An Improved Setting for Generalized Functions: Fine Ultrafunctions

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Abstract. Ultrafunctions are a particular class of functions defined on a Non Archimedean field \( E \supset \mathbb{R} \). They have been introduced and studied in some previous works (Benci, in Adv Nonlinear Stud 13:461–486, 2013; Benci and Luperi Baglini, in Electron J Differ Equ Conf 21:11–21, 2014; Benci et al., in Adv Nonlinear Anal 10. https://doi.org/10.1515/anona-2017-0225.2; Benci et al., in Adv. Nonlinear Anal 9, 2018). In this paper we develop the notion of fine ultrafunctions which improves the older definitions in many crucial points. Some applications are given to show how ultrafunctions can be applied in studying Partial Differential Equations. In particular, it is possible to prove the existence of ultrafunction solutions to ill posed evolution problems.

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1. Introduction

In many circumstances, the notion of real function is not sufficient to the needs of a theory and it is necessary to extend it. The ultrafunctions are a kind of generalized functions based on a field \( \mathbb{E} \) containing the field of real numbers \( \mathbb{R} \). The field \( \mathbb{E} \) (called field of Euclidean numbers) is a peculiar hyperreal field which satisfies some properties useful for the purposes of this paper.

The ultrafunctions provide generalized solutions to certain equations which do not have any solution, not even among the distributions.

We list some of the main properties of the ultrafunctions:

- the ultrafunctions are defined on a set \( \Gamma \),
  \[ \mathbb{R}^N \subset \Gamma \subset \mathbb{E}^N, \]
  and take values in \( \mathbb{E} \); actually they form an algebra \( V^o \) over the field \( \mathbb{E} \);
- to every function \( f : \mathbb{R}^N \to \mathbb{R} \) corresponds a unique ultrafunction \( f^o : \Gamma \to \mathbb{E} \) that extends \( f \) to \( \Gamma \) and satisfies suitable properties described below;
• there exists a linear functional 
\[ \oint : V^\circ \to \mathbb{E} \]
called pointwise integral such that \( \forall f \in C^0_{c,1} \left( \mathbb{R}^N \right) \),
\[ \oint \circ f(x) dx = \int f(x) dx \]

• there are \( N \) operators 
\[ D_i : V^\circ \to V^\circ, \quad i = 1, \ldots, N \]
called generalized partial derivatives such that \( \forall f \in C^1_{c,1} \left( \mathbb{R}^N \right) \),
\[ \left( \frac{\partial f}{\partial x_i} \right)^\circ = D_i f^\circ \]

• to every distribution \( T \in \mathcal{D}' \) corresponds an ultrafunction \( T^\circ \) such that \( \forall \varphi \in \mathcal{D} \)
\[ \oint T^\circ(x) \varphi^\circ dx = \langle T, \varphi \rangle \]
and
\[ \left( \frac{\partial T}{\partial x_i} \right)^\circ = D_i T^\circ \]

• if \( u \) is the solution of a PDE, then \( u^\circ \) is the solution of the same PDE "translated" in the framework of ultrafunctions.

• \( \Gamma \) is a hyperfinite set (see Sect. 2.4) so that we have enough compactness to prove the existence of a solution for a very large class of equations which includes many ill posed problems.

The ultrafunctions have been recently introduced in [6] and developed in [11, 20],.., [24]. In these papers different models of ultrafunctions have been analyzed and several applications have been provided. In the present paper, we introduce an improved model: the space of fine ultrafunctions. The fine ultrafunctions form an algebra in which the pointwise integral and the generalized derivative satisfy most of the familiar properties that are consistent with the algebraic structure of \( V^\circ \).

In particular, these properties allow to solve many evolution problem in the space \( C^1(\mathbb{E}, V^\circ) \) (see Sects. 3.6 and 5.5).

This paper is organized as follow: in the rest of this introduction, we frame the theory of ultrafunction and expose our point of view on Non-Archimedean Mathematics and on the notion of generalized functions.

In Sect. 2, we present the preliminary material necessary to the rest of the paper. In particular we present an approach to Non Standard Analysis (NSA) suitable for the theory of ultrafunctions. This approach is based on the notion of \( \Lambda \)-limit (see also [6,18]) which leads to the field of Euclidean numbers (see also [19]). This part has been written in such a way to be understood also by a reader who is not familiar with NSA.

In Sect. 3 we recall the notion of ultrafunction, we define the spaces of fine ultrafunctions and of time dependent ultrafunctions.

The main properties of the fine ultrafunctions are examined in Sect. 4.
Section 5 is devoted to some applications that exemplify the use of ultrafunctions in PDE’s.

Section 6 is devoted to the proof that the definition of ultrafunctions is consistent. In fact, even if this definition is based on notions which appear quite natural, the consistency of these notions is a delicate issue. We prove this consistency by the construction of a very involved model; we do not know if a simpler model exists. This section is very technical and we assume the reader to be used with the techniques of NSA.

1.1. Few Remarks on Non-Archimedean Mathematics

The scientific community has always accepted new mathematical entities, especially if these are useful in the modeling of natural phenomena and in solving the problems posed by the technique. Some of these entities are the infinitesimals that have been a carrier of the modern science since the discovery of the infinitesimal calculus at the end of XVII century. But despite the successes achieved with their employment, they have been opposed and even fought by a considerable part of the scientific community (see e.g. [3, 7, 25]). At the end of the 19th century they were placed on a more rigorous basis thanks to the works of Du Bois-Reymond [28], Veronese [42], Levi-Civita [33] and others, nevertheless they were fought (and defeated) by the likes of Russell (see e.g. [35]) and Peano [36]. Also the reception of the Non Standard Analysis created in the ’60s by Robinson has not been as good as it deserved, even though a minority of mathematicians of the highest level has elaborated interesting theories based on it (see e.g. [1, 34, 40]).

Personally, I am convinced that the Non-Archimedean Mathematics is branch of mathematics very rich and allows to construct models of the real world in a more efficient way. Actually, this is the main motivation of this paper.

1.2. Few Remarks on Generalized Function

The intensive use of the Laplace transform in engineering led to the heuristic use of symbolic methods, called operational calculus. An influential book on operational calculus was Oliver Heaviside’s Electromagnetic Theory of 1899 [30]. During the late 1920s and 1930s further steps were taken, very important to future work. The Dirac delta function was boldly defined by Paul Dirac as a central aspect of his scientific formalism. Jean Leray and Sergei Sobolev, working in partial differential equations, defined the first adequate theory of generalized functions and generalized derivative in order to work with weak solutions of partial differential equations. Sobolev’s work was further developed in an extended form by Laurent Schwartz. Today, among people working in partial differential equations, the theory of distributions of L. Schwartz is the most commonly used, but also other notions of generalized functions have been introduced by J.F. Colombeau [27] and M. Sato [37].

After the discovery of Non Standard Analysis, many models of generalized functions based on hyperreal fields appeared. The existence of infinite and infinitesimal numbers allows to relate the delta of Dirac δ to a function which takes an infinite value in a neighborhood of 0 and vanishes in the other points. So, in this context, expression such as $\sqrt{\delta_a}$ or $\delta_a^2$ make absolutely sense.
The literature in this context is quite large and, without the hope to be exhaustive, we refer to the following papers and their references: Albeverio, Fenstad, Hoegh-Krohn [1], Nelson [34], Arkeryd, Cutland, Henson [2], Bottazzi [26], Todorov [41].

1.3. Notations

For the sake of the reader, we list the main notation used in this paper. If $X$ is any set and $\Omega$ is a measurable subset of $\mathbb{R}^N$, then

- $\wp(X)$ denotes the power set of $X$ and $\wp_{fin}(X)$ denotes the family of finite subsets of $X$;
- $|X|$ will denote the cardinality of $X$;
- $B_r(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < r\}$;
- $N_\varepsilon(\Omega) = \{x \in \mathbb{R}^N \mid \text{dist}(x, \Omega) < \varepsilon\}$;
- $\text{int}(\Omega)$ denotes the interior part of $\Omega$ and $\overline{\Omega}$ denotes the closure of $\Omega$;
- $\mathfrak{F}(X, Y)$ denotes the set of all functions from $X$ to $Y$ and $\mathfrak{F}(\Omega) = \mathfrak{F}(\Omega, \mathbb{R})$;
- if $W$ is any function space, then $W_c$ will denote the function space of functions in $W$ having compact support;
- if $W \subset \mathfrak{F}(\mathbb{R}^N)$ is any function space, then $W(\Omega)$ will denote the space of their restriction to $\Omega$.

- $C_0^k(\Omega)$ denotes the set of functions defined on $\Omega$ which have continuous derivatives up to the order $k$;
- $C^{k,1}(\Omega)$ denotes the set of $C^k(\Omega)$-functions whose $k$-derivative is Lipschitz continuous.

- $L^p(\Omega)$ ($L^p_{loc}(\Omega)$) denotes the set of the functions $u$ such that $|u|^p$ is integrable (locally integrable) functions in $\Omega$;
- $L^1(\Omega)$ ($L^1_{loc}(\Omega)$) denotes the usual equivalence classes of integrable (locally integrable) functions in $\Omega$; these classes will be denoted by:

$$[f]_{L^1} = \{g \in L^1_{loc}(\Omega) \mid g(x) = f(x) \text{ a.e.}\}$$

- $H^{k,p}(\Omega)$ denotes the usual Sobolev space of functions defined on $\Omega$;
- $BV(\Omega)$ denotes space of functions of bounded variation defined on $\Omega$;
- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on $\Omega$; $\mathcal{D}'(\Omega)$ denotes the topological dual of $\mathcal{D}(\Omega)$, namely the set of distributions on $\Omega$;
- if $\Omega = \mathbb{R}^N$, when no ambiguity is possible, we will write $\mathcal{D}$, $\mathcal{D}'$, $L^1$, $L^1$, ... instead of $\mathcal{D}(\mathbb{R}^N)$, $\mathcal{D}'(\mathbb{R}^N)$, $L^1(\mathbb{R}^N)$, $L^1(\mathbb{R}^N)$, ...;
- $\mathbb{E}$ will denote the field of Euclidean number which will be defined in Sect. 2.2;
- for any $\xi \in \mathbb{E}^N, \rho \in \mathbb{E}$, we set $\mathfrak{B}_\rho(\xi) = \{x \in \mathbb{E}^N \mid |x - \xi| < \rho\}$;
- $\text{supp}(f)$ denotes the usual notion of support of a function or a distribution in $\mathbb{E}^N$;
- $\text{supp}(f) = \{x \in \Gamma : f(x) \neq 0\}$ denotes the support of a grid function.
\[ \text{mon}(x) = \{ y \in \mathbb{E}^N : x \sim y \} \] where \( x \sim y \) means that \( x - y \) is infinitesimal; the set \( \text{mon}(x) \) is called monad of \( x \) (see Def. 2.8);

\[ \text{gal}(x) = \{ y \in \mathbb{E}^N : x - y \text{ is finite} \} \] the set \( \text{gal}(x) \) is called galaxy of \( x \) (see Def. 2.8);

we denote by \( \chi_X \) the indicator (or characteristic) function of \( X \), namely

\[
\chi_X(x) = \begin{cases} 
1 & \text{if } x \in X \\
0 & \text{if } x \notin X 
\end{cases}
\]

If \( X = \{a\} \) and \( a \) is an atom, then, in order to simplify the notation, we will write \( \chi_a(x) \) instead of \( \chi_{\{a\}}(x) \).

\( \nabla = (\partial_1, \ldots, \partial_N) \) denotes the usual gradient of standard functions;

\( D = (D_1, \ldots, D_N) \) will denote the extension of the gradient in the sense of the ultrafunctions;

\( \nabla \cdot \phi \) will denote the usual divergence of standard vector fields \( \phi \in \mathbb{C}^1 \); 

\( D \cdot \phi \) will denote the extension of the divergence in the sense of ultrafunctions;

\( \Delta \) denotes the usual Laplace operator of standard functions;

\( D^2 \) will denote the extension of the Laplace operator in the sense of ultrafunctions.

2. Preliminary Notions

As we have already remarked in the introduction, in this section, we present the material necessary to the rest of the paper. In particular we present an approach to NSA based on \( \Lambda \)-theory. This part has been written in such a way to be understood also by a reader who is not familiar with NSA. \( \Lambda \)-theory can be considered a different approach to Nonstandard Analysis. It can be introduced via the notion of \( \Lambda \)-limit, and it can be easily used for the purposes of this paper.

2.1. Non Archimedean Fields

Here, we recall the basic definitions and some facts regarding non Archimedean fields.

**Definition 2.1.** A field \( \mathbb{K} \) is called ordered if there is a set \( \mathbb{K}^+ \subset \mathbb{K} \) such that

1. \( x, y \in \mathbb{K}^+ \Rightarrow x + y, xy \in \mathbb{K}^+ \)
2. \( \mathbb{K} = \mathbb{K}^+ \cup \{0\} \cup \mathbb{K}^- \) where \( \mathbb{K}^- = \{ x \in \mathbb{K} | -x \in \mathbb{K}^+ \} \)

In an ordered field the order relation is defined as follows:

\[ x < y \iff y - x \in \mathbb{K}^+ \]

In the following, \( \mathbb{K} \) will denote an ordered field. Its elements will be called numbers. It is well known that every ordered field contains (a copy of) the rational numbers; hence the following definitions makes sense:

**Definition 2.2.** Let \( \mathbb{K} \) be an ordered field. Let \( \xi \in \mathbb{K} \). We say that:

- \( \xi \) is infinitesimal if, for all positive \( n \in \mathbb{N} \), \( |\xi| < \frac{1}{n} \);
- \( \xi \) is finite if there exists \( n \in \mathbb{N} \) such that \( |\xi| < n \);
• $\xi$ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if $\xi$ is not finite).

**Definition 2.3.** An ordered field $\mathbb{K}$ is called Non-Archimedean if it contains an infinite number.

It’s easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number. Infinitesimal numbers can be used to formalize a new notion of “closeness”:

**Definition 2.4.** We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.

Clearly, the relation “$\sim$” of infinite closeness is an equivalence relation.

**Theorem 2.5.** If $\mathbb{K} \supseteq \mathbb{R}$ is an ordered field every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$. $r$ is called the the standard part of $\xi$ and denoted by $st(\xi)$.

Proof. Given a finite number $\xi \in \mathbb{K}$, we set

$$A := \{r \in \mathbb{R} \mid r < \xi\}, B := \{r \in \mathbb{R} \mid r \geq \xi\}$$

We have that $(A, B)$ is a section of $\mathbb{R}$; moreover, since $\xi$ is finite, $A \neq \emptyset$ and $B \neq \emptyset$. Then, by the completeness of the reals $\exists c \in \mathbb{R}$

$$\forall a \in A, \forall b \in B, a \leq c \leq b.$$ 

Now, it is not difficult to check that $c \sim \xi$. \hfill $\blacksquare$

**Corollary 2.6.** If $\mathbb{K} \supseteq \mathbb{R}$, $(\mathbb{K} \neq \mathbb{R})$ is an ordered field, then it is Non-Archimedean.

Proof. Take $\xi \in \mathbb{K}\setminus\mathbb{R}$. If $\xi$ is infinite, then $\mathbb{K}$ is Non-Archimedean by definition. If $\xi$ is finite then

$$\zeta := \frac{1}{\xi - st(\xi)}$$

is infinite, and hence $\mathbb{K}$ is Non-Archimedean. \hfill $\blacksquare$

Now let us examine some (obvious) properties of the function $st(\cdot)$.

**Proposition 2.7.** Let $\xi$ and $\zeta$ be finite numbers, then

1. if $\xi \in \mathbb{R}$, $st(\xi) = \xi$;
2. $\xi \leq \zeta \Rightarrow st(\xi) \leq st(\zeta)$;
3. $st(\xi + \zeta) = st(\xi) + st(\zeta)$;
4. $st(\xi \cdot \zeta) = st(\xi) \cdot st(\zeta)$;
5. if $st(\zeta) \neq 0$, then $st\left(\frac{\xi}{\zeta}\right) = \frac{st(\xi)}{st(\zeta)}$.

Proof. The first four statements can be proved easily. In order to prove (5), we put

$$\xi = r + \varepsilon$$
$$\zeta = s + \theta$$

where $r, s \in \mathbb{E}_{\kappa}$, $\varepsilon \sim \theta \sim 0$. Then,

$$st(\xi \cdot \zeta) = st[(r + \varepsilon)(s + \theta)] = st[rs + (\varepsilon s + \theta r + \varepsilon \theta)]$$
Since \( \varepsilon s + \theta r + \varepsilon \theta \sim 0 \), we have that 
\[
\text{st} (\xi \cdot \zeta) = rs = \text{st} (\xi) \cdot \text{st} (\zeta).
\]
Finally
\[
\text{st} (\frac{\xi}{\zeta}) = \text{st} \left( \frac{\xi}{\zeta} \right) = \text{st} (\xi);
\]
hence
\[
\text{st} \left( \frac{\xi}{\zeta} \right) = \frac{\text{st} (\xi)}{\text{st} (\zeta)}.
\]

**Definition 2.8.** Let \( \mathbb{K} \) be a Non-Archimedean field, and \( \xi \in \mathbb{K} \) a number. The monad of \( \xi \) is the set of all numbers that are infinitely close to it:
\[
\text{mon}(\xi) = \{ \zeta \in \mathbb{K} : \xi \sim \zeta \},
\]
and the galaxy of \( \xi \) is the set of all numbers that are finitely close to it:
\[
\text{gal}(\xi) = \{ \zeta \in \mathbb{K} : \xi - \zeta \text{ is finite} \}.
\]
By definition, it follows that the set of infinitesimal numbers is \( \text{mon}(0) \) and that the set of finite numbers is \( \text{gal}(0) \). Moreover, the standard part can be regarded as a function:
\[
st : \text{gal}(0) \rightarrow \mathbb{R}.
\] (1)

**2.2. A-Theory**

In order to construct a space of ultrafunctions it is useful to take the set \( \Lambda \) sufficiently large; for example a superstructure over \( \mathbb{R} \) defined as follows:
\[
\Lambda = V_\infty (\mathbb{R}) = \bigcup_{n \in \mathbb{N}} V_n (\mathbb{R}),
\]
where the sets \( V_n (\mathbb{R}) \) are defined by induction:
\[
V_0 (\mathbb{R}) = \mathbb{R}
\]
and, for every \( n \in \mathbb{N} \),
\[
V_{n+1} (\mathbb{R}) = V_n (\mathbb{R}) \cup \mathcal{P} (V_n (\mathbb{R})).
\] (2)
Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that \( V_\infty (\mathbb{R}) \) contains every mathematical entities used in PDE’s.

Let
\[
\mathcal{L} = \mathcal{P}_{fin} (\Lambda)
\] (3)
be the family of finite subsets of \( \Lambda \). \( \mathcal{L} \) equipped with the partial order structure “\( \subset \)” is a directed set. A function \( \varphi : \mathcal{L} \rightarrow E \) will be called net (with values in \( E \)). The limit of a net is well defined: for example if \( \varphi : \mathcal{L} \rightarrow \mathbb{R} \), we set
\[
L = \lim_{\lambda \rightarrow \Lambda} \varphi (\lambda)
\] (4)
if and only if, \( \forall \varepsilon \in \mathbb{R}^+ \), \( \exists \lambda_0 \in \mathcal{L} \), such that \( \forall \lambda \supset \lambda_0 \),
\[
|\varphi (\lambda) - L| \leq \varepsilon
\]
Notice that in the notation (4), $\Lambda$ can be regarded as the “point at infinity” of $\mathcal{L}$. A typical example of a limit of a net defined on $\mathcal{L}$ is provided by the definition of the Cauchy integral:

$$\int_{a}^{b} f(x)dx = \lim_{\lambda \to \Lambda} \sum_{x \in [a,b] \cap \lambda} f(x)(x^+ - x); \quad x^+ = \min \{ y \in \mathbb{R} \cap \lambda \mid y > x \}.$$ 

Now we will introduce axiomatically a new notion of limit:

**Axiom 2.9.** There is a field $\mathbb{E} \supset \mathbb{R}$, called field of Euclidean numbers, such that every net

$$\varphi : \mathcal{L} \to V_n(\mathbb{R}), \quad n \in \mathbb{N},$$

has a unique $\Lambda$-limit

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \in V_n(\mathbb{E})$$

which satisfies the following properties:

1. if eventually $\varphi(\lambda) = \psi(\lambda)$, \(^1\) then

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = \lim_{\lambda \uparrow \Lambda} \psi(\lambda);$$

2. if $\varphi_1(\lambda), ... , \varphi_n(\lambda)$ are nets, then

$$\lim_{\lambda \uparrow \Lambda} \{ \varphi_1(\lambda), ... , \varphi_n(\lambda) \} = \left\{ \lim_{\lambda \uparrow \Lambda} \varphi_1(\lambda), ... , \lim_{\lambda \uparrow \Lambda} \varphi_n(\lambda) \right\};$$

3. if $E_\lambda$ is a net of sets, then

$$\lim_{\lambda \uparrow \Lambda} E_\lambda = \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \mid \forall \lambda \in \mathcal{L}, \varphi(\lambda) \in E_\lambda \right\};$$

4. we have that

$$\mathbb{E} := \left\{ \lim_{\lambda \uparrow \Lambda} x_\lambda \mid \forall \lambda \in \mathcal{L}, \ x_\lambda \in \mathbb{R} \right\}$$

and if $x_\lambda, y_\lambda \in \mathbb{R}$, then,

$$\lim_{\lambda \uparrow \Lambda} (x_\lambda + y_\lambda) = \lim_{\lambda \uparrow \Lambda} x_\lambda + \lim_{\lambda \uparrow \Lambda} y_\lambda,$$

$$\lim_{\lambda \uparrow \Lambda} (x_\lambda \cdot y_\lambda) = \lim_{\lambda \uparrow \Lambda} x_\lambda \cdot \lim_{\lambda \uparrow \Lambda} y_\lambda;$$

$$x_\lambda \geq y_\lambda \Rightarrow \lim_{\lambda \uparrow \Lambda} x_\lambda \geq \lim_{\lambda \uparrow \Lambda} y_\lambda. \quad (6)$$

Notice that in order to distinguish the limit (2.10) (which we will call *Cauchy limit*) from the $\Lambda$-limit, we have used the symbols “$\lambda \to \Lambda$” and “$\lambda \uparrow \Lambda$” respectively.

In the rest of this section, we will make some remarks for the readers who are not familiar with NSA. The points 1,2,4 are not surprising since we expect them to be satisfied by any notion of limit provided the target space be equipped with a reasonable topology. The point 3 can be considered as a definition. In axiom 2.9, the new (and, for someone, surprising) fact is that every net has a $\Lambda$-limit. Nevertheless

\(^1\) We say that a relation $\varphi(\lambda) \not\equiv \psi(\lambda)$ eventually holds if $\exists \lambda_0 \in \mathcal{L}$ such that $\forall \lambda \supset \lambda_0, \varphi(\lambda) \not\equiv \psi(\lambda)$. 
Axiom 2.9 is not contradictory and a model for it can be constructed in ZFC (see Sect. 6.1 or [6] or [19] for further details). Probably the first question which a newcomer to the world of NSA would ask is the following: what is the limit of a sequence such that \( \varphi(\lambda) := (-1)^{\mid \lambda \mid} \) since it takes the values +1 if \( \mid \lambda \mid \) is even or -1 if \( \mid \lambda \mid \) is odd. Let us see what Axiom 2.9 tells us. By 2.9.2 and 2.9.1,

\[
\lim_{\lambda \uparrow \Lambda} \{1, -1\} = \left\{ \lim_{\lambda \uparrow \Lambda} 1, \lim_{\lambda \uparrow \Lambda} (-1) \right\} = \{1, -1\}
\]

and by 2.9.3, we have that

\[
\lim_{\lambda \uparrow \Lambda} (-1)^{\mid \lambda \mid} \in \lim_{\lambda \uparrow \Lambda} \{1, -1\};
\]

and hence

\[
\lim_{\lambda \uparrow \Lambda} (-1)^{\mid \lambda \mid} \in \{1, -1\}
\]

Then either \( \lim_{\lambda \uparrow \Lambda} (-1)^{\mid \lambda \mid} = 1 \) or \( \lim_{\lambda \uparrow \Lambda} (-1)^{\mid \lambda \mid} = -1 \). Which alternative occurs cannot be deduced by axiom 2.9; each alternative can be added as an independent axiom. In the models constructed in [19] this limit is +1; however this and similar questions are not relevant for this paper and we refer to the mentioned references for a deeper discussion of this point (in particular see [18] and [19]). Axiom 2.9 is sufficient for our applications.

