FRACTIONAL SUPERSYMMETRIC QUANTUM MECHANICS, TOPOLOGICAL INVARIANTS AND GENERALIZED DEFORMED OSCILLATOR ALGEBRAS

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Abstract

Fractional supersymmetric quantum mechanics of order $\lambda$ is realized in terms of the generators of a generalized deformed oscillator algebra and a $\mathbb{Z}_\lambda$-grading structure is imposed on the Fock space of the latter. This realization is shown to be fully reducible with the irreducible components providing $\lambda$ sets of minimally bosonized operators corresponding to both unbroken and broken cases. It also furnishes some examples of $\mathbb{Z}_\lambda$-graded uniform topological symmetry of type $(1, 1, \ldots, 1)$ with topological invariants generalizing the Witten index.

Running head: Fractional Supersymmetric Quantum Mechanics

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1 Introduction

Supersymmetric quantum mechanics (SSQM), which was originally introduced as a testing bench for some new ideas in quantum field theory [1], has found a lot of applications in various fields (for reviews see e.g. [2]). In such a theory, there is a $\mathbb{Z}_2$-grading of the Hilbert space and the Hamiltonian $H$ is written as the square of (at least) one conserved supercharge: $Q^2 = H$, with $[H,Q]=0$.

Since its introduction, it has been extended in various ways, thus giving rise for instance to parasupersymmetric (PSSQM) [3, 4], orthosupersymmetric (OSSQM) [5], pseudosupersymmetric [6], and fractional supersymmetric quantum mechanics (FSSQM) [7, 8, 3, 10, 11, 12, 13, 14]. In this letter, our main interest will be the latter, where one replaces the $\mathbb{Z}_2$-grading characterizing SSQM by a $\mathbb{Z}_\lambda$-grading in such a way that the Hamiltonian becomes the $\lambda$th power of a conserved fractional supercharge:

$$Q^\lambda = H, \quad (1)$$

with

$$[H,Q]=0 \quad (2)$$

and $\lambda \in \{3,4,5,\ldots\}$.

It is usual to realize FSSQM of order $\lambda$ in terms of bosonic creation and annihilation operators together with some operators generalizing the fermionic ones. The latter, which are distinct from the parafermionic operators of order $\lambda$ [15], are related instead to the $q$-deformed harmonic oscillator [10] with $q$ a primitive $\lambda$th root of unity, e.g., $q = \exp(2\pi i/\lambda)$ such that $q^\lambda = 1$.

Here, in line with our previous studies of some other variants of SSQM [17, 18], we shall adopt another viewpoint and realize FSSQM of order $\lambda$ in terms of generalized deformed oscillator algebra (GDOA) generators ([16] and references quoted therein) by imposing a $\mathbb{Z}_\lambda$-grading structure on the corresponding Fock space. As a result, FSSQM of order $\lambda$ will prove fully reducible and we shall get a minimal bosonization of this theory in terms of a single bosonic degree of freedom.
Another purpose of this letter is connected with the concept of topological symmetries, which has recently been introduced \[20\] in an attempt to construct generalizations of supersymmetry sharing its topological properties in the sense that they involve integer-valued topological invariants similar to the Witten index \[1\]. Since a special case of \(\mathbb{Z}_\lambda\)-graded topological symmetries has been shown to be related to FSSQM of order \(\lambda\) \[20\], our realization of the latter in terms of GDOA generators will provide us with simple examples of the former and of its topological invariants.

2 Generalized Deformed Oscillator Algebras

Let us start with a brief review of GDOAs \[19\].

A GDOA may be defined as a nonlinear associative algebra \(\mathcal{A}(G(N))\) generated by the operators \(N = N^\dagger\), \(a^\dagger\), and \(a = (a^\dagger)^\dagger\), satisfying the commutation relations

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = G(N),
\]

where \(G(N) = [G(N)]^\dagger\) is some Hermitian function of \(N\).

