How Important Is Secular Evolution for Black Hole and Neutron Star Mergers in 2+2 and 3+1 Quadruple-star Systems?

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Abstract

Mergers of black holes (BHs) and neutron stars (NSs) result in the emission of gravitational waves that can be detected by LIGO. In this paper, we look at 2+2 and 3+1 quadruple-star systems, which are common among massive stars, the progenitors of BHs and NSs. We carry out a detailed population synthesis of quadruple systems using the Multiple Stellar Evolution code, which seamlessly takes into consideration stellar evolution, binary and tertiary interactions, N-body dynamics, and secular evolution. We find that, although secular evolution plays a role in compact object (BH and NS) mergers, (70–85)% (depending on the model assumptions) of the mergers are solely due to common envelope evolution. Significant eccentricities in the LIGO band (higher than 0.01) are only obtained with zero supernova (SN) kicks and are directly linked to the role of secular evolution. A similar outlier effect is seen in the $\chi_{eff}$ distribution, with negative values obtained only with zero SN kicks. When kicks are taken into account, there are no systems that evolve into a quadruple consisting of four compact objects. For our fiducial model, we estimate the merger rates (in units of Gpc$^{-3}$ yr$^{-1}$) in 2+2 quadruples (3+1 quadruples) to be $10.8 \pm 0.9$ ($2.9 \pm 0.5$), $5.7 \pm 0.6$ ($1.4 \pm 0.4$), and $0.6 \pm 0.2$ ($0.7 \pm 0.3$) for BH–BH, BH–NS, and NS–NS mergers, respectively. The BH–BH merger rates represent a significant fraction of the current LIGO rates, whereas the other merger rates fall short of LIGO estimates.

Unified Astronomy Thesaurus concepts: Binary stars (154); Multiple stars (1081); Stellar dynamics (1596); Stellar evolution (1599); Gravitational waves (678); Black holes (162); Neutron stars (1108)

1. Introduction

In the past few years, there have been extensive studies of gravitational wave (GW) sources and their progenitors. These have been motivated by recent detections of GWs by LIGO/VIRGO, starting in 2015. Abbott et al. (2021a) introduced the second and latest version of the Gravitational Wave Transient catalog (GWTC-2), which also includes the GW detections from the previous catalog (GWTC-1) of Abbott et al. (2019).

GWs are emitted during the merger of neutron stars (NSs) and black holes (BHs). These compact objects are the final stages in the evolution of massive stars ($\gtrsim 8$ $M_\odot$, assuming solar metallicity and single-star evolution). Hence, for individual BHs and NSs to merge, the progenitor massive stars must avoid merging before evolving into a compact object binary. Various stages in a star’s life (radius expansion in giant phases, mass-loss due to stellar winds, external encounters, supernova (SN) kicks) tend to destroy binary systems before compact object formation. Therefore, systems with binary BHs and NSs are expected to be very rare. Mergers within a Hubble time ($\sim 14$ Gyr) are even rarer.

Any proposed channel for the merger of compact objects (henceforth used to refer only to BHs and NSs, and not white dwarfs (WDs)) must, hence, explain the presence of such systems and their merger within a Hubble time. There have been a number of merger channels proposed in the recent past, which can be divided into: (1) isolated binary evolution (e.g., Belczynski et al. 2002, 2016; Dominik et al. 2012; de Mink & Mandel 2016; Chruslinska et al. 2018; Giacobbo & Mapelli 2018; Giacobbo et al. 2018; Spera et al. 2019); (2) dynamical interactions in star clusters (e.g., Portegies Zwart & McMillan 2000; Banerjee et al. 2010; Rodriguez et al. 2016; Ba
erjee 2017; Chatterjee et al. 2017; Hamers & Samsing 2019), galactic nuclei (e.g., O’Leary et al. 2009; Antonini & Perets 2012; Antonini & Rasio 2016; Petrovich & Antonini 2017; Arca-Sedda & Gualandris 2018; Hamers et al. 2018; Hoang et al. 2018; Fragione et al. 2019), and triple and quadruple systems (e.g., Antonini et al. 2017; Silsbee & Tremaine 2017; Liu & Lai 2018, 2019; Fragione & Kosinski 2019; Fragione & Loeb 2019; Fragione et al. 2020; Arca Sedda et al. 2021; Hamers et al. 2021a); (3) in active galactic nucleus (AGN) disks (e.g., Bartos et al. 2017; Stone et al. 2017; McKernan et al. 2018; Secunda et al. 2019; Tagawa et al. 2020); or (4) primordial BH mergers (e.g., Bird et al. 2016; Sasaki et al. 2016; Ali-Haïmoud et al. 2017; Raidal et al. 2017).

Compact object mergers in triples are interesting for several reasons. First, studies (Raghavan et al. 2010; Moe & Di Stefano 2017) have found that massive stars, which are the progenitors of BHs and NSs, are most likely found in high-multiplicity star systems. Moe & Di Stefano (2017) showed that for stellar systems in the field with primary components more massive than 10 $M_\odot$, the triple and quadruple fractions each exceed 20%. Second, the presence of companion stars can significantly affect the dynamics of triple- and quadruple-star systems. Unlike isolated binaries, triple-star systems can undergo eccentricity enhancements (if mutual inclinations are large) in the inner binary orbits due to the presence of tertiary companions. These perturbations, to the lowest order, are known as Lidov–Kozai (LK) oscillations1 (Kozai 1962; Lidov 1962; see for reviews Naoz 2016; Shevchenko 2017).

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1 Also referred to as von Zeipel–Lidov–Kozai oscillations after Ito & Ohtsuka (2019) noted the contribution of von Zeipel (1910).
LK oscillations can accelerate compact object mergers since enhanced eccentricity shortens the coalescence time due to GW-driven orbital inspiral (Blaes et al. 2002; Thompson 2011). Thus, the study of GW progenitors is incomplete without accounting for multiple-star systems.

Population synthesis is a useful tool to study the statistical properties of such systems. Antonini et al. (2017) used a population synthesis code TRES (Toonen et al. 2016) to combine the effects of orbital dynamics and stellar evolution, in order to estimate merger rates in triples. They derived BH–BH merger rates of \( \sim (0.3–1.3) \text{Gpc}^{-3} \text{yr}^{-1} \), and showed that mergers from the triple channel have much higher eccentricities in the LIGO band (10 Hz) compared to the isolated binary channel.

In this paper, we concentrate on quadruple-star systems. Quadruples allow for a larger parameter space than triples for eccentricity excitation due to secular (long-term) evolution (Pejcha et al. 2013; Hamers et al. 2015; Hamers 2017, 2018a, 2019; Hamers & Lai 2017; Grishin et al. 2018; Liu & Lai 2019). Smaller mutual inclinations can lead to chaotic behavior and extreme eccentricity enhancements if various secular timescales are commensurate. This, coupled with the fact that massive stars are likely to reside in triple and quadruple systems, justifies a detailed investigation into quadruples.

Based on their hierarchical configuration, quadruples can be classified into two types: (1) the 2+2, where two binaries orbit each other; and (2) the 3+1, where a triple system is orbited by a distant fourth body (see Figure 1).

There have been a few studies on 2+2 quadruples. Fragione & Kocsis (2019) carried out a population synthesis study assuming four BHs in a 2+2 configuration. In reality, the survival of bound quadruple systems consisting of four BHs is rare since gravitational dynamics and stellar evolution tend to destabilize orbits. SN natal kicks are a major cause of the destruction of potential BH quadruple systems. Therefore, a thorough study self-consistently should combine both these effects to predict merger rates. Nevertheless, the authors found that merger fractions in quadruples can be \( \sim (3–4) \) times higher than in triples. Thus, the quadruple channel cannot be ignored. Hamers et al. (2021a) performed a simplified evolution of 2+2 quadruples, where two binaries are evolved independently, and secular evolution is considered only after compact object formation. They inferred a compact object merger rate of \( \sim (10–100) \text{Gpc}^{-3} \text{yr}^{-1} \), which they mentioned is most likely an overestimation.

In this paper, we go a step further and use the recently developed population synthesis code Multiple Stellar Evolution (MSE), which combines stellar evolution, binary interaction, gravitational dynamics, fly-bys in the field, and other processes seamlessly (Hamers et al. 2021b). Additionally, we study both 2+2 and 3+1 quadruples and compare their merger rates with other channels.

