Nonlocal de Sitter gravity and its exact cosmological solutions

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ABSTRACT: This paper is devoted to a simple nonlocal de Sitter gravity model and its exact vacuum cosmological solutions. In the Einstein-Hilbert action with $\Lambda$ term, we introduce nonlocality by the following way: $R-2\Lambda = \sqrt{R-2\Lambda} \sqrt{R-2\Lambda} \to \sqrt{R-2\Lambda} F(\Box) \sqrt{R-2\Lambda}$, where $F(\Box) = 1 + \sum_{n=1}^{\infty} (f_n \Box^n + f_{-n} \Box^{-n})$ is an analytic function of the d’Alembert-Beltrami operator $\Box$ and its inverse $\Box^{-1}$. By this way, $R$ and $\Lambda$ enter with the same form into nonlocal version as they are in the local one, and nonlocal operator $F(\Box)$ is dimensionless. The corresponding equations of motion for gravitational field $g_{\mu\nu}$ are presented. The first step in finding some exact cosmological solutions is solving the equation $\Box \sqrt{R-2\Lambda} = q \sqrt{R-2\Lambda}$, where $q = \zeta \Lambda$ ($\zeta \in \mathbb{R}$) is an eigenvalue and $\sqrt{R-2\Lambda}$ is an eigenfunction of the operator $\Box$. We presented and discussed several exact cosmological solutions for homogeneous and isotropic universe. One of these solutions mimics effects that are usually assigned to dark matter and dark energy. Some other solutions are examples of the nonsingular bounce ones in flat, closed and open universe. There are also singular and cyclic solutions. All these cosmological solutions are a result of nonlocality and do not exist in the local de Sitter case.
Contents

1 Introduction 2

2 Nonlocal de Sitter gravity 4
  2.1 Action 4
  2.2 Equations of motion 4

3 Cosmological solutions 7
  3.1 Cosmological solutions in the flat universe \((k = 0)\) 7
    3.1.1 Solutions of the form \(a(t) = A t^n e^{\gamma t}, \ (k = 0)\) 7
    3.1.2 New solutions of the form \(a(t) = (\alpha e^{\lambda t} + \beta e^{-\lambda t})^\gamma, \ (k = 0)\) 8
    3.1.3 New solutions of the form \(a(t) = (\alpha \sin \lambda t + \beta \cos \lambda t)^\gamma, \ (k = 0)\) 10
  3.2 Cosmological solutions in the closed and open universe \((k = \pm 1)\) 10
    3.2.1 Cosmological solution of the form \(a(t) = A e^{\pm \sqrt{\Lambda} t}, \ (k = \pm 1)\) 10
    3.2.2 New solutions of the form \(a(t) = (\alpha e^{\lambda t} + \beta e^{-\lambda t})^\gamma, \ (k = \pm 1)\) 11

4 Discussion 11
  4.1 Solutions \(a_1(t) = A t^2 e^{\frac{\Lambda_1}{2} t^2}\) and \(a_2(t) = A e^{\frac{\Lambda_2}{2} t^2}, \ (k = 0, \Lambda_{1,2} \neq 0)\) 12
    4.1.1 Cosmological solution \(a_1(t) = A t^2 e^{\frac{\Lambda_1}{2} t^2}\) 12
    4.1.2 Cosmological solution \(a_2(t) = A e^{\frac{\Lambda_2}{2} t^2}\) 14
  4.2 Solutions \(a_3(t) = A \cosh \left(\frac{\sqrt{3\Lambda}}{8} t\right)\) and \(a_4(t) = A \sinh \left(\frac{\sqrt{3\Lambda}}{8} t\right), \ (k = 0, \Lambda > 0)\) 15
  4.3 Solutions \(a_5(t) = A \left(1 + \sin \left(\frac{\sqrt{3\Lambda}}{8} t\right)\right)^\frac{1}{2}\) and \(a_6(t) = A \left(1 - \sin \left(\frac{\sqrt{3\Lambda}}{8} t\right)\right)^\frac{1}{2}, \ (k = 0, \Lambda < 0)\) 15
  4.4 Solutions \(a_7(t) = A \sin \left(\frac{\sqrt{3\Lambda}}{8} t\right)\) and \(a_8(t) = A \cos \left(\frac{\sqrt{3\Lambda}}{8} t\right), \ (k = 0, \Lambda < 0)\) 16
  4.5 Solutions \(a_9(t) = A e^{\pm \sqrt{\frac{3\Lambda}{8}} t}, \ a_{10}(t) = A \cosh \left(\frac{\sqrt{3\Lambda}}{2} t\right), \text{ and } a_{11}(t) = A \sinh \left(\sqrt{\frac{3\Lambda}{2}} t\right), \ (k = \pm 1, \Lambda > 0)\) 17

5 Concluding Remarks 19

A Derivation of the equations of motion 20
1 Introduction

According to the Standard Model of Cosmology (SMC), at the current cosmic time the universe approximately consists of 68% of dark energy (DE), 27% of dark matter (DM) and only 5% of visible (ordinary) matter known from the Standard model of particle physics, see [1]. It is worth noting that the SMC assumes General Relativity (GR) as theory of the gravitational interaction, not only in the Solar system but also at the galactic and cosmological scales. According to the SMC point of view, dark matter is responsible for observational dynamics inside galaxies and their clusters, while dark energy acts as repulsive force and causes accelerated expansion of the universe. The SMC is also known as the ΛCDM model, what means that DE corresponds to the cosmological constant and that DM is in a cold state. However, despite many efforts to confirm existence of DM and DE either in the sky or in the laboratory experiments, they are still not discovered and remain hypothetical.

There is a common opinion that general relativity is one of the most beautiful and successful physical theories [2]. From phenomenological point of view, GR has had several significant confirmed predictions. However, GR as a theory of gravitation has not been verified at the galactic and cosmological scales. Hence, in spite of remarkable successes, it is reasonable to doubt in its validity in description and understanding of all astrophysical and cosmological gravitational phenomena. Moreover, GR solutions for the black holes as well as for the beginning of the universe contain singularity and it means that GR should be modified in the vicinity of these singularities, see, e.g. [3, 4]. It is also well known that GR is nonrenormalizable theory from the quantum point of view. Let us also mention that any other physical theory has its domain of validity which is usually constrained by spacetime scale, complexity of the system under consideration, or by some parameters. There is no a priori reason that GR should be appropriate at all spacetime scales. Keeping all this in mind, it follows that general relativity is not a final theory of gravitation and that its extension is desirable.

Since no new physical principle is presently known that could tell us in what direction to look for theoretical generalization of GR, there are many approaches to its modification, see [5–10] as some reviews. Despite many attempts, there is not yet generally accepted modification of general relativity. One of the current and attractive approaches is nonlocal modified gravity, see, e.g. [11–16]. The idea behind nonlocality is that dynamics of the gravitational field may depend not only on its first and second spacetime derivatives but also on all higher derivatives. Practically, it means that the Einstein-Hilbert action should be modified by an additional term that contains higher order degrees of d’Alembert-Beltrami operator $\Box = \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$. So far $\Box$ has been mainly used in one of the following two forms: (i) non-polynomial analytic expansion $F(\Box) = \sum_{n=0}^{+\infty} f_n \Box^n$, see [17–24] as some examples or (ii) some inverse manner $\Box^{-1}$, see, e.g. [16, 25–29].

A part of motivation to use nonlocal operator in the form $F(\Box) = \sum_{n=0}^{+\infty} f_n \Box^n$ comes from string field theory [30] and $p$-adic string theory [31]. Also this type of nonlocality improves quantum renormalizability [32–34].

Some of the nonlocal gravity models that have been so far considered with analytic
nonlocality are particular cases of the action

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - 2\Lambda + P(R)F(\Box)Q(R)\right), \tag{1.1} \]

where \( \Lambda \) is the cosmological constant, \( P(R) \) and \( Q(R) \) are some differentiable functions of the Ricci scalar \( R \), see [35–46] and references therein. The case \( P(R) = Q(R) = R \) and \( \Lambda = 0 \) has attracted a lot of attention, i.e. nonlocal \( R^2 \) gravity, see [11, 18] and references therein.

A very special case of (1.1) is

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - 2\Lambda + \sqrt{R - 2\Lambda}F(\Box)\sqrt{R - 2\Lambda}\right), \tag{1.2} \]

\[ = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \sqrt{R - 2\Lambda}F(\Box)\sqrt{R - 2\Lambda} \tag{1.3} \]

where \( F(\Box) = 1 + F(\Box) = 1 + \sum_{n=1}^{+\infty} f_n \Box^n \), see [44]. In fact, in the Einstein-Hilbert action a nonlocal term is introduced so that dependence on \( R \) and \( \Lambda \) remains the same as in the local \( (F(\Box) = 1) \) case. By this way, nonlocal model (1.2) is introduced by minimal change of the Einstein-Hilbert action with nonlocal dimensionless operator \( F(\Box) = \sum_{n=1}^{+\infty} f_n \Box^n \).

Importance of model (1.2) is not only in its simple form but also in its cosmological solutions in flat space-time: (i) \( a_1(t) = A t^2 e^{\frac{2\Lambda}{3\pi}} \) and (ii) \( a_2(t) = A t^2 e^{\frac{2\Lambda}{3\pi}} \). Solution \( a_1(t) \) mimics interplay of dark matter and dark energy in a good agreement with cosmological observations [1, 44]. Solution \( a_2(t) \) is the nonsingular bounce one. The corresponding explicit expressions of the nonlocal operator are: \( F(\Box_1) = \frac{7}{3\pi} \frac{\Box}{\Lambda} \exp\left(-\frac{7}{3}\frac{\Box}{\Lambda}\right) \) and \( F(\Box_2) = \frac{\Box}{\Lambda} \exp\left(-\frac{\Box}{\Lambda}\right) \), respectively, see also Section 2. Note that \( \frac{\Box}{\Lambda} \) is dimensionless operator and no new mass (energy) parameter is introduced in this model.

