A ROTOR CONFIGURATION WITH MAXIMUM ESCAPE RATE

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Abstract. Rotor walk is a deterministic analogue of simple random walk. For any given graph, we construct a rotor configuration for which the escape rate of the corresponding rotor walk is equal to the escape rate of simple random walk, and thus answer a question of Florescu, Ganguly, Levine, and Peres (2014).

1. Introduction

Let $G := (V, E)$ be a graph that is connected, simple (i.e., no loops and multiple edges) and locally finite (i.e., every vertex has finitely many neighbors). In a rotor walk [WLB96, PDDK96], each vertex has a rotor, which is an outgoing edge of the vertex. All of the rotors together constitute a rotor configuration, which is encoded by a function $\rho$ that maps every vertex of $G$ to one of its outgoing edges. To each vertex $x$ we assign a fixed rotor mechanism, which is a cyclic ordering on the set of outgoing edges $\mathcal{E}_x$ of $x$, and is encoded by a bijection $m_x : \mathcal{E}_x \to \mathcal{E}_x$ that has only one orbit.

Rotor walk evolves in the following manner. A particle is initially located at a fixed vertex $o$. At each time step, the rotor at the particle’s current location is first incremented to the next edge in the cyclic order, and the particle moves to the target vertex of the new rotor.

Propp [Pro03] proposed rotor walk as a derandomized version of simple random walk, and this naturally invited a comparison between the two walks. One such comparison is given by the following experiment. Start with an initial rotor configuration $\rho$, and with $n$ particles initially located at $o$. At each time step, each of these $n$ particles will take turns in performing one step of rotor walk, and the particle is removed if it ever returns to $o$. Denote by $I(\rho, n)$ the number of particles that never return to $o$.
Schramm [HP10, FGLP14] showed that the escape rate of rotor walk is always bounded above by the escape rate of simple random walk. That is to say, for any rotor configuration $\rho$:

$$\limsup_{n \to \infty} \frac{I(\rho, n)}{n} \leq \alpha_G,$$  \hspace{1cm} (1)

where $\alpha_G$ is the probability for simple random walk starting at $o$ to never return to $o$.

The result of Schramm inspired Florescu, Ganguly, Levine, and Peres [FGLP14] to ask if there is always a rotor configuration with escape rate equal to $\alpha_G$. Such a configuration has been constructed for certain choices of $G$, such as for the binary tree [LL09]; for transient trees [AH11]; for $\mathbb{Z}^d$ with $d \geq 3$ [He14]; and for transient vertex-transitive graphs [Cha18].

In this paper, we resolve the question of Florescu et al. by constructing a rotor configuration with maximum escape rate for any given graph. We focus on the case when $G$ is a transient graph, as any rotor configuration on a recurrent graph has escape rate equal to 0 by (1).

Let $G : V \to \mathbb{R}_{\geq 0}$ be the Green function of $G$, which maps $x \in V$ to the expected number of visits to $x$ by the simple random walk on $G$ starting at $o$. The weight of a directed edge $(x, y)$ of $G$ is

$$w(x, y) := \frac{-1}{\deg(x)} \sum_{i=0}^{\deg(x)-1} i \frac{G(y_{i+1})}{\deg(y_{i+1})},$$  \hspace{1cm} (2)

where $(x, y_i) := m_i(x, y)$ is the edge obtained by incrementing the edge $(x, y)$ for $i$ consecutive times by using the rotor mechanism at $x$.

**Theorem 1.** Let $G$ be a transient graph that is connected, simple, and locally finite. If $\rho_{\min}$ is a rotor configuration such that, for any vertex $x$ and any outgoing edge $(x, y)$ of $x$,

$$w(\rho_{\min}(x)) \leq w(x, y),$$  \hspace{1cm} (3)

then

$$\lim_{n \to \infty} \frac{I(\rho_{\min}, n)}{n} = \alpha_G.$$

Theorem 1 is proved by constructing an invariant of the rotor walk that balances between the Green function of the location of the particles and the weight of the edges in the rotor configuration at any given time.

