The second stable homotopy groups of motivic spheres

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Abstract

We compute the 2-line of stable homotopy groups of motivic spheres over fields of characteristic not two in terms of motivic cohomology and hermitian $K$-groups.

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1 Introduction

One of the fundamental problems in motivic homotopy theory over a field $F$ is to calculate the homotopy groups $\pi_{t,w}1$ of the motivic sphere spectrum $1$. Here the integers $t, w \in \mathbb{Z}$ refer to the topological degree and the weight, respectively. In a precise sense, these groups are the universal motivic invariants because the motivic sphere is the unit for the tensor product on motivic spectra. All the relations witnessed in $\pi_{*,*}1$ hold in every other theory representable in the stable motivic homotopy category, such as algebraic cobordism, algebraic and hermitian $K$-theory, motivic cohomology, and higher Witt theory. By work of Morel [Mor06], the group $\pi_{s,t}1 = 0$ if $s < t$ and the so-called 0-line $\bigoplus_{n \in \mathbb{Z}} \pi_{n,n}1$ is the Milnor-Witt $K$-theory $K_{MW}(F)$ of $F$, which incorporates a great deal of arithmetic information about the base field. It turns out that the Grothendieck-Witt group of isomorphism classes of symmetric bilinear forms features very naturally in this calculation; a surprising fact since the construction of motivic homotopy theory does not involve quadratic forms. Over the complex numbers, the Betti realization functor witnesses all the classical stable homotopy groups as the weight zero part of the corresponding groups of the motivic sphere [Lev14]. The key inputs enabling the connection between motivic homotopy groups and quadratic forms are the resolutions of the Bloch-Kato, and Milnor conjectures on Galois cohomology and quadratic forms in [Voe03], [OVV07], and [Voe11].

In [RSØ19] we extended these calculations to the 1-line $\bigoplus_{n \in \mathbb{Z}} \pi_{n+1,n}1$ of the stable motivic homotopy groups, for every field $F$ of characteristic unequal to two. Our approach makes a systematic analysis of the slice spectral sequences for the sphere and hermitian $K$-theory [Voe02a], [RØ16]. To ”turn the pages” in the slice spectral sequence for the sphere, we calculate sufficiently many differentials to deduce that it collapses at its $E^2$-page, at least as far as the 1-line is concerned. A case-by-case analysis achieves this, which combines Voevodsky’s description of the motivic Steenrod algebra [Voe03], [HKØ17], with the observation that the slice differentials induce graded Milnor $K$-theory $K^M(F)$-module maps [Mil70].

This paper deepens our understanding of the stable motivic homotopy groups by means of an explicit calculation of the 2-line $\bigoplus_{n \in \mathbb{Z}} \pi_{n+2,n}1$ over fields of characteristic different from two. To that end, we develop powerful computational techniques taking into account the Milnor-Witt $K$-theory module structure on successive stages of the slice filtration. Owing to a comparison between the motivic sphere and very effective hermitian $K$-theory, we also strengthen the main result in [RSØ19]. Moving up from the 1- to the 2-line increases the computational complexity, which is rooted in the problem of controlling mod-\(n\) motivic cohomology groups as $n$ increases.

Suppose $S$ is a base scheme and $\Lambda$ is a localization of $\mathbb{Z}$ such that $(S, \Lambda)$ is a compatible pair in the sense of [RSØ19] §2.1], e.g., $S$ is smooth over a field or a Dedekind domain of mixed characteristic such that every positive residue characteristic of $S$ is invertible in $\Lambda$. Later we will restrict to the case of a field $F$ of exponential characteristic $c \geq 1$ and $\Lambda = \mathbb{Z}[\frac{1}{c}]$. By Theorem 2.12 in [RSØ19] the slices of the $\Lambda$-local sphere spectrum are given by a finite wedge product of suspensions of motivic Eilenberg-MacLane spectra

\[ s_q(1_\Lambda) \cong \bigvee_{p \geq 0} \Sigma^{2q-p,q}M(\Lambda \otimes \text{Ext}_{\text{MU}_*}^{2q}(\text{MU}_*, \text{MU}_*)) . \quad (1.1) \]
Here, the extension groups are calculated in comodules over the Hopf algebroid for the cobordism spectrum $\text{MU}$; these form the $E^2$-page of the classical Adams-Novikov spectral sequence $[\text{Rav86}]$. Voevodsky’s slice spectral sequence $[\text{Voe02a}]$ takes (1.1) as its input

$$E^1_{t,q,w} = \pi_{t,w} s_q(1_\Lambda) \implies \pi_{t,w} 1_\Lambda.$$  

To formulate our results, we compare the motivic sphere spectrum with hermitian $K$-theory $KQ$, or rather its very effective cover $kq$. Over a perfect field, $kq$ coincides with the effective cover $f_0(KQ_{\geq 0})$ with respect to the slice filtration by $kq$. Here $KQ_{\geq 0}$ is the connective cover of $KQ$ with respect to the homotopy $t$-structure on the stable motivic homotopy category. For a compatible pair $(S, \Lambda)$ as above, the slices of $kq$ have the following form.

$$s_q(kq_\Lambda) \cong \begin{cases} 
\Sigma^{2i,q} M\Lambda \lor \bigvee_{0 \leq i < \frac{q}{2}} \Sigma^{2i+q,q} M\Lambda/2 & 0 \leq q \equiv 0 \text{ mod } 2 \\
\bigvee_{0 \leq i < \frac{q}{2} + 1} \Sigma^{2i+q,q} M\Lambda/2 & 0 \leq q \equiv 1 \text{ mod } 2 \\
* & q < 0 
\end{cases}$$

The canonical map $kq_\Lambda \to KQ_\Lambda$ induces an inclusion on all slices. The infinite wedge sum in the identification of $s_q(KQ_\Lambda)$ in $[\text{RO16} \text{ Theorem 4.18}], [\text{RSO19} \text{ Theorem 2.15}]$ explains to some extent why we prefer to use $kq_\Lambda$ in our comparison with $1_\Lambda$.

A refinement of $[\text{RSO19} \text{ Theorem 5.6}]$, see Section 4 yields the following calculation.

**Theorem 1.1:** Let $F$ be a field of characteristic zero. For every integer $n$, the unit map $1 \to kq$ induces a short exact sequence

$$0 \to K^M_{2-n}(F)/24 \to \pi_{n+1,n} 1 \to \pi_{n+1,n} kq \to 0.$$  

In particular, this proves Morel’s $\pi_1$-conjecture, i.e., there is a short exact sequence

$$0 \to K^M_2(F)/24 \to \pi_{1,0} 1 \to F^x/2 \oplus \mathbb{Z}/2 \to 0.$$  

(1.4)

With reference to the $K^{MW}(F)$-module structure, the second motivic Hopf map $\nu \in \pi_{3,2} 1$ generates $K^M_2(F)/24$ and the topological Hopf map $\eta_{\text{top}} \in \pi_{1,0} 1$ generates $F^x/2 \oplus \mathbb{Z}/2$. These generators are subject to the relations $2\eta_{\text{top}} = 0$ and $\eta^2_{\text{top}} = 6(1-\epsilon)\nu = 12\nu$, implying $24\nu = 0$ (as predicted by a motivic analogue of the Adams conjecture in this degree). Here $\eta \in \pi_{1,1} 1$ and $h = 1 - \epsilon \in \pi_{0,0} 1$ are the first and zeroth motivic Hopf maps. By specializing to the field of real numbers $\mathbb{R}$, we can distinguish between the elements $\rho^2\eta_{\text{top}}$ and $\rho^4\nu$ in the integral group $\pi_{-1,-2} 1_{\mathbb{R}}$, which cannot be witnessed after 2-completion according to $[\text{DP17}]$. On the other hand we have $\rho^3\eta_{\text{top}} = \rho^5\nu$ in $\pi_{-2,-3} 1_{\mathbb{R}}$.

Theorem [1.1] holds more generally over any field $F$ of exponential characteristic $c \neq 2$. For $\Lambda = \mathbb{Z}[\frac{1}{c}]$ there is an exact sequence

$$0 \to K^M_{2-n}(F) \otimes \Lambda/24 \to \pi_{n+1,n} 1_\Lambda \to \pi_{n+1,n} kq_\Lambda \to 0.$$  

(1.5)
It is unclear whether the restriction on $c$ is necessary. To begin with, an identification of the motivic Steenrod algebra at the characteristic, extending [Voe03] and [HKØ17], may be required for answering this question.

In this paper we continue the work in [RSØ19] by calculating the 2-line.

**Theorem 1.2:** Let $F$ be a field of characteristic zero. For every integer $n$, the unit map $1 \to kq$ induces a short exact sequence

$$0 \to H^{1-n,2-n}(F)/24 \oplus K^{M}_{4-n}(F)/2 \to \pi_{n+2,n}1 \to \pi_{n+2,n}kq.$$ 

In particular, $\pi_{n+2,n}1 = 0$ for $n \geq 5$ and $\pi_{6,4}1 \cong \mathbb{Z}/2$.

Here $H^{1-n,2-n}(F)$ refers to integral motivic cohomology. Its mod-24 reduction maps canonically to $\pi_{n+2,n}1$, which contains a quotient of $h_{24}^{1-n,2-n}$. Theorem 1.2 holds more generally for fields with $\Lambda = \mathbb{Z}[\frac{1}{2}]$-coefficients, $c \neq 2$, as in (1.5). The $K^{MW}(F)$-module $K^{M}_{4-n}/2$ is generated by the motivic Hopf map $\nu^2$. On the 2-line, we find the relation

$$\rho^2 \nu^2 = \eta_{\text{top}}\nu \in \pi_{4,2}1.$$ 

For $n = 2$, the rightmost map in Theorem 1.2 is not surjective since $\pi_{4,2}1$ is torsion and $\pi_{4,2}kq$ is isomorphic to the zeroth symplectic Grothendieck group $K^{Sp}_{0}(F) \cong 2\mathbb{Z}$ of even integers. For $n = 1$, there is a split short exact sequence

$$0 \to K^{M}_{3}(F)/2 \to \pi_{3,1}1 \to \mu_{24}(F) \to 0.$$ 

Here $\pi_{3,1}1 \to \pi_{3,1}kq = H^{1,1}$ is given by the inclusion of the 24-th roots of unity into $F^\times$. Since $\pi_{0,-1}1_{C} = 0$ and $\pi_{0,1}1_{C} = \pi_{2,1}1_{C} = \mathbb{Z}/2$, the product map from the 1-line to the 2-line is not surjective over the complex numbers, contrary to the topological situation.

We carry out our calculations of stable motivic stems on $F$-points, but the results hold on the level of motivic homotopy sheaves since these are strictly $A^1$-invariant and hence unramified sheaves on $\text{Sm}_F$, see [Mor05, Theorem 6.2.7], [Mor12, Theorem 2.11].

Throughout the paper, we employ the following notation.
$F$, $S$

$\text{Sm}_S$

$S^{s,t} = S^{s-t+(t)}$

$\Sigma^{s,t} = \Sigma^{s-t+(t)}$

$\text{SH}(S)$

$E$, $1 = S^{0,0}$

$R$, $MR$

$\text{MGL, MU, BP}$

field, finite dimensional separated Noetherian scheme

smooth schemes of finite type over $S$

motivic $(s, t) = s - t + (t)$-sphere

suspension with $S^{s,t} = S^{s-t+(t)}$

motivic stable homotopy category over $S$

generic motivic spectrum, the motivic sphere spectrum

ring, motivic Eilenberg-MacLane spectrum of $R$

algebraic and complex cobordism,

Brown-Peterson spectrum

algebraic and hermitian $K$-theory, Witt-theory

gth effective cover and slice functors

Milnor-Witt $K$-theory, Milnor $K$-theory (modulo 2)

integral, mod-2, mod-$n$ motivic cohomology groups

connecting homomorphism, $a, b \in \mathbb{N} \cup \{\infty\}$, $h_\infty = H$

inclusion, projection homomorphism,

$a, b \in \mathbb{N} \cup \{\infty\}$, $h_\infty = H$.

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that the suspension functor $\Sigma$ is suspension with the simplicial circle. Set

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denote the direct sum, considered as a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\pi_{0+(+)}1.$

The notation $\pi_{s-}(+):=\pi_{s+(-)}E$ will be used frequently, as well as the abbreviation

$\pi_{E}$

Note that $\pi_{E}$ is canonically a (left) $\pi_{0}1$-module for every $s \in \mathbb{Z}$. Over

field, the graded ring $\pi_{0}1_{F} \cong \text{KMW}(F)$ is the Milnor-Witt $K$-theory of $F$ by Morel’s theorem \[Mor04\]. Hence $\pi_{E}$ is naturally a $\text{KMW}$-module for every $s \in \mathbb{Z}$. The strictly $\mathbb{A}^{1}$-invariant sheaf obtained as the associated Nisnevich sheaf of $U \mapsto \pi_{s,w}E_{U}$ for $U \in \text{Sm}_{F}$ is denoted

$\pi_{s,w}E$, which gives rise to the homotopy module $\pi_{s,w}E_{U}$. As mentioned already, our main results can be suitably reinterpreted as statements regarding homotopy modules.

We write $d_{(q)}: s_{(q)}(E) \to \Sigma^{1,0}s_{q+1}(E)$ or simply $d_{(q)}$ for the first slice differential of $E \ [RO16, \ \S 2]$, \[Voc02a, \ \S 7\]. For $r \geq 1$, we write

$d_{pr,q,w}(E): E_{pr,q,w}(E) \to E_{pr-1,q+r,w}(E)$ or simply $d_{r}(E)$ for the $r$th differential in the weight $w$ slice spectral sequence.

## 2 Very effective hermitian $K$-theory

The path to computations for the sphere spectrum proceeds via a comparison with the very effective cover $kq \to KQ$ of the motivic ring spectrum representing hermitian $K$-theory that is classifying vector bundles equipped with quadratic forms. Over perfect
fields of characteristic not two, it can be identified as \( kq = f_0(KQ_{≥0}) \), the effective cover of the connective cover of \( KQ \) \[Bac17\], although it can be defined formally over any base scheme for which \( KQ \) is defined \[SO12\]. Since 1 is very effective, the unit map for \( KQ \) factors uniquely over \( kq \). The computations for \( kq \) require an analysis of the slice spectral sequence for \( kq \) extending the results in \[ARØ20\]. Its convergence will be necessary for applying these slice spectral sequence computations and hence is discussed first. When the base scheme \( S \) has no points of residue field characteristic two, let \( \tau \) denote the generator of \( h^0, 1 \sim = \mu_2(\mathcal{O}_S) \) and \( \rho \) denote the class of \(-1\) in \( h^{1, 1} \sim = \mathcal{O}_S^×/2 \). We denote motivic Steenrod operations by \( Sq^i \) \[Voe03, §9\]. Let \( kgl := f_0KGL \) be the (very) effective cover of the motivic ring spectrum \( KGL \) classifying vector bundles. See \[ARØ20\] for a proof of the next result.

**Theorem 2.1:** Multiplication with the Hopf map \( \eta \) yields a homotopy cofiber sequence

\[
\Sigma^{1, 1}kq \xrightarrow{\eta} kq \xrightarrow{\text{forget}} kgl \xrightarrow{\text{hyper}} \Sigma^{2, 1}kq.
\]

**Corollary 2.2:** There is a canonical equivalence between the \( \eta \)-completion of \( kq \) and its slice completion

\[
kq^\wedge \simeq \text{sc}(kq).
\]

**Proof.** By \[RSO19\] Lemma 3.13 and Theorem 2.1 it suffices to prove that \( kgl \) is slice complete. This follows via \( \text{holim}_q f_q kgl \simeq \text{holim}_q f_q KGL \simeq * \) since \( KGL \) is slice complete by \[Voe02b\]. \( \square \)

The Ph.D. thesis \[LA17\] provides an \( E_\infty \)-ring structure on \( KQ \), which by \[GRSØ12\] lifts to \( kq \) for formal reasons. The graded slices of \( kq \) then admit the multiplicative description

\[
s_*(kq) \cong \mathbb{MZ}[\eta, \sqrt{\alpha}]/(2\eta = 0, \eta^2 \delta \rightarrow \sqrt{\alpha}). \tag{2.1}
\]

Here \( |\eta| = (1, 1) \) and \( |\sqrt{\alpha}| = (4, 2) \). Figure 2.1 displays the slice \( d^1 \)-differential for \( kq \).

**Definition 2.3:** Let \( kw := kq[1/\eta] \) be shorthand for the \( \eta \)-inversion of \( kq \).

Figure 2.2 displays the slices together with the slice \( d^1 \)-differentials for \( kw \).

**Proposition 2.4:** The canonical map \( kq \rightarrow KQ \rightarrow KQ[1/\eta] = KW \) yields an identification \( kw \simeq KW_{≥0} \) between \( kw \) and the connective cover of \( KW \).

**Proof.** There is an induced map \( kw \rightarrow KW_{≥0} \) since \( kw \) is a homotopy colimit of connective motivic spectra (\( kq \) is connective by definition). We show that it is a \( \varpi_{t,w} \)-isomorphism. The homotopy sheaves are trivial for \( t < w \). When \( t \geq w \), then \( \varpi_{t,w} kw \) is isomorphic to

\[
\text{colim}_n \varpi_{t+n,w+n}kq \cong \text{colim}_n \varpi_{t+n,w+n}KQ \cong \varpi_{t,w}KW
\]

via the canonical map, which concludes the proof. \( \square \)

\(^1\)The Ph.D. thesis \[Kum20\] provides a motivic spectrum over \( \text{Spec}(\mathbb{Z}) \), and hence any base scheme, which pulls back to \( KQ \) over any scheme in which 2 is invertible.
Figure 2.1: The first slice differential for $kq$. 
Figure 2.2: The first slice differential for $k\mathbb{W}$. 
As a consequence, the following proposition provides that the slice spectral sequence for \( kq \) computes the correct data in suitable degrees.

**Proposition 2.5:** The canonical map \( kq \rightarrow sc(kq) \) induces an isomorphism

\[
\pi_k kq \cong \pi_k sc(kq)
\]
of \( K^{MW} \)-modules for all \( k \in \mathbb{N} \) with \( k \not\equiv 0, 3 \mod 4 \).