The structure of this paper is as follows. Section 2 describes the methods used, Section 3 has a few examples of compact object mergers in quadruple systems selected from a large set of population synthesis calculations, Section 4 discusses the initial conditions and assumptions made in the population synthesis in detail, Section 5 presents the results, Section 6 contains a discussion, and Section 7 concludes.

2. Methods

For our population synthesis simulations, we use the MSE code (Version 0.84), described thoroughly in Hamers et al. (2021b). A brief overview of MSE is given in the following sub-sections.

2.1. Gravitational Dynamics

MSE uses two methods to model the dynamics of a multi-body system: secular and direct N-body integration. The code dynamically switches between these two modes depending on the stability of the configuration at a given time step.

The secular (orbit-averaged) approximation is used when the orbit is dynamically stable. It is faster than direct N-body integration since an orbit-averaged and expanded Hamiltonian is used and orbital phases are not resolved. MSE uses the code SECULARMULTIPLE (Hamers & Portegies Zwart 2016; Hamers 2018b, 2020) for this purpose. Tides are included following the equilibrium tide model (Hut 1981; Eggleton et al. 1998, with the efficiency of tidal dissipation determined by the prescription of Hurley et al. 2002). Post-Newtonian (PN) terms up to orders 1PN and 2.5PN (ignoring orbit–orbit interaction terms) are considered in the secular integration mode.

In certain situations, however, the secular approximation can break down. This can be due to changes in the orbital parameters due to wind mass-loss from evolving stars, SN natal kicks, fly-bys, or secular evolution in multiple-star systems (the last applies particularly to 3+1 quadruple systems). The code then switches to the direct N-body integration mode, where Newton’s equations of motions are solved using the code

![Figure 1. Mobile diagrams of the two types of quadruples, with a triple for comparison. Here, the \( m_i \) are masses and \( b_i \) are nested binaries.](image-url)
MSTAR (Rantala et al. 2020). When the switch occurs, the positions and velocities of all bodies are computed under the assumption that the mean anomalies of all orbits evolve linearly with time. The MSTAR code uses algorithmic chain regularization for highly accurate integration for a wide range of mass ratios. It includes PN terms, although tidal effects are currently not included.

MSE switches modes from secular to direct N-body in the following cases.

1. The system becomes dynamically unstable according to the stability criterion of Mardling & Aarseth (2001).
2. The system enters the “semisecular regime” (Antonini & Perets 2012; Luo et al. 2016; Grishin et al. 2018; Lei et al. 2018; Hamers 2020) i.e., the timescale of angular momentum change due to secular evolution is shorter than the orbital timescale.
3. Any orbits become unbound due to SN kicks or mass-loss.
4. After common envelope (CE) evolution and directly following collisions.

The code switches back to the secular mode if it is deemed stable (see Hamers et al. 2021b for details).

2.2. Stellar Evolution

The evolution of isolated stars follows the stellar tracks from the code Single Star Evolution (SSE; Hurley et al. 2000). The evolution track of a star with given mass and metallicity is fit analytically from a grid of pre-computed tracks of standard masses and metallicities. MSE uses SSE at each time step, and the orbital response to stellar mass-loss is calculated, assuming adiabatic wind mass-loss. When a star evolves into an NS or a BH, MSE accounts for the mass-loss (assumed to be instantaneous, and with no feedback of the mass lost on the rest of the system) and any natal kicks from the SN explosion.

In multiple-star systems, however, interactions between stars can become important and binary evolution can play an important role. Many of the assumptions for mass transfer and CE evolution in MSE are based on the code Binary Star Evolution (Hurley et al. 2002). An exception to this is the way that mass transfer is treated in eccentric orbits. Instead of enforcing circular orbits at the onset of mass transfer, we assume the following model for the secular orbital changes due to mass transfer in an orbit $k$:

$$\frac{\dot{a}_k}{a_k} = -2\dot{m}_d \frac{1 - \beta m_d}{m_d} \sqrt{1 - e_k^2}$$

$$\dot{e}_k = -2\dot{m}_d \frac{1 - \beta m_d}{m_d} \sqrt{1 - e_k^2} (1 - e_k) e_k$$

where $a_k$ and $e_k$ are the orbital semimajor axis and eccentricity, respectively, $m_d$ and $m_a$ are the donor and accretor mass, respectively, and $\beta$ is the mass transfer efficiency. This model is adopted from Sepinsky et al. (2007) and Dosopoulou & Kalogera (2016), ignoring finite-size effects, and with an additional factor of $e_k$ in the equation for $\dot{e}_k$ to resolve the problem that the equations of motion would otherwise break down as the orbit circularizes due to mass transfer. Our model, although simplified, accommodates the onset and self-consistent treatment of eccentric mass transfer in multiple-star systems.

In addition, MSE includes prescriptions for triple mass transfer and triple CE evolution in the case when an outer star fills its Roche lobe around an inner binary, motivated by more detailed simulations. It also takes into consideration the effect of fly-bys in the field, under the impulsive approximation (see Hamers et al. 2021b for details).

3. Examples

In this section, we provide examples of quadruple systems that undergo compact object mergers, using the MSE code. These examples are taken from the population synthesis simulations (Section 4). It is important to note that the same systems can evolve differently if different random numbers are generated by the code (for example, the magnitudes and directions of the SN natal kicks). We describe three qualitatively different scenarios of mergers, in 2+2 quadruples, with examples 1–3. For completeness, we also provide an example of a 3+1 quadruple system undergoing a merger (Scenario 3). Other examples of mergers are briefly mentioned in Section 6.

1. **Scenario 1.** Only CE evolution (most of the cases). Figure 2 (Model 0) shows a 2+2 quadruple with the inner binaries having relatively small initial semimajor axes ($a_{in} \approx 17$ au and $a_{in} \approx 0.1$ au), but a much larger outer separation ($a_{out} \approx 3509$ au, with a periapsis of 807 au). Owing to the very hierarchical configuration of this quadruple, the inner binaries more or less evolve independently of each other. The “interesting” inner binary, in which the merger occurs, has two massive stars of masses $16 M_\odot$ and $17 M_\odot$. Within $t \approx 12.2$ Myr, both stars reach their giant phases, and a Roche lobe overflow (RLOF) event ensues shortly. After two SN explosions (which unbind the two binaries from each other, but not the binaries themselves) and CE evolution, the separation between the two new NSs reduces significantly. Finally, at $t \approx 17.4$ Myr, the NS–NS binary merges due to GW emission. In the meantime, the two stars in the other binary evolve into giants, go through a CE phase of their own, and merge into a single massive star. Eventually, this star evolves into an NS. The bound 2+2 quadruple therefore finally evolves into two single compact objects.

2. **Scenario 2.** Only secular evolution (extremely rare, only possible with zero SN kicks). Figure 3 (Model 1) shows a 2+2 quadruple with inner binaries with semimajor axes $a_{in} \approx 39$ au and $a_{in} \approx 204$ au (but very high eccentricity of 0.95), and an outer orbit with semimajor axis $a_{out} \approx 3142$ au and a periapsis of 1288 au. In the evolution of this system, there are RLOF events, but no CE event to bring inner companions close to each other. The “interesting” binary has two very massive stars of $41 M_\odot$ and $37 M_\odot$. Since SN kicks are disabled in this model, the quadruple system remains bound with three BHs and one low-mass main-sequence (MS) star at $t \approx 6.0$ Myr. By this time, the BH–BH inner binary is still wide, with a semimajor axis of 690 au and a periapsis of 90 au. This is too wide for a merger in a Hubble time. However, secular evolution now shows its capability. The orbit eccentricity is enhanced significantly from 0.87 to almost 1.0 ($1 - e \approx 5 \times 10^{-7}$) after $t \approx 143$ Myr, which leads to a secular breakdown. The two BHs end up making extremely close passes to each other for many
millions of years and ultimately collide at $t \approx 1157$ Myr. The LIGO band eccentricity $e_{\text{LIGO}} = 5 \times 10^{-3}$ is higher than that of Scenario 1. The GW recoil unbinds the merger remnant from the companion binary without strongly interacting with it (i.e., the scenario proposed by Fragione et al. 2020 and Hamers et al. 2021a does not occur here). The companion binary finally becomes a wide BH+WD binary without further interaction (when it was still bound to its companion, it experienced eccentricity excitation).

