SELF-DUAL VORTICES IN CHERN-SIMONS HYDRODYNAMICS

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Abstract

The classical theory of non-relativistic charged particle interacting with $U(1)$ gauge field is reformulated as the Schrödinger wave equation modified by the de-Broglie-Bohm quantum potential nonlinearity. For, $(1 - \hbar^2)$ deformed strength of quantum potential the model is gauge equivalent to the standard Schrödinger equation with Planck constant $\hbar$, while for the strength $(1 + \hbar^2)$, to the pair of diffusion-anti-diffusion equations. Specifying the gauge field as Abelian Chern-Simons (CS) one in 2+1 dimensions interacting with the Nonlinear Schrödinger field (the Jackiw-Pi model), we represent the theory as a planar Madelung fluid, where the Chern-Simons Gauss law has simple physical meaning of creation the local vorticity for the fluid flow. For the static flow, when velocity of the center-of-mass motion (the classical velocity) is equal to the quantum one (generated by quantum potential velocity of the internal motion), the fluid admits $N$-vortex solution. Applying the Auberson-Sabatier type gauge transform to phase of the vortex wave function we show that deformation parameter $\hbar$, the CS coupling constant and the quantum potential strength are quantized. Reductions of the model to 1+1 dimensions, leading to modified NLS and DNLS equations with resonance soliton interactions are discussed.

1. Introduction

Nonlinear extension of the Schrödinger equation by the ”quantum potential” nonlinear term has been considered long time ago in connection with a stochastic quantization problem [1], and corrections to quantum mechanics from quantum gravity [2]. It appears also in the wave theoretical formulation of classical mechanics [3], and the dispersionless limit of nonlinear wave dynamics [4]. As was shown by Sabatier [5] this extension preserves the Lagrangian structure. Moreover, by proper transformation of the wave function’s phase, Auberson and Sabatier [6] obtained linearization of the model, which depending on the strength of the quantum potential, has to appear in the form of the Schrödinger equation with rescaled potential or as the pair of the time-reversed diffusion equations. Due to this linearization no soliton type solutions were found [5,6]. Meanwhile, recently we have been considering the nonlinear version of the Bohm’s formulation of the quantum mechanics [7], namely the problem of the Nonlinear Schrödinger (NLS) soliton under the
influence of the quantum potential [4,8]. Application of the Auberson-Sabatier type phase transform to this problem with over-critical strength of the quantum potential, allowed us reduce the problem to the pair of time-reversed reaction-diffusion equations, representing an imaginary time version of the real q-r NLS type system [9] (SL(2,R) reduction of the Zhakharov-Shabat problem). Then, constructing two soliton solution we found a resonance character of their mutual interaction [4,8].

In the present paper we consider the influence of quantum potential on the planar vortex in 2+1 dimensional problem for the Nonlinear Schrödinger equation interacting with the Chern-Simons gauge field. Application of the Auberson-Sabatier type transform, affecting the phase of the wave function, dramatically changes parameters of the vortex configurations. In the Madelung representation we reformulate the model as a rotational planar hydrodynamics. Then, the self-dual limit, admitting N-vortex solutions, has simple physical interpretation as a condition of equality between ”classical velocity” (velocity of the center-of-mass) and the ”quantum” one (velocity of the motion in the center-of-mass frame associated with the internal ”spin motion” or zitterbewegung).

In Section 2 we reformulate the classical dynamics of charged particle interacting with Abelian gauge field as a nonlinear Schrödinger type wave equation. Deforming properly the strength of quantum potential we recover the standard Schrödinger equation, where the deformation parameter plays role of the Planck constant. In Section 3 we specify the gauge field as an Abelian Chern-Simons one, interacting with the Nonlinear Schrödinger equation (NLS), and derive corresponding rotational Madelung type hydrodynamics, its dispersionless limit and deformations. Section 4 devoted to the quantum velocity and its properties. For the static flow moving with a velocity equal to the quantum one we reduce the problem to the Liouville equation and describe corresponding vortex configurations. From conditions of non-singularity and single valuedness we find quantization condition for coupling constants. Dimensional reduction to one dimensional NLS equation and its modification by quantum potential are considered in Section 5. In Conclusions we briefly discuss our results.

