L∞ A-PRIORI ESTIMATES FOR SUBCRITICAL
p-LAPLACIAN EQUATIONS WITH A
CARATHÉODORY NONLINEARITY

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Abstract. We present new L∞ a priori estimates for weak solutions of a wide class of subcritical p-laplacian equations in bounded domains. No hypotheses on the sign of the solutions, neither of the non-linearities are required. This method is based in elliptic regularity for the p-laplacian combined either with Gagliardo-Nirenberg or Caffarelli-Kohn-Nirenberg interpolation inequalities.

Let us consider a quasilinear boundary value problem

\[-\Delta^p u = f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega,\]

with \( p < N \), is a bounded smooth domain strictly convex, and \( f \) is a subcritical Carathéodory non-linearity. We provide \( L^\infty \) a priori estimates for weak solutions, in terms of their \( L^p \)-norm, where

\[ p^* = \frac{Np}{N-p} \]

is the critical Sobolev-Hardy exponent. Our non-linearities includes non-power non-linearities.

In particular we prove that when

\[ f(x, s) = |x|^{-\mu} \tilde{f}(s), \quad \text{with } \mu \in (0, p), \quad \text{and } \tilde{f}(s)|s|^{p^*-1} \to 0 \text{ as } |s| \to \infty, \]

there exists a constant \( C_\varepsilon > 0 \) such that for any solution \( u \in H^1_0(\Omega) \), the following holds

\[ \left[ \log \left( e + \| u \|_\infty \right) \right] ^\alpha \leq C_\varepsilon \left( 1 + \| u \|_{p^*} \right)^{(p^*-2)(1+\varepsilon)}, \]

where \( C_\varepsilon \) is independent of the solution \( u \).

A priori estimates, subcritical nonlinearity, changing sign weight, \( L^\infty \) a priori bound, singular elliptic equations.

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1. Introduction

Let us consider the following quasilinear boundary value problem involving the p-Laplacian

\[ -\Delta^p u = f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega, \]
where $\Delta_p(u) = \text{div}(|Du|^{p-2}Du)$ is the $p$-Laplace operator, $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$, $N > p$, is a bounded, strictly convex, open subset with $C^2$ boundary $\partial\Omega$, and the non-linearity $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function (that is, the mapping $f(\cdot, s)$ is measurable for all $s \in \mathbb{R}$, and the mapping $f(x, \cdot)$ is continuous for almost all $x \in \Omega$), and subcritical (see definition 1.1).

We analyze the effect of the smoothness of the subcritical non-linearity $f = f(x,u)$ on the $L^\infty(\Omega)$ a priori estimates of weak solutions to (1.1). This study is usually focused on positive classical solutions, see the classical references of de Figueiredo-Lions-Nussbaum, and of Gidas-Spruck [15, 20], see also [8, 9].

A natural question concerning the class of uniformly bounded solutions is the following one,

(Q1) those $L^\infty(\Omega)$ estimates apply also to a bigger class of solutions, in particular to weak solutions (and to changing sign solutions)?

Another natural question with respect to the class of subcritical non-linearities, can be stated as follows,

(Q2) those $L^\infty(\Omega)$ estimates are valid into a bigger class of non-linearities (not asymptotically powers), and in particular to non-smooth non-linearities (with possibly changing sign weights)?

In this paper we extend the previous work in [27] for $p = 2$, and provide sufficient conditions guaranteeing uniform $L^\infty(\Omega)$ a priori estimates for any $u \in W_{1,p}^{0}(\Omega)$ weak solution to (1.1), in terms of their $L^{p^{*}}(\Omega)$ bounds, in the class of Caratheodory subcritical generalized problems. In this class, we state that any set of weak solutions uniformly $L^{p^{*}}(\Omega)$ a priori bounded is universally $L^\infty(\Omega)$ a priori bounded. Our theorems allow changing sign weights, and singular weights, and also apply to changing sign solutions.

Problem (1.1) with $f(x,s) = |x|^{-\mu}|s|^{q-1}s$, $\mu > 0$, is known as Hardy’s problem, due to its relation with the Hardy-Sobolev inequality. The Caffarelli-Kohn-Nirenberg interpolation inequality for radial singular weights [6], states that whenever $0 \leq \mu \leq p$,

$$p_\mu^* := \frac{p(N-\mu)}{N-p},$$

is the critical exponent of the Hardy-Sobolev embedding $W_{0}^{1,p}(\Omega) \hookrightarrow L^{p_\mu^*}(\Omega, |x|^{-\mu})$ (this embedding is continuous but not compact). For the case $0 \leq \mu \leq p$, using a Pohozaev type identity, Pucci and Servadei
prove some non-existence results in $\mathbb{R}^N$. Some existence and non-existence results for power like nonlinearities can be found in [1,18,19,22], see also [31] for the case $p = N$.

Usually the term subcritical non-linearity is reserved for power like non-linearities. Next, we expand this concept to nonlinearities including the class $o(|s|^{p^*_N - 1})$.

**Definition 1.1.** By a subcritical non-linearity we mean that $f$ satisfies one of the following two growth conditions:

(H0) $$|f(x, s)| \leq |a(x)| \tilde{f}(s)$$

where $a \in L^r(\Omega)$ with $r > N/p$, $\tilde{f} : \mathbb{R} \to [0, +\infty)$ is continuous and satisfy

(1.4) $\tilde{f}(s) > 0$ for $|s| > s_0$, and $\lim_{s \to \pm \infty} \frac{\tilde{f}(s)}{|s|^{p^*_N - 1}} = 0$,

where

(1.5) $$p^*_{N/r} := \frac{p^*}{r'} = p^* \left(1 - \frac{1}{r}\right),$$

and where $r'$ is the conjugate exponent of $r$, $1/r + 1/r' = 1$, or

(H0)' $|f(x, s)| \leq |x|^{-\mu} \tilde{f}(s)$,

where $\mu \in (0, p)$, and $\tilde{f} : \mathbb{R} \to [0, +\infty)$ is continuous and satisfy

(1.7) $\tilde{f}(s) > 0$ for $|s| > s_0$, and $\lim_{|s| \to \infty} \frac{\tilde{f}(s)}{|s|^{p^*_N - 1}} = 0$.

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1 Since $r > N/p$, then $p^*_{N/r} > p$. Moreover, thanks to Sobolev embeddings, for any $u \in W^{1, p}_{0}(\Omega)$,

$$\tilde{f}(u) \in L^{p^*/r'}(\Omega) \quad \text{with} \quad \frac{p^*_{N/r} - 1}{p^*} = 1 - \frac{1}{r} + \frac{1}{N} - \frac{1}{p},$$

and $f(\cdot, u) \in L^{(p^*)'}(\Omega)$.

