DMV-strong uniqueness principle for the compressible Navier-Stokes system with potential temperature transport

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June 25, 2021

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Abstract

We establish a DMV-strong uniqueness result for the compressible Navier-Stokes system with potential temperature transport. The concept of generalized, the so-called dissipative measure-valued (DMV), solutions was proposed in [7], where their global-in-time existence was proved. Here we show that strong solutions are stable in the class of DMV solutions. More precisely, a DMV solution coincides with a strong solution emanating from the same initial data as long as the strong solution exists.

Keywords: compressible Navier-Stokes system · measure-valued solution · DMV-strong uniqueness principle

1 Introduction

In meteorological applications the following system of compressible Navier-Stokes equations governing the motion of viscous Newtonian fluid is often used, see, e.g., [5, 3, 6, 1],

\[ \partial_t \rho + \text{div}_x (\rho u) = 0, \quad (1.1) \]

\[ \partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x p(\rho \theta) = \text{div}_x (S(\nabla_x u)), \quad (1.2) \]

\[ \partial_t (\rho \theta) + \text{div}_x (\rho \theta u) = 0, \quad (1.3) \]
where $\varrho \geq 0$, $\mathbf{u}$, and $\theta \geq 0$, denote the fluid density, velocity, and potential temperature, respectively. The viscous stress tensor $\mathbb{S}(\nabla \mathbf{u})$ is determined by the stipulation

$$
\mathbb{S}(\nabla \mathbf{u}) = \mu \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{d} \text{div}_x(\mathbf{u}) \mathbb{I} \right) + \lambda \text{div}_x(\mathbf{u}) \mathbb{I},
$$

(1.4)

where the viscosity constants $\mu$ and $\lambda$ satisfy $\mu > 0$ and $\lambda \geq -\frac{2}{d} \mu$. The state equation for the pressure $p$ reads

$$
p(\varrho \theta) = a(\varrho \theta)^\gamma, \quad a = \text{const.} > 0,
$$

(1.5)

where $\gamma > 1$ is the so-called adiabatic index. System (1.1)–(1.3) is solved on $(0, T) \times \Omega$, where $T > 0$ is a given time and $\Omega \subset \mathbb{R}^d$ a bounded domain, $d \in \{2, 3\}$. It is accompanied with the initial data

$$
\varrho(0, \cdot) = \varrho_0, \quad \theta(0, \cdot) = \theta_0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0,
$$

(1.6)

and no-slip boundary conditions

$$
\mathbf{u}|_{(0,T) \times \partial \Omega} = \mathbf{0}.
$$

(1.7)

In the sequel, we shall call system (1.1)–(1.5) Navier-Stokes system with potential temperature transport. For a brief overview of analytical results for this system we refer to our recent paper [7]. It is to be pointed out that the existence of global-in-time weak solutions to (1.1)–(1.5) is available in three space dimensions only for $\gamma \geq 9/5$, see Maltese et al. [8, Theorem 1 with $T(s) = s^\gamma$]. We note in passing that a specific choice of the function $T$ in [8] yields $s = \theta$ and thus the Navier-Stokes equations with potential temperature transport. More importantly, physically relevant values of the adiabatic index $\gamma$ lie in the interval $(1, 5/3]$ for $d = 3$. However, this is not the case when the existence of global-in-time weak solutions is available. This drawback motivated our recent paper [7], where we have identified a larger class of generalized solutions—dissipative measure-valued (DMV) solutions to the Navier-Stokes system with potential temperature transport. Analyzing the convergence of a suitable numerical scheme, the mixed finite element–finite volume method, we have proved global-in-time existence of DMV solutions for all adiabatic indices $\gamma > 1$ for $d = 2, 3$.

The goal of the present paper is to show that the strong solutions to the Navier-Stokes system with potential temperature transport are stable in the class of DMV solutions. To this end we establish a DMV-strong uniqueness principle. This result states that the DMV and strong solutions emanating from the same initial data coincide. The key concept for the proof of this principle is the relative energy: Once a suitable relative energy is identified and the corresponding relative energy inequality is derived, the proof of the DMV-strong uniqueness principle is essentially a consequence of Gronwall’s lemma. This strategy for proving DMV/weak-strong uniqueness is not new; see, e.g., [2], where DMV-strong uniqueness is proven for the Navier-Stokes system, and [4, Chapter 6], where DMV-strong uniqueness is proven for the barotropic Euler system, the complete Euler system, and the Navier-Stokes system. However, till now the weak-strong uniqueness principle
was not available for the Navier-Stokes equations with potential temperature transport (1.1)–(1.5).

The key difficulty lies in the pressure law that only depends on the total potential temperature \( \rho \theta \), without any independent control of the density \( \rho \). To cure this problem, we will rewrite the pressure as a function of the density and total physical entropy. This allows us to separate the effects of the density and potential temperature in the derivation of the relative energy and finally to show the DMV-strong uniqueness principle.

The paper is organized as follows: In Section 2, we briefly repeat the relevant notation and our definition of DMV solutions to Navier-Stokes system with potential temperature transport proposed in [7]. Section 3 is devoted to the proof of the DMV-strong uniqueness principle.

