INTER-CRITICAL NLS: CRITICAL $\dot{H}^s$-BOUNDS IMPLY SCATTERING

JASON MURPHY

Abstract. We consider a class of power-type nonlinear Schrödinger equations for which the power of the nonlinearity lies between the mass- and energy-critical exponents. Following the concentration-compactness approach, we prove that if a solution $u$ is bounded in the critical Sobolev space throughout its lifespan, that is, $u \in L_t^\infty \dot{H}_x^s$, then $u$ is global and scatters.

Contents

1. Introduction 1
1.1. Outline of the proof of Theorem 1.3 3
Acknowledgements 6
2. Notation and useful lemmas 10
2.1. Some notation 10
2.2. Basic harmonic analysis 10
2.3. Strichartz estimates 12
2.4. Concentration-compactness 14
3. Local well-posedness 15
4. Reduction to almost periodic solutions 18
5. Long-time Strichartz estimates 30
6. The rapid frequency-cascade scenario 39
7. The frequency-localized interaction Morawetz inequality 45
8. The quasi-soliton scenario 55
Appendix A. Some basic estimates 59
References 60

1. Introduction

We consider the initial-value problem for defocusing nonlinear Schrödinger equations of the form

$$\begin{cases} (i\partial_t + \Delta)u = |u|^p u \\ u(0, x) = u_0(x), \end{cases}$$

(1.1)

where $p > 0$ is chosen to lie between the mass- and energy-critical exponents, that is, $\frac{d}{2} < p < \frac{d}{d-2}$. Here $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is a complex-valued function of time and space.

The class of solutions to (1.1) is left invariant by the scaling

$$u(t, x) \mapsto \lambda^\frac{2}{p} u(\lambda^2 t, \lambda x)$$

for $\lambda > 0$. We will show that if $u \in L_t^\infty \dot{H}_x^s$ is a solution to (1.1) that is bounded in the critical Sobolev space, then it scatters as $t \to \pm \infty$.
for \( \lambda > 0 \). This scaling defines a notion of criticality. In particular, one can check that the only homogeneous \( L^2 \)-based Sobolev space that is left invariant by this scaling is \( H^{s_c}_x(\mathbb{R}^d) \), where the critical regularity \( s_c \) is given by \( s_c := \frac{d}{2} - \frac{2}{p} \). If we take \( u_0 \in H^{s_c}_x(\mathbb{R}^d) \) in (1.1), then for \( s = s_c \), we call the problem critical. For \( s > s_c \), we call the problem subcritical.

We consider the critical problem for (1.1) in the inter-critical regime, that is, \( 0 < s_c < 1 \). For \((d, s_c)\) satisfying an appropriate set of constraints, we prove that any maximal-lifespan solution that is uniformly bounded (throughout its lifespan) in \( H^{s_c}_x(\mathbb{R}^d) \) must be global and scatter.

We begin by making the notion of a solution more precise.

**Definition 1.1 (Solution).** A function \( u : I \times \mathbb{R}^d \to \mathbb{C} \) on a non-empty time interval \( I \ni 0 \) is a solution to (1.1) if it belongs to \( C_t H^{s_c}_x(K \times \mathbb{R}^d) \cap L^{p(d+2)/2}(K \times \mathbb{R}^d) \) for every compact \( K \subset I \) and obeys the Duhamel formula

\[
u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-s)\Delta}(|u|^p u)(s) \, ds
\]
for all \( t \in I \). We call \( I \) the lifespan of \( u \); we say \( u \) is a maximal-lifespan solution if it cannot be extended to any strictly larger interval. If \( I = \mathbb{R} \), we say \( u \) is global.

We define the scattering size of a solution \( u \) to (1.1) on a time interval \( I \) by

\[
S_I(u) := \int_I \int_{\mathbb{R}^d} |u(t, x)|^\frac{p(d+2)}{2} \, dx \, dt.
\]

(1.2)

Standard arguments show that if a solution \( u \) to (1.1) is global, with \( S_\mathbb{R}(u) < \infty \), then it scatters; that is, there exist unique \( u_\pm \in H^{s_c}_x(\mathbb{R}^d) \) such that

\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u_\pm\|_{H^{s_c}_x(\mathbb{R}^d)} = 0.
\]

The goal of this paper is to address some cases of the following

**Conjecture 1.2.** Let \( d \geq 1 \) and \( p > 0 \) such that \( s_c := \frac{d}{2} - \frac{2}{p} \geq 0 \). Let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a maximal-lifespan solution to (1.1) such that \( u \in L^\infty_t H^{s_c}_x(I \times \mathbb{R}^d) \). Then \( u \) is global and scatters, with

\[
S_\mathbb{R}(u) \leq C(\|u\|_{L^\infty_t H^{s_c}_x(\mathbb{R} \times \mathbb{R}^d)}),
\]

for some function \( C : [0, \infty) \to [0, \infty) \).

For two special cases, it is unnecessary to take \( u \in L^\infty_t H^{s_c}_x \) as an additional hypothesis, as this bound follows from conservation laws. In particular, in the mass-critical case, \( s_c = 0 \) (i.e. \( p = \frac{4}{d-2} \)), the fact that \( u \in L^\infty_t L^2_x \) follows from the conservation of mass, defined by

\[
M[u(t)] := \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx,
\]

while in the energy-critical case, \( s_c = 1 \) (i.e. \( p = \frac{4}{d-2}, \ d \geq 3 \)), the fact that \( u \in L^\infty_t H^1_x \) follows from the conservation of energy, defined by

\[
E[u(t)] := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{p+2} |u(t, x)|^{p+2} \, dx.
\]

Due to the presence of conserved quantities at critical regularity, the mass- and energy-critical equations have been the most widely studied; in fact, Conjecture
The defocusing energy-critical case was handled first by Bourgain [3], Grillakis [23], and Tao [49] for radial data, and subsequently by Colliander–Keel–Staffilani–Takaoka–Tao [15], Ryckman–Vişan [40], and Vişan [53, 56] for arbitrary data. (See also [25, 32] for results in the focusing case.) The primary obstacle to establishing these results was the lack of any a priori estimates with critical scaling (besides the conservation of energy); that is to say, none of the known monotonicity formulae for NLS (i.e. Morawetz inequalities) scale like the energy (in contrast to the energy-critical nonlinear wave equation, for example). It was ultimately the ‘induction on energy’ technique of Bourgain that showed how one can move beyond this difficulty: by finding a bubble of concentration inside a solution, one introduces a characteristic length scale into a scale-invariant problem. Having ‘broken’ the scaling symmetry in this way, the available Morawetz inequalities come back into play, despite their non-critical scaling. All subsequent work for NLS at critical regularity has built upon this fundamental idea.

In the energy-critical case, the critical regularity associated to the available Morawetz estimates is lower than the critical regularity of the problem. Bourgain was able to make use of the Lin–Strauss Morawetz inequality (appearing first in [38]), which scales like $\dot{H}^{1/2}$ and is well-suited to the radial case; to remove the radial assumption, Colliander–Keel–Staffilani–Takaoka–Tao introduced the interaction Morawetz inequality (see [14]), which has the scaling of $\dot{H}^{1/4}$ (but still requires control over at least half of a derivative, cf. (1.3) below). These considerations lead us to believe that the techniques developed to handle the energy-critical problem should be applicable to resolve Conjecture 1.2 in the case $s_c \geq \frac{1}{2}$.

In particular, we will be making use of the concentration-compactness (or ‘minimal counterexample’) approach to induction on energy. Minimal counterexamples were introduced over the course of several papers in the context of the mass-critical problem (see, for example, [1, 2, 7, 28, 29, 39]); however, the first application of minimal counterexamples to establish a global well-posedness result was carried out by Kenig–Merle [25], who developed the technique in the focusing, energy-critical setting.

Minimal counterexamples were also used to establish Conjecture 1.2 in the mass-critical setting, first for spherically-symmetric data in dimensions $d \geq 2$ (see [30, 37, 52]), and subsequently for arbitrary data in all dimensions by Dodson [17, 18, 19]. (For results in the focusing case, see [20, 30, 37].) Notice that in the mass-critical case, the critical regularity of the problem is lower than that of the available Morawetz estimates; thus one needs to prove additional regularity (instead of decay) to access these estimates. We pause here to point out [38, 57], as well, which revisit the defocusing energy-critical problem from the perspective of minimal counterexamples.
The first case of Conjecture 1.2 at non-conserved critical regularity to be addressed was the case \( d = 3 \) and \( s_c = \frac{1}{2} \), in which case the nonlinearity is cubic. Kenig–Merle \([26]\) were able to handle this case by using their concentration-compactness technique (as in \([25]\)), together with the Lin–Strauss Morawetz inequality (which is scaling-critical in this case). As we will see, this case also falls into the range of cases that we consider, although we will opt to use the interaction Morawetz inequality instead.

Some cases of Conjecture 1.2 in the energy-supercritical regime (i.e. \( s_c > 1 \)) have also been handled by Killip–Vidan \([33]\), also through the use of minimal counterexamples. In particular, they deal with the case of a cubic nonlinearity for \( d \geq 5 \), along with some other cases for which \( s_c > 1 \) and \( d \geq 5 \). Their restriction to high dimensions stems ultimately from their use of the so-called ‘double Duhamel trick’; for more details, see \([33]\) and the references cited therein.

Before we discuss our contribution, we note that the analogous conjecture has also been studied for the nonlinear wave equation. For progress in the energy-supercritical case, one can refer to works of Kenig–Merle \([27]\), Killip–Vidan \([34, 35]\), and Bulut \([4, 5, 6]\). For some results in the energy-subcritical case with radial data, see \([42, 43]\).

Finally, we are in a position to describe the cases of Conjecture 1.2 that we will address in this paper. As mentioned above, we will work in the inter-critical regime, \( 0 < s_c < 1 \) (that is, \( \frac{4}{d} < p < \frac{4}{d-2} \)). Our primary restriction is technical; namely, we only consider cases for which \( p \geq 1 \). This restriction serves to simplify the analysis, which still becomes a bit complicated. For example, when we need to estimate a fractional number of derivatives of the function \( G(z) = \vert z \vert^p \), things are quite a bit simpler when \( G \) is locally Lipschitz, rather than merely Hölder continuous.

Next, in order to make use of the interaction Morawetz inequality, we restrict to the cases \( d \geq 3 \) and \( s_c \geq \frac{1}{2} \) (cf. (1.3) below). For this restriction to be compatible with \( p \geq 1 \), we must then restrict to \( d \leq 5 \). The use of the interaction Morawetz inequality ultimately leads to a further (more severe) restriction: when \( d = 3 \), we must exclude the cases \( \frac{3}{4} < s_c < 1 \). Let us briefly describe the reason for this additional restriction.

The standard interaction Morawetz inequality may be written as follows: for \( u \) solving (1.1), we have

\[
- \int_I \int_{\mathbb{R}^d \times \mathbb{R}^d} \vert u(t,x) \vert^2 \Delta (\frac{1}{\vert x \vert})(x-y) \vert u(t,y) \vert^2 \, dx \, dy \, dt \\
\lesssim \left\| u \right\|^2_{\mathcal{L}_t^{\infty} L_\mathcal{L}^2(I \times \mathbb{R}^d)} \left\| \nabla \right\|_{\mathcal{L}_t^{\infty} L_\mathcal{L}^2(I \times \mathbb{R}^d)}^{1/2} \left\| u \right\|^2_{\mathcal{L}_t^p L_\mathcal{L}^2(I \times \mathbb{R}^d)}. \tag{1.3}
\]

As we are in the case \( s_c \geq \frac{1}{2} \), we see that to guarantee that the right-hand side is finite, we must truncate the solution \( u \) to high frequencies (that is, we must work with \( u_{\geq N} \) for some \( N > 0 \)). In Section 7 we do exactly this. However, \( u_{\geq N} \) is no longer a solution to (1.1); thus, the truncation results in error terms for (1.3) that must be handled to arrive at a useful estimate. When \( d = 3 \) and \( s_c > \frac{1}{2} \), we find that there is one error term that we cannot handle unless we also impose a spatial truncation on the weight we use to derive (1.3); see, for example, \([15, 36]\), which address the case \( d = 3, s_c = 1 \). This spatial truncation, however, results in even more error terms; it turns out that some of these additional error terms then require control over the \( \dot{H}^1_x \)-regularity of the solution. Thus, in the energy-critical
As we proceed, we will keep track of which restrictions are necessary for which results.

Our main result is the following:

**Theorem 1.3.** Suppose \((d, s_c)\) satisfies (1.4). Let \(u : I \times \mathbb{R}^d \to \mathbb{C}\) be a maximal-lifespan solution to (1.1) such that \(u \in L_t^\infty H_x^{s_c}(I \times \mathbb{R}^d)\). Then \(u\) is global and scatters, with

\[
S_R(u) \leq C(\|u\|_{L_t^\infty H_x^{s_c}(\mathbb{R} \times \mathbb{R}^d)})
\]

for some function \(C : [0, \infty) \to [0, \infty)\).

To establish Theorem 1.3 we will model our approach after several sources, including 31, 32, 33, 36, 57. In particular, Section 3 follows the presentation in 31, 33; Section 4 draws heavily from 32, and the presentation of the remaining sections is inspired largely by 36, 57. We note also that we rely on 31 for several standard results regarding almost periodic solutions in the outline below.

The first step towards a global-in-time theory for (1.1) is to develop a good local-in-time theory for this equation. In particular, building off arguments of Cazenave–Weissler 31, one can prove the following

**Theorem 1.4** (Local well-posedness). Let \((d, s_c)\) satisfy (1.5). Then, given \(u_0 \in H_x^{s_c}(\mathbb{R}^d)\), there exists a unique maximal-lifespan solution \(u : I \times \mathbb{R}^d \to \mathbb{C}\) to (1.1). Moreover, this solution satisfies the following:

- (Local existence) \(I\) is an open neighborhood of 0.
- (Blowup criterion) If \(\sup I\) is finite, then \(u\) blows up forward in time, in the sense that \(S_{[0, \sup I]}(u) = \infty\). If \(\inf I\) is finite, then \(u\) blows up backwards in time, in the sense that \(S_{[\inf I, 0]}(u) = \infty\).
- (Scattering) If \(\sup I = +\infty\) and \(u\) does not blow up forward in time, then \(u\) scatters forward in time; that is, there exists a unique \(u_+ \in H_x^{s_c}(\mathbb{R}^d)\) such that
  \[
  \lim_{t \to +\infty} \|u(t) - e^{it\Delta}u_+\|_{H_x^{s_c}(\mathbb{R}^d)} = 0.
  \]

Conversely, for any \(u_+ \in H_x^{s_c}(\mathbb{R}^d)\), there is a unique solution \(u\) to (1.1) so that (1.6) holds. The analogous statements hold backward in time.
- (Small-data global existence) There exists \(\eta_0 = \eta_0(d, p)\) such that if
  \[
  \|u_0\|_{H_x^{s_c}(\mathbb{R}^d)}^2 < \eta_0,
  \]
then \( u \) is global and scatters, with \( S_\mathbb{R}(u) \lesssim \|u_0\|^p_{\dot{H}^{s_c}_{x_t}(\mathbb{R}^d)} \).

**Remark 1.5.** We note here that the notion of blowup described above corresponds exactly to the impossibility of extending the solution to a larger time interval in the class described in Definition 1.1.

In Section 3, we will establish this theorem as a corollary of a local well-posedness result of Cazenave–Weissler [8] (Theorem 3.1) and a stability result (Theorem 3.4). This stability result plays an essential role in the argument we present, specifically in the proof of Theorem 1.12.

### 1.1. Outline of the proof of Theorem 1.3

The proof is by contradiction. We first recall that Theorem 1.3 holds if we restrict to sufficiently small initial data (cf. Theorem 1.4); thus, the failure of Theorem 1.3 would imply the existence of a ‘threshold’ size, below which Theorem 1.3 holds, but above which we can find (almost) counterexamples. Using a limiting argument, we then find blowup solutions at this threshold, so-called ‘minimal blowup solutions’. By carefully analyzing such solutions, we can show that they must have so many nice properties that in fact, they cannot exist at all.

The main property of these special counterexamples is that of almost periodicity modulo symmetries:

**Definition 1.6** (Almost periodic solutions). Let \( s_c > 0 \). A solution \( u \) to (1.1) with lifespan \( I \) is said to be almost periodic (modulo symmetries) if \( u \in L^\infty_t \dot{H}^{s_c}_{x_t}(I \times \mathbb{R}^d) \) and there exist functions \( N : I \to \mathbb{R}^+ \), \( x : I \to \mathbb{R}^d \), and \( C : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for all \( t \in I \) and all \( \eta > 0 \),

\[
\int_{|x-x(t)| \geq \frac{C(t)}{N(t)}} |\nabla|^s u(t, x)|^2 \, dx + \int_{|s| \geq C(t)} |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta.
\]

We call \( N \) the frequency scale function, \( x \) the spatial center function, and \( C \) the compactness modulus function.

**Remark 1.7.** By the Arzelà–Ascoli theorem, a family of functions is precompact in \( \dot{H}^{s_c}_{x_t}(\mathbb{R}^d) \) if and only if it is norm-bounded and there exists a compactness modulus function \( C \) such that

\[
\int_{|x| \geq C(\eta)} |\nabla|^s f(x)|^2 \, dx + \int_{|s| \geq C(\eta)} |\xi|^2 |\hat{f}(\xi)|^2 \, d\xi \leq \eta
\]

uniformly for all functions \( f \) in the family. Thus, an equivalent formulation of Definition 1.6 is the following: \( u \) is almost periodic (modulo symmetries) if and only if

\[
\{u(t) : t \in I\} \subset \{\lambda \hat{\tilde{f}}(\lambda(x + x_0)) : \lambda \in (0, \infty), \ x_0 \in \mathbb{R}^d, \ f \in K\}
\]

for some compact \( K \subset \dot{H}^{s_c}_{x_t}(\mathbb{R}^d) \).

Furthermore, Sobolev embedding gives that every compact set in \( \dot{H}^{s_c}_{x_t}(\mathbb{R}^d) \) is also compact in \( L^p_{x_t}(\mathbb{R}^d) \); thus, for any almost periodic solution \( u : I \times \mathbb{R}^d \to \mathbb{C} \), we also have

\[
\int_{|x-x(t)| \geq \frac{C(t)}{N(t)}} |u(t, x)| \frac{d^2}{dx^2} \, dx \leq \eta
\]

for all \( t \in I \) and \( \eta > 0 \).
Remark 1.8. Another consequence of almost periodicity modulo symmetries is the existence of a function $c : \mathbb{R}^+ \to \mathbb{R}^+$ so that for all $t \in I$ and all $\eta > 0$,
\[
\int_{|x-\hat{x}(t)| \leq \frac{\eta}{\eta(t)}} |\nabla|^s c u(t, x)|^2 \, dx + \int_{|\xi| \leq c(\eta) N(t)} |\xi|^2 c |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta.
\]

One can show (see [31] Lemma 5.18, for example) that the modulation parameter of almost periodic solutions obey the following local constancy property:

Lemma 1.9 (Local constancy). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan almost periodic solution to (1.1). Then there exists $\delta = \delta(u) > 0$ such that if $t_0 \in I$, then
\[
[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I,
\]
with
\[
N(t) \sim_u N(t_0) \quad \text{for} \quad |t - t_0| \leq \delta N(t_0)^{-2}.
\]

Given a maximal-lifespan almost periodic solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1), Lemma 1.9 allows us to subdivide $I$ into characteristic subintervals $J_k$ on which $N(t)$ is constant and equal to some $N_k$, with $|J_k| \sim_u N_k^{-2}$. To do this, we need to modify the compactness modulus function by a time-independent multiplicative factor.

The local constancy property also has the following consequence (see [31] Corollary 5.19):

Corollary 1.10 (N(t) at blowup). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan almost periodic solution to (1.1). If $T$ is any finite endpoint of $I$, then $N(t) \gtrsim_u |T - t|^{-1/2}$; in particular, $\lim_{t \to T} N(t) = \infty$. If $I$ is infinite or semi-infinite, then for any $t_0 \in I$, we have $N(t) \gtrsim (t - t_0)^{-1/2}$.

Finally, we need the following result, which relates the frequency scale function of an almost periodic solution to its Strichartz norms.

Lemma 1.11 (Spacetime bounds). Let $(d, s_c)$ satisfy (1.5), and suppose $u$ is an almost periodic solution to (1.1) on a time interval $I$. Then
\[
\int_I N(t)^2 \, dt \lesssim_u \left\| \nabla |^{s_c} u \right\|^2_{L^2_t L^\infty_x(I \times \mathbb{R}^d)} \lesssim_u 1 + \int_I N(t)^2 \, dt.
\]

One can prove this result by adapting the proof of [31] Lemma 5.21]; the key is to notice that $\int_I N(t)^2 \, dt$ is approximately the number of characteristic subintervals inside $I$. The restriction on $(d, s_c)$ is not actually necessary, but it covers our cases of interest.

With these preliminaries established, we can now describe the first major step in the proof of Theorem 1.13.

Theorem 1.12 (Reduction to almost periodic solutions). Suppose that Theorem 1.3 fails for $(d, s_c)$ satisfying (1.5). Then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) such that $u \in L^\infty_t \dot{H}^{s_c}_x(I \times \mathbb{R}^d)$, $u$ is almost periodic modulo symmetries, and $u$ blows up both forward and backward in time. Moreover, $u$ has minimal $L^\infty_t \dot{H}^{s_c}_x$-norm among all blowup solutions; i.e.,
\[
\sup_{t \in I} ||u(t)||_{\dot{H}^{s_c}_x(\mathbb{R}^d)} \leq \sup_{t \in J} ||v(t)||_{\dot{H}^{s_c}_x(\mathbb{R}^d)}
\]
for all maximal-lifespan solutions $v : J \times \mathbb{R}^d \to \mathbb{C}$ that blow up in at least one time direction.
We sketch a proof of Theorem 1.12 in Section 4. By now, the reduction to almost periodic solutions is a fairly standard technique in the study of dispersive equations at critical regularity. Keraani [29] first proved the existence of minimal blowup solutions (in the mass-critical setting), while Kenig–Merle [25] were the first to use them as a tool to prove global well-posedness (in the energy-critical setting). For many more examples of these techniques, one can refer to [26, 27, 30, 32, 31, 33, 34, 35, 37, 53, 52], for example.

Still, while the underlying ideas are well-established, we will see that to carry out the reduction in the cases we consider will require some new ideas and careful analysis; indeed, the resolution of this problem is the chief novelty of this paper. One of the principal difficulties arises in the proof of Lemma 4.2, in which we establish a decoupling of nonlinear profiles in order to show that a sequence of approximate solutions to (1.1) converges in some sense to a true solution.

For the mass- and energy-critical cases, one can use pointwise estimates and well-known arguments of [28] to establish this decoupling; in our setting, the nonlocal nature of $|\nabla|^s$ prevents the direct use of any pointwise estimates. The authors of [33], who dealt with some cases in the energy-supercritical regime, overcame this difficulty by establishing analogous pointwise estimates for a square function of Strichartz that shares estimates with $|\nabla|^s$ (see [46]). With such estimates in hand, the usual arguments can then be pushed through. Their approach does not work in our setting, however, as it strongly relies on the fact that $s_c > 1$. In [26], the authors treat a cubic nonlinearity in dimension $d = 3$ (in which case $s_c = \frac{1}{2}$); by exploiting the polynomial nature of the nonlinearity and employing a paraproduct estimate, they too overcome the nonlocal nature of $|\nabla|^s$ and put themselves in a position where the standard arguments are applicable. In our setting, the combination of fractional derivatives and non-polynomial nonlinearities presents a new technical challenge. Ultimately, the resolution of this problem comes from a careful reworking of the proof of the fractional chain rule, in which the Littlewood–Paley square function allows us to work at the level of individual frequencies. By making use of some tools from harmonic analysis (including maximal and vector maximal inequalities), we are eventually able to adapt the standard arguments to establish the decoupling in our setting. For further discussion, see Section 4.

After establishing Theorem 1.12, we make some further refinements to the class of solutions we consider. First, we can use a rescaling argument to restrict our attention to almost periodic solutions that do not escape to arbitrarily low frequencies on at least half of their maximal lifespan, say $[0, T_{\text{max}})$. We will not include the details here; one can instead refer to Section 4 in any of [30, 32, 52]. Next, following [17], we will divide these solutions into two classes that depend on the control given by the interaction Morawetz inequality; these classes will correspond to the ‘rapid frequency-cascade’ and ‘quasi-soliton’ scenarios. Finally, as described above, we use Lemma 1.9 to subdivide $[0, T_{\text{max}})$ into characteristic subintervals $J_k$ and set $N(t)$ to be constant and equal to $N_k$ on each $J_k$, with $|J_k| \sim u N_k^{-2}$. In this way, we arrive at

**Theorem 1.13 (Two special scenarios for blowup).** Suppose that Theorem 1.3 fails for $(d, s_c)$ satisfying (1.15). Then there exists an almost periodic solution $u : [0, T_{\text{max}}) \times \mathbb{R}^d \to \mathbb{C}$ that blows up forward in time, with

$$N(t) \equiv N_k \geq 1$$
for each $t \in J_k$, where $[0,T_{\text{max}}) = \bigcup_k J_k$. Furthermore,

$$
\text{either } \int_0^{T_{\text{max}}} N(t)^{3-4s_c} \, dt < \infty \quad \text{or} \quad \int_0^{T_{\text{max}}} N(t)^{3-4s_c} \, dt = \infty.
$$

Thus, to establish Theorem 1.3 it remains to preclude the existence of the almost periodic solutions described in Theorem 1.13. The main technical tool we will use to achieve this will be a long-time Strichartz inequality, Proposition 5.1. Such inequalities were originally developed by Dodson [17] for almost periodic solutions in the mass-critical setting; for variants in the energy-critical setting, see [36, 57]. In this paper, we establish a long-time Strichartz estimate for the first time in the inter-critical regime. The proof of Proposition 5.1 is by induction; the recurrence relation is derived with the aid of Strichartz estimates, together with a paraproduct estimate (Lemma 2.6) and a bilinear Strichartz inequality (Lemma 2.10).

In Section 6, we preclude the rapid frequency-cascade scenario, that is, almost periodic solutions as in Theorem 1.13 for which $\int_0^{T_{\text{max}}} N(t)^{3-4s_c} \, dt < \infty$. This proof requires two ingredients. The first ingredient is the long-time Strichartz inequality (Proposition 5.1), while the second is the following

**Proposition 1.14** (No-waste Duhamel formula). Let $u : [0,T_{\text{max}}) \times \mathbb{R}^d \to \mathbb{C}$ be an almost periodic solution to (1.1) with $N(t) \equiv N_k \geq 1$ on each characteristic subinterval $J_k$. Then for all $t \in [0,T_{\text{max}})$, we have

$$
u(t) = i \lim_{T \to T_{\text{max}}} \int_t^T e^{i(t-s)\Delta}(|u|^p u)(s) \, ds,$$

where the limits are taken in the weak $\dot{H}^s_{x}$ topology.

To prove Proposition 1.14 one can adapt the proof of [31, Proposition 5.23]; we omit the details. Using Proposition 1.14 and Strichartz estimates, we can upgrade the information given by Proposition 5.1 to see that a rapid frequency-cascade solution must have finite mass. In fact, we can show that the solution has zero mass, which contradicts that the solution blows up.

