HIGHER SCHLÄFLI FORMULAS AND APPLICATIONS II
VECTOR-VALUED DIFFERENTIAL RELATIONS

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ABSTRACT. The classical Schlafli formula, and its “higher” analogs given in [SS03], are relations between the variations of the volumes and “curvatures” of faces of different dimensions of a polyhedra (which can be Euclidean, spherical or hyperbolic) under a first-order deformation. We describe here analogs of those formulas which are vector-valued rather than scalar. Some consequences follow, for instance constraints on where cone singularities can appear when a constant curvature manifold is deformed among cone-manifolds.

1. INTRODUCTION AND RESULTS

1.1. The Schlafli formula. The celebrated Schlafli formula (see e.g. [Mil94]) relates in a simple way the variations of the volume and of the dihedral angles, at codimension 2 faces, of a polyhedron under a first-order deformation. In a $n+1$-dimensional space of constant curvature $K$, the formula reads as:

$$K dV = \frac{1}{n} \sum_e W_e d\theta_e ,$$

where the sum is over the codimension 2 faces of the polyhedron, $W_e$ is the volume of the face, and $\theta_e$ is the interior dihedral angle at that face.

This simple formula is a key tool in different fields of mathematics, where it has found and still finds important applications, for instance in hyperbolic geometry (see e.g. [Bon98, BP01, BLP05]) or in discrete geometry (see e.g. [BC02, BC04, C06]). Smooth versions of the Schlafli formula have also appeared as interesting tools for hyperbolic 3-manifolds or higher dimensional Einstein manifolds (e.g. in [RS99, And01, KS06]).

The Schlafli formula is also important in parts of physics, where it is known as the “Regge formula” and is a key tool in the “Regge calculus”, a discretization of gravity based on simplicial metrics (see [Reg61], or [CMS84] for some related mathematical questions).

1.2. The higher Schlafli formulas. In a previous paper [SS03] we gave “higher” analogs of the Schlafli formula: for each $1 \leq p \leq n - 1$,

$$K(n - p) \sum_j K_j dW_j + p \sum_i W_i dK_i = 0 ,$$

where $j$ runs over the faces of codimension $p$ and $i$ runs over those of codimension $p + 2$, $W_i$ and $W_j$ denote the volumes of the faces, and $K_i$, $K_j$ their curvatures.

Our goal here is to give extensions of those formulas which, rather than being scalar equations as the Schlafli formula above, are vector-valued. We will also derive some consequences. The new formulas are not only extensions of those earlier ones, but also explain them, since the scalar (higher) Schlafli formulas follow from the vector-valued ones by a simple translation invariance argument. The vector-valued formulas presented here contain much more information, however, on the possible variations of the face areas or curvatures of a polyhedron under a deformation.

1.3. A simple example. We state first an elementary example, which should help understand the higher-dimensional cases considered below.

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Conversely, any first-order variation \( \frac{d}{dt} \sum_{i=1}^{n} v_i d\theta_i - w_i dl_i = 0 \).

The special case of isometric deformations — for which the terms \( l_i' \) are zero — has been well-known for some time, see e.g. [Ale05, Glu75]. The proof, which is given in section 7, follows from Theorem 1.8 below and from some simple dimension-counting arguments. The only part of this statement that will be generalized in higher dimensions is the first part, namely the vector-valued equation which holds during deformations.

It is interesting to note that this formula appears in the context of classifying constant mean curvature (CMC) surfaces in the Euclidean space. It was indeed derived independently by K. Grosse-Brauckmann, N. Korevaar, R. Kusner, J. Ratzkin and J. Sullivan and used to estimate the dimension of the space of appropriate \( \text{CMC} \) surfaces in the Euclidean space. It was indeed derived independently by K. Grosse-Brauckmann, N. Korevaar, R. Kusner, J. Ratzkin and J. Sullivan and used to estimate the dimension of the space of appropriate \( \text{CMC} \) surfaces in the Euclidean space.

Very similar statements hold in the hyperbolic and in the de Sitter space, the Euclidean 3-dimensional space appearing in Theorem 1.1 is then replaced by the 3-dimensional Minkowski space. A similar statement also holds for Euclidean polygons in the plane, we state it here since it is slightly different from the spherical statement.

**Theorem 1.2.** Let \( p \) be a polygon in \( \mathbb{R}^2 \), with vertices \( v_1, v_2, \ldots, v_n \), with edge lengths \( l_1, \ldots, l_n \neq 0 \) and (exterior) angles \( \theta_1, \ldots, \theta_n \). For each \( i \in \{1, \ldots, n\} \), let \( v_i \) be the point in \( \mathbb{R}^2 \) which is dual to the oriented edge \( e_i := (v_i, v_{i+1}) \). Under any first-order deformation of \( p \), the variations of its edge lengths and dihedral angles satisfy the equations:

1. \( \sum_{i=1}^{n} l_i' = 0 \).
2. \( \sum_{i=1}^{n} l_i' v_i - l_i' n_i = 0 \).

Conversely, any first-order variation \( (l_1', \ldots, l_n', \theta_1', \ldots, \theta_n') \) which satisfies those equations corresponds to a first-order deformation of \( p \), which is uniquely defined (up to the addition of a trivial deformation).

1.4. **Some definitions.** In the sequel we view \( S^{n+1} \) as the unit sphere centered at the origin in \( \mathbb{R}^{n+2} \) and we let \( x \) denote the position vector in \( \mathbb{R}^{n+2} \).

**Definition 1.3.** Let \( F \) be a face of \( P \). Then:

\[
\pi(F) := \int_F x dv \in \mathbb{R}^{n+2},
\]

while \( V(F) \) is the \( n+1-k \)-dimensional volume of \( F \).

So \( \pi(F)/V(F) \) is the barycenter of \( F \), considered as a vector of \( \mathbb{R}^{n+2} \). We also recall here the classical notion of polar duality for spherical polyhedra.

**Definition 1.4.** Let \( P \subset S^{n+1} \) be a (convex) polyhedron, the dual polyhedron \( P^* \) is the set of points \( x \in S^{n+1} \) such that \( P \) is contained in the half-space bounded by the hyperplane orthogonal to \( x \) and not containing \( x \). Given a face \( F \) of \( P \), the face of \( P^* \) dual to \( F \) is the set of points \( x \in S^{n+1} \) such that the oriented plane orthogonal to \( x \) is a support plane of \( P \) at \( F \).

It is well-known that \( P^* \) is combinatorially dual to \( P \) — faces of dimension \( k \) of \( P \) are dual to faces of dimension \( n-k \) of \( P^* \) — and that \( (P^*)^* = P \).

Very similar definitions can be used for polyhedra in the hyperbolic space. Instead of considering \( S^{n+1} \subset \mathbb{R}^{n+2} \), one then considers \( H^{n+1} \) as a quadric (a so-called pseudo-sphere) in the \( n+2 \)-dimensional Minkowski space. The dual polyhedron is defined as in the spherical case, but it is contained in de Sitter space \( S_1^{n+1} \), a simply connected (for \( n \geq 2 \)) Lorentz space of constant curvature 1 defined as:

\[
S_1^{n+1} = \{ x \in \mathbb{R}_1^{n+2} \mid \langle x, x \rangle = 1 \}.
\]

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1We are grateful to R. Kusner for pointing out those references.
More details can be found in particular in [RH93] (or, for smooth surfaces, in [Sch06]).

1.5. The first formula. We can now state the most general form of the first vector-valued Schläfli-type formulas presented here.

We consider convex polyhedra $P$ in a constant curvature space $M^{n+1}_K$, with $K \in \{-1, 1\}$, so that $M^{n+1}_K$ is either the hyperbolic space or the sphere. For each $p \in \{0, \cdots, n+1\}$, we denote by $F_p$ the set of faces of codimension $p$ of $P$. Similar formulas for the Euclidean space are mentioned below. We will use a simple notation.

**Definition 1.5.** Let $P$ be a spherical (resp. hyperbolic) polyhedron, let $F$ be a face of $P$. We call $\Pi_F$ the orthogonal projection on $\text{vect}(F)$, and $\Pi_{F^*}$ the orthogonal projection on $\text{vect}(F^*)$. If $P$ is an Euclidean polyhedron, $\Pi_F$ is the orthogonal projection on the codimension $p$ linear space parallel to $F$, denoted by $F_0$, and $\Pi_{F^*}$ is the orthogonal projection on $F^*_0$ (the orthogonal complement of $F_0$).

**Theorem 1.6.** In any first-order deformation of $P$ and for any $p \in \{1, 2, \cdots, n-1\}$:

\[(E_p) \quad (n-p+1)K \sum_{H \in F_p} V(H^*)\Pi_H(\pi'(H)) + p \left( \sum_{F \in F_{p+2}} V'(F^*)\pi(F) + \sum_{G \in F_{p+1}} V(G)\Pi_G(\pi'(G^*)) \right) = 0 . \]

There is a geometric interpretation of $\Pi_H(\pi'(H))$: it is the component of $\pi'(H)$ corresponding to the deformation of $H$ in the vector subspace of $\mathbb{R}^{n+2}$ that it generates (rather than the component corresponding to the displacement of $\text{vect}(H)$ in $\mathbb{R}^{n+2}$). For any face $F$ of $P$, $V(F^*)$ also has a simple interpretation: it is the volume of the set of normals to the support planes of $P$ along $F$, considered as a subset of the dual of the link of $P$ at $F$.

1.6. The corresponding formulas for $p = 0$. Formula $(E_p)$ given in Theorem 1.6 takes a simpler form for $p = 0$, where some terms disappear.

**Theorem 1.7.** Let $P \subset M^{n+1}_K$ be a convex polyhedron. Call $\theta(F)$ the exterior dihedral angle at a face $F$ of codimension 2, and $G^* \subset S^{n+1}$, if $K = 1$ (resp. $G^* \subset S^{n+1}_1$, if $K = -1$), the point dual to the face $G$ of codimension 1. Then, under any first-order deformation of $P$:

\[(E_0) \quad \sum_{F \in F_2} \theta'(F)\pi(F) - \sum_{G \in F_1} V'(G)G^* = 0 . \]

The proof can be found in section 5.

1.7. Another type of formula. As a consequence of those formulas, we find other vector-valued Schläfli type formulas, which are simpler in that they do not involve the projection terms in formula $(E_p)$ but, on the other hand, they involve faces of four different dimensions rather than only three in $(E_p)$.

**Theorem 1.8.** Let $P \subset M^{n+1}_K$ be a convex polyhedron, and let $p \in \{2, \cdots, n-1\}$. For any first-order deformation of $P$:

\[(H_p) \quad (n-p+1)K \left( \sum_{F \in F_{p-1}} V'(F)\pi(F^*) + \sum_{G \in F_p} V(G^*)\pi'(G) \right) + p \left( \sum_{H \in F_{p+1}} V(H)\pi'(H^*) + \sum_{L \in F_{p+2}} V'(L^*)\pi(L) \right) = 0 . \]

There is an apparent contradiction between this formula and formula $(E_p)$ above since, for $K = 0$, the remaining terms look different. In fact $K = 0$ corresponds to the Euclidean case, then formula $(M'_p)$ below shows that the two terms actually correspond.

Those formulas can be better understood, and formulated in slightly different ways, using the following simple remark. The proof is given in section 5.

**Remark 1.9.** (1) For any polyhedron $P \subset M^{n+1}_K$ oriented by the exterior unit normal, we have:

\[(n+1)K \int_{\text{int}(P)} xdv + \sum_{F \in F_i} V(F)F^* = 0 . \]
(2) Suppose $P \subset M_{R}^{n+1}$ is a convex polyhedron. Then for $p = 1, 2, \cdots, n$, we have:

$$
(n-p+1)K \sum_{F \in F_{p}} V(F^*)\pi(F) + p \sum_{G \in F_{p+1}} V(G)\pi(G)^* = 0,
$$

with the convention that for a face $G$ of codimension $n+1$, that is, for a vertex of $P$, $V(G) = 1$.

It is now possible to remove from formula $(H_p)$ the terms involving $\pi'$.