2 DMV solutions

We start by introducing the pressure potential \( P : [0, \infty) \to \mathbb{R} \) as

\[
P(z) = \frac{a}{\gamma - 1} z^\gamma. \tag{2.1}
\]

In what follows we write \( \Omega_t = (0,t) \times \Omega \) whenever \( t > 0 \). If \( \mathcal{V} = \{ \mathcal{V}(t,x) \}_{(t,x) \in \Omega_T} \) is a space-time parametrized probability measure acting on \( \mathbb{R}^{d+2} \), we write

\[
\langle \mathcal{V}(t,x); g \rangle \equiv \int_{\mathbb{R}^{d+2}} g \, d\mathcal{V}(t,x) \equiv \int_{\mathbb{R}^{d+2}} g(\tilde{\rho}, \tilde{\theta}, \tilde{u}) \, d\mathcal{V}(t,x)(\tilde{\rho}, \tilde{\theta}, \tilde{u})
\]

whenever \( g \in C(\mathbb{R}^{d+2}) \). In particular, we tend to write out the function \( g \) in terms of the integration variables \( (\tilde{\rho}, \tilde{\theta}, \tilde{u}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \equiv \mathbb{R}^{d+2} \): if, for example, \( g(\tilde{\rho}, \tilde{\theta}, \tilde{u}) = \tilde{\rho} \tilde{u} \), then we also write

\[
\langle \mathcal{V}(t,x); \tilde{\rho} \tilde{u} \rangle \quad \text{instead of} \quad \langle \mathcal{V}(t,x); g \rangle.
\]

We recall the definition of dissipative measure-valued solutions to the Navier-Stokes system with potential temperature transport (1.1)–(1.5) from [7].

**Definition 2.1** (DMV solutions, [7, Definition 2.1]). A parametrized probability measure \( \mathcal{V} = \{ \mathcal{V}(t,x) \}_{(t,x) \in \Omega_T} \) that satisfies

\[
\mathcal{V} \in L^{\infty}_{\text{weak}^*}(\Omega_T; \mathcal{P}(\mathbb{R}^{d+2})) \tag{[1]}, \quad \mathbb{R}^{d+2} = \left\{ (\tilde{\rho}, \tilde{\theta}, \tilde{u}) \mid \tilde{\rho}, \tilde{\theta} \in \mathbb{R}, \tilde{u} \in \mathbb{R}^d \right\},
\]

and for which there exists a constant \( c_* > 0 \) such that

\[
\mathcal{V}(t,x) \left( \{ \tilde{\rho} \geq 0 \} \cap \{ \tilde{\theta} \geq c_* \} \right) = 1 \quad \text{for a.a.} \,(t, x) \in \Omega_T,
\]

is called a **dissipative measure-valued (DMV) solution** to the Navier-Stokes system with potential temperature transport (1.1)–(1.5) with initial and boundary conditions (1.6) and (1.7) if it satisfies:

\[\text{[1]} \mathcal{P}(\mathbb{R}^{d+2}) \text{ denotes the space of probability measures on } \mathbb{R}^{d+2}.\]
• energy inequality

\[ \mathbf{u}_V \equiv \langle V; \tilde{u} \rangle \in L^2(0, T; W^{1,2}_0(\Omega)^d), \quad \langle V; \frac{1}{2} \tilde{\rho} |\tilde{u}|^2 + P(\tilde{\rho} \tilde{\theta}) \rangle \in L^1(\Omega_T), \]

and the integral inequality

\[
\int_\Omega \left\langle V_{(\tau, \cdot)}; \frac{1}{2} \tilde{\rho} |\tilde{u}|^2 + P(\tilde{\rho} \tilde{\theta}) \right\rangle \, dx + \int_0^\tau \int_\Omega \mathcal{S}(\nabla_x \mathbf{u}_V) : \nabla_x \mathbf{u}_V \, dx \, dt

+ \int_\Pi \mathcal{E}(\tau) + \int_\Pi \mathcal{D} \leq \int_\Omega \left[ \frac{1}{2} \rho_0 |u_0|^2 + P(\rho_0 \theta_0) \right] \, dx
\]

holds for a.a. \( \tau \in (0, T) \) with the energy concentration defect

\[ \mathcal{E} \in L_{\text{weak}}^\infty(0, T; \mathcal{M}^+(\Omega)) \]

and the dissipation defect

\[ \mathcal{D} \in \mathcal{M}^+(\Omega_T); \]

• continuity equation

\[ \langle V; \tilde{\rho} \rangle \in C_{\text{weak}}([0, T]; L^\gamma(\Omega)), \quad \langle V_{(0, x)}; \tilde{\rho} \rangle = \rho_0(x) \text{ for a.a. } x \in \Omega \]

and the integral identity

\[
\left[ \int_\Omega \langle V_{(t, \cdot)}; \tilde{\rho} \rangle \varphi(t, \cdot) \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \langle V; \tilde{\rho} \rangle \partial_t \varphi + \langle V; \tilde{\rho} \tilde{u} \rangle \cdot \nabla_x \varphi \right] \, dx \, dt
\]

holds for all \( \tau \in [0, T] \) and all \( \varphi \in W^{1,\infty}(\Omega_T) \)[2];

• momentum equation

\[ \langle V; \tilde{\rho} \tilde{u} \rangle \in C_{\text{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)^d), \quad \langle V_{(0, x)}; \tilde{\rho} \tilde{u} \rangle = \rho_0(x) u_0(x) \text{ for a.a. } x \in \Omega \]

and the integral identity

\[
\left[ \int_\Omega \langle V_{(t, \cdot)}; \tilde{\rho} \tilde{u} \rangle \cdot \varphi(t, \cdot) \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \langle V; \tilde{\rho} \tilde{u} \rangle \cdot \partial_t \varphi + \langle V; \tilde{\rho} \tilde{u} \otimes \tilde{u} + p(\tilde{\rho} \tilde{\theta}) \Pi \rangle : \nabla_x \varphi \right] \, dx \, dt

