Global well-posedness for axisymmetric Boussinesq system with horizontal viscosity

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Abstract

In this paper, we are concerned with the tridimensional anisotropic Boussinesq equations which can be described by

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \kappa \Delta_h u + \nabla \Pi = \rho e_3, & \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho = 0, & \\
\text{div} \ u = 0.
\end{aligned}
\]

Under the assumption that the support of the axisymmetric initial data $\rho_0(r,z)$ does not intersect the axis ($Oz$), we prove the global well-posedness for this system with axisymmetric initial data. We first show the growth of the quantity $\rho_r$ for large time by taking advantage of characteristic of transport equation. This growing property together with the horizontal smoothing effect enables us to establish $H^1$-estimate of the velocity via the $L^2$-energy estimate of velocity and the Maximum principle of density. Based on this, we further establish the estimate for the quantity $\|\omega(t)\|_{L^p(\mathbb{R}^3)}$ which implies $\|\nabla u(t)\|_{L^p(\mathbb{R}^3)} < \infty$. However, this regularity for the flow admits forbidden singularity since $L$ seems be the minimum space for the gradient vector field $u(x,t)$ ensuring uniqueness of flow. To bridge this gap, we exploit the space-time estimate about $\|\omega(t)\|_{L^p(\mathbb{R}^3)} < \infty$ by making good use of the horizontal smoothing effect and micro-local techniques. The global well-posedness for the large initial data is achieved by establishing a new type space-time logarithmic inequality.

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1 Introduction

The Boussinesq system is used as a toy model in the dynamics of the ocean or of the atmosphere, and play an important role in the study of Raleigh-Bernard convection. One may refer to [28] for more details about its physical background. It takes the form:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \kappa \Delta_h u + \nabla \Pi = \rho e_n, & \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \quad n = 2, 3, \\
\partial_t \rho + u \cdot \nabla \rho - \nu \Delta \rho = 0, \\
\text{div} \ u = 0, \\
(u, \rho)|_{t=0} = (u_0, \rho_0),
\end{aligned}
\]
where, the velocity \( u = (u_1, \cdots, u_n) \) is a vector field and the scalar unknown \( \rho \) denotes quantity such as the concentration of a chemical substance or the temperature variation in a gravity fields, in which case \( \rho e_n \) represents the buoyancy force. And the nonnegative parameters \( \kappa \) an \( \nu \) stands for the viscosity and the molecular diffusion respectively. In addition, the scalar function \( \Pi \) is pressure which can be recovered from the unknowns \( u \) and \( \rho \) via Riesz operator.

This system have been intensively studied due to their physical background and mathematical significance. In dimension two, the standard energy method enables us to establish the global existence of regular solutions for the case where \( \nu \) and \( \kappa \) are nonnegative constants. But, for the inviscid Boussinesq system (1.1), the global well-posedness for some nonconstant \( \rho_0 \) is still an challenge open problem. When \( \nu \) is a positive constant and \( \kappa = 0 \) or \( \nu = 0 \) and \( \kappa \) is a positive constant, the global well-posedness was independently obtained in \([8, 22]\) for the two-dimensional Boussinesq system, see also \([17]\) for the global well-posedness in the critical spaces. For the fractional case, Hmidi, Keraani and Rousset \([19]\) showed the global well-posedness for the critical case by using a hidden cancellation given by the coupling. Moreover, for the case where fractional viscosity and thermal diffusion the fractional powers satisfy mild condition, the global results on the two-dimensional Boussinesq equations were obtained in \([23, 26]\).

For the tri-dimensional Boussinesq equations, R. Danchin and M. Paicu \([13]\) showed the global existence of weak solution for \( L^2 \)-data and the global well-posedness for small initial data. They \([14]\) also obtained a existence and uniqueness result for small initial data belonging to some critical Lorentz spaces. But there is little study about the global well-posedness result for large initial data, even for the tri-dimensional Navier-Stokes equations. Inspired by the study of Navier-Stokes equations for large data in special case, more recent works target to consider the tri-dimensional axisymmetric Boussinesq system without swirl case. In \([1]\), a global existence and uniqueness result for the following system

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \kappa \Delta u + \nabla \Pi &= \rho e_3, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho &= 0, \\
\text{div} \ u &= 0, \\
(u, \rho)|_{t=0} &= (u_0, \rho_0),
\end{aligned}
\]

was obtained by establishing the following quadratic growth estimate

\[
\left\| \frac{\rho}{r}(t) \right\|_{L^2} \leq \left\| \frac{\rho_0}{r} \right\|_{L^2} + C_0 \left\| \frac{u}{r} \right\|_{L^1_t L^\infty} \left( 1 + \|u\|_{L^1_t L^\infty} \right),
\]

under assumption that the support of the initial density does not intersect the axis \( r = 0 \). From that time on, much effects has been made to show the global well-posedness for the three-dimensional axisymmetric Boussinesq system without swirl case, when the dissipation only occurs one equation or is present only in one direction (anisotropic dissipation). In a series of paper \([20, 21]\), T. Hmidi and F. Rousset \([20]\) proved the global well-posedness for the Navier-Stokes-Boussinesq system by virtue of the structure of the coupling between two equations of (1.1) with \( \nu = 0 \). In \([21]\), they also showed the global well-posedness for the tridimensional Euler-Boussinesq system with axisymmetric initial data without swirl. And their proofs strongly relies on the fact the dissipation occurs in three directions.

As pointed out by J.-Y. Chemin et al in \([11]\), the anisotropic dissipation assumption is natural and physical. In fact, in certain regimes and after suitable rescaling, the vertical dissipation (or the horizontal dissipation) is negligible as compared to the horizontal dissipation (or the vertical dissipation). In the past years, there are several works devoted to study of the two-dimensional Boussinesq system with anisotropic dissipation. When the horizontal viscosity occurs in only one equation, the global well-posedness result for the two-dimensional Boussinesq system was obtained in \([15]\). Moreover, A. Adhikari, C. Cao and J. Wu also established some global results under various
assumption for the two-dimensional Boussinesq system with dissipation occurs in vertical direction in a series of recent papers \[2, 3\]. More recently, C. Cao and J. Wu \[7\] successfully proved the global well-posedness for the two-dimensional Boussinesq system (HBS) in terms of a Log-type inequality

\[
\|u_2\|_{L^\infty} \leq C \|u_2\|_{L^\log L} \left( \log \left( e + \|u_2\|_{H^2} \right) \log \log \left( e + \|u_2\|_{H^2} \right) \right)^{\frac{1}{2}}.
\]

together with a control of \(\|u_2\|_{L^\log L}\), where the space \(L^\log L\) stands for the space of functions \(f\) in \(\cap_{2 \leq p < \infty} L^p\) such that

\[
\|f\|_{L^\log L} := \sup_{2 \leq p < \infty} (p \log p)^{-\frac{1}{2}} \|f\|_{L^p} < \infty.
\]

In addition, under the assumption that the initial data is axisymmetric without swirl, we stated the global well-posedness for the two-dimensional Boussinesq system (1.1) in terms of a Log-type inequality in a series of recent papers \[2, 3\]. More recently, C. Cao and J. Wu \[7\] successfully proved the global well-posedness for the two-dimensional Boussinesq system with dissipation occurs in vertical direction in terms of a Log-type inequality

First of all, let us recall some algebraic and geometric properties of the axisymmetric vector fields (cf. \[20, 27\]) and discuss the special structure of the vorticity of (HBS). Let

\[
\omega = \left( \frac{\partial_x \Delta^{-1} \left( \frac{\omega_y}{r} \right)}{r} - 2 \frac{\partial_r \Delta^{-1} \partial_z \Delta^{-1} \left( \frac{\omega_y}{r} \right)}{r} \right).
\]

In the presented paper, we take effect to investigate the global well-posedness for tridimensional Boussinesq system with horizontal viscosity in the whole space with axisymmetric initial data. This system is described as follows:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \kappa \Delta_h u + \nabla P &= \rho e_3, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t \rho + u \cdot \nabla \rho &= 0, \\
\text{div} \nu &= 0, \\
(u, \rho)|_{t=0} &= (u_0, \rho_0),
\end{align*}
\]

where \(\Delta_h := \partial_r^2 + \partial_z^2\). In the following parts, we assume that \(\kappa = 1\) for the sake of convenience.

Let \(u\) is an axisymmetric vector field without swirl, that is, \(u(t, x) = u_r(r, z)e_r + u_z(r, z)e_z\). Then a simple calculation yields that the vorticity \(\omega := \text{curl} u\) of the vector field has the form

\[
\omega = (\partial_x u_r - \partial_r u_z)e_\theta := \omega_\theta e_\theta,
\]

and

\[
u \cdot \nabla = u_r \partial_r + u_z \partial_z, \quad \text{div} u = \partial_r u_r + \frac{u_r}{r} + \partial_z u_z \quad \text{and} \quad \omega \cdot \nabla u = \frac{u_r}{r} \omega
\]

in the cylindrical coordinates. As a consequence, the vorticity \(\omega\) solves

\[
\partial_t \omega + u \cdot \nabla \omega - \Delta_h \omega = -\partial_r \rho e_\theta + \frac{u_r}{r} \omega. \tag{1.5}
\]

This together with the fact that \(\Delta_h = \partial_r^2 + \frac{1}{r} \partial_r\) in the cylindrical coordinates enables us to conclude the quantity \(\omega_\theta\) satisfies

\[
\partial_t \omega_\theta + u \cdot \nabla \omega_\theta - \Delta_h \omega_\theta + \frac{\omega_\theta}{r^2} = -\partial_r \rho + \frac{u_r}{r} \omega_\theta. \tag{1.6}
\]

The target of this paper is to study the global existence and the uniqueness for the system (HBS) with axisymmetric initial data, which means that the velocity \(u_0\) is assumed to be an axisymmetric vector field without swirl and the density \(\rho_0\) depends only on \((r, z)\). Now we shall briefly discuss the difficulties and outline the main ingredient in our proof. First, the quadratic growth estimate \[13\].
which plays the key role in the proof of \cite{H} does not work for the system \cite{HBS} due to the absence of vertical viscosity. Rough speaking, the difficulty arises in dealing with vorticity equation. Taking the $L^2$-inner product of \eqref{1.6} with $\omega_\theta$ and integrating by parts, we readily get

$$
\frac{1}{2} \frac{d}{dt} \|\omega_\theta(t)\|_{L^2}^2 + \|\nabla_h \omega_\theta(t)\|_{L^2}^2 = \int_{\mathbb{R}^3} \frac{u_r}{r} \omega_\theta \omega_\theta dx + \int_{\mathbb{R}^3} \frac{\omega_\theta}{r} \rho dx + \int_{\mathbb{R}^3} \rho \partial_r \omega_\theta dx.
$$

Taking advantage of the anisotropic inequality of Lemma \ref{lem:anisotropic}, the first integral term in the right side of the above equality can be bounded by

$$
C \|u\|_{L^2} \|\omega_\theta\|_{L^2}^2 + \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^2 \|\nabla_h \left( \frac{\omega_\theta}{r} \right)\|_{L^2}^2 + \frac{1}{4} \|\nabla_h \omega_\theta\|_{L^2}^2.
$$

On the other hand, the unknown $\omega_\theta$ satisfies the following equation

$$
(\partial_t + u \cdot \nabla) \frac{\omega_\theta}{r} - (\Delta_h + \frac{2}{r} \partial_r) \frac{\omega_\theta}{r} = -\frac{1}{\rho} \partial_r \rho.
$$

In a similar fashion as in \cite{H}, one can conclude by the virtue of the estimate \eqref{1.3} that

$$
\left\| \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 + \frac{1}{2} \int_0^t \left\| \nabla_h \left( \frac{\omega_\theta}{r} \right)(\tau) \right\|_{L^2}^2 d\tau \leq C_0 (1 + t^5) \left(1 + \int_0^t \left\| \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 d\tau \right) + \frac{1}{4} \int_0^t \|\nabla_h \omega(\tau)\|_{L^2}^2 d\tau.
$$

From this, it seems impossible to use the quantities in the left side of the above inequality to control the integral term $\int_0^t \left\| \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 \|\nabla_h \left( \frac{\omega_\theta}{r} \right)(\tau)\|_{L^2}^2 d\tau$. This require us to refine this quadratic growth estimate to make up for the shortage of vertical diffusion. To do this, we establish the following estimate

$$
\left\| \frac{\rho}{r}(t) \right\|_{L^2} \leq C \left\| \frac{\rho}{r} \right\|_{L^2} + C \left\| \frac{u_r}{r} \right\|_{L^1_t L^\infty} \left(1 + \int_0^t \left\| \nabla_h u(\tau) \right\|_{L^2} \left\| \nabla_h \omega(\tau) \right\|_{L^2} d\tau \right).
$$

by deeply using the axisymmetric structure and the incompressible condition. As an consequence, we have

$$
\left\| \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 + \frac{1}{2} \int_0^t \left\| \nabla_h \left( \frac{\omega_\theta}{r} \right)(\tau) \right\|_{L^2}^2 d\tau \leq C_0 (1 + t^5) \left(1 + \int_0^t \left\| \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 d\tau \right) + \frac{1}{4} \left( \int_0^t \|\nabla_h \omega(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{2}}.
$$

