Numerically exact treatment of dissipation in a driven two-level system

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Recent progress in the experimental implementations of controlled quantum systems and especially of tunable dissipation for them provides an ideal starting point for precision studies of open quantum systems and their models. However, typically the standard weak-coupling approximations are very accurate in describing the dynamics of the simplest and most easily experimentally approachable system, the qubit, since it is challenging to engineer dissipation that is strong relative to the qubit frequency while preserving a high level of control. In this paper, we propose to overcome such challenges by studying a driven qubit, for which the weak-coupling approaches may fail at dissipation rates comparable to the Rabi frequency of the drive which is typically much lower than the qubit frequency. Consequently, we observe such failure of the temporally local Lindblad master equation, the induced dynamics of which are compared with a numerically exact method. To this end, we propose a metric that may be used in experiments to verify our predictions. In addition, we study the well-known Mollow triplet and observe its meltdown owing to dissipation in an experimentally feasible parameter regime. Thus we expect our results to inspire future experimental research on open quantum systems and expand the understanding on this intriguing field of physics.

I. INTRODUCTION

Driven quantum systems are ubiquitous in quantum technologies. They appear, for example, in the control and measurement of quantum systems as well as in studies of non-equilibrium dynamics [1, 2]. One of the simplest paradigmatic examples encompasses a two-level quantum system, a qubit, subjected to a classical transverse drive field which promotes population dynamics in the eigenbasis of the bare qubit. Despite of its simplicity, such a model has been applied in many contexts ranging from the coherent control in quantum computing to the simulation of a number of important photochemical reactions [3, 4].

Remarkably, Mollow theoretically showed that the emission spectrum of a driven qubit may turn into a triplet in the presence of weak dissipation [13]. If the Rabi frequency of the classical field well exceeds the dissipation rate, two sidebands emerge in the spectrum with an offset equal to the Rabi frequency from the center peak at the drive frequency. A more sophisticated explanation of such a phenomenon was later provided using a quantum treatment also for the drive field [14]. In this so-called dressed-state picture, the energy levels of the composite qubit-field system are split due to the dynamic Stark effect promoted by the strong drive. The Mollow triplet has been verified experimentally in many different physical scenarios [15–23].

The approach for solving the open-quantum-system dynamics in Mollow’s study [13] and in the follow-up work in Refs. [14] assumes a weak coupling between the system, i.e., the qubit, and its bath of quantized bosonic modes, thus allowing for a perturbative treatment of the dissipation [24, 25]. In such a technique, the influence of the system on the state of the bath due to the interactions is neglected, which is often referred to as the Born approximation. In the case of a thermal bath with a high cutoff frequency, the temporal bath correlation functions decay much faster than the characteristic times involved in the evolution of the open small system [26]. This may allow the use of a master equation for the open system which depends only on the present time and leads to temporal evolution that may be represented as a divisible map. Such a treatment is generally referred to as the Markov approximation. Furthermore, the weak coupling motivates the elimination of fast-oscillating terms arising either in the system-bath interaction Hamiltonian, i.e., the rotating-wave approximation, or in the reduced master equation for the system, i.e., the secular approximation [24, 25].

Under the above-mentioned assumptions, the typically non-unitary evolution of the system is usually expressed by a master equation in the Lindblad form [27] with positive decay, excitation, and dephasing rates, in addition to which the environment introduces a rescaling of the transition frequency of the free system. However, the presence of a time-dependent drive field may rise questions on the validity of such approximations since the interplay between the drive and dissipation is not fully contemplated in the derivation of the Lindblad master equation. Other perturbative strategies have approached such cases differently, for instance, by deriving the master equation in the time-dependent basis of the driven system [28, 29] or by combining the weak-coupling assumptions with the Floquet theory for periodically driven systems [31].

The recent experimental progress in the implementation of tunable and engineered environments, for instance in circuit quantum electrodynamics (cQED) [32–34], has
allowed the exploration of new frontiers which are not well captured by the weak-coupling Markovian dynamics encoded in the Lindblad master equation. Consequently, a precise proposal for experiments and corresponding parameters in such scenarios calls for a model contemplating both the drive–dissipation interplay and high-order corrections to the system–bath correlations.

In this paper, we use the stochastic Liouville equation with dissipation (SLED) [33] to study the dynamics and steady-state properties of a dissipative and driven qubit, beyond all of the above-mentioned weak-coupling approximations. Assuming linear interaction between the system and its thermal bath, such a method has the advantage of being non-perturbative and numerically exact provided that the spectral density of the bath is Ohmic with a high cutoff frequency. Within SLED, time-dependent drive fields can be included without further assumptions. The SLED method has been applied, for instance, in the optimal control of quantum systems [14, 15], and more recently in the initialization of a non-driven superconducting qubit [46].

In order to illustrate the shortcomings of the weak-coupling approximations, we focus on the comparison between the SLED and Lindblad master equations. In particular, we show that the often overlooked Lamb-shift term in the Lindblad master equation may give an important contribution to the dynamics and greatly alter the steady state of the driven qubit, in stark contrast to non-driven systems. Taking advantage of the non-perturbative characteristics of SLED, we also propose a scheme to witness the failure of the Lindblad master equation in predicting the steady state. In addition, we study signatures of the Mollow triplet using SLED in parameter regimes which may be experimentally implemented in the framework of cQED.

This manuscript is organized as follows. In Sec. II we provide the theoretical and numerical models, emphasizing the main differences between the Lindblad and SLED master equations for a driven system. In Sec. III we apply both methods for the case of a monochromatic transverse drive field and show that the overlap fidelity between the SLED and Lindblad solutions is drastically reduced if the Lamb-shift term is not taken into account in the Lindblad master equation. Furthermore, we analyze the steady-state properties of the driven system and describe a protocol to witness the failure of the Lindblad master equation. In Sec. IV we numerically study the meltdown of the Mollow triplet within SLED and Lindblad formalisms for experimentally feasible parameters for a cQED implementation. Our conclusions are drawn in Sec. V.

II. MODEL

Consider a qubit with the bare transition frequency $\omega_q$, driven by a time-dependent transverse field with Hamiltonian $H_d(t)$. The free Hamiltonian of the qubit can be expressed with the help of its eigenbasis $\{|0\rangle, |1\rangle\}$ as $\hat{H}_S = -\hbar \omega_q \hat{\sigma}_z / 2$, where $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, and the drive Hamiltonian as $\hat{H}_d(t) = \hbar f(t) \hat{\sigma}_x$, where $f(t)$ is an arbitrary time-dependent real-valued function and $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$. We choose the qubit to interact linearly with a dissipative bosonic bath which is modeled by an infinite set of quantum harmonic oscillators. The $j$-th oscillator has creation and annihilation operators $\hat{b}_j$ and $\hat{b}_j^\dagger$, respectively, and an angular frequency $\omega_j$. The free Hamiltonian of the bath can thus be written as $\hat{H}_B = \hbar \sum_j \omega_j \hat{b}_j^\dagger \hat{b}_j$, and the system-bath interaction Hamiltonian as $\hat{H}_{SB} = \hbar \delta \sum_j g_j (\hat{b}_j + \hat{b}_j^\dagger)$, with $\{g_j\}$ being coupling strengths. The total Hamiltonian is given by

$$\hat{H}(t) = \hat{H}_S + \hat{H}_d(t) + \hat{H}_B + \hat{H}_{SB}. \quad (1)$$

Conveniently, the system-bath interaction can be fully characterized by the spectral density function

$$J(\omega) = 2\pi \sum_j g_j^2 \delta(\omega - \omega_j), \quad (2)$$

where $\delta(\omega - \omega_j)$ is Dirac’s delta function. In the continuum limit, the spectral density becomes a smooth function of $\omega$ that approaches zero as $\omega \to \infty$. This so-called ultraviolet cutoff is physically motivated in cQED, for example, by the finite bandwidth of the transmission lines coupled to the qubit. Specifically, we use an Ohmic environment with a high cutoff frequency $\omega_c$, characterized by the spectral density

$$J(\omega) = \frac{2 \eta \omega}{(1 + \frac{\omega}{\omega_c})^2}, \quad (3)$$

where $\eta$ is an effective dimensionless coupling constant. This spectral density can capture the relevant physics of a tunable resistor coupled to a superconducting transmon qubit [49].

