DE RHAM DECOMPOSITION FOR RIEMANNIAN MANIFOLDS WITH BOUNDARY*

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Abstract In this paper, we extend the classical de Rham decomposition theorem to the case of Riemannian manifolds with boundary by using the trick of the development of curves.

Key words de Rham decomposition; parallel distribution; development of curve

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1 Introduction

Let \((M^n, g)\) be a simply connected complete Riemannian manifold with two nontrivial parallel distributions, \(T_1\) and \(T_2\), that are orthogonal complements of each other. Then, \(M\) is isometric to a Riemannian product \(M_1 \times M_2\) with \(M_i\) being a maximal integral submanifold of \(T_i\) for \(i = 1, 2\). This is a classical result in differential geometry obtained by de Rham [5] in 1952. In 1962, Wu [16] extended the result to simply connected complete semi-Riemannian manifolds. The strategy of de Rham’s proof is to patch up local product decompositions to a global one. This strategy was taken up and presented in a modern form by Maltz [12] using an idea for patching up local isometries that was introduced by O’Neil [13]. Wu’s strategy of proof is different. He used the theorem of Cartan-Ambrose-Hicks to construct a global isometry from \(M_1 \times M_2\) to \(M\). In fact, Maltz [12] extended de Rham’s decomposition theorem to complete affine manifolds. The de Rham decomposition theorem was also extended to non-simply connected manifolds by Eschenburg-Heintz [6] and to geodesic spaces by Foertsch-Lytchak [7]. The uniqueness of Wu’s de Rham decomposition for indefinite metrics was shown just recently by Chen [4].

In this paper, we extend de Rham’s decomposition theorem to complete Riemannian manifolds with boundary. Here, by the completeness of a Riemannian manifold with boundary, we mean metric completeness. Note that if a Riemannian manifold with boundary is decomposable, then it must be decomposed as a product of a Riemannian manifold with boundary and a Riemannian manifold without boundary, because of the smoothness of the boundary. This implies that the normal vectors on the boundary must all be contained in one of the two parallel distributions. Our result confirms the converse of the above observation.

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Theorem 1.1 Let $(M^n, g)$ be a simply connected complete Riemannian manifold with boundary. Let $T_1$ and $T_2$ be two nontrivial parallel distributions that are orthogonal complements of each other. Suppose that $T_1$ contains the normal vectors on $\partial M$. Let $p$ be an interior point of $M$ and let $\iota_i : (M_i, p_i) \rightarrow (M, p)$ be the simply connected leaf of the foliation $T_i$ passing through $p$ for $i = 1, 2$. Then, $M_1$ is a manifold with boundary and $M_2$ is a manifold without boundary; moreover, there is an isometry $f : M_1 \times M_2 \rightarrow M$ such that $f(p_1, p_2) = p$ and $f^* (p_1, p_2) = \iota_1^* p_1 + \iota_2^* p_2$.

Here, a leaf of a foliation is a maximal integrable submanifold of the foliation (see [15]), and for a simply connected leaf we mean the universal cover of the maximal integrable submanifold for the foliation. We would like to mention that the assumption on the simply connectedness of $M$ cannot be removed. For example, let $M = [0, 1] \times \mathbb{R}^2 / \mathbb{Z}^2$, equipped with the standard product metric, and let

\[ T_1 = \text{span} \left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial x} + r \frac{\partial}{\partial y} \right\} \]  \hspace{1cm} (1.1)

and

\[ T_2 = \text{span} \left\{ -r \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right\} \]  \hspace{1cm} (1.2)

with $r$ being an irrational number, where $t$ is the natural coordinate on $[0, 1]$ and $(x, y)$ is the natural coordinate on $\mathbb{R}^2$. Then, $T_1$ and $T_2$ are parallel distributions on $M$ that are orthogonal complements of each other with $T_1$ containing the normal vectors. However, we cannot have a decomposition of $M$ according the distributions $T_1$ and $T_2$ because $r$ is an irrational number.

Because Wu’s proof used geodesics to connect two different points and Maltz’s proof relied heavily on convex normal neighborhoods, their proofs will not work for Riemannian manifolds with boundary without any convexity assumption on the boundary. We will prove the result by combining the idea of Kobayashi-Nomizu [11, P.187], using development of curves, and the idea of Wu, using the Cartan-Ambrose-Hicks theorem.

Let us recall the notion of developments of curves in [11, P.130]. The original definition in [11] was given in the language of connections for principal fibre bundles. We will present here an equivalent notion in a more elementary form.

Definition 1.2 Let $(M^n, g)$ be a Riemannian manifold and let $v : [0, T] \rightarrow T_p M$ be a curve in $T_p M$. A curve $\gamma : [0, T] \rightarrow M$ such that

\[ \gamma(0) = p \quad \text{and} \quad \gamma'(t) = P^v_0(\gamma)(v(t)) \quad \text{for any} \quad t \in [0, T] \]

is called a development of the curve $v$. Here $P^v_{t_1}(\gamma) : T_{\gamma(t_1)} M \rightarrow T_{\gamma(t_2)} M$ means the parallel displacement along $\gamma$ from $\gamma(t_1)$ to $\gamma(t_2)$.

Note that when $v$ is constant, the development of $v$ is just a geodesic, and when $v$ is piecewise constant, the development of $v$ is just a broken geodesic. It can be shown that when $v$ is smooth, the development of $v$ is unique if it exists, and when the Riemannian manifold is complete, the development of $v$ exists for any $v$.

