We point out an interesting connection between fluid dynamics and minimal surface theory: When gluing helicoids into a minimal surface, the limit positions of the helicoids correspond to a ‘vortex crystal’, an equilibrium of point vortices in two-dimensional fluid that move together as a rigid body. While vortex crystals have been studied for almost 150 years, the gluing construction of minimal surfaces is relatively new. As a consequence of the connection, we obtain many new minimal surfaces and some new vortex crystals by simply comparing notes.

1. Introduction

In 2005, Traizet & Weber [1] glued helicoids into screw-motion invariant minimal surfaces with helicoidal ends. For the glue construction to succeed, the limit positions of the helicoids must satisfy a balancing condition and a non-degenerate condition. For simplicity, they assumed that the helicoids are aligned along a straight line, and noted that the roots of Hermite polynomials provide examples of balanced and non-degenerate configurations. Recently, the second named author implemented a similar construction without this assumption [2].

The main goal of this note is to point out an interdisciplinary connection: the balanced configurations of helicoids correspond to binary vortex crystals. Here, a vortex crystal [3], also known as vortex equilibrium, is a configuration of vortices in two-dimensional fluids that moves as a rigid body, i.e. without any change of shape and size. A vortex crystal is binary if the circulations of vortices are \pm 1. We will recall vortex dynamics in §1. For an example of this
connection: the ‘definite’ configurations in [1], given by roots of Hermite polynomials, correspond to vortex crystals that trace back to the nineteenth century [4].

Rotating vortex crystals correspond to screw-motion invariant minimal surfaces; this connection was readily established in [1,2]. Our main results in §2 establish the other cases of the claimed connection. The construction will be given in §4, where we glue helicoids into translation-invariant minimal surfaces, corresponding to translating or periodic stationary vortex crystals. The construction will only be sketched because similar constructions have been repeated many times in the literature [5–7].

In view of the connection, we will compare notes between fluid dynamics and minimal surface theory, and obtain new examples in §3 for both vortex crystals and minimal surfaces.

The minimal surface theory would benefit a lot because, in about 150 years, the fluid dynamics community has accumulated a large collection of examples of binary vortex crystals. In particular, stationary and translating vortex crystals have been analytically obtained with the help of Adler–Moser polynomials. When symmetries are imposed, non-degeneracy was recently verified for translating Adler–Moser examples [8]. They then lead to many examples.

On the other hand, the idea to glue helicoids into minimal surfaces is only about 18 years old. Nevertheless the minimal surface community is also in possession of a few examples that would lead to new vortex crystals. For instance, it is very hard to observe periodic vortex crystals in experiments, hence there have been very few relevant physics studies; but our knowledge on periodic minimal surfaces is quite advanced [9,10].

It is worth mentioning that, in natural sciences, minimal surfaces have been used to model bicontinuous structures in liquid crystals, block copolymers, biological membranes and nuclear pasta, etc. (see e.g. [11]). In particular, helicoid structures are used to model topological defects (screw dislocations) in layered structures [12–18]. On the other hand, analogy between these structures and vortices in superfluids and superconductors has long been noted [19,20].

Nevertheless, rigorous mathematical treatment of helicoidal minimal surfaces was missing until Traizet developed the node-opening technique [1,5,6,21,22] for constructing minimal surfaces. The present work is the first to state a well-defined mathematical analogy between point vortices and helicoids. We are able to push the analogy to its full generality and to provide rigorous constructions using the node-opening technique.

Remark 1.1. Our work is not the first to connect minimal surfaces with fluid dynamics. In [23], a correspondence was established between hollow vortices and minimal surfaces bounded by horizontal symmetry curves.

2. Vortex dynamics

We recommend [3] for general reference on vortex crystals.

Incompressible and inviscid flow in zero gravity is governed by the Euler equation

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u = -\nabla p
\]  

(2.1)

under the incompressible condition \( \nabla \cdot u = 0 \), where \( u \) is the flow velocity field and \( p \) is the pressure. The pressure term can be eliminated by taking the curl of the Euler equation, which results in the reformulation

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) w = w \cdot \nabla u
\]  

(2.2)

in terms of the vorticity field \( w := \nabla \times u \). For two-dimensional flows, the right-hand side of (2.2) vanishes, giving the transport equation

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) w = 0.
\]

We see that vorticity is transported in the velocity field as material elements.
A point vortex in the plane is given by the vorticity field
\[ w(x) = \frac{\sigma}{2\pi} \delta(x), \]
where \( \delta \) is the Dirac delta function, and \( \sigma \in \mathbb{R} \) is the circulation of the vortex. In the complex coordinate, a point vortex gives rise to a velocity field
\[ \frac{u(z)}{2\pi i} = \frac{\sigma}{z}. \]
We use \((p_k, \sigma_k)_{1 \leq k \leq n}\) to denote a configuration of \(n\) point vortices located at \(p_k\) with circulation \(\sigma_k\), \(k = 1, \ldots, n\). Each vortex is advected as a material particle by the velocity field produced by other vortices. So the dynamics of the vortex configuration is governed by the ordinary differential equation
\[ \frac{d}{dt} p_j = -\frac{1}{2\pi i} \sum_{k \neq j} \frac{\sigma_k}{p_j - p_k}, \quad \forall 1 \leq j \leq n, \quad (2.3) \]
also known as the Kirchhoff–Routh dynamics [24, 25].

We say that the configuration \((p_k, \sigma_k)_{1 \leq k \leq n}\) is a vortex crystal if it moves as a rigid body. If this is the case, we have \(dp_j/dt = v + i\omega p_j\), where \(v \in \mathbb{C}\) and \(\omega \in \mathbb{R}\) are constant for all vortices. Hence vortex crystals are characterized by the balance equations
\[ F_j := -v + i\omega p_j + \frac{1}{2\pi i} \sum_{k \neq j} \frac{\sigma_k}{p_j - p_k} = 0, \quad \forall 1 \leq j \leq n. \quad (2.4) \]
Multiplying (2.4) by \(\sigma_j\), and summing the conjugates over \(j\), we obtain
\[ v \sum_{j=1}^{n} \sigma_j + i\omega \sum_{j=1}^{n} \sigma_j |p_j|^2 = 0. \quad (2.5) \]
Multiplying (2.4) by \(\sigma_j p_j\), and taking the sum over \(j\), we obtain
\[ v \sum_{j=1}^{n} \sigma_j p_j - i\omega \sum_{j=1}^{n} \sigma_j |p_j|^2 = \frac{1}{4\pi} \left[ \left( \sum_{k=1}^{n} \sigma_k \right)^2 - \sum_{k=1}^{n} \sigma_k^2 \right]. \quad (2.6) \]

**Remark 2.1.** Recently, the balance equation (2.4) for vortex crystals also appeared in scenarios such as three-dimensional travelling waves of the Gross–Pitaskii equation [26] and clustered vortex rings for axis-symmetric three-dimensional Euler flow [27].