The second question which a newcomer would ask is about the limit of a divergent sequence such that \( \varphi(\lambda) := \mid \lambda \cap \mathbb{N} \mid \). Let us put

\[
\alpha := \lim_{\lambda \uparrow \Lambda} \mid \lambda \cap \mathbb{N} \mid \quad (7)
\]

What can we say about \( \alpha \)? By (6), \( \alpha \not\in \mathbb{R} \). In order to give a feeling of the “meaning” of \( \alpha \), we will put it in relation with other infinite numbers. If \( E \in \Lambda \backslash \mathbb{R} \), we put

\[
\text{num}(E) = \lim_{\lambda \uparrow \Lambda} \mid E \cap \lambda \mid \quad (8)
\]

If \( E \) is a finite set, the sequence is eventually equal to the number of elements of \( E \); then, by axiom 2.9.1,

\[
\text{num}(E) = \mid E \mid \in \mathbb{N}
\]

If \( E \) is an infinite set, \( \text{num}(E) \not\in \mathbb{N} \). Hence, the limits like (8) give mathematical entities that extend the notion of “number of elements of a set” to infinite sets and it is legitimate to call them “infinite numbers”. The infinite number \( \text{num}(E) \) is called numerosity of \( E \). The theory of numerosities can be considered as an extension of the Cantorian theory of cardinal and ordinal numbers. The reader interested to the details and the developments of this theory is referred to [4,10,19,21,22].

If a real net \( x_\lambda \) admits the Cauchy limit, the relation between the two limits is given by the following identity:

\[
\lim_{\lambda \rightarrow \Lambda} x_\lambda = \text{st} \left( \lim_{\lambda \uparrow \Lambda} x_\lambda \right) \quad (9)
\]

An other important relation between the two limits is the following:
Proposition 2.10. If
\[ \lim_{\lambda \uparrow \Lambda} x_\lambda = \xi \in E \]
and \( \xi \) is bounded, then there exists a sequence \( \lambda_n \in \mathcal{L} \) such that
\[ \lim_{n \to \infty} x_{\lambda_n} = st(\xi). \]

Proof. Set \( x_0 = st(\xi) \) and for every \( n \in \mathbb{N} \), take \( \lambda_n \) such that \( x_{\lambda_n} \in B_{1/n}(x_0) \). □

Remark 2.11. As we have already remarked, the field of Euclidean numbers is a hyperreal field in the sense of Non Standard Analysis. We do not use the name “hyperreal numbers” to emphasize the fact that \( E \) has been defined by the notion of \( \Lambda \)-limit and hence it satisfies some properties which are not shared by other hyperreal fields. These properties are relevant in the definitions of ultrafunctions. The explanation of the choice of the name “Euclidean numbers” can be found in [21].

2.3. Extension of Sets and Functions
In this section we recall some basic notions of Non Standard Analysis presented in the framework of \( \Lambda \)-theory. Given a set \( A \in \Lambda \), we define
\[ A^* = \left\{ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \mid \forall \lambda, \varphi(\lambda) \in A \right\}; \quad (10) \]
Following Keisler [31], \( A^* \) will be called the natural extension of \( A \). By (5), we have that \( \mathbb{R}^* = E \). If we identify a relation \( R \) or a function \( f \) with its graph, then, by (10) \( R^* \) and \( f^* \) are well defined.

In particular any function
\[ f : A \to B, \quad A, B \in \Lambda, \]
can be extended to \( A^* \) and we have that
\[ f^* \left( \lim_{\lambda \uparrow \Lambda} x_\lambda \right) = \lim_{\lambda \uparrow \Lambda} f(x_\lambda); \quad (11) \]
the function
\[ f^* : A^* \to B^*, \]
will be called natural extension of \( f \). More in general, if
\[ u_\lambda : A \to B \]
is a net of functions, we have that for any \( x = \lim_{\lambda \uparrow \Lambda} x_\lambda \), \( x_\lambda \in B \),
\[ u(x) = \lim_{\lambda \uparrow \Lambda} u_\lambda(x_\lambda) \]
(12)
is a function from \( A^* \) to \( B^* \).

Example. Let \( f : \mathbb{R}^N \to \mathbb{R} \) be a differentiable function, then
\[ \partial_i^* f^* = (\partial_i f)^* \]
where the operator
\[ \partial_i = \frac{\partial}{\partial x_i} : C^1(\mathbb{R}^N) \to C^0(\mathbb{R}^N) \]
is regarded as a function between functional spaces and hence
\[ \partial^*_i : C^1(\mathbb{R}^N)^* \to C^0(\mathbb{R}^N). \]

Following the current literature in NSA, we give the following definition:

**Definition 2.12.** A set \( E_\Lambda \) obtained as \( \Lambda \)-limit of a net of sets \( E_\lambda \in \Lambda \) is called internal.

In particular, if \( E \in \Lambda \), if you compare Axiom 2.9.3 with (10), then you see that
\[ E^* = \lim_{\lambda \uparrow \Lambda} E_\lambda \]
in the case in which \( E_\lambda \) is the net identically equal to \( E \).

Let us see an example of external set i.e. of a set which is not internal. By axiom 2.9.3, the set \( E^* \) contains a unique \textit{“copy”} \( x^* \) of every element \( x \in E \). Now set
\[ E^\sigma := \{ x^* \in E^* \mid x \in E \}. \tag{13} \]
We have that \( E^\sigma \subseteq E^* \) and the equality holds if and only if \( E \) is finite. It is easy to see that if \( E \) is infinite \( E \) and \( E^\sigma \) are external.

**Example.** Let \( E = C^0(\mathbb{R}) \); then \( C^0(\mathbb{R})^\sigma \subset C^0(\mathbb{R})^* \). If we take
\[ \sin^*(x) = \lim_{\lambda \uparrow \Lambda} \sin(x_\lambda), \quad x = \lim_{\lambda \uparrow \Lambda} x_\lambda \]
and, using the notation (7),
\[ \sin^*(\alpha x) = \lim_{\lambda \uparrow \Lambda} \sin(|\lambda \cap \mathbb{N}| \cdot x_\lambda), \]
we have that \( \sin^*(x) \in C^0(\mathbb{R})^\sigma \subset C^0(\mathbb{R})^* \) and \( \sin^*(\alpha x) \in C^0(\mathbb{R})^* \setminus C^0(\mathbb{R})^\sigma \).

If \( V \) is a vector space, making some abuse of notation, we set
\[ V^\odot = \text{span} \left[ \lim_{\lambda \uparrow \Lambda} (E \cap \lambda) \right] \]
namely \( V^\odot \) is a hyperfinite vector space containing \( V^\sigma \)

### 2.4. Hyperfinite Sets

An other fundamental notion in NSA is the following:

**Definition 2.13.** We say that a set \( F \in \Lambda \) is hyperfinite if there is a net \( \{F_\lambda\}_{\lambda \in \Lambda} \) of finite sets such that
\[ F = \lim_{\lambda \uparrow \Lambda} F_\lambda = \left\{ \lim_{\lambda \uparrow \Lambda} x_\lambda \mid x_\lambda \in F_\lambda \right\} \]

The hyperfinite sets share many properties of finite sets. For example, a hyperfinite set \( F \subset \mathbb{E} \) has a maximum \( x_M \) and a minimum \( x_m \) respectively given by
\[ x_M = \lim_{\lambda \uparrow \Lambda} \max F_\lambda; \quad x_m = \lim_{\lambda \uparrow \Lambda} \min F_\lambda \]
Moreover, it is possible to “add” the elements of an hyperfinite set of numbers. If \( F \) is an hyperfinite set of numbers, the hyperfinite sum of the elements of \( F \) is defined as follows:

\[
\sum_{x \in F} x = \lim_{\lambda \uparrow \Lambda} \sum_{x \in F_{\lambda}} x.
\]

One of the advantage to use the field of Euclidean numbers rather than a generic hyperreal field lies in the possibility to associate a unique hyperfinite set \( E^{\circledast} \) to any set \( E \in V_{\infty}(\mathbb{R}) \) according to the following definition:

**Definition 2.14.** Given a set \( E \subset \Lambda \), the set

\[
E^{\circledast} := \lim_{\lambda \uparrow \Lambda} (E \cap \lambda)
\]

is called hyperfinite extension of \( E \).

To any set \( E \) we can associate the sets \( E^{\sigma} \), \( E^{\circledast} \) and \( E^{*} \) which are ordered as follows:

\[
E^{\sigma} \subseteq E^{\circledast} \subseteq E^{*};
\]

the inclusions are strict if and only if the set \( E \) is infinite. If \( F = \lim_{\lambda \uparrow \Lambda} F_{\lambda} \) is a hyperfinite set, its hypercardinality is defined by

\[
|F|^{*} := \lim_{\lambda \uparrow \Lambda} |F_{\lambda}|
\]

Notice that the hypercardinality of \( E^{\circledast} \), defined by

\[
|E^{\circledast}|^{*} = \lim_{\lambda \uparrow \Lambda} |E \cap \lambda|,
\]

is the numerosity of \( E \) as it has been defined by (8).

If \( V \) is a vector space, making some abuse of notation, we set

\[
V^{\circledast} = \text{span} \left[ \lim_{\lambda \uparrow \Lambda} (E \cap \lambda) \right]
\]

namely \( V^{\circledast} \) is a hyperfinite vector space containing \( V^{\sigma} \).

### 2.5. Grid Functions

**Definition 2.15.** A hyperfinite set \( \Gamma \) such that \( \mathbb{R}^{N} \subset \Gamma \subset \mathbb{E}^{N} \) is called hyperfinite grid.

For example the set \( \mathbb{R}^{N}^{\circledast} \) is an hyperfinite grid.

**Definition 2.16.** A space of grid functions is a family \( \mathcal{G}(\Gamma) \) of internal functions

\[
u : \Gamma \to \mathbb{R}
\]

If \( w \in \mathcal{F}(\mathbb{R}^{N})^{*} \), the restriction of \( w \) to \( \Gamma \) is a grid function which we will denote by \( w^{\circledast} \) namely, if \( w = \lim_{\lambda \uparrow \Lambda} w_{\lambda} \) and \( x = \lim_{\lambda \uparrow \Lambda} x_{\lambda} \in \Gamma \), we have that

\[
w^{\circledast}(x) = \lim_{\lambda \uparrow \Lambda} w_{\lambda}(x_{\lambda}).
\]

For every \( a \in \Gamma \),

\[
\chi_{a}(x) \in \mathcal{G}(\Gamma)
\]
is a grid function, and hence every grid function can be represented by the following sum:

\[ u(x) = \sum_{a \in \Gamma} u(a) \chi_a(x) \]  

(15)

\[ \text{namely } \{ \chi_a \}_{a \in \Gamma} \text{ is a basis for } \mathcal{G}(\Gamma) \text{ considered as a vector space over } \mathbb{E}. \]

Given \( f \in \mathcal{F}(\mathbb{R}^N) \), we will write \( f^\circ \) instead of \( (f^\star)^\circ \), namely

\[ f^\circ(x) := \lim_{\lambda \uparrow \Lambda} f(x_\lambda) = \sum_{a \in \Gamma} f^\star(a) \chi_a(x). \]  

(16)

so, \( \mathcal{G}(\Gamma) \) contains a unique copy \( f^\circ \) of every function \( f \in \mathcal{F}(\mathbb{R}^N) \). If a function, such as \( 1/|x| \) is not defined in some point and \( x \in \Gamma \), we put \( (1/|x|)^\circ \) equal to 0 for \( x = 0 \); in general, if \( \Omega \) is a subset of \( \mathbb{R}^N \) and \( f \) is defined in \( \Omega \), we set

\[ f^\circ(x) = \sum_{a \in \Omega^\circ} f^\star(a) \chi_a(x) \]  

(17)

where for every set \( E \subset \mathbb{R}^N \), we define

\[ E^\circ = E^\star \cap \Gamma. \]  

(18)

3. Ultrafunctions

If we have a differential equation, it is relatively easy to find an approximated solution in a suitable space of grid functions. If this equation has a “classic” solution, this solution, in some sense, approximates the classic solution. Then, if we take a grid, the “grid solutions” is almost equal to the classic solution. However the “grid solutions” cannot be considered as generalizations of the classic solutions since they do not coincide with them. The theory of ultrafunctions is based on the idea of a space of functions (defined on a hyperfinite grid) in which the generalized derivative and the generalized integral coincide with the usual ones for every function \( f \) in \( C^1 \) and in \( C^0 \) respectively. This fact implies that a “ultrafunction solution” coincides with the classical one if the latter exists and hence it is legitimate to be considered a generalized solution.

3.1. Definition of Ultrafunctions

Let \( V = V(\mathbb{R}^N) \) be a function space such that

\[ \mathcal{D}(\mathbb{R}^N) \subset V \subset \mathcal{L}^1_{loc}(\mathbb{R}^N) \]

and let \( \{ V_\lambda \}_{\lambda \in \mathcal{L}} \) be a net of finite dimensional subspaces of \( V \) such that

\[ \bigcup_{\lambda \in \mathcal{L}} V_\lambda = V. \]

Now we set

\[ V_\Lambda = V_\Lambda(\mathbb{E}^N) = \lim_{\lambda \uparrow \Lambda} V_\lambda = \left\{ \lim_{\lambda \uparrow \Lambda} u_\lambda \mid u_\lambda \in V_\lambda \right\}; \]
$V_\Lambda$ is an internal vector space of hyperfinite dimension. Clearly $V_\Lambda \subset V^*$ since

$$V^* = \left\{ \lim_{\lambda \uparrow \Lambda} u_\lambda \mid u_\lambda \in V \right\}.$$  

The space $V_\Lambda$ allows to equip a space of grid functions $\mathfrak{G}(\Gamma)$ of a richer structure:

**Definition 3.1.** A space of ultrafunctions $V^\circ(\Gamma)$ modelled on $V_\Lambda$ is a family of grid functions $G(\Gamma)$ such that the restriction map

$$^\circ : V_\Lambda \to V^\circ(\Gamma)$$  

is an internal isomorphism between hyperfinite dimensional vector spaces.

So, $u \in V^\circ(\Gamma)$ if and only if there exists a net $u_\lambda \in V_\Lambda(\mathbb{R}^N)$ such that

$$u = \left( \lim_{\lambda \uparrow \Lambda} u_\lambda \right)^\circ$$

In the following of this paper, if $u \in V^\circ(\Gamma)$, such a net will be denoted by $u_\lambda$. We will denote by $\sigma_a(x)$ the only function in $V_\Lambda(\mathbb{E}^N)$ such that

$$\sigma_a^\circ = \chi_a$$

Clearly $\{\sigma_a(x)\}_{a \in \Gamma}$ is a basis of $V_\Lambda(\mathbb{E}^N)$ which will be called $\sigma$-basis. The $\sigma_a$’s allow to write the inverse of the map (19)

$$(\cdot)_\Lambda : V^\circ(\Gamma) \to V_\Lambda(\mathbb{E}^N)$$

as follows: if $u \in V^\circ(\Gamma)$,

$$u_\Lambda(x) := \sum_{a \in \Gamma} u(a)\sigma_a(x).$$

If $f \in \mathfrak{F}(\mathbb{E}^N)$, in order to simplify the notation, we will write $f_\Lambda$ instead of $(f^\circ)_\Lambda$, namely we have that

$$f_\Lambda(x) = \sum_{a \in \Gamma} f^*(a)\sigma_a(x).$$

Notice that

$$f_\Lambda = f^* \iff f \in V(\mathbb{R}^N)$$

More in general, if $w \in \mathfrak{F}(\mathbb{E}^N)^*$, we will write

$$w_\Lambda(x) = \sum_{a \in \Gamma} w(a)\sigma_a(x),$$

In this case, the map

$$^\circ : \mathfrak{F}(\mathbb{R}^N)^* \to V_\Lambda(\mathbb{E}^N)$$

is just a projection.

If $u \in V^\circ(\Gamma)$ and $u_\Lambda \in L^1(\mathbb{R}^N)^*$, the integral can be defined as follows:

$$\int u(x)dx := \int^* u_\Lambda(x)dx = \lim_{\lambda \uparrow \Lambda} \int u_\lambda(x)dx.$$  

$$\int u(x)dx := \int^* u_\Lambda(x)dx = \lim_{\lambda \uparrow \Lambda} \int u_\lambda(x)dx.$$
We will refer to

\[ \oint : V^\circ(\Gamma) \to \mathbb{E} \]

as to the \textbf{pointwise integral}. The reason of this name is due to the fact that (15) and (25) imply that

\[ \oint u(x)dx = \sum_{a \in \Gamma} u(a)d(a) \quad (26) \]

where

\[ d(a) := \oint \chi_a(x)dx = \int^* \sigma_a(x)dx. \quad (27) \]

We may think of \( d(a) \) as the “measure” of the point \( a \in \Gamma \). The pointwise integral extends the usual Lebesgue integral from \( V \) to \( V^\circ \), more exactly, if \( f \in V \cap L^1 \), then

\[ \oint f^\circ(x)dx = \int f(x)dx \quad (28) \]

However the equality above is not true for every Lebesgue integrable function. In fact, if \( a \in \mathbb{R}^N \),

\[ \int \chi_a(x)dx = 0 \]

but, by (26), we have that

\[ \oint \chi_a^\circ(x)dx > 0 \]

at least for some \( a \in \mathbb{R}^N \). This fact is quite natural, in fact when we work in a non-Archimedean world infinitesimals matter and cannot be forgotten as the Riemann and the Lebesgue integrals do. Also the above inequality shows that it is necessary to use a different symbol to distinguish the pointwise integral from the Lebesgue integral (here we have used \( \oint \)).

Given \( u \in V^\circ \), if \( u_\lambda \in C^1 \cap V_\lambda \), and \( \partial_i u_\lambda \in V_\lambda \), it is natural to define the \textbf{partial derivative} in a point \( x = \lim_{\lambda \uparrow \Lambda} x_\lambda \in \Gamma \), as follows

\[ D_iu(x) := [\partial_i^* u_\lambda(x)]^\circ = \lim_{\lambda \uparrow \Lambda} \partial_i u_\lambda(x_\lambda). \quad (29) \]

So, if \( f \in C^1 \cap V \), and \( \partial f \in C^0 \cap V \), we have

\[ D_if^\circ(x) = [\partial_i^* f^\ast (x)]^\circ \]

and hence, if \( x \in \mathbb{R} \), \( D_if^\circ(x) = \partial_i f(x) \).

In particular, if we choose \( V = C^1_c \), we have that the integral and the derivative are defined for every ultrafunction in \( V^\circ \).
3.2. Epilogic Functions and the Space $V$

There are many ultrafunction spaces which depend on the choice of the space $V$, the net $\{V_\lambda\}$ and the grid $\Gamma$. However there are some basic properties which should be satisfied by a “good” space of ultrafunctions which make the theory rich and flexible. In order to get such a space, the first step consists in choosing an appropriate space $V$. The simplest choice is to take $V = C^0_1$ so that the pointwise integral and the generalized derivative be well defined for every ultrafunctions just using (36) and (29). However this choice is too restrictive for many applications. In fact it is useful to work with the characteristic function $\chi_\Omega$ of an open set at least when $\partial \Omega$ is sufficiently smooth. Then, it seems reasonable to work in $BV \cap L^\infty$. Unfortunately this space is not suitable since a function $f \in L^\infty$ is not pointwise defined and hence the map (9) is not well defined. However we can overcome this difficulty taking a space isomorphic to $L^\infty$ by choosing one function in each equivalence class of $L^\infty$.

This choice must be done in a way consistent with the main operations in $L^\infty$. Let us see how to do it.

We start recalling the following standard terminology: for every function $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ we say that a point $x \in \mathbb{R}^N$ is a Lebesgue point for $f$ if

$$f(x) = \lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)dy,$$

where $m(B_r(x))$ is the Lebesgue measure of the ball $B_r(x)$; we recall the very important Lebesgue theorem (see e.g. [32]), that we will need in the following:

**Theorem 3.2.** If $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ then a.e. $x \in \mathbb{R}^N$ is a Lebesgue point for $f$.

Now, we fix once for ever an infinitesimal number $\eta > 0$; for example we can choose $\eta = \alpha^{-1}$ (see (7)). Given a function $f \in L^\infty_{\text{loc}}$, we set

$$\overline{f}(x) = \text{st} \left( \frac{1}{m(B_\eta(x))} \int_{B_\eta(x)} f(y)dy \right),$$

(30)

where $m(B_\eta(x))$ is the Lebesgue measure of the ball $B_\eta(x)$. By this trick we can choose a unique function $\overline{f}$ in the equivalence class $[f]_{L^\infty} \in L^\infty$.

**Lemma 3.3.** The operator $f \mapsto \overline{f}$ satisfies the following properties:

1. if $x$ is a Lebesgue point for $f$ then $\overline{f}(x) = f(x)$;
2. $f(x) = \overline{f}(x)$ a.e.;
3. if $f(x) = g(x)$ a.e. then $\overline{f}(x) = \overline{g}(x)$;
4. $\overline{f}(x) = \overline{f}(x)$.
5. $\overline{f} + g = \overline{f} + \overline{g}$.

**Proof.** (1) If $x$ is a Lebesgue point for $f$ then

$$\frac{1}{m(B_\eta(x))} \int_{B_\eta(x)} f(y)dy \sim f(x),$$

so $\overline{f}(x) = f(x)$.

(2) This follows immediately by Theorem 3.2 and (1).
(3) Let \( x \in \mathbb{R}^N \). Since \( f(x) = g(x) \) a.e., we obtain that \( \int_{B_{\eta}(x)} f(y) dy = \int_{B_{\eta}(x)} g(y) dy \), so 
\[
\overline{f}(x) = \text{st} \left( \frac{1}{m(B_{\eta}(x))} \int_{B_{\eta}(x)} f(y) dy \right) 
\]
\[
= \text{st} \left( \frac{1}{m(B_{\eta}(x))} \int_{B_{\eta}(x)} g(y) dy \right) = \overline{g}(x).
\]

(4) This follows from (2) and (3).

(5) Follows from the linearity of the function \( \text{st} \).

We now define the space of epilogic functions as follows:

\[
EPL := \{ u \in L^\infty_{\text{loc}} \mid \overline{u}(x) = u(x) \}
\]

This name comes from the greek \( \epsilon\pi\lambda\gamma\eta = \text{“choice”} \) since each function in \( EPL \) has been chosen in an equivalence class of \( L^\infty_{\text{loc}} \).

We list some properties of \( EPL \) that will be useful in the following:

**Proposition 3.4.** The following properties hold:

1. if \( u, v \in EPL \) then \( u = v \) a.e. if and only if \( u = v \);
2. \( EPL \) is a module over the ring \( C^0 \), namely, \( \varphi \in C^0 \) and \( f \in EPL \) implies \( \varphi f \in EPL \);
3. the \( L^2 \) norm is a norm for \( EPL \cap L^2 \) (and not a pseudonorm).

**Proof.** (1) Let \( u, v \in EPL \). If \( u = v \) then clearly \( \overline{u} = \overline{v} \) a.e.; conversely, let us suppose that \( u = v \) a.e.; by Lemma 3.3, (3) we deduce that \( \overline{u}(x) = \overline{v}(x) \). But \( u, v \in EPL \), so \( u(x) = \overline{u}(x) = \overline{v}(x) = v(x) \).

(2) if \( f(x) \in EPL \), then for every \( \varphi \) in \( C^k \),

\[
\overline{\varphi}(x)f(x) = \text{st} \left( \frac{1}{m(B_{\eta}(x))} \int_{B_{\eta}(x)} \varphi(y)f(y) dy \right) 
\]
\[
\sim \text{st} \left( \frac{1}{m(B_{\eta}(x))} \int_{B_{\eta}(x)} \varphi(x)f(y) dy \right) 
\]
\[
= \varphi(x) \text{st} \left( \frac{1}{m(B_{\eta}(x))} \int_{B_{\eta}(x)} f(y) dy \right) 
\]
\[
= \varphi(x) \overline{f(x)} = \varphi(x)f(x).
\]

(3) Let \( u \in EPL \) be such that \( \|u\|_{L^2} = 0 \). Then \( u = 0 \) a.e. and since \( 0 \in EPL \), by (1), we deduce that \( u = 0 \).