**Corollary 1.10.** Let $P \subset M_{R}^{n+1}$ be a convex polyhedron, and let $p \in \{2, \cdots, n-1\}$. For any first-order deformation of $P$:

$$(K_p)
(n-p+1)K \left( \sum_{F \in F_{p-1}} V'(F)\pi(F^*) - \sum_{G \in F_{p}} V'(G^*)\pi(G) \right) + p \left( - \sum_{H \in F_{p+1}} V'(H)\pi(H^*) + \sum_{L \in F_{p+2}} V'(L^*)\pi(L) \right) = 0.
$$

This corollary follows directly from Theorem 1.8 and from the second point in Remark 1.9.

### 1.8. The Euclidean space.

As in many cases one can obtain Euclidean versions of the spherical (or hyperbolic) results given above by considering Euclidean polyhedra, scaling them by a factor going to 0, and sending them on $S_{n+1}$ by the projective map. Precise arguments can be found in section 5, we state the results here.

**Theorem 1.11.** Let $P \subset \mathbb{R}^{n+1}$ be a polyhedron. Call $\theta(F)$ the exterior dihedral angle at a face $F$ of codimension 2, and, for any codimension 1 face $G$ of $P$, let $N(G) \in S_{n+1}$ be the unit vector orthogonal to $G$ towards the exterior of $P$. Then, under any first-order deformation of $P$:

$$
\sum_{F \in F_{2}} \theta'(F)\pi(F) - \sum_{G \in F_{3}} V'(G)N(G) = 0.
$$

There is also a direct analog of Theorem 1.6. We need to adapt some of the definitions to the Euclidean setting. Let $P \subset \mathbb{R}^{n+1}$ be a polyhedron, let $F$ be a codimension $p$ face of $P$, and consider a first-order deformation of $P$ (and thus of $F$). $F^*$ is defined mostly as in the spherical (or hyperbolic) context, it is now a subset of the unit sphere $S^{n}$, actually the interior of a spherical polyhedron.

We denote by $F_{0}$ the codimension $p$ plane parallel to $F$ going through the origin (in other terms, $F_{0}$ is a linear subspace, rather than an affine subspace) and by $F_{\perp}$ its orthogonal complement.

**Theorem 1.12.** In any first-order deformation of $P$ and for any $p \in \{1, 2, \cdots, n-1\}$:

$$
\sum_{F \in F_{p+2}} V'(F^*)\pi(F) + \sum_{G \in F_{p+1}} V(G)\Pi_{G}(\pi(G^*)) = 0.
$$

It is not quite evident that this formula is invariant under translation of $P$ (a condition that is obviously necessary). Actually the right-hand term has this invariance property by the definition, while the left-hand term changes, under a translation of vector $a$, by

$$
a \sum_{F \in F_{p+2}} V'(F^*)V(F).
$$

The fact that this sum is zero is precisely the Euclidean version of the higher Schl"afli formula of [SS03].

Other examples can be obtained, in particular analogs of the spherical formulas in Theorem 1.8 and in Corollary 1.10 can be obtained by just taking $K = 0$ in those statements.

### 1.9. Non-convex polyhedra.

It is interesting to point out the following difference between the classical Schl"afli formula and ours (either scalar or vector-valued). The classical Schl"afli formula is readily seen to be linear. As a consequence it needs only to be proved for simplices and extends to arbitrary polyhedra by linearity. Our generalized formulas do not have this linearity property. The reason is that, unlike the dihedral angle at codimension 2 faces, the volume and the mapping $\pi$ evaluated on polar duals are not linear. Nevertheless it is possible to extend our formulas to general polyhedra once they are known for simplices. Indeed, consider a convex polyhedron $P$ which is cut into two convex pieces $P'$ and $P''$. Then given a face $F$ common to $P'$ and $P''$, it is possible to relate the volumes of its polar duals (resp. the mapping $\pi$ evaluated on its polar duals) in
the polyhedra $P, P', P''$ and $P' \cap P''$, the latter being seen as a polyhedron in a hyperplane (cf. Proposition \ref{th:0.1}).

Furthermore this observation allows us to extend our formulas to non-convex polyhedra. We explain this with more details in section 6, some of the definitions have to be made with some care. This might be interesting in view of possible applications to questions of rigidity for non-convex polyhedra (in dimension 3 or higher).

1.10. Applications to deformations of cone-manifolds. It is possible to deduce from the vector-valued Schl"afli formulas described here some interesting properties of singular deformations of spherical, Euclidean or hyperbolic manifolds. The statements given here are simple and direct applications of the vector-valued Schl"afli formulas that can be found above.

We describe first the simplest situation, in which we deform $S^2$ so that cone singularities appear, to show how the well-known Kazdan-Warner obstruction appears naturally.

We consider a one-parameter family of spherical metrics $(g_t)_{t \in [0,1]}$ on $S^2$, with cone singularities at points $v_1, \ldots, v_n$, where the cone angle is equal to $\theta_i(t)$, and we suppose that $g_0$ is the canonical metric, i.e., $\theta_i(0) = 2\pi$ for all $i \in \{1, \ldots, n\}$.

**Theorem 1.13.** Under those hypothesis,

\[
\sum_{i=1}^{n} \theta'_i(0)v_i = 0 ,
\]

where the $v_i$ are considered as points in $S^2 \subset \mathbb{R}^3$. Moreover, given points $v_i$ and numbers $t_i$ such that $\sum_{i=1}^{n} t_i v_i = 0$, there is a unique first-order deformation of $S^2$ as a cone-manifold with cone points at the $v_i$ and first-order variations of the cone angles equal to the $t_i$.

The previous theorem is actually also a direct consequence of (a singular version of) the Kazdan-Warner relation [KW74]. In higher dimensions, there is a simple relation involving only faces of codimension 2. For a face $F$ of codimension 2 of a cell decomposition $C$ of $S^{n+1}$, it is quite natural to call $K(F)$ the “curvature” at $F$, i.e., $2\pi$ minus the sum of the angles at $F$ of the cells of $C$ containing it. This singular curvature is 0 for the spherical cone metric obtained from a cell decomposition of the sphere, since the angles then add up to $2\pi$, but it becomes non-zero as soon as this metric is deformed so that singularities occur along codimension 2 faces.

**Theorem 1.14.** Let $C$ be a cell decomposition of $S^{n+1}$, let $C_k$ be the set of faces of codimension $k$ of $C$. Consider a first-order deformation of the canonical metric on $S^{n+1}$ among spherical cone-manifolds with cone singularities at the faces of $C$. Then:

\[
\sum_{F \in C_2} K'(F)\pi(F) = 0 .
\]

The simplest illustration is obtained for $n = 2$, for a deformation of the sphere for which cone singularities appear along a graph. The theorem then states that the singular graph, “weighted” by the first-order variations of the singular curvatures at the edges, has to be “balanced”.

It is possible to extend the previous statement to higher codimensions, based on Theorem 1.8 yielding a wealth of relations between the possible variations of the volumes and “curvatures” of faces of different dimensions of $C$. A simple definition is needed.

**Definition 1.15.** Let $C$ be a cell decomposition of $S^{n+1}$, i.e., a decomposition of $S^{n+1}$ as the union of spherical polyhedra (glued isometrically along their faces). Let $F$ be a codimension $p$ face of $C$, $1 \leq p \leq n+1$. We define:

- $W^*(F) := \sum_{C \supseteq F} V(F_C^*)$, 
- $F^* := \sum_{C \supseteq F} \pi(F_C^*)$,

where the sum is over the (maximal dimension) cells of $C$ containing $F$, and $F_C^*$ is the face dual of $F$, considered as a face of $C$ (so that $F_C^* \supseteq F_C$ is a face of $C$).

By construction $F^*$ is a vector in $\mathbb{R}^{n+2}$ orthogonal to $\text{vect}(F)$.
Theorem 1.16. Under the same hypothesis as for Theorem 1.14 and for all \( p \in \{2, \ldots, n-1\} \):

\[
(n-p+1) \left( \sum_{F \in F_{p-1}} V'(F) F^\circ - \sum_{G \in F_p} W'(G) \pi(G) \right) + p \left( - \sum_{H \in F_{p+1}} V'(H) H^\circ + \sum_{L \in F_{p+2}} W'(L) \pi(L) \right) = 0 .
\]

The proof, which is given in section 7, follows quite directly from Theorem 1.8. It is also possible to use Theorem 1.6 or Corollary 1.10 to obtain other, more or less similar statements, we leave this point to the reader.

Those statements are not restricted to the spherical setting, however the fact that the manifold one starts with is simply connected is important. It remains easy to state similar results for deformations, again among hyperbolic or Euclidean polyhedra; in that case boundary terms appear, unless one makes relevant hypothesis to the extent that some boundary quantities are fixed.

Theorem 1.17. Let \( p \) be a convex hyperbolic polygon, with vertices \( v_1, \ldots, v_n \) and edges \( e_1, \ldots, e_p \). Let \( w_1, \ldots, w_q \) be distinct points in the interior of \( p \), and let \( l_1', \ldots, l_n' \), \( \alpha_1', \ldots, \alpha_p' \) and \( \theta_1', \ldots, \theta_q' \) be real numbers. There exists a first-order deformation of \( p \) among hyperbolic polygons with cone singularities at the \( w_i \), with:

- the first-order variation of the length of \( e_i \) equal to \( l_i' \), \( 1 \leq i \leq n \),
- the first-order variation of the angle at \( v_j \) equal to \( \alpha_j' \), \( 1 \leq j \leq p \),
- the first-order variation of the total curvature at \( w_k \) equal to \( \theta_k' \), \( 1 \leq k \leq q \), if and only if

\[
\sum_{i=1}^{n} l_i' e_i + \sum_{j=1}^{p} \alpha_j' v_j + \sum_{k=1}^{q} \theta_k' w_k = 0 .
\]

This first-order deformation is then unique.

This simple statement could be considered as a kind of extension of the Kazdan-Warner obstruction to conformal deformations of hyperbolic polygons. The proof is in section 7. It should also be possible to give hyperbolic analogs of Theorems 1.13 and 1.16 in this context of deformation of hyperbolic polyhedra with cone singularities appearing in the interior.

It also appears possible that related statements exist for deformations of hyperbolic or Euclidean manifolds (or cone-manifolds). However stating such statements would certainly require some care. We believe that it would be necessary to consider formulas taking values not in \( \mathbb{R}^{n+2} \) or \( \mathbb{R}^{n+2}_+ \), as in the simpler cases considered here where the fundamental group is trivial, but rather in a \( n + 2 \)-dimensional flat bundle over the manifold considered (or over the regular part of the cone-manifold under considerations). We do not go further in this direction here, since it would take us too far from our original motivations.

2. Low-dimensional examples

The geometric meaning of formulas \( (E_p),(E'_p) \) and \( (H_p) \) is not obvious at first sight. In this section we describe in more details what happens in dimension 3 and dimension 4, in the hope of offering the reader a better grasp of those formulas.

2.1. 3-dimensional polyhedra. Obviously formula \( (H_p) \) does not apply to polyhedra in 3-dimensional spaces, since then \( n = 2 \) and \( p \) has to be between 2 and \( n-1 \). So we consider in more details formulas \( (E_1) \) for 3-dimensional polyhedra in this subsection, then formula \( (H_2) \) for 4-dimensional polyhedra in the next subsection. Formula \( (E_0) \) is quite simple anyway so we do not consider it here.

Let \( P \subset \mathcal{M}_3^3 \) be a convex polyhedron. We first note that if \( v \) is a vertex of \( P \), then \( V(v^*) \) has a simple interpretation.

Remark 2.1. \( V(v^*) \) is the singular curvature at \( v \) of the induced metric on the boundary of \( P \).

Proof. By definition \( V(v^*) \) is the area of the face \( v^* \) of \( P^* \) dual to \( v \). But, by the definition of the duality, \( v^* \) is isometric to the dual (in \( S^2 \)) of the link of \( P \) at \( v \). So, by the Gauss formula, its area is \( 2\pi \) minus the sum of the angle so the dual of the link of \( P \) at \( v \). The angles of the dual of the link are equal to the lengths of the edges of the link of \( P \) at \( v \), that is, to the angles at \( v \) of the faces of \( P \) containing it. So \( V(v^*) \) is equal to \( 2\pi \) minus the sum of the total angle at \( v \), or in other terms to the singular curvature at \( v \) of the induced metric on the boundary of \( P \). \( \square \)
Note that this argument extends to the case where \( v \) is a codimension 3 face of \( P \subset M_{K}^{n+1} \), for \( n \geq 2 \). In this subsection, we use the notation \( k(v) \) for the singular curvature at \( v \) of the induced metric on \( P \), so that \( k(v) = V(v^*) \).

