Analytical and Numerical Solutions of a TB-HIV/AIDS Co-infection Model via Fractional Derivatives Without Singular Kernel

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Abstract Human beings, who have encountered a lot of diseases and viruses in the last century, conduct a lot of studies to defeat such diseases, particularly HIV, known as the human immunodeficiency virus, and AIDS disease known as advanced immunodeficiency syndrome. The said diseases can be quite dangerous for countries and even for the whole world as the mode of transmission and transmission rate of such diseases increase. The literature needs mathematical modeling and subsequent analysis of such diseases. In the light of this information, TB-HIV/AIDS co-infection model was analyzed. To this end, the model was firstly extended to the Caputo-Fabrizio fractional derivative obtained using the exponential function. Then, uniqueness of solution was investigated by the help of the fixed-point theorem. Thereafter, the uniqueness of solution of the model was made by assuming certain parameters, and its stability analyzes were examined. Finally, numerical solutions of the mathematical model were made and its numerical simulations were shown.

Keywords Co-infection · Treatment · Caputo-Fabrizio fractional derivative · Numerical approximation

1 Introduction

In epidemiology, epidemic refers to a disease, which seems to be new cases, but has more than expected effects compared to previous experiences in a certain human population within a certain period. Epidemics have greatly affected cities, countries, continents and sometimes the entire world throughout human history. Epidemics have occurred in almost all ages. The examples include the bubonic plague known...
as the black plague that occurred in the middle ages and the Spanish flu that emerged at the end of the First World War. Viruses are the greatest factor in occurrence and spread of epidemics. For most of infectious diseases, the easiest way to catch the disease is direct contact with an infected person or animal. Types of direct contract can be listed as follows:

- **From person to person**: Infectious diseases usually spread by direct transfer of bacteria, viruses or other germs from one person to another. Bacteria or viruses can be transported if a person carrying bacteria or viruses sneezes, coughs, touches or hugs a healthy person.

- **From animal to human**: Being bitten or scratched by an infected animal, including pets, can cause disease. Some infectious diseases, such as rabies or tetanus, can be fatal if left untreated.

- **To unborn babies**: A pregnant woman can pass on germs that cause infectious diseases to her unborn baby. Some microbes can be transmitted through the placenta or breast milk.

The new corona virus identified on 13/01/2020 and named COVID-19, which is thought to have emerged from Wuhan city of China, has an impact on the whole world. Although many countries try to take precautions, the virus, which is thought to have been transmitted from animal to human, is also passed on from person to person very simply and quickly. According to two-month studies, its mortality rate is around 3%. Although its transmission rate is quite high, the mortality is lower compared to many other epidemics. Nowadays, many scientists are making great efforts to find an antivirus. Unfortunately, there is no definitive treatment for each epidemic, particularly the HIV-TB coinfection disease.

Tuberculosis (TB) is an infectious disease that can spread from person to person. TB is caused by bacteria called Mycobacterium tuberculosis. TB bacteria spread into the air and usually affect the lungs. However, the bacteria that cause TB can attack any part of the body, including the kidneys, spine, or brain. If left untreated, TB can cause death. HIV stands for human immunodeficiency virus, which is the virus that leads to HIV infection. AIDS stands for Acquired Immune Deficiency Syndrome. AIDS is the most advanced stage of HIV infection. HIV attacks and destroys the immune system’s infection-fighting CD4 cells. Loss of CD4 cells makes it difficult for the body to fight infections and some cancers. Without treatment, HIV can gradually destroy the immune system and progress to AIDS. ART is recommended for everyone with HIV. ART cannot cure HIV, but HIV drugs may help people with HIV live longer and healthier. ART also reduces the risk of HIV transmission. Infection with both HIV and TB is called HIV/TB coinfection. Latent TB is likely to progress to TB disease in people with HIV. TB disease may cause HIV to worsen. Treatment with HIV drugs is called antiretroviral therapy (ART). ART protects the immune system and prevents HIV infection from progressing to AIDS. In people with HIV/TB coinfection, ART reduces the chance of latent TB to switch to TB. TB disease is one of the leading causes of death among people with HIV worldwide. In the United States, where HIV drugs are widely used, people with HIV receive less TB compared to many other countries. However, TB affects many people with HIV in the United States, especially those who were born outside the United States.
The literature includes very serious and important studies in this respect. Some of those studies are as follows: Sharma et al. [1] suggest that the treatment of HIV-TB coinfection requires strong commitment and a focused approach. They also concluded that proper use of highly active antiretroviral therapy (HAART) to maintain immunity and treat HIV infection requires a high level of containment and adjustment to prevent TB. In their analysis, Gao et al. [2] showed that the prevalence of HIV/TB co-infection in China deserves special attention, and more attention should be paid to TB screening among HIV/AIDS populations for treatment of both diseases. Pawlowski et al. [3] showed that, in light of the information on the interaction mechanisms of the two pathogens, there are many gaps to be filled in order to develop preventive measures against the both diseases. McShane [4] has examined HAART’s effects on TB in many aspects. Furthermore, an increased risk of TB in HIV-positive patients and differences in the clinical picture in the HIV-positive population suggest that a high degree of clinical suspicion should remain. Naresh [5] analyzed a nonlinear mathematical model for transmission dynamics of HIV and a curable TB pathogen in a variable-size population. It was shown that the positive coinfection balance is always locally stable, but it may be globally stable, indicating that the disease becomes endemic due to the continuous migration of the population into the habitat under certain circumstances. Datiko et al. [6] showed that HIV infection rate in TB patients and pregnant women was higher among study participants in urban areas, and found that the HIV infection rate in TB patients is associated with the prevalence of HIV infection in pregnant women participating in attending antenatal care (ANC).

Moreover, some studies were conducted on the mathematical modeling and analysis of HIV-TB coinfection disease. Some of those studies are as follows: Agusto and Adekunle [7] showed that the most cost-effective control strategy is to practice combination strategy involving the prevention of treatment failure in drug-sensitive TB-infected individuals and treat individuals with drug-resistant TB. Okosun and Makinde [8] created a mathematical model to investigate synergistic relationships in the presence of treatments for malaria-cholera coinfection. They also analyzed the steady state of single infection.

The literature includes following studies conducted by extending the HIV-TB coinfection disease model to fractional derivative: Carvalho and Pinto [9] studied a delayed fractional mathematical model for malaria and human immunodeficiency virus infection, taking into account personal protection and vaccination against malaria. The reproduction number of the model was calculated to examine its balance stability. Zafar et al. [10] studied a fractional order nonlinear mathematical model to analyze and control the spread of HIV/AIDS. Both disease-free equilibrium $E_0$ and endemic equilibrium $E^*$ were found and stability results were obtained using the stability theorem of fractional order differential equations. Kheiri and Jafari [11] presented a general formulation for a fractional optimal control problem (FOCP) in which state and co-state equations are given in terms of left fraction derivatives. Then, the forward/backward sweeping method (FBSM) was developed using Adams-type predictive-correction method to solve FOCP. The numerical calculations of Mallela et al. [12] model solution showed that the results of treatment programs aimed at reducing the total load of this co-infection largely depend on both the strength and the start time of antiretroviral therapy (ART). Khan et al. [13] discussed an analysis
of the HIV-TB-infected model in Atangana—Baleanu fractional differential form, maintaining the importance of the HIV model. Besides, the model was also examined for existence and uniqueness of solution, Hyers–Ulam (HU) stability and numerical simulations assuming certain parameters.

In the literature, many mathematical modeling and their analysis related to disease models have been made. For instance, Dokuyucu and Dutta [14] extended the Ebola virus model to the Caputo-Fabrizio fractional derivative. Then they examined the existence and uniqueness solutions. Finally the results are compared with numerical simulations. Dokuyucu et al. [15] investigated cancer treatment model via new Caputo fractional derivative. They proved the existence and uniqueness solution for the mathematical model. Singh et al. [16] explore a fractional smoking model of a new non-singular fractional derivative. The existence of the solution was investigated with the fixed-point theorem. Then the uniqueness of the solution is shown. Basavarajaiah and Murthy [17] have published a book on the statistical modeling of HIV transmission. In this book, they organized the information available in the literature in a frame. They examined the relationship between the social and biological mechanisms that affect AIDS and its spread.

There are many fractional operators in the literature. Some of these are Grünwald-Letnikov, Riemann-Liouville, Caputo, Caputo-Fabrizio, Atangana-Baleanu. In addition to the advantages of operators, they also have some disadvantages. For example, the Caputo fractional derivative operator has a singularity problem. Due to this problem, the Caputo-Fabrizio fractional derivative has been identified [18]. This operator eliminated the problem of singularity using the exponential kernel. In addition, solutions made with the exponential kernel give results very close to the exact solution. The motivation of using this kernel in the analysis of our model is the reasons we have listed above.

This study will analyze the HIV-TB coinfection disease model. The model will first be expanded to the Caputo-Fabrizio fractional derivative containing an exponential core. Then, the existence and uniqueness solutions of the extended model will be made. Finally, numerical results will be obtained and numerical simulations will be done.

2 Preliminaries

This section will give necessary definitions, theorems and lemmas with the new Caputo fractional derivative and integral operator.

**Definition 2.1.** The Caputo’s fractional derivative are defined as follows [19],

\[ C_a D_t^\rho \phi(t) = \frac{1}{\Gamma(n - \rho)} \int_a^t \frac{\phi^{(n)}(\omega)}{(t - \omega)^{\rho+1-n}} d\omega, \quad n - 1 < \rho < n \in \mathbb{N}. \] (1)

**Definition 2.2.** Let \( f \in H^1(a, b) \), \( b > a \), \( \rho \in (0, 1) \) then, the definition of the new Caputo fractional derivative is [18],
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\[
\frac{\text{CF}}{a} D_t^\rho (\phi(t)) = \frac{\rho M(\rho)}{1 - \rho} \int_a^t \frac{d\phi(\omega)}{d\omega} \exp \left[ -\frac{\rho \omega - t}{1 - \rho} \right] d\omega, \quad (2)
\]

where \( M(\rho) \) is a normalization function. Furthermore, \( M(0) = M(1) = 1. \) The definition is also written if the function does not belong to \( H^1(a,b), \)

\[
\frac{\text{CF}}{a} D_t^\rho (\phi(t)) = \frac{\rho M(\rho)}{1 - \rho} \int_a^t (\phi(t) - \phi(\omega)) \exp \left[ -\frac{\rho \omega - t}{1 - \rho} \right] d\omega. \quad (3)
\]

Remark 2.1. If \( \eta = \frac{1 - \rho}{\rho} \in (0, \infty), \rho = \frac{1}{1 + \eta} \in [0, 1], \) in this case the following equation can be written according to the above equation.

\[
D_t^\eta (\phi(t)) = \frac{N(\eta)}{\eta} \int_a^t \frac{d\phi(\omega)}{d\omega} \exp \left[ -\frac{t - \omega}{\eta} \right] d\omega, \quad N(0) = N(\infty) = 1. \quad (4)
\]

In addition,

\[
\lim_{\rho \to 0} \frac{1}{\rho} \exp \left[ -\frac{t - \omega}{\rho} \right] = \delta(\omega - t).
\]

Thus, Nieto and Losada proposed that the new Caputo derivative of order \( 0 < \rho < 1 \) can be reformulated as below,

**Definition 2.3.** The Caputo-Fabrizio fractional derivative of order \( \rho \) is as follows [20],

\[
\frac{\text{CF}}{a} D_t^\rho (\phi(t)) = \frac{1}{1 - \rho} \int_0^t f'(\omega) \exp \left[ -\frac{\rho t - \omega}{1 - \rho} \right] d\omega. \quad (5)
\]