3. **Scenario 3.** (2+2 quadruple) Mixture of both (intermediary in occurrence; see Section 5.4). Figure 4 (Model 0) shows a 2+2 quadruple with wide inner binaries (semimajor axes of $a_{\text{in}} \approx 217$ au and $a_{\text{in}} \approx 405$ au) with a large outer separation ($a_{\text{out}} \approx 5296$ au, with a periapsis of 2224 au). The mutual inclination between the inner and outer orbits is high enough to excite the inner eccentricities from $e_{\text{in}} \approx 0.04$ to 0.60 and from $e_{\text{in}} \approx 0.41$ to 0.96 respectively. Chaotic quadruple secular evolution could also play a role here since the two inner semimajor axes are not too distinct from each other (so the LK timescales are similar). This eccentricity enhancement is a key factor for the eventual compact object merger. The periapsis of the “interesting” binary (with masses $41M_\odot$ and $17M_\odot$) reduces drastically from 243 au to a mere $18$ au, which is close enough for RLOF and CE evolution to occur. This is what happens after $t \approx 4.8$ Myr, and the orbit shrinks even further and circularizes. Again, SN kicks unbind the quadruple companions, but the inner binary remains intact. The final CE phase brings the two objects to a close separation of $0.01$ au, and a BH–NS merger occurs at $t \approx 13.0$ Myr. Meanwhile, the other inner binary does not survive an SN kick when its more massive star evolves into a BH. The
two stars then evolve independently, with the companion becoming a high-mass O–Ne WD.

4. Scenario 3 (3+1 quadruple) Figure 5 (Model 0) shows a 3+1 quadruple with a very close inner binary ($a_{in} \approx 0.6$ au), a fairly distant intermediate star ($a_{mid} \approx 72$ au) and a very distant, but very eccentric, outer star ($a_{out} \approx 8014$ au, with a periapsis of 480 au). Secular evolution excites the intermediate eccentricity from a low 0.12 to a very high 0.95, with the intermediate periapsis decreasing from 64 au to just 4 au. This triggers a phase of dynamical instability; due to the tight inner semimajor axis, the massive inner stars (of masses $25 M_e$ and $20 M_e$) collide early and become a 44 $M_e$ star. The resulting system is a triple, with a high inner (previously intermediate) eccentricity. The previously intermediate 20 $M_e$ star becomes a giant and transfers mass to its very massive companion. Note that this type of evolution (the onset of RLOF of a star onto a companion which is twice as massive) is not expected in isolated binary evolution, and is unique to higher-order multiple systems. After another dynamical instability phase, a CE event follows and the two stars end up in a near-circular orbit with a semimajor axis $0.01$ au. The inner stars survive two SN events (although the outer companion is kicked out), become a BH–NS binary, and merge within $t \approx 13$ Myr. Meanwhile, the ejected outer 8 $M_e$ star eventually becomes an NS.

4. Population Synthesis

We perform a population synthesis of both 2+2 and 3+1 quadruples. Each system is run for 14 Gyr or a Hubble time. In the case of the 2+2 quadruples, we also compare results with isolated binaries having the same initial conditions as the inner binaries of the quadruples. Hence, we have twice as many
isolated binaries as 2+2 quadruples for comparison. It is important to note that these isolated binaries are not distributed like real binaries in the field, mainly due to the quadruple stability requirements. The latter implies that the inner binaries in 2+2 quadruples are, on average, more compact compared to “truly” isolated binaries in the field. We include this additional set to directly investigate the effect of secular evolution in quadruples.

For each of the three above-mentioned cases, we use seven different models which vary certain distributions or parameters (see Section 4.1 for details). In each model, we run $10^5$ quadruples (or $2 \times 10^5$ binaries in the isolated binaries cases). Hence, in total, we run $7 \times 10^5$ 2+2 and 3+1 quadruples each, and $14 \times 10^5$ isolated binaries, making a total of $2.8 \times 10^6$ systems.

The following sub-sections describe the different models used, the initial conditions, and the other important parameters.

### 4.1. Different Models

To better comprehend our results, we vary different parameters to see how they affect compact object merger rates. We use seven different models, numbered 0, 1, 2, 3a, 3b, 4a, 4b...
Model 0 is taken to be our fiducial model, and the others are compared against it. In Model 1, SN kicks during NS and BH formation are excluded. In Model 2, fly-bys, which are enabled by default, are excluded. In Models 3a and 3b, the stellar metallicities $Z$ are changed from solar ($Z_{\odot} = 2 \times 10^{-2}$) to one-tenth and one-hundredth of solar metallicity respectively. In Models 4a and 4b, the CE mass-loss timescale $t_{CE}$, which is chosen to be $10^3$ yr by default, is varied to one-tenth and $10$ times that of the default respectively. $t_{CE}$ parameterizes the timescale at which mass is lost during a CE event. Specifically, if $t_{CE}$ is short compared to the orbital period of the companions to the two stars undergoing CE evolution, then the mass-loss can be considered to be instantaneous for these orbits, and the companions can become unbound. In contrast, if $t_{CE}$ is long compared to the orbital motion of the outer companions, the effect can be considered to be adiabatic and the outer orbits remain bound and become wider. In MSE, these regimes, and the transitional regime, are taken into account by carrying out short-term $N$-body integrations in which the mass of the binary undergoing CE evolution (modeled as a point mass) gradually loses mass (see Hamers et al. 2021b for details).

Here, we stress that the parameters discussed above are among the many which could have been altered. For example,

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Here, we stress that the parameters discussed above are among the many which could have been altered. For example,
the CE evolution prescription, the α-CE model (Paczynski 1976; van den Heuvel 1976; Webbink 1984; Livio & Soker 1988; de Kool 1990; see for a review Ivanova et al. 2013), is uncertain. In this paper, we choose to alter $t_{\text{CE}}$ since this effect has not been studied much, whereas it can determine whether or not outer companions remain bound after a CE event (e.g., Michaely & Perets 2019). However, other parameters, most notably the $a_{\text{CE}}$ parameter, also significantly affect binary evolution and compact object merger rates in particular (Dominik et al. 2012; Broekgaarden et al. 2021; Fragione et al. 2021). Another example is the choice of a model for the SN kick distribution. We choose a Maxwell–Boltzmann distribution for both NSs and BHs. While the NS kick distribution has been widely studied (e.g., Hobbs et al. 2005), the BH kick distribution is largely unknown. Other assumptions we make include the initial distributions for stellar masses, semimajor axes, and eccentricities. They are still poorly constrained for quadruples; here, we chose to focus on model assumptions, rather than different initial conditions.

The parameters varied in the different models are given in Table 1. The details of important parameters are given in Section 4.2.

4.2. Initial Conditions

For each of the quadruple systems, we sample four zero-age main-sequence (ZAMS) masses $m_i$, and three each of eccentricities $e_i$, semimajor axes $a_i$, orbital inclinations $i_i$, and arguments of periastron $\omega_i$. Stellar metallicities have constant values in each model, as given in Table 1. After sampling, we check for dynamical stability of the orbital configuration using the triple stability criterion described by Mardling & Aarseth (2001). For 2+2 quadruples, the stability criterion is evaluated for both inner binaries, considering the other binary to be the “tertiary” star. In the case of 3+1 quadruples, the two “triple” systems are: (1) the innermost binary, with the intermediate star as the companion; and (2) the intermediate star–inner two-star binary, with the outermost star as the companion. Furthermore, we also check that the ZAMS stars (with a mass–radius relation given by $m \sim R^{3/7}$, in solar units) do not fill their Roche lobes at periastron (approximate Roche lobe radii are given by Eggleton 1983). If any of these criteria are not fulfilled, the sampling is restarted.

1. Mass sampling. The primary mass (most massive star in the quadruple system) $m_{\text{pri}}$ is sampled from a Kroupa distribution ($dN/dm \propto m^{-2.3}$) in the high-mass tail; Kroupa (2001). However, we put an additional constraint that $m_{\text{pri}} > 8 M_\odot$, which is justified since we look specifically for BH and NS mergers. In principle, it could be possible to form BHs and NSs in multiple-star systems with primary masses $m_{\text{pri}} < 8 M_\odot$; if stellar mergers occur during the evolution, e.g., due to secular eccentricity excitation. However, in post-processing, we find zero compact object mergers in systems with primary masses lower than 10 $M_\odot$. The maximum possible mass of each star is 100 $M_\odot$, a constraint set by SSE.