- \int_0^\tau \int_\Omega \mathcal{S}(\nabla_x \mathbf{u}_V) : \nabla_x \varphi \, dx \, dt + \int_0^\tau \int_\Omega \nabla_x \varphi : d\mathcal{R}(t) \, dt
\]

[2] Here, the (Lipschitz) continuous representative of \( \varphi \in W^{1,\infty}(\Omega_T) \) is meant.
holds for all $\tau \in [0,T]$ and all $\varphi \in C^1([0,T] \times \partial \Omega)$ satisfying $\varphi|_{[0,T] \times \partial \Omega} = 0$, where the Reynolds concentration defect fulfills
\[ \mathfrak{R} \in L^\infty_{\text{weak}}(0,T; \mathcal{M}(\Omega)_{\text{sym},+}) \] \[ \text{and} \quad dE \leq \text{tr}(\mathfrak{R}) \leq dE \quad \text{for some constants} \quad d \geq \overline{d} > 0; \]

- **potential temperature equation**
  \[ \langle V; \overline{\theta} \rangle \in C_{\text{weak}}([0,T]; L^2(\Omega)) , \quad \langle V_{(0,x)}; \overline{\theta} \rangle = \varrho_0(x)\theta_0(x) \quad \text{for a.a.} \quad x \in \Omega \]
  and the integral identity
  \[ \left[ \int_\Omega \langle V(t,\cdot); \overline{\theta} \rangle \varphi(t,\cdot) \, dx \right]^{t=\tau}_{t=0} = \int_0^\tau \int_\Omega \left[ \langle V; \overline{\theta} \rangle \partial_t \varphi + \langle V; \overline{\theta} \overline{u} \rangle \cdot \nabla_x \varphi \right] \, dx \, dt \quad (2.5) \]
  holds for all $\tau \in [0,T]$ and all $\varphi \in W^{1,\infty}(\Omega_T)$;

- **entropy inequality**
  \[ \langle V_{(0,x)}; \overline{\ln(\theta)} \rangle = \varrho_0(x) \ln(\theta_0(x)) \quad \text{for a.a.} \quad x \in \Omega \]
  and for any $\psi \in W^{1,\infty}(\Omega_T), \psi \geq 0$, the integral inequality
  \[ \left[ \int_\Omega \langle V(t,\cdot); \overline{\ln(\theta)} \rangle \psi(t,\cdot) \, dx \right]^{t=\tau}_{t=0} \geq \int_0^\tau \int_\Omega \left[ \langle V; \overline{\ln(\theta)} \rangle \partial_t \psi + \langle V; \overline{\theta} \overline{u} \rangle \cdot \nabla_x \psi \right] \, dx \, dt \quad (2.6) \]
  is satisfied for a.a. $\tau \in (0,T)$;

- **Poincaré’s inequality**
  there exists a constant $C_P > 0$ such that
  \[ \int_0^\tau \int_\Omega \langle V; |\overline{u} - U|^2 \rangle \, dx \, dt \leq C_P \left( \int_0^\tau \int_\Omega |\nabla_x(u_V - U)|^2 \, dx \, dt + \int_0^\tau \int_{\Omega_T} dE(t) \, dt + \int_{\Omega_T} d\mathfrak{D} \right) \quad (2.7) \]
  for a.a. $\tau \in (0,T)$ and all $U \in L^2(0,T; W^{1,2}_0(\Omega)^d)$.

**Remark 2.2.** As we shall see in the next section, the entropy inequality (2.6) and Poincaré’s inequality (2.7) included in the definition of DMV solutions to the Navier-Stokes system with potential temperature transport are fundamental to guarantee DMV-strong uniqueness.

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\[ \mathcal{M}(\Omega)^{d \times d}_{\text{sym},+} \] denotes the set of bounded Radon measures defined on $\Omega$ and ranging in the set of symmetric positive semi-definite matrices, i.e., $\mathcal{M}(\Omega)^{d \times d}_{\text{sym},+} = \{ \mu \in \mathcal{M}(\Omega)^{d \times d} \mid \int_{\Omega} \phi(\xi \otimes \xi) : d\mu \geq 0 \text{ for all } \xi \in \mathbb{R}^d, \phi \in C(\Omega), \phi \geq 0 \}$. 

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5
3 DMV-strong uniqueness

The aim of this section is to derive a DMV-strong uniqueness principle for our measure-valued solutions. For this purpose, we rely on the concept of relative energy. We introduce the total (physical) entropy $S$ as

$$S = S(\varrho, \theta) = \begin{cases} \varrho \ln((a\theta)^{\frac{1}{\gamma}}) & \text{if } \theta > 0, \\ \infty & \text{if } \theta = 0 \end{cases}$$

(3.1)

and realize that the pressure $p = a(\varrho\theta)^{\gamma}$ can be rewritten with respect to $\varrho$, $S$ as

$$p(\varrho, S) = \begin{cases} \varrho^{\gamma} \exp\left((\gamma - 1) \frac{S}{\varrho}\right) & \text{if } \varrho > 0 \text{ and } S \in \mathbb{R}, \\ 0 & \text{if } \varrho = 0 \text{ and } S \leq 0, \text{ or } S = -\infty, \\ \infty & \text{if } \varrho = 0 \text{ and } S > 0. \end{cases}$$

