Boundedness analysis of stochastic integro-differential systems with Lévy noise

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ABSTRACT
This paper is concerned with the \( p \)-th moment globally exponential ultimate boundedness of stochastic integro-differential systems with Lévy noise. With the help of the Lyapunov function methods and the inequality techniques, several sufficient criteria on the exponential ultimate boundedness are presented for the systems. The results show that the boundedness was determined by the coefficients in the estimation of the Itô operator \( \mathcal{L}V \) of the energy function \( V \) along the trajectories of the addressed systems.

1. Introduction
Integro-differential system is a kind of very important type of an ordinary differential equation which has not only integrals of unknown function but also derivatives of unknown function. Integro-differential system has received much attention from researchers since this system has a wide application in many fields, such as biological system [1], switched systems [2] and neural networks [3].

Boundedness is a very important asymptotic property of dynamical systems. Recently, the study of the boundedness of dynamical systems has attracted increasing attention from researchers since it plays a key role in the analysis of the stability, existence, persistence, chaotic behaviour, attracting properties and other properties of the solutions. Many significant results on the boundedness have been established for various systems such as stochastic delay differential systems [4–7], non-autonomous neural networks [8,9], non-linear switched systems [10], integro-differential systems [11–13], impulsive stochastic systems [14–16], impulsive fractional differential systems [17–19], impulsive fractional difference systems [20], integro-differential systems [21,22].

Stochastic systems with Lévy noise are used to describe the evolutionary processes in which the structures are subject to stochastic abrupt changes. Recently, these systems received remarkable attention from the researchers. Many important results can be found in the literature concerning the stability [23–25], the periodic solution [26] and the existence and uniqueness [27] of these systems. However, the problem of the boundedness of stochastic integro-differential systems with Lévy noise is more complicated and still open.

Inspired by the aforementioned discussions, the present paper is focused on the \( p \)-th moment globally exponential ultimate boundedness of stochastic integro-differential systems with Lévy noise. With the help of the Lyapunov function methods and the inequality techniques, several sufficient conditions are presented for the \( p \)-th moment globally exponential ultimate boundedness and the \( p \)-th moment globally exponential stability of the systems. The main contributions of this paper are highlighted as follows: (i) concerned with the problem of the boundedness for a class of stochastic integro-differential systems and take fully into account the effects of Lévy noise and infinite time-delay; (ii) some sufficient conditions, including both Lyapunov type and coefficients type, are derived for the exponential ultimate boundedness; and (iii) the estimation of the ultimate bound sets is also given out.

2. Preliminaries
Let \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{R}_0 = [0, \infty) \). For given real numbers \( a_1 \) and \( a_2 \), let \( a_1 \land a_2 = \min(a_1, a_2) \). Let \( \lambda_{\min}(\cdot) \) \((\lambda_{\max}(\cdot))\) denote the smallest [largest] eigenvalue of a symmetric matrix. Let \( \mathbf{B}(t) = (B_1(t), \ldots, B_m(t))^T \) be an \( m \)-dimensional Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). Let \( \mathcal{C}([-\infty, 0], \mathbb{R}_0^m) \) denote the family of bounded continuous functions \( \phi \) from \((-\infty, 0] \) to...
$R^n$ equipped with the norm $|\phi|_\infty = \sup_{-\infty < \theta < 0} |\phi(\theta)|$, where $|\cdot|$ denote the Euclidean norm in $R^n$. Denote by $C^b_{F}([-\infty, 0], [0, \infty))$ the family of bounded $F_{t_0}$-measurable, $C([-\infty, 0], R^n)$-valued random variables $\phi$, satisfying $E|\phi|_\infty < \infty$, where $E$ denotes the expectation of the stochastic process. Let $L = \{\phi(s) : R^+ \to R|\phi(s) \text{ is continuous and satisfies } \int_0^\infty e^{\lambda t}|\phi(s)| \, ds < \infty, \text{where } \lambda_0 > 0 \}$ be a constant positive). Let $C_{F}([-\infty, 0], R^n, R^+)$ be the family of all nonnegative functions $U(t, y)$ from $R_{t_0} \times R^n$ to $R^+$, which are once continuously differentiable in $t \in R^+$ and twice in $y \in R^n$. The symbol $\mathcal{L}$ stands for the infinitesimal operator given by the generalized Itô formula (for more information on it, cf. [25]).

