CONVERGENCE THEOREMS FOR GENERALIZED FUNCTIONAL SEQUENCES OF DISCRETE-TIME NORMAL MARTINGALES

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Abstract. The Fock transform recently introduced by the authors in a previous paper is applied to investigate convergence of generalized functional sequences of a discrete-time normal martingale $M$. A necessary and sufficient condition in terms of the Fock transform is obtained for such a sequence to be strong convergent. A type of generalized martingales associated with $M$ are introduced and their convergence theorems are established. Some applications are also shown.

1. Introduction

Hida’s white noise analysis is essentially a theory of infinite dimensional calculus on generalized functionals of Brownian motion [9, 12, 14, 16]. In 1988, Ito [13] introduced his analysis of generalized Poisson functionals, which can be viewed as a theory of infinite dimensional calculus on generalized functionals of Poisson martingale. It is known that both Brownian motion and Poisson martingale are continuous-time normal martingales. There are theories of white noise analysis for some other continuous-time processes (see, e.g., [1, 2, 4, 11, 15]).

Discrete-time normal martingales [18] also play an important role in many theoretical and applied fields. For example, the classical random walk (a special discrete-time normal martingale) is used to establish functional central limit theorems in probability theory [5, 19]. It would then be interesting to develop a theory of infinite dimensional calculus on generalized functionals of discrete-time normal martingales.

Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale satisfying some mild conditions. In a recent paper [20], we constructed generalized functionals of $M$, and introduced a transform, called the Fock transform, to characterize those functionals.

In this paper, we apply the Fock transform [20] to investigate generalized functional sequences of $M$. First, by using the Fock transform, we obtain a necessary and sufficient condition for a generalized functional sequence of $M$ to be strong convergent. Then we introduce a type of generalized martingales associated with $M$, called $M$-generalized martingales, which are a special class of generalized functional sequences of $M$ and include as a special case the classical martingales with respect to the filtration generated by $M$. We establish a strong-convergent criterion in terms of the Fock transform for $M$-generalized martingales. Some other convergence criteria are also obtained. Finally we show some applications of our main results.

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Our one interesting finding is that for an $M$-generalized martingale, its strong convergence is just equivalent to its strong boundedness.

Throughout this paper, $\mathbb{N}$ designates the set of all nonnegative integers and $\Gamma$ the finite power set of $\mathbb{N}$, namely
\begin{equation}
\Gamma = \{ \sigma \mid \sigma \subset \mathbb{N} \text{ and } \#(\sigma) < \infty \},
\end{equation}
where $\#(\sigma)$ means the cardinality of $\sigma$ as a set. In addition, we always assume that $(\Omega, \mathcal{F}, P)$ is a given probability space with $E$ denoting the expectation with respect to $P$. We denote by $L^2(\Omega, \mathcal{F}, P)$ the usual Hilbert space of square integrable complex-valued functions on $(\Omega, \mathcal{F}, P)$ and use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to mean its inner product and norm, respectively. By convention, $\langle \cdot, \cdot \rangle$ is conjugate-linear in its first argument and linear in its second argument.

2. Generalized functionals

Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale on $(\Omega, \mathcal{F}, P)$ that has the chaotic representation property and $Z = (Z_n)_{n \in \mathbb{N}}$ the discrete-time normal noise associated with $M$ (see Appendix). We define
\begin{equation}
Z_\emptyset = 1; \quad Z_\sigma = \prod_{i \in \sigma} Z_i, \quad \sigma \in \Gamma, \sigma \neq \emptyset.
\end{equation}
And, for brevity, we use $L^2(M)$ to mean the space of square integrable functionals of $M$, namely
\begin{equation}
L^2(M) = L^2(\Omega, \mathcal{F}_\infty, P),
\end{equation}
which shares the same inner product and norm with $L^2(\Omega, \mathcal{F}, P)$, namely $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. We note that $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms a countable orthonormal basis for $L^2(M)$ (see Appendix).

