Generalized Reflection Coefficients and Inverse Factorization of Hermetian Block Toeplitz Matrix.

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Abstract

A factorization of the inverse of a Hermetian positive definite matrix based on a diagonal by diagonal recurrence formulae permits the inversion of Block Toeplitz matrices, using only matrix-vector products, and with a complexity of $O(n_1^3n_2^2)$, where $n_1$ is the block size, and $n_2$ is the block matrix block-size.

Introduction.

The techniques developed here are based on a generalization of the reflection coefficients or partial autocorrelation presented in [1], they appear when the well know Levinson recursion for inverting matrix Toeplitz [4][5] is extended to two families of bi orthogonal polynomials.

These generalized coefficients result to be two triangular arrays of numbers, which completely describe the structure of the related matrix. In the Toeplitz case, the coefficients along each diagonal are identical, so they are determined by two sequences of pairs of numbers. In the Toeplitz Hermetian case, the coefficients collapse to the classical reflection coefficients.

In this paper we consider Hermetian positive definite matrices. In this case, the reflection coefficients result to be a triangular array of pairs of numbers whose product is a positive number less than one. The families of related orthogonal polynomials determine two Cholesky factorizations of the inverse of the matrix.

These generated coefficients are computed recursively diagonal by diagonal. Setting some initial conditions on the principal diagonal, at each step of the recursion a whole diagonal is computed from the previous results.

In the Block Toeplitz case, the related two triangular arrays of the generalized coefficients keep the block structure. The levinson algorithm has been generalized to this case and the reflection coefficients results to be matrices which definition involves a square root of a matrix. Several methods for choosing a convenient square root have been developed, see Dégerine[2] and Delsartre [3].

Applying the algorithm obtained for p.d. matrices to the Block Toeplitz case, at each step of the recursion, instead of computing a whole diagonal, only $n_1$ coefficients are computed from the previous results. We find a factorization of the inverse with a complexity $O(n_1^3n_2^2)$, where $n_2$ is the number of blocks in the matrix.

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Notations.

We consider the Hermetian positive definite (covariance) matrix,
\[ R := \{ r_{i,j} \}_{i,j=0,1,\ldots,n-1}, \]  
(1)
\( r_{i,j} \in C, \)
\( n \in \mathbb{N}. \)

For each matrix \( R \) we consider the sub matrices family:
\[ R^{k,l} := \{ b_{i,j} \}_{i,j=0,\ldots,n-k-l}, b_{i,j} := r_{i+k,j+k} \quad k < l, \quad k,l := 0,\ldots,n-1, \]

for each \( R^{k,l} \) matrix we define the following block diagonal matrix :
\[ 0_{k \times k} B_{k,l} R^{k,l} \]
which contains \( R^{k,l} \) diagonally positioned, with zero filled elsewhere.

1 Reflection Coefficients and Cholesky Factorization of inverse for Hermetian Positive definite matrices.

For any Hermetian p.d matrix the inverse can be factorized into the form [1]
\[ R^{-1} = (R^p)^* \left( D^p \right)^{-1} (R^p)^T = (R^q)^* \left( D^q \right)^{-1} (R^q)^T, \]
(2)
where both of \( D^p = (R^p)^T (R) (R^p)^* \), and \( D^q = (R^q)^T (R) (R^q)^* \),
(3)
are diagonal, and \( R^p, R^q \) are lower triangular and upper triangular respectively, both formed from Vectors \( p_{k,l}, q_{k,l} \) \( k,l = 0,1,\ldots,n-1, k \leq l \) according to
\[ R^p = [p_{0,0}, p_{1,1}, \ldots, p_{n-1,n-1}], \]
(4)
\[ R^q = [q_{0,0}, q_{0,1}, \ldots, q_{0,n-1}]. \]

These vectors (or polynomials) are mutually orthogonal relatively to the matrix defined cross product:
\[ \langle v_1, v_2 \rangle := v_1^T R (v_2)^*, \]
this can be expressed in the conditions:
\[ \forall 0 \leq k, 0 \leq k', k < l, k' < l, l \leq n-1, \]
\[ \langle p_{k,l}, p_{k',l} \rangle = 0 \text{ if } k \neq k', \]
\[ \langle p_{k,l}, p_{k,l} \rangle > 0 \text{ if } k = k'. \]

