Area distribution and the average shape of a Lévy bridge

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Received 28 April 2010
Accepted 10 July 2010
Published 3 August 2010

Abstract. We consider a one-dimensional Lévy bridge \( x_B \) of length \( n \) and index \( 0 < \alpha < 2 \), i.e. a Lévy random walk constrained to start and end at the origin after \( n \) time steps, \( x_B(0) = x_B(n) = 0 \). We compute the distribution \( P_B(A, n) \) of the area \( A = \sum_{m=1}^{n} x_B(m) \) under such a Lévy bridge and show that, for large \( n \), it has the scaling form \( P_B(A, n) \sim n^{-1-1/\alpha} F_\alpha(A/n^{1+1/\alpha}) \), with the asymptotic behavior \( F_\alpha(Y) \sim Y^{-2(1+\alpha)} \) for large \( Y \). For \( \alpha = 1 \), we obtain an explicit expression of \( F_1(Y) \) in terms of elementary functions. We also compute the average profile \( \langle \tilde{x}_B(m) \rangle \) at time \( m \) of a Lévy bridge with fixed area \( A \). For large \( n \) and large \( m \) and \( A \), one finds the scaling form \( \langle \tilde{x}_B(m) \rangle = n^{1/\alpha} H_\alpha(m/n, A/n^{1+1/\alpha}) \), where at variance with the Brownian bridge, \( H_\alpha(X, Y) \) is a non-trivial function of the rescaled time \( m/n \) and rescaled area \( Y = A/n^{1+1/\alpha} \). Our analytical results are verified by numerical simulations.

Keywords: exact results, fluctuations (theory), stochastic processes (theory)
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1. Introduction

Random walks, and the associated continuous-time Brownian motion (BM), are ubiquitous in nature. As such, they are not only the cornerstones of statistical physics [1]–[3] but have also found many applications in a variety of areas such as biology [4], computer science [5, 6] and finance [7]. Continuous-time Brownian motion is simply defined by the equation of motion

$$x(0) = 0, \quad \frac{dx(t)}{dt} = \eta(t),$$

(1)

where $\eta(t)$ is a Gaussian white noise of zero mean $\langle \eta(t) \rangle = 0$ and short range correlations $\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t')$ where $D$ is the diffusion constant (in the following we set $D = 1$). An interesting variant of Brownian motion in a given time interval $[0, T]$ is the so-called Brownian bridge $x_B(t)$, which is a Brownian motion conditioned to start and end at zero, i.e. $x_B(T) = x_B(0) = 0$. Here we focus on two interesting observables associated with this bridge, namely

- the distribution of the area $A$ under the bridge (see figure 1(a))

$$A = \int_0^T x_B(t) \, dt,$$

(2)

doi:10.1088/1742-5468/2010/08/P08005 2
which is obviously a random variable, being the sum of (strongly correlated) random variables. For the Brownian bridge, the distribution of $A$ can easily be computed using the fact that $x_B(t)$ is a Gaussian random variable. This can be seen from the well known identity in law [2]

$$x_B(t) := x(t) - \frac{t}{T} x(T),$$

where $x(t)$ is a standard Brownian motion (1). For the Brownian bridge, $A$ is thus also a centered Gaussian random variable. A direct computation of the second moment $\langle A^2 \rangle$ yields straightforwardly

$$P_B(A, T) = \sqrt{\frac{3}{\pi T^3}} \exp \left(-\frac{3A^2}{T^3}\right)$$

the average shape of the bridge, $\langle \tilde{x}_B(t) \rangle$ for a fixed area $A$ (see figure 1(b)). For the Brownian bridge, it takes a simple form [8]

$$\langle \tilde{x}_B(t) \rangle = \frac{A}{T} f \left( \frac{t}{T} \right), \quad f(x) = 6x(1-x).$$

The distribution of the area under a Brownian bridge (4) is a standard result and its extension to various constrained Brownian motions has recently attracted much attention [9]–[15]. For instance, the distribution of the area under a Brownian excursion (i.e. a Brownian motion conditioned to start and end at 0 and constrained to stay positive in-between), the so-called Airy distribution, describes the statistics of the maximal relative height of one-dimensional elastic interfaces [9, 11, 15]. Another example is the area $A$ under a Brownian motion till its first-passage time $t_f$ [10], which has an interesting application to the description of the avalanches in the directed Abelian sandpile model proposed in [16], such that $t_f$ relates to the avalanche duration and $A$ to the size of the avalanche cluster.

doi:10.1088/1742-5468/2010/08/P08005
Related quantities were recently studied in the statistics of avalanches near the depinning transition of elastic manifolds in random media [17]. On the other hand, the average shape of random walk bridges with a fixed area $A$ (5) was studied some time ago in the context of wetting [8] to prove the validity of the Wulff construction in 1 + 1 dimensions, and more recently in the context of mass transport models [18]. In these models, where the transport rules depend on the environment of the departure site, the steady state has a pair-factorized form [19], which generalizes the factorized steady states found in simpler system such as the zero range process [20]–[22]. As the mass density crosses some critical value, the system exhibits a condensation transition which is governed by interactions, which in turn give rise to a spatially extended condensate. It was shown in [18] that the shape of this condensate can be described by the average shape of a random walk bridge with fixed area, and the results of [8] were recovered.

While these quantities are well understood for a Brownian bridge, much less is known for the case of a Lévy bridge. The aim of the present paper is to compute the distribution of the area and the average shape for a fixed area $A$ in that case. To this purpose, it is convenient to consider a random walk $x(m)$, in discrete time (see figure 2(a)), starting at $x(0) = x_0$ at time 0 and evolving according to

$$x(m) = x(m-1) + \eta(m),$$

where $\eta(m)$ are independent and identically distributed (i.i.d.) random variables distributed according to a common distribution $\phi(\eta)$. Here we focus on the case where $\phi(\eta) = S_{\alpha}(\eta)$, where $S_{\alpha}(\eta)$ is a symmetric $\alpha$-stable (Lévy) distribution. Its characteristic function is given by $\int_{-\infty}^{\infty} S_{\alpha}(\eta)e^{ik\eta}d\eta = e^{-|k|^\alpha}$. In particular, for large $\eta$, the distribution of $\eta$ has a power law tail $\phi(\eta) \sim \eta^{-(1+\alpha)}$, with $0 < \alpha < 2$. The Lévy bridge $x_B(m)$, on the interval $[0,n]$, is a Lévy random walk conditioned to start and end at 0, i.e. $x_B(n) = x_B(0) = 0$. In the following we will compute the distribution $P_B(A,n)$ of the area $A$ under the bridge of length $n$, i.e. $A = \sum_{m=0}^{n} x_B(m)$, and the average shape of a bridge $\langle \hat{x}_B(m) \rangle$, with $0 \leq m \leq n$, for fixed $A$. Here we consider the natural scaling limit where $x_B \sim n^{1/\alpha}$, while $A \sim n^{1+1/\alpha}$, whereas the aforementioned previous works [8,18] focused on a different scaling limit which, for Brownian motion, corresponds to $x_B \sim n$ and $A \sim n^2$. Note that, even in this natural scaling limit, the identity law valid for the
Brownian bridge (3) does not hold for a Lévy bridge [23, 24] and one thus expects the distribution of the area $A$ to be non-trivial. Our results can be summarized as follows:

- for a free Lévy random walk starting at $x_0 = 0$, one finds that the distribution of the area $P(A, n)$ takes the form

$$P(A, n) = \frac{1}{\gamma_n} S_\alpha \left( \frac{A}{\gamma_n} \right), \quad \gamma_n = \left( \sum_{m=1}^{n} m^\alpha \right)^{1/\alpha},$$

(7)

where $S_\alpha(x)$ is a symmetric $\alpha$-stable distribution (21). For a Lévy bridge, one finds, in the scaling limit $A \to \infty$, $n \to \infty$, keeping $A/n^{1+1/\alpha}$ fixed, that the distribution of the area $P_B(A, n)$ takes the scaling form

$$P_B(A, n) \sim \frac{1}{n^{1+1/\alpha}} F_\alpha \left( \frac{A}{n^{1+1/\alpha}} \right),$$

(8)

where $F_\alpha(y)$ is a monotonically decreasing function, with asymptotic behavior

$$F_\alpha(Y) \sim \begin{cases} F_\alpha(0), & Y \to 0, \\ \frac{a_\alpha}{Y^{2(1+\alpha)}}, & Y \to \infty, \end{cases}$$

(9)

where $F_\alpha(0)$, see equations (36), and $a_\alpha$, see equation (37), are computable constants. For $\alpha = 1$, we obtain an explicit expression for $F_1(Y)$ in terms of elementary functions (C.6).

- on the other hand, in the aforementioned scaling limit, one obtains the average profile $\langle \tilde{x}_B(m) \rangle$ for a Lévy bridge as well as the average profile $\langle \tilde{x}(m) \rangle$ for a free Lévy walk with a fixed area $A$. For $\langle \tilde{x}(m) \rangle$ one obtains a fairly simple expression

$$\langle \tilde{x}(m) \rangle = \frac{A \alpha + 1}{n^{1+1/\alpha}} \left[ 1 - \left( \frac{1-m}{n} \right)^\alpha \right].$$

(10)

For a Lévy bridge, the expression is more involved. For generic $\alpha$ one finds the scaling form

$$\langle \tilde{x}_B(m) \rangle = n^{1/\alpha} H_\alpha \left( \frac{m}{n^{1+1/\alpha}}, \frac{A}{n^{1+1/\alpha}} \right),$$

(11)

which, in general, has a non-trivial dependence on $A$. One recovers a linear dependence in $A$ (as for the Brownian bridge in equation (5)) only in the limits $A \to 0$ (68) and $A \to \infty$ (69).

