The study of thermodynamic fluctuations allows one to relate the free energy difference between two equilibrium states with the work done on a system through processes far from equilibrium. This finding plays a crucial role in the quantum regime, where the definition of work becomes non-trivial. Based on these relations, here we develop a simple interferometric method allowing a direct estimation of the work distribution and the average dissipative work even without full control over the thermodynamic process, and we propose methodological variations depending on the possible experimental limitations encountered. Finally, we exemplify its applicability by an experimental proposal for implementing our method on a quantum photonics system, on which the thermodynamic process is performed through polarization rotations induced by liquid crystals acting in a discrete temporal regime.

I. INTRODUCTION

While microscopic dynamical physical laws of both classical and quantum physics are time-symmetric, and hence reversible, the dynamics of macroscopic quantities exhibit a preferred temporal direction. The physical law formalizing this concept is the second law of thermodynamics, whereby the “arrow of time” [1] is associated with a production of entropy [2]. According to this law, for instance, if we take a vessel divided by a wall, and put a gas in only one half of the vessel, when we remove the wall we will observe with a near-unity probability the gas expanding and occupying the whole vessel. Because of its unidirectional temporal evolution, this phenomenon has often been used to differentiate between past and future. There is, however, a non-zero probability that at a time all the molecules may happen to visit one half of the vessel. In this regard, the development of so-called “fluctuation theorems”, both for classical [3–7] and quantum [8–18] systems, has led to the sharpening of our understanding of the second law as a statistical law, where the entropy of a system away from equilibrium can spontaneously decrease rather than increase with non-zero probability. As specified by those theorems, the ratio between the probability of entropy-decreasing events and that of entropy-increasing ones vanishes exponentially with the size of the fluctuations, and can hence be neglected in the macroscopic limit [6].

The fundamental and empirical basis for the study of entropy production and thermodynamic irreversibility in driven systems is typically provided by the notion of dissipative work, \( W_{\text{diss}} \equiv W - \Delta F \) (namely, the work invested in a thermodynamic transformation between equilibrium states having a free energy difference \( \Delta F \), which cannot be recovered by reversing the driving protocol) [4, 5, 19–21]. The fluctuations of the dissipative work in the process can be characterized by constructing the work probability distribution, \( P(W) \), associated to the observation of a particular value of \( W \) in a single realization of the driving protocol. Such fluctuations are constrained by a refined version of the second law: namely, Crook’s fluctuation theorem, according to which

\[
\frac{P(W)}{P(-W)} = e^{\beta W_{\text{diss}}},
\]

where \( P(-W) \) is the probability of performing a work \( W \) in the time-reversal dynamics, \( \beta = 1/\kappa_B T \) is the inverse temperature of the surrounding thermal environment, and \( \kappa_B \) is the Boltzmann constant. According to Eq. (1), this probability ratio decreases exponentially with the amount of dissipative work, \( W_{\text{diss}} \), in the realization. Furthermore, Eq. (1) implies the famous Jarzynski equality \( e^{-\beta W_{\text{diss}}} = 1 \), where the brackets denote the statistical average with respect to \( P(W) \). Jarzynski’s equality has severe implications by itself, such as the exponential decay of the probability to observe negative values of \( W_{\text{diss}} \) in the forward dynamics (explicitly, \( P(W_{\text{diss}} < -\zeta) \leq e^{-\beta \zeta} \) for any \( \zeta \geq 0 \) [6].

Work fluctuations have been measured in small classical systems leading to both testing the Crook’s theorem and the Jarzynski equality, and developing applications like measurements of free-energy [22–27]. In quantum physics, since work is not associated to any observable [28], its definition becomes more complex, and it usually demands the use of the so-called “two-point measurement (TPM) scheme” [9]. In the TPM scheme, work is represented as the difference between the ini-
tial and final energies of the system, obtained by performing two projective measurements of the Hamiltonian at the begin-
ning and at the end of the forward as well as of the time-reversal process. Extensions to non-ideal measurements [29–
31] and variants of the TPM scheme [32–37] have been also con-
sidered recently. The TPM approach has been directly im-
plemented in several experiments [38–42]. However, since implementating projective energy measurements before and af-
ter an arbitrary process may be challenging in certain exper-
imental scenarios, and the measurement might annihilate the system measured, alternative methods for extracting the work distribution were proposed to circumvent this requirement. For example, in Refs. [43, 44], a scheme based on Ramsey interferometry using a single probe qubit was proposed, and subsequently implemented [45, 46], to extract the characteris-
tic function of work in an NMR platform. A similar method to sample the work probability distribution from a general-
ized measurement scheme was introduced in Refs. [47–49], and tested experimentally on an ensemble of cold atoms [50].

Despite their many advantages and proven efficacy, previous schemes due to their robustness, individual addressability and
intrinsic mobility, we propose a photonic implementation of our scheme where the Hamiltonian of the thermodynamic system is represented by the polarization of a single photon. Other platforms that may be used to realize the scheme in-
clude ultracold atoms [48, 50], or NMR spectroscopy of nu-
clear spins [45, 46]. Also a methodologically related scheme was proposed recently to investigate the thermodynamic ar-
row of time in a quantum superposition of the forward and time-reversal processes [51].

II. PROCEDURE OVERVIEW

Consider a thermodynamic system $S$ that is driven by a
time-dependent Hamiltonian $H(\Lambda(t))$ depending on some control parameter $\Lambda(t)$ which varies from $t = 0$ to $t = \tau$,
according to a protocol $\Lambda = \{\Lambda(t) : 0 \leq t \leq \tau\}$. The system starts the evolution in a thermal state $\rho_0^n = 
\exp[-\beta(H_0 - F_0)]$ in equilibrium with a thermal reservoir at inverse temperature $\beta$, where $F_0$ is the free energy corre-
sponding to the initial Hamiltonian $H_0 \equiv H(\Lambda(0))$. The system is then isolated from the environment, and the driv-
ing protocol $\Lambda$ is applied, bringing the system to an out-of-
equilibrium state $\rho(t) = U(t, 0) \rho_0^n U^\dagger(t, 0)$, where $U(t, 0) = 
\tilde{T} \exp[-i \int_0^t H(\Lambda(t')) dt']$, $\tilde{T}$ being the so-called “time-
ordering” operator resulting from the Dyson decomposition. Once the driving protocol is ended at time $\tau$, the system may eventually equilibrate again from $\rho_\tau = \rho(\tau)$ to the reservoir
vapor, thereby reaching the thermal state $\rho_\tau^n = \exp[-\beta(H_\tau - F_\tau)]$, corresponding to the final Hamiltonian $H_\tau \equiv H(\Lambda(\tau))$ and the free energy $F_\tau$.

