A PRIMAL FINITE ELEMENT SCHEME OF THE HODGE LAPLACE PROBLEM

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Abstract. In this paper, a unified family, for any \( n \geq 2 \) and \( 1 \leq k \leq n-1 \), of nonconforming finite element schemes are presented for the primal weak formulation of the \( n \)-dimensional Hodge-Laplace equation on \( H^k \cap H^0_0 \) and on the simplicial subdivisions of the domain. The finite element scheme possesses an \( O(h) \)-order convergence rate for sufficiently regular data, and an \( O(h^s) \)-order rate on any \( s \)-regular domain, \( 0 < s \leq 1 \), no matter what topology the domain has.

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1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be a domain with Lipschitz boundary. In this paper, we consider the primal weak formulation of the Hodge-Laplace problem: given \( f \in L^2 \Lambda^k(\Omega) \), find \( \omega \in H^k(\Omega) \cap H^0_0(\Omega) \), such that

\[
\langle \omega, \vartheta \rangle_{L^2 \Lambda^k} = 0, \quad \forall \vartheta \in \delta \Lambda^k, \quad \text{and} \quad \langle \delta_k \omega, \delta_k \mu \rangle_{L^2 \Lambda^{k+1}} = \langle f - P_0 f, \mu \rangle_{L^2 \Lambda^k}, \quad \forall \mu \in H^k(\Omega) \cap H^0_0(\Omega).
\]

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Here, following [1], we denote by $\Lambda^k(\Xi)$ the space of differential $k$-forms on an $n$-dimensional domain $\Xi$, and $L^2\Lambda^k(\Xi)$ consists of differential $k$-forms with coefficients in $L^2(\Xi)$ component by component, and $\langle \cdot, \cdot \rangle_{L^2(\Xi)}$ is the inner product of the Hilbert space $L^2\Lambda^k(\Xi)$. In the sequel, we will occasionally drop $\Omega$ for differential forms on $\Omega$. The exterior differential operator $d^k : \Lambda^k(\Xi) \to \Lambda^{k+1}(\Xi)$ is an unbounded operator from $L^2\Lambda^k(\Xi)$ to $\Lambda^{k+1}(\Xi)$. Denote

$$H\Lambda^k(\Xi) := \{ \omega \in L^2\Lambda^k(\Xi) : d^k \omega \in L^2\Lambda^{k+1}(\Xi) \},$$

and $H\Lambda^k(\Xi)$ is a Hilbert space with the norm $||\omega||_{L^2(\Xi)} + ||d^k \omega||_{L^2(\Xi)}$. Denote by $H_0\Lambda^k(\Xi)$ the closure of $C^0_0\Lambda^k(\Xi)$ in $H\Lambda^k(\Xi)$. The Hodge star operator $\star$ maps $L^2\Lambda^k(\Xi)$ isomorphically to $L^2\Lambda^{n-k}(\Xi)$ for each $0 \leq k \leq n$. The codifferential operator $\delta_k$ defined by $\delta_k \mu = (-1)^{k+1} \star d^{n-k} \star \mu$ is unbounded from $L^2\Lambda^k(\Xi)$ to $L^2\Lambda^{k-1}(\Xi)$. Denote

$$H^*\Lambda^k(\Xi) := \{ \mu \in L^2\Lambda^k(\Xi) : \delta_k \mu \in L^2\Lambda^{k-1}(\Xi) \},$$

and $H_0^*\Lambda^k(\Xi)$ the closure of $C^0_0\Lambda^k(\Xi)$ in $H^*\Lambda^k(\Xi)$. Then $H^*\Lambda^k(\Xi) = \star H\Lambda^{n-k}(\Xi)$, and $H_0^*\Lambda^k(\Xi) = \star H_0\Lambda^{n-k}(\Xi)$. Denote spaces of harmonic forms by

$$\mathcal{H}(\Xi) := N(d^k, H_0\Lambda^k(\Xi)) \oplus^1 R(d^{n-k}, H_0\Lambda^{n-k}(\Xi)), $$

where $N(\cdot, \cdot)$ and $R(\cdot, \cdot)$ denote the kernel and range spaces of certain operators, and $\oplus^1$ denotes the orthogonal difference, namely $N(d^k, H_0\Lambda^k(\Xi)) = R(\delta_{n-k}, H_0\Lambda^{n-k}(\Xi)) \oplus^1 \mathcal{H}(\Xi)$. Similarly

$$\mathcal{H}_0(\Xi) := N(\delta_k, H^*\Lambda^k(\Xi)) \oplus^1 R(\delta_{n-k}, H^*\Lambda^{n-k}(\Xi)).$$

Then $\star \mathcal{H}(\Xi) = \mathcal{H}_0(\Xi)$, and further the Poincaré-Lefschetz duality holds as $\mathcal{H}^\perp = \star \mathcal{H}(\Xi)$. Besides, $P_\mathcal{H}$ denotes the $L^2$ projection to $\mathcal{H}$.

The model problem (1.1) corresponds to a strong form that

(1.2) $d^k \omega \in H^*_0\Lambda^k(\Omega)$, $\delta_k \omega \in H\Lambda^{k-1}(\Omega)$,

and

(1.3) $\omega \perp \mathcal{H}\Lambda^k(\Omega)$, and $\delta_{k+1} d^k \omega + d^{k-1} \delta_k \omega = f - P_\mathcal{H} f$.

The Hodge-Laplace problem arises in many applied sciences, including electromagnetics [18, 24], fluid-structure interaction [7, 8, 17], and others. Particularly, the numerical solution of the Hodge Laplace equation is a central subject of the theory of finite element exterior calculus (FEEC), and we refer to [1, 3, 4] for a thorough introduction to FEEC.

A major feature in the discretization of Hodge Laplace problem is that, the conforming finite element scheme for (1.1) may lead to a spurious solution that converges to a wrong limit when the exact solution $\omega$ is not regular enough. Indeed, as is well known, for domains which are not smooth enough, the singular part of $\omega$ can not be captured by the conforming finite element space. To cope with this situation, a well-developed approach is to use mixed finite element method. Again, the main approach can be found in detail in [1, 3, 4], for which the structure of de Rham complex plays a crucial role, and another key ingredient is that spaces of discrete harmonic forms
are established isomorphic to the space of continuous harmonic forms. Besides, some recent progress can be found in [15, 20] for \textit{a posteriori} error estimation and adaptive methods, and in [19] for a detailed analysis of Discontinuous Galerkin (DG) methods in FEEC in the newly-presented eXtended Galerkin (XG) framework.

On the other hand, to discretize directly the primal formulation (1.1) has been still attracting research interests. Virtual element methods are designed for the three dimensional vector potential formulation of magnetostatic problems [14], with the major interests restricted to cases where the computational domain has no re-entrant corner, and the space of harmonic forms is not concerned for these cases. Nonconforming element methods and discontinuous Galerkin methods are also designed which can lead to a correct approximation of the nonsmooth solution for the $H(\text{curl}) \cap H(\text{div})$ problem in two dimension polygonal domains, particularly for (1.1) on domains with connected boundary on which harmonic forms vanish; readers are referred to [11] for an interior penalty method, to [10] for a nonconforming finite element method, and to [9] for a non-conforming finite element used with inter-element penalties. Recent works also include [5, 6, 23].

In this paper, we present a unified family, for any $n \geq 2$ and $1 \leq k \leq n - 1$, of nonconforming finite element schemes for the primal formulation (1.1) and on the subdivision of the domain by simplexes. The main feature of the finite element schemes is a nonconforming finite element space for $H\Lambda^k \cap H^\ast_0 \Lambda^k$ where all the finite element functions are defined by local shape function spaces and the continuity conditions, while for $\delta \Lambda^k$, we use the well-studied discrete space of harmonic forms as, e.g., in [1, 3, 4], and no penalty term or stabilization is used in the schemes. The local shape function space is a slight enrichment based on $P_0 \Lambda^k(T) + \kappa (P_0 \Lambda^{k+1}(T)) + \ast \ast \ast (P_0 \Lambda^{k-1}(T))$, namely the minimal local space for $\delta_{k+1} \mathbf{d}^k + \mathbf{d}^{k-1}\delta_k$, but, not similar to [5, 6, 9-11, 23], it does not contain the complete linear polynomial space. Another difference from these existing works, particularly [10, 23], is that the finite element functions in this present paper possess a different kind of inter-element continuity. As precisely described in (3.4), the continuity is imposed in a dual way; the dual way for imposing continuity has been suggested by the theory of partially adjoint operators in [31], and used for constructing nonconforming $H(\mathbf{d})$ Whitney form spaces as well as commutative diagrams. The way makes the finite element functions not correspond to a “finite element” in the sense of Ciarlet’s triple [12], and the analysis thus relies on non-standard techniques. In this paper, for the analysis, the discrete Poincaré inequality, which is crucial with respect to nontrivial topologies, is proved by the theory of partially adjoint operators developed in [31], and different from [31], the error estimation is accomplished by an indirect way; namely, we first show as Lemma 4.8 that certain primal scheme (4.12) is, in some sense, equivalent to a classical mixed element scheme (4.11), and the error estimation of the classical mixed scheme can be used as a midway step for the analysis. We finally show that, the finite element scheme possesses an $O(h)$-order convergence rate for sufficiently regular data, and an $O(h^s)$-order rate on any $s$-regular domain, $0 < s \leq 1$, no matter the topology is trivial or not.

It is interesting to clarify again that, the schemes given in this paper are primal ones, even though the continuity conditions for the finite element functions are imposed in a dual way, and
for some special cases, the schemes can be equivalent, in some sense, to mixed schemes. In principle, the finite element scheme aims at (1.1) (as the primal weak formulation (4.15) of [1]) and utilizes only a single field. Meanwhile, the equivalence between primal and mixed finite element schemes has been found as to, e.g., the Poisson and the biharmonic equations [2, 22]. For practical implementation, in the present paper, a precise set of basis functions can be figured out for the newly-designed finite element space, the supports of which are each contained in a vertex patch, and the programming of the scheme can be done in a standard routine as for the standard “primal finite element” method. The figuration of basis functions is quite similar to the procedure given in [31], but also with essentially different steps. The basis functions are presented in a unified way, and an illustration is given for the two-dimensional case for example. Locally supported basis functions have been also found and implemented for many other specific non-Ciarlet type finite element spaces [16, 21, 25-30].

The remaining of the paper is organized as follows. In Section 2, some preliminaries are collected. Particularly, some key points of the theory of partially adjoint operator, which is developed in [31] and is fundamental in the present paper, are reviewed, including the definitions of base operator pair (Definition 2.1) and of partially adjoint operators (Definition 2.2), and the quantified closed range theorem for partially adjoint operators (Theorem 2.4). In Section 3, the finite element space is constructed, and the discrete Poincaré inequality of the space is proved by the theory of partially adjoint operator. In Section 4, a unified family of finite element schemes are defined, and the error estimation and the implementation are presented.

2. Preliminaries

2.1. Theory of partially adjoint operators. Let \( \mathbf{X} \) and \( \mathbf{Y} \) be two Hilbert spaces with respective inner products \( \langle \cdot, \cdot \rangle_{\mathbf{X}} \) and \( \langle \cdot, \cdot \rangle_{\mathbf{Y}} \), and respective norms \( \| \cdot \|_{\mathbf{X}} \) and \( \| \cdot \|_{\mathbf{Y}} \). Let \( (T, \tilde{M}) : \mathbf{X} \to \mathbf{Y} \) and \( (T, \tilde{N}) : \mathbf{Y} \to \mathbf{X} \) be two closed operators, not necessarily densely defined. Denote, for \( v \in \tilde{M} \), \( \|v\|_T := (\|v\|_{\mathbf{X}}^2 + \|Tv\|_{\mathbf{Y}}^2)^{1/2} \), and for \( w \in \tilde{N} \), \( \|w\|_T := (\|w\|_{\mathbf{X}}^2 + \|Tw\|_{\mathbf{Y}}^2)^{1/2} \). Denote

\[
M := \{v \in \tilde{M} : \langle v, Tvw \rangle_{\mathbf{X}} - \langle Tv, w \rangle_{\mathbf{Y}} = 0, \ \forall w \in \tilde{N}\},
\]

(2.1)

\[
N := \{v \in \tilde{N} : \langle v, Tvw \rangle_{\mathbf{X}} - \langle Tv, w \rangle_{\mathbf{Y}} = 0, \ \forall v \in \tilde{M}\},
\]

(2.2)

\[
M_B := \{v \in \tilde{M} : \langle v, w \rangle_{\mathbf{X}} = 0, \ \forall w \in \mathcal{N}(T, \tilde{M}) ; \langle Tv, w \rangle_{\mathbf{Y}} = 0, \ \forall w \in \tilde{M}\},
\]

(2.3)

and

\[
N_B := \{v \in \tilde{N} : \langle v, w \rangle_{\mathbf{Y}} = 0, \ \forall w \in \mathcal{N}(T, \tilde{N}) ; \langle Tvw, w \rangle_{\mathbf{X}} = 0, \ \forall w \in \tilde{N}\}.
\]

(2.4)

We call \( (M_B, N_B) \) the twisted part of \( (\tilde{M}, \tilde{N}) \).