In this work, we assume that the total density operator is initially factorized, i.e., $\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_B(0)$, with $\hat{\rho}_B(0) = e^{-\beta \hat{H}_B} / \text{Tr}[e^{-\beta \hat{H}_B}]$ being the Gibbs state of the bath at temperature $T = (kB_B)^{-1}$ and with mean excitation number $\bar{n}(\omega) = (e^{\beta \omega} - 1)^{-1}$. Consequently, the internal dynamics of the heat bath can be represented by its autocorrelation function

$$L(\tau) = \langle \hat{\xi}(\tau) \hat{\xi}(0) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega \tau} S(\omega), \quad (4)$$

where $\hat{\xi} = \sum_j g_j (\hat{b}_j + \hat{b}_j^\dagger)$ and $S(\omega) = J(\omega) [\bar{n}(\omega) + 1]$ is the power spectrum of the noise. The behavior of $L(\tau)$ also plays an important role on the reduced dynamics of the qubit, with its real and imaginary parts, denoted hereafter as $L_r(\tau)$ and $L_i(\tau)$, usually being referred to as noise and dissipation kernel, respectively.

Motivated by experiments, for example, in nuclear magnetic resonance [47, 49], a perturbative description...
of the reduced qubit dynamics under the Hamiltonian in Eq. (1) and the choice of $\hat{\rho}(0)$ has been extensively studied in the weak-coupling case, i.e., for $|g_j/\omega_q| \ll 1$ \cite{24, 25, 50, 52}. Such an approach relies on assuming that the system-bath correlations created dynamically are negligible for the dynamics of the system so that in the reduced master equation of the system, we may use the thermal equilibrium state of the environment, i.e., $\hat{\rho}_B(t) \approx \hat{\rho}_B(0)$. In addition, memory effects on the reduced dynamics arising from the finite decay time of $L(\tau)$ are usually neglected. These assumptions constitute the Born-Markov approximations, and along with the elimination of quickly oscillating terms, produce a time-local master equation for the reduced density operator $\hat{\rho}_S(t) = \text{Tr}_B[\hat{\rho}(t)]$ of the qubit which can be cast into a Lindblad form as \cite{24, 25}

$$\frac{d}{dt} \hat{\rho}_S(t) = -\frac{i}{\hbar}[\hat{H}_S + \hat{H}_a + \hat{H}_d(t), \hat{\rho}_S(t)] + \Gamma(\omega_q) D_{01}[\hat{\rho}_S(t)] + \Gamma(-\omega_q) D_{10}[\hat{\rho}_S(t)], \quad (5)$$

where the superoperators $D_{ij}(\hat{\rho}) = |i\rangle\langle j|\hat{\rho}|j\rangle\langle i| - |j\rangle\langle j|\hat{\rho}|i\rangle\langle i|/2$ express incoherent transitions between the eigenstates of the bare system with positive rates $\Gamma(\pm\omega_q) = 2\text{Re}\int_0^\infty d\tau e^\pm i\omega_q\tau L(\tau)$ and $\hat{H}_a = -\hbar\Delta_s\hat{\sigma}_z/2$ is the bath-induced energy shift of the qubit characterized by the correction

$$\Delta_s = 2\int_0^\infty d\tau \sin(\omega_q\tau)L_\tau(\tau), \quad (6)$$

to its bare frequency $\omega_q$. For a non-driven system characterized by $\hat{H}_d(t) = 0$, Eq. (5) describes the thermalization process of the qubit with the heat bath, the superoperator of which commutes with the unitary part of the master equation, and consequently the specific value of the shift $\Delta_s$ does not have an effect on the steady-state quantities. Neglecting the shift for a driven system, however, may lead to incorrect predictions both dynamically and in the steady state as we show in Section III. The assumption of a stationary thermal bath implies the drive field to contribute only to the unitary part of the master equation (see Appendix A).

One can overcome the limitations imposed by the above-mentioned approximations using a non-perturbative treatment of the system-bath coupling that relies on a stochastic unraveling of the reduced density operator $\hat{\rho}_S(t)$. In this procedure, one resorts to the path integral description of quantum mechanics \cite{25} to establish a numerically exact model of the open-quantum-system dynamics with the help of a classical stochastic process \cite{24, 25}. For the Ohmic spectral density in Eq. (2), and in the limit $\omega_c \rightarrow \infty$, a single trajectory for the state of the system, $\hat{\rho}_S(t)$, is given by the so-called stochastic Liouville equation with dissipation, SLED, as

$$\frac{d}{dt} \hat{\rho}_S(t) = -\frac{i}{\hbar}[\hat{H}_S + \hat{H}_d(t) - h\xi(t)\hat{\sigma}_x, \hat{\rho}_S(t)] - \frac{\eta}{\hbar^2} \big[\hat{\sigma}_z, \big[\hat{\sigma}_z, \hat{\rho}_S(t)\big]\big] - i\frac{\omega_q}{2} \big[\hat{\sigma}_z, \{\hat{\sigma}_y, \hat{\rho}_S(t)\}\big], \quad (7)$$

where $\xi(t)$ is defined in Appendix B. By denoting $E[.]$ as the ensemble average over the trajectories, the actual reduced density operator of the system is obtained as $\hat{\rho}_S(t) = E[\hat{\rho}_S(t)]$ in the limit of infinite trajectories. The correspondence between the transition rates in Eq. (5) and Eq. (7) is obtained by writing $\Gamma(\omega_q) = \gamma[\bar{n}(\omega_q) + 1]$ and $\Gamma(-\omega_q) = \gamma\bar{n}(\omega_q)$, with $\gamma = 2\hbar\omega_q a$ being an effective qubit dissipation rate calculated in the limit $\omega_c \gg \omega_q$. Consequently, $\gamma = \gamma/(2\omega_q)$ in Eq. (7).

Below, we study the accuracy of the approximative Lindblad master equation (5) against the numerically exact SLED method which specifically, does not rely on any weak-coupling assumptions. We define the fidelity of a density operator of an approximate solution $\hat{\rho}_1$ against the numerically exact SLED solution $\hat{\rho}_2$ for qubits as \cite{55}

$$F = \text{Tr}[\hat{\rho}_1\hat{\rho}_2] + 2\sqrt{\det(\hat{\rho}_1)\det(\hat{\rho}_2)}. \quad (8)$$

The infidelity, $1 - F$, describes the distance between the two states, and hence provides information on the amount of error introduced by the approximations used in the inexact approaches. In particular, we compare the fidelities of the Lindblad solutions obtained with and without the inclusion of the bath-induced energy shift defined in Eq. (6).

### III. MONOCHROMATIC PERIODIC FIELD

In this section, we compare the Lindblad and SLED approaches for the dissipative dynamics of a driven qubit. Namely, we solve the Lindblad equation (5) using very accurate deterministic numerical integration and compare the results with those of SLED (7) using a sufficiently large number of noise realizations $\xi(t)$ to obtain the reduced system density operator as an average over individual trajectories. As an illustrative example, we study the case in which a qubit is driven by a monochromatic and periodic transverse field at angular frequency $\omega_d$ as

$$f(t) = \Omega_d \cos(\omega_d t), \quad (9)$$

where $\Omega_d$ is referred to as the Rabi frequency. The full Hamiltonian is given by Eq. (1).

