Classification of the reducible Verma modules over the Jacobi algebra $\mathcal{G}_2$

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Abstract

In the present paper we study the representations of the Jacobi algebra. More concretely, we define, analogously to the case of semi-simple Lie algebras, the Verma modules over the Jacobi algebra $\mathcal{G}_2$. We study their reducibility and give explicit construction of the reducible Verma modules exhibiting the corresponding singular vectors. Using this information we give a complete classification of the reducible Verma modules. More than this we exhibit their interrelation of embeddings between these modules. These embeddings are illustrated by diagrams of the embedding patterns so that each reducible Verma module appears in one such diagram.

Keywords: Jacobi algebra, representations, reducible Verma modules

1. Introduction

Invariant differential operators play a very important role in the description of physical symmetries—recall, e.g. the examples of Dirac, Maxwell, Klein–Gordon, d’Almbert, Schrödinger, equations. Thus, when studying the applications of some symmetry object, always a very interesting question is to find the invariant differential operators and equations with that symmetry.

The role of nonrelativistic symmetries in theoretical physics was always important. Currently one of the most popular fields in theoretical physics—string theory, pretending to be a universal theory—encompasses together relativistic quantum field theory, classical gravity, and certainly, nonrelativistic quantum mechanics, in such a way that it is not even necessary to separate these components.

Since the cornerstone of quantum mechanics is the Schrödinger equation then it is not a surprise that the Schrödinger group—the group that is the maximal group of symmetry of
the Schrödinger equation—was the first to play a prominent role in theoretical physics. The latter is natural since originally the Schrödinger group, actually the Schrödinger algebra, was introduced in [1, 2] as a nonrelativistic limit of the vector-field realization of the conformal algebra.

Another interesting non-relativistic example is the Jacobi algebra [3, 4] which is the semi-direct sum of the Heisenberg algebra and the \( sp(n) \) algebra. Actually the lowest case of the Jacobi algebra coincides with the lowest case of the Schrödinger algebra which makes it interesting to apply to the Jacobi algebra the methods applied to the Schrödinger algebra [5, 6].

For our approach, we recall that in the case of Lie algebras and groups there are several methods of finding the corresponding invariant differential operators. We shall follow the method developed in [7, 8]. In this method there is a correspondence between invariant differential operators and singular vectors of Verma modules over the (complexified) Lie algebra in question.

Thus, in the present paper we develop the first stage of the project, namely, we give the complete classification of the reducible Verma modules over the Jacoby algebra \( G_2 \). This is achieved by the study and explicit construction the singular vectors of the Verma modules. We should note that this was started in [9] where were given some examples the low level singular vectors.

The paper is organized as follows. In the next section we give the preliminaries needed. Then we develop in detail the theory of lowest weight Verma modules over \( G_2 \). Then we study the singular vectors of Verma modules \( G_2 \). Further, we study the question of multiple reducibilities of Verma modules. In the final section we give the classification of the reducible Verma modules. For this we need to study the complete embedding pictures since some some reducible Verma modules contain chains of embedded Verma modules. All results are illustrated by diagrams of the embedding patterns.

2. Preliminaries

2.1. Jacoby algebra \( G_n \): general case

The Jacobi algebra is the semi-direct sum \( G_n := H_n \rtimes sp(n, \mathbb{R}) \subset [3, 4] \). The Heisenberg algebra \( H_n \) is generated by the boson creation (respectively, annihilation) operators \( a^+_i, a^-_i \), \( i, j = 1, \ldots, n \), which verify the canonical commutation relations

\[
[a^-_i, a^-_j] = \delta_{ij}, \quad [a^+_i, a^+_j] = 0.
\] (2.1)

\( H_n \) is an ideal in \( G_n \), i.e. \( [H_n, G_n] = H_n \), determined by the commutation relations (following the notation of [10]):

\[
[a^+_i, K^+_j] = [a^-_i, K^-_j] = 0,
\] (2.2a)

\[
[a^+_i, K^-_j] = \frac{1}{2} \delta_{ij} a^+_j + \frac{1}{2} \delta_{ji} a^+_i, \quad [K^-_j, a^+_j] = \frac{1}{2} \delta_{ij} a^-_i + \frac{1}{2} \delta_{ji} a^-_j,
\] (2.2b)

\[
[K^0_j, a^+_i] = \frac{1}{2} \delta_{ij} a^+_i, \quad [a^-_j, K^0_i] = \frac{1}{2} \delta_{ij} a^-_j.
\] (2.2c)
$K_{ij}^{±,0}$ are the generators of the $S_n \equiv sp(n, \mathbb{R})_C$ algebra:

\[ [K_{ij}, K_{kl}] = [K_{ij}^+, K_{kl}^+] = 0, \quad 2[K_{ij}, K_{kl}^0] = K_{jl}^0 \delta_{ik} + K_{il}^0 \delta_{jk}, \quad (2.3a) \]

\[ 2[K_{ij}^+, K_{kl}^+] = K_{lj}^0 \delta_{ik} + K_{il}^0 \delta_{jk} + K_{ij}^0 \delta_{kl}, \quad (2.3b) \]

\[ 2[K_{ij}^+, K_{kl}^0] = -K_{ij}^+ \delta_{kl} - K_{kl}^+ \delta_{ij}, \quad 2[K_{ij}^0, K_{kl}^+] = K_{ij}^0 \delta_{kl} - K_{ij}^0 \delta_{ik}. \quad (2.3c) \]

In order to implement our approach we introduce a triangular decomposition of $\mathcal{G}_n$:

\[ \mathcal{G}_n = \mathcal{G}_n^+ \oplus \mathcal{K}_n \oplus \mathcal{G}_n^-, \quad (2.4) \]

using the triangular decomposition $S_n = S_n^+ \oplus \mathcal{K}_n \oplus S_n^-$, where:

\[ \mathcal{G}_n^+ = \mathcal{H}_n^+ \oplus S_n^+ \]

\[ \mathcal{H}_n^+ = \text{l.s.}\{ a_i^+ : i = 1, \ldots, n \}, \]

\[ S_n^+ = \text{l.s.}\{ K_{ij}^+ : 1 \leq i < j \leq n \} \oplus \text{l.s.}\{ K_{ij}^0 : 1 \leq i < j \leq n \} \]

\[ S_n^- = \text{l.s.}\{ K_{ij}^- : 1 \leq i < j \leq n \} \oplus \text{l.s.}\{ K_{ij}^0 : 1 \leq i < j \leq n \} \]

\[ \mathcal{K}_n = \text{l.s.}\{ K_{ij}^0 : 1 \leq i \leq n \}. \quad (2.5) \]

Note that the subalgebra $\mathcal{K}_n$ is abelian and is a Cartan subalgebra of $\mathcal{S}_n$. Furthermore, not only $S_n^+$, but also $\mathcal{G}_n^+$ are its eigenspaces:

\[ [\mathcal{K}_n, \mathcal{G}_n^+] = \mathcal{G}_n^+. \quad (2.6) \]

Thus, $\mathcal{K}_n$ plays for $\mathcal{G}_n$ the role that Cartan subalgebras are playing for semi-simple Lie algebras.

Note that the algebra $\mathcal{G}_1$ is isomorphic to the $(1 + 1)$-dimensional Schrödinger algebra (without central extension). The representations of the latter are well known, cf [5, 6, 11, 12]. Thus, below in this paper we study the first new case of the $\mathcal{G}_n$ series, namely, $\mathcal{G}_2$.

### 2.2. Jacoby algebra: case of $\mathcal{G}_2$

First, for simplicity, we introduce the following notations for the basis of $\mathcal{S}_2$:

\[ S^+ : \quad b_i^+ \equiv K_{ii}^+, \quad i = 1, 2; \quad c^+ \equiv K_{12}^+, \quad d^+ \equiv K_{12}^0 \quad (2.7a) \]

\[ S^- : \quad b_i^- \equiv K_{ii}^-, \quad i = 1, 2; \quad c^- \equiv K_{12}^-, \quad d^- \equiv K_{21}^0 \quad (2.7b) \]

\[ \mathcal{K}^+ : \quad h_i \equiv K_{ii}^0, \quad i = 1, 2. \quad (2.7c) \]

Define $\delta_1 := (\frac{1}{2}, 0)$ and $\delta_2 := (0, \frac{1}{2})$. Next, using (2.2) and (2.3) we give the eigenvalues of the basis of $\mathcal{G}_2^+$ w.r.t. $\mathcal{K}$, namely, $\text{ad}h_1$ and $\text{ad}h_2$ are expressed in terms of $\delta_1, \delta_2$ as follows:

\[ h_1 : (b_1^+, b_2^+, c^+, d^+, a_1^+, a_2^+) : (1, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0), \]

\[ h_2 : (b_1^+, b_2^+, c^+, d^+, a_1^+, a_2^+) : (0, 1, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}). \quad (2.8) \]
(e.g. $[h_1, b_1^+] = b_1^+$, $[h_2, d^+] = -\frac{1}{2}d^+$, etc). Naturally, the eigenvalues of the basis of $G^-$ w.r.t. $K$ are obtained from (2.8) by multiplying every eigenvalue by $(-1)$.

Thus, we can introduce the following grading of the basis of $G^+_2$:

$$(b_1^+ , b_2^+ , c^+ , d^+ , a_1^+ , a_2^+ ) : (2\delta_1 , 2\delta_2 , \delta_1 + \delta_2 , \delta_1 - \delta_2 , \delta_1 , \delta_2 )$$ (2.9)

and the corresponding formula for $G^+_2$ is obtained by multiplying $-1$.

The grading of the $S_2^+$ part of the basis follows from the root system of $S_2^+$, while the grading of the $H^+_2$ part of the basis is determined by consistency with commutation relations (2.2). It is consistent also with formulae (2.8).