Note that one can always construct a rotor configuration $\rho$ satisfying (3), by defining $\rho(x)$ for any $x \in V$ to be the edge for which its weight is the minimum among all outgoing edges of $x$. Also note that (3) is not a necessary condition, as all configurations with maximum escape rate from other works (mentioned above) do not satisfy (3).
2. Proof of Theorem 1

We now give a formal definition to the experiment in Section 1. Let $\rho$ be the initial rotor configuration, and let $n$ be the number of particles. The location of the particles $X_t^{(0)}, X_t^{(1)}, \ldots, X_t^{(n-1)}$ and the rotor configuration $\rho_t$ at the $t$-th step of the experiment ($t \geq 0$) are given by the following recurrence:

- Initially, $X_0^{(i)} := o$ for $i \in \{0, 1, \ldots, n-1\}$ and $\rho_0 := \rho$;
- Write $i_t := t + 1 \mod n$. If the $i_t$-th particle has returned to $o$ (i.e. $X_t^{(i_t)} = o$ and $X_s^{(i_t)} \neq o$ for some $s < t$), then $\rho_{t+1} := \rho_t$, and $X_{t+1}^{(i)} := X_t^{(i)}$ for $i \in \{0, \ldots, n-1\}$.
- If the $i_t$-th particle has not returned to $o$, then
  \[
  \rho_{t+1}(x) := \begin{cases} 
  m_x(\rho_t(x)) & \text{if } x = X_t^{(i_t)}; \\
  \rho_t(x) & \text{otherwise}.
  \end{cases}
  \]
  \[
  X_{t+1}^{(i)} := \begin{cases} 
  \text{target vertex of } \rho_{t+1}(X_t^{(i)}) & \text{if } i = i_t; \\
  X_t^{(i)} & \text{otherwise}.
  \end{cases}
  \]

That is, at time $t$, the $i_t$-th particle performs one step of a rotor walk if it has not returned to $o$, and does nothing if it has returned to $o$.

We denote by $R_t := R_t(\rho, n)$ the range of the experiment at time $t$,
\[
R_t := \{X_s^{(i)} | i \in \{0, 1, \ldots, n-1\} \text{ and } s \leq t\}.
\]

We now define an invariant of the rotor walk that is a special case of the invariant introduced in [HP10, Proposition 13]; a related invariant has been used in [HS11] and [HS12] to study the rotor-router aggregation of comb lattices. Let $M_t := M_t(\rho, n) (t \geq 0)$ be given by:
\[
M_t := \sum_{i=0}^{n-1} \frac{G(X_t^{(i)})}{\deg(X_t^{(i)})} + \min\{t, n\} \frac{\deg(o)}{\deg(o)} + \sum_{x \in R_t} (w(\rho_t(x)) - w(\rho(x))).
\] (4)

**Proposition 2.1.** For any initial rotor configuration $\rho$, any $n \geq 1$, and any $t \geq 0$, we have
\[
M_t = n \frac{G(o)}{\deg(o)}.
\]

We will use the fact that the Green function is a voltage function when a unit current enters $G$ through $o$ [LP16, Proposition 2.1]. That is, for any $x \in V$,
\[
\frac{1}{\deg(x)} \sum_{y \sim x} \frac{G(y)}{\deg(y)} = \frac{G(x)}{\deg(x)} - \frac{1}{\deg(o)},
\] (5)
where \( y \sim x \) means that \( y \) is a neighbor of \( x \) in \( G \).

**Proof of Proposition 2.1.** It follows directly from the definition that \( M_0 = n \frac{G(o)}{\deg(o)} \). Therefore it suffices to show that, for any \( t \geq 0 \):

\[
M_{t+1} - M_t = 0.
\]

Recall that \( i_t := t + 1 \mod n \). Write \( \alpha_t := X_t^{(i_t)} \) and \( \beta_t := X_{t+1}^{(i_t)} \). If the \( i_t \)-th particle has returned to \( o \) by time \( t \), then no action is performed at time \( t \), and \( M_{t+1} = M_t \). If the \( i_t \)-th particle has not returned to \( o \) by time \( t \), then it follows from the definition of \( M_t \) and \( M_{t+1} \) in (4) that

\[
M_{t+1} - M_t = \frac{G(\beta_t)}{\deg(\beta_t)} - \frac{G(\alpha_t)}{\deg(\alpha_t)} + \frac{1}{\deg(o)} \{ t \leq n - 1 \} - \sum_{y \sim \alpha_t} \frac{G(y)}{\deg(y)}.
\]