In Fig. 1A, we show the dynamics of the Bloch vector components $\sigma_i(t) = \text{Tr}[\sigma_i \rho(t)]$ for $\rho(t)$ in a frame rotating with the drive frequency $\omega_d$ and for a qubit initially prepared in the excited state $|1\rangle$. The drive frequency is set at the bare qubit frequency $\omega_1 = \omega_q$ and the other parameters are chosen in compliance with the current state.
of art of cQED implementations, as shown in Table I. Note that the temperature of the bath is chosen such that $\hbar/\omega_{\text{bath}} = 5$, corresponding to $\bar{n}(\omega_{\text{bath}}) \approx 6.8 \times 10^{-3}$. For a typical superconducting qubit frequency of $\omega_{\text{q}}/(2\pi) = 5.0$ GHz, such temperature corresponds to approximately 48 mK. Moreover, the Rabi frequency is kept fixed at 1% of the qubit frequency, $\Omega_{d}/(2\pi) = 50$ MHz, and the environment cutoff frequency at $\omega_{c}/(2\pi) = 250$ GHz. We simulate the qubit dynamics for a broad range of effective dissipation rates $\gamma$, comprising values of $2\pi \times 2.5$ MHz ($0.05\%$ of $\omega_{\text{q}}$) up to $2\pi \times 500$ MHz ($10\%$ of $\omega_{\text{q}}$); such tunability has been demonstrated in recent protocols for engineered environments [32–37]. For simplicity of comparison between Lindblad and SLED in the present model, we assume that the intrinsic dephasing and decay rates of the qubit are low compared to $\Omega_{d}$ and $\gamma$, such that they have a negligible effect on the qubit dynamics.

In Fig. 1a, we find a very good agreement between the SLED and Lindblad methods if the bath-induced energy shift is included and if we use a relatively small value of $\gamma = 5\omega_{\text{q}} \times 10^{-3}$. Thus the Born-Markov approximations in this weak-coupling regime seem valid. Such an agreement is manifested by the considerably high value of $\bar{F}$ shown in Fig. 1. However, neglecting the bath-induced energy shift alters the dynamics significantly, reducing $\bar{F}$ by approximately 10%. The reduction is even more pronounced in the case $\gamma = \omega_{\text{q}} \times 10^{-2}$, where the dissipation rate and the Rabi frequency are equal in magnitude.

In addition to giving rise to faster stabilization time scales, the progressive increase of dissipation over Rabi frequency ($\gamma/\Omega_{d} > 1$) attenuates the relevance of the bath-induced energy shift in the Lindblad dynamics, as shown in Fig. 1a for $\gamma = \omega_{\text{q}} \times 10^{-3}$, and in Fig. 1b for several different dissipation rates. We attribute this intriguing behavior of the dissipative dynamics of the driven qubit to the amount of coherent superposition between the eigenstates of the bare qubit promoted by the drive, which is increased for a resonant drive but inhibited in the strongly dissipative regime. Thus, for the chosen drive frequency $\omega_{d} = \omega_{\text{q}}$ in Fig. 1, the inclusion of the bath-induced energy shift in the Lindblad master equation renders the drive nonresonant with the actual qubit frequency modified by the bath. This in turn leads to a non-vanishing $x$ component of the Bloch vector in the rotating frame. With increasingly strong environmental coupling however, the amount of coherence promoted by the drive field decreases, even in the resonant case, and the decay towards the thermal steady state is favored. This state commutes with $\sigma_{z}$, and hence is dynamically unaffected by the bath-induced energy shift.

Despite the negligible effect of the bath-induced energy shift on the steady state of the Lindblad equation in the dissipation-dominated regime $\gamma \gg \Omega_{d}$, the overall validity of the Lindblad equation is compromised. Namely, Fig. 1 shows a progressive reduction of the fidelity $\bar{F}$ as a function of increasing $\gamma \gtrsim \Omega_{d}$. Manifestations of this breakdown are studied for the $z$ component of the Bloch vector in Fig. 1c where we observe non-exponential short-time dynamics and a shift in the steady-state values for the SLED method. Such behavior suggests the existence of non-Markovian dynamics and of asymptotic system-bath correlations [36], both being not contemplated by the Born-Markov approximations. Note that owing to the low bath temperature, the effect of the bath-induced energy shift on the thermal populations in SLED solution is expected to be negligible here.

### A. Steady state: bath-induced energy shift and failure of the Lindblad master equation

Let us study in detail the importance of the bath-induced energy shift and the failure of the Born–Markov approximations in the Lindblad master equation [5] by carefully inspecting steady-state properties of the qubit driven by a weak field ($\Omega_{d} \ll \omega_{\text{q}}$). In this case, the drive Hamiltonian can be written in the rotating-wave approximation such that $H_{d}(t) \approx \hbar/\Omega_{d}(\langle 0|e^{i\omega_{d}t} + |1\rangle\langle 1|e^{-i\omega_{d}t})/2$. Consequently, the asymptotic components of the Bloch vector can be found analytically for an arbitrary detuning $\Delta_{q} = \omega_{q} + \Delta_{d} - \omega_{d}$. Denoting these components of the steady-state system density operator $\hat{\rho}_{ss}$ in the rotating frame as $\sigma_{ii}^{ss}(\Delta_{q}) = \text{Tr} [\hat{\sigma}_{i} e^{-i\omega_{d}t/2} \hat{\rho}_{ss} e^{i\omega_{d}t}/2] (i = x, y, z)$, we define the quantity

$$\Delta \sigma_{ii}^{ss} = \sigma_{ii}^{ss}(\Delta_{q}) - \sigma_{ii}^{ss}(0),$$

(10)

to express how a nonresonant drive modifies the steady state of the qubit in comparison to the resonant-drive case $\Delta_{q} = 0$. Analytical expressions for $\sigma_{ii}^{ss}(\Delta_{q})$ are shown in the Appendix [A1] and according to Eq. (10).
one finds

\[
\Delta \sigma_x^{ss} = -\frac{4 \left( \frac{\gamma}{\gamma_D} \right) \left( \frac{\Delta q}{\Omega_D} \right)}{2 + \left( \frac{\gamma}{\gamma_D} \right)^2 + 4 \left( \frac{\Delta q}{\Omega_D} \right)^2},
\]

\[
\Delta \sigma_y^{ss} = \frac{8 \left( \frac{\gamma}{\gamma_D} \right) \left( \frac{\Delta q}{\Omega_D} \right)^2}{2 + \left( \frac{\gamma}{\gamma_D} \right)^2 + 4 \left( \frac{\Delta q}{\Omega_D} \right)^2},
\]

\[
\Delta \sigma_z^{ss} = \frac{8 \left( \frac{\gamma}{\gamma_D} \right) \left( \frac{\Delta q}{\Omega_D} \right)^2}{2 + \left( \frac{\gamma}{\gamma_D} \right)^2 + 4 \left( \frac{\Delta q}{\Omega_D} \right)^2}.
\] \tag{11}

where \(\gamma_D = \gamma [2 \hbar (\omega_q) + 1]\).

In Fig. 2a, we show the dependence of \(\Delta \sigma_x^{ss}\) on the qubit decay rate \(\gamma\) and on the angular frequency detuning of the drive \(\Delta q\) for a low-temperature environment \((\gamma_D \approx \gamma)\). A non-resonant drive on the qubit \((\Delta q \neq 0)\) affects the different components of the Bloch vector individually. The difference in the \(x\) component changes its sign with that of the detuning and is clearly pronounced in the region of moderate dissipation, tending to vanish at \(|\Delta q/\Omega_D| \gg 1\). On the other hand, the detuning barely affects the \(y\) component for \(\gamma/\Omega_D \ll 1\), and in this regime, \(\Delta \sigma_y^{ss}\) saturates to a high value for a sufficiently large \(|\Delta q/\Omega_D|\). Despite the different behaviors, the difference in all components of the Bloch vector decreases and becomes independent of the detuning with increasing \(\gamma/\Omega_D \gg 1\), thus being another manifestation of dissipation dominating over the drive dynamics.