In Section 8 we preclude the quasi-soliton scenario, that is, almost periodic solutions as in Theorem 1.13 for which $\int_0^{T_{\text{max}}} N(t)^{3-4s_c} \, dt = \infty$. The main ingredient is a frequency-localized interaction Morawetz inequality (Proposition 7.1), which we prove in Section 7. To establish this estimate, we begin with the usual interaction Morawetz inequality, truncate to high frequencies, and use Proposition 5.1 to control the resulting error terms. (As described above, one of these error terms eventually forces us to exclude the cases $(d,s_c) \in \{3\} \times (\frac{3}{4},1)$ from Theorem 1.3; see Remark 7.4.) To rule out the quasi-soliton scenario and thereby complete the proof of Theorem 1.3 we notice that the frequency-localized interaction Morawetz inequality provides uniform control over $\int_I N(t)^{3-4s_c} \, dt$ for all compact time intervals $I \subset [0,T_{\text{max}})$; thus we can derive a contradiction by taking $I$ to be sufficiently large inside of $[0,T_{\text{max}})$.

**Acknowledgements.** I owe many thanks to my advisors, Rowan Killip and Monica Vişan, for all of their guidance and support. I am very grateful to them, not only for bringing this problem to my attention, but also for engaging in many helpful discussions, and for a careful reading of the manuscript. This work was supported in part by NSF grant DMS-1001531 (P.I. Rowan Killip).
2. Notation and useful lemmas

2.1. Some notation. We write $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some $C > 0$. If $X \lesssim Y \lesssim X$, we write $X \sim Y$. If the implicit constant $C$ depends on the dimension $d$ or the power $p$, this dependence will be suppressed; dependence on additional parameters will be indicated with subscripts. For subscripts, $X \lesssim_u Y$ indicates that $X \leq CY$ for some $C = C(u)$.

For a spacetime slab $I \times {\mathbb R}^d$, we write $L^q_t L^r_x(I \times {\mathbb R}^d)$ to denote the Banach space of functions $u: I \times {\mathbb R}^d \to {\mathbb C}$ equipped with the norm

$$
\|u\|_{L^q_t L^r_x(I \times {\mathbb R}^d)} := \left(\int_I \|u(t)\|_{L^r_x({\mathbb R}^d)}^q dt\right)^{1/q},
$$

with the usual conventions when $q$ or $r$ is infinity. If $q = r$, we abbreviate $L^q_t L^q_x = L^q_{x,t}$. At times we will also abbreviate $\|f\|_{L^q_x}$ to $\|f\|_{L^q_x}$ for some $C = C(u)$.

We define the Fourier transform on $\mathbb{R}^d$ by

$$
\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx.
$$

For $s \in \mathbb{R}$, we can then define the fractional differentiation operator $|\nabla|^s$ via

$$
|\nabla|^s f(\xi) := |\xi|^s \hat{f}(\xi),
$$

which in turn defines the homogeneous Sobolev norm

$$
\|f\|_{H^s_x(\mathbb{R}^d)} := \| |\nabla|^s f \|_{L^2_x(\mathbb{R}^d)}.
$$

2.2. Basic harmonic analysis. Let $\varphi$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{1}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For $N \in 2\mathbb{Z}$, we define the Littlewood–Paley projection operators via

$$
\overline{P_{\leq N}} f(\xi) := \hat{f}_{\leq N}(\xi) := \varphi(\xi/N) \hat{f}(\xi),
$$

$$
\overline{P_{> N}} f(\xi) := \hat{f}_{> N}(\xi) := (1 - \varphi(\xi/N)) \hat{f}(\xi),
$$

$$
\overline{P_N} f(\xi) := \hat{f}_N(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \hat{f}(\xi).
$$

We define $P_{< N}$ and $P_{\geq N}$ similarly. We also define

$$
P_{M < \leq N} := P_{\leq N} - P_{\leq M} = \sum_{M < N' \leq N} P_{N'},
$$

for $M < N$. All such summations are understood to be over $N \in 2\mathbb{Z}$. Being Fourier multiplier operators, these Littlewood–Paley projection operators commute with $e^{it\Delta}$, as well as differential operators (for example, $i\partial_t + \Delta$). We will need the following standard estimates for these operators:

**Lemma 2.1** (Bernstein estimates). For $1 \leq r \leq q \leq \infty$ and $s \geq 0$,

$$
\| |\nabla|^s P_N f \|_{L^q_x(\mathbb{R}^d)} \lesssim N^s \| P_N f \|_{L^r_x(\mathbb{R}^d)};
$$

$$
\| |\nabla|^s P_{\leq N} f \|_{L^q_x(\mathbb{R}^d)} \lesssim N^s \| P_{\leq N} f \|_{L^r_x(\mathbb{R}^d)};
$$

$$
\| P_{> N} f \|_{L^q_x(\mathbb{R}^d)} \lesssim N^{-s} \| |\nabla|^s P_{> N} f \|_{L^r_x(\mathbb{R}^d)};
$$

$$
\| P_{N} f \|_{L^q_x(\mathbb{R}^d)} \lesssim N^{s-\frac{d}{q}} \| P_{\leq N} f \|_{L^r_x(\mathbb{R}^d)};
$$

$$
\| P_{\leq N} f \|_{L^q_x(\mathbb{R}^d)} \lesssim N^{s-\frac{d}{r}} \| P_{\leq N} f \|_{L^r_x(\mathbb{R}^d)}.
$$
Lemma 2.2 (Littlewood–Paley square function estimates). For $1 < r < \infty$,
\[
\left\| \left( \sum |P_n f(x)|^2 \right)^{1/2} \right\|_{L^r_x} \sim \|f\|_{L^r_x}.
\]
\[
\left\| \left( \sum N^{2s}|f(x)|^2 \right)^{1/2} \right\|_{L^r_x} \sim \|\nabla^s f\|_{L^r_x} \quad \text{for } s \in \mathbb{R},
\]
\[
\left\| \left( \sum N^{2s}|f_N(x)|^2 \right)^{1/2} \right\|_{L^r_x} \sim \|\nabla^s f\|_{L^r_x} \quad \text{for } s > 0.
\]

We will also need the following general inequalities, which appear originally in [10]. For a textbook treatment, one can refer to [48].

Lemma 2.3 (Fractional product rule, [10]). Let $s \in (0, 1]$ and $1 < r, r_1, r_2, q_1, q_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then
\[
\left\| \nabla^s (f g) \right\|_r \lesssim \|f\|_{r_1} \left\| \nabla^s g \right\|_{q_1} + \left\| \nabla^s f \right\|_{r_2} \|g\|_{q_2}.
\]

Lemma 2.4 (Fractional chain rule, [10]). Suppose $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, and $1 < r, r_1, r_2 < \infty$ are such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then
\[
\left\| \nabla^s G(u) \right\|_r \lesssim \|G'(u)\|_{r_1} \|\nabla^s u\|_{r_2}.
\]

We will also make use of the following refinement of the fractional chain rule, which appears in [35]:

Lemma 2.5 (Derivatives of differences, [35]). Fix $p > 1$ and $0 < s < 1$. Then for $1 < r, r_1, r_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, we have
\[
\left\| \nabla^s (|u + v|^p - |u|^p) \right\|_r \lesssim \left\| \nabla^s u \right\|_{r_1} \|v\|_{r_2}^{p-1} + \left\| \nabla^s v \right\|_{r_1} \|u + v\|_{r_2}^{p-1}.
\]

Finally, we prove a paraproduct estimate, very much in the spirit of Lemma 2.3 in [57].

Lemma 2.6 (Paraproduct estimate). Fix $d \in \{3, 4, 5\}$.
(a) For $p > 0$ such that $s_c := \frac{d}{2} - \frac{2}{p} \in (\frac{1}{2}, 1)$, we have
\[
\left\| \nabla^{-\frac{s_c}{2}} (f g) \right\|_{L^p_x} \lesssim \left\| \nabla^{-\frac{s_c}{2}} f \right\|_{L^p_x} \left\| \nabla^{-\frac{s_c}{2}} g \right\|_{L^{2p/(d+8)}_x}.
\]
(b) We also have
\[
\left\| \nabla^{-\frac{s_c}{2}} (f g) \right\|_{L^{2d}_x} \lesssim \left\| \nabla^{-\frac{s_c}{2}} f \right\|_{L^{2d}_x} \left\| \nabla^{-\frac{s_c}{2}} g \right\|_{L^{2d}_x}.
\]

Proof. For (a), we will prove the equivalent estimate
\[
\left\| \nabla^{-\frac{s_c}{2}} \left( |\nabla|^{\frac{s_c}{2}} f |\nabla|^{-\frac{s_c}{2}} g \right) \right\|_{L^{2d}_x} \lesssim \left\| f \right\|_{L^{2d}_x} \left\| g \right\|_{L^{2d}_x}
\]
by decomposing the left-hand side into low-high and high-low frequency interactions. More precisely, we introduce the projections $\pi_{l,h}$ and $\pi_{h,l}$, defined for any pair of functions $\phi, \psi$ by
\[
\pi_{l,h}(\phi, \psi) := \sum_{N \leq M} \phi_N \psi_M \quad \text{and} \quad \pi_{h,l}(\phi, \psi) := \sum_{N \gg M} \phi_N \psi_M.
\]

We first consider the low-high interactions: by Sobolev embedding, we have
\[
\left\| \nabla^{-\frac{s_c}{2}} \pi_{l,h}(|\nabla|^{\frac{s_c}{2}} f, |\nabla|^{-\frac{s_c}{2}} g) \right\|_{L^{2d}_x} \lesssim \left\| \pi_{l,h}(|\nabla|^{\frac{s_c}{2}} f, |\nabla|^{-\frac{s_c}{2}} g) \right\|_{L^{2d}_x}.
\] (2.1)
We note here that when \( d = 3 \), the assumption \( \sigma_c < 1 \) guarantees that
\[
\frac{4dp}{p(3d+4)-d} > 1.
\]
If we now consider the multiplier of the operator given by
\[
T(f, g) := \pi_{t,h}(\langle \nabla \rangle^{\frac{1}{2}} f, \langle \nabla \rangle^{\frac{1}{2}} g),
\]
that is,
\[
\sum_{N \leq M} |\xi_1|^{\frac{1}{2}} \hat{f}_N(\xi_1) \xi_2|^{\frac{1}{2}} g_M(\xi_2),
\]
then we see that this multiplier is a symbol of order zero with \( \xi = (\xi_1, \xi_2) \). Thus, continuing from (2.1), we can cite a theorem of Coifman–Meyer (see [11, 13], for example) to conclude
\[
\| |\nabla|^{-\frac{1}{2}} \pi_{t,h}(\langle \nabla \rangle^{\frac{1}{2}} f, \langle \nabla \rangle^{\frac{1}{2}} g)\|_{L^p_x \rightarrow L^q_x} \lesssim \| f \|_{L^2_t L^{\frac{2q}{q-2}}_x} \| g \|_{L^{\frac{4p}{4p-2d}}_x}.
\]

We now consider the high-low interactions: if we consider the multiplier of the operator given by
\[
S(f, h) := |\nabla|^{-\frac{1}{2}} \pi_{h,t}(\langle \nabla \rangle^{\frac{1}{2}} f, h),
\]
that is,
\[
\sum_{N \gg M} |\xi_1 + \xi_2|^{-\frac{1}{2}} \xi_1^{\frac{1}{2}} \hat{f}_N(\xi_1) \hat{h}_M(\xi_2),
\]
then we see that this multiplier is also a symbol of order zero. Thus, using the result cited above, along with Sobolev embedding, we can estimate
\[
\| |\nabla|^{-\frac{1}{2}} \pi_{h,t}(\langle \nabla \rangle^{\frac{1}{2}} f, h)\|_{L^p_x \rightarrow L^q_x} \lesssim \| f \|_{L^2_t L^{\frac{2q}{q-2}}_x} \| |\nabla|^{-\frac{1}{2}} g\|_{L^{\frac{4p}{4p-2d}}_x}.
\]

Combining the low-high and high-low interactions, we recover (a). Mutis mutandis, the exact same proof gives (b). \( \square \)

2.3. Strichartz estimates. Let \( e^{it\Delta} \) be the free Schrödinger propagator,
\[
[e^{it\Delta} f](x) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) \, dy
\]
for \( t \neq 0 \). This explicit formula immediately implies the dispersive estimate
\[
\|e^{it\Delta} f\|_{L^\infty_t L^2_x(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \| f \|_{L^1_t L^\infty_x(\mathbb{R}^d)}
\]
for \( t \neq 0 \). Interpolating with \( \|e^{it\Delta} f\|_{L^r_t L^s_x(\mathbb{R}^d)} = \| f \|_{L^r_t L^s_x(\mathbb{R}^d)} \) (cf. Plancherel), one arrives at
\[
\|e^{it\Delta} f\|_{L^r_t L^s_x(\mathbb{R}^d)} \lesssim |t|^{-\left(\frac{d}{r} - \frac{d}{s}\right)} \| f \|_{L^r_t L^s_x(\mathbb{R}^d)}
\]
for \( t \neq 0 \) and \( 2 \leq r \leq \infty \), with \( \frac{1}{r} + \frac{d}{s} = 1 \). This estimate can be used to prove the standard Strichartz estimates, which we state below. First, we need the following

**Definition 2.7 (Admissible pairs).** For \( d \geq 3 \), we call a pair of exponents \((q, r)\) Schrödinger admissible if
\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \text{and} \quad 2 \leq q, r \leq \infty.
\]
For a spacetime slab \( I \times \mathbb{R}^d \), we define
\[
\|u\|_{S^0(I)} := \sup \left\{ \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)} : (q, r) \text{ admissible} \right\}.
\]
We define $S^0(I)$ to be the closure of the test functions under this norm, and denote the dual of $S^0(I)$ by $N^0(I)$. We note
\[ \|u\|_{N^0(I)} \lesssim \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)} \quad \text{for any admissible pair } (q, r). \]

We now state the Strichartz estimates in the form we will need them.

**Lemma 2.8 (Strichartz).** Let $s \geq 0$, let $I$ be a compact time interval, and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to the forced Schrödinger equation
\[ (i\partial_t + \Delta)u = F. \]
Then
\[ \|\nabla|^s u\|_{S^0(I)} \lesssim \|\nabla|^s u(t_0)\|_{L^2_x} + \|\nabla|^s F\|_{N^0(I)} \]
for any $t_0 \in I$.

**Proof.** As mentioned, the key ingredient is (2.2). For the endpoint $(q, r) = (2, \frac{2d}{d-2})$ in $d \geq 3$, see [24]. For the non-endpoint cases, see [22] [17], for example.

The free propagator also obeys some local smoothing estimates (see [16] [14] [54] for the original results). We will make use of the following, which appears as Proposition 4.14 in [31]:

**Lemma 2.9 (Local smoothing).** For any $f \in L^2_x(\mathbb{R}^d)$ and any $\varepsilon > 0$, \[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\nabla|^\varepsilon f(x)^2 \|_{L^2}^2 \|dx dt \lesssim \|f\|_{L^2}^2. \]

Next, we record the following bilinear Strichartz estimates. The version we need can be deduced from (the proof of) Corollary 4.19 in [31].

**Lemma 2.10 (Bilinear Strichartz).** Let $0 < s_c < \frac{d-1}{2}$. For any spacetime slab $I \times \mathbb{R}^d$ and any frequencies $M > 0$ and $N > 0$, we have
\[ \|u_{\leq M} v_{\geq N}\|_{L^2_{t,x}(I \times \mathbb{R}^d)} \lesssim M^{\frac{d}{d+1} - s_c} N^{-\frac{1}{2} + s_c} \|\nabla|^s u_{\leq M}\|_{S^0(I)} \|\nabla|^s v_{\geq N}\|_{S^0(I)}, \]
where
\[ \|u\|_{S^0(I)} := \|u\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)} + \|(i\partial_t + \Delta)u\|_{L^{2\frac{d+2}{d-4}}(I \times \mathbb{R}^d)}^{\frac{d+2}{d-4}}. \]

**Remark 2.11.** We will use Lemma 2.10 in the proof of Proposition 3.1. In that context, we will have $u = v$ an almost periodic solution to (1.1) and $I = J_k$, a characteristic subinterval. In this case, interpolating between $u \in L^\infty_t H^s_x$ and Lemma 1.11 gives
\[ \|\nabla|^s u\|_{S^0(J_k)} \lesssim_a 1, \]
so that we can use the fractional chain rule and Sobolev embedding to estimate
\[ \|\nabla|^s (|u|^p u)\|_{L^{\frac{2(d+2)}{d-4}}(J_k \times \mathbb{R}^d)} \lesssim\|u\|_{L^{\frac{2(d+2)}{d-4}}(J_k \times \mathbb{R}^d)}^{\frac{2(d+2)}{d-4}} \|\nabla|^s u\|_{L^{2\frac{d+2}{d-4}}(J_k \times \mathbb{R}^d)}^{\frac{2(d+2)}{d-4}} \lesssim \|\nabla|^s u\|_{S^0(J_k)}^{p+1} \lesssim_a 1. \]
Thus, in this setting, an application of Lemma 2.10 gives
\[ \|u_{\leq M} u_{\geq N}\|_{L^2_{t,x}(J_k \times \mathbb{R}^d)} \lesssim M^{\frac{d-1}{d+1} - s_c} N^{-\frac{1}{2} + s_c}. \]
2.4. Concentration-compactness. We record here the linear profile decomposition that we will use to prove the reduction in Theorem 1.12. We begin with the following

Definition 2.12 (Symmetry group). For any position $x_0 \in \mathbb{R}^d$ and parameter $\lambda > 0$, we define a unitary transformation $g_{x_0, \lambda} : H^s_x(\mathbb{R}^d) \to H^s_x(\mathbb{R}^d)$ by

$$[g_{x_0, \lambda} f](x) := \lambda^{-\frac{d}{2}} f(\lambda^{-1}(x - x_0))$$

(recall $s_c := \frac{d}{2} - \frac{n}{p}$). We let $G$ denote the collection of such transformations. For a function $u : I \times \mathbb{R}^d \to \mathbb{C}$, we define $T_{g_{x_0, \lambda}} u : \lambda^2 I \times \mathbb{R}^d \to \mathbb{C}$ by the formula

$$[T_{g_{x_0, \lambda}} u](t, x) := \lambda^{-\frac{d}{2}} u(\lambda^{-2} t, \lambda^{-1}(x - x_0)),$$

where $\lambda^2 I := \{\lambda^2 t : t \in I\}$. Note that if $u$ is a solution to (1.1), then $T_g u$ is a solution to (1.1) with initial data $gu_0$.

Remark 2.13. It is easily seen that $G$ is a group under composition. The map $u \mapsto T_g u$ maps solutions to (1.1) to solutions with the same scattering size, that is, $S(T_g u) = S(u)$. Furthermore, $u$ is a maximal-lifespan solution if and only if $T_g u$ is a maximal-lifespan solution.

We now state the linear profile decomposition in the form that we need. For $s_c = 0$, this result was originally proven in [1, 7, 39], while for $s_c = 1$ it was established in [28]. In the generality we need, a proof can be found in [41].

Lemma 2.14 (Linear profile decomposition, [41]). Fix $0 < s_c < 1$ and let $\{u_n\}_{n \geq 1}$ be a bounded sequence in $H^s_x(\mathbb{R}^d)$. Then, after passing to a subsequence if necessary, there exist functions $\{\phi_j\}_{j \geq 1} \subset H^s_x(\mathbb{R}^d)$, group elements $g^j_n \in G$, and times $t^j_n \in \mathbb{R}$ such that for all $J \geq 1$, we have the decomposition

$$u_n = \sum_{j=1}^J g^j_n e^{it^j_n \Delta} \phi_j + w^j_n$$

with the following properties:

- For all $n$ and all $J \geq 1$, we have $w^j_n \in H^s_x(\mathbb{R}^d)$, with

$$\lim_{J \to \infty} \limsup_{n \to \infty} \left\| e^{it^j_n \Delta} w^j_n \right\|_{L^2_x(\mathbb{R}^{d+2})} = 0. \quad (2.3)$$

- For any $j \neq k$, we have the following asymptotic orthogonality of parameters:

$$\frac{\lambda^j_n}{\lambda^k_n} + \frac{\lambda^k_n}{\lambda^j_n} + \frac{x^j_n - x^k_n}{2} + \frac{|f^j_n(\lambda^j_n)^2 - f^k_n(\lambda^k_n)^2|}{\lambda^j_n \lambda^k_n} \to \infty \quad \text{as } n \to \infty. \quad (2.4)$$

- For any $J \geq 1$, we have the decoupling properties:

$$\lim_{n \to \infty} \left[ \left\| \nabla \phi_u u_n \right\|^2 - \sum_{j=1}^J \left\| \nabla \phi_j \right\|^2 - \left\| \nabla w^j_n \right\|^2 \right] = 0, \quad \text{for any } 1 \leq j \leq J,$$

$$e^{-it_n^j \Delta} (g^j_n)^{-1} w^j_n \to 0 \quad \text{weakly in } H^s_x \quad \text{as } n \to \infty. \quad (2.6)$$

Remark 2.15. In this linear profile decomposition, we may always choose the scaling parameters $\lambda^j_n$ so that they belong to $2^\mathbb{Z}$. 


3. LOCAL WELL-POSEDNESS

In this section, we develop a local theory for \((1.1)\). We begin by recording a standard local well-posedness result proven by Cazenave–Weissler \([8]\); see also \([9, 31, 50]\). We will also need to establish a stability result (appearing as Theorem 3.4), which will be essential in the reduction to almost periodic solutions in Section 4. For stability results in the mass- and energy-critical settings, see \([15, 40, 51, 53]\).

For the following local well-posedness result, one must assume that the initial data belongs to the inhomogeneous Sobolev space \(H^s_{\text{loc}}(\mathbb{R}^d)\). This assumption serves to simplify the proof (allowing for a contraction mapping argument in mass-critical spaces); we can remove it a posteriori by using the stability result we prove below.

**Theorem 3.1** (Standard local well-posedness \([8]\)). Let \(d \geq 1\), \(0 < s_c < 1\), and \(u_0 \in H^{s_c}_{\text{loc}}(\mathbb{R}^d)\). Then there exists \(\eta_0 > 0\) so that if \(0 < \eta \leq \eta_0\) and \(I\) is an interval containing zero such that
\[
\||\nabla|^{s_c} e^{it\Delta} u_0\|_{L_t^{p+2} L_x^{\frac{2(d+p+2)}{d+2+2p}} (I \times \mathbb{R}^d)} \leq \eta,
\]
then there exists a unique solution \(u\) to \((1.1)\) that obeys the following bounds:
\[
\||\nabla|^{s_c} u\|_{L_t^{p+2} L_x^{\frac{2d}{d+2+2p}} (I \times \mathbb{R}^d)} \leq 2\eta,
\]
\[
\||\nabla|^{s_c} u\|_{L^p(I)} \lesssim \||\nabla|^{s_c} u_0\|_{L^2(\mathbb{R}^d)} + \eta^{p+1},
\]
\[
\|u\|_{L^p(I)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}.
\]

**Remark 3.2.** By Strichartz, we have
\[
\||\nabla|^{s_c} e^{it\Delta} u_0\|_{L_t^{p+2} L_x^{\frac{2d}{d+2+2p}} (I \times \mathbb{R}^d)} \lesssim \||\nabla|^{s_c} u_0\|_{L^p(\mathbb{R}^d)},
\]
so that \((3.1)\) holds with \(I = \mathbb{R}\) for sufficiently small initial data. One can also guarantee that \((3.1)\) holds by taking \(|I|\) sufficiently small (cf. monotone convergence).

We now turn to the question of stability for \((1.1)\). We will prove a stability result for \((d, s_c)\) satisfying \((1.5)\), in which case we always have \(p \geq 1\). As we will see, this assumption allows for a very simple stability theory. On the other hand, when \(p < 1\), developing a stability theory can become quite delicate. For a discussion in the energy-critical case, see \([31\text{ Section 3.4}]\) and the references cited therein. See also \([33]\) for a stability theory in the energy-supercritical regime, as well as \([21]\) for a stability theory in the inter-critical regime in high dimensions.

Following the arguments in \([31]\), we begin with the following

**Lemma 3.3** (Short-time perturbations). Fix \((d, s_c)\) satisfying \((1.5)\). Let \(I\) be a compact interval and \(\tilde{u} : I \times \mathbb{R}^d \to \mathbb{C}\) a solution to
\[
(i \partial_t + \Delta)\tilde{u} = |\tilde{u}|^p \tilde{u} + e
\]
for some function \(e\). Assume that
\[
\|\tilde{u}\|_{L_t^{\infty} \dot{H}_{\text{loc}}^{s_c} (I \times \mathbb{R}^d)} \leq E.
\]
Let \( t_0 \in I \) and \( u_0 \in \dot{H}^s_x(\mathbb{R}^d) \). Then there exist \( \varepsilon_0, \delta > 0 \) (depending on \( E \)) such that for all \( 0 < \varepsilon < \varepsilon_0 \), if

\[
\left\| \nabla |^s \tilde{u} \right\|_{L^p(\mathbb{R}^d) \cap L^{\infty} \times \mathbb{R}^d)} \leq \delta, \quad (3.3)
\]

\[
\| u_0 - \tilde{u}(t_0) \|_{\dot{H}^s_x(\mathbb{R}^d)} \leq \varepsilon, \quad (3.4)
\]

\[
\left\| \nabla |^s e \right\|_{N^0(I)} \leq \varepsilon, \quad (3.5)
\]

then there exists \( u : I \times \mathbb{R}^d \rightarrow \mathbb{C} \) solving \((i\partial_t + \Delta)u = |u|^p u \) with \( u(t_0) = u_0 \) satisfying

\[
\left\| \nabla |^s(u - \tilde{u}) \right\|_{S^0(I)} \lesssim \varepsilon, \quad (3.6)
\]

\[
\left\| \nabla |^s u \right\|_{S^0(I)} \lesssim E, \quad (3.7)
\]

\[
\left\| \nabla |^s(|u|^pu - |\tilde{u}|^p \tilde{u}) \right\|_{N^0(I)} \lesssim \varepsilon. \quad (3.8)
\]

Proof. We prove the lemma under the additional hypothesis \( u_0 \in L^2_x(\mathbb{R}^d) \); this allows us (by Theorem 3.1) to find a solution \( u \), so that we are left to prove all of the estimates as a priori estimates. Once the lemma is proven, we can use approximation by \( H^s_x(\mathbb{R}^d) \) functions (along with the lemma itself) to see that the lemma holds for \( u_0 \in \dot{H}^s_x(\mathbb{R}^d) \).