Next we give the explicit commutation relations:

- **Heisenberg algebra $H_2$**

$$[a_i^-, a_j^+] = \delta_{ij}, \quad [a_i^+, a_j^-] = 0.$$ (2.10)

- **Symplectic algebra $S_2 = C_2$**

$$[h_1, b_1^+] = \pm b_1^+, \quad [h_1, b_2^+] = 0, \quad [h_1, c^+] = \pm \frac{1}{2}c^+, \quad [h_1, d^+] = \pm \frac{1}{2}d^+, \quad [h_2, b_1^+] = \pm b_2^+, \quad [h_2, b_2^+] = 0, \quad [h_2, c^+] = \pm \frac{1}{2}c^+, \quad [h_2, d^+] = \mp \frac{1}{2}d^+, \quad [h_1 , h_2] = 0, \quad [b_1^+, b_2^+] = 0, \quad [b_1^+, b_2^+] = 2\delta_1 h_1, \quad [b_1^+, c^+] = 0, \quad [b_1^+, d^+] = 0, \quad [b_1^+, d^+] = \mp c^+, \quad [b_2^+, d^+] = \mp d^+, \quad [b_2^+, d^+] = 0, \quad [b_2^+, d^+] = \mp d^+, \quad [c^-, d^+] = \frac{1}{2}(h_1 + h_2), \quad [d^-, d^+] = \frac{1}{2}(h_2 - h_1).$$ (2.11)

- **Semidirect sum of $H_2$ and $S_2$**

$$[h_i , a_j^+] = \frac{1}{2} a_j^+, \quad [a_j^+, b_j^+] = 0, \quad [a_j^+, b_j^+] = \mp a_j^+, \quad [a_j^+, c^+] = 0, \quad [a_j^+, d^+] = \mp a_j^+, \quad [a_j^+, d^+] = 0, \quad [a_j^+, d^+] = \mp a_j^+, \quad [a_j^+, d^+] = 0,$$ (2.12)

and

$$[h_i , a_j^+] = [a_i^+, b_j^+] = 0 \quad \text{for } i \neq j.$$ (2.13)
3. Lowest weight Verma modules over $G_2$

As in [9] we introduce Verma modules over the Jacobi algebra analogously to the case of semi-simple algebras. Thus, we define a lowest weight Verma module $V^\Lambda$ over $G_2$ as the lowest weight module over $G_2$ with lowest weight $\Lambda \in \mathcal{K}_n^+$ and lowest weight vector $v_0 \in V^\Lambda$, induced from the one-dimensional representation $V_0 \cong \mathbb{C}v_0$ of $U(B_u)$, where $B_u = \mathcal{K}_n \oplus G_2$ is a Borel subalgebra of $G_n$, such that:

$$Xv_0 = 0, \quad \forall X \in G_2^-, \quad H v_0 = \Lambda(H) v_0, \quad \forall H \in \mathcal{K}_n.$$  \hspace{1cm} (3.1)

We introduce

$$\hat{b}_k^+ := b_k^+ - \frac{1}{2}(a_k^+)^2, \quad \hat{c}_k^- := c^+ - \frac{1}{2}a_1^+ a_2^-.$$  \hspace{1cm} (3.2)

As a basis of $V^\Lambda$ one may take ($|0\rangle = v_0$)

$$|k_1, k_2, m_1, m_2, n_1, n_2\rangle := (\hat{b}_k^+)^{k_1}(\hat{b}_k^+)^{k_2}(a_1^+)^{m_1}(a_2^-)^{m_2}(\hat{c}_k^-)^{n_1}(d^-)^{n_2}|0\rangle$$  \hspace{1cm} (3.3)

where $k_i, m_i, n_i \in \mathbb{Z}_{\geq 0}$. This is an eigenvector of $h_1, h_2$ and its eigenvalue is given by

$$\Lambda = \Lambda_1 + p_1 \delta_1 + p_2 \delta_2, \quad \Lambda_1 := \Lambda(h_i), \quad p_1 = K_1 + n_1 + n_2, \quad p_2 = K_2 + n_1 - n_2, \quad (K_i := 2k_i + m_i).$$  \hspace{1cm} (3.4)

We also use the shorthand notation $|k, m, n\rangle$ with $k = (k_1, k_2)$, etc.

We also introduce

$$d^+ := d^+ - \frac{1}{2}a_1^+ a_2^-, \quad d^- := d^- - \frac{1}{2}a_1^- a_2^+, \quad N_k := a_k^+ a_k^-.$$  \hspace{1cm} (3.6)

$$\hat{b}_k^- := b_k^- - \frac{1}{2}(a_k^-)^2, \quad \hat{c}_k^+ := c^- - \frac{1}{2}a_1^- a_2^+.$$  \hspace{1cm} (3.7)

Next we compute the action of $G_2^+$ on $|k, m, n\rangle$. The action of $a_k^-$ is given by

$$a_k^- |k, m, n\rangle = m_1 |k, m_1 - 1, m_2, n\rangle,$$  \hspace{1cm} (3.8)

$$a_k^- |k, m, n\rangle = m_2 |k, m_1, m_2 - 1, n\rangle.$$  \hspace{1cm} (3.9)

The action of $\hat{b}_k^-$ is given by

$$\hat{b}_k^- |k, m, n\rangle = k_1 \left(2\Lambda_1 + k_1 + m_1 + n_2 - \frac{3}{2}\right) |k_1 - 1, k_2, m, n\rangle + \frac{1}{2} n_1 n_2 \left(\Lambda_2 - \Lambda_1 - \frac{1}{2} (n_2 - 1)\right) |k, m, n_1 - 1, n_2 - 1\rangle + \frac{1}{4} n_1 (n_1 - 1) |k_1, k_2 + 1, m, n_1 - 2, n_2\rangle.$$  \hspace{1cm} (3.10)
\[ \hat{b}_2 \vert \vec{k}, \vec{m}, n \rangle = k_2 \left( 2 \Lambda_2 + k_2 + n_1 - n_2 - \frac{3}{2} \right) \vert k_1, k_2 - 1, m, n \rangle + n_1 \vert k, m, n_1 - 1, n_2 + 1 \rangle + \frac{1}{4} n_1(n_1 - 1) \vert k_1 + 1, k_2, m, n_1 - 2, n_2 \rangle. \] (3.11)

The action of $\hat{d}^-$, $\hat{c}^-$ is as follows:
\[ \hat{d}^- \vert \vec{k}, \vec{m}, n \rangle = \frac{n_2}{2} \left( \Lambda_2 - \Lambda_1 - \frac{1}{2}(n_2 - 1) \right) \vert \vec{k}, m, n_1, n_2 - 1 \rangle + k_1 \vert k_1 - 1, k_2, m, n_1 + 1, n_2 \rangle + \frac{n_1}{2} \vert k_1, k_2 + 1, m, n_1 - 1, n_2 \rangle, \] (3.12)
\[ \hat{c}^- \vert \vec{k}, \vec{m}, n \rangle = \frac{n_2}{2} \left( \sum_{j=1}^{2} (\Lambda_j + k_j) + \frac{1}{2}(n_1 - 2) \right) \vert \vec{k}, m, n_1 - 1, n_2 \rangle + \frac{1}{2} k_2 n_2 \left( \Lambda_2 - \Lambda_1 - \frac{1}{2}(n_2 - 1) \right) \vert k_1, k_2 - 1, m, n_1, n_2 - 1 \rangle + k_1 \vert k_1 - 1, k_2, m, n_1 + 1 \rangle + k_1 k_2 \vert k_1 - 1, k_2 - 1, m, n_1 + 1, n_2 \rangle. \] (3.13)

The Verma module $V^\Lambda$ has a weight space decomposition:
\[ V^\Lambda = \bigoplus_{p_1, p_2} V^\Lambda_{p_1, p_2}, \] (3.14)
where $V^\Lambda_{p_1, p_2}$ is a subspace of $V^\Lambda$ spanned by the vectors with the weight $\Lambda + p_1 \delta_1 + p_2 \delta_2$.

Remarks

(a) $p_1 \in \mathbb{Z}_{\geq 0}$, $p_2 \in \mathbb{Z}$.

(b) For a fixed value of $p_1$, the smallest possible value of $p_2$ is $p_2 = -p_1$. This is seen from the fact that the largest value of $n_2$ for a fixed $p_1$ is $n_2 = p_1$.

4. Singular vectors in Verma modules

4.1. Definitions and summary of singular vectors

We are interested in the cases when the Verma modules are reducible. Namely, we are interested in the cases when a Verma module $V^\Lambda$ contains an invariant submodule which is also a Verma module $V^{\Lambda'}$, where $\Lambda' \neq \Lambda$, and holds the analog of
\[ X \vert 0' \rangle = 0, \quad \forall X \in G^- \] (4.1a)
\[ H \vert 0' \rangle = \Lambda'(H) \nu_0', \quad \forall H \in \mathcal{K}_n. \] (4.1b)

Since $V^{\Lambda'}$ is an invariant submodule then there should be a mapping such that $\vert 0' \rangle$ is mapped to a singular vector $\vert \nu_0' \rangle \in V^{\Lambda'}$ fulfilling exactly (4.1). Thus, as in the semi-simple case there should be a polynomial $P$ of $G^+_n$ elements which is eigenvector of $\mathcal{K}_n$: $[H, P] = \Lambda'(H)P$, ($\forall H \in \mathcal{K}_n$), and then we would have: $\vert \nu_0' \rangle = P \vert 0 \rangle$.  

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The above situation we shall depict by the following diagram:

\[ V^\Lambda \rightarrow \rightarrow V^{\Lambda'} \]

Note that in the diagram the arrow points to the embedded Verma module.

Now we present the result of the complete search of singular vectors. We found five types of singular vectors and they exist in Verma modules with a particular value of the lowest weight. To specify the lowest weight, we introduced positive integers \( p^1, p^2, p^3, p^4, p^5 \) and \( q^3 \).