2. Nonlinear Wave Equation of Classical Dynamics

The classical dynamics of charged non-relativistic particle in U(1) gauge field \( A_\mu = (A_0, A) \), with the Hamiltonian function

\[
H = \frac{p^2}{2m} + \frac{e}{c} A_0 + U,
\]

(2.1)

is described by the Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial t} + H(\nabla S, A_0, A, U) = 0,
\]

(2.2)

where for the momentum \( p \) we substitute

\[
p = \nabla S + \frac{e}{c} A.
\]

(2.3)
Combining (2.2) with the Liouville equation

\[ \frac{\partial \rho}{\partial t} + \nabla (\rho V) = 0, \tag{2.4} \]

for the density \( \rho \) of integral invariant in the gradient dynamical system

\[ \dot{x} = V = \frac{1}{m} p = \frac{1}{m} [\nabla S + \frac{e}{c} A], \tag{2.5} \]

we have the system of equations

\[
\begin{cases}
\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S + \frac{e}{c} A)^2 + \frac{e}{c} A_0 + U = 0,
\frac{\partial \rho}{\partial t} + \nabla (\rho V) = 0.
\end{cases}
\tag{2.6}
\]

This classical system is representable in the wave form. Introducing the complex wave function ("order parameter")

\[ \psi = \sqrt{\rho e^{iS}}, \tag{2.7} \]

we rewrite equations (2.6) as the single nonlinear wave equation

\[ i D_0 \psi + \frac{1}{2m} D^2 \psi - U \psi = \frac{1}{2m} \frac{\Delta |\psi|}{|\psi|} \psi, \tag{2.8} \]

where \( D_0 = \partial_t + \frac{e}{c} A_0, \ D = \nabla + \frac{e}{c} A \). The last equation has form of the Schrödinger equation (without any Planck constant) modified by the so called "quantum potential" term in the right hand side. It admits all the usual solutions of classical mechanics but does not allow superpositions of these solutions. Since the system (2.6) describes the formal semiclassical limit of the quantum mechanical Schrödinger equation, Eq. (2.8) can be considered as its dispersionless limit. In fact, the wave equation (2.8) is covariant under the gauge transformations

\[ \psi \rightarrow \psi e^{i\alpha}, \ A \rightarrow A - \frac{e}{c} \nabla \alpha, \tag{2.9} \]

generating shift of the classical action

\[ S \rightarrow S + \alpha. \]

Then, being U(1) gauge invariant, the additional term on the r.h.s. of (2.8) completely compensates the corresponding gauge invariant contribution from dispersion in the l.h.s.

As we mentioned above, Eq.(2.8) does not contain the Planck constant. But if we consider contribution from quantum potential in the r.h.s. of Eq. (2.8) deformed by a constant \( \hbar^2 \)

\[ i D_0 \psi + \frac{1}{2m} D^2 \psi - U \psi = \frac{1}{2m} (1 - \hbar^2) \frac{\Delta |\psi|}{|\psi|} \psi, \tag{2.10} \]

then in terms of the new wave function

\[ \chi = \sqrt{\rho e^{\frac{i}{\hbar^2} S}}, \tag{2.11} \]
we recover the standard linear Schrödinger equation

\[ i\hbar D_0\chi + \frac{\hbar^2}{2m}D^2\chi - U\chi = 0, \quad (2.12) \]

where \(\hbar\) plays the role of the Planck constant. Thus for \(\hbar \neq 0\) Eq.(2.10) is gauge equivalent to the Schrödinger equation and for \(\hbar = 0\) reduces to the nonlinear wave equation of classical mechanics (2.8). Moreover, for \(\hbar = \pm 1\) it reduces directly to the linear Schrödinger equation and its complex conjugation. From another site if the deformation of Eq.(2.8) appears with an opposite sign as

\[ iD_0\psi + \frac{1}{2m}D^2\psi - U\psi = \frac{1}{2m}(1 + \hbar^2)\frac{\Delta|\psi|}{|\psi|}\psi, \quad (2.13) \]

nevertheless to the same classical limit \(\hbar = 0\) as for Eq.(2.10), it cannot be linearized in the form of the Schrödinger equation by transformation (2.11). However, if we notice that Eq.(2.13) can be reduced to Eq.(2.10) by formal analytical substitution for the Planck constant to the pure imaginary value \(\hbar \rightarrow i\hbar\) (in quantum mechanics similar continuation to the classically inaccessible region leads to the exponentially decaying (growing) wave function). Then, written in terms of two real functions

\[ Q^\pm = \sqrt{\rho e^{\mp \frac{i}{\hbar}S}}, \quad (2.14) \]

