CLT for continuous random processes under approximations terms.

E.Ostrovsky\textsuperscript{a}, L.Sirota\textsuperscript{b}

\textsuperscript{a} Corresponding Author. Department of Mathematics and computer science, Bar-Ilan University, 84105, Ramat Gan, Israel.

E - mail: galo@list.ru eugostrovsky@list.ru

\textsuperscript{b} Department of Mathematics and computer science. Bar-Ilan University, 84105, Ramat Gan, Israel.

E - mail: sirota3@bezeqint.net

ABSTRACT.

We formulate and prove a new sufficient conditions for Central Limit Theorem (CLT) in the space of continuous functions in the terms typical for the approximation theory.

We prove that the sufficient conditions for continuous CLT obtained by N.C.Jain and M.B.Marcus are under some natural additional conditions necessary.

We provide also some examples in order to show the exactness of obtained results and illustrate briefly the applications in the Monte-Carlo method.

\textit{Key words and phrases:} Algebraic and trigonometrical approximation, random processes (r.p.) and random fields (r.f.), metric entropy, De la Vallee-Poussin kernel and approximation, Central Limit Theorem in the space of continuous functions, majorizing and minorizing measures, module of continuity, lacunar series, generating functional, uniform equiconvergense.

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1 Notations. Statement of problem.

Let $\eta(t), t \in T = [0, 2\pi]$ be separable centered (zero mean) random process (r.p.) with finite covariation function $R(t,s) = \text{cov}(\eta(t), \eta(s)) = E\eta(t)\eta(s), \{\eta_i(t)\}, \ i = 1, 2, \ldots$ be a sequence of independent copies (identical distributed) of $\eta(t)$. We will denote as usually the probabilistic notions by $P$, $E$, $\text{cov}$, $\text{Var}$ etc.

Let as introduce the following sequence of a random processes:

$$
\zeta_n(t) = n^{-1/2} \sum_{j=1}^{n} \eta_j(t).
$$

(1.1)

Let also $C(T)$ be a Banach space of all periodical: $f(0) = f(2\pi)$ continuous functions equipped with usually uniform norm $||f|| = \max_{t \in T} |f(t)|$, $\zeta_{\infty}(t) = \zeta(t)$ be a
centered separable Gaussian process with at the same covariation function $R(t,s)$. It is clear that the finite-dimensional distributions of r.p. $\zeta_n(t)$ converges as $n \to \infty$ to the ones for $\zeta_\infty(t)$.

**Definition 1.1.** We will say as ordinary that the r.p. $\eta(t)$ satisfies the Central Limit Theorem in the space $C(T)$, notation: $\eta(\cdot) \in CLT(C(T))$, if for any continuous bounded functional $F : C(T) \to \mathbb{R}$

$$
\lim_{n \to \infty} E F(\zeta_n(\cdot)) = EF(\zeta_\infty(\cdot)).
$$

Our purpose in this article is to formulate and prove sufficient conditions for CLT in the space $C(T)$ under the terms which are habitual in the theory of approximation.

This problem has a long history; see e.g. [11], [23], [22], [25], [30], [37]. About applications CLT in the space $C(T)$ in the Monte-Carlo method and in statistics see, e.g. [15], [19], [37].

Indeed, if $\eta(\cdot) \in CLT(C(T))$, then for all positive values $u$

$$
\lim_{n \to \infty} P \left( \max_{t \in T} |\zeta_n(t)| > u \right) = P \left( \max_{t \in T} |\zeta_\infty(t)| > u \right).
$$

This problem contains as a subproblem the finding of a sufficient condition for continuity with probability one of random processes, Gaussian or not, see [2] - [5], [12] - [14], [16], [20], [21], [24], [22] - [27], [31] - [35], [38] - [41], [43], [47], [49] - [53], [55].

Note that in the articles [11], [23], [22], [30] the CLT is formulated in the so-called metric entropy terms, in [11], [20], [37], [43] by means of the notions majorizing measure.