We restrict ourselves here to GDOAs possessing a bosonic Fock space representation. In the latter, we may write \(a^\dagger a = F(N),\ aa^\dagger = F(N + 1)\), where the structure function \(F(N) = [F(N)]^\dagger\) is such that

\[
G(N) = F(N + 1) - F(N)
\]

and is assumed to satisfy the conditions

\[
F(0) = 0, \quad F(n) > 0 \quad \text{if} \quad n = 1, 2, 3, \ldots
\]

The carrier space \(\mathcal{F}\) of such a representation can be constructed from a vacuum state \(|0\rangle\) (such that \(a|0\rangle = N|0\rangle = 0\)) by successive applications of the creation operator \(a^\dagger\). Its basis states

\[
|n\rangle = \left(\prod_{i=1}^{n} F(i)\right)^{-1/2} (a^\dagger)^n|0\rangle, \quad n = 0, 1, 2, \ldots
\]
where we set $\prod_{i=1}^{0} \equiv 1$, satisfy the relations $N|n\rangle = n|n\rangle$, $a^\dagger|n\rangle = \sqrt{F(n+1)}|n+1\rangle$, and $a|n\rangle = \sqrt{F(n)}|n-1\rangle$.

For $G(N) = I$, we obtain $F(N) = N$ and the algebra $\mathcal{A}(G(N))$ reduces to the standard (bosonic) oscillator algebra $\mathcal{A}(I)$, for which the creation and annihilation operators may be written as $a^\dagger = (x - iP)/\sqrt{2}$, $a = (x + iP)/\sqrt{2}$, where $P$ denotes the momentum operator ($P = -id/dx$).

A $\mathbb{Z}_{\lambda}$-grading structure can be imposed on $\mathcal{F}$ by introducing a grading operator

$$T = e^{2\pi i N/\lambda}, \quad \lambda \in \{2, 3, 4, \ldots\},$$

which is such that

$$T^\dagger = T^{-1}, \quad T^\lambda = I.$$  \hspace{1cm} (8)

It has $\lambda$ distinct eigenvalues $q^\mu$, $\mu = 0, 1, \ldots, \lambda - 1$, with corresponding eigenspaces $\mathcal{F}_\mu$ spanned by $|n\rangle = |k\lambda + \mu\rangle$, $k = 0, 1, 2, \ldots$, and such that $\mathcal{F} = \sum_{\mu=0}^{\lambda-1} \oplus \mathcal{F}_\mu$. Here $q \equiv \exp(2\pi i/\lambda)$. From (3), it results that $T$ satisfies the relations

$$[N, T] = 0, \quad a^\dagger T = q^{-1}T a^\dagger, \quad aT = qTa,$$ \hspace{1cm} (9)

expressing the fact that $N$ preserves the grade, while $a^\dagger$ (resp. $a$) increases (resp. decreases) it by one unit.

The operators

$$P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} q^{-\mu\nu} T_\nu, \quad \mu = 0, 1, \ldots, \lambda - 1,$$ \hspace{1cm} (10)

project on the various subspaces $\mathcal{F}_\mu$, $\mu = 0, 1, \ldots, \lambda - 1$, and therefore satisfy the relations

$$P^\dagger_\mu = P_\mu, \quad P_\mu P_\nu = \delta_{\mu\nu} P_\mu, \quad \sum_{\mu=0}^{\lambda-1} P_\mu = I$$ \hspace{1cm} (11)

in $\mathcal{F}$. As a consequence of (3), they also fulfil the relations

$$[N, P_\mu] = 0, \quad a^\dagger P_\mu = P_{\mu+1} a^\dagger, \quad aP_\mu = P_{\mu-1} a,$$ \hspace{1cm} (12)

where we use the convention $P_{\mu'} = P_\mu$ if $\mu' - \mu = 0 \bmod \lambda$. 

A special case of GDOA with a built-in $\mathbb{Z}_\lambda$-grading structure is provided by the GDOA associated with a $C_\lambda$-extended oscillator algebra $A^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$, where the cyclic group $C_\lambda = \mathbb{Z}_\lambda$ is generated by $T$, i.e., $C_\lambda = \{T, T^2, \ldots, T^{\lambda-1}, T^\lambda = I\}$ [17]. It corresponds to the choice $G(N) = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu$, where $\alpha_\mu$, $\mu = 0, 1, \ldots, \lambda - 1$, are some real parameters constrained by $\sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0$, and it reduces to the Calogero-Vasiliev algebra [21] in the $\lambda = 2$ limit.