Eq.(2.13) and its complex conjugation become the pair of decoupled diffusion-antidiffusion equations

\[ \pm\hbar D_0Q^\pm + \frac{\hbar^2}{2m}D^2Q^\pm - UQ^\pm = 0, \quad (2.15) \]

similar to the one considered by Schrödinger in 1931 [10]. From the above consideration we see that Schrödinger equation perturbed by quantum potential includes as a particular cases the classical mechanics (\(\hbar = 0\)), the quantum mechanics (\(\hbar = \pm |\hbar|\)) and the pair of diffusion-antidiffusion equations (\(\hbar = i|\hbar|\)).

3. Chern-Simons Hydrodynamics

The semiclassical limit has been applied recently to defocusing Nonlinear Schrödinger (NLS) equation

\[ i\hbar \partial_t\chi + \frac{\hbar^2}{2m}\Delta\chi + 2g|\chi|^2\chi = 0, \quad (3.1) \]

\((g < 0)\) in one [11] and two space dimensions [12] and provides an analytical tool to describe shock waves in nonlinear optics and vortices in superfluid. Decomposing the wave function like in (2.11) one derives quantum deformation of the Hamilton-Jacobi equation by quantum potential, or after differentiation according to space coordinates, the Madelung fluid. In the formal semiclassical limit \(\hbar \rightarrow 0\) (before shocks appear), one neglects contribution
from the quantum potential and fluid becomes the Euler system. Then in terms of the wave function (2.7) we have dispersionless NLS equation

\[ i \partial_t \psi + \frac{1}{2m} \Delta \psi + 2g|\psi|^2 \psi = \frac{1}{2m} \frac{\Delta |\psi|}{|\psi|} \psi. \]  

(3.2)

The quantum deformation of the last equation in the form

\[ i \partial_t \psi + \frac{1}{2m} \Delta \psi + 2g|\psi|^2 \psi = (1 - \hbar^2) \frac{1}{2m} \frac{\Delta |\psi|}{|\psi|} \psi, \]  

(3.3)

refORMulated for the wave function (2.11), leads us again to the original equation (3.1).

The NLS model (3.1) interacting with Chern-Simons gauge field in 2+1 dimensions is called the Jackiw-Pi (JP) model and describes anyonic phenomena [13]. The semiclassical limit of anyons requires to study this model in the limit when \( \hbar \to 0 \), or similarly to the case of Eq. (3.2), its perturbations by quantum potential.

To describe the deformed theory we consider the Lagrangian

\[ L = \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{i}{2} (\bar{\psi}D_0 \psi - \psi \bar{D}_0 \bar{\psi}) - \frac{1}{2m} |D\psi|^2 + (1 - \hbar^2) \frac{1}{2m} (\nabla |\psi|)^2 + g|\psi|^4, \]  

(3.4)

where \( D_\mu = \partial_\mu + \frac{ie}{\hbar c} A_\mu \), leading to the system of equations of motion

\[ iD_0 \psi + \frac{1}{2m} D^2 \psi + 2g|\psi|^2 \psi = (1 - \hbar^2) \frac{1}{2m} \frac{\Delta |\psi|}{|\psi|} \psi, \]  

(3.5a)

\[ \partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa c} \tilde{\psi} \psi, \]  

(3.5b)

\[ \partial_0 A_j - \partial_j A_0 = -\frac{ie}{2mc\kappa} \epsilon_{jk} (\bar{\psi}D_k \psi - \psi \bar{D}_k \bar{\psi}). \]  

(3.5c)