It is clear that a local isometry of Riemannian manifolds will preserve curvature tensors. It was Cartan [3] who first gave a converse of this fact in local settings. This result is now called Cartan’s isometry theorem (see [9, Theorem 1.12.8]). The conclusion was extended to a global setting by Ambrose [1] under the assumptions of simply connectedness and that curvature tensors are preserved by parallel displacements along broken geodesics. Finally, Hicks [8] extended
the conclusion to complete affine manifolds. A more general form of the Cartan-Ambrose-Hicks theorem can be found in [2]. In [13], O’Neil gave an alternative proof of Ambrose’s result.

In this paper, to implement Wu’s idea of proving de Rham decomposition by using the Cartan-Ambrose-Hicks theorem, we need the following version of Cartan-Ambrose-Hicks theorem:

**Theorem 1.3** Let \((M^n, g)\) and \((\tilde{M}^n, \tilde{g})\) be two Riemannian manifolds (where are not necessarily complete and may have boundaries). Let \(p\) be an interior point of \(M\), \(\tilde{p} \in \tilde{M}\) and let \(\varphi : T_p M \to T_{\tilde{p}} \tilde{M}\) be a linear isometry. Suppose that \(M\) is simply connected and that for any smooth interior curve \(\gamma : [0,1] \to M\) with \(\gamma(0) = p\), the development \(\tilde{\gamma}\) of \(\varphi(\gamma')\) exists in \(\tilde{M}\). Here \(v_\gamma(t) = \nabla^0_1(\gamma)(\gamma'(t))\) for \(t \in [0,1]\). Moreover, suppose that \(\tau_\gamma^* R_M = R_M\) for any smooth interior curve \(\gamma : [0,1] \to M\) with \(\gamma(0) = p\), where

\[
\tau_\gamma = P^0_0(\tilde{\gamma}) \circ \varphi \circ P^0_1(\gamma) : T_{\gamma(1)} M \to T_{\tilde{\gamma}(1)} \tilde{M}.
\]

Here a curve \(\gamma : [0,1] \to M\) is said be an interior curve if \(\gamma(t)\) is in the interior of \(M\) for any \(t \in [0,1]\), and \(R_M\) and \(R_{\tilde{M}}\) are the curvature tensors of \(M\) and \(\tilde{M}\), respectively. Then, the map \(f(\gamma(1)) = \tilde{\gamma}(1)\) from \(M\) to \(\tilde{M}\) is well defined and \(f\) is the local isometry from \(M\) to \(\tilde{M}\) with \(f(p) = \tilde{p}\) and \(f_* p = \varphi\).

Our proof of Theorem 1.3 is similar to the proof of Cartan’s isometry theorem using the Jacobi field equation (see [9, Theorem 1.12.8]). Because we are considering variations for the development of curves, we need the equations of the variation fields for variations of the development of curves that may be considered as a generalization of the equation for Jacobi fields. Here we require the curve \(\gamma\) to be interior in Theorem 1.3 (because of a technical reason for its application in proving Theorem 1.1). One can see from the proof of Theorem 1.3 that the conclusion of Theorem 1.3 is still true if the assumption that \(\gamma\) is interior is removed.

We would like to mention that by using the trick developed in this paper, we are able to obtain a decomposition result in [14] when replacing the assumption of the simply connectedness of the manifold by the simply connectedness of one of the factors. We have also used this trick to extend the fundamental theorem for submanifolds to general ambient spaces in [17]. Note that the product of two manifolds with boundary is not a manifold with boundary: it is a manifold with corners (see [10]). Thus, it is an interesting challenge to extend the de Rham decomposition theorem to Riemannian manifolds with corners. The argument in this paper may help to solve this problem. However, because our argument relies heavily on the smoothness of the boundary (see the proof of Lemma 3.2), our proof does not work for the case of Riemannian manifolds with corners.

The rest of the paper is organized as follows: in Section 2, we prove the local existence and uniqueness for developments, and prove Theorem 1.3. In Section 3, we prove Theorem 1.1.
Let $v : [0, T] \rightarrow T_pM$ be a smooth curve in $T_pM$. Then, there is a positive number $\epsilon$ and a unique smooth curve $\gamma : [0, \epsilon] \rightarrow M$ such that $\gamma(0) = p$ and

$$\gamma'(t) = P^t_0(\gamma)(v(t)) \quad (2.1)$$

for $t \in [0, \epsilon]$.

**Proof** We only need to derive the equation of $\gamma$. The conclusion will follow directly by the existence and uniqueness of the solution for Cauchy problems of ordinary differential equations. In the next computation, we adopt Einstein’s summation convention for repeated indices.

Let $(x^1, x^2, \ldots, x^n)$ be a local coordinate at $p$ with $x^i(p) = 0$ for $i = 1, 2, \ldots, n$. Suppose that $v(t) = v^j(t) \frac{\partial}{\partial x^j} \bigg|_p$. (2.2)

Let $\gamma(t) = (x^1(t), x^2(t), \ldots, x^n(t))$ be a development of $v$ and let $E_i$ be the parallel extension of $\frac{\partial}{\partial x^i} \big|_p$ along $\gamma$. Suppose that $E_i(t) = x^j_i(t) \frac{\partial}{\partial x^j}$. (2.3)

Then,

$$\frac{dx^j_i}{dt} + x^k_i \frac{dx^l_k}{dt} \Gamma^j_{kl}(x^1, x^2, \ldots, x^n) = 0 \quad (2.4)$$

for any $i, j = 1, 2, \ldots, n$, since $\nabla_v E_i = 0$.

Moreover, note that $P^t_0(\gamma)(v(t)) = v^j(t) E_i = v^j x^j_i \frac{\partial}{\partial x^j}$, (2.5)

so

$$\frac{dx^j_i}{dt} = v^j x^j_i \quad (2.6)$$

for $i = 1, 2, \ldots, n$, by which $\gamma'(t) = P^t_0(\gamma)(v(t))$.