and
The orthogonal polynomials $p_{k,l}$, $q_{k,l}$ are obtained for $R$ in a recursive manner. The positive definiteness of the matrix $R$ permits us to establish a recurrence system in which the famous reflection coefficients are used. The Generalized reflection coefficients define the relation between a group of vectors $p_{k,k+d}$, $q_{k,k+d}$ and the next step group of vectors $p_{k,k+1}$, $q_{k,k+1}$, $d := 0, 1, \ldots, n - 2$, according to

$$p_{k,l} = p_{k,l-1} - a_{k,l} q_{k,l+1}, \quad (5)$$

$$q_{k,l} = q_{k+1,l} - a'_{k,l} p_{k,l-1}, \quad (6)$$

starting from the canonical basis vectors $p_{k,k} := e_k$, $q_{k,k} := e_k$, $k = 0, 1, \ldots, n - 1$.

The Generalized reflection coefficients are complex numbers, obtained in each step according to:

$$a_{k,l} = \frac{p^T_{k,l-1} R e_l}{q^T_{k+1,l} R e_l}, \quad (7)$$

$$a'_{k,l} = \frac{q^T_{k+1,l} R e_k}{p^T_{k,l-1} R e_k}. \quad (8)$$

The definition of $v_{k,l} := q^T_{k,l} R e_l$, $v'_{k,l} := p^T_{k,l} R e_k$ installs the following recurrence system:

$$v_{k,l} = v_{k+1,l} \left(1 - a_{k,l} a'_{k,l}\right) \in \mathbb{R}^+, \quad (9)$$

$$v'_{k,l} = v'_{k+1,l} \left(1 - a_{k,l} a'_{k,l}\right) \in \mathbb{R}^+. \quad (10)$$

This permit us to avoid applying the product in the denominator in (7) and (8) at each step, while for numerators, the following hold:

$$p^*_{k,l-1} R e_l = \left(q^T_{k+1,l} R e_k\right)^*. \quad (11)$$

2 Generalized Reflection Coefficients in Hermitian Block Toeplitz Case.

In this section we will consider the more limiting case of Block Toeplitz matrix in which each block is not necessarily Toeplitz. Each matrix block is of size $n_1 \times n_1$, while the total block matrix is of size $n_1 n_2 \times n_1 n_2$. To help our approach, we notice that the covariance matrix elements $r_{i,j}$ in this context admit the following property.

**Property 1.**

$$r_{i,j} = r_{\text{mod } n_1 \cdot j - \text{sec } n_1} \quad \text{if } i \leq j,$$

$$r_{i,j} = \left(r_{j,i}\right)^* \quad \text{if } i > j.$$  

where \( a \ \text{sec } b := b \ \text{int}(a/b), \ \text{int}(x) \) is the integral part of $x$, and \( a \ \text{mod } b \) is equal to the remainder of division of $a$ by $b$.

**Proof.** From the T-B-T structure we got for $k := 0, 1, \ldots, n_1 - 1, l := 0, 1, \ldots, n_1 n_2 - 1$:...
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\[ r_{k+n_1,l+n_1} = r_{k,j}, \quad t = 0, 1, \ldots, n_2 - 1 - \text{int}(l / n_1), \]

Setting \( i := k + tn_1, j := l + tn_1 \), we notice that \( (i, j) \in \{0, n_1n_2-1\} \times \{0, n_1n_2-1\} \), and:

\[ r_{i,j} = r_{i-n_1,j-n_1}. \]

By the definition of \( i \), \( i - tn_1 = i \mod n_1 \) and \( tn_1 = i \sec n_1 \).

The previous property is to simply, the direct mathematical representation, of the Hermitian Block Toeplitz Matrix case, the following properties; \textit{Property 2} and \textit{Property 3} will help us to consider the main recurrent system in the Block-Toeplitz case, segmented into recurrent sub matrices.

\textbf{Property 2.}

\[
\begin{pmatrix}
  p_{k,j}, a_{k,j}, a_{k,j}', v_{k,j}, v_{k,j}'
\end{pmatrix} = \begin{pmatrix}
  p^R_{k,j}, q^R_{k,j}, a^R_{k,j}, a^{R'}_{k,j}, v^R_{k,j}, v^{R'}_{k,j}
\end{pmatrix},
\]

with condition that \( s \leq k < l \leq d \).

The entities \( p^M_{k,j}, q^M_{k,j}, a^M_{k,j}, a^{M'}_{k,j}, v^M_{k,j}, v^{M'}_{k,j} \) that corresponds to a given covariance matrix \( M \), are noted respectively as \( p^M_{k,j}, q^M_{k,j}, a^M_{k,j}, a^{M'}_{k,j}, v^M_{k,j}, v^{M'}_{k,j} \).