(3.2)

We proceed by defining the relative energy between a triplet of arbitrary functions $(\varrho, \theta, u)$ belonging to a regularity class

$$\varrho, \theta \in C^1(\overline{\Omega_T}), \quad \varrho, \theta > 0, \quad u \in C^1(\overline{\Omega_T}) \cap L^2(0,T;W^{2,\infty}(\Omega)), \quad u|_{[0,T] \times \partial\Omega} = 0,$$

(3.3)

and a DMV solution $V$ to the Navier-Stokes system with potential temperature transport (1.1)–(1.5) as

$$E(V|\varrho, \theta, u) = \left\langle V; \frac{1}{2} \bar{\varrho} |\bar{u} - u|^2 + P(\bar{\varrho}, \bar{S}) - \frac{\partial P(\varrho, S)}{\partial \varrho} (\bar{\varrho} - \varrho) - \frac{\partial P(\varrho, S)}{\partial S} (\bar{S} - S) - P(\varrho, S) \right\rangle,$$

(3.4)

where $P(\varrho, S) = \frac{1}{\gamma - 1} p(\varrho, S)$ is the pressure potential expressed in terms of $\varrho$ and $S$, $S = S(\varrho, \theta)$, and $\bar{S} = S(\bar{\varrho}, \bar{\theta})$. We note that the relative energy defined in (3.4) is the generalization of the relative energy used in [4, Formula (4.59)] in the context of weak solutions. The corresponding relative energy inequality reads as follows.

**Lemma 3.1 (Relative energy inequality).** Let $(\varrho, \theta, u)$ be a triplet of test functions, cf. (3.3), and $V$ a DMV solution to (1.1)–(1.5) in the sense of Definition 2.1. Then the relative energy defined in (3.4) satisfies the inequality

$$\left[ \int_{\Omega} E(V(t, \cdot)|\varrho, \theta, u) \, dx \right]_{t=0}^{t=\tau} + \int_{\Pi} d\mathcal{E}(\tau) + \int_{\Pi} d\mathcal{D} + \int_{0}^{\tau} \int_{\Omega} \mathcal{S}(\nabla x(u_V - u) : \nabla x(u_V - u)) \, dx \, dt$$

$$\leq -\int_{0}^{\tau} \int_{\Omega} \left\langle V; \bar{\varrho} (\bar{u} - u)^T \cdot \nabla x u \cdot (\bar{u} - u) \right\rangle \, dx \, dt$$
\[- \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; p(\bar{\rho}, \bar{S}) - \frac{\partial p(\rho, S)}{\partial \rho} (\bar{\rho} - \rho) - \frac{\partial p(\rho, S)}{\partial S} (\bar{S} - S) - p(\rho, S) \right\rangle \text{div}_x(u) \, dx \, dt \]

\[+ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; \mathcal{V}(\bar{\rho})(u - \bar{u}) \right\rangle \cdot \left[ \rho \partial_t u + \rho \nabla_x u \cdot u + \nabla_x p(\rho, S) - \text{div}_x (S (\nabla_x u)) \right] \, dx \, dt \]

\[+ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; (\bar{\rho} - \rho) \frac{1}{\rho} \frac{\partial p(\rho, S)}{\partial \rho} \right\rangle \left[ \partial_t S + \text{div}_x (S u) \right] \, dx \, dt \]

\[+ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; (\bar{\rho} - \rho) \frac{1}{\rho} \frac{\partial p(\rho, S)}{\partial S} \right\rangle \left[ \partial_t S + \text{div}_x (S u) \right] \, dx \, dt \]

\[+ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; (\bar{\rho} - \rho) \frac{1}{\rho} \text{div}_x (S (\nabla_x u)) \cdot (u - \bar{u}) \right\rangle \, dx \, dt \]

for a.a. \( \tau \in (0, T) \). Here,

\[\bar{\vartheta} = \frac{1}{\gamma - 1} \frac{\partial p(\rho, S)}{\partial S}\]

denotes the absolute temperature.

**Proof.** Using Gauss’s theorem we easily verify that

\[- \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; \nabla_x (u \mathcal{V} - u) \right\rangle \, dx \, dt = - \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; \frac{\bar{\rho}}{\rho} (u - \bar{u}) \right\rangle \cdot \text{div}_x (S (\nabla_x u)) \, dx \, dt \]

\[+ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}; (\bar{\rho} - \rho) \frac{1}{\rho} \text{div}_x (S (\nabla_x u)) \cdot (u - \bar{u}) \right\rangle \, dx \, dt . \]

Thus, to prove inequality (3.5), it suffices to realize that

\[ \left[ \int_{0}^{\tau} \int_{\Omega} E(\mathcal{V}(t, \cdot), \rho, \theta, u) \, dx \right]_{t=0}^{t=\tau} = \left[ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}(t, \cdot); \frac{1}{2} \bar{\rho} |\bar{u}|^2 + P(\bar{\rho}, \bar{S}) \right\rangle \, dx \right]_{t=0}^{t=\tau} + \left[ \int_{0}^{\tau} \int_{\Omega} p(\rho, S) \, dx \right]_{t=0}^{t=\tau} \]

\[ - \left[ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}(t, \cdot); \frac{\partial P(\rho, S)}{\partial \rho} \right\rangle \, dx \right]_{t=0}^{t=\tau} - \left[ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}(t, \cdot); \bar{\rho} \bar{u} \right\rangle \cdot u \, dx \right]_{t=0}^{t=\tau} \]

\[+ \left[ \int_{0}^{\tau} \int_{\Omega} \left\langle \mathcal{V}(t, \cdot); \bar{\rho} \left( \frac{1}{2} |u|^2 - \frac{\partial P(\rho, S)}{\partial \rho} \right) \right\rangle \, dx \right]_{t=0}^{t=\tau} .\]
and utilize (2.2)–(2.6) to rewrite the terms on the right-hand side. We omit the necessary computations since they are straightforward and very similar to those leading to [4, (4.66)].