(i) \( \Lambda_1 - \Lambda_2 = \frac{1}{2} (1 - p^1) \)

\[ |v_1^\Lambda\rangle = (d^+) p^1 |0\rangle, \quad \Lambda' = \Lambda + p^1 (\delta_1 - \delta_2). \tag{4.2} \]

(ii) \( \Lambda_1, \Lambda_2 = \frac{3}{2} - \frac{p^2}{2} \)

\[ |v_2^\Lambda\rangle = (b_+^d) p^2 |0\rangle, \quad \Lambda' = \Lambda + 2p^2 \delta_2. \tag{4.3} \]

(iii) \( \Lambda_1 = \frac{5}{4} - \frac{1}{2} (p^3 - q^3), \quad \Lambda_2 = \frac{5}{4} - \frac{1}{2} q^3, \quad (p^3 \neq q^3, p^3 \neq 2q^3) \)

There are three subcases depending on the range of \( p^3, q^3 \). The form of \( |v_3^\Lambda\rangle \) and \( \Lambda' \) are common for all the subcases:

\[ |v_3^\Lambda\rangle = \sum c(k,n) |k, q^3 - k - n, n, p^3 - 2k - n\rangle, \quad \Lambda' = \Lambda + p^3 \delta_1 + (2q^3 - p^3) \delta_2. \tag{4.4} \]

They differ only the range of summation:

1. \( p^3 < q^3 \) \( \Rightarrow \sum = \sum_{k=0}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{p^3 - 2k} \)

2. \( q^3 < p^3 < 2q^3 \) \( \Rightarrow \sum = \sum_{k=0}^{\lfloor p^3/q^3 \rfloor} \sum_{n=0}^{p^3 - 2k} + \sum_{k=p^3/q^3 + 1}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{q^3 - k} \)

3. \( 2q^3 < p^3 \) \( \Rightarrow \sum = \sum_{k=0}^{p^3/q^3 - k} \sum_{n=0}^{q^3 - k} \)

(iv) \( \Lambda_1 + \Lambda_2 = 2 - \frac{p^4}{2} \)

\[ |v_4^\Lambda\rangle = \sum_{k=0}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{p^3 - 2k} c(k,n) |k, p^4 - k - n, n, p^4 - 2k - n\rangle, \quad \Lambda' = \Lambda + p^4 (\delta_1 + \delta_2). \tag{4.6} \]

(v) \( \Lambda_1 = \frac{5}{4} - \frac{p^5}{4}, \quad \Lambda_2 \)

\[ |v_5^\Lambda\rangle = \sum_{k=0}^{p^5 - k} \sum_{n=0}^{p^5 - k} c(k,n) |k, p^5 - k - n, 2p^5 - 2k - n\rangle, \quad \Lambda' = \Lambda + 2p^5 \delta_1. \tag{4.7} \]
The coefficient \( c(k, n) \) is given by

\[
c(k, n) = \frac{(-1)^n}{4^n k! n!} \frac{\Gamma(2\Lambda_1^* + p - \frac{k}{2})}{\Gamma(2\Lambda_1^* + p - \frac{k}{2} - n)} \frac{\Gamma(2\Lambda_2^* - p + q - \frac{k}{2} + 2k + n)}{\Gamma(2\Lambda_2^* - p + q - \frac{k}{2})},
\]

(4.8)

where \( \Lambda_1^*, \, p \) and \( q \) are given as follows:

- Type (iii): \( \Lambda_1^* = \frac{3}{2} - \frac{1}{2}(p^3 - q^3), \, \Lambda_2^* = \frac{3}{4} - \frac{q^2}{4}, \, p = p^3, \, q = q^3 \)
- Type (iv): \( \Lambda_1^* = \Lambda_1, \, \Lambda_2^* = \frac{3}{4} - \frac{p^2}{2}, \, p = q = p^4 \)
- Type (iv): \( \Lambda_1^* = \frac{3}{2} - \frac{p^2}{2}, \, \Lambda_2^* = \Lambda_2, \, p = 2p^5, \, q = p^5 \)

We note that the weight \( \Lambda' \) has a unified expression:

\[
\Lambda' = \Lambda + p\delta_1 + (2q - p)\delta_2,
\]

(4.9)

where \( p \) and \( q \) for type (i) (ii) are given by

- Type (i): \( p = p^4, \, q = 0 \)
- Type (ii): \( p = 0, \, q = p^5 \)

We prove the result in the rest of this section.

4.2. General facts on the used singular vectors

To perform a complete search of singular vectors, we need to specify the vectors in the weight space decomposition \( V^{\Lambda}_{p_1, p_2} \), cf (3.14). To this end, observe first that

\[
p_1 + p_2 \quad \text{and} \quad K_1 + K_2 \quad \text{have the same parity.}
\]

(4.10)

This is due to the relation \( p_1 + p_2 = K_1 + K_2 + 2n_1 \).

The basis \( |k, m, n\rangle \) of \( V^\Lambda \) is labelled by six non-negative integers. However, if we fix the value of the pair \( (p_1, p_2) \), then two of the six non-negative integers become dependent parameters. We take \( K_1, k_1, k_2 \) and \( n_1 \) as independent parameters. Then we have

\[
m_1 = K_1 - 2k_1, \quad m_2 = p_1 + p_2 - K_1 - 2k_2 - 2n_1, \quad n_2 = p_1 - K_1 - n_1.
\]

(4.11)

The possible range of these parameters depends on the value of \( (p_1, p_2) \). We will specify the range later. At this stage, one may write a singular vector as follows:

\[
|v_i^m_{p_1, p_2}\rangle = \sum_{K_1, k_1, k_2, n_1} c(K_1, k_1, k_2, n_1) |k, m, n\rangle
\]

(4.12)

where \( m_1, m_2, n_2 \) are given by (4.11) and the sum runs all possible values of \( K_1, k_1, k_2 \) and \( n_1 \).

With this expression of \( |v_i^m_{p_1, p_2}\rangle \) one may readily show the following:

If \( p_1 + p_2 \) is an odd integer, then there exist no singular vectors.
Proof. The condition
\[ a_2^+ |\psi^{p_1, p_2}_{k_1, k_2} \rangle = \sum_{k_1, k_2, n_1} c(K_1, k_1, k_2, n_1) m_1 |k, m, n\rangle = 0 \] (4.13)
means that \( m_1 = K_1 - 2k_1 \) must vanish to have \( c(K_1, k_1, k_2, n_1) \neq 0 \). Obviously, \( m_1 \neq 0 \) if \( K_1 \) is odd. Thus \( K_1 \) must be even.

The condition
\[ a_2^- |\psi^{p_1, p_2}_{k_1, k_2} \rangle = \sum_{k_1, k_2, n_1} c(K_1, k_1, k_2, n_1) m_2 |k, m, n\rangle = 0 \] (4.14)
requires \( m_2 = p_1 + p_2 - K_1 - 2k_2 - 2n_1 = 0 \), but \( m_2 \) never vanish if \( p_1 + p_2 \) is odd. □

Even when \( p_1 + p_2 \) is even, the conditions (4.13) and (4.14) require \( m_1 = m_2 = 0 \) which gives
\[ K_1 = 2k, \quad k_2 = \rho - k - n, \quad \rho = \frac{1}{2}(p_1 + p_2) \] (4.15)
where we set \( k := k_1, n := n_1 \). Therefore, the singular vector has the form of
\[ |\psi^{p_1, p_2}_{k_1, k_2} \rangle = \sum_{k,n} c(k,n) |k, \rho - k - n, n, p_1 - 2k - n\rangle \] (4.16)
where we introduced the new notation for the basis vector:
\[ |k, \ell, n, m\rangle := (\hat{b}_+)^\ell (\hat{b}_-)^n (\hat{c}^+)^m (\hat{d}^+)^n |0\rangle \]

By definition of \( p_1 \) and (4.16) we see that
\[ 0 \leq k \leq \left\lfloor \frac{p_1}{2} \right\rfloor, \quad 0 \leq n \leq p_1, \quad 0 \leq \rho - k - n, \quad 0 \leq p_1 - 2k - n \] (4.17)

These relations determine the possible range of summation over \( k, n \) in (4.16).

In the sequel we show the followings:
If \( p_1 + p_2 \) is even, then there exists only one singular vectors for particular values of the lowest weights.

First of all, we consider the two special cases for which we have \( k = n = 0 \), namely, there is no summation in (4.16):

i) \( \rho = 0 \Leftrightarrow p_1 + p_2 = 0 \) i.e. \( |\psi^{p_1, p_2}_{0, 0} \rangle = |0, 0, 0, p_1\rangle = (\hat{d}^+)^{p_1} |0\rangle \).

ii) \( p_1 = 0 \) i.e. \( |\psi^{p_1, p_2}_{0, n} \rangle = |0, 0, 0, 0\rangle = (\hat{b}_+)^n (\hat{c}^+)^{p_2/2} |0\rangle \).

Then the following is immediate:

(a) If \( \rho = 0 \), then there exists only one singular vector for \( \Lambda_1 - \Lambda_2 = \frac{1}{4}(1 - p_1) \) which is given by
\[ |\psi^{p_1, p_2}_{\rho = 0} \rangle = (\hat{d}^+)^{p_1} |0\rangle \] (4.18)

(b) If \( p_1 = 0 \), then there exists only one singular vector for \( \Lambda_2 = \frac{1}{4}(3 - p_2) \) which is given by
\[ |\psi^{p_1, p_2}_{\rho = 0} \rangle = (\hat{b}_+)^n (\hat{c}^+)^{p_2/2} |0\rangle \] (4.19)
Proof

i) It is immediate to see that $\hat{b}_1^-, \hat{b}_2^-$ and $\hat{c}^-$ annihilate $|v_{i}^{p_1(p_1-2)}\rangle$. Also the following is readily seen

$$d^- |v_{i}^{p_1(p_1-2)}\rangle = \frac{p_1}{2} \left( \Lambda_2 - \Lambda_1 - \frac{1}{2}(p_1 - 1) \right) [0, 0, 0, p_1 - 1]. \tag{4.20}$$

ii) One may immediately see that $\hat{b}_1^-, \hat{c}^-$ and $d^-$ annihilate $|v_{i}^{p_2}\rangle$. One also see

$$\hat{b}_2^- |v_{i}^{p_2}\rangle = \frac{p_2}{2} \left( 2\Lambda_2 + \frac{p_2}{2} - \frac{3}{2} \right) |0, \frac{p_2}{2} - 1, 0, 0\rangle. \tag{4.21}$$

Now we specify the possible range of the parameters $k, n$. The range, which is determined by (4.17), is classified into three patterns:

iii) $0 < p_1 \leq \rho \Rightarrow 0 \leq k \leq \left\lfloor \frac{p_1}{2} \right\rfloor, 0 \leq n \leq p_1 - 2k$

iv) $0 < \rho < p_1$

(a) $p_2 \leq 0 \Rightarrow 0 \leq k \leq \rho, 0 \leq n \leq \rho - k$

(b) $p_2 > 0 \Rightarrow 0 \leq k \leq \rho, 0 \leq n \leq \rho - k$ and $\rho + 1 \leq k \leq \left\lfloor \frac{p_1}{2} \right\rfloor, 0 \leq n \leq p_1 - 2k$

where $\rho = \frac{1}{2}(p_1 - p_2)$. This classification is easily seen by considering the intersection of two lines $\ell_1 : n = -k + \rho$ and $\ell_2 : n = -2k + p_1$ in the $kn$-plane as depicted below:

Before investigating the cases iii) and iv), we derive recurrence relations for $c(k, n)$ which are common for all the cases. The singular vector (4.16) must be annihilated by $\hat{b}_1^-, \hat{b}_2^-, \hat{c}^-, d^-$. The condition $\hat{b}_1^- |v_{i}^{p_1(p_1-2)}\rangle = 0$ gives

$$4(k + 1) \left( 2\Lambda_1 + p_1 - k - \frac{5}{2} \right) c(k + 1, n - 1)$$

$$+ 2n(p_1 - 2k - n)\tilde{\Lambda} c(k, n) + n(n + 1)c(k, n + 1) = 0. \tag{4.22}$$
From the condition \( \hat{b}^{-}_{2} |\psi^{p_{1}^{1} + p_{1}^{2}}\rangle = 0 \) we have

\[
4(\rho - k - n) \left( 2\Lambda_{2} - \bar{\rho} + k + n - \frac{3}{2} \right) c(k, n) \\
+ (n + 1)(n + 2)c(k - 1, n + 2) + 4(n + 1)c(k, n + 1) = 0. \tag{4.23}
\]

The condition \( \hat{c}^{-}_{y} |\psi^{p_{1}^{1} + p_{1}^{2}}\rangle = 0 \) gives

\[
(n + 1) \left( \Lambda_{1} + \Lambda_{2} + \rho - \frac{n}{2} - \frac{3}{2} \right) c(k, n + 1) + (\rho - k - n)(p_{1} - 2k - n) \tilde{\Lambda} c(k, n) \\
+ 2(k + 1)c(k + 1, n) + 2(k + 1)(\rho - k - n)c(k + 1, n - 1) = 0. \tag{4.24}
\]

From the condition \( d^{-}_{y} |\psi^{p_{1}^{1} + p_{1}^{2}}\rangle = 0 \) we have

\[
2(k + 1)c(k + 1, n - 1) + (n + 1)c(k, n + 1) + (p_{1} - 2k - n) \tilde{\Lambda} c(k, n) = 0. \tag{4.25}
\]

The possible range of \( k \) and \( n \) depends on the cases, so we will specify it later. In addition to these relations, there exit more recurrence relations stems from the ‘boundary values’ (i.e. minimum and maximum values) of \( k \), \( n \) which also depend on the cases. They will also be presented when we discuss the each cases.

We find that the relations (4.22) and (4.25) are solved for arbitrarily \( \Lambda_{1}, \Lambda_{2} \) and the solution is given by

\[
c(k, n) = \frac{(-1)^{p_{1}} k!}{4^{k} k! (p_{1} - 2k)!} \frac{\Gamma \left( 2\Lambda_{1} + p_{1} - \frac{3}{2} \right)}{\Gamma \left( 2\Lambda_{1} + k - n - \frac{3}{2} \right)} \times \frac{\Gamma \left( 2\Lambda_{2} - 2\Lambda_{1} - p_{1} + 2k + n + 1 \right)}{\Gamma \left( 2\Lambda_{2} - 2\Lambda_{1} - p_{1} + 1 \right)} \frac{\Gamma \left( 4\Lambda_{1} + 2p_{1} - 2k - n - 4 \right)}{\Gamma \left( 4\Lambda_{1} + 2p_{1} - 4 \right)} c(0, 0).
\]

(4.26)

However, (4.26) does not solve the relations (4.23) and (4.24) unless \( \Lambda_{1}, \Lambda_{2} \) take particular values. As we will see later, (4.23) and (4.24) together with other recurrence relations fix the value of the lowest weight.

In the next subsections we are finding more specific results.

4.3. Case iii) \( 0 < p_{1} \leq \rho \)

There exists only one singular vector of the form

\[
|\psi^{p_{1}^{1} + p_{1}^{2}}\rangle = \sum_{k=0}^{\lfloor p_{1}/2 \rfloor} \sum_{n=0}^{p_{1} - 2k} c(k, n) |k, \rho - k - n, n, p_{1} - 2k - n \rangle \tag{4.27}
\]

for the following two cases

- \( 0 < p_{1} < \rho \) and \( \Lambda_{1} = \frac{3}{4} - \frac{1}{2}\tilde{\rho}, \Lambda_{2} = \frac{3}{4} - \frac{1}{2}\rho \)

\[
c(k, n) = \frac{p_{1}! \rho!}{4^{k} k! (p_{1} - 2k - n)! (\rho - k - n)!}.
\]

(4.28)
• $0 < p_1 = \rho$ and $\Lambda_1 + \Lambda_2 = 2 - \frac{\mu}{\rho}$

$$c_{(k,n)} = \frac{p_1!}{4^k k! n!(p_1 - 2k - n)!} \frac{\Gamma(2\Lambda_1 + p_1 - \frac{\mu}{\rho})}{\Gamma(2\Lambda_1 + p_1 - k - n - \frac{\mu}{\rho})}.$$ (4.29)

The proof of the above statement is straightforward by applying the conditions of annihilation of (4.27) by the negative generators and we omit it.

4.4. Case iv) (a) $0 < \rho < p_1, p_2 \leq 0$

There exists only one singular vector of the form

$$|\psi_{p_1^1 + p_2^2}^{p_1^0 + p_2^0} \rangle = \sum_{k=0}^{\rho} \sum_{n=0}^{\rho - k} c_{(k,n)} |k, \rho - n, n, p_1 - 2k - n \rangle$$ (4.30)

for the following two cases

- $p_2 < 0$ and $\Lambda_1 = \frac{\rho}{2} - \frac{1}{2} \rho$; $\Lambda_2 = \frac{\rho}{4} - \frac{1}{2} \rho$; $c(k,n)$ is given by (4.28).
- $p_2 = 0$ and $\Lambda_1 = \frac{\rho}{2} - \frac{1}{4} p_1$, $\forall \Lambda_2$

$$c_{(k,n)} = \frac{(-1)^n \left(\frac{p_2}{2}\right)!}{4^k k! n! \left(\frac{p_2}{2} - k - n\right)!} \frac{\Gamma(2\Lambda_2 - \frac{p_2}{2} - \frac{\mu}{\rho} + 2k + n)}{\Gamma(2\Lambda_2 - \frac{p_2}{2} - \frac{\mu}{\rho})}.$$ (4.31)

As in the previous case (iii) the proof is straightforward.

4.5. Case iv) (b) $0 < \rho < p_1, p_2 > 0$

There exists only one singular vector of the form

$$|\psi_{p_1^1 + p_2^2}^{p_1^0 + p_2^0} \rangle = \left(\sum_{k=0}^{\rho} \sum_{n=0}^{\rho - k} \sum_{k=\rho + 1}^{p_1} \sum_{n=0}^{p_1 - 2k} c_{(k,n)} |k, \rho - n, n, p_1 - 2k - n \rangle \right)$$ (4.32)

for $\Lambda_1 = \frac{\rho}{2} - \frac{1}{4} \rho$, $\Lambda_2 = \frac{\rho}{4} - \frac{1}{2} \rho$ and $c(k,n)$ is given by (4.28).

The proof is the same as the previous cases.

We successfully carried out the search of all singular vectors in $\mathcal{V}^\Lambda$. For the sake of simplicity and for the convenience of later computation, we change the notations of parameters specifying the lowest weight. By this change, all the new parameters $p_1^1, p_2^2, p_3^3, p_4^4, p_5^5$ and $q^3$ take a positive integer.

Case i) $p_1 \rightarrow p_1^1$ in (4.18)
Case ii) $p_2 \rightarrow 2p_2^2$ in (4.19)

These two cases corresponds to the type (i) (ii), respectively. We have the common constraint on $\Lambda_1, \Lambda_2$ and the common form of $c(k,n)$ for case iii) ($0 < p_1 < \rho$), case iv) (a) ($p_2 < 0$) and case (iv) (b) ($0 < \rho < p_1, p_2 > 0$). By the change

$$p_1 \rightarrow p_3^3, \quad \rho \rightarrow q^3$$ (4.33)
which leads $\bar{\rho} = p^3 - q^3$, $p_2 = 2q^3 - p^3$, these three cases correspond to the three subcases of the type (iii):

Case iii) $0 < p_1 < \rho \rightarrow 0 < p^3 < q^3$,

Case iv)(a) $p_2 < 0 \rightarrow 2q^3 < p^3$

Case iv)(b) $0 < \rho < p_1, 0 < p_2 \rightarrow q^3 < p^3 < 2q^3$. (4.34)

For case iii) with $p_1 = \rho$, we make the change $p_1 \rightarrow p_4$ and this corresponds to the type (iv).

Finally, for case iv) (a) with $p_2 = 0$ we make the change $p_1 \rightarrow 2p_5$ and this gives the type (v).

In this way, we have completed the proof of the result in section 4.1.

5. Elementary reducibilities

In the precious section we have established that there are five types of singular vectors. Here we shall study the cases when some Verma modules have SV of various types.

A12345. We start with Verma modules which are reducible under all five types. We denote these as A12345. We shall see that these are related to two more types: A234, which have SVs of types (ii), (iii), (iv); further: A2345 which have SVs of types (ii), (iii), (iv), (v).

We take the LW of type (iii) as representative and denote it $\Lambda^0$:

$$\Lambda^0 = \left( \frac{5}{4} - \frac{1}{2}(p^3 - q^3), \frac{3}{4} - \frac{q^3}{2} \right),$$

$$p^1 = p^3 - 2q^3, \quad p^2 = q^3, \quad p^4 = p^3, \quad p^5 = p^3 - q^3, \quad (5.1a)$$

where by $p^j$, $j = 1, 2, 4, 5$ we denote the parameters of the corresponding types of reducibility and we give how these are related with the parameters of type (iii) $p^3, q^3$.