Another interesting example of (1.1) gets when \( P(R) = Q(R) = R - 4\Lambda \), which can be derived from (1.2) under approximation \( |R| \ll 2\Lambda \) [45, 46]. In the paper [45] two cosmological solutions are presented: (i) \( a(t) = A t^2 e^{\frac{2\Lambda}{3\pi}} \), which mimics properties similar to an interference between radiation and the dark energy, and (ii) nonsingular bounce solution \( a(t) = A e^{\frac{2\Lambda}{3\pi}} \). The paper [46] contains cosmological solutions which demonstrate how this kind of gravitational nonlocality can change background topology.

In this paper we consider model (1.2) with extension of nonlocal operator \( F(\Box) = \sum_{n=1}^{+\infty} f_n \Box^n \) to \( F(\Box) = \sum_{n=1}^{+\infty} (f_n \Box^n + f_{-n} \Box^{-n}) \) and finding some new cosmological solutions.

Structure of the paper is as follows. In Section 2, we introduce a simple nonlocal de Sitter gravity model giving its action and present the corresponding equations of motion. We also consider the suitable form of nonlocal operator \( F(\Box) \) as an analytic function with respect to \( \Box \) and \( \Box^{-1} \). Presentation of two known solutions and finding some new exact vacuum cosmological solutions is subject of Section 3. All obtained cosmological solutions of this nonlocal de Sitter gravity are discussed in Section 4. Some concluding remarks, with list of main results and plan for future works, are contained in Section 5. At the end, there is also an appendix related to the derivation of equations of motion.
2 Nonlocal de Sitter gravity

2.1 Action

Our nonlocal gravity model is given by its action

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \sqrt{\tilde{R} - 2\Lambda} \ F(\Box) \sqrt{\tilde{R} - 2\Lambda}, \quad (2.1) \]

where \( F(\Box) \) is the following formal expansion in terms of the d’Alembert-Beltrami operator \( \Box \):

\[ F(\Box) = 1 + \mathcal{F}(\Box) = 1 + \mathcal{F}_+(\Box) + \mathcal{F}_-(\Box), \quad \mathcal{F}_+(\Box) = \sum_{n=1}^{+\infty} f_n \Box^n, \quad \mathcal{F}_-(\Box) = \sum_{n=1}^{+\infty} f_{-n} \Box^{-n}. \quad (2.2) \]

When \( F(\Box) = 1 \), i.e. \( \mathcal{F}(\Box) = 0 \), then model (2.1) becomes local and coincides with Einstein-Hilbert action with cosmological constant \( \Lambda \):

\[ S_0 = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \sqrt{\tilde{R} - 2\Lambda} \sqrt{\tilde{R} - 2\Lambda} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda). \quad (2.3) \]

It is worth pointing out that action (2.1) can be obtained in a very simple and natural way from action (2.3) by embedding nonlocal operator (2.2) in symmetric product form of \( R - 2\Lambda \), that is \( \sqrt{\tilde{R} - 2\Lambda} \sqrt{\tilde{R} - 2\Lambda} \). Action (2.1) does not contain matter term and this is intentionally done to better see possible role of this nonlocal model in effects usually assigned to dark matter and dark energy.

At the beginning, let the nonlocal operator \( \mathcal{F}(\Box) \) be in general form (2.2), that is in all powers of \( \Box \) and \( \Box^{-1} \) with unknown real coefficients \( f_n \) and \( f_{-n} \), where \( 1 \leq n < +\infty \). These coefficients \( f_n \) and \( f_{-n} \) will be presented later in the simple form which satisfies necessary conditions for exact cosmological solutions.

2.2 Equations of motion

The next step is obtaining the equations of motion (EoM) for model (2.1). To this end, we will start with more general case, that is

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left( R - 2\Lambda + P(R) \mathcal{F}(\Box) Q(R) \right), \quad (2.4) \]

where \( P(R) \) and \( Q(R) \) are some differentiable functions of \( R \), while \( \mathcal{F}(\Box) = \sum_{n=1}^{+\infty} f_n \Box^n + \sum_{n=1}^{+\infty} f_{-n} \Box^{-n} \) remains the same as in (2.2). A short version of derivation of EoM for model (2.4) is presented in Appendix, a similar case can be seen in [43]. According to expressions (A.23) and (A.24) in the Appendix, equations of motion for nonlocal model (2.4) are

\[ \hat{G}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} P \mathcal{F}(\Box) Q + R_{\mu\nu} W - K_{\mu\nu} W + \frac{1}{2} \Omega_{\mu\nu} = 0, \quad (2.5) \]
where

\[ W = P'(R) \mathcal{F}(\Box) Q(R) + Q'(R) \mathcal{F}(\Box) P(R), \quad K_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box, \quad (2.6) \]

\[ \Omega_{\mu\nu} = \sum_{n=1}^{+\infty} f_n \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^\ell P, \Box^{n-1-\ell} Q) - \sum_{n=1}^{+\infty} f_{-n} \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^{-(\ell+1)} P, \Box^{-(n-\ell)} Q), \quad (2.7) \]

and \( P' \) (\( Q' \)) means derivative of \( P \) (\( Q \)) with respect to scalar curvature \( R \).

Equations of motion (2.5) look very complicated comparing to their local (Einstein) counterpart \( G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \). Finding some solutions of (2.5) is a difficult task. Nevertheless, as we will see, one can find some exact cosmological solutions in the case when \( P = Q = \sqrt{R - 2\Lambda} \).

Now, let us first consider the case \( Q(R) = P(R) \). The corresponding EoM are

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{g_{\mu\nu}}{2} P(R) \mathcal{F}(\Box) P(R) + R_{\mu\nu} W - K_{\mu\nu} W + \frac{1}{2} \Omega_{\mu\nu} = 0, \quad (2.9) \]

\[ W = 2P'(R) \mathcal{F}(\Box) P(R), \quad K_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box, \quad (2.10) \]

\[ \Omega_{\mu\nu} = \sum_{n=1}^{+\infty} f_n \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^\ell P, \Box^{n-1-\ell} P) - \sum_{n=1}^{+\infty} f_{-n} \sum_{\ell=0}^{n-1} S_{\mu\nu}(\Box^{-(\ell+1)} P, \Box^{-(n-\ell)} P). \quad (2.11) \]

If \( P(R) \) is an eigenfunction of the corresponding d’Alembert-Beltrami operator \( \Box \), and consequently also of its inverse \( \Box^{-1} \), i.e. holds

\[ \Box P(R) = q P(R), \quad \Box^{-1} P(R) = q^{-1} P(R), \quad \mathcal{F}(\Box) P(R) = \mathcal{F}(q) P(R), \quad q \neq 0, \quad (2.12) \]

then

\[ W = 2\mathcal{F}(q) P' P, \quad \mathcal{F}(q) = \sum_{n=1}^{+\infty} f_n q^n + \sum_{n=1}^{+\infty} f_{-n} q^{-n}, \quad (2.13) \]

\[ S_{\mu\nu}(\Box^\ell P, \Box^{n-1-\ell} P) = q^{n-1} S_{\mu\nu}(P, P), \quad (2.14) \]

\[ S_{\mu\nu}(\Box^{-(\ell+1)} P, \Box^{-(n-\ell)} P) = q^{-n+1} S_{\mu\nu}(P, P), \quad (2.15) \]

\[ S_{\mu\nu}(P, P) = g_{\mu\nu} (\nabla_{\alpha} P \nabla_{\alpha} P + P \Box P) - 2 \nabla_{\mu} P \nabla_{\nu} P, \quad (2.16) \]

\[ \Omega_{\mu\nu} = \mathcal{F}'(q) S_{\mu\nu}(P, P), \quad \mathcal{F}'(q) = \sum_{n=1}^{+\infty} n f_n q^{n-1} - \sum_{n=1}^{+\infty} n f_{-n} q^{-n+1}, \quad (2.17) \]

and

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{g_{\mu\nu}}{2} \mathcal{F}(q) P'^2 + 2 \mathcal{F}(q) R_{\mu\nu} P P' - 2 \mathcal{F}(q) K_{\mu\nu} P P' + \frac{1}{2} \mathcal{F}'(q) S_{\mu\nu}(P, P) = 0. \quad (2.18) \]

The last equation transforms to

\[ (G_{\mu\nu} + \Lambda g_{\mu\nu}) \left(1 + 2 \mathcal{F}(q) P P'\right) + \mathcal{F}(q) g_{\mu\nu} \left(-\frac{1}{2} P^2 + PP'(R - 2\Lambda)\right) \]

\[ - 2 \mathcal{F}(q) K_{\mu\nu} P P' + \frac{1}{2} \mathcal{F}'(q) S_{\mu\nu}(P, P) = 0. \quad (2.19) \]
Let now $P = \sqrt{R - 2\Lambda}$, then $PP' = \frac{1}{2}$ and
\[\Box \sqrt{R - 2\Lambda} = q\sqrt{R - 2\Lambda} = \zeta \Lambda \sqrt{R - 2\Lambda}, \quad \zeta \Lambda \neq 0,\] (2.20)

where $q = \zeta \Lambda$ and $q^{-1} = \zeta^{-1} \Lambda^{-1}$ ($\zeta$ – dimensionless) follows from dimensionality of equalities (2.20). Since $P = \sqrt{R - 2\Lambda}$, EoM (2.19) simplify to
\[(G_{\mu\nu} + \Lambda g_{\mu\nu}) (1 + \mathcal{F}(q)) + \frac{1}{2} \mathcal{F}'(q) S_{\mu\nu}(\sqrt{R - 2\Lambda}, \sqrt{R - 2\Lambda}) = 0.\] (2.21)

It is evident that EoM (2.21) are satisfied if
\[\mathcal{F}(q) = -1 \quad \text{and} \quad \mathcal{F}'(q) = 0.\] (2.22)