The paper is organized as follows. In the next section we formulate and prove a new sufficient conditions for Central Limit Theorem (CLT) in the space of continuous functions. In the third section we prove that the sufficient conditions for continuous CLT obtained by N.C.Jain and M.B.Marcus are still under some natural additional conditions necessary.

We provide also some examples in the 4th section in order to show the exactness of obtained results and to illustrate briefly the applications in the Monte-Carlo method. The last section contains some concluding remarks.

We need to introduce some useful notations. We will use the well-known Vallee-Poussin sums, which play a very important role in the approximation theory, see e.g. [9], [54], chapter 5.

Recall that the Vallee-Poussin kernel $K_{n,p}(t)$ is defined as follows:

$$
K_{n,p}(t) = \frac{\sin((2n + 1 - p)t/2) \cdot \sin((p + 1)t/2)}{2(p + 1) \sin^2 t/2}, \quad p \in [1, 2, \ldots, n].
$$

It is known that $K_{n,p}(t)$ is trigonometrical polynomial of degree $n$: $K_{n,p}(t) \in A(n)$, where $A(n)$ denotes the set of all $2\pi$ periodical trigonometrical polynomials of degree $\leq n$. 


The Vallee-Poussin approximation (sum) $V_{n,p}[f](t)$ for a periodical integrable function $f$ may be defined as follows:

$$V_{n,p}[f](t) := [f * K_{n,p}](t)$$

(periodical convolution). Recall that

$$V_{n,p}[f](t) = \frac{1}{p + 1} \sum_{k=n-p}^{n} S_k[f](t),$$

where $S_k[f](t)$ is the $k^{th}$ partial Fourier sum for the function $f$.

We pick hereafter for definiteness for the values $n \geq 4$ $p = p(n) \overset{def}{=} \text{Ent}(n/2) := [n/2]$, (integer part), so that $V_{n,p}[f](t) \in A(n)$ and

$$||f(\cdot) - V_{n,p(n)}[f](\cdot)|| \leq C \cdot E([n/2], f),$$

where $E(m, f)$ denotes the error in the uniform norm of the best trigonometrical approximation of the function $f$ by means of trigonometrical polynomials with degree $\leq m$:

$$E(m, f) = \inf_{g \in A(m)} \max_{t \in T} |f(t) - g(t)| = \inf_{g \in A(m)} ||f(t) - g(t)||,$$

see [29], chapter 6.

We define for arbitrary r.p. $\xi(t)$ the so-called *generating functional* $\Phi(\xi; \psi)$ as follows:

$$\Phi_{\xi} = \Phi(\xi; \psi) = E e^{\int_T \xi(s) \, d\psi(s)},$$

if there exists. Here $\psi(s), s \in [0, 2\pi]$ is any deterministic function of bounded variation.

Further, we denote by $\mathcal{N}$ a set of all strictly increasing sequences of natural numbers \{n(k)\}, $k = 1, 2, \ldots$, $n(1) = 1$; and define for each such a sequence $\vec{n} = \{n(k)\} \in \mathcal{N}$

$$W^{(k)}(t) = W_{\vec{n}}(n(k+1), n(k))(t) := V_{n(n(k+1), n(k))}(t) - V_{n(n(k), n(k+1))}(t),$$

and we define for arbitrary periodical random process $\xi = \xi(t), t \in T$

$$Z_k[\xi](t) = Z_{\vec{n}, k}[\xi](t) = [W^{(k)} \ast \xi](t) = [V_{n(n(k+1), n(k))} \ast \xi](t) - [V_{n(n(k), n(k+1))} \ast \xi](t),$$

$$\Psi_{\vec{n}}(\xi; \lambda, n(k), n(k + 1)) = \Psi(\xi; \lambda, n(k), n(k + 1)) = (2\pi)^{-1} \int_T E e^{\lambda Z_k[\xi](t)} \, dt, \ \lambda = \text{const} > 0;$$

