TOPOLOGICAL PROPERTIES OF FUNCTION SPACES $C_k(X,2)$ OVER ZERO-DIMENSIONAL METRIC SPACES $X$

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Abstract. Let $X$ be a zero-dimensional metric space and $X'$ its derived set. We prove the following assertions: (1) the space $C_k(X,2)$ is an Ascoli space iff $C_k(X,2)$ is $k_{\mathbb{R}}$-space iff either $X$ is locally compact or $X$ is not locally compact but $X'$ is compact, (2) $C_k(X,2)$ is a $k$-space iff either $X$ is a topological sum of a Polish locally compact space and a discrete space or $X$ is not locally compact but $X'$ is compact, (3) $C_k(X,2)$ is a sequential space iff $X$ is a Polish space and either $X$ is locally compact or $X$ is not locally compact but $X'$ is compact, (4) $C_k(X,2)$ is a Fréchet–Urysohn space iff $C_k(X,2)$ is a Polish space iff $X$ is a Polish locally compact space, (5) $C_k(X,2)$ is normal iff $X'$ is separable, (6) $C_k(X,2)$ has countable tightness iff $X$ is separable. In cases (1)-(3) we obtain also a topological and algebraical structure of $C_k(X,2)$.

1. Introduction

Topological properties of function spaces are of great importance and have been intensively studied from many years (see [1], [16], [18], and references therein). Various topological properties generalizing metrizability are intensively studied by topologists and analysts. Let us mention the Fréchet–Urysohn property, sequentiality, $k$-space property, $k_{\mathbb{R}}$-space property and Ascoli property (all relevant definitions are given in Section 2). It is known that

$$
\text{metric} \implies \text{Fréchet–Urysohn} \implies \text{sequential} \implies \text{k-space} \implies \text{k}_{\mathbb{R}}\text{-space} \implies \text{Ascoli space},
$$

and none of these implications is reversible.

For topological spaces $X$ and $Y$, we denote by $C_k(X,Y)$ the space $C(X,Y)$ of all continuous functions from $X$ into $Y$ endowed with the compact-open topology. The space $C_k(X,\mathbb{R})$ of all real-valued functions on $X$ is denoted by $C_k(X)$. If $G$ is a topological group, then so is $C_k(X,G)$ under the pointwise operation.

For a metrizable space $X$ and $Y = \mathbb{R}$ or $Y = \mathbb{I} := [0,1]$, the spaces $C_k(X)$ and $C_k(X,\mathbb{I})$ are typically not a $k$-space. R. Pol proved the following remarkable result:

Theorem 1.1 (22). Let $X$ be a first countable paracompact space. Then the space $C_k(X,\mathbb{I})$ is a $k$-space if and only if $X = L \cup D$ is the topological sum of a locally compact Lindelöf space $L$ and a discrete space $D$.

It is well known (see [18]) that $C_k(X)$ is metrizable if and only if $X$ is hemicompact, and $C_k(X)$ is completely metrizable if and only if $X$ is a hemicompact $k$-space. Taking into account that $C_k(X,\mathbb{I})$ is a closed subspace of $C_k(X)$ and the fact that the space $\mathbb{R}^{\omega_1}$ is not a $k$-space by [17] Problem 7.J(b)], Theorem 1.1 implies

Corollary 1.2. For a metric space $X$, the space $C_k(X)$ is a $k$-space if and only if $C_k(X)$ is a Polish space if and only if $X$ is a Polish locally compact space.

Metric spaces $X$ for which $C_k(X)$ and $C_k(X,\mathbb{I})$ are $k_{\mathbb{R}}$-spaces or Ascoli spaces were completely characterized in [13]: $C_k(X)$ is an Ascoli space iff $C_k(X,\mathbb{I})$ is an Ascoli space iff $C_k(X)$ is a $k_{\mathbb{R}}$-space iff $C_k(X,\mathbb{I})$ is a $k_{\mathbb{R}}$-space iff $X$ is locally compact.
If \( Y = 2 = \{0, 1\} \) is the doubleton, the situation changes. Recall that the derived set \( X' \) of a topological space \( X \) is the set of all non-isolated points of \( X \). G. Gruenhage et al. proved the following result.

**Theorem 1.3** ([15]). Let \( X \) be a zero-dimensional Polish space. Then \( C_k(X, 2) \) is sequential if and only if \( X \) is either locally compact or the derived set \( X' \) is compact.

T. Banakh and S. Gabriyelyan [4] posed the following problem: Characterize separable metrizable spaces \( X \) for which the function space \( C_k(X, 2) \) is an Ascoli space. We prove the following result which also shows that the Fréchet–Urysohn property, sequentiality and \( k \)-space property differ on spaces of the form \( C_k(X, 2) \). While these properties coincide for \( C_k(X) \) by Pytkeev’s theorem [25]: for a Tychonoff space \( X \), the space \( C_k(X) \) is a \( k \)-space if and only if it is Fréchet–Urysohn.

**Theorem 1.4.** Let \( X \) be a zero-dimensional metric space \( X \). Then:

(i) \( C_k(X, 2) \) is Ascoli if and only if \( C_k(X, 2) \) is a \( k_{\mathbb{R}} \)-space if and only if either \( X \) is locally compact or \( X \) is not locally compact but the derived set \( X' \) is compact;

(ii) \( C_k(X, 2) \) is a \( k \)-space if and only if either \( X = L \cup D \) is a topological sum of a separable metrizable locally compact space \( L \) and a discrete space \( D \) or \( X \) is not locally compact but the derived set \( X' \) is compact;

(iii) \( C_k(X, 2) \) is sequential if and only if \( X \) is a Polish space and either \( X \) is locally compact or \( X \) is not locally compact but the derived set \( X' \) is compact;

(iv) \( C_k(X, 2) \) is a Fréchet–Urysohn space if and only if \( C_k(X, 2) \) is a Polish space if and only if \( X \) is a Polish locally compact space.

