Lie symmetries of radiation natural convection flow past an inclined surface

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Abstract

Applying Lie symmetry method, we find the most general Lie point symmetry groups of the radiation natural convection flow equation (RNC). Looking the adjoint representation of the obtained symmetry group on its Lie algebra, we will find the preliminary classification of its group-invariant solutions.

Keywords: Lie-point symmetries, invariant, optimal system of Lie sub-algebras.

Mathematics Subject Classification: 70G65, 58K70, 34C14.

1 Introduction

This paper can be viewed as a continuation of the work of paper [7] where the authors by using Lie group analysis for the PDE system corresponding to radiation effects on natural convection heat transfer past an inclined surface (RNC) could reduce this system to an ODE one and obtain numerical solutions by applying Runge-Kutta method.

There are various techniques for finding solutions of differential equations but most of them are useful for a few classes of equations and applying these techniques for unknown equations is impossible. Fortunately symmetries of differential equations remove this problem and give exact solutions.

Symmetry methods for differential equations were discussed by S.Lie at first. One of the most important Lie surveys was finding relationships between continuous transformation group and their infinitesimal generators that allows us to reduce invariant conditions of differential equations.

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under group action, that are complicated because of non linearity, to linear ones.

The main results of applying Lie symmetry method for differential equations is finding group invariant solutions. A useful way for reducing equations is finding any subgroup of the symmetry group and writing invariant solutions with respect to this subalgebra. This reduced equation is of fewer variables and is easier to solve. In fact for many important equations arising from geometry and physics these invariant solutions are the only ones which can be studied thoroughly.

This type of equations as (RNC) system plays an important role in engineering and industrial fields. In this paper Lie symmetry method is applied to find the most general Lie symmetry group and optimal system of (RNC). Finally, we gain group-invariant solutions of the reduced system.

2 Symmetries of RNC System

Consider the (RNC) system:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\partial^2 u}{\partial y^2} + Gr \theta \cos \alpha, \\
u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} &= \frac{1}{Pr} (1 + 4R) \frac{\partial^2 \theta}{\partial y^2}.
\end{align*}
\]

(1)

where \( Gr = \frac{g\beta(T_W - T_\infty)}{U_\infty^2} \) is the Grashof number, \( Pr = \frac{\rho c p \nu k}{k} \) is the Prandtl number and \( R = \frac{4\sigma_0 T_\infty^4}{3k^8} \) is the radiation parameter [7]. Now the infinitesimal method is implied as follow:

The infinitesimal generator \( X \) on total space of the form:

\[
X = \sum_{i=1}^{p} \xi^i(x, u, v) \partial_x^i + \sum_{j=1}^{q} \eta^j(x, u, v) \partial_u^j,
\]

(2)

has the \( n^{th} \) order prolongation \( Pr^{(n)}X \) ([6], Th 4.16). Applying the fundamental infinitesimal symmetry criterion ([6], Th 6.5) on \( X \) as follow:

\[
Pr^{(n)}X \left[ \Delta_\nu(x, u^{(n)}) \right] = 0, \quad \nu = 1, \ldots, l, \quad \text{whenever} \quad \Delta_\nu(x, u^{(n)}) = 0.
\]

(3)

obtains the infinitesimal generators of the symmetry group.

The vector field associated with RNC system is of the form:

\[
X = \xi_1(x, y, u, v) \partial_x + \xi_2(x, y, u, v) \partial_y \\
+ \varphi_1(x, y, u, v) \partial_u + \varphi_2(x, y, u, v) \partial_v + \varphi_3(x, y, u, v) \partial_\theta.
\]

(4)

Since in RNC system, second order derivatives are appeared, symmetry generators are obtained by applying (3) for the second prolongation of \( X \).
The vector field $X$ generates a one parameter symmetry group of RNC $= (\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ if and only if equations (3) hold for $\nu = 1, 2, 3, 4$. So symmetry group of RNC system is spanned by following infinitesimal generators:

$$X_1 = \partial_x,$$
$$X_2 = \partial_y,$$
$$X_3 = x \partial_x + u \partial_u + t \partial_t,$$
$$X_4 = 2x \partial_x + y \partial_y - v \partial_v - 2t \partial_t.$$

