SOME REMARKS ON THE GOTTMAN-MURRAY MODEL OF MARITAL DISSOLUTION AND TIME DELAYS

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Abstract. In the paper we consider mathematical model proposed by Gottman, Murray and collaborators to describe marital dissolution. This model is described in the framework of discrete dynamical system reflecting emotional states of wife and husband during consecutive rounds of talks between spouses. The model is, however, non-symmetric. To make it symmetric, one need to assume that the husband reacts with delay. Following this idea we consider the influence of time delays in the reaction terms of wife and husband.

The delay means that one or both of spouses split their attention between present and previous rounds of talks. We study possibility of the change of stability with increasing delay. Surprisingly, it occurs that the delay has no impact on the stability, that is the condition of stability proposed by Murray remains unchanged.

1. Introduction. Mathematical description of dyadic interactions has started from so-called Romeo and Juliet models, and S. Strogatz put a milestone in this topic by publishing in 1988 one-page article [23]; cf. also [24]. Typically, “R&J” models are described by a system of two ordinary differential equations (ODEs), as proposed by S. Strogatz and then followed by many researchers; cf. e.g. [1, 6, 9, 11, 12, 14, 16, 17, 18, 19, 20, 21] and the references therein. In more general context, dyadic interactions are described by dynamical systems, discrete or continuous, finite or infinite-dimensional. S. Strogatz used a linear system of ODEs reflecting intensity of emotions of partners being in a close relationship. He named the partners “Romeo” and “Juliet”, firstly in the context of oscillatory emotions between them described in the literature. Therefore, his model is typically referred as to the Romeo and Juliet model. Very interesting interpretation of the results of modelling in the context of various kinds of couples known from the literature and movies could be found in the series of papers by S. Rinaldi and co-authors [17, 18, 19, 20].

Mathematical modeling of marital interactions is rather a young branch of applications. Pioneering research done by J.M. Gottman together with J.D. Murray, who is one of the best recognized biomathematicians nowadays, was described in [9]. J.M. Gottman, J.D. Murray and collaborators used discrete dynamical system to reflect an experiment conducted in the Gottman’s clinic. Throughout this paper the model introduced in [9] is called shortly Gottman-Murray model.

Married couples with problems discussed problematic topics during sessions of talks in the Gottman’s clinic. Their emotional states and reactions were described by the following

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equations

\[ x_{n+1} = Ax_n + B + f(y_n), \]
\[ y_{n+1} = Cy_n + D + g(x_{n+1}), \]

where \( x_n, y_n \) denote intensity of emotions (love/hate) of wife and husband, respectively, during \( n \)th session of conversation on a given topic being problematic for them. The system is non-symmetric because the wife talks first during each round of the speech. For Equations (1), linear parts describe uninfluenced (or inner in other words) emotional dynamics of each of the spouses, while \( f \) and \( g \) are influence functions, that is describe the influence of wife to husband’s emotions and vice versa. It is obvious that the uninfluenced dynamics stabilizes on so-called uninfluenced steady state, which is \( B/(1-A) \) for the wife and \( D/(1-C) \) for the husband. Moreover, in [9] it is assumed that this convergence is monotonic, which leads to one of the main assumptions as regards the model parameters, that is

\[ A \in (0,1) \quad \text{and} \quad C \in (0,1). \]

Although Gottman et al. did not assumed any specific properties of the influence functions, they noticed that this functions should be partially monotonic. Moreover, in general such functions should be non-linear, but in [9] the authors used the simplest functions that allow to fit the model to the data, that is partially linear influence functions. Using this model Gottman et al. divided diagnosed married couples into subclasses for which they made a prediction for divorce. The results of that prediction were interesting; cf. [9, 15] for more details.

It seems that in the description of real everyday life continuous time models are more relevant to reflect emotions of married couples, because such emotions are present every time moment. On the other hand, the experiment of Gottman et al. was very specific, and therefore, for the description of it discrete time model is better than continuous one.

