On Narrow Operators from $L_p$ into Operator Ideals

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Abstract. It is well known that every $l_2$-strictly singular operator from $L_p$, $1 < p < \infty$ to any Banach space $X$ with an unconditional basis is narrow. In this article, we extend this result to the setting of Banach spaces without an unconditional basis. We show that if $1 \leq p, r < \infty$, then every $\ell_2$-strictly singular operator $T$ from $L_p$ into the Schatten–von Neumann $r$-class $C_r$ is narrow. This is a noncommutative complement to results in Mykhaylyuk et al. (in Israel J Math 203:81–108, 2014).

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1. Introduction

Let $F(0,1)$ be a separable symmetric function space of (classes of equivalent) Lebesgue measurable functions on $(0,1)$. Suppose that $A$ is a measurable subset of $(0,1)$. By a sign on $A$ we mean an element $x \in L_\infty(0,1)$ with $\text{supp } x = A$ which takes values in $\{-1, 0, 1\}$. We say that $x \in F(0,1)$ is a mean zero sign on $A$ if $x$ is a sign on $A$ and $\int_0^1 x \, d\mu = 0$. Let $X$ be a Banach space and $L(F(0,1), X)$ be the space of all bounded linear operators from $F(0,1)$ into $X$. We say that a linear operator $T \in L(F(0,1), X)$ is narrow if for every $\varepsilon > 0$ and every measurable set $A \subset (0,1)$, there exists a mean zero sign $x$ on $A$, such that $\|Tx\|_X < \varepsilon$. The notion of narrow operators was formally introduced by Plichko and Popov [18] for operators acting on symmetric function spaces (in fact, the study of operators of this type goes back to Bourgain, Ghoussoub and Rosenthal, see [18,19,22] and references therein).

In 1990, Plichko and Popov [18] asked whether every $\ell_2$-strictly singular operator $T : L_p \rightarrow X$ is necessarily narrow (see also [19] and [20], Problem...
1.6]). Here, an operator $T$ is said to be $\ell_2$-strictly singular if it is not an isomorphism when restricted to any isomorphic copy of $\ell_2$ in $L_p$.

When $1 \leq p < 2$ and $p < r$, it is proved in [13, Theorem 5] that every operator $T : L_p \to \ell_r$ is narrow. It is also known that when $1 \leq r < 2$ and $1 \leq p < \infty$, every operator $T : L_p \to \ell_r$ is narrow [18, Proposition 2].

The remaining cases were settled in [16]. Precisely, [16, Theorem A] shows that every $T : L_p \to \ell_r$ is narrow when $1 \leq p < \infty$ and $1 \leq r \neq 2 < \infty$ or $1 \leq p < 2$ and $r = 2$. Moreover, when $p > 2$ and $r = 2$, there exists a non-narrow operator $T : L_p \to \ell_2$ [16, Example 1.1]. Finally, [16, Theorem B] asserts that every $\ell_2$-strictly singular operator $T : L_p \to X$ is necessarily narrow provided that $X$ has an unconditional basis.

The main objective of this paper is to establish a noncommutative generalization of [16, Theorems A and B] for the situation where we deal with $\ell_2$-strictly singular operators $T : L_p \to X$ if the space $X$ does not possess an unconditional basis. For all unexplained notions and notations we refer the reader to [1,11,12].

Let $E$ be a separable symmetric sequence space $[2,14]$, that is, a Banach space of sequences such that the standard unit vectors $e_n$’s, $n = 1, 2, 3, \ldots$, (defined by $e_n(j) = \delta_{n,j}$) form a normalized, 1-symmetric basis of $E$. Let $C_E$ be the ideal in $B(\ell_2)$ corresponding to $E$ (see [2,14]), i.e., the Banach space of all compact operators $x$ on $\ell_2$ for which $s(x) \in E$, normed by

$$\|x\|_{C_E} = \|s(x)\|_E.$$ 

Here, $s(x) = \{s_n(x)\}_{n=1}^\infty$ is the sequence of $s$-numbers of $x$, i.e., the eigenvalues of $|x| = (x^*x)^{1/2}$ arranged in a non-increasing ordering, counting multiplicity. In the case when $E = \ell_p$, the ideal $C_{\ell_p}$ is denoted simply by $C_p$.

Let $\{e_{ij}\}_{i,j \geq 1}$ be the matrix unit of $C_E$. Let $T_E$ be the upper triangular part of $C_E$, i.e., $x \in T_E$ if and only if $x := (x_{ij}) \in C_E$ with $x_{ij} = 0$ when $i > j$.

Note that all preceding results in this area are established only for spaces $X$ either with unconditional bases or for Banach lattices [19], whereas the spaces $C_E$ do not even possess the local unconditional structure [7]. In particular, it follows from [10] that the ideal $C_E$ has an unconditional basis if and only if it coincides with the Hilbert–Schmidt ideal.

Our approach to the study of narrow operators $T : L_p \to X$ when $X = T_E$ or $X = C_E$ is based on a fundamental fact that the spaces $T_E$ admit the finite-dimensional unconditional (Schauder) decomposition (UFDD) given by elements $\{\text{span}\{e_{ij}\}_{1 \leq i \leq j}\}_{j \geq 1}$ (see for details [2, Proposition 4.9]). Recall also that $T_E$ is isomorphic to $C_E$ when $E$ has non-trivial Boyd indices [2, Theorem 4.7] (see also [3, Proposition 2] and [15]).