The Lie table of Lie algebra $g$ for RNC system is given below:

|     | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|-----|-------|-------|-------|-------|
| $X_1$ | 0     | 0     | $X_1$ | $2X_1$|
| $X_2$ | 0     | 0     | 0     | $X_2$ |
| $X_3$ | $-X_1$| 0     | 0     | 0     |
| $X_4$ | $-2X_1$| $-X_2$| 0     | 0     |

**Theorem 1.** If $g_k(h)$ be the one parameter group generated by $X_k$, $k = 1, \ldots, 4$, then

$$g_1 : (x, y, u, v, t) \mapsto (h + x, y, u, v, t),$$
$$g_2 : (x, y, u, v, t) \mapsto (x, h + y, u, v, t),$$
$$g_3 : (x, y, u, v, t) \mapsto (xe^h, y, u/e^h, v, te^h),$$
$$g_4 : (x, y, u, v, t) \mapsto (xe^{2h}, ye^h, u, v/e^{-h}, te^{-2h}).$$

3 Optimal System

The aim of Lie theory is classifying the invariant solutions and reducing equations. Some of Lie algebras contain different subalgebras, so classifying them plays an important role in transforming equations into easier ones. Therefore we are classifying these subalgebras up to adjoint representation and finding an optimal system of subalgebras instead of finding an optimal system of subgroups.

The adjoint action is given by the Lie series

$$\text{Ad}(\exp(\varepsilon X))X_j = X_j - \varepsilon [X_i, X_j] + \frac{\varepsilon^2}{2} [X_i, [X_i, X_j]] - \cdots,$$

where $[X_i, X_j]$ is the commutator for the Lie algebra, $\varepsilon$ is a parameter, and $i, j = 1, \ldots, 4$ ([5], ch 3.3).

**Theorem 2.** A one-dimensional optimal system of RNC results as follow:

1) $X_1$, 4) $X_1 + X_2$, 7) $X_3 - X_2$,
2) $X_2$, 5) $X_2 - X_1$, 8) $X_3 + X_4$,
3) $X_3$, 6) $X_2 + X_3$, 9) $X_4 - X_3$. 

3
Proof: Consider the symmetry algebra $\mathfrak{g}$ of the RNC system with adjoint representation demonstrated in the below table:

|   | $X_1$   | $X_2$   | $X_3$   | $X_4$   |
|---|---------|---------|---------|---------|
| $X_1$ | $X_1 + \varepsilon X_3 + 2\varepsilon X_4$ | $X_2$   | $X_3$   | $X_4$   |
| $X_2$ | $X_1$   | $X_2 + \varepsilon X_4$ | $X_3$   | $X_4$   |
| $X_3$ | $e^{-\varepsilon} X_1$ | $X_2$   | $X_3$   | $X_4$   |
| $X_4$ | $e^{-2\varepsilon} X_1$ | $e^{-\varepsilon} X_2$ | $X_3$   | $X_4$   |

Let

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4.$$

(7)

In this stage our goal is to simplify as many of the coefficient $a_i$ as possible.

The following results are obtained from adjoint application on $X$:

1. Let $a_4 \neq 0$. Scaling $X$ if necessary, suppose that $a_4 = 1$. Then

$$X' = a_1 X_1 + a_2 X_2 + a_3 X_3 + X_4.$$

(8)

If we act on $X'$ with $\text{Ad}(\exp(a_3 X_3))$, the coefficient of $X_1$ can be vanished:

$$X'' = a_2 X_2 + a_3 X_3 + X_4.$$

(9)

By applying $\text{Ad}(\exp(a_4 X_4))$ on $X''$, we will obtain $X'''$ as follow:

$$X''' = a_3 X_3 + X_4.$$

(10)

1-a. If $a_3 \neq 0$, then the coefficient of $X_3$ can be -1 or 1. Therefore every one-dimensional subalgebra generated by $X$ in this case is equivalent to $X_3 + X_4, X_4 - X_3$.

1-b. If $a_3 = 0$, then every one-dimensional subalgebra generated by $X$ in this case is equivalent to $X_4$.