From the relative energy inequality we can deduce DMV-strong uniqueness.

**Theorem 3.2 (DMV-strong uniqueness).** Let \( \gamma > 1, \Omega \subset \mathbb{R}^d, d \in \{2, 3\} \), be a bounded Lipschitz-continuous domain. Further, let \( T^* > 0 \) and \((\varrho, \theta, \mathbf{u})\) be a strong solution to system (1.1)–(1.5) on \( \Omega_{T^*} \), belonging to the regularity class (3.3). Let \( \mathbf{V} \) be a DMV solution in the sense of Definition 2.1 emanating from the same initial data. Then

\[
\mathcal{E} = 0, \quad \mathcal{D} = 0, \quad \mathcal{R} = 0,
\]

and the DMV and strong solutions coincide on \([0, T^*]\), i.e.

\[
\mathbf{V}_{(t, x)} = \delta_{(\varrho(t, x), \theta(t, x), \mathbf{u}(t, x))} \quad \text{for a.a.} \ (t, x) \in \Omega_{T^*}.
\]

**Proof.** Plugging the strong solution \((\varrho, \theta, \mathbf{u})\) into the relative energy inequality (3.5), we obtain

\[
\left[ \int_{\Omega} E(\mathbf{V}(\cdot) | \varrho, \theta, \mathbf{u}) \, d\mathbf{x} \right]_{t=0}^{t} + \int_0^T \int_{\Omega} \mathcal{E}(\tau) + \int_0^T \int_{\Omega} \mathcal{D} + \int_0^T \int_{\Omega} \mathcal{R}(\nabla_x(\mathbf{u}_V - \mathbf{u})) : \nabla_x(\mathbf{u}_V - \mathbf{u}) \, d\mathbf{x} \, dt
\]

\[
\leq -\int_0^T \int_{\Omega} \left< \mathbf{V}; \bar{\varrho}(\bar{\mathbf{u}} - \mathbf{u})^T \cdot \nabla_x \mathbf{u} \cdot (\bar{\mathbf{u}} - \mathbf{u}) \right> d\mathbf{x} \, dt
\]

\[
- \int_0^T \int_{\Omega} \left< \mathbf{V}; p(\bar{\varrho}, \bar{S}) - \frac{\partial p(\varrho, S)}{\partial \varrho} (\bar{\varrho} - \varrho) - \frac{\partial p(\varrho, S)}{\partial S} (\bar{S} - S) - p(\varrho, S) \right> \text{div}_x(\mathbf{u}) \, d\mathbf{x} \, dt
\]

\[
+ \int_0^T \int_{\Omega} \left< \mathbf{V}; \left( \frac{\bar{\varrho}}{\varrho} \bar{S} - \bar{S} \right) (\bar{\mathbf{u}} - \mathbf{u}) \right> \cdot \nabla_x \vartheta \, d\mathbf{x} \, dt - \int_0^T \int_{\Omega} \mathbf{V} \cdot d\mathbf{R}(t) \, dt
\]

\[
+ \int_0^T \int_{\Omega} \left< \mathbf{V}; (\bar{\varrho} - \varrho) \frac{1}{\varrho} \text{div}_x(\mathbf{S}(\nabla_x \mathbf{u})) \cdot (\mathbf{u} - \bar{\mathbf{u}}) \right> \, d\mathbf{x} \, dt
\]

\[
\lesssim \int_0^T \left[ \int_{\Omega} E(\mathbf{V} | \varrho, \theta, \mathbf{u}) \, d\mathbf{x} + \int_{\Omega} \mathcal{E}(t) \right] \, dt + \int_0^T \int_{\Omega} \left< \mathbf{V}; \left( \frac{\bar{\varrho}}{\varrho} \bar{S} - \bar{S} \right) (\bar{\mathbf{u}} - \mathbf{u}) \right> \cdot \nabla_x \vartheta \, d\mathbf{x} \, dt
\]

\[
+ \int_0^T \int_{\Omega} \left< \mathbf{V}; (\bar{\varrho} - \varrho) \frac{1}{\varrho} \text{div}_x(\mathbf{S}(\nabla_x \mathbf{u})) \cdot (\mathbf{u} - \bar{\mathbf{u}}) \right> \, d\mathbf{x} \, dt
\]

(3.6)

for a.a. \( \tau \in (0, T^*) \). To handle the last two integrals, we first observe that

\[
\int_0^T \int_{\Omega} \mathbf{S}(\nabla_x(\mathbf{u}_V - \mathbf{u})) : \nabla_x(\mathbf{u}_V - \mathbf{u}) \, d\mathbf{x} \, dt = \int_0^T \int_{\Omega} \left[ \mu |\nabla_x(\mathbf{u}_V - \mathbf{u})|^2 + \nu |\text{div}_x(\mathbf{u}_V - \mathbf{u})|^2 \right] \, d\mathbf{x} \, dt
\]

\[
\geq \mu \int_0^T \int_{\Omega} |\nabla_x(\mathbf{u}_V - \mathbf{u})|^2 \, d\mathbf{x} \, dt.
\]