We now consider what Theorem 1.6 means, first in the Euclidean 3-dimensional space. Then \( n = 2 \), and the only possible value of \( p \) is \( p = 1 \). The first term in formula \((E_1)\) vanishes since \( K = 0 \), the second sum is over vertices, and the right-hand side is a sum over edges. We call \( V, E \) the sets of vertices and edges of \( P \), respectively. For each edge \( e \), \( \Pi_e \) is a linear map from \( e^0 \) (the linear line parallel to \( e \)) to \( e^\perp \), and \((E_1)\) reads as follows.

**Corollary 2.2.** In any first-order deformation of a polyhendedron in \( \mathbb{R}^3 \),

\[
\sum_{v \in V} k'(v)v = -\sum_{e \in E} l(e)\Pi_e(\pi'(e^*)).
\]

The meaning of the first sum is clear, while the second sum involve the way the edges “rotate”. The statement is illustrated in Figure 1, with signs indicating how the curvature at a vertex varies.

![Figure 1. The deformation of a Euclidean simplex.](image)

In the sphere or the hyperbolic space the first term in formula \((E_1)\) kicks in, but in a simpler form since \( V(H^*) = 1 \) when \( H \) is codimension 1 face (\( H^* \) is then a point). The terms \( \pi(H) \) have a very simple meaning, they are just the barycenter of the faces (in the ambient 4-dimensional space, \( \mathbb{R}^4 \) or \( \mathbb{H}^4 \)) times the area of \( H \). Theorem 1.6 then translates as follows.

**Corollary 2.3.** Let \( P \subset S^3 \) (resp. \( H^3 \)), let \( F \) be the set of 2-dimensional faces of \( P \). Then

\[
2K \sum_{f \in F} \Pi_f(\pi'(f)) + \sum_{v \in V} k'(v)v = -\sum_{e \in E} l(e)\Pi_e(\pi'(e^*)).
\]

Theorem 1.8 is empty in dimension 3 since \( p \) has to be chosen between 2 and \( n-1 = 1 \). We do not elaborate on Theorem 1.11 since it is quite clear.

2.2. **Dimension four.** We now turn to 4-dimensional polyhedra, first in the Euclidean and then in the spherical (or hyperbolic) setting.

In \( \mathbb{R}^4 \), Theorem 1.6 can be applied either with \( p = 1 \) or with \( p = 2 \). Now \( F_2 \) the set of 2-faces of \( P \), and, for \( f \in F_2 \), \( a(f) \) is its area. When \( e \) is an edge of \( P \), we still use the notation \( k(e) = V(e^*) \) since, as pointed out above, \( V(e^*) \) is the singular curvature at \( e \) of the induced metric on the boundary of \( P \).

**Corollary 2.4.** Let \( P \subset \mathbb{R}^4 \), formula \((E_1)\) becomes

\[
\sum_{e \in E} k'(e)\pi(e) = -\sum_{f \in F_2} a(f)\Pi_f(\pi'(f^*))
\]

while formula \((E_2)\) can be stated as

\[
\sum_{v \in V} V'(v^*)v = -\sum_{e \in E} l(e)\Pi_e(\pi'(e^*)).
\]
Theorem 1.8 can be applied in $\mathbb{R}^4$ with $p = 2$ only, it gives a formula similar to the one obtained in the previous corollary for $p = 2$ but slightly simpler, the two formulas are actually equivalent once one takes into account formula (M'$_2$) below. In $S^4$ or $H^4$, Theorem 1.6 takes a slightly more complicated form. For each edge $e$ of $P$, we call $\theta(e)$ the exterior dihedral angle of $P$ at $e$, and we also call $F_3$ the set of 3-dimensional faces of $P$.

**Corollary 2.5.** Let $P \subset S^4$ (resp. $P \subset H^4$), formula (E$_1$) becomes

$$3K \sum_{f \in F_3} \Pi_f(\pi'(f)) + \sum_{e \in E} k(e)\pi(e) = - \sum_{f \in F_2} a(f)\Pi_f(\pi'(f^*)) \ ,$$

while (E$_2$) can be written as

$$2K \sum_{f \in F_2} \theta(f)\Pi_f(\pi'(f)) + \sum_{v \in V} V'(v^*)v = - \sum_{e \in E} l(e)\Pi_e(\pi'(e^*)) \ .$$

Finally in this 4-dimensional context Theorem 1.8 can be applied non-trivially for $p = 2$ to obtain the following result.

**Corollary 2.6.** Let $P \subset S^4$ (resp. $P \subset H^4$), then, in a first-order deformation,

$$K \left( \sum_{f \in F_3} V'(f)f^* + \sum_{h \in F_2} \theta(h)\pi'(h) \right) + \left( \sum_{e \in E} l(e)\pi'(e^*) + \sum_{v \in V} V'(v^*)v \right) = 0 \ .$$

### 3. Smooth differential formulas

The polyhedral formulas given in the introduction are proved here using analogous formulas for first-order deformations of smooth hypersurfaces. Let $\phi : \Sigma \to M_K^{n+1}$ be an immersed oriented hypersurface in $M_K$, $K \in \{-1,0,1\}$, and let $N$ be an oriented unit normal vector field along $\Sigma$. Without loss of generality, we will sometimes, when dealing with local matters, implicitly identify the immersion $\phi$ with an inclusion map. Let $\nabla$ denote the connection on $M_K^{n+1}$. We define the shape operator of $\Sigma$ as:

$$B u = \nabla u N \ ,$$

where $u$ is a vector tangent to $\Sigma$. Note that this sign convention is opposite to the one we used in [SS03] and is such that $\langle Bu, x \rangle$ is non-negative for convex hypersurfaces if $N$ is oriented towards the exterior (i.e., towards the concave side of $\Sigma$).

#### 3.1. Notations.

In all the paper, we call $\nabla$ the Levi-Cività connection of the hypersurface $\Sigma \subset M_K^{n+1}$, as well as its natural extension to tensor fields on $\Sigma$. So, given a function $f : \Sigma \to \mathbb{R}$, $\nabla f$ is its gradient on $\Sigma$. Given a vector field $Y$ tangent to $\Sigma$, its divergence is defined as:

$$\text{div}(Y) := \sum_{i=1}^n \langle \nabla_{e_i} Y, e_i \rangle \ ,$$

where $(e_i)_{i=1, \ldots, n}$ is an orthonormal moving frame on $\Sigma$. Given a vector-valued 1-form $T$ on $\Sigma$, its divergence is similarly defined as:

$$\text{div}(T)(Z) = \left( \sum_{i=1}^n \langle \nabla_{e_i} T(Z), e_i \rangle \right) \ .$$

We also call $D$ the flat connection on $\mathbb{R}^{n+2}$ and $\mathbb{R}^{1+2}_0$. Let $X$ be a vector field tangent to $M_K^{n+1}$ defined on $\Sigma$, which we consider as a first-order deformation of $\Sigma$. It induces a first-order deformation of the extrinsic invariants defined on $\Sigma$.  

3.2. The Newton operators. We will use the elementary symmetric functions $S_r$ of the principal curvatures $k_1, k_2, \ldots, k_n$ of the immersion $\phi$:

$$S_r = \sum_{i_1 < \ldots < i_r} k_{i_1} \ldots k_{i_r} \quad (1 \leq r \leq n),$$

as well as the Newton operators, defined as follows.

**Definition 3.1.** The Newton operators of $\Sigma$ are defined for $0 \leq r \leq n$, as:

$$T_r = S_r \text{Id} - S_{r-1}B + \ldots + (-1)^r B^r,$$

or, inductively, by $T_0 = \text{Id}, T_r = S_r \text{Id} - BT_{r-1}$.

**Lemma 3.2.** The Newton operators satisfy the following formulas for $0 \leq r \leq n-1$:

1. $\text{div}(T_r) = 0$.
2. $\text{Trace}(T_r) = (n-r)S_r$.
3. $\text{Trace}(BT_r) = (r+1)S_{r+1}$.
4. $S'_r = \text{Trace}(B'T_{r-1})$, the derivative being taken with respect to any first-order deformation of the hypersurface.

The proofs can be found in [Rei77, Ros93].

3.3. Smooth formulas in non-zero curvature. Let $\xi$ be the orthogonal projection of $X$ on $\Sigma$, and let $f := \langle X, N \rangle$, where $N$ is the unit vector field normal to $\Sigma$, so that $X = \xi + fN$. Here we denote by $I$ the induced metric on $\Sigma$, which is also classically known as its first fundamental form.

**Lemma 3.3.** For $p = 0, \ldots, n-1$:

$$(F_p) \quad \int_\Sigma T_p \xi + \frac{x}{2} \langle I', T_p \rangle - (p+1)xfS_{p+1} dv = 0.$$

**Proof.** Let $a \in \mathbb{R}^{n+2}$ if $K = 1$ (resp. $a \in \mathbb{R}^{n+2}$ if $K = -1$) be a fixed vector. The proof is based on the integration over $\Sigma$ of the function defined at a point $x \in \Sigma$ as $\text{div}(\langle x, a \rangle T_p \xi)$. The vector field $X$ is defined only on $\Sigma$, but we extend it as a smooth vector field in a neighborhood of $\Sigma$.

First note that the gradient of $\langle x, a \rangle$ on $\Sigma$ is equal to the orthogonal projection of $a$ on $\Sigma$, which we call $\overline{a}$. Indeed, given an orthonormal moving frame $(e_i)_{i=1, \ldots, n}$ on $\Sigma$, we have:

$$\nabla \langle x, a \rangle = \sum_{i=1}^n \langle e_i, \langle x, a \rangle \rangle e_i = \sum_{i=1}^n \langle De_i x, a \rangle e_i = \sum_{i=1}^n \langle e_i, a \rangle e_i = \overline{a}.$$

It follows that:

$$\text{div}(\langle x, a \rangle T_p \xi) = \langle \nabla \langle x, a \rangle, T_p \xi \rangle + \langle x, a \rangle \text{div}(T_p \xi) = \langle a, T_p \xi \rangle + \langle x, a \rangle \text{div}(T_p \xi).$$
But:

\[
\operatorname{div}(T_p \xi) = \sum_{i=1}^{n} \langle \nabla_{e_i}(T_p \xi), e_i \rangle
\]

\[
= \sum_{i=1}^{n} (\langle \nabla_{e_i} T_p \xi, T_p \nabla_{e_i} \xi, e_i \rangle
\]

\[
= \left\langle \sum_{i=1}^{n} (\nabla_{e_i} T_p \xi, e_i, \xi) \right\rangle + \langle \nabla_{e_i} \xi, T_p e_i \rangle
\]

\[
= \operatorname{div}(T_p)(\xi) + \sum_{i=1}^{n} \langle \nabla_{e_i} (X - fN), T_p e_i \rangle
\]

\[
= \sum_{i=1}^{n} (\nabla_{e_i} X - fB e_i, T_p e_i) = f \text{Trace}(B T_p) .
\]

Let \(u, v\) be two vector fields tangent to \(\Sigma\). Let \(\phi_t, t \in (-\epsilon, \epsilon), \epsilon > 0\), be a deformation of the immersion \(\phi\) having \(X\) as a deformation vector field. Then we can write

\[
X(\langle u, v \rangle) = \langle I'(u, v) \rangle
\]

\[
= \left( \frac{d}{dt} \right)_{t=0} \langle \phi_t(u), \phi_t(v) \rangle
\]

\[
= \left\langle \left( \frac{D}{dt} \right)_{t=0} \phi_t(u), v \right\rangle + \left\langle u, \left( \frac{D}{dt} \right)_{t=0} \phi_t(v) \right\rangle
\]

\[
= \langle \nabla_u X, v \rangle + \langle u, \nabla_v X \rangle .
\]

It follows that the symmetrization of \(\nabla X\) is equal to half the first-order variation of \(I\). Therefore:

\[
\text{(2)} \quad \operatorname{div}(\langle x, a \rangle T_p \xi) = \langle a, T_p \xi + \left( \frac{1}{2} I'(T_p) - f(p + 1)xS_{p+1} \right) \rangle .
\]

The result now follows by integration of this equality over \(\Sigma\). \qed

**Lemma 3.4.** Let \(\Sigma \in M^{n+1}_K, K \in \{-1, 1\}\). For \(p = 0, \cdots, n - 1\):

\[
(G_p) \quad \int_{\Sigma} \left( S_{p+1} + \frac{1}{2} (I', BT_p) + (n - p)KfS_p \right) x + T_p \nabla_X N dv = 0 ,
\]

where \(\nabla_X N\) is the vector field defined on \(\Sigma\), as the first-order variation of the unit normal vector field.

