Global attractivity for a nonautonomous Nicholson’s equation with mixed monotonicities

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Received 21 June 2021, revised 25 October 2021
Accepted for publication 22 November 2021
Published 7 December 2021

Abstract
We consider a Nicholson’s equation with multiple pairs of time-varying delays and nonlinear terms given by mixed monotone functions. Sufficient conditions for the permanence, local stability and global attractivity of its positive equilibrium $K$ are established. The main novelty here is the construction of a suitable auxiliary difference equation $x_{n+1} = h(x_n)$ with $h$ having negative Schwarzian derivative, and its application to derive the attractivity of $K$ for a model with one or more pairs of time-dependent delays. Our criteria depend on the size of some delays, improve results in recent literature and provide answers to open problems.

Keywords: Nicholson equation, mixed monotonicity, global attractivity, stability, permanence, Schwarzian derivative
Mathematics Subject Classification numbers: 34K12, 34K20, 34K25, 92D25.

1. Introduction

The classical Nicholson’s blowflies equation

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-a(x(t-\tau)} \quad (p, \delta, \tau > 0) \quad (1.1)$$

was introduced in 1980 by Gurney et al [13] to model the life cycle of the Australian sheep blowfly, \textit{Lucilla cuprina}, responsible for a serious sheep plague. This equation was immediately accepted by the scientific community, since the proposed model fit the experimental data

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Recommended by Dr Alexander Kiselev.
collected and published by the Australian entomologist Nicholson in the 1950s, see e.g. [19]. Here, \( x(t) \) stands for the size of the adult blowfly population at time \( t \), \( \delta \) is the adult mortality rate, \( p \) the maximal egg production rate, \( 1/a \) the size at which the population produces eggs at maximal rate; the birth function is given by the Ricker nonlinearity \( xe^{-ax} \) with \( x \) evaluated at a time \( t - \tau \), where \( \tau \) is the time of the life-cycle of the fly (from egg to the final adult form). Since its introduction, Nicholson’s equation has been extensively studied, and many scalar and multi-dimensional variants analysed in different contexts of mathematical biology. These studies have had an enormous impact, not only from a mathematical viewpoint but also in terms of applications to real world problems.

In population dynamics and other applications, some authors long ago realised that more accurate models are obtained if the growth or production function incorporates several delays with different roles, for instance, in neural network systems [12], models of hematopoiesis (production and specialisation of blood cells in the bone marrow) [1] and other regulatory physiological processes. Along these lines, the Nicholson’s equation with multiple delays,

\[
x'(t) = -\delta x(t) + \sum_{j=1}^{m} p_j x(t - \tau_j)e^{-a(t-\tau_j)},
\]

has been proposed and intensively studied. Clearly, more realistic models are obtained with nonautonomous equations, where parameters and delays are not fixed constants.

The following nonautonomous variant of (1.1) with two time-varying delays was suggested in [4]:

\[
x'(t) = \beta(t) \left( px(t - \tau(t))e^{-a(t-\sigma(t))} - \delta x(t) \right),
\]

where \( p, a, \delta \in (0, \infty), \beta(t) > 0, \) and \( \tau(t), \sigma(t) \) represent respectively the incubation and the maturation delays for the blowfly. Obtaining conditions for the stability of the equilibria for this equation was stated as a relevant open problem in [4]. In fact, Berezansky and Braverman [4] provided criteria for the local and global stability of the positive equilibrium of the Mackey–Glass-type equation \( x'(t) = \beta(t) \left[ -x(t) + \frac{a(t-\sigma(t))}{1+x(t-\sigma(t))} \right] \) (with \( \nu > 0, a > 1 \)), and set as an open problem a similar analysis for the Nicholson model (1.3).

Recently, Long and Gong [17] considered a Nicholson’s equation with multiple pairs of time-varying delays of the form

\[
x'(t) = \beta(t) \left( \sum_{j=1}^{m} p_j x(t - \tau_j(t))e^{-a(t-\sigma(t))\beta(t)} - \delta x(t) \right), \quad t \geq t_0
\]

where \( p_j, a, \delta \in (0, \infty) \) and \( \beta(t), \sigma(t), \tau_j(t) \) are continuous, non-negative and bounded, with \( \beta(t) \) bounded from below by a positive constant. If \( \sum_j p_j \leq \delta \), it was shown in [17] that the equilibrium 0 is a global attractor of all positive solutions, and moreover is globally exponentially stable if \( \sum_j p_j < \delta \). Typically, equation (1.5) is used in population dynamics, regulatory
physiological systems and other contexts, where the situation of 0 being a global attractor corresponds to the extinction of the population represented by \(x(t)\).

The study of the global attractivity of a positive equilibrium, when it exists, is more interesting from a biological viewpoint. Since \(f(x) := \delta^{-1} \sum_j p_j e^{-a_j x} \) is strictly decreasing with \(f(0) = \delta^{-1} \sum_j p_j \) and \(\lim_{x \to +\infty} f(x) = 0\), there is a unique positive equilibrium \(K\) if and only if \(\sum_{j=1}^m p_j > \delta\). In the present paper, we always assume \(p = \sum_{j=1}^m p_j > \delta\), in which case the carrying capacity \(K > 0\) is defined by the identity

\[
\sum_{j=1}^m p_j e^{-a_j K} = \delta.
\]  

(1.6)

For (1.3), if \(p > \delta\) the positive equilibrium is explicitly evaluated as

\[
K = \frac{1}{a} \log \left( \frac{p}{\delta} \right),
\]  

(1.7)

whereas for (1.5) it is implicitly given by (1.6).

Within the set of modified Nicholson’s blowflies equations, we mention the so-called ‘neoclassical growth model’ \(x'(t) = -\delta x(t) + px(t - \tau) e^{-a(t - \gamma)}\), with \(\gamma \in (0, 1)\), often used in economics, where \(\tau\) is incorporated to account for a reaction delay to market changes. In a recent work, Huang et al [15] have also addressed the global attractivity of the positive equilibrium of the neoclassical growth model obtained by replacing in (1.5) \(x(t - \tau_j(t))\) by \(x(t - \tau_j(t))(1 \leq j \leq m)\) for some fixed \(\gamma \in (0, 1)\). See [15, 18] for additional references on real-world applications of such models.

In recent years, there has been a growing interest in delay differential equations (DDEs) with two or more delays appearing in the same nonlinear function, possibly with a mixed monotonicity. The presence of a unique small delay in each nonlinear term of a DDE is in general harmless, i.e. the global dynamics properties of the equation without delays are maintained with small delays. However, the situation can be drastically different if two or more different delays are involved in each nonlinearity, as exemplified in [4] for a variant of the celebrated Mackey–Glass equation. These generalisations are interesting, not simply as a theoretical scenario but also as possible refinements of biological models. Such models appear naturally, e.g. in gene regulatory systems with both transmission and translation delays [3, 4], as well as other real-world problems such as the Nicholson’s equation (1.3). How to generalise known results for classical DDEs to some modified versions with two or more different delays in the same nonlinear term is a subject that lately has caught the attention of a number of researchers [3, 4, 6, 14, 23]. Several different techniques have been proposed, such as Lyapunov functions or the theory of monotone systems [21], which are difficult to apply in the context of mixed monotonicity. The method in [9], based on an auxiliary monotone difference equation associated with (1.4), has inspired our work. However, the application of this method to nonautonomous equations with possibly more than one pair of delays is not apparent, and has not been addressed.

Motivated by the above mentioned works [4, 9, 17], the main goal of this paper is to establish sufficient conditions for the equilibrium \(K\) of (1.5) to be a global attractor of all positive solutions.