In Theorems 3.3–3.5 below we obtain also a topological and algebraic structure of function spaces \( C_k(X, 2) \) for cases (i)-(iii). Note that (iii) of Theorem 1.4 generalizes Theorem 1.3 by essentially differs from the proof of Theorem 1.3.

For topological spaces \( X \) and \( Y \), we denote by \( C_p(X, Y) \) the space \( C(X, Y) \) endowed with the topology of pointwise convergence. If \( Y = \mathbb{I} \), R. Pol proved the following theorem.

**Theorem 1.5** ([22]). For a metrizable space \( X \), the following assertions are equivalent: (i) \( C_k(X, \mathbb{I}) \) is normal, (ii) \( C_p(X, \mathbb{I}) \) is normal, (iii) \( C_k(X, \mathbb{I}) \) is Lideljf, (iv) \( X' \) is separable.

It turns out that the same holds also for \( Y = 2 \) with an identical proof.

**Theorem 1.6.** For a zero-dimensional metric space \( X \), the following assertions are equivalent: (i) \( C_k(X, 2) \) is normal, (ii) \( C_p(X, 2) \) is normal, (iii) \( C_k(X, 2) \) is Lideljf, (iv) \( X' \) is separable.

In the next theorem we characterize metric spaces \( X \) for which \( C_k(X, 2) \) has countable tightness.

**Theorem 1.7.** For a zero-dimensional metric space \( X \) the following assertions are equivalent: (i) \( C_k(X, 2) \) has countable tightness, (ii) \( C_k(X, 2) \) is a \( \mathfrak{P}_0 \)-space, (iii) \( C_p(X, 2) \) has countable tightness, (iv) \( X \) is separable.

2. Auxiliary results

We start from the definitions of the following well-known notions. A topological space \( X \) is called

- Fréchet–Urysohn if for any cluster point \( a \in X \) of a subset \( A \subset X \) there is a sequence \( \{a_n\}_{n \in \mathbb{N}} \subset A \) which converges to \( a \);
• *sequential* if for each non-closed subset $A \subset X$ there is a sequence $\{a_n\}_{n \in \mathbb{N}} \subset A$ converging to some point $a \in \bar{A} \setminus A$;

• a *$k$-space* if for each non-closed subset $A \subset X$ there is a compact subset $K \subset X$ such that $A \cap K$ is not closed in $K$;

• a *$k_R$-space* if a real-valued function $f$ on $X$ is continuous if and only if its restriction $f|_K$ to any compact subset $K$ of $X$ is continuous.

For topological spaces $X$ and $Y$, denote by $\psi : X \times C_k(X,Y) \to Y$, $\psi(x,f) := f(x)$, the evaluation map. Recall that a subset $K$ of $C_k(X,Y)$ is *evenly continuous* if the restriction of $\psi$ onto $X \times K$ is jointly continuous, i.e. for any $x \in X$, each $f \in K$ and every neighborhood $O_f(x) \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset K$ of $f$ and $O_x \subset X$ of $x$ such that $U_f(O_x) := \{g(y) : g \in U_f, \ y \in O_x\} \subset O_f(x)$. Following [4], a regular (Hausdorff) space $X$ is called an *Ascoli space* if each compact subset $K$ of $C_k(X)$ is evenly continuous. It is easy to see that a space is Ascoli if and only if the canonical valuation map $X \hookrightarrow C_k(C_k(X))$ is an embedding, see [4]. By Ascoli’s theorem [9, 3.4.20], each $k$-space is Ascoli. N. Noble [20] proved that any $k_R$-space is Ascoli.

For a sequence $\{G_n\}_{n \in \mathbb{N}}$ of groups, the *direct sum* of $G_n$ is denoted by

$$\bigoplus_{n \in \mathbb{N}} G_n := \left\{ (g_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} G_n : \ g_n = e_n \text{ for almost all } n \right\}.$$ 

Let $(G, \tau)$ be a topological group. The filter of all open neighborhoods of the identity $e$ is denoted by $\mathcal{N}(G)$. Let $\{((G_n, \tau_n))_{n \in \mathbb{N}}$ be a sequence of (Hausdorff) topological groups. For every $n \in \mathbb{N}$ fix $U_n \in \mathcal{N}(G_n)$ and put

$$\prod_{n \in \mathbb{N}} U_n := \left\{ (g_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} G_n : \ g_n \in U_n \text{ for all } n \in \mathbb{N} \right\}.$$ 

Then the sets of the form $\prod_{n \in \mathbb{N}} U_n$, where $U_n \in \mathcal{N}(G_n)$ for every $n \in \mathbb{N}$, form a neighborhood basis at the unit of a (Hausdorff) group topology $\mathcal{T}_b$ on $\prod_{n \in \mathbb{N}} G_n$ that is called the *box topology*. The items (i) and (ii) of the next proposition are well-known.