Decomposing the wave function \( \psi = \sqrt{\rho} \exp(iS) \) as in Eq.(2.7), and introducing the new function \( \chi = \sqrt{\rho} \exp(iS/\hbar) \) as in Eq.(2.11), we have the Jackiw-Pi model

\[ i\hbar (\partial_0 + \frac{ie}{\hbar c} A_0) \chi + \frac{\hbar^2}{2m} (\nabla + \frac{ie}{\hbar c} A)^2 \chi + 2g|\chi|^2 \chi = 0, \]  

(3.6a)

\[ \partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa c} \bar{\chi} \chi, \]  

(3.6b)

\[ \partial_0 A_j - \partial_j A_0 = -\frac{ie\hbar}{2mc\kappa} \epsilon_{jk} [\bar{\chi}(\partial_k + \frac{ie}{\hbar c} A_k) \chi - \chi(\partial_k - \frac{ie}{\hbar c} A_k) \bar{\chi}]. \]  

(3.6c)

Corresponding Lagrangian follows from (3.4) as

\[ L = \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \]
\[
\frac{i\hbar}{2} [\bar{\chi}(\partial_0 + \frac{ie}{\hbar c} A_0)\chi - \chi(\partial_0 - \frac{ie}{\hbar c} A_0)\bar{\chi}] - \frac{\hbar^2}{2m}(\nabla - \frac{ie}{\hbar c} A)\bar{\chi}(\nabla + \frac{ie}{\hbar c} A)\chi + g|\chi|^4. 
\] (3.7)

In the above system of equations (3.6) the deformation parameter \( \hbar \) plays the role similar to the Planck constant. Both of the systems (3.5) and (3.6) admit the same hydrodynamical (Madelung type) representation. From equation (3.5a) we obtain the quantum Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial t} + \frac{mV^2}{2} + \frac{e}{c} A_0 - 2g\rho - \frac{\hbar^2}{2m} \Delta \sqrt{\rho} \sqrt{\rho} = 0, 
\] (3.8)

and the continuity one

\[
\frac{\partial \rho}{\partial t} + \nabla(\rho V) = 0, 
\] (3.9)

where like in Eq.(2.5) we introduced the local velocity field

\[
V = \frac{1}{m} [\nabla S + \frac{e}{c} A]. 
\] (3.10)

Then equations (3.6b,c) become of the form

\[
\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa c} \rho, 
\] (3.11)

\[
\partial_0 A_j - \partial_j A_0 = \frac{e}{\kappa c} \epsilon_{jk} \rho V_k. 
\] (3.12)

Now we can completely exclude from consideration the vector potentials \( A \) in favor of the velocity field (3.10). It is worth to note that the last one is an explicitly gauge invariant variable. Thus from Eqs.(3.12) and (3.8) we derive the Euler equation for velocity \( V \),

\[
\frac{\partial V}{\partial t} + (V \nabla) V = -\frac{1}{m} \nabla P, 
\] (3.13)

with the pressure

\[
P = -2g\rho - \frac{\hbar^2}{2m} \Delta \sqrt{\rho} \sqrt{\rho}. 
\] (3.14)

The Chern-Simons Gauss law (3.11) in terms of our hydrodynamical variables becomes

\[
\nabla \times V = \frac{e^2}{m\kappa c^2} \rho. 
\] (3.15)

The last condition has simple meaning of a rotational fluid, such that at any point of the fluid with nonvanishing density \( \rho \) the local vorticity is nonzero. The system of equations (3.9), (3.13-15) determines the Madelung fluid for our model. As well as the velocity field mentioned above in (3.10) it is explicitly \( U(1) \) gauge invariant. Moreover the continuity equation (3.9) from this system is not independent. It appears as a consistency condition for the Chern-Simons Gauss law (3.15) during the time evolution. To check this it is
sufficient simply differentiate (3.15) according to the time variable and use Eq.(3.13). Thus we have hydrodynamical model defined by two equations

\[
\begin{align*}
\frac{\partial V}{\partial t} + (V \nabla) V &= -\frac{1}{m} \nabla(-2g\rho - \frac{\hbar^2}{2m} \nabla\sqrt{\rho}), \\
\nabla \times V &= \frac{e^2}{m\kappa c^2} \rho.
\end{align*}
\] (3.16)

The semiclassical or dispersionless limit of this model when \( \hbar \to 0 \), is given by

\[
\begin{align*}
\frac{\partial V}{\partial t} + (V \nabla) V &= -\frac{1}{m} \nabla(-2g\rho), \\
\nabla \times V &= \frac{e^2}{m\kappa c^2} \rho.
\end{align*}
\] (3.17)

The nonlinear wave form of these equations follows directly from the system (3.5) and Lagrangian from Eq.(3.4) with \( \hbar = 0 \).

