Transport signatures of a quantum spin Hall - chiral topological superconductor junction

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We investigate transport through a normal-superconductor (NS) junction made from a quantum spin Hall (QSH) system and a chiral topological superconductor (TSC) using a two-dimensional extended four-band model for HgTe-based quantum wells in a magnetic (Zeeman) field and subject to s-wave superconductivity. We show using the Bogoliubov-de Gennes scattering formalism that this structure provides a striking transport signal of a chiral Majorana edge mode. The helical edge states of the QSH side act as modes of spatially separated Kramers pairs that are weakly coupled to the chiral Majorana edge mode on the superconducting side. Due to the finite width of the ribbon geometry, a zero-energy Majorana bound state appears at the NS-interface which we identify via a sharp $2e^2/h$ conductance resonance, that turns over into a quantized $4e^2/h$ conductance signal above the resonance. These signatures are a manifestation of the topological nature of the QSH effect and the TSC.

The entrance of topology in characterizing the features of materials is a rather recent event [1–12]. After the discovery of the quantum Hall effect (QHE) [1] and its theoretical description in terms of a topological Chern number relating a bulk property to the existence of chiral edge channels [3, 4], complementary effects in two-dimensional materials were predicted and discovered, like the quantum spin Hall (QSH) effect [5, 10], possessing helical edge states, the quantum anomalous Hall effect (QAH) [13], exhibiting chiral edge states, and topological superconductors (TSC) with chiral Majorana edge modes [11, 12].

The latter system recently got considerable attention theoretically [14–17] and experimentally [18] in a QAH-TSC-QAH hybrid system showing evidence of a distinct $e^2/2h$ conductance step. This signature was propagated as an indication of a chiral Majorana edge channel at the boundary of the TSC region. However, subsequent theoretical works [19, 20] put forward alternative explanations not related to the existence of a chiral Majorana edge mode. Furthermore, the QAH-TSC system was proposed as a platform for non-Abelian braiding [21, 22]. It is therefore of utmost importance to find additional means to probe chiral Majorana edge channels. In the seminal work by Law et al. [23], the signature of a chiral Majorana edge mode was proposed—in a closed system with finite mode quantization—via $2e^2/h$ tunneling resonances reflecting the Majorana nature of the chiral edge mode. Similarly, Majorana bound states (MBS) at the ends of proximitized topological semiconducting quantum wires [24–29] and in chains of magnetic adatoms on superconductors with spin-orbit coupling [30, 31] have been probed by tunneling experiments.

We propose a new system presented by an NS-junction composed of a QSH insulator (N-side) with helical edge channels and a TSC (S-side) with a chiral Majorana edge channel in the same material system. In a ribbon geometry using an extended Bernevig-Hughes-Zhang (BHZ) model for an inverted HgTe quantum well (QW) in proximity to an s-wave superconductor and in the presence of a Zeeman field [32, 33], we show that the presence of the chiral TSC is represented by regions of conductance quantization at zero energy being either $4e^2/h$ in a trivial phase of the TSC, reflecting the two spatially separated helical edge channels acting as sources of perfect Andreev reflection, or $2e^2/h$ being the indication of a non-trivial phase for the TSC. The latter phase can be traced back to the existence of a MBS at the NS-interface coupling to a single but spatially separated spinful channel composed of the two helical edge states in the N-region (see Fig. 1). The non-trivial phases are identified by the parity of the number of bands crossing the Fermi energy in the S-region, reminiscent of a multichannel topological quantum wire [33, 34]. We stress that the existence of these two distinguishing quantized conductance values is independent of the geometrical details of the setup like the sample width and other imperfections and are therefore a decisive signature of the presence of a chiral Majorana edge channel. We contrast our calculations with the HgTe QW in the non-inverted (without helical edge states) regime where in general multiple channels exist in the N-region and the above mentioned quantization in the inverted case becomes blurred by non-generic and non-quantized conductance values.

Model.— We model the NS junction in a HgTe-based QW by the Bogoliubov-de Gennes (BdG) formalism based on the BHZ model [9] including the effects of Rashba and Dresselhaus spin-orbit interaction [35, 37].
the Dresselhaus spin-orbit term becomes $H_{\text{D}}$.

We decompose the reduced s-wave pairing potential and energy. We decompose the reduced s-wave pairing potential and energy. We decompose the reduced s-wave pairing potential and energy. We decompose the reduced s-wave pairing potential and energy. We decompose the reduced s-wave pairing potential and energy.

edge states appear within the bulk gap of the QSH insulator. We note that the sample is tunable by an electric field $V$ and transport proceeds via the helical edge states (see Fig. 1b)). On the S-side, we tune the Fermi level via gate voltage) and Zeeman energy $\Delta E_{\text{Zeeman}}$ (see also the SM for further discussions).

