ON THE POSSIBILITY OF SPINORIAL QUANTIZATION
IN THE SKYRME MODEL

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Abstract
We consider the configuration space of the Skyrme model and give a simple proof that loops generated by $2\pi$-rotations are contractible in the even-, and non-contractible in the odd-winding-number sectors.

Introduction
As is well known, the Skyrme model and related models allow quantizations with odd-half angular momentum due to the non-contractibility of some loops in the configuration space (represented by an infinite dimensional mapping-space). Standard expositions, treating the quantization of the collective degrees of freedom in the unit winding-number sector, usually argue only within the (finite dimensional) subspace of collective coordinates (e.g. [Ba][Na-Wi]). But non-contractibility of certain loops in a subspace does not imply non-contractibility when allowed to be deformed outside this space. Valid proofs for the unit winding-number sector have been given first in [Wi] and later in [Wi-Zv][Sk][Pa-Tz]. From [Pa-Tz] one might get the impression that this is all there is to prove. But this is not the case and other winding number sectors require in principle a separate treatment. To fill this gap, this article presents a proof for the statement made in the abstract. We tried to design it as elementary and self-contained as possible.

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Section 1: General Setting

Let us start by a general consideration of a configuration space $Q$ together with a right $SO(3)$-action which we write

$$R: (SO(3) \times Q) \to Q$$

$$(g, q) \mapsto R_g(q) = q \cdot g.$$  

(1)

A curve $\gamma: [0, 1] \to SO(3)$ then defines the flow $\phi_q(t) = q \cdot \gamma(t)$ in $Q$. We are interested in the case where $\gamma$ is a closed loop in $SO(3)$ representing a non-contractible loop. Since $\pi_1(SO(3)) \cong \mathbb{Z}_2$, we want $\gamma$ (denoting its homotopy class) to be the generator of $\mathbb{Z}_2$. In particular, any $\gamma$ describing a $2\pi$-rotation about a fixed axis will do. Now, the question is whether the loop at $q$, $\phi_q: t \mapsto \phi_q(t)$, is contractible in $Q$. Clearly, a necessary condition is that $\pi_1(Q, q)$ posesses a $\mathbb{Z}_2$ subgroup ($4\pi$-rotations are necessarily contractible). If $q, p \in Q$ are connected by a path $c: t \mapsto c(t)$, $c(0) = q$, $c(1) = p$, there is the standard isomorphism $c^\#: \pi_1(Q, p) \to \pi_1(Q, q)$ (see [St] §16). The one-parameter family of loops,

$$\lambda_s: t \mapsto \begin{cases} 
    c(3st) & \text{for } t \in [0, \frac{1}{3}] \\
    \phi_{c(s)}(3t - 1) & \text{for } t \in [\frac{1}{3}, \frac{2}{3}] \\
    c(3s(1 - t)) & \text{for } t \in [\frac{2}{3}, 1], 
\end{cases}$$

(2)

define a continuous deformation between representatives of $[\phi_q]$ and $c^#[\phi_p]$, showing that $[\phi_q]$ is trivial in $\pi_1(Q, q)$, if and only if $[\phi_p]$ is trivial in $\pi_1(Q, p)$. Since we certainly assume the space $Q$ to be locally pathwise connected, the path-components coincide with the connected components, and one has the

Lemma 1. Contractibility of $\phi_q$ depends only on the connected component of $q$ in $Q$ and not on the choice of $q$ within that component.

If we decompose $Q$ into its connected components $Q_n$, we can now ask the question, whether for a given $SO(3)$-action $R$ on $Q$ the homotopy class of loops $[\phi_n]$ (due to Lemma 1 we omit indicating the basepoint) is contractible or not. Following [Fi-Ru], we make the following

Definition 1. The configuration space $Q$ is said to allow for spinorial states in the sector $Q_n$ with respect to the action $R_g$ of the spatial group $SO(3)$, if and only if the
loop \( t \mapsto R_{gt}(q) \) is non-contractible, for \( t \mapsto g_t \) any representative of the generator of \( \mathbb{Z}_2 \cong \pi_1(\text{SO}(3)) \), and any \( q \in Q_n \).

If the components \( Q_n \) are mutually homeomorphic, as it is for example the case if \( Q \) is a topological group, the homeomorphisms \( h_{mn} : Q_n \to Q_m \) then also define isomorphisms on all homotopy groups, hence, in particular, on the fundamental groups \( \pi_1(Q_n) \). This implies

**Lemma 2.** The loop \( \phi_n \) is contractible in \( Q_n \), if and only if the loop \( h_{mn} \circ \phi_n \) is contractible in \( Q_m \).