Provided that the drive frequency is set to \(\omega_d = \omega_q\), we have \(\Delta q = \Delta_\alpha\). In this case, Fig. 2a shows the bath induced energy-shift as function of the qubit decay rate. Whereas the perturbative approach of the dissipative dynamics allows one to obtain \(\Delta_\alpha\) directly from Eq. (6), we obtain it for the SLED method by fitting an exponentially damped cosine function to the early decay of the qubit coherence as illustrated in Fig. 2a. For the chosen parameters, the relation between \(\Delta_\alpha\) and \(\gamma\) is well approximated by a linear fit in both methods. As shown in [10], this dependence ceases to be linear for strong enough system-bath coupling strength.

Interestingly, Eqs. (11) along with Fig. 2a show that \(\Delta \sigma_x^{ss}\) and \(\Delta \sigma_z^{ss}\) are symmetric with respect to \(\Delta_\alpha\). Focusing our attention to the \(x\) component, we observe that \(\Delta \sigma_x^{ss} \geq 0\) in Eqs. (11) for any choice of parameters. Based

Figure 1. (a) Dynamics of the Bloch vector components in a frame rotating at \(\omega_d = \omega_q\) as functions of time \(t\) for different indicated values of the dissipation rate \(\gamma\). (b) Fidelity of the Lindblad solution against SLED averaged over an interval \(t \in [0, 10] / \gamma\) as function of \(\gamma\), with (green squares) and without (orange circles) the bath-induced energy shift. (c) Short-time and long-time dynamics of the \(z\) component of the Bloch vector obtained from the right panel of (a). Parameters not given here are chosen as in Table [II].
on this result, one can construct the measure
\[ \Delta \tilde{\sigma}_{ss}^z = \sigma_{ss}^z(\Delta_q) - \sigma_{ss,Li}^z(\gamma), \]  
(12)
where \( \sigma_{ss,Li}^z(\gamma) \) is the steady-state \( z \) component of the Bloch vector given by the Lindblad master equation at resonance and \( \sigma_{ss}^z(\Delta_q) \) is obtained by our method of choice or even experimentally. This measure can be used to identify regimes where the perturbative approach encoded in the Lindblad master equation is not sufficient to correctly predict the steady state of the weakly driven qubit. In an experiment, \( \sigma_{ss,Li}^z(\gamma) \) can be inferred from the characterization of parameters involved in the dynamics and a subsequent analytical calculation of the asymptotic \( z \) component [see Eq. (A4) of Appendix A1]. On the other hand, \( \sigma_{ss}^z(\Delta_q) \) can be obtained through usual steady-state readout of the qubit driven out of resonance. The negativity of \( \Delta \tilde{\sigma}_{ss}^z \) violates the lower bound imposed by Eq. (11), being a sufficient condition for the failure of the time-local Lindblad master equation.

Figure 2c shows \( \Delta \tilde{\sigma}_{ss}^z \) for selected values of \( \gamma/\Omega _d \) and choice \( \Delta_q = \Delta_q \). The values of \( \tilde{\sigma}_{ss}^z(\Delta_q) \) are calculated from long-time solutions of the Lindblad master equation and SLED, which are intended to simulate the dynamics of the qubit in an experiment. The good agreement between Eq. (11) and the numerical results from the Lindblad equation highlights the validity of RWA in the drive Hamiltonian \( \hat{H}_d(t) \). In these cases, as expected from Eq. (11), \( \Delta \tilde{\sigma}_{ss}^z \) is positive for all decay rates and achieves its maximum for \( \gamma \approx \Omega _q \). However, the detuned asymptotic \( z \) component given by SLED produces \( \Delta \tilde{\sigma}_{ss}^z < 0 \) for dissipation rates \( \gamma > 0.05 \times \omega _q \), or in terms of the parameters of Table I, for \( \gamma > 2\pi \times 250 \text{ MHz} \). This is a clear evidence of incompatibility with the used weak-coupling approximations. As pointed out in Ref. [46] for a non-driven qubit, the shift of \( \sigma_z(t) \) given by SLED compared to that by the Lindblad equation cannot be fully
attributed to a bath-induced energy shift since the correlations between the qubit and the bath created during the dynamics contributes as well. By turning on a very weak drive field, one is able to obtain the threshold $\Delta \sigma_z^T = 0$ indicating the conditions, in which such correlations are significant.

**IV. PUMP-PROBE SPECTROSCOPY**

The second example of a driven dissipative system presented in this work consists of a qubit driven by a bichromatic field of the form

$$f(t) = \Omega_\text{d} \cos(\omega_\text{d} t) + \Omega_\text{p} \cos(\omega_\text{p} t + \pi/2),$$

(13)

which is a sum of the monochromatic drive field of Eq. (9), referred to as the primary drive, and a probe field with angular frequency $\omega_\text{p}$ and an associated Rabi angular frequency $\Omega_\text{p}$. Below, we employ both the Lindblad and the SLED formalism to study signatures of the qubit fluorescence spectrum by means of a pump-probe approach [17]. Specifically, assuming $\Omega_\text{p} < \Omega_\text{d}$ so that the probe field acts as a weak perturbation to the driven qubit, information about the spectrum under the primary drive is obtained from the measurement of the field transmitted through a readout resonator coupling.

Within this approach, for $\gamma/\Omega_\text{d} \ll 1$, the fluorescence spectrum of a dissipative qubit that is driven by a resonant field of the form of Eq. (9) in the RWA presents three peaks centered at frequencies $\omega_0 = \omega_\text{d}$ and $\omega_{\pm} = \omega_\text{d} \pm \Omega_\text{d}$ [13]. The sideband peaks have a Lorentzian shape that becomes broadened and flattened as the ratio $\gamma/\Omega_\text{d}$ increases. These features are present in the top and middle panels of Fig. 3, where we fit the data provided by SLED with Lorentzian functions peaked roughly at

$$A = \frac{\Omega_\text{m}}{\kappa} \frac{1}{\sqrt{1 + \left(\frac{2\Delta s}{\kappa}\right)^2}},$$

(15)

where $\Omega_\text{m}$ is the amplitude of a weak measurement drive continuously applied on the input port of the resonator, $\kappa$ is the resonator energy decay rate that is assumed to be dominated by leakage to the output port, and $\gamma$ is the so-called dispersive shift associated to the qubit-resonator coupling.

Figure 3 shows $\bar{s}_z$ and $A$ as functions of the probe frequency for various dissipation rates $\gamma$. Similar to Sec. III, the parameters are chosen according to Table 1 unless otherwise stated. Here, the drive frequency $\Omega_\text{d}$ is adjusted to the resonance with the frequency of the qubit including any bath-induced energy shifts for each $\gamma$. The bath-induced frequency shift is calculated as in Sec. III A that is, through Eq. (6) for the Lindblad master equation and through a fit to the damped decay of the qubit coherence for the SLED.