**Definition 2.1** (Definition 2.13 of [31]). A pair of closed operators \( (T, \tilde{M}) : \mathbf{X} \to \mathbf{Y} \) is called a base operator pair, if, with notations (2.1), (2.2), (2.3) and (2.4),

1. \( \mathcal{R}(T, \tilde{M}), \mathcal{R}(\mathbf{T}, \tilde{N}), \mathcal{R}(T, \tilde{M}) \) and \( \mathcal{R}(\mathbf{T}, \tilde{N}) \) are all closed;
\( \mathcal{N}(T, M_B) \) and \( \mathcal{R}(T, N_B) \) are isomorphic, and \( \mathcal{N}(T, N_B) \) and \( \mathcal{R}(T, M_B) \) are isomorphic.

For \([T, \widetilde{M}], (T, \widetilde{N}) \) a base operator pair, for nontrivial \( \mathcal{R}(T, N_B) \) and \( \mathcal{N}(T, M_B) \), denote

\[
\alpha := \inf_{0 \neq v \in \mathcal{N}(T, M_B)} \sup_{w \in \mathcal{R}(T, N_B)} \frac{\langle v, w \rangle_X}{\|v\|_X\|w\|_X} = \inf_{0 \neq w \in \mathcal{R}(T, M_B)} \sup_{v \in \mathcal{N}(T, N_B)} \frac{\langle v, w \rangle_X}{\|v\|_X\|w\|_X},
\]

and for nontrivial \( \mathcal{N}(T, N_B) \) and \( \mathcal{R}(T, M_B) \), denote

\[
\beta := \inf_{0 \neq v \in \mathcal{N}(T, M_B)} \sup_{w \in \mathcal{R}(T, N_B)} \frac{\langle v, w \rangle_Y}{\|v\|_Y\|w\|_Y} = \inf_{0 \neq w \in \mathcal{R}(T, M_B)} \sup_{v \in \mathcal{N}(T, N_B)} \frac{\langle v, w \rangle_Y}{\|v\|_Y\|w\|_Y}.
\]

Then \( \alpha > 0 \) and \( \beta > 0 \). We further make a convention that,

\[
\begin{cases}
\alpha = 1, & \text{if } \mathcal{N}(T, M_B) = \mathcal{R}(T, N_B) = \{0\}; \\
\beta = 1, & \text{if } \mathcal{N}(T, N_B) = \mathcal{R}(T, M_B) = \{0\}.
\end{cases}
\]

**Definition 2.2** (Definition 2.15 of [31]). For \([T, \widetilde{M}] : X \to Y, (T, \widetilde{N}) : Y \to X \) a base operator pair, two operators \((T, D) \subset (T, \widetilde{M}) \) and \((T, \mathcal{I}) \subset (T, \widetilde{N}) \) are called partially adjoint based on \([T, \widetilde{M}], (T, \widetilde{N}) \), if

\[
D = \{ v \in \widetilde{M} : \langle v, T w \rangle_X = \langle Tv, w \rangle_Y = 0, \quad \forall w \in \mathcal{I} \},
\]

and

\[
\mathcal{I} = \{ v \in \widetilde{N} : \langle v, T w \rangle_X = \langle Tv, w \rangle_Y = 0, \quad \forall v \in D \}.
\]

**Definition 2.3** (Definition 2.8 of [31]). For \((T, D) : X \to Y \) a closed operator, define

\[
D^\perp := \{ v \in D : \langle v, w \rangle_X = 0, \quad \forall w \in \mathcal{N}(T, D) \}.
\]

Define the index of closed range of \((T, D)\) as

\[
icr(T, D) := \begin{cases}
\sup_{0 \neq v \in D^\perp} \frac{\|v\|_X}{\|Tv\|_Y}, & \text{if } D^\perp \neq \{0\}; \\
0, & \text{if } D^\perp = \{0\}.
\end{cases}
\]

Note that \(\nicr(T, D)\) evaluates in \([0, +\infty)\), and \(\mathcal{R}(T, D)\) is closed if and only if \(\nicr(T, D) < \infty\). Further, \(\nicr(T, D^\perp)\) plays like the constant for Poincaré inequality in the sense that \(\|v\|_X \leq \nicr(T, D^\perp)\|Tv\|_Y\) for \(v \in D^\perp\).

**Theorem 2.4** (quantified closed range theorem, Theorem 2.21 of [31]). For \([T, D], (T, \mathcal{I}) \) partially adjoint based on \([T, \widetilde{M}], (T, \widetilde{N}) \), with notations given in (2.1), (2.2), (2.5), (2.6) and (2.7), if \(\nicr(T, \mathcal{I}) < \infty\),

\[
\nicr(T, D) \leq (1 + \alpha^{-1}) \cdot \nicr(T, \widetilde{M}) + \alpha^{-1} \nicr(T, \mathcal{I}) + \nicr(T, M);
\]

if \(\nicr(T, D) < \infty\),

\[
\nicr(T, \mathcal{I}) \leq (1 + \beta^{-1}) \cdot \nicr(T, \widetilde{N}) + \beta^{-1} \nicr(T, D) + \nicr(T, \widetilde{N}).
\]
2.2. Polynomial spaces on a simplex. Denote the set of $k$-indices as
\[ \mathcal{I}_k := \{ \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n, \mathbb{N} \text{ the set of integers} \}. \]

Then
\[ d^k(\kappa(\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k})) = (k + 1) \mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k+1}, \]
and
\[ \delta_k(\kappa \kappa \star (\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k-1})) = (-1)^{\alpha_n-k}(n + 1)(\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k-1}), \]
where $\kappa$ is the Koszul operator
\[ \kappa(\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}) = \sum_{\gamma} (-1)^{\gamma+1} x^{\gamma_j} \mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_{j-1}} \wedge \mathrm{dx}^{\alpha_j+1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}. \]

Given $T$ a simplex, denote on $T$ $\bar{x}^j = x^j - c_j$ where $c_j$ is a constant such that $\int_T \bar{x}^j = 0$. Denote a simplex dependent Koszul operator
\[ \kappa_T(\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}) := \sum_{\gamma} (-1)^{\gamma+1} \bar{x}^{\gamma_j} \mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_{j-1}} \wedge \mathrm{dx}^{\alpha_j+1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}, \]
for $\alpha \in \mathcal{I}_k$. Then
\[ d^{k-1} \kappa_T(\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}) = k \mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}. \]

In this part and in the sequel, we make the convention that, for $\alpha \in \mathcal{I}_{k,n}$, we use $\beta$ for one in $\mathcal{I}_{n-k,n}$, such that $\alpha$ and $\beta$ partition $\{1, 2, \ldots, n\}$. For $\alpha \in \mathcal{I}_{k,n}$, following [32], denote
\[ \bar{\mu}_{\delta, T}^\alpha = \sum_{j=1}^{k} [((\bar{x}^\alpha)^2 - c^\alpha_j) \mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_2} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}], \]
and
\[ \bar{\rho}_{a, T}^\alpha = \sum_{j=1}^{n-k} [((\bar{x}^\beta)^2 - c^\beta_j) \mathrm{dx}^{\alpha_1} \wedge \mathrm{dx}^{\alpha_2} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}], \]
where $c^\alpha_j$ and $c^\beta_j$ are constants such that $\int_T [(\bar{x}^\alpha)^2 - c^\alpha_j] = 0$ and $\int_T [(\bar{x}^\beta)^2 - c^\beta_j] = 0$. Then, ( [32])
\[ d^k \bar{\mu}_{\delta, T}^\alpha = 0, \quad \delta_k \bar{\mu}_{\delta, T}^\alpha = 2(-1)^n \kappa_T(\mathrm{dx}^{\alpha_1} \wedge \mathrm{dx}^{\alpha_2} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}), \]
\[ \delta_k \bar{\rho}_{a, T}^\alpha = 0, \quad \text{and} \quad d^k \bar{\rho}_{a, T}^\alpha = 2(-1)^{n(1+k)+1} \kappa_T(\kappa(\star(\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}))). \]

Denote, following [32],
\begin{align*}
(2.13) \quad \mathcal{H}_d^k \Lambda^k(T) &:= \text{span}\left\{ \bar{\mu}_{\delta, T}^\alpha : \alpha \in \mathcal{I}_{k,n} \right\}, \\
(2.14) \quad \mathcal{H}_d^k \Lambda^k(T) &:= \text{span}\left\{ \bar{\mu}_{a, T}^\alpha : \alpha \in \mathcal{I}_{k,n} \right\}.
\end{align*}

**Lemma 2.5.** ( [32])
(1) $d^k$ is bijective from $\kappa_T(P_0 \Lambda^{k+1})$ onto $P_0 \Lambda^{k+1}$, and bijective from $H^2_d \Lambda^k(T)$ onto $\star \kappa_T \star (P_0 \Lambda^k(T))$.

(2) $\delta_k$ is bijective from $\star \kappa_T \star (P_0 \Lambda^{k-1})$ onto $P_0 \Lambda^{k-1}$, and bijective from $H^2_d \Lambda^k(T)$ onto $\kappa_T(P_0 \Lambda^k(T))$.

**Lemma 2.6.** [32] There exists a constant $C_{k,n}$, depending on the regularity of $T$, such that

\[
\| \mu \|_{L^2 \Lambda^1(T)} \leq C_{k,n} h_T \| \delta_k \mu \|_{L^2 \Lambda^{k-1}(T)}, \text{ for } \mu \in \star \kappa_T \star (P_0 \Lambda^{k-1}(T)) + H^2_d \Lambda^k(T),
\]

and

\[
\| \mu \|_{L^2 \Lambda^1(T)} \leq C_{k,n} h_T \| d^k \mu \|_{L^2 \Lambda^{k-1}(T)}, \text{ for } \mu \in \kappa_T(P_0 \Lambda^{k+1}(T)) + H^2_d \Lambda^k(T).
\]

Denote by $P_{0,T}^k$ the $L^2$ projection onto $P_0 \Lambda^k(T)$. The lemma follows by Lemma 2.6 directly.

**Lemma 2.7.** There exists a constant $C_{k,n}$, depending on the regularity of $T$, such that

\[
\| \mu \|_{L^2 \Lambda^1(T)} \leq C_{k,n} h_T (\| \delta_k \mu \|_{L^2 \Lambda^{k-1}(T)} + \| d^k \mu \|_{L^2 \Lambda^{k-1}(T)}),
\]

for $\mu \in \star \kappa_T \star (P_0 \Lambda^{k-1}(T)) + \kappa_T(P_0 \Lambda^{k+1}(T)) + H^2_d \Lambda^k(T) + H^2_d \Lambda^k(T)$,

and

\[
\| \mu - P_{0,T}^k \mu \|_{L^2 \Lambda^1(T)} \leq C_{k,n} h_T (\| \delta_k \mu \|_{L^2 \Lambda^{k-1}(T)} + \| d^k \mu \|_{L^2 \Lambda^{k-1}(T)}),
\]

for $\mu \in P_0 \Lambda^k(T) + \star \kappa_T \star (P_0 \Lambda^{k-1}(T)) + \kappa_T(P_0 \Lambda^{k+1}(T)) + H^2_d \Lambda^k(T) + H^2_d \Lambda^k(T)$.

### 2.3. Conforming and nonconforming Whitney forms for exterior differential forms.

The space of Whitney forms, the lowest-degree trimmed polynomial $k$-forms, associated with the operator $d^k$ is denoted by ([1, 3, 4])

\[
P_{1-}^k \Lambda^k = P_0 \Lambda^k + \kappa(P_0 \Lambda^{k+1}).
\]

We denote the space associated with the operator $\delta_k$ by

\[
P_{1-}^{\star-} \Lambda^k := \star (P_{1-} \Lambda^{n-k}) = P_0 \Lambda^k + \star \kappa \star \star (P_0 \Lambda^{k-1}).
\]

For $\Xi$ a subdomain of $\Omega$, we denote by $E^\Omega_{\Xi}$ the extension from $L^1_{\text{loc}}(\Xi)$ to $L^1_{\text{loc}}(\Omega)$, the spaces of locally integrable functions, respectively. Namely,

\[
E^\Omega_{\Xi} : L^1_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\Omega), \quad E^\Omega_{\Xi} v = \begin{cases} v, & \text{on } \Xi, \\ 0, & \text{else}, \end{cases} \text{ for } v \in L^1_{\text{loc}}(\Xi).
\]

We use the same notation $L^1_{\text{loc}}$ for both scalar and non-scalar locally integrable functions, and, here and in the sequel, use the same notation $E^\Omega_{\Xi}$ for both scalar and non-scalar functions.

