Perimeter Variance of Uniform Random Triangles

Steven Finch

July 1, 2010

Abstract. Let $T$ be a random triangle in a disk $D$ of radius $R$ (meaning that vertices are independent and uniform in $D$). We determine the bivariate density for two arbitrary sides $a, b$ of $T$. In particular, we compute that $E(ab) = (0.837...) R^2$, which implies that $\text{Var}(\text{perimeter}) = (0.649...) R^2$. No closed-form expression for either coefficient is known. The Catalan numbers also arise here.

Let $A, B, C$ denote three independent uniformly distributed points in the disk

$$D = \{(\xi, \eta) : \xi^2 + \eta^2 \leq R^2\}.$$  

Let $T$ denote the triangle with sides $a, b, c$ opposite the vertices $A, B, C$. We are interested in the perimeter $a + b + c$ of triangle $T$. The univariate density $f(x)$ for side $a$ is

$$f(x) = \frac{4x}{\pi R^2} \arccos \left( \frac{x}{2R} \right) - \frac{x^2}{\pi R^4} \sqrt{4R^2 - x^2}, \quad 0 < x < 2R$$

and

$$E(a) = \frac{128}{45\pi} R = (0.9054147873672267990407609...) R, \quad E(a^2) = R^2.$$ 

Clearly

$$E(\text{perimeter}) = 3E(a) = \frac{128}{15\pi} R = (2.7162443621016803971222828...) R$$

but to compute $\text{Var}(\text{perimeter}) = E(\text{perimeter}^2) - E(\text{perimeter})^2$, we will further need to consider cross-correlation $\rho$ between sides.

The bivariate density $f(x, y)$ for sides $a, b$ is

$$f(x, y) = \begin{cases} 
\varphi(x, y) & \text{if } x + y \leq 2R, \\
\psi(x, y) & \text{if } x + y > 2R \text{ and } x \leq 2R
\end{cases}$$

when $0 \leq y \leq x$ (use symmetry otherwise) where

$$\varphi(x, y) = \frac{2xy}{\pi R^6} \left\{-\sqrt{(2R - x - y)(x - y)(2R + x - y)(x + y)} + 
2(R - y)^2 \arccos \left( \frac{x^2 - 2Ry + y^2}{2x(R - y)} \right) + 2R^2 \arccos \left( \frac{x^2 + 2Ry - y^2}{2Rx} \right) \right\} +$$

$$\frac{8xy}{\pi^2 R^6} \int_{R-y}^{R} t \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \arccos \left( \frac{t^2 + y^2 - R^2}{2ty} \right) dt,$$

$^0$ Copyright © 2010 by Steven R. Finch. All rights reserved.
\[ \psi(x, y) = \frac{8xy}{\pi^2 R^6} \int_{x-R}^{x+R} t \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \arccos \left( \frac{t^2 + y^2 - R^2}{2ty} \right) dt. \]

It follows by numerical integration that

\[ \mathbb{E}(ab) = (0.8378520652962219016710654...)R^2 \]

hence

\[ \rho(a, b) = \frac{\mathbb{E}(ab) - \mathbb{E}(a) \mathbb{E}(b)}{\sqrt{\text{Var}(a) \text{Var}(b)}} = 0.100298083565900175822627..., \]

\[ \mathbb{E}((\text{perimeter})^2) = 3\mathbb{E}(a^2) + 6\mathbb{E}(ab) = (8.027112391777314100263929...)R^2, \]

\[ \text{Var}(\text{perimeter}) = (0.6491289571281667551974101...)R^2. \]

Exact evaluation of \( \mathbb{E}(ab) \) remains an open problem. We review derivation of the univariate case in section 1, imitating the analysis in [8, 9] very closely. (Parry’s thesis [8] is concerned with triangles in three-dimensional space; it is surprising that our two-dimensional analog has not yet been examined.) The bivariate case is covered in section 2. An experimental consequence of our work is the formula

\[ \mathbb{E}(a^2 b^2) = \frac{13}{12}R^4 \]

which we prove via a different approach in section 3. Finally, in section 4, the Catalan numbers from combinatorics appear rather unexpectedly.

