SHARP STABILITY OF THE LOGARITHMIC SOBOLEV INEQUALITY IN THE CRITICAL POINT SETTING

JUNCHENG WEI AND YUANZE WU

Abstract. In this paper, we consider the Euclidean logarithmic Sobolev inequality
\[ \int_{\mathbb{R}^d} |u|^2 \log |u| \, dx \leq \frac{d^4}{4} \log \left( \frac{2}{\pi d e} \right), \]
where \( u \in W^{1,2}(\mathbb{R}^d) \) with \( d \geq 2 \) and \( \|u\|_{L^2(\mathbb{R}^d)} = 1 \). It is well known that extremal functions of this inequality are precisely the Gaussians
\[ g_{\sigma,z}(x) = \left( \frac{\pi \sigma}{2} \right)^{-d/2} g_{\star}(\sqrt{\sigma}(x-z)) \]
with \( g_{\star}(x) = e^{-|x|^2/2} \).

We prove that if \( u \geq 0 \) satisfying \( (\nu - \frac{1}{2})c_0 < \|u\|_{H^1(\mathbb{R}^d)} < (\nu + \frac{1}{2})c_0 \) and \( \| -\Delta u + u - 2u \log |u| \|_{H^{-1}} \leq \delta \), where \( c_0 = \|g_{1,0}\|_{H^1(\mathbb{R}^d)} \), \( \nu \in \mathbb{N} \) and \( \delta > 0 \) sufficiently small, then
\[ \text{dist}_{H^1}(u, M^\nu) \leq \| -\Delta u + u - 2u \log |u| \|_{H^{-1}} \]
which is optimal in the sense that the order of the right hand side is sharp, where
\[ M^\nu = \{(g_{1,0}(\cdot-z_1), g_{1,0}(\cdot-z_2), \ldots, g_{1,0}(\cdot-z_\nu)) | z_i \in \mathbb{R}^d\}. \]

Our result provides an optimal stability of the Euclidean logarithmic Sobolev inequality in the critical point setting.

Keywords: Euclidean logarithmic Sobolev inequality; Optimal stability; Critical point setting.

AMS Subject Classification 2010: 35A23; 35B35.

1. Introduction

A fundamental task in understanding functional inequalities, arise in the calculus of variations, geometry, etc, is to study the best constants, the classification of extremal functions, as well as their qualitative properties for parameters in the full region, since such functional inequalities are crucial in understanding nonlinear partial differential equations (Nonlinear PDEs for short) by virtue of the complete knowledge of the best constants, extremal functions, and qualitative properties. The most well-studied functional inequality in the community of Nonlinear PDEs is the Sobolev inequality, whose classical one with exponent 2 states that for any \( u \in H^1(\mathbb{R}^d) \) with \( d \geq 3 \), there holds
\[ S \left( \int_{\mathbb{R}^d} |u|^{2d \over d-2} \, dx \right)^{d-2 \over d} \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dx, \]
\[ S \left( \int_{\mathbb{R}^d} |u|^{2d \over d-2} \, dx \right)^{d-2 \over d} \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dx, \]
where $S > 0$ is a constant which is only dependent of the dimension and $H^1(\mathbb{R}^d)$ is the classical Sobolev space given by

$$H^1(\mathbb{R}^d) = \{ u \in L^{\frac{2d}{d-2}}(\mathbb{R}^d) \mid |\nabla u| \in L^2(\mathbb{R}^d) \}.$$  

It has been proved in [2, 55] that

$$S = \pi d(d-2) \left( \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)} \right)^\frac{2}{d}$$

is optimal and the extremal functions, which are called the Aubin-Talenti bubbles in the literature, are given by

$$U_{\lambda,z,c}(x) = c \left( \frac{1}{1 + \lambda^2 |x-z|^2} \right)^{\frac{d-2}{2}},$$

where $\lambda > 0$, $c \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

Once a functional inequality is well understood for its best constant and extremal functions, it is natural to concern its stability, which is growingly interested in recent years by its important applications in understanding many Nonlinear PDEs, such as the fast diffusion equation, the Keller-Segel equation and so on. The basic question one wants to address in this aspect is the following (cf. [30]):

(Q) Suppose we are given a functional inequality for which minimizers are known. Can we prove, in some quantitative way, that if a function “almost attains the equality” then it is close (in some suitable sense) to one of the minimizers?

Such study was first raised by Brezis and Lieb in [8] for the classical Sobolev inequality (1.1) as an open question, which was settled by Bianchi and Egnell in [6] by proving that

$$dist_{H^1}(u, Z) \lesssim \| \nabla u \|^2_{L^2(\mathbb{R}^d)} - S \| u \|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)},$$

where $\| \cdot \|_{L^p(\mathbb{R}^d)}$ is the usual norm in the Lebesgue space $L^p(\mathbb{R}^d)$ and

$$Z = \{ U_{\lambda,z,c} \mid (\lambda, z, c) \in \mathbb{R}^+ \times \mathbb{R}^{d+1} \}.$$

Unlike Bianchi and Egnell’s study ([6]) in the functional inequality setting, in recent years, Figalli et al. initiated the study on the stability of the classical Sobolev inequality (1.1) in the critical point setting, that is, studying the stability of the Euler-Lagrange equation of the classical Sobolev inequality (1.1). The study on the stability of the classical Sobolev inequality (1.1) in the critical point setting is more challenging since the Euler-Lagrange equation of the classical Sobolev inequality (1.1) has sign-changing solutions. Moreover, the “almost” solutions of the Euler-Lagrange equation of the classical Sobolev inequality (1.1) may decompose into several parts at infinity (cf. [54]). By setting the study in a suitable way, Figalli et al. proved that

1. (Ciraolo-Figalli-Maggi [16]) Let $d \geq 3$ and $u \in D^{1,2}(\mathbb{R}^d)$ be positive such that $\| \nabla u \|^2_{L^2(\mathbb{R}^d)} \leq \frac{4}{S^2} S^2$ and $\| \Delta u + |u|^\frac{4}{d-2} u \|_{H^{-1}} \leq \delta$ for some $\delta > 0$ sufficiently small. Then,

$$dist_{D^{1,2}}(u, M_0) \lesssim \| \Delta u + |u|^\frac{4}{d-2} u \|_{H^{-1}}.$$
where $\mathcal{M}_0 = \{(U[z, \lambda] \mid z \in \mathbb{R}^d, \lambda > 0)\}.$
(2) (Figalli-Glaudo [31]) Let $u \in D^{1,2}(\mathbb{R}^d)$ be nonnegative such that
\[
(\nu - \frac{1}{2})S^{\frac{2}{\nu}} < \|u\|_{D^{1,2}(\mathbb{R}^d)} < (\nu + \frac{1}{2})S^{\frac{2}{\nu}}
\]
and $\|\Delta u + |u|^{\frac{2}{\nu-2}} u\|_{H^{-1}} \leq \delta$ for some $\delta > 0$ sufficiently small. Then, for $3 \leq d \leq 5,$
\[
dist_{D^{1,2}}(u, \mathcal{M}^v_0) \lesssim \|\Delta u + |u|^{\frac{4}{d-2}} u\|_{H^{-1}}
\]
where
\[
\mathcal{M}^v_0 = \{(U[z_1, \lambda_1], U[z_2, \lambda_2], \ldots, U[z_v, \lambda_v]) \mid z_i \in \mathbb{R}^d, \lambda_i > 0\}.
\]
All the above results are optimal in the sense that the orders of the right hand sides in the above estimates are sharp, while the optimal stability of the classical Sobolev inequality (1.1) in the critical point setting for the case $N \geq 6$ was left in [31] as an open problem which was solved by Deng, Sun and Wei in [21], very recently, by proving the following optimal stability:
\[
dist_{D^{1,2}}(u, \mathcal{M}_0) \lesssim \begin{cases} 
\|\Delta u + |u|u\|_{H^{-1}} \left( \ln(\|\Delta u + |u|u\|_{H^{-1}}) \right)^{\frac{d}{4}} , & d = 6; \\
\|\Delta u + |u|^{\frac{2}{\nu-2}} u\|_{H^{-1}}^{\frac{d+2}{2d-2}} , & d \geq 7.
\end{cases}
\]

According to the important applications in understanding many Nonlinear PDEs, several other famous functional inequalities, such as the Gagliardo-Nirenberg-Sobolev inequality (cf. [11, 12, 25, 26, 49, 51, 52]), the Hardy-Littlewood-Sobolev inequality (cf. [5, 18, 34, 37, 40, 43, 50]), the Caffarelli-Kohn-Nirenberg inequality (cf. [9, 13, 17, 22–24, 28, 45, 57, 58, 61]), the $L^p$-Sobolev inequality (cf. [14, 15, 32, 33, 35, 36, 48]) and so on, are also studied on the best constants, the classification of extremal functions and stability. However, according to their multi-parameters or no-Hilbert, most of these functional inequalities are far from well understood except for some special cases.

It is worth pointing out that, besides the classical Sobolev inequality (1.1), the Euclidean logarithmic Sobolev inequality,
\[
\int_{\mathbb{R}^d} |u|^2 \log |u| dx \leq \frac{d}{4} \log \left( \frac{2}{\pi e} \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \right)
\]  \hspace{1cm} (1.2) eq0001
where $u \in W^{1,2}(\mathbb{R}^d)$ with $d \geq 2$ and $\|u\|_{L^2(\mathbb{R}^d)} = 1,$ is also well understood for the best constants and the classification of extremal functions. It is well known (cf. [20, 60]) that this inequality is optimal and extremal functions of (1.2) are precisely the Gaussians
\[
g_{\sigma, z}(x) = (\pi \sigma)^{-\frac{d}{2}} g_{\sigma} \left( \sqrt{\frac{\sigma}{2}} (x - z) \right) \quad \text{with} \quad g_{\sigma}(x) = e^{-\frac{|x|^2}{2\sigma}} .
\]  \hspace{1cm} (1.3) eq0002
Moreover, it is equivalent to the Gross logarithmic inequality (cf. [38]) with respect to Gaussian weight
\[
\int_{\mathbb{R}^d} |g|^2 \log |g| d\mu \leq \int_{\mathbb{R}^d} |\nabla g|^2 d\mu \quad \text{with} \quad \int_{\mathbb{R}^d} |g|^2 d\mu = 1 \text{ and } d\mu = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}} .
\]
The Euclidean logarithmic Sobolev inequality has another two different forms:

\[
\int_{\mathbb{R}^d} |v|^2 \log |v|^2 \, dx \leq \frac{d^2}{\pi} \| \nabla v \|^2_{L^2(\mathbb{R}^d)} \left( \log \| v \|^2_{L^2(\mathbb{R}^d)} - d(1 + \log a) \right) \| v \|^2_{L^2(\mathbb{R}^d)} \tag{1.4} \]

for all \( a > 0 \) and any function \( v \in W^{1,2}(\mathbb{R}^d) \), and

\[
\int_{\mathbb{R}^d} |w|^2 \log |w|^2 \, dx + \frac{d}{2}(1 + \log(2\pi)) \leq \| \nabla w \|^2_{L^2(\mathbb{R}^d)} \tag{1.5}
\]

for any function \( w \in W^{1,2}(\mathbb{R}^d) \), with \( \| w \|_{L^2(\mathbb{R}^d)} = 1 \). (1.4) and (1.5) are equivalent by the relation \( w(x) = \sqrt{\frac{a}{2\pi}} \), and they are established in [44, Theorem 8.14] and [59, Theorem 1.2], respectively. However, (1.2) is optimal since it can be obtained by minimizing the right hand side of (1.4) for \( a > 0 \) (cf. [26]).

