On Graphs with Minimal Eternal Vertex Cover Number

Veena Prabhakaran
Department of Computer Science and Engineering,
Indian Institute Of Technology, Palakkad

Co-authors: Jasine Babu, L. Sunil Chandran, Mathew Francis, Deepak Rajendraprasad, J. Nandini Warrier

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Outline

1 Introduction

2 Characterization for \( evc(G) = mvc(G) \) for some graph classes

3 Algorithms using the characterization

4 Conclusion and Open problems
Eternal Vertex Cover ($EVC$) problem

- Introduced by Klostermeyer et al.\textsuperscript{1} in 2009
- Attacker-defender game in which $k$ guards are placed on distinct vertices of $G$
- In each round, attacker chooses an edge to attack
- As a response to the attack, defender has to move guards such that
  - At least one guard must move across the attacked edge.
  - Others can either remain in the current position or move to an adjacent vertex.
  - At most one guard exists on any vertex.
- If an attack cannot be defended, the attacker wins.
- The defender wins if he can defend any sequence of infinite attacks.
- Eternal vertex cover number (evc) of a graph $G$: The minimum number $k$ such that the defender has a winning strategy with $k$ guards on $G$.
- For any graph $G$, $mvc(G) \leq evc(G)$
- Given a graph $G$ and an integer $k$, checking if $evc(G) \leq k$ is NP-hard\textsuperscript{2}

\textsuperscript{1}William F. Klostermeyer and C. M. Mynhardt. Australas. J. Combin, 2009
\textsuperscript{2}Fedor V. Fomin, Serge Gaspers, Petr A. Golovach, Dieter Kratsch, and Saket Saurabh, Inf. Process. Lett., 2010
Eternal Vertex Cover Number (evc)-Some Examples

- \( \text{mvc}(P_4) = 2 \) and \( \text{evc}(P_4) = 3 \)
Eternal Vertex Cover Number (evc)-Some Examples

- $\text{mvc}(P_4) = 2$ and $\text{evc}(P_4) = 3$

**Configuration 1:**

![Configuration Diagram]

Veena Prabhakaran
IIT Palakkad
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Eternal Vertex Cover Number (evc)-Some Examples

- mvc($P_4$) = 2 and evc($P_4$) = 3

**Configuration 1:**

![Diagram showing two configurations of vertices v1, v2, v3, v4, and v5, with one configuration marked with an X.]
Eternal Vertex Cover Number (evc)-Some Examples

- \( \text{mvc}(P_4) = 2 \) and \( \text{evc}(P_4) = 3 \)

Configuration 1:

Configuration 2:
mvc\( (P_4) = 2 \) and evc\( (P_4) = 3 \)

**Configuration 1:**

\[
\begin{array}{cccc}
 & v_1 & v_2 & v_3 & v_4 \\
\hline
v_1 & \blackbullet & \circ & \blackbullet & \circ \\
v_2 & \blackbullet & \circ & \blackbullet & \circ \\
v_3 & \blackbullet & \circ & \blackbullet & \circ \\
v_4 & \blackbullet & \circ & \blackbullet & \circ \\
\end{array}
\]

**Configuration 2:**

\[
\begin{array}{cccc}
 & v_1 & v_2 & v_3 & v_4 \\
\hline
v_1 & \circ & \blackbullet & \blackbullet & \circ \\
v_2 & \blackbullet & \circ & \blackbullet & \circ \\
v_3 & \blackbullet & \circ & \blackbullet & \circ \\
v_4 & \blackbullet & \circ & \blackbullet & \circ \\
\end{array}
\]

\[
\text{evc}(C_n) = \text{mvc}(C_n) = \lceil \frac{n}{2} \rceil
\]
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\[
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  v_1 & v_2 & v_3 & v_4 \\
  \bullet & \circ & \bullet & \circ \\
  v_1 & v_2 & v_3 & v_4 \\
  \bullet & \bullet & \bullet & \circ \\
\end{array}
\]