**Proposition 2.1.** Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of compact topological groups and let $G := (\bigoplus_{n \in \mathbb{N}} G_n, \mathcal{T}_b)$. Then:

(i) the group $G$ is a $k$-space (even a $k_\omega$-space);

(ii) if $G_n$ is metrizable for every $n \in \mathbb{N}$, then $G$ is a sequential space;

(iii) if $G_n$ is metrizable for every $n \in \mathbb{N}$, then $G$ is a Fréchet–Urysohn space if and only if $G_n$ is a separable locally compact metrizable space if and only if $G_n$ is finite for all but finitely many $n$.

**Proof.** (i) It is easy to see that the topology $\mathcal{T}_b$ of $G$ is defined by the sequence $\prod_{n \leq n} G_i$ of compact groups. So $G$ is a $k_\omega$-space (in particular, $G$ is hemicompact). (ii) follows from (i) and [6, Lemma 1.5] (note that every compact subset of $G$ is contained in $\prod_{n \leq n} G_i$ for some $n \in \mathbb{N}$). (iii) follows from Proposition 3.18 and Corollary 2.17 of [12].

If $G_n = G$ for every $n \in \mathbb{N}$, we set $G^\infty := (\bigoplus_{n \in \mathbb{N}} G_n, \mathcal{T}_b)$. Proposition 2.1 immediately implies the next result.

**Corollary 2.2.** For every infinite metrizable compact group $G$ the group $G^\infty$ (in particular, $(2^\omega)^\infty$) is a sequential non-Fréchet–Urysohn space.

Recall that the family of subsets

$$[C; \epsilon] := \{ f \in C_k(X) : |f(x)| < \epsilon \ \forall x \in C \},$$
where \( C \) is a compact subset of \( X \) and \( \epsilon > 0 \), forms a basis of open neighborhoods at the zero function \( 0 \in C_k(X) \).

We need the following generalization of the countably infinite metric fan \( M \). Fix a sequence \( \kappa = (\kappa_n)_{n \in \mathbb{N}} \) of (non-zero) cardinal numbers. Denote by \( M_{\kappa} \) the space

\[
M_{\kappa} := \left( \bigcup_{n \in \mathbb{N}} \kappa_n \times \{n\} \right) \cup \{\infty\},
\]

where the points of \( \bigcup_{n \in \mathbb{N}} \kappa_n \times \{n\} \) are isolated, and the basic neighborhoods of \( \infty \) are

\[
U(n) = \left( \bigcup_{i \geq n} \kappa_i \times \{i\} \right) \cup \{\infty\}, \quad n \in \mathbb{N}.
\]

If \( \alpha \times \{n\} \neq \beta \times \{m\} \) and \( n \leq m \), set

\[
\rho(\alpha \times \{n\}, \beta \times \{m\}) = \rho(\alpha \times \{n\}, \infty) = \frac{1}{n}.
\]

Then \( \rho \) is a complete metric on \( M_{\kappa} \). So the space \( M_{\kappa} \) is a complete metrizable space. It is easy to see that \( M_{\kappa} \) is locally compact at \( \infty \) if and only if all but finitely many of cardinals \( \kappa_n \) are finite. If \( \kappa_n = \omega \) for every \( n \in \mathbb{N} \), the space \( M_{\kappa} = (\omega \times \mathbb{N}) \cup \{\infty\} \) is called the countably infinite metric fan and is denoted by \( M \); so \( M \) is a Polish space which is not locally compact at \( \infty \). Set

\[
C^0_k(M_{\kappa}, 2) := \{ f \in C_k(M_{\kappa}, 2) : f(\infty) = 0 \}.
\]

So \( C^0_k(M_{\kappa}, 2) \) is a clopen subgroup of \( C_k(M_{\kappa}, 2) \) and \( C_k(M_{\kappa}, 2) = \mathbb{Z}(2) \times C^0_k(M_{\kappa}, 2) \).

**Proposition 2.3.** Let \( \kappa = (\kappa_n)_{n \in \mathbb{N}} \) be a sequence of cardinal numbers. Then:

(i) The group \( C_k(M_{\kappa}, 2) \) is topologically isomorphic to the direct sum \( \mathbb{Z}(2) \oplus \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n} \) endowed with the box topology.

(ii) \( C_k(M_{\kappa}, 2) \) is a \( \kappa_\omega \)-space.

(iii) \( C_k(M_{\kappa}, 2) \) is a sequential space if and only if \( \kappa_n \leq \omega \) for every \( n \in \mathbb{N} \).

(iv) The following assertions are equivalent:

\begin{itemize}
  \item[(iv_1)] \( C_k(M_{\kappa}, 2) \) is a Fréchet–Urysohn space;
  \item[(iv_2)] \( C_k(M_{\kappa}, 2) \) is a locally compact Polish abelian group;
  \item[(iv_3)] \( \kappa_n \leq \omega \) for every \( n \in \mathbb{N} \) and \( \kappa_n \) is finite for almost all indices \( n \).
\end{itemize}

**Proof.** (i): It is enough to show that \( C^0_k(M_{\kappa}, 2) \) is topologically isomorphic to the direct sum \( G := \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n} \) endowed with the box topology. Define the map \( F : C^0_k(M_{\kappa}, 2) \to G \) by the rule

\[
F(f) = (f_n), \quad \text{where} \quad f_n := f|_{\kappa_n \times \{n\}}, \quad n \in \mathbb{N}.
\]

We claim that \( F \) is a desired topological isomorphism.

First we note that \( F \) is well-defined. Indeed, since \( f \) is continuous, there is \( m \in \mathbb{N} \) such that \( f(U(m)) = \{0\} \). So \( f_n = 0 \) for every \( n \geq m \), and \( F(f) \) belongs to \( \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n} \). Clearly, \( F \) is an algebraic isomorphism.