In this case, nonlocal action (2.1) has null value. Nonlocal operator $\mathcal{F}(\Box)$, that satisfies conditions (2.22) in model (2.1), can be taken in the symmetric form
\[\mathcal{F}(\Box) = 1 + \mathcal{F}(\Box), \quad \mathcal{F}(\Box) = \sum_{n=1}^{+\infty} \tilde{f}_n \left[ \left( \frac{\Box}{q} \right)^n + \left( \frac{q}{\Box} \right)^n \right],\] (2.23)

where $\tilde{f}_n$ are now dimensionless coefficients. It is easy to prove that $\mathcal{F}(\Box)$ presented in the following simple symmetric form:
\[\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} \tilde{f}_n \left[ \left( \frac{\Box}{q} \right)^n + \left( \frac{q}{\Box} \right)^n \right] = -\frac{1}{2e} \left( \frac{\Box}{q} e^{\frac{\Box}{q}} + \frac{q}{\Box} e^{\frac{q}{\Box}} \right), \quad q \neq 0,\] (2.24)

satisfies conditions (2.22). The symmetry is with respect to the transformation $\left( \frac{\Box}{q} \right) \leftrightarrow \left( \frac{q}{\Box} \right)$. Operator $\mathcal{F}(\Box)$, defined in (2.24), has $\sqrt{R - 2\Lambda}$ as its eigenfunction with eigenvalue $-1$, that is
\[-\frac{1}{2e} \left( \frac{\Box}{q} e^{\frac{\Box}{q}} + \frac{q}{\Box} e^{\frac{q}{\Box}} \right) \sqrt{R - 2\Lambda} = -\sqrt{R - 2\Lambda} \quad \text{whenever} \quad \Box \sqrt{R - 2\Lambda} = q \sqrt{R - 2\Lambda}.\] (2.25)

Since the d’Alembert-Beltrami operator $\Box$ and Ricci scalar $R$ depend on spacetime metric, it is essential to find suitable metric that will satisfy (2.25). As we will see in the next section, there are several exact cosmological solutions that realize (2.25) and eigenvalue $q$ is proportional to $\Lambda$, i.e. $q = \zeta \Lambda$, where $\zeta \neq 0$ is a definite dimensionless constant for each concrete case. Moreover, it has to be $q = \zeta \Lambda$, since there is no other parameter than $\Lambda$ which is of the same dimension as $\Box$ in this nonlocal gravity model. Hence, nonlocal operator (2.24) can be rewritten as
\[\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} \tilde{f}_n \left[ \left( \frac{\Box}{\zeta \Lambda} \right)^n + \left( \frac{\zeta \Lambda}{\Box} \right)^n \right] = -\frac{1}{2e} \left( \frac{\Box}{\zeta \Lambda} e^{\frac{\Box}{\zeta \Lambda}} + \frac{\zeta \Lambda}{\Box} e^{\frac{\zeta \Lambda}{\Box}} \right), \quad \zeta \Lambda \neq 0,\] (2.26)

where for some specific $\Box$ holds $\Box \sqrt{R - 2\Lambda} = \zeta \Lambda \sqrt{R - 2\Lambda}$.

Note that representation of $\mathcal{F}(\Box)$ by exponential function in the form (2.26) is not unique and can be written in the following more general form
\[\mathcal{F}(\Box) = -\frac{1}{2} e^{(\pm 1)} \left( \frac{\Box}{\zeta \Lambda} e^{\pm \frac{\Box}{\zeta \Lambda}} + \frac{\zeta \Lambda}{\Box} e^{\pm \frac{\zeta \Lambda}{\Box}} \right).\] (2.27)
3 Cosmological solutions

We are now interested in some exact cosmological solutions of nonlocal gravity model (2.1) with equations of motion (2.21).

At the cosmological scale the universe is homogeneous and isotropic with the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right), \quad (c = 1), \quad k = 0, \pm 1,$$

(3.1)

where $a(t)$ is the cosmic scale factor. Our primary interest is to find cosmological scale factor $a(t)$ that satisfies equations (2.20), i.e.

$$\Box \sqrt{R - 2\Lambda} = q \sqrt{R - 2\Lambda}, \quad \Box^{-1} \sqrt{R - 2\Lambda} = q^{-1} \sqrt{R - 2\Lambda}, \quad q = \zeta \Lambda \neq 0,$$

(3.2)

where $\Box$, $R$ and the Hubble parameter $H$ in the FLRW universe depend on $a(t)$ as follows:

$$\Box = -\frac{\partial^2}{\partial t^2} - 3H(t)\frac{\partial}{\partial t}, \quad H(t) = \frac{\dot{a}}{a},$$

$$R(t) = 6\left(\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}\right), \quad \dot{a} = \frac{\partial a}{\partial t}.$$  

(3.3)

(3.4)

If $a(t)$ is a solution of equation (3.2), then it is also solution of equations of motion (2.25) with the corresponding two conditions (2.22) on nonlocal operator: $F(q) = -1$ and $F'(q) = 0$.

Before to start concrete investigation of the possible time dependent scale factors $a(t)$, let us note that there is not the Minkowski space solution. It follows from the fact that one can not take $\Lambda = 0$ in nonlocal operator $F(\Box)$, see (2.26) and (3.2). According to (3.2), there is not also (anti-)de Sitter solution, since in (anti-)de Sitter case $\zeta = 0$ and consequently $q = 0$. However, if there is the absence of terms with $\Box^{-1}$, then there exist both Minkowski and (anti-)de Sitter solutions. Namely, in the Minkowski case $\Lambda = R = k = 0$ and EoM (2.21) are satisfied since $F(0) = 0$ and $F'(0) = 0$.

3.1 Cosmological solutions in the flat universe ($k = 0$)

3.1.1 Solutions of the form $a(t) = A t^n e^{\gamma t^2}$, ($k = 0$)

According to our previous paper [44], there are two solutions of the form

$$a(t) = A t^n e^{\gamma t^2}, \quad k = 0,$$

(3.5)

where $n$ and $\gamma$ are some definite real constants. The corresponding $\Box, H(t)$ and $R(t)$ are:

$$\Box = -\frac{\partial^2}{\partial t^2} - 3(nt^{-1} + 2\gamma t)\frac{\partial}{\partial t}, \quad H(t) = nt^{-1} + 2\gamma t,$$

$$R(t) = 6\left(-nt^{-2} + 2\gamma + 2(nt^{-1} + 2\gamma t)\right).$$

(3.6)

(3.7)
Inserting (3.6) and (3.7) in (3.2) one obtains a system of six equations:

\[ n^2(2 - 3n)(2n - 1)^2 = 0, \]
\[ n(2n - 1)(-nq + 2n^2q - \Lambda + n\Lambda + 6\gamma + 24n\gamma - 36n^2\gamma) = 0, \]
\[ n(2n - 1)(-nq\Lambda + 6nq\gamma + 3\Lambda\gamma + 54\gamma^2 - 72n\gamma^2) = 0, \]
\[ q\Lambda^2 - 12q\Lambda\gamma - 48nq\Lambda\gamma + 36q\gamma^2 + 144nq\gamma^2 + 864n^2q\gamma^2 - 24\Lambda\gamma^2 - 72n\Lambda\gamma^2 + 144\Lambda^3 + 1008n\Lambda^3 + 1728n^2\gamma^3 = 0, \]
\[ \gamma^2(-q\Lambda + 6nq\gamma - 3\Lambda\gamma + 18\gamma^2 + 108n\gamma^2) = 0, \]
\[ \gamma^4(q + 6\gamma) = 0. \]

The above system of equations is satisfied in the following two cases:

1. \( n = \frac{2}{3}, \gamma = \frac{\Lambda}{14}, q = -\frac{3}{7}\Lambda, \)

2. \( n = 0, \gamma = \frac{1}{6}\Lambda, q = -\Lambda. \)

According to (3.8) and (3.9), there are two solutions in flat space:

\[ a_1(t) = A t^\frac{2}{3} e^{\frac{\Lambda t}{14}}, \quad k = 0, \quad F(-\frac{3}{7}\Lambda) = -1, \quad F'(-\frac{3}{7}\Lambda) = 0, \]
\[ a_2(t) = A e^{\frac{\Lambda t}{6}}, \quad k = 0, \quad F(-\Lambda) = -1, \quad F'(-\Lambda) = 0. \]

### 3.1.2 New solutions of the form \( a(t) = (\alpha e^{\lambda t} + \beta e^{-\lambda t})^\gamma, \quad (k = 0) \)

We are now going to find some new cosmological solutions of the form

\[ a(t) = (\alpha e^{\lambda t} + \beta e^{-\lambda t})^\gamma, \quad \gamma \in \mathbb{R}, \]

that satisfy equation

\[ \Box \sqrt{R} = q\sqrt{R} - 2\Lambda, \quad q = \zeta\Lambda \neq 0. \]

Note that the corresponding Hubble parameter and the Ricci scalar are:

\[ H(t) = \left(1 - \frac{2\beta}{\alpha e^{2\lambda t} + \beta}\right)\gamma \lambda, \]
\[ R(t) = 6k(\alpha e^{\lambda t} + \beta e^{-\lambda t})^{-2\gamma} + \frac{12\gamma^2(\alpha^2\gamma e^{4\lambda t} - 2\alpha\beta(\gamma - 1)e^{2\lambda t} + \beta^2\gamma)}{(\alpha e^{2\lambda t} + \beta)^2}. \]