**Proof.** Set \(Y := \nabla_X N\). We need the following well known formula, of which we include a proof for the reader’s convenience:

\[
\text{(3)} \quad \operatorname{div}(T_p Y) = S_{p+1} + \frac{1}{2} (I', BT_p) + (n - p)KfS_p(X, N) .
\]

Indeed, consider a deformation \(\phi_t, t \in (-\epsilon, \epsilon), \epsilon > 0\), of the immersion \(\phi\) having \(X\) as deformation vector field. Denote by \(N_t\) a unit normal field to \(\phi_t\) depending in a differentiable way on \(t\). Call \(I_t\) the metric induced on \(\Sigma\) by \(\phi_t\) and \(B_t\) its shape operator. Let \(u, v\) be tangent vectors to \(\Sigma\). Taking the derivative at \(t = 0\) in the equation:

\[
I_t(B_t u, v) = \langle \nabla_{\phi_t}(u), N_t, d\phi_t(v) \rangle ,
\]

we obtain:

\[
I'(B u, v) + I(B' u, v) = \left\langle \nabla_{\phi_{0+}} \nabla_{d\phi_{0+}(N_t)} \right|_{t=0}, v \right\rangle + \langle \nabla_u N, \nabla_v X \rangle .
\]
Now, on the one hand we have
\[ I'(Bu, v) = \langle \nabla Bu, v \rangle + \langle Bu, \nabla v \rangle. \]
On the other hand, denoting by \( R \) the curvature tensor of \( M_{n+1} \), we can write:
\[ \left( \langle \nabla_{\phi(u)} \nabla_{\phi_i} N \rangle_{t=0}, v \right) = \langle \nabla_u \nabla_X N, v \rangle - \langle R(X, u)N, v \rangle. \]
Collecting the terms we finally get:
\[ I'(Bu, v) = \langle \nabla u \nabla X N, v \rangle - \langle \nabla Bu \nabla X, v \rangle - \langle R(X, u)N, v \rangle. \]

(4)

Take now a local orthonormal frame \( \{e_1, \ldots, e_n\} \) on \( \Sigma \). Then:
\[ \text{div}(T_p Y) = \sum_{i=1}^n I'(e_i, T_p e_i) = \sum_{i=1}^n \langle \nabla_{e_i} T_p Y + T_p \nabla_{e_i} Y, e_i \rangle = \text{div}(T_p Y) + \sum_{i=1}^n \langle \nabla_{e_i} Y, T_p e_i \rangle = \sum_{i=1}^n \langle \nabla_{e_i} Y, T_p e_i \rangle. \]

We now use equation (4) above to write:
\[ \text{div}(T_p Y) = \sum_{i=1}^n I'(e_i, T_p e_i) + \langle \nabla_{Be_i} X, e_i \rangle + \langle R(X, e_i)N, e_i \rangle = \text{tr}(B'T_p) + \frac{1}{2} \langle I', BT_p \rangle + K \sum_{i=1}^n \langle X, N \rangle \langle e_i, T_p e_i \rangle = \text{tr}(B'T_p) + \frac{1}{2} \langle I', BT_p \rangle + K \langle X, N \rangle \text{tr}(T_p). \]

Finally using properties (2) and (4) of Lemma 3.2 we obtain the desired equation (3) above. Now, for any fixed vector \( a \in \mathbb{R}^{n+2} \):
\[ \text{div}(\langle a, x \rangle T_p Y) = \langle \nabla(\langle a, x \rangle), T_p Y \rangle + \langle a, x \rangle \text{div}(T_p Y), \]
and so using (3) we obtain:
\[ \text{div}(\langle a, x \rangle T_p Y) = \langle a, T_p Y \rangle + \langle a, x \rangle \left( S'_{p+1} + \frac{1}{2} \langle I', BT_p \rangle + (n - p)KS_p \langle X, N \rangle \right), \]
and the result follows by integration over \( \Sigma \).

4. Polyhedral localization formulas

This section contains some technical statements on the limits of integral quantities defined on a hypersurface, when one considers them on the boundary of the (outer) \( \epsilon \)-neighborhood of a polyhedron \( P \), as \( \epsilon \to 0 \). The general idea is that some integral quantities have limits that depend only on what happens on faces of \( P \) of given codimension. For any \( \epsilon > 0 \) small enough, we call \( P_\epsilon \) the boundary of the set of points at distance at most \( \epsilon \) from the interior of \( P \). Given a face \( F \) of \( P_\epsilon \) of codimension \( p+1 \), we call \( F_\epsilon \) the set of points \( x \in P_\epsilon \) such that the point of \( P \) which is closest to \( x \) is in \( F \). We orient \( P \) and \( P_\epsilon \) by the exterior unit normal.
4.1. The spherical case. We now concentrate on the case where the ambient space is $S^{n+1}$. It turns out that the same results apply in the hyperbolic space, and the proofs can be extended from the spherical to the hyperbolic context with only very little changes. The local geometry of $F_\epsilon$ is simple. Its shape operator has two eigenvalues:

- $1/\tan(\epsilon)$, which has multiplicity $p$, on directions corresponding (by parallel transport along shortest geodesics) to directions orthogonal to $F$.
- $-\tan(\epsilon)$, which has multiplicity $n-p$, on directions corresponding by parallel transport to directions tangent to $F$.

Actually $F_\epsilon$ is isometric to the product $\cos(\epsilon)F \times \sin(\epsilon)F^*$, so that its volume is $\cos(\epsilon)^{n-p}\sin(\epsilon)^p V(F)V(F^*)$.

**Remark 4.1.**

$$\lim_{\epsilon \to 0} \int_{F_\epsilon} S_p' x \, dv = 0.$$  

**Proof.** This is clear since, for each face $F$ of $P$, $S_p' = 0$ on $F_\epsilon$. $\square$

We now consider in more details how some important quantities behave on the surfaces $F_\epsilon$ which “approximates” a codimension $p + 1$ face of $P$. At each point of such a surface, the tangent space $T_x F_\epsilon$ has a natural decomposition as the direct sum of two subspaces, corresponding under the parallel transport along the geodesics segments orthogonal to $F_\epsilon$ to the spaces tangent (resp. orthogonal) to $F$. We call those subspaces $V$ (resp. $W$) and also call $\Pi_V$ (resp. $\Pi_W$) the orthogonal projection from $T_x F_\epsilon$ to $V$ (resp. to $W$). The dimension of $W$ is equal to $p$, while the dimension of $V$ is equal to the dimension of $F$, which is $n - p$ since $F$ has codimension $p + 1$.

**Lemma 4.2.** Let $F \in F_{p+1}$ be a codimension $p + 1$ face. For all $q \in \{1, \ldots, n\}$, $S_q$ is constant over $F_\epsilon$. Moreover:

1. for $q \neq p$, $S_q = o(\epsilon^{-p})$ as $\epsilon \to 0$, while $S_p = \epsilon^{-p} + o(\epsilon^{-p})$,
2. $T_q = o(\epsilon^{-p})$ if $q \neq p$, while $T_p = \epsilon^{-p} \Pi_V + o(\epsilon^{-p})$,
3. $BT_q = o(\epsilon^{-p})$ for $q \neq p - 1$, while $BT_{p-1} = \epsilon^{-p} \Pi_W + o(\epsilon^{-p})$.

**Proof.** At each point of $F_\epsilon$ the shape operator has only two eigenvalues (corresponding to two different principal curvatures): $-\tan(\epsilon)$, with eigenspace $V$, and $1/\tan(\epsilon)$ with eigenspace $W$. The result concerning $S_p$ then follows from the explicit expression of $S_p$, since at most $p$ principal curvatures can be equal to $1/\tan(\epsilon)$ while the others are equal to $-\tan(\epsilon)$. To obtain the result on $T_p$ and on $BT_p$ it is necessary to remark that, for each eigenvector $e_i$ of the shape operator at a point $x \in F_\epsilon$, $e_i$ is also an eigenvector of $T_p$ with eigenvalue equal to the $p$-th symmetric function in the other eigenvalues of $B$ at $x$:

$$T_p e_i = \left( \sum_{1 \leq i_1 < \cdots < i_p \leq n; k_1 + \cdots + k_p = p} k_1 \cdots k_p \right) e_i.$$  

This formula, which can be checked by a direct induction argument using the definition of the Newton operators, leads directly to the second result: $T_q$ is maximal for $q = p$, and is then equivalent to $\epsilon^{-p} \Pi_V$. Similarly, $BT_q$ is maximal for $q = p - 1$, and is then equivalent to $\epsilon^{-p} \Pi_W$. $\square$

It is now possible to estimate how some interesting quantities behave as $\epsilon \to 0$. We introduce for this a simple notation.

**Definition 4.3.** Consider a first-order deformation of the polyhedron $P$ with deformation field a vector field $X$ tangent to $S^{n+1}$ defined on $P$. Let $F$ be a face of $P$. For each point $x \in F$, we call $\nu_F(x)$ the component of $X$ at $x$ which is orthogonal to the subspace of $\mathbb{R}^{n+2}$ generated by $F$, called $\text{vect}(F)$ here. Then $\nu_F$ is the restriction to $F$ of a linear map, which we still call $\nu_F$, from $\text{vect}(F)$ to $\text{vect}(F^*)$.

**Remark 4.4.** The adjoint operator $\nu_F^*: \text{vect}(F^*) \to \text{vect}(F)$ is equal to $-\nu_F$. 


Proof. Let \( x \in F, y \in F^* \), then \( \langle x, y \rangle = 0 \), and this remains true under a first-order deformation. So, under this deformation
\[
\langle x, y \rangle' = \langle \nu_F(x), y \rangle + \langle x, \nu_F^*(y) \rangle = 0 ,
\]
and the result follows. \( \square \)

**Lemma 4.5.** For all \( p \in \{0, \cdots, n - 1\} \):
\[
\lim_{\epsilon \to 0} \int_{P_\epsilon} f \sum_{F \in \mathcal{F}_{p+1}} \int_F \langle \nu_F(x), \pi(F^*) \rangle dx dv = \sum_{F \in \mathcal{F}_{p+1}} \int_F \langle \nu_F(x), \pi(F^*) \rangle dv .
\]

Proof. Clearly \( f \) is bounded over \( P_\epsilon \), and the bound is uniform as \( \epsilon \to 0 \). We have seen above that, given a face \( F \) of \( P \) of codimension \( q + 1 \), the volume of \( F_\epsilon \) is equal to \( \sin(\epsilon)^q \cos(\epsilon)^{n-q} V(F) V(F^*) \). Given the estimates on \( S_p \) described above, it follows that:
\[
\int_{F_\epsilon} f S_p dv \to 0
\]
unless \( q = p \), and that in that case, since \( f := \langle X, N \rangle \):
\[
\int_{F_\epsilon} f S_p dv \to \int_{x \in F, n \in F^*} \langle n, X \rangle dv(x) dv^*(n),
\]
where \( dv^* \) is used to denote the volume element on \( F^* \). But by definition of \( \nu_F \) we have: \( \langle n, X \rangle = \langle \nu_F(x), n \rangle \), at each point \( x \in F \) and for each normal vector \( n \in F^* \), and it follows that:
\[
\int_{F_\epsilon} f S_p dv \to \int_{x \in F, n \in F^*} \langle \nu_F(x), n \rangle dv(x) dv^*(n) = \int_{x \in F} \langle \nu_F(x), \pi(F^*) \rangle dv(x),
\]
and the result follows by summing over faces of codimension \( p + 1 \). \( \square \)

It will be useful below to note that this formula extends to deformations of polyhedra which, rather than being contained in the “usual” unit sphere \( S^{n+1} \subset \mathbb{R}^{n+2} \), are contained in a unit sphere centered at another point \( a \in \mathbb{R}^{n+2} \). In that case, call \( \tau_{-a} \) the translation by \( -a \) in \( \mathbb{R}^{n+2} \), and let \( P_0 := \tau_{-a}(P) \), so that \( P_0 \subset S^{n+1} \). For each face \( F \) of \( P \), define the dual \( F^* \) of \( F \) as \( \tau_{-a}(F)^* \) (which is well-defined since \( \tau_{-a}(F) \) is a face of \( P_0 \subset S^{n+1} \)). It is also necessary to replace the function \( \nu_F \) defined above (for any face \( F \) of \( P \)) by another function, which we call \( \nabla_F \), and which is defined as \( \nu_{\tau_{-a}(F)} \circ \tau_{-a} \), where \( \nu_{\tau_{-a}(F)} \) is defined with respect to the first-order deformation of \( P_0 \) corresponding to the chosen deformation of \( P \).