In this paper, we only consider binary vortex crystals, so \(\sigma_k = \pm 1\). Let \(n_{\pm}\) be, respectively, the number of vortices with circulation \(\pm 1\), and write \(m = n_+ - n_- = \sum \sigma_k\). We distinguish three cases.

(i) We say that the vortex crystal is *rotating* if \(\omega \neq 0\). In this case, the governing equation (2.4) is invariant under Euclidean rotations. Moreover, we may assume that \(v = 0\) up to a translation, and that \(\omega = 1\) up to a Euclidean scaling. After these normalizations, (2.5) and (2.6) give
\[ \sum_{k=1}^{n} \sigma_k p_k = 0 \quad \text{and} \quad \sum_{k=1}^{n} \sigma_k |p_k|^2 = \frac{m^2 - n}{4\pi}. \]

(ii) We say that the vortex crystal is *translating* if \(\omega = 0\) but \(v \neq 0\). Then (2.5) implies that \(m = 0\) so \(n_+ = n_- = n/2\). In this case, (2.4) is invariant under translations. We may assume that \(v = 1\) up to a complex scaling (Euclidean scaling and rotation). After this normalization, (2.6) implies that
\[ \sum_{k=1}^{n} \sigma_k p_k = -\frac{n}{4\pi}. \]
(iii) We say that the vortex crystal is stationary if \( \omega = 0 \) and \( v = 0 \). Then (2.6) implies that \( m^2 = n \), hence \( n_+ \) and \( n_- \) must be successive triangular numbers. In this case, (2.4) is invariant under Euclidean similarities (translations, rotations and scalings).

A vortex crystal is said to be stable if any sufficiently small perturbation does not diverge. Linear and nonlinear analyses have been carried out on the stability. We say that a vortex crystal is non-degenerate\(^1\) if the Jacobian matrix \( D\mathbf{F} = (\partial F_i/\partial p_j)_{ij} \) for the autonomous system (2.3) has the maximum possible rank. We have seen that, for a rotating (resp. translating, stationary) vortex crystal, the dynamics are invariant under Euclidean rotations (resp. translations, similarities), so the maximum possible real rank of its Jacobian is \( 2n - 1 \) (resp. \( 2n - 2, 2n - 4 \)).

We may also consider singly or doubly periodic vortex crystals. In this case, it is convenient to consider a co-rotating reference frame in which the periods are fixed. Then the vortex crystal is

\[ F_j := -\tau + \frac{1}{2\pi i} \sum_{k \neq j} \pi \sigma_k \cot \pi (p_j - p_k) = 0, \quad \forall 1 \leq j \leq n \]

in an appropriate reference frame.

— Assume that the vortex crystal is singly periodic, i.e. invariant under a single translation \( T \in \mathbb{C} \). Up to rotations and scalings, we may fix \( T = 1 \). Then the vortices can be seen as lying in the annulus \( \mathbb{C}/(1) \). Up to translations, a vortex crystal in the annulus is governed by

\[ F_j := -\tau + \frac{1}{2\pi i} \sum_{k \neq j} \sigma_k (\zeta(p_j - p_k; \tau) - \xi(p_j - p_k; \tau)) = 0, \quad \forall 1 \leq j \leq n \]

in an appropriate reference frame, where \( \zeta(z; \tau) \) is the Weierstrass zeta function on the torus \( \mathbb{C}/(1, \tau) \) and \( \xi(z; \tau) = 2x\zeta(1/2; \tau) + 2y\zeta(\tau/2; \tau) \) with \( z = x + y\tau, x, y \in \mathbb{R} \).

In either case, the maximum possible real rank of the Jacobian is \( 2n - 2 \), which defines the degeneracy of these vortex crystals.

### 3. Main results

The connection between rotating vortex crystals and screw-motion invariant minimal surfaces was already established in the following theorem [2].

**Theorem 3.1 (Rotating vortex crystal).** Let \((p_k, \sigma_k)_{1 \leq k \leq n}\) be a normalized non-degenerate rotating vortex crystal with \( n \) vortices at \( p_1, \ldots, p_n \in \mathbb{C} \). There then exists a one-parameter family \((M_\varepsilon)_{0 < \varepsilon < \delta}\) of embedded minimal surfaces in \( \mathbb{R}^3 \) such that:

(i) \( M_\varepsilon \) admits a screw symmetry \( S_\varepsilon \) composed of a vertical translation \( 2\pi(0, 0, 1) \) and a rotation around the vertical axis by an angle \( 2\pi \varepsilon \).

(ii) The quotient \( M_\varepsilon / S_\varepsilon \) is of genus \( n - 1 \) and has two ends. The ends are helicoidal if \( m = \sum \sigma_k \neq 0 \), or planar if \( m = 0 \).

(iii) As \( \varepsilon \to 0 \), up to a translation, \( M_\varepsilon \) converges to a helicoid of period \( (0, 0, 2\pi) \) in the neighbourhood of \((p_i/\varepsilon, 0)\) for each \( 1 \leq i \leq n \). The helicoid is right-handed (resp. left-handed) if \( \sigma_i = 1 \) (resp. \(-1\)).

\(^1\)In fluid dynamics literature, degenerate vortex crystals were often said to be 'neutrally stable'.
When a symmetry group is imposed, the vortex crystal is said to be non-degenerate if the symmetry is a Euclidean isometry that maps vortices to identical (resp. opposite) vortices.

3.2 with imposed symmetry. Euclidean translations for translating vortex crystals. We have the following version of theorem normalized translating vortex crystal with \( n \) vortices at \( p \quad \in \mathbb{C} - \{ p_1^0, \ldots, p_n^0 \} \),

\[
 f(z) = \sum_{i=1}^{n} \sigma_i \arg(z - p_i^0), \quad z \in \mathbb{C} - \{ p_1^0, \ldots, p_n^0 \},
\]

the multigraph of \( f(z) + \pi \), and vertical lines over the points \( p_i^0 \).

Conversely, for any such family of screw-motion invariant minimal surfaces, in the limit \( \varepsilon \to 0 \), the positions and handednesses of the helicoids give a rotating vortex crystal (possibly degenerate).