Define \( w = u - \tilde{u} \), so that \((i\partial_t + \Delta)w = |u|^p u - |\tilde{u}|^p \tilde{u} - \varepsilon \). Without loss of generality, assume \( t_0 = \inf I \), and define

\[
A(t) = \left\| \nabla |^s(|u|^pu - |\tilde{u}|^p \tilde{u}) \right\|_{N^0([t_0, t])}.
\]

We first note that by Duhamel, Strichartz, \( (3.4) \), and \( (3.5) \), we get

\[
\left\| \nabla |^s w \right\|_{S^0([t_0, t])}
\lesssim \left\| \nabla |^s w(t_0) \right\|_{L^2(\mathbb{R}^d)} + \left\| \left| \nabla |^s u \right|_{N^0([t_0, t])} + \left\| \nabla |^s e \right\|_{N^0(I)}
\lesssim \varepsilon + A(t). \quad (3.9)
\]

Using this fact, together with Lemma \( 2.25 \) \( (3.3) \), and Sobolev embedding, we can estimate (with all spacetime norms over \([t_0, t] \times \mathbb{R}^d \))

\[
A(t) \lesssim \left\| \nabla |^s(|\tilde{u} + w|^p(\tilde{u} + w) - |\tilde{u}|^p \tilde{u}) \right\|_{L^{p\epsilon} \times L^{p\epsilon + 2d\epsilon + 2d} \times L^{p\epsilon + 2d\epsilon + 2d}} \text{,}
\]

\[
\lesssim \left\| \nabla |^s \tilde{u} \right\|_{L^{p\epsilon} \times L^{p\epsilon + 2d\epsilon + 2d}} \left\| \tilde{u} \right\|_{L^{p\epsilon} \times L^{p\epsilon + 2d\epsilon + 2d}} \text{,}
\]

\[
+ \left\| \nabla |^s w \right\|_{L^{p\epsilon} \times L^{p\epsilon + 2d\epsilon + 2d}} \left\| \tilde{u} + w \right\|_{L^{p\epsilon} \times L^{p\epsilon + 2d\epsilon + 2d}} \text{,}
\]

\[
\lesssim \delta \varepsilon + A(t) + \varepsilon + A(t) \| \delta \varepsilon + \varepsilon + A(t) \|.
\]

Thus, recalling \( p \geq 1 \) and choosing \( \delta, \varepsilon \) sufficiently small, we conclude \( A(t) \lesssim \varepsilon \) for all \( t \in I \), which gives \( (3.8) \). Combining \( (3.8) \) with \( (3.9) \), we also get \( (3.6) \). Finally, we can prove \( (3.7) \) as follows: by Strichartz, \( (3.6) \), \( (3.2) \), \( (3.5) \), \( (3.3) \), the fractional
Theorem 3.4 (Stability). Fix $(d, s_c)$ satisfying (1.3). Let $I$ be a compact time interval and $\tilde{u} : I \times \mathbb{R}^d \to \mathbb{C}$ a solution to

$$(i\partial_t + \Delta)\tilde{u} = |\tilde{u}|^p\tilde{u} + e$$

for some function $e$. Assume that

$$\|\tilde{u}\|_{L^p_t H^{s_c}(I \times \mathbb{R}^d)} \leq E,$$  \hspace{1cm} (3.10)

$$S_I(\tilde{u}) \leq L.$$  \hspace{1cm} (3.11)

Let $t_0 \in I$ and $u_0 \in \dot{H}^{s_c}_x(\mathbb{R}^d)$. Then there exists $\varepsilon_1 = \varepsilon_1(E, L)$ such that if

$$\|u_0 - \tilde{u}(t_0)\|_{\dot{H}^{s_c}_x(\mathbb{R}^d)} \leq \varepsilon,$$  \hspace{1cm} (3.12)

$$\|\nabla|^{s_c}e\|_{N^0(I)} \leq \varepsilon$$  \hspace{1cm} (3.13)

for some $0 < \varepsilon < \varepsilon_1$, then there exists a solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to $(i\partial_t + \Delta)u = |u|^p u$ with $u(t_0) = u_0$ satisfying

$$\|\nabla|^{s_c}(u - \tilde{u})\|_{S^0(I)} \leq C(E, L)\varepsilon,$$  \hspace{1cm} (3.14)

$$\|\nabla|^{s_c}u\|_{S^0(I)} \leq C(E, L).$$  \hspace{1cm} (3.15)

Proof. Once again, we may assume $t_0 = \inf I$. To begin, we let $\eta > 0$ be a small parameter to be determined shortly. By (3.11), we may subdivide $I$ into (finitely many, depending on $\eta$ and $L$) intervals $J_k = [t_k, t_{k+1})$ so that

$$\|\tilde{u}\|_{L_{t,x}^{\frac{d+2}{d+2}}(J_k \times \mathbb{R}^d)} \sim \eta$$

for each $k$. Then by Strichartz, (3.10), (3.13), and the fractional chain rule, we have

$$\|\nabla|^{s_c}\tilde{u}\|_{S^0(J_k)} \lesssim \|\nabla|^{s_c}\tilde{u}(t_k)\|_{L^2_x(\mathbb{R}^d)} + \|\nabla|^{s_c}(|\tilde{u}|^p\tilde{u})\|_{N^0(J_k)} + \|\nabla|^{s_c}e\|_{N^0(J_k)}$$

$$\lesssim E + \|\nabla|^{s_c}\tilde{u}\|_{S^0(I)} \|\tilde{u}\|_{L_{t,x}^{\frac{d+2}{d+2}}(J_k \times \mathbb{R}^d)}^p + \varepsilon$$

$$\lesssim E + \varepsilon + \eta^p\|\nabla|^{s_c}\tilde{u}\|_{S^0(I)}.$$  \hspace{1cm} (3.16)

Thus for $\varepsilon \leq E$ and $\eta$ sufficiently small, we find

$$\|\nabla|^{s_c}\tilde{u}\|_{S^0(J_k)} \lesssim E.$$  \hspace{1cm} (3.17)

Adding these bounds, we find

$$\|\nabla|^{s_c}\tilde{u}\|_{S^0(I)} \leq C(E, L).$$  \hspace{1cm} (3.18)
Now, we take \( \delta > 0 \) as in Lemma 3.3 and subdivide \( I \) into finitely many, say \( J_0 = J_0(C(E, L), \delta) \) intervals \( I_j = [t_j, t_{j+1}) \) so that
\[
\| |\nabla| \psi u_j \|_{L_t^\infty \cdot L_x^{p(d+2)} \left( \frac{\delta p^{d+2}}{2 \delta p} \right)} \leq \delta
\]
for each \( j \). We now wish to proceed inductively. We may apply Lemma 3.3 on each \( I_j \), provided we can guarantee
\[
\| u(t_j) - \tilde{u}(t_j) \|_{\dot{H}_x^s(\mathbb{R}^d)} \leq \varepsilon
\]  
(3.17)
for some \( 0 < \varepsilon < \varepsilon_0 \) and each \( j \) (where \( \varepsilon_0 \) as is in Lemma 3.3). In the event that (3.17) holds for some \( j \), applying Lemma 3.3 on \( I_j = [t_j, t_{j+1}) \) gives
\[
\| |\nabla| \psi (u - \tilde{u}) \|_{S^0(I_j)} \leq C(j)\varepsilon, \quad (3.18)
\]
\[
\| |\nabla| \psi u \|_{S^0(I_j)} \leq C(j)\varepsilon, \quad (3.19)
\]
Now, we first note that (3.17) holds for \( j = 0 \), provided we take \( \varepsilon_1 < \varepsilon_0 \). Next, assuming that (3.17) holds for \( 0 \leq k \leq j - 1 \), we can use Strichartz, (3.12), (3.13), and the inductive hypothesis (3.20) to estimate
\[
\| u(t_j) - \tilde{u}(t_j) \|_{\dot{H}_x^s(\mathbb{R}^d)} \leq \| u(t_0) - \tilde{u}(t_0) \|_{\dot{H}_x^s(\mathbb{R}^d)} + \| |\nabla| \psi (|u|^p u - |\tilde{u}|^p \tilde{u}) \|_{N^0([t_0, t_j])} + \| |\nabla| \psi e \|_{N^0([t_0, t_j])}
\]
\[
\leq \varepsilon + \sum_{k=0}^{j-1} C(k)\varepsilon + \varepsilon
\]
\[
< \varepsilon_0,
\]
provided \( \varepsilon_1 = \varepsilon_1(\varepsilon_0, J_0) \) is taken sufficiently small. Thus, by induction, we get (3.18) and (3.19) on each \( I_j \). Adding these bounds over the \( I_j \) yields (3.14) and (3.15).

**Remark 3.5.** As mentioned above, with this stability result in hand, we can see that Theorem 3.1 holds without the assumption \( u_0 \in L^2_x(\mathbb{R}^d) \). Using this updated version of Theorem 3.1 (along with the original proof of Theorem 3.1), one can then derive Theorem 1.12. We omit the standard arguments; one can refer instead to [32].

4. REDUCTION TO ALMOST PERIODIC SOLUTIONS

The goal of this section is to sketch a proof of Theorem 1.12. We will follow the argument presented in [32] Section 3. By now, the general procedure is fairly standard; see, for example, [25, 26, 27, 31, 53] for other instances in different contexts. Thus, we will merely outline the main steps of the argument, providing full details only when significant new difficulties arise in our setting.

We suppose that Theorem 1.3 fails for some \((d, s, c)\) satisfying (1.5). We then define the function \( L : [0, \infty) \to [0, \infty] \) by
\[
L(E) := \sup \{ S_I(u) : u : I \times \mathbb{R}^d \to \mathbb{C} \text{ solving (1.1)} \text{ with } \sup_{t \in I} \| u(t) \|_{\dot{H}_x^s(\mathbb{R}^d)}^2 \leq E \},
\]
where \( S_I(u) \) is defined as in (1.2). We note that \( L \) is a non-decreasing function, and that Theorem 1.4 implies
\[
L(E) \lesssim E^{\frac{p(d+2)}{d}} \quad \text{for} \quad E < \eta_0,
\]
where \( \eta_0 \) is the small-data threshold. Thus, there exists a unique ‘critical’ threshold \( E_c \in (0, \infty) \) such that \( L(E) < \infty \) for \( E < E_c \) and \( L(E) = \infty \) for \( E > E_c \). The failure of Theorem 1.3 implies that \( 0 < E_c < \infty \).

The key ingredient to proving Theorem 1.12 is a Palais–Smale condition modulo the symmetries of the equation; indeed, once the following proposition is proven, deriving Theorem 1.12 is standard (see [32]).

**Proposition 4.1** (Palais–Smale condition modulo symmetries). Let \((d, s_c)\) satisfy (1.3). Let \( u_n : I_n \times \mathbb{R}^d \to \mathbb{C} \) be a sequence of solutions to (1.1) such that
\[
\limsup_{n \to \infty} \sup_{t \in I_n} \| u_n(t) \|_{H^s(\mathbb{R}^d)}^2 = E_c,
\]
and suppose \( t_n \in I_n \) are such that
\[
\lim_{n \to \infty} S_{[t_n, \sup t_n]}(u_n) = \lim_{n \to \infty} S_{[\inf t_n, t_n]}(u_n) = \infty.
\]
Then the sequence \( u_n(t_n) \) has a subsequence that converges in \( H^s(\mathbb{R}^d) \) modulo symmetries; that is, there exist \( g_n \in G \) such that \( g_n[u_n(t_n)] \) converges along a subsequence in \( H^s(\mathbb{R}^d) \), where \( G \) is as in Definition 2.14.

We now sketch the proof of this proposition, following the argument as it appears in [32]. As in that setting, the main ingredients will be a linear profile decom- position (Lemma 2.14 in our case) and a stability result (Theorem 3.4 in our case). However, as we will see, combining fractional derivatives with non-polynomial nonlinearities will present some significant new difficulties in our setting when it comes time to apply the stability result.

We begin by translating so that each \( t_n = 0 \), and apply the linear profile decomposition Lemma 2.14 to write
\[
u_n(0) = \sum_{j=1}^J g_n^j e^{it_n^j \Delta} \phi^j + w_n^J
\]
along some subsequence. Refining the subsequence for each \( j \) and diagonalizing, we may assume that for each \( j \), we have \( t_n^j \to t^j \in [-\infty, \infty] \). If \( t^j \in (-\infty, \infty) \), then we replace \( \phi^j \) by \( e^{it^j \Delta} \phi^j \), so that we may take \( t^j = 0 \). Moreover, we can absorb the error \( e^{it_n^j \Delta} \phi^j - \phi^j \) into the error term \( w_n^J \), and so we may take \( t_n^j \equiv 0 \). Thus, without loss of generality, either \( t_n^j \equiv 0 \) or \( t_n^j \to \pm \infty \).

Next, appealing to Theorem 1.4 for each \( j \) we define \( \nu^j : I^j \times \mathbb{R}^d \to \mathbb{C} \) to be the maximal-lifespan solution to (1.1) such that
\[
\begin{cases}
\nu^j(0) = \phi^j & \text{if } t_n^j \equiv 0, \\
\nu^j \text{ scatters to } \phi^j \text{ as } t \to \pm \infty & \text{if } t_n^j \to \pm \infty.
\end{cases}
\]
We now define the nonlinear profiles \( v_n^j : I_n^j \times \mathbb{R}^d \to \mathbb{C} \) by
\[
v_n^j(t) = g_n^j \nu^j((\lambda_n^j)^{-2} t + t_n^j),
\]
where \( I_n^j = \{ t : (\lambda_n^j)^{-2} t + t_n^j \in I^j \} \). Now, the \( H^s \) decoupling of the profiles \( \phi^j \), (2.6), immediately tells us that the \( v_n^j \) are global and scatter for \( j \) sufficiently large,
say \( j \geq J_0 \); indeed, for large enough \( j \), we are in the small-data regime. We want to show that there exists some \( 1 \leq j_0 < J_0 \) such that

\[
\limsup_{n \to \infty} S_{[0, \sup t_j^n)}(v_{j_0}^n) = \infty. \tag{4.4}
\]

Once we obtain at least one such ‘bad’ nonlinear profile, we can show that in fact, there can only be one profile. To see this, one needs to adapt the argument in [32, Lemma 3.3] to see that the \( \dot{H}_c^{s-c} \)-decoupling of the profiles persists in time (this does not follow immediately, as the \( \dot{H}_c^{s-c} \)-norm is not a conserved quantity for (1.1)). Then, the ‘critical’ nature of \( E_c \) can be used to rule out the possibility of multiple profiles.

Comparing with (4.3), one sees that once we show that there is only one profile \( \phi_{j_0} \), the proof of Proposition 4.1 is nearly complete; one needs only to rule out the cases \( t_j^n \to \pm \infty \). This can be easily done by applying the stability theory; we omit the details and instead refer the reader to [32].

We turn now to proving that there is at least one bad profile. We suppose towards a contradiction that there are no bad nonlinear profiles. In this case, we can show

\[
\sum_{j \geq 1} S_{[0, \infty)}(v_j^n) \lesssim E_c. \tag{4.5}
\]

for \( n \) sufficiently large (to control the tail of the sum, for example, we recall that for \( j \geq J_0 \), we are in the small-data regime; thus we can use (2.5) and (4.1) to bound the tail by \( E_c \) for \( n \) sufficiently large). We would like to use (4.5) and the stability result (Theorem 3.4) to deduce a bound on the scattering size of the \( u_n \), thus contradicting (4.2).

To this end, we define the approximations

\[
u_j^n(t) = \sum_{j=1}^J v_j^n(t) + e^{it\Delta} w_j^n.
\]

By the construction of the \( v_j^n \), it is easy to see that

\[
\limsup_{n \to \infty} \| u_n(0) - u_j^n(0) \|_{\dot{H}_c^{s-c}(\mathbb{R}^d)} = 0. \tag{4.6}
\]

We also claim that we have

\[
\lim_{J \to \infty} \limsup_{n \to \infty} S_{[0, \infty)}(u_j^n) \lesssim E_c. \tag{4.7}
\]

To see why (4.7) holds, first note that by (4.5) and (4.6), it suffices to show

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left| S_{[0, \infty)} \left( \sum_{j=1}^J v_j^n \right) - \sum_{j=1}^J S_{[0, \infty)}(v_j^n) \right| = 0. \tag{4.8}
\]

To establish (4.8), we can first use the pointwise inequality

\[
\left| \sum_{j=1}^J v_j^n \right|_{L_t^{\frac{p(d+2)}{p-2}}(\mathbb{R}^d)} \lesssim \sum_{j \neq k} \left| v_j^n \right|_{L_t^{\frac{p(d+2)}{p-2}}(\mathbb{R}^d)} \leq J \sum_{j \neq k} \left| v_j^n \right|_{L_t^{\frac{p(d+2)}{p-2}}(\mathbb{R}^d)}.
\]

along with H"older's inequality to see

\[
\text{LHS (4.8)} \lesssim J \sum_{j \neq k} \left\| v_j^n \right\|_{L_t^{\frac{p(d+2)}{p-2}}(\mathbb{R}^d)} \left\| v_k^n \right\|_{L_t^{\frac{p(d+2)}{p-2}}(\mathbb{R}^d)} = 0. \tag{4.9}
\]
Then, following an argument of Keraani (cf. [28, Lemma 2.7]), given \( j \neq k \), we can approximate \( v^j \) and \( v^k \) by compactly supported functions in \( \mathbb{R} \times \mathbb{R}^d \) and use the asymptotic orthogonality of parameters \( \frac{(4.4)}{d} \) to show

\[
\lim_{n \to \infty} \lim_{N \to \infty} \| \nabla |\cdot|^s c \left( (i\partial_t + \Delta)u^j_n - |u^j_n|^p u^j_n \right) \|_{\dot{H}^s L^2(\mathbb{R}^d)} = 0.
\]

Thus, continuing from (4.10), we get that (4.18) (and therefore (4.7)) holds.

With (4.16) and (4.17) in place, we see that if we can show that the \( u^j_n \) asymptotically solve (1.1), that is,

\[
\lim_{n \to \infty} \lim_{N \to \infty} \| \nabla |\cdot|^s c \left( (i\partial_t + \Delta)u^j_n - |u^j_n|^p u^j_n \right) \|_{L^2(\mathbb{R}^d)} = 0,
\]

then we will be able to apply Theorem 3.3 to deduce bounds on the scattering size of the \( u_n \). Writing \( F(z) = |z|^p z \), the proof of Proposition 4.1 therefore reduces to showing the following

**Lemma 4.2** (Decoupling of nonlinear profiles).

\[
\lim_{j \to \infty} \lim_{n \to \infty} \left\| \nabla |\cdot|^s c \left( F \left( \sum_{j=1}^J v^j_n \right) - \sum_{j=1}^J F(v^j_n) \right) \right\|_{\dot{H}^{s/2}(\mathbb{R}^d)} = 0,
\]

(4.11)

\[
\lim_{j \to \infty} \lim_{n \to \infty} \left\| \nabla |\cdot|^s c \left( F(u^j_n e^{-it\Delta} u^j_n) - F(u^j_n) \right) \right\|_{\dot{H}^{s/2}(\mathbb{R}^d)} = 0.
\]

(4.12)

While many of the ideas needed to establish this lemma may be found in [28], we will see that new difficulties appear in our setting. Consider, for example, (4.11). In the mass-critical setting, i.e. \( s_c = 0 \), one has the pointwise estimate

\[
\left| F \left( \sum_{j=1}^J v^j_n \right) - \sum_{j=1}^J F(v^j_n) \right| \lesssim J \sum_{j \neq k} |v^j_n| |v^k_n|^p.
\]

(4.13)

To see that the contribution of the terms on the the right-hand side of (4.13) is acceptable, one can follow the argument of Keraani just described above: that is, one can use the asymptotic orthogonality of parameters to derive an estimate like (4.10), which in turn gives (4.11).

In the energy-critical setting, i.e. \( s_c = 1 \), one can instead use the pointwise estimate

\[
\left| \nabla \left( F \left( \sum_{j=1}^J v^j_n \right) - \sum_{j=1}^J F(v^j_n) \right) \right| \lesssim J \sum_{j \neq k} |\nabla v^j_n| |v^k_n|^p.
\]

A similar argument can then be used to prove (4.11); the key in both cases is to exhibit terms that all contain some \( v^j_n \) paired against some \( v^k_n \) for \( j \neq k \).

In the energy-supercritical case, the authors of [33] were able to establish analogous pointwise estimates (in terms of the Hardy–Littlewood maximal function) for a square function of Strichartz that shares estimates with fractional differentiation operators (see [40]). With the appropriate pointwise estimates in place, the usual arguments can then be applied; in this way, a potentially complicated analysis is handled quite efficiently. The approach of [33], however, does not work in our setting, as it relies fundamentally on the fact that \( s_c > 1 \).

See also [26], which deals with the case \( d = 3 \) and \( s_c = \frac{1}{3} \) (in which case \( p = 2 \)). In that setting, one also has to face the nonlocal nature of \( |\nabla|^\frac{1}{2} \); however, by using the polynomial nature of the nonlinearity, along with the well-developed theory of
paraproducts (see [12, 18]), the authors are able to place themselves back into a situation where the usual arguments apply. In this way, they are able to overcome the difficulty of fractional derivatives while still providing a very clean analysis.

In our case, we must deal simultaneously with a non-polynomial nonlinearity and a fractional number of derivatives; as we will see, this necessitates a fairly delicate and technical analysis. The main difficulty of our task stems from the fact the nonlocal operator $|\nabla|^{sc}$ does not respect pointwise estimates in the spirit of (4.13). We will deal with this problem by opening up the proof of the fractional chain rule (Lemma 2.4) as given in [48, §2.4]; in particular, we will employ the Littlewood–Paley square function (specifically, Lemma 2.2), which allows us to work at the level of individual frequencies. By making use of maximal function and vector maximal function estimates, we can then find a way to adapt the standard arguments.

Proof of (4.11). By induction, it will suffice to treat the case of two summands; to simplify notation, we write $f = v_n^j$ and $g = v_n^k$ for some $j \neq k$, and we are left to show

$$\left\| |\nabla|^{sc} \left( |f + g|^p(f + g) - |f|^p f - |g|^p g \right) \right\|_{L^p_{t,x}(0,\infty)} \to 0$$

as $n \to \infty$.

As alluded to above, the key will be to perform a decomposition in such a way that all of the resulting terms we need to estimate have $f$ paired against $g$ inside of a single integrand; for such terms, we will be able to use the asymptotic orthogonality of parameters (2.4) to our advantage.

We first rewrite

$$|f + g|^p(f + g) - |f|^p f - |g|^p g = (|f + g|^p - |f|^p) f + (|f + g|^p - |g|^p) g.$$  

By symmetry, it will suffice to treat the first term. We turn therefore to estimating

$$\left\| |\nabla|^{sc} ((|f + g|^p - |f|^p)f) \right\|_{L^\infty_{t,x}}.$$  

By Lemma 2.2 it will suffice to consider

$$\left\| \left( \sum |N^{sc} P_N \left( (|f + g|^p - |f|^p)f \right) \right) \right\|_{L^\infty_{t,x}}.$$  

Thus, we restrict our attention to a single frequency $N \in 2^\mathbb{Z}$. We let $\delta_y f(x) := f(x - y) - f(x)$, and let $\hat{\psi}$ denote the convolution kernel of the Littlewood–Paley projection $P_1$. As $\psi(0) = 0$, we have

$$\int \hat{\psi}(y) dy = 0,$$

so that exploiting cancellation, we can write

$$P_N \left( (|f(x) + g(x)|^p - |f(x)|^p) f(x) \right) = \int N^d \hat{\psi}(Ny) \delta_y \left( (|f(x) + g(x)|^p - |f(x)|^p) f(x) \right) dy.$$  

(4.16)
We now rewrite
\[
\delta_y \left( \left| f(x) + g(x) \right|^p - \left| f(x) \right|^p f(x) \right) = \delta_y f(x) \left| f(x-y) + g(x-y) \right|^p - \left| f(x-y) \right|^p f(x) + f(x) \left[ f(x) + g(x-y) \right]^p - \left| f(x) + g(x) \right|^p f(x) + f(x-y)^p - f(x-y)^p f(x-y) \right]^p. \tag{4.19}
\]

We estimate each term individually. First, we have
\[
\left| f(x) \right|^p \lesssim \left| \delta_y f(x) \right| \left| g(x-y) \right| \left\{ \left| f(x-y) \right|^{p-1} + \left| g(x-y) \right|^{p-1} \right\}. \tag{4.17}
\]

Next, we see
\[
\left| f(x) \right|^p \lesssim \left| \delta_y f(x) \right| \left| g(x-y) \right| \left\{ \left| f(x) \right|^{p-1} + \left| g(x) \right|^{p-1} + \left| g(x-y) \right|^{p-1} \right\}. \tag{4.18}
\]

We now turn to (4.19). First, if \( 1 < p \leq 2 \), a simple argument using the fundamental theorem of calculus implies
\[
\left| f(x) \right|^p \lesssim \left| \delta_y f(x) \right| \left| g(x-y) \right| \left\{ \left| f(x) \right|^{p-1} + \left| g(x) \right|^{p-1} + \left| g(x-y) \right|^{p-1} \right\}. \tag{4.19}
\]

(see Lemma A.2 for details). For \( p > 2 \), one instead finds
\[
\left| f(x) \right|^p \lesssim \left| \delta_y f(x) \right| \left| g(x-y) \right| \left\{ \left| f(x) \right|^{p-2} + \left| f(x-y) \right|^{p-2} + \left| g(x-y) \right|^{p-2} \right\}. \tag{4.20}
\]

**Remark 4.3.** Let us pause here to note that if \( p = 1 \), the approach above breaks down. Notice that each term in the bounds for (4.17), (4.18), and (4.19) has two essential properties: (i) it features \( f \) paired against powers of \( g \), and (ii) the derivative (in the form of \( \delta_y \)) lands on either \( f \) or \( g \). When \( p = 1 \), the same approach does not yield a decomposition that is satisfactory in this sense; it is for this reason that we have excluded the case \( (d, s_c) = (5, \frac{1}{2}) \) from this paper.

To ease the exposition, we will restrict our attention here and below to the more difficult case \( 1 < p \leq 2 \); once we have dealt with this case, it should be clear how to proceed when \( p > 2 \).

Collecting terms, we continue from (4.16) to see
\[
\left| P_N \left( \left[ f(x) + g(x) \right]^p - \left| f(x) \right|^p f(x) \right) \right| \lesssim \int \left| T \right| \left| \delta_y f(x) \right| \left| g(x-y) \right| \left\{ \left| f(x-y) \right|^{p-1} + \left| g(x-y) \right|^{p-1} \right\} \left(4.20\right)
\]
\[
+ \int \left| T \right| \left| \delta_y g(x) \right| \left\{ \left| f(x) \right|^{p-1} + \left| g(x) \right|^{p-1} + \left| g(x-y) \right|^{p-1} \right\} \left(4.21\right)
\]
\[
+ \int \left| T \right| \left| \delta_y f(x) \right| \left| g(x-y) \right|^{p-1} \left(4.22\right).
\]

One can see that we are already faced with several terms to estimate; moreover, to estimate any single term will require further decomposition. However, in the end, the same set of tools will suffice to handle every term that appears. Thus, let us deal with only (4.20) in detail; once we have seen how to handle this term, it should be clear that the same techniques apply to handle (4.21) and (4.22).

