Abstract. We show that if a $PD_3$-group $G$ splits as an HNN extension $A \ast_C \varphi$ where $C$ is a $PD_2$-group then the Poincaré dual in $H^1(G; \mathbb{Z}) = Hom(G, \mathbb{Z})$ of the homology class $[C]$ is the epimorphism $f : G \to \mathbb{Z}$ with kernel the normal closure of $A$. We also make several other observations about $PD_3$-groups which split over $PD_2$-groups.

In this note we shall give algebraic analogues of some properties of Haken 3-manifolds. We are interested in the question “when does a $PD_3$-group split over a $PD_2$-group?”. In §2 we show that such splittings are minimal in a natural partial order on splittings over more general subgroups. In the next two sections we consider $PD_3$-groups $G$ which split as an HNN extension $A \ast_C \varphi$ with $A$ and $C$ finitely generated. In §3 we show that $A$ and $C$ have the same number of indecomposable factors. Our main result is in §4, where we show that if $C$ is a $PD_2$-group then the Poincaré dual in $H^1(G; \mathbb{Z}) = Hom(G, \mathbb{Z})$ of the homology class $[C]$ is the epimorphism $f : G \to \mathbb{Z}$ with kernel the normal closure of $A$. In §5 we extend an argument from [7] to show that no $FP_2$ subgroup of a $PD_3$-group is a properly ascending HNN extension, and in §6 we show that if $G$ is residually finite and splits over a $PD_2$-group then $G$ has a subgroup of finite index with infinite abelianization. Our arguments extend readily to $PD_n$-groups with $PD_{n-1}$-subgroups, but as our primary interest is in the case $n = 3$, we shall formulate our results in such terms.

1. terminology

We mention here three properties of 3-manifold groups that are not yet known for all $PD_3$-groups: coherence, residual finiteness and having subgroups of finite index with infinite abelianization. Coherence may often be sidestepped by requiring the subgroups in play to be $FP_2$ rather than finitely generated. If every finitely generated subgroup of a group $G$ is $FP_2$ we say that $G$ is almost coherent.

We shall say that a group $G$ is split over a subgroup $C$ if it is either a generalized free product with amalgamation (GFPA) $G = A \ast_C B$, where $C < A$ and $C < B$, or an HNN extension $G = HNN(A; \alpha, \gamma : C \to A)$, where $\alpha$ and $\gamma$ are monomorphisms. (We may also write $G = A \ast_C \varphi$, where $\varphi = \gamma \circ \alpha^{-1}$.) An HNN extension is ascending if one of the associated subgroups is the base. In that case we may assume that $\alpha = id_A$, and $\varphi = \gamma$ is an injective endomorphism of $A$.

The virtual first Betti number $\nu \beta(G)$ of a finitely generated group is the least upper bound of the first Betti numbers $\beta_1(N)$ of normal subgroups $N$ of finite index in $G$. Thus $\nu \beta(G) > 0$ if some subgroup of finite index maps onto $\mathbb{Z}$.

A group $G$ is large if it has a subgroup of finite index which maps onto a non-abelian free group. It is clear that if $G$ is large then $\nu \beta(G) = \infty$.

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2. COMPARISON OF SPLITTINGS

Let $G$ be a group which is a GFPA $A \ast_C B$ or an HNN extension $A \ast_C \varphi$. If we identify the groups $A$ and $B$ with subgroups of $G$ then inclusion defines a partial order on such splittings: $A \ast_C B \leq A' \ast_C B'$ if $A \leq A'$, $B \leq B'$ and $C \leq C'$, and $A \ast_C \varphi \leq A' \ast_C \varphi'$ if $A \leq A'$, $C \leq C'$ and $\varphi'|_C = \varphi$, and the stable letters coincide. (In the HNN case we are really comparing splittings compatible with a given epimorphism $G \to Z \cong G/\langle \langle A \rangle \rangle$.)

**Lemma 1.** Let $G = A' \ast_C \varphi$ be an HNN extension, with stable letter $t$, and let $A \leq A'$ be a subgroup such that $C \cup \varphi(C) \leq A$. If $G = \langle A, t \rangle$ then $A = A'$.