We establish the following:

* $p^3 > 2q^3 \Rightarrow$ A12345

$$\Lambda^1 = \left( \frac{5}{4} - \frac{q^3}{2}, \frac{3}{4} - \frac{1}{2}(p^3 - q^3) \right). \quad (5.2)$$

* $q^3 > p^3 \Rightarrow$ A234

$$\Lambda^2 = \left( \frac{5}{4} + \frac{1}{2}(p^3 - q^3), \frac{3}{4} + \frac{q^3}{2} \right), \quad (5.3a)$$

$$\Lambda^3 = \left( \frac{5}{4} + \frac{q^3}{2}, \frac{3}{4} - \frac{1}{2}(p^3 - q^3) \right), \quad (5.3b)$$

$$\Lambda^4 = \left( \frac{5}{4} + \frac{q^3}{2}, \frac{3}{4} + \frac{1}{2}(p^3 - q^3) \right). \quad (5.3c)$$
\[ 2q^3 > p^3 > q^3 \Rightarrow A2345 \]

\[ \Lambda^0 = \left( \frac{5}{4} + \frac{1}{2}(p^3 - q^3), \frac{3}{4} - \frac{q^3}{2} \right). \] (5.4)

Since \( \Lambda^0 \) is the LW of case (iii), this result covers all the LW coincide with \( \Lambda^0 \), e.g. A13, A123.

A1245. Next we consider VMs which are reducible under conditions (i), (ii), (iv), (v) but not under case (iii). Actually, we start with the coincidence of the LWs for cases (ii) and (v) which gives the relations:

\[ \Lambda_1 = \frac{5}{4} - \frac{p^3}{2}, \]
\[ \Lambda_1 + \frac{1}{2}(p^3 - 1) = \frac{3}{4} - \frac{p^3}{2} = -\Lambda_1 + 2 - \frac{p^4}{2} = \Lambda_2. \] (5.5)

Take \( p^3, p^5 \) as independent parameters (i.e. the LW of A25), then

\[ \Lambda_{1245}^0 = \Lambda_{25}^0 = \left( \frac{5}{4} - \frac{p^3}{2}, \frac{3}{4} - \frac{p^5}{2} \right), \quad p^1 = p^5 - p^3, \quad p^4 = p^5 + p^2. \] (5.6)

We have still to exclude case (iii) so we divide into two cases:

(a) \( p^5 \neq p^2 \)

For this case, \( \Lambda_{1245}^0 \) takes a same value as \( \Lambda^0 \). This is seen by setting

\[ p^5 = p^3 - q^3 > 0, \quad p^3 = q^3 \] (5.7)

which brings \( \Lambda_{1245}^0 \) to the same form as \( \Lambda^0 \). The conditions \( p^1 > 0 \) and \( p^4 > 0 \) becomes

\[ p^1 = p^3 - 2q^3 > 0, \quad p^4 = p^3 > 0 \] (5.8)

which shows that A1245 corresponds to \( p^3 > 2q^3 \) case of \( \Lambda^0 \), i.e. A12345. Thus, A1245 for \( p^5 \neq p^2 \) is a subcase of A12345.

(b) \( p^5 = p^2 \)

For this case, \( \Lambda_{1245}^0 \) never coincide with \( \Lambda^0 \). Because the replacement (5.7) gives \( p^3 = 2q^3 \) which is not allowed for \( \Lambda^0 \). Also, \( p^1 \) is not admissible since \( p^1 = 0 \) for \( p^5 = p^2 \) (cf \( p^1 = 2p^2 \) is admissible). Namely, the case A1 is decoupled from A1245, since if we apply the A1 conditions to the signature of A245 we shall obtain A1 with \( p^1 = 0 \) which is not allowed.

Thus, we conclude:

The VM with the LW

\[ \Lambda_{245}^0 = \left( \frac{5}{4} - \frac{p^2}{2}, \frac{3}{4} - \frac{p^2}{2} \right) \] (5.9)
has the SVs of the weights:

\[
\Lambda_{1245}^1 = \left( \frac{5}{4} - \frac{p^2}{2}, \frac{3}{4} + \frac{p^3}{2} \right),
\]
\[
\Lambda_{1245}^2 = \left( \frac{5}{4} + \frac{p^2}{2}, \frac{3}{4} - \frac{p^3}{2} \right),
\]
\[
\Lambda_{1245}^3 = \left( \frac{5}{4} + \frac{p^2}{2}, \frac{3}{4} + \frac{p^3}{2} \right),
\]

where the type of SVs are (ii), (v), (iv), respectively. Since \( \Lambda_{1245}^0 = \Lambda_{25}^0 \), \( \Lambda_{25}^0 \) are the subcases of \( \Lambda_{12345}^0 \) or \( \Lambda_{245}^0 \).

**A124.** Next we try to obtain VMs which are reducible under conditions (i), (ii), (iv) but not under cases (iii), (v). Actually, we start with the coincidence of the LWs for cases (i) and (ii) which gives the relations:

\[
\Lambda_1 + \frac{1}{2}(p^1 - 1) = \frac{3}{4} - \frac{p^2}{2} = -\Lambda_1 + 2 - \frac{p^4}{2}.
\]

Take \( p^1, p^2 \) as independent parameters, then

\[
\Lambda_{124}^0 = \Lambda_{12}^0 = \left( \frac{5}{4} - \frac{1}{2}(p^1 + p^2), \frac{3}{4} - \frac{p^2}{2} \right), \quad p^4 = p^1 + 2p^2.
\]

Obviously, this is a subcase of \( \Lambda_{1245}^0 \). Also \( \Lambda_{12}^0 \) is a subcase of \( \Lambda_{1245}^0 \).

**A145.** Next we try to obtain VMs which are reducible under conditions (i), (iv), (v) but not under cases (ii), (iii). Actually, we start with the coincidence of the LWs for cases (iv) and (v) which gives the relations:

\[
\Lambda_1 = \frac{5}{4} - \frac{p^5}{2},
\]
\[
\Lambda_1 + \frac{1}{2}(p^1 - 1) = -\Lambda_1 + 2 - \frac{p^4}{2} = \Lambda_2.
\]

Take \( p^4, p^5 \) as independent parameters, then

\[
\Lambda_{145}^0 = \Lambda_{15}^0 = \left( \frac{5}{4} - \frac{p^5}{2}, \frac{3}{4} - \frac{1}{2}(p^4 - p^5) \right), \quad p^1 = 2p^5 - p^4.
\]

Compare this with \( \Lambda_{1245}^0 \), then one may see \( \Lambda_{145}^0 = \Lambda_{1245}^0 \) if \( p^4 - p^5 > 0 \). Because by setting \( p^5 = p^4 - p^5 > 0 \) we have:

\[
\Lambda_{145}^0 = \left( \frac{5}{4} - \frac{p^5}{2}, \frac{3}{4} - \frac{p^3}{2} \right), \quad p^1 = p^5 - p^2,
\]

which is identical to (5.6).
However, if \( p^4 \leq p^5 \) then (5.14) is never identical to \( \Lambda_{145}^0 \). Note that \( p^1 > 0 \) for \( p^4 \leq p^5 \) so that \( p^1 \) takes an admissible value. Thus, we shall use:

\[
\Lambda_{145}^0 = \left( \frac{5}{4} - \frac{p^5}{2}, \frac{3}{4} - \frac{1}{2}(p^4 - p^5) \right), \quad p^4 \leq p^5. \tag{5.15}
\]

We find:

The VM with the LW (5.15) has the SVs with weights:

\[
\begin{align*}
\Lambda^1_{145} &= \left( \frac{5}{4} + \frac{1}{2}(p^5 - p^4), \frac{3}{4} - \frac{p^5}{2} \right), \\
\Lambda^2_{145} &= \left( \frac{5}{4} - \frac{1}{2}(p^5 - p^4), \frac{3}{4} + \frac{p^5}{2} \right), \\
\Lambda^5_{145} &= \left( \frac{5}{4} + \frac{p^5}{2}, \frac{3}{4} - \frac{1}{2}(p^4 - p^5) \right),
\end{align*}
\tag{5.16a}
\]

where the type of SVs are (i), (iv), (v), respectively. The somewhat unnatural enumeration of \( \Lambda_{145}^0 \)'s is for later consistency with section 6. Since \( \Lambda_{45}^0 = \Lambda_{145}^0 \), \( \textbf{A45} \) with \( p^4 \leq p^5 \) always implies \( \textbf{A14} \).

\[\textbf{A14}.\] Next we try to obtain VMs which are reducible under conditions (i), (iv) but not under cases (ii), (iii), (v). After some analysis we find that the VM with the LW

\[
\Lambda^0_{14} = \left( 1 - \frac{q}{2}, \frac{1}{2}(1 + p^1 - q) \right), \quad p^1, q \in \mathbb{Z}_{\geq 1}, \ p^1 < 1 + 2q \tag{5.17}
\]

has the SVs of weights:

\[
\begin{align*}
\Lambda^1_{14} &= \left( 1 + \frac{1}{2}(p^1 - q), \frac{1}{2}(1 - q) \right), \\
\Lambda^4_{14} &= \left( \frac{1}{2}(3 - p^1 + q), 1 + \frac{q}{2} \right).
\end{align*}
\tag{5.18a}
\]

\[\textbf{A15}.\] Next we try to obtain VMs which are reducible under conditions (i), (v) but not under cases (ii), (iii), (iv). The coincidence of the LWs gives the relations:

\[
\Lambda_1 = \frac{5}{4} - \frac{p^5}{2}, \quad \Lambda_1 + \frac{1}{2}(p^1 - 1) = \Lambda_2. \tag{5.19}
\]

Thus

\[
\Lambda^0_{15} = \left( \frac{5}{4} - \frac{p^5}{2}, \frac{3}{4} - \frac{1}{2}(p^4 - p^5) \right). \tag{5.20}
\]

Comparing this with \( \Lambda_{145}^0 \) we see that if \( 2p^5 > p^4 \), then \( \Lambda^0_{15} = \Lambda^0_{145} \).
On the other hand, if \( 2p^5 \leq p^1 \), then \( \Lambda_{15}^0 \neq \Lambda_{145}^0 \) and \( \Lambda_{15}^0 \) is not identical to any LW considered so far. Thus, we shall use:

\[
\Lambda_{15}^0 = \left( \frac{5}{4} - \frac{p^5}{2}, \frac{3}{4} - \frac{1}{2}(p^2 - p^1) \right), \quad 2p^5 \leq p^1.
\]

We find:

The VM with LW (5.21) has SVs of weights:

\[
\Lambda_{15}^1 = \left( \frac{5}{4} + \frac{1}{2}(p^1 - p^5), \frac{3}{4} - \frac{p^5}{2} \right),
\]

\[
\Lambda_{15}^2 = \left( \frac{5}{4} + \frac{p^5}{2}, \frac{3}{4} - \frac{1}{2}(p^2 - p^1) \right).
\]

**A24.** Next we look for VMs which are reducible under conditions (ii), (iv) but not under cases (i), (iii), (v). The coincidence of the LWs gives the relation:

\[
\frac{3}{4} - \frac{p^2}{2} = -\Lambda_1 + 2 - \frac{p^4}{2}.
\]

Thus

\[
\Lambda_{24}^0 = \left( \frac{5}{4} + \frac{1}{2}(p^1 - p^5), \frac{3}{4} - \frac{p^5}{2} \right).
\]

Comparing this with \( \Lambda^0 \), we see that \( \Lambda_{24}^0 = \Lambda^0 \) if \( p^4 \neq p^2 \) or \( p^4 \neq 2p^2 \) by the correspondence

\[
p^4 \leftrightarrow p^3, \quad p^2 \leftrightarrow q^3.
\]

Thus if \( p^4 \neq p^2 \) or \( p^4 \neq 2p^2 \), then A24 is a subcase of A12345.

