Commutative Local Rings whose Ideals are Direct Sums of Cyclic Modules*†‡

M. Behboodiä,b§ and S. H. Shojaeea

daDepartment of Mathematical Sciences, Isfahan University of Technology
P.O.Box: 84156-83111, Isfahan, Iran

bSchool of Mathematics, Institute for Research in Fundamental Sciences (IPM)
P.O.Box: 19395-5746, Tehran, Iran
mbehbood@cc.iut.ac.ir
hshojaee@math.iut.ac.ir

Abstract

A well-known result of Köthe and Cohen-Kaplansky states that a commutative ring $R$ has the property that every $R$-module is a direct sum of cyclic modules if and only if $R$ is an Artinian principal ideal ring. This motivated us to study commutative rings for which every ideal is a direct sum of cyclic modules. Recently, in [M. Behboodi, A. Ghorbani, A. Moradzadeh-Dehkordi, Commutative Noetherian local rings whose ideals are direct sums of cyclic modules, J. Algebra 345 (2011) 257–265] the authors considered this question in the context of finite direct products of commutative Noetherian local rings. In this paper, we continue their study by dropping the Noetherian condition.

1. Introduction

The study of rings over which modules are direct sums of cyclic modules has a long history. The first important contribution in this direction is due to Köthe [6] who considered rings over which all modules are direct sums of cyclic modules. Köthe showed that over an Artinian principal ideal ring, each module is a direct sum of cyclic modules. Furthermore, if a commutative Artinian ring has the property that all its modules are direct sums of cyclic modules, then it is an Artinian principal ideal ring. Continuing this line of thought, we study commutative rings for which every ideal is a direct sum of cyclic modules.

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§Corresponding author.
cyclic modules, then it is necessarily a principal ideal ring. Later, Cohen and Kaplansky [3] obtained the following.

Result 1.1. (Cohen and Kaplansky, [3]) If \( R \) is a commutative ring such that each \( R \)-module is a direct sum of cyclic modules, then \( R \) must be an Artinian principal ideal ring.

An interesting natural question arises. Instead of considering rings for which all modules are direct sums of cyclic modules, we weaken this condition and study rings \( R \) for which it is assumed only that the ideals of \( R \) are direct sums of cyclic modules. The study of such commutative rings was initiated by Behboodi, Ghorbani and Moradzadeh-Dehkordi in [1]. In particular, they established the following theorem.

Result 1.2. ([1, Theorem 2.11]) Let \((R, \mathcal{M})\) be a commutative Noetherian local ring, where \( \mathcal{M} \) denotes the unique maximal ideal of \( R \). Then the following statements are equivalent:

1. Every ideal of \( R \) is a direct sum of cyclic \( R \)-modules.
2. There exist an positive integer \( n \) and a set of elements \( \{w_1, \cdots, w_n\} \subseteq R \) such that \( \mathcal{M} = Rw_1 \oplus \cdots \oplus Rw_n \) with at most two of \( Rw_i \)’s not simple.
3. There exists an positive integer \( n \) such that every ideal of \( R \) is a direct sum of at most \( n \) cyclic \( R \)-modules.
4. Every ideal of \( R \) is a direct summand of a direct sum of cyclic \( R \)-modules.

In this paper we consider commutative local rings for which every ideal is a direct sum of cyclic modules, that is, we drop the Noetherian condition from [1]. In particular, we describe the ideal structure of such rings.

In the sequel all rings are commutative with identity and all modules are unital. For a ring \( R \), we denote (as usual) the set of prime ideals of \( R \) by \( \text{Spec}(R) \). Also, \( \text{Nil}(R) \) is the ideal of all nilpotent elements of \( R \). We denote the classical Krull dimension of \( R \) by \( \dim(R) \). Let \( X \) be either an element or a subset of \( R \). The annihilator of \( X \) is the ideal \( \text{Ann}(X) = \{a \in R \mid aX = 0\} \). A ring \( R \) is local in case \( R \) has a unique maximal ideal. In this paper \((R, \mathcal{M})\) will be a local ring with maximal ideal \( \mathcal{M} \). An \( R \)-module \( N \) is called simple if \( N \neq (0) \) and it has no submodules except \((0)\) and \( N \). An \( R \)-module \( M \) is a semisimple module if it is a direct sum of simple modules. Also, an \( R \)-module \( M \) is called a homogeneous semisimple \( R \)-module if it is a direct sum of isomorphic simple \( R \)-modules, i.e., \( \text{Ann}(M) \) is a maximal ideal of \( R \).

It will be shown (see Theorems 3.1 and 3.3) that if a local ring \((R, \mathcal{M})\) has the property that every ideal of \( R \) is a direct sum of cyclic \( R \)-modules, then \( \dim(R) \leq 1 \) and \( |\text{Spec}(R)| \leq 3 \). Moreover, there is an index set \( \Lambda \) and a set of elements \( \{x, y\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R \) such that \( \mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \) with: each \( Rw_\lambda \) a simple \( R \)-module, \( R/\text{Ann}(x) \), \( R/\text{Ann}(y) \) principal ideal rings, and \( \text{Spec}(R) \subseteq \{(0), \mathcal{M}, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda), Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\} \). Also,
we prove the following main theorem.

**Result 1.3.** (See Theorem 3.7) For a local ring \((R,M)\) the following statements are equivalent:

1. Every ideal of \(R\) is a direct sum of cyclic \(R\)-modules.
2. Every ideal of \(R\) is a direct sum of cyclic \(R\)-modules, at most two of which are not simple.
3. There is an index set \(\Lambda\) and a set of elements \(\{x,y\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R\) such that \(M = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\) with: each \(Rw_\lambda\) a simple \(R\)-module, \(R/\text{Ann}(x)\) and \(R/\text{Ann}(y)\) principal ideal rings.
4. Every ideal of \(R\) is a direct summand of a direct sum of cyclic \(R\)-modules.

Finally, some relevant examples and counterexamples are indicated in Section 4.

### 2. Preliminaries

We begin this section with the following result from commutative algebra due to I. M. Isaacs which states that to check whether every ideal in a ring is principal, it suffices to test only the prime ideals.

**Lemma 2.1.** (Attributed to I. M. Isaacs in [5, p. 8, Exercise 10]) A commutative ring \(R\) is a principal ideal ring if and only if every prime ideal of \(R\) is a principal ideal.

**Proposition 2.2.** Let \(R\) be a ring. If every prime ideal of \(R\) is a direct sum of cyclic \(R\)-modules, then \(R/P\) is a principal ideal domain (PID) for each prime ideal \(P \in \text{Spec}(R)\). Consequently, \(\dim(R) \leq 1\).