**Proof.** Since \( kw = kq[\frac{1}{\eta}] \), \( \eta \) acts invertibly on \( s_*(kw) \) and the columns in the slice spectral sequences for \( kq \) and \( kw \) agree outside a finite range. The calculation of the \( E^2 = E^\infty \)-page for \( KW \) in [RØ16, Theorem 6.3] implies that \( \pi_k sc(kq)[\frac{1}{\eta}] = 0 \) if \( k \not\equiv 0 \mod 4 \). Corollary 2.2 shows \( sc(kq) \simeq 1^{\wedge} \eta \). It remains to apply the \( \eta \)-arithmetic square for \( kq \) and recall that \( kq[\frac{1}{\eta}] = kw \) is the connective cover of \( KW \), see Proposition 2.4. The result follows since \( \pi_k kw = 0 \) for \( k < 0 \) or \( k \not\equiv 0 \mod 4 \). \( \square \)

Having clarified the range in which the slice spectral sequence for \( kq \) actually computes \( kq \), it is now time to present these computations, starting with the following summary.

**Theorem 2.6:** The nontrivial rows in the 1st and 2nd columns of the \( E^2 \)-page of the \(-n\)th slice spectral sequence for \( kq \) are given as:

| \( q \) | \( E^2_{-n+1,q,-n}(kq) \) | \( E^2_{-n+2,q,-n}(kq) \) |
|---|---|---|
| 3 | \( h^{n+1,n+3}/(Sq^2 h^{n-1,n+2} + \tau pr^\infty_2 H^{n+1,n+2}) \) | |
| 2 | \( h^{n+1,n+2}/Sq^2 h^{n-1,n+1} \) | \( 2H^{n+2,n+2} \oplus (h^{n,n+2}/Sq^2 h^{n-2,n+1}) \) |
| 1 | \( h^{n+1}/Sq^2 pr^\infty_2 H^{n-2,n} \) | \( \ker(h^{n-1,n+1} \xrightarrow{Sq^2} h^{n+1,n+2}) \) |
| 0 | \( H^{n-1,n} \) | \( \ker(H^{n-2,n} \xrightarrow{Sq^2 pr} h^{n,n+1}) \) |

The proof of Theorem 2.6 is based on the identification of the slices of \( kq \) in (1.3). Various arguments are required to conclude \( E^2 = E^\infty \) for all of these tridegrees. First, we need to record how \( 1 \rightarrow kq \) affects slices.

**Lemma 2.7:** On 0-slices, the unit map \( 1 \rightarrow kq \) induces the identity

\[
s_0(1) = MZ \xrightarrow{1} MZ = s_0(kq).
\]

**Proof.** The zero slice functor preserves the ring structure; see also [RSØ19, Lemma 2.29]. \( \square \)

**Lemma 2.8:** On 1-slices, the unit map \( 1 \rightarrow kq \) induces the identity

\[
s_1(1) = \Sigma^{1,1} MZ/2 \{\alpha_1\} \xrightarrow{1} \Sigma^{1,1} MZ/2 = s_1(kq).
\]

**Proof.** This follows from Lemma 2.7 and the multiplicative structure since the graded slice functor preserves the ring structure, see also [RSØ19, Lemma 2.29]. \( \square \)
Lemma 2.9: On 2-slices the unit map $1 \to kq$ induces

$$s_2(1) = \Sigma^{2,2} \mathbb{M}/2\{\alpha_1^2\} \vee \Sigma^{3,2} \mathbb{M}/12\{\alpha_{2/2}\} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \partial_{\infty}^{12} \end{pmatrix}} \Sigma^{2,2} \mathbb{M}/2 \vee \Sigma^{4,2} \mathbb{M} = s_2(kq).$$

Proof. The first column follows from the multiplicative structure or [RSO19, Lemma 2.29]. By [RSO19, Lemma 2.30] the second diagonal entry has the form $n \cdot \partial_{\infty}^{12}$ for $n$ an odd integer. We show that $n$ is not divisible by 3. The commutative diagram

$$\begin{array}{ccc}
1 & \longrightarrow & \text{MGL} \\
\downarrow & & \downarrow \\
kq & \text{forget} & kgl
\end{array}$$

of maps of motivic ring spectra induces a commutative diagram

$$\begin{array}{ccc}
s_21 & \longrightarrow & s_2\text{MGL} \\
\downarrow & & \downarrow \\
s_2kq & s_2\text{forget} & s_2kgl
\end{array}$$

on slices. The Wood cofiber sequence from Theorem 2.1 implies that $s_2\text{forget}$ is the canonical projection, as is the map $s_2\text{MGL} \to s_2\text{kgl}$. Hence $n$ can be identified from the map on 2-slices induced by the unit map $1 \to \text{MGL}$. This computation follows from the proof of (1.1), and shows that $n$ is not divisible by 3. Thus $n$ is a unit in $\mathbb{Z}/12\mathbb{Z}$. The result follows by applying a suitable isomorphism to $s_2(1)$. \qed

Lemma 2.10: For $q \geq 3$ the induced map $s_q(1 \to kq)$ is the identity on the summands generated by $\alpha_1^q$ and $\alpha_3\alpha_1^{q-3}$, i.e., there is a commutative diagram with vertical inclusions

$$\begin{array}{ccc}
\Sigma^{q,q} \mathbb{M}/2 \vee \Sigma^{q+2,q} \mathbb{M}/2 & \xrightarrow{id} & \Sigma^{q,q} \mathbb{M}/2 \vee \Sigma^{q+2,q} \mathbb{M}/2 \\
\downarrow & & \downarrow \\
s_q(1) & \longrightarrow & s_q(kq).
\end{array}$$

In particular, the unit map $1 \to kq$ induces the identity on $s_3$. \qed

Proof. The argument in [RSO19, Lemma 2.30] for $1 \to \text{KQ}$ applies also to $1 \to kq$. \qed

Corollary 2.11: For all $n \in \mathbb{Z}$ the unit map $1 \to kq$ induces a surjection on $E^1_{-n+1,2,-n}$ and $E^1_{-n+2,4,-n}$, and for all $m, k \in \mathbb{Z}$ an isomorphism on:

- $E^1_{-n,m,-n}$
- $E^1_{-n+1,m,-n}$ $m \neq 2$
- $E^1_{-n+2,m,-n}$ $m \neq 2, 4$
- $E^1_{-n+k,m,-n}$ $m \leq 1$. 

Proof. This follows from Lemmas 2.7, 2.8, 2.9, and 2.10.

Lemma 2.12: The groups $E^1_{-n,m,-n}(kq)$ and $E^1_{-n+1,m,-n}(kq)$ consist of permanent cycles.

Proof. This follows from Corollary 2.11 and the corresponding result for 1, or alternatively from the computation of the $E^2 = E^\infty$ page for $kw \simeq K\mathbb{W}_{\geq 0}$ and the fact that $\pi_{-n,-n}kw \cong \pi_{-n,-n}kw \cong W(F)$ is the Witt ring for $n < 0$.

Lemma 2.13: The group $E^2_{-n+1,m,-n}(kq)$ is trivial for $m \geq 3$.

Proof. We show the $d^1$-differential entering $E^1_{-n+1,m,-n}(kq) = h^{n+m-1,n+m}$ is surjective. For $m \geq 4$ the differential is given by

$$E^1_{-n+2,m-1,-n}(kq) = h^{n+m-3,n+m-1} \oplus h^{n+m-1,n+m-1} \rightarrow h^{n+m-1,n+m} = E^1_{-n+1,m,-n}(kq)$$

$$(b_2, b_0) \mapsto Sq^2b_2 + \tau b_0.$$ 

The claim follows since $\tau : h^{n+m-1,n+m-1} \rightarrow h^{n+m-1,n+m}$ is surjective. For $m = 3,$

$$E^1_{-n+2,2,-n}(kq) = h^{n,n+2} \oplus H^{n+2,n+2} \rightarrow h^{n+2,n+3} = E^1_{-n+1,3,-n}(kq)$$

$$(b_2, B_0) \mapsto Sq^2b_2 + \tau pr_\infty^2 B_0$$

is surjective, since $\tau$ and $pr_\infty^2 : H^{n+2,n+2} \rightarrow h^{n+2,n+2}$ are both surjective maps.

Lemma 2.14: The possibly nontrivial groups in the first column $E^2_{-n+1,m,-n}(kq)$ are:

$$E^2_{-n+1,0,-n}(kq) \cong H^{n-1,n}$$
$$E^2_{-n+1,1,-n}(kq) \cong h^{n,n+1}/Sq^2pr_\infty^2 H^{n-2,n}$$
$$E^2_{-n+1,2,-n}(kq) \cong h^{n+1,n+2}/Sq^2h^{n-1,n+1}$$

Proof. This follows from Lemma 2.13 and the determination of the slice $d^1$-differential.

Lemma 2.15: The group $E^2_{-n+2,2,-n}(kq)$ is trivial for $m \geq 4$.

Proof. The argument is analogous to the part of the proof of Lemma 2.13 for $m \geq 4.$

Lemma 2.16: The group $E^2_{-n+2,3,-n}(kq)$ is isomorphic to

$$h^{n+1,n+3}/\tau^2 \rho^2 h^{n-1,n-1} + \tau pr_\infty^2 H^{n+1,n+2}$$

generated by $\alpha_1^3$.

Proof. The kernel of the $d^1$-differential

$$h^{n+1,n+3} \oplus h^{n+3,n+3} \rightarrow h^{n+3,n+4}, \quad (b_2, b_0) \mapsto Sq^2b_2 + \tau b_0$$

is isomorphic to $h^{n+1,n+3}$ via the map $c_2 \mapsto (c_2, \tau^{-1}Sq^2c_2).$ The image of the $d^1$-differential

$$h^{n-1,n+2} \oplus H^{n+1,n+2} \rightarrow h^{n+1,n+3} \oplus h^{n+3,n+3}, \quad (a_3, A_1) \mapsto (Sq^2a_3 + \tau pr_\infty^2 A_1, Sq^2 Sq^1 a_3)$$

is generated by $\tau^2 \rho^2 h^{n-1,n-1}$ and $\tau pr_\infty^2 H^{n+1,n+2} = \ker(\partial^2_\infty : h^{n+1,n+2} \rightarrow H^{n+2,n+2})$. 


Lemma 2.17: The group $E^2_{-n+2,2,-n}(kq)$ is isomorphic to the direct sum

\[ h^{n,n+2}/\mathrm{Sq}^2h^{n-2,n+1} \oplus 2H^{n+2,n+2}. \]

Proof. The kernel of $d^1$ maps to $2H^{n+2,n+2}$ via the surjection

\[ \psi_n: \{(b_2, b_0) \in h^{n,n+2} \oplus H^{n+2,n+2}: \mathrm{Sq}^2b_2 = \tau \mathrm{pr}_2^\infty b_0\} \to \ker(H^{n+2,n+2} \xrightarrow{\mathrm{pr}_2^\infty} h^{n+2,n+2}) \]

\[ (b_2, b_0) \mapsto \partial_\infty^2(\tau^{-1}b_2) \cdot \{-1\} + b_0. \]

Here $\psi_n(0, b_0) = b_0$ for all $b_0 \in \ker(H^{n+2,n+2} \xrightarrow{\mathrm{pr}_2} h^{n+2,n+2})$, and $\psi_n$ is well-defined since

\[ \mathrm{pr}_2^\infty(\partial_\infty^2(\tau^{-1}b_2)) \cdot \{-1\} + b_0 = \mathrm{pr}_2^\infty(\partial_\infty^2(\tau^{-1}b_2)) \cdot \mathrm{pr}_2^\infty(\{-1\}) + \mathrm{pr}_2^\infty b_0 = \mathrm{Sq}^1(\tau^{-1}b_2) \cdot c + \mathrm{pr}_2^\infty b_0 = \tau^{-1}\mathrm{Sq}^2b_2 + \mathrm{pr}_2^\infty b_0 = 0. \]

We can identify $\ker(\psi_n)$ with $h^{n,n+2}$ via the split injection $a_2 \mapsto (a_2, \partial_\infty^2(\tau^{-1}a_2) \cdot \{-1\})$. Hence the kernel of the exiting $d^1$ is isomorphic to the direct sum

\[ h^{n,n+2} \oplus \ker(H^{n+2,n+2} \xrightarrow{\mathrm{pr}_2} h^{n+2,n+2}). \]

The entering $d^1$ has image generated by $\mathrm{Sq}^2h^{n-2,n+1}$ in $\ker(\psi_n)$, while its image in the target of $\psi_n$ is trivial since

\[ \psi_n \circ (\mathrm{Sq}^2, \partial_\infty^2\mathrm{Sq}^2\mathrm{Sq}^1)(h^{n-2,n+1}) = 0. \]

This follows from the calculation

\[ \psi_n(\mathrm{Sq}^2(\tau^{-3}c_0), \partial_\infty^2\mathrm{Sq}^2\mathrm{Sq}^1(\tau^{-3}c_0)) = \psi_n(\tau^2\rho^2c_0, \partial_\infty^2\tau\rho^3c_0) = \partial_\infty^2(\tau\rho^2c_0) \cdot \{-1\} + \partial_\infty^2\tau\rho^3c_0 = C_0 \cdot \{-1\}^4 + C_0 \cdot \{-1\}^4 = 0, \]

where $C_0 \in H^{n-2,n-2}$ satisfies $\mathrm{pr}_2^\infty C_0 = c_0$. Hence there is a split short exact sequence

\[ 0 \to h^{n,n+2}/\mathrm{Sq}^2h^{n-2,n+1} \to E^2_{-n+2,2,-n}(kq) \to 2H^{n+2,n+2} \to 0, \]

which concludes the proof. \[ \square \]

Lemma 2.18: The group $E^2_{-n+2,1,-n}(kq)$ coincides with the kernel of $\mathrm{Sq}^2: h^{n-1,n+1} \to h^{n+1,n+2}$. Proof. This follows from Corollary 2.20 \[ \square \]

Lemma 2.19: The composition $\mathrm{Sq}^2\mathrm{pr}_2^4: h^{n-3,n}_4 \to h^{n-1,n+1}$ is zero.
Proof. An element \( x \in h_{n-3,n} \) maps via \( \text{pr}_2^4 \) to an element \( \tau^3 y \), where \( y \in h_{n-3,n-3} \). Since \( Sq^1 \text{pr}_2^4 = 0 \), the equality

\[
0 = Sq^1 (\text{pr}_2^4 (x)) = Sq^1 (\tau^3 y) = \tau^2 \rho y
\]

follows, whence \( Sq^2 (\text{pr}_2^4 (x)) = Sq^2 (\tau^3 y) = \tau^2 \rho^2 y = \rho \cdot \tau^2 \rho y = \rho \cdot 0 = 0. \)

**Corollary 2.20:** The composition \( Sq^2 \text{pr}_2^\infty : H_{n-3,n} \to h_{n-1,n+1} \) is zero.

**Proof.** This follows, because \( \text{pr}_2^\infty = \text{pr}_4^2 \text{pr}_2^\infty \) and already \( Sq^2 \text{pr}_2^4 \) is zero by Lemma 2.19.

**Lemma 2.21:** The group \( E_{-n+2,0,-n}^2 (kq) \) coincides with the kernel of \( Sq^2 \text{pr}_2^\infty : H_{-n,n} \to h_{n,n+1} \).

**Proof.** Use the description of the slice \( d^1 \)-differential.

**Lemma 2.22:** The group \( E_{n+3,m,-n}^2 (kq) \) is trivial for \( m \geq 4 \).

**Proof.** The argument is analogous to the part of the proof of Lemma 2.13 for \( m \geq 4 \).

**Lemma 2.23:** The group \( E_{-n+3,3,-n}^2 (kq) \) is isomorphic to

\[
h_{n,n+2}/\text{pr}_2^\infty H_{n,n+2} \cong 2H_{n+1,n+2}
\]

generated by \( \alpha_3^3 \).

**Proof.** The kernel of the \( d^1 \)-differential

\[
h_{n,n+3} \oplus h_{n+2,n+3} \to h_{n+2,n+4}, \quad (b_3, b_1) \mapsto Sq^2 b_3 + \tau b_1
\]

is isomorphic to \( h_{n,n+3} \) via the map \( c_3 \mapsto (c_3, \tau^{-1} Sq^2 c_3) \). The image of the \( d^1 \)-differential

\[
h_{n-2,n+2} \oplus H_{n,n+2} \to h_{n,n+3} \oplus h_{n+2,n+3}, \quad (a_4, A_2) \mapsto (Sq^2 a_4 + \tau \text{pr}_2^\infty A_2, Sq^3 Sq^1 a_4) = (\tau \text{pr}_2^\infty A_2, 0)
\]

is generated by \( \tau \text{pr}_2^\infty H_{n,n+2} \). The result follows, where the last isomorphism in the statement is induced by the short exact sequence

\[
0 \to H_{n,n+2}/2H_{n,n+2} \to h_{n,n+2} \to 2H_{n+1,n+2} \to 0.
\]

**Lemma 2.24:** The group \( E_{-n+3,2,-n}^2 (kq) \) is isomorphic to the kernel of

\[
d^1 : h_{n-1,n+2} \oplus H_{n+1,n+2}, (b_3, B_1) \mapsto (\tau \text{pr}_2^\infty B_1 + Sq^2 b_3, Sq^2 Sq^1 b_3).
\]

**Proof.** This follows from the description of the slice \( d^1 \)-differential since the incoming differential is zero.
**Lemma 2.25:** The group $E^2_{-n+3,1,-n}(kq)$ is isomorphic to the kernel of $\text{Sq}^2: h^{n-2,n+1} \to h^{n,n+2}$.

*Proof.* This follows from the description of the slice $d^1$-differential since the incoming differential is zero. □

**Lemma 2.26:** The group $E^2_{-n+3,0,-n}(kq)$ is equal to $H^{n-3,n}$.

*Proof.* This follows from the description of the slice $d^1$-differential and Corollary 2.20. □

The above concludes our identification of the first four nontrivial columns of the second page of the slice spectral sequence for $kq$. To determine the first three nontrivial columns on the third page, it remains to identify the second slice differentials in the relevant range.

**Lemma 2.27:** Every element in $E^2_{n+1,m,-n}(kq)$ is a permanent cycle.

*Proof.* This follows from Lemma 2.12. □

Lemma 2.27 shows that $E^1_{n,m,-n}(kq) = E^\infty_{n,m,-n}(kq)$ for all $m, n \in \mathbb{Z}$. Hence the first potentially nonzero second slice differential for $kq$ is:

$$d^2: \ker(\text{Sq}^2\pr_2^\infty: H^{n-2,n} \to h^{n,n+1}) \to h^{n+1,n+2}/\text{Sq}^2h^{n-1,n+1}$$

(2.2)

Instead of referring to [RS019] Lemma 4.15, which implies that the differential (2.2) is zero, an alternative argument based on [Rön20] will be employed. The advantage is the identification of $K^{MW}$-module structures. For this purpose, consider the slice filtration

$$\cdots \to f_2kq \to f_1kq \to f_0kq = kq$$

and the induced filtration

$$\cdots \subset f_2\pi_mkq \subset f_1\pi_mkq \subset \pi_mkq$$

on $K^{MW}$-modules; here $f_n\pi_mkq$ is the image of $\pi_mf_nkq \to \pi_mkq$. Proceed now to the first stage of the identification of the $K^{MW}$-modules $\pi_1kq$ and $\pi_2kq$. If $A$ is a graded $K^{MW}$-module and $n, d$ are integers, then $A(n)$ is the graded $K^{MW}$-module with $A(n)_d = A_{n+d}$.

