L} = \{ (f \ast wv)^\sim : f \in C_{00}(\mathbb{R}), w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}), \{ \| f \|_\infty, \| w \|_2, \| v \|_2 \} \leq 1 \} \subset A_0(\mathbb{C}_+).
\]

Recall (see the proof of Lemma 4.3) that any $h \in H^1(\mathbb{R})$ can be written as a product $h = wv$, with $w, v \in H^2(\mathbb{R})$ and $\| w \|_2^2 = \| v \|_2^2 = \| h \|_1$, and that $H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ is dense in $H^2(\mathbb{R})$. Going back to Definition 3.2 we derive that
\[
\text{Ball}(A_0(\mathbb{C}_+)) = \overline{\text{Conv}} \{ \mathcal{L} \}.
\]

This implies that
\[
\rho_{0,A}(\text{Ball}(A_0(\mathbb{C}_+))) \subset \overline{\text{Conv}} \{ \rho_{0,A}(\mathcal{L}) \}.
\]

Since $\rho_{0,A}((f \ast wv)^\sim) = \Gamma(A, (2\pi)^{-1} \hat{f} \hat{w} \hat{v})$ for any $f \in C_{00}(\mathbb{R})$ and any $w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$, Lemma 6.3 says that $\rho_{0,A}(\mathcal{L})$ is $\gamma$-bounded, with $\gamma$-bound $\leq C^2$. Owing to the fact that $\gamma$-boundedness and $\gamma$-bounds are preserved by convex hulls (see e.g. [21 Proposition 8.1.21]) and uniform limits, we infer that $\rho_{0,A}$ is $\gamma$-bounded, with $\gamma(\rho_{0,A}) \leq C^2$. This proves (ii).

Conversely assume (ii). The proof of Corollary 4.6 shows the existence of a unique bounded homomorphism $\rho_A : A(\mathbb{C}_+) \to B(X)$ extending $\rho_{0,A}$ as well as the fact that $\rho_A(\text{Ball}(A(\mathbb{C}_+)))$ belongs to the strong closure of $\rho_{0,A}(\text{Ball}(A_0(\mathbb{C}_+)))$. Since $\gamma$-boundedness and $\gamma$-bounds are preserved by strong limits, we obtain that $\rho_A$ is $\gamma$-bounded, with $\gamma(\rho_A) = \gamma(\rho_{0,A})$.

Finally the argument in Remark 5.5 (1) shows that for any $t > 0$,
\[
T_t \in \rho_A(\text{Ball}(A(\mathbb{C}_+))).
\]

This implies (i), with $\gamma(T_A) \leq \gamma(\rho_A)$. \qed

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\section*{References}

[1] L. Arnold. $\gamma$-boundedness of $C_0$-semigroups and their $H^\infty$-functional calculi. \textit{Studia Math.}, 254(1):77–108, 2020.

[2] L. Arnold. Derivative bounded functional calculus of power bounded operators on Banach spaces. \textit{Acta Sci. Math. (Szeged)}, 87(1-2):251–280, 2021.

[3] L. Arnold, C. Le Merdy, and S. Zadeh. $S^1$-bounded Fourier multipliers on $H^1(\mathbb{R})$ and functional calculus for semigroups. \textit{Preprint} 2022, \texttt{arXiv:2203.16829}.

[4] J.-B. Baillon and P. Clément. Examples of unbounded imaginary powers of operators. \textit{J. Funct. Anal.}, 100(2):419–434, 1991.

[5] C. Batty, A. Gomilko, and Y. Tomilov. A Besov algebra calculus for generators of operator semigroups and related norm-estimates. \textit{Math. Ann.}, 379(1-2):23–93, 2021.

[6] C. Batty, A. Gomilko, and Y. Tomilov. The theory of Besov functional calculus: developments and applications to semigroups. \textit{J. Funct. Anal.}, 281(6):Paper No. 109089, 60, 2021.

[7] C. Batty, M. Haase, and J. Mubeen. The holomorphic functional calculus approach to operator semigroups. \textit{Acta Sci. Math. (Szeged)}, 79(1-2):289–323, 2013.

[8] J.-E. Björn. $L^p$ estimates for convolution operators defined by compactly supported distributions in $\mathbb{R}^n$. \textit{Math. Scand.}, 34:129–136, 1974.

[9] A. Derighetti. \textit{Convolution operators on groups}, volume 11 of \textit{Lecture Notes of the Unione Matematica Italiana}. Springer, Heidelberg; UMI, Bologna, 2011.