It is important to note that contractibility of \( h_{mn} \circ \phi_n \) has generally nothing to do with contractibility of \( \phi_m \), the first loop being just a homeomorphic image in \( Q_m \) of the loop \( \phi_n \) in \( Q_n \), which might not reflect how the group acts in \( Q_m \). The use of Lemma 2 lies in the fact that for standard applications there is a distinguished component, say \( Q_0 \), which is analytically more accessible, so that all the constructions can be performed on \( Q_0 \) using \( h_{0n} \).

**Section 2: The \( SU(2) \) Skyrme Model Admits Spinorial States**

In this section we continue our notation from the previous section. In particular, \( q, p, \ldots \) still denote points in the configuration space which is now a mapping space so that e.g. \( q \) denotes both, a point in \( Q \) and a map. Integer subscripts usually refer to the connected component the point lies in, i.e., \( q_n \in Q_n \).

The \( SU(2) \) Skyrme model is described by a map \( q \) from \( R^3 \) into \( SU(2) \cong S^3 \), dynamically stabilized by adding higher order terms in the first derivatives of the field variables. A geometric interpretation is given in [Ma]. Finite energy requires the map to be extendable to the compactification \( S^3 \) of \( R^3 \) by adding a point, called \( \infty \), at infinity, which must be mapped to the identity element \( e \) in \( SU(2) \). The configuration space is then the space of basepoint preserving maps

\[
Q = \{ q : (R^3 \cup \infty, \infty) \to (SU(2), e) \}
\]

whose path-components are just the homotopy classes

\[
\pi_0(Q) = \pi_3(S^3) = \mathbb{Z}.
\]
Although the domain- and target-space can both be identified with $S^3$, we notationally separate them by using $R^3 \cup \infty$ or $S^3$ for the domain-, and $SU(2)$ for the target-space, also stressing the additional group structure of the latter.

A static configuration minimizes the energy functional which is bounded below by a real number proportional to the winding number $W[q]$ of the map

$$R^3 \cup \infty \xrightarrow{q} SU(2),$$

which can be identified with the integer appearing in (4). The winding number has the analytic expression [Bo-Se]

$$W[q] = \frac{1}{24\pi^2} \int_{S^3} \text{tr} \left( q^{-1} dq \right)^3,$$

and can be considered as the piecewise constant function $W : Q \to Z$, $W[q] = n \Leftrightarrow q \in Q_n$. Since the target space is a topological group, the space $Q$ can also be given the structure of a topological group by pointwise multiplication, i.e. $(q_1 q_2)(x) := q_1(x) q_2(x) \forall x \in S^3$, and a suitable choice of topology on $Q$ (e.g. compact-open or finer). From expression (6) it easily follows that

$$W[q_1 q_2] = W[q_1] + W[q_2],$$

so that we can define homeomorphism from $Q_n$ to $Q_m$ by right (conventionally) multiplication with an element $p_{m-n} \in Q_{m-n}$

$$h_{mn} : Q_n \to Q_m$$

$$q_n \mapsto h_{mn}(q_n) := q_n p_{m-n} \quad \forall q_n \in Q_n.$$  

In order to find the fundamental group of $Q$ (i.e. of each component $Q_n$), we only need to find $\pi_1(Q_0, b)$ for some basepoint $b$. We may therefore choose $b$ to be the constant map $S^3 \mapsto e \in SU(2)$. On the other hand, a loop in $Q_0$ at $b$ defines a one parameter family (labelled by the unit intervall $I$) of maps from the solid three-dimensional cube $K$ to $SU(2)$, starting and ending at $b$, and such that the boundary $\partial K$ of $K$ is mapped to $e \in SU(2)$:

$$\alpha : I \times K \to SU(2)$$

where $\alpha(0 \times K) = \alpha(1 \times K) = e$  

and $\alpha(I \times \partial K) = e$.  

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But this is a map from the solid four-dimensional cube $I \times K$ to $SU(2)$ whose boundary is mapped to $e \in SU(2)$. Homotopic loops in $Q$ at $b$ thus define homotopic maps from the four-cube with the boundary mapped into $e$, and vice versa. Therefore, there is an isomorphism:

$$\pi_1(Q, b) \cong \pi_4(SU(2), e) \cong \mathbb{Z}_2. \quad (10)$$

Since this group is abelian, we need not indicate the basepoint$^{(1)}$ and we can unambiguously write

$$\pi_1(Q) = \pi_1(Q_n) = \mathbb{Z}_2. \quad (11)$$

Next we need to specify $R$, the $SO(3)$–action on $Q$. It has the obvious action on the spatial $R^3$ and extends to the compactification $S^3 = R^3 \cup \infty$ by fixing $\infty$. Denoting the $SO(3)$-matrix by $M$ and by $Mx$ the action of $M$ on $x \in R^3$, we define $R$ via