2 Observe that $p^*_\mu > p$ for $\mu \in (0, p)$. Let $a(x) = |x|^{-\mu}$, then $a \in L^p(\Omega)$ for any $p < N/\mu$. Moreover, the sharp Caffarelli-Kohn-Nirenberg inequality implies that if $u \in W^{1, p}_{0}(\Omega)$, then $f(\cdot, u) \in L^{(p^*)'}(\Omega)$. 

Our analysis shows that non-linearities satisfying either (H0): \((1.3)-(1.4)\) (either (H0)': \((1.6)-(1.7)\)), widen the class of subcritical non-linearities to non-power non-linearities, sharing with power like non-linearities properties such as \(L^\infty\) a priori estimates. Our definition of a subcritical non-linearity includes non-linearities such as

\[
f^{(1)}(x, s) := \frac{a(x)|s|^{p_{N/r}-2}s}{\log(e+|s|)^\alpha}, \quad \text{or} \quad f^{(2)}(x, s) := \frac{|x|^{-\mu}|s|^{p^*_s-2}s}{\log [e + \log(1 + |s|)]^\alpha},
\]

for any \(\alpha > 0\), and \(\mu \in (0, p)\), or any \(a \in L^r(\Omega)\), with \(r > N/p\).

In particular, if \(f(x, s) = f^{(1)}(x, s)\) with \(a \in L^r(\Omega)\) for \(r \in (N/p, N]\), then for any \(\varepsilon > 0\) there exists a constant \(C > 0\) depending only on \(\varepsilon, \Omega, r\) and \(N\) such that for any \(u \in W^{1,p}_0(\Omega)\) solution to \((1.1)\), the following holds:

\[
\left[ \log \left( e + \|u\|_\infty \right) \right]^\alpha \leq C \|a\|_r^{1+\varepsilon} \left( 1 + \|u\|_p^* \right)^{(p_{N/r}^-p)(1+\varepsilon)},
\]

where \(C\) is independent of the solution \(u\), see Theorem 1.3. Related results concerning those non-power non-linearities can be found in [14], and for \(p = 2\) in [12] analyzing what happen when \(\alpha \to 0\), in [13] with changing sign weights, in [23] for systems, and in [28] for the radial case.

Moreover, if \(f(x, s) = f^{(2)}(x, s)\) with \(\mu \in [1, p)\), then for any \(\varepsilon > 0\) there exists a constant \(C > 0\) depending on \(\varepsilon, \mu, N\), and \(\Omega\), such that for any \(u \in W^{1,p}_0(\Omega)\) solution to \((1.1)\), the following holds:

\[
\left[ \log \left[ e + \log \left( 1 + \|u\|_\infty \right) \right] \right]^\alpha \leq C \left( 1 + \|u\|_p^* \right)^{(p_s^*-p)(1+\varepsilon)},
\]

and where \(C\) is independent of the solution \(u\), see Theorem 1.4.

**Definition 1.2.** By a solution we mean a weak solution \(u \in W^{1,p}_0(\Omega)\) such that \(f(\cdot, u) \in L^{p^*(\cdot)}(\Omega)\), and

\[
(1.8) \quad \int_\Omega |\nabla|^{p-2}\nabla u \cdot \nabla \varphi = \int_\Omega f(x, u) \varphi, \quad \forall \varphi \in W^{1,p}_0(\Omega).
\]

To state our main results, for a non-linearity \(f\) satisfying \((1.3)-(1.4)\), let us define

\[
(1.9) \quad h(s) = h_{N/r}(s) := \frac{|s|^{p_{N/r}-1}}{\max \{f(-s), f(s)\}} \quad \text{for} |s| > s_0.
\]
And for a non-linearity $f$ satisfying (1.6)-(1.7), let us now define

$$h(s) = h_\mu(s) := \frac{|s|^{p^* - 1}}{\max \{\tilde{f}(-s), \tilde{f}(s)\}}, \quad \text{for } |s| > s_0.$$  

By sub-criticality, (see (1.4) or (1.7) respectively),

$$h(s) \to \infty \quad \text{as } s \to \infty.$$

Let $u$ be a solution to (1.1). We estimate $h(\|u\|_\infty)$, in terms of the $L^p^*$-norm of $u$.

Our main results are Theorem 1.3 and Theorem 1.4 stated in the following two subsections for (H0) or (H0)' respectively.

1.1. Estimates of the $L^\infty$-norm of the solutions to (1.1) in presence of a Carathéodory nonlinearity.

We assume that the non-linearity $f$ satisfies the growth condition (H0),
and that $\tilde{f} : \mathbb{R} \to (0, +\infty)$ satisfies the following hypothesis:

(H1) there exists a uniform constant $c_0 > 0$ such that

$$\limsup_{s \to +\infty} \frac{\max_{[-s,s]} \tilde{f}}{\max \{\tilde{f}(-s), \tilde{f}(s)\}} \leq c_0.$$  

Under hypothesis (H0)-(H1), we establish an estimate for the function $h$ applied to the $L^\infty(\Omega)$-norm of any $u \in W^{1,p}_0(\Omega)$ solution to (1.1), in terms of their $L^p^*$($\Omega$)-norm.

From now on, $C$ denotes several constants that may change from line to line, and are independent of $u$.

Our first main results is the following theorem.

**Theorem 1.3.** Assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying (H0)-(H1).

Then, for any $u \in W^{1,p}_0(\Omega)$ weak solution to (1.1), the following holds:

(i) either there exists a constant $C > 0$ such that $\|u\|_\infty \leq C$, where $C$ is independent of the solution $u$,

(ii) either for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$h(\|u\|_\infty) \leq C\|a\|_r^{A+\varepsilon} \left(1 + \|u\|_{p^*}^{(p^*_N/p)A+\varepsilon}\right).$$

Observe that in particular, if $\tilde{f}(s)$ is monotone, then (H1) is obviously satisfied with $c_0 = 1$. 

\[\]
where $h$ is defined by (1.9),

\begin{equation}
A := \begin{cases} 
1, & \text{if } r \leq N, \\
\frac{p^*_{N/r} - 1}{p^*_{N/p}}, & \text{if } r > N,
\end{cases}
\end{equation}

and $C$ depends only on $\varepsilon$, $c_0$ (defined in (1.12)), $r$, $N$, and $\Omega$, and it is independent of the solution $u$.

As an immediate consequence, as soon as we have a universal a priori $L^{p^*}$-norm for weak solutions in $W^{1,p}_0(\Omega)$, then solutions are a priori universally bounded in the $L^\infty$-norm, see Corollary 3.1.

1.2. Estimates of the $L^\infty$-norm of the solutions to (1.1) in presence of radial singular weights.

Now, assuming that $0 \in \Omega$ and that $|f(x,s)| \leq |x|^{-\mu} \tilde{f}(s)$ for some $\mu \in (0,p)$, we state our second main result concerning weak solutions for singular subcritical non-linearities, see the following theorem.