The paper is organized as follows. In section 2, we compute the joint distribution of the position and the area under a Lévy random walk. In section 3, we use these results to compute the distribution of the area $A$ under a Lévy bridge of size $n$, while in section 4, we use them to compute the average profile of a Lévy walk with a fixed area. Finally, in section 5 we present a numerical method, based on a Monte Carlo algorithm, to compute numerically $P_B(A, n)$ and $\langle \tilde{x}_B(m) \rangle$ before we conclude in section 6. Some technical (and useful) details have been left to appendices A, B and C.
2. Free Lévy walk: joint distribution of the position and the area

We start with the computation of the joint distribution $P(x, A, m|x_0, x_0, 0)$ of the position and the area after $m$ steps given that $x(0) = x_0$ (see figure 2(a)). If we denote by $A(m)$ the area under the random walk after $m$ time steps, this random variable evolves according to the equation

\[
A(0) = x_0,
\]

\[
A(m) = A(m - 1) + x(m).
\]

Therefore $P(x, A, m|x_0, x_0, 0)$ satisfies the following recursion relation:

\[
P(x, A, 0|x_0, x_0, 0) = \delta(x - x_0)\delta(A - x_0),
\]

\[
P(x, A, m|x_0, x_0, 0) = \int_{-\infty}^{\infty} P(x - \eta, A - x, m - 1|x_0, x_0, 0)\phi(\eta)\,d\eta.
\]

Introducing $\hat{\phi}(k) = \int_{-\infty}^{\infty} \phi(\eta)e^{ik\eta}\,d\eta$ the Fourier transform of $\phi(\eta)$, and thus $\hat{\phi}(k) = e^{-|k|^\alpha}$ for a Lévy random walk, and $\hat{P}(k_1, k_2, m|x_0, x_0, 0)$ the double Fourier transform of $P(x, A, m|x_0, x_0, 0)$ with respect to both $x$ and $A$, i.e. $\hat{P}(k_1, k_2, m|x_0, x_0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dA \, AP(x, A, m|x_0, x_0, 0)e^{ik_1 x + ik_2 A}$, the recursion relation (14) reads

\[
\hat{P}(k_1, k_2, 0|x_0, x_0, 0) = e^{i(k_1 + k_2)x_0},
\]

\[
\hat{P}(k_1, k_2, m|x_0, x_0, 0) = \hat{\phi}(k_1 + k_2)\hat{P}(k_1 + k_2, k_2, m - 1|x_0, x_0, 0),
\]

which can be solved, yielding

\[
\hat{P}(k_1, k_2, n|x_0, x_0, 0) = \prod_{m=1}^{n} \hat{\phi}(k_1 + mk_2)e^{i(k_1 + (n+1)k_2)x_0}.
\]

Hence for a Lévy walk of index $\alpha$ one has simply

\[
P(x, A, n|x_0, x_0, 0) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{-\sum_{m=1}^{n}|k_1+mk_2|^\alpha} e^{-ik_1 x_0} e^{-ik_2(A-(n+1)x_0)}.
\]

Note that this expression (18) can also be obtained directly by noticing that $x(n) = x_0 + \sum_{m=1}^{n} \eta(m)$ and thus $A(n) - (n + 1)x_0 = \sum_{m=1}^{n} x(m) = \sum_{m=1}^{n} \sum_{l=1}^{m} \eta(l)$ such that

\[
P(x, A, n|x_0, x_0, 0) = \prod_{m=1}^{n} \int_{-\infty}^{\infty} \delta(x_0)\int_{-\infty}^{\infty} \delta(x_0 - \sum_{m=1}^{n} \eta(m)) \phi[\eta(m)]\delta(x - x_0 - \sum_{m=1}^{n} \eta(m))
\]

\[
\times \delta(A - (n + 1)x_0 - \sum_{m=1}^{n} m\eta(n + 1 - m))
\]

After a double Fourier transform with respect to $x$ and $A$, this equation (19) yields immediately the expression $\hat{P}(k_1, k_2, n|x_0, x_0, 0)$ in equation (17). Of course the marginal distribution of the position $P(x, n|x_0, 0)$ and of the area $P(A, n|x_0, 0)$ are also stable laws.
Indeed one has
\[ P(x, n|x_0, 0) = \int_{-\infty}^{\infty} P(x, A, n|x_0, x_0, 0) \, dA = \frac{1}{n^{1/\alpha}} S_\alpha \left( \frac{x - x_0}{n^{1/\alpha}} \right), \]
\[ P(A, n|x_0, 0) = \int_{-\infty}^{\infty} P(x, A, n|x_0, x_0, 0) \, dx = \frac{1}{\gamma_n} S_\alpha \left( \frac{A - (n + 1)x_0}{\gamma_n} \right), \quad (20) \]
\[ \gamma_n = \left( \sum_{m=1}^{n} n_m^\alpha \right)^{1/\alpha} \sim \frac{n^{1+1/\alpha}}{(\alpha + 1)^{1/\alpha}}, \quad n \gg 1, \]
where
\[ S_\alpha(x) = \int_{-\infty}^{\infty} e^{-|k|^\alpha ikx} \frac{dk}{2\pi}. \quad (21) \]
For example, \( S_1(x) \) is the Cauchy distribution while \( S_2(x) \) is a Gaussian distribution:
\[ S_1(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \quad S_2(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}. \quad (22) \]
Note also the explicit expression
\[ S_\alpha(0) = \int_{-\infty}^{\infty} e^{-|k|^\alpha} \frac{dk}{2\pi} = \frac{\Gamma(1 + \alpha^{-1})}{\pi}, \quad (23) \]
which will be useful in the following.

We now want to study \( P(x, A, n|x_0, x_0, 0) \) in the limit of large \( n \). The marginal distributions in equations (20) suggest the scaling \( x \sim n^{1/\alpha} \) and \( A \sim n^{1/\alpha+1} \). From the expression in equation (18) one checks explicitly that in the limit \( n \to \infty \), keeping \( X = x/n^{1/\alpha} \) and \( Y = A/n^{1/\alpha+1} \) fixed, the joint distribution takes the scaling form
\[ P(x, A, n|x_0, x_0, 0) = \frac{1}{n^{2/\alpha+1}} G \left( \frac{x}{n^{1/\alpha}}, \frac{A}{n^{1/\alpha+1}} \bigg| \frac{x_0}{n^{1/\alpha}} \right), \quad (24) \]
where the function \( G(X, Y|X_0) \) is given by
\[ G(X, Y|X_0) = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{-\int_{0}^{1} |k_1 + k_2 z|^\alpha d z - i k_1 (X - X_0) - i k_2 (Y - X_0)}. \quad (25) \]
After the change of variable \( k_2 = k \) and \( k_1 = kr \), we obtain
\[ G(X, Y|X_0) = \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| e^{-|k|^\alpha \gamma(r) - i k r (X - X_0) - i k (Y - X_0)}, \quad (26) \]
\[ \gamma(r) = \int_{0}^{1} |r + z|^\alpha d z, \]
where the function \( \gamma(r) \) is explicitly given by
\[ \gamma(r) = \begin{cases} \frac{1}{\alpha + 1}((-r)^{\alpha+1} - (-1 - r)^{\alpha+1}), & r < -1, \\ \frac{1}{\alpha + 1}((r + 1)^{\alpha+1} + (-r)^{\alpha+1}), & -1 \leq r \leq 0, \\ \frac{1}{\alpha + 1}((r + 1)^{\alpha+1} - r^{\alpha+1}), & r > 0. \end{cases} \quad (27) \]
For $\alpha = 2$, one has $\gamma(r) = r^2 + r + 1/3$ and one finds

$$G(X, Y|X_0) = \frac{\sqrt{3}}{2\pi} \exp \left(-3(Y - X_0)(Y - X) - (X - X_0)^2 \right),$$

which yields back the propagator of the so-called random acceleration process [25].

### 3. Lévy bridge

In the absence of any constraint for the walker, the area under a Lévy random walk is the sum of Lévy random variables and is thus again a Lévy random variable. However, if one considers constrained Lévy walks, this is no longer true and the area may become different from a simple Lévy random variable. In section 3.1, we compute the distribution of the position for a Lévy bridge, while section 3.2 is devoted to the distribution of the area under this bridge.