In the case of 2+2 quadruples, the secondary mass (immediate companion star of the primary) $m_{\text{sec}}$ is sampled from a flat mass ratio distribution $\sim U(0, 1)$ with respect to the primary ($q = m_{\text{sec}}/m_{\text{pri}} < 1$). The companion binary is sampled similarly with respect to the primary mass.

2. Semimajor axis sampling. The $a_i$ are sampled independent of each other, from a log flat distribution (Sana et al. 2012), ranging from $10^{-2}$ to $10^4$ au. However, the final stability check is biased toward more hierarchical orbits, which is seen in Figures 8 (for 2+2 quadruples; all models and 9 (for 3+1 quadruples; all models).

3. Eccentricity sampling. The $e_i$ are also sampled independent of each other, from a flat distribution (Duchêne & Kraus 2013), ranging from 0.01 to 0.99. The final stability check is biased toward lower eccentricities, especially the outer ones, which is seen in Figures 8 (for 2+2 quadruples; all models) and 9 (for 3+1 quadruples; all models). The flat distribution of eccentricities is chosen over the commonly adopted thermal distribution (Jeans 1919).

4. Orbital angle sampling. The orbital angles are sampled such that the orbits are distributed isotropically in 3D space. The $i_i$ are hence distributed uniformly in $\cos i$.
whereas the $\Omega_i$ and the $\omega_i$ are distributed randomly from 0 to $2\pi$.

Studies have shown that orbital alignment in multiple-star systems is not always isotropic. Tokovinin (2017) showed that, while low-mass triples with wider outer orbits are nearly isotropic, tighter triples are more aligned to an orbital plane. However, the study also showed that high-mass triples do not show as significant an alignment. Hence, our assumption of isotropy is justified.

4.3. Other Parameters and Assumptions

There are various parameters, other than the initial conditions, which significantly affect MSE’s results. Since MSE does not model the detailed stellar structure, many parameter values are either prescription-dependent or assumed. Some of the important parameters, and the assumptions involved, are mentioned below.

1. Supernova kicks. The default SN kick distribution in MSE is Maxwellian for both NSs and BHs. Here, $\sigma_{\text{NS}} = 265$ km s$^{-1}$ (Hobbs et al. 2005) and $\sigma_{\text{BH}} = 50$ km s$^{-1}$. In our models with SN kicks (all except Model 1), we use the default distribution.

2. Fly-bys. MSE samples stars passing by the multiple-star system assuming a homogeneous stellar background of solar density ($n = 0.1$ pc$^{-3}$) and a Maxwellian distribution of stellar velocities, with dispersion $\sigma_v = 30$ km s$^{-1}$, consistent with the solar neighborhood (Binney & Tremaine 2008; Hamers & Tremaine 2017). We adopt
Table 2
Initial Conditions and Parameters

| Quantity                          | Distribution or Value                                      |
|-----------------------------------|------------------------------------------------------------|
| Masses $m$                        | Primary from Kroupa distribution (>8 $M_\odot$); others as mass ratios of primary |
| Metalicities $Z$                  | 0.02 (changed in models 3a and 3b)                        |
| Semimajor axes $a$                | Log flat distribution ($10^{-2}$–$10^4$ au); only stable systems |
| Eccentricities $e$                | Flat distribution (0.01–0.99); only stable systems         |
| Inclinations $i$                  | Flat in cos $i$                                            |
| Longitudes of ascending node $\Omega$ | Flat distribution (0 – 2$\pi$)                              |
| Arguments of periapsis $\omega$   | Flat distribution (0 – 2$\pi$)                              |
| Supernova kicks                   | Maxwellian distribution; $\sigma_{NS}$ = 265 km s$^{-1}$ and $\sigma_{BH}$ = 50 km s$^{-1}$ (0 for model 1) |
| Fly-bys                           | $n_i$ = 0.1 pc$^{-3}$; $\sigma_*$ = 30 km s$^{-1}$ (Maxwellian); $R_{enc}$ = $10^4$ au (0 for model 2) |
| CE parameters                     | $\alpha_{CE} = 1$; $\tau_{CE} = 10^1$ yr (changed in models 4a and 4b) |

Note. See the text for detailed descriptions.

an encounter sphere radius of $R_{enc} = 10^5$ au, with the perturber masses following the Kroupa distribution. Only impulsive encounters (the orbital velocity is much lower than the velocity of the external star) are assumed to affect the orbits since the effects of secular encounters are usually unimportant in low-density systems in the field (this is different for dense stellar systems, see Heggie 1975; Heggie & Rasio 1996; Hamers & Sanders 2019; and also Section 6.2). In our models with fly-bys (all except Model 2), we use this method.

3. CE parameters. MSE uses the energy argument-based $\alpha$-CE prescription. The common envelope efficiency $\alpha_{CE} = 1$ by default. In this paper, we use three different values for the CE mass-loss timescale $\tau_{CE}$, as seen in Table 1.

4. Collisions. In MSE, a “collision” between stars is assumed to have occurred when their mutual separation is less than the sum of their effective radii. The effective radius is the same as the stellar radius for non-compact objects, whereas it is a factor 100 more for compact objects. This is done since it is computationally expensive to integrate the equations of motion just before a compact object merger, and justified by the very short remaining merger time (see also Hamers et al. 2021b, Equation (108)).

Table 2 summarizes the initial conditions and adopted parameters.

5. Results

In Section 5.1, we define quantities of interest pertaining to compact object mergers. In Section 5.2, we present the compact object merger numbers for both 2+2 quadruples and isolated binaries, for direct comparison. In Section 5.3, we do the same for 3+1 quadruples. In Sections 5.4 and 5.5, we discuss in detail merger scenarios and rates respectively. Finally, in Section 5.6, we talk about systems which are ignored in this study.

5.1. Definitions of Certain Quantities

One of the most important features of a compact object merger is the eccentricity $e_{LIGO}$ in the LIGO band. First, we have the analytical relation between $a$ and $e$ due to GW emission given by Peters (1964):

$$a(e) = C_0 \frac{e^{12/19}}{1 - e^2} \left[ 1 + \frac{121 e^2}{304} \right]^{870/2299}$$

(2)

where $C_0$ depends on the initial values $a_0$ and $e_0$. We also have a relation for the GW peak frequency for given $a$, $e$, and total mass $M = m_1 + m_2$ from Wen (2003):

$$f_{GW}(a, e, M) = \frac{\sqrt{GM}}{\pi} \left( \frac{1 + e}{1 - e^2} \right)^{1.1954}$$

(3)

Using these equations and adopting $f_{LIGO} = 10$ Hz, we can calculate $e_{LIGO}$.

Another feature of a LIGO detection is the effective spin parameter $\chi_{eff}$:

$$\chi_{eff} = \chi_1 m_1 (\hat{S}_1 \cdot \hat{L}) + \chi_2 m_2 (\hat{S}_2 \cdot \hat{L})$$

$$m_1 + m_2$$

(4)

where $\hat{S}_1$ and $\hat{S}_2$ are the unit spin angular momentum vectors of the two compact objects with $\chi_i = c ||S_i|| / Gm_i^2$ lying between 0 and 1 ($c$ and $G$ have their usual meaning), and $\hat{L}$ is the unit Newtonian orbital angular momentum. We assume that the spins during compact object formation $\chi_1$ and $\chi_2$ are sampled uniformly between 0 and 1. It should be noted that assuming a different range for $\chi$ results only in a horizontal re-scaling, while the distribution shape remains the same (e.g., Hamers et al. 2021a show this for the range 0–0.1).

We also calculate another spin parameter, the spin precession parameter $\chi_P$ (Schmidt et al. 2015; Abbott et al. 2020):

$$\chi_P = \max \{ \chi_1 ||\hat{S}_{1\perp}||, \kappa \chi_2 ||\hat{S}_{2\perp}|| \}$$

(5)

where $\hat{S}_{i\perp} = \hat{S}_i - (\hat{S}_i \cdot \hat{L}) \hat{L}$ (component of $\hat{S}_i$ perpendicular to $\hat{L}$) and $\kappa = \frac{q (q + 3)}{4 + 3q}$.

A final important quantity is the merger mass ratio $q = m_2 / m_1$, where $m_1$ and $m_2$ are the heavier and lighter compact object masses respectively ($0 < q \leq 1$).

