Total reflectance of a random medium with the Reynolds-McCormick phase function at grazing incidence of light

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Abstract. We study the total reflectance of an absorbing, multiply scattering medium with large (as compared to the light wavelength) inhomogeneities at grazing incidence of light. To model highly forward scattering in the medium, we take advantage of the two-parameter Reynolds-McCormick scattering phase function. Using the scaling analysis for the small-angle radiative transfer equation, we derive simple analytic formulae for the dependence of the reflectance on the medium transport coefficients and the angle of incidence. The results obtained are verified by comparison with results of a direct numerical integration of the radiative transfer equation.

1. Introduction

Reflection of light from a multiply scattering medium with large inhomogeneities is of significant theoretical and practical interest for many problems of atmospheric and oceanic optics \cite{1–5} and biomedical optics \cite{6,7}. Important information about the bulk optical properties of turbid media can be extracted from the measured reflectances \cite{7–10}.

In reflection of light from a highly forward scattering medium, two multiple scattering regimes can occur. The diffusive regime develops over spatial scales exceeding the transport mean free path $l_{tr}$. If the direction of incidence does not differ much from the normal one and absorption is weak (the absorption coefficient $\kappa$ is much less than the transport scattering coefficient $\sigma_{tr} = l_{tr}^{-1}$), the reflected flux is governed by diffusely scattered light \cite{1}. At grazing angles of incidence ($\pi/2 - \theta_0 \ll 1$, $\theta_0$ is the angle between the direction of light incidence and the inward normal to the medium boundary), the light can be effectively reflected due to small-angle multiple scattering. In this case, even for a medium with rather strong absorption ($\kappa \sim \sigma_{tr}$), the total reflectance can be close to unity \cite{11–13}.

Small-angle reflection of light from an absorbing medium at grazing angles of incidence was studied in Refs. \cite{11,12} (see also Ref. \cite{14}) on the basis of the Fokker-Planck approximation (small-angle diffusion approximation). The latter, however, imposes a restriction on the scattering phase function, the results \cite{11,12} are valid only for the phase functions decreasing with scattering angle $\vartheta$ more rapidly than $1/\vartheta^4$.

In the present work, within the small-angle approximation, we calculate the total reflectance of an absorbing medium with the Reynolds-McCormick phase function. This function decreases...
with the scattering angle by the power law $1/\vartheta^\alpha$ and unifies a number of realistic scattering models. For grazing angles of incidence, we derive analytic expressions for the total reflectance in the range $2 < \alpha < 4$. To verify the validity of our results, we perform a direct numerical integration of the radiative transfer equation for the Henyey-Greenstein phase function ($\alpha = 3$) by the discrete ordinate method. The obtained analytic expressions are in very good agreement with the numerical data.

2. General relations
Consider a broad collimated beam of light incident on a turbid medium with large-scale inhomogeneities. The medium is assumed to occupy the half-space $z > 0$, the $z$-axis coincides with the inward normal to the surface. The intensity of light $I(z, \Omega)$ inside the medium obeys the radiative transfer equation (see, e.g., Ref. [4])

$$
\left( \mu \frac{\partial}{\partial z} + \kappa \right) I(z, \Omega) = \sigma \int d\Omega' \rho(\Omega, \Omega') [I(z, \Omega') - I(z, \Omega)]
$$

(1)

where $\sigma$ and $\kappa$ are the scattering and the absorption coefficients, respectively; $\Omega = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ is the unit vector in the direction of light propagation, $\mu = \Omega_z = \cos \theta$, $\varphi$ is the azimuthal angle; $\rho(\cos \vartheta)$ is the single-scattering phase function normalized by the condition $\int_0^{2\pi} 2\pi \sin \vartheta d\vartheta \rho(\cos \vartheta) = 1$.

Boundary condition for Eq. (1),

$$
I(z = 0, \mu, \varphi) = \delta(\mu - \mu_0) \delta(\varphi), \quad \mu > 0,
$$

(2)

determines the intensity within the half-range of angles $0 < \theta < \pi/2$ which correspond to light incident upon the medium; $\mu_0 = \cos \theta_0$, $\theta_0$ is the angle between the direction of incidence $\Omega_0$ and the $z$-axis. The azimuth angle of $\Omega_0$ is chosen to be zero. We normalize the intensity in such a way that the incident flux is equal to $\mu_0$.