Consequently, we obtain the estimate of

$$
\|\omega_\theta(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \omega_\theta(\tau)\|_{L^2}^2 d\tau + \left\| \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 + \left( \int_0^t \left\| \nabla_h \left( \frac{\omega_\theta}{r} \right)(\tau) \right\|_{L^2}^2 d\tau \right)^2.
$$

This entails us to obtain the estimate of $\|\omega(t)\|_{\mathcal{L}^n} := \sup_{t\leq T<\infty} \frac{\|\omega(t)\|_{L^p(R^3)}}{\sqrt{p}}$, which together with the well-known fact

$$
\|\nabla u\|_{L^p} \leq C \frac{p^2}{p-1} \|\omega\|_{L^p} \quad \text{with} \quad p \in ]1, \infty[\]
$$

gives $\|\nabla u(t)\|_{L^p}^{\frac{1}{4}} := \sup_{t\leq T<\infty} \frac{\|\nabla u(t)\|_{L^p(R^3)}}{p^{\frac{1}{4}}}$ < $\infty$. Unfortunately, the function $p^{\sqrt{p}}$ does not belong to the dual Osgood modulus of continuity, which prevents us trying to obtain higher-order estimates of $(\rho, u)$, where an dual Osgood modulus of continuity $\omega(p)$ is the non-decreasing function satisfying $\int_0^\infty \frac{1}{\omega(\tau)} d\tau = \infty$ for some $a > 0$. To bridge the gap between the dual Osgood modulus of continuity and $\mathbb{L}^\frac{3}{2}$, inspired by the Boot-Strap argument, we exploit the space-time estimate
about $\sup_{2 \leq p < \infty} \int_0^t \| \nabla u(\tau) \|_{L^p(\mathbb{R}^3)} \frac{d\tau}{\sqrt{p}}$ based on the estimate of $\| \nabla u(t) \|_{L^p(\mathbb{R}^3)}$ by making good use of the horizontal smoothing effect and micro-local techniques. Combining this with a new type space-time logarithmic inequality established in Section 2 entails us to obtain the desired result.

Before stating our main result we denote by $\Pi_z$ the orthogonal projector over the axis $(Oz)$. We define the distance from a point $x$ to a subset $A \subset \mathbb{R}^3$ by

$$d(x, A) := \inf_{y \in A} \| x - y \|,$$

where $\| \cdot \|$ is the usual Euclidian norm. The distance between two subsets $A$ and $B$ of $\mathbb{R}^3$ is defined by

$$d(A, B) := \inf_{x \in A, y \in B} \| x - y \|.$$

And $\text{diam} \ A = \sup_{x, y \in A} \| x - y \|$ denotes the diameter of a bounded subset $A \subset \mathbb{R}^3$. Moreover, let us introduce $L^a(\mathbb{R}^3)(a \in (0, 1])$ of those functions $f$ which belong to every space $L^p(\mathbb{R}^3)$ with $2 \leq p < \infty$ and satisfy

$$\| f \|_{L^a(\mathbb{R}^3)} := \sup_{p \geq 2} \frac{\| f \|_{L^p(\mathbb{R}^3)}}{p^a} < \infty. \quad (1.7)$$

We denote $L^\frac{3}{2}(\mathbb{R}^3)$ by $\sqrt{L}(\mathbb{R}^3)$ for the sake of simplicity.

Our result reads as follows.

**Theorem 1.1.** Assume that $u_0 \in H^1(\mathbb{R}^3)$ be an axisymmetric vector field with zero divergence, and its vorticity satisfies $\frac{\omega}{r} \in L^2(\mathbb{R}^3)$ and $\partial_z \omega \in L^2$. Let $\rho_0 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ depending only on $(r, z)$ and such that $\text{Supp} \ \rho_0$ does not intersect the axis $(Oz)$ and $\Pi_z(\text{Supp} \ \rho_0)$ is a compact set. Then the Boussinesq system (HBS) has a unique global solution $(\rho, u)$ such that

$$u \in C(\mathbb{R}^+; H^1(\mathbb{R}^3)), \quad \nabla_h u \in L^2_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^3)), \quad \partial_z \omega, \frac{\omega}{r} \in C(\mathbb{R}^+; L^2(\mathbb{R}^3)),

\nabla_h \partial_z \omega, \nabla_h \left( \frac{\omega}{r} \right) \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3)), \quad \rho \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^3)), \quad \rho \in C(\mathbb{R}^+; H^1(\mathbb{R}^3)),

\nabla u \in L^1_{\text{loc}}(\mathbb{R}^+; L^\infty(\mathbb{R}^3)).$$

**Remark 1.1.** Our proof strongly relies on the growth estimate of $\frac{\rho}{r}$ and the horizontal smoothing effect.

The rest of the paper is organized as follows. In Section 2 we review Littlewood-Paley theory and establish a new type space-time logarithmic inequality which is an important ingredient in the proof of Theorem 1.1. Next, we study analytic properties of the flow associated to an axisymmetric vector field. In Section 3 we obtain a priori estimates for sufficiently smooth solutions of the system (HBS) by using the procedure that we have just described in introduction. Section 4 is devoted to the proof of Theorem 1.1. Finally, an appendix is devoted to two useful lemmas.

## 2 Preliminaries

In the first subsection, we first provide the definition of some function spaces and properties based on the so-called Littlewood-Paley decomposition that will be used constantly in the following sections. Next, we give a space-time logarithmic inequality in view of the low-high decomposition techniques. In the last subsection, we main establish the grow estimate of the quantity $\frac{\rho}{r}$ by taking advantage of some geometric and analytic properties of the generalized flow map associated to an axisymmetric vector field.
2.1 Littlewood-Paley Theory and a space-time logarithmic inequality

Let \((\chi, \varphi)\) be a couple of smooth functions with values in \([0, 1]\) such that \(\text{Supp}\chi \subset \{\xi \in \mathbb{R}^n | |\xi| \leq \frac{4}{3}\}\), \(\text{Supp}\varphi \subset \{\xi \in \mathbb{R}^n | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}\) and

\[
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{for each} \quad \xi \in \mathbb{R}^n.
\]

For every \(u \in S'(\mathbb{R}^n)\), we define the littlewood-Paley operators as follows:

\[
S_j u := \chi(2^{-j}D)u \quad \text{and} \quad \Delta_j u := \varphi(2^{-j}D)u \quad \text{for all} \quad j \geq 0.
\]

From this, it is easy to verify that

\[
u = \sum_{j \geq -1} \Delta_j u, \quad \text{in} \quad S'(\mathbb{R}^n),
\]

and

\[
\Delta_j \Delta_{j'} u \equiv 0 \quad \text{if} \quad |j - j'| \geq 2.
\]

Next, we recall the classical Bernstein lemma which will be useful throughout this paper (cf. [10]).

**Lemma 2.1** (Bernstein). Let \(1 \leq p \leq q < \infty\) and \(u \in L^p(\mathbb{R}^n)\). There exists a positive constant \(C\) such that for \(j, k \in \mathbb{N}\), we have

\[
\sup_{|\alpha| = k} \|\partial^\alpha S_j u\|_{L^q(\mathbb{R}^n)} \leq C^k 2^{j(k+n/2 - \frac{1}{q})} \|S_j u\|_{L^p(\mathbb{R}^n)},
\]

and

\[
C^{-k} 2^{kq} \|\Delta_j u\|_{L^p(\mathbb{R}^n)} \leq \sup_{|\alpha| = k} \|\partial^\alpha \Delta_j u\|_{L^p(\mathbb{R}^n)} \leq C^k 2^{jk} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}.
\]

Let us now introduce Bony’s decomposition (see for example [4]) which is a basic tool of the para-differential calculus. Specifically, one can split a product \(uv\) into three parts as follows:

\[
uv = T_u v + T_v u + \mathcal{R}(u, v),
\]

where

\[
T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \text{and} \quad \mathcal{R}(u, v) = \sum_q \Delta_q u \bar{\Delta}_q v,
\]

with \(\bar{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}\).

In usual, \(T_u v\) is called para-product of \(v\) by \(u\) and \(\mathcal{R}(u, v)\) denotes the remainder term. In addition, it is worthwhile to point out that \(\bar{\Delta}_q \Delta_q = \Delta_q\) for \(q \geq 0\) by using the property of support of \(\varphi\).

**Definition 2.1.** For \(s \in \mathbb{R}\), \((p, q) \in [1, +\infty)^2\) and \(u \in S'(\mathbb{R}^n)\), the inhomogeneous Besov spaces are defined by

\[
B^s_{p, q}(\mathbb{R}^n) := \{u \in S'(\mathbb{R}^n) | \|u\|_{B^s_{p, q}(\mathbb{R}^n)} < \infty\}.
\]

Here

\[
\|u\|_{B^s_{p, q}(\mathbb{R}^n)} := \begin{cases} \left(\sum_{j \geq -1} 2^{jsq} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}^q\right)^{1/q} & \text{if} \quad r < \infty, \\
\sup_{j \geq -1} 2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^n)} & \text{if} \quad r = \infty.
\end{cases}
\]
Since the dissipation only occurs in the horizontal direction to Equations [HBS], we need introduce the following anisotropic space.

**Definition 2.2.** For \( s, t \in \mathbb{R}, (p, q) \in [1, +\infty]^2 \) and \( u \in S'(\mathbb{R}^3) \), we define the anisotropic Besov spaces as

\[
B_{p,q}^{s,t}(\mathbb{R}^3) := \{ u \in S'(\mathbb{R}^3) \mid \| u \|_{B_{p,q}^{s,t}(\mathbb{R}^3)} < \infty \},
\]

where

\[
\| u \|_{B_{p,q}^{s,t}(\mathbb{R}^3)} := \left( \sum_{j,k \geq 1} 2^{jsq} 2^{ktq} \left\| \Delta_j^h \Delta_k^v u \right\|^q_{L^p(\mathbb{R}^3)} \right)^{\frac{1}{q}} \text{ if } r < \infty,
\]

\[
= \sup_{j,k \geq 1} 2^{jsq} 2^{ktq} \| \Delta_j^h \Delta_k^v u \|_{L^p(\mathbb{R}^3)} \text{ if } r = \infty.
\]

Here and in what follows,

\[
\Delta_j^h f(x_h) := 2^{2j} \int_{\mathbb{R}^2} \varphi(x_h - 2^j y) f(y) \, dy
\]

and

\[
\Delta_j^v f(z) := 2^{2j} \int_{\mathbb{R}} \varphi(z - 2^j y) f(y) \, dy.
\]

In the following, we briefly review some basic properties for \( H^{s,t} \) spaces which will be useful later.

**Lemma 2.2 (27).** There hold that

(i) For \( s_2 \geq s_1 \) and \( t_2 \geq t_1 \), one has \( \| u \|_{H^{s_2,t_2}(\mathbb{R}^3)} \hookrightarrow \| u \|_{H^{s_1,t_1}(\mathbb{R}^3)} \).

(ii) For \( s_1, s_2, t_1, t_2 \in \mathbb{R} \), there exists \( \theta \in [0, 1] \) such that

\[
\| u \|_{H^{s_1+(1-\theta)s_2, t_1+(1-\theta)t_2}(\mathbb{R}^3)} \leq \| u \|_{H^{s_2,t_2}(\mathbb{R}^3)} \| u \|_{H^{s_1,t_1}(\mathbb{R}^3)}^{1-\theta}.
\]

(iii) For \( s, t \geq 0 \), \( \| u \|_{H^{s,t}(\mathbb{R}^3)} \) is equivalent to

\[
\| u \|_{L^2(\mathbb{R}^3)} + \| \Lambda^h_1 u \|_{L^2(\mathbb{R}^3)} + \| \Lambda^l_1 u \|_{L^2(\mathbb{R}^3)} + \| \Lambda^l_1 \Lambda^h_1 u \|_{L^2(\mathbb{R}^3)}.
\]

(iv) \( \| u \|_{H^{s,t}(\mathbb{R}^3)} \simeq \| u \|_{H^s(\mathbb{R}^3)} \| u \|_{H^t(\mathbb{R}^3)} \leq \| u \|_{H^{s,t}(\mathbb{R}^3)} \| u \|_{H^s(\mathbb{R}^3)} \| u \|_{H^t(\mathbb{R}^3)} \).