**Lemma 2.28:** There are isomorphisms $\pi_1f_2kq \cong k^M(2)$ generated in $\pi_1(2)f_3kq \cong k_0^M$ and $\pi_2f_3kq \cong k^M(1)$ generated in $\pi_2(1)f_3kq \cong k_0^M$.

*Proof.* Slice convergence for $kq$ implies slice convergence for $f_3kq$, as in Corollary 2.2. Similar to Lemma 2.13 and 2.22, the columns of the second page of the slice spectral sequence for $f_3kq$ corresponding to $\pi_1$ and $\pi_2$ are concentrated in the lowest possible slice, namely 3. There is no room for higher differentials, whence $\pi_1f_3kq \cong \pi_1s_3kq \cong h^{+2,*,+3}$.

The computation of the slice $d^1$ differential $\pi_2s_3kq \to \pi_1s_4kq$ provides as kernel $\pi_2f_3kq \cong h^{+1,*,+3}$. □

**Lemma 2.29:** There is an isomorphism $\pi_1f_2kq \cong k^M(1)$ generated in $\pi_1(1)f_2kq \cong k_0^M$.

The map $\pi_1f_3kq \to \pi_1f_2kq$ is zero.
Proof. Slice convergence for $\text{kq}$ implies slice convergence for $f_2\text{kq}$, as in Corollary 2.2. The column of the second page of the slice spectral sequence for $f_2\text{kq}$ corresponding to $\pi_1$ is concentrated in the lowest possible slice, namely 2. There is no room for higher differentials, whence $\pi_1f_2\text{kq} \cong \pi_1s_2\text{kq} \cong h^{+1,*+2}$. The long exact sequence on homotopy modules induced by the cofiber sequence
\[
f_3\text{kq} \to f_2\text{kq} \to s_2\text{kq} \to \Sigma f_3\text{kq}
\]
provides the last statement.

The following statement is essentially [Røn20, Lemma 2.3].

Lemma 2.30: There is an isomorphism $\pi_1f_1\text{kq} \cong K^{\text{MW}}/(2, \eta^2)$ generated by the image of the topological Hopf map $\eta_{\text{top}}$ in $\pi_{1+(0)}f_1\text{kq}$. The image of $\pi_1f_2\text{kq} \to \pi_1f_1\text{kq}$ is the submodule generated by $\eta_{\text{top}}$, which is isomorphic to $k^M(1)/\rho^2k^M(-1)$.

Proof. Slice convergence for $\text{kq}$ implies slice convergence for $f_1\text{kq}$, as in Corollary 2.2. The column of the second page of the slice spectral sequence for $f_1\text{kq}$ corresponding to $\pi_1$ is concentrated in slices 1 and 2. The slice $d^1$ differential $\text{Sq}^2: s_1\text{kq} \to \Sigma s_2\text{kq}$ induces $\text{Sq}^2: k^M(-1) \cong \pi_2s_1\text{kq} \to \pi_1f_2\text{kq} \cong \pi_1s_2\text{kq} \cong k^M(1)$, whence the image in $\pi_1f_2\text{kq}$ coincides with $\rho^2k^M(-1)$. There is no room for higher differentials, thus the long exact sequence
\[
\cdots \to \pi_2s_1\text{kq} \to \pi_1f_2\text{kq} \to \pi_1f_1\text{kq} \to \pi_1s_1\text{kq} \to 0
\]
induced by the cofiber sequence
\[
f_2\text{kq} \to f_1\text{kq} \to s_1\text{kq} \to \Sigma f_2\text{kq}
\]
induces a short exact sequence
\[
0 \to k^M(1)/\rho^2k^M(-1) \to \pi_1f_1\text{kq} \to \pi_1s_1\text{kq} \cong k^M \to 0 \tag{2.3}
\]
of $K^{\text{MW}}$-modules. The target of the extension (2.3) is generated in $\pi_{1+(0)}s_1\text{kq}$ by the image of the topological Hopf map $\eta_{\text{top}}$, as (étale or complex) realization implies. The multiplicative structure of the slices implies that the image of $\eta_{\text{top}}$ in $\pi_{1+(1)}f_1\text{kq}$ is non-trivial, hence coincides with the (unique) generator of $k^M(1)/\rho^2k^M(-1)$. In particular, the map $K^{\text{MW}} \to \pi_1f_1\text{kq}$ sending 1 to the image of $\eta_{\text{top}}$ is surjective. As in the proof of [Røn20, Lemma 2.3], one concludes the desired isomorphisms.

Under the isomorphism given in Lemma 2.30, the short exact sequence (2.3) corresponds to the short exact sequence
\[
0 \to \eta K^{\text{MW}}/(2, \eta^2) \to K^{\text{MW}}/(2, \eta^2) \to K^{\text{MW}}/(2, \eta) = k^M \to 0
\]
of $K^{\text{MW}}$-modules. Hence $\eta K^{\text{MW}}/(2, \eta^2) \cong K^{\text{MW}}/(2, \eta \cdot [-1]^2) \cong k^M(1)/\rho^2k^M(-1)$. This furnishes an identification
\[
\eta(K^{\text{MW}}/(2, \eta^2)) \cong k^M(1)/\rho^2k^M(-1) \oplus \rho^2(k^M(-2)) \tag{2.4}
\]
of $K^{\text{MW}}$-modules. The summands are generated by $\eta_{\text{top}}$ and $[-1]^2\eta_{\text{top}}$, respectively. This will be used in the proof of the following statement.
Lemma 2.31: The slice $d^2$ differential $E^2_{-n+2,0,−n}(kq) \to E^2_{−n+1,2,−n}(kq)$ is zero.

Proof. The cofiber sequence

$$f_1 kq \to kq \to s_0 kq \to \Sigma f_1 kq$$

induces a long exact sequence of homotopy modules containing the portion

$$\cdots \to \pi_2 s_0 kq \to \pi_1 f_1 kq \to \pi_1 kq \to \pi_1 s_0 kq \to 0.$$  

Since $s_0 kq \cong \mathbb{M}Z$, on which $\eta$ acts as zero, the homotopy module $\pi_2 s_0 kq$ is a $K^M$-module. Hence the $K^{MW}$-module homomorphism $\pi_2 s_0 kq \to \pi_1 f_1 kq$ factors as

$$\pi_2 s_0 kq \to \eta(\pi_1 f_1 kq) \to \pi_1 f_1 kq$$

where the first map is a $K^M$-homomorphism. The image of the $K^{MW}$-module $\eta(\pi_1 f_1 kq) \cong k^M(1)/\rho^2 k^M(−1) \oplus \rho^2(\mathbb{M}^M(−2))$ in $\pi_2 s_1 kq$ coincides with $\rho^2(\mathbb{M}^M(−2))$ generated by the element $[-1]^2 \eta_{\text{top}}$, because $\eta_{\text{top}}$ maps trivially to $\pi_1(1) s_1 kq$. The image of the map $\pi_2 \mathbb{M}Z/2 \to \pi_1 s_1 kq$ given by the first slice differential $Sq^2 pr_{\infty}^1$: $s_0 kq \to \Sigma s_1 kq$ is (strictly) contained in $\rho^2(\mathbb{M}^M(−2))$. It remains to observe that the map $\pi_2 \mathbb{M}Z \to k_M(1)/\rho^2 k_M(−1)$ to the first summand of $(2,4)$ is zero. Since the target is 2-torsion, $\psi$ factors over $(\pi_2 \mathbb{M}Z)/2$. Consider the short exact sequence

$$0 \to (\pi_2 \mathbb{M}Z)/2 \to \pi_2 \mathbb{M}Z/2 \to \pi_1 \mathbb{M}Z \to 0$$

of $K^{MW}$-modules, where the first map is induced by $pr_{\infty}^1: \mathbb{M}Z/ \to \mathbb{M}Z/2$. This map coincides with $s_0(kq \xrightarrow{\text{can}} kq/2)$ up to equivalence. Since $\pi_1 f_1 kq$ is 2-torsion, the map $\pi_1 f_1 kq \to f_1 kq/2$ is injective. As a consequence of the commutative diagram

$$\begin{array}{ccc}
\pi_2 \mathbb{M}Z & \longrightarrow & (\pi_2 \mathbb{M}Z)/2 \\
\downarrow & & \downarrow \\
\pi_1 f_1 kq & \longrightarrow & \pi_1 f_1 kq/2
\end{array}$$

it suffices to prove that the map $\pi_2 \mathbb{M}Z/2 \cong k_M(−2) \to \pi_1 f_1 kq/2$ hits the summand $k_M(1)/\rho^2 k_M(−1)$ trivially. The latter map is determined by the image of the (unique) $K^{MW}$-module generator $g = g_F \in \pi_2(−2) \mathbb{M}Z/2$, which is decorated with notation for the base field, because a base change argument is about to happen. This generator $g_F$ is the image of the generator $g_{F_0} \in \pi_2(−2) \mathbb{M}Z/2$, where $F_0 \subset F$ is the prime field of $F$. The commutative diagram

$$\begin{array}{ccc}
\pi_{2+(-2)}{(\mathbb{M}Z/2)}_{F_0} & \longrightarrow & \pi_{2+(-2)}{(\mathbb{M}Z/2)}_F \\
\downarrow & & \downarrow \\
k^M_3(F_0)/\rho^2 k^M_1(F_0) & \longrightarrow & k^M_3(F)/\rho^2 k^M_1(F)
\end{array}$$

implies that the vertical map on the right hand side is zero, because $k^M_3(F_0)/\rho^2 k^M_1(F_0)$ is zero [Mil70, Example 1.5, Appendix].  

\[\square\]
A consequence of Lemma 2.31 is that the $K^{MW}$-module $\pi_1kq$ is given by the short exact sequence
\[0 \to \pi_1f_1kq/Sq^2pr_2^{\infty}MZ \to \pi_1kq \to \pi_1MZ \to 0\]
which does not split in general, neither as an extension of $K^{MW}$-modules, nor degreewise as an extension of abelian groups.

**Theorem 2.32:** The canonical map $kq \to KQ$ induces an isomorphism $\pi_{1-(w)}kq \to \pi_{1-(w)}KQ$ for all integers $w < 4$. In particular, there result isomorphisms
\[\pi_{1-(w)}KQ \cong (K^{MW}\eta_{\text{top}}/(2,\eta^2))_w\]
for $w < 2$ (including a vanishing for $w < -1$), and short exact sequences
\[0 \to k^M/\rho^2k^M \oplus k^M \to \pi_{1-(2)}KQ \to H^{1,2} \to 0\]
\[0 \to k^M/\rho^2k^M \oplus k^M/\tau^{-1}\pi^2pr_2^{\infty}H^{1,3} \to \pi_{1-(3)}KQ \to H^{2,3} \to 0.\]

**Proof.** The first statement holds by definition for $w$ nonpositive and follows in the remaining cases by inspection of the slice spectral sequence for $KQ$ [RO16]. The remaining assertions follow from Lemma 2.31 and Lemma 2.30.

Having determined $\pi_1kq$, the next challenge is $\pi_2kq$.

**Lemma 2.33:** There is a short exact sequence
\[0 \to K^M(2)/\{-1\}^3K^M(-1) \to \pi_2f_2kq \to k^M \to 0\]
of $K^{MW}$-modules, where $k^M$ is generated in $\pi_{2+(0)}f_2kq$ by the image of $\eta_{\text{top}}^2$, and $K^M$ is generated in $\pi_{2+(2)}k^2$ by the unit in symplectic $K$-theory. The extension is uniquely determined by the fact that $\eta_{\text{top}}^2 \in \{1\} \in K^M \cong \pi_{1+(1)}f_2kq$. The image of the map $\pi_2f_3kq \to \pi_2f_2kq$ coincides with $2K^M(2)/\{-1\}^3K^M(-1)$.

**Proof.** Lemma 2.28 provides $\pi_2f_3kq \cong k^M(1)$, a sub-$K^{MW}$-module of $\pi_2s_3kq$, generated in $\pi_{2+(1)}f_3kq$. The slice $d^1$ differential $\pi_3s_2kq \to \pi_2s_3kq$ – see for comparison Lemma 2.16 – and the short exact sequence
\[0 \to (\pi_1MZ)/2 \to \pi_1MZ/2 \to 2\pi_0MZ \to 0\]
imply that the image of $\pi_2f_3kq$ in $\pi_2f_2kq$ coincides with $2K^M(2)/\{-1\}^3K^M(-1)$. The proof of Lemma 2.17 shows that the kernel of the slice $d^1$ differential $\pi_2s_2kq \to \pi_1s_3kq$ is isomorphic to $2K^M(2) \oplus k^M$. There results a short exact sequence
\[0 \to 2K^M(2)/\{-1\}^3K^M(-1) \to \pi_2f_2kq \to 2K^M(2) \oplus k^M \to 0\]
(2.5)
of $K^{MW}$-modules. The image of $\eta_{\text{top}}^2$ in $\pi_{2+(0)}kq$ lifts to a unique element $\eta_{\text{top}}^2 \in \pi_{2+(0)}f_2kq$ generating $k^M$ in the extension (2.5). The multiplicative structure of the slices, or rather, its behaviour with respect to the Hopf map, implies that $\eta\eta_{\text{top}}^2$ is the unique nonzero element $\{1\} \in 2K^M_1$. Alternatively, one may use (étale or complex) realization to conclude
this fact. To specify the extension (2.5) with respect to the summand $2K^M(2)$, observe that

$$\text{Ext}_{K^MW}(2K^M, 2K^M/\{-1\}^3K^M) = \text{Hom}_{K^MW}(2K^M, 2K^M/\{-1\}^3K^M)$$

has two elements by [Ron] Theorem A.1 and Lemma A.3], namely the zero map and the projection. These correspond to the trivial and a unique nontrivial extension. The Wood cofiber sequence from Theorem 2.1 supplies a long exact sequence

$$\cdots \to \pi_{3+2}(\text{id}) \to \pi_{3+2}(\text{hyper}) \to \pi_{2+1}\text{gl} \to \pi_{2+2}\text{gl} \to \cdots$$

in which $\pi_{3+2}(\text{gl}) \cong \pi_{3+2}(\text{KQ}) \cong K^S_1 \cong 0$ (also deducible from the slice spectral sequence), whence $\pi_{3+2}(\text{gl}) \cong \pi_{3+2}(\text{KGL}) \cong K^M_1$ \text{hyper} $\pi_{2+1}\text{gl}$. Indeed the latter map is bijective, because $K^S_0 \cong \pi_{2+2}(\text{gl}) \cong K^M_0$ is the inclusion of the even integers, thus injective. Hence the extension in question is nontrivial, which concludes the proof.

\begin{lemma}
\textbf{Lemma 2.34:} The slice $d^2$ differentials $E^2_{-n+3,j,-n}(Kq) \to E^2_{-n+2,j+2,-n}(Kq)$ are zero for all $j$.
\end{lemma}

\begin{proof}
The statement is true for $j > 1$ by Lemma 2.15. Consider first the canonical map $\pi_3s_1kq \to \pi_2f_2kq$. Its source is isomorphic to $K^M(-2)$ generated by a unique element in $\pi_{-3-2}s_1kq$, because $s_1kq \simeq \Sigma^1MZ/2$. Lemma 2.33 identifies its target. However, since the source is a (free) $K^M$-module, its image is contained in $2\eta\pi_2f_2kq$. Hence the map $\pi_3s_1kq \to \pi_2f_2kq$ is determined by sending the generator $g = \tau^3 \in \pi_{-3-2}s_1kq$ to an element in $2\eta\pi_2f_2kq$. The generator $g$ is already defined over the prime field $F_0 \subset F$ of $F$, whence its image has to lie in (the image of) $2\eta\pi_{-2-2}f_2kq_{F_0}$. Lemma 2.33 provides in particular that the kernel of the canonical map $\pi_2f_2kq \to \pi_2gqkq$ is $K^M(2)/\{-1\}^3K^M(-2)$.

In the specific degree $2-(2)$, this kernel is $2K^M(2)/\{-1\}^3K^M_3$, which is the zero group for prime fields, as [Mil70] implies. Hence the map $\pi_3s_1kq \to \pi_2f_2kq$ is determined by the slice $d^1$ differential $\pi_3s_1kq \to \pi_2gqkq$. In other words, the slice $d^2$ differential $E^2_{-n+3,1,-n}(kq) \to E^2_{-n+2,1,-n}(kq)$ is zero. The statement for the slice $d^2$ differential $E^2_{-n+3,0,-n}(kq) \to E^2_{-n+2,2,-n}(kq)$ follows from Corollary 3.23 the corresponding statement for the slice $d^2$ differential $E^2_{-n+3,0,-n}(1) \to E^2_{-n+2,2,-n}(1)$, because of the commutative diagram

$$
\begin{array}{ccc}
E^2_{-n+3,0,-n}(1) & \xrightarrow{id} & E^2_{-n+3,0,-n}(kq) \\
\downarrow & & \downarrow \\
E^2_{-n+2,2,-n}(1) & \xrightarrow{} & E^2_{-n+2,2,-n}(kq)
\end{array}
$$

in which the top horizontal map is the identity.

\begin{lemma}
\textbf{Lemma 2.35:} The slice $d^3$ differentials $E^3_{-n+3,j,-n}(kq) \to E^3_{-n+2,j+3,-n}(kq)$ are zero for all $j$.
\end{lemma}
Proof. The statement is true for \( j > 0 \) by Lemma \( 2.15 \). The statement for the slice \( d^3 \) differential \( E^3_{-n+3,0,-n}(kq) \to E^3_{-n+2,3,-n}(kq) \) follows from Corollary \( 3.23 \). The corresponding statement for the slice \( d^3 \) differential \( E^3_{-n+3,0,-n}(1) \to E^3_{-n+2,3,-n}(1) \), because of the commutative diagram

\[
\begin{array}{ccc}
E^3_{-n+3,0,-n}(1) & \xrightarrow{id} & E^3_{-n+3,0,-n}(kq) \\
\downarrow & & \downarrow \\
E^3_{-n+2,3,-n}(1) & \to & E^3_{-n+2,3,-n}(kq)
\end{array}
\]

in which the top horizontal map is the identity. \( \square \)

**Lemma 2.36:** The image of \( \pi_3 s^1 kq \to \pi_2 f_2 kq \) coincides with \( \rho^2 k^M \). The kernel of \( \pi_2 s^1 kq \to \pi_1 f_2 kq \) coincides with \( \rho^2 k^M \). There results a short exact sequence

\[
0 \to \pi_2 f_2 kq/\rho^2 k^M \to \pi_2 f_1 kq \to \rho^2 k^M(-1) \to 0
\]

of \( K^{MW} \)-modules.