**Example.** If \( \Omega \subset \mathbb{R}^N \) is a measurable set, the density function of \( \Omega \) is defined as follows:

\[
\Theta_{\Omega}(x) = \lim_{r \to 0^+} \frac{m(B_r(x) \cap \Omega)}{m(B_r(x))},
\]
Hence, by the Lebesgue theorem we have that $\Theta_\Omega(x)$ is defined a.e. Using the operator (30), we can define $\Theta_\Omega$ in every point by setting

$$\Theta_\Omega(x) = \bar{\chi}_\Omega(x).$$  \hspace{1cm} (31)

Clearly $\bar{\chi}_\Omega(x)$ is a function whose value is 1 in $\text{int}(\Omega)$ and 0 in $\mathbb{R}^N \setminus \overline{\Omega}$. If $\Omega$ is an open set with smooth boundary, we have that $\forall x \in \mathbb{R}^N$

$$\bar{\chi}_\Omega(x) = \begin{cases} 
1 & \text{if } x \in \Omega; \\
0 & \text{if } x \notin \Omega; \\
\frac{1}{2} & \text{if } x = \partial \Omega.
\end{cases} \hspace{1cm} (32)$$

The next ingredient necessary to define $V$ is the space of the function of bounded variation $BV$. We recall that $f \in BV$ if $f \in L^1$ and its derivative $\partial_i f$ (in the sense of distributions) is a Radon measure, namely, for every continuous function $\varphi$, the functional $\varphi \mapsto \langle \partial_i f, \varphi \rangle$ is well defined; it is well known that this measure can be extended to every Borellian function and, with some abuse of notation, we will write

$$\int g(x) \partial_i f \, dx$$  \hspace{1cm} (33)

rather than $\int g(x) \, d(\partial_i f)$, since for $\partial_i f \in L^1$, $\partial_i f$ coincides with a measure density and hence

$$\int g(x) \, d(\partial_i f) = \int g(x) \partial_i f(x) \, dx$$

Finally, we can define the space $V$ by setting

$$V = BV \cap EPL \cap R$$  \hspace{1cm} (34)

where $R$ is the space of Riemann integrable function (we recall that in the usual definition the function in $R$ have compact support). From now on $V$ will denote the space (34) and $\partial_i f$ will denote the $BV$-derivative of $f$.

The space $V$ is suitable for our purposes; we have that

$$C_c^{0,1} \subset V$$

and that $\bar{\chi}_\Omega \in V$ provided that $\chi_\Omega \in R$ and $\Omega$ is a Caccioppoli set, namely $\chi_\Omega \in BV$. Moreover, by Prop. 3.4.2, $V$ is a $C_c^{0,1}$ module:

$$\varphi \in C^{0,1}, \ f \in V \Rightarrow \varphi f \in V.$$  

### 3.3. Definition of Fine Ultrafunctions

Roughly speaking a space of ultrafunctions is fine if many of the properties of standard functions are satisfied.

**Definition 3.5.** A space of ultrafunctions $V^\circ$, $V = BV \cap EPL \cap R$, is called fine if

1. if $u, v \in V^\circ$, then

$$\overline{u \lambda v \lambda} \in V_\lambda$$  \hspace{1cm} (35)
2. there is a linear internal functional
\[ \oint : V^o \to E \]
called \textbf{pointwise integral} which satisfies the following properties:
(a) if \( u = \lim_{\lambda \uparrow \Lambda} u_\lambda, \ u_\lambda \in V_\lambda, \) then
\[ \oint u(x)dx = \lim_{\lambda \uparrow \Lambda} \int u_\lambda(x)dx = \int^* u_\Lambda dx. \] (36)
(b) for every \( a \in \Gamma, \)
\[ \oint \chi_a(x)dx > 0; \] (37)
(c) if \( f \in L^1, \) then,
\[ \oint f^o(x) \ dx \sim \int f(x) \ dx. \] (38)

3. there are \( N \) internal operators
\[ D_i : V^o \to V^o, \ i = 1, \ldots, N \]
called \textbf{generalized partial derivatives} such that the following properties are fulfilled:
(a) for every \( u, v \in V^o, \)
\[ \oint D_i u(x)v(x)dx = \lim_{\lambda \uparrow \Lambda} \int \partial_i u_\lambda(x)v_\lambda(x)dx = \int^* (\partial^*_i u_\Lambda) v_\Lambda dx; \] (39)
(b) for every \( a \in \Gamma, \)
\[ \text{supp} [D_i \chi_a] \subset \text{mon}(a). \] (40)

Now some comments on Def. 3.5.
Assumptions (1) states that \( V^o \) is sufficiently large with respect to
\[ V^o = \text{span} \lim_{\lambda \uparrow \Lambda} V \cap \lambda. \]
This is a technical assumption which simplifies the definition of regular ultrafunctio-ns (see section 3.5).
Assumption (2a) is nothing else but the definition of the pointwise integral. Assumption (2b) is quite natural, but it is not satisfied by every space of ultrafunctio-ns; it is very important and, among other things, it implies that the bilinear form \( (u, v) \mapsto \oint uv \ dx \) defines a scalar product (see section 3.4).

Assumption (3a) defines the generalized derivative. By (39) we get that for every function \( f \in C^{1,1} \) and every \( x \in \mathbb{R}^N, \) (see Corollary 3.10),
\[ D_i f^o (x) = \partial_i f(x); \] (41)
hence the generalized derivative extends the usual derivative; by (16) we have that, \( \forall x \in \Gamma, \)
\[ D_i f^o (x) = \sum_{a \in \Gamma} \partial^*_i f^* (a) \chi_a (x). \]
Moreover,
\[ \int D_i u(x)v(x)dx = - \int D_i v(x)u(x)dx \]  
(42)

since
\[ \int D_i u(x)v(x)dx = \lim_{\lambda \uparrow \Lambda} \int \partial_i u_\lambda(x)v_\lambda(x)dx \]
\[ = - \lim_{\lambda \uparrow \Lambda} \int u_\lambda(x)\partial_i v_\lambda(x)dx = - \int D_i v(x)u(x)dx \]

Equation (42) is of primary importance in the theory of weak derivatives, distributions, calculus of variations etc. Usually this equality is deduced by the Leibniz rule

\[ D(fg) = Df g + f Dg \]

However, it is inconsistent to assume that Leibniz rule be satisfied by every pair of ultrafunction (see [39] and the discussion in Sect. 3.5). Nevertheless the identity (42) holds for the fine ultrafunctions. In particular, by (42) and (41), we have that \( \forall u \in V^\circ, \forall \varphi \in D, \)
\[ \int D_i u \varphi^\circ dx = - \int u D_i \varphi^\circ dx = - \int u (\partial_i \varphi)^\circ dx; \]
this equality relates the generalized derivative to the notion of weak derivative.

Property (3b) is a natural request and you expect that it is satisfied by (39); on the contrary, it does not follows from the other properties and it needs to be stated explicitly.

While the construction of a generic space of ultrafunction is a relatively easy task, the construction of a space of fine ultrafunctions is much more delicate. We have the following theorem which is one of the main results of this paper:

**Theorem 3.6.** The requests of Def. 3.5 are consistent.

The proof of this theorem is rather involved and it will be given in section 6 (see Th. 6.23). The rest of this section and section 4 will be devoted in showing that a fine space ultrafunctions provides a quite rich structure and many interesting and natural properties can be proved in a relative simple way.

**Remark 3.7.** A space of ultrafunctions cannot be uniquely defined, since its existence depends on \( \mathbb{E} \) and hence on Zorn’s Lemma. However, if we exclude the choice of the space \( V \), the properties required in Def. 3.5 are quite natural; hence, if a physical phenomenon is modelled by ultrafunctions, the properties which can be deduced can be considered reliable.

From now on, we will treat only with fine ultrafunctions and the word “fine” will be usually omitted.
3.4. The Pointwise Scalar Product of Ultrafunctions

By (37), we have that \( d(a) > 0 \) and hence, the pointwise integral allows to defines the following scalar product which we will call **pointwise scalar product**:

\[
\oint u(x)v(x)dx = \sum_{x \in \Omega} u(x)v(x)d(x).
\]

(43)

If \( f, g, fg \in V \), we have that

\[
\oint f \circ g \circ dx = \int f^* g^* dx = \int fg \ dx;
\]

(44)

however we must be careful since, for some \( u, v \in V^o \) such that \( u_\Lambda v_\Lambda \notin V_\Lambda \), we might have that

\[
\oint uv \ dx \neq \int u_\Lambda v_\Lambda \ dx.
\]

(45)

even if these two quantities are not too different.

**Example.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set. We have that

\[
\oint \chi_\Omega \chi_\Omega \ dx \neq \oint \chi^*_\Omega \chi^*_\Omega \ dx;
\]

in fact

\[
\oint \chi^*_\Omega \chi^*_\Omega \ dx - \oint \chi_\Omega \chi_\Omega \ dx = \oint (\chi^*_\Omega - \chi_\Omega) \ dx = \oint \chi_{\partial\Omega} \ dx > 0
\]

since \( \chi_{\partial\Omega} > 0 \). Hence, it is not possible that

\[
\oint f^\circ g^\circ \ dx = \int fg \ dx
\]

for all measurable functions.

In any case, by (35), if \( u, v \in (V \cap C^0)^\otimes \)

\[
u, v \in V_\Lambda \Rightarrow \oint uv \ dx = \lim_{\lambda \uparrow \Lambda} \int u_\lambda v_\lambda \ dx = \int u_\Lambda v_\Lambda \ dx.
\]

(46)

The pointwise scalar product allows to get the ultrafunction analogous of the Riesz representation theorem in the following form:

**Theorem 3.8.** If

\[
\Phi : V^o \to \mathbb{E}
\]

is an internal linear functional, there exists an unique ultrafunction \( u_\Phi \) such that, \( \forall v \in V^o \),

\[
\Phi(v) = \oint u_\Phi v \ dx
\]

**Proof.** The scalar product (43) is the \( \Lambda \)-limit of a net of scalar products \( \langle \cdot, \cdot \rangle_\Lambda \) defined over \( V_\Lambda \). Since \( \Phi \) is an internal functional, there exists a net of functionals \( \Phi : V_\Lambda \to \mathbb{R} \) such that

\[
\Phi = \lim_{\lambda \uparrow \Lambda} \Phi_\lambda,
\]
and hence \( \exists u_{\Phi, \lambda}, \forall v \in V_\lambda, \)
\[
\Phi_\lambda(v) = \int u_{\Phi, \lambda} \cdot v \, dx
\]
Taking
\[
u_\Phi := \lim_{\lambda \uparrow \Lambda} u_{\Phi, \lambda}
\]
we get that \( \forall v \in V^\circ, \)
\[
\Phi(v) = \lim_{\lambda \uparrow \Lambda} \Phi_\lambda(v) = \lim_{\lambda \uparrow \Lambda} \int u_{\Phi, \lambda} \cdot v \, dx = \oint u_\Phi v \, dx.
\]

The pointwise product provides the **pointwise (Euclidean) norm** of an ultrafunction:
\[
\|u\| = \left( \sum_{a \in \Gamma} |u(a)|^2 d(a) \right)^{\frac{1}{2}} = \left( \oint |u(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]
Also, we can define other norms which might be useful in the applications: for \( p \in [1, \infty), \)
set
\[
\|u\|_p = \left( \sum_{a \in \Gamma} |u(a)|^p \, d(a) \right)^{\frac{1}{p}} = \left( \oint |u(x)|^p \, dx \right)^{\frac{1}{p}}
\]
and, for \( p = \infty, \)
obviously, we set
\[
\|u\|_\infty = \max_{a \in \Gamma} |u(a)|.
\]
Notice that all these norms are equivalent in the sense that given any two norms \( \|u\|_p \) and \( \|u\|_q \), there exists two numbers \( m \) and \( M \) such that \( \forall u \in V^\circ, \)
\[
m \leq \frac{\|u\|_p}{\|u\|_q} \leq M
\]
Of course, if \( p \neq q, \)
\( m \) is an infinitesimal number and \( M \) is an infinite number.

The scalar product (43) also allows to define the **delta (or the Dirac) ultrafunction** as follows: for every \( a \in \Gamma, \)
\[
\delta_a(x) = \frac{\chi_a(x)}{d(a)}.
\]
As it is natural to expect, for every \( u \in V^\circ, \) we have that
\[
\oint \delta_a(x) u(x) \, dx = \sum_{x \in \Gamma} u(x) \delta_a(x) \, d(x) = \sum_{x \in \Gamma} u(x) \frac{\chi_a(x)}{d(a)} \, d(x) = u(a)
\]
The delta ultrafunctions are orthogonal with each other with respect to the pointwise scalar product; hence, if normalized, they provide an orthonormal basis, called **delta-basis**, given by
\[
\{ \sqrt{\delta_a} \}_{a \in \Gamma} = \left\{ \frac{\chi_a}{\sqrt{d(a)}} \right\}_{a \in \Gamma}
\]
So, the identity (15) can be rewritten as follows:
\[ u(x) = \sum_{a \in \Gamma} \left( \oint u(\xi) \sqrt{\delta_a(\xi)} d\xi \right) \sqrt{\delta_a(x)}. \]

### 3.5. Regular and Smooth Ultrafunctions

**Theorem 3.9.** If \( f \in C_c^{1,1} \)
\[ D_i f^\circ(x) = (\partial_i^* f^*)^\circ. \]

**Proof.** If \( f \in C_c^{1,1} \), then \( \partial_i f \in C_c^{0,1} \subseteq V \subseteq V^\circ \); then, \( \forall v \in V^\circ \),
\[ \oint D_i f^\circ v\, dx = \int \partial_i^* f^* v^\Lambda dx \]
Since \( (\partial_i f)^\circ, v \in V^\circ \), by (35), \( \partial_i f v^\lambda = \partial_i f v^\lambda \in V^\lambda \), and hence, by (36)
\[ \oint D_i f^\circ v\, dx = \int \partial_i^* f^* v^\Lambda dx = \oint (\partial_i^* f^*)^\circ v\, dx \]
Since this equality holds for every \( v \in V^\circ \), the conclusion follows. \( \square \)

**Corollary 3.10.** If \( f \in C^{1,1} \) in a neighborhood \( B_\varepsilon(x_0) \) of \( x_0 \in \mathbb{R}^N \); then
\[ D_i f^\circ(x_0) = \partial_i f(x_0). \]

**Proof.** Let \( \phi \in C_c^\infty \) be \( = 1 \) for \( x \in B_{\varepsilon/2}(x) \) and null for \( x \not\in B_{\varepsilon/2}(x) \). Then \( \phi f \in C_c^{1,1} \)
and by Th. 3.9,
\[ D_i (\phi f)^\circ(x) = (\partial_i^* \phi^* f^*)^\circ(x) \]
By (40), \( D_i (\phi f)^\circ(x) \) depends only on the values in \( \text{mon}(x_0) \subseteq B_\varepsilon(x_0)^* \) and hence, for every \( x \in \text{mon}(x_0) \),
\[ D_i f^\circ(x) = (\partial_i^* f^*)^\circ(x) \]
and from here the conclusion. \( \square \)

**Example 1.** If
\[ f(x) = \min(0, x) \]
then \( Df(0) = \frac{1}{2} \).

**Example 2.** If
\[ f(x) = \int_0^x t \sin \frac{1}{t^2} dt \]
by Cor. 3.10, for \( x \in \mathbb{R} \setminus \{0\} \), \( Df(x) = x \sin \frac{1}{x^2} \), but the properties of Def. 3.5 do not guarantee that \( Df(0) = 0 \), since
\[ x \sin \frac{1}{x^2} \not\in BV. \]

In any case, we have that
\[ Df(0) = \oint D_i u(x) \delta_0(x) dx = \lim_{\lambda \searrow} \int \partial_i u_\lambda(x) \delta_{0,\lambda}(x) dx \sim 0. \]
As we already remarked the Leibniz rule does not hold for ultrafunctions and it is not possible to define a generalized derivative which has this property. It is easy to check this fact for idempotent functions, in fact by the Leibniz rule, we should have

\[ D\chi^2_E = 2\chi_E D\chi_E \]

and since \( \chi^2_E = \chi_E \), we deduce that

\[ D\chi_E = 2\chi_E D\chi_E \]

and hence, for every \( x \in E \),

\[ D\chi_E = 0 \quad (49) \]

and this fact contradicts any reasonable generalization of the notion of derivative. Actually the Schwartz impossibility theorem states that the Leibniz rule cannot be satisfied by any algebra which contains, not only the idempotent functions, but also the continuous functions (see [14, 39]). So it is interesting to investigate the subspaces of ultrafunctions for which the Leibniz rule holds and more in general to determine spaces in which many of the usual properties of smooth functions be satisfied. This is important for the applications when we want to study the qualitative properties (and in particular the regularity) of the solutions of an equation.

**Definition 3.11.** We set

\[ U^0 := \left\{ u \in V^\circ \mid u_\Lambda \in (C^{0,1} \cap V) \right\} \]

and for \( m > 1 \), we define by induction

\[ U^m := \left\{ u \in U^{m-1} \mid \forall i = 1, \ldots, N, \; D_i u \in U^{m-1} \right\} \]

If \( u \in U^m \), we will say that \( u \) is \( m \)-regular.

Let us see the main properties of the spaces \( U^m \).

**Theorem 3.12.** The spaces of \( m \)-regular ultrafunctions satisfy the following properties:

1. if \( u \in U^1 \), then
   \[ D_i u = (\partial^*_i u_\Lambda)^\circ; \]
2. if \( u \in U^m \), then
   \[ D_i u = (\partial^*_i u_\Lambda(x))^\circ \]
3. if \( f \in C_c^{m,1} \), then \( f^\circ \in U^m; \)
4. if \( u, v, uv \in U^1 \) then the Leibniz rule holds:
   \[ D_i (uv) = D_i uv + uD_i v. \]

**Proof.** 1. If \( u \in U^1 \), \( D_i u \in U^0 \subset V^\circ \), then, by (35), \( \forall v \in U^0 \subset V^\circ \), \( (D_i u)_\Lambda v_\Lambda \in V_\Lambda \). Since \( D_i u \) is continuous, \( (D_i u)_\Lambda v_\Lambda = (D_i u)_\Lambda v_\Lambda \in V_\Lambda \) and hence \( (D_i u)_\Lambda v_\Lambda \in V_\Lambda \) and

\[ \int D_i uv \; dx = \int (D_i u)_\Lambda v_\Lambda \; dx \]
Then, by (39) we have that
\[
\int^* (D_i u)_\Lambda v_\Lambda \, dx = \int^* (\partial_i^* u_\Lambda) v_\Lambda \, dx
\]
and so,
\[
(D_i u)_\Lambda = \partial_i^* u_\Lambda.
\]
2. It follows trivially by (1) since \(U^m \subset U^1\).
3. For \(m = 0, f \in C^{0,1}_c\), and hence \(f^0 \in U^0\). For \(m \geq 1\), since \(\partial_i f \in C^{m-1,1}_c\), by (50), \((\partial_i f)^0 = D_i f^0\); hence \(D_i f^0 \in U^{m-1}\). So \(f^0 \in U^m\).
4. If \(u, v \in U^1\) then \(u, v, D_i u, D_i v \in U^0\) and so, using (1) and (35), we have that
\[
(D_i (uv)) = (D_i u)v + u(D_i v)
\]
\[\square\]
Since the \(U^m\)'s have hyperfinite dimension then there exist finite dimensional spaces \(U^m\)'s such that
\[
U^m = \{u^0 \mid u \in U^m_\Lambda\}; \quad U^m_\Lambda := \lim_{\lambda \uparrow \Lambda} U^m_\lambda.
\] (51)
The space of smooth ultrafunctions (or \(\infty\)-regular ultrafunctions) is defined as follows:
\[
U^\infty := \bigcap_{m \leq \alpha} U^m = \left[\lim_{\lambda \uparrow \Lambda} \left(\bigcap_{m \leq |\lambda|} U^m_\Lambda\right)\right]^0.
\] (52)
where \(\alpha\) has been defined by (7). Clearly \(U^\infty \neq \emptyset\), since
\[
f \in D \Rightarrow f^0 \in U^\infty.
\] (53)
For \(m \in \mathbb{N} \cup \{\infty\}\), we set
\[
(U^m)^\perp = \left\{u \in V^0 \mid \forall \psi \in U^m, \quad \int u \psi \, dx = 0\right\}
\]
and we denote by \(\Pi_m u\) and \(\Pi^\perp_m u\) the relative “orthogonal” projection of \(u\) on \(U^m\) and \((U^m)^\perp\). Then every ultrafunction \(u\) can be split as follows
\[
u = \Pi_m u + \Pi^\perp_m u
\] (54)
**Definition 3.13.** Given the splitting (54), \(\forall m \in \mathbb{N} \cup \{\infty\}\), \(\Pi_m u\) will be called the \(m\)-regular part of \(u\) and \(\Pi^\perp_m u\) the \(m\)-singular part of \(u\).

**Remark 3.14.** It is possible to define different types of regular ultrafunction namely we can choose different subspaces of \(V^0\) that satisfy suitable conditions. For example, we can set
\[
C^m_\Lambda = \text{span} \lim_{\lambda \uparrow \Lambda} C^m_0 \cap \lambda
\]
We have that \( C^m_\Lambda \supset U^m \) and hence the functions in \( C^m_\Lambda \) satisfy less properties. Similarly, we can choose more regular spaces such as

\[
U^{m,p} := \left\{ u \in U^m \mid |u|^{p-2} u \in U^0 \right\}; \quad p \geq 2. \tag{55}
\]

Of course the choice of a particular space depends on the problems that we would like to treat. We can make an analogy with the theory of distributions; in this case the spaces \( C^m \)'s and the Sobolev spaces \( H^{m,p} \)'s can be considered as subspaces of \( D' \) which present different kinds of regularity.

### 3.6. Time-Dependent Ultrafunctions

In evolution problems the time variable plays a different role that the space variables; then the functional spaces used in these problems (e.g. \( C^k([0,T],H^0_0(\Omega)) \), \( L^p_{loc}(\mathbb{R},H^k(\Omega)) \) etc.) reflect this fact. The same is true in the frame of ultrafunctions. This section is devoted in the description of the appropriate ultrafunction-spaces for evolution problems.

First of all we need to recall some well known facts about free modules:

**Definition 3.15.** Given a ring \( R \) and a module \( M \) over \( R \), the set \( B \subset M \) is a basis for \( M \) if:

- \( B \) is a generating set for \( M \); that is to say, every element of \( M \) is a finite sum of elements of \( B \) multiplied by coefficients in \( R \);
- \( B \) is linearly independent.

**Definition 3.16.** A free module is a module with a basis.

The following is a well known theorem:

**Theorem 3.17.** If \( R \) is a commutative ring and \( M \) is a free \( R \)-module, then all the bases of \( E \) have the same cardinality. The cardinality of a basis is called **rank** of \( M \).

We will describe some free modules which will be used in the following part of this paper.