Remark 2.2. The Laplace transform of the new Caputo fractional derivative with \( s \) variable

\[
LT[D_t^\rho (\phi(t))] = \frac{1}{1 - \nu} \int_0^\infty \exp(-st) \int_0^t \frac{d\phi(x)}{dx} \exp \left[ -\frac{\rho \omega - x}{1 - \rho} \right] d\omega d\omega = \frac{(sLT(\phi(t)) - f(0))}{s + \nu(1-s)}. \quad (6)
\]

Theorem 2.1. For \( NFD_t, \) if the function \( \phi(t) \) is such that [18],

\[
\phi^{(s)}(a) = 0, \quad s = 1, 2, \ldots, n,
\]

then, we have

\[
D_t^{(n)} (D_t^\rho \phi(t)) = D_t^{(\rho)} (D_t^n \phi(t)). \quad (8)
\]

Proof 2.1. We begin considering \( n = 1, \) then \( D_t^{(\rho+1)} \phi(t), \) we obtain...
\[
\mathcal{D}_{t}^{(\rho)}(\mathcal{D}_{t}^{(1)} \phi(t)) = \frac{M(\rho)}{1 - \rho} \int_{a}^{t} f'(\omega) \exp \left[ -\frac{\rho(t - \omega)}{1 - \rho} \right] d\omega.
\] (9)

Hence, after an integration by parts and assuming \( f'(a) = 0 \), we have,

\[
\mathcal{D}_{t}^{(\rho)}(\mathcal{D}_{t}^{(1)} \phi(t)) = \frac{M(\rho)}{1 - \rho} \left[ \int_{a}^{t} \frac{d}{d\omega} \phi'(\omega) \exp \left( -\frac{\rho(t - \omega)}{1 - \rho} \right) d\omega - \frac{\rho}{1 - \rho} \int_{a}^{t} \phi'(\omega) \exp \left( -\frac{\rho(t - \omega)}{1 - \rho} \right) d\omega \right].
\] (10)

Otherwise,

\[
\mathcal{D}_{t}^{(1)}(\mathcal{D}_{t}^{(\rho)} \phi(t)) = \frac{d}{dt} \left( \frac{M(\rho)}{1 - \rho} \int_{a}^{t} \phi'(\omega) \exp \left( -\frac{\rho(t - \omega)}{1 - \rho} \right) d\omega \right) = \frac{M(\rho)}{1 - \rho} \left[ \phi'(t) - \frac{\rho}{1 - \rho} \int_{a}^{t} \phi'(\omega) \exp \left( -\frac{\rho(t - \omega)}{1 - \rho} \right) d\omega \right].
\] (11)

It is easy to generalize the proof for any \( n > 1 \) [18].

**Definition 2.4.** Let \( 0 < \rho < 1 \). The fractional integral of order \( \rho \) of a function \( f \) is defined by [18],

\[
\text{CF} I^{\rho} \phi(t) = \frac{2(1 - \rho)}{(2 - \rho)M(\rho)} u(t) + \frac{2\rho}{(2 - \rho)M(\rho)} \int_{0}^{t} u(s) ds, \quad t \geq 0.
\] (12)

**Definition 2.5.** The Sobolev space of order 1 in \((a, b)\) is defined as [21]:

\[
H^{1}(a, b) = \{ u \in L^{2}(a, b) : u' \in L^{2}(a, b) \}.
\]

3 Materials and Methods

3.1 Classical Model

Our model consisting of 11 differential equations (13) is given in the system of equations. The population is divided into certain groups as \( A(t), B_{T}(t), C_{T}(t), E(t), C_{H}(t), F(t), G_{H}(t), B_{TH}(t), C_{TH}(t), E_{H}(t) \) and \( F_{T}(t) \), namely susceptible individuals \( A(t), \)
individuals with no TB disease symptom and non-infected individuals $B_T(t)$, TB-infected individuals with active TB disease and infectious TB $C_T(t)$, individuals with recurrent TB $E(t)$, individuals who do not show AIDS symptoms but who are infected with HIV $C_H(t)$, individuals who are being treated for HIV infections $F(t)$, HIV-infected individuals with AIDS symptoms $G_H(t)$, HIV-infected TB-latent individuals (pre-AIDS) $B_{TH}(t)$, HIV-infected individuals with active TB disease (pre-AIDS) $C_{TH}(t)$, HIV-infected individuals who are recovered without TB and AIDS symptoms $E_H(t)$, and individuals infected with HIV and also infected with active TB, showing AIDS symptoms $F_T(t)$, respectively. Total population denoted by $N(t)$ at time $T$,

$$
N(t) = A(t) + B_T(t) + C_T(t) + E(t) + C_H(t) + F(t) + G_H(t) + B_{TH}(t) + C_{TH}(t) + E_H(t) + F_T(t).
$$

The following system differential equations can be taken as the TB-HIV/AIDS model [22]:

$$
\begin{align*}
\frac{dA(t)}{dt} &= \Theta - \mu_T(t)A(t) - \mu_H(t)A(t) - \xi A(t), \\
\frac{dB_T(t)}{dt} &= \mu_T(t)A(t) + \alpha'\mu_T(t)E(t) - (m_1 + \chi_1 + \xi)B_T(t), \\
\frac{dC_T(t)}{dt} &= m_1B_T(t) - (\chi_2 + r_T + \xi + \theta\mu_H(t))C_T(t), \\
\frac{dE(t)}{dt} &= \chi_1B_T(t) + \chi_2C_T(t) - (\alpha'\mu_T(t) + \mu_H(t) + \xi)E(t), \\
\frac{dC_H(t)}{dt} &= \mu_H(t)A(t) - (\theta_1 + \psi + \nu\mu_T(t) + \xi)C_H(t) + \beta_1F(t) + \mu_H(t)E(t) + \nu_1C_H(t), \\
\frac{dF(t)}{dt} &= \nu_1C_H(t) + \nu_2E_H(t) - \beta_1F(t) - (\xi + \nu_A)F(t), \\
\frac{dG_H(t)}{dt} &= \psi C_H(t) + q\theta_2C_{TH}(t) + \epsilon\chi_3B_{TH}(t) - (\nu_1 + \xi)G_H(t), \\
\frac{dB_{TH}(t)}{dt} &= \alpha'\mu_T(t)E_H(t) - (m_2 + \chi_3 + \xi)B_{TH}(t), \\
\frac{dC_{TH}(t)}{dt} &= \theta\mu_H(t)C_T(t) + \nu\mu_T(t)C_H(t) + \beta_2F_T(t) + m_2B_{TH}(t) - (\theta_2 + \xi + r_T)C_{TH}(t), \\
\frac{dE_H(t)}{dt} &= \rho\theta_2C_{TH}(t) + (1 - \epsilon)\chi_3B_{TH}(t) - (\alpha'\mu_T(t) + \nu_2 + \xi)E_H(t), \\
\frac{dF_T(t)}{dt} &= (1 - (q + \rho))\theta_2C_{TH}(t) - (\beta_2 + \xi + r_T\lambda)F_T(t).
\end{align*}
$$

Approximately 10% of people infected with mycobacterium tuberculosis develop active TB disease. Therefore, it remains hidden in about 90% of infected people. Latent infected TB patients are asymptomatic and do not transmit TB [23]. The transmission rate in treatment of individuals in the latent TB category is $\chi_1$ and the recovery rate is $m_1$. Recovery rate of active individuals infected with TB is $\chi_2$.

Supposing that individuals with recurrent TB have become partially immune and the transmission rate of this class is limited to $\alpha' \leq 1$. Individuals with active TB disease are exposed to evoked death at the rate of $r_T$. We assume that individuals in class $E$ are susceptible to HIV infection at the rate of $\mu_H$. On the other hand, TB-active infected individuals are susceptible to HIV infection at the rate of $C_T$, $\theta\mu_H(t)$, whereas the modification parameter $\theta \leq 1$ explains that individuals in the class $C_T$ are more likely to be HIV positive. HIV-infected individuals in the class
F (without AIDS symptoms) have an AIDS ratio of \( \vartheta_1 \) and the recovery rate of the HIV-infected individuals progresses to the rate of \( \psi \). Individuals in the class \( G_H \) differ from individuals in the class \( C_H \) by \( \nu_1 \). HIV-infected individuals with AIDS symptoms have an HIV recovery rate of \( \beta_1 \) and they experience induced death at the rate of \( r_A \). Individuals in the class \( C_H \) are sensitive to TB infection at the rate of \( \nu \mu_t \) [24]. HIV-positive individuals infected with individuals with TB disease (pre-AIDS) leave the class \( C_{TH} \) at the rate of \( \vartheta_2 \). \( C_{TH} \) Individuals in the class \( G_H \) progress to the class at the rate of \( q \vartheta_2 \) and to the class \( E_H \) at the rate of \( q \vartheta_2 \).

Individuals in the class who do not receive any treatment for TB or HIV progress to the class at the rate of \( (1 - (q + p)) \vartheta_2 \) Individuals leave the class \( B_{TH} \) at the rate of \( \chi_3 \). Individuals in the class \( B_{TH} \) are more likely to progress to active TB disease compared to individuals infected with latent TB only. The progress rate in the model is expressed in \( m_2 \). Similarly, HIV-infected individuals become more susceptible to TB infection compared to the patients who are not HIV positive. For individuals in the class \( E_H \) with recurrent TB, the modification parameter associated with the rate of infection is expressed in \( \alpha_2' \) and \( \alpha_2' \leq 1 \). Individuals in this class progress to the class \( F \) at the rate of \( \nu_2 \). Individuals with both HIV and TB (with AIDS symptoms) are treated at the rate of \( \beta_2 \) for HIV. The individuals in the class \( F_T \) have a mortality rate of \( r_{TA} \) in AIDS-TB co-infection.

### 3.2 Existence of Solution to Fractional Model

In this section, existence solution will be obtained by using fixed point technique of fractional TB-HIV/AIDS mathematical model. Dokuyucu and Dutta [14] have proven the Ebola virus mathematical model with a fixed point theorem. Similarly, when the system (13) is extended to the Caputo-Fabrizio fractional derivative, we have the following system:

\[
\begin{align*}
\frac{d}{dt} C_{TH}^\alpha(t) &= \Theta - \mu_T(t)A(t) - \mu_H(t)A(t) - \xi A(t), \\
\frac{d}{dt} C_{TH}^\alpha(t) &= \mu_T(t)A(t) + \alpha'_1 \mu_T(t)E(t) - (m_1 + \chi_1 + \xi)B_T(t), \\
\frac{d}{dt} C_T(t) &= m_1 B_T(t) - (\chi_2 + r_T + \xi + \theta_H(t))C_T(t), \\
\frac{d}{dt} E(t) &= \chi_1 B_T(t) + \chi_2 C_T(t) - (\alpha'_1 \mu_T(t) + \mu_H(t) + \xi)E(t), \\
\frac{d}{dt} C_H(t) &= \mu_H(t)A(t) - (\vartheta_1 + \psi + \nu_H(t) + \xi)C_H(t) + \beta_1 F(t) + \mu_H(t)E(t) + \nu_1 C_H(t), \\
\frac{d}{dt} F(t) &= \vartheta_1 C_H(t) + \nu_2 E_H(t) - \beta_1 F(t) - (\xi + r_A)F(t), \\
\frac{d}{dt} G_H(t) &= \psi C_H(t) + q \vartheta_2 C_{TH}(t) + \epsilon \chi_3 B_{TH}(t) - (\nu_1 + \xi)G_H(t), \\
\frac{d}{dt} B_{TH}(t) &= \alpha'_2 \mu_T(t)E_H(t) - (m_2 + \chi_3 + \xi)B_{TH}(t), \\
\frac{d}{dt} C_T(t) &= \psi \mu_T(t)C_H(t) + \nu_H(t)C_H(t) + \beta_2 F_T(t) + m_2 B_{TH}(t) - (\vartheta_2 + \xi + r_T)C_{TH}(t), \\
\frac{d}{dt} E_H(t) &= p \vartheta_2 C_{TH}(t) + (1 - e) \chi_3 B_{TH}(t) - (\alpha'_2 \mu_T(t) + \nu_2 + \xi)E_H(t), \\
\frac{d}{dt} F_T(t) &= (1 - (q + p)) \vartheta_2 C_{TH}(t) - (\vartheta_2 + \xi + r_{TA})F_T(t).
\end{align*}
\]
In addition, the initial values are as follows,

\[ A_0(t) = A(0), \quad B_0(t) = B(0), \quad C_0(t) = C(0), \quad E_0(t) = E(0), \quad C_H(0) = C_H(0), \quad F_0(t) = F(0) \]

\[ G_{H_0}(t) = G_{H_0}(0), \quad B_{TH_0}(t) = B_{TH_0}(0), \quad C_{TH_0}(t) = C_{TH_0}(0), \quad E_{H_0}(t) = E_{H_0}(0), \quad F_{T_0}(t) = F_T(0). \]