2. The remaining one-dimensional subalgebras are spanned by vector fields of the form $X$ with $a_4 = 0$.

2-a. If $a_3 \neq 0$, let $a_3 = 1$. By acting of $\text{Ad}(\exp(a_3 X_3))$ on $X$, we will have

$$\hat{X} = a_2 X_2 + X_3.$$

(11)

2-a-1. If $a_2 \neq 0$, then the coefficient of $X_2$ can be -1 or 1. So every one-dimensional subalgebra generated by $\hat{X}$ in this case is equivalent to $X_2 + X_3, X_3 - X_2$.

2-a-2. If $a_2 = 0$ then every one-dimensional subalgebra generated by $X_1$.

2-b. Let $a_4 = 0, a_3 = 0$.

2-b-1. If $a_2 \neq 0$, then we can make the coefficient of $X_1$ either -1 or 1 or 0. Every one-dimensional subalgebra generated by $\hat{X}$ is equivalent to $X_1 + X_2, X_2 - X_1, X_2$.

2-b-2. If $a_2 = 0$, then every one-dimensional subalgebra generated by $\hat{X}$ is equivalent to $X_1$. \qed
4 Invariant Solutions and Reduction

Now for finding invariant solutions using characteristic method [2]:

\[ Q_\alpha \big|_{u=u(x,t)} = X[u^\alpha - u^\alpha(x,t)] \big|_{u=u(x,t)} \quad \alpha = 1, \ldots, M \]  

where \( M \) is the number of dependant variables. Now invariant solutions for RNC result as follow:

Consider \( X_1 = \partial_u \). Applying characteristic method on \( X_1 \) obtains: \( Q_u = -\frac{\partial}{\partial x} u(x,y), Q_v = -\frac{\partial}{\partial y} v(x,y), Q_t = -\frac{\partial}{\partial t} t(x,y) \). Solutions of this system are of the form: \( u = F_1(y), v = F_1(y), t = F_1(y) \).

Consider \( X_2 = \partial_y \). Applying characteristic method on \( X_2 \) obtains: \( Q_u = -\frac{\partial}{\partial x} u(x,y), Q_v = -\frac{\partial}{\partial y} v(x,y), Q_t = -\frac{\partial}{\partial t} t(x,y) \). Solutions of this system are of the form: \( u = F_1(y), v = F_1(y), t = F_1(x) \).

Consider \( X_3 = x\partial_x + u\partial_u + t\partial_t \). Applying characteristic method on \( X_3 \) obtains: \( Q_u = u(x,y) - x\frac{\partial}{\partial x} u(x,y), Q_v = -x\frac{\partial}{\partial y} v(x,y), Q_t = t(x,y) - x\frac{\partial}{\partial t} t(x,y) \).

Solutions of this system are of the form: \( u = xF_1(y), v = F_1(y), t = xF_1(y) \).

Consider \( X_4 = 2x\partial_x + y\partial_y - v\partial_v - 2t\partial_t \). Applying characteristic method on \( X_4 \) obtains: \( Q_u = -2x\frac{\partial}{\partial x} u(x,y) - y\frac{\partial}{\partial y} u(x,y), Q_v = -v(x,y) - 2x\frac{\partial}{\partial t} v(x,y) - y\frac{\partial}{\partial t} v(x,y), Q_t = t(x,y) - 2x\frac{\partial}{\partial t} t(x,y) - y\frac{\partial}{\partial t} t(x,y) \).

Solutions of this system are of the form: \( u = F_1 \left( \frac{x}{\sqrt{2}} \right), v = F_1 \left( \frac{y}{\sqrt{2}} \right), t = xF_1(y) \).

Consider \( X_1 + X_2 = \partial_x + \partial_y \). Applying characteristic method on \( X_1 + X_2 \) obtains: \( Q_u = \frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial y} u(x,y), Q_v = \frac{\partial}{\partial x} v(x,y) + \frac{\partial}{\partial y} v(x,y), Q_t = \frac{\partial}{\partial t} t(x,y) + \frac{\partial}{\partial t} t(x,y) \). Solutions of this system are of the form: \( u = F_1(-x+y), v = F_1(-x+y), t = F_1(-x+y) \).