Additionally, we consider a Zeeman field $\mathcal{H}_{\text{Zeeman}}$ and incorporate superconductivity by the proximity effect with an s-wave bulk superconductor. The BdG Hamiltonian for the NS-hybrid structure is then written as $H_{\text{NS}} = \int d^2 r \Psi^\dagger(r) \mathcal{H}_s \Psi(r)/2$ with

$$H = \begin{pmatrix} H_e - E_F & \Delta \\ \Delta^* & E_F - H_h \end{pmatrix}$$

where $H_h = TH_0T^{-1}$ is the Hamiltonian for holes with $T = i\gamma_0\sigma_0C$ the time-reversal operator. $\Delta$ is the induced s-wave pairing potential and $E_F$ is the Fermi energy. We decompose $H_e = H_0 + H_R + H_D + H_Z$ in the basis $(|E+,|H+),|E-),|H-)$ $| E, H \rangle$ denotes the electron (heavy hole) subband (SB) and $+$ ($-$) stands for spin up (down). The bare BHZ Hamiltonian reads $H_0 = A(k_x s_x + k_y s_y) + \xi(k) + M(k)\sigma_z$ with $\xi(k) = C - Dk^2$ and $M(k) = M - Bk^2$, the Rashba spin-orbit term is $H_R = \alpha(k_y s_x - k_x s_y)(\sigma_0 + \sigma_x)/2$, and the Dresselhaus spin-orbit term becomes $H_D = \delta_0 s_y \sigma_y + \delta_0 s_z \sigma_z$. The Rashba spin-orbit coupling strength $\alpha$ is tunable by an electric field $V$. BIA parameters $\delta_0$, $\delta_h$, $\delta_0$ are material specific $[31, 32]$ and $A, B, C, D, M$ are band structure parameters $[32]$, where the sign of $M$ distinguishes the inverted ($M < 0$) regime with helical edge states from the non-inverted (trivial) regime. We also consider the effect of a Zeeman field perpendicular to the plane of the QW $[33]$ (see Fig. 1) described by $\mathcal{H}_Z = s_z(B_x + B_y - s_z \sigma_z)$ with $B_{\pm} = (\Delta_E \pm \Delta_H)/2$ where $\Delta_E$ and $\Delta_H$ are the Zeeman energies of the $E$ and $H$ bands, respectively. The Pauli matrices $\sigma_i$ and $\sigma_0$ act on spin ($\pm$) and orbital ($E, H$) degrees of freedom, respectively, and $\sigma_0$ denotes the $2 \times 2$ identity matrix and $k = -i\hbar \nabla$.

**Transport properties of a QSH-chiral TSC junction.**—

We consider transport in $x$-direction of an NS structure and assume hard-wall boundary conditions in $y$-direction (see Fig. 1). The normal (N) region ($x < 0$) has $\Delta = 0$ and $C = C_N$ whereas the superconducting (S) region ($x > 0$) has $\Delta = \Delta_0 e^{i\phi}$ $[33]$ and $C = C_S$. We use a generalized wave-matching method $[36, 44]$ in order to solve the Andreev scattering problem for an incoming normal electron with a given excitation energy $\varepsilon$. The corresponding scattering states $\Phi(r)$ solve the BdG equation $\mathcal{H}_s \Phi(r) = \varepsilon \Phi(r)$. More details on the approach are given in the supplementary material (SM).

On the N-side of the junction, $C_N$ is chosen such that the Fermi level lies in the bulk gap of the inverted QW and transport proceeds via the helical edge states (see Fig. 1a)). On the S-side, we tune the Fermi level via $C_S$ to lie in the vicinity of the valence band maximum ($C_S \approx -M$) where most of the weight is on the $E$-SB for low energies $[32]$. Since the Zeeman splitting in the $E$-SB is much larger than the one in the $H$-SB $[35]$ the Zeeman effect is maximized. On the contrary, the helical edge states are mainly localized on the $H$-SB $[35, 36]$, so there the Zeeman effect is negligible. For a transparent presentation of our results, we set $\Delta_H = 0$ in the main text in the following (see also the SM for further discussions).