3 Fractional Supersymmetric Quantum Mechanics

Let us now look for a $\lambda \times \lambda$-matrix realization of the fractional supercharge and supersymmetric Hamiltonian of the type

$$Q = \sum_{i=1}^{\lambda-1} A_i e_{i+1,i} + A_\lambda e_{1,\lambda}, \quad H = \sum_{i=1}^{\lambda} h_i e_{i,i},$$

(13)

where

$$A_i = f_i(N + i) a, \quad i = 1, 2, \ldots, \lambda - 1, \quad A_\lambda = f_\lambda(N) (a^\dagger)^{\lambda-1},$$

(14)

$$h_i = h_i(N), \quad i = 1, 2, \ldots, \lambda,$$

(15)

are defined in terms of the generators $N, a^\dagger, a$ of a GDOA $A(G(N))$. Here $e_{i,j}$ denotes the $\lambda$-dimensional matrix with entry 1 at the intersection of row $i$ and column $j$ and zeros everywhere else, while $f_i(N)$ and $h_i(N), i = 1, 2, \ldots, \lambda$, are some complex and real functions of $N$, respectively. The functions $f_i(N)$ are furthermore restricted by the condition that

$$\varphi(N) \equiv \prod_{i=1}^{\lambda} f_i(N)$$

(16)

be such that

$$\varphi(n) \in \mathbb{R}^+ \quad \text{if} \quad n = \lambda - 1, \lambda, \lambda + 1, \ldots.$$  

(17)

On inserting (13) into (1), we obtain $\lambda$ conditions

$$A_\lambda A_{\lambda-1} \ldots A_1 = h_1,$$

$$A_{i-1} A_{i-2} \ldots A_1 A_\lambda A_{\lambda-1} \ldots A_i = h_i, \quad i = 2, 3, \ldots, \lambda,$$

(18)
which, on taking (14) and (15) into account, reduce to

\[ h_i(N) = h_1(N + i - 1) = \varphi(N + i - 1) \prod_{j=1}^{\lambda-1} F(N + i - j), \quad i = 2, 3, \ldots, \lambda. \]  

(19)

It is then straightforward to check that with this choice, \( H \) and \( Q \) also satisfy Eq. (2). Hence \( H \) is completely determined by the function \( \varphi(N) \), defined in (16), and by the GDOA structure function \( F(N) \).

In FSSQM, it is well known that there exists another conserved fractional supercharge, the fractional covariant derivative \( D \) (see e.g. [11, 14]). It satisfies relations similar to (1) and (2),

\[ D^\lambda = H, \quad [H, D] = 0, \]  

(20)

as well as a \( q \)-commutation relation with \( Q \),

\[ [D, Q]_q \equiv DQ - qQD = 0. \]  

(21)

A \( \lambda \times \lambda \)-matrix realization of \( D \) can be obtained in the form

\[ D = \sum_{i=1}^{\lambda-1} B_i e_{i+1,i} + B_\lambda e_{1,\lambda}, \]  

(22)

where

\[ B_i = g_i(N + i)a, \quad i = 1, 2, \ldots, \lambda - 1, \quad B_\lambda = g_\lambda(N)(a^\dag)^{\lambda-1}, \]  

(23)

and \( g_i(N), i = 1, 2, \ldots, \lambda \), are some complex functions of \( N \). Equation (20) is satisfied provided

\[ \prod_{i=1}^\lambda g_i(N) = \prod_{i=1}^\lambda f_i(N), \]  

(24)

while Eq. (21) imposes the conditions

\[ f_i(N)g_{i+1}(N) = qf_{i+1}(N)g_i(N), \quad i = 1, 2, \ldots, \lambda - 1, \]

\[ f_\lambda(N)g_1(N) = qf_1(N)g_\lambda(N). \]  

(25)

The general solution of the latter is given by \( g_i(N) = q^{i-1}f_i(N)k(N), i = 1, 2, \ldots, \lambda \), in terms of some complex function \( k(N) \), which, from Eq. (24), is restricted
by the condition $k^\lambda(N) = q^{-\lambda(\lambda-1)/2}$. Up to some $N$-dependent $\lambda$th root of unity, which for simplicity’s sake we assume equal to 1, $k(N)$ is therefore obtained as $k(N) = q^{-\lambda(\lambda-1)/2}$, so that the functions $g_i(N)$ are finally given by

$$g_i(N) = q^{-(\lambda-2i+1)/2} f_i(N), \quad i = 1, 2, \ldots, \lambda. \quad (26)$$