The stability of the Euclidean logarithmic Sobolev inequality (1.2) in the functional inequality setting has also been widely studied, cf. [7, 10, 11, 26, 27, 29, 41, 42, 60] and the references therein. It is worth pointing out that most of these results are devoted to more general version of the logarithmic Sobolev inequality in the probability setting which contains the Euclidean logarithmic Sobolev inequality (1.2) as a special case for the Gaussians measure \( d\mu = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}} \). Thus, it is natural to consider the stability of the Euclidean logarithmic Sobolev inequality (1.2) in the critical point setting, as that for the classical Sobolev inequality (1.1). Let

\[
\mathcal{E}(u) = \frac{d}{4} \log \left( \frac{2}{\pi d} \nabla u \right) + \int_{\mathbb{R}^d} |u|^2 \log |u| \, dx.
\]

Then critical points of \( \mathcal{E}(u) \) in \( H^1(\mathbb{R}^d) \) with the finite energy on the smooth manifold

\[
\mathcal{S} = \{ u \in H^1(\mathbb{R}^d) \mid \| u \|_{L^2(\mathbb{R}^d)} = 1 \}
\]
satisfy \( \int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx < +\infty \) and the following Euler-Lagrange equation

\[
-(\frac{d}{2\| \nabla u \|^2_{L^2(\mathbb{R}^d)}})\Delta u - (1 + 2\sigma_u)u = 2u \log |u| \quad \text{in } \mathbb{R}^d, \tag{1.6}
\]

where \( \sigma_u \) is a part of unknown and appears in (1.6) as the Lagrange multiplier. Thus, (1.6) is the Euler-Lagrange equation of (1.2). Let

\[
u(\lambda x) = \lambda^\frac{d}{2} u(\lambda x) \quad \text{where } \lambda = \left( \frac{d}{2\| \nabla u \|^2_{L^2(\mathbb{R}^d)}} \right)^{\frac{1}{2}},
\]

then by (1.6) and a direct calculation, \( u_\lambda \) satisfies

\[
-\Delta u_\lambda + (d \log \lambda - 1 - 2\sigma_u)u_\lambda = 2u_\lambda \log |u_\lambda| \quad \text{in } \mathbb{R}^d. \tag{1.7}
\]

Let

\[
\nu^*_\lambda(x) = \alpha_u u_\lambda(x) \quad \text{where } \alpha_u = e^{\frac{d}{2} \log \lambda - \sigma_u},
\]

then by (1.7) and a direct calculation, \( u^*_\lambda \) satisfies the equation

\[
-\Delta u + u = 2u \log |u| \quad \text{in } \mathbb{R}^d. \tag{1.8}
\]

Thus, the logarithmic Schrödinger equation (1.8) can be seen as the Euler-Lagrange equation of the Euclidean logarithmic Sobolev inequality (1.2).
It has been proved in [19, 56] that the Gaussian
\[ g = e^{-\frac{1}{2}|x|^2} \] (1.9)

is the unique positive solution of the logarithmic Schrödinger equation (1.8) which satisfies \( u(x) \to 0 \) as \( |x| \to +\infty \), where \( g_* \) is given by (1.3). We remark that any solution of (1.8) satisfying \( \int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx < +\infty \) must exponentially decay to zero as \( |x| \to +\infty \) by the standard applications of the maximum principle. Moreover, it has been proved in [19, Theorem 1.3] (see also [39, Theorem 7.5]) that \( g \) is nondegenerate in the sense that \( \text{Ker}(\mathcal{L}) = \text{span}\{\partial_x, g\} \)

is the linearized operator of (1.8) at \( g \). Note that (1.8) is invariant under translations, thus, the smooth manifold
\[ \mathcal{M} = \{g(\cdot - z) \mid z \in \mathbb{R}^d\} \]
contains all positive solutions of (1.8) satisfying \( \int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx < +\infty \).

Let \( \{u_n\}, u_n \geq 0 \) for all \( n \), be bounded in \( H^1(\mathbb{R}^d) \) and almost solves (1.8), that is, \( \| -\Delta u_n + u_n - 2u_n \log |u_n|\|_{H^{-1}} \to 0 \) as \( n \to \infty \). Then it is easy to see that \( \{u_n\} \) is bounded in \( L^1(\mathbb{R}^d) \). Thanks to the Brezis-Lieb lemma [53, Lemma 3.1] and the positivity of the energy of the unique positive solution of (1.8) (cf. [53, Lemma 3.3]), it is standard (cf. [1, Proposition 3.1]) to prove the following Struwe’s decomposition of \( \{u_n\} \).

**Proposition 1.1.** There exists \( \nu \in \mathbb{N} \) and \( y_1, n, y_2, n, \cdots, y_{\nu}, n \subset \mathbb{R}^d \) such that
\[ \|u_n - \sum_{i=1}^{\nu} g(\cdot - y_{i,n})\|_{H^1(\mathbb{R}^d)} \to 0 \quad \text{as} \quad n \to \infty. \]

By Proposition 1.1, for nonnegative functions \( u_n \) with uniform \( H^1 \)-bound, we know that there exists \( \nu \in \mathbb{N} \) such that \( \{u_n\} \) will be close to the manifold
\[ \mathcal{M}' = \{(g(\cdot - z_1), g(\cdot - z_2), \cdots, g(\cdot - z_{\nu})) \mid z_i \in \mathbb{R}^d\} \]
in the \( H^1 \)-topology. Thus, it is natural to ask, if \( u \geq 0 \) satisfy
\[ (\nu - \frac{1}{2})c_0 < \|u\|^2_{H^1(\mathbb{R}^d)} < (\nu + \frac{1}{2})c_0 \quad \text{and} \quad \| -\Delta u + u - 2u \log |u|\|_{H^{-1}} \leq \delta, \]
where \( c_0 = \|g\|^2_{H^1(\mathbb{R}^d)}, \nu \in \mathbb{N} \) and \( \delta > 0 \) sufficiently small, can we obtain a quantitative version of Proposition 1.1 by optimally controlling \( \text{dist}_{H^1}(u, \mathcal{M}') \) by \( \| -\Delta u + u - 2u \log |u|\|_{H^{-1}} \)? The purpose of this paper is to give a positive answer to this natural question and our main result reads as follows.

**Theorem 1.1.** Let \( u \geq 0 \) satisfy
\[ (\nu - \frac{1}{2})c_0 < \|u\|^2_{H^1(\mathbb{R}^d)} < (\nu + \frac{1}{2})c_0 \quad \text{and} \quad \| -\Delta u + u - 2u \log |u|\|_{H^{-1}} \leq \delta, \]
where \( c_0 = \|g\|^2_{H^1(\mathbb{R}^d)}, \nu \in \mathbb{N} \) and \( \delta > 0 \) sufficiently small. Then
\[ \text{dist}_{H^1}(u, \mathcal{M}') \lesssim \| -\Delta u + u - 2u \log |u|\|_{H^{-1}}, \]
where
\[ \mathcal{M}' = \{(g_1, 0(\cdot - z_1), g_1, 0(\cdot - z_2), \cdots, g_1, 0(\cdot - z_{\nu})) \mid z_i \in \mathbb{R}^d\}. \]
Moreover, (1.1) is optimal in the sense that the order of the right hand side is sharp.
Remark 1.1. The main ideas in proving Theorem 1.1 is similar to that of [21, 61], that is, we choose special $y_1, \delta, y_2, \delta, \ldots, y_\nu, \delta \subset \mathbb{R}^d$ and decompose $u - \sum_{i=1}^{\nu} g(\cdot - y_i, \delta)$ into two parts, where the first part is very regular which can be well estimated and the second part is much smaller than the first part. However, the logarithmic nonlinearity is very different from the power-type one dealt with in [21, 61], thus, we need to careful estimate it to make sure that the strategy in [21, 61] work for (1.8).

Notations. Throughout this paper, $C$ and $C'$ are indiscriminately used to denote various absolutely positive constants. $\sigma, \sigma', \sigma''$ are indiscriminately used to denote various absolutely positive constants which can be taken arbitrary small. $a \sim b$ means that $Cb \leq a \leq Cb$ and $a \lesssim b$ means that $a \leq Cb$.

