Relaxation limit for Aw-Rascle system

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Abstract
We study the relaxation limit for the Aw-Rascle system of traffic flow. For this we apply the theory of invariant regions and the compensated compactness method to get global existence of Cauchy problem for a particular Aw-Rascle system with source, where the source is the relaxation term, and we show the convergence of this solutions to the equilibrium state

1 Introduction
In [1] the author introduces the system
\[
\begin{align*}
\rho_t + (\rho v)_x &= 0, \\
(v + P(\rho))_t + v(v + P(\rho))_x &= 0,
\end{align*}
\]
(1)
as a model of second order of traffic flow. It was proposed by the author to remedy the deficiencies of second order model or car traffic pointed in [5] by the author. The system (1) models a single lane traffic where the functions \(\rho(x,t)\) and \(v(x,t)\) represent the density and the velocity of cars on the road way and \(P(\rho)\) is a given function describing the anticipation of road conditions in front of the drivers. In [1] the author solves the Riemann problem for the case in which the vacuum appears and the case in which the vacuum does not. Making the change of variable
\[w = v + P(\rho),\]
the system (1) is transformed in to the system
\[
\begin{align*}
\rho_t + (\rho(w - P(\rho)))_x &= 0, \\
w_t + w(w - P(\rho))w_x &= 0.
\end{align*}
\]
(2)
Multiplying the second equation in (2) by \(\rho\) we have the system
\[
\begin{align*}
\rho_t + (\rho(w - P(\rho))) &= 0, \\
(w\rho)_t + (w\rho(w - P(\rho)))_x &= 0.
\end{align*}
\]
(3)
Now making the substitution \( m = \omega \rho \), system (3) is transformed in to system

\[
\begin{align*}
\rho_t + (\rho \phi(\rho, m))_x &= 0, \\
(m)_t + (m \phi(\rho, m))_x &= 0.
\end{align*}
\] (4)

where \( \phi(\rho, m) = \frac{\omega}{\rho} - P(\rho) \), this is a system of non symmetric Keyfitz-Kranzer type. In \[9\], the author, using the Compensate Compactness Method, shows the existence of global bounded solutions for the Cauchy problem for the homogeneous system\[4\. In this paper we are concerned with the Cauchy problem for the following Aw-Rascle system

\[
\begin{align*}
\rho_t + (m - \rho P(\rho))_x &= 0, \\
m_t + \left( \frac{\omega^2}{\rho} + m P(\rho) \right)_x &= \frac{1}{\tau}(h(p) - m).
\end{align*}
\] (5)

with bounded measurable initial data

\[
(\rho(x,0), m(x,0)) = (\rho_0(x) + \epsilon, m_0(x)),
\] (6)

Where the source term \( \frac{1}{\tau}(h(p) - m) \) is called relaxation term, \( \tau \) is the relaxation time and \( h(p) \) is an equilibrium velocity. To see important issues in this model see \[10\] and references therein. Let us put \( F(\rho, m) = (\rho \phi(\rho, m), m \phi(\rho, m)) \), with \( \phi(\rho, m) = \frac{\omega}{\rho} - P(\rho) \) by direc calculations we have that the eigenvalues and corresponding eigenvectors of system (5) are given by

\[
\begin{align*}
\lambda_1(\rho, m) &= \frac{m}{\rho} - P(\rho) - \rho P'(\rho), \quad r_1 = (1, \frac{m}{\rho}), \\
\lambda_2(\rho, m) &= \frac{m}{\rho} - P(\rho), \quad r_2 = (2, \frac{m}{\rho} + \rho P'(\rho)).
\end{align*}
\] (7)

The Riemann’s invariants are given by

\[
W(\rho, m) = \frac{m}{\rho}, \quad Z(\rho, m) = \frac{m}{\rho} - P(\rho).
\] (8)