So we only need to construct translation-invariant minimal surfaces, corresponding to translating or stationary vortex crystals. More specifically, we will prove the theorems below.

**Theorem 3.2 (Finite translating vortex crystals).** Let \( (p_k^0, \sigma_k)_{1 \leq k \leq n} \) be a non-degenerate translating vortex crystal normalized with velocity \( v = 1 \), with \( n \) vortices at \( p_1^0, \ldots, p_n^0 \in \mathbb{C} \). Then there exists a one-parameter family \( (M_\varepsilon)_{0 < \varepsilon < \delta} \) of embedded minimal surfaces in \( \mathbb{R}^3 \) such that:

(i) \( M_\varepsilon \) admits a translational symmetry \( T_{0, \varepsilon} = 2\pi(-2\pi\varepsilon,0,1) \). So \( M_\varepsilon \) is a singly periodic minimal surface in \( \mathbb{R}^3 \).

(ii) The quotient \( M_\varepsilon / T_{0, \varepsilon} \) is of genus \( n - 1 \) and has two helicoidal ends if the vortex crystal is translating.

(iii) As \( \varepsilon \to 0 \), up to a translation, \( M_\varepsilon \) converges to a helicoid of period \( (0,0,2\pi) \) in the neighbourhood of \( (p_1^0 / \varepsilon, 0) \) for each \( 1 \leq i \leq n \). The helicoid is right-handed (resp. left-handed) if \( \sigma_i = 1 \) (resp. -1).

(iv) After rescaling the horizontal coordinates by \( \varepsilon \), the resulting surface (no longer minimal) converges to the union of the multigraph of the multivalued function

\[
 f(z) = \sum_{i=1}^{n} \sigma_i \arg(z - p_i^0), \quad z \in \mathbb{C} - \{ p_1^0, \ldots, p_n^0 \},
\]

the multigraph of \( f(z) + \pi \), and vertical lines over the points \( p_i^0 \).

Conversely, for any such family of singly periodic minimal surfaces, in the limit \( \varepsilon \to 0 \), the positions and handednesses of the helicoids give a translating vortex crystal (possibly degenerate).

**Remark 3.3.** The helicoid limits of minimal surfaces are necessarily balanced, hence they correspond to vortex crystals. In the opposite direction, for a vortex crystal to give rise to a minimal surface, we need the non-degeneracy condition that allows the use of Implicit Function Theorem in our construction. The connection is not clear for degenerate vortex crystals, which may or may not give rise to minimal surfaces.

Unfortunately, we are aware of very few finite, non-degenerate, translating vortex crystals (example 3.5). So the theorem above does not bring us many new minimal surfaces.

Nevertheless, in example 3.5, we will obtain new minimal surfaces from translating vortex crystals with imposed symmetry. For a vortex crystal, a *circulation-preserving* (resp. -reversing) symmetry is a Euclidean isometry that maps vortices to identical (resp. opposite) vortices. When a symmetry group is imposed, the vortex crystal is said to be *non-degenerate* if the only perturbations that preserve the balance as well as the symmetry are the trivial ones, namely Euclidean translations for translating vortex crystals. We have the following version of theorem 3.2 with imposed symmetry.

**Theorem 3.4 (Translating vortex crystals with imposed symmetry).** Let \( (p_k^0, \sigma_k)_{1 \leq k \leq n} \) be a normalized translating vortex crystal with \( n \) vortices at \( p_1^0, \ldots, p_n^0 \in \mathbb{C} \). Let \( G \) be a symmetry group of \( (p_k^0, \sigma_k) \). If the vortex crystal is non-degenerate with the symmetry group \( G \) imposed, then the conclusion of theorem 3.2 holds. Moreover, the symmetry group \( G \) induces a symmetry group of the resulting minimal surfaces.
In particular, a circulation-reversing reflection in the vortex crystal induces an order-2 rotational symmetry around a straight line in the minimal surface, and a circulation-preserving reflection induces a reflection symmetry for the minimal surface.

We did not manage to establish a similar connection between singly periodic minimal surfaces and finite stationary vortex crystals. See remark 5.5 for a detailed explanation.

Periodic vortex crystals will give rise to doubly or triply periodic minimal surfaces, as stated in the following theorems.

**Theorem 3.5 (Singly periodic vortex crystals).** Let \((p^0_k, \sigma_k)_{1 \leq k \leq n}\) be a non-degenerate vortex crystal with \(n\) vortices at \(p^0_k, \ldots, p^0_n \in \mathbb{C}/(1, \tau)\). Then there exists a one-parameter family \((M_{\varepsilon})_{0 < \varepsilon < \delta}\) of embedded minimal surfaces in \(\mathbb{R}^3\) such that:

(i) \(M_{\varepsilon}\) admits translational symmetries along the vectors
\[
T_{0, \varepsilon} = 2\pi (\varepsilon \text{Re}\nu, \varepsilon \text{Im}\nu, 1) \quad \text{and} \quad T_{1, \varepsilon} = (\varepsilon^{-1}, 0, m\pi),
\]
where \(m = \sum \sigma_k\), \(\nu\) is related to the velocity \(\nu\) of the translating vortex crystal by \(\nu = -2\pi \nu\), and \(\nu = 0\) if the vortex crystal is stationary. So \(M_{\varepsilon}\) is a doubly periodic minimal surface.

(ii) The quotient \(M_{\varepsilon}/(T_{0, \varepsilon}, T_{1, \varepsilon})\) is of genus \(n - 1\) and has four Scherk ends (asymptotic to half-planes).

(iii) The flux vector along any closed curve in \(M_{\varepsilon}/(T_{0, \varepsilon}, T_{1, \varepsilon})\) has no vertical component.

(iv) As \(\varepsilon \to 0\), up to a translation, \(M_{\varepsilon}/(T_{1, \varepsilon})\) converges to a helicoid of period \((0, 0, 2\pi)\) in the neighbourhood of \((p^0_i/\varepsilon, 0)\) for each \(1 \leq i \leq n\). The helicoid is right-handed (resp. left-handed) if \(\sigma_i = 1\) (resp. \(-1\)).

Conversely, for any such family of doubly periodic minimal surfaces, in the limit \(\varepsilon \to 0\), the positions and handednesses of the helicoids give a singly periodic vortex crystal (possibly degenerate).

Recall that the flux vector along a closed curve \(\Gamma\) is defined as the integral of the conormal vector along \(\Gamma\) [28]. It can be physically interpreted as the surface tension force along the curve [29].