2. Preliminaries

Let us consider the equation

$$-\Delta u + u = 2u \log |u| + f, \quad \text{in } \mathbb{R}^d, \quad (2.1)$$

where $f \in H^{-1}$ and $u \in H^1(\mathbb{R}^d)$ is nonnegative and satisfies

$$(\nu - \frac{1}{2})c_0 < \|u\|_{H^1(\mathbb{R}^d)}^2 < (\nu + \frac{1}{2})c_0$$

for some fixed $\nu \in \mathbb{N}$ with $c_0 = \|g\|^2_{H^1(\mathbb{R}^d)}$. By multiplying (2.1) with $u$ and integrating by parts, it is easy to see that $\int_{\mathbb{R}^d} |u|^2 \log |u|^2 dx \lesssim 1$. Moreover, by Proposition 1.1, there exists $z_1, f, z_2, f, \ldots, z_\nu, f \subset \mathbb{R}^d$ such that

$$\|u - \sum_{i=1}^{\nu} g(\cdot - z_i, f)\|_{H^1(\mathbb{R}^d)} \to 0 \quad \text{as } \|f\|_{H^{-1}} \to 0.$$ 

It follows that

$$c_f = \inf_{z_i \in \mathbb{R}^d} \|u - \sum_{i=1}^{\nu} g(\cdot - z_i)\|_{H^1(\mathbb{R}^d)}^2 \to 0 \quad \text{as } \|f\|_{H^{-1}} \to 0. \quad (2.2)$$

Thus, by solving the minimizing problem (2.2) in a standard way (cf. [4]), we can write $u = \sum_{i=1}^{\nu} g(\cdot - y_i, f) + \rho_f$, where $\{y_i, f\}$ is the solution of (2.2) and the remaining term $\rho_f$ satisfies

$$\|\rho_f\|_{H^1(\mathbb{R}^d)}^2 = c_f \to 0 \quad \text{as } \|f\|_{H^{-1}} \to 0$$

and the orthogonal conditions

$$\langle \rho_f, \partial x_l g_{i,j,f} \rangle_{H^1(\mathbb{R}^d)} = 0, \quad l = 1, 2, \ldots, d \text{ and } j = 1, 2, \ldots, \nu. \quad (2.3)$$

For the sake of simplicity, we denote $g(\cdot - y_i, f)$ by $g_{i,f}$. Clearly, by (2.1), $\rho_f$ satisfies

$$\left\{ \begin{array}{l}
-\Delta \rho_f + \rho_f = 2\left(\sum_{i=1}^{\nu} g_{i,f} + \rho_f\right) \log(\sum_{i=1}^{\nu} g_{i,f} + \rho_f) - 2 \sum_{i=1}^{\nu} g_{i,f} \log g_{i,f} + f, \quad \text{in } H^{-1}, \\
\langle \rho_f, \partial x_l g_{i,f} \rangle_{H^1(\mathbb{R}^d)} = 0, \quad l = 1, 2, \ldots, d \text{ and } i = 1, 2, \ldots, \nu. \quad (2.4) \end{array} \right.$$
It is convenient to write (2.4) as follows:

\[
\begin{align*}
\mathcal{L}_f(\rho_f) &= E + N(\rho_f) + f, \quad \text{in } H^{-1}, \\
\langle \rho_f, \partial_x g_i, f \rangle_{H^1(\mathbb{R}^d)} &= 0, \quad l = 1, 2, \ldots, d \text{ and } i = 1, 2, \ldots, \nu,
\end{align*}
\]

where

\[
\mathcal{L}_f = -\Delta - 1 - 2 \log(\sum_{i=1}^\nu g_{i,f})
\]

is the linear operator,

\[
E = 2(\sum_{i=1}^\nu g_{i,f}) \log(\sum_{i=1}^\nu g_{i,f}) - 2(\sum_{i=1}^\nu g_{i,f}) \log g_{i,f}
\]

is the error and

\[
N(\rho_f) = 2(\sum_{i=1}^\nu g_{i,f} + \rho_f) \log(\sum_{i=1}^\nu g_{i,f} + \rho_f) - 2(\sum_{i=1}^\nu g_{i,f}) \log(\sum_{i=1}^\nu g_{i,f})
\]

\[-2(1 + \log(\sum_{i=1}^\nu g_{i,f})) \rho_f
\]

is the nonlinear part.

3. Proof of (1.11)

Let

\[
\eta_i = \min_{j \neq i} \eta_{i,j} \quad \text{and} \quad \eta = \min_i \eta_i,
\]

where \(\eta_{i,j} = |y_{j,f} - y_{i,f}|\). Then by (2.1), (2.2) and the uniqueness of \(g\), we can easy to see that

\[
\eta = \min_{i \neq j} \{|y_{i,f} - y_{j,f}|\} \to +\infty \quad \text{as } \delta \to 0,
\]

where we denote \(\|f\|_{H^{-1}}\) by \(\delta\) for the sake of simplicity. Let

\[
\Omega_i = \{x \in \mathbb{R}^d \mid g_{i,f} \geq g_{j,f} \quad \text{for all } j \neq i\}.
\]

Then \(\mathbb{R}^d = \bigcup_{i=1}^\nu \Omega_i\) and \(\sum_{j=1}^\nu g_{j,f} \sim g_{i,f}\) in \(\Omega_i\). Moreover, we also introduce

\[
\Pi_{c,i,j} : 2(y_{i,f} - y_{j,f})x + c = 0 \quad \text{and} \quad \mathcal{L}_{i,j} : (x - y_{i,f}) \times (y_{i,f} - y_{j,f}) = 0,
\]

where \(c \in \mathbb{R}\) are constants. We take \(x_{c,i,j} \in \Pi_{c,i,j} \cap \Omega_i \cap \Omega_j\) and denote

\[
|x_{c,i,j} - y_{i,f}| = \pm \alpha_{c,i,j} \eta_{i,j},
\]

where \(\alpha_{c,i,j} > -\frac{1}{2}\) with

\[
\alpha_{c,i,j} \begin{cases} 
> 0, & \langle x_{c,i,j} - y_{i,f}, y_{i,f} - y_{j,f} \rangle > 0, \\
< 0, & \langle x_{c,i,j} - y_{i,f}, y_{i,f} - y_{j,f} \rangle < 0.
\end{cases}
\]

Then

\[
|x_{c,i,j} - y_{j,f}| = (1 + \alpha_{c,i,j}) \eta_{i,j}.
\]
Lemma 3.1. We have,

\[ E \sim \sum_{i=1}^{\nu} \chi_{\Omega_i} g_{i,f}(\sum_{j \neq i} \chi_{\Pi_{c,i,j}} e^{-(\alpha_{c,i,j} + \frac{1}{2})\eta_{i,j}^2} \log(1 + e^{(\alpha_{c,i,j} + \frac{1}{2})\eta_{i,j}^2})) \quad (3.6) \]

as \( \|f\|_{H^{-1}} \to 0 \).

Proof. By (2.7), we have

\[ E = 2 \sum_{i=1}^{\nu} g_{i,f} \log(1 + \sum_{j \neq i} g_{j,f}) \quad (3.7) \]

By (3.7), we write

\[ E = 2 g_{i,f} \log(1 + \sum_{j \neq i} g_{j,f}) + 2 \sum_{j \neq i} g_{j,f} \log(1 + \sum_{l \neq j} g_{l,f}) =: I + II \]

in \( \Omega_i \), where \( \Omega_i \) is given by (3.3). For \( I \), we have

\[ I \sim \sum_{j \neq i} g_{j,f} \quad \text{in } \Omega_i. \]

For \( g_{j,f} \log(1 + \sum_{l \neq j} \frac{g_{l,f}}{g_{j,f}}) \) with \( j \neq i \), we write

\[ g_{j,f} \log(1 + \sum_{l \neq j} \frac{g_{l,f}}{g_{j,f}}) = g_{j,f} \log(1 + \frac{g_{i,f} + \sum_{l \neq j} g_{l,f}}{g_{j,f}}). \]

Then in \( \Omega_i \), we have

\[ E \sim II \sim \sum_{j \neq i} g_{j,f} \log(1 + \frac{g_{i,f}}{g_{j,f}}). \quad (3.8) \]

Let us consider the function

\[ \frac{g_{i,f}}{g_{j,f}}, \quad \forall j \neq i. \]

By (1.3), (1.9) and direct calculations, we have

\[ \nabla \left( \frac{g_{i,f}}{g_{j,f}} \right) = (y_{i,f} - y_{j,f}) e^{-\frac{1}{2}((x-y_{i,f})^2 - (x-y_{j,f})^2)}. \quad (3.9) \]

Then by (3.9),

\[ \frac{g_{i,f}}{g_{j,f}} = \text{const.} \quad \text{on the hyperplane } \Pi_{c,i,j} \text{ for all } c \in \mathbb{R}, \]

where \( \Pi_{c,i,j} \) is given by (3.4). Thus, we have

\[ \frac{g_{i,f}}{g_{j,f}} = e^{(\alpha_{c,i,j} + \frac{1}{2})|y_{i,f} - y_{j,f}|^2}, \quad \forall x \in \Pi_{c,i,j} \cap \Omega_i, \quad (3.10) \]

where \( \alpha_{c,i,j} > -\frac{1}{2} \) is given by (3.5). It follows from (3.8), (3.10) and the fact that \( \Pi_{c,i,j} \perp L_{i,j} \) for all \( c \in \mathbb{R} \) that (3.6) holds.
As that in [21, 61], we decompose $\rho_f = \phi_f + \varphi_f$, where $\phi_f$ is the solution of the following equation:

$$
\begin{align*}
\mathcal{L}_f(\phi_f) &= E + N(\phi_f) - \sum_{j=1}^{d} \sum_{i=1}^{\nu} a_{j,i} \partial_{x_j} g_{i,f}, \quad \text{in } H^{-1}, \\
\langle \phi_f, \partial_{x_j} g \rangle_{H^1(\mathbb{R}^d)} &= 0, \quad l = 1, 2, \ldots, d 
\end{align*}
$$

(3.11) eqnn0034

with $a_{j,i}$ being the Lagrange multipliers given by

$$
a_{j,i} \sim \langle E + N(\phi_f), \partial_{x_j} g_{i,f} \rangle_{L^2}.
$$

To solve (3.11), let us first establish a good linear theory. Let

$$
g_{i,f,d-1,j} = e^{\frac{1+d-|x_{i,f}|^2}{2}}
$$

and

$$
D_{R,i,j} = \{ x \in \Pi_{c,i,j} \mid |x_{c,i,j} - \frac{y_{i,f} + y_{j,f}}{2}| \leq R \},
$$

where $z_{i,j} \perp L_{i,j}$ and $x_{c,i,j}$ is given by (3.5). We define

$$
\Omega_{i,j,d-1} = \{ x \in \Pi_i \mid g_{i,f,d-1,j} \geq g_{i,f,d-1,l}, \forall l \neq i, j \}.
$$

Then as above, $\Omega_i = \bigcup_{j \neq i} \Omega_{i,j,d-1}$ and $\sum_{l \neq i} g_{i,f,d-1,l} \sim g_{i,f,d-1,j}$ in $\Omega_{i,j,d-1}$ for all $j$, where $\Omega_i$ is given by (3.3). We introduce the norms

$$
\|u\|_1 = \sum_{i=1}^{\nu} \sum_{j \neq i} \left( \sup_{D_{\nu_{i,j},d-1} \cap \Omega_{i,j,d-1}} \frac{|u|}{\nu_{i,j}} + \sup_{D_{\nu_{i,j},d-1} \cap \Omega_{i,j,d-1}} e^{-\frac{\eta^2}{2} |\eta_{i,j}|^2} \frac{\eta^2 |u|}{g_{i,f,d-1,j}} \right)
$$

and

$$
\|u\|_2 = \sum_{i=1}^{\nu} \sum_{j \neq i} \left( \sup_{D_{\nu_{i,j},d-1} \cap \Omega_{i,j,d-1}} \frac{|u|}{\nu_{i,j}} + \sup_{D_{\nu_{i,j},d-1} \cap \Omega_{i,j,d-1}} e^{-\frac{\eta^2}{2} |\eta_{i,j}|^2} \frac{\eta^2 |u|}{g_{i,f,d-1,j}} \right).
$$

Then

$$
\mathcal{X} = \{ u \in H^1(\mathbb{R}^d) \mid \|u\|_1 < +\infty \} \quad \text{and} \quad \mathcal{Y} = \{ u \in L^2(\mathbb{R}^d) \mid \|u\|_2 < +\infty \},
$$

are Banach spaces.