In this paper, we study the following stochastic integro-differential systems with Lévy noise:

$$
dy(t) = F(t, y(t), \int_{-\infty}^{t} h(t-s)y(s) \, ds) \, dt + G(t, y(t), \int_{-\infty}^{t} h(t-s) \, ds) \, dB(t) + \int_{|u| < c} I(t, u, y(t^-)) \, dN(t, du) + \int_{|u| \geq c} I(t, u, y(t^-)) \, dN(t, du),
$$

where the initial value $\phi(s) \in C^b_{F}([-\infty, 0], R^n), h(s) \in L_\nu(y(s^-)) = \lim_{|\theta| \to 0} y(\theta), F : R_{t_0} \times R^n \times R^n \to R^n, G : R_{t_0} \times R^n \times R^n \to R^n, I : R_{t_0} \times R^n \times R^n \times R^n \to R^n$ is the scalar $c \in (0, \lambda_0)$ stands for the maximum allowable jump size, $N$ stands for a Poisson random measure defined on $R_{t_0} \times (R^n \setminus \{0\})$ with intensity measure $\nu$ and compensator $N$. Moreover, $N$ is independent of $B$ and $\nu$ is a Levy measure such that $N(\, dt, \, du) = N(\, dt, \, du) - \nu(\, du) \, dt$ and $\int_{|u| < c} (|u|^2 \wedge 1) \nu(\, du) < \infty$. The pair $(B, N)$ generally referred to as a Lévy noise. We assume that for any $\phi(s) \in C^b_{F}([-\infty, 0], R^n)$, there exists at least one solution of system (1).

**Lemma 2.1 ([28]):** For $a_t \geq 0, b_t > 0$ and $\sum_{i=1}^{n} a_i = 1$,$$
\int_{t_0}^{t} a_{t_i} \, dt_i \leq \sum_{i=1}^{n} b_i a_t,$$

**3. Exponential ultimate boundedness**

It is well known that Lyapunov function methods play a key role in the study of the stability [29–35] and boundedness [15,21,36–38] of dynamical systems. In this section, we will derive several sufficient criteria on the exponential ultimate boundedness for the systems (1) by Lyapunov function methods.

Theorem 3.1: Assume that there are functions $U(t, y) \in C_{F}([t_0, t^\infty)) \text{ and } \lambda_2(s) \in L$ and constants $p > 0, \phi_1 > 0(i = 1, 2, \ldots, 4)$ and $\varphi_2 \geq 0$ with $\phi_1 \varphi_2 > \varphi_2 \varphi_4 \int_0^\infty \lambda_2(s) \, ds$ such that for all $(t, y) \in R_{t_0} \times R^n$,

1. $\phi_1 |y|^p \leq U(t, y) \leq \varphi_2 |y|^p$;(3)
2. $\mathcal{L}U(t, y) \leq -\varphi_3 U(t, y(t))$
   $+ \varphi_4 \int_0^\infty \lambda_2(s)U(t, y(s^-)) \, ds + \varphi_5.$ (4)

Then system (1) is $p$th moment globally exponentially ultimately bounded (p-GEUB), and the trajectory of system (1) converges into the ultimate bound set

$$
\mathcal{D} = \{ \xi \in C^b_{F}([-\infty, 0], R^n) | E|\xi|^p \leq \frac{\varphi_5}{\phi_1 \lambda} \},
$$

where the scalar $\lambda \in (0, \lambda_0)$ satisfies the following relation

$$
\varphi_2 \lambda - \phi_1 \varphi_3 + \varphi_2 \varphi_4 \int_0^\infty \lambda_2(s)e^{\lambda s} \, ds < 0. \quad (6)
$$

**Proof:** With the help of the Itô formula, one can derive that

$$
e^{\lambda(t-t_0)}U(t \wedge r_n, y(t \wedge r_n)) - e^{\lambda t_0}U(t_0, y(t_0))
$$

$$= \int_{t_0}^{t \wedge r_n} e^{\lambda \varphi} [U(\varphi, y(\varphi)) + \mathcal{L}U(\varphi, y(\varphi))] \, d\varphi + \int_{t_0}^{t \wedge r_n} e^{\lambda \varphi} U(\varphi, y(\varphi)) \, d\varphi,
$$

