SEMICLASSICAL STATES FOR A STATIC SUPERCRITICAL
KLEIN-GORDON-MAXWELL-PROCA SYSTEM ON A CLOSED
RIEMANNIAN MANIFOLD

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Abstract. We establish the existence of semiclassical states for a nonlinear
Klein-Gordon-Maxwell-Proca system in static form, with Proca mass 1, on a
closed Riemannian manifold.
Our results include manifolds of arbitrary dimension and allow supercrit-
cical nonlinearities. In particular, we exhibit a large class of 3-dimensional
manifolds on which the system has semiclassical solutions for every exponent
p ∈ (2, ∞). The solutions we obtain concentrate at closed submanifolds of
positive dimension as the singular perturbation parameter goes to zero.

1. Introduction
Let (M, g) be a closed (i.e. compact and without boundary) smooth Riemannian
manifold of dimension m ≥ 2. Given real numbers ε > 0, q > 0, ω ∈ R and
p ∈ (2, ∞), and a real-valued C1-function α such that α(x) > ω2 on M, we consider
the system
\[
\begin{aligned}
-\varepsilon^2 \Delta_g u + \alpha(x) u &= u^{p-1} + \omega^2 (q v - 1)^2 u & \text{on } M, \\
-\Delta_g v + (1 + q^2 u^2) v &= q u^2 & \text{on } M, \\
u, v &\in H^1_g(M), \quad u, v > 0.
\end{aligned}
\]
(1.1)
The space $H^1_g(M)$ is the completion of $C^\infty(M)$ with respect to the norm defined by
\[ \|v\|_g^2 := \int_M (|\nabla_g v|^2 + v^2) \, d\mu_g. \]
Solutions to this system correspond to standing waves of a Klein-Gordon-Maxwell-
Proca (KGMP) system in static form (i.e. one in which the external Proca field is
time-independent) with Proca mass 1.

KGMP-systems are massive versions of the more classical electrostatic Klein-
Gordon-Maxwell (KGM) systems: KGM-systems are KGMP-systems with Proca
mass 0, i.e. the second equation in (1.1) is replaced by
\[ -\Delta_g v + q^2 u^2 v = q u^2. \]
Note that $v = 1/q$ solves this last equation and reduces the KGM-system to a
single Schrödinger equation in u. So for the system on a closed manifold the Proca

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formalism is more interesting and more appropriate. We refer to [11] for a detailed discussion on KGMP-systems and their physical meaning.

For $\varepsilon = 1$ existence of solutions to system (1.1), which are stable with respect to the phase $\omega$, was established by Druet and Hebey [7] and Hebey and Truong [10] for manifolds of dimension $m = 3$ and 4, and subcritical ($2 < p < \frac{2m}{m-2}$) or critical ($p = \frac{2m}{m-2}$) nonlinearities, under certain assumptions. For critical systems in dimension 3 Hebey and Wei [11] showed the existence of standing waves with multispike amplitudes, which are unstable with respect to the phase, i.e. they blow up with $k$ singularities as the phase $\omega$ approaches some phase $\omega_0$.

Here we are interested in semiclassical states, i.e. in solutions to system (1.1) for $\varepsilon$ small. The existence of semiclassical states for similar systems in flat domains $\Omega$ in $\mathbb{R}^m$ has been investigated e.g. in [4, 5, 15]. On closed 3-dimensional manifolds, the existence of semiclassical states to system (1.1), which concentrate at a single point as $\varepsilon \to 0$, was established in [8] and [9] for subcritical exponents $p \in (2, 6)$.

The results we present in this paper apply to manifolds of arbitrary dimension and include supercritical nonlinearities $p > 2^*_m$, where $2^*_m := \frac{2m}{m-2}$ is the critical Sobolev exponent in dimension $m \geq 3$ and $2^*_2 := \infty$. In particular, we shall exhibit a large class of 3-dimensional manifolds on which the system (1.1) has semiclassical solutions for every exponent $p \in (2, \infty)$. The solutions $u$ we obtain concentrate at closed submanifolds of $\mathfrak{M}$ of positive dimension. Moreover, for fixed $\varepsilon$, they are stable with respect to the phase in the sense of [7].

Our approach consists in reducing system (1.1) to a system of a similar type on a manifold $M$ of lower dimension but with the same exponent $p$. This way, if $n := \dim M < \dim \mathfrak{M} := m$ and $p \in [2^*_m, 2^*_n)$, then $p$ is subcritical for the new system but it is critical or supercritical for the original one. Moreover, solutions of the new system which concentrate at a point in $M$ as $\varepsilon \to 0$ will give rise to solutions of the original system concentrating at a closed submanifold of $\mathfrak{M}$ of dimension $m - n$ as $\varepsilon \to 0$.

This approach was introduced by Ruf and Srikanth in [13], where a Hopf map is used to obtain the reduction. Reductions may also be performed by means of other maps which preserve the Laplace-Beltrami operator, or by considering warped products, or by a combination of both, see [3, 14] and the references therein. We describe these reductions in the following two subsections.

1.1. Warped products. If $(M, g)$ and $(N, h)$ are closed smooth Riemannian manifolds of dimensions $n$ and $k$ respectively, and $f : M \to (0, \infty)$ is a $C^1$-map, the warped product $M \times_f N$ is the cartesian product $M \times N$ equipped with the Riemannian metric $g := g + f^2 h$.

For example, if $M$ is a closed Riemannian submanifold of $\mathbb{R}^\ell \times (0, \infty)$, then

$$\mathfrak{M} := \{(y, z) \in \mathbb{R}^\ell \times \mathbb{R}^{k+1} : (y, |z|) \in M\},$$

with the induced euclidian metric, is isometric to the warped product $M \times_f S^k$, where $S^k$ is the standard $k$-sphere and $f(x_1, \ldots, x_{\ell+1}) = x_{\ell+1}$.

Let $\pi_M : M \times_f N \to M$ be the projection. A straightforward computation gives the following result, cf. [9].

**Proposition 1.1.** Let $\beta : M \to \mathbb{R}$ and $\alpha = \beta \circ \pi_M$. Then $u_\varepsilon, v_\varepsilon : M \to \mathbb{R}$ solve

\begin{align*}
\begin{cases}
-\varepsilon^2 \text{div}_g \left(f^k(x) \nabla_g u\right) + f^k(x) \beta(x) u = f^k(x) u^{p-1} + \omega^2 f^k(x)(qv - 1)^2 u & \text{on } M, \\
-\text{div}_g \left(f^k(x) \nabla_g v\right) + f^k(x) \left(1 + qv^2\right) v = q f^k(x) u^2 & \text{on } M,
\end{cases}
\end{align*}
iff $u_c := u_c \circ \pi_M, v_c := v_c \circ \pi_M : M \times_f^2 N \to \mathbb{R}$ solve
\begin{align*}
(1.3) \quad \begin{cases}
-\varepsilon^2 \Delta_g u + \alpha(x) u = u^{p-1} + \omega^2 (qv - 1)^2 u & \text{on } M \times_f^2 N, \\
-\Delta_g v + (1 + qu^2) v = qu^2 & \text{on } M \times_f^2 N.
\end{cases}
\end{align*}

Note that the exponent $p$ is the same for both systems. So if $p \in (2^*, 2^*_n)$ then $p$ is subcritical for (1.2) but supercritical for (1.3). Moreover, if the functions $u_c$ concentrate at a point $\xi_0 \in M$ as $\varepsilon \to 0$, then the functions $u_c := u_c \circ \pi_M$ concentrate at the submanifold $S^{-1}_M(\xi_0) \equiv (N, f^2(\xi_0) h)$ as $\varepsilon \to 0$.

1.2. Harmonic morphisms. Let $(\mathcal{M}, g)$ and $(M, g)$ be closed Riemannian manifolds of dimensions $m$ and $n$ respectively. A harmonic morphism is a horizontally conformal submersion $\pi : \mathcal{M} \to M$ with dilation $\lambda : \mathcal{M} \to [0, \infty)$ which satisfies
\begin{align*}
(1.4) \quad (n - 2) \mathcal{H}(\nabla_{\mathcal{M}} \ln \lambda) + (m - n) \kappa^V = 0,
\end{align*}
where $\kappa^V$ is the mean curvature of the fibers of $\pi$ and $\mathcal{H}$ is the projection of the tangent space of $\mathcal{M}$ onto the space orthogonal to the fibers, see [1].

So for $n = 2$ a harmonic morphism is just a horizontally conformal submersion $\pi : \mathcal{M} \to M$ with minimal fibers. Typical examples are the Hopf fibration $S^1 \to S^2$ whose fiber is $S^1$, and the induced fibration $\mathbb{R}P^1 \to S^2$ with fiber $\mathbb{R}P^1$, see [1] Example 2.4.15). They are, in fact, Riemannian submersions (i.e. $\lambda \equiv 1$).

Harmonic morphisms preserve the Laplace-Beltrami operator, i.e.
\begin{align*}
\Delta_g (u \circ \pi) = \lambda^2 (\Delta_g u) \circ \pi
\end{align*}
for every $C^2$-function $u : M \to \mathbb{R}$. This fact yields the following result.

**Proposition 1.2.** Assume there exist $\beta : M \to \mathbb{R}$ and $\mu : M \to (0, \infty)$ such that $\beta \circ \pi = \alpha$ and $\mu \circ \pi = \lambda^2$. Then $u_c, v_c : M \to \mathbb{R}$ solve the system
\begin{align*}
(1.5) \quad \begin{cases}
-\varepsilon^2 \Delta_g u + \frac{\beta(x)}{\mu(x)} u = \frac{1}{\mu(x)} u^{p-1} + \frac{\omega^2}{\mu(x)} (qv - 1)^2 u & \text{on } M, \\
-\Delta_g v + \frac{1}{\mu(x)} (1 + qu^2) v = \frac{\omega^2}{\mu(x)} u^2 & \text{on } M,
\end{cases}
\end{align*}
iff $u_c := u_c \circ \pi_M, v_c := v_c \circ \pi_M : \mathcal{M} \to \mathbb{R}$ solve the system
\begin{align*}
(1.6) \quad \begin{cases}
-\varepsilon^2 \Delta_g u + \alpha(x) u = u^{p-1} + \omega^2 (qv - 1)^2 u & \text{on } \mathcal{M}, \\
-\Delta_g v + (1 + qu^2) v = qu^2 & \text{on } \mathcal{M}.
\end{cases}
\end{align*}

Again, if $p \in (2^*_n, 2^*_m)$, the system (1.5) is subcritical and the system (1.6) is supercritical and, if the functions $u_c$ concentrate at a point $\xi_0 \in M$ as $\varepsilon \to 0$, the functions $u_c := u_c \circ \pi_M$ concentrate at the $(m - n)$-dimensional submanifold $S^{-1}_M(\xi_0)$ of $\mathcal{M}$ as $\varepsilon \to 0$.

