System of roots of very special sandwich algebras\(^\dagger\)  
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This paper continues the study of very special sandwich algebras begun in Cushman [2]. We show that a class of very special sandwich algebras has an analogue of a root system of a semisimple Lie algebra [4, p.42]. This leads to an analogue of a Weyl group, which we study in another paper [3].

1 System of roots

Summary §1

In §1.1 we give the axioms of a system of roots, which unlike the axioms for a root system does not use an inner product. In §1.2 we prove some consequences of these axioms, one of which verifies that every root system is a system of roots.

1.1 Axioms for a system of roots

Let \( V \) be a finite dimensional real vector space with \( \Phi \) a finite subset of nonzero vectors, which satisfy the following axioms.

1. \( V = \text{span}_\mathbb{R} \Phi \), using addition + of vectors in \( V \).
2. \( \Phi = -\Phi \), where \(-\) is the additive inverse of +.
3. For every \( \beta, \alpha \in \Phi \cup \{0\} \) there is an extremal root chain \( S_\alpha^\beta \) through \( \beta \) in the direction \( \alpha \) given by \( \{\beta + j\alpha \in \Phi \cup \{0\} \mid \text{for every } j \in \mathbb{Z}, -q \leq j \leq p\} \). Here \( q, p \in \mathbb{Z}_{\geq 0} \) and are as large as possible. The pair \( (q, p) \) is the integer pair associated to \( S_\alpha^\beta \). The integer \( \langle \beta, \alpha \rangle = q - p \) is called the Killing integer of \( S_\alpha^\beta \).
4. Fix \( \alpha \in \Phi \) and suppose that \( \beta_1, \beta_2, \) and \( \beta_1 + \beta_2 \in \Phi \cup \{0\} \). Then
   \[ \langle \beta_1 + \beta_2, \alpha \rangle = \langle \beta_1, \alpha \rangle + \langle \beta_2, \alpha \rangle. \]
5. For every \( \alpha \in \Phi \) we have \( \langle \alpha, \alpha \rangle = 2 \).

We call \( \Phi \cup \{0\} \) a system of roots in \( V \).

The main point in the axioms for a system of roots is that there is no Euclidean inner product on \( V \). In fact, this distinguishes a system of roots from a root system of a semisimple complex Lie algebra.

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1.2 Some consequences

In this section we draw some consequences from the axioms of a system of roots.

Lemma 1.2.1 Axiom 3 is follows from axiom 1.

Proof. Axiom 3 holds because for every $\beta, \alpha \in \Phi$ the affine line $\mathbb{R} \rightarrow V : t \mapsto \beta + t\alpha$ through $\beta$ in the direction of $\alpha$ intersects $\Phi \cup \{0\}$ in at most a finite number of distinct values of the parameter $t \in \mathbb{Z}$. Since $\beta \in \Phi$ it follows that 0 is such a parameter value. Thus there is a maximal $q$ and $p \in \mathbb{Z}_{\geq 0}$ such that for every $j \in \mathbb{Z}$ with $-q \leq j \leq p$ we have $\beta + j\alpha \in \Phi \cup \{0\}$. So an extremal root chain through $\beta$ in the direction $\alpha$ exists. By definition $\langle \beta, \alpha \rangle = q - p$. Thus axiom 3 is a consequence of axiom 1.

Claim 1.2.2 Let $\alpha, \beta \in \Phi$. If $\langle \beta, \alpha \rangle < 0$, then $\beta + \alpha \in \Phi \cup \{0\}$; while if $\langle \beta, \alpha \rangle > 0$, then $\beta - \alpha \in \Phi \cup \{0\}$.

Proof. From axiom 3 we deduce that the extremal root chain $S_\alpha^\beta$ through $\beta$ in the direction $\alpha$ with integer pair $(q, p)$ exists. If $q - p = \langle \beta, \alpha \rangle < 0$, then $p > q \geq 0$. So $S_\alpha^\beta$ contains $\beta + \alpha$. Therefore $\beta + \alpha \in \Phi \cup \{0\}$. If $q - p = \langle \beta, \alpha \rangle > 0$, then $q > p \geq 0$. So the extremal root chain $S_\alpha^\beta$ through $\beta$ in the direction $\alpha$ contains $\beta - \alpha$. Therefore $\beta - \alpha \in \Phi \cup \{0\}$. □

Lemma 1.2.3 For every $\beta, \alpha \in \Phi$ we have $\langle \beta, -\alpha \rangle = -\langle \beta, \alpha \rangle$ and $\langle -\beta, \alpha \rangle = -\langle \beta, \alpha \rangle$.

Proof. From axiom 3 the extremal root chain $S_\alpha^\beta : \beta - q\alpha, \ldots, \beta + p\alpha$ through $\beta$ in the direction $\alpha$ with integer pair $(q, p)$ exists. From axiom 2 it follows that $-\beta$ and $-\alpha \in \Phi$. The following argument shows that the extremal root chain $S_\alpha^{-\beta}$ has integer pair $(p, q)$. From the extremal root chain $S_\alpha^\beta$ we see that

$$\beta - p(-\alpha), \beta - (p - 1)(-\alpha), \ldots, \beta + (q - 1)(-\alpha), \beta + q(-\alpha)$$

is an extremal root chain through $\beta$ in the direction $-\alpha$ with integer pair $(p, q)$. In other words, $\langle \beta, -\alpha \rangle = p - q = -\langle \beta, \alpha \rangle$.

Next we show that the extremal root chain $S_\alpha^{-\beta}$ through $-\beta$ in the direction $\alpha$ has integer pair $(p, q)$. Multiplying the elements of the extremal root chain $S_\alpha^\beta$ by $-1$ and using axiom 2, we see that

$$-\beta - po\alpha, -\beta - (p - 1)\alpha, \ldots, -\beta + (q - 1)\alpha, -\beta + q\alpha$$

is an extremal root chain $S_\alpha^{-\beta}$ through $-\beta$ in the direction $\alpha$ with integer pair $(p, q)$. In other words, $\langle -\beta, \alpha \rangle = p - q = -\langle \beta, \alpha \rangle$. □.
We now consider root systems, see [4, p.42]. Let \((U, (\ , \ ))\) be a finite dimensional real vector space with a Euclidean inner product \((\ , \ )\). Let \(\Phi\) be a finite subset of nonzero vectors in \(U\). Suppose that the following axioms hold.

1. \(U = \text{span}_R \Phi\).
2. If \(\alpha \in \Phi\) and \(\lambda \in \mathbb{R}\), then \(\lambda \alpha \in \Phi\) if and only if \(|\lambda| = 1\).
3. If \(\alpha \in \Phi\), then the reflection \(\sigma_\alpha : U \to U : v \mapsto v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha\) is an orthogonal real linear mapping of \((U, (\ , \ ))\) into itself, which preserves \(\Phi\).
4. If \(\beta, \alpha \in \Phi\), then \(\langle \beta, \alpha \rangle = \frac{2(v, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}\).

Then \(\Phi\) is a root system.

**Claim 1.2.4** Every root system is a system of roots.

**Proof.** Suppose that \(\Phi\) is a root system. Then axiom 1 of root system is the same as axiom 1 of a system of roots. If \(\alpha \in \Phi\), then from axiom 2 of a root system it follows that \(-\alpha \in \Phi\) and thus \(\alpha = -(-\alpha)\). So \(\Phi = -\Phi\), which is axiom 2 of a system of roots. From lemma 5.2.1 it follows that for every \(\beta, \alpha \in \Phi\) there is an extremal root chain \(S^\beta_\alpha\) through \(\beta\) in the direction \(\alpha\) with integer pair \((q, p)\). Thus the first statement in axiom 3 for a system of roots holds. This does not complete its verification, because we still have to show that \(\langle \beta, \alpha \rangle = q - p\), where \(\langle \beta, \alpha \rangle = \frac{2(v, \alpha)}{(\alpha, \alpha)}\). Axiom 5 of a system of roots follows because for every \(\alpha \in \Phi\), which is nonzero by hypothesis, we have \(\langle \alpha, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2\). From the definition \(\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\) it follows that for each fixed \(\alpha \in \Phi\) the function \(K_\alpha : \Phi \to \mathbb{Z} : \beta \mapsto \langle \beta, \alpha \rangle\) is linear and is \(\mathbb{Z}\)-valued by axiom 4 of a root system. This proves axiom 4 of a system of roots.