**Proof.** Assume that \(P \in \text{Spec}(R)\) and \(Q/P\) is a prime ideal of \(R/P\). Since \(Q \in \text{Spec}(R)\), \(Q = \bigoplus_{i \in I} Rx_i\) for some index set \(I\) and \(x_i \in R\) for each \(i \in I\). If \(Q/P\) is nonzero, then there exists \(j \in I\) such that \(x_j \notin P\). Since for each \(i \in I\), \(Rx_i Rx_j = (0) \subseteq P\), we conclude that \(Rx_i \subseteq P\) for each \(i \neq j\). It follows that \(x_j + P \in R/P\) is a generator for \(Q/P\). Thus by Lemma 2.1, \(R/P\) is a PID. Since this holds for all prime ideals \(P\) of \(R\), we conclude that \(\dim(R) \leq 1\). \(\Box\)

The following two results from [1] are crucial to our investigation.

**Lemma 2.3.** ([1, Proposition 2.2]) Let \((R,M)\) be a local ring. Suppose that \(M = Rx \oplus Ry \oplus Rz \oplus K\) for nonzero elements \(x, y,\) and \(z\) and an ideal \(K\) of \(R\). Further suppose that neither of \(Rx, Ry,\) or \(Rz\) is a simple \(R\)-module. Then the ideal \(J := R(x+y)+R(x+z)\) is not a direct sum of cyclic \(R\)-modules.

**Lemma 2.4.** ([1, Corollary 2.3]) Suppose \((R,M)\) is a local ring such that every ideal of...
$R$ is a direct sum of cyclic $R$-modules. Then there is an index set $\Lambda$ and a set of elements $\{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $M = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$ with at most two of the $Rw_\lambda$’s not simple.

Lemma 2.5. (See [8, Proposition 3]) Let $R$ be a local ring and let $M$ be an $R$-module. If there is an index set $\Lambda$ and a set of ideals $\{I_\lambda\}_{\lambda \in \Lambda}$ such that $M = \bigoplus_{\lambda \in \Lambda} R/I_\lambda$, then every direct summand of $M$ is also a direct sum of cyclic $R$-modules, each isomorphic to one of the $R/I_\lambda$.

Lemma 2.6. Let $R$ be a ring and let $M$ be a homogenous semisimple $R$-module. Then there is an index set $\Lambda$ and a set of elements $\{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $M = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$ where $Rw_\lambda$’s are isomorphic simple $R$-modules and every submodule of $M$ is also of the form $N = \bigoplus_{\gamma \in \Gamma} Rw'_\gamma$ where $\Gamma$ is an index set with $|\Gamma| \leq |\Lambda|$ and $Rw'_\gamma$’s are isomorphic simple $R$-modules.

Proof. The proof is clear from the fact that $\text{Ann}(M)$ is a maximal ideal of $R$ and $M$ is an $R/\text{Ann}(M)$-vector space. □

We conclude this section with the following proposition from [1] that provides an analogue of the Invariant Base Number of a free module over a commutative ring.

Lemma 2.7. ([1, Proposition 2.15]) Let $R$ be a ring. The following statements are equivalent:

1. $R$ is a local ring.
2. If $\bigoplus_{i=1}^n Rx_i \cong \bigoplus_{j=1}^m Ry_j$ where $n, m \in \mathbb{N}$ and $\forall i, j$, $Rx_i, Ry_j$ are nonzero cyclic $R$-modules, then $n = m$.
3. If $\bigoplus_{i \in I} Rx_i \cong \bigoplus_{j \in J} Ry_j$ where $I, J$ are index sets and $Rx_i, Ry_j$ are nonzero cyclic $R$-modules, then $|I| = |J|$.

3. Main Results

Theorem 3.1. Let $(R, M)$ be a local ring such that every ideal of $R$ is a direct sum of cyclic $R$-modules. Then $\dim(R) \leq 1$, $|\text{Spec}(R)| \leq 3$, and there is an index set $\Lambda$ and a set of elements $\{x, y\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $M = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ with: each $Rw_\lambda$ a simple $R$-module. Moreover,

(a) If $x, y \in \text{Nil}(R)$, then $\text{Spec}(R) = \{M\}$.
(b) If $M = Rz$ and $z \not\in \text{Nil}(R)$, then $\text{Spec}(R) = \{(0), M\}$.
(c) If $M$ is not cyclic, $x \not\in \text{Nil}(R)$ and $y \in \text{Nil}(R)$, then $\text{Spec}(R) = \{M, Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$.
(d) If $M$ is not cyclic, $x \in \text{Nil}(R)$ and $y \not\in \text{Nil}(R)$, then $\text{Spec}(R) = \{M, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$.
(e) If $M$ is not cyclic and $x, y \not\in \text{Nil}(R)$, then
\[ \text{Spec}(R) = \{ M, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda), Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \}. \]

**Proof.** By Proposition 2.2, \( \dim(R) \leq 1 \). Also, by Lemma 2.4, there is an index set \( \Lambda \) and a set of elements \( \{ x, y \} \cup \{ w_\lambda \}_{\lambda \in \Lambda} \subseteq R \) such that \( M = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \) with: each \( Rw_\lambda \) a simple \( R \)-module. Thus by Lemma 2.1, \( w_\lambda^2 = 0 \) for each \( \lambda \in \Lambda \). We consider the following five cases.

Case (a): Suppose that \( x, y \in \text{Nil}(R) \). Since \( w_\lambda^2 = 0 \) for each \( \lambda \in \Lambda \), we conclude that \( \text{Nil}(R) = M \) and thus \( \text{Spec}(R) = \{ M \} \).

Case (b): Suppose that \( M = Rz \) and \( z \notin \text{Nil}(R) \). Then \( \dim(R) = 1 \). Let \( P \in \text{Spec}(R) \setminus \{ M \} \). Since \( P \subseteq M = Rz \), \( P = Pz \) and, by Nakayama’s lemma, \( P = (0) \) (since Nakayama’s lemma holds for any direct sum of finitely generated modules). Thus, by Lemma 2.1, \( R \) is a principal ideal domain and \( \text{Spec}(R) = \{ (0), M \} \).

Case (c): Suppose that \( M \) is not cyclic, \( x \notin \text{Nil}(R) \) and \( y \notin \text{Nil}(R) \). Then \( \text{Nil}(R) \neq M \) and \( \dim(R) = 1 \). Let \( P \) be a prime ideal of \( R \) such that \( P \subseteq M \). Since \( w_\lambda^2 = 0 \) for each \( \lambda \in \Lambda \), we conclude that \( Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \subseteq P \). Thus \( P = (Rx \cap P) \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \). Since \( M \nsubseteq P \), it follows that \( x \notin P \) and so \( Rx \cap P = Px \). Thus \( P = Px \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \) and hence, \( Px = Px^2 = RzPz \). By Nakayama’s lemma, \( Px = 0 \). Thus \( P = Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \).