Turning to (4.20), we first write
\[
\left(4.20\right) = \int \left| T \right| \left| \delta_y f(x) \right| \left| g(x-y) \right| \left| f(x-y) \right|^{p-1} \left(4.23\right)
\]
\[
+ \int \left| T \right| \left| \delta_y f(x) \right| \left| g(x-y) \right| \left| f \right|^{p} \left(4.24\right).
\]

For both of these terms, we will need to make use of some auxiliary inequalities in the spirit of [48, §2.3], which we record in Lemma A.2.
We turn to (4.23). If we first write
\[ |\delta_y f(x)| \lesssim |f_{>N}(x)| + |f_{>N}(x-y)| + \sum_{K \leq N} |\delta_y f_K(x)|, \tag{4.25} \]
then putting Lemma A.2 to use, we arrive at
\[ (4.23) \lesssim |f_{>N}(x)| M(g |f|^{p-1})(x) \tag{4.26} \]
\[ + M(f_{>N} g |f|^{p-1})(x) \tag{4.27} \]
\[ + \sum_{K \leq N} K M(f_K)(x) M(g |f|^{p-1})(x) \tag{4.28} \]
\[ + \sum_{K \leq N} K M(M(f_K) g |f|^{p-1})(x). \tag{4.29} \]

Similarly, we can decompose
\[ (4.24) \lesssim |f_{>N}(x)| M(|g|^p)(x) \tag{4.30} \]
\[ + M(f_{>N} |g|^p)(x) \tag{4.31} \]
\[ + \sum_{K \leq N} K M(f_K)(x) M(|g|^p)(x) \tag{4.32} \]
\[ + \sum_{K \leq N} K M(M(f_K) |g|^p)(x). \tag{4.33} \]

Let us now consider the contribution of (4.26) to the left-hand side of (4.14). Comparing with (4.15), we see it will suffice to estimate
\[ \left\| \left( \sum_N \left| N^{s_c} f_{>N} M(g |f|^{p-1}) \right|^2 \right)^{1/2} \right\|_{L_{t,x}^{2(d+2)\delta}}. \tag{4.34} \]

Using Hölder’s inequality and maximal function estimates, we can control this term by
\[ \left\| \left( \sum_N \left| N^{s_c} f_{>N} \right|^2 \right)^{1/2} \right\|_{L_{t,x}^{2(d+2)\delta}} \left\| |g| |f|^{p-1} \right\|_{L_{t,x}^{2d+4}}. \tag{4.35} \]

We now recall that \( f = v_j^u \) and \( g = v_k^u \) for some \( j \neq k \). Then, the first term is controlled by \( \left\| \nabla^{s_c} v_j^u \right\|_{S^0} \) (cf. Lemma 2.2), which in turn is bounded (recall that by assumption, all of the \( v_j^u \) have scattering size \( \lesssim E^c \)). The second term can be handled in the standard way; that is, this term vanishes in the limit due to the asymptotic orthogonality of parameters (2.4) (cf. [28, Lemma 2.7]). Thus, we see that (4.26) is under control. A similar approach (this time using the vector maximal inequality) handles (4.27).

To estimate the contribution of (4.28) to the left-hand side of (4.14), we need to estimate
\[ \left\| \left( \sum_N \left| N^{s_c} \sum_{K \leq N} K M(f_K) M(g |f|^{p-1}) \right|^2 \right)^{1/2} \right\|_{N^0([0,\infty))}. \tag{4.34} \]

For this term, we need to make use of the following basic inequality: for a nonnegative sequence \( \{a_K\}_{K \in \mathbb{Z}^2} \) and \( 0 < s < 1 \), one has
\[ \sum_{N \in \mathbb{Z}^2} N^{2s}|\sum_{K \leq N} K a_K|^2 \lesssim \sum_{K \in \mathbb{Z}^2} K^{2s}|a_K|^2 \tag{4.35} \]
(cf. [18] Lemma 4.2). Using this inequality, along with Hölder, we can estimate

\[
\begin{align*}
\mathbb{E} \preceq & \left\| \left( \sum_{K} |K^{\varphi} M(f_{K})|^{2} \right)^{1/2} M(g |f|^{p-1}) \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \\
\preceq & \left\| \left( \sum_{K} |K^{\varphi} M(f_{K})|^{2} \right)^{1/2} \right\|_{L_{t,x}^{2(d+2)}} \left\| g |f|^{p-1} \right\|_{L_{t,x}^{\frac{d+2}{d-2}}} \to 0
\end{align*}
\]

as \( n \to \infty \), exactly as before. Thus, (4.28) is under control; the same approach handles (4.29) (after an application of the vector maximal inequality).

Let us now turn to (4.30). As before, we sum over \( N \in 2\mathbb{Z} \) and find that we need to estimate

\[
\left\| \left( \sum |N^{\varphi} f_{N}^{t} |^{2} \right)^{1/2} M(|g|^{p}) \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)}. \tag{4.36}
\]

Recalling that \( f = v_{n}^{t} \) and \( g = v_{n}^{k} \) for some \( j \neq k \), we see that we are once again in a position to use the argument from [28]. To begin, we may assume without loss of generality that both

\[
\Phi_{1} := \left( \sum |N^{\varphi} P_{N} v_{N}^{t} |^{2} \right)^{1/2} \quad \text{and} \quad \Phi_{2} := M(|v_{N}^{k}|^{p})
\]

belong to \( C_{c}^{\infty} (\mathbb{R} \times \mathbb{R}^{d}) \); indeed, \( C_{c}^{\infty} \)-functions are dense in both \( L_{t,x}^{2(d+2)} \) and \( L_{t,x}^{\frac{d+2}{d-2}} \). We now wish to use the asymptotic orthogonality of parameters, that is,

\[
\frac{\lambda_{n}^{j}}{\lambda_{n}^{k}} + \frac{\lambda_{n}^{k}}{\lambda_{n}^{j}} + \frac{|x_{n}^{j} - x_{n}^{k}|^{2}}{\lambda_{n}^{j} \lambda_{n}^{k}} + \frac{|t_{n}^{j} (\lambda_{n}^{j})^{2} - t_{n}^{k} (\lambda_{n}^{k})^{2}|}{\lambda_{n}^{j} \lambda_{n}^{k}} \to \infty \quad \text{as} \quad n \to \infty, \tag{4.37}
\]

to show (4.30) \( \to 0 \).

Consider first the case \( \frac{\lambda_{n}^{j}}{\lambda_{n}^{k}} \to c > 0 \) (along a subsequence, say). If we unravel the definition of the nonlinear profiles and change variables to move the symmetries onto \( \Phi_{2} \), we arrive at

\[
\begin{align*}
\frac{2(d+2)}{2(d+2)} &= \left( \frac{\lambda_{n}^{j}}{\lambda_{n}^{k}} \right) \frac{4(d+2)}{2(d+2)} \\
&= \frac{\lambda_{n}^{j}}{\lambda_{n}^{k}} \int_{\mathbb{R}^{d}} \int \Phi_{1}(s, y) \Phi_{2}(t_{n}^{j} (\lambda_{n}^{j})^{2} s - t_{n}^{j} \lambda_{n}^{j} y + x_{n}^{j} - x_{n}^{k}) \ dy \ ds.
\end{align*}
\]

Then, recalling (4.37), we see that as \( n \to \infty \), either the spatial or temporal argument of \( \Phi_{2} \) must escape the support of \( \Phi_{1} \). Thus, in this case, we get (4.36) \( \to 0 \).

If instead we have \( \frac{\lambda_{n}^{j}}{\lambda_{n}^{k}} \to 0 \), then continuing from above, we can estimate

\[
\begin{align*}
\mathbb{E} \preceq & \left( \frac{\lambda_{n}^{j}}{\lambda_{n}^{k}} \right)^{2} \left\| \Phi_{1} \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \left\| \Phi_{2} \right\|_{L_{t,x}^{\infty}}.
\end{align*}
\]

As \( \Phi_{1}, \Phi_{2} \in C_{c}^{\infty} (\mathbb{R} \times \mathbb{R}^{d}) \), we see that (4.36) \( \to 0 \) in this case, as well.

Finally, we can treat the case \( \frac{\lambda_{n}^{j}}{\lambda_{n}^{k}} \to \infty \) just like the previous case; the only difference is that we change variables to move the symmetries onto \( \Phi_{1} \), instead of \( \Phi_{2} \). Thus, we have that (4.30) \( \to 0 \) in this third and final case.

We have now shown that (4.30) is under control. The same ideas can be used to handle (4.31), (4.32), and (4.33).

As mentioned above, this same set of ideas suffices to deal with all the remaining terms stemming from (4.11).
Proof of (1.12). For this term, we will need to make use of (2.3). As we will see, the terms in which \( e^{it\Delta} u_n^J \) appears without derivatives will be relatively easy to handle, as (2.3) will apply directly. On the other hand, the terms that only contain \( |\nabla|^s e^{it\Delta} u_n^J \) will require a more careful analysis; in particular, we will need to carry out a local smoothing argument before we can make effective use of (2.3).

Defining \( g := \sum_{j=1}^J v_n^j \) and \( h := e^{it\Delta} u_n^J \), we are left to show

\[
\lim_{j \to \infty} \limsup_{n \to \infty} \| |\nabla|^s (|g + h|^p (g + h) - |g|^p g) \|_{N^0([0, \infty))} = 0. 
\]

(4.38)

We write

\[
|g + h|^p (g + h) - |g|^p g = |g + h|^p h + (|g + h|^p - |g|^p) g 
\]

(4.39)

and first restrict our attention to (4.39). We proceed as before, working at a single frequency and exploiting cancellation to write

\[
|P_N (|g + h|^p h)(x)| = \left| \int N^d \hat{\psi}(Ny) \delta_y \left[ (|g(x) + h(x)|^p h(x) \right] dy \right|
\]

\[
\leq \int N^d |\hat{\psi}(Ny)| |g(x - y) + h(x - y)|^p |\delta_y h(x)| dy + \int N^d |\hat{\psi}(Ny)| |\delta_y (|g(x) + h(x)|^p) |h(x)| dy.
\]

(4.41)

We will deal only with (4.41), which is the more difficult term. Indeed, in all of the terms that stem from (4.42), we will have a copy of \( e^{it\Delta} u_n^J \) appearing without derivatives, so that (2.3) will suffice. (For completeness, we will later show how to handle such a term; cf. (4.54) below.)

Proceeding as in (4.25), we write

\[
\int N^d |\hat{\psi}(Ny)| |g(x - y) + h(x - y)|^p |h_N(x)| dy \leq \int N^d |\hat{\psi}(Ny)| |g(x - y) + h(x - y)|^p |\delta_y h_N(x)| dy + \sum_{K \leq N} \int N^d |\hat{\psi}(Ny)| |g(x - y) + h(x - y)|^p |\delta_y h_K(x)| dy.
\]

(4.43)

(4.44)

Let us now deal only with (4.45): in doing so, we will see all of the ideas necessary to handle (4.43) and (4.44), as well. We first write

\[
\int N^d |\hat{\psi}(Ny)| |g(x - y) + h(x - y)|^p |\delta_y h_K(x)| dy \leq \sum_{K \leq N} \int N^d |\hat{\psi}(Ny)| |g(x - y)|^p |\delta_y h_K(x)| dy.
\]

(4.46)

We only consider (4.46), as the contribution of (4.47) is easier to estimate (again, due to the presence of \( e^{it\Delta} u_n^J \) without derivatives). Employing the inequalities of Lemma (A.2) we find

\[
\int N^d |\hat{\psi}(Ny)| |g(x - y)|^p |\delta_y h_K(x)| dy \leq \sum_{K \leq N} \frac{K}{N} |M(|g|^p)(x) M(h_K)(x)| + \sum_{K \leq N} \frac{K}{N} |M(|g|^p M(h_K))(x)|.
\]

(4.47)

(4.48)

Let us now concern ourselves only with the first term above, as the second is similar. As before, to estimate the contribution of this term to (4.38) (and thereby complete our treatment of (4.39)), we need to sum over \( N \in 2\mathbb{Z} \). Using (4.35) and
recalling the definitions of $g$ and $h$, we write

$$\| \left( \sum_{N} |N^{s_c} \sum_{K \leq N} \frac{K}{N^2} M(|g|^p)M(h_K)|^2 \right)^{1/2} \|_{L^{0,t,x}} \lesssim \left\| \left( \sum_{N} |N^{s_c} M(h_N)|^2 \right)^{1/2} M(|g|^p) \right\|_{L^{2d+2,\infty,t,x}} \lesssim \left\| \left( \sum_{N} |N^{s_c} M(P Не^{it\Delta} u_n^J)|^2 \right)^{1/2} M\left( \left( \sum_{j=1}^{J} v_{n,j}^J \right)^p \right) \right\|_{L^{2d+2,\infty,t,x}}.$$  

Thus, to complete our treatment of (4.39), we are left to show

$$\lim_{J \to \infty} \lim_{n \to \infty} \left\| \left( \sum_{N} |N^{s_c} M(P Не^{it\Delta} u_n^J)|^2 \right)^{1/2} M\left( \left( \sum_{j=1}^{J} v_{n,j}^J \right)^p \right) \right\|_{L^{2d+2,\infty,t,x}} = 0. \quad (4.48)$$

To begin, we let $\eta > 0$; then using (4.5), we see that there exists some $J_1 = J_1(\eta)$ so that

$$\sum_{j \geq J_1} \left\| v_{n,j}^J \right\|_{L^{2d+2,\infty,t,x}} < \eta.$$

Using Hölder’s inequality, maximal function and vector maximal function estimates, and Lemma 2.2, we can argue as we did to obtain (4.3) to see

$$\lim_{n \to \infty} \sup_{t,x} \left\| \left( \sum_{N} |N^{s_c} M(P Не^{it\Delta} u_n^J)|^2 \right)^{1/2} M\left( \left( \sum_{j \geq J_1} v_{n,j}^J \right)^p \right) \right\|_{L^{2d+2,\infty,t,x}} \lesssim \lim_{n \to \infty} \sup_{t,x} \left\| \nabla |N^{s_c} e^{it\Delta} u_n^J| \right\|_{L^{d+2,\infty,t,x}} \sum_{j \geq J_1} \left\| v_{n,j}^J \right\|_{L^{2d+2,\infty,t,x}} \lesssim \eta.$$  

As $\eta > 0$ was arbitrary, we see that to establish (4.48), it will suffice to show

$$\lim_{J \to \infty} \lim_{n \to \infty} \left\| \left( \sum_{N} |N^{s_c} M(P Не^{it\Delta} u_n^J)|^2 \right)^{1/2} M\left( \left( \sum_{j \geq J_1} v_{n,j}^J \right)^p \right) \right\|_{L^{2d+2,\infty,t,x}} = 0 \quad (4.49)$$

for $1 \leq j < J_1$.

Restricting our attention to a single $j$ and recalling the definition of $v_{n,j}^J$, we change variables and find we need to estimate

$$\left\| \left( \sum_{N} |(\lambda_n^J)^{\frac{2}{p}} N^{s_c} MP_n \left[ e^{i(\lambda_n^J)^2(t-t_{n,j})}\Delta u_n^J(\lambda_n^J x + x_{n,j}) \right]|^2 \right)^{1/2} M\left( |v_{n,j}^J|^p \right) \right\|_{L^{2d+2,\infty,t,x}}.$$  

We will now carry out some reductions, inspired by the proof of [28, Proposition 3.4]: as $M(|v_{n,j}^J|^p)$ shares bounds with $|v_{n,j}^J|^p$, and $v_{n,j}^J$ obeys good bounds (it has scattering size $\lesssim E_c$), we may replace $M(|v_{n,j}^J|^p)$ with some function $\Phi$ in $C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$. If we then use Hölder’s inequality, we find it suffices to estimate the first term in $L^{2,\infty}_t(K)$, where $K$ is the (compact) support of this function $\Phi$. The next step will be to use a local smoothing estimate on this (fixed) set $K$. Now, the norms that will appear in these estimates will have critical scaling; that is, they will be invariant under the change of variables that eliminates the parameters $\lambda_n^J$, $x_{n,j}$, and $t_{n,j}$. Thus, without loss of generality, we will ignore them from the start.

To establish (4.49) and complete our treatment of (4.39), we are therefore left to show

$$\lim_{J \to \infty} \lim_{n \to \infty} \left\| \left( \sum_{N} |M(N^{s_c} P Не^{it\Delta} u_n^J)|^2 \right)^{1/2} \right\|_{L^{2,\infty}_t(K)} = 0 \quad (4.50)$$
for a fixed compact set $K \subset \mathbb{R} \times \mathbb{R}^d$.

To establish (4.50), we will need to rely on the fact that we are working on a compact set, so that we can carry out a local smoothing argument. Indeed, the term appearing above is morally like $|\nabla|^s e^{it\Delta} u_n^J$, over which we do not have sufficient control (cf. (2.3)). However, we do have good control over $e^{it\Delta} u_n^J$, in the form of (2.3). Thus, to succeed, we need to find a way to estimate the term above using fewer than $s_c$ derivatives; this is exactly the role of local smoothing.

For the proof of (4.50), we will use a standard local smoothing result for the free propagator (Lemma 2.9), along with a few results from Chapter V]. In particular, we need the following: if we choose $\varepsilon > 0$ so that $-\frac{d}{2} < -1 - \varepsilon$, then $|x|^{-1-\varepsilon}$ is an $A_2$ weight, so that $M$ is bounded on $L^2(|x|^{-1-\varepsilon} dx)$.

**Proof of (4.50).** We can write $K \subset [-T, T] \times \{|x| \leq R\}$ for some $T, R > 0$. We fix some $N_0 \in \mathbb{Z}^2$ and break into low and high frequencies:

$$\int_\mathbb{R} \int_\mathbb{R} \sum_{N} |N^{s_c} M(N e^{it\Delta} u_n^J)|^2 dx dt \leq \sum_{N \leq N_0} \int_K |M(N^{s_c} P_N e^{it\Delta} u_n^J)|^2 dx dt$$

$$+ \sum_{N > N_0} \int_K |M(N^{s_c} P_N e^{it\Delta} u_n^J)|^2 dx dt.$$

For the low frequencies, we use Hölder and maximal function estimates to write

$$\sum_{N \leq N_0} \int_K |M(N^{s_c} P_N e^{it\Delta} u_n^J)|^2 dx dt$$

$$\lesssim \sum_{N \leq N_0} T^{\frac{d(d+2)}{2p(d+2)} - \frac{d}{2}} R^{\frac{d(d+2)}{2p(d+2)} - \frac{d}{2}} \|M(N^{s_c} P_N e^{it\Delta} u_n^J)\|_L^{2(d+2)}_{t,x}$$

$$\lesssim_K N^{2s_c} \|e^{it\Delta} u_n^J\|_L^{2(d+2)}_{t,x}^2.$$
Optimizing in the choice of $N_0$ now yields
\[
\left\| \left( \sum_N \left| M(N^s P_N e^{it \Delta} u_n^J) \right|^2 \right)^{1/2} \right\|_{L^2_t L_x^2(K)} \lesssim_K \left\| e^{it \Delta} u_n^J \right\|_{\dot{H}^{s+} L^2_t L_x^2}^{2^{s+}},
\]
which, by (2.3), gives (4.50). \hfill \square

We have now dealt with (4.39), and so we finally turn to (4.40). As usual, we first restrict our attention to a single frequency $N$. We have dealt with a term of this form before (cf. (4.10)); proceeding in exactly the same way, we arrive at
\[
|P_N(\{|g(x) + h(x)|^p - |g(x)|^p|g(x)\}|)
\lesssim \int N^d |\tilde{\psi}(Ny)| \left| |\delta_y g(x)| \right| |h(x-y)| \left\{ \{|g(x-y)|^{p-1} + |h(x-y)|^{p-1}\} \right\} dy \quad (4.51)
\]
\[
+ \int N^d |\tilde{\psi}(Ny)| \{|g(x)| \left| |\delta_y g(x)| \right| |h(x-y)|^{p-1} \} dy \quad (4.52)
\]
\[
+ \int N^d |\tilde{\psi}(Ny)| \{|g(x)| \left| |\delta_y h(x)| \right| \left\{ \{|g(x)|^{p-1} + |h(x)|^{p-1} + |h(x-y)|^{p-1}\} \right\} dy, \quad (4.53)
\]
at least in the case $p \leq 2$ (as above, we will only consider this case).

Note that all of the terms above are similar to terms we have handled before. Thus, we proceed in the same way, decomposing terms exactly as before. Whenever a term includes a copy of $e^{it \Delta} u_n^J$ without derivatives, things will be relatively straightforward, as one can rely on (2.3) (see (4.54) below for details); for the one term stemming from (4.53) in which $e^{it \Delta} u_n^J$ only appears with derivatives, we have to go through the same local smoothing argument given above (cf. the proof of (4.50)).

Thus, to conclude the proof of (4.12), we will see how to estimate the contribution of the term
\[
\int N^d |\tilde{\psi}(Ny)| \left| |\delta_y g(x)| \right| |h(x-y)| |g(x-y)|^{p-1} \} dy. \quad (4.54)
\]
Estimating $|\delta_y g(x)|$ as before, we find we need to bound the terms
\[
M(h|g|^{p-1} g \geq N) + M(h|g|^{p-1} g > N)
\]
\[
+ \sum_{K \leq N} \frac{K}{N} M(h|g|^{p-1} K) M(g_k) + \sum_{K \leq N} M(h|g|^{p-1} M(g_k)).
\]

Let us now see how to handle the contribution of the first term only, as the other three are similar. We begin by summing over $N \in 2^\mathbb{Z}$ and recalling the definitions of $g$ and $h$; then, using Hölder, maximal function estimates, and Lemma 2.2, we can argue as we did to obtain (4.8) to see
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left\| \left( \sum_N \left| N^s g \geq N \right|^2 \right)^{1/2} M(h|g|^{p-1}) \right\| \frac{2^d}{2^{d+2}} \frac{\dot{H}^{s+}}{L^2_t L_x^2}
\]
\[
\lesssim \lim_{J \to \infty} \limsup_{n \to \infty} \left\| \nabla |^s \left( \sum_{j=1}^J v_j^J \right) \right\| \frac{2^d}{2^{d+2}} \frac{\dot{H}^{s+}}{L^2_t L_x^2} \left\| e^{it \Delta} u_n^J \right\| \frac{2^d}{2^{d+2}} \frac{\dot{H}^{s+}}{L^2_t L_x^2} \sum_{j=1}^J \left\| v_j^J \right\| \frac{2^d}{2^{d+2}} \frac{\dot{H}^{s+}}{L^2_t L_x^2}.
\]
\[
(4.55)
\]
We turn to estimating the first term above. We first write
\[ \lim_{J \to \infty} \limsup_{n \to \infty} \left\| \nabla |^{s_c} \left( \sum_{j=1}^J v_n^j \right) \right\| _{L_t^2 \times \mathbb{R}^d}^{2d+2} \leq \lim_{J \to \infty} \limsup_{n \to \infty} \left( \sum_{j=1}^J \left\| \nabla |^{s_c} v_n^j \right\| _{L_t^2 \times \mathbb{R}^d}^{2d+2} + \sum_{j \neq k} \left\| \nabla |^{s_c} v_n^j \nabla |^{s_c} v_n^k \right\| _{L_t^2 \times \mathbb{R}^d}^{d+2} \right) . \] (4.56)

Arguing as we did to obtain (4.10), we immediately get that
\[ \lim_{J \to \infty} \limsup_{n \to \infty} \sum_{j \neq k} \left\| \nabla |^{s_c} v_n^j \nabla |^{s_c} v_n^k \right\| _{L_t^2 \times \mathbb{R}^d}^{d+2} = 0. \] (4.57)

Next, we let \( \eta > 0 \); then, using (2.5), we can find \( J(\eta) > 0 \) so that
\[ \sum_{j > J(\eta)} \left\| \nabla |^{s_c} \phi^j \right\| _{L_t^2 \times \mathbb{R}^d}^{2} < \eta. \] (4.58)

On the other hand, the fact that each \( v_n^j \) has scattering size \( \lesssim E_c \) implies
\[ \sum_{j=1}^{J(\eta)} \left\| \nabla |^{s_c} v_n^j \right\| _{L_t^2 \times \mathbb{R}^d}^{2} \lesssim E_c. \] (4.59)

Combining (4.57), (4.58), and (4.59), we can continue from (4.56) to see
\[ \lim_{J \to \infty} \limsup_{n \to \infty} \left\| \nabla |^{s_c} \left( \sum_{j=1}^J v_n^j \right) \right\| _{L_t^2 \times \mathbb{R}^d}^{2d+2} \lesssim E_c. \] (4.60)

Thus, continuing from (4.55) and using (4.5) and (2.3), we find
\[ \lim_{J \to \infty} \limsup_{n \to \infty} \left\| \nabla |^{s_c} \left( \sum_{j=1}^J v_n^j \right) \right\| _{L_t^2 \times \mathbb{R}^d}^{2d+2} \lesssim E_c. \] (4.61)

Having established (4.11) and (4.12), we are now done with the proof of Lemma 4.2, as well as the sketch of the proof of Proposition 4.1.

5. Long-time Strichartz estimates

In this section, we prove a long-time Strichartz estimate. Such estimates were first developed by Dodson [17] in the study of the mass-critical NLS, but have since appeared in the energy-critical setting (see [36, 57]). In this paper, we establish a long-time Strichartz estimate for the first time in the inter-critical setting, modeling our approach after [17, 36, 57]. The long-time Strichartz estimate will be an important technical tool in Section 6, in which we rule out rapid frequency-cascade solutions, as well as in Section 7, in which we establish a frequency-localized interaction Morawetz inequality.
We will prove long-time Strichartz estimates for \((d,s)\) satisfying (5.3). This guarantees \(p > 1\), which simplifies the proof. Actually, as we will point out below, the same ideas can be used to handle \((d,s) = (5, \frac{1}{2})\), in which case \(p = 1\).

**Proposition 5.1** (Long-time Strichartz estimates). Take \((d,s)\) satisfying (5.3). Let \(u : [0,T_{\max}) \times \mathbb{R}^d \to \mathbb{C}\) be an almost periodic solution to \((\mathbb{I})\) with \(N(t) \equiv N_k \geq 1\) on each characteristic subinterval \(J_k \subset [0,T_{\max})\). Then on any compact time interval \(I \subset [0,T_{\max})\), which is a union of contiguous characteristic subintervals \(J_k\), and for any \(N > 0\), we have

\[
\| \nabla |^{s_c} u \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim_u 1 + N^{2s_c - \frac{d}{2}} K^{\frac{1}{2}},
\]  

(5.1)

where \(K := \int_I N(t)^{3-4s_c} \, dt\). Moreover, for any \(\varepsilon > 0\), there exists \(N_0 = N_0(\varepsilon)\) such that for all \(N \leq N_0\),

\[
\| \nabla |^{s_c} u \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim_u \varepsilon(1 + N^{2s_c - \frac{d}{2}} K^{\frac{1}{2}}).
\]  

(5.2)

We also note that the implicit constants in (5.1) and (5.2) are independent of \(I\).

**Proof.** Fix a compact interval \(I \subset [0,T_{\max})\), which is a contiguous union of characteristic subintervals \(J_k\); throughout the proof, all spacetime norms will be taken over \(I \times \mathbb{R}^d\) unless stated otherwise. Let \(\eta_0 > 0\) and \(\eta > 0\) be small parameters to be chosen later, and note that by Remark 1.8 we may find \(c = c(\eta)\) so that

\[
\| |^{s_c} u \|_{L_t^\infty L_x^2} \leq \eta.
\]  

(5.3)

For \(N > 0\), we define

\[
A(N) := \| |^{s_c} u \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \quad \text{and} \quad A_k(N) := \| |^{s_c} u \|_{L_t^\infty L_x^2(J_k \times \mathbb{R}^d)}
\]

for an individual characteristic subinterval \(J_k\). We first note that by Lemma 1.11 (5.1) holds whenever \(N \geq \sup_{J_k \subset I} N_k\). Indeed, in this case, we have

\[
A(N) \lesssim_u 1 + \left( \int_I N(t)^2 \, dt \right)^{\frac{1}{2}} \\
\lesssim_u 1 + \left( \int_I N(t)^{3-4s_c} N^{4s_c-1} \, dt \right)^{\frac{1}{2}} \\
\lesssim_u 1 + N^{2s_c - \frac{d}{2}} K^{\frac{1}{2}}.
\]

(5.4)

We will establish (5.4) for arbitrary \(N > 0\) by induction, beginning by establishing a recurrence relation for \(A(N)\):

**Lemma 5.2** (Recurrence relation for \(A(N)\)).

\[
A(N) \lesssim_u \inf_{t \in I} \| |^{s_c} u \|_{L_t^2(\mathbb{R}^d)} + B(\eta, \eta_0) N^{2s_c - \frac{d}{2}} K^{\frac{1}{2}} + \eta^\nu A(N_{\eta_0}) + \sum_{M > N/\eta_0} \left( \frac{N}{M} \right)^{\frac{5}{2} s_c} A(M)
\]

(5.5)

uniformly in \(N\), for some positive constants \(B(\eta, \eta_0)\) and \(\nu\).
Proof of Lemma 5.2: We first apply Strichartz to see
\[ A(N) \lesssim \inf_{t \in I} \| |\nabla|^s u|_{L_t^p L_x^q} \|_{L_t^1 L_x^{r\ast}} + \| |\nabla|^s P_{\leq N} (|u|^p u)\|_{L_t^2 L_x^{r\ast}}. \] (5.5)

Our next step is to decompose the nonlinearity $|u|^p u$ and estimate the resulting pieces; the particular decomposition we choose depends on the ambient dimension.