Consider \( X_2 - X_1 = \partial_y - \partial_x \). Applying characteristic method on \( X_2 - X_1 \) obtains: \( Q_u = \frac{\partial}{\partial x} u(x,y) - \frac{\partial}{\partial y} u(x,y), Q_v = \frac{\partial}{\partial x} v(x,y) - \frac{\partial}{\partial y} v(x,y), Q_t = \frac{\partial}{\partial t} t(x,y) - \frac{\partial}{\partial t} t(x,y) \). Solutions of this system are of the form: \( u = F_1(x+y), v = F_1(x+y), t = F_1(x+y) \).

Consider \( X_2 + X_3 = x\partial_x + \partial_y + u\partial_u + t\partial_t \). Applying characteristic method on \( X_2 + X_3 \) obtains: \( Q_u = u(x,y) - x\frac{\partial}{\partial x} u(x,y) - \frac{\partial}{\partial y} u(x,y), Q_v = -x\frac{\partial}{\partial t} v(x,y) - y\frac{\partial}{\partial t} v(x,y), Q_t = t(x,y) - x\frac{\partial}{\partial t} t(x,y) - \frac{\partial}{\partial t} t(x,y) \). Solutions of this system are of the form: \( u = xF_1(-\ln(x+y)), v = F_1(-\ln(x+y)), t = xF_1(-\ln(x+y)) \).

Consider \( X_3 - X_2 = \partial_y - \partial_x + u\partial_u + t\partial_t \). Applying characteristic method on \( X_3 - X_2 \) obtains: \( Q_u = u(x,y) - x\frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial y} u(x,y), Q_v = -x\frac{\partial}{\partial t} v(x,y) - y\frac{\partial}{\partial t} v(x,y), Q_t = -x\frac{\partial}{\partial t} t(x,y) - \frac{\partial}{\partial t} t(x,y) \). Solutions of this system are of the form: \( u = xF_1(-\ln(x+y)), v = F_1(-\ln(x+y)), t = xF_1(-\ln(x+y)) \).
\[-x \frac{\partial}{\partial x} v(x, y) + \beta \frac{\partial}{\partial y} v(x, y), Q_1(t(x, y) - x \frac{\partial}{\partial x} t(x, y) + \frac{\partial}{\partial y} t(x, y).\]

Solutions of this system are of the form: \(u = x F_1(\ln(x) + y), v = F_1(\ln(x) + y), t = x F_1(\ln(x) + y).\)

Consider \(X_3 + X_4 = 3x \partial_x + y \partial_y + u \partial_u - v \partial_v - t \partial_t.\) Applying characteristic method on \(X_3 + X_4\) obtains: \(Q_u = u(x, y) - 3x \frac{\partial}{\partial x} u(x, y) - y \frac{\partial}{\partial y} u(x, y), Q_v = -v(x, y) - 3x \frac{\partial}{\partial x} v(x, y) - y \frac{\partial}{\partial y} v(x, y), Q_t = -t(x, y) - 3x \frac{\partial}{\partial x} t(x, y) - y \frac{\partial}{\partial y} t(x, y).\) Solutions of this system are of the form: \(u = F_1 \left( \frac{x}{\sqrt{y}} \right), v = F_1 \left( \frac{x}{\sqrt{y}} \right), t = F_1 \left( \frac{x}{\sqrt{y}} \right).\)

Consider \(X_4 - X_3 = x \partial_x + y \partial_y - u \partial_u - v \partial_v - 3t \partial_t.\) Applying characteristic method on \(X_4 - X_3\) obtains: \(Q_u = -u(x, y) - x \frac{\partial}{\partial x} u(x, y) - y \frac{\partial}{\partial y} u(x, y), Q_v = -v(x, y) - x \frac{\partial}{\partial x} v(x, y) - y \frac{\partial}{\partial y} v(x, y), Q_t = -3t(x, y) - x \frac{\partial}{\partial x} t(x, y) - y \frac{\partial}{\partial y} t(x, y).\) Solutions of this system are of the form: \(u = F_1 \left( \frac{x}{y} \right)^{-1}, v = F_1 \left( \frac{x}{y} \right)^{-1}, t = F_1 \left( \frac{x}{y} \right)x^{-3}.\)

5 Conclusion

In this paper by applying infinitesimal symmetry methods, we find optimal system and finally could reduce the RNC system and find invariant solutions.

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