$$+ \int_0^{t \wedge r_n} I(\varphi, y(\varphi)) \, dB(\varphi) + \int_{-\infty}^{t} h(\varphi-s)y(s^-) \, ds \, dB(\varphi) + M_1(t) + M_2(t), \quad (7)
$$

where $r_n = \inf\{t > t_0 : |y(t)| \geq n\}$ denotes the stopping time,

$$M_1(t) = \int_{t_0}^{t \wedge r_n} \int_{|u| < c} e^{\lambda \varphi} [U(y(\varphi^-)) + H(\varphi, u, y(\varphi^-))]
$$

$$\int_{-\infty}^{t} h(\varphi-s)y(s^-) \, ds) - U(\varphi^-, y(\varphi^-)) \, N(\, d\varphi, \, du) \quad (8)
$$

and

$$M_2(t) = \int_{t_0}^{t \wedge r_n} \int_{|u| \geq c} e^{\lambda \varphi} [U(y(\varphi^-)) + I(\varphi, u, y(\varphi^-)) \int_{-\infty}^{t} h(\varphi-s)y(s^-) \, ds) - U(\varphi^-, y(\varphi^-)) \, N(\, d\varphi, \, du) \quad (9)
$$

are two martingales with $M_1(t_0) = M_2(t_0) = 0$. Consequently, taking expectation on both sides, we arrive
at
\[
\mathbb{E} \left( e^{\lambda (t-n)} U(t \wedge r_n, y(t \wedge r_n)) - e^{\lambda t_0} U(t_0, y(t_0)) \right)
= \mathbb{E} \left( \int_{t_0}^{t \wedge r_n} e^{\lambda \theta} [\nu(U(\theta, y(\theta))) + L(U(\theta, y(\theta)))] d\theta \right). 
\]
(10)

Using the condition (ii) and (10), we obtain the estimate
\[
\mathbb{E} \left( e^{\lambda (t-n)} U(t \wedge r_n, y(t \wedge r_n)) \right)
\leq \varrho e^{\lambda t_0} \mathbb{E}|\phi|^p 
+ \mathbb{E} \left( \int_{t_0}^{t \wedge r_n} e^{\lambda \theta} (\varrho \xi(t) |y(\theta)|^p - \xi_1 \xi_3 |y(\theta)|^p) \right. 
+ \left. \int_0^\infty \varrho \xi_4 \lambda_2(s) |y(\theta - s)|^p ds + \xi_5 \right) d\theta 
\leq \varrho e^{\lambda t_0} \mathbb{E}|\phi|^p 
+ (\varrho \lambda - \xi_1 \xi_3) \mathbb{E} \left( \int_{t_0}^{t \wedge r_n} e^{\lambda \theta} |y(\theta)|^p d\theta \right) 
+ \mathbb{E} \left( \int_{t_0}^{t \wedge r_n} e^{\lambda \theta} \int_0^\infty \varrho \xi_4 \lambda_2(s) |y(\theta - s)|^p ds d\theta \right) 
+ \frac{\xi_5}{\lambda} (e^{\lambda (t-n)} - e^{\lambda t_0}) 
\]
(11)

On the other hand,
\[
\mathbb{E} \left( \int_{t_0}^{t \wedge r_n} e^{\lambda \theta} \int_0^\infty \varrho \xi_4 \lambda_2(s) |y(\theta - s)|^p ds d\theta \right) 
= \mathbb{E} \left( \int_0^{\infty} \varrho \xi_4 \lambda_2(s) e^{\lambda \theta} (\int_{t_0}^{t \wedge r_n - s} e^{\lambda \theta} |y(\theta)|^p d\theta) \right. 
+ \left. \int_{t_0}^{t \wedge r_n} e^{\lambda \theta} |y(\theta)|^p d\theta \right) ds d\theta 
\leq \mathbb{E} \left( \int_0^{\infty} \varrho \xi_4 \lambda_2(s) e^{\lambda \theta} (\int_{t_0}^{t \wedge r_n - s} e^{\lambda \theta} |y(\theta)|^p d\theta) \right. 
+ \left. \int_{t_0}^{t \wedge r_n} e^{\lambda \theta} |y(\theta)|^p d\theta \right) ds 
= (\varrho \xi_4 \int_0^\infty \lambda_2(s) e^{\lambda \theta} ds) \mathbb{E} \left( \int_0^{t \wedge r_n} e^{\lambda \theta} |y(\theta)|^p d\theta \right) 
+ \left( \int_0^{\infty} \lambda_2(s) e^{\lambda \theta} ds \left( \frac{\varrho \xi_4}{\lambda} e^{\lambda t_0} \right) \right) \mathbb{E}|\phi|^p 
\]
(12)