5.2. 2+2 Quadruples and Isolated Binaries

Table 3 shows the number of compact object mergers, and the Poisson errors, in $10^5$ and $2 \times 10^5$ sampled systems, for each model of 2+2 quadruples and isolated binaries, respectively. It also shows the merger rates for the 2+2 quadruples. It is important to note that these isolated binaries
are not distributed like “real” binaries, and thus it is irrelevant to consider their rates. Their numbers should be considered only as a direct comparison to $2+2$ quadruples, and not out of this context.

The table shows that the corresponding number of mergers in both cases (bound versus unbound $2+2$ quadruples) are mostly within the Poisson error margin of each other. However, we can see that isolated binaries consistently produce a higher number of mergers in all models except Model 1 (where SN kicks are disabled). This may be due to secular evolution driving mergers in $2+2$ quadruples in the pre-compact object phases. For example, an isolated inner binary of a $2+2$ quadruple could lead to a BH–BH merger, whereas the same inner binary in a bound system could see a merger before the component stars evolve into BHS due to eccentricity excitation.

Let us look at the numbers for each model in detail.

1. **Model 0.** The number of BH–BH mergers is higher than that of BH–NS mergers, which in turn is significantly higher than that of NS–NS mergers. The very low merger rates for NS–NS binaries can be attributed to their high SN kicks, which tend to unbind orbits.

2. **Model 1.** Excluding SN kicks has a drastic effect on the number of compact object mergers, especially the NS–NS mergers, for both $2+2$ quadruples and isolated binaries. This is expected since SN kicks almost always result in the unbinding of orbits, more so for NSs since their kick distribution has a higher $\sigma$ than that of BHs. Moreover, in $2+2$ quadruples, having no SN kicks means that there can be secular evolution even in the compact object phase. This eccentricity excitation can lead to much higher $e_{\text{LIGO}}$ values and negative $\chi_{\text{eff}}$ values. These outlier systems can be seen in Figures 10 and 11(b) and (f) (Model 1 quadruples), but not in (a) and (e) (Model 0 quadruples) or (d) (Model 1 binaries).

3. **Model 2.** Excluding fly-bys systematically reduces the total number of mergers. This is most prominent in the BH–BH merger numbers for both $2+2$ quadruples and isolated binaries. The similar numbers for BH–NS and NS–NS mergers (in the quadruples case) may be attributed to the higher SN kicks for NSs, which can diminish the effect of fly-bys. It is important to note that, while an external perturbation can destabilize a wide orbit, it can also decrease the outer periapsis distance, triggering stronger secular evolution with its inner orbits. In short, the general consequence of fly-bys is not immediately clear. Parameter distributions for Model 2 are roughly similar to those from Model 0.

4. **Models 3a and 3b.** Reducing $Z$ from the default $Z_{\odot} = 2 \times 10^{-2}$ significantly increases the number of mergers. This is expected since lower-$Z$ stars are more compact than higher-$Z$ stars, thereby reducing the chances of pre-compact object phase merger events. Moreover, the maximum BH masses are also significantly higher than those in Model 0 since lower-$Z$ stars lose less mass due to stellar winds. This effect is seen by comparing Figures 14(b) (Model 3a) and (c) (Model 3b) with (a) (Model 0).

5. **Models 4a and 4b.** Changing $t_{\text{CE}}$ from the default $10^{3}$ yr does not seem to change the overall statistics of the mergers. This may be somewhat unexpected since $t_{\text{CE}}$ affects the number of bound stars after CE evolution. However, since secular evolution does not play a dominant role (see Figure 16), whether or not the outer orbits in the multiple system remain bound is not important. Hence, the impact of $t_{\text{CE}}$ tends to be small. Merger numbers for Models 4a and 4b are roughly similar to those from Model 0.

Tables 3

| Model | Description | $2+2$ Quadruples BH–BH | BH–NS | NS–NS | Isolated Binaries BH–BH | BH–NS | NS–NS | 3+1 Quadruples BH–BH | BH–NS | NS–NS |
|-------|-------------|-------------------------|-------|-------|-------------------------|-------|-------|-----------------------|-------|-------|
| 0     | Fiducial    | 156 ± 13               | 82 ± 9 | 9 ± 3  | 168 ± 13               | 80 ± 9 | 14 ± 4 | 32 ± 6                | 15 ± 4 | 8 ± 3  |
| 1     | 0 kicks     | 285 ± 17               | 215 ± 15 | 351 ± 19 | 272 ± 16               | 194 ± 14 | 346 ± 19 | 164 ± 13               | 120 ± 11 | 44 ± 7 |
| 2     | No fly-bys  | 108 ± 10              | 79 ± 9  | 10 ± 3 | 130 ± 11               | 52 ± 7  | 8 ± 3   | 24 ± 5                | 17 ± 4  | 8 ± 3  |
| 3a    | 0.1 Z_{\odot} | 274 ± 17              | 191 ± 14 | 20 ± 4 | 309 ± 18               | 197 ± 14 | 28 ± 5 | 72 ± 8                | 64 ± 8  | 17 ± 4 |
| 3b    | 0.01 Z_{\odot} | 429 ± 21             | 525 ± 23 | 51 ± 7 | 477 ± 22               | 619 ± 25 | 45 ± 7 | 87 ± 9                | 199 ± 14 | 51 ± 6 |
| 4a    | 0.1 t_{\text{CE,0}} | 148 ± 12             | 72 ± 8   | 14 ± 4 | 171 ± 13               | 79 ± 9  | 14 ± 4 | 30 ± 5                | 19 ± 4  | 8 ± 3  |
| 4b    | 10 t_{\text{CE,0}} | 149 ± 12             | 68 ± 8   | 10 ± 3 | 171 ± 13               | 79 ± 9  | 14 ± 4 | 35 ± 6                | 17 ± 4  | 8 ± 3  |

Note. Refer to Table 1 for detailed model specifications.
The $\chi_p$ (Figure 12) distribution is dependent on the $q$ values. $\chi_p$ is related to the spin component in the orbital plane (unlike $\chi_{\text{eff}}$, which is related to the perpendicular component), and hence is relevant in quantifying the precession of the orbit. Our distribution shows that $\chi_p$ is spread throughout the parameter space, but with a preference for small $\chi_p$ ($\sim 0$). For Model 0 of

![2+2 quadruples; Model 0](image)

![2+2 quadruples; Model 1](image)

![Binaries; Model 0](image)

![Binaries; Model 1](image)

![3+1 quadruples; Model 0](image)

![3+1 quadruples; Model 1](image)

**Figure 10.** Frequency polygon of LIGO-band eccentricities $e_{\text{LIGO}}$. (Rows correspond to 2+2 quadruples, binaries and 3+1 quadruples respectively; Columns correspond to Models 0 and 1 respectively.) The quadruples have a significant tail at high $e_{\text{LIGO}}$ values for Model 1, whereas the binaries do not.
3+1 quadruples, high $\chi_p$ values are not seen, which may be due to the few mergers in this case. The preference for $\chi_p \sim 0$ can be attributed to the dominance of isolated binary evolution: in our simulations, the stellar spins were assumed to be aligned with the orbit, implying initially zero component of the spin to the orbit. Dynamical evolution can change the orbital
orientations and hence increase $\chi_p$, and there are indeed more systems with larger $\chi_p$ in Model 1 (no kicks; secular evolution is more important) compared to Model 0.

The $q$ distribution (Figure 13) itself is dependent on the type of merger. NS–NS mergers have $q \gtrsim 0.5$, peaking at $\sim 1$, since all NSs have masses greater than the Chandrasekhar mass $\sim 1.4 \, M_\odot$ and lower than $\sim 3 \, M_\odot$ (the approximate lower limit for the BH mass). BH–NS mergers typically have low $q$ values because a BH is much heavier than an NS. The BH–BH distribution seems to be roughly flat, with not much of a skew ($q \gtrsim 0.3$). Metallicity is important in determining the final compact object masses. This is seen in Figure 14, where the models with lower metallicities (Models 3a and 3b) consistently produce very high-mass BHs ($\gtrsim 17 \, M_\odot$, up to $\sim 27 \, M_\odot$).

The final distribution we look at is of the delay time $t_{\text{delay}}$ (Figure 15). All compact object mergers take $\gtrsim 10$ Myr to merge. As mentioned before, we limit our simulations up to 14 Gyr (Hubble time). We do not notice any significant preferences relating to $t_{\text{delay}}$.