Now as

\[
\nabla \lambda_1 \cdot r_1 = -(2P'(\rho) + \rho P''(\rho)), \quad \nabla \lambda_2 \cdot r_2 = 0,
\]

we see that the the second wave family is always linear degenerate and the behavior of the second family wave depends to the values of \( \rho P(\rho) \). In fact, if \( \theta(\rho) = \rho P(\rho) \) concave or convex then the second family wave is genuinely non linear, see \[2\] to the case in which the two families wave are linear degenerate,

\section{The positive invariant regions}

In this section we show the theorem for invariant regions for find a estimates a priori of the parabolic system \[22\]

Let
Proposition 2.1. Let $\overline{\Omega} \subset \Omega \subset \mathbb{R}^2$ be a compact, convex region whose boundary consists of a finite number of level curves $\gamma_j$ of Riemann invariants, $\xi_j$, such that

$$(U - Y)\nabla \xi_j(U) > 0 \quad \text{for } U \in \gamma_j, \ Y \in \Omega. \quad (10)$$

where $U = (u, v)$, $Y = (y_1, y_2)$. If $U_0(x) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ and $U_0(x) \in K \subset \subset \Omega$ for all $x \in \mathbb{R}$, with $U_0(x) = (u_0(x), v_0(x))$ then for any $\epsilon > 0$ the solution of the system

$$
\begin{aligned}
u' + f(u', v') &= \epsilon u'_{xx}, \\
\psi' + g(u', v') &= \epsilon \psi'_{xx},
\end{aligned} \quad (11)
$$

with initial data

$$u'(x, 0) = u_0(x), \quad v'(x, 0) = v_0(x), \quad (12)$$

exists in $[0, \infty) \times \mathbb{R}$ and $(u'(x, t), v'(x, t)) \in \overline{\Omega}$.

Proof. It is sufficient to prove the result for $u_0(x), v_0(x) \in C^\infty(\mathbb{R})$. Let $U^{\epsilon, \delta} = (u^{\epsilon, \delta}, v^{\epsilon, \delta})$ be the unique solution of the Cauchy problem

$$
\begin{aligned}
\partial_t U^{\epsilon, \delta} + \partial_x F(U^{\epsilon, \delta}) &= \epsilon \partial_x^2 U^{\epsilon, \delta} - \delta \nabla P(U^{\epsilon, \delta}), \\
U^{\epsilon, \delta}(x, 0) &= U_0(x),
\end{aligned} \quad (13)
$$

where $F = (f, g)$ and $P(U) = |U - Y|^2$ for some fix $Y \in \Omega$. If we suppose that $U^{\epsilon, \delta} \notin \Omega$ for all $(x, t)$, then there exist some $t_0 > 0$ and $x_0$ such as

$$U^{\epsilon, \delta}(x_0, t_0) \in \partial \Omega.$$

Since $x \in \partial \Omega = \bigcup \gamma_j$, $U^{\epsilon, \delta}(x_0, t_0) \in \gamma_j$ for some $j$. Then, multiplying by $\nabla \xi_j$ in (13) we have,

$$
\begin{aligned}
\partial_t \xi(U^{\epsilon, \delta}) + \lambda_i(U^{\epsilon, \delta}) \partial_x \xi(U^{\epsilon, \delta}) &= \partial_x^2 H(U^{\epsilon, \delta})(\partial_x U^{\epsilon, \delta}) - \delta \partial_t \xi(U^{\epsilon, \delta}) - \delta \nabla \xi(U^{\epsilon, \delta}) \nabla P(U^{\epsilon, \delta}).
\end{aligned} \quad (14)
$$

Now by (10) we have that

$$
(\partial_x U^{\epsilon, \delta})^T H(U^{\epsilon, \delta})(\partial_x U^{\epsilon, \delta}) \geq 0, \quad (15)
$$

and

$$\delta \nabla \xi(U^{\epsilon, \delta}) \nabla P(U^{\epsilon, \delta}) = 2(U^{\epsilon, \delta} - Y) \nabla \xi(U^{\epsilon, \delta}) > 0. \quad (16)$$