Let $\mathcal{G}_\Omega = \{ \mathcal{G}_h \}$ be a set of shape regular simplicial subdivisions of $\Omega$. On a $\mathcal{G}_h$, define formally the product of a set of function spaces $\{ \Gamma(T) \}_{T \in \mathcal{G}_h}$ defined cell by cell such that $E^\Omega_{\Xi} \Gamma(T)$ for all $T \in \mathcal{G}_h$ are compatible,

\[
\prod_{T \in \mathcal{G}_h} \Gamma(T) := \sum_{T \in \mathcal{G}_h} E^\Omega_{\Xi} \Gamma(T).
\]
and the summation is direct. The \( \prod_{T \in \mathcal{G}_h} \mathcal{Y}(T) \) defined this way is essentially the tensor product of all \( \mathcal{Y}(T) \). Denote
\[
\mathcal{P}_1^{-} \Lambda^k (\mathcal{G}_h) := \prod_{T \in \mathcal{G}_h} \mathcal{P}_1^{-} \Lambda^k (T), \quad \mathcal{P}_1^{+} \Lambda^k (\mathcal{G}_h) := \prod_{T \in \mathcal{G}_h} \mathcal{P}_1^{+} \Lambda^k (T), \quad \mathcal{P}_0^{+} \Lambda^k (\mathcal{G}_h) = \prod_{T \in \mathcal{G}_h} \mathcal{P}_0^{+} \Lambda^k (T).
\]

Denote the conforming finite element spaces with Whitney forms by
\[
W_h \Lambda^k := \mathcal{P}_1^{-} (\mathcal{G}_h) \cap H \Lambda^k, \quad W_{h0} \Lambda^k := \mathcal{P}_1^{-} (\mathcal{G}_h) \cap H_0 \Lambda^k,
\]
and
\[
W^* \Lambda^k := \mathcal{P}_1^{+} (\mathcal{G}_h) \cap H^* \Lambda^k, \quad W^*_h \Lambda^k := \mathcal{P}_1^{+} (\mathcal{G}_h) \cap H^*_0 \Lambda^k.
\]
Then
\[
W^*_h \Lambda^k = \star W_h \Lambda^{n-k}, \quad \text{and} \quad W^*_h \Lambda^k = \star W_{h0} \Lambda^{n-k}.
\]
Denote
\[
\mathcal{S}_h \Lambda^k := N(d_h, W_h \Lambda^k) \ominus \mathcal{R}(d_h^{-1}, W_h \Lambda^{k-1}), \quad \mathcal{S}_{h0} \Lambda^k := N(d_h, W_{h0} \Lambda^k) \ominus \mathcal{R}(d_h^{-1}, W_{h0} \Lambda^{k-1}),
\]
\[
\mathcal{S}_* \Lambda^k := N(\delta_h, W_* \Lambda^k) \ominus \mathcal{R}(\delta_{h+1}, W_* \Lambda^{k+1}), \quad \text{and} \quad \mathcal{S}_{h0} \Lambda^k := N(\delta_h, W_{h0} \Lambda^k) \ominus \mathcal{R}(\delta_{h+1}, W_{h0} \Lambda^{k+1}).
\]
Then
\[
\mathcal{S}_h \Lambda^k = \mathcal{S}_{h0} \Lambda^k, \quad \text{and} \quad \mathcal{S}_{h0} \Lambda^k = \mathcal{S}_{h} \Lambda^k.
\]

**Lemma 2.8.** ([11]) \( \mathcal{S}_h \Lambda^k \) and \( \mathcal{S}_{h0} \Lambda^k \) are isomorphic to \( \mathcal{S} \Lambda^k \) and \( \mathcal{S}_{h} \Lambda^k \), respectively.

Following [31], denote the accompanied-by-conforming (ABC) finite element spaces with Whitney forms by
\[
W^{abc}_h \Lambda^k := \left\{ \mu_h \in \mathcal{P}_1^{-} \Lambda^k (\mathcal{G}_h) : \sum_{T \in \mathcal{G}_h} \langle d_h \mu_h, \eta_h \rangle_{L^2 \Lambda^{k+1}(T)} - \langle \mu_h, \delta_{h+1} \eta_h \rangle_{L^2 \Lambda^k(T)} = 0, \forall \eta_h \in W_{h0} \Lambda^{k+1} \right\},
\]
\[
W^{abc}_{h0} \Lambda^k := \left\{ \mu_h \in \mathcal{P}_1^{-} \Lambda^k (\mathcal{G}_h) : \sum_{T \in \mathcal{G}_h} \langle d_h \mu_h, \eta_h \rangle_{L^2 \Lambda^{k+1}(T)} - \langle \mu_h, \delta_{h+1} \eta_h \rangle_{L^2 \Lambda^k(T)} = 0, \forall \eta_h \in W_{h} \Lambda^{k+1} \right\},
\]
\[
W^{,abc}_h \Lambda^k := \left\{ \mu_h \in \mathcal{P}_1^{-} \Lambda^k (\mathcal{G}_h) : \sum_{T \in \mathcal{G}_h} \langle d_h \mu_h, \tau_h \rangle_{L^2 \Lambda^{k+1}(T)} - \langle \mu_h, d_h^{-1} \tau_h \rangle_{L^2 \Lambda^k(T)} = 0, \forall \tau_h \in W_{h0} \Lambda^{k+1} \right\},
\]
and
\[
W^{,abc}_{h0} \Lambda^k := \left\{ \mu_h \in \mathcal{P}_1^{-} \Lambda^k (\mathcal{G}_h) : \sum_{T \in \mathcal{G}_h} \langle d_h \mu_h, \tau_h \rangle_{L^2 \Lambda^{k+1}(T)} - \langle \mu_h, d_h^{-1} \tau_h \rangle_{L^2 \Lambda^k(T)} = 0, \forall \tau_h \in W_{h} \Lambda^{k+1} \right\}.
\]
Note that, \( W^{abc}_{h(0)} \Lambda^0 \) and \( W^{,abc}_{h(0)} \Lambda^n \) are basically the famous lowest-degree Crouzeix-Raviart element spaces [13]. Besides, we have, for example,
\[
W^* \Lambda^k := \left\{ \mu_h \in \mathcal{P}_1^{+} \Lambda^k (\mathcal{G}_h) : \sum_{T \in \mathcal{G}_h} \langle d_h \mu_h, \eta_h \rangle_{L^2 \Lambda^{k+1}(T)} - \langle \mu_h, \delta_{h+1} \eta_h \rangle_{L^2 \Lambda^k(T)} = 0, \forall \eta_h \in W^{*} \Lambda^{k+1} \right\}.
\]
Lemma 2.9. ([31]) There exists a constant $C_{k,n}$, depending on the regularity of $G_h$, such that

$$\text{icr}(d^k, W_h \Lambda^k) \leq C_{k,n}, \quad \text{and} \quad \text{icr}(d^k, W_{h^2 \Lambda^k}) \leq C_{k,n},$$

and

$$\text{icr}(d^k, W_{h^2 \Lambda^k}) \leq C_{k,n}, \quad \text{and} \quad \text{icr}(d^k, W_{h^2 \Lambda^k}) \leq C_{k,n}.$$ 

Here and in the sequel, we use the subscript “$h$” to denote the piecewise operation on $G_h$.

Remark 2.10. By Lemma 2.9, it follows that

$$\text{icr}(\delta_k, W^*_h \Lambda^k) \leq C_{n-k,n}, \quad \text{and} \quad \text{icr}(\delta_k, W_{h^2 \Lambda^k}) \leq C_{n-k,n},$$

and

$$\text{icr}(\delta_k, W^*_h \Lambda^k) \leq C_{n-k,n}, \quad \text{and} \quad \text{icr}(\delta_k, W_{h^2 \Lambda^k}) \leq C_{n-k,n}.$$ 

Denote

$$\mathcal{S}_{h}^{\text{abc} \Lambda^k} := N(\delta_{k,h}, W^*_h \Lambda^k) \oplus R(\delta_{k-1,h}, W^*_h \Lambda^{k-1}), \quad \mathcal{S}_{h}^{\text{abc} \Lambda^k} := N(d^k, W^*_h \Lambda^k) \oplus R(d^{k-1}, W^*_h \Lambda^{k-1}),$$

$$\mathcal{S}_{h^2}^{\text{abc} \Lambda^k} := N(\delta_{k,h}, W^*_{h^2 \Lambda^k} \oplus R(\delta_{k-1,h}, W^*_{h^2 \Lambda^{k-1}}), \quad \text{and} \quad \mathcal{S}_{h^2}^{\text{abc} \Lambda^k} := N(d^k, W^*_{h^2 \Lambda^k} \oplus R(d^{k-1}, W^*_{h^2 \Lambda^{k-1}}).$$

Lemma 2.11 (Discrete Poincaré-Lefschetz duality, [31]).

$$\mathcal{S}_{h} \Lambda^k = * \mathcal{S}_{h^2} \Lambda^{n-k}, \quad \text{and} \quad \mathcal{S}_{h^2} \Lambda^k = * \mathcal{S}_{h} \Lambda^{n-k}.$$ 

Lemma 2.12. ([31]) The Hodge decompositions hold:

$$\mathcal{P}_0 \Lambda^k(G_h) = R(d^{k-1}, W_h \Lambda^{k-1}) \oplus \perp \mathcal{S}_{h} \Lambda^k \oplus \perp R(\delta_{k+1,h}, W^*_{h^2 \Lambda^{k+1}})$$

$$= R(d^{k-1}, W^*_{h^2 \Lambda^{k-1}}) \oplus \perp \mathcal{S}_{h^2} \Lambda^k \oplus \perp R(\delta_{k+1,h}, W^*_{h^2 \Lambda^{k-1}}).$$

3. Finite element space and Poincaré inequality

3.1. A pair of adjoint operators associated with Hodge Laplacian. Denote

$$T : L^2 \Lambda^k \rightarrow L^2 \Lambda^{k+1} \times L^2 \Lambda^{k-1}, \quad \text{by} \quad T\mu := (d^k \mu, \delta_k \mu),$$

and

$$\mathcal{T} : L^2 \Lambda^{k+1} \times L^2 \Lambda^{k-1} \rightarrow L^2 \Lambda^k, \quad \text{by} \quad \mathcal{T}(\eta, \tau) := \delta_{k+1} \eta + d^{k-1} \tau.$$ 

Then formally, the Hodge Laplacian $\delta_{k+1} d^k + d^{k-1} \delta_k = \mathcal{T} \circ T$.

Lemma 3.1. The operator $(T, H\Lambda^k \cap H_0^0 \Lambda^k)$ is closed, and its range $\mathcal{R}(T, H\Lambda^k \cap H_0^0 \Lambda^k)$ is closed.

Proof. We begin with the structure of $H\Lambda^k \cap H_0^0 \Lambda^k$. Firstly, decompose

$$H\Lambda^k = N(d^k, H\Lambda^k) \oplus \perp (N(d^k, H\Lambda^k))^\perp = R(d^{k-1}, H\Lambda^{k-1}) \oplus \perp \mathcal{S}_{h} \Lambda^k \oplus \perp (N(d^k, H\Lambda^k))^\perp.$$ 

By the Helmholtz decomposition, $(N(d^k, H\Lambda^k))^\perp \subset H\Lambda^k \cap N(\delta_k, H_0^0 \Lambda^k)$. Namely,

$$\mathcal{R}(d^k, (N(d^k, H\Lambda^k))^\perp) = R(d^k, H\Lambda^k), \quad \text{and} \quad \mathcal{R}(\delta_k, (N(d^k, H\Lambda^k))^\perp) = \{0\}.$$
Similarly, decompose $H_0^k \Lambda^k = N(\delta_k, H_0^k \Lambda^k) \oplus \perp (N(\delta_k, H_0^k \Lambda^k))^\perp$, and

$$\mathcal{R}(d^k, (N(\delta_k, H_0^k \Lambda^k))^\perp) = \{0\}, \quad \text{and} \quad \mathcal{R}(\delta_k, (N(\delta_k, H_0^k \Lambda^k))^\perp) = \mathcal{R}(\delta_k, H_0^k \Lambda^k).$$

It then follows that

$$\mathcal{R}(T, H\Lambda^k \cap H_0^k \Lambda^k) = \mathcal{R}(d^k, H\Lambda^k) \times \mathcal{R}(\delta_k, H_0^k \Lambda^k).$$

Simultaneously,

$$\mathcal{N}(T, H\Lambda^k \cap H_0^k \Lambda^k) = \mathcal{N}(d^k, H\Lambda^k) \cap \mathcal{N}(\delta_k, H_0^k \Lambda^k)$$

$$= [\mathcal{R}(d^{k-1}, H\Lambda^{k-1}) \oplus \perp \mathcal{H} \Lambda^k] \cap [\mathcal{R}(\delta_{k+1}, H_0^k \Lambda^{k+1}) \oplus \perp \mathcal{H}_0^k \Lambda^k] = \mathcal{H} \Lambda^k.$$

The closeness of $(T, H\Lambda^k \cap H_0^k \Lambda^k)$ can be proved by definition. The proof is completed.

\[\square\]

**Lemma 3.2.** The adjoint operator of $(T, H\Lambda^k \cap H_0^k \Lambda^k)$ is $(\mathbb{T}, H_0^k \Lambda^{k+1} \times H\Lambda^{k-1})$.