$$[R_M q](x) := q(Mx) \quad \forall x \in R^3. \quad (12)$$

A closed curve $\gamma$ in $SO(3)$, represented by the one-parameter family of matrices $M_t$, then defines the free (i.e. unbased) homotopy class $[\phi_n]$ of loops in $Q_n$ represented by the loops $\phi_{q_n}$ in $Q_n$ given by

$$\phi_{q_n}(t) := q_n \circ M_t. \quad (13)$$

Right multiplication by $[q_n]^{-1}$ then defines a loop $\varphi_n$ in $Q_0$ based at the constant map $R^3 \cup \infty \mapsto e$:

$$\varphi_n(t) = \phi_{q_n}(t)[q_n]^{-1} \quad \text{or} \quad \varphi_n^t(x) = q_n(M_t x)[q_n(x)]^{-1}. \quad (14)$$

According to what has been said above, the $W = n$ sector admits spinorial states, if and only if $\varphi_n$ is non-contractible, or, as expressed in (10), if and only if $\varphi_n$ generates $\mathbb{Z}_2 = \pi_4(SU(2))$.

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$^{(1)}$ Note that there are generally no canonical isomorphisms of the fundamental groups at different basepoints. Isomorphisms are only defined up to inner automorphisms. A canonical identification therefore exists for abelian groups.

$^{(2)}$ If $i$ denotes the map $g \mapsto i(g) := g^{-1}$ on $SU(2)$, then we write $i \circ q := [q]^{-1}$ or $(i \circ q)(x) =: [q(x)]^{-1}$. 

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The Truncated Model

Before proceeding with the general case, let us very briefly review the standard arguments for the truncated model in the $W=1$ sector. Here one introduces collective coordinates, $A(t)$, around the energy minimizing map $m \in Q_1$ in the $W=1$ sector, using the ansatz

$$ q(t, x) = A(t)m(x)A^{-1}(t), \quad (15) $$

where $m(x) = \cos[f(r)] + i\vec{n} \cdot \vec{\tau} \sin[f(r)]$, \quad (16)

where, $\vec{\tau}$ denotes the triplet of Pauli matrices and $\vec{n} = \vec{x}/|\vec{x}|$. In this way one truncates the configuration space $Q$ to

$$ Q_{tr} = SU(2)/Z_2 \cong RP^3, \quad (17) $$

since $A(t)$ and $-A(t)$ are to be identified. It is obvious that a spatial rotations with $SO(3)$ matrix $M$ and corresponding $SU(2)$ covering element $S$ obeys

$$ q(t, Mx) = ASmS^{-1}A^{-1}, \quad (18) $$

i.e., rotations act on $Q_{tr}$ by right multiplication. A 1-parameter family $S_t$ generating a $2\pi$-rotation therefore generates $Z_2 = \pi_1(Q_{tr})$. Hence the truncated model admits spin in the sense of the definition given above. But clearly, this does not imply the same statement in the full theory.

The General Case

Let us first consider the sector $Q_1$. As basepoint we choose the map $q_1 \in Q_1$ corresponding to the inverse stereographic projection:

$$ q_1(\vec{x}) = \frac{1-r^2}{1+r^2} + i\vec{\tau} \cdot \vec{n}(\vec{x}) \frac{2r}{r^2+1} \quad \text{(where $\vec{n}(\vec{x}) = \vec{x}/r$)} \quad (19) $$

where the last line just introduces $a$ and $\vec{a}$ as abbreviations of the expressions above. As loop $\gamma$ in $SO(3)$ we choose