The characterization of $(x_0, t_0)$ implies

$$\partial_t \xi(U^{\epsilon, \delta}(x_0, t_0)) = 0, \quad \partial_x^2 \xi(U^{\epsilon, \delta}(x_0, t_0)). \quad (17)$$

Replacing (15), (16), (17) in (14), we have that

$$\partial_t \xi(U^{\epsilon, \delta}(x_0, t_0)) < 0,$$

which is a contradiction. Now we show that $U^{\epsilon, \delta} \to U^{\delta}$ as $\delta \to 0$. For this let $W^{\epsilon, \delta, \sigma}$ be the solution of

$$
\begin{aligned}
\partial_t W^{\epsilon, \delta, \sigma} + \partial_x (G^{\epsilon, \delta, \sigma} W^{\epsilon, \delta, \sigma}) &= \epsilon W^{\epsilon, \delta, \sigma} - \delta \nabla P(U^{\epsilon, \delta}) + \sigma \nabla P(U^{\epsilon, \sigma}),
\end{aligned} \quad (18)$$

with initial data
where
\[ G^{\epsilon, \delta, \sigma} = \int_0^1 DF(sU^{\epsilon, \delta} + (1 - s)U^{\epsilon, \sigma}) ds. \]

Multiplying by \( W^{\epsilon, \delta, \sigma} \) in (18), and integrating over \( \mathbb{R} \), we have
\[
\frac{d}{dt} \| W^{\epsilon, \delta, \sigma}(t) \|_{L^2(\mathbb{R}^2)}^2 \leq K \| W^{\epsilon, \delta, \sigma}(t) \|_{L^2(\mathbb{R}^2)}^2 + K(\delta + \sigma). \tag{19}
\]
Then, integrating respect to variable \( t \) over interval \((0, t)\), we have that
\[
\| W^{\epsilon, \delta, \sigma}(t) \|_{L^2(\mathbb{R}^2)}^2 \leq K(\delta + \sigma)t + \int_0^1 \frac{K}{\epsilon} \| W^{\epsilon, \delta, \sigma}(t) \|_{L^2(\mathbb{R}^2)}^2 dt. \tag{20}
\]
Finally by applying Gronwall’s inequality we obtain
\[
\| W^{\epsilon, \delta, \sigma}(t) \|_{L^2(\mathbb{R}^2)}^2 \leq K(\delta + \sigma)te^{\frac{Kt}{\epsilon}}
\]
then, for \( \sigma = 0 \) and \( \delta \to 0 \) we have that \( U^{\epsilon, \delta} \to U^{\epsilon} \) as \( \delta \to 0 \).

## 3 Relaxation limit

Based in the Theory of Invariant Regions and Compensated Compactness Method we can obtain the following result.

**Theorem 3.1.** Let \( h(\rho) \in C(\mathbb{R}) \). Suppose that there exists a region
\[ \Sigma = \{ (\rho, m) : W(\rho, m) \leq C_1, Z(\rho, m) \geq C_2, \rho \geq 0 \} \]
were \( C_1 > 0, C_2 > 0 \). Assume that \( \Sigma \) is such that the curve \( m = h(\rho) \) as \( 0 \leq \rho < \rho_1 \) and the initial data are inside \( \Sigma \) and \( (\rho_1, m_1) \) is the intersection of the curve \( W = C_1 \) with \( Z = C_2 \). Then, for any fixed \( \epsilon > 0 \), \( \tau > 0 \) the solution \( (\rho^{\epsilon, \tau}(x, t), m^{\epsilon, \tau}(x, t)) \) of the Cauchy problem globally exists and satisfies
\[
0 \leq \rho^{\epsilon, \tau}(x, t) \leq M, \ 0 \leq m^{\epsilon, \tau}(x, t) \leq M, \ (x, t) \in [0, \infty) \times \mathbb{R}. \tag{21}
\]
Moreover, if \( \tau = o(\epsilon) \) as \( \epsilon \to 0 \) then there exists a subsequence \( (\rho^{\epsilon, \tau}, m^{\epsilon, \tau}) \) converging a.e. to \( (\rho, m) \) as \( \epsilon \to 0 \), where \( (\rho, m) \) is the equilibrium state uniquely determined by

I. The function \( m(x, t) \) satisfies \( m(x, t) = h(\rho(x, t)) \) for almost all \( (x, t) \in [0, \infty) \times \mathbb{R} \).

