AN INSENSITIZING CONTROL PROBLEM FOR A LINEAR STABILIZED KURAMOTO-SIVASHINSKY SYSTEM

KUNTAL BHANDARI* AND VÍCTOR HERNÁNDEZ-SANTAMARÍA†

Abstract. In this work, we address the existence of insensitizing controls for a coupled system of fourth- and second-order parabolic equations known as the stabilized Kuramoto-Sivashinsky model. The main idea is to look for control functions such that some functional of the state is locally insensitive to the perturbations of initial data. Let \( O \) be a nonempty observation set for the solution component(s) w.r.t. the \( L^2 \)-norm(s) and \( \omega \) be another nonempty set where the interior controls are acting. Then, we study the associated insensitizing control problem under the assumption \( O \cap \omega \neq \emptyset \) and as usual, it can be shown that this is equivalent to a null-controllability problem for a cascade system where the number of equations are doubled. This new problem is studied by means of the classical duality arguments and Carleman estimates, but unlike other insensitizing problems for scalar systems, the election of the Carleman tools depends on the number of components of the system to be insensitized.

Key words. Parabolic system, insensitizing controls, Carleman inequalities, observability.

AMS subject classifications. 35K52 - 93B05 - 93B07 - 93B35.

1. Introduction and the main result. Let \( T > 0 \) and \( \omega \subset (0,1) \) be any nonempty open set. We introduce the space \( Q_T := (0,T) \times (0,1) \) and its boundary \( \Sigma_T := (0,T) \times \{0,1\} \). We also take another nonempty open set \( O \subset (0,1) \) hereinafter referred as the observation set.

Let us consider the following control system with incomplete data

\[
\begin{aligned}
&y_t + y_{xxxx} + \gamma y_{xx} = z_x + h_1 \mathbf{1}_\omega + \xi_1 & \text{in } Q_T, \\
&z_t - z_{xx} + \beta z_x = y_x + h_2 \mathbf{1}_\omega + \xi_2 & \text{in } Q_T, \\
&y(0) = y_0 + \tau \bar{y}_0, \quad z(0) = z_0 + \tau \bar{z}_0 & \text{in } (0,1),
\end{aligned}
\]

where \( \gamma > 0 \) and \( \beta \) is any real number.

In (1.1), \( y = y(t,x) \) and \( z = z(t,x) \) are the state variables, \( h_i = h_i(t,x), \ i = 1,2 \) are control functions acting on the control set \( \omega, \ \xi_i = \xi_i(t,x), \ i = 1,2, \) are given external source terms and the initial state \((y(0),z(0))\) is partially unknown in the following sense:

- \((y_0,z_0) \in [L^2(0,1)]^2\) are given,
- \((\bar{y}_0,\bar{z}_0) \in [L^2(0,1)]^2\) are unknown and satisfy \( \|\bar{y}_0\|_{L^2(0,1)} = \|\bar{z}_0\|_{L^2(0,1)} = 1 \). They represent some uncertainty on the initial data.
- \( \tau \in \mathbb{R} \) is unknown and small enough.

From the modeling point of view, when \( h_1 \equiv \xi_i \equiv 0, \ i = 1,2, \) and \( \tau = 0 \), system (1.1) is the so-called stabilized Kuramoto-Sivashinsky system, which was proposed in [29] as a model of front propagation in reaction-diffusion phenomena and combines dissipative features with dispersive ones. For chosen initial data as above and given \((h_1,h_2) \in [L^2((0,T) \times \omega)]^2, \ (\xi_1,\xi_2) \in [L^2(Q_T)]^2\), the existence and uniqueness of solution

\[(y,z) \in [C^0([0,T]; L^2(0,1))]^2 \cap L^2(0,T; H^2_0(0,1) \times H^1_0(0,1))\]

to (1.1) can be guaranteed by Proposition B.1. This system and some variants have been studied from a null-controllability point of view in several papers, see, for instance, [10,11,14,15,24].

In this work, our main goal is to study an insensitizing control problem for (1.1). This problem, originally introduced by J.-L. Lions in [27], can be stated as follows: we observe the solution of system (1.1) through a functional \( J_\tau \) (the so-called sentinel) defined on the set of solutions to (1.1), which is in this case given by

\[
J_\tau (y,z) = \frac{\alpha}{2} \int_{\mathcal{O} \times (0,T)} |y|^2 \, dx \, dt + \frac{1 - \alpha}{2} \int_{\mathcal{O} \times (0,T)} |z|^2 \, dx \, dt, \quad \alpha \in [0,1].
\]
Then, the insensitizing control problem is to find controls \((h_1, h_2)\) such that the uncertainty in the initial data does not affect the measurement \(J_\tau\), that is

\[
\frac{\partial J_\tau(y, z)}{\partial \tau} \bigg|_{\tau=0} = 0 \quad \forall (\bar{y}_0, \bar{z}_0) \in [L^2(0, 1)]^2 \quad \text{with} \quad \|\bar{y}_0\|_{L^2(0, 1)} = \|\bar{z}_0\|_{L^2(0, 1)} = 1.
\]

When (1.3) holds, the sentinel \(J_\tau\) is said to be locally insensitive to the perturbations of the initial data.

The parameter \(\alpha\) has been introduced in (1.2) to take into account the contribution of each state variable in the sentinel. Note that the insensitivity condition (1.3) should be satisfied for any perturbation of the initial datum of both components, hence removing one observation in (1.2) (i.e. taking \(\alpha = 1\) or \(\alpha = 0\)) reduces the information available and will introduce additional difficulties in the problem.

The first results concerning the existence of insensitizing controls were obtained for linear and semilinear heat equations in [2,16]. After that, many works have been devoted to study the insensitizing problem from different perspectives: in [3,4,5], the authors study such problem for linear and semilinear heat equations with different types of nonlinearities and/or boundary conditions, while in [21] the problem of insensitizing a sentinel depending on the gradient of the solution of a linear parabolic equation is addressed. For insensitizing problems of equations in fluid mechanics we refer to the works [9,12,13,22] and for a phase field system to [7]. Most recently, the insensitizing control problem has been addressed from a numerical point of view in [6], for fourth-order parabolic equations in [25] and with respect to shape variations in [18,28].

Following the well-known arguments (see e.g. [2, Proposition 1] or [17, Appendix]), it can be proved that the insensitivity condition (1.3) is equivalent to a null-control problem for an extended system. More precisely, we have the following.

**Proposition 1.1.** Let \(O \cap \omega \neq \emptyset\). Consider the following extended system

\begin{equation}
\begin{aligned}
& y_t + y_{xxxx} + \gamma y_{xx} = z_x + h_1 \mathbb{1}_O + \xi_1 \quad \text{in } QT, \\
& z_t - z_{xx} + \beta z_x = y_x + h_2 \mathbb{1}_O + \xi_2 \quad \text{in } QT, \\
& y = y_x = z = 0 \quad \text{in } \Sigma_T, \\
& y(0) = y_0, \quad z(0) = z_0 \quad \text{in } (0,1), \\
& -p_t + p_{xxx} + \gamma p_{xx} = -q_x + \alpha y \mathbb{1}_O \quad \text{in } QT, \\
& -q_t - q_{xx} - \beta q_x = -p_x + (1 - \alpha)z \mathbb{1}_O \quad \text{in } QT, \\
& p = p_x = q = 0 \quad \text{in } \Sigma_T, \\
& p(T) = 0, \quad q(T) = 0 \quad \text{in } (0,1),
\end{aligned}
\end{equation}

Then, the controls \((h_1, h_2)\) verify the insensitivity condition (1.3) for the sentinel (1.2) if and only if the associated solution to (1.4)–(1.5) satisfies

\[
(p(0), q(0)) = (0,0) \quad \text{in } (0,1).
\]

In view of this result, in what follows we only focus on studying controllability properties for the extended system (1.4)–(1.5).

We note that the controls \((h_1, h_2)\) act indirectly on the state \((p, q)\) by means of the couplings terms exerted on the observation set \(O\), that is, we have more equations than controls. As it has been pointed out in [1], this situation is more complicated than controlling scalar systems and, as we have said before, the introduction of the parameter \(\alpha\) introduces an additional difficulty. Note that when \(\alpha = 0\) or \(\alpha = 1\) one of the couplings in system (1.5) is removed and the action of the controls \((h_1, h_2)\) enters indirectly on the backward system only through one coupling term. As we will see later, this translates into using different Carleman tools for studying the observability of the corresponding adjoint system and establishing the controllability of (1.4)–(1.5).

In this spirit, our main control result is the following.

**Theorem 1.2.** Assume that \(O \cap \omega \neq \emptyset\) and \(y_0 = z_0 = 0\). Then for any \(\alpha \in [0, 1]\), there exists a positive function \(\rho = \rho(t)\) blowing up at \(t = 0\), such that for any \((\xi_1, \xi_2) \in [L^2(Q_T)]^2\) verifying

\[
\int_{Q_T} \rho^2 |\xi_i|^2 \, dx \, dt < +\infty, \quad i = 1, 2,
\]

where \(\rho^2 \,|\xi_i|^2 \, dx \, dt \quad i = 1, 2\)
there exists controls \((h_1, h_2) \in [L^2((0, T) \times \omega)]^2\) such that the corresponding solution to (1.4)–(1.5) satisfies (1.6).

As in other insensitizing problems, the assumption on the zero initial condition is roughly related to the fact that system (1.4)–(1.5) is composed by forward and backward equations. As noticed in the work [17], even for the simple heat equation is not an easy task to characterize the space of initial datums that can be insensitized.

The assumption \(O \cap \omega \neq \emptyset\) is essential to prove an observability inequality (see Proposition 3.2 for instance), which is the main ingredient in the proof of Theorem 1.2. Notwithstanding, in [26], the authors have proved that in the simpler case of the heat equation this condition is not necessary if one considers an \(\epsilon\)-insensitizing problem (i.e., \(|\frac{\partial J_r(w, z)}{\partial x}|_{r=0} \leq \epsilon\)). In our case, this remains as an open question.

To prove Theorem 1.2, we follow standard duality arguments and we reduce the problem to study observability properties of the adjoint system associated to (1.4)–(1.5). Written in a more compact notation, the adjoint system reads as

\[
\begin{align*}
-u_t + u_{xxx} + \gamma u_{xx} &= -w_x + \alpha \zeta \mathbb{1}_{\partial \Omega} \quad \text{in } Q_T, \\
-w_t - w_{xx} - \beta w_x &= -u_x + (1 - \alpha)\theta \mathbb{1}_{\partial \Omega} \quad \text{in } Q_T, \\
\zeta_t + \zeta_{xxx} + \gamma \zeta_{xx} &= \theta_x \quad \text{in } Q_T, \\
\theta_t - \theta_{xx} + \beta \theta_x &= \zeta_x \quad \text{in } Q_T, \\
u = u_x &= w = \zeta = \zeta_x = \theta = 0 \quad \text{in } \Sigma_T, \\
u(T) = 0, \ w(T) = 0, \ \zeta(0) = \zeta_0, \ \theta(0) = \theta_0 \quad \text{in } (0, 1),
\end{align*}
\]

(1.8)

As usual, the strategy amounts to apply Carleman estimates for each equation of system (1.8) and then use the first and second equations to estimate locally the terms related to \(\zeta\) and \(\theta\). Note that for \(\alpha \in (0, 1)\), we have a natural way to estimate such terms thanks to the hypothesis \(O \cap \omega \neq \emptyset\), but as soon as \(\alpha = 0\) or \(\alpha = 1\), we lose information on either \(\zeta\) or \(\theta\) and we have to use the first-order couplings from the third and fourth equation of (1.8) to do local energy estimates. To circumvent this, we shall use some Carleman tools from the works [15] and [10] allowing us to derive with respect to \(x\) the equations verified by \(\zeta\) or \(\theta\) and then estimate locally the first-order derivative of these variables.

Before going into detail, it is worth mentioning the following well-posedness and regularity results associated to the adjoint system (1.8), thanks to the Propositions B.1 and B.2.

1. For given data \(\zeta_0, \theta_0 \in L^2(0, 1)\), there exists unique weak solution \((u, w, \zeta, \theta)\) to the system (1.8) such that

\[
\begin{align*}
u, \zeta \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1)) \cap H^1(0, T; H^{-1}(0, 1)), \\
\nu, \theta \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1)) \cap H^1(0, T; H^{-1}(0, 1)),
\end{align*}
\]

(1.9)

(1.10)

satisfying

\[
\|u\|_{C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1)) \cap H^1(0, T; H^{-1}(0, 1))} + \|w\|_{C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1)) \cap H^1(0, T; H^{-1}(0, 1))} + \|\zeta\|_{C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1)) \cap H^1(0, T; H^{-1}(0, 1))} + \|\theta\|_{C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^2(0, 1)) \cap H^1(0, T; H^{-1}(0, 1))} \leq C\|\zeta_0, \theta_0\|_{L^2(0, 1)}^2.
\]

2. If we choose the data \(\zeta_0 \in H^2_0(0, 1)\) and \(\theta_0 \in H^1_0(0, 1)\), the solution to (1.8) satisfies the following regularity results:

\[
\begin{align*}
u, \zeta \in C^0([0, T]; H^2_0(0, 1)) \cap L^2(0, T; H^4(0, 1) \cap H^2_0(0, 1)) \cap H^1(0, T; L^2(0, 1)), \\
\nu, \theta \in C^0([0, T]; H^1_0(0, 1)) \cap L^2(0, T; H^2(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; L^2(0, 1)),
\end{align*}
\]

(1.11)

(1.12)

with in addition,

\[
\|u\|_{C^0([0, T]; H^2(0, 1) \cap H^2(0, 1)) \cap H^1(0, T)} + \|w\|_{C^0([0, T]; H^2(0, 1) \cap H^2(0, 1)) \cap H^1(0, T)} + \|\zeta\|_{C^0([0, T]; H^2(0, 1) \cap H^2(0, 1)) \cap H^1(0, T)} + \|\theta\|_{C^0([0, T]; H^2(0, 1) \cap H^2(0, 1)) \cap H^1(0, T)} \leq C\|\zeta_0, \theta_0\|_{H^2_0(0, 1) \times H^1_0(0, 1)}.
\]
Paper organization. The rest of the paper is organized as follows. In Section 2 we address several Carleman estimates for different $\alpha$. More precisely, the subsections 2.1, 2.2 and 2.3 contain the Carleman inequalities associated to the adjoint system (1.8) for $\alpha \in (0,1)$, $\alpha = 0$ and $\alpha = 1$ respectively. Thereafter, in Section 3 we discuss about the required observability inequalities and as a consequence, we prove the controllability result Theorem 1.2. Finally, Section 4 is devoted to present some concluding remarks.