Using the Definition (2.4), the system of equations above can be written as follows:

\[ A(t) - A(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \Theta - \mu_t(t)A(t) - \mu_H(t)A(t) - \xi A(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \Theta - \mu_T(y)A(y) - \mu_H(y)A(y) - \xi A(y) \right) dy, \]

\[ B_T(t) - B_T(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mu_t(t)A(t) + \alpha'_1\mu_T(t)E(t) - (m_1 + \chi_1 + \xi) B_T(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mu_T(y)A(y) + \alpha'_1\mu_T(y)E(y) - (m_1 + \chi_1 + \xi) B_T(y) \right) dy, \]

\[ C_T(t) - C_T(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( m_1 B_T(t) - (\chi_2 + r_T + \xi + \theta \mu_H(t)) C_T(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( m_1 B_T(y) - (\chi_2 + r_T + \xi + \theta \mu_H(y)) C_T(y) \right) dy, \]

\[ E(t) - E(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \alpha'_1 B_T(t) + \chi_2 C_T(t) - (\alpha'_1 \mu_T(t) + \mu_H(t) + \xi) E(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \chi_2 B_T(y) + \chi_2 C_T(y) - (\alpha'_1 \mu_T(y) + \mu_H(y) + \xi) E(y) \right) dy, \]

\[ C_H(t) - C_H(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \theta_1 C_H(t) + \nu_2 E_H(t) - \beta_1 F(t) - (\xi + r_A) F(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \theta_1 C_H(y) + \nu_2 E_H(y) - \beta_1 F(y) - (\xi + r_A) F(y) \right) dy, \]

\[ F(t) - F(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \theta_1 C_H(t) + \nu_2 E_H(t) - \beta_1 F(t) - (\xi + r_A) F(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \theta_1 C_H(y) + \nu_2 E_H(y) - \beta_1 F(y) - (\xi + r_A) F(y) \right) dy, \]

\[ G_H(t) - G_H(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \theta_1 C_H(t) + \nu_2 E_H(t) + \mu_H(t) C_H(t) + \xi_3 B_{TH}(t) - (\nu_1 + \xi) G_H(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \theta_1 C_H(y) + \nu_2 E_H(y) + \mu_H(y) C_H(y) + \xi_3 B_{TH}(y) - (\nu_1 + \xi) G_H(y) \right) dy, \]

\[ B_{TH}(t) - B_{TH}(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \theta_1 C_H(t) + \nu_2 E_H(t) + \mu_H(t) C_H(t) + \xi_3 B_{TH}(t) - (\nu_2 + \xi + r_T) C_{TH}(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \theta_1 C_H(y) + \nu_2 E_H(y) + \mu_H(y) C_H(y) + \xi_3 B_{TH}(y) + \mu_H(y) C_H(y) - (\nu_2 + \xi + r_T) C_{TH}(y) \right) dy. \]

\[ C_{TH}(t) - C_{TH}(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \theta_1 C_H(t) + \nu_2 E_H(t) + \mu_H(t) C_H(t) + \beta_2 F_T(t) + m_2 B_{TH}(t) - (\nu_2 + \xi + r_T) C_{TH}(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \theta_1 C_H(y) + \nu_2 E_H(y) + \mu_H(y) C_H(y) + \beta_2 F_T(y) + m_2 B_{TH}(y) - (\nu_2 + \xi + r_T) C_{TH}(y) \right) dy. \]

\[ E_{H}(t) - E_{H}(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \theta_1 C_H(t) + \nu_2 E_H(t) + \mu_H(t) C_H(t) + \beta_2 F_T(t) + m_2 B_{TH}(t) - (\nu_2 + \xi + r_T) C_{TH}(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \theta_1 C_H(y) + \nu_2 E_H(y) + \mu_H(y) C_H(y) + \beta_2 F_T(y) + m_2 B_{TH}(y) - (\nu_2 + \xi + r_T) C_{TH}(y) \right) dy. \]

\[ F_T(t) - F_T(0) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \theta_1 C_H(t) + \nu_2 E_H(t) + \mu_H(t) C_H(t) + \beta_2 F_T(t) + m_2 B_{TH}(t) - (\nu_2 + \xi + r_T) C_{TH}(t) \right) \]
\[ + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \theta_1 C_H(y) + \nu_2 E_H(y) + \mu_H(y) C_H(y) + \beta_2 F_T(y) + m_2 B_{TH}(y) - (\nu_2 + \xi + r_T) C_{TH}(y) \right) dy. \]
To simplify, the kernels can be written as follows,

\[
\begin{align*}
\mathcal{S}_1(t, A) &= \Theta - \mu_T(t)A(t) - \mu_H(t)A(t) - \xi A(t), \\
\mathcal{S}_2(t, B_T) &= \mu_T(t)A(t) + \alpha_1 \mu_T(t)E(t) - (m_1 + \chi_1 + \xi)B_T(t), \\
\mathcal{S}_3(t, C_T) &= m_1 B_T(t) - (\chi_2 + r_T + \xi + \theta \mu_H(t))C_T(t), \\
\mathcal{S}_4(t, E) &= \chi_1 B_T(t) + \chi_2 C_T(t) - (\alpha_1 \mu_T(t) + \mu_H(t) + \xi)E(t), \\
\mathcal{S}_5(t, C_H) &= \mu_H(t)A(t) - (q_1 + \psi + \nu \mu_T(t) + \xi)C_H(t) + \beta_1 F(t) + \mu_H(t)E(t) + \nu_1 C_H(t), \\
\mathcal{S}_6(t, F) &= \theta_1 C_H(t) + \nu_2 E_H(t) - \beta_1 F(t) - (\xi + r_A)F(t), \\
\mathcal{S}_7(t, G_H) &= \psi C_H(t) + q_2 \chi_3 B_{TH}(t) + \epsilon \chi_3 B_{TH}(t) - (\nu_1 + \xi)G_H(t), \\
\mathcal{S}_8(t, B_{TH}) &= \alpha_2 \mu_H(t)E_H(t) - (m_2 + \chi_3 + \xi)B_{TH}(t), \\
\mathcal{S}_9(t, C_{TH}) &= \theta \mu_H(t)C_{TH}(t) + \nu_2 F_T(t) + m_2 B_{TH}(t) - (\theta_2 + \xi + r_T)C_{TH}(t), \\
\mathcal{S}_{10}(t, E_H) &= p \theta_2 \chi_3 B_{TH}(t) + (1 - e)\chi_3 B_{TH}(t) - (\alpha_2 \mu_H(t) + \nu_2 + \xi)E_H(t), \\
\mathcal{S}_{11}(t, F_T) &= (1 - (q + p))\theta_2 C_{TH}(t) - (\beta_2 + \xi + r_A)F_T(t).
\end{align*}
\]

and

\[
\begin{align*}
\Psi_1 &= \mu_T - \mu_H - \xi, \\
\Psi_2 &= m_1 + \chi_1 + \xi, \\
\Psi_3 &= \chi_2 + r_T + \xi + \theta \mu_H, \\
\Psi_4 &= \alpha_1 \mu_T + \mu_H + \xi, \\
\Psi_5 &= \theta_1 + \psi + \nu \mu_T + \xi, \\
\Psi_6 &= \beta_1 + \xi + r_A, \\
\Psi_7 &= \nu_1 + \xi, \\
\Psi_8 &= m_2 + \chi_3 + \xi, \\
\Psi_9 &= \theta_2 + \xi + r_T, \\
\Psi_{10} &= \alpha_2 \mu_H + \nu_2 + \xi, \\
\Psi_{11} &= \beta_2 + \xi + r_A.
\end{align*}
\]

It can be assumed that the following assumption \(H\): \(A(t), B_T(t), C_T(t), E(t), C_H(t), F(t), G_H(t), B_{TH}(t), C_{TH}(t), E_H(t), F_T(t), A_1(t), B_{T_1}(t), C_{T_1}(t), E_{1}(t), C_{H_1}(t), F_1(t), G_{H_1}(t), B_{TH_1}(t), C_{TH_1}(t), E_{H_1}(t), F_{T_1}(t) \in L[0, 1]\) are continuous functions, so that \(|A(t)| \leq \mathcal{L}_1, |B(t)| \leq \mathcal{L}_2, |C_T(t)| \leq \mathcal{L}_3, |E(t)| \leq \mathcal{L}_4, |C_H(t)| \leq \mathcal{L}_5, |F(t)| \leq \mathcal{L}_6, |G_H(t)| \leq \mathcal{L}_7, |B_{TH}(t)| \leq \mathcal{L}_8, |C_{TH}(t)| \leq \mathcal{L}_9, |E_H(t)| \leq \mathcal{L}_{10}, |F_T(t)| \leq \mathcal{L}_{11}.\)

**Theorem 3.1.** The kernels \(\mathcal{S}_i, i = 1, \ldots 11\) are satisfying the Lipschitz condition if the assumption \(H\) is true and are contractions provided that \(\Psi_i < 1\) for \(\forall i = 1 \ldots 11.\)

**Proof 3.1.** First of all, we will begin to proof that \(\mathcal{S}_i(t, A)\) satisfies Lipschitz condition. Let \(A(t)\) and \(A_1(t)\) are two functions, then
Next, we will prove that $\mathcal{S}_2(t, B_T)$ satisfies Lipschitz condition. Let $B_T(t)$ and $B_{T_1}(t)$ are two functions, then

$$
||\mathcal{S}_2(t, B_T) - \mathcal{S}_2(t, B_{T_1})|| = ||(\mu_T A + \alpha'_1 \mu_T E - (m_1 + \chi_1 + \xi) B_T)
- ((\mu_T A + \alpha'_1 \mu_T E - (m_1 + \chi_1 + \xi) B_{T_1})||
\leq (m_1 + \chi_1 + \xi)||B_T - B_{T_1}||
= \Psi_2||B_T - B_{T_1}||.
$$