**Proof.** This follows from the identification of the slice \( d^1 \) differential, Lemma \( 2.34 \) and Lemma \( 2.35 \). \( \square \)

**Lemma 2.37:** The map \( \pi_3 s_0 kq \to \pi_2 f_1 kq \) is the zero map. Hence there is a short exact sequence

\[
0 \to \pi_2 f_1 kq \to \pi_2 kq \to \ker(\text{Sq}^2 p^\infty_2 \colon \pi_2 MZ \to \pi_1 \Sigma^{(1)} MZ/2) \to 0
\]

of \( K^{MW} \)-modules, where \( \eta \) operates on \( \ker(\text{Sq}^2 p^\infty_2 \colon \pi_2 MZ \to \pi_1 \Sigma^{(1)} MZ/2) \) via the projection \( p^\infty_2 \to \rho^1 k^M \).

**Proof.** The first statement follows from Lemma \( 3.22 \). The second statement is then a consequence of the slice \( d^1 \) differential on \( \pi_2 s_0 kq \), Lemma \( 2.34 \) and Lemma \( 2.35 \). The last statement follows from the multiplicative structure on the slices of \( kq \). \( \square \)

**Theorem 2.38:** The canonical map \( kq \to KQ \) induces an isomorphism \( \pi_2_{(-w)} kq \to \pi_2_{(-w)} KQ \) for all integers \( w < 4 \). In particular, there results a vanishing \( \pi_2_{(-w)} KQ \cong 0 \) for \( w < -2 \), isomorphisms \( \pi_2_{(+2)} KQ \cong K^M_0, \pi_2_{(1)} KQ \cong K^M_1, \pi_2_{(0)} KQ \cong K^M_2 \oplus k^M_0 \) of abelian groups, and short exact sequences

\[
0 \to K^M_3/[\{-1\}^3 K^M_0 \oplus k^M_1] \to \pi_2_{(-1)} KQ \to \rho^2 k^M_0 \to 0
\]

\[
0 \to K^M_4/[\{-1\}^3 K^M_1 \oplus k^M_2/\rho^2 k^M_0 \to \pi_2_{(-2)} KQ \to \rho^2 k^M_1 \to 0
\]

\[
0 \to K^M_5/[\{-1\}^3 K^M_2 \oplus k^M_3/\rho^2 k^M_1 \to \pi_2_{(-3)} KQ \to \rho^2 k^M_2 \times \ker(\text{Sq}^2 p^\infty_2 : H^{1,3} \to h^{3,4}) \to 0
\]

of abelian groups.

**Proof.** The first statement holds by definition for \( w \) nonpositive and follows in the remaining cases by inspection of the slice spectral sequence for \( KQ \) [RO16]. The remaining assertions follow from Lemma \( 2.33 \) Lemma \( 2.36 \) and Lemma \( 2.37 \). \( \square \)
Remark 2.39: Theorem 2.38 contains in particular a "symbolic" description of the fourth symplectic $K$-group $K^3_p(F) \cong \pi_{2-}(KQ_F)$ of a field of characteristic not 2, in the sense that only symbols in Milnor $K$-theory are involved. It relies on the vanishing $H^{0,2} = 0$, which can be deduced from [Voe11, Theorem 6.18] and [Mer16, Equation (4.2)].

3 The slice filtration on the sphere spectrum

The group $\text{Ext}^{1,2i}_{MU, MU}(MU_+, MU_+)$ in the slices of $I\Lambda$, see [1.1], is finite cyclic of order $a_2$; let $\tau_i$ denote the generator of its $p$-primary component $\text{Ext}^{1,2i}_{BP, BP}(BP_+, BP_+)$ [Rav86]. Still $\tau$ denotes the unique nonzero element in $h^{0,1} \cong \mu_2$ and $\rho$ denotes the class of $-1$ in $h^{1,1}$. We denote by $\partial^{a_2}$ the unique nontrivial map from $\Lambda/a_2$ to $\Sigma^{1,0} \Lambda/2$ in $\text{SH}$. For reference we recall [RSO19, Theorem 4.1].

Lemma 3.1: In slice degrees $0 \leq q \leq 3$, $d_1^q(\lambda): s_q(\lambda) \rightarrow \Sigma^{1,0} s_{q+1}(\lambda)$ is given by:

- $d_1^0(0) = \text{Sq}^{2pr}: \Lambda \rightarrow \Lambda/2 \rightarrow \Sigma^{1,1} \Lambda/2$
- $d_1^1(1) = (\text{Sq}^2, \text{inc}_2, \text{Sq}^1): \Sigma^{1,1} \Lambda/2 \rightarrow \Sigma^{3,2} \Lambda/2 \vee \Sigma^{4,2} \Lambda/2$
- $d_1^2(1) = (\text{Sq}^2, \tau \partial_2^{a_2}, \text{Sq}^{1,2}): \Sigma^{2,2} \Lambda/2 \vee \Sigma^{3,2} \Lambda/2 \rightarrow \Sigma^{4,3} \Lambda/2 \vee \Sigma^{6,3} \Lambda/2$
- $d_1^3(1) = (\text{Sq}^2, \tau \partial_{a_2}, \text{Sq}^{1,2}): \Sigma^{3,3} \Lambda/2 \vee \Sigma^{5,3} \Lambda/2 \rightarrow \Sigma^{5,4} \Lambda/2 \vee \Sigma^{7,4} \Lambda/2$

For $q \geq 4$, $d_1^q(\lambda)$ restricts to the direct summand $\Sigma^{q,2} \Lambda/2 \vee \Sigma^{q+2,2} \Lambda/2$ of $s_q(\lambda)$ by

$$\left(\frac{\text{Sq}^2}{\text{Sq}^{3,2}}, \frac{\tau}{\text{Sq}^2 + \rho \text{Sq}^1}\right): \Sigma^{q,2} \Lambda/2 \vee \Sigma^{q+2,2} \Lambda/2 \rightarrow \Sigma^{q+2,q+1} \Lambda/2 \vee \Sigma^{q+4,2} \Lambda/2.$$

Here $\Sigma^{q+2,2} \Lambda/2$ is generated by $\alpha^{q-3}_2 \alpha_3 \in \text{Ext}_{BP, BP}^{q-2,2}(BP_+, BP_+)$.

Moreover, $d_1^q(\lambda)$ restricts as follows on direct summands of $s_q(\lambda)$:

- $\text{inc}_{a_2} : 2\text{Sq}^2 \text{Sq}^{1,2} : \Sigma^{4q-3,2q-1} \Lambda/2 \rightarrow \Sigma^{4q,2q} \Lambda/a_2q$
- $\text{Sq}^2 \partial_{a_2} : \Sigma^{4q-1,2q} \Lambda/a_2q \rightarrow \Sigma^{4q+2,2q+1} \Lambda/2$
- $\tau \partial_{a_2} : \Sigma^{4q-1,2q} \Lambda/a_2q \rightarrow \Sigma^{4q,2q+1} \Lambda/2$
- $0 : \Sigma^{4q-1,2q} \Lambda/a_2q \rightarrow \Sigma^{4q,2q+1} \Lambda/2$

Here $\Sigma^{4q-3,2q-1} \Lambda/2$ and $\Sigma^{4q-1,2q} \Lambda/a_2q$ are generated by $\alpha_{2q-1}$ and $\alpha_{q/2}$, respectively.

Figure 3 summarizes these calculations by pairing the slices of $1\Lambda$ with their corresponding generators. Each direct summand of a fixed slice is labeled by the difference between the simplicial suspension degree and the slice degree. The colors of the $d^1$-differentials correspond to elements of the motivic Steenrod algebra, here ordered by simplicial degree. An open square refers to integral (or rather $\Lambda$-) coefficients and solid dots to $\Lambda/2$-coefficients. The open circle in the second slice indicates $\Lambda/12$-coefficients, and
Figure 3.1: The first slice differential for $\mathbf{1}_A$. 
similarly for $\Lambda/240$, $\Lambda/6$, and $\Lambda/504$-coefficients. Lemma 3.3 discusses the $d^1$-differentials exiting and entering the direct summand $\Sigma^6\Lambda/2\{\beta_{2/2}\}$ corresponding to the generator $\beta_{2/2} = \alpha_2^3$.

Fix a compatible pair $(F, \Lambda = \mathbb{Z}[\frac{1}{n}])$, where $F$ is a field of exponential characteristic $c \neq 2$. For legibility we leave $\Lambda$ out of the notation, and write $h^*,^*$ for motivic cohomology groups with $(\Lambda/2 \times \Lambda/2)$-coefficients. Figure 3.2 displays the $E^1$-page of the weight $w = -n$ slice spectral sequence, $n \in \mathbb{Z}$. Here $E^1_{p,q,-n}(1) = 0$ for $p < -n$ or $q < 0$, and $H^{p,n} = 0$ for $n < 0$.

Next we show vanishing in a certain range of the terms contributing to the 2-line.

**Lemma 3.2:** The group $E^2_{-n+2,m,-n}(1)$ is trivial for $m \geq 5$.

**Proof.** For $m \geq 5$ the kernel of the differential

$$E^1_{-n+2,m,-n}(1) = h^{n+m-2,n+m} \oplus h^{n+m,n+m} \to h^{n+m,n+m+1} = E^1_{-n+1,m+1,-n}(1)$$

is isomorphic to $h^{n+m-2,n+m}$ via the map sending $c_2$ to $(c_2, \tau^{-1}S\tau c_2)$. The image of the differential

$$E^1_{-n+3,m-1,-n}(1) \to E^1_{-n+2,m,-n}(1) = h^{n+m-2,n+m} \oplus h^{n+m,n+m}$$

hits this group, since it restricts to multiplication with $\tau$ on the summand $h^{n+m-2,n+m-1}$ generated by $\alpha_1^{m-1}$.

**Lemma 3.3:** The group $E^2_{-n+2,4,-n}(1)$ is isomorphic to $h^{n+4,n+4}$ generated by $\beta_{2/2}$.

**Proof.** The $d^1$-differential on $h^{n+4,n+4}\{\beta_{2/2}\}$ is not $\tau$ by comparison with complex points; the corresponding $d^3$-differential in the Adams-Novikov spectral sequence on $\beta_{2/2}$ vanishes. Hence the $d^1$-differential restricted to $\Sigma^6\mathbb{M}Z/2\{\beta_{2/2}\}$ maps trivially to $\Sigma^6\mathbb{M}Z/\{\alpha_1^5\}$. This implies that on $s_4$ the unit map $1 \to KQ$ restricts trivially on $\Sigma^6\mathbb{M}Z/2\{\beta_{2/2}\}$. In turn, the $d^1$-differential restricted to $\Sigma^6\mathbb{M}Z/2\{\beta_{2/2}\}$ maps trivially to $\Sigma^6\mathbb{M}Z/\{\alpha_1^5\alpha_3\}$ as well. Since there is no other possible $d^1$-differential on $h^{n+4,n+4}\{\beta_{2/2}\}$ for weight reasons, this group survives to the $E^2$-page.

The $d^1$-differentials on the other two direct summands are the same as in the proof of Lemma 3.2 whence the kernel of

$$E^1_{-n+2,4,-n}(1) \to E^1_{-n+1,5,-n}(1)$$

is isomorphic to $h^{n+2,n+3} \oplus h^{n+4,n+4}$. The entering $d^1$-differential has the form

$$E^1_{-n+3,3,-n}(1) \to h^{n+4,n+4}\{\beta_{2/2}\} \oplus h^{n+4,n+4} \oplus h^{n+2,n+4}$$

$$(a_1, a_3) \mapsto (\phi S\phi(a_1) + bS3S\phi(a_3), \rho S\phi(a_1) + S3S\phi(a_3), \tau(a_1) + S2(a_3)),$$

for some $\phi \in h^{1,1}$ and $b \in h^{0,0}$. By naturality, $\phi \in h^{1,1}(F_0)$ and $b \in h^{0,0}(F_0)$, where $F_0$ is the prime field of $F$. The homomorphism $\rho a_0 \mapsto \phi S\phi(\tau a_0) = \phi \rho a_0$ thus
Figure 3.2: The $E^1$-page for \(1\).
amounts to multiplication by $\phi \rho \in h^{2,2}(F_0)$ on $h^{n+2,n+3}(F)$. If $c > 2$, it is thus trivial, since $\phi \rho \in h^{2,2}(F_0) = 0$ for $F_0$ a finite field. If $c = 1$, the square class $\phi$ can be determined via comparison with $\mathbb{Z}[\frac{1}{2}]$: the slice computation for 2-primary torsion summands holds for that base scheme, as well as the identification of the motivic Steenrod algebra at 2. Hence we can conclude $\phi \in \{ \pm 1, \pm 2 \}$. Considering the real numbers rules out the cases $\phi < 0$. To this end, observe that real realization sends $\nu$ to $\eta_{top}$ and $\rho$ to 1. Hence $[-1]^{-n}\nu^2 \in \pi_{n+6,n+4}$ is a nontrivial element for every subfield of $\mathbb{R}$. Setting $n = -2$ implies that $\phi = 1$ over $\mathbb{R}$. To exclude the case $\phi = 2$ observe that the symbol $\{ -1, 2 \}$ is the zero element in $K^M_2$. In particular, the homomorphism given by multiplication by $\phi \rho$ is the zero homomorphism.

Similarly, but easier, the homomorphism $a_3 = \tau^3a_0 \mapsto b\Sigma^3\Sigma^1(a_3) = b\rho^4a_0$ amounts to multiplication by $b\rho^4$. It is thus trivial over fields of odd characteristic. To conclude for fields of characteristic zero we consider the real numbers. Recall that $[-1]^{-n}\nu^2 \in \pi_{n+6,n+4}$ is a nontrivial element for every subfield of $\mathbb{R}$. Setting $n = -4$ implies $b = 0$ over $\mathbb{R}$, and hence over any field of characteristic zero. Comparison with $\mathbb{Z}[\frac{1}{2}]$ implies $b = 0$ over any field with $c \neq 2$. Hence the entering $d^1$-differential has the form

$$h^{n+2,n+3} \oplus h^{n,n+3} = E^1_{n+1,3,n}(1) \to h^{n+4,n+4}(\beta_2/2) \oplus h^{n+4,n+4} \oplus h^{n+2,n+4}$$

$$(a_1,a_3) \mapsto (0,\rho\Sigma^3(a_1) + \Sigma^3\Sigma^1(a_3), \tau(a_1) + \Sigma^3(a_3))$$

We may identify this map’s image with the direct summand $h^{n+2,n+4}$.

**Corollary 3.4:** The $d^1$-differential $s_3(1) \to \Sigma^1s_4(1)$ maps trivially to the direct summand $\Sigma^6\mathbb{M}/2$ generated by $\beta_2/2$.

**Proof.** The proof of Lemma 3.3 shows the map $\Sigma^3\mathbb{M}/2 \to \Sigma^7\mathbb{M}/2$ (given by $b\Sigma^3\Sigma^3$) is zero, and that $\Sigma^5\mathbb{M}/2 \to \Sigma^7\mathbb{M}/2$ (given by $a\Sigma^2 + \phi\Sigma^1$, where $a \in h^{0,0}, \phi \in h^{1,1}$) is of the form $a\Sigma^2 + \phi\Sigma^1$, where $a \in h^{0,0}, \phi \in \{1, 2\}$. The composite of $d^1$-differentials

$$s_2(1) \to \Sigma^1s_3(1) \to \Sigma^2s_4(1)$$

is trivial. Since $a\Sigma^5\Sigma^1 = \phi\Sigma^1\Sigma^2\partial_2^2 = 0$, it follows that $a = 0$ and $\phi = 1$.

**Lemma 3.5:** The group $E^2_{n+2,3,n}(1)$ is isomorphic to

$$h^{n+1,n+3}/\tau^2\rho^2h^{n-1,n-1}$$

generated by $\alpha^3_1$.

**Proof.** As before, the kernel of the $d^1$-differential

$$h^{n+1,n+3} \oplus h^{n+3,n+3} \to h^{n+3,n+4}, \quad (b_2,b_0) \mapsto \Sigma^2b_2 + \tau b_0$$

is isomorphic to $h^{n+1,n+3}$ via the map $c_2 \mapsto (c_2, \tau^{-1}\Sigma^2c_2)$. The image of the $d^1$-differential

$$h^{n-1,n+2} \oplus h^{n,n+2} \to h^{n+1,n+3} \oplus h^{n+3,n+3}, \quad (a_2, a_3) \mapsto (\Sigma^2a_3 + \tau\partial_2^1 a_2, \Sigma^3\Sigma^1a_3 + \Sigma^2\partial_2^1 a_2)$$

is the subgroup $\Sigma^2h^{n-1,n+2}$ by Corollary [A.6]. It can be identified with $\tau^2\rho^2h^{n-1,n-1}$.
Lemma 3.6: The group $E_{-n+2,2,-n}^2(1)$ is isomorphic to the direct sum

$$(h^{n,n+2}/\text{Sq}^2 h^{n-2,n+1}) \oplus \text{im}(h^{n+1,n+2}_2 \to h^{n+1,n+2}_{12})$$

generated by $\alpha_2^2$ and $\alpha_{2/2}$, respectively.