Lemma 2.1. \cite{22} Let $\sigma \mapsto \lambda_\sigma$ be the $\mathbb{N}$-valued function on $\Gamma$ given by
\begin{equation}
\lambda_\sigma = \begin{cases}
\prod_{k \in \sigma} (k + 1), & \sigma \neq \emptyset, \sigma \in \Gamma; \\
1, & \sigma = \emptyset, \sigma \in \Gamma.
\end{cases}
\end{equation}
Then, for $p > 1$, the positive term series $\sum_{\sigma \in \Gamma} \lambda_\sigma^{-p}$ converges and moreover
\begin{equation}
\sum_{\sigma \in \Gamma} \lambda_\sigma^{-p} \leq \exp \left( \sum_{k=1}^{\infty} k^{-p} \right) < \infty.
\end{equation}

Using the $\mathbb{N}$-valued function defined by (2.3), we can construct a chain of Hilbert spaces consisting of functionals of $M$ as follows. For $p \geq 0$, we define a norm $\| \cdot \|_p$ on $L^2(M)$ through
\begin{equation}
\| \xi \|_p^2 = \sum_{\sigma \in \Gamma} \lambda_\sigma^{2p} \langle Z_\sigma, \xi \rangle^2, \quad \xi \in L^2(M)
\end{equation}
and put
\begin{equation}
S_p(M) = \{ \xi \in L^2(M) \mid \| \xi \|_p < \infty \}.
\end{equation}
It is not hard to check that $\| \cdot \|_p$ is a Hilbert norm and $S_p(M)$ becomes a Hilbert space with $\| \cdot \|_p$. Moreover, the inner product corresponding to $\| \cdot \|_p$ is given by
\begin{equation}
\langle \xi, \eta \rangle_p = \sum_{\sigma \in \Gamma} \lambda_\sigma^{2p} \overline{\langle Z_\sigma, \xi \rangle} \langle Z_\sigma, \eta \rangle, \quad \xi, \eta \in S_p(M).
\end{equation}
Here $(Z_{\sigma}, \xi)$ means the complex conjugate of $(Z_{\sigma}, \xi)$.

**Lemma 2.2.** [20] For each $p \geq 0$, one has $\{Z_{\sigma} \mid \sigma \in \Gamma\} \subset S_p(M)$ and moreover the system $\{\lambda_{\sigma}^p Z_{\sigma} \mid \sigma \in \Gamma\}$ forms an orthonormal basis in $S_p(M)$.

It is easy to see that $\lambda_{\sigma} \geq 1$ for all $\sigma \in \Gamma$. This implies that $\|\cdot\|_p \leq \|\cdot\|_q$ and $S_q(M) \subset S_p(M)$ whenever $0 \leq p \leq q$. Thus we actually get a chain of Hilbert spaces of functionals of $M$:

\[
\cdots \subset S_{p+1}(M) \subset S_p(M) \subset \cdots \subset S_1(M) \subset S_0(M) = L^2(M).
\]

We now put

\[
S(M) = \bigcap_{p=0}^{\infty} S_p(M)
\]

and endow it with the topology generated by the norm sequence $\{\|\cdot\|_p\}_{p \geq 0}$. Note that, for each $p \geq 0$, $S_p(M)$ is just the completion of $S(M)$ with respect to $\|\cdot\|_p$. Thus $S(M)$ is a countably-Hilbert space [3 8]. The next lemma, however, shows that $S(M)$ even has a much better property.

**Lemma 2.3.** [20] The space $S(M)$ is a nuclear space, namely for any $p \geq 0$, there exists $q > p$ such that the inclusion mapping $i_{pq}: S_q(M) \rightarrow S_p(M)$ defined by $i_{pq}(\xi) = \xi$ is a Hilbert-Schmidt operator.

For $p \geq 0$, we denote by $S^*_p(M)$ the dual of $S_p(M)$ and $\|\cdot\|_\ast_p$ the norm of $S^*_p(M)$. Then $S^*_p(M) \subset S^*_q(M)$ and $\|\cdot\|_\ast_p \geq \|\cdot\|_\ast_q$ whenever $0 \leq p \leq q$. The lemma below is then an immediate consequence of the general theory of countably-Hilbert spaces (see, e.g., [3] or [8]).