**Theorem 1.4.** Assume that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying (H0)' and (H1).

Then, for any $u \in W^{1,p}_0(\Omega)$ solution to (1.1), the following holds:

(i) either there exists a constant $C > 0$ such that $\|u\|_\infty \leq C$, where $C$ is independent of the solution $u$,

(ii) either for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$h(\|u\|_\infty) \leq C\varepsilon \left(1 + \|u\|_{p^*}^{(p^*_\mu - p)(B + \varepsilon)}\right),$$

where $h$ is defined by (1.10),

\begin{equation}
B := \begin{cases} 
\frac{p^*_\mu - 1}{p^*_{N/p}}, & \text{if } \mu \in (0,1), \\
1, & \text{if } \mu \in [1,p),
\end{cases}
\end{equation}

and $C$ depends only on $\varepsilon$, $c_0$ (defined in (1.12)), $\mu$, $N$, and $\Omega$, and it is independent of the solution $u$.

This results hold for positive, negative and changing sign non-linearities, and also for positive, negative and changing sign solutions. The techniques and ideas introduced in [27] are robust enough to be used for proving analogues of our results in other non-linear problems. Here we

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\[ \text{4Observe that } \frac{p^*_\mu - 1}{p^*_{N/p}} = 1 + \frac{\mu}{p-1} \frac{1-\mu}{N} = B \text{ if } \mu \in (0,1). \]
present the work for the \( p \)-Laplacian. The work for nonlinear boundary conditions is actually in preparation by Chhetri, Mavinga, and the author.

This paper is organized in the following way. Section 2 collects some well known results. In Section 3, using Gagliardo–Nirenberg inequality, we analyze the case when \( a \in L^r(\Omega) \) with \( r > N/p \), see Theorem 1.3. In Section 4, we analyze the more involved case of a radial singular weight, see Theorem 1.4. It yields on the Caffarelli-Kohn-Nirenberg inequality.

2. Preliminaires and known results

2.1. Gradient Regularity.

We are going to use the following result about the summability of the gradient for solutions to equations involving the \( p \)-Laplace operator.

**Theorem 2.1 (Gradient Regularity).** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), and let \( u \in W^{1, p}_0(\Omega) \), \( 1 < p < \infty \), be a solution of the problem

\[
\begin{aligned}
-\Delta_p(u) &= g \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

with \( g \in L^q(\Omega) \). We assume that

\[
\begin{aligned}
1 < q < \infty & \quad \text{if } p \geq N, \\
(p^*)' \leq q < \infty & \quad \text{if } 1 < p < N.
\end{aligned}
\]

Here \( p^* = \frac{Np}{N - p} \) is the critical exponent for Sobolev embedding, and \( (p^*)' = \frac{p^*}{p^* - 1} = \frac{Np}{Np - N + p} \), is its conjugate exponent.

i) If \( q < N \), then \( \|\nabla u\|_{L^{p^*}(\Omega)} \leq C \|g\|_{L^q(\Omega)}^{1 - \frac{1}{p^* - 1}} \)

ii) If \( q \geq N \), then \( \|\nabla u\|_{L^q(\Omega)} \leq C \|g\|_{L^q(\Omega)}^{\frac{-1}{p^* - 1}} \) for any \( \sigma < \infty \).

Here \( C \) is a constant that depends on \( p, N, q \).

**Remark 2.2.** The exponent \( (p^*)' \) is called the duality exponent, and the condition \( q \geq (p^*)' \) if \( 1 < p < N \) guarantees by Sobolev’s embeddings that \( g \in L^q(\Omega) \) belongs to the dual space \( W^{-1,p'}(\Omega) \). If other cases, we enter into the field of problems with measure data, and other definitions of solutions have to be considered (see [2], [25]).

The previous theorem follows from different results proved in several papers (see [2], [5], [10], [16], [17], [21], [25], the survey [11], and the references therein), where more general situations are also considered.
2.2. Improved regularity of the weak solutions.

We first collect a regularity Lemma for any weak solution to (1.1) with a non-linearity of sub-critical growth, in fact weak solutions in $W^{1,p}_0(\Omega)$ are in $L^q$ for any finite $q \geq 1$, see [29, Theorem 2.1, Theorem 2.2].

**Theorem 2.3** (Improved regularity). Assume that $u \in W^{1,p}_0(\Omega)$ weakly solves (1.1) for a Carathéodory non-linearity $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ with sub-critical groth, either (H0), either (H0)', see (1.3)-(1.4) or (1.6)-(1.7) respectively.

Then, $u \in L^q(\Omega)$ for any $1 \leq q < \infty$.

Moreover, $u \in L^\infty(\Omega)$.

**Proof.** We first adapt to the $p$-laplacian the technique used in [15] (based in Brezis-Kato, see [4]) to get the $L^q$ estimates for any finite $q \geq 1$.

Testing the equation $-\Delta_p u = f(x, u)$ with $|u|^t$, $t \geq 1$, we get that

$$t \int_\Omega |\nabla u|^p |u|^t \, dx = \int \int f(x, u) |u|^t \, dx.$$

Since

$$|\nabla (|u|^{p^{-1}+t})|^p = \left(\frac{p-1+t}{p}\right)^p |\nabla u|^p |u|^{t-1},$$

we can write the previous equation as

$$(2.3) \quad t \left(\frac{p}{p-1+t}\right)^p \int_\Omega |\nabla (|u|^{p^{-1}+t})|^p = \int_\Omega f(x, u) |u|^t.$$

(i) We start assuming (H0), see (1.3)-(1.4). By sub-criticality, (see (1.4)), for any $\varepsilon > 0$, there exists $s'_\varepsilon$ such that

$$|f(x, s)| s^t \leq \varepsilon |a(x)| s^{p^*_N/r-1+t} \quad \text{if} \quad s \geq s'_\varepsilon,$$

so that denoting by $C_t$ a uniform constant depending also on $t$, we get that

$$\int_\Omega |\nabla (|u|^{p^{-1}+t})|^p \leq C_t \left(C_1 + \varepsilon \int_\Omega |u|^{p^*_N/r-1+t} \, dx\right)$$

$$= C_t + \varepsilon C_t \int_\Omega |u|^{p-1+t} |u|^{p^*_N/r-p} \, dx.$$
By Sobolev’s inequality, and Hölder’s inequality with exponents $\frac{p_{N/r}}{p_{N/r}-p}$, we get that

$$\left( \int_{\Omega} |u|^{\frac{p_{N/r}-1+t}{p} p_{N/r}} dx \right)^{\frac{p}{p_{N/r}}} \leq C \int_{\Omega} \left| \nabla \left( |u|^{\frac{p_{N/r}-1+t}{p}} \right) \right|^p$$

$$\leq C_t + \varepsilon C_t \int_{\Omega} |u|^{p-1+t} |u|^{p_{N/r}-p} dx$$

$$\leq C_t + \varepsilon C_t \left( \int_{\Omega} \left| |u|^{\frac{p-1+t}{p} p_{N/r}} \right| \right)^{\frac{p}{p_{N/r}}} \left( \int_{\Omega} |u|^{p_{N/r}} dx \right)^{\frac{p_{N/r}-p}{p_{N/r}}}$$