Substituting (12) into (11) yields
\[
\mathbb{E} \left( e^{\lambda (t-n)} U(t \wedge r_n, y(t \wedge r_n)) \right)
\leq \varrho e^{\lambda t_0} \mathbb{E}|\phi|^p 
+ \varrho \xi_4 \lambda \mathbb{E} \left( \int_0^{t \wedge r_n} e^{\lambda \theta} |y(\theta)|^p d\theta \right) 
\leq \varrho e^{\lambda t_0} \mathbb{E}|\phi|^p 
+ \varrho \xi_4 \lambda \mathbb{E} \left( \int_0^{t \wedge r_n} e^{\lambda \theta} |y(\theta)|^p d\theta \right) 
+ \frac{\xi_5}{\lambda} (e^{\lambda (t-n)} - e^{\lambda t_0}) 
\]
(13)

such that the inequality (6) holds. Then the following inequality follows by letting \( n \to \infty \).
\[
\mathbb{E} \left( e^{\lambda t} U(t, y(t)) \right) \leq \varrho e^{\lambda t_0} \mathbb{E}|\phi|^p 
+ \left( \int_0^\infty \varrho \lambda_2(s) e^{\lambda s} ds \left( \frac{\varrho \xi_4}{\lambda} e^{\lambda t_0} \right) \right) \mathbb{E}|\phi|^p 
+ \frac{\xi_5}{\lambda} (e^{\lambda t} - e^{\lambda t_0}) 
\]
(14)

Using the condition (i), we have
\[
\mathbb{E}|y(t)|^p \leq \varrho \left( 1 + \frac{\varrho \lambda_2(s)}{e^{\lambda t}} \right) \mathbb{E}|\phi|^p 
\times \mathbb{E}|\phi|^p e^{-\lambda (t-t_0)} + \frac{\xi_5}{\lambda} e^{\lambda t_0} 
\]
(15)

This ends the proof of Theorem 3.1.

**Assumption 3.1:** There exist a symmetric positive-definite matrix \( S \), a function \( h(s) \in L^2 \), and several constants \( p > 0 \) and \( \xi_i (i = 1, 2, \ldots, 12) \) such that
(i) \( y^T SF + \frac{1}{2} \text{tr}(G^T SG) \leq \xi_1 y^T SY + \xi_2 \)
\times \int_0^\infty h(s)(y^T (t - s)) \times SY(t - s)) \times dy + \xi_3; 
(16)
(ii) \( |y^T SG|^2 \leq \xi_4 (y^T SY)^2 + \xi_5 \int_0^\infty h(s) (y^T (t - s)) \times SY(t - s)^2 \times dy + \xi_6; 
(17)
(iii) \( \int_{|u| < \xi_1} \left( (y + H)^T S(y + H) - y^T SY \right)^{p/2} \times dy \leq \xi_7 (y^T SY)^{p/2} + \xi_8 \int_0^\infty h(s) (y^T (t - s)) \times SY(t - s)^{p/2} \times dy + \xi_9; 
(18)
(iv) \( \int_{|u| < \xi_1} \left( (y + H)^T S(y + H) - y^T SY \right)^{p/2} \times dy \leq \xi_10 (y^T SY)^{p/2} + \xi_11 \int_0^\infty h(s) (y^T (t - s))SY(t - s)^{p/2} \times dy + \xi_{12}; 
(19)
(v) \( (\lambda_{\text{min}}(S))^{p/2} \xi_3 > (\lambda_{\text{max}}(S))^{p/2} \xi_4 \int_0^\infty h(s) \times dy > 0, 
(20)
(vi) \( \xi_5 = 2\xi_3 + \xi_6(2p - 4) \times + \xi_9 + \xi_{12} \geq 0. 
(21)