Next, we will prove that $\mathcal{S}_3(t, C_T)$ satisfies Lipschitz condition. Let $C_T(t)$ and $C_{T_1}(t)$ are two functions, then

$$
||\mathcal{S}_3(t, C_T) - \mathcal{S}_3(t, C_{T_1})|| = ||(m_1 B_T - (\chi_2 + r_T + \xi + \theta \mu_H) C_T)
- (m_1 B_{T_1} - (\chi_2 + r_T + \xi + \theta \mu_H) C_{T_1})||
\leq (\chi_2 + r_T + \xi + \theta \mu_H)||C_T - C_{T_1}||
= \Psi_3||C_T - C_{T_1}||.
$$

Next, we will prove that $\mathcal{S}_4(t, E)$ satisfies Lipschitz condition. Let $E(t)$ and $E_1(t)$ are two functions, then

$$
||\mathcal{S}_4(t, E) - \mathcal{S}_4(t, E_1)|| = ||(\chi_1 B_T + \chi_2 C_T - (\alpha'_1 \mu_+ \mu_H + \xi) E)
- (\chi_1 B_{T_1} + \chi_2 C_{T_1} - (\alpha'_1 \mu_+ \mu_H + \xi) E_1)||
\leq (\alpha'_1 \mu_+ \mu_H + \xi)||E - E_1||
= \Psi_4||E - E_1||.
$$

Next, we will prove that $\mathcal{S}_5(t, C_H)$ satisfies Lipschitz condition. Let $C_H(t)$ and $C_{H_1}(t)$ are two functions, then

$$
||\mathcal{S}_5(t, C_H) - \mathcal{S}_5(t, C_{H_1})|| = ||(\mu_H A - (\theta_1 + \psi + \nu \mu_T + \xi) C_H + \beta_1 F + \mu_H E + \nu_1 C_H)
- (\mu_H A - (\theta_1 + \psi + \nu \mu_T + \xi) C_{H_1} + \beta_1 F + \mu_H E + \nu_1 C_{H_1})||
\leq (\theta_1 + \psi + \nu \mu_T + \xi)||C_H - C_{H_1}||
= \Psi_5||C_H - C_{H_1}||.
$$

Next, we will prove that $\mathcal{S}_6(t, F)$ satisfies Lipschitz condition. Let $F(t)$ and $F_1(t)$ are two functions, then
Next, we will proof that $S_7(t, G_H)$ satisfies Lipschitz condition. Let $G_H(t)$ and $G_{H_1}(t)$ are two functions, then

$$||S_7(t, G_H) - S_7(t, G_{H_1})|| = \|(\psi C_H + q \partial_2 C_{TH} + e \chi_3 B_{TH} - (v_1 + \xi)G_H) - (\psi C_H + q \partial_2 C_{TH} + e \chi_3 B_{TH} - (v_1 + \xi)G_{H_1})||$$

$$\leq (v_1 + \xi)||G_H - G_{H_1}||$$

$$= \Psi_7||G_H - G_{H_1}||.$$

Next, we will proof that $S_8(t, B_{TH})$ satisfies Lipschitz condition. Let $B_{TH}(t)$ and $B_{TH_1}(t)$ are two functions, then

$$||S_8(t, B_{TH}) - S_8(t, B_{TH_1})|| = \|((\alpha' \mu T E_H - (m_2 + \chi_3 + \xi)B_{TH}) - ((\alpha' \mu T E_H - (m_2 + \chi_3 + \xi)B_{TH_1})||$$

$$\leq (m_2 + \chi_3 + \xi)||B_{TH} - B_{TH_1}||$$

$$= \Psi_8||B_{TH} - B_{TH_1}||.$$

Next, we will proof that $S_9(t, C_{TH})$ satisfies Lipschitz condition. Let $C_{TH}(t)$ and $C_{TH_1}(t)$ are two functions, then

$$||S_9(t, C_{TH}) - S_9(t, C_{TH_1})|| = \|((\theta \mu T C_T + \nu \mu T C_H + \beta_2 F_T + m_2 B_{TH} - (\vartheta_2 + \xi + \tau r)C_{TH}) - ((\theta \mu T C_T + \nu \mu T C_H + \beta_2 F_T + m_2 B_{TH} - (\vartheta_2 + \xi + \tau r)C_{TH_1})||$$

$$\leq (\vartheta_2 + \xi + \tau r)||C_{TH} - C_{TH_1}||$$

$$= \Psi_9||C_{TH} - C_{TH_1}||.$$

Next, we will proof that $S_{10}(t, E_H)$ satisfies Lipschitz condition. Let $E_H(t)$ and $E_{H_1}(t)$ are two functions, then

$$||S_{10}(t, E_H) - S_{10}(t, E_{H_1})|| = \|((\partial_2 C_{TH} + (1 - e)\chi_3 B_{TH} - (\alpha' \mu T + \nu_2 + \xi)E_H) - ((\partial_2 C_{TH} + (1 - e)\chi_3 B_{TH} - (\alpha' \mu T + \nu_2 + \xi)E_{H_1})||$$

$$\leq (\alpha' \mu T + \nu_2 + \xi)||E_{H} - E_{H_1}||$$

$$= \Psi_{10}||E_{H} - E_{H_1}||.$$

Finally, we will proof that $S_{11}(t, F_T)$ satisfies Lipschitz condition. Let $F_T(t)$ and $F_{T_1}(t)$ are two functions, then
\[ ||S_{11}(t, F_T) - S_{11}(t, F_{T_i})|| = \|((1 - (q + p)) \phi_2 C_{TH} - (\beta_2 + \xi + r_{TA}) F_T) \\
- ((1 - (q + p)) \phi_2 C_{TH} - (\beta_2 + \xi + r_{TA}) F_{T_i})|| \\
\leq (\beta_2 + \xi + r_{TA})||F_T - F_{T_i}|| \\
= \Psi_{11}||F_T - F_{T_i}||.\]

All kernels which \( S_i, i = 1 \ldots 11 \) are satisfying conditions, so that they are contractions with \( \Psi_i < 1, i \in 1 \ldots 11 \). As a result, this completes the proof.

Let all the initial values be zero. If the system (15) is then rewritten,

\[
A(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_1(t, A(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_1(y, A(y)) dy, \\
B_T(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_2(t, B_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_2(y, B_T(y)) dy, \\
C_T(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_3(t, C_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_3(y, C_T(y)) dy, \\
E(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_4(t, E(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_4(y, E(y)) dy, \\
C_H(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_5(t, C_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_5(y, C_H(y)) dy, \\
F(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_6(t, F(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_6(y, F(y)) dy, \\
G_H(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_7(t, G_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_7(y, G_H(y)) dy, \\
B_{TH}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_8(t, B_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_8(y, B_{TH}(y)) dy, \\
C_{TH}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_9(t, C_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_9(y, C_{TH}(y)) dy, \\
E_H(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_{10}(t, E_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_{10}(y, E_H(y)) dy, \\
F_T(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} S_{11}(t, F_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t S_{11}(y, F_T(y)) dy.\]

Then the following system of equations can be defined with the help of a recursive formula.
\[A_n(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_1(t, A_n-1(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_1(y, A_{n-1}(y)) dy,\]

\[B_{T_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_2(t, B_{T_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_2(y, B_{T_{n-1}}(y)) dy,\]

\[C_{T_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_3(t, C_{T_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_3(y, C_{T_{n-1}}(y)) dy,\]

\[E_n(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_4(t, E_{n-1}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_4(y, E_{n-1}(y)) dy,\]

\[C_{H_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_5(t, C_{H_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_5(y, C_{H_{n-1}}(y)) dy,\]

\[F_n(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_6(t, F_{n-1}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_6(y, F_{n-1}(y)) dy,\]

\[G_{H_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_7(t, G_{H_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_7(y, G_{H_{n-1}}(y)) dy,\]

\[B_{TH_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_8(t, B_{TH_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_8(y, B_{TH_{n-1}}(y)) dy,\]

\[C_{TH_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_9(t, C_{TH_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_9(y, C_{TH_{n-1}}(y)) dy,\]

\[E_{H_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_{10}(t, E_{H_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_{10}(y, E_{H_{n-1}}(y)) dy,\]

\[F_{T_n}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_{11}(t, F_{T_{n-1}}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_{11}(y, F_{T_{n-1}}(y)) dy,\]

Also, the differences of each equation can be written as follows:

\[(A_{n+1} - A_n)(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_1(t, A_n(t)) - \mathcal{S}_1(t, A_{n-1}(t)) \right) \]
\[+ \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_1(y, A_n(y)) - \mathcal{S}_1(y, A_{n-1}(y)) \right) dy,\]

\[(B_{T_{n+1}} - B_{T_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_2(t, B_{T_n}(t)) - \mathcal{S}_2(t, B_{T_{n-1}}(t)) \right) \]
\[+ \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_2(y, B_{T_n}(y)) - \mathcal{S}_2(y, B_{T_{n-1}}(y)) \right) dy,\]

\[(C_{T_{n+1}} - C_{T_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_3(t, C_{T_n}(t)) - \mathcal{S}_3(t, C_{T_{n-1}}(t)) \right) \]
\[+ \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_3(y, C_{T_n}(y)) - \mathcal{S}_3(y, C_{T_{n-1}}(y)) \right) dy,\]

\[(E_{n+1} - E_n)(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_4(t, E_n(t)) - \mathcal{S}_4(t, E_{n-1}(t)) \right) \]
\[+ \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_4(y, E_n(y)) - \mathcal{S}_4(y, E_{n-1}(y)) \right) dy,\]
\((C_{H_{n+1}} - C_{H_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_5(t, C_{H_n}(t)) - \mathcal{S}_5(t, C_{H_{n-1}}(t)) \right) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_5(t, C_{H_n}(y)) - \mathcal{S}_5(t, C_{H_{n-1}}(y)) \right) dy,\)

\((F_{n+1} - F_n)(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_6(t, F_n(t)) - \mathcal{S}_6(t, F_{n-1}(t)) \right) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_6(t, F_n(y)) - \mathcal{S}_6(t, F_{n-1}(y)) \right) dy,\)

\((G_{H_{n+1}} - G_{H_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_7(t, G_{H_n}(t)) - \mathcal{S}_7(t, G_{H_{n-1}}(t)) \right) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_7(t, G_{H_n}(y)) - \mathcal{S}_7(t, G_{H_{n-1}}(y)) \right) dy,\)

\((B_{TH_{n+1}} - B_{TH_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_8(t, B_{TH_n}(t)) - \mathcal{S}_8(t, B_{TH_{n-1}}(t)) \right) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_8(t, B_{TH_n}(y)) - \mathcal{S}_8(t, B_{TH_{n-1}}(y)) \right) dy,\)

\((C_{TH_{n+1}} - C_{TH_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_9(t, C_{TH_n}(t)) - \mathcal{S}_9(t, C_{TH_{n-1}}(t)) \right) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_9(t, C_{TH_n}(y)) - \mathcal{S}_9(t, C_{TH_{n-1}}(y)) \right) dy,\)

\((E_{H_{n+1}} - E_{H_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_{10}(t, E_{H_n}(t)) - \mathcal{S}_{10}(t, E_{H_{n-1}}(t)) \right) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_{10}(t, E_{H_n}(y)) - \mathcal{S}_{10}(t, E_{H_{n-1}}(y)) \right) dy,\)

\((F_{T_{n+1}} - B_{T_n})(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \left( \mathcal{S}_{11}(t, F_{T_n}(t)) - \mathcal{S}_{11}(t, F_{T_{n-1}}(t)) \right) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \mathcal{S}_{11}(t, F_{T_n}(y)) - \mathcal{S}_{11}(t, F_{T_{n-1}}(y)) \right) dy.\)