**Proof.** Let $\alpha \in A'$. Then we may write $\alpha = a_0 t^{\varepsilon_1} a_1 \ldots t^{\varepsilon_n} a_n$ where $a_i \in A$ and $\varepsilon_i = \pm 1$, for all $i$, since $G = \langle A, t \rangle$. We may clearly assume that $n$ is minimal. Hence there are no substrings of the form $tct^{-1}$ or $t^{-1} \varphi(c)t$, with $c \in C$, in this expression for $\alpha$ (since any such may be replaced by $\varphi(c)$ or $c$, respectively). But it then follows from Britton’s Lemma for the HNN extension $A' \ast_C \varphi$ that $n = 0$, and so $\alpha = a_0$ is in $A$. $\square$

If $G$ is a $PD_3$-group then we would like to know when $C$ can be chosen to be a $PD_2$-group.

**Lemma 2.** Let $G$ be a $PD_3$-group which is a generalized free product with amalgamation $A \ast_C B$ or an HNN extension $A \ast_C \varphi$, with $C$ a $PD_2$-group. Then the splitting is minimal in the partial order determined by inclusions.

**Proof.** Suppose that $A' \ast_C B' \leq A \ast_C B$ or $A' \ast_C \varphi' \leq A \ast_C \varphi$ (respectively), is another splitting for $G$. Then $C'$ is either a free group or has finite index in $C$. The inclusions induce a commuting diagram relating the Mayer-Vietoris sequences associated to the splittings. In each case, the left hand end of the diagram is

$$
\begin{array}{ccc}
0 \to H_3(G; \mathbb{Z}) & \overset{\delta'}{\longrightarrow} & H_2(C'; \mathbb{Z}) \\
\downarrow & & \downarrow \\
0 \to H_3(G; \mathbb{Z}) & \overset{\delta}{\longrightarrow} & H_2(C; \mathbb{Z}).
\end{array}
$$

Since the connecting homomorphisms $\delta'$ is injective, $H_2(C'; \mathbb{Z}) \neq 0$, and so $C'$ cannot be a free group. Hence it is a $PD_2$-group, and so $\delta$ and $\delta'$ are isomorphisms [3]. Since the inclusion of $C'$ into $C$ has degree $1$, we see that $C' = C$. If $G = A' \ast_C \varphi$ it then follows from Lemma 1 that $A' = A$. If $G = A \ast_C B$ and $G = A' \ast_C B'$ then a similar argument based on normal forms shows that $A' = A$ and $B' = B$. $\square$

If $f : G \to Z$ is an epimorphism then $G \cong A \ast_C \varphi$ with $\text{Ker}(f) = \langle \langle A \rangle \rangle$ and stable letter represented by $t \in G$ with $f(t) = 1$. For instance, we may take $C = A = \text{Ker}(f)$ and $\varphi$ to be conjugation by $t$. If $\text{Ker}(f)$ is finitely generated, this is the only possibility (up to the choice of $t$ with $f(t) = \pm 1$), but in general there are other ways to do this. If $G$ is $FP_2$ then we may choose $A$ and $C$ finitely generated [4], and if $G$ is almost coherent then $A$ and $C$ are also $FP_2$. The construction of [4] gives a pair $(A, C)$ with $A$ generated by $C \cup \varphi(C)$, which is usually far from minimal in this partial order. (See below for an example.) If $G$ is $FP$ then $A$ is $FP_k$ if and only if $C$ is $FP_k$, for any $k \geq 1$ [2, Proposition 2.13].

If $G$ is $FP_2$ and $\text{Ker}(f)$ is not finitely generated then any HNN structure for $G$ with finitely generated base and associated subgroups is the initial term of an
has a doubly infinite chain of HNN structures, obtained by applying the construction of [4]. If \( G = A * A \) \( \phi \) is a properly ascending HNN extension, so that \( \phi(A) < A \), then \( G \) has a doubly infinite chain of HNN structures, with bases the subgroups \( n \in \mathbb{Z} \). However \( PD_3 \)-groups are never properly ascending HNN extensions. (See Theorem 5 below.) Does every descending chain of HNN structures for a \( PD_3 \)-group terminate? Do any \( PD_3 \)-groups which are HNN extensions have minimal splittings over \( FP_2 \)-groups which are not \( PD_2 \)-groups?