The equality (3.13) can be expanded into equation

\[ -9(\beta + \alpha e^{2\lambda t})^4(A_0 + A_1 e^{2\lambda t} + A_2 e^{4\lambda t}) + 3(\alpha e^{\lambda t} + \beta e^{-\lambda t})^{2\gamma} \left(\beta + \alpha e^{2\lambda t}\right)^2 \left(B_0 + B_1 e^{2\lambda t} + B_2 e^{4\lambda t} + B_3 e^{6\lambda t} + B_4 e^{8\lambda t}\right) - \left(\alpha e^{\lambda t} + \beta e^{-\lambda t}\right)^{4\gamma} \left(C_0 + C_1 e^{2\lambda t} + C_2 e^{4\lambda t} + C_3 e^{6\lambda t} + C_4 e^{8\lambda t} + C_5 e^{10\lambda t} + C_6 e^{12\lambda t}\right) = 0, \]

\[ (3.14) \]
where

\[ A_0 = k^2 \beta^2 \left( q - 2\gamma^2 \lambda^2 \right), \]
\[ A_1 = 2k^2 \alpha \beta \left( 2(\gamma - 1) \gamma \lambda^2 + q \right), \]
\[ A_2 = k^2 \alpha^2 \left( q - 2\gamma^2 \lambda^2 \right), \]
\[ B_0 = -k\beta^4 \left( 2q - \gamma^2 \lambda^2 \right) \left( 6\gamma^2 \lambda^2 - \Lambda \right), \]
\[ B_1 = -4k\alpha \beta^3 \left( 3\gamma \left( 2\gamma^3 + 7\gamma^2 - 9\gamma + 2 \right) \lambda^4 + \gamma \Lambda \lambda^2 + q \left( 6\gamma \lambda^2 - 2\Lambda \right) \right), \]
\[ B_2 = 2k\alpha^2 \beta^2 \left( 3\gamma \left( 6\gamma^3 + 28\gamma^2 - 44\gamma + 16 \right) \lambda^4 + \gamma (\gamma - 4) \Lambda \lambda^2 + 6q \left( 2\gamma^2 \lambda^2 - 4\gamma \lambda^2 + \Lambda \right) \right), \]
\[ B_3 = -4k\alpha^3 \beta \left( 3\gamma \left( 2\gamma^3 + 7\gamma^2 - 9\gamma + 2 \right) \lambda^4 + \gamma \Lambda \lambda^2 + q \left( 6\gamma \lambda^2 - 2\Lambda \right) \right), \]
\[ B_4 = -k\alpha^4 \left( 2q - \gamma^2 \lambda^2 \right) \left( 6\gamma^2 \lambda^2 - \Lambda \right), \]

and

\[ C_0 = q \beta^6 \left( 6\gamma^2 \lambda^2 - \Lambda \right)^2, \]
\[ C_1 = -6\alpha \beta^5 \left( 6\gamma^2 \lambda^2 - \Lambda \right) \left( 2\gamma \left( -6\gamma^2 + 7\gamma - 2 \right) \lambda^4 + q \left( 2\gamma^2 \lambda^2 - 4\gamma \lambda^2 + \Lambda \right) \right), \]
\[ C_2 = -3\alpha^2 \beta^4 \left( 48\gamma^2 \left( 12\gamma^3 - 20\gamma^2 + 9\gamma - 1 \right) \lambda^6 + 16\gamma \left( 2\gamma - 1 \right) \Lambda \lambda^4 \right. \]
\[ + 4q \left( 3\gamma^4 \lambda^4 - 12\gamma^2 \lambda^4 \right) + q \Lambda \left( -4\gamma^2 \lambda^2 + 32\gamma \lambda^2 - 5\Lambda \right) \right), \]
\[ C_3 = 4\alpha^3 \beta^3 \left( 108\gamma^2 \left( 6\gamma^3 - 11\gamma^2 + 8\gamma - 2 \right) \lambda^6 + 6\gamma \left( 6\gamma^2 - 15\gamma + 6 \right) \Lambda \lambda^4 \right. \]
\[ + 36q \left( \gamma^4 \lambda^4 - 2\gamma^2 \lambda^4 + 2\gamma^2 \lambda^4 \right) + q \Lambda \left( 12\gamma^2 \lambda^2 - 36\gamma \lambda^2 + 5\Lambda \right) \right), \]
\[ C_4 = -3\alpha^4 \beta^2 \left( 48\gamma^2 \left( 12\gamma^3 - 20\gamma^2 + 9\gamma - 1 \right) \lambda^6 + 16\gamma \left( 2\gamma - 1 \right) \Lambda \lambda^4 \right. \]
\[ + 4q \left( 3\gamma^4 \lambda^4 - 12\gamma^2 \lambda^4 \right) + q \Lambda \left( -4\gamma^2 \lambda^2 + 32\gamma \lambda^2 - 5\Lambda \right) \right), \]
\[ C_5 = -6\alpha^5 \beta \left( 6\gamma^2 \lambda^2 - \Lambda \right) \left( 2\gamma \left( -6\gamma^2 + 7\gamma - 2 \right) \lambda^4 + q \left( 2\gamma^2 \lambda^2 - 4\gamma \lambda^2 + \Lambda \right) \right), \]
\[ C_6 = q \alpha^6 \left( 6\gamma^2 \lambda^2 - \Lambda \right)^2. \]

It is evident that equation (3.14) is satisfied if

\[ A_0 = A_1 = A_2 = 0, \quad B_0 = B_1 = B_2 = B_3 = B_4 = 0, \quad (3.18) \]
\[ C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0. \quad (3.19) \]

If \( \alpha \beta \neq 0, \ R \neq 2\Lambda \) and \( q \neq 0 \) then equations (3.18) and (3.19) are satisfied in the flat space when

\[ \gamma = \frac{2}{3}, \quad q = \frac{3}{8} \Lambda, \quad \lambda = \pm \sqrt{\frac{3}{8} \Lambda}, \quad k = 0. \quad (3.20) \]
\[ \gamma = \frac{2}{3}. \quad (3.21) \]

When \( \alpha \neq 0 \) and \( \beta \neq 0 \), we have the following two special solutions:

\[ a_3(t) = A \cosh^2 \left( \sqrt{\frac{3}{8} \Lambda} \ t \right), \quad k = 0, \quad \mathcal{F} \left( \frac{3}{8} \Lambda \right) = -1, \quad \mathcal{F}' \left( \frac{3}{8} \Lambda \right) = 0, \quad (3.22) \]
\[ a_4(t) = A \sinh^2 \left( \sqrt{\frac{3}{8} \Lambda} \ t \right), \quad k = 0, \quad \mathcal{F} \left( \frac{3}{8} \Lambda \right) = -1, \quad \mathcal{F}' \left( \frac{3}{8} \Lambda \right) = 0. \quad (3.23) \]
Solution similar to $a_3(t)$ is also obtained in [47] nonlocal gravity model of the form $R - 2\Lambda + RF(\Box)R$.

3.1.3 New solutions of the form $a(t) = (\alpha \sin \lambda t + \beta \cos \lambda t)\gamma$, $(k = 0)$

Note that $a(t) = \alpha \sin \lambda t + \beta \cos \lambda t$ can be presented as $a(t) = \tilde{\alpha} e^{i\lambda t} + \tilde{\beta} e^{-i\lambda t}$ with connections $\tilde{\alpha} = \frac{\alpha}{2} + \frac{\beta}{2\lambda}$, $\tilde{\beta} = \frac{\alpha}{2} - \frac{\beta}{2\lambda}$, i.e. $\tilde{\alpha} = \tilde{\beta}^*$. As a consequence of this property, one can use results obtained in the previous subsubsection by replacement $\lambda \rightarrow i\lambda$.

When $\alpha \neq 0$ and $\beta \neq 0$ there are two possibilities for $\gamma$: $\gamma = \frac{2}{3}$ and $\gamma = \frac{1}{2}$. Since $\alpha \sin \lambda t + \beta \cos \lambda t$ has negative as well as positive values, hence only $\gamma = \frac{2}{3}$ can be applied. Taking $\beta = \alpha$ or $\beta = -\alpha$, and $\alpha^2 = A$, we can write the following two solutions:

$$a_5(t) = A \left(1 + \sin \left(\sqrt{-\frac{3}{2}\Lambda} t\right)\right)^{\frac{1}{4}}, \quad k = 0, \quad F\left(\frac{3}{8}\Lambda\right) = -1, \quad F'\left(\frac{3}{8}\Lambda\right) = 0,$$  

$$a_6(t) = A \left(1 - \sin \left(\sqrt{-\frac{3}{2}\Lambda} t\right)\right)^{\frac{1}{4}}, \quad k = 0, \quad F\left(\frac{3}{8}\Lambda\right) = -1, \quad F'\left(\frac{3}{8}\Lambda\right) = 0.$$  

When $\alpha = 0$ or $\beta = 0$, we have also two cosmological solutions with $\gamma = \frac{2}{3}$ and $k = 0$. These solutions are:

$$a_7(t) = A \sin \frac{2}{3} \sqrt{-\frac{3}{8}\Lambda} t, \quad k = 0, \quad F\left(\frac{3}{8}\Lambda\right) = -1, \quad F'\left(\frac{3}{8}\Lambda\right) = 0,$$  

$$a_8(t) = A \cos \frac{2}{3} \sqrt{-\frac{3}{8}\Lambda} t, \quad k = 0, \quad F\left(\frac{3}{8}\Lambda\right) = -1, \quad F'\left(\frac{3}{8}\Lambda\right) = 0.$$  

Note that there are the following connections:

$$a_5(t) = \sqrt{2} a_8\left(\frac{\pi}{4} \sqrt{-\frac{8}{3\Lambda}} - t\right), \quad a_6(t) = \sqrt[3]{2} a_7\left(\frac{\pi}{4} \sqrt{-\frac{8}{3\Lambda}} - t\right).$$  

3.2 Cosmological solutions in the closed and open universe $(k = \pm 1)$

We found three vacuum solutions in the closed and open FLRW space.

3.2.1 Cosmological solution of the form $a(t) = A e^{\pm\sqrt{\frac{2}{3}\Lambda} t}$, $(k = \pm 1)$

If in the above Section 3.1.2 $\alpha \neq 0, \beta = 0$ or $\alpha = 0, \beta \neq 0$ then there is a new solution which is the same as already obtained one, $a_9(t) = A e^{\pm\sqrt{\frac{2}{3}\Lambda} t}$, $k = \pm 1$.