evidently, $\Psi_{\vec{n}}(\cdot; \cdot, \cdot, \cdot)$ may be easily expressed through the generating functional $\Phi(\cdot)$. Namely, let
\[ \psi_t^{(k)}(s) = \int_0^s W^{(k)}(t-x) \, dx, \]

then

\[ \Psi(\xi; \lambda, n(k), n(k+1)) = (2\pi)^{-1} \int_T E e^{\lambda \int_T \xi(s) W^{(k)}(t-s) \, ds} \, dt = \]

\[ (2\pi)^{-1} \int_T dt \, E \, e^{\lambda \int_T \xi(s) \, \psi_t^{(k)}(s)} = (2\pi)^{-1} \int_T \Phi(\xi; \lambda \psi_t^{(k)}(\cdot)) \, dt. \]  

Define also at last

\[ U(\Phi_\xi; n(k), n(k+1)) = U_\infty(\xi; n(k), n(k+1)) = \]

\[ \inf_{\lambda > 0} \left[ \log n(k+1) + \log \Psi_\infty(\xi; \lambda, n(k), n(k+1)) \right]. \]  

2 Main result: sufficient conditions for CLT for continuous processes.

A. Weak compactness of a family of random processes.

Let \( \xi_\alpha(t), \alpha \in \mathcal{A} \) be a family of separable stochastically continuous periodical processes, \( \mathcal{A} \) be arbitrary set. Assume that for some non-random point \( t_0 \in T \) the family of one-dimensional r.v. \( \xi_\alpha(t_0) \) is tight.

Theorem 2.1.
1. Let \( \alpha \in \mathcal{A} \) be a given. If for some sequence \( \{n(k)\} \in \mathcal{N} \)

\[ \sum_{k=1}^{\infty} U(\Phi_{\xi_\alpha}; n(k), n(k+1)) < \infty, \]  

then almost all trajectories of r.p. \( \xi_\alpha(t) \) are continuous.

2. Suppose that there exists a single sequence \( \{n(k)\} \) such that for all the set \( \mathcal{A} \) the series \( (2.1) \) are uniform equiconvergent:

\[ \lim_{m \to \infty} \sup_{\alpha \in \mathcal{A}} \sum_{k=m}^{\infty} U(\Phi_{\xi_\alpha}; n(k), n(k+1)) = 0. \]  

Then the family of distributions \( \mu_\alpha(\cdot) \) in the space \( C(T) \) generated by \( \xi_\alpha(\cdot) \):

\[ \mu_\alpha(A) = P(\xi_\alpha(\cdot) \in A) \]

is weakly compact in this space.

Proof.
Let \( \alpha \in \mathcal{A} \) be a fix. We will use one of the main results of the article [43]:
\[ E\|Z_k[\xi]\| \leq C \ U(\Phi\xi; n(k), n(k+1)), \quad (2.3) \]

where \( C \) is an absolute constant.

We conclude by virtue of condition (2.1) that the following series converges:

\[ \sum_{k=1}^{\infty} E\|Z_k[\xi]\| < \infty, \]

with him

\[ \sum_{k=1}^{\infty} \|Z_k[\xi]\| < \infty \quad (\text{mod } P). \]

Therefore, the partial sums of \( Z_k[\xi](t) \), namely, the sequence of the r.p.

\[ \sum_{m=1}^{k} Z_m[\xi](t) = [V_n(k+1), p(n(k+1)) \ast \xi](t) \quad (2.4) \]

converges uniformly on \( t, \ t \in T \) also with probability one. The limiting as \( k \to \infty \) process coincides with \( \xi(t) \) since it is continuous in probability; thus, it is sample part continuous. We proved the first proposition of theorem 1.1.