In this paper we discuss the Gottman-Murray model in the context of time delays. Using delay equations, like in [2, 3] or [13], is not common in the description of dyadic interactions. On the other hand, there are many psychological evidences (we have discussed that topic in Bielczyk et al. 2013) that one of the partners is deliberative and his/her emotions are delayed with respect to stimulus. Therefore, it seems to be interesting to discuss the influence of delays on the dynamics of Equations (1). Although introducing delays in discrete systems seems to be easier than in the case of differential equations because it does not lead to infinite-dimensional systems, however it is sometimes more difficult to analyze large finite systems than infinite dimensional low-variable one. Moreover, there are many standard tools to study stability of delay differential equations, cf. e.g. [4, 7, 22], but very few and complex in the case of large discrete systems, compare formulas presented and proved in [25]. The main mathematical tool used in this paper is Rouché’s Theorem [10]. This theorem states that if we consider two complex-valued functions \( f \) and \( g \) which are holomorphic (i.e. complex differentiable in a neighborhood of every point in its domain) inside some region \( K \) with closed contour \( \partial K \) and \(|g(z)| < |f(z)| \) on \( \partial K \), then \( f \) and \( f + g \) have the same number of zeros inside \( K \) (counting multiplicity of zeros).

It should be also marked that even simple discrete time models could have very reach dynamics comparing to the continuous time analogue, like in the case of the logistic equation (cf. e.g. [15]). It is then not surprising that strange attractors could be found in the models of love affairs [5].

2. Some remarks on the stability of Equations (1). J.D. Murray [9, 15] presented mathematical analysis of Equations (1). Below we stress the main points of that analysis.
Steady states of Equations (1) satisfy
\[ \bar{x} = A\bar{x} + B + f(\bar{y}), \]
\[ \bar{y} = C\bar{y} + D + g(\bar{x}), \]
that is
\[ \bar{x} = \frac{B + f(\bar{y})}{1 - A}, \]
\[ \bar{y} = \frac{D + g(\bar{x})}{1 - C}. \]

(2)

It could be easily shown (cf. [15, 8]) that if
\[ f'(\bar{y})g'(\bar{x}) < (1 - A)(1 - C), \]

(3)

then the steady state \((\bar{x}, \bar{y})\) is stable. When considering non-linear influence functions, this stability is local, obviously. However, in the case considered by Gottman et al., under the assumption that this steady state is unique, the stability is global (due to the fact that the system is partially linear). Condition (3) has clear geometrical interpretation which is associated with the location of “null-clines” proposed by J.D. Murray [15] in the phase space \((x, y)\). According to [15] these null-clines are described by the functions
\[
\begin{align*}
  x &= \frac{B + f(y)}{1 - A}, & \text{for the variable } x, \\
  y &= \frac{D + g(x)}{1 - C}, & \text{for the variable } y,
\end{align*}
\]

that is any steady state satisfying (2) lies on the intersection of these curves. It should be noticed that \(\frac{dx}{dy} = \frac{f'(y)}{1 - A}\) on the null-cline for \(x\) and \(\frac{dy}{dx} = \frac{g'(x)}{1 - C}\) on the null-cline for \(y\), while the condition of stability described by (3) could be rewritten as
\[ \frac{g'(x)}{1 - C} < \frac{1}{f'(y)}, \]

such that we have the slope for \(x\) null-cline is less than the slope for \(y\) null-cline in the space \((x, y)\). However, this condition needs additional assumption on the derivatives of influence functions at the steady state. Namely, it is necessary to assume that these derivatives are positive. Otherwise, the condition of stability is different, which was discussed in [8]. However, in the following we assume that Inequality (3) is satisfied and check the influence of delays into the model dynamics.

Notice, that the notion of null-clines for discrete systems has no other meaning than mentioned above, and they do not play the same role as for continuous systems. Therefore, the name “null-cline” used by Murray could be a little misleading here.