**Lemma 2.4.** [20] Let $S^*(M)$ the dual of $S(M)$ and endow it with the strong topology. Then

\[
S^*(M) = \bigcup_{p=0}^{\infty} S^*_p(M)
\]

and moreover the inductive limit topology on $S^*(M)$ given by space sequence $\{S^*_p(M)\}_{p \geq 0}$ coincides with the strong topology.

We mention that, by identifying $L^2(M)$ with its dual, one comes to a Gel’fand triple

\[
S(M) \subset L^2(M) \subset S^*(M),
\]

which we refer to as the Gel’fand triple associated with $M$.

**Lemma 2.5.** [20] The system $\{Z_{\sigma} \mid \sigma \in \Gamma\}$ is contained in $S(M)$ and moreover it serves as a basis in $S(M)$ in the sense that

\[
\xi = \sum_{\sigma \in \Gamma} \langle Z_{\sigma}, \xi \rangle Z_{\sigma}, \quad \xi \in S(M),
\]

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(M)$ and the series converges in the topology of $S(M)$.

**Definition 2.1.** [20] Elements of $S^*(M)$ are called generalized functionals of $M$, while elements of $S(M)$ are called testing functionals of $M$. 

Denote by $\langle \langle \cdot, \cdot \rangle \rangle$ the canonical bilinear form on $S^*(M) \times S(M)$, namely
\begin{equation}
\langle \langle \Phi, \xi \rangle \rangle = \Phi(\xi), \quad \Phi \in S^*(M), \xi \in S(M),
\end{equation}
where $\Phi(\xi)$ means $\Phi$ acting on $\xi$ as usual. Note that $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(M)$, which is different from $\langle \langle \cdot, \cdot \rangle \rangle$.

**Definition 2.2.** For $\Phi \in S^*(M)$, its Fock transform is the function $\hat{\Phi}$ on $\Gamma$ given by
\begin{equation}
\hat{\Phi}(\sigma) = \langle \langle \Phi, Z_{\sigma} \rangle \rangle, \quad \sigma \in \Gamma,
\end{equation}
where $\langle \langle \cdot, \cdot \rangle \rangle$ is the canonical bilinear form.

It is easy to verify that, for $\Phi, \Psi \in S^*(M)$, $\Phi = \Psi$ if and only if $\hat{\Phi} = \hat{\Psi}$. Thus a generalized functional of $M$ is completely determined by its Fock transform. The following theorem characterizes generalized functionals of $M$ through their Fock transforms.

**Lemma 2.6.** Let $F$ be a function on $\Gamma$. Then $F$ is the Fock transform of an element $\Phi$ of $S^*(M)$ if and only if it satisfies
\begin{equation}
|F(\sigma)| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma
\end{equation}
for some constants $C \geq 0$ and $p \geq 0$. In that case, for $q > p + \frac{1}{2}$, one has
\begin{equation}
\|\Phi\|_{-q} \leq C \left[ \sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p)} \right]^{\frac{1}{2}}
\end{equation}
and in particular $\Phi \in S^*_p(M)$.

### 3. Convergence Theorems for Generalized Functional Sequences

Let $M = (M_n)_{n \in \mathbb{N}}$ be the same discrete-time normal martingale as described in Section 2. In the present section, we apply the Fock transform (see Definition 2.2) to establish convergence theorems for generalized functionals of $M$.

In order to prove our main results in a convenient way, we first give a preliminary proposition, which is an immediate consequence of the general theory of countably normed spaces, especially nuclear spaces [3, 7, 8], since $S(M)$ is a nuclear space (see Lemma 2.3).

**Proposition 3.1.** Let $\Phi, \Phi_n \in S^*(M), n \geq 1$, be generalized functionals of $M$. Then the following conditions are equivalent:

(i) The sequence $(\Phi_n)$ converges weakly to $\Phi$ in $S^*(M)$;

(ii) The sequence $(\Phi_n)$ converges strongly to $\Phi$ in $S^*(M)$;

(iii) There exists a constant $p \geq 0$ such that $\Phi, \Phi_n \in S^*_p(M), n \geq 1$, and the sequence $(\Phi_n)$ converges to $\Phi$ in the norm of $S^*_p(M)$.