$$ M : [0, 2\pi] \ni t \mapsto M_t = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (20) $$
Its $SU(2)$-lift is given by
\[ S_t = \exp \left( \frac{i}{2} \tau_3 t \right) , \]
so that according to (14) and (19)
\[ \varphi_t^1(x) = q_1(M_t x) [q_1(x)]^{-1} = S_t q_1(x) S_t^{-1} [q_1(x)]^{-1} . \] Inserting (19), we obtain
\[ \varphi_t^1 = (a + i \vec{\tau} \cdot M_t \vec{a}) (a - i \vec{\tau} \cdot \vec{a}) =: (a' + i \vec{\tau} \vec{a}') \]
\[ = a^2 + \vec{a} \cdot M_t \vec{a} + i \tau_3 (a_2^2 + a_1^2) \sin t + \text{terms} \propto \tau_1 \text{ and } \tau_2 . \]
For $0 \leq t \leq \pi$ this maps onto the hemisphere $a'_3 \geq 0$, for $\pi \leq t \leq 2\pi$ onto the hemisphere $a'_3 \leq 0$, and for $t = \pi$ onto the equator $a'_3 = 0$. It is therefore a suspension (defined by these conditions) of the “equator-map”
\[ \varphi_1^=\pi : S^3 \rightarrow S^2 := \{(a'_1, a'_2, a') | a_1'^2 + a_2'^2 + a'^2 = 1\} , \]
which, using expressions (21-22), is given by
\[ \varphi_1^\pi = S_\pi q_1 S_\pi^{-1} [q_1]^{-1} = \tau_3 q_1 \tau_3 [q_1]^{-1} . \] Now, there is a special map from $S^3$ to $S^2$, called the Hopf map, which generates $\pi_3(S^2) = \mathbb{Z}$. It is given by the projection map $S^3 \rightarrow S^2$ of the Hopf bundle that fibres $S^3$ by $S^1$ over $S^2$. Identifying $SU(2) \cong S^3$, it is simply given by$^{(3)}$
\[ h(g) = g \tau_3 g^{-1} =: \tilde{h}(g) \cdot \tau , \text{ where } \tilde{h} \cdot \tilde{h} = 1 , \]
so that (25) now reads
\[ \varphi_1^\pi = \tau_3 \circ h \circ q_1 = \tau_3 \tilde{\tau} \circ \tilde{h} \circ q_1 , \]
which shows that this is the Hopf map onto the two-Sphere in $(a'_1, a'_2, a')$ coordinates, composed with a rotation in the $a'_1$-$a'_2$ plane about an angle $\frac{\pi}{2}$. In particular, it is homotopic to the Hopf map. We now use the standard result in homotopy theory (see [St] 21.4, 21.6) that any suspension of the Hopf map, and therefore all its

$^{(3)}$ In Euler angles on $S^3$ and polar angles on $S^2$ this corresponds to $(\psi, \theta, \varphi) \rightarrow (\theta, \varphi)$. 7
homotopies, generate $Z_2 = \pi_4(S^3)$. This then proves the existence of spinorial quantizations in the sector $Q_1$.

For $Q_n$ we can proceed as above by choosing as basepoint $q_n \in Q_n$ a slight modification of the ansatz (19), in which $\tilde{n}(\vec{x})$ is replaced by a map $\tilde{s}_n := w_n \circ \tilde{n}$, where $w_n$ is a winding number $n$ map of the two-sphere onto itself. In spherical polar coordinates $(\theta, \varphi)$ ($\theta$ being as usual the angle to the z-axis) it is defined by

$$\begin{align*}
w_n : S^2 &\to S^2 \\
(\theta, \varphi) &\mapsto w_n(\theta, \varphi) := (\theta, n\varphi),
\end{align*}$$

(28)

so that the basepoint $q_n$ is now given by

$$
q_n(\vec{x}) = \frac{1 - r^2}{1 + r^2} + i\vec{\tau} \cdot \tilde{s}_n(\vec{x}) \frac{2r}{r^2 + 1} = : a_n + i\vec{\tau} \cdot \tilde{a}_n.
$$

(29)

It is obvious that $q_n \in Q_n$ (i.e. it satisfies $W[q_n] = n$). Choosing the same rotation map $M_t$ as in the $n = 1$ case, we now have the crucial property

$$
\tilde{s}_n(M_t \vec{x}) = M_{nt} \tilde{s}_n(\vec{x}),
$$

(30)

which follows readily from (20) (in polar angles: $M_t(\theta, \varphi) = (\theta, \varphi + t)$) and (28). Mapping back the loop $q_n^t$ to a loop $\varphi_n^t$ in $Q_0$ via right $[q_n]^{-1}$-multiplication, we obtain

$$
\varphi_n^t = (a_n + i\vec{\tau} \cdot M_{nt} \tilde{a}_n)(a_n - i\vec{\tau} \cdot \tilde{a}_n) = : (a_n' + i\vec{\tau} \tilde{a}_n')
= a_n'^2 + \tilde{a}_n \cdot M_{nt} \tilde{a}_n + i\tau_3(a_n'^2 + a_n''^2) \sin nt + \text{terms } \propto \tau_1 \text{ and } \tau_2.
$$

(31)

As above, we conclude that the partial loop for $0 \leq t \leq 2\pi/n$ represents the generator of $Z_2 = \pi_4(SU(2))$. The $n$-fold loop therefore represents the generator for $n = \text{odd}$, and the trivial element for $n = \text{even}$. We have thus proven the

**Theorem.** The Skyrme model admits spinorial states in the sectors of odd winding number, and no spinorial states in the sectors of even winding number.

Due to the topological spin-statistics theorem, proven in [Fi-Ru] (see also [So] for an elegant formulation), we have at the same time the corresponding statement for exchange in all winding number sectors: The Skyrme model allows for fermionic quantizations in the odd winding number sectors, and bosonic quantizations only in the even winding number sectors.

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