We show in Fig. 3 that for very weak dissipation there is a good agreement between the two methods for $\bar{s}_z$ and $A$. As $\gamma$ increases, the solutions given by the two approaches tend to separate, indicating that the steady state of the Lindblad master equation significantly deviates from the one given by SLED. Despite numerical fluctuations caused by the finite number of noise trajectories used for SLED, we observe that some resonance-like features tend to be preserved even for the dissipation rate of the order of the Rabi angular frequency of the drive. The main differences arise from the probe-frequency-independent shift of the response. Similar to Sec. III, this effect is caused by a shift of $\sigma_z(t)$ given by SLED as compared to that produced by the Lindblad master equation, becoming more pronounced as $\gamma$ increases. In contrast to Fig. 1, in which the drive frequency is fixed at the bare qubit frequency in both methods ($\omega_\text{d} = \omega_0$), the shift appears in Fig. 3 for drive frequencies $\omega_\text{p}$ matching the qubit transition frequency shifted by the bath. This suggests that such a phenomenon is not primarily caused by the renormalization of the qubit frequency, but rather it may be attributed to the appearance of asymptotic system-bath correlations as the strong coupling is approached.

A qualitative analysis may connect the presented results with the actual qubit fluorescence spectrum predicted by Mollow in Ref. [13]. Typically, the radiation spectrum is proportional to the Fourier transform of a two-time correlation function of the system evaluated at its steady state. For a weak environmental coupling, the calculation of such correlation functions is usually obtained through the quantum regression theorem [24, 25], where one resorts to the Born–Markov approximations.

Within this approach, for $\gamma/\Omega_\text{d} \ll 1$, the fluorescence spectrum of a dissipative qubit that is driven by a resonant field of the form of Eq. (9) in the RWA presents three peaks centered at frequencies $\omega_0 = \omega_\text{d}$ and $\omega_{\pm} = \omega_\text{d} \pm \Omega_\text{d}$ [13]. The sideband peaks have a Lorentzian shape that becomes broadened and flattened as the ratio $\gamma/\Omega_\text{d}$ increases. These features are present in the top and middle panels of Fig. 3, where we fit the data provided by SLED with Lorentzian functions peaked roughly at
In the bottom panels, however, the sideband peaks are absent due to the high qubit dissipation rate. The small oscillations in this case, which are noticeable in both SLED and Lindblad data, may be attributed to the different number of periods $n_p$ used in the numerical integration of Eq. (14).

In contrast to the studies of Ref. [13], no central peak at $\omega_0 = \omega_{d}$ is observed in Fig. 3, since the probe field does not excite the qubit at resonance, according to the definition of $f(t)$ in Eq. (13). This and the above-mentioned qualitative features of the qubit fluorescence spectrum can also be checked in Fig. 4 where we show the amplitude $h_z$ of the probe-induced oscillations of $\sigma_z(t)$ as a function of the probe frequency $\omega_p$ and a broad range of qubit dissipation rates $\gamma$. We clearly observe that the regions of high amplitude indicate the sideband peaks of the Mollow triplet, these being pronounced and narrow at small $\gamma/\Omega_d$. These peaks become flat and broad at high qubit dissipation rates, eventually disappearing at $\gamma/\Omega_d \gg 1$.

Similar damped oscillations as shown here have also been observed in different physical setups, for instance, in two coupled degenerate resonators with significantly different leakage rates [33]. In addition, the radiation spectrum of a qubit under a bichromatic field in the RWA has also been obtained through the quantum regression theorem and presents a rich variety of phenomena depending on the choice of parameters [58]. However, the features of the qubit spectrum under the drive field of Eq. (9) are preserved assuming that it is much stronger than the probe field, the case considered in this work.

\[ \omega_- = 0.99 \omega_{d} \text{ and } \omega_+ = 1.01 \omega_{d}. \]

**V. CONCLUSIONS**

We carried out a quantitative comparison of the dissipative dynamics of a driven qubit based on two different approaches, namely the Lindblad equation and the stochastic Liouville–von Neumann equation with dissipation, or SLED. We showed that the often overlooked bath-induced energy shift in the Lindblad master equation becomes less relevant for the dynamics as the strength of the dissipation increases. However, new effects arising from the failure of the weak-coupling as-
assumptions emerge in these regimes, being captured by the non-perturbative treatment of the drive and dissipation given by SLED. In addition, we proposed a measure based on the sensitivity of the qubit population to the drive frequency. Consequently, we identified regimes where the Lindblad equation is not able to correctly predict the steady state of the qubit dynamics. Moreover, we have used the above-mentioned approaches to study signatures of the qubit fluorescence spectrum for different dissipation rates that may be produced by a tunable environment in cQED. In conclusion, our results may guide new experiments to probe driven open quantum systems and the validity of the weak-coupling approximations in describing their dynamics.

\[ \frac{d}{dt} \hat{\rho}_S(t) = -\frac{i}{\hbar} \left[ \hat{H}_S(t), \hat{\rho}_S(0) \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[ \left[ \hat{H}_S(t), \left( \hat{H}_S(t') + \hat{\rho}_S(t') \right) \right] - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left( \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\} \right. \]

\[ - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\} \]

\[ = -\frac{i}{\hbar} \left[ \hat{H}_S(t), \hat{\rho}_S(0) \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[ \left[ \hat{H}_S(t), \left( \hat{H}_S(t') + \hat{\rho}_S(t') \right) \right] - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\} \]

\[ - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\}, \quad (A1) \]

where \( \hat{\rho}_S(0) = \text{Tr}_B[\hat{\rho}'] \) is the reduced density operator of the system at time instant \( t \) and we have assumed an initially factorized state as mentioned in the main text. Since the first moments of observables calculated in the state \( \hat{\rho}_S(0) \) may be chosen to vanish, one naturally obtains \( \text{Tr}_B[\hat{H}_{SB}^2(t), \hat{\rho}(0)] = 0 \), which is not shown in Eq. (A1). In the absence of drive \( \{f(t) = 0\} \), only the third term of the first line on the right-hand side of Eq. (A1) remains.

The structure of Eq. (A1) is rather complicated as it is an integro-differential exact equation. However, it can be simplified under a series of assumptions. First, one writes the total density operator as \( \hat{\rho}_S(t) = \hat{\rho}_S(t) \otimes \hat{\rho}_B(t) + \hat{\omega}_S(t) \), where \( \hat{\omega}_S(t) \) is present only when system–bath correlations are created during the dynamics. Naturally, \( \text{Tr}[\hat{\omega}(t)] = \text{Tr}[\hat{\omega}(t)] = 0 \) in order to preserve the normalization. The so-called Born approximation physically asserts that the influence of the system on the bath dynamics is small so that it essentially stays in the Gibbs state throughout the interaction. Consequently, one neglects the correlation term \( \hat{\omega}_S(t') \) when considering \( \hat{\rho}(t') \) in Eq. (A1), and hence we obtain

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Appendix A: Lindblad master equation

Below, we review the derivation of the Lindblad master equation for the driven dissipative qubit described in Sec. III of the main text. Similar derivations have been reported in the existing literature [24, 25, 51], and hence the discussion in this Appendix A is given mainly for the sake of completeness of our notation.