Together with the above thermodynamic process, we con-
side its time-reversal twin. In the reverse process, the sys-
tem starts the evolution at time $t = 0$ with Hamiltonian $\Theta H_\tau \Theta^\dagger$ in equilibrium with the thermal reservoir, that is,
$\rho_{\tau}^{th} = \Theta \rho_\tau^n \Theta^\dagger = \exp[-\beta(\Theta H_\tau \Theta^\dagger - F_\tau)]$. Here, $\Theta$ is the (anti-unitary) time-reversal operator, responsible for chang-
ing the sign of observables with odd parity (such as momen-
tum, or spin under time-reversal). The time-reversal op-
erator fulfills $\Theta \xi = -i \partial \Theta$ and $\Theta \Theta^\dagger = \Theta^\dagger \Theta = 1$. The system is then driven according to the time-reversal pro-
col $\tilde{\Lambda} = \{\Lambda(t) \equiv \Lambda(\tau - t) : 0 \leq t \leq \tau\}$, cor-
responding to the inverse sequence of values of the control parameter. This brings the system out-of-equilibrium to the state $\tilde{\rho}(t) = \tilde{U}(t, 0) \tilde{\rho}_{\tau}^{th} \tilde{U}^\dagger(t, 0)$ at intermediate times, where now $\tilde{U}(t, 0) = \tilde{T} \exp[-i \int_0^t \Theta H(\tilde{\Lambda}(t')) \Theta^\dagger dt]$. Af-
ter completing the protocol $\tilde{\Lambda}$, the system may return back
to equilibrium at time $t = \tau$, reaching $\tilde{\rho}_\tau^n = \Theta \rho_{\tau}^{th} \Theta^\dagger = \exp[-\beta(\Theta H_\tau \Theta^\dagger - F_\tau)]$.

We denote by $|E_m^n(0)\rangle$ the initial energy eigenstates of the system in the forward process, and by $p_{m}^{(0)} = e^{-\beta(E_m^n(0) - F_0)}$ the probability that the system has energy $E_m^n(0)$. Analogously, the initial eigenstates of the system in the time-reversal process read $\Theta |E_m^{(\tau)}\rangle$, with $p_\tau^n = e^{-\beta(E_m^{(\tau)} - F_\tau)}$ being the corre-
spanding probabilities to measure the energy $E_m^{(\tau)}$. The work probability distribution in the TPM scheme results then [9]:

$$P(W) = \sum_{m,n} p_\tau^n n p_{m|n} \delta(W - (E_m^{(\tau)} - E_n^{(0)})),$$ 

(2)

where we introduced the conditional probabilities $p_{m|n} = \langle E_m^{(\tau)} | E_n^{(0)} \rangle$.
An initial thermal state $\rho^0$ with Hamiltonian $H_0$ is driven into a final, non-equilibrium state $\rho_T$. It then eventually equilibrates at the reservoir temperature, reaching the thermal state $\rho^\beta$. (If the driving process was reversible, quasi-static, the system would have ended in the state $\rho^\beta$, immediately after the drive.) Along the driving process the Hamiltonian is changed from $H_0$ to $H_T$. Analogously, in the process’ time-reversal twin a thermal state $\rho^\beta = \rho^0$ with Hamiltonian $H_r$ evolves into a state $\rho^\tau_r$ and then it eventually equilibrates to the state $\rho^\tau_r = \rho^0$.

$$\langle E^{(\tau)}_m | U(\tau, 0) | E^{(0)}_n \rangle^2$$

The operation is then applied to $\rho(\tau)$, while in the other state $|\psi\rangle$ the preparation is $|E^{(\tau)}_m\rangle$ for a certain choice of $n$ and $m$. Consequently, the initial state is the pure state $\rho(\tau)$, the state of system and path $|\psi\rangle$.

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

where $S'(\rho(\tau) || \Theta^\dagger \tilde{U}(\tau - t) \Theta)$ in Eq. (4).

III. INTERFEROMETRIC SCHEME

The main idea of our scheme is to entangle the system of interest with a two-level “auxiliary system”, and implement different dynamics (forward and time-reversal) on each of the two states of the auxiliary system. To fix ideas, we assume that the auxiliary system is the path of a single photon in a Mach-Zehnder interferometer such as the one depicted in Fig. 2, and denote by $\{|0\rangle_A, |1\rangle_A\}$ the basis of the two possible paths. (We stress, however, that this auxiliary system does not have to be encoded in the path, but can be any degree of freedom which can be suitably controlled.) Suppose now that, in one of the two states of the superposition (say, $|0\rangle_A$), the system is prepared in the state $|E^{(0)}_n\rangle$, while in the other state of the superposition $|1\rangle_A$ the preparation is $|E^{(\tau)}_m\rangle$ for a certain choice of $n$ and $m$. Consequently, the initial state is the pure state

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

where $S'(\rho(\tau) || \Theta^\dagger \tilde{U}(\tau - t) \Theta)$ in Eq. (4).

The operation $U(\tau/2, 0)$ is then applied to $S$ in the path $|0\rangle_A$, while, on the path $|1\rangle_A$, the operation $\tilde{U}(\tau/2, 0)$ is performed, followed by the time-inversion operation $\Theta^\dagger$. The total evolution is given by

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

The operation $U(\tau/2, 0)$ is then applied to $S$ in the path $|0\rangle_A$, while, on the path $|1\rangle_A$, the operation $\tilde{U}(\tau/2, 0)$ is performed, followed by the time-inversion operation $\Theta^\dagger$. The total evolution is given by