(v) For \( s > 1 \) and \( t > \frac{1}{2} \), \( \| u \|_{H^{s,t}(\mathbb{R}^3)} \) is an algebra.

Let us point out that the usual Sobolev spaces \( H^s \) and \( H^{s,t} \) coincide with Besov spaces \( B^s_{2,2} \) and \( B^{s,t}_{2,2} \), respectively.

**Lemma 2.3** (Morse estimate). Let \( s > 0, q \in [1, \infty] \). Then there exists a constant \( C \) such that

\[
\| fg \|_{B^s_{p,q}} \leq C \left( \| f \|_{L^{p_1}} \| g \|_{B^{r_1}_{p_2,q}} + \| g \|_{L^{r_1}} \| f \|_{B^{s}_{1,q}} \right),
\]

where \( p_1, r_1 \in [1, \infty] \) satisfy \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2} \).

**Proof.** The proof of Lemma 2.3 is standard, here we omit the details. One also refer to [25] for the proof.

**Lemma 2.4.** For any \( p \in ]1, \infty[ \), there holds that

\[
\| \nabla u \|_{L^p(D)} \leq C \frac{p^2}{p - 1} \| \omega \|_{L^p(D)},
\]

where \( C \) depending only on the domain \( D \), and not on \( p \).
Proof. The classical result of Calderon-Zygmund [5] together with the Biot-Savart law allows to obtain the desired result for simple domains such as the whole space $\mathbb{R}^n$, the half space or the ball. As for the general case the proof is based on a rather awkward technique, developed in [30, 31]. □

Proposition 2.5 ([10]). Let $u$ be solution of the classical heat equations

$$\begin{align*}
\{ & \partial_t u - \Delta u = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\
& u|_{t=0} = u_0.
\end{align*}$$

Then there exists a constant $C > 0$ such that for each $j \geq 0$,

$$\|\Delta_j u(t)\|_{L^p(\mathbb{R}^n)} = \|e^{t\Delta} \Delta_j u_0\|_{L^p(\mathbb{R}^n)} \leq Ce^{-ct2^j} \|u_0\|_{L^p(\mathbb{R}^n)}.$$  

Lemma 2.6. Let $s_i > 1$ and $t_i > \frac{1}{2}$ with $i = 1, 2$. Then there exists a constant $C$ such that

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C\left(\|u\|_{L^2(\mathbb{R}^3)} + \|\Lambda_{h}^{s_1} u\|_{L^2(\mathbb{R}^3)} + \|\Lambda_{v}^{t_1} u\|_{L^2(\mathbb{R}^3)} + \|\Lambda_{h}^{s_2} \Lambda_{v}^{t_2} u\|_{L^2(\mathbb{R}^3)}\right).$$  

(2.3)

Proof. Thanks to Littewood-Paley decomposition, one has

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq \sum_{i,j \geq -1} \|\Delta_i^{h} \Delta_j^{v} u\|_{L^\infty(\mathbb{R}^3)}$$

$$\leq \|\Delta_{-1}^{h} \Delta_{-1}^{v} u\|_{L^\infty(\mathbb{R}^3)} + \sum_{i \geq 0} \|\Delta_i^{h} \Delta_{i-1}^{v} u\|_{L^\infty(\mathbb{R}^3)} + \sum_{j \geq 0} \|\Delta_{j+1}^{h} \Delta_{j}^{v} u\|_{L^\infty(\mathbb{R}^3)}$$

$$+ \sum_{i,j \geq 0} \|\Delta_i^{h} \Delta_j^{v} u\|_{L^\infty(\mathbb{R}^3)}$$

$$:= I_1 + I_2 + I_3 + I_4.$$  

It is clear that

$$I_1 \leq C\|u\|_{L^2}.$$  

By using the Bernstein inequality, the Minkowski inequality and the fact that $s_1 > 1$, we infer that

$$I_2 \leq C \sum_{i \geq 0} 2^i \|\Delta_i^{h} \Delta_{i-1}^{v} u\|_{L^2(\mathbb{R}^3)}$$

$$\leq C \sum_{i \geq 0} 2^{i(1-s_1)} \|\Lambda_i^{s_1} u\|_{L^2(\mathbb{R}^3)}$$

$$\leq C\|\Lambda_{h}^{s_1} u\|_{L^2(\mathbb{R}^3)}.$$  

And similarly, we can conclude that for $t_1 > \frac{1}{2}$,

$$I_3 \leq C\|\Lambda_{v}^{t_1} u\|_{L^2(\mathbb{R}^3)}.$$  

As for the term $I_4$, the Bernstein inequality and the Minkowski inequality enable us to conclude that for $s_2 > 1$ and $t_2 > \frac{1}{2}$,

$$I_4 \leq C \sum_{i,j \geq 0} 2^i 2^j \|\Delta_i^{h} \Delta_j^{v} u\|_{L^2(\mathbb{R}^3)}$$

$$\leq C \sum_{i,j \geq 0} 2^{i(1-s_2)2^j(\frac{1}{2}-t_2)} \|\Delta_i^{h} \Delta_j^{v} \Lambda_{h}^{s_2} \Lambda_{v}^{t_2} u\|_{L^2(\mathbb{R}^3)}$$

$$\leq C\|\Lambda_{h}^{s_2} \Lambda_{v}^{t_2} u\|_{L^2(\mathbb{R}^3)}.$$  

Collecting these estimates yields the desired result (2.3). □
Next, we will give a new type space-time logarithmic inequality which is a key component in our analysis, by using the low-high frequency decomposition technique.

**Lemma 2.7.** Let $a \in [0, 1]$, $p, q \in [1, \infty]$ and $s > \frac{n}{p}$. Assume that $f \in L_T^1(B_{p,q}^s(\mathbb{R}^n))$ such that

$$\sup_{2 \leq j < \infty} \int_0^T \frac{\|S_j f(t)\|_{L^\infty(\mathbb{R}^n)}}{j^a} \, dt \leq \infty.$$ 

Then the following inequality holds:

$$\int_0^T \|f(t)\|_{L^\infty(\mathbb{R}^n)} \, dt \leq C \left( 1 + \sup_{2 \leq j < \infty} \int_0^T \frac{\|S_j f(t)\|_{L^\infty(\mathbb{R}^n)}}{j^a} \, dt \right) \left( \log \left( e + \|f\|_{L_T^1(B_{p,q}^s(\mathbb{R}^n))} \right) \right)^a. \tag{2.4}$$

Here the constant $C$ independent of $f$ and $T$.

**Proof.** According to the Littlewood-Paley decomposition, we decompose $f$ into two parts as follows:

$$f = S_N f + \sum_{k \geq N} \Delta_k f,$$

where $N$ is a positive integer to be fixed later.

For the low-frequency part $S_N f$, it is clear that for $N \geq 2$

$$\int_0^T \|S_N f(t)\|_{L^\infty(\mathbb{R}^n)} \, dt \leq N^a \sup_{2 \leq j < \infty} \int_0^T \frac{\|S_j f(t)\|_{L^\infty(\mathbb{R}^n)}}{j^a} \, dt.$$

Next, for the high-frequency part, in view of the Bernstein inequality and the definition of Besov space, we have

$$\int_0^T \left\| \sum_{k \geq N} \Delta_k f(t) \right\|_{L^\infty(\mathbb{R}^n)} \, dt \leq C \int_0^T \sum_{j \geq N} 2^{j(s-n/p)} 2^js \|\Delta_k f(t)\|_{L^p(\mathbb{R}^n)} \, dt \leq C 2^{-N(s-n/p)} \int_0^T \|f(t)\|_{B_{p,q}^s(\mathbb{R}^n)} \, dt,$$

in the last line we have used the Hölder inequality.

Collecting these estimates, we thus get

$$\int_0^T \|f(t)\|_{L^\infty(\mathbb{R}^n)} \, dt \leq N^a \sup_{2 \leq j < \infty} \int_0^T \frac{\|S_j f(t)\|_{L^\infty(\mathbb{R}^n)}}{j^a} \, dt + C 2^{-N(s-n/p)} \int_0^T \|f(t)\|_{B_{p,q}^s(\mathbb{R}^n)} \, dt. \tag{2.5}$$

Now we choose $N$ which more than 2 such that $2^{-N(s-n/p)} \int_0^T \|f(t)\|_{B_{p,q}^s(\mathbb{R}^n)} \, dt \leq 1$, i.e.,

$$N \geq \max \left\{ 2, \frac{\log \|f\|_{L_T^1(B_{p,q}^s(\mathbb{R}^n))}}{(s - \frac{n}{p}) \log 2} \right\}.$$

This together with (2.5) yields the desired result (2.4). $\square$
2.2 Study of the flow map

In this subsection, we first review some basic results about the flow in \([1]\). And we give another form of a growth estimate of \(\rho\) which be suitable to our problems according to the generalized flow map associated to an axisymmetric vector field.

\[
\psi(t, s, x) = x + \int_s^t u(\tau, \psi(\tau, s, x)) d\tau.
\]  

(2.6)

It is well-known that if the vector field \(u\) lies in \(L^1_{\text{loc}}(\mathbb{R}; \text{Lipschitz})\) then the generalized flow is uniquely determined and exists globally in time. In addition, the incompressible condition that \(\text{div} \ u = 0\) guarantees that for every \(t, s \in \mathbb{R}\), \(\psi(t, s)\) is a diffeomorphism that preserves Lebesgue measure and

\[
\psi^{-1}(t, s, x) = \psi(s, t, x).
\]

Now we begin to show a new form of estimate for the quantity \(\|\rho(t)\|_{L^2}\) which different from the quadratic growth estimate \([1.3]\) in \([1]\). This is the cornerstone for establishing of the quantity \(\|\omega(t)\|_{L^2}\).

**Proposition 2.8.** Let \(u\) be a smooth axisymmetric vector field with zero divergence and \(\rho\) be a solution of the transport equation

\[
\partial_t \rho + u \cdot \nabla \rho = 0, \quad \rho_{|t=0} = \rho_0.
\]

Assume in addition that

\[
\rho_0 \in L^2 \cap L^\infty, \quad d(\text{Supp} \ \rho_0, (Oz)) := r_0 > 0 \quad \text{and} \quad \text{diam} (\Pi_z(\text{Supp} \ \rho_0)) := d_0 < \infty.
\]

Then we have

\[
\left\| \frac{\rho(t)}{r} \right\|_{L^2}^2 \leq \frac{1}{r_0^2} \| \rho_0 \|_{L^2}^2 + 2\pi \| \rho_0 \|_{L^\infty}^2 \int_0^t \left\| \frac{u_x}{r} (\tau) \right\|_{L^\infty}^2 d\tau + \frac{1}{r_0^2} \int_0^t \left\| \nabla_h \omega(\tau) \right\|_{L^2}^2 d\tau,
\]

(2.7)

where \(r = (x^2_1 + x^2_2)^{\frac{1}{2}}\).

**Proof.** We have from the definition that

\[
\left\| \frac{\rho(t)}{r} \right\|_{L^2}^2 = \int_{r \geq r_0} \frac{\rho^2(t, x)}{r^2} dx + \int_{r \leq r_0} \frac{\rho^2(t, x)}{r^2} dx \\
\leq \frac{1}{r_0^2} \| \rho(t) \|_{L^2}^2 + \| \rho(t) \|_{L^\infty}^2 \int_{r \leq r_0} \cap \text{Supp} \ \rho(t) \frac{1}{r^2} dx \\
\leq \frac{1}{r_0^2} \| \rho_0 \|_{L^2}^2 + \| \rho_0 \|_{L^\infty}^2 \int_{r \leq r_0} \cap \text{Supp} \ \rho(t) \frac{1}{r^2} dx.
\]

(2.8)

On the other hand, according to \([1]\ Proposition 3.2\), we know

\[
\text{diam} (\Pi_z \text{Supp} \rho(t)) \leq \text{diam} (\Pi_z \text{Supp} \rho_0) + 2 \int_0^t \| u_z(\tau) \|_{L^\infty} d\tau.
\]

This together with the maximum principle of \(\rho\) gives that

\[
\int_{r \leq r_0} \cap \text{Supp} \ \rho(t) \frac{1}{r^2} dx \leq 2\pi \left( \int_{r_0}^{r_0 - \int_0^t \| \omega(\tau) \|_{L^\infty} d\tau} \frac{1}{r} d\tau \right) \left( \int_{\Pi_z \text{Supp} \ \rho(t)} dz \right)
\]

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Putting this estimate into (3.1) gives
\[ \leq 2\pi \int_0^t \| \frac{u_r}{r} (\tau) \|_{L^\infty} d\tau \left( d_0 + 2 \int_0^t \| u_z (\tau) \|_{L^\infty} d\tau \right). \]

The divergence free condition guarantees that \( \| \Delta u_z \|_{L^2} \leq C \| \nabla \nabla_h u \|_{L^2} \) and \( \| \nabla u_z \|_{L^2} \leq C \| \nabla_h u \|_{L^2} \). Thus, we can deduce by using the interpolation theorem that
\[ \| u_z \|_{L^\infty} \leq \| u_z \|_{L^0} \| u_z \|_{L^2} \leq C \| \nabla \nabla_h u \|_{L^2} \| \nabla \omega \|_{L^2} \leq C \| \nabla_h u \|_{L^2} \| \nabla \omega \|_{L^2}. \]

Inserting this inequality in (2.8), we eventually obtain the desired result (2.7).

\[ \square \]

3 A priori estimates

This section is devoted to a priori estimates which can be viewed as a preparation for the proof of Theorem 1.1.