Therefore, \( \text{Spec}(R) = \{ M, Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \} \).

Case (d): If \( M \) is not cyclic, \( x \in \text{Nil}(R) \) and \( y \notin \text{Nil}(R) \) and a similar argument allows us to conclude that \( \text{Spec}(R) = \{ M, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \} \).

Case (e): Suppose that \( M \) is not cyclic and \( x, y \notin \text{Nil}(R) \). Thus \( \text{Nil}(R) \neq M \) and so \( \dim(R) = 1 \). Let \( P \) be a prime ideal of \( R \) such that \( P \subseteq M \). Since \( xy = 0 \), \( x \in P \) or \( y \in P \). If \( x \in P \), then \( y \notin P \) and \( P = Rx \oplus (P \cap Ry) \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \). As in Case (c), we have \( Rx \cap P = Py = (0) \) and \( P = Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \). Similarly, if \( y \in P \), then \( P = Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \). On the other hand, since \( x, y \notin \text{Nil}(R) \), there exist \( P_1, P_2 \in \text{Spec}(R) \setminus \{ M \} \) such that \( x \in P_1 \), \( y \notin P_1 \) and \( x \notin P_2 \), \( y \in P_2 \). Therefore, \( \text{Spec}(R) = \{ M, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda), Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \} \). \( \square \)

We can now state the following corollary, an analog of both Kaplansky’s theorem [4], Theorem 12.3] (which states that a commutative Noetherian ring \( R \) is a principal ideal ring if and only if every maximal ideal of \( R \) is principal) and Cohen’s theorem [2] (which states that \( R \) is Noetherian if and only if every prime ideal of \( R \) is finitely generated).

**Corollary 3.2.** Let \( (R, M) \) be a local ring such that every ideal of \( R \) is a direct sum of cyclic \( R \)-modules. Then \( M \) is cyclic (resp. finitely generated) if and only if \( R \) is a principal ideal ring (resp. Noetherian ring).

**Proof.** From Theorem 3.1, we see that if \( M \) is principal (resp. finitely generated), then
the same holds for every prime ideal of \( R \). The conclusion follows from Lemma 2.1 (resp. Cohen’s theorem). □

Next, we sharpen Corollary 2.3 of [1] (see Lemma 2.4).

**Theorem 3.3.** Let \((R, M)\) be a local ring such that every ideal of \( R \) is a direct sum of cyclic \( R \)-modules. Then, with the notation of Theorem 3.1, both \( R/\text{Ann}(x) \) and \( R/\text{Ann}(y) \) are principal ideal rings.

**Proof.** Using Theorem 3.1, we see that each prime ideal of \( S := R/(Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_{\lambda})) \) is principal. Now Lemma 2.1 implies that \( S \) is a principal ideal ring. Since \( Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_{\lambda}) \subseteq \text{Ann}(x) \), \( R/\text{Ann}(x) \) is a homomorphic image of \( S \) and is thus a principal ideal ring. Similarly, \( R/\text{Ann}(y) \) is a principal ideal ring. □

Let us now outline the proof of the main theorem of this paper. We have divided the proof into a sequence of propositions.

**Proposition 3.4.** Let \((R, M)\) be a local ring such that \( M = Rx \oplus L \) for some ideal \( L \) of \( R \) and some \( x \in R \). Suppose further that \( R/\text{Ann}(x) \) is a principal ideal ring. Then every nonzero element of \( Rx \) is of the form \( ax^n \) for some unit \( a \) and positive integer \( n \).

**Proof.** We identify \( Rx \) with \( \bar{R} := R/\text{Ann}(x) \). Then an element \( a \in R \) is a unit whenever \( \bar{a} \) is a unit in \( \bar{R} \) (in fact, if \( \bar{a} \bar{b} = \bar{1} \) for some \( b \in R \), then \( ab - 1 \in \text{Ann}(x) \subseteq M \) and so \( a \) is a unit in \( R \)). Since \( \bar{R} \) is a local principal ideal ring, with maximal ideal \( \bar{R} \bar{x} \), \( \bar{R} \) is a Noetherian ring and so by Krull’s intersection theorem \( \bigcap_{i=1}^{\infty} \bar{R}\bar{x}^i = (0) \). Suppose that \( 0 \neq rx \in Rx \) where \( r \) is not a unit. Thus \( 0 \neq \bar{r} \in \bar{R}\bar{x} \). We claim that \( \bar{r} = \bar{a}\bar{x}^n \) for some unit \( \bar{a} \) and \( n \in \mathbb{N} \). If not, for each \( i \in \mathbb{N} \) there exists \( \bar{r}_i \in \bar{R}\bar{x} \) such that \( \bar{r} = \bar{r}_i\bar{x}^i \), i.e., \( \bar{r} \in \bigcap_{i=1}^{\infty} \bar{R}\bar{x}^i = (0) \), a contradiction. Therefore, \( \bar{r} = \bar{a}\bar{x}^n \) for some unit \( \bar{a} \) and \( n \in \mathbb{N} \). Consequently, \( a \) is a unit in \( R \) and \( rx = ax^{n+1} \), which completes the proof. □

**Proposition 3.5.** Let \((R, M)\) be a local ring such that there is an index set \( \Lambda \) and a set of elements \( \{x\} \cup \{w_{\lambda}\}_{\lambda \in \Lambda} \subseteq R \) such that \( M = Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_{\lambda}) \) with: each \( Rw_{\lambda} \) a simple \( R \)-module and \( R/\text{Ann}(x) \) principal ideal ring. Then every proper ideal of \( R \) is of the form \( I = Rx' \oplus (\bigoplus_{\gamma \in \Gamma} Rw_{\gamma}) \) where \( \Gamma \) is an index set with \( |\Gamma| \leq |\Lambda| \), \( Rw_{\gamma} \)'s \( (\gamma \in \Gamma) \) are simple \( R \)-modules and \( x' \in R \) is such that \( R/\text{Ann}(x') \) is a principal ideal ring.