Here we mention that “$(\Phi_n)$ converges strongly (resp. weakly) to $\Phi$” means that $(\Phi_n)$ converges to $\Phi$ in the strong (resp. weak) topology of $S^*(M)$. For details about various topologies on the dual of a countably normed space, we refer to [3, 7].

The next theorem is one of our main results, which offers a criterion in terms of the Fock transform for checking whether or not a sequence in $S^*(M)$ is strongly convergent.

**Theorem 3.2.** Let $\Phi, \Phi_n \in S^*(M), n \geq 1$, be generalized functionals of $M$. Then the sequence $(\Phi_n)$ converges strongly to $\Phi$ in $S^*(M)$ if and only if it satisfies:
(1) \( \hat{\Phi}_n(\sigma) \to \hat{\Phi}(\sigma) \) for all \( \sigma \in \Gamma \);

(2) There are constants \( C \geq 0 \) and \( p \geq 0 \) such that

\[
\sup_{n \geq 1} |\hat{\Phi}_n(\sigma)| \leq C \lambda_\sigma^p, \quad \sigma \in \Gamma.
\]

Proof. The “only if” part. Let \( (\Phi_n) \) converge strongly to \( \Phi \) in \( S^*(M) \). Then, we obviously have

\[
\hat{\Phi}_n(\sigma) = \langle \Phi_n, Z_\sigma \rangle \to \langle \Phi, Z_\sigma \rangle = \hat{\Phi}(\sigma), \quad \sigma \in \Gamma,
\]

because \( \{Z_\sigma \mid \sigma \in \Gamma \} \subset S(M) \) and \( (\Phi_n) \) also converges weakly to \( \Phi \). On the other hand, by Proposition 3.1, we know that there exists \( p \geq 0 \) such that \( \Phi, \Phi_n \in S^*_p(M) \), \( n \geq 1 \), and \( (\Phi_n) \) converges to \( \Phi \) in the norm of \( S^*_p(M) \), which implies that \( C \equiv \sup_{n \geq 1} \|\Phi_n\|_p < \infty \). Therefore

\[
\sup_{n \geq 1} |\hat{\Phi}_n(\sigma)| = \sup_{n \geq 1} \|\langle \Phi_n, Z_\sigma \rangle\| \leq \sup_{n \geq 1} \|\Phi_n\|_p \|Z_\sigma\|_p = C \lambda_\sigma^p, \quad \sigma \in \Gamma.
\]

The “if” part. Let \( (\Phi_n) \) satisfy conditions (1) and (2). Then, by taking \( q > p + \frac{1}{2} \) and using Lemma 2.4, we get

\[
\sup_{n \geq 1} \|\Phi_n\|_{-q} \leq C \left[ \sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p)} \right]^{\frac{1}{2}},
\]

in particular \( \Phi_n \in S^*_q(M) \), \( n \geq 1 \). On the other hand, \( \{Z_\sigma \mid \sigma \in \Gamma \} \) is total in \( S_q(M) \), which, together with (3.2) as well as the property

\[
\langle \Phi_n, Z_\sigma \rangle = \hat{\Phi}_n(\sigma) \to \hat{\Phi}(\sigma) = \langle \Phi, Z_\sigma \rangle, \quad \sigma \in \Gamma,
\]

implies that \( \Phi \in S^*_q(M) \) and

\[
\langle \Phi_n, \xi \rangle \to \langle \Phi, \xi \rangle, \quad \forall \xi \in S_q(M).
\]

Thus \( (\Phi_n) \) converges weakly to \( \Phi \) in \( S^*(M) \), which together with Proposition 3.1 implies that \( (\Phi_n) \) converges strongly to \( \Phi \) in \( S^*(M) \). \( \square \)

In a similar way we can prove the following theorem, which is slightly different from Theorem 3.2 but more convenient to use.

**Theorem 3.3.** Let \( (\Phi_n) \subset S^*(M) \) be a sequence of generalized functionals of \( M \). Suppose \( (\hat{\Phi}_n(\sigma)) \) converges for all \( \sigma \in \Gamma \), and moreover there are constants \( C \geq 0 \) and \( p \geq 0 \) such that

\[
\sup_{n \geq 1} |\hat{\Phi}_n(\sigma)| \leq C \lambda_\sigma^p, \quad \sigma \in \Gamma.
\]

Then there exists a generalized functional \( \Phi \in S^*(M) \) such that \( (\Phi_n) \) converges strongly to \( \Phi \).