#### 3.1. Distribution of the position

Here we study the Lévy bridge $\{x_B(m)\}_{0 \leq m \leq n}$ which starts at 0 at time 0, $x_B(0) = 0$, and is constrained to come back to 0 after $n$ time steps, i.e. $x_B(n) = 0$. In that case, one can compute the distribution of the position $P_B(x, m)$ after $m$ time steps for such a bridge as $P_B(x, m) = P(x, m|0, 0)P(x, n-m|0, 0)/P(0, n|0, 0)$, such that in the scaling limit one has

$$P_B(x, m) = \frac{1}{n^{1/\alpha}} G_B \left( \frac{x}{n^{1/\alpha}}, \frac{m}{n} \right),$$

$$G_B(X, \tau) = \frac{\pi}{\Gamma(1 + \alpha^{-1}) (\tau(1 - \tau))^{1/\alpha}} S_\alpha \left( \frac{X}{\tau^{1/\alpha}} \right) S_\alpha \left( \frac{X}{(1 - \tau)^{1/\alpha}} \right),$$

where we have used $S_\alpha(0) = \Gamma(1 + \alpha^{-1})/\pi$, see equation (23). For $\alpha = 2$ it is easy to see from equations (29) that $x_B(m)$ is a Gaussian variable. However, for $\alpha < 2$, the Lévy bridge is no longer a Lévy random variable. For instance, for $\alpha = 1$ one obtains a non-trivial distribution

$$G_B(X, \tau) = \frac{1}{\tau(1 - \tau)} \frac{\tau(1 - \tau)}{\tau^2 + X^2}.$$  

It is also easy to see that for any $\alpha < 2$ one has the asymptotic behavior

$$G_B(X, \tau) \sim c'_\alpha \tau(1 - \tau) X^{-2(\alpha + 1)}, \quad X \gg 1,$$

where $c'_\alpha$ is independent of $\tau$, which implies that $\langle (x_B(m))^2 \rangle$ is well defined for $\alpha > 1/2$ (of course $\langle x_B(m) \rangle = 0$ by symmetry for all $\alpha$). A straightforward calculation shows that

$$\frac{\langle x_B^2(m) \rangle}{n^{2/\alpha}} = \tilde{a}_\alpha \frac{m}{n} \left( 1 - \frac{m}{n} \right), \quad \tilde{a}_\alpha = \frac{\alpha \Gamma(2 - \alpha^{-1})}{\Gamma(1 + \alpha^{-1})}.$$  

It is interesting to notice that $\langle x_B^2(m) \rangle/n^{2/\alpha}$ depends on $\alpha$ only through the amplitude $\tilde{a}_\alpha$ but the parabolic shape in $(m/n)(1 - m/n)$ holds for all values of $2 \geq \alpha > 1/2$. Besides $\tilde{a}_\alpha$ is diverging for $\alpha \to 1/2^+$, while one has $\tilde{a}(1) = 1$ and $\tilde{a}(2) = 2$. One finds, curiously, that it reaches a minimum for a non-trivial value $\alpha^* = 0.74122 \ldots$ for which $\tilde{a}(\alpha^*) = 0.85264 \ldots$. In view of these properties (29), (30) one expects that, for $\alpha < 2$, the area under such a Lévy bridge has a non-trivial distribution, which we now focus on.
3.2. Distribution of the area

In this subsection, we consider a Lévy bridge, i.e. a Lévy walk which starts at the origin \(x_0 = 0\) and is conditioned to come back to the origin after \(n\) steps and we ask: what is the distribution \(P_B(A, n)\) of the area \(A\) under this Lévy bridge? One can obtain \(P_B(A, n)\) from equations (18) and (20) as

\[
P_B(A, n) = \frac{P(0, A, n|0, 0, 0)}{P(x = 0, n|0, 0, 0)}.
\]  

Therefore, using \(P(x = 0, n|0, 0, 0) = n^{-1/\alpha}S_\alpha(0) = n^{-1/\alpha}\Gamma(1 + \alpha^{-1})/\pi\), see equation (23), together with the scaling form (24) and (26) one obtains, in the limit \(n \to \infty\), keeping \(A/n^{1+1/\alpha}\) fixed:

\[
P_B(A, n) = \frac{1}{n^{1+1/\alpha}}F_\alpha \left( \frac{A}{n^{1+1/\alpha}} \right),
\]

\[
F_\alpha(Y) = \frac{1}{2\Gamma(1 + \alpha^{-1})} \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k| e^{-|k|^\alpha Y - i k r}.
\]

Using the explicit expression of \(\gamma(r)\) above (27), one computes the Fourier transform \(\hat{F}_\alpha(k)\) of \(F_\alpha(Y)\) as

\[
\hat{F}_\alpha(k) = \int_{-\infty}^{\infty} F_\alpha(Y) e^{ikY} dY = \frac{|k|}{\Gamma(1 + \frac{1}{\alpha})} \int_{0}^{\infty} e^{-|k|^\alpha/(\alpha+1)(1+\alpha^{-1})} dr
\]

\[
+ \int_{0}^{1/2} e^{-|k|^\alpha/(\alpha+1)((1/2+r)^{\alpha+1}+(1/2-r)^{\alpha+1})} dr.
\]

For generic \(\alpha\), it seems quite difficult to perform explicitly the integrals over \(r\) and \(k\) in the expression for the distribution \(F_\alpha(Y)\) in equation (34). One can however extract from this expression the asymptotic behaviors both for \(Y \to 0\) and \(\infty\).

Asymptotic behavior for small argument. For small arguments, it is straightforward to see from the expression (34) that the leading behavior of \(F_\alpha(Y)\) when \(Y \to 0\) is given by

\[
F_\alpha(Y) \sim F_\alpha(0), \quad Y \to 0,
\]

with

\[
F_\alpha(0) = \frac{(\alpha + 1)^{2/\alpha}}{2\pi} \frac{\Gamma(1 + \frac{2}{\alpha})}{\Gamma(1 + \frac{1}{\alpha})} \left( \int_{0}^{\infty} \frac{dr}{(r + 1)^{\alpha+1} - r^{\alpha+1}} \right)^{2/\alpha}
\]

\[
+ \int_{0}^{1/2} \frac{dr}{(1/2 + r)^{\alpha+1} + (1/2 - r)^{\alpha+1}}.
\]

A study of this function \(F_\alpha(0)\) shows that it is a decreasing function of \(\alpha\) on the interval \([0, 2]\), which is diverging when \(\alpha \to 0\). For \(\alpha = 1\) and 2, \(F_\alpha(0)\) assumes simple values

\[
F_1(0) = 1 + \frac{4}{\pi} = 2.27324 \ldots, \quad F_2(0) = \sqrt{\frac{3}{\pi}} = 0.977205 \ldots.
\]

Asymptotic behavior for large argument. The analysis of the large argument behavior of \(F_\alpha(Y)\) is more involved. A careful analysis, left to appendix (B), shows that for large \(Y\)
one has:
\[
F_\alpha(Y) \sim \frac{a_\alpha}{Y^{2(1+\alpha)}}, \quad Y \gg 1, \quad (37)
\]
\[
a_\alpha = \frac{2^{-2(2+\alpha)} \sqrt{\pi} \Gamma(2+2\alpha) \tan(\alpha \pi/2)}{\Gamma(2+\alpha^2) \Gamma(1-\alpha) \Gamma(\frac{3}{2}+\alpha)}. \quad (38)
\]
When \( \alpha \to 2 \), one has from (37), \( a_\alpha \sim (\sqrt{\pi}/21)(\alpha - 2)^2 \). From equation (37) one obtains that the second moment of the distribution \( \langle Y^2 \rangle \) is defined only for \( \alpha > 1/2 \), where it takes the value (see equation (B.23))
\[
\langle Y^2 \rangle = \frac{\tilde{a}_\alpha}{12} = \frac{\alpha \Gamma(2-\alpha^{-1})}{12 \Gamma(1+\alpha^{-1})}, \quad \alpha > 1/2, \quad (39)
\]
where the amplitude \( \tilde{a}_\alpha \) appears in the expression for \( \langle x_B^2(m) \rangle \) computed above (32). This power law tail of the area distribution (37), with an exponent \( 2(1+\alpha) \), is quite interesting. Indeed, the area itself is the sum of non-identical and strongly correlated variables \( x_B(m) \) all having a similar power law tail, also with exponent \( 2(1+\alpha) \) (31). For \( \alpha > 1/2 \), their variance is finite and the non-Gaussianity of \( A \) can a priori be due both to the correlations between the \( x_B(m) \)’s and to the fact that \( A \) is the sum of non-identical random variables. To test which of these features is responsible for the non-Gaussianity of \( A \), we study the sum of \( n \) random variables \( X_m \), which are independent and such that \( X_m \) has the same distribution as \( x_B(m) \). Defining \( S_n = \sum_{m=1}^{n} X_m \) and \( \Sigma_n = \sum_{m=1}^{n} \langle X_m^2 \rangle \), it is known that \( S_n/\Sigma_n \) converges to a centered Gaussian variable of unit variance if the following condition (known as the Lindeberg condition) is satisfied [2]
\[
\lim_{n \to \infty} \frac{1}{\Sigma_n^2} \int_{|x| > \epsilon \Sigma_n} x^2 \text{Proba}(X_m = x) \, dx = 0, \quad \forall \epsilon > 0. \quad (40)
\]
Intuitively, this Lindeberg condition (40) ensures that the probability that any term \( X_m \) will be of the same order of magnitude as the sum \( S_n \) must tend to zero (see appendix (A) for an example of non-identical independent random variables which do not satisfy the Lindeberg condition). In the present case of the Lévy bridge, one can check (see appendix (A)) that if \( X_m \) is distributed like \( x_B(m) \) then the above Lindeberg condition (40) is satisfied (see appendix (A)). Therefore the deviations from Gaussianity (37) are purely due to the strong correlations between the positions of the walker \( x_B(m) \)’s.