It results from Eq. (19) and from the assumptions (5) and (17) that the spectrum of $H$ is nonnegative. A complete set of eigenvectors is given in terms of the Fock space basis states (6) by

$$|\phi_0, i\rangle = |i - d_{j-1} - 1\rangle e_{j,1}, \quad i = 1, 2, \ldots, \frac{1}{2}\lambda(\lambda - 1), \quad (27)$$

for $E_0 = 0$ and

$$|\phi_n, i\rangle = |n + \lambda - 1 - i\rangle e_{i,1}, \quad i = 1, 2, \ldots, \lambda, \quad (28)$$

for $E_n = \varphi(n + \lambda - 2) \prod_{j=1}^{\lambda-1} F(n + \lambda - 1 - j) > 0$, $n = 1, 2, \ldots$. In (27), $j = j(i) \in \{1, 2, \ldots, \lambda - 1\}$ is determined by the condition $d_{j-1} + 1 \leq i \leq d_j$, where $d_j \equiv j(2\lambda - j - 1)/2$.

All the excited states are therefore $\lambda$-fold degenerate and the fractional supercharge $Q$ acts cyclically on them: $|\phi_n, 1\rangle \rightarrow |\phi_n, 2\rangle \rightarrow \cdots \rightarrow |\phi_n, \lambda\rangle \rightarrow |\phi_n, 1\rangle$. For the $\frac{1}{2}\lambda(\lambda - 1)$-fold degenerate ground state, the action of $Q$ is more complicated since $Q|\phi_0, d_{j-1} + 1\rangle = 0$, while for $d_{j-1} + 2 \leq i \leq d_j$, $Q|\phi_0, i\rangle \propto f_j(i + j - d_{j-1} - 2)|\phi_0, i + \lambda - j - 1\rangle$ may be vanishing or not according to the value assumed by $f_j(i + j - d_{j-1} - 2)$. Since $i + j - d_{j-1} - 2 \leq \lambda - 2$, condition (17) does not indeed ensure the nonvanishing of the latter.

The $\lambda \times \lambda$-matrix realization (13), (22) of $H$, $Q$, and $D$ can be diagonalized through a unitary transformation $U = \sum_{i,j=1}^{\lambda} P_{i\rightarrow j} e_{i,j}$, expressed in terms of the projection operators $P_{\mu}$ defined in (10). The results read

$$H' \equiv UHU^\dagger = \text{diag}(H_0, H_1, \ldots, H_{\lambda-1}),$$

$$Q' \equiv UQU^\dagger = \text{diag}(Q_0, Q_1, \ldots, Q_{\lambda-1}), \quad (29)$$

$$D' \equiv UDU^\dagger = \text{diag}(D_0, D_1, \ldots, D_{\lambda-1}),$$
where

\[ H_{\mu} = \sum_{i=1}^{\lambda} h_i(N) P_{\mu-i+1}, \quad Q_{\mu} = \sum_{i=1}^{\lambda} A_i P_{\mu-i+1}, \quad D_{\mu} = \sum_{i=1}^{\lambda} B_i P_{\mu-i+1}, \]  

(30)

for \( \mu = 0, 1, \ldots, \lambda - 1 \), and \( h_i(N) \), \( A_i \), and \( B_i \) are respectively given by Eqs. (19), (14), and (23), together with Eqs. (16) and (26). Each of the \( \lambda \) sets of operators \( \{ H_{\mu}, Q_{\mu}, D_{\mu} \} \) satisfies the FSSQM relations (1), (2), (20), and (21), and is written in terms of a single bosonic degree of freedom through the operators \( N, a^\dagger, a \) of \( A(G(N)) \). We have therefore proved that FSSQM of order \( \lambda \) is fully reducible and we have obtained a minimal bosonization thereof.

The eigenvalues \( E_{\mu}^{(\mu)} \) of the bosonized fractional supersymmetric Hamiltonian \( H_{\mu} \), defined in (30), can be written as

\[
E_{\lambda k+\nu}^{(\mu)} = \begin{cases} 
\varphi(\lambda k + \mu) \prod_{i=1}^{\lambda-1} F(\lambda k + \mu - i + 1) & \text{if } \nu = 0, 1, \ldots, \mu, \\
\varphi[\lambda(k+1) + \mu] \prod_{i=1}^{\lambda-1} F[\lambda(k+1) + \mu - i + 1] & \text{if } \nu = \mu + 1, \mu + 2, \ldots, \lambda - 1,
\end{cases}
\]  