### 4. \( M \)-generalized martingales and their convergence theorems

In this section, we first introduce a type of generalized martingales associated with \( M \), which we call \( M \)-generalized martingales, and then we use the Fock transform to give necessary and sufficient conditions for such a generalized martingale to be strongly convergent. Some other convergence results are also obtained.

For a nonnegative integer \( n \geq 0 \), we denote by \( \Gamma_{\lceil n \rceil} \) the power set of \( \{0, 1, \cdots, n\} \), namely

\[
\Gamma_{\lceil n \rceil} = \{ \sigma \mid \sigma \subset \{0, 1, \cdots, n\} \}.
\]

(1) \( \hat{\Phi}_n(\sigma) \to \hat{\Phi}(\sigma) \) for all \( \sigma \in \Gamma \);

(2) There are constants \( C \geq 0 \) and \( p \geq 0 \) such that

\[
\sup_{n \geq 1} |\hat{\Phi}_n(\sigma)| \leq C \lambda_\sigma^p, \quad \sigma \in \Gamma.
\]
Clearly $\Gamma_n \subset \Gamma$. We use $I_{n|}$ to mean the indicator of $\Gamma_{n|}$, which is a function on $\Gamma$ given by

(4.2) \[ I_{n|}(\sigma) = \begin{cases} 1, & \sigma \in \Gamma_{n|}; \\ 0, & \sigma \notin \Gamma_{n|}. \end{cases} \]

**Definition 4.1.** A sequence $(\Phi_n)_{n \geq 0} \subset S^*(M)$ is called an $M$-generalized martingale if it satisfies that

(4.3) \[ \hat{\Phi}_n(\sigma) = I_{n|}(\sigma)\hat{\Phi}_{n+1}(\sigma), \quad \sigma \in \Gamma, \; n \geq 0, \]

where $I_{n|}$ mean the indicator of $\Gamma_{n|}$ as defined by (4.2).

Let $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$ be the filtration on $(\Omega, \mathcal{F}, P)$ generated by $Z = (Z_n)_{n \geq 0}$, namely

(4.4) \[ \mathcal{F}_n = \sigma \{ Z_k \mid 0 \leq k \leq n \}, \quad n \geq 0. \]

The following theorem justifies Definition 4.1.

**Theorem 4.1.** Suppose $(\xi_n)_{n \geq 1} \subset \mathcal{L}^2(M)$ is a martingale with respect to filtration $\mathfrak{F} = (\mathcal{F}_n)_{n \geq 0}$. Then $(\xi_n)_{n \geq 1}$ is an $M$-generalized martingale.

**Proof.** By the assumptions, $(\xi_n)_{n \geq 1}$ satisfies that the following conditions

(4.5) \[ \xi_n = E[\xi_{n+1} \mid \mathcal{F}_n], \quad n \geq 0, \]

where $E[\cdot \mid \mathcal{F}_n]$ means the conditional expectation given $\sigma$-algebra $\mathcal{F}_n$. Note that

\[ E[Z_\tau \mid \mathcal{F}_n] = I_{n|}(\tau)Z_\tau, \quad \tau \in \Gamma, \]

which, together with (4.5) and the expansion $\xi_{n+1} = \sum_{\tau \in \Gamma} (Z_\tau, \xi_{n+1})Z_\tau$, gives

\[ \xi_n = E[\xi_{n+1} \mid \mathcal{F}_n] = \sum_{\tau \in \Gamma} (Z_\tau, \xi_{n+1})E[Z_\tau \mid \mathcal{F}_n] = \sum_{\tau \in \Gamma} (Z_\tau, \xi_{n+1})I_{n|}(\tau)Z_\tau. \]

Taking Fock transforms yields

\[ \hat{\xi}_n(\sigma) = \sum_{\tau \in \Gamma} (\xi_{n+1}, Z_\tau)I_{n|}(\tau)\hat{Z}_\tau(\sigma) = (\xi_{n+1}, Z_\sigma)I_{n|}(\sigma) = I_{n|}(\sigma)\hat{\xi}_{n+1}(\sigma), \]

where $\sigma \in \Gamma$. Thus $(\xi_n)_{n \geq 1}$ is an $M$-generalized martingale.