Configuration 2:

\[
\begin{array}{cccc}
  v_1 & v_2 & v_3 & v_4 \\
  \bullet & \bullet & \bullet & \circ \\
  v_1 & v_2 & v_3 & v_4 \\
  \bullet & \bullet & \bullet & \circ \\
  v_1 & v_2 & v_3 & v_4 \\
  \bullet & \bullet & \bullet & \circ \\
\end{array}
\]

- \( evc(C_n) = mvc(C_n) = \left\lceil \frac{n}{2} \right\rceil \)
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**Configuration 1:**

**Configuration 2:**

- $\text{evc}(C_n) = \text{mvc}(C_n) = \left\lceil \frac{n}{2} \right\rceil$
Contribution

- It is known\(^3\) that for any graph \(G\), \(\text{mvc}(G) \leq \text{evc}(G) \leq 2 \text{mvc}(G)\).
- Klostermeyer et al. gave a characterization of graphs with \(\text{evc}(G) = 2 \text{mvc}(G)\).
- Characterization of graphs with \(\text{evc}(G) = \text{mvc}(G)\) remains open.
- We achieve such a characterization for a subclass of graphs.
- This subclass include chordal graphs and internally triangulated planar graphs.

\(^3\)William F. Klostermeyer and C. M. Mynhardt. Australas. J. Combin, 2009
Contribution

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- We achieve such a characterization for a subclass of graphs.
- This subclass include chordal graphs and internally triangulated planar graphs.

Overview of the Approach

- A simple necessary condition for $\text{evc}(G) = \text{mvc}(G)$ is proposed here.
- For many graph classes including chordal and internally triangulated planar graphs, the necessary condition is also shown to be sufficient.
- The characterization leads to the computation of $\text{evc}(G)$ in polynomial time for some graph classes like biconnected chordal graphs.
- For some graphs including chordal graphs, if $\text{mvc}(G) = \text{evc}(G)$, we have a polynomial time strategy for guard movements.

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Characterization for evc($G$) = mvc($G$) for some graph classes

 Necessary condition for any graph

If evc($G$) = mvc($G$), then for every vertex $v \in V(G)$, $\exists$ a min $VC$ of $G$ containing $v$.

Proof:

- Suppose there are $mvc$ guards and $\exists$ a vertex $v$ that does not belong to any min $VC$ of $G$.
- When an edge incident to $v$ is attacked, $v$ has to be occupied in the next configuration.
- Since there is no min $VC$ containing $v$, attack cannot be handled.
Characterization for $\text{evc}(G) = \text{mvc}(G)$ for some graph classes

### Necessary condition for any graph

If $\text{evc}(G) = \text{mvc}(G)$, then for every vertex $v \in V(G)$, $\exists$ a min $VC$ of $G$ containing $v$.

**Proof:**

- Suppose there are $\text{mvc}$ guards and $\exists$ a vertex $v$ that does not belong to any min $VC$ of $G$.
- When an edge incident to $v$ is attacked, $v$ has to be occupied in the next configuration.
- Since there is no min $VC$ containing $v$, attack cannot be handled.

### Sufficiency condition for some graph classes

- The necessary condition is also sufficient for graphs in which all min $VC$s are connected
- Biconnected chordal and biconnected internally triangulated graphs are some examples of such graphs.
- The characterization can be generalized for handling more graph classes.
How are connected vertex covers helpful?

- The *connected vertex cover number*, $\text{cvc}(G)$, is the minimum cardinality of a connected vertex cover of $G$.

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**Lemma (Klostermeyer et al.)**

*Let $G$ be a nontrivial, connected graph and $D$ be a vertex cover of $G$ such that $G[D]$ is connected. Then, $\text{evc}(G) \leq \text{cvc}(G) + 1 \leq |D|+1$.***

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Figure: Handling attack using connected VC

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\[4\] William F. Klostermeyer and C. M. Mynhardt. Australas. J. Combin, 2009.
Characterization for $\text{evc}(G) = \text{mvc}(G)$ for graphs with all min VCs connected

**Theorem**

Let $G(V, E)$ be a connected graph with $|V| \geq 2$ such that every min VC of $G$ is connected. Then $\text{evc}(G) = \text{mvc}(G)$ if and only if for every vertex $v \in V$, there exists a min VC of $G$ containing $v$.