where
\[
\xi_3 = - \left( p \xi_1 + p \left( \frac{p}{2} - 1 \right) \xi_4 + (p - 2) \xi_3 \right) 
+ \left( \frac{p}{2} - 1 \right) (p - 4) \xi_8 + \xi_{10} 
+ \left( \xi_3(p - 2) + \left( \frac{p}{2} - 1 \right) (p - 4) \xi_5 \right) \times \int_0^\infty h(s) \times dy > 0, 
(22)
\]
and
\[
\xi_4 = \xi_8 + \xi_{11} + 2\xi_2 + 2(p - 2) \xi_5. 
(23)
\]
Theorem 3.2: Suppose that the Assumption 3.1 holds. If \( \hat{\vartheta}_3 > \hat{\vartheta}_4 > 0 \), then system (1) is p-GEUS, and the solutions \( y(t) \) of (1) will converge to the ultimate bound set

\[
\mathcal{D} = \left\{ \xi \in C_{x}([0, \infty), \mathbb{R}^n) \mid \xi|_{\mathbb{R}^n} \leq \frac{\hat{\vartheta}_5}{(\lambda_{\min}(s))^{p/2}} \right\},
\]  

(24)

where the scalar \( \lambda \in (0, \lambda_0) \) satisfies the following relation

\[
(\lambda_{\max}(S))^{p/2} \lambda - (\lambda_{\min}(S))^{p/2} \hat{\vartheta}_3
\]

\[
+ (\lambda_{\max}(S))^{p/2} \hat{\vartheta}_4 \int_0^\infty h(s) \xi(s)^{1/2} ds < 0.
\]

(25)

Proof: Consider the Lyapunov function candidate

\[
U(t, y(t)) = (y^T(t)S^2 y(t))^{p/2}.
\]

(26)

Obviously, we have

\[
(\lambda_{\min}(S))^{p/2} \mathbb{E}[|y|^p] \leq \mathbb{E}U(t, y(t)) \leq (\lambda_{\max}(S))^{p/2} \mathbb{E}[|y|^p].
\]

(27)

Using (16) to (19), we arrive successively

\[
\mathcal{L}U(t, y) = p(y^T Sy)^{p/2 - 1} \left[ y^T SF + \frac{1}{2} \text{trace}(G^T SG) \right]
\]

\[
+ p \left( \frac{p}{2} - 1 \right) (y^T Sy)^{p/2} - y^T SG^2
\]

\[
+ \int_{|u| < \xi} ((y + H)^T S(y + H))^{p/2} - (y^T Sy)^{p/2}
\]

\[
- p(y^T Sy)^{p/2 - 1} y^T Sh \nu du
\]

\[
+ \int_{|u| > \xi} ((y + I)^T S(y + I))^{p/2} - (y^T Sy)^{p/2} \nu du
\]

\[
\leq (p \hat{\vartheta}_1 + p \left( \frac{p}{2} - 1 \right) \hat{\vartheta}_4)(y^T Sy)^{p/2} + p \hat{\vartheta}_2
\]

\[
\times \int_0^\infty h(s)(y^T Sy)^{p/2 - 1} y^T(t - s) Sy(t - s) ds
\]

\[
+ \hat{\vartheta}_3 p(y^T Sy)^{p/2 - 1} + p \left( \frac{p}{2} - 1 \right) \hat{\vartheta}_5
\]

\[
\times \int_0^\infty h(s)(y^T Sy)^{p/2 - 2} ((y^T(t - s) Sy(t - s)))^2 ds
\]

\[
+ \hat{\vartheta}_6 p \left( \frac{p}{2} - 1 \right) (y^T Sy)^{p/2 - 2}
\]

\[
+ (\hat{\vartheta}_7 + \hat{\vartheta}_10)(y^T Sy)^{p/2}/2
\]

\[
+ (\hat{\vartheta}_8 + \hat{\vartheta}_11) \int_0^\infty h(s)(y^T(t - s) Sy(t - s))^{p/2}
\]

\[
\times \nu + \hat{\vartheta}_9 + \hat{\vartheta}_12
\]

(28)

Applying Lemma 2.1 to (28) yields

\[
\mathcal{L}U(t, y) \leq (p \hat{\vartheta}_1 + p \left( \frac{p}{2} - 1 \right) \hat{\vartheta}_4)(y^T Sy)^{p/2} + \hat{\vartheta}_2(p - 2)
\]

\[
\times \int_0^\infty h(s)(y^T Sy)^{p/2} ds
\]

\[
+ 2 \hat{\vartheta}_2 \int_0^\infty h(s)(y^T(t - s) Sy(t - s))^{p/2} ds
\]

\[
+ \hat{\vartheta}_3(p - 2)(y^T Sy)^{p/2} + 2 \hat{\vartheta}_3
\]