Proof. If $\rho^2 = 0$, the kernel of the $d^1$-differential

$$h^{n,n+2} \oplus h^{n+1,n+2}_{12} \to h^{n+2,n+3}, \quad (b_2,b_1) \mapsto \text{Sq}^2 b_2 + \tau \partial_{12}^2 b_1$$

is isomorphic to $h^{n,n+2} \oplus \ker(h^{n+1,n+2}_{12} \to h^{n+2,n+2}) = h^{n,n+2} \oplus \text{im}(h^{n+1,n+2}_2 \to h^{n+1,n+2}_{12})$, and the image of

$$h^{n-2,n+1} \to h^{n,n+2} \oplus h^{n+1,n+2}_{12}, \quad a_3 \mapsto (\text{Sq}^2 a_3, \text{inc} \text{Sq}^2 \text{Sq}^1 a_3)$$

is trivial. In general, consider the map

$$\phi_n : \{(b_2,b_1) \in h^{n,n+2} \oplus h^{n+1,n+2}_{12} : \text{Sq}^2 b_2 = \tau \partial_{12}^2 b_1 \} \to \ker(h^{n+1,n+2}_{12} \to h^{n+2,n+2})$$

$$(b_2,b_1) \mapsto \partial_{12}^2 (\tau^{-1} b_2) \cdot \tau_{12} + b_1,$$

where $\tau_{12}$ denotes $-1 \in F$ viewed as an element in $h^{0,1}_{12}$. The kernel of $\text{Sq}^2 h^{n-2,n+2}$, mapping via the split injection $a_2 \mapsto (a_2, \partial_{12}^2 (\tau^{-1} a_2) \cdot \tau_{12})$.

The image of the $d^1$-differential in $\ker(\phi_n)$ coincides with $\text{Sq}^2 h^{n-2,n+1}$, which can be identified with $\tau^2 \rho^2 h^{n-2,n-2}$. The image of the $d^1$-differential in the target of $\phi_n$ is trivial since

$$\phi_n(\text{Sq}^2(\tau^3 \text{pr}_2^\infty C_0), \text{inc}_{12} \text{Sq}^2 \text{Sq}^1(\tau^3 \text{pr}_2^\infty C_0)) = \phi_n(\tau^2 \rho^2 \text{pr}_2^\infty C_0, \text{inc}_{12}^2(\tau \rho^3 \text{pr}_2^\infty C_0))$$

$$= \partial_{12}^2(\tau \rho^2 \text{pr}_2^\infty C_0) \cdot \tau_{12} + \text{inc}_{12}^2(\tau \rho^3 \text{pr}_2^\infty C_0)$$

$$= \rho_{12}^3 \text{pr}_2^\infty C_0 \cdot \tau_{12} + \rho_{12}^3 \text{pr}_2^\infty C_0 \cdot \tau_{12} = 0.$$ 

Here $\rho_{12} = \text{pr}_{12}^\infty\{-1\}$, which implies that $\text{pr}_{12}^2(\rho_{12}) = \rho$. From the above we obtain a split short exact sequence

$$0 \to h^{n,n+2}/\text{Sq}^2(h^{n-2,n+1}) \to E_{-n+2,2,-n}^2(1) \to \text{im}(h^{n+1,n+2}_2 \to h^{n+1,n+2}_{12}) \to 0,$$

which completes the argument.
Lemma 3.7: The group $E^2_{-n+2,1,-n}(1)$ is isomorphic to

$$\{a \in h^{n-1,n+1} : Sq^2a = 0\} \cong \rho_2 h^{n-1,n-1}$$

generated by $\alpha_1$.

Proof. This follows from the description of the $d^1$-differential and Lemma 2.19 below, since $Sq^2 pr_2^\infty = Sq^2 pr_4^\infty$. \hfill \square

Lemma 3.8: The group $E^2_{-n+2,0,-n}(1)$ is isomorphic to

$$\ker(H^{n-2,n} \xrightarrow{\partial^1} h^{n,n+1})$$

generated by $\alpha_0^1 = 1$.

Proof. This follows directly from the description of the $d^1$-differential. \hfill \square

We summarize our calculations by displaying the groups contributing to the 1- and 2-line:

Theorem 3.9: The nontrivial rows in the 1st and 2nd columns of the $E^2$-page of the $-n$th slice spectral sequence for 1 are given as:

| $q$ | $E^2_{-n+1,q,-n}(1)$ | $E^2_{-n+2,q,-n}(1)$ |
|-----|----------------------|----------------------|
| 4   | $h^{n+2,n+3}/\tau\partial^2 h^{n+1,n+2}_{12}$ | $h^{n+1,n+3}/Sq^2 h^{n-1,n+2}$ |
| 3   | $h^{n+2,n+2}/\tau\partial^2 h^{n+1,n+2}_{12}$ | $\ker(h^{n+1,n+2}/\partial^1_{12} h^{n+2,n+2}) \oplus h^{n,n+2}/Sq^2 h^{n-2,n+1}$ |
| 2   | $h^{n+1,n+1}/Sq^2 pr h^{n-2,n}$ | $\ker(h^{n-1,n+1} \xrightarrow{Sq^2} h^{n+1,n+2})$ |
| 1   | $h^{n-1,n}/Sq^2 pr h^{n-1,n}$ | $\ker(H^{n-2,n} \xrightarrow{Sq^2} h^{n+1,n+1})$ |

Proof. For the first column, see the table in the proof of [RSO19, Theorem 5.5]. Note that $E^2_{-n+1,3,-n}(1)$ coincides with the group $h^{n+2,n+3}/\tau\partial^2 h^{n+1,n+2}_{12}$, because of the inclusion $Sq^2(h^{n,n+2}) \subset \tau\partial^2 h^{n+1,n+2}_{12}$, which is identified with $\tau\rho_2 h^{n,n-1}$. Since $\partial^2_{12} \circ \text{inc}_{12}^2 = Sq^2$, the second subgroup contains $\tau Sq^2(h^{n+1,n+2}) = \tau \rho h^{n+1,n+1}$. The second column follows from Lemmas 3.2, 3.3, 3.5, 3.6, 3.7, and 3.8. \hfill \square

We claim the higher slice differentials entering the second column vanish. Due to Lemma 3.2, it suffices to identify three terms on the second page of 1’s slice spectral sequence.

Lemma 3.10: The group $E^2_{n-3,2,-n}(1)$ coincides with the direct sum

$$h^{n,n+2}_{12} \oplus \ker(Sq^2 : h^{n-1,n+2} \to h^{n+1,n+3}).$$
Proof. The incoming differential $\text{Sq}^2: h^{n-3,n+1} \to h_{12}^{n,n+2} \oplus h^{n-1,n+2}$ is zero. The connecting map $\partial_2^{12}: h_{12}^{n,n+2} \to h_{12}^{n+1,n+2}$ is zero as, well, by Corollary A.6 which implies the statement.

Lemma 3.11: The group $E^2_{n+3,1,-n}(1)$ coincides with $\ker(\text{Sq}^2: h^{n-2,n+1} \to h^{n,n+2})$.

Proof. The incoming differential $\text{Sq}^2 \text{pr}_2^\infty: H^{n-4,n} \to h^{n-2,n+1}$ is zero. The statement then follows, because the kernel of $\text{Sq}^2: h^{n-2,n+1} \to h^{n,n+2}$ is contained in the kernel of $\text{Sq}^2 \text{Sq}^1: h^{n-2,n+1} \to h^{n+1,n+2}$.

Lemma 3.12: The group $E^2_{n+3,0,-n}(1)$ coincides with $H^{n-3,n}$.

Proof. This follows from Corollary 2.20.

Lemma 3.13: The second differential

$$E^2_{n+3,2,-n}(1) \to E^2_{n+2,4,-n}(1)$$

in the $-n$th slice spectral sequence for 1 is trivial.

Proof. The target is computed as $E^2_{n+2,4,-n}(1) \cong h^{n+4,n+4}\{\alpha_2^{2/2}\}$ in Lemma 3.3. For $n = -4$, the element $\{\alpha_2^{2/2}\} \in h^{0,0}$ detects $\nu^2 \in \pi_{6,4}$. If $F$ is a subfield of the real numbers, the complex realization of $\nu^2$ is $\nu^2_{\text{top}} \neq 0$, and the real realization of $\nu^2$ is $\eta^2_{\text{top}} \neq 0$.

Lemma 3.10 determines the source of this differential as

$$E^2_{n+3,2,-n}(1) = \{(a_2, a_3) \in h_{12}^{n,n+2} \oplus h^{n-1,n+2}: \tau \partial_2^{12}(a_2) = \text{Sq}^2(a_3)\} = h_{12}^{n,n+2} \oplus \{a_3 \in h^{n-1,n+2}: \text{Sq}^2(a_3) = 0\}$$

because of Corollary A.6. Since the target of this differential is 2-torsion, it will be trivial on the direct summand $h_{12}^{n,n+2}$ of $E^2_{n+3,2,-n}(1)$. In order to prove that it is trivial on the direct summand $h_{12}^{n,n+2}$ of $E^2_{n+3,2,-n}(1)$, it suffices to prove that a generator $g \in h_{12}^{0,2}$ provided by Lemma A.4 is mapped to zero. As remarked already in Lemma A.3, a generator of $h_{12}^{0,2}$ already exists over the prime field. For prime fields of odd characteristic, the target of the differential is the trivial group $h^{1,4}$. For the prime field $\mathbb{Q}$, the unique nontrivial element in $h^{1,4}$ indexed by $\beta_{2,2}$ is $\rho^3$. Since the real realization of $\rho^3 \nu^2$ is the nontrivial element $1^4 \eta^2_{\text{top}}$, the element $\rho^4$ cannot be hit by any differential.

It remains to prove that the differential is zero on the direct summand

$$\{a_3 \in h^{n-1,n+2}: \text{Sq}^2(a_3) = 0\} \subset E^2_{n+3,2,-n}(1).$$

Considering the direct sum over all $n$, one obtains a graded module over Milnor $K$-theory by Lemma A.1 which is isomorphic to $\{a \in h^{n-1,n-1}: \rho^2 a = 0\}$. Hence it is generated by elements in degree at most 1 (corresponding to $1 \leq n \leq 2$) by [OVV07, Theorem 3.3]. Consider these cases.
$n = 2$ In this case, a generator $(0, \overline{a})$ is represented by a (square class of a) unit $a \in F$ such that $\rho^2 \overline{a} = 0 \in h^{5,3}(F)$. By [OV07, Theorem 3.2] such an element is in the image of the sum of finitely many transfer maps for quadratic field extensions $F \subset L_i$, each of them satisfying $\rho^2 = 0 \in h^{2,2}(L_i)$. By naturality of slice differentials with respect to transfer maps, it suffices to consider such fields, bringing us to the case

$n = 1$ Here a generator can only have the form $(0, \tau^3)$, which is possible if and only if $\rho^2 = 0 \in h^{2,2}(F)$. On the other hand $\rho^2 = 0 \in h^{2,2}(F)$ if and only if there exist two elements $a, b \in F$ such that $-1 = a^2 + b^2$ [EL72, Corollary 3.5]. Then $(0, \tau^3)$ is the image of $(0, \tau^3)$ over the field $F_0(a, b)$, where $F_0 \subset F$ is the prime field of $F$. This field has 2-cohomological dimension at most 3. In that case, the target of the second differential under consideration, $h^{5,5}$, vanishes.

This case by case analysis shows that also the last direct summand of $E^2_{-n+3,2,-n}(1)$ maps trivially under this differential, implying that the whole differential is zero. □

Similar to the treatment of very effective hermitian $K$-theory $kq$, the homotopy groups of the successive slice filtrations $f_q1$ will be discussed as $K^{MW}$-modules, starting with $\pi_1 f_q1$ whose image in $\pi_1 1$ will be zero. The treatment for $\pi_1 1$ given here can also be found in [Rön20].

**Lemma 3.14:** There is an isomorphism of $K^{MW}$-modules $\pi_1 f_q1 \cong k^M(3)$ generated by $\eta^3 \eta_{top} \in \pi_{1+(3)} f_q1$.

**Proof.** Since $\eta_{top} \in \pi_{1+(0)} 1$ lifts uniquely to $\eta_{top} \in \pi_{1+(0)} f_q1$, the multiplicativity of the slice filtration (see [GRSO12]) provides that $\eta^3 \eta_{top} \in \pi_{1+(3)} f_q1$. The multiplicative structure on the slices (with respect to multiplication with $\alpha_1$ only) shows that the image of $\eta^3 \eta_{top}$ in $\pi_{1+(3)} f_q1 \cong h^{0,1} \cong k^M$ is the unique nonzero element. Since the map $\pi_1 s_q1 \to \pi_0 f_q1$ is zero, and the map $\pi_2 s_q1 \to \pi_1 f_q1$ is surjective, there results an identification $\pi_1 f_q1 \cong \pi_1 s_q1 \cong k^M(3)$.

□

**Lemma 3.15:** There is an isomorphism of $K^{MW}$-modules $\pi_1 f_31 \cong k^M(2)$ generated by $\eta^2 \eta_{top} \in \pi_{1+(2)} f_31$. The image of the canonical map $\pi_1 f_q1 \to \pi_1 f_31$ is zero.

**Proof.** A proof follows by inspection of the slice spectral sequence for $f_31$, similar to the argument in Lemma 3.14. □

**Lemma 3.16:** There is a short exact sequence

$$0 \to K^M(2)/24 \to \pi_1 f_21 \to k^M(1) \to 0$$

of $K^{MW}$-modules generated by $\nu \in \pi_{1+(2)} f_21$ and $\eta^2 \eta_{top} \in \pi_{1+(1)} f_21$, respectively. It is uniquely determined by the fact that $\eta^2 \eta_{top} = 12\nu$. The image of the canonical map $\pi_1 f_31 \to \pi_1 f_21$ coincides with $k^M(2)$ generated by $12\nu \in \pi_{1+(2)} 1$.

**Proof.** The crucial part is to identify the extension as a $K^{MW}$-module. As [Rön] Lemma A.3] implies, a unique nontrivial extension with the property that $\eta^2 \eta_{top} = 12\nu$ exists. See, for example, the proof of [RSO19 Theorem 5.5] for this identification. □
For the discussion of higher slice differentials, considering the second slice filtration suffices. Therefore the presentation

\[
K^{MW}(2)\{\nu\} \oplus K^{MW}\{\eta_{\top}\} \cong \pi_1 f_1 \mathbb{1}
\]  

(3.1)

given in [Rön20, Theorem 2.5] will not be directly relevant. Instead the path to \( \pi_2 \mathbb{1} \) will be followed, starting with \( \pi_2 f_1 \mathbb{1} \), as higher filtrations do not contribute – see Lemma 3.2.

**Lemma 3.17:** There is an isomorphism of \( K^{MW} \)-modules \( \pi_2 f_1 \mathbb{1} \cong k^M(4) \oplus k^M(2) \) generated by \( \nu^2 \in \pi_{2+(4)} f_1 \mathbb{1} \) and \( \eta^2 \eta^2_{\top} \in \pi_{2+(2)} f_1 \mathbb{1} \), respectively.

**Proof.** Since \( \nu \in \pi_{1+(2)} \mathbb{1} \) lifts (uniquely) to \( \pi_{1+(2)} f_2 \mathbb{1} \), the multiplicativity of the slice filtration provides that \( \nu^2 \in \pi_{2+(4)} f_1 \mathbb{1} \). Lemma 5.1 shows that \( 2\nu^2 = 0 \). The multiplicative structure on the slices implies that \( \nu^2 \) maps to the unique nonzero element in the group \( E^2_{2,4,4}(1) \) computed in Lemma 6.3. Similarly, the element \( \eta^2 \eta^2_{\top} \in \pi_{2+(2)} f_1 \mathbb{1} \) hits the element \( (\tau^2, \rho^2) \in h^{0,2} \oplus h^{2,2} \) of the kernel of the slice \( d^1 \) differential. There results an identification as stated. \( \square \)

**Lemma 3.18:** There is an isomorphism of \( K^{MW} \)-modules \( \pi_2 f_3 \mathbb{1} \cong k^M(4) \oplus k^M(1) \) generated by \( \nu^2 \in \pi_{2+(4)} f_2 \mathbb{1} \) and \( \eta^2 \eta^2_{\top} \in \pi_{2+(2)} f_2 \mathbb{1} \), respectively. The image of the canonical map \( \pi_2 f_1 \mathbb{1} \rightarrow \pi_2 f_3 \mathbb{1} \) coincides with \( k^M(4) \) generated by \( \nu^2 \in \pi_{2+(4)} \mathbb{1} \).

**Proof.** The slice filtration provides a short exact sequence

\[
0 \rightarrow k^M(4) \rightarrow \pi_2 f_3 \mathbb{1} \rightarrow k^M(1) \rightarrow 0
\]

where \( k^M(4) \) is the cokernel of \( \pi_3 s_3 \mathbb{1} \rightarrow \pi_2 f_1 \mathbb{1} \). The element \( \eta^2 \eta^2_{\top} \in \pi_{2+(2)} f_3 \mathbb{1} \) lifts the unique generator of \( k^M(1) \). The equation \( \eta^2 \eta^2_{\top} = 12\nu \) implies that \( \eta^2 \eta^2_{\top} = 12\nu \eta_{\top} = 0 \), since already \( 2\nu \eta_{\top} = 0 \). It follows that \( \eta \) operates as zero on \( \pi_2 f_3 \mathbb{1} \), whence it is in fact a \( K^M \)-module. The equation \( 2\eta^2 \eta_{\top} = 0 \) implies that \( \pi_2 f_3 \mathbb{1} \) splits as described. \( \square \)

**Lemma 3.19:** There is a short exact sequence

\[
0 \rightarrow k^M(4) \oplus (\pi_1 \Sigma^{(2)} \mathbb{M}Z/24)/\text{inc}^2_{24} \rho^2 \tau k^M \rightarrow \pi_2 f_2 \mathbb{1} \rightarrow k^M \rightarrow 0
\]

of \( K^{MW} \)-modules, where \( k^M \) is generated by \( \eta^2_{\top} \in \pi_{2+(0)} f_2 \mathbb{1} \). This extension is uniquely determined by the fact that \( \eta^2 \eta_{\top} = -1 \in h_{24}^{0,1} \cong \pi_{1+(1)} \Sigma^{(2)} \mathbb{M}Z/24 \). The image of the canonical map \( \pi_2 f_3 \mathbb{1} \rightarrow \pi_2 f_2 \mathbb{1} \) coincides with \( k^M(4) \oplus k^M(1) / \rho^2 k^M(-1) \) generated by \( \nu^2 \in \pi_{2+(4)} f_2 \mathbb{1} \) and \( \eta^2 \eta^2_{\top} \in \pi_{2+(2)} f_2 \mathbb{1} \), respectively.