**Proof.** Firstly,

$$\mathcal{R}(\mathbb{T}, H_0^k \Lambda^{k+1} \times H\Lambda^{k-1}) = \mathcal{R}(\delta_{k+1}, H_0^k \Lambda^{k+1}) \oplus \perp \mathcal{R}(d^{k-1}, H\Lambda^{k-1}),$$

and

$$\mathcal{N}(\mathbb{T}, H_0^k \Lambda^{k+1} \times H\Lambda^{k-1}) = \mathcal{N}(\delta_{k+1}, H_0^k \Lambda^{k+1}) \times \mathcal{N}(d^{k-1}, H\Lambda^{k-1}).$$

Namely,

$$L^2 \Lambda^k = \mathcal{N}(T, H\Lambda^k \cap H_0^k \Lambda^k) \oplus \perp \mathcal{R}(\mathbb{T}, H_0^k \Lambda^{k+1} \times H\Lambda^{k-1})$$

and

$$L^2 \Lambda^{k+1} \times L^2 \Lambda^{k-1} = \mathcal{R}(T, H\Lambda^k \cap H_0^k \Lambda^k) \oplus \perp \mathcal{N}(\mathbb{T}, H_0^k \Lambda^{k+1} \times H\Lambda^{k-1}).$$

Therefore, $(\mathbb{T}, H_0^k \Lambda^{k+1} \times H\Lambda^{k-1})$ has the same range and kernel spaces as the adjoint operator of $(T, H\Lambda^k \cap H_0^k \Lambda^k)$. Further, it is easy to verify that, for any $\mu \in H\Lambda^k \cap H_0^k \Lambda^k$, $\eta \in H_0^k \Lambda^{k+1}$, and $\tau \in H\Lambda^{k-1}$,

$$\langle d^k \mu, \eta \rangle_{L^2 \Lambda^{k+1}} + \langle \delta_k \mu, \tau \rangle_{L^2 \Lambda^{k-1}} - \langle \mu, (\delta_{k+1} \eta + d^{k-1} \tau) \rangle_{L^2 \Lambda^k} = 0.$$

Hence $(\mathbb{T}, H_0^k \Lambda^{k+1} \times H\Lambda^{k-1})$ is the adjoint operator of $(T, H\Lambda^k \cap H_0^k \Lambda^k)$.

\[\square\]

### 3.2. Base operator pair for discretization.

We use the following notation: (\(\cdot_\cdot^m\) for minimal)

- \(P_{m+d}^{\delta, 0} \Lambda^k(T) := \mathcal{P}_0 \Lambda^k(T) + \kappa_T(\mathcal{P}_0 \Lambda^{k+1}(T)) + \kappa_T \ast (\mathcal{P}_0 \Lambda^{k-1}(T))\);
- \(P_{m+d}^{\delta, d} \Lambda^k(T) := P_{m+d}^{\delta, 0} \Lambda^k(T) + \mathcal{H}_0^k(T); \quad P_{m+d}^{\delta, \delta} \Lambda^k(T) := P_{m+d}^{\delta, 0} \Lambda^k(T) + \mathcal{H}_0^k(T); \quad P_{\delta, \delta}^k(T) := \mathcal{P}_1^{\delta, \delta} \Lambda^{k+1}(T) \times \mathcal{P}_1^{\delta, \delta} \Lambda^{k-1}(T).\)

Direct calculation leads to the lemma below.

**Lemma 3.3.** Given $\mu \in P_{m+d}^{\delta, 0} \Lambda^k(T)$, $\mu = 0$ if and only if, for any $\eta, \tau \in P_{\delta, \delta}^k(T)$,

$$\langle d^k \omega, \eta \rangle_{L^2 \Lambda^{k+1}(T)} + \langle \delta_k \omega, \tau \rangle_{L^2 \Lambda^{k-1}(T)} - \langle \omega, (\delta_{k+1} \eta + d^{k-1} \tau) \rangle_{L^2 \Lambda^k(T)} = 0.$$

**Lemma 3.4.** (1) \(\mathcal{N}(T, P_{m+d}^{\delta, 0} \Lambda^k(T)) = \mathcal{R}((\mathbb{T}, P_{\delta, \delta}^k(T)).\)
(2) There exists a constant $C_{\kappa,n} > 0$, depending on the regularity of $T$, such that

\[
\inf_{(\eta,\tau)\in \mathcal{R}(T, P_{\delta \times d}^{m+\delta} \Lambda^k(T))} \sup_{(\zeta,\sigma)\in \mathcal{N}(\mathbb{T}, P_{\delta \times d}^{m+\delta} \Lambda^k(T))} \frac{\langle \zeta, \eta \rangle_{L^2 \Lambda^{k+1}(T)} + \langle \zeta, \tau \rangle_{L^2 \Lambda^{k-1}(T)}}{\|\zeta\|_{L^2 \Lambda^{k+1}(T)} + \|\zeta\|_{L^2 \Lambda^{k-1}(T)}} \geq C_{\kappa,n}.
\]

Proof. For the first item, evidently, $\mathcal{N}(\mathbb{T}, P_{\delta \times d}^{m+\delta} \Lambda^k(T)) = \mathcal{P}_0 \Lambda^k(T) = \mathcal{R}(\mathbb{T}, P_{\delta \times d}^k(T))$.

For the second item, note that $\mathcal{R}(T, P_{\delta \times d}^{m+\delta} \Lambda^k(T)) = \mathcal{P}_0 \Lambda^{k+1}(T) \times \mathcal{P}_0 \Lambda^{k-1}(T)$ and

\[
\mathcal{N}(\mathbb{T}, P_{\delta \times d}^k(T)) = \mathcal{P}_0 \Lambda^{k+1}(T) \times \mathcal{P}_0 \Lambda^{k-1}(T)
\]

\[\oplus \text{ span } \left\{ \left\langle (-1)^{\ell_n} \frac{k}{n-k} \star \kappa_T \star (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}), \kappa_T(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) \right\rangle, \alpha \in \mathbb{X}_{\kappa,n} \right\} \]

Given $(\eta, \tau) \in \mathcal{R}(T, P_{\delta \times d}^{m+\delta} \Lambda^k(T))$, $\eta \in \mathcal{P}_0 \Lambda^{k+1}(T)$, and $\tau = \tau_0 + \tau_1$, such that $\tau_0 \in \mathcal{P}_0 \Lambda^{k-1}(T)$ and $\tau_1 \in \kappa_T(\mathcal{P}_0 \Lambda^k(T))$, we choose $\zeta_0 = \eta, \sigma_0 = \tau_0$, and $(\zeta_1, \sigma_1) \in \text{span}\left\{ \left\langle (-1)^{\ell_n} \frac{k}{n-k} \star \kappa_T \star (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}), \kappa_T(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}) \right\rangle, \alpha \in \mathbb{X}_{\kappa,n} \right\}$ such that $\sigma_1 = \tau_1$, and set $(\zeta, \sigma) = (\zeta_0 + \zeta_1, \sigma_0 + \sigma_1)$. Then

\[
\langle \zeta, \eta \rangle_{L^2 \Lambda^{k+1}(T)} + \langle \zeta, \tau \rangle_{L^2 \Lambda^{k-1}(T)} = \langle \eta, \eta \rangle_{L^2 \Lambda^{k+1}(T)} + \langle \tau, \tau \rangle_{L^2 \Lambda^{k-1}(T)},
\]

and

\[
\|\zeta\|_{L^2 \Lambda^{k+1}(T)} + \|\eta\|_{L^2 \Lambda^{k-1}(T)} \leq C_{\kappa,n}(\|\eta\|_{L^2 \Lambda^{k+1}(T)} + \|\tau\|_{L^2 \Lambda^{k-1}(T)}),
\]

for some $C_{\kappa,n}$ depending on the shape regularity of $T$. This proves

\[
\inf_{(\eta,\tau)\in \mathcal{R}(T, P_{\delta \times d}^{m+\delta} \Lambda^k(T))} \sup_{(\zeta,\sigma)\in \mathcal{N}(\mathbb{T}, P_{\delta \times d}^{m+\delta} \Lambda^k(T))} \frac{\langle \zeta, \eta \rangle_{L^2 \Lambda^{k+1}(T)} + \langle \zeta, \tau \rangle_{L^2 \Lambda^{k-1}(T)}}{\|\zeta\|_{L^2 \Lambda^{k+1}(T)} + \|\zeta\|_{L^2 \Lambda^{k-1}(T)}} \geq C_{\kappa,n} > 0.
\]

The other part of the assertion follows the same way. The proof is completed. \qed

Denote $P_{\delta \times d}^{m+\delta} \Lambda^k(\mathcal{G}_h) := \bigcap_{T \in \mathcal{G}_h} P_{\delta \times d}^{m+\delta} \Lambda^k(T)$ and $P_{\delta \times d}^k(\mathcal{G}_h) := \bigcap_{T \in \mathcal{G}_h} P_{\delta \times d}^k(T)$.

Lemma 3.5. The pair $[(\mathbf{T}_h, P_{\delta \times d}^{m+\delta} \Lambda^k(\mathcal{G}_h)) : L^2 \Lambda^k \to L^2 \Lambda^{k+1} \times L^2 \Lambda^{k-1}, (\mathbf{T}_h, P_{\delta \times d}^k(\mathcal{G}_h)) : L^2 \Lambda^{k+1} \times L^2 \Lambda^{k-1} \to L^2 \Lambda^k]$ is a base operator pair.

Proof. Firstly, given $\mu_h \in P_{\delta \times d}^{m+\delta} \Lambda^k(\mathcal{G}_h), \mu_h = 0$ if and only if

\[
\sum_{T \in \mathcal{G}_h} \langle d^k \mu_h, \eta_h \rangle_{L^2 \Lambda^{k+1}(T)} + \langle d^k \mu_h, \tau_h \rangle_{L^2 \Lambda^{k-1}(T)} - \langle \mu_h, (d^k \eta_h + d^{k-1} \tau_h) \rangle_{L^2 \Lambda^k(T)} = 0, \forall (\eta_h, \tau_h) \in P_{\delta \times d}^k(\mathcal{G}_h).
\]

Similarly, given $(\eta_h, \tau_h) \in P_{\delta \times d}^k(\mathcal{G}_h), (\eta_h, \tau_h) = (0, 0)$ if and only if

\[
\sum_{T \in \mathcal{G}_h} \langle d^k \mu_h, \eta \rangle_{L^2 \Lambda^{k+1}(T)} + \langle d^k \mu_h, \tau \rangle_{L^2 \Lambda^{k-1}(T)} - \langle \mu_h, (d^k \eta + d^{k-1} \tau) \rangle_{L^2 \Lambda^k(T)} = 0, \forall \mu_h \in P_{\delta \times d}^{m+\delta} \Lambda^k(\mathcal{G}_h).
\]
Therefore, the twisted part of \((P_{d+\delta}^n \Lambda^k(G_h), P_{\delta \times d}^k(G_h))\) is the pair itself.

It is easy to obtain:

\[
\mathcal{R}(T, P_{d+\delta}^n \Lambda^k(G_h)) = \prod_{T \in G_h} \mathcal{R}(T, P_{d+\delta}^n \Lambda^k(T)), \quad \mathcal{N}(T, P_{d+\delta}^n \Lambda^k(G_h)) = \prod_{T \in G_h} \mathcal{N}(T, P_{d+\delta}^n \Lambda^k(T)),
\]

\[
\mathcal{R}(T, P_{\delta \times d}^k(G_h)) = \prod_{T \in G_h} \mathcal{R}(T, P_{\delta \times d}^k(T)), \quad \mathcal{N}(T, P_{\delta \times d}^k(G_h)) = \prod_{T \in G_h} \mathcal{N}(T, P_{\delta \times d}^k(T)).
\]

Therefore, by Lemma 3.4,

\[
\mathcal{N}(T, P_{d+\delta}^n \Lambda^k(G_h)) = \mathcal{R}(T, P_{\delta \times d}^k(G_h)),
\]

and

\[
\inf_{(\eta, \sigma) \in \mathcal{R}(T, P_{d+\delta}^n \Lambda^k(G_h))} \sup_{(\zeta, \sigma) \in \mathcal{N}(T, P_{\delta \times d}^k(G_h))} \left( \langle \zeta, \eta \rangle_{L^2 \Lambda^{k+1}} + \langle \sigma, \tau \rangle_{L^2 \Lambda^{k-1}} \right) \geq C_{k,h},
\]

Therefore \([T, P_{d+\delta}^n \Lambda^k(G_h)] : L^2 \Lambda^k \to L^2 \Lambda^{k+1} \times L^2 \Lambda^{k-1}, (T, P_{\delta \times d}^k(G_h)) : L^2 \Lambda^{k+1} \times L^2 \Lambda^{k-1} \to L^2 \Lambda^k\) is a base operator pair by Definition 2.1. The proof is completed.

\[\square\]

**Lemma 3.6.** For \([(T_h, D_h), (T_h, D_h)]\) partially adjoint based on \([(T_h, P_{d+\delta}^n \Lambda^k(G_h)), (T_h, P_{\delta \times d}^k(G_h))]\), if \(\text{icr}(T_h, D_h) < \infty\),

\[\text{(3.3) } \text{icr}(T_h, D_h) < 2 \cdot \text{icr}(T_h, P_{d+\delta}^n \Lambda^k(G_h)) + \text{icr}(T_h, D_h).\]

**Proof.** The lemma can be proved by noting \((T, \widetilde{M}) = (T_h, P_{d+\delta}^n \Lambda^k(G_h)), (T, \widetilde{N}) = (T_h, P_{\delta \times d}^k(G_h))\) and \(\widetilde{M} = \{0\}\) in Theorem 2.4 with \(\alpha = 1\).

\[\square\]