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

$$\rho_{S,A}(\tau/2) = \frac{1}{2} \left\{ |0\rangle_A \langle 0|_A \otimes U(\tau/2, 0) |E^{(0)}_n\rangle \langle E^{(0)}_n| U^\dagger(\tau/2, 0) + |1\rangle_A \langle 1|_A \otimes \Theta^\dagger \tilde{U}(\tau/2, 0) \right\}$$

where $S'(\rho(\tau) || \Theta^\dagger \tilde{U}(\tau - t) \Theta)$ in Eq. (4).
In an interferometer, the difference \( |p_+ - p_-|/2 \) is called interferometric visibility or fringe, and is related to our capacity to identify the path followed by the auxiliary system \([53]\):

\[
\mathcal{V}_{m,n} = \left| \langle 0 \rangle_A \rho_A(\tau/2) |1\rangle_A \right| = \left| \text{Tr}_S \left[ U(\tau/2, 0) |E_n^{(0)}\rangle \langle E_n^{(0)}| \tilde{U}^\dagger(\tau/2, 0) \Theta \right] \right|. \tag{11}
\]

Now, we crucially apply the micro-reversibility relation in Eq. (3), to realize that \( \Theta^\dagger \tilde{U}^\dagger(\tau/2, 0) \Theta = U(\tau, \tau/2) \). Inserting this into Eq. (11), and using the cyclic property of the trace, we obtain the main result of our proposal:

\[
\mathcal{V}_{m,n} = \left| \text{Tr}_S \left[ U(\tau/2, 0) |E_n^{(0)}\rangle \langle E_n^{(0)}| \tilde{U}^\dagger(\tau/2, 0) \Theta \right] \right| = \left| \langle E_n^{(\tau)} | U(\tau, 0) |E_n^{(0)}\rangle \right| = \sqrt{p_m}, \tag{12}
\]

where, in the last equality, we identified the expression of the conditional probabilities \( p_{m,n} \) of the TPM scheme, and where we identified \( U(\tau, \tau/2) \tilde{U}(\tau/2, 0) = U(\tau, 0) \).

Running this scheme for the \( N^2 \) different initial states, \( n, m = 1, 2, \ldots, N \) (where \( N \) is the dimension of the system Hilbert space), and assuming that we know the eigenenergies \( E_n^{(0)}, E_n^{(\tau)} \), and the equilibrium free energies \( F_0 \) and \( F_\tau \) (or, equivalently, the initial probabilities \( p_{m,n}^{(0)} \) and \( \tilde{p}_m^{(0)} \)), we can readily reconstruct the full probability distribution in Eq. (2):

\[
P(W) = \sum_{m,n} p_{m,n}^{(0)} \mathcal{Y}_{m,n}^2 \delta(W - \langle E_m^{(\tau)} - E_m^{(0)} \rangle), \tag{13}
\]

and its time-reversal twin \( \tilde{P}(W) \). We notice that in practice only \((N-1)^2\) of the \( N^2 \) initial preparations need to be considered, since the properties of the conditional probability imply \( \sum_n \mathcal{Y}_{m,n}^2 = 1 \) for all \( n = 1, \ldots, N \), and for any unital process \( p_{m,n} \) becomes doubly stochastic thus we also have \( \sum_n \mathcal{Y}_{m,n}^2 = 1 \) for all \( m = 1, \ldots, N \), as also noticed in Ref. [50]. Furthermore, we can rewrite the r.h.s. of Eq. (4) in terms of known quantities:

\[
\beta(W_{\text{diss}}) = S(\rho(t)) \| \Theta^\dagger \tilde{\rho}(t - \tau) \Theta \|
\]

\[
= \sum_n p_n^{(0)} \log p_n^{(0)} - \sum_{m,n} p_{m,n}^{(0)} \mathcal{Y}_{m,n}^2 \log \tilde{p}_m^{(0)}, \tag{14}
\]

which can be alternatively obtained from the average of the work probability distribution in Eq. (13), \( \langle W \rangle = \int_{-\infty}^{\infty} W P(W) dW \), and the free energy difference between the initial equilibrium states, \( \Delta F = F_\tau - F_0 \). As a consequence, this scheme allows, through Eqs. (13) and (14), the direct estimation of the work dissipation, and the testing of the Jarzynski equality.

**IV. LIMITED PREPARATION AND BOUNDS ON WORK DISSIPATION**

In the previous section, we assumed that we have the ability to prepare a superposition of pairs of energy eigenstates of the initial and final Hamiltonians of the system. Nonetheless, it
could be the case that, due to technical limitations, one may not be able to prepare these pure states in the laboratory. For instance, if we do not have full control over the system in its preparation stage, and cannot isolate it from the reservoir, we may only be able to prepare the thermal states $\rho_0^{th}$ and $\rho_2^{th}$. In the following we explore what we can still learn about the work dissipation by exploiting our interferometric scheme in such a situation. We anticipate that, although the full work probability distribution is no longer recoverable in this case, we are still able to provide useful upper bounds on the dissipative work done in the process.

As before, we prepare our auxiliary degree of freedom in a quantum superposition $\frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$ at $t < 0$. The initial states for the system in the two branches will now be, in general, the mixed thermal states $\rho_0^{th}$ and $\rho_2^{th}$. However, hereafter we will make use of their “purifications”, which can be considered as useful mathematical tools, and may correspond physically to all the environmental degrees of freedom $E$, such that the overall joint state of the system and these degrees of freedom is pure. (Notice that here the environment includes, but is not limited to, the thermal reservoir. Furthermore, our scheme does not require to have access to the environmental degrees of freedom.) We denote the purifications of the thermal states, respectively, as $|\psi(0)\rangle_{S,E}$ and $|\tilde{\psi}(0)\rangle_{S,E}$, and they verify $\text{Tr}_E[\psi(0)_{S,E} \psi(0)_{S,E}] = \rho_0^{th}$ and $\text{Tr}_E[\tilde{\psi}(0)_{S,E} \tilde{\psi}(0)_{S,E}] = \rho_2^{th} = \tilde{\rho}_2^{th}$.