\textbf{Property 3.} \( R^{k,j} = R^k \mod n_1, l-k \sec n_1 \)

\textit{Proof.} For each of both matrices elements we apply \textit{property 1}:

\[ r_{i,j} = r_{(k+i) \mod n_1, (j+i) \mod n_2, i} = 0, 1, \ldots, l - k, \]

\[ r_{i,j}' = r_{k \mod n_1, l - k \sec n_1, i} = 0, 1, \ldots, l - k, \]

from which our proof is completed.

Now we are ready to state the main result of this paper.

\textbf{Theorem 1.} In the case of Hermitian Block Toeplitz Matrix we got:

\[
\begin{align*}
  a_{k,j} &= a_{k \mod n_1, l-k \sec n_1}, \\
  a_{k,j}' &= a_{k \mod n_1, l-k \sec n_1}, \\
  p_{k,j} &= U^{k \sec n_1} p_{k \mod n_1, l-k \sec n_1}, \\
  q_{k,j} &= U^{k \sec n_1} q_{k \mod n_1, l-k \sec n_1}, \\
  v_{k,j} &= V_{k \mod n_1, l-k \sec n_1}, \\
  v_{k,j}' &= V_{k \mod n_1, l-k \sec n_1},
\end{align*}
\]

where \( U \) is the \((n_1n_2 \times n_1n_2)\) shift matrix defined as,

\[
U = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}
\]

with \( U^0 = 1 \).
Proof. We start by noticing that
\[ \forall 0 \leq k < l \leq n_n_2 - 1 \quad \exists s, d : s \leq k < l \leq d , \text{ so that we can write} \]
\[ (p^{a_d}_{k,j}, q^{a_d}_{k,j}, a^{a_d}_{k,j}, v^{a_d}_{k,j}, v^{a_d}_{k,l}, a^{a_d}_{k,l}, v^{a_d}_{k,l}) = \]
\[ (U^d_{d-s} p^{a_d}_{k-s,j-s}, U^d_{d-s} q^{a_d}_{k-s,j-s}, a^{a_d}_{k-s,j-s}, a^{a_d}_{k-j-s}, v^{a_d}_{k-s,j-s}, v^{a_d}_{k-j-s}) , \]

where \( U_s \) is a Matrix of size \((n_n_2) \times (x + 1)\), defined as:
\[
U_s = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & & \\
0 & 1 & & \\
0 & 0 & & \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0
\end{bmatrix},
U_s^0 = \begin{bmatrix}
I & \vdots & \vdots & \vdots \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots
\end{bmatrix}.
\]
coupling this with property 2, we obtain:
\[ (U^d_{d-s} p^{a_d}_{k-s,j-s}, U^d_{d-s} q^{a_d}_{k-s,j-s}, a^{a_d}_{k-s,j-s}, a^{a_d}_{k-j-s}, v^{a_d}_{k-s,j-s}, v^{a_d}_{k-j-s}) = \]
\[ (p^{R_d}_{k,j}, q^{R_d}_{k,j}, a^{R_d}_{k,j}, v^{R_d}_{k,j}, v^{R_d}_{k,j}) \].

By setting \( s = k \), \( \sec_n_1 \), admitting that \( k - \sec_n_1 = k \mod n_1 \), and using property 3, we obtain:
\[ (U^d_{d-s} p^{a_d}_{k-s,j-s}, U^d_{d-s} q^{a_d}_{k-s,j-s}, a^{a_d}_{k-s,j-s}, a^{a_d}_{k-j-s}, v^{a_d}_{k-s,j-s}, v^{a_d}_{k-j-s}) = \]
\[ (p^{R_d}_{k,j}, q^{R_d}_{k,j}, a^{R_d}_{k,j}, v^{R_d}_{k,j}, v^{R_d}_{k,j}) \],

Finally, our proof is completed by noticing that:
\[ (U^d_{d-s} p^{a_d}_{k-s,j-s}, U^d_{d-s} q^{a_d}_{k-s,j-s}, a^{a_d}_{k-s,j-s}, a^{a_d}_{k-j-s}, v^{a_d}_{k-s,j-s}, v^{a_d}_{k-j-s}) = \]
\[ (p^{R_d}_{k,j}, q^{R_d}_{k,j}, a^{R_d}_{k,j}, v^{R_d}_{k,j}, v^{R_d}_{k,j}) \].