(31)

where \( k = 0, 1, \ldots \). From Eqs. (5) and (17), it follows that

\[
E_{0}^{(\mu)} = E_{1}^{(\mu)} = \cdots = E_{\mu}^{(\mu)} = 0,
\]

\[
E_{\lambda k+\mu+1}^{(\mu)} = E_{\lambda k+\mu+2}^{(\mu)} = \cdots = E_{\lambda(k+1)+\mu}^{(\mu)} > 0, \quad k = 0, 1, 2, \ldots, \]  

(32)

if \( \mu = 0, 1, \ldots, \) or \( \lambda - 2 \), and that

\[
E_{\lambda k}^{(\lambda-1)} = E_{\lambda k+1}^{(\lambda-1)} = \cdots = E_{\lambda(k+1)-1}^{(\lambda-1)} > 0, \quad k = 0, 1, 2, \ldots, \]  

(33)

if \( \mu = \lambda - 1 \).

In the former case, the corresponding eigenvectors may be written as

\[
|\phi_{0}^{(\mu)}, i\rangle = |\mu + 1 - i\rangle, \quad i = 1, 2, \ldots, \mu + 1,
\]  

(34)

for \( E = E_{0}^{(\mu)} = 0 \) and

\[
|\phi_{k}^{(\mu)}, i\rangle = |\lambda k + \mu + 1 - i\rangle, \quad i = 1, 2, \ldots, \lambda,
\]  

(35)

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for \( E = E^{(\mu)}_{\lambda k + \mu} > 0, k = 1, 2, \ldots \). Furthermore, \( Q_\mu |\phi_0^{(\mu)}, i\rangle \propto f_i(\mu) |\phi_0^{(\mu)}, i+1\rangle \), \( i = 1, 2, \ldots, \mu \), and \( Q_\mu |\phi_0^{(\mu)}, \mu + 1\rangle = 0 \), while \( Q_\mu \) acts cyclically on \( |\phi_k^{(\mu)}, 1\rangle, |\phi_k^{(\mu)}, 2\rangle, \ldots, |\phi_k^{(\mu)}, \lambda\rangle \) for \( k = 1, 2, \ldots \). On the contrary, in the latter case, one has \( |\phi_k^{(\lambda-1)}, i\rangle = |\lambda(k + 1) - i\rangle, \quad i = 1, 2, \ldots, \lambda \) for \( E = E^{(\lambda-1)}_{\lambda k} > 0, k = 0, 1, 2, \ldots, \) and \( Q_{\lambda - 1} \) has a cyclic action on all the sets of states \( |\phi_k^{(\lambda-1)}, 1\rangle, |\phi_k^{(\lambda-1)}, 2\rangle, \ldots, |\phi_k^{(\lambda-1)}, \lambda\rangle \).

We therefore conclude that in the present realization, for \( \mu = 0 \) FSSQM is unbroken with a nondegenerate ground state at a vanishing energy. For \( \mu = 1, 2, \ldots, \) or \( \lambda - 2 \), the ground state is still at a vanishing energy but is \((\mu + 1)\)-fold degenerate and FSSQM is unbroken or broken according to whether \( f_1(\mu) = f_2(\mu) = \cdots = f_\mu(\mu) = 0 \) or at least one of the \( f_i(\mu), i = 1, 2, \ldots, \mu \), is different from zero. Finally, for \( \mu = \lambda - 1 \), the \( \lambda \)-fold degenerate ground state lies at a positive energy and FSSQM is broken. In all the cases, the excited states are \( \lambda \)-fold degenerate.

It is worth noting that for the standard realization of FSSQM in terms of ordinary bosonic operators and \( q \)-deformed ones \([11]\), only the counterparts of the \( \mu = 0 \) case and of the \( \mu = \lambda - 2 \) one with broken FSSQM are obtained. The present realization therefore leads to a much richer picture.