The subgap conductance at zero temperature can be expressed via the Andreev reflection matrix $r_{he}$ only

$$G = \frac{2e^2}{h} \text{Tr}[r_{he} r_{he}^\dagger]$$

evaluated at a given excitation energy $\varepsilon = eV$ with $V$ the bias voltage applied to the normal contact and $e$ the elementary charge. In Fig. 2(e), we present the zero voltage conductance as a function of Fermi energy (or gate voltage) and Zeeman energy $\Delta_E$.

For vanishing or small Zeeman splittings ($\Delta_E < \Delta_0$) we observe a constant value of $G = 4e^2/h$. This is consistent with previous studies $[34]$ at zero magnetic field and in the absence of spin axial symmetry breaking terms which is a consequence of the spin helicity of the edge states in the QSH insulator. We note that the sample

**FIG. 1:** a) Scheme of an inverted HgTe QW based NS-junction with width $W$. Superconductivity is induced in the S-region via the proximity effect with a bulk s-wave superconductor ($x > 0$). The helical edge channels (blue) are present on the N-side with the Fermi level in the bulk gap. A bias voltage $V$ is applied to the N-contact. In the presence of a magnetic field $B$, the S-side becomes a TSC with a chiral Majorana edge channel (red). b) Dispersion relation of the BdG-spectrum for electrons (black) and holes (red) for the N-side. Only helical edge states appear within the bulk gap of the QSH insulator. The arrows denote the propagation- and spin-directions of electrons, respectively. c) For $|\Delta| > 0$, the QW is turned into a TSC with chiral Majorana edge modes (propagating along the red arrows). The miniband around $k \neq 0$ is due to the mode quantization in the ribbon geometry and can be opened and closed by tuning the magnetic field changing the topological character of the ribbon geometry. In the topologically non-trivial phases a single MBS at $\varepsilon = 0$ appears at the NS-interface, whereas the helical edge channels become gapped out by $\Delta$. We choose $\Delta_E = 1.5$ meV, $\Delta_H = 0$, $\alpha = 0$, $C_N = 0$, $C_S = 9.7$ meV, $M = -10$ meV, $E_F = 0$, $W = 250$ nm.
width in our case is finite ($W = 250$ nm ($W = 1000$ nm) in Fig. 2 a-d) (e) and f)) leading to a small overlap of the helical edge states near the Dirac point. At finite $E_F$ ($C_N = 0$), this hybridization, however, is effectively suppressed leading to two separate channels with perfect Andreev reflection.

If $\Delta_E > \Delta_0$, the zero bias conductance switches between $4e^2/h$ and $2e^2/h$ (see Fig. 2 c,e)), depending on the number $N$ of bulk subbands crossing the Fermi level on the S-side in the absence of $\Delta$ (see Figs. 2 a, f)). These changes of spectral as well as transport features are associated with the closing and reopening of mini gaps of the S-dispersion with finite $\Delta$ (see Figs. 2 b) and 1 c)).

*Chiral Majorana edge modes.*— We now explain why the switching from a $4e^2/h$ to a $2e^2/h$ conductance plateau at $V = 0$ is a direct qualitative and quantitative transport signature of a TSC induced by the presence of Dresselhaus and/or Rashba spin orbit coupling in the HgTe QWs when $\Delta_E > \Delta_0$ 32 33. The hallmark of 2D TSCs is the appearance of chiral Majorana edge modes. In the ribbon geometry, the two counterpropagating chiral Majorana edge channels at the opposite edges develop a minigap around $\varepsilon = 0$ due to the finite width of the ribbon (see Fig. 1 b)). The topological character of this minigap depends on the number $N$ of bulk subbands present at the Fermi level (see Figs. 2 a) and the SM for examples of dispersion relations). Note that the tails of the helical edge states at higher $\varepsilon$-values, are gapped by the superconductor. If $N$ is odd the S-ribbon is topologically non-trivial with an associated MBS at exactly zero energy ($\varepsilon = 0$) at the boundary to the normal side of the NS-junction at $x = 0$ 33. The normal (N)-lead couples to the MBS such that the eigenvalues of $r_{le}^\alpha r_{ce}^\alpha$ become non-degenerate and equal to 1 and 0. A non-degenerate unit Andreev reflection eigenvalue results here in a quantized conductance of $2e^2/h$, which is a signature of the TSC.