In the previous paper [44], we presented the following exact solution:

$$a_9(t) = A e^{\pm\sqrt{\frac{2}{3}\Lambda} t}, \quad k = \pm 1, \quad F\left(\frac{1}{3}\Lambda\right) = -1, \quad F'\left(\frac{1}{3}\Lambda\right) = 0, \quad \Lambda > 0,$$  

of the same EoM (2.21) with property (3.2). In this case

$$\Box = -\frac{\partial^2}{\partial t^2} - 3H(t) \frac{\partial}{\partial t}, \quad H(t) = \pm \frac{\Lambda}{6},$$  

$$R(t) = \frac{6k}{A^2} \exp \left(\mp \frac{\sqrt{\frac{2}{3}\Lambda}}{\Lambda} t\right) + 2\Lambda,$$  

$$\Box \sqrt{R - 2\Lambda} = \frac{\Lambda}{3} \sqrt{R - 2\Lambda}, \quad \Box^{-1} \sqrt{R - 2\Lambda} = \frac{3}{\Lambda} \sqrt{R - 2\Lambda},$$  

$$F\left(\frac{1}{3}\Lambda\right) = -1, \quad F'\left(\frac{1}{3}\Lambda\right) = 0.$$  

– 10 –
3.2.2 New solutions of the form $a(t) = (\alpha \, e^{\lambda t} + \beta \, e^{-\lambda t})^\gamma$, $(k = \pm 1)$

Here we have the same general consideration until Eqs. (3.18) and (3.19). When $\alpha \neq 0$, $\beta \neq 0$, $R \neq 2\Lambda$, $q \neq 0$ and $k \neq 0$ we have that Eqs. (3.18) and (3.19) are satisfied if

$$\gamma = \frac{1}{2}, \quad q = \frac{1}{3} \Lambda, \quad \lambda = \pm \sqrt{\frac{2}{3} \Lambda}, \quad k \neq 0. \quad (3.34)$$

The corresponding cosmological solutions are:

$$a_{10}(t) = A \cosh^\frac{1}{2} \left( \sqrt{\frac{2}{3} \Lambda} \, t \right), \quad k = \pm 1, \quad \mathcal{F}(\frac{1}{3} \Lambda) = -1, \quad \mathcal{F}'(\frac{1}{3} \Lambda) = 0, \quad (3.35)$$

$$a_{11}(t) = A \sinh^\frac{1}{2} \left( \sqrt{\frac{2}{3} \Lambda} \, t \right), \quad k = \pm 1, \quad \mathcal{F}(\frac{1}{3} \Lambda) = -1, \quad \mathcal{F}'(\frac{1}{3} \Lambda) = 0. \quad (3.36)$$

4 Discussion

Here, we discuss cosmological solutions presented in the previous section. We have practically 11 exact vacuum background solutions of our nonlocal gravity model (2.1). These solutions can be divided into two classes: 1) $a_1(t), a_2(t), a_3(t), a_4(t), a_5(t), a_6(t), a_7(t), a_8(t)$ which are solutions in the flat universe $(k = 0)$, and 2) $a_9(t), a_{10}(t), a_{11}(t)$ that are the same solutions in both closed $(k = +1)$ and open space $(k = -1)$.

It is useful to introduce effective Friedmann equations to the above solutions:

$$\frac{\ddot{a}_i}{a_i} = -\frac{4\pi G}{3} (\bar{\rho}_i + 3\bar{p}_i) + \frac{\Lambda_i}{3}, \quad \frac{\dot{a}_i^2 + k}{a_i^2} = \frac{8\pi G}{3} \bar{p}_i + \frac{\Lambda_i}{3}, \quad i = 1, 2, ..., 11, \quad (4.1)$$

where $\bar{\rho}_i$ and $\bar{p}_i$ are counterparts of the energy density and pressure in the standard model of cosmology, respectively. $\Lambda_i$ is an effective cosmological constant, which differ from $\Lambda$ in $\Lambda$CDM cosmological model. From (4.1) we have

$$\bar{\rho}_i(t) = \frac{3}{8\pi G} \left( \frac{\dot{a}_i^2 + k}{a_i^2} - \frac{\Lambda_i}{3} \right), \quad \bar{p}_i(t) = \frac{1}{8\pi G} \left( \Lambda_i - 2 \frac{\dot{a}_i}{a_i} - \frac{\dot{a}_i^2 + k}{a_i^2} \right). \quad (4.2)$$

Then the equation of state is

$$\bar{w}_i(t) = \bar{\rho}_i(t) / \bar{p}_i(t), \quad (4.3)$$

where $\bar{w}_i(t)$ is the corresponding effective state parameter.

We also use the following Planck-2018 results for the $\Lambda$CDM parameters [1]:

$$t_0 = (13.801 \pm 0.024) \times 10^9 \text{yr} - \text{age of the universe}, \quad (4.4)$$

$$H(t_0) = (67.40 \pm 0.50) \text{ km/s/Mpc} - \text{Hubble parameter}, \quad (4.5)$$

$$\Omega_m = 0.315 \pm 0.007 - \text{matter density parameter}, \quad (4.6)$$

$$\Omega_{\Lambda} = 0.685 - \Lambda \text{ density parameter}, \quad (4.7)$$

$$w_0 = -1.03 \pm 0.03 - \text{ratio of pressure to energy density}. \quad (4.8)$$
4.1 Solutions \( a_1(t) = A \ t^{\frac{2}{3}} \ e^{\frac{\Lambda_1 t^2}{2}} \) and \( a_2(t) = A \ e^{\frac{\Lambda_2 t^2}{2}}, \ (k = 0, \Lambda_{1,2} \neq 0) \)

These two solutions, with \( k = 0, \Lambda_{1,2} \neq 0 \), were found earlier and presented in [44]. We are going to discuss them again and point out some new features. Both are even functions of cosmic time \( t \). As we will see, especially significant is solution \( a_1(t) = A \ t^{\frac{2}{3}} \ e^{\frac{\Lambda_1 t^2}{2}} \).

4.1.1 Cosmological solution \( a_1(t) = A \ t^{\frac{2}{3}} \ e^{\frac{\Lambda_1 t^2}{2}} \)

This solution is singular at cosmic time \( t = 0 \). There are two constraints on nonlocal operator: \( \mathcal{F}( - \frac{2}{3} \Lambda_1 ) = -1, \mathcal{F}'( - \frac{2}{3} \Lambda_1 ) = 0. \)

Note that the scale factor \( a_1(t) \) is a product of \( t^\frac{2}{3} \) and \( e^{\frac{\Lambda_1 t^2}{2}} \), where \( t^\frac{2}{3} \) is scale factor of the matter-dominated universe in Einstein theory of gravity, while \( e^{\frac{\Lambda_1 t^2}{2}} \) is a new scale factor that induces accelerated expansion of the universe for \( \Lambda_1 > 0 \). This exponential acceleration is similar to the ordinary de Sitter case. The effective cosmological constant \( \Lambda_1 \) contains the dark energy. Since \( \Lambda_1 \) is introduced in this model by its construction, a solution with accelerated expansion of the universe is not a surprise. The surprise is emergence of the effects which are usually assigned to the dark matter in the \( \Lambda \)CDM model. This nonlocal de Sitter nature of the dark matter is illustrated by analysis below.

The corresponding Hubble parameter, acceleration and the Ricci scalar are:

\[
H_1(t) = \frac{\dot{a}_1}{a_1} = \frac{2}{3} \frac{1}{t} + \frac{1}{7} \Lambda_1 t, \tag{4.9}
\]

\[
\ddot{a}_1(t) = \left( - \frac{2}{9} \frac{1}{t^2} + \frac{1}{3} \Lambda_1 + \frac{1}{49} \Lambda_1^2 t^2 \right) a_1(t), \tag{4.10}
\]

\[
R_1(t) = 6 \left( \frac{\ddot{a}_1}{a_1} + \left( \frac{\dot{a}_1}{a_1} \right)^2 \right) = \frac{4}{3} \frac{1}{t^2} + \frac{22}{7} \Lambda_1 + \frac{12}{49} \Lambda_1^2 t^2, \tag{4.11}
\]

where subscript "1" means that these quantities are related to the solution \( a_1(t) \). The first terms expressed as function of time in all equations (4.9), (4.10) and (4.11) are just what is in the case of the matter-dominated universe in the Einstein theory of gravity. Since \( t^\frac{2}{3} \) mimics effects of matter without matter at the cosmic scale, we see that in this case (dark) matter emerges in nonlocal gravity model (2.1) through vacuum solution \( a_1(t) \).

It is also worth mentioning that at present cosmic time \( t_0 = 13.801 \times 10^9 \text{yr} \) expression (4.9) gives an interesting connection between \( H(t_0) = 67.40 \text{ km/s/Mpc} \) and the effective cosmological constant \( \Lambda_1 \) [44], i.e. the following equation holds:

\[
H_1(t_0) = \frac{2}{3} \frac{1}{t_0} + \frac{1}{7} \Lambda_1 t_0, \tag{4.12}
\]

where here we take \( H_1(t_0) = H(t_0) \). Given \( H(t_0) \) and \( t_0 \) we computed \( \Lambda_1 = 1.05 \times 10^{-35} \text{s}^{-2} \) that differs from \( \Lambda \) in \( \Lambda \)CDM model, where \( \Lambda = 3 H^2(t_0) \Omega_\Lambda = 0.98 \times 10^{-35} \text{s}^{-2} \). We also computed \( \ddot{a}_1(t_0)/a_1(t_0) = 2.7 \times 10^{-36} \text{s}^{-2} \) and \( R(t_0) = 4.5 \times 10^{-35} \text{s}^{-2} \). Consequently \( R_1(t_0) - 2 \Lambda_1 = 2.4 \times 10^{-35} \text{s}^{-2} \).

Another aspect of this cosmological solution, that is useful to discuss, concerns effective energy density \( \bar{\rho}_1(t) \) and pressure \( \bar{p}_1(t) \). Replacing solution \( a_1(t) \) in (4.2) with \( k = 0 \), we
\[ \bar{\rho}_1(t) = \frac{3}{8\pi G} \left( H_1^2(t) - \frac{\Lambda_1}{3} \right) = \frac{3}{8\pi G} \left( \frac{4}{9} t^{-2} - \frac{1}{4} \Lambda_1 + \frac{1}{49} \Lambda_1^2 t^2 \right), \quad (4.13) \]

\[ \bar{p}_1(t) = \frac{\Lambda_1}{56\pi G} \left( 1 - \frac{3}{7} \Lambda_1 t^2 \right). \quad (4.14) \]

The first term in \( \bar{\rho}_1(t) \) is proportional to \( \frac{4}{9} t^2 \) and consequently behaves in the same way as in the usual matter-dominated universe. Computation of (4.13) in \( t = t_0 \) gives \( \bar{\rho}_1(t_0) = 2.26 \times 10^{-30} \frac{g}{cm^3} \). Note that energy density \( \rho(t_0) \) in Einstein theory of gravity with \( \Lambda \)-term is

\[ \rho(t_0) = \frac{3}{8\pi G} \left( H_2(t_0) - \frac{\Lambda}{3} \right) = 2.68 \times 10^{-30} \frac{g}{cm^3}, \quad (4.15) \]

Then we have

\[ \rho(t_0) - \bar{\rho}_1(t_0) = \frac{\Lambda_1 - \Lambda}{8\pi G} = \rho_{\Lambda_1} - \rho_{\Lambda} = 0.42 \times 10^{-30} \frac{g}{cm^3}, \quad (4.16) \]

where

\[ \rho_{\Lambda_1} = \frac{\Lambda_1}{8\pi G} = 6.25 \times 10^{-30} \frac{g}{cm^3}, \quad \rho_{\Lambda} = \frac{\Lambda}{8\pi G} = 5.83 \times 10^{-30} \frac{g}{cm^3} \quad (4.17) \]

are effective vacuum energy density of background solution \( a_1(t) \) and \( \Lambda CD M \) model, respectively.