Throughout the paper, $C > 0$ denotes a generic constant that may vary line to line but is independent in the Carleman parameters $s$ or $\lambda$.

2. Carleman estimates for different $\alpha$. This section is devoted to obtain several Carleman inequalities depending on the parameter $\alpha$. In the sequel, we shall prove the required Carleman estimates for the three different cases:

(i) $\alpha \in (0,1)$,
(ii) $\alpha = 0$,
(iii) $\alpha = 1$.

Note that, the cases $\alpha = 0$ and $\alpha = 1$ are more interesting since we remove one of the two observed variables. In fact, depending on the above three choices of $\alpha$, the strategy to obtain a suitable Carleman estimate also changes.

Let us define some Carleman weights which have been introduced in the articles [15,21].

Weight functions. Recall that $O \cap \omega \neq \emptyset$ and therefore, there is an open set $\omega_0 \subset \subset O \cap \omega$. In what follows, we establish the Carleman estimates with the observation domain $\omega_0$.

Consider a function $\nu \in C^4([0,1])$ satisfying
\[
\begin{cases}
\nu(x) > 0 \quad \forall x \in (0,1), \\
\nu(0) = \nu(1) = 0, \\
|\nu'(x)| \geq c > 0 \quad \forall x \in (0,1) \setminus \omega_0 \quad \text{for some } c > 0.
\end{cases}
\]

In particular, we have $\nu'(0) > 0$ and $\nu'(1) < 0$.

Now, for some constants $\lambda > 1$ and $k > m > 0$, we define the weight functions
\[
\varphi_m(t,x) = \frac{e^{\lambda(1+\frac{1}{m})k||\nu||_\infty - e^{\lambda(k||\nu||_\infty + \nu(x))}}}{t^m(T-t)^m}, \quad \xi_m(t,x) = \frac{e^{\lambda(k||\nu||_\infty + \nu(x))}}{t^m(T-t)^m}, \quad \forall (t,x) \in Q_T.
\]

Here, observe that $\varphi_m$ and $\xi_m$ are positive functions in $[0,1]$ due to the choices of $\lambda$, $k$ and $m$.

Some immediate results associated with the weights are following.

- For any $b > 0$, there exists some constant $C > 0$ such that
\[
\left|\left(e^{-2s\varphi_m}\xi_m^b\right)_t\right| \leq Cs\lambda\xi_m(e^{-2s\varphi_m}\xi_m),
\]
where $\lambda > 1$ and $k > m > 0$.

- Similarly, for any $b > 0$, there is some constant $C > 0$ such that
\[
\left|\left(e^{-2s\varphi_m}\xi_m^b\right)_x\right| \leq C\xi_m^{1+\frac{1}{m}}(e^{-2s\varphi_m}\xi_m),
\]
with $\lambda > 1$ and $k > m > 0$.

- For any $m > 1$, we observe that
\[
t^{m-1}(T-t)^{m-1} \leq CT^{2m-2},
\]
i.e., $1 \leq CT^{2m-2} \frac{t(T-t)}{t^m(T-t)^m} \leq CT^{2m-2}\xi_m^{1-1/m}$
\[
i.e., \quad \xi_m^{1/m} \leq CT^{2m-2}\xi_m.
\]

Using this in (2.4), one has
\[
\left|\left(e^{-2s\varphi_m}\xi_m^b\right)_x\right| \leq CT^{2m-2}\xi_m(e^{-2s\varphi_m}\xi_m), \quad \text{for } m > 1.
\]
Some useful notations. Let us write some useful notations which are used frequently in our present work.

- By \( \int_I \), we denote the integral on \( QT \) and by \( \int_O \), we denote the integral on \((0, T) \times O \) for any non-empty open set \( O \subset (0, 1) \).
- We also declare the following notations which will simplify the expressions of our Carleman inequalities.

1. For any function \( q \in C^2(Q_T) \) and \( r > 0 \), we denote
   \[
   I_H(q; r) := \int_I e^{-2s\varphi_m} \left[ (s\xi_m)^r - 4 \lambda^r - 3 (|q_t|^2 + |q_{xx}|^2) + (s\xi_m)^r - 2 \lambda^r - 1 |q_x|^2 + (s\xi_m)^r - 1 |q|^2 \right].
   \]
2. For any \( q \in C^2([0, T]; C^4([0, 1])) \), we denote
   \[
   I_KS(q) := s^7 \lambda^5 \int_I e^{-2s\varphi_m} \xi_m^7 |\xi_m| |u|^2 + \lambda^3 \int_I e^{-2s\varphi_m} \xi_m^5 |q_{xx}|^2 + \lambda \int_I e^{-2s\varphi_m} \xi_m^3 |q_{xxx}|^2.
   \]

2.1. A Carleman estimate for the case when \( \alpha \in (0, 1) \). Recall the adjoint system (1.8) for any \( \alpha \in (0, 1) \). Then, our goal is to prove the following Carleman inequality.

**Theorem 2.1** (Carleman inequality: the case \( \alpha \in (0, 1) \)). Let the weight functions \((\varphi_m, \xi_m)\) be given by (2.2) with \( m \geq 1, k > m \). Then, there exist positive constants \( \lambda, \tilde{\sigma} := \tilde{\sigma}(T^m + T^{2m-2/5} + T^{2m-1} + T^{2m} + T^{2m+2}) \) with some \( \tilde{\sigma} > 0 \) and \( C \) such that we have the following estimate satisfied by the solution to (1.8):

\[
I_KS(u) + I_H(w; 3) + I_KS(\zeta) + I_H(\theta; 3) \leq C s^{15} \lambda^{16} \int \omega \left[ e^{-2s\varphi_m} \xi_m^{15} |u|^2 + s^3 \lambda^4 \int_i e^{-2s\varphi_m} \xi_m^3 |w|^2 \right],
\]

for all \( \lambda \geq \tilde{\lambda} \) and \( s \geq \tilde{\sigma} \), where \( I_KS(\cdot) \) and \( I_H(\cdot; \cdot) \) are given by (2.8) and (2.7) respectively.

To prove the above Carleman inequality, we start by providing the partial Carleman inequalities for the adjoint states \( u, w, \zeta \) and \( \theta \) of the equation (1.8).

(i) For the variables \( u \) and \( \zeta \), satisfying fourth order parabolic equations, we use a Carleman estimate from the work [30]; a similar estimate has been applied for instance in [15].

(ii) For \( w \) and \( \theta \), we use the classical Carleman inequalities for the heat equations, thanks to the pioneer work [20].

**Lemma 2.2** (Carleman inequality for \( u \), the case \( \alpha \in (0, 1) \)). Let \( \varphi_m \) and \( \xi_m \) be as given by (2.2) with \( m \geq 2/5 \). Then, there exist positive constants \( \tilde{\lambda}_1, \tilde{\sigma}_1 := \tilde{\sigma}_1(T^{2m} + T^{2m-2/5}) \) with some \( \tilde{\lambda}_1 > 0 \) and \( C \), such that we have the following estimate for \( u \in L^2(0, T; L^2(0, 1)) \cap H^1(0, T; L^2(0, 1)) \),

\[
I_KS(u) \leq C \left( \int \omega \left[ e^{-2s\varphi_m} (|u_t|^2 + \alpha^2 |\zeta|^2) + \lambda^3 \int \omega \left[ e^{-2s\varphi_m} \xi_m^3 |u|^2 \right] \right),
\]

for all \( \lambda \geq \tilde{\lambda}_1, s \geq \tilde{\sigma}_1 \), where \( I_KS(\cdot) \) is defined by (2.8).

**Lemma 2.3** (Carleman inequality for \( w \), the case \( \alpha \in (0, 1) \)). Let \( \varphi_m \) and \( \xi_m \) be defined as in (2.2) with \( m \geq 1 \). Then, there exist positive constants \( \tilde{\lambda}_2, \tilde{\sigma}_2 := \tilde{\sigma}_2(T^{2m} + T^{2m-1}) \) with some \( \tilde{\lambda}_2 > 0 \) and \( C \), such that we have the following estimate for \( w \in L^2(0, T; H^2(0, 1)) \cap H^1(0, T; L^2(0, 1)) \),

\[
I_H(w; 3) \leq C \left( \int \omega \left[ e^{-2s\varphi_m} (|u_t|^2 + (1 - \alpha)^2 |\theta|^2) + \lambda^4 \int \omega \left[ e^{-2s\varphi_m} \xi_m^3 |w|^2 \right] \right),
\]

for all \( \lambda \geq \tilde{\lambda}_2 \) and \( s \geq \tilde{\sigma}_2 \), where \( I_H(\cdot; \cdot) \) is defined by (2.7).
Lemma 2.4 (Carleman inequality for $\zeta$, the case $\alpha \in (0, 1)$). Let $\varphi_m$ and $\xi_m$ be as given by (2.2) with $m \geq 2/5$. Then, there exist positive constants $\tilde{\lambda}_3$, $\tilde{s}_3 := \tilde{s}_3(T^{2m} + T^{2m-2/5})$ with some $\tilde{s}_3 > 0$ and $C$, such that we have the following estimate for $u \in L^2(0, T; H^1(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; L^2(0, 1))$,

$$I_{KS}(\zeta) \leq C \left( \int_{\omega} e^{-2s\varphi_m}|\theta_x|^2 + s^7\lambda^8 \int_{\omega} e^{-2s\varphi_m}\xi_m^7|\zeta|^2 \right),$$

for all $\lambda \geq \tilde{\lambda}_3$, $s \geq \tilde{s}_3$, where $I_{KS}(\cdot)$ is defined by (2.8).

Lemma 2.5 (Carleman inequality for $\theta$, the case $\alpha \in (0, 1)$). Let $\varphi_m$ and $\xi_m$ be defined as in (2.2) with $m \geq 1$. Then, there exist positive constants $\tilde{\lambda}_4$, $\tilde{s}_4 := \tilde{s}_4(T^{2m} + T^{2m-1})$ with some $\tilde{s}_4 > 0$ and $C$, such that we have the following estimate for $\theta \in L^2(0, T; H^2(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; L^2(0, 1))$,

$$I_H(\theta; 3) \leq C \left( \int_{\omega} e^{-2s\varphi_m}|\xi_x|^2 + s^3\lambda^4 \int_{\omega} e^{-2s\varphi_m}\xi_m^3|\theta|^2 \right),$$

for all $\lambda \geq \tilde{\lambda}_4$ and $s \geq \tilde{s}_4$, where $I_H(\cdot; \cdot)$ is defined by (2.7).

Using the above four Carleman inequalities, we now prove our joint Carleman inequality (2.9).

Proof of Theorem 2.1. Adding the Carleman inequalities (2.10), (2.11), (2.12) and (2.13), we have the following: there is some $C_\alpha > 0$, such that,

$$I_{KS}(u) + I_H(w; 3) + I_{KS}(\zeta) + I_H(\theta; 3)$$

$$\leq C_\alpha \left[ \int_{\omega} e^{-2s\varphi_m}(|w_x|^2 + |\zeta|^2 + |u_x|^2 + |\theta|^2 + |\theta_x|^2 + |\xi_x|^2) + s^7\lambda^8 \int_{\omega} e^{-2s\varphi_m}\xi_m^7|u|^2 \right.$$

$$+ s^3\lambda^4 \int_{\omega} e^{-2s\varphi_m}\xi_m^3|w|^2 + s^7\lambda^8 \int_{\omega} e^{-2s\varphi_m}\xi_m^7|\theta|^2 + s^3\lambda^4 \int_{\omega} e^{-2s\varphi_m}\xi_m^3|\theta|^2 \left. \right],$$

for all $\lambda \geq \tilde{\lambda} := \max\{\tilde{\lambda}_3 : 1 \leq j \leq 4\}$, $s \geq b_1(T^{2m} + T^{2m-1} + T^{2m-2/5})$ for some $b_1 \geq \max\{\tilde{s}_j : 1 \leq j \leq 4\}$.

**Step 1**: Absorbing the lower order integrals. Using the fact that $1 \leq CT^{2m}\xi_m$, we can deduce that

$$\int_{\omega} e^{-2s\varphi_m}(|w_x|^2 + |\zeta|^2 + |u_x|^2 + |\theta|^2 + |\theta_x|^2 + |\xi_x|^2)$$

$$\leq C \int_{\omega} e^{-2s\varphi_m}(T^{2m}\xi_m|w_x|^2 + T^{10m}\xi_m^3|u_x|^2 + T^{2m}\xi_m|\theta_x|^2 + T^{10m}\xi_m^3|\zeta|^2)$$

$$+ C \int_{\omega} e^{-2s\varphi_m}(T^{14m}\xi_m^7|\theta|^2 + T^{6m}\xi_m^3|\theta|^2).$$

Thus, for any $\lambda \geq \tilde{\lambda}$ and $s \geq b_2T^{2m}$, for some $b_2 > 0$, the integrals appearing in the right hand side of (2.15) can be absorbed by the associated integrals in the left hand side of (2.14).

**Step 2**: Absorbing the observation integral associated to $\zeta$. Let us choose a nonempty set $\omega_1 \subset \subset \omega_0$ and a function

$$\phi \in C_c^\infty(\omega_0) \quad \text{with} \quad 0 \leq \phi \leq 1 \quad \text{in} \quad \omega_0, \quad \phi = 1 \quad \text{in} \quad \omega_1.$$

Without loss of generality, we consider the Carleman estimate (2.14) with the observation domain $\omega_1$.