3. Introducing one-unit delay into the model. In [8] we made the first attempt to study the influence of delays into dynamics of Equations (1). First, we should notice that symmetrizing Equations (1) we obtain the system in which the husband reacts with “delay”, that is he respond not to the the last statement of the spouse but to the last but one. However, it is difficult to imagine that in such a case the respond is “pure delayed", and it considers both spouses, obviously. It is much more reasonable to assume, that one or both of the partners also remembers the previous round of speech. Hence, we can assume that during the \((n + 1)\)th round of talk the spouses split their attention both into the present and previous rounds. The case when only wife or husband reacts with one-unit delay was considered in [8]. Now, we consider the case when both the wife and the husband are reflexive, that is in the \(n\)th round of the talk they react directly not only to the last statement of the spouse but also to the last but one. This means that we study the following system of equations
\[
\begin{align*}
  x_{n+1} &= Ax_n + B + \alpha f(y_{n-1}) + (1 - \alpha)f(y_n), \\
  y_{n+1} &= Cy_n + D + \beta g(x_n) + (1 - \beta)g(x_{n+1}),
\end{align*}
\]

(4)
where $\alpha, \beta \in [0, 1]$ denote parameters of splitting the attention of spouses between two last statements of their partner. We are interested in checking robustness of the system stability with respect to delays under Condition (3). We start from presenting Equations (4) in the normal form, which means that the right-hand side depends only on the $n$th time step. Therefore, we introduce a new variable

$$(x_n, y_n, y_{n-1}),$$

and denote

$$G_1(x_n, y_n, y_{n-1}) = Ax_n + B + \alpha f(y_{n-1}) + (1 - \alpha)f(y_n),$$
$$G_2(x_n, y_n, y_{n-1}) = Cy_n + D + \beta g(x_n) + (1 - \beta)g(G_1(x_n, y_n, y_{n-1})).$$

It is obvious that the dynamics of Equations (4) is described by iterations of the function

$$F(x_n, y_n, y_{n-1}) = \left( G_1(x_n, y_n, y_{n-1}), G_2(x_n, y_n, y_{n-1}), y_n \right). \quad (5)$$

**Proposition 3.1.** If $f'(\bar{y}) > 0, g'(\bar{x}) > 0$ and Inequality (3) holds, then stable steady states of Equations (1) are also stable for Equations (4).

**Proof.** Let us consider the stable steady state $(\bar{x}, \bar{y})$ of Equations (1). For Equations (4) we need to study $(\bar{x}, \bar{y}, \bar{y})$. We have

$$DF(\bar{x}, \bar{y}, \bar{y}) = \begin{pmatrix}
A & (1 - \alpha)\bar{f}' & \alpha\bar{f}' \\
0 & (1 - \alpha)\bar{g}' & C + (1 - \alpha)(1 - \beta)\bar{f}'\bar{g}' \\
1 & \beta\bar{g}' & 0
\end{pmatrix} \quad (6)$$

with the characteristic polynomial

$$W(\lambda) = \lambda^3 - (A + C + (1 - \alpha)(1 - \beta)\bar{f}'\bar{g}')(1) + (AC - (\alpha(1 - \beta) + \beta(1 - \alpha))\bar{f}'\bar{g}')(\lambda - \alpha\beta\bar{f}'\bar{g}'). \quad (7)$$

where we denote $f'(\bar{y}) = \bar{f}'$, $g'(\bar{x}) = \bar{g}'$, to shorten the notation.

We use Rouché’s Theorem [10]. Using this Theorem we divide the characteristic polynomial $W(\lambda)$ described by (7) into two polynomials

$$W_1(\lambda) = \lambda(\lambda^2 - (A + C)\lambda + AC)$$

and

$$W_2(\lambda) = -(1 - \alpha)(1 - \beta)\bar{f}'\bar{g}'\lambda^2 - (\alpha(1 - \beta) + \beta(1 - \alpha))\bar{f}'\bar{g}'\lambda - \alpha\beta\bar{f}'\bar{g}' .$$

It is obvious that $W_1$ and $W_2$ are holomorphic. We easily see that $W_1(\lambda) = \lambda(\lambda - A)(\lambda - C)$ which means that it has all roots inside the unite circle. Moreover, $W_2(\lambda) = -\bar{f}'\bar{g}'((1 - \alpha)\lambda + \alpha)((1 - \beta)\lambda + \beta)$. Comparing modulus of $W_1$ with modulus of $W_2$ on the unit circle we obtain

$$\left| W_2(e^{i\omega}) \right| \leq \bar{f}'\bar{g}'\left( |1 - \alpha||\lambda| + \alpha \right)\left( |1 - \beta||\lambda| + \beta \right) \bigg|_{\lambda = e^{i\omega}} = \bar{f}'\bar{g}'$$

and

$$\left| W_1(e^{i\omega}) \right| = |\lambda - A||\lambda - C| \bigg|_{\lambda = e^{i\omega}} \geq (1 - A)(1 - C).$$

Hence, under the assumption (3), $|W_1(e^{i\omega})| > |W_2(e^{i\omega})|$, independently of $\omega \in [0, 2\pi]$, and all roots of $W$ are inside the unite circle according to Rouché’s Theorem.