3.1 Weak a priori estimates

**Proposition 3.1.** Let \( u_0 \in L^2 \) be a vector field with zero divergence and \( \rho_0 \in L^2 \cap L^\infty \). Then every smooth solution of (HBS) satisfies
\[ \| \rho(t) \|_{L^p} \leq \| \rho_0 \|_{L^p} \quad \text{for} \quad p \in [2, \infty]. \]
\[ \| u(t) \|_{L^2}^2 + 2 \int_0^t \| \nabla_h u(t) \|_{L^2}^2 d\tau \leq 2 (\| u_0 \|_{L^2}^2 + t^2 \| \rho_0 \|_{L^2}^2). \]

**Proof.** According to (2.6), one can write
\[ \rho(t, x) = \rho_0 (\psi^{-1}(t, x)). \]
This together with the incompressible condition implies the first estimate.

Next, we take the \( L^2 \)-inner product of the velocity equation with \( u \). Integrating by parts with respect to space leads to
\[ \frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2}^2 + \| \nabla_h u(t) \|_{L^2}^2 \leq \| u(t) \|_{L^2} \| \rho(t) \|_{L^2}. \tag{3.1} \]

This means that
\[ \frac{d}{dt} \| u(t) \|_{L^2} \leq \| \rho(t) \|_{L^2}. \]

Integrating in time this inequality yields
\[ \| u(t) \|_{L^2} \leq \| u_0 \|_{L^2} + \int_0^t \| \rho(\tau) \|_{L^2} d\tau. \]

Since \( \| \rho(t) \|_{L^2} = \| \rho_0 \|_{L^2} \), then
\[ \| u(t) \|_{L^2} \leq \| u_0 \|_{L^2} + t \| \rho_0 \|_{L^2}. \]

Putting this estimate into (3.1) gives
\[ \| u(t) \|_{L^2}^2 + 2 \int_0^t \| \nabla_h u(\tau) \|_{L^2}^2 d\tau \leq \| u_0 \|_{L^2}^2 + 2 \int_0^t (\| u_0 \|_{L^2}^2 + \| \rho_0 \|_{L^2}^2) \| \rho_0 \|_{L^2} d\tau \]
\[ \leq \| u_0 \|_{L^2}^2 + 2 (\| u_0 \|_{L^2}^2 + \frac{1}{2} t \| \rho_0 \|_{L^2}^2) \| \rho_0 \|_{L^2} t. \]

This gives the second desired estimate. \[ \square \]
Next, we intend to review the behavior of the operator $\frac{\partial}{\partial r} \Delta^{-1}$ over axisymmetric functions and the algebraic relation between $\frac{u_r}{r}$ and $\frac{\omega_{\theta}}{r}$.

**Lemma 3.2** ([21]). Assume that $u$ is an axisymmetric smooth scalar function, then there holds that

$$
\left( \frac{\partial}{\partial r} \right) \Delta^{-1} u(x) = \frac{x_2^2}{r^2} R_{11} u(x) + \frac{x_1^2}{r^2} R_{22} u(x) - 2 \frac{x_1 x_2}{r^2} R_{12} u(x), \tag{3.2}
$$

where $R_{ij} = \partial_i \Delta^{-1}$.

**Lemma 3.3.** Assume that $u$ be an axisymmetric vector-field without swirl satisfying $\text{div} u = 0$ and $\omega = \text{curl} u$. Then

$$
\frac{u_r}{r} = \partial_2 \Delta^{-1} \left( \frac{\omega_{\theta}}{r} \right) - 2 \frac{\partial_r}{r} \Delta^{-1} \partial_2 \Delta^{-1} \left( \frac{\omega_{\theta}}{r} \right) \tag{3.3}
$$

Furthermore, one has

$$
\left| \frac{u_r}{r} \right| \leq \left| \partial_2 \Delta^{-1} \left( \frac{\omega_{\theta}}{r} \right) \right| + \sum_{i,j=1}^{2} \left| R_{ij} \partial_2 \Delta^{-1} \left( \frac{\omega_{\theta}}{r} \right) \right|, \tag{3.4}
$$

and

$$
\left\| \partial_2^2 \left( \frac{u_r}{r} \right) \right\|_{L^p} \leq C \left\| \partial_2 \left( \frac{\omega_{\theta}}{r} \right) \right\|_{L^p}, \quad \text{for} \quad p \in ]1, \infty[. \tag{3.5}
$$

**Proof.** One can refer to [27] for the proof of (3.3) and (3.4). Next, we turn to show (3.5). According to [30], we obtain

$$
\partial_2^2 \frac{u_r}{r} = \partial_2 \Delta^{-1} \partial_2^2 \left( \frac{\omega_{\theta}}{r} \right) - 2 \frac{\partial_r}{r} \Delta^{-1} \partial_2 \Delta^{-1} \left( \frac{\omega_{\theta}}{r} \right). \tag{3.6}
$$

Using Lemma 3.2 and applying the $L^p$-boundedness of Riesz operator, we eventually obtain (3.5). $\square$

The next proposition describes some estimates linking the velocity to the vorticity in virtue of the Biot-Savart law and Lemma 3.3.

**Proposition 3.4** ([27]). Assume that $u$ is an axisymmetric vector-field without swirl satisfying $\text{div} u = 0$ and vorticity $\omega = \omega_{\theta} e_{\theta}$. Then

$$
\left\| u \right\|_{L^\infty(\mathbb{R}^3)} \leq C \left\| \omega_{\theta} \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla_h \omega_{\theta} \right\|_{L^2(\mathbb{R}^3)}, \tag{3.7}
$$

and

$$
\left\| \frac{u_r}{r} \right\|_{L^\infty(\mathbb{R}^3)} \leq C \left\| \frac{\omega_{\theta}}{r} \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla_h \left( \frac{\omega_{\theta}}{r} \right) \right\|_{L^2(\mathbb{R}^3)}. \tag{3.8}
$$

### 3.2 Strong a priori estimates

In this subsection, our task is to obtain the global Lipschitz estimates of the vector field. Now, let us begin with the estimate of $\left\| u(t) \right\|_{H^1}$.

**Proposition 3.5.** Let $u_0 \in H^1$ be an axisymmetric vector field with zero divergence such that $\frac{\omega_{\theta}}{r} \in L^2$. Let $\rho_0 \in L^2 \cap L^\infty$ depending only on $(r, z)$ such that $\text{Supp} \rho_0$ does not intersect the axis $(Oz)$ and $\Pi_z(\text{Supp} \rho_0)$ is a compact set. Then every smooth solution $(u, \rho)$ of the system (HBS) satisfies for every $t \geq 0$,

$$
\left\| \omega_{\theta}(t) \right\|_{L^2}^2 + \int_0^t \left( \left\| \nabla_h \omega_{\theta}(\tau) \right\|_{H^1}^2 + \left\| \frac{\omega_{\theta}}{r}(\tau) \right\|_{L^2}^2 d\tau + \left( \left\| \frac{\omega_{\theta}}{r}(t) \right\|_{L^2}^2 + \int_0^t \left\| \nabla_h \left( \frac{\omega_{\theta}}{r} \right)(\tau) \right\|_{L^2}^2 d\tau \right) \right) \leq C e^{\exp Ct^2},
$$

where the constant $C$ depends on the initial data.
Proof. Taking the $L^2$-inner product of the equation (3.9) with $\omega_\theta$ we get
\[
\frac{1}{2} \frac{d}{dt} \|\omega_\theta(t)\|_{L^2}^2 + \|\nabla_h \omega_\theta(t)\|_{L^2}^2 + \left\| \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 = \int_{\mathbb{R}^3} u_r \frac{\omega_\theta}{r} \omega_r dx - \int_{\mathbb{R}^3} \partial_r \rho \omega_r dx. \tag{3.9}
\]
For the first integral term in the right side of (3.9), by using Lemma E.1 and the Young inequality, we get
\[
\int_{\mathbb{R}^3} u_r \frac{\omega_\theta}{r} \omega_r dx \leq \|u_r\|_{L^2} \left( \left\| \frac{\omega_\theta}{r} \right\|_{L^2} + \|\partial_r \omega_r\|_{L^2} \right) \leq \|\rho_0\|_{L^2}^2 + \frac{1}{4} \left( \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^2 + \|\partial_r \omega_r\|_{L^2}^2 \right).
\]
Indeed, integration by parts gives
\[
-\int_{\mathbb{R}^3} \partial_r \rho \omega_\theta dx = -2\pi \int \partial_r \rho \omega_\theta r dr dz = 2\pi \int \rho \partial_r \omega_\theta r dr dz + 2\pi \int \rho \frac{\omega_\theta}{r} r dr dz = \int_{\mathbb{R}^3} \rho \left( \partial_r \omega_\theta + \frac{\omega_\theta}{r} \right) dx.
\]
Putting together these estimates and using the fact that $\|\nabla_h \omega_\theta\|_{L^2}^2 = \|\partial_r \omega_\theta\|_{L^2}^2 + \|\omega_\theta/r\|_{L^2}^2$ yield
\[
\frac{d}{dt} \|\omega_\theta(t)\|_{L^2}^2 + \|\nabla_h \omega_\theta(t)\|_{L^2}^2 + \left\| \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 \leq \|u(t)\|_{L^2} \|\omega_\theta(t)\|_{L^2}^2 + \frac{1}{2} \left\| \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 \left\| \nabla_h \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 + 2\|\rho_0\|_{L^2}^2.
\]
Next, integrating the above inequality with respect to time, we readily get
\[
\|\omega_\theta(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \omega_\theta(\tau)\|_{L^2}^2 d\tau + \int_0^t \left\| \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 d\tau \leq \|\omega_\theta(0)\|_{L^2}^2 + \|u_0\|_{L^2} \int_0^t \|\omega_\theta(\tau)\|_{L^2}^2 d\tau + \frac{1}{2} \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^2 \int_0^t \left\| \nabla_h \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 d\tau + 2t\|\rho_0\|_{L^2}^2.
\]
To show the estimate for the quantity $\Gamma := \frac{\omega_\theta}{r}$. We observe that the quantity $\Gamma$ satisfies the following equation
\[
(\partial_t + u \cdot \nabla) \Gamma - (\Delta_h + \frac{2}{r} \partial_r) \Gamma = -\frac{\partial_r \rho}{r}. \tag{3.11}
\]
Taking the $L^2$-inner product of (3.11) with $\Gamma$ and integrating by parts, we get
\[
\frac{1}{2} \frac{d}{dt} \|\Gamma(t)\|_{L^2}^2 + \|\nabla_h \Gamma(t)\|_{L^2}^2 - 4\pi \int \partial_r (\Gamma) \Gamma dr dz = -2\pi \int \partial_r \rho \Gamma dr dz.
\]
For the term in the right side of equality above, integrating by parts and using the fact \( \rho(t, 0, z) = 0 \), we obtain
\[
-2\pi \int \partial_r \rho \Gamma r dr dz = 2\pi \int \frac{\rho}{r} \partial_r \Gamma r dr dz = \int_{\mathbb{R}} \frac{\rho}{r} \partial_r \Gamma dx \leq \frac{1}{2} \| \rho \|_{L^2}^2 \| \partial_r \Gamma \|_{L^2}^2 + \frac{1}{2} \| \partial_r \Gamma \|_{L^2}^2.
\]
Since
\[
4\pi \int \partial_r (\Gamma) r dr dz = 2\pi \int_{0}^{\tau} \partial_r (\Gamma)^2 dr dz \leq 0,
\]
then one can conclude that
\[
\frac{d}{dt} \| \frac{\omega \theta}{r} (t) \|_{L^2}^2 + \| \nabla h \left( \frac{\omega \theta}{r} \right) (t) \|_{L^2}^2 \leq \| \frac{\rho (0)}{r} \|_{L^2}^2 + \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^2}^2 d\tau.
\]
Integrating this inequality in time, we immediately get
\[
\| \frac{\omega \theta}{r} (t) \|_{L^2}^2 + \int_{0}^{t} \| \nabla h \left( \frac{\omega \theta}{r} \right) (\tau) \|_{L^2}^2 d\tau \leq \| \frac{\omega \theta (0)}{r} \|_{L^2}^2 + \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^2}^2 d\tau.
\] (3.12)
Plugging (2.7) of Proposition 2.8 into (3.12), we have
\[
\| \frac{\omega \theta}{r} (t) \|_{L^2}^2 + \int_{0}^{t} \| \nabla h \left( \frac{\omega \theta}{r} \right) (\tau) \|_{L^2}^2 d\tau \leq \frac{1}{2} \| \frac{\omega \theta (0)}{r} \|_{L^2}^2 + \| \rho_0 \|_{L^2}^2 + 2\pi \| \rho_0 \|_{L^\infty} d_0 t \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^\infty} d\tau
\]
\[
+ 4\pi \| \rho_0 \|_{L^\infty} t \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^\infty} d\tau \int_{0}^{t} \| \nabla h u (\tau) \|_{L^2}^2 d\tau \leq C (1 + t^4) \left( 1 + \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^\infty}^2 d\tau \right) + \frac{1}{4} \left( \int_{0}^{t} \| \nabla h \omega (\tau) \|_{L^2}^2 d\tau \right)^\frac{1}{2}.
\]
It follows from Proposition 3.1 that,
\[
\| \frac{\omega \theta}{r} (t) \|_{L^2}^2 + \int_{0}^{t} \| \nabla h \left( \frac{\omega \theta}{r} \right) (\tau) \|_{L^2}^2 d\tau \leq C (1 + t^4) \left( 1 + \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^\infty}^2 d\tau \right) + \frac{1}{4} \left( \int_{0}^{t} \| \nabla h \omega (\tau) \|_{L^2}^2 d\tau \right)^\frac{1}{2}.
\] (3.13)
By using (3.8) and the Young inequality, one can conclude that
\[
C (1 + t^4) \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^\infty}^2 d\tau \leq C (1 + t^4) \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^2}^2 d\tau \leq C (1 + t^8) \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^2}^2 d\tau.
\]
Plugging this estimate into (3.13), we obtain
\[
\| \frac{\omega \theta}{r} (t) \|_{L^2}^2 + \int_{0}^{t} \| \nabla h \left( \frac{\omega \theta}{r} \right) (\tau) \|_{L^2}^2 d\tau \leq \frac{1}{2} \| \frac{\omega \theta (0)}{r} \|_{L^2}^2 + C (1 + t^8) \left( 1 + \int_{0}^{t} \| \frac{\rho (\tau)}{r} \|_{L^2}^2 d\tau \right) + \frac{1}{4} \left( \int_{0}^{t} \| \nabla h \omega (\tau) \|_{L^2}^2 d\tau \right)^\frac{1}{2}.
\] (3.14)
For the first term in the last line of the above equality, we deduce by the Hölder inequality that

