Precise determination of the inflationary epoch and constraints for reheating

Gabriel Germán
Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Cuernavaca, Morelos, 62210, Mexico
(Dated: March 12, 2020)

We present a simple formula which allows to calculate the value of the inflaton field, denoted by \( \phi \), at the scale \( k \). From here all inflationary observables follow. We illustrate the procedure with Starobinsky model of inflation. This gives an spectral index \( n_s = 0.96534 \) with running \( \alpha = -6.1 \times 10^{-3} \), tensor-to-scalar ratio \( r = 0.0034 \) within reach of future experiments and an inflationary energy scale of \( 7.8 \times 10^{15} \text{GeV} \). We also discuss the reheating epoch finding a constraint equation for the effective number of degrees of freedom \( g_{re} \). This constraint translates into constraints for the reheating temperature \( T_{re} \) and for the number of e-folds during reheating and also during the radiation era. For the Starobinsky model we find that \( g_{re} \lesssim 70.39 \) giving a bound \( T_{re} \lesssim 500 \text{MeV} \) with \( 40.84 > N_{re} > 36.30 \) and \( 16.69 < N_{rd} < 21.23 \) for the number of e-folds during reheating and during radiation domination, respectively. Power-law (monomials) models are also studied where an exact analytical solution for \( \phi_k \) is given. It is shown that this type of models together with Natural Inflation are inconsistent with Planck bounds.

I. INTRODUCTION

There have been several recent attempts to establish constraints on inflationary models \([1, 2, 3, 4]\), particularly during the epochs of inflation and reheating, with varying degrees of success \([5] - [18]\). For reviews on reheating see e.g., \([19, 20, 21]\). Here, we present a simple approach where the inflationary epoch is precisely determined by constructing an equation for \( \phi_k \), the value of the inflation field at the scale \( k \) which we set equal to the pivot scale \( k_* \). We use \( k_* = 0.05 M_{\text{Pl}} \), where most parameter values are reported in particular by the Planck collaboration \([22, 23]\). Once we have determined \( \phi_k \), all inflationary quantities of interest follow. The outline of the paper is as follows: in Section II we provide a brief discussion on the obtention of the \( \phi_k \)-equation an establish it in Eq. (1). In Section III we study the reheating epoch where we get a constraint equation for the number of effective degrees of freedom during reheating. Section IV contains a study of the Starobinsky, Power-law and Natural Inflation models along the lines described previously. Finally, Section V contains the main conclusions of the paper.

II. THE INFLATIONARY EPOCH

We work in Planck mass units, where \( M_{\text{pl}} = 2.44 \times 10^{18} \text{GeV} \) and set \( M_{\text{pl}} = 1 \), the pivot scale \( k_* = a_* H_* = 0.05 M_{\text{Pl}} \), used in particular by the Planck collaboration, becomes a dimensionless number given by \( k_* \approx 1.31 \times 10^{-58} \). This can be compared with \( k_0 = a_0 H_0 \approx 8.74 \times 10^{-61} \). To find the value of \( a_* \) and from there the number of e-folds \( N_* = \ln \frac{a_*}{d_*} \), from \( k_* \) up to \( k_0 \) we solve the Friedmann equation for \( a_* \)

\[
k_* = H_0 \sqrt{\frac{\Omega_{md,0}}{a_*}} + \frac{\Omega_{rd,0}}{a_*^2} + \Omega_{de} a_*^2 ,
\]

where \( \Omega_{md,0} = 0.315 \), \( \Omega_{rd,0} = 7.9 \times 10^{-5} \), and \( \Omega_{de} = 0.685 \). To calculate \( a_* \) we have to specify \( h \) for the Hubble parameter \( H_0 \) at the present time. We take the value given by Planck \( h = 0.67 \) for definitiveness and check that no important changes occur in \( N_* \), for \( h \) in the interval \( 0.67 < h < 0.73 \). The solution of Eq. (1) for \( a_* \) is \( a_f = 4.264 \times 10^{-25} \) from where we get \( N_* = 10.05 \) for the number of e-folds from \( a_* \) to \( a_0 \).