The total reflectance (total reflection coefficient or plane albedo) $r$ is a ratio of the reflected flux to the incident one:

$$
r = \mu_0^{-1} \int_{-\pi}^{\pi} \frac{d\varphi}{\varphi} \int_{-1}^{1} \frac{d\mu}{|\mu|} \int_{z = 0, \mu, \varphi} I(z = 0, \mu, \varphi).
$$

(3)

In nonabsorbing media ($\kappa = 0$), the total flux is conserved, and $r = 1$.

To model scattering in the medium we take advantage of the two-parameter phase function proposed in Ref. [15],

$$
\rho(\cos \vartheta) = \frac{p_\alpha}{2\pi(\vartheta_0^2 + 2(1 - \cos \vartheta))^{\alpha/2}},
$$

(4)

where $p_\alpha$ is the normalization factor. Parameter $\vartheta_0$ is the characteristic angle of single scattering. For large scatterers, the forward scattering dominates and $\vartheta_0 \ll 1$. The phase function (4) unifies a number of scattering models [4], including scattering of light by a fractal [16]. For $\alpha = 3$, Eq. (4) coincides with the widely used Henyey-Greenstein phase function.

The small-angle form of the phase function (4) at $\alpha > 2$ is given by

$$
p(\vartheta) = \frac{\alpha - 2}{2\pi} \frac{\vartheta_0^{\alpha - 2}}{(\vartheta_0^2 + \vartheta^2)^{\alpha/2}}.
$$

(5)

For $\alpha < 2$, multiple scattering of light can not be described within the small-angle approximation (rigorously speaking, the regime of small-angle multiple scattering of light exists for $2 - \alpha \ll 1$ [17]). In what follows, we consider only the case $\alpha > 2$. 


At grazing angles of incidence, $\pi/2 - \theta_0 \ll 1$, the radiative transfer equation (1) can be simplified using the following relations (see, e.g., [11, 14, 18]):

$$\mu = \cos \theta = \sin \zeta \approx \zeta, \quad 2(1 - \Omega \Omega') \approx (\zeta - \zeta')^2 + (\varphi - \varphi')^2$$

where $\zeta = \pi/2 - \theta$ is the angle between the direction $\Omega$ and the medium boundary. With allowance for the explicit form of the phase function (5), the radiative transfer equation (1) takes the form

$$\left( \zeta \frac{\partial}{\partial z} + \kappa \right) I(z, \zeta, \varphi) = \frac{\sigma_0^{\alpha-2}}{2\pi} \int_{-\infty}^{\infty} d\zeta' \int_{-\infty}^{\infty} d\varphi' \frac{I(z, \zeta', \varphi') - I(z, \zeta, \varphi)}{(\vartheta_0^2 + (\zeta - \zeta')^2 + (\varphi - \varphi')^2)^{\alpha/2}},$$

$$I(z = 0, \zeta, \varphi) = \delta(\zeta - \zeta_0) \delta(\varphi), \quad \zeta > 0.$$ \hspace{1cm} (8)

As is usual for the small-angle approximation (see, e.g., Ref. [4]), the angles $\zeta$ and $\varphi$ are considered to vary within infinite limits provided that the intensity decreases rapidly as the values of $|\zeta|$ and $|\varphi|$ increase.

### 3. Scaling laws for the total reflectance at grazing angles of incidence

Within the small-angle approximation, the total reflectance (3) takes the form

$$r = \zeta_0^{-1} \int_{-\infty}^{\infty} d\varphi \int_{-\infty}^{0} |\zeta| I(z = 0, \zeta, \varphi) d\zeta.$$ \hspace{1cm} (9)

The solution to Eqs.(7) and (8) is expressed as a function of dimensionless variables,

$$I(z, \zeta, \varphi) = \frac{1}{\sigma_0^2} F_\alpha \left( \frac{\sigma_s}{\vartheta_0}; \frac{\zeta}{\sigma_s}, \frac{\varphi}{\vartheta_0}, \frac{\zeta_0}{\vartheta_0}, \frac{\kappa}{\sigma} \right).$$ \hspace{1cm} (10)

From Eqs.(9) and (10) we find the following scaling law for the total reflectance:

$$r = r_\alpha \left( \frac{\zeta_0}{\vartheta_0}, \frac{\kappa}{\sigma} \right).$$ \hspace{1cm} (11)