When we take the norm of both sides of the above equations,
\[
|| (C_{T_{n+1}} - C_{T_n}) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_3 (t, C_{T_n} (t)) - S_3 (t, C_{T_{n-1}} (t)) \right) \right) \, dy,
\]

\[
\int (E_{n+1} - E_n) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_4 (t, E_n (t)) - S_4 (t, E_{n-1} (t)) \right) \right) \, dy,
\]

\[
|| (C_{H_{n+1}} - C_{H_n}) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_5 (t, C_{H_n} (t)) - S_5 (t, C_{H_{n-1}} (t)) \right) \right) \, dy,
\]

\[
|| (F_{n+1} - F_n) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_6 (t, F_n (t)) - S_6 (t, F_{n-1} (t)) \right) \right) \, dy,
\]

\[
|| (G_{H_{n+1}} - G_{H_n}) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_7 (t, G_{H_n} (t)) - S_7 (t, G_{H_{n-1}} (t)) \right) \right) \, dy,
\]

\[
|| (B_{TH_{n+1}} - B_{TH_n}) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_8 (t, B_{TH_n} (t)) - S_8 (t, B_{TH_{n-1}} (t)) \right) \right) \, dy,
\]

\[
|| (C_{TH_{n+1}} - C_{TH_n}) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_9 (t, C_{TH_n} (t)) - S_9 (t, C_{TH_{n-1}} (t)) \right) \right) \, dy,
\]

\[
|| (E_{H_{n+1}} - E_{H_n}) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_{10} (t, E_{H_n} (t)) - S_{10} (t, E_{H_{n-1}} (t)) \right) \right) \, dy,
\]

\[
|| (F_{H_{n+1}} - B_{H_n}) (t) || = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \int_0^t \left( \left( S_{11} (t, F_{H_n} (t)) - S_{11} (t, F_{H_{n-1}} (t)) \right) \right) \, dy.
\]

**Theorem 3.2.** The TB-HIV/AIDS coinfection model has a solution if the following inequality is achieved:
Similarly, Proof 3.2. Let us consider the following equations,

$$\mathcal{R}_{1n}(t) = A_{n+1}(t) - A_n(t), \quad \mathcal{R}_{2n}(t) = B_{Tn+1}(t) - B_{Tn}(t), \quad \mathcal{R}_{3n}(t) = C_{Tn+1}(t) - C_{Tn}(t), \quad \mathcal{R}_{4n}(t) = E_{n+1}(t) - E_n(t),$$

$$\mathcal{R}_{5n}(t) = C_{Hn+1}(t) - C_{Hn}(t), \quad \mathcal{R}_{6n}(t) = F_{n+1}(t) - F_n(t), \quad \mathcal{R}_{7n}(t) = G_{Hn+1}(t) - G_{Hn}(t),$$

$$\mathcal{R}_{8n}(t) = B_{Tn+1}(t) - B_{Tn}(t), \quad \mathcal{R}_{9n}(t) = C_{Tn+1}(t) - C_{Tn}(t).$$

Similarly, $\mathcal{R}_{11n}(t) = E_{n+1}(t) - E_n(t), \quad \mathcal{R}_{12n}(t) = F_{Tn+1}(t) - F_T(t).$

Firstly, we will start with $\mathcal{R}_{1n}(t),

$$||\mathcal{R}_{1n}(t)|| \leq \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} ||\mathcal{S}_1(t, A_n(t)) - \mathcal{S}_1(t, A(t))|| + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t ||\mathcal{S}_1(y, A_n(y)) - \mathcal{S}_1(y, A(y))|| dy \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right) \Psi_1 ||A_n - A|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||A - A_1||. \quad (20)$$

Similarly,

$$||\mathcal{R}_{2n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||B_T - B_{T1}||,$$

$$||\mathcal{R}_{3n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||C_T - C_{T1}||,$$

$$||\mathcal{R}_{4n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||E - E_1||,$$

$$||\mathcal{R}_{5n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||C_H - C_{H1}||,$$

$$||\mathcal{R}_{6n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||F - F_1||,$$

$$||\mathcal{R}_{7n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||G_H - G_{H1}||,$$

$$||\mathcal{R}_{8n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||B_{TH} - B_{TH1}||,$$

$$||\mathcal{R}_{9n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||C_{TH} - C_{TH1}||,$$

$$||\mathcal{R}_{10n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||E_H - E_{H1}||,$$

$$||\mathcal{R}_{11n}(t)|| \leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right)^n \Upsilon^n ||F_T - F_{T1}||. \quad (21)$$
So that, it can be said that, we can find \( \mathfrak{S}_i(t) \to 0, i = 1, 2, \ldots, 11 \), as \( n \to \infty \). Thus, the proof is complete.

### 3.3 Uniqueness of Solution to Fractional Model

In this section we will show you the unique solution of TB-HIV/AIDS mathematical model as [14].

**Theorem 3.3.** The TB-HIV/AIDS mathematical model shown in system (14) will have a unique solution if the following inequality hold true:

\[
\left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right) \Psi_i \leq 1, \quad i = 1, 2, \ldots, 11.
\]

**Proof 3.3.** Let us assume that the system (16) has solutions \( A(t), B_T(t), C_T(t), E(t), C_H(t), F(t), G_H(t), B_{TH}(t), C_{TH}(t), E_H(t), F_T(t) \), as well as \( A(t), B_T(t), C_T(t), E(t), C_H(t), F(t), G_H(t), B_{TH}(t), C_{TH}(t), E_H(t), F_T(t) \). So that, the following system can be written,

\[
\begin{align*}
\dot{A}(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_1(t, \dot{A}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_1(y, \dot{A}(y)) dy, \\
\dot{B}_T(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_2(t, \dot{B}_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_2(y, \dot{B}_T(y)) dy, \\
\dot{C}_T(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_3(t, \dot{C}_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_3(y, \dot{C}_T(y)) dy, \\
\dot{E}(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_4(t, \dot{E}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_4(y, \dot{E}(y)) dy, \\
\dot{C}_H(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_5(t, \dot{C}_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_5(y, \dot{C}_H(y)) dy.
\end{align*}
\]

\[
\begin{align*}
\dot{F}(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_6(t, \dot{F}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_6(y, \dot{F}(y)) dy, \\
\dot{G}_H(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_7(t, \dot{G}_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_7(y, \dot{G}_H(y)) dy, \\
\dot{B}_{TH}(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_8(t, \dot{B}_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_8(y, \dot{B}_{TH}(y)) dy, \\
\dot{C}_{TH}(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_9(t, \dot{C}_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_9(y, \dot{C}_{TH}(y)) dy, \\
\dot{E}_H(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_{10}(t, \dot{E}_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_{10}(y, \dot{E}_H(y)) dy, \\
\dot{F}_T(t) &= \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathfrak{S}_{11}(t, \dot{F}_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathfrak{S}_{11}(y, \dot{F}_T(y)) dy.
\end{align*}
\]
When the norm is taken from both sides of the system of equations above, firstly

\[ ||A(t) - \tilde{A}(t)|| \leq \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} ||\mathcal{S}_1(t, A(t)) - \mathcal{S}_1(t, \tilde{A}(t))|| + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t ||\mathcal{S}_1(y, A(y)) - \mathcal{S}_1(y, \tilde{A}(y))|| dy \]

\[ \leq \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \Psi_1 ||A - \tilde{A}|| + \frac{2\rho \Psi_1}{2M(\rho) - \rho M(\rho)} ||A - \tilde{A}||. \]

The following inequality can be written,

\[ \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \Psi_1 + \frac{2\rho \Psi_1}{2M(\rho) - \rho M(\rho)} - 1 \right) ||A - \tilde{A}|| \geq 0. \]

Thus \( ||A - \tilde{A}|| = 0 \). This implies \( A(t) = \tilde{A}(t) \). When the same method is applied that \( B_T(t) = \hat{B}_T(t), C_T(t) = \hat{C}_T(t), E(t) = \hat{E}(t), C_H(t) = \hat{C}_H(t), F(t) = \hat{F}(t), G_H(t) = \hat{G}_H(t), B_T H(t) = \hat{B}_T H(t), C_T H(t) = \hat{C}_T H(t), E_H(t) = \hat{E}_H(t) \) and \( F_T(t) = \hat{F}_T(t) \). According to these results, the model has a unique solution.

4 Stability Analysis

In this section, we will examine the stability of the TB-HIV/AIDS model. First of all, the following definition should be given.

**Definition 4.1.** The system (18) Hyers-Ulam stable [25] if exists constants \( \Upsilon_i > 0, i = 1, 2, \ldots 11 \) satisfying for every \( \varsigma_i > 0, i = 1, 2, \ldots 11 \),

\[ |A(t) - \tilde{A}(t)| \leq \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} |\mathcal{S}_1(t, A(t))| + \frac{2\rho}{2M(\rho) + \rho M(\rho)} \int_0^t |\mathcal{S}_1(y, A(y))| dy \leq \varsigma_1, \]

\[ |B_T(t) - \tilde{B}_T(t)| \leq \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} |\mathcal{S}_2(t, B_T(t))| + \frac{2\rho}{2M(\rho) + \rho M(\rho)} \int_0^t |\mathcal{S}_2(y, B_T(y))| dy \leq \varsigma_2, \]

\[ |C_T(t) - \tilde{C}_T(t)| \leq \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} |\mathcal{S}_3(t, C_T(t))| + \frac{2\rho}{2M(\rho) + \rho M(\rho)} \int_0^t |\mathcal{S}_3(y, C_T(y))| dy \leq \varsigma_3, \]
\[ E(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_4(t, E(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_4(y, E(y))dy \leq \xi_4, \]
\[ C_H(t) = \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} \mathcal{S}_5(t, C_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_5(y, C_H(y))dy \leq \xi_5, \]
\[ F(t) = \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} \mathcal{S}_6(t, F(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_6(y, F(y))dy \leq \xi_6. \]
\[ G_H(t) = \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} \mathcal{S}_7(t, G_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_7(y, G_H(y))dy \leq \xi_7. \]
\[ B_{TH}(t) = \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} \mathcal{S}_8(t, B_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_8(y, B_{TH}(y))dy \leq \xi_8. \]
\[ C_{TH}(t) = \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} \mathcal{S}_9(t, C_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_9(y, C_{TH}(y))dy \leq \xi_9. \]
\[ E_{H}(t) = \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} \mathcal{S}_{10}(t, E_{H}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_{10}(y, E_{H}(y))dy \leq \xi_{10}. \]
\[ F_{H}(t) = \frac{2(1 - \rho)}{2M(\rho) + \rho M(\rho)} \mathcal{S}_{11}(t, F_{H}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_{11}(y, F_{H}(y))dy \leq \xi_{11}. \]