Let \( T_2 \) be the orientable surface of genus 2. The \( PD_2 \)-group \( H = \pi_1(T_2) \) has a standard presentation

\[
\langle a, b, c, d \mid [a, b][c, d] = 1 \rangle.
\]

We may rewrite this presentation as

\[
\langle a, b, c, t \mid tct^{-1} = aba^{-1}b^{-1}c \rangle,
\]

which displays \( H \) as an HNN extension \( F(a, b, c) *_{\langle c \rangle} \phi \), split over the \( PD_1 \)-group \( \langle c \rangle \cong \mathbb{Z} \). The associated epimorphism \( f : H \to \mathbb{Z} \) is determined by \( f(a) = f(b) = f(c) = 0 \) and \( f(d) = 1 \). In this case the algorithm from [4] would suggest taking \( C = \langle a, b, c \rangle \) and \( A = \langle a, b, c, t \rangle \), giving an HNN extension with base \( A \cong F(5) \) and split over \( C \cong F(3) \). Taking products, we see then that the \( PD_3 \)-group \( G = \pi_1(T_2 \times S^1) = H \times \mathbb{Z} \) splits over the \( PD_2 \)-group \( \mathbb{Z}^2 \), and is also an HNN extension with base \( F(5) \times \mathbb{Z} \) and associated subgroups \( F(3) \times \mathbb{Z} \). The latter groups have one end, but are not \( PD_2 \)-groups.

Splittings over \( PD_2 \)-groups need not be unique. Let \( W \) be an aspherical orientable 3-manifold with incompressible boundary and two boundary components \( U, V \). Let \( M = DW \) be the double of \( W \) along its boundary. Then \( M \) splits over copies of \( U \) and \( V \), and \([U] = [V] \) in \( H_2(M; \mathbb{Z}) \). If \( U \) and \( V \) are not homeomorphic the corresponding (minimal) splittings of \( G = \pi_1(M) \) are evidently distinct. For instance, we may start with the hyperbolic 3-manifold of [11, Example 3.3.12], which is the exterior of a knotted \( \theta \)-curve \( \Theta \subset S^3 \). Let \( W \) be obtained by deleting an open regular neighbourhood of a meridian of one of the arcs of \( \Theta \). Then \( W \) is aspherical, \( \partial W = T \cup T_2 \) and each component of \( \partial W \) is incompressible in \( W \).

3. INDECOMPOSABLE FACTORS

If \( G \) is a \( PD_3 \)-group then \( c.d.A = c.d.C = 2 \), since these subgroups have infinite index in \( G \), and \( H_2(C; \mathbb{Z}) \neq 0 \), as observed in Lemma 2. A simple Mayer-Vietoris argument shows that \( H^1(A; \mathbb{Z}[G]) \cong H^1(C; \mathbb{Z}[G]) \) as right \( \mathbb{Z}[G] \)-modules, since \( H^i(G; \mathbb{Z}[G]) = 0 \) for \( i \geq 2 \). (Note that the latter condition fails for \( PD_2 \)-groups.) The isomorphism is given by the difference \( \alpha \phi - \gamma \phi \) of the homomorphisms induced by \( \alpha \) and \( \gamma \).

We shall assume henceforth that \( A \) and \( C \) are finitely generated. Then these modules may be obtained by extension of coefficients from the “end modules” \( H^1(A; \mathbb{Z}[A]) \) and \( H^1(C; \mathbb{Z}[C]) \). If one is 0 so is the other, and so \( A \) has one end if and only if \( C \) has one end. If \( A \) and \( C \) are \( FP_2 \) and have one end then they are 2-dimensional duality groups, and we may hope to apply the ideas of [9].

Can \( G \) have splittings with base and associated subgroups having more than one end? The next lemma implies that the subgroups \( A \) and \( C \) must have the same numbers of indecomposable factors. (The analogous statement for \( PD_2 \)-groups is false, as may be seen from the example in §2 above!)


Lemma 3. Let $K = (*_{i=1}^m K_i) * F(n)$ be the free product of $m \geq 1$ finitely generated groups $K_i$ with one end and $n \geq 0$ copies of $\mathbb{Z}$. Then $H^1(K; \mathbb{Z}[K]) \cong \mathbb{Z}[K]^{m-1}$, where $r = m + n$ is the number of indecomposable factors of $K$.