Recall that the critical energy density \( \rho_c \) is

\[ \rho_c = \frac{3}{8\pi G} H^2(t_0) = 8.51 \times 10^{-30} \frac{g}{cm^3}, \quad (4.18) \]

and consequently

\[ \Omega_{\Lambda_1} = \frac{\rho_{\Lambda_1}}{\rho_c} = 0.734, \quad \Omega_{\Lambda} = \frac{\rho_{\Lambda}}{\rho_c} = 0.685, \quad \Delta \Omega_{\Lambda} = \Omega_{\Lambda_1} - \Omega_{\Lambda} = 0.049 \quad (4.19) \]

\[ \Omega_m = \frac{\rho(t_0)}{\rho_c} = 0.315, \quad \Omega_{m_1} = \frac{\rho_\lambda(t_0)}{\rho_c} = 0.266, \quad \Delta \Omega_m = \Omega_m - \Omega_{m_1} = 0.049. \quad (4.20) \]

According to (4.19) and (4.20), we obtain that \( \Omega_m = 26.6\% \) corresponds to dark matter and \( \Delta \Omega_m = \Delta \Omega_{\Lambda} = 4.9\% \) is related to visible matter, what is in a very good agreement with the standard model of cosmology [1]. One can also conclude that the effective cosmological constant \( \Lambda_1 \) contains the standard cosmological constant of the \( \Lambda CD M \) and something else what is related to the standard visible matter.

It is worth to summarize the above analysis. We used the cosmological parameters listed in (4.4) – (4.7) to compute the effective cosmological constant \( \Lambda_1 \) and the effective energy density \( \bar{\rho}_1(t) \) in this nonlocal de Sitter model. Comparing calculated \( \Omega_{m_1} = \frac{\rho_\lambda(t_0)}{\rho_c} \) with dark matter density parameter in the \( \Lambda CD M \) model we concluded that \( \bar{\rho}_1(t_0) \) corresponds to the dark energy density. We also found that \( \Omega_m - \Omega_{m_1} = \Omega_{\Lambda_1} - \Omega_{\Lambda} \) and is related to the visible matter density parameter.

Now we can discuss effective pressure (4.14). At the beginning, effective pressure is positive, i.e. \( \bar{p}_1(0) = \frac{\Lambda_1}{56\pi G} \), then decreases with time and equals zero at \( t = \sqrt{\frac{7}{3\Lambda_1}} = \)
4.71 \times 10^{17} \text{ s} = 14,917 \times 10^9 \text{ yr}. According to (4.3), we have parameter \( \bar{w}_1(t) = \frac{\ddot{a}_1(t)}{\dot{a}_1(t)} \) which has future behavior in agreement with standard model of cosmology, i.e. \( \bar{w}_1(t \rightarrow \infty) \rightarrow -1 \).

Note that the Hubble parameter (4.9) has minimum at \( t_{min} = 21.1 \times 10^9 \text{ yr} \) and it is \( H_1(t_{min}) = 61.72 \text{ km/s/Mpc} \). From (4.10) follows that the universe experiences decelerated expansion during matter dominance, that turns to acceleration at time \( t_{acc} = 7.84 \times 10^9 \text{ yr} \) when \( \ddot{a} = 0 \).

Presently, it is not clear what role should play our other vacuum cosmological solutions in the evolution of the universe. Hence, we cannot estimate effective cosmological constants \( \Lambda_2, \Lambda_3, \ldots, \Lambda_{11} \) and we will use for them the standard cosmological constant \( \Lambda \) in their graphic illustration and write almost everywhere in the sequel \( \Lambda \) instead of any \( \Lambda_2, \Lambda_3, \ldots, \Lambda_{11} \).

### 4.1.2 Cosmological solution \( a_2(t) = A e^{\frac{\Lambda}{6} t^2} \)

As it is already said, we will use \( \Lambda_2 = \Lambda \) and write \( \Lambda \) instead of \( \Lambda_2 \). This is a nonsingular bounce solution symmetric under time change \( t \rightarrow -t \), related to the flat space-time and similar to \( e^{\frac{\Lambda}{6} t^2} \) factor of \( a_1(t) \) cosmological solution. Equations of motion are satisfied when \( F(-\Lambda) = -1, \quad F'(-\Lambda) = 0. \)

The corresponding Hubble parameter is

\[
H_2(t) = \frac{1}{3} \dot{t},
\]

with \( H_2(0) = 0 \) and depends linearly on the cosmic time \( t \). Equating \( H_2(t) \) with \( H_1(t) \) given by (4.9), and taking \( \Lambda = 0.98 \times 10^{-35} \text{ s}^{-2} \) from the \( \Lambda \text{CDM} \) model, we compute time \( t = 5.98 \times 10^{17} \text{ s} = 18.97 \times 10^9 \text{ yr} \) when \( H_2(t) \) reaches \( H_1(t) \) (see Fig. 1).

The corresponding effective energy density and pressure are:

\[
\bar{\rho}_2(t) = \frac{\Lambda (\Lambda t^2 - 3)}{24\pi G}, \quad \bar{p}_2(t) = \frac{\Lambda (1 - \Lambda t^2)}{24\pi G}, \quad \bar{\omega}_2(t)_{t \rightarrow \infty} \rightarrow -1.
\]

The equality \( \bar{\rho}_2(t) = \bar{\rho}_1(t) \) gives the same value of the time as \( H_2(t) = H_1(t) \), that is \( t = 18.97 \times 10^9 \text{ yr} \). When \( \Lambda > 0 \) then \( \bar{p}_2(t) < 0 \) for \( t < \frac{\sqrt{3}}{\Lambda} \) and it is unlike that solution \( a_2(t) \) plays some cosmological role in this cosmic interval. However \( a_2(t) \) could be of some interest when \( t > \frac{\sqrt{3}}{\Lambda} \) and \( \Lambda > 0 \). Note that \( \bar{\rho}_2(t) > 0 \) when \( \Lambda < 0 \) for any time \( t \).
4.2 Solutions $a_3(t) = A \cosh^{\frac{2}{3}} (\sqrt{\frac{3\Lambda}{8}} t)$ and $a_4(t) = A \sinh^{\frac{2}{3}} (\sqrt{\frac{3\Lambda}{8}} t)$, $(k = 0, \Lambda > 0)$

Solution $a_3(t) = A \cosh^{\frac{2}{3}} (\sqrt{\frac{3\Lambda}{8}} t)$ satisfies equations of motion if $F(\frac{3}{8}\Lambda) = -1$, $F'(\frac{3}{8}\Lambda) = 0$. The corresponding Hubble parameter, effective energy dense and pressure, are:

$$H_3(t) = \sqrt{\frac{\Lambda}{6}} \tanh(\sqrt{\frac{3\Lambda}{8}} t),$$

$$\bar{\rho}_3(t) = \frac{\Lambda}{16\pi G} \left( \tanh^2(\sqrt{\frac{3\Lambda}{8}} t) - 2 \right), \quad \bar{w}_3(t) = \frac{\Lambda}{16\pi G}, \quad \bar{w}_3(t)_{t \to \infty} \to -1. \quad (4.23)$$

In the similar way, solution $a_4(t) = A \sinh^{\frac{2}{3}} (\sqrt{\frac{3\Lambda}{8}} t)$ requires conditions $F(\frac{3}{8}\Lambda) = -1$, $F'(\frac{3}{8}\Lambda) = 0$ to satisfy EoM. The related Hubble parameter, and the analogs of energy density and pressure, are:

$$H_4(t) = \sqrt{\frac{\Lambda}{6}} \coth(\sqrt{\frac{3\Lambda}{8}} t),$$

$$\bar{\rho}_4(t) = \frac{\Lambda}{16\pi G} \left( \coth^2(\sqrt{\frac{3\Lambda}{8}} t) - 2 \right), \quad \bar{w}_4(t) = \frac{\Lambda}{16\pi G}, \quad \bar{w}_4(t)_{t \to \infty} \to -1. \quad (4.24)$$

Both solutions $a_3(t)$ and $a_4(t)$ are even functions of cosmic time $t$. Solution $a_3(t)$ is a nonsingular bounce solution, while $a_4(t)$ is a singular one. For large cosmic times, these solutions become asymptotically close, see also Fig. 2.

4.3 Solutions $a_5(t) = A \left( 1 + \sin \left( \sqrt{\frac{3\Lambda}{2}} t \right) \right)^{\frac{1}{3}}$ and $a_6(t) = A \left( 1 - \sin \left( \sqrt{\frac{3\Lambda}{2}} t \right) \right)^{\frac{1}{3}}$, $(k = 0, \Lambda < 0)$

Both $a_5(t)$ and $a_6(t)$ are bounce solutions. They are also periodic solutions with periodicity $T = 2\pi \sqrt{-\frac{3}{\Lambda}}$. These two solutions could be interesting as vacuum backgrounds to study a toy cyclic cosmology. Both solutions satisfy equations of motion if $F(\frac{3}{8}\Lambda) = -1$ and $F'(\frac{3}{8}\Lambda) = 0$. Solutions $a_5(t)$ and $a_6(t)$ replace each other under change $t \to -t$. 