Since $u \in W^{1,p}_0(\Omega)$, we have that $\int |\nabla u|^p$ is bounded. Taking $\varepsilon$ small we get that $\int |u|^{p_{N/r}-1+t}$ is bounded for any fixed $1 \leq t < \infty$, so that $\int |u|^q$ is bounded for any fixed $q \geq p_{N/r}$ (and since $\Omega$ is bounded in fact for any $q \in [1, \infty]$).

(ii) We now assume (H0)', see (1.6)-(1.7). By sub-criticality, (see (1.7)), for any $\varepsilon > 0$, there exists $s_{\varepsilon}$ such that $|f(x,s)| |s|^{t} \leq \varepsilon |x|^{-\mu} |s|^{p_{N/r}-1+t}$ if $s \geq s_{\varepsilon}$, so that denoting by $C_t$ a uniform constant depending also on $t$, and by Hölder’s inequality with exponents $\frac{p_{N/r}}{p}, \frac{p_{N/r}-p}{p_{N/r}-p}$, we get that

$$\int_{\Omega} \left| \nabla \left( |u|^{\frac{p_{N/r}-1+t}{p}} \right) \right|^p \leq C_t \left( C_1 + \varepsilon \int_{\Omega} |x|^{-\mu} |u|^{p_{N/r}-1+t} dx \right)$$

$$= C_t + \varepsilon C_t \int_{\Omega} |x|^{-\mu} |u|^{p_{N/r}-p} |u|^{p-1+t} dx$$

$$\leq C_t + \varepsilon C_t \left( \int_{\Omega} \left( |x|^{-\mu} |u|^{p_{N/r}-p} \right)^{\frac{p_{N/r}-p}{p_{N/r}-p}} \left( \int_{\Omega} |u|^{p-1+t} dx \right)^{\frac{p}{p_{N/r}}}, \right.$$

$$\left. \right.$$}

where $\gamma := \frac{\mu}{p_{N/r}-p}$, and $\rho := \frac{(p_{N/r}-p)p^*}{p^* - p}$.

Now, since Caffarelli-Kohn-Nirenberg interpolation inequality,

$$\left( \int_{\Omega} \left| x^{-\gamma} u \right|^{\rho} \right)^{\frac{1}{\rho}} \leq C \left( \int_{\Omega} |\nabla u|^p \right)^{\frac{1}{p}} \left( \int_{\Omega} |u|^{p_{N/r}} \right)^{\frac{1}{p_{N/r}}}, \right.$$}

$$\left. \right.$$}

(2.4)

$$\left( \int_{\Omega} |x|^{-\gamma} u \right|^\rho \leq C \left( \int_{\Omega} |\nabla u|^p \right)^{\frac{1}{p}} \left( \int_{\Omega} |u|^{p_{N/r}} \right)^{\frac{1}{p_{N/r}}}, \right.$$}

$$\left. \right.$$}
where
\begin{equation}
\frac{1}{\rho} - \frac{\gamma}{N} = \sigma \left( \frac{1}{p} - \frac{1}{N} \right) + \frac{1 - \sigma}{p^*} = \frac{1}{p^*},
\end{equation}
which trivially holds for any $\sigma \in (0, 1)$. Then, the above can be written as
\begin{equation}
\int_{\Omega} \left| \nabla \left( |u|^{\frac{p-1+\gamma}{p}} \right) \right|^{p} \leq C_{t} + \varepsilon C_{t} \left\| \nabla u \right\|_{p}^{p^*-p} \left( \int_{\Omega} |u|^{\frac{p-1+\gamma}{p}} \right)^{\frac{p}{p^*}}.
\end{equation}
Firstly, by Sobolev’s inequality, and secondly by (2.7), we get that
\begin{equation}
\left( \int_{\Omega} |u|^{\frac{p-1+\gamma}{p}} \right)^{\frac{p}{p^*}} \leq C \int_{\Omega} \left| \nabla \left( |u|^{\frac{p-1+\gamma}{p}} \right) \right|^{p}
\end{equation}
\begin{equation}
\leq C_{t} + \varepsilon C_{t} \left\| \nabla u \right\|_{p}^{p^*-p} \left( \int_{\Omega} |u|^{\frac{p-1+\gamma}{p}} \right)^{\frac{p}{p^*}}.
\end{equation}
Since $u \in W^{1,p}_{0}(\Omega)$, we have that $\int |\nabla u|^{p}$ is bounded. Taking $\varepsilon$ small we get that $\int |u|^{\frac{p-1+\gamma}{p}}$ is bounded for any fixed $1 \leq t < \infty$, so that $\int |u|^{q}$ is bounded for any fixed $q \geq p^*_\mu$ (and since $\Omega$ is bounded in fact for any $q \in [1, \infty)$).

Finally, combining the above estimates, with the gradient regularity of Theorem 2.1 and the Sobolev embeddings, we deduce that $u \in L^\infty(\Omega)$.

3. Carathéodory non-linearities

We start this section with an immediate corollary of Theorem 1.3: any sequence of solutions in $W^{1,p}_{0}(\Omega)$, uniformly bounded in the $L^{p^*}(\Omega)$-norm, is also uniformly bounded in the $L^\infty(\Omega)$-norm.

**Corollary 3.1.** Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (H0)–(H1).

Let $\{u_k\} \subset W^{1,p}_{0}(\Omega)$ be any sequence of solutions to (1.1) such that there exists a constant $C_0 > 0$ satisfying
\begin{equation}
\left\| u_k \right\|_{p^*} \leq C_0.
\end{equation}
Then, there exists a constant $C > 0$ such that
\begin{equation}
\left\| u_k \right\|_{\infty} \leq C.
\end{equation}

**Proof.** We reason by contradiction, assuming that (3.1) does not hold. So, at least for a subsequence again denoted as $u_k$, $\left\| u_k \right\|_{\infty} \to \infty$ as $k \to \infty$. Now part (ii) of the Theorem 1.3 implies that
\begin{equation}
h(\left\| u_k \right\|_{\infty}) \leq C.
\end{equation}
From hypothesis (H0)(see in particular (1.11)), for any \( \varepsilon > 0 \) there exists \( s_1 > 0 \) such that \( h(s) \geq 1/\varepsilon \) for any \( s \geq s_1 \), and so \( h(\|u_k\|_\infty) \geq 1/\varepsilon \) for any \( k \) big enough. This contradicts (3.2), ending the proof. □

3.1. Proof of Theorem 1.3.

The arguments of the proof use Gagliardo-Nirenberg interpolation inequality (see [26]), and are inspired in the equivalence between uniform \( L^{p^*}(\Omega) \) a priori bounds and uniform \( L^\infty(\Omega) \) a priori bounds for solutions to subcritical elliptic equations, see [7, Theorem 1.2] for the quasilinear case and \( f = f(u) \), and [24, Theorem 1.3] for the \( p \)-laplacian and \( f = f(x, u) \). We first use elliptic regularity, and next, we invoke the Gagliardo-Nirenberg interpolation inequality (see [26]).