**Proof.** Assume that \( I \) is a proper ideal of \( R \) and \( L = \bigoplus_{\lambda \in \Lambda} Rw_{\lambda} \). Clearly, \( L \) is a homogenous semisimple \( R \)-module with \( L^2 = (0) \). If \( I \subseteq Rx \), then every ideal contained in \( I \) is principal, and we are done since \( R/\text{Ann}(x) \) is a principal ideal ring. If \( I \nsubseteq L \), then by Lemma 2.6, \( I \) is a direct sum of at most \( |\Lambda| \) simple modules. Thus we can assume that \( I \nsubseteq Rx \), \( I \nsubseteq L \) and \( (0) \nsubseteq I \nsubseteq M \). By Proposition 3.4, there exist \( n \in \mathbb{N} \) and \( l \in L \) such that \( x^n + l \in I \). Among all such expressions, choose one, \( x^{n_0} + l_0 \), with \( n_0 \) minimal. We
set \( x' = x^{\nu_0} \) if \( x^{\nu_0} \in I \), otherwise we set \( x' = x^{\nu_0} + l_0 \). Set

\[
J := \{ l \in L \mid ax + l \in I, \text{ for some } a \in R \}.
\]

Then \( J \) is an ideal of \( R \) with \( J \subseteq L \). We next prove that \( Rx' \cap (I \cap J) = (0) \). For see this, let \( rx' \in Rx' \cap (I \cap J) \) where \( r \in R \). If \( x^{\nu_0} \in I \), then \( x' = x^{\nu_0} \) and so \( rx' = rx^{\nu_0} \in Rx \cap L = (0) \). If \( x^{\nu_0} \not\in I \), then \( rx' = r(x^{\nu_0} + l_0) = l \) for some \( l \in I \cap J \subseteq L \). Hence we have \( rx^{\nu_0} = l - rl_0 \in Rx \cap L = \{0\} \). Since \( x^{\nu_0} \not\in I \), \( r \) is not a unit and so by Proposition 3.4, \( r = ax^n + l_1 \) where \( a \) is a unit, \( l_1 \in L \) and \( n \in \mathbb{N} \). Since \( L^2 = (0) \), we conclude that \( rx' = (ax^n + l_1)(x^{\nu_0} + l_0) = ax^{n+\nu_0} = l \in Rx \cap L = \{0\} \). Thus \( Rx' \cap (I \cap J) = (0) \) and \( Rx' \oplus (I \cap J) \subseteq I \). In fact, we will show that \( I = Rx' \oplus (I \cap J) \). Assume that \( u = ax^s + l \in I \) where \( a \in (R \setminus \mathcal{M}) \cup \{0\} \), \( s \in \mathbb{N} \) and \( l \in L \). If \( a = 0 \), then \( l \in I \cap J \) and so \( u = l \in Rx' \oplus (I \cap J) \). Thus we can assume that \( a \in R \setminus \mathcal{M} \). Therefore,

\[
u - ax^{s-\nu_0}x' = (ax^n + l) - ax^{s-\nu_0}(x^{\nu_0} + l_0) = l + ax^{s-\nu_0}l_0 \in I.
\]

Since \( l \) and \( l_0 \) contained in \( J \), \( l + ax^{s-\nu_0}l_0 \in J \). Hence it follows that \( u \in Rx' \oplus (I \cap J) \). Therefore, \( I = Rx' \oplus (I \cap J) \). By Lemma 2.6, \( I \cap L \) is a direct sum of at most \( |\Lambda| \) simple \( R \)-modules. Since \( \mathcal{M}l_0 = (0) \), we conclude that \( \text{Ann}(x^{\nu_0} + l_0) = \text{Ann}(x^{\nu_0}) \). This implies that \( Rx' \cong Rx^{\nu_0} \subseteq Rx \). Since \( R/\text{Ann}(x) \) is a principal ideal ring, \( R/\text{Ann}(x') \) is also a principal ideal ring. □

**Proposition 3.6.** Let \((R, \mathcal{M})\) be a local ring such that there is an index set \( \Lambda \) and a set of elements \( \{x, y\} \cup \{w_{\lambda}\}_{\lambda \in \Lambda} \subseteq R \) such that \( \mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_{\lambda}) \) with: each \( Rw_{\lambda} \) a simple \( R \)-module and \( R/\text{Ann}(x), R/\text{Ann}(y) \) principal ideal rings. If \( I \) is an ideal of \( R \), then one of the following holds:

(i) There is an index set \( \Gamma \) and a set of elements \( \{x', y'\} \cup \{w'_{\gamma}\}_{\gamma \in \Gamma} \subseteq R \) such that \( I = Rx' \oplus Ry' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_{\gamma}) \) with: \( |\Gamma| \leq |\Lambda| \), each \( Rw'_{\gamma} \) a simple \( R \)-module, and \( R/\text{Ann}(x') \), \( R/\text{Ann}(y') \) principal ideal rings.

(ii) There is an index set \( \Gamma \) and a set of elements \( \{z'\} \cup \{w'_{\gamma}\}_{\gamma \in \Gamma} \subseteq R \) such that \( I = Rz' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_{\gamma}) \) with: \( |\Gamma| \leq |\Lambda| \), each \( Rw'_{\gamma} \) a simple \( R \)-module, and each ideal of \( R/\text{Ann}(z') \) a direct sum of at most two principal ideals.

**Proof.** By Proposition 3.5 we can assume that \( Rx \) and \( Ry \) are not simple \( R \)-modules. Let \( L = \bigoplus_{\lambda \in \Lambda} Rw_{\lambda} \). Clearly both \( R/Rx \) and \( R/Ry \) are local rings with maximal ideals \( \mathcal{M}_x = \mathcal{M}/Rx \cong Ry \oplus L \) and \( \mathcal{M}_y = \mathcal{M}/Ry \cong Rx \oplus L \). Assume that \( I \) is an ideal of \( R \). First, note that if \( I \subseteq Rx \oplus L \) (resp. \( I \subseteq Ry \oplus L \)), then \( I \cong (I \oplus Rx)/Rx \) (resp. \( I \cong (I \oplus Ry)/Ry \)) and so, by Proposition 3.5, \( I = Rx' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_{\gamma}) \) where \( \Gamma \) is an index set with \( |\Gamma| \leq |\Lambda| \), the \( Rw'_{\gamma} \)’s (\( \gamma \in \Gamma \)) are simple \( R \)-modules, and \( x' \in R \) is such that \( R/\text{Ann}(x') \) is a principal ideal ring. Therefore, according to the above remark, we can assume that \( I \not\subseteq Rx \oplus L, I \not\subseteq Ry \oplus L \) and \( (0) \not\subseteq I \not\subseteq \mathcal{M} \).
By Proposition 3.4, every element of $I$ has the form $ax^s + by^t + l$ where $a \in (R \setminus \mathcal{M}) \cup \{0\}$, $b \in (R \setminus \mathcal{M}) \cup \{0\}$, $s$, $t \in \mathbb{N}$ and $l \in L$. Since $I \not\subseteq Rx \oplus L$, and $I \not\subseteq Ry \oplus L$, it follows that there exist $e_1, e_2 \in I$ where $e_1 = a_1 x^n + b_1 y^m + l_1$, and $e_2 = a_2 x^s + b_2 y^t + l_2$ for some $a_1, b_2 \in R \setminus \mathcal{M}$, $a_2, b_1 \in R$, $l_1, l_2 \in L$ and $n, s, t, m \in \mathbb{N}$ (in fact $e_1 \in I \setminus Rx \oplus L$ and $e_2 \in I \setminus Ry \oplus L$). Thus $xe_1 = a_1 x^{n+1} \in I$ and $ye_2 = b_2 y^{m+1} \in I$. Since $a_1, b_2 \in R \setminus \mathcal{M}$, $x^{n+1} \in I$ and $y^{m+1} \in I$. Suppose that $n_0$ (resp. $m_0$) is the smallest natural number such that $x^{n_0} + l_1 \in I$ (resp. $y^{m_0} + l_2 \in I$) for some $l_1 \in L$ (resp. $l_2 \in L$). We set $x' = x^{n_0}$ if $x^{n_0} \in I$, otherwise we set $x' = x^{n_0} + l_1$. Also, we set $y' = y^{m_0}$ if $y^{m_0} \in I$, otherwise we set $y' = y^{m_0} + l_2$. Set