The special case \( \alpha = 1 \). In the Cauchy case, \( \alpha = 1 \), the integral over \( k \) can be performed in equation (26) to obtain
\[
G(X, Y|0) = \int_{-\infty}^{\infty} \frac{\gamma^2(r) - (rX + Y)^2}{\gamma^2(r) + (rX + Y)^2} \frac{dr}{2\pi}, \quad (41)
\]
where the function \( \gamma(r) \) (27) here takes a rather simple form
\[
\gamma(r) = \begin{cases} 
-\frac{r - \frac{1}{2}}{2}, & r < -1 \\
-\frac{r^2 + r + \frac{1}{2}}{2}, & -1 \leq r \leq 0 \\
\frac{r + \frac{1}{2}}{2}, & r > 0.
\end{cases} \quad (42)
\]
The distribution of the area under a Cauchy bridge is thus given by

\[ P_B(A, n) = \frac{1}{n^2} F_1 \left( \frac{A}{n^2} \right), \tag{43} \]

where

\[ F_1(Y) = \frac{2}{\pi (1 + 4Y^2)} + \frac{1}{\pi} \int_0^{1/2} \frac{(u^2 + 1/4)^2 - 1}{((u^2 + 1/4)^2 + Y^2)^2} \, du. \]

Under this form (43), one can easily obtain the asymptotic behavior as

\[ F_1(Y) \sim \begin{cases} 1 + \frac{4}{\pi Y}, & Y \to 0 \\ \frac{1}{20\pi Y^4}, & Y \to \infty \end{cases} \tag{44} \]

in agreement with the asymptotic behavior obtained above (36), (37). In fact the integral over \( u \) in the expression (43) can be performed explicitly yielding the expression

\[ F_1(Y) = \frac{1}{\pi} \frac{2}{1 + 4Y^2} + \frac{1}{\pi} \left( \frac{2(1 - 8Y^2)}{(1 + 4Y^2)(1 + 16Y^2)} + \frac{4}{(1 + 16Y^2)^{3/2}} \right. \]

\[ \times \, \text{Re} \left[ (1 - 4iY)^{3/2} \arctan \left( (1 + 4iY)^{-1/2} \right) \right] \], \tag{45} \]

where \( \text{Re}(z) \) denotes the real part of the complex number \( z \). In appendix (C) we show how this expression (45) can be written explicitly in terms of elementary functions (C.4)–(C.6).

### 4. Average profile for fixed area

#### 4.1. The case of a free Lévy walk

We first consider the case of a free Lévy walk of constrained area. We compute the probability \( \tilde{P}(x, m| A, n) \) that the position of the random walker, starting at \( x(0) = x_0 = 0 \) at time 0, is \( x \) after \( m \) time steps, given that the area after \( n \) time steps is fixed to \( A \).

From this probability, one obtains the average profile as

\[ \langle \tilde{x}(m) \rangle = \int_{-\infty}^{\infty} x \tilde{P}(x, m| A, n) \, dx. \]

To compute this probability \( \tilde{P}(x, m| A, n) \), we divide the interval \([0, n]\) into two intervals \([0, m]\) and \([m, n]\). Over \([0, m]\) the process starts at \( x_0 = 0 \) with area \( A_0 = 0 \) and reaches \( x \) with area \( A_1 \) (see the light area on figure 2(b)). Over the interval \([m, n]\), the process starts at \( x \) and reaches \( x_F \) with area \( A - A_1 \) (see the shaded area on figure 2(b)). Therefore this probability \( \tilde{P}(x, m| A, n) \) can be simply expressed in terms of the propagator \( P(x, A, n|x_0, x_0, 0) \) computed above (18) as (see figure 2(b)):

\[ \tilde{P}(x, m| A, n) = \frac{1}{P(A, n)} \int_{-\infty}^{\infty} dx_F \int_{-\infty}^{\infty} dA_1 P(x, A_1, m|0, 0, 0) \]

\[ \times \, P(x_F, A - A_1, n - m|x, x, 0), \tag{46} \]

where we have used the Markov property of the Lévy random walk. In the above expression (46), \( x_F \) is the end point of the walk (see figure 2(b)), which is free here. Hence \( \tilde{P}(x, m| A, n) \) is obtained by integration over this end point \( x_F \). Notice that it is normalized according to \( \int_{-\infty}^{\infty} \tilde{P}(x, m| A, n) \, dx = 1 \) (and therefore we have divided by \( P(A, n) \) in the expression (46) because the measure is restricted to random walks of fixed
area $A$ after $n$ time steps). Using the explicit expressions computed above (18) one obtains after integration over $x_F$ and $A_1$:

$$
\hat{P}(x, m|A, n) = \frac{1}{P(A, n|0, 0)} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{-\sum_{\nu=0}^{m} |k_1+\nu k_2|^\alpha - |k_2|^\alpha} e^{-ik_1 x - ik_2 (A-(n-m)x)}.
$$

(47)

In the large $n$ limit, keeping $X = x/n^{1/\alpha}$, $Y = A/n^{1+1/\alpha}$ and $\tau = m/n$ fixed one has

$$
\hat{P}(x, m|A, n) = \frac{1}{n^{1/\alpha}} \hat{G}(X, \tau|Y),
$$

$$
\hat{G}(X, \tau|Y) = \frac{(1+\alpha)^{-1/\alpha}}{S_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-|k|^\alpha (\hat{\gamma}(r, \tau) + \hat{\gamma}(0,1-\tau))} e^{-ikr X - ik(Y-(1-\tau)X)},
$$

where we have used the expression of $P(A, n|0, 0)$ given in equations (20) and we have introduced

$$
\hat{\gamma}(r, \tau) = \int_{0}^{\tau} |r + z|^\alpha \, dz,
$$

(49)

which is a generalization of the function $\gamma(r) \equiv \hat{\gamma}(r, 1)$ in (26). It reads

$$
\hat{\gamma}(r, \tau) = \begin{cases}
\frac{1}{\alpha+1} (-(r)^{\alpha+1} - (-\tau - r)^{\alpha+1}), & r \leq -\tau, \\
\frac{1}{\alpha+1} ((r + \tau)^{\alpha+1} + (-r)^{\alpha+1}), & -\tau \leq r \leq 0, \\
\frac{1}{\alpha+1} ((r + \tau)^{\alpha+1} - r^{\alpha+1}), & r \geq 0.
\end{cases}
$$

(50)

We can now compute $\langle \hat{x}(m) \rangle$, which, in the large $n$ limit, takes the scaling form

$$
\langle \hat{x}(m) \rangle = h_\alpha \left( \frac{m}{n^{1+1/\alpha}} \frac{A}{n^{1/\alpha}} \right)
$$

$$
h_\alpha(\tau, Y) = \frac{(1+\alpha)^{-1/\alpha}}{S_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^\alpha g(r, \tau) e^{-ikr X - ik(Y-(1-\tau)X)},
$$

(51)

with $g(r, \tau) = \hat{\gamma}(r, \tau) + \hat{\gamma}(0,1-\tau)$. This function $h_\alpha(\tau, Y)$ can be written as

$$
h_\alpha(\tau, Y) = \frac{(1+\alpha)^{-1/\alpha}}{S_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^\alpha g(r, \tau)
$$

$$
\times \frac{\partial}{\partial \tau} (e^{-ikr X - ik(Y-(1-\tau)X)})
$$

(52)

which suggests performing an integration by parts in the integral over $r$, yielding (one can check that the boundary terms vanish)

$$
h_\alpha(\tau, Y) = \frac{(1+\alpha)^{-1/\alpha}}{S_\alpha((\alpha+1)^{1/\alpha} Y)} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^{1+\alpha} \frac{\partial g(r, \tau)}{\partial \tau}
$$

$$
\times e^{-|k|^\alpha g(r, \tau)} e^{-ikr X - ik(Y-(1-\tau)X)}.
$$

(53)

doi:10.1088/1742-5468/2010/08/P08005
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In expression (53), the integral over $X$ can be performed to simply yield a delta function of $r$, namely $2\pi|k|^{-1}\delta(r - (1 - \tau))$. This allows us then to perform the integral over $r$ to obtain

$$h_{\alpha}(\tau, Y) = \frac{(1 + \alpha)^{-1/\alpha}}{S_{\alpha}((\alpha + 1)^{1/\alpha}Y)} \left( \frac{\partial g(r, \tau)}{\partial r} \right)_{r=1-\tau} \int_{-\infty}^{\infty} |k|^\alpha k^{-1} e^{-(|k|^{\alpha}/(1+\alpha))} e^{-ikY} \frac{dk}{2\pi}, \quad (54)$$

where we have used the relation $\hat{\gamma}(1 - \tau, \tau) + \hat{\gamma}(0, 1 - \tau) = 1/(1 + \alpha)$. It is then easy to check that

$$\int_{-\infty}^{\infty} |k|^\alpha k^{-1} e^{-(|k|^{\alpha}/(1+\alpha))} e^{-ikY} \frac{dk}{2\pi} = \frac{\alpha + 1}{\alpha} Y(1 + \alpha)^{1/\alpha} S_{\alpha} [(1 + \alpha)^{1/\alpha}Y], \quad (55)$$

so that finally one obtains the simple result

$$h_{\alpha}(\tau, Y) = Y\frac{\alpha + 1}{\alpha} (1 - (1 - \tau)^{\alpha}), \quad (56)$$

where we have used $(\partial g(r, \tau)/\partial r)_{r=1-\tau} = 1 - (1 - \tau)^{\alpha}$. In figure 3(a), we show a plot of $h_{\alpha}(\tau, Y)/Y$ as a function of $\tau$.