4. Quantum velocity and stationary flow

Recently in the set of papers, an interpretation of quantum potential in terms of velocity of internal motion or the *zitterbewegung* was given [14]. In that approach starting from the Pauli current a decomposition of the nonrelativistic local velocity in two parts, one parallel and the other orthogonal to the impulse has been obtained. The first part, determined by \( \nabla S \), is recognized as the ”classical” part, corresponding to the velocity of the center-of-mass. While the second one, called the ”quantum” one, is the velocity of motion in the center-of-mass frame (the internal ”spin motion” or Schrödinger’s *zitterbewegung*). Then the contribution of the quantum potential to the Lagrangian (3.7) has simple physical meaning of the kinetic energy for this last motion

\[
\frac{\hbar^2}{2m} (\nabla|\chi|^2) = \frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 = \frac{mV_q^2}{2},
\] (4.1)

where the ”quantum” velocity is defined by

\[
V_q = \frac{\nabla \rho \times s}{m\rho}.
\] (4.2)

For the planar motion in the \( x - y \) plane, \( \nabla z = 0 \) and \( s_x = s_y = 0 \), so that \( s_z = \frac{\hbar}{2} \), and we have for components of quantum velocity

\[
(V_q)_x = \frac{\hbar}{2m} \frac{\partial_y \rho}{\rho}, \quad (V_q)_y = -\frac{\hbar}{2m} \frac{\partial_x \rho}{\rho},
\] (4.3)

or

\[
(V_q)_i = \frac{\hbar}{2m} \epsilon_{ij} \frac{\partial_j \rho}{\rho}.
\] (4.4)

Differentiating in time the last equation and using the continuity equation (3.9) we obtain

\[
\partial_0 V_q + (V \nabla) V_q = 0,
\] (4.5)
which means that $V_q$ is propagating with the main flow velocity $V$, i.e. it is a velocity of the inner motion. Moreover, by direct computation from Eq. (4.4) we have divergenceless condition for the quantum velocity flow
\[
\nabla(\rho V_q) = 0. \tag{4.6}
\]
This condition applied to the continuity equation (3.9) for a flow propagating with quantum velocity
\[
V = \pm V_q, \tag{4.7}
\]
having meaning of a special planar motion when velocities of classical (center-of-mass) motion and quantum (internal) motion coincide, leads to the stationary flow
\[
\partial_0 \rho = 0. \tag{4.8}
\]
But from another site for the stationary flow when
\[
\partial_0 V = 0, \tag{4.9}
\]
and
\[
\frac{\kappa g}{e^2} = \pm \frac{\hbar}{2mc^2}, \tag{4.10}
\]
the Madelung fluid equation (3.16) can be rewritten as
\[
\frac{m}{2} \nabla_j (V - V_q)(V + V_q) - \frac{e^2}{\kappa c^2} \rho \epsilon_{jk}(V \mp V_q)_k \mp \frac{\hbar}{2} \nabla_j [\nabla \times (V \mp V_q)] = 0, \tag{4.11}
\]
which is identically satisfied by Eq. (4.7). Deriving this equation we explored the identity
\[
\frac{\hbar^2}{2m} \nabla \sqrt{\rho} = \frac{mV_q^2}{2} - \frac{\hbar}{2} [\partial_1(V_q)_2 - \partial_2(V_q)_1],
\]
and the Chern-Simons Gauss law (3.15). Thus, under condition (4.7) it remains to satisfy only the vorticity condition (3.16) for the quantum velocity
\[
\nabla \times V_q = \pm \frac{e^2}{\kappa mc^2} \rho, \tag{4.12}
\]
acquiring by definition (4.4) Liouville’s equation form
\[
\Delta \ln \rho = \mp \frac{2e^2}{\kappa hc^2} \rho. \tag{4.13}
\]
We stress again that the Liouville equation in our model has meaning of the vorticity condition for quantum flow. Solutions of the model are well known [13,15]. We just mention the polar symmetric case (for the sign minus)
\[
\rho = 4 \frac{\kappa hc^2 N^2}{e^2 r^2} \left[ \left( \frac{r}{r_0} \right)^N + \left( \frac{r_0}{r} \right)^N \right]^{-2}, \tag{4.14}
\]
which is regular for \( N \geq 1 \) and appears from the general solution

\[
\rho = \alpha \frac{|\zeta'(z)|^2}{(1 + |\zeta(z)|^2)^2},
\]