Let \( K \) be a compact subset of \( M_{\kappa} \). Then, for every \( n \in \mathbb{N} \), the intersection \( K_n := K \cap (\kappa_n \times \{n\}) \) is also compact in the discrete space \( \kappa_n \times \{n\} \), so \( K_n \) is finite. Hence every compact subset of \( M_{\kappa} \) is contained in a compact subset of the form

\[
K = \left( \bigcup_{n \in \mathbb{N}} K_n \times \{n\} \right) \cup \{\infty\},
\]

where \( K_n \) is a finite subset of \( \kappa_n \) for every \( n \in \mathbb{N} \). Since for \( 0 < \epsilon < 1 \)

\[
[K; \epsilon] = \{ f \in C^0_k(M_{\kappa}, 2) : f(K) = \{0\} \},
\]
we see that \( F(K; \epsilon) = \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n \setminus K_n} \). Taking into account that the sets \( \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n \setminus K_n} \) form an open basis at 0 in the group \( G \), this equality means that \( F \) is a homeomorphism. Thus \( F \) is a topological isomorphism.

(ii) follows from (i) and Proposition 2.4(ii).

(iii): If \( G \) is sequential, then all compact groups \( 2^{\kappa_n} \) are sequential. So \( \kappa_n \leq \omega \) for every \( n \in \mathbb{N} \), as a compact abelian group is sequential if and only if it is metrizable. Conversely, if \( \kappa_n \leq \omega \) for every \( n \in \mathbb{N} \), then \( G \) is sequential by Proposition 2.1(ii).

(iv) follows from (i), (iii) and Proposition 2.1(iii). \( \square \)

We shall use the following sufficient condition on a space \( X \) to be a non-Ascoli space.

**Proposition 2.4** ([13]). Assume a Tychonoff space \( X \) admits a family \( \mathcal{U} = \{ U_i : i \in I \} \) of open subsets of \( X \), a subset \( A = \{ a_i : i \in I \} \subset X \) and a point \( z \in X \) such that

1. \( a_i \in U_i \) for every \( i \in I \);
2. \( |\{ i \in I : C \cap U_i \neq \emptyset \}| < \infty \) for each compact subset \( C \) of \( X \);
3. \( z \) is a cluster point of \( A \).

Then \( X \) is not an Ascoli space.

We need also the following construction from [24]. Denote by \( s = \{0, 1/2, 1/3, \ldots \} \subset \mathbb{R} \) the convergent sequence and set

\[ L := M \cup (\mathbb{N} \times s). \]

Then \( L \) is a zero-dimensional Polish space which is not locally compact at the point \( \infty \in M \). For every \( p, q \in \mathbb{N} \), set

\[
a_{p,q} := (p, p + q) \in M, \quad b_{p,q} := (p, 1/(p + q)) \in \mathbb{N} \times s, \quad c_p := (p, 0) \in \mathbb{N} \times s,
\]
and define the function

\[
f_{p,q}(a_{p,q}) = f_{p,q}(b_{p,q}) := 1, \quad \text{and} \quad f_{p,q}(x) := 0 \text{ if } x \not\in \{a_{p,q}, b_{p,q}\}.
\]

To use Proposition 2.4 we define also the open neighborhood \( U_{p,q} \) of \( f_{p,q} \) in \( C_k(L, 2) \) by

\[ U_{p,q} := \{ h \in C_k(L, 2) : h(a_{p,q}) = h(b_{p,q}) = 1, h(\infty) = h(c_p) = 0 \}, \]
and set \( A := \{ f_{p,q} : p, q \in \mathbb{N} \} \) and \( \mathcal{U} := \{ U_{p,q} : p, q \in \mathbb{N} \} \).

**Lemma 2.5.** The set \( A \) and the family \( \mathcal{U} \) satisfy the following conditions:

1. the zero function \( 0 \) is a unique cluster point of \( A \);
2. if \( K \subset C_k(L, 2) \) is compact, the set \( \{ (p, q) \in \mathbb{N} \times \mathbb{N} : U_{p,q} \cap K \neq \emptyset \} \) is finite.

**Proof.** (i) It is easy to see that any compact subset \( C \) of \( L \) is contained in a compact subset of \( L \) of the form

\[
C_{t,n} = \{ (p, s) \in M : s \leq t(p), p \in \mathbb{N} \} \cup \{ \infty \} \cup \bigcup_{i \leq n} \{ i \} \times s,
\]
where \( n \in \mathbb{N} \) and \( t : \mathbb{N} \to \mathbb{N} \) is a function. If we take \( q > t(n + 1) \), then \( f_{n+1,q} \in [C; \epsilon] \cap A \) for every \( \epsilon > 0 \). Thus \( 0 \in \overline{A} \).

Denote by \( \text{supp}(g) = \{ x \in L : g(x) \neq 0 \} \) the support of a function \( g \in C_k(L, 2) \). Note that the supports of the functions \( f_{p,q} \in A \) are pairwise disjoint. So \( f_{p,q} \) is an isolated point of \( A \). Now if \( g \) is a cluster point of \( A \) and \( g \not\in A \), then \( g \) is in the closure of \( A \) in the topology \( \tau_p \) of pointwise convergence (note that \( \tau_p \) is metrizable since \( L \) is countable). Thus \( g \) must be the zero function \( 0 \).