In summary, by substituting (2.6) into (2.4), we know that the curve $\gamma$ must satisfy the following ODEs:

$$\begin{cases}
\frac{dx^i}{dt} = v^j x^j_i & \text{for } i = 1, 2, \ldots, n, \\
\frac{dx^j_i}{dt} + v^m x^j_i x^m_k \Gamma^j_{kl}(x^1, x^2, \ldots, x^n) = 0 & \text{for } i, j = 1, 2, \ldots, n, \\
x^i(0) = \delta^i_j & \text{for } i, j = 1, 2, \ldots, n, \\
x^j_i(0) = 0 & \text{for } i = 1, 2, \ldots, n.
\end{cases} \quad (2.7)$$

By the standard theory of ODEs, the equation has a unique solution for a short time. This completes the proof of the lemma. □

By combining the local uniqueness and a standard trick in extending solutions for ODEs, one has the following global existence and uniqueness of the development of curves for complete Riemannian manifolds without boundary (one can find the proof in [11, P. 175]):

**Theorem 2.2** Let $(M^n, g)$ be a complete Riemannian manifold without boundary. Then, each smooth curve $v : [0, T] \rightarrow T_pM$ has a unique development $\gamma : [0, T] \rightarrow M$. 

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We will denote the development of $v$ as \( \text{dev}(p, v) \). When $v$ is constant, it is clear that
\[
\text{dev}(p, v)(t) = \exp_p(tv).
\] (2.8)

Moreover, it is clear that
\[
\text{dev}(p, v)(t) = \text{dev}(\text{dev}(p, v)(t_0), P_{t_0}(\text{dev}(p, v))v_{t_0})(t - t_0)
\] (2.9)
for any $t_0 \in [0, T]$ and $t \in [t_0, T]$. Here $v_{t_0}(t) = v(t_0 + t)$ for $t \in [0, T - t_0]$. For simplicity, we will denote a vector and its parallel displacement by the same symbol when it causes no confusion. Under this convention, the identity (2.9) can simply be written as
\[
\text{dev}(p, v)(t) = \text{dev}(\text{dev}(p, v)(t_0), v_{t_0})(t - t_0).
\] (2.10)

Next, we derive the equation for the variation field of a variation for the development of curves, which can be viewed as a generalization of the Jacobi field equation:

**Lemma 2.3** Let \((M^n, g)\) be a Riemannian manifold and $p \in M$. Let $v(u, t) : [0, 1] \times [0, 1] \to T_p M$ be a smooth map and let
\[
\Phi(u, t) = \text{dev}(p, v(u, \cdot))(t).
\]

Let $e_1, e_2, \cdots, e_n$ be an orthonormal basis of $T_p M$ and let $E_i(u, t)$ be the parallel translation of $e_i$ along $\Phi(u, \cdot)$ for $i = 1, 2, \cdots, n$. Suppose that
\[
v(u, t) = \sum_{i=1}^n v_i(u, t)e_i
\] (2.11)
and
\[
\frac{\partial}{\partial u} := \frac{\partial \Phi}{\partial u} = \sum_{i=1}^n U_i E_i.
\] (2.12)

Moreover, suppose that
\[
\nabla \frac{\partial}{\partial t} E_i(u, t) = \sum_{j=1}^n X_{ij} E_j.
\] (2.13)

Then,
\[
\begin{align*}
U''_i &= \sum_{j,k,l=1}^n v_k v_l R(E_k, E_i, E_l, E_j) U_j + \partial_u \partial_t v_i + \sum_{j=1}^n \partial_t v_j X_{ji} \quad i = 1, 2, \cdots, n, \\
X'_{ij} &= \sum_{k,l=1}^n v_l R(E_i, E_j, E_l, E_k) U_k \quad i, j = 1, 2, \cdots, n, \\
X_{ij}(u, 0) &= 0 \quad i, j = 1, 2, \cdots, n, \\
U_i(u, 0) &= 0 \quad i = 1, 2, \cdots, n, \\
U'_i(u, 0) &= \partial_u v_i(u, 0) \quad i = 1, 2, \cdots, n.
\end{align*}
\] (2.14)

Here the symbol $'$ means taking the derivative with respect to $t$.

**Proof** Note that
\[
\frac{\partial}{\partial t} := \frac{\partial \Phi}{\partial t} = \sum_{i=1}^n v_i E_i
\] (2.15)
and
\[
\nabla \frac{\partial}{\partial t} = \sum_{i=1}^n \partial_t v_i(u, t) E_i(u, t).
\] (2.16)
Then,
\[
\nabla_x \nabla_t \frac{\partial}{\partial t} = \sum_{i=1}^{n} \partial_i \partial_t v_i(u, t) E_i(u, t) + \sum_{i=1}^{n} \partial_i v_i(u, t) \nabla_x \frac{\partial}{\partial t} E_i(u, t)
\]
\[
= \sum_{i=1}^{n} \left( \partial_i \partial_t v_i(u, t) + \sum_{j=1}^{n} \partial_t v_j X_{ji} \right) E_i(u, t).
\] (2.17)

Thus,
\[
\sum_{i=1}^{n} U_i'' E_i = \nabla_x \nabla_t \frac{\partial}{\partial u} = \nabla_x \nabla_t \frac{\partial}{\partial \tau} = \nabla_x \nabla_t \frac{\partial}{\partial t} + R \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial \tau} \right) \frac{\partial}{\partial \tau} E_i
\]
\[
= \sum_{i=1}^{n} \left( \sum_{j,k,l=1}^{n} v_{kl} R(E_k, E_i, E_l, E_j) + \partial_u \partial_t v_i + \sum_{j=1}^{n} \partial_t v_j X_{ji} \right) E_i.
\] (2.18)