On the other hand, we have from the definition of \( w \) in (2) that

\[
w(\rho_{t+1}(\alpha_t)) - w(\rho_t(\alpha_t)) = -\frac{1}{\deg(\alpha_t)} \left( \deg(\alpha_t) \frac{G(\beta_t)}{\deg(\beta_t)} - \sum_{y \sim \alpha_t} \frac{G(y)}{\deg(y)} \right)
\]

\[
= -\frac{G(\beta_t)}{\deg(\beta_t)} + \frac{1}{\deg(\alpha_t)} \sum_{y \sim \alpha_t} \frac{G(y)}{\deg(y)}.
\]

Applying (5) to the equation above then gives us

\[
w(\rho_{t+1}(\alpha_t)) - w(\rho_t(\alpha_t)) = -\frac{G(\beta_t)}{\deg(\beta_t)} + \frac{G(\alpha_t)}{\deg(\alpha_t)} - \frac{1}{\deg(o)} \{ \alpha_t = o \}.
\]

Combining (6) and (7), we then get

\[
M_{t+1} - M_t = \frac{\{ t \leq n - 1 \}}{\deg(o)} - \frac{\{ \alpha_t = o \}}{\deg(o)}.
\]

Since the \( i_t \)-th particle has not returned to \( o \) yet by time \( t \), this means that the \( \alpha_t = o \) if and only if \( t \leq n - 1 \) (i.e., the \( i_t \)-th particle has not left \( o \) yet). This then implies \( M_{t+1} - M_t = 0 \) by the equation above, and the proof is complete. \( \square \)

Let \( I_t(\rho, n) \) be the number of particles that have not returned to \( o \) by time \( t \).

**Proposition 2.2.** If \( \rho_{min} \) is a configuration that satisfies (3), then for any \( n \geq 1 \) and any \( t \geq n \),

\[
\frac{I_t(\rho_{min}, n)}{n} \geq \alpha_G.
\]
Proof. Let $S_t \subseteq \{0, 1 \ldots, n-1\}$ be the set of particles that has returned to $o$ by time $t$. Since the Green function is a nonnegative function, we have:

$$\sum_{i=0}^{n-1} \frac{G(X^{(i)}_t)}{\deg(X^{(i)}_t)} \geq \sum_{i \in S_t} \frac{G(X^{(i)}_t)}{\deg(X^{(i)}_t)} = (n - I_t(\rho_{\min}, n)) \frac{G(o)}{\deg(o)}. \quad (8)$$

Since $\rho_{\min}$ satisfies (3), we also have

$$\sum_{x \in R_t} (w(\rho_t(x)) - w(\rho_{\min}(x))) \geq 0. \quad (9)$$

Plugging (8) and (9) into the definition of $M_t$ in (4), we have that, for $t \geq n$,

$$M_t \geq (n - I_t(\rho_{\min}, n)) \frac{G(o)}{\deg(o)} + \frac{n}{\deg(o)},$$

which is equivalent to

$$\frac{I_t(\rho_{\min}, n)}{n} \geq 1 + \frac{1}{G(o)} - \frac{\deg(o)}{n G(o)} M_t.$$

Plugging in the value of $M_t$ from Proposition 2.1, we then have:

$$\frac{I_t(\rho_{\min}, n)}{n} \geq \frac{1}{G(o)},$$

and the conclusion now follows by noting that $\alpha_G = 1/G(o)$. \qed

We now present the proof of Theorem 1.

Proof of Theorem 1. We have for any $n \geq 1$,

$$\frac{I(\rho_{\min}, n)}{n} = \lim_{t \to \infty} \frac{I_t(\rho_{\min}, n)}{n} \geq \alpha_G,$$

where the inequality is due to Proposition 2.2. The theorem now follows by combining the inequality above with (1). \qed

3. Open problems

(i) Classify all $c \geq 0$ for which there exists a rotor configuration $\rho$ such that $\lim_{n \to \infty} I(\rho, n)/n = c$.

To date the only known result of this kind is due to Landau and Levine [LL09], which shows that, for the complete binary tree, the constant $c$ can range from 0 to $\alpha_G$.

(ii) Consider the random rotor configuration $\rho$ where $(\rho(x))_{x \in V}$ are independent and uniformly distributed among the outgoing edges of $x$. What is the probability that $\rho$ has escape rate equal to $\alpha_G$?
Angel and Holroyd [AH11] showed that this probability is 1 if $G$ is a complete $b$-ary tree. The author [Cha18] also showed the same result if $G$ is a transient vertex-transitive graph and the configuration is sampled from the oriented wired spanning forest measure.

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