We now finish the proof of axiom 3 of a system of roots. Using axiom 3 of a root system we show that for every \(\alpha \in \Phi\) the orthogonal reflection \(\sigma_\alpha : U \to U : \beta \mapsto \langle \beta, \alpha \rangle\) maps the extremal root chain \(S^\beta_\alpha\) into itself. For every \(j \in \mathbb{Z}\) with \(-q \leq j \leq p\) let \(\beta + j\alpha \in S^\beta_\alpha\). Then

\[
\sigma_\alpha(\beta + j\alpha) = \beta + j\alpha - K_\alpha(\beta + j\alpha)\alpha \\
= \beta + j\alpha - K_\alpha(\beta)\alpha - jK_\alpha(\alpha)\alpha \\
= \beta - (\langle \beta, \alpha \rangle + j)\alpha. \quad \square
\]

Since \(\beta + j\alpha \in \Phi \cup \{0\}\) and \(\sigma_\alpha\) maps \(\Phi\) into itself and 0 into 0, it follows that \(\beta - (\langle \beta, \alpha \rangle + j)\alpha \in \Phi \cup \{0\}\). Because \(\langle \beta, \alpha \rangle \in \mathbb{Z}\), for every \(j \in \mathbb{Z}\) with \(-q \leq j \leq p\) we have \(-(\langle \beta, \alpha \rangle + j)\alpha \in F = \{j \in \mathbb{Z}, \ -q \leq j \leq p\}\).
So the orthogonal reflection $\sigma_\alpha$ maps the extremal root chain $S^\beta_\alpha$ into itself. Moreover $-((-\beta,\alpha)+p)$ is the smallest element of $F$. But $S^\beta_\alpha$ is an extremal root chain with integer pair $(q,p)$. So $-q = -(-\beta,\alpha) - p$, that is, $(\beta,\alpha) = q - p$. This proves axiom 3 of a system of roots and thereby the claim.

2 Very special sandwich algebras of class $C$

Recall that a complex sandwich algebra $\tilde{g} = g \oplus \tilde{n}$ is a complex Lie algebra where $g$ is a simple Lie algebra, $\tilde{n}$ is the nilpotent radical of $\tilde{g}$, which is a sandwich, that is, $[\tilde{n},[\tilde{n},\tilde{n}]] = 0$, and for $\tilde{h}$, a Cartan subalgebra of $g$, the set $\text{ad}_{\tilde{h}}$ is a maximal toral subalgebra of $\text{gl}(\tilde{n},\mathbb{C})$. The sandwich algebra $\tilde{g}$ is very special if $\tilde{g}$ is a subalgebra of a complex simple Lie algebra $g$, which has rank 1 greater than the rank of the simple Lie algebra $g$. Let $h$ be a Cartan subalgebra of $g$. We assume that the Cartan subalgebras $\tilde{h}$ and $h$ are aligned, that is, there is a vector $H^* \in \tilde{h}$ such that $R = R^0 = \{\alpha \in R \mid \alpha(H^*) = 0\}$. Here $R$ and $\tilde{R}$ are the root systems associated to the Cartan subalgebras $h$ and $\tilde{h}$ of the complex simple Lie algebras $g$ and $\tilde{g}$, respectively. For later purposes let $R^- = \{\alpha \in R \mid \alpha(H^*) < 0\}$. A very special complex sandwich algebra is of class $C$ if the center $Z$ of the nilradical $\tilde{n}$ has complex dimension 1. see Cushman [2].

Summary of §2

In §2.1 we recall the definition of the collection of roots $\hat{R}$ for the nilpotent radical $\tilde{n}$. We define an operation of addition on $\hat{R}$ and construct a set of positive simple roots $\hat{\Pi}$, which is, an basis for the real vector space $V = \text{span}_R \{\hat{\alpha} \mid \hat{\alpha} \in \hat{\Pi}\}$. This allows us to show that axioms 1 and 2 for a system of roots holds for the set $\Phi$ of nonzero roots in $\hat{R}$. Associated to a positive root $\hat{\alpha} \in \hat{\Pi}$ is a root algebra $\tilde{g}(\hat{\alpha})$, which is isomorphic to a 3-dimensional Heisenberg algebra $h_3$. In §2.3 we associate to every extremal root chain $S^\beta_\alpha$ the space $\tilde{g}^{\beta}_{\alpha}$, formed by taking the direct sum of root spaces corresponding to elements of $S^\beta_\alpha$. In claim 2.3.1 we show that the adjoint representation of $\tilde{g}^{(\alpha)}$ on $\tilde{n}$ is completely reducible and classify its irreducible summands. The goal of §2.4 is to verify that $\Phi$ satisfies axioms 4 and 5 of a system of roots. Lemm 2.4.1 shows that $\Phi$ satisfies axiom 5 of a system of roots. The main result §2 is claim 2.4.2, which shows that for every fixed $\hat{\alpha} \in \hat{R}$ the function $K_{\hat{\alpha}} : \hat{R} \to \mathbb{Z} : \hat{\beta} \mapsto (\hat{\beta}, \hat{\alpha})$ is linear. In other words, $\Phi$ satisfies axiom 4 of a system of roots. Our main tool in proving claim 2.4.2 is claim 2.4.4, which
states that if for $i = 1, 2$ the extremal root chain $S^β_α$ has an integer pair $(q_i, p_i)$ and if $S^β_α + β_2$ is also an extremal root chain with integer pair $(r, s)$, then the Killing integer $r - s$ of $S^β_α + β_2$ is equal to $(q_1 + q_2) - (p_1 + p_2)$.

A list of very special sandwich algebras of class $C$ is given in the appendix.

2.1 Roots and root spaces

Let $\tilde{g} = g \oplus \tilde{n}$ be a very special sandwich algebra with nilpotent radical $\tilde{n}$ and a complex simple Lie algebra $g$ with Cartan subalgebra $\mathfrak{h}$. By definition of sandwich algebra $\text{ad}_\mathfrak{h}$ is a maximal torus of $\text{gl}(\tilde{n}, \mathbb{C})$. Thus $\tilde{n} = \sum_{\alpha \in \tilde{R}} \tilde{g}_\alpha$, where $\tilde{g}_\alpha$ is the $\text{ad}_\mathfrak{h}$-invariant subspace $\{X \in \tilde{n} \mid [H, X] = \tilde{\alpha}(H)X, \text{for all } H \in \mathfrak{h}\}$, which is called the root space corresponding to the root $\tilde{\alpha} \in \tilde{R}$. The set $R$ of roots associated to $\mathfrak{h}$ is finite. Since $\text{ad}_\mathfrak{h}$ is a maximal torus, $\dim_{\mathbb{C}} \tilde{g}_\alpha = 1$ for every $\tilde{\alpha} \in \tilde{R}$. So $\tilde{g}_\alpha$ is spanned by the nonzero root vector $X_{\tilde{\alpha}}$. Since $\tilde{g}$ is a very special sandwich algebra of class $C$, the center $Z$ of its nilpotent radical $\tilde{n}$ is $\text{ad}_\mathfrak{h}$-invariant subspace of $\tilde{n}$, which is 1-dimensional. Hence $Z$ is spanned by the nonzero root vector $X_{\tilde{\zeta}}$ for some $\tilde{\zeta} \in \tilde{R}$. Let $Z = \{\tilde{\zeta}\}$ and set $Y = \tilde{R} \setminus Z$. Let $Y = \sum_{\tilde{\alpha} \in \mathcal{Y}} \tilde{g}_{\tilde{\alpha}}$. Then $Y$ is $\text{ad}_\mathfrak{h}$-invariant and $\tilde{n} = Y \oplus Z$. Because $\tilde{n}$ is a sandwich, it follows that $[Y, Y] = Z$.