Before concluding this section, it is interesting to consider the \( \lambda = 2 \) limit, wherein FSSQM reduces to ordinary SSQM. In such a case, \( q = \exp(\pi i) = -1 \), so that the \( q \)-commutator of Eq. (21) becomes an anticommutator, the functions \( g_i(N) \) are given by \( g_1(N) = -i f_1(N), g_2(N) = i f_2(N) \), and \( Q, D \) are the usual supercharge and covariant derivative, respectively. In the special case where \( f_1(N) = f_2(N) = f(N) \) and \( f(N) \) is a real function of \( N \) such that \( f(n) \in \mathbb{R}^+ \) for \( n \in \mathbb{N}^+ \), \( Q \) and \( D \) are two Hermitian conserved supercharges. From them, one can construct non-Hermitian ones,

\[
Q \equiv \frac{1}{\sqrt{2}}(Q + iD) = \sqrt{2} \begin{pmatrix} 0 & 0 \\ f(N+1)a & 0 \end{pmatrix},
\]

\[
Q^\dagger \equiv \frac{1}{\sqrt{2}}(Q - iD) = \sqrt{2} \begin{pmatrix} 0 & f(N)a^\dagger \\ 0 & 0 \end{pmatrix},
\] (37)
satisfying the usual SSQM defining relations $Q^2 = (Q^\dagger)^2 = 0$, $\{Q, Q^\dagger\} = H$, with $H = \text{diag}(h_1(N), h_2(N))$ and $h_1(N) = f^2(N)F(N)$, $h_2(N) = h_1(N + 1)$. Such a realization of SSQM coincides with that considered in Ref. [18] for PSSQM of order $p = 1$.

4 $\mathbb{Z}_\lambda$-Graded Uniform Topological Symmetries of Type $(1, 1, \ldots, 1)$ and Topological Invariants

Before applying the concept of topological symmetries to the new realization of FSSQM obtained in the previous section, let us briefly review the former.

According to Ref. [20], a quantum system is said to possess a $\mathbb{Z}_\lambda$-graded topological symmetry (TS) of type $(m_1, m_2, \ldots, m_\lambda)$ if and only if the following conditions are satisfied.

1. The quantum system is $\mathbb{Z}_\lambda$-graded. This means that the Hilbert space $\mathcal{H}$ of the quantum system is the direct sum of $\lambda$ of its (nontrivial) subspaces $\mathcal{H}_i$, $i = 1, 2, \ldots, \lambda$, whose vectors are said to have a definite grading $c_i$. In addition, the Hamiltonian $H$ of the system has a complete set of eigenvectors with definite grading.

2. The energy spectrum is nonnegative.

3. For every positive energy eigenvalue $E$, there is a positive integer $d_E$ such that $E$ is $d_E m$-fold degenerate and the corresponding eigenspaces are spanned by $d_E m_1$ vectors of grade $c_1$, $d_E m_2$ vectors of grade $c_2$, $\ldots$, and $d_E m_\lambda$ vectors of grade $c_\lambda$ (hence $m = \sum_{i=1}^{\lambda} m_i$).

One speaks of uniform topological symmetries (UTS) whenever $d_E = 1$ for all positive energy eigenvalues $E$.

For a system with a $\mathbb{Z}_\lambda$-graded TS of type $(m_1, m_2, \ldots, m_\lambda)$, one can introduce a set of integer-valued topological invariants $\Delta_{ij} \equiv m_i n_j^{(0)} - m_j n_i^{(0)}$, where $i, j \in \{1, 2, \ldots, \lambda\}$ and $n_k^{(0)}$ denotes the number of zero-energy states of grade $c_k$. 

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From the definition of TS, it is possible to obtain the underlying operator algebras supporting such symmetries [20]. In particular, Z₂-graded TS of type (1, 1) has been shown to yield the SSQM algebra with $\Delta_{12}$ reducing to the Witten index.

Here we shall restrict ourselves to a special case of $Z_\lambda$-graded UTS of type (1, 1, . . . , 1), whose algebra coincides with that of FSSQM of order $\lambda$. For a quantum system with Hamiltonian $H$ to have a $Z_\lambda$-graded TS of type (1, 1, . . . , 1), it is indeed sufficient that the following conditions be fulfilled.

1. There exist a grading operator $\tau$ and a TS generator $Q$ satisfying the relations

\[
\tau^\lambda = 1, \quad \tau^\dagger = \tau^{-1}, \quad [H, \tau] = 0, \quad [\tau, Q]_q = 0, \quad \text{(38)}
\]

with $q = \exp(2\pi i/\lambda)$, as well as Eqs. (1) and (2).