In theorem 3.5, we actually constructed a 3-parameter family of minimal surfaces, depending on a real parameter \(\varepsilon\) and a complex parameter \(\nu\).

**Theorem 3.6 (Doubly periodic vortex crystals).** Let \((p^0_k, \sigma_k)_{1 \leq k \leq n}\) be a non-degenerate vortex crystal with \(n\) vortices at \(p^0_1, \ldots, p^0_n \in \mathbb{T} = \mathbb{C}/(1, \tau)\). Assume that \(m = \sum \sigma_k = 0\). Then there exists a one-parameter family \((M_{\varepsilon})_{0 < \varepsilon < \delta}\) of embedded minimal surfaces in \(\mathbb{R}^3\) such that:

(i) \(M_{\varepsilon}\) admits translational symmetry along the vectors
\[
T_{0, \varepsilon} = 2\pi (\varepsilon \text{Re}\nu, \varepsilon \text{Im}\nu, 1),
\]
\[
T_{1, \varepsilon} = (\varepsilon^{-1}, 0, \Psi_1(\varepsilon))
\]
\[
T_{2, \varepsilon} = (\varepsilon^{-1} \text{Re}\nu, \varepsilon^{-1} \text{Im}\nu, \Psi_2(\varepsilon)),
\]
where \(\nu\) is related to the velocity \(\nu\) of the translating vortex crystal by \(\nu = -2\pi \nu\), and \(\nu = 0\) if the vortex crystal is stationary. So \(M_{\varepsilon}\) is a triply periodic minimal surface.

(ii) The quotient \(M_{\varepsilon}/(T_{0, \varepsilon}, T_{1, \varepsilon}, T_{2, \varepsilon})\) is of genus \(n + 1\).

(iii) The flux vector along any closed curve in \(M_{\varepsilon}/(T_{0, \varepsilon}, T_{1, \varepsilon}, T_{2, \varepsilon})\) has no vertical component.

(iv) As \(\varepsilon \to 0\), up to a translation, \(M_{\varepsilon}/(T_{1, \varepsilon}, T_{2, \varepsilon})\) converges to a helicoid of period \((0, 0, 2\pi)\) in the neighbourhood of \((p^0_i/\varepsilon, 0)\) for each \(1 \leq i \leq n\). The helicoid is right-handed (resp. left-handed) if \(\sigma_i = 1\) (resp. \(-1\)). Moreover, we have
\[
\Psi_1(\varepsilon) \to -2\pi y \quad \text{and} \quad \Psi_2(\varepsilon) \to 2\pi x \quad \text{as} \quad \varepsilon \to 0,
\]
where \((x, y) \in \mathbb{R}^2\) are defined by \(\sum \sigma_k p_k = x + y \tau\).
Conversely, for any such family of triply periodic minimal surfaces, in the limit $\varepsilon \to 0$, the positions and handednesses of the helicoids give a doubly periodic vortex crystal (possibly degenerate).

In theorem 3.6, we actually constructed a 5-parameter family of minimal surfaces, depending on a real parameter $\varepsilon$ and two complex parameters $\nu$ and $\tau$.

Remark 3.7. In the theorems, the surface is rotated into a position so that the flux vectors are horizontal. We find this choice best to reveal the connection to vortex crystals. The price is that $\Psi_1$ and $\Psi_2$ are left to vary with $\varepsilon$. One could also rotate the surface to fix $\Psi_1 = \Psi_2 \equiv 0$. Then the flux vectors are not horizontal, and the connection to vortex crystals is less direct.

4. Examples

The study on vortex crystals traces back to about 150 years ago, and has accumulated plenty of examples, many of which are binary and hence would imply minimal surfaces. A nice survey of these examples is provided by Aref et al. [3]. Here, we examine the known vortex crystals and their corresponding minimal surfaces. Occasionally, we also have minimal surfaces that lead to new vortex crystals.

Example 4.1 (Linear configuration). Traizet & Weber [1] considered configurations of helicoids along a straight line. In particular, when $p_i^\nu$, $1 \leq i \leq n$, are the roots of $H_n$, the Hermite polynomial of degree $n$, and $\sigma_i = -1$ for all $i$, then the configuration $(p_i^\nu, \sigma_i)_{1 \leq i \leq n}$ is balanced and non-degenerate. In fluid dynamics, the corresponding rotating vortex crystal was first found by Stieltjes in 1885 [4], and has been rediscovered and revisited many times [30–32].

Traizet & Weber [1] also considered a configuration with $n = 2m + 1$ helicoids, $m + 1$ of which lie at the roots of $H_{m+1}$ and have positive handedness, and the remaining $m$ lie at the roots $H_m$ and have negative handedness. This configuration is proved to be balanced and non-degenerate. We are not aware of any discussion of corresponding vortex crystals in the fluid dynamics community. So this is a new example of vortex crystals inspired by the minimal surfaces.

Example 4.2 (Polygonal configuration). A Karcher–Scherk tower with $2n$ wings can be twisted into a configuration of $n$ negatively handed helicoids lying at the vertices of a regular polygon (see figure 1 and [2, Prop. 8.11]). The Fischer–Koch surfaces can be twisted into a similar configuration, only with an extra helicoid, positively or negatively handed, at the centre of the polygon ([2, Prop. 8.13]).

The corresponding rotating vortex crystals were first investigated by Thomson in 1883 [34]. In particular, he famously proved that identical vortices at the vertices of a regular $n$-gon are linearly stable if $n \leq 6$, and linearly unstable if $n \geq 8$, and ‘neutrally stable’ if $n = 7$. The proof was improved and modified many times [35–37].
For our minimal surfaces, this stability analysis means that the helicoid limit of twisted Scherk surface is non-degenerate for $n \neq 7$, for which theorem 3.2 applies. The $n = 7$ case is degenerate, but becomes non-degenerate if we impose the dihedral symmetry of the heptagon [2], so theorem 3.4 applies. Even without imposed symmetry, it was proved that the nonlinear stability still holds for $n = 7$ [38], so a more elaborated version of the implicit function theorem might apply.

**Example 4.3 (Nested polygonal configurations).** Fluid dynamists also investigated vortex crystals where vortices lie on the vertices of several concentric regular polygons [35,39,40]. In some cases, e.g. when the vortices lie on two concentric polygons, with or without an extra vortex at the centre, an algebraic approach is possible. However, we are not aware of any systematic investigation on such configurations.