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### 5.3. 3+1 Quadruples

Table 3 shows the number of compact object mergers, along with the Poisson error, in $10^5$ 3+1 quadruple systems. The table shows that the number of mergers in 3+1 quadruples is significantly lower than in 2+2 quadruples (almost by a factor of 5 for Model 0). This is not surprising since stable 3+1 configurations need to be more hierarchical and can get destabilized more easily. Additionally, we can see that Models 1 (zero SN kicks) and 3a/3b (lower metallicities) host a much higher proportion of mergers than in the case of the 2+2 quadruples. This, again, is due to the susceptibility of 3+1 quadruples to get unbound, for example, due to even small SN kicks. The details of individual models have similar explanations as in the case of 2+2 quadruples while keeping the above points in mind.

We refer to Figures 10, 11, 12, 13 and 15 for the distributions of $\varepsilon_{\text{LIGO}}$, $\chi_{\text{eff}}$, $\chi_p$, $q$ and $t_{\text{delay}}$. The $\varepsilon_{\text{LIGO}}$ and $\chi_{\text{eff}}$ distributions for Model 1 show the outlier effect (seen in 2+2 quadruples; see Section 5.2) characteristic to secular
evolution. The other distributions also qualitatively follow the $2^+2^+$ quadruples, albeit with fewer data points. Thus, it is not possible to distinguish between $2^+2^+$ and $3^+1^+$ quadruples based on the parameter distributions alone.

5.4. Scenarios of Mergers

In this section, we look back on the three merger scenarios referred to in Section 3: only CE evolution (Scenario 1), only secular evolution (Scenario 2) and mixture of both (Scenario 3). An attempt is made to estimate the fraction of merger systems that belong in each of these scenarios. It should, however, be noted that the distinction made between Scenarios 1 and 3 may not be unique, since it is generally difficult to quantify whether or not secular evolution played a decisive role.

The classification is as follows. Any merger system which does not undergo any CE or triple CE event throughout its evolution is considered in Scenario 2. The rest of the systems have a CE event at some point in their evolution. The distinction between Scenarios 1 and 3 is more arbitrary. To see if a system has been affected by secular evolution, we check for changes in the periapsis distance $r_p$ in the early stages of evolution. More specifically, the following conditions need to be satisfied by the inner or intermediate (in $3^+1^+$ quadruples) binaries to be classified under Scenario 3: $r_p < 0.8r_p,0$ and $r_p,0 > 5$ au. Here, $r_p,0$ is the inner or intermediate periapsis at $t = 0$, and this condition is checked for the first three log entries of MSE (a time step in MSE is “logged” when there are important events—stellar type changes, SNe, RLOF or CE events, collisions, dynamical instabilities, etc.). Any system which does not satisfy these conditions is considered in Scenario 1.

The resulting scenario fractions are presented in Figure 16. In every model other than Model 1, Scenario 3 contributes to (15–30)% of all compact object mergers. The rest are contributed by Scenario 1. $3^+1^+$ quadruples have a systematically higher contribution from Scenario 2 as well, most likely because there is more room for secular interactions. Unlike in $2^+2^+$ quadruples, the $3^+1^+$ quadruples have an intermediate orbit whose eccentricity and inclination can change, which can, in turn, affect the eccentricity and inclination of the inner orbit. Model 1 is the only one where Scenario 2 is observed—3% and 10%, respectively, in $2^+2^+$ and $3^+1^+$ quadruples. This exemplifies how much SN kicks affect the evolution of quadruple-star systems.

Figure 13. Frequency polygon of mass ratios $q$ pre-merger. (Rows correspond to $2^+2^+$ and $3^+1^+$ quadruples respectively; Columns correspond to Models 0 and 1 respectively. Isolated binary profiles are similar to $2^+2^+$ quadruples.)
We stress that our estimated percentages may not represent the true fraction of systems affected by secular evolution. First, our periapsis condition is checked only for the first few log entries. Any future secular evolution effects due to changed inclinations are not taken into account. Thus, the Scenario 3 fraction might be underestimated. Second, secular evolution does not aid all the merger systems in Scenario 3. For example, in 2+2 quadruples, one of the inner binary eccentricities might be enhanced, while the actual merger takes place in the other inner binary. In these cases, the Scenario 3 fraction might be overestimated. Nonetheless, Scenario 2 is unambiguously defined and, in particular, shows that the overall fraction of systems in which “clean” secular evolution leads to compact object mergers is very small.

5.5. Merger Rate Calculation

Our population synthesis systems do not represent the whole parameter space of quadruple-star systems, let alone all types of stellar systems. Thus, we need to take into consideration the assumptions made to get a realistic estimate for compact object merger rates. We employ a rate calculation method similar to the one used by Hamers et al. (2013).

The star formation rate (SFR) at redshift $z = 0$ is assumed to be $R_{\text{SF}} = 1.5 \times 10^7 \ M_{\odot} \text{Gpc}^{-3} \text{yr}^{-1}$ (given by Madau & Dickinson 2014). The overall merger rate $R_{\text{merge;conf}}$ for a given model of a given quadruple configuration can be expressed by

$$R_{\text{merge;conf}} \sim R_{\text{SF}} \frac{N_{\text{merge;conf}}}{M_{\text{sim}}} \sim \frac{R_{\text{SF}} N_{\text{merge;conf}} F_{\text{quad;conf}}}{N_{\text{sample}} M_{\text{avg}}}$$

where $N_{\text{sample}} = 10^5$ (total number of sampled systems in each model), $N_{\text{merge;conf}}$ values are given in Table 3 for given model and configuration (2+2 or 3+1) of quadruples, $M_{\text{sim}}$ is the total mass represented by our simulation, $M_{\text{avg}}$ is the average system mass of all types (singles, binaries, triples, and quadruples) and $F_{\text{quad;conf}}$ is the fraction of parameter space represented by a configuration of quadruples. In our rate normalization calculations, we neglect the contribution of quintuples and higher-order systems.

Let us first start with the masses. We assume that the universal initial mass function (IMF) is the Kroupa distribution $(dN/dm \propto m^{-2.3}$, $m > 0.5 \ M_{\odot}$ and $dN/dm \propto m^{-1.3}$, $m < 0.5 \ M_{\odot}$; Kroupa 2001). We then sample single, binary, triple, and quadruple (both types) systems from this IMF as done in Section 4.2 and calculate their average masses $\mu_{\text{sim}}$, $\mu_{\text{bin}}$, $\mu_{\text{trip}}$, and $\mu_{\text{quad}}$ in different mass bins. Each mass bin also occupies a fraction $f$ of the IMF. Next, we use interpolated and extrapolated values of multiplicity fractions from Moe & Di Stefano (2017) to calculate the contributions of singles $\alpha_{\text{bin}}$, binaries $\alpha_{\text{bin}}$, triples $\alpha_{\text{trip}}$, and quadruples $\alpha_{\text{quad}}$ to the average system mass $M_{\text{avg}}$ in these mass bins. These fractions are shown in Figure 17. In the case of quadruples, we further separate the contributions $\lambda_{2+2}$ and $\lambda_{3+1}$ of 2+2 and 3+1 quadruples respectively. To do this, we analyze the quadruple systems in the comprehensive Multiple Star catalog (MSC) (Tokovinin 1997, 2018) and calculate fractions of 2+2 and 3+1 quadruples in four mass bins. This is shown in Figure 18. It should be mentioned that the MSC suffers from observational biases, but it is currently the best source to infer fractions of the two types of quadruples. Finally, we put

![Figure 14. Scatter plot of heavier ($m_{\text{big}}$) and lighter ($m_{\text{small}}$) compact object masses pre-merger. (Rows correspond to Models 1, 3a and 3b respectively.)](image-url)
Figure 15. Frequency polygon of delay time $t_{\text{delay}}$ of merger. (Rows correspond to 2+2 and 3+1 quadruples respectively; Columns correspond to Models 0 and 1 respectively. Isolated binary profiles are similar to 2+2 quadruples.)

Figure 16. Bar graph of percentages of the three scenarios of compact object mergers in 2+2 and 3+1 quadruples.