Proof. If $n = 0$ the result follows from the Mayer-Vietoris sequence for the free product, with coefficients $\mathbb{Z}[K]$.

In general, let $J = (*_{i=1}^m K_i)$ and let $C_*(J)$ be a resolution of the augmentation module $\mathbb{Z}$ by free $\mathbb{Z}[J]$-modules, with $C_0(J) = \mathbb{Z}[J]$. Then there is a corresponding resolution $C_*(K)$ of $\mathbb{Z}$ with $C_q(K) \cong \mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_q(J)$ if $q \neq 1$ and $C_1(K) \cong \mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_q(J) \oplus \mathbb{Z}[K]$. Hence there is a short exact sequence of chain complexes (of left $\mathbb{Z}[K]$-modules)

$$0 \to \mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_*(J) \to C_*(K) \to \mathbb{Z}[K]^n \to 0,$$

where the third term is concentrated in degree 1. The exact sequence of cohomology with coefficients $\mathbb{Z}[K]$ gives a short exact sequence of right $\mathbb{Z}[K]$-modules

$$0 \to \mathbb{Z}[K]^n \to H^1(K; \mathbb{Z}[K]) \to H^1(\text{Hom}_{\mathbb{Z}[K]}(\mathbb{Z}[K] \otimes_{\mathbb{Z}[J]} C_*(J), \mathbb{Z}[K])) \to 0.$$

We may identify the right-hand term with $H^1(J; \mathbb{Z}[J]) \otimes_{\mathbb{Z}[J]} \mathbb{Z}[K] \cong \mathbb{Z}[K]^{m-1}$, since $J$ is finitely generated. The lemma follows easily. $\square$

The lemma applies to $A$ and $C$, since they are finitely generated and torsion-free. The indecomposable factors of $C$ are either conjugate to subgroups of indecomposable factors of $A$ or are infinite cyclic, by the Kurosh subgroup theorem. If $A$ and $C$ have no free factors and the factors of $C$ are conjugate into distinct factors of $A$ then, after modifying $\varphi$ appropriately, we may assume that $\alpha(C_i) \leq A_i$, for all $i$. However, we cannot expect to also normalize $\gamma$ in a similar fashion.

4. THE DUAL CLASS

If $M$ is a closed 3-manifold with $\beta_1(M) > 0$ then there is an essential map $f : M \to S^3$. Transversality and the Loop Theorem together imply that there is a closed incompressible surface $S \subset M$ such that $M \setminus S$ is connected. Hence $\pi_1(M)$ is an HNN extension with base $\pi_1(M \setminus S)$ and associated subgroups copies of $\pi_1(S)$. Moreover, the stable letter of the extension is represented by a simple closed curve in $M$ which intersects $S$ transversely in one point. Let $w = w_1(M)$. Then $w_1(S) = w|_S$ and the image of the fundamental class $[S]$ in $H_2(M; \mathbb{Z}^w)$ is Poincaré dual to the image of $f$ in $H^1(M; \mathbb{Z}) = [M, S^1]$.

There is no obvious analogue of transversality in group theory. Nevertheless a similar result holds for $PD_3$-groups. (We consider only the orientable case, for simplicity.)

Theorem 4. Let $G = \text{HNN}(A; \alpha, \gamma : C \to A)$ be an orientable $PD_3$-group which is an HNN extension split over a $PD_2$-group $C$. Let $f \in H^1(G; \mathbb{Z})$ be the epimorphism with kernel $\langle \langle A \rangle \rangle$. Then $f \sim [G]$ is the image of $[C]$ in $H_2(G; \mathbb{Z})$, up to sign.