On the other hand for \( p^4 = 2p^2 \) we have:

\[
\Lambda_{24}^0 = \left( \frac{5}{4} \frac{3}{2}, \frac{3}{4} - \frac{p^2}{2} \right)
\]

which is identical to the LW of A245.

Finally, if \( p^4 = p^2 \), then the LW:

\[
\Lambda_{24}^0 = \left( \frac{5}{4} \frac{3}{2}, \frac{p^2}{2} \right)
\]

is not identical to any LW considered so far. Thus, we have:

The VM with the LW (5.27) has the SVs of weights:

\[
\Lambda_{24}^1 = \left( \frac{5}{4} \frac{3}{2} + \frac{p^2}{2} \right),
\]

\[
\Lambda_{24}^2 = \left( \frac{5}{4} \frac{3}{2} \right).
\]
A1. Next we try to obtain VMs which are reducible only under condition (i). We need to exclude all the coincidence with the type (i) LW. After some tedious analysis we obtain that it is enough to exclude coincidence with case A14. The latter is given by:

\[ \Lambda_1, \Lambda_1 + \frac{1}{2}(p^i - 1) \], \quad \Lambda_1 = \frac{1}{4}(5 - p^i - p^j). \quad (5.29) \]

Thus, we have:

The VM of LW

\[ \Lambda_0^0 = \left( \Lambda_1, \Lambda_1 + \frac{1}{2}(p^i - 1) \right), \quad (5.30a) \]
\[-4\Lambda_1 + 5 - p^i \notin \mathbb{N} \quad (5.30b)\]

has only the type (i) SV of weight:

\[ \Lambda_1^1 = \left( \Lambda_1 + \frac{p^i}{2}, \Lambda_1 - \frac{1}{2} \right). \quad (5.31) \]

(The condition in (5.30b) excludes the SVs of type (iv)—the excluded positive integer would be identified with \( p^j \).)

A2. Next we try to obtain VMs which are reducible only under conditions (ii). We need to exclude all the coincidence with the type (i) LW. After some tedious analysis we obtain that it is enough to exclude coincidence with case A24. The latter is given by:

\[ \left( \Lambda_1, \frac{3}{4} - \frac{p^2}{2} \right). \quad (5.32) \]

Thus, we have:

The VM of LW

\[ \Lambda_0^0 = \left( \Lambda_1, \frac{3}{4} - \frac{p^2}{2} \right), \quad (5.33a) \]
\[-2\Lambda_1 + \frac{5}{2} + p^2 \notin \mathbb{N} \quad (5.33b)\]

has only the type (ii) SV of weight:

\[ \Lambda_1^1 = \left( \Lambda_1, \frac{3}{4} + \frac{p^2}{2} \right). \quad (5.34) \]

(The condition in (5.33b) excludes the SVs of type (iv).)

A3. We have seen that the LW \( \Lambda_0^0 \) has at least A234 (see the part A12345), so there exists no VM having only type (iii) SV.
A4. Next we try to obtain VMs which are reducible only under conditions (iv). We need to exclude all the coincidence with the type (iv) LW. After some tedious analysis we obtain that it is enough to exclude coincidence with case A14. The latter is given by:

\[
(\Lambda_1, -\Lambda_1 + 2 - \frac{p^4}{2}), \quad \Lambda_1 = \frac{1}{4}(5 - p^1 - p^4).
\]  

(5.35)

Thus, we have

The VM of the LW

\[
\Lambda^0_4 = \left(\Lambda_1, -\Lambda_1 + 2 - \frac{p^4}{2}\right),
\]

\[-4\Lambda_1 + 5 - p^4 \notin \mathbb{N}
\]  

(5.36a)

(5.36b)

has only the type (iv) SV of weight:

\[
\Lambda^1_4 = \left(\Lambda_1 + \frac{p^4}{2}, -\Lambda_1 + 2\right).
\]  

(5.37)

(The condition in (5.36b) excludes the SVs of type (i).)

A5. Next we try to obtain VMs which are reducible only under conditions (v). We need to exclude all the coincidence with the type (v) LW. After some tedious analysis we obtain that it is enough to exclude coincidence with cases A15 and A45 since they contain all other cases. With an abuse of notation we represent them as:

\[
\Lambda^0_{15} \cup \Lambda^0_{45} = \left(\frac{5}{4} - \frac{p^5}{2}, \frac{3}{4} - \frac{r}{2}\right), \quad r \in \mathbb{Z}.
\]  

(5.38)

The relation which gives the coincidence with the type (v) LW

\[
\left(\frac{5}{4} - \frac{p^5}{2}, \Lambda_2\right) = \Lambda^0_{15} \cup \Lambda^0_{45}
\]  

(5.39)

is reduced to \(\Lambda_2 = \frac{3}{4} - \frac{r}{2}\).

Thus, we have:

The VM of LW

\[
\Lambda^0_5 = \left(\frac{5}{4} - \frac{p^5}{2}, \Lambda_2\right), \quad \Lambda_2 \neq \frac{3}{4} - \frac{r}{2}, \quad r \in \mathbb{Z}
\]  

(5.40)

has only the type (v) SV of weight:

\[
\Lambda^1_5 = \left(\frac{5}{4} + \frac{p^5}{2}, \Lambda_2\right).
\]  

(5.41)
In conclusion, we have the following types of elementary embeddings: A12345, A234, A2345, A245, A145, A14, A15, A24, A1, A2, A4, A5, which are distinguished from one another by conditions given above. In the next section we shall find their complete embedding pictures.

6. Complete embedding pictures

We shall find the complete embedding pictures of the reducible Verma modules. These are obtained using the elementary picture from section 4.1 not only to the initial modules but also to their invariant submodules.

6.1. Case A12345

As we have seen the VM which is reducible under case (iii) may have all five reducibilities (some under some conditions). The initial embedding picture is as follows:

where the LWs $\Lambda^a$, $a = 0, 1, \ldots, 5$ are given in (5.1)–(5.4). We now obtain the embeddings of each of these five cases.

6.1.1. Invariant modules in $V^{\Lambda^1}$. After some analysis we find that $V^{\Lambda^1}$ has the type (ii), (iii), (iv) and (v) SVs for all possible values of $p^3, q^3$. Their weights are given by $\Lambda^{12}, \Lambda^5, \tilde{\Lambda}^0, \Lambda^3$, respectively, where:

$$\Lambda^{12} = \left(\frac{5}{4} - \frac{q^3}{2}, \frac{3}{4} + \frac{1}{2}(p^3 - q^3)\right),$$

$$\tilde{\Lambda}^0 = \left(\frac{5}{4} + \frac{1}{2}(p^3 - q^3), \frac{3}{4} + \frac{q^3}{2}\right),$$

and use notation $\tilde{\Lambda}^0$ for the LW which differs from $\Lambda^0$ by the signs of the additions to $5/4, 3/4$.

Next, we seek SVs in $V^{\Lambda^{12}}$. We find that it has types (i) and (v) SVs for all possible values of $p^3, q^3$ and the weights of SVs are given by respectively by $\Lambda^5, \Lambda^4$.

Finally, we seek SVs in $V^{\Lambda^3}$. We find that it has no SVs.
The results are summarized in the diagram:

6.1.2. Invariant modules in $V^{\Lambda^2}$. After some analysis we find that $V^{\Lambda^2}$ has the following SVs:

- $2q^3 < p^3 \Rightarrow$ type (i) (iv) (v) SVs of weights $\Lambda^3, \Lambda^{12}, \tilde{\Lambda}^0$.
- $q^3 < p^3 < 2q^3 \Rightarrow$ type (i) (v) SVs of weights $\Lambda^3$ and $\tilde{\Lambda}^0$, respectively.
- $p^3 < q^3 \Rightarrow$ type (i) SV of weight $\Lambda^3$.

Further embedding in $V^{\Lambda^{12}}$ when $2q^3 < p^3$ are given by type (i) and (v) SVs of weight $\Lambda^5$ and $\Lambda^4$, respectively.

Further embedding in $V^{\tilde{\Lambda}^0}$ where $q^3 < p^3$ is given by type (i) SV of the weight $\Lambda^4$.

There are no further embeddings and thus, the embedding picture of $V^{\Lambda^2}$ is given as follows:

6.1.3. Invariant modules in $V^{\Lambda^3}$. After some analysis we find that $V^{\Lambda^3}$ has the following SVs:

- $2q^3 < p^3 \Rightarrow$ type (ii) (iii) (iv) SVs of weights $\Lambda^4, \tilde{\Lambda}^0, \Lambda^5$, resp.
- $q^3 < p^3 < 2q^3 \Rightarrow$ type (ii) SV of weights $\Lambda^4$. 