Proof. Let \( \{u_k\} \subset W^{1,p}_0(\Omega) \) be any sequence of weak solutions to (1.1). Since Theorem 2.3, in fact \( \{u_k\} \subset W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \). If \( \|u_k\|_\infty \leq C \), then (i) holds. Now, we argue on the contrary, assuming that there exists a sequence \( \|u_k\|_\infty \to +\infty \) as \( k \to \infty \).

We split the proof in two steps. Firstly, we write an \( W^{1,q^*(p-1)} \) estimate for \( q \in \left(N/p, \min\{r, N\}\right) \), with \( q^*(p-1) > N \). Secondly, we invoke the Gagliardo-Nirenberg interpolation inequality for the \( L^\infty \)-norm in terms of its \( W^{1,q^*(p-1)} \)-norm and its \( L^{p^*} \)-norm.

Step 1. \( W^{1,q^*(p-1)} \) estimates for \( q \in \left(N/p, \min\{r, N\}\right) \).

Let us denote by

\[
M_k := \max \left\{ \tilde{f}(\|u_k\|_\infty), \tilde{f}(\|u_k\|_\infty) \right\} \geq \frac{1}{2c_0} \max_{\|u_k\|_\infty - \|u_k\|_\infty} \tilde{f},
\]

where the inequality holds by hypothesis (H1), see (1.12).

Let us take \( q \) in the interval \( (N/p, N) \cap (N/p, r) \). Growth hypothesis (H0)(see (1.3)-(1.4)), hypothesis (H1) (see (1.12)), and Hölder inequality, yield the following

\[
\int_\Omega |f(x, u_k(x))|^q \, dx \leq \int_\Omega |a(x)|^q \left( \tilde{f}(u_k(x)) \right)^q \, dx \\
= \int_\Omega |a(x)|^q \left( \tilde{f}(u_k(x)) \right)^t \left( \tilde{f}(u_k(x)) \right)^{q-t} \, dx \\
\leq C \left[ \int_\Omega |a(x)|^q \left( \tilde{f}(u_k(x)) \right)^t \, dx \right] M_k^{q-t} \\
\leq C \left( \int_\Omega |a(x)|^{q_s} \, dx \right)^{\frac{t}{q}} \left( \int_\Omega \left( \tilde{f}(u_k(x)) \right)^{ts'} \, dx \right)^{\frac{1}{s'}} M_k^{q-t} \\
\leq C \|a\|_r^q \left( \|\tilde{f}(u_k)\|_{p^{N/r-1}} \right)^t M_k^{q-t},
\]

(3.4)
where $\frac{1}{s} + \frac{1}{q} = 1$, $qs = r$, $C = c_0^{q^{-t}}$ (for $c_0$ defined in (1.12)), and $t s' = \frac{p^*}{p_{N/r}^{p^*}} - 1$, so

$$(3.5) \quad t := \frac{p^*}{p_{N/r}^{p^*}} - 1 \left(1 - \frac{q}{r}\right) < q$$

$$\iff \frac{1}{q} - \frac{1}{r} < \frac{p_{N/r}^{p^*} - 1}{p^*} = 1 - \frac{1}{r} - \frac{1}{p} + \frac{1}{N}$$

$$\iff \frac{1}{q} < 1 - \frac{1}{p} + \frac{1}{N} \iff \frac{1}{q} < 1 - \frac{1}{p^*} = \frac{1}{(p^*)'}$$

since $p/N < 1 - \frac{1}{p} \iff p < N$, and $q > N/p > (p^*)'$.

By the gradient regularity for the $p$-laplacian (see Theorem 2.1) we have that

$$(3.6) \quad \|\nabla u_k\|_{L^{q^* (p-1)}(\Omega)} \leq C \|f(\cdot, u_k(\cdot))\|_q^{\frac{1}{p^* - 1}},$$

where $1/q^* = 1/q - 1/N$, and $C = C(c_0, N, p, q, |\Omega|)$ and it is independent of $u$. Since $q > N/p$, then

$$(3.7) \quad r := q^*(p - 1) > N.$$ 

Now, substituting (3.4) into (3.6)

$$\|\nabla u_k\|_{L^{q^* (p-1)}(\Omega)} \leq C \left(\|a\|_r \left(\|\tilde{f}(u_k)\|_{p_{N/r}^{p^*} - 1} \right)^{\frac{1}{q}} M_k^{\frac{1}{q} - \frac{1}{p}} \right)^{\frac{1}{p^* - 1}},$$

Observe that since $q > N/p$, then $q^*(p - 1) > N$.

**Step 2. Gagliardo-Nirenberg interpolation inequality.**

Thanks to the Gagliardo-Nirenberg interpolation inequality, there exists a constant $C = C(N, q, |\Omega|)$ such that

$$\|u_k\|_\infty \leq C\|\nabla u_k\|_{q^* (p-1)}^{\sigma} \|u_k\|_{p^*}^{1-\sigma}$$

where

$$\frac{1 - \sigma}{p^*} = \sigma \left(\frac{1}{N} - \frac{1}{q^* (p-1)}\right)$$

$$= \frac{\sigma}{p - 1} \left(\frac{p - 1}{N} - \frac{1}{q} + \frac{1}{N}\right) = \frac{\sigma}{p - 1} \left(\frac{p}{N} - \frac{1}{q}\right)$$

$$= \frac{\sigma}{p - 1} \left[1 - \frac{1}{q} - p \left(\frac{1}{p} - \frac{1}{N}\right)\right] = \frac{\sigma}{(p-1)p^*} \left(p_{N/q}^* - p\right).$$

$$(3.8)$$
Hence

\[(3.9) \quad \|u_k\|_\infty \leq C \left[ \|a\|_r \left( \|\tilde{f}(u_k)\|_p^* \right)^\frac{\sigma}{q^*} M_k^{1-\frac{\sigma}{p^*}} \right] \|u_k\|_p^{1-\sigma},\]

where \( C = C(c_0, r, N, q, |\Omega|). \)