Proof. The subgroup $C$ is orientable and the pair $(A; \alpha, \gamma)$ is a $PD_3^e$-pair [3, Theorem 8.1], and so there is an exact sequence

$$H_3(A; \partial; \mathbb{Z}) \xrightarrow{(1, 1)} H_2(C; \mathbb{Z}) \oplus H_2(C; \mathbb{Z}) \xrightarrow{\alpha \gamma_2 \cdots \gamma_1} H_2(A; \partial; \mathbb{Z}) \to H_2(A; \partial; \mathbb{Z}).$$

Hence $\alpha \gamma[C] = \gamma_1[C]$, and the subgroup they generate is an infinite cyclic direct summand of $H_2(A; \mathbb{Z})$, since $H_2(A; \partial; \mathbb{Z}) \cong H^1(A; \mathbb{Z})$ is free abelian.
Let \( t \in G \) correspond to the stable letter for the HNN extension, and let \( A_j = t^j At^{-j} \), \( \alpha_j(c) = t^j \alpha(c)t^{-j} \) and \( \gamma_j(c) = t^j \gamma(c)t^{-j} \), for all \( c \in C \) and \( j \in \mathbb{Z} \). Let \( K_p \) be the subgroup generated by \( \bigcup_{j \in \mathbb{Z}} A_j \), for \( p \geq 0 \). Then \( K_0 = A \) and

\[
K_{p+1} = A_{-p-1} \ast_{\alpha_{-p-1}=\gamma_{-p-1}} K_p \ast_{\alpha_{p+1}=\gamma_{p+1}} A_{p+1}, \quad \text{for all } p \geq 0,
\]
and \( K = \langle \langle A \rangle \rangle_G = \text{Ker}(f) \) is the increasing union \( K = \bigcup K_p \) of iterated amalgamations with copies of \( A \) over copies of \( C \). Each pair \( (K_p; \alpha_{-p}, \gamma_{-p}) \) is again a \( PD_3^+ \)-pair, and so the images of \( [C] \) in \( H_2(K; \mathbb{Z}) \) under the homomorphisms induced by the \( \alpha_n \)'s all agree.

Let \( \Lambda = \mathbb{Z}[G/K] = \mathbb{Z}[t, t^{-1}] \), and let \( \varepsilon : \Lambda \to \mathbb{Z} \) be the augmentation. Then \( H_i(K; \mathbb{Z}) = H_2(G; \Lambda) \) is a finitely generated \( \Lambda \)-module, with action deriving from the action of \( G \) on \( K \) by conjugation. Then \( H_2(G; \Lambda) = H_2(K; \mathbb{Z}) \). Since \( t.\alpha_n[C] = \alpha_{n+1}[C] = \alpha_n[C] \), for all \( n \), the image of \( [C] \) in \( H_2(K; \mathbb{Z}) \) generates an infinite cyclic direct summand.

Poincaré duality gives an isomorphism \( H_2(G; \Lambda) \cong \overline{H^1(G; \Lambda)} \), and this is in turn an extension of \( \overline{\text{Hom}}(K/K', \Lambda) \) by \( \text{Ext}^1_\Lambda(\mathbb{Z}, \Lambda) \), by the Universal Coefficient spectral sequence. Note that \( \overline{\text{Hom}}(K/K', \Lambda) \) has no non-trivial \( \Lambda \)-torsion, while \( \text{Ext}^1_\Lambda(\mathbb{Z}, \Lambda) \) is \( \Lambda/(t-1)\Lambda = \mathbb{Z} \).

We have a commutative diagram

\[
\begin{array}{ccc}
H^1(\mathbb{Z}; \Lambda) & \xrightarrow{H^1(f)} & H^1(G; \Lambda) \\
\downarrow \varepsilon \# & & \downarrow \varepsilon \# \\
H^1(\mathbb{Z}; \mathbb{Z}) & \xrightarrow{id_\mathbb{Z} \cdot f} & H^1(G; \mathbb{Z}) \\
\end{array}
\]

\[
\sim_{[G]} \quad \sim_{[G]} \quad \sim_{[G]}
\]

in which the vertical homomorphisms are induced by the change of coefficients \( \varepsilon \) and the two right hand horizontal homomorphisms are Poincaré duality isomorphisms. Since \( H^1(f) \) carries \( H^1(\mathbb{Z}; \Lambda) = \text{Ext}^1_\Lambda(\mathbb{Z}, \Lambda) \cong \mathbb{Z} \) onto the \( \Lambda \)-torsion submodule of \( H^1(G; \Lambda) \), a diagram chase shows that \( f \sim [G] \) is the image of \( [C] \) in \( H_2(G; \mathbb{Z}) \), up to sign.