Define an addition operation $\tilde{+}$ on $\tilde{R}$ by saying that if $\tilde{\alpha}, \tilde{\beta} \in \tilde{R}$, then $\tilde{\alpha} \tilde{+} \tilde{\beta} \in \tilde{R}$ if there are $\alpha, \beta \in \mathbb{R}^- \subseteq R$ such that $\alpha + \beta \in \mathbb{R}^-$ and $\tilde{\alpha} = \alpha |\mathfrak{h}$ and $\tilde{\beta} = \beta |\mathfrak{h}$. In particular, we have $(\tilde{\alpha} \tilde{+} \tilde{\beta}) |\mathfrak{h} = (\alpha + \beta) |\mathfrak{h}$. Clearly the operation of addition $\tilde{+}$ is commutative and associative. An inspection of the table for the very special sandwich algebras of class $C$ in the appendix shows that

Observation 2.1.1 For every very special sandwich algebra the linear function $\tilde{\zeta} \in \tilde{R}$ is identically zero on $\mathfrak{h}$.

Thus $\tilde{\zeta}$ is the additive identity element for the addition operation $\tilde{+}$. In fact, this observation shows that the operation $\tilde{+}$ on $\tilde{R}$ the ordinary addition operation of addition of linear functions on $\mathfrak{h}$. Thus $\tilde{R}$ is an abelian group under $\tilde{+}$. We will use the notation $+$ for $\tilde{+}$ and $0$ interchangeably with $\tilde{\zeta}$. We now show that

Lemma 2.1.2 The set $\tilde{R}_{\tilde{\zeta}} = \{(\tilde{\alpha}, \tilde{\beta}) \in \mathcal{Y} \times \mathcal{Y} \mid \tilde{\alpha} + \tilde{\beta} = \tilde{\zeta}\}$ is nonempty.

Proof. Suppose that $x = \sum_{\tilde{\alpha} \in \mathcal{Y}} a_{\tilde{\alpha}} X_{\tilde{\alpha}}$ and $y = \sum_{\tilde{\beta} \in \mathcal{Y}} b_{\tilde{\beta}} X_{\tilde{\beta}}$ are nonzero
This recursion stops after a finite number of repetitions because 
finite set. Then

\[ \mathcal{R}_\zeta \text{ is nonempty.} \]

Construct the finite set \( \mathcal{R}_\zeta \) as follows. Let \( \hat{\alpha}_1 \in \mathcal{R}_\zeta \) if there is \((\hat{\alpha}_1, \hat{\beta}_1) \in \mathcal{R}_\zeta \). Note that \((\hat{\alpha}_1, \hat{\beta}_1) \in \mathcal{R}_\zeta \) implies that \((\hat{\beta}_1, \hat{\alpha}_1) \in \mathcal{R}_\zeta \), because addition is commutative. Recursively let \( \hat{\alpha}_{i+1} \in \mathcal{R}_\zeta \) if there is

\[
(\hat{\alpha}_{i+1}, \hat{\beta}_{i+1}) \in \mathcal{R}_\zeta \setminus \{(\hat{\alpha}_j, \hat{\beta}_j) \& (\hat{\beta}_j, \hat{\alpha}_j) \in \mathcal{Y} \times \mathcal{Y} \text{ for } 1 \leq j \leq i \}.
\]

This recursion stops after a finite number of repetitions because \( \mathcal{R}_\zeta \) is a finite set. Then \( \mathcal{R}_\zeta = \{\hat{\alpha}_i \in \mathcal{Y} | 1 \leq i \leq M\} \).

**Lemma 2.1.3** \( \mathcal{Y} = \mathcal{R}_\zeta \Pi (-\mathcal{R}_\zeta) \).

**Proof.** Suppose that there is a root \( \hat{\alpha} \in \mathcal{Y} \setminus (\mathcal{R}_\zeta \Pi -\mathcal{R}_\zeta) \). Then \( -\hat{\alpha} \) is an element of \( \mathcal{R} \) by definition of addition on \( \mathcal{R} \). But \( \hat{\alpha} + (-\hat{\alpha}) = \hat{0} \), that is, \( \hat{0} \in \mathcal{R}_\zeta \) by definition. This is a contradiction. So \( \mathcal{Y} = \mathcal{R}_\zeta \Pi (-\mathcal{R}_\zeta) \). \( \square \)

The elements of \( \mathcal{R}_\zeta \) are *simple*. For if for some \( j \in \{1, \ldots, M\} \) the vector \( \hat{\alpha}_j \) can be written as \( \hat{\alpha} + \hat{\beta} \) for some \( \hat{\alpha}, \hat{\beta} \in \mathcal{Y} \), then for some nonzero \( c_{\hat{\alpha},\hat{\beta}} \in \mathbb{C} \) we have \( X_{\hat{\alpha}+\hat{\beta}} = c_{\hat{\alpha},\hat{\beta}} X_{\hat{\alpha},X_{\hat{\beta}}} \in \mathbb{Z} \), since \( |\mathcal{Y}||\mathcal{Y}| = \mathbb{Z} \). So \( \hat{\alpha}_j \in \mathbb{Z} \), which contradicts the fact that \( \hat{\alpha}_j \in \mathcal{R}_\zeta \subseteq \mathcal{Y} \) by construction and \( \mathcal{Y} \cap \mathbb{Z} = \emptyset \) by definition. We call \( \mathcal{R}_\zeta \) the set \( \Pi \) of *simple positive roots* of \( \mathcal{R} \).

Using lemma 2.1.3 we get

**Corollary 2.1.4** \( \mathcal{Y} = \sum_{\hat{\alpha} \in \Pi} \oplus(\hat{0}_{\hat{\alpha}} \oplus \hat{0}_{-\hat{\alpha}}) = \text{span}_\mathbb{C}\{X_{\hat{\alpha}_j}, X_{-\hat{\alpha}_j}\}_{j=1}^M \).

Since \( \mathcal{\bar{n}} = \mathcal{Y} \oplus \text{span}_\mathbb{C}\{X_{\zeta}\} \), it follows that the real vector space \( V = \text{span}_\mathbb{R}\{\hat{\alpha}_j \in \mathcal{Y}, 1 \leq j \leq M\} \) with nonzero roots \( \Phi = \mathcal{Y} \) satisfies the axiom 1 and axiom 2 (and thus axiom 3) of a system of roots.
2.2 Root subalgebras

For each root \( \tilde{\alpha} \in \tilde{\Pi} \subseteq \tilde{R} \), the root subalgebra \( \tilde{\mathfrak{g}}^{(\tilde{\alpha})} \) = \( \text{span}\{X_{\tilde{\alpha}}, X_{-\tilde{\alpha}}, H_{\tilde{\alpha}} = X_{\tilde{\alpha}}^2\} \) is the Heisenberg algebra \( h_3 \) with bracket relations

\[
[H_{\tilde{\alpha}}, X_{\tilde{\alpha}}] = [H_{\tilde{\alpha}}, X_{-\tilde{\alpha}}] = 0 \quad \& \quad [X_{\tilde{\alpha}}, X_{-\tilde{\alpha}}] = H_{\tilde{\alpha}}.
\]

Claim 2.2.1 For every \( \alpha \in \tilde{\Pi} \) the adjoint representation of \( \tilde{\mathfrak{g}}^{(\tilde{\alpha})} \) on \( \tilde{\mathfrak{n}} \) decomposes into a finite sum of irreducible \( \tilde{\mathfrak{g}}^{(\tilde{\alpha})} \) representations. This decomposition is unique up to a reordering of the summands.