$$J := \{ l \in L \mid ax + by + l \in I, \text{ for some } a, b \in R \}.$$ 

Then $J$ is an ideal of $R$ with $J \subseteq L$. On can easily see that $Rx' + Ry' + (I \cap J) = Rx' \oplus Ry' \oplus (I \cap J) \subseteq I$. Now we proceed by cases.

Case (a): Suppose that $I = Rx' \oplus Ry' \oplus (I \cap J)$. By Lemma 2.6, $I \cap J$ is a direct sum of at most $|\Lambda|$ simple $R$-modules. It follows that $I$ is a direct sum of at most $|\Lambda| + 2$ cyclic modules by Lemma 2.7. Also, if $x' \neq 0$, then $x^{n_0} \neq 0$ and so $\text{Ann}(x') = \text{Ann}(x^{n_0})$ since $\mathcal{M}l_1 = (0)$. It follows that $R/\text{Ann}(x')$ is a principal ideal ring (since $Rx^{n_0} \subseteq Rx$). We conclude similarly that either $Ry' = (0)$ or $R/\text{Ann}(y')$ is a principal ideal ring.

Case (b): Suppose that $Rx' \oplus Ry' \oplus (I \cap J) \not\subseteq I$. We claim that every element of $I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$ is of the form $cx^{n_0-1} + dy^{m_0-1} + l$ where $c, d \in R \setminus \mathcal{M}$ and $l \in L$. Let $z = cx^s + dy^t + l \in I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$ where $c, d \in (R \setminus \mathcal{M}) \cup \{0\}$ and $l \in L$. If $c = 0$, then $z = dy^t + l \in I$ and so $t \geq m_0$. If $t > m_0$, then $z = dy^{t-m_0}(y^{m_0} + l_2) + l \in Ry' \oplus (I \cap J)$, a contradiction. Thus $t = m_0$ and this implies that $z = dy^t + (l - dl_1) \in Ry' \oplus (I \cap J)$, a contradiction. Thus $c \in R \setminus \mathcal{M}$. We conclude similarly that $d \in R \setminus \mathcal{M}$. Our next claim is that $s = n_0 - 1$ and $t = m_0 - 1$. If not, to obtain a contradiction, as we see by considering the following three subcases:

Subcase (i): Suppose that $s < n_0 - 1$ or $t < m_0 - 1$. There is no loss of generality in assuming that $s < n_0 - 1$. Then $x^{n_0-1} = x^{n_0-1-s}c^{-1}z \in I$ which contradicts the minimality of $n_0$.

Subcase (ii): Suppose that $s \geq n_0$ and $t \geq m_0$. Then $z = cx^{s-n_0}x' + dy^{t-m_0}y' - (cx^{s-n_0}l_1 - dy^{t-m_0}l_2) + l \in I$. Since $z - cx^{s-n_0}x' + dy^{t-m_0}y' \in I$, $(cx^{s-n_0}l_1 - dy^{t-m_0}l_2) + l \in (I \cap J)$ and hence $z \in Rx' \oplus Ry' \oplus (I \cap J)$, a contradiction.

Subcase (iii): Suppose that $s \geq n_0$, $t = m_0 - 1$ or $s = n_0 - 1$, $t \geq m_0$. Without loss of generality we can assume $s \geq n_0$ and $t = m_0 - 1$. Then $z = cx^{s-n_0}x' + dy^{m_0-1} - cx^{s-n_0}l_1 + l$ and so $z - cx^{s-n_0}x' = dy^{m_0-1} - cx^{s-n_0}l_1 + l \in I$. Thus $y^{m_0-1} + d^{-1}(l - cx^{s-n_0}l_1) \in I$ which contradicts the minimality of $m_0$.

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Therefore, every element of $I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$ has the form $cx^{\alpha_0} + dy^{\beta_0} + l$ where $c, d \in R \setminus \mathcal{M}$ and $l \in L$. Let $z' = cx^{\alpha_0} + dy^{\beta_0} + l \in I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$ where $c, d \in R \setminus \mathcal{M}$ and $l \in L$. Since $L^2 = (0)$, it is easy to check that $Rz' \cap (I \cap J) = (0)$. We note that if $x' = x^{\alpha_0}$, then $x' = xc^{-1}z'$ and so $x' \in Rz'$. Also, if $x' = x^{\alpha_0} + l_1$ for some nonzero element $l_1$ of $L$, then $x' = xc^{-1}z' + l_1$. Therefore, since $l_1 \in J$ and $l_1 = x' - xc^{-1}z' \in I$, so $l_1 \in I \cap J$. Hence $x' \in Rz' \oplus (I \cap J)$.

