New developments in model-independent Partial-Wave Analysis

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Abstract

Partial-wave analyses (PWA) are an essential tool for studying resonance structures in decays with hadronic multi-body final states. For several years, more model-independent approaches to such analyses have been used for various decay final states. However, up to now, these methods have mostly been applied to sub-sets of partial waves, also called freed waves. In this article, we explore possibilities and limitations of extended model-independent approaches. We systematically apply various different fit models to the analysis of pseudo data sets, to study both the impact of the mathematical description used for the freed waves and the choice of simultaneously freed waves. We can show that suitable methods exist, which lift restrictions to only sub-sets of freed partial waves and demonstrate hidden caveats present in previous works.

Keywords: PWA, partial-wave analysis, model independent, spectroscopy, Dalitz plot, light mesons, heavy meson decay, ambiguities

1. Introduction

Amplitude analyses of hadronic multi-body final states prove to be a valuable tool to disentangle the excitation spectrum of QCD and to determine strong phases in weak decays. Small contributions to the decay amplitude can interfere with dominant ones and thereby become visible and accessible through amplitude analysis. However, Partial-Wave Analyses (PWA) of multi-body final states, which constitute the most common tool for amplitude analyses, require amplitude modelling. The assumptions underlying the models can introduce a model bias and possibly obscure small signal components. In this article we propose improvements to the so-called “model-independent PWA” approach often used to alleviate the necessity for such model assumptions; other names for this method include “freed-isobar PWA” or “quasi-model-independent PWA”. In particular, we explore the suppression of ambiguities, arising when extending the model-independent approach to multiple parts of the amplitude model.

2. Model-independent partial-wave analysis

To understand the complex-valued amplitudes governing a particular meson decay, we use the framework of partial-wave analysis. Here, the measured intensity distribution $I(\bar{\theta})$—which for three-body decays often is depicted in a Dalitz plot—depends on the kinematic variables $\bar{\theta}$ and is described as the modulus square of the full modeled amplitude:

$$ I(\bar{\theta}) = \left| \sum_{w \in \text{waves}} T_w A_w(\bar{\theta}) \right|^2, \quad (1) $$

where the full amplitude is decomposed into a sum over a set of partial waves $w$. The complex-valued transition amplitudes $T_w$ encode the strength and relative phases of the individual partial waves $w$; they are the free parameters of a PWA. In contrast, the complex-valued decay amplitudes $A_w(\bar{\theta})$ usually do not contain any free
parameters, but encode the dependence of the individual partial waves on \( \bar{\theta} \). They have to be fully known beforehand, which requires modelling and prior knowledge.

For the description of the decay amplitudes, we use the isobar model, which describes a multi-body decay process as a sequence of subsequent two-particle decays, which includes the appearance of resonant subsystems, the isobars. In this article, we focus on a three-body final state, where the process is described by two two-body decays introducing an additional two-particle sub-system \( \xi \). Within this isobar model, we express the decay amplitudes \( \mathcal{A}_\nu (\bar{\theta}) \) as:

\[
\mathcal{A}_\nu (\bar{\theta}) = \psi_\nu (\bar{\theta}) \Delta_\nu (m_\xi^2).
\]  

The angular amplitudes \( \psi_\nu (\bar{\theta}) \) describe the dependence of the partial waves on the decay angles, which is given by first principles and fully determined by the spin and angular momentum quantum numbers of the corresponding partial wave \( \nu \). The dynamic amplitude \( \Delta_\nu (m_\xi^2) \) is a complex-valued function of only the invariant mass of the isobar system and describes its resonance content. They are most commonly parameterized through Breit-Wigner functions describing isolated resonances appearing far from thresholds. However, the appearance of more than one such resonance with the same \( J^{PC} \) quantum numbers or interactions between \( \xi \) and the third spectator particle require more advanced modelling. The actual model—and its model parameters—are not given by first principles. At any rate, the dynamic amplitude is a necessary input for a partial wave model.