Case 1. When $d = 3$, we have $2 \leq p < 4$, and we decompose as follows:
\[ |u|^p u = (|u|^p + |u|^{p-2} \tilde{u}u_{\leq N/\eta_0})u_{> N/\eta_0} + |u|^{p-2} \tilde{u} (P_{> N}(u) u_{\leq N/\eta_0})u_{\leq N/\eta_0} + |u|^{p-2} \tilde{u} (P_{\leq N}(u) u_{\leq N/\eta_0})u_{\leq N/\eta_0}. \] (5.6)

To estimate the contribution of the first term on the right-hand side of (5.6) to (5.5), we let
\[ G := |u|^p + |u|^{p-2} \tilde{u}u_{\leq N/\eta_0} \]
and use Bernstein, Lemma 2.6, and Hölder to estimate
\[ \| |\nabla|^s P_{\leq N} (G u_{> N/\eta_0})\|_{L_t^1 L_x^{r\ast}} \lesssim N^{2s} \| |\nabla|^s P_{\leq N} (G u_{> N/\eta_0})\|_{L_t^1 L_x^{r\ast}} \lesssim N^{2s} \| |\nabla|^s G\|_{L_t^6 L_x^6} \lesssim \| |\nabla|^s G\|_{L_t^6 L_x^6} \sum_{M > N/\eta_0} \left( \frac{N}{M} \right)^{2s} A(M). \] (5.7)

To estimate the contribution of the first term above, we first use the fractional chain rule and Sobolev embedding to see
\[ \| |\nabla|^s |u|^p\|_{L_t^p L_x^q} \leq \| |u|^{p-1}\|_{L_t^p L_x^q} \| |\nabla|^s u\|_{L_t^q L_x^{r\ast}} \lesssim \| |\nabla|^s u\|_{L_t^q L_x^{r\ast}}, \]
while by the fractional product rule, the fractional chain rule, and Sobolev embedding we get
\[ \| |\nabla|^s (u^p - \tilde{u}u_{\leq N/\eta_0})\|_{L_t^p L_x^q} \lesssim \| |\nabla|^s (u^p - \tilde{u}u_{\leq N/\eta_0})\|_{L_t^p L_x^q} \lesssim \| |\nabla|^s u\|_{L_t^q L_x^{r\ast}}. \]

Thus, continuing from (5.7), we see
\[ \| |\nabla|^s P_{\leq N} \left( |u|^p + |u|^{p-2} \tilde{u}u_{\leq N/\eta_0} \right)\|_{L_t^2 L_x^{r\ast}} \lesssim \sum_{M > N/\eta_0} \left( \frac{N}{M} \right)^{2s} A(M). \] (5.8)

Next, we turn to estimating the contribution of the second term in (5.6) to (5.5). We begin by restricting our attention to an individual $J_k \times \mathbb{R}^d$. Note that we only
need to consider the case $cN_k \leq N/\eta_0$; in this case, we can use Bernstein, Hölder, Sobolev embedding, Lemma 1.11 and the fact that $s_c \geq \frac{1}{2}$ to estimate

$$
\| |\nabla|^{s_c}P_{\leq N}( |u|^{p-2}\tilde{u}(P_{>cN_k}u_{\leq N/\eta_0})u_{\leq N/\eta_0}) \|_{L_t^4L_x^6/5} \\
\lesssim N^{s_c} \| |u|^{p-2}\tilde{u}(P_{>cN_k}u_{\leq N/\eta_0})u_{\leq N/\eta_0}) \|_{L_t^4L_x^6/5} \\
\lesssim N^{s_c} \| u \|_{\mathcal{L}^1(L_x^\infty L_x^6)} \| P_{>cN_k}u_{\leq N/\eta_0} \|_{L_t^1L_x^6} \| u_{\leq N/\eta_0} \|_{L_t^{\infty}L_x^p} \\
\lesssim u N^{s_c} (cN_k)^{-s_c} \| |\nabla|^{s_c}u_{\leq N/\eta_0}) \|_{L_t^4L_x^3}^2 \\
\lesssim u B(\eta, \eta_0) \left( \frac{N}{N_k} \right)^{2s_c - \frac{1}{2}} \tag{5.9}
$$

for some positive constant $B(\eta, \eta_0)$. Summing the estimates (5.9) over the characteristic subintervals $J_k \subset I$ then gives

$$
\| |\nabla|^{s_c}P_{\leq N}( |u|^{p-2}\tilde{u}(P_{>cN_k}u_{\leq N/\eta_0})u_{\leq N/\eta_0}) \|_{L_t^4L_x^6/5} \\
\lesssim u B(\eta, \eta_0) N^{2s_c - \frac{1}{2} K^{1/2}}. \tag{5.10}
$$

Before proceeding to the next term in (5.8), we note that in obtaining estimate (5.10), we could have held onto the term $\| |\nabla|^{s_c}u_{\leq N/\eta_0} \|_{L_t^4L_x^3}$, which (by interpolation) we can estimate by

$$
\| |\nabla|^{s_c}u_{\leq N/\eta_0} \|_{L_t^4L_x^3} \lesssim \| |\nabla|^{s_c}u_{\leq N/\eta_0} \|_{\mathcal{L}^1(L_x^\infty L_x^6)}^{1/2} \| |\nabla|^{s_c}u_{\leq N/\eta_0} \|_{L_t^{\infty}L_x^p}^{1/2} \\
\lesssim u \| |\nabla|^{s_c}u_{\leq N/\eta_0} \|_{L_t^4L_x^3}. \tag{5.11}
$$

In this case, summing the estimates yields

$$
\| |\nabla|^{s_c}P_{\leq N}( |u|^{p-2}\tilde{u}(P_{>cN_k}u_{\leq N/\eta_0})u_{\leq N/\eta_0}) \|_{L_t^4L_x^6/5} \\
\lesssim u \sup_{J_k \subset I} \| |\nabla|^{s_c}u_{\leq N/\eta_0} \|_{\mathcal{L}^1(L_x^\infty L_x^6)(J_k \times \mathbb{R}^d)} B(\eta, \eta_0) N^{2s_c - \frac{1}{2} K^{1/2}}. \tag{5.12}
$$

This variant of (5.10) will be important when we eventually need to exhibit smallness in (5.2).

To estimate the contribution of the final term in (5.8) to (5.9), we begin with an application of the fractional product rule and Hölder to see

$$
\| |\nabla|^{s_c}( |u|^{p-2}\tilde{u}(P_{<cN_k}u)u_{\leq N/\eta_0}) \|_{L_t^2L_x^6/5} \\
\lesssim \| |\nabla|^{s_c}(|u|^{p-2}\tilde{u}) \|_{L_t^\infty L_x^6} \| P_{<cN_k}u_{\leq N/\eta_0} \|_{L_t^1L_x^6} \| u_{\leq N/\eta_0} \|_{L_t^{\infty}L_x^p} \tag{5.12}
$$

$$
+ \| |u|^{p-1} \|_{L_t^\infty L_x^\infty} \| |\nabla|^{s_c}P_{<cN_k}u \|_{L_t^1L_x^6} \| u_{\leq N/\eta_0} \|_{L_t^{\infty}L_x^p} \tag{5.13}
$$

$$
+ \| |u|^{p-1} \|_{L_t^\infty L_x^\infty} \| P_{<cN_k}u \|_{L_t^1L_x^6} \| |\nabla|^{s_c}u_{\leq N/\eta_0} \|_{L_t^4L_x^3}. \tag{5.14}
$$

We first note that by the fractional chain rule and Sobolev embedding, we get

$$
\| |\nabla|^{s_c}( |u|^{p-2}\tilde{u}) \|_{L_t^\infty L_x^6} \|_{L_t^{\infty}L_x^p} \lesssim \| u \|_{L_t^{\infty}L_x^p} \| |\nabla|^{s_c}u \|_{L_t^4L_x^3} \lesssim u. 1
$$
Using Sobolev embedding, interpolation, and (5.3), we also see
\[
\left\| P_{\leq \epsilon N(t)} u \leq \frac{\eta}{\nu} \right\|_{L^1_t L^\frac{d+1}{2}} \lesssim \left\| \nabla |^\frac{s}{s} P_{\leq \epsilon N(t)} u \leq \frac{\eta}{\nu} \right\|_{L^1_t L^\frac{d+1}{2}} \lesssim \left\| \nabla |^\frac{s}{s} P_{\leq \epsilon N(t)} u \leq \frac{\eta}{\nu} \right\|_{L^1_t L^\frac{d+1}{2}} \lesssim \eta \frac{d}{2} A\left(\frac{\eta}{\nu}\right).\]

Estimating similarly gives
\[
\left\| u \leq \frac{\eta}{\nu} \right\|_{L^1_t L^\frac{d+1}{p}} \lesssim \eta \frac{d}{2} A\left(\frac{\eta}{\nu}\right).\]

Plugging these last three estimates into (5.12), (5.13), and (5.14) and employing a few more instances of Sobolev embedding and (5.3) finally gives
\[
\left\| \nabla |^\frac{s}{s} P_{\leq \epsilon N(t)} (u \leq \frac{\eta}{\nu}) \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim \eta \frac{d}{2} A\left(\frac{\eta}{\nu}\right).\]  

Collecting the estimates (5.8), (5.10), and (5.15), we see that in the case \(d = 3\), the estimate (5.5) becomes
\[
A(N) \lesssim u \inf_{t \in I} \left\| \nabla |^\frac{s}{s} u \leq \frac{\eta}{\nu} (t) \right\|_{L^2(\mathbb{R}^d)} + B(\eta, \nu) N^{2^s - \frac{d}{2}} K^\frac{d}{2}\]
\[+ \eta \frac{d}{2} A\left(\frac{\eta}{\nu}\right) + \sum_{M > N/\nu} \left(\frac{N}{M}\right)^{\frac{d}{2} s} A(M).\]  

Comparing (5.10) to (5.11), we see that Lemma 5.2 holds for \(d = 3\).

**Case 2.** In this case, we have \(d \in \{4, 5\}\) and \(\frac{4}{d-1} \leq p < \frac{4}{d-2}\), with both inequalities strict for \(d = 5\). In particular, we have \(1 < p < 2\).

Again, we wish to decompose the nonlinearity and continue from (5.5). This time, we decompose as follows:
\[
|u|^p u = |u|^p u > \frac{\eta}{\nu} + |u| > \frac{\eta}{\nu} P_{\leq \epsilon N(t)} u \leq \frac{\eta}{\nu} + |u| > \frac{\eta}{\nu} P_{\leq \epsilon N(t)} u \leq \frac{\eta}{\nu} + (|u|^p - |u| > \frac{\eta}{\nu} |) u \leq \frac{\eta}{\nu}.\]

We estimate the contribution of the first term on the right-hand side of (5.17) to (5.18) similarly to the case \(d = 3\); in particular, by Bernstein, Hölder, and Lemma 2.6, we have
\[
\left\| \nabla |^\frac{s}{s} P_{\leq \epsilon N} (|u|^p)^{-1} u \leq \frac{\eta}{\nu} \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim N^{\frac{s}{2}} \left\| \nabla |^{-\frac{s}{2}} (|u|^p)^{-1} u \leq \frac{\eta}{\nu} \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim N^{\frac{s}{2}} \left\| \nabla |^{-\frac{s}{2}} |u|^p \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim \left\| \nabla |^{-\frac{s}{2}} |u|^p \right\|_{L^1_t L^\frac{d+1}{d}} \sum_{M > N/\nu} \left(\frac{N}{M}\right)^{\frac{d}{2} s} A(M).\]

As we can use the fractional chain rule and Sobolev embedding to estimate
\[
\left\| \nabla |^{-\frac{s}{2}} |u|^p \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim \left\| |u|^p \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim \left\| |u|^p \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim \left\| |u|^p \right\|_{L^1_t L^\frac{d+1}{d}} \lesssim 1.\]
we can continue from (5.18) to get
\[
\left\| \nabla^{s_c} P_{\leq N} \left( |u|^p P_{\leq cN_k u \leq N/\eta_0} \right) \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \lesssim_u \sum_{M > N/\eta_0} \left( \frac{N}{M} \right)^{\frac{s_c}{2}} A(M). \tag{5.19}
\]

Next, we turn to estimating the second term in (5.17) to (5.3). Restricting our attention to an individual characteristic subinterval \( J_k \), we first apply Bernstein, Hölder, and the fractional product rule to see
\[
\left\| \nabla^{s_c} P_{\leq N} \left( |u|^p P_{\leq cN_k u \leq N/\eta_0} \right) \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \lesssim N^{s_c-\frac{1}{2}} \left\| \nabla^{\frac{s_c}{2}} |u|^p P_{\leq cN_k u \leq N/\eta_0} \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \tag{5.20}
\]
\[
\lesssim N^{s_c-\frac{1}{2}} \left\| \nabla^{\frac{s_c}{2}} |u|^{cN_k} \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \left\| \nabla^{s_c} P_{\leq cN_k u \leq N/\eta_0} \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \tag{5.21}
\]
Using Hölder, the fractional chain rule, Sobolev embedding, Bernstein, interpolation, (5.3), and Young’s inequality, we can estimate
\[
(5.20) \lesssim N^{s_c-\frac{1}{2}} \left( \frac{N}{N_k} \right)^{s_c-\frac{1}{2}} \frac{\eta}{\eta} A_k \left( \frac{N}{\eta_0} \right)^4 \lesssim_u B(\eta) \left( \frac{N}{N_k} \right)^{2s_c-\frac{1}{2}} + \eta A_k \left( \frac{N}{\eta_0} \right),
\]
for some positive constant \( B(\eta) \). Using Lemma 1.11 as well, we can estimate similarly
\[
(5.21) \lesssim N^{s_c-\frac{1}{2}} (cN_k)^{\frac{1}{2}-s_c} \left\| \nabla^{\frac{s_c}{2}} |u|^{cN_k} \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \times \left\| \nabla^{s_c} P_{\leq cN_k u \leq N/\eta_0} \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \lesssim B(\eta) \left( \frac{N}{N_k} \right)^{s_c-\frac{1}{2}} \left\| \nabla^{s_c} |u|^{cN_k} \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \times \left\| \nabla^{s_c} P_{\leq cN_k u \leq N/\eta_0} \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \lesssim_u B(\eta) \left( \frac{N}{N_k} \right)^{2s_c-\frac{1}{2}} + \eta A_k \left( \frac{N}{\eta_0} \right).
\]
Collecting the estimates for (5.20) and (5.21) and summing over the intervals \( J_k \subset I \), we arrive at
\[
\left\| \nabla^{s_c} P_{\leq N} \left( |u|^{cN_k} P_{\leq cN_k u \leq N/\eta_0} \right) \right\|_{L_t^2 L_x^{\frac{4dp}{2dp-1}}} \lesssim_u B(\eta) N^{2s_c-\frac{1}{2}} K + \eta A \left( \frac{N}{\eta_0} \right). \tag{5.22}
\]
Before proceeding, we note that for both $[5.20]$ and $[5.21]$, we could have instead estimated
\[
\|\|\nabla\|^{\eta} P_{\leq cN_{t}} u_{\leq N/\eta_{0}}\|_{L_{t}^{\infty} L_{x}^{2}(J_{k} \times \mathbb{R}^{d})} \\
\lesssim \|\|\nabla\|^{\eta} u_{\leq cN_{k}}\|_{L_{t}^{\infty} L_{x}^{2}(J_{k} \times \mathbb{R}^{d})} \|\|\nabla\|^{\eta} u_{\leq N/\eta_{0}}\|_{L_{t}^{\infty} L_{x}^{2}(J_{k} \times \mathbb{R}^{d})} \\
\lesssim \eta^{\frac{4}{5}} \|\|\nabla\|^{s_{c}} u_{\leq N/\eta_{0}}\|_{L_{t}^{\infty} L_{x}^{2}(J_{k} \times \mathbb{R}^{d})}.
\]
If we had done this, upon summing we could have ended up with the alternate estimate
\[
\|\|\nabla\|^{\eta} P_{\leq N}(u_{\geq N(t)} P_{\leq cN(t)} u_{\leq N/\eta_{0}})\|_{L_{t}^{4} L_{x}^{4}} \\
\lesssim \sup_{J_{k} \subset I} \|\|\nabla\|^{s_{c}} u_{\leq N/\eta_{0}}\|_{L_{t}^{\infty} L_{x}^{2}(J_{k} \times \mathbb{R}^{d})} B(\eta) N^{2s_{c} - \frac{1}{2} K + \frac{7}{2}} + \eta^{\frac{4}{5}} A \left(\frac{N}{\eta_{0}}\right). \tag{5.23}
\]
This variant of $[5.22]$ will be important when we need to exhibit smallness in $[5.2]$. To estimate the contribution of the third term in $[5.17]$ to $[5.5]$, we first define the following:
\[
\begin{align*}
\theta &:= \frac{d_{p} - d - p}{4 - p} \in [0, 1), \\
\sigma &:= \frac{p^{2}(d^{2} - 2d - 2) - 4p(4d + 1) + 48}{4p(d - 8)} \in (0, s_{c}), \\
r_{1} &:= \frac{4dp(dp - 8)}{p^{2}(d^{2} - 2d - 2) + p(28 - 8d) - 16}, \\
r_{2} &:= \frac{4dp(dp - 8)}{p^{2}(d^{2} - 2d - 2) + p(2d + 1) + 16}.
\end{align*}
\]
With this choice of parameters, we have
\[
\begin{align*}
s_{c} + \theta \left(\frac{d_{p} - 1}{2} - s_{c}\right) &= 2s_{c} - \frac{1}{2}, \\
-\theta \left(s_{c} + \frac{1}{2}\right) - 2\sigma(1 - \theta) &= -(2s_{c} - \frac{1}{2})
\end{align*}
\]
and (by Sobolev embedding)
\[
H^{s_{c} - \frac{1}{2}} \hookrightarrow H^{\sigma, r_{1}}, \quad H^{s_{c} - \frac{1}{2}} \hookrightarrow H^{\sigma, r_{2}}.
\]
Then restricting our attention to an individual $J_{k}$, we can use Bernstein, H"{o}lder, the bilinear Strichartz estimate (Lemma $2.10$), and Sobolev embedding to estimate
\[
\|\|\nabla\|^{s_{c} - \frac{1}{2}} P_{\leq N}(u_{\geq cN_{k}} P_{\geq cN_{k}} u_{\leq N/\eta_{0}})\|_{L_{t}^{4} L_{x}^{4}} \\
\lesssim N^{s_{c}} \left|\|u_{\geq cN_{k}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{\frac{d_{p} - d - p}{4 - p}} \left|\|u_{\geq cN_{k}} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{\theta} \\
\times \left|\|u_{\geq cN_{k}} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \left|\|u_{\geq cN_{k}} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \\
\lesssim u \left|\|u_{\geq cN_{k}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{\frac{d_{p} - d - p}{4 - p} - s_{c}} \left(c_{N_{k}}\right)^{-\theta(s_{c} + \frac{1}{2})} \\
\times \left|\|u_{\geq cN_{k}} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \left|\|u_{\geq cN_{k}} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \\
\lesssim u \left|\|u_{\geq cN_{k}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{\frac{d_{p} - d - p}{4 - p} - s_{c}} \left(c_{N_{k}}\right)^{-\theta(s_{c} + \frac{1}{2}) - 2\sigma(1 - \theta)} \\
\times \left|\|\nabla\|^{\eta} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \left|\|\nabla\|^{\eta} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \\
\lesssim u B(\eta, \eta_{0}) \left|\|u_{\geq cN_{k}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{\frac{2s_{c} - \frac{1}{2}}{\eta}} \left|\|\nabla\|^{\eta} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \\
\lesssim u B(\eta, \eta_{0}) \left(\frac{N}{\eta_{0}}\right)^{2s_{c} - \frac{1}{2}} \left|\|\nabla\|^{\eta} P_{\geq cN_{k}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \\
\lesssim u B(\eta, \eta_{0}) \left(\frac{N}{\eta_{0}}\right)^{2s_{c} - \frac{1}{2}} \left|\|\nabla\|^{s_{c}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \left|\|\nabla\|^{s_{c}} u_{\leq N/\eta_{0}}\|_{L_{t}^{4} L_{x}^{4}}\right|^{1 - \theta} \tag{5.24}
\]

for some positive constant $B(\eta, \eta_0)$. If we sum the estimates (5.24) over the intervals $J_k \subset I$, we arrive at

$$
\| |\nabla|^{s_e} P_{\leq N} (\{ |u| > cN(t) \}^p P_{> cN(t)} u_{\leq N/\eta_0}) \|_{L^2_t L^2_x} \lesssim_u B(\eta, \eta_0) N^{2s-e-\frac{1}{2}} K^{\frac{1}{2}}. \quad (5.25)
$$

Before moving on to the fourth (and final) term in (5.17), we note that if we had held on to the term $\| |\nabla|^{s_e} u_{\leq N/\eta_0} \|_{L^\infty_t L^2_x}$ when deriving (5.24), then upon summing we would get

$$
\| |\nabla|^{s_e} P_{\leq N} (\{ |u| > cN(t) \}^p P_{> cN(t)} u_{\leq N/\eta_0}) \|_{L^2_t L^2_x} \lesssim_u \sup_{J_k \subset I} \| |\nabla|^{s_e} u_{\leq N/\eta_0} \|_{L^2_t L^{\frac{2d}{d+2}}_x}^\frac{2d}{d+2}.
$$

This variant of (5.25) will be important when we eventually need to exhibit smallness in (5.2).

We now turn to the final term in (5.17), beginning with an application of the fractional product rule and Hölder:

$$
\| |\nabla|^{s_e} P_{\leq N} (\{ |u| - |u| > cN(t) \}^p) u_{\leq N/\eta_0} \|_{L^2_t L^2_x}^\frac{2d}{d+2} \lesssim_u \| |\nabla|^{s_e} (|u| - |u| > cN(t))^p \|_{L^\infty_t L^2_x} \| u_{\leq N/\eta_0} \|_{L^2_t L^{\frac{2d}{d+2}}_x} + \| |u| - |u| > cN(t) \|^p \| |\nabla|^{s_e} u_{\leq N/\eta_0} \|_{L^2_t L^{\frac{2d}{d+2}}_x}.
$$

By Lemma 2.5 Sobolev embedding, and (5.3), we first estimate

$$
(5.27) \quad \| |\nabla|^{s_e} P_{\leq N} (\{ |u| - |u| > cN(t) \}^p) u_{\leq N/\eta_0} \|_{L^2_t L^2_x}^\frac{2d}{d+2} \lesssim_u \| u \|_{L^p_t L^\infty_x} \| u \|_{L^\infty_t L^2_x} \| u_{\leq N/\eta_0} \|_{L^2_t L^{\frac{2d}{d+2}}_x} \lesssim_u (p^{-1} + \eta) A(N). \quad (5.28)
$$

On the other hand, by Sobolev embedding, Hölder, and (5.3), we get

$$
(5.28) \quad \| |\nabla|^{s_e} P_{\leq N} (\{ |u| - |u| > cN(t) \}^p) u_{\leq N/\eta_0} \|_{L^2_t L^2_x}^\frac{2d}{d+2} \lesssim_u \eta \| u \|_{L^p_t L^\infty_x} \| u \|_{L^\infty_t L^2_x} \| u_{\leq N/\eta_0} \|_{L^2_t L^{\frac{2d}{d+2}}_x} \lesssim_u \eta A(N). \quad (5.29)
$$

Thus we can estimate the contribution of the final term in (5.17) by

$$
(5.29) \quad \| |\nabla|^{s_e} P_{\leq N} (\{ |u| - |u| > cN(t) \}^p) u_{\leq N/\eta_0} \|_{L^2_t L^2_x}^\frac{2d}{d+2} \lesssim_u \eta \| u \|_{L^p_t L^\infty_x} \| u \|_{L^\infty_t L^2_x} \| u_{\leq N/\eta_0} \|_{L^2_t L^{\frac{2d}{d+2}}_x} \lesssim_u \eta A(N). \quad (5.29)
$$

Collecting the estimates (5.19), (5.22), (5.25), and (5.29), we see that in Case 2, the estimate (5.3) becomes

$$
A(N) \lesssim_u \inf_{t \in I} \| |\nabla|^{s_e} u_{\leq N} (t) \|_{L^2_x(R^d)} + B(\eta, \eta_0) N^{2s-e-\frac{1}{2}} K^{\frac{1}{2}} + \eta \min(\frac{2-d}{2}, 1-p) A(N) + \sum_{M>N/\eta_0} (\frac{N}{M})^{\frac{d+2}{2}} A(M). \quad (5.30)
$$

Comparing (5.30) to (5.4), we see that Lemma 5.2 holds for $d \in \{4, 5\}$. \hfill \Box

**Remark 5.3.** We have omitted the case $(d, s_e) = (5, \frac{1}{2})$, in which $p = 1$; this scenario is not handled under Case 2 due to the use of Lemma 2.5. However, by using the alternate decomposition

$$
|u| = |u|_{\leq N/\eta_0} + (|u| - |u|_{\leq cN(t)}) u_{\leq N/\eta_0} + |u|_{\leq cN(t)} u_{\leq N/\eta_0},
$$
one can use the same ideas as above to establish the recurrence relation in this case.

With the recurrence relation (5.4) in hand, we can now use induction to complete the proof of Proposition 5.1. First, recall that (5.4) holds for \( N \geq \sup_{J_k \subset I} N_k \); i.e. we have

\[
A(N) \leq C(u) \left[ 1 + N^{2s_c - \frac{1}{2}} K^1 \right] \tag{5.31}
\]

for \( N \geq \sup_{J_k \subset I} N_k \). Of course, this inequality remains true if we replace \( C(u) \) by any larger constant.