**Remark 4.6.** With those notations, the formula becomes:
\[
\lim_{\epsilon \to 0} \int_{P_\epsilon} f S_p dv = \sum_{F \in \mathcal{F}_{p+1}} \int_F \langle \nabla_F(x), \pi(F^*) \rangle dx dv .
\]

Proof. The proof can be obtained as in the previous lemma. Alternatively it is possible to apply the previous lemma to \( \tau_{-a}P \), using the fact that:
\[
\int_{P_\epsilon} f S_p dv = \int_{\tau_{-a}(P_\epsilon)} f S_p dv + a \int_{\tau_{-a}(P_\epsilon)} f S_p dv ,
\]
while:
\[
\sum_{F \in \mathcal{F}_{p+1}} \int_F \langle \nabla_F(x), \pi(F^*) \rangle dx dv = \sum_{F \in \mathcal{F}_{p+1}} \int_{\tau_{-a}(F)} \langle \nabla_F(x), \pi(F^*) \rangle dx dv + a \sum_{F \in \mathcal{F}_{p+1}} \int_{\tau_{-a}(F)} \langle \nabla_F(x), \pi(F^*) \rangle dv ,
\]
The result then follows from the fact that, for any polyhedron in \( S^{n+1} \):
\[
\lim_{\epsilon \to 0} \int_{P_\epsilon} f S_p dv = \sum_{F \in \mathcal{F}_{p+1}} \int_F \langle \nabla_F(x), \pi(F^*) \rangle dv ,
\]
which can be proved exactly like the previous lemma (removing some “\( a \)'s” from the equations). \( \square \)
Lemma 4.7. For any first-order deformation of $P$ and any $p \in \{0, \cdots, n-1\}$:

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} \frac{1}{2} (I', BT_P)xdv = \sum_{F \in F_{p+2}} V'(F^*)\pi(F) .$$

**Proof.** The same argument as in the proof of Lemma 4.5 shows that the contribution to the limit of faces of codimension $q$ vanishes unless $q = p + 2$. For faces of codimension $p + 2$, $BT_P \simeq \epsilon^{-p-1} \Pi_W$, so that:

$$\frac{1}{2} (I', BT_P)dv^* \simeq \epsilon^{-p-1}(dv^*)' ,$$

where $(dv^*)'$ is the first-order variation of the volume element on $V^*$. Therefore:

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} \frac{1}{2} (I', BT_P)xdv = \sum_{F \in F_{p+2}} \int_{x \in F, n \in F^*} xdv(x)(dv^*)'(n) = \sum_{F \in F_{p+2}} V'(F^*)\pi(F) .$$

□

**Remark 4.8.** The same formula is valid if one considers a sphere centered at a point $a \in \mathbb{R}^{n+2}$ which is not the origin.

**Proof.** Again the proof can be obtained either by checking that the proof of the previous lemma extends to a sphere centered at a point $a \in \mathbb{R}^{n+2}$, or by remarking that the difference in the two terms that arises by centering the sphere at $a$ is equal to:

$$a \left( \lim_{\epsilon \to 0} \int_{P_\epsilon} \frac{1}{2} (I', BT_P)dv - \sum_{F \in F_{p+2}} V'(F^*)\pi(F) \right) ,$$

a term which is shown to vanish by the arguments used to prove the previous lemma, with some \textit{“x”} suppressed from the equations.

**Lemma 4.9.** Again for any first-order deformation of $P$ and for any $p \in \{0, \cdots, n-1\}$:

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} T_P \nabla_X Ndv = \sum_{F \in F_{p+1}} \left( -V(F)\nu_F^*(\pi(F^*)) + K \int_F \langle \nu_F(x), \pi(F^*) \rangle xdv \right) .$$

**Proof.** First note that $\nabla_X N$ is bounded, so that, by the second point of Lemma 4.2, only faces of codimension $p + 1$ give a non-zero contribution to the limit on the left-hand side. It follows that:

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} T_P \nabla_X Ndv = \lim_{\epsilon \to 0} \sum_{F \in F_{p+1}} \int_{F_p} \Pi_V(\nabla_X N)dv .$$

This can also be written as:

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} T_P \nabla_X Ndv = \sum_{F \in F_{p+1}} \int_F \int_{F^*} \Pi_{T_x F}(\nu_{F^*}(n))dv(x)dv^*(n) .$$

(6)

Note however that $\nu_{F^*}(n) \in vect(F)$, so that

$$\Pi_{T_x F}(\nu_{F^*}(n)) = \nu_{F^*}(n) - K \langle \nu_{F^*}(n), x \rangle x ,$$

where the term \textit{“K”} is necessary because, if $K = -1$, $x$ is a time-like unit vector in the Minkowski space $\mathbb{R}^{n+2}$. So it follows from (6) that:

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} T_P \nabla_X Ndv = \sum_{F \in F_{p+1}} \int_F \nu_{F^*}(n) - K \langle \nu_{F^*}(n), x \rangle xdv(x)dv^*(n) .$$

(7)

After integration in $n$ this becomes:

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} T_P \nabla_X Ndv = V(F)\nu_{F^*}(\pi(F^*)) - K \int_F \langle \nu_{F^*}(\pi(F^*)), x \rangle xdv .$$
We have seen above that $\nu_{F^*} = -(\nu_F)^*$. So:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \nabla_X N \, dv = -V(F)\nu_{F^*}(\pi(F^*)) + K \int_F \langle \pi(F^*), \nu_F(x) \rangle \, dx,
$$

and the result follows.

\begin{remark} \textbf{Remark 4.10.} Consider a unit sphere centered at a point $a \in \mathbb{R}^{n+2}$. The formula becomes:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \nabla_X N \, dv = \sum_{F \in P_{p+1}} \left( -V(F)\nu_{F^*}(\pi(F^*)) + \int_F (\nabla_F(x), \pi(F^*)) \, dx - \langle \int_F \nabla_F(x) \, dv, \pi(F^*) \rangle a \right).
$$

\end{remark}

\begin{proof} \textit{Proof.} The proof proceeds as above, except that equation (7) is replaced by:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \nabla_X N \, dv = \sum_{F \in P_{p+1}} \int_F \nu_F(n) - \langle \nu_F(n), (x-a)(x-a) \rangle (n) \, dv(x).
$$

Therefore:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \nabla_X N \, dv = V(F)\nu_{F^*}(\pi(F^*)) - \int_F \langle \nu_{F^*}(\pi(F^*)), x-a \rangle (x-a) \, dv.
$$

So:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \nabla_X N \, dv = -V(F)\nu_{F^*}(\pi(F^*)) + \int_F \langle \pi(F^*), \nabla_F(x) \rangle (x-a) \, dv,
$$

and the result follows.
\end{proof}

\begin{lemma} \textbf{Lemma 4.11.} For all $p \in \{0, \ldots, n-1\}$:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \xi + \frac{x}{2} (I', T_p) \, dv = \sum_{F \in P_{p+1}} V(F^*)(\pi'(F) - \nu_F(\pi(F))).
$$

\end{lemma}

\begin{proof} \textit{Proof.} The same argument as above can be used to show that only codimension $p+1$ faces have a non-zero contribution to the limit. So:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \xi + \frac{x}{2} (I', T_p) \, dv = \sum_{F \in P_{p+1}} V(F^*) \int_F \Pi_{T_x F}(\xi) + \frac{x}{2} (I', I) \, dv
$$

$$
= \sum_{F \in P_{p+1}} V(F^*) \int_F \Pi_{T_x F}(X) + \frac{x}{2} (I', I) \, dv
$$

$$
= \sum_{F \in P_{p+1}} V(F^*) \int_F X - \nu_F(x) + \frac{x}{2} (I', I) \, dv
$$

$$
= \sum_{F \in P_{p+1}} V(F^*) \left( \left( \int_F x \, dv \right)' - \int_F \nu_F(x) \, dx \right)
$$

$$
= \sum_{F \in P_{p+1}} V(F^*) (\pi'(F) - \nu_F(\pi(F))),
$$

which is the desired result.
\end{proof}

\begin{remark} \textbf{Remark 4.12.} Consider a sphere centered at a point $a$ which is not the origin. Then the formula becomes:

$$
\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \xi + \frac{x}{2} (I', T_p) \, dv = \sum_{F \in P_{p+1}} V(F^*) \left( \pi'(F) - \int_F \nabla_F(x) \right) \, dv.
$$

\end{remark}

\begin{proof} \textit{Proof.} The proof follows directly from the computations above.
\end{proof}

5. \textbf{Proof of the main results}

In this section we use the results of section 4 to take the limit of the smooth formulas of section 3 for (outer) $\epsilon$-neighborhoods of polyhedra, to obtain the proofs of the main results.
5.1. **The spherical and hyperbolic settings.** We first explain why the formulas in Lemma 3.3 and Lemma 3.4 do apply to outer $\epsilon$-neighborhoods ($\epsilon > 0$ small enough) of convex polyhedra although they are only $C^{1,1}$. Indeed, $P_\epsilon$ can be decomposed as

$$
P_\epsilon = \bigcup_{m=1}^{m=n+1} F_\epsilon,
$$

where $F_\epsilon$ denotes, as before, the set of points where the normal to $P_\epsilon$ meets $P$ at the codimension $m$ face $F$. The parts $F_\epsilon$ are smooth and have piecewise smooth boundaries. We can thus apply to each $F_\epsilon$ the formulas (2) and (3) in the proofs of Lemmas 3.3 and 3.4 respectively. In order to show that the formulas of Lemma 3.3 and Lemma 3.4 apply to $P_\epsilon$, we have to check that the boundary terms that appear after integrating, and using the divergence theorem, in (2) and (3), cancel out two by two after summing up. These boundary terms are (using the symmetry of the Newton transformations) of the form $\int_{\partial F_\epsilon} \langle a, x \rangle \langle Z, T_r(\nu) \rangle$, for some $1 \leq r \leq n$, $Z$ being a (continuous) vector field on $P_\epsilon$ and $\nu$ the unit exterior conormal to $\partial F_\epsilon$. Let $F \in F_m$ and $Q \in F_q$, then $F_\epsilon$ and $Q_\epsilon$ have a common boundary of non-zero $(n-1)$-measure if and only if $q = m + 1$ or (symmetrically) $q = m - 1$. Take then $F \in F_m$ and $Q \in F_{m+1}$. Along the common boundary of $F_\epsilon$ and $Q_\epsilon$, the unit cononormal $\nu$ corresponds under parallel transport along geodesic segments orthogonal to $F_\epsilon$ (resp. $Q_\epsilon$) to the space tangent to $F$ (resp. orthogonal to $Q$). Then a straightforward computation, using properties (1), (2) and (3) in Lemma 4.2, shows that the vectors $T_r(\nu)$, where $\nu$ is a vector orthogonal to the common boundary, are equal on the two sides of the common boundary (cf. [SS03]).

We can now state two lemmas obtained by applying the polyhedral localization formulas to Lemma 3.3 and Lemma 3.4.