### 3.3. Finite element space \(V_{d+\delta}^m \Lambda^k\) and discrete Poincaré inequality

On the subdivision \(G_h\) of \(\Omega\), define

\[\text{(3.4) } V_{d+\delta}^m \Lambda^k := \left\{ \mu_h \in P_{d+\delta}^m \Lambda^k(G_h) : \langle d^k \mu_h, \eta_h \rangle_{L^2 \Lambda^{k+1}} - \langle \mu_h, \delta_{k+1,h} \eta_h \rangle_{L^2 \Lambda^k} = 0, \forall \eta_h \in W_{h_0}^{a,b,c} \Lambda^{k+1}, \quad \text{and} \quad \langle \delta_{k,h} \mu_h, \tau_h \rangle_{L^2 \Lambda^{k-1}} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2 \Lambda^k} = 0, \forall \tau_h \in W_h \Lambda^{k-1} \right\}.\]

**Lemma 3.7.** The pair \([T, V_{d+\delta}^m \Lambda^k], (T, W_{h_0}^{a,b,c} \Lambda^{k+1} \times W_h \Lambda^{k-1})\) is partially adjoint based on \([T_h, P_{d+\delta}^n \Lambda^k(G_h)), (T_h, P_{\delta \times d}^k(G_h))]\)
Proof. By the definition (3.4),
\[ V_{d^c\sigma}^{m+\delta} \Lambda^k = \left\{ \mu_h \in P_{d^c\sigma}^{m+\delta} \Lambda^k (G_h) : \langle T_h \mu_h, (\eta_h, \tau_h) \rangle - \langle \mu_h, \nabla (\eta_h, \tau_h) \rangle = 0, \quad \forall (\eta_h, \tau_h) \in W_{h^0}^{\tau, abc} \Lambda^{k+1} \times W_{h^0} \Lambda^{k-1} \right\}. \]

On the other hand, by Lemma 3.3,
\[ W_{h^0}^{\tau, abc} \Lambda^{k+1} \times W_{h^0} \Lambda^{k-1} = \left\{ (\eta_h, \tau_h) \in P_{d^c\sigma}^k (G_h) : \langle T_h \mu_h, (\eta_h, \tau_h) \rangle - \langle \mu_h, \nabla (\eta_h, \tau_h) \rangle = 0, \quad \mu_h \in V_{d^c\sigma}^{m+\delta} \Lambda^k \right\}. \]

The assertion follows by Definition 2.2. The proof is completed. \hfill \Box

Lemma 3.8. \( N(T_h, V_{d^c\sigma}^{m+\delta} \Lambda^k) = \mathfrak{S}_h \Lambda^k \).

Proof. By Lemma 2.12,
\[ \mathfrak{S}_h \Lambda^k = \left\{ \mu_h \in P_0 \Lambda^k (G_h) : \langle \mu_h, \delta_{k+1,h} \eta_h \rangle_{L^2 \Lambda} = 0, \quad \forall \eta_h \in W_{h^0}^{\tau, abc} \Lambda^{k+1}, \quad \right\} \]

and (3.4))

The proof is completed. \hfill \Box

Lemma 3.9. There exists a constant \( C_{k,n} \), depending on the regularity of \( G_h \), such that
\[ \| \mu_h \|_{L^2 \Lambda^1} \leq C_{k,n}(\| d^c_k \mu_h \|_{L^2 \Lambda^{k+1}} + \| \delta_{k,h} \mu_h \|_{L^2 \Lambda^{k-1}}), \quad \forall \mu_h \in V_{d^c\sigma}^{m+\delta} \Lambda^k \cap \mathfrak{S}_h \Lambda^k. \]

Proof. By Lemma 2.7 and by the definition of \( \text{icr} \), \( \text{icr}(T, P_{d^c\sigma}^{m+\delta} \Lambda^k(T)) \leq C_{k,n}^\prime hT \), with \( C_{k,n}^\prime \) depending on the regularity of \( T \). Noting that \( P_{d^c\sigma}^{m+\delta} \Lambda^k (G_h) = \prod_{T \in G_h} P_{d^c\sigma}^{m+\delta} \Lambda^k (T) \), \( \text{icr}(T_h, P_{d^c\sigma}^{m+\delta} \Lambda^k (G_h)) = \sup_{T \in G_h} \text{icr}(T, P_{d^c\sigma}^{m+\delta} \Lambda^k(T)) \leq C_{k,n}^\prime > 0 \), with \( C_{k,n}^\prime \) depending on the shape regularity of \( G_h \), and if \( G_h \) is quasi-uniform, \( \text{icr}(T_h, P_{d^c\sigma}^{m+\delta} \Lambda^k (G_h)) \leq C_{k,n}^\prime \).

By Lemma 2.9, noting that \( N(T_h, W_{h^0}^{\tau, abc} \Lambda^{k+1} \times W_{h^0} \Lambda^{k-1}) = N(\delta_{k+1,h}, W_{h^0}^{\tau, abc} \Lambda^{k+1} \times N(d^c_k, W_{h^0} \Lambda^{k-1})) \), we have \( \text{icr}(T_h, W_{h^0}^{\tau, abc} \Lambda^{k+1} \times W_{h^0} \Lambda^{k-1}) \leq C_{k,n}^\prime > 0 \). Therefore, \( \text{icr}(T_h, V_{d^c\sigma}^{m+\delta}) \leq 2 \text{icr}(T_h, P_{d^c\sigma}^{m+\delta} \Lambda^k (G_h)) + \text{icr}(T_h, W_{h^0}^{\tau, abc} \Lambda^{k+1} \times W_{h^0} \Lambda^{k-1}) + C_{k,n}^\prime \leq C_{k,n}. \)

Namely
\[ \| \mu_h \|_{L^2 \Lambda^1} \leq C_{k,n}(\| d^c_k \mu_h \|_{L^2 \Lambda^{k+1}} + \| \delta_{k,h} \mu_h \|_{L^2 \Lambda^{k-1}}), \quad \forall \mu_h \in V_{d^c\sigma}^{m+\delta} \Lambda^k \cap \mathfrak{S}_h \Lambda^k. \]

The proof is completed. \hfill \Box

4. A Primal Finite Element Scheme of the Hodge Laplace Problem

We consider the finite element problem: find \( \omega_h \in V_{d^c\sigma}^{m+\delta} \Lambda^k \), such that
\[ \left\{ \begin{array}{l}
\langle \omega_h, \mathfrak{s}_h \rangle_{L^2 \Lambda^1} = 0, \\
\langle d^c_k \omega_h, d^c_k \mu_h \rangle_{L^2 \Lambda^1} + \langle \delta_{k,h} \omega_h, \delta_{k,h} \mu_h \rangle_{L^2 \Lambda^1} = \langle f - P_{\mathfrak{S}_h \Lambda^k} f, \mu_h \rangle_{L^2 \Lambda^1}, \quad \forall \mathfrak{s}_h \in \mathfrak{S}_h \Lambda^k, \\
\end{array} \right. \]

\( P_{\mathfrak{S}_h \Lambda^k} \) denotes the projection to \( \mathfrak{S}_h \Lambda^k \).
Lemma 4.1. The system (4.1) is well posed.

Proof. The existence of a unique solution to (4.1) is straightforward by Lemma 3.9 the discrete Poincaré inequality. Further, it holds that
\[
\|\omega_h\|_{L^2(A^k)} + \|d_h^k\omega_h\|_{L^2(A^k+1)} + \|\delta_h\omega_h\|_{L^2(A^k+1)} \leq C_{k,h} \frac{\langle f, \mu_h \rangle}{\|d_h^k\mu_h\|_{L^2(A^k+1)} + \|\delta_h\mu_h\|_{L^2(A^k+1)}},
\]
for any \(\mu_h \in V^{m,+}_h \cap \mathcal{H}_h \Lambda^k\). The proof is completed. \(\Box\)

4.1. Error estimation. Given \(f \in L^2(A^k)\), an equivalent formulation of (1.1) is to find \(\omega \in H\Lambda^k(\Omega) \cap H^0_0(\Omega)\) and \(\vartheta \in \mathcal{H}_h \Lambda^k(\Omega)\), such that
\[
\langle \omega, \vartheta \rangle_{L^2(A^k)} = 0, \quad \forall \vartheta \in \mathcal{H}_h \Lambda^k, \quad \text{and}
\]
\[
\langle \vartheta, \mu \rangle_{L^2(A^k)} + \langle d^k\vartheta, d^k\mu \rangle_{L^2(A^k+1)} + \langle \delta_h\vartheta, \delta_h\mu \rangle_{L^2(A^k+1)} = \langle f, \mu \rangle_{L^2(A^k)}, \quad \forall \mu \in H\Lambda^k \cap H^0_0\Lambda^k.
\]

We consider the finite element problem for (4.2): find \(\omega_h \in V^{m,+}_d \Lambda^k\) and \(\vartheta_h \in \mathcal{H}_h \Lambda^k\), such that
\[
\langle \omega_h, \vartheta_h \rangle_{L^2(A^k)} = 0, \quad \forall \vartheta_h \in \mathcal{H}_h \Lambda^k,
\]
\[
\langle \vartheta_h, \mu_h \rangle_{L^2(A^k)} + \langle d_h^k\omega_h, d_h^k\mu_h \rangle_{L^2(A^k+1)} + \langle \delta_h\omega_h, \delta_h\mu_h \rangle_{L^2(A^k+1)} = \langle f, \mu_h \rangle_{L^2(A^k)}, \quad \forall \mu_h \in V^{m,+}_d \Lambda^k.
\]
Then evidently (4.1) and (4.3) are equivalent, as the solutions (the part of \(\omega_h\)) are equal. The main technical results of this paper are Lemmas 4.2 and 4.4 below.

Lemma 4.2. Let \((\vartheta, \omega)\) and \((\vartheta_h, \omega_h)\) be the solutions of (4.2) and (4.3), respectively. Then
\[
\|\omega - \omega_h\|_{L^2(A^k)} + \|d_h^k(\omega - \omega_h)\|_{L^2(A^k+1)} + \|\delta_h(\omega - \omega_h)\|_{L^2(A^k+1)} + \|\vartheta - \vartheta_h\|_{L^2(A^k)} \leq C \left( \inf_{\tau_h \in W_h \Lambda^k} \|\delta_h\omega - \tau_h\|_{L^2(A^k)} + \inf_{\mu_h \in W_h \Lambda^k} \|\omega - \mu_h\|_{L^2(A^k)} + \inf_{\vartheta_h \in \mathcal{H}_h \Lambda^k} \|\vartheta - \vartheta_h\|_{L^2(A^k)} + h\|f\|_{L^2(A^k)} \right).
\]

Here and in the sequel, denote \(\|\cdot\|_{A^k} \equiv (\|\cdot\|_{L^2(A^k)}^2 + \|d^k\cdot\|_{L^2(A^k+1)}^2)^{1/2}\), and similar is \(\|\cdot\|_{A^k}\).

A domain \(\Omega\) is called \(s\)-regular if, for some \(0 < s \leq 1\), for \(\omega \in H^s(\Omega) \cap H^0_0(\Omega)\) or \(H^s(\Omega) \cap H^0(\Omega)\),
\[
\|\omega\|_{H^s(A^k)} \leq C(\|\omega\|_{L^2(A^k)} + \|d_h^k\omega\|_{L^2(A^k+1)} + \|\delta_h\omega\|_{L^2(A^k+1)}),
\]
A smoothly bounded domain is 1-regular and a Lipschitz domain is 1/2-regular. [3]

Lemma 4.3. [3] Let \(\Omega\) be \(s\)-regular. Let \(\omega\) be the solution of (4.1). Then
\[
\|\omega\|_{H^s(A^k)} + \|d^k\omega\|_{H^s(A^k+1)} + \|\delta_h\omega\|_{H^s(A^k+1)} \leq C\|f\|_{L^2(A^k)}.
\]

Lemma 4.4. Let \(\Omega\) be \(s\)-regular. Let \((\vartheta, \omega)\) and \((\vartheta_h, \omega_h)\) be the solutions of (4.2) and (4.3), respectively. Then
\[
\|\omega - \omega_h\|_{L^2(A^k)} + \|d_h^k(\omega - \omega_h)\|_{L^2(A^k+1)} + \|\delta_h(\omega - \omega_h)\|_{L^2(A^k+1)} + \|\vartheta - \vartheta_h\|_{L^2(A^k)} \leq Ch^s\|f\|_{L^2(A^k)}.
\]
We postpone the proofs of Lemmas 4.2 and 4.4 into Section 4.1.2 after some technical preparations. The main result of the paper, the theorem below, follows by Lemmas 4.2 and 4.4 directly.
\textbf{Theorem 4.5.} Let $\omega$ and $\omega_h$ be the solutions of (1.1) and (4.1), respectively. Then
\begin{equation}
\|\omega - \omega_h\|_{L^2\Lambda^k} + \|d^k(\omega - \omega_h)\|_{L^2\Lambda^{k+1}} + \|\delta_k(\omega - \omega_h)\|_{L^2\Lambda^{k-1}} \\
\leq C \left( \inf_{\tau_h \in W_{h}\Lambda^k} \|\delta_k \omega - \tau_h\|_{d^{k-1}} + \inf_{\mu_h \in W_{h}\Lambda^k} \|\omega - \mu_h\|_{d^k} + \inf_{\varsigma_h \in \mathcal{G}_h} \|P_{\mathcal{G}_h} f - \varsigma_h\|_{L^2\Lambda^k} + h\|f\|_{L^2\Lambda^k} \right).
\end{equation}

Let $\Omega$ be $s$-regular. Then
\begin{equation}
\|\omega - \omega_h\|_{L^2\Lambda^k} + \|d^k(\omega - \omega_h)\|_{L^2\Lambda^{k+1}} + \|\delta_k(\omega - \delta_k(\omega_h))\|_{L^2\Lambda^{k-1}} \leq Ch^s\|f\|_{L^2\Lambda^k}.
\end{equation}

\textbf{Remark 4.6.} We here discuss the global finite element space based on local shape function space $P_{d^{\delta}}\Lambda^k(T)$ and for the continuous space $H^1(\Omega) \cap H^1_0\Lambda^k$. The discussions on global finite element space based on local shape function space $P_{d^{\delta}}\Lambda^k(T)$ and for the continuous space $H^1(\Omega) \cap H^1_0\Lambda^k$ can be carried out in quite a symmetric way.