This line of research overlaps with [2], where the second named author considered configurations with dihedral symmetry, and helicoids all lie on the symmetry lines (including the centre). Each dihedral configuration of helicoids can be seen as corresponding to a nested polygonal vortex crystal.

For instance, the Callahan–Hoffman–Meeks (CHM) surface [41] of genus $2k + 1$, $k \geq 1$ can be twisted into a family of minimal surfaces invariant under screw-motions [42]. Numerical computation suggests a smooth deformation with the angle $\theta$ of the screw symmetry varying within the open interval $(-2\pi/(k + 1), 2\pi/(k + 1))$ [43]. As $\theta$ approaches $\pm 2\pi/(k + 1)$, the surface seems to diverge to a degenerate limit [43]. Computer images suggest that this is a helicoid limit corresponding to a vortex crystal with positive vortices forming a regular $(k + 1)$-gon, and negative vortices forming a concentric regular $(k + 1)$-gon. One may determine from the balance equations (2.4) that the ratio $r$ of the circumradii is one of the two solutions (inverse to each other) to $k((1 - r^k)/(1 + r^k)) = (1 + r^2)/(1 - r^2)$. See [2, fig. 1] for an image with $k = 1$. We are not aware of any discussion of these vortex crystals in the fluid dynamics community, so they might be new.

**Example 4.4 (Numerical examples).** Campbell & Ziff [39,44] have obtained numerical examples of vortex crystals, and claimed to have found linearly stable configurations with up to 30 identical vortices. Their 1978 report is often referred to as the Los Alamos Catalog. Many of their examples were experimentally observed in superfluid helium [33] (figure 1, right). By our connection, they all correspond to screw-motion invariant minimal surfaces.

In many of their examples, the vortices seem to lie on concentric rings, but this impression is not precise. Rather, all their examples admit an axis of symmetry. Asymmetric examples were not found until [45].

**Example 4.5 (Translating vortex crystal).** A translating vortex crystal must consist of an even number $n$ of vortices, with $n/2$ positive vortices and $n/2$ negative ones. The simplest case is a pair of opposite vortices, corresponding to Riemann minimal examples (figure 2). Surprisingly, there is no solution for $n/2 = 2$. Some examples for $n/2 = 3$ and $n/2 = 6$ can be found in [47].

More generally, when $n/2 = j(j + 1)/2$ is a triangular number, binary translating vortex crystals have been found [48–50] with positive vortices at the roots of a $j$th Adler–Moser polynomial $\Theta_j$, and negative vortices at the roots of the corresponding modified Adler–Moser polynomial $\tilde{\Theta}_j$.

The Adler–Moser polynomial $\Theta_j(z)$ actually depends on $n$ complex parameters $\kappa_1 = z, \kappa_2, \ldots, \kappa_j$. Changing these parameters preserves the balance. In fact, these perturbations are linearly independent and span the kernel of the Jacobian [8]. As a consequence, the Adler–Moser translating configurations are degenerate except for the trivial case $n = 2$.

However, under the assumption that $\Theta_j$ has only simple roots, these configurations are proved to be non-degenerate [8] if we impose (up to Euclidean motions) a reflection symmetry in the real axis that preserves circulations and a reflection symmetry in the imaginary axis that reverses circulations. In fact, this symmetry is realized by a unique choice of parameters $\kappa_2, \ldots, \kappa_j$. With this choice, the assumption that $\Theta_j$ has only simple roots is verified for $j \leq 34$ [8].

By theorem 3.4, these symmetric translating Adler–Moser configurations give rise to singly periodic minimal surfaces. The reflection in the real axis becomes a rotational symmetry in the
Figure 2. (a) A Riemann minimal example near the limit consisting of a pair of opposite helicoids (source: 3D-XplorMath Minimal Surface Gallery). (b) A pair of opposite vortices seen in a cross-section of the vortex sheet behind a wing. (source: H. Bippes [46, p. 50]). (Online version in colour.)

Figure 3. (a) An Adler–Moser translating vortex crystal with $n/2 = 6$. (b) Top view of a singly periodic minimal surface arising from the corresponding helicoid configuration. (c) Side view of the same minimal surface. (Online version in colour.)

$x$-axis, and the reflection in the imaginary axis becomes a reflection in the $yz$-plane. See figure 3 for an example with $n/2 = 6$.

Remark 4.6. There have been some generalizations of the Adler–Moser polynomials. See [26,51] for instance.

Example 4.7 (Singly periodic vortex crystals). The doubly periodic Scherk surface can be deformed to a periodic helicoid limit. It corresponds to a singly periodic vortex crystal with a single vortex in the period.

The famous vortex street of von Kármán [52], and more general cases of Dolaptschiew & Maue [53], are the only singly periodic vortex crystals with two (opposite) vortices in the period. They correspond to the helicoid limits of Karcher–Meeks–Rosenberg surfaces [9,54] (see figure 4).

Example 4.8 (Doubly periodic vortex crystals). If a doubly periodic vortex crystal has two vortices in the period, they must be of opposite circulations. Such a configuration is generically non-degenerate. They give rise to triply periodic minimal surfaces of genus 3 (TPMSg3) (figure 5).
for an example. In fact, our construction will assume an orientation-reversing translation, hence the produced examples must all belong to the 5-parameter family of TPMSg3s constructed by Meeks [10].

5. Sketched construction

In [1,2], the main technical issue was the multivaluedness of the Weierstrass data. For translating invariant minimal surfaces, the Weierstrass data is single-valued, so the construction is much easier. The construction below is adapted from [5]. This approach has been repeated many times in the literature [5–7] and is only simpler in our context; hence we will only present a sketch.
(a) Weierstrass data

Let \( \Sigma_+ \) be

- the Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) if the vortex crystal is finite;
- the annuli \( \mathbb{C}/\langle 1 \rangle \) if the vortex crystal is singly periodic;
- the torus \( T_+ = \mathbb{C}/\langle 1, \tau_+ \rangle \) if the vortex crystal is doubly periodic.

Moreover, let \( \Sigma_- \) be

- another copy of \( \Sigma_+ \) if the vortex crystal is finite or singly periodic;
- the torus \( T_- = \mathbb{C}/\langle 1, \tau_- \rangle \) if the vortex crystal is doubly periodic.

Consider \( n \) points \( p = (p_k)_{1 \leq k \leq n} \) in \( \Sigma_+ \) and \( n \) points \( q = (q_k)_{1 \leq k \leq n} \) in \( \Sigma_- \).