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4. **Larger delays.** In this section we consider the delay larger than one unit. We start from the delay in the husband’s reaction as it is the simplest case as regards the model analysis.
4.1. Delays in the husband’s reaction. Now, we introduce larger delay into the husband’s reaction. Therefore, the husband splits his attention into $k$ rounds of talks and Equations (1) changes into

$$x_{n+1} = Ax_n + B + f(y_n),$$

$$y_{n+1} = Cy_n + D + \sum_{i=0}^{k} \beta_i g(x_{n+1-i}),$$

where $k \geq 1$ and $\sum_{i=0}^{k} \beta_i = 1$ with $\beta_i \geq 0, i = 0, \ldots, k$. To present Equations (8) in the normal form we need to increase the dimension of our dynamical system introducing new multi-dimensional variable $(x_n, x_{n-1}, \ldots, x_{n+1-k}, y_n)$. Then Equations (8) are described by iterations of

$$F_h(x_n, x_{n-1}, \ldots, x_{n+1-k}, y_n) = \left( x_{n+1}, x_n, \ldots, x_{n+2-k}, y_{n+1} \right)$$

$$= (Ax_n + B + f(y_n), x_n, \ldots, x_{n+2-k}, Cy_n + D + \beta_0 g(Ax_n + B + f(y_n)) + \sum_{i=1}^{k} \beta_i g(x_{n+1-i})).$$

As for the case studied in the previous section we can prove that stability does not change for $k \geq 1$. Clearly, the steady state $(\bar{x}, \bar{y})$ changes to $(\bar{x}, \ldots, \bar{x}, \bar{y})$ and we obtain the following characteristic matrix:

$$DF_{h}(\bar{x}, \ldots, \bar{x}, \bar{y}) - \bar{\lambda} =$$

$$= \begin{pmatrix}
A - \lambda & 0 & 0 & \ldots & 0 & \bar{f}' \\
1 & -\lambda & 0 & \ldots & \ldots & 0 \\
0 & 1 & -\lambda & 0 & \ldots & 0 \\
\vdots & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & 1 & -\lambda & 0 \\
(A\beta_0 + \beta_1) \bar{g}' & \beta_2 \bar{g}' & \ldots & \beta_k \bar{g}' & C + \beta_0 \bar{f}' \bar{g}' - \lambda
\end{pmatrix}. \quad (9)$$

Notice, that matrix (9) is $(k+1)$-dimensional. Calculating $d_1 = \det \left(DF_{h}(\bar{x}, \ldots, \bar{x}, \bar{y}) - \bar{\lambda} \right)$ we obtain

$$d_1 = (A - \lambda) \det M^I_k + (-1)^{k+2} \bar{f}' \det M^II_k,$$

where

$$M^I_k = \begin{pmatrix}
-\lambda & 0 & \ldots & \ldots & 0 \\
1 & -\lambda & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 \\
\vdots & \ldots & 1 & -\lambda & 0 \\
\beta_2 \bar{g}' & \ldots & \beta_k \bar{g}' & C + \beta_0 \bar{f}' \bar{g}' - \lambda
\end{pmatrix},$$