II. The function \( \rho(x, t) \) is the \( L^\infty \) entropy solution of the Cauchy problem
\[ \rho_t + (\rho h(\rho))_x = 0, \ \rho(x, 0) = \rho_0(x). \]

The proof of this theorem is postponed for later, first we collect some preliminary estimates in the following lemmas.

**Lemma 3.2.** Let \( (\rho_\epsilon, m_\epsilon) \) be solutions of the system (5), with bounded measurable initial data (6), and the condition \( S \) given in (ref) holds. Then \( (\rho_\epsilon, m_\epsilon) \) is uniformly bounded in \( L^\infty \) with respect \( \epsilon \) and \( \tau \).
Proof. First, we show that the region
\[ \Sigma = \{(\rho, m) : W(\rho, m) \leq C_1, Z(\rho, m) \geq C_2, \rho \geq 0\} \]
is invariant for the parabolic system (see Figure 1)
\[ \begin{align*}
\rho_t + (m - \rho P(\rho))_x &= \epsilon \rho_{xx}, \\
m_t + \left( \frac{m^2}{\rho} + mP(\rho) \right)_x &= \epsilon m_{xx}.
\end{align*} \tag{22} \]
If \( \gamma_1 \) is given for \( m(\rho) = C_1 \rho \) and \( \gamma_2 \) is given for \( m(\rho) = \rho C_2 + \rho P(\rho) \), it is easy to show that if \( u = (\overline{\rho}, \overline{m}) \in \gamma_1 \) and \( y = (\rho, m) \in \Sigma \) it then holds
\[ (u - y) \nabla W(u) > 0 \]
and if \( u = (\overline{\rho}, \overline{m}) \in \gamma_2 \) and \( y = (\rho, m) \in \Sigma \) then we have
\[ (u - y) \nabla W(u) > 0. \]
Using Proposition (2.1), we have that \( \Sigma \) is an invariant region for (22). \( \square \)

![Figure 1: Riemann Invariant Region I](image)

For the case in which the system 5 contains relaxation term we use the ideas of the authors in [4]
\[ \begin{align*}
R(y, s) &= \rho(\tau y, \tau s) \tag{23} \\
M(y, s) &= m(\tau y, \tau s), \tag{24}
\end{align*} \]
the system (5) is transformed into the system
\[ \begin{align*}
R_y + (M + RP(R))_y &= \epsilon R_{yy}, \\
M_y + \left( \frac{M^2}{R} - RP(R) \right)_y &= h(R) - M + \epsilon M_{yy}
\end{align*} \tag{25} \]
which does not depend on \( \tau \), and taken \( C_1 = W(1, 0), C_2 = Z(1, 0) \) the curves \( M = Q(R), W = C_1, Z = C_2 \) intersect with the \( R \) axis at the same point in \( \tilde{R} = 0, R = 1 \) (see Figure 2). Using the stability conditions
\[ \lambda_1(\rho, h(\rho)) < h'(\rho) < \lambda_1(\rho, h(\rho)), \tag{26} \]
it is easy to show that the vector \( (0, h(R) - M) \) points inwards the region \( \Sigma_2 \) and from [4] it follows that \( \Sigma_2 \) is an invariant region.
Lemma 3.3. If the solutions of (5), (6) have an a priori $L^\infty$ bounds, and $h \in C^2$, then $\epsilon(p x)2$, $\epsilon(mx)2$, $\frac{(h(\rho)-m)}{\tau}$ are bounded in $L_{loc}^1$ on the case that $\tau = o(\epsilon(\epsilon \to 0))$, when $\epsilon \to 0$.