(ii) Suppose for a contradiction that the set \( J := \{ (p, q) \in \mathbb{N} \times \mathbb{N} : U_{p,q} \cap K \neq \emptyset \} \) is infinite for some compact subset \( K \subset C_k(L, 2) \). Note that the space \( C_k(L, 2) \) is a \( \sigma \)-space and hence \( K \) is metrizable (see [14]). So there is a sequence \( F = \{ h_{p,q} \}_{i \in \mathbb{N}} \subset K \) converging
to a function \( h \in K \), where \( h_{p_i,q_i} \in U_{p_i,q_i} \) and all pairs \( (p_i, q_i) \) are distinct. In particular, \( h(\infty) = 0 \).

We claim that the sequence \( (p_i) \) is bounded. Indeed, assuming the converse we would find \( p_1 < p_2 < \ldots \) such that the points \( a_{p_i,q_i} \) converge to \( \infty \). But since \( h_{p_i,q_i}(a_{p_i,q_i}) = 1 \) and \( h_{p_i,q_i}(\infty) = 0 \) for every \( s \in \mathbb{N} \), this contradicts the equicontinuity of the compact set \( F \cup \{h\} \) on the compact set \( \{a_{p_i,q_i} : s \in \mathbb{N}\} \cup \{\infty\} \).

So without loss of generality we can assume that \( p_i = n \) for every \( i \in \mathbb{N} \). Set \( C := \{b_{n,q}, c_n : q \in \mathbb{N}\} \) and \( \epsilon = 1/2 \). Since \( h_{n,q}(b_{n,q}) = 1 \) and \( h_{n,q}(c_n) = 0 \) for every \( i \in \mathbb{N} \), we obtain that \( h(c_n) = 0 \) and the compact set \( F \cup \{h\} \) is not equicontinuous on the compact set \( C \). This contradiction shows that the set \( J \) is finite. \( \square \)

Lemma 2.5 and Proposition 2.4 immediately imply

**Proposition 2.6** \([23]\). The space \( C_k(L, 2) \) is not Ascoli.

3. **Proofs of Theorems 1.4–1.7**

We start from the following proposition proved by R. Pol.

**Proposition 3.1** \([23]\). Let \( X \) be a zero-dimensional metric space. If the space \( C_k(X, 2) \) is Ascoli, then either \( X \) is locally compact or \( X \) is not locally compact but the derived set \( X' \) is compact.

**Proof.** Suppose for a contradiction that \( X \) is not locally compact and \( X' \) is not compact. By Lemma 8.3 of \([7]\), \( X \) contains as a closed subspace the countable metric fan \( M \). Since \( X' \) is not compact, \( X \) contains also an isomorphic copy of \( \mathbb{N} \times s \). Thus \( X \) contains a closed subspace \( Y \) which is homeomorphic to the space \( L \). As \( Y \) is a retract of \( X \) (see \([8]\) for a more general assertion) and \( X \) has the Dugundji extension property by \([5]\), \( C_k(Y, 2) \) is a retract of \( C_k(X, 2) \). So \( C_k(X, 2) \) is not Ascoli by Proposition 2.6 and \([4\), Proposition 5.2]. \( \square \)

Below and in the proof of Theorem 3.3 we shall use the following fact: a separable metrizable space \( X \) such that \( X' \) is compact is Polish (this easily follows from Cantor’s theorem \([9\), 4.38]). The next result gives a partial answer to Problem 6.8 in \([4]\).

**Corollary 3.2** \([23]\). Let \( X \) be a zero-dimensional separable metric space. Then the space \( C_k(X, 2) \) is Ascoli if and only if either \( X \) is locally compact or \( X \) is not locally compact but the derived set \( X' \) is compact.

**Proof.** The necessity follows from Proposition 3.1 and the sufficiency follows from Corollary 1.2 and Theorem 1.3. \( \square \)

In the next theorem we strengthen Theorem 1.4(i) by showing also the algebraic structure of the group \( C_k(X, 2) \) which is essentially used in the proof of Theorem 3.4 below.

**Theorem 3.3.** For a zero-dimensional metric space \( X \), the following assertions are equivalent:

1. the space \( C_k(X, 2) \) is Ascoli;
2. the space \( C_k(X, 2) \) is a \( k_\mathbb{R} \)-space;
3. one of the following conditions holds
   3a. \( X \) is locally compact; in this case \( C_k(X, 2) \) is the product of a family of Polish abelian groups;
   3b. \( X \) is not locally compact but the derived set \( X' \) is compact; in this case the open subgroup
   \[ H := \{f \in C_k(X, 2) : f|_{X'} \equiv 0\} \]
of \( C_k(\mathcal{X}, 2) \) is a k-space and is topologically isomorphic to the direct sum \( \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n} \) endowed with the box topology for some sequence \( \kappa = (\kappa_n)_{n \in \mathbb{N}} \) of cardinal numbers.

**Proof.** (1)\( \Rightarrow \) (3): Proposition 3.1 implies that either \( X \) is locally compact or \( X \) is not locally compact but the derived set \( X' \) is compact.

Assume that \( X \) is locally compact. Then \( X \) is a disjoint union of a family \( \{X_i\}_{i \in I} \) of separable metrizable locally compact spaces by [9, 5.1.27]. So

\[
C_k(\mathcal{X}, 2) = \prod_{i \in I} C_k(X_i, 2),
\]

where each space \( C_k(X_i, 2) \) is Polish by Corollary 1.2.