Figure 2. Scale factors $a_3(t) = A \cosh^{\frac{2}{3}} (\sqrt{\frac{3\Lambda}{8}} t)$, (red) and $a_4(t) = A \sinh^{\frac{2}{3}} (\sqrt{\frac{3\Lambda}{8}} t)$, (blue) are on the left side. The corresponding Hubble parameters $H_3(t)$ and $H_4(t)$ are on the right side.
Figure 3. Scale factors $a_5(t) = A \left(1 + \sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right)\right)^\frac{1}{3}$, (red) and $a_6(t) = A \left(1 - \sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right)\right)^\frac{1}{3}$, (blue) are on the left side. The corresponding Hubble parameters $H_5(t)$ and $H_6(t)$ are on the right side. In these figures, we used $\Lambda = -0.98 \times 10^{-35} \text{s}^{-2}$.

For solutions $a_5(t) = A \left(1 + \sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right)\right)^\frac{1}{3}$ and $a_6(t) = A \left(1 - \sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right)\right)^\frac{1}{3}$, we have, respectively:

$$H_5(t) = \sqrt{-\frac{\Lambda}{6}} \frac{\cos \left(\sqrt{-\frac{3\Lambda}{2}} t\right)}{\sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right) + 1},$$

$$\bar{\rho}_5(t) = -\frac{\Lambda}{16\pi G} \left(\frac{\cos^2 \left(\sqrt{-\frac{3\Lambda}{2}} t\right)}{\sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right) + 1} + 2\right), \quad \bar{\rho}_5(t) = \frac{\Lambda}{16\pi G},$$

$$H_6(t) = \sqrt{-\frac{\Lambda}{6}} \frac{\cos \left(\sqrt{-\frac{3\Lambda}{2}} t\right)}{\sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right) - 1},$$

$$\bar{\rho}_6(t) = -\frac{\Lambda}{16\pi G} \left(\frac{\cos^2 \left(\sqrt{-\frac{3\Lambda}{2}} t\right)}{\sin \left(\sqrt{-\frac{3\Lambda}{2}} t\right) - 1} + 2\right), \quad \bar{\rho}_6(t) = \frac{\Lambda}{16\pi G}.$$

Solutions $a_5(t)$ and $a_6(t)$, as well as related $H_5(t)$ and $H_6(t)$, are illustrated in Fig. 3.

4.4 Solutions $a_7(t) = A \sin^2 \left(\sqrt{-\frac{3\Lambda}{8}} t\right)$ and $a_8(t) = A \cos^2 \left(\sqrt{-\frac{3\Lambda}{8}} t\right)$, $(k = 0, \Lambda < 0)$

Solutions $a_7(t)$ and $a_8(t)$ are bounce ones with constraints on nonlocal operator $\mathcal{F} \left(\frac{3}{8} \Lambda\right) = -1$, $\mathcal{F}' \left(\frac{3}{8} \Lambda\right) = 0$. Both solutions are even functions of $t$ and periodic with period $T = 2\pi \sqrt{-\frac{2}{3\Lambda}}$. These solutions can be interesting as backgrounds for toy cyclic universes in the flat space.
The related Hubble parameters, energy density and pressure are:

\[
H_7(t) = \sqrt{-\frac{\Lambda}{6}} \cot \left( \sqrt{-\frac{3\Lambda}{8}} t \right),
\]
(4.31)

\[
\bar{\rho}_7(t) = -\frac{\Lambda}{16\pi G} \left( \cot^2 \left( \sqrt{-\frac{3\Lambda}{8}} t \right) + 2 \right), \quad \bar{p}_7(t) = \frac{\Lambda}{16\pi G}.
\]
(4.32)

\[
H_8(t) = -\sqrt{-\frac{\Lambda}{6}} \tan \left( \sqrt{-\frac{3\Lambda}{8}} t \right),
\]
(4.33)

\[
\bar{\rho}_8(t) = -\frac{\Lambda}{16\pi G} \left( \tan^2 \left( \sqrt{-\frac{3\Lambda}{8}} t \right) + 2 \right), \quad \bar{p}_8(t) = \frac{\Lambda}{16\pi G}.
\]
(4.34)

See also Fig. 4.

4.5 Solutions \( a_9(t) = Ae^{\pm\sqrt{\frac{\Lambda}{6}} t} \), \( a_{10}(t) = A \cosh^{\frac{3}{2}} \left( \sqrt{\frac{3\Lambda}{2}} t \right) \), and \( a_{11}(t) = A \sinh^{\frac{1}{2}} \left( \sqrt{\frac{3\Lambda}{2}} t \right) \), \((k = \pm 1, \Lambda > 0)\)

First of all, note that all these three solutions are valid with the same form in both closed and open FLRW space. Solutions \( a_9(t) \) and \( a_{10}(t) \) are nonsingular bounce ones, while \( a_{11}(t) \) is valid only for \( t \geq 0 \). All these solutions satisfy equations of motion under conditions \( \mathcal{F}'(\frac{1}{3}\Lambda) = -1 \) and \( \mathcal{F}'(\frac{1}{3}\Lambda) = 0 \).

The corresponding Hubble parameter, energy density and pressure are as follows:

\[
H_9(t) = \pm \sqrt{\frac{\Lambda}{6}},
\]
(4.35)

\[
\bar{\rho}_9(t) = \frac{6ke^{\mp\sqrt{\frac{3\Lambda}{2}} t} - A^2\Lambda}{16\pi A^2G}, \quad \bar{p}_9(t) = \frac{A^2\Lambda - 2ke^{\mp\sqrt{\frac{3\Lambda}{2}} t}}{16\pi A^2G}.
\]
(4.36)
Figure 5. Scale factors $a_9(t) = A e^{\pm \sqrt{3} t}$, (red) $a_{10}(t) = A \cosh^{\frac{3}{2}} (\sqrt{\frac{2A}{3}} t)$, (blue) and $a_{11}(t) = A \sinh^{\frac{3}{2}} (\sqrt{\frac{2A}{3}} t)$, (green) are on the left side. The corresponding Hubble parameters $H_9(t)$, $H_{10}(11)$ and $H_{11}(t)$ are on the right side.

The above solutions and the corresponding Hubble parameters are depicted in Fig. 5.

At the end of this section, it is worth noting that we have found no cosmological solutions of the form:

\[ a(t) = A t^n e^{\gamma t^m}, \quad k = +1, -1. \]
\[ a(t) = A t^n \cosh(\gamma t^m), \quad k = 0, +1, -1. \]
\[ a(t) = A t^n \sin(\gamma t^m), \quad k = 0, +1, -1. \]
\[ a(t) = A t^n \sinh(\gamma t^m), \quad k = 0, +1, -1. \]
\[ a(t) = A t^n \cos(\gamma t^m), \quad k = 0, +1, -1. \]
\[ a(t) = A t^n \tanh(\gamma t^m), \quad k = 0, +1, -1. \]

where $m$ and $n$ are rational numbers. Also, let us notice that there are cosmological investigation with nonlocal matter fields, see, e.g. [48–50].
5 Concluding Remarks

We want to emphasize once again that this nonlocal de Sitter gravity model (2.1) maintains $R-2\Lambda$ dependence like in the local case through equality $R-2\Lambda = \sqrt{R-2\Lambda} \sqrt{R-2\Lambda}$ and that nonlocalization is introduced replacing $\sqrt{R-2\Lambda} \sqrt{R-2\Lambda}$ by $\sqrt{R-2\Lambda} F(\square) \sqrt{R-2\Lambda}$. Dimensionless operator $F(\square)$ is given by expression (2.2) and in the more explicit form by Eq. (2.26) or (2.27). In this nonlocal de Sitter model, $\Lambda \neq 0$ is a parameter with dimensionality of the cosmological constant and is an effective cosmological constant which may differ for different cosmological solutions. There is not the Minkowski space solution in the general case of this model and it should not make problem in its use to the Solar System, since $\Lambda$ has practically a very small value.

Another remarkable feature of this nonlocal de Sitter model consists in the fact that it contains $a_1(t) = A t^2 e^{\frac{\Lambda_1 t^2}{2}}$ solution which mimics effects related to dark matter and dark energy at the cosmological scale. Results presented in the section 4.1.1 are in good agreement with parameters of the standard model of cosmology. Dark matter emerges in this cosmological vacuum state as an effect of nonlocality in the presence of the corresponding effective cosmological constant $\Lambda_1$. Dark energy is related to the standard cosmological constant $\Lambda$.

Main results presented in this article can be summarized as follows.

- Nonlocal de Sitter gravity model (1.3), introduced in [44] with operator $F(\square) = \sum_{n=1}^{\infty} f_n \Box^n$, is generalized by involving $F(\square) = \sum_{n=1}^{\infty} (f_n \Box^n + f_{-n} \Box^{-n})$. This generalized $F(\square)$ is presented in two explicit forms in (2.27).

- When $\Box \sqrt{R-2\Lambda} = q \sqrt{R-2\Lambda}$ holds in (2.20), then equations of motion have very simple form (2.21) which are satisfied with two conditions (2.22) on nonlocal operator: $F(q) = -1$, $F'(q) = 0$.

- Vacuum solution $a_1(t) = A t^2 e^{\frac{\Lambda_1 t^2}{2}}$ is investigated in details. Effective cosmological constant $\Lambda_1$ is computed using expression (4.12) for the Hubble parameter. Then this $\Lambda_1$ is employed to calculate the effective energy density $\bar{\rho}_1(t_0)$ according formula (4.13). Finally, we obtained $\Omega_{m_1} = \frac{\bar{\rho}_1(t_0)}{\rho_c} = 26.6\%$, which is counterpart of the dark matter density parameter, see (4.20).

- Eight new exact vacuum cosmological solutions were found in Section 3 by detail analysis of two classes of functions: $a(t) = (\alpha \ e^{\lambda t} + \beta \ e^{-\lambda t})^\gamma$ and $a(t) = (\alpha \ \sin \lambda t + \beta \ \cos \lambda t)^\gamma$. These solutions are discussed in Section 4.

- All solutions are illustrated by depicting the scale factors and Hubble parameters in figures. Also, effective energy density and pressure are presented for all vacuum solutions.

We plan to continue investigation of this nonlocal de Sitter gravity model. Among the first tasks will be:

- investigation of cosmological solutions in the presence of matter,
analysis of the solution around spherically symmetric body,
research towards possible inflation aspects,
investigation of the fluctuations,
进一步分析非局域算子 $F(triskelion)$ 从一般理论要求的角度，特别是在可能的幽灵的缺席的情况下。

At the end, it is worth noting that there is also cosmological research in application of nonlocal fields in the matter sector of general relativity, see, e.g. [48–50] and references therin.