Proof. Let $Q(\rho, m) = \frac{\rho^2}{2} - h(\rho)m + \frac{C_1}{\rho^2}$, since $(\rho, m)$ is bounded we can choose $C_1$ such that

$$Q_{\rho\rho}(\rho) \phi_{xx} + 2Q_{\rho m}(\rho) \rho_{xx} + Q_{mm}(m) \geq C_2(\rho^2 + m^2).$$

Multiplying the system (5) by $(Q_{\rho}(\rho, m), Q_{m}(\rho, m))$ we have

$$Q_{\rho}(\rho, m)t + Q_{\rho}(\rho, m)\phi_{\rho}(\rho, m)x + Q_{m}(\rho, m)\phi_{m}(\rho, m)x \leq \epsilon \left( Q_{xx} - C_2(\rho^2 + m^2) \right).$$

Adding terms and applying the mean value theorem in the $m$ variable to the functions $\phi_1(\rho, m) = \rho \phi(\rho, m)$ and $Q(\rho, m)$ we have that

$$Q_{\rho}(\rho, m)(\rho \phi(\rho, m)) = T_1 + T_2 + T_3,$$

where

$$T_1 = \left( Q_{\rho}(\rho, m)(\phi_1(\rho, m) - \phi_1(\rho, h\rho)) + \int_{\rho}^\rho \phi_1(s, h(s) \frac{d}{ds} \phi_1(s, h(s))ds \right),$$

$$T_2 = - \left( Q_{\rho m}(\rho, m)\rho_{xx} + Q_{mm}(\rho, m)m_{xx} \right) \phi_{1m}(\rho, \alpha_1)(m - h(\rho))$$

and

$$T_3 = Q_{mm}(\rho, \beta_1)(m - h(\rho)) \left( \phi_{1m}(\rho, h(\rho)) + \phi_{1m}(\rho, h(\rho))h'(\rho) \right) \rho_x.$$

Putting $\phi_2(\rho, m) = m \phi(\rho, m)$ and proceeding as above we have that

$$Q_{m}(\rho, m)(m \phi(\rho, m)) = \bar{T}_1 + \bar{T}_2 + \bar{T}_3,$$

where

$$\bar{T}_1 = \left( Q_{m}(\rho, m)(\phi_2(\rho, m) - \phi_2(\rho, h\rho)) + \int_{\rho}^\rho \phi_2(s, h(s) \frac{d}{ds} \phi_2(s, h(s))ds \right),$$

$$\bar{T}_2 = - \left( Q_{mm}(\rho, m)\rho_{xx} + Q_{m m}(\rho, m)m_{xx} \right) \phi_{2m}(\rho, \alpha_1)(m - h(\rho))$$