Again, we perform the operation $U(\tau/2,0)$ in the path $|0\rangle_A$ according to the protocol $\Lambda$, and $\tilde{U}(\tau/2,0)$ in the path $|1\rangle_A$ according to $\tilde{\Lambda}$, followed by $\Theta^\dagger$. Notice that the unitaries $U(\tau/2,0)$ and $\tilde{U}(\tau/2,0)$ only act on the system of interest, with no effect on the environment. We can then compute the global state of the system, the environment and the auxiliary system at $\tau/2$ similarly as before, and obtain the marginal states for the auxiliary degree of freedom and the composite system consisting of the system and environment. For the latter, we obtain a mixture over the states of the system and the environment in $\tau/2$ in the forward and time-reversal dynamics:

$$\rho_{S,E}(\tau/2) = \frac{1}{2} \left[ \rho_{S,E}^{(+)} + \rho_{S,E}^{(-)} \right], \quad (15)$$

where

$$\rho_{S,E}^{(+)} = \left( U(\tau/2,0) \otimes 1_E \right) |\psi(0)\rangle_{S,E} \langle \psi(0)|_{S,E} \left( U^\dagger(\tau/2,0) \otimes 1_E \right), \quad (16a)$$

$$\rho_{S,E}^{(-)} = \left( \Theta^\dagger \tilde{U}(\tau/2,0) \otimes 1_E \right) |\tilde{\psi}(0)\rangle_{S,E} \langle \tilde{\psi}(0)|_{S,E} \left( \tilde{U}^\dagger(\tau/2,0) \Theta \otimes 1_E \right). \quad (16b)$$

The corresponding state of the system only will be then an equal probability mixture of the states $\rho_{S}(\tau/2) = \text{Tr}_E[\rho_{S,E}^{(+)}]$ and $\Theta^\dagger \tilde{\rho}_{S}(\tau/2) \Theta = \text{Tr}_E[\rho_{S,E}^{(-)}]$.

The visibility, determined by the off-diagonal elements of the auxiliary degree of freedom, reads in this case:

$$V = \left| \text{Tr}_{S,E} \left[ (U(\tau/2,0) \otimes 1_E) |\psi(0)\rangle_{S,E} \langle \psi(0)|_{S,E} \left( \tilde{U}^\dagger(\tau/2,0) \Theta \otimes 1_E \right) \right] \right|$$

$$= \left| \langle \tilde{\psi}(0)|_{S,E} \left( U(\tau,0) \otimes 1_E \right) |\psi(0)\rangle_{S,E} \right|, \quad (17)$$

which can no longer be related to the different outcomes of a TPM scheme. This notwithstanding, as we will shortly see, one can still make use of this information in an alternative way.

From Ref. [53], we know that the visibility $V$ of the interferometer fringes and the distinguishability $D(\rho, \sigma)$ between two “which-path detector states” $\rho$ and $\sigma$ (i.e., two states from which we can optimally infer the which-path information, would we perform a measurement to distinguish between them) are mutually exclusive. In particular, it has been shown that these two quantities respect the complementarity relationship

$$V^2 + D^2(\rho, \sigma) \leq 1, \quad (18)$$

and that this relation becomes an equality if the “detector states” are in pure states, as it is in our case. The distinguishability between the two states is given by the trace-norm distance between them, i.e., $D(\rho, \sigma) := \frac{1}{2} ||\rho - \sigma|| := \frac{1}{2} \text{Tr}[\sqrt{\rho - \sigma}^\dagger (\rho - \sigma)]$.

In our case, $D(\rho_{S,E}^{(+)}; \rho_{S,E}^{(-)})$ gives us an estimation of how well one can distinguish between the two paths in the interferometer by measuring the system and the environment. However, we are interested in the trace-norm distance between the marginal states of the system only. We can therefore use the fact that the trace distance is non-increasing under partial trace, i.e., $D(\rho_{S,E}^{(+)}; \rho_{S,E}^{(-)}) \geq D(\rho_S(\tau/2), \Theta^\dagger \tilde{\rho}_S(\tau/2) \Theta)$, to get:

$$V^2 + D^2(\rho_S(\tau/2), \Theta^\dagger \tilde{\rho}_S(\tau/2) \Theta) \leq V^2 + D^2(\rho_{S,E}^{(+)}; \rho_{S,E}^{(-)}) = 1. \quad (19)$$

Finally, we relate the distinguishability between the system states at $\tau/2$ in the forward and time-reversal dynamics with the relative entropy in Eq. (4), and hence to the average dissipative work during the protocol $\Lambda$. This can be done using the upper bounds obtained in Eqs. (17) and (19) of Ref. [54]. Minor manipulations of these equations lead to the formulation of the following theorem:

**Theorem.** Let $\rho$ and $\sigma$ be two strictly positive density operators in a finite-dimensional Hilbert space $\mathcal{H}$. Then

$$S(\rho||\sigma) \leq \frac{||\rho - \sigma||_2^2}{\alpha_\sigma} \leq \frac{||\rho - \sigma||^2}{\alpha_\sigma}, \quad (20)$$

where $\alpha_\sigma \in (0, 1]$ is the smallest eigenvalue of $\sigma$, and $||\varrho||_2 = \sqrt{\text{Tr}[^2\varrho]}$ denotes the Frobenius (or Euclidean) norm, which verifies $||\varrho||_2 \leq ||\varrho||$. \]
Furthermore, setting the dimension of the Hilbert space to \( \dim(H) \equiv d \), we also have:

\[
S(\rho|\sigma) \leq ||\rho - \sigma|| \log(d/\sqrt{\alpha_\sigma}) + e^{-1} = D(\rho, \sigma) \log(d^2/\alpha_\sigma) + e^{-1}. \tag{21}
\]

Combining Eqs. (19) and the bounds (20)-(21), we obtain the following two bounds for the dissipative work during the original thermodynamic process:

\[
(W_{\text{diss}}) \leq k_B T \left( 1 - \mathcal{V}^2 \right)/\alpha \equiv \mathcal{B}_2, \tag{22}
\]

\[
(W_{\text{diss}}) \leq k_B T \left[ \sqrt{1 - \mathcal{V}^2} \log(d^2/\alpha) + e^{-1} \right] \equiv \mathcal{B}_{\log}, \tag{23}
\]

where we have used the relation between the dissipative work and the relative entropy in Eq. (22). Additionally, we denoted \( \alpha \equiv \alpha_\rho = \alpha_\rho = e^{-\beta(E_{\text{max}} - E_f)} \), where \( E_{\text{max}} \) is the maximum eigenvalue of the Hamiltonian \( H_f \). This follows from the fact that the states \( \rho_S(\tau/2) \) and \( \rho_S(0) = \rho_0^2 \) have the same spectrum due to their unitary equivalence, that is, \( \rho_S(\tau/2) = U(\tau/2, 0) \rho_S(0) U^\dagger(\tau/2, 0) \).