We begin by employing the interaction picture with respect to the free Hamiltonian \( \hat{H}_S + \hat{H}_B \), such that the exact temporal evolution of the total density operator \( \hat{\rho}(t) \) is given by the Liouville–von Neumann equation \( d\hat{\rho}(t)/dt = -i[\hat{H}(t), \hat{\rho}(t)]/\hbar \), where \( \hat{H}(t) = \hat{H}_S(t) + \hat{H}_{SB}(t) \) and the superscript \( i \) stands for the interaction picture. The recursive integration up to the second order and the trace over the bath degrees of freedom yield

\[ \frac{d}{dt} \hat{\rho}_S(t) = -\frac{i}{\hbar} \left[ \hat{H}_S(t), \hat{\rho}_S(0) \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[ \left[ \hat{H}_S(t), \left( \hat{H}_S(t') + \hat{\rho}_S(t') \right) \right] - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\} \right. \]

\[ - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\} \]

\[ \left. = -\frac{i}{\hbar} \left[ \hat{H}_S(t), \hat{\rho}_S(0) \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[ \left[ \hat{H}_S(t), \left( \hat{H}_S(t') + \hat{\rho}_S(t') \right) \right] - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\} \right. \]

\[ - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^2(t), \left[ \hat{H}_{SB}^2(t'), \hat{\rho}'(t') \right] \right] \right\}, \quad (A1) \]
\[
\frac{d}{dt} \hat{\rho}_S(t) = -\frac{i}{\hbar} \left[ \hat{H}_d(t), \hat{\rho}_S(t) \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[ \hat{H}_d(t'), \left[ \hat{H}_d(t), \hat{\rho}_S(t') \right] \right] - \frac{1}{\hbar^2} \int_0^t dt'' \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^+(t), \left[ \hat{H}_{SB}(t'), \hat{\rho}_S(t') \hat{\rho}_B(0) \right] \right] \right\}.
\] (A2)

Note that the Born approximation implemented inside the double commutators of Eq. (A1) does not guarantee the conservation of the system entropy as in a unitary evolution.

We note that in Eq. (A2), the two last terms in Eq. (A1) have been dropped. This is a consequence of the Born approximation, i.e., \( \dot{\omega}(t) \to 0 \). Therefore, the Born approximation does not account for the drive-dissipation interplay promoted by such terms in the case of linear approximation does not account for the drive-dissipation oscillating terms. This is referred as secular approximation, and returning to Schrödinger’s picture, one obtains the Lindblad master equation (5) of the main text.

\[
\frac{d}{dt} \hat{\rho}_S(t) = -\frac{i}{\hbar} \left[ \hat{H}_d(t), \hat{\rho}_S(t) \right] - \frac{1}{\hbar^2} \int_0^\infty d\tau \text{Tr}_B \left\{ \left[ \hat{H}_{SB}^+(t), \left[ \hat{H}_{SB}(t - \tau), \hat{\rho}_S(t) \hat{\rho}_B(0) \right] \right] \right\}.
\] (A3)

In general, the double commutator in Eq. (A3) gives rise to a correction of the system’s energy and non-unitary dynamics. However, a master equation of the form (A3) typically does not generate a completely positive map. One can overcome this problem, and consequently write Eq. (A3) in the so-called Lindblad form, by removing fast oscillating terms. This is referred as secular approximation and it is usually justified in the weak system-bath coupling regime. For the case under study, such terms oscillate according to \( e^{\pm 2i\omega t} \) and do not contribute to energy shifts. Consequently, by performing the integration in the right-hand side of Eq. (A3), using the secular approximation, and returning to Schrödinger’s picture, one obtains the Lindblad master equation (5) of the main text.

### 1. Steady-state and fidelity between Lindblad solutions

Here, we show the analytical expressions for the components of the steady-state Bloch vector in the rotating frame according to the Lindblad equation, \( \sigma_{x,y,z}^{SS}(\Delta_q) \), which can be found following the procedure described in Sec. IIIA. They read

\[
\begin{align*}
\sigma_x^{SS}(\Delta_q) &= -\frac{4\gamma \Omega_0 \Delta_q}{\gamma_\beta(2\Omega_0^2 + \gamma_\beta^2 + 4\Delta_q^2)}, \\
\sigma_y^{SS}(\Delta_q) &= -\frac{2\gamma \Omega_0}{2\Omega_0^2 + \gamma_\beta^2 + 4\Delta_q^2}, \\
\sigma_z^{SS}(\Delta_q) &= -\frac{\gamma(\gamma_\beta^2 + 4\Delta_q^2)}{\gamma_\beta(2\Omega_0^2 + \gamma_\beta^2 + 4\Delta_q^2)}.
\end{align*}
\] (A4)

Using Eqs. (A4) in the definition for \( \Delta_q^{SS} \) in Eq. (10), yields Eqs. (11) of the main text.

One may also be interested in the fidelity between the steady states of the Lindblad master equation for the qubit driven at the bare frequency \( \Delta_q = \Delta_q \) and driven at the frequency shifted by the system-bath interactions \( \Delta_q = 0 \). Such a fidelity can be obtained analytically by writing the steady states in terms of the Bloch vector components in Eq. (A4) and using the definition in Eq. (5). Assuming \( \gamma_\beta \approx \gamma \), we find

\[
F(\Delta_q) = 1 - \frac{4\Omega_0^2 \Delta_q^2}{(2\Omega_0^2 + \gamma^2 + 4\Delta_q^2)(2\Omega_0^2 + \gamma^2 + 4\Delta_q^2)}.
\] (A5)

As shown in Fig. 2a and can be directly checked from Eq. (6), the energy shift \( \Delta_q \) in the Lindblad master equation is negative and depends linearly on \( \gamma \). By writing \( \Delta_q = -\alpha \gamma \), with \( \alpha > 0 \), it is possible to show that the
fidelity in Eq. (A5) is minimized at the critical ratio
\[ \frac{\gamma}{\Omega_4} = \bar{\gamma}_c = \frac{\sqrt{2}}{1 + 4\alpha^2} \] (A6)

For values of \( \alpha \approx 1 \), one obtains \( \bar{\gamma}_c \approx 1 \). This condition can be qualitatively observed in Figs. 11 and 25 of the main text.

**Appendix B: Stochastic Liouville-von Neumann equation for dissipation (SLED)**

Here we briefly review the main aspects of the so-called SLED formalism, in which one resorts to the path integral description of quantum mechanics. Consider a closed quantum system with total position (momentum) operator \( \hat{q} (\hat{p}) \) and corresponding eigenstates \( |q \rangle (|p \rangle) \). Under the influence of a temporally dependent Hamiltonian \( \hat{H}(t) \), the propagator associated to the closed dynamics \( K(q_f, q_i) = \langle q_f|T \exp(i \int_0^t dt' \hat{H}(t')/\hbar)|q_i \rangle \) can be given in terms of the classical action functional \( S[q, t'] = \int_0^t dt' \mathcal{L}(q, t) \) as follows [2]:

\[ K(q_f, q_i) = \int_{q_i}^{q_f} Dq \exp \left\{ \frac{i}{\hbar} S[q, t'] \right\}, \] (B1)

where the integration measure
\[ \int_{q_i}^{q_f} Dq = \lim_{N \to \infty} \int_{-\infty}^{\infty} d^N q \frac{1}{c}, \] (B2)

with \( c \) being a normalization factor, goes along all possible paths \( q(t') \) between \( q_i = q(0) \) and \( q_f = q(t) \).

Here, the temporal dependence of \( \hat{H}(t') \) is manifested by the explicit dependence on \( t' \) in the classical Lagrangian \( \mathcal{L}(q, t') = \hat{H}(q, t') - V(q, t') \), with \( \hat{H}(q, t') \) and \( V(q, t') \) being the kinetic-like and potential energy of the system, respectively.

Suppose now that the considered system is multipartite and one is just interested in the reduced dynamics of a subpartition with position (momentum) operator \( \hat{q} (\hat{p}) \) and eigenvalues \( |q \rangle (|p \rangle) \). Moreover, suppose that the temporal dependence of \( \hat{H}(t) \) comes strictly from the free Hamiltonian of this main subsystem, assigned here as \( \hat{H}_S(t) \). By assuming that the state of the main subsystem is initially factorized from the rest, the path integral formalism allows one to write the reduced density operator in position representation, \( \rho_S(q_f, q_i) = \langle q_f|\rho_S(t)|q_i \rangle \), as

\[ \rho_S(q_f, q_i) = \int dq dq'J(q_f, q', q_i, q)\rho_S(q, q'), \]

\[ J(q_f, q', q_i, q) = \int_{q_i}^{q_f} Dq \int_{q_i}^{q_f} Dq' e^{\pm S_S[q, t]} e^{-\pm S_S[q', t]} \]
\[ \times F[q, q'], \] (B3)

with classical action \( S_S[q, t] \) associated to \( \hat{H}_S(t) \). All the dynamical effects of the secondary subsystems on the main one are encoded in the so-called influence functional \( F[q, q'] \), which equals unity in absence of interaction.