We now describe briefly the major steps and highlights of the procedure developed here. Firstly, assuming that \(p > \delta\), the permanence of (1.5) is shown without additional restrictions by using results in [3], and is followed by the local stability analysis. We emphasise that here the permanence is relevant as an auxiliary result, therefore we are not worried about getting refined explicit uniform lower and upper bounds for all positive solutions.
Next, we establish global attractivity criteria for the mixed monotonicity equation (1.5) with the help of a suitable associated difference equation $x_{n+1} = h(x_n)$ constructed here, where $h$ has negative Schwarzian derivative and for which we prove that $K$ is a global attractor. Moreover, we show that this fact implies that there are no solutions of (1.5) with an oscillatory behaviour (i.e. with $\liminf_{t \to \infty} x(t) < \limsup_{t \to \infty} x(t)$), and that this leads to the global attractivity of $K$, now as a solution of (1.5). The construction of such a difference equation is a major novelty of this paper, since, as far as the authors know, this methodology is used here for the first time to handle the situation of more than one pair of delays, which moreover appear in a nonautonomous equation.

The idea of using properties of functions with negative Schwarzian derivatives to exhibit conditions for the global attractivity of a positive equilibrium is not new in autonomous settings (see some additional comments in section 3). For Nicholson models, we simply refer to Liz et al [16], who applied Schwarzian derivatives to a family of autonomous scalar DDEs which encompasses (1.2) as a particular case, and El-Morshedy and Ruiz-Herrera [9], who introduced new ingredients in the method in order to deal with the mixed monotonicity in (1.4). Typically, these works established delay independent criteria as well as conditions depending on the size of the delays. The construction of a suitable autonomous difference equation of type $x_{n+1} = h(x_n)$, with $h$ having negative Schwarzian derivative, is not apparent and requires some careful estimates to deal with the mixed monotonicity character of the nonlinear terms and the multiple pairs of time-dependent delays. To simplify the exposition, we have chosen to treat equation (1.3) with one pair of delays separately, and then generalise the construction of $h$ for (1.5), in which case additional difficulties come from not having an explicit value for the carrying capacity $K$. To summarise, our criteria largely generalise the results obtained in [9] for (1.4) and in particular provide an answer to the above mentioned open problem concerning (1.3) raised in [4]. We anticipate that the method developed here can be applied to other modified classical scalar delayed models, such as logistic, Michaelis–Menten and Mackey–Glass type equations.

The paper is organised as follows: with $p > \delta$, in section 2 we show that (1.5) is permanent and derive sufficient conditions for the local asymptotic stability of its positive equilibrium $K$. Relying on the permanence previously established and with the methodology in [9] as a key ingredient, the main results on the global attractivity of $K$ are obtained in section 3. Two examples illustrate our results. Conclusions are contained in section 4.

2. Permanence and local stability

We introduce briefly some notation and the abstract framework to deal with the DDE model (1.5). In what follows, we always assume the general assumptions below:

\[ p, a, \delta \in (0, \infty), \beta, \sigma, \tau : [t_0, \infty) \to [0, \infty) \text{ are continuous and bounded (} 1 \leq j \leq m \text{), with} \]

\[ 0 < \beta^- := \inf_{t \geq t_0} \beta(t) \leq \beta(t) \leq \sup_{t \geq t_0} \beta(t) =: \beta^+. \]  

(2.1)

Hereafter, denote

\[ p = \sum_{j=1}^m p_j, \quad a^+ = \max_j a_j, \quad a^- = \min_j a_j. \]
assume $p > \delta$ and let $K > 0$ be the equilibrium defined by equation (1.6). Define

$$
\tau = \max \left\{ \sup_{t \geq t_0} \tau_j(t), \sup_{t \geq t_0} \sigma_j(t) : j = 1, \ldots, m \right\}.
$$

For (1.5), the space $C := C([-\tau, 0]; \mathbb{R})$ equipped with the supremum norm $||\phi|| = \max_{t \in [-\tau, 0]} |\phi(t)|$ will be taken as the phase space [22]. Furthermore, due to the biological motivation of the model, we are only interested in non-negative solutions of (1.5). Thus, we consider $C_0^+ := \{ \phi \in C : \phi(\theta) \geq 0 \text{ on } [-\tau, 0), \phi(0) > 0 \}$ as the set of admissible initial conditions. For $t \geq t_0$, as usual $x_t$ designates the element in $C$ given by $x_t(\theta) = x(t + \theta), -\tau \leq \theta \leq 0$. In this way, we write $x(t_0, \phi) \in C$, or $x(t; t_0, \phi) \in \mathbb{R}$, to denote the solution of (1.5) with the initial condition

$$
x_{t_0} = \phi \in C_0^+,
$$

defined on a maximal interval $[t_0, \eta(\phi))$. Presently, the usual concepts of stability always refer to solutions with initial conditions in $C_0^+$. Hence, the equilibrium $K$ is called a global attractor of (1.5) if it is globally attractive in $C_0^+$, i.e. $\lim_{t \to \infty} x(t) = K$ for every solution $x(t) = x(t; t_0, \phi)$ of (1.5) with initial condition (2.2); $K$ is globally asymptotically stable if it is stable and globally attractive.

In abstract form, (1.5) is written as $x'(t) = F(t, x_t), t \geq t_0$, where

$$
F(t, \phi) = \beta(t) \left( \sum_{j=1}^{m} p_j \phi(-\tau_j(t))e^{-\beta(-\sigma_j(t))} - \delta \phi(0) \right), \quad t \geq t_0, \phi \in C_0^+.
$$

For $t \geq t_0, \phi \in C_0^+$, $F(t, \phi) \geq 0$ if $\phi(0) = 0$ and $F(t, \phi) \leq g(t, \phi)$, where the function $g(t, \phi) := \beta(t) \left( \sum_{j=1}^{m} p_j \phi(-\tau_j(t)) - \delta \phi(0) \right)$ satisfies the quasi-monotone condition in [21, p78]. From results of comparison of solutions [21], solutions of (1.5) with initial conditions (2.2) are defined and positive on $[t_0, \infty)$.

We recall the standard definitions of (uniform) persistence and permanence. The DDE (1.5) is said to be persistent if there exists $m > 0$ such that for every $\phi \in C_0^+$ there is $t_* = t_*(\phi)$ such that the solution $x(t; t_0, \phi)$ satisfies

$$
x(t; t_0, \phi) \geq m \quad \text{for } t \geq t_*.
$$

Analogously, (1.5) is said to be dissipative if there exists an $M > 0$ such that for every $\phi \in C_0^+$ there is $t_* = t_*(\phi)$ such that

$$
x(t; t_0, \phi) \leq M \quad \text{for } t \geq t_*.
$$

Equation (1.5) is said to be permanent if it is both dissipative and persistent, i.e. there are constants $m, M > 0$ such that for $\phi \in C_0^+$ there is $t_* = t_*(\phi)$ such that $m \leq x(t; t_0, \phi) \leq M$ for $t \geq t_*$. We now establish the permanence of (1.5), noting however that we are mostly interested in the permanence as an auxiliary result to derive conditions for the global attractivity of $K$.

**Theorem 2.1.** If $p > \delta$, then (1.5) is permanent. Moreover,

$$
Ke^{-2\beta^+\tau} \leq \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) \leq Ke^{2(\delta + \beta^+)\tau} \quad (2.3)
$$

for any solution $x(t)$ of (1.5) with initial condition in $C_0^+$. 