**Proof.** The slice filtration provides a short exact sequence

\[
0 \rightarrow k^M(4) \oplus k^M(1) / \rho^2 k^M(-1) \rightarrow \pi_2 f_2 \mathbb{1} \rightarrow d \oplus k^M \rightarrow 0
\]

of \( K^{MW} \)-modules, where \( d \) is the \( K^M \)-module with \( d_1 \) coming from \( \pi_2 f_3 \mathbb{1} \) is generated by \( \eta^2_{\top} \),
and the $K^\text{MW}$-module $k^M$ is generated by (the image of) $\eta_{\text{top}}^2$. Already $2\eta_{\text{top}} = 0$, whence the extension with respect to $k^M$ is uniquely determined. It remains to determine the extension with respect to $d$. For this purpose, consider the commutative diagram

$$
\begin{array}{c}
k^M(4) \oplus k^M(1)/\rho^2 k^M(-1) \twoheadrightarrow \pi_2 f_2 1 \twoheadrightarrow d \oplus k^M \\
\downarrow \quad \quad \quad \quad \downarrow \\
2K^M(2)/\{-1\}^3 K^M(-1) \twoheadrightarrow \pi_2 f_2 kq \twoheadrightarrow 2K^M(2) \oplus k^M
\end{array}
$$

(3.2)

induced by the unit map $1 \to kq$ on the slice filtration. Lemma 2.10 implies that the vertical map on the left hand side in diagram (3.2) projects away from $k^M(4)$ and is induced by the boundary map

$$k^M(1) \cong \oplus_{n \in \mathbb{N}} H^{n,n+1} \xrightarrow{\partial_\infty} \oplus_{n \in \mathbb{N}_2} H^{n+1,n+1} \cong 2K^M(2);$$

in particular it is surjective. The vertical map on the right hand side in diagram (3.2) is the identity on the summand $k^M$ and induced by the boundary $\partial_\infty: M\mathbb{Z}/12 \to \Sigma M\mathbb{Z}$ by Lemma 2.9 (in particular, it is injective. As the proof of Lemma 2.33 shows, there is a unique nontrivial extension

$$0 \to 2K^M(2)/\{-1\}^3 K^M(-1) \to \pi_2 f_2 kq \to 2K^M(2) \to 0$$

corresponding to the unique nonzero element in

$$\text{Hom}_{K^\text{MW}}(2K^M(2)/\{-1\}^3 K^M, 2K^M(2)/\{-1\}^3 K^M) \cong \text{Hom}_{K^\text{MW}}(k^M(1)/\rho^2 k^M, k^M(1)/\rho^2 k^M).$$

Hence there is a unique nontrivial extension

$$0 \to k^M(1)/\rho^2 k^M(1) \to (\pi_1 \Sigma(1) M\mathbb{Z}/24)/\text{inc}^2_{24} \rho^2 \tau k^M \to d \to 0$$

mapping to the extension

$$0 \to 2K^M(2)/\{-1\}^3 K^M(-1) \to \pi_2 f_2 kq \to 2K^M(2) \to 0$$

as prescribed by the diagram (3.2).

As a consequence of Lemma 3.19 every element of $\pi_2 f_2 1$ is 24-torsion, as well as 12h-torsion.

**Lemma 3.20:** The second differential

$$E_{-n+3,1,-n}^2(1) \to E_{-n+2,3,-n}^2(1)$$

in the $-n$th slice spectral sequence for 1 is trivial.
Proof. The cofiber sequence
\[ f_2 \mathbf{1} \to f_1 \mathbf{1} \to s_1 \mathbf{1} \to \Sigma f_2 \mathbf{1} \]
induces a long exact sequence on \( \mathbf{K}^{\text{MW}} \)-modules containing the portion
\[ \cdots \to \pi_3 s_1 \mathbf{1} \to \pi_2 f_2 \mathbf{1} \to \pi_2 f_1 \mathbf{1} \to \pi_2 s_1 \mathbf{1} \to \cdots . \]
The map \( k^M(-2) \cong \pi_3 s_1 \mathbf{1} \to \pi_2 f_1 \mathbf{1} \) has as source a \( k^M \)-module on a (unique) generator \( g \) in degree \( \pi_3-(2)s_1 \mathbf{1} \), which comes from the prime field \( F_0 \subset F \). Hence the map is determined by the image of \( g \) in \( 2n \pi_2-(2)f_21F_0 \). Lemma 3.19 provides that this group is zero in odd characteristic. For \( F_0 = \mathbb{Q} \), the group \( 2(h_{34}^{3,4}/\text{inc}_{24}^2\rho^2h_{1,2})(\mathbb{Q}) = 0 \) is zero, and the group \( k_0^M(\mathbb{Q}) \) contains a unique nonzero element \( \rho^2\nu^2 \), which is mapped to \( \eta_2^2\top \) under real realization. It follows that the image of \( g \) is given by \( \rho^2\eta_2^2 \), as determined by the slice \( d^1 \) differential. The result follows. □

Lemma 3.21: The slice \( d^3 \) differential
\[ E_{n+3,1,n}^3(1) \to E_{n+2,4,n}^3(1) \]
in the \( -n \)-th slice spectral sequence for \( \mathbf{1} \) is trivial.
Proof. The proof of Lemma 3.20 verifies this claim. □

Lemma 3.22: The canonical map \( \pi_3 s_0 \mathbf{1} \to \pi_2 f_1 \mathbf{1} \) is the zero map.
Proof. The composition \( \pi_3 s_0 \mathbf{1} \to \pi_2 f_1 \mathbf{1} \to \pi_2 s_1 \mathbf{1} \) is the slice \( d^1 \) differential, and zero by Corollary 3.20. Hence the map \( \pi_3 s_0 \mathbf{1} \to \pi_2 f_1 \mathbf{1} \) lifts to a map \( \pi_3 s_0 \mathbf{1} \to \pi_2 f_2 \mathbf{1}/\pi_3 s_1 \mathbf{1} \), where the target denotes the cokernel of the (not necessarily injective) canonical map \( \pi_3 s_1 \mathbf{1} \to \pi_2 f_2 \mathbf{1} \). The slice \( d^1 \) differential \( \pi_3 s_1 \mathbf{1} \to \pi_2 f_2 \mathbf{1} \to \pi_2 s_1 \mathbf{1} \) sends the (unique) generator \( g \in \pi_3-(2)s_1 \mathbf{1} \cong k_0^M \) to \( (\tau^2\rho^2/\text{inc}_{12}^2(\tau\rho^2)) \in h_{2,4}^* \oplus h_{12}^{3,4} \). Lemma 3.19 then provides a short exact sequence
\[ 0 \to k^M(4) \oplus (\pi_1 \Sigma^{(2)} \mathbb{M}Z/24)/\text{inc}_{24}^2\rho^2 \tau k^M \to \pi_1 f_2 \mathbf{1}/\pi_3 s_1 \mathbf{1} \to k^M/\rho^2 k^M(-2) \to 0 \]
of \( \mathbf{K}^{\text{MW}} \)-modules which is determined by the equality
\[ \eta_{\top}^2 = -1 \in h_{24}^{0,1} \cong \pi_1(\Sigma^{(2)} \mathbb{M}Z/24). \]
Since every element in the \( \mathbf{K}^{\text{MW}} \)-module \( \pi_2 f_2 \mathbf{1}/\pi_3 s_1 \mathbf{1} \) is 24-torsion\(^2\) the map \( \pi_3 s_0 \mathbf{1} \to \pi_2 f_2 \mathbf{1}/\pi_3 s_1 \mathbf{1} \) factors as
\[ \pi_3 s_0 \mathbf{1} \to (\pi_3 \mathbb{M}Z)/24 \to \pi_2 f_2 \mathbf{1}/\pi_3 s_1 \mathbf{1} . \]
The second map fits into a commutative diagram
\(^2\)The \( k^M \)-module \( \pi_3 \mathbb{M}Z/24 \) is more accessible than the quite mysterious \( k^M \)-module \( \pi_3 \mathbb{M}Z \).
Consider now the 2-primary torsion part by elements in $\mathcal{M}_2$. Suppose now that $F$ has virtual cohomological dimension at the prime $2$ strictly less than four. The relevant target group $\pi_3$ simplifies to a group with either one element, or the two elements $\{0, \rho^7 \nu^2\}$. In the latter case, real realization provides that the element $\rho^7 \nu^2$, realizing to $\eta_{\text{top}}^2$, cannot be hit.

An element in $h_4^{1,4}(F)$ already lives over a subfield $F_0(u)$, where $u$ is a unit in $F$. In particular, $F_0(u)$ has virtual cohomological dimension at the prime 2 strictly less than four. The relevant target group

$$\pi_{2-(4)f_2}1/\pi_{3-(4)}s_11(F_0(u)) \cong k_8^M \oplus h_2^{4,5}/\text{inc}_{24}^2 \rho^2 h^{2,3} \oplus k_3^M/\rho k_1^M \cong k_7^M(F_0(u))$$
Hence it suffices to prove that \( y \) module of \( \pi \) maps is the canonical map. This canonical map induces an injection on the sub-

\[
\begin{align*}
\pi_3 s_0 &\to \pi_2 f_1 / 12h \\
\pi_3 s_0 12h &\to \pi_2 f_1 / 12h
\end{align*}
\]

By the preceding argument, \( 2x = 0 \). Lemma \[3.19\] implies that the 2-torsion in \( \pi_2 f_2 1 / \pi_3 s_1 1 \), which injects into the 2-torsion in \( \pi_2 f_1 1 / 12h \), is generated by \( \nu^2 \) and \( \eta_{top}^2 \) as a \( \text{K}^{MW} \)-module. In order to show that \( x = 0 \), let \( y \) be its image in \( \pi_2 f_1 1 / 2 \) under the canonical map \( 1 / 12h \to 1 / 2 \). It has the property that the composition \( 1 \to 1 / 12h \to 1 / 2 \) of canonical maps is the canonical map. This canonical map induces an injection on the sub-\( \text{K}^{MW} \)-module of \( \pi_2 f_2 1 / \pi_3 s_1 1 \) generated by \( \nu^2 \) and \( \eta_{top}^2 \), because these elements are 2-torsion. Hence it suffices to prove that \( y = 0 \). The canonical maps \( 1 \to 1 / 12h \to 1 / 2 \) induce a commutative diagram

\[
\begin{align*}
\pi_3 s_0 1 &\xrightarrow{pr_2 2^4} \pi_3 s_0 1 / 12h \\
\pi_3 s_0 12h &\xrightarrow{pr_2 2^4} \pi_3 s_0 1 / 2 \\
\pi_2 f_1 &\xrightarrow{} \pi_2 f_1 1 / 12h & \pi_2 f_1 1 / 2.
\end{align*}
\]

Naturality implies that \( y \) is in the image of \( \pi_3 s_0 1 / 2 \to \pi_2 f_1 1 / 2 \), the vertical map on the right hand side. It is determined by the image of the (unique) generator in \( \pi_3 - (3) \mathbb{M} \mathbb{Z} / 2 \), which already lives over the prime field \( F_0 \subset F \). In this degree, \( \pi_3 - (3) f_2 1 / \pi_3 - (3) s_1 1 (F_0) \cong k^M_2 (F_0) = \{0, \rho^7 \nu^2\} \), as mentioned before. Since \( 2 \nu^2 = 0 \), the element \( \rho^7 \nu^2 \) maps to a nonzero element in \( \pi_2 - (3) f_1 1 \mathbb{Q} / 2 \); real realization may be invoked as well to see this. As a consequence, \( y = 0 \). It follows that \( x \) is already zero. This concludes the proof that

\[
\text{Im}(\pi_3 s_0 12h \to \pi_2 f_1 1 / 12h) \cap \text{Im}(\pi_2 f_2 1 \to \pi_2 f_1 1 / 12h) = \{0\}.
\]

In particular, the map \( \pi_3 s_0 1 \to \pi_2 f_1 1 \) is zero.

**Corollary 3.23:** The slice \( d^j \) differential

\[
E^j_{n+3,0,-n}(1) \to E^j_{n+2,j,-n}(1)
\]

in the \( -n \)th slice spectral sequence for \( 1 \) is zero.

**Proof.** This follows from Corollary \[2.20\] and Lemma \[3.22\] \( \square \)

As a consequence of Corollary \[3.23\], the remaining statements of this section address actually the \( E^\infty \)-page of the slice spectral sequence for \( 1 \).

**Corollary 3.24:** The kernel of \( E^2_{n+2,4,-n}(1) \to \mathbb{k}q \) is \( h^{n+4,n+4} \) generated by \( \beta_{2/2} \).

**Proof.** This follows from Lemmas \[3.3\] and \[2.15\] \( \square \)
Corollary 3.25: The kernel of the induced map $E^2_{-n+2,3,-n}(1 \to kq)$ is isomorphic to the image of $pr_\infty^2 H^{n+1,n+2} = H^{n+1,n+2}/2$ in $h^{n+1,n+2}/Sq^2 h^{n-1,n+1}$.

Proof. This follows from Lemmas 3.5, 2.10, and 2.16.

Corollary 3.26: The kernel of $E^2_{-n+2,3,-n}(1 \to kq)$ is isomorphic to $H^{n+1,n+2}/12$.

Proof. This follows from Lemmas 3.6, 2.9, 2.17, and 3.27.

Lemma 3.27: The kernel $H^{n+1,n+2}/12$ of $\partial_1: h^{n+1,n+2}/12 \to H^{n+2,n+2}$ is contained in the kernel of $\partial_2: h^{n+1,n+2} \to h^{n+2,n+2}$.

Proof. If $x \in h^{n+1,n+2}$ satisfies $\partial_1 x = 0$ then $\partial_2 x = pr_\infty^2 \partial_1 x = 0$.

Lemma 3.28: For $n,k \in \mathbb{Z}$ the unit map $1 \to kq$ induces an isomorphism on

$$
\begin{align*}
E^2_{-n,m,-n} \\
E^2_{-n+1,m,-n} & \quad m \neq 2, 3 \\
E^2_{-n+2,m,-n} & \quad m \neq 2, 3, 4 \\
E^2_{-n+k,m,-n} & \quad m \leq 1,
\end{align*}
$$

and a surjection on $E^2_{-n+1,m,-n}$ for $2 \leq m \leq 3$.

Proof. This follows by inspection of $d^1$ in Figure 2.1 and Corollary 2.11 with the exception of $E^2_{-n+k,1,-n}$ for $k \equiv 3 \mod 4$. In the latter case, if an element of $h^{n+1-k,n+1}$ is in ker($Sq^2$) then it is also in ker($Sq^2 Sq^1$). The result for the direct summand $\Sigma^{2,2} \mathbb{M}\mathbb{Z}/2$ of $s^2(1)$ in Lemma 2.9 implies now the isomorphism on $E^2_{-n+k,1,-n}$ for all $k \in \mathbb{Z}$.

4 The 1-line

Let $F$ be a field of exponential characteristic $c \neq 2$, and set $\Lambda := \mathbb{Z}[[\frac{1}{c}]]$. Then the slice spectral sequence for $1_F$ converges conditionally to the homotopy groups of the $\eta$-completion $1^\wedge_\eta$ [RSO19, Theorem 3.50]. The calculation of the $E^2$-page given in Theorem 3.9 shows that $\pi_{1+(n)} 1^\wedge_\eta = 0$ for $n \geq 3$, and $\pi_{2+(n)} 1^\wedge_\eta = 0$ for $n \geq 5$. As a consequence,

$$
\pi_1 1^\wedge_{\eta,\frac{1}{c}} = \pi_2 1^\wedge_{\eta,\frac{1}{c}} = 0.
$$

The vanishing $\pi_1 1^\wedge_{\eta,\frac{1}{c}} = \pi_2 1^\wedge_{\eta,\frac{1}{c}} = 0$ from [Røn18, Theorem 8.3] and the $\eta$-arithmetic square imply there is a short exact sequence

$$
0 \to \pi_0 1 \to \pi_0 1^\wedge_{\eta,\frac{1}{c}} \oplus \pi_0 1^\wedge_{\eta,\frac{1}{c}} \to \pi_0 1^\wedge_{\eta,\frac{1}{c}} \to 0,
$$

an isomorphism

$$
\pi_1 1 \cong \pi_1 1^\wedge_\eta,
$$

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and an exact sequence
\[ \cdots \to \pi_3 \Lambda \to \pi_2 \Lambda \to \pi_2 \Lambda \to 0 \]
of \text{K}^{\text{MW}}\text{-modules.}

**Theorem 4.1:** Let \( F \) be a field of exponential characteristic \( c \neq 2 \). The unit map \( 1 \to \text{kq} \) induces an isomorphism \( \pi_{0+(*)}1 \to \pi_{0+(*)}\text{kq} \) and a surjection \( \pi_{1+(*)}1 \to \pi_{1+(*)}\text{kq} \) whose kernel coincides with \( \text{K}^{\text{MW}}_{2*,/24} \) after inverting \( c \). In particular, since \( \pi_{1+(n)}\text{kq} = 0 \) for \( n \geq 2 \), \( \pi_{1+(2)}\Lambda \sim \Lambda/24 \) and \( \pi_{1+(n)}\Lambda = 0 \) for \( n \geq 3 \).

**Proof.** Surjectivity of \( \pi_{1+(*)}1 \to \pi_{1+(*)}\text{kq} \) follows from Lemma 3.28 for the \( \pi_{1+} \)-column of the \( E^2 = E^\infty \)-page; see Lemma 2.31 for \( E^2_{-n+1,j,-n}(\text{kq}) = E^\infty_{-n+1,h,-n}(\text{kq}) \) and [RSÖ19, Lemma 4.16 and 4.17] for \( E^2_{-n+1,j,-n}(1) = E^\infty_{-n+1,h,-n}(1) \). The statement on the kernel of \( \pi_{1+(n)}1 \to \pi_{1+(n)}\text{kq} \) follows from [RSÖ19, Theorem 5.5], which applies to \( f_0 \text{KQ} \) instead of \( \text{kq} \). Finally, for \( n > 1 \), \( \pi_{1+(n)}\text{kq} = \pi_{1+(n)}\text{KQ} = W^{2n-1}(F) = 0 \), although it follows as well from the presentation of \( \pi_{1+1}1 \text{f}_1 \text{KQ} \) given in Lemma 2.30.

At least after inverting the exponential characteristic \( c \) of the base field, the kernel of \( \pi_{1+(*)}1 \to \pi_{1+(*)}\text{kq} \) is generated as a \text{K}^{\text{MW}}\text{-module} by the element \( \nu \in \pi_{1+(2)}1 \). An unstable representative of \( \nu \), already defined over the integers, is obtained as the Hopf construction on \( \text{SL}_2 \simeq \mathbb{A}^2 \setminus \{0\} \). Another description of this representative is the attaching map \( \nu: S^7,4 \simeq A^4 \setminus \{0\} \to \text{HP}^1 \simeq S^{4,2} \) for the quaternionic plane, again defined over any base scheme. The equation \( \eta \nu = 0 \) from [DI13, Theorem 1.4] (which follows from Theorem 4.1 as well) implies that the kernel of \( \pi_{1+(*)}1 \to \pi_{1+(*)}\text{kq} \) is in fact a \text{K}^{\text{M}}\text{-module.}

### 5 The 2-line

The obvious three elements in \( \pi_2 1 \) are
\[
\nu^2 \in \pi_{2+(4)}1, \quad \nu \eta \top \in \pi_{2+(2)}1, \quad \eta^2 \top \in \pi_{2+(0)}1.
\]

Even the first one is 2-torsion by the following statement.