This setting has been already explored in [8], where assuming the so-called 2-co-lacunary property (see [25]), the authors of the present article obtained a noncommutative version of [18, Proposition 2]. Precisely, since $C_r$, $r \leq 2$, is 2-co-lacunary, it follows from [8, Theorem 4.3 and Remark 4.4] that every $\ell_2$-strictly singular operator $T : L_p \to C_r$ is narrow when $1 \leq p < \infty$ and $1 \leq r \leq 2$. However, the case of $r > 2$ remains unresolved and this case cannot be treated by methods from [8].
The following theorem is a noncommutative analogue of [16, Theorem A], which answers Plichko and Popov’s question for Banach spaces $C_r$, $r > 2$ and resolves the unanswered cases in [8]. Our proof is motivated by an extended version of reproducibility hatched within noncommutative analysis in [4]. Based on a careful analysis of approach used in [16], we obtain a slight extension of [16, Proposition 3.1] concerning on the reproducibility (with respect to non-narrow operators) of the Haar basis in $L_p$ (see Proposition 2.2 below). Our approach is applicable to a much wider class of operator ideals than the class of (Schatten–von Neumann ideals) $C_r$, $2 < r < \infty$. The main result of the present paper, Theorem 1.1 below, is stated for ideals $C_E$, for which the symmetric space $E$ is satisfying an upper $r$-estimate (see, e.g. [12]), which extends and complements results in [6,8,13,16,18].

**Theorem 1.1.** Let $2 < r < \infty$ and let $F(0,1)$ be a separable symmetric function space having the Khintchine property. If $E$ is a separable symmetric sequence space satisfying an upper $r$-estimate, then every $\ell_2$-strictly singular operator $T : F(0,1) \to T_E$ is narrow.

Recall that $T_p := T_{C_p}$ is isomorphic to $C_p$ when $p > 1$ [2, Theorem 4.7] (see also [3, Proposition 2] and [15]) and $L_p$ has the Khintchine property when $p \in [1,\infty)$ (see, e.g. [12] or [21]). Combining Theorem 1.1 with [8, Theorem 4.3], we obtain the following corollary. Note that [16, Theorem A] does not hold if we replace $\ell_2$ with $C_r$ (i.e., there exist operators $T : L_p \to C_2$, $p > 2$, which are not narrow, see e.g. Remark 3.4 below or [16, Theorem A and Example 1.1]).

**Corollary 1.2.** Let $1 \leq p, r < \infty$. Every $\ell_2$-strictly singular operator $T : L_p \to C_r$ is narrow.

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## 2. Proof of Theorem 1.1

Let $S(0,1)$ be the space of all Lebesgue measurable functions on $(0,1)$ equipped with Lebesgue measure $m$ i.e. functions which coincide almost everywhere are considered identical.

For $x \in S(0,1)$, we denote by $x^*$ the decreasing rearrangement of the function $|x|$. That is,

$$x^*(t) = \inf \{s \geq 0 : m(\{|x| > s\}) \leq t\}, \quad t > 0.$$ 

**Definition 2.1.** Let $E(0,1) \subset S(0,1)$ be a Banach space. We say that $(E(0,1), \|\cdot\|_E)$ is a symmetric function space on $(0,1)$ if whenever $x \in E(0,1)$ and $y \in S(0,1)$ are such that $y^* \leq x^*$, then $y \in E(0,1)$ and $\|y\|_E \leq \|x\|_E$.

We recall some basic terminology concerning Schauder decomposition [11, Chapter 1, Section g]. Let $X$ be a Banach space. A sequence $(X_n)_{n=1}^{\infty}$
of closed subspaces of $X$ is called a Schauder decomposition of $X$ if every $x \in X$ has a unique representation of the form

$$x = \sum_{n=1}^{\infty} x_n$$

with $x_n \in X_n$ for every $n \geq 1$.

If $\{X_n\}_{n=1}^{\infty}$ is a Schauder decomposition of $X$ and if for any $x = \sum_{n=1}^{\infty} x_n \in X$ and any sequence $\epsilon = \{\epsilon_n = \pm 1\}_{n=1}^{\infty}$, the series

$$\sum_{n=1}^{\infty} \epsilon_n x_n$$

converges in $X$ [1, Lemma 2.4.2], then the sequence $\{X_n\}_{n=1}^{\infty}$ is said to form an unconditional Schauder decomposition of $X$. Moreover, the operator $M_\epsilon x := \sum_{i=1}^{\infty} \epsilon_n x_n$ is bounded and $\sup_\epsilon \|M_\epsilon\| < \infty$ (see e.g. [4] or [1, Proposition 3.1.3]).

Recall that a Schauder basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space $X$ is said to be $K$-reproducible for some $K \geq 1$, if for every isometric embedding of $X$ into a space $Y$ with a basis $\{y_k\}_{k=1}^{\infty}$ and every $\epsilon > 0$, there exists a block basis $\{z_n\}_{n=1}^{\infty}$ of $\{y_k\}_{k=1}^{\infty}$ which is $K + \epsilon$-equivalent to $\{x_n\}_{n \geq 1}$. When $K = 1$, the basis is said to be precisely reproducible. It is well-known that the Haar system in an arbitrary separable symmetric function space on $(0,1)$ is precisely reproducible [12, Theorem 2.c.8] (see also [17]).

In noncommutative analysis, there is a more useful notion of reproducibility. A Schauder basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space $X$ is said to be precisely finite-dimensional decomposition (FDD)-reproducible, if for every isometric embedding of $X$ into a space $Y$ with a finite-dimensional decomposition $\{Y_n\}_{n \geq 1}$ and every $\epsilon > 0$, there exists an increasing sequence $\{q_n\}_{n \geq 1}$ of positive integers and a basic sequence of elements $z_n = \sum_{q_n \leq k \leq q_{n+1} - 1} \lambda_k y_k$ ($y_k \in Y_k, \lambda_k \in \mathbb{R}$), which is $(1 + \epsilon)$-equivalent to $\{x_n\}$. It was observed in [4, Theorem 5.2] that [12, Theorem 2.c.8] can be improved, that is, the Haar system in an arbitrary separable symmetric function space is precisely FDD-reproducible.