Now, recall the adjoint system (1.8), one has

$$\alpha\zeta = -u_t + u_{xxx} + \gamma u_{xx} + w_x \quad \text{in} \quad \omega_0, \quad \text{since} \quad \omega_0 \subset \subset \mathcal{O}.$$

Using this, we see

$$\int_{\omega_1} s^7\lambda^8 \int_{\omega_1} e^{-2s\varphi_m}\xi_m^7|\zeta|^2 \leq s^7\lambda^8 \int_{\omega_0} \phi e^{-2s\varphi_m}\xi_m^7|\zeta|^2 \leq \frac{1}{\alpha} s^7\lambda^8 \int_{\omega_0} \phi e^{-2s\varphi_m}\xi_m^7(-u_t + u_{xxx} + \gamma u_{xx} + w_x) := \frac{1}{\alpha} (A_1 + A_2 + A_3 + A_4).$$
Then, for any $\epsilon > 0$, we observe that for any $\epsilon > 0$, that

$$|A_1| \leq \epsilon s^{-1} \int e^{-2s\varphi_m} \xi_{m-1} \frac{1}{|\xi|} + \epsilon s^{7} \int e^{-2s\varphi_m} \xi_m \frac{1}{|\xi|} + C \epsilon s^{15} \int e^{-2s\varphi_m} \xi_{m+1} |u|^2.$$

- Estimate for $A_2$. Next, by performing a successive number of integration by parts on $A_2$ with respect to $x$, we get

$$|A_2| := s^7 \lambda^8 \int \phi e^{-2s\varphi_m} \xi_m^7 \xi_{uxxx} u | \leq C \left( s^7 \lambda^8 \int \phi e^{-2s\varphi_m} \xi_m^7 \xi_{uxxx} u \right) + s^7 \lambda^8 \int \phi s e^{-2s\varphi_m} \xi_m^7 \xi_{xx} u + s^7 \lambda^8 \int \phi \xi e^{-2s\varphi_m} \xi_m^7 \xi_{xu} u + s^7 \lambda^8 \int \phi e^{-2s\varphi_m} \xi_m^7 \xi_{uxxx} u.$$
for any small $\epsilon > 0$.

Thus, using the estimates (2.18), (2.21), (2.22) and (2.23), the integral in (2.17) can be estimated as

\begin{equation}
(2.24) \quad s^7 \lambda^8 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^2 |\zeta|^2 \\
\leq C_\omega \epsilon \left( s^7 \lambda^8 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^2 |\zeta|^2 + s^5 \lambda^6 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^5 |\zeta_x|^2 + s^3 \lambda^4 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |\zeta_{xx}|^2 \right) \\
+ s^2 \lambda^2 \int_{\omega_0} e^{-2s\varphi_m} \xi_m |\zeta_{xxxx}|^2 + s^{-1} \int_{\omega_0} e^{-2s\varphi_m} \xi_m^{-1} (|\zeta_t|^2 + |\zeta_{xxxx}|^2) \\
+ \frac{C_\omega}{\epsilon} s^{15} \lambda^{16} \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |u|^2 + \frac{C_\omega}{\epsilon} s^3 \lambda^{10} \int_{\omega_0} e^{-2s\varphi_m} \xi_m^9 |w|^2.
\end{equation}

**Step 3: Absorbing the observation integral associated to $\theta$.** Recall the cut-off function $\phi$ given by (2.16) and the adjoint system (1.8). One has

\[(1 - \alpha) \theta = - w_t - w_{xx} - \beta w_x + u_x \quad \text{in $\omega_0$},\]

and so

\begin{equation}
(2.25) \quad s^3 \lambda^4 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^3 |\theta|^2 \leq s^3 \lambda^4 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 |\theta|^2 \\
= \frac{1}{1 - \alpha} s^3 \lambda^4 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 \theta (- w_t - w_{xx} - \beta w_x + u_x) := \frac{1}{1 - \alpha} (B_1 + B_2 + B_3 + B_4).
\end{equation}

- **Estimate for $B_1$.** We look into the term $B_1$,

\[B_1 = - s^3 \lambda^4 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 \theta w_t \\
= s^3 \lambda^4 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 \theta w + s^3 \lambda^4 \int_{\omega_0} \phi (e^{-2s\varphi_m} \xi_m^3) \theta w.
\]

Now, recall the estimate (2.6), so that one has

\[\left| (e^{-2s\varphi_m} \xi_m^3) \right| \leq CT^{2m-2} e^{-2s\varphi_m} \xi_m^5.
\]

Thanks to this and for any $\epsilon > 0$, we have (by using the Young’s inequality) that

\begin{equation}
(2.26) \quad |B_1| \leq \epsilon s^{-1} \int_{\omega_0} e^{-2s\varphi_m} \xi_m^4 |\theta_t|^2 + \epsilon s^3 \lambda^4 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |\theta|^2 + \frac{C_\epsilon}{\epsilon} s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^7 |w|^2.
\end{equation}

- **Estimate for $B_2$.** Let us focus on $B_2$; after a consecutive number of integration by parts with respect to $x$, one can deduce that

\begin{equation}
(2.27) \quad |B_2| \leq s^3 \lambda^4 \int_{\omega_0} \left| \left( \phi e^{-2s\varphi_m} \xi_m^3 \right)_x \theta w \right| + 2 s^3 \lambda^4 \int_{\omega_0} \left| \left( \phi e^{-2s\varphi_m} \xi_m^3 \right)_x \theta_x w \right| \\
+ s^3 \lambda^4 \int_{\omega_0} \left| \phi e^{-2s\varphi_m} \xi_m^3 \theta_{xx} w \right| \leq C s^5 \lambda^6 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^5 |\theta w| + C s^4 \lambda^5 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^4 |\theta_x w| \\
+ C s^3 \lambda^4 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |\theta_{xx} w|,
\end{equation}

thanks to the result (2.20) (with $b = 3$ and $n = 2$).

Now, for any $\epsilon > 0$, applying the Young’s inequality, we get from (2.27),

\begin{equation}
(2.28) \quad |B_2| \leq \epsilon s^3 \lambda^4 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |\theta|^2 + \epsilon s^3 \lambda^4 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2 + \epsilon s^{-1} \int_{\omega_0} e^{-2s\varphi_m} \xi_m^{-1} |\theta_{xx}|^2 \\
+ \frac{C_\epsilon}{\epsilon} s^7 \lambda^8 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^7 |w|^2.
\end{equation}
– Estimate for $B_3$. Now, we compute some estimate for $B_3$.

\[
B_3 = -\beta s^3 \lambda^4 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 \theta w_x
= s^3 \lambda^4 \int_{\omega_0} (\phi e^{-2s\varphi_m} \xi_m^3) \theta w + s\lambda^4 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 \theta w_x.
\]

Using the estimates for $n$-th derivative of $(\phi e^{-2s\varphi_m} \xi_m^3)$, given by (2.20), and applying the Young’s inequality for any $\epsilon > 0$, we get

\[
(2.29) \quad |B_3| \leq C s^5 \lambda^6 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 |\theta w| + s^3 \lambda^4 \int_{\omega_0} \phi e^{-2s\varphi_m} \xi_m^3 |\theta_x w| \leq \epsilon s^4 \lambda^4 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |\theta|^2 + \epsilon s^2 \lambda^2 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2 + \frac{C}{\epsilon} s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^7 |w|^2.
\]

– Estimate for $B_4$. Analogously, the term $B_4$ satisfies

\[
(2.30) \quad |B_4| \leq \epsilon s^3 \lambda^4 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^3 |\theta|^2 + \epsilon s^2 \lambda^2 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2 + \frac{C}{\epsilon} s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^7 |w|^2.
\]

Now, using the estimates (2.26), (2.28), (2.29) and (2.30), eventually we have the following from (2.25),

\[
(2.31) \quad s^3 \lambda^4 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^3 |\theta|^2 \leq C \alpha \left( s^3 \lambda^4 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^3 |\theta|^2 + s \lambda^2 \int_{\omega_1} e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2 \right)
+ s^{-1} \int_{\omega_1} e^{-2s\varphi_m} \xi_m^{-1} (|\theta|^2 + |\theta_x|^2) + \frac{C \alpha}{\epsilon} s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^7 (|w|^2 + |u|^2).
\]

Finally, we choose $\epsilon > 0$ small enough, so that all the integrals in the right hand side of (2.24) and (2.31) with the coefficient $C \alpha \epsilon$, can be absorbed by the leading integrals of the estimate (2.14). Hence, the required Carleman inequality (2.9) follows.

\[
\Box
\]

2.2. A Carleman estimate for the case when $\alpha = 0$. The adjoint system (1.8) for the case $\alpha = 0$ reads as

\[
\begin{aligned}
-u_t + u_{xxxx} + g u_{xx} &= -w_x & & \text{in } Q_T, \\
-w_t - u_{xx} - \beta w_x &= -u_x + \theta \mathbf{1}_{\partial} & & \text{in } Q_T, \\
\zeta_t + \zeta_{xxxx} + g \zeta_{xx} &= \theta_x & & \text{in } Q_T, \\
\theta_t - \theta_{xx} + \beta \theta_x &= \zeta_x & & \text{in } Q_T, \\
u &= u_x = w = \zeta = \zeta_x = \theta = 0 & & \text{in } \Sigma_T, \\
u(T) &= 0, \quad w(T) &= 0, & & \text{in } (0,1), \\
\zeta(0) &= \zeta_0, \quad \theta(0) &= \theta_0 & & \text{in } (0,1).
\end{aligned}
\]

Before going to prove the required Carleman inequality associated to the system (2.32), we recall the weight functions $\varphi_m$ and $\xi_m$ as given by (2.2). We also define

\[
(2.33) \quad
\begin{cases}
\varphi_m(t) = \max_{x \in [0,1]} \varphi_m(t,x) = \varphi_m(t,0) = \varphi_m(t,1), \\
\xi_m(t) = \min_{x \in [0,1]} \xi_m(t,x) = \xi_m(t,0) = \xi_m(t,1).
\end{cases}
\]

We now prove the following Carleman inequality for the adjoint system (2.32).

\textbf{Theorem 2.6} (Carleman inequality: the case $\alpha = 0$). Let $(\varphi_m, \xi_m)$ and $(\varphi_m^*, \xi_m^*)$ be given by (2.2) and (2.33) respectively with $m \geq 2$, $k > m$. Then, there exist positive constants $\lambda^*, s^* :=
\( \sigma^* (T^m + T^{2m-2/5} + T^{2m-1} + T^{2m} + T^{2m+2}) \) with some \( \sigma^* > 0 \) and \( C \) such that we have the following estimate satisfied by the solution to (2.32):

\[
\begin{align*}
I_{K5}(w) + I_H(w; 3) + s7^8 & \int \int e^{-2s\varphi_m - \xi_m^7} |\xi_x|^2 + s7^8 \int \int e^{-2s\varphi_m (\xi_m^7)^T} |\xi|^2 + I_H(\theta; 9) \\
\leq & \quad C s8^{39} \lambda^{24} \int \int_{\omega_0} e^{-10s\varphi_m + 8s\varphi_m} \xi_m^{39} |u|^2 + C s41^{36} \int \int_{\omega_0} e^{-10s\varphi_m + 8s\varphi_m} \xi_m^{31} |u|^2
\end{align*}
\]

for all \( \lambda \geq \lambda^* \) and \( s \geq s^* \), where \( I_{K5}(\cdot) \) and \( I_H(\cdot; \cdot) \) are given by (2.8) and (2.7) respectively.

Let us shortly point out the idea behind the proof for the Carleman inequality (2.34).

(i) In this case, observe that the Carleman estimate for \( \xi \) will always be associated with an observation integral of \( \xi \) and there is no chance to absorb it by any of the leading integrals. In fact, there is a coupling by \( \xi_x \) to the equation of \( \theta \) and therefore, we are going to use a Carleman estimate for the variable \( \xi_x \), see Lemma 2.10.

In this context, we recall the work [10], where such a Carleman estimate has been established.

(ii) For the variable \( u \) satisfying fourth order parabolic equation, we use the Carleman estimate given by Lemma 2.8.

(iii) For \( w \) and \( \theta \), we use the classical Carleman inequalities (see [20]) for the heat equation, possibly with different powers of Carleman parameters.

**Remark 2.7.** Observe that, in the Carleman estimate (2.34), there appears the weight function \( e^{-10s\varphi_m + 8s\varphi_m} \). But, by Lemma A.1, there exists some \( \delta > 0 \) such that

\[
-10s\varphi_m + 8s\varphi_m \leq \frac{-\delta s}{t_m(T-t)^m},
\]

which ensures obtaining a suitable observability inequality from the Carleman estimate (2.34), see Proposition 3.2.

Let us give the required Carleman estimates for the quantities \( u, w, \xi_x \) and \( \theta \). As earlier, we shall use the Carleman estimate for \( u \) ( [15,30]) which is the adjoint state for the KS equation.

**Lemma 2.8** (Carleman inequality for \( u \), the case \( a = 0 \)). Let \( \varphi_m \) and \( \xi_m \) be as given by (2.2) with \( m \geq 2/5 \). Then, there exist positive constants \( \lambda_1, \tilde{s}_1 := \sigma_1(T^{2m} + T^{2m-2/5}) \) with some \( \sigma_1 > 0 \) and \( C \), such that we have the following estimate for \( u \in L^2(0, T; H^4(0, 1) \cap H_0^1(0, 1) \cap H^1(0, T; L^2(0, 1)), )

\[
I_{K5}(u) \leq C \left( \int \int e^{-2s\varphi_m} |w_x|^2 + s7^8 \int \int \xi_m^{37} |u|^2 \right)
\]

for all \( \lambda \geq \lambda_1 \), \( s \geq \tilde{s}_1 \), where \( I_{K5}(\cdot) \) is defined by (2.8).

For the component \( w \), we shall use the standard Carleman inequality for heat equations.

**Lemma 2.9** (Carleman inequality for \( w \), the case \( a = 0 \)). Let \( \varphi_m \) and \( \xi_m \) be defined as in (2.2) with \( m \geq 1 \). Then, there exist positive constants \( \lambda_2, \tilde{s}_2 := \sigma_2(T^{2m} + T^{2m-1}) \) with some \( \sigma_2 > 0 \) and \( C \), such that we have the following estimate for \( w \in L^2(0, T; H^2(0, 1) \cap H_0^1(0, 1) \cap H^1(0, T; L^2(0, 1)), )

\[
I_H(w; 3) \leq C \left( \int \int e^{-2s\varphi_m} (|w_x|^2 + |\theta|^2) + s3^4 \int \int \xi_m^{31} |w|^2 \right)
\]

for all \( \lambda \geq \lambda_2 \) and \( s \geq \tilde{s}_2 \), where \( I_H(\cdot; \cdot) \) is defined by (2.7).

We now write the Carleman estimate for \( \xi_x \), thanks to the work [10].