If the angle $\zeta_0$ is greater than the characteristic single-scattering angle, $\zeta_0 > \vartheta_0$, the reflected flux is governed by light scattered through angles exceeding the value $\vartheta_0$, and $\vartheta_0$ can be neglected in the denominator of the RHS of Eq.(7). For $\alpha < 4$, no divergency appears in Eq.(7) because the singularity in the denominator at $\zeta = \zeta_0$ and $\varphi = \varphi_0$ is canceled by the difference of the intensities in the numerator [18]. Within this approximation, a solution to the radiative transfer equation can be presented in the following self-similar form

$$I(z, \zeta, \varphi) = \frac{1}{\sigma_{tr}^2} \tilde{F}_\alpha \left( \frac{\sigma_{tr}s}{\zeta_0^2}; \frac{\zeta}{\zeta_0}, \frac{\varphi}{\zeta_0}, \frac{\zeta_0}{\zeta_0}, \frac{\kappa}{\sigma_{tr}} \right),$$ \hspace{1cm} (12)

where $\zeta_0 = (\sigma_{tr}/\kappa)^{1/(\alpha-2)}$; $\sigma_{tr} = \sigma(1 - \langle \cos \vartheta \rangle) = \frac{2(\alpha-2)}{4-\alpha} (\vartheta_0/2)^{\alpha-2}$ is the transport scattering coefficient. In this case, for a given value $\alpha$, the total reflectance proves to be a universal function of a single parameter,

$$r = \tilde{r}_\alpha \left( \frac{\zeta_0^{\alpha-2}}{\sigma_{tr}} \right), \quad 2 < \alpha < 4.$$ \hspace{1cm} (13)

The explicit form of the functions $r_\alpha(x, y)$ and $\tilde{r}_\alpha(x)$ appearing in Eqs. (11) and (13) is governed by the specific angular profile of the scattering phase function (i.e. by the exponent $\alpha$).
For $\alpha = 4$, relation (13) has a logarithmic accuracy. For $\alpha > 4$, the characteristic angle $\varphi_0$ appearing in Eq. (7) can not be neglected in the denominator. Instead, the intensity $I(z, \zeta', \varphi')$ can be expanded in angles of single-scattering deflection $\zeta' - \zeta$ and $\varphi' - \varphi$, resulting in the Fokker-Planck approximation (the small-angle diffusion approximation) in Eq.(7) [18]. Then, in going to dimensionless variables, we obtain $r$ in the form of Eq.(13), with the only difference that exponent $\alpha$ should be put $\alpha = 4$. So, for rapidly decreasing phase functions, the total reflectance takes the universal form $r = r(\kappa/\sigma_{tr})$.

Using the method proposed in Refs. [13, 19, 20], we derive the following asymptotic expression $(1 - r \ll 1)$ for the total reflectance in the range $2 < \alpha \leq 4$:

$$1 - r = f_\alpha \left( \vartheta_0 / \varphi_0 \right) (\kappa / \sigma)^{1/\alpha}$$

for

$$\kappa < \min\{\sigma, \sigma (\vartheta_0 / \varphi_0)^{\alpha - 2} \}. \quad (15)$$

Note that the condition (15) is weaker than the analogous condition at normal incidence of light. In the latter case, the condition $1 - r \ll 1$ implies that absorption is weak, $\kappa < \sigma_{tr}$ [1](i.e. $\kappa < \sigma \varphi_0^{\alpha - 2}$).

For $\varphi_0 > \vartheta_0$, Eq.(14) takes the form (see, Ref. [13]):

$$1 - r = c_\alpha \left( \kappa \varphi_0^{\alpha - 2} / \sigma_{tr} \right)^{1/\alpha} \quad \text{for} \quad \kappa < \sigma_{tr} / \varphi_0^{\alpha - 2}. \quad (16)$$

where the numeric constant $c_\alpha$ depends only on the parameter $\alpha$. Comparing Eqs.(14) and (16), we find the asymptotic behavior of the $f_\alpha$-function: $f_\alpha(x) \sim x^{1 - 2/\alpha}$ for $x > 1$.

If the phase function decreases more rapidly than $1/\varphi^4$ $(\alpha > 4)$, the radiative transfer equation can be solved analytically within the small-angle diffusion approximation [11, 12]. In this case, $1 - r = 1.42 \cdots (\kappa \varphi_0^{2} / \sigma_{tr})^{1/4}$ for $\kappa < \sigma_{tr} / \varphi_0^{2}$.