There exist, \( \overline{A}(t), \overline{B}_T(t), \overline{C}_T(t), \overline{E}(t), \overline{C}_H(t), \overline{F}(t), \overline{G}_H(t), \overline{B}_{TH}(t), \overline{C}_{TH}(t), \overline{E}_{H}(t), \overline{F}_{H}(t) \) which are satisfying

\[ \overline{A}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_1(t, \overline{A}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_1(y, \overline{A}(y))dy, \]
\[ \overline{B}_T(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_2(t, \overline{B}_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_2(y, \overline{B}_T(y))dy, \]
\[ \overline{C}_T(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_3(t, \overline{C}_T(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_3(y, \overline{C}_T(y))dy, \]
\[ \overline{E}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_4(t, \overline{E}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_4(y, \overline{E}(y))dy, \]
\[ \overline{C}_H(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_5(t, \overline{C}_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_5(y, \overline{C}_H(y))dy, \]
\[ \overline{F}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_6(t, \overline{F}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_6(y, \overline{F}(y))dy, \]
\[ \overline{G}_H(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_7(t, \overline{G}_H(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_7(y, \overline{G}_H(y))dy, \]
\[ \overline{B}_{TH}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_8(t, \overline{B}_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_8(y, \overline{B}_{TH}(y))dy, \]
\[ \overline{C}_{TH}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_9(t, \overline{C}_{TH}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_9(y, \overline{C}_{TH}(y))dy, \]
\[ \overline{E}_{H}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_{10}(t, \overline{E}_{H}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_{10}(y, \overline{E}_{H}(y))dy, \]
\[ \overline{F}_{H}(t) = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} \mathcal{S}_{11}(t, \overline{F}_{H}(t)) + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t \mathcal{S}_{11}(y, \overline{F}_{H}(y))dy, \]
such that

\[ |A(t) - \overline{A}(t)| \leq \sigma_1 \varsigma_1, |B_T(t) - \overline{B}_T(t)| \leq \sigma_2 \varsigma_2, |C_T(t) - \overline{C}_T(t)| \leq \sigma_3 \varsigma_3, |E(t) - \overline{E}(t)| \leq \sigma_4 \varsigma_4. \]
\[ |C_H(t) - \overline{C}_H(t)| \leq \sigma_5 \varsigma_5, |F(t) - \overline{F}(t)| \leq \sigma_6 \varsigma_6, |G_H(t) - \overline{G}_H(t)| \leq \sigma_7 \varsigma_7, |B_{TH}(t) - \overline{B}_{TH}(t)| \leq \sigma_8 \varsigma_8. \]
\[ |C_{TH}(t) - \overline{C}_{TH}(t)| \leq \sigma_9 \varsigma_9, |E_H(t) - \overline{E}_H(t)| \leq \sigma_{10} \varsigma_{10}, |F_T(t) - \overline{F}_T(t)| \leq \sigma_{11} \varsigma_{11}. \]

**Theorem 4.1.** The fractional system (18) is Hyers-Ulam stable with assumption H.

**Proof 4.1.** In Theorem (3.3), \( A(t), B_T(t), C_T(t), E(t), C_H(t), F(t), G_H(t), B_{TH}(t), C_{TH}(t), E_H(t) \) and \( F_T(t) \) were shown to have a unique solution. Let \( \overline{A}(t), \overline{B}_T(t), \overline{C}_T(t), \overline{E}(t), \overline{C}_H(t), \overline{F}(t), \overline{G}_H(t), \overline{B}_{TH}(t), \overline{C}_{TH}(t), \overline{E}_H(t) \) be an approximate solution of system (13) satisfying system (18). After, we can say that

\[
\begin{align*}
||A(t) - \overline{A}(t)|| &\leq \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} ||\mathcal{S}_1(t, A(t)) - \mathcal{S}_1(t, \overline{A}(t))|| \\
&+ \frac{2\rho}{2M(\rho) - \rho M(\rho)} \int_0^t ||\mathcal{S}_1(y, A(y)) - \mathcal{S}_1(y, \overline{A}(y))|| dy \\
&\leq \left( \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)} \right) \Psi_1 ||A - \overline{A}||
\end{align*}
\]

(26)

when we take \( \varsigma_1 = \Psi_1, \varsigma_1 = \frac{2(1 - \rho)}{2M(\rho) - \rho M(\rho)} + \frac{2\rho}{2M(\rho) - \rho M(\rho)}, \) we have

\[ ||A(t) - \overline{A}(t)|| \leq \varsigma_1 \varsigma_1. \]

In this way, the following inequalities can be easily written.

\[
\begin{align*}
||B_T(t) - \overline{B}_T(t)|| &\leq \varsigma_2 \varsigma_2 \\
||C_T(t) - \overline{C}_T(t)|| &\leq \varsigma_3 \varsigma_3 \\
||E(t) - \overline{E}(t)|| &\leq \varsigma_4 \varsigma_4 \\
||C_H(t) - \overline{C}_H(t)|| &\leq \varsigma_5 \varsigma_5 \\
||F(t) - \overline{F}(t)|| &\leq \varsigma_6 \varsigma_6 \\
||G_H(t) - \overline{G}_H(t)|| &\leq \varsigma_7 \varsigma_7 \\
||B_{TH}(t) - \overline{B}_{TH}(t)|| &\leq \varsigma_8 \varsigma_8 \\
||C_{TH}(t) - \overline{C}_{TH}(t)|| &\leq \varsigma_9 \varsigma_9 \\
||E_H(t) - \overline{E}_H(t)|| &\leq \varsigma_{10} \varsigma_{10} \\
||F_T(t) - \overline{F}_T(t)|| &\leq \varsigma_{11} \varsigma_{11}.
\end{align*}
\]

(27)

With the help of Eqs. (26) and (27), the system (18) Hyers-Ulam is stable. Thus, the theorem is proved.
5 Numerical Simulations

Atangana and Owolabi [26] have found a new numerical approach using the new Caputo fractional derivative for the discretization of fractional differential equations. The authors consider the following fractional differential equation first.

\[ C^\rho_0 \mathcal{D} t^\rho x(t) = (f(t, x(t))), \]  

(28)

or

\[ (f(t, x(t))) = \frac{M(\rho)}{1 - \rho} \int_0^t x'(\tau) \exp \left[ -\frac{\rho}{1 - \rho}(t - \tau) \right] d\tau. \]  

(29)

When they edit the above equation using the fundamental theorem of analysis, they get,

\[ x(t) - x(0) = \frac{1 - \rho}{M(\rho)} f(t, x(t)) + \frac{\rho}{M(\rho)} \int_0^t f(\tau, x(\tau)) d\tau. \]  

(30)

Therefore,

\[ x(t_{n+1}) - x(0) = \frac{1 - \rho}{M(\rho)} f(t_n, x(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} f(t, x(t)) dt, \]  

(31)

and

\[ x(t_{n+1}) - x(0) = \frac{1 - \rho}{M(\rho)} f(t_{n-1}, x(t_{n-1})) + \frac{\rho}{M(\rho)} \int_0^{t_n} f(t, x(t)) dt. \]  

(32)

When they removing (32) from (31), the following equation system is obtained.

\[ x(t_{n+1}) - x(t_n) = \frac{1 - \rho}{M(\rho)} \left[ f(t_n, x(t_n)) - f(t_{n-1}, x_{n-1}) \right] + \frac{\rho}{M(\rho)} \int_0^{t_n} f(t, x(t)) dt, \]  

(33)

where

\[ \int_0^{t_{n+1}} f(t, x(t)) dt = \int_0^{t_{n+1}} \left\{ \frac{f(t_n, x_n)}{h}(t - t_{n-1}) - \frac{f(t_{n-1}, x_{n-1})}{h}(t - t_n) \right\} dt 
\]

\[ = \frac{3h}{2} f(t_n, x_n) - \frac{h}{2} f(t_{n-1}, x_{n-1}). \]  

(34)
Thus,

\[ x(t_{n+1}) - x(t_n) = \frac{1 - \rho}{M(\rho)} \left[ f(t_n, x_n) - f(t_{n-1}, x_{n-1}) \right] + \frac{3\rho h}{2M(\rho)} f(t_n, x_n) - \frac{\rho h}{2M(\rho)} f(t_{n-1}, x_{n-1}), \]

(35)

which implies that

\[ x(t_{n+1}) - x(t_n) = \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) f(t_n, x_n) + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) f(t_{n-1}, x_{n-1}). \]

(36)

Hence,

\[ x_{n+1} = x_n + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) f(t_n, x_n) + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) f(t_{n-1}, x_{n-1}), \]

(37)

which is the corresponding two-step Adams-Bashforth method for the Caputo-Fabrizio fractional derivative.

**Theorem 5.1.** Let \( x(t) \) be a solution of \( ^C^F D_t^\rho (x(t)) = f(t, x(t)) \) where \( f \) is a continuous function bounded for the Caputo-Fabrizio fractional derivative [26],

\[ x_{n+1} = x_n + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) f(t_n, x_n) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) f(t_{n-1}, x_{n-1}) + R_n^\rho, \]

(38)

where \( ||R_n^\rho|| \leq M \).

### 5.1 Numerical Simulations for the Model

The expanded TB-HIV/AIDS model for the Caputo-Fabrizio fractional derivative was introduced in system (15). When the system (15) is rearranged by the fundamental theorem of analysis, the next system of equations is obtained for \( S_i, i = 1, 2, \ldots 11 \) kernels as follows:
\begin{align}
A(t) - A(0) &= \frac{1 - \rho}{M(\rho)} S_1(t, A(t)) + \frac{\rho}{M(\rho)} \int_0^t S_1(\omega, A(\omega))d\omega, \\
B_T(t) - B_T(0) &= \frac{1 - \rho}{M(\rho)} S_2(t, B_T(t)) + \frac{\rho}{M(\rho)} \int_0^t S_2(\omega, B_T(\omega))d\omega, \\
C_T(t) - C_T(0) &= \frac{1 - \rho}{M(\rho)} S_3(t, C_T(t)) + \frac{\rho}{M(\rho)} \int_0^t S_3(\omega, C_T(\omega))d\omega, \\
E(t) - E(0) &= \frac{1 - \rho}{M(\rho)} S_4(t, E(t)) + \frac{\rho}{M(\rho)} \int_0^t S_4(\omega, E(\omega))d\omega, \\
C_H(t) - C_H(0) &= \frac{1 - \rho}{M(\rho)} S_5(t, C_H(t)) + \frac{\rho}{M(\rho)} \int_0^t S_5(\omega, C_H(\omega))d\omega, \\
F(t) - F(0) &= \frac{1 - \rho}{M(\rho)} S_6(t, F(t)) + \frac{\rho}{M(\rho)} \int_0^t S_6(\omega, F(\omega))d\omega. \\
G_H(t) - G_H(0) &= \frac{1 - \rho}{M(\rho)} S_7(t, G_H(t)) + \frac{\rho}{M(\rho)} \int_0^t S_7(\omega, G_H(\omega))d\omega, \\
B_{TH}(t) - B_{TH}(0) &= \frac{1 - \rho}{M(\rho)} S_8(t, B_{TH}(t)) + \frac{\rho}{M(\rho)} \int_0^t S_8(\omega, B_{TH}(\omega))d\omega, \\
C_{TH}(t) - C_{TH}(0) &= \frac{1 - \rho}{M(\rho)} S_9(t, C_{TH}(t)) + \frac{\rho}{M(\rho)} \int_0^t S_9(\omega, C_{TH}(\omega))d\omega, \\
E_{H}(t) - E_{H}(0) &= \frac{1 - \rho}{M(\rho)} S_{10}(t, E_{H}(t)) + \frac{\rho}{M(\rho)} \int_0^t S_{10}(\omega, E_{H}(\omega))d\omega, \\
F_{T}(t) - F_{T}(0) &= \frac{1 - \rho}{M(\rho)} S_{11}(t, F_{T}(t)) + \frac{\rho}{M(\rho)} \int_0^t S_{11}(\omega, F_{T}(\omega))d\omega.
\end{align}