**Lemma 2.10** (Carleman inequality for \( \xi_x \), the case \( a = 0 \)). Let \( (\varphi_m, \xi_m) \) and \( (\varphi_m, \xi_m^*) \) be given by (2.2) and (2.33) respectively with \( m \geq 2, k > m \). Then, there exist positive constants \( \lambda_3, \tilde{s}_3 := \sigma_3(T^m + T^{2m}) \) with some \( \sigma_3 > 0 \) and \( C \) such that we have the following estimate for \( \xi_x \in L^2(0, T; H^3(0, 1) \cap H_0^1(0, 1)), )

\[
s7^8 \int \int \left( e^{-2s\varphi_m} \xi_m^7 |\xi_x|^2 + e^{-2s\varphi_m (\xi_m^7)^T} |\xi|^2 \right) + s5^{4} \lambda^{5} \int \int e^{-s\varphi_m} \xi_m^{3} \xi_m^{3} |\xi_x|^2 + \frac{4}{s} \xi_m^{3} \xi_m^{3} |\xi|^2 + \frac{4}{s} \xi_m^{3} \xi_m^{3} |\xi|^2 \int L^2(0, T; H^4(0, 1))
\]

\[
\leq C \left[ s5^4 T^{4m+4} \int \int e^{-2s\varphi_m} \xi_m^7 |\theta_x|^2 + T^{10m} \int \int e^{-2s\varphi_m} \xi_m^{5} |\theta_x|^2 + s7^8 \int \int e^{-2s\varphi_m} \xi_m^{3} |\xi_x|^2 \right]
\]

for all \( \lambda \geq \lambda_3 \) and \( s \geq \tilde{s}_3 \).
The proof of Lemma 2.10 can be verified using the techniques developed in [10, Section 3.2]. For sake of completeness, we give a short sketch below.

**Proof of Lemma 2.10.** Recall the equation of $\zeta$ from the system (2.32) and consider the following equation for $\zeta := \zeta_s$, given by

$$
\begin{cases}
\tilde{\zeta}_t + \tilde{\zeta}_{xxx} + \gamma \tilde{\zeta}_{xx} = \theta_{xx} & \text{in } Q_T, \\
\tilde{\zeta} = 0, \quad \tilde{\zeta}_x = \zeta_{xx} & \text{in } \Sigma_T, \\
\zeta(0, \cdot) = 0 & \text{in } (0,1),
\end{cases}
$$

(2.38)

where it is clear that $\theta_{xx} \in L^2(Q_T)$ and we have $\tilde{\zeta}_s(\cdot) \in L^2(0, T)$ in $\Sigma_T$.

Now, thanks to [10, Theorem 3.5], we have the following auxiliary estimate for $\zeta_s$,

$$
\begin{align*}
s^7 \lambda^8 \iint e^{-2s\varphi_m} \tilde{\zeta}^7 \langle \zeta \rangle^2 & \leq C \left[ \iint e^{-2s\varphi_m} |\theta_{xx}|^2 \
+ s^5 \lambda^5 \int_0^T e^{-2s\varphi_m} (\xi_m^*)^5 (|\langle \zeta \rangle_0 (t, 0) |^2 + |\langle \zeta \rangle (t, 1) |^2) + s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi_m} \tilde{\zeta}^7 \langle \zeta \rangle^2 \right],
\end{align*}
$$

(2.39)

for every $\lambda \geq c_1$ and $s \geq c_2(T^m + T^{2m-2/5})$ for some $c_1, c_2 > 0$.

First observe that,

$$
\begin{align*}
s^7 \lambda^8 \iint e^{-2s\varphi_m} (\xi_m^*)^7 |\zeta|^2 & \leq C s^7 \lambda^8 \iint e^{-2s\varphi_m} \xi_m^7 |\zeta|^2,
\end{align*}
$$

(2.40)

since $\zeta(t, 0) = \zeta(t, 1) = 0$, $\forall t \in [0, T]$.

Next, we define $\rho(t) = s^{5/2-1/m} \lambda^{5/2} e^{-s\varphi_m} (\xi_m^*)^{5/2-1/m}$, $\forall t \in [0, T]$, and consider the equation satisfied by $\zeta_\rho := \rho \zeta_s$,

$$
\begin{cases}
(\zeta_{\rho})_t + (\zeta_{\rho})_{xxx} + \gamma (\zeta_{\rho})_{xx} = \rho \theta_{xx} - \rho t \zeta & \text{in } Q_T, \\
(\zeta_{\rho}) = (\zeta_{\rho})_x = 0 & \text{in } \Sigma_T, \\
\zeta_{\rho}(0, \cdot) = 0 & \text{in } (0,1).
\end{cases}
$$

Then, using the regularity result for KS equations given by Proposition B.2, we have

$$
\begin{align*}
||\zeta_{\rho}||^2_{L^2(0,T;H^4(0,1))} & \leq C \left( s^{5-2/m} \lambda^{5} ||e^{-s\varphi_m} (\xi_m^*)^{5/2-1/m}\theta_{xx}||^2_{L^2(Q_T)} \
+ s^7 \lambda^8 ||e^{-s\varphi_m} (\xi_m^*)^{7/2} \zeta||^2_{L^2(0,T;H^4(0,1))} \right),
\end{align*}
$$

(2.41)

for all $s \geq C(T^m + T^{2m})$.

Also, by some standard interpolation result and Young’s inequality, one has

$$
\begin{align*}
s^5 \lambda^5 \left( ||e^{-s\varphi_m} (\xi_m^*)^{5/2} \zeta_{xx}(t,0)||^2_{L^2(0,T)} + ||e^{-s\varphi_m} (\xi_m^*)^{5/2} \zeta_{xx}(t,1)||^2_{L^2(0,T)} \right) \\
\leq C s^7 \lambda^7 ||e^{-s\varphi_m} (\xi_m^*)^{7/2} \zeta||^2_{L^2(0,T;H^4(0,1))} + C s^4 \lambda^4 ||e^{-s\varphi_m} (\xi_m^*)^2 \zeta||^2_{L^2(0,T;H^4(0,1))}.
\end{align*}
$$

(2.42)

So, thanks to the estimates (2.40), (2.41) and (2.42), we have from (2.39),

$$
\begin{align*}
s^7 \lambda^8 \iint \left( e^{-2s\varphi_m} \xi_m^7 |\zeta|^2 + e^{-2s\varphi_m} (\xi_m^*)^7 |\zeta|^2 \right) + s^5 \lambda^5 ||e^{-s\varphi_m} (\xi_m^*)^{5/2} - \frac{\tilde{\zeta}}{2} ||^2_{L^2(0,T;H^4(0,1))} \\
\leq C \left[ \iint \left( s^{5-2/m} \lambda^{5} \theta_{xx}^2 + e^{-2s\varphi_m} (\xi_m^*)^{5/2} \theta_{xx}^2 \right) + s^7 \lambda^8 \iint e^{-2s\varphi_m} (\xi_m^*)^7 |\zeta|^2 \right] \\
+ s^7 \lambda^8 \iint e^{-2s\varphi_m} (\xi_m^*)^7 |\zeta|^2 + s^5 \lambda^5 ||e^{-s\varphi_m} (\xi_m^*)^{2} \zeta||^2_{L^2(0,T;H^4(0,1))} + s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi_m} \xi_m^7 |\zeta||^2,
\end{align*}
$$

(2.43)

It is clear that the terms associated to $\zeta$ and $\zeta_\rho$ in $Q_T$ can be easily absorbed in terms of the l.h.s. of (2.43). Then, using the facts that $\xi_m^{-2/m} \leq CT^4$, $1 \leq CT^{2m} \xi_m$ (and consequently, $\xi_m^5 \leq CT^{4m} \xi_m^5$), we can deduce the required Carleman inequality (2.37). \(\square\)
Next, for the variable $\theta$, we write the following Carleman inequality (usual for the heat equation).

**Lemma 2.11** (Carleman inequality for $\theta$, the case $\alpha = 0$). Let $\varphi_m$ and $\xi_m$ be defined as in (2.2) with $m \geq 1$. Then, there exist positive constants $\lambda_3$, $s_4 := 2^3(T^{2m} + T^{2m-1})$ with some $\sigma_1 > 0$ and $C$, such that we have the following estimate for $\theta \in H^2(0, T; H^4(0, 1)) \cap H^1(0, T; L^2(0, 1))$,

\begin{equation}
I_H(\theta; \delta) \leq C \left( \int_{\omega_0} |w_0|^2 + |u_0|^2 + |\theta|^2 e^{-4 \varphi_m} \xi_m^7 |\xi|^2 + e^{-3 \varphi_m} \xi_m^7 |\xi|^2 \right) + s^9 \lambda^{10} \int_{\omega_0} e^{-2 \varphi_m} \xi_m^9 |\theta|^2, \end{equation}

for all $\lambda \geq \lambda_3$ and $s \geq s_4$, where $I_H(\cdot, \cdot)$ is defined by (2.7).

**Proof of Theorem 2.6.** To obtain the Carleman estimate (2.34), let us first add all four Carleman estimates (2.35), (2.36), (2.37) and (2.44), which yields

\begin{equation}
I_{KS}(u) + I_H(w; 3) + s^3 \lambda^8 \int_{\omega_0} \left( e^{-2 \varphi_m} \xi_m^7 |\xi|^2 + e^{-2 \varphi_m} \xi_m^7 |\xi|^2 \right)
+ s^5 \lambda^5 \int_{\omega_0} e^{-3 \varphi_m} \xi_m^7 |\xi|^2 + s^9 \lambda^{10} \int_{\omega_0} e^{-2 \varphi_m} \xi_m^9 |\theta|^2, \end{equation}

for all $\lambda \geq \lambda^* := \max\{\lambda_1 : 1 \leq j \leq 4\}$ and $s \geq \check{c}_1(T^m + T^{2m} + T^{2m-1} + T^{2m-2/3})$ for some $\check{c}_1 \geq \max\{\check{c}_j : 1 \leq j \leq 4\}$.

**Step 1:** Absorbing the lower order integrals. Using the fact that $1 \leq CT^{2m} \xi_m$, we can deduce that

\begin{equation}
\int_{\omega_0} e^{-2 \varphi_m} \left( |w_0|^2 + |u_0|^2 + |\theta|^2 \right) + s^5 \lambda^5 T^{4m+4} \int_{\omega_0} e^{-2 \varphi_m} \xi_m^7 |\theta|^2 \right)
+ s^9 \lambda^{10} \int_{\omega_0} e^{-2 \varphi_m} \xi_m^9 |\theta|^2, \end{equation}

and thus by choosing $\lambda \geq \lambda^*$ and $s \geq \check{c}_2(T^{2m} + T^{2m+2})$ for some $\check{c}_2 > 0$, one can absorb all the source integrals appearing in the r.h.s. of (2.46) by the associated higher order integrals in the l.h.s. of (2.45).

**Step 2:** Absorbing the observation integral associated to $\xi_x$. Using the equation $\xi_x = \theta_t - \theta_{xx} + \beta \theta_{xx}$, and then following the techniques developed in [10, Step 4–Proposition 3.6], we can eliminate the observation integral of $\xi_x$. The resultant estimate can be then written as

\begin{equation}
I_{KS}(u) + I_H(w; 3) + s^3 \lambda^8 \int_{\omega_0} \left( e^{-2 \varphi_m} \xi_m^7 |\xi|^2 + e^{-2 \varphi_m} \xi_m^7 |\xi|^2 \right) + s^9 \lambda^{10} \int_{\omega_0} e^{-2 \varphi_m} \xi_m^9 |\theta|^2, \end{equation}

for every $\lambda \geq \lambda^*$ and $s \geq \check{c}_3(T^m + T^{2m} + T^{2m-1} + T^{2m-2/3} + T^{2m+2})$, for some $\check{c}_3 > 0$.

**Step 3:** Absorbing the observation integral associated to $\theta$. We choose a smooth function $\phi \in C_c^\infty(\omega_0)$ such that $0 \leq \phi \leq 1$ in $\omega_0$ and $\phi = 1$ in $\tilde{\omega} \subset \subset \omega_0$ for some $\tilde{\omega} \neq \emptyset$ open set and without loss of generality, let us consider the Carleman estimate (2.47) with the observation domain $\tilde{\omega}$.
Then, we focus on the term

\[(2.48) \quad J := s^{23} \lambda^{16} \int_{\Omega} e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} |\theta|^2 \leq s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} |\theta|^2.\]

From the second equation in (2.32), we have

\[\theta = -u_t - u_{xx} - \beta w_x + u_x \quad \text{in } \Omega \quad \text{(consequently in } \omega_0)\]

Using this, we see

\[s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} |\theta|^2 = s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} (u_x - u_t - w_{xx} - \beta w_x)\]

\[= \sum_{1 \leq i \leq 4} J_i.\]

\[\text{– Estimate for } J_1. \quad \text{We first look into the term } J_1.\]

\[J_1 = s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} \theta u_x = -s^{23} \lambda^{16} \int_{\omega_0} (\phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}}) \theta u - s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} \theta_x u.\]

At this point, we recall the estimate (2.20), so that, one has

\[(2.49) \quad |(\phi e^{-6s\varphi_m \xi_m^{23}})_{n,x}| \leq C s^n \lambda^n e^{-6s\varphi_m \xi_m^{23+n}}, \quad \text{for } n \in \mathbb{N}^+,\]

and applying the Young’s inequality with any \(\epsilon > 0\), we obtain

\[(2.50) \quad |J_1| \leq C s^{24} \lambda^{17} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{24}} |\theta u| + s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} |\theta_x u|\]

\[\leq \epsilon s^9 \lambda^{10} \int_{\omega_0} e^{-2s\varphi_m \xi_m^7} |\theta|^2 + \epsilon s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi_m \xi_m^7} |\theta_x|^2 + \frac{C}{\epsilon} s^{39} \lambda^{24} \int_{\omega_0} e^{-10s\varphi_m + 8s\tilde{\varphi}_m \xi_m^39} |u|^2.\]

\[\text{– Estimate for } J_2. \quad \text{Next, we look into the term } J_2,\]

\[J_2 = -s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} \theta w_t = s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} \theta_t w + s^{23} \lambda^{16} \int_{\omega_0} \phi e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}} \theta_x w,\]

since \(e^{-6s\varphi_m + 4s\tilde{\varphi}_m}\) is 0 at \(t = 0\) and \(t = T\), due to the result in Lemma A.1.