4.1.1. An auxiliary scheme. We use the classical mixed method of the Hodge-Laplace problem as an auxiliary scheme. The mixed formulation of the Hodge-Laplace problem reads: find $(\tilde{\theta}, \tilde{\sigma}, \tilde{\omega}) \in \mathcal{G}_h^k \times H^1(\Omega) \times H^1_0\Lambda^k$ such that
\begin{equation}
\begin{cases}
\langle \tilde{\omega}, \varsigma_2 \rangle_{L^2\Lambda^k} = 0 & \forall \varsigma_2 \in \mathcal{G}_h^k \\
-\langle \tilde{\omega}, d^{-1}\tau \rangle_{L^2\Lambda^k} = 0 & \forall \tau \in H^1(\Omega) \\
\langle \tilde{\sigma}, \mu \rangle_{L^2\Lambda^k} + \langle d^{-1}\tilde{\sigma}, \mu_h \rangle_{L^2\Lambda^k} + \langle d^k\tilde{\omega}, d^k\mu \rangle_{L^2\Lambda^{k+1}} = \langle f, \mu \rangle_{L^2\Lambda^k} & \forall \mu \in H^1(\Omega),
\end{cases}
\end{equation}

and the corresponding discretization is to find $(\tilde{\theta}_h, \tilde{\sigma}_h, \tilde{\omega}_h) \in \mathcal{G}_h^k \times W_{h\Lambda^k} \times W_{h\Lambda^k}$ such that
\begin{equation}
\begin{cases}
\langle \tilde{\omega}_h, \varsigma_2 \rangle_{L^2\Lambda^k} = 0 & \forall \varsigma_2 \in \mathcal{G}_h^k \\
-\langle \tilde{\omega}_h, d^{-1}\tau_h \rangle_{L^2\Lambda^k} = 0 & \forall \tau_h \in W_{h\Lambda^k} \\
\langle \tilde{\sigma}_h, \mu_h \rangle_{L^2\Lambda^k} + \langle d^{-1}\tilde{\sigma}_h, \mu_h \rangle_{L^2\Lambda^k} + \langle d^k\tilde{\omega}_h, d^k\mu_h \rangle_{L^2\Lambda^{k+1}} = \langle f, \mu_h \rangle_{L^2\Lambda^k} & \forall \mu_h \in W_{h\Lambda^k},
\end{cases}
\end{equation}

\textbf{Lemma 4.7.} Let $(\theta, \omega)$ and $(\tilde{\theta}, \tilde{\sigma}, \tilde{\omega})$ be the solutions of (4.2) and (4.9), respectively; then
(1) $\omega = \tilde{\omega}$, $\theta = \tilde{\theta}$, and $\sigma = \delta_k\omega$;
(2) let $(\tilde{\theta}_h, \tilde{\sigma}_h, \tilde{\omega}_h)$ be the solution of (4.10); then
\begin{equation}
\begin{aligned}
||\tilde{\sigma} - \tilde{\sigma}_h||_{d^{k-1}} + ||\tilde{\omega} - \tilde{\omega}_h||_{d^k} + ||\tilde{\vartheta} - \tilde{\vartheta}_h||_{L^2\Lambda^k} \\
\leq C \left( \inf_{\tau_h \in W_{h}\Lambda^k} \|\tilde{\sigma} - \tau_h\|_{d^{k-1}} + \inf_{\mu_h \in W_{h}\Lambda^k} \|\tilde{\omega} - \mu_h\|_{d^k} + \inf_{\varsigma_h \in \mathcal{G}_h} \|\tilde{\vartheta} - \varsigma_h\|_{L^2\Lambda^k} + h\|f\|_{L^2\Lambda^k} \right);
\end{aligned}
\end{equation}
(3) for any $f \in L^2\Lambda^k$ and $\Omega$ being $s$-regular,
\begin{equation}
||\tilde{\sigma} - \tilde{\sigma}_h||_{L^2\Lambda^k} + ||\tilde{\omega} - \tilde{\omega}_h||_{d^k} + ||\tilde{\vartheta} - \tilde{\vartheta}_h||_{L^2\Lambda^k} \leq Ch^s\|f\|_{L^2\Lambda^k}.
\end{equation}

\textbf{Proof.} The proof can be found in [3], particularly (7.17), (7.30) and Theorem 7.10 therein. $\square$

We here note that $\tilde{\vartheta} = P_{\mathcal{G}_h^k}f$. Denote $P_{0\Lambda^k}$ the $L^2$ projection to $P_{0\Lambda^k}(\mathcal{G}_h).$
Lemma 4.8. Given $f_0 \in P_0 \Lambda^k(\mathcal{G}_h)$, let $(\tilde{\theta}_h, \tilde{\sigma}_h, \tilde{\omega}_h) \in \mathcal{Y}_h \Lambda^k \times W_h \Lambda^{k-1} \times W_h \Lambda^k$ be such that

$$
(4.11) \begin{cases}
\langle \tilde{\omega}_h, \zeta \rangle_{L^2, \Lambda^k} = 0 & \forall \zeta \in \mathcal{Y}_h \Lambda^k \\
-\langle \tilde{\omega}_h, \tau \rangle_{L^2, \Lambda^k} = 0 & \forall \tau \in W_h \Lambda^{k-1} \\
\langle \tilde{\theta}_h, \mu \rangle_{L^2, \Lambda^k} + \langle d^k \tilde{\omega}_h, d^k \mu \rangle_{L^2, \Lambda^{k+1}} = \langle f_0, \mu \rangle_{L^2, \Lambda^k} & \forall \mu \in W_h \Lambda^k,
\end{cases}
$$

and let $(\bar{\theta}_h, \bar{\omega}_h) \in \mathcal{Y}_h \Lambda^k \times V_{d^k \delta}^{m+\delta} \Lambda^k$ be such that

$$
(4.12) \begin{cases}
\langle \bar{\omega}_h, \zeta \rangle_{L^2, \Lambda^k} = 0 & \forall \zeta \in \mathcal{Y}_h \Lambda^k \\
\langle \bar{\theta}_h, \mu \rangle_{L^2, \Lambda^k} + \langle d^k \bar{\omega}_h, d^k \mu \rangle_{L^2, \Lambda^{k+1}} + \langle \delta_{k,h} \bar{\omega}_h, \delta_{k,h} \mu \rangle_{L^2, \Lambda^{k-1}} = \langle f_0, \mu \rangle_{L^2, \Lambda^k} & \forall \mu \in V_{d^k \delta}^{m+\delta} \Lambda^k.
\end{cases}
$$

Then

$$
(4.13) \quad \bar{\theta}_h = \bar{\theta}_h, \quad \delta_{k,h} \bar{\omega}_h = \tilde{\sigma}_h, \quad d^k \bar{\omega}_h = d^k \tilde{\omega}_h, \quad \text{and} \quad P_0^k \bar{\omega}_h = P_0^k \tilde{\omega}_h.
$$

Proof. Firstly,

$$
\langle d^k \tilde{\omega}_h, \eta \rangle_{L^2, \Lambda^k} = 0, \quad \forall \eta \in W_{h0}^{r,abc} \Lambda^{k+1},
$$

and it holds with some $\zeta \in W_{h0}^{r,abc} \Lambda^{k+1}$ and for any $\mu \in P_1 \Lambda^k(\mathcal{G}_h)$ that

$$
\langle \tilde{\theta}_h, \mu \rangle_{L^2, \Lambda^k} + \langle d^k \tilde{\omega}_h, \mu \rangle_{L^2, \Lambda^k} = \langle \zeta, \mu \rangle_{L^2, \Lambda^{k+1}} = \langle f_0, \mu \rangle_{L^2, \Lambda^k}.
$$

Now we choose arbitrarily $\mu \in P_0 \Lambda^k(\mathcal{G}_h)$, and obtain

$$
\bar{\theta}_h + d^k \tilde{\omega}_h = f_0,
$$

and thus

$$
\langle d^k \bar{\omega}_h, d^k \mu \rangle_{L^2, \Lambda^{k+1}} = 0, \quad \forall \mu \in P_1 \Lambda^k(\mathcal{G}_h),
$$

which leads to further that

$$
\bar{\theta}_h = d^k \bar{\omega}_h = P_0^k \zeta.
$$

Therefore, it holds for $\zeta \in \mathcal{Y}_h \Lambda^k$, $\tau \in W_h \Lambda^{k-1}$, $\eta \in W_{h0}^{r,abc} \Lambda^{k+1}$ and $\mu \in P_0 \Lambda^k(\mathcal{G}_h)$ that

$$
(4.14) \begin{cases}
\langle \tilde{\sigma}_h, \tau \rangle_{L^2, \Lambda^k} = 0 \\
-\langle \tilde{\omega}_h, \tau \rangle_{L^2, \Lambda^k} = 0 \\
\langle \tilde{\theta}_h, \mu \rangle_{L^2, \Lambda^k} + \langle d^k \tilde{\omega}_h, \mu \rangle_{L^2, \Lambda^{k+1}} + \langle \delta_{k+1,h} \zeta, \mu \rangle_{L^2, \Lambda^{k-1}} = \langle f_0, \mu \rangle_{L^2, \Lambda^k}.
\end{cases}
$$

By Lemma 2.12 the discrete Hodge decomposition, (4.14) is well-posed. By Lemma 2.5, there exists a unique $\bar{\omega}_h \in V_{d^k \delta}^{m+\delta} \Lambda^k(\mathcal{G}_h)$, such that

$$
(4.15) \quad P_0^k \tilde{\omega}_h = P_0^k \bar{\omega}_h, \quad \delta_{k,h} \bar{\omega}_h = \tilde{\sigma}_h, \quad \text{and} \quad d^k \bar{\omega}_h = P_0^k \zeta.
$$

Then by (4.14), $\tilde{\omega}_h \in V_{d^k \delta}^{m+\delta} \Lambda^k$, as $\langle d^k \tilde{\omega}_h, \eta \rangle_{L^2, \Lambda^{k+1}} = 0, \forall \eta \in W_{h0}^{r,abc} \Lambda^{k+1}$, and $\langle \delta_{k+1,h} \zeta, \tau \rangle_{L^2, \Lambda^{k-1}} = 0, \forall \tau \in W_h \Lambda^{k-1}$. Further, by Lemma 3.8, it is easy to verify that $(\bar{\theta}_h, \bar{\omega}_h)$ satisfies (4.12). The proof is completed. \qed