Let us consider the following linear equation:

$$
\begin{align*}
\mathcal{L}_f(\psi) &= h - \sum_{i=1}^{\nu} \sum_{j=1}^{d} b_{j,i} \partial_{x_j} g_{i,f}, \quad \text{in } \mathbb{R}^d, \\
\phi &\in \mathcal{X}^1,
\end{align*}
$$

(3.12) eqnn0018

where $\mathcal{L}_f$ is the linear operator given by (2.6),

$$
\mathcal{X}^1 = \{ u \in \mathcal{X} \mid \langle u, \partial_{x_j} g_{i,f} \rangle_{H^1(\mathbb{R}^d)} = 0, \quad j = 1, 2, \ldots, d; \quad i = 1, 2, \ldots, \nu \}
$$

(3.13) eq0029

and $b_{j,i}$ are the Lagrange multipliers given by

$$
b_{j,i} \sim \langle h, \partial_{x_j} g_{i,f} \rangle_{L^2}.
$$

(3.14) eq0028

(lem0004) Lemma 3.2. As $\|f\|_{H^{-1}} \to 0$, (3.12) is unique solvable for every $h \in \mathcal{Y}$ with

$$
\|\psi\|_2 + \sum_{i=1}^{\nu} \sum_{j=1}^{d} |b_{j,i}| \lesssim \|h\|_2.
$$
Proof. By (1.3) and (1.9), it is easy to see that for $R > 0$ sufficiently large,

$$-(1 + 2 \log(\sum_{j=1}^{\nu} g_{j,f})) \geq 0$$

in $(\bigcup_{j=1}^{\nu} B_R(y_{j,f}))^c$. It follows from $\sum_{j=1}^{\nu} g_{j,f} \sim g_{i,f}$ in $\Omega_i$ that

$$-(1 + 2 \log(\sum_{j=1}^{\nu} g_{j,f})) = |x - y_{i,f}|^2 + O(1) \quad \text{in } D_{\sigma \eta^{+2},i,j} \cap \Omega_i. \quad (3.15)$$

eq 0025

Note that by the definition of $L$ given by (3.4) and rotations, $g_{i,f,d-1,j}$ is the unique solution of (1.8) in $\mathbb{R}^{d-1}$. Thus, by (3.15), rotations, the definitions of $\Pi_{c,i,j}$ and $L$, given by (3.4),

$$-\Delta g_{i,f,d-1,j} - (1 + 2 \log(\sum_{j=1}^{\nu} g_{j,f}))g_{i,f,d-1,j} \gtrsim \eta g_{i,f,d-1,j} \quad (3.16)$$

eq 0n0909

in $D_{\sigma \eta^{+2},i,j} \cap \Omega_{i,j,d-1}$. Similarly,

$$-\Delta g_{i,f}^{1-\sigma} - (1 + 2 \log(\sum_{j=1}^{\nu} g_{j,f}))g_{i,f}^{1-\sigma} \gtrsim g_{i,f}^{1-\sigma} \quad (3.17)$$

eq 0n0910

in $(\bigcup_{j=1}^{\nu} B_R(y_{j,f}))^c$. For every $x \in \Omega_{i,j,d-1}$, by the fact that $\Pi_{c,i,j} \perp \mathbb{L}_{i,j}$ for all $c \in \mathbb{R}$, we can re-write $x = (\alpha_{c,i,j}, z^{i,j})$, where $\alpha_{c,i,j} > -\frac{1}{2}$ is given by (3.5) and $z^{i,j} \perp \mathbb{L}_{i,j}$. Now, let

$$\phi(\alpha_{c,i,j}, z^{i,j}) = g_{i,f}^{1-\sigma} \varphi(\alpha_{c,i,j}) + \eta^{-2} e^{-\frac{1}{2}(\sigma\eta^{2}-\eta^{2})} g_{i,f,d-1,j}(1 - \varphi(\alpha_{c,i,j}))$$

where $\varphi(\alpha_{c,i,j})$ is the unique solution of the following equation:

$$\begin{cases}
\varphi'' - \varphi' + \varphi = 1, & \text{in } (\frac{\sigma\eta}{\eta_{i,j}} - 1) \frac{\sigma\eta}{\eta_{i,j}} - \frac{1}{2}, \\
\varphi'(\frac{\sigma\eta}{\eta_{i,j}} - 1) \frac{\sigma\eta}{\eta_{i,j}} - \frac{1}{2} = 0.
\end{cases}$$

Then by (3.16) and (3.17), in $(\bigcup_{j=1}^{\nu} B_R(y_{j,f}))^c$,

$$-\Delta \phi - (1 + 2 \log(\sum_{j=1}^{\nu} g_{j,f}))\phi \gtrsim \begin{cases}
eq 0025
\frac{e^{-\frac{1}{2}(\sigma\eta^{2}-\eta^{2})} g_{i,f,d-1,j}, \quad \alpha_{c,i,j} \leq \frac{\sigma\eta}{\eta_{i,j}} - \frac{1}{2}, \\
\eta_{i,j}^{1-\sigma}, \quad \alpha_{c,i,j} \geq \frac{\sigma\eta}{\eta_{i,j}} - \frac{1}{2},
\end{cases}$$

which, together with the maximum principle, implies that

$$|\psi| \lesssim (\|h\|_2 + \|\psi\|_{L^\infty(\partial(\bigcup_{j=1}^{\nu} B_R(y_{j,f}))^c)}) \phi(\alpha_{c,i,j}, z^{i,j}) \quad (3.18)$$

eq 0023

in $(\bigcup_{j=1}^{\nu} B_R(y_{j,f}))^c$ for $R > 0$ sufficiently large. Based on the a-priori estimate (3.18), we shall prove the a-priori estimate

$$\|\psi\|_2 \lesssim \|h\|_2 \quad \text{uniformly as } \delta \to 0. \quad (3.19)$$

eq 0026

Since the proofs for (3.19), based on the blow-up arguments, are standard nowadays (cf. [46, 47, 61]), we only sketch it here. We assume the contrary that there exists $\delta_n \to 0$, $\{\psi_n\}$ solves (3.12) with $\{h_n\} \subset L^2(\mathbb{R}^d)$ satisfying $\|\psi_n\|_2 = 1$ and $\|h_n\|_2 = o_n(1)$ as $n \to \infty$. Since $\delta_n \to 0$, by (3.2),

$$\log(\sum_{j=1}^{\nu} g_{j,f}(x + y_{j,f})) \to \log g = -\frac{1}{2} |x|^2 + \frac{d + 1}{2}.$$
in $\mathbb{R}^d$ as $n \to \infty$. Now, let
\[ \psi_{i,n}(x) = \phi_n(x + y_{i,n}), \]
then by $\delta_n \to 0$ as $n \to \infty$ and (3.14), it is standard to prove that $\psi_{i,n} \to \overline{\psi}_i$ uniformly in every compact set of $\mathbb{R}^d$ for every $i$, where $\overline{\psi}_i$ are bounded solutions of the following equation
\[ -\Delta \overline{\psi}_i + (|x|^2 - d - 2)\overline{\psi}_i = 0 \quad \text{in } \mathbb{R}^d. \]
By [19, Theorem 1.3] (see also [39, Theorem 7.5]), $\overline{\psi}_i = \sum_{j=1}^d a_j \partial_{x_j} \mathcal{g}$. On the other hand, by (1.3) and (1.9), we can pass to the limit in the orthogonal conditions of $\psi_{i,n}$ given by (3.13), which implies that
\[
0 = \lim_{n \to +\infty} \langle \psi_{i,n}, \partial_{x_j} \mathcal{g} \rangle_{H^1(\mathbb{R}^d)} = \lim_{n \to +\infty} \int_{\mathbb{R}^d} (2 + 2 \log(\sum_{j=1}^d \mathcal{g}_{j,i})) \psi_n \partial_{x_j} \mathcal{g}_{i,n} \, dx = \int_{\mathbb{R}^d} (3 + d - |x|^2) \overline{\psi}_i \partial_{x_j} \mathcal{g} \, dx = \langle \overline{\psi}_i, \partial_{x_j} \mathcal{g} \rangle_{H^1(\mathbb{R}^d)}
\]
for all $j = 1, 2, \ldots, d$. Since
\[
\langle \partial_{x_j} \mathcal{g}, \partial_{x_j} \mathcal{g} \rangle_{H^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (d + 3 + |x|^2) \partial_{x_j} \mathcal{g} \partial_{x_j} \mathcal{g} \, dx = 0,
\]
we have $\overline{\psi}_i = 0$, which implies that $\psi_{i,n} \to 0$ uniformly in every compact set of $\mathbb{R}^d$ for all $i$. Thus, by (3.18), we have $\|\psi_n\|_2 = o_n(1)$ which is a contradiction. Thanks to the a-priori estimate (3.19), by the Fredholm alternative, we know that the linear equation (3.12) is unique solvable for all $h \in \mathcal{Y}$. The estimate $\sum_{i=1}^\nu \sum_{j=1}^d |b_{j,i}| \lesssim \|h\|_2$ comes from (3.1) and (3.14).