From definition of \( M_k \) (see (3.3)), and definition of \( h \) (see (1.9)), we deduce that

\[M_k = \frac{\|u_k\|_p^{p^*_N/r-1}}{h(\|u_k\|_\infty)}.\]

From (3.8)

\[\frac{1}{\sigma} = 1 + p^* \left( \frac{1}{N} - \frac{1}{q^*(p-1)} \right) \]

\[= \frac{1}{(N-p)q(p-1)} \left[(N-p)q(p-1) + pq(p-1) - p(N-q)\right] \]

\[= \frac{1}{(N-p)q(p-1)} \left[Nq(p-1) - p(N-q)\right] \]

\[= \frac{1}{(N-p)q(p-1)} \left[Np(q-1) - q(N-p)\right] \]

\[= \frac{1}{p-1} \left[p^* - \frac{p^*}{q} - 1\right] = \frac{1}{p-1} \left(p_N^*/q - 1\right). \]

Moreover, since definition of \( t \) (see (3.5)), and definition of \( p^*_N/r \) (see (1.5))

\[\frac{1}{q} - \frac{1}{r} = \frac{p^* - \frac{p^*}{q} - 1}{p^*_N/r - 1} \]

\[\frac{1}{p-1} \left[p^*_N/r - 1\right] = \frac{p^*_N/q - 1}{p^*_N/r - 1}, \]

which, joint with (3.10), yield

\[\frac{\sigma}{p-1} \left[1 - \frac{t}{q}\right] (p_N^*/r - 1) = 1. \]

Now (3.9) can be rewritten as

\[h(\|u_k\|_\infty) \left(\frac{1}{q^*} \right)^{\frac{\sigma}{p^*}} \leq C \left[ \|a\|_r \left( \|\tilde{f}(u_k)\|_p^* \right)^\frac{\sigma}{q^*} M_k^{1-\frac{\sigma}{p^*}} \right] \|u_k\|_p^{1-\sigma},\]

or equivalently

\[h(\|u_k\|_\infty) \leq C\|a\|_r^\theta \left( \|\tilde{f}(u_k)\|_p^* \right)^{\theta - 1} \|u_k\|_p^\theta, \]
where

\[
\theta := (1 - t/q)^{-1} = \frac{p_{N/r}^* - 1}{p_{N/q}^* - 1},
\]

(3.12)

\[
\vartheta := \frac{1 - \sigma}{\sigma} (1 - t/q)^{-1} (p - 1) = \theta \left( p_{N/q}^* - p \right),
\]

(3.13)

see (3.11) and (3.8). Observe that since \( q < r \), then \( \theta > 1 \). Moreover, since (3.12)

\[
\theta - 1 = \frac{p_{N/r}^* p_{N/q}^* - 1}{p_{N/q}^* - 1}.
\]

(3.14)

Furthermore, from sub-criticality, see (1.4)

\[
\int_{\Omega} |\tilde{f}(u_k)| \frac{p_{N/r}^*}{p_{N/q}^* - 1} \leq C \left( 1 + \int_{\Omega} |u_k| p^* \, dx \right),
\]

so

\[
\|\tilde{f}(u_k)\|_{p_{N/q}^* - 1} \leq C \left( 1 + \|u_k\|_{p_{N/r}^*} \right).
\]

Consequently

\[
h\left( \|u_k\|_\infty \right) \leq C \|a\|_r^\theta \left( 1 + \|u_k\|_{p_{N/r}^*}^\Theta \right),
\]

with

\[
\Theta := (p_{N/r}^* - 1)(\theta - 1) + \vartheta = (p_{N/r}^* - p)\theta,
\]

where we have used (3.14), (3.13), and (3.12).

Fixed \( N > p \) and \( r > N/p \), the function \( q \to \theta = \theta(q) \) for \( q \in (N/p, \min\{r, N\}) \), is decreasing, so

\[
\inf_{q \in (N/p, \min\{r, N\})} \theta(q) = \theta\left( \min\{r, N\} \right) = A := \begin{cases} 1, & \text{if } r \leq N, \\ \frac{p_{N/r}^* - 1}{p_{N/r}^*}, & \text{if } r > N. \end{cases}
\]

Finally, and since the infimum is not attained in \( (N/p, \min\{r, N\}) \), for any \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that

\[
h\left( \|u_k\|_\infty \right) \leq C \|a\|_r^{A + \varepsilon} \left( 1 + \|u_k\|_{p_{N/r}^* - p}^{(A + \varepsilon)(A + \varepsilon)} \right),
\]

where \( C = C(\varepsilon, c_0, r, N, |\Omega|) \), ending the proof. \( \blacksquare \)
4. Radial singular weights

We start this section with their corresponding immediate corollary of Theorem 1.4: any sequence of solutions in $W^{1,p}_0(\Omega)$, uniformly bounded in the $L^p(\Omega)$-norm, is also uniformly bounded in the $L^\infty(\Omega)$-norm. Their proof is identical to that of Corollary 3.1, we omit it.

Corollary 4.1. Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (H0)'–(H1).

Let $\{u_k\} \subset W^{1,p}_0(\Omega)$ be any sequence of solutions to (1.1) such that there exists a constant $C_0 > 0$ satisfying

$$\|u_k\|_{p^*} \leq C_0.$$

Then, there exists a constant $C > 0$ such that

$$\|u_k\|_{\infty} \leq C.$$

4.1. Proof of Theorem 1.4.

We split the proof in two steps, in the first one we use elliptic regularity, in the second one, the Caffarelli-Kohn-Nirenberg interpolation inequality for singular weights (see [6]).

Proof. Let $\{u_k\} \subset W^{1,p}_0(\Omega)$ be any sequence of solutions to (1.1). Since Theorem 2.3, $\{u_k\} \subset W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. If $\|u_k\|_{\infty} \leq C$, then (i) holds.

Now, we argue on the contrary, assuming that there exists a sequence $\{u_k\} \subset W^{1,p}_0(\Omega)$ of solutions to (1.1), such that $\|u_k\|_{\infty} \to +\infty$ as $k \to \infty$.

By Morrey’s Theorem (see [3, Theorem 9.12]), observe that also

$$\|\nabla u_k\|_q \to +\infty \quad \text{as} \quad k \to \infty,$$

for any $q > N$.

Step 1. $W^{1,q^*(p-1)}$ estimates for $q \in \left(\frac{N}{p}, \min\{N, N/\mu\}\right)$.

As in the proof of Theorem 1.3, let us denote by

$$M_k := \max \left\{ \tilde{f} \left( -\|u_k\|_{\infty} \right), \tilde{f} \left( \|u_k\|_{\infty} \right) \right\} \geq \frac{1}{2c_0} \max_{[-\|u_k\|_{\infty}, \|u_k\|_{\infty}]} \tilde{f},$$

where the inequality is due to hypothesis (H1), see (1.12).