We conclude similarly that $y' \in Rz' \oplus (I \cap J)$. Thus $Rx' \oplus Ry' \oplus (I \cap J) \subseteq Rz' \oplus (I \cap J) \subseteq I$. Now we are in a position to prove the main theorem of this paper. In fact we describe $\Lambda$-modules. It follows that $\Lambda = \Lambda/\Lambda(0)$ and so

$$\Lambda = \Lambda/\Lambda(0).$$

Therefore

$$A = 0, \quad l = 0, \quad u = c'x^{\alpha_0} + d'y^{\beta_0} + l'$$

for some $c', d' \in R \setminus \mathcal{M}$ and $l' \in L$. Then

$$u - c'c^{-1}z' = (c'x^{\alpha_0} + d'y^{\beta_0} + l') - c'c^{-1}(cx^{\alpha_0} + dy^{\beta_0} + l) = d'y^{\beta_0} + l' \in I$$

where $d' = d' - c'c^{-1}d$ and $l'' = l' - c'c^{-1}l$. If $d'' = 0$, then $l'' \in (I \cap J)$ and so $u = c'e^{-1}z' + l'' \in Rz' \oplus (I \cap J)$, a contradiction. If $d'' \in R \setminus \mathcal{M}$, then $y^{\beta_0} + d''^{-1}l'' \in I$ which contradicts the minimality of $m_0$. Thus $d'' \in \mathcal{M}$ and there exists $r \in R$ such that $d''y^{\beta_0} = ry^{\beta_0}$. Therefore

$$u - c'c^{-1}z' = ry^{\beta_0} + l'' = ry' - rl_2 + l'' \in I,$$

and since $l'' - rl_2 \in I \cap J$, it follows that $u \in Rz' \oplus (I \cap J)$, a contradiction.

Therefore $I = Rz' \oplus (I \cap J)$. By Lemma 2.6, $I \cap J$ is a direct sum of at most $|\Lambda|$ simple $R$-modules. It follows that $I$ is a direct sum of at most $|\Lambda| + 1$ cyclic $R$-modules by Lemma 2.7. We need only consider the structure of each ideal of $R/\text{Ann}(z')$. It is easy to check that $\text{Ann}(z') = \text{Ann}(x^{\alpha_0}) \cap \text{Ann}(y^{\beta_0})$. Also, a trivial verification shows that

$$\text{Ann}(x^{\alpha_0}) = H_1 \oplus Ry \oplus L \quad \text{and} \quad \text{Ann}(y^{\beta_0}) = Rx \oplus H_2 \oplus L$$

where

$$H_1 = \{rx \in Rx \mid rx + sy + l \in \text{Ann}(x^{\alpha_0}), \text{ for some } s \in R, \ l \in L\}$$

$$H_2 = \{sy \in Ry \mid rx + sy + l \in \text{Ann}(y^{\beta_0}), \text{ for some } r \in R, \ l \in L\}.$$

Therefore we conclude that $\text{Ann}(z') = H_1 \oplus H_2 \oplus L$. Put $R = R/\text{Ann}(z')$. Then $R$ is a local ring with the maximal ideal $\mathcal{M} := \mathcal{M}/\text{Ann}(z') \cong Rx/H_1 \oplus Ry/H_2$. It follows that $\mathcal{M}$ is a direct sum of at most two cyclic $\mathcal{M}$-modules. Thus, by Corollary 3.2, the ring $\mathcal{M}$ is Noetherian. Therefore, by [1, Theorem 2.11], every ideal of $\mathcal{M}$ is a direct sum of at most two cyclic modules (see Result 1.2) which completes the proof. □

Now we are in a position to prove the main theorem of this paper. In fact we describe the ideal structure of local rings $R$ for which every ideal is a direct sum of cyclic modules.

**Theorem 3.7.** Let $(R, \mathcal{M})$ be a local ring. The following statements are equivalent:

(1) $\mathcal{M}$ is a direct sum of cyclic $R$-modules.

(2) Every ideal of $\mathcal{M}$ is a direct sum of cyclic $\mathcal{M}$-modules.

(3) Every ideal of $R$ is a direct sum of cyclic $R$-modules.

(4) Every ideal of $R$ is a direct sum of cyclic modules.
Every ideal of $R$ is a direct sum of cyclic $R$-modules.

(2) Every ideal of $R$ is a direct sum of cyclic $R$-modules, at most two of which are not simple.

(3) There is an index set $\Lambda$ and a set of elements $\{x, y\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $M = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ with: each $Rw_\lambda$ a simple $R$-module and $R/\text{Ann}(x)$, $R/\text{Ann}(y)$ principal ideal rings.

(4) There is an index set $\Lambda$ such that every ideal $I$ of $R$ is one of the following forms:

(i) $I = Rx' \oplus Ry' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where $\Gamma$ is an index set with $|\Gamma| \leq |\Lambda|$ where $Rw'_\gamma$'s are simple $R$-modules and $x', y' \in R$ such that $R/\text{Ann}(x')$, $R/\text{Ann}(y')$ are principal ideal rings.

(ii) $I = Rz' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where $\Gamma$ is an index set with $|\Gamma| \leq |\Lambda|$, $Rw'_\gamma$'s are simple $R$-modules and $z' \in R$ is such that every ideal of $R/\text{Ann}(z')$ is a direct sum of at most two principal ideals.

(5) Every ideal of $R$ is a direct summand of a direct sum of cyclic $R$-modules.

(We note that the index sets $\Lambda$’s in the above statements (3) and (4) are the same and $|\Lambda| + 2$ is a bound for the direct sum decompositions of all ideals of $R$).

Proof. (1) $\Rightarrow$ (3). Follows from Theorem 3.1 and Theorem 3.3.

(3) $\Rightarrow$ (4). Follows from Propositions 3.5 and 3.6.

(4) $\Rightarrow$ (2), (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1). Follows from Lemma 2.5. □

By Corollary 3.2, Theorem 3.7 and [1, Theorem 2.11], we have the following ideal structure description for local rings $(R, M)$ where $M$ is finitely generated and every ideal of $R$ is a direct sum of cyclic modules. Also, this result is an analogue of Kaplansky’s theorem [4, Theorem 12.3].

Corollary 3.8. Let $(R, M)$ be a local ring. Then the following statements are equivalent:

(1) $M$ is finitely generated and every ideal of $R$ is a direct sum of cyclic $R$-modules.

(2) $R$ is Noetherian and every ideal of $R$ is a direct sum of cyclic $R$-modules.

(3) There exist an positive integer $n$ and a set of elements $\{v_1, \cdots, v_n\} \subseteq R$ such that $M = Rv_1 \oplus \cdots \oplus Rv_n$ with: each $R/\text{Ann}(v_i)$ a principal ideal ring and at most two of $Rv_i$’s not simple.

(4) There exists an positive integer $n$ such that every ideal of $R$ is of the form $I = Rv_1 \oplus \cdots \oplus Rv_m$ where $m \leq n$, $\{v_1, \cdots, v_m\} \subseteq R$ and at most two of $Rv_i$’s are not simple.
There exists an positive integer \( n \) such that every ideal of \( R \) is a direct sum of at most \( n \) cyclic \( R \)-modules.

There exists an positive integer \( n \) such that every ideal of \( R \) is a direct summand of a direct sum of at most \( n \) cyclic \( R \)-modules.

(We note that the integers \( n \)'s in the above statements are the same and it is a bound for the direct sum decompositions of all ideals of \( R \)).