**Proof:**

$\implies$ Trivial from necessary condition

$\impliedby$ **Claim 1:** For any min VC $S_i$ of $G$, an attack on any edge $uv$ with $u \in S_i$ and $v \not\in S_i$ can be defended by moving to a min VC $S_j$ such that $v \in S_j$ and $|S_i \triangle S_j|$ is minimum.

- $X$ and $Y$ are independent sets
- $H = G[X \cup Y]$ is a bipartite graph
- Since $|S_i| = |S_j|$, $|X| = |Y|$
Claim 1.1: $H = G[X \cup Y]$ has a perfect matching. (Recall: $H = G[X \cup Y]$ is a bipartite graph),

Proof strategy:
- Consider $Y' \subseteq Y$
- $X' = N_H(Y')$
- Suppose $|X'| < |Y'|$. 
- Let $S' = Z \cup (Y \setminus Y') \cup X'$
- $|S'| < \text{mve}(G)$. $\Rightarrow \Leftarrow$

$$\forall Y' \subseteq Y, |N_H(Y')| \geq |Y'|$$ and by Hall’s theorem $H$ has a perfect matching.
Proof of Claim 1...

Claim 1.2: \( \forall w \in X, \) the bipartite graph \( H \setminus \{w, v\} \) has a perfect matching.
(Recall: \( S_j \) is a min VC such that \( v \in S_j \) and \( |S_i \triangle S_j| \) is minimum)

\[
\begin{align*}
X &= S_i \setminus S_j \\
Y &= S_j \setminus S_i \\
Z &= S_i \cap S_j \\
W &\subseteq (Y \setminus \{v\}) \\
|X'| &= |N_H(Y')| \\
&\text{By Claim 1.1, } |X'| \geq |Y'|. \\
&\text{Suppose } |X'| = |Y'|. \\
&\text{Let } S' = Z \cup (Y \setminus Y') \cup X' \\
&\text{Therefore, } |X'| > |Y'| \\
\end{align*}
\]

\( \forall Y' \subseteq (Y \setminus \{v\}), |N_H(Y') \setminus \{w\}| \geq |Y'| \) and by Hall’s theorem, \( H \setminus \{w, v\} \) has a perfect matching.
Handling attack on $uv$ by moving to $S_j$

**Claim 1:** For any min VC $S_i$ of $G$, an attack on any edge $uv$ with $u \in S_i$ and $v \notin S_i$ can be defended by moving to a min VC $S_j$ such that $v \in S_j$ and $|S_i \Delta S_j|$ is minimum.

1. $u \in X$: (Using perfect matching $M$ in $H \setminus \{u, v\}$)

   ![Diagram of $u \in X$, $S_i$ to $S_j$]

2. $u \notin X$: (Using perfect matching $M$ in $H \setminus \{w, v\}$)

   Connectivity of $S_i$ is crucial here
   - $w$: nearest vertex of $u$ in $X$
   - $P$: shortest path from $u$ to $w$ in $S_i$
Deciding \( \text{evc}(G) \) when all min \( VC \)'s are connected

**Theorem**

Let \( G(V, E) \) be a graph for which every min VC is connected. If for every vertex \( v \in V \), there exists a min VC \( S_v \) of \( G \) such that \( v \in S_v \), then \( \text{evc}(G) = \text{mvc}(G) \). Otherwise, \( \text{evc}(G) = \text{mvc}(G) + 1 \).