1.3. The main result for the general system. Propositions 1.1 and 1.2 suggest studying a more general KGMP-system.

Let $(M, g)$ be a closed Riemannian manifold of dimension $n = 2$ or $3$, $a, b, c \in C^1(M, \mathbb{R})$ be strictly positive functions, $\varepsilon, q \in (0, \infty)$, $p \in (2, 2^*_n)$, and $\omega \in \mathbb{R}$ be such that $a(x) > \omega^2 b(x)$ on $M$. We consider the subcritical system
\begin{align*}
(1.7) \quad \begin{cases}
-\varepsilon^2 \text{div}_g (c(x) \nabla_g u) + a(x) u = b(x) u^{p-1} + b(x) \omega^2 (qv - 1)^2 u & \text{in } M, \\
-\text{div}_g (c(x) \nabla_g v) + b(x)(1 + qu^2) v = b(x)qu^2 & \text{in } M, \\
u, v \in H^1_0(M), & u, v > 0.
\end{cases}
\end{align*}
Theorem 1.3. Let \( K \) be a \( C^1 \)-stable critical set of the function \( \Gamma : M \to \mathbb{R} \) given by
\[
\Gamma(x) := \frac{c(x)^2 \alpha(x) \frac{c}{p} - \frac{n}{q}}{b(x) \frac{c}{q}}.
\]
Then, for \( \varepsilon \) small enough, the system (1.7) has a solution \((u_\varepsilon, v_\varepsilon)\) such that \( u_\varepsilon \) concentrates at a point \( \xi_0 \in K \) as \( \varepsilon \to 0 \).

Recall that \( K \) is a \( C^1 \)-stable critical set of a function \( f \in C^1(M, \mathbb{R}) \) if \( K \subset \{ x \in M : \nabla_g f(x) = 0 \} \) and for any \( \mu > 0 \) there exists \( \delta > 0 \) such that, if \( h \in C^1(M, \mathbb{R}) \) with
\[
\max_{d_g(x,K) \leq \mu} |f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)| \leq \delta,
\]
then \( h \) has a critical point \( x_\mu \) with \( d_g(x_\mu, K) \leq \mu \). Here \( d_g \) denotes the geodesic distance associated to the Riemannian metric \( g \).

1.4. The main results for the KGMP-system. Theorem 1.3 together with Propositions 1.1 and 1.2 yields the following results.

Theorem 1.4. Let \( \mathcal{M} \) be the warped product \( M \times f^2 \mathbb{R} \) of two closed Riemannian manifolds \((M, g)\) and \((N, h)\) with \( n := \dim M = 2 \) or 3. Set \( k := \dim N \), and let \( p \in (2, \infty) \) if \( n = 2 \) and \( p \in (2, 6) \) if \( n = 3 \). Assume there exists \( \beta \in C^1(M, \mathbb{R}) \) such that \( \alpha = \beta \circ \pi_M \) and let \( K \) be a \( C^1 \)-stable critical set for the function \( \Gamma := f^k \beta \frac{c}{p} \cdot \frac{n}{q} - \frac{n}{q} \) on \( M \). Then, for \( \varepsilon \) small enough, the KGMP-system (1.6) has a solution \((u_\varepsilon, v_\varepsilon)\) such that \( u_\varepsilon \) concentrates at the submanifold \( \pi^{-1}_M(\xi_0) \cong (N, f^2(\xi_0)h) \) for some \( \xi_0 \in K \) as \( \varepsilon \to 0 \).

Theorem 1.5. Assume there exist a closed Riemannian manifold \( M \) with \( n := \dim M = 2 \) or 3 and a harmonic morphism \( \pi : \mathcal{M} \to M \) whose dilation \( \lambda \) is such that \( \mu \circ \pi = \lambda^2 \). Assume further that \( \alpha = \beta \circ \pi_M \) with \( \beta \in C^1(M, \mathbb{R}) \). Let \( p \in (2, \infty) \) if \( n = 2 \) and \( p \in (2, 6) \) if \( n = 3 \), and let \( K \) be a \( C^1 \)-stable critical set for the function \( \Gamma := \beta \frac{c}{p} \cdot \frac{n}{q} - \frac{n}{q} \mu^2 - 1 \) on \( M \). Then, for \( \varepsilon \) small enough, the KGMP-system (1.6) has a solution \((u_\varepsilon, v_\varepsilon)\) such that \( u_\varepsilon \) concentrates at the submanifold \( \pi^{-1}(\xi_0) \) of \( \mathcal{M} \) for some \( \xi_0 \in K \) as \( \varepsilon \to 0 \).

This last result applies, in particular, to the standard 3-sphere \( \mathbb{S}^3 \) and the real projective space \( \mathbb{R}P^3 \) for all \( p \in (2, \infty) \) with \( \mu = \lambda = 1 \), see subsection 1.2.

The rest of the paper is devoted to the proof of Theorem 1.3. In section 2 we reduce the system to a single equation and give the outline of the proof of Theorem 1.3 which follows the well-known Lyapunov-Schmidt reduction procedure. In section 3 we establish the Lyapunov-Schmidt reduction and in section 4 we derive the expansion of the reduced energy functional. Section 5 is devoted to the proof of some technical results.

2. OUTLINE OF THE PROOF OF THEOREM 1.3

2.1. Reduction to a single equation. First, we reduce the system to a single equation. To overcome the problems caused by the competition between \( u \) and \( v \), using an idea of Benci and Fortunato [2], we consider the map \( \Psi : H^1_g(M) \to H^1_g(M) \) defined by the equation
\[
(2.1) \quad -\text{div}_g (c(x)\nabla_g \Psi(u)) + b(x)(1 + q^2 u^2)\Psi(u) = b(x)qu^2.
\]
It follows from standard variational arguments that $\Psi$ is well-defined in $H^1_g(M)$.

Using the maximum principle and regularity theory it is not hard to prove that
\begin{equation}
0 < \Psi(u) < 1/q \quad \text{for all } u \in H^1_g(M).
\end{equation}

For the proofs of the following two lemmas we refer to \cite{7}.

**Lemma 2.1.** The map $\Psi : H^1_g(M) \to H^1_g(M)$ is of class $C^1$, and its differential $V_u := \Psi'(u)$ at $u$ is defined by
\begin{equation}
-\text{div}_g(c(x)\nabla_g V_u[h]) + b(x) (1 + q^2 u^2) V_u[h] = 2b(x)qu(1 - q\Psi(u))h
\end{equation}
for every $h \in H^1_g(M)$. Moreover,
\begin{equation*}
0 \leq \Psi'(u)[u] \leq \frac{2}{q} \quad \text{for all } u \in H^1_g(M).
\end{equation*}

**Lemma 2.2.** The map $\Theta : H^1_g(M) \to \mathbb{R}$ given by
\[ \Theta(u) := \frac{1}{2} \int_M b(x)(1 - q\Psi(u))u^2d\mu_g \]
is of class $C^1$ and
\[ \Theta'(u)[h] = \int_M b(x)(1 - q\Psi(u))^2uhd\mu_g \quad \text{for all } u, h \in H^1_g(M). \]

Next, we introduce the functionals $I_\epsilon, J_\epsilon, G_\epsilon : H^1_g(M) \to \mathbb{R}$ given by
\begin{equation}
I_\epsilon(u) := J_\epsilon(u) + \frac{\omega^2}{2}G_\epsilon(u),
\end{equation}
where
\[ J_\epsilon(u) := \frac{1}{2\epsilon^2} \int_M \left[ \epsilon^2 c(x)\nabla_g u^2 + d(x)u^2 \right]d\mu_g - \frac{1}{p\epsilon^2} \int_M b(x)(u^+)^p d\mu_g \]
with $d(x) := a(x) - \omega^2b(x)$, and
\[ G_\epsilon(u) := \frac{q}{\epsilon^2} \int_M b(x)\Psi(u)u^2d\mu_g. \]

From Lemma 2.2 we deduce that
\[ \frac{1}{2}G'_\epsilon(u)[\varphi] = \frac{1}{\epsilon^2} \int_M b(x)[2q\Psi(u) - q^2\Psi^2(u)]u\varphi d\mu_g. \]
Hence,
\[ I'_\epsilon(u)[\varphi] = \frac{1}{\epsilon^2} \int_M \epsilon^2 c(x)\nabla_g u\nabla_g \varphi + a(x)u\varphi - b(x)(u^+)^{p-1}\varphi - b(x)\omega^2(1 - q\Psi(u))^2u\varphi d\mu_g. \]

Therefore, if $u$ is a critical point of the functional $I_\epsilon$, then $u$ solves the problem
\begin{equation}
\begin{cases}
-\epsilon^2\text{div}_g(c(x)\nabla_g u) + (a(x) - \omega^2b(x))u + \omega^2qb(x)\Psi(u)(2 - q\Psi(u))u = b(x)(u^+)^{p-1}, \\
u \in H^1_g(M).
\end{cases}
\end{equation}
If $u \neq 0$ by the maximum principle and regularity theory we have that $u > 0$. Thus the pair $(u, \Psi(u))$ is a solution of the system (1.7). This reduces the existence problem for the system (1.7) to showing that the functional $I_\epsilon$ has a nontrivial critical point.
2.2. The limit problems. Theorem 1.3 concerns manifolds of dimensions 2 and 3. To simplify the exposition we shall treat in full detail only the case $n = 2$. Everything can be extended in a straightforward way to the case $n = 3$, except for the estimates in section 5. These estimates, however, were computed in the appendix of [9] for $n = 3$.