We notice that the bounds (22)-(23) cannot be saturated in general when the initial state of the system is mixed due to the complementarity relation in Eq. (19), which involves a partial trace over the environmental degrees of freedom. (Conversely, saturation would require either measuring the whole environment or a pure initial state of the system as in the previous sections.) Nevertheless, there is a single case where the bound \( \mathcal{B}_2 \) in Eq. (22) is saturated, namely, by verifying the reversibility conditions (quasi-static evolution) where \( \mathcal{V} \to 1 \) and \( \langle W_{\text{diss}} \rangle \to 0 \). On the other hand, the bound in Eq. (23) is not saturated even in the reversible case, since it is designed to work better in irreversible conditions for \( \mathcal{V} < 1 \) and \( \alpha \to 0 \), where Eq. (19) becomes a strict inequality.

Further practical limitations on the ability to split the protocol or to implement the time-reversal operation \( \Theta \) are addressed in appendix A.

Although in our discussion we supposed that the auxiliary degree-of-freedom is the path of the particle which encodes the system of interest, this is not a requirement of our proposal. The only three requirements on the auxiliary degree-of-freedom are the following. (1) The state in Eq. (5) should be initially prepared. (2) Depending on the state of this auxiliary degree-of-freedom, the forward and time-reversal evolutions should then be implemented. (3) Finally, the auxiliary degree-of-freedom should be measured in the basis \( \{ \frac{1}{\sqrt{2}} |0 \rangle_A, \frac{1}{\sqrt{2}} e^{i\phi} |1 \rangle_A \} \), while scanning the phase \( \phi \) to estimate the visibility. This auxiliary degree-of-freedom could be encoded in the same particle, perhaps in additional energy levels of an atomic system, in which case the visibility measurement would take the form of atomic interferometry. Alternatively, a second particle could be used to condition the forward and time-reversal evolution. For example, if the target system is a qubit encoded in a single trapped ion, one could place a second ion in the trap and then couple the two via the collective vibrational mode. More concretely, Ref. [55] shows explicitly how one can implement the controlled evolution of different unitary operations using trapped ions. In this case, the initial state [Eq. (5)] would require entangling operations for its preparation, and the visibility could be easily measured on the internal degree-of-freedom of the second ion.

V. EXAMPLE OF A PHOTONIC IMPLEMENTATION

We apply our scheme to an illustrative experimental set-up in which the thermodynamic system is represented by a single qubit realized through the polarization degree of freedom of a single photon, its thermality is given by the degree of entanglement with an additional photon, the auxiliary qubit is encoded in its path, and the time-dependent thermodynamic process is performed in \( N \) discrete time-steps \( \tau \) by sending the photon through a sequence of liquid crystal waveplates each executing a quench on the (time-independent) Hamiltonian \( H(\Lambda(t_k)) \) with \( k = 1, ..., N \), as sketched in Fig. 3.

The Hamiltonian of the qubit system can be defined as:

\[
H(\Lambda) = \frac{\hbar \omega}{2} \left[ 1 + \cos(\Lambda) \sigma_z + \sin(\Lambda) \sigma_x \right], \tag{24}
\]

where \( \omega \) is the qubit’s natural frequency, and the control parameter implements \( \Lambda \) sudden changes in the range \( \Lambda(0) = 0 \) to \( \Lambda(\pi) = \frac{\pi}{2} \). Consequently, the Hamiltonian is given by the spin operator within the \( x-z \) plane, which rotates by an angle of \( \frac{\pi}{2N} \) at each step around the \( y \)-axis. At the initial and final times of the protocol, the Hamiltonian is diagonal in the \( \sigma_x \) and \( \sigma_z \) bases, respectively. Therefore, \( \{ E_n(0) = \{ |z- \rangle, |z+ \rangle \} \} \), with corresponding energies \( E_n(0) = \{ 0, \hbar \omega \} \), and \( E_n(\tau) = \{ |x- \rangle, |x+ \rangle \} \), where \( |x- \rangle = 1/\sqrt{2}(|z- \rangle - |z+ \rangle) \) and \( |x+ \rangle = 1/\sqrt{2}(|z- \rangle + |z+ \rangle) \), with same eigenvalues \( E_n(\tau) = \{ 0, \hbar \omega \} \). This implies that \( F_0 = F_\tau = -\log(1 + e^{-\beta \hbar \omega}) \), and thus \( \Delta F = F_\tau - F_0 = 0 \) such that \( W_{\text{diss}} = W \).

In the \( k \)-th step, the control parameter takes a fixed value \( \Lambda_k = \Omega \tau k/N \), where \( \Omega = \frac{\pi}{2N} \) is the angular frequency of the rotation. Therefore, any initial state \( |\psi(0)\rangle \) evolves according to

\[
e^{-\frac{i}{\hbar} H(N \Delta t)} \ldots e^{-\frac{i}{\hbar} H_2 \Delta t} e^{-\frac{i}{\hbar} H_1 \Delta t} |\psi(0)\rangle, \tag{25}\]

where \( \Delta t = \frac{\pi}{2N \Omega} \), and for each step \( k = 1, ..., N \):

\[
H_k = \frac{\hbar \omega}{2} \left[ 1 + \cos \left( \frac{k \pi}{2N} \right) \sigma_z + \sin \left( \frac{k \pi}{2N} \right) \sigma_x \right]. \tag{26}
\]

The Hamiltonian at each step \( H_k \) induces a rotation on the system state of an angle \( \theta = \sqrt{\frac{k \pi}{2N}} \) around the axis whose direction \( \hat{d}_k = (\cos \frac{k \pi}{2N}, 0, \sin \frac{k \pi}{2N}) \) changes from step to step. This evolution can be implemented by means of a sequence of \( N \) liquid crystal wave-plates (LCWP). The \( k \)-th LCWP rotates the photon’s polarization about an axis \( \hat{d}_k \), and the angle of rotation is given by the retardance which we can change by an externally applied voltage. Hence, to implement the full evolution we can use a series of \( N \) LCWPs, each with an optic axis set at \( \theta_k = \sqrt{\frac{k \pi}{2N}} \in [0, \pi/2] \), and with the same retardance for all LCWP s (i.e., \( \theta \)).