**Remark 3.9.** Let \( R_1, \ldots, R_k \), where \( k \in \mathbb{N} \), be nonzero rings, and let \( R \) denote the direct product ring \( \prod_{i=1}^{k} R_i \). It is well-known that, if \( I_i \) is an ideal of \( R_i \) for each \( i = 1, \ldots, k \), then \( I = \prod_{i=1}^{k} I_i \) is an ideal of \( R \). Furthermore each ideal of \( R \) has this form. It is straightforward to check that the ideal \( I = \prod_{i=1}^{k} I_i \) of \( R \) is a direct sum of cyclic ideals of \( R \) if and only if the ideal \( I_i \) is a direct sum of cyclic ideals of \( R_i \) for each \( i = 1, \ldots, k \).

We are thus led to the following strengthening of Theorem 3.7 and Remark 3.9.

**Corollary 3.10.** Let \( R = \prod_{i=1}^{k} R_i \), where \( k \in \mathbb{N} \) and where each \( R_i \) is a local ring with maximal ideal \( M_i \) (1 \( \leq i \leq k \)). The following statements are equivalent:

1. Every ideal of \( R \) is a direct sum of cyclic \( R \)-modules.
2. For each \( i \), every ideal of \( R_i \) is a direct sum of cyclic \( R_i \)-modules, at most two of which are not simple.
3. For each \( i \), there is an index set \( \Lambda_i \) and a set of elements \( \{x, y\} \cup \{w_\lambda\}_{\lambda \in \Lambda_i} \subseteq R_i \) such that \( M_i = R_i x \oplus R_i y \oplus (\bigoplus_{\lambda \in \Lambda_i} R_i w_\lambda) \) with: each \( R_i w_\lambda \) a simple \( R_i \)-module and \( R_i/\text{Ann}(x), R_i/\text{Ann}(y) \) principal ideal rings.
4. There exit index sets \( \Lambda_1, \ldots, \Lambda_k \) such that for each \( 1 \leq i \leq k \), every ideal of \( R_i \) is of the forms \( I = R_i x' \oplus R_i y' \oplus (\bigoplus_{\gamma \in \Gamma_i} R_i w'_\gamma) \) or \( I = R_i z' \oplus (\bigoplus_{\gamma \in \Gamma_i} R_i w'_\gamma) \) where \( \Gamma_i \) is an index set with \( |\Gamma_i| \leq |\Lambda_i| \), \( R_i w'_\gamma \)'s (\( \gamma \in \Gamma_i \)) are simple \( R_i \)-modules, \( x', y', z' \in R_i \) such that \( R_i/\text{Ann}(x'), R_i/\text{Ann}(y') \) are principal ideal rings and every ideal of \( R_i/\text{Ann}(z') \) is a direct sum of at most two principal ideals.
5. Every ideal of \( R \) is a direct summand of a direct sum of cyclic \( R \)-modules.

**4. Examples**

In this section we provide several examples illustrating the results of Section 3 as well as the necessity of certain hypotheses in these results. We begin with the following interesting example. In fact, the following example shows that the corresponding of the above Corollary 3.10, is not true in general for the case \( R = \prod_{\lambda \in \Lambda} R_\lambda \) where \( \Lambda \) is an infinite index set and each \( R_\lambda \) is a local ring (even if for each \( \lambda \in \Lambda, R_\lambda \cong \mathbb{Z}_2 \)).
**Example 4.1.** Let $\Lambda$ be an infinite index set and $\{F_\lambda \mid \lambda \in \Lambda\}$ be a set of fields. We put $R = \prod_{\lambda \in \Lambda} F_\lambda$. Thus for each $\lambda \in \Lambda$, $F_\lambda$ is a local ring and every ideal of $F_\lambda$ is cyclic (that is, $(0)$ or $F_\lambda$). Clearly, $I = \bigoplus_{\lambda \in \Lambda} F_\lambda$ is a non-maximal ideal of $R$ and hence there exists a maximal ideal $P$ of $R$ such that $I \subseteq P$. It was shown by Cohen and Kaplansky \[3\] Lemma 1] that $P$ is not a direct sum of principal ideals. Thus the corresponding of the above Corollary 3.10, is not true in general for the case $R = \prod_{\lambda \in \Lambda} R_\lambda$ where $\Lambda$ is an infinite index set.

Let $(R, \mathcal{M})$ be a local ring. By Theorem 3.7, every ideal of $R$ is a direct sum of cyclic $R$-modules if and only if there is an index set $\Lambda$ and a set of elements $\{x, y\} \cup \{w_\lambda\}_{\lambda \in \Lambda} \subseteq R$ such that $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ with: (i) each $Rw_\lambda$ a simple $R$-module, (ii) $R/\text{Ann}(x)$, $R/\text{Ann}(y)$ principal ideal rings. Moreover, in this case, every ideal $I$ of $R$ has the form $I = R\gamma' \oplus Ry' \oplus (\bigoplus_{\gamma \in \Gamma} Rw_\gamma)$ where $\Gamma$ is an index set with $|\Gamma| \leq |\Lambda|$, $Rw_\gamma$s ($\gamma \in \Gamma$) are simple $R$-modules and $x', y' \in R$. The following example shows that property (ii) does not hold for all ideals of $R$, even if $R$ is Artinian and $\mathcal{M}$ is two generated.

**Example 4.2.** Let $F$ be a field, let $n \geq 3$ and let $R$ be the $F$-algebra with generators $x, y$ subject to the relations $x^n = y^n = xy = 0$, $R \cong F[X, Y]/(X^n, Y^n, XY)$. The ring $R$ is a Noetherian local ring with maximal ideal $\mathcal{M} = Rx \oplus Ry$. Since $\mathcal{M}^n = (0)$, $\dim(R) = 0$ and so $R$ is an Artinian local ring. By Theorem 3.7, every ideal of $R$ is a direct sum of at most two cyclic $R$-modules and $R/\text{Ann}(x)$, and $R/\text{Ann}(y)$ are principal ideal rings. Let $I = Rz$ where $z = x + y$. We note that if $I = \bigoplus_{i=1}^n Rz_i$ where $n \in \mathbb{N}$ and $Rz_i$ are nonzero cyclic $R$-modules, then by Lemma 2.7, $n = 1$. Set $\bar{R} = R/\text{Ann}(z)$. Clearly $\text{Ann}(z) = Rx^{n-1} \oplus Ry^{n-1}$. Since $\bar{M} := \mathcal{M}/\text{Ann}(z) \cong (Rx/Rx^{n-1}) \oplus (Ry/Ry^{n-1})$, it follows that the maximal ideal $\bar{M}$ of $\bar{R}$ is a direct sum of two nonzero cyclic $\bar{R}$-modules and hence by Lemma 2.7, $\mathcal{M}$ is not principal, i.e., $\bar{R}$ is not a principal ideal ring.