Assume that \( X \) is not locally compact but the derived set \( X' \) is compact. As in the first paragraph of the proof of Theorem 3.1 in [15], there is a clopen outer base \( \{U_n\}_{n \in \mathbb{N}} \) of \( X' \) such that \( U_1 = X, U_{n+1} \subset U_n \), and \( U_n \setminus U_{n+1} \) is infinite for all \( n \in \mathbb{N} \). Noting that \( U_n \setminus U_{n+1} \) is a clopen discrete subspace of \( X \), we set \( \kappa = (\kappa_n), \) where \( \kappa_n = |U_n \setminus U_{n+1}| \) for every \( n \in \mathbb{N} \).

Let \( T: X \to M_{\kappa} \) be a map such that \( T(X') = \{\infty\} \) and \( T|_{U_n \setminus U_{n+1}} \) is an injective map onto \( \kappa_n \times \{\infty\} \). Then \( T \) is a continuous map such that \( T^{-1}(K) \) is compact for every compact subset \( K \subset M_{\kappa} \) (see the structure of compact subsets of \( M_{\kappa} \) given in the proof of Proposition 2.3).

The map

\[
T^*: C_k^0(M_{\kappa}, 2) \to H, \quad T^*(f) := f \circ T,
\]

is easily seen to be a continuous isomorphism of abelian groups. As \( T^*(\{K; \varepsilon\}) = (T^{-1}(K) + \varepsilon) \) for every compact subset \( K \subset M_{\kappa} \) and \( 0 < \varepsilon < 1 \), \( T^* \) is open and hence it is a topological isomorphism. Thus, by Proposition 2.3, \( H \) is a k-space which is topologically isomorphic to the direct sum \( \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n} \) endowed with the box topology. Therefore also \( C_k(\mathcal{X}, 2) \) is a k-space.

(3)\( \Rightarrow \) (2): If \( X \) is locally compact, then \( C_k(\mathcal{X}, 2) = \prod_{i \in I} C_k(X_i, 2) \) is a \( k_{\mathbb{R}} \)-space by [21, Theorem 5.6]. The second case is immediate.

(2)\( \Rightarrow \) (1) follows from [20]. \( \square \)

**Theorem 3.4.** Let \( X \) be a zero-dimensional metric space. Then the space \( C_k(\mathcal{X}, 2) \) is a k-space if and only if one of the following conditions holds

(a) \( X \) is a topological sum of a separable metrizable locally compact space \( L \) and a discrete space \( D \); in this case \( C_k(\mathcal{X}, 2) \) is the product of a Polish abelian group and a compact abelian group;

(b) \( X \) is not locally compact but the derived set \( X' \) is compact, in this case \( C_k(\mathcal{X}, 2) \) contains an open subgroup which is topologically isomorphic to the direct sum \( \bigoplus_{n \in \mathbb{N}} 2^{\kappa_n} \) endowed with the box topology for some sequence \( \kappa = (\kappa_n)_{n \in \mathbb{N}} \) of cardinal numbers.

**Proof.** Assume that \( C_k(\mathcal{X}, 2) \) is a k-space. Then, by Theorem 3.3, either \( X \) is locally compact or \( X \) is not locally compact but \( X' \) is compact. In the second case \( C_k(\mathcal{X}, 2) \) is a k-space and (b) holds by (3b) of Theorem 3.3.

If \( X \) is locally compact, then \( X \) is a disjoint union of a family \( \{X_i\}_{i \in I} \) of separable metrizable locally compact spaces by [9, 5.1.27]. So

\[
C_k(\mathcal{X}, 2) = \prod_{i \in I} C_k(X_i, 2),
\]

where each space \( C_k(X_i, 2) \) is Polish by Corollary 1.2.

We claim that \( X_i \) is discrete for all but countably many \( i \in I \). Indeed, suppose for a contradiction that \( X_i \) is not discrete for an uncountable subset \( J \) of \( I \). So \( X_i \) contains a clopen infinite compact subset \( K_i, i \in J \). As \( K_i \) is compact and metric, \( C_k(K_i, 2) \) is
discrete and countable, so it is homeomorphic to \(\mathbb{N}\). Since \(C_k(K_i, 2)\) is homeomorphic to a closed subset of \(C_k(X_i, 2)\), we obtain that \(C_k(X, 2)\) contains a closed subspace \(Z\) which is homeomorphic to \([1]^{|J|}\). As \(Z\) is not a \(k\)-space by \([9, 2.7.16]\), we get a contradiction.

Denote by \(L\) the direct topological sum of all non-discrete spaces \(X_i\) and by \(D\) the direct topological sum of all discrete spaces \(X_i\) (if they exist). So \(L\) is a Polish locally compact space, and \(X = L \cup D\) is a topological union of \(L\) and \(D\). Then

\[
C_k(X, 2) = C_k(L, 2) \times 2^D,
\]

where \(C_k(L, 2)\) is a Polish abelian group group by Corollary \([12]\) and \(2^D\) is a compact abelian group.

Conversely, assume that (a) holds and \(X = L \cup D\), where \(L\) is a separable metrizable locally compact space and \(D\) is a discrete space. Then \(C_k(X, 2) = C_k(L, 2) \times 2^D\) is a \(k\)-space by \([9, 3.3.27]\). If (b) holds and \(X\) is not locally compact but the derived set \(X'\) is compact, then \(C_k(X, 2)\) is a \(k\)-space by Theorem \([3.3(3b)]\).

\[\square\]

**Theorem 3.5.** Let \(X\) be a zero-dimensional metric space. Then \(C_k(X, 2)\) is sequential if and only if \(X\) is a Polish space and one of the following conditions holds

(a) \(X\) is not locally compact but the derived set \(X'\) is compact, in this case \(C_k(X, 2)\) has an open subgroup \(H\) which is topologically isomorphic to \((2^\omega)^\infty\) and \(C_k(X, 2)\) is homeomorphic to \((2^\omega)^\infty\);

(b) \(X\) is locally compact, in this case \(C_k(X, 2)\) is a Polish space.