**Lemma 5.1:** Over any base scheme we have \((1 - \varepsilon) \nu^2 = 2 \nu^2 = 0 \).

**Proof.** The first equality follows by \( \varepsilon \nu = -\nu [\text{DI13, Corollary 1.5}] \). The commutative diagram
\[
\begin{array}{ccc}
S^7,4 \land S^7,4 & \xrightarrow{\text{twist} = (-1)^3 e^4 = -1} & S^7,4 \land S^7,4 \\
\downarrow \nu \land \nu & & \downarrow \nu \land \nu \\
S^{4,2} \land S^{4,2} & \xrightarrow{\text{twist} = (-1)^2 e^4 = 1} & S^{4,2} \land S^{4,2}
\end{array}
\]
shows that \( \nu^2 = -\nu^2 \), which implies the second equation. \( \square \)
The passage to \( \pi_2 \mathbf{1} \) via the slice spectral sequence requires a discussion of the exact sequence
\[
\cdots \to \pi_3 \mathbf{1}_q^\wedge \left[ \frac{1}{q} \right] \to \pi_2 \mathbf{1} \to \pi_2 \mathbf{1}_q^\wedge \to 0
\]
of \( \mathbf{K}_{\mathcal{MW}} \)-modules given in Section 3. The equality \( \pi_2 \mathbf{1} = \pi_2 \mathbf{1}_q^\wedge \) is equivalent to the surjectivity of the map \( \pi_3 \mathbf{1}_q^\wedge \oplus \pi_3 \mathbf{1}_q \left[ \frac{1}{q} \right] \to \pi_3 \mathbf{1}_q^{\wedge} \left[ \frac{1}{q} \right] \) and the latter \( \mathbf{K}_{\mathcal{MW}} \)-module can be understood completely via the slice spectral sequence.

**Lemma 5.2:** For \( n \geq 5 \), multiplication with the first Hopf map \( \eta \) induces a surjection \( \pi_{3+(n)} \mathbf{1}_q^\wedge \to \pi_{3+(n+1)} \mathbf{1}_q^\wedge \). It is an isomorphism when \( n \geq 7 \).

**Proof.** The slice spectral sequence, the weak equivalence \( \mathcal{S}(\mathbf{1}) \simeq \mathbf{1}_q^\wedge \), and the data in [Zah72, Proposition 6.1] and [Rav86, Tables A.3.3, A.3.4] to identify the relevant slices, imply this result. Here are the details.

Recall that \( s_0(\mathbf{1}) \equiv \mathbf{M} \mathcal{Z} \) is generated by \( \alpha_1^0 \). For \( q \geq 1 \), the \( q \)-dimensional part of \( s_q(\mathbf{1}) \) is \( \Sigma^q \mathbf{M} \mathcal{Z}/2 \) on the generator \( \alpha_1^q \). For \( q \geq 0 \) and \( q \neq 2 \), the \( q+1 \)-dimensional part of \( s_q(\mathbf{1}) \) is trivial. We note that \( s_2(\mathbf{1}) \) contains \( \Sigma^3 \mathbf{M} \mathcal{Z}/12 \) as a direct summand. For \( q \geq 3 \) and \( q \neq 4 \), the \( q+2 \)-dimensional part of \( s_q(\mathbf{1}) \) is \( \Sigma^{q+2} \mathbf{M} \mathcal{Z}/2 \) generated by \( \alpha_1^{q-3} \alpha_3 \). We note that \( s_4(\mathbf{1}) \) has an additional direct summand of dimension \( 4+2 \); it is \( \Sigma^6 \mathbf{M} \mathcal{Z}/2 \) generated by \( \beta_2^2 \). For \( q \geq 7 \), the \( q+3 \)-dimensional part of \( s_q(\mathbf{1}) \) is \( \Sigma^{q+3} \mathbf{M} \mathcal{Z}/2 \) generated by \( \alpha_1^{q-4} \alpha_4 \). The \( 7 \)-dimensional part \( \Sigma^7 \mathbf{M} \mathcal{Z}/240 \) of \( s_7(\mathbf{1}) \) is generated by \( \alpha_1^0 \alpha_4 \). The \( 8 \)-dimensional part of \( s_8(\mathbf{1}) \) is \( \Sigma^8 \mathbf{M} \mathcal{Z}/2 \{ \alpha_1^1 \alpha_4 \} \cup \Sigma^8 \mathbf{M} \mathcal{Z}/2 \{ \beta_2 \} \). The \( 9 \)-dimensional part of \( s_9(\mathbf{1}) \) is \( \Sigma^9 \mathbf{M} \mathcal{Z}/2 \{ \alpha_1^2 \alpha_4 \} \cup \Sigma^9 \mathbf{M} \mathcal{Z}/2 \{ \alpha_1 \beta_2 \} \). For \( q \geq 5 \) and \( q \neq 6 \), the \( q+4 \)-dimensional part of \( s_q(\mathbf{1}) \) is \( \Sigma^{q+4} \mathbf{M} \mathcal{Z}/2 \) generated by \( \alpha_1^{q-5} \alpha_5 \). Finally, the \( 10 \)-dimensional part of \( s_{10}(\mathbf{1}) \) is \( \Sigma^{10} \mathbf{M} \mathcal{Z}/2 \{ \alpha_1^3 \alpha_5 \} \cup \Sigma^{10} \mathbf{M} \mathcal{Z}/3 \{ \beta_1 \} \).

For \( q \geq 5 \), the second column of the \( E^2 \)-page of the \( q \)-th slice spectral sequence for \( \mathbf{1} \) contains only zeros. Hence all elements in the third column of the corresponding \( E^2 \)-page are permanent cycles. The description of the generators of the slice summands shows that multiplication by \( \eta \) – which is represented by \( \alpha_1 \) – induces an isomorphism between the third column of the \( q \)-th and the \( q+1 \)-th slice spectral sequence, provided \( q \geq 7 \). For \( q \in \{5, 6\} \) it is surjective. Since the same is true for the fourth column, and since the differentials are \( \eta \)-linear, there results in an isomorphism resp. surjection between the third column of the infinite terms in the \( q \)-th and the \( q+1 \)-th slice spectral sequences.

**Lemma 5.3:** If \( \operatorname{char}(\mathcal{F}) \neq 2 \) then \( \pi_{3+(n)} \mathbf{1}_q^\wedge \to \pi_{3+(n+1)} \mathbf{1}_q^\wedge \) is surjective for \( n \geq 4 \).

**Proof.** This follows directly from Lemma 5.2 for \( n \geq 5 \). Although multiplication with \( \eta \) is not surjective as a map \( \pi_{3+(4)} \mathbf{1}_q^\wedge \to \pi_{3+(5)} \mathbf{1}_q^\wedge \), an inspection of the slices described in detail in the proof of Lemma 5.2 shows that multiplication with \( \eta^3 \) induces a surjection
\[
\pi_{3+(4)} \mathbf{1}_q^\wedge \to \pi_{3+(7)} \mathbf{1}_q^\wedge.
\]

The following statement, which generalizes Lemma 5.3, uses the fundamental ideal \( \mathbf{I} \subset \mathbf{W} \) in the Witt ring of \( \mathcal{F} \), and its powers \( \mathbf{I}^\ell \subset \mathbf{W} \) which by definition coincide with \( \mathbf{W} \) for \( \ell \leq 0 \).
Lemma 5.4: For $n \in \mathbb{Z}$, the cokernel of
\[ \pi_{3+(n)} \mathbb{1}_{\eta}^{\wedge} \to \pi_{3+(n)} \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}] \]
is a quotient of $\mathbb{W}/I^{-(n-4)}$.

Proof. If $n \geq 7$, then the column on the $E^2$-page of the $n$-th slice spectral sequence computing $\pi_3 \mathbb{1}_{\eta}^{\wedge}$ is independent of $n$ by the proof of Lemma 5.2. It consists of the groups $h_{0,0}, h_{1,1}, \ldots$, where $h_{q,a}$ is generated by $\alpha_{q+a}^{3-n} \alpha_3$. By Milnor’s conjecture on quadratic forms these groups form the associated graded of the $I$-adic filtration on $\mathbb{W}$. The multiplicative structure on the generators imply that for $n \geq 7$, $\pi_{3+(n)} \mathbb{1}_{\eta}^{\wedge}$ is a quotient of the $I$-adic completion $\mathbb{W}_I^{\wedge}$. The same holds thus for $\pi_{3+(n)} \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}]$, which is independent of $n \in \mathbb{Z}$, by Lemma 5.2. This quotient is the zero quotient for $n \geq 4$ by Lemma 5.3.

If $n \leq 4$ there are no motivic cohomology groups of weight $< 4 - n$ in the column on the $E^2$-page of the $n + 4$-th slice spectral sequence computing $\pi_3 \mathbb{1}_{\eta}^{\wedge}$. That is, by the proof of Lemma 5.2, multiplication with $\alpha_3^{3-n}$ hits $h_{-n+4,-n+4}, h_{-n+5,-n+5}, \ldots$, but not $h_{0,0}, h_{1,1}, \ldots, h_{-n+3,-n+3}$. Thus the cokernel of
\[ \pi_{3+(n)} \mathbb{1}_{\eta}^{\wedge} \xrightarrow{\eta^{7-n}} \pi_{3+(7)} \mathbb{1}_{\eta}^{\wedge} \cong \pi_{3+(n)} \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}] \]
is a quotient of $\mathbb{W}/I^{-(n-4)}$. \qed

Lemma 5.4 implies that the cokernel of
\[ \pi_3 \mathbb{1}_{\eta}^{\wedge} \to \pi_3 \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}] \]
is generated, as a $\mathbb{K}^{MW}$-module (in fact even a $\mathbb{K}^{MW}[\eta^{-1}] = \mathbb{W}[\eta, \eta^{-1}]$-module), by a single element. What is it?

Lemma 5.5: The image of the third Hopf map $\sigma \in \pi_{3+(4)} \mathbb{1}$ generates the cokernel of
\[ \pi_3 \mathbb{1}_{\eta}^{\wedge} \to \pi_3 \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}] \]
as a $\mathbb{K}^{MW}$-module.

Proof. The third Hopf map $\sigma \in \pi_{3+(4)} \mathbb{1}$ lifts (uniquely) to $\pi_{3+(4)} f_{4} \mathbb{1}$ for weight reasons. This lift maps to a generator in the cyclic group $\pi_{3+(4)} s_{4} \mathbb{1}$ and hence by constitutes by Lemma 5.3 a generator of $\pi_3 \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}]$, hence also of its cokernel. \qed

As a consequence we can improve on Lemma 5.3

Lemma 5.6: The canonical map
\[ \pi_3 \mathbb{1}_{\eta}^{\wedge} \oplus \pi_3 \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}] \to \pi_3 \mathbb{1}_{\eta}^{\wedge}[\frac{1}{\eta}] \]
is surjective for all $n \in \mathbb{Z}$.
Proof. Lemma \ref{5.5} says that the cokernel of the map \( \pi_3\mathcal{B}_n^1 \to \pi_3\mathcal{B}_n^{1\frac{1}{n}} \) is generated by the image of \( \sigma \), which in turn is in the image of \( \pi_3\mathcal{B}_n^{1\frac{1}{2}} \to \pi_3\mathcal{B}_n^{1\frac{1}{n}} \).

**Theorem 5.7:** If \( \text{char}(F) \neq 2 \) the canonical map
\[
\pi_{n+2,n}1 \to \pi_{n+2,n}1^\wedge
\]
is an isomorphism for all \( n \in \mathbb{Z} \).

Proof. Lemma \ref{5.6} implies the map \( \pi_{n+2,n}1 \to \pi_{n+2,n}1^\wedge \oplus \pi_{n+2,n}1^{\frac{1}{n}} \) from the \( \eta \)-arithmetic square is injective. The group \( \pi_{n+2,n}1^{\frac{1}{n}} \) is trivial by [Rön18, Theorem 8.3].

The Twoline Convergence Theorem \ref{5.7} provides that the slice spectral sequence for \( 1 \) computes the 2-line of \( 1 \). Proposition \ref{2.5} implies the same for \( kq \). Hence the kernel of \( \pi_{2+}\mathfrak{g} \to \pi_{2+}\mathfrak{g} kq \) can be read off, up to extensions, from the kernel of the map
\[
E^2_{n+2,*,n}(1) = E^\infty_{n+2,*,n}(1) \to E^\infty_{n+2,*,n}(kq) = E^2_{n+2,*,n}(kq),
\]
described in Corollaries \ref{3.24} \( (*) = 4 \), \ref{3.25} \( (*) = 3 \) and \ref{3.26} \( (*) = 2 \). It remains to clarify extensions, which a priori are \( \mathbb{K} \text{MW} \)-module extensions by varying \( n \). The crucial contributions to the kernel of \( \pi_{2+}\mathfrak{g} \to \pi_{2+}\mathfrak{g} kq \) occur in the fourth, third and second filtration, as Lemma \ref{3.28} already indicates.

**Theorem 5.8:** If \( F \) is a field of exponential characteristic \( c \neq 2 \), the kernel of
\[
\pi_{2+}(\mathfrak{g}) \to \pi_{2+}(\mathfrak{g}) kq
\]
is isomorphic to the \( \mathbb{K} \text{MW} \)-module \( \pi^\mathbb{M}(4) \oplus (\pi_1\Sigma(2)\mathbb{M}\mathbb{Z})/24 \) after inverting \( c \). The (unique) generator for \( \pi^\mathbb{M}(4) \) is \( \nu^2 \in \pi_{2+}(1) \).

Proof. Lemma \ref{2.33} and Lemma \ref{3.19} describe the \( \mathbb{K} \text{MW} \)-modules \( \pi_{2\mathfrak{g}} kq \) and \( \pi_{2\mathfrak{g}} 1 \), respectively. The behaviour of the unit map on slices, as given in Lemma \ref{2.9} and Lemma \ref{2.10}, provides the commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K}^M(4) & \oplus & (\pi_1\Sigma(2)\mathbb{M}\mathbb{Z}/24) & \oplus & \mathbb{K}^M & \longrightarrow & \pi_{2\mathfrak{g}} 1 & \longrightarrow & \mathbb{K}^M & \longrightarrow & 0 \\
& & \downarrow \delta & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\
0 & \longrightarrow & (\pi_0\Sigma(2)\mathbb{M}\mathbb{Z}) & \oplus & \mathbb{K}^M & \longrightarrow & \pi_{2\mathfrak{g}} kq & \longrightarrow & \mathbb{K}^M & \longrightarrow & 0
\end{array}
\]
where the map \( \delta \) is the composition
\[
\begin{array}{c}
\mathbb{K}^M(4) & \oplus & (\pi_1\Sigma(2)\mathbb{M}\mathbb{Z}/24) & \oplus & \mathbb{K}^M \\
\downarrow \text{pr} & & \downarrow \text{pr} & & \downarrow \text{pr} \\
(\pi_1\Sigma(2)\mathbb{M}\mathbb{Z}/24) & \oplus & \mathbb{K}^M \\
\downarrow \partial_{24} & & \downarrow \partial_{24} & & \downarrow \partial_{24} \\
(\pi_0\Sigma(2)\mathbb{M}\mathbb{Z}) & \oplus & \mathbb{K}^M
\end{array}
\]

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The snake lemma implies that the kernel of $\pi_2 f_2 \mathbf{1} \to \pi_2 f_2 k\mathbb{Q}$ coincides with the kernel of $\delta$. Since the sequence

$$0 \to H^{n+1,n+2}/24 \to h^{n+1,n+2}_{24} \to H^{n+2,n+2}$$

is exact, another application of the snake lemma shows that the kernel of $\delta$ is thus the $K^M$-module $k^M(4) \oplus (\pi_1 \Sigma^2 M\mathbb{Z})/24$. The kernel of $\pi_2 f_2 \mathbf{1} \to \pi_2 f_2 k\mathbb{Q}$ coincides with the kernel of $\pi_2 \mathbf{1} \to \pi_2 k\mathbb{Q}$ because of Lemma 2.7 and Lemma 2.8.

It is worthwhile to revisit the three obvious elements listed at the beginning of this section. The element $\nu^2 \in \pi_2(\mathbf{1})$ figured prominently in Theorem 5.8 already. The element $\eta_{\text{top}} \nu \in \pi_2(\mathbf{1})$ lifts by construction to $\pi_2 f_3 \mathbf{1} \cong k^M_2$, whence there exists a unique element $\varphi \in k^M_2$ with $\eta_{\text{top}} \nu = \varphi \nu^2$. Since these elements are defined over Spec($\mathbb{Z}$), base change implies that $\varphi = \rho^2$, giving us the equation

$$\eta_{\text{top}} \nu = \rho^2 \nu^2 \in \pi_2(\mathbf{1})$$

over every field of characteristic not two. The element $\nu^2 \in \pi_2(\mathbf{1})$ lifts to an element in $\pi_2 f_2 \mathbf{1}$, as described in Lemma 3.19 by construction. This element has the property that $\eta_{\text{top}}^2 \in \pi_2 f_2 (\mathbf{1})$ is the unique nonzero element of order 2 in $h^{0,1}_{24}$. Moreover, $\nu^2 \eta_{\text{top}} = 0$. The group $h^{0,1}_{24} = \mu_{24}$ of 24-th roots of unity contributing to $\pi_2 f_2 (\mathbf{1})$ has the following property.