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Theorem 2.1 largely extends the permanence result obtained in [9] for the Remark 2.1. The functions \( f_j \) are monotone increasing in \( u \) and monotone decreasing in \( v \). Furthermore,

\[
\sum_j f_j(t, u, v) = \beta(t) \sum_j p_j u e^{-a_j(v_j)} \leq \beta^+ \sum_j p_j \bar{u} = A \bar{u}
\]

where \( \bar{u} = \max_{1 \leq j \leq m} \{u_j\} \), \( A := \beta^+ p > 0 \), and \( B \bar{u} \leq w(t, u) \leq C \bar{u} \) where \( B := \delta \beta^- \), \( C := \delta \beta^+ \).

Let \( M > 0 \). Then,

\[
L = L(M) := \limsup_{t \to \infty} \frac{\sum_j f_j(t, u, M)}{w(t, u)} = \frac{\sum_j p_j e^{-a_M}}{\delta}.
\]

Thus, \( L < 1 \) as long as \( M > K \). From [3, theorem 5.6], for any \( M > K \) we obtain

\[
\limsup_{t \to \infty} x(t) \leq M e^{(\delta + p) \beta^+ t},
\]

yielding that (1.5) is dissipative. Next, for \( \mu > 0 \) we have

\[
l = l(\mu) := \liminf_{t \to \infty} \frac{\sum_j f_j(t, u, \mu)}{w(t, u)} = \frac{\sum_j p_j e^{-a_M \mu}}{\delta}.
\]

Thus, \( l > 1 \) if \( \mu < K \). From [3, theorem 5.6], for \( \mu < K \) we derive

\[
\liminf_{t \to \infty} x(t) \geq \mu e^{-2K \beta^+ t},
\]

hence (1.5) is persistent. Combining these results for \( M = K + \varepsilon \) and \( \mu = K - \varepsilon \) and letting \( \varepsilon \to 0 \), (2.3) follows.

In the case of a single pair of time-varying delays, as in (1.3), we remark that the alternative approach presented in [14] might lead to better uniform lower and upper bounds than the ones obtained in (2.3).

**Remark 2.1.** Theorem 2.1 largely extends the permanence result obtained in [9] for the Nicholson’s equation (1.4), where, besides a single pair of constant delays \( \tau(t) \equiv \tau, \sigma(t) \equiv \sigma \), the additional constraint \( 0 \leq \sigma \leq \tau \) is imposed.

The local stability of the positive equilibrium \( K \) is now studied. By setting \( u(t) = x(t) - K \) and dropping the nonlinear terms, one obtains the linearised equation about \( K \) given by

\[
u'(t) = \beta(t) \left( \sum_j p_j e^{-a_j K} [u(t - \tau_j(t)) - a_j K u(t - \sigma_j(t))] - \delta u(t) \right).
\]

**Theorem 2.2.** If \( p > \delta \) and

\[
\limsup_{t \to +\infty} \sum_{j=1}^m a_j p_j e^{-a_j K} \int_{1-\sigma_j(t)}^{\tau_j(t)} \beta(s)\,ds < \frac{\sum_{j=1}^m a_j p_j e^{-a_j K}}{2\delta + K \sum_{j=1}^m a_j p_j e^{-a_j K}},
\]

then
then $K$ is locally exponentially stable for (1.5).

Proof. Write (2.4) as

$$u'(t) + \sum_{j=0}^{2m} b_j(t)u(r_j(t)) = 0$$

where $b_j(t) = B_j\beta(t)$,

$$B_j = \begin{cases} \delta & j = 0 \\ -p_j e^{-a_j K} & j = 1, \ldots, m \\ a_j K p_j e^{-a_j K} & j = m + 1, \ldots, 2m \end{cases}$$

for all $j = 0, \ldots, 2m$ and $\beta(t) \geq \beta^* > 0$. Note that (2.4) has the form required in [2, corollary 1.3]. Let $I = \{0, m + 1, \ldots, 2m\}$ and $J = \{1, \ldots, m\}$. Then,

$$\sum_{j \in I} B_j = \delta + K \sum_{j=1}^{m} a_j p_j e^{-a_j K} > 0, \quad \sum_{j \in J} B_j = -\delta.$$

Following [2], we have exponential stability for (2.4) if

$$\limsup_{t \to \infty} \sum_{j \in I} |B_j| \int_{r_j(t)}^{t} \beta(s) \, ds < \frac{\sum_{j \in I} B_j - \sum_{j \in J} |B_j|}{\sum_{j=0}^{2m} |B_j|}.$$

Hence, (2.4) is exponentially stable if (2.5) holds. By the principle of linearised stability [22], this yields the result.

Considering a uniform upper bound for the integrals in (2.5), we obtain the criterion below.

Corollary 2.1. Let $p > \delta$ and define

$$\zeta_M := \max_{1 \leq j \leq m} \limsup_{t \to \infty} \int_{t - \sigma_j(t)}^{t} \beta(s) \, ds. \quad (2.6)$$

If

$$\zeta_M < \frac{1}{2\delta + K \left(\sum_{j=1}^{m} p_j e^{-a_j K}\right)},$$

then $K$ is locally exponentially stable for (1.5). In particular, this holds if

$$\zeta_M < \frac{1}{\delta(2 + a + K)}. \quad (2.7)$$

In the case of only one pair of mixed delays, from the identity $aK = \log(p/\delta)$ the next corollary follows immediately. For an alternative proof, one can use [4, lemma 3.4].

Corollary 2.2. If $p > \delta$ and

$$\zeta_M := \limsup_{t \to \infty} \int_{t - \sigma(t)}^{t} \beta(s) \, ds < \frac{1}{\delta(2 + \log(p/\delta))}, \quad (2.8)$$

then $K$ is locally exponentially stable for (1.3).
3. Global attractivity of the positive equilibrium

The next goal is to establish sufficient conditions for the global attractivity of the positive equilibrium $K$ (in the set of all positive solutions).

As in [9], to preclude the existence of solutions for which $\liminf_{t \to \infty} x(t) < \limsup_{t \to \infty} x(t)$, we shall use global attractivity results for difference equations of the form

$$x_{n+1} = f(x_n), \quad n \in \mathbb{N}_0,$$

where $f: I \to I$ is a continuous function on a real interval $I$ and the initial data $x_0 \in I$.

For a compact interval $I = [a, b] (a < b)$, the pioneering work of Coppel [7] shows that if the equation $f^2(x) = x$ (where $f^2 = f \circ f$) has no roots except the roots of $f(x) = x$, then all solutions of the difference equation (3.1) converge to a fixed point of $f$. In particular, if there is a unique fixed point $x^*$ of $f$ and $f^2$ has no other fixed points, then $x^*$ is a global attractor for (3.1). In applications, however, it is often rather difficult to find the fixed points of $f^2$. In the case of $C^1$ functions with a unique fixed point $x^*$, the assumption $|f'(x)| < 1$ would show that $x^*$ is the unique fixed point of $f^2$, but again this condition is usually not verifiable in a simple way.

Lately, the use of Schwarzian derivatives has proven to be very helpful in this context. For instance, Singer [20] extended to the family (3.1), where $f$ has negative Schwarzian derivative $Sf(x)$ on $I = [0, 1]$, previous results known only for the case of a rational function. If $Sf(x) < 0$ on $I$, to rule out the existence of other fixed points of $f^2$ besides a unique fixed point $x^*$ of $f$, it is useful to have in mind some properties of $f^2$ established in [20]. This was the approach followed by El-Morshedy and López in [8], where some powerful criteria (and easier to verify) were given. For convenience of the reader, we include below some selected results derived from [8, lemma 2.5 and corollary 2.9]. (With $a \in \mathbb{R}$ and $b = \infty$, $I = [a, b]$ denotes the interval $[a, \infty)$.)