Now, recall the estimate (2.5), so that one has

\[\left| (e^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}}) \right| \leq CT^{2m-2} s^{-6s\varphi_m + 4s\tilde{\varphi}_m \xi_m^{23}}.\]

Thanks to this and for any \(\epsilon > 0\), we have (again, by using the Young’s inequality) that

\[(2.51) \quad |J_2| \leq \epsilon s^5 \lambda^6 \int_{\omega_0} e^{-2s\varphi_m \xi_m^5} |\theta_t|^2 + \epsilon s^9 \lambda^{10} \int_{\omega_0} e^{-2s\varphi_m \xi_m^7} |\theta|^2 + \frac{C}{\epsilon} s^{41} \lambda^{20} \int_{\omega_0} e^{-10s\varphi_m + 8s\tilde{\varphi}_m \xi_m^41} |u|^2.\]
– Estimate for $J_3$. Let us now focus on $J_3$; upon consecutive integration by parts with respect to $x$, one can deduce that

$$
(2.52) \quad |J_3| \leq s^{23} \lambda^6 \int_{\omega_0} \left| \left( e^{-6s\varphi m + 4s\varphi_m \xi_m^2} \right)_{x} \right| \theta_x w | + 2s^{23} \lambda^6 \int_{\omega_0} \left| \left( e^{-6s\varphi m + 4s\varphi_m \xi_m^2} \right)_{x} \right| \theta_x w |
$$

$$
\quad + s^{23} \lambda^6 \int_{\omega_0} \left| \left( e^{-6s\varphi m + 4s\varphi_m \xi_m^2} \right)_{x} \right| \theta_x w |
$$

$$
\leq C s^{25} \lambda^{18} \int_{\omega_0} e^{-6s\varphi m + 4s\varphi_m \xi_m^2} \left| \theta_x w \right| + C s^{24} \lambda^{17} \int_{\omega_0} e^{-6s\varphi m + 4s\varphi_m \xi_m^2} \left| \theta_x w \right|
$$

$$
+ C s^{23} \lambda^6 \int_{\omega_0} e^{-6s\varphi m + 4s\varphi_m \xi_m^2} \left| \theta_x w \right|,
$$

thanks to the result (2.49).

Now, for any $\epsilon > 0$, applying the Young’s inequality, we have from (2.52),

$$
(2.53) \quad |J_3| \leq \epsilon s^3 \lambda^{10} \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left| \theta \right|^2 + \epsilon s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left| \theta_x \right|^2 + \epsilon s^{15} \lambda^6 \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left| \theta_{xx} \right|^2
$$

$$
+ \epsilon \int_{\omega_0} e^{-10s\varphi m + 8s\varphi_m \xi_m^4} \left| \theta \right|^2.
$$

– Estimate for $J_4$. Analogous to the estimate of $J_1$ in (2.50), the term $J_4$ satisfies

$$
(2.54) \quad |J_4| \leq \epsilon s^9 \lambda^{10} \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left| \theta \right|^2 + \epsilon s^7 \lambda^8 \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left| \theta_x \right|^2
$$

$$
+ \epsilon \int_{\omega_0} e^{-10s\varphi m + 8s\varphi_m \xi_m^4} \left| \theta \right|^2.
$$

Finally, gathering the estimates of $J_1$, $J_2$, $J_3$ and $J_4$ given by (2.50), (2.51), (2.53) and (2.54) respectively, we have

$$
(2.55) \quad s^{23} \lambda^6 \int_{\omega_0} \left| \left( e^{-6s\varphi m + 4s\varphi_m \xi_m^2} \right)_{x} \right| \theta_x w | \leq C s^9 \lambda^{10} \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left| \theta \right|^2 + C s^{24} \lambda^6 \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left| \theta_{xx} \right|^2
$$

$$
+ C s^{35} \lambda^6 \int_{\omega_0} e^{-2s\varphi m + \xi_m^2 s^6} \left( \left| \theta \right|^2 + \left| \theta_{xx} \right|^2 \right) + \frac{C}{\epsilon} \int_{\omega_0} e^{-10s\varphi m + 8s\varphi_m \xi_m^4} \left( s^{39} \lambda^{24} \left| \theta \right|^2 + s^{11} \lambda^{26} \left| \theta \right|^2 \right).
$$

Fix $\epsilon > 0$ small enough so that, the integrals in $Q_T$ can be absorbed in terms of the l.h.s. in (2.47), and this yields the required Carleman inequality

$$(2.54) \quad \text{for every } \lambda \geq \lambda^* \text{ and } s \geq \sigma^*(T^m + T^{2m-1} + T^{2m-3/2} + T^{2m-1} + T^{2m+2}), \text{ for some } \lambda^* > 0 \text{ and } \sigma^* > 0 \text{ chosen largely enough.}
$$

Hence, the proof of Theorem 2.6 is finished. \(\square\)

2.3. A Carleman estimate for the case when $\alpha = 1$. The adjoint system (1.8) for the case $\alpha = 0$ reads as

$$
(2.56) \quad \begin{cases}
-u_x + u_{xxx} + \gamma u_{xx} = -w_x + \zeta \theta_0 & \text{in } Q_T, \\
-w_1 - u_{xx} - \beta w_x = -u_x & \text{in } Q_T, \\
\zeta_x + \zeta_{xxx} + \gamma \zeta_{xx} = \theta_x & \text{in } Q_T, \\
\theta_1 - \theta_{xx} + \beta \theta_x = \zeta_x & \text{in } Q_T, \\
u = u_x = w = \zeta = \zeta_x = \theta = 0 & \text{in } \Sigma_T, \\
u(T) = 0, \ w(T) = 0, & \text{in } (0, 1), \\
\zeta(0) = \zeta_0, \ \theta(0) = \theta_0 & \text{in } (0, 1).
\end{cases}
$$

We now write the Carleman inequality associated to the system (2.56) in terms of the following theorem.
Theorem 2.12 (Carleman inequality: the case $\alpha = 1$). Let $\varphi_m$ and $\xi_m$ be given by (2.2) with $m \geq 2/5$ and $k > m$. Then, there exist positive constants $\lambda_0$, $s_0 := \sigma_0(T^{m} + T^{2m} + T^{2m-1} + T^{2m-2/5} + T^{4m/3} + T^{3m/2})$ with some $\sigma_0 > 0$ and $C$, such that we have the following inequality satisfied by the solution to (2.56):

\[
I_{KS}(u) + I_H(w; 3) + I_{KS}(\zeta) + s^3 \lambda^4 \int \int e^{-2s\varphi_m} \xi_m^3 |\theta|_x|^2 + s\lambda^2 \int \int e^{-2s\varphi_m} \xi_m^3 |\theta|_x|^2 
\leq C s^{79} \lambda^{80} \int \int e^{-2s\varphi_m} \xi_m^3 \xi_m^3 |\theta|^2 + C s^{79} \lambda^{74} \int \int e^{-2s\varphi_m} \xi_m^3 \xi_m^3 |\theta|^2,
\]

for all $\lambda \geq \lambda_0$ and $s \geq s_0$, where $I_{KS}(\cdot)$ is introduced in (2.8) and $I_H(\cdot; \cdot)$ in (2.7) respectively.

To prove the above Carleman inequality with the observations only on $u$ and $w$, we do the following steps.

(i) First, observe that the usual Carleman estimate for $\theta$ will always give an observation integral of $\theta$ and there is no chance to absorb it by any leading integrals. In fact, there is a coupling by $\theta$ to the equation of $\zeta$ and therefore, it is relevant to seek for a Carleman estimate associated with the variable $\theta$.

In this context, we recall the work [15], where they proved such a Carleman estimate to demonstrate a joint Carleman inequality for the adjoint to the KS system coupled with a heat equation and we shall use it in our present article.

(ii) For the variables $u$ and $\zeta$, we use a Carleman estimate from the work [30], as mentioned earlier.

(iii) Finally, for $w$, we make use the standard Carleman inequality for the heat equation, thanks to the pioneer work [20].

Below, we prescribe the individual Carleman estimates for each of $u$, $w$, $\zeta$ and $\theta$.

Lemma 2.13 (Carleman inequality for $u$, the case $\alpha = 1$). Let $\varphi_m$ and $\xi_m$ be given by (2.2) with $m \geq 2/5$. Then, there exist positive constants $\lambda_1$, $s_1 := \sigma_1(T^{m} + T^{2m} + T^{2m-1} + T^{2m-2/5})$ with some $\sigma_1 > 0$ and $C$, such that we have the following estimate for $u \in L^2(0, T; H^2(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; L^2(0, 1))$,

\[
I_{KS}(u) \leq C \left( \int \int e^{-2s\varphi_m} (|w|_x^2 + |\zeta|^2) + s^7 \lambda^4 \int \int e^{-2s\varphi_m} \xi_m^3 |\theta|^2 \right)
\]

for all $\lambda \geq \lambda_1$, $s \geq s_1$, where $I_{KS}(\cdot)$ is defined by (2.8).

The next lemma is concerned with a Carleman estimate for $w$.

Lemma 2.14 (Carleman inequality for $w$, the case $\alpha = 1$). Let $\varphi_m$ and $\xi_m$ be defined as in (2.2) with $m \geq 1$. Then, there exist positive constants $\lambda_2$, $s_2 := \sigma_2(T^{2m} + T^{2m-2/5})$ with some $\sigma_2 > 0$ and $C$, such that we have the following estimate for $w \in L^2(0, T; H^2(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; L^2(0, 1))$,

\[
I_H(w; 3) \leq C \left( \int \int e^{-2s\varphi_m} |u|_x^2 + s^3 \lambda^4 \int \int e^{-2s\varphi_m} \xi_m^3 |\theta|^2 \right)
\]

for all $\lambda \geq \lambda_2$ and $s \geq s_2$, where $I_H(\cdot; \cdot)$ is defined by (2.7).

We also write a Carleman estimate for $\zeta$ which is similar with the one for $u$.

Lemma 2.15 (Carleman inequality for $\zeta$, the case $\alpha = 1$). Let $\varphi_m$ and $\xi_m$ be defined as in (2.2) with $m \geq 2/5$. Then, there exist positive constants $\lambda_3$, $s_3 := \sigma_3(T^{m} + T^{2m} + T^{2m-2/5})$ with some $\sigma_3 > 0$ and $C$, such that we have the following estimate for $\zeta \in L^2(0, T; H^2(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; L^2(0, 1))$,

\[
I_{KS}(\zeta) \leq C \left( \int \int e^{-2s\varphi_m} |\theta|_x^2 + s^7 \lambda^8 \int \int e^{-2s\varphi_m} \xi_m^3 |\theta|^2 \right)
\]

for all $\lambda \geq \lambda_3$ and $s \geq s_3$, where $I_{KS}(\cdot)$ has been defined in (2.8).

Lastly, by following [15, Theorem 3.1], we have a Carleman estimate for $\theta$ as given below, which can be proved using a result given by [22, Lemma 6]. At this point, we can start with regular enough initial data $\theta_0$ in the equation (2.56) so that $\theta \in L^2(0, T; H^2(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; H^1(0, 1))$. 
Lemma 2.16 (Carleman inequality for \( \theta_s \), the case \( \alpha = 1 \)). Let \( \varphi_m \) and \( \xi_m \) be defined by (2.2) with \( m > 3, k > m \). Then, there exist positive constants \( \lambda_4, s_4 := \sigma_4(T^{2m} + T^{2m-1}) \) with some \( \sigma_4 > 0 \) and \( C \), such that we have the following estimate for \( \theta \in L^2(0, T; H^3(0, 1) \cap H^1_0(0, 1)) \cap H^1(0, T; H^1(0, 1)) \),

\[
(2.61) \quad s^3 \lambda^4 \int e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2 + s\lambda^2 \int e^{-2s\varphi_m} \xi_m |\theta_x|^2 \\
\leq Cs \lambda^2 \int e^{-2s\varphi_m} \xi_m^2 (|\xi|^2 + |\xi_x|^2) + Cs^3 \lambda^4 \int e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2,
\]

for all \( \lambda \geq \lambda_4, s \geq s_4 \).

Now, we are in the situation to prove our main Carleman inequality, that is Theorem 2.12.

Proof of Theorem 2.12. We divide it into several steps.

Step 1: Absorbing the lower order integrals. Observe the following result:

\[
1 \leq T^{2pm} \xi_m^p, \quad \forall p \in \mathbb{N}^*.
\]

- Using this, the lower order integrals in the r.h.s. of the Carleman inequality (2.58) can be estimated as

\[
(2.62) \quad \int e^{-2s\varphi_m} \left( |w_x|^2 + |\zeta|^2 \right) \leq T^{2m} \int e^{-2s\varphi_m} \xi_m |w_x|^2 + T^{14m} \int e^{-2s\varphi_m} \xi_m^7 |\zeta|^2,
\]

whereas, the source integral in the r.h.s. of (2.59) satisfies

\[
(2.63) \quad \int e^{-2s\varphi_m} |u_x|^2 \leq T^{10m} \int e^{-2s\varphi_m} \xi_m^5 |u_x|^2.
\]

Thus, by choosing any \( \lambda \geq \lambda_0 := \max\{\lambda_j : 1 \leq j \leq 4\} \) fixed and \( s \geq \sigma_5 T^{2m} \) for some \( \sigma_5 > 0 \), one can absorb all the integrals appearing in the r.h.s. of (2.62) and (2.63) by the associated leading integrals in the l.h.s. of (2.58) (2.59) and (2.60).

- Next, the source integral in the r.h.s. of (2.60) enjoys

\[
(2.64) \quad \int e^{-2s\varphi_m} |\theta_x|^2 \leq T^{6m} \int e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2,
\]

and the r.h.s. of the Carleman inequality (2.61) (for \( \theta_s \)) can be estimated as

\[
(2.65) \quad s\lambda^2 \int e^{-2s\varphi_m} \xi_m^2 (|\xi|^2 + |\xi_x|^2) \\
\leq s\lambda^2 \left( T^{6m} \int e^{-2s\varphi_m} \xi_m^5 |\xi|^2 + T^{2m} \int e^{-2s\varphi_m} \xi_m^3 |\xi_x|^2 \right).
\]

Then, by choosing any \( \lambda \geq \lambda_0 \) and \( s \geq \sigma_6 (T^{m} + T^{3m/2}) \) for some \( \sigma_6 > 0 \), the quantity in (2.64) can be absorbed by the 1st leading integral in the l.h.s. of (2.61) and the quantities appearing in the r.h.s. of (2.65), by the associated leading integrals in the l.h.s. of (2.59) and (2.60).