The node-opening is parameterized by \( n \) complex numbers \( t = (t_k)_{1 \leq k \leq n} \). If \( t = 0 \), we identify \( p_k \in \mathbb{C}_+ \) and \( q_k \in \mathbb{C}_- \) to form a node. The resulting singular Riemann surface with nodes is denoted \( \Sigma_0 \). If \( t \neq 0 \), we open the nodes as follows. Let \( z_\pm \) be the standard coordinates of \( \mathbb{C}_\pm \). Consider local coordinates \( w_+^k = z_+ - p_k \) in the neighbourhood of \( p_k \) and \( w_-^k = z_- - q_k \) in the neighbourhood of \( q_k \). Fix a small \( \delta > 0 \) such that the disks \( |w_\pm^k| < \delta \) are all disjoint. Then for every \( 1 \leq k \leq n \), we remove the disk \( |w_{k\pm}| < |t_k|/\delta \), and identify the annuli

\[
\frac{|t_k|}{\delta} \leq |w_+^k| \leq \delta \quad \text{and} \quad \frac{|t_k|}{\delta} \leq |w_-^k| \leq \delta
\]

by

\[
w_+^k w_-^k = t_k.
\]

This produces a Riemann surface that we denote \( \Sigma_t \).

We construct the minimal surface using the Weierstrass parameterization

\[
\Sigma_t \ni z \mapsto \text{Re} \int_{z_0}^z (\phi_1, \phi_2, \phi_3)
\]

where \( \phi_1, \phi_2 \) and \( \phi_3 \) are meromorphic forms on \( \Sigma_t \) satisfying the conformality condition

\[
Q = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0. \tag{5.1}
\]

Then the flux vector along a closed curve \( \Gamma \) is given by Pérez & Ros [28]

\[
\text{Im} \int_{\Gamma} (\phi_1, \phi_2, \phi_3).
\]

(b) Equations

We define

\[
\Omega_\pm = \{ z \in \Sigma_\pm | |w_\pm^k| > \delta \text{ for all } 1 \leq k \leq n \}.
\]

Let \( A_k \) be an anticlockwise circle in \( \Omega_+ \) around \( p_k \) and \( A'_k \) be an anticlockwise circle in \( \Omega_- \) around \( q_k \). Note that \( A_k \) is homologous in \( \Sigma_t \) to \( -A'_k \). Let \( B_k \) be a cycle in \( \Sigma_e \) which goes ‘half-way up’ from \( \Sigma_+ \) to \( \Sigma_- \) through the helicoid near \( p_1 \) then ‘half-way down’ through the helicoid near \( p_k \), as in [2]. We need to solve the period problems

\[
\begin{align*}
\text{Re} \int_{A_k} (\phi_1, \phi_2, \phi_3) &= 2\pi a_k (\epsilon \text{Re} v, \epsilon \text{Im} v, 1) \\
\text{Re} \int_{B_k} (\phi_1, \phi_2, \phi_3) &= (0, 0, 0).
\end{align*} \tag{5.2}
\]

We close the \( A \)-period by defining \( \phi \)'s as the unique meromorphic forms satisfying
\[ \int_{A_k} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) = 2 \pi i (\alpha_k - i \sigma_k \varepsilon \Re Nu, \beta_k - i \sigma_k \varepsilon \Im Nu, \gamma_k - i \sigma_k \varepsilon), \]

where \( \tilde{\phi}_i := \varepsilon \phi_i \) are the rescaled Weierstrass data. Depending on the type of the vortex crystal, we also require the following.

— If the vortex crystal is finite and translating, we want \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) to have double poles at \( \infty \). Up to rotations and scalings, we assume that

\[ \tilde{\phi}_1 \sim \frac{dz_+ + O(z_+^{-2})}{dz_+} \quad \text{and} \quad \tilde{\phi}_2 \sim \mp i \frac{dz_+ + O(z_+^{-2})}{dz_+} \quad \text{at } \infty. \tag{5.3} \]

On the other hand, since the minimal surfaces have planar ends, \( \tilde{\phi}_3 \) must be holomorphic at \( \infty \). Consequently, we must have

\[ m = \sum \sigma_k = 0. \]

— If the vortex crystal is singly periodic, let \( A_\pm \subset \Sigma_\pm \) be the segments \( \{ t \mp K_i | 0 \leq t \leq 1 \} \), where \( K > |\text{Im} \phi_k| \) and \( K > |\text{Im} \phi_k| \) for all \( k \). We want that

\[ \Re \int_{A_\pm} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) = (1, 0, \varepsilon \Psi_1). \]

So we require that

\[ \int_{A_\pm} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) = (1 + i \alpha_\pm, i \beta_\pm, \varepsilon \Psi_1 + i \gamma_\pm). \tag{5.4} \]

The flux vectors \((\alpha_\pm, \beta_\pm, \gamma_\pm)\) (or rather their inverse) can be physically interpreted as the surface tension forces along the Scherk ends. Up to a rotation around the \( x \)-axis (corresponding to the real axis), we may assume that \( \gamma_+ = 0 \).

— If the vortex crystal is doubly periodic, let \( A_\pm \subset \Omega_\pm \) be curves homologous in \( \Sigma_\pm \) to the segment from 0 to 1, and \( B_\pm \subset \Omega_\pm \) be curves homologous to the segment from 0 to \( \tau_\pm \). We want that

\[ \Re \int_{A_\pm} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) = (1, 0, \varepsilon \Psi_1) \quad \text{and} \quad \Re \int_{B_\pm} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) = (\Re \tau, \Im \tau, \varepsilon \Psi_2). \]

So we require that

\[ \int_{A_\pm} (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) = (1 + i \alpha_\pm, i \beta_\pm, \varepsilon \Psi_1 + i \gamma_\pm). \tag{5.4} \]

Again, the flux vectors can be physically interpreted as the surface tension forces. Up to a Euclidean rotation, we may assume that the periods of \( \tilde{\phi}_3 \) over \( A_+ \) and \( B_+ \) are real. That is

\[ \gamma_+ = \Im \int_{A_+} \tilde{\phi}_3 = 0 \quad \text{and} \quad \Im \int_{B_+} \tilde{\phi}_3 = 0. \]

In any of these cases, the \( \phi \)’s are uniquely determined by the requirements above.