\[
+ \left( \frac{p}{2} - 1 \right)(p - 4) \hat{\vartheta}_5 \int_0^\infty h(s) (y^T Sy)^{p/2} ds
\]

\[
+ 2(p - 2) \hat{\vartheta}_6 \int_0^\infty h(s)((y^T(t - s) Sy(t - s)))^{p/2} ds
\]

\[
+ \hat{\vartheta}_6 \left( \frac{p}{2} - 1 \right)(p - 4)(y^T Sy)^{p/2} + \hat{\vartheta}_6(2p - 4)
\]

\[
+ (\hat{\vartheta}_7 + \hat{\vartheta}_10)(y^T Sy)^{p/2}/2
\]

\[
+ (\hat{\vartheta}_8 + \hat{\vartheta}_11) \int_0^\infty h(s)(y^T(t - s) Sy(t - s))^{p/2} ds
\]

\[
+ \hat{\vartheta}_9 + \hat{\vartheta}_12
\]

\[
= -\hat{\vartheta}_3(y^T Sy)^{p/2}
\]

\[
+ \hat{\vartheta}_4 \int_0^\infty h(s)(y^T(t - s) Sy(t - s))^{p/2} ds + \hat{\vartheta}_5
\]

(29)

where

\[
\hat{\vartheta}_5 = 2\hat{\vartheta}_3 + \hat{\vartheta}_6(2p - 4) + \hat{\vartheta}_9 + \hat{\vartheta}_12.
\]

(30)

Since

\[
(\lambda_{\min}(S))^{p/2} \hat{\vartheta}_3 > (\lambda_{\max}(S))^{p/2} \int_0^\infty h(s) \ ds,
\]

(31)

using the continuity and noting \( h(s) \in L^p \), there exist a scalar \( \lambda \in (0, \lambda_0) \) such that (25) holds. Consequently, from (27), (29) and Theorem 3.1, one derives the following estimate

\[
\mathbb{E}[|y(t)|^p] \leq \left( \lambda_{\max}(S) \right)^{p/2} \left( \frac{1 + \hat{\vartheta}_4}{\lambda_{\min}(S)} \right)^{p/2} \left( \int_0^\infty h(s) \xi(s)^{1/2} ds \right)
\]

\[
\times \mathbb{E}[|\varphi|^p e^{-\lambda(t-t_0)}] + \frac{\hat{\vartheta}_5}{\lambda_{\min}(S)^{p/2}}
\]

(32)

where the scalar \( \lambda \) satisfies \( 0 < \lambda < \lambda_0 \) and is determined by (25). This concludes the proof of the result.

\( \square \)

Corollary 3.3: Suppose that the hypotheses of Theorem 3.1 hold. If \( \hat{\vartheta}_5 = 0 \), then system (1) is pth moment globally exponentially stable (p-GEUS).

Proof: This result follows directly from Theorems 3.1.

\( \square \)

Corollary 3.4: Suppose that the Assumption 3.1 with \( \hat{\vartheta}_3 = \hat{\vartheta}_6 = \hat{\vartheta}_9 = \hat{\vartheta}_12 = 0 \) holds. If \( \hat{\vartheta}_3 > \hat{\vartheta}_4 > 0 \), then system (1) is p-GEUS.

Proof: This result follows directly from Theorems 3.2.

\( \square \)
Remark 3.1: In [39], several exponential ultimate boundedness criteria have been obtained for stochastic integro-differential systems. However, the criteria proposed in [39] are unable to detect the exponential ultimate boundedness of system (1) because Lévy noise was ignored in [39].

Remark 3.2: In [25], authors derived some interesting sufficient conditions for the pth moment exponential stability of stochastic delay differential systems with Lévy noise. Compared with [25], system (1) here is an infinite distributed delay system. Moreover, the results in [25] are limited to stability. Up till now, the boundedness problem has not been studied for stochastic integro-differential systems with Lévy noise, this indicates that our results are new.

4. An example

Example 4.1: Let us consider the one-dimensional stochastic integro-differential system with Lévy noise as follows

\[ dy(t) = \left[ -12y(t) + \int_{-\infty}^{t} e^{-(t-s)} y(s) \, ds + 2 \right] dt + \sqrt{2}y(t) \, dB(t) \]

\[ + \int_{|u| < 1} \int_{-\infty}^{t} e^{-(t-s)} y(s^-) \, ds \, d\mathcal{N}(dt, du) \]

\[ + \int_{|u| \geq 1} y(t^-) + \int_{-\infty}^{t} e^{-(t-s)} y(s^-) \times ds \, d\mathcal{N}(dt, du), \quad t \geq 0, \]

where the Lévy measure \( \nu \) satisfies \( \nu(du) = du/1 + |u|^2 \).