Defining \( N_k = \ln \frac{a_k}{a_*} \), \( N_{re} = \ln \frac{a_{eq}}{a_*} \), \( N_{rd} = \ln \frac{a_{rd}}{a_{eq}} \) and \( N_{md} = \ln \frac{a_{md}}{a_{eq}} \), it is easy to show that the number of e-folds from reheating plus the radiation dominated epochs can be written as

\[
N_{re} + N_{rd} = N_k + N_{re} + N_{rd} - N_k = \ln \left[ \frac{a_{eq} H_k}{k} \right] - N_k ,
\]

we see that the r.h.s only depends on \( \phi_k \), the value of the inflaton at \( k \) (and also of parameters of the model, if any). The equation which determines \( \phi_k \) with the assumption that \( k = k_* \), is

\[
N_{re} + N_{rd} + N_{md} - N_* = N_k ,
\]

which can be written as

\[
\ln \left[ \frac{a_* H_k}{k_*} \right] = 2 N_k .
\]

Solving Eq. (3) requires specifying a model of inflation; \( H_k \) and \( N_k \) are model dependent quantities. Thus, after finding \( \phi_k \), we can proceed to determine all inflationary quantities like the scale of inflation, Hubble parameter \( H_k \), tensor-to-scalar ratio \( r \), spectral index \( n_s \), running \( \alpha \), etc. Notice how the determination of \( \phi_k \) requires not only the knowledge of the present universe through quantities like \( \Omega_{i,0} \) and \( H_0 \) but also of the early universe through

\* e-mail: gabriel@icf.unam.mx
the scalar power spectrum amplitude given here by $A_s$ and contained in $H_k = \sqrt{8\pi^2\epsilon_kA_s}$.

An equivalent way of obtaining Eq. (4) is by connecting the epoch where the scale $k$ left the horizon during inflation to the pivot scale $k_*$ where we measure the horizon reentry of precisely the same scale $k$. This can be expressed by

$$\ln\left(\frac{a_*}{a_k}\right) = 2N_k, \quad (5)$$

where $2N_k$ is the number of e-folds from the scale $k$ up to the end of inflation plus the number of e-folds from the end of inflation up to the scale $k_*$. Multiplying the term inside the parenthesis above and below by $H_k$ and setting $a_kH_k \equiv k = k_* \equiv a_*H_k$, we get again Eq. (4).

From Eq. (4) and the upper bound for $r$ also follows a bound for the number of e-folds $N_k$. It is easy to see that the tensor-to-scalar ratio $r$ at $\phi_k$ is given by

$$r = \frac{2k^2_*}{\pi^2a^2_*A_s}e^{4N_k} \approx e^{3(N_k-57.016)}. \quad (6)$$

From the bound $r < 0.063$ follows that $N_k < 56.3$.

III. THE REHEATING EPOCH

Having specified a formula from where $\phi_k$ can, in principle, be obtained and from there all relevant quantities characteristic of the inflationary epoch we then turn to the reheating era. Assuming a constant equation of state parameter $\omega$ during reheating the fluid equation gives $\rho \propto a^{-3(1+\omega)}$ from where the number of e-folds during reheating follows

$$N_{re} \equiv \ln\left(\frac{a_{re}}{a_e}\right) = [3(1+\omega)]^{-1}\ln\left[\frac{\rho_{re}}{\rho_{e}}\right], \quad (7)$$

where $\rho_e$ denotes the energy density at the end of inflation and $\rho_{re}$ the energy density at the end of reheating where $\rho_{re} = \frac{\pi^2g_{*re}}{30}\rho_{e}$, with $g_{*re}$ the number of degrees of freedom of relativistic species at the end of reheating. Assuming entropy conservation after reheating

$$g_{*re}T_{re}^3 = \left(\frac{a_0}{a_{eq}}\right)\left(\frac{a_{eq}}{a_{re}}\right)\left(2T_0^3 + 6 \times \frac{7}{8}T_{\nu,0}^3\right), \quad (8)$$

where $T_0 = 2.725K$ and the neutrino temperature is $T_{\nu,0} = (4/11)^{1/3}T_{0}$. The number of e-folds during radiation domination $N_{rd} \equiv \ln\left(\frac{a_{eq}}{a_{re}}\right)$ follows from Eqs. (7) and (8)

$$N_{rd} = -\frac{3(1+\omega)}{4}N_{re} + \frac{1}{4}\ln\left[\frac{30}{g_{*re}\pi^2}\right] + \frac{1}{4}\ln\left[\frac{\rho_{e}}{T_0^4}\right] + \frac{1}{3}\ln\left[\frac{11g_{*re}}{43}\right] + \ln\left[\frac{a_{eq}}{a_0}\right]. \quad (9)$$

Combining Eqs. (4) and (9) we get an expression for the number of e-folds during reheating $N_{re}$

$$N_{re} = \frac{4}{1-3\omega}\left(N_k - \frac{1}{3}\ln\left[\frac{11g_{*re}}{43}\right] - \frac{1}{4}\ln\left[\frac{30}{\pi^2g_{*re}}\right] - \ln\left[\frac{a_{eq}}{a_0}\right]\right)^{1/4}. \quad (10)$$