4.2. The case of a Lévy bridge

Here we consider a Lévy bridge, i.e. a Lévy random walker starting at 0 at the initial time and constrained to come back to the origin after $n$ time steps. We compute the probability $\tilde{P}_B(x, m|A, n)$ that the position of the random walker is $x$ after $m$ time steps given that the area, after $n$ time steps, is fixed to $A$. From this probability, one obtains the average profile as $\langle \tilde{x}_B(m) \rangle = \int_{-\infty}^{\infty} x\tilde{P}_B(x, m|A, n) \, dx$. This probability $\tilde{P}_B(x, m|A, n)$ can be expressed, as in equation (46), in terms of the propagator $P(x, A, n|x_0, x_0, 0)$ computed

doi:10.1088/1742-5468/2010/08/P08005

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above in equation (18) as:
\[
\tilde{P}_B(x, m|A, n) = \frac{1}{P(0, A, n|0, 0)} \int_{-\infty}^{\infty} P(x, A_1, m|0, 0, 0)P(x, A - A_1, n - m|0, 0, 0) \, dA_1,
\]
(57)
where we have used the Markov property of the Lévy random walk. It is normalized according to \( \int_{-\infty}^{\infty} \tilde{P}_B(x, m|A, n) \, dx = 1 \) (and therefore we have divided by \( P(0, A, n|0, 0, 0) \) because the measure is restricted to bridges of fixed area \( A \)). Using the explicit expressions obtained above (18) one has
\[
\tilde{P}_B(x, m|A, n) = \frac{1}{P(0, A, n|0, 0, 0)} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk'_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{-i(k_1+kk')x-ikA}
\times e^{-\sum_{m=1}^{\infty} [k_1^{\alpha}+k2^{\alpha}]-\sum_{m=1}^{\infty} [k_1^{\alpha}+k2^{\alpha}]}. \tag{58}
\]
In the large \( n \) limit, keeping \( X = x/n^{1/\alpha}, Y = A/n^{1+1/\alpha} \) and \( \tau = m/n \) fixed one has
\[
\tilde{P}_B(x, m|Y, n) = \frac{1}{n^{1/\alpha}} \tilde{G}_B(X, \tau|Y), \tag{59}
\]
\[
\tilde{G}_B(X, \tau|Y) = \frac{\pi}{\Gamma(1+\alpha)}F_\alpha(Y) \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dr'}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} k^2 e^{-ik(r+r')X-ikY}
\times e^{-|k|^{\alpha}\tilde{\gamma}(\tau, r)+\tilde{\gamma}(\tau',1-r)} \tag{60}
\]
We can now compute \( \langle \tilde{x}_B(m) \rangle \), which in the large \( n \) limit takes the scaling form
\[
\frac{\langle \tilde{x}_B(m) \rangle}{n^{1/\alpha}} = H_\alpha \left( \frac{m}{n}, \frac{A}{n^{1+1/\alpha}} \right), \tag{61}
\]
where
\[
H_\alpha(\tau, Y) = \frac{1}{2\Gamma(1+\alpha)F_\alpha(Y)} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dr |k|^{\alpha-1} \partial_r \tilde{\gamma}(r, \tau)e^{-|k|^{\alpha}\tilde{\Gamma}(\tau, r)-ikY}
\]
\[
= -\left. \frac{1}{2\Gamma(1+\alpha)F_\alpha(Y)} \frac{\partial}{\partial Y} \left( \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dr |k|^{\alpha-1} \partial_r \tilde{\gamma}(r, \tau)e^{-|k|^{\alpha}\tilde{\Gamma}(\tau, r)-ikY} \right) \right|_{Y=0}, \tag{62}
\]
where we have introduced the notation \( \tilde{\Gamma}(r, \tau) = \tilde{\gamma}(r, \tau) + \tilde{\gamma}(-r, 1-\tau) \) which we compute straightforwardly from equation (50) as:
\[
\tilde{\gamma}(r, \tau) = \tilde{\gamma}(r, \tau) + \tilde{\gamma}(-r, 1-\tau) = \begin{cases} 
\frac{1}{\alpha+1}((-r+1-\tau)^{\alpha+1} - (-r - \tau)^{\alpha+1}), \\
\quad r \leq -\tau \\
\frac{1}{\alpha+1}((r + \tau)^{\alpha+1} + (-r + 1 - \tau)^{\alpha+1}), \\
\quad -\tau \leq r \leq 1-\tau \\
\frac{1}{\alpha+1}((r + \tau)^{\alpha+1} - (r - 1 + \tau)^{\alpha+1}), \\
\quad r \geq 1-\tau.
\end{cases} \tag{63}
\]
Note that this function $\hat{\Gamma}(r, \tau)$ satisfies the identity
\[
\hat{\Gamma}(1 - \tau - r, \tau) = \int_{0}^{1} |z - r|^{\alpha} \, dz = \gamma(-r),
\] (64)
independently of $\tau$.

For generic $\alpha$, the expression (62) is quite difficult to handle. For $\alpha = 2$ (Brownian motion) and $\alpha = 1$, further analytical progress is however possible. For $\alpha = 2$, one has $\partial_{r}\hat{\gamma}(r, \tau) = \tau(2r + \tau)$ and $\hat{\Gamma}(r, \tau) = r^{2} + r(2\tau - 1) - \tau(1 - \tau) + 1/3$ and therefore one finds
\[
\int_{-\infty}^{\infty} \partial_{r}\hat{\gamma}(r, \tau)e^{-k^{2}\hat{\Gamma}(r, \tau)} \, dr = \frac{\sqrt{\pi}}{|k|}e^{-(k^{2}/12)(1 - \tau)}.
\] (65)
Using this identity (65), integrating over $r$ in the expression (62), and using $F_{2}(Y) = \sqrt{3/\pi} \exp(-3Y^{2})$ one obtains for $\alpha = 2$:
\[
H_{2}(\tau, Y) = 6Y\tau(1 - \tau).
\] (66)

Another interesting case where analytical progress is possible is $\alpha = 1$. Given the expression for $\hat{\gamma}(r, \tau)$ in equation (50) and $\hat{\Gamma}(r, \tau)$ in equation (63), one observes that the integral over $r$ in equation (62) gives rise to four different terms, corresponding to $r \in ]-\infty, -\tau]$, $r \in ]-\tau, 0]$, $r \in [0, 1 - \tau]$ and finally $r \in [1 - \tau, +\infty]$. For $\alpha = 1$ it turns out that the first and fourth terms, corresponding to $r \in ]-\infty, -\tau]$ and $r \in [1 - \tau, +\infty]$ cancel each other (which is the case only for $\alpha = 1$) resulting in the following expression:
\[
H_{1}(\tau, Y) = \frac{Y}{\pi \Gamma(1 + \alpha)}F_{1}(Y)\left(\int_{-\tau}^{0} (2r + \tau) \frac{\hat{\Gamma}(r, \tau)}{(\hat{\Gamma}(r, \tau)^{2} + Y^{2})^{2}} \, dr + \frac{\tau}{\hat{\Gamma}(r, \tau)^{2} + Y^{2}}\right),
\] (67)
with $\hat{\Gamma}(r, \tau) = \frac{1}{2}(r + \tau)^{2} + (r + \tau - 1)^{2})$. In the asymptotic limit $Y \to 0$ one obtains
\[
H_{1}(\tau, Y) \sim \frac{Y}{4 + \pi} \left(3\pi - 2 + \frac{2}{1 + 2\tau(\tau - 1)} + 12(2\tau - 1) \arctan(1 - 2\tau)\right).
\] (68)
In the opposite limit $Y \to \infty$ one obtains
\[
H_{1}(\tau, Y) \sim Y\frac{10}{\pi}\tau(1 - \tau)(2 - \tau(1 - \tau)).
\] (69)
Note that although the two functions of $\tau$ entering these asymptotic expansions in equations (68) and (69) have very different analytical expressions, they are actually quite close to each other on the interval $[0, 1]$ (see figure 3(b)).