To minimize the number of assumptions for the dynamic amplitudes, model-independent PWA approaches are often used. Here, the mass distribution of the isobar \( m_\xi^2 \) is discretized into bins \( b \) and the complex valued dynamic amplitude is replaced by a set of basis functions \( \Lambda^b (m_\xi^2) \), defined within a mass bin \( b \):

\[
\Delta_\nu (m_\xi^2) = \sum_{b \in \text{bins}} C^b_\nu \Lambda^b (m_\xi^2; m_{\text{low}}^2, m_{\text{center}}^2, m_{\text{up}}^2).
\]

The full dynamic amplitude is obtained by summing of all bins \( b \) in the invariant mass-squared \( m_\xi^2 \) of the isobar, that continuously span the full kinematically allowed range. In the past, piecewise constant functions were most commonly used as function basis \([1]\).

However, in our studies we found spike functions, \( \Lambda^b (m_\xi^2; m_{\text{low}}^2, m_{\text{center}}^2, m_{\text{up}}^2) \) growing linearly from 0 to 1 as \( m_\xi^2 \) goes from \( m_{\text{low}}^2 \) to \( m_{\text{center}}^2 \) and within a mass bin then returning to 0 as \( m_\xi^2 \) reaches \( m_{\text{up}}^2 \) to result in a more favorable convergence behavior. The spike functions vanish outside the range \([m_{\text{low}}^2, m_{\text{up}}^2] \).

The complex-valued coefficients \( C^b_\nu \) in Eq. (3) encode the value of the dynamic amplitude in a given \( m_\xi^2 \) bin \( b \). Note, that the replacement given in Eq. (3) does not alter the over-all structure of Eq. (1). Covering the kinematically allowed \( m_\xi^2 \) range with overlapping spike functions results in an approximation to the dynamic amplitude, which is linear in \( m_\xi^2 \).

The use of higher order polynomials as basis functions instead of a linear interpolation across the isobar mass bins, ensures also derivatives of the dynamic amplitudes to be continuous at the bin borders. Such polynomials of degree \( n \) have to be non-vanishing in \( n + 1 \) neighboring bins, which results in \( n + 1 \) non-vanishing basis functions at any given value of \( m_\xi^2 \) and thus to large overlaps between the functions. Such large overlaps tend to produce artifacts, large correlations among the fit parameters in the analysis, and a worse convergence towards the input model. Function bases, that don’t rely on a fixed binning in \( m_\xi^2 \), like e.g. the Fourier basis, or Chebyshev polynomials suffer from similar problems.

3. The pseudo data sets

To study the viability and limitations of the model-independent PWA approach, we chose the most simple case in form of a three-body decay of a spin 0 initial state into three spin 0 final-state particles, which can be depicted in a Dalitz plot. However, we stress that the methods discussed are viable for a large variety of PWA three-body final states. We will exemplarily consider the decay

\[
D^+ \rightarrow K^- + \pi^+ + \pi^+
\]

for which we generated two pseudo data sets using Monte Carlo methods. The decay is modeled with three partial waves with the spin of the \([K^- \pi^+] \) isobar ranging from zero to two. Since all initial and final state particles are spinless, the spin of the isobar already fixes all quantum numbers of a partial wave and the set of kinematic variables is given by:

\[
\bar{\theta} = \{m_{K^+}^2, m_{\pi^+}^2, m_{\pi^-}^2, m_{\pi^0}^2\},
\]

\[3\]

For the first and the last bin, \( m_{\pi^0}^2 \) and \( m_{\pi^-}^2 \) lie outside the kinematically allowed range and are therefore neglected.

\[3\]

The free parameters in such a fit are the products \( T_\nu C^b_\nu \), where the \( T_\nu \) are a global complex-valued factor for the dynamic isobar amplitude, which acts as a global phase- and normalization factor.
For a data set (A) of a million events, the S-, P-, and D-wave we used the following \([ \text{K}^- \pi^+ ]\) resonances to describe the isobars:

\[
\text{K}_0^*(700) ; \quad \text{K}^*(892) ; \quad \text{K}^*_2(1430). \tag{6}
\]

The appearance of two identical \(\pi^+\) in the final state, requires the amplitude in Eq. (4) to be symmetrized under the exchange of the two \(\pi^+\) to fulfill Bose symmetry. We do not use doubly charged isobars.