The next theorem gives a necessary and sufficient condition in terms of the Fock transform for an $M$-generalized martingale to be strongly convergent.

**Theorem 4.2.** Let $(\Phi_n)_{n \geq 1} \subset S^*(M)$ be an $M$-generalized martingale. Then the following two conditions are equivalent:

(1) $(\Phi_n)_{n \geq 1}$ is strongly convergent in $S^*(M)$;

(2) There are constants $C \geq 0$ and $p \geq 0$ such that

(4.6) \[ \sup_{n \geq 1} |\hat{\Phi}_n(\sigma)| \leq C \lambda_\sigma^p, \quad \sigma \in \Gamma. \]

**Proof.** By Theorem 4.1 we need only to prove “(2) $\Rightarrow$ (1)”. Let $\sigma \in \Gamma$ be taken. Then by the definition of $M$-generalized martingales (see Definition 4.1) we have

\[ \hat{\Phi}_m(\sigma) = I_{m|}(\sigma)\hat{\Phi}_{m+k}(\sigma), \quad m, k \geq 0. \]

Now take $n_0 \geq 0$ such that $\sigma \in \Gamma_{n_0}$. Then $I_{n_0}(\sigma) = 1$ and moreover

\[ \hat{\Phi}_{n_0}(\sigma) = I_{n_0}(\sigma)\hat{\Phi}_n(\sigma) = \hat{\Phi}_n(\sigma), \quad n > n_0, \]

where $\sigma \in \Gamma_{n_0}$.
which implies $(\widehat{\Phi}_n(\sigma))$ converges. Thus, by Theorem 3.3 $(\Phi_n)_{n \geq 1}$ is strongly convergent in $S^*(M)$.

\[ \square \]

**Theorem 4.3.** Let $D$ be a subset of $S^*(M)$. Then the following two conditions are equivalent:

1. There is a constant $p \geq 0$ such that $D$ is contained and bounded in $S^*_p(M)$;
2. There are constants $C \geq 0$ and $p \geq 0$ such that

\[ \sup_{\Phi \in D} |\widehat{\Phi}(\sigma)| \leq C \lambda_\sigma^p, \quad \sigma \in \Gamma. \]

**Proof.** Obviously, condition (1) implies condition (2). We now verify the inverse implication relation. In fact, under condition (2), by using Lemma 2.6 we have

\[ \sup_{\Phi \in D} \|\Phi\|_{-q} \leq C \left( \sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p)} \right)^{\frac{1}{2}}, \]

where $q > p + \frac{1}{2}$, which clearly implies condition (1). \[ \square \]

The next theorem shows that for an $M$-generalized martingale, its strong (weak) convergence is just equivalent to its strong (weak) boundedness.

**Theorem 4.4.** Let $(\Phi_n)_{n \geq 1} \subset S^*(M)$ be an $M$-generalized martingale. Then the following conditions are equivalent:

1. $(\Phi_n)_{n \geq 1}$ is strongly convergent in $S^*(M)$;
2. $(\Phi_n)_{n \geq 1}$ is weakly bounded in $S^*(M)$;
3. $(\Phi_n)_{n \geq 1}$ is strongly bounded in $S^*(M)$;
4. $(\Phi_n)_{n \geq 1}$ is bounded in $S^*_p(M)$ for some $p \geq 0$.

**Proof.** Clearly, conditions (2), (3) and (4) are equivalent each other because $S(M)$ is a nuclear space (see Lemma 2.3). Using Theorems 4.2 and 4.3 we immediately know that conditions (1) and (4) are also equivalent. \[ \square \]

5. **Applications**

In the last section we show some applications of our main results.