1. **Univariate Case**

We omit geometric details, referring to [8, 9] instead. The distance \( t \) between point \( C \) and the origin has density \( 2t/R^2 \) for \( 0 < t < R \). Let \( f(x \mid t) \) be the conditional density for distance \( x \) between points \( C \) and \( B \), given \( t \). We will compute the sought-after density \( f(x) \) for side \( a \) via

\[ f(x) = \int_0^R f(x \mid t) \frac{2t}{R^2} dt. \]

There are two subcases.

1.1. \( 0 < x < R \).

\[ f(x) = \int_0^{x} \frac{2x}{R^2} \frac{2t}{R^2} dt + \int_{-x}^{R-x} \frac{2x}{\pi R^2} \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \frac{2t}{R^2} dt \]

which corresponds to formula (1.12) in Parry’s thesis [8]. The arccos term arises since, if the portion of a circle of radius \( x \), center \( C \) contained within \( D \) has arclength \( 2\theta x \), then \( f(x \mid t) = (2\theta x)/(\pi R^2) \); the Law of Cosines gives \( \theta \).
1.2. \( R < x < 2R \).

\[
f(x) = \int_{x-R}^{R} \frac{2x}{\pi R^2} \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \frac{2t}{R^2} dt
\]

which corresponds to Parry’s (1.18). Straightforward integration provides the desired result (valid in both of the preceding regions).

2. **Bivariate Case**

We omit geometric details, referring to [8] instead. The distance \( t \) between point \( C \) and the origin has density \( \frac{2}{R^2} \) for \( 0 < t < R \). Let \( f(x,y|t) \) be the conditional density for distance \( x \) between points \( B \) and \( C \), and distance \( y \) between points \( A \) and \( C \), given \( t \). We will compute the sought-after density \( f(x,y) \) for sides \( a, b \) via

\[
f(x,y) = \int_{0}^{R} f(x,y|t) \frac{2t}{R^2} dt.
\]

There are six subcases.

2.1. \( y < x \) and \( 0 < x < R \).

\[
f(x,y) = \int_{0}^{R-y} \frac{2x}{R^2} \frac{2y}{R^2} \frac{2t}{R^2} dt + \int_{R-x}^{R-y} \frac{2x}{\pi R^2} \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \frac{2y}{R^2} \frac{2t}{R^2} dt + \int_{R-y}^{R} \frac{2x}{\pi R^2} \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \frac{2y}{\pi R^2} \arccos \left( \frac{t^2 + y^2 - R^2}{2ty} \right) \frac{2t}{R^2} dt
\]

which corresponds to formula (4.26) in Parry’s thesis [8].

2.2. \( R < x < 2R \) and \( 0 < y < 2R - x \).

\[
f(x,y) = \int_{x-R}^{R-y} \frac{2x}{\pi R^2} \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \frac{2y}{R^2} \frac{2t}{R^2} dt + \int_{R-y}^{R} \frac{2x}{\pi R^2} \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \frac{2y}{\pi R^2} \arccos \left( \frac{t^2 + y^2 - R^2}{2ty} \right) \frac{2t}{R^2} dt
\]

which corresponds to Parry’s (4.29). Straightforward integration gives \( \varphi(x,y) \) (valid in both of the preceding regions).
2.3. \( R < x < 2R \) and \( 2R-x < y < x \).

\[
f(x, y) = \int_{x-R}^{R} 2x \frac{2y}{\pi R^2} \frac{2y}{\pi R^2} \arccos \left( \frac{t^2 + x^2 - R^2}{2tx} \right) \frac{2t}{R^2} dt
\]

which corresponds to Parry’s (4.32). This is, of course, \( \psi(x, y) \).

2.4. \( x < y \) and \( 0 < y < R \).

\[
f(x, y) = \int_{0}^{R-y} \frac{2x}{R^2} \frac{2y}{R^2} \frac{2t}{R^2} dt + \int_{R-y}^{R} \frac{2x}{R^2} \frac{2y}{R^2} \frac{2t}{R^2} dt + \int_{R-x}^{R} 2 \frac{x}{\pi R^2} \frac{2y}{\pi R^2} \frac{2t}{R^2} dt
\]

which corresponds to Parry’s (4.35). This is, of course, \( \phi(y, x) \).