**Example.** (i) If \( \Gamma \) is a finite set then \( C^k(\mathbb{R})^\Gamma \) is a free module over \( C^k(\mathbb{R}) \) of rank \( |\Gamma| \) and a basis is given by

\[
\{ \chi_a \}_{a \in \Gamma}
\]

(ii) Let \( W \subset \mathfrak{F}(\mathbb{R}^N) \) be a vector space of finite dimension, then

\[
C^k(\mathbb{R},W) := C^k(\mathbb{R}) \otimes_{\mathbb{R}} W \subset \mathfrak{F}(\mathbb{R}^{N+1})
\]

is a free \( C^k \)-module of rank equal to \( \dim W \), namely every function \( f \in C^k(\mathbb{R},W) \) can be written as follows:

\[
f(t,x) := \sum_{k=1}^{\dim W} c_k(t)e_k(x)
\]

where \( c_k \in C^k(\mathbb{R}) \) and \( \{e_k\} \) is any basis in \( W \).
(iii) If \( W = \lim_{\Lambda \uparrow} \Lambda W \subset \mathcal{F}(\mathbb{R}^N)^* \) is an internal vector space of hyperfinite dimension, then by \( C^k(\mathbb{E}, W) \) we denote the internal \( C^k(\mathbb{R})^* \)-module defined by

\[
C^k(\mathbb{E}, W) = \lim_{\Lambda \uparrow} C^k(\mathbb{R}, W_{\Lambda}).
\]

Now we are ready to define the time-dependent ultrafunctions:

**Definition 3.18.** The space of **time dependent ultrafunctions** of order \( k \in \mathbb{N} \) is the free \( C^k(\mathbb{R})^* \)-module given by

\[
C^k(\mathbb{E}, V^\circ) := \{ u^\circ = u|_{\mathbb{E} \times \Gamma} \mid u \in C^k(\mathbb{E}, V_{\Lambda}) \}. 
\]

Every time-dependent ultrafunction can be represented by the following hyperfinite sum:

\[
u(t, x) = \sum_{a \in \Gamma} c(t) \chi_a(x)\]

where \( c(t) \in C^k(\mathbb{R})^* \) and \( \{\chi_a(x)\}_{a \in \Gamma} \) is the canonical basis of \( V^\circ \). The map

\[
(\cdot^\circ) : C^k(\mathbb{E}, V_{\Lambda}) \to C^k(\mathbb{E}, V^\circ) ; \quad u^\circ = u|_{\mathbb{E} \times \Gamma}
\]

is an isomorphism between free \( C^k(\mathbb{R})^* \)-modules: in fact, by using (20), we have that \( \{\sigma_a\}_{a \in \Gamma} \) is a basis of \( C^k(\mathbb{E}, V_{\Lambda}) \) and we have that

\[
\left( \sum_{a \in \Gamma} c(t)\sigma_a(x) \right)^\circ = \sum_{a \in \Gamma} c(t)\chi_a(x) \quad (56)
\]

The restriction map \((\cdot^\circ)\) can be extended to a \( C^k(\mathbb{R})^* \)-module homomorphism

\[
(\cdot^\circ) : C^k(\mathbb{R}, \mathcal{F}(\mathbb{R}^N))^* \to C^k(\mathbb{E}, V^\circ)
\]

by setting

\[
w^\circ(t, x) = \sum_{a \in \Gamma} w(t, a)\chi_a(x)
\]

and, of course, by (51),

\[
w_{\Lambda}(t, x) = \sum_{a \in \Gamma} w(t, a)\sigma_a(x) \in C^k(\mathbb{E}, V^\circ)
\]

In particular, if \( f \in C^k(\mathbb{R}, \mathcal{F}(\mathbb{R}^N)) \), we have that

\[
f^\circ(t, x) = \sum_{a \in \Gamma} f^*(t, a)\chi_a(x) \quad \text{and} \quad f_{\Lambda}(t, x) = \sum_{a \in \Gamma} f^*(t, a)\sigma_a(x) \quad (57)
\]

Observe that, unlikely of (22), in this case \( f_{\Lambda}(t, x) \neq f^*(t, x) \) for some \( t \in \mathbb{E} \setminus \mathbb{R} \); this fact does not prevent the theory to work, but, in some circumstances, it is necessary to be careful.

The notion of generalized derivative in the space variable is trivially defined by linearity:

\[
D_i u(t, x) = D_i \left( \sum_{a \in \Gamma} c(t)\chi_a(x) \right) = \sum_{a \in \Gamma} c(t)D_i\chi_a(x). \quad (58)
\]
It is not necessary to introduce a generalized time-derivative, since the natural derivative
\[ \partial^*_t : C^{k+1}(E, V^\circ) \to C^k(E, V^\circ), \quad k \geq 0, \]
is well defined by setting
\[ \partial^*_t u(t, x) = \partial^*_t \left( \sum_{a \in \Gamma} c(t) \chi_a(x) \right) = \sum_{a \in \Gamma} \partial^*_t c(t) \chi_a(x). \quad (59) \]

In our applications, we do not need a generalized nor a weak time-derivative for the functions in \( C^0(E, V^\circ) \).

**Theorem 3.19.** If \( f \in C^k(\mathbb{R}, C^0_c) \) and \( m \geq 0 \), then, \( \partial^*_t f^\circ \in C^{k-1}(E, U^m) \) and
\[ \partial^*_t f^\circ(t, x) = \sum_{a \in \Gamma} \partial^*_t f^*(t, a) \chi_a(x). \]

Moreover, for \( i = 1, \ldots, N \) and \( m \geq 1 \), \( D_i f^\circ \in C^k(E, U^{m-1}) \), and
\[ D_i f^\circ(t, x) = \sum_{a \in \Gamma} \partial^*_i f^*(t, a) \chi_a(x). \]

**Proof.** The first equality follows immediately from (57) and (59).

Let us prove the second statement. By the definition 3.11, if \( w \in C^k(E, U^m) \),
\[ w_\Lambda(t, x) = \sum_{a \in \Gamma} w(t, a) \sigma_a(x) \in C^k(E, U^m_\Lambda); \]
and by 3.12.2, \( \forall t \in E \), \( D_i w(t, \cdot) = [\partial^*_i w_\Lambda(t, \cdot)]^\circ \in U^{m-1} \) and hence
\[ D_i w(t, x) = [\partial^*_i w_\Lambda(t, x)]^\circ = \sum_{a \in \Gamma} \partial^*_i w(t, a) \chi_a(x) \]
In particular, by (57), if \( f \in C^k(\mathbb{R}, C^0_c) \),
\[ f_\Lambda(t, x) = \sum_{a \in \Gamma} f^*(t, a) \sigma_a(x) \in C^k(E, U^m_\Lambda) \]
and hence
\[ D_i f^\circ(t, x) = [\partial^*_i f^*(t, x)]^\circ = \sum_{a \in \Gamma} \partial^*_i f^*(t, a) \chi_a(x). \]

\[ \square \]

### 4. Basic Properties of Ultrafunctions

In this section we analyze some properties of the fine ultrafunction that seems interesting in themselves and/or relevant in the applications.
4.1. Ultrafunctions and Measures

As we have seen, if $f \in V$, in general
\[ \int f^\circ(x) \, dx \neq \int f(x) \, dx \]
since $\int f^\circ(x) \, dx$ takes account of the value of $f$ in any single point. Thus it is a natural question to ask if there exists an ultrafunction $u$ some way related to $f$ such that
\[ \int u \, dx = \int f \, dx. \tag{60} \]

This question has an easy answer if we think of $f$ as the density of a measure $\mu_f$.

In fact, the following definition appears quite natural:

**Definition 4.1.** If $\mu$ is a Radon measure, we define an ultrafunction $\mu^\circ$ as follows: for every $v \in V$, we set
\[ \int v \mu^\circ(x) \, dx = \lim_{\lambda \uparrow \Lambda} \int v(x) \, d\mu \]
Notice that the existence of $\mu^\circ$ is guaranteed by Th. 3.8, since
\[ \Phi(v) := \lim_{\lambda \uparrow \Lambda} \int v(x) \, d\mu \]
is an internal linear functional over $V^\circ$.

**Example.** 1 - If $\mu_f$ is a measure whose density is $f \in L^1_{loc}$, then, $\forall v \in V$
\[ \int v \mu_f^\circ \, dx = \lim_{\lambda \uparrow \Lambda} \int f(x) v(x) \, dx = \int^* f^* v \, dx. \]
then, taking $f \in L^1$, and $v = 1^\circ$
\[ \int \mu^\circ f \, dx = \int f \, dx; \]
then (60) holds with $u = \mu_f^\circ$.

**Example.** 2 - If $\delta_a$ is the Dirac measure, then $\delta_a^\circ = \delta_a$ where $\delta_a$ is the Dirac ultrafunction defined by (48).

**Example.** 3 - For all $u \in V^\circ$, $\partial_i u_\Lambda$ is a measure and we will denote by $\mu_{\partial_i u}$ the related ultrafunction, namely, using the notation (33), we have that, $\forall v \in V^\circ$
\[ \int v \mu_{\partial_i u} \, dx = \lim_{\lambda \uparrow \Lambda} \int \partial_i u_\lambda \, v \, d\mu \]
and hence, by (39)
\[ \int v \mu_{\partial_i u} \, dx = \int (\partial_i^* u_\Lambda) \, v_\Lambda \, dx = \int D_i u \, v \, dx \tag{61} \]
namely
\[ D_i u = \mu_{\partial_i u}. \tag{62} \]
Example. 4 - If $\Omega \subset \mathbb{R}^N$ is a set of finite measure, and $\mu_\Omega$ is the measure whose density is $\chi_\Omega$, then $\forall u \in V^\circ$, we have that
\[
\oint u(x)\mu_\Omega^\circ(x)dx = \lim_{\lambda \downarrow \Lambda} \int u_\lambda(x)d\mu_\Omega = \lim_{\lambda \downarrow \Lambda} \int u_\lambda(x)\chi_\Omega dx = \int_\Omega u_\Lambda(x)\chi_\Omega dx = \int_{\Omega^*} u_\Lambda(x)dx.
\] (63)
In particular, if $f \in V$,
\[
\int_{\Omega^*} f(x)dx = \int_{\Omega^*} f^*(x)\chi_\Omega dx = \oint f^\circ(x)\mu_\Omega^\circ(x)dx
\] (64)

Example. 5 - If $\Omega \subset \mathbb{R}^N$ has a smooth boundary $\partial \Omega$ whose $(N-1)$-dimensional measure is denoted by $S_{\partial \Omega}$, we have that
\[
\oint u(x)S_{\partial \Omega}^\circ(x)dx = \lim_{\lambda \downarrow \Lambda} \int u_\lambda(x)dS_{\partial \Omega} = \int_{\partial \Omega^*} u_\Lambda(x) dS^*_{\partial \Omega}
\] (65)

Caveat: It is absolutely natural to generalize Eq. (26) by the following definition:
\[
\oint_E u(x)dx := \sum_{a \in E} u(a)d(a)
\] (66)
for every set $E \subset \Gamma$. At this point in is important to notice that for a measurable set $\Omega \subset \mathbb{R}^N$ and $f \in C^0$
\[
\oint f^\circ(x)dx \neq \oint f^\circ(x)\mu_\Omega^\circ(x)dx = \int_{\Omega} f(x)dx.
\] (67)
and
\[
\oint f^\circ(x)dx \neq \oint f^\circ(x)S_{\partial \Omega}^\circ(x)dx = \int_{\partial \Omega^*} f(x)dS_{\partial \Omega}
\]

The next proposition shows a useful way to represent $\mu_f^\circ$:

Proposition 4.2. If $f \in L^1_{\text{loc}}$, then
\[
\mu_f^\circ(x) = \sum_{a \in \Gamma} \left( \int f^*\sigma_a(y)dy \right) \delta_a(x)
\] (68)
where $\{\sigma_a(x)\}_{a \in \Gamma}$ is the $\sigma$-basis (see (20)).

Proof. Let $\sigma_{a,\lambda}$ be the net such that $\sigma_a = \lim_{\lambda \downarrow \Lambda} \sigma_{a,\lambda}$. Then, by (48),
\[
\mu_f^\circ(a) = \oint \mu_f^\circ(x)\delta_a(y)dy = \frac{1}{d(a)} \oint \mu^\circ(x)\chi_a(y)dy = \frac{1}{d(a)} \lim_{\lambda \downarrow \Lambda} \int f(y)\sigma_{a,\lambda}(y)dy
\]
\[
= \frac{1}{d(a)} \int f^*(y)\sigma_a(y)dy
\]
Hence,
\[
\mu_f^\circ(x) = \sum_{a \in \Gamma} \mu_f^\circ(a)\chi_a(x) = \sum_{a \in \Gamma} \left( \frac{1}{d(a)} \int f^*(y)\sigma_a(y)dy \right) \chi_a(x)
\]
\[
= \sum_{a \in \Gamma} \left( \int f^*(y)\sigma_a(y)dy \right) \delta_a(x)
\]
\[
\square
\]
Prop. 4.2 suggests to generalize the operator $f^* \mapsto \mu_f^*$ to an operator $w \mapsto \mu_w$ defined $\forall w \in (L^1_{\text{loc}})^*$ by setting

$$\mu_w(x) := \sum_{a \in \Gamma} \left( \int^* w(y) \sigma_a(y) dy \right) \delta_a(x)$$

$\mu_w(x)$ can be considered as a sort of measure density defined on $\Gamma$.

Moreover, to simplify the notation, we set

$$\mu_E(x) = \mu_{\chi_E}(x). \quad (69)$$

By the pointwise representation of $u$ (see (15)), we get

$$\oint u(x) \mu_E(x) dx = \sum_{a \in \Gamma} u(a) \oint \chi_a(x) \mu_E(x) dx;$$

then setting $d_E(a) = \oint \chi_a(x) \mu_E(x) dx$,

$$\oint u(x) \mu_E(x) dx = \sum_{a \in \Gamma} u(a) d_E(a) \quad (70)$$

generalizing Eq. (26). Notice the difference between the equalities (66) and (70). These equalities suggest the following notation:

$$\oint u(x) d_E x = \oint u(x) \mu_E(x) dx \quad (71)$$

In particular, if we take $f \in V$,

$$\int_\Omega f(x) dx = \sum_{a \in \Gamma} f^\circ(a) d_\Omega(a) = \oint f^\circ(x) d_\Omega x$$

$$\int_{\partial \Omega} f(x) dx = \sum_{a \in \Gamma} f^\circ(a) d_{\partial \Omega}(a) = \oint f^\circ(x) d_{\partial \Omega} x.$$

We end this section with the following

**Proposition 4.3.** $\mu_w(x)$ is the element of $V^\circ$ characterized by the following identity: $\forall u \in V^\circ$,

$$\oint u(x) \mu_w(x) dx = \int^* u_A(x) w_A(x) dx = \oint w(x) \mu_u(x) dx \quad (72)$$

**Proof.** We have that

$$\oint \mu_w(x) u(x) dx = \oint \sum_{a \in \Gamma} \left( \int^* w_A(y) \sigma_a(y) dy \right) \delta_a(x) u(x) dx$$

$$= \sum_{a \in \Gamma} \left( \int^* w_A(y) \sigma_a(y) dy \right) \oint \delta_a(x) u(x) dx$$

$$= \sum_{a \in \Gamma} \left( \int^* w_A(y) \sigma_a(y) u(a) dy \right) = \int^* w_A(y) \left[ \sum_{a \in \Gamma} \sigma_a(y) u(a) \right] dy$$

$$= \int^* w_A(y) u_A(y) dy.$$
The last equality follows by symmetry. \( \square \)

4.2. The Vicinity of a Set

Given an open set \( \Omega \subset \mathbb{R}^N \) and \( f \in C^1 \), then the value of \( \nabla f(x_0) \) in a point \( x_0 \in \partial \Omega \) depends only on the values which \( f \) takes in \( \Omega \) since

\[
\nabla f(x_0) = \lim_{x \to x_0} \nabla f(x_0)
\]

If \( f \) is not continuous, \( \nabla f \) is not defined, but \( Df \) makes sense; however \( Df(x_0) \) in a point \( x_0 \in \partial \Omega \) depends on the values which \( f \) takes in suitable points \( y \sim x_0 \) even if \( y \notin \Omega^{o} \). Roughly speaking, the vicinity of a set \( E \) consists of the set of points which influence the derivative of the points in \( E \).

In order to make this definition precise, we need to define a function \( \theta_E \in V^{o} \) for every \( E \subset \Gamma \). If \( \Omega \subset \mathbb{R}^N \) is a set such that \( \bar{\chi}_{\Omega} \in V \), we set

\[
\theta_{\Omega^{c}} = \bar{\chi}_{\Omega}
\]

where the operator \( u \mapsto \bar{u} \) has been defined by (30). We want to generalize the above formula to every set \( E \subset \Gamma \). We put

\[
(\chi_{E})_{\Lambda} = \sum_{a \in E} \sigma_a(x)
\]

and

\[
\theta_E = \left[ (\chi_{E})_{\Lambda} \right]^{o}
\]

In the above formula, the operator \( u \mapsto \bar{u} \) who has been defined by (30) for \( u \in L^{\infty}_{loc} \), has been extended to \( (L^{\infty}_{loc})^{*} \). \( \theta_E \) is similar to the measure density \( \mu_E \) but it is slightly different; for example we have that \( \forall f \in C^{1;1} \) and \( \forall \Omega \subset \mathbb{R}^N \), bounded and open,

\[
\oint f^{o}\mu_{\Omega^{c}}dx = \int fdx = \oint f^{o}\theta_{\Omega^{c}}dx
\]

since \( f^{o}\theta_{\Omega^{c}} \in V \); however, if you take \( f = \theta_{\Omega^{c}} \) you have that

\[
\oint \theta_{\Omega^{c}}\mu_{\Omega^{c}}dx = \int \bar{\chi}_{\Omega} dx = m(\Omega)
\]

and, if we assume that \( \partial \Omega \) is smooth, by (32), we have that

\[
\oint \theta_{\Omega^{c}}\mu_{\Omega^{c}}dx = \oint \theta_{\Omega^{c}}\bar{\chi}_{\Omega}dx = \oint \theta_{\Omega^{c}}dx - \frac{1}{2} \oint \chi_{\partial \Omega^{c}} dx = m(\Omega) - \frac{1}{2} \oint \chi_{\partial \Omega^{c}} dx.
\]

Then \( \theta_{\Omega^{c}}\mu_{\Omega^{c}} \neq \theta_{\Omega^{c}}\theta_{\Omega^{c}} \) and hence \( \mu_{\Omega^{c}} \neq \theta_{\Omega^{c}} \).

Now we can state the following definitions:

**Definition 4.4.** Given an internal set \( E \subset \Gamma \), we define the vicinity of \( E \) as follows

\[
\text{vic}(E) := \text{supp} \left( |D\theta_E| + \theta_E \right)
\]

The operator \( \Omega \mapsto \text{vic}(\Omega) \) reminds the closure operator \( \Omega \mapsto \bar{\Omega} \) but it is not a closure operator in the topological sense; in fact, in general, \( \text{vic}^2(E) := \text{vic}(\text{vic}(E)) \supset \text{vic}(E) \) in a strict sense. However, this similarity with the closure operator suggests the following
**Definition 4.5.** For any internal set $E \subset \Gamma$, we define
\[ \text{bd}(E) := \text{supp}(|D\theta_E|) \]
and we will call $\text{bd}(E)$ the **pointwise boundary** of $E$ and
\[ \text{int}(E) := \{ x \in \Gamma \mid D\theta_E(x) = 0 \text{ and } \theta_E(x) > 1 \} = \text{vic}(E) \setminus \text{bd}(E) \]
and we will call $\text{int}(E)$ the **pointwise interior** of $E$.

If we take $E = (\mathbb{R}^N)^c = \Gamma$, then $\text{bd}(\Gamma)$ is called the **boundary at infinity**.

**4.3. The Gauss’ Divergence Theorem**

In this section we want to generalize the Gauss’ divergence theorem in the framework of the ultrafunction; in particular it is interesting to analyze the case when $\partial \Omega$ is not smooth.

First, we will examine the smooth case. If $\Omega \subset \mathbb{R}^N$ is a bounded set with smooth boundary and $\phi$ is a smooth vector field, we have that
\[ \int_{\Omega} \nabla \cdot \phi \, dx = \int_{\partial \Omega} \phi \cdot n_{\Omega} \, dS_{\partial \Omega} \]
where $S_{\partial \Omega}$ denotes the $(N - 1)$-dimensional measure over $\partial \Omega$ and $n_{\Omega}(x)$ is the exterior normal derivative. For future purposes, we assume that $n_{\Omega}(x)$ is a $C^1_c$ function defined in all $\mathbb{R}^N$ which coincides with the exterior normal derivative in $\partial \Omega$. We have the following result:

**Theorem 4.6.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $\bar{\chi}_\Omega \in V$; then
\[ S_{\partial \Omega}^0(x) = |D\theta_{\Omega^c}(x)| \]
and $\forall x \in \text{bd}(\Omega)$
\[ n_{\Omega}(x) = -\frac{D\theta_{\Omega^c}(x)}{|D\theta_{\Omega^c}(x)|}. \]

**Proof.** The Gauss divergence theorem can be generalized to vector fields $\phi \in V^N$ by writing
\[ \int \nabla \cdot \phi \, \bar{\chi}_\Omega \, dx = \int_{\partial \Omega} \phi \cdot n_{\Omega} \, dS_{\partial \Omega} \]
In this case $\nabla \cdot \phi$ is a measure and we have used the notation (33). By the transfer principle, we have that $\forall \phi \in (V^N)^*$
\[ \int^* \nabla^* \cdot \phi \, \bar{\chi}_\Omega^* \, dx = \int_{\partial \Omega^*} \phi \cdot n_{\Omega}^* \, dS_{\partial \Omega^*} \tag{74} \]
By (39), for every $v \in V_\Lambda$,
\[ \int^* \nabla^* \cdot \phi \, v \, dx = \sum_{i=1}^N \int^* \partial_i \phi_i \, v \, dx = \sum_{i=1}^N \phi \, D_i \phi_i \, v^\circ \, dx = \int \phi \, D \cdot \phi \, v^\circ \, dx; \]
then, by (73) and (42), we have that
\[ \int^* \nabla^* \cdot \phi \, \bar{\chi}_\Omega^* \, dx = \int D \cdot \phi \, \bar{\chi}_\Omega^* \, dx = \int D \cdot \phi \, \theta_{\Omega^c} \, dx = -\int \phi \, D \cdot \theta_{\Omega^c} \, dx \tag{75} \]
Moreover, by (65)
\[ \int_{\partial \Omega}^* \phi^* \cdot n^*_\Omega dS_{\partial \Omega} = \oint \phi \cdot n^*_\Omega S^\circ_{\partial \Omega} dx; \] (76)

Then, by (74), (75) and (76), \( \forall \phi \in (V^\circ)^N \)
\[ \oint \phi \cdot D\theta_{\Omega^\circ} dx = - \oint \phi \cdot n^*_\Omega S^\circ_{\partial \Omega} dx \]

Now we take \( \phi(x) = \delta_a(x)e_i \) where \( a \in \Gamma \) and \( \{e_i\}_{i=1,\ldots,N} \) is the canonical basis in \( \mathbb{R}^N \) (and hence in \( \mathbb{E}^N \)), we replace \( \phi \) in the above formula:
\[ \oint \delta_a(x)e_i \cdot n^\circ_\Omega(x) S_{\partial \Omega}(x) dx = - \oint \delta_a(x)e_i \cdot D\theta_{\Omega^\circ}(x) dx. \]

and we get
\[ (n^\circ_\Omega(a) \cdot e_i) S_{\partial \Omega}(a) = -e_i \cdot D\theta_{\Omega^\circ}(a) = -D_i\theta_{\Omega^\circ}(a); \] (77)
then
\[ S_{\partial \Omega}(a) = \sqrt{\sum_{i=1}^N [(n^\circ_\Omega(a) \cdot e_i) S_{\partial \Omega}(a)]^2} = \sqrt{\sum_{i=1}^N [D_i\theta_{\Omega^\circ}(a)]^2} = |D\theta_{\Omega^\circ}(a)| \]

Moreover, using again (77) we have that
\[ (n^\circ_\Omega(a) \cdot e_i) |D_i\theta_{\Omega^\circ}(a)| = D_i\theta_{\Omega^\circ}(a); \]
if \( a \in \partial \Omega \), we have that \( |D_i\theta_{\Omega^\circ}(a)| \neq 0 \) and hence
\[ n^\circ_\Omega(a) = -\frac{D\theta_{\Omega^\circ}(a)}{|D\theta_{\Omega^\circ}(a)|}. \]

Theorem 4.6 suggests the “right” generalization of the Gauss’ theorem; given any set \( E \subseteq \Gamma \), we define
\[ n_E(x) = \begin{cases} -\frac{D\theta_E(x)}{|D\theta_E(x)|} & \text{if } x \in \partial \Omega(E) \\ 0 & \text{if } x \notin \partial \Omega(E) \end{cases} \] (78)
It is amazing that \( n_E(x) \) makes sense even if \( E \) consists of a single point \( x_0 \). Clearly in this case, \( \text{supp} (n_{\{x_0\}}) \subseteq \text{mon} (x_0) \).