Recall that the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ is an orthonormal basis of $L^2(M)$. Now if we write

\[ \Psi_n^{(0)} = \sum_{\tau \in \Gamma_n} Z_\tau, \quad n \geq 0, \]

then $(\Psi_n^{(0)})_{n \geq 0} \subset L^2(M)$, and moreover $(\Psi_n^{(0)})_{n \geq 0}$ is a martingale with respect to filtration $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$. However, $(\Psi_n^{(0)})_{n \geq 0}$ is not convergent in $L^2(M)$ since

\[ \|\Psi_n^{(0)}\| = \sqrt{\#(\Gamma_n)} = 2^{n+1} \rightarrow \infty \quad (\text{as } n \rightarrow \infty), \]

where $\#(\Gamma_n)$ means the cardinality of $\Gamma_n$ as a set and $\| \cdot \|$ the norm in $L^2(M)$.

**Proposition 5.1.** The sequence $(\Psi_n^{(0)})_{n \geq 0}$ defined above is an $M$-generalized martingale, and moreover it is strongly convergent in $S^*(M)$. 

\[ \square \]
Thus (Φ_{n}(5.6) and (5.7) that Definition A.1. product and norm, respectively. normal martingales. For details we refer to [18, 21].

On the other hand, by using (5.3), we get (5.7) Φ_{n} ∈ S^{*} \times S(M) and the inner product in L^{2}(M), we have

\( \hat{Ψ}_{n}^{(0)}(σ) = ⟨Ψ_{n}^{(0)}, Z_{σ}⟩ = ⟨Ψ_{n}^{(0)}, Z_{σ}⟩ = I_{n}[σ], \quad σ ∈ Γ, \quad n ≥ 0, \)

which implies that

\[ \sup_{n ≥ 0} |\hat{Ψ}_{n}^{(0)}(σ)| ≤ C\lambda_{σ}^{p}, \quad σ ∈ Γ \]

with \( C = 1 \) and \( p = 0 \). It then follows from Theorem 4.2 that \( (Ψ_{n}^{(0)})_{n ≥ 0} \) is strongly convergent in \( S^{*}(M) \).

Recall that [20], for two generalized functionals Φ_{1}, Φ_{2} ∈ S^{*}(M), their convolution \( Φ_{1} * Φ_{2} \) is defined by

\( \hat{Φ}_{1} * \hat{Φ}_{2}(σ) = \hat{Φ}_{1}(σ)\hat{Φ}_{2}(σ), \quad σ ∈ Γ. \)

The next theorem provides a method to construct an \( M \)-generalized martingale through the \( M \)-generalized martingale \( (Ψ_{n}^{(0)})_{n ≥ 0} \) defined in (5.1).

**Theorem 5.2.** Let \( Φ ∈ S^{*}(M) \) be a generalized functional and define

\( Φ_{n} = Ψ_{n}^{(0)} * Φ, \quad n ≥ 0. \)

Then \( (Φ_{n})_{n ≥ 0} \) is an \( M \)-generalized martingale, and moreover it converges strongly to \( Φ \) in \( S^{*}(M) \).

**Proof.** By Lemma 2.6 there exist some constants \( C ≥ 0 \) and \( p ≥ 0 \) such that

\( |\hat{Φ}(σ)| ≤ C\lambda_{σ}^{p}, \quad σ ∈ Γ. \)

On the other hand, by using (5.3), we get

\( \hat{Φ}_{n}(σ) = \hat{Ψ}_{n}(σ)\hat{Φ}(σ) = I_{n}[σ]\hat{Φ}(σ), \quad σ ∈ Γ, \quad n ≥ 0, \)

which, together with the fact \( I_{n}[σ] I_{n+1}[σ] = I_{n}[σ] \), gives

\( \hat{Φ}_{n}(σ) = I_{n}[σ]\hat{Φ}_{n+1}(σ), \quad σ ∈ Γ, \quad n ≥ 0. \)

Thus \( (Φ_{n})_{n ≥ 0} \) is an \( M \)-generalized martingale. Additionally, it easily follows from (5.6) and (5.7) that \( \hat{Φ}_{n}(σ) → \hat{Φ}(σ) \) for each \( σ ∈ Γ \) and

\[ \sup_{n ≥ 0} |\hat{Φ}_{n}(σ)| = \sup_{n ≥ 0} [I_{n}[σ]|\hat{Φ}(σ)| ≤ C\lambda_{σ}^{p}, \quad σ ∈ Γ. \]

Therefore, by Theorem 3.3 we finally find \( (Φ_{n})_{n ≥ 0} \) converges strongly to \( Φ \). \( \square \)

**Appendix**

In this appendix, we provide some basic notions and facts about discrete-time normal martingales. For details we refer to [18, 21].