Let \( U(t,y) = y^2 \), we have

\[ U_y(t,y(t))F \leq -22y^2(t) + \int_{0}^{\infty} e^{-s} y^2(t - s) \, ds + 4, \quad t \geq 0. \]

\[ \left( \frac{1}{2} U_{yy}(t,y(t)) \right) G^2 = 5y^2(t), \quad t \geq 0. \]

\[ \int_{|u| < 1} \left[ U(t,y + H) - U(t,y) - HU_y(t,y) \right] \nu(du) \]

\[ = \int_{|u| < 1} \left[ \left( \frac{y}{2} + \int_{-\infty}^{t} e^{-(t-s)} y(s) \, ds \right)^2 \right. \]

\[ \left. - y \int_{-\infty}^{t} e^{-(t-s)} y(s) \, ds \right] \frac{du}{1 + |u|^2} \]

\[ = \int_{|u| < 1} \left[ \frac{y^2}{4} + \left( \int_{-\infty}^{t} e^{-(t-s)} y(s) \, ds \right)^2 \right] \frac{du}{1 + |u|^2} \]

\[ \leq \frac{\pi y^2}{8} + \frac{\pi}{2} \int_{0}^{\infty} e^{-s} y^2(t - s) \, ds. \]

\[ \int_{|u| \geq 1} \left[ U(t,y + l) - U(t,y) \right] \nu(du) \]

\[ = \int_{|u| \geq 1} \left[ (y(t) + \int_{0}^{t} e^{-(t-s)} y(s) \, ds)^2 - y^2 \right] \nu(du) \]

\[ \leq \frac{\pi y^2}{2} + \pi \int_{0}^{\infty} e^{-s} y^2(t - s) \, ds. \]

Hence

\[ \mathcal{L}U(t,y(t)) \]

\[ = -22y^2(t) + \int_{0}^{\infty} e^{-s} y^2(t - s) \, ds + 4 + 5y^2(t) \]

\[ + \frac{\pi y^2}{8} + \frac{\pi}{2} \int_{0}^{\infty} e^{-s} y^2(t - s) \, ds + \frac{\pi y^2}{2} \]

\[ + \pi \int_{0}^{\infty} e^{-s} y^2(t - s) \, ds \]

\[ \leq \left( -17 + \frac{5\pi}{8} \right) y^2(t) + \left( 1 + \frac{3\pi}{2} \right) \]

\[ \times \int_{0}^{\infty} e^{-s} y^2(t - s) \, ds + 4, \quad t \geq 0. \]

All conditions of Theorem 3.1 are satisfied by taking \( p = 2, \varrho_1 = \varrho_2 = 1, \varrho_3 = 17 - (5\pi / 8), \varrho_4 = 1 + (3\pi / 2), \lambda_2(s) = e^{-s} \) and \( \varrho_5 = 4 \). On the other hand, it is easy to verify that \( e^{-s} \in \mathbb{L} \), provided that \( 0 < \lambda_0 < 1 \). Here, we choose \( \lambda_0 = 0.9 \) and \( \lambda = 0.5 < \lambda_0 \) which satisfies the inequality (6). According to Theorem 3.1, system (33) is GEUB in the mean square, and the solutions \( y(t) \) will converge to the ultimate bound set

\[ \mathcal{D} = \{ \xi \in C_{T_{1,0}} [(-\infty,0), \mathbb{R}) | E[|\xi|^2_{T} \leq \frac{\varrho_5}{\varrho_1 \lambda} = 8 \}. \]

5. Conclusions

This paper has investigated the pth moment globally exponential ultimate boundedness issue of stochastic integro-differential systems with Lévy noise. Using the Lyapunov function methods and combining the inequality techniques, some sufficient criteria on the exponential ultimate boundedness have been presented for the systems. The method presented in this paper may be applied to some other kinds of stochastic systems such as stochastic systems with exogenous disturbances [40], semi-Markov switched stochastic systems [41] and stochastic systems with additive noise [42].

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