Our scheme can be executed, following Fig. 3, by inserting pairs of the eigenstates of the Hamiltonians \( H_0 \) and \( H_N \) to the
interferometer. In particular, we take \( N = 7 \), and apply the discretised Hamiltonian \( H_k \) for \( k = 1, 2, 3 \) along path \( |0\rangle_A \), while along path \( |1\rangle_A \) we perform \( H_k \) for \( k = 6, 5, 4 \). In this case, we recover the whole work probability distribution, together with the average dissipative work during the process, which can be used to test the fluctuation relations. We note that the work evaluated does not correspond to the intrinsic photonic energy (which is given by its frequency) but to the generator of the evolution (24).

In addition, our scheme can also be used to test the upper bounds on the dissipative work obtained in Eqs. (22) and (23) by inserting the thermal states of the two Hamiltonians. More precisely, we may insert a single photon from a pair of photons in a partially entangled state \( |\psi\rangle_0 = a |z_+\rangle |z_+\rangle + b |z_-\rangle |z_-\rangle \), where \( a, b \in \mathbb{C} \). The state of the injected photon is obtained by tracing out the second photon, \( \rho_0^{\text{th}} = |a|^2 |z_+\rangle \langle z_+| + |b|^2 |z_-\rangle \langle z_-|. \) This corresponds to a thermal state for the choice \( |a|^2 = \frac{\exp(-\beta \hbar \omega)}{Z_0} \) and \( |b|^2 = \frac{1}{Z_0} \), with \( Z_0 = 1 + \exp(-\beta \hbar \omega) \). Specifically, if \( T \to 0 \), \( \rho = |z_-\rangle \langle z_-| \), while if \( T \to \infty \), \( \rho = 1/2 \).

In Fig. 4(a), we show the expected work probability distribution associated to our discretized protocol with \( N = 7 \) steps for a fixed inverse temperature \( \beta = 1.2(\hbar \Omega)^{-1} \) and three values of the frequency \( \omega = \{0.5, 1.5, 3.0\} \Omega \) (light blue, dark blue, yellow) and fixed duration \( \tau = 2\pi/\Omega \). Since the eigenvalues of the Hamiltonian \( H(A) \) are constant, the work probability distribution consists of three peaks placed at work values \( W = \{-\hbar \omega, 0, \hbar \omega \} \). Low temperatures favor an asymmetric distribution with a higher peak for positive work \( W = \hbar \omega \) with respect to \( W = -\hbar \omega \), while for higher temperatures the two lateral peaks approach equal heights. Moreover, as we observe, for faster protocols the three peaks in the distribution are comparable, while slower protocols \( \omega \gg \Omega \) lead to the suppression of lateral peaks in favor of a high central peak at \( W = 0 \). In this case, we approach an adiabatic evolution where the initial populations of Hamiltonian eigenstates remain almost constant in time, hence leading to zero energy changes and zero work. On the contrary, in the opposite limit \( \omega \ll \Omega \), we approach a sudden quench of the Hamiltonian. In this case, the state of the system remains unchanged by the evolution in Eq. (25), and the lateral peaks associated with the overlaps \( |x_-\rangle |z_+\rangle \) and \( |x_+\rangle |z_-\rangle \) become maximal.

In Fig. 4(b) we show the performance of the bounds for the dissipative work in Eqs. (22) and (23). We assume the equality in Eq. (19), and take a rotation frequency \( \Omega = 1.5\omega \). As can be appreciated in the plot, in the low temperature regime (right side) the logarithmic bound \( B_{\text{log}} \) becomes the best option while \( B_3 \) diverges due to the exponential decrease of \( \alpha_{rh} = e^{-\beta_{\text{max}} \log F(\tau)} \) with temperature. On the contrary,
when temperature is increased (left part), $B_2$ starts to perform better as soon as $k_B T$ becomes higher than the system energy splitting ($k_B T > \hbar \omega$). When increasing $\Omega$ (not shown in the Figure), logarithmic and quadratic bounds become tighter in their respective temperature regimes of performance. In the opposite limit of a near adiabatic process (where the dissipative work vanishes), the quadratic bound still performs good for high temperatures, but, contrary to previous cases, the logarithmic bound becomes worst even in the limit of small temperatures. Nevertheless, the bounds do not appear to become saturated in any of the parameters’ regime.

In the limit of many steps $N \gg 1$ the discrete rotation protocol can be approximated by a continuous rotation, with $\Lambda(t) = \Omega t$ for arbitrary $\Omega$ and $t \in [0, \tau]$. Experimentally, this could be realized using “twisted nematic liquid crystals” (TNLC). These are devices where the optic axis is continuously rotated (typically by 90°) along the beam propagation [56]. For our proposal, we would require two devices with 45° rotation, one for the forward arm and one for the time-reversed arm. Note that one could directly implement the final unitary operation using a set of three waveplates [56], which should also be achievable with current technology [57]. In the limit $\omega/\Omega \gg 1$, we obtain a fully adiabatic process, where the populations of Hamiltonian eigenstates remain constant through the entire evolution (see App. B for a detailed analysis). Moreover, the Hamiltonian $H(\Lambda(t))$ has the same eigenvalues at all times, we conclude that, under adiabatic evolution, a system starting in a thermal state at $t = 0$ will remain in equilibrium at the same temperature at all later times.

VI. CONCLUSIONS

In this work, we have developed a new method based on interferometric tools to measure the work probability distribution and the thermodynamic irreversibility of a generic driving process acting on a quantum system. The method utilizes the interference between two paths, one along which the system is driven out of thermal equilibrium in the forward, and one where it is driven in the time-reversal process. We demonstrated that inserting the energy eigenstates of the initial and final Hamiltonians of the system in the two paths of the interferometer and measuring the fringe visibility enable us to directly reconstruct the work distribution and the average dissipative work. The latter is known to be equal to a production of positive average entropy, and it is a measure of the thermodynamic irreversibility.

Our proposal offers a faster implementation speed than TPM schemes as it halves the duration of each execution. A speed enhancement in each run is a considerable advantage since, in TPM schemes, sufficient statistics must be acquired to reconstruct the order of $N^2$ instances (i.e., probabil-
cles) of the work probability distribution from the results of projective measurements. Furthermore, in the TPM scheme the results of the projective measurements are randomly sampled. Due to finite size effects in sampling, the TPM scheme can have a significant delay in acquiring sufficient data, especially for low-probability instances in the work distribution. In contrast, in our scheme one can control which instance in the work distribution to measure by choosing the appropriate input states in the forward and time-reversal amplitudes, making the scheme much less affected by finite-size statistics.