Related behavior has also been found in spin-orbit coupled nanowires with proximity-induced s-wave superconductivity where the degeneracy of Andreev reflection eigenvalues has been shown to play a major role in the determination of the topological character of the hybrid system 38. A topological quantum number

$$Q = (-1)^M = \text{sgn(Det } r\text{)}$$

(3)
can be used to determine the number of the topologically protected (quasi)bound states $M$ at the end of the TSC in the presence of particle-hole symmetry and in the absence of time-reversal and spin-rotation symmetry 49 50. $Q$ is determined by the reflection matrix $r$ which has the property $r = (-1)^{d_u}r^\dagger$, where $d_u$ is the degeneracy of the unit Andreev reflection eigenvalue 51. In the limit of large $W$ (Fig. 2 d) and f)), conductance plateaux with $G = 2e^2/h$ ($Q = -1$) become dense and fall into the region of a non-trivial Chern number ($C = -1$ 32 corresponding to the region to the right of the full line in Fig. 2 e)) of the 2D TSC. Outside this region, the conductance is $G = 4e^2/h$ ($Q = +1$), independent on wether the parity of occupied bulk subbands at the Fermi energy is odd or even (Fig. 2 f)).

Information on the MBS is also contained in the scattering states of our NS-system. We plot the probability density of these scattering states as obtained in the SM in Fig. 5. Contrary, in the trivial case $Q = 1$, these bound states are absent and only the incoming electron-like and the reflected hole-like helical edge states are visible (see Fig. 3)). We also depict the presence of these bound states in Fig. 3 b) consistent with the corresponding conductance values in Fig. 2 f) and spectral properties in Figs. 2 a), b).

*NN S-junction.*— The helical edge states do only couple weakly to the MBS. This is expressed via a sharp
resonance as a function of $\varepsilon$ which we display in Fig. 4 (full line). To observe the conductance quantization due to such resonances the energy broadening should exceed the temperature. In this sense, the resonance width sets the temperature at which the experiment should be performed. We observe that the overlap between the MBS and the helical edge states can be enhanced by an intermediate N’ layer of length $L$ that has a different $C_N$ parameter (see Fig. 4). The states in this N’ layer couple more efficiently to the MBS which allows to observe the conductance quantization towards higher excitation energies $\varepsilon$. By increasing the Fermi level into the bulk states in the N’ layer, we observe a two-orders-of-magnitude increase in $\varepsilon$ at which the resonance is still seen (dashed line). This should make it feasible to observe the MBS resonances in current state of the art experiments in HgTe QWs.

**Non-inverted HgTe QWs.** — In the non-inverted QWs, the TSC phases are still possible (the sign of $M$ does not influence the topology of the TSC) but the normal lead ceases to have helical edge states. Similar to the case of the QSH insulator (see Fig. 2) the S-phase is related to the number $N$ of the occupied bulk SBs in the absence of the pairing potential. But in contrast to the QSH-S junction, the conductance takes on quantized values $G = (2e^2/h)n$, where $n$ is odd (even) for the topologically non-trivial (trivial) S-phase with $Q = -1$ ($Q = 1$). Moreover, the conductance quantization in the trivial phase is not protected due to imperfect Andreev reflection in the absence of helical edge modes in non-inverted QWs (see SM for more details). Similar behavior has been reported for multichannel semiconducting nanowires in proximity to an s-wave S [15]. This shows that the QSH insulator-TSC junction has rather unique and stable quantized conductance features not seen in other systems which rely on the combination of two topological phases — the QSH insulator and the chiral TSC.

We note in passing that the propagating states of the chiral Majorana edge mode above the minigap (see Fig. 1 c)), can be probed in transport in the non-inverted regime (or in the inverted regime, when the Fermi level is above the gap of the QSH insulator) leading to rather common non-quantized conductance values.

In conclusion, we have shown how the chiral Majorana edge mode of a chiral TSC can be observed in an NS-junction based on a QSH insulator and a TSC made of the same material in contact to an s-wave bulk superconductor and subjected to a magnetic field. Using the extended two-dimensional BHZ model (including axial spin symmetry breaking terms, induced s-wave superconductivity and a Zeeman field) in a ribbon geometry – which takes into account bulk as well as edge states – we show that the signature of a chiral Majorana edge mode in the TSC part expresses itself in quantized $2e^2/h$ conductance plateaux at zero voltage. These resonances can be traced back to Majorana bound states (MBS) appearing at the NS interface in this ribbon geometry. Moreover, a gate voltage can be used to tune the topological phase of the TSC, resulting in clearly separated quantized conductance plateaux of $2e^2/h$ (topologically non-trivial with MBS) and $4e^2/h$ (topologically trivial without MBS) which is the unique signature of this setup. This constitutes a new way to detect a chiral TSC in two dimensions via a transport experiment.