In [16, Proposition 3.1], Mykhaylyuk et al. considered the reproducibility of the $(L_p$-normalized) Haar system $\{h_n\}_{n \geq 1}$ with respect to non-narrow operators. Note that [16, Proposition 3.1] can be easily generalized from the case of $L_p$-space, $1 \leq p < \infty$, to the case where $F(0,1)$ is an arbitrary separable symmetric space on $(0,1)$. The following proposition is a slight generalization of [16, Proposition 3.1] with respect to the FDD-reproducibility.

**Proposition 2.2.** [16, Proposition 3.1] Suppose that $F(0,1)$ is an arbitrary separable symmetric function space, $X$ is a Banach space with a basis $(e_n)$, $T \in L(F(0,1), X)$ satisfies $\|Tx\|_X \geq 2\delta$ for each mean zero sign $x \in F(0,1)$ on $(0,1)$ and some $\delta > 0$. Let $\{X_n = \text{span}\{e_k\}_{k=1}^{n-1}\}_{n=1}^{\infty}$ be a finite-dimensional decomposition of $X$, where $\{q_n\}_{n \geq 1}$ is an arbitrary increasing sequence of positive integers.
Then, for each $\varepsilon > 0$, there exist an operator $S \in L(F(0,1), X)$, an increasing sequence $\{p_n\}_{n \geq 1}$ of positive integers, a normalized basis $(u_n)$ such that $u_n = \sum_{p_k \leq n \leq p_n + 1 - 1} \lambda_k x_k$ ($x_k \in X_k, \lambda_k \in \mathbb{R}$), and real numbers $(a_n)$ such that

1. $\|S u_n\|_X = a_n$ for each $n \in \mathbb{N}$ with $a_1 = 0$;
2. $\|S x\|_X \geq \delta$ for each mean zero sign $x \in F(0,1)$ on $(0,1)$;
3. there exists a linear isometry $V : F(0,1) \to F(0,1)$, which sends signs to signs, so that $\|S x\|_X \leq \|TV x\|_X + \varepsilon$ for every $x \in F(0,1)$ with $\|x\|_{F(0,1)} = 1$;
4. there are finite codimensional subspaces $X_n$'s of $F(0,1)$ such that $\|S x\|_X \leq \|TV x\|_X + \frac{1}{n}$ for every sign $x$, then $|a_n| \geq \delta$ for each $n \geq 2$.

Proof. The proof is almost a verbatim repetition of that in [16, Proposition 3.1]. The only difference is that we need to consider a subsequence of the basis projections in the Banach space $X$ while the proof in [16, Proposition 3.1] simply proceeded with the set of all basis projections.

More precisely, let $(P_n)_{n=1}^\infty$ be the basis projections in $X$ with respect to the basis $\{e_k\}_{k \geq 1}$ and $P_0 = 0$. Recall that $X_n = \text{span}\{e_k\}_{k=q_n}^{q_{n+1}-1}$ for every $n \geq 1$. Let

$$Q_n = P_{q_n+1-1} \quad \text{and} \quad Q_0 = 0.$$ 

In particular, $Q_n$ is the projection onto $\text{clm}\{X_k\}_{k=1}^n$. Now, the claim of Proposition 2.2 can be obtained from the proof of [16, Proposition 3.1] (see also [19, Proposition 9.10]) by simply replacing $P_n$ with $Q_n$ throughout. Since our other arguments repeat [16, Proposition 3.1] we omit further details. \hfill \Box

The following proposition is well known to experts [16,19]. We include a proof below for completeness.

**Proposition 2.3.** Let $T$ and $S$ be as in Proposition 2.2. If $T$ is $\ell_2$-strictly singular, then $S$ is also $\ell_2$-strictly singular.

Proof. Assume by contradiction that $S$ is not $\ell_2$-strictly singular. That is, there exists a sequence $(x_n)_{n=1}^\infty$ in $F(0,1)$ which is equivalent to the natural basis of $\ell_2$, and a constant $c > 0$ such that

$$c^{-1} \|(\lambda_n)\|_{\ell_2} \leq \left\| \sum_{n=1}^\infty \lambda_n x_n \right\|_{F(0,1)} \leq c \|(\lambda_n)\|_{\ell_2}, \quad \forall (\lambda_n) \in \ell_2 \quad (2.1)$$

and a constant $C > 0$ such that

$$C^{-1} \|(\lambda_n)\|_{\ell_2} \leq \left\| S \left( \sum_{n=1}^\infty \lambda_n x_n \right) \right\|_X \leq C \|(\lambda_n)\|_{\ell_2}, \quad \forall (\lambda_n) \in \ell_2. \quad (2.2)$$

Let $k \in \mathbb{N}$ be so large that $\frac{1}{k} (1 + \frac{1}{k})^2 \leq \frac{1}{2} c^{-1} C^{-1}$. By (4) of Proposition 2.2, there is a finite codimensional subspace $X_k$ of $F(0,1)$ such that

$$\|S x\| \leq \|TV x\| + \frac{1}{k}, \quad (2.3)$$
for every $x \in X_k$ with $\|x\| = 1$.