So, after adding the inequalities: (2.58), (2.59), (2.60) and (2.61), and using the above absorption techniques, we obtain the following auxiliary estimate:

\[
(2.66) \quad I_{KS}(u) + I_H(w, 3) + I_{KS}(\zeta) + s^3 \lambda^4 \int e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2 + s\lambda^2 \int e^{-2s\varphi_m} \xi_m |\theta_x|^2 \\
\leq Cs^7 \lambda^8 \int e^{-2s\varphi_m} \xi_m^2 |u|^2 + Cs^3 \lambda^4 \int e^{-2s\varphi_m} \xi_m^3 |w|^2 + Cs^7 \lambda^8 \int e^{-2s\varphi_m} \xi_m^7 |\zeta|^2 \\
+ Cs^3 \lambda^4 \int e^{-2s\varphi_m} \xi_m^3 |\theta_x|^2,
\]

for all \( \lambda \geq \lambda_0 \) and \( s \geq \sigma_0 (T^m + T^{2m} + T^{2m-1} + T^{2m-2}/2 + T^{3m}/2 + T^{4m}/3) \) where \( \sigma_0 := \max\{\sigma_j : 1 \leq j \leq 6\} \).
Now, our duty is to absorb the observation integrals associated with $\theta_z$ and $\zeta$ by some leading integrals in the l.h.s. of (2.66).

**Step 2: Absorbing the observation integral associated to $\theta_z$.** In the sequel, we choose a nonempty set $\tilde{\Omega}_2 \subset \subset \tilde{\Omega}_1 \subset \subset \omega_0$ and a function

$$\phi \in C_0^\infty(\tilde{\Omega}_1) \with \ 0 \leq \phi \leq 1 \in \tilde{\Omega}_1, \ \phi = 1 \in \tilde{\Omega}_2,$$

and we consider the auxiliary Carleman estimate (2.66) with the observation domain $\tilde{\Omega}_2$.

Our goal is to eliminate the observation integral of $\theta_z$. Using the equation of $\zeta$ from (2.56), we have

\begin{equation}
(2.67) \quad \begin{aligned}
s^3 \lambda^4 \int_{\tilde{\Omega}_2} e^{-2s\varphi_m \xi^3_m} |\theta_z|^2 \\
\leq s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_z (\zeta_x + \zeta_{xxx} + \gamma \zeta_x) \\
= s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_z (\zeta_x + \zeta_{xxx}) + s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_x (\zeta_{xxx} + (\gamma - 1) \zeta_x) \\
:= I_1 + I_2.
\end{aligned}
\end{equation}

- **Estimate for $I_1$.** We have

\begin{equation}
(2.68) \quad I_1 = s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_x (\zeta_x + \zeta_{xxx}) \\
= -s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi (e^{-2s\varphi_m \xi^3_m}) \theta_x \zeta - s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_x \zeta \\
+ s^3 \lambda^4 \int_{\tilde{\Omega}_1} (\phi e^{-2s\varphi_m \xi^3_m}) \theta_{xxx} \zeta + s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_{xx} \zeta.
\end{equation}

In above, there is no boundary terms at $t = 0, T$ (while integrating by parts in time), since $e^{-2s\varphi_m(t,z)} \to 0$ as $t \to 0^+$ or $T^-$. We also perfomed two consecutive integration by parts in space as follows

\begin{equation}
\begin{aligned}
s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_x \zeta_{xx} &= -s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_x \zeta - s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_{xx} \zeta \\
&= -s^3 \lambda^4 \int_{\tilde{\Omega}_1} (\phi e^{-2s\varphi_m \xi^3_m}) \theta_x \zeta_x + s^3 \lambda^4 \int_{\tilde{\Omega}_1} (\phi e^{-2s\varphi_m \xi^3_m}) \theta_{xx} \zeta + s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_{xx} \zeta.
\end{aligned}
\end{equation}

Now, going back to (2.68), and using the equation $\theta_t - \theta_{xx} = -\beta \theta_x + \xi_x$, we have

\begin{equation}
(2.69) \quad I_1 = -s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} (-\beta \theta_x + \xi_x) \zeta + X_1,
\end{equation}

where

\begin{equation}
X_1 := -s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi (e^{-2s\varphi_m \xi^3_m}) \theta_x \zeta - s^3 \lambda^4 \int_{\tilde{\Omega}_1} \phi e^{-2s\varphi_m \xi^3_m} \theta_x \zeta + s^3 \lambda^4 \int_{\tilde{\Omega}_1} (\phi e^{-2s\varphi_m \xi^3_m}) \theta_{xx} \zeta + X_1.
\end{equation}

Now, recall (2.6) (for $m > 1$) to write

$$|(e^{-2s\varphi_m \xi^3_m})t| \leq C T^{2m-2} s \phi_m^2 (e^{-2s\varphi_m \xi^3_m}).$$

We use this fact in the first integral of $X_1$, and then thanks to the Cauchy-Schwarz inequality, we have for any $\epsilon > 0$, that

\begin{equation}
(2.70) \quad |X_1| \leq \epsilon s^3 \lambda^4 \int_{\tilde{\Omega}_1} e^{-2s\varphi_m \xi^3_m} |\theta_x|^2 + \epsilon s \lambda^2 \int_{\tilde{\Omega}_1} e^{-2s\varphi_m \xi^3_m} |\theta_{xx}|^2 \\
+ \frac{C}{\epsilon} s^5 \lambda^6 \int_{\tilde{\Omega}_1} e^{-2s\varphi_m \xi^3_m} |\zeta_x|^2 + \frac{C}{\epsilon} s^5 \lambda^6 \int_{\tilde{\Omega}_1} e^{-2s\varphi_m \xi^3_m} |\zeta|^2.
\end{equation}
Let us estimate the other integrals of $I_1$ in (2.69). We have for any $\epsilon > 0$ (again by applying Cauchy-Schwarz inequality)

\[(2.71) \quad s^3\lambda^4 \left| \iint_{\Omega_1} \phi e^{-2s\varphi_m} C_3 \left( -\beta \frac{\delta x}{\lambda} + \zeta x \right) \right| \leq \epsilon s\lambda^2 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m|\theta_{xx}|^2 + \epsilon s^5\lambda^6 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^5|\zeta|^2.
\]

Therefore, the estimates (2.70) and (2.71) yields

\[(2.72) \quad |I_1| \leq \epsilon s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\theta_{xx}|^2 + 2s\lambda^2 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m|\theta_{xx}|^2 + \epsilon s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\zeta|^2.
\]

\[+ \frac{C}{\epsilon} s^5\lambda^6 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^5|\zeta|^2 + \frac{C}{\epsilon} s^5\lambda^6 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^7|\zeta|^2.
\]

**Estimate for $I_2$.** Let us recall the quantity $I_2$ from (2.67) and we have

\[I_2 = s^3\lambda^4 \iint_{\Omega_1} \phi e^{-2s\varphi_m} \xi_m^3 \theta_x (\xi_{xxx} + (\gamma - 1)\zeta_{xx})
\]

\[= -s^3\lambda^4 \iint_{\Omega_1} \phi e^{-2s\varphi_m} \xi_m^3 \theta_x \xi_{xxx} - s^3\lambda^4 \iint_{\Omega_1} \phi e^{-2s\varphi_m} \xi_m^3 \theta_x \xi_{xxx} - (\gamma - 1)\theta_x \xi_{xx},\]

where we perform an integration by parts on the term involving fourth order derivative in $\zeta$. It follows that

\[(2.73) \quad |I_2| \leq \epsilon s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\theta_{xx}|^2 + \epsilon s\lambda^2 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m|\theta_{xx}|^2 + \epsilon s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\zeta|^2.
\]

\[+ \frac{C}{\epsilon} s^5\lambda^6 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^5|\zeta|^2 + \frac{C}{\epsilon} s^5\lambda^6 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^7|\zeta|^2.
\]

Now, using the estimates of $I_1$ and $I_2$ given by (2.72) and (2.73) respectively, we have from (2.67) that

\[(2.74) \quad s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\theta_{xx}|^2 \leq C s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\theta_{xx}|^2 + C s\lambda^2 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m|\theta_{xx}|^2 \]

\[+ C s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\zeta|^2 + \frac{C}{\epsilon} s^5\lambda^6 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^5|\zeta|^2 + \frac{C}{\epsilon} s^5\lambda^6 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^7|\zeta|^2.
\]

Fix $\epsilon > 0$ small enough so that we can absorb the first three integrals in the r.h.s. of (2.74) by the associated leading terms in the l.h.s. of (2.66).

Next, one can deduce the following result: for any $\epsilon > 0$, there exists $C > 0$ such that,

\[(2.75) \quad \left( s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\theta_{xx}|^2 + s^3\lambda^4 e^{-2s\varphi_m} \xi_m^3|\zeta|^2 + s^3\lambda^4 e^{-2s\varphi_m} \xi_m^5|\zeta_{xxx}|^2 \right)
\]

\[\leq \epsilon \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\theta_{xx}|^2 + s^3\lambda^4 e^{-2s\varphi_m} \xi_m^3|\zeta|^2 + s^3\lambda^4 e^{-2s\varphi_m} \xi_m^3|\zeta|^2 + \frac{C}{\epsilon} s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\zeta|^2,
\]

assuming that there is a set $\Omega_0$ such that $\Omega_1 \subset \Omega_0 \subset \Omega_0$.

The proof can be done by performing some integration by parts in space and applying the Cauchy-Schwarz inequality accordingly. We refer [15, Claim 3, eq. (3.30)] where a similar result has been obtained. We omit the details here.

Then, for $\epsilon > 0$ small enough, we can absorb the first three integrals in the r.h.s. of (2.75) by the corresponding leading integrals in the l.h.s. of (2.66) and as a consequence, we have

\[(2.76) \quad I_{KS}(u) + I_H(w, 3) + I_{KS}(\zeta) + s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\theta_{xx}|^2 + s^3\lambda^4 \iint_{\Omega_1} e^{-2s\varphi_m} \xi_m^3|\zeta|^2
\]

\[\leq C s^3\lambda^4 \iint_{\Omega_0} e^{-2s\varphi_m} \xi_m^3|u|^2 + C s^3\lambda^4 \iint_{\Omega_0} e^{-2s\varphi_m} \xi_m^3|w|^2 + C s^3\lambda^4 \iint_{\Omega_0} e^{-2s\varphi_m} \xi_m^3|\zeta|^2.
\]
Step 3: Absorbing the observation integral associated to $\zeta$. In the previous step, one can assume a couple of nonempty sets $\widehat{\omega}_2 \subset \subset \widehat{\omega}_1 \subset \subset \omega_1 \subset \subset \omega_0$ and prove the auxiliary inequality (2.76) with the observation domain $\omega_1$.

Then, choose a function $\phi_1 \in C^\infty_c(\omega_0)$ with $0 \leq \phi_1 \leq 1$ in $\omega_0$ and $\phi_1 = 1$ in $\omega_1$. Also, one has from the $4 \times 4$ adjoint system (2.56) that

$$
\zeta = -u_t + u_{xxx} + \gamma u_{xx} + w_x \quad \text{in} \; \omega_0, \; \text{since} \; \omega_0 \subset \subset \mathcal{O}.
$$

Therefore, we observe that

$$
(2.77) \quad s^{39} \lambda^{40} \int_{\omega_1} e^{-2s\varphi_m} \zeta^{39} |\zeta|^2 \leq s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} |\zeta|^2
$$

$$
= s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} (-u_t + u_{xxx} + \gamma u_{xx} + w_x) := I_3 + I_4 + I_5 + I_6.
$$

- Estimate for $I_3$. First, we compute

$$
(2.78) \quad I_3 := -s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} u_t = s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} u_t
$$

$$
+ s^{39} \lambda^{40} \int_{\omega_0} \phi_1 (e^{-2s\varphi_m} \zeta^{39}) u.
$$

Let us recall (2.6) (for $m > 1$), to write

$$
|(e^{-2s\varphi_m} \zeta^{39})_t| \leq C T^{2m-2} e^{-2s\varphi_m} \zeta^{41}_m.
$$

Using the above fact and the Cauchy-Schwarz inequality, we have for any $\epsilon > 0$ that

$$
(2.79) \quad |I_3| \leq \epsilon s^{-1} \int e^{-2s\varphi_m} \zeta^{-1} \zeta |\zeta|^2 + \epsilon s^{7} \lambda^{8} \int e^{-2s\varphi_m} \zeta^{7} |\zeta|^2 + \frac{C}{\epsilon} s^{79} \lambda^{80} \int e^{-2s\varphi_m} \zeta^{79} |u|^2.
$$

- Estimate for $I_4$. Next, by performing a successive number of integration by parts on $I_4$ w.r.t. $x$, we get

$$
(2.80) \quad |I_4| := \left| s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} u_{xxxx} \right| \leq C \left( s^{39} \lambda^{40} \int_{\omega_0} (\phi_1 e^{-2s\varphi_m} \zeta^{39})_xxxx u
$$

$$
+ s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} u_{xx} \zeta u + s^{39} \lambda^{40} \int_{\omega_0} (\phi_1 e^{-2s\varphi_m} \zeta^{39})_xx u
$$

$$
+ s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} u_{xxx} u + s^{39} \lambda^{40} \int_{\omega_0} \phi_1 e^{-2s\varphi_m} \zeta^{39} u_{xxxx} u \right).
$$

Let us recall the result (2.20), so that we have the following:

$$
\left| \phi_1 e^{-2s\varphi_m} \zeta^{39} \right|_{n,x} \leq C s^n \lambda^n e^{-2s\varphi_m} \zeta^{39+n}_{m,x}, \quad \text{for} \; n \in \mathbb{N}^*,
$$

and the estimate (2.80) follows

$$
(2.81) \quad |I_4| \leq C \left( s^{43} \lambda^{44} \int_{\omega_0} e^{-2s\varphi_m} \zeta^{43} \zeta u + s^{42} \lambda^{43} \int_{\omega_0} e^{-2s\varphi_m} \zeta^{42} \zeta u + s^{41} \lambda^{42} \int_{\omega_0} e^{-2s\varphi_m} \zeta^{41} \zeta u
$$

$$
+ s^{40} \lambda^{41} \int_{\omega_0} e^{-2s\varphi_m} \zeta^{40} \zeta u + s^{39} \lambda^{40} \int_{\omega_0} e^{-2s\varphi_m} \zeta^{39} \zeta u \right).
$$

Then, for any $\epsilon > 0$, we obtain by using Cauchy-Schwarz inequality that

$$
(2.82) \quad |I_4| \leq \epsilon \left( s^{7} \lambda^{6} \int e^{-2s\varphi_m} \zeta^{7} |\zeta|^2 + s^{6} \lambda^{5} \int e^{-2s\varphi_m} \zeta^{5} |\zeta|^2 + s^{5} \lambda^{4} \int e^{-2s\varphi_m} \zeta^{3} |\zeta|^2
$$

$$
+ s^{4} \lambda^{3} \int e^{-2s\varphi_m} \zeta |\zeta|^2 + s^{-1} \int e^{-2s\varphi_m} \zeta^{-1} (|\zeta|^2 + |\zeta_{xxxx}|^2) \right)
$$

$$
+ \frac{C}{\epsilon} s^{79} \lambda^{80} \int_{\omega_0} e^{-2s\varphi_m} \zeta^{79} |u|^2.
$$
In a similar manner, one can obtain an estimate for $I_5$, given by

$$\tag{2.83} |I_5| \leq \epsilon \left( s^2 \lambda^2 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + s^5 \lambda^6 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + s^3 \lambda^4 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + \frac{C}{\epsilon} s^{75} \lambda^{76} \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} \right).$$

Estimate for $I_6$. Finally, we see

$$\tag{2.84} |I_6| \leq \left| -s^{39} \lambda^{40} \int_{\omega} (\phi_1 e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} - s^{39} \lambda^{40} \int_{\omega} \phi_1 e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} \right|$$

$$\leq \epsilon \left( s^7 \lambda^8 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + s^5 \lambda^6 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + s^3 \lambda^4 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + \frac{C}{\epsilon} s^{73} \lambda^{74} \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} \right).$$

Thus, using the estimates (2.79), (2.82), (2.83) and (2.84) in (2.77), we have

$$\tag{2.85} s^{39} \lambda^{40} \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} \leq C \left( s^7 \lambda^8 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + s^5 \lambda^6 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + s^3 \lambda^4 \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} \right)$$

$$+ \frac{C}{\epsilon} s^{79} \lambda^{80} \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} + \frac{C}{\epsilon} s^{73} \lambda^{74} \int_{\omega} e^{-2s \rho \cdot \xi^T \rho \cdot \xi \cdot |\xi|^2} \right).$$

Fix $\epsilon > 0$ in (2.85) small enough so that all the integrals in $QT$ can be absorbed by the quantity $I_{KS}(\xi)$ in (2.76), and this yields the required Carleman inequality (2.57).