Thus,
\begin{align}
A_{n+1} - A(0) &= \frac{1 - \rho}{M(\rho)} S_1(t_n, A(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_1(\omega, A(\omega))d\omega, \\
B_{Tn+1} - B_T(0) &= \frac{1 - \rho}{M(\rho)} S_2(t_n, B_T(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_2(\omega, B_T(\omega))d\omega, \\
C_{Tn+1} - C_T(0) &= \frac{1 - \rho}{M(\rho)} S_3(t_n, C_T(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_3(\omega, C_T(\omega))d\omega, \\
E_{n+1} - E(0) &= \frac{1 - \rho}{M(\rho)} S_4(t_n, E(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_4(\omega, E(\omega))d\omega, \\
C_{Hn+1} - C_H(0) &= \frac{1 - \rho}{M(\rho)} S_5(t_n, C_H(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_5(\omega, C_H(\omega))d\omega, \\
F_{n+1} - F(0) &= \frac{1 - \rho}{M(\rho)} S_6(t_n, F(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_6(\omega, F(\omega))d\omega, \\
G_{Hn+1} - G_H(0) &= \frac{1 - \rho}{M(\rho)} S_7(t_n, G_H(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_7(\omega, G_H(\omega))d\omega, \\
B_{THn+1} - B_{TH}(0) &= \frac{1 - \rho}{M(\rho)} S_8(t_n, B_{TH}(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_8(\omega, B_{TH}(\omega))d\omega, \\
C_{THn+1} - C_{TH}(0) &= \frac{1 - \rho}{M(\rho)} S_9(t_n, C_{TH}(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_9(\omega, C_{TH}(\omega))d\omega, \\
E_{Hn+1} - E_{H}(0) &= \frac{1 - \rho}{M(\rho)} S_{10}(t_n, E_{H}(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_{10}(\omega, E_{H}(\omega))d\omega, \\
F_{Tn+1} - F_{T}(0) &= \frac{1 - \rho}{M(\rho)} S_{11}(t_n, F_{T}(t_n)) + \frac{\rho}{M(\rho)} \int_0^{t_{n+1}} S_{11}(\omega, F_{T}(\omega))d\omega,
\end{align}

and
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When we removing (41) from (40), the following equation system is obtained.

\[
\begin{align*}
A_{n+1} - A(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_1(t_{n-1}, A(t_{n-1})) - \mathcal{S}_1(t_{n-1}, A(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_1(t, A(t)) dt, \\
B_{Tn+1} - B_T(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_2(t_{n-1}, B_T(t_{n-1})) - \mathcal{S}_2(t_{n-1}, B_T(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_2(t, B_T(t)) dt, \\
C_{Tn+1} - C_T(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_3(t_{n-1}, C_T(t_{n-1})) - \mathcal{S}_3(t_{n-1}, C_T(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_3(t, C_T(t)) dt, \\
E_{n+1} - E(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_4(t_{n-1}, E(t_{n-1})) - \mathcal{S}_4(t_{n-1}, E(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_4(t, E(t)) dt, \\
C_{Hn+1} - C_H(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_5(t_{n-1}, C_H(t_{n-1})) - \mathcal{S}_5(t_{n-1}, C_H(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_5(t, C_H(t)) dt, \\
F_{n+1} - F(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_6(t_{n-1}, F(t_{n-1})) - \mathcal{S}_6(t_{n-1}, F(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_6(t, F(t)) dt, \\
G_{Hn+1} - G_H(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_7(t_{n-1}, G_H(t_{n-1})) - \mathcal{S}_7(t_{n-1}, G_H(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_7(t, G_H(t)) dt, \\
B_{THn+1} - B_{TH}(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_8(t_{n-1}, B_{TH}(t_{n-1})) - \mathcal{S}_8(t_{n-1}, B_{TH}(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_8(t, B_{TH}(t)) dt, \\
C_{THn+1} - C_{TH}(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_9(t_{n-1}, C_{TH}(t_{n-1})) - \mathcal{S}_9(t_{n-1}, C_{TH}(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_9(t, C_{TH}(t)) dt, \\
E_{Hn+1} - E_H(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_{10}(t_{n-1}, E_H(t_{n-1})) - \mathcal{S}_{10}(t_{n-1}, E_H(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_{10}(t, E_H(t)) dt, \\
F_{THn+1} - F_{TH}(0) &= \frac{1 - \rho}{M(\rho)} \left( \mathcal{S}_{11}(t_{n-1}, F_{TH}(t_{n-1})) - \mathcal{S}_{11}(t_{n-1}, F_{TH}(t_{n-1})) \right) + \frac{\rho}{M(\rho)} \int_{t_n}^{t_{n+1}} \mathcal{S}_{11}(t, F_{TH}(t)) dt,
\end{align*}
\]
\[
\begin{align*}
\int_{t_n}^{t_{n+1}} \mathcal{G}_1(t, A(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_1(t_n, A_n)}{h} (t - t_{n-1}) - \frac{\mathcal{G}_1(t_{n-1}, A_{n-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_2(t_n, B_{Tn}) - \frac{h}{2} \mathcal{G}_2(t_{n-1}, B_{Tn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_2(t, B_T(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_2(t_n, B_{Tn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_2(t_{n-1}, B_{Tn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_2(t_n, B_{Tn}) - \frac{h}{2} \mathcal{G}_2(t_{n-1}, B_{Tn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_3(t, C_T(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_3(t_n, C_{Tn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_3(t_{n-1}, C_{Tn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_3(t_n, C_{Tn}) - \frac{h}{2} \mathcal{G}_3(t_{n-1}, C_{Tn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_4(t, E(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_4(t_n, E_n)}{h} (t - t_{n-1}) - \frac{\mathcal{G}_4(t_{n-1}, E_{n-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_4(t_n, E_{Tn}) - \frac{h}{2} \mathcal{G}_4(t_{n-1}, E_{Tn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_5(t, C_H(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_5(t_n, C_{Hn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_5(t_{n-1}, C_{Hn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_5(t_n, C_{Hn}) - \frac{h}{2} \mathcal{G}_5(t_{n-1}, C_{Hn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_6(t, F(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_6(t_n, F_n)}{h} (t - t_{n-1}) - \frac{\mathcal{G}_6(t_{n-1}, F_{n-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_6(t_n, F_{Tn}) - \frac{h}{2} \mathcal{G}_6(t_{n-1}, F_{Tn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_7(t, G_H(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_7(t_n, G_{Hn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_7(t_{n-1}, G_{Hn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_7(t_n, G_{Hn}) - \frac{h}{2} \mathcal{G}_7(t_{n-1}, G_{Hn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_8(t, B_{TH}(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_8(t_n, B_{THn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_8(t_{n-1}, B_{THn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_8(t_n, B_{THn}) - \frac{h}{2} \mathcal{G}_8(t_{n-1}, B_{THn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_9(t, C_{TH}(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_9(t_n, C_{THn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_9(t_{n-1}, C_{THn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_9(t_n, C_{THn}) - \frac{h}{2} \mathcal{G}_9(t_{n-1}, C_{THn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_{10}(t, E_H(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_{10}(t_n, E_{Hn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_{10}(t_{n-1}, E_{Hn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_{10}(t_n, E_{Hn}) - \frac{h}{2} \mathcal{G}_{10}(t_{n-1}, E_{Hn-1}), \\
\int_{t_n}^{t_{n+1}} \mathcal{G}_{11}(t, F_T(t)) dt &= \int_{t_n}^{t_{n+1}} \left\{ \frac{\mathcal{G}_{11}(t_n, F_{Tn})}{h} (t - t_{n-1}) - \frac{\mathcal{G}_{11}(t_{n-1}, F_{Tn-1})}{h} (t - t_n) \right\} \\
&= \frac{3h}{2} \mathcal{G}_{11}(t_n, F_{Tn}) - \frac{h}{2} \mathcal{G}_{11}(t_{n-1}, F_{Tn-1}).
\end{align*}
\]
Therefore,

\[
\begin{align*}
A_{n+1} - A_n &= \frac{1 - \rho}{M(\rho)} \left| \mathcal{S}_1(t_n, A_n) - \mathcal{S}_1(t_{n-1}, A_{n-1}) \right| + \frac{3\rho h}{2M(\rho)} \mathcal{S}_1(t_n, A_n) - \frac{\rho h}{2M(\rho)} \mathcal{S}_1(t_{n-1}, A_{n-1}), \\
B_{Tn+1} - B_{Tn} &= \frac{1 - \rho}{M(\rho)} \left| \mathcal{S}_2(t_n, B_{Tn}) - \mathcal{S}_2(t_{n-1}, B_{Tn-1}) \right| + \frac{3\rho h}{2M(\rho)} \mathcal{S}_2(t_n, B_{Tn}) - \frac{\rho h}{2M(\rho)} \mathcal{S}_2(t_{n-1}, B_{Tn-1}), \\
C_{Tn+1} - C_{Tn} &= \frac{1 - \rho}{M(\rho)} \left| \mathcal{S}_3(t_n, C_{Tn}) - \mathcal{S}_3(t_{n-1}, C_{Tn-1}) \right| + \frac{3\rho h}{2M(\rho)} \mathcal{S}_3(t_n, C_{Tn}) - \frac{\rho h}{2M(\rho)} \mathcal{S}_3(t_{n-1}, C_{Tn-1}), \\
E_{n+1} - E_n &= \frac{1 - \rho}{M(\rho)} \left| \mathcal{S}_4(t_n, E_n) - \mathcal{S}_4(t_{n-1}, E_{n-1}) \right| + \frac{3\rho h}{2M(\rho)} \mathcal{S}_4(t_n, E_n) - \frac{\rho h}{2M(\rho)} \mathcal{S}_4(t_{n-1}, E_{n-1}), \\
C_{H_{n+1}} - C_{Hn} &= \frac{1 - \rho}{M(\rho)} \left| \mathcal{S}_5(t_n, C_{Hn}) - \mathcal{S}_5(t_{n-1}, C_{Hn-1}) \right| + \frac{3\rho h}{2M(\rho)} \mathcal{S}_5(t_n, C_{Hn}) - \frac{\rho h}{2M(\rho)} \mathcal{S}_5(t_{n-1}, C_{Hn-1}), \\
F_{n+1} - F_n &= \frac{1 - \rho}{M(\rho)} \left| \mathcal{S}_6(t_n, F_n) - \mathcal{S}_6(t_{n-1}, F_{n-1}) \right| + \frac{3\rho h}{2M(\rho)} \mathcal{S}_6(t_n, F_n) - \frac{\rho h}{2M(\rho)} \mathcal{S}_6(t_{n-1}, F_{n-1}), \\
G_{H_{n+1}} - G_{Hn} &= \frac{1 - \rho}{M(\rho)} \left| \mathcal{S}_7(t_n, G_{Hn}) - \mathcal{S}_7(t_{n-1}, G_{Hn-1}) \right| + \frac{3\rho h}{2M(\rho)} \mathcal{S}_7(t_n, G_{Hn}) - \frac{\rho h}{2M(\rho)} \mathcal{S}_7(t_{n-1}, G_{Hn-1}).
\end{align*}
\]

which implies that,

\[
\begin{align*}
A_{n+1} &= A_n + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) \mathcal{S}_1(t_n, A_n) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) \mathcal{S}_1(t_{n-1}, A_{n-1}), \\
B_{Tn+1} &= B_{Tn} + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) \mathcal{S}_2(t_n, B_{Tn}) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) \mathcal{S}_2(t_{n-1}, B_{Tn-1}), \\
C_{Tn+1} &= C_{Tn} + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) \mathcal{S}_3(t_n, C_{Tn}) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) \mathcal{S}_3(t_{n-1}, C_{Tn-1}), \\
E_{n+1} &= E_n + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) \mathcal{S}_4(t_n, E_n) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) \mathcal{S}_4(t_{n-1}, E_{n-1}), \\
C_{H_{n+1}} &= C_{Hn} + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) \mathcal{S}_5(t_n, C_{Hn}) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) \mathcal{S}_5(t_{n-1}, C_{Hn-1}), \\
F_{n+1} &= F_n + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) \mathcal{S}_6(t_n, F_n) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) \mathcal{S}_6(t_{n-1}, F_{n-1}), \\
G_{H_{n+1}} &= G_{Hn} + \left( \frac{1 - \rho}{M(\rho)} + \frac{3\rho h}{2M(\rho)} \right) \mathcal{S}_7(t_n, G_{Hn}) + \left( \frac{1 - \rho}{M(\rho)} + \frac{\rho h}{2M(\rho)} \right) \mathcal{S}_7(t_{n-1}, G_{Hn-1}).
\end{align*}
\]
According to Theorem (4.1), we get,