\[
\frac{1}{2} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty L^2}^2 \int_0^t \left\| \nabla h \left( \frac{\omega_\theta}{r} \right)(\tau) \right\|_{L^2}^2 \, d\tau \leq \frac{1}{8} \left( \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty L^2}^2 + \int_0^t \left\| \nabla h \left( \frac{\omega_\theta}{r} \right)(\tau) \right\|_{L^2}^2 \, d\tau \right)^2 \\
\leq \frac{1}{8} \left\| \frac{\omega_\theta(0)}{r} \right\|_{L^4}^4 + C(1 + t^{16}) \left( 1 + \int_0^t \left\| \frac{\omega_\theta}{r} \right\|_{L^2}^2 \, d\tau \right)^2 \\
+ \frac{1}{64} \left( \int_0^t \left\| \nabla h(\tau) \right\|_{L^2}^2 \, d\tau \right).
\]

This together with (3.10) and (3.14) give

\[
\left\| \omega_\theta(t) \right\|_{L^2}^2 + \int_0^t \left( \left\| \nabla h \frac{\omega_\theta}{r} \right\|_{L^2}^2 + \left\| \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 \right) \, d\tau + \left\| \frac{\omega_\theta}{r}(t) \right\|_{L^2}^2 \leq C \left( \left\| u \right\|_{L^\infty L^2} \int_0^t \left\| \omega_\theta(\tau) \right\|_{L^2}^2 \, d\tau + C_0(1 + t^{16}) \left( 1 + \int_0^t \left\| \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 \, d\tau \right)^2 \\
+ \frac{1}{2} \int_0^t \left\| \nabla h \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 \, d\tau + 2t \left\| \rho_0 \right\|_{L^2}^2 + \left\| \frac{\omega_\theta(0)}{r} \right\|_{L^2}^2 + \frac{1}{8} \left\| \frac{\omega_\theta(0)}{r} \right\|_{L^2}^2.
\]

The Gronwall inequality ensures that

\[
\left\| \omega_\theta(t) \right\|_{L^2}^2 \leq C \exp \left( \int_0^t \left\| \nabla h \frac{\omega_\theta}{r}(\tau) \right\|_{L^2}^2 \, d\tau \right) \leq C e^{\exp C t^{17}} \left( \left\| \omega_\theta(0) \right\|_{L^2} + 1 \right).
\]

This completes the proof. \( \square \)

**Proposition 3.6.** Assume that \( u_0 \in H^1 \), with \( \frac{\omega_\theta}{r} \in L^2 \) and \( \omega_0 \in \sqrt{1} \). Let \( \rho_0 \in L^2 \cap L^\infty \) depending only on \((r, z)\) such that \( \text{Supp} \rho_0 \) does not intersect the axis \((Oz)\) and \( \Pi_z(\text{Supp} \rho_0) \) is a compact set. Then any smooth axisymmetric without swirl solution \((r, u)\) of (HBS) satisfies

\[
\left\| \omega_\theta(\tau) \right\|_{L^2} \leq C \exp \left( \int_0^t \left\| \omega_\theta(\tau) \right\|_{L^2} \, d\tau \right) \leq C e^{\exp C t^{17}} \left( \left\| \omega_\theta(0) \right\|_{L^2} + 1 \right).
\]

Here the positive constant \( C \) depends on the initial data.

**Proof.** Multiplying the vorticity equation (1.6) with \( |\omega_\theta|^{p-2} \omega_\theta \) and performing integration in space, we get

\[
\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^3} |\omega_\theta|^p \, dx + (p - 1) \int_{\mathbb{R}^3} |\nabla h \omega_\theta|^2 |\omega_\theta|^{p-2} \, dx + \int_{\mathbb{R}^3} |\omega_\theta|^{p-2} \frac{\omega_\theta^2}{r^2} \, dx \\
= \int_{\mathbb{R}^3} \frac{1}{r} |\omega_\theta|^p \, dx - \int_{\mathbb{R}^3} \partial_r \rho |\omega_\theta|^{p-2} \, dx.
\]

For the first term in the last line of the above equality, we deduce by the Hölder inequality that

\[
\int_{\mathbb{R}^3} \frac{1}{r} |\omega_\theta|^p \, dx \leq \left\| \frac{1}{r} \right\|_{L^\infty} \left\| \omega_\theta \right\|_{L^p}^p.
\]

For the second term, by the Hölder inequality, we have

\[
-\int_{\mathbb{R}^3} \partial_r \rho |\omega_\theta|^{p-2} \omega_\theta \, dx \\
\leq \left\| \rho \right\|_{L^p} \left\| \omega_\theta \right\|_{L^\infty \frac{p-2}{2}} \left\| \omega_\theta \right\|_{L^2} + (p - 1) \left\| \rho \right\|_{L^p} \left\| \omega_\theta \right\|_{L^\frac{p-2}{2}} \left\| \omega_\theta \right\|_{L^2} \frac{\omega_\theta^2}{r^2} \left\| \partial_r \omega_\theta \right\|_{L^2}
\]

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\[ \leq \frac{p-1}{2} \int_{\mathbb{R}^3} |\nabla_k \omega_\theta|^2 |\omega_\theta|^{p-2} \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\omega_\theta|^{p-2} \frac{\omega_\theta^2}{r^2} \, dx + \frac{p}{2} \| \rho \|^2_{L^p} \| \omega_\theta \|^2_{L^p}. \]

Indeed, integrating by parts leads to
\[ - \int_{\mathbb{R}^3} \partial_\tau \rho |\omega_\theta|^{p-2} \omega_\theta \, dx = - 2\pi \int_{\mathbb{R}^3} \partial_\tau \rho |\omega_\theta|^{p-2} \omega_\theta r \, dz \]
\[ = \int \rho |\omega_\theta|^{p-2} \omega_\theta r \, dz + (p-1) \int \rho |\omega_\theta|^{p-2} \partial_r \omega_\theta r \, dz \]
\[ = \int_{\mathbb{R}^3} \rho |\omega_\theta|^{p-2} \frac{\omega_\theta}{r} \, dx + (p-1) \int_{\mathbb{R}^3} \rho |\omega_\theta|^{p-2} \partial_r \omega_\theta \, dx. \]

Therefore, by virtue of Proposition 3.1 one has
\[ \frac{d}{dt} \| \omega_\theta(t) \|^2_{L^p} \leq \left\| \frac{u_p}{r}(t) \right\|_{L^\infty} \| \omega_\theta(t) \|^2_{L^p} + \frac{p}{2} \| \rho_0 \|^2_{L^p}. \]

The Gronwall inequality yields that
\[ \| \omega_\theta(t) \|^2_{L^p} \leq e^{\int_0^t \left\| \frac{u_p}{r}(\tau) \right\|_{L^\infty} \, d\tau} \left( \| \omega_\theta(0) \|^2_{L^p} + \frac{p}{2} \| \rho_0 \|^2_{L^p} \right). \]

According to (3.8) of Proposition 3.4 and to the fact that \( \| \omega \|_{L^p} = \| \omega_\theta \|_{L^p} \), one can deduce that
\[ \| \omega(t) \|^2_{L^\infty} \leq e^{C \exp(C_1 t)} \left( \| \omega(0) \|^2_{L^\infty} + \frac{t}{2} \| \rho_0 \|^2_{L^2 \cap L^\infty} \right). \]

This completes the proof.

Proposition above together with the well-known fact that \( \| \nabla u \|_{L^2} \leq C \frac{\omega_\theta^2}{r^{p-1}} \| \omega \|_{L^p} \) for \( p \in ]1, \infty[ \) yields that \( \sup_{p \geq 2} \frac{\| \nabla u(t) \|_{L^p}}{\rho_{L^p}} \) is locally bounded with respect to time. But, the growth rate \( p\sqrt{p} \) does not satisfies the dual Osgood modulus of continuity. This requires us to further refine the growth estimate of \( \| \nabla u \|_{L^p} \) to get the growth estimate we require. Inspired by the Boot-Strap argument, we will establish the below proposition which is the heart in our proof based on the estimate of \( \sup_{p \geq 2} \frac{\| \nabla u(t) \|_{L^p}}{\rho_{L^p}} \).

**Proposition 3.7.** Assume that \( u_0 \in H^1 \), with \( \frac{u_0}{r} \in L^2 \) and \( \omega_0 \in \sqrt{\mathcal{L}} \). Let \( \rho_0 \in L^2 \cap L^\infty \) depending only on \( (r, z) \) such that \( \text{Supp} \rho_0 \) does not intersect the axis \( (Oz) \) and \( \Pi_z(\text{Supp} \rho_0) \) is a compact set. Then any smooth axisymmetric without swirl solution \( (\rho, u) \) of (HBS) satisfies
\[ \sup_{p \geq 2} \int_0^t \sum_{q \geq 0} 2^{qs} \| u_q(\tau) \|_{L^p}^p \, d\tau \leq C_1 e^{C_1 \exp(C_1 t)} \quad \text{for} \quad s \in [0, 2]. \] (3.15)

In particular, we have
\[ \sup_{2 \leq p < \infty} \int_0^t \| \nabla u(\tau) \|_{L^p}^p \, d\tau \leq C_2 e^{C_2 \exp(C_2 t)}. \] (3.16)

Here constants \( C_1 \) and \( C_2 \) depend on the initial data.