Figure 17. Distribution of multiplicity fraction as a function of primary mass. The solid lines are approximate interpolated values from Moe & Di Stefano (2017), while the dotted lines are extrapolations.
these contributions together to give
\[
M_{\text{avg}} = \sum_{m_{\text{bin}}} \left( f_{\text{sin}} \alpha_{\text{sin}} \mu_{\text{sin}} + f_{\text{bin}} \alpha_{\text{bin}} \mu_{\text{bin}} + f_{\text{trip}} \alpha_{\text{trip}} \mu_{\text{trip}} + f_{\text{quad;2}+2} \alpha_{\text{quad}} \lambda_{2+2} \mu_{\text{quad}} + f_{\text{quad;3}+1} \alpha_{\text{quad}} \lambda_{3+1} \mu_{\text{quad}} \right).
\]

The other quantity in Equation (6) is \( F_{\text{quad;conf}} \):
\[
F_{\text{quad;conf}} = \sum_{m_{\text{bin}}} f_{\text{quad;conf}} \alpha_{\text{quad}} \lambda_{\text{conf}}.
\]

The final calculated rates are shown in Table 4. The errors indicate the Poisson errors.

### 5.6. Systems Not Considered

Not all of the systems run completely. For such systems, the code gets stuck either in the direct \(N\)-body or secular integration modes. Thus, we terminate any system which takes longer than a CPU wall time of \(t_{\text{wall}} = 10\) hr to run. We note that this is a reasonable time limit given that many systems take a few seconds to a few minutes to run. For completeness, in Figure 19, we present the percentage of systems that take longer than \(t_{\text{wall}}\) to run.

The isolated binary models have the least proportion of such systems (0.1%–0.3%) and the 3+1 quadruples have the most (2.9%–3.5%). The proportion for 2+2 quadruples is midway (0.6%–1.6%). This is expected since dynamical integration is straightforward in binaries while it is most complicated in 3+1 quadruples. In the case of the quadruples, it can be seen that Model 1 has the highest proportion of systems with \(t_{\text{wall}} > 10\) hr. This is because zero SN kicks aid in keeping more systems bound, and hence there is a higher chance for the gravitational dynamical integration to be slow.

However, more than 96% of the systems in all our models do run completely. Moreover, the offending systems are not clustered but spread throughout our initial parameter space. Hence, our overall statistics are not affected.

### 6. Discussion

#### 6.1. Merger Rate Comparisons

To put our estimated compact object merger rates in quadruples in context, we mention the compact object merger rates derived from LIGO detections. Abbott et al. (2021b) determined the GWTC-2 BH–BH and NS–NS merger rates to be \(23.9_{-8.6}^{+14.3} \text{ Gpc}^{-3} \text{ yr}^{-1}\) and \(320_{-240}^{+390} \text{ Gpc}^{-3} \text{ yr}^{-1}\) respectively. Abbott et al. (2021c) carried out a similar analysis for BH–NS mergers and found the rate to be \(45_{-31}^{+113} \text{ Gpc}^{-3} \text{ yr}^{-1}\) (assuming the two BH–NS detections are representative of the whole population) or \(130_{-40}^{+122} \text{ Gpc}^{-3} \text{ yr}^{-1}\) (assuming a broader distribution of masses). The rates we observe are either in agreement with the LIGO rates (for BH–BH mergers) or much lower (for NS–NS mergers, and BH–NS mergers to a lesser extent).
extent). Merger rates also depend on SFRs, which depend on redshift (Madau & Dickinson 2014).

Let us now compare with the other theoretical channels of GW emission. Most of these studies only consider BH–BH mergers, of which quadruples can constitute a significant fraction. The high NS–NS merger rates inferred from LIGO detections are not reproduced in many population synthesis studies due to the high SN kicks attributed to NSs. This is true for our study as well. Generally, dynamical merger channels can have high LIGO band eccentricities $e_{\text{LIGO}}$ and possibly anti-aligned (negative) effective spins $\chi_{\text{eff}}$. Many of the below-mentioned rates are uncertain and should be taken as order of magnitude estimates.

1. **Isolated binary channel.** Belczynski et al. (2016) estimated a high BH–BH merger rate of $\sim 200$ Gpc$^{-3}$ yr$^{-1}$ within redshift $z \sim 0.1$, for their standard model of isolated binary evolution of very massive stars ($\geq 40$ $M_\odot$). In contrast, de Mink & Mandel (2016) predicted a local BH–BH merger rate of $\sim 10$ Gpc$^{-3}$ yr$^{-1}$, and a rate of $\sim 20$ Gpc$^{-3}$ yr$^{-1}$ at redshift $z \sim 0.4$, for chemically homogeneous evolution in tidally distorted massive binaries. Since there is no secular evolution in isolated binaries, LIGO band eccentricities $e_{\text{LIGO}}$ are very small and $\chi_{\text{eff}}$ is aligned with their orbits.

2. **Dynamical channel in star clusters.** Rodriguez et al. (2016) predicted a local BH–BH merger rate of $\sim 5$ Gpc$^{-3}$ yr$^{-1}$ in globular clusters using a Monte Carlo approach, with 80% of them being in the mass range $(30–60) M_\odot$. They also found that nearly all of the BH–BH systems circularize and have $e_{\text{LIGO}} \lesssim 10^{-3}$, similar to isolated binaries. In comparison, Banerjee (2017) used direct N-body evolution to estimate a LIGO merger BH–BH rate of $\sim 13$ yr$^{-1}$ within a radius of 1.5 Gpc, which is equivalent to $\sim 3$ Gpc$^{-3}$ yr$^{-1}$. BH–NS and NS–NS mergers are unlikely since NSs do not efficiently segregrate to the center (as BHs do).

3. **Dynamical channel in galactic nuclei.** Petrovich & Antonini (2017) estimated BH–BH, BH–NS and NS–NS merger rates of $\lesssim 15$ Gpc$^{-3}$ yr$^{-1}$, $\lesssim 0.4$ Gpc$^{-3}$ yr$^{-1}$ and $\lesssim 0.02$ Gpc$^{-3}$ yr$^{-1}$ respectively for binaries in the sphere of influence of the central massive BHs in galactic nuclei. They also noted that the fraction of systems that reach extremely high eccentricities (1–$e \sim 10^6$–10$^9$) is $\sim (10–100)$ higher than in spherical clusters. The predictions of Hamers et al. (2018) agree with the above, with their most optimistic BH–BH merger rate being $\sim 12$ Gpc$^{-3}$ yr$^{-1}$, Hoang et al. (2018) predicted lower BH–BH merger rates of $\sim (1–3)$ Gpc$^{-3}$ yr$^{-1}$ using Monte Carlo simulations.

4. **Dynamical channel in triple systems.** Silsbee & Tremaine (2017) used a simple triple BH assumption to estimate a merger rate of $\sim 6$ Gpc$^{-3}$ yr$^{-1}$. On the other hand, Antonini et al. (2017) combined stellar evolution and dynamics to predict BH–BH merger rates of $\sim (0.3–1.3)$ Gpc$^{-3}$ yr$^{-1}$, with many of the mergers having $e_{\text{LIGO}} > 10^{-2}$. For BH–NS mergers, Fragione & Loeb (2019) estimated rates of $\sim (1.0 \times 10^{-3}–3.5 \times 10^{-2})$ Gpc$^{-3}$ yr$^{-1}$ when natal kicks are included.

5. **Dynamical channel in quadruple systems.** Liu & Lai (2019) showed that interactions between the two inner binaries in 2+2 quadruples can result in resonances which can increase the number of mergers by almost an order of magnitude compared to similar triples. Another study by Fragione & Kocsis (2019) (assuming all components are BHs) showed that this factor can be $\sim (3–4)$. Hamers et al. (2021a), who carried a simplified population synthesis of 2+2 quadruples, inferred optimistic rates of $\sim (10–100)$ Gpc$^{-3}$ yr$^{-1}$, of which the lower limits are consistent with our study. They also predicted second-generation mergers (see below) at a rate $\sim 10^{-5}$ Gpc$^{-3}$ yr$^{-1}$, which is not seen in our population synthesis (possibly due to low resolution). As in the case of triples, all these studies see a non-negligible fraction of mergers with high $e_{\text{LIGO}}$. Moreover, Hamers et al. (2021a) noted some merger products with negative $\chi_{\text{eff}}$.

6. **AGN disk channel.** Bartos et al. (2017) and Stone et al. (2017) calculated BH–BH merger rates of $\sim 1.2$ and $\sim 3$ Gpc$^{-3}$ yr$^{-1}$ in AGN disks. This channel is interesting due to potential electromagnetic counterparts in the form of X-rays or $\gamma$-rays, emitted due to super-Eddingtion accretion.