**Lemma 3.1.** Let $f: I \to I$ be a continuous function on a real interval $I = [a, b]$ ($a < b$ with $b \in \mathbb{R}$ or $b = \infty$) with a unique fixed point $x^* \in (a, b)$ and such that

$$(f(x) - x)(x - x^*) < 0 \quad \text{for all } x \in (a, b), x \neq x^*.$$

The following assertions hold:

(a) If $x^*$ is a global attractor for (3.1), then there are no points $c, d \in I$ with $c < d$ and $f([c, d]) \supset [c, d]$.

(b) If $f$ is a $C^3$ function, decreasing on $I$ and such that:

1. $Sf(x) < 0$ for $x \in I$, where $Sf(x)$ is the Schwarzian derivative of $f$, defined by

$$Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2,$$

2. $-1 \leq f'(x^*) < 0$,

then $x^*$ is a global attractor for (3.1).

**Remark 3.1.** In fact, the result on the global attractivity of $x^*$ in [8, corollary 2.9] is stronger than the above lemma 3.1(b) and applies to more general functions $f$, not necessarily decreasing. For our purposes, we can restrict our attention to the setting of $f$ decreasing with $Sf(x) < 0$ on $I$. In this situation, in [8] the authors showed that $|f^2(x) - x^*| < |x - x^*|$ for $x \in I, x \neq x^*$ (thus $f^2(x) \neq x$ for $x \neq x^*$) by using some properties of $f^2$, namely that $f^2$ has also negative Schwarzian derivative.
We are now in the position to state the main results of this section on the global attractivity of $K$. Although proofs are inspired in the techniques of El-Morshedy and Ruiz-Herrera [9], we need to carefully adjust the major arguments in [9] to the present situation.

Recall that a priori it is not possible to write explicitly the equilibrium $K$ if $m > 1$, see (1.6). This alone justifies considering separately, for the sake of simplicity of exposition, the case of a single pair of delays, prior to the general situation (1.5).

3.1. The case of one pair of delays

Theorem 3.1. Assume $p > \delta$, denote $\zeta_M := \lim_{r \to \infty} \int_{t-r}^{t} \beta(s) \, ds$ as in (2.8) (with $m = 1$), and further assume

(h1) $(e^{\zeta_M} - 1) \log \frac{p}{\delta} \leq 1.$

Then the equilibrium $K$ of (1.3) is globally attractive (in $C^0_+$).

Proof. Consider any solution of (1.3) with initial condition in $C^0_+$. We need to prove that there exists $\lim_{t \to \infty} x(t) = K.$ By the permanence in theorem 2.1, there exist

\[ l := \liminf_{t \to \infty} x(t) \quad \text{and} \quad \limsup_{t \to \infty} x(t) =: L \quad (3.2) \]

with $0 < l \leq L$.

Case 1: suppose that $l = L$, i.e. there exists $\lim_{t \to \infty} x(t) = C > 0.$

From (1.3), we may write $x'(t) = \beta(t)g(x_t)$, where $\lim_{t \to \infty} g(x_t) = \delta C (f(C) - 1)$ and

\[ f(x) := \frac{p}{\delta} e^{-ax}, \quad x \geq 0. \]

Let $\eta = f(C) - 1$. If $0 < C < K$, then for $t$ large we have $x'(t) \geq \beta \delta C_2 > 0$, thus $x(t) \to \infty$ as $t \to \infty$, which is a contradiction. Similarly, $\eta < 0$ if $C > K$, in which case $x(t) \to -\infty$ as $t \to \infty$, which is not possible. Hence $C = K$.

Case 2: suppose that there is a solution $x(t)$ with $l < L$ in (3.2).

We preclude this situation in several steps. To simplify the exposition, we first suppose that there exists

\[ \zeta := \lim_{t \to \infty} \int_{t-r}^{t} \beta(s) \, ds, \quad (3.3) \]

leaving the general situation, where $\zeta$ is replaced by $\zeta_M$, for the last step of the proof. In this case, assumption (h1) reads as

\[ (e^{\zeta_M} - 1) \log \frac{p}{\delta} \leq 1. \quad (3.4) \]

Step 1. As in the proof of [9, theorem 3.1], consider sequences $(t_n), (s_n)$ such that $t_n, s_n \to \infty$ and $x'(t_n) = x'(s_n) = 0, x(t_n) \to L, x(s_n) \to l$. Taking subsequences if needed, there exist $l_r, l_L, l_r, l_L \in [l, L]$ such that

\[ x(t_n - \tau(t_n)) \to L_r, \quad x(t_n - \sigma(t_n)) \to L_L, \]
\[ x(s_n - \tau(s_n)) \to l_r, \quad x(s_n - \sigma(s_n)) \to l_L \]

(3.5)

and $\lim_{n} \beta(t_n) > 0, \lim_{n} \beta(s_n) > 0.$ From (1.3), it follows that

\[ 0 = \beta(t_n) \left( p x(t_n - \tau(t_n)) e^{-ax(t_n - \tau(t_n))} - \delta x(t_n) \right), \]
\[ 0 = \beta(s_n) \left( p x(s_n - \tau(s_n)) e^{-ax(s_n - \tau(s_n))} - \delta x(s_n) \right). \]
Using the limits in (3.5) leads to $0 = p L_\epsilon e^{-al_\epsilon} - \delta l_\epsilon$, $0 = p l_\epsilon e^{-al_\epsilon} - \delta l$, thus
\[
L = \frac{p}{\delta} L_\epsilon e^{-al_\epsilon} \quad \text{and} \quad l = \frac{p}{\delta} l_\epsilon e^{-al_\epsilon}. \tag{3.6}
\]
As $L_\epsilon \leq L$ and $l_\epsilon \geq l$, we have $L \leq \frac{p}{\delta} L e^{-al_\epsilon}$ and $l \geq \frac{p}{\delta} l e^{-al_\epsilon}$. Furthermore,
\[
\left(\frac{p}{\delta} x e^{-ax} - x\right)(x - K) < 0, \quad x > 0, x \neq K,
\]
which combined with the above inequalities yields
\[
L_\sigma \leq K \leq l_\sigma. \tag{3.7}
\]
By multiplying the equation by $\mu(t) = e^{\int_{0}^{t} b(s) \, ds}$ and integrating over $[t - \sigma(t), t]$, one obtains
\[
x(t) = x(t) e^{-\int_{t-\sigma(t)}^{t} b(s) \, ds} + \int_{t-\sigma(t)}^{t} \mu(s) x(s - \sigma(s)) e^{-\int_{s-\sigma(s)}^{s} b(v) \, dv} \, ds.
\]
Let $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $x(t) \in [1 - \epsilon, L + \epsilon]$ for $t \geq t_n - 2\tau$ and $x(t_n - \sigma(t_n)) \leq L_\sigma + \epsilon$, for all $n \geq n_0$. From the previous equation, it follows that
\[
x(t_n) = x(t_n - \sigma(t_n)) e^{-\int_{t_n-\sigma(t_n)}^{t_n} b(s) \, ds} + \int_{t_n-\sigma(t_n)}^{t_n} \mu(s) x(s - \sigma(s)) e^{-\int_{s-\sigma(s)}^{s} b(v) \, dv} \, ds
\leq (L_\sigma + \epsilon) e^{-\int_{t_n-\sigma(t_n)}^{t_n} b(s) \, ds} + \int_{t_n-\sigma(t_n)}^{t_n} \mu(s)(L + \epsilon) e^{-\int_{s-\sigma(s)}^{s} b(v) \, dv} \, ds.
\]
Using the mean value theorem for integrals, there exists \( \tilde{I}_n(\epsilon) \in [l - \epsilon, L + \epsilon] \) such that
\[
x(t_n) \leq (L_\sigma + \epsilon) e^{-\int_{t_n-\sigma(t_n)}^{t_n} b(s) \, ds} + \frac{p}{\delta}(L + \epsilon) e^{-\tilde{I}_n(\epsilon)}(1 - e^{-\tilde{I}_n(\epsilon)}).
\]
By taking subsequences if needed, we may suppose that $\tilde{I}_n(\epsilon) \to \tilde{I}$ as $n \to \infty$. Taking limits and using (3.3), we obtain
\[
L \leq e^{-\tilde{I}}(L_\sigma + \epsilon) + \frac{p}{\delta}(L + \epsilon) e^{-\tilde{I}_n}(1 - e^{-\tilde{I}_n}).
\]
Once again, we may suppose that $\tilde{I}(\epsilon) \to \tilde{I}$ as $\epsilon \to 0$ and by passing to the limit it follows that
\[
L \leq e^{-\tilde{I}}L_\sigma + \frac{p}{\delta} L e^{-\tilde{I}_n}(1 - e^{-\tilde{I}_n}) \leq e^{-\tilde{I}}K + \frac{p}{\delta} L e^{-\tilde{I}_n}(1 - e^{-\tilde{I}_n}). \tag{3.8}
\]
Analogously, by using the sequence $(s_n)$, one also shows that
\[
l \geq e^{-\tilde{I}}l_\sigma + \frac{p}{\delta} l e^{-\tilde{I}_n}(1 - e^{-\tilde{I}_n}) \geq e^{-\tilde{I}}K + \frac{p}{\delta} l e^{-\tilde{I}_n}(1 - e^{-\tilde{I}_n}), \tag{3.9}
\]
where $\tilde{L}, \tilde{I} \in [l, L]$.