This gives us the first equation of (2.14). Moreover,
\[
\sum_{j=1}^{n} X'_{ij} E_j = \nabla_x \nabla_t \frac{\partial}{\partial \tau} E_i(u, t) = \nabla_x \nabla_t \frac{\partial}{\partial \tau} E_i(u, t) + R \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \tau} \right) E_i
\]
\[
= \sum_{j=1}^{n} \left( \sum_{k,l=1}^{n} v_{k,l} R(E_i, E_j, E_k, E_l) U_k \right) E_j.
\] (2.19)

This gives us the second equation in (2.14). Finally, note that
\[
\left. \frac{\partial}{\partial u} \right|_{t=0} = 0, \quad \left. \nabla_x \frac{\partial}{\partial \tau} E_i \right|_{t=0} = 0
\]
and
\[
\left. \nabla_x \frac{\partial}{\partial u} \right|_{t=0} = \lim_{t \to 0^+} \nabla_x \frac{\partial}{\partial t} = \sum_{i=1}^{n} \lim_{t \to 0^+} \nabla_x \frac{\partial}{\partial \tau} (v_i E_i) = \sum_{i=1}^{n} \partial_0 v_i(u, 0) e_i.
\] (2.20)

Therefore, \(X_{ij}(u, 0) = 0\), \(U_i(u, 0) = 0\) and \(U'_i(u, 0) = \partial_0 v_i(u, 0)\) for \(i, j = 1, 2, \ldots, n\). This completes the proof of the lemma. \(\square\)

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3** For \(x \in M\), let \(\gamma_0, \gamma_1 : [0, 1] \to M\) be two interior smooth curves joining \(p\) to \(x\). Since \(M\) is simply connected, there is a smooth map \(\Phi : [0, 1] \times [0, 1] \to M\) such that
\[
\left\{
\begin{array}{ll}
\Phi(0, t) = \gamma_0(t) & \text{for } t \in [0, 1], \\
\Phi(1, t) = \gamma_1(t) & \text{for } t \in [0, 1], \\
\Phi(u, 0) = p & \text{for } u \in [0, 1], \\
\Phi(u, 1) = x & \text{for } u \in [0, 1],
\end{array}
\right.
\]
(2.21)

and \(\gamma_u(t) = \Phi(u, t)\) is an interior curve for any \(u \in [0, 1]\). Let
\[
v(u, t) = P^u(t)(\gamma_u(t)).
\] (2.22)

Then \(v_u\) is the development of \(v(u, \cdot)\). Let \(e_1, e_2, \ldots, e_n\) be an orthonormal basis of \(T_p M\) and let \(E_i(u, t)\) be the parallel extension of \(e_i\) along \(\gamma_u\). Suppose that
\[
v(u, t) = \sum_{i=1}^{n} v_i(u, t) e_i
\]
and
\[
\frac{\partial \Phi}{\partial u} = \sum_{i=1}^{n} U_i E_i. \tag{2.23}
\]
Let \( \tilde{e}_i = \varphi(e_i) \) for \( i = 1, 2, \cdots, n \). Then
\[
\varphi(v(u, t)) = \sum_{i=1}^{n} v_i(u, t) \tilde{e}_i. \tag{2.24}
\]
Let \( \tilde{\Phi}(u, t) = \text{dev}(\tilde{p}, \varphi(v(u, \cdot)))(t) \) and \( \tilde{E}_i \) be the parallel extension of \( \tilde{e}_i \) along \( \tilde{\Phi}(u, \cdot) \). Suppose that
\[
\frac{\partial \tilde{\Phi}}{\partial u} = \sum_{i=1}^{n} \tilde{U}_i \tilde{E}_i. \tag{2.25}
\]
Note that
\[
R_M(E_i, E_j, E_k, E_l) = R_{\tilde{M}}(\tilde{E}_i, \tilde{E}_j, \tilde{E}_k, \tilde{E}_l),
\]
by assumption. Thus, by Lemma 2.3, \( \tilde{U}_i \)'s and \( \tilde{U}_i \)'s satisfy the same Cauchy problem of ODEs. By the uniqueness of the solution for Cauchy problems, we know that
\[
\tilde{U}_i = U_i \tag{2.26}
\]
for \( i = 1, 2, \cdots, n \). In particular, \( \tilde{U}_i(u, 1) = U_i(u, 1) = 0 \) for \( i = 1, 2, \cdots, n \), so \( \tilde{\Phi}(0, 1) = \tilde{\Phi}(1, 1) \).
This implies that \( f \) is well defined.
Moreover, note that \( f_\ast (\frac{\partial \Phi}{\partial u}) = \frac{\partial \tilde{\Phi}}{\partial u} \) since \( f(\Phi(u, t)) = \tilde{\Phi}(u, t) \), so
\[
\left\| f_\ast \left( \frac{\partial \Phi}{\partial u} \right) \right\| = \left\| \frac{\partial \tilde{\Phi}}{\partial u} \right\| = \sqrt{\sum_{i=1}^{n} \tilde{U}_i^2} = \sqrt{\sum_{i=1}^{n} U_i^2} = \left\| \frac{\partial \Phi}{\partial u} \right\|. \tag{2.27}
\]
This means that \( f \) is a local isometry. It is not hard to see that \( f(p) = \tilde{p} \) and \( f_\ast p = \varphi \). This completes the proof of the theorem. \( \square \)

3 De Rham Decomposition

In this section, we prove Theorem 1.1. First, we have the following simple conclusion for products of Riemannian manifolds:

**Lemma 3.1** Let \( (M_1, g_1) \) and \( (M_2, g_2) \) be two Riemannian manifolds and let \( M = M_1 \times M_2 \) be the product Riemannian manifold. Let \( p = (p_1, p_2) \in M \) and \( v_i : [0, T] \to T_{p_i}M_i \) for \( i = 1, 2 \). Suppose that the developments of \( v_1 \) and \( v_2 \) exist. Then,
1. (for any \( t_i \in [0, T] \) with \( i = 1, 2 \),
   \[
   \text{dev}(\text{dev}(p, v_1)(t_1), v_2)(t_2) = (\text{dev}(p_1, v_1)(t_1), \text{dev}(p_2, v_2)(t_2)) = \text{dev}(\text{dev}(p_2, v_2)(t_2), v_1)(t_1),
   \]
   and the parallel displacement along the closed curve,
   \[
   \text{dev}(p, v_2)_{|t_2} \cdot \text{dev}(\text{dev}(p, v_2)(t_2), v_1)_{|t_1} \cdot \text{dev}(\text{dev}(p, v_1)(t_1), v_2)_{|t_0} \cdot \text{dev}(p, v_1)_{|t_0} = \text{id}_{T_pM};
   \]
2. (for any \( t \in [0, T] \),
   \[
   \text{dev}(\text{dev}(p, v_1)(t), v_2)(t) = \text{dev}(p, v)(t) = \text{dev}(\text{dev}(p, v_2)(t), v_1)(t).
   \]
and the parallel displacements along the closed curves
\[ \text{dev}(p, v)^0_t \cdot \text{dev}(\text{dev}(p, v_1)(t), v_2)^0_t = \text{dev}(p, v_1)^0_t \cdot \text{dev}(p, v_2)^0_t \]
and
\[ \text{dev}(p, v)^0_t \cdot \text{dev}(\text{dev}(p, v_2)(t), v_1)^0_t = \text{dev}(p, v^2_1)^0_t \cdot \text{dev}(p, v_2)^0_t \]
are the identity map of \( T_pM \), where \( v = (v_1, v_2) \in T_pM \).

The conclusion of the lemma is clearly true, from the product structure. For brevity, we will not give the details of the proof here. Next, we show that conclusions similar to that of Lemma 3.1 hold on Riemannian manifolds with two nontrivial parallel distributions that are orthogonal complements of each other.

**Lemma 3.2** Let \((M^n, g)\) be a complete Riemannian manifold with boundary, and let \(T_1\) and \(T_2\) be two nontrivial parallel distributions on \(M\) that are orthogonal complements of each other, with \(T_1\) containing the normal vectors of \( \partial M \) when \( \partial M \neq \emptyset \). Let \(p\) be an interior point of \(M\) and let \(v^i(t) : [0, 1] \to T_i(p)\) be a smooth curve for \(i = 1, 2\). Let \(v = v^1 + v^2\). Then,

1. The development of \(v^2\) exists and stays in the interior of \(M\);
2. If \(\text{dev}(p, v^1)\) exists and is an interior curve, then so is \(\text{dev}(p, v)\), and

\[ \text{dev}(\text{dev}(p, v^1)(t), v^2)(t) = \text{dev}(p, v)(t) = \text{dev}(\text{dev}(p, v^2)(t), v^1)(t) \] (3.1)

for any \(t \in [0, 1]\). Moreover the parallel displacements along the closed curves
\[ \text{dev}(p, v)^0_t \cdot \text{dev}(\text{dev}(p, v^1)(t), v^2)^0_t = \text{dev}(p, v^1)^0_t \cdot \text{dev}(p, v^2)^0_t \]
and
\[ \text{dev}(p, v)^0_t \cdot \text{dev}(\text{dev}(p, v^2)(t), v_1)^0_t = \text{dev}(p, v^2)^0_t \cdot \text{dev}(p, v_1)^0_t \]
are the identity map for any \(t \in [0, 1]\);
3. If \(\text{dev}(p, v)\) exists and is an interior curve, then so is \(\text{dev}(p, v^1)\).

**Proof**

1. Let \(I\) be the maximal interval at which \(\text{dev}(p, v^2)\) exists. By the completeness of \(M\), it is clear that \(I\) is closed and that \(\text{dev}(p, v^2)(b) \in \partial M\) with \(I = [0, b]\) when \(b < 1\). This implies that \(p\) is contained in the leaf of the foliation \(T_2\) passing through \(\text{dev}(p, v^2)(b)\), because \(\text{dev}(p, v^2)'(t) \in T_2\) for \(t \in I\). However, because \(T_2\) is tangential to \(\partial M\), we know that the leaf of \(T_2\) passing through \(\text{dev}(p, v^2)(b) \in \partial M\) must be contained in \(\partial M\). This implies that \(p \in \partial M\), and contradicts the fact that \(p\) is an interior point. For the same reason, \(\text{dev}(p, v^2)(t) \in M \setminus \partial M\) for any \(t \in [0, 1]\).

2. Let \(b > 0\) be such that the development \(\text{dev}(p, v)\) of \(v\) exists on \([0, b]\), and \(\text{dev}(p, v)(t)\) is in the interior of \(M\) for \(t \in [0, b]\). We first show

**Claim 1** The statement (2) is true for \(t \in [0, b]\).

Proof of Claim 1 Note that, for any interior point \(x \in M\), there is an open neighborhood \(U\) of \(x\) in \(M\) such that \(U = U_1 \times U_3\), and each copy of \(U_i\) is an integral submanifold of \(T_i\) for \(i = 1, 2\); we call \(U\) a product neighborhood of \(p\). Let \(B_p(\delta)\) be contained in some product neighborhood. Then \(\text{dev}(p, v)(t) \in B_p(\delta)\) is contained in some product neighborhood for any \(t < \frac{\delta}{A}\) where \(A = \max_{t \in [0, 1]} \|v(t)\|\). By Lemma 3.1, the statement (2) is true for \(t < \frac{\delta}{A}\).