By direct calculations,
\[
\max_{\alpha, \epsilon, \sigma > -\frac{3}{4}} e^{-\frac{1}{2} \alpha (\epsilon + 1) \mathcal{V}^2_{\eta, \sigma}} \log(1 + e^{(\epsilon + 1) \mathcal{V}^2_{\eta, \sigma}}) \sim e^{-\frac{1}{8} \mathcal{V}^2_{\eta, \sigma}}. \quad (3.20)
\]
Thus, by taking $\sigma > 0$ sufficiently small and Lemma 3.1, we have $\|E\|_2 \lesssim e^{-\frac{1}{8} \mathcal{V}^2}$. We define
\[
\mathcal{E} = \{ \phi \in X^\perp \mid \|\phi\|_2 \leq Me^{-\frac{1}{8} \mathcal{V}^2} \} \quad (3.21)
\]
where $M > 0$ is a sufficiently large constant.

**Lemma 3.3.** There exists $M > 0$ sufficiently large such that (3.11) has a unique solution $\phi_f \in \mathcal{E}$ with $\|\phi_f\|_2 + \sum_{j=1}^d \sum_{i=1}^\nu |a_{j,i}| \lesssim e^{-\frac{1}{8} \mathcal{V}^2}$ as $\|f\|_{H^{-1}} \to 0$. Moreover, $\|\phi_f\|_{H^1(\mathbb{R}^d)} \lesssim e^{-\frac{1}{8} \mathcal{V}^2}$ as $\|f\|_{H^{-1}} \to 0$.

**Proof.** The proof is standard nowadays, so we also sketch it here. For $\phi \in \mathcal{E}$, by (2.8) and the Taylor expansion,
\[
N(\phi_+) = 2 \log(1 + \frac{\theta \phi}{\sum_{j=1}^\nu \mathcal{g}_{j,f}}) \phi_+ \text{ and } N(\phi_-) = -2 \log(1 - \frac{\theta \phi}{\sum_{j=1}^\nu \mathcal{g}_{j,f}}) \phi_, \quad (3.22)
\]
Thus, by (3.22) and the symmetry of \( \mathcal{g}_{i,f} \),

\[
\| N(\phi) \|_2 \lesssim \eta^{-2} e^{-\frac{\eta^2}{2}},
\]

which implies that \( \| N(\phi) \|_2 \lesssim \eta^{-2} e^{-\frac{\eta^2}{2}} \). Now, we can solve (4.1) by the standard fix-point arguments in \( \mathbb{B} \) by choosing a sufficiently large \( M > 0 \). The estimate

\[
\| \phi_f \|_2 + \sum_{j=1}^d \sum_{i=1}^\nu \| a_{j,i} \| \lesssim \eta^{-2} e^{-\frac{\eta^2}{2}}
\]

comes from Lemma 3.2. By multiplying (3.11) with \( \phi_f \) on both sides and integrating by parts and using the fact that \( \phi_f \in \mathbb{B} \), we have \( \| \phi_f \|_{H^1(\mathbb{R}^d)} \lesssim e^{-\frac{\eta^2}{2}} \) as \( ||f||_{H^{-1}} \to 0 \) which completes the proof. \( \square \)

We recall that by (2.5) and (3.11), the remaining term \( \varphi_f = \rho_f - \phi_f \) satisfies

\[
\begin{align*}
L_f(\varphi_f) &= N(\phi_f + \varphi_f) - N(\phi_f) + \sum_{j=1}^d \sum_{i=1}^\nu a_{j,i} \partial_{x_i} \mathcal{g}_{i,f} + f, \quad \text{in } H^{-1}, \\
\langle \varphi_f, \partial_{x_j} \mathcal{g}_{i,f} \rangle_{H^1(\mathbb{R}^d)} &= 0, \quad j = 1, 2, \ldots, d \quad \text{and} \quad i = 1, 2, \ldots, \nu.
\end{align*}
\]

Moreover, it is well known (cf. [39, Theorem 7.5 and Remark 7.7]) that the eigenfunctions of the linear operator \( L \) given by (1.10) forms an orthogonal basis in \( L^2(\mathbb{R}^d) \), where the first eigenvalue is \(-2\) with eigenspace \( \text{span}\{ g \} \) and the second eigenvalue is \( 0 \) with eigenspace \( \text{span}\{ \partial_x g \} \). Thus, we can write

\[
\varphi_f = \sum_{j=1}^{\nu} (c_j g_{j,f} + b_{i,j} \partial_{x_i} g_{j,f}) + \varphi_f^\perp,
\]

where \( \varphi_f^\perp \) is orthogonal to \( \text{span}_{j,i} \{ g_{j,f}, \partial_{x_i} g_{j,f} \} \) in \( L^2(\mathbb{R}^d) \).

**Lemma 3.4.** As \( ||f||_{H^{-1}} \to 0 \), we have

\[
||\varphi_f||^2_{L^2(\mathbb{R}^d)} \sim \sum_{j=1}^{\nu} |c_j|^2 + ||\varphi_f^\perp||^2_{L^2(\mathbb{R}^d)}.
\]

**Proof.** Note that we have

\[
\langle \mathcal{g}_{j,f}, \nabla \mathcal{g}_{j,f} \rangle_{H^1(\mathbb{R}^d)} = -\frac{1}{2} \nabla_{\mathcal{g}_{j,f}} ||\mathcal{g}_{j,f}||_{H^1(\mathbb{R}^d)} = 0
\]

for all \( j = 1, 2, \ldots, \nu \). Thus, by (2.3), (3.25) and (3.27),

\[
0 = \sum_{i \neq j} (c_i \mathcal{g}_{i,f}, \partial_{x_i} \mathcal{g}_{j,f})_{H^1} + \sum_{m=1}^d b_{i,m} (\partial_{x_m} \mathcal{g}_{i,f}, \partial_{x_i} \mathcal{g}_{j,f})_{H^1} + b_{i,j}
\]

for all \( l = 1, 2, \ldots, d \) and \( j = 1, 2, \ldots, \nu \). Since

\[
||\partial_{x_l} \mathcal{g}_{i,f}|| \lesssim r_{i,l} \mathcal{g}_{i,f} \lesssim \mathcal{g}_{i,f}^{1-\sigma}
\]
for all \( l \), where \( \sigma > 0 \) can be taken arbitrary small if necessary and \( r_{i,f} = |x - y_{i,f}| \), by [3, Lemma 3.7],

\[
|\langle g_{i,f}, \partial_{x_l} g_{j,f} \rangle_{H^1}| + |\langle \partial_{x_m} g_{i,f}, \partial_{x_l} g_{j,f} \rangle_{H^1}| \lesssim \int_{\mathbb{R}^d} \frac{1}{|x_{i,f} - x_{j,f}|^{\sigma}} dx_{j,f} \lesssim e^{-\frac{1}{2}\sigma'' \eta^2} \tag{3.29} \]

for all \( i,j,l,m \), which, together with (3.28), implies that

\[
\sum_{j=1}^{\nu} \sum_{l=1}^{d} |b_{l,j}| \lesssim e^{-\frac{1}{2}\sigma'' \eta^2} \sum_{j=1}^{\nu} |c_j|. \tag{3.30} \]

Here, \( \sigma', \sigma'' > 0 \) are constants which can be taken arbitrary small. (3.26) then follows from (3.25) and (3.30).

We denote \( N(\phi_f + \varphi_f) - N(\phi_f) \) by \( N_{\phi_f}(\varphi_f) \) for the sake of simplicity. Then by Lemma 3.3 and by multiplying (3.24) with \( \varphi_f \) and integrating by parts, we have

\[
\langle L_f(\varphi_f) \varphi_f \rangle_{L^2} \lesssim \int_{\mathbb{R}^d} N_{\phi_f}(\varphi_f) \varphi_f dx + (\|f\|_{H^{-1}} + e^{-\frac{1}{8}\eta^2})\|\varphi_f\|_{H^1(\mathbb{R}^d)}. \tag{3.31} \]

**Lemma 3.5.** As \( \|f\|_{H^{-1}} \to 0 \), we have

\[
\|\varphi_f\|_{H^1(\mathbb{R}^d)} \lesssim \|f\|_{H^{-1}} + e^{-\theta \eta^2} \tag{3.32} \]

**Proof.** For the sake of clarity, we divide the proof into several parts.

**Step 1** The estimate of \( N_{\phi_f}(\varphi_f) \varphi_f \).

By the Taylor expansion and \( u = \sum_{j=1}^{\nu} g_{j,f} + \phi_f + \varphi_f \geq 0 \),

\[
N_{\phi_f}(\varphi_f) = 2\left( \sum_{j=1}^{\nu} g_{j,f} + \phi_f + \varphi_f \right) \log \left( \sum_{j=1}^{\nu} g_{j,f} + \phi_f + \varphi_f \right) - 2\left( \sum_{j=1}^{\nu} g_{j,f} + \phi_f \right) \log \left( \sum_{j=1}^{\nu} g_{j,f} + \phi_f \right) - 2(1 + \log \left( \sum_{j=1}^{\nu} g_{j,f} \right)) \varphi_f \tag{3.33} \]

\[
= 2 \log \left( 1 + \frac{\phi_f + \varphi_f}{\sum_{j=1}^{\nu} g_{j,f}} \right) \varphi_f \tag{3.34} \]

where \( \theta, \theta' \in (0, 1) \). By (3.23), we have

\[
\left| \frac{\phi_f}{\sum_{j=1}^{\nu} g_{j,f}} \right| \lesssim \eta^{-2}. \tag{3.35} \]

Thus,

\[
\sum_{i=1}^{\nu} g_{i,f} + \phi_f = (1 + O(\eta^{-2})) \sum_{i=1}^{\nu} g_{i,f}. \]
For every $R > 0$, let
\[ \Upsilon_{R, \alpha} = \{ x \in \partial(\cup_{j=1}^\nu B_R(y_{j,f}) \mid \varphi_f = \alpha \sum_{i=1}^\nu g_{i,f} \} \]
where $\alpha \geq -1 + O(\eta^{-2})$. Since $0 \leq \alpha \leq e^{\frac{\nu^2 - (d+5)}{2}} - 1 + O(\eta^{-2})$ implies that
\[ 2(1 + \log(1 + O(\eta^{-2}) + \theta \alpha)) \leq R^2 - (d + 3) \]
and $-1 + O(\eta^{-2}) + e^{\frac{\nu^2 - (d+5)}{2}} \leq \alpha < 0$ implies that
\[ -2(1 + \log(1 + O(\eta^{-2}) + \theta \alpha)) \leq R^2 - (d + 3) \]
by $\theta \in (0, 1)$, (1.9), (3.33) and (3.35), for every $R > 0$ and every $x \in \partial(\cup_{j=1}^\nu B_R(y_{j,f}))$, one of the following cases must happen:

(a) $N_{\Phi_f}(\varphi_f) \varphi_f \leq (R^2 - (d + 3))\varphi_f^2$,
(b) $\varphi_f \gtrsim 1$,
(c) $\varphi_f \sim - \sum_{i=1}^\nu g_{i,f}$.