Henceforth, we assume that $\dim M = 2$. We fix $r > 0$ smaller than the injectivity radius of $M$. We identify the tangent space of $M$ at $\xi$ with $\mathbb{R}^2$ and denote by $B(x, r)$ the ball in $\mathbb{R}^2$ centered at $x$ of radius $r$ and by $B_\xi (\xi, r)$ the ball in $M$ centered at $\xi$ of radius $r$, with respect to the distance induced by the Riemannian metric $g$. The exponential map $\exp_\xi : B(0, r) \to B_\xi (\xi, r)$ provides local coordinates on $M$, which are called normal coordinates. We denote by $g_\xi$ the Riemannian metric at $\xi$ given in normal coordinates by the matrix $(g_{ij})$. We denote the inverse matrix by $(g^{ij}(z)) := (g_{ij}(z))^{-1}$ and write $|g_\xi(z)| := \det (g_{ij}(z))$. Then, we have that

\begin{equation}
(2.6) \quad g^{ij}(\varepsilon z) = \delta_{ij} + \frac{\varepsilon^2}{2} \sum_{r,k=1}^{n} \frac{\partial^2 g^{ij}}{\partial \xi_r \partial \xi_k} (0) \varepsilon^r z^k + O(\varepsilon^3 |z|^3) = \delta_{ij} + o(\varepsilon),
\end{equation}

\begin{equation}
(2.7) \quad |g(\varepsilon z)|^{\frac{1}{2}} = 1 - \frac{\varepsilon^2}{4} \sum_{i,r,k=1}^{n} \frac{\partial^2 g^{ii}}{\partial \xi_r \partial \xi_k} (0) \varepsilon^r z^k + O(\varepsilon^3 |z|^3) = 1 + o(\varepsilon).
\end{equation}

Here $\delta_{ij}$ denotes the Kronecker symbol.

For $p \in (2, \infty)$ and $\xi \in M$, set

\[ A(\xi) := \frac{a(\xi)}{c(\xi)}, \quad B(\xi) := \frac{b(\xi)}{c(\xi)}, \quad \gamma(\xi) := \left( \frac{a(\xi)}{b(\xi)} \right)^{\frac{1}{p-2}}. \]

We consider the problem

\[-c(\xi) \Delta V + a(\xi)V = b(\xi) V^{p-1}, \quad V \in H^1(\mathbb{R}^2),
\]

and denote by $V^\xi$ its unique positive spherically symmetric solution. This problem is equivalent to

\[-\Delta V + A(\xi)V = B(\xi) V^{p-1}, \quad V \in H^1(\mathbb{R}^2).\]

The function $V^\xi$ and its derivatives decay exponentially at infinity. $V^\xi$ can be written as

\[ V^\xi(z) = \gamma(\xi) U(\sqrt{A(\xi)} z), \]

where $U$ is the unique positive spherically symmetric solution to

\[-\Delta U + U = U^{p-1}, \quad U \in H^1(\mathbb{R}^2).\]

For $\xi \in M$ and $\varepsilon > 0$ we define $W_{\varepsilon, \xi} \in H^1_0(M)$ by

\[ W_{\varepsilon, \xi}(x) := \begin{cases} 
\varepsilon \xi \exp^{-1}(x) & \text{if } x \in B_\varepsilon (\xi, r), \\
0 & \text{otherwise},
\end{cases}
\]

where $\chi \in C_\infty(\mathbb{R}^n)$ is a radial cut-off function such that $\chi(z) = 1$ if $|z| \leq r/2$ and $\chi(z) = 0$ if $|z| \geq r$. Setting $V_\varepsilon(z) := V \left( \frac{z}{\varepsilon} \right)$ and $y := \exp_\xi^{-1} x$ we have that

\[ W_{\varepsilon, \xi}(\exp_\xi(y)) = V^\xi \left( \frac{y}{\varepsilon} \right) \chi(y) = V_\varepsilon(y) \chi(y), \]

so the function $W_{\varepsilon, \xi}$ is simply the function $V^\xi$ rescaled, cut off and read in normal coordinates at $\xi$ in $M$. 
Similarly, for $i = 1, 2$ we define
\[
Z^i_{\epsilon, \xi}(x) = \begin{cases} 
\psi^i_\xi \left( \frac{1}{\epsilon} \exp^{-1}_\xi(x) \right) \chi \left( \exp^{-1}_\xi(x) \right) & \text{if } x \in B_g(\xi, r), \\
0 & \text{otherwise,}
\end{cases}
\]
where
\[
\psi^i_\xi(\eta) = \frac{\partial}{\partial \eta_i} V^\xi(\eta) = \gamma(\xi) \sqrt{A(\xi)} \frac{\partial U}{\partial \eta_i} (\sqrt{A(\xi)} \eta).
\]
The functions $\psi^i_\xi$ are solutions of the linearized equation
\[
-\Delta \psi + A(\xi) \psi = (p - 1)B(\xi) \left( V^\xi \right)^{p-2} \psi \quad \text{in } \mathbb{R}^2.
\]

**Proposition 2.3.** There is a positive constant $C$ such that
\[
\langle Z^h_{\epsilon, \xi}, Z^k_{\epsilon, \xi} \rangle = C \delta_{hk} + o(1),
\]
as $\epsilon \to 0$.

**Proof.** From the Taylor expansions of $g^{ij}(\epsilon z), |g(\epsilon z)|^{\frac{1}{2}}, a(\exp_\xi(\epsilon z))$ and $c(\exp_\xi(\epsilon z))$ we obtain
\[
\langle Z^h_{\epsilon, \xi}, Z^k_{\epsilon, \xi} \rangle = \frac{1}{\epsilon^2} \int M c(x) \nabla_g Z^h_{\epsilon, \xi}(x) \nabla_g Z^k_{\epsilon, \xi}(x) + d(x) Z^h_{\epsilon, \xi}(x) Z^k_{\epsilon, \xi}(x) d\mu_g
\]
\[
= \int_{B(0, r/\epsilon)} \sum_{ij} c(\exp_\xi(\epsilon z)) g^{ij}(\epsilon z) \frac{\partial}{\partial z_i} (\psi^h_\xi(z) \chi(\epsilon z)) \frac{\partial}{\partial z_j} (\psi^k_\xi(z) \chi(\epsilon z)) |g_\xi(\epsilon z)|^{\frac{1}{2}} dz
\]
\[
+ \int_{B(0, r/\epsilon)} d(\exp_\xi(\epsilon z)) \psi^h_\xi(z) \psi^k_\xi(z) \chi^2(\epsilon z) |g_\xi(\epsilon z)|^{\frac{1}{2}} dz
\]
\[
= c(\xi) \int_{\mathbb{R}^2} \nabla h \psi^h_\xi \nabla k \psi^k_\xi dz + d(\xi) \int_{\mathbb{R}^2} h k \psi^h_\xi \psi^k_\xi dz + o(1) = C \delta_{hk} + o(1),
\]
as claimed. \qed

Next, we compute the derivatives of $W_{\epsilon, \xi}$ with respect to $\xi$ in normal coordinates. Fix $\xi_0 \in M$. We write the points $\xi \in B_g(\xi_0, r)$ as
\[
\xi = \xi(y) = \exp_{\xi_0}(y) \quad \text{with } y \in B(0, r).
\]
We define
\[
\mathcal{E}(y, x) = \exp^{-1}_{\xi(y)}(x) = \exp^{-1}_{\exp_{\xi_0}(y)}(x),
\]
where $x \in B_g(\xi(y), r)$ and $y \in B(0, r)$. Then we can write
\[
W_{\epsilon, \xi}(y) = \gamma(\xi(y)) U_{\epsilon} \left( \sqrt{A(\xi(y))} \exp^{-1}_{\xi(y)}(x) \chi(\exp^{-1}_{\xi(y)}(x)) \right)
\]
\[
= \tilde{\gamma}(y) U_{\epsilon} \left( \sqrt{A(y) \mathcal{E}(y, x)} \chi(\mathcal{E}(y, x)) \right)
\]
where $\tilde{A}(y) = A(\exp_{\xi_0}(y))$ and $\tilde{\gamma}(y) = \gamma(\exp_{\xi_0}(y))$. Thus we have
\[
\left. \frac{\partial}{\partial y_s} W_{\epsilon, \xi}(y) \right|_{y=0} = \left. \left( \frac{\partial}{\partial y_s} \tilde{\gamma}(y) \right) \right|_{y=0} U \left( \frac{1}{\epsilon} \sqrt{\tilde{A}(y) \mathcal{E}(y, x)} \right) \chi(\mathcal{E}(0, x))
\]
\[
+ \tilde{\gamma}(0) U \left( \frac{1}{\epsilon} \sqrt{\tilde{A}(0) \mathcal{E}(0, x)} \right) \frac{\partial}{\partial y_s} \chi(\mathcal{E}(y, x)) \left|_{y=0} \right.
\]
\[
+ \tilde{\gamma}(0) \chi(\mathcal{E}(0, x)) \left. \frac{\partial}{\partial y_s} U \left( \frac{1}{\epsilon} \sqrt{A(y) \mathcal{E}(y, x)} \right) \right|_{y=0}.
\]
If $x = \exp_{z_0} \varepsilon z$, $z_0 = \xi(0)$, then $\mathcal{E}(0, x) = \varepsilon z$ and we have
\[
\left. \frac{\partial}{\partial y_s} W_{z, \xi(y)} \right|_{y=0} = \left( \frac{\partial}{\partial y_s} \tilde{\gamma}(y) \right)_{y=0} U(\sqrt{A(0)} z) \chi(\varepsilon z)
\]
\[
+ \tilde{\gamma}(0) U \left( \sqrt{A(0)} z \right) \frac{\partial \chi}{\partial \eta_k}(\varepsilon z) \frac{\partial \varepsilon_k(y, \exp_{z_0} \varepsilon z)}{\partial y_s} \left|_{y=0} \right.
\]
\[
+ \tilde{\gamma}(0) \chi(\varepsilon z) \frac{\partial U}{\partial \eta_k} \left( \sqrt{A(0)} z \right) \frac{\partial \varepsilon_k(y, \exp_{z_0} \varepsilon z)}{\partial y_s} \left|_{y=0} \right.
\]
\[
(2.8)
\]