Let \(J = \{t \in [0, b] \mid \text{the statement (2) is true up to } t\}\) and let \(t_0 = \sup J\). By continuity, it is clear that \(t_0 \in J\). Suppose that \(t_0 < b\). By compactness, there is an \(\epsilon > 0\) such that, for any
$t \in [0, t_0]$, $B_{\text{dev}(p, v^1)(t_0), v^2}(t) \epsilon$ is contained in some product neighborhood. Let $t_1 \in [0, b]$ with $0 < t_1 - t_0 < \frac{1}{N}$. We want to show that $t_1 \in J$. This is a contradiction. Thus, we have proven Claim 1.

Let $N$ be a natural number such that $\frac{t_1}{N} < \frac{1}{2\pi}$, and let $\xi_i = \frac{t_1}{N}$ for $i = 0, 1, \ldots, N$. Note that $\text{dev}(p, v)(t_0), v_\epsilon(t)$ for $t \in [0, t_1 - t_0]$ is contained in $B_{\text{dev}(p, v)(t_0)}(\epsilon)$, which is contained in a product neighborhood. By Lemma 3.1 and (2.9), we know that

\[
\text{dev}(p, v)(t_1) = \text{dev}(p, v)(t_0), v_{\epsilon_0}(t_1 - t_0)
\]

\[
= \text{dev}(\text{dev}(p, v)(t_0), v_{\epsilon_0}^1(t_1 - t_0), v_{\epsilon_0}^2(t_1 - t_0))
\]

\[
= \text{dev}(\text{dev}(\text{dev}(p, v^1)(t_0), v^2(t_0), v_{\epsilon_0}^1(t_1 - t_0), v_{\epsilon_0}^2(t_1 - t_0)).
\]

This last equality is by the fact that $t_0 \in J$. We claim that

\[
\text{dev}(\text{dev}(p, v^1)(t_0), v^2(t_0), v_{\epsilon_0}^1(t_1 - t_0) = \text{dev}(p, v^1)(t_1), v^2(t_0).
\]

In fact, we will show that

\[
\text{dev}(\text{dev}(p, v^1)(t_0), v^2(t_0), v_{\epsilon_0}^1(t_1 - t_0) = \text{dev}(p, v^1)(t_1), v^2(t_0)\).
\]

for $i = 0, 1, \ldots, N$, inductively. The equality (3.3) is just (3.4) with $i = N$.

First, (3.4) is clearly true for $i = 0$, by (2.9). Suppose that (3.4) is true for some $i$ less than $N$. Note that

\[
\text{dev}(\text{dev}(p, v^1)(t_0), v^2(t_0), v_{\epsilon_0}^1(t_1 - t_0) \in B_{\text{dev}(p, v^1)(t_0), v^2(t_0)}(\epsilon)
\]

and

\[
\text{dev}(\text{dev}(p, v^1)(t_0), v^2(t_0), v_{\epsilon_0}^2(t) \in B_{\text{dev}(p, v^1)(t_0), v^2(t)}(\epsilon)
\]

for $t \in [0, t_1 - t_0]$ and $t \in [0, t_0/N]$, respectively. By Lemma 3.1, we know that

\[
\text{dev}(\text{dev}(p, v^1)(t_0), v^2(t_0), v_{\epsilon_0}^1(t_1 - t_0), v_{\epsilon_0}^2(\xi_{i+1} - \xi_i) = \text{dev}(\text{dev}(p, v^1)(t_0), v^2(t_0), v_{\epsilon_0}^1(t_1 - t_0)).
\]

By this, and that the fact that (3.4) is true for $i$, we know that (3.4) is true for $i + 1$. This proves (3.3) (see Figure 1 for an illustration of this idea).

![Figure 1](image)

Figure 1 Each Small “rectangle” is contained in a product neighborhood so that Lemma 3.1 can be applied.
Substituting (3.3) into the last equality of (3.2) and using (2.9), we know that
\[ dev(p, v)(t_1) = dev(dev(p, v^1)(t_1), v^2)(t_1). \]  
(3.7)

Similarly, one has that
\[ dev(p, v)(t_1) = dev(dev(p, v^2)(t_1), v^1)(t_1). \]  
(3.8)

Moreover, by a similar argument, one can show that the parallel displacements along the two closed curves in statement (2) form the identity map for \( t = t_1 \). This implies that \( t_1 \in J \), and we have completed the proof of Claim 1.

We next show that the development of \( v \) exists. Otherwise, by the completeness of \( (M, g) \), there is a \( b \in (0, 1) \) such that \( dev(p, v) \) exists on \([0, b]\), where \( dev(p, v)(t) \) is in the interior of \( M \) for \( t \in [0, b] \) and \( dev(p, v)(b) \in \partial M \). By Claim 1, we know that \( dev(p, v)(b) \) can be joined to \( dev(p, v^1)(b) \) by the curve \( dev(dev(p, v^1)(b), v^2) \cdot \cdot \cdot \), which is tangential to \( T_2 \). This implies that \( dev(p, v^1)(b) \in \partial M \). Because \( dev(p, v^1) \) is an interior curve, we know that \( b = 1 \). This is a contradiction. By the same argument, we know that \( dev(p, v) \) is an interior curve. This completes the proof of (2).

(3) This can be shown by the same argument to that in the last paragraph of the proof for statement (2).

Next, we have the following simple properties of curvature tensors for Riemannian manifolds with two nontrivial parallel distributions that are orthogonal complements of each other.