In [10] it is shown that if a \( PD_3 \)-group \( G \) has a subgroup \( S \) which is a \( PD_2 \)-group then \( G \) splits over a subgroup commensurate with \( S \) if and only if an invariant \( \text{sing}(S) \in \mathbb{Z}/2\mathbb{Z} \) is 0, and then \( S \) is maximal among compatibly oriented commensurate subgroups. Theorem 4 suggests a slight refinement of this splitting criterion.

**Theorem** (Kropholler-Roller [10]). *Let \( G \) be an orientable \( PD_3 \)-group and \( S \subset G \) a subgroup which is an orientable \( PD_2 \)-group. Then*

1. \( G \cong A \ast_T B \) for some \( T \) commensurate with \( S \) if and only if \( \text{sing}(S) = 0 \) and \( [S] = 0 \) in \( H_2(G; \mathbb{Z}) \);
2. \( G \cong A \ast_T \varphi \) for some \( T \) commensurate with \( S \) if and only if \( \text{sing}(S) = 0 \) and \( [S] \) has infinite order in \( H_2(G; \mathbb{Z}) \);
3. \( G \cong A \ast_S \varphi \) if and only if \( \text{sing}(S) = 0 \) and \( [S] \) generates an infinite direct summand of \( H_2(G; \mathbb{Z}) \).

**Proof.** The group \( G \) splits over a subgroup \( T \) commensurate with \( S \) if and only if \( \text{sing}(S) = 0 \) [10], and \( [S] \) and \([T]\) are then proportional. If \( G = A \ast_T B \) is a generalized free product with amalgamation over a \( PD_2 \)-group \( T \) then the pairs \((A, T)\) and \((B, T)\) are again \( PD_3^+ \) pairs [3]. The image of \([T]\) in \( H_2(G; \mathbb{Z}) \) is trivial, since \( T \) bounds each of \((A, T)\) and \((B, T)\), and so \([S] = 0 \) also.
If $G \cong A *_{T} \varphi$ is an HNN extension then the Poincaré dual of $[T]$ is an epimorphism $f : G \to \mathbb{Z}$, by the theorem, and so $[T]$ generates an infinite cyclic direct summand of $H_2(G; \mathbb{Z})$. Hence $[S]$ also has infinite order. □

If $[C] = [S]$ and $\text{sing}(S) = 0$ is $\text{sing}(C) = 0$ also?

5. NO PROPERLY ASCENDING HNN EXTENSIONS

Cohomological arguments imply that no $PD_3$-group is a properly ascending HNN extension [7, Theorem 3]. A stronger result holds for 3-manifold groups: no finitely generated subgroup can be conjugate to a proper subgroup of itself [6]. We shall adapt the argument of [7] to prove the corresponding result for $FP_2$ subgroups of $PD_3$-groups.

Theorem 5. Let $H$ be an $FP_2$ subgroup of a $PD_3$-group $G$. If $gHg^{-1} \leq H$ for some $g \in G$ then $gHg^{-1} = H$.

Proof. Suppose that $gHg^{-1} < H$. Then $g \not\in H$. Let $\theta(h) = ghg^{-1}$, for all $h \in H$, and let $K = H \ast_{T} \theta$ be the associated HNN extension, with stable letter $t$. The normal closure of $H$ in $K$ is the union $\bigcup_{x \in \mathbb{Z}} t^{x}Ht^{-x}$, and so every element of $K$ has a normal form $k = t^{m}ht^{-n}$, where $m$ is uniquely determined by $k$, and $h$ is determined by $k, m$ and $r$. Let $f : K \to G$ be the homomorphism defined by $f(h) = h$ for all $h \in H$ and $f(t) = g$. If $f(t^{m}ht^{-n}) = f(t^{n}ht^{-s})$ for some $m, n, r, s$ then $g^{n-m} = g^{r}h^{s}g^{-1}h^{-1}g^{-t}$. After conjugating by a power of $g$ if necessary, we may assume that $s, t \geq 0$, and so $g^{n-m} \in H$. But then $H = g^{n-m}Hg^{-[n-m]}$. Since $gHg^{-1}$ is a proper subgroup of $H$, we must have $n = m$. It follows easily that $f$ is an isomorphism from $K$ to the subgroup of $G$ generated by $g$ and $H$.