**Proof.** Assume that \(C_k(X, 2)\) is a sequential space. By the proof of Theorem \([3.4]\) we have to cases.

**Case (a):** \(X\) is not locally compact but the set \(X'\) is compact. So \(C_k(X, 2)\) contains an open subgroup \(H\) which is topologically isomorphic to the direct sum \(\bigoplus_{n \in \mathbb{N}} 2^{\kappa_n}\) endowed with the box topology for some sequence \(\kappa = (\kappa_n)_{n \in \mathbb{N}}\) of cardinal numbers, see (3b) of Theorem \([3.3]\). Since every \(2^{\kappa_n}\) is also sequential, we obtain that \(\kappa_n \leq \omega\) for every \(n \in \mathbb{N}\). By the proof of the implication \((3) \Rightarrow (2)\) of Theorem \([3.3]\) this means that \(X \setminus X'\) is countable. Thus \(X\) is Polish. Moreover, for infinitely many indices \(n\), the cardinal \(\kappa_n := |U_n \setminus U_{n+1}|\) is equal to \(\omega\) since, otherwise, the space \(X\) would be locally compact. Since the groups \(2^{\omega}\) and \(\mathbb{Z}(2)^\infty\) are topologically isomorphic for every natural number \(r\), we obtain that the group \(H\) is topologically isomorphic to \((2^\omega)^\infty\). Further, since \(X\) is Polish, \(C_k(X, 2)\) is separable. Hence the discrete group \(S := C_k(X, 2)/H\) is a countable abelian group of order 2. Thus \(S = \bigoplus_{i \in \mathbb{N}} S_i\), where \(S_i = \mathbb{Z}(2)\) for every \(i \in \mathbb{N}\) (see \([10, 11.2]\)). Therefore \(C_k(X, 2)\) is homeomorphic to \(S \times (2^\omega)^\infty\). Since the groups \(S \times (2^\omega)^\infty\), \((\mathbb{Z}(2) \times 2^\omega)^\infty\) and \((2^\omega)^\infty\) are topologically isomorphic, the item (a) is proven.

**Case (b):** \(X\) is locally compact and \(C_k(X, 2) = C_k(L, 2) \times 2^D\), where \(L\) is a Polish locally compact space and \(D\) is a discrete space. Since \(C_k(X, 2)\) is sequential, \(D\) is countable. So \(X\) is a Polish locally compact space and \(C_k(X, 2)\) is a Polish group.

\[\square\]

Now we prove Theorem \([1.4]\).

**Proof of Theorem \([1.4]\).** Items (i)-(iii) follow from Theorems \([3.3]\) and \([3.5]\) respectively. Let us prove (iv).

If \(C_k(X, 2)\) is Fréchet–Urysohn, then \(X\) is a Polish locally compact space and \(C_k(X, 2)\) is Polish by Theorem \([3.5]\) and Corollary \([22]\). If \(X\) is a Polish locally compact space, then \(C_k(X, 2)\) is Fréchet–Urysohn by Corollary \([1.2]\).

For (metrizable) Tychonoff spaces \(X\) and \(Y\), it would be interesting to obtain an analogue of Theorem \([1.4]\) for the spaces \(C_p(X, 2)\) and \(C_p(X, Y)\).

To prove Theorem \([1.6]\), we need the following lemma whose proof is identical with the proof of Lemma 1 in \([22]\). For the sake of completeness we prove this lemma.
Lemma 3.6. Let $X$ be a zero-dimensional metric space such that $X'$ is not Lindelöf. Then $C_k(X, 2)$ and $C_p(X, 2)$ contain $\mathbb{N}^{\omega_1}$ as a closed set.

Proof. As the pointwise topology is weaker than the compact-open one, it is enough to prove that $C_p(X, 2)$ contains $\mathbb{N}^{\omega_1}$ as a closed set. Since $X$ is paracompact and zero-dimensional and $X'$ is not Lindelöf, there exists a family, discrete in the space $X$, of clopen sets $\{F_i : i < \omega_1\}$ and a family $\{z_i : i < \omega_1\}$ such that $z_i \in F_i \cap X'$ for every $i < \omega_1$. Let

$$A_i := \{ f \in C_k(X, 2) : f(X \setminus F_i) = \{0\}\}$$

and

$$A := \left\{ f \in C_k(X, 2) : f(X \setminus \bigcup_{i \in I} F_i) = \{0\}\right\}.$$

Then $A = \prod_{i \in I} A_i$, and since $A$ is closed in $C_p(X, 2)$, it is enough to prove that every $A_i$ contains a discrete closed countable subset. Fix arbitrarily $i < \omega_1$. Take a one-to-one sequence $\{x_{n,i} : n \in \mathbb{N}\} \subseteq F_i$ such that $x_{n,i} \to z_i$ and $x_{n,i} \neq z_i$ for every $n \in \mathbb{N}$, and chose a function $f_{n,i} \in A_i$ such that

$$f_{n,i}(x_{m,i}) = \begin{cases} 0, & m > n, \\ 1, & m \leq n, \quad f_{n,i}(z_i) = 0. \end{cases}$$

Clearly, the open neighborhood $\{ f \in A_i : f(x_{n,i}) = 1, f(x_{n+1,i}) = 0\}$ of $f_{n,i}$ does not contain $f_{k,i}$ for $k \neq n$. So the set $B := \{f_{n,i} : n \in \mathbb{N}\}$ is discrete in $A_i$. Let us show that $B$ is also closed in $A_i$.