We also recall the following Taylor expansions:
\[
(2.9)
\]

\[ \frac{\partial}{\partial y_k} \mathcal{E}_k(0, \exp_{z_0} \varepsilon z) = -\delta_{hk} + O(\varepsilon^2 |z|^2). \]

2.3. **Outline of the proof of Theorem 1.3.** Let $H_\varepsilon$ denote the Hilbert space $H^1_g(M)$ equipped with the inner product
\[
\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^2} \left( \varepsilon^2 \int_M c(x) \nabla_g u \nabla_g v \, d\mu_g + \int_M d(x) uv \, d\mu_g \right),
\]
which induces the norm
\[
\|u\|_\varepsilon^2 := \frac{1}{\varepsilon^2} \left( \varepsilon^2 \int_M c(x) |\nabla_g u|^2 \, d\mu_g + \int_M d(x) u^2 \, d\mu_g \right),
\]
with $d(x) := a(x) - \omega^2 b(x) > 0$. Similarly, let $L^q_\varepsilon$ be the Banach space $L^q_g(M)$ with the norm
\[
|u|_{q, \varepsilon} := \left( \frac{1}{\varepsilon^2} \int_M |u|^q \, d\mu_g \right)^{1/q}.
\]

Since we are assuming that $\dim M = 2$, for each $q \geq 2$ the embedding $H_\varepsilon \hookrightarrow L^q_\varepsilon$ is continuous. In fact, there is a positive constant $C$, independent of $\varepsilon$, such that
\[
(2.10) \quad |u|_{q, \varepsilon} \leq C \|u\|_\varepsilon \quad \forall u \in H_\varepsilon,
\]

Moreover, this embedding is compact.

Fix $p \in (2, \infty)$. The adjoint operator $i^*_\varepsilon : L^p_\varepsilon' \to H_\varepsilon$, $p' := \frac{p}{p-1}$, to the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L^p_\varepsilon$ is defined by
\[
\langle u \rangle_\varepsilon \leftrightarrow \langle u, \varphi \rangle_\varepsilon = \frac{1}{\varepsilon^2} \int_M uv \varphi \quad \forall \varphi \in H_\varepsilon
\]
\[
\leftrightarrow - \varepsilon^2 \text{div}_g (c(x) \nabla_g u) + d(x) u = v, \quad u \in H^1_g(M).
\]

One has that
\[
i^*_\varepsilon(v) \|_\varepsilon \leq C|v|_{p', \varepsilon} \quad \forall v \in L^p_\varepsilon',
\]

where the constant $C$ does not depend on $\varepsilon$.

Using the adjoint operator we can rewrite problem (2.5) as
\[
u = i^*_\varepsilon \left[ b(x) f(u) + \omega^2 b(x) g(u) \right], \quad u \in H_\varepsilon,
\]
where
\[
f(u) := (u^+)^{p-1} \quad \text{and} \quad g(u) := (q^2 \Psi^2(u) - 2q \Psi(u)) u.
\]

Let
\[
K_{\varepsilon, \xi} := \text{Span} \left\{ Z^{\varepsilon, \xi}_1, Z^{\varepsilon, \xi}_2 \right\}
\]
and
\[ K^\perp_{\varepsilon, \xi} := \{ \phi \in H_\varepsilon : \langle \phi, Z^i_{\varepsilon, \xi} \rangle = 0, \ i = 1, 2 \}. \]

We denote the projections onto these subspaces by
\[ \Pi_{\varepsilon, \xi} : H_\varepsilon \to K_{\varepsilon, \xi} \quad \text{and} \quad \Pi^\perp_{\varepsilon, \xi} : H_\varepsilon \to K^\perp_{\varepsilon, \xi}. \]

We look for a solution of \((2.5)\) of the form
\[ u_\varepsilon := W_{\varepsilon, \xi} + \phi \quad \text{with} \quad \phi \in K^\perp_{\varepsilon, \xi}. \]

This is equivalent to solving the pair of equations
\begin{align*}
(2.13) & \quad \Pi^\perp_{\varepsilon, \xi} \left\{ W_{\varepsilon, \xi} + \phi - i^*_\varepsilon \left\{ \begin{array}{l}
\xi_x \\
\xi_y
\end{array} \right. \right\} \left( b(x) f(W_{\varepsilon, \xi} + \phi) + \omega^2 b(x) g(W_{\varepsilon, \xi} + \phi) \right) = 0, \\
(2.14) & \quad \Pi_{\varepsilon, \xi} \left\{ W_{\varepsilon, \xi} + \phi - i^*_\varepsilon \left\{ \begin{array}{l}
\xi_x \\
\xi_y
\end{array} \right. \right\} \left( b(x) f(W_{\varepsilon, \xi} + \phi) + \omega^2 b(x) g(W_{\varepsilon, \xi} + \phi) \right) = 0.
\end{align*}

The first step of the proof of Theorem 1.3 is to solve equation \((2.13)\). More precisely, for any fixed \( \xi \in M \) and \( \varepsilon \) small enough, we will show that there is a function \( \phi \in K^\perp_{\varepsilon, \xi} \) such that \((2.13)\) holds. To do this we consider the linear operator \( L_{\varepsilon, \xi} : K^\perp_{\varepsilon, \xi} \to K^\perp_{\varepsilon, \xi} \) given by
\[ L_{\varepsilon, \xi} (\phi) := \Pi^\perp_{\varepsilon, \xi} \{ \phi - i^*_\varepsilon \left\{ \begin{array}{l}
\xi_x \\
\xi_y
\end{array} \right. \right\} \left( b(x) f(W_{\varepsilon, \xi} + \phi) \right). \]

For the proof of the following statement we refer to Lemma 4.1 of [12] (see also Proposition 3.1 of [12]).

**Proposition 2.4.** There exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for every \( \varepsilon \in (0, \varepsilon_0) \), \( \xi \in M \) and \( \phi \in K^\perp_{\varepsilon, \xi} \),
\[ \| L_{\varepsilon, \xi}(\phi) \|_\varepsilon \geq C \| \phi \|_\varepsilon. \]

This result allows to use a contraction mapping argument to solve equation \((2.13)\). The following statement is proved in section 3.

**Proposition 2.5.** There exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for each \( \xi \in M \) and each \( \varepsilon \in (0, \varepsilon_0) \), there exists a unique \( \phi_{\varepsilon, \xi} \in K^\perp_{\varepsilon, \xi} \) which solves equation \((2.13)\). Moreover,
\[ \| \phi_{\varepsilon, \xi} \|_\varepsilon \leq C \varepsilon. \]

The map \( \xi \mapsto \phi_{\varepsilon, \xi} \) is a \( C^1 \)-map.

The second step is to solve equation \((2.14)\). More precisely, for \( \varepsilon \) small enough we will find a point \( \xi \in M \) such that equation \((2.14)\) is satisfied. To this end we introduce the reduced energy functional \( \tilde{I}_\varepsilon : M \to \mathbb{R} \) defined by
\[ \tilde{I}_\varepsilon(\xi) := I_\varepsilon(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}), \]
where \( I_\varepsilon \) is the variational functional defined in \((2.3)\) whose critical points are the solutions to problem \((2.5)\). It is easy to verify that \( \xi_\varepsilon \) is a critical point of \( \tilde{I}_\varepsilon \) if and only if the function \( u_\varepsilon := W_{\varepsilon, \xi_\varepsilon} + \phi_{\varepsilon, \xi_\varepsilon} \) is a critical point of \( I_\varepsilon \).

In Lemmas 4.1 and 4.2 we compute the asymptotic expansion of the reduced functional \( \tilde{I}_\varepsilon \) with respect to the parameter \( \varepsilon \). We prove the following result.

**Proposition 2.6.** The expansion
\[ \tilde{I}_\varepsilon(\xi) = C \frac{c(\xi)^2 a(\xi)^{p-2} - \frac{a}{b(\xi)^{p-2}}}{b(\xi)^{p-2}} + o(1) = C \Gamma(\xi) + o(1), \]
holds true \( C^1 \)-uniformly with respect to \( \xi \) as \( \varepsilon \to 0 \), where \( C = \left( \frac{1}{2} - \frac{2}{p} \right) \int_{\mathbb{R}^n} U^p dz \).
Using the previous propositions we now prove Theorem 1.3.