**Proof.** Applying the operator \( \Delta_h^h \) to (HBS) and using Duhamel formula we get
\[ u_q(t, x) = e^{t \Delta_h^h} u_q(0) - \int_0^t e^{(t-\tau) \Delta_h^h} \mathcal{P} (u \cdot \nabla u)(\tau, x) \, d\tau - \int_0^t e^{(t-\tau) \Delta_h^h} \Delta_h^h \mathcal{P} \rho(\tau, x) \, dz \, d\tau, \]
\[ = e^{t \Delta_h^h} u_q(0) - \int_0^t e^{(t-\tau) \Delta_h^h} \mathcal{P} (u \cdot \nabla u)(\tau, x) \, d\tau - \int_0^t e^{(t-\tau) \Delta_h^h} \Delta_h^h \mathcal{P} \rho(\tau, x) \, dz \, d\tau, \]
where \( u_q = \Delta_q^h u \) and \( \mathcal{P} \) is the Leray projection on divergence free vector fields.

Notice that
\[
  u \cdot \nabla u = \omega \times u + \frac{1}{2} \nabla |u|^2.
\]

Therefore,
\[
  \mathcal{P}(u \cdot \nabla u) = \mathcal{P}(\omega \times u).
\]

According to Proposition 2.5, we have the following estimate for \( q \geq 0 \)
\[
  \left\| e^{t \Delta_q^h} \Delta_q^h f \right\|_{L^p} \leq \left\| e^{t \Delta_q^h} \Delta_q^h f \right\|_{L^p(R^3)} \leq Ce^{-\epsilon 2q} \left\| \Delta_q^h f \right\|_{L^p(R^3)}.
\]

Therefore, we have that for \( q \geq 0 \)
\[
  \left\| u_q \right\|_{L^1_t L^p} \leq C 2^{-2q} \left\| u_q(0) \right\|_{L^p} + p 2^{-2q} \int_0^t \left\| \Delta_q^h (\omega \times u)(\tau) \right\|_{L^p} \, d\tau + C p 2^{-2q} \left\| \Delta_q^h \rho \right\|_{L^1_t L^p}.
\]

Multiplying the above inequality by \( 2^{qs} \) and summing over \( q \geq 0 \), we readily obtain that
\[
  \sum_{q \geq 0} 2^{qs} \left\| u_q \right\|_{L^1_t L^p} \leq C \sum_{q \geq 0} 2^{q(s-2)} \left\| u_q(0) \right\|_{L^p} + p \int_0^t \sum_{q \geq 0} 2^{q(s-2)} \left\| \Delta_q^h (\omega \times u)(\tau) \right\|_{L^p} \, d\tau + p \sum_{q \geq 0} 2^{q(s-2)} \left\| \Delta_q^h \rho \right\|_{L^1_t L^p} \leq I_1 + I_2 + I_3.
\]

For the first term \( I_1 \), by virtue of the imbedding theorem, we have that for \( s \in [1, 2] \)
\[
  I_1 \leq C \left\| u_0 \right\|_{L^p} \leq C \left( \left\| u_0 \right\|_{L^2} + \left\| u_0 \right\|_{L^\infty} \right) \leq C \left( \left\| u_0 \right\|_{L^2} + \left\| \omega_0 \right\|_{L^4} \right).
\]

For the third term \( I_3 \), by Proposition 3.1, we obtain that for \( s \in [1, 2] \),
\[
  I_3 \leq C pt \left\| \rho \right\|_{L^\infty_t L^p} \leq C pt \left\| \rho_0 \right\|_{L^2 \cap L^\infty}.
\]

Now we tackle the second parentheses of \( I_2 \). the H"older inequality and the interpolation inequality allow us to conclude that for \( s \in [1, 2] \)
\[
  \sum_{q \geq 0} 2^{q(s-2)} \left\| \Delta_q^h (\omega \times u) \right\|_{L^p} \leq C \left\| \omega \times u \right\|_{L^p} \leq C \left\| \omega \right\|_{L^p} \left( \left\| u \right\|_{L^2} + \left\| \omega \right\|_{L^4} \right)
\]

Whence,
\[
  I_2 \leq C p \left( \left\| u_0 \right\|_{L^2} + \left\| \omega \right\|_{L^\infty_t L^4} \right) \int_0^t \left\| \omega(\tau) \right\|_{L^p} \, d\tau.
\]

Putting together (3.19), (3.20) and (3.21), we finally get for \( s \in [1, 2] \)
\[
  \sum_{q \geq 0} 2^{qs} \left\| u_q \right\|_{L^1_t L^p} \leq C \left( \left\| u_0 \right\|_{L^2} + \left\| \omega_0 \right\|_{L^4} \right) + C pt \left\| \rho_0 \right\|_{L^2 \cap L^\infty} + C p \left( \left\| u_0 \right\|_{L^2} + \left\| \omega \right\|_{L^\infty_t L^4} \right) \int_0^t \left\| \omega(\tau) \right\|_{L^p} \, d\tau.
\]
It follows from Proposition 3.1, Proposition 3.5 and Proposition 3.6 that
\[
\sup_{\rho \geq 2} \int_0^t \sum_{q \geq 0} 2^{qs} \left\| u_q(\tau) \right\|_{L^p} \frac{d\tau}{\rho^\frac{s}{2}} \leq C \left( \| u_0 \|_{L^2} + \| \omega_0 \|_{L^4} \right) + Ct \| \rho_0 \|_{L^2 \cap L^\infty} + C \left( \| u_0 \|_{L^2} + \| \omega \|_{L^\infty L^4} \right) \int_0^t \| \omega(\tau) \|_{\sqrt{\Gamma}} d\tau 
\leq Ce^{\exp Ct^{17}}. \tag{3.22}
\]
On the other hand, by the sharp interpolation inequality, we have
\[
\sum_{q \geq 0} 2^q \| \Delta_q u \|_{L^p} \leq C \left( \| \Delta_{\frac{3}{2}} u \|_{L^p} \left( \sum_{q \geq 0} 2^{\frac{3}{2}q} \| \Delta_q u \|_{L^p} \right) \right)^{\frac{1}{3}} \leq C \left( \| \Delta_{\frac{3}{2}} u \|_{L^2} + \| \nabla u \|_{L^{15}} \right) \left( \sum_{q \geq 0} 2^{\frac{3}{2}q} \| \Delta_q u \|_{L^p} \right)^{\frac{1}{3}} \leq C \left( \| u \|_{H^1} + \| \omega \|_{L^{15}} \right) \left( \sum_{q \geq 0} 2^{\frac{3}{2}q} \| \Delta_q u \|_{L^p} \right)^{\frac{1}{3}}.
\]
Consequently, by using (3.22), Proposition 3.1, Proposition 3.5 and Proposition 3.6, we get that
\[
\int_0^t \frac{\| \nabla u(\tau) \|_{L^p}}{\sqrt{p}} d\tau \leq C \int_0^t \sum_{q \geq 0} 2^q \| \Delta_q u(\tau) \|_{L^p} \frac{d\tau}{\sqrt{p}} \leq C \int_0^t \| u(\tau) \|_{L^p} d\tau + C \int_0^t \| u(\tau) \|_{L^p} d\tau \leq Ct^{\frac{2}{3}} \left( \| u \|_{L^\infty H^1} + \| \omega \|_{L^\infty L^{15}} \right) \left( \sum_{q \geq 0} 2^{\frac{3}{2}q} \| \Delta_q u(\tau) \|_{L^p} \right)^{\frac{1}{3}} \leq C e^{\exp Ct^{17}}. \tag{3.23}
\]
Note that
\[
\frac{\partial_z u - \partial_z (u_1 e_r + u_2 e_z)}{\partial_z (u_1 e_r + u_2 e_z)} = \partial_z u_1 e_r + \partial_\tau u_2 e_z = -\omega_0 e_r + \partial_\tau u_2 e_z - \left( \partial_1 u_1 + \partial_2 u_2 \right) e_z.
\]
Thus, by using (3.23) and Proposition 3.6, we end up with
\[
\sup_{p \geq 2} \int_0^t \frac{\| \nabla u(\tau) \|_{L^p}}{\sqrt{p}} d\tau \leq C \sup_{p \geq 2} \int_0^t \frac{\| \nabla h \nabla u(\tau) \|_{L^p}}{\sqrt{p}} d\tau + C \int_0^t \sup_{p \geq 2} \| \omega(\tau) \|_{L^p} \frac{d\tau}{\sqrt{p}} \leq C e^{\exp Ct^{17}}.
\]
This implies the desired result (3.16).

\textbf{Proposition 3.8.} Let \((\rho, u)\) be a smooth solution of the system (HBS). Assume the initial data \((\rho_0, u_0)\) satisfies the conditions stated in Theorem 1.1. Then there exists a constant \(C\) such that for all \(t \in [0, T]\)
\[
\| \nabla \rho(t) \|_{L^2}^2 + \| \partial_z \omega_0(t) \|_{L^2}^2 + \int_0^t \| \nabla h \partial_z \omega_0(\tau) \|_{L^2}^2 d\tau + \int_0^t \frac{\| \partial_z \omega_0(\tau) \|_{L^2}^2}{p} d\tau \leq C_3 e^{\exp C_4 t^{17}}. \tag{3.24}
\]
Moreover, we have
\[
\| \nabla u \|_{L^1 L^\infty} \leq C_4 e^{\exp C_4 t^{17}}. \tag{3.25}
\]
Here the positive constants \(C_3\) and \(C_4\) depend on the initial data \((u_0, \rho_0)\).
Proof. Applying the operator $\partial_z$ to (1.6) yields

$$(\partial_t + u \cdot \nabla) \partial_z \omega_\theta - \Delta_h \partial_z \omega_\theta + \frac{\partial_z \omega_\theta}{r^2} = -\partial_z \partial_t \rho + \partial_z \left( \frac{u_r}{r} \right) \omega_\theta + \frac{u_r}{r} \partial_z \omega_\theta - \partial_z u_r \partial_r \omega_\theta - \partial_z u_z \partial_\theta \omega_\theta. \quad (3.26)$$

By the standard energy method, one can conclude that

$$\frac{1}{2} \frac{d}{dt} \|\partial_z \omega_\theta(t)\|_{L^2}^2 + \|\nabla_h \partial_z \omega_\theta(t)\|_{L^2}^2 + \frac{1}{r} \frac{\|\partial_z \omega_\theta(t)\|_{L^2}^2}{r}$$

$$= -\int_{\mathbb{R}^3} \partial_z \partial_t \rho \partial_z \omega_\theta \, dx + \int_{\mathbb{R}^3} \partial_z \left( \frac{u_r}{r} \right) \omega_\theta \partial_z \omega_\theta \, dx + \int_{\mathbb{R}^3} \frac{u_r}{r} \partial_z \omega_\theta \partial_z \omega_\theta \, dx$$

$$-\int_{\mathbb{R}^3} \partial_z u_r \partial_r \omega_\theta \partial_z \omega_\theta \, dx - \int_{\mathbb{R}^3} \partial_z u_z \partial_\theta \omega_\theta \partial_z \omega_\theta \, dx$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$

Since the support of $\rho$ ensures

$$-\int_{\mathbb{R}^3} \partial_z \partial_t \rho \partial_z \omega_\theta \, dx = -2\pi \int_{0}^{\infty} \int_{\mathbb{R}} \partial_z \partial_t \rho \partial_z \omega_\theta \, r \, dr \, dz$$

$$= 2\pi \int_{0}^{\infty} \int_{\mathbb{R}} \partial_z \rho \partial_z \omega_\theta \, r \, dr \, dz + 2\pi \int_{0}^{\infty} \int_{\mathbb{R}} \partial_z \rho \partial_r \omega_\theta \rho \, r \, dr \, dz$$

$$= \int_{\mathbb{R}^3} \partial_z \rho \frac{\partial_z \omega_\theta}{r} \, dx + \int_{\mathbb{R}^3} \partial_z \rho \partial_r \omega_\theta \, dx,$$

then the first term $I_1$ may be bounded by

$$\|\partial_z \rho\|_{L^2} \left\| \frac{\partial_z \omega_\theta}{r} \right\|_{L^2} + \|\partial_z \rho\|_{L^2} \|\partial_r \partial_z \omega_\theta\|_{L^2}$$

$$\leq 2 \|\partial_z \rho\|_{L^2}^2 + \frac{1}{4} \left( \left\| \frac{\partial_z \omega_\theta}{r} \right\|_{L^2}^2 + \|\partial_r \partial_z \omega_\theta\|_{L^2}^2 \right).$$

We observe that

$$\int_{\mathbb{R}^3} \partial_z \left( \frac{u_r}{r} \right) \omega_\theta \partial_z \omega_\theta \, dx = \frac{1}{2} \int_{\mathbb{R}^3} \partial_z \left( \frac{u_r}{r} \right) \partial_z (\omega_\theta)^2 \, dx$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} \partial_z^2 \left( \frac{u_r}{r} \right) (\omega_\theta)^2 \, dx.$$ 