(3.7)
Next, we set
\[(\varrho, \overline{\varrho}, \theta, \overline{\theta}) = \left( \inf_{(t,x) \in \Omega^+} \{\varrho(t,x)\}, \sup_{(t,x) \in \Omega^+} \{\varrho(t,x)\}, \inf_{(t,x) \in \Omega^+} \{\theta(t,x)\}, \sup_{(t,x) \in \Omega^+} \{\theta(t,x)\} \right)\]
and apply Lemma A.1 to find constants \(c_1, c_2, c_3 > 0\) that only depend on \(\varrho, \overline{\varrho}, \theta, \overline{\theta}, c_*, \) and \(\gamma\), and corresponding sets
\[
\mathcal{R} = \left\{(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \in \mathbb{R}^{d+2} \mid c_1 \varrho \leq \tilde{\varrho} \leq c_2 \varrho, \ c_* \leq \tilde{\theta} \leq c_3 \overline{\theta} \right\},
\]
\[
\mathcal{S} = \left\{(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \in \mathbb{R}^{d+2} \mid \tilde{\varrho} \geq 0, \ \tilde{\theta} \geq c_* \right\} \setminus \mathcal{R}
\]
such that
\[
\int_0^T \int_{\Omega} E(\mathcal{V}|\varrho, \theta, u) \, dx \, dt \gtrsim \int_0^T \int_{\Omega} \left( \mathcal{V} \cdot \mathbf{1}_{\mathcal{R}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \right) (|\tilde{u} - u|^2 + |\tilde{\varrho} - \varrho|^2 + |\tilde{S} - S|^2) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega} \left( \mathcal{V} \cdot \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \right) \left( 1 + \tilde{\varrho} |\tilde{u} - u|^2 + (\tilde{\theta})^\gamma \right) \, dx \, dt.
\]
(3.8)

Seeing that
\[
\left| \left( \frac{\partial}{\partial \varrho} S - \tilde{S} \right) (\tilde{u} - u) \right| \lesssim |(\tilde{\varrho} S - \tilde{S} \varrho)(\tilde{u} - u)| \lesssim |S(\tilde{\varrho} - \varrho)(\tilde{u} - u)| + |\varrho(S - \tilde{S})(\tilde{u} - u)|
\]
\[
\lesssim |\tilde{u} - u|^2 + |\tilde{\varrho} - \varrho|^2 + |\tilde{S} - S|^2
\]
as well as
\[
\left| \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \left( \frac{\partial}{\partial \varrho} S - \tilde{S} \right) (\tilde{u} - u) \right|
\]
\[
\lesssim \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \left( \tilde{\varrho} |\tilde{u} - u| + \tilde{S} |\tilde{u} - u| \right) \lesssim \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \left( \tilde{\varrho} |\tilde{u} - u| + \tilde{\varrho}^{1/2} |\tilde{u} - u| \right)
\]
\[
\lesssim \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \left( \tilde{\varrho} + \tilde{\varrho}^{\gamma} + \tilde{\varrho} |\tilde{u} - u|^2 \right) \lesssim \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u}) \left( 1 + \tilde{\varrho} |\tilde{u} - u|^2 + (\tilde{\varrho})^\gamma \right),
\]
we may use (3.8) to deduce
\[
\left| \int_0^T \int_{\Omega} \left( \mathcal{V} \cdot \left( \frac{\partial}{\partial \varrho} S - \tilde{S} \right) (\tilde{u} - u) \right) \cdot \nabla x \varrho \, dx \, dt \right| \lesssim \int_0^T \int_{\Omega} E(\mathcal{V}|\varrho, \theta, u) \, dx \, dt.
\]
(3.9)

We proceed by observing that
\[
| (\tilde{\varrho} - \varrho)(\tilde{u} - u) | \lesssim |\tilde{\varrho} - \varrho|^2 + |\tilde{u} - u|^2
\]
and
\[
\left| \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u})(\tilde{\varrho} - \varrho)(\tilde{u} - u) \right| \lesssim \mathbf{1}_{\mathcal{S}}(\tilde{\varrho}, \tilde{\theta}, \tilde{u})(\tilde{\varrho} + 1)|\tilde{u} - u|
\]
9
\[
\lesssim 1_S(\bar{\rho}, \bar{\theta}, \bar{u}) \left( |\bar{\rho} - \bar{\rho}|^2 + \alpha |\bar{u} - \bar{u}|^2 + \alpha^{-1} \right)
\]
\[
\lesssim 1_S(\bar{\rho}, \bar{\theta}, \bar{u}) \left( 1 + (\sqrt{\bar{\theta}})^2 + |\bar{u} - \bar{u}|^2 + \alpha |\bar{u} - \bar{u}|^2 + \alpha^{-1} \right)
\]
for all \( \alpha > 0 \), where here and in the sequel the constant hidden in "\( \lesssim \)" does not depend on \( \alpha \).