**Theorem 4.7.** (Generalized Gauss’ divergence theorem) Let \( \phi : \Gamma \rightarrow (V^\circ)^N \) be a (ultrafunctions) vector field and let \( E \subseteq \Gamma \) be an internal set; then
\[ \oint D \cdot \phi \ dE x = \oint \phi \cdot n_E \ |D\theta_E| \ dx \]

**Proof.** By (71), and (42).
\[ \oint D \cdot \phi \ dE x = \oint D \cdot \phi \ |\theta_E| \ dx = -\oint \phi \cdot D\theta_E \ dx \]
\[ = -\oint \phi \cdot \frac{D\theta_E}{|D\theta_E|} \ |D\theta_E| \ dx = \oint \phi \cdot n_E \ |D\theta_E| \ dx. \]
It is interesting to compare the above results with the notion of Caccioppoli set:

**Definition 4.8.** A Caccioppoli set $\Omega$ is a Borel set such that $\chi_\Omega \in BV$, namely such that $\nabla(\chi_\Omega)$ (the distributional gradient of $\chi_\Omega$) is a finite Radon measure. If $\Omega$ is a Caccioppoli set, then the measure $|\nabla(\chi_\Omega)|$ is defined as follows: $\forall f \in C^1_c$, $f \geq 0$,

$$\int f \ d (|\nabla(\chi_\Omega)|) := \sup \left\{ \int_\Omega \nabla \cdot (f\phi) \, dx \mid \phi \in (C^1)^N, \|\phi\|_{L^\infty} \leq 1 \right\}$$

The number

$$p(\Omega) := \int d (|\nabla(\chi_\Omega)|)$$

(79)

is called Caccioppoli perimeter of $\Omega$. If the reduced boundary of $\Omega$ coincides with $\partial \Omega$, we have that (see [29, Sect. 5.7])

$$\int f(x) \ d (|\nabla(\chi_\Omega)|) = \int_{\partial \Omega} f(x) \, d\mathcal{H}^{N-1}$$

(80)

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure of $\partial \Omega$. The Gauss’ theorem for a Caccioppoli set takes the following form:

$$\int_{\partial \Omega} \phi \cdot n_\Omega \, d (|\nabla(\chi_\Omega)|) = \int_{\partial \Omega^*} \nabla \cdot \phi \, dx$$

By transfer, we have that for every $*$-Borellian vector field $\phi$,

$$\int_{\partial \Omega^*} \phi \cdot n_\Omega^* \, d (|\nabla(\chi_\Omega)|)^* = \int_{\partial \Omega^*} \nabla^* \cdot \phi \, dx$$

and hence $\forall v \in V_\Lambda$, and by (65) and Th. 4.6,

$$\oint \phi^\circ \cdot n_\Omega^\circ \mu_{|\nabla(\chi_\Omega)|} \, dx = \oint \phi \cdot n_\Omega \ |D\theta_\Omega^\circ| \, dx$$

and by the arbitrariness of $\phi$,

$$|D\theta_\Omega^\circ(x)| = \mu_{|\nabla(\chi_\Omega)|}$$

However, the measure $|D\mu_\Omega^\circ(x)|$ is more general than $\mu_{|\nabla(\chi_\Omega)|}$. For example if $\partial \Omega$ is a set with Hausdorff dimension $d > N - 1$, $\mu_{|\nabla(\chi_\Omega)|}$ is not defined, while $|D\mu_\Omega^\circ(x)|$ is well defined. Moreover, by the generalization of (79)

$$p(\Omega) := \oint |D\theta_\Omega^\circ| \, dx$$

$|D\mu_\Omega^\circ(x)|$ allows to define the perimeter of every set $\Omega \subset \mathbb{R}^N$ even when $\partial \Omega$ is very wild such as e.g. $\Omega = \partial \Omega$. With this respect, the theory of fine ultrafunctions is a good improvement of the theory developed in [20].
4.4. Ultrafunctions and Distributions

One of the most important properties of the ultrafunctions is that they can be seen (in some sense that we will make precise in this section) as a generalizations of the distributions.

**Definition 4.9.** We say that an ultrafunction $u$ is **distribution-like (DL)** if there exist a distribution $T$ such that for any $\varphi \in \mathcal{D}$

$$\int u(x)\varphi^\circ(x)dx = \langle T, \varphi \rangle$$

We say that an ultrafunction $u$ is **almost distribution like** if there exist a distribution $T$ such that for any $\varphi \in \mathcal{D}$

$$\int u(x)\varphi^\circ(x)dx \sim \langle T, \varphi \rangle$$

**Example.** The measures $|D\mu_\Omega^\circ(x)|$ and $S_{\partial\Omega}(x)$ are distribution-like since for any $\varphi \in \mathcal{D}$

$$\int |D\mu_\Omega^\circ|\varphi^\circ dx = \int S_{\partial\Omega}\varphi^\circ dx = \langle T_{S_{\partial\Omega}}, \varphi \rangle$$

where $T_{S_{\partial\Omega}}$ is the distribution related to the measure $S_{\partial\Omega}$.

It is easy to see that:

**Proposition 4.10.** For any distribution $T$ there is a distribution-like ultrafunction $u_T$.

**Proof.** Let us consider any projection $P_\lambda : V_\lambda \to V_\lambda \cap \mathcal{D}$ and set

$$\int u_T(x)v(x)dx = \lim_{\lambda \uparrow \Lambda} \langle T, P_\lambda v_\lambda \rangle$$

By Th. 3.8, $u_T$ is well defined. So, $\forall \varphi \in \mathcal{D}$

$$\int u_T(x)\varphi^\circ(x)dx = \lim_{\lambda \uparrow \Lambda} \langle T, P_\lambda \varphi \rangle = \lim_{\lambda \uparrow \Lambda} \langle T, \varphi \rangle = \langle T, \varphi \rangle$$

Clearly $u_T$ is not univocally defined since, in the proof of Prop. 4.10, the projection $P_\lambda$ can be defined arbitrarily. So it make sense to set

$$[u]_{\mathcal{D}^\circ} = \{ v \in V^\circ \mid v \approx_{\mathcal{D}^\circ} u \}$$

where

$$v \approx_{\mathcal{D}^\circ} u :\Leftrightarrow \forall \varphi \in \mathcal{D}, \int (u - v)\varphi^\circ dx = 0$$

Then there is a bijective map

$$\Psi : \mathcal{D}^\prime \to V_{DL}^\circ / \approx_{\mathcal{D}^\prime}$$

where $V_{DL}^\circ$ is the set of distribution like ultrafunction and

$$\Psi(T) = \left\{ u \in V^\circ \mid \forall \varphi \in \mathcal{D}, \int u\varphi^\circ dx = \langle T, \varphi \rangle \right\}$$

The linear map is $\Psi$ consistent with the distributional derivative, namely:
Proposition 4.11. If $\Psi(T) = [u]_{\mathcal{D}'}$, then $\Psi(\partial_i T) = [D_i u]_{\mathcal{D}'}$.

Proof. If $\Psi(T) = [u]_{\mathcal{D}'}$, then
\[
\oint D_i u \varphi^* \, dx = -\oint u D_i \varphi^* \, dx
\]
Since $\varphi^* \in U^\infty$, then by Th. 3.12.2, $D_i \varphi^* = (\partial_i \varphi)^*$ and so
\[
\oint D u \varphi^* \, dx = \oint u D \varphi^* \, dx = -\oint u (\partial_i \varphi)^* \, dx
= -\langle T, \partial_i \varphi \rangle = \langle \partial_i T, \varphi \rangle
\]
Hence $[D_i u]_{\mathcal{D}'} = \Psi(\partial_i T)$. 

At this point it is a natural question to ask if there exists a linear map

$\Phi : \mathcal{D}' \rightarrow V^\circ$

which selects in any equivalence class $[u]_{\mathcal{D}'}$ a distribution-like ultrafunction $\phi(u)$ in a way consistent with the distributional derivative, namely

$\Phi(\partial_i T) = D_i \Phi(T)$ (82)

Actually this goal can be achieved in several ways. We will describe one of them by using Th. 3.8:

Definition 4.12. For every $T \in \mathcal{D}'$, we denote by $T^\circ$ the unique ultrafunction in $U^\infty$ (see (52)) such that $\forall \psi \in U^\infty$

$\oint T^\circ(x) \psi(x) \, dx = \lim_{\lambda \uparrow \Lambda} \langle T, \psi_\lambda \rangle = \langle T^*, \psi_\lambda \rangle^*$ (83)

Clearly, $T^\circ$ is a $DL$-ultrafunction since $\forall \varphi \in \mathcal{D}$, $\varphi^* \in U^\infty$ and

$\oint T^\circ(x) \varphi^* \, dx = \lim_{\lambda \uparrow \Lambda} \langle T, \varphi \rangle = \langle T, \varphi \rangle$.

Theorem 4.13. The map $T \mapsto T^\circ$ defined by (83) satisfies (82), namely

$(\partial_i T)^\circ = D_i T^\circ$ (84)

Proof. By (53) and Th. 3.12.50, we have that $\forall \psi \in U^\infty$,

$\oint D_i T^\circ \psi \, dx = -\oint T^\circ D_i \psi \, dx = -\oint T^\circ \partial_i^* \psi \, dx$

$= -\langle T^*, \partial_i^* \psi_\lambda \rangle = \langle \partial_i^* T^*, \psi_\lambda \rangle$

$= \oint \langle \partial_i^* T \rangle^\circ \psi \, dx$

Every function $f \in L^1_{loc}$ defines a distribution $T_f$; then, given $f$, we can define three ultrafunctions: $f^\circ$, $\mu_f^\circ$ and $T_f^\circ$. What is the relation between them? By Prop. 4.2, we have that $\mu_f^\circ$ is a projection of $f^*$ over $V^\circ$, namely

$\mu_f^\circ(x) = \sum_{a \in \Gamma} \left( \int f^*(x) \sigma_a(y) \, dy \right) \delta_a(x)$
Similarly also $T^\circ_f$ is a projection, but over a smaller space as the following proposition shows:

**Proposition 4.14.** If $f \in L^1_{\text{loc}}$, then

$$T^\circ_f = \Pi_\infty f^\circ,$$

where $\Pi_\infty$ has been defined by (54).

**Proof.** By (53), we have that $\psi \in U^\infty$,

$$\oint \Pi_\infty f^\circ \psi \, dx = \lim_{\lambda \uparrow \Lambda} \int f\psi_\lambda \, dx = \lim_{\lambda \uparrow \Lambda} \langle T, \psi_\lambda \rangle = \oint T^\circ f^\circ \psi \, dx$$

Since both $T^\circ_f$ and $\Pi_\infty f^\circ \in U^\infty$, the conclusion follows. \( \square \)

So, if $f \in L^1_{\text{loc}}$, $T^\circ_f$, similarly to $\mu^\circ_f$, destroys some information contained in $f$; namely $T^\circ_f$ (resp. $\mu^\circ_f$) cannot be distinguished by $T^\circ_g$ (resp. $\mu^\circ_g$) if $f$ and $g$ agree almost everywhere. Similarly, if $\mu$ is any Radon measure and $T_\mu$ is the corresponding distribution, then $T^\circ_\mu$ destroys some information contained in $\mu^\circ$ since

$$T^\circ_\mu = \Pi_\infty \mu^\circ$$

**Example.** The $\delta_a$ ultrafunction is distribution like since for every $\varphi \in D$, we have

$$\oint \delta_a \varphi^\circ(x) \, dx = \varphi(a) = \langle \delta_a, \varphi \rangle;$$

(here we have used the boldface to distinguish the ultrafunction $\delta_a$ from the distribution $\delta_a$). However

$$\delta_a \neq \delta_a^\circ$$

Actually, according to Def. 3.13,

$$\delta_a = \delta_a^\circ + \Pi_{\infty}^\perp \delta_a,$$

namely $\delta_a^\circ$ is the smooth part of $\delta_a$.

5. Some Applications

In this section we will sketch how the theory of fine ultrafunctions can be used in the study of Partial Differential Equations. In the framework of ultrafunctions, a very large class of problems is well posed and has solutions. Very often, hard *a priori* estimates are not necessary in proving the existence, but only in understanding the properties of a solution (qualitative analysis). In particular, if you have a problem from Physics or from Geometry, it is interesting to investigate whether the generalized solutions describe the Physical or the Geometric phenomenon. We refer to [6, 7, 11, 20],..., [24] where such kind of problems have been treated in the framework of ultrafunction. *A fortiori*, these problems can be treated using fine ultrafunctions. In this section, we limit ourselves to give some new examples just to illustrate the use of fine ultrafunctions with a particular emphasis in the study of ill posed problems. Obviously, each example is treated superficially. A deep analysis of each case, probably, would deserve a full paper.
5.1. Second Order Equations in Divergence Form
Let \( \Omega \subseteq \mathbb{R}^N \) be a bounded open set with regular boundary and let us consider the following boundary value problem:

\[
-\nabla \cdot [k(x, u)\nabla u] + f(x, u) = 0 \quad \text{in} \quad \Omega \\
u(x) = 0 \quad \text{for} \quad x \in \partial \Omega
\]  
(84)

where \( f \) is a function and \( k(x, u) \) is a function or a \((N \times N)\)-matrix.

A function which satisfies (84) and (85) is called classical solution if \( u \in C^2(\Omega) \).

The natural “translation” of this problem in the world of ultrafunctions is the following:

\[
-\mathcal{D} \cdot [k^\ast(x, u)Du] + f^\ast(x, u) = 0 \quad \text{in} \quad \Omega^\circ \\
u(x) = 0 \quad \text{if} \quad x \in \partial \Omega^\circ
\]  
(86)

We must be very careful in the interpretation of Eq. (86). In fact \( Du \) makes sense if \( u \) is defined in \( \text{vic}(\Omega) \) and \( \mathcal{D} \cdot [k^\ast(x, u)Du] \) makes sense if \( u \) is defined in \( \text{vic}^2(\Omega) := \text{vic}(\text{vic}(\Omega)) \). Thus the problem (86), (87) is well posed if \( f(x, \cdot) \) and \( k(x, \cdot) \) are defined in a neighborhood of \( \Omega \).

In conclusion, we are lead to the following definition:

**Definition 5.1.** A solution of (86), (87), is the restriction to \( \Omega^\circ \) of an ultrafunction \( u^\ast \). \( u^\ast|_{\Omega^\circ} \) will be called ultrafunction solution of the problem (84), (85).

This is the right definition as the following theorem shows:

**Theorem 5.2.** If \( w \) is a classical solution of (84), (85) then \( u = w^\ast|_{\Omega^\circ} \) is an ultrafunction solution.

**Proof.** By our definition of \( C^2(\overline{\Omega}) \), we have that \( w \) can be extended in a neighborhood \( \mathcal{N}_\varepsilon(\Omega) \) of \( \Omega \); consequently also \( f(x, \cdot) \) and \( k(x, \cdot) \) can be extended in \( \mathcal{N}_\varepsilon(\Omega) \) and hence, by the transfer principle, \( w^\ast \) satisfies the equation

\[
-\nabla^\ast \cdot [k^\ast(x, w^\ast)\nabla^\ast w^\ast] + f^\ast(x, w^\ast) = 0 \quad \forall x \in \mathcal{N}^\ast(\Omega)
\]

where, with some abuse of notation we have denoted with \( f(x, \cdot) \) and \( k(x, \cdot) \) the extension of the homonymous functions. Then, \( u = w^\ast|_{\Omega^\circ} \) satisfies (86), (87). \( \Box \)

If a problem does not have a classical solution, we can look for weak solutions in some Sobolev space or in a space of distribution. However if there are not weak solutions, we can find a ultrafunction solution exploiting the following theorem:

**Theorem 5.3.** Assume (88). If the \( \exists M, R \in \mathbb{E}^+ \) such that

\[
\|u\| \geq R \Rightarrow \int_{\Omega^+} [k^\ast(x, u)Du \cdot Dv + f^\ast(x, u)u] \, dx \geq M \cdot \sqrt{\int_{\Omega^+} |u|^2 \, dx},
\]  
(89)

then problem (86), (87) has at least one solution.
Proof. First of all we set
\[ V_{\text{dir}}^\circ (\Omega^\circ) := \{ u \in V^\circ \mid \forall x \in \partial \Omega^\circ, \ u(x) = 0 \} ; \tag{90} \]
\( V_{\text{dir}}^\circ (\Omega^\circ) \) is a hyperfinite Hilbert space equipped with the norm
\[ \|u\| := \sqrt{\int \|u\|^2 \, dx} \]
Now let \( \Psi \) be a continuous function such that \( \Psi(x) = 1 \) if \( x \in N_{\epsilon/2}(\Omega) \), \( \Psi(x) = 0 \) if \( x \notin N_{\epsilon}(\Omega) \) and \( 0 \leq \Psi \leq 1 \). We put
\[ \hat{f}(x, u) = f^*(x, u)\Psi^*(x) \]
\[ \hat{k}(x, u) = k^*(x, u)\Psi^*(x) + [1 - \Psi^*(x)] \]
and we define an operator
\[ \mathcal{A} : V_{\text{dir}}^\circ (\Omega^\circ) \to V_{\text{dir}}^\circ (\Omega^\circ) \]
by setting, \( \forall v \in V_{\text{dir}}^\circ (\Omega^\circ) \),
\[ \oint \mathcal{A}[u] v \, dx = \oint \left[ \hat{k}(x, u)Du \cdot Dv + \hat{f}(x, u)v \right] dx. \]
By (89), we have that \( \exists M, R \in \mathbb{R}^+ \) such that
\[ \|u\| \geq R \Rightarrow \oint \mathcal{A}[u] u \, dx \geq M\|u\|^2. \]
Then, by the Brower fixed point theorem and the fact that \( V^\circ (\Omega^\circ) \) has hyperfinite dimension the equation \( \mathcal{A}[u] = 0 \) has at least a solution \( \bar{u} \in V_{\text{dir}}^\circ (\Omega^\circ) \), and hence \( \forall v \in V_{\text{dir}}^\circ (\Omega^\circ) \),
\[ 0 = \oint \mathcal{A}[\bar{u}] v \, dx = \oint \left[ \hat{k}(x, \bar{u})Du \cdot Dv + \hat{f}(x, \bar{u})v \right] dx \]
\[ = \oint \left[ -D \cdot (\hat{k}(x, \bar{u})Du) + \hat{f}(x, \bar{u}) \right] v \, dx \tag{91} \]
and hence, for every \( x_0 \in \Omega^\circ \), taking \( v = \delta_{x_0} \),
\[ -D \cdot (k^*(x_0, \bar{u})Du) + f^*(x_0, \bar{u}) = 0. \]
since for \( x_0 \in \Omega^\circ \), \( \hat{k} = k^* \) and \( \hat{f} = f^* \). Then \( \bar{u}|_{\Omega^\circ} \) solves (86), (87). \( \square \)

Remark 5.4. By the above theorem we have that \( u|_{\Omega^\circ} \) might depend on the values that \( k^* \) and \( f^* \) assume in \( \Omega^+ \) and hence, assumption (88) seems essential. However Th. 5.2 shows that when \( u|_{\Omega^\circ} \) is sufficiently regular, this dependence disappears.

The solution of (86), (87) is the restriction \( u|_{\Omega^\circ} \) of an ultrafunction \( u \in V^\circ \). It is interesting to examine the nature of \( u \) near \( \partial \Omega^\circ \). By the definition of \( V_{\text{dir}}^\circ (\Omega^\circ) \), we have that
\[ v \in V^\circ \iff v\chi_{\partial \Omega^\circ} \in V_{\text{dir}}^\circ (\Omega^\circ) \]
So, the Eq. (91) is equivalent to
\[ \forall v \in V^\circ, \oint \left[ \hat{k}(x, u)Du \cdot D(v\chi_{\partial \Omega^\circ}) + f^*(x, u)(v\chi_{\partial \Omega^\circ}) \right] dx = 0 \]
Then, we have that, $\forall v \in V^\circ$,

$$\oint -D \cdot \left[ \hat{k}(x,u)Du \right] \chi_{\partial \Omega^\circ} + f^*(x,u)\chi_{\partial \Omega^\circ} \chi_{\partial \Omega} vdx = 0 \quad (92)$$

and hence

$$\left( -D \cdot \left[ \hat{k}(x,u)Du \right] + f^*(x,u) \right) \chi_{\partial \Omega^\circ} = 0. \quad (93)$$

If we set

$$[V^\circ_{\text{dir}} (\Omega^\circ)]^\perp := \left\{ \Phi \in V^\circ \mid \forall v \in V^\circ_{\text{dir}} (\Omega^\circ), \oint \Phi(x)v(x)dx = 0 \right\}$$

$$= \left\{ \Phi \in V^\circ \mid \text{supp}(\Phi) \subseteq \partial \Omega^\circ \right\}$$

Equation (93) can be rewritten as follows:

$$- D \cdot \left[ \hat{k}(x,u)Du \right] + \hat{f}(x,u) = \Phi(x) \quad \text{with} \quad \Phi \in [V^\circ_{\text{dir}} (\Omega^\circ)]^\perp. \quad (94)$$

This equation describes in terms of infinitesimal analysis what happens near $\partial \Omega^\circ$. $\Phi(x)$ can be regarded as a force which constrains $u$ to vanish on $\partial \Omega^\circ$. Essentially $\Phi$ describes the reaction of a constraint.

**Remark 5.5.** If we want to consider equation (84) with non-homogeneous boundary condition, i.e.

$$u(x) = g(x) \quad \text{for} \quad x \in \partial \Omega$$

we can adopt the standard trick to set

$$w(x) = u(x) - \bar{g}(x)$$

where $\bar{g}(x)$ is any function which extends $g$ in $\Omega$ and to solve the resulting equation in $w$ with the homogeneous boundary conditions.

**Remark 5.6.** The equation (84) with the homogeneous Neumann boundary conditions, i.e.

$$\frac{du}{dn}(x) = \nabla u \cdot n_\Omega = 0 \quad \text{for} \quad x \in \partial \Omega \quad (95)$$

can be treated in a very similar way. The boundary condition, in the framework of the ultrafunctions becomes

$$Du \cdot n_E = 0 \quad \text{for} \quad x \in \partial \Omega^\circ \quad (96)$$

where $n_E$ has been defined by (78). The space $V^\circ_{\text{dir}} (\Omega^\circ)$ must be replaced by

$$V^\circ_{\text{neu}} (\Omega^\circ) := \left\{ u \in V^\circ \mid \forall x \in \partial \Omega^\circ, \ Du \cdot n_E = 0 \right\}.$$

Everything else proceeds in a similar way.
5.2. Examples

Example 1. Let us consider the following problem:

\[-\nabla \cdot [k(x, u)\nabla u] + u = f(x) \quad \text{in} \quad \Omega \tag{97}\]
\[u(x) = 0, \quad \text{for} \quad x \in \partial \Omega \tag{98}\]

where \(k \in C^1(\mathbb{R})\). We set

\[k_0 := \inf \{k(x, s) \mid (x, s) \in \Omega \times \mathbb{R}\}\]

and

\[k_0 > 0, \tag{99}\]

then, if

\[k \text{ does not depend on } u, \tag{100}\]

\(A = -\nabla \cdot [k(x) \nabla u] + u\) is a strictly monotone operator and it is immediate to check that Eq. (97) has a unique weak solution in \(H^1_0(\Omega) \cap L^4(\Omega)\) for every \(f \in H^{-1}(\Omega) + L^{4/3}(\Omega)\). Moreover, if \(f\) and \(\partial \Omega\) are smooth, by the usual regularity results, problem (97),(98) has a classical solution. If (99) or (100) is not satisfied, this problem is more delicate. In particular, if for some (but not all) value of \(u\)

\[k(x, u) < 0,\]

the problem is not well posed and, in general, it has no solution in any distribution space. Nevertheless we have the following result:

**Theorem 5.7.** If (88) holds and

\[
\min_{|u| \to \infty} \lim k(x, u) \geq 0 \tag{101}
\]

then problem (97), (98) has at least a ultrafunction solution, namely, \(\forall f \in V^\circ\), there exists \(u \in V^\circ_{\text{dir}}(\Omega)\), such that

\[-D \cdot [k^*(x, u)Du] + u = f, \quad \forall x \in \Omega^\circ. \tag{102}\]

**Proof.** By Th. 5.3 we have to prove that the operator is (102) coercive, namely that (89) holds. By (101) and the continuity of \(k\), there exists a constant \(M > 0\) such that for \(|u| > M\),

\[k(x, u) \geq -\frac{1}{2\|D\|^2}; \quad \|D\| = \max_{u \neq 0} \frac{\|Du\|}{\|u\|}\]

Then, setting

\[k_\infty = \sup \{-k(x, u) \mid |u| < M\}\]
for any $M \geq 0$,
\[
\int_{\Omega^+} \left[ k(x, u) |Du|^2 + |u|^2 \right] dx \\
\geq \int_{\Omega^+, |u| \geq M} k(x, u) |Du|^2 dx + \int_{\Omega^+, |u| < M} k(x, u) |Du|^2 dx + \|u\|^2 \\
\geq -\frac{1}{2 \|D\|^2} \int |Du|^2 dx - k_\infty \max_{u \neq 0, |u| \leq M} \|Du\| + \|u\|^2 \\
\geq -\frac{\|Du\|^2}{2 \|D\|^2} - C + \|u\|^2 \geq \frac{1}{2} \|u\|^2 - C \]

If $k$ is not positive, our problem might have infinitely many solutions and they can be quite wild. For example, if we consider the problem
\[
u \in V^\circ (\Omega^\circ), \quad -D \cdot \left[(u^3 - u)Du\right] = 0. \tag{103}
\]
we can check directly that, for any internal set $E \subset \text{int} (\Omega^\circ)$ the function $u(x) = \chi_E(x)$ is a legitimate solution in the frame of ultrafunction.