2. The spectrum of $H$ is nonnegative.

The presence of this particular TS in turn implies the existence of $l = [\lambda/2]$ Hermitian operators $M_i$, $i = 1, 2, \ldots, l$, commuting with $\tau$ and $Q$ and fulfilling the equations

\[
(Q_1^2 - M_1)(Q_1^2 - M_2)\cdots(Q_1^2 - M_l) = 2^{-l+1}H, \\
(Q_2^2 - M_1)(Q_2^2 - M_2)\cdots(Q_2^2 - M_l) = (-1)^l 2^{-l+1}H, \quad \text{if } \lambda = 2l, \quad \text{(39)}
\]

or

\[
(Q_1^2 - M_1)(Q_1^2 - M_2)\cdots(Q_1^2 - M_l)Q_1 = 2^{-l+1/2}H, \\
(Q_2^2 - M_1)(Q_2^2 - M_2)\cdots(Q_2^2 - M_l)Q_2 = 0, \quad \text{if } \lambda = 2l + 1, \quad \text{(40)}
\]

where

\[
Q_1 = \frac{1}{\sqrt{2}}(Q + Q^\dagger), \quad Q_2 = \frac{1}{i\sqrt{2}}(Q - Q^\dagger). \quad \text{(41)}
\]

For the $\lambda \times \lambda$-matrix realization of FSSQM of order $\lambda$ in terms of GDOA generators considered in Sec. 3, the Hilbert space $\mathcal{H}$ is the direct sum of $\lambda$ copies of
the GDOA Fock space $\mathcal{F}$: $\mathcal{H}_i = \mathcal{F}$, $i = 1, 2, \ldots, \lambda$. With the grading operator $\tau$ realized by the $\lambda \times \lambda$ matrix

$$\tau = \sum_{i=1}^{\lambda} q^i e_{i,i},$$

(42)
a grade $c_i = q^i$ is assigned to the $i$th Fock space $\mathcal{H}_i$. It is straightforward to check that $\tau$, as defined in (12), satisfies Eq. (38) with $H$ and $Q$ as expressed in (13) and that $\tau|\phi_0, i\rangle = q^i|\phi_0, i\rangle$, $\tau|\phi_n, i\rangle = q^i|\phi_n, i\rangle$ for the energy eigenstates (27) and (28), respectively. Since it has been shown in Sec. 3 that the spectrum of $H$ is nonnegative and that all the positive-energy eigenvalues are $\lambda$-fold degenerate, it follows that all the conditions for having a $\mathbb{Z}_\lambda$-graded UTS of type $(1, 1, \ldots, 1)$ are fulfilled. The topological invariants for the present system are

$$\Delta_{ij} = -\Delta_{ji} = i - j, \quad 1 \leq i < j \leq \lambda.$$

(43)

On using Eqs. (13) – (16) and (19), we can also obtain an explicit form for the operators $M_i$ of Eq. (39) or (40),

$$M_i = \frac{1}{2} \sum_{j=1}^{\lambda} m_{ij}(N)e_{j,j}, \quad m_{ij}(N) = m_{i1}(N + j - 1),$$

(44)

where $m_{i1}(N)$, $i = 1, 2, \ldots, l$, are real solutions of an $l$th-degree algebraic equation. For $\lambda = 3, 4$, and 5, the latter are given by

$$\begin{align*}
\alpha_1(N) & = \alpha_2(N) + \alpha_3(N), \\
m_{i1}(N) & = \frac{1}{2}[\alpha_1(N) + \alpha_2(N) + \alpha_3(N) + \alpha_4(N) + (-1)^i\delta(N)], \quad i = 1, 2, \\
\delta(N) & \equiv \left\{ [\alpha_1(N) + \alpha_2(N) + \alpha_3(N) + \alpha_4(N)]^2 \\
& \quad - 4[\alpha_1(N)\alpha_3(N) + \alpha_2(N)\alpha_4(N)] \right\}^{1/2}, \\
m_{i1}(N) & = \frac{1}{2}[\alpha_1(N) + \alpha_2(N) + \alpha_3(N) + \alpha_4(N) + \alpha_5(N) + (-1)^i\delta(N)], \quad i = 1, 2, \\
\delta(N) & \equiv \left\{ [\alpha_1(N) + \alpha_2(N) + \alpha_3(N) + \alpha_4(N) + \alpha_5(N)]^2 \\
& \quad - 4[\alpha_1(N)\alpha_3(N) + \alpha_2(N)\alpha_4(N) + \alpha_3(N)\alpha_5(N) + \alpha_4(N)\alpha_1(N) + \alpha_5(N)\alpha_2(N)] \right\}^{1/2}.
\end{align*}$$

(45) (46)