and

$$M^II_k = \begin{pmatrix}
1 & -\lambda & 0 & \ldots & 0 \\
0 & 1 & -\lambda & 0 & \ldots \\
0 & \ldots & \ldots & \ldots & 0 \\
\vdots & \ldots & 0 & 1 & -\lambda \\
(A\beta_0 + \beta_1) \bar{g}' & \beta_2 \bar{g}' & \beta_2 \bar{g}' & \ldots & \beta_k \bar{g}'
\end{pmatrix}.$$
Both $M^I_k$ and $M^{II}_k$ are $k$-dimensional, and it is obvious that $M^I_k$ is lower-triangular, so that its determinant equals
\[
\det M^I_k = (-\lambda)^{k-1}(C + \beta_0 \tilde{f}\tilde{g} - \lambda),
\]
while for $M^{II}_k$ we can use the following recurrent formula
\[
\det M^{II}_j = \det M^{II}_{j-1} + \det M^{I}_{j-1} \tilde{g} \det M^{I}_{j-1}, \quad a_i = \begin{cases} A\beta_0 & \text{for } j = k, \\ 0 & \text{for } j < k, \end{cases}
\]
for $j = k, \ldots, 2$, where $M^I_l$ is $l$-dimensional matrix with the terms $-\lambda$ on the main diagonal, and 1 below the main diagonal, while the rest terms are equal to 0. Hence, $\det M^I_k = (-\lambda)^k$.

Eventually, we obtain
\[
d_1 = (-\lambda)^{k-1}(A - \lambda)(C + \beta_0 \tilde{f}\tilde{g} - \lambda) + \det M^{I}_{1-1} \tilde{g} \det M^{I}_{1-1},
\]
which leads to the following characteristic polynomial
\[
W^h(\lambda) = \lambda^{k+1} - \lambda^k (A + \beta_0 \tilde{f}\tilde{g}) + \lambda^{k-1} (AC - \beta_1 \tilde{f}\tilde{g}) - \tilde{f}\tilde{g} \sum_{j=2}^{k} \beta_j \lambda^{k-j}.
\]
Using Rouché’s Theorem [10] we again divide $W^h$ described by (10) into two polynomials
\[
W^h_1(\lambda) = \lambda^{k+1} - \lambda^k (A + \beta_0 \tilde{f}\tilde{g}) + \lambda^{k-1} (AC - \beta_1 \tilde{f}\tilde{g}) = \lambda^{k-1} (A - \lambda)(A - C),
\]
and again zeros of $W^h_1$ are located inside the unit circle and
\[
\left| W^h_2 \left( e^{i\omega} \right) \right| \leq \tilde{f}\tilde{g} \sum_{j=0}^{k} \beta_j = \tilde{f}\tilde{g} < (1 - A)(1 - C) \leq \left| W^h_1 \left( e^{i\omega} \right) \right|,
\]
assuming that Inequality (3) holds. Therefore, we have obtained the following result.

**Theorem 4.1.** If Inequality (3) holds for $f'(\tilde{y}) > 0$ and $g'(\tilde{x}) > 0$, then the steady state $(\tilde{x}, \tilde{y})$ is stable independently of the magnitude of the delay in the husband’s reaction.

4.2. Delays in the wife’s reaction. It is obvious that we can consider also delayed reaction of the wife, cf. the discussion on the delays in such a context in [3]. Although calculations in this case are more complicated, but the result is the same – stability does not change.

We can check that the characteristic polynomial does not differ much comparing to those for the husband, namely it is just the same polynomial multiplied by $-\lambda$. It could be also checked on the basis of calculations presented in the next section, where delays in both partners reactions are considered.

**Corollary 4.2.** If Inequality (3) holds for $f'(\tilde{y}) > 0$ and $g'(\tilde{x}) > 0$, then for any delay in the wife’s reaction the steady state is stable.
4.3. Delays in both partners reaction. Eventually, we consider the case when both partners split their attention between several rounds of talks. Although the length of delay could be different for the partners, we assume that it is the same, as we can put some splitting coefficients equal to 0. This means that we study the following system of equations

\[
x_{n+1} = Ax_n + B + \sum_{i=0}^{k} \alpha_i f(y_{n-i}),
\]

\[
y_{n+1} = Cy_n + D + \sum_{i=0}^{k} \beta_i g(x_{n+1-i}),
\]

where \( k \geq 1 \), \( \sum_{i=0}^{k} \alpha_i = 1 \) and \( \sum_{i=0}^{k} \beta_i = 1 \) with \( \alpha_i, \beta_i \geq 0 \), \( i = 0, \ldots, k \).