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There are no further embeddings. Thus, the embedding picture of $V^\Lambda^3$ is given as follows:

6.1.4. Invariant submodules in $V^\Lambda^4$. After some analysis we find that $V^\Lambda^4$ has the following SVs:

- $2q^3 < p^3 \Rightarrow$ type (i) SV of weight $\tilde{\Lambda}^0$;
- $p^3 < q^3 \Rightarrow$ type (ii) SV of weight $\Lambda^3$;

There are no further embeddings. Thus, the embedding picture of $V^\Lambda^4$ is given as follows:

6.1.5. Invariant modules in $V^\Lambda^5$. After some analysis we find that $V^\Lambda^5$ has the following SVs:

- $q^3 < p^3 < 2q^3 \Rightarrow$ type (ii) (iii) (iv) SVs of weights $\tilde{\Lambda}^0$, $\Lambda^4$, $\Lambda^3$, resp.
- $2q^3 < p^3 \Rightarrow$ type (ii) SV of weight $\tilde{\Lambda}^0$.

Further embedding in $V^{\tilde{\Lambda}^0}$ where $q^3 < p^3$ shows that it has the type (i) SV of weight $\Lambda^4$ for $q^3 < p^3 < 2q^3$.

There are no further embeddings. Thus, the embedding picture of $V^\Lambda^5$ is given as follows:

This completes the embeddings in $\textbf{A12345}$. 
6.1.6. Complete diagram for A12345 and subcases. Doing the complete diagram it may be better to distinguish the cases, since pairs of modules have submodules of opposite directions depending on the values of the parameters $p^3, q^3$. The distinction of cases was already done in (5.2)–(5.4) and so we proceed:

**A234.** The LW is $\Lambda^0$ with the constraint $p^3 < q^3$ which we denote by $\Lambda^0_{234}$. For the constraint $p^3 < q^3$ we have the following embedding patterns.

![Diagram](image)

$V^{\Lambda^1}$ does not have SV. Combining these diagrams, we obtain the following diagram.

![Combined Diagram](image) (6.2)

**A2345.** The LW is $\Lambda^0$ with the constraint $q^3 < p^3 < 2q^3$ which we denote by $\Lambda^0_{2345}$. For the constraint $q^3 < p^3 < 2q^3$ we have the following embedding patterns.

![Diagram](image)
$V^{A^4}$ does not have SV.

Combining these diagrams, we obtain the following embedding diagram:

$A12345$. The LW is $\Lambda^0$ with the constraint $2q^3 < p^3$ for which we keep using the notation $\Lambda^0$. For the constraint $2q^3 < p^3$ we have the following embedding patterns:
$\tilde{\Lambda}_0$ does not have SV.
Combining these diagrams, we obtain the complete embedding diagram of this case:
6.2. Case A245

In (5.10) we have found the initial embeddings of this case which are given by the diagram:

\[
\begin{array}{c}
\Lambda_{245}^0 \\
(\text{(ii)}) \\
\Lambda_{245}^1 \\
(\text{(iv)}) \\
\Lambda_{245}^2 \\
(\text{(v)}) \\
\Lambda_{245}^3
\end{array}
\]

where \(\Lambda_{245}^1, \Lambda_{245}^2, \Lambda_{245}^3\) are given in (5.10).

Next we look for further embeddings in the above invariant submodules. After some analysis we find that the VM \(V^{\Lambda_{245}^1}\) has the type (i) and (v) SVs of weights \(\Lambda_{245}^2, \Lambda_{245}^3\), thus we have:

\[
\begin{array}{c}
\Lambda_{245}^1 \\
(\text{(i)}) \\
\Lambda_{245}^2 \\
(\text{(v)}) \\
\Lambda_{245}^3
\end{array}
\]

Further we find that the VM \(V^{\Lambda_{245}^3}\) has no SV.

Next we find that the VM \(V^{\Lambda_{245}^2}\) has the type (ii) SV of weight \(\Lambda_{245}^3\).
Combining the last three subcases we give the complete diagram for $A_{245}$:

$$
\lambda_{245}^0 \quad \lambda_{245}^1 \quad \lambda_{245}^2 \quad \lambda_{245}^3
$$

6.3. Case A145

In (5.16) we have found the initial embeddings of this case which are given by the diagram:

$$
\lambda_{145}^0 \quad \lambda_{145}^1 \quad \lambda_{145}^2 \quad \lambda_{145}^3
$$

where $\lambda_{145}^1, \lambda_{145}^2, \lambda_{145}^3$ are given in (5.16). Next we find the further embeddings in the above invariant submodules.

First we find that the VM $\lambda_{145}^1$ has the type (iii) (ii) (iv) SVs respectively of weights $\lambda_{145}^5$ and:

$$
\begin{align*}
\lambda_{145}^3 &= \left( \frac{5}{4} + \frac{1}{2}(p^5 - p^4), \frac{3}{4} + \frac{p^5}{2} \right), \\
\lambda_{145}^4 &= \left( \frac{5}{4} + \frac{p^5}{2}, \frac{3}{4} - \frac{1}{2}(p^5 - p^4) \right).
\end{align*} \quad (6.6a, 6.6b)
$$

Further we find that the VMs $\lambda_{145}^3$ and $\lambda_{145}^4$ have the type (i) SV of weight $\lambda_{145}^5$. 

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Thus, the embedding picture for $\Lambda_{145}^1$ is:

Next we find that the VM $\Lambda_{145}^2$ has the types (i) and (v) SVs of weights (6.6b), (6.6a) (for $p^4 < p^5$), respectively. Thus, we have the diagram:

Finally we find that the VM $\Lambda_{145}^3$ has no SV. Thus, the complete diagram for $\Lambda_{145}$ is:

(6.7)
6.4. Case A14

In (5.18) we have found the initial embeddings of this case which are given by the diagram:

see (5.17) and (5.18) for $\Lambda_{a}^{14}$. Further we distinguish two cases depending whether $p^j$ is bigger or smaller than $\frac{1}{2} + q$ since the embedding diagrams are different.

Case 1: $p^j < \frac{1}{2} + q$ we have the following embeddings:

where

\[
\Lambda_{14}^{2} = \left(1 + \frac{1}{2}(p^j - q), 1 + \frac{q}{2}\right), \quad (6.8a)
\]

\[
\Lambda_{14}^{3} = \left(\frac{3}{2} + \frac{1}{2}(1 + p^j - q)\right), \quad (6.8b)
\]

\[
\Lambda_{14}^{5} = \left(\frac{2}{2} + \frac{1}{2}, 1 - \frac{1}{2}(p^j - q)\right), \quad (6.8c)
\]

\[
\Lambda_{14}^{6} = \left(\frac{3}{2} - \frac{1}{2}(p^j - q), \frac{1}{2}(1 - q)\right). \quad (6.8d)
\]
The complete embedding pattern of case 1 is:

\[
\begin{align*}
\Lambda_1^1 & \rightarrow \Lambda_1^0, \\
\Lambda_1^2 & \rightarrow \Lambda_1^3, \\
\Lambda_1^3 & \rightarrow \Lambda_1^4, \\
\Lambda_1^4 & \rightarrow \Lambda_1^5.
\end{align*}
\] (6.9)

**Case 2:** $\frac{1}{2} + q < p^1$ we have the following embeddings. The LWs are the same as case 1.)
The complete embedding pattern of case 2 is:

\[
\Lambda^1_{14} \xrightarrow{(i)} \Lambda^9_{14} \xrightarrow{(iv)} \Lambda^4_{14} \xrightarrow{(v)} \Lambda^1_{14} \xrightarrow{(i)} \Lambda^3_{14} \xrightarrow{(ii)} \Lambda^5_{14} \xrightarrow{(iii)} \Lambda^1_{14} \,
\]

(6.10)

6.5. Case A15

In (5.22) we have found the initial embeddings of this case which are given by the diagram:

\[
\Lambda^0_{15} \xrightarrow{(i)} \Lambda^1_{15} \xrightarrow{(v)} \Lambda^2_{15}
\]

Next we find the further embeddings in the above invariant submodules. We find that \(\Lambda^1_{15}\) has the type (ii) SV of weight:

\[
\Lambda^3_{15} = \left( \frac{5}{4} + \frac{1}{2}(p^l - p^s), \frac{5}{4} + \frac{1}{2}p^s \right).
\]

(6.11)

Next we find that \(\Lambda^3_{15}\) has the type (i) SV if \(2p^s < p^l\) and its weight is given by (6.11). We also find that \(\Lambda^3_{15}\) has no SV.
Finally, we find the complete diagram for A15:

\[ \Lambda^0_{15} \rightarrow \Lambda^1_{15} \rightarrow \Lambda^2_{15} \rightarrow (\text{diagram}) \]

(6.12)

6.6. Case A24

In (5.28) we have found the initial embeddings of this case which are given by the diagram:

\[ \Lambda^0_{24} \rightarrow \Lambda^1_{24} \rightarrow \Lambda^2_{24} \rightarrow (\text{diagram}) \]

Next we find the further embeddings in the above invariant submodules. We find that \( \Lambda^1_{24} \) has the type (i) SV of weight \( \Lambda^2_{24} \), while \( \Lambda^1_{24} \) has no SV. Thus, the complete diagram for A24 is:

\[ \Lambda^0_{24} \rightarrow \Lambda^1_{24} \rightarrow \Lambda^2_{24} \rightarrow (\text{diagram}) \]

(6.13)

6.7. Cases A1, A2, A4, A5

In section 5 we have found the initial embeddings of these cases which are given by the diagram:
A1, A2, A4, A5

\[ \Lambda_{a}^{0} \xrightarrow{(a)} \Lambda_{a}^{1} \quad \alpha = 1, 2, 4, 5 \]

Next we find the further embeddings in the above invariant submodules.

6.7.1. Embedding diagram of \( \mathcal{V}_{A1}^{q} \) (case A1). We first recall from section 5 the weights in the initial embedding diagram above for this case:

\[ \Lambda_{i}^{0} = \left( \Lambda_{1}, \Lambda_{1} + \frac{1}{2} (p^{1} - 1) \right), \quad \Lambda_{1} \neq \frac{1}{4} (5 - p^{1} - p^{2}) \] (6.14a)

\[ \Lambda_{i}^{1} = \left( \Lambda_{1} + \frac{p^{1}}{2}, \Lambda_{1} - \frac{1}{2} \right). \] (6.14b)

We draw the embedding diagram by distinguishing the four cases.