Further, let the condition (2.2) be satisfied. We denote

\[ \epsilon(m) = \sup_{\alpha \in A} \sum_{k=m}^{\infty} U(\Phi\xi; n(k), n(k+1)), \quad (2.5) \]

then \( \lim_{m \to \infty} \epsilon(m) = 0. \)

As long as

\[ \sum_{m=1}^{k} Z_m[\xi](t) = [V_n(k+1), p(n(k+1)) \ast \xi](t) \]

is as the function on the variable \( t \) the trigonometrical polynomial of degree less than \( n(k+1) \), we conclude

\[ \sup_{\alpha \in A} E E(n(k+1), \xi) \leq C_1 \epsilon(n(k)) \to 0, \ k \to \infty. \quad (2.6) \]

We can use the inverse theorems of approximation theory; see for example, Stechkin’s estimate ([54], chapter 6, section 6.1):

\[ \omega(f, 1/n) \leq \frac{C_2}{n} \cdot \sum_{m=0}^{n} E_m(f). \]

We ensue for the module of continuity \( \omega(\xi, 1/n) \) for periodical r.p. \( \xi(t) \):

\[ \sup_{\alpha \in A} E \omega(\xi, 1/n) \leq \frac{C_3}{n} \cdot \sum_{m=0}^{n} E E_m(\xi). \]

Note as a consequence taking into account the monotonicity of the function \( m \to E(m, f) \):
\[
\lim_{n \to \infty} \sup_{\alpha \in A} E \omega(\xi_\alpha(\cdot), 1/n) = 0 \quad (2.7)
\]

and therefore
\[
\forall \epsilon > 0 \Rightarrow \lim_{\delta \to 0^+} \sup_{\alpha \in A} P(\omega(\xi_\alpha(\cdot), \delta) > \epsilon) = 0.
\]

Our proposition follows now from theorem 1 in the book of I.I.Gikhman and A.V.Skorohod [17], chapter 9, section 2; see also [46], after applying the Tchebychev’s inequality.

**Remark 2.1.** The conditions of theorem 2.1 are essentially non-improvable still for the Gaussian processes, see [12], [27], [40], [41], [42], [43].

**B. CLT for random processes.**

**Theorem 2.2.**

Suppose that there exists a single strictly increasing sequence \( \{s(k)\} \) of natural numbers such that the following series (2.8) are uniform equiconvergent:

\[
\lim_{m \to \infty} \sup_{n} \sum_{k=m}^{\infty} U \left( \Phi_{n}(\psi)/\sqrt{n}; s(k), s(k+1) \right) = 0.
\]

Then the r.p. \( \eta(t), t \in T \) satisfies CLT in the space \( C(T) \).

**Proof.** The generating functional for the r.p. \( \zeta_n(\cdot) \), i.e. \( \Phi_{\zeta_n}(\psi) \) may be calculated and uniformly estimated as follows:

\[
\Phi_{\zeta_n}(\psi) = \Phi_{\eta}(\psi/\sqrt{n}),
\]

\[
\Phi_{\zeta_n}(\psi) \leq \sup_{n} \left[ \Phi_{\eta}(\psi/\sqrt{n}) \right] < \infty.
\]

It remains to apply the second assertion of theorem 2.1, choosing \( \{\alpha\} = \{1, 2, \ldots\} \).

### 3 Necessary conditions for CLT for continuous processes.

In this section the set \( T \) be instead \([0, 2\pi]\) the arbitrary compact metric space equipped with the distance function \( d = d(s, t) \). The r.p. \( \eta(t) \) remains to be separable and centered.

N.C.Jain and M.B.Marcus in [22] are formulated and proved in particular the following famous result (we retell its in our notations).

**Theorem of Jain and Marcus.** Assumptions: there exists a random variable \( M \) such that
\begin{equation}
|\eta(t) - \eta(s)| \leq M \cdot \rho(t, s), \tag{3.1}
\end{equation}

condition of factorization; where \( \rho(t, s) \) is continuous deterministic distance, more exactly, semi-distance, possible different on the source distance \( d \) on the set \( T \);

\[ \mathbb{E}M^2 < \infty, \tag{3.2} \]

moment condition;

\[ \int_0^1 H^{1/2}(T, \rho, z) \, dz < \infty, \tag{3.3} \]

entropy condition. Here \( H(\rho, T, \epsilon) \) denotes a metric entropy function of the set \( T \) relative the distance \( \rho(\cdot, \cdot) \) at the point \( \epsilon \), \( 0 < \epsilon < 1 \).