(4.15)

when

\[
\zeta(z) = \frac{c_N}{(z - z_0)^N}, \quad z = x + iy.
\]

(4.16)

Now there are two physical conditions in the original \((\psi, A)\) formulation, restricting our solution. From regularity of the gauge potential \(A\) we fix the phase of \(\chi = \sqrt{\rho}\exp(iS/\hbar)\) (see Eqs.(3.6)) as \(S/\hbar = (N - 1)\theta, \quad \theta = \tan^{-1}(x_2/x_1)\), and restrict \(N\) to be an integer for single-valued \(\chi\). But single-valuedness of the original function \(\psi = \sqrt{\rho}\exp(iS)\) in (3.5) requires integer valuedness for the product

\[
(N-1)\hbar = \text{integer},
\]

(4.17)

valuedness of which for any integer \(N\) leads to an integer valuedness of deformation parameter

\[
\hbar = n,
\]

(4.18)

and as a consequence of (4.10), we find the quantization condition

\[
\frac{\kappa g}{e^2} = \pm \frac{n}{2mc^2}, \quad (n = 1, 2, 3, \ldots).
\]

(4.19)

The last relation means that the Chern-Simons coupling constant and the quantum potential strength must be quantized

\[
\kappa = n \frac{e^2}{2gmc^2}, \quad \hbar = 1 - n^2 = (1 - n)(1 + n).
\]

(4.20)

At the end of this section we present the Lagrangian formulation of our fluid model (3.16). After excluding the vector potentials \(A_\mu\) from (3.7) we get

\[
L = \frac{\kappa m^2 c^2}{2e^2} \epsilon_{\mu\nu\lambda} V_\mu \partial_\nu V_\lambda - \rho V_0 - \rho \frac{mV^2}{2} - \rho \frac{mV_\perp^2}{2} + g\rho^2,
\]

(4.21)

where \(V_0\) plays the role of Lagrange multiplier. The Hamiltonian (constrained by Chern-Simons Gauss law)

\[
H = \int \rho \frac{mV^2}{2} + \rho \frac{mV_\perp^2}{2} - g\rho^2,
\]

(4.22)

has simple interpretation as the sum of kinetic energies of the classical and quantum motions, plus self-interaction energy. As easy to check it vanishes for the self-dual flow (4.7) with fixed constants (4.10).

5. 1+1 dimensional reduction
For the one directional flow, assume in the $x$ direction when $\partial_2 = 0$, the system (3.16) reduces to

$$\partial_0 V_1 + V_1 \partial_1 V_1 = \frac{1}{m} \partial_1 (2g\rho + \frac{\hbar^2}{2m} \frac{\partial^2}{\sqrt{\rho}}), \quad (5.1a)$$

$$\partial_0 V_2 + V_1 \partial_1 V_2 = 0, \quad (5.1b)$$

$$\partial_1 V_2 = \frac{e^2}{\hbar mc^2}\rho. \quad (5.1c)$$

Substituting $\partial_1 V_2$ from the last equation to the second one we find that velocity field component $V_2$ is decoupled completely from Eq.(5.1a) and is determined by the first order system of equations

$$\partial_1 V_2 = \frac{e^2}{\hbar mc^2}\rho, \quad \partial_0 V_2 = -\frac{e^2}{\hbar mc^2}\rho V_1. \quad (5.2)$$