4.1.2. Proofs of Lemmas 4.2 and 4.4.
Proof of Lemma 4.2. Denote $f_0 := P_h^k f$. Let $(\bar{\varphi}_h, \bar{\omega}_h)$ be such that
\begin{equation}
\left\{ \begin{array}{l}
\langle \bar{\omega}_h, \bar{\varphi}_h \rangle_{L^2 \Lambda^k} = 0, \forall \bar{\varphi}_h \in S_h \Lambda^k, \\
\langle \delta_{\Lambda^k \bar{\omega}_h} - \delta_{\Lambda^k \varphi_h} \rangle_{L^2 \Lambda^k} = \langle f_0, \mu_h \rangle_{L^2 \Lambda^k}, \forall \mu_h \in V_{d,0}^m \Lambda^k.
\end{array} \right.
\end{equation}
Then $\bar{\varphi}_h = \varphi_h$, and, for any $\mu_h \in V_{d,0}^m \Lambda^k$, by Lemma 2.7,
\begin{equation}
\| \omega_h - \bar{\omega}_h \|_{L^2 \Lambda^k} + \| \delta_{\Lambda^k \omega_h} - \delta_{\Lambda^k \bar{\omega}_h} \|_{L^2 \Lambda^k} \leq C h \| f \|_{L^2 \Lambda^k}.
\end{equation}
Let $(\bar{\varphi}_h, \bar{\sigma}_h, \bar{\omega}_h) \in S_h \Lambda^k \times W_h \Lambda^{k-1} \times W_h \Lambda^k$ be such that
\begin{equation}
\left\{ \begin{array}{l}
\langle \bar{\omega}_h, \bar{\varphi}_h \rangle_{L^2 \Lambda^k} = 0 \forall \bar{\varphi}_h \in S_h \Lambda^k, \\
\langle \delta_{\Lambda^k \bar{\omega}_h} - \delta_{\Lambda^k \varphi_h} \rangle_{L^2 \Lambda^k} = \langle f_0, \mu_h \rangle_{L^2 \Lambda^k} \forall \mu_h \in W_h \Lambda^k.
\end{array} \right.
\end{equation}
Then, by Lemma 4.8,
\begin{equation}
\bar{\varphi}_h = \bar{\varphi}_h, \quad \delta_{\Lambda^k \bar{\omega}_h} = \delta_{\Lambda^k \bar{\omega}_h}, \quad \delta_{\Lambda^k \bar{\omega}_h} = \delta_{\Lambda^k \bar{\omega}_h}, \quad \text{and} \quad P_h^k \bar{\omega}_h = P_h^k \bar{\omega}_h.
\end{equation}
Let $(\bar{\varphi}_h, \bar{\sigma}_h, \bar{\omega}_h) \in S_h \Lambda^k \times W_h \Lambda^{k-1} \times W_h \Lambda^k$ be such that
\begin{equation}
\left\{ \begin{array}{l}
\langle \bar{\omega}_h, \bar{\sigma}_h \rangle_{L^2 \Lambda^k} = 0 \forall \bar{\sigma}_h \in S_h \Lambda^k, \\
\langle \delta_{\Lambda^k \bar{\omega}_h} - \delta_{\Lambda^k \varphi_h} \rangle_{L^2 \Lambda^k} = \langle f, \mu_h \rangle_{L^2 \Lambda^k} \forall \mu_h \in W_h \Lambda^k.
\end{array} \right.
\end{equation}
Then $\bar{\varphi}_h = \bar{\varphi}_h$, and, by Lemma 2.7,
\begin{equation}
\| \bar{\omega}_h - \bar{\omega}_h \|_{d^k \Lambda^k} + \| \bar{\varphi}_h - \bar{\varphi}_h \|_{d^k \Lambda^k} \leq C \sup_{\mu_h \in W_h \Lambda^k} \frac{\langle f - f_0, \mu_h \rangle_{L^2 \Lambda^k}}{\| \mu_h \|_{d^k \Lambda^k}} = C \sup_{\mu_h \in W_h \Lambda^k} \frac{\langle f, \mu_h - P_h^k \mu_h \rangle_{L^2 \Lambda^k}}{\| \mu_h \|_{d^k \Lambda^k}} \leq C h \| f \|_{L^2 \Lambda^k}.
\end{equation}
Now, by Lemma 4.7,
\begin{equation}
\| \bar{\sigma} - \bar{\sigma}_h \|_{d^k \Lambda^k} + \| \bar{\varphi} - \bar{\omega}_h \|_{d^k \Lambda^k} + \| \bar{\varphi} - \bar{\varphi}_h \|_{d^k \Lambda^k} \leq C \left( \inf_{\tau_h \in W_h \Lambda^{k-1}} \| \bar{\sigma} - \tau_h \|_{d^k \Lambda^k} + \inf_{\mu_h \in W_h \Lambda^k} \| \bar{\omega} - \mu_h \|_{d^k \Lambda^k} + \inf_{\omega_h \in S_h \Lambda^k} \| \bar{\varphi} - \omega_h \|_{L^2 \Lambda^k} + h \| f \|_{L^2 \Lambda^k} \right).
\end{equation}
and thus
\begin{equation}
\| \bar{\sigma} - \bar{\sigma}_h \|_{d^k \Lambda^k} + \| \bar{\varphi} - \bar{\omega}_h \|_{d^k \Lambda^k} + \| \bar{\varphi} - \bar{\varphi}_h \|_{d^k \Lambda^k} \leq C \left( \inf_{\tau_h \in W_h \Lambda^{k-1}} \| \bar{\sigma} - \tau_h \|_{d^k \Lambda^k} + \inf_{\mu_h \in W_h \Lambda^k} \| \bar{\omega} - \mu_h \|_{d^k \Lambda^k} + \inf_{\omega_h \in S_h \Lambda^k} \| \bar{\varphi} - \omega_h \|_{L^2 \Lambda^k} + h \| f \|_{L^2 \Lambda^k} \right).
\end{equation}
By standard techniques,
\begin{equation}
\| \omega - \bar{\omega}_h \|_{L^2 \Lambda^k} \leq \| \omega - \bar{\omega}_h \|_{L^2 \Lambda^k} + C h \| f \|_{L^2 \Lambda^k}.
\end{equation}
Therefore,

\[ ||\omega - \tilde{\omega}_h||_{L^2(A^k)} + ||d^k_h(\omega - \tilde{\omega}_h)||_{L^2(A^{k+1})} + ||\delta_{k,h}(\omega - \tilde{\omega}_h)||_{L^2(A^{k-1})} \]

\[ = ||\omega - \tilde{\omega}_h||_{L^2(A^k)} + ||d^k\omega - d^k\tilde{\omega}_h||_{L^2(A^{k+1})} + ||\tilde{\sigma} - \sigma_h||_{L^2(A^{k-1})} \quad \text{(by (4.19))} \]

\[ \leq ||\omega - \tilde{\omega}_h||_{L^2(A^k)} + ||d^k\omega - d^k\tilde{\omega}_h||_{L^2(A^{k+1})} + ||\tilde{\sigma} - \sigma_h||_{L^2(A^{k-1})} + Ch||f||_{L^2(A^k)}. \]

And finally,

\[ ||\omega - \omega_h||_{L^2(A^k)} + ||d^k_h(\omega - \omega_h)||_{L^2(A^{k+1})} + ||\delta_{k,h}(\omega - \omega_h)||_{L^2(A^{k-1})} + ||\tilde{\sigma} - \sigma_h||_{L^2(A^k)} \]

\[ \leq ||\omega - \tilde{\omega}_h||_{L^2(A^k)} + ||d^k\omega - d^k\tilde{\omega}_h||_{L^2(A^{k+1})} + ||\tilde{\sigma} - \sigma_h||_{L^2(A^{k-1})} + ||\tilde{\sigma} - \sigma_h||_{L^2(A^k)} + Ch||f||_{L^2(A^k)} \quad \text{(by (4.17))} \]

\[ \leq C(\inf_{\tau_h \in W_hA^{k-1}} ||\tilde{\sigma} - \tau_h||_{A^k} + \inf_{\tilde{\mu}_h \in W_hA^{k-1}} ||\tilde{\omega} - \mu_h||_{A^k} + \inf_{\tilde{\omega}_h \in \tilde{W}_hA^k} ||\tilde{\sigma} - \tilde{\omega}_h||_{L^2(A^k)} + h||f||_{L^2(A^k)}). \]

The proof is completed.

**Proof of Lemma 4.4.** For general \( f \), we would have, by Lemma 4.7, with notations defined in the proof of Lemma 4.2,

\[(4.23) \quad ||\tilde{\sigma} - \sigma_h||_{L^2(A^{k-1})} + ||\tilde{\omega} - \omega_h||_{A^k} + ||\tilde{\theta} - \theta_h||_{L^2(A^k)} \leq Ch||f||_{L^2(A^k)}.
\]

Take this into the place of (4.21), and (4.6) can be obtained by repeating the proof of Lemma 4.2. We omit the details here.

\[ \Box \]

4.2. Implementation of the scheme: locally supported basis functions of \( V_{d\omega}^{m+\delta}A^k \). The finite element space \( V_{d\omega}^{m+\delta}A^k \) does not correspond to a “finite element” defined as Ciarlet’s triple [12]. Though, in this section, we present a set of basis functions of \( V_{d\omega}^{m+\delta}A^k \) which are tightly supported. Therefore, with the space \( \tilde{W}_hA^k \) well studied, the finite element scheme can be implemented by the standard routine.

A general procedure is given in Section 4.2.1, and, for an illustration of the procedure, a two-dimensional example is given in Section 4.2.2, where we particularly refer to Figures 2 and 3 for the illustration of the local supports of the basis functions.

4.2.1. A general procedure. On a simplex \( T \), denote

\[ P_{d\omega,\delta}^{m+\delta}A^k(T) := \left\{ \mu \in P_{d\omega,\delta}^{m+\delta}A^k(T) : \langle \delta_k\mu, \tau \rangle_{L^2(A^{k-1})} = 0, \forall \tau \in P_1^{-}A^{k-1}(T) \right\}, \]

and

\[ P_{d\omega,\delta}^{m+\delta}A^k(T) := \left\{ \mu \in P_{d\omega,\delta}^{m+\delta}A^k(T) : \langle d^k\mu, \eta \rangle_{L^2(A^{k+1})} = 0, \forall \eta \in P_1^{-}A^{k+1}(T) \right\}. \]

Namely, with \( A^\alpha = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \),

\[ P_{d\omega,\delta}^{m+\delta}A^k(T) = \kappa_T (P_0A^{k+1}(T)) \oplus \left\{ A^\alpha + \sum_{\alpha' \in D_{2,n}} C_{\alpha' \alpha} \tilde{\mu}_{\alpha',T} : \alpha \in D_{2,n} \right\}, \]
where $C_{\alpha \alpha'}$ are chosen such that

$$
\left( \delta_h \left( \Lambda^\alpha + \sum_{\alpha' \in \mathcal{E}_d} C_{\alpha \alpha'} \overline{\mu}_{d,T}^{\alpha'} \right), \tau \right)_{L^2(\Lambda^k(T))} - \left( \Lambda^\alpha + \sum_{\alpha' \in \mathcal{E}_d} C_{\alpha \alpha'} \overline{\mu}_{d,T}^{\alpha'}, d^{k-1} \tau \right)_{L^2(\Lambda^k(T))} = 0, \quad \forall \tau \in \mathcal{P}_1 \Lambda^{k-1}(T),
$$

and

$$
\mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T) = \star \mathbf{K}_T \star (\mathcal{P}_0 \Lambda^{k-1}(T)) + \mathcal{H}^{2}_{\delta}(T).
$$

Then $\mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T)$ is unisolvent with respect to $\langle d^k \mu, \eta \rangle_{L^2(\Lambda^k)} - \langle \mu, \delta_{k+1} \eta \rangle_{L^2(\Lambda^k)}$ for $\eta \in \mathcal{P}^\ast_{-1} \Lambda^{k+1}(T)$, $\mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T)$ is unisolvent with respect to $\langle \delta_k \mu, \tau \rangle_{L^2(\Lambda^k)} - \langle \mu, d^{k-1} \tau \rangle_{L^2(\Lambda^k)} = 0$ for $\tau \in \mathcal{P}_1 \Lambda^{k-1}(T)$. Further,

$$
\mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T) = \mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T) \oplus \mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T).
$$

Denote $\mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(\mathcal{G}_h) = \prod_{T \in \mathcal{G}_h} \mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T)$ and $\mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(\mathcal{G}_h) = \prod_{T \in \mathcal{G}_h} \mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(T)$. Then

$$
(4.24) \quad \mathbf{V}^{m+\delta}_{d \cap \delta} \Lambda^k = \left\{ \mu_h \in \mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(\mathcal{G}_h) : \langle d^k \mu_h, \eta_h \rangle_{L^2(\Lambda^k)} - \langle \mu_h, \delta_{k+1} \eta_h \rangle_{L^2(\Lambda^k)} = 0, \quad \forall \eta_h \in W_{h0}^{r,abc} \Lambda^{k+1}, \right. \\
and \left. \langle \delta_k \mu_h, \tau_h \rangle_{L^2(\Lambda^k)} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2(\Lambda^k)} = 0, \quad \forall \tau_h \in \mathcal{P}_1 \Lambda^{k-1} \right\}
$$

$$
= \left\{ \mu_h \in \mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(\mathcal{G}_h) : \langle d^k \mu_h, \eta_h \rangle_{L^2(\Lambda^k)} - \langle \mu_h, \delta_{k+1} \eta_h \rangle_{L^2(\Lambda^k)} = 0, \quad \forall \eta_h \in W_{h0}^{r,abc} \Lambda^{k+1} \right\} \\
\oplus \left\{ \mu_h \in \mathbf{P}^{m+\delta}_{d \cap \delta} \Lambda^k(\mathcal{G}_h) : \langle \delta_k \mu_h, \tau_h \rangle_{L^2(\Lambda^k)} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2(\Lambda^k)} = 0, \quad \forall \tau_h \in \mathcal{P}_1 \Lambda^{k-1} \right\} =: \mathbf{V}_d + \mathbf{V}_\delta.
$$

Now we figure out the basis functions of $\mathbf{V}_d$ and $\mathbf{V}_\delta$ respectively. Their combination is the set of basis functions of $\mathbf{V}^{m+\delta}_{d \cap \delta} \Lambda^k$.