Proof. We argue as follows. Since \( \{X, Y, H\} \) span the nilpotent Lie algebra \( h_3 \) with bracket relations

\[
[H, X] = [H, Y] = 0, \quad \text{and} \quad [X, Y] = H,
\]

their image \( \rho(X) \), \( \rho(Y) \), and \( \rho(H) \) under the finite dimensional Lie algebra representation \( \rho : h_3 \rightarrow \text{gl}(\tilde{\mathfrak{n}}, \mathbb{C}) \) are nilpotent linear maps of the finite dimensional complex vector space \( \tilde{\mathfrak{n}} \) into itself. Let \( v \in \ker \rho(Y) \) and let \( f = \{w, \rho(Y)w, \ldots, (\rho(Y))^nw = v\} \) be the longest Jordan chain in \( \tilde{\mathfrak{n}} \), which ends at \( v \). Suppose that \( n \geq 2 \). Then the matrix of \( \rho(Y) \) with respect to the basis \( f \) of the vector space \( U = \text{span}\{w, \rho(Y)w, \ldots, (\rho(Y))^nw = v\} \) is the lower \((n + 1) \times (n + 1)\) Jordan block. With respect to the basis \( f \) the matrices of \( \rho(Y) \), \( \rho(X) \), and \( \rho(H) \) are

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
2 & 0 & \cdots & 0
\end{pmatrix},
\]

respectively. When \( n = 0 \) let \( \rho(Y) \), \( \rho(X) \), and \( \rho(H) \) be the zero matrix; while when \( n = 1 \) let \( \rho(Y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and let \( \rho(X) \) and \( \rho(H) \) be the \( 2 \times 2 \) zero matrix. For every \( n \geq 0 \) it is straightforward to check that the following bracket relations hold

\[
[\rho(H), \rho(X)] = [\rho(H), \rho(Y)] = 0 \quad \text{and} \quad [\rho(X), \rho(Y)] = \rho(H).
\]

This verifies that on \( U \) the mapping \( \rho \) is an \( n+1 \)-dimensional representation of \( h_3 \). It is irreducible. Suppose not. Then the only subspace of \( U \), which is invariant under \( \rho(h_3) \), is \( \text{span}\{v\} \). By hypothesis there is a \( \rho(h_3) \)-invariant subspace \( V \) such that \( U = V \oplus \text{span}\{v\} \). But \((\rho(Y))^nV \subseteq \text{span}\{v\} \), which
contradicts the $\rho(h_3)$-invariance of $V$. Therefore on $U$ the representation $\rho$ is irreducible.

Repeat the above construction for each vector in a basis $\{v, v_2, \ldots, v_m\}$ of $\ker \rho(Y)$ in $\hat{n}$. This determines the Jordan normal form of $\rho(Y)$ and decomposes the representation $\rho$ into a sum of finite dimensional irreducible representations of $h_3$. The summands are unique up to a reordering, because the Jordan blocks of $\rho(Y)$ are unique up to a reordering. \hfill \Box

### 2.3 Extremal root chains

In this section we identify the representation space of the irreducible summands of the adjoint representation of the root subalgebra $\hat{\mathfrak{g}}^{(\hat{\alpha})}$, $\hat{\alpha} \in \hat{\Pi}$, on $\hat{n}$ with an extremal root chain $S^\beta_\hat{\alpha}$ through $\hat{\beta} \in \Phi$ in the direction $\hat{\alpha}$.

We begin by recalling the concept of an extremal root chain. Let $\hat{\alpha}, \hat{\beta} \in \hat{\mathcal{R}}$ and let $V = \text{span}_R\{\hat{\alpha} \mid \hat{\alpha} \in \Phi\}$. A collection of the form $\{\hat{\beta} + j \hat{\alpha} \in V \mid j \in F\}$, where $F$ is a finite subset of $\mathbb{Z}$, is called a chain. If a chain is of the form

$$\hat{\beta} - q \hat{\alpha}, \hat{\beta} - (q - 1)\hat{\alpha}, \ldots, \hat{\beta} - \hat{\alpha}, \hat{\beta}, \hat{\beta} + \hat{\alpha}, \ldots, \hat{\beta} + p \hat{\alpha},$$

where $q, p \in \mathbb{Z}_{\geq 0}$ and $\ell \in F$ if and only if $\ell \in \mathbb{Z}$ and $-q \leq \ell \leq p$, then \textbf{[2]} is an unbroken chain through $\hat{\beta}$ in the direction $\hat{\alpha}$ with integer pair $(q, p)$. Its length is $q + p + 1$. Given a chain $\{\hat{\beta} + j \hat{\alpha} \in V \mid j \in F\}$, a chain of the form $\{\hat{\beta} + j \hat{\alpha} \in V \mid j \in F' \subseteq F\}$ is a subchain of the given chain. This subchain is proper if $F'$ is a proper subset of $F$. If each element of a chain lies in $\Phi \cup \{0\}$, then the chain is called a root chain through $\hat{\beta}$ in the direction $\hat{\alpha}$. If \textbf{[2]} is an unbroken root chain with integer pair $(q, p)$ chosen as large as possible so that this chain is not a proper subchain of another unbroken root chain through $\hat{\beta}$ in the direction $\hat{\alpha}$, then the unbroken root chain $S^\beta_\hat{\alpha}$ \textbf{[2]} is called an extremal root chain through $\hat{\beta}$ in the direction $\hat{\alpha}$ with integer pair $(q, p)$. For every $\hat{\beta}, \hat{\alpha} \in \hat{\mathcal{R}}$ there is an external roots chain with a suitable integer pair $(q, p)$ an extremal root chain exists.

We now identify the irreducible summands of the adjoint representation of the root subalgebra $\hat{\mathfrak{g}}^{(\hat{\alpha})}$, $\hat{\alpha} \in \hat{\Pi}$ on the nilradical $\hat{n}$. Corresponding to the extremal root chain \textbf{[2]} is the subspace

$$\hat{\mathfrak{g}}^{\beta}_{\hat{\alpha}} = \bigoplus_{-q \leq j \leq p} \hat{\mathfrak{g}}_{\hat{\beta} + j \hat{\alpha}}.$$

of $\hat{n}$. If $\hat{0}$ appears in the root chain \textbf{[2]}, then $\hat{\mathfrak{g}}^{\beta}_{\hat{\alpha}} = Z = \text{span}_C\{\hat{\zeta}\}$. If $\hat{\beta} \pm \hat{\alpha} \notin \hat{\mathcal{R}}$, then $\hat{\mathfrak{g}}^{\beta}_{\hat{\alpha}} = \hat{\mathfrak{g}}_{\hat{\beta}}$. Using the fact that $[\hat{\mathfrak{g}}_{\hat{\alpha}}, \hat{\mathfrak{g}}_{\hat{\beta}}] = \hat{\mathfrak{g}}_{\hat{\alpha} + \hat{\beta}}$ if $\hat{\alpha} + \hat{\beta} \in \hat{\mathcal{R}}$. 

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and 0 if \( \hat{\alpha} + \hat{\beta} \notin \hat{R} \) and the definition of extremal root chain, it follows that \( \hat{g}_\alpha \) is an \( \text{ad}\hat{g}_{(\alpha)} \)-invariant subspace of \( \hat{n} \).