Let us take $q$ in the interval $(N/p, N) \cap (N/p, N/\mu)$. Using growth hypothesis (H0)' (see (1.6)), hypothesis (H1) (see (1.12)), and Hölder
inequality, we deduce
\[
\int_{\Omega} |f(x, u_k(x))|^q \, dx \leq \int_{\Omega} |x|^{-\mu q} \left( f(u_k(x)) \right)^q \, dx \\
= \int_{\Omega} |x|^{-\mu q} \left( f(u_k(x)) \right)^{\frac{1}{p_{\mu}^{-1}}} \left( f(u_k(x)) \right)^{q - \frac{1}{p_{\mu}^{-1}}} \, dx \\
\leq C \left[ \int_{\Omega} |x|^{-\mu q} (1 + u_k(x)^t) \, dx \right] M_k^{q - \frac{1}{p_{\mu}^{-1}}} \\
\leq C \left( 1 + |x|^{-\gamma} u_k \right)^t M_k^{q - \frac{1}{p_{\mu}^{-1}}},
\]
where \( \gamma = \frac{\mu q}{t}, \ t \in (0, q(p_{\mu}^{-1} - 1)) \), \( C = c_0 \frac{1}{p_{\mu}^{-1}} \) (for \( c_0 \) defined in (1.12)), and where \( M_k \) is defined by (4.3).

Since elliptic regularity see Theorem 2.1, we have that
\[
\|\nabla u_k\|_{q^*(p-1)} \leq C \left[ \left( 1 + |x|^{-\gamma} u_k \right)^t \right]^{\frac{1}{q}} M_k^{1 - \frac{t}{q(p_{\mu}^{-1} - 1)}} \left( \frac{1}{p_{\mu}^{-1}} \right)^{\frac{1}{q - p_{\mu}^{-1}}} (p_{\mu}^{-1} - 1)_{p_{\mu}^{-1}}.
\]
where \( \frac{1}{q^*} = \frac{1}{q} - \frac{1}{N} \) (since \( q > N/p \), then \( q^*(p - 1) > N \)), and \( C = C(N, q, |\Omega|) \).

**Step 2. Caffarelli-Kohn-Nirenberg interpolation inequality.**

Since the Caffarelli-Kohn-Nirenberg interpolation inequality, there exists a constant \( C > 0 \) depending on the parameters \( N, q, \mu, \) and \( t \), such that
\[
\|x|^{-\gamma} u_k \|_t \leq C \|\nabla u_k\|_{q^*(p-1)} \|u_k\|_{p^*}^{1-\theta},
\]
where
\[
\frac{1}{t} - \frac{\frac{1}{p_{\mu}^{-1}}}{Nt} = -\theta \left( \frac{1}{N} - \frac{1}{q^*(p-1)} \right) + (1 - \theta) \frac{1}{p^*}
\]
\[
= \frac{1}{p^*} - \theta \left( \frac{1}{p} - \frac{1}{q^*(p-1)} \right)
\]
\[
= \frac{1}{p^*} - \frac{\theta}{p-1} \left( 1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{N} \right) = \frac{1}{p^*} - \frac{\theta(p_{N/q}^* - 1)}{(p-1)p^*},
\]
for \( \theta \in (0, \frac{p_{\mu}^{-1}}{p_{\mu}^{-1} - 1}) \), and \( \frac{\mu q}{t} = \gamma. \)

\(^5\)Observe that since \( t < q(p_{\mu}^{-1} - 1) \), then the r.h.s. of (4.6) is bounded from below,
\[
\frac{1}{p^*} - \frac{\theta}{(p-1)p^*} (p_{N/q}^* - 1) > \frac{1}{p_{\mu}^{-1} - 1} \left( \frac{1}{q} - \frac{\mu}{N} \right),
\]
Substituting now (4.5) into (4.4) we can write
\[
\|\nabla u_k\|_{q^*(p-1)} \leq C \left[ \left( 1 + \|\nabla u_k\|_{q^*(p-1)} \|u_k\|_{p^*}^{(1-\theta)t} \right)^{1 \over q} M_k^{1 - \theta t / q(p^*-1)} \right]^{1 \over p-1}. 
\]
Now, dividing by \(\|\nabla u_k\|_{q^*(p-1)}\) and using (4.2) we obtain
\[
(4.7) \quad \left( \|\nabla u_k\|_{q^*(p-1)} \right)^{1 \over q^*(p-1)} \leq C \left[ \left( 1 + \|u_k\|_{p^*}^{(1-\theta)t} \right)^{1 \over q} M_k^{1 - \theta t / q(p^*-1)} \right]^{1 \over p-1}. 
\]

Let us check that
\[
(4.8) \quad 1 - \frac{\theta t}{q(p-1)} > 0 \quad \text{for any} \quad t < q(p^*-1).
\]
Indeed, observe first that (4.6) is equivalent to
\[
(4.9) \quad \theta = \frac{1}{p^* - 1} + \frac{\mu}{N}. 
\]
Moreover, from (4.9)
\[
(4.10) \quad \frac{\theta t}{q(p-1)} = \frac{1}{q} \left( \frac{t}{p^* - 1} + \frac{\mu}{N} \right) = \frac{1}{q} \left( \frac{t}{p^* - 1} - 1 \right) + \frac{\mu}{N}, 
\]
consequently
\[
\frac{\theta t}{q(p-1)} < 1 \iff \frac{1}{q} \left( \frac{t}{p^* - 1} + \frac{\mu}{N} \right) < 1 - \frac{1}{q} - \frac{1}{p^*} 
\iff \frac{1}{p^* - 1} < 1 - \frac{1}{p^* - 1} - \frac{\mu}{N} 
\iff \frac{t}{q} < p^* \left( 1 - \frac{\mu}{N} \right) - 1 = p^*_\mu - 1 
\iff t < q(p^*_\mu - 1), 
\]
so, (4.8) holds.
Consequently,
\[
(4.11) \quad \|\nabla u_k\|_{q^*(p-1)} \leq C \left( 1 + \|u_k\|_{p^*}^{A_0} \right) M_k^{B_0}, 
\]
so
\[
\frac{\theta}{(p-1)}(p^*_N/q - 1) < 1 - \frac{1}{1 - \frac{\mu}{N}} \left( \frac{1}{q} + \frac{\mu}{N} \right) = \frac{1 - \frac{1}{q}}{1 - \frac{\mu}{N}}, 
\]
and we get the upper bound of \(\theta\).
where

$$A_0 := \frac{(1-\theta)t}{q(p-1)} \frac{1}{1-\frac{\theta t}{q(p-1)}}, \quad B_0 := \frac{1-\frac{t}{q(p-1)}}{1-\frac{\theta t}{q(p-1)}}. \tag{4.12}$$