**Lemma 5.9:** Complex realization induces an isomorphism $\pi_2 f_2 (\mathbf{1})_\mathbb{C} \cong \pi_3 \mathbb{S}$.

**Proof.** By [Lev13] the slice spectral sequence for $\mathbf{1}_\mathbb{C} = \mathbf{1}$ is strongly convergent. The nontrivial groups on the $E^1$-page contributing to $\pi_2 f_2 (\mathbf{1})$ are $\pi_2 f_2 s_2 (\mathbf{1}) = h^{0,1}_{12}$ and $\pi_2 f_2 s_3 (\mathbf{1}) = h^{0,2} = \pi_2 f_2 s_3 (\mathbf{1})$ generated by $\eta_{\text{top}}^2$. Since $\pi_3 f_2 s_2 (\mathbf{1}) = \pi_1 f_3 s_2 (\mathbf{1}) = 0$, there is a short exact sequence

$$0 \to \pi_2 f_2 s_3 (\mathbf{1}) \to \pi_2 f_2 s_2 (\mathbf{1}) \to \pi_2 f_2 s_2 (\mathbf{1}) \to 0.$$ 

The main result in [Lev15] shows its complex Betti realization is the short exact sequence

$$0 \to \mathbb{Z}/2 \to \pi_3 \mathbb{S} \to \mathbb{Z}/12 \to 0$$

obtained from the Adams-Novikov filtration. The map $\pi_2 f_2 s_3 (\mathbf{1}) \to \mathbb{Z}/2$ is an isomorphism since the complex Betti realization of $\eta_{\text{top}}^2$ is $\eta_{\text{top}}^2$. It remains to show that $\pi_2 f_2 s_2 (\mathbf{1}) \to \mathbb{Z}/12$ is an isomorphism. This follows from Lemma 5.11 below.

**Remark 5.10:** More generally than Lemma 5.9, the complex Betti realization induces an isomorphism $\pi_2(\mathbf{1})_\mathbb{C} \cong \pi_2(\mathbf{1}) \mathbb{S}$ for $-1 \leq n \leq 4$. For $n = 4$, the generator $\nu^2$ maps to $\nu_{\text{top}}^2$. The groups $\pi_2 f_2 s_3 (\mathbf{1})_\mathbb{C}$ and $\pi_2 f_2 s_2 (\mathbf{1})_\mathbb{C}$ are trivial by Theorem 1.2, the vanishing of $\pi_2 f_2 k\mathbb{Q} = \pi_2 f_2 K\mathbb{Q}$, and since $\pi_2 f_2 k\mathbb{Q} = \pi_2 f_2 K\mathbb{Q}$ is the group of even integers. For $n = 0$, $\pi_2 f_2 \mathbf{1}_\mathbb{C} \to \pi_2 f_2 k\mathbb{Q}_\mathbb{C}$ is injective (both $H^1,2(\mathbb{C})$ and $K^M(\mathbb{C})$ are divisible), and the image is $h^{0,2}$ generated by $\eta_{\text{top}}^2$. The case $n = -1$ is quite interesting. The slice

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spectral sequence for $1_C$ contains a single nonzero element $\eta_{top}^2/\eta$ on the column of the $E^1$-page contributing to $\pi_{2(1)}1_C$. It comes from $\pi_{2(1)}S_11 \cong h^{0,2}$, which is isomorphic to $\pi_{2(1)}S_11$. Since $\rho = 0$, and in particular $\rho^2 = 0$, the element $\eta_{top}^2/\eta$ is a permanent cycle, both for $1$ and $kq$. The multiplicative structure on the slices of $1$ or $kq$ shows that $\eta \cdot \eta_{top}^2/\eta$ coincides with the unique element in $\pi_{2+0}S_11 \cong h^{0,2}$ detecting $\eta_{top}^2$, whence the awkward name for this element. Consider the commutative diagram

$$
\begin{array}{cccccccc}
\cdots \rightarrow & \pi_{3-0}kgl_C & \rightarrow & \pi_{2(-1)}kq_C & \rightarrow & \pi_{2-0}kq_C & \rightarrow & \pi_{2-0}kgl_C & \rightarrow & \cdots \\
& b_4 & & b_3 & & b_2 & & b_1 & \\
\cdots \rightarrow & \pi_2ku & \rightarrow & \pi_1ko & \rightarrow & \eta_{top} & \rightarrow & \pi_2ko & \rightarrow & \pi_2ku & \rightarrow & \cdots \\
\end{array}
$$

(5.2)

of long exact sequences, where Theorem 2.1 provides the top sequence, and the vertical maps are given by complex Betti realization. The map labelled “$\eta_{top}$” is an isomorphism. The map $b_2: \pi_{2(0)}kq_C \cong K_2^M(C) \oplus K_0^M \rightarrow \pi_2ko$ is surjective, with the nonzero element in $k_0^M$ given by the image of $\eta_{top}^2$ realizing to $\eta_{top}^2$. Since this element is mapped to $0 \in \pi_{2+0}kgl_C$, via the forgetful map, it is in the image of multiplication with $\eta$. Hence also the map labelled “$b_3$” is surjective, implying the same for $\pi_{2(1)}1_C \rightarrow \pi_1S \cong \pi_1ko$. Complex Betti realization $\pi_{2(1)}1_C \rightarrow \pi_1S$ is the zero map.

**Lemma 5.11:** Complex Betti realization induces an isomorphism

$$
\pi_{1(1)}MZ/n \cong \pi_0HZ/n \cong Z/nZ
$$

of groups.

**Proof.** The complex Betti realization of $MZ/n$ is $HZ/n$ by [Lev14, Proposition 5.10]. A primitive $n$-th root of unity $\xi \in \mathbb{C}^\times$ provides a stable map $\xi: S^{1(1)} \rightarrow MZ/n$ or alternatively an unstable map $\xi: Spec(C)_+ \rightarrow \mu_n$, with target the fiber of multiplication by $n$ on $BG_m = P^\infty$ (the first space in the $P^1$-spectrum $MZ$). The complex Betti realization of $Spec(C)_+ \rightarrow \mu_n$ is the choice of a generator of $Z/n$. Hence it sends $\xi: S^{1+(-1)} \rightarrow MZ/n$ to the unit map $S \rightarrow HZ/n$. The result follows now from [Lev15, Corollary 5.12] after multiplication with $\xi$. 

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**A Modules over Milnor $K$-Theory**

The following result from [RSO19] permeates the differential calculations.

**Lemma A.1:** The $r$th slice differentials induce a $K^n = \bigoplus_{n \in \mathbb{N}} H^{n,n}$-module map

$$
\bigoplus_{n \in \mathbb{Z}} d^r_{p+n,q,n}(E): \bigoplus_{n \in \mathbb{Z}} E^r_{p+n,q,n}(E) \rightarrow \bigoplus_{n \in \mathbb{Z}} E^r_{p-1+n,q+r,n}(E).
$$

The triviality of the higher slice differentials for the relevant column follows from arguing with $K^n$-module generators. An important example is the $K^n$-module of mod-2
Milnor $K$-theory

$$k^M = \bigoplus_{n \in \mathbb{Z}} h^{n,n}.$$  

It is generated by the nontrivial element $1 \in h^{0,0}$. More generally, Voevodsky’s solution of the Milnor conjecture implies that for a fixed integer $k \geq 0$, the $K^M$-module

$$\bigoplus_{n \in \mathbb{Z}} h^{n-k,n}$$

is generated by the unique nontrivial element $\tau^k \in h^{0,k}$ in bidegree $(0,k)$. Other examples of generators for $K^M$-modules are given by [OVV07, Theorem 3.3] for fields of characteristic zero and [MS10, Theorem 2.1] for fields of odd characteristic. A central example is the kernel of $\text{Sq}^2: h^{n-2,n} \to h^{n,n+1}$, since it corresponds to multiplication with the pure symbol $\{-1,-1\}$.

**Lemma A.2:** Let $\alpha \in \pi_{s+(w)}1$, and $E$ a motivic spectrum. The cofiber sequence

$$\Sigma^{s+(w)}E \xrightarrow{\alpha} E \to E/\alpha \to \Sigma^{s+1+(w)}E$$

provides a short exact sequence

$$0 \to (\pi_{m+(*)}E)/\alpha \to \pi_{m+(*)}E/\alpha \to \alpha \pi_{m-s-1+(*)}E$$

of $K^M_W$-modules for every $m \in \mathbb{Z}$.

**Proof.** This claim follows directly from the definition of $E/\alpha$. \hfill $\Box$

**Lemma A.3:** Let $n > 0$ be a natural number. The canonical maps induce a short exact sequence

$$0 \to (\pi_{1+(*)}\mathbb{Z})/2^n \to \pi_{1+(*)}\mathbb{Z}/2^n \to 2^n K^M \to 0$$

of $K^M$-modules.

**Proof.** This result is a particular case of Lemma A.2. \hfill $\Box$

Specializing even further, let us consider mod-4 coefficients. Let $F_0$ denote the prime field of $F$.

**Lemma A.4:** For $k \geq 0$ a fixed natural number, consider the $K^M$-module

$$\bigoplus_{n \in \mathbb{Z}} h_{4}^{n-k,n}.$$  

If $k = 2n$ is even, it is generated by a single element $\theta^n \in h_{4}^{0,k} = \mathbb{Z}/4$ where $\theta \in h_{4}^{0,2}$. The element $\theta$ is defined over $F_0$. If $k$ is odd, it is generated by elements in bidegrees $(0,k)$ and $(1,k+1)$; every generator in bidegree $(0,k)$ is defined over $F_0$ or $F_0[\sqrt{-1}]$, whereas for every generator $\overline{g} \in h_{4}^{1,k+1}(F)$ there exists a unit $g \in F^\times$ such that $\overline{g}$ is defined over the field $F_0(g)$.
Proof. The boundary map in motivic cohomology for the change of coefficients exact sequence

\[ 0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0 \]

is the motivic Steenrod operation \( \text{Sq}^1 : h^{n-k,n} \to h^{n-k+1,n} \). This map is trivial if \( k \) is even, and if \( k \) is odd it exchanges one factor of \( \tau \) for \( \rho \).

Assuming \( k \) is even there is a short exact sequence

\[ 0 \to h^{n-k,n}/\text{Sq}^1 h^{n-k-1,n} \to h^{n-k,n}_4 \to h^{n-k,n} \to 0. \]

Both of the outer modules are generated by a single element in bidegree \((0,k)\), and there is a short exact sequence

\[ 0 \to \mathbb{Z}/2 = h^{0,k}_0 \to h^{0,k}_4 \to h^{0,k}_0 = \mathbb{Z}/2 \to 0. \]

Comparison with the case \( k = 2 \) shows this is a non-split extension over any field. The above proves the statement for \( k \) even.

Assuming \( k \) is odd, there is a short exact sequence

\[ 0 \to h^{n-k,n} \to h^{n-k,n}_4 \to \ker(\text{Sq}^1 : h^{n-k,n} \to h^{n-k+1,n}) \to 0. \]

As modules, \( h^{n-k,n} \) is generated in bidegree \((0,k)\) and \( \ker(\text{Sq}^1 : h^{n-k,n} \to h^{n-k+1,n}) \) is generated in \((0,k)\) or in \((1,k+1)\). More precisely, if \( \rho = 0 \) over \( F \), it is generated in bidegree \((0,k)\), and the generator already lives over the smallest subfield of \( F \) containing \( \sqrt{-1} \). If \( \rho \neq 0 \) over \( F \), there exists a set of units \( G \subset F \) such that for \( g \in G \), \( \text{Sq}^1(\tau^k g) = \tau^{k-1} \rho g = 0 \) and every element in \( \ker(\text{Sq}^1 : h^{n-k,n} \to h^{n-k+1,n}) \) is a finite \( \mathbf{K}^M \)-linear combination of elements \( \tau^k g \), for \( g \in G \). Finally, in the \( \mathbf{K}^M \)-theory long exact sequence for the field extension \( F \subset F[\sqrt{-1}] \) every \( g \) is the image of the transfer map of some \( h, \ \text{h} \in F[\sqrt{-1}] \). The field \( F_0(g') \), where \( g' \) is the transfer of \( h \), has the property that \( \tau^k g' = \tau^k g \) over \( F \) and \( \text{Sq}^1(\tau^k g') = 0. \)

Next, we consider mod-8 coefficients. Let \( F_0 \) denote the prime field of \( F \).

Lemma A.5: For \( k \geq 0 \) a fixed natural number, consider the \( \mathbf{K}^M \)-module

\[ \bigoplus_{n \in \mathbb{Z}} h^{n-k,n}_8. \]

If \( k = 2n \) is even, it is generated by a single element \( \omega^n \in h^{0,k}_8 = \mathbb{Z}/8 \), where \( \omega \in h^{0,2}_8 \) is defined over \( F_0 \).

Proof. The change of coefficients exact sequence

\[ 0 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \to \mathbb{Z}/2 \to 0 \]

induces a long exact sequence of motivic cohomology groups

\[ 0 \to h^{0,k}_4 \to h^{0,k}_8 \to h^{0,k}_0 \to h^{1,k}_4 \to \cdots. \]
Assuming \( k \) is even, Lemma A.4 implies \( h_{4}^{0,k} \cong \mathbb{Z}/4 \). Hence \( h_{8}^{0,k} \) has order at most 8, and order precisely 8 if \( h_{8}^{0,k} \to h_{4}^{1,k} \) is trivial. It suffices to prove this map is trivial over \( F_0 \). For \( k = 2 \) note that \( h_{8}^{0,2} \), the kernel of multiplication by 8 on \( H^{1,2} \), is \( \mathbb{Z}/8 \) when \( \text{char}(F_0) \neq 2 \). The group \( h_{4}^{0,2} \), i.e., the kernel of multiplication by 4 on \( H^{1,2} \), is \( \mathbb{Z}/4 \) when \( \text{char}(F_0) \neq 2 \). The projection map \( h_{8}^{0,2} \to h_{4}^{0,2} \) is induced by multiplication by 2 on \( H^{1,2} \), and hence surjective over prime fields. For \( k > 2 \) even, using the commutative diagram
\[
\begin{array}{ccc}
h_{8}^{0,2} \times h_{8}^{0,k-2} & \xrightarrow{\text{pr}_4} & h_{8}^{0,k-2} \\
\downarrow \text{mult} & & \downarrow \text{mult} \\
h_{4}^{0,2} & \rightarrow & h_{4}^{0,2}
\end{array}
\]
we conclude the claimed surjection by induction and Lemma A.4. It follows that \( \partial^2_4 h_{4}^{0,k} = \{0\} \). To prove \( \partial^2_4 h_{4}^{n-k,n} = \{0\} \) for all \( n > k \), let \( \tau^k x \in h_{4}^{n-k,n} \), with \( x \in h_{4}^{n-k,n-k} \). We have
\[
\partial^2_4 (\tau^k x) = \partial^2_4 (\tau^k) \cdot x = 0 \cdot x = 0,
\]
since \( \partial^2_4 \) is a \( K^M \)-module map. Thus there is a short exact sequence of \( K^M \)-modules
\[
0 \to \bigoplus_{n \in \mathbb{Z}} h_{8}^{n-k,n} / \partial^2_4 h_{8}^{n-k-1,n} \to \bigoplus_{n \in \mathbb{Z}} h_{8}^{n-k,n} \to \bigoplus_{n \in \mathbb{Z}} h_{4}^{n-k,n} \to 0
\]
with outer terms generated in bidegree \((0,k)\). Hence so is the middle term, concluding the case where \( k \) is even.

**Corollary A.6:** Let \( k \) be an even natural number. Then \( \partial^2_4 : h_{4}^{n-k,n} \to h_{4}^{n-k+1,n} \) is the zero map.

**Proof.** This follows as in the proof of Lemma A.5 using a generator in \( h_{8}^{0,k} \) from A.4.

**Lemma A.7:** Multiplication with a generator \( \omega \in h_{8}^{0,2} \) induces an isomorphism
\[
\bigoplus_{n \in \mathbb{Z}} h_{8}^{n-k,n} \cong \bigoplus_{n \in \mathbb{Z}} h_{8}^{n-k,n+2}
\]
of \( K^M \)-modules for every natural number \( k \).

**Proof.** First, we show that multiplication with a generator \( \theta \in h_{4}^{0,2} \) induces an isomorphism
\[
\bigoplus_{n \in \mathbb{Z}} h_{4}^{n-k,n} \cong \bigoplus_{n \in \mathbb{Z}} h_{4}^{n-k,n+2}.
\]
In effect, for \( \tau^2 \in h_{8}^{0,2} \) and \( \theta \in h_{4}^{0,2} \), consider the diagram of Bockstein sequences
\[
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\text{Sq}^1} & h_{4}^{m,n+1} & \xrightarrow{\tau^2} & h_{4}^{m,n+1} & \xrightarrow{\theta} & h_{4}^{m,n+1} & \xrightarrow{\text{Sq}^1} & h_{4}^{m+1,n+1} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\text{Sq}^1} & h_{4}^{m,n+3} & \xrightarrow{\tau^2} & h_{4}^{m,n+3} & \xrightarrow{\theta} & h_{4}^{m,n+3} & \xrightarrow{\text{Sq}^1} & h_{4}^{m+1,n+3} & \cdots
\end{array}
\]
Recall \(\mathbb{M}Z/4 \to \mathbb{M}Z/2\) is a map of motivic ring spectra, and the image of \(\theta\) is \(\tau^2\). Whence the middle square commutes. Moreover, for \(\text{Sq}^1: h^{n-k,n} \to h^{n-k+1,n}\) we have

\[
\text{Sq}^1(\tau^k x) = \begin{cases} 
\tau^{k-1} \rho x & k \equiv 1 \mod 2 \\
0 & k \equiv 0 \mod 2,
\end{cases}
\]

which implies the right-hand square commutes. For commutativity of the left-hand square, we use that \(\mathbb{M}Z/2 \to \mathbb{M}Z/4\) is a \(\mathbb{M}Z/4\)-module map. Since multiplication with \(\tau^2\) is an isomorphism, so is multiplication with \(\theta\) according to the five lemma.

The above feeds into the case of \(\mathbb{Z}/8\)-coefficients via the diagram of Bockstein sequences

\[
\begin{array}{cccccccc}
\cdots & \frak{d}_2^4 & h^{m,n+1}_{4} & \rightarrow & h^{m,n+1}_{8} & \rightarrow & h^{m,n+1}_{2} & \frak{d}_4^4 \rightarrow & h^{n+1,n+1}_{4} & \rightarrow & \cdots \\
& \theta & \downarrow \omega & & \tau^2 & \downarrow \theta & & \\
\cdots & \frak{d}_2^4 & h^{m,n+3}_{4} & \rightarrow & h^{m,n+3}_{8} & \rightarrow & h^{m,n+3}_{2} & \frak{d}_4^4 \rightarrow & h^{n+1,n+3}_{4} \rightarrow & \cdots .
\end{array}
\]

Commutativity of this diagram follows for the same reasons as above since the \(\mathbb{K}^M\)-module map \(\frak{d}_2^4: h^{n-k,n} \to h^{n-k+1,n}_{4}\) satisfies

\[\theta \cdot \frak{d}_2^4(\tau^k x) = \frak{d}_4^4(\tau^{k+2} x).\]

Using the five lemma we conclude that multiplication with \(\theta \in h^0_{4.2}\) is an isomorphism.

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