**Basis functions of $\mathbf{V}_d$.** Note that

$$
\mathbf{V}_d = \left\{ \mu_h \in \prod_{T \in \mathcal{G}_h} \left[ \mathcal{P}_0 \Lambda^k(T) + \mathbf{K}_T (\mathcal{P}_0 \Lambda^{k+1}(T)) + \mathcal{H}^2_{\delta}(T) \right] : \\
\langle d^k \mu_h, \eta_h \rangle_{L^2(\Lambda^k)} - \langle \mu_h, \delta_{k+1} \eta_h \rangle_{L^2(\Lambda^k)} = 0, \quad \forall \eta_h \in W_{h0}^{r,abc} \Lambda^{k+1} \right. \\
\left. \langle \delta_k \mu_h, \tau_h \rangle_{L^2(\Lambda^k)} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2(\Lambda^k)} = 0, \quad \forall \tau_h \in \mathcal{P}_1 \Lambda^{k-1}(T), \forall T \in \mathcal{G}_h \right\}
$$

$$
= \left\{ \mu_h \in W_h \Lambda^k + \prod_{T \in \mathcal{G}_h} \mathcal{H}^2_{\delta}(T) : \langle \delta_k \mu_h, \tau_h \rangle_{L^2(\Lambda^k)} - \langle \mu_h, d^{k-1} \tau_h \rangle_{L^2(\Lambda^k)} = 0, \quad \forall \tau_h \in \mathcal{P}_1 \Lambda^{k-1}(T), \forall T \in \mathcal{G}_h \right\}.
$$

On any simplex $T$, given $\mu \in \mathcal{P}_0 \Lambda^k(T) + \mathbf{K}_T (\mathcal{P}_0 \Lambda^{k+1}(T))$, there is always a unique $\mu' \in \mathcal{H}^2_{\delta}(T)$, such that $\langle \delta_k (\mu + \mu'), \tau \rangle_{L^2(\Lambda^k)} - \langle \mu + \mu', d^{k-1} \tau \rangle_{L^2(\Lambda^k)} = 0$, for $\forall \tau \in \mathcal{P}_1 \Lambda^{k-1}(T)$. Therefore, there is a bijection between $\mathbf{V}_d$ and $W_h \Lambda^k$. This way, the basis functions of $\mathbf{V}_d$ are determined by this 2-step procedure:

1. find $\mathbf{B}_{\mathcal{W}}^k$ a set of linearly independent basis functions of $W_h \Lambda^k$;
(2) for every $\psi_W \in B_W$, choose $\bar{\mu}_\phi \in \prod_{T \in \mathcal{G}_h} \mathcal{H}^2_\delta(T)$, such that

$$\langle \delta T \psi_W + \bar{\mu}_\phi, \tau \rangle_{L^2_\delta \Lambda^{k-1}(T)} - \langle \psi_W + \bar{\mu}_\phi, \mathbf{d}^{k-1} \tau \rangle_{L^2_\delta \Lambda^k(T)} = 0, \quad \forall \tau \in \mathcal{P}^1_1 \Lambda^{k-1}(T), \quad \forall T \in \mathcal{G}_h,$$

and set $\psi_V := \psi_W + \bar{\mu}_\phi$.

Then $\{\psi_V \}_{\psi_W \in B_W}$ is a set of basis functions of $V_\delta$. Evidently, the support of $\psi_V$ is contained in the support of $\psi_W$.

**Basis functions of $V_\delta$.** To determine the basis functions of $V_\delta$, we adopt a different 3-step procedure.

**Step 1:** find $B_{W}^{k-1} = \{\psi_j\}_{j=1}^{\text{dim}(W_h \Lambda^{k-1})}$ a set of nodal basis functions of $W_h \Lambda^{k-1}$, and on every simplex $T$, the restrictions $\psi_j|_T$ of those $\psi_j$ that are nonzero on $T$ are linearly independent;

**Step 2:** given $T \in \mathcal{G}_h$, set $I^T := \{1 \leq i \leq \text{dim}(W_h \Lambda^{k-1}) : \hat{T} \cap \text{supp}(\psi_j) \neq \emptyset\}$, and there exist a set of functions $\{\mu^T_i : i \in I^T\} \subset \mathcal{P}^{n+\delta}_{d \delta, \delta} \Lambda^k(T)$, such that $\langle \delta T \mu^T_i, \psi_j|_T \rangle_{L^2_\delta \Lambda^{k-1}(T)} - \langle \mu^T_i, \mathbf{d}^{k-1} \psi_j|_T \rangle_{L^2_\delta \Lambda^k(T)} = \delta_{ij}, \quad i, j \in I^T$. Then $\mathcal{P}^{n+\delta}_{d \delta, \delta} \Lambda^k(T) = \text{span}(\mu^T_i : i \in I^T)$.

**Step 3:** a set of basis functions of $V_\delta$ consists of, for $1 \leq j \leq \text{dim}(W_h \Lambda^{k-1})$, functions

$$\mu_h \in \sum_{T \cap \text{supp}(\psi_j) \neq \emptyset} \text{span}\{L_{\mu}^T \mu_j^T\}, \quad \text{such that} \quad \langle \delta_{k,h} \mu_h, \psi_j \rangle_{L^2_\delta \Lambda^{k-1}} - \langle \mu_h, \mathbf{d}^{k-1} \psi_j \rangle_{L^2_\delta \Lambda^k} = 0.$$

Actually,

$$V_\delta = \left\{ \mu_h \in \mathcal{P}^{n+\delta}_{d \delta, \delta} \Lambda^k(\mathcal{G}_h) : \langle \delta_{k,h} \mu_h, \tau_h \rangle_{L^2_\delta \Lambda^{k-1}} - \langle \mu_h, \mathbf{d}^{k-1} \tau_h \rangle_{L^2_\delta \Lambda^k} = 0, \quad \forall \tau_h \in W_h \Lambda^{k-1} \right\}$$

$$= \left\{ \mu_h \in \sum_{T \in \mathcal{G}_h} \sum_{i \in I^T} \text{span}\{L_{\mu}^T \mu_j^T\} : \langle \delta_{k,h} \mu_h, \psi_j \rangle_{L^2_\delta \Lambda^{k-1}} - \langle \mu_h, \mathbf{d}^{k-1} \psi_j \rangle_{L^2_\delta \Lambda^k} = 0, \quad \forall \psi_j \in B_W^{k-1} \right\}$$

$$= \sum_{1 \leq j \leq \dim(W_h \Lambda^{k-1})} \left\{ \mu_h \in \sum_{\forall \psi_j \in B_W^{k-1}} \text{span}\{L_{\mu}^T \mu_j^T\} : \langle \delta_{k,h} \mu_h, \psi_j \rangle_{L^2_\delta \Lambda^{k-1}} - \langle \mu_h, \mathbf{d}^{k-1} \psi_j \rangle_{L^2_\delta \Lambda^k} = 0 \right\}.$$

Note that, again, the support of such functions are contained in the support of $\psi_j$.

4.2.2. **Examples.** We take the two-dimensional Hodge-Laplacian problem of 1-form for example. Let $\Omega$ be a polygon. Denote by $\mathcal{H}(\Omega)$ the space of harmonic forms. The problem reads: find $\omega \in H(\text{rot, } \Omega) \cap H_0(\text{div, } \Omega)$, such that $\omega \perp \mathcal{H}(\Omega)$, and

$$\text{(rot, } \omega + \text{div, } \mu) = (f - \mathbf{P}_h f, \mu), \quad \forall \mu \in H(\text{rot, } \Omega) \cap H_0(\text{div, } \Omega).$$

The corresponding spaces are $H^1(\Omega) = H(\text{grad, } \Omega)$ for 0-forms and $H^1_0(\Omega) = H_0(\text{curl, } \Omega)$ for 2-forms, respectively. We use the conforming linear element space $V_1^h$ for $H(\text{grad, } \Omega)$ and the linear Crouzeix-Raviart element space $V_{h0}^{CR}$ for $H_0(\text{curl, } \Omega) = H^1_0(\Omega)$.

Let $T_h$ be a shape-regular triangular subdivision of $\Omega$ with mesh size $h$, such that $\overline{\Omega} = \cup_{T \in T_h} \overline{T}$, and every boundary vertex is connected to at least one interior vertex. Denote by $E_h, E_h^i, E_h^b, \chi_h$, ...
\( \mathcal{X}_h^i, \mathcal{X}_h^b \) and \( \mathcal{X}_h^c \) the set of edges, interior edges, boundary edges, vertices, interior vertices, boundary vertices and corners, respectively. Evidently, \( V_h^1 \) admits locally supported basis functions, denoted by \( \phi_a \) associated with vertices \( a \in \mathcal{X}_h^i \). The restrictions of \( \phi_a \) on a triangle are each one of the barycentric coordinates on the triangle. We refer to Figure 1 for an illustration of the triangulation, and also the supports of \( \phi_a \).

In the setting,

\[
P_{m+\delta}^{d,\delta} A^1(T) = P_{\text{rot}/\text{div}}^{m+\delta}(T) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \begin{pmatrix} \tilde{x}^2 \\ -\tilde{x} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{y}^2 \end{pmatrix} \right\},
\]

and

\[
P_{m+\delta}^{d,\delta,\delta} A^1(T) = P_{\text{rot}/\text{div},\text{div}}^{m+\delta}(T) = \text{span} \left\{ \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \begin{pmatrix} \tilde{x}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{y}^2 \end{pmatrix} \right\}.
\]

![Figure 1](image_url)

**Figure 1.** Illustration of supports of \( \phi_A, \phi_B \) and \( \phi_C \). A (as well as B) denotes an interior vertex, and C denotes a boundary vertex.

The finite element space is defined by

\[
(4.26) \quad V_{d,\delta}^{m,\delta}(\mathcal{T}_h) := \{ \mu_h \in P_{\text{rot}/\text{div}}^{m+\delta}(\mathcal{T}_h) : (\text{rot}_h \mu_h, \eta_h) - (\mu_h, \text{curl}_h \eta_h) = 0, \forall \eta_h \in V_{\text{CR}}^{\text{rot} h},
\]

\[
\text{and} \quad (\text{div}_h \mu_h, \tau_h) - (\mu_h, \text{grad} \tau_h) = 0, \forall \tau_h \in V_{h}^{1} \}\}
\]

Now, following the general procedure, we present the basis functions of \( V_d = V_{\text{rot}} \) and \( V_\delta = V_{\text{div}} \), respectively. The a set of basis functions of \( V_{d,\delta}^{m,\delta}(\mathcal{T}_h) \) is a direct summation of the set of basis functions of \( V_{\text{rot}} \) and \( V_{\text{div}} \).

**Basis functions of \( V_{\text{rot}} \).** The basis functions are determined by 2 steps.

**Step 1:** Choose \( \{\tau_j\}_{j=1}^{\dim(V_{\text{rot}} h)} \) to be a set of nodal basis functions of \( V_{\text{rot}}^h \), the lowest degree conforming Raviart-Thomas finite element space for \( H(\text{rot}, \Omega) \).

**Step 2:** For any \( \tau_j \), on every \( T \in \mathcal{T}_h \), find \( \tilde{\mu}^T_j \in \text{span} \left\{ \begin{pmatrix} \tilde{x}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tilde{y}^2 \end{pmatrix} \right\} \), such that

\[
(\text{div}(\tau_j|_T + \tilde{\mu}^T_j), \lambda_j|_T) + ((\tau_j|_T + \tilde{\mu}^T_j), \nabla \lambda_j|_T) = 0, \quad j = 1, 2, 3.
\]
Set $\tilde{\tau}_j := \tau_j + \prod_{T \in T_h} \mu_T^\div$. Then $\{\tilde{\tau}_j\}_{j=1}^{\dim(V_h^\rot)}$ is a set of basis functions of $V_\rot$. The support of $\tilde{\tau}_j$ is the same as that of $\tau_j$.

**Basis functions of $V_\div$**. The basis functions are determined by 2 steps.

**Step 1**: On a cell $T$, with $a_i \in X_h$ being its vertices, let $\lambda_{a_i}^T$ be the barycentric coordinates of $T$, and find $\mu_{a_i,T}^\div \in P_{m+\div,\div}(T)$, $1 \leq i \leq 3$, such that

$$(\div \mu_{a_i,T}^\div, \lambda_{a_i}^T)_T + (\mu_{a_i,T}^\div, \grad \lambda_{a_i}^T)_T = \delta_{ij}, \quad 1 \leq i, j \leq 3.$$  

**Step 2**: Find functions in $\prod_{T \in T_h} \text{span}\{\mu_{a,T}^\div : a \in X_h\}$ such that conditions in (4.26) are satisfied with respect to every $\phi_a \in V_1^1$ a basis function. Particularly, with respect to any vertex $a$, the associated basis functions of $V_\div$ are all these functions in $\text{span}\{\mu_{a,T}^\div : \hat{T} \cap \text{supp}(\phi_a) \neq \emptyset\}$, namely, functions of the form $\omega_h = \sum_{\partial T \ni a} c_T \tilde{E}_T^\Omega \mu_{a,T}^\div$, such that

$$\sum_{\partial T \ni a} (\div \omega_h, \phi_a|_T)_T + (\omega_h, \grad \phi_a|_T)_T = 0.$$  

Those $\omega_h$ for all $\phi_a$ form a set of basis functions of $V_\div$. They are each supported in a two-successive-cell patch. We refer to Figure 2 for the case $a \in X_h^l$, and to Figure 3 for an illustration that $a \in X_h^b$.

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Figure 3. C is boundary vertex with a three-cell patch; cf. Figure 1. Two basis functions are associated with the interior vertex C. They are each supported on the shadowed parts.

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