On the other hand, by (3.32), (3.34) and (3.35), in $\Upsilon_{R, \alpha}$ with $-1 \leq \alpha < 0$,
\[ 2((1 + O(\eta^{-2}) + \alpha) \log(1 + O(\eta^{-2}) + \alpha) - \alpha) = O(\eta^{-2})\alpha + \frac{\alpha^2}{1 + O(\eta^{-2}) + \theta \alpha}. \]

It follows that $\theta' \to \frac{1}{2} + O(\eta^{-\sigma})$ as $\alpha \to -1$. Thus, by (1.9),
\[ N_{\Phi_f}(\varphi_f) \varphi_f \leq (\max\{-(2 + 2 \log(\sum_{j=1}^\nu g_{j,f}), 0\}) + O(\eta^{-\sigma}))|\varphi_f|^2 + O(\varphi_f^3). \]

**Step. 2** The estimate of $\|\varphi_f^\perp\|_2^2$.

By (3.25), (3.30), (3.31), (3.36) and Lemma 3.4,
\[ \langle \mathcal{L}_f(\varphi_f), \varphi_f \rangle_{L^2} \lesssim \int_{\mathbb{R}^d} N_{\Phi_f}(\varphi_f) \varphi_f dx + (\|f\|_{H^{-1}} + e^{-\frac{1-\eta''}{2}\eta^2} \sum_{j=1}^\nu |c_j|)\|\varphi_f\|_{H^1(\mathbb{R}^d)} \]
\[ \leq \int_{\mathbb{R}^d} \max\{-(2 + 2 \log(\sum_{j=1}^\nu g_{j,f}), 0\}) \varphi_f^2 dx \]
\[ + O(\|\varphi_f\|_{H^1(\mathbb{R}^d)}) + (\|f\|_{H^{-1}} + e^{-\frac{1-\eta''}{2}\eta^2} \sum_{j=1}^\nu |c_j|)\|\varphi_f\|_{H^1(\mathbb{R}^d)}. \]

It follows that
\[ \int_{\cup_{j=1}^\nu B_R(y_{j,f})} (|\nabla \varphi_f|^2 - (1 + 2 \log(\sum_{j=1}^\nu g_{j,f})))|\varphi_f|^2 dx \]
\[ + \int_{\mathbb{R}^d \setminus (\cup_{j=1}^\nu B_R(y_{j,f}))} |\nabla \varphi_f|^2 + |\varphi_f|^2 dx \]
\[ \leq O(\|\varphi_f\|_{H^1(\mathbb{R}^d)})^3 + (\|f\|_{H^{-1}} + e^{-\frac{1-\eta''}{2}\eta^2} \sum_{j=1}^\nu |c_j|)\|\varphi_f\|_{H^1(\mathbb{R}^d)}. \]
for a sufficiently large $R > 0$. For every $j$,
\[
\int_{B_R(y_{j,f})} (|\nabla f_j|^2 - (1 + 2 \log (\sum_{j=1}^{N} g_{j,f})))|f_j|^2 \, dx
\geq \int_{B_R(y_{j,f})} [(1 - \sigma)(d + 1) - r_{i,f}^2 - 2 \log (\sum_{j=1}^{N} g_{j,f}))]|f_j|^2 \, dx
\]
\[
+ \int_{B_R(y_{j,f})} |\nabla f_j|^2 + (r_{j,f}^2 - d - 2)|f_j|^2 \, dx
\]
(3.38) eqnn0006

where $\sigma > 0$ is sufficient small. By [39, Theorem 7.5 and Remark 7.7], it is easy to show that
\[
\int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + (r_{j,f}^2 - d - 2)|\tilde{f}_{j,R}|^2 \, dx \geq \int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + |\tilde{f}_{j,R}|^2 \, dx
\]
(3.39) eqnn0007

where $R > 0$ sufficiently large and $\tilde{f}_{j,R} = \psi_R \tilde{f}_j$ with $\psi_R$ being a smooth cut-off function such that $\psi_R = 1$ in $B_\frac{R}{\epsilon}(0)$ and $\psi_R = 0$ in $B_R(0) \setminus B_\frac{R}{\epsilon}(0)$. Note that
\[
\int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + (r_{j,f}^2 - d - 2)|\tilde{f}_{j,R}|^2 \, dx
\leq \int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + (r_{j,f}^2 - d - 2)|\tilde{f}_{j,R}|^2 \, dx
\]
\[
+ \int_{B_R(y_{j,f}) \setminus B_{\frac{R}{\epsilon}}(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + |\tilde{f}_{j,R}|^2 \, dx
\]
\[
\leq \int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + (r_{j,f}^2 - d - 2)|\tilde{f}_{j,R}|^2 \, dx
\]

and
\[
\int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + |\tilde{f}_{j,R}|^2 \, dx = \int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + |\tilde{f}_{j,R}|^2 \, dx
\]
\[
+ O(1) \int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + (r_{j,f}^2 - d - 2)|\tilde{f}_{j,R}|^2 \, dx,
\]

thus, by (3.39),
\[
\int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + (r_{j,f}^2 - d - 2)|\tilde{f}_{j,R}|^2 \, dx \geq \int_{B_R(y_{j,f})} |\nabla \tilde{f}_{j,R}|^2 + |\tilde{f}_{j,R}|^2 \, dx.
\]
(3.40) eqnn0006

On the other hand, we have $|x - y_{i,f}| \geq \eta + o(1)$ for all $j \neq i$ in $B_R(y_{j,f})$ which implies that
\[
\sum_{i \neq j} g_{i,f} \lesssim e^{-\frac{\eta^2}{\text{dist}^2}} g_{j,f} \quad \text{in } B_R(y_{j,f}).
\]

It follows that
\[
-2 \log \left(\sum_{j=1}^{N} g_{j,f} \right) = -(d + 1) + r_{j,f}^2 + O(e^{-\frac{\eta^2}{\text{dist}^2}}) \quad \text{in } B_R(y_{j,f}),
\]
which implies that
\[
| \int_{B_{R}(y_{j,f})} (1 - \sigma)(d + 1) - r_{i,f}^{2} - 2 \log(\sum_{j=1}^{\nu} \vartheta_{i,f})|^{2} dx | \lesssim \sigma \| \varphi_{f} \|_{H^{1}(\mathbb{R}^{d})}^{2}.
\]

It follows from (3.37), (3.38), (3.40) and Lemma 3.4 that
\[
\| \varphi_{f} \|_{H^{1}(\mathbb{R}^{d})}^{2} \lesssim \| f \|_{H^{-1}}^{2} + e^{- (1 - \sigma') \sigma^{2}} + \sum_{j=1}^{\nu} |c_{j}|^{2}.
\]

**Step 3** The estimate of \( \sum_{j=1}^{\nu} |c_{j}|^{2} \).

Let
\[
\tilde{\psi}_{j} = \vartheta_{j,f} \psi_{j},
\]
where \( \tilde{\psi}_{j} \) is a smooth cut-off function satisfying
\[
\tilde{\psi}_{j} = \begin{cases} 
 1, & |x - y_{j,f}| \leq \left( \frac{1}{2} - \sigma \right) \eta, \\
 0, & |x - y_{j,f}| \geq \left( \frac{1}{2} - \sigma \right) \eta + 1.
\end{cases}
\]

Note that \( \vartheta_{j,f} \) is the unique positive solution of (1.8), thus, by multiplying (3.24) with \( - \text{sgn}(c_{j}) \tilde{\psi}_{j} \) and integrating by parts, we have
\[
\begin{align*}
-\text{sgn}(c_{j}) & \int_{\mathbb{R}^{d}} (N_{\varphi_{f}}(\varphi_{f}) + \sum_{j=1}^{d} \nu \sum_{i=1}^{\nu} a_{j,i} \partial_{x_{i}} \vartheta_{j,f} \vartheta_{j,f} + \tilde{\psi}_{j} + f \tilde{\psi}_{j} dx \\
= & -\text{sgn}(c_{j}) \int_{\mathbb{R}^{d}} -\Delta \vartheta_{j,f} \vartheta_{j,f} - 2 \varphi_{f} \nabla \tilde{\psi}_{j} \nabla \vartheta_{j,f} - \Delta \vartheta_{j,f} \vartheta_{j,f} \varphi_{f} dx \\
& + \text{sgn}(c_{j}) \int_{\mathbb{R}^{d}} (1 + 2 \log(\sum_{j=1}^{\nu} \vartheta_{j,f})) \tilde{\psi}_{j} \vartheta_{j,f} \varphi_{f} dx \\
= & \text{sgn}(c_{j}) \int_{\mathbb{R}^{d}} (r_{j,f}^{2} + 1 - d + 2 \log(\sum_{j=1}^{\nu} \vartheta_{j,f})) \tilde{\psi}_{j} \vartheta_{j,f} \varphi_{f} dx \\
& + \text{sgn}(c_{j}) \int_{\mathbb{R}^{d}} 2 \varphi_{f} \nabla \tilde{\psi}_{j} \nabla \vartheta_{j,f} + \Delta \vartheta_{j,f} \vartheta_{j,f} \varphi_{f} dx.
\end{align*}
\]

By (3.29) and similar estimates of (3.36),
\[
\int_{\mathbb{R}^{d}} N_{\varphi_{f}}(\varphi_{f}) \tilde{\psi}_{j} dx \leq |c_{j}| \int_{\mathbb{R}^{d}} \max\{-2 + 2 \log(\sum_{j=1}^{\nu} \vartheta_{j,f}), 0\} \| \vartheta_{j,f} \|^{2} \tilde{\psi}_{j} dx \\
+ o \left( \sum_{i=1}^{\nu} |c_{i}| + \| \varphi_{f} \|_{H^{1}(\mathbb{R}^{d})} + \sum_{i=1}^{\nu} |c_{i}|^{2} + \| \varphi_{f} \|_{H^{1}(\mathbb{R}^{d})}^{2} \right).
\]