In this work, one assumes that the secondary subsystems form a thermal bosonic bath and its interaction with the main subsystem is linear through position coordinates as described by the Caldeira-Leggett model [59].

If the main subsystem is a driven qubit as described in Sec. 1, \( \hat{H}_S(t) = \hat{H}_S + \hat{H}_D(t) \), the Caldeira-Leggett model reduces to Hamiltonian in Eq. (1), and the influence functional can be cast into the form \( F[u, v] = e^{-\Phi[u, v]} \), where \( \Phi[u, v] \) is a phase functional with real and imaginary parts [33 60]

\[ \Phi_{\tau}[u] = \int_0^t dt' \int_0^{t'} dt'' u(t'')L(t' - t''), \] (B4)

\[ \Phi_{I}[u, v] = \int_0^t dt' \int_0^{t'} dt'' u(t'')v(t'')L(t' - t''), \] (B5)

where we defined new integration path variables \( u = q - q' \) and \( v = q + q' \).

In spite of the exact expression for the influence functional \( F[u, v] \), its calculation is nontrivial since it is non-local in time. For the choice of spectral density \( J(\omega) \) in Eq. (3) and in the limit of high cutoff frequency \( (\omega_c \to \infty) \), such temporal nonlocality can be only attributed to the finite temperature of the bath. In this case, one can rewrite the phase functional as the sum of temporally nonlocal and temporally local phases, i.e., \( \Phi[u, v] = \Phi_{\text{int}}[u] + \Phi_{\text{loc}}[u, v] \), with

\[ \Phi_{\text{int}}[u] = \int_0^t dt' \int_0^{t'} dt'' u(t'')L'(t' - t''), \] (B6)

\[ \Phi_{\text{loc}}[u, v] = \frac{\gamma}{\hbar^2} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' u(t'')v(t')L(t' - t''), \] (B7)

In Eq. (B6) one has defined

\[ L'(\tau) = \int_0^\infty \frac{d\omega}{2\pi} J(\omega) \coth(h\beta\omega/2) \]
\[ -2/(h\beta\omega) \cos(\omega\tau) \] (B8)

as the white noise deducted from the real part of \( L(\tau) \).

Here, one can establish a numerically exact correspondence of \( F[u, v] \) with a temporally local averaged functional \( \mathbb{E}\{F_\epsilon[u, v]\} \) arising from a classical stochastic process. This process is described by the real-valued classical random variable \( \xi(t) \) with null mean and autocorrelation

\[ \mathbb{E}\{\xi(t')\xi(t'')\} = L'(t' - t''). \] (B9)

Placing Eq. (B3) into Eq. (B6) and using a Hubbard-Stratonovich transformation [51], one can then write the influence functional \( F[u, v] \) as

\[ \mathbb{E}\{F_\epsilon[u, v]\} = e^{-\Phi_{\text{int}}[u, v]} \mathbb{E}\left[ e^{\xi F_\epsilon[u, v]} \right]. \] (B10)
and consequently Eq. (B3) reduces to \( \rho_S(q_f, q_f') = \mathbb{E}[\rho_S, \xi, q_f, q_f'] \). Namely, the actual density operator \( \rho_S(q_f, q_f') \) can be regarded as the initial state \( \rho_S(q_i, q_i') \) evolving according to the stochastic influence functional \( F[\xi, \nu] \) and averaged over a large number of noise trajectories. By returning to the operator representation and making the replacements \( \hat{q} \rightarrow \hat{\sigma}_x, \hat{p} \rightarrow \omega_q \hat{\sigma}_y, \) a single realization of \( \xi(t) \) is given by Eq. (7) of the main text. Such equation is deterministic and a single realization of \( \xi(t) \) can be generated from an arbitrary Gaussian random variable [see Appendix B1]. Interestingly, Eq. (7) has the form of the Caldeira–Leggett master equation [25] with \( -\xi(t)\hat{\sigma}_x \) added to the unitary part. Physically, such a term is responsible to account for the quantum fluctuations neglected in the classical treatment of the dissipative environment [43].

It has been shown [51] [60] that the full stochastic unraveling of \( \rho_S(q_f, q_f') \) for an arbitrary spectral density \( J(\omega) \) requires the inclusion of two complex-valued random variables, therefore making numerical convergence slower than that of SLED. Analysis involving such an extended method is out of the scope of this work.

1. Noise generation in SLED

Here we describe the procedure for the generation of the stochastic noise \( \xi(t) \) which appears in Eq. (7). The initial point is to consider a Gaussian random variable \( r(t) \) whose autocorrelation function is

\[
\mathbb{E}[r(t)r(t')] = \delta(t - t').
\]

(B11)

Then one can define a real-valued convolution kernel \( G(t) \) in such a way that the noise \( \xi(t) \) is written as

\[
\xi(t) = \int_{-\infty}^{\infty} d\tau G(t - \tau) r(\tau).
\]

(B12)

By using the definition \( [B12] \) and the property \( [B11] \), the autocorrelation function of the noise \( \xi(t) \) becomes

\[
\mathbb{E}[\xi(t)\xi(t')] = \int_{-\infty}^{\infty} d\tau G(t - \tau) G(t' - \tau).
\]

(B13)

The relation between Eq. (B13) and Eq. (B9) can here be established by writing \( G(t) \) and \( L'_e(t) \) as the inverse associated to their Fourier transforms

\[
\hat{G}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} G(t),
\]

\[
\hat{L}'_e(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t} L'_e(t),
\]

(B14)

(B15)

that is to say,

\[
G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \hat{G}(\omega),
\]

\[
L'_e(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \hat{L}'_e(\omega).
\]

(B16)

(B17)

Equating Eq. (B13) with Eq. (B9) allows one to obtain

\[
\hat{G}(\omega) = \sqrt{L'(\omega)}.
\]

(B18)

For an odd spectral density \( J(\omega) \), as the one defined in Eq. (3), the Fourier transform of \( L'_e(t) \) acquires the form

\[
L'_e(\omega) = J(\omega)[\coth(h\beta\omega/2) - 2/(h\beta\omega)]/2.
\]

(B19)

Finally, by denoting the Fourier transform of \( r(t) \) as \( \hat{r}(\omega) \), Eq. (B12) can be rewritten as

\[
\xi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \hat{G}(\omega) \hat{r}(\omega).
\]

(B20)

Therefore, \( \xi(t) \) can be regarded as the inverse transform of \( \hat{G}(\omega)\hat{r}(\omega) \), with \( \hat{G}(\omega) \) being obtained through Eq. (B18) and \( \hat{r}(\omega) \) being generated from a Gaussian random variable \( r(t) \). In this work, we have used Python built-in functions for generating \( r(t) \), calculating Fourier and inverse transforms.

Appendix C: Phenomenological description of the experimental setup

In this section, we phenomenologically describe a cQED setup where pump-probe measurements could be performed in the presence of tunable dissipation. For simplicity, we consider a transmon qubit with the transition frequency of \( \omega_q \) between its two lowest levels \( |i\) \) \((i = 0, 1)\), which is capacitively coupled to microwave drive and probe lines characterized by a time dependent voltage [56]. In addition, the qubit is capacitively coupled to a tunable resistor at inverse temperature \( \beta \), which can be implemented either through a quantum circuit refrigerator or heat sink [10]. These features may be modeled by the Hamiltonian [1] of the main text with the form of \( f(t) \) given by Eq. (13).