Our scheme also offers advantages over existing alternatives to the TPM scheme [44–46] as it enables a direct measure of the conditional probabilities that make up the work probability distribution. For example, in Refs. [44–46], the proposed scheme measures the characteristic function of work, i.e., the Fourier transform of the work probability distribution, from which the work probability distribution must then be recovered indirectly. In the implementations of Refs. [45, 46], this problem required a large sampling of a continuous function (the characteristic function) to recover a discrete probability distribution with only a few peaks. In our proposal, these drawbacks are overcome by directly obtaining the conditional probabilities associated with the peaks.

In the case of limited experimental control, when only the thermal states of the initial and final Hamiltonians of the system can be prepared, our method provides useful upper bounds on the average dissipative work. The scheme involves no entangling operations with external auxiliary systems and no energy measurements, and thus offers an accessible and versatile playground for studying the thermodynamics of quantum processes.

To provide a concrete example of implementation of our scheme, we have developed an experimental proposal of our scheme using an all-optical platform, and standard tools for single- and entangled-photon manipulation. The out-of-equilibrium quantum dynamics is realized via a series of liquid crystals wave-plates splitting the thermodynamic process in a series of discrete time-steps $t_k$, each represented by a liquid crystal with an optical axis set at a different angle of rotation $\vartheta_k$.

Although here we have focused for simplicity on the case of initial equilibrium states, we stress that our method can be used to determine the work probability distribution for generic nonequilibrium initial states. Systems with initial coherence in the energy basis or composite systems shared following the proposal in Ref. [42] by augmenting the probability distribution. Finally, work probability distributions (Bayesian networks) techniques [37, 58] to infer the work probability distribution. For example, in Refs. [44–46], the proposed scheme measures the characteristic function of work, i.e., the Fourier transform of the work probability distribution, from which the work probability distribution must then be recovered indirectly. In the implementations of Refs. [45, 46], this problem required a large sampling of a continuous function (the characteristic function) to recover a discrete probability distribution with only a few peaks. In our proposal, these drawbacks are overcome by directly obtaining the conditional probabilities associated with the peaks.

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lowing, we present the scheme in the case of interference at time \( t \) in the forward dynamics (corresponding to \( t = 0 \) in the time-reversal dynamics), but an analogous scheme can be developed for interference at time \( t = 0 \) in the forward dynamics (corresponding to \( t = \tau \) in the time-reversal one).

As in the previous case, we start by preparing the auxiliary degree of freedom in the quantum superposition \( \frac{1}{\sqrt{2}} (|0\rangle_A + |1\rangle_A) \) at \( t < 0 \). Once again, the initial states of the system in the two branches may either be the pure states \( |E_n(t)\rangle \) along the path \( |0\rangle_A \) and \( |E_m(t)\rangle \) along \( |1\rangle_A \), or the mixed thermal states \( \rho_{0t}^{\text{th}} \) and \( \rho_{1t}^{\text{th}} \), respectively, depending on whether we have full control over the system in the preparation stage. However, in contrast to the previous case, we implement the whole protocol \( \Lambda \) over the system in the path \( |0\rangle_A \), while the branch \( |1\rangle_A \) remains unaffected.

Assuming, for concreteness, initial pure states, the global state of the system and the auxiliary system after time \( \tau \) can be evaluated and, tracing the system degrees of freedom, we obtain:

\[
\rho_A(\tau) := \text{Tr}_S[\rho_{S,A}(\tau)] = \frac{1}{2} \left[ |0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A + |0\rangle_A \langle 1|_A \text{Tr}_S[U(\tau, 0) \langle E_n(0) \rangle \langle E_m(\tau)|] + |1\rangle_A \langle 0|_A \text{Tr}_S[|E_m(\tau)\rangle \langle E_n(0)| U^d(\tau, 0)] \right].
\]

Consequently, in this case the visibility directly gives us the conditional probabilities for the work probability distribution:

\[
\mathcal{V}_{m,n} := \left| \text{Tr}_S \left[ U(\tau, 0) \langle E_n(0) \rangle \langle E_m(\tau)| \right] \right| = \sqrt{p_{mn}}, \tag{A2}
\]

and we recover Eqs. (13) and (14).

Likewise, when the initial states in the two interferometer paths are the mixed thermal states, we find again, for the visibility:

\[
\mathcal{V} = \left| \text{Tr}_{S,E}[U(\tau, 0) \otimes 1_E] \langle \psi(0) \rangle_{S,E} \langle \psi(0)|_{S,E} \right| = \left| \langle \psi(0)|_{S,E} \langle \psi(0) \rangle_{S,E} \right|, \tag{A3}
\]

which is equivalent to Eq. (17). Consequently, the bounds developed in Eqs. (22) and (23) for the dissipative work apply also in this situation.

**Appendix B: Continuous Rotation Protocol**

We analytically obtain the evolution generated by the time-dependent Hamiltonian in Eq. (24) for the continuous variation of the control parameter \( \Lambda(t) = \Omega t \) for a constant angular velocity \( \Omega \) in the interval \( t \in [0, \tau] \). In order to reach a description in terms of a time-independent Hamiltonian, we use a picture in which the states rotate at the same rate as the Hamiltonian around the \( \vec{y} \)-axis: \( \psi(t) = e^{-i\Omega \cdot \sigma_y t} |\psi(0)\rangle \).