The validity of the model-independent PWA method, which aims at reconciling the isobar structure, is studied through a second pseudo data set (B) of the same size now also including a second resonance in both the S- and P-wave, namely the \(\text{K}_0^*(1430)\) and \(\text{K}^*(1410)\) isobars. For all appearing resonances we use Breit-Wigner parameterizations with resonance parameters taken from Ref. [4].

4. Model-independent PWA results

We now perform an analysis on the two pseudo data sets generated in Sec. 3.

4.1. Fit model

We model the dynamic amplitudes of the S- and P-wave in Eq. (1) by spike functions as given in Eq. (2). However, instead of an equidistant binning in the isobar mass, and apply this fit using the identical model input and fit results, as seen in Fig. 2 for the S-wave. The width of the band gives the statistical uncertainty of the fit.

4.2. Base fit

We performed partial-wave analyses with the model independent formulation. It is now interesting to probe the method for cross talk between waves; namely we use an incomplete fixed model description for the S-wave and investigate the P-wave in a model-independent way. The results of this fit are shown in Fig. 3. Using an incomplete model description for the
fixed S-wave impacts the results for the model independent P-wave, which does not reproduce the input data anymore.

The large freedom present in the model-independent wave allows to accommodate part of the difference between the input model and the fit model for the S-wave thereby distorting the model-independent P-wave. It thus generates cross talk. For most meson decay analyses, the shapes of the amplitudes are not known a priori. Thus, they must be extracted from data, which in turn only gives consistent results, if this determination is done simultaneously in all waves, not in a partial or iterative way as often performed before.

However, such simultaneous approaches often generate unphysical results and large uncertainties due to exact mathematical cancellations of different amplitudes or parts of them, as can be seen for example in Ref. [5]. However, when correctly identified, these cancellations can be removed while still avoiding potential leakage effects from improper parameterizations of fixed waves in the model.

The origin of these cancellations are the particular angular amplitudes $\psi_{12}^S (\vec{\theta})$, for an isobar formed by particles 1 and 2. For the S- and P-wave they are given by:

$$ \psi_{12}^S (\vec{\theta}) \propto 1 $$

and

$$ \psi_{12}^P (\vec{\theta}) \propto m_{23}^2 - m_{13}^2 - \left( m_{123}^2 - m_1^2 \right) \frac{m_1^2 - m_2^2}{m_{12}^2} \quad (9) $$

Due to Bose symmetrization, the total contribution of the S- and P-wave to the amplitude $\mathcal{A}_{S+P} (\vec{\theta})$ is given by:

$$ \mathcal{A}_{S+P} (\vec{\theta}) = \psi_{12}^S (\vec{\theta}) \Delta_S \left( m_{12}^2 \right) \quad (10) $$

$$ + \psi_{12}^P (\vec{\theta}) \Delta_P \left( m_{12}^2 \right) \quad (11) $$

$$ + \psi_{13}^S (\vec{\theta}) \Delta_S \left( m_{13}^2 \right) \quad (12) $$

$$ + \psi_{13}^P (\vec{\theta}) \Delta_P \left( m_{13}^2 \right), \quad (13) $$

where $\psi_{13}^{S/P}$ are the Bose-symmetrized versions of $\psi_{12}^{S/P}$, with the like-sign pions being interchanged.