2.5. \( R < y < 2R \) and \( 0 < x < 2R-y \).

\[
f(x, y) = \int_{y-R}^{R-x} \frac{2x}{R^2} \frac{2y}{\pi R^2} \frac{2t}{R^2} dt + \int_{y-R}^{R} \frac{2x}{R^2} \frac{2y}{\pi R^2} \frac{2t}{R^2} dt
\]

which corresponds to Parry’s (4.38). This is, of course, \( \phi(y, x) \).

2.6. \( R < y < 2R \) and \( 2R-y < x < y \).

\[
f(x, y) = \int_{y-R}^{R} \frac{2x}{R^2} \frac{2y}{\pi R^2} \frac{2t}{R^2} dt
\]

which corresponds to Parry’s (4.41). This is, of course, \( \psi(y, x) \).

3. Characteristic Function

We follow an approach found in [10, 11]. Let \( u, v, w \) denote the squared distances between \( A, B, C \) and the origin \( O \). Let \( \varphi \) denote the angle between vectors \( \overrightarrow{OA}, \overrightarrow{OB} \) and \( \psi \) denote the angle between vectors \( \overrightarrow{OA}, \overrightarrow{OC} \). We have

\[
a^2 = v + w - 2\sqrt{vw} \cos(\psi - \varphi),
\]
by the Law of Cosines, where \(u, v, w\) are independent uniform on \([0, R]^2\) and \(\varphi, \psi\) are independent uniform on \([0, 2\pi]\). The characteristic function for \((a^2, b^2, c^2)\) is thus

\[
g(r, s, t) = \frac{1}{R^6} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\varphi d\psi \int_0^{R^2} \int_0^{R^2} du dv dw \exp \left[ ir \left( v + w - 2\sqrt{uv} \cos(\psi - \varphi) \right) + is \left( u + w - 2\sqrt{uw} \cos(\psi) \right) + it \left( u + v - 2\sqrt{uv} \cos(\varphi) \right) \right].
\]

It is well-known that

\[
E(c^2) = \frac{1}{i} \frac{\partial g}{\partial t} \bigg|_{r=s=t=0}, \quad E(b^2 c^2) = \frac{1}{i^2} \frac{\partial^2 g}{\partial s \partial t} \bigg|_{r=s=t=0}
\]

and the former becomes

\[
E(c^2) = \frac{1}{i} \frac{\partial}{\partial t} R^4 \int_0^{2\pi} \int_0^{2\pi} d\varphi d\psi \int_0^{R^2}\int_0^{R^2} du dv \exp \left( it \left( u + v - 2\sqrt{uv} \cos(\varphi) \right) \right) J_0 \left( 2t\sqrt{uv} \right) du dv \bigg|_{t=0}
\]

where \(J_0(\theta)\) is the zeroth Bessel function of the first kind; hence

\[
E(c^2) = \frac{1}{i} \frac{1}{R^4} \int_0^{2\pi} \int_0^{2\pi} \exp \left( it \left( u + v \right) \right) J_0 \left( 2t\sqrt{uv} \right) du dv \bigg|_{t=0}
\]

\[
= \frac{1}{i} \frac{1}{R^4} \int_0^{2\pi} \int_0^{2\pi} \left( u + v \right) du dv = R^2
\]

which is consistent with before. The latter becomes

\[
E(b^2 c^2) = \frac{1}{i^2} \frac{\partial^2}{\partial s \partial t} R^6 \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\varphi d\psi \int_0^{R^2} \int_0^{R^2} du dv dw \exp \left[ is \left( u + w - 2\sqrt{uw} \cos(\psi) \right) + it \left( u + v - 2\sqrt{uv} \cos(\varphi) \right) \right] \bigg|_{s=t=0}
\]