**Lemma 3.3** Let \( (M^n, g) \) be a Riemannian manifold, and that \( T_1 \) and \( T_2 \) be two nontrivial parallel distributions that are orthogonal complements of each other on \( M \). Then,

1. for any \( X, Y, Z, W \in T_pM \), suppose that \( X = X_1 + X_2, Y = Y_1 + Y_2, Z = Z_1 + Z_2 \) and \( W = W_1 + W_2 \), with \( X_1, Y_1, Z_1, W_1 \in T_1(p) \) and \( X_2, Y_2, Z_2, W_2 \in T_2(p) \). Then,
   \[ R(X, Y, Z, W) = R(X, Y_1, Z_1, W_1) + R(X_2, Y_2, Z_2, W_2); \]

2. let \( \gamma : [0, 1] \to M \) be a curve in \( M \) that is tangential to \( T_2 \). Then, for any \( X_1, Y_1, Z_1, W_1 \in T_1(\gamma(0)) \),
   \[ R(X_1, Y_1, Z_1, W_1) = R(P_0^1(\gamma)X_1, P_0^1(\gamma)Y_1, P_0^1(\gamma)Y_1, P_0^1(\gamma)Z_1). \]  
(3.9)

**Proof**

(1) Since \( T_i \) is parallel, \( \nabla_\xi \eta \in T_i \) for any vector field \( \xi \) and any vector field \( \eta \) in \( T_1 \) with \( i = 1, 2 \). Thus,
\[ R(X_1, Y_2, Z, W) = \langle \nabla_Z \nabla_W X_1 - \nabla_Z \nabla_W X_1 - \nabla_{[Z, W]} X_1, Y_2 \rangle = 0, \]

since \( \nabla_Z \nabla_W X_1 - \nabla_Z \nabla_W X_1 - \nabla_{[Z, W]} X_1 \in T_1 \). Similarly,
\[ R(X, Y, Z_1, W_2) = 0. \]

This gives us (1).

(2) Let \( X_1(t), Y_1(t), Z_1(t), W_1(t) \) be the parallel extensions of \( X_1, Y_1, Z_1, W_1 \) along \( \gamma \). Because \( T_1 \) is parallel, \( X_1(t), Y_1(t), Z_1(t), W_1(t) \in T_1 \). Thus, by the second Bianchi identity and 

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\[\frac{d}{dt} R(X_1(t), Y_1(t), Z_1(t), W_1(t)) = (\nabla_{\gamma(t)} R)(X_1(t), Y_1(t), Z_1(t), W_1(t)) = - (\nabla_{W_1(t)} R)(X_1(t), Y_1(t), \gamma'(t), Z_1(t)) - (\nabla_{Z_1(t)} R)(X_1(t), Y_1(t), W_1(t), \gamma'(t)) = 0,\]

since \(\gamma' \in T_2\). This gives us (2). \(\Box\)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1** By (1) of Lemma 3.2, we know that \(M_2\) is a manifold without boundary. On the other hand, because \(T_1\) is transverse to \(\partial M\), we know that \(M_1\) is a manifold with boundary.

Let \(\gamma : [0, 1] \to M_1 \times M_2\) be an interior curve with \(\gamma(0) = (p_1, p_2)\). Suppose that \(\gamma(t) = (\gamma_1(t), \gamma_2(t))\). Let \(v^i(t) = P^0_i(\gamma_i)(\gamma'_i(t))\) for any \(t \in [0, 1]\), with \(i = 1, 2\), and let \(v = (v^1, v^2)\). Then, \(\gamma\) is the development of \(v\).

Let \(\tilde{v}^i = (v^i)_{(0)}(v^3)\) for \(i = 1, 2, 3\), and let \(\tilde{\tilde{v}} = \tilde{v}^1 + \tilde{v}^2 = ((v^1)_{p_1} + (v^2)_{p_2})v\). It is clear that \(\tilde{\gamma}_i = \gamma_i \circ \gamma_i\) is the development of \(\tilde{v}^i\) because \(M_i\) is totally geodesic for \(i = 1, 2\). By (2) of Lemma 3.2, we know that the development \(\tilde{\gamma}\) of \(\tilde{v}\) exists.

Let \(X_i(t)\) be parallel vector fields along \(\gamma\) for \(i = 1, 2, 3, 4\). Suppose that

\[X_i(t) = (X^1_i(t), X^2_i(t)) \quad (3.10)\]

for \(i = 1, 2, 3, 4\). Then, it is clear that \(X^j_i\) is parallel along \(\gamma_j\) for \(i = 1, 2, 3, 4\) and \(j = 1, 2\).

Let \(X^i_j(t) = (\gamma_j)_{(0)}X^i_j(t)\) for \(i = 1, 2, 3, 4\) and \(j = 1, 2\). Again, by the fact that \(M_i\) is totally geodesic, we know that \(X^j_i\) is parallel along \(\tilde{\gamma}_j\).

Let \(X_i(0) = ((v_1)_{p_1} + (v_2)_{p_2})(X_i(0)) = \tilde{X}^1_i(0) + \tilde{X}^2_i(0)\) and let \(\tilde{X}_i(t)\) be the parallel extension of \(\tilde{X}_i(0)\) along \(\tilde{\gamma}\). By (2) of Lemma 3.2, we know that

\[\tilde{X}_i(1) = P^0_i(\sigma_2)(\tilde{X}^1_i(1)) + P^0_i(\sigma_1)(\tilde{X}^2_i(1)). \quad (3.11)\]