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we expand the wave functions in terms of Fourier modes. Following the procedure from Refs. [R3, R6], we consider transport in the N-region (N) and S-sides of the junction and discuss in more detail the behavior of the N-TSC junction for the non-inverted HgTe quantum wells without helical edge states.

\begin{equation}
\mathcal{H} = \begin{pmatrix}
\xi(\tilde{k}) + M(\tilde{k}) & A\tilde{k}_+ & i\alpha\tilde{k}_- & -\delta_0 & \Delta & 0 & 0 & 0 \\
-A\tilde{k}_- & \xi(\tilde{k}) - M(\tilde{k}) & \delta_0 & -\delta_h\tilde{k}_- & 0 & \Delta & 0 & 0 \\
i\alpha\tilde{k}_+ + \delta_e\tilde{k}_- & \delta_0 & \xi(\tilde{k}) + M(\tilde{k}) & -A\tilde{k}_- & 0 & 0 & 0 & \Delta \\
-\delta_0 & \delta_h\tilde{k}_+ & -A\tilde{k}_+ & \xi(\tilde{k}) - M(\tilde{k}) & 0 & 0 & 0 & 0 \\
\Delta^* & 0 & 0 & 0 & \Delta & 0 & 0 & 0 \\
0 & \Delta^* & 0 & 0 & 0 & \Delta & 0 & 0 \\
0 & 0 & \Delta^* & 0 & 0 & 0 & \Delta^* & 0 \\
0 & 0 & 0 & \Delta^* & 0 & 0 & 0 & \Delta^* \\
\end{pmatrix}
\end{equation}

with $\xi(\tilde{k}) = C - D\tilde{k}^2$, $M(\tilde{k}) = M - B\tilde{k}^2$ and $\tilde{k}_\pm = \tilde{k}_x \pm i\tilde{k}_y$. We assume a step-like profile for the pairing potential and the doping parameter in the NS structure, i.e. $\Delta = 0$ and $C = C_N$ in the N-region ($x < 0$) whereas $\Delta = \Delta_0 e^{i\phi}$ and $C = C_S$ in the S-region ($x \geq 0$). The operator hat “^” implies that $\mathbf{k}$ should be replaced by $-i\hbar \nabla_x$ whenever it acts on the spinor $\Phi(r)$.

We consider transport in $x$ direction of the NS structure and choose hard-wall boundary conditions in $y$ direction. Following the procedure from Refs. [R3, R6], we expand the wave functions in terms of Fourier modes $\varphi_n(y) = \sqrt{2/W} \sin(n\pi y/W)$:

\begin{equation}
\Phi_m(x, y) = e^{ik_x^m x} \sum_{n=1}^{N_{\text{max}}} a_n^m \varphi_n(y),
\end{equation}

where $m$ is an index for different values of the longitudinal momentum for a given excitation energy $\varepsilon$, the number of transverse modes $N_{\text{max}}$ is chosen to be large enough to ensure the convergence of the numerical solution, and an eight-component spinor $a_n^m$ and momentum $k_x^m$ are determined by solving the eigenvalue problem:

\begin{equation}
\begin{pmatrix}
1 & 0 & 0 \\
0 & (H_{k_x}^m)^{-1} & 0 \\
0 & H_{k_x}^m + H_{k_y}^m & 1 \\
\end{pmatrix}
\begin{pmatrix}
a_1^m \\
a_2^m \\
a_3^m \\
\end{pmatrix}
= k_x^m \begin{pmatrix}
a_1^m \\
a_2^m \\
a_3^m \\
\end{pmatrix},
\end{equation}

Here, $a_n^m = (a_1^m, a_2^m, ..., a_8^m)^T$, $a_n^m = (a_1^m, a_2^m, ..., a_8^m)^T$, $a_n^m = k_x^m a_n^m$, and the $8 \times 8$ sub-matrices in Eq. (S3) have the
Here, to take into account different current densities for the transmission amplitudes should be renormalized in order to consider the influence of the Zeeman splitting on the energy effect due to an s-wave superconductor. First, we consider the NS junctions for the case of the inverted and non-inverted HgTe QWs.