5. Numerical results

We now come to numerical simulations of Lévy bridges. As mentioned above, one can not use the relation (3), which is only valid for $\alpha = 2$ [23] to simulate a Lévy bridge. Instead, we consider the joint probability distribution function (pdf) of the increments $\eta(m)$ for a Lévy bridge of size $n$. Indeed, these increments are independent random variables, distributed according to $\phi(\eta)$ with the global constraint that $x(n) = \sum_{m=1}^{n} \eta(m) = 0$. doi:10.1088/1742-5468/2010/08/P08005
Therefore the joint pdf of the increments $P_B(\eta(1), \eta(2), \ldots, \eta(n))$ is simply given by

$$P_B(\eta(1), \eta(2), \ldots, \eta(n)) \propto \prod_{m=1}^{n} \phi[\eta(m)] \delta \left( \sum_{m=1}^{n} \eta(m) \right) \exp \left[ \sum_{m=1}^{n} \ln[\phi[\eta(m)]] \right] \delta \left( \sum_{m=1}^{n} \eta(m) \right).$$

(70)

This joint distribution can thus be considered as a Boltzmann weight with an effective energy $E = -\sum_{m=1}^{n} \ln[\phi[\eta(m)]]$ and effective inverse temperature $\beta = 1$. This thus leads us to use a Monte Carlo algorithm, with a global constraint, to generate ‘configurations’ of the increments distributed according to the distribution (70). We implement it in the following way. We start with a random initial configuration of the $\eta(m)$’s which satisfies the global constraint $\sum_{m=1}^{n} \eta(m) = 0$ (it can also be $\eta(m) = 0$, for all $m$). At each time step we choose randomly two sites $i$ and $j$ among $1, 2, \ldots, n$ and the simple following moves are proposed

$$\eta(i) \rightarrow \eta'(i) = \eta(i) + \Delta \eta, \quad \eta(j) \rightarrow \eta'(j) = \eta(j) - \Delta \eta,$$

(71)

such that the global constraint of zero sum is automatically satisfied. This move is then accepted, in Metropolis algorithm that we use here, with a probability $P_{ij}$ given by

$$P_{ij} = \min \left( 1, \frac{\phi[\eta'(i)] \phi[\eta'(j)]}{\phi[\eta(i)] \phi[\eta(j)]} \right) \exp \left( -\Delta E \right), \quad \Delta E = \log \left( \frac{\phi[\eta(i)] \phi[\eta(j)]}{\phi[\eta'(i)] \phi[\eta'(j)]} \right).$$

(72)

(73)

This Monte Carlo algorithm is thus very similar to the Kawasaki dynamics for ferromagnetic spin systems relaxing towards equilibrium with a conserved global magnetization [26]. Once the increments $\eta(k)$’s are generated according to this joint probability (70), we can generate the random walk bridge $x_B(m) = \sum_{k=1}^{m} \eta(k)$ and compute the distribution of the area $A = \sum_{m=1}^{n} x_B(m)$ under the Lévy bridge. In figure 4(a), we show a plot of this distribution $P_B(A, n)$ for $\alpha = 1$ and $n = 100$. To compute it we have first run $10^7$ Monte Carlo steps to equilibrate the system, and the distribution was then computed as an average over $10^7$ samples generated in the time interval $[10^7, 2 \times 10^7]$. In figure 4, we also show a plot of the exact explicit expression for $F_1(Y)$ given in equation (C.6), showing a very good agreement with our numerics. We have also computed numerically this distribution for other values of $\alpha \in [0, 2]$, showing a good agreement with the power law tail obtained in equation (37). Note, however, that for small $\alpha$, to equilibrate the system such that a precise estimate of the exponent characterizing the power law tail of $P_B(A, n)$ is obtained is quite difficult for $\alpha < 1$.

We can use a similar Monte Carlo approach to generate a random walk bridge with a fixed area $A = \sum_{m=1}^{n} (n + 1 - m) \eta(m)$. In that case, the joint pdf of the increments $P_B(\eta(1), \eta(2), \ldots, \eta(n))$ is simply given by

$$P_B(\eta(1), \eta(2), \ldots, \eta(n)) \propto \prod_{m=1}^{n} \phi[\eta(m)] \delta \left( \sum_{m=1}^{n} \eta(m) \right) \delta \left( \sum_{m=1}^{n} (n + 1 - m) \eta(m) - A \right).$$

(74)
We start with an initial configuration of the $\eta(m)$’s which satisfies both the global constraints. In practice, we start with $\eta(m) = 6A(N + 1 - 2m)/(N - 1)N(N + 1)$. Then, to satisfy both constraints (74), at each time step we choose randomly three sites $i, j$ and $k$ among $1, 2, \ldots, n$ and the simple following moves are proposed

\begin{align}
\eta(i) &\to \eta'(i) = \eta(i) + \Delta \eta, \\
\eta(j) &\to \eta'(j) = \eta(j) + \frac{i - k}{k - j} \Delta \eta, \\
\eta(k) &\to \eta'(k) = \eta(k) + \frac{j - i}{k - j} \Delta \eta.
\end{align}

(75)

Note that to converge to the correct probability measure (74) one has to choose $\eta$ either positive or negative with equal probability. This move (75) is then accepted with a probability $P_{ijk}$ given by

\begin{equation}
P_{ijk} = \min \left( 1, \frac{\phi[\eta'(i)] \phi[\eta'(j)] \phi[\eta'(k)]}{\phi[\eta(i)] \phi[\eta(j)] \phi[\eta(k)]} \right)
= \min(1, \exp(-\Delta E)), \quad \Delta E = \log \left( \frac{\phi[\eta(i)] \phi[\eta(j)] \phi[\eta(k)]}{\phi[\eta'(i)] \phi[\eta'(j)] \phi[\eta'(k)]} \right).
\end{equation}

(76)

Once the increments $\eta(k)$’s are generated according to this joint probability (74), we can generate the random walk bridge $\tilde{x}_B(m) = \sum_{k=1}^m \eta(k)$ with fixed area $A$ and compute the profile $\langle \tilde{x}_B(m) \rangle$. In figure 4(b), we show a plot of this average profile for $\alpha = 1$, $n = 100$ and $A/n^2 \sim 20$. To compute it we have first run $10^7$ Monte Carlo steps to equilibrate the system and the average was then computed over $10^7$ samples generated in the time interval $[10^7, 2 \times 10^7]$. In figure 4(b), we also plot, with a solid line, our asymptotic result in equation (69), showing a relatively good agreement with our numerics (note that here $A/n^2 = 20$). Note that the small discrepancies are mainly due to the slow convergence of the Monte Carlo algorithm to the ‘equilibrium’ distribution given in equation (74). On the
same plot, figure 4(b), we also show in dotted line, the result for the Brownian bridge (5), which is independent of $A$. It is quite remarkable that these two profiles are very similar which show that the global constraints that we impose here have strong consequences on the statistics of the Lévy random walk.

6. Conclusion

To conclude, we have studied two main properties of a Lévy bridge $x_B(m)$ of length $n$: (i) the distribution $P_B(A,n)$ of the area under a Lévy bridge and (ii) the average profile $\langle \tilde{x}_B(m) \rangle$ of a Lévy bridge with fixed area $A$.

- For $P_B(A,n)$ we have found the scaling form, valid for large $n$, $P_B(A,n) \sim n^{-1-1/\alpha} F_\alpha(Y)$ with an interesting power law behavior $F_\alpha(Y) \sim Y^{-2(1+\alpha)}$. For $\alpha = 1$, we have obtained an explicit expression for $F_1(Y)$ in terms of elementary functions (C.6). We have also shown, using the Lindeberg condition that the non-Gaussianity of $P_B(A,n)$, for $\alpha > 1/2$, is due only to the correlations between the positions of the walkers $x_B(m)$.

- For the average profile, $\langle \tilde{x}_B(m) \rangle$, we have found the scaling form $\langle \tilde{x}_B(m) \rangle \sim n^{1/\alpha} H_\alpha(m/n,A/n^{1+1/\alpha})$ where, at variance with Brownian motion, $H_\alpha(X,Y)$ is a non-trivial function of the rescaled area $Y$. For $\alpha = 1$, we have obtained simple analytical expressions for $H_1(X,Y)$ in both limits $Y \to 0$ and $\infty$. In particular, we have shown that the average profile of the Lévy random walk with a fixed area is not very far from the profile of a Brownian bridge with fixed area.

- We have finally compared our analytical results with Monte Carlo simulations of these Lévy random walks with global constraints.

In view of recent developments in the study of area distributions for variants of Brownian motions [9,11,13,15], it would be very interesting to extend the results presented here to other constrained Lévy walks, including in particular Lévy random walks conditioned to stay positive (Lévy excursion), which is a challenging open problem.

Appendix A. On the use of the Lindeberg condition

Let us consider independent and non-identical random variables $X_1, \ldots, X_n$ which have the same distribution as the Lévy bridge $x_B(m)$ (29), i.e.

$$\text{Proba}(X_m = x) = \frac{\pi}{\Gamma(1 + \alpha^{-1/2})} m^{1/\alpha} (n-m)^{1/\alpha} S_\alpha \left( \frac{x}{m^{1/\alpha}} \right) S_\alpha \left( \frac{x}{(n-m)^{1/\alpha}} \right).$$ (A.1)

For $\alpha > 1/2$, $\sigma_m^2 = \langle X_m^2 \rangle$ is well defined and one has (32)

$$\sigma_m^2 = \langle X_m^2 \rangle = \langle x_B(m)^2 \rangle = \tilde{a}_an^{2/\alpha-2}m(n-m).$$ (A.2)

Given that the variables $X_i$ are not identical, one can not apply directly the Central Limit Theorem. However, one can show that these random variables $X_m$ do satisfy the Lindeberg condition, which guarantees that their sum $A_n = \sum_{m=0}^n X_m$ is distributed
which implies \( \Sigma \) holds. One computes straightforwardly, for
\[
\langle X_m^2 \rangle = \int_{|x| > \epsilon \Sigma} x^2 \text{Proba}(X_m = x) \, dx
\]
\[
= 2 \frac{\pi}{\Gamma(1 + \alpha^{-1/2})} m^{1/\alpha} \int_{\epsilon \Sigma} \frac{n^{1/\alpha}}{(n - m)^{1/\alpha}} \, x^2 S_\alpha \left( \frac{x}{m^{1/\alpha}} \right) S_\alpha \left( \frac{x}{(n - m)^{1/\alpha}} \right) \, dx
\]
\[
\sim c'_m n^{2/\alpha} m^{-3(2/\alpha)} (n - m),
\]
where \( c'_m \) is independent of \( m \) and \( n \). Therefore one has
\[
\frac{\sum_{n=1}^m \langle X_m^2 \rangle}{\Sigma_n^2} \sim n^{-(\alpha - 1/2)}.
\]
For random variables for which the above ratio \((A.6)\) goes to zero in the limit \( n \to \infty \)
(which is the case here for \( \alpha < 1/2 \)), a theorem due to Lindeberg (thus called the
‘Lindeberg condition’) [2], says that their sum \( A_n/\Sigma_n = \Sigma_n^{-1} \sum_{m=0}^n X_m \) is distributed, in the limit \( n \to \infty \), according to a Gaussian distribution of unit variance. The fact that, for a Lévy bridge, the area is not a Gaussian distribution \((37)\) is thus, for \( \alpha > 1/2 \), a consequence of the correlations between the random variables \( x_B(m) \).