\[
= - \frac{\partial^2}{\partial s \partial t} R^6 \int_0^{2\pi} \int_0^{2\pi} \exp \left( is \left( u + w \right) \right) J_0 \left( 2s\sqrt{uw} \right) \exp \left( it \left( u + v \right) \right) J_0 \left( 2t\sqrt{uv} \right) du dv dw \bigg|_{s=t=0}
\]
\[ \int \int \frac{\partial^2}{\partial s \partial t} \exp \left( is (u + w) \right) J_0 \left( 2s \sqrt{uw} \right) \exp \left( it (u + v) \right) J_0 \left( 2t \sqrt{uv} \right) \bigg|_{s=t=0} \, du \, dv \, dw \]

\[ = - \frac{1}{R^6} \int \int \int \frac{\partial^2}{\partial s \partial t} \exp \left( is (u + w) \right) J_0 \left( 2s \sqrt{uw} \right) \exp \left( it (u + v) \right) J_0 \left( 2t \sqrt{uv} \right) \bigg|_{s=t=0} \, du \, dv \, dw \]

\[ = - \frac{1}{R^6} \int \int \int - (u + v)(u + w) du \, dv \, dw = \frac{13}{12} R^4 \]

as was to be shown. The fact that \( \frac{13}{12} - 1 = \frac{1}{12} \neq 0 \) offers the simplest proof we know that arbitrary sides of a random triangle in \( D \) must be dependent.

4. Catalan Numbers

Let \( R = 1 \) for the remainder of our discussion. From

\[ P \left( a^2 < x \right) = P \left( a < \sqrt{x} \right) = \int_0^{\sqrt{x}} \left( \frac{4\xi}{\pi} \arccos \left( \frac{\xi}{2} \right) - \frac{\xi^2}{\pi} \sqrt{4 - \xi^2} \right) \frac{1}{2\sqrt{x}} \, d\xi \]

we obtain that the density for \( a^2 \) is

\[ \left( \frac{4\sqrt{x}}{\pi} \arccos \left( \frac{\sqrt{x}}{2} \right) - \frac{x}{\pi} \sqrt{4 - x} \right) \frac{1}{2\sqrt{x}}, \quad 0 < x < 4. \]

On the one hand, the characteristic function for \( a^2 \) is

\[ \int_0^1 \int_0^1 \exp \left( it (u + v) \right) J_0 \left( 2t \sqrt{uv} \right) \, du \, dv \]

by the preceding section; on the other hand, it is

\[ \int_0^4 \exp(itx) \left( \frac{4\sqrt{x}}{\pi} \arccos \left( \frac{\sqrt{x}}{2} \right) - \frac{x}{\pi} \sqrt{4 - x} \right) \frac{1}{2\sqrt{x}} \, dx \]

\[ = \frac{i}{t} \left[ 1 - \exp(2it) \left( J_0(2t) - iJ_1(2t) \right) \right] \]

\[ = \frac{i}{t} \left[ 1 - h(t) \right] \]

where \( J_1(\theta) = -J_0'(\theta) \). A direct evaluation of the double integral seems to be difficult. Boersma \[12\], using work of Zernike & Nijboer \[13, 14, 15\], gave a rapidly-convergent series for the inner integral:

\[ \int_0^1 \exp(itu) J_0 \left( 2t \sqrt{uv} \right) \, du = \frac{\sqrt{\pi}}{t^{3/2}v^{1/2}} \exp \left( \frac{it}{2} \right) \sum_{n=0}^{\infty} (-i)^n (2n+1) J_{n+1/2} \left( \frac{t}{2} \right) J_{2n+1} \left( 2t \sqrt{v} \right) \]
but this apparently does not help with the outer integral.

Let $I_0(\theta)$ be the zeroth modified Bessel function of the first kind and $I_1(\theta) = I_0'(\theta)$. We note that the exponential generating function for the Catalan numbers [16]:

$$
\exp(2t) (I_0(2t) - I_1(2t)) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{2n}{n}\right) t^n
$$

is remarkably similar to the expression for $h(t)$. Replacing $t$ by $it$, we obtain

$$
h(t) = \exp(2it) (J_0(2t) - iJ_1(2t)) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{2n}{n}\right) (it)^n
$$

because $J_0(i\theta) = I_0(\theta)$, $J_1(i\theta) = iI_1(\theta)$. Therefore the Catalan numbers are associated with the characteristic function for $a^2$. We wonder if a two-dimensional integer array, suitably generalizing the Catalan numbers, can be associated with the characteristic function for $(a^2, b^2)$:

$$
\int_0^1 \int_0^1 \int_0^1 \exp(is(u+w)) J_0(2s\sqrt{uw}) \exp(it(u+v)) J_0(2t\sqrt{uv}) \, du \, dv \, dw.
$$

Since the bivariate density $f(x, y)$ for $(a, b)$ is much more complicated than the univariate density $f(x)$ for $a$, an answer to our question may be a long time coming.