In this section, we consider some aspects of the band structure of the N and S-sides of the QSHI-S junction as well as transport properties of the NS junctions for the case of the inverted and non-inverted HgTe QWs.

\[ H_{n1,n2}^{kz} = \delta_{n1,n2}[(D + B_{sz})\tau_z], \]

\[ H_{const}^{n1,n2} = \delta_{n1,n2}[(C + M_{sz} - \epsilon_F + \delta_0 s_y s_y)\tau_z - \epsilon] \]

\[ + (B_+ + B_- s_z) s_z + (\Delta_+ \tau_x + i\Delta_- \tau_y)], \]

\[ H_{n1,n2}^{kx} = \delta_{n1,n2}[A s_x \tau_x + (-\alpha s_y + \delta s_z)(\sigma_0 + \sigma_z)/2 + \delta_0 s_y (\sigma_0 - \sigma_z)/2] \tau_z], \]

\[ H_{n1,n2}^{ky} = \{-(D + B_{sz})\tau_z\}P_{n1,n2} \]

\[ + [(A s_y + (\alpha s_x - \delta s_z))(\sigma_0 + \sigma_z)/2 + \delta_0 s_y (\sigma_0 - \sigma_z)/2] \tau_z G_{n1,n2}. \] (S4)

Here, \( P_{n1,n2} = \left(\frac{n_1 n_2}{W}\right)^2 \delta_{n1,n2}, G_{n1,n2} = \langle \varphi_{n1}(y) - i\partial_y \varphi_{n2}(y) \rangle \), \( B_{sz} = (\Delta E_{\pm} 3\Delta H)/2; \Delta_{\pm} = (\Delta \pm \Delta^*)/2 \) and the Pauli matrices \( s_i, \sigma_i \) and \( \tau_i \) act on spin (\( \pm \)), orbital \((E, H)\) and electron-hole degrees of freedom, respectively and \( \sigma_0 \) denotes the 2 \( \times \) 2 identity matrix.

The wave function in the N-layer \((x < 0)\) can be taken in the form:

\[ \Phi^N(x, y) = \Phi_{N_{Re}}(x, y) + \sum_{N_{Le}} r_{N_{Le},N_{Re}} \Phi_{N_{Le}}(x, y) \]

\[ + \sum_{N_{Lh}} r_{N_{Lh},N_{Re}} \Phi_{N_{Lh}}(x, y) \]

\[ + \sum_{N_{Ev}} r_{N_{Ev},N_{Re}} \Phi_{N_{Ev}}(x, y), \] (S5)

and includes propagating states, i.e. incoming electrons (with index \( N_{Re} \) moving to the right (R) along the positive x-axis, see Fig. 1 in the main text) and reflected electrons (with indices \( N_{Le} \) moving to the left (L)) and holes (with indices \( N_{Lh} \) moving to the left (L)), respectively, as well as evanescent solutions decaying to the left (with indices \( N_{Ev} \)). Note that, in general, there are several reflected and evanescent modes for a given incoming mode \( N \).

The wave function in the S-region \((x \geq 0)\) takes the form:

\[ \Phi^S(x, y) = \sum_{S_{Re}} t_{S_{Re},N_{Re}} \Phi_{S_{Re}}(x, y) \]

\[ + \sum_{S_{Ev}} t_{S_{Ev},N_{Re}} \Phi_{S_{Ev}}(x, y), \] (S6)

with the evanescent solutions exponentially decaying for \( x \to \infty \) (with index \( S_{Ev} \)) and transmitted propagating states (with index \( S_{Re} \)). Like in Refs. \[R3, R6\], we determine the reflection amplitudes of the electron \( (r_{N_{Le},N_{Re}}) \) and hole \( (r_{N_{Lh},N_{Re}}) \) states in the left lead and transmission amplitudes \( (t_{S_{Re},N_{Re}}) \) of the states in the right lead by matching the wave functions \( \Phi(x, y) \) and the currents \[\partial_x \mathcal{H}\] \( \Phi(x, y) \) at the NS-interface \( x = 0 \). Reflection and transmission amplitudes should be renormalized in order to take into account different current densities for the incident, reflected and transmitted states:

\[ r_{N_{Le},N_{Re}} = r_{N_{Le},N_{Re}} \sqrt{\frac{v_{N_{Re}}}{v_{N_{Re}}}}, \]

\[ r_{N_{Lh},N_{Re}} = r_{N_{Lh},N_{Re}} \sqrt{\frac{v_{N_{Re}}}{v_{N_{Re}}},} \]

\[ t_{S_{Re},N_{Re}} = t_{S_{Re},N_{Re}} \sqrt{\frac{v_{S_{Re}}}{v_{N_{Re}}}}, \] (S7)

where the velocity \( v_m \) is given by

\[ v_m = \frac{\hbar}{m} \int_0^W dy \Phi_m^\dagger(x, y) \partial_x \mathcal{H} |x_s \rightarrow k_x \Phi_m(x, y). \] (S8)

With this renormalization, the propagating states all carry the same particle current. Here, \( r_{N_{Le},N_{Re}}, r_{N_{Lh},N_{Re}}, t_{S_{Re},N_{Re}} \) are the associated reflection probability amplitudes for an incoming electron in mode \( N_{Re} \) into an electron in mode \( N_{Le} \) or a hole in mode \( N_{Lh} \), respectively, and \( t_{S_{Re},N_{Re}} \) is the probability amplitude for the transmission of the incoming electron into mode \( S_{Re} \) in S.