**Proof of Theorem 1.3.** Since $K$ is a $C^1$-stable critical set for $\Gamma$, by Proposition 2.6 $I_\varepsilon$ has a critical point $\xi_\varepsilon \in M$ such that $d_\varepsilon(\xi_\varepsilon, K) \to 0$ as $\varepsilon \to 0$. Hence, $u_\varepsilon = W_{\varepsilon, \xi_\varepsilon} + \phi_{\varepsilon, \xi_\varepsilon}$ is a solution of (2.3), and the pair $(u_\varepsilon, \Psi(u_\varepsilon))$ is a solution to the system (1.7) such that $u_\varepsilon$ concentrates at a point $\xi_0 \in K$ as $\varepsilon \to 0$. \hfill \square

3. The finite dimensional reduction

This section is devoted to the proof of Proposition 2.5. We denote by

$$L_{\varepsilon, \xi}(\phi) = N_{\varepsilon, \xi}(\phi) + S_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi},$$

where

$$N_{\varepsilon, \xi}(\phi) := \Pi_{\varepsilon, \xi} \{ i_{\varepsilon}^* [b(x) (f(W_{\varepsilon, \xi}) + \phi) - f(W_{\varepsilon, \xi})] \phi \},$$

$$S_{\varepsilon, \xi}(\phi) := \varepsilon^2 \Pi_{\varepsilon, \xi} \{ i_{\varepsilon}^* [b(x) (q^2 \Psi^2(W_{\varepsilon, \xi}) + \phi) - 2q\Psi(W_{\varepsilon, \xi}) + \phi)] (W_{\varepsilon, \xi}) + \phi) \},$$

$$R_{\varepsilon, \xi} := \Pi_{\varepsilon, \xi} \{ i_{\varepsilon}^* [b(x)f(W_{\varepsilon, \xi})] - W_{\varepsilon, \xi} \}. $$

In order to solve equation (2.13) we will show that the operator $T_{\varepsilon, \xi} : K_{\varepsilon, \xi}^+ \to K_{\varepsilon, \xi}^+$ defined by

$$T_{\varepsilon, \xi}(\phi) := L_{\varepsilon, \xi}^{-1} (N_{\varepsilon, \xi}(\phi) + S_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi})$$

has a fixed point. To this end we prove that $T_{\varepsilon, \xi}$ is a contraction mapping on suitable ball in $H_{\varepsilon}$. We start with an estimate for $R_{\varepsilon, \xi}$.

**Lemma 3.1.** There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for any $\xi \in M$ and any $\varepsilon \in (0, \varepsilon_0)$, the inequality

$$\|R_{\varepsilon, \xi}\|_\varepsilon \leq C\varepsilon$$

holds true.

**Proof.** See Lemma 4.2 in [3]. \hfill \square

Next, we give an estimate for $N_{\varepsilon, \xi}(\phi)$.

**Lemma 3.2.** There exist $\varepsilon_0 > 0$, $C > 0$ and $\tilde{C} \in (0, 1)$ such that, for any $\xi \in M$, $\varepsilon \in (0, \varepsilon_0)$ and $R > 0$, the inequalities

$$(\varepsilon) \|N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2)\|_\varepsilon \leq \tilde{C}\|\phi_1 - \phi_2\|_\varepsilon,$$

hold true for $\phi, \phi_1, \phi_2 \in \{ \phi \in H_{\varepsilon} : \|\phi\|_\varepsilon \leq R\varepsilon \}$.

**Proof.** By direct computation we obtain

$$|f'(W_{\varepsilon, \xi} + v) - f'(W_{\varepsilon, \xi})| \leq \begin{cases} CW_{\varepsilon, \xi}^{p-3}|v| & 2 < p < 3, \\ C(W_{\varepsilon, \xi})^{p-2}|v| + |v|^{p-2} & p \geq 3. \end{cases}$$

From the mean value theorem and inequality (2.11) we derive

$$\|N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2)\|_\varepsilon \leq C|f'(W_{\varepsilon, \xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon, \xi})| \rightarrow_{\varepsilon, \varepsilon} \|\phi_1 - \phi_2\|_\varepsilon.$$
Using (3.5) we conclude that
\[ C |f'(W_{ε,ξ} + φ_1 + t(φ_1 - φ_2)) - f'(W_{ε,ξ})| \frac{dt}{t^2} < 1 \]
provided \( \|φ_1\|_ε \) and \( \|φ_2\|_ε \) are small enough. The same estimates yield (3.6). \[ \square \]

Now we estimate \( S_{ε,ξ}(φ) \).

**Lemma 3.3.** There exists \( ε_0 > 0 \) and \( C > 0 \) such that, for any \( ξ \in M, ε \in (0, ε_0) \) and \( R > 0 \), the inequalities
\begin{align*}
(3.6) & \quad \|S_{ε,ξ}(φ)\|_ε \leq Cε, \\
(3.7) & \quad \|S_{ε,ξ}(φ_1) - S_{ε,ξ}(φ_2)\|_ε \leq ε\|φ_1 - φ_2\|_ε,
\end{align*}
hold true for \( φ, φ_1, φ_2 \in \{ φ \in H_ε : \|φ\|_ε \leq Rε \} \), where \( ε \to 0 \) as \( ε \to 0 \).

**Proof.** Let us prove (3.6). From the definition of \( i^* \) and inequality (2.11) we derive
\[ \|S_{ε,ξ}(φ)\|_ε \leq C \left( \left| \Psi^2 (W_{ε,ξ} + φ) (W_{ε,ξ} + φ) \right|_{p',ε} + \left| \Psi (W_{ε,ξ} + φ) (W_{ε,ξ} + φ) \right|_{p',ε} \right) \\
= : I_1 + I_2. \]

For any \( t \in (2, \infty) \), setting \( s := \frac{4p'}{p'} \) and \( θ := \frac{2}{t} \in (1, 2) \) and applying Lemma 5.3 and Remark 5.2 we obtain
\begin{align*}
I_2 & \leq C \frac{1}{ε^2/p'} \left( \int_M |Ψ (W_{ε,ξ} + φ)|^t dμ_ε \right)^{1/t} \left( \int_M |W_{ε,ξ} + φ|^s dμ_ε \right)^{1/s} \\
& \leq C \frac{1}{ε^2/p'} |Ψ (W_{ε,ξ} + φ)|_g \left( ε^2 \left( \frac{1}{ε^2} \int_M |W_{ε,ξ}|^s dμ_ε \right)^{1/s} + |φ|_{g^*} \right) \\
& \leq C \left( ε^{θ+\frac{θ}{p'}} + ε^{θ+1-\frac{θ}{p'}} \right) = C \left( ε^{θ-\frac{θ}{p'}} + ε^{θ+1-\frac{θ}{p'}} \right) \\
& \leq Cε
\end{align*}
for all \( \|φ\|_ε \leq Rε \). From this estimate we deduce that \( I_1 \leq Cε \) and, hence, (3.6) follows.

Next, we prove (3.7). From inequality (2.11) we obtain that
\begin{align*}
\|S_{ε,ξ}(φ_1) - S_{ε,ξ}(φ_2)\|_ε & \leq C \left[ |Ψ (W_{ε,ξ} + φ_1) - Ψ (W_{ε,ξ} + φ_2)| W_{ε,ξ}|_{p',ε} \right. \\
& \quad + C \left[ |Ψ^2 (W_{ε,ξ} + φ_1) - Ψ^2 (W_{ε,ξ} + φ_2)| W_{ε,ξ}|_{p',ε} \right. \\
& \quad + C \left[ |Ψ (W_{ε,ξ} + φ_1) φ_1 - Ψ (W_{ε,ξ} + φ_2) φ_2| W_{ε,ξ}|_{p',ε} \right. \\
& \quad + C \left| Ψ^2 (W_{ε,ξ} + φ_1) φ_1 - Ψ^2 (W_{ε,ξ} + φ_2) φ_2 W_{ε,ξ}|_{p',ε} \right. \\
& \quad = : I_1 + I_2 + I_3 + I_4.
\end{align*}

By Remark 5.2 and Lemma 5.4 with \( s := \frac{3}{2} \), for some \( θ \in (0, 1) \) we have that
\begin{align*}
I_1' & \leq C \frac{1}{ε^2} \left( \int_M |Ψ' (W_{ε,ξ} + θφ_1 + (1 - θ)φ_2) (φ_1 - φ_2)|^p \right)^{1/p} \left( \frac{1}{ε^2} \int_M |W_{ε,ξ}|^{p'} \right)^{p/(p-1)} \\
& \leq C \frac{ε^{2(θ-\frac{θ}{p'})}}{ε^2} \left( ε^{\frac{θ}{p}} + |φ_1|_g + |φ_2|_g \right)^{p'} \|φ_1 - φ_2\|_g^{p'} \\
& \leq Cε \|φ_1 - φ_2\|_ε^{p'}. \end{align*}
Lemma 5.4 with $s \parallel \vartheta$ and $R \varepsilon$ for theorem to the $C I$ recalling that $0 \leq C(\varepsilon) \leq 1$ we derive

$$I_2' = \frac{1}{\varepsilon^2} \int_M |\Psi(W_{\varepsilon, \xi} + \phi_1) + \Psi(W_{\varepsilon, \xi} + \phi_2)|^{p'} |\Psi(W_{\varepsilon, \xi} + \phi_1) - \Psi(W_{\varepsilon, \xi} + \phi_2)|^{p'} |W_{\varepsilon, \xi}|^{p'}$$

$$\leq C I_1'.$$

On the other hand, choosing $\vartheta \in (1, 2)$ in Lemma 5.3 such that $\vartheta p' > 2$ and applying Lemma 5.4 with $s := \frac{1}{2}$, we obtain

$$I_3' \leq \frac{1}{\varepsilon^2} \int_M |\Psi'(W_{\varepsilon, \xi} + \theta \phi_1 + (1 - \theta) \phi_2) (\phi_1 - \phi_2)|^{p'} |\phi_1||p'$$

$$+ \frac{1}{\varepsilon^2} \int_M |\Psi(W_{\varepsilon, \xi} + \phi_2)|^{p'} |\phi_1 - \phi_2|^p$$

$$\leq C \frac{1}{\varepsilon^2} \left( \int_M |\Psi'(W_{\varepsilon, \xi} + \theta \phi_1 + (1 - \theta) \phi_2) (\phi_1 - \phi_2)|^p \right)^{\frac{p'}{p}} \left( \int_M |\phi_1|^p \right)^{\frac{p - p'}{p}}$$

$$+ C \frac{1}{\varepsilon^2} \left( \int_M |\phi_1 - \phi_2|^p \right)^{\frac{p'}{p}}$$

$$\leq C \frac{1}{\varepsilon^2} \left( \frac{4}{\varepsilon^2} + \|\phi_1\|_g + \|\phi_2\|_g \right)^{\frac{p'}{p}} \|\phi_1 - \phi_2\|^p \|\phi_1\|_g^{p'}$$

$$+ C \frac{\varepsilon^{p'}}{\varepsilon^2} (1 + \|\phi_2\|^2) \|\phi_1 - \phi_2\|^{p'}$$

$$\leq C \left( \frac{2^{p'}}{\varepsilon^2} + \frac{\varepsilon^{p'}}{\varepsilon^2} \right) \|\phi_1 - \phi_2\|^{p'} = l_{\varepsilon} \|\phi_1 - \phi_2\|^{p'}.$$

for $\|\phi_1\|_\varepsilon, \|\phi_2\|_\varepsilon \leq R \varepsilon$, where $l_{\varepsilon} \to 0$ as $\varepsilon \to 0$.