Therefore, by the Hölder inequality, the Young inequality and Lemma 3.3, we obtain

$$I_2 \leq \left\| \partial_z^2 \left( \frac{u_r}{r} \right) \right\|_{L^2} \|\omega_\theta\|_{L^4}^2$$

$$\leq C \left\| \partial_z \left( \frac{\omega_\theta}{r} \right) \right\|_{L^2} \|\omega_\theta\|_{L^4}^2$$

$$\leq C \|\omega_\theta\|_{L^4}^2 + \frac{1}{8} \left\| \partial_z \left( \frac{\omega_\theta}{r} \right) \right\|_{L^2}^2.$$

By the Hölder inequality and Proposition 3.4, we have

$$I_3 \leq \left\| \frac{u_r}{r} \right\|_{L^\infty} \|\partial_z \omega_\theta\|_{L^2} \leq C \left\| \frac{\omega_\theta}{r} \right\|_{L^2} \|\nabla_h \left( \frac{\omega_\theta}{r} \right) \right\|_{L^2} \|\partial_z \omega_\theta\|_{L^2}^2.$$

As for the term $I_4$, we observe that

$$-\int_{\mathbb{R}^3} \partial_z u_r \partial_r \omega_\theta \partial_z \omega_\theta \, dx = \int_{\mathbb{R}^3} \omega_\theta \partial_r \omega_\theta \partial_z \omega_\theta \, dx - \int_{\mathbb{R}^3} \partial_r \partial_z \omega_\theta \partial_\theta \omega_\theta \, dx \quad (3.28)$$
For the last term

\[ I \]

Consequently, the H"older inequality and the Young inequality enable us to conclude that

\[ \text{Integrating by parts gives} \]

In the light of Lemma E.1, we have

\[ \text{By the H"older inequality and the Young inequality, we get} \]

\[ \text{In the light of Lemma E.1 we have} \]

\[ \text{For the last term } I_5, \text{ the incompressible condition that } \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0 \text{ guarantees that} \]

\[ \text{Integrating by parts gives} \]

\[ \text{Consequently, the H"older inequality and the Young inequality enable us to conclude that} \]

\[ \text{Here we have used the fact that} \]

\[ \|u\|_{L^\infty} \leq \sum_{q \geq -1} \|\Delta_q u\|_{L^\infty} \leq C \sum_{q \geq -1} 2^{q} \|\Delta_q u\|_{L^6} \]

\[ \leq C\|u\|_{L^6} + C \sum_{q \geq 0} 2^{-\frac{q}{2}} \|\Delta_q \nabla u\|_{L^6} \]

\[ \leq C(\|\omega\|_{L^2} + \|\omega\|_{L^6}) \leq C\|\omega\|_{\sqrt{\nabla}}. \]
Gathering these estimates, we finally obtain that

$$
\frac{d}{dt} \| \partial_z \omega(t) \|_{L^2}^2 + \| \nabla_h \partial_z \omega(t) \|_{L^2}^2 + \left\| \frac{\partial_z \omega(t)}{r} \right\|_{L^2}^2
\leq C \left( \| \frac{\omega_\rho}{r} \|_{L^2}^2 \| \nabla_h \left( \frac{\omega_\rho}{r} \right) \|_{L^2}^2 + \| \nabla_h \omega \|_{L^2}^2 + \| \omega \|_{L^2}^2 \right) \| \partial_z \omega(t) \|_{L^2}^2
\quad + C \left( \| \omega \|_{L^2}^2 \| \nabla_h \omega \|_{L^2}^2 + \| \omega \|_{L^2}^2 \right) \| \partial_z \omega(t) \|_{L^2}^2 + C \| \nabla \rho \|_{L^2}^2.
$$

(3.27)

Next, applying the differential operator $\partial_t$ to the density equation with $i = 1, 2, 3$, one has

$$
(\partial_t + u \cdot \nabla) \partial_t \rho = -\partial_t u_r \partial_t \rho - \partial_t u_z \partial_t \rho_i.
$$

Taking $L^2$-inner product of the above equation with $\partial_t \rho$, we immediately obtain that

$$
\frac{1}{2} \frac{d}{dt} \| \partial_t \rho(t) \|_{L^2}^2 = \frac{1}{2} \int_{\mathbb{R}^3} \partial_t u_r \partial_t \rho \partial_t \rho \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \partial_t u_z \partial_t \rho \partial_t \rho \, dx
\leq 2 \| \nabla u \|_{L^\infty} \| \nabla \rho \|_{L^2}^2.
$$

This implies that

$$
\frac{d}{dt} \| \nabla \rho(t) \|_{L^2}^2 \leq C \| \nabla \rho \|_{L^2}^2 e^{C \int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau}.
$$

(3.28)

On the other hand, by taking advantage of Lemma 3.6, we know that for all $\epsilon \in ]0, \frac{1}{10}[$,

$$
\int_0^t \| \nabla u(\tau) \|_{L^\infty(\mathbb{R}^3)} \, dt \leq C + \frac{1}{2} \sup_{2 \leq p < \infty} \int_0^t \frac{\| S_j \nabla u(\tau) \|_{L^\infty(\mathbb{R}^3)}}{\sqrt{\tau}} \, dt \left( \log (e + \| \nabla u \|_{L^1_1(B_{0,\infty}(\mathbb{R}^3))}) \right)^{\epsilon}
\leq C + C \left( \sup_{2 \leq p < \infty} \int_0^t \frac{\| \nabla u(\tau) \|_{L^p(\mathbb{R}^3)}}{\sqrt{\tau}} \, dt \right)^{\epsilon} + \log (e + \| \nabla u \|_{L^1_1(B_{0,\infty}(\mathbb{R}^3))}^\epsilon).
$$

Inserting this into (3.28), we get

$$
\begin{align*}
\| \nabla \rho(t) \|_{L^2}^2 & \leq \| \nabla \rho_0 \|_{L^2} e^{C e^{Ch(t)}} (e + \| \nabla u \|_{L^1_1(B_{0,\infty}(\mathbb{R}^3))}) \\
& \leq C \| \nabla \rho_0 \|_{L^2} e^{Ch(t)} + C \| \nabla \rho_0 \|_{L^2} e^{Ch(t)} \| \nabla u \|_{L^1_1(B_{0,\infty})} \\
& \leq C e^{Ch(t)} + C e^{Ch(t)} \| \nabla u \|_{L^1_1(B_{0,\infty})},
\end{align*}
$$

(3.29)

where $\sqrt{h(t)} := \sup_{2 \leq p < \infty} \int_0^t \frac{\| \nabla u(\tau) \|_p}{\sqrt{\tau}} \, d\tau$.

Putting this estimate together with (3.27) yields

$$
\begin{align*}
\frac{d}{dt} \| \partial_z \omega(t) \|_{L^2}^2 + \| \nabla_h \partial_z \omega(t) \|_{L^2}^2 + \left\| \frac{\partial_z \omega(t)}{r} \right\|_{L^2}^2
\leq C \left( \| \frac{\omega_\rho}{r} \|_{L^2}^2 \| \nabla_h \left( \frac{\omega_\rho}{r} \right) \|_{L^2}^2 + \| \nabla_h \omega \|_{L^2}^2 + \| \omega \|_{L^2}^2 \right) \| \partial_z \omega(t) \|_{L^2}^2
\quad + C \left( \| \omega \|_{L^2}^2 \| \nabla_h \omega \|_{L^2}^2 + \| \omega \|_{L^2}^2 \right) \| \partial_z \omega(t) \|_{L^2}^2 + C e^{Ch(t)} \| \nabla u \|_{L^1_1(B_{0,\infty})}.
\end{align*}
$$

(3.30)

Employing the Hölder inequality, Proposition 3.5 and Proposition 3.6, we get that

$$
\int_0^t \left( \| \frac{\omega_\rho}{r} (\tau) \|_{L^2}^2 \| \nabla_h \left( \frac{\omega_\rho}{r} \right)(\tau) \|_{L^2}^2 + \| \nabla_h \omega(\tau) \|_{L^2}^2 + \| \omega(\tau) \|_{L^2}^2 \right) \, d\tau
$$

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\[
\leq \sqrt{t} \left| \frac{\omega^g}{r} \right| L^2_{L^2} + \left| \nabla_h \left( \frac{\omega^g}{r} \right) \right| L^2_{L^2} + \left| \nabla_h \omega \right| L^2_{L^2} + t \left| \omega \right| L^2_{\infty} \sqrt{t}
\]
\[
\leq C e^{\exp C t^{17}}.
\]

By using the Gronwall inequality, Proposition 3.5, Proposition 3.6 and Proposition 3.7, we readily obtain
\[
\left\| \partial_2 \omega \theta(t) \right\|_2^2 + \int_0^t \left\| \nabla_h \partial_2 \omega \theta(\tau) \right\|_2^2 d\tau + \int_0^t \left\| \frac{\partial_2 \omega \theta(\tau)}{r} \right\|_2^2 d\tau 
\leq C e^{\exp C t^{17}} \left( \left\| \partial_2 \omega \theta(0) \right\|_2^2 + \left\| \omega \right\|_2^2 \left\| \nabla_h \omega \right\|_2^2 L^2_{L^2} + \left\| \omega \right\|_2^4 L^\infty_{\infty} + C t e^{Ch(t)} + C t e^{Ch(t)} \left\| \nabla u \right\| L^1_1 (B_{s_0}^{s_0}) \right)
\leq C e^{\exp C t^{17}} + C e^{\exp C t^{17}} \left\| \nabla u \right\| L^1_1 (B_{s_0}^{s_0}).
\]

(3.31)

This together with (3.29) and Proposition 3.6 yields
\[
\left\| \partial_2 \omega \theta(t) \right\|_2^2 + \left\| \nabla \rho(t) \right\|_2^2 + \int_0^t \left\| \nabla_h \partial_2 \omega \theta(\tau) \right\|_2^2 d\tau + \int_0^t \left\| \frac{\partial_2 \omega \theta(\tau)}{r} \right\|_2^2 d\tau 
\leq C e^{\exp C t^{17}} + C e^{\exp C t^{17}} \left\| \nabla u \right\| L^1_1 (B_{s_0}^{s_0}).
\]

(3.32)

Now, we tackle with the integral term \( \int_0^t \left\| \nabla u(\tau) \right\|_{B_{s_0}^{s_0}} d\tau \). It is clear that
\[
\left\| \nabla u(t) \right\|_{B_{s_0}^{s_0}} \leq \left\| \partial_2 u(t) \right\|_{B_{s_0}^{s_0}} + \left\| \nabla_h u(t) \right\|_{B_{s_0}^{s_0}}.
\]

(3.33)

On the one hand, with the help of Lemma 2.6 and the fact that \( \omega = \omega_\theta e_\theta \), one can infer that for \( \epsilon \in [0, \frac{1}{10}] \),
\[
\left\| \partial_2 u(\tau) \right\|_{B_{s_0}^{s_0}} \leq C \left\| \partial_2 u \right\|_{L^2} + C \left\| \Lambda^\epsilon \partial_2 u \right\|_{L^\infty}
\leq C \left\| \partial_2 u \right\|_{L^2} + C \left\| \Lambda^\epsilon \partial_2 u \right\|_{L^2} + C \left\| \Lambda^\epsilon \Lambda_h \partial_2 u \right\|_{L^2} + C \left\| \Lambda^\epsilon \Lambda_h \partial_2 u \right\|_{L^2} + C \left\| \Lambda^\epsilon \Lambda_h \partial_2 u \right\|_{L^2} + C \left\| \Lambda^\epsilon \Lambda_h \partial_2 u \right\|_{L^2}
\leq C \left\| \partial_2 u \right\|_{H^1} + C \left\| \partial_2 u \right\|_{L^2} + C \left\| \nabla_h \partial_2 u \right\|_{L^2} + C \left\| \partial_2 \left( \frac{\omega_\theta}{r} \right) \right\|_{L^2}.
\]

Therefore, we immediately obtain
\[
\left\| \partial_2 u(\tau) \right\|_{L^1_1 (B_{s_0}^{s_0})} \leq C t \left\| \partial_2 u \right\|_{L^2} + C \sqrt{t} \left\| \partial_2 u \right\|_{L^2} + C \sqrt{t} \left\| \nabla_h \partial_2 u \right\|_{L^2} + C \sqrt{t} \left\| \partial_2 \left( \frac{\omega_\theta}{r} \right) \right\|_{L^2}.
\]

(3.34)

Next, we need to introduce an useful lemma in order to tackle with another part \( \left\| \nabla_h u(t) \right\|_{B_{s_0}^{s_0}} \).