Write \( \tilde{Q} := \varepsilon^2 Q \). The conformality condition (5.1) is equivalent to

\[ \mathcal{E}_k := \int_{A_k} \frac{\tilde{\phi}_w^+}{dz_+} = 0, \quad 1 \leq k \leq n, \tag{5.5} \]

\[ \mathcal{F}_k := \int_{A_k} \frac{\tilde{Q}}{dz_+} = 0, \quad 1 \leq k \leq n, \tag{5.6} \]

and

\[ \mathcal{F}'_k := \int_{A'_k} \frac{\tilde{Q}}{dz_-} = 0, \quad 1 \leq k \leq n, \tag{5.7} \]
and, if the vortex crystal is periodic,

\[ \int_{A_{\pm}} \frac{\bar{Q}}{dz_{\pm}} = 0. \tag{5.8} \]

Note that (5.6) are not independent. One dependence comes from the residue theorem, namely that

\[ \sum_{k=1}^{n} F_k = 0. \tag{5.9} \]

(c) Solutions

To facilitate the solution, we will construct minimal surfaces with an orientation-reversing translational symmetry \( R_\varepsilon \) such that \( R_0^2 = T_{0,\varepsilon} \). We want \( R_\varepsilon \) to correspond to the symmetry

\[ t : \Sigma \ni z \mapsto \bar{z} \in \Sigma. \]

More specifically, we want that

\[ t^*(\phi_1, \phi_2, \phi_3) = (\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3). \]

This can be achieved by assuming that

\[ q_k \equiv \bar{p}_k, \quad t_k \in \mathbb{R} \quad \text{and} \quad (\alpha_k, \beta_k, \gamma_k) \equiv 0. \]

In the doubly periodic case, we also assume that \( \tau_\pm \equiv \tau_{\pm0} \). As a consequence, the \( B \)-period problems (5.2) are automatically solved. Indeed, we can choose \( B \)-curves such that \( B_k + (A_k) \) is homologous to \( \sigma_1 A_1 - \sigma_k A_k \), for which all \( \phi \) periods are pure imaginary [2]. In the periodic cases, since \( t(A_{\pm}) = A_{\pm} \), we also have \( \gamma_\pm = \gamma_\pm \equiv 0 \). In the doubly periodic case, since \( t(B_+) = B_- \), the period of \( \bar{\phi}_3 \) over \( B_- \) must also be real. Note that the \( A \)-curves and \( B \)-curves form a homology basis. Now that the vertical components of the flux vectors vanish along all these curves, they must vanish along any closed curve.

Moreover, since \( t^*(Q) = \bar{Q} \) and \( t(A_k) = A_k = -A_k \), we have

\[ E_k = \int_{A_k} \frac{\bar{Q}w_k^+}{dz_+} = \int_{t(A_k)} t^* \left( \frac{\bar{Q}w_k^+}{dz_+} \right) = -\int_{A_k} \bar{Q}w_k^- \frac{dz_-}{dz_+} = -E_k \]

and

\[ F_k = \int_{A_k} \frac{\bar{Q}}{dz_-} = \int_{t(A_k)} t^* \left( \frac{\bar{Q}}{dz_+} \right) = -\int_{A_k} \frac{\bar{Q}}{dz_+} = -F_k. \]

This means that \( \text{Re} E_k = 0 \) and (5.7) is automatically solved if (5.6) is solved.

Remark 5.1. Without the symmetry \( t \), we can use the Implicit Function Theorem to prove the existence of \( q_k, \text{Im} t_k, \alpha_k, \beta_k, \gamma_k \) and \( \tau_\pm \) that solve the \( B \)-period problem and the conformality equation (5.7).

Similar arguments as in [5], using the Implicit Function Theorem, show that

Proposition 5.2. For \( \varepsilon \) in a neighbourhood of 0 and \( p_k \) in a neighbourhood of its central values \( p_k^0 \), there exist unique values for parameters \( t_k, \alpha_\pm, \beta_\pm \) and \( \tau_\pm \), depending smoothly on \( \varepsilon \) and \( p_k \), such that the imaginary part of (5.5), as well as (5.8) in the periodic cases, and (5.4) in the doubly periodic case, are solved. At \( \varepsilon = 0 \), we have \( t_k = 0 \) in all cases, \( \alpha_\pm = 0 \), \( \beta_\pm = \mp 1 \) in the periodic cases, and \( \tau_\pm = \tau \) in the doubly periodic case, where \( \tau \) is the torus parameter for the given doubly periodic vortex crystal.

Sketched proof. When \( \varepsilon = 0 \), all the \( A_k \)-periods of \( \bar{\phi}'s \) vanish, and thus \( \bar{\phi}_3 \) converges to 0. If the vortex crystal is finite, \( \bar{\phi}_1 \) and \( \bar{\phi}_2 \) converge to holomorphic forms in \( \mathbb{C} \), and their limits are determined by their behaviour at \( \infty \). In view of the assumptions (5.3), we have

\[ \bar{\phi}_1 \rightarrow \text{dz}_\pm \quad \text{and} \quad \bar{\phi}_2 \rightarrow \mp i \text{dz}_\pm \]

in \( \Sigma_\pm \) as \( \varepsilon \rightarrow 0 \), and one easily verifies that \( \bar{Q} \) converges to 0.
If the vortex crystal is periodic, \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) converge to holomorphic forms in \( \Sigma_{\pm} \) and are determined by their \( A_{\pm} \) periods. More specifically, at \( \epsilon = 0 \), we have

\[
\tilde{\phi}_1 \to (1 + i\alpha_{\pm}) d\zeta_{\pm} \quad \text{and} \quad \tilde{\phi}_2 \to i\beta_{\pm} d\zeta_{\pm}
\]

in \( \Sigma_{\pm} \) as \( \epsilon \to 0 \). Whence we have,

\[
\tilde{Q} = (\phi_1)^2 + (\phi_2)^2 \to (1 + 2i\alpha_{\pm} - \alpha_{\pm}^2 - \beta_{\pm}^2) d\zeta_{\pm}^2.
\]

In order for this to vanish, we need \( \alpha_{\pm} = 0 \) and \( \beta_{\pm} = \mp 1 \). The sign of \( \beta_{\pm} \) is chosen so the surface has the desired orientation. Hence, again, we have the limit

\[
\tilde{\phi}_1 \to d\zeta_{\pm} \quad \text{and} \quad \tilde{\phi}_2 \to \mp i d\zeta_{\pm}
\]

in \( \Sigma_{\pm} \) as \( \epsilon \to 0 \).