**Claim 2.3.1** For every \( \hat{\beta} \in \hat{R} \) and every \( \hat{\alpha} \in \hat{P} \) the subspace \( \hat{g}_\alpha \) of \( \hat{n} \) is a representation space for the adjoint representation of \( \hat{g}_{(\hat{\alpha})} \), which is irreducible.

**Proof.** Let \((q,p)\) be the integer pair associated to the extremal root chain through \( \hat{\beta} \) in the direction of \( \hat{\alpha} \). Then \( \hat{g}_{\beta+p\hat{\alpha}} \) is the top root space corresponding to the top root \( \hat{\beta} + p\hat{\alpha} \) in this chain. The linear operator \( \text{ad}_{X-\hat{\alpha}} \) steps down the extremal root chain from \( \hat{g}_{\beta+j\hat{\alpha}} \) to

\[
\begin{cases}
\hat{g}_{\beta+(j-1)\hat{\alpha}}, & \text{when } -q < j \leq p, j \in \mathbb{Z} \\
\{0\}, & \text{when } j = -q.
\end{cases}
\]

Since \( \text{ad}_{X-\hat{\alpha}} \hat{g}_{\beta+q\hat{\alpha}} v = \{0\} \), the linear map \( \text{ad}_{X-\hat{\alpha}} : \hat{g}_\beta \to \hat{g}_\beta \) is nilpotent of height \( p + q \), because \( \text{ad}_{X-\hat{\alpha}} \hat{g}_{\beta+p\hat{\alpha}} = \hat{g}_{\beta+q\hat{\alpha}} \), while \( \text{ad}_{X-\hat{\alpha}} \hat{g}_{\beta+p\hat{\alpha}} = \{0\} \). Taking a nonzero vector \( v \in \hat{g}_{\beta+p\hat{\alpha}} \), the above argument shows that \( f = \{v, \text{ad}_{X-\hat{\alpha}} v, \ldots, \text{ad}_{X-\hat{\alpha}}^p v\} \) is a Jordan chain in \( \hat{g}_\beta \) of length \( p + q + 1 \). So \( f \) is a basis of \( W = \text{span}_C \{v, \text{ad}_{X-\hat{\alpha}} v, \ldots, \text{ad}_{X-\hat{\alpha}}^p v\} \) with respect to which the matrix of \( \text{ad}_{X-\hat{\alpha}}|W \) is a \((p + q + 1) \times (p + q + 1)\) lower Jordan block. By the classification of irreducible \( \hat{g}_{(\alpha)} \)-representations, \( W \) is a representation space for a \( p + q + 1 \)-dimensional irreducible representation of \( \text{ad}_{\hat{g}_{(\alpha)}} \). But \( \dim_C \hat{g}_{\beta+j\hat{\alpha}} = 1 \) for every \( -q \leq j \leq p, j \in \mathbb{Z} \). Therefore \( \dim_C \hat{g}_\beta = p+q+1 = \dim W \), which shows that \( W = \hat{g}_\beta \). \( \square \)

### 2.4 Verification of axioms 4 and 5

In this subsection we verify that axioms 4 and 5 for a system of roots holds for the set of roots \( \hat{R} \) of a very special sandwich algebra.

First we verify that axiom 5 holds. In other words,

**Lemma 2.4.1** For every \( \hat{\alpha} \in \Phi = \mathcal{Y} \) the Killing integer \( \langle \hat{\alpha}, \hat{\alpha} \rangle \) associated to the extremal root chain through \( \hat{\alpha} \) in the direction of \( \hat{\alpha} \) is 2.

**Proof.** From lemma 1.2.3 it follows that for every \( \hat{\beta}, \hat{\alpha} \in \mathcal{Y} \) we have

\[
\langle \hat{\beta}, -\hat{\alpha} \rangle = -\langle \hat{\beta}, \hat{\alpha} \rangle \text{ and } \langle -\hat{\beta}, \hat{\alpha} \rangle = -\langle \hat{\beta}, \hat{\alpha} \rangle,
\]

(4)
respectively. Thus it suffices to assume that \( \hat{\beta}, \hat{\alpha} \in \hat{\Pi} \). We now determine the extremal root chain in \( \hat{\mathcal{R}} \) through \( \hat{\alpha} \in \hat{\Pi} \) in the direction \( \hat{\alpha} \). Such an unbroken root chain is

\[
\hat{\alpha} - 2\hat{\alpha} = -\hat{\alpha}, \quad \hat{\alpha} - \hat{\alpha} = 0, \quad \hat{\alpha} + 0\hat{\alpha} = \hat{\alpha}.
\]

This root chain is extremal, since \( \pm 2\hat{\alpha} = \pm \hat{\alpha} \pm \hat{\alpha} \in \mathcal{Z} \) and thus does not lie in \( \mathcal{Y} \). It has associated integer pair \((2, 0)\). Hence by definition its Killing integer \( \left< \alpha, \alpha \right> \) is 2. \(\square\)

We now verify that axiom 4 of a system of roots holds for the collection of roots \( \Phi \). This amounts to proving Claim 2.4.2

For each fixed \( \hat{\alpha} \in \Phi = \mathcal{Y} \) the function

\[
K_{\hat{\alpha}} : \hat{\mathcal{R}} \to \mathbb{Z} : \hat{\beta} \mapsto \left< \hat{\beta}, \hat{\alpha} \right>
\]

is linear. By linear we mean: if \( \hat{\gamma}, \hat{\delta} \in \hat{\mathcal{R}} \) such that \( \hat{\gamma} + \hat{\delta} \in \hat{\mathcal{R}} \), then

\[
K_{\hat{\alpha}}(\hat{\gamma} + \hat{\delta}) = K_{\hat{\alpha}}(\hat{\gamma}) + K_{\hat{\alpha}}(\hat{\delta}).
\]

We will need a few preliminary results. Let \( \hat{\alpha} \in \hat{\Pi} \). Suppose that \( \hat{\beta}_1, \hat{\beta}_2 \) and \( \hat{\beta}_1 + \hat{\beta}_2 \) lie in \( \mathcal{R} \). For \( i = 1, 2 \) let

\[
\hat{\beta}_i - q_i \hat{\alpha}, \ldots, \hat{\beta}_i - 1\hat{\alpha}, \hat{\beta}_i, \hat{\beta}_i + 1\hat{\alpha}, \ldots, \hat{\beta}_i + p_i \hat{\alpha}
\]

be an extremal root chain through \( \hat{\beta}_i \) in the direction \( \hat{\alpha} \) with integer pair \((q_i, p_i) \in (\mathbb{Z}_{\geq 0})^2 \). Let

\[
\hat{\beta}_1 + \hat{\beta}_2 - r\hat{\alpha}, \ldots, \hat{\beta}_1 + \hat{\beta}_2 - (r - 1)\hat{\alpha}, \ldots, \hat{\beta}_1 + \hat{\beta}_2 + (s - 1)\hat{\alpha}, \hat{\beta}_1 + \hat{\beta}_2 + s\hat{\alpha}
\]

be an extremal root chain through \( \hat{\beta}_1 + \hat{\beta}_2 \) in the direction \( \hat{\alpha} \) with integer pair \((r, s) \). Consider the chain

\[
\hat{\beta}_1 + \hat{\beta}_2 - (q_1 + q_2)\hat{\alpha}, \ldots, \hat{\beta}_1 + \hat{\beta}_2 - 0\hat{\alpha}, \ldots, \hat{\beta}_1 + \hat{\beta}_2 + (p_1 + p_2)\hat{\alpha}
\]

Lemma 2.4.3 The chain (6) is an extremal root subchain of the unbroken chain (7).