Then the random process (field) \( \eta(t) \) satisfies the CLT in the space \( C(T, d) \).

We discuss in this section the necessity of all conditions 3.1; 3.2; 3.3 for the CLT in the space \( C(T, d) \).

So, it will be presumed sometimes further that the r.p. \( \eta(t) \) satisfies this CLT.

**Proposition 3.1.** Assume that the r.p. \( \eta(t) \) is sample part continuous, still without CLT. Then the condition of factorization (3.1) is satisfied.

This assertion follows immediately from the main result of the articles [7], [44]; see also [8].

**Proposition 3.2.** Suppose in addition \( \eta(t) \) satisfies the CLT in \( C(T, d) \); then in the factorization (3.1) the metric \( \rho(\cdot, \cdot) \) may be selected such that

\[ \forall p \in (0, 2) \Rightarrow \mathbb{E}|M|^p < \infty. \tag{3.4} \]

**Proof.** It is proved in the famous book of M.Ledoux and M.Talagrand [25], chapter 10, section 1, p. 274-277 that if the r.p. \( \eta(\cdot) \) satisfies the CLT in the separable Banach space \( X \) with the norm \( || \cdot ||_X \), then \( \mathbb{E}||\eta||^p < \infty. \) Therefore

\[ \mathbb{E}\sup_{t \in T} |\eta(t)|^p < \infty, \quad p \in (0, 2). \]

Note that the function \( u \rightarrow |u|^p \) is Young - Orlicz function satisfying the well-known \( \Delta_2 \) condition.

It remains to use on of the main result of aforementioned article [44].

Analogously may be proved the following assertion.

**Proposition 3.3.** Assume that the r.p. \( \eta(t) \) is sample part continuous, still without CLT. Moreover, let

\[ \mathbb{E}\sup_{t \in T} |\eta(t)|^2 < \infty. \]

Then the distance \( \rho \) in the factorization representation (3.1) may be selected such that the r.v. \( M \) has finite second moment: \( \mathbb{E}M^2 < \infty. \)
Before formulating the next result, we introduce together with R.M. Dudley [11], N.C. Jain and M.B. Marcus [22] the third distance \( \tau = \tau(t, s) \) on the set \( R \), also natural, as follows:

\[
\tau(t, s) = \sqrt{\text{Var}(\eta(t) - \eta(s))} = \left[ \mathbb{E}(\eta(t) - \eta(s))^2 \right]^{1/2}. \tag{3.5}
\]

It follows from the condition (3.1) that \( \tau(t, s) \leq C \cdot \rho(t, s) \).

**Proposition 3.4.** Suppose the r.p. \( \eta(t), t \in [0, 2\pi] \) satisfies the CLT in \( C(T, d) \), is stationary in wide sense. Assume also that the metric functions \( \rho(r, s) \) and \( \tau(t, s) \) are linear equivalent:

\[
C_1 \rho(t, s) \leq \tau(t, s) \leq C_2 \rho(t, s), \quad C_1, C_2 = \text{const} \in (0, \infty). \tag{3.6}
\]

Then the entropy condition (3.3) is satisfied.

**Proof.** Since the limit process \( \zeta_\infty(t) \) is continuous, Gaussian and has at the same covariation function as \( \eta(t) \), it is stationary in strong sense and also continuous. Our assertion follows from the famous necessary condition for sample part continuity of Gaussian stationary process belonging to X. Fernique [12].

**Remark 3.1.** At the same result is true even without assumption of stationarity with at the same proof if the function \( t \to \tau(t, 0) \) is monotonic in some neighborhood of zero, see [27].

4 Some examples.

1. Let here \( T = [0, e^{-4}] \) and let \( \delta = \text{const} \in (0, 1/4) \). We define the r.p. \( \eta_0(t), t \in T \) as follows.

\[
\eta_0(t) := \frac{w(t)}{(2t)^{1/2} \left( \log | \log t | \right)^{1/2 + \delta/2}}, \quad 0 < t \leq e^{-4}, \tag{4.1}
\]

and \( \eta_0(0) := 0; \ w(t) \) is ordinary Brownian motion.