Note that \( \text{supp}(\tilde{\psi}_{j}) \subset \Omega_{j} \) with \( \sum_{i \neq j} \vartheta_{i,f} \lesssim e^{- \sigma \vartheta^{2}} \vartheta_{j,f} \) in \( \text{supp}(\tilde{\psi}_{j}) \), thus, by (1.9), (3.25) and Lemma 3.4,
\[
\int_{\mathbb{R}^{d}} 2 \varphi_{f} \nabla \tilde{\psi}_{j} \nabla \vartheta_{j,f} + \Delta \vartheta_{j,f} \vartheta_{j,f} \varphi_{f} dx = \alpha \left( \sum_{i=1}^{\nu} |c_{i}| + \| \varphi_{f} \|_{H^{1}(\mathbb{R}^{d})} \right)
\]
and
\[
sgn(c_j) \int_{\mathbb{R}^d} \left( r_j^2 f + 1 - d + 2 \log \left( \sum_{j=1}^{\nu} g_{j,f} \right) \right) \bar{\psi}_j g_{j,f} \phi_f dx - \left| \int_{\mathbb{R}^d} N_{\phi_f} (\varphi_f) \bar{\psi}_j dx \right|
\]
\[
\sim |c_j| + o \left( \sum_{i \neq j} |c_i| + \|\phi_f\|_{H^1(\mathbb{R}^d)} \right),
\]
which, together with (3.42) and Lemma 3.3, implies that
\[
|c_j| + o \left( \sum_{i \neq j} |c_i| + \|\phi_f\|_{H^1(\mathbb{R}^d)} \right) \lesssim O(e^{-\left( \frac{3}{2} - \sigma \right) \eta^2}) + \|f\|_{H^{-1}}.
\]
(3.43) eq0052

Since (3.43) holds for all \( j = 1, 2, \cdots, \nu \) and the linear part of this inequality is diagonally dominant, we have
\[
\sum_{j=1}^{\nu} |c_j|^2 \lesssim O(e^{-\left( \frac{3}{2} - 2\sigma \right) \eta^2}) + \|f\|_{H^{-1}}^2.
\]
(3.44) eq0056

**Step. 4** The estimate of \( \|\varphi_f\|_{H^1(\mathbb{R}^d)} \).

(3.26) comes from (3.41), (3.44) and Lemma 3.4. □

We also need the following lemma.

**Lemma 3.6.** As \( \|f\|_{H^{-1}} \to 0 \), we have \( e^{-\frac{1}{2} \eta^2} \lesssim \|f\|_{H^{-1}} \).

**Proof.** Without loss of generality, we may assume that \( \eta = |y_{1,f} - y_{2,f}| \). We define
\[
\hat{\psi} = g_{1,f} \overline{\psi},
\]
(3.45) eq0089
where \( \overline{\psi} \) is a smooth cut-off function satisfying
\[
\overline{\psi} = \begin{cases} 1, & x \in B_1 \left( \frac{1}{2} (y_{1,f} + y_{2,f}) + \frac{2 \log \eta}{\eta^2} (y_{2,f} - y_{1,f}) + 5 \right) \\
0, & x \notin B_1 \left( \frac{1}{2} (y_{1,f} + y_{2,f}) + \frac{2 \log \eta}{\eta^2} (y_{2,f} - y_{1,f}) + 5 \right).
\end{cases}
\]

By similar estimates of (3.34), we know that \( N(\rho_f) \geq 0 \). Thus, similar to (3.42), by multiplying (2.5) with \( -\bar{\psi} \) and integrating by parts and (3.34), we have
\[
\|\hat{\psi}\|_{H^1(\mathbb{R}^d)} \|f\|_{H^{-1}} \geq \int_{\mathbb{R}^d} \left( r_{1,f}^2 + 1 - d + 2 \log \left( \sum_{j=1}^{\nu} g_{j,f} \right) \right) \bar{\psi}_1 g_{1,f} \rho_f dx
\]
\[
+ \int_{\mathbb{R}^d} (E + N(\rho_f)) \hat{\psi} dx + \int_{\mathbb{R}^d} 2 \rho_f \nabla \hat{\psi} \nabla g_{1,f} dx + \Delta \overline{\psi} g_{1,f} \rho_f dx
\]
\[
\geq \int_{\mathbb{R}^d} \left( r_{1,f}^2 + 1 - d + 2 \log \left( \sum_{j=1}^{\nu} g_{j,f} \right) \right) \bar{\psi}_1 g_{1,f} \rho_f dx
\]
\[
\int_{\mathbb{R}^d} 2 \rho_f \overline{\psi} \nabla g_{1,f} + \Delta \overline{\psi} g_{1,f} \rho_f dx.
\]
(3.46) eq0157
By (1.9) and (3.20), $\int_{\mathbb{R}^d} E \hat{\psi} dx \sim e^{-\frac{4}{7} \eta^2}$. By Lemmas 3.3 and 3.5,

\[
\left| \int_{\mathbb{R}^d} 2 \rho f \nabla \nabla g_{1,f} + \Delta \hat{g}_{1,f} \rho f dx \right| \\
+ \left| \int_{\mathbb{R}^d} (r_{1,f}^2 + 1 - d + 2 \log(\sum_{j=1}^{\nu} g_{j,f})) \hat{g}_{1,f} \rho f dx \right| \\
\lesssim \eta^{-2} e^{-\frac{4}{7} \eta^2} + e^{-\sigma \eta^2} \| f \|_{H^{-1}}.
\]

Note that $\| \hat{g}_{1,f} \|_{H^1(\mathbb{R}^d)} \lesssim e^{-\frac{8}{7} \eta^2}$, thus, we obtain $e^{-\frac{8}{7} \eta^2} \lesssim \| f \|_{H^{-1}}$. \hfill $\Box$

We close this section by the following proposition.

Proposition 3.1. As $\| f \|_{H^{-1}} \to 0$, we have $\| \rho f \|_{H^1(\mathbb{R}^d)} \lesssim \| f \|_{H^{-1}}$.

Proof. The conclusion follows immediately from Lemmas 3.3, 3.5 and 3.6. \hfill $\Box$

4. Optimality of (1.11)

In this section, we shall construct an example to show that (1.11) is optimal. Let $g_{\pm L} = g(x \pm \frac{L}{2} e_1)$ with $e_1 = (1, 0, \cdots, 0)$ and we consider the following equation:

\[
\begin{cases}
L_L(\rho_L) = E_L + N(\rho_L) - \sum_{j=1}^{d} (a_{j,+L} \partial_{x_j} g_L + a_{j,-L} \partial_{x_j} g_{-L}), & \text{in } H^{-1}, \\
\langle \rho_L, \partial_{x_j} g_{\pm L} \rangle_{H^1(\mathbb{R}^d)} = 0, & j = 1, 2, \cdots, d,
\end{cases}
\]

where

\[
L_L = -\Delta - 1 - 2 \log(g_L + g_{-L})
\]

is the linear operator,

\[
E_L = 2(g_L + g_{-L}) \log(g_L + g_{-L}) - 2(g_{L} \log g_L + g_{-L} \log g_{-L})
\]

is the error,

\[
N(\rho_L) = 2(g_L + g_{-L} + \rho_L) \log(g_L + g_{-L} + \rho_L) - 2(g_{L} + g_{-L}) \log(g_L + g_{-L})
\]

\[
-2(1 + \log(g_L + g_{-L})) \rho_L
\]

is the nonlinear part, and $a_{j,\pm L}$ are Lagrange multipliers given by

\[
a_{j,\pm L} \sim \langle E_L + N(\rho_L), \partial_{x_j} g_{\pm L} \rangle_{L^2}.
\]

By Lemma 3.3, (4.1) is unique solvable in $B$, where $B$ is given by (3.21). Moreover, by Lemma 3.1,

\[
\| \rho_L \|_2 + \sum_{j=1}^{d} \sum_{i=1}^{\nu} |a_{j,\pm L}| \lesssim e^{-\frac{4}{7} L^2}.
\]

Let

\[
u_L = g_{L} + g_{-L} + \rho_L.
\]
Then by similar estimates for (3.35), \( u_L \sim g_L + g_{-L} > 0 \). Moreover, by the classical regularity, \( \rho_L \in C^{1,\alpha}(\mathbb{R}^d) \) for all \( \alpha \in (0,1) \). It follows from (4.1) that

\[
\begin{align*}
f_L &:= -\Delta u_L + u_L - 2u_L \log |u_L| \\
&= -\sum_{j=1}^{d}(a_{j,+} \partial_{x_j} g_L + a_{j,-} \partial_{x_j} g_{-L}).
\end{align*}
\]  

(\text{eq0083})

**Proposition 4.1.** As \( L \to +\infty \), we have

\[
\inf_{z_i \in \mathbb{R}^d} \|u_L - \sum_{i=1}^{2} g(\cdot - z_i)\|_{H^1(\mathbb{R}^d)}^2 \sim \|f_L\|_{H^{-1}}^2.
\]

Proof. Thanks to Lemma 3.6 and (4.2), we have

\[
\|f_L\|_{H^{-1}} \sim e^{-\frac{1}{4}L^2}.
\]

(\text{eq0085})

Now, we consider the minimizing problem

\[
c_L = \inf_{z_i \in \mathbb{R}^d} \|u_L - \sum_{i=1}^{2} g(\cdot - z_i)\|_{H^1(\mathbb{R}^d)}^2.
\]

(\text{eq0086})

Clearly, by (4.2), \( c_L \lesssim \|\rho_L\|_{H^1(\mathbb{R}^d)}^2 \lesssim e^{-\frac{1}{4}L^2} \). As before, we can also write

\[
u_L = \sum_{i=1}^{2} g(\cdot - y_i) + \rho_L^*
\]

(\text{eq0087})

where \( \{y_i\} \) is the solution of (4.5) and the remaining term \( \rho_L^* \) satisfies

\[
\|\rho_L^*\|_{H^1(\mathbb{R}^d)}^2 = c_L \lesssim e^{-\frac{1}{4}L^2}.
\]

(\text{eq0088})

and the orthogonal conditions

\[
\langle \rho_L^*, \partial_{x_j} g_{j,L} \rangle_{H^1(\mathbb{R}^d)} = 0, \quad l = 1, 2, \ldots, d \text{ and } j = 1, 2,
\]

where \( g_{j,L} = g(\cdot - y_{j,L}) \). By (4.6) and (4.7), we may assume that \( y_{1,L} = \frac{L}{2} \varepsilon_1 + o(1) \)

and \( y_{2,L} = -\frac{L}{2} \varepsilon_1 + o(1) \). By (4.3) and (4.6), \( \rho_L^* \) satisfies the equation

\[
\begin{align*}
\begin{cases}
-\Delta \rho_L^* + \rho_L^* &= 2u_L \log u_L - 2\sum_{j=1}^{2} g_{j,L} \log g_{j,L} + f_L, \quad \text{in } \mathbb{R}^d, \\
\langle \rho_L^*, \partial_{x_j} g_{j,L} \rangle_{H^1(\mathbb{R}^d)} &= 0, \quad l = 1, 2, \ldots, d \text{ and } j = 1, 2.
\end{cases}
\end{align*}
\]