**Proof.** The following argument shows that the chain (7) is unbroken. Write

\[
\hat{\beta}_2 - q_2\hat{\alpha} + \hat{\beta}_1 - q_1\hat{\alpha}, \hat{\beta}_2 - q_2\hat{\alpha} + \hat{\beta}_1 - (q_1 - 1)\hat{\alpha}, \ldots, \hat{\beta}_2 - q_2\hat{\alpha} + \hat{\beta}_1 + p_1\hat{\alpha}
\]

\[
\hat{\beta}_2 - (q_2 - 1)\hat{\alpha} + \hat{\beta}_1 + p_1\hat{\alpha}, \ldots, \hat{\beta}_2 + p_2\hat{\alpha} + \hat{\beta}_1 + p_1\hat{\alpha}.
\]

Using the fact that the chains in (5) are unbroken, it follows that the chain (8) is unbroken. Clearly the chains (7) and (8) are equal.

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To prove that the chain (6) is a subchain of the chain (7) suppose that
\[ q_1 + q_2 < r \text{ or } p_1 + p_2 < s. \] (9)
Recall that \( \overrightarrow{g}(\hat{\alpha}) \) is the root subalgebra of \( \overrightarrow{n} \) associated to the positive simple root \( \hat{\alpha} \in \hat{\Pi} \). Then \( \overrightarrow{g}(\hat{\alpha}) \) is equal to span\{\( X_{\hat{\alpha}}, X_{-\hat{\alpha}}, H_{\hat{\gamma}} \)\}, which is isomorphic to \( h_3 \). Let
\[
W = \sum_{-(q_1+q_2) \leq \ell \leq (p_1+p_2)} \mathbb{Z} \overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2+\ell\hat{\alpha}},
\] where \( \overrightarrow{g}_{\hat{\gamma}} \) is the root space of \( \overrightarrow{n} \) corresponding to \( \hat{\gamma} \) in the chain (7). Since the root chains in (5) are unbroken and extremal, we have
\[
\text{ad}^{q_1+p_1+1}_{X_{-\hat{\alpha}}}{\overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2}} = \{0\}
\]
and therefore
\[
\text{ad}^{p_1+p_2+q_1+q_2+1}_{X_{-\hat{\alpha}}}{\overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2+(p_1+p_2)\hat{\alpha}}} = \{0\}.
\]
This implies that \( W \) is \( \text{ad}_{\overrightarrow{g}(\hat{\alpha})} \)-invariant. By claim 2.3.1
\[
\mathbb{Z} \overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2} = \sum_{-r \leq m \leq s} \mathbb{Z} \overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2+m\hat{\alpha}}
\]
is a representation space for an irreducible representation of \( \text{ad}_{\overrightarrow{g}(\hat{\alpha})} \) on \( \overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2} \), since (6) is an extremal root chain. From the hypothesis (9) it follows that \( W \) is a proper \( \text{ad}_{\overrightarrow{g}(\hat{\alpha})} \)-invariant subspace of \( \overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2} \). (Recall that equation (3) implies that \( \overrightarrow{g}_{\hat{\gamma}} = \{0\} \), if \( \hat{\gamma} = \hat{\beta}_1 + \hat{\beta}_2 + m\hat{\alpha} \) lies in the chain (7) and either \( m < -r \) or \( m > s \).) But this contradicts the irreducibility of the \( \text{ad}_{\overrightarrow{g}(\hat{\alpha})} \)-representation on \( \overrightarrow{g}_{\hat{\beta}_1+\hat{\beta}_2} \). Therefore the statement in (9) is false, that is,
\[
q_1 + q_2 \geq r \text{ and } p_1 + p_2 \geq s. \] (10)
This shows that the chain (6) is a subchain of the chain (7). Since the chain (6) is extremal by hypothesis, we have proved the lemma. \( \square \)

We now define the notion of the sum of the extremal root chains
\[
\hat{\beta}_1 - q_i \hat{\alpha}, \ldots, \hat{\beta}_1 + p_i \hat{\alpha}
\] (11)
with integer pair \((q_i, p_i)\) for \( i = 1, 2 \). Suppose that \( \hat{\beta}_1 + \hat{\beta}_2 \in \Phi \). The sum of the extremal root chains (11) is the root chain obtained from the unbroken chain
\[
\hat{\beta}_1 + \hat{\beta}_2 - (q_1 + q_2)\hat{\alpha}, \ldots, \hat{\beta}_1 + \hat{\beta}_2 + (p_1 + p_2)\hat{\alpha}
\] (12)
by removing all the elements $\hat{\beta}_1 + \hat{\beta}_2 + j \hat{\alpha}$ of (12) which do not lie in $\Phi \cup \{0\}$. By lemma 2.4.3 the extremal root chain

$$\hat{\beta}_1 + \hat{\beta}_2 - r \hat{\alpha}, \ldots, \hat{\beta}_1 + \hat{\beta}_2 + s \hat{\alpha}$$

(13)

with integer pair $(r, s)$ is an unbroken root subchain of (12). From the definition of the nonnegative integer $r$ it follows that $\hat{\beta}_1 + \hat{\beta}_2 - k \hat{\alpha} \notin \Phi \cup \{0\}$ for every $k \in \mathbb{Z}$ with $r + 1 \leq k \leq (q_1 + q_2)$. Similarly, from the definition of the nonnegative integer $s$ it follows that $\{\hat{\beta}_1 + \hat{\beta}_2 + m \hat{\alpha} \notin \Phi \cup \{0\}\}$ for every $m \in \mathbb{Z}$ with $s + 1 \leq m \leq (p_1 + p_2)$. Consequently, the sum of the extremal root chains (11) is the extremal root chain (13).

Next we prove

Claim 2.4.4 The integer $(q_1 + q_2) - (p_1 + p_2)$ associated to the unbroken chain (12) with integer pair $(q_1 + q_2, p_1 + p_2)$ is equal to the Killing integer $r - s$ of the extremal root chain (13).

Proof. For $i = 1, 2$ let $\hat{\alpha} \in \hat{\Pi}$ and $\hat{\beta}_i \in \hat{\mathcal{R}}$. Recall that $\hat{\beta}_i - q_i \hat{\alpha}, \ldots, \hat{\beta}_i + p_i \hat{\alpha}$ is an extremal root chain with integer pair $(q_i, p_i)$ and length $q_i + p_i + 1$. Since (11) is an extremal root chain, $\hat{\mathcal{g}}_{\hat{\alpha}}$ is a complex $N_i = q_i + p_i + 1$-dimensional representation space for an irreducible $\text{ad}_{\hat{\mathcal{g}}_{\hat{\alpha}}}$. Let $n_i = \begin{cases} \frac{i}{2} (q_i + p_i), & \text{if } N_i \text{ is odd} \\ \frac{i}{2} (q_i + p_i + 1), & \text{if } N_i \text{ is even} \end{cases}$ Then $n_i$ is a nonnegative integer.