To conclude this paragraph, we discuss a simple case where the Lindeberg condition
does not hold. Consider the case where \( X_1, \dotsc, X_n \) are independent random variables
distributed according to [27]
\[
\text{Proba}(X_m = x) = me^{-mx},
\]
such that one has
\[
\langle X_m \rangle = \frac{1}{m}, \quad \langle (X_m - \langle X_m \rangle)^2 \rangle = \frac{1}{m^2}.
\]
Then in that case one has immediately
\[
\Sigma_n^2 = \sum_{m=1}^n \langle (X_m - \langle X_m \rangle)^2 \rangle = \sum_{m=1}^n \frac{1}{m^2} \to \frac{\pi^2}{6}, \quad n \to \infty.
\]
One computes straightforwardly, for \( \epsilon > 0 \)
\[
\int_{\epsilon \Sigma}^\infty (X_m - \langle X_m \rangle)^2 \text{Proba}(X_m = x) \, dx = \frac{e^{-\epsilon \Sigma m}}{m^2} (1 + m^2 (e \Sigma m)^2),
\]
such that here one has
\[
\frac{1}{\Sigma_n^2} \sum_{m=1}^\infty \int_{\epsilon \Sigma}^\infty (X_m - \langle X_m \rangle)^2 \text{Proba}(X_m = x) \, dx \to \frac{6}{\pi^2} \sum_{m=1}^\infty \frac{e^{-\epsilon m}}{m^2} (1 + (\epsilon m)^2) > 0,
\]
with \( \epsilon = \epsilon \pi^2/6 \). Therefore the Lindeberg condition \((40)\) does not hold here. In fact, it
can be shown that the distribution of the variable \( (S_n - \sum_{k=1}^n k^{-1})/\Sigma_n \) converges to a
Gumbel distribution [27].

\[\text{doi:10.1088/1742-5468/2010/08/P08005}\]
Appendix B. Asymptotic behavior of $F_{\alpha}(Y)$ for large $Y$

To analyze the large argument behavior of $F_{\alpha}(Y)$, we analyze the small $k$ behavior of its Fourier transform $\hat{F}_{\alpha}(k)$ given in the text in equation (35):

$$\hat{F}_{\alpha}(k) = \int_{-\infty}^{\infty} dY F_{\alpha}(Y) e^{ikY} = \hat{F}_{\alpha,1}(k) + \hat{F}_{\alpha,2}(k), \quad (B.1)$$

$$\hat{F}_{\alpha,1}(k) = \frac{|k|}{\Gamma(1 + \alpha^{-1})} \int_{0}^{\infty} e^{-|k|^\alpha/(\alpha+1)(r+1)^{\alpha+1} - r^{\alpha+1}} \, dr, \quad (B.2)$$

$$\hat{F}_{\alpha,2}(k) = \frac{|k|}{\Gamma(1 + \alpha^{-1})} \int_{0}^{1/2} e^{-|k|^\alpha/(\alpha+1)(1/2+r)^{\alpha+1} + (1/2-r)^{\alpha+1}} \, dr. \quad (B.3)$$

The analysis of the small $k$ behavior of $\hat{F}_{\alpha,2}(k)$ is simply obtained by expanding the exponential under the integral. It yields straightforwardly:

$$\hat{F}_{\alpha,2}(k) = \frac{1}{\Gamma(1 + \alpha^{-1})} \left[ \frac{|k|}{2} \left( \frac{|k|^{1+\alpha}}{\alpha + 1} + \frac{|k|^{1+2\alpha}}{2(\alpha + 1)^2} \left( \frac{1}{3 + 2\alpha} + \frac{\sqrt{2\pi}}{\Gamma(\frac{5}{2} + \alpha)^2} \right) \right) \right] + O(|k|^{1+3\alpha}). \quad (B.4)$$

The asymptotic expansion of $\hat{F}_{\alpha,1}(k)$ is a bit a more subtle. To get the two first terms of the expansion, one performs the change of variable $z = |k|r$ and then expand $(r + 1)^{\alpha+1}$ using the binomial formula,

$$\frac{1}{\alpha + 1}((r + 1)^{\alpha+1} - r^{\alpha+1}) = r^\alpha + \frac{\alpha}{2} r^{\alpha-1} + \cdots. \quad (B.5)$$

This yields

$$\hat{F}_{\alpha,1}(k) \sim \frac{1}{\Gamma(1 + \alpha^{-1})} \int_{0}^{\infty} e^{-r^\alpha - |k|r^{\alpha-1}} \, dr + O(k^{1+\eta}), \quad (B.6)$$

where $\eta > 0$ is yet unknown (see below). From this expression (B.6), one immediately obtains the two first terms of the expansion of $\hat{F}_{\alpha,1}(k)$ as

$$\hat{F}_{\alpha,1}(k) = 1 - \frac{|k|}{2\Gamma(1 + \alpha^{-1})} + O(k^{1+\eta}). \quad (B.7)$$

Combining equations (B.4) and (B.7) one sees that the first non-trivial term, proportional to $|k|$ cancels in $\hat{F}_{\alpha}(k)$. Therefore, one needs to develop $\hat{F}_{\alpha,1}(k)$ beyond the first terms (B.7). To this purpose, we need to study separately the cases $0 < \alpha \leq 1/2$, $1/2 < \alpha \leq 1$ and $1 < \alpha \leq 2$.

B.1. The case $0 < \alpha \leq 1/2$

Let us first analyze the term $\hat{F}_{\alpha,1}(k)$ which we decompose as

$$\hat{F}_{\alpha,1}(k) = B_1(k) + B_2(k) + B_3(k),$$

$$B_1(k) = \frac{|k|}{\Gamma(1 + \alpha^{-1})} \int_{0}^{1} dr \, e^{-|k|^\alpha/(\alpha+1)(r+1)^{\alpha+1} - r^{\alpha+1}}, \quad (B.8)$$

doi:10.1088/1742-5468/2010/08/P08005
To expand $B_1(k)$ for small $k$ as

$$B_1(k) = \frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})} \left[ \int_1^\infty dr \, e^{-(|k|^\alpha/\alpha+1)(r+1)^{\alpha+1}-r^{\alpha+1}} - e^{-|k|^\alpha(r^{\alpha+1}/2)\alpha-1} \right], \quad \text{(B.9)}$$

$$B_3(k) = \frac{k}{\Gamma(1+\alpha^{-1})} \int_1^\infty dr \, e^{-|k|^\alpha(r^{\alpha+1}/2)\alpha-1}.$$  \hspace{1cm} \text{(B.10)}

It is easy to expand $B_1(k)$, for small $k$ as

$$B_1(k) = \frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})} \left[ \frac{2 - 2^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{|k|^{1+2\alpha}}{2\Gamma(1+\alpha^{-1})} \right]$$

$$\times \int_0^1 \left[ \frac{dr}{\alpha+1} \left( (r+1)^{\alpha+1} - r^{\alpha+1} \right) \right]^2 + O(k^{1+3\alpha}). \quad \text{(B.11)}$$

To expand $B_2(k)$, one checks the asymptotic behavior, for large $r$

$$\left( \frac{1}{\alpha+1} \left[ (r+1)^{\alpha+1} - r^{\alpha+1} \right] \right)^2 - \left( r^{\alpha+1} - \frac{\alpha}{2} r^{\alpha-1} \right)^2 = O(r^{2\alpha-2}), \quad \text{(B.12)}$$

so that, for $\alpha < 1/2$ one can safely expand the exponentials in the integrand of $B_2(k)$ (B.9) up to second order to obtain

$$B_2(k) = -\frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})} \int_1^\infty dr \left[ \frac{1}{\alpha+1} \left[ (r+1)^{\alpha+1} - r^{\alpha+1} \right] \right]$$

$$+ \frac{|k|^{1+2\alpha}}{2\Gamma(1+\alpha^{-1})} \int_1^\infty dr \left( \frac{1}{\alpha+1} \left[ (r+1)^{\alpha+1} - r^{\alpha+1} \right] \right)^2 - \left( r^{\alpha+1} - \frac{\alpha}{2} r^{\alpha-1} \right)^2$$

$$+ O(|k|^{\min(2,1+3\alpha)}). \quad \text{(B.14)}$$

Summing up the contributions from $B_1(k)$ and $B_2(k)$ and performing the integrals yields

$$B_1(k) + B_2(k) = \frac{|k|^{1+\alpha}}{\Gamma(1+\alpha^{-1})} \left( \frac{1}{\alpha+1} - \frac{1}{\alpha+2} - \frac{1}{2} \right)$$

$$+ \frac{|k|^{1+2\alpha}}{2\Gamma(1+\alpha^{-1})} \left[ \frac{1}{2} + \frac{1}{1+2\alpha} - \frac{1}{(1+\alpha^2)(3+2\alpha)} + \frac{\alpha^2}{8\alpha-4} \right]$$

$$+ 2 \frac{\Gamma(-3-2\alpha)\Gamma(1+\alpha)}{\Gamma(-\alpha)}, \quad \text{(B.15)}$$

which we have carefully checked using Mathematica.