For Equations (11) we introduce \((2k+1)\)-dimensional variable \((x_n, \ldots, x_{n+1-k}, y_n, \ldots, y_{n-k})\). Then our system is described by iterations of \( G(x_n, x_{n+1-k}, y_n, \ldots, y_{n-k}) = (G_1, \ldots, G_{2k+1}) \), where

\[
G_1 (x_n, \ldots, x_{n+1-k}, y_n, \ldots, y_{n-k}) = Ax_n + B + \sum_{i=0}^{k} \alpha_i f(y_{n-i}),
\]

\[
G_i (x_n, \ldots, x_{n+1-k}, y_n, \ldots, y_{n-k}) = x_{n+2-i}, \quad \text{for } i = 2, \ldots, k,
\]

\[
G_{k+1} (x_n, \ldots, x_{n+1-k}, y_n, \ldots, y_{n-k}) = Cy_n + D + \sum_{i=0}^{k} \beta_i g(x_{n+1-i}),
\]

\[
G_{k+i} (x_n, \ldots, x_{n+1-k}, y_n, \ldots, y_{n-k}) = y_{n+2-i}, \quad \text{for } i = 2, \ldots, k + 1,
\]

which yields the characteristic matrix

\[
DG - \lambda I = \begin{pmatrix}
A - \lambda & 0 & \cdots & 0 & \alpha_0 \tilde{f} & \alpha_1 \tilde{f} & \cdots & \alpha_k \tilde{f} \\
1 & -\lambda & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(\beta_0 + \beta_1) \tilde{g} & \beta_2 \tilde{g} & \cdots & \beta_k \tilde{g} & C + \alpha_0 \tilde{f} & \alpha_1 \tilde{f} & \cdots & \alpha_k \tilde{f} \\
0 & \cdots & \cdots & \cdots & 1 & -\lambda & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1 & -\lambda & \cdots \\
\end{pmatrix}
\]

(12)

To calculate the determinant of matrix (12) we first make a series of elementary operations. Clearly, after such operations the determinant remains the same. Let us multiply the first row of this matrix by \( \beta_0 \tilde{g} \) and subtract it from the \((k + 1)\)th row. Denoting \( \det(DG - \lambda I) \) by \( d_G \) we then see that

\[
d_G = \det \begin{pmatrix}
A - \lambda & 0 & \cdots & 0 & \alpha_0 \tilde{f} & \alpha_1 \tilde{f} & \cdots & \alpha_k \tilde{f} \\
1 & -\lambda & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(\beta_0 + \beta_1) \tilde{g} & \beta_2 \tilde{g} & \cdots & \beta_k \tilde{g} & C - \lambda & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 1 & -\lambda & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1 & -\lambda & \cdots \\
\end{pmatrix}
\]
Next, multiplying the second row by \((\beta_0 \lambda + \beta_1) \tilde{g}'\) and subtracting it from the \((k + 1)th\) row one gets

\[
d_G = \det \begin{pmatrix}
A - \lambda & 0 & \ldots & 0 & \alpha_0 \tilde{f}' & \alpha_1 \tilde{f}' & \ldots & \alpha_k \tilde{f}' \\
1 & -\lambda & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ldots & 1 & -\lambda & 0 & 0 & \ldots & 0 \\
0 & \left(\beta_0 \lambda^2 + \beta_1 \lambda + \beta_2\right) \tilde{g}' & \ldots & \beta_k \tilde{g}' & C - \lambda & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 1 & -\lambda & 0 \\
\vdots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & 0 & 1 & -\lambda & \end{pmatrix}
\]

It is easy to see that making a sequence of elementary operations of the form: multiplying the \(i\)th row, \(i = 2, \ldots, k - 1,\) by \(\tilde{g}'_{i-1} \sum_{j=0}^{i-1} \beta_j \lambda^{i-1-j}\) and subtracting it from the \((k + 1)th\) row eventually yields

\[
d_G = \det \begin{pmatrix}
A - \lambda & 0 & \ldots & 0 & \alpha_0 \tilde{f}' & \alpha_1 \tilde{f}' & \ldots & \alpha_k \tilde{f}' \\
1 & -\lambda & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ldots & 1 & -\lambda & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \tilde{g}' \sum_{j=0}^{k-1} \beta_j \lambda^{k-j} & C - \lambda & 0 & \ldots & 0 \\
0 & \ldots & \ldots & 0 & 1 & -\lambda & 0 & \ldots \\
\vdots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & 0 & 1 & -\lambda & \end{pmatrix}
\]