Since every subspace with finite codimension in a Banach space is complemented [23, Lemma 4.21], it follows that there exists a bounded projection $P$ from $F(0, 1)$ onto $X_k$. Since $(x_n)$ is weakly null, it follows that $(1 - P)x_n$ is weakly null. Since $X_k$ is finite-codimensional, it follows that $(1 - P)x_n \to 0$ in $\|\cdot\|_{F(0, 1)}$. Therefore, passing to a subsequence if necessary, we may assume that the sequence

$$\{y_n := P(x_n)\}_{n \geq 1} \subset X_k$$

is $(1 + \frac{1}{k})$-equivalent to the basic sequence $\{x_n\}$ and is $c(1 + \frac{1}{k})$-equivalent to the natural basis of $\ell_2$. By (2.1) and (2.2), we have

$$c^{-1}(1 + \frac{1}{k})^{-1} \|\lambda\|_{\ell_2} \leq \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0, 1)} \leq c(1 + \frac{1}{k}) \|\lambda\|_{\ell_2}, \forall (\lambda_n) \in \ell_2.$$  (2.4)

Passing to a subsequence if necessary, we may assume that

$$C^{-1}(1 + \frac{1}{k})^{-1} \|\lambda\|_{\ell_2} \leq \left\| S(\sum_{n=1}^{\infty} \lambda_n y_n) \right\|_X \leq C(1 + \frac{1}{k}) \|\lambda\|_{\ell_2}, \forall (\lambda_n) \in \ell_2.$$  (2.5)

For any $\sum_{n=1}^{\infty} \lambda_n y_n \in F(0, 1)$ such that $\|\sum_{n=1}^{\infty} \lambda_n y_n\|_{F(0, 1)} = 1$, we have

$$c^{-1}C^{-1} \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0, 1)} \leq (1 + \frac{1}{k})C^{-1} \|\lambda\|_{\ell_2} \leq (1 + \frac{1}{k})^2 \left\| S(\sum_{n=1}^{\infty} \lambda_n y_n) \right\|_X \leq (1 + \frac{1}{k})^2 \left\| TV(\sum_{n=1}^{\infty} \lambda_n y_n) \right\|_X + \frac{1}{k}(1 + \frac{1}{k})^2.$$  (2.6)

Recall that $\frac{1}{k}(1 + \frac{1}{k})^2 \leq \frac{1}{2}c^{-1}C^{-1}$. For any $\sum_{n=1}^{\infty} \lambda_n y_n \in F(0, 1)$ such that $\|\sum_{n=1}^{\infty} \lambda_n y_n\|_{F(0, 1)} = 1$, we have

$$\frac{1}{2}c^{-1}C^{-1} \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0, 1)} = c^{-1}C^{-1} \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0, 1)} - \frac{1}{2}c^{-1}C^{-1} \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0, 1)}$$

$$= c^{-1}C^{-1} \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0, 1)} - \frac{1}{2}c^{-1}C^{-1}$$
\[ \leq c^{-1}C^{-1} \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0,1)} - \frac{1}{k}(1 + \frac{1}{k})^2 \]

\[ \leq (1 + \frac{1}{k})^2 \left\| TV \left( \sum_{n=1}^{\infty} \lambda_n y_n \right) \right\|_X, \quad (2.6) \]

and, therefore,

\[ \frac{1}{2}c^{-1}C^{-1} \left\| V \left( \sum_{n=1}^{\infty} \lambda_n y_n \right) \right\|_{F(0,1)} = \frac{1}{2}c^{-1}C^{-1} \left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\|_{F(0,1)} \]

\[ \leq \left( 1 + \frac{1}{k} \right)^2 \left\| TV \left( \sum_{n=1}^{\infty} \lambda_n y_n \right) \right\|_X \]

\[ \leq \left( 1 + \frac{1}{k} \right)^2 \left\| T \right\| \left\| V \left( \sum_{n=1}^{\infty} \lambda_n y_n \right) \right\|_{F(0,1)}. \]

Hence, \( T \) is an isomorphism on \( \operatorname{span} \{ Vy_n \}_{n=1}^{\infty} \). However, by (2.4) and the fact that \( V \) is an isometry, \( \operatorname{span} \{ Vy_n \}_{n=1}^{\infty} \) is isomorphic to \( \ell_2 \), which contracts the assumption that \( T \) is \( \ell_2 \)-strictly singular. \( \square \)

Throughout this section by \( (h_n)_{n=1}^{\infty} \) is denoted the \( L_\infty \)-normalized Haar system. Let

\[ r_m := \sum_{k=1}^{2^m} h_{2^m+k}, \quad m = 0, 1, \ldots \]

be \( m \)th Rademacher function.

We say that a symmetric function space \( E(0,1) \) has the Khintchine property if the Rademacher system in \( E(0,1) \) is equivalent to the natural basis of \( \ell_2 \). In particular, \( L_p(0,1), p \geq 1 \), has the Khintchine property (see e.g. [12,21]).

### 2.1. Proof of Theorem 1.1

Before proceeding to the proof of Theorem 1.1, we need one more auxiliary result.

**Proposition 2.4.** Let \( F(0,1) \) be a separable symmetric function space having the Khintchine property and \( E \) be a separable symmetric sequence space. Let \( T_E \) be the upper triangular part of the separable symmetric ideal \( C_E \) corresponding to \( E \). Let \( S: F(0,1) \to T_E \) be defined as in Proposition 2.2 by taking \( X = T_E \) with FDD \( X_n := \operatorname{span} \{ e_{in} \}_{i \leq n} \).

Assume, in addition, that \( S \) is an \( \ell_2 \)-strictly singular operator. There exists a subsequence \( (n_m)_{m \geq 1} \) of \( \mathbb{N} \) such that

\[ \|Sr_{n_m} - y_m\|_{T_E} \leq \frac{\delta}{2m}. \]
where \((y_m)\) is a sequence of elements in \(T_E\) which are disjointly supported from the left and the right.\(^1\)

**Proof.** Let

\[
P_N = \sum_{i=1}^{N} e_{ii}. \tag{2.7}
\]

By Proposition 2.2, \((Sr_n)\) is a sequence in \(T_E\), which are disjointly supported from the right. Indeed, every non-zero element in \(X_n\) has its right support equal to \(e_{nn}\). By Proposition 2.2, \(r(Sh_n) \leq \sum_{k=p_{n+1}-1}^{p_n} e_{ii}\) for some strictly increasing sequence \(\{p_n\}_{n \geq 1}\). Therefore, \(Sr_n\)'s are disjointly supported from the right.