We find, that $\mathcal{A}_{S+P} (\vec{\theta})$ vanishes everywhere in $\vec{\theta}$ for a special choice of the dynamic isobar amplitudes. Those are described through the following shapes:

$$ \Delta_S \left( m_2^2 \right) = Z \left[ 3m_2^2 - m_K^2 - 2m_\pi^2 \right. $$

$$ + \left. \left( m_{D^*}^2 - m_{\pi^*}^2 \right) \frac{m_{D^*}^2 - m_{\pi^*}^2}{m_2^2} \right] \quad (14) $$

$$ \Delta_P \left( m_2^2 \right) = Z, $$

with an arbitrary complex-valued coefficient $Z$. Since these two peculiar choices for the dynamic isobar amplitude cancel exactly, a shift of the S-wave dynamic amplitude following Eq. (14) can be compensated by a corresponding shift in the P-wave and vice versa. Their exact cancellation does not impact the total amplitude $\mathcal{A}_{S+P}$ anywhere in phase space and a PWA fit cannot differentiate these solutions. We call a set of dynamic
amplitudes for a sub-set of partial waves, that result in such exact cancellations “zero mode”.

Note, that the particular origin of the zero mode in our example is caused by the Bose symmetrization and the resulting interference of two different combinations of two-particle sub-systems. However, such interferences cannot only be caused through Bose symmetrization of the final states, but may occur in every process, which allows for two or three different two-particle sub-systems to form an isobar. If isobars can be formed by a unique two-particle sub-system only, the orthogonality of the angular amplitudes ensures the linear independence of the fit model. In that case, however, a model-independent PWA is not even necessary, since a kinematic binning in $m^2_{c}$ suffices to extract the resonance content of all partial waves.

Although the zero-mode ambiguity given by the parameter $Z$ of Eq. (14) can in principle be resolved, as demonstrated in Ref. [5], it is advantageous to remove it from the fit model from the start. This can be achieved through the optimized binning described in Sec. [4.1]. The optimized binning reflects the existence of regions in $m^2_{c}$ with rapidly varying dynamic amplitudes, but does not carry information on the amplitudes themselves. The best approximation of the shapes in Eq. (14) by the basis functions in discrete intervals of $m^2_{c}$ results in partial cancellations that do not suffice to cause ambiguities in the fit. In particular, the mass binning for the various isobaric waves will be different. Still, the dynamic binning in $m^2_{c}$ allows for narrow resonance structures within an isobar amplitude to be described sufficiently well at a price of affordable information loss in regions of little variations. With the zero mode now being suppressed, we can reproduce the full dynamic amplitudes in the pseudo data containing ground-state and excited resonance for the S-wave and the P-waves simultaneously, as shown in Fig. 4 for the P-wave exemplarily, avoiding leakage effects and zero-mode ambiguities.

5. Summary & conclusions

We have demonstrated, that model-independent partial wave analysis of three-body final states is a valid method to extract any dynamic isobar amplitude of intermediate two-particle sub-systems. However, several caveats have to be respected:

A) cross-talk between various partial waves, may distort the fit results if only a sub-set of waves employed are subjected to the model-independent extraction. This cross talk causes distortions of the dynamic amplitudes extracted from the data itself by waves with imperfect fixed dynamic amplitudes for the isobars.

B) As demonstrated in Ref. [5], the presence of more than one freed partial wave can cause exact cancellations of their amplitudes at every point in phase space. This manifests as continuous ambiguities in the fit.

In this paper we have demonstrated, that an improved model-independent functions basis, which relies on an optimized binning in $m^2_{c}$ for its construction, may suppress these ambiguities and at the same time allows to resolve complex resonant features in the isobaric amplitude underlying the partial waves. The reason for this is that the binning is relatively large in regions without rapidly changing resonance content and therefore the model-independent amplitudes are unable to approximate the zero mode well enough to cause ambiguities.

Although we exemplarily demonstrated the proposed method for the decay $D^+ \rightarrow K^- + π^+ + π^+$, we stress that it is applicable to a great number of partial-wave analyses of three-body final states.

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