Step 2. We claim that $\zeta > 0$ and $l < K < L$.

We first observe that, if $\zeta = 0$, (3.7)–(3.9) imply
\[
L \leq L_\sigma \leq K \leq l_\sigma \leq l,
\]
then $l = L$, which contradicts our assumption. Next, we show that $\tilde{I} < K < \tilde{L}$. 

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If \( \tilde{I} > K \), then \( \frac{e^{-\alpha I}}{K} < 1 \) and from (3.8) \( L < e^{-\delta K} + L(1 - e^{-\delta K}) \leq L \), which is not possible. Hence \( I \leq \tilde{I} \leq K \). Analogously, from (3.9) we deduce that \( K \leq \tilde{K} \leq L \).

If \( \tilde{I} = K \), again from (3.8) we derive

\[
L \leq e^{-\delta K} + L(1 - e^{-\delta K}) \leq L,
\]

hence \( L = \tilde{L} = K \). Inserting this into (3.9) yields \( L > e^{-\delta K} + L(1 - e^{-\delta K}) \geq I \), thus \( I = K \), which contradicts the assumption that \( l < L \). This shows that \( \tilde{I} < K \). Similarly, one shows that \( \tilde{L} > K \). Thus,

\[
l \leq \tilde{I} < K < \tilde{L} \leq L.
\]

Step 3. From steps 1 and 2, note that \( I > e^{-\delta K} = \theta \). Now, let \( f(x) := \frac{e^{-ax}}{\mu} \) and \( \mu := 1 - e^{-\delta K} = 1 - \frac{\theta}{\mu} \). Next, we prove that (h1) implies

\[
f(x) < \frac{1}{\mu}, \quad x > \theta. \tag{3.10}
\]

Since \( f \) is decreasing, (3.10) holds if \( f(\theta) < \mu^{-1} \), which is equivalent to

\[
\theta > \frac{1}{a} \log \frac{P}{\delta} + \frac{1}{a} \log(1 - e^{-\delta K}) =: \theta_1. \tag{3.11}
\]

Inserting the identity \( \theta = \frac{1}{a} \log \frac{P}{\delta} \) in (3.11), we have \( \theta > \theta_1 \) if and only if

\[
(1 - e^{-\delta K}) \log \frac{P}{\delta} + \log(1 - e^{-\delta K}) < 0.
\]

Since \( x + \log(1 - x) < 0 \) for all \( x \in (0, 1) \), from (3.4) we conclude that

\[
(1 - e^{-\delta K}) \log \frac{P}{\delta} + \log(1 - e^{-\delta K}) \leq e^{-\delta K} + \log(1 - e^{-\delta K}) < 0,
\]

thus (3.10) holds. Therefore, the function

\[
h(x) := \frac{\theta}{1 - \mu f(x)} \tag{3.12}
\]

is well-defined on \( I := [\theta, \infty) \). Note that \( h(K) = K \) and \( (h(x) - K)(x - K) < 0 \) for \( x \geq \theta, x \neq K \). From (3.8) and (3.9), it follows that

\[
L \leq h(\tilde{I}) \leq h(I) \tag{3.13a}
\]

\[
l \geq h(\tilde{L}) \geq h(L). \tag{3.13b}
\]

We have \( f \in C^0 \), \( f'(x) < 0 \) and \( Sf(x) = -\frac{e^x}{2} < 0 \) for \( x \geq \theta \). Write \( h(x) = h_1(f(x)) \) where \( h_1(x) = \frac{\theta}{x - \mu x} \), and note that \( Sh_1(x) \equiv 0 \). From the formula \( Sh(x) = Sh_1(f(x))(f'(x))^2 + Sf(x) \) (see e.g. [20]), we have \( Sh(x) < 0 \) for all \( x \geq \theta \). On the other hand, \( h'(K) = -\alpha K e^{-\delta K} - 1 \), and therefore \(-1 \leq h'(K) < 0 \) is equivalent to

\[
0 < \log \frac{P}{\delta} \leq \frac{1}{e^\delta - 1}.
\]
which, with \( \zeta > 0 \), is our hypothesis (h1). Thus, from lemma 3.1(b), we derive that \( K \) is a global attractor of the difference equation

\[
x_{n+1} = h(x_n)
\]  
(3.14)

with initial conditions \( x_0 \geq \theta \). From lemma 3.1(a), this implies that there are no points \( c, d \in [\theta, \infty) \), \( c < d \), such that \( h([c, d]) \supset [c, d] \). This however contradicts (3.13), since clearly \( h([l, L]) = [h(L), h(l)] \supset [l, L] \) with \( l < K < L \). This shows that there are no solutions with \( l < L \) when the above limit \( \zeta \) in (3.3) exists.

Step 4. The assumption that the limit in (3.3) exists is now removed. Bearing this in mind, the proof follows as above up to conditions (3.8) and (3.9), now taking subsequences of \((t_n)\) such that

\[
\lim_{t_n \to t_\infty} \beta(s) = \zeta_1, \quad \lim_{t_n \to t_\infty} \beta(s) = \zeta_2
\]

for some \( \zeta_1, \zeta_2 \in [\underline{\zeta}_M, \overline{\zeta}_M] \), with \( \underline{\zeta}_M \) as in (2.8) and \( \overline{\zeta}_M := \lim_{r \to +\infty} \int_{r-\sigma(t)}^t \beta(s) \, ds \). In this way, for a solution \( x(t) \) with \( l < L \) in (3.2), the inequalities (3.8) and (3.9) become

\[
L \leq e^{-\delta_1} K + \frac{p}{\delta} K e^{-\delta_1} (1 - e^{-\delta_1}) \quad \text{and} \quad 1 \geq e^{-\delta_2} K + \frac{p}{\delta} K e^{-\delta_2} (1 - e^{-\delta_2}).
\]  
(3.15a, 3.15b)

As in step 2, one sees that it is not possible to have \( \zeta_1 = \zeta_2 = 0 \). If \( \zeta_1 > 0 \) and \( \zeta_2 = 0 \), proceeding as in step 2 would lead to \( l = l_\infty = K \) and \( l \leq K \), thus also \( L = K \), which contradicts the assumption \( l < L \). Thus, we deduce that \( \zeta_1 > 0 \) (i = 1, 2) and, arguing as above, that \( l \leq l < K < L \leq L \).