The remaining proof focuses on \( \Sigma_+ \). To ease the notations, we write \( d\zeta \) in the place of \( d\zeta_+ \). For periodic vortex crystals, we compute the partial derivatives

\[
\frac{\partial}{\partial \alpha_+} \int_{A_+} \tilde{Q} d\zeta = 2i \quad \text{and} \quad \frac{\partial}{\partial \beta_+} \int_{A_+} \tilde{Q} d\zeta = 2,
\]

and all other partial derivatives of \( \int_{A_+} \tilde{Q}/d\zeta \) vanish. By \([5, \text{lemma 3}]\), we have the partial derivatives

\[
\frac{\partial E_k}{\partial t_k} = \int_{A_k} (z - p_k)(\tilde{Q} d\zeta) = -8\pi i,
\]

and all other partial derivatives of \( E_k \) vanish.

Finally, we compute, at \( \epsilon = 0 \),

\[
(\text{Re} \tau, \text{Im} \tau) = \text{Re} \int_{B_+} (\tilde{\phi}_1, \tilde{\phi}_2) = \text{Re} \int_0^{\tau_+} (d\zeta, -i d\zeta) = (\text{Re} \tau_+, \text{Im} \tau_+),
\]

which determines \( \tau_+ = \tau \).

The proposition then follows from the Implicit Function Theorem.

At \( \epsilon = 0 \), we have

\[
\phi_1 = d\zeta_{\pm} \quad \text{and} \quad \phi_2 = \mp i d\zeta_{\pm} \quad \text{on} \quad \Sigma_{\pm}.
\]

On \( \Sigma_+ \), \( \phi_3 = \phi_3/\epsilon \) extends smoothly to \( \epsilon = 0 \) with the explicit form

\[
\sum -i\sigma_k \Upsilon(z - p_k) d\zeta
\]

where

\[
\Upsilon(z) := \begin{cases} 
1/z & \text{if } (p_k, \sigma_k) \text{ is finite;} \\
\pi \cot(\pi z) & \text{if } (p_k, \sigma_k) \text{ is singly periodic;} \\
(\xi(z; \tau) - \xi(z; \tau)) & \text{if } (p_k, \sigma_k) \text{ is doubly periodic.}
\end{cases}
\]

One then verifies that, at \( \epsilon = 0 \), we have indeed

\[
\int_{A_+} \phi_3 = m\pi \in \mathbb{R}
\]

if \( (p_k, \sigma_k) \) is singly periodic and (cf. \([5, \text{§4.3.1}]\) and \([7, \text{§5}]\))

\[
\int_{A_+} \phi_3 = -2\pi y \in \mathbb{R} \quad \text{and} \quad \int_{B_+} \phi_3 = 2\pi x \in \mathbb{R}
\]

if \( (p_k, \sigma_k) \) is doubly periodic, as we have assumed. Here \( (x, y) \in \mathbb{R}^2 \) are defined by \( \sum \sigma_k p_k = x + y\tau \).

Note that, in the doubly periodic case, \( \Upsilon \) is not meromorphic, but \( \phi_3 \) is.
Then, as $\varepsilon \to 0$, $F_{j,+}/\varepsilon^2$ converges smoothly to the value

$$\left. \frac{\partial F_{j,+}}{\partial \varepsilon^2} \right|_{\varepsilon=0} = \int_{\mathcal{A}_j} \left( \frac{2\phi_1}{\varepsilon^2} \frac{\partial \phi_1}{\partial z} + \frac{2\phi_2}{\varepsilon^2} \frac{\partial \phi_2}{\partial z} \right) + 2\pi i \text{Res}_{\phi_j} \frac{\phi_3^2}{\partial z}$$

$$= \frac{\partial}{\partial \varepsilon^2} \int_{\mathcal{A}_j} (2\phi_1 - 2i\phi_2) - 2\pi i \text{Res}_{\phi_j} \left( \sum_k \sigma_k \mathcal{Y}(z - p_k) \right)^2$$

$$= 4\pi \sigma_j \text{Re} \nu - 4\pi \sigma_j \text{Im} \nu - 4\pi i \sum_{k \neq j} \sigma_k \mathcal{Y}(p_j - p_k)$$

$$= 4\pi \sigma_j \left[ \nu - i \sum_{k \neq j} \sigma_k \mathcal{Y}(p_j - p_k) \right] = 8\pi^2 F_j,$$

which vanishes if and only if $(p_k)$ and $(\sigma_k)$ are the positions and circulations of a binary vortex crystal with velocity $v = -\nu/2\pi$. This proves that the helicoid limits are necessarily vortex crystals.

Recall that $F_k$ are related by (5.9). Therefore, if the vortex crystal is not a finite stationary one, and is non-degenerate (possibly with imposed symmetry), we may apply the Implicit Function Theorem to prove the following proposition that concludes the construction of a family of immersed surfaces.

**Proposition 5.3.** If $(p_k^*)$ and $(\sigma_k)$ are the positions and circulations of a binary vortex crystal that is non-degenerate (possibly with imposed symmetry). Then for $\varepsilon$ in a neighbourhood of 0, there exist unique values for $p_k$, depending smoothly on $\varepsilon$, such that $p_k(0) = p_k^*$ and the conformality condition (5.6) is solved.

Finally, a similar argument as in [1,2] proves that the constructed surfaces are all embedded for $\varepsilon$ sufficiently small.

**Remark 5.4.** The same construction can be carried out without the symmetry $\iota$ (see remark 5.1) and the conclusion is the same. By uniqueness of the implicit functions, this implies that all minimal surfaces sufficiently close to a balanced and non-degenerate configuration of helicoids admit an orientation-reversing symmetry.

**Remark 5.5.** For finite stationary vortex crystals, we may fix two vortices to quotient out Euclidean similarities, leaving $n - 2$ free complex parameters. But in the last step of the construction, we have $n$ complex equations $F_k = 0$ to solve. The relation (5.9) eliminates one complex equation, Riemann Bilinear relation can eliminate a real equation (see [6,21]), but we still have one real equation too many. Hence the construction does not work. This is compatible with Traizet’s observation that catenoids cannot be glued into a single periodic minimal surface with vertical periods [22].

**Remark 5.6.** In [23], the height differential of the minimal surface corresponds to the velocity field of the fluid. A similar correspondence can be seen in our construction by noticing from (5.10) that, at $\varepsilon = 0$, $\phi_3 = 2\pi \tilde{u} dz$, where $u$ is the flow velocity field generated by the vortex crystal (away from the vortices).

**Data accessibility.** This article has no additional data.

**Authors’ contributions.** H.C.: conceptualization, formal analysis, funding acquisition, investigation, methodology, project administration, validation, visualization, writing—original draft, writing—review and editing; D.F.: conceptualization, formal analysis, investigation, validation, visualization, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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