Here \(\sigma_1(t) = \text{dev}(\text{dev}(p, v^2)(1), v^1)(t)\), which is tangential to \(T_1\), and

\[\sigma_2 = \text{dev}(\text{dev}(p, v^1)(1), v^2)(t),\]

which is tangential to \(T_2\). Then, by Lemma 3.3, we have that

\[R_M(\tilde{X}_1(1), \tilde{X}_2(1), \tilde{X}_3(1), \tilde{X}_4(1)) = R_M(\tilde{X}^1_1(1), \tilde{X}^2_1(1), \tilde{X}^3_1(1), \tilde{X}^4_1(1)) + R_M(\tilde{X}^2_1(1), \tilde{X}^3_1(1), \tilde{X}^4_1(1)) + R_M(\tilde{X}^3_1(1), \tilde{X}^1_1(1), \tilde{X}^2_1(1)) + R_M(\tilde{X}^4_1(1), \tilde{X}^1_1(1), \tilde{X}^2_1(1)) = R_{M_1}(\tilde{X}_1^1(1), X^1_2(1), X^3_1(1), X^4_1(1)) + R_{M_2}(X^1_3(1), X^2_3(1), X^3_2(1), X^4_3(1)) + R_{M_1 \times M_2}(X_1(1), X_2(1), X_3(1), X_4(1)). \quad (3.12)\]

Hence, by Theorem 1.3, there is a local isometry \(f : M_1 \times M_2 \to M\) such that \(f(p_1, p_2) = p\) and \(f_{\gamma(0)}(p_1, p_2) = t_1 + t_2 + p_2\).

Conversely, for each interior curve \(\tilde{\gamma} : [0, 1] \to M\) in \(M\), let

\[\tilde{v}(t) = P^0_\gamma(\tilde{\gamma})(\tilde{\gamma}'(t)) \quad (3.13)\]

for \(t \in [0, 1]\). Suppose that \(\tilde{v} = \tilde{v}^1 + \tilde{v}^2\) with \(\tilde{v}^i \in T_i(p)\) for \(i = 1, 2\). By Lemma 3.2, we know that the development \(\tilde{\gamma}_i\) of \(\tilde{v}_i\) exists for \(i = 1, 2\). Because \(M_i\) is the leaf of the foliation \(T_i\) passing through \(p\), there is a unique curve \(\gamma_i : [0, 1] \to M_i\) such that \(\gamma_i(0) = p_i\) and
(τ_i)_sp_i(\hat{\gamma}_i(t)) = \hat{\gamma}_i'(t) for \( i = 1, 2 \). Because \( M_i \) is totally geodesic in \( M \), \( \gamma_i \) is the development of \( v_i \) with \( \eta_i \ast p_i(v_i) = \hat{v}^i \) for \( i = 1, 2 \). Let \( \gamma = (\gamma_1, \gamma_2) : [0,1] \to M_1 \times M_2 \). Then, \( \gamma \) is the development of \( v = (v^1, v^2) = ((\tau_1)_sp_1 + (\tau_2)_sp_2)^{-1}(\hat{v}) \). By the same argument as before, using Lemmas 3.2 and 3.3, one can show that

\[
R_M = \tau^*_i R_{M_1 \times M_2}.
\]

(3.14)

Hence, by Theorem 1.3, there is a local isometry \( h : M \to M_1 \times M_2 \) such that \( h(p) = (p_1, p_2) \) and \( h_{sp} = ((\tau_1)_sp_1 + (\tau_2)_sp_2)^{-1} \). Then, \( f \circ h : M \to M \) is a local isometry with \( f \circ h(p) = p \) and \( (f \circ h)_{sp} = id \). This implies that \( f \circ h = id \), and similarly, \( h \circ f = id \). Thus, \( f \) is in fact an isometry. This completes the proof of the theorem.

\[\Box\]

References

[1] Ambrose W. Parallel translation of Riemannian curvature. Ann of Math, 1956, 64(2): 337–363
[2] Blumenthal R A, Hebda J J. The generalized Cartan-Ambrose-Hicks theorem. C R Acad Sci Paris Sér I Math, 1987, 305(14): 647–651
[3] Cartan É. Leçons sur la Géométrie des Espaces de Riemann. 2nd ed. Paris: Gauthier-Villars, 1946 (French)
[4] Chen Zhiqi. The uniqueness in the de Rham–Wu decomposition. J Geom Anal, 2015, 4: 2687–2697
[5] de Rham Georges. Sur la reductibilité d’un espace de Riemann. Comment Math Helv, 1952, 26: 328–344 (French)
[6] Eschenburg J -H, Heintze E. Unique decomposition of Riemannian manifolds. Proc Amer Math Soc, 1998, 126(10): 3075–3078
[7] Foertsch T, Lychak A. The de Rham decomposition theorem for metric spaces. Geom Funct Anal, 2008, 18(1): 120–143
[8] Hicks N. A theorem on affine connexions. Illinois J Math, 1959, (3): 242–254
[9] Klingenberg Wilhelm P A. Riemannian Geometry. De Gruyter Studies in Mathematics, 1. 2nd ed. Berlin: Walter de Gruyter & Co, 1995
[10] Joyce D. On manifolds with corners//Advances in Geometric Analysis. Adv Lect Math 21. Somerville, MA: Int Press, 2012
[11] Kobayashi S, Nomizu K. Foundations of Differential Geometry. Vol I. Reprint of the 1963 original. Wiley Classics Library. A Wiley-Interscience Publication. New York: John Wiley & Sons, Inc, 1996
[12] Maltz R. The de Rham product decomposition. J Differential Geometry, 1972, 7: 161–174
[13] O’Neill B. Construction of Riemannian coverings. Proc Amer Math Soc, 1968, 19: 1278–1282
[14] Shi Yongjie, Yu Chengjie. Rigidity of a trace estimate for Steklov eigenvalues. J Differential Equations, 2021, 278: 50–59
[15] Warner Frank W. Foundations of Differentiable manifolds and Lie Groups. Corrected reprint of the 1971 edition. Graduate Texts in Mathematics, 94. New York, Berlin: Springer-Verlag, 1983
[16] Wu H. On the de Rham decomposition theorem. Illinois J Math, 1964, 8: 291–311
[17] Yu Chengjie. Fundamental theorem for submanifolds in general ambient spaces. Preprint