3. Observability inequality and controllability of the extended system. As we have mentioned, the main ingredient to prove Theorem 1.2 is to obtain a suitable observability inequality for system (1.8). Due to the presence of the parameter $\alpha$ in the system, we have used different strategies leading to different Carleman estimates, see Theorems 2.1, 2.6, and 2.12. How to obtain an observability inequality from those can be done by following classical arguments (see e.g. [19] or [22] for a similar insensitizing problem), therefore we only present a brief proof of the observability inequality derived from Theorem 2.6. The others can be done in a similar fashion by making minor adjustments.

To do this, we shall first prove a refined Carleman inequality with weight functions that do not vanish at $t = T$. More precisely, let us consider

$$\ell(t) = \begin{cases} t(T - t), & 0 \leq t \leq T/2, \\ T^2/4, & T/2 \leq t \leq T, \end{cases}$$

and the following associated weight functions

$$\tag{3.2} \mathcal{G}_m(t, x) = \frac{e^{\lambda (k + 1/2) |k| \cdot |\nu| \cdot \infty} - e^{\lambda (k + 1/2) |k| \cdot |\nu| \cdot \infty + v(x)}}{\ell(t)^m}, \quad \mathfrak{M}_m(t, x) = \frac{e^{\lambda (k + 1/2) |k| \cdot |\nu| \cdot \infty + v(x)}}{\ell(t)^m}, \quad \forall (t, x) \in QT.$$ 

for any constants $\lambda > 1$ and $k > m > 0$. Additionally, we define

$$\tag{3.3} \hat{\mathcal{G}}_m(t) = \max_{x \in [0, 1]} \mathcal{G}_m(t, x), \quad \mathfrak{M}_m(t) = \min_{x \in [0, 1]} \mathcal{G}_m(t, x).$$

We have the following.

**Proposition 3.1** (A refined Carleman estimate: the case $\alpha = 0$). Let $m$, $k$, and $\lambda$ be fixed constants according to Theorem 2.6. Then, there exists a positive constant $\hat{C}$ depending at most on $\omega, \mathcal{O}, T, m, k, s$, and $\lambda$ such that

$$\tag{3.4} \int_{\omega} e^{-2s \hat{\mathcal{G}}_m(\mathfrak{M}_m^*)^2} |\nu|^2 + \int_{\omega} e^{-2s \hat{\mathcal{G}}_m(\mathfrak{M}_m^*)^2} |\nu|^2 + \int_{\omega} e^{-2s \hat{\mathcal{G}}_m(\mathfrak{M}_m^*)^2} |\xi|^2 + \int_{\omega} e^{-2s \hat{\mathcal{G}}_m(\mathfrak{M}_m^*)^2} |\theta|^2 \leq \hat{C} \int_{\omega} e^{-10s \hat{\mathcal{G}}_m(\mathfrak{M}_m^*)^2} |\nu|^2 + \hat{C} \int_{\omega} e^{-10s \hat{\mathcal{G}}_m(\mathfrak{M}_m^*)^2} |\theta|^2,$$

where $(u, w, \zeta, \theta)$ is the solution associated to (2.32), for any given $(\zeta_0, \theta_0) \in [L^2(0, 1)]^2$. 


Proof. First recall that the Carleman estimates have been proved with sufficiently regular data but the same hold true for any given data \((\zeta_0, \theta_0) \in [L^2(0,1)]^2\) by applying the usual density argument.

Now, we start by the construction \(\varphi = \mathcal{G}\) and \(\xi = \mathfrak{F}\) in \((0, T/2) \times (0,1)\), hence

\[
\begin{align*}
&\int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^2 |u|^2 + \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^3 |w|^2 \\
&+ \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^2 |\zeta|^2 + \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^9 |\theta|^2 \\
&= \int_0^{T/2} \int_0^1 e^{-2s\varphi_m} \varphi_m^2 |u|^2 + \int_0^{T/2} \int_0^1 e^{-2s\varphi_m} \varphi_m^3 |w|^2 \\
&+ \int_0^{T/2} \int_0^1 e^{-2s\varphi_m} \varphi_m^2 |\zeta|^2 + \int_0^{T/2} \int_0^1 e^{-2s\varphi_m} \varphi_m^9 |\theta|^2.
\end{align*}
\]

Therefore, from (2.34), (2.7), (2.8), and the definitions of \(\mathcal{G}\) and \(\mathfrak{F}\), we readily get

\[
\begin{align*}
&\int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^2 |u|^2 + \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^3 |w|^2 \\
&+ \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^2 |\zeta|^2 + \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^9 |\theta|^2 \\
&\leq C_1 \left( \int_{\omega_0} \int_0^{1} e^{-10s\mathfrak{F}_m + 8s\mathfrak{F}_m} \mathfrak{F}_m^9 |u|^2 + \int_{\omega_0} \int_0^{1} e^{-10s\mathfrak{F}_m + 8s\mathfrak{F}_m} \mathfrak{F}_m^{41} |w|^2 \right),
\end{align*}
\]

for a constant \(C_1 > 0\) depending on \(s\) and \(\lambda\).

For the domain \((T/2, T) \times (0,1)\), we argue as follows. Let us introduce a function \(\eta \in C^1([0,T])\) such that

\[
\eta = 0 \text{ in } [0, T/4], \quad \eta = 1 \text{ in } [T/2, T], \quad |\eta'| \leq C / T.
\]

Using Proposition B.1, we apply the energy estimate to the equation verified by \((\eta\zeta, \eta\theta)\), so that one can deduce that

\[
\begin{align*}
\|\eta\zeta\|^2_{L^2((T/4,T) \times (0,1))} + \|\eta\theta\|^2_{L^2((T/4,T) \times (0,1))} \\
&\leq \frac{C}{T^2} \left( \|\zeta\|^2_{L^2((T/4,T) \times (0,1))} + \|\theta\|^2_{L^2((T/4,T) \times (0,1))} \right).
\end{align*}
\]

Analogously, we have for the equation verified by \((\eta u, \eta w)\) that

\[
\begin{align*}
\|\eta u\|^2_{L^2((T/4,T) \times (0,1))} + \|\eta w\|^2_{L^2((T/4,T) \times (0,1))} \\
&\leq C \left( \|\eta\theta\|^2_{L^2((T/4,T) \times (0,1))} + \frac{1}{T^2} \|\eta\zeta\|^2_{L^2((T/4,T) \times (0,1))} + \frac{1}{T^2} \|\theta\|^2_{L^2((T/4,T) \times (0,1))} \right).
\end{align*}
\]

Using Poincaré’s inequality and having in mind the definition of \(\eta\), we can combine estimates (3.7)–(3.8) to obtain

\[
\begin{align*}
\|u\|^2_{L^2((T/2,T) \times (0,1))} + \|w\|^2_{L^2((T/2,T) \times (0,1))} + \|\zeta\|^2_{L^2((T/2,T) \times (0,1))} + \|\theta\|^2_{L^2((T/2,T) \times (0,1))} \\
&\leq C \left( \|u\|^2_{L^2((T/4,T/2) \times (0,1))} + \|w\|^2_{L^2((T/4,T/2) \times (0,1))} \right) \\
&+ C \left( \|\zeta\|^2_{L^2((T/4,T/2) \times (0,1))} + \|\theta\|^2_{L^2((T/4,T/2) \times (0,1))} \right).
\end{align*}
\]

Since the weight functions \(\mathcal{G}\) and \(\mathfrak{F}\) are bounded (by below and above) in \([T/4, T]\), we can introduce them in both sides of the above inequality and using (3.5) to estimate in the right-hand side we get

\[
\begin{align*}
&\int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^2 |u|^2 + \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^3 |w|^2 \\
&+ \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^2 |\zeta|^2 + \int_0^{T/2} \int_0^1 e^{-2s\mathfrak{F}_m} \mathfrak{F}_m^9 |\theta|^2 \\
&\leq \tilde{C} \left( \int_{\omega_0} \int_0^{1} e^{-10s\mathfrak{F}_m + 8s\mathfrak{F}_m} \mathfrak{F}_m^9 |u|^2 + \int_{\omega_0} \int_0^{1} e^{-10s\mathfrak{F}_m + 8s\mathfrak{F}_m} \mathfrak{F}_m^{41} |w|^2 \right),
\end{align*}
\]
for some $\hat{C} > 0$. This, together with inequality (3.5) and definition (3.3), yield the desired result. □

As a consequence, we have the following.

**Proposition 3.2 (Observability inequality: the case $\alpha = 0$).** There exists a constant $C > 0$ and a weight function $\rho = $ $\rho(t)$ blowing up at $t = 0$ only depending on $\omega$, $\mathcal{O}$, $\gamma$, $\beta$ and $T$ such that, for any $(\zeta_0, \theta_0) \in [L^2(0,1)]^2$, the solution $(u, w, \zeta, \theta)$ to (2.32) satisfies

$$
\int_0^1 \int_0^1 \rho^{-2} (|u|^2 + |w|^2) \leq C \int_0^1 \int_0^1 (|u|^2 + |w|^2).
$$

**Proof.** The observability inequality follows immediately from Proposition 3.1. Indeed, we fix $m \geq 2$, $k > m$, and take $s = s^*, \lambda = \lambda^*$ with $s^*, \lambda^*$ as in Theorem 2.6 and define $\rho(t) = e^{\hat{\phi}_m}$. Thus $\rho(t)$ is a strictly positive function blowing up at $t = 0$ only depending on $\omega$, $\mathcal{O}$, $\gamma$, $\beta$, and $T$. We note that Lemma A.1 is also valid if we replace $\varphi_m$ (resp. $\hat{\varphi}_m$) by $\bar{\mathcal{G}}_m$ (resp. $\hat{\mathcal{G}}_m$) and thus $e^{-10s\hat{\phi}_m + 8s\hat{\phi}_m} \lesssim C$ for all $\mu > 0$ and some $C > 0$. This allows us to bound the right-hand side of (3.4). Finally, since $(3^*)^\nu \geq c > 0$ for all $\nu > 0$ we can bound the left-hand side of (3.4) in terms of $\rho$. This ends the proof. □

**Remark 3.3.** As we have mentioned, here we only obtain the observability inequality for the case $\alpha = 0$. However, it is important to mention that for $\alpha = 1$ and $\alpha \in (0,1)$, the structure of the observability inequality (3.11) will not change and only the weight $\rho$ and the constant $C$ should be determined naturally from their corresponding Carleman estimate (see Theorems 2.1 and 2.12).

With the inequality (3.11) at hand, the proof of Theorem 1.2 follows by a standard duality argument. For completeness, we give a brief proof. Also, in view of Remark 3.3, we give a proof that can be used in all cases.

**Proof of Theorem 1.2.** We introduce the following linear subspace of $[L^2((0,T) \times \omega)]^2$,

$$
L = \{(u|_{(0,T) \times \omega}), (w|_{(0,T) \times \omega}) : (u, w, \zeta, \theta) \text{ solve (1.8)}\} \text{ with some } (\zeta_0, \theta_0) \in [L^2(0,1)]^2,
$$

and define a linear functional on $L$ as

$$
\mathcal{L}((u|_{(0,T) \times \omega}), (w|_{(0,T) \times \omega})) = - \left( \int_0^1 \int_0^1 u \xi_1 + \int_0^1 \int_0^1 w \xi_2 \right).
$$

Using (3.11), for any $\xi_i \in L^2(Q_T)$ ($i = 1,2$) verifying (1.7), we can see that $\mathcal{L}$ is a bounded linear functional on $L$ and by means of Hahn-Banach Theorem, $\mathcal{L}$ can be extended to a bounded linear functional on $[L^2((0,T) \times \omega)]^2$. To avoid introducing additional notation, we use the same notation for this extension. Thus, by Riesz representation theorem, we can find $(h_1, h_2) \in [L^2((0,T) \times \omega)]^2$ such that

$$
\int_0^1 \int_0^1 h_1 u + \int_0^1 \int_0^1 h_2 w = - \left( \int_0^1 \int_0^1 u \xi_1 + \int_0^1 \int_0^1 w \xi_2 \right).
$$

Now, we shall prove that $(h_1, h_2)$ are our desired controls. Indeed, for any $(\zeta_0, \theta_0) \in [L^2(0,1)]^2$, using system (1.4)–(1.5) (with $(y_0, z_0) = (0,0)$) and the adjoint equation (1.8), we obtain after a straightforward computation that

$$
\int_0^1 p(0) \zeta_0 + \int_0^1 q(0) \theta_0 = \int_0^1 \int_0^1 h_1 u + \int_0^1 \int_0^1 h_2 w + \int_0^1 \int_0^1 u \xi_1 + \int_0^1 \int_0^1 w \xi_2.
$$

Combining the above identity with (3.14), we obtain

$$
\int_0^1 p(0) \zeta_0 + \int_0^1 q(0) \theta_0 = 0, \text{ for any } (\zeta_0, \theta_0) \in [L^2(0,1)]^2,
$$

whence $p(0) = q(0) = 0$ in $(0,1)$. This ends the proof. □
4. Concluding remarks and comments. In this work, we have proved the existence of insensitizing controls for a coupled system of fourth- and second-order parabolic PDEs. As usual, this problem can be reformulated as a null-control problem for an extended system in cascade form (see (1.4)–(1.5)) but due to the presence of the parameter $\alpha$ in the sentinel (1.2), this system may change its structure and different Carleman tools should be employed for studying the observability of the corresponding adjoint equation. We present two concluding remarks concerning the problems addressed in this work.