Let us now expand $B_3(k)$ (B.10) for small $k$. It is easily seen from equation (B.7) that the first terms of this expansion are indeed given by

$$B_3(k) = 1 - \frac{|k|}{2\Gamma(1+\alpha^{-1})} + O(|k|^{1+\mu}), \quad \text{(B.16)}$$

\text{doi:10.1088/1742-5468/2010/08/P08005}
with \( \mu > 0 \). To go beyond the lowest orders, we first perform a change of variable \( x = |k| r \), then compute \( B_3''(k) \) and finally expand it for small \( k \). This yields

\[
B_3''(k) = \frac{1}{\Gamma(1 + \alpha - 1)} \left( \alpha \left( \frac{1}{2} + \left( 1 + \frac{\alpha}{2} \right) k^{\alpha - 1} e^{-(1 + (\alpha/2)) k^{\alpha}} + I_3(k) \right) \right), \tag{B.17}
\]

\[
I_3(k) = \left( \frac{\alpha}{2} \right)^2 \int_k^\infty e^{-(\alpha/2) k x^{\alpha - 1}} x^{2(\alpha - 1)} \, dx, \tag{B.18}
\]

where \( 2(\alpha - 1) < -1 \) for \( \alpha < 1/2 \). One then obtains the small \( k \) behavior of \( I_3(k) \) by simply expanding the term \( e^{-(\alpha/2) k x^{\alpha - 1}} \) in the integrand. This yields, to lowest order

\[
I_3(k) = \frac{1}{1 - 2\alpha} \left( \frac{\alpha}{2} \right)^2 k^{2\alpha - 1} + \mathcal{O}(k^{3\alpha - 1}). \tag{B.19}
\]

From equations (B.17) and (B.19), one obtains straightforwardly

\[
B_3(k) = 1 - \frac{|k|^{1 + \alpha}}{\Gamma(1 + \alpha - 1)} + \frac{|k|^{1 + \alpha}}{\Gamma(1 + \alpha - 1)} \left( \frac{1}{2} + \frac{1}{\alpha + 1} \right)
+ \frac{|k|^{1 + 2\alpha}}{\Gamma(1 + \alpha - 1)} \left( -\frac{1 + \alpha/2}{2(1 + 2\alpha)} \left( \frac{1}{2} + (1 + \alpha/2) \right) + \frac{1}{2(1 + 2\alpha)} \frac{1}{1 - 2\alpha} \left( \frac{\alpha}{2} \right)^2 \right).	ag{B.20}
\]

Finally, combining equations (B.15) and (B.20) together with the small \( k \) expansion of \( \hat{F}_{\alpha,2}(k) \) above (B.4), one sees that the term proportional to \( |k|^{1 + \alpha} \) actually cancels, yielding

\[
\hat{F}_{\alpha}(k) = 1 + c_{\alpha} |k|^{1 + 2\alpha} + \mathcal{O}(k^{3\alpha}), \tag{B.21}
\]

\[
c_{\alpha} = \frac{1}{\Gamma(1 + \alpha - 1) \cos (\alpha \pi)(1 + \alpha) \Gamma(-\alpha) \Gamma(5/2 + \alpha)} \left( -2^{-(2+\alpha)} \pi^{3/2} \tan (\alpha \pi/2) \right). \tag{B.22}
\]

This singular behavior of \( \hat{F}_{\alpha}(k) \) for small \( k \) (B.21) yields the power law behavior of \( F_{\alpha}(Y) \) for large \( Y \)

\[
F_{\alpha}(Y) \propto \frac{a_{\alpha}}{Y^{2(1+\alpha)}}, \quad Y \gg 1, \tag{B.23}
\]

\[
a_{\alpha} = -\frac{1}{\pi} \Gamma(2 + 2\alpha) \cos (\alpha \pi) c_{\alpha} = \frac{2^{-(2+\alpha)} \sqrt{\pi} \Gamma(2 + 2\alpha) \tan (\alpha \pi/2)}{\Gamma(2 + \alpha) \Gamma(1 - \alpha) \Gamma(5/2 + \alpha)} \tag{B.24}
\]

**B.2. The case \( \frac{1}{2} < \alpha \leq 1 \)**

This case can be studied along the same line as above except here, \( 1 + 2\alpha > 2 \) and therefore one has to handle with care the analysis of terms which are proportional to \( k^2 \), while the coefficient proportional to \( k^{1 + 2\alpha} \) has the same form (B.21). We will not repeat the analysis and simply give the result. One finds that \( \hat{F}_{\alpha}(k) \) behaves for small \( k \) as

\[
\hat{F}_{\alpha}(k) = 1 - \frac{b_{\alpha}}{2} k^2 + c_{\alpha} k^{1 + 2\alpha} + \mathcal{O}(k^{3(1 + 3\alpha)}), \quad b_{\alpha} = \frac{\alpha \Gamma(2 - \alpha - 1)}{12 \Gamma(1 + \alpha - 1)} \tag{B.25}
\]

doi:10.1088/1742-5468/2010/08/P08005
The expression for $b_\alpha$ given above (B.25) yields the expression for $\langle Y^2 \rangle$ given in the text in equation (39).

**B.3. The case $1 < \alpha \leq 2$**

In this case one can again perform a similar analysis but in this case one has $1 + 2\alpha > 3$. And therefore one has to handle carefully the term proportional to $|k|^3$. A quite lengthy calculation shows that this term actually vanishes for $\alpha > 1$, while the coefficients of the terms proportional to $k^2$ and $|k|^{1+2\alpha}$ are still given by the expressions above (B.25). This yields again as above (B.25)

$$\hat{F}_\alpha(k) = 1 - \frac{b_\alpha}{2} k^2 + c_\alpha k^{1+2\alpha} + O(k^5). \quad \text{(B.26)}$$

**Appendix C. Explicit expression of $F_\alpha(Y)$ for $\alpha = 1$**

In this appendix, we give an explicit expression of $F_1(Y)$ for $\alpha = 1$. The starting point of our analysis is the expression (45) given in the text:

$$F_1(Y) = \frac{1}{\pi} \frac{2}{1+4Y^2} + \frac{1}{\pi} \left( \frac{2(1-8Y^2)}{(1+4Y^2)(1+16Y^2)} + \frac{4}{(1+16Y^2)^{3/2}} \right. \left. \times \Re[(1-4iY)^{3/2} \arctan((1+4iY)^{-1/2})] \right). \quad \text{(C.1)}$$

For $z$ a complex number, the following elementary relations are useful:

$$\arctan z = \frac{1}{2i} \left( \log (1+iz) - \log (1-iz) \right), \quad \text{(C.2)}$$

$$\log (x+iy) = \log (\sqrt{x^2+y^2}) + 2i \arctan \left( \frac{y}{x+\sqrt{x^2+y^2}} \right). \quad \text{(C.3)}$$

On the other hand has

$$\frac{1}{\sqrt{1+4iY}} = a + ib, \quad a = \frac{1}{(1+16Y^2)^{1/4}} \cos \left( \frac{\theta}{2} \right), \quad b = \frac{-1}{(1+16Y^2)^{1/4}} \sin \left( \frac{\theta}{2} \right), \quad \theta = \arctan(4Y). \quad \text{(C.4)}$$

Defining $\lambda$ and $\mu$ as

$$\lambda = \arctan \left[ \frac{a^2 + b^2 - 1 + \sqrt{a^2 + (1-b)^2} \sqrt{a^2 + (1+b)^2}}{2a} \right],$$

$$\mu = -\frac{1}{4} \log \left[ \frac{(1-b)^2 + a^2}{(1+b)^2 + a^2} \right]. \quad \text{(C.5)}$$
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in terms of \(a, b\) defined above \((C.4)\), one obtains finally (after straightforward algebra)

\[
F_1(Y) = \frac{1}{\pi} \frac{2}{1 + 4Y^2} + \frac{1}{\pi} \left( \frac{2(1 - 8Y^2)}{(1 + 4Y^2)(1 + 16Y^2)} \right) + \frac{4}{(1 + 16Y^2)^{3/4}} \left( \lambda \cos \left( \frac{3\theta}{2} \right) + \mu \sin \left( \frac{3\theta}{2} \right) \right). \tag{C.6}
\]

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