If (99) and (100) hold, problem (97), (98) has a classical solution provided that $f$ is sufficiently good. Nevertheless, our problem has a unique generalized solution even if $f$ is a wild function (e.g. $f \not\in H^{-1}(\Omega) + L^{4/3}(\Omega)$ or $f$ not measurable or $f$ is a distribution not in $H^{-1}$). For example if $N \geq 2$, then $\delta_\alpha \not\in H^{-1}(\Omega)$; in this case the ultrafunction solution, for any delta-like $f$, concentrates in $\text{mon}(a)$.

**Example 2.** Let us consider the following problem:
\[
u \in C^2 (\overline{\Omega}) : \quad -\nabla \cdot [k(u)\nabla u] + u^3 = f(x) \quad \text{in} \quad \Omega \tag{104}
\]
\[
u(x) = 0, \quad \text{for} \quad x \in \partial \Omega
\]
where $k$ is a matrix such that
\[
k(u)\xi \cdot \xi \geq -k_0 |\xi|^2, \quad k_0 \geq 0. \tag{105}
\]

**Theorem 5.8.** If (105) holds, problem (97), (98) has at least a ultrafunction solution, namely, $\forall f \in V^\circ$, there exists $u \in V^\circ_{\text{dir}} (\Omega^\circ)$, such that
\[
-D \cdot \left[k^*(u)Du\right] + u^3 = f \quad \forall x \in \Omega^\circ. \tag{106}
\]

**Proof.** By Th. 5.3 we have to prove that the operator is (102) coercive namely that it satisfies (89). By (47), all the norms on $V^\circ_{\text{dir}} (\Omega^\circ)$ are equivalent; hence, we have
that

\[ \int_{\Omega} (-D \cdot [k(u)Du + u^3]) \, u \, dx = \int_{\Omega} [k(u)Du \cdot Du + u^4] \, dx \]

\[ \geq -k_0 \| Du \|^2 + \| u \|^4 \]

\[ \geq -k_0 \| D \|^2 \| u \|^2 + \| u \|^4. \]

an hence the operator (102) is coercive. \( \square \)

As a particular case, we can take

\[ k = \begin{bmatrix} -1 & 0 \\ 0 & c^2 \end{bmatrix}; \]

so equation (104) reduces to the nonlinear wave equation:

\[ \Box u + u^3 = f; \quad \Box = \partial_1^2 - c^2 \partial_2^2 \]

If we take \( \Omega = [0, T] \times [0, 1] \) and we impose periodic boundary conditions in \( x_1 \) this problem reduces to the classical problem relative to the existence of periodic solution of the nonlinear wave equation. In general, this problem does not have classical solutions because of the presence of small divisors which prevent the approximate solutions to converge. So this is a problem that can be studied in the framework of ultrafunctions where the existence is guaranteed.

### 5.3. Regular Weak Solutions

The expression *regular weak* sounds like an oxymoron, nevertheless it well describes the notion we are going to present now.

For example, let us consider the equation (103). It is possible that the function of the form \( \chi_E \) are not acceptable as solutions of a physical model described by (103). Then we may ask if there exist “approximate” solutions of equation (103) which exhibit some form of regularity.

More in general, given the second order operator (84) and (85), we might be interested in regular solutions \( u \in U^m \) for some \( m \in \mathbb{N} \cup \{\infty\} \) (see section 3.5).

For example, let us consider problem (84), (85). We set

\[ U^1_{\text{dir}}(\Omega^c) := \{ u \in U^1 \mid \forall x \in \partial \Omega^c, u(x) = 0 \} \]  

(107)

we translate problem (84),(85) as follows

\[ u \in U^1_{\text{dir}}(\Omega^c) \]  

(108)

such that \( \forall v \in U^1_{\text{dir}}(\Omega^c), \)

\[ \int_{\Omega^c} [k^*(x, u)Du \cdot Dv + f^*(x, u)v] \, dx = 0 \]  

(109)

Arguing as in theorem 5.3, we have the following result:

**Theorem 5.9.** If the operator

\[ A : U^1_{\text{dir}}(\Omega^c) \rightarrow U^1_{\text{dir}}(\Omega^c) \]

defined by

\[ \int_{\Omega} A[u] \, v \, dx = \int_{\Omega^c} [k^*(x, u)Du \cdot Dv + f^*(x, u)v] \, dx \quad \forall v \in U^1(\Omega), \]
is coercive, the equation (109) has at least one solution.

In general a solution of (108), (109) does not satisfy equation (95), but the equation

\[- D \cdot [k^*(x, u)Du] + f^*(x, u) = \psi(x) \quad \text{in} \quad \Omega^\circ \quad (110)\]

where

\[\psi(x) \in U^{1}_{\text{dir}}(\Omega^\circ)^\perp = \{ w \in U^{1}_{\text{dir}}(\Omega^\circ) | \forall v \in U^{1}_{\text{dir}}(\Omega^\circ), \oint wv \, dx = 0 \} \quad (111)\]

can be considered as a sort of error. The error \( \psi \) can be considered negligible since

\[\forall v \in U^{1}(\Omega^\circ), \oint \psi v \, dx = 0\]

Probably, in many situation, the regular weak solutions are more relevant than the solution of type (95).

**Remark 5.10.** If we adopt the strategy of using regular weak solutions, the more appropriate functional framework is the use of the quotient space

\[\tilde{U}^{1} := V^{\circ} / I\]

where

\[I := U^{1}_{\text{dir}}(\Omega^\circ)^\perp = \{ \psi \in V^{\circ} | \forall v \in U^{1}, \oint \psi v \, dx = 0 \}\]

The space \( \tilde{U}^{1} \) is the analogous, in the world of ultrafunctions, of the Sobolev space \( H^{1} \); in fact in both cases the functions are not defined pointwise, but they are classes of equivalence of functions defined up to negligible functions.

**Remark 5.11.** For suitable choices of \( k, f \) and \( \Omega \), it is possible that the regular weak solutions coincide with the ultrafunctions solutions, however for ill posed problem in general this fact does not happens. Clearly, in most cases the regular weak solutions are infinitely close to ultrafunctions solutions in the appropriate topology (e.g. in the topology of \( H^{1}(\Omega) \)). However, if \( \partial \Omega \) is very wild they can differ from each other in a relevant way. Then, in dealing with a problem relative to a physical phenomenon, the choice of the space in which to work might be very relevant (see also Remark 3.14).

### 5.4. Calculus of Variations

Let us consider the minimization problem of the functional

\[J(u) = \int_{\Omega} F(x, u, \nabla u) \, dx \quad (112)\]

If we assume the Dirichlet boundary condition, the natural space where to work is \( C^{1}(\Omega) \cap C^{0}_{0}(\overline{\Omega}) \). If we translate this problem in the framework of ultrafunctions the natural space is \( V^{1}_{\text{dir}}(\Omega^\circ) \) defined by (90) and the condition (88) is translated in the assumption that \( F(x, u, \xi) \) be defined in \( \mathcal{N}_{\varepsilon}(\Omega) \times \mathbb{R} \times \mathbb{R}^{N} \). Then the functional (112) becomes

\[J^{\circ}(u) := \oint F^{*}(x, u, Du) \, d_{\Omega}x. \quad (113)\]
and we have the following result:

**Theorem 5.12.** If (113) is coercive, then \( J^\circ(u) \) has a minimizer \( \bar{u} \) in \( V^\circ_{\text{dir}}(\Omega^\circ) \). Moreover, if \( J(u) \) has a minimizer \( w \in C^1(\overline{\Omega}) \), then, \( \forall x \in \overline{\Omega} \),

\[
\bar{u}(x) = w^*(x).
\]

**Proof.** Trivial. \( \square \)

Also in this case it is interesting to see the form assumed by The Euler-Lagrange equations. For simplicity, we assume that \( F^*(x, u, Du) = \frac{1}{2}k^*(x, u)|Du|^2 + h^*(x, u) \)
then,

\[
dJ^\circ(u)[v] := \oint [k^*(x, u)Du \cdot Dv + f(x, u)v] \, d\Omega x; \quad f(x, u) = \frac{\partial h(x, u)}{\partial u}.
\]

By (63), we have that, \( \forall v \in V^1_{\text{dir}}(\Omega^\circ) \)

\[
dJ^\circ(u)[v] := \oint [k^*(x, u)Du \cdot Dv + f(x, u)v] \, \mu(x) \, dx.
\]

and arguing as we did at the end of Sect. 5.1 (see Eq. (94)) we get the equation

\[
- D \cdot [\mu^\circ_{\overline{\Omega}}(x)k^*(x, u)Du] + f^*(x, u)\mu^\circ_{\overline{\Omega}}(x) = \Phi(x) \quad \text{with} \quad \Phi \in [V^\circ_{\text{dir}}(\Omega^\circ)]^\perp.
\]

(114)

It is interesting to note that Eqs. (94) and (114) are slightly different; they coincide for the \( x \in \Omega^\circ \) when \( \text{mon}(x) \subset \Omega^\circ \), but they differ for some \( x \sim \partial \Omega^\circ \). This fact depend on the fact that in the framework of the ultrafunction a weak solution formulated with the measure \( \mu^\circ_{\overline{\Omega}} \) does not coincide with a solution defined by (95), (87). In any case, we have seen that the regular solutions are the same.

The minimization problem in \( C^1(\overline{\Omega}) \), with no constraint on the boundary, in the regular case leads to Eq. (84) with the Neumann boundary conditions; in the world of ultrafunctions, this problem gives the equation

\[
- D \cdot [\mu^\circ_{\overline{\Omega}}(x)k^*(x, u)Du] + f^*(x, u)\mu^\circ_{\overline{\Omega}}(x) = 0.
\]

(115)

Thus the boundary conditions are included in this equation.

When we deal with the calculus of variations the notion of regular weak solutions arises in a natural way. In fact, it makes perfect sense to minimize the functional (113) in the spaces

\[
U^m_{\text{dir}}(\Omega^\circ) := V^\circ_{\text{dir}}(\Omega^\circ) \cap U^m; \quad m \geq 0.
\]

If \( J^\circ \) is coercive, the minimizer exists in each space and if the minimizer in \( V^\circ_{\text{dir}}(\Omega^\circ) \) is not \( m \)-regular it is different from the minimizer in \( U^m_{\text{dir}}(\Omega^\circ) \); this phenomenon is typical of the ultrafunctions and it does not have any analogous thing in the framework of \( C^m \)-functions or in the Sobolev spaces.
Example. Let us consider the functional defined in $C^1(0,1) \cap C^0_0([0,1])$

$$J(u) = \int_{\Omega} \left[ \left( |\partial u|^2 - 1 \right)^2 + u^2 \right] \, dx$$

(116)

which presents the well known Lavrentiev phenomenon, namely every minimizing sequence $u_n$ converges uniformly to 0 but

$$0 = \lim_{n \to \infty} J(u_n) \neq J(0) = 1.$$ 

If we use Th. 5.12, the minimizing sequence $u_\lambda \in U^1_\lambda(\Omega)$ has an infinitesimal $\Lambda$-limit $\bar{u} \in U^1_{\text{dir}}(\Omega^\circ)$ and $J^\circ(\bar{u})$ is a positive infinitesimal which satisfies the following Euler-Lagrange equations in $\text{int}(\Omega^\circ)$:

$$D \left[ \left( |D\bar{u}|^2 - 1 \right) D\bar{u} \right] - \frac{1}{2} \bar{u} = \psi$$

(117)

where

$$\forall v \in U^1(\Omega)^\perp, \int \psi v \, dx = 0.$$ 

However, it is possible to minimize the functional

$$J^\circ(u) := \int \left[ \left( |Du|^2 - 1 \right)^2 + u^2 \right] \, dx$$

in $V^\circ_{\text{dir}}(\Omega^\circ)$; in this case we get a minimizer $\hat{u}$ such that $J^\circ(\hat{u}) < J^\circ(\bar{u})$ and in $\text{int}(\Omega^\circ)$ we have:

$$D \left[ \left( |D\hat{u}|^2 - 1 \right) D\hat{u} \right] - \frac{1}{2} \hat{u} = 0.$$ 

The function $\psi$ in equation (117) can be interpreted as a structural “force” which prevents $u$ to form “angles”. The presence of this force, increases the energy level and we have that

$$J^\circ(\bar{u}) > J^\circ(\hat{u}) > 0$$

even if $J^\circ(\bar{u})$ remains infinitesimal.

Also this example shows how the choice of the space where to work changes the solution of the problem and hence, as observed in Remarks 5.11 and 3.14, the choice of a particular space depends on the phenomenon which we want to describe.

Remark 5.13. We have assumed $F(x,u,\xi)$ to be continuous in $u$ and $\xi$; however also the case of discontinuous functions can be easily analyzed (see e.g. [20]).

5.5. Evolution Problems

Let $\Omega \subset \mathbb{R}^N$ be an open set and let

$$A(x, \partial_i) : D_A(\Omega) \to C(\Omega) ;$$

be a differential operator. Here $D_A(\Omega)$denotes the domain of $A(x, \partial_i)$.

We consider the following Cauchy problem: given $u_0(x) \in C^0(\Omega)$, find

$$u \in C^1(I,D_A(\Omega)) \cap C^0(I,C(\Omega)), \ 0 \in I \subseteq \mathbb{R} :$$

(118)

$$\partial_t u = A(x, \partial_i)[u]$$

(119)

$$u(0,x) = u_0(x).$$

(120)
A function which satisfies (118), (119), (120) is called classical solution. We want to translate this problem in the world of ultrafunctions. Because of the nature of the Cauchy problem, it is not convenient to translate this problem in the space $V^\circ (\mathbb{R} \times \Omega)$. It is better to use the internal space of functions $C^1(E,V^\circ)$ defined in Sect. 3.6. We assume that the boundary conditions which are “contained” in the definition of the domain $D_A(\Omega)$ can be translated in a domain $D_A^\circ(\Omega) \subset V^\circ$ (in Sect. 5.1 we have seen how this can be done for second order operators with Dirichlet and Neumann boundary conditions). Then, setting

$$C^1(I^*, D_A^\circ(\Omega)) = \{ u \in C^1(E,V^\circ) \mid \forall t \in I^*, u(t,x) \in D_A^\circ(\Omega) \}$$

The problem (118), (119), (120) translates in the following one:

1. $u \in C^1(I, D_A^\circ(\Omega))$ (121)
2. $\partial_t^* u = A^*(x,D_i)[u], t \in I^*$ (122)
3. $u(0,x) = u_0(x)$ (123)

Remember that the time derivative $\partial_t^*$ is not the generalized derivative, but the natural extension of $\partial_t$ defined by (59).

A solution of (121), (122), (123) will be called ultrafunction solution of the problem (118), (119), (120).

Using this notation, we can state the following fact:

**Theorem 5.14.** If $w$ is a classical solution of (118), (119), (120) and we assume that $w$ be extended continuously in a neighborhood $\mathcal{N}(\Omega)$; then $u = w^\circ$ is a ultrafunction solution of (121), (122), (123).

**Proof.** It is the same than the proof of Th. 5.2. □

Also in the evolution case, the conditions which guarantee the existence of a ultrafunction solution are very weak.

**Theorem 5.15.** Assume that $A(x, D_i)[u]$ restricted to $V_{A}(\Omega)$ is locally Lipschitz continuous in $u$; then there exists $T_{A}$ such that the problem (121), (122), (123) has a unique ultrafunction solution for $t \in [0,T_{A})$. Moreover, if there is an a priori bound for such a solution, then there exists a unique ultrafunction solution in $C^1(I^*, D_A^\circ(\Omega))$.

**Proof.** For every $\lambda$ consider the following system of ODE's in $\mathfrak{F}(\Gamma_{\lambda} \cap \Omega)$

$$\partial_t u_{\lambda}(t,a_{\lambda}) = A(x,D_{i\lambda})[u_{\lambda}](t,a_{\lambda}), \ a_{\lambda} \in \Gamma_{\lambda}$$

(124)

Since $\mathfrak{F}(\Gamma_{\lambda})$ is a finite dimensional vector space and $A(x,D_{i\lambda})[u]$ restricted to $V_{A}(\Omega)$ is locally Lipschitz continuous the above system has a local solution for $t \in [0,T_{\lambda})$. Then, setting

$$T_{A} = \lim_{\lambda \uparrow A} T_{\lambda}$$

and taking the $\Lambda$-limit in (124), $\forall t \in [0,T_{A})$, we get

$$\partial_t^* u(t,a) = A^*(x,D_i)[u](t,a)$$
Moreover, if there is an \textit{a priori} bound for such a solution, then there is a bound for the approximate solution \((124)\) in \([0, T_\lambda)\) (which might also depend on \(\lambda\)); then, it is well known that \([0, T_\lambda) = I\) and hence \([0, T_\lambda) = I^*\) \(\square\)

5.6. Some Examples of Evolution Problems

\textit{Example 1.} Let \(\Omega\) be a bounded open set and let us consider the following problem:

\[
 u \in C^1(I, C^0_0(\Omega)) \cap C^0(I, C^2(\Omega)) : \tag{125}
\]

\[
 \partial_t u = \nabla \cdot [k(u)\nabla u] \quad \text{for} \quad x \in \Omega \tag{126}
\]

\[
 u(0, x) = u_0(x) \tag{127}
\]

If \(k(u)\) satisfies (99), then (126) is a parabolic equation and the problem, under suitable condition, has a classical solution. If \(k(u) < 0\) for some \(u \in \mathbb{R}\), then the problem is ill-posed, and in general classical solutions do not exist.

Let us translate this problem in the world of ultrafunctions; taking account of the results of Sect. 5.1, we get:

\[
 u \in C^1(I^*, V^\circ_{\text{dir}}(\Omega^\circ)) \tag{128}
\]

\[
 \partial^*_t u = D \cdot [k^*(u)Du] \quad \text{for} \quad x \in \Omega^\circ \tag{129}
\]

\[
 u(0, x) = u_0^\circ(x) \tag{130}
\]

So we can apply the theorems 5.15 and we get the existence of a unique (local in time) solution. It is not difficult to get sufficient conditions which guarantee the existence of a solution for every \(t \in I^*\). For example

\textbf{Theorem 5.16.} \textit{If the set}

\[
 B = \{ r \in \mathbb{R} \mid k(r) < 0 \} \tag{131}
\]

is bonded, problem (128), (129), (130) has a global solution.

\textit{Proof.} We set \(k(r) = k^+(r) - k^-(r)\) and with some abuse of notation we write, for \(r \in \mathbb{E}\), \(k^*(r) = k^+(r) - k^-(r)\). Then, we have that

\[
 \partial^*_t \oint u^2 d\Omega x = 2 \oint u \partial^*_t u \ d\Omega x = 2 \oint u D \cdot [k^*(u)Du] d\Omega x \]

\[
 = -2 \oint k^*(u) |Du|^2 d\Omega x \leq 2 \oint k^-(u) |Du|^2 d\Omega x \]

\[
 \leq 2 \|D\|^2 \oint k^-(u) |u|^2 d\Omega x
\]

If we set

\[
 M = \max_{r \in \mathbb{R}} k^-(r) \|r\|^2
\]

then, we have that

\[
 \partial^*_t \oint u^2 d\Omega x \leq 2M \|D\|^2 \oint d\Omega x
\]

and this implies the existence of a global solution. \(\square\)
In many applications of Eq. (126) \( u \) represent a density and \( k(u) \nabla u \) its flow. Then, thanks to the generalized Gauss’ theorem 4.7, it is easy to prove that the “mass” \( \int_\Omega u(x) dx \) is preserved up to the flow crossing \( \partial \Omega \):

\[
\partial_t^* \int u(x) d\Omega x = \int \partial_t^* u(x) d\Omega x
= \int D \cdot [k^*(u) Du] d\Omega x
= \int k^*(u) Du \cdot n_\Omega |D\theta_E| dx
\]

If we want to model a situation where the flow of \( u \) cannot cross \( \partial \Omega \), \( \Omega \) bounded, in a classical context, the Neumann boundary conditions are imposed:

\[
\forall x \in \partial \Omega, \quad \frac{du}{dn}(x) = 0; \quad (132)
\]

in the framework of ultrafunctions this situation can be easily described using the analog of Eq. (115) for equation (126):

\[
\partial_t^* u = D \cdot [\mu_\Omega^*(x) k^*(u, Du)] \quad (133)
\]

In this case, we have the flow \( \mu_\Omega^*(x) k^*(u) Du \) vanishes out of \( \text{vic}^2(\Omega^\circ) \) and we have the conservation of the “mass” \( \int u(x) dx \). We can prove this fact directly, in fact, since \( \Omega \) is bounded, \( \forall x \in \text{vic}(\Omega^\circ), D1^\circ = 0; \) hence

\[
\partial_t^* \int u(x) dx = \int D \cdot [\mu_\Omega^*(x) k^*(u) Du] dx
= \int D \cdot [\mu_\Omega^*(x) k^*(u) Du] 1^\circ dx
= -\int k^*(u) \mu_\Omega^* Du \cdot D1^\circ dx = 0.
\]

Remark 5.17. This problem has been studied with nonstandard methods by Bottazzi in the framework of grid functions [26]. One of the main differences is that in the context of [26], theorem 5.14 does not hold.

Example 2. Now let us consider the following “conservation law”:

\[
u \in C^1(I, C^0(\Omega)) \cap C^0(I, C^1(\Omega)) : \quad (134)
\]

\[
\partial_t u = \nabla \cdot F(x, u) \quad (135)
\]

\[
u(0, x) = u_0(x) \quad (136)
\]

where \( F : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N \) is a smooth function with support in \( \overline{\Omega} \) as well as the initial data \( u_0(x) \). This problem is not well posed and when \( N > 1 \) and very little is known. Nevertheless this problem is well posed in the frame of ultrafunctions:

\[
u \in C^1(I, V^\circ(\text{vic}(\Omega^\circ))) : \quad (137)
\]

\[
\partial_t^* u = D \cdot F^*(x, u) \quad (138)
\]

\[
u(0, x) = u_0^*(x) \quad (139)
\]
Theorem 5.18. If
\[
\left| \frac{\partial F}{\partial u} \right| \leq c_1 + c_2 |u| , \tag{140}
\]
problem (134), (135), (136) has a unique (global in time) ultrafunction solution \( u(t, x) \) which satisfies the following properties:

\[
\text{supp}(u(t, x)) \subset \text{vic}(\Omega^\circ) \tag{141}
\]
\[
\partial_t^* \oint u \, dx = 0 \tag{142}
\]

Proof. The existence of a unique solution follows from Th. 5.15. \( (141) \) follows from the fact that for every \( x \not\in \text{vic}(\Omega^\circ) \), \( D \cdot F(x, u) = 0 \). \( (142) \) is an immediate consequence of the Gauss’ theorem 4.7. \( \square \)

A particular case of Eq. (138) is the Burger’s equation
\[
\partial_t u = -u \partial_x u \\
u(0, x) = u_0(x) \geq 0.
\]
It is well known that this equation has infinitely many weak solutions which preserve the mass. One of them, the entropy solution, describes the phenomena occurring in fluid mechanics. It is convenient to write the Burger’s equation in the framework of ultrafunctions as follows:

\[
u \in C^1(E, V^\circ) \tag{143}
\]
\[
\partial_t^* u = -D_x \left( \frac{u |u|}{2} \right) \tag{144}
\]
\[
u(0, x) = u_0(x) \tag{145}
\]

Since the right hand side of this equation does not satisfy \( (140) \), the formulation \( (144) \) grants \textit{a priori} bounds and hence the existence of a global solutions. In fact, by \( (39) \),
\[
\partial_t^* \oint |u|^3 \, dx = 3 \oint (u |u|) \partial_t u \, dx = -\frac{3}{2} \oint (u |u|) D_x (u |u|) \, dx = 0
\]
The solutions of Eq. (144) are different from the entropy solution since they preserves also the quantity \( \oint |u|^3 \, dx \). The viscosity solution can be modelled in the frame of ultrafunctions by the equation
\[
\partial_t^* u = -D_x \left( \frac{u^2}{2} \right) + \nu D_x^2 u
\]
where \( \nu \) is a suitable infinitesimal (see [17]).