5. Acknowledgement

I am thankful to Michelle Parry for her correspondence. Much more relevant material can be found at [17, 18], including experimental computer runs that aided theoretical discussion here.

References

[1] R. Deltheil, Probabilités Géométriques, t. 2, Traité du calcul des Probabilités et de ses Applications, f. 2, ed E. Borel, Gauthier-Villars, 1926, pp. 40–42, 114–120.

[2] J. M. Hammersley, The distribution of distance in a hypersphere, Annals Math. Statist. 21 (1950) 447–452; MR0037481 (12,268e).

[3] R. D. Lord, The distribution of distance in a hypersphere, Annals Math. Statist. 25 (1954) 794–798; MR0065048 (16,377d).

[4] V. S. Alagar, The distribution of the distance between random points, J. Appl. Probab. 13 (1976) 558–566; MR0418183 (54 2#6225).

[5] H. Solomon, Geometric Probability, SIAM, 1978, pp. 35–36, 128–129; MR0488215 (58 #7777).
[6] S. R. Dunbar, The average distance between points in geometric figures, *College Math. J.* 28 (1997) 187–197; MR1444006 (98a:52007).

[7] S.-J. Tu and E. Fischbach, Random distance distribution for spherical objects: general theory and applications to physics, *J. Phys. A* 35 (2002) 6557–6570; MR1928848.

[8] M. Parry, *Application of Geometric Probability Techniques to Elementary Particle and Nuclear Physics*, Ph.D. thesis, Purdue Univ., 1998.

[9] M. Parry and E. Fischbach, Probability distribution of distance in a uniform ellipsoid: theory and applications to physics, *J. Math. Phys.* 41 (2000) 2417–2433; MR1751899 (2001j:81267).

[10] Y. Isokawa, Limit distributions of random triangles in hyperbolic planes, *Bull. Faculty Educ. Kagoshima Univ. Natur. Sci.* 49 (1997) 1–16; MR1653095 (99k:60020).

[11] Y. Isokawa, Geometric probabilities concerning large random triangles in the hyperbolic plane, *Kodai Math. J.* 23 (2000) 171–186; MR1768179 (2001f:60014).

[12] J. Boersma, On the computation of Lommel’s functions of two variables, *Math. Comp.* 16 (1962) 232–238; MR0146419 (26 #3941).

[13] F. Zernike and B. R. A. Nijboer, Théorie de la diffraction des aberrations, *La Théorie des Images Optiques*, Proc. 1946 Paris colloq., ed. P. Fleury, A. Maréchal and C. Anglade, La Revue d’Optique, 1949, pp. 227–235.

[14] B. R. A. Nijboer, *The Diffraction Theory of Aberrations*, Ph.D. thesis, Univ. of Groningen, 1942, available online at [http://www.nijboerzernike.nl/_html/intro.html](http://www.nijboerzernike.nl/_html/intro.html).

[15] A. J. E. M. Janssen, J. J. M. Braat and P. Dirksen, On the computation of the Nijboer-Zernike aberration integrals at arbitrary defocus, *J. Mod. Optics* 51 (2004) 687–703; available online at [http://www.nijboerzernike.nl/_html/biblio.html](http://www.nijboerzernike.nl/_html/biblio.html).

[16] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A144186.

[17] S. Finch, *Random triangles. I–IV*, unpublished essays (2010), [http://algo.inria.fr/bsolve/](http://algo.inria.fr/bsolve/)
[18] S. Finch, Simulations in R involving triangles and tetrahedra, 
http://algo.inria.fr/csolve/rsimul.html

Steven Finch
Dept. of Statistics
Harvard University
Cambridge, MA, USA
Steven.Finch@inria.fr