**Lemma 5.1.** Let $P \subset M_K^{n+1}$, $K \in \{-1, 1\}$, and let $X$ be a first-order deformation vector field on $P$. For all $p \in \{1, 2, \cdots, n\}$:

$$(F_p') \quad \sum_{F \in F_p} V(F^*)(\pi'(F) - \nu_F(\pi(F))) - p \sum_{G \in F_{p+1}} \int_G \langle \nu_G(x), \pi(G^*) \rangle dv = 0 .$$

**Proof.** According to Lemma 3.3 we have for all $p \in \{0, \cdots, n-1\}$:

$$\int_{P_p} T_p \xi + \frac{x}{2} \langle I', T_p \rangle - (p + 1) x f S_{p+1} dv = 0 .$$

However we know by Lemma 4.5 that

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} f S_{p+1} dv = \sum_{F \in F_{p+2}} \int_F \langle \nu_F(x), \pi(\xi) \rangle dv ,$$

while Lemma 4.11 shows that

$$\lim_{\epsilon \to 0} \int_{P_\epsilon} T_p \xi + \frac{x}{2} \langle I', T_p \rangle dv = \sum_{F \in F_{p+1}} V(F^*)(\pi'(F) - \nu_F(\pi(F))) .$$

The proof follows by replacing $p$ by $p - 1$. \hfill $\square$

**Remark 5.2.** If the sphere is centered at a point $a \in \mathbb{R}^{n+2}$ then the formula becomes:

$$\sum_{F \in F_p} V(F^*)(\pi'(F) - \int_F \nu_F(x) dv) - p \sum_{G \in F_{p+1}} \int_G \langle \nu_G(x), \pi(G^*) \rangle dv = 0 .$$

**Proof.** As the proof of the previous Lemma, with Remark 4.4 instead of Lemma 4.5 and Remark 4.12 rather than Lemma 4.11 One should then remark that, by the definition of $\nu_H$,

$$\nu_H^*(\pi(H^*)) = - \Pi_H(\pi'(H^*)) .$$

\hfill $\square$
Lemma 5.3. For all $p = 0, \cdots, n - 1$:
\[
\sum_{F \in F_{p+2}} G_p' F' (F^*) \pi(F) + (n - p + 1) K \sum_{G \in F_{p+1}} \int_G \langle \nu_G(x), \pi(G^*) \rangle x dv - \sum_{H \in F_{p+1}} V(H) \nu_H^*(\pi(H^*)) = 0.
\]

Proof. Recall that by Lemma 3.4 above,
\[
\int_{P} (S_{p+1} + \frac{1}{2} (T', BT_p) + (n - p) K f S_p) x + T_p \nabla_X N dv = 0.
\]
However, by Lemma 4.5
\[
\lim_{\epsilon \to 0} \int_{P} f S_p x dv = \sum_{F \in F_{p+1}} \int_F \langle \nu_F(x), \pi(F^*) \rangle x dv,
\]
while Lemma 4.7 shows that
\[
\lim_{\epsilon \to 0} \int_{P} \frac{1}{2} (T', BT_p) x dv = \sum_{F \in F_{p+2}} V'(F^*) \pi(F),
\]
and Lemma 4.9 indicates that
\[
\lim_{\epsilon \to 0} \int_{P} T_p \nabla_X N dv = \sum_{F \in F_{p+1}} (-V(F) \nu_F^*(\pi(F^*)) + K \int_F \langle \nu_F(x), \pi(F^*) \rangle x dv).
\]
The proof follows.

Remark 5.4. If the sphere is centered at a point $a \in \mathbb{R}^{n+2}$ then the formula becomes:
\[
\sum_{F \in F_{p+2}} \int F' (F^*) \pi(F) + (n - p + 1) K \sum_{G \in F_{p+1}} \int_G \langle \nu_G(x), \pi(G^*) \rangle x dv - \sum_{H \in F_{p+1}} \left( V(H) \nu_H^*(\pi(H^*)) + \left( \int_H \nabla(x) dv, \pi(H^*) \right) a \right) = 0.
\]

Proof of Theorem 1.6. Formula $(E_p)$ is obtained as the sum of $(n - p + 1) K$ times formula $(F_p)$ in Lemma 5.1 and $p$ times formula $(G_p)$ in Lemma 5.3.

Proof of Theorem 1.7. Lemma 5.3 applied with $p = 0$, translates as:
\[
\sum_{F \in F_2} \theta_p \pi(F) + (n + 1) K \sum_{G \in F_1} \int_G \langle \nu_G(x), N_G \rangle x dv + \sum_{H \in F_1} V(H) N_H^* = 0.
\]
But it is easy to check that:
\[
\left( \int_{int(P)} x dv \right)' = \sum_{G \in F_1} \int_G \langle \nu_G(x), N_G \rangle x dv,
\]
so that:
\[
\sum_{F \in F_2} \theta_p \pi(F) + (n + 1) K \left( \int_{int(P)} x dv \right)' + \sum_{H \in F_1} V(H) N_H^* = 0.
\]
Using the first equation in Remark 1.9 leads to:
\[
\sum_{F \in F_2} \theta_p \pi(F) - \left( \sum_{H \in F_1} V(H) N_H \right)' + \sum_{H \in F_1} V(H) N_H^* = 0,
\]
and the result follows.
5.2. The second main formula. The proof of Theorem 1.8 follows from establishing a formula which is dual to the one in Theorem 1.6 (see the Remark below). This is the content of the following

**Proposition 5.5.** In any first-order deformation of a convex polyhedron \( P \) in \( M_{K}^{n+1} \) and for any \( p \in \{1, 2, \cdots, n-1\} \):

\[
(L_p) \quad (p + 1) \sum_{F \in F_{p+2}} V(F)(\pi'(F^*) - \nu_{F^*}(\pi(F^*))) + K(n - p) \sum_{G \in F_{p}} V'(G)\pi(G^*) = -K(n - p) \sum_{H \in F_{p+1}} V(H^*)\nu_{H}(\pi(H)) .
\]

**Proof.** The idea is to replace in the proof of Theorem 1.6 the position vector \( x \) by the (exterior) normal vector \( N \). We will indicate only the main steps in the proof and omit the details which are very similar to the arguments involved in the proof of Theorem 1.6.

Consider a compact hypersurface \( \Sigma \) in \( M_{K}^{n+1} \), oriented by a unit normal field \( N \). Consider a first order deformation of \( \Sigma \) with deformation vector field \( X \) and let \( X = \xi + fN \) be its decomposition into its tangent part \( \xi \) to \( \Sigma \) and normal one \( fN \). For any fixed vector \( a \), in \( \mathbb{R}^{n+2} \) in case \( K = 1 \) and in \( \mathbb{R}^{n+2} \) in case \( K = -1 \), and any \( p \in \{a, \cdots, n-1\} \), we have:

\[
div((a, N)T_p \xi) = \left< a, \left( \frac{1}{2}(I', T_p) - (p + 1)fS_{p+1} \right) N + BT_p \xi \right> .
\]

Integrating over \( \Sigma \) leads to the equation, valid for \( p = 0, \cdots, n-1 \):

\[
(Q_p) \quad \int_{\Sigma} BT_p \xi + \frac{N}{2}(I', T_p) - (p + 1)fS_{p+1} dv = 0 .
\]

The localization technique as in Section 3 gives the following formula, valid for all \( p = 0, \cdots, n-1 \), for first order deformations of a convex polyhedron \( P \subset M_{K}^{n+1} \):

\[
(Q'_p) \quad \sum_{F \in F_{p+2}} V(F^*)\nu_F(\pi(F)) - (p + 2) \sum_{F \in F_{p+2}} \int_{F^*} (\nu_F(\pi(F)), n) ndv^*(n) + \sum_{G \in F_{p+1}} V'(G)\pi(G^*) = 0 .
\]

Next, we have for all \( p = 0, \cdots, n-1 \):

\[
div((a, N)T_p \nabla_X N) = \left< a, \left( S_{p+1} + \frac{1}{2}(I', BT_p) + (n - p)KfS_p \right) N + BT_p \xi \right> .
\]

Integrating over \( \Sigma \) leads to the equation

\[
(M_p) \quad \int_{\Sigma} \left( S_{p+1} + \frac{1}{2}(I', BT_p) + (n - p)KfS_p \right) N + BT_p \nabla_X N dv = 0 ,
\]

Again, the localization technique gives that, for all \( p \in \{0, \cdots, n-1\} \):

\[
(M'_p) \quad \sum_{F \in F_{p+2}} V(F)(\pi'(F^*) - \nu_{F^*}(\pi(F^*))) + (n - p)K \sum_{G \in F_{p+1}} \int_{G^*} (\nu_{G}(\pi(G)), n) ndv^*(n) = 0 .
\]

Formula \( (L_p) \) follows taking, for \( p \in \{1, \cdots, n-1\} \), the combination \( K(n - p)(Q'_{p-1}) + (p + 1)(M'_p) \). \( \square \)

**Proof of Theorem 1.8** The proof is obtained adding, for \( p \in \{2, \cdots, n-1\} \), equation \( (E_p) \) in Theorem 1.6 to equation \( (L_{p-1}) \) in Proposition 5.5 above and using the fact that \( \nu_{G^*} = -\nu_{G'} \). \( \square \)

**Remark 5.6.** In the spherical case, formula \( (L_p) \) in Proposition 5.5 follows in fact directly under duality from Theorem 1.6.

**Proof.** Indeed, let \( p \in \{1, \cdots, n-1\} \). By Theorem 1.6,

\[
(n - p + 1) \sum_{H^* \in F_q^*} V(H^*)(\pi'(H^*) - \nu_{H^*}(\pi(H^*))) + \sum_{F \in F_{p+2}} V'(F^*)\pi(F) = p \sum_{G \in F_{p+1}} V(G)\nu_{G}(\pi(G^*)) .
\]

We can apply this formula to the dual polyhedron \( P^* \) for an integer \( q \in \{1, \cdots, n-1\} \). We call \( F_i^* \) the set of faces of codimension \( l \) of \( P^* \), and find that:

\[
(n - q + 1) \sum_{H^* \in F_q^*} V(H^*)(\pi'(H^*) - \nu_{H^*}(\pi(H^*))) + q \sum_{F^* \in F_{q+2}} V'(F^*)\pi(F^*) = q \sum_{G^* \in F_{q+1}} V(G^*)\nu_{G^*}(\pi(G)) .
\]
But a face of codimension $l$ of $P^*$ is the dual of a face of codimension $n + 2 - l$ of $P$. Moreover, we have seen that $\nu_{G^*} = -\nu_G$. So the formula above can be written as follows:

$$
(n - q + 1) \sum_{H \in F_{n+2-q}} V(H)(\pi'(H^*) - \nu_{H^*}(\pi(H^*))) + q \sum_{F \in F_{n-q}} V'(F)\pi(F^*) = -q \sum_{G \in F_{n+1-q}} V(G^*)\nu_G(\pi(G)) .
$$

Take $p \in \{2, \cdots, n\}$, then $n + 1 - p \in \{1, \cdots, n - 1\}$, so that it is possible to set $q = n + 1 - p$, so that the equation becomes:

$$
p \sum_{H \in F_{p+1}} V(H)(\pi'(H^*) - \nu_{H^*}(\pi(H^*))) + (n + 1 - p) \sum_{F \in F_{p-1}} V'(F)\pi(F^*) = -(n + 1 - p) \sum_{G \in F_p} V(G^*)\nu_G(\pi(G)) .
$$

Replacing $p$ by $p + 1$ leads to $(L_p)$.

**Proof of Remark 5.4** For the first formula, note that, for any fixed vector $a$ which belongs to $\mathbb{R}^{n+2}$ when $K = 1$ and to $\mathbb{R}^n$ when $K = -1$, its orthogonal projection on $M_{K+1}^n$ is $a - K\langle a, x \rangle x$, which satisfies:

$$
\text{div}(a - K\langle a, x \rangle x) = -(n+1)K(x, a) .
$$

Integrating this equality over the interior of an oriented hypersurface $\Sigma$ yields:

$$
(n+1)K \int_{\text{int}(\Sigma)} x dv = - \int_{\Sigma} N dv ,
$$

where $N$ is the unit exterior normal to $\Sigma$. The formula follows by taking the limit when $\Sigma$ converges to the boundary of a polyhedron in $M_{K+1}^n$.

For the second formula, choose $p \in \{0, \cdots, n\}$ and take the limit, as $\Sigma$ converges to the boundary of $P$, of the Minkowski formula:

$$
\int_{\Sigma} (n-p)KS_p x + (p+1)S_{p+1} N dv = 0 .
$$

We include a proof of the Minkowski formulae for the reader’s convenience. Let $a$ be any fixed vector in $\mathbb{R}^{n+2}$ in case $K = 1$ and in $\mathbb{R}^n$ in case $K = -1$. Call $Z := a - K\langle a, x \rangle x - \langle a, N \rangle N$ the projection of $a$ on $\Sigma$. Finally take a local orthonormal frame $\{e_1, \cdots, e_n\}$ on $\Sigma$. Since $\text{div}(T_p) = 0$, we have:

$$
\text{div}(T_p(Z)) = \sum_{i=1}^n \langle \nabla_{e_i} Z, T_p e_i \rangle
$$

$$
= \sum_{i=1}^n \langle D_{e_i} Z, T_p e_i \rangle
$$

$$
= \sum_{i=1}^n -K\langle a, x \rangle \langle T_p(e_i), e_i \rangle - \langle a, N \rangle \langle \nabla_{e_i} N, T_p(e_i) \rangle
$$

$$
= -(n-p)KS_p(a, x) - (p+1)S_{p+1}(a, N) ,
$$

where we used properties (2) and (3) in Lemma 2.2. Integrating over $\Sigma$ and replacing $p$ by $p - 1$ yields the result. 