Finally we remark that in the frame of ultrafunctions the equation \( (144) \) is different from the equation
\[
\partial_t^* u = -uD_x u \tag{146}
\]
even if \( u \geq 0 \); in fact, in the point where \( u \) is singular \( D_x (u^2) \neq 2uD_x u \) and \textit{a priori} Eq. (146) is not a conservation law since it does not have the form (138).
Nevertheless the mass is preserved since, by (39),
\[ \partial_t \oint u \, dx = - \oint u D_x u \, dx = 0 \]
It is immediate to see that \( u_0^\circ(x) \geq 0 \) implies that, \( \forall t, u(t, x) \geq 0 \) and hence, by the fact that \( \oint |u| \, dx \) is constant, we get an a priori bound and the existence of a global solution.

Moreover, if \( \text{supp}(u_0) \subset [a, b] \) then \( \text{supp}(u(t, \cdot)) \subset \text{vic}([a, b] \circ) \) since for every \( x \notin \text{vic}([a, b] \circ), \partial_t u(x) = 0 \) for every \( t \in \mathbb{E} \). So a new phenomenon occurs: the mass concentrates in the front of the shock waves. For example consider the initial condition
\[ u_0(x) = \begin{cases} x & \text{if } x \in [0, 1] \circ \\ 0 & \text{if } x \in \Gamma \setminus [0, 1] \circ \end{cases} \]
In this case the solution, for \( t \geq 0 \), is
\[ u(t, x) = \begin{cases} \frac{x}{t+1} & \text{if } x \in [0, 1] \circ \setminus \text{mon} \,(1) \\ 0 & \text{if } x \leq 0 \text{ or } x > 0. \end{cases} \]
and hence, as \( t \to \infty \), all the mass concentrates in \( \text{mon} \,(1) \).

In conclusion, the translation of the Burger’s equation in the frame of ultrafunctions, leads to several different situation which might reproduce different physical models and different solutions (related to different weak solutions).

**Example 3.** Let us consider the following problem:
\[ u \in C^2(I, C^0) \cap C^0(I, C^2) \]
\[ \Box u + |u|^{p-2} u = 0 \quad \text{in} \quad \mathbb{R}^N; \quad p > 2 \]
\[ u(0, x) = u_0(x), \]
\[ \partial_t u(0, x) = u_1(x); \]
where
\[ \Box u = \partial_t^2 u - \Delta u. \]
For simplicity we assume that \( u_0(x) \) and \( u_1(x) \) have compact support.

It is well known that this problem has a unique weak solution in suitable Sobolev spaces, provided that, for \( N \geq 3 \),
\[ p \leq 2^* = \frac{2N}{N-2} \]
and, in this case, the energy
\[ E(u(t, \cdot)) = \int \left( \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{p} |u|^p \right) \, dx \]
is a constant of motion.
If \( p > 2^* \), the problem has a weak solution but it is an open question if it is unique and if the energy is preserved. This problem can be translated and generalized in the framework of ultrafunctions:

\[
\begin{align*}
  &u \in C^2(I, V^\circ) \quad (147) \\
  &\partial_t^2 u - D^2 u + |u|^{p-2} u = 0 \quad \text{for } x \in \Gamma \quad (148) \\
  &u(0, x) = u_0^\circ(x); \; \partial_t u(0, x) = u_1^\circ(x). \quad (149)
\end{align*}
\]

**Theorem 5.19.** *The problem (147), (148), (149) has a unique solution. Moreover the energy equality holds, namely*

\[
\partial_t^* E(u(t, \cdot)) = \partial_t^* \int \left( \frac{1}{2} |\partial_t^* u|^2 + \frac{1}{2} |D u|^2 + \frac{1}{p} |u|^p \right) \, dx = 0
\]

**Proof.** Although the solution of this problem could be easily proved directly, we will reduce this problem to the form (121), (122), (123) by setting

\[
\begin{align*}
  &\partial_t^* u = w \\
  &\partial_t^* w = D^2 u - |u|^{p-2} u
\end{align*}
\]

so problem (147), (148), (149) can be reformulated as follows:

\[
\begin{align*}
  &u \in C^1(I^*, (V^\circ)^2) : \\
  &A^\circ(D) [u] = 0 \\
  &u(0, x) = u_0(x); \; w(0, x) = u_1(x)
\end{align*}
\]

where

\[
\begin{align*}
  u = \begin{bmatrix} u \\ w \end{bmatrix} \quad \text{and} \quad A^\circ(D) [u] = \begin{bmatrix} w \\ D^2 u - |u|^{p-2} u \end{bmatrix};
\end{align*}
\]

Then the existence of a unique solution is guaranteed by Th. 5.15 provided that we get an *a priori* estimate. This is given by the conservation of the energy:

\[
\begin{align*}
  &\partial_t^* E(u(t, \cdot)) = \partial_t^* \int \left( \frac{1}{2} |w|^2 + \frac{1}{2} |D u|^2 + \frac{1}{p} |u|^p \right) \, dx \\
  &= \int \left( w \partial_t^* w + D u \cdot \partial_t^* D u + |u|^{p-1} \partial_t^* u \right) \, dx \\
  &= \int \left( w \partial_t^* w + D u \cdot D w + |u|^{p-1} w \right) \, dx \\
  &= \int \left( \partial_t^* w - D^2 u + |u|^{p-1} \right) w \, dx = 0
\end{align*}
\]

Notice that \( \partial_t^* \) and \( D \) commute, since \( \partial_t^* \) “behaves” as an ordinary derivative and \( D \) “behaves” as a matrix in a finite dimensional space with the coefficients independent of time. \( \square \)

### 5.7. Linear Problems

Let us consider the following linear boundary value problem:

\[
\begin{align*}
  &u \in C^2(\Omega) \cap C^0_0(\bar{\Omega}) \quad (150) \\
  &-\nabla \cdot [k(x)\nabla u] + \lambda u = f(x) \quad \text{in} \; \Omega, \; f \in C^0(\bar{\Omega}). \quad (151)
\end{align*}
\]
For what we have discussed in Sect. 5.1, it is convenient to translate it in the framework of ultrafunctions as follows:

\[ u \in V_\text{dir}^\circ (\Omega^\circ) \]

\[ -D \cdot [k^*(x)Du] + \lambda u = f^*(x), \]  

Since the operator \(-D \cdot [k^*(x)Du]\) is symmetric it has a hyperfinite spectrum \(\Sigma = \{\lambda_k\}_{k \in K}\) with an orthonormal basis of eigenvalues \(\{e_k\}_{k \in K}\). Then the Fredholm alternative holds and if \(\lambda \notin \Sigma\), problem \((152), (153)\) has a unique solution given by

\[ u(x) = \sum_{k \in K} \frac{f_k}{\lambda_k + \lambda} e_k \]  

where \(f(x) = \sum_{k \in K} f_k e_k(x)\).

The operator \(L^o u := -D \cdot [k^*(x)Du]\) can be regarded as a sort of selfadjoint realization of \(L\) with respect to the scalar product \((u, v) \mapsto \int uv\, d\Omega\).

Let us examine the spectrum of \(L^o\) in some cases. If \(k(x)\) is a strictly positive smooth function and \(\Omega\) is bounded, then the classical operator

\[ Lu = -\nabla \cdot [k(x)\nabla u] \]

has a discrete spectrum of positive eigenvalues which correspond to an orthonormal basis \(\{e_k\}_{k \in \mathbb{N}}\) of smooth functions. Then the spectrum of \(L^2 u\) has an orthonormal basis \(\{h_k\}_{k \leq \dim(V_\text{dir}^\circ (\Omega^\circ))}\). For some infinite number \(k < \dim(V_\text{dir}^\circ (\Omega^\circ))\), \(\{h_k\}\) coincides with the spectrum of \(L^*\); in particular, if \(k \in \mathbb{N}\), the eigenvalues of \(L^2\) coincide with the eigenvalues of \(L\) and \(h_k = e_k^\circ\).

If \(k(x)\) is negative in a subset of \(\Omega\) of positive measure, then \(Lu\) has a continuum unbounded spectrum. In this case the eigenvalues of \(L^o\) are infinitely close to each other.

**Example.** Let us consider the following ill posed problem relative to the Tricomi equation:

\[ \partial_1^2 u + x_1 \partial_2^2 u = 0 \text{ in } \Omega; \quad 0 \in \Omega. \]

\[ u = g(x) \text{ for } x \in \partial \Omega \]

In this case

\[ L^o u := - (D_1^2 u + x_1 D_2^2 u) \theta_\Omega^o(x). \]

It is not difficult to prove that for a “generic” open set \(\Omega\), 0 is not in the spectrum of \(L^o u\). In this case, using Remark 5.5, this problem has a unique ultrafunction solutions.

### 6. A Model for Ultrafunctions

This section is devoted to prove Th. 6.23 namely to the construction a space of fine ultrafunctions. The construction presented here is the simplest which we have been able to find. Nevertheless it is quite involved but we do not know if a substantially simpler one exists. The main difficulty relies in the fact that all the properties of Def. 3.5 need to be satisfied simultaneously; in particular, properties (37) and (40) are quite difficult to be obtained simultaneously.
Our model of fine ultrafunctions combines the theory of ultrafunctions with the techniques related to step functions. Roughly speaking we can say that, in this model, the fine ultrafunctions, as well as the standard continuous functions, can be well approximated by step functions and this fact is a cornerstone of our construction.

6.1. A Construction of the Euclidean Numbers

As we have seen, a basic tool in the theory of ultrafunctions is the field of Euclidean numbers and the notion of Λ-limit. Although a construction of a field which satisfies Axiom 2.9 can be found in several papers, we repeat this construction here for the sake of the reader (see e.g. [6,12], see also [8,19] for richer models).

**Theorem 6.1.** Let \( \mathcal{L} \) be defined by (3). There exists a field \( \mathbb{E} \supset \mathbb{R} (\mathbb{E} \neq \mathbb{R}) \) and a surjective field homomorphism,

\[
J : \mathfrak{F}(\mathcal{L}, \mathbb{R}) \to \mathbb{E}
\]

namely a map with the following properties:

\[
J (\varphi + \psi) = J (\varphi) + J (\psi), \\
J (\varphi \cdot \psi) = J (\varphi) \cdot J (\psi).
\]

**Proof.** Let \( \mathcal{U} \) be a fine ultrafilter on \( \mathcal{L} \), namely a filter of sets such that

- Maximalitv: \( Q \in \mathcal{U} \iff L \setminus Q \notin \mathcal{U} \);
- Finess: \( \forall \lambda \in \mathcal{L}, Q [\lambda] \in \mathcal{U} \), where

\[
Q [\lambda] := \{ \mu \in \mathcal{L} | \mu \supseteq \lambda \}.
\]  

The existence of \( \mathcal{U} \) is a well known and easy consequence of Zorn’s Lemma. We use \( \mathcal{U} \) to introduce an equivalence relation on nets, by letting for all \( \psi, \varphi \in \mathfrak{F}(\mathcal{L}, \mathbb{R}) \)

\[
\varphi \approx_\mathcal{U} \psi \iff \exists Q \in \mathcal{U}, \forall \lambda \in Q, \varphi (\lambda) = \psi (\lambda).
\]

We set

\[
\mathbb{E} := \mathfrak{F}(\mathcal{L}, \mathbb{R}) / \approx_\mathcal{U}
\]

and we denote by \([\varphi]_\mathcal{U}\) the equivalence classes. The operations on \( \mathbb{E} \) can be easily defined by letting

\[
[\varphi + \psi]_\mathcal{U} = [\varphi + \psi]_\mathcal{U} ; \quad [\varphi]_\mathcal{U} \cdot [\psi]_\mathcal{U} = [\varphi \cdot \psi]_\mathcal{U}.
\]

It is very well known (see e.g. [31]) and simple to show that, thanks to \( \mathcal{U} \) being an ultrafilter, \( \mathbb{E} \) endowed with the above operations is a field. The operator \( J \) is defined by the canonical projection

\[
J (\varphi) := [\varphi]_\mathcal{U}.
\]  

**Proposition 6.2.** The set

\[
\mathbb{E}^+ = \{ J (\varphi) | \forall \lambda \in \mathcal{L}, \varphi (\lambda) > 0 \}
\]

provides \( \mathbb{E} \) of the linear order structure.
Proof. We need to prove that
\[ E = E^+ \cup \{0\} \cup E^- \]
namely, if we take \( \xi \in E \setminus \{0\} \), then \( \exists \varphi \in E^+ \) such that
\[ \xi = J(\varphi) \text{ or } \xi = J(-\varphi). \]
By the surjectivity of \( J \) there exists \( \psi \) such that
\[ \xi = J(\psi). \]
If we set
\[ R^+ = \{ \lambda \in \mathfrak{L} \mid \psi(\lambda) > 0 \} \]
\[ R^- = \{ \lambda \in \mathfrak{L} \mid \psi(\lambda) < 0 \} \]
\[ R^0 = \{ \lambda \in \mathfrak{L} \mid \psi(\lambda) = 0 \} \]
then
\[ \mathfrak{L} = R^+ \cup R^- \cup R^0 \]
Since \( \mathcal{U} \) is an ultrafilter, only one of the sets \( R^+, R^-, R^0 \) is in \( \mathcal{U} \). It is not possible that \( R^0 \in \mathcal{U} \) since
\[ 0 \neq \xi = J(\psi) = [\psi]_\mathcal{U}. \]
If \( R^+ \in \mathcal{U} \), then \( \xi \in E^+ \) since
\[ \xi = J(\psi) = J(\psi^+) \]
where
\[ \psi^+(\lambda) := \begin{cases} \psi(\lambda) & \text{if } \lambda \in R^+ \\ 1 & \text{if } \lambda \in R^- \cup R^0 \end{cases} \]
If \( R^- \in \mathcal{U} \), then, arguing in a similar way, \( \xi \in E^- \).

Now, we can define the notion of \( \Lambda \)-limit. For every net \( \varphi : \mathfrak{L} \to V_0(\mathbb{R}) \), we set
\[ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = J(\varphi) \quad (155) \]
and for a net \( \varphi : \mathfrak{L} \to V_n(\mathbb{R}) \), \( n \geq 0 \), we define the \( \Lambda \)-limit by induction. If \( n = 0 \), \( \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \) has been defined above; if \( n > 0 \), we set
\[ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) := \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \forall \lambda \in \mathfrak{L}, \psi(\lambda) \in \varphi(\lambda) \right\}. \quad (156) \]

**Theorem 6.3.** The \( \Lambda \)-limit defined by (155) and (156) satisfies the requests of Axiom 2.9.

Proof. Let us check each request separately.

2.9.1 - If eventually
\[ \varphi(\lambda) = \psi(\lambda), \]
the above relation is satisfied by every \( \lambda \in Q[\lambda_0] \) for a suitable \( \lambda_0 \). By (154), \( Q[\lambda_0] \in \mathcal{U} \) and hence \( J(\varphi) = J(\psi) \).
2.9.2 - By (156), it is immediate to see that
\[
\lim_{\lambda \uparrow \Lambda} \{ \varphi_1(\lambda), ..., \varphi_n(\lambda) \} \subseteq \left\{ \lim_{\lambda \uparrow \Lambda} \varphi_1(\lambda), ..., \lim_{\lambda \uparrow \Lambda} \varphi_n(\lambda) \right\}
\]
It is more delicate to show that
\[
\left\{ \lim_{\lambda \uparrow \Lambda} \varphi_1(\lambda), ..., \lim_{\lambda \uparrow \Lambda} \varphi_n(\lambda) \right\} \subseteq \lim_{\lambda \uparrow \Lambda} \{ \varphi_1(\lambda), ..., \varphi_n(\lambda) \}
\]
namely that for every \( \xi \in \lim_{\lambda \uparrow \Lambda} \{ \varphi_1(\lambda), ..., \varphi_n(\lambda) \} \), \( \exists k \) such that
\[
\xi = \lim_{\lambda \uparrow \Lambda} \varphi_k(\lambda)
\]
Take
\[
\xi \in \lim_{\lambda \uparrow \Lambda} \{ \varphi_1(\lambda), ..., \varphi_n(\lambda) \}
\]
then
\[
\xi = \lim_{\lambda \uparrow \Lambda} \psi(\lambda)
\]
where \( \forall \lambda \in \mathcal{L}, \exists k \leq n \), such that
\[
\psi(\lambda) = \varphi_k(\lambda)
\]
Now, if we set, for \( k \leq n \),
\[
R_k = \{ \lambda \in \mathcal{L} \mid \psi(\lambda) = \varphi_k(\lambda) \}
\]
it turns out that
\[
R_1 \cup R_2 \cup ... \cup R_n = \mathcal{L}.
\]
Since \( \mathcal{U} \) is a ultrafilter, then one and only one of the \( R_k \)'s is in \( \mathcal{U} \). If \( R_k \in \mathcal{U} \), then
\[
\xi = \lim_{\lambda \uparrow \Lambda} \psi(\lambda) = J(\psi) = J(\varphi_k) = \lim_{\lambda \uparrow \Lambda} \varphi_k(\lambda)
\]
2.9.3 - It is nothing else but the definition (156).
2.9.4 - It follows directly by the definition of \( \mathbb{E} \) and Prop. 6.2. \( \square \)

6.2. Hyperfinite Step Functions

The step functions are easy to handle ad hence they are largely used many branches of mathematics and in particular in nonstandard analysis. Let us describe the kind of step functions that will be considered in this paper.

Given an infinite hypercube
\[
Q = [-L, L] \times \mathbb{N} \subseteq \mathbb{E}^{\mathbb{N}} \ni [-L, L] = \{ \xi \in \mathbb{E} \mid -L \leq \xi < L \}
\]
we say that a partition \( \{ S_a \}_{a \in \Gamma} \) of \( Q \) is **fine** if
- \( \Gamma \) is an hyperfinite set,
- the index \( a \) is a point in \( int(S_a) \) and if \( S_a \cap \mathbb{R}^N = \{ r \} \), then \( a = r \).
- \( \forall a \in \Gamma, \chi_{S_a} \in V^* \).
- the **size** of the partition, namely the number
\[
\eta = \max_{a \in \Gamma} \text{diam} \left( S_a \right)
\]
is infinitesimal; here \( \text{diam} \left( S_a \right) \) denotes the diameter of \( S_a \).
**Definition 6.4.** Given a fine partition \( \{ S_a \}_{a \in \Gamma} \), a **step function** \( u : \mathbb{E}^N \to \mathbb{E} \) is defined as follows:

\[
u(x) = \sum_{a \in \Gamma} u_a \chi_{S_a}(x).
\]

where \( u(\cdot) : \Gamma \to \mathbb{E} \) is an internal function namely a grid function.

The integral of a step function \( u \) is given by

\[
\int^* u(x) \, dx = \sum_{a \in \Gamma} u(a) m(S_a);
\]

Given a fine partition \( \{ S_a \}_{a \in \Gamma} \), any internal function \( u \in \mathfrak{F}(\mathbb{E}^N, \mathbb{E}) \) can be approximated by a step function \( \check{u} \) by setting

\[
\check{u}(x) = \sum_{a \in \Gamma} u(a) \chi_{S_a}(x).
\quad (157)

The next theorem shows a very interesting property of the infinitesimal partition in the contest of the Euclidean numbers.

**Theorem 6.5.** Let \( \{ S_a \}_{a \in \Gamma} \) be a fine partition: then, if \( f \in L^1 \),

\[
\int f(x) \, dx \sim \int^* \check{f}(x) \, dx
\]

First we need the following Lemma.

**Lemma 6.6.** Let \( \{ S_a \}_{a \in \Gamma} \) be as in Th. 6.5; if \( f \) is a measurable bounded function with compact support, then

\[
\int f(x) \, dx \sim \int^* \check{f}(x) \, dx
\]

**Proof.** Since \( \{ S_a \}_{a \in \Gamma} \) is an internal set, then

\[
\{ S_a \}_{a \in \Gamma} = \lim_{\lambda \uparrow \Lambda} \{ S^\lambda_a \}_{a \in \Gamma^\lambda},
\]

where \( \{ S^\lambda_a \}_{a \in \Gamma^\lambda} \) is a net of finite measurable partitions. Then

\[
\check{f} = \lim_{\lambda \uparrow \Lambda} f^\lambda,
\]

where

\[
f^\lambda(x) = \sum_{a \in \Gamma^\lambda} f(a) m(S^\lambda_a),
\]

are standard measurable step functions. Since \( f(a) \) is bounded and with compact support, the \( f^\lambda \)'s are uniformly bounded and with compact support. Moreover, since \( \mathbb{R}^N \subset \Gamma = \lim_{\lambda \uparrow \Lambda} \Gamma^\lambda \), if you take \( x \in \mathbb{R}^N \), we have that, for \( \lambda \) sufficiently large, \( x \in \Gamma^\lambda \) and hence

\[
f^\lambda(x) = f(x)
\]

namely the \( f^\lambda \)'s converge pointwise to \( f \):

\[
\lim_{\lambda \uparrow \Lambda} f^\lambda(x) = f(x)
\]
Then by the Lebesgue Dominated Convergence Theorem

\[
\lim_{\lambda \to \Lambda} \int f_{\lambda}(x) \, dx = \int f(x) \, dx.
\]

On the other hand, by (9),

\[
\lim_{\lambda \to \Lambda} \int f_{\lambda}(x) \, dx = st \left( \lim_{\lambda \uparrow \Lambda} \int f_{\lambda}(x) \, dx \right) = st \left( \int^* \lim_{\lambda \downarrow \Lambda} f_{\lambda}(x) \, dx \right) = st \left( \int^* \tilde{f}(x) \, dx \right)
\]

\[\Box\]

**Proof of Th. 6.5.** Since \( L^\infty_c \) is dense in \( L^1 \) there is a sequence \( f_n \) of measurable bounded functions with compact support such that

\[
f(x) = \sum_{n=0}^{\infty} f_n(x); \quad \int f(x) \, dx = \sum_{n=0}^{\infty} \int f_n(x) \, dx
\]

Then, by transfer

\[
f^*(x) = \sum_{n=0}^{\infty^*} f^*_n(x); \quad \int^* f^*(x) \, dx = \sum_{n=0}^{\infty^*} \int^* f^*_n(x) \, dx
\]

Since \( \{S_a\}_{a \in \Gamma} \) has been fixed we have that

\[
\tilde{f}(x) = \sum_{n=0}^{\infty^*} \tilde{f}_n(x); \quad \int^* \tilde{f}(x) \, dx = \sum_{n=0}^{\infty^*} \int^* \tilde{f}_n \, dx
\]

Hence, for every \( N \in \mathbb{N} \),

\[
\left| \int^* f^*(x) \, dx - \int^* \tilde{f}(x) \, dx \right| \leq \sum_{n=0}^{\infty^*} \left| \int^* f^*_n(x) \, dx - \int^* \tilde{f}_n \, dx \right|
\]

\[
\leq \sum_{n=0}^{\infty^*} \left| \int^* f^*_n(x) \, dx - \int^* \tilde{f}_n \, dx \right|
\]

\[
= \sum_{n=0}^{N} \left| \int^* f^*_n(x) \, dx - \int^* \tilde{f}_n \, dx \right|
\]

\[
+ \sum_{n=N+1}^{\infty^*} \left| \int^* f^*_n(x) \, dx - \int^* \tilde{f}_n \, dx \right|
\]

\[
\leq \sum_{n=0}^{N} \left| \int^* f^*_n(x) \, dx - \int^* \tilde{f}_n \, dx \right|
\]

\[
+ \sum_{n=N+1}^{\infty^*} \left| \int^* f^*_n(x) \, dx \right| + \sum_{n=N+1}^{\infty^*} \left| \int^* \tilde{f}_n \, dx \right|
\]

(158)
Now choose $\varepsilon \in \mathbb{R}^+$ arbitrarily and $N \in \mathbb{N}$ such that
\[
\sum_{n=N+1}^{\infty} \left| \int f_n(x) \, dx \right| = \sum_{n=N+1}^{\infty} \left| \int f_n^*(x) \, dx \right| < \varepsilon
\]
consequently,
\[
\sum_{n=N+1}^{\infty} \left| \int f_n^* \, dx \right| < 2\varepsilon
\]
Since $N$ is a finite number, by lemma 6.6, we have that
\[
\sum_{n=0}^{N} \left| \int f_n^*(x) \, dx - \int f_n^* \, dx \right| \sim 0
\]
Then, by (158), we have that
\[
\left| \int f_n^*(x) \, dx - \int f_n^* \, dx \right| \leq 4\varepsilon.
\]
The conclusion follows from the arbitrariness of $\varepsilon$. \hfill \square

6.3. $\sigma$-Bases

Definition 6.7. Let $W \subset \mathfrak{F}(\mathbb{R}^N)$ be a function space of finite dimension; we say that a family of functions $\{\sigma_a\}_{a \in \mathcal{A}}$, $\mathcal{A} \subset \mathbb{R}^N$ is a $\sigma$-basis for $W$ if every function $u$ in $W$ can be written as follows:
\[
u(x) = \sum_{a \in \mathcal{A}} u(a) \sigma_a(x).
\]
in a unique way.