We are also interested in the algorithmic aspect of theorem 1; more precisely the following theorem explains the resulting algorithmic details and calculus coast.

**Theorem 2.** In Hermetian Block Toeplitz case, the application of Recurrence relations (5)-(10) resume in the following Algorithm, with complexity \( O(n^3 n_2^2) \):

First we rewrite Equations (5)-(10) as the subroutine,
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Subroutine 1

\[ a_{k,l} = \frac{p_{k,l} R e_1}{v} \]
\[ a'_{k,l} = \frac{q'_{k,l} R e_1}{v'} \]
\[ p_{k,l} = p_{k,l} - a_{k,l} \hat{q} \]
\[ q_{k,l} = \hat{q} - a_{k,l} p_{k,l} \]
\[ v_{k,l} = v (1-a_{k,l} a'_{k,l}) \]
\[ v'_{k,l} = v' (1-a_{k,l} a'_{k,l}) \]

and proceed as the following.

Initialization:
For \( k = 0 \) to \( n_1 - 1 \):
\[ p_{k,k} = q_{k,k} = e_k \]
\[ v_{k,k} = v'_{k,k} = r_{0,0} \]

Main routine:
For \( d = 0 \) to \( n_2 - 1 \) do (loop 1)
\{ If \( d \) not equal to zero:
\{ For \( d = 0 \) to \( n_1 - d - 1 \) do (Lower triangle loop)
\{ For \( u = 0 \) to \( n_1 - d - 1 \) do:
\{ \( k = u + d \)
\( l = d n_1 + u \)
\( (k^0, l^-) = (k, l - 1) \)
if \( u \) equal to \( n_1 - d - 1 \):
\( (k^+, l^0) = ((k + 1) \mod n_1, l - (k + 1) \sec n_1) \)
\( \hat{q} = U w q_{k^+, l^0}, v = v_{k^+, l^0} \)
\} else
\( (k^+, l^0) = (k + 1, l) \)
\( \hat{q} = q_{k^+, l^0}, v = v_{k^+, l^0} \)
\} Apply Subroutine 1
\}
\}
\}
For \( d = 1 \) to \( n_1 - 1 \) do (Upper triangle loop)
\{ For \( u = 0 \) to \( n_1 - d - 1 \) do:
\{ \( k = u \)
\( l = d n_1 + u + d \)
\( (k^0, l^-) = (k, l - 1) \)
\}
\[(k^+, l^0) = (k + 1, l)\]
\[\hat{q} = q_{k^+, l^0}, \nu = v_{k^+, l^0}\]

Apply Subroutine 1

Proof. For any element \((i, j) \in T_d\),
\[T_d = \{(k, l) : k < l, (k, l) \in \left[0, n_d - 1\right] \times \left[0, n_d - 1\right], d = 1, 2, \ldots, n_2;\]
\[(i \mod d, j - i \sec d) \in P_d,\]
\[P_d = \{(k, l) : k < l, (k, l) \in \left[0, n_i - 1\right] \times \left[0, n_d - 1\right]\}\]
From Theorem 1, by obtaining \(p_{k,j}, q_{k,j}, a_{k,j}, a'_{k,j}, v_{k,j}, v'_{k,j}\) for all subscripts \((k, l) \in P_d\) we obtain directly these values for \((k, l) \in T_d\).
The algorithm proceeds in calculating values for all subscripts contained in the following groups, with respective order:
\[\Lambda_1, \Lambda_2, \ldots, \Lambda_{n_2}, \text{ where } \Lambda_d := P_d \setminus P_{d-1}, \text{ and } P_0 := \emptyset.\]

In Lower triangle loop we obtain values for
\[\forall (k, l) \in \Lambda_d^-, \text{ where } \Lambda_d^- := \{(k, l) \in \Lambda_d : k \geq l \mod n_i\}.\]

To demonstrate this, first we notice that for any group defined as
\[\Lambda_{d^-}^{d_1} := \{(k, l) : (k, l) \in \Lambda_d^-, k \mod n_i = d_1\},\]
\[\Lambda_d^- = \bigcup_{d_1=0}^{n_i-1} \Lambda_{d^-}^{d_1},\]
we got, for \(d_1 \in [0, n_i - 1]\):
\[\Lambda_{d^-}^{d_1} = \{(k, l) : l \mod n_i \in [0, n_i - 1 - d_1], k = l \mod n_i + d_1\}.