\begin{align}
A_{n+1} &= A_n + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_1(t_n, A_n) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_1(t_{n-1}, A_{n-1}) + R^n_{p}, \\
B_{Tn+1} &= B_Tn + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_2(t_n, B_{Tn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_2(t_{n-1}, B_{Tn}) + 2R^n_{p}, \\
C_{Tn+1} &= C_{Tn} + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_3(t_n, C_{Tn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_3(t_{n-1}, C_{Tn}) + 3R^n_{p}, \\
E_{n+1} &= E_n + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_4(t_n, E_n) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_4(t_{n-1}, E_{n-1}) + 4R^n_{p}, \\
C_{Hn+1} &= C_{Hn} + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_5(t_n, C_{Hn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_5(t_{n-1}, C_{Hn}) + 5R^n_{p}, \\
F_{n+1} &= F_n + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_6(t_n, F_n) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_6(t_{n-1}, F_{n-1}) + 6R^n_{p}, \\
G_{Hn+1} &= G_{Hn} + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_7(t_n, G_{Hn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_7(t_{n-1}, G_{Hn}) + 7R^n_{p}, \\
B_{THn+1} &= B_{THn} + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_8(t_n, B_{THn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_8(t_{n-1}, B_{THn}) + 8R^n_{p}, \\
C_{THn+1} &= C_{THn} + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_9(t_n, C_{THn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_9(t_{n-1}, C_{THn}) + 9R^n_{p}, \\
E_{Hn+1} &= E_{Hn} + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_{10}(t_n, E_{Hn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_{10}(t_{n-1}, E_{Hn}) + 10R^n_{p}, \\
F_{Tn+1} &= F_{Tn} + \left(1 - \frac{\rho}{M(\rho)} + \frac{3\rho}{2M(\rho)}\right) \mathcal{E}_{11}(t_n, F_{Tn}) + \left(1 - \frac{\rho}{M(\rho)} + \frac{\rho h}{2M(\rho)}\right) \mathcal{E}_{11}(t_{n-1}, F_{Tn}) + 11R^n_{p},
\end{align}

where

\[|R^n_{p}|_{\infty} < \frac{\rho}{M(\rho)}(n + 1)!h^{n+1}, \quad i = 1, 2, \ldots, 11.

The numerical simulations of the system (17) are performed below for each function below. The behavior of the model is analyzed in detail using fractional order derivative different \(\rho\) values. The parameters used in the simulations are \(\Theta = 425000\), \(\xi = 2.13 \times 10^{-5}\), \(\alpha_1 = 1.12 \times 10^{-4}\), \(m_1 = 3.45 \times 10^{-4}\), \(\chi_1 = 2.19 \times 10^{-3}\), \(\chi_2 = 2.69 \times 10^{-3}\), \(\tilde{\chi}_1 = 4.78 \times 10^{-4}\), \(\psi = 6.12 \times 10^{-5}\), \(\upsilon = 2.89 \times 10^{-4}\), \(\beta_1 = 3.81 \times 10^{-5}\), \(\upsilon_1 = 1.29 \times 10^{-4}\), \(\upsilon_2 = 2.15 \times 10^{-4}\), \(q = 2.31 \times 10^{-3}\), \(\tilde{\chi}_2 = 3.64 \times 10^{-4}\), \(e = 7.14 \times 10^{-3}\), \(\chi_3 = 1.96 \times 10^{-3}\), \(\alpha'_2 = 1.42 \times 10^{-4}\), \(m_2 = 2.78 \times 10^{-4}\), \(\beta_2 = 4.19 \times 10^{-5}\), \(p = 3.18 \times 10^{-3}\) and \(\alpha = 3.14 \times 10^{-2}\) with initial conditions \(A(0) = 129526000\), \(B_T(0) = 150000\), \(C_T(0) = 150000\), \(E(0) = 129050000\), \(C_H(0) = 1200000\), \(F(0) = 125325000\), \(G_H(0) = 1200000\), \(B_{TH}(0) = 500000\), \(C_{TH}(0) = 500000\), \(E_H(0) = 1200000\) and \(F_T(t) = 150000\).

The numerical results given in Figs. 1(a)—(b), 2(a)—(b), 3(a)—(b), 4(a)—(b), 5(a)—(b) and 6 show the numerical simulations of the specific solution of the model for different \(\rho\) values. Mathematical models are useful for investigating TB-HIV/AIDS coinfection population dynamics and evaluating treatment programs, the process of evolution and control of the outbreak is memory related. We simulate the fractional model and investigate the effects of fractional orders placed on the model on the spread of the TB-HIV/AIDS coinfection model. The results showed
Fig. 1 Numerical simulation of TB-HIV/AIDS model given by Eq. (17) for several values of $\rho$ for $A(t)$ and $B_T(t)$

Fig. 2 Numerical simulation of TB-HIV/AIDS model given by Eq. (17) for several values of $\rho$ for $C_T(t)$ and $E(t)$

that fractional ranking provides more information about the effect of treatment on TB-HIV/AIDS coinfection compared to the classical model with $\rho$, integer-order derivatives. The new model with non-singular exponential kernel takes into account abnormal spread like this infection biological models. Our generalized model also describes two different waiting time distributions, which are an ideal waiting time distribution, as observed in many biological events, such as the spread of human infections. The figures obtained show that outbreak models defined by fractional order differential equations have rich dynamics and describe biological systems better than traditional integer order models.
Fig. 3  Numerical simulation of TB-HIV/AIDS model given by Eq. (17) for several values of $\rho$ for $C_H(t)$ and $F(t)$

Fig. 4  Numerical simulation of TB-HIV/AIDS model given by Eq. (17) for several values of $\rho$ for $G_H(t)$ and $B_{TH}(t)$

Fig. 5  Numerical simulation of TB-HIV/AIDS model given by Eq. (17) for several values of $\rho$ for $C_{TH}(t)$ and $E_H(t)$
6 Conclusion

The new non-singular exponential core model takes into account abnormal spread like this infection biological models. This new model does not allow the incorporation of artificial singularities with the Caputo-Fabrizio fractional derivative operator, which allows a better explanation of the biological process. In addition, the generalized model allows to explain the ideal waiting time distribution for the spread of infection. The Caputo-Fabrizio operator allows us to obtain precise information about how the biological process proceeds for long-term observations.

In this study, the TB-HIV/AIDS mathematical model was extended to the Caputo-Fabrizio fractional derivative operator and analyzed. Firstly, the existence of solution to the new model was examined with the help of the fixed point theorem. Then, the conditions under which the uniqueness of solution can be realized were analyzed and the stability analysis of the model was performed. Finally, numerical solutions were obtained and shown with simulations. Finally, we believe that we have extended the HIV-TB/AIDS coinfection model to the Caputo-Fabrizio fractional derivative, creating a more complete and more realistic model.

References

1. Sharma, S.K., Mohan, A., Kadhiravan, T.: HIV-TB co-infection: epidemiology, diagnosis & management. Indian J. Med. Res. 121(4), 550–567 (2005)
2. Gao, L., Zhou, F., Li, X., Jin, Q.: HIV/TB co-infection in mainland China: a meta-analysis. PloS ONE 5(5), e10736 (2010)
3. Pawlowski, A., Jansson, M., Sköld, M., Rottenberg, M.E., Källenius, G.: Tuberculosis and HIV co-infection. PLoS Pathogens 8(2), e1002464 (2012)
4. McShane, H.: Co-infection with HIV and TB: double trouble. Int. J. STD & AIDS 16(2), 95–101 (2005)
5. Naresh, R., Tripathi, A.: Modelling and analysis of HIV-TB co-infection in a variable size population. Math. Model. Anal. 10(3), 275–286 (2005)
6. Datiko, D.G., Yassin, M.A., Chekol, L.T., Kabeto, L.E., Lindtjørn, B.: The rate of TB-HIV co-infection depends on the prevalence of HIV infection in a community. BMC Public Health 8(1), 266 (2008)
7. Agusto, F.B., Adekunle, A.I.: Optimal control of a two-strain tuberculosis-HIV/AIDS co-infection model. Biosystems 119, 20–44 (2014)
8. Okosun, K.O., Makinde, O.D.: A co-infection model of malaria and cholera diseases with optimal control. Math. Biosci. 258, 19–32 (2014)
9. Carvalho, A., Pinto, C.M.: A delay fractional order model for the co-infection of malaria and HIV/AIDS. Int. J. Dyn. Control 5(1), 168–186 (2017)
10. Zafar, Z.U., Rehan, K., Mushtaq, M.: HIV/AIDS epidemic fractional-order model. J. Differ. Equ. Appl. 23(7), 1298–1315 (2017)
11. Kheiri, H., Jafari, M.: Optimal control of a fractional-order model for the HIV/AIDS epidemic. Int. J. Biomath. 11(07), 1850086 (2018)
12. Mallela, A., Lenhart, S., Vaidya, N.K.: HIV-TB co-infection treatment: modeling and optimal control theory perspectives. J. Comput. Appl. Math. 307, 143–161 (2016)
13. Khan, H., Gómez-Aguilar, J.F., Alkhazzan, A., Khan, A.: A fractional order HIV-TB coinfection model with nonsingular Mittag-Leffler Law. Math. Methods Appl. Sci. 43(6), 3786–3806 (2020)
14. Dokuyucu, M.A., Dutta, H.: A fractional order model for Ebola Virus with the new Caputo fractional derivative without singular kernel. Chaos, Solitons Fractals 134, 109717 (2020)
15. Dokuyucu, M.A., Celik, E., Bulut, H., Baskonus, H.M.: Cancer treatment model with the Caputo-Fabrizio fractional derivative. Eur. Phys. J. Plus 133(3), 92 (2018)
16. Singh, J., Kumar, D., Al Qurashi, M., Baleanu, D.: A new fractional model for giving up smoking dynamics. Adv. Differ. Equat. 2017(1), 88 (2017)
17. Basavarajiah, D.M., Murthy, B.N.: HIV Transmission: Statistical Modelling. Springer, Singapore (2019)
18. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Progr. Fract. Differ. Appl. 1(2), 1–3 (2015)
19. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent-II. Geophys. J. Int. 13(5), 529–39 (1967)
20. Losada, J., Nieto, J.J.: Properties of a new fractional derivative without singular kernel. Progr. Fract. Differ. Appl. 1(2), 87–92 (2015)
21. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)
22. Silva, C.J., Torres, D.F.: A TB-HIV/AIDS co-infection model and optimal control treatment. arXiv preprint arXiv:1501.03322, 14 January 2015
23. Styblo, K.: Epidemiology of tuberculosis. Bull. Int. Union Tuberc. 53, 141–152 (1978)
24. Kwan, C.K., Ernst, J.D.: HIV and tuberculosis: a deadly human syndemic. Clin. Microbiol. Rev. 24(2), 351–376 (2011)
25. Hyers, D.H.: The stability of homomorphisms and related topics. Glob. Anal. Anal. Manifolds 57, 140–153 (1983)
26. Atangana, A., Owolabi, K.M.: New numerical approach for fractional differential equations. Math. Modell. Nat. Phenom. 13(1), 3 (2018)