The readout of the qubit state can be achieved by measuring the transmitted field through a resonator coupled dispersively to the transmon. Namely, for a resonator with angular frequency \( \omega_r \), photon decay rate \( \kappa \), and annihilation operator \( \hat{a} \), coupled linearly to the transmon via a Jaynes–Cummings interaction \( \hat{H}_{JC} = \hbar g (|0\rangle\langle 0|\hat{a} + |1\rangle\langle 1|\hat{a}^\dagger) \), the dispersive condition \( (g/\Delta_{qr} \ll 1, \Delta_{qr} = \omega_q - \omega_r) \) assures that no energy is exchanged between them; only frequency shifts are induced [32]. The phenomenological inclusion of the resonator in the dispersive regime then produces the effective system Hamiltonian

\[
\hat{H}_{e}(t) = -\frac{\hbar \tilde{\omega}_q}{2} \hat{\sigma}_z + \hbar f(t)\hat{\sigma}_x + \hbar (\omega_r + \chi \sigma_z) \hat{a}^\dagger \hat{a}
\]

\[
+ \frac{\hbar \Omega_{in}}{2} (\hat{a} e^{-i\omega_{in} t} + \hat{a}^\dagger e^{i\omega_{in} t}),
\]

(C1)

where \( \tilde{\omega}_q = \omega_q + \chi \) and \( \chi = g^2/\Delta_r \) is the frequency shift induced by the dispersive interaction. Notice that we also
consider a weak measurement drive of frequency \( \omega_m \) and amplitude \( \Omega_m \) applied to the input port of the resonator.

Here, we make the assumption that the open dynamics of the qubit-resonator system is provided by local quantum environments. This is augmented by the fact that the incoherent dynamics of the whole system is caused by independent sources. For simplicity, we neglect intrinsic uncontrollable dephasing and dissipation rates of the qubit in the treatment by assuming that they are much smaller than the dissipation rate \( \gamma \) produced by its coupling to the tunable resistor, the latter one thus being the main source of dissipation for the qubit within the considered times scales. On the other hand, the dissipative dynamics of the resonator is caused by its finite quality factor so that it behaves as a lossy cavity where photons can leak out incoherently. These features can be represented by a master equation of the form

\[
\dot{\rho}_{qr}(t) = -\frac{i}{\hbar} \left[ \hat{H}_{qr}(t), \rho_{qr}(t) \right] + \mathcal{L}_\gamma [\rho_{qr}(t)] + \mathcal{L}_\kappa [\rho_{qr}(t)]
\]

where the commutator describes the unitary dynamics determined by the Hamiltonian in Eq. (C2), and \( \mathcal{L}_\gamma/\kappa [\rho_{qr}(t)] \) describe the non-unitary dynamics promoted by the tunable resistor and the lossy resonator, respectively. Hence, \( \mathcal{L}_\gamma/\kappa [\rho_{qr}(t)] \) contains only operators in the qubit/resonator subspace and already includes all possible bath-induced energy shifts and dissipative effects.

Based on the arguments presented above, Eq. (C2) allows one to separate the non-unitary effects produced by each local bath and describe the mutually induced frequency shifts in the qubit-resonator system by the effective Hamiltonian \( \hat{H}_{qr}(t) \). While the average number of photons in the resonator contributes to the shift of the qubit frequency, the qubit-dependent frequency of the resonator changes the behavior of the transmitted field providing an indirect measurement of the driven qubit spectrum as a function of the probe frequency \( \omega_q \). In order to visualize such a phenomenon, we study the temporal evolution of the expectation value of \( \hat{a} \) in a frame rotating at \( \omega_m \), \( a(t) = \text{Tr}[\hat{a}e^{i\omega mt}] e^{-i\omega mt} \). First, we assume that \( \mathcal{L}_\kappa [\rho_{qr}(t)] \) is phenomenologically described in the Lindblad form

\[
\mathcal{L}_\kappa [\rho_{qr}(t)] = \kappa \left[ \hat{a} \rho_{qr}(t) \hat{a}^\dagger - \frac{1}{2} \left\{ \hat{a}^\dagger \hat{a}, \rho_{qr}(t) \right\} \right],
\]

where for shortness of notation we omitted the energy shift term, assuming that it is already incorporated to the definition of \( \omega_m \). Using Eq. (C2), we can write the dynamical equation for \( a(t) \) as

\[
\dot{a}(t) = -i \frac{\Omega_m}{2} - \left( i \Delta_{rm} + \frac{\kappa}{2} \right) a(t) - i \chi a_z(t),
\]

where we have defined \( \Delta_{rm} = \omega_t - \omega_m \) and \( a_z(t) = \text{Tr}[\sigma_z \hat{a} e^{i\omega mt} \hat{a}^\dagger \rho_{qr}(t)e^{-i\omega mt}] \). In the semiclassical approximation \([67]\), one can neglect the influence of qubit-resonator entanglement on the temporal evolution of \( a_z(t) \), in such a way that \( a_z(t) \approx a(t) \sigma_z(t) \), with \( \sigma_z(t) = \text{Tr}[\sigma_z \rho_{qr}(t)] \). Consequently, Eq. (C4) can be rewritten as

\[
\dot{a}(t) = -i \frac{\Omega_m}{2} - \left[ i \Delta_{rm} + i \chi \sigma_z(t) + \frac{\kappa}{2} \right] a(t).
\]

Here, we assume that the measurement field \( \Omega_m \) is turned on at a sufficiently long time after the initial transient dynamics of the dissipative driven qubit. Consequently, provided that the probe is much weaker than the drive, i.e., \( \Omega_p \ll \Omega_d \), we can write the solution to Eq. (C5) as

\[
a(t) = a(0) - i \Omega_m \left[ e^{i(\Delta_{rm}+\chi \sigma_z)+\kappa t} - 1 \right] \kappa + 2 i \left( \Delta_{rm} + \chi \sigma_z \right) + 2 i \kappa \sigma_z e^{-i(\Delta_{rm}+\chi \sigma_z)+\kappa t},
\]

where \( \sigma_z \) is the temporal average of \( \sigma_z(t) \). For the choice of \( t = n_p t_p \) as in Sec. IV, \( \sigma_z \) may be written as in Eq. (14).

In our approach, we solve the dissipative dynamics of the driven qubit and feed the solution of \( a(t) \) with the values of \( \sigma_z \). As explained in the main text, in this work \( \sigma_z(t) \) is obtained either from the Lindblad master equation \([5]\) or by the average solution of SLED in Eq. (7). Regardless on the method of solution of the qubit dynamics, Eq. (C6) clearly shows that \( a(t) \) contains information about the qubit population and, therefore, contains information about its spectrum. One can access features of the spectrum, for instance, through the amplitude, phase or Fourier transform of the transmitted field. Defining the field quadratures as \( I(t) = \text{Re}[a(t)] \) and \( Q(t) = \text{Im}[a(t)] \), the amplitude \( A(t) \) and phase \( \phi(t) \) of the transmitted signal can be expressed as

\[
A(t) = \sqrt{T^2(t) + Q^2(t)}, \quad \phi(t) = \text{Arg}[a(t)].
\]

Finally, by setting \( \Delta_{rm} = 0 \), assuming \( \kappa t/2 = n_p t_p / 2 \gg 1 \), and using the definition for \( A(t) \) in Eq. (C7), one finds the transmitted field amplitude

\[
A = \frac{\Omega_m}{\kappa} \sqrt{\frac{1}{1 + \left( \frac{2 \pi \kappa}{\omega_m} \right)^2}},
\]

as defined in Eq. (15).

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