**5.3. The Euclidean formulas.** The proof of the Euclidean version of our vector-valued Schl"afli formula follows by an elementary scaling argument from Theorem 1.6.

**Proof of Theorem 1.7** Consider an Euclidean polyhedron $P$, along with a first-order deformation. For each $t \in \mathbb{R}$, $t > 0$, we consider the spherical polyhedron $P_t$ which is the inverse image of the scaled polyhedron $tP$ by the projective map:

$$
S_{+}^{n+1} := \{(x_1, \cdots, x_{n+1}) \in S^{n+1} | x_{n+1} > 0 \} \rightarrow \mathbb{R}^{n+1}
$$

$$
x \mapsto (x_1, x_2, \cdots, x_n, 1).$$

Then $P_t \rightarrow N$ as $t \rightarrow 0$, where $N$ is the “north pole” of $S^{n+1}$, the point of coordinates $(0, 0, \cdots, 0, 1)$.

Let $H$ be a codimension $p$ face of $P$, and let $H_t$ be the corresponding faces of $P_t$. As $t \rightarrow 0$, $V(H_t^*)$ tends to a constant, while $\pi'(H_t)$ behaves as $V(H_t)$, i.e., as $t^{n+1-p}$. If $F$ is a codimension $p+2$ face of $P$ and $F_t$ is the corresponding face of $P_t$, then $V'(F_t^*)$ tends to a constant, while $\pi(F_t)$ scales as $t^{n-1-p}$; but $\pi(F_t)$ is almost
parallel to $N$, so that the component of $\pi(F_i)$ orthogonal to $N$ scales as $t^{n-p}$. Finally, if $G$ is a codimension \( p + 1 \) face of $P$ and $G_i$ is the corresponding face of $P_i$, then $V(G_i)$ scales as $t^{n-p}$, while $\nu_{G_i}^r(\pi(G_i^+))$ tends to a constant vector, which is easily seen to be $v_{G_i}^r(\pi(G^+))$. So the equation in the statement of Theorem 1.12 follows from equation $(E_p)$ by removing the negligible terms as $t \to 0$.

The proof of Theorem 1.11 follows by a very similar scaling argument from Theorem 1.7; we leave the details to the reader.

Remark 5.7. In the same way, Corollary 1.10 implies that for any first order deformation of a polyhedron $P \subset \mathbb{R}^{n+1}$ and any $p \in \{2, \ldots, n - 1\}$:

$$- \sum_{H \in F_p} V'(H) \pi(H^*) + \sum_{L \in F_{p+2}} V'(L^*) \pi(L) = 0.$$

6. Non-convex polyhedra

In this section we show how our formulas extend to non-convex polyhedra. We will do this with some details for the formulas appearing in Theorem 1.8 and Corollary 1.10. The other formulas, like for instance formula $(E_p)$ in Theorem 1.10 can be extended in a similar way. Since the notion of a dual polyhedron makes sense only for convex polyhedra, some care is needed in defining the quantities involved in those formulas in the non-convex case.

First, we define a non-convex polyhedron as the union of a finite number of convex polyhedra $P_1, \ldots, P_N$, with disjoint interiors, but such that $P_i$ shares a codimension 1 face with $P_{i+1}$ for $i \in \{1, \ldots, n - 1\}$.

Let now $F$ be a face of a convex polyhedron $P$. For reasons that will become clear, we change here our notation and denote its dual face, in the dual polyhedron $P'$, $F^*$, by $F^*_p$ in order to emphasize the dependence on the polyhedron $P$.

Consider a convex polyhedron $P$ which can be cut into two convex pieces $P', P''$, that is $P = P' \cup P''$. Let $F$ be a codimension $p$ face of $P, P'$ and $P''$. Note that $P' \cap P''$ is a convex polyhedron in a hyperplane $\Pi$ and $F^*_{P' \cap P''}$ will refer to the dual of the face $F$ in $P' \cap P''$ as a polyhedron in $\Pi$.

Proposition 6.1. We have:

$$V(F^*_p) + V(F^*_p) = V(F^*_p) + \frac{V(S^{p-1})}{V(S^{p-2})} V(F^*_{P' \cap P''})$$

$$\pi(F^*_p) + \pi(F^*_p) = \pi(F^*_p) + \frac{V(S^p)}{V(S^{p-1})} \pi(F^*_{P' \cap P''}),$$

where $V(S^k)$ is the volume of the canonical $k$-sphere.

Proof. In this setting, $P' \cap P''$ can be considered either as a polyhedron with non-empty interior in the hyperplane $\Pi$ of $M^{n+1}_K$ which contains it or as a (degenerate) polyhedron in the whole $(n + 1)$-dimensional ambient space $M^{n+1}_K$. Denote by $\overline{F}_{P' \cap P''}$ the dual face of $F$ in the latter case. Note that this does not coincide with its dual $F^*_{P' \cap P''}$ in the former case.

Recall that given a face $F$ of a convex polyhedron $P$ in $S^{n+1}$ (resp. in $H^{n+1}$), then $F^*_p$ is the set of points $x \in S^{n+1}$ (resp. in de Sitter space $S^{n+1}_1$) such that the oriented plane orthogonal to $x$ is a support plane of $P$ at $F$. It follows that:

$$F^*_p \cap F^*_p = F^*_{P' \cap P''},$$

$$F^*_p \cup F^*_p = \overline{F}_{P' \cap P''}.$$ 

Consequently:

$$V(F^*_p) + V(F^*_p) = V(F^*_p) + V(\overline{F}_{P' \cap P''})$$

$$\pi(F^*_p) + \pi(F^*_p) = \pi(F^*_p) + \pi(\overline{F}_{P' \cap P''}).$$

Now observe that $F^*_{P' \cap P''}$ is a polyhedron (with nonempty interior) in an equatorial sphere $S^{p-2}$ of dimension $(p - 2)$ in the unit sphere $S^{p-1}$ of the linear space $\text{vect}(\overline{F}_{P' \cap P''})$ generated by $\overline{F}_{P' \cap P''}$ and that $\overline{F}_{P' \cap P''}$ is
obtained by joining the corresponding poles (by shortest geodesic arcs) to each point of \( F^*_{P \cap P'} \). It is then easily checked that:

\[
V(F^*_{P \cap P'}) = \frac{V(S^p-1)}{V(S^p-2)} V(F^*_P \cap F^*_P) \quad \text{and} \quad \pi(F^*_{P \cap P'}) = \frac{V(S^p)}{V(S^p-1)} \pi(F^*_P \cap F^*_P)
\]

and the formulas follow.

The proof would actually work in the same way for a convex polyhedron cut into more than two convex pieces, with some additional terms corresponding to the intersections of more than two of the pieces.

It is clear from this proposition that, if the formulas in Theorem 1.8 and Corollary 1.10 apply to \( P' \) and to \( P'' \), then they must apply to \( P \). For instance, in the formula in Theorem 1.8 the first term is:

\[
(n + 1 - p) K \sum_{F \in F_{p-1}} V'(F) \pi(F^*_P) = (n + 1 - p) K \sum_{F \in F_{p-1}} V'(F) \pi(F^*_P) + (n + 1 - p) P \sum_{F \in F_{p-1}} V'(F) \pi(F^*_P) -
\]

\[
- (n + 1 - p) K \frac{V(S^p-1)}{V(S^p-2)} \sum_{F \in F_{p-2}} V'(F) \pi(F^*_P) \cap F^*_P.
\]

where the sums are over \( n + 2 - p \)-dimensional faces of \( P, P', P'' \) and \( P' \cap P'' \) respectively. The second term is:

\[
(n + 1 - p) K \sum_{G \in F_p} V(G^*_P) \pi'(G) = (n + 1 - p) K \sum_{G \in F_p} V(G^*_P) \pi'(G) + (n + 1 - p) K \sum_{G \in F_p} V(G^*_P) \pi'(G) -
\]

\[
- (n + 1 - p) K \frac{V(S^p-1)}{V(S^p-2)} \sum_{G \in F_{p-1}} V(G^*_P) \pi'(G),
\]

where the sums are now over \( n + 1 - p \)-dimensional faces of \( P, P', P'' \) and \( P' \cap P'' \) respectively. The third term is:

\[
p \sum_{H \in F_{p+1}} V(H) \pi'(H^*_P) = p \sum_{H \in F_{p+1}} V(H) \pi'(H^*_P) + p \sum_{H \in F_{p+1}} V(H) \pi'(H^*_P, P'') -
\]

\[
- p \frac{V(S^p+1)}{V(S^p)} \sum_{H \in F_p} V(H) \pi'(H^*_P, P''),
\]

where the sums are over \( n - p \)-dimensional faces of \( P, P', P'' \) and \( P' \cap P'' \) respectively. Finally the fourth term is:

\[
p \sum_{L \in F_{p+2}} V'(L^*_P) \pi(L) = p \sum_{L \in F_{p+2}} V'(L^*_P) \pi(L) + p \sum_{L \in F_{p+2}} V'(L^*_P) \pi(L) -
\]

\[
- p \frac{V(S^p+1)}{V(S^p)} \sum_{L \in F_{p+1}} V'(L^*_P) \pi(L),
\]

where the sums are now over \( n - p - 1 \)-dimensional faces of \( P, P', P'' \) and \( P' \cap P'' \) respectively. Adding the four terms and using the formula in Theorem 1.8 for \( P' \) and for \( P'' \) leaves us with:

\[
(n - p - 1) K \frac{V(S^p-1)}{V(S^p-2)} \left( \sum_{F \in F_{p-2}} V'(F) \pi(F^*_P) \cap F^*_P + \sum_{G \in F_{p-1}} V(G^*_P) \pi'(G) \right) -
\]

\[
- p \frac{V(S^p+1)}{V(S^p)} \left( \sum_{H \in F_p} V(H) \pi'(H^*_P) + \sum_{L \in F_{p+1}} V'(L^*_P) \pi(L) \right).
\]

Now an elementary computation shows that, for any \( k \geq 2 \):

\[
\frac{V(S^k+1)}{V(S^k)} = \frac{k - 1}{k} \frac{V(S^k-1)}{V(S^k-2)}.
\]
so that the remaining term is simply:

\[
-V(S^{p-1}) \frac{1}{V(S^{p-2})} \left[ (n - (p - 1)) \left( \sum_{F \in F_{p-2}} V'(F) \pi(F_{p-1}) + \sum_{G \in F_{p-1}} V(G_{p-1}) \pi'(G) \right) + \right.
\]

\[
\left. + (p - 1) \left( \sum_{H \in F_p} V(H) \pi'(H_{p-1}) + \sum_{L \in F_{p+1}} V'(L_{p-1}) \pi(L) \right) \right].
\]

So this vanishes because of the formula in Theorem 1.8 applied to \( P' \cap P'' \).

Now we can prove Theorem 1.8 for non-convex polyhedra. First, we have to define the analogues of the quantities \( V(F^*) \) and \( \pi(F^*) \) for a face \( F \) of a non-convex polyhedron \( P \). The polyhedron \( P \) can be decomposed into a finite number of convex polyhedra \( P_1, \ldots, P_N \), i.e. \( P = \cup_i P_i \) and the \( P_i \) have pairwise disjoint interiors. Then we can apply the formula of Proposition 6.1 (actually the analogous formula for more than two polyhedra if necessary) and call \( V(F^*) \) and \( \pi(F^*) \) the results. \( V(F^*) \) and \( \pi(F^*) \) are independent of the decomposition \( P = \cup_i P_i \) of \( P \) into convex polyhedra \( P_i \); if \( P = \cup_j P'_j \) is another decomposition into convex pieces, taking the finer decomposition \( P = \cup_{i,j} P_i \cap P'_j \) and applying Proposition 6.1 shows that the values of \( V(F^*) \) and \( \pi(F^*) \) obtained by \( \cup_i P_i \cup_j P'_j \) and \( \cup_{i,j} P_i \cap P'_j \) are identical.