Recall that the root subalgebra $\hat{\mathcal{g}}_{(\hat{\alpha})}$ is isomorphic to $h_3$. Let $\iota: \hat{\mathcal{g}} \to \mathfrak{g}$ be the inclusion map, which is defined since $\hat{\mathcal{g}}$ is a subalgebra of the simple Lie algebra $\mathfrak{g}$. The image of the root vector $X_{-\hat{\alpha}}$ under $\iota$ is the root vector $X_{-\alpha}$ in $\mathfrak{g}$. This latter root vector embeds into the root subalgebra $\mathcal{g}_{\alpha} = \text{span}\{X_{-\alpha}, X_{\alpha}, H_{\alpha}\}$ of $\mathfrak{g}$, which is isomorphic to $\text{sl}(2, \mathbb{C})$. Since the root chains in (11) are extremal, it follows that the complex $N_i = q_i + p_i + 1$-dimensional $\text{ad}_{\mathcal{g}_{\alpha}}$ representation on $\hat{\mathcal{g}}_{\hat{\alpha}}$ is irreducible. Thus there is a vector $v_i \in \hat{\mathcal{g}}_{\hat{\alpha}}$ such that $V_i = \text{span}\{v_i, \text{ad}_{X_{-\alpha}} v_i, \ldots, \text{ad}_{X_{-\alpha}}^{q_i + p_i} v_i\} = \hat{\mathcal{g}}_{\hat{\alpha}}$. Consider the vector $w_i = \iota(v_i) \in \mathfrak{g}$. Let $W_i = \text{span}\{w_i, \text{ad}_{X_{-\alpha}} w_1, \ldots, \text{ad}_{X_{-\alpha}}^{q_i + p_i} w_i\}$. Because $X_{-\alpha} = \iota(X_{-\hat{\alpha}})$ and $\text{ad}_{X_{-\alpha}} \circ \iota = \iota \circ \text{ad}_{X_{-\hat{\alpha}}}$, it follows that $W_i = \iota(V_i)$ and $\text{ad}_{X_{-\alpha}}^{q_i + p_i + 1} w_i = 0$. Since $V_i$ is an $N_i$-dimensional irreducible representation of $\hat{\mathcal{g}}_{(\hat{\alpha})}$ on $\mathfrak{n}$, we find that the root subalgebra $\mathcal{g}_{\alpha}$ of $\mathfrak{g}$, which is isomorphic to $\text{sl}(2, \mathbb{C})$ and contains $\iota(\mathfrak{n}_{(\alpha)})$, acts irreducibly on $W_i$. Therefore $\text{ad}_{H_{\alpha}} w_i = n_i w_i$, where $n_i = \begin{cases} \frac{i}{2} (q_i + p_i), & \text{if } N_i \text{ is odd} \\ \frac{i}{2} (q_i + p_i + 1), & \text{if } N_i \text{ is even} \end{cases}$ is a nonnegative integer. This shows that

$$-n_i, -n_i + 2, -n_i + 2(2), \ldots, -n_i + 2(n_i - 1) = n_i - 2, n_i$$

(14)
lists the elements of the extremal root chains in (11). The root chain (13) with integer pair \((r, s)\) and length \(M = r + s + 1\) is extremal. Therefore \(\hat{g}_{\hat{\alpha} + \hat{\beta}}\) is an \(M\)-dimensional representation space for an \(\text{ad}_{\hat{\gamma}(\hat{\alpha})}\)-irreducible representation. The eigenvalues of \(\text{ad}_{\hat{H}_{\hat{\alpha}}}\) on \(i(\hat{\gamma}_{\hat{\beta}}\hat{\alpha})\) are listed as follows.

\[-m, -m + 2, -m + 2(2), \ldots, -m + 2(m - 1) = m - 2, m.\]  

(15)

Since the vector spaces \(i(\hat{\gamma}_{\hat{\beta}}\hat{\alpha})\) and \(\hat{g}_{\hat{\beta}}\hat{\alpha}\) are isomorphic, equation (15) lists the elements of the extremal root chain (13) provided that the positive integer \(m = \begin{cases} \frac{1}{2}(r + s), & \text{if } M \text{ is odd} \\ \frac{1}{2}(r + s + 1), & \text{if } M \text{ is even} \end{cases}\).

From claim 2.4.3 the extremal root chain (13) is a subchain of the unbroken chain (12). Thus the list (15) is a sublist of the list

\[-\ell = -(n_1 + n_2), -\ell + 2, -\ell + 2(2), \ldots, -\ell + 2(n_1 + n_2 - 1) = \ell - 2, \ell,\]  

(16)

which labels the elements of the unbroken chain (12). Suppose that the nonnegative integers \(\ell\) and \(m\) have a different parity. In particular, suppose that \(\ell\) is even and \(m\) is odd. Then 1 appears in the list (15) but not in the list (16). This contradicts the fact that (15) is a sublist of (16). Suppose that \(\ell\) is odd and \(m\) is even. Then 0 appears in the list (15) but not in the list (16). Again this contradicts the fact that (15) is a sublist of (16). Therefore \(\ell\) and \(m\) must have the same parity. Conversely, if \(\ell\) and \(m\) have the same parity, then the list (15) is a sublist of the list (16). Consequently, for \(i = 1, 2\) there is a \(j_i \in \mathbb{Z}, 0 \leq j_i \leq \ell\), such that

\[-\ell + 2j_1 = -m \text{ and } \ell - 2j_2 = m.\]

Therefore \(j_1 = j_2 = j\). Thus \(j\) is the number of elements of the unbroken chain (12) which need to be removed from its left and right ends to obtain the extremal root chain (13). Hence the integer \((q_1 + q_2) - (p_1 + p_2)\) associated to the unbroken chain (12) with integer pair \((q_1 + q_2, p_1 + p_2)\) is equal to the Killing integer \(r - s\) of the extremal root chain (13) with integer pair \((r, s)\). This proves claim 2.4.4. □

Claim 2.4.4 may be reformulated as

**Claim 2.4.5** Let \(\hat{\alpha} \in \hat{\Pi}\) with \(\hat{\beta}_1\), and \(\hat{\beta}_2 \in \hat{\mathcal{R}}\) such that \(\hat{\beta}_1 + \hat{\beta}_2 \in \hat{\mathcal{R}}\). For \(i = 1, 2\) let

\[\hat{\beta}_i - q_i \hat{\alpha}, \ldots, \hat{\beta}_i + p_i \hat{\alpha}\]  

(17)
be extremal root chains with integer pair \((q_i, p_i)\) and Killing integer \(\langle \hat{\beta}_i, \hat{\alpha} \rangle = q_i - p_i\). Suppose that the extremal root chain
\[ \hat{\beta}_1 + \hat{\beta}_2 - r \hat{\alpha}, \ldots, \hat{\beta}_1 + \hat{\beta}_2 + s \hat{\alpha} \]
with integer pair \((r, s)\) and Killing integer \(\langle \hat{\beta}_1 + \hat{\beta}_2, \hat{\alpha} \rangle = r - s\) is the sum of the extremal root chains in (17). Then
\[ \langle \hat{\beta}_1 + \hat{\beta}_2, \hat{\alpha} \rangle = \langle \hat{\beta}_1, \hat{\alpha} \rangle + \langle \hat{\beta}_2, \hat{\alpha} \rangle. \quad (18) \]

**Proof.** If \(-\hat{\beta} \in \hat{\Pi} \) and \(\hat{\alpha} \in \hat{\Pi}\), then by lemma 1.2.3 we have \(\langle \hat{\beta}, \hat{\alpha} \rangle = -\langle -\hat{\beta}, \hat{\alpha} \rangle = -K_{\hat{\alpha}}(-\hat{\beta})\). Similarly, if \(\hat{\beta} \in \hat{\Pi} \) and \(\hat{\alpha} \in \hat{\Pi}\), then by lemma 1.2.3 \(\langle \hat{\beta}, \hat{\alpha} \rangle = -\langle \hat{\beta}, -\hat{\alpha} \rangle = -K_{-\hat{\alpha}}(\hat{\beta})\). Therefore it suffices to prove the claim when \(\hat{\alpha}, \hat{\beta} \in \hat{\Pi}\). Using claim 2.4.4 we deduce that for every root \(\hat{\alpha}\) in \(\hat{\Pi}\) the function \(K_{\hat{\alpha}} : \hat{\mathcal{R}} \to \mathbb{Z} : \hat{\beta} \mapsto \langle \hat{\beta}, \hat{\alpha} \rangle\) is linear. \(\square\)

This completes the verification of

**Theorem 2.4.6** Let \(\tilde{g} = g \oplus \tilde{n}\) be a very special sandwich algebra of class \(\mathcal{C}\). Then the collection \(\Phi\) of nonzero roots in \(\hat{\mathcal{R}}\) associated to the nilradical \(\tilde{n}\) is a system of roots.