It is convenient to rewrite this equation in the form

$$N_{re} = \frac{4}{1-3\omega}\tilde{N}_{re}, \quad (11)$$

where $\tilde{N}_{re}$ is the term in the brackets of Eq. (10) and is independent of $\omega$. A final quantity of physical relevance is the thermalization temperature at the end of the reheating phase

$$T_{re} = \left(\frac{30\rho_e}{\pi^2g_{*re}}\right)^{1/4}e^{-\frac{1}{4}(1+\omega)\tilde{N}_{re}}, \quad (12)$$

This is a function of the number of e-folds during reheating. It can also be written as an equation for the parameter $\omega$, using Eq. (11)

$$\omega = \frac{1}{3} + \frac{4\tilde{N}_{re}}{3\left(-\tilde{N}_{re} + \frac{1}{3}\ln\left[\frac{30g_{*re}}{\pi^2T_{re}^4}\right]\right)}, \quad (13)$$

from here we can rewrite the equations for $N_{rd}$ and $N_{re}$ as functions of $T_{re}$

$$N_{rd} = \ln\left[\frac{T_{re}}{(4\pi)^{1/4}T_0}\right] + \frac{1}{3}\ln\left[\frac{g_{*re}}{10.75}\right] + \ln\left[\frac{a_{eq}}{a_0}\right], \quad (14)$$

and
\begin{equation}
N_{re} = \dot{N}_{re} + \frac{1}{4} \ln \left( \frac{\pi^2 g_{re}}{30 \rho_c} \right) - \ln [T_{re}], \tag{15}
\end{equation}

from where we can see that \(N_{re} + N_{rd}\) is \(T_{re}\) independent (equivalently \( \omega \) independent)

\begin{equation}
N_{re} + N_{rd} = \dot{N}_{re} + \frac{1}{3} \ln \left( \frac{11 g_{s, re}}{43} \right) + \ln \left( \frac{12 \rho_{re}}{\pi^2 g_{re}} \right) + \frac{1}{4} \ln \left( \frac{a_{eq} \rho_{re}^{1/4}}{a_{0} T_{0}} \right). \tag{16}
\end{equation}

Using Eqs. (10) and (11) we check that \(N_{re} + N_{rd} + N_{rd} - N_s = N_k\) and we are back in Eq. (3). Instead of Eq. (10) for the number of e-folds during reheating we can study the dependence of the degrees of freedom \(g_{re}\) on \(N_{re}\) and \(\omega\): for as long as species have the same temperature and \(p \approx \frac{3}{2} \rho\) we have that \(g_{s, re} \approx g_{re}\). Thus, we set \(g_{s, re} = g_{re}\) in Eq. (10) and proceed to solve for \(g_{re}\), the result is

\begin{equation}
g_{re} = g_{re} (\phi_k) e^{-3(1 - 3 \omega)N_{re}}, \tag{17}
\end{equation}

where

\begin{equation}
g_{re} (\phi_k) = \left( \frac{43}{11} \right)^{4/3} \left( \frac{\pi^2}{30} \right)^{\frac{H_k}{e^{N_k} \rho_{re}^{1/4} k_s}} \left( \frac{a_0 T_0}{k_s} \right)^{12}, \tag{18}
\end{equation}

and

\begin{equation}
\rho_{c} = \frac{3}{2} V_{e} = \frac{9}{2} V_{k} H_{k}^2. \tag{19}
\end{equation}

The last result follows considering the end of inflation when \(\omega = -1/3\). We see that \(g_{re} (\phi_k)\) is a number which depends on \(\phi_k\). Numerical studies of the thermalization phase during reheating suggest that \(0 \leq \omega \leq 0.25\) \cite{23}. Here we extended our discussion up to \(\omega \approx 1/3\).

For \(\omega \approx \frac{1}{3}\) the exponential in Eq. (17) is always less than one and \(g_{re} (\phi_k)\) gives the largest possible value of \(g_{re}\). Thus, \(g_{re} \leq g_{re} (\phi_k)\). (20)

We can expect that in certain models \(g_{re} (\phi_k)\) and, as a consequence, \(g_{re}\) is less than the number of species from the Standard Model of Particles, if this is the case then a constraint on the reheating temperature at the end of reheating \(T_{re}\) follows immediately. We will see that this is indeed the case for the Starobinsky model.