With this definition, it is not too difficult to prove that the formulas in Theorem 1.8 and Corollary 1.10 apply to deformations of a non-convex polyhedron \( P \). Start by choosing a decomposition of \( P \) into convex pieces \( (P_i)_{1 \leq i \leq N} \), and use the corresponding formulas for the \( P_i \). Then do as in the proof above for \( P = P' \cup P'' \) to get rid of the terms involving the intersection of two or more of the \( P_i \)'s, and the result follows.

**Remark 6.2.** Similarly, the corresponding formulas make sense for non-convex polyhedra in the Euclidean case.

### 7. Applications

#### 7.1. Deformations of polygons

We consider first the simple result on the possible first-order variations of the lengths and angles of spherical (or hyperbolic) polygons.

**Proof of Theorem 1.7** It follows from Theorem 1.7 that, under any first-order deformation of a spherical convex polygon, the equation in the statement is satisfied. The formula also applies to non-convex polygons. Indeed, a general polygon can be cut up into convex ones and we can apply the formula to each of them and sum up the results. The terms corresponding to added vertices sum up to zero (since the total angle remains equal to \( 2\pi \)) and so do the terms corresponding to the added edges as each such edge appears twice with opposite orientations.

Let \( P_n \) be the space of polygons with \( n \) vertices in \( S^2 \) for which not two vertices are the same point, considered up to global isometries. It is a manifold of dimension \( 2n - 3 \). The functions \( l_i \) and \( \theta_i, 1 \leq i \leq n \), are defined on \( P_n \). At each polygon \( p \in P \), the differentials of the \( l_i \) and the \( \theta_i \) generate \( T_p P \), because a polygon is uniquely determined by its edge lengths and its dihedral angles. So the \( dl_i \) and the \( d\theta_i \) generate a vector subspace of dimension \( 2n - 3 \). However they satisfy three linear relations (those in Theorem 1.13 which are obviously linearly independent as soon as \( p \) has at least 3 vertices). This shows that those differentials satisfy no other relation.

#### 7.2. Recovering the scalar Schlafli formulas

We now show how the previous considerations allow to recover the higher Schlafli formulas [SS03]. We explain this in the spherical case, the hyperbolic case is similar, see the next remark. This will follow from the following two formulas.

**Lemma 7.1.** For any first order deformation of a convex polyhedron \( P \subset S^{n+1} \subset \mathbb{R}^{n+2} \) we have:

\[
\forall p \in \{1, \ldots, n\}, \sum_{F \in F_p} V(F^*) V'(F) - p \sum_{G \in F_{p+1}} \langle \nu_G(\pi(G)), \pi(G^*) \rangle = 0,
\]

\[
\forall p \in \{0, \ldots, n-1\}, (n-p) \sum_{G \in F_{p+1}} \langle \nu_G(\pi(G)), \pi(G^*) \rangle + \sum_{H \in F_{p+2}} V'(H^*) V(H) = 0.
\]
Proof. Let \( a \in \mathbb{R}^{n+2} \) be any point. Apply Remark 5.2 to the polyhedron \( \tau_n(P) \) which is contained in the unit sphere centered at \( a \). This gives:

\[
\sum_{F \in F_p} V(F^*)(\pi'(F) + V'(F) a - \nu_F(\pi(F))) - p \sum_{G \in F_{p+1}} \int_{\tau_n(G)} (\nu_F(G), \pi(G^*)) dv = 0.
\]

Applying \((F_p)\) to \( P \) and taking into account the equation:

\[
\int_{\tau_n(G)} (\nu_F(G), \pi(G^*)) dv = \int_G (\nu_F(x), \pi(G^*)) dv + \langle \nu_F(G), \pi(G^*) \rangle a,
\]

we conclude that

\[
\left( \sum_{F \in F_p} V(F^*) V'(F) - p \sum_{G \in F_{p+1}} \langle \nu_F(G), \pi(G^*) \rangle \right) a = 0.
\]

We get the first formula. The second formula is obtained in the same way using Remark 5.4. \( \square \)

An immediate consequence is

**Corollary 7.2** (Higher scalar Schl"afli formulas). For any first order deformation of a convex polyhedron \( P \subset S^{n+1} \) we have, for each \( p \in \{1, \ldots, n-1\} \)

\[
(n-p) \sum_{F \in F_p} V(F^*) V'(F) + p \sum_{H \in F_{p+2}} V'(H^*) V(H) = 0.
\]

**Remark 7.3.** We can recover in a similar way the higher scalar Schl"afli formulas in the hyperbolic case. Indeed, recall that in the hyperboloid model of the hyperbolic space, the latter is the pseudosphere of imaginary radius \( i \), centered at the origin, in the Minkowski space \( \mathbb{R}^{1,n+2}_1 \). Then one may also consider the same sphere but now centered at any point \( a \in \mathbb{R}^{1,n+2}_1 \) and mimic the arguments in the spherical case. We omit the details.

### 7.3. Deformations of the sphere as a cone-manifold.

The formulas found for the deformations of spherical polyhedra can be applied to deformations of the sphere which let cone singularities appear along a stratified subset. We first consider the two-dimensional case, the singular locus is then a finite set of points.

**Proof of Theorem 7.13** We can choose a decomposition of the sphere \( (S^2, g_0) \) as a union of interiors of spherical polygons, with a set of vertices which includes the \( v'_i \)s. Under the deformation \( g_t \) of \( g_0 \), those polygons are deformed, and it is possible to apply to each of them Theorem 1.1. However, the terms corresponding to the first-order variations of the lengths of the edges cancel out – each edge appears twice and the terms are opposite to each other – so only the terms corresponding to the variations of the angles remain. This shows that equation (1) has to be satisfied.

Conversely, let \( v_1, \ldots, v_n \) be points in \( S^2 \), and let \( t_1, \ldots, t_n \in \mathbb{R} \), chosen so that \( \sum_{i=1}^n t_i v_i = 0 \). Consider a closed simple polygonal line \( p \) which has the \( v_i \) among its vertices. We can actually suppose that its vertices are \( v_1, \ldots, v_p \), with \( p \geq n \). Then the complement of \( p \) is the disjoint union of two disks \( D_+ \) and \( D_- \). We can apply Theorem 1.13 to each of those disks, taking \( \theta_i = -t_i/2 \) if \( i \leq n \) and \( \theta_i = 0 \) if \( n < i \leq p \), and \( t_i' = 0 \). Theorem 1.13 shows that there exist first-order deformations of \( D_+ \) and \( D_- \) such that the first-order variation of the total angle at \( v_i \) is \( -t_i \). This leads to a first-order deformation of the sphere – obtained by gluing \( D_+ \) and \( D_- \) isometrically along their boundary – as needed. \( \square \)

**Proof of Theorem 1.14** The argument used in the proof of Theorem 1.13 can also be used here, based on Theorem 1.7 applied to the top dimensional cells of a cell decomposition of \( S^{n+1} \). The terms corresponding to codimension 1 faces vanish since each appears twice, once for each codimension 0 cell containing the codimension 1 cell considered, and those two terms vanish. \( \square \)

**Proof of Theorem 1.16** Applying formula \((K_p)\) to each of all (maximal dimension) cells of \( C \), we obtain that:

\[
\sum_{C \in C_0} \left( n - p + 1 \right) \left( \sum_{F \in F_{p-1}} V'(F) \pi(F^*) - \sum_{G \in F_p} V'(G^*) \pi(G) \right).
\]
and Corollary 7.4. Remark 5.7 is:

consequence of Theorem 1.7 and Corollary 1.10 (more precisely their non-convex versions, cf. Section 6) and

the volume of all faces (except the interior of the polyhedron) invariant up to the first order, an immediate

some constraints on deforming polyhedra isometrically. Indeed, as a first order isometric deformation keeps

keeps the distance on the codimension 1 faces invariant up to the first order. We can deduce from our formulas

isometric first order deformation of $P$ deformations have infinitesimal versions: a first order deformatio n of a polyhedron

where $C_0$ is the set of codimension 0 faces of $C$, i.e., the set of the maximal dimension cells of $C$. The previous

formula can be written as

$$(n - p + 1) \left( \sum_{F \in F_p} V'(F) \sum_{C \supseteq F} \pi(F^*_C) - \sum_{G \in F_p} \sum_{C \supseteq G} V'(G^*_C) \pi(G) \right) +$$

$$\sum_{H \in F_{p+1}} V'(H) \sum_{C \supseteq H} \pi(H^*_C) + \sum_{L \in F_{p+2}} \sum_{C \supseteq L} V'(L^*_C) \pi(L) = 0 .$$

The proof therefore follows by identifying the different terms with the quantities introduced before the statement

of the theorem. \qed

7.4. Cone singularities in hyperbolic polygons. The arguments used above to describe first-order vari-ations of the 2-dimensional sphere among cone-surfaces can be applied also for deformations of a hyperbolic polygon among hyperbolic polygons containing cone singularities at some points. This is of course based on the hyperbolic version of Theorem 1.1 applied to each polygon in a cell decomposition of the hyperbolic polygon considered, with the singular points among the vertices. The terms corresponding to the interior edges of the cell decomposition cancel, and the terms at the interior vertices sum up to the first-order variations of the total angles at those vertices. This yields Theorem 1.17.

As mentioned in the introduction, it is also possible to state corresponding results in higher dimensions, for deformations of hyperbolic polyhedra in which cone singularities appear along a stratified subset. We leave the details to the interested reader.

7.5. Isometric deformations. Let $P$ be a polyhedron in $\mathbb{R}^{n+1}, H^{n+1}$ or $S^{n+1}$. A deformation of $P$ is called isometric if each codimension 1 face of $P$ remains congruent to itself (through rigid motions) during the deformation. For example deforming a polyhedron through ambient rigid motions induces trivial isometric deformations. $P$ is said flexible if it admits non trivial isometric deformations and rigid otherwise. A result going back to Legendre and Cauchy states that convex polyhedra are rigid. R. Connelly [Con77] constructed the first example of a flexible polyhedron in $\mathbb{R}^3$. To our knowledge there are no known examples in higher dimensions. Isometric deformations have infinitesimal versions: a first order deformation of a polyhedron $P$ is called isometric if it keeps the distance on the codimension 1 faces invariant up to the first order. We can deduce from our formulas some constraints on deforming polyhedra isometrically. Indeed, as a first order isometric deformation keeps the volume of all faces (except the interior of the polyhedron) invariant up to the first order, an immediate consequence of Theorem 1.17 and Corollary 1.10 (more precisely their non-convex versions, cf. Section 9) and Remark 5.7 is:

Corollary 7.4. Let $P$ be a polyhedron in $\mathbb{R}^{n+1}$ ($K = 0$), $H^{n+1}$ ($K = -1$) or $S^{n+1}$ ($K = 1$). Under any isometric first order deformation of $P$ we have, for any $p \in \{2, \ldots, n - 1\}$:

$$(n - p + 1)K \left( - \sum_{G \in F_p} V'(G^*) \pi(G) \right) + p \left( \sum_{L \in F_{p+2}} V'(L^*) \pi(L) \right) = 0 ,$$

and

$$\sum_{F \in F_2} \theta'(F) \pi(F) = 0 .$$

An analogous formula can be derived using Theorem 1.8.
7.6. Possible future extensions. It is quite natural to wonder whether the results stated here on the first-order deformations of the sphere – or of hyperbolic polygons or polyhedra – extend to corresponding deformations of spherical, Euclidean or hyperbolic manifolds – for which cone singularities appear – or of cone-manifolds. We do not wish to elaborate much on this here, except to mention that this indeed seems to be possible, although more elaborate constructions are needed. It appears in particular necessary to consider a flat bundle over the manifold which is deformed, which “contains” the relations which have to be satisfied under the deformations. For flat (cone-)manifolds this is simply the tangent bundle, but for spherical or hyperbolic $n + 1$-dimensional manifolds it is a bundle of dimension $n + 2$, obtained as the pull-back of the ambient space by the developing map sending the (non-singular part of) the (cone-)manifold to $S^{n+1}$ or $H^{n+1}$, respectively. We hope to come back to this question in a subsequent work.

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