AN INTEGRALITY THEOREM OF GROSSHANS
OVER ARBITRARY BASE RING

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Abstract. We revisit a theorem of Grosshans and show that it holds over arbitrary
commutative base ring $k$. One considers a split reductive group scheme $G$ acting on a
$k$-algebra $A$ and leaving invariant a subalgebra $R$. Let $U$ be the unipotent radical of
a split Borel subgroup scheme. If $R^U = A^U$ then the conclusion is that $A$ is integral
over $R$.

Introduction

In [G92] Grosshans considered a reductive algebraic group $G$ defined over an
algebraically closed field $k$ acting algebraically on a commutative $k$-algebra $A$.
Fix a Borel subgroup $B$ with unipotent radical $U$. Then Grosshans considered
the smallest $G$-invariant $k$-subalgebra $G \cdot A^U$ of $A$ that contains the fixed point
algebra $A^U$. He showed that $A$ is integral over $G \cdot A^U$. If $R$ is any other $G$-invariant
$k$-subalgebra of $A$ that contains $A^U$ it then follows that $A$ is integral over $R$. One
of the tools used by Grosshans is what is called power reductivity in [FvdK]. As it
is shown in [FvdK] that power reductivity holds over arbitrary commutative base
ring $k$, we now set out to prove the integrality result of Grosshans in the same
generality. We need a little care, as we are not even assuming that the ground ring
is noetherian.

1. Preliminaries

We use an arbitrary commutative ring $k$ as base ring. Let $A$ be a commutative
$k$-algebra. We say that an affine algebraic group scheme $G$ acts on $A$ if $A$ is a
$G$-module [J] and the multiplication map $A \otimes_k A \rightarrow A$ is a $G$-module map. Then
the coaction $A \rightarrow A \otimes_k k[G]$ is an algebra homomorphism. One also says that $G$
acts rationally on $A$ by algebra automorphisms. Geometrically it means that $G$
acts from the right on $\text{Spec } A$.

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Lemma 1. Let $G$ be a smooth affine algebraic group scheme over $k$. Let $G$ act on the commutative $k$-algebra $A$. Then the nilradical of $A$ is a $G$-submodule.

Proof. (Thanks to Angelo Vistoli http://mathoverflow.net/questions/68366/ for explaining to me that smoothness is the right condition.)

As the base change map $A_{\text{red}} \to A_{\text{red}} \otimes_k k[G]$ is a smooth map, $A_{\text{red}} \otimes_k k[G]$ is reduced, by [EGA4, Prop.(17.5.7)] or by [Stacks, Lemma 033B] with URL http://stacks.math.columbia.edu/tag/033B.

Now let $N$ denote the nilradical of $A$. The coaction $A \to A \otimes_k k[G]$ sends $N$ to the nilradical $N \otimes_k k[G]$ of $A \otimes_k k[G]$. □

From now on let $G = G_k$, where $G_Z$ is a Chevalley group over $\mathbb{Z}$. In other words, $G$ is a split reductive group scheme over $k$ under the conventions of [SGA3]. Choose a split maximal torus $T$, a standard Borel subgroup $B$ and its unipotent radical $U$.

Lemma 2. The coordinate ring $k[G]$ is a free $k$-module.

Proof. As $k[G] = \mathbb{Z}[G_Z] \otimes_{\mathbb{Z}} k$, it suffices to treat the case $k = \mathbb{Z}$. Now the coordinate ring of $G$ is a subring of the coordinate ring of the big cell. And the coordinate ring of the big cell is clearly free as a $\mathbb{Z}$-module. Now use that a submodule of a free $\mathbb{Z}$-module is free [HS, Chap. I, Thm. 5.1]. □

Lemma 3. If $V$ is a $G$-module and $v \in V$, then the $G$-submodule generated by $v$ exists and is finitely generated as a $k$-module.

Proof. As $k[G]$ is a free $k$-module, this follows from [SGA3, Exposé VI, Lemme 11.8]. □

See also [S, Prop. 3]. Note that the existence result in the Lemma does not follow from the fact that $G$ is flat over $k$ [SGA3, Exposé VI, Édition 2011, Remarque 11.10.1].

Definition 1. Recall that we call a homomorphism of $k$-algebras $f : A \to B$ power surjective [FvdK, Def. 2.1] if for every $b \in B$ there is an $n \geq 1$ so that the power $b^n$ is in the image of $f$.

A flat affine group scheme $H$ over $k$ is called power reductive [FvdK, Def. 2] if the following holds.

Property (Power Reductivity). Let $L$ be a cyclic $k$-module with trivial $H$-action. Let $M$ be a rational $H$-module, and let $\varphi$ be an $H$-module map from $M$ onto $L$. Then there is a positive integer $d$ such that the $d$th symmetric power of $\varphi$ induces a surjection:

$$(S^d M)^H \to S^d L.$$ 

Here $V^H = H^0(H,V)$ denotes the submodule of invariants in an $H$-module $V$.

Proposition 4. Let $H$ be a flat affine algebraic group scheme over $k$. The following are equivalent:

1. $H$ is power reductive,
2. for every power surjective $H$-homomorphism of commutative $k$-algebras $f : A \to B$ the map $A^H \to B^H$ is power surjective.