Together with (3.8) and Poincaré’s inequality (2.7), these observations yield
\[
\left| \int_0^\tau \int_\Omega \left( \mathcal{V}_t \mathbf{(\bar{\rho} - \rho)} \frac{1}{\rho} \text{div}_x (\mathbb{S}_x \mathbf{u}) \cdot (\mathbf{u} - \bar{\mathbf{u}}) \right) d\mathbf{x} dt \right| 
\lesssim (1 + \alpha^{-1}) \int_0^\tau \int_\Omega E(\mathcal{V}_t, \mathbf{\theta}, \mathbf{u}) d\mathbf{x} dt + \alpha \left( \int_0^\tau \int_\Omega |\nabla_x (\mathbf{u}_v - \mathbf{u})|^2 d\mathbf{x} dt + \int_0^\tau \int_{\mathbb{P}} d\mathcal{E}(t) dt + \int_{\mathbb{P}} d\mathcal{D} \right).
\]

(3.10)

Finally, combining (3.6), (3.7), (3.9), and (3.10), we arrive at
\[
\left[ \int_\Omega E(\mathcal{V}_t, \mathbf{\theta}, \mathbf{u}) d\mathbf{x} \right]_{t=0}^{t=\tau} + \int_{\mathbb{P}} d\mathcal{E}(\tau) + \int_{\mathbb{P}} d\mathcal{D} + \mu \int_0^\tau \int_\Omega |\nabla_x (\mathbf{u}_v - \mathbf{u})|^2 d\mathbf{x} dt
\lesssim (1 + \alpha^{-1}) \int_0^\tau \int_\Omega E(\mathcal{V}_t, \mathbf{\theta}, \mathbf{u}) d\mathbf{x} dt + (1 + \alpha) \int_0^\tau \int_{\mathbb{P}} d\mathcal{E}(t) dt
\]
\[
+ \alpha \left( \int_0^\tau \int_\Omega |\nabla_x (\mathbf{u}_v - \mathbf{u})|^2 d\mathbf{x} dt + \int_{\mathbb{P}} d\mathcal{D} \right)
\]
for a.a. \( \tau \in (0, T^*) \) and all \( \alpha > 0 \). In particular, there exists a constant \( C > 0 \) such that
\[
\left[ \int_\Omega E(\mathcal{V}_t, \mathbf{\theta}, \mathbf{u}) d\mathbf{x} \right]_{t=0}^{t=\tau} + \int_{\mathbb{P}} d\mathcal{E}(\tau) + \int_{\mathbb{P}} d\mathcal{D}
\leq C \left( \int_0^\tau \int_\Omega E(\mathcal{V}_t, \mathbf{\theta}, \mathbf{u}) d\mathbf{x} dt + \int_0^\tau \int_{\mathbb{P}} d\mathcal{E}(t) dt + \int_{\mathbb{P}} d\mathcal{D} dt \right)
\]
for a.a. \( \tau \in (0, T^*) \). Consequently, the desired result follows from Gronwall’s lemma.

\[\square\]

4 Conclusions

In the present paper, we proved the DMV-strong uniqueness principle for the Navier-Stokes system with potential temperature transport (1.1)–(1.5). In fact, this result shows that strong solutions are stable in the class of DMV solutions introduced in [7]. We have derived the relative energy by taking the total physical entropy into account. More precisely, the pressure was rewritten as a function of the density and entropy, instead of the total potential temperature only. Moreover, we also require the entropy inequality (2.6) that is included in our definition of DMV solutions. The importance of Poincaré’s inequality (2.7) became clear during the proof of DMV-strong uniqueness: It allowed us to rewrite viscosity terms in such a way that Gronwall’s lemma was applicable. Finally, the DMV-strong uniqueness result follows by applying Gronwall’s lemma.
The DMV-strong uniqueness principle was used in our recent work [7]. In Theorem 6.1 we relied on this result to prove the strong convergence of the numerical solutions of our mixed finite element–finite volume scheme [7, Definition 3.2] to the classical solution of the system as long as the latter exists.

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A Appendix

A.1 An auxiliary result concerning the relative energy

Here, we prove the auxiliary result used in the proof of DMV-strong uniqueness.
Lemma A.1. Let \( \tilde{\theta} \geq 0, \tilde{\theta} \geq c_\ast > 0, 0 < \bar{\theta} \leq \theta \leq \tilde{\theta}, \) and \( \gamma > 1. \) Then there exist constants \( c_1, c_2, c_3, c_4 > 0 \) that only depend on \( \bar{\theta}, \tilde{\theta}, \tilde{\theta}, \bar{\theta}, \) and \( \gamma, \) and corresponding sets

\[
\mathcal{R} = \left\{ (\tilde{\theta}, \theta) \in \mathbb{R}^2 \mid c_1 \tilde{\theta} \leq \tilde{\theta} \leq c_2 \theta, \ c_\ast \leq \tilde{\theta} \leq c_3 \theta \right\},
\]

\[
\mathcal{S} = \left\{ (\tilde{\theta}, \theta) \in \mathbb{R}^2 \mid \tilde{\theta} \geq 0, \ \tilde{\theta} \geq c_\ast \right\} \setminus \mathcal{R}
\]

such that

\[
F(\tilde{\theta}, \tilde{\theta} | \theta, S) = P(\tilde{\theta}, \tilde{\theta}) - \frac{\partial P(\theta, S)}{\partial \theta} (\tilde{\theta} - \theta) - \frac{\partial P(\theta, S)}{\partial S} (\tilde{\theta} - S) - P(\theta, S)
\]

\[
\geq c_4 \left[ 1_{\mathcal{S}}(\tilde{\theta}, \theta) \left( |\tilde{\theta} - \theta|^2 + |\tilde{\theta} - S|^2 \right) + 1_{\mathcal{R}}(\tilde{\theta}, \theta) \left( 1 + (\tilde{\theta} \theta)^\gamma \right) \right], \quad (A.1)
\]

where \( P(\theta, S) = \frac{1}{\gamma - 1} p(\theta, S) \) with \( p \) from (3.2), \( S = S(\theta, S) \) is defined in (3.1), and \( \tilde{\theta} = S(\theta, \tilde{\theta}) \).