It follows from the classical Law of Iterated Logarithm (LIL) for Wiener process that the r.p. \( \eta_0(t) \) is continuous a.e. Since it is "per se" Gaussian, it satisfies the CLT(\( C(T) \)). The conditions (3.1) and (3.2) are also satisfied, but the entropy integral (3.3) is divergent.

Indeed, we have denoting

\[
\tau_0(t, s) = \sqrt{\text{Var}(\eta_0(t) - \eta_0(s))} : \tau_0(t, 0) \sim \frac{C}{[\log | \log t |]^{1+\delta}},
\]

therefore

\[
h_+(\epsilon) := \sup_{t \in T} \mu(B(t, \epsilon)) \geq \mu(B(0, \epsilon)) \geq C \cdot \exp \left( \exp \left( C_1 \epsilon^{1/(1+\delta)} \right) \right), \quad 0 < \epsilon < 1/8, \tag{4.2}
\]
where
\[ B(t, \epsilon) = \{ s : s \in T, \tau_0(t, s) \leq \epsilon \} \]
and \( \mu \) is ordinary Lebesgue measure on the real axis.

We can apply the following inequality, see [30], chapter 3, section 3.2:
\[ \exp H(T, \tau_0, \epsilon) \geq \frac{\mu(T)}{H_+(\epsilon)}, \]
or equally
\[ H(T, \tau_0, \epsilon) \geq \exp \left( C_2 \epsilon^{1/(1+\delta)} \right). \quad (4.3) \]

It is easy to verify that for such an entropy the condition (3.3) is not satisfied.

2. The following example appears in the Monte-Carlo method for computation and error estimation of multiple multivariate parametric integrals, see [15], [19], [11], [30], [37] etc. Namely, let us consider the problem of computation of the following parametric integral
\[ I(t) = \int_D v(t, x) \mu(dx), \]
where \( \mu \) is probability measure: \( \mu(D) = 1 \), by means of Monte-Carlo method:
\[ I(t) \approx I_n(t) := n^{1-n} \sum_{j=1}^n v(t, \beta_j), \]
where \( \{ \beta_j \} \) are independent r.v. with distribution \( \mu \).

Consider for error estimation the following variable:
\[ \gamma_n(u) := P \left( \sqrt{n} \sup_t |I_n(t) - I(t)| > u \right). \]
We put here \( \eta_I(t) = v(t, \beta_1) - I(t) \). If \( \eta_I(t) \) satisfies the CLT(\( C(T) \)), then as \( n \to \infty \)
\[ \gamma_n(u) \to P \left( \max_{t \in T} |\zeta_\infty(t)| > u \right), \]
therefore
\[ \gamma_n(u) \approx P \left( \max_{t \in T} |\zeta_\infty(t)| > u \right) =: \gamma_\infty(u). \]

The asymptotical behavior or exact estimations as \( u \to \infty \) for the right-hand side of the last equality are well known, see [30], chapter 3; [45]. As a rule as \( u \to \infty \)
\[ \gamma_\infty(u) \sim K \cdot u^{\kappa-1} \cdot \exp \left( -\frac{u^2}{2\sigma^2} \right), \]
where
\[ \sigma^2 = \max_{t \in T} \text{Var} \eta(t) = \max_{t \in T} R(t, t), \quad K, \kappa = \text{const} \in (0, \infty). \]

Let \( \varepsilon \) be some "small" number, for instance, 0.05 or 0.01 etc. Denote by \( U(\varepsilon) \) the maximal root of equation
\[ \gamma_\infty(U(\varepsilon)) = \varepsilon, \]

we deduce that if \( \eta(t) \) satisfies CLT(C(T)), then with probability \( \approx 1 - \varepsilon \)

\[ \sup_{t \in T} |I_n(t) - I(t)| \leq \frac{U(\varepsilon)}{\sqrt{n}}; \]

which is a twice asymptotical: as \( n \to \infty \) and as \( u \to \infty \) confidence region for \( I(t) \) in the uniform norm.