Now, we are in a position to calculate \(d_G.\) Clearly, \(d_G = (A - \lambda) \det N_I - \det N_{II},\) where \(N_I\) is lower triangular and has the block form

\[
N_I = \begin{pmatrix}
M^I_{k-1} & 0 \\
M^I_{k} & \Sigma + M^I_{k+1}
\end{pmatrix},
\]

where

\[
M^I_{l} = \begin{pmatrix}
-\lambda & 0 & \ldots & 0 \\
1 & -\lambda & 0 & \ldots \\
0 & \ldots & \ldots & \ldots \\
\vdots & 0 & 1 & -\lambda
\end{pmatrix} \in \mathbb{R}^{l \times l}, \quad l = k - 1, \quad l = k + 1,
\]

and

\[
M^\Sigma = \begin{pmatrix}
0 & \ldots & \tilde{g}' \sum_{j=0}^{k-1} \beta_j \lambda^{k-j} \\
0 & \ldots & 0 \\
0 & \ldots & 0
\end{pmatrix},
\]

while \(N_{II}\) has the block form

\[
N_{II} = \begin{pmatrix}
M^I_{k-1} + \lambda I & M^\Sigma \\
M^I_{k} & \Sigma + M^I_{k+1}
\end{pmatrix},
\]

where

\[
M^\Sigma = \begin{pmatrix}
\alpha_0 \tilde{f}' & \ldots & \alpha_k \tilde{f}' \\
0 & \ldots & 0 \\
0 & \ldots & 0
\end{pmatrix}.
As $N_I$ is triangular and has dimension $2k$, we have $\det N_I = (-\lambda)^{2k-1}(C - \lambda)$. Calculating $\det N_{II}$ we need to use the recurrent formula

$$\det \begin{pmatrix} M_l^I + \lambda I & M^\alpha \\ M^\alpha & C_l + M_{k+1}^I \end{pmatrix} = -\det \begin{pmatrix} M_{l-1}^I + \lambda I & M^\alpha \\ M^\alpha & C_l + M_{k+1}^I \end{pmatrix},$$

for $l = k - 1, \ldots, 2$, because every such matrix has only one non-zero term in the first column, which is equal to 1 and appears in the second row. This recurrence ends with

$$\det \begin{pmatrix} 0 & V^\alpha \\ C_l + M_{k+1}^I \end{pmatrix},$$

where $V^\alpha = (\alpha_0, \tilde{f}^\alpha, \ldots, \tilde{f}^\alpha)$, $V^\alpha = (g^\alpha_j \sum_{j=0}^k \beta_j \lambda^{k-j}, 0, \ldots, 0)^T$, which means that

$$\det N_{II} = (-1)^k \tilde{g}^\alpha \sum_{j=0}^k \beta_j \lambda^{k-j} \det \begin{pmatrix} \alpha_0 \tilde{f}^\alpha & \alpha_1 \tilde{f}^\alpha & \ldots & \alpha_k \tilde{f}^\alpha \\ 1 & -\lambda & 0 & \ldots \\ 0 & \ldots & 0 \\ \vdots & 0 & 1 & -\lambda \end{pmatrix},$$

and the last determinant could be calculated as $\det M_{k+1}^{II}$ in Subsection 4.1. Eventually

$$\det N_{II} = (-1)^k \tilde{f}^\alpha \sum_{j=0}^k \alpha_j \lambda^{k-j} \sum_{j=0}^k \beta_j \lambda^{k-j},$$

and

$$d_G = (-\lambda)^{2k-1}(A - \lambda)(C - \lambda) - (-1)^k \tilde{f}^\alpha \sum_{j=0}^k \alpha_j \lambda^{k-j} \sum_{j=0}^k \beta_j \lambda^{k-j}.$$ 

Therefore, splitting $d_G$ in the same way as in Subsection 4.1 we easily prove the main result.