We now define

\[
h_i(x) = \frac{e^{-\delta_i} K}{1 - \mu_i f(x)}, \quad i = 1, 2,
\]

for \( x > f^{-1}(1/\mu_i) \), where \( \mu_i = 1 - e^{-\delta_i} \). Let

\[
j(x, \zeta) := \frac{1 - \mu}{1 - \mu f(x)} \quad \text{where} \quad \mu = \mu(\zeta) = 1 - e^{-\delta},
\]

and note that \( (x - K) j (x, \zeta) < 0 \) for all \( x \neq K \). Thus \( \zeta \mapsto j(x, \zeta) \) is increasing for \( x < K \) and decreasing for \( x > K \), hence for all \( x \) such that \( f(x) \mu(\zeta_M) < 1 \), it follows that

\[
h_1(x) = K j(x, \zeta_1) \leq K j(x, \zeta_M), \quad \text{if} \ x < K, \quad \text{(3.16a)}
\]

\[
h_2(x) = K j(x, \zeta_2) \geq K j(x, \zeta_M), \quad \text{if} \ x > K. \quad \text{(3.16b)}
\]

Define

\[
h(x) = \frac{e^{-\delta M} K}{1 - (1 - e^{-\delta M}) f(x)} = K j(x, \zeta_M).
\]  
(3.17)
From (3.15),
\[ L \leq h_1(\tilde{l}) \leq h(l), \]
\[ l \geq h_2(\tilde{L}) \geq h(L). \]

From this point onwards, we can resume the proof of step 3, and conclude that there are no solutions \( x(t) \) with \( l < L \) in (3.2). The proof that \( K \) is a global attractor is complete. □

**Remark 3.2.** For (1.4) with \( p > \delta > 0 \) and constant delays \( \tau \geq \sigma \geq 0 \), it was shown in [9] that \( K \) is a global attractor of all positive solutions if \((e^{p\sigma} - 1) \log(p/\delta) \leq 1\). This criterion is a very particular case of the result established in theorem 3.1.

Combining corollary 2.2 and theorem 3.1, we obtain:

**Theorem 3.2.** With the above notations, let \( p > \delta \) and assume that one of the following conditions holds:

(a) \( \log(p/\delta) \leq \frac{2\delta \zeta_M}{e^{\delta \zeta_M} - 1} \) and \( \delta \zeta_M (2 + \log(p/\delta)) < 1 \);

(b) \( \log(p/\delta) > \frac{2\delta \zeta_M}{e^{\delta \zeta_M} - 1} \) and \( (e^{\delta \zeta_M} - 1) \log \frac{p}{\delta} \leq 1 \).

Then the equilibrium \( K \) of (1.3) is globally asymptotically stable (in \( C^+_0 \)).

**Proof.** Define \( c = \frac{2\delta \zeta_M}{e^{\delta \zeta_M} - 1} \), \( g_1(x) = (e^x - 1) \log(p/\delta) \), \( g_2(x) = (2 + \log(p/\delta))x \). Since \( g_1(\delta \zeta_M) \leq g_2(\delta \zeta_M) \) if and only if \( \log(p/\delta) \leq c \), the latter condition and (2.8) imply (h1), and thus \( K \) is stable and globally attractive; if \( \log(p/\delta) > c \) and (h1) are satisfied, then (2.8) holds as well. □

**Remark 3.3.** The global attractivity in theorem 3.1 suggests that the criterion for local stability in corollary 2.2 is not sharp. In fact, depending on the size of the coefficients in (2.4), sharper results for the local asymptotic stability of \( K \) are occasionally obtained from the criteria in [5].

### 3.2. The case of \( m \) pairs of delays

Next, we address the global attractivity of the positive equilibrium \( K \) of (1.5).

**Theorem 3.3.** Consider (1.5) and define \( \zeta_M \) as in (2.6). Assume \( p > \delta \) and that:

(H1) \( \frac{a^+}{a^-} < \frac{1}{\zeta_M} \);

(H2) \( a^+ K (e^{\delta \zeta_M} - 1) \leq 1 \).

Then the equilibrium \( K \) of (1.5) is globally attractive.

**Proof.** The main arguments are as in the proof of theorem 3.1, thus some details will be omitted.

Let \( x(t) \) be a solution with initial condition in \( C^+_0 \). If there exists \( C = \lim_{t \to \infty} x(t) \), proceeding as in the above mentioned theorem it follows that \( C = K \).

Define \( l, L \) as in (3.2), and suppose that \( l < L \). Since afterwards one may proceed as in step 4 above and obtain ((3.16) and (3.17)), without loss of generality assume already that there exist

\[ \zeta_j := \lim_{t \to \infty} \int_{t-\sigma_j(t)}^t \beta(s) \, ds, \quad j = 1, \ldots, m. \]
Step 1. We argue along the lines of step 1 of the above proof. Consider sequences \((t_n), (s_n)\) with \(t_n, s_n \to \infty\), \(x(t_n) \to L, x(s_n) \to l, x'(t_n) = x'(s_n) = 0\), such that (by taking subsequences, if necessary), for each \(j \in \{1, \ldots, m\} \),

\[
\begin{align*}
x(t_n - \tau_f(t_n)) & \to L_{\gamma_j}, \quad x(t_n - \sigma_f(t_n)) \to L_{\sigma_j}, \\
x(s_n - \tau_f(s_n)) & \to l_{\gamma_j}, \quad x(s_n - \sigma_f(s_n)) \to l_{\sigma_j}.
\end{align*}
\]

(3.18)

As before, from \(x'(t_n) = x'(s_n) = 0\) and taking limits, we deduce that

\[
L = \frac{1}{\delta} \sum_j p_j L_{\gamma_j} e^{-a_j L_{\gamma_j}} \quad \text{and} \quad l = \frac{1}{\delta} \sum_j p_j L_{\sigma_j} e^{-a_j L_{\sigma_j}},
\]

and consequently

\[
\min_j L_{\sigma_j} \leq K \leq \max_j l_{\sigma_j},
\]

which leads to \(l \leq K \leq L\). Moreover, for each \(j = 1, \ldots, m\), there are \(\tilde{L}_j, \tilde{L}_m \in [l, L] (1 \leq i \leq m)\) for which

\[
\begin{align*}
L \leq e^{-\delta K} l_{\sigma_j} + \frac{1}{\delta} \sum_{i=1}^m p_i e^{-a_i \tilde{\xi}} (1 - e^{-\delta K}), \\
l \geq e^{-\delta K} l_{\sigma_j} + \frac{1}{\delta} \sum_{i=1}^m p_i e^{-a_i \tilde{\xi}} (1 - e^{-\delta K}).
\end{align*}
\]

(3.19)

For

\[
\min_{1 \leq i, j \leq m} \tilde{L}_j =: \tilde{\tilde{L}}_j, \quad \max_{1 \leq i, j \leq m} \tilde{L}_j =: \tilde{L}_m
\]

from the inequalities in (3.19) we deduce that

\[
\begin{align*}
L & \leq e^{-\delta K} l_{\sigma_j} + L(1 - e^{-\delta K}) f(\tilde{\tilde{L}}), \\
l & \geq e^{-\delta K} l_{\sigma_j} + l(1 - e^{-\delta K}) f(\tilde{L}), \quad j = 1, \ldots, m,
\end{align*}
\]

(3.20)

where, as before, \(f(x) = \frac{1}{\delta} \sum_{i=1}^m p_i e^{-a_i x}, x \geq 0\).