The non-asymptotical exponential exact estimation for \( \gamma_\infty(u) \) in the modern terms of majorizing measures see in the article [33].

Suppose there exist a r.v. \( \theta \) and a non-random monotonically decreasing sequence \( \delta(n) \), such that

\[ E(n, g) \leq \theta \cdot \delta(n), \quad \lim_{n \to \infty} \delta(n) = 0. \quad (4.4) \]

We impose the following condition on the r.v. \( \theta \):

\[ \exists m = \text{const} > 1, \ C = \text{const} \in (0, \infty), \ P(\theta > x) \leq e^{-C x^m}, \ x \geq 0, \quad (4.5) \]

or equally

\[ |\theta|_p \leq C_2 \ p^{1/m}, \ p \geq 1, \]

and denote \( \tilde{m} = \min(m, 2), \ m' = \tilde{m}/(\tilde{m} - 1) \). The conditions of theorem 2.2 are satisfied if for example

\[ \sum_{r=1}^{\infty} \frac{\delta(2^r)}{r^{1/\tilde{m}}} < \infty. \quad (4.6) \]

The condition (4.6) is satisfied in turn if for instance for some positive value \( \Delta = \text{const} > 0 \)

\[ \delta(n) \leq \frac{C_3}{[\log(n + 2)]^{1/m' + \Delta}}. \quad (4.7) \]

We used the following proposition, see [23], [30], chapter 2, section 2.1, page 50-53: if \( \xi_i, \ i = 1, 2, \ldots \) are centered, i.e., i.d. r.v. such that

\[ P(\{|\xi_i| > x\}) \leq \exp(-x^m), \ x > 0, \]

then

\[ \sup_n P \left( n^{-1/2} \left| \sum_{i=1}^{n} \xi_i \right| > x \right) \leq \exp \left( -C(m) \ x^m \right), \ x > 0, \]

and the last inequality is exact.

The examples of a (random) functions \( g(\cdot) \) for which
\[ \delta(n) \approx \frac{C_4}{[\log(n + 2)]^{1/m' + \Delta}} \]

may be constructed by means of lacunary random series, see [28], [43].

Notice that the conditions (4.6) and (4.7) are so weak so that the entropy condition [22] and majorizing measures conditions [20] are not satisfied.

**Remark 4.1.** The condition (4.4) is satisfied if for example

\[ \omega(g, \epsilon) := \sup_{h : |h| \leq \epsilon} \| g(\cdot + h) - g(\cdot) \| \leq C \cdot \theta \cdot \delta(\text{Ent}[1/\epsilon]), \]

Jackson’s inequality; see [1], [10], [26], [29]; but the inverse inequality is non true, see aforementioned Stechkin’s estimate ([54], chapter 6, section 6.1).

3. Let now the set \( T = \{1, 2, \ldots; \infty \} \) equipped with the distance

\[ d(i, j) = \left| \frac{1}{j} - \frac{1}{i} \right|, \quad d(i, \infty) = \frac{1}{i}, \quad d(\infty, \infty) = 0. \tag{4.8} \]

The set \( T \) is compact set relatively the distance \( d \).

We choose in the capacity of probability space the ordinary interval \((0, 1)\) with Lebesgue measure.