We now suppose (5.31) holds at frequency \( N \) and use the recurrence relation (5.4) to show it holds at frequency \( N/2 \). Let us first rewrite (5.4) as

\[
A(N) \leq \tilde{C}(u) \left[ 1 + B(\eta, \eta_0) N^{2s_c - \frac{1}{2}} K^1 + \eta^\alpha A\left(\frac{N}{\eta_0}\right) + \sum_{M > N/\eta_0} \left(\frac{N}{M}\right)^{2s_c} A(M) \right]. \tag{5.32}
\]

To simplify notation, we will let \( \alpha := 2s_c - \frac{1}{2} \) and write \( B(\eta, \eta_0) = B \); then, if we take \( \eta_0 < \frac{1}{2} \) and use the inductive hypothesis, (5.32) becomes

\[
A\left(\frac{N}{2}\right) \leq \tilde{C}(u) \left[ 1 + B\left(\frac{N}{2}\right)^\alpha K^\frac{1}{2} + \eta^\alpha C(u) (1 + \eta_0^{-\alpha} \left(\frac{N}{2}\right)^\alpha K^\frac{1}{2}) \right.
+ C(u) \sum_{M > N/2\eta_0} \left(\frac{N}{M}\right)^{2s_c} (1 + M^\alpha K^\frac{1}{2}) \left.ight]
\leq \tilde{C}(u) \left[ 1 + B\left(\frac{N}{2}\right)^\alpha K^\frac{1}{2} + \eta^\alpha C(u) (1 + \eta_0^{-\alpha} \left(\frac{N}{2}\right)^\alpha K^\frac{1}{2}) \right.
+ C(u)\eta_0^{\frac{1}{2} s_c} + C(u)\eta_0^{\frac{1}{2} (1 - s_c) \left(\frac{N}{2}\right)^\alpha K^\frac{1}{2}} \right]
= \tilde{C}(u) \left[ 1 + B\left(\frac{N}{2}\right)^\alpha K^\frac{1}{2} \right] + C(u)\left[ (\eta^\alpha + \eta_0^{\frac{1}{2} s_c}) \tilde{C}(u) \right.
+ \left(\eta_0^{-\alpha} \eta^\alpha + \eta_0^{\frac{1}{2} (1 - s_c)} \right) \tilde{C}(u) \left(\frac{N}{2}\right)^\alpha K^\frac{1}{2}) \right]. \tag{5.33}
\]

Notice that we had convergence of the sum above precisely because \( s_c < 1 \). If we choose \( \eta_0 \) possibly even smaller depending on \( \tilde{C}(u) \), and \( \eta \) sufficiently small depending on \( \tilde{C}(u) \) and \( \eta_0 \), we can guarantee

\[
\left(\eta_0^{-\alpha} \eta^\alpha + \eta_0^{\frac{1}{2} (1 - s_c)} \right) \tilde{C}(u) \left(\frac{N}{2}\right)^\alpha K^\frac{1}{2}) \leq \tilde{C}(u) \left[ 1 + B(\eta, \eta_0) \left(\frac{N}{2}\right)^\alpha K^\frac{1}{2} \right] + \tilde{C}(u) \left[ 1 + (\frac{N}{2})^\alpha K^\frac{1}{2} \right]. \tag{5.34}
\]

If we now choose \( C(u) \) possibly larger so that \( C(u) \geq 2 (1 + B(\eta, \eta_0)) \tilde{C}(u) \), then this inequality implies that (5.31) holds at \( N/2 \), as was needed to show. This completes the proof of (5.1).

It remains to establish (5.2). To begin, fix \( \varepsilon > 0 \). To exhibit the smallness in (5.2), we need to revisit the proof of the recurrence relation for \( A(N) \), paying closer attention to the terms that gave rise to the expression \( N^{2s_c - \frac{1}{2}} K^1 \). More precisely, if we use (5.11) instead of (5.10); (5.23) instead of (5.22); and (5.26) instead of (5.25); then continuing from (5.3), the recurrence relation for \( A(N) \) takes the form

\[
A(N) \lesssim_{u} f(N) + f(N) N^{2s_c - \frac{1}{2}} K^\frac{1}{2} + \eta^\alpha A\left(\frac{N}{\eta_0}\right) + \sum_{M > N/\eta_0} \left(\frac{N}{M}\right)^{2s_c} A(M), \tag{5.34}
\]

where \( f(N) \) has the form

\[
f(N) = \left\| |\nabla|^s u \right\|_{L^\infty_t L^2_x(I \times \mathbb{R}^d)}
+ B(\eta, \eta_0) \sum_{i=1}^{4} \sup_{J_k \subset I} \left\| |\nabla|^s u \right\|_{L^\infty_t L^2_x(J_k \times \mathbb{R}^d)} \tag{5.35}
\]

\[\text{For } N \geq \sup_{J_k \subset I} N_k, \text{ and } \eta \text{ sufficiently small.} \]
for some \( \theta_i \in (0,1) \). Here the particular values of the \( \theta_i \) are not important; we will only need the fact that each \( \theta_i > 0 \). Combining the updated recurrence relation \((5.34)\) with the newly proven estimate \((5.1)\) and once again simplifying notation via \( \alpha = 2s_c - \frac{1}{2} \), we see

\[
A(N) \lesssim_u f(N) + f(N)N^\alpha K^\frac{1}{2} + \eta^\nu (1 + \eta_0^{-\alpha} N^\alpha K^\frac{1}{2}) + \eta_0^{\frac{1}{2}s_c} (1 + \eta_0^{-\alpha} N^\alpha K^\frac{1}{2})
\]

\[
\lesssim u f(N) + \eta^\nu + \eta_0^{\frac{1}{2}s_c} + \left[f(N) + \eta^\nu \eta_0^{-\alpha} + \eta_0^{(1-s_c)}\right] N^\alpha K^\frac{1}{2}. \tag{5.36}
\]

To complete the argument, we will need the fact that for fixed \( \eta, \eta_0 > 0 \), we have

\[
\lim_{N \to 0} f(N) = 0, \tag{5.37}
\]

which is a consequence of almost periodicity and the fact that

\[
\inf_{t \in [0,T_{\text{max}}]} N(t) \geq 1.
\]

Then, continuing from \((5.36)\), we choose \( \eta_0 \) small enough that \( \eta_0^{\frac{1}{2}s_c} + \eta_0^{(1-s_c)} < \varepsilon \), and choose \( \eta \) sufficiently small depending on \( \eta_0 \) so that \( \eta^\nu + \eta_0^{-\alpha} \eta^\nu < \varepsilon \). Finally, using \((5.37)\), we choose \( N_0 = N_0(\varepsilon) \) so that \( f(N) < \varepsilon \) for \( N \leq N_0 \). With this choice of parameters, \((5.36)\) becomes

\[
A(N) \lesssim u \varepsilon (1 + N^{2s_c-\frac{1}{2}} K^\frac{1}{2})
\]

for \( N \leq N_0 \), which completes the proof of \((5.2)\).

\[
\square
\]

6. The rapid frequency-cascade scenario

In this section, we preclude the existence of almost periodic solutions as in Theorem\((1.13)\) for which \( \int_0^{T_{\text{max}}} N(t)^{3-4s_c} \ dt < \infty \). We show that their existence is inconsistent with the conservation of mass. The main tool we will use is the long-time Strichartz estimate established in the previous section; as such, we will prove the following result for \((d, s_c)\) satisfying \((13)\).

**Theorem 6.1** (No rapid frequency-cascades). *Let \((d, s_c)\) satisfy \((13)\). Then there are no almost periodic solutions \( u : [0,T_{\text{max}}] \times \mathbb{R}^d \to \mathbb{C} \) to \((1.1)\) with \( N(t) \equiv N_k \geq 1 \) on each characteristic subinterval \( J_k \subset [0,T_{\text{max}}) \) such that

\[
\|u\|_{L_t^{p(x)}(\mathbb{R}^d)}^{\frac{p(x)}{2}} = \infty \tag{6.1}
\]

and

\[
\int_0^{T_{\text{max}}} N(t)^{3-4s_c} \ dt < \infty. \tag{6.2}
\]

**Proof.** We argue by contradiction. Suppose \( u \) were such a solution; then by Corollary\((1.10)\) we have

\[
\lim_{t \to T_{\text{max}}} N(t) = \infty,
\]

whether \( T_{\text{max}} \) is finite or infinite (recall \( s_c > \frac{1}{4} \)). Thus by Remark\((1.8)\) we see

\[
\lim_{t \to T_{\text{max}}} \|\nabla|s_c u| N(t)\|_{L_x^2(\mathbb{R}^d)} = 0 \quad \text{for any } N > 0. \tag{6.3}
\]

Now, we let \( I_n \) be a nested sequence of compact subintervals of \([0,T_{\text{max}})\), each of which is a contiguous union of characteristics intervals \( J_k \). On each \( I_n \), we will
now apply Proposition 5.1 specifically, for fixed $\eta, \eta_0 > 0$, we use the recurrence
relation (5.4), the estimate (5.1), and the hypothesis (6.2) to see

$$A_n(N) := \norm{\nabla^s u \leq N}_{L^2 T^\frac{2d}{p} (I_n \times \mathbb{R}^d)}$$

$$\lesssim_u \inf_{t \in I_n} \norm{\nabla^s u \leq N(t)}_{L^2_T (\mathbb{R}^d)} + B(\eta, \eta_0) N^{2s_c - \frac{d}{p}} \left( \int_{I_n} N(t)^{3-4s_c} \, dt \right)^{\frac{1}{s_c}}$$

$$+ \sum_{M > N/\eta_0} \left( \frac{N}{M} \right)^{\frac{2s_c}{p}} A_n(M)$$

$$\lesssim_u \inf_{t \in I_n} \norm{\nabla^s u \leq N(t)}_{L^2_T (\mathbb{R}^d)} + B(\eta, \eta_0) N^{2s_c - \frac{d}{p}} \left( \int_0^{T_{\max}} N(t)^{3-4s_c} \, dt \right)^{\frac{1}{s_c}}$$

$$+ \sum_{M > N/\eta_0} \left( \frac{N}{M} \right)^{\frac{2s_c}{p}} A_n(M)$$

Arguing as we did to obtain (5.1), we conclude

$$A_n(N) \lesssim_u \inf_{t \in I_n} \norm{\nabla^s u \leq N(t)}_{L^2_T (\mathbb{R}^d)} + N^{2s_c - \frac{d}{p}}.$$

Letting $n \to \infty$ and using (6.3) then gives

$$\norm{\nabla^s u \leq N}_{L^2 T^\frac{2d}{p} (0, T_{\max}) \times \mathbb{R}^d} \lesssim_u N^{2s_c - \frac{d}{p}} \quad \text{for all } N > 0. \quad (6.4)$$

We now claim that (6.4) implies

**Lemma 6.2.**

$$\norm{\nabla^s u \leq N}_{L^\infty T^\frac{2d}{p} (0, T_{\max}) \times \mathbb{R}^d} \lesssim_u N^{2s_c - \frac{d}{p}} \quad \text{for all } N > 0. \quad (6.5)$$

**Proof of Lemma 6.2** Fix $N > 0$; we first use Proposition 1.14 and Strichartz to estimate

$$\norm{\nabla^s u \leq N}_{L^\infty T^\frac{2d}{p} (0, T_{\max}) \times \mathbb{R}^d} \lesssim_u \norm{\nabla^s P \leq N |u|^p u}_{L^\infty T^\frac{2d}{p} (0, T_{\max}) \times \mathbb{R}^d}. \quad (6.6)$$

To proceed, we decompose the nonlinearity and estimate the individual pieces; as before, the particular decomposition we use depends on the ambient dimension. In the estimates that follow, spacetime norms will be taken over $[0, T_{\max}] \times \mathbb{R}^d$.

**Case 1.** When $d = 3$, we decompose

$$|u|^p u = |u|^{p-2} |u|^2 u + |u|^{p-2} |u| u + 2 |u|^{p-2} u u u + |u|^{p-2} u u u.$$
\[
\begin{align*}
\left\| \nabla^s P_{\leq N} \left( |u|^{p-2} \bar{u} u_{\leq N} \right) \right\|_{L^2_t L^5_x} & \lesssim \left\| \nabla^s \left( |u|^{p-2} \bar{u} \right) \right\|_{L^{3,\infty}_t L^{3,\infty}_x} \left\| u \right\|_{L^4_t L^\infty_x}^2 \left\| u \right\|_{L^4_t L^\infty_x}^6 \\
& + \left\| u \right\|_{L^\infty_t L^2_x}^{p-1} \left\| \nabla^s (u^2) \right\|_{L^2_t L^\infty_x} \\
& \lesssim \left\| u \right\|_{L^\infty_t L^\infty_x}^{p-2} \left\| \nabla^s u \right\|_{L^2_t L^\infty_x} \left\| \nabla^s u_{\leq N} \right\|_{L^2_t L^\infty_x}^2 \\
& + \left\| \nabla^s u \right\|_{L^2_t L^\infty_x}^{p-1} \left\| u_{\leq N} \right\|_{L^\infty_t L^2_x} \left\| \nabla^s u_{\leq N} \right\|_{L^2_t L^\infty_x} \\
& \lesssim_u \left\| \nabla^s u \right\|_{L^2_t L^\infty_x}^{p-1} \left\| u \right\|_{L^\infty_t L^\infty_x} \left\| \nabla^s u \right\|_{L^\infty_t L^\infty_x} \left\| \nabla^s u_{\leq N} \right\|_{L^2_t L^\infty_x} + N^{2s_c - \frac{1}{2}} \\
& \lesssim_u N^{2s_c - \frac{1}{2}}.
\end{align*}
\]

To estimate the contribution of the second piece, we denote

\[ G = |u|^{p-2} \bar{u} u_{> N} + 2 |u|^{p-2} \bar{u} u_{\leq N} \]

and use Bernstein, Hölder, Lemma 2.0, and (6.4) to see

\[
\begin{align*}
\left\| \nabla^s P_{\leq N} (G u_{> N}) \right\|_{L^2_t L^{5/2}_x} & \lesssim N^{2s_c} \left\| \nabla^s (G u_{> N}) \right\|_{L^2_t L^{5/2}_x} \\
& \lesssim N^{2s_c} \left\| \nabla^s G \right\|_{L^{\infty}_t L^{12p/5}_x} \left\| u \right\|_{L^\infty_t L^2_x} \left\| \nabla^s u_{> N} \right\|_{L^2_t L^\infty_x} \\
& \lesssim \left\| \nabla^s G \right\|_{L^{\infty}_t L^{12p/5}_x} \sum_{M > N} \left( \frac{N}{M} \right)^{2s_c} \left\| \nabla^s u_M \right\|_{L^2_t L^\infty_x} \\
& \lesssim_u \left\| \nabla^s G \right\|_{L^{\infty}_t L^{12p/5}_x} N^{2s_c - \frac{1}{2}}. \tag{6.7}
\end{align*}
\]

A few applications of the fractional product rule, fractional chain rule, and Sobolev embedding give

\[
\left\| \nabla^s G \right\|_{L^{\infty}_t L^{12p/5}_x} \lesssim \left\| \nabla^s u \right\|_{L^\infty_t L^2_x}^{p-1} \lesssim 1,
\]

so that continuing from (6.7), we get

\[
\left\| \nabla^s P_{\leq N} \left( \left( |u|^{p-2} \bar{u} u_{> N} + 2 |u|^{p-2} \bar{u} u_{\leq N} \right) u_{> N} \right) \right\|_{L^2_t L^{5/2}_x} \lesssim_u N^{2s_c - 1/2}.
\]

Thus we see that the claim holds in this first case.

**Case 2.** When \( d \in \{4, 5\} \), we decompose

\[ |u|^p u = |u|^p u_{\leq N} + |u|^p u_{> N}. \]

We employ Hölder, the fractional product rule, the fractional chain rule, Sobolev embedding, and (6.3) to estimate the contribution of the first piece as follows:

\[
\begin{align*}
\left\| \nabla^s P_{\leq N} (|u|^p u_{\leq N}) \right\|_{L^2_t L^{2d/2d}_x} & \lesssim \left\| \nabla^s |u|^p \right\|_{L^{\infty}_t L^{2d/2d}_x} \left\| u_{\leq N} \right\|_{L^2_t L^\infty_x}^{2d} + \left\| u \right\|_{L^{2d/2d}_x} \left\| \nabla^s u_{\leq N} \right\|_{L^2_t L^\infty_x}^{2d} \\
& \lesssim_u \left\| u \right\|_{L^\infty_t L^2_x}^{p-1} \left\| \nabla^s u \right\|_{L^\infty_t L^2_x} \left\| \nabla^s u_{\leq N} \right\|_{L^2_t L^\infty_x} + N^{2s_c - \frac{1}{2}} \\
& \lesssim_u N^{2s_c - \frac{1}{2}}.
\end{align*}
\]
For the second piece, we use Hölder, Bernstein, Lemma 2.6, the fractional chain rule, and Sobolev embedding to see

\[
\left\| \nabla^{s_c} P_{\leq N}(|u|^p u_{>N}) \right\|_{L_t^2 L_x^{2p}} \lesssim N^{\frac{1}{2} s_c} \left\| \nabla^{\frac{s_c}{2}} (|u|^p u_{>N}) \right\|_{L_t^2 L_x^{2p}} \\
\lesssim N^{\frac{3}{2} s_c} \left\| \nabla^{\frac{s_c}{2}} |u|^p \right\|_{L_t^2 L_x^{\frac{4dp}{d+sp-4}}} \left\| \nabla^{\frac{s_c}{2}} u_{>N} \right\|_{L_t^2 L_x^{2p}} \\
\lesssim \left\| u \right\|_{L_t^\infty L_x^p}^{p-1} \left\| \nabla^{\frac{s_c}{2}} u \right\|_{L_t^\infty L_x^p} \sum_{M > N} \left( \frac{N}{M} \right)^{\frac{s_c}{2}} \left\| \nabla^{s_c} u_M \right\|_{L_t^2 L_x^{2p}} \\
\lesssim u \left\| \nabla^{s_c} u \right\|_{L_t^\infty L_x^p}^p N^{2s_c - \frac{p}{2}} \\
\lesssim u N^{2s_c - \frac{p}{2}}.
\]

Thus we see that the claim holds in this second case, completing the proof of Lemma 6.2.

We now wish to use (6.5) to prove

Lemma 6.3.

\[ u \in L_t^\infty \dot{H}_x^{s_c}([0,T_{\text{max}}) \times \mathbb{R}^d) \quad \text{for some } \varepsilon > 0. \]

Proof of Lemma 6.3. For \( s_c > \frac{1}{2} \), this is easy; indeed, choosing \( \varepsilon > 0 \) such that \( s_c - \frac{1}{2} - \varepsilon > 0 \), we can use Bernstein and (6.5) to see

\[
\left\| \nabla^{s_c} u \right\|_{L_t^\infty L_x^p} \lesssim \sum_{N \leq 1} N^{-s_c - \varepsilon} \left\| \nabla^{s_c} u_N \right\|_{L_t^\infty L_x^p} + \sum_{N > 1} N^{-s_c - \varepsilon} \left\| \nabla^{s_c} u_N \right\|_{L_t^\infty L_x^p} \\
\lesssim u \sum_{N \leq 1} N^{-s_c - \varepsilon} N^{2s_c - \frac{p}{2}} + 1 \\
\lesssim u 1.
\]

When \( s_c = \frac{1}{2} \) (that is, \( p = \frac{4}{d-1} \)), we need to work a bit harder. To begin, we note that by Bernstein and (6.5), we have

\[
\left\| \nabla^{\frac{s_c}{2}} u \right\|_{L_t^\infty L_x^p} \lesssim \sum_{N \leq 1} N^{-\frac{p}{4}} \left\| \nabla^{\frac{s_c}{2}} u_N \right\|_{L_t^\infty L_x^p} + \sum_{N > 1} N^{-\frac{p}{4}} \left\| \nabla^{\frac{s_c}{2}} u_N \right\|_{L_t^\infty L_x^p} \\
\lesssim u \sum_{N \leq 1} N^{\frac{p}{4}} + 1 \\
\lesssim u \left\| \nabla^{\frac{s_c}{2}} u \right\|_{L_t^\infty L_x^p} + 1 \quad \text{(6.8)}
\]

We wish to show that in fact, we have the more quantitative statement

\[
\left\| \nabla^{\frac{s_c}{2}} u_{\leq N} \right\|_{L_t^\infty L_x^p([0,T_{\text{max}}) \times \mathbb{R}^d)} \lesssim u N^{\frac{p}{4}} \quad \text{for all } N > 0.
\]

(6.9)
Once we have established (6.9), we can complete the proof of Lemma 6.3 as follows: choosing $0 < \varepsilon < \frac{1}{10}$, we use Bernstein, (6.8), and (6.9) to estimate

$$\|\nabla^{-\varepsilon} u\|_{L^\infty_t L^2_x} \lesssim \sum_{N \leq 1} N^{-\frac{2}{3} - \varepsilon} \|\nabla^{\frac{1}{2}} u_N\|_{L^\infty_t L^2_x} + \sum_{N > 1} N^{-\frac{2}{3} - \varepsilon} \|\nabla^{\frac{1}{2}} u_N\|_{L^\infty_t L^2_x} \lesssim u \sum_{N \leq 1} N^{\frac{1}{3} - \varepsilon} + 1 \lesssim u. $$

Thus, to complete the proof of Lemma 6.3, it remains to establish (6.9). We begin by fixing $N > 0$. The proof of (6.9) will be a second iteration of the arguments that gave (6.5), this time using (6.8) as additional input.

We first use (6.8) (and the uniqueness of weak limits) to see that the no-waste Duhamel formula (Proposition 1.14) also holds in the weak $\dot{H}^{\frac{1}{2}}$ topology; thus, using Strichartz as well, we can estimate

$$\|\nabla^\frac{1}{2} u\|_{L^\infty_t L^2_x([0,T_{\text{max}}] \times \mathbb{R}^d)} \lesssim \|\nabla^\frac{1}{2} P_{\leq N}(|u|^2 u)\|_{L^\infty_t L^\frac{4}{3+3N}([0,T_{\text{max}}] \times \mathbb{R}^d)}.$$ 

Once again, we decompose the nonlinearity, and again the decomposition depends on the ambient dimension. The estimates that follow will be very similar in spirit to the estimates that gave (6.5); all estimates will be taken over $[0,T_{\text{max}}]$.

**Case 1.** When $d = 3$, we decompose

$$|u|^2 u = \bar{u} u \lesssim N + (\bar{u}_N + 2\bar{u}_N)u_N.$$ 

We estimate the first piece as follows: by Hölder, the fractional product rule, the fractional chain rule, Sobolev embedding, interpolation, (6.4) and (6.8),

$$\|\nabla^\frac{1}{2} P_{\leq N}(\bar{u} u \lesssim N)\|_{L^3_t L^6_x} \lesssim \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^2_x} \|\bar{u}\|_{L^6_t L^\infty_x} + \|\bar{u}\|_{L^\infty_t L^{30/11}_x} \|\nabla^\frac{1}{2} (\bar{u} u \lesssim N)\|_{L^\frac{5}{4}_t L^{15/7}_x} \lesssim \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^6_x} \|\bar{u}\|_{L^6_t L^\infty_x} \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^6_x} \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^6_x} \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^6_x} \lesssim u \frac{N}{\bar{u}}.$$ 

For the second piece, we first let $G := \bar{u} u \lesssim N + 2\bar{u} u \lesssim N$, and use Bernstein, Hölder, Lemma 2.6, Sobolev embedding, and (6.4) to see

$$\|\nabla^\frac{1}{2} P_{\leq N}(G)\|_{L^3_t L^6_x} \lesssim N \|\nabla^\frac{1}{2} (G)\|_{L^\infty_t L^6_x} \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^6_x} \lesssim \|\nabla^\frac{1}{2} G\|_{L^\infty_t L^\frac{15}{2}_x} \sum_{M > N} \left(\frac{N}{M}\right)^\frac{1}{2} \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^\frac{15}{2}_x} \lesssim \|\nabla^\frac{1}{2} G\|_{L^\infty_t L^\frac{15}{2}_x} \sum_{M > N} \left(\frac{N}{M}\right)^\frac{1}{2} \|\nabla^\frac{1}{2} u\|_{L^\infty_t L^\frac{15}{2}_x} \lesssim \|\nabla^\frac{1}{2} G\|_{L^\infty_t L^\frac{15}{2}_x} \frac{N}{\bar{u}.} \quad (6.10)\]
A few applications of the fractional product rule, Sobolev embedding, and (6.8) give

\[
\| |\nabla|^\frac{\alpha}{2} G\|_{L^p_t L^q_x} \lesssim \| G \|_{L^p_t L^q_x} \| |\nabla|^\frac{\alpha}{2} u\|_{L^p_t L^q_x} \lesssim u \| |\nabla|^\frac{\alpha}{2} u\|_{L^p_t L^q_x} \| |\nabla|^\frac{\alpha}{2} u\|_{L^p_t L^q_x} \lesssim u 1,
\]

so that (6.10) becomes

\[
\| |\nabla|^\frac{\alpha}{2} P_{\leq N}(G u \geq N)\|_{L^p_t L^q_x} \lesssim u N^\frac{\alpha}{2}.
\]

We see that (6.9) holds in this first case.

**Case 2.** When \( d \in \{4, 5\} \), we decompose

\[
|u|^{\frac{1}{d-\tau}} = |u|^{\frac{1}{d-\tau}} u_{\leq N} + |u|^{\frac{1}{d-\tau}} u_{> N}.
\]

We first note that interpolating between \( u \in L^\infty_t \dot{H}^{\frac{\alpha}{2}}_x \) and \( u \in L^\infty_t \dot{H}^{\frac{1}{d-\tau}}_x \), we have

\[
u \in L^\infty_t \dot{H}^{\frac{2\alpha}{d-\tau}}_x.
\] (6.11)

Thus, to estimate the contribution of the first piece, we can use Hölder, the fractional product rule, the fractional chain rule, Sobolev embedding, (6.4), (6.5), and (6.8) to see

\[
\| |\nabla|^\frac{\alpha}{2} (|u|^{\frac{1}{d-\tau}} u_{\leq N})\|_{L^p_t L^q_x} \lesssim \| |\nabla|^\frac{\alpha}{2} (|u|^{\frac{1}{d-\tau}} u_{\leq N})\|_{L^p_t L^q_x} \lesssim u \| |\nabla|^\frac{\alpha}{2} (|u|^{\frac{1}{d-\tau}} u_{\leq N})\|_{L^p_t L^q_x} \lesssim u N^\frac{\alpha}{2}
\]

For the second piece, we use Bernstein, Hölder, Lemma 2.6 the fractional chain rule, Sobolev embedding, (6.4), (6.5), and (6.8) to see

\[
\| |\nabla|^\frac{\alpha}{2} P_{\leq N}(|u|^{\frac{1}{d-\tau}} u)\|_{L^p_t L^q_x} \lesssim N^\frac{\alpha}{2} \| |\nabla|^\frac{\alpha}{2} (|u|^{\frac{1}{d-\tau}} u_{\geq N})\|_{L^p_t L^q_x} \lesssim N^\frac{\alpha}{2} \| |\nabla|^\frac{\alpha}{2} (|u|^{\frac{1}{d-\tau}} u_{\geq N})\|_{L^p_t L^q_x} \lesssim N^\frac{\alpha}{2} \| |\nabla|^\frac{\alpha}{2} (|u|^{\frac{1}{d-\tau}} u_{\geq N})\|_{L^p_t L^q_x} \lesssim N^\frac{\alpha}{2}
\]

Thus (6.9) holds in this second case. This completes the proof of Lemma 6.3.
With Lemma 6.3 at hand, we are ready to complete the proof of Theorem 6.1. Fix $t \in [0, T_{\text{max}})$ and $\eta > 0$. By Remark 1.8 we may find $c(\eta) > 0$ so that
\[
\int_{|\xi| \leq c(\eta)N(t)} |\xi|^{2sc} |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta.
\]
Interpolating with $u \in L^\infty_x H^{-s}_t$, we get
\[
\int_{|\xi| \leq c(\eta)N(t)} |\hat{u}(t, \xi)|^2 \, d\xi \lesssim_u \eta^{\frac{1}{7sc}}.
\]
On the other hand, we have
\[
\int_{|\xi| \geq c(\eta)N(t)} |\hat{u}(t, \xi)|^2 \, d\xi \leq (c(\eta)N(t))^{-2sc} \int |\xi|^{2sc} |\hat{u}(t, \xi)|^2 \, d\xi \lesssim_u (c(\eta)N(t))^{-2sc}.
\]
Adding these last estimates and using Plancherel, we conclude that for all $t \in [0, T_{\text{max}})$, we have
\[
0 \leq M(u(t)) := \int |u(t, x)|^2 \, dx \lesssim_u \eta^{\frac{1}{7sc}} + (c(\eta)N(t))^{-2sc}.
\]
Thus, recalling $\lim_{t \to T_{\text{max}}} N(t) = \infty$, we can conclude that for all $\eta > 0$, we may find $t_0 \in [0, T_{\text{max}})$ so that for all $t \in (t_0, T_{\text{max}})$, we have $M(u(t)) \leq \eta$. But by conservation of mass, $M(u(t)) \equiv M(u_0)$, and so we find that $M(u_0) \leq \eta$ for all $\eta > 0$. Of course, this gives $u \equiv 0$, which contradicts that $u$ blows up (cf. (6.1)). \qed

**Remark 6.4.** We have omitted the case $(d, s_c) = (5, \frac{1}{2})$ from Theorem 6.1 only because we omitted this case from the long-time Strichartz estimate, Proposition 5.1. Of course, as remarked in the proof of Proposition 5.1, the long-time Strichartz estimates continue to hold when $(d, s_c) = (5, \frac{1}{2})$; thus we see that Theorem 6.1 holds in this case as well.