Then compatibility condition for the last system is just the continuity equation for one dimensional flow

$$\partial_0 \rho + \partial_1 (\rho V_1) = 0. \quad (5.3)$$

Equations (5.1a) and (5.3) determine the Madelung fluid in one space dimension. Rewriting them for the wave function

$$\chi = \sqrt{\rho} e^{\frac{i}{\hbar} \int_{-\infty}^{x} V_1}, \quad (5.4)$$

we obtain the NLS model

$$i\hbar \partial_0 \chi + \frac{\hbar^2}{2m} \partial_1^2 \chi + 2g|\chi|^2\chi = 0. \quad (5.5)$$

We note that for the negative sign of the quantum deformation in the system (3.5), corresponding to the replacement $\hbar^2 \rightarrow -\hbar^2$, the one dimensional reduction of the fluid is given by the system (5.3), (5.1a), where in the last equation we must change the sign of the quantum potential contribution. The result is that for the wave function

$$\psi = \sqrt{\rho} e^{\frac{i}{\hbar} \int_{-\infty}^{x} V_1}, \quad (5.6)$$

we get NLS modified by quantum potential

$$i\partial_0 \psi + \frac{1}{2m} \partial_1^2 \psi + 2g|\psi|^2\psi = (1 + \hbar^2) \frac{1}{2m} \frac{\partial^2}{|\psi|} |\psi|^2 \psi. \quad (5.7)$$

At the same time in terms of two real functions

$$Q^\pm = \sqrt{\rho} e^{\pm \frac{i}{\hbar} \int_{-\infty}^{x} V_1}, \quad (5.8)$$

we have "dissipative" (reaction-diffusion) version of NLS

$$\pm \hbar \partial_0 Q^\pm + \frac{\hbar^2}{2m} \partial_1^2 Q^\pm + 2gQ^+ Q^- Q^\pm = 0. \quad (5.9)$$
This analogy allows us to derive bilinear representation for (5.7). Solution for the wave function \( \psi \) is represented in terms of three real functions \( G^\pm, F \),

\[
\psi = \frac{(G^+)^{\frac{1+\imath \hbar}{2}}(G^-)^{\frac{1-\imath \hbar}{2}}}{F}, \quad \bar{\psi} = \frac{(G^+)^{\frac{1+\imath \hbar}{2}}(G^-)^{\frac{1-\imath \hbar}{2}}}{F}
\] (5.9)

satisfying bilinear system of equations

\[
(\pm \hbar D_t - \frac{\hbar^2}{2m} D_x^2)(G^\pm * F) = 0, \quad (5.11a)
\]

\[
\frac{\hbar^2}{2m} D_x^2(F * F) = 2gG^+G^-.
\] (5.11b)

Then, hydrodynamical variables are given by formulae

\[
V_1 = \frac{\hbar}{2m} \partial_1 \ln \frac{G^-}{G^+}, \quad \rho = \frac{\hbar^2}{2mg} \partial_1^2(\ln F).
\] (5.12)

Constructing one and two-soliton solutions we find that in contrast to the NLS case, soliton dynamics of Eq.(5.7) have a reach resonance phenomenology [4,8]. Another reduction of 2+1 dimensional model (3.5) leads to the DNLS type equation and its reaction-diffusion analog. Details of reduction procedure and resonance soliton interactions would be published elsewhere.

6. Conclusions

We reformulated the classical dynamics of non-relativistic particle interacting with Abelian gauge field as a nonlinear wave equation with quantum potential. Then we considered deformations of this equation and found two cases depending on the sign of deformation. For one of the signs the standard Schrödinger model with deformation parameter playing the role of the Planck constant was obtained. While for the second sign we got diffusion-anti-diffusion equation. Specifying the gauge field as Chern-Simons one and including cubic nonlinear term to the Schrödinger equation we found the dispersionless limit of the Jackiw-Pi model which could be useful descriptive of the anyon’s semiclassical limit. The deformation of this model is equivalent to the standard JP model which we represented as a rotational hydrodynamics of Madelung type fluid. Special flow in this fluid, when velocities of the classical and quantum motion coincide, lead to the Liouville equation admitting vortex configurations. The similar equation, as we found before [4,8], defines the event horizon for black hole type solution in the one dimensional NLS with quantum potential (5.7). Moreover, in terms of the wave function it is exactly the Chern-Simons self-(anti-self)duality condition [13]. In fact the self-duality equations are first order equations why they can be interpreted in terms of velocity fields and we hope that our interpretation could be applied to other models as well.

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