and

$$\bar{T}_3 = Q_{m m}(\rho, \beta_1)(m - h(\rho)) \left( \phi_{2m}(\rho, h(\rho)) + \phi_{2m}(\rho, h(\rho))h'(\rho) \right) \rho_x.$$
Now replacing the values of $Q$ and $\phi_1$ in $T_2$ we have that
\[ T_2 = -\left(-h''(\rho)m + C_1 \rho_x^2 - h'(\rho)m_x^2\right)(m - h\rho), \]
then
\[ |T_2| = C|\rho_x^2 + m_x^2||h(\rho) - m|, \]
where $C = \max(|h''(\rho)m - C_1|, |h'(\rho)|)$. Using the Young's $\delta$-inequality
\[ |ab| \leq \delta a^2 + \frac{b^2}{4\delta}, \] (30)
we have that
\[ |T_2| \leq C_2(\delta)\tau(\rho_x^2 + m_x^2) + \delta \frac{(m - h(\rho))^2}{\tau}. \] (31)
For $\hat{T}_2 = -\left(-h'(\rho)\rho_x^2 + m_x^2\right)(2\frac{\partial h}{\partial \rho} - P(\rho))(m - h(\rho))$, using (30) we have
\[ |\hat{T}_2| \leq \hat{C}_2(\delta)\tau(\rho_x^2 + m_x^2) + \hat{\delta} \frac{(h(\rho) - m)^2}{4\tau}. \] (32)
For $T_3$ and $\hat{T}_3$ we have
\[ |T_3| \leq \frac{(h(\rho) - m)^2}{\tau} + C_3(\delta)\rho_x^2 \text{ and} \] (33)
\[ |\hat{T}_3| \leq \frac{(h(\rho) - m)^2}{\tau} + \hat{C}_3(\delta)\rho_x^2. \] (34)
Let us introducing the following
\[ A = C_2(\delta)\tau(\rho_x^2 + m_x^2) + \delta \frac{(m - h(\rho))^2}{\tau}, \] (35)
\[ \hat{A} = \hat{C}_2(\delta)\tau(\rho_x^2 + m_x^2) + \hat{\delta} \frac{(m - h(\rho))^2}{\tau}, \] (36)
\[ B = \frac{(h(\rho) - m)^2}{\tau} + C_3(\delta)\rho_x^2, \] (37)
\[ \hat{B} = \frac{(h(\rho) - m)^2}{\tau} + \hat{C}_3(\delta)\rho_x^2. \] (38)
and $R(\rho, m) = T_1 + \hat{T}_2$. Then substituting (28), (29) in (27) and using (30) we have
\[ Q(\rho, m)_{\tau} + R(\rho, m)_{\tau} + (\epsilon C_2 - \tau C_4)(\rho_x^2 + m_x^2) + \frac{(h(\rho) - m)^2}{2\tau} \leq \epsilon Q(\rho, m)_{\tau}. \] (39)
For $\epsilon$ sufficiently small we can choose $C_2, C_4$ such that $C_4 \tau \leq (C_2 - T)\epsilon$ for $T > 0$. Let $K$ be a compact subset of $\mathbb{R} \times \mathbb{R}^+$ and $\Phi(x, t) \in D(\mathbb{R} \times \mathbb{R}^+)$, sucha that $\Phi = 1$ in $K$, $0 \leq \Phi \leq 1$. Then, multiplying (39) by $\Phi(x, t)$ and integrating by parts we have
\[ \int_{\mathbb{R} \times \mathbb{R}^+} \left(2T\epsilon(\rho_x^2 + m_x^2)\Phi + \frac{(h(\rho) - m)^2}{\tau}\Phi\right) dxdt \leq M(\Phi). \] (40)
Lemma 3.4. If \((\eta(\rho), q(\rho))\) is any entropy-entropy flux pair for the scalar equation
\[
\rho_t (\rho \phi(\rho, h(\rho)))_x = 0,
\]
then
\[
\eta(\rho)_t + q(\rho)_x
\]
is compact in \(H^{-1}(\mathbb{R} \times \mathbb{R}^+)\).