Let \(n_1 = 1\) and let \(y_1 = Sr_1\). By Proposition 2.2, there exists a positive integer \(N_1\) large enough such that

\[
P_{N_1}Sr_1P_{N_1} = Sr_1.
\]

We claim that \(P_{N_1}Sr_n \to 0\) in \(T_E\) as \(n \to \infty\). Otherwise, if

\[
\liminf_{n \to \infty} \|P_{N_1}Sr_n\|_{T_E} > \delta', > 0,
\]

then, passing to a subsequence of \((r_n)\) if necessary, we obtain the existence of an integer \(m\) with \(m \leq N_1\) such that

\[
\liminf_{n \to \infty} \|(P_m - P_{m-1})Sr_n\|_{T_E} \geq \frac{\delta'}{N_1} > 0.
\]

Indeed, if \(\liminf_{n \to \infty} \|(P_m - P_{m-1})Sr_n\|_{T_E} < \frac{\delta'}{N_1}\) for all \(m \leq N_1\), then by the triangle inequality, we have \(\liminf_{n \to \infty} \|P_mSr_n\|_{T_E} < \delta\), which is a contradiction. Passing to a subsequence, we may assume that \(\|(P_m - P_{m-1})Sr_n\|_{T_E} \geq \frac{\delta'}{N_1}\) for all \(n\). Denoting \(e_{mm}(Sr_n)(e_{mm}(Sr_n))^* = e_{mm}(Sr_n)(Sr_n)^*e_{mm} = ae_{mm}\) for some \(a \geq 0\), we have

\[
\frac{\delta'}{N_1} \leq \|(P_m - P_{m-1})Sr_n\|_{T_E} = \|e_{mm}Sr_n\|_{T_E}
= \|s(e_{mm}Sr_n)\|_{E} = \|s(e_{mm}Sr_n(e_{mm}Sr_n)^*)^{1/2}\|_{E}
= \|s(ae_{mm})^{1/2}\|_{E} = a^{1/2}.
\]

Hence, \(e_{mm}(Sr_n)(Sr_n)^*e_{mm} = ae_{mm} \geq \left(\frac{\delta'}{N_1}\right)^2 e_{mm}\). Moreover, since \(Sr_n\)'s are disjointly supported from the right, it follows that for any \((\alpha_n) \in \ell_2\), we have

\(^1\) Let \(z \in C_E\). We denote by \(l(z)\) (resp. \(r(z)\)) the left (resp., right) support of \(x\). Let \(x, y \in C_E\). If \(l(x)l(y) = y\) (resp., \(r(x)r(y) = 0\)), then \(x\) and \(y\) are said to be disjointly supported from the left (resp., disjointly supported from the right).
\[ s \left( (P_m - P_{m-1}) \sum_{n=1}^{\infty} \alpha_n Sr_n \right) \overset{(2.7)}{=} s \left( e_{mm} \sum_{n=1}^{\infty} \alpha_n Sr_n \right) \]
\[ = s \left( e_{mm} \sum_{n=1}^{\infty} \alpha_n Sr_n (e_{mm} \sum_{n=1}^{\infty} \alpha_n Sr_n)^* \right)^{1/2} \]
\[ = s \left( e_{mm} \sum_{n=1}^{\infty} |\alpha_n|^2 (Sr_n)^* e_{mm} \right)^{1/2} \]
\[ \geq s \left( \sum_{n=1}^{\infty} |\alpha_n|^2 \left( \frac{\delta'}{N_1} \right)^2 e_{mm} \right)^{1/2} \]
\[ = \left( \frac{\delta'}{N_1} \| (\alpha_n) \|_{\ell_2} : 0, 0, 0, \ldots \right). \quad (2.8) \]

Hence, for any \((\alpha_n) \in \ell_2\), we have
\[
\frac{\delta'}{N_1} \| (\alpha_n) \|_{\ell_2} \overset{(2.8)}{\leq} \left\| (P_m - P_{m-1}) \sum_{n=1}^{\infty} \alpha_n Sr_n \right\|_{T_E} \leq \left\| P_{N_1} \sum_{n=1}^{\infty} \alpha_n Sr_n \right\|_{T_E} \leq \left\| \sum_{n=1}^{\infty} \alpha_n S r_n \right\|_{T_E} \leq \| S \| \left\| \sum_{n=1}^{\infty} \alpha_n r_n \right\|_{F(0,1)} \leq c(p) \| (\alpha_n) \|_{\ell_2},
\]
where \(c(p)\) is a positive constant depending on \(p\). This contradicts the fact that \(S\) is \(\ell_2\)-strictly singular. Hence,
\[ P_{N_1} S r_n \to 0, \]
in \(T_E\) as \(n \to \infty\). There exists a positive integer \(n_2\) such that
\[ \| P_{N_1} S r_{n_2} \|_{T_E} \leq \frac{\delta}{4}. \]

By Proposition 2.2, there exists a positive integer \(N_2\) sufficiently large such that
\[ Sr_{n_2} = P_{N_2} S r_{n_2}. \]

We have
\[ \| S r_{n_2} - (P_{N_2} - P_{N_1}) S r_{n_2} \|_{T_E} = \| P_{N_1} S r_{n_2} \|_{T_E} \leq \frac{\delta}{4}. \]

Observe that \((P_{N_2} - P_{N_1}) S r_{n_2}\) and \(y_1\) are disjointly supported from the left and the right. We define \(y_2 := (P_{N_2} - P_{N_1}) S r_{n_2}\).