Assuming the converse we choose $h \in \overline{B} \setminus B$. Then $h(z_i) = 0$ and the open neighborhood $h + \{\{x_{1,i}, \ldots, x_{n,i}\} : 1/2\}$ of $h$ contains infinitely many $f_{m,i} \in B$. By the construction of $f_{n,i}$ we obtain that $h(x_{n,i}) = 1$ for every $n \in \mathbb{N}$, and hence $h(z_i) = 1$. This contradiction shows that $B$ is closed in $A_i$.

We are ready to prove Theorem 1.6.

Proof of Theorem 1.6. The implications (i)⇒(iv) and (ii)⇒(iv) follow from Lemma 3.6 and from the fact that the space $\mathbb{N}^{\omega_1}$ is not normal by [9, 2.7.16]. The implication (iv)⇒(iii) follows from Proposition 1 of [22]. The implications (iii)⇒(i) and (iii)⇒(ii) are clear. $\square$

Recall that a topological space $X$ has countable tightness at a point $x \in X$ if whenever $x \in A$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$; $X$ has countable tightness if it has countable tightness at each point $x \in X$.

Recall also (see [19]) that a topological space $X$ is called cosmic, if $X$ is a regular space with a countable network (a family $\mathcal{N}$ of subsets of $X$ is called a network in $X$ if, whenever $x \in U$ with $U$ open in $X$, then $x \in N \subseteq U$ for some $N \in \mathcal{N}$). It is trivial that any cosmic space has countable tightness.

Following [3], a family $\mathcal{N}$ of subsets of a topological space $X$ is called a Pytkeev network at a point $x \in X$ if $\mathcal{N}$ is a network at $x$ and for every open set $U \subseteq X$ and a set $A$ accumulating at $x$ there is a set $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite. The space $X$ is called a a Pytkeev network space $\mathcal{N}$ has a countable Pytkeev network. Any $\mathcal{N}$-space is cosmic.

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. (i)⇒(iv) and (iii)⇒(iv): Suppose for a contradiction that $X$ is not separable. Then $X$ has a discrete family $A := \{A_i\}_{i \in \omega_1}$ of clopen subsets by [9, 4.1.15 and 5.1.12] (recall that $X$ is zero-dimensional). Define the monomorphism $T : 2^{\omega_1} \to C(X, 2)$ by

$$T((z_i)_{i \in \omega_1}) := \sum_{i < \omega_1} z_i \chi_{A_i},$$

where $\chi_A$ is the characteristic function of a subset $A$ of $X$. Clearly, $T$ is continuous as a map from the compact group $2^{\omega_1}$ into $C_p(X, 2)$. Also $T$ is an embedding from $2^{\omega_1}$ into $C_k(X, 2)$. \(\square\)
Indeed, since $A$ is discrete, any compact subset $K$ of $X$ intersects only with a finite subfamily $\{A_{i_1}, \ldots, A_{i_m}\}$ of $A$. Then, for every $\epsilon > 0$, we obtain

$$T \left( \prod_{k=1}^{m} \{0,1\} \times 2^{\omega_1 \setminus \{i_1, \ldots, i_m\}} \right) \subseteq [K; \epsilon] \subset C_k(X, 2).$$

Thus $T : 2^{\omega_1} \to C_k(X, 2)$ is an embedding.

So $C_k(X, 2)$ and $C_p(X, 2)$ contain an isomorphic copy of the compact abelian group $2^{\omega_1}$. As $2^{\omega_1}$ is not metrizable, $2^{\omega_1}$ and hence also $C_k(X, 2)$ and $C_p(X, 2)$ have uncountable tightness by Corollary 4.2.2 in [2]. This contradiction shows that $X$ is separable.

(iv)⇒(ii): If $X$ is separable, then $C_k(X, 2)$ is a $\Psi_0$-space by [3] (see also [11 Corollary 6.4]). (ii)⇒(i) is clear, see [11]. (iv)⇒(iii): By Proposition 10.4 of [19], the space $C_p(X, 2)$ is cosmic, so it has countable tightness. □

**Remark 3.7.** R. Pol and F. Smentek [21] proved the following interesting result: If $X$ is a zero-dimensional realcompact $k$-space, then the group $C_k(X, 2)$ is reflexive. Note also that, for the group of rational numbers $\mathbb{Q} \subset \mathbb{R}$ and the convergent sequence $s$, the metric spaces $\mathbb{Q} \times \omega_1$ and $s \times \omega_1$ are realcompact by [9, 3.11.5 and 5.5.10(b)]. These results and Theorems 1.4 and 1.6 show the following: (i) the group $C_k(\mathbb{Q} \times \omega_1, 2)$ is reflexive but it is neither normal nor Ascoli, (ii) the group $C_k(\mathbb{Q}, 2)$ is a reflexive $\Psi_0$-group but is not Ascoli, (iii) the group $C_k(s \times \omega_1, 2)$ is a reflexive $k_\mathbb{R}$-space but is not normal.

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