Step 2. We claim that \(l < K < L\).

As for the case \(m = 1\), one easily shows that \(\tilde{L} \leq K \leq \tilde{L} \leq \tilde{L} \leq \tilde{L}\). Next suppose that \(\tilde{L} = K\), so that \(f(\tilde{L}) = 1\). From (3.20), one obtains \(l_{\sigma_j} = L\) for all \(j\), thus \(L = \tilde{L} = K\). Inserting \(L = K\) in the second inequality of (3.20) for \(i\) such that \(l_{\sigma_i} = \max_{1 \leq j \leq m} l_{\sigma_j}\), since \(l_{\sigma_i} \geq K\) we arrive to

\[
l \geq e^{-\delta K} l_{\sigma_i} + l(1 - e^{-\delta K}) \geq e^{-\delta K} K + l(1 - e^{-\delta K}) \geq l,
\]

thus \(l = K\), which is not possible. A similar contradiction is obtained if \(\tilde{L} = K\). Therefore, \(l \leq \tilde{L} < K \leq L \leq L\).

Step 3. Define

\[
h_j(x) = \frac{e^{-\delta K} K}{1 - (1 - e^{-\delta K}) f(x)} \quad \text{for} \quad x > \theta_j, \quad j = 1, \ldots, m,
\]
and
\[ h(x) = \frac{e^{-\delta M}K}{1 - (1 - e^{-\delta M})f(x)} \quad \text{for } x > \theta_1, \]
where \( \theta_1 > 0 \) is such that \( f(\theta_1) = (1 - e^{-\delta M})^{-1} \). From (3.20), for \( j \) such that \( L_j \leq K \), we have
\[ L \leq h_j(\tilde{L}) \leq h_j(L) \leq h(L); \]
and for \( j \) such that \( L_j \geq K \), we obtain
\[ l \geq h_j(L) \geq h(L). \]

From this point onward, to resume the proof of theorem 3.1 it suffices to show that our hypotheses imply the claims (i)–(iii) below.

Claim (i): \( Sf(x) < 0 \) for \( x > 0 \).

In fact, by virtue of assumption (H1),
\[
S f(x) = \sum_j p_j a_j^3 e^{-a_j x} \sum_j p j a_j e^{-a_j x} - \frac{3}{2} \left( \sum_j p j a_j e^{-a_j x} \right)^2
\]
\[ = \frac{1}{2 \left( \sum_j p j e^{-a_j x} \right)^2} \left( 2 \left( \left( \sum_j p j a_j^3 e^{-a_j x} \right) \left( \sum_j p j a_j e^{-a_j x} \right) - 3 \left( \sum_j p j a_j^3 e^{-a_j x} \right)^2 \right) \right)
\]
\[ = \frac{1}{2 \left( \sum_j p j e^{-a_j x} \right)^2} \sum_{j, \mu} p j p \mu e^{-a_j + a_\mu} a_j a_\mu (2a_j - 3a_\mu) < 0. \]

Claim (ii): \( \| h'(K) \| \leq 1 \).

The assertion follows by observing that \( h'(K) = K(e^{\delta M} - 1)f'(K) \), thus (H2) leads to
\[
\| h'(K) \| = K(e^{\delta M} - 1) \leq (e^{\delta M} - 1)a^+ K \leq 1.
\]

Claim (iii): \( \theta := Ke^{-\delta M} > \theta_1 \).

This claim is equivalent to \( f(\theta) \leq (1 - e^{-\delta M})^{-1}. \) From (H2), \( \theta = Ke^{-\delta M} \geq K - \frac{1}{a^+} e^{-\delta M} \) and, since \( f \) is decreasing,
\[
(1 - e^{-\delta M})f(\theta) \leq (1 - e^{-\delta M})f \left( K - \frac{1}{a^+} e^{-\delta M} \right)
\]
\[ = (1 - e^{-\delta M}) \frac{1}{a^+} \sum_j p_j \exp \left( -a_j K + \frac{a_j}{a^+} e^{-\delta M} \right) \]
\[ \leq (1 - e^{-\delta M}) \exp \left( e^{-\delta M} \right). \quad (3.21) \]

Next, again observe that
\[
\log(1 - e^{-\delta M}) + e^{-\delta M} < 0,
\]
Proposition 3.1. Let \( \gamma \in (0,1) \) and \( \delta, \beta, \tau, \sigma_p, \tau_j(\cdot) \) be continuous functions with \( 0 < \beta^- \leq \beta(\cdot) \leq \beta^+, 0 < \sigma_p(\cdot) \leq \tau \) for all \( j = 1, \ldots, m, t \geq t_0 \), and denote \( a^+ = \max_{1 \leq j \leq m} a_j \). Assume that:

(A1) \( a^- K_{\gamma}(e^{\beta^+ t} - 1) \leq 1 \);

(A2) \( e(1 - e^{-\beta^+ \tau}) < 1 \);

(A3) \( \frac{1 - e^{-\beta^+ \tau}}{1 - e^{-\beta^+ \tau}} a^+ K_{\gamma} \leq 1 \).

Then, \( K_{\gamma} \) is a global attractor for (3.22).

In [15], the main technique to prove the above result is the fluctuation lemma, combined with the classical lemma of So and Yu [24], which tells us that two non-negative real numbers \( \lambda, \mu \) satisfying the inequalities \( e^{-\mu} - 1 \leq -\lambda \) and \( \mu \leq e^\lambda - 1 \) must be zero.

We stress that the case \( \gamma = 1 \) in (3.22) was not covered in [15], however for \( p > \delta \) the positive equilibrium \( K_{\gamma} \) of (3.22) converges to the equilibrium \( K > 0 \) of (1.5), as \( \gamma \to 1 \). Clearly, by letting \( \gamma \to 1 \), in (A1) one obtains a stronger version of our hypothesis (H2), since \( \zeta_M \leq \beta^+ \). Note also that (A2) translates as \( \beta^+ \tau < 1 - \log(e - 1) \approx 0.46 \), which is a severe restriction to be imposed on the size of the delays, regardless of the size of the equilibrium \( K_{\gamma} \) that one

\[ x(t) = -\delta x(t) + p x(t - \tau) e^{-\alpha(t - \tau)} - \delta x(t) \]

\[ x(t) = \beta(t) \left( \sum_{j=1}^{m} p_j x(t - \tau_j(t)) e^{-a_j(t - \tau_j(t))} - \delta x(t) \right), \]  

(3.22)

Remark 3.4. Clearly the case \( m = 1 \) in theorem 3.1 is recovered by theorem 3.3. To this end, as done in theorem 3.2, the assumptions in theorem 3.3 together with (2.5) or (2.7) provide sufficient conditions for the global asymptotic stability of \( K \).

Remark 3.5. Clearly, theorem 3.3 could be stated under slightly weaker hypotheses, as long as they allow to conclude that claims (i)–(iii) in the proof above are satisfied.

Since in general \( K \) is not explicitly computable when there are multiple pairs of delays, a criterion with (H2) not depending on \( K \) is useful.

Corollary 3.1. With the previous notations, assume \( p > \delta \), (H1) and \( (H2^*) \alpha^-(e^{\delta \mu} - 1) \log \frac{\delta}{\mu} \leq 1 \).

Then the equilibrium \( K \) of (1.5) is globally attractive.