Given $W \subset \mathfrak{F}(\mathbb{R}^N)$ and a set of points $\mathcal{A} = \{a_1, \ldots, a_k\} \subset \mathbb{R}^N$, we can define “restriction” map
\[
\Phi : W \rightarrow \mathfrak{F}(\mathcal{A})
\]
\[
\Phi(f) = (f(a_1), \ldots, f(a_k))
\]
If $\{e_1, \ldots, e_m\}$ is a basis of $W$, then $\Phi$ can be “represented” by the matrix
\[
\{e_n(a_l)\}_{n \leq m, l \leq k}
\]

Definition 6.8. Let $W \subset \mathfrak{F}(\mathbb{R}^N)$ be a function space of finite dimension; we say that a set $\mathcal{A} = \{a_1, \ldots, a_k\} \subset \mathbb{R}^N$ is:
\begin{itemize}
\item **independent** in $W$ if the map $\Phi$ is surjective and hence for any $k$-ple of points $(c_1, \ldots, c_k) \in \mathbb{R}^N$, there exists $f \in W$ such that
\[
f(a_l) = c_l; \quad l = 1, \ldots, k.
\]
in this case the matrix (160) has rank $k$.
\item **complete** in $W$ if the map $\Phi$ is bijective and hence there exists a unique $f \in W$ which satisfies (161); in this case
\[
det [e_k(a_l)] \neq 0
\]
\end{itemize}
• redundant in $W$ if the map $\Phi$ is injective and hence

$$(\forall f \in W, \forall a \in \mathbb{R}, f(a) = 0) \Rightarrow (f = 0);$$

in this case the matrix (160) has rank $m$.

Notice that a set of points is complete in $W$ if and only if it is independent and redundant.

**Lemma 6.9.** Let $W \subset \mathcal{F}(\mathbb{R}^N)$ be a function space of finite dimension and let $\mathcal{R} = \{a_1, ..., a_m\} \subset \mathbb{R}^N$. Then it is complete if and only if there exists a $\sigma$-basis $\{\sigma_a(x)\}_{a \in \mathcal{R}}$.

**Proof.** Let $\mathcal{R} = \{a_1, ..., a_n\} \subset \mathbb{R}^N$ be a complete set of points and let $\{\zeta_a\}_{a \in \mathcal{R}}$ be the “canonical” basis in $\mathcal{F}(\mathcal{R})$ namely

$$\zeta_a(b) = \delta_{ab}$$

Then, $\{\Phi^{-1}(\zeta_a)\}_{a \in \mathcal{R}}$ is a $\sigma$-basis. If $\{\sigma_a(x)\}_{a \in \mathcal{R}}$ is a $\sigma$-basis, then the map defined by $\Phi(\sigma_a) = \zeta_a$ is bijective. □

It is evident that every finite dimensional vector space has a redundant set of points. The existence of a complete set is a consequence of the following theorem.

**Theorem 6.10.** Let $\mathcal{R}$ be redundant in $W$. Then there exists a set $\mathcal{C} \subseteq \mathcal{R}$, complete in $W$.

**Proof.** The matrix (160) has rank $m = \dim(W)$. Let $M_m$ be a nonsingular $m \times m$ submatrix of the matrix $M$ and let $\mathcal{C}$ be the image of the operator defined by $M_m$. So, the set $\{\sigma_a(x)\}_{a \in \mathcal{C}}$ is a $\sigma$-basis and by lemma 6.9, $\mathcal{C}$ is complete. □

**Corollary 6.11.** Every finite or hyperfinite dimensional vector space $W$ has a complete system of points.

**Proof.** If $W$ is finite dimensional, take any redundant set of points in $W$ and apply Th. 6.10. If the dimension of $W$ is hyperfinite, take the $\Lambda$-limit. □

**Corollary 6.12.** Every finite or hyperfinite dimensional vector space $W$ has a $\sigma$-basis.

**Proof.** It is an immediate consequence of Corollary 6.11 and Lemma 6.9. □

We end this section with a technical lemma which will be used in the proof of Lemma 6.14.

**Lemma 6.13.** Let $W$ be a finite or hyperfinite dimensional vector space and let $r \in \mathbb{R}^N$ be a point such that

$$\exists w \in W, \ w(r) \neq 0.$$ 

Then $W$ has a complete set of points $\mathcal{R}$ with $r \in \mathcal{R}$.

**Proof.** Let $\{a_1, ..., a_m\}$ be complete set of points and let $\{e_n(a_l)\}_{n,l \leq m}$ be the relative matrix defined by (160). Since $w(r) \neq 0$, the vector $\{e_n(r)\}_{n \leq m}$ is $\neq 0$. Then there exists a point $a_l$ such that the vector $\{e_n(a_l)\}_{n \leq m}$ is a linear combination of the vectors $\{e_n(a_l)\}_{l \neq l} \cup \{e_n(r)\}_{n \leq m}$, and hence the set

$$\mathcal{R} := \{a_l \mid l \neq l\} \cup \{r\}$$

is complete. □
6.4. The Space $V_\Lambda$

This and the next sections are devoted to construct the space $V_\Lambda$ so that it is possible to define a pointwise integral and a generalized derivative which satisfy the requests of Def. 3.5. This will be done in three steps building three spaces $W_0^\Lambda, W_\Lambda, V_\Lambda$ with

$$V^\circ \subseteq W_0^\Lambda \subseteq W_\Lambda \subseteq V_\Lambda.$$  \hspace{1cm} (163)

We fix

$$Q = [L, L)^N$$  \hspace{1cm} (164)

where $L \in \mathbb{E}$ is an infinite number such that for every $u_\lambda \in V \cap \lambda$, we have that

$$\text{supp} \, (u) \subseteq (-L, L)^N.$$  \hspace{1cm} (165)

We set

$$W_0^\Lambda := \text{span} \{ uv \mid u, v \in V^\circ \cup \{ \bar{\chi}_Q \} \}$$  \hspace{1cm} (166)

We recall that the operator $u \mapsto \bar{u}$ has been defined by (30).

By Cor. 6.12, the space $W_0^\Lambda$ has a $\sigma$-basis; however we need to have a $\sigma$-basis which satisfies suitable properties. For this reason we need the following lemma.

Lemma 6.14. There exists a hyperfinite dimensional vector space $W_\Lambda \supseteq W_0^\Lambda$ which admits a $\sigma$-basis $\{ \tau_a \}_{a \in \Gamma_W}$ such that

$$\mathbb{R}^N \subset \Gamma_W;$$  \hspace{1cm} (167)

and

$$\text{supp} \, (\tau_a) \subset B_{2\varepsilon} (x_j)$$  \hspace{1cm} (168)

where $B_{2\varepsilon} (x_j) \subset \mathbb{E}^N$ is a ball of radius $2\varepsilon \sim 0$.

Proof. Let $\{ E_k \}_{k \in \mathbb{N}}$ be a fine partition of $Q$ such that, for every $k$, $diam \, (E_k) < \varepsilon \sim 0$. Then, $\forall u \in W_0^\Lambda$, we have that

$$u(x) = \sum_{k \in \mathbb{N}} u(x) \chi_{E_k}(x)$$

Now, we set

$$W(E_k) = \left\{ \overline{u(x) \chi_{E_k}(x)} \mid u \in W_0^\Lambda \right\}$$

and

$$W_\Lambda = W(E_1) \oplus \ldots \oplus W(E_r)$$

Clearly $W_0^\Lambda \subset W_\Lambda$. Now, for every $k \in \mathbb{N}$, take a $\sigma$-basis $\{ \tau_a \}_{a \in \Gamma_k}$ of $W(E_k)$ which exists by Cor. 6.11. If $\mathbb{R}^N \cap E_k = \{ r_k \} \neq \emptyset$, we can take $\Gamma_k$ such that $r_k \in \Gamma_k$; this is possible by lemma 6.13. Now, if we set

$$\Gamma_W = \bigcup_{k \in \mathbb{N}} \Gamma_k$$

we have that $\{ \tau_a \}_{a \in \Gamma_W}$ is a $\sigma$-basis which satisfies our requests. \hfill $\square$
Now, we fix once for ever a positive infinitesimal
\[ \gamma < |\Gamma_W|^{-1} \]  
and we set
\[ \Omega_W := \bigcup_{a \in \Gamma_W} B_\varepsilon(a) \]
where, for \( a \) in \( \Gamma_W \), \( B_\varepsilon(a) \) is the ball of center \( a \) and radius \( \varepsilon \).

**Lemma 6.15.** We can take \( \varepsilon \) so small that \( \forall a, b \in \Gamma_W, a \neq b \)
\[ B_\varepsilon(a) \cap B_\varepsilon(b) = \emptyset \]
\[ (1 - \gamma^2) m(B_\varepsilon) \leq \int_{B_\varepsilon(a)} \tau_a(x)dx \leq (1 - \gamma^2) m(B_\varepsilon) \]
\[ \int_{\Omega_W \setminus B_\varepsilon(a)} |\tau_a(x)|dx \leq \gamma^2 m(B_\varepsilon) \]
where \( m(B_\varepsilon) := m(B_\varepsilon(0)) \) denotes the Lebesgue measure of \( B_\varepsilon(0) \).

**Proof.** Since the functions \( \tau_a \) are epilogic the conclusion follows from the fact that \( \tau_a(a) = 1 \) and, for \( b \neq a \), \( \tau_a(b) = 0 \). \( \square \)

Now, we set
\[ \Omega_Z := Q \setminus \Omega_W \]
and we denote by \( \{S_a\}_{a \in \Gamma_Z} \) a fine partition of \( \Omega_Z \) of size \( \eta \leq \varepsilon \); moreover we set
\[ Z_\eta := \text{span} \{\bar{\chi}_{S_a} \mid a \in \Gamma_Z\} \].

**Lemma 6.16.** \( W_\Lambda \cap Z_\eta = \{0\} \)

**Proof.** Let \( u \in W_\Lambda \cap Z_\eta \). Since \( u \in Z_\eta \),
\[ \forall a \in \Gamma_W, \ u(a) = 0. \]
\( \Gamma_W \) is a complete set for \( W_\Lambda \) (see Def. 6.8) and \( u \in W_\Lambda \), then \( u = 0 \). \( \square \)

We set
\[ V^n_\Lambda := W_\Lambda \oplus Z_\eta; \quad \Gamma^n := \Gamma_W \cup \Gamma^n_Z \]
Notice that \( \{\tau_a\}_{a \in \Gamma_W} \cup \{\bar{\chi}_{S_a}\}_{a \in \Gamma_Z} \) is a basis of \( V_\Lambda \) but it is not a \( \sigma \)-basis even if \( \{\tau_a\}_{a \in \Gamma_W} \) is a \( \sigma \)-basis of \( W_\Lambda \) and \( \{\bar{\chi}_{S_a}\}_{a \in \Gamma^*_Z} \) is a \( \sigma \)-basis of \( Z_\eta^1 \). If we put
\[ S_a = B_\varepsilon(a) \]
we have that
\[ \{S_a\}_{a \in \Gamma^n} = \{B_\varepsilon(a)\}_{a \in \Gamma_W} \cup \{S_a\}_{a \in \Gamma^n_Z} ; \quad \Gamma = \Gamma_W \cup \Gamma^n_Z \]
is a fine partition of \( Q \) of size \( 2\varepsilon \).
Using the notation (157), we set,
\[ \bar{\tau}_a(x) = \sum_{s \in \Gamma^\eta} \tau_a(s) \chi_{S_a}(x) \]

**Lemma 6.17.** If \( \eta \) is sufficiently small, \( \forall a \in \Gamma_W \),
\[ \int_{\Omega_Z^*} |\tau_a(x) - \bar{\tau}_a(x)| \leq \gamma^2 m(B_{\varepsilon}). \]

**Proof.** Since the \( \tau_a \)'s are a hyperfinite number of Riemann integrable functions with compact support, then we can choose \( \eta \) so small that our request be satisfied. \( \square \)

**Remark 6.18.** The proof of lemma 6.17 is the only point in which it is required that the functions in \( V \) be Riemann-integrable.

From now on, we will fix \( \eta \) in such a way that Lemma 6.17 be satisfied and we will write \( V_{\Lambda}, Z, \Gamma \) and \( \Gamma_Z \) instead of \( V_{\Lambda}^\eta, Z, \Gamma^\eta \) and \( \Gamma_Z^\eta \).

**Lemma 6.19.** We set
\[ \sigma_a(x) = \begin{cases} \bar{\chi}_{S_a}(x) & \text{if } a \in \Gamma_Z \\ \tau_a(x) - \bar{\tau}_a(x) \chi_{\Omega_Z}(x) & \text{if } a \in \Gamma_W \end{cases}, \quad (173) \]
then \( \{\sigma_a\}_{a \in \Gamma} \) is a \( \sigma \)-basis of \( V_{\Lambda} \) such that
\[ \text{supp} \ (\sigma_a) \subset B_{2\varepsilon}(a). \quad (174) \]

**Proof.** First of all let us check that
\[ \sigma_a(c) = \delta_{ac} \quad (175) \]
for every \( c \in \Gamma_W \cup \Gamma_Z \). If \( a \in \Gamma_Z \), then, \( \sigma_a(c) = \bar{\chi}_{S_a}(c) = \delta_{ac} \).

Let us see the case in which \( a \in \Gamma_W \). In this case, (173) takes the following form:
\[ \sigma_a(x) = \tau_a(x) - \sum_{b \in \Gamma_Z} \tau_a(b) \bar{\chi}_{S_b}(x); \]
- if \( c \in \Gamma_Z \) then
  \[ \sigma_a(c) = \tau_a(c) - \sum_{b \in \Gamma_Z} \tau_a(b) \bar{\chi}_{S_b}(x) = \tau_a(c) - \tau_a(c) = 0 \]
- if \( c \in \Gamma_W \), since \( \{\tau_a\}_{a \in \Gamma_W} \) is a \( \sigma \)-basis, then
  \[ \sigma_a(c) = \tau_a(c) - \sum_{b \in \Gamma_Z} \tau_a(b) \bar{\chi}_{S_b}(x) = \delta_{ac} - 0 = \delta_{ac} \]

So the \( \sigma_a \)'s are \(|\Gamma| = |\Gamma_W| + |\Gamma_Z|\) elements independent in \( V_{\Lambda} \) and hence they span all \( V_{\Lambda} \). So, they form a \( \sigma \)-basis.

Now let us prove (174). If \( a \in \Gamma_Z \),
\[ \text{supp} \ (\sigma_a) = Q_a \setminus \Omega_W \subset B_{2\eta}(a) \subset B_{2\varepsilon}(a); \]
if \( a \in \Gamma_Z^\eta \), by (168)
\[
\text{supp } \sigma_a \subseteq \text{supp } \tau_a \subset B_{2\varepsilon}(a).
\]

**Lemma 6.20.** For every \( a \in \Gamma' \),
\[
(1 - 3\gamma^2) m(S_a) \leq \int \sigma_a dx \leq (1 + 3\gamma^2) m(S_a);
\]  
(176)

*Proof.* If \( a \in \Gamma_Z \), \( \sigma_a = \bar{\chi}_{S_a} \) and hence
\[
\int \sigma_a dx = \int \bar{\chi}_{S_a} dx = m(S_a)
\]

Now let us consider the case \( a \in \Gamma_W \); since \( \bar{\chi}_Q = \bar{\chi}_{B_{\varepsilon}(a)} + \bar{\chi}_{\Omega \setminus B_{\varepsilon}(a)} + \bar{\chi}_\Omega \), we have that
\[
\tau_a(x) = \tau_a(x)\chi_{B_{\varepsilon}(a)} + \tau_a(x)\chi_{\Omega \setminus B_{\varepsilon}(a)} + \tau_a(x)\chi_{\Omega_z}
\]
then, by Lemma 6.19,
\[
\int \sigma_a dx = \int [\tau_a(x) - \bar{\tau}_a(x) \cdot \chi_{\Omega_z}(x)] dx
\]
\[
= \int_{B_{\varepsilon}(a)} \tau_a dx + \int_{\Omega \setminus B_{\varepsilon}(a)} \tau_a dx + \int_{\Omega_z} [\tau_a - \bar{\tau}_a] dx
\]

By Lemma 6.15 and Lemma 6.17, we have that
\[
\left\| \int_{\Omega \setminus B_{\varepsilon}(a)} \tau_a dx + \int_{\Omega_z} [\tau_a - \bar{\tau}_a] dx \right\| \leq 2\gamma^2 m(B_{\varepsilon})
\]

Then, using (170)
\[
\int \sigma_a dx \geq (1 - \gamma^2) m(B_{\varepsilon}) - 2\gamma^2 m(B_{\varepsilon}) = (1 - 3\gamma^2) m(B_{\varepsilon})
\]
and
\[
\int \sigma_a dx \leq (1 + \gamma^2) m(B_{\varepsilon}) + 2\gamma^2 m(B_{\varepsilon}) = (1 + 3\gamma^2) m(B_{\varepsilon})
\]
\]

**Corollary 6.21.** If \( u \in V_\Lambda \), then
\[
\int |u(x) - \bar{u}(x)| dx \leq 3\gamma \|u\|_{L^\infty}.
\]

*Proof.* By Lemma 6.20 and (169) we have that
\[
\int |u(x) - \bar{u}(x)| dx = \int \sum_{a \in \Gamma} [u(a)(\sigma_a(x) - \bar{\chi}_{S_a}(x))] dx
\]
\[
\leq \sum_{a \in \Gamma_W} |u(a)| \int |\sigma_a(x) - \bar{\chi}_{S_a}(x)| dx
\]
\[
\leq \sum_{a \in \Gamma_W} (\|u\|_{L^\infty} \cdot 3\gamma^2) = 3\gamma^2 \|u\|_{L^\infty} |\Gamma_W|
\]
\[
\leq 3\gamma \|u\|_{L^\infty}
\]
\]

\[\Box\]
6.5. The Pointwise Integral and the Generalized Derivative

Finally we set

\[ V^\circ = \{ w|_\Gamma \mid w \in V_{\Lambda} \}. \quad (178) \]

and we define the pointwise integral for the functions in \( V^\circ \) as follows:

\[ \oint u \, dx = \int^* u_{\Lambda} \, dx, \quad (179) \]

where \( u_{\Lambda} \), as usual, is defined by (23). Since \( V_{\Lambda} \) is a hyperfinite space, then

\[ V_{\Lambda} = \lim_{\Lambda \uparrow \Lambda} V_{\lambda} \]

for a suitable net of finite dimensional spaces \( V_{\lambda} \subset V \) and hence this definition agrees with (36). In particular we have that

\[ \oint u \, dx = \sum_{a \in \Gamma} u(a) d(a); \quad d(a) := \int^* \sigma_a(x) \, dx. \]

So, if \( w \in V_{\Lambda} \), then

\[ \oint w \circ \, dx = \int^* w \, dx, \]

Moreover we have also the following interesting result:

**Theorem 6.22.** If \( f \in L^1 \), then \( \oint f^\circ \, dx \sim \int f \, dx. \)

**Proof.** First let us assume that \( f \) be bounded. By corollary 6.21, we have that

\[ \int^* |f^\circ(x) - \tilde{f}(x)| \, dx \leq 3\gamma \| f \|_{L^\infty} \sim 0 \]

Then, by Th. 6.5,

\[ \left| \oint f^\circ \, dx - \int f \, dx \right| \leq \left| \oint f^\circ \, dx - \int^* \tilde{f} \, dx \right| + \int^* \tilde{f} \, dx - \int f \, dx \sim 0. \]

If \( f \) is not bounded we argue as in the proof of Th. 6.5. \( \square \)

For every \( u \in V_{\Lambda} \) we define the “generalized derivative”

\[ D_i : V_{\Lambda} \rightarrow V_{\Lambda} \]

as follows: given \( u \in V_{\Lambda} \), \( D_i u \) is the only element in \( V_{\Lambda} \) such that

\[ \oint D_i u v \, dx = \int^* \partial_i^* u v \, dx \quad \forall v \in V_{\Lambda} \quad (180) \]

namely

\[ D_i u = P \partial_i^* u \]

where

\[ P : \mathcal{M}^* \rightarrow V_{\Lambda} \quad (181) \]
is the “projection” of the natural extension of the space Radon measures $\mathcal{M}$ over $V_\Lambda$ with respect to the duality

$$\langle w, v \rangle = \oint vw \, dx.$$ 

**Theorem 6.23.** $V^\circ$ is a fine space of ultrafunctions.

**Proof.** We will check that $V^\circ$ verifies the requests of Def. 3.5.

3.5.1 follows from the construction of $V^\circ$; in fact, by (166), we have that

$$u, v \in V^\circ \Rightarrow uv \in W^\circ_\Lambda \subset W_\Lambda \subset V_\Lambda.$$

3.5.2a follows from (179).

3.5.2b follows from Lemma 6.20 since $\chi_a = (\sigma_a)^\circ$.

3.5.2c follows from Th. 6.22.

3.5.3a follows from (180).

3.5.3b follows from the definition (180) of $D_i$ since also $\partial_i^* \sigma$ is a local operator. □

**7. Conclusive Remarks**

The definition 3.5 of fine ultrafunctions extends the notion of real functions in such a way that:

- (a) “almost” all the partial differential equations (and other functional problems) have a solution;
- (b) some of the main properties of the smooth functions are preserved.

The assumptions included in definition 3.5 seems quite natural; however they do not define a unique model and there is a lot of room to require others properties that allow to prove more facts about the solutions of a given problem.

Let’s illustrate this point with an example: let us consider equation (148); we know that it has a unique solution which preserves the energy; however there are a lot of questions that are relevant for the physical interpretation such as:

- If $f(u)$ grows more than $|u|^{(N+2)/(N-2)}$ most likely, this solution has singular regions $S \subset \Gamma$ where the density of energy is an infinite number; is there a precise theorem? What can we say about the properties of $S$?
- In general the momentum is not “exactly” preserved since the “space” $\Gamma$ is not invariant for translation (and hence we cannot apply the Noether’s theorem); then the natural question is to know the initial conditions for which the momentum is preserved and/or the initial conditions for which the momentum is preserved up to an infinitesimal.
- Under which assumptions the solutions converge to 0 as $t \to \infty$?
- And so on....

Some of these questions have an answer in the context of ultrafunctions provided that you work enough on a single question. However there are questions which do not have a YES/NO answer since there are models of ultrafunctions in which the answer is YES and other in which the answer is NO. A possible development of the theory is to add to the definition of ultrafunctions other properties which allows a
more detailed description of the physical phenomenon described by the mathematical model. For example, in Def. 3.5 we have not included all the properties of the generalized derivative which can be deduced by the choice of $V^\circ$ and Def. (180). The main difficulty, if you want to add a new property to the ultrafunctions, is the proof of its consistency, namely the construction of a model. For example it is easy to prove that the Leibniz rule is not consistent with an algebra of functions which includes idempotent functions (see the discussion in section 3.5) but in general, it is difficult to guess if a “reasonable property” is consistent with all the others (for example the identity $D_i D_j = D_j D_i$).

At this points, it is interesting compare the ultrafunction theory (and more in general a theory which includes infinitesimals) and a traditional theory based on real valued function spaces. In both cases there exist questions which do not have a YES/NO answer because of the Goedel incompleteness theorem, and, in both cases, we can add new axioms which allow to solve the problem. The difficult issue is the proof of the consistency of these new axioms. However, this issue is much easier in the world of Non-Archimedean mathematics, since the mathematical universe is wider and there is more room to construct a model.

Let us consider an example that clarifies this point. We do not know if the Navier-Stokes equations have a smooth solution; however they have a unique ultrafunction solution. Probably this fact is not relevant for an applied mathematician or for an engineer since it does not help to discover new fact relative to the motion of a real fluid. However, it is possible to add new axioms which allow to prove properties of the solutions consistent with the experiments and to end with a richer mathematical model. This model can be used to get new practical results even if the problem of the existence of a smooth solutions remains unsolved.

For this reasons we think that it is worthwhile to investigate the potentialities of the Non-Archimedean mathematics.

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