7. **The frequency-localized interaction Morawetz inequality**

In this section, we prove spacetime bounds for the high-frequency portions of almost periodic solutions to (1.1); these bounds can be used to preclude the existence of quasi-solitons (Section 8). As we will see, establishing these bounds will lead to the most non-trivial restrictions on the set of $(d, s_c)$ to which our main theorem (Theorem 1.3) applies; see the proof below for a more detailed discussion. The main result of this section is the following

**Proposition 7.1** (Frequency-localized interaction Morawetz inequality). Let $(d, s_c)$ satisfy (1.4). Suppose $u : [0, T_{\text{max}}) \times \mathbb{R}^d \to \mathbb{C}$ is an almost periodic solution to (1.1) such that $N(t) \equiv N_k \geq 1$ on each characteristic subinterval $J_k \subset [0, T_{\text{max}})$, and let $I \subset [0, T_{\text{max}})$ be a compact time interval, which is a union of contiguous subintervals $J_k$. Then for any $\eta > 0$, there exists $N_0 = N_0(\eta)$ such that for any $N \leq N_0$, we have
\[
- \int_I \int_{\mathbb{R}^d \times \mathbb{R}^d} |u_{\geq N}(t, y)|^2 \Delta(\frac{1}{|t|})(x-y)|u_{\geq N}(t, x)|^2 \, dx \, dy \, dt \lesssim_u \eta(N^{1-4sc} + K),
\]
where $K := \int_I N(t)^{3-4sc} \, dt$. Furthermore, $N_0$ and the implicit constants above do not depend on $I$.\[7.1\]
Before we begin the proof of Proposition 7.1, we recall a general form of the interaction Morawetz inequality, introduced originally in [14] (for more discussion, see also [31] and the references cited therein). We will essentially follow the presentation in [55, Section 5].

For a fixed function \( a : \mathbb{R}^d \to \mathbb{R} \) and \( \varphi \) solving \( (i \partial_t + \Delta)\varphi = N \), we define the interaction Morawetz action by

\[
M(t) = 2 \text{Im} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(t, y)|^2 a_k(x - y) (\varphi_k \bar{\varphi})(t, x) \, dx \, dy,
\]

where subscripts denote spatial derivatives and repeated indices are summed. If we define the mass bracket

\[
\{f, g\}_m := \text{Im}(f \bar{g})
\]

and the momentum bracket

\[
\{f, g\}_P := \text{Re}(f \nabla \bar{g} - g \nabla \bar{f}),
\]

then one can show

\[
\begin{align*}
\partial_t M(t) &= - \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(t, y)|^2 a_{jjkk}(x - y)|\varphi(t, x)|^2 \, dx \, dy \\
&\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(t, y)|^2 4a_{jk}(x - y)\text{Re}(\bar{\varphi}_j \varphi_k)(t, x) \, dx \, dy \\
&\quad - \iint_{\mathbb{R}^d \times \mathbb{R}^d} 2 \text{Im}(\bar{\varphi} \varphi_k)(t, y)a_{jk}(x - y)2 \text{Im}(\bar{\varphi}_j)(t, x) \, dx \, dy \\
&\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} 2\{N, \varphi\}_m(t, y)a_j(x - y) \cdot \{N, \varphi\}_P(t, x) \, dx \, dy \\
&\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(t, y)|^2 2 \nabla a(x - y) \cdot \{N, \varphi\}_P(t, x) \, dx \, dy.
\end{align*}
\]

To prove Proposition 7.1 we will use \( a(x) = |x| \). Note that in this case, we have

\[
\begin{align*}
a_j(x) &= \frac{x_j}{|x|}, \\
a_{jk}(x) &= \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3}, \\
\Delta a(x) &= \frac{d-1}{|x|}, \\
\Delta \Delta a(x) &= -(d - 1)\Delta \left( \frac{1}{|x|} \right).
\end{align*}
\]

For this choice of \( a \), one can also show (7.2) + (7.3) \( \geq 0 \) (for details, see for example [55, Lemma 5.4]). Thus, integrating \( \partial_t M \) over \( I \), we arrive at the following
Lemma 7.2 (Interaction Morawetz inequality).

\[- \int_1 \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(t, y)|^2 \Delta \left( \frac{1}{|z|} \right) |x - y| |\varphi(t, x)|^2 \, dx \, dy \, dt \right.\]

\[+ \int_1 \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(t, y)|^2 \frac{x - y}{|x - y|} \cdot \{N, \varphi\}_p (t, x) \, dx \, dy \, dt \right.\]

\[\leq \sup_{t \in I} \int_1 \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(t, y)|^2 \frac{x - y}{|x - y|} \cdot \nabla \varphi(t, x) \bar{\varphi}(t, x) \, dx \, dy\]

\[+ \left| \int_1 \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \{N, \varphi\}_m (t, y) \frac{x - y}{|x - y|} \cdot \nabla \varphi(t, x) \bar{\varphi}(t, x) \, dx \, dy \, dt \right| .\]

To prove Proposition 7.1, we will apply this estimate with \( \varphi = u_{\geq N} \), with \( N \) chosen small enough to capture ‘most’ of the solution. To make this idea more precise, we first need to record the following corollary of Proposition 5.1.

Corollary 7.3 (Low and high frequencies control). Let \((d, s_c)\) satisfy (1.5), and let \( u : [0, T_{\text{max}}) \times \mathbb{R}^d \to \mathbb{C} \) be an almost periodic solution to (1.1) with \( N(t) \equiv N_k \geq 1 \) on each characteristic subinterval \( J_k \subset (0, T_{\text{max}}) \). Then on any compact time interval \( I \subset (0, T_{\text{max}}) \), which is a union of continuous subintervals \( J_k \), and for any frequency \( N > 0 \), we have

\[ \left\| u_{\geq N} \right\|_{L^p_t L^{q,r}_x (I \times \mathbb{R}^d)} \lesssim u \ N^{-s_c} (1 + N^{4s_c-1} K)^{\frac{1}{q}} \]  \tag{7.4} \]

for all \( \frac{2}{q} + \frac{d}{r} = \frac{d}{4} \) with \( q > 4 - \frac{28d}{dp-4} \), where \( K := \int_1 N(t)^{3-4s_c} \, dt \).

Moreover, for any \( \eta > 0 \), there exists \( N_0 = N_0(\eta) \) such that for all \( N \leq N_0 \), we have

\[ \left\| \left\| \nabla \right\|^{s_c} u_{\leq N} \right\|_{L^p_t L^{q,r}_x (I \times \mathbb{R}^d)} \lesssim u \ (1 + N^{4s_c-1} K)^{\frac{1}{q}} \]  \tag{7.5} \]

for all \( \frac{2}{q} + \frac{d}{r} = \frac{d}{4} \) with \( q \geq 2 \).

Furthermore, \( N_0 \) and the implicit constants in (7.4) and (7.5) do not depend on \( I \).

Proof of Corollary 7.3. We first show (7.4). For fixed \( \alpha > s_c - \frac{1}{2} \), we can use Bernstein and (5.1) to see

\[ \left\| \nabla \right\|^{\alpha} u_{\geq N} \right\|_{L^p_t L^{q,r}_x (I \times \mathbb{R}^d)} \lesssim \sum_{M \geq N} M^{-\alpha-s_c} \left\| \nabla^{s_c} u_M \right\|_{L^p_t L^{q,r}_x (I \times \mathbb{R}^d)} \]

\[\lesssim u \sum_{M \geq N} M^{-\alpha-s_c} (1 + M^{2s_c-\frac{d}{2} K}) \]

\[\lesssim u \ N^{-\alpha-s_c} (1 + N^{4s_c-1} K)^{\frac{1}{2}}. \tag{7.6} \]

Now, take \((q, r)\) with \( 2 < q \leq \infty \) and \( \frac{2}{q} + \frac{d}{r} = \frac{d}{4} \), and define \( \alpha = \frac{(q-2)(dp-4)}{4p} \). Notice that \( \alpha > s_c - \frac{1}{2} \) exactly when \( q > 4 - \frac{28d}{dp-4} \). Thus, in this case, we get by interpolation and (7.6) that

\[ \left\| u_{\geq N} \right\|_{L^p_t L^{q,r}_x (I \times \mathbb{R}^d)} \lesssim \left\| \nabla \right\|^{\alpha} u_{\geq N} \right\|_{L^p_t L^{q,r}_x (I \times \mathbb{R}^d)}^{\frac{1}{q}} \left\| \nabla^{s_c} u_{\geq N} \right\|_{L^p_t L^{q,r}_x (I \times \mathbb{R}^d)}^{\frac{1}{r}} \]

\[\lesssim u \left[ N^{-\frac{\alpha}{2}} (1 + N^{4s_c-1} K)^{\frac{1}{2}} \right]^\frac{1}{q} , \]
which gives (7.4). As for (7.5), we first note that since \( \inf_{t \in I} N(t) \geq 1 \), for any \( \eta > 0 \) we may find \( N_0(\eta) \) so that

\[
\| |\nabla|^{s_c} u \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \leq \eta
\]

for all \( N \leq N_0 \) (cf. Remark 1.8). The estimate (7.6) then follows by interpolating with (5.1).

We are now ready for the

**Proof of Proposition 7.1.** Take \( I \subset [0, T_{max}) \), a compact time interval, which is a contiguous union of subintervals \( J_k \), and let \( K := \int \int N(t)^{3-4s_c} \, dt \). Throughout the proof, all spacetime norms will be taken over \( I \times \mathbb{R}^d \).

Fix \( \eta > 0 \), and choose \( N_0 = N_0(\eta) \) small enough that (7.5) holds; recall that (7.4) holds without any restriction on \( N \). Next, we claim that for \( N_0 \) possibly even smaller, we can guarantee that for \( N \leq N_0 \), we have

\[
\| u_{N} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim N^{-s_c};
\]

and

\[
\| |\nabla|^{1-s_c} u_{N} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim \eta^1 N^{1-2s_c}.
\]

Indeed, by Remark 1.8 and the fact that \( \inf_{t \in I} N(t) \geq 1 \), we may find \( c(\eta) > 0 \) so that

\[
\| |\nabla|^{s_c} u_{\leq c(\eta)} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim \eta^1;
\]

combining this inequality with Bernstein, we get

\[
N^{s_c} \| u_{\geq N} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \lesssim N^{s_c} \| u_{\leq N} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + N^{s_c} \| u_{> c(\eta)} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)}
\]

\[
\lesssim \| |\nabla|^{1-s_c} u_{\leq c(\eta)} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} + \frac{N^{s_c}}{c(\eta)^{s_c}} \| |\nabla|^{s_c} u_{> c(\eta)} \|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)}
\]

\[
\lesssim N^{-s_c} + \eta^1.
\]

Thus, taking \( N \) sufficiently small, we recover (7.4). A similar argument yields (7.6).

Next, we record the following inequality that will be useful below:

\[
\sup_{y \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \frac{x-y}{|x-y|} \cdot \nabla \varphi(x) \tilde{\varphi}(x) \, dx \right| \lesssim \| |\nabla|^{s} \varphi \|_2 \| |\nabla|^{1-s} \tilde{\varphi} \|_2
\]

for \( 0 \leq s \leq 1 \). Indeed, for fixed \( y \in \mathbb{R}^d \), we can first write

\[
\int_{\mathbb{R}^d} \frac{x-y}{|x-y|} \cdot \nabla \varphi(x) \tilde{\varphi}(x) \, dx \lesssim \| |\nabla|^{s} \frac{x-y}{|x-y|} \varphi \|_2 \| |\nabla|^{1-s} \varphi \|_2
\]

\[
\sim \| |\nabla|^{s} \frac{x-y}{|x-y|} \varphi \|_2 \| |\nabla|^{1-s} \varphi \|_2.
\]

Thus, to prove (7.9), we need to see that the operator \( |\nabla|^{s} \frac{x-y}{|x-y|} |\nabla|^{-s} \) is bounded on \( L_x^2 \) (uniformly in \( y \)). When \( s = 0 \), this is clear. When \( s = 1 \), this follows from the chain rule, Hardy’s inequality, and the boundedness of Riesz transforms. The general case then follows from complex interpolation.

We now wish to apply the interaction Morawetz inequality (Lemma 7.2) with \( \varphi = u_{\geq N} \) and \( \mathcal{N} = P_{\geq N}(|u|^p u) \), with \( N \leq N_0 \). Together with (7.4), (7.5), (7.9),
Thus, to prove Proposition 7.1, we need to get sufficient control over the mass and momentum bracket terms appearing above.

To begin, we consider the contribution of the momentum bracket term. We can write

\[
\{ P_{\geq N}(|u|^pu), u_{\geq N} \} \quad = \quad \{ |u|^pu, u \} - \{ |u|^pu - |u|^pu_{\leq N}, u_{\leq N} \} - \{ |u|^pu_{\leq N}, u_{\leq N} \} + \{ P_{\leq N}(|u|^pu), u_{\geq N} \} - \{ P_{\leq N}(|u|^pu), u_{\geq N} \}.
\]

After an integration by parts, we see that term I contributes to the left-hand side of (7.10) a multiple of

\[
\iint \left| u_{\geq N}(t,y) \right|^2 \left( \frac{\Delta}{|x-y|} \right) (x-y) |u_{\geq N}(t,x)|^2 \, dx \, dy \, dt.
\]

For term II, we use \( \{ f, g \} = \nabla O(fg) + O(f \nabla g) \); when the derivative hits the product, we integrate by parts, while for the second term we simply bring absolute values inside the integral. In this way, we find that term II contributes to the right-hand side of (7.10) a multiple of

\[
\iint \left| u_{\geq N}(t,y) \right|^2 (|u|^pu - |u|^pu_{\leq N}) u_{\leq N}(t,x) \, dx \, dy \, dt + \iint \left| u_{\geq N}(t,y) \right|^2 (|u|^pu - |u|^pu_{\leq N}) u_{\leq N}(t,x) \, dx \, dy \, dt.
\]

Finally, for term III, we integrate by parts when the derivative falls on \( u_{\geq N} \); in this way, we see that term III contributes to the right-hand side of (7.10) a multiple of

\[
\iint \left| u_{\geq N}(t,y) \right|^2 \left( u_{\geq N}(t,x) \right) \left( P_{\leq N}(|u|^pu)(t,x) \right) \, dx \, dy \, dt + \iint \left| u_{\geq N}(t,y) \right|^2 \, \left( u_{\geq N}(t,x) \right) \left( \nabla P_{\leq N}(|u|^pu)(t,x) \right) \, dx \, dy \, dt.
\]
We next consider the mass bracket term in \((7.10)\). Exploiting the fact that
\[
\{u_{\geq N}|^p u_{\geq N}, u_{\geq N}\}_m = 0,
\]
we can write
\[
\{P_{\geq N}(|u|^p u), u_{\geq N}\}_m = \{P_{\geq N}(|u|^p u) - |u_{\geq N}|^p u_{\geq N}, u_{\geq N}\}_m
+ \{P_{\geq N}(|u_{\leq N}|^p u_{\leq N}), u_{\geq N}\}_m - \{P_{\leq N}(|u_{\geq N}|^p u_{\geq N}), u_{\geq N}\}_m.
\]

We will now collect the contributions of the mass and momentum bracket terms and insert them back into \((7.10)\). We will also make use of the pointwise inequalities
\[
|f + g|^p (f + g) - |f|^p f &\lesssim |g|^p + |f|^p, \\
|f + g|^{p+2} - |f|^{p+2} - |g|^{p+2} &\lesssim |f|^p + |g|^p + |f|^{p+1} |g|.
\]
In this way, \((7.10)\) becomes
\[
-\iint\int |u_{\geq N}(t, y)|^2 \Delta \left( \frac{1}{10} \right) (x - y) u_{\geq N}(t, x)|^2 dx dy dt
+ \iint\int \frac{|u_{\geq N}(t, y)|^2 |u_{\geq N}(t, x)|^{p+2}}{|x - y|} dx dy dt
\leq_{\text{u}} \eta^{20} N^{1 - 4s_c}
+ \eta^{10} N^{1 - 2s_c} \left\| |u_{\leq N}|^p u_{\geq N} \right\|_{L^1_{t,x}}
+ \eta^{10} N^{1 - 2s_c} \left\| |u_{\geq N}|^{p+1} u_{\leq N} \right\|_{L^1_{t,x}}
+ \eta^{10} N^{1 - 2s_c} \left\| P_{\leq N}(|u_{\leq N}|^p u_{\leq N}) u_{\geq N} \right\|_{L^1_{t,x}}
+ \eta^{10} N^{1 - 2s_c} \left\| P_{\leq N}(|u_{\geq N}|^p u_{\geq N}) u_{\leq N} \right\|_{L^1_{t,x}}
+ \iint\int \frac{|u_{\geq N}(t, y)|^2 |u_{\geq N}(t, x)|^{p+1}}{|x - y|} dx dy dt
+ \iint\int \frac{|u_{\geq N}(t, y)|^2 |u_{\leq N}(t, x)|^{p+1}}{|x - y|} dx dy dt
+ \iint\int \frac{P_{\leq N}(|u|^p u)(t, x)}{|x - y|} dx dy dt
+ \iint\int |u_{\geq N}(t, y)|^2 |u_{\leq N}(t, x)|^{p-1} |\nabla u_{\leq N}(t, x)| dx dy dt
+ \iint\int |u_{\geq N}(t, y)|^2 |u_{\leq N}(t, x)|^{p+1} |\nabla u_{\leq N}(t, x)| dx dy dt
+ \iint\int |u_{\geq N}(t, y)|^2 |u_{\leq N}(t, x)| |\nabla P_{\leq N}(|u|^p u)(t, x)| dx dy dt.
\]

To complete the proof of Proposition \((7.1)\) we need to show that the error terms \((7.12)\) through \((7.22)\) are acceptable, in the sense that they can be controlled by \(\eta(N^{1 - 4s_c} + K)\). Clearly, \((7.12)\) is acceptable.
Next, we consider (7.13). Using Hölder, Sobolev embedding, (7.3), and (7.5), we get

\[ \|u_{\leq N}|^p u_{\geq N}^2\|_{L^p_x} \lesssim \|u_{\leq N}\|_{L^p_t L^s_x}^{p-1} u_{\kappa x} \|u_{\leq N}\|_{L^p_t L^s_x}^{\frac{1}{p}} \]

\[ \lesssim \|\nabla|^{\kappa x} u_{\leq N}\|_{L^p_t L^s_x} \|u_{\geq N}\|_{L^p_t L^s_x}^{\frac{1}{p}} \]

\[ \lesssim u \eta N^{-2\kappa x} (1 + N^{4\kappa x-1} K), \]

which renders (7.13) acceptable.

We now turn to (7.14). For this term, we can again use Hölder, Sobolev embedding, (7.3), and (7.5) to see

\[ \|P_{\leq N}(u_{\leq N}|^p u_{\leq N})\|_{L^p_t L^s_x} \lesssim \|u_{\leq N}\|_{L^p_t L^s_x}^{p-1} u_{\kappa x} \|u_{\leq N}\|_{L^p_t L^s_x} \]

\[ \lesssim \|u_{\geq N}\|_{L^p_t L^s_x} \|\nabla|^{\kappa x} u_{\leq N}\|_{L^p_t L^s_x} \]

\[ \lesssim u \eta N^{-2\kappa x} (1 + N^{4\kappa x-1} K), \]

Thus this term is acceptable as well. Before proceeding, however, we note that it is this term that has forced us to exclude the cases \((d, s_c) \in \{3\} \times \left(\frac{1}{2}, 1\right)\) from this paper; we postpone further discussion until Remark (7.4) below.

We next turn to (7.15): using Hölder, Bernstein, the fractional chain rule, Sobolev embedding, (7.3), and (7.5), we see

\[ \|P_{\geq N}(u_{\leq N}|^p u_{\leq N})u_{\geq N}\|_{L^p_t L^s_x} \lesssim \|u_{\geq N}\|_{L^p_t L^s_x} \|\nabla|^{\kappa x} (u_{\leq N}|^p u_{\leq N})\|_{L^p_t L^s_x} \]

\[ \lesssim N^{-\kappa x} \|u_{\geq N}\|_{L^p_t L^s_x} \|\nabla|^{\kappa x} u_{\leq N}\|_{L^p_t L^s_x} \]

\[ \lesssim N^{-\kappa x} \|u_{\geq N}\|_{L^p_t L^s_x} \|\nabla|^{\kappa x} u_{\leq N}\|_{L^p_t L^s_x} \]

\[ \lesssim u \eta N^{-2\kappa x} (1 + N^{4\kappa x-1} K), \]

so that (7.15) is also acceptable.

For the final term originating from the mass bracket, (7.10), we use Hölder, Bernstein, Sobolev embedding, (7.3), and (7.5) to see

\[ \|P_{\leq N}(u_{\geq N}|^p u_{\geq N})\|_{L^p_t L^s_x} \lesssim \|u_{\geq N}\|_{L^p_t L^s_x} \|\nabla|^{\kappa x} (u_{\geq N}|^p u_{\geq N})\|_{L^p_t L^s_x} \]

\[ \lesssim N^{\kappa x} \|u_{\geq N}\|_{L^p_t L^s_x} \|\nabla|^{\kappa x} u_{\geq N}\|_{L^p_t L^s_x} \]

\[ \lesssim u \eta N^{-2\kappa x} (1 + N^{4\kappa x-1} K), \]

which shows that (7.10) is acceptable.
We now turn to the terms originating from the momentum bracket. First, consider (7.17). By Hölder, Hardy–Littlewood–Sobolev, Sobolev embedding, Bernstein, (7.24), (7.5), and (7.7), we can estimate

\[
\begin{align*}
(7.17) & \lesssim \left\| |u| \right\|_{L^6} + \left\| |u| \right\|_{L^6}^2 \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta} \\
& \quad \times \left\| |u| \right\|_{L^6} \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta-1} \\
& \lesssim \left\| |u| \right\|_{L^6} \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta} \\
& \quad \times \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta-1} \\
& \lesssim \langle u \rangle \left\| |u| \right\|_{L^6} \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta} \\
& \lesssim \theta \langle u \rangle \left( 1 + N^{4s_c-1} K \right),
\end{align*}
\]

so that (7.17) is acceptable.

For (7.18), we consider two cases. If \(|u| \leq 10^{-100} \left| u \right|_{N \geq N}|\), then we can absorb this term into the left-hand side of the inequality, provided we can show

\[
\begin{align*}
\int \int \frac{|u|_{N \geq N}(t, y)|^2 |u|_{N \geq N}(t, x)|^{p+2}}{|x-y|} \, dx \, dy \, dt < \infty. 
\end{align*}
\]

(7.23)

On the other hand, if \(|u| \geq 10^{100} |u|_{N \geq N}|\), then we are back in the situation of (7.17), which we have already handled. Thus, to render (7.18) acceptable, it remains to prove (7.23). To this end, we define

\[
\theta = \frac{4dp-16-3p}{2(dp-4)} \in (0, p+2),
\]

and use Hölder, Hardy–Littlewood–Sobolev, Sobolev embedding, Lemma 1.11, and interpolation to estimate

\[
\begin{align*}
\text{LHS (7.23)} & \lesssim \left\| |u| \right\|_{L^6} \left\| |u| \right\|_{L^6}^2 \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta} \\
& \quad \times \left\| |u| \right\|_{L^6} \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta-1} \\
& \lesssim \left\| |u| \right\|_{L^6} \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta} \\
& \quad \times \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta-1} \\
& \lesssim \langle u \rangle \left\| |u| \right\|_{L^6} \left\| \nabla |u| \right\|_{L^{6/5}} \left\| |u| \right\|_{L^6}^{\theta} \\
& \lesssim \theta \langle u \rangle \left( 1 + \int_I N(t) \right)^2 \frac{dt}{|x-y|}. 
\end{align*}
\]

which gives (7.23), and thereby shows that (7.18) is acceptable.
we begin by writing

\[(7.19)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.24)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.25)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.26)\]

For \[(7.24)\], we can write

\[(7.24)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.25)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.26)\]

by the same arguments that dealt with \[(7.17)\].

For \[(7.25)\], we can use Hölder, Hardy–Littlewood–Sobolev, Bernstein, Sobolev embedding, \[(7.4)\], \[(7.5)\], and \[(7.7)\] to estimate

\[(7.25)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.26)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.27)\]

which renders \[(7.25)\] acceptable.

For \[(7.26)\], we first note

\[(7.26)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.27)\]

so that using Hölder, Hardy–Littlewood–Sobolev, Bernstein, Sobolev embedding, \[(7.4)\], \[(7.5)\], and \[(7.7)\], we get

\[(7.27)\]
\[
\int \int \int \frac{|u_{N}(t, y)|^2}{|x - y|} \left| P_{\leq N} \left( \left| u \right| |u|_{N}^{P} u_{N} \right)(t, x) \right| \ d x \ d y \ d t.
\]
\[(7.28)\]

Thus \[(7.28)\], and so \[(7.19)\], is acceptable.
We now turn to (7.20). By Hölder, Sobolev embedding, Bernstein, (7.3), and (7.7), we estimate

\[
\begin{align*}
(7.20) & \lesssim \|u \geq N\|_{L^3_t L^2_x}^3 \|u \leq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim N^{1-s_c} \|u \geq N\|_{L^p_t L^p_x}^3 \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim u \eta^{p+31} N^{1-4s_c} (1 + N^{4s_c-1} K),
\end{align*}
\]

so that (7.20) is acceptable.

For (7.21), we use Hölder, Sobolev embedding, Bernstein, (7.3), (7.5), and (7.7) to get

\[
\begin{align*}
(7.21) & \lesssim \|u \geq N\|_{L^2_t L^2_x}^2 \|u \leq N\|_{L^{p-1}_t L^{p'}_x}^{p-1} \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim u \|u \geq N\|_{L^2_t L^2_x}^2 \|u \geq N\|_{L^2_t L^2_x}^{3/2} \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim u \eta^{21} N^{1-4s_c} (1 + N^{4s_c-1} K),
\end{align*}
\]

which renders (7.21) acceptable.

Finally, we consider (7.22). We begin by writing

\[
\begin{align*}
(7.22) & \lesssim \|u \geq N\|_{L^2_t L^2_x}^2 \|u \geq N\|_{L^{p-1}_t L^{p'}_x} \|u \leq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \quad + \|u \geq N\|_{L^2_t L^2_x}^2 \|u \geq N\|_{L^{p-1}_t L^{p'}_x} \|u \leq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \quad + \|u \geq N\|_{L^2_t L^2_x}^2 \|u \geq N\|_{L^{p-1}_t L^{p'}_x} \|u \leq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \quad + \|u \geq N\|_{L^2_t L^2_x}^2 \|u \geq N\|_{L^{p-1}_t L^{p'}_x} \|u \leq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2}.
\end{align*}
\]

To begin, we use Hölder, the chain rule, and the arguments that gave (7.20) to see

\[
\begin{align*}
(7.24) & \lesssim \|u \geq N\|_{L^3_t L^2_x}^3 \|u \leq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim u \eta^{p+31} N^{1-4s_c} (1 + N^{4s_c-1} K),
\end{align*}
\]

so that (7.27) is acceptable.