Finally, from the estimate of $I_2$ we derive $I_4' \leq C I_3'$. Collecting the previous estimates we obtain (14).

Proof of Proposition 2.5 From Proposition 2.4 we deduce

$$\|T_{\varepsilon, \xi}(\phi)\|_\varepsilon \leq C \left( \|N_{\varepsilon, \xi}(\phi)\|_\varepsilon + \|S_{\varepsilon, \xi}(\phi)\|_\varepsilon + \|R_{\varepsilon, \xi}\|_\varepsilon \right)$$

and

$$\|T_{\varepsilon, \xi}(\phi_1) - T_{\varepsilon, \xi}(\phi_2)\|_\varepsilon \leq C \|N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2)\|_\varepsilon + C \|S_{\varepsilon, \xi}(\phi_1) - S_{\varepsilon, \xi}(\phi_2)\|_\varepsilon.$$

Lemmas 3.1, 3.3, and 5.4 imply that $T_{\varepsilon, \xi}$ is a contraction in the ball centered at 0 of radius $R \varepsilon$ in $K_{\varepsilon, \xi}$ for a suitable constant $R$. Hence, $T_{\varepsilon, \xi}$ has a unique fixed point.

In order to prove that the map $\xi \mapsto \phi_{\varepsilon, \xi}$ is $C^1$ we apply the implicit function theorem to the $C^1$-function $G : M \times H_{\varepsilon} \to H_{\varepsilon}$ defined by

$$G(\xi, u) := \Pi_{\varepsilon, \xi} \left\{ W_{\varepsilon, \xi} + \Pi_{\varepsilon, \xi}^1 u - i_{\varepsilon}^1 \left[ b(x) f (W_{\varepsilon, \xi} + \Pi_{\varepsilon, \xi}^1 u) + \omega^2 b(x) g (W_{\varepsilon, \xi} + \Pi_{\varepsilon, \xi}^1 u) \right] \right\}$$

+ $\Pi_{\varepsilon, \xi} u.$
Note that $G(\xi, \phi_{\varepsilon, \xi}) = 0$. Next we show that the linearized operator $\frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon, \xi}) : H_\varepsilon \to H_\varepsilon$ defined by

$$
\frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon, \xi})(u) = \Pi_{\varepsilon, \xi}^{-1} \{ \Pi_{\varepsilon, \xi}^{-1}(u) - i_\varepsilon^* \left[ b(x) f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^{-1}(u) + \omega^2 b(x) g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^{-1}(u) \right] \}
+ \Pi_{\varepsilon, \xi}(u)
$$

is invertible, provided $\varepsilon$ is small enough. For any $\phi$ with $\|\phi\|_{\varepsilon} \leq C\varepsilon$ we have that

$$
\left\| \frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon, \xi})(u) \right\|_{\varepsilon} \geq C \left\| \Pi_{\varepsilon, \xi}(u) \right\|_{\varepsilon}
+ C \left\| \Pi_{\varepsilon, \xi}^{-1}(u) - i_\varepsilon^* \left[ f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^{-1}(u) + \omega^2 g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^{-1}(u) \right] \right\|_{\varepsilon}
\geq C \left\| \Pi_{\varepsilon, \xi}(u) \right\|_{\varepsilon} + C \left\| L_{\varepsilon, \xi} (\Pi_{\varepsilon, \xi}^{-1}(u)) \right\|_{\varepsilon}
- C \left\| \Pi_{\varepsilon, \xi}^{-1} \left[ i_\varepsilon^* \left[ f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) - f'(W_{\varepsilon, \xi}) \right] \Pi_{\varepsilon, \xi}^{-1}(u) \right] \right\|_{\varepsilon}
- C \left\| \Pi_{\varepsilon, \xi}^{-1} \left[ i_\varepsilon^* \left[ \omega^2 g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^{-1}(u) \right] \right] \right\|_{\varepsilon}
\geq C \left\| \Pi_{\varepsilon, \xi}(u) \right\|_{\varepsilon} + C \left\| \Pi_{\varepsilon, \xi}^{-1}(u) \right\|_{\varepsilon} - o(1) \left\| \Pi_{\varepsilon, \xi}(u) \right\|_{\varepsilon}
\geq C \left\| u \right\|_{\varepsilon}.
$$

Indeed, by (3.5) we have

$$
\left\| \Pi_{\varepsilon, \xi}^{-1} \left[ f'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) - f'(W_{\varepsilon, \xi}) \right] \Pi_{\varepsilon, \xi}^{-1}(u) \right\|_{\varepsilon} \leq C \left( \|\phi\|_{\varepsilon}^{p-2} + \|\phi\|_{\varepsilon} \right) \left\| \Pi_{\varepsilon, \xi}^{-1}(u) \right\|_{\varepsilon}
\leq o(1) \left\| \Pi_{\varepsilon, \xi}(u) \right\|_{\varepsilon}.
$$

Moreover,

$$
\left\| \Pi_{\varepsilon, \xi}^{-1} \left[ i_\varepsilon^* \left[ \omega^2 g'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^{-1}(u) \right] \right] \right\|_{\varepsilon}
\leq C \left| (W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) (2q - 2q \Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi})) \Psi'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \right| \left\| \Pi_{\varepsilon, \xi}^{-1}(u) \right\|_{p', \varepsilon}
\leq C \left[ 2q \Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) - q^2 \Psi^2(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \right] \left\| \Pi_{\varepsilon, \xi}(u) \right\|_{p', \varepsilon}
\leq \Pi_{1} + \Pi_{2}.
$$

From Lemma 5.3 we derive

$$
\Pi_{1} \leq \frac{C}{\varepsilon^{2q'}} |W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}|_{g, 2q} \left| \Psi'(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \Pi_{\varepsilon, \xi}^{-1}(u) \right|_{g, 2q'} \left| 2q - 2q \Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \right|_{g, 2q'}
\leq C \frac{1}{\varepsilon^{2q'}} \varepsilon^{\frac{4}{q'}} \left\| \Pi_{\varepsilon, \xi}^{-1} \right\|_{g} \leq \varepsilon^{2-\frac{2}{q'}} \left\| \Pi_{\varepsilon, \xi}^{-1} \right\|_{g} = o(1) \left\| \Pi_{\varepsilon, \xi}^{-1} \right\|_{g},
$$

and, since $0 \leq \Psi(u) \leq 1/q$, from Lemma 5.3 with $\partial p' > 2$ we get

$$
\Pi_{2} \leq \frac{C}{\varepsilon^{2q'}} \left| \Pi_{\varepsilon, \xi}^{-1} \right|_{g, p'} \left| \Psi(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}) \right|_{g, p'}
\leq C \frac{\varepsilon^{0}}{\varepsilon^{p'}} \left( 1 + \|\phi_{\varepsilon, \xi}\|_{g}^2 \right) \left\| \Pi_{\varepsilon, \xi}^{-1} \right\|_{g} = o(1) \left\| \Pi_{\varepsilon, \xi}^{-1} \right\|_{g}
$$

This concludes the proof. \qed
4. The reduced energy

This section is devoted to the proof of Proposition 2.6.

**Lemma 4.1.** The following estimate

\[ \tilde{I}_\varepsilon (\xi) = I_\varepsilon (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = I_\varepsilon (W_{\varepsilon,\xi}) + o(1) = J_\varepsilon (W_{\varepsilon,\xi}) + \frac{\omega^2}{2} G_\varepsilon (W_{\varepsilon,\xi}) + o(1) \]

holds true \( C^0 \)-uniformly with respect to \( \xi \) as \( \varepsilon \) goes to zero. Moreover, setting \( \xi(y) := \exp_\varepsilon (y), \ y \in B(0, r) \), we have that

\[ \left( \frac{\partial}{\partial y_h} \tilde{I}_\varepsilon (\xi(y)) \right)_{|y=0} = \left( \frac{\partial}{\partial y_h} I_\varepsilon (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right)_{|y=0} = \left( \frac{\partial}{\partial y_h} J_\varepsilon (W_{\varepsilon,\xi(y)}) \right)_{|y=0} + \frac{\omega^2}{2} \left( \frac{\partial}{\partial y_h} G_\varepsilon (W_{\varepsilon,\xi(y)}) \right)_{|y=0} + o(1) \]

\( C^0 \)-uniformly with respect to \( \xi \) as \( \varepsilon \) goes to zero.

