Disentangling scaling arguments to empower complex systems analysis

Scaling arguments provide valuable analysis tools across physics and complex systems yet are often employed as one generic method, without explicit reference to the various mathematical concepts underlying them. A careful understanding of these concepts empowers us to unlock their full potential.

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The notion of scaling plays a central role across physics as well as complex systems analysis. It offers powerful tools for answering fundamental questions about a system without considering all its details. However, scaling arguments are often used without careful consideration of their various distinct mathematical definitions. Their full potential can be exploited only by respecting these different concepts. We may be able to classify system structure and dynamics, to understand mechanisms underlying (collective) phenomena that complex systems display and, sometimes, to quantitatively predict system behaviour.

In statistical physics, scaling exponents naturally appear in the analysis of continuous phase transitions. Here, an observable of the system — an order parameter — quantifies properties of the system that change between two qualitatively different macroscopic states. At a critical value of an external control parameter such as temperature, the order parameter continuously grows from zero, indicating the emergence of a partially ordered state. Near the phase transition point, the order parameter typically grows as a power law with the distance from that critical point.

For instance, in ferromagnetic systems, as characterized by the Ising model\(^1,2\), the macroscopic magnetization is zero above some critical temperature. The magnetic moments (spins) are not aligned and thus do not induce macroscopic magnetization. Below the critical temperature, the system’s symmetry is broken as spins align with each other macroscopically in one preferred spatial direction. The overall magnetization of the system changes continuously from zero to non-zero values at the critical temperature. In the mean field solution of the Ising model, the average magnetization in the ordered phase grows with the square root of the distance to the critical temperature. In this sense, power-law scaling constitutes an essential indicator of continuous phase transitions, with the value of the scaling exponent distinguishing between different subtypes of transitions.

Intriguing theoretical and empirical insights starting in the twentieth century demonstrated that these critical exponents characterizing continuous phase transitions often are universal. Their value is largely independent of the material properties but instead characterizes the similarity of the underlying mechanisms and the effective dimensionality of the underlying interactions across different systems. For our example of ferromagnetic systems, these critical exponents essentially depend on the dimension of the system and the degrees of freedom of the magnetic moments, yet not on the atoms or molecules carrying these moments or details of their interactions. Moreover, the same mean field exponent \(\frac{1}{2}\) is shared with a host of entirely different systems and phenomena\(^1\). As a result of this universality, how order emerges and thus how order parameters scale near critical points enables us to classify these systems by the scaling exponents they exhibit.

Although (universal) scaling exponents are key to characterize and classify different systems, there is more to power law scaling than the exponent. Even if the exponent of a power law is known, its knowledge is insufficient to quantitatively predict the value of an order parameter (or any other quantity exhibiting the power law scaling). Unfortunately, unclear and inconsistent mathematical notation employed across

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**Fig. 1** | Two distinct types of scaling analysis. a, b. Universal power-law scaling in continuous phase transitions (a) and asymptotic expansions (b) constitute two powerful tools of scaling analysis for physics and complex systems. We emphasize that the two panels illustrate two distinct notions of scaling, each with a different meaning of the symbol ‘\(\sim\)’ according to Eqs. (1) and (2), respectively.
physics and complex systems analysis further mystifies the issue. To characterize the square-root scaling of the magnetization $M$ with the temperature difference $T_c - T$ to a critical temperature $T_c$ in ferromagnets, we often write $M \sim (T_c - T)^{\beta}$. Mathematically, this notation is short hand for

$$\lim_{\varepsilon \to 0} \frac{\ln M(\varepsilon)}{\ln \varepsilon} = \beta \quad (1)$$

where $\varepsilon = (T_c - T)/T_c$ quantifies the normalized distance to the critical temperature (Fig. 1a). On first glance, it may appear as if we could theoretically predict $M$, at least approximately, for small temperature differences $(T_c - T)$ given the exact definition (Eq. (1)) if we know the exponent $\beta$ (along with the critical temperature $T_c$). However, we are missing an overall constant. If $M \sim A \times [(T_c - T)/T_c]^\beta$, also the constant $A$ determines the actual value of $M$ for $T < T_c$. Without it, no prediction is possible. In contrast to the scaling exponents, the value of $A$ is not universal and typically depends on detailed material properties. For instance, the prefactor $A$ may depend not only on the type of spin and the structure of the interactions but also on the specific type of atoms carrying these spins. The same holds for all scaling laws across systems, settings and subjects. In asymptotic analysis, a subfield of Mathematics, the notation $\sim$ has the mathematical meaning the respective author assigns to the mathematical symbol ‘~’. Finally, their interpretation is context dependent; in the above example, $\varepsilon$ is a function of the distance from the perturbed unit and others. Therefore, their spreading properties are controlled by the graph-theoretical networks of dynamical elements that result from permanent fixing the state of a single unit to a new value. They find qualitatively different types of propagation patterns in the dynamics of such elements that result from permanently fixing the state of a single unit to a new value.

In summary, the authors of the original article and the Reply argue that the mathematical analysis of the asymptotic scaling of the magnetization $M$ is insufficient to quantitatively predict absolute magnitudes of observables, especially when controlling for the temperature difference and the specific type of atoms carrying these spins. The same holds for all scaling laws across systems, settings and subjects. In asymptotic analysis, a subfield of Mathematics, the notation $\sim$ has the mathematical meaning the respective author assigns to the mathematical symbol ‘~’. Finally, their interpretation is context dependent; in the above example, $\varepsilon$ is a function of the distance from the perturbed unit and others. Therefore, their spreading properties are controlled by the graph-theoretical networks of dynamical elements that result from permanent fixing the state of a single unit to a new value. They find qualitatively different types of propagation patterns in the dynamics of such elements that result from permanently fixing the state of a single unit to a new value. They find qualitatively different types of propagation patterns in the dynamics of such elements that result from permanent fixing the state of a single unit to a new value. They find qualitatively different types of propagation patterns in the dynamics of such elements that result from permanent fixing the state of a single unit to a new value.
Fully embracing the different aspects of scaling thus helps in analysing and predicting phenomena across physics and complex systems. The tools of scaling analysis become particularly powerful once we exactly distinguish the different notions of scaling and their distinct implications.

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