Continuing the procedure, we construct a sequence of integers \((n_m)\) and a sequence \((y_m)\) of elements in \(C_E\) which are disjointly supported from the left and the right such that
\[ \| S r_{n_m} - y_m \|_{T_E} \leq \frac{\delta}{2m}. \]

This completes the proof. \(\square\)

Now, we are ready to prove Theorem 1.1.
Proof of Theorem 1.1. Let \( r > 2 \). Let \( \{e_{ij}\}_{j \geq 1} \) be the natural Schauder basis of \( T_E \) (e.g. in the induced rectangular ordering [15]). Suppose that \( T \in L(F(0,1), T_E) \) is not narrow. Without loss of generality, we may assume that there exists a positive number \( \delta \) such that \( \|Tx\|_{T_E} \geq 2\delta \) for any mean zero sign \( x \) on \((0,1)\). Applying Proposition 2.2 by taking \( \varepsilon < \|T\| \), \( X = T_E \) and \( X_n = \text{span}\{e_{i,n}\}_{1 \leq i \leq n} \), we can choose an operator \( S \in L(F(0,1), T_E) \) such that

\[
\|S\| \leq 2 \|T\|, \tag{2.9}
\]

and for which conditions (1)–(4) in Proposition 2.2 hold. By Proposition 2.3, \( S \) is \( \ell_2 \)-strictly singular. By Proposition 2.4, we may assume that there exists a subsequence \( (n(m))_{m \geq 1} \) of \( \mathbb{N} \) such that \( n(m) \)'s are odd numbers and \( (Sr_{n(m)}) \) is equivalent to a basic sequence of elements in \( T_E \) which are disjointly supported from the left and the right. We consider

\[
\{h_{2^n(m)+i}\}_{m \geq 1, 1 \leq i \leq 2^n(m)}.
\]

Let \( C > 0 \) and \( N \in \mathbb{N} \). We denote

\[
I_m^k := \text{supp} \ h_{2^n+k} = \left[ \frac{k-1}{2^m}, \frac{k}{2^m} \right).
\]

We define

\[
f := C \sqrt{\frac{2^{n+1}}{N}} \sum_{m=1}^{2^n+1} \sum_{k=1}^{2^n+1} \bar{r}_{2^n+1+k} = C \sqrt{\frac{2^{n+1}}{N}} \sum_{m=1}^{2^n+1} r_{n(m)}.
\]

Since \( Sr_{n(m)} \) is equivalent to a sequence of elements in \( T_r \) which are disjointly supported from the left and the right and \( E \) satisfies an upper \( r \)-estimate with \( r > 2 \), we have

\[
\|Sf\|_{T_E} = \left\| \frac{C}{\sqrt{N}} \sum_{m=1}^{2^{n+1}} S(r_{n(m)}) \right\|_{T_E} \leq \frac{C}{\sqrt{N}} \left( \sum_{m=1}^{2^{n+1}} \|S(r_{n(m)})\|_{T_E}^r \right)^{1/r}
\]

\[
\leq \frac{C}{\sqrt{N}} \|S\| \left( \sum_{m=0}^{N} \|r_{n(m)}\|_{F(0,1)}^r \right)^{1/r} \leq 2C \|T\| (N+1)^{1/r} N^{-\frac{1}{r}}. \tag{2.9}
\]

Since \( r > 0 \), it follows that

\[
\|Sf\|_{T_E} \to 0, \text{ as } N \to \infty.
\]

Our goal is to select a subset \( J \) of \( \{2^{n(m)} + k\}_{m \geq 1, 0 \leq i \leq 2^n(m) - 1} \) such that

\[
g = \frac{C}{\sqrt{N}} \sum_{n \in J} \bar{r}_n \text{ is close enough to a sign.}
\]

Set

\[
A := \left\{ \omega \in [0,1] : \max_{j \in \{2^{n(m)} + k\}_{m \geq 1, 1 \leq k \leq 2^n(m)}} \left| \frac{C}{\sqrt{N}} \sum_{i=2^{n(m)}+k}^{2^{n(m)}+k} \bar{r}_i(\omega) \right| > 1 \right\},
\]

and

\[
\tau(\omega) = \begin{cases} 
\min \left\{ j \in \{2^{n(m)} + k\}_{m \geq 1, 1 \leq k \leq 2^n(m)} : \left| \frac{C}{\sqrt{N}} \sum_{i=2^{n(m)}+k}^{2^{n(m)}+k} \bar{r}_i(\omega) \right| > 1 \right\}, & \text{if } \omega \in A, \\
2^n(N) + k, & \text{if } \omega \notin A \text{ and } \omega \in I_{n(N)}^k. 
\end{cases}
\]
Observe that if $\tau(\omega) = 2^{n(m)} + k$, then $\omega \in I^k_{n(m)}$ (see, e.g. the argument in [16, p.94] and [19]). Further, if there exists $\omega \in I^k_{n(m)}$ with $\tau(\omega) \geq 2^{n(m)} + k$, then we have

$$\tau(\xi) \geq 2^{n(m)} + k$$  \hspace{2cm} (2.10)

for every $\xi \in I^k_{n(m)}$. Indeed, since $\omega \in I^k_{n(m)}$, for every $z < 2^{n(m)} + k$ and every $\xi \in I^k_{n(m)}$, we have $h_z(\omega) = h_z(\xi)$. Thus,

$$\left| \frac{C}{\sqrt{N}} \sum_{p=2^n+1, j \leq 2^{n(m)}+k-1} \bar{h}_p(\xi) \right| = \left| \frac{C}{\sqrt{N}} \sum_{p=2^n+1, j \leq 2^{n(m)}+k-1} \bar{h}_p(\omega) \right| \leq 1.$$

Hence, $\tau(\xi) \geq 2^{n(m)} + k$.