[10] J. Diestel and J. J. Uhl, Jr. \textit{Vector measures}. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 15.

[11] J. García-Cuerva and J. L. Rubio de Francia. \textit{Weighted norm inequalities and related topics}, volume 116 of \textit{North-Holland Mathematics Studies}. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[12] J. B. Garnett. \textit{Bounded analytic functions}, volume 96 of \textit{Pure and Applied Mathematics}. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[13] G. I. Gaudry. $H^p$ multipliers and inequalities of Hardy and Littlewood. \textit{J. Austral. Math. Soc.}, 10:23–32, 1969.

[14] A. Gomilko. On conditions for the generating operator of a uniformly bounded $C_0$-semigroup of operators. \textit{Funktsional. Anal. i Prilozhen.}, 33(4):66–69, 1999.

[15] L. Grafakos. \textit{Classical Fourier analysis}, volume 249 of \textit{Graduate Texts in Mathematics}. Springer, New York, second edition, 2008.

[16] M. Haase. \textit{The functional calculus for sectorial operators}, volume 169 of \textit{Operator Theory: Advances and Applications}. Birkhäuser Verlag, Basel, 2006.

[17] M. Haase. Transference principles for semigroups and a theorem of Peller. \textit{J. Funct. Anal.}, 261(10):2959–2998, 2011.

[18] M. Haase and J. Rozendaal. Functional calculus for semigroup generators via transference. \textit{J. Funct. Anal.}, 265(12):3345–3368, 2013.

[19] E. Hille and R. S. Phillips. \textit{Functional analysis and semi-groups}. American Mathematical Society, Providence, R. I., 1974.

[20] K. Hoffman. \textit{Banach spaces of analytic functions}. Prentice-Hall Series in Modern Analysis. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962.

[21] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. \textit{Analysis in Banach spaces. Vol. II}, volume 67 of \textit{Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics}. Springer, Cham, 2017.

[22] N. Kalton and L. Weis. The $H^\infty$-functional calculus and square function estimates. In Nigel J. Kalton \textit{Selecta}, volume 1. Springer-Verlag, 2016.

[23] P. Koosis. \textit{Introduction to $H_p$ spaces}, volume 40 of \textit{London Mathematical Society Lecture Note Series}. Cambridge University Press, Cambridge-New York, 1980.
[24] C. Le Merdy. The similarity problem for bounded analytic semigroups on Hilbert space. *Semigroup Forum*, 56(2):205–224, 1998.

[25] C. Le Merdy. $H^\infty$-functional calculus and applications to maximal regularity. In *Semi-groupes d’opérateurs et calcul fonctionnel (Besançon, 1998)*, volume 16 of *Publ. Math. UFR Sci. Tech. Besançon*, pages 41–77. Univ. Franche-Comté, Besançon, 1999.

[26] C. Le Merdy. $\gamma$-Bounded representations of amenable groups. *Adv. Math.*, 224(4):1641–1671, 2010.

[27] A. McIntosh and A. Yagi. Operators of type $\omega$ without a bounded $H_\infty$ functional calculus. In *Mini-conference on Operators in Analysis (Sydney, 1989)*, volume 24 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 159–172. Austral. Nat. Univ., Canberra, 1990.

[28] V. V. Peller. Estimates of functions of power bounded operators on Hilbert spaces. *J. Operator Theory*, 7(2):341–372, 1982.

[29] G. Pisier. Interpolation between $H^p$ spaces and noncommutative generalizations. I. *Pacific J. Math.*, 155(2):341–368, 1992.

[30] G. Pisier. *Similarity problems and completely bounded maps*, volume 1618 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 2001.

[31] R. A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[32] D.-H. Shi and D.-X. Feng. Characteristic conditions of the generation of $C_0$ semigroups in a Hilbert space. *J. Math. Anal. Appl.*, 247(2):356–376, 2000.

[33] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[34] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[35] P. Vitse. A Besov class functional calculus for bounded holomorphic semigroups. *J. Funct. Anal.*, 228(2):245–269, 2005.

[36] J. H. Wells. Some results concerning multipliers of $H^p$. *J. London Math. Soc. (2)*, 2:549–556, 1970.

[37] S. J. White. *Norm-estimates for functions of semigroups of operators*, volume PhD thesis,Univ. of Edinburgh. https://www.era.lib.ed.ac.uk/handle/1842/11552, 1989.

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