(\text{eq0091})

As before, we can write

\[
2u_L \log u_L - 2\sum_{j=1}^{2} g_{j,L} \log g_{j,L} = E_L + N_L(\rho_L^*) + 2(1 + \log(\sum_{j=1}^{2} g_{j,L}))\rho_L^*,
\]

which, together with (4.3), implies that we can re-write (4.8) as follows:

\[
\begin{align*}
\begin{cases}
-\mathcal{L}(\rho_L^*) &= E_L + N_L(\rho_L^*) - \sum_{j=1}^{d}(a_{j,+} \partial_{x_j} g_L + a_{j,-} \partial_{x_j} g_{-L}), \quad \text{in } \mathbb{R}^d, \\
\langle \rho_L^*, \partial_{x_j} g_{j,L} \rangle_{H^1(\mathbb{R}^d)} &= 0, \quad l = 1, 2, \ldots, d \text{ and } j = 1, 2.
\end{cases}
\end{align*}
\]

By Lemma 3.2 and similar estimates in the proof of Lemma 3.3, we have

\[
\|\rho_L^*\|_{L^2} \lesssim e^{-\frac{1}{4}L^2}.
\]
Now, using \( \hat{\psi} \), given by (3.45), as the test function of (4.8) and estimating as that for (3.46), we have \( \| \rho_k^+ \|_{H^1(\mathbb{R}^d)} \gtrsim e^{-\frac{d}{2} k^2} \). It follows from (4.7) that \( \| \rho_k^+ \|_{H^1(\mathbb{R}^d)} \sim e^{-\frac{d}{2} k^2} \), which, together with (4.4) and (4.7), implies that \( \| \rho_k^+ \|_{H^1(\mathbb{R}^d)} \sim \| f \|_{H^{-1}} \).

It completes the proof.

We close this section by the proof of Theorem 1.1.

**Proof of Theorem 1.1:** The conclusion follows immediately from Propositions 3.1 and 4.1.

\[ \square \]

**References**

[1] C. O. Alves, C. Ji, Multi-peak positive solutions for a logarithmic Schrödinger equation via variational methods, *Israel J. Math.*, 2022, to appear.

[2] T. Aubin, Problèmes isopérimétriques de Sobolev, *J. Differential Geometry*, 11 (1976), 573–598.

[3] A. Ambrosetti, E. Colorado, D. Ruiz, Multi-bump solutions to linearly coupled systems of nonlinear Schrödinger equations, *Calc. Var.*, 30 (2007), 85–112.

[4] A. Bahri, J. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Commun. Pure Appl. Math.*, 41 (1988), 253–294.

[5] M. Badiale, M. Guida, On some properties of the Sobolev inequality. *Ann. Mat. Pura Appl.*, 163 (1993), 1–24.

[6] J. Dolbeault, M. J. Esteban, M. Loss, Symmetry of extremals of functional inequalities via spectral estimates for linear operators. *J. Math. Phys.*, 53 (2012), article 095204, 18 pp.

[7] T. Aubin, Problèmes isopérimétriques de Sobolev. *J. Differential Geometry*, 11 (1976), 573–598.

[8] H. Brezis, E. Lieb, Sobolev inequalities with remainder terms.

[9] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with applications to nonlinear Schrödinger equations, *Comm. Pure Appl. Math.*, 41 (1988), 253–294.

[10] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, The sharp Sobolev inequality in quantitative form. *J. Funct. Anal.*, 259 (2001), 229–258.

[11] A. Cianchi, A quantitative Sobolev inequality in BV, *J. Funct. Anal.*, 237 (2006), 466–481.

[12] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, The sharp Sobolev inequality in quantitative form. *J. Eur. Math. Soc.*, 11 (2009), 1105–1139.

[13] G. Ciraolo, A. Figalli, F. Maggi, A quantitative analysis of metrics on \( \mathbb{R}^N \) with almost constant positive scalar curvature, with applications to fast diffusion flows. *Int. Math. Res. Not.*, 2018 (2017), 6780–6797.

[14] K. Chou, W. Chu, On the best constant for a weighted Sobolev-Hardy inequality. *J. London Math. Soc.*, 48 (1993), 137–151.

[15] J. Dávila, L. Dupaigne, Hardy-type inequalities. *J. Eur. Math. Soc.*, 6 (2004), 335–365.

[16] P. d’Avenia, E. Montefusco, M. Squassina, On the logarithmic Schrödinger equation, *Comm. Contemp. Math.*, 16 (2014), 1350032.

[17] M. del Pino, J. Dolbeault, The optimal Euclidean \( L^p \)-Sobolev logarithmic inequality, *J. Funct. Anal.*, 197 (2003), 151–161.

[18] G. Ciraolo, A. Figalli, F. Maggi, A quantitative analysis of metrics on \( \mathbb{R}^N \) with almost constant positive scalar curvature, with applications to fast diffusion flows. *Int. Math. Res. Not.*, 2018 (2017), 6780–6797.

[19] K. Chou, W. Chu, On the best constant for a weighted Sobolev-Hardy inequality. *J. London Math. Soc.*, 48 (1993), 137–151.
J. Dolbeault, M. J. Esteban, M. Loss, Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces, *Invent. math.*, 206 (2016), 397–440.

J. Dolbeault, G. Toscani, Improved interpolation inequalities, relative entropy and fast diffusion equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30 (2013), 917–934.

J. Dolbeault, G. Toscani, Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities, *Int. Math. Res. Not.*, 2016, (2015), 473–498.

M. Fathi, E. Indrei, M. Ledoux, Quantitative logarithmic Sobolev inequalities and stability estimates, *Discrete Contin. Dyn. Syst.*, 36 (2016), 6835–6853.

V. Felli, M. Schneider, Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type, *J. Differential Equations*, 191 (2003), 121–142.

A. Figalli, M. R. Posteraro, Some Remarks on the Stability of the Log-Sobolev Inequality for the Gaussian Measure, *Potential Anal.*, 47 (2017), 37–52.

A. Figalli, Stability in geometric and functional inequalities, European Congress of Mathematics, 585–599, Eur. Math. Soc., Zurich, 2013.

A. Figalli, F. Glaudo, On the Sharp Stability of Critical Points of the Sobolev Inequality, *Arch. Rational Mech. Anal.*, 237 (2020), 201–258.

A. Figalli, R. Neumayer, Gradient stability for the Sobolev inequality: the case $p \geq 2$, *J. Eur. Math. Soc.*, 21 (2019), 319–354.

A. Figalli, Y. Zhang, Sharp gradient stability for the Sobolev inequality, preprint, arXiv:2003.04037v1.

A. Figalli, The quantitative isoperimetric inequality and related topics, *Bull. Math. Sci.*, 5 (2015), 517–607.

A. Figalli, F. Maggi, A. Pratelli, The sharp quantitative Sobolev inequality for functions of bounded variation, *J. Funct. Anal.*, 244 (2007), 315–341.

V. Glaser, H. Grosse, A. Martin, W. Thirring, A Family of Optimal Conditions for the Absence of Bound States in a Potential, *Studies in Math. Phys.*, Princeton University Press, New Jersey, 1976, essays in Honor of Valentine Bargmann.

E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, 118 (1983), 349–374.

E. H. Lieb, M. Loss, *Analysis*, Graduate Studies in Mathematics, 14, Second edition. American Mathematical Society, Providence (2001).

C.-S. Lin, Z.-Q. Wang, Symmetry of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities, *Proc. Amer. Math. Soc.*, 132 (2004), 1685–1691.

F. Lin, W. Ni, J. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem, *Comm. Pure Appl. Math.*, 60 (2007), 252–281.

M. Musso, F. Pacard, J. Wei, Finite energy sgh-changing solutions with dihedral symmetry for the stationary nonlinear schrodinger equation, *J. Eur. Math. Soc.*, 14 (2012), 1923–1953.

R. Neumayer, A note on strong-form stability for the Sobolev inequality, *Calc. Var.*, 59 (2020), Paper No. 25.

V. Nguyen, The sharp Gagliardo-Nirenberg-Sobolev inequality in quantitative form, *J. Funct. Anal.*, 277 (2019), 2179–2208.

V. Radulescu, D. Smets, M. Willem, Hardy-Sobolev inequalities with remainder terms, *Topol. Methods Nonlinear Anal.*, 20 (2002), 145–149.

B. Rufini, Stability theorems for Gagliardo-Nirenberg-Sobolev inequalities: a reduction principle to the radial case, *Rev. Mat. Complut.*, 27 (2014), 509–539.

F. Seuffert, A stability result for a family of sharp Gagliardo-Nirenberg inequalities, preprint, arXiv:1610.06869v1.
22 J. WEI AND Y. WU

[53] W. Shuai, Multiple solutions for logarithmic Schrödinger equations, *Nonlinearity*, 32 (2019), 2201.

[54] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187 (1984), 511–517.

[55] G. Talenti, Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110 (1976), 353–372.

[56] W. Troy, Uniqueness of positive ground state solutions of the logarithmic Schrödinger equation. *Arch. Ration. Mech. Anal.*, 222 (2016), 1581–1600.

[57] Z.-Q. Wang, M. Willem, Singular Minimization Problems, *J. Differential Equations*, 161 (2000), 307–320.

[58] Z.-Q. Wang, M. Willem, Caffarelli-Kohn-Nirenberg inequalities with remainder terms, *J. Funct. Anal.*, 203 (2003), 550–568.

[59] Z.-Q. Wang, C. Zhang, Convergence from power-law to logarithm-law in nonlinear scalar field equations, *Arch. Ration. Mech. Anal.*, 231 (2019), 45–61.

[60] F.B. Weissler, Logarithmic Sobolev inequalities for the heat-diffusion semigroup, *Trans. Amer. Math. Soc.*, 237 (1978), 255–269.

[61] J. Wei, Y. Wu, On the stability of the Caffarelli-Kohn-Nirenberg inequality, *Mathematische Annalen*, 2022, Doi:10.1007/s00208-021-02325-0.

**Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2**

*Email address: jwei@math.ubc.ca*

**School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, P.R. China**

*Email address: wuyz850306@cumt.edu.cn*