The differential conductance of the NS structure can be calculated using the Blonder-Tinkham-Klapwijk formula

\[ G = \int d\varepsilon (\partial_x f(\varepsilon - eV)) \] (S9)

\[ \times \text{Tr} [1 - r_{ee}^\dagger(\varepsilon) r_{ee}(\varepsilon) + r_{he}^\dagger(\varepsilon) r_{he}(\varepsilon)], \] (S10)

DETAILS OF THE BAND STRUCTURE AND TRANSPORT PROPERTIES

Here, we present additional information concerning the band structure of the N and S-sides of the QSHI-S superconductor junction as well as transport properties of the NS junctions for the case of the inverted and non-inverted HgTe QWs.

QSHI-S junctions

In this section, we consider some aspects of the band structure of inverted HgTe QWs including the proximity effect due to an s-wave superconductor. First, we consider the influence of the Zeeman splitting on the energy dispersion of the edge states. In the absence of the
pairing potential $\Delta$, an inverted HgTe QW has a band gap in the bulk and helical edge states within this gap \[\text{[R11]}\]. To illustrate the main effects of the influence of the Zeeman term and the induced superconductivity on the helical edge states, the spin-axial breaking Rashba and Dresselhaus terms can be neglected as their influence is rather weak \[\text{[R8 R9]}\] in HgTe QWs for not too strong Rashba coefficients $\alpha$. Therefore, we can use the following equations for the spin-up ($\uparrow$) and the spin-down ($\downarrow$) edge states in a large $W$ limit in the bandgap region \[\text{[R10]}\]:

$$
E_{e\pm}^{\uparrow} = C - \frac{MD}{B} \pm A\sqrt{\frac{B^2 - D^2}{B^2}} k_x + \Delta Z,
$$

$$
E_{e\pm}^{\downarrow} = C - \frac{MD}{B} \pm A\sqrt{\frac{B^2 - D^2}{B^2}} k_x - \Delta Z,
$$

where $\Delta Z$ is the Zeeman term and we assume that $\Delta Z = \Delta_E = \Delta_H$. It should be noted that we choose the position of the Fermi energy in N- an S-layers away from the Dirac point, thus we do not consider here the questions concerning the opening of the gap in the helical edge-state spectrum by finite size effects \[\text{[R10]}\]. In the proximity to an s-wave superconductor electron states and their time-reversed partners (i.e. holes with opposite spin orientations) are coupled by the pairing potential $\Delta$. A scheme of the electron and hole edge-state dispersion without their coupling is shown in Fig. S1. One can see that the electron and hole bands for the states with the same spin direction, which are not coupled by the pairing potential, cross at $E = E_F$ (crossing points are marked by the open circles in the figure). In contrast, around the energy values $E = E_F \pm \Delta Z$ (black circles in the figure), where the electron and hole states with the opposite spin direction are coupled by the the pairing potential, a gap of $2\Delta_0$ will be opened. A non-trivial superconducting state supporting Majorana zero-modes requires $\Delta_0 < \Delta_Z$ \[\text{[R11]}\]. Under this condition the helical edge states will not be gapped at the Fermi energy, which would render the probing of Majorana zero modes more difficult. As an example, Fig. S2d shows the numerically calculated energy dispersion in a QW without [Fig. S2a)] and with [Fig. S2b)] proximity to an s-wave superconductor for $\Delta_0 = 0.5$ meV < $\Delta_E = \Delta_H = 1.5$ meV. Taking into account that in HgTe QWs the $g$ factor of the $H$ subband is negligibly small in comparison with that of the $E$-subband \[\text{[R5]}\] we can set $\Delta_H = 0$ in our calculations. Moreover, the edge states are composed of mainly the $H$-component which leads to the small Zeeman splitting of the edge states even in the regime of non-trivial superconductivity with $\Delta_0 < \Delta_E$ (see Fig. S2 for the case without and Fig. S2d) with proximity to an s-wave superconductor).