Proof. To begin with, let \( 0 < c_1 \leq c_2, \) and \( c_3 \geq c_\ast / \tilde{\theta} \) be arbitrary numbers. Further, let \( \mathcal{R}, \mathcal{S} \) be defined as described in the lemma. We decompose \( \mathcal{S} \) into the sets

\[\mathcal{S}^+ = \left\{ (\tilde{\theta}, \theta) \in \mathcal{S} \mid \tilde{\theta} < c_1 \theta \right\}, \quad \mathcal{S}^- = \left\{ (\tilde{\theta}, \theta) \in \mathcal{S} \mid \tilde{\theta} > c_2 \theta \right\}, \quad \mathcal{S}^0 = \mathcal{S} \setminus (\mathcal{S}^+ \cup \mathcal{S}^-)\]

and observe that

\[
F(\tilde{\theta}, \tilde{\theta} | \theta, S) = \left( \theta \right) \left( 1 - \ln (\theta) + \ln (\tilde{\theta}) \right) + \frac{1}{\gamma - 1} (\tilde{\theta} \theta)^\gamma\left( 1 + \ln (\theta) \right)
\]

\[
\geq \left( \theta \right) \left( 1 - \ln (\theta) + \ln (\tilde{\theta}) \right) + \frac{1}{\gamma - 1} (\tilde{\theta} \theta)^\gamma\left( 1 + \ln (\theta) \right)
\]

wherefore

\[
1_{\mathcal{S}^-}(\tilde{\theta}, \theta) F(\tilde{\theta}, \tilde{\theta} | \theta, S)
\]

\[
\geq \left( \theta \right) + \frac{a \gamma}{\gamma - 1} \left( 1 + \max \left\{ |\ln (\theta)|, |\ln (\tilde{\theta})| \right\} \right) (\tilde{\theta} \theta)^\gamma - c_1^2 \left( 2^{(r - 1)} \frac{a (2^{(r - 1)})}{2^{(r - 1)}} \right) \left( \theta \right) \left( 1 - \ln (\theta) \right)
\]

\[
+ \frac{a \gamma}{\gamma - 1} \left( 1 - \frac{c_1^2}{2} \right) \left( \tilde{\theta} \theta)^\gamma\right) (\tilde{\theta} \theta)^\gamma,
\]

\[
1_{\mathcal{S}^0}(\tilde{\theta}, \theta) F(\tilde{\theta}, \tilde{\theta} | \theta, S)
\]

\[
\geq \left( \theta \right) + \frac{a \gamma}{\gamma - 1} \left( 1 - \ln (\theta) \right) \left( 1 + \max \left\{ |\ln (\theta)|, |\ln (\tilde{\theta})| \right\} \right) \left( \theta \right) \left( 1 - \ln (\theta) \right)
\]

\[
+ \frac{a \gamma}{\gamma - 1} \left( 1 - \ln (\theta) \right) \left( 1 + \max \left\{ |\ln (\theta)|, |\ln (\tilde{\theta})| \right\} \right) \left( \theta \right) \left( 1 - \ln (\theta) \right)
\]
Here, the first inequality is obtained using Young’s inequality. Together, the above observations show that we can specify $c_1, c_2, c_3$ in dependence of $\tilde{\theta}, \theta, \tilde{\theta}, \theta, c, \gamma$ such that
\[
1_{S}(\tilde{\theta}, \theta)F(\tilde{\theta}, \tilde{S}|\theta, S) \geq a(\tilde{\theta})\gamma + \frac{a}{\gamma - 1} \left(1 - \gamma c_3 \gamma \left(\frac{\tilde{\theta}}{c_2 \theta}\right)^{\gamma} \left(1 + \max \{|\ln(\tilde{\theta})|, |\ln(\theta)|\} + (c_3 \theta)^{1/2}\right)\right)(\tilde{\theta})\gamma.
\]
where $c_4 > 0$ solely depends on $\tilde{\theta}, \theta, \tilde{\theta}, \theta, c, \gamma$. Having fixed $c_1, c_2, c_3$ as described above, it remains to show that
\[
1_{R}(\tilde{\theta}, \theta)F(\tilde{\theta}, \tilde{S}|\theta, S) \geq c_{4,1} 1_{S}(\tilde{\theta}, \theta) \left(1 + (\tilde{\theta})\gamma\right),
\]
where $c_{4,1} > 0$ only depends on $\tilde{\theta}, \theta, \tilde{\theta}, \theta, c, \gamma$. However, this inequality is a direct consequence of the fact that $P = P(\theta, S)$ is strongly convex on every compact convex subset of $(0, \infty) \times \mathbb{R}$ which, in turn, follows from the positive definiteness of the Hessian of $P$ on $(0, \infty) \times \mathbb{R}$. \qed