Proof. From \( f(K) = 1 \), we get \( a^- K = \log(p/\delta) \leq a^+ K \) and consequently \( (H2^*) \) implies (H2). \( \square \)

Remark 3.6. The economic model \( x'(t) = -\delta x(t) + px(t - \tau) e^{-a(t - \tau)} \) with \( \gamma \in (0,1) \) is often referred to as a neoclassical growth model, where \( \tau \) takes into account the time delay in the process of production of capital or in the reaction to market changes and fluctuations [18]. Recently, the model

\[ x'(t) = \beta(t) \left( \sum_{j=1}^{m} p_j x(t - \tau_j(t)) e^{-a_j(t - \tau_j(t))} - \delta x(t) \right), \]  

(3.22)

with \( \gamma \in (0,1) \) and \( p_j, a_j, \delta, \beta(t), \sigma_p(t), \tau_j(t) \) as in (1.5), was studied in [15]. It is worth emphasising that there is always a positive equilibrium \( K_{\gamma} \), for (3.22) given by the equation \( \sum_{j} p_j e^{-a_j K_{\gamma}} = \delta K_{\gamma} \). Huang et al [15, theorem 3.3] proved the criterion for the global attractivity of \( K_{\gamma} \) given below.
may have. These observations suggest that the result in proposition 3.1 might be improved by using different techniques.

Example 3.1. Inspired by an example in [15], consider (1.5) with $m = 2$, $p_1 = \frac{1}{50}e^4$, $p_2 = \frac{1}{25}e^5$, $a_1 = \frac{4}{5}$, $a_2 = 1$, $\delta = 0.1$, and $\beta(t), \tau(f), \sigma(f)$ $(j = 1, 2)$ continuous, non-negative and bounded, with $\beta(t) \geq \beta^* > 0$ on $[0, \infty)$. In this situation, the positive equilibrium is explicitly computed as $K = 5$. With the above notations, $f(x) = \frac{1}{2}e^{4/(1-x/5)} + \frac{e^5}{2}x$, $\theta = 5e^{-0.1}/m$, $\frac{e^5}{\theta} = \frac{5}{2} < \frac{1}{2}$ and $|h'(5)| = 4.4(e^{0.1/5} - 1)$. Hence $|h'(5)| \leq 1$ if

$$\zeta_M \leq 10 \log \left( \frac{27}{22} \right) \approx 2.05.$$  

(3.23)

If (3.23) holds, then $(1 - e^{-0.1/5})h(\theta) \leq (1 - \frac{22}{27})h(\theta) \leq \frac{1}{2}(3e^{10/2} + 2e^{15/7}) \approx 0.42 < 1$. Theorem 3.3 and remark 3.5 allow concluding that $K = 5$ is a global attractor of all positive solutions. On the other hand, note that (H2) reads as $\zeta_M \leq 10 \log(1,2) \approx 1.82$, a more restrictive imposition.

For instance, choose $\beta(t) = 1 + \sin^2 t$, $\sigma(f) = |\cos^2 t|$ and any $\tau(f)$ satisfying the general conditions above, $j = 1, 2$. Clearly $\zeta_M \leq 2$, thus (3.23) holds. With the same $\beta(t)$ and $\sigma(f) = 2/\pi(j = 1, 2)$, one sees that $\zeta_M \leq 3/2$, thus $\delta \zeta_M(2 + \log(p/\delta)) \approx 0.98 < 1$. From corollary 2.1, $K$ is globally asymptotically stable.

Example 3.2. Consider again (1.5) with $m = 2$, but now suppose that $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{25}, a_1 = \frac{4}{5}, a_2 = 1$, $\delta = 0.09$, and $\beta(t), \tau(f), \sigma(f)$ $(j = 1, 2)$ satisfy the general conditions above. In this situation, $p = 0.1 > 0.09$, but the positive equilibrium $K$ is not explicitly given. Instead, we can assert that (H2*) is satisfied if

$$\zeta_M \leq \frac{100}{9} \log \left( 1 + \frac{4}{5 \log \frac{10}{24}} \right) \approx 23.90,$$

in which case $K$ is globally attractive.

4. Conclusions

In the present paper, we have considered a Nicholson-type equation with multiple pairs of delays, equation (1.5). The presence of multiple mixed monotone nonlinear terms, each one with two different time-varying delays, may drastically alter the dynamics and stability of equilibria, when compared with either the associated differential equation without delays, or the DDE with only one delay $\tau(f) = \sigma(f)$ $(1 \leq j \leq m)$ in each nonlinear term [4].

In [17], the authors showed the global stability of the equilibrium 0 when $p = \sum_j p_j \leq \delta$. Here, we have assumed $p > \delta$, proved the permanence without additional conditions, and studied the local stability and global attractivity of the positive equilibrium $K$ of (1.5). Inspired by the techniques in [9], we provide sufficient conditions for the global attractivity of $K$ which are easily verifiable (see the main result theorem 3.3). Our criteria do not involve the delays $\tau(f)$, depend on the size of the delays $\sigma(f)$ in the sense that the limit $\zeta_M$ in (2.6) must not be too large so that (H2) (or alternatively the set of claims (ii) and (iii) in the proof of theorem 3.3) holds, and impose the constraint expressed by (H1) on the relative sizes of the several coefficients $a_j$. Whether one can replace (H1), or alternatively the requirement of $S(f(x))$ negative, by a weaker condition is a question deserving further analysis. The results presented here largely generalise the ones obtained in [9] for (1.4). Also, they provide an answer to an
The results in this paper also raise some interesting problems. As stated in the introduction, we anticipate that the present method works if applied to versions of other classical scalar delayed models with mixed monotonocities, in particular to Mackey–Glass type equations. Moreover, we believe that the application of our method would lead to original results if adopted to other Nicholson-type equations, such as model (3.22) with \( \gamma \in (0, 1) \), studied in [15]. Also, instead of (1.5) one may consider the general nonautonomous model
\[
x'(t) = \sum_{j=1}^{m} p_j(t)x(t - \tau_j(t))e^{-a_j(t)(t - \sigma_j(t))} - \delta(t)x(t).
\]

Equation (4.1) is said to be \textit{globally attractive} (in \( C_0^+ \)) if all its positive solutions are globally attractive, in the sense that
\[
x(t, 0, \phi) - x(t, 0, \psi) \to 0 \quad \text{as } t \to \infty,
\]
for any \( \phi, \psi \in C_0^+ \); if moreover all positive solutions are stable, then (4.1) is \textit{globally asymptotically stable}. For (4.1), it is natural to inquire about sufficient conditions for extinction, permanence, global attractivity and (asymptotic or exponential) stability. These will be the aims of our future work, where, since the procedure developed here is not directly applicable to (4.1), we intend to exploit different techniques such as the ones proposed in [11]. Another goal is to establish global stability criteria for systems of DDEs with mixed monotonocities with the help of Schwarzian derivatives and difference equations. This difficult task has been initiated in [10] for a particular family of autonomous systems \( x'_i(t) = -d_i x_i(t) + f_i(x_i(t - \tau_i), x_i(t - \sigma_i)) (1 \leq i \leq n) \), but we expect to use our method to address more general frameworks. The treatment of mixed monotonocity Nicholson systems with patch structure, given by \( n \)-dimensional versions of (1.5), is clearly a challenge deserving attention and another strong motivation for future research.

Acknowledgments

This work was supported by Fundação para a Ciência e a Tecnologia (Portugal) under project UIDB/04561/2020. The research of H Prates was funded by a BII fellowship of the research center CMAFCIO, under the above project. The authors are very grateful to the referees and editors, whose valuable comments led to significant improvements in the exposition of the paper.

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