**Proof.** In Lemma 5.1 of [3] we have proved the following two estimates:

\[ J_\varepsilon (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J_\varepsilon (W_{\varepsilon,\xi(y)}) = o(1), \]

\[ \left( J'_\varepsilon (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J'_\varepsilon (W_{\varepsilon,\xi(y)}) \right) \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right] = o(1). \]

To complete the proof we shall prove the three following estimates:

\[ G_\varepsilon (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon (W_{\varepsilon,\xi}) = o(1), \]

\[ \left[ G'_\varepsilon (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - G'_\varepsilon (W_{\varepsilon,\xi_0}) \right] \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right] = o(1), \]

\[ \left( J'_\varepsilon (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) + \frac{\omega^2}{2} G'_\varepsilon (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right) \left[ \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right] = o(1). \]

We start with (4.2). For some \( \theta \in [0, 1] \) we have

\[ G_\varepsilon (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon (W_{\varepsilon,\xi}) = \int_M b(x) \left[ \Theta (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})^2 - \Theta (W_{\varepsilon,\xi}) (W_{\varepsilon,\xi})^2 \right]. \]

Since \( \| \phi_{\varepsilon,\xi} \| \leq C_\varepsilon \), from Lemma 5.3 we obtain (4.2).
Next, we prove (4.3). For some $\theta \in [0, 1]$ we have

$$
[C'_{\varepsilon}(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - G'_{\varepsilon}(W_{\varepsilon,\xi_0})] \left[ \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right]_{y=0} 
\leq \frac{q}{2\varepsilon^2} \int_M b(x) \left\{ [2\Psi(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - \Psi(W_{\varepsilon,\xi_0})] - [q\Psi^2(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - q\Psi^2(W_{\varepsilon,\xi_0})] \right\} 
\cdot W_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{y=0} 
\leq \frac{q}{2\varepsilon^2} \int_M 2b(x) \left[ \Psi(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - q\Psi^2(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) \right] \phi_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{y=0} 
\leq \frac{q}{2\varepsilon^2} \int_M 2b(x) \Psi'(W_{\varepsilon,\xi_0} + \theta\phi_{\varepsilon,\xi_0})(\phi_{\varepsilon,\xi_0})W_{\varepsilon,\xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{y=0} 
\leq I_1^1 + I_2^2 + I_3 + I_4 + I_5
$$

From Lemma 5.4, Remark 5.2 and equations (2.8), (2.9), (2.6), (2.7), recalling that $\|\phi_{\varepsilon,\xi(y)}\|_{\varepsilon} \leq C\varepsilon$, we get

$$
I_1 \leq C\varepsilon^4 \left( \frac{1}{\varepsilon^2} \int_M [\Psi'(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0})(\phi_{\varepsilon,\xi_0})]^3 \right) \frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi_0}^3 \left( \frac{1}{\varepsilon^2} \int_M \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{y=0} \right]^3 \right)^{\frac{1}{3}} \leq C\varepsilon^4 \left( \frac{1}{\varepsilon^2} \sum_{k=1}^{2} \left| \frac{\partial U}{\partial z_k}(z) \right| (\varepsilon z) + \left( \chi(\varepsilon z) + \frac{\partial \chi}{\partial z_k}(\varepsilon z) \right) U(z) \right)^3 dz \leq C\varepsilon^4 \frac{1}{\varepsilon} = O(\varepsilon^4)
$$

In a similar way, using Lemma 5.3 and embedding the first and the second term in $L^6$ and the third one in $L^{3/2}$, we get

$$
I_4 \leq C\varepsilon^4 \frac{1}{\varepsilon^2} [\varepsilon^{4/3} \|\phi_{\varepsilon,\xi}\|_{\varepsilon}^2 + \|\phi_{\varepsilon,\xi}\|_{\varepsilon}^2 \|\phi_{\varepsilon,\xi}\|_{\varepsilon} \varepsilon^{4/3 - 1}] = O(\varepsilon^4).
$$
Again, as \( 0 < \Psi(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) < 1/q \), we obtain
\[ I_5 \leq C I_3 = O(\varepsilon). \]

Finally, we prove (4.3). Following the proof of Lemma 5.1 in [3], we need only to prove that
\[ \left| G'_{\varepsilon} \left( W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)} \right) Z_{\varepsilon,\xi(y)} \right| = o(1), \]
that is
\[ \left| \frac{1}{2} \varepsilon \int_M \left[ \Psi(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - q\Psi^2(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right] (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) Z_{\varepsilon,\xi(y)} \right| = o(1). \]
We have
\[ \left| \frac{1}{2} \varepsilon \int_M \left[ \Psi(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - q\Psi^2(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right] (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) Z_{\varepsilon,\xi(y)} \right| \]
\[ \leq C \varepsilon^2 \int_M \left[ \Psi(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) Z_{\varepsilon,\xi(y)} \right] \]
\[ + C \varepsilon^2 \int_M \left[ \Psi^2(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) Z_{\varepsilon,\xi(y)} \right] = I_1 + I_2. \]
By Proposition 2.3, we have that \( \| Z_{\varepsilon,\xi(y)} \|_\varepsilon = O(1) \). So, by Lemma 5.3 and Remark 4.2, we have
\[ I_1 \leq C \frac{\varepsilon^4}{\varepsilon^2} \left( \int_M [\Psi(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0})]^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0})^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M |Z_{\varepsilon,\xi(y)}|^3 \right)^{\frac{1}{3}} \]
\[ \leq C \frac{\varepsilon^4}{\varepsilon^2} \| \Psi(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) \|_\varepsilon (\| W_{\varepsilon,\xi_0} \|_3 \varepsilon + \| \phi_{\varepsilon,\xi_0} \|_\varepsilon) \| Z_{\varepsilon,\xi(y)} \|_\varepsilon = O(\varepsilon). \]
Again, as \( 0 < \Psi(W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) < 1/q \), we obtain
\[ I_2 \leq C I_1 = O(\varepsilon). \]
This concludes the proof.  

\[ \text{Lemma 4.2. The expansion} \]
\[ I_\varepsilon(W_{\varepsilon,\xi}) = \left( \frac{1}{2} - \frac{1}{p} \right) \frac{c(\xi)^{\frac{1}{p}^2} a(\xi)^{\frac{p}{p-2}} - \frac{1}{2}}{b(\xi)^{\frac{p}{p-2}}} \int_{\mathbb{R}^n} U^p dz + o(1) \]
holds true \( C^1 \)-uniformly with respect to \( \xi \in M \).
Proof. In Lemma 5.2 of [3] we proved that
\[ J_\varepsilon(W_{\varepsilon,\xi}) = \left( \frac{1}{2} - \frac{1}{p} \right) \frac{c(\xi) \frac{2}{p} a(\xi) \frac{2}{p} \varepsilon^{-\frac{2}{p}}}{b(\xi) \frac{1}{p} - 2} \int_{\mathbb{R}^n} U' dz + O(\varepsilon). \]

Hence, it suffices to show now that \(|G_\varepsilon(W_{\varepsilon,\xi})| = o(1)|, C^1\)-uniformly with respect to \(\xi \in M.\)

Regarding the \(C^0\)-convergence, by Remark 5.2 and Lemma 5.3, we have that
\[ |G_\varepsilon(W_{\varepsilon,\xi})| \leq \frac{C}{\varepsilon^2} \int_M \Psi(W_{\varepsilon,\xi}) W_{\varepsilon,\xi}^2 d\mu_g \]
\[ \leq C \frac{\varepsilon}{\varepsilon^2} \left( \int_M \Psi(W_{\varepsilon,\xi})^2 \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi}^4 \right)^{\frac{1}{2}} \]
\[ \leq C \frac{1}{\varepsilon} \|\Psi(W_{\varepsilon,\xi})\|_g \leq \frac{\varepsilon^{\frac{3}{2}}}{\varepsilon} = O(\varepsilon^{\frac{3}{2}}). \]

Regarding the \(C^1\)-convergence observe that
\[ \left| \frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi}) \right|_{y=0} \leq \frac{C}{\varepsilon^2} \left| \frac{\partial}{\partial y_h} \int_M \Psi(W_{\varepsilon,\xi}(y)) W_{\varepsilon,\xi}^2(y) d\mu_g \right|_{y=0} \]
\[ + \left| \frac{C}{\varepsilon^2} \int_M \Psi(W_{\varepsilon,\xi}(y)) 2 W_{\varepsilon,\xi}(y) \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi}(y) \right)_{y=0} \right| d\mu_g \]
\[ + \left| \frac{C}{\varepsilon^2} \int_M W_{\varepsilon,\xi}(y) \Psi'(W_{\varepsilon,\xi}(y)) \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi}(y) \right)_{y=0} \right| d\mu_g \]
\[ := I_1 + I_2. \]

Now, from Remark 5.2, Lemma 5.3 and the estimates (2.8) and (2.9), we derive
\[ I_1 \leq C \frac{\varepsilon^{\frac{3}{2}}}{\varepsilon^2} \left( \frac{1}{\varepsilon^2} \int_M \Psi(W_{\varepsilon,\xi}(y))^2 \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi}(y) \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon^2} \int_M \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi}(y) \right)_{y=0} \right)^{\frac{1}{2}} \]
\[ \leq C \frac{\varepsilon^{\frac{3}{2}}}{\varepsilon^2} \frac{1}{\varepsilon} = o(1). \]
On the other hand, from Remark 5.2, the proof of Lemma 5.4, and the estimates (2.8) and (2.9), for some $t \in (1,3/2)$ we obtain

$$I_2 \leq C \frac{\varepsilon^2}{\varepsilon^2} \left( \frac{1}{\varepsilon^2} \int_M W_{t,\xi(h)}^2 \right)^{\frac{1}{2}} \left( \int_M \left( \frac{\partial}{\partial y_h} W_{t,\xi(h)} \left|_{y=0} \right. \middle)^{t'} \right)^{\frac{1}{t'}} \right)^{\frac{1}{2}}$$

$$\leq C \frac{\varepsilon^2}{\varepsilon^2} \left\| \Psi'(W_{t,\xi(y)}) \left[ \frac{\partial}{\partial y_h} W_{t,\xi(h)} \left|_{y=0} \right. \right] \right\|_g$$

$$\leq C \frac{\varepsilon^2}{\varepsilon^2} \frac{1}{\varepsilon} \left( \frac{1}{\varepsilon^2} \int_M \left( \frac{\partial}{\partial y_h} W_{t,\xi(h)} \left|_{y=0} \right. \middle)^{6} \right)^{\frac{1}{6}} \right)$$

$$\leq C \frac{\varepsilon^2}{\varepsilon^2} \frac{1}{\varepsilon^2} = C \frac{\varepsilon^{2} - \frac{1}{6}}{\varepsilon^2} = o(1).$$

This concludes the proof. □

5. **Some estimates involving $\Psi$**

We start by pointing out the following facts.