As mentioned in the main text of the paper, if $\Delta_0 < \Delta_E$, the zero bias conductance of NS junctions switches between $4e^2/h$ and $2e^2/h$, depending on the number $N$ of bulk subbands crossing the Fermi level in the S-region in the absence of the pairing potential $\Delta$ (see Fig. 2 in the main text). Fig S3 shows the energy dispersion in the QW without $\Delta$ corresponding to the black points in Fig. 2a) for different values of $N$. Here, for a fixed value of the Zeeman term $\Delta_E$, $N$ increases with increasing negative value of the Fermi energy. It should be noted that the topological phase in the S depends also on the width of the QW $W$ as well as the structure inversion asymmetry, see Fig. S4a) and b) where the conductance in a QSHI-TSC junction switches between $4e^2/h$ and $2e^2/h$ depending on the values of $\Delta_E$, $W$ and the Rashba term $\alpha$.

**FIG. S1:** Schemes of the electron (solid lines) and hole (dashed lines) edge-state dispersion (without coupling due to the pairing potential) in an inverted HgTe QW. Spin-up (spin-down) states are shown by red (blue) lines.

**NS junctions based on non-inverted QWs**

Here, we will consider NS junctions based on non-inverted QWs (since non-trivial TSC phases are still possible in these QWs) and compare these structures with the QSHI-S junctions. In analogy with the inverted QW regime the conductance of the NS structure depends on the number $N$ of bulk subbands crossing the Fermi level in the S-region in the absence of the pairing potential (see Fig. S5). In contrast to the QSHI-S junction, the conductance takes quantized values $G = (2e^2/h)n$, where $n$ is not restricted to be 1 or 2 but takes odd (even) values for the topologically non-trivial (trivial) S phase with $Q = -1$ ($Q = 1$). As it can bee seen from Fig. S5, $n = 0, 1, 2, 3$ for the white, blue, red and yellow regions in the plot. However, conductance quantization in the non-
FIG. S2: Energy dispersion in a HgTe QW without [plots a) and c)] and with [plots b) and d)] proximity to an s-wave superconductor. The calculations have been done for $M = -10$ meV, $C_{N,S} = 9.7$ meV, $W = 250$ nm, $E_F = 0$, $\alpha = 0$. Zeeman splitting $\Delta E = \Delta H = 1.5$ meV in a) and b) plots, and $\Delta E = 1.5$ meV, $\Delta H = 0$ in c) and d) plots. In a) and c) black (red) lines correspond to the electron (hole) energy dispersion, and the arrows represent the spin direction of the corresponding states.

FIG. S3: Energy dispersion in a HgTe QW with different number $N$ of occupied bulk subbands. The calculations have been performed for $M = -10$ meV, $C_N = 9.7$ meV, $W = 250$ nm, $\alpha = 0$, $\Delta E = 1.5$ meV, $\Delta H = 0$. The dashed line shows the position of the Fermi level for $\epsilon = 0$.

inverted structures is not protected because of imperfect Andreev reflection of bulk states at the NS interface.
FIG. S4: Differential conductance (in units of $e^2/h$) in the NS structure based on the QSHI in a) and b) and on the non-inverted QW in c) as a function of Zeeman term $\Delta E$ and width $W$ in a) and c) or Rashba spin-orbit term $\alpha$ in b). The calculations are presented for $\varepsilon = 0$ and $E_F = 0$. Other parameters are $M = -10$ meV, $C_N = 0$, $C_S = 9.7$ meV in a) and b) and $M = 10$ meV, $C_N = -11$ meV, $C_S = -9.7$ meV in c), $W = 500$ nm in b) and $\alpha = 0$ in a) and c).

[see Fig. S4c)]. An alternation of the trivial and non-trivial topological superconducting phases has been reported also in spin-orbit-coupled multichannel semiconducting nanowires in proximity to an s-wave superconductor, where protected Majorana modes are predicted to appear at the ends of the wire with an odd number of channels, whereas an even number of the occupied subbands corresponds to the trivial superconducting phase [R12–R15]. But from the experimental point of view it could be difficult to determine the number of occupied subbands in the system, similar to the case of an NS junction based on a non-trivial QW, and, as a consequence, correctly identify the topological character of the superconductor.
FIG. S5: a) Diagram representing the number of the occupied subbands in a non-inverted QW at the Fermi level as a function of the Zeeman term and Fermi energy. b) Conductance (in units of $e^2/h$) in the NS structure based on a non-inverted QW. All plots are presented as a function of the Zeeman term $\Delta E$ and the Fermi energy $E_F$ for the QW with a width $W = 250$ nm, $M = 10$ meV, $\varepsilon = 0$, $\alpha = 0$. The doping parameter is $C_N = -9.7$ meV in a), $C_N = -11$, $C_S = -9.7$ meV in b).

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