By defining \(u := l \mod n_2\) we conclude that we did consider all \((k, l) \in \Lambda_d^-\).
Using the same logic we conclude that in Upper triangle loop we obtain values for
\[\forall (k, l) \in \Lambda_d^+, \text{ where } \Lambda_d^+ := \{(k, l) \in \Lambda_d : k < l \mod n_i\}.\]
It is clear that \(\Lambda_d = \Lambda_d^- \cup \Lambda_d^+\).

In the upper triangle loop it is easy to verify that:
\[\forall (k, l) \in \Lambda_d^+, \{(k, l-1), (k + 1, l)\} \subseteq \Lambda_d,\]
while in the Lower triangle loop this is not always true, more precisely for \(k = n_i - 1,\)
or/and \(l \mod n_i = 0\). So we can write:
\[\forall (k, l) \in \Lambda_d^-, k \neq n_i - 1 \Rightarrow \{(k, l-1), (k + 1, l)\} \subseteq P_d.\]
Finally for the case of \(k = n_i - 1,\) we apply theorem 1 to obtain the needed values,
since: \(\forall (k, l) \in \Lambda_d^-, \{(k, l-1), ((k + 1) \mod n_i, l \sec n_i)\} \subseteq P_d.\)

For computing the complexity, we proceed as the following:
Each Entry in the routine will require operations of \(O(l - k)\), by noting \(c_i\) as the step constant, \(opc\) as the total number of operation, the calculus cost take the form of:
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\[ \text{opc} = \sum_{d_1=1}^{n-1} \sum_{d_2=1}^{n-1} c_1 (n_d^2 + d_1) + \sum_{d_1=1}^{n-1} \sum_{d_2=1}^{n-1} c_1 (n_d^2 - d_1), \]
\[ = \sum_{d_1=1}^{n-1} (n_1 - d_1 - 1) c_1 (d_1) + \sum_{d_2=1}^{n-1} (n_1 - d_1 - 1) c_1 (n_d^2 + d_1) + \sum_{d_1=1}^{n-1} (n_1 - d_1 - 1) c_1 (n_d^2 - d_1), \]
\[ = \sum_{d_1=1}^{n-1} (n_1 - d_1 - 1) c_1 (d_1) + \sum_{d_2=1}^{n-1} \sum_{d_1=1}^{n-1} [(n_1 - d_1 - 1) c_1 (n_d^2 + d_1) + (n_1 - d_1 - 1) c_1 (n_d^2 - d_1) + \]
\[ \sum_{d_1=1}^{n-1} (n_1 - 1) c_1 (n_d^2) \]
\[ = \sum_{d_1=1}^{n-1} (n_1 - d_1 - 1) c_1 (d_1) + \sum_{d_2=1}^{n-1} 2 c_1 (n_d^2) \sum_{d_1=1}^{n-1} (n_1 - d_1 - 1) + \sum_{d_1=1}^{n-1} (n_1 - 1) c_1 (n_d^2), \]
\[ = (n_1 - 1) c_1 \sum_{d_1=1}^{n-1} d_1 - c_1 \sum_{d_2=1}^{n-1} (d_1)^2 + \sum_{d_2=1}^{n-1} 2 c_1 (n_d^2) (n_1 - 1)^2 - \sum_{d_2=1}^{n-1} 2 c_1 (n_d^2) \sum_{d_1=1}^{n-1} d_1 + \]
\[ (n_1 - 1) c_1 n_1 \sum_{d_1=1}^{n-1} d_1, \]
\[ = (n_1 - 1) c_1 \frac{1}{2} (n_1 - 1) n_1 - c_1 \frac{1}{6} (n_1 - 1) (n_1) (2n_1 - 1) + \]
\[ 2 c_1 n_1 \frac{1}{2} (n_2 - 1) n_1 (n_1 - 1)^2 - 2 c_1 n_1 \frac{1}{2} (n_2 - 1) n_2 \frac{1}{2} (n_1 - 1) n_1 + \]
\[ (n_1 - 1) c_1 n_1 \frac{1}{2} (n_2 - 1) n_2. \]

From which we conclude the complexity order of \( n_1^3 n_2^2 \).

**Bezoutian Inversion Formulae.**

One can directly develop a bezoutian factorization of the inverse, depending on \( (2) \). Instead of that, and for the benefit of the reader, we will express our new formulae depending on the already existing work of [8].