For (7.28), we argue essentially as we did for (7.10). That is, we use Hölder, Bernstein, Sobolev embedding, (7.3), and (7.7) to estimate

\[
\begin{align*}
(7.28) & \lesssim \|u \geq N\|_{L^2_t L^2_x}^2 \|u \geq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim N^{1+s_c} \|u \geq N\|_{L^p_t L^p_x}^2 \|u \geq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim N^{1+s_c} \|u \geq N\|_{L^p_t L^p_x}^2 \|u \geq N\|_{L^p_t L^p_x}^p \|\nabla u \leq N\|_{L^2_t L^2_x}^{3/2} \\
& \lesssim u \eta^{20} N^{1-4s_c} (1 + N^{4s_c-1} K),
\end{align*}
\]

which gives that (7.28) is acceptable.
For (7.29), we argue similarly to the case of (7.20). In particular, we use Hölder, Bernstein, Sobolev embedding, (7.4), (7.5), and (7.7) to see

\[ (7.29) \lesssim N \left\| u_{\leq N} \right\|_{L^{4/3} L^2_x}^2 \left\| u_{\geq N} \right\|_{L^1_t L^{4/3} x} \left\| \mathcal{O} \left( u_{\leq N} u_{\geq N} |u|^{-1} \right) \right\|_{L^3_t L^{12/7} x} \]

\[ \lesssim N \left\| u_{\leq N} \right\|_{L^{4/3} L^2_x}^2 \left\| u_{\geq N} \right\|_{L^1_t L^{4/3} x} \left\| \mathcal{O} \left( u_{\leq N} u_{\geq N} |u|^{-1} \right) \right\|_{L^3_t L^{12/7} x} \]

\[ \lesssim N \left\| u_{\leq N} \right\|_{L^{4/3} L^2_x} \left\| u_{\geq N} \right\|_{L^1_t L^{4/3} x} \left\| \nabla |u| u_{\leq N} \right\|_{L^3_t L^{12/7} x} \left\| \nabla |u| u_{\leq N} \right\|_{L^3_t L^{12/7} x} \]

\[ \lesssim u \eta^{21} N^{1-4s_0} (1 + N^{4s_0-1} K), \]

which gives that (7.20). Collecting the estimates for (7.27), (7.28), and (7.29), we see that (7.22) is acceptable. This completes the proof of Proposition 7.1. \( \square \)

Remark 7.4. Let us discuss why (7.14) has forced us to exclude the cases \((d, s_c) \in \{3 \times (\frac{4}{3}, 1)\) from this paper. As one can see in the proof above, in the cases we consider, this term is fairly harmless. However, once \(s_c > \frac{4}{3}\) in dimension \(d = 3\) (which corresponds to \(p > \frac{4}{3}\)), this term becomes a problem; put simply, we end up with too many copies of \(u_{\geq N}\) to deal with.

This problem has already been encountered in the energy-critical setting \((s_c = 1)\) in dimension \(d = 3\); in this case, one can overcome the hurdle by applying a spatial truncation to the weight \(u\). One can refer to [15] for the original argument, wherein spatial truncation is applied at various levels and subsequently averaged. The authors of [36] revisit the result of [15] in the context of minimal counterexamples; at this point in the argument, they choose to work with a more carefully designed spatial truncation, which removes the need for any subsequent averaging argument.

This discussion begs the question: why doesn’t spatial truncation work in our setting? To answer this, we need to understand how spatial truncations affect the argument that leads to Proposition 7.1. What we find is that spatial truncations ruin the convexity properties of \(a\) that made some of the terms in the proof of Lemma 7.2 positive; thus, to establish Proposition 7.1 with a further spatial truncation, we have to control additional error terms. It turns out that one of these additional error terms requires uniform control over \(\|u\|_{L^{p+2}_x}\), while another requires uniform control over \(\|\nabla u\|_{L^2_x}\) (see [36] Lemma 6.5 and Lemma 6.6). In the energy-critical case, one can use the conservation of energy to push the argument through, while in our cases, we cannot proceed without some significant new input. We have therefore abandoned the cases \((d, s_c) \in \{3 \times (\frac{4}{3}, 1)\) in this paper.

For a further discussion of these issues, refer to [36], especially Remark 6.9 therein.

8. The quasi-soliton scenario

In this section we preclude the existence of almost periodic solutions as in Theorem 1.13 for which \(\int_{0}^{T_{\text{max}}} N(t)^{3-4s_0} \, dt = \infty\). We will show that their existence is inconsistent with the frequency-localized interaction Morawetz inequality (Proposition 7.1).

Before we begin, we note that

\[ -\Delta \left( \frac{1}{|x|^d} \right) = \begin{cases} 4\pi\delta & d = 3 \\ \frac{d-3}{|x|^d} & d \geq 4. \end{cases} \]
Thus, if \( d = 3 \), the conclusion of Proposition 7.1 reads
\[
\int_I \int_{\mathbb{R}^3} |u_{\geq N}(t,x)|^4 \, dx \, dt \lesssim_u \eta (N^{1-4s_c} + K) \tag{8.1}
\]
for \( N \leq N_0(\eta) \), while if \( d \in \{4, 5\} \), the conclusion of Proposition 7.1 reads
\[
\int_I \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u_{\geq N}(t,x)|^2 |u_{\geq N}(t,y)|^2}{|x-y|^3} \, dx \, dy \, dt \lesssim_u \eta (N^{1-4s_c} + K) \tag{8.2}
\]
for \( N \leq N_0(\eta) \).

We now turn to Theorem 8.1 (No quasi-solitons). Let \((d, s_c)\) satisfy (1.4). Then there are no almost periodic solutions \( u : [0, T_{\text{max}}) \times \mathbb{R}^d \to \mathbb{C} \) to (1.1) with \( N(t) \equiv N_k \geq 1 \) on each characteristic subinterval \( J_k \subset [0, T_{\text{max}}) \) that satisfy both
\[
\inf_{t \in [0, T_{\text{max}})} N(t)^{2s_c} \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} |u_{\geq N}(t,x)|^2 \, dx \gtrsim_u 1 \quad \text{for all } N \leq N_0. \tag{8.4}
\]

We provide the proof of Lemma 8.2 below; let us first take it for granted and use it to complete the proof of Theorem 8.1. We let \( I \) be a compact time interval, which is a union of contiguous subintervals \( J_k \), and let \( \eta > 0 \) be a small parameter. We take \( C(u) \) and \( N_0 \) as in Lemma 8.2, then choosing \( N_0 \) possibly smaller, we can guarantee by Proposition 7.1 that (8.1) or (8.2) holds (depending on the dimension) for all \( N \leq N_0 \).

We now consider two cases:

**Case 1.** When \( d = 3 \), we first note by Hölder’s inequality that
\[
\left( \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} |u_{\geq N}(t,x)|^2 \, dx \right)^2 \gtrsim_u N(t)^{-3} \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} |u_{\geq N}(t,x)|^4 \, dx.
\]
Using this inequality, followed by (8.4), we find
\[
\int_I \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} |u_{\geq N}(t,x)|^4 \, dx \, dt \gtrsim_u \int_I \left( \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} |u_{\geq N}(t,x)|^2 \, dx \right)^2 N(t)^3 \, dt \gtrsim_u \int_I N(t)^{3-4s_c} \, dt.
\]
for all \( N \leq N_0 \). Thus, appealing to (8.1), we find

\[
\eta \left[ N^{1-4\epsilon_c} + \int_I N(t)^{3-4\epsilon_c} \right] \geq u \int \int_{\mathbb{R}^3} |u_{\geq N}(t, x)|^4 \, dx \, dt \\
\geq u \int \int_{|x-x(t)| \leq \frac{c(u)}{N(t)^4}} |u_{\geq N}(t, x)|^4 \, dx \, dt \\
\geq u \int \int I(t)^{3-4\epsilon_c} \, dt 
\]

for all \( N \leq N_0 \).

**Case 2.** If \( d \in \{4, 5\} \), we once again use (8.3), but use (8.2) instead of (8.1). We find

\[
\eta \left[ N^{1-4\epsilon_c} + \int_I N(t)^{3-4\epsilon_c} \, dt \right] \\
\geq u \int \int \int_{\mathbb{R}^d} \left| \frac{u_{\geq N}(t, x)}{|x-y|^3} \right|^2 |u_{\geq N}(t, y)|^2 \, dy \, dx \, dt \\
\geq u \int \int |x-x(t)| \leq \frac{c(u)}{N(t)^4} \left[ \frac{N(t)^4}{c(u)} \right] \left| u_{\geq N}(t, x) \right|^2 |u_{\geq N}(t, y)|^2 \, dy \, dx \, dt \\
\geq u \int \left[ \frac{N(t)^4}{c(u)} \right] \left( \int |x-x(t)| \leq \frac{c(u)}{N(t)^4} \left| u_{\geq N}(t, x) \right|^2 \, dx \right)^2 \, dt \\
\geq u \int \int I(t)^{3-4\epsilon_c} \, dt 
\]

for all \( N \leq N_0 \). Thus, continuing from (8.5) or (8.6), we see that in either case, for \( \eta \) sufficiently small depending on \( u \), we get

\[
\int I(t)^{3-4\epsilon_c} \, dt \geq u N^{1-4\epsilon_c} \quad \text{for all } I \subset [0, T_{\max}) \text{ and all } N \leq N_0.
\]

Recalling (8.3), we now reach a contradiction by taking \( I \) sufficiently large inside \([0, T_{\max})\). \( \square \)

Finally, we turn to the

**Proof of Lemma 8.2.** Let \( \eta_0 > 0 \) be a small parameter to be determined later. As \( \inf_{t \in [0, T_{\max})} N(t) \geq 1 \), for \( N_0 = N_0(\eta_0) \) sufficiently small, we can guarantee

\[
\|u_{\leq N}\|_{L^\infty_t L^2_x([0, T_{\max}) \times \mathbb{R}^d)} \leq \eta_0 \quad \text{for } N \leq N_0.
\]

Then, given any \( C(u) > 0 \) and \( N \leq N_0 \), we can use Hölder’s inequality and Sobolev embedding to estimate

\[
\left| \int |x-x(t)| \leq \frac{C(u)}{N(t)^4} \left| u_{\geq N}(t, x) \right|^2 - |u(t, x)|^2 \, dx \right| \leq u \|N(t)^{-2\epsilon_c} \|_{L^\infty_t L^\infty_x} \|u\|_{L^\infty_t L^\infty_x} \leq \eta_0 \left| N(t)^{-2\epsilon_c} \right|
\]

for all \( t \in [0, T_{\max}) \). Thus, if we can show that for \( C(u) \) sufficiently large, we have

\[
\inf_{t \in [0, T_{\max})} N(t)^{2\epsilon_c} \int |x-x(t)| \leq \frac{C(u)}{N(t)^4} \left| u(t, x) \right|^2 \, dx \geq u 1,
\]

we will have (8.3) by choosing \( \eta_0 = \eta_0(u) \) sufficiently small.
Finally, by Hölder and Bernstein, we estimate

\[ \left\| \nabla |x|^s u_{cN(t)} \right\|_{L_p^\infty L_2^2([0,T_{\max}) \times \mathbb{R}^d)} < \eta_0. \]  
(8.8)

We then notice that by Hölder, Bernstein, Sobolev embedding, and (8.8), we have

\[ \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u(t,x)|^2 - |u_{cN(t)}(t,x)|^2 \, dx \lesssim N(t)^{-s} \left\| u_{cN(t)} \right\|_{L_2^\infty |x|^{-s} (\mathbb{R}^d)} \| u(t) \|_{L_2^\infty (\mathbb{R}^d)} \lesssim_{\eta} N(t)^{-2s_c} \]  
(8.9)

for all \( t \in [0,T_{\max}) \). Thus, if we can show that for \( C(u) \) sufficiently large, we have

\[ \inf_{t \in [0,T_{\max})} N(t)^{2s_c} \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u_{cN(t)}(t,x)|^2 \, dx \gtrsim_{\eta} 1, \]  
(8.10)

then (8.7) will follow by taking \( \eta_0 = \eta_0(u) \) sufficiently small.

Let us therefore turn to establishing (8.10). We begin by choosing \( C(u) \) sufficiently large that

\[ \inf_{t \in [0,T_{\max})} \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u(t,x)|^\frac{d}{d-1} \, dx \gtrsim_{\eta} 1 \]

(cf. Remark 1.7). Then, with \( c = c(\eta_0) \) as above, we see by Hölder’s inequality, Sobolev embedding, and (8.8) that

\[ \left| \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u(t,x)|^\frac{d}{d-1} - |u_{cN(t)}(t,x)|^\frac{d}{d-1} \, dx \right| \lesssim \left\| u_{cN(t)} \right\|_{L_2^\infty |x|^{-s} (\mathbb{R}^d)} \| u(t) \|_{L_2^\infty (\mathbb{R}^d)} \lesssim_{\eta} \eta_0 \]  
for all \( t \in [0,T_{\max}) \). Thus for \( \eta_0 = \eta_0(u) \) sufficiently small, we have

\[ \inf_{t \in [0,T_{\max})} \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u_{cN(t)}(t,x)|^\frac{d}{d-1} \, dx \gtrsim_{\eta} 1. \]  
(8.11)

Finally, by Hölder and Bernstein, we estimate

\[ \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u_{cN(t)}(t,x)|^\frac{d}{d-1} \, dx \lesssim \left\| u_{cN(t)}(t) \right\|_{L_2^\infty (\mathbb{R}^d)}^\frac{d}{d-2} \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u_{cN(t)}(t,x)|^2 \, dx \lesssim_{\eta} N(t)^{2s_c} \left\| u(t) \right\|_{L_2^\infty (\mathbb{R}^d)}^\frac{d}{d-2} \left[ \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u_{cN(t)}(t,x)|^2 \, dx \right] \lesssim_{\eta} N(t)^{2s_c} \int_{|x-x(t)| \leq \frac{c(\eta_0)}{N(t)}} |u_{cN(t)}(t,x)|^2 \, dx \]  
(8.12)

for all \( t \in [0,T_{\max}) \). Combining (8.11) and (8.12) now yields (8.10), which completes the proof of Lemma 8.2.
Appendix A. Some Basic Estimates

We collect here some basic estimates that are useful in Section 4. We begin with

**Lemma A.1.** Let $1 < p \leq 2$. Then

$$|a + c|^p - |a|^p - |b + c|^p + |b|^p \lesssim |a - b| |c|^{p-1}$$

(A.1)

for all $a, b, c \in \mathbb{C}$.

**Proof.** Defining $G(z) := |z + c|^p - |z|^p$, the fundamental theorem of calculus gives

$$\text{LHS}(A.1) = \left| (a - b) \int_0^1 G_z(b + \theta(a - b)) d\theta + \bar{G}_z(b + \theta(a - b)) d\theta \right|.$$

Thus, to establish (A.1), it will suffice to establish

$$|G_z(z)| + |\bar{G}_z(z)| \lesssim |c|^{p-1}$$

uniformly for $z \in \mathbb{C}$. That is, we need to show

$$\left| |z + c|^{p-2}(z + c) - |z|^{p-2}z \right| \lesssim |c|^{p-1}$$

uniformly in $z$. If $c = 0$, this inequality is obvious. Otherwise, setting $z = c\zeta$ reduces the problem to showing

$$\left| |z + 1|^{p-2}(z + 1) - |z|^{p-2}z \right| \lesssim 1$$

(A.2)

uniformly in $z$. For $|z| \lesssim 1$, we immediately get (A.2) from the triangle inequality. For $|z| \gg 1$, we can use the fundamental theorem of calculus and the fact that $p \leq 2$ to see

$$\left| |z + 1|^{p-2}(z + 1) - |z|^{p-2}z \right| \lesssim |z|^{p-2} \lesssim 1.$$

Thus, we see that (A.2) holds, which completes the proof of Lemma A.1.

Next, we record a few inequalities in the spirit of [48, §2.3].

**Lemma A.2.** Let $M$ denote the Hardy–Littlewood maximal function, and let $\tilde{\psi}$ denote the convolution kernel of the Littlewood–Paley projection $P_1$. For a fixed function $f, y \in \mathbb{R}^d$, and $N \in 2\mathbb{Z}$, we have

$$\int_{\mathbb{R}^d} N^d |\tilde{\psi}(Ny)| |f(x - y)| \, dy \lesssim M(f)(x),$$

(A.3)

$$\left| \delta_y f_N(x) \right| \lesssim N |y| \{ M(f_N)(x) + M(f_N)(x - y) \},$$

(A.4)

$$\int_{\mathbb{R}^d} N^d |y| |\tilde{\psi}(Ny)| \, dy \lesssim \frac{1}{N}.$$  

(A.5)

**Proof.** We begin with (A.3). Note first that

$$\eta := N^d |\tilde{\psi}(Ny)|$$

is a spherically symmetric, decreasing function of radius; thus, we can write

$$\eta(y) = \int_0^\infty \chi_{B(0,r)}(y)(-\eta'(r)) \, dr,$$
where \( \eta' := \frac{\partial \eta}{\partial r} \). We can then use the definition of the Hardy–Littlewood maximal function and integrate by parts to estimate

\[
LHS(A.3) \lesssim \int_0^\infty \left( \int_{|y| \leq r} |f(x - y)| \, dy \right) (-\eta'(r)) \, dr
\]

\[
\lesssim \left( \int_0^\infty \eta(r)r^{d-1} \, dr \right) M(f)(x)
\]

\[
\lesssim \psi M(f)(x).
\]

For (A.4), we begin by defining \( \psi_0(\xi) = \psi(2\xi) + \psi(\xi) + \psi(\xi/2) \), the ‘fattened’ Littlewood–Paley multiplier. Then we can write

\[
|\delta_y f_N(x)| = \left| \int N^d \tilde{\psi}_0(N(z - y)) - N^d \tilde{\psi}_0(Nz) f_N(x - z) \, dz \right|. \tag{A.6}
\]

If \( N|y| \geq 1 \), we can use the triangle inequality and argue as above to see that

\[
|\delta_y f_N(x)| \leq M(f_N)(x - y) + M(f_N)(x),
\]

giving (A.4) in this case. If instead \( N|y| \leq 1 \), we can use the fact that \( \tilde{\psi}_0 \) is Schwartz to estimate

\[
|\tilde{\psi}_0(N(z - y)) - \tilde{\psi}_0(Nz)| \lesssim N|y|(1 + N|z|)^{-100d}.
\]

Then continuing from (A.4) and once again arguing as for (A.3), we find

\[
|\delta_y f_N(x)| \lesssim N|y|M(f_N)(x),
\]

which gives (A.4) in this case.

Finally, we note that since \( \tilde{\psi} \) is Schwartz, we have

\[
\int_{\mathbb{R}^d} N^d |N y| \, |\tilde{\psi}(N y)| \, dy \lesssim 1,
\]

which immediately gives (A.5). \( \square \)

References

[1] P. Béqout and A. Vargas, Mass concentration phenomena for the \( L^2 \)-critical nonlinear Schrödinger equation. Trans. Amer. Math. Soc. 359 (2007), 5257–5282. MR2327030

[2] J. Bourgain, Refinements of Strichartz’ inequality and applications to 2d-NLS with critical nonlinearity. Int. Math. Res. Not. (1998), 253–283. MR1616917

[3] J. Bourgain, Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. J. Amer. Math. Soc. 12 (1999), 145–171. MR1626257

[4] A. Bulut, Global well-posedness and scattering for the defocusing energy-supercritical cubic nonlinear wave equation. J. Funct. Anal. 263 (2012), 1609–1660. MR2948225

[5] A. Bulut, The radial defocusing energy-supercritical cubic nonlinear wave equation in \( \mathbb{R}^{1+5} \). Preprint [arXiv:1104.2002]

[6] A. Bulut, The defocusing energy-supercritical cubic nonlinear wave equation in dimension five. Preprint [arXiv:1112.0629]

[7] R. Carles and S. Keraani, On the role of quadratic oscillations in nonlinear Schrödinger equations. II. The \( L^2 \)-critical case. Trans. Amer. Math. Soc. 359 (2007), 33-62. MR2247881

[8] T. Cazenave and F. B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in \( H^s \). Nonlinear Anal. 14 (1990), 807–836. MR1055532

[9] T. Cazenave, Semilinear Schrödinger equations. Courant Lecture Notes in Mathematics, 10. American Mathematical Society, 2003. MR1902047

[10] M. Christ and M. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 100 (1991), 87–109. MR1124294

[11] R. R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals. Trans. Amer. Math. Soc. 212 (1975), 315–331. MR0380244
INTER-CRITICAL NLS: CRITICAL $\dot{H}^s$-BOUNDS IMPLY SCATTERING

[12] R. R. Coifman and Y. Meyer, Au-delà des opérateurs pseudo-différentiels, Astérisque 57 (1979). MR0518170
[13] R. R. Coifman and Y. Meyer, Ondlettes et opérateurs III, Ondelettes. Actualités Mathématiques, Hermann, Paris (1991). MR1160989
[14] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$. Comm. Pure Appl. Math. 57 (2004), 987–1014. MR2053757
[15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. Ann. Math. 167 (2008), 767–865. MR2415387
[16] P. Constantin, J.-C. Saut, Local smoothing properties of dispersive equations. J. Amer. Math. Soc. 1 (1988), 413–439. MR0928265
[17] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d \geq 3$. J. Amer. Math. Soc. 25 (2012), 429–463. MR2680923
[18] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 2$. Preprint [arXiv:1006.1375]
[19] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 1$. Preprint [arXiv:1010.0040]
[20] B. Dodson, Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. Preprint [arXiv:1104.1114]
[21] L. Dong, X. Zhang, Stability of solutions for nonlinear Schrödinger equations in critical spaces. Sci. China Math. 54 (2011), 973–986. MR2800921
[22] J. Ginibre and G. Velo, Smoothing properties and retarded estimates for some dispersive evolution equations. Comm. Math. Phys. 144 (1992), 163–188. MR1151250
[23] G. Grillakis, On nonlinear Schrödinger equations. Comm. Partial Differential Equations 15 (1990), 137–150. MR1041466
[24] M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), 955–980. MR1646048
[25] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. Invent. Math. 166 (2006), 645–675. MR2257393
[26] C. E. Kenig and F. Merle, Scattering for $H^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions. Trans. Amer. Math. Soc. 362 (2010), 1937–1962. MR2574882
[27] C. E. Kenig and F. Merle, Non-dispersive radial solutions to energy super-critical non-linear wave equations, with applications. Amer. J. Math. 133 (2011), 1029–1065. MR2823870
[28] S. Keraani, On the defect of compactness for the Strichartz estimates for the Schrödinger equations. J. Diff. Eq. 175 (2001), 353–392. MR1855973
[29] S. Keraani, On the blow up phenomenon of the critical nonlinear Schrödinger equation. J. Funct. Anal. 235 (2006), 171–192. MR2216444
[30] R. Killip, T. Tao, and M. Vişan, The cubic nonlinear Schrödinger equation in two dimensions with radial data. J. Eur. Math. Soc. (JEMS) 11 (2009), 1203–1258. MR2557134
[31] R. Killip and M. Vişan, Nonlinear Schrödinger equations at critical regularity. To appear in proceedings of the Clay summer school “Evolution Equations”, June 23–July 18, 2008, Eidgenössische Technische Hochschule, Zürich.
[32] R. Killip and M. Vişan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. Amer. J. Math. 132 (2010), 361–424. MR2654778
[33] R. Killip and M. Vişan, Energy-supercritical NLS: critical $H^s$-bounds imply scattering. Comm. Partial Differential Equations 35 (2010), 945–987. MR2753625
[34] R. Killip and M. Vişan, The defocusing energy-supercritical nonlinear wave equation in three space dimensions. Trans. Amer. Math. Soc. 363 (2011), 3893–3934. MR2775831
[35] R. Killip and M. Vişan, The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions. Proc. Amer. Math. Soc. 139 (2011), 1805–1817. MR2763767
[36] R. Killip and M. Vişan, Global well-posedness and scattering for the defocusing quintic NLS in three dimensions. To appear in Analysis and PDE. Preprint [arXiv:math/1102.1192]
[37] R. Killip, M. Vişan, X. Zhang, The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher. Analysis and PDE 1 (2008), 229–266. MR2472890
[38] J. Lin and W. Strauss, Decay and scattering of solutions of a nonlinear Schrödinger equation. J. Funct. Anal. 30 (1978), no. 2, 245–263. MR0515228
[39] F. Merle and L. Vega, Compactness at blow-up time for $L^2$ solutions of the critical nonlinear Schrödinger equation in 2D. Int. Math. Res. Not. (1998), 399–425. MR1628235

[40] E. Ryckman and M. Vișan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$. Amer. J. Math. 129 (2007), 1–60. MR2288737

[41] S. Shao, Maximizers for the Strichartz inequalities and Sobolev-Strichartz inequalities for the Schrödinger equation. Electron. J. Differential Equations (2009), 1–13. MR2471112

[42] R. Shen, Global well-posedness and scattering of defocusing energy subcritical nonlinear wave equation in dimension 3 with radial data. Preprint [arXiv:1111.2565]

[43] R. Shen, On the energy subcritical, non-linear wave equation with radial data for $p \in (3,5)$. Preprint [arXiv:1206.2108]

[44] P. Sjölin, Regularity of solutions to the Schrödinger equation. Duke Math. J. 55 (1987), 699–715. MR0904948

[45] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, 43. Princeton University Press, Princeton, NJ, 1993. MR1232192

[46] R. S. Strichartz, Multipliers on fractional Sobolev spaces. J. Math. Mech. 16 (1967), 1031–1060. MR0215084

[47] R. S. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations. Duke Math. J. 44 (1977), 705–774. MR0512086

[48] M. E. Taylor, Tools for PDE. Mathematical Surveys and Monographs, 81. American Mathematical Society, Providence, RI, 2000. MR1766415

[49] T. Tao, Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data. New York J. of Math. 11 (2005), 57–80. MR2154347

[50] T. Tao, Nonlinear dispersive equations. Local and global analysis. CBMS Regional Conference Series in Mathematics, 106. American Mathematical Society, Providence, RI, 2006. MR2233925

[51] T. Tao and M. Vișan, Stability of energy-critical nonlinear Schrödinger equations in high dimensions. Electron. J. Diff. Eqns. (2005), 1–28 MR2174550

[52] T. Tao, M. Vișan, and X. Zhang, Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions. Duke Math. J. 140 (2007), 165–202. MR2355070

[53] T. Tao, M. Vișan, and X. Zhang, Minimal-mass blowup solutions of the mass-critical NLS. Forum Math. 20 (2008), 881–919. MR2445122

[54] L. Vega, Schrödinger equations: pointwise convergence to the initial data. Proc. Amer. Math. Soc. 102 (1988), 874–878. MR0934859

[55] M. Vișan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. Duke Math. J. 138 (2007), 281–374. MR2318286

[56] M. Vișan, The defocusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. Ph.D. Thesis, UCLA, 2006.

[57] M. Vișan, Global well-posedness and scattering for the defocusing cubic NLS in four dimensions. Int. Math. Res. Not. (2011), 1037–1067.