Let \( p_0 = \text{const} \in (1, 2), \)

\[ \alpha = \text{const} \in (0, \min(1, p_0/(2 - p_0))), \quad a(n) = 1 - 0.5n^{-\alpha}, \quad n = 1, 2, \ldots; \]

\[ \Delta(n) = a(n + 1) - a(n), \quad c(n) = n^{\alpha/p_0}. \]

Let also

\[ f_{1/2}(x) = \sqrt{|\log x|}, \quad x \in (0, 1); \quad f_{1/2}(x) = 0, \quad x \notin (0, 1). \tag{4.9} \]

We define for the values \( n \in T \) the following r.p.

\[ \eta_n(x) = c(n) \cdot \epsilon_n \cdot f_{1/2} \left( \frac{x - a(n)}{\Delta(n)} \right), \quad \eta_\infty(x) = 0, \tag{4.10} \]

where \( \{\epsilon_n\} \) is a Rademacher’s sequence defined on some other probabilistic space and independent on \( f_{1/2}(\cdot) \), so that \( E\eta_n = 0 \).

Recall that

\[ ||\eta(\cdot)|| = \sup_n \eta_n. \]

This example was offered by the authors in [35] for another purpose. It was proved in particular in [35] that the r.p. \( \eta_n \) is continuous with probability one, this imply:

\[ P \left( \lim_{n \to \infty} \eta_n \to 0 \right) = 1, \]
and

\[ E[\|\eta\|^p] < \infty, \forall p > p_0 \Rightarrow E[\|\eta\|^p] = \infty. \tag{4.11} \]

Let us prove in addition that the sequence of a r.v. \{\eta_n\} is pre-gaussian in the considered space \(C(T,d)\). This imply by definition that the mean zero Gaussian distributed sequence \{\nu_n\} with at the same covariation function \(Q(n_1,n_2)\) as \{\eta_n\} belong to the set \(C(T,d)\) with probability one.

Namely, since the Rademacher’s sequence contains from independent r.v. \(\epsilon_n\),

\[ Q(n_1,n_2) = 0, \quad n_1 \neq n_2. \]

It suffices to prove \(P(\nu_n \to 0) = 1\). We have:

\[ \text{Var} \nu_n = Q(n,n) = \text{Var} \eta_n = E|\eta_n|^2. \]

It is proved in [35] that

\[ E|\eta_n|^2 \leq C_1 \cdot n^{-C_2}, \quad C_2 > 0, \]

therefore \(\text{Var} \nu_n \leq C_1 \cdot n^{-C_2}\) and by virtue of Gassiness of the sequence \{\nu_n\}

\[ \forall \epsilon > 0 \Rightarrow \sum_{n=1}^{\infty} P(|\nu_n| > \epsilon) < \infty. \]

Thus,

\[ P \left( \lim_{n \to \infty} \nu_n \to 0 \right) = 1. \]

The r.p. \(\eta_n\) does not satisfy the majorizing measure condition, [35]. It follows from (4.11) that the condition (3.4) is also not satisfied. Therefore, \(\eta_n\) does not satisfy CLT in our space \(C(T,d)\).

5 Concluding remarks.

1. Note that the r.v. \(M\) and the distance \(\rho = \rho(t,s)\) in (3.1) may be introduced constructively in two stages as follows. First step: define the r.v. \(L\)

\[ L := \sup_{t,s \in T} |\eta(t) - \eta(s)|. \]

Second step:

\[ q(t,s) := \text{vraisup}_\Omega \left[ \frac{|\eta(t) - \eta(s)|}{L} \right]; \]

(the case \(L = 0\) is trivial). Then the function \((t,s) \to q(t,s)\) is continuous bounded distance (more exactly, semi-distance) on the set \(T\) and

\[ |\eta(t) - \eta(s)| \leq L \cdot q(t,s). \]

Obviously, this choice of the variables \(L, q\) is optimal.
2. It is clear that we can use instead trigonometrical approximation the approximation by means of algebraic polynomials.

3. Suppose that instead the condition (3.6) is true the following inequality:

\[ \rho(t, s) \leq C \cdot \tau^{1/\beta}(t, s), \quad \beta = \text{const} \in (0, 1). \]

Then the necessary condition of a view (3.3) must be transformed as follows:

\[ \int_0^1 H^{1/2}(T, \rho, z^\beta) \, dz < \infty. \]

4. It is no hard to generalize our results into the multidimensional case \( T = [0, 2\pi]^d \) and into the non-periodical continuous function space.

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