A Derivation of the equations of motion

In this section we are going to present some main steps in derivation of the equations of motion for nonlocal gravity model given by action

$$ S = \frac{1}{16\pi G} \int (R - 2\Lambda + P(R) \, \mathcal{F}(\Box) \, Q(R)) \sqrt{-g} \, d^4x, \quad (A.1) $$

where $\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} f_n \, \Box^n + \sum_{n=1}^{+\infty} f_{-n} \, \Box^{-n}$. The special case $P(R) = Q(R) = \sqrt{R - 2\Lambda}$ is nonlocal de Sitter gravity, that is under consideration in the previous sections. For some details in the case $\mathcal{F}(\Box) = \sum_{n=1}^{+\infty} f_n \, \Box^n$ we refer to [43].

To derive the expression for variation $\delta S$ with respect to the variation $\delta g_{\mu\nu}$, let us first recall the variation of $\sqrt{-g}$, the Christoffel symbol $\Gamma_{\mu\nu}^\lambda$, the Ricci tensor $R_{\mu\nu}$ and scalar curvature $R$:

$$ \delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \, \delta g^{\mu\nu}, \quad (A.2) $$

$$ \delta \Gamma_{\mu\nu}^\lambda = -\frac{1}{2} \left( g_{\nu\alpha} \nabla_\mu \delta g^{\lambda\alpha} + g_{\mu\alpha} \nabla_\nu \delta g^{\lambda\alpha} - g_{\mu\alpha} g_{\nu\beta} \nabla^\alpha \delta g^{\beta\lambda} \right), \quad (A.3) $$

$$ \delta R_{\mu\nu} = \nabla_\lambda \delta \Gamma_{\mu\nu}^\lambda - \nabla_\nu \delta \Gamma_{\mu\lambda}^\lambda, \quad (A.4) $$

$$ \delta R = R_{\mu\nu} \delta g^{\mu\nu} - K_{\mu\nu} \delta g^{\mu\nu}, \quad K_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box. \quad (A.5) $$

Moreover, by application of the Stokes’ theorem one obtains that every scalar function $P(R)$ satisfies

$$ \int P K_{\mu\nu} \delta g^{\mu\nu} \, \sqrt{-g} \, d^4x = \int K_{\mu\nu} P \, \delta g^{\mu\nu} \, \sqrt{-g} \, d^4x. \quad (A.6) $$

From equation (A.5) we get

$$ \int P \delta R \sqrt{-g} \, d^4x = \int \left( R_{\mu\nu} P \delta g^{\mu\nu} - P K_{\mu\nu} \delta g^{\mu\nu} \right) \, \sqrt{-g} \, d^4x = \int \left( R_{\mu\nu} P - K_{\mu\nu} P \right) \delta g^{\mu\nu} \, \sqrt{-g} \, d^4x. \quad (A.7) $$

Let us introduce the operator $\delta \Box$ by

$$ (\delta \Box) X = \delta (\Box X) - \Box (\delta X). \quad (A.8) $$
A straightforward calculation proves the following identity for all scalar functions \( P \) and \( Q \).

\[
\int P(\delta \Box)Q\sqrt{-g} \, d^4x = \frac{1}{2} \int S_{\mu\nu}(P,Q) \delta g^{\mu\nu}\sqrt{-g} \, d^4x, 
\]  
(A.9)

where \( S_{\mu\nu}(P,Q) = g_{\mu\nu} \nabla_{\lambda} P \nabla^\lambda Q + g_{\mu\nu} P \Box Q - 2 \nabla_{\mu} P \nabla_{\nu} Q \).

On the other hand from \( \delta (\Box^{-1}) = 0 \), we conclude that the variation of the inverse operator \( \Box^{-1} \) is given by

\[
\delta \Box^{-1} = -\Box^{-1}(\delta \Box)\Box^{-1}. 
\]  
(A.10)

Moreover, operator \( \Box^{-1} \) can be written as a formal integral

\[
\Box^{-1} = \int_{\infty}^{0} e^{-\alpha \Box} d\alpha. 
\]

The integral representation of \( \Box^{-1} \) and repeated application of the Stoke’s theorem yields

\[
\int P \Box^{-1} Q \sqrt{-g} d^4x = \int P \int_{0}^{\infty} \sum_{n=0}^{+\infty} \frac{(-\alpha)^n}{n!} \Box^n Q \sqrt{-g} d^4x \, d\alpha 
\]

\[
= \int \int_{0}^{\infty} \left( \sum_{n=0}^{+\infty} \frac{(-\alpha)^n}{n!} \Box^n P \right) Q \sqrt{-g} d^4x \, d\alpha 
\]

\[
= \int (\Box^{-1} P) Q \sqrt{-g} d^4x. 
\]  
(A.11)

The variation \( \delta S \) can be split into five parts

\[
\delta S = \frac{1}{16\pi G} (I_0 + I_1 + I_2 + I_3 + I_4), 
\]  
(A.12)

where

\[
I_0 = \int \delta (R - 2\Lambda) \sqrt{-g} \, d^4x, 
\]

\[
I_1 = \int \delta P \mathcal{F}(\Box) Q \sqrt{-g} \, d^4x, 
\]

\[
I_2 = \int P \delta (\mathcal{F}(\Box)) Q \sqrt{-g} \, d^4x, 
\]  
(A.13)

\[
I_3 = \int P \mathcal{F}(\Box) \delta Q \sqrt{-g} \, d^4x, 
\]

\[
I_4 = \int P \mathcal{F}(\Box) \delta (\sqrt{-g}) \, d^4x. 
\]

The first term \( I_0 \) is a well known

\[
I_0 = \int G_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \, d^4x. 
\]  
(A.14)

Integral \( I_4 \) is calculated directly and we get

\[
I_4 = -\frac{1}{2} \int g_{\mu\nu} P \mathcal{F}(\Box) Q \delta g^{\mu\nu} \sqrt{-g} \, d^4x. 
\]  
(A.15)

From equation (A.7) integral \( I_1 \) gives,

\[
I_1 = \int P' \mathcal{F}(\Box) Q \delta R \sqrt{-g} \, d^4x 
\]

\[
= \int (R_{\mu\nu} - K_{\mu\nu})(P' \mathcal{F}(\Box) Q) \delta g^{\mu\nu} \sqrt{-g} \, d^4x. 
\]  
(A.16)
Integral $I_3$ after partial integration takes the same form as $I_1$ and therefore

\[
I_3 = \int Q' \mathcal{F}(\Box) P \delta R\sqrt{-g} \, d^4x
= \int (R_{\mu\nu} - K_{\mu\nu})(Q' \mathcal{F}(\Box) P)\delta g^{\mu\nu}\sqrt{-g} \, d^4x. \tag{A.17}
\]

To calculate the last part $I_2$ we first introduce $J_n = \int P \delta(\Box^n) Q \sqrt{-g} \, d^4x$, then

\[
I_2 = \sum_{n=-\infty}^{+\infty} f_n J_n. \tag{A.18}
\]

Since $\Box^0 = \text{Id}$ and $\delta\text{Id} = 0$ we conclude $J_0 = 0$. We have to treat cases $n > 0$ and $n < 0$ separately. Let $n \in \mathbb{N}$ then

\[
J_n = \int P \delta(\Box^n) Q \sqrt{-g} \, d^4x
= \sum_{l=0}^{n-1} \int \Box^l P (\delta\Box)\Box^{n-1-l} Q \sqrt{-g} \, d^4x \tag{A.19}
= \frac{1}{2} \sum_{l=0}^{n-1} \int S_{\mu\nu}(\Box^l P, \Box^{n-1-l} Q)\delta g^{\mu\nu} \sqrt{-g} \, d^4x.
\]

To calculate $J_{-n}$ we do as follows

\[
J_{-n} = \int P \delta(\Box^{-n}) Q \sqrt{-g} \, d^4x
= \sum_{l=0}^{n-1} \int \Box^{-l} P (\delta\Box)\Box^{-(n-1-l)} Q \sqrt{-g} \, d^4x
= -\sum_{l=0}^{n-1} \int \Box^{-(l+1)} P (\delta\Box)\Box^{-(n-l)} Q \sqrt{-g} \, d^4x \tag{A.20}
= -\sum_{l=0}^{n-1} \frac{1}{2} \int S_{\mu\nu}(\Box^{-(l+1)} P, \Box^{-(n-l)} Q)\delta g^{\mu\nu} \sqrt{-g} \, d^4x.
\]

Hence,

\[
I_2 = \frac{1}{2} \int \Omega_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \, d^4x, \tag{A.21}
\]

\[
\Omega_{\mu\nu} = \sum_{n=1}^{+\infty} \sum_{l=0}^{n-1} f_n \int S_{\mu\nu}(\Box^l P, \Box^{n-1-l} Q)
- \sum_{n=1}^{+\infty} \sum_{l=0}^{n-1} f_{-n} \int S_{\mu\nu}(\Box^{-(l+1)} P, \Box^{-(n-l)} Q). \tag{A.22}
\]
Hence, we obtained the variation of the action (A.1) in the following form

\[ \delta S = \frac{1}{16\pi G} \int \hat{G}_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \, d^4x, \]

\[ \hat{G}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} P F(\Box) Q + R_{\mu\nu} W - K_{\mu\nu} W + \frac{1}{2} \Omega_{\mu\nu}, \]

\[ W = 2 P' F(\Box) Q, \]

\[ \Omega_{\mu\nu} = \sum_{n=1}^{+\infty} \sum_{l=0}^{n-1} S_{\mu\nu}(\Box^n P, \Box^{n-1-l} Q) - \sum_{n=1}^{+\infty} \sum_{l=0}^{n-1} S_{\mu\nu}(\Box^{-(l+1)} P, \Box^{-(n-l)} Q). \]

Therefore equations of motion are given by

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} P F(\Box) Q + R_{\mu\nu} W - K_{\mu\nu} W + \frac{1}{2} \Omega_{\mu\nu} = 0. \]  

(A.24)

It is interesting to note that \( \nabla^{\mu} \hat{G}_{\mu\nu} = 0. \)

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