In [9], the inverse of a Hermetian Block Toeplitz matrix can be expressed in the form of:

\[ R^{-1} = \left( L_A \right)^H D \left( P_j^{-1} \right) L_A - \left( L_B \right)^H D \left( P_p^{-1} \right) L_B, \]  \( \text{eqn. 13} \)

where both of \( L_A \) , and \( L_B \) are block Toeplitz in the form:

\[ L_A = \begin{bmatrix} I & A_1 & \cdots & A_{n_e-1} \\ 0 & I & \cdots & \vdots \\ \vdots & \ddots & \ddots & A_i \\ 0 & \cdots & 0 & I \end{bmatrix}, \]
\[ L_B = \begin{bmatrix} 0 & B_{n_e-1} & \cdots & B_1 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & B_{n_e-1} \\ 0 & \cdots & 0 & 0 \end{bmatrix}. \] \( \text{eqn. 14} \)

While the entities \( \{ A_i \}, \{ B_i \}, P_j, P_p \) , are defined as the unique solution to the equations [6],[7]:

\[ \begin{bmatrix} I & A_1 & \cdots & A_{n_e-1} \end{bmatrix} R = [P_j \ 0 \ \cdots], \]
\[ \begin{bmatrix} B_{n_e-1} & \cdots & B_1 \end{bmatrix} R = [0 \ \cdots \ P_p], \]
\[ \text{eqn. 15, 16} \]
\( D(M) \) is a block diagonal matrix containing \( M \) on its main block diagonal with zeros elsewhere.

The next theorem will help establish the correspondence between the WWR algorithm, and the Generalized Reflection Coefficients algorithm.

**Theorem.** In the case of strongly regular Hermetian Block Toeplitz matrix, the inverse is equal to

\[
R^{-1} = (L_p)^H \, D \left( V^{-1} \right) L_p - (L_q)^H \, D \left( V^{-1} \right) L_q, \quad (13)
\]

where

\[
L_p = \begin{bmatrix} P_0^T & P_1^T & \ldots & P_{n_2-1}^T \\ 0 & P_0^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & P_0^T \end{bmatrix}, \quad L_q = \begin{bmatrix} 0 & Q_{n_2-1}^T & \cdots & Q_1^T \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (14)
\]

While the entities \( \{P_i\}, \{Q_i\}, V', V \) are defined as:

\[
\begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_{n_2-1} \end{bmatrix} := \begin{bmatrix} p_{0,n_2-1} & p_{1,n_2-1} & \cdots & p_{n_2-1,n_2-1} \end{bmatrix}, \quad \begin{bmatrix} Q_{n_2-1} \\ \vdots \\ Q_1 \\ Q_0 \end{bmatrix} := \begin{bmatrix} q_{0,n_2-1} & q_{0,n_2-2} & \cdots & q_{0,n_2-1} \end{bmatrix},
\]

\[
V' := \begin{bmatrix} v_{0,n_2-1} & 0 & \cdots & 0 \\ 0 & v_{1,n_2-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & v_{n_2-1,n_2-1} \end{bmatrix}, \quad V := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \end{bmatrix}.
\]

**Proof.** It is clear that both of \( P_0 \), and \( Q_0 \) are lower, and upper triangular respectively. From (2), and (3) we can write that:

\[
R \cdot V^{-1} (P_0)^T = I, \quad R \cdot V^{-1} (Q_0)^T = 0.
\]

From which, since \( R \) is Hermetian, we can write that:

\[
(P_0)^T V^{-1} = [I 0 \cdots 0], \quad (Q_0)^T V^{-1} Q_0^T = [0 \cdots 0 0].
\]

This leads us to the equalities of

\[
(P_0)^T V^{-1} P_0^T = (P_f)^{-1}, \quad (Q_0)^T V^{-1} Q_0^T = (P_0)^{-1},
\]

and eventually to

\[
A_i = (P_0)^{-1} P_i^T, \quad B_i = (Q_0)^{-1} Q_i^T.
\]
since \( P_f = \left( P_0^T \right)^{-1} V \left( P_0^* \right)^{-1} \), and \( P_b = \left( Q_0^T \right)^{-1} V \left( Q_0^* \right)^{-1} \).

By replacing (14)-(17) into (13) , (14) we obtain the proof of (13).

**Conclusion.**

The previous presented results, explain clearly the structure of the Generalized Reflection coefficients, and their relevant orthogonal polynomials in the Hermetian Block-Toeplitz Case, while the reflection coefficients admit the same Block Toeplitz status, the polynomials \( p_{k,j}, q_{k,j} \), show a block Toeplitz recurrence with an added shift between Blocks polynomials counterparts.

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