Proof. Adding \(\psi(\rho) = \rho \phi(\rho, h(\rho))\) in the first equation of \((\mathbf{4})\) we have
\[
\rho_t + \psi(\rho)_x = \epsilon \rho_{xx} + (\rho \phi(\rho, h(\rho)) - \rho \phi(\rho, m)),
\]
and multiplying by \(\eta\) 8q) in \((\mathbf{42})\) we have that
\[
\eta(\rho)_t + q(\rho)_x = \epsilon \eta(\rho)_{xx} - \epsilon \eta^2(\rho) \rho_{xx}
\]
\[\quad + \left( \eta'(\rho)(\psi(\rho) - \rho \phi(\rho, m)) \right) - \eta^2(\rho)(\psi(\rho) - \rho \phi(\rho, m)) \rho_x.
\]
Let \(A^* = \epsilon \eta(\rho)_{xx} - \epsilon \eta^2(\rho) \rho_{xx}\), and \(B^* = \left( \eta'(\rho)(\psi(\rho) - \rho \phi(\rho, m)) \right) - \eta^2(\rho)(\psi(\rho) - \rho \phi(\rho, m)) \rho_x\), we state that \(A^* \in H^{1}_{loc}(\mathbb{R} \times \mathbb{R}^+)\) and \(B^*\) is bounded in \(M(\mathbb{R} \times \mathbb{R}^+)\), then for Murat’s lemma we have that
\[
\eta(\rho)_t + q(\rho)_x
\]
is compact in \(H^{1}_{loc}(\mathbb{R} \times \mathbb{R}^+)\). For the first affirmation see \(\mathbf{11}\), we show that \(B^*\) is bounded in \(L^1_{loc}\). Applying the mean value theorem in the second variable to the function \(\phi(\rho, m)\) in \([h(\rho), m]\) and using Lemma \(\mathbf{3.3}\) we have
\[
\int_{\Omega} \eta''(\rho)(\psi(\rho) - \rho \phi(\rho, m)) \rho_x dx dt
\]
\[\leq M \int_{\Omega} (h(\rho) - m) \rho_x dx dt \leq M \left( \int_{\Omega} \frac{(h(\rho) - m)^2}{\tau} dx dt \right)^{\frac{1}{2}} \left( \int_{\Omega} \tau \rho_x^2 dx dt \right)^{\frac{1}{2}},
\]
and
\[
\left| \int_{\Omega} (\eta'(\rho)(\psi(\rho) - \rho \phi(\rho, m))) \Phi(x, t) dx dt \right| = \left| \int_{\Omega} \eta'(\rho)(\psi(\rho) - \rho \phi(\rho, m)) \Phi(x, t) dx dt \right|
\]
\[\leq M \left( \int_{\Omega} \frac{(h(\rho) - m)^2}{\tau} dx dt \right)^{\frac{1}{2}} \left( \int_{\Omega} \tau \Phi^2 x dx dt \right)^{\frac{1}{2}}. \]

Now we prove Theorem 3.1. By the Lemma 3.2 we have the a priori bounds \((\mathbf{24})\), and we also have that there is a subsequence of \((\rho^*, m^*)\) such as
\[
\rho(x, t) = w^* - \lim \rho^*(x, t), \quad m(x, t) = w^* - \lim m^*(x, t)
\]
(\mathbf{43})

Let us introduce the following
\[
\eta_1(\theta) = \theta - k, \quad q_1(\theta) = \psi(\theta) - \psi(k), \quad \eta_2(\theta) = \psi(\theta) - \psi(k) \quad \text{and} \quad q_2(\theta) = \int_{\rho} \rho^*(\psi'(s))^2 ds.
\]
(\mathbf{46})
(\mathbf{47})
Then by the weak convergence of determinant \[8\] page 15, we have that
\[
\eta_1(\rho')q_2(\rho') - \eta_2(\rho')q_1(\rho') = \eta_1(\rho')q_1(\rho') - \eta_2(\rho')q_1(\rho'),
\] (48)
by direct calculations, replacing \(\rho'\) in 44-47 we have that
\[
(\rho' - \rho) \int_{\rho}^{\rho'} (\psi(\rho))^2 - (\psi(\rho') - \psi(\rho))^2 + \left(\psi(\rho') - \psi(\rho)\right)^2 = (\rho' - \rho) \int_{\rho}^{\rho'} (\psi'(s))^2 ds,
\]
an since by \[43\]
\[
(\rho' - \rho) \int_{\rho}^{\rho'} (\psi'(s))^2 ds = 0.
\]
we have that
\[
\psi(\rho') = \psi(\rho),
\] (49)
\[
(\rho' - \rho) \int_{\rho}^{\rho'} (\psi'(s))^2 ds - (\psi(\rho') - \psi(\rho))^2 = 0 \tag{50}
\]
Now, using Minty’s argument \[7\] or arguments of author in \[3\] it’s finished the proof of the Theorem 3.1

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