- The second case follows from \( \text{evc}(G) \leq \text{cvc}(G) + 1 \)
Deciding \( \text{evc}(G) \) when all \( \text{min VC}s \) are connected

**Theorem**

Let \( G(V, E) \) be a graph for which every \( \text{min VC} \) is connected. If for every vertex \( v \in V \), there exists a \( \text{min VC} \) \( S_v \) of \( G \) such that \( v \in S_v \), then \( \text{evc}(G) = mvc(G) \). Otherwise, \( \text{evc}(G) = mvc(G) + 1 \).

- The second case follows from \( \text{evc}(G) \leq \text{cvc}(G) + 1 \)

**Consequence:**

- If all \( \text{min VC}s \) of \( G \) are connected, then deciding \( \text{evc}(G) \leq k \) is in \( \text{NP} \).
- For biconnected chordal graphs and biconnected internally triangulated graphs, all \( \text{min VC}s \) are connected and hence deciding \( \text{evc}(G) \leq k \) is in \( \text{NP} \).
- If all \( \text{min VC}s \) of \( G \) are connected and the necessary condition can be checked in polynomial time, then \( \text{evc}(G) \) can be computed in polynomial time.
- For biconnected chordal graphs, \( \text{evc}(G) \) can be computed in polynomial time.
Generalization of the characterization

Necessary condition

Let \( G(V, E) \) be any connected graph. Let \( X \subseteq V \) be the set of cut vertices of \( G \). If \( \text{evc}(G) = \text{mvc}(G) \), then for every vertex \( v \in V \setminus X \), there exists a min VC \( S_v \) of \( G \) such that \( (X \cup \{v\}) \subseteq S_v \).

**proof idea:**
- All vertices of \( X \) have to be occupied in all configurations.
- When an edge incident to \( v \) is attacked, \( (X \cup \{v\}) \) has to be occupied.

Sufficiency condition for some class of graphs

Let \( G(V, E) \) be a connected graph with \( |V| \geq 2 \) and \( X \subseteq V \) be the set of cut vertices of \( G \). Suppose every min VC \( S \) of \( G \) with \( X \subseteq S \) is connected. If for every vertex \( v \in V \setminus X \), there exists a min VC \( S_v \) of \( G \) such that \( (X \cup \{v\}) \subseteq S_v \), then \( \text{evc}(G) = \text{mvc}(G) \).
A class of graphs $\mathcal{H}$ is called hereditary, if deletion of vertices from any graph $G$ in $\mathcal{H}$ would always yield another graph in $\mathcal{H}$.

Chordal graphs form a hereditary graph class.

**Theorem**

If $\mathcal{H}$ is a hereditary graph class such that:

- for every graph $G$ in $\mathcal{H}$, $\text{mvc}(G)$ can be computed in polynomial time and
- for every biconnected graph $H$ in $\mathcal{H}$, all vertex covers of $H$ are connected.

Then,

1. for any graph $G$ in $\mathcal{H}$, in polynomial time we can decide whether $\text{evc}(G) = \text{mvc}(G)$
2. for any graph $G$ in $\mathcal{H}$ with $\text{evc}(G) = \text{mvc}(G)$, there is a polynomial time strategy for guard movements using $\text{evc}(G)$ guards.
3. for any biconnected graph $G$ in $\mathcal{H}$, in polynomial time we can compute $\text{evc}(G)$. Moreover, there is a polynomial time strategy for guard movements using $\text{evc}(G)$ guards.
For any chordal graph $G$, we can decide in polynomial-time whether $\text{evc}(G) = \text{mvc}(G)$. Also, if $\text{mvc}(G) = \text{evc}(G)$, there is a polynomial-time strategy for guard movements using $\text{evc}(G)$ guards.

If $G$ is a biconnected chordal graph, then we can determine $\text{evc}(G)$ in polynomial-time. Moreover, there is a polynomial-time strategy for guard movements using $\text{evc}(G)$ guards.
In certain graph classes, we gave a condition for characterizing graphs with $\text{evc}(G) = \text{mvc}(G)$.

The characterization does not hold for biconnected bipartite planar graphs.

Obtaining a characterization for bipartite graphs is an interesting open problem.

Identify other graph classes for which this characterization holds.
Thank You!