Because simple sandwich algebras of class \(\mathcal{C}\) are Lie algebras which are not semisimple and have a system of roots, the notion of a system of roots is a conservative generalization of the concept of a root system of a semisimple Lie algebra.

**Appendix**

From the classification of very special sandwich algebras, see [2], a very special sandwich algebra of \(\mathcal{C}\) is one of the following: \(\tilde{C}_{\ell+1}\), \(\tilde{G}_2^1\), \(\tilde{G}_2^2\), \(\tilde{F}_4\), \(\tilde{E}_6\) or \(\tilde{E}_7\). The following list gives \(\tilde{g}, g, \tilde{g}\) the Cartan subalgebra \(\mathfrak{h}\) of \(g\), and the roots \(\hat{\mathcal{R}}\) of the adjoint action of \(\mathfrak{h}\) on \(\tilde{n}\). By \(X_n^{(\ell)}\) we mean the simple Lie algebra of Cartan type \(X\) of rank \(n - 1\) whose Dynkin diagram is obtained by removing the node numbered \(\ell\) from the Dykin diagram of the simple Lie algebra of Cartan type \(X\) of rank \(n\). We use the following notation. Let \(\{\epsilon_i\}_{i=1}^n\) be the standard basis for \((\mathbb{R}^n)^*\) and let \(\{e_i\}_{i=1}^n\) be the standard dual basis of \(\mathbb{R}^n\). \(h_{2n+1}\) denotes the Lie algebra of the \(2n + 1\) dimensional Heisenberg group.
Very special sandwich algebras of class $C$

1. $\tilde{C}_{\ell+1}, \ G^{(1)}_{\ell+1}, \ C^{(1)}_{\ell+1}; \ \tilde{n} = h_{2\ell+1};$
   
   $h = \text{span}_C \{ h_i = e_i - e_{i+1}, \ 2 \leq i \leq \ell; \ h_{\ell+1} = e_{2\ell+1} \}$
   
   $\tilde{\mathcal{R}} = \left\{ \begin{array}{l}
   \hat{\zeta} = -2\varepsilon_1|h, \\
   \hat{\alpha}_k = -(\varepsilon_1 - \varepsilon_{k+1})|h, \\
   \hat{\alpha}^k = -(\varepsilon_1 + \varepsilon_{k+1})|h, \ 1 \leq k \leq \ell
\end{array} \right\}$

2. $\tilde{G}^1_2, \ G^{(1)}_2, \ G_2; \ \tilde{n} = h_3;$
   
   $h = \text{span}_C \{ h_2 = -2e_1 - e_2 - e_3 \}$
   
   $\tilde{\mathcal{R}} = \left\{ \begin{array}{l}
   \hat{\zeta} = (\varepsilon_2 - \varepsilon_3)|h, \\
   \hat{\alpha}_1 = (\varepsilon_1 - \varepsilon_3)|h, \\
   \hat{\alpha}^1 = (\varepsilon_2 - \varepsilon_3)|h
\end{array} \right\}$

3. $\tilde{G}^2_2, \ G^{(2)}_2, \ G_2; \ \tilde{n} = h_5;$
   
   $h = \text{span}_C \{ h_1 = e_1 - e_2 \}$
   
   $\tilde{\mathcal{R}} = \left\{ \begin{array}{l}
   \hat{\zeta} = (\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3)|h, \\
   \hat{\alpha}_1 = (-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3)|h, \\
   \hat{\alpha}^1 = (2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)|h \\
   \hat{\alpha}_2 = (e_2 - e_3)|h, \\
   \hat{\alpha}^2 = (\varepsilon_1 - \varepsilon_3)|h
\end{array} \right\}$

4. $\tilde{F}_4, \ F^{(1)}_4, \ F_4; \ \tilde{n} = h_{15};$
   
   $h = \text{span}_C \{ h_2 = e_3 - e_4, \ h_3 = 2e_4, \ h_4 = e_1 - e_2 - e_3 - e_4 \}$
   
   $\tilde{\mathcal{R}} = \left\{ \begin{array}{l}
   \hat{\zeta} = (\varepsilon_1 + \varepsilon_2)|h, \\
   \hat{\alpha}_1 = -\varepsilon_1|h, \ \hat{\alpha}^1 = -\varepsilon_2|h \\
   \hat{\alpha}_{k+1} = (-\varepsilon_1 + \varepsilon_{k+2})|h, \\
   \hat{\alpha}^{k+1} = (\varepsilon_1 - \varepsilon_{k+2})|h, \ k = 1, 2 \\
   \hat{\alpha}_{k+3} = (-\varepsilon_1 - \varepsilon_{k+2})|h, \\
   \hat{\alpha}^{k+3} = (\varepsilon_1 + \varepsilon_{k+2})|h, \ k = 1, 2 \\
   \hat{\alpha}_6 = \frac{1}{2} (-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)|h, \\
   \hat{\alpha}^6 = \frac{1}{2} (-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)|h, \\
   \hat{\alpha}_7 = \frac{1}{2} (-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)|h, \\
   \hat{\alpha}^7 = \frac{1}{2} (-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)|h
\end{array} \right\}$
5. \( \tilde{E}_6, \ E_6^{(2)}, \ E_5; \ \tilde{n} = h_{21}; \)
\[ h = \text{span}_C \{ h_1 = \frac{1}{2} (e_1 - \sum_{i=2}^{5} e_i + 3e_6), h_i = e_{i-1} - e_{i-2}, \ 3 \leq i \leq 6 \} \]
\[
\hat{R} = \left\{ \begin{array}{c}
\hat{\zeta} = -\frac{1}{2} (\sum_{i=1}^{6} \varepsilon_i)|h, \\
\hat{\alpha}_{i<j} = -(\varepsilon_i + \varepsilon_j)|h, \ 1 \leq i < j \leq 5 \\
\hat{\alpha}_{j\ell m} = -\frac{1}{2} (\sum_{i=1}^{5} (-1)^{k(i)} \varepsilon_i - \varepsilon_6)|h \\
\text{if } i \notin \{j, \ell, m\}; \ j, \ell, m \in \{1, \ldots, 5\} \text{ distinct} 
\end{array} \right. 
\]

6. \( \tilde{E}_7, \ E_7^{(1)}, \ E_7; \ \tilde{n} = h_{33}; \)
\[ h = \text{span}_C \{ h_2 = e_2 - e_1, h_3 = e_2 + e_1, h_i = e_{i-1} - e_{i-2}, \ 4 \leq i \leq 7 \} \]
\[
\hat{R} = \left\{ \begin{array}{c}
\hat{\zeta} = -\varepsilon_7|h, \\
\hat{\alpha}_1 = -\frac{1}{2} (\sum_{i=1}^{6} \varepsilon_i - \varepsilon_7)|h, \\
\hat{\alpha}^1 = -\frac{1}{2} (\sum_{i=1}^{7} \varepsilon_i)|h, \\
\hat{\alpha}_{j\ell} = -\frac{1}{2} (\sum_{i=1}^{6} (-1)^{k(i)} \varepsilon_i - \varepsilon_7)|h \\
\text{if } i \notin \{j, \ell\}; \ j, \ell \in \{1, \ldots, 5\} \text{ distinct} \\
\end{array} \right. 
\]

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