\section{IV. The Starobinsky, Power-Law and Natural Inflation Models}

\textbf{The Starobinsky model: inflation.}– The potential of the Starobinsky model \cite{26,27,28} is given by \cite{29}:

\begin{equation}
V = V_0 \left( 1 - e^{-\sqrt{2} \phi} \right)^2. \tag{21}
\end{equation}

From here we calculate the number of e-foldings from the scale \(k\) up to the end of inflation

\begin{equation}
N_k = - \int_{\phi_k}^{\infty} d\phi = \frac{1}{2} \left( 3k_0 \sqrt{V_{re}} - \sqrt{V_{ph}} \right) - \frac{1}{2} \left( 3k_0 \sqrt{V_{re}} - \sqrt{V_{ph}} \right), \tag{22}
\end{equation}

where the end of inflation is given by the solution to the equation \(\epsilon = 1\) at \(\phi_e\):

\(\phi_e = \sqrt{\frac{3}{2}} \ln \left( 1 + \frac{2}{\sqrt{\pi}} \right)\). The Hubble function is

\begin{equation}
H_k = \sqrt{8 \pi^2 \epsilon_k A_s} = \sqrt{\frac{32 A_s}{3}} \frac{\pi}{e^{\sqrt{2} \phi_k} - 1}, \tag{23}
\end{equation}

where \(\epsilon_k\) is the slow-roll parameter \(\epsilon \approx \frac{1}{2} \left( \frac{V'}{V} \right)^2\) at \(k\) and the scalar power spectrum amplitude is \(A_s = 2.0968 \times 10^{-9}\). Taking \(a_s = 4.264 \times 10^{-5}\), \(k_s = 1.31 \times 10^{-58}\) and solving Eq. (4) we get

\(\phi_k = 5.365\). \tag{24}

From here and from the usual expressions for the observables in terms of the slow-roll parameters follow the numbers quoted in the abstract. Also, the number of e-folds for reheating plus radiation is fixed by \(\phi_k\) (see Eq. (3)), the result is

\(N_{re} + N_{rd} \approx 57.5\), \tag{25}

while \(N_k \approx 55.6\).

\textbf{The Starobinsky model: reheating.}– Having determined the value of \(\phi_k\) in Eq. (24) we evaluate \(g_{re} (\phi_k)\) in Eq. (18) with the result

\(g_{re} (\phi_k) = 70.39\). \tag{26}

This is the maximum number of species during reheating for \(w \lesssim 1/3\) and approximately corresponds to a temperature around \(T = 500\ MeV \approx 5.8 \times 10^{12}\ \text{K}\). Thus, the temperature at the end of reheating is bounded as

\(10\ MeV \lesssim T_{re} \lesssim 500\ MeV\), \tag{27}

where the left hand side comes from nucleosynthesis considerations \cite{24} (even a lower \(T_{re}\) for the lower bound has been discussed in the literature \cite{25}).

From here and from Eqs. (14) and (15) we can obtain bounds for the number of e-folds during reheating and during the radiation dominated eras with a fixed number of e-folds for the sum as given by Eq. (25)

\(40.84 > N_{re} > 36.30\), \tag{28}

\(16.69 < N_{rd} < 21.23\). \tag{29}

We can obtain from Eq. (11) an equation for \(\omega\)

\(\omega = \frac{1}{3} \left( 1 - 4 \frac{N_{re}}{N_{re}} \right). \tag{30}
\)

Having determined bounds for \(N_{re}\), Eq. (30) allows to get the corresponding bounds for \(\omega\), as follows

\(0.333264 > \omega_{re} > 0.333255\). \tag{31}
we have proposed an equation which allows to calculate the value of the inflation \( \phi \) at the observable scale \( k \), this is given by Eq. (33). The determination of all inflationary quantities of interest follows from \( \phi_k \) (and from model dependent parameters, if any). We have also found a constraint equation for the number of species during reheating \( g_{re} \) such that \( g_{re}(\phi_k) \). If \( g_{re}(\phi_k) \) and, as a consequence, \( g_{re} \) is less than the expected number of species from the Standard Model of Particles then a constraint on the reheating temperature at the end of reheating \( T_{re} \) follows immediately. We have illustrated our approach using Starobinsky, Power-law and Natural Inflation models of inflation although the strategy presented here can be applied to any model of inflation where \( H_k \) and \( N_k \) can be obtained.

\[ V(\phi) = V_0 \left( 1 + \cos \left( \frac{\phi}{f} \right) \right), \]

We solve Eq. (4) so as to get the upper bound on \( r = 0.0630 \), obtaining \( f = 6.285 \), in this case the spectral index has the value \( n_s = 0.95893 \). As \( f \) decreases from the value \( f = 6.285 \), \( r \) falls from its upper bound but \( n_s \) also decreases from the already low value of \( n_s = 0.95893 \). No reheating constraint arises for Power-law or Natural Inflation models since \( g_{re} \) is no less than the number (106.75) of species of the Standard Model of Particles.

\section{Conclusions}

We would like to thank Juan Carlos Hidalgo and Ariadna Montiel for reading the manuscript and comments. It is also a pleasure to thank N. Sánchez and M. Dirzo for encouraging conversations. We acknowledge financial support from UNAM-PAPIIT, IN104119, Estudios en gravitación y cosmología.