**Theorem 4.3.** If $f^\alpha(\bar{\gamma}) > 0$, $g^\alpha(\bar{\xi}) > 0$ and Inequality (3) holds, then stable steady states of Equations (1) are also stable for Equations (11).

**Proof.** We have $d_G = W_1(\lambda) + W_2(\lambda)$, where $W_1(\lambda) = (-\lambda)^{2k-1}(A - \lambda)(C - \lambda)$ has zeros inside the unite circle, while for $W_2(\lambda) = \tilde{f}^\alpha \sum_{j=0}^k \alpha_j \lambda^{k-j} \sum_{j=0}^k \beta_j \lambda^{k-j}$ we have $|W_2(\lambda)|_{|\lambda| = 0} \leq \tilde{f}^\alpha < (A - 1)(C - 1) \leq |W_1(\lambda)|_{|\lambda| = 0}$, and Rouché’s Theorem [10] implies that $d_G$ also has zeros inside the unite circle. $\square$

5. **Discussion.** In this paper we made an attempt to introduce delays into the Gottman-Murray model reflecting dyadic interactions in the context of marital dissolution. This model is described by discrete dynamical system. As we mentioned in Introduction, dyadic interactions are often described by continuous dynamical systems. Hence, it is interesting to compare these two types of description. In general, when delay is introduced into a system of differential equations, the model dynamics could change from stable to unstable, depending on the model parameters. It is easily visible for the linear equation $\frac{dx}{dt} = -ax(t - \tau)$, $a > 0$ (see e.g. [7] and the references therein), for which $x = 0$ is stable for $\tau = 0$, and loses stability for larger delays, namely for $\tau > \pi/2$, while at $\tau = \pi/2$ a Hopf bifurcation occurs. On the other hand, considering discrete version of this differential equation, i.e. $x_{n+1} = ax_{n-k}$, $k \in \mathbb{N}$, $a \in (0, 1)$, no change of stability occurs. Clearly, for any initial data $(\phi_{-k}, \phi_{-k+1}, \ldots, \phi_0)$ we can easily write general solution in the form $x_{m(k+1)+1} = a^{m+1} \phi_{-k}$, $x_{m(k+1)+2} = a^{m+1} \phi_{-k+1}$, $x_{m(k+1)+k+1} = a^{m+1} \phi_0$ for $n = m(k + 1) + l$, $l = 1, \ldots, k + 1$, which leads to convergence of the sequence $(x_n)_{n \in \mathbb{N}}$ to 0 independently of the magnitude of $k$ reflecting the delay. It is then clear that in general we should expect
different behavior of delayed discrete equation comparing to the corresponding differential one. It is not surprising as discrete equations have usually different dynamics than continuous, however we typically expect that this dynamics could be more rich, like for the logistic equation.

In this paper we have proved that for the Gottman-Murray model of marital interactions introducing delay does not influence stability of the steady state. More precisely, if both interaction functions are increasing, then if the steady state is stable in the case without delay, then it remains stable for any larger delay in interactions of both partners. What is interesting, similar, however simpler, result was obtained in [3], where we showed that when one of the reaction terms is delayed and the product of slopes of interaction functions is less than the product of forgetting coefficients, then the delay does not influence the stability. On the other hand, from the analysis presented in this paper it is obvious that for the product of slopes greater than 1 we always have unstable steady state (as this product is equal to the free term of the characteristic polynomial, and when it is greater than 1, at least one eigenvalue has modulus greater than 1), while for the continuous system, analysis in [3] showed that sometimes we should expect stability for small delays, but it is changed with increasing delay. It shows that the Gottman-Murray model is much more robust comparing to the continuous-time model.

Moreover, it should be noticed that the uninfluenced dynamics could be also non-monotonic, which is another interesting topic to discuss in the future. It could be also interesting to include the delay into this dynamics and compare discrete and continuous case.

On the other hand, it is also important to notice that the model considered in this paper is more general and could be used to describe another situations than those described by Gottman et al., for example we can discuss the changes in a person’s opinion under the influence of his/her partner. More generally, we can also consider not single persons and interactions between them but also interactions between two groups of persons. Therefore, such a simple model could have much wider range of applications than this considered in the present paper.

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