**Step 3. Gagliardo-Nirenberg interpolation inequality.**

Thanks to the Gagliardo-Nirenberg interpolation inequality (see [26]), there exists a constant $C = C(N, q, |\Omega|)$ such that

$$\|u_k\|_\infty \leq C \|\nabla u_k\|_{p^*(p-1)}^\sigma \|u_k\|_{p^*/(p-1)}^{1-\sigma}, \tag{4.13}$$

where

$$\frac{1-\sigma}{p^*} = \sigma \left[ \frac{1}{N} - \frac{1}{(p-1)q^*} \right]. \tag{4.14}$$

Hence, substituting (4.11) into (4.13) we deduce

$$\|u_k\|_\infty \leq C \left( 1 + \|u_k\|_{p^*/(p-1)}^{\sigma A_0 + 1-\sigma} \right) M_k^{\sigma B_0}. \tag{4.15}$$

From definition of $M_k$ (see (3.3)), and of $h$ (see (1.10)), we obtain

$$M_k = \frac{\|u_k\|_{p^*}^{p^* - 1}}{h(\|u_k\|_\infty)}. \tag{4.16}$$

Now we check that

$$\sigma B_0 (p^*_\mu - 1) = 1. \tag{4.17}$$

Indeed, from (4.14)

$$\frac{1}{\sigma} = 1 + p^* \left( \frac{1}{N} - \frac{1}{q^*(p-1)} \right) = \frac{p^*}{p} - \frac{p^*}{q^*(p-1)}$$

$$= \frac{p^*}{p-1} \left[ 1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{N} \right] = \frac{1}{p-1} (p^*_{N/q} - 1). \tag{4.18}$$

From (4.10), we deduce

$$1 - \frac{\theta t}{q(p-1)} = \frac{1 - \frac{1}{q} - \frac{1}{p^*} - \frac{1}{q} \left( \frac{t}{p^*} - 1 \right)}{1 - \frac{1}{q} - \frac{1}{p^*}}$$

$$= \frac{(1 - \frac{\mu}{N}) - \frac{1}{q} - \frac{t}{q p^*}}{1 - \frac{1}{q} - \frac{1}{p^*}} = \frac{p^*_\mu - 1 - \frac{t}{q}}{p^*_{N/q} - 1} = \frac{p^*_\mu - 1 - \frac{t}{q}}{p^*_{N/q} - 1}. \tag{4.19}$$
Moreover, since (4.19),

\[
\left( 1 - \frac{t}{q(p^*_\mu - 1)} \right) \left( p^*_\mu - 1 \right) \frac{1}{\left( 1 - \frac{\theta t}{q(p-1)} \right)} \left[ \left( p^*_\mu - 1 - \frac{t}{q} \right) \frac{1}{\left( 1 - \frac{\theta t}{q(p-1)} \right)} = p^*_{N/q} - 1. \right.
\]

Hence

\[
B_0 (p^*_\mu - 1) = \frac{p^*_{N/q} - 1}{p - 1}.
\]

Taking into account (4.18) and (4.21), we deduce that (4.17) holds.

Consequently, we can rewrite (4.15) in the following way

\[
h\left( \|u_k\|_\infty \right) \leq C \left( 1 + \|u_k\|_{p^*_\sigma} \right)^{\frac{\sigma}{\sigma + 1} - 1},
\]

or equivalently

\[
h\left( \|u_k\|_\infty \right) \leq C \left( 1 + \|u_k\|_{p^*_\Theta} \right),
\]

where

\[
\Theta := (p^*_\mu - 1) \left[ 1 + \frac{\sigma}{q(p-1)} - 1 \right].
\]

Since (4.18)-(4.19), \( \sigma \left( 1 - \frac{\theta t}{q(p-1)} \right)^{-1} = (p - 1) \left( p^*_\mu - 1 - \frac{t}{q} \right)^{-1} \), and substituting it into the above equation we obtain

\[
\Theta = (p^*_\mu - 1) \left( \frac{p^*_\mu - p}{p^*_\mu - 1 - \frac{t}{q}} \right).
\]

Fixed \( p > N \) and \( \mu \in (0, p) \), the function \( (t, q) \to \Theta = \Theta(t, q) \) for \( (t, q) \in (0, q(p^*_\mu - 1)) \times (N/p, \min\{N, N/\mu\}) \), is increasing in \( t \) and decreasing in \( q \).

For \( \mu \in [1, p) \), \( \min\{N, N/\mu\} = N/\mu \). Equation (4.6) with \( q = q_k = N/\mu(1 - 1/k) \to N/\mu \), \( t = t_k = \frac{1}{2kp^*_\sigma} \) and \( \theta = \theta_k = \frac{p-1}{2(pN/q_k - 1)} \) is satisfied.

Hence, when \( \mu \in [1, p) \),

\[
p^*_\mu - p \leq \inf_{t \in (0, (p^*_\mu - 1))} \Theta(t, q) \leq \Theta(t_k, q_k) \to p^*_\mu - p.
\]

On the other hand, for \( \mu \in (0, 1) \), \( \min\{N, N/\mu\} = N \). For any \( \varepsilon_k \to 0 \), equation (4.6) with \( q = q_k = N(1 - \varepsilon_k) \to N \), and \( t = t_k \to t_0 \in (0, p^*_\mu - 1) \)
\[ (0, (p_\mu^* - 1)N], \text{ yields } \theta = \theta_k = \left( \frac{(p-1)p_\mu^*}{p_{N/\alpha_k}^* - 1} \left[ \frac{1}{p^*} - \frac{1}{t_k} (1 - \mu (1 - 1/k)) \right] \right) \rightarrow \]

\[ p \left[ \frac{1}{p^*} - \frac{1}{t_0} (1 - \mu) \right] \geq 0, \text{ so } t_0 \geq p^* (1 - \mu). \text{ Hence, when } \mu \in (0, 1), \]

\[ \inf_{t \in [p^* (1 - \mu), (p_\mu^* - 1)N], q \in \left( \frac{N}{p}, N \right)} \Theta(t, q) = \Theta(p^* (1 - \mu), N) = (p_\mu^* - p)B, \]

where \( B \) is defined by (1.14).

Since the infimum is not attained, for any \( \varepsilon > 0 \), there exists a constant \( C = C(\varepsilon, c_0, \mu, N, \Omega) \) such that

\[ (4.24) \quad h\left( \|u_k\|_\infty \right) \leq C \left( 1 + \|u_k\|_{p^*}^{(p_\mu^* - p)(B+\varepsilon)} \right), \]

which ends the proof. \( \square \)

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