Here $\alpha_i(N) \equiv |f_i(N)|^2 F(N + 1 - i)$ if $i = 1, 2, \ldots, \lambda - 1$, and $\alpha_\lambda(N) \equiv |f_\lambda(N)|^2 \prod_{j=1}^{\lambda-1} F(N + 1 - j)$. It can be checked that the operators within the square roots in (10) and (17) are nonnegative in $\mathcal{F}$ as it should be.
Let us finally consider the fully reduced form of FSSQM given in Eqs. (29) and (30). The transformed operators corresponding to \( \tau \) and \( M_i \), defined in (42) and (44), are given by

\[
\tau' \equiv U \tau U^\dagger = \text{diag}(\tau_0, \tau_1, \ldots, \tau_{\lambda - 1}),
\]

\[
M_i' \equiv U M_i U^\dagger = \text{diag}(M_{i,0}, M_{i,1}, \ldots, M_{i,\lambda - 1}),
\] (48)

where

\[
\tau_\mu = q^{\mu + 1}T^{-1}, \quad M_{i,\mu} = \sum_{j=1}^{\lambda} m_{ij}(N)P_{\mu-j+1},
\] (49)

for \( \mu = 0, 1, \ldots, \lambda - 1 \), and \( T \) is defined in Eq. (7). From Eqs. (42), (48), and (49), it can also be shown that \( \tau' = \tau T^{-1} \).

For each \( \mu \) value, \( \tau_\mu, Q_\mu \), and \( H_\mu \) satisfy the defining assumptions of a \( \mathbb{Z}_\lambda \)-graded UTS of type (1, 1, \ldots, 1) with \( \mathcal{H} \) now coinciding with \( \mathcal{F} \). We therefore get \( \lambda \) realizations of such a UTS in the same space \( \mathcal{F} \), differing from one another by the grade assigned to the subspaces \( \mathcal{F}_\nu, \nu = 0, 1, \ldots, \lambda - 1 \), which according to (7) and (49), is given by \( c(\mu)_{\nu} = q^{\mu - \nu + 1} \). Hence, the subspaces \( \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_\lambda \) of \( \mathcal{H} \) with grade \( q, q^2, \ldots, q^{\mu + 1}, q^{\mu + 2}, \ldots, q^{\lambda - 1}, 1 \) are to be identified with \( \mathcal{F}_\mu, \mathcal{F}_{\mu-1}, \ldots, \mathcal{F}_0, \mathcal{F}_{\lambda-1}, \ldots, \mathcal{F}_{\mu+2}, \mathcal{F}_{\mu+1} \), respectively. For all the energy eigenstates (34) – (36), we then obtain \( \tau_\mu |\phi_k^{(\mu)}, i\rangle = q^i |\phi_k^{(\mu)}, i\rangle, k = 0, 1, 2, \ldots \). As a consequence, the topological invariants are now

\[
\Delta_{ij}^{(\mu)} = -\Delta_{ji}^{(\mu)} = \begin{cases} -1 & \text{if } 1 \leq i \leq \mu + 1 < j \leq \lambda, \\ 0 & \text{if } 1 \leq i < j \leq \mu + 1 \text{ or } \mu + 2 \leq i < j \leq \lambda, \end{cases}
\] (50)

for \( \mu = 0, 1, \ldots, \lambda - 2 \), and

\[
\Delta_{ij}^{(\lambda-1)} = -\Delta_{ji}^{(\lambda-1)} = 0 \quad \text{if } 1 \leq i < j \leq \lambda,
\]

(51)

for \( \mu = \lambda - 1 \).

5 Conclusion

In this letter, we have extended to FSSQM of order \( \lambda \) the approach to PSSQM and OSSQM in terms of GDOAs that we had previously proposed and we have
obtained both a fully reducible realization and a minimal bosonization of the theory. Furthermore, we have provided some explicit examples of $\mathbb{Z}_\lambda$-graded UTS of type $(1, 1, \ldots, 1)$ and we have evaluated the corresponding topological invariants.

As in the cases of Beckers-Debergh PSSQM and of OSSQM, it turns out that in the limit $G(N) \rightarrow I$, the fractional supersymmetric Hamiltonian and supercharge contain powers of $P^2$ and $P$, respectively. Such features, characteristic of higher-derivative SSQM [22] and $\mathcal{N}$-fold SSQM [23], hint at possible connections with such theories.
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