**Remark 5.1.** There exists a constant $C > 0$ such that, for every $\varphi \in H^1_g(M)$ and every $0 < \varepsilon < 1$, we have

$$C \| \varphi \|^2_g = C \int_M \left( |\nabla_g \varphi|^2 + \varphi^2 \right) d\mu_g$$

$$\leq \int_M \left( c(x)|\nabla_g \varphi|^2 + \frac{d(x)}{\varepsilon^2} \varphi^2 \right) d\mu_g = \| \varphi \|^2_\varepsilon.$$

**Remark 5.2.** The following estimates

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} |W_{t,\xi}|_{p,g}^p \leq C |U|_p^p, \quad p \geq 2,$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left| \frac{\partial}{\partial y_h} W_{t,\xi} \right|_{g,2}^2 \leq C |\nabla U|_2^2$$

hold true uniformly with respect to $\xi \in M$.

Abusing notation we write

$$\| u \|^2_g = \int_M \left( c(x)|\nabla_g \varphi|^2 + b(x)u^2 \right) d\mu_g.$$ 

This norm is equivalent to the standard norm (3.1) of $H^1_g(M)$. From equations (2.1), (2.2) and (2.3) we obtain

$$\| \Psi(u) \|^2_g = \int_M b(x)qu^2 \Psi(u) d\mu_g - \int_M b(x)q^2 u^2 (\Psi(u))^2 d\mu_g$$

$$\leq C \int_M u^2 \Psi(u) d\mu_g.$$
From inequality (5.2) we obtain,

\[ \| \Psi'(u) [h] \|_g^2 = \int_M 2b(x)qu(1 - q\Psi(u))h\Psi'(u) [h] \, d\mu_g \]

\[ - \int_M b(x)q^2u^2 (\Psi'(u) [h])^2 \, d\mu_g \]

\[ \leq C \int_M |u| |h| \|\Psi'(u) [h]\|_g \, d\mu_g, \]

for all \( u, h \in H^1_g(M) \).

**Lemma 5.3.** Given \( \vartheta \in (1, 2) \) there is a constant \( C > 0 \) such that the inequality

\[ \| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \leq C (\varepsilon^\vartheta + \| \varphi \|_g^2) \]

holds true for every \( \varphi \in H^1_g(M), \xi \in M \) and small enough \( \varepsilon > 0 \).

**Proof.** Let \( t \in (2, \infty) \) be such that \( \frac{2}{t} = \vartheta \) where \( t' \) is the exponent conjugate to \( t \).

From inequality (5.1) we obtain

\[ \| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g^2 \leq C \left( \int_M [\Psi(W_{\varepsilon, \xi} + \varphi)]^t \, d\mu_g \right)^{1/t} \]

\[ \times \left( \int_M (W_{\varepsilon, \xi} + \varphi)^{2t'} \right)^{1/t'} \]

\[ \leq C \| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \| W_{\varepsilon, \xi} + \varphi \|_{g, 2t'}^2. \]

Thus, by Remark 5.2

\[ \| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \leq C \left( \varepsilon^{2/t'} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi}^{2t'} \right)^{1/t'} \right) \]

\[ + \left( \int_M \varphi^{2t'} \right)^{1/t'} \]

\[ \leq C (\varepsilon^\vartheta + \| \varphi \|_g^2), \]

as claimed. \( \square \)

**Lemma 5.4.** Given \( s \in (1, 2) \) there is a constant \( C > 0 \) such that the inequality

\[ \| \Psi'(W_{\varepsilon, \xi} + k)[h] \|_g \leq C \| h \|_g \left( \varepsilon^{2/s} + \| k \|_g \right) \]

holds true for every \( k, h \in H^1_g(M), \xi \in M \) and small enough \( \varepsilon > 0 \).

**Proof.** From inequality (5.2) we obtain,

\[ \| \Psi'(W_{\varepsilon, \xi} + k)[h] \|_g^2 \leq C \int_M \| \Psi'(W_{\varepsilon, \xi} + k)[h] \| \, d\mu_g \]

\[ \leq C \left( \int_M |W_{\varepsilon, \xi}| |h| \| \Psi'(W_{\varepsilon, \xi} + k)[h] \| \, d\mu_g + \int_M |k| |h| \| \Psi'(W_{\varepsilon, \xi} + k)[h] \| \, d\mu_g \right) \]

\[ =: I_1 + I_2. \]

Set \( t := 2s' \in (4, \infty) \), where \( s' \) is the conjugate exponent to \( s \). Using Remark 5.2 we conclude that

\[ I_1 \leq C \| \Psi'(W_{\varepsilon, \xi} + k)[h] \|_{g, t} \| h \|_{g, t} \| W_{\varepsilon, \xi} \|_{g, s} \]

\[ = C \| \Psi'(W_{\varepsilon, \xi} + k)[h] \|_{g} \| h \|_{g} \varepsilon^{2/s} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi}^s \right)^{1/s} \]

\[ = C \| \Psi'(W_{\varepsilon, \xi} + k)[h] \|_{g} \| h \|_{g} \varepsilon^{2/s}. \]

Since

\[ I_2 \leq C \| \Psi'(W_{\varepsilon, \xi} + k)[h] \|_{g, 3} \| h \|_{g, 3} |k|_{g, 3} \leq C \| \Psi'(W_{\varepsilon, \xi} + k)[h] \|_{g} \| h \|_{g} \| k \|_{g}, \]

A STATIC SUPERCRITICAL KGMP SYSTEM ON A CLOSED MANIFOLD 19
the claim follows.

Lemma 5.5. Consider the functions

$$\tilde{v}_{\varepsilon, \xi}(z) := \begin{cases} \Psi(W_{\varepsilon, \xi})(\exp(\xi z)) & \text{for } z \in B(0, r/\varepsilon), \\ 0 & \text{for } z \in \mathbb{R}^2 \setminus B(0, r/\varepsilon). \end{cases}$$

Then, for any $\theta \in (1, 2)$, there exists a constant $C > 0$, independent of $\varepsilon, \xi$, such that

$$|\tilde{v}_{\varepsilon, \xi}(z)|_{L^2(\mathbb{R}^3)} \leq C \varepsilon^{\theta-1},$$

$$|\nabla \tilde{v}_{\varepsilon, \xi}(z)|_{L^2(\mathbb{R}^3)} \leq C \varepsilon^{\theta}.$$

Proof. After a change of variables we have that

$$\int_{B_{\theta}(\xi, r)} |\nabla \Psi(W_{\varepsilon, \xi})|^2 + |\Psi(W_{\varepsilon, \xi})|^2 d\mu_g$$

$$= \varepsilon^2 \int_{B(0, r/\varepsilon)} |g_{\xi}(\varepsilon z)|^{1/2} \left( \sum_{ij} g_{ij}(\varepsilon z) \frac{1}{\varepsilon^2} \frac{\partial \tilde{v}_{\varepsilon, \xi}(z)}{\partial z_i} \frac{\partial \tilde{v}_{\varepsilon, \xi}(z)}{\partial z_j} + \tilde{v}_{\varepsilon, \xi}(z)^2 \right) dz.$$

Thus

$$\|\Psi(W_{\varepsilon, \xi})\|_g^2 \geq C \left( |\nabla \tilde{v}_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)}^2 + \varepsilon^2 |\tilde{v}_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)}^2 \right).$$

This, combined with Lemma 5.3 gives

$$|\nabla \tilde{v}_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)} + \varepsilon |\tilde{v}_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)} \leq C \varepsilon^\theta,$$

as claimed.

References

[1] P. Baird and J.C. Wood. Harmonic morphisms between Riemannian manifolds. London Mathematical Society Monographs. New Series 29. The Clarendon Press, Oxford University Press, Oxford, 2003.

[2] V. Benci and D. Fortunato. Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations. Rev. Math. Phys. 14 (2002), 409–420.

[3] M. Clapp, M. Ghimenti, and A.M. Micheletti. Solutions to a singularly perturbed supercritical elliptic equation on a Riemannian manifold concentrating at a submanifold. Preprint 2013.

[4] T. D’Aprile and J. Wei. Layered solutions for a semilinear elliptic system in a ball. J. Differential Equations 226 (2006), 269–294.

[5] T. D’Aprile and J. Wei. Clustered solutions around harmonic centers to a coupled elliptic system. Ann. Inst. H. Poincaré Anal. Non Linéaire 226 (2007), 605–628.

[6] F. Dobarro and E. Lami Dozo. Scalar curvature and warped products of Riemann manifolds. Trans. Amer. Math. Soc. 303 (1987), 161–168.

[7] O. Druet and E. Hebey. Existence and a priori bounds for electrostatic Klein-Gordon-Maxwell systems in fully inhomogeneous spaces. Commun. Contemp. Math. 12 (2010), 831–869.

[8] M. Ghimenti and A.M. Micheletti. Number and profile of low energy solutions for singularly perturbed Klein-Gordon-Maxwell systems on a riemannian manifold. arXiv preprint [http://arxiv.org/abs/1303.6449] in press.

[9] M. Ghimenti, A.M. Micheletti, and A. Pistoia. The role of the scalar curvature in some singularly perturbed coupled elliptic systems on Riemannian manifolds. Discr. Cont. Dyn. Syst. in press.

[10] E. Hebey and T.T. Truong. Static Klein-Gordon-Maxwell-Proca systems in 4-dimensional closed manifolds. J. Reine Angew. Math. 667 (2012), 221–248.

[11] E. Hebey and J. Wei. Resonant states for the static Klein-Gordon-Maxwell-Proca system. Math. Res. Lett. 19 (2012). 953–967.
[12] A.M. Micheletti and A. Pistoia. The role of the scalar curvature in a nonlinear elliptic problem on Riemannian manifolds. *Calc. Var. Partial Differential Equations* **34** (2009), 233–265.

[13] B. Ruf and P.N. Srikanth. Singularly perturbed elliptic equations with solutions concentrating on a 1-dimensional orbit. *J. Eur. Math. Soc. (JEMS)* **12** (2010), 413–427.

[14] B. Ruf and P.N. Srikanth, Concentration on Hopf fibres for singularly perturbed elliptic equations. *Preprint* 2013.

[15] D. Ruiz. Semiclassical states for coupled Schrödinger-Maxwell equations: Concentration around a sphere. *Math. Models Methods Appl. Sci.* **15** (2005), 141–164.

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