6.2. Caveats

As emphasized several times, the main takeaway from our estimated merger rate is that SN kicks are a major deciding factor. The kick distribution we have chosen is a Maxwellian one. Even though NS kicks have been constrained, to some extent, from the motion of observed isolated pulsars (though it is not clear how this generalizes to natal kicks in multiple-star systems), BH kicks are still poorly constrained. Assuming different distributions can hence significantly affect merger rates, and the values of $e_{\text{LIGO}}$ and $\chi_{\text{eff}}$.

Another caveat is that we looked at quadruple-star systems in the field, where stellar encounters are typically weak and infrequent. The evolution of quadruples in high-density environments, such as star clusters or galactic nuclei, will ostensibly be more dependent on fly-bys. For example, in the cores of globular clusters, binary–binary scatterings (e.g., Mikkola 1983; Sigurdsson & Hernquist 1993) can dynamically form triple systems, with one of the stars escaping. Meanwhile, stable triples in clusters are much more likely to be destroyed by strong interactions. Similarly, quadruples can be dynamically formed by scattering processes (e.g., van den Berk et al. 2007) and can be destroyed by strong encounters. Antognini & Thompson (2016) performed scattering experiments of binary–binary, triple–single, and triple–binary interactions to quantify and characterize the formation of triples in clusters. Antognini et al. (2016) studied dynamically formed triple systems in globular clusters using Monte Carlo models for the cluster coupled with an $n$-body integrator. They found that the timescales for angular momentum change can become comparable to the orbital periods, thereby making the secular approximation inaccurate. They also found eccentric LIGO band mergers. Martinez et al. (2020) performed a similar study and derived conservative BH–BH merger rates of $\sim 0.35$ Gpc$^{-3}$ yr$^{-1}$. However, a study on quadruples in clusters is yet to be carried out and is beyond the scope of this paper.

We also looked at the effect of the CE mass-loss timescale $\tau_{\text{CE}}$ and concluded that it does not affect merger rates meaningfully. Other studies (Dominik et al. 2012; Broekgaarden et al. 2021; Fragione et al. 2021) have shown that the CE efficiency parameter $\alpha_{\text{CE}}$ has a more significant effect. Moreover, CE evolution still faces significant uncertainties.
The main question this work has tried to address is the contribution of secular evolution to compact object mergers in quadruple-star systems. In Section 5.4, we described the different scenarios of mergers and concluded that secular evolution indeed has an effect, although perhaps not as significant as could be expected based on secular dynamics alone. However, our classification does not represent systems that undergo pre-compact object phase stellar collisions or have dynamical instabilities. Here, we briefly mention a few other examples from our population synthesis, leading to compact object mergers, not shown in Section 3.

1. In 2+2 quadruples, the two inner binaries can be close enough that they merge in their MS or giant phases. Alternatively, eccentricity enhancements cause them to merge prematurely. Now we have a binary system of merger products, typically very massive stars, which can evolve into BHs and merge. There can also be systems in which there is only one inner binary merger in the pre-compact object phase. The resulting system evolves as a triple-star system.

2. In 3+1 quadruples, the intermediate orbit’s eccentricity can be enhanced to an extent that the resulting configuration becomes dynamically unstable. One of two things can happen—either there is a collision, or one of the stars gets ejected from the system, resulting in a triple hierarchy. This triple system can now evolve and produce a merger.

Such examples show the wide complexity in the evolution of quadruples, and also explain why second-generation compact object mergers (see Fragione et al. 2020; Hamers et al. 2021a for examples) are rare. In fact, in our simulations, none such mergers have been observed, not even in the zero SN kicks case. Nevertheless, we can expect to see a few if the number of systems run is significantly higher. For a second-generation merger to occur, we need to have at least three compact objects in a bound stable configuration, which in itself never happens in our non-zero SN kick models. Then the first merger should occur either due to interaction with the tertiary or due to a prior CE evolution and GW emission-aided orbit shrinkage. Now the two resulting compact objects are far enough that they will not merge within a Hubble time, and there is no companion to enhance the merger.

The possibility of having four bound compact objects is even more scarce than having three of them. Even then, the orbit alignments and separations need to be just right to have a second-generation merger within a Hubble time. Thus, quadruple systems probably cannot explain recent GW detections of higher-mass BH (≥40 M_☉) mergers.

7. Conclusion

We used the population synthesis code MSE (Hamers et al. 2021b) to look for BH and NS mergers in quadruple-star systems, from their birth as MS stars to their death as compact objects, taking into account a wide range of physical processes. We looked at the two configurations of quadruples—2+2 and 3+1—and compared the 2+2 quadruples with “isolated” binaries. We also compared seven different models (with altered parameters) for each of the configurations. Our main conclusions are listed below.

1. Quadruple-star systems contribute to a significant fraction of BH–BH mergers, with their merger rates being on the order of the LIGO rate. BH–NS mergers also do occur in quadruples, but not as many as LIGO rates predict. On the other hand, NS–NS merger rates are extremely low due to SN kicks in these systems. The measured rates for our fiducial model (includes both types of quadruples) are 13.7 Gpc⁻³ yr⁻¹, 7.1 Gpc⁻³ yr⁻¹, and 1.3 Gpc⁻³ yr⁻¹ respectively for BH–BH, BH–NS, and NS–NS mergers.

2. 2+2 quadruples have similar merger numbers to the isolated binaries, although the latter have higher merger numbers in all models except our model with zero SN kicks. This indicates that, although the effects of secular evolution are seen in (15–30)% of systems, they can either aid or hinder compact object mergers.

3. Only in ~(3–10)% of cases (when SN kicks are excluded) is a compact object merger not associated with CE evolution, and a compact object multiple system forms successfully. In such cases, the mergers are due to dynamically induced high eccentricities.

4. A comparison of the two types of quadruples 2+2 and 3+1 shows that the former has many more BH–BH and BH–NS than the latter (by a factor of 3–4). The NS–NS merger rates are comparable (with SN kicks).

5. SN kicks are the most important factor for determining merger rates. Excluding kicks increases the number of mergers by factors of ~2–40 for 2+2 quadruples, and ~5–8 for 3+1 quadruples. The large outlier (factor ~40 increase) is the NS–NS merger rate in 2+2 quadruples. The increase in BH–BH and BH–NS mergers is more significant in 3+1 quadruples, however.

6. Metallicity Z is another parameter that significantly affects the merger numbers, similar to isolated binaries. For Z = 0.1 Z_☉, merger numbers are scaled up by factors of ~2–4. For Z = 0.01 Z_☉, they are scaled up by factors of ~3–10. The greatest increase is seen in BH–NS mergers (~3–4 and ~6–10 for the two metallicities respectively) for both types of quadruples, but more drastically in the 3+1 quadruples.

7. For quadruples, the LIGO band (f_{GW} ~ 10 Hz) eccentricities ε_{LIGO} lie in the range 10⁻³⁻¹⁻² in all the models where SN kicks are included. In the latter case, ε_{LIGO} can be high, even up to 0.3 (for both 2+2 and 3+1 systems). This can be attributed to secular evolution in the compact object phase.

8. Effective spin parameters χ_{eff} of the compact object mergers, for quadruples, mostly lie in the range 0–1 in all the models where SN kicks are included. In the latter case, a significant fraction of systems have negative χ_{eff}. Similarly to ε_{LIGO}, the outliers, with negative χ_{eff}, can be explained by secular evolution.

9. BH masses pre-merger can go up to ~17 M_☉ for solar metallicity systems. Lower metallicities can produce even higher masses, up to ~27 M_☉ for Z = 0.01 Z_☉. We find no second-generation mergers, which could be due to their low formation rate and the limited number of systems considered here.

10. Excluding fly-bys decreases the number of BH–BH mergers by a factor of ~0.7 for both types of quadruples. However, this decrease is not seen for BH–NS and NS–NS mergers. In the case of binaries, there is a noticeable decrease in all three types of mergers.
11. The CE mass-loss timescale $t_{CE}$ does not alter merger rates much. Scaling or descaling $t_{CE}$ by a factor of 10 does not change merger numbers beyond the Poisson error deviation. This can be understood by noting that isolated binary evolution (in particular, CE evolution) is the dominant factor in driving compact object mergers in quadruplets. Therefore, whether or not outer orbits in the multiple system remain bound after a CE event is not very important, since secular evolution on its own only drives a small fraction of compact object mergers.

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