Since $K$ is an ascending HNN extension with $FP_2$-base, $H^1(K; \mathbb{Z}[K])$ is a quotient of $H^0(H; \mathbb{Z}[K]) = 0$ [5, Theorem 0.1]. Hence it has one end. Since no $PD_3$-group is an ascending HNN extension [7, Theorem 3], $K$ is a 2-dimensional duality group. Hence it is the ambient group of a $PD_3$-pair $(K, S)$ [9]. Doubling this pair along its boundary gives a $PD_3$-group. But this is again a properly ascending HNN extension, and so cannot happen. Therefore the original supposition was false, and so $gHg^{-1} = H$. □

6. RESIDUAL FINITENESS, SPLITTING AND LARGENESS

The fundamental group of an aspherical closed 3-manifold is either solvable or large [1, Flowcharts 1 and 4]. This is also so for residually finite $PD_3$-groups containing $\mathbb{Z}^2$ [8, Theorem 11.19]. Here we shall give a weaker result for $PD_3$-groups which split over other $PD_2$-groups.

Theorem 6. Let $G$ be a residually finite orientable $PD_3$-group which splits over an orientable $PD_2$-group $C$. Then either $\beta_1(G) > 0$, or $G$ maps onto $D_\infty$, or $G$ is large. Hence $v_3(G) > 0$. If $G$ is LERF and $\chi(C) < 0$ then $G$ is large.

Proof. For the first assertion, we may assume that $\beta_1(G) = 0$, and that $G \cong A *_C B$. Then $(A, C)$ and $(B, C)$ are $PD_3$-pairs, and so $\beta_1(C) \leq 2\beta_1(A)$ and $\beta_1(C) \leq 2\beta_1(B)$. Since $\beta_1(C) > 0$, we must have $\beta_1(A) > 0$ and $\beta_1(B) > 0$ also. Moreover $\beta_1(C) = \beta_1(A) + \beta_1(B)$, since $H_1(G)$ is finite and $H_2(G) = 0$. Hence $\beta_1(C) > \beta_1(A)$ and $\beta_1(C) > \beta_1(B)$.

Let $\{\Delta_n | n \geq 1\}$ be a descending filtration of $G$ by normal subgroups of finite index. Then $A_n = A/A \cap \Delta_n$, $B_n = B/B \cap \Delta_n$ and $C_n = C/C \cap \Delta_n$ are finite, and
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$G$ maps onto $A_n \ast_{C_n} B_n$, for all $n$. If $A_n \ast_{C_n} B_n$ is finite then $C_n = A_n$ or $B_n$. Thus if all these quotients of $G$ are finite we may assume that $C_n = A_n$ for all $n$. But then the inclusion of $C$ into $A$ induces an isomorphism on profinite completions, and so $\beta_1(C) = \beta_1(A)$, contrary to what was shown in the paragraph above.

If $C_n$ is a proper subgroup of both $A_n$ and $B_n$ then either $[A_n : C_n] = [B_n : C_n] = 2$, in which case $G$ maps onto $D_\infty$, or one of these indices is greater than 2, in which case $A_n \ast_{C_n} B_n$ is virtually free of rank $> 1$, and so $G$ is large. In each case, it is clear that $v\beta(G) \geq 1$.

Suppose now that $G$ is LERF. If $[A_n : C_n] \leq 2$ then $C_n$ is normal in $A_n$, and so $C(A \cap \Delta_n)$ is normal in $A$. Hence if $[A_n : C_n] \leq 2$ for all $n$ then $\cap_n C(A \cap \Delta_n)$ is normal in $A$. Since $G$ is LERF, this intersection is $C$. Hence if both $[A_n : C_n] \leq 2$ and $[B_n : C_n] \leq 2$ for all $n$ then $C$ is normal in $G$, so $G$ is virtually a semidirect product $C \rtimes \mathbb{Z}$, and is a 3-manifold group. If $\chi(C) < 0$ then $G$ is large [1, Flowcharts 1 and 4].

Remark. The lower central series of $D_\infty = \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ gives a descending filtration by normal subgroups of finite index which meets each of the free factors trivially.

Is every $PD_3$-group either solvable or large?

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