4. Results of a direct numerical integration of the radiative transfer equation

To illustrate the range of validity of the results obtained above we perform a direct numerical integration of the radiative transfer equation for a medium with the Henyey-Greenstein phase function ($\alpha = 3, \vartheta_0 = 1 - g, g$ is the mean cosine of the single-scattering angle). Without resorting to the small-angle approximation, we calculate the total reflectance $r$ from Eqs.(1)-(3) with a numerical code based on the discrete-ordinate method [21, 22]. Results of our numerical calculations are presented in Fig.1a. The figure shows the dependence of the difference $1 - r$ on the ratio of the absorption coefficient to the scattering one. Figure 1b shows the numerical data of Ref. [1].

The scaling law (11) is well illustrated by the presented numerical data, the graphs for $g = 0.9$, $\mu_0 = 0.1$ and $g = 0.95, \mu_0 = 0.05$ are practically indistinguishable in a wide range of values of the absorption coefficient (see Fig.1a), and the total reflectance becomes independent of the anisotropy factor $g$ in the limit $\zeta_0 \rightarrow +0$ (see Fig.1b). In all the presented graphs it is easy to distinguish the range of values of the absorption coefficient, where the difference $1 - r$ follows the law

$$1 - r = f(\zeta_0 / \vartheta_0) (\kappa / \sigma)^{1/3} \quad (17)$$

in accordance with Eq.(14). Some values of the function $f(\zeta_0 / \vartheta_0)$ can be extracted from our numerical data (see Fig.2a). With increasing the ratio $\zeta_0 / \vartheta_0$, the $f$-function tends to the dependence $f = 1.25 \cdots (\zeta_0 / \vartheta_0)^{1/3}$, and Eq.(17) takes the form (see Eq.(16) with $\alpha = 3$)

$$1 - r = 1.25 \cdots (\kappa \zeta_0 / \sigma_{tr})^{1/3}. \quad (18)$$
Figure 1. Total reflectance as a function of a ratio of the absorption coefficient to the scattering one. (a) The results of our numerical calculations with Eqs.(1) - (3) by the discrete-ordinate method. Empty and filled symbols correspond to $g = 0.95$ and $g = 0.9$, respectively. From lower to upper graphs, $\mu_0 = 0.03$ (squares), $0.05$ (triangles), $0.1$ (circles) and $0.5$ (stars). (b) Total reflectance in the limit $z_0 = +0$. Symbols are the numerical data of Ref. [1] for the Henyey-Greenstein phase function with $g = 0.875$ (circles), $g = 0.75$ (triangles) and $g = 0.5$ (squares). Solid lines correspond to the dependence $1 - r \sim (\kappa/\sigma)^{1/3}$.

Figure 2. (a) The $f$-function appearing into the relation (17). Crosses are the values of $f$ extracted from the data shown in Fig.1a,b. The dashed curve shows the dependence $f = 1.25(\xi_0/\theta_0)^{1/3}$. (b) Total reflectance as a function of dimensionless variable $\kappa\mu_0/\sigma_t$. Symbols are our numerical results for $g = 0.95$, $\mu_0 = 0.1$ (triangles), $g = 0.95$, $\mu_0 = 0.5$ (squares) and $g = 0.9$, $\mu_0 = 0.5$ (circles). The solid line is Eq.(18).

For $\mu_0 > 1 - g$, in accordance with the scaling law (13), the total reflectance proves to be a universal function of a single variable $\kappa\mu_0/\sigma_t$ (see Fig.2b). Remarkably, Eq.(18) turns out to be a good approximation even for rather moderate values of the total reflectance, $1 - r \leq 0.6$ (ratio $\kappa\xi_0/\sigma_t \leq 0.1$).

The scaling laws (11) and (13) fail in the case of very weak absorption, where the small-angle approximation becomes inapplicable, and the contribution from light scattered through large (of the order of unity) angles should be taken into account.
5. Conclusions
In conclusion, we have presented a theoretical study of light reflection from a turbid medium with highly forward single scattering at grazing angles of incidence. Within the small-angle approximation, we have calculated the total reflectance of an absorbing medium with the Reynolds-McCormick phase function. A range of values of the medium transport coefficients is found where the total reflectance proves to be a universal function of a single parameter which is expressed in terms of the incidence angle, the absorption coefficient and the transport scattering one. The explicit form of this function is governed by the specific angular profile of the scattering phase function. Our results are validated by comparison with results (both our and other authors’) of direct numerical integration of the radiative transfer equation.

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