Now, we define a set

$$J := \{ j = 2^{n(m)} + k \leq 2^{n(N)} + 1 : \exists \omega \in I^k_{n(m)} \text{ with } \tau(\omega) \geq j \}$$

$$= \{ j = 2^{n(m)} + k \leq 2^{n(N)} + 1 : \forall \xi \in I^k_{n(m)} \text{ with } \tau(\xi) \geq j \}.$$

Let $g : [0, 1] \rightarrow \mathbb{R}$ be defined as

$$g(\omega) = \frac{C}{\sqrt{N}} \sum_{2^{n(m)} + k \leq \tau(\omega)} \bar{h}_{2^{n(m)} + k}(\omega).$$

Observe that for every $\omega \in [0, 1]$, we have

$$\left\{ j = 2^{n(m)} + k : \omega \in I^k_{n(m)} \right\} \cap J$$

$$= \left\{ j = 2^{n(m)} + k : \omega \in I^k_{n(m)} \right\} \cap$$

$$\times \left( \bigcup_{\omega_1 \in [0, 1]} \left\{ j = 2^{n(m)} + k \leq \tau(\omega_1) : \omega_1 \in I^k_{n(m)} \right\} \right)$$

$$= \bigcup_{\omega_1 \in [0, 1]} \left\{ j = 2^{n(m)} + k \leq \tau(\omega_1) : \omega_1, \omega \in I^k_{n(m)} \right\}$$

$$= \left\{ j = 2^{n(m)} + k \leq \tau(\omega) : \omega \in I^k_{n(m)} \right\}.$$

Since

$$\bar{h}_{2^{n(m)} + k}(\omega) = 0,$$  \hspace{2cm} (2.11)

for any $\omega \notin I^k_{n(m)}$, it follows that for any $\omega \in [0, 1]$, we have

$$g(\omega) = \frac{C}{\sqrt{N}} \sum_{j \in \{ j = 2^{n(m)} + k \leq \tau(\omega) : \omega \in I^k_{n(m)} \}} \bar{h}_j(\omega)$$

$$= \frac{C}{\sqrt{N}} \sum_{j \in \{ j = 2^{n(m)} + k : \omega \in I^k_{n(m)} \} \cap J} \bar{h}_j(\omega)$$

$$\stackrel{(2.11)}{=} \frac{C}{\sqrt{N}} \sum_{j \in \{ j = 2^{n(m)} + k : \omega \in I^k_{n(m)} \} \cap J} \bar{h}_j(\omega)$$

$$+ \frac{C}{\sqrt{N}} \sum_{j \in \{ j = 2^{n(m)} + k : \omega \notin I^k_{n(m)} \} \cap J} \bar{h}_j(\omega) = \frac{C}{\sqrt{N}} \sum_{j \in J} \bar{h}_j(\omega).$$  \hspace{2cm} (2.12)
By the unconditionality of the decomposition \( \{X_n = \text{span}\{e_{i,n}\}\}_{1 \leq i \leq n} \) of \( T_E \) [2, Lemma 4.5], we have

\[
\|Sg\|_{TE} \overset{(2.12)}{=} \left\| \frac{C}{\sqrt{N}} \sum_{j \in J} S \hat{h}_j \right\|_{TE} \leq c_E \left\| \frac{C}{\sqrt{N}} \sum_{m=1}^{2N+1} \left( \sum_{k=1}^{2n(m)} S \hat{h}_{2n(m)+k} \right) \right\|_{TE} = c_E \|Sf\|_{TE} \leq c_E \cdot 2C \|T\| (N+1)^{1/r} N^{-1/2},
\]

where \( c_E > 0 \) depends on \( E \) only.

Assume that \( N \) is an odd number. By the definition of \( \tau(\omega) \) and \( g(\omega) \), for every \( \omega \in A \), we have

\[
1 < |g(\omega)| < 1 + \frac{C}{\sqrt{N}}, \tag{2.14}
\]

and for every \( \omega \in [0,1] \setminus A \),

\[
g(\omega) \neq 0. \tag{2.15}
\]

Indeed, since \( N \) is odd (that is, \( N-1 \) is even), it follows that \( \sum_{j=1}^{N-1} r_{n_j} \neq \pm 1 \) everywhere, and therefore, for \( \omega \in I_k^{n(N)} \), we have

\[
g(\omega) = \frac{C}{\sqrt{N}} \sum_{1 \leq k_1 \leq 2^{n(m)}} \hat{h}_{2^{n(m)}+k_1} (\omega)
\]

\[
= \frac{C}{\sqrt{N}} \sum_{1 \leq k_1 \leq 2^{n(N)}} \hat{h}_{2^{n(m)}+k_1} (\omega)
\]

\[
= \frac{C}{\sqrt{N}} \sum_{1 \leq k_1 \leq 2^{n(m)}} \hat{h}_{2^{n(m)}+k_1} (\omega) + \frac{C}{\sqrt{N}} \sum_{2^{n(N)}+1 \leq j \leq 2^{n(N)}+k} \hat{h}_j (\omega)
\]

\[
= \frac{C}{\sqrt{N}} \sum_{j=1}^{N-1} r_{n_j} (\omega) + \frac{C}{\sqrt{N}} \sum_{2^{n(N)}+1 \leq j \leq 2^{n(N)}+k} \hat{h}_j (\omega) \neq 0.
\]

Now, we define a function \( g \) by setting

\[
\tilde{g}(\omega) = \text{sgn}(g(\omega)).
\]

By (2.14), we have

\[
\|g - \tilde{g}\|_{F(0,1)} \leq \left\| \frac{C}{\sqrt{N}} \chi_A + \chi_{[0,1] \setminus A} \right\|_{F(0,1)}.
\]