Case (1) \( 5 \leq 4 \Lambda_{1} \)
\( \Lambda_{1} \) is irreducible.
Case (2) \( 4 \Lambda_{1} < 5 - 2p^{1} \)
We have the following embedding patterns:

\[ \Lambda_{1}^{1} \]

\[ \Lambda_{1}^{2} \]

\[ \Lambda_{1}^{3} \]

\[ \Lambda_{1}^{4} \]

\[ \Lambda_{1}^{5} \]

\[ \Lambda_{1}^{6} \]

where

\[ \Lambda_{1}^{1} = \left( \Lambda_{1} + \frac{p^{1}}{2}, -\Lambda_{1} + 2 \right) \] (6.15a)

\[ \Lambda_{1}^{2} = \left( -\Lambda_{1} + \frac{5}{2}, \Lambda_{1} + \frac{1}{2} (p^{1} - 1) \right) \] (6.15b)

\[ \Lambda_{1}^{3} = \left( -\Lambda_{1} + \frac{5}{2}, -\Lambda_{1} - \frac{p^{1}}{2} + 2 \right) \] (6.15c)

\[ \Lambda_{1}^{4} = \left( -\Lambda_{1} - \frac{1}{2} (p^{1} - 5), \Lambda_{1} - \frac{1}{2} \right) \] (6.15d)
\[ \Lambda_6^i = \left( -\Lambda_1 - \frac{1}{2}(p^i - 5), -\Lambda_1 + 2 \right). \quad (6.15e) \]

Thus the complete pattern for \( A_1 \) case 2 is given as follows:

![Diagram](image)

Case (3) \( 5 - 2p^1 < 4\Lambda_1 < 5 - p^i \)

In this case, the embedding pattern is simplified:

![Diagram](image)
Then the complete embedding pattern A1 case 3 is given by:

\[
\Lambda_1^0 \
\Lambda_1^1 \\
\Lambda_1^2 \\
\Lambda_1^3 \\
\Lambda_1^4
\]

(6.17)

Case (4) \( 5 - p^1 < 4\Lambda_1 < 5 \)
In A1 case 4 the embedding pattern is very simple:

\[
\Lambda_1^0 \quad (i) \quad \Lambda_1^1 \quad (ii) \quad \Lambda_1^2
\]

(6.18)

Note that the diagram of case (4) may be considered as part of the diagram of case (3) taking into account only the VMs \( \Lambda_0^1, \Lambda_1^1, \Lambda_2^1 \) and the two embeddings connecting them, also noting in for case (4) the VM \( \Lambda_1^4 \) is irreducible.

6.7.2. Embedding diagram of \( V^{A_2^3} \) (case A2). We first recall from (5.33) the weights in the initial embedding diagram above for this case:

\[
\Lambda_2^0 = \left( \Lambda_1, \frac{3}{4} - \frac{p^2}{2} \right), \\
-2\Lambda_1 + \frac{5}{2} + p^2 \notin \mathbb{N}
\]

(6.19a)

\[
\Lambda_2^1 = \left( \Lambda_1, \frac{3}{4} + \frac{p^2}{2} \right).
\]

(6.19b)

We draw the embedding diagrams by distinguishing the four cases.
Case (1) \( 5 + 2p^2 \leq 4\Lambda_1 \)
\( \Lambda_3^1 = \Lambda_3^{(ii)} \) is irreducible.
Case (2) \( 4\Lambda_1 < 5 - 2p^2 \)
We have the following patterns:

\[ \Lambda_2^1 = \left( \frac{5}{4} + \frac{p^2}{2}, \Lambda_1 - \frac{1}{2} \right), \]  
\[ \Lambda_2^3 = \left( \frac{5}{4} - \frac{p^2}{2}, -\Lambda_1 + 2 \right), \]  
\[ \Lambda_2^6 = \left( -\Lambda_1 + \frac{5}{2}, -\Lambda_1 + \frac{3}{4} + \frac{p^2}{2} \right), \]  
\[ \Lambda_2^4 = \left( \frac{5}{4} + \frac{p^2}{2}, -\Lambda_1 + 2 \right), \]  
\[ \Lambda_2^5 = \left( -\Lambda_1 + \frac{5}{2}, -\Lambda_1 + \frac{3}{4} - \frac{p^2}{2} \right). \]
Thus the complete embedding pattern of $A_2$ case 2 is given by:

\[
\text{(6.25)}
\]

Case (3) $5 - 2p^2 < 4\Lambda_1 < 5$

Here for $A_2$ case 3 the complete embedding pattern is simple:

\[
\text{(6.26)}
\]

Case (4) $5 < 4\Lambda_1 < 5 + 2p^2$
For A2 case 4 the complete embedding pattern is simple:

\[
\Lambda_2^0 \xrightarrow{(i)} \Lambda_2^5 \xrightarrow{(j)} \Lambda_2^2
\]  (6.27)

Like A1, one may combine cases (3) and (4) into single diagram.

6.7.3. Embedding diagram of \( V^{\Lambda_4} \) (case A4). We first recall from (5.36) the weights in the initial embedding diagram above for this case:

\[
\Lambda_4^0 = \left( \Lambda_1, -\Lambda_1 + 2 - \frac{p^4}{2} \right),
\]

\[-4\Lambda_1 + 5 - p^4 \notin \mathbb{N}, \]  (6.28a)

\[
\Lambda_4^1 = \left( \Lambda_1 + \frac{p^4}{2}, -\Lambda_1 + 2 \right). \]  (6.28b)

We find that \( V^{\Lambda_4} \) has the following SVs:

- \( 5 < 4\Lambda_4 \Rightarrow \) type (ii) SV of weight

\[
\Lambda_5^2 = \left( \Lambda_1 + \frac{p^4}{2}, \Lambda_1 - \frac{1}{2} \right). \]  (6.29)

- \( 4\Lambda_4 < 5 - 2p^4 \Rightarrow \) type (v) SVs of weights:

\[
\Lambda_3^4 = \left( -\Lambda_1 + \frac{5}{2}, \frac{1}{2}(p^4 - 1) \right), \]  (6.30a)

\[
\Lambda_4^4 = \left( -\Lambda_1 - \frac{1}{2}(p^4 - 5), -\Lambda_1 + 2 \right). \]  (6.30b)

- \( 5 - 2p^4 < 4\Lambda_4 < 5 - p^4 \Rightarrow \) type (i) SV of weight \( \Lambda_4^3 \).

Further embeddings for case A4:

We find that for \( 5 < 4\Lambda_4 \) the VM \( \Lambda_4^2 \) has no SV.

Next we find that if \( 4\Lambda_4 < 5 - p^4 \) the VM \( \Lambda_4^3 \) has the type (ii) SV of weight

\[
\Lambda_4^5 = \left( -\Lambda_1 + \frac{5}{2}, -\Lambda_1 - \frac{p^4}{2} + 2 \right). \]  (6.31)

Next we find that the VM \( \Lambda_4^4 \) has the type (i) SV of weight (6.31).

Finally, we find that if \( 4\Lambda_4 < 5 - 2p^4 \) the VM \( \Lambda_4^5 \) has no SV.

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Thus, we find the complete diagram for $A_4$:

$$
\Lambda_4^0 \quad \Lambda_4^1 \quad \Lambda_4^2 \quad \Lambda_4^3 \quad \Lambda_4^4 \quad \Lambda_4^5
$$

\begin{align*}
\Lambda_4^0 & \quad 5 < 4\Lambda_1 \quad \text{(iv)} \\
\Lambda_4^1 & \quad 4\Lambda_1 < 5 - p^4 \quad \text{(i)} \\
\Lambda_4^2 & \quad 4\Lambda_1 < 5 - 2p^4 \quad \text{(ii)} \\
\Lambda_4^3 & \quad \Lambda_4^4 \\
\Lambda_4^5 & \quad 4\Lambda_1 < 5 - 2p^4 \quad \text{(ii)} \\
\end{align*}

(6.32)

6.74. Embedding diagram of $V^{\Lambda_0}$ (case A5). We first recall from (5.40) the weights in the initial embedding diagram above for this case:

$$
\Lambda_0^0 = \left( \frac{5}{4} - \frac{p^5}{2}, \Lambda_2 \right),
$$

$$
\Lambda_2 \neq \frac{3}{4} - \frac{r}{2}, \quad r \in \mathbb{Z}
$$

(6.33a)

$$
\Lambda_3^1 = \left( \frac{5}{4} + \frac{p^5}{2}, \Lambda_2 \right)
$$

(6.33b)

We find that $V^{\Lambda_0}$ has the following SVs:

- $3 + 2p^5 < 4\Lambda_2 \Rightarrow$ type (i) SV of weight

$$
V^{\Lambda_0} = \left( \Lambda_2 + \frac{1}{2}, \frac{3}{4} + \frac{p^5}{2} \right).
$$

(6.34)

- $4\Lambda_2 < 3 - 2p^5 \Rightarrow$ type (ii) (iii) (iv) SVs of weights

$$
V^{\Lambda_1} = \left( \frac{5}{4} + \frac{p^5}{2}, -\Lambda_2 + \frac{3}{2} \right),
$$

(6.35a)

$$
V^{\Lambda_1} = \left( -\Lambda_2 + 2, \frac{3}{4} + \frac{p^5}{2} \right).
$$

(6.35b)
\[ V^{A_3^3} = \left( -\Lambda_2 + 2, \frac{3}{4} - p^5 \right) . \] (6.35c)

\[ 3 - 2p^5 < 4\Lambda_2 < 3 \Rightarrow \text{type (ii) SV of weight } \Lambda_3^3. \]

Next we find that if \( 3 + 2p^5 < 4\Lambda_2 \) the VM \( \Lambda_3^3 \) has no SV.

Next we find that if \( 4\Lambda_2 < 3 \) the VM \( \Lambda_3^3 \) has the type (i) SV of weight (6.35b).

Next we find that if \( 4\Lambda_2 < 3 - 2p^5 \) the VM \( \Lambda_3^3 \) has no SV.

Next we find that if \( 4\Lambda_2 < 3 - 2p^5 \) the VM \( \Lambda_3^3 \) has the type (ii) SV of weight (6.35b).

Combining all results for case A5 we find that it has the following diagram:

![Diagram](image)

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Data availability statement

No new data were created or analysed in this study.

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