Let \( (Ω, F, P) \) be a given probability space with \( E \) denoting the expectation with respect to \( P \). We denote by \( L^{2}(Ω, F, P) \) the usual Hilbert space of square integrable complex-valued functions on \( (Ω, F, P) \) and use \( ⟨·, ·⟩ \) and \( ∥ · ∥ \) to mean its inner product and norm, respectively.

**Definition A.1.** A stochastic process \( M = (M_{n})_{n ∈ N} \) on \( (Ω, F, P) \) is called a discrete-time normal martingale if it is square integrable and satisfies:
(i) $\mathbb{E}[M_0|\mathcal{F}_{-1}] = 0$ and $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$ for $n \geq 1$;
(ii) $\mathbb{E}[M_0^2|\mathcal{F}_{-1}] = 1$ and $\mathbb{E}[M_n^2|\mathcal{F}_{n-1}] = M_{n-1}^2 + 1$ for $n \geq 1$,
where $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(M_k; 0 \leq k \leq n)$ for $n \in \mathbb{N}$ and $\mathbb{E}[\cdot|\mathcal{F}_k]$ means the conditional expectation.

Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale on $(\Omega, \mathcal{F}, P)$. Then one can construct from $M$ a process $Z = (Z_n)_{n \in \mathbb{N}}$ as

$$Z_0 = M_0, \quad Z_n = M_n - M_{n-1}, \quad n \geq 1. \quad (A.1)$$

It can be verified that $Z$ admits the following properties:

$$\mathbb{E}[Z_n|\mathcal{F}_{n-1}] = 0 \quad \text{and} \quad \mathbb{E}[Z_n^2|\mathcal{F}_{n-1}] = 1, \quad n \in \mathbb{N}. \quad (A.2)$$

Thus, it can be viewed as a discrete-time noise.

**Definition A.2.** Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale. Then the process $Z$ defined by (A.2) is called the discrete-time normal noise associated with $M$.

The next lemma shows that, from the discrete-time normal noise $Z$, one can get an orthonormal system in $L^2(\Omega, \mathcal{F}, P)$, which is indexed by $\sigma \in \Gamma$.

**Lemma A.1.** Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale and $Z = (Z_n)_{n \in \mathbb{N}}$ the discrete-time normal noise associated with $M$. Define $Z_\emptyset = 1$, where $\emptyset$ denotes the empty set, and

$$Z_\sigma = \prod_{i \in \sigma} Z_i, \quad \sigma \in \Gamma, \sigma \neq \emptyset. \quad (A.3)$$

Then $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms a countable orthonormal system in $L^2(\Omega, \mathcal{F}, P)$.

Let $\mathcal{F}_\infty = \sigma(M_n; n \in \mathbb{N})$, the $\sigma$-field over $\Omega$ generated by $M$. In the literature, $\mathcal{F}_\infty$-measurable functions on $\Omega$ are also known as functionals of $M$. Thus elements of $L^2(\Omega, \mathcal{F}_\infty, P)$ can be called square integrable functionals of $M$. For brevity, we usually denote by $L^2(M)$ the space of square integrable functionals of $M$, namely

$$L^2(M) = L^2(\Omega, \mathcal{F}_\infty, P). \quad (A.4)$$

**Definition A.3.** The discrete-time normal martingale $M$ is said to have the chaotic representation property if the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ defined by (A.3) is total in $L^2(M)$.

So, if the discrete-time normal martingale $M$ has the chaotic representation property, then the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ is actually an orthonormal basis for $L^2(M)$, which is a closed subspace of $L^2(\Omega, \mathcal{F}, P)$ as is known.

**Remark A.1.** Émery [6] called a $\mathbb{Z}$-indexed process $X = (X_n)_{n \in \mathbb{Z}}$ satisfying (A.2) a novation and introduced the notion of the chaotic representation property for such a process.

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