By \[1\] Example 3.1, for each integer $n \geq 3$, there exists an Artinian (Noetherian) local ring $(R, \mathcal{M})$ such that $\mathcal{M}$ is a direct sum of $n$ cyclic $R$-modules, but there exists a two generated ideal of $R$ such that it is not a direct sum of cyclic $R$-modules. Next, the following example shows that there exists also a non-Noetherian local ring $R$ such that every prime ideal of $R$ is a direct sum of cyclic $R$-modules, but some of the ideals of $R$ are not direct sums of cyclic $R$-modules.

**Example 4.3.** Let $F$ be a field and let $R$ be the $F$-algebra with generators $\{x_i \mid i \in \mathbb{N}\}$ subject to the following relations $x_i^3 = x_j^3 = x_k^2 = 0$, $k \geq 4$ and $x_ix_j = 0$ for all $i \neq j$, $R \cong F[\{X_i \mid i \in \mathbb{N}\}]/(X_i^3, X_j^3, X_k^2, X_iX_j \mid 4 \leq k \in \mathbb{N}, i \neq j \geq 1)$. Then $R$ is a non-Noetherian local ring with the maximal ideal $\mathcal{M} = \bigoplus_{i \in \mathbb{N}} Rx_i$. Clearly $\text{Spec}(R) = \{\mathcal{M}\}$ since $\mathcal{M}^3 = (0)$. Thus every prime ideal of $R$ is a direct sum of cyclic $R$-modules, but by Lemma 2.3, the ideal $J = R(x_1 + x_2) + R(x_1 + x_3)$ is not a direct sum of cyclic $R$-modules.
By Theorem 3.1, if a local ring \((R, \mathcal{M})\) has the property that every ideal \(I\) of \(R\) is a direct sum of cyclic \(R\)-modules, then \(\dim(R) \leq 1\) and \(|\text{Spec}(R)| \leq 3\). Clearly, \(|\text{Spec}(R)| = 1\) whenever \(\dim(R) = 0\) and \(|\text{Spec}(R)| = 2\) or \(3\) whenever \(\dim(R) = 1\). The following examples cover all the different cases mentioned above for \(\dim(R)\) and \(|\text{Spec}(R)|\).

**Example 4.4.** Let \(F\) be a field and \(n \in \mathbb{N}\). Consider the following rings:

1. \(R_1 = F[[X]]\) (formal power series ring).
2. \(R_2 = F[[X_1, X_2, \ldots, X_n]]/\langle X_iX_j \mid 1 \leq i, j \leq n \rangle\).
3. \(R_3 = F[[X_1]]/\langle X_1X_j \mid i, j \in \mathbb{N} \rangle\).
4. \(R_4 = F[[X, Y]]/\langle XY, Y^2 \rangle\).
5. \(R_5 = F[[X_1, X_2]]/\langle X_1X_j \mid i \neq j \rangle \cup \{X_1^n \mid i \geq 2\}\).
6. \(R_6 = F[[X, Y]]/\langle XY \rangle\).
7. \(R_7 = F[[X_i \mid i \in \mathbb{N}]]/\langle X_iX_j \mid i \neq j \rangle \cup \{X_1^n \mid i \geq 3\}\).

It is easy to check that all above rings are local and by Theorem 3.7, one can easily see that for each \(i \in \{1, 2, \ldots, 7\}\) the ring \(R_i\) has the property that every ideal is a direct sum of cyclic ideals. Let \(\mathcal{M}_i\) denote the maximal ideal of \(R_i\) for \(i \in \{1, 2, \ldots, 7\}\). Then we easily obtain the following:

1. \(R_1\) is a domain (in fact \(R_1\) is a local PID) with \(\dim(R_1) = 1\), \(\mathcal{M}_1 = \langle X \rangle\) and \(\text{Spec}(R_1) = \{(0), \mathcal{M}_1\}\).
2. \(R_2\) is a non-domain Artinian ring with \(\dim(R_2) = 0\), \(\mathcal{M}_2 = \oplus_{i=1}^{n} R_2x_i\) (where \(x_i = X_i + \langle X_iX_j \mid 1 \leq i, j \leq n \rangle\) and \(\text{Spec}(R_2) = \{\mathcal{M}_2\}\).
3. \(R_3\) is a non-domain, non-Noetherian ring with \(\dim(R_3) = 0\), \(\mathcal{M}_3 = \bigoplus_{i=1}^{\infty} R_3x_i\) (where \(x_i = X_i + \langle X_iX_j \mid i, j \in \mathbb{N} \rangle\) and \(\text{Spec}(R_3) = \{\mathcal{M}_3\}\).
4. \(R_4\) is a non-domain, Noetherian ring with \(\dim(R_4) = 1\), \(\mathcal{M}_4 = R_4x \oplus R_4y\) (where \(x = X + \langle XY, Y^2 \rangle\) and \(y = X + \langle XY, Y^2 \rangle\) and \(\text{Spec}(R_4) = \{\mathcal{M}_4, R_4y\}\).
5. \(R_5\) is a non-domain, non-Noetherian ring with \(\dim(R_5) = 1\), \(\mathcal{M}_5 = \bigoplus_{i=1}^{\infty} R_5x_i\) (where \(x_i = X_i + \langle X_iX_j \mid i \neq j \rangle \cup \{X_1^2 \mid i \geq 2\}\) and \(\text{Spec}(R_5) = \{\mathcal{M}_5, \bigoplus_{i=2}^{\infty} R_5x_i\}\).
6. \(R_6\) is a non-domain, Noetherian ring with \(\dim(R_6) = 1\), \(\mathcal{M}_6 = R_6x \oplus R_6y\) (where \(x = X + \langle XY \rangle\) and \(y = Y + \langle XY \rangle\) and \(\text{Spec}(R_6) = \{\mathcal{M}_6, R_6x, R_6y\}\).
7. \(R_7\) is a non-domain, non-Noetherian ring with \(\dim(R_7) = 1\), \(\mathcal{M}_7 = \bigoplus_{i=1}^{\infty} R_7x_i\) (where \(x_i = X_i + \langle X_iX_j \mid i \neq j \rangle \cup \{X_1^2 \mid i \geq 3\}\) and \(\text{Spec}(R_7) = \{\mathcal{M}_7, \bigoplus_{i=2}^{\infty} R_7x_i, R_7x_1 \oplus \bigoplus_{i=3}^{\infty} R_7x_i\}\).
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