By the Central Limit Theorem, for a sufficiently large odd number \( N \), we have

\[
\mu([0,1] \setminus A) = \mu \left\{ \omega : \left| \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \sum_{k=1}^{2^{n(m)}} \hat{h}_{2^{n(m)}+k} \right| \leq \frac{1}{C} \right\}
\]
Thus,
\[
\|\chi_{[0,1]}|A\|_{F(0,1)} \leq \|\chi_{(0,\frac{1}{C})}\|_{F(0,1)}.
\]
Hence, we have
\[
\|g - \tilde{g}\|_{F(0,1)} \leq \frac{C}{\sqrt{N}} + \|\chi_{(0,\frac{1}{C})}\|_{F(0,1)}.
\]
Thus, by (2.9) and (2.13), we have
\[
\|S\tilde{g}\|_{T_E} \leq \|Sg\|_{T_E} + \|S\| \|g - \tilde{g}\|_{F(0,1)}
\leq c_E \cdot 2^C \|T\| (N + 1)^{1/r} N^{-1/2} + 2 \|T\| \left( \frac{C}{\sqrt{N}} + \|\chi_{(0,\frac{1}{C})}\|_{F(0,1)} \right).
\]
Since \(r > 2\), it follows that for every \(\delta > 0\), there exists a sufficiently large positive number \(C\) and a sufficiently large odd number \(N\) such that
\[
\|S\tilde{g}\|_{T_E} \leq \delta.
\]
It remains to observe that \(\tilde{g}\) is a mean zero sign on \([0,1]\). Indeed, since \(N\) is an odd number, it follows from (2.14) and (2.15) that the support of \(\tilde{g}\) is equal to \([0,1]\). Observe that for every \(\omega \in [0,1]\) and every \(2^n(m) + k \leq 2^{n(N)+1}, k \leq 2^{n(m)}\), we have
\[
\tilde{h}_{2^n(m) + k}(\omega) = -\tilde{h}_{2^n(m) + (2^n(m) - k)+1}(1 - \omega).
\]
Thus, \(g(\omega) = -g(1 - \omega)\) for every \(\omega \in [0,1]\), and
\[
\mu(\{\omega \in [0,1] : g(\omega) > 0\}) = \mu(\{\omega \in [0,1] : g(\omega) < 0\}).
\]
Thus, \(\tilde{g}\) is a mean zero sign on \([0,1]\), which contradicts (2) of Proposition 2.2.

3. Remarks

It is shown in [6, Theorem 1.1] that if \(T\) is a regular operator from \(L_p\) into an order continuous Banach lattice \(F\), then \(T\) is \(\ell_2\)-strictly singularity” \(\Rightarrow\) “\(T\) is narrow”. Note that [16, Theorem B] is stated for \(\ell_2\)-strictly singular operator \(T : L_p \to X\) when \(1 < p < \infty\) and \(X\) has an unconditional basis. However, a careful analysis of its proof shows that it still holds for any \(p \in [1, \infty)\) (see also [19, p.110]). By a verbatim repetition of the proof in [16, Theorem B] by replacing [16, Proposition 3.1] with Proposition 2.2, we obtain the following result for general Banach spaces, which provides an alternative proof for Theorem 1.1.

**Theorem 3.1.** Let \(F(0,1)\) be a separable symmetric function space having the Khintchine property and let \(X\) be a Banach space having an unconditional finite-dimensional decomposition. Then every \(\ell_2\)-strictly singular operator \(T : F(0,1) \to X\) is narrow.
The class of Banach spaces having unconditional finite-dimensional decompositions is very wide. For example, the operator ideal $C_E$ has an UFDD when $E$ has non-trivial Boyd indices [2, Corollary 4.6]. Let $E(0, 1)$ be a symmetric function space on the unit interval $(0, 1)$ and let $E(R)$ be the non-commutative operator space [14] affiliated with the hyperfinite $II_1$-factor $R$. Then, $E(R)$ has an UFDD [4, 24]. This opens an avenue for further extensions of Theorem 1.1.

**Definition 3.2.** Let $E(\mathcal{M}, \tau)$ be a noncommutative symmetric space and $Y$ be an $F$-space. We call $T : E(\mathcal{M}, \tau) \to Y$ a narrow operator if for each projection $p \in \mathcal{M}$ and $\varepsilon > 0$, there exists a self-adjoint element $x \in E(\mathcal{M}, \tau)$ such that $x^2 = p$, $\tau(x) = 0$ and $\|T(x)\|_Y < \varepsilon$.

Assume that $E(0, 1)$ is a symmetric function and $X$ is a Banach space. It is clear (see e.g. the proof of [8, Corollary 4.5]) that if all elements of $L(E(0, 1), X)$ are narrow, then every element of $L(E(\mathcal{M}), X)$ is narrow for an arbitrary atomless finite von Neumann algebra $\mathcal{M}$ equipped with a faithful normal tracial state $\tau$. The following result is a direct consequence of Corollary 1.2.

**Corollary 3.3.** Let $\mathcal{M}$ be an atomless finite von Neumann algebra equipped with a faithful normal tracial state $\tau_1$. Let $1 \leq p, r < \infty$. Every $\ell_2$-strictly singular operator $T : L_p(\mathcal{M}) \to C_r$ is narrow.

Recall that a Banach space $X$ is said to have infratype $q > 1$ [19, p.216] if there exists a constant $C > 0$ such that for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have

$$\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\| \leq C \left( \sum_{k=1}^{n} \|x_k\|^q \right)^{1/q}.$$  

It is clear that if a Banach space has type $q$ then it has infratype $q$. Note that if $q < 2$, then the notions of type and infratype coincide [26].

**Remark 3.4.** Assume that $1 \leq p < 2$ and $r > p$. Recall that $C_r$ has type $\min\{r, 2\}$ (see, e.g. [5]), and therefore, has infratype $\min\{r, 2\}$. By [19, Theorem 9.8], we obtain that all operators from $L_p$ into $C_r$ are narrow.

Assume that $p \geq 2$. Note that there exists a non-narrow operator $T : L_p \to \ell_2$ [16, Example 1.1]. Since $\ell_2$ is a complemented subspace of $C_r$, it follows that there exists a non-narrow operator from $L_p$ into $C_r$.

When $p = r = 1$, it is shown in [9] that all operators from $L_1$ into $C_1$ are Dunford–Pettis, and therefore, narrow.

The case for $1 < p < 2$ and $1 \leq r \leq p$ seems to be open, i.e., we do not know whether there exists a non-narrow operator from $L_p$ into $C_r$ when $1 < p < 2$ and $1 \leq r \leq p$.

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