1. Less controls than equations. An important question related to the controllability of coupled systems is: what is (are) the minimum number of control(s) required to accomplish a given task. In the works [15] and [10], it has been shown that for proving the null-controllability of the system

$$
\begin{cases}
y_t + yxxxx + \gamma yxx + yy_x = z_x & \text{in } Q_T, \\
z_t - zxx + \beta z_x = y_x & \text{in } Q_T, \\
y = y_x = z = 0 & \text{in } \Sigma_T, \\
y(0) = y_0, & z(0) = z_0 & \text{in } (0,1),
\end{cases}
$$

(4.1)

it is needed only one control localized in either equation.

In our insensitizing problem we have used two controls, one for each component, but determining if we can reduce its number it is not so clear. As far as we know, there are only a few papers devoted to the insensitizing control problems for coupled systems, see [7,9,12]. Similar to our case, in those works, the original problem is transformed into a control problem for an extended system of four equations (two forward and two backward in time). In particular, in [7], the authors have used only one control to prove their result by using a sentinel depending (explicitly) only on one of the components of the system. This is comparable to choosing $\alpha = 0$ in our case (1.2). At a first glance, it seems that we can follow the ideas similar to [7] to eliminate the extra control, but in our case, the first order couplings make things difficult. Indeed, recalling our adjoint system (2.32) (i.e. the case $\alpha = 0$), we see that the only way to remove an observation related to $w$ is to differentiate the second equation of (2.32) (changing $\Omega$ by some suitable smooth approximation) and use the coupling in the first one. By doing so we can obtain a Carleman estimate with localized terms depending on $u$, $\zeta$ and $\theta$. Then, the only way to estimate $\theta$ locally is by using the second equation of (2.32) which reintroduces a local term of $w$ (see eq. (2.55)), and we failed! Therefore, dealing with cascade systems (forward-backward) of the original coupled systems can be tricky and its controllability and observability properties deserve further attention. This fact has been also pointed out in [23].

2. The nonlinear case. A natural question that arises in this framework is to study the insensitizing control problem for the nonlinear system (4.1), more precisely

$$
\begin{cases}
y_t + yxxxx + \gamma yxx + yy_x = z_x + h_1 \mathbb{I}_\omega + \xi_1 & \text{in } Q_T, \\
z_t - zxx + \beta z_x = y_x + h_2 \mathbb{I}_\omega + \xi_2 & \text{in } Q_T, \\
y = y_x = z = 0 & \text{in } \Sigma_T, \\
y(0) = y_0 + \tau \tilde{y}_0, & z(0) = z_0 + \tau \tilde{z}_0 & \text{in } (0,1).
\end{cases}
$$

(4.2)

Under the same assumptions of Theorem 1.2, it is possible to use similar strategies as in Sections 2 and 3 for studying the observability of the corresponding linearized (extended) adjoint system (cf. (1.8)) and then use a local inversion theorem as in [7,8] to conclude that if $(\xi_1, \xi_2)$ verify

$$
\int_{Q_T} \rho^2 |\xi_i|^2 \, dx dt \leq \delta, \quad i = 1, 2,
$$

(4.3)

for some $\delta > 0$ small enough, there exists (local) insensitizing controls for (4.2). Of course, details remain to be given but most of the computations are similar to the ones shown in Section 2 and to maintain this work at a reasonable length we do not pursue this goal here.

Appendix A. An auxiliary result.
Lemma A.1. Recall the weight functions \( \varphi_m \) and \( \tilde{\varphi}_m \) defined by (2.2) and (2.33) respectively for \( \lambda > 1 \) and \( k > m > 0 \). Then, for any \( s > 0 \) and \( p \in \mathbb{N}^* \), there exists \( \delta > 0 \), such that we have

\[
- ps \varphi_m + (p - 2)s \tilde{\varphi}_m \leq \frac{-\delta s}{m(T-t)^m}.
\]

Proof. The proof can be deduced from the explicit expressions of the weight functions. We see

\[
- ps \varphi_m + (p - 2)s \tilde{\varphi}_m = -ps e^{\lambda(1+\frac{1}{x})}{\|v\|}_{\infty} - e^{\lambda(\frac{k}{x}) + \nu(x)} + (p - 2)s e^{\lambda(1+\frac{1}{x})}{\|v\|}_{\infty} - e^{\lambda}\nu{\|v\|}_{\infty}
\]

\[
= -2se^{\lambda(1+\frac{1}{x})}{\|v\|}_{\infty} + se^{\lambda\nu}{\|v\|}_{\infty} \left( p(e^{\lambda\nu(x)} - 1) + 2 \right)
\]

\[
= -2e^{\lambda\nu}{\|v\|}_{\infty} \left( 2e^{\lambda\nu}{\|v\|}_{\infty} - pe^{\lambda\nu(x)} + p - 2 \right)
\]

Now, for fixed \( m > 0 \) and any \( \lambda > 0 \), one may choose \( k > 0 \) large enough such that the quantity \( \left( 2e^{\lambda\nu}{\|v\|}_{\infty} - pe^{\lambda\nu(x)} + p - 2 \right) \geq \delta > 0 \), for some \( \delta > 0 \), \( \forall x \in [0,1] \). Hence, the result (A.1) follows.

Appendix B. Well-posedness results. Let us consider the following coupled system

\[
\begin{align*}
\begin{cases}
y_t + y_{xxx} + \gamma y_{xx} &= z_x + f_1 & \text{in } Q_T, \\
z_t - z_{xx} + \beta z_x &= y_x + f_2 & \text{in } Q_T, \\
y &= y_x = z = 0 & \text{in } \Sigma_T, \\
y(0) = y_0, z(0) = z_0 & \text{in } (0,1),
\end{cases}
\end{align*}
\]

where \((f_1, f_2) \in [L^2(Q_T)]^2\), \(\gamma > 0\) and \(\beta\) is any real number.

Below, we write the standard well-posedness and some regularity result concerning the prototype of coupled system (B.1). The proof will be omitted.

Proposition B.1 (Well-posedness & energy estimate). For any given \((y_0, z_0) \in [L^2(0,1)]^2\) and \((f_1, f_2) \in [L^2(Q_T)]^2\), there exists unique weak solution

\[
(y, z) \in C^0([0,T]; L^2(0,1))^2 \cap L^2(0,T; H^2_0(0,1) \times H^1_0(0,1))
\]

to (B.1), such that it satisfies

\[
\| (y, z) \|_{C^0([0,T]; L^2(0,1))^2} + \| (y, z) \|_{L^2(0,T; H^2_0(0,1) \times H^1_0(0,1))} \leq C \left( \left\| (y_0, z_0) \right\|_{L^2(0,1)}^2 + \| (f_1, f_2) \|_{L^2(Q_T)^2}^2 \right),
\]

for some \( C > 0 \).

Proposition B.2 (Regularity). For any given \((y_0, z_0) \in H^2_0(0,1) \times H^1_0(0,1)\) and \((f_1, f_2) \in [L^2(Q_T)]^2\), the solution

\[
(y, z) \in C^0([0,T]; H^2_0(0,1) \times H^1_0(0,1)) \cap L^2(0,T; H^4(0,1) \times H^2(0,1))
\]

to (B.1) satisfies

\[
\| (y, z) \|_{C^0([0,T]; H^2_0(0,1) \times H^1_0(0,1))} + \| (y, z) \|_{L^2(0,T; H^4(0,1) \times H^2_0(0,1))} \leq C \left( \left\| (y_0, z_0) \right\|_{H^2_0(0,1) \times H^1_0(0,1)} + \| (f_1, f_2) \|_{L^2(Q_T)^2} \right),
\]

for some \( C > 0 \).

A formal proof for the above regularity result can be found in [15, Prop. 2.1].

Acknowledgements. The work of the first author is partially supported by the French government research program "Investissements d’Avenir" through the IDEX-ISITE initiative 16-IDEX-0001 (CAP 20-25). The work of the second author has been partially supported by the program “Estancias posdoctorales por México” of CONACyT, Mexico and by Project IN100919 of DGAPA-UNAM, Mexico.
REFERENCES

[1] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa, Recent results on the controllability of linear coupled parabolic problems: a survey, Math. Control Relat. Fields, 1 (2011), pp. 267–306.

[2] O. Bodart and C. Fabre, Controls insensitizing the norm of the solution of a semilinear heat equation, J. Math. Anal. Appl., 195 (1995), pp. 658–683.

[3] O. Bodart, M. González-Burgos, and R. Pérez-García, Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity, Comm. Partial Differential Equations, 29 (2004), pp. 1017–1050.

[4] ———, Insensitizing controls for a heat equation with a nonlinear term involving the state and the gradient, Nonlinear Anal., 57 (2004), pp. 687–711.

[5] ———, A local result on insensitizing controls for a heat equation with nonlinear boundary Fourier conditions, SIAM J. Control Optim., 43 (2004), pp. 955–969.

[6] F. Boyer, V. Hernández-Santamaría, and L. de Teresa, Insensitizing controls for a semilinear parabolic equation: a numerical approach, Math. Control Relat. Fields, 9 (2019), pp. 117–158.

[7] B. M. R. Calsavara, N. Carreño, and E. Cerpa, Insensitizing controls for a phase field system, Nonlinear Anal., 143 (2016), pp. 120–137.

[8] R. Capistrano-Filho and T. Y. Tanaka, Controls insensitizing the norm of solution of a Schrödinger type system with mixed dispersion, arXiv preprint arXiv:2010.15104, (2020).

[9] N. Carreño, Insensitizing controls for the Boussinesq system with no control on the temperature equation, Adv. Differential Equations, 22 (2017), pp. 235–258.

[10] N. Carreño and E. Cerpa, Local controllability of the stabilized Kuramoto-Sivashinsky system by a single control on the heating equation, J. Math. Pures Appl. (9), 106 (2016), pp. 670–694.

[11] N. Carreño, E. Cerpa, and A. Mercado, Boundary controllability of a cascade system coupling fourth- and second-order parabolic equations, Systems Control Lett., 133 (2019), pp. 104542, 7.

[12] N. Carreño, S. Guerrero, and M. Gueye, Insensitizing controls with two vanishing components for the three-dimensional Boussinesq system, ESAIM Control Optim. Calc. Var., 21 (2015), pp. 73–100.

[13] N. Carreño and M. Gueye, Insensitizing controls with one vanishing component for the Navier-Stokes system, J. Math. Pures Appl. (9), 101 (2014), pp. 27–53.

[14] E. Cerpa, A. Mercado, and A. F. Pazoto, On the boundary control of a parabolic system coupling KS-KdV and heat equations, Sci. Ser. A Math. Sci. (N.S.), 22 (2012), pp. 55–74.

[15] ———, Null controllability of the stabilized Kuramoto-Sivashinsky system with one distributed control, SIAM J. Control Optim., 53 (2015), pp. 1543–1568.

[16] L. de Teresa, Insensitizing controls for a semilinear heat equation, Comm. Partial Differential Equations, 25 (2000), pp. 39–72.

[17] L. de Teresa and E. Zuazua, Identification of the class of initial data for the insensitizing control of the heat equation, Commum. Pure Appl. Anal., 8 (2009), pp. 457–471.

[18] S. Ervedoza, P. Lissy, and Y. Privat, Insensitizing controls for the heat equation with respect to boundary variations, arXiv preprint arXiv:2012.14327, (2020).

[19] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel, Local exact controllability of the Navier-Stokes system, J. Math. Pures Appl. (9), 83 (2004), pp. 1501–1542.

[20] A. V. Fursikov and O. Y. Imanuvilov, Controllability of evolution equations, vol. 34 of Lecture Notes Series, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[21] S. Guerrero, Null controllability of some systems of two parabolic equations with one control force, SIAM J. Control Optim., 46 (2007), pp. 379–394.

[22] M. Gueye, Insensitizing controls for the Navier-Stokes equations, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), pp. 825–844.

[23] V. Hernández-Santamaría and L. de Teresa, Some remarks on the hierarchic control for coupled parabolic PDEs, in Recent advances in PDEs: analysis, numerics and control, vol. 17 of SEMA SIMAI Springer Ser., Springer, Cham, 2018, pp. 117–137.

[24] V. Hernández-Santamaría and L. Peralta, Controllability results for stochastic coupled systems of fourth-and second-order parabolic equations, J. Evol. Equ., 22 (2022).

[25] K. Kassar, Negative and positive controllability results for coupled systems of second and fourth order parabolic equations, (2020).

[26] O. Kavian and L. de Teresa, Unique continuation principle for systems of parabolic equations, ESAIM Control Optim. Calc. Var., 16 (2010), pp. 247–274.

[27] J.-L. Lions, Quelques notions dans l’analyse et le contrôle de systèmes à données incomplètes, in Proceedings of the Xth Congress on Differential Equations and Applications/First Congress on Applied Mathematics (Spanish) (Málaga, 1989), Univ. Málaga, Málaga, 1990, pp. 43–54.

[28] P. Lissy, Y. Privat, and Y. Simporé, Insensitizing control for linear and semi-linear heat equations with partially unknown domain, ESAIM Control Optim. Calc. Var., 25 (2019), pp. Paper No. 50, 21.

[29] B. A. Malomed, B.-F. Feng, and T. Kawahara, Stabilized kuramoto-Sivashinsky system, Physical Review E, 64 (2001), p. 046304.

[30] Z. Zhou, Observability estimate and null controllability for one-dimensional fourth order parabolic equation, Taiwanese J. Math., 16 (2012), pp. 1991–2017.