We write the time-dependent Schrödinger equation \( i\hbar \frac{d}{dt} |\psi(t)\rangle = H(\Lambda(t)) |\psi(t)\rangle \) with this substitution for \(|\psi(t)\rangle\), as

\[
\frac{\hbar \Omega}{2} \sigma_y |\psi(0)\rangle + i\hbar \frac{d}{dt} |\psi(0)\rangle = e^{i\frac{\Omega}{2} \sigma_y} H(\Lambda(t)) e^{-i\frac{\Omega}{2} \sigma_y} |\psi(0)\rangle. \tag{B1}
\]

We focus now on the r. h. s. of this equation. By substituting the expression Eq. (24) for the Hamiltonian, we get

\[
e^{i\frac{\Omega}{2} \sigma_y} H(\Lambda(t)) e^{-i\frac{\Omega}{2} \sigma_y} |\psi(0)\rangle = \frac{\hbar \omega}{2} \left( \mathbb{1} + \cos(\Omega t) e^{i\frac{\Omega}{2} \sigma_y} \sigma_x e^{-i\frac{\Omega}{2} \sigma_y} + \sin(\Omega t) e^{i\frac{\Omega}{2} \sigma_y} \sigma_y e^{-i\frac{\Omega}{2} \sigma_y} \right) |\psi(0)\rangle \tag{B2}
\]

We now write the Pauli matrices in the \( \sigma_y \) operator’s eigensystem, and we correspondingly evaluate the two terms in Eq. (B2):

\[
\cos(\Omega t) \sigma_z - \sin(\Omega t) \sigma_x = e^{i\frac{\Omega}{2} \sigma_y} \sigma_z e^{-i\frac{\Omega}{2} \sigma_y} \tag{B3a}
\]

\[
\cos(\Omega t) \sigma_x + \sin(\Omega t) \sigma_z = e^{i\frac{\Omega}{2} \sigma_y} \sigma_x e^{-i\frac{\Omega}{2} \sigma_y} \tag{B3b}
\]

From this, Eq. (B2) becomes \( e^{i\frac{\Omega}{2} \sigma_y} H(\Lambda(t)) e^{-i\frac{\Omega}{2} \sigma_y} = \frac{\hbar \omega}{2} (\mathbb{1} + \sigma_z) \). By substituting this result into Eq. (B1), we obtain

\[
i\hbar \frac{d}{dt} |\psi(t)\rangle = \frac{\hbar}{2} \left[ \omega (\mathbb{1} + \sigma_z) - \Omega \sigma_y \right] |\psi(t)\rangle. \tag{B4}
\]

We have thus reduced the Schrödinger equation with a time-dependent Hamiltonian into one with a time-independent Hamiltonian. By calling

\[
\sin \xi = \frac{\omega}{\sqrt{\omega^2 + \Omega^2}} \quad \cos \xi = \frac{-\Omega}{\sqrt{\omega^2 + \Omega^2}} \tag{B5}
\]

where by \( \xi \) we defined the angle between the direction \( \vec{n} = (0, \cos \xi, \sin \xi) \) and \( \vec{y} \)-axis within \( z \)-\( y \) plane, we can therefore rewrite Eq. (B4) as

\[
i\hbar \frac{d}{dt} |\psi(t)\rangle = \frac{\hbar \omega}{2} \mathbb{1} + \frac{\hbar}{2} \sqrt{\omega^2 + \Omega^2} \vec{n} \cdot \vec{\sigma} |\psi(t)\rangle. \tag{B6}
\]

The solution of this equation is \( |\psi(t)\rangle = e^{-i\frac{\Omega}{2} t \sqrt{\omega^2 + \Omega^2} \vec{n} \cdot \vec{\sigma}} |\psi(0)\rangle \). Neglecting the global phase \( e^{-i\frac{\Omega}{2} t} \), we thus get

\[
|\psi(t)\rangle = e^{-i\frac{\Omega}{2} \sigma_y} |\psi(0)\rangle \tag{B7}
\]

\[
= \exp \left( -i \frac{\Omega}{2} \sigma_y \right) \exp \left( -i \frac{\Omega}{2} \sqrt{\omega^2 + \Omega^2} \vec{n} \cdot \vec{\sigma} t \right) |\psi(0)\rangle.
\]

We now introduce an eigenstate basis \( \{|\vec{n}_\pm\rangle\} \) of \( \vec{n} \cdot \vec{\sigma} \) (i.e., \( \vec{n} \cdot \vec{\sigma} \mid \vec{n}_\pm \rangle = \pm |\vec{n}_\pm \rangle \)), where \( |\vec{n}_+ \rangle = \cos(\xi/2) |y_+\rangle + \sin(\xi/2) |y_-\rangle \), and \( |\vec{n}_- \rangle = -\sin(\xi/2) |y_+\rangle + \cos(\xi/2) |y_-\rangle \), with \( \{|y_\pm\rangle\} \) being the eigenbasis of \( \sigma_y \).

If we write the initial state \(|\psi(0)\rangle\) in terms of this new basis in the general form \(|\psi(0)\rangle = c_1 |\vec{n}_+ \rangle + c_2 |\vec{n}_- \rangle \), where
Finally, to set $c_1$ and $c_2$, we suppose that the initial state $|\psi_0(0)\rangle$ was an eigenstate of the initial Hamiltonian $H(t = 0) = \frac{\hbar}{2}(\hat{1} + \sigma_z)$, i.e., $|z_+\rangle = (|y_+\rangle + |y_-angle)/\sqrt{2}$ and $|z_-\rangle = -i(|y_+\rangle - |y_-angle)/\sqrt{2}$. Then $c_1 = [\cos(\xi/2) + \sin(\xi/2)]/\sqrt{2}$ and $c_2 = -[\sin(\xi/2) - \cos(\xi/2)]/\sqrt{2}$ for $|z_+\rangle$ and $c_1 = -i[\cos(\xi/2) - \sin(\xi/2)]/\sqrt{2}$ and $c_2 = i[\sin(\xi/2) + \cos(\xi/2)]/\sqrt{2}$ for $|z_-\rangle$.

Let us denote by $|\psi_{\pm}(\tau)\rangle$ the final states evolved from the two initial states $|z_{\pm}\rangle$. If initially the system was in the thermal state, i.e., $\rho_0^{\Omega} = \frac{1}{Z_0^{\Omega}}|z_-\rangle \langle z_-| + e^{-\hbar \omega \beta} |z_+\rangle \langle z_+|$, then the final state will be

$$\rho_{\tau} = \frac{1}{Z_0^{\Omega}} [\langle \psi_-(\tau) | \psi_-(\tau) \rangle + e^{-\hbar \omega \beta} \langle \psi_+(\tau) | \psi_+(\tau) \rangle].$$

(B9)

Let us separate the two cases of a slow-varying Hamiltonian $\omega \gg \Omega$ (which will correspond to an adiabatic thermodynamic process), and that of a rapidly-varying Hamiltonian $\omega \ll \Omega$. We will assume below that the initial state is always $|z_+\rangle$.

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