**Lemma 3.9** ([27]). Let \( s_1, s_2 \in \mathbb{R} \) and \( p \in [2, \infty] \). Assume that \( (\rho, u) \) be a smooth solution of the system [HBS], then there holds that
\[
\left\| u \right\|_{L^p_{1} B_{p,1}^{s_1+s_2}} \lesssim \left\| u_0 \right\|_{B_{p,1}^{s_1+s_2}} + \left\| u \right\|_{L^1_1 B_{p,1}^{s_1+s_2}} + \left\| u \otimes u \right\|_{L^1_1 B_{p,1}^{s_1+s_2}} + \left\| u \right\|_{L^1_1 B_{p,1}^{s_1+s_2}} + \left\| \rho \right\|_{L^1_1 B_{p,1}^{s_1+s_2}}.
\]

With the help of Lemma 3.9 and the Bernstein inequality, the term \( \left\| \nabla_h u(\tau) \right\|_{B_{s_0}^{s_0}} \) can be bounded as follows:
\[
\left\| \nabla_h u \right\|_{L^1_1 B_{s_0}^{s_0}} \leq C \left\| u \right\|_{L^1_1 B_{p,1}^{s_1+s_2}}.
\]
According to the Banach algebra property of Lemma 2.2, one has that for \( \epsilon \in (0, \frac{1}{10}] \),

\[
\| u \|^2_{L^2_t H^{\frac{1}{2}+2\epsilon} \cap L^\infty_t B^{\frac{3}{2}+2\epsilon}_2} \leq C \| u \|_{L^1_t B^{\frac{3}{2}+\epsilon}_2}^2 \leq C \| u \|_{L^1_t H^{\frac{1}{2}+\epsilon}}^2 + \| \rho \|_{L^1_t H^1}.
\]

(3.35)

On the other hand, Lemma 2.2 allows us to conclude that

\[
\| u \|_{L^1_t B^{\frac{3}{2}+\epsilon}_2} \leq C \| u \|_{L^1_t H^{\frac{1}{2}+\epsilon}}.
\]

Since the interpolation theorem and the fact that \( \omega = \omega_0 \epsilon_0 \) guarantee that there exist \( \beta \in (0, 1) \) such that for \( \epsilon \in (0, \frac{1}{10}] \),

\[
\| u \|_{H^{\frac{1}{2}+\epsilon} \cap L^\infty_t B^{\frac{3}{2}+\epsilon}_2} \leq C \| u \|_{L^2_t} + C \| \Lambda^1_\epsilon u \|_{L^2_t} + C \| \Lambda^2_\epsilon u \|_{L^2_t} + C \| \Lambda^3_\epsilon u \|_{L^2_t} + C \| \Lambda^4_\epsilon u \|_{L^2_t}
\]

\[
\leq C \| u \|_{H^1} + C \| \nabla \omega \|_{L^2_t} + C \| \partial_\omega \|_{L^2_t} + C \| \Lambda^1_\epsilon u \|_{L^2_t} + C \| \Lambda^2_\epsilon u \|_{L^2_t} + C \| \Lambda^3_\epsilon u \|_{L^2_t} + C \| \Lambda^4_\epsilon u \|_{L^2_t}
\]

\[
\leq C \| u \|_{H^1} + C \| \nabla \omega \|_{L^2_t} + C \| \partial_\omega \|_{L^2_t} + C \| \Lambda^1_\epsilon u \|_{L^2_t} + C \| \Lambda^2_\epsilon u \|_{L^2_t} + C \| \Lambda^3_\epsilon u \|_{L^2_t} + C \| \Lambda^4_\epsilon u \|_{L^2_t}
\]

Collecting these estimates together with (3.35) yields that for \( \epsilon \in (0, \frac{1}{10}] \),

\[
\| \nabla u \|_{L^1_t B^{\infty}_{2, \infty}} \leq C \| u \|_{H^1} + C \| u \|_{H^1} + C \| u \|_{L^2_t H^1} + C \| \partial_\omega \|_{L^2_t} + C \| \nabla \omega \|_{L^2_t} + C \| \partial_\omega \|_{L^2_t} + C \| \nabla \omega \|_{L^2_t}
\]

\[
\leq C \| u \|_{H^1} + C \| \nabla \omega \|_{L^2_t} + C \| \partial_\omega \|_{L^2_t} + C \| \nabla \omega \|_{L^2_t} + C \| \partial_\omega \|_{L^2_t} + C \| \nabla \omega \|_{L^2_t}
\]

(3.36)

Plugging (3.34) and (3.35) into (3.32), and using Proposition 3.1, Proposition 3.5 and Proposition 3.6 we finally obtain that

\[
\| \partial_\omega \|_{L^2_t} + \| \nabla \rho \|_{L^2_t} + \int_0^t \| \nabla \partial_\omega \|_{L^2_t} d \tau + \int_0^t \| \partial_\omega \|_{L^2_t} d \tau \leq C e^{\exp C t} + C e^{\exp C t} \left( C \| u \|_{L^\infty_t H^1} + C \sqrt{t} \| \partial_\omega \|_{L^2_t} + C \sqrt{t} \| \nabla \partial_\omega \|_{L^2_t} + C \sqrt{t} \| \partial_\omega \|_{L^2_t} + C \| \nabla \omega \|_{L^2_t} \right)^2 + C \| u \|_{L^2_t H^1} + C \| u \|_{L^2_t H^1} + C \| u \|_{L^2_t H^1} + C \| \partial_\omega \|_{L^2_t} + C \| \nabla \omega \|_{L^2_t} + C \| \nabla \omega \|_{L^2_t}
\]

(3.37)
In this section, we restrict our attention to show Theorem 1.1. Before proving, we first give an useful proposition which plays an important role in proof of Theorem 1.1.

**Proposition 4.1.** Let \((u_0, \rho_0) \in H^{s+1} \times H^s\) with \(s > \frac{3}{2}\). Assume that \(\text{div} u_0 = 0\). Then (HBS) admits a unique global solution \((\rho, u)\) satisfying \(u \in C(\mathbb{R}^+; H^{s+1})\) and \(\rho \in C(\mathbb{R}^+; H^{s})\).

Here we adopt the classical Friedrichs method (see for example [4]) to prove the existence part of Proposition 4.1. For \(n \geq 1\), the spectral cut-off \(J_n\) be defined by

\[ J_n f(\xi) = \chi_{[0,n]}(|\xi|) \hat{f}(\xi), \quad \text{for each } \xi \in \mathbb{R}^3. \]

Thus, by the same argument as in [27], one can consider the following system in the spaces \(L^2_n := \{ f \in L^2(\mathbb{R}^3) \mid \text{Supp} f \subset B(0,n) \} \):

\[
\begin{aligned}
\partial_t u_n + \mathcal{P} J_n \text{div}(u_n \otimes u_n) - \Delta_h u_n &= 0, \\
\partial_t \rho_n + J_n \text{div}(u_n \rho_n) &= 0, \\
\rho_n|_{t=0} = \rho_0,
\end{aligned}
\]

Note that the operators \(J_n\) and \(\mathcal{P} J_n\) are the orthogonal projectors for the \(L^2\)-inner product. Combining this with the stability result of [1] Lemma 5.1 ensures that the above formal calculations remain unchanged.

Next, our first target is to show the local well-posedness for the system (HBS). Applying the operator \(\Delta_q\) to the density equation, we thus get

\[ \partial_t \Delta_q \rho + S_{q+1} u \cdot \nabla \Delta_q \rho = S_{q+1} u \cdot \nabla \Delta_q \rho - \Delta_q (u \cdot \nabla \rho) := F_q(u, \rho). \]

Taking the \(L^2\)-inner product to the above equality with \(\Delta_q \rho\) and using the incompressible condition, we thus obtain

\[ \frac{1}{2} \frac{d}{dt} \| \Delta_q \rho(t) \|_{L^2}^2 \leq \| F_q(u, \rho) \|_{L^2} \| \Delta_q \rho \|_{L^2}. \] (4.1)
By using Lemma \(\text{E.2}\) multiplying both sides by \(2^{2qs}\) and summing up over \(q \geq -1\), we have
\[
\frac{d}{dt} \|\rho(t)\|^2_{H^s} \leq C \left( \|\nabla u(t)\|_{L^\infty} \|\rho(t)\|^2_{H^s} + \|\rho(t)\|_{L^\infty} \|\omega(t)\|_{H^s} \right). \tag{4.2}
\]

Next, we turn to show the estimate of \(\omega\). Applying \(\Delta_q\) to the velocity equations, we get
\[
\partial_t \Delta_q u + S_{q+1} u \cdot \nabla \Delta_q u - \Delta_h \Delta_q u + \Delta_q \nabla q = S_{q+1} u \cdot \nabla \Delta_q u - \Delta_q (u \cdot \nabla u) := F_q(u, u).
\]
Taking the \(L^2\)-inner product to this equation with \(\Delta_q u\) and using the divergence free condition, we can conclude that
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q u(t)\|^2_{L^2} + \|\nabla \Delta_q u(t)\|^2_{L^2} \leq \|F_q(u, u)\|_{L^2} \|\Delta_q u\|_{L^2}.
\]

By using Lemma \(\text{E.2}\) again, multiplying both sides by \(2^{q(1+s)}\) and summing up over \(q \geq -1\), we have
\[
\frac{d}{dt} \|u(t)\|^2_{H^{1+s}} \leq C \|\nabla u(t)\|_{L^\infty} \|u(t)\|^2_{H^{1+s}}. \tag{4.3}
\]

In a similar way as above, we can conclude that the approximate solution \((\rho_n, u_n)\) to \(\text{11}\) satisfies
\[
\|(\rho_n, \omega_n)(t)\|^2_{H^{s}} + \int_0^t \|\nabla \omega_n(\tau)\|^2_{H^{s}} d\tau \leq \|J_n(\rho_0, \omega_0)\|^2_{H^{s}} e^{C \int_0^t \|u_n(\tau)\|_{L^\infty} d\tau}.
\]

Since \(s > \frac{3}{2}\), the space \(H^s(\mathbb{R}^3)\) continuously embeds in \(L^\infty(\mathbb{R}^3)\). Thus, the well-known fact that \(\|\nabla u_n\|_{H^s}\) is equivalent to \(\|\omega_n\|_{H^s}\) entails us to conclude that
\[
X_n(t) \leq \|(\rho_0, \omega_0)\|^2_{H^{s}} e^{\frac{t}{2} C \int_0^t \|u_n(\tau)\|_{L^\infty} d\tau} \text{ and } X_n^2(t) := \|(\rho_n, \omega_n)(t)\|^2_{H^{s}}.
\]

This inequality may be easily integrated into
\[
\exp \left( - C \int_0^t X_n(\tau) d\tau \right) \geq 1 - 2C X_0 e^{\frac{t}{2}}, \text{ for all } t \geq 0.
\]

The energy estimate yields the \(L^2\)-bound of \(u_n\). Therefore, there exists a constant \(c > 0\) such that if we set
\[
T := 2 \log \left( \frac{c}{\|(\rho_0, \omega_0)\|_{H^{s}}} \right). \tag{4.4}
\]

Therefore, we get
\[
\rho_n \in L^\infty([0, T]; H^s), \quad u_n \in L^\infty([0, T]; H^{s+1}) \quad \text{and} \quad \nabla_h u_n \in L^2([0, T]; H^{s+1}). \tag{4.5}
\]

We now turn to proof of the local existence of a solution. By virtue of Equations \(\text{11}\) and uniform estimates \(\text{4.5}\), it is easy to check that \(\partial_t \rho_n \in L^\infty([0, T]; H^{s-1})\) and \(\partial_t u_n \in L^2([0, T]; H^{s+1})\). On the other hand, we know that \(H^s \hookrightarrow H^{s-1}\) and \(H^{s+1} \hookrightarrow H^s\) are locally compact. Therefore, by the classical Aubin-Lions argument and Cantor’s diagonal process, we can conclude that there exists a solution \((\rho, u)\) in \(L^\infty([0, T]; H^s \times H^{s+1})\) such that \(\nabla_h u \in L^2([0, T]; H^{s+1})\) and the time continuity follows from the fact that \(\rho\) and \(\omega\) satisfy transport equations with \(H^s\) initial data and a \(L^2([0, T]; H^s)\) source term. In addition, the standard energy method allows us to obtain the uniqueness of solution for Lipschitz vector field.

Now, it remains for us to show that the local smooth solutions may be extended to all positive time. Put together the lower bound for the lifespan of \((\rho, u)\) give by \(\text{4.3}\) and the uniqueness of smooth solutions, it suffices to state that under the assumption of the theorem, we have
\[
\sup_{0 \leq t \leq T} (\|\rho(t)\|_{H^s} + \|\omega(t)\|_{H^s}) < \infty. \tag{4.6}
\]
First, as $\rho$ is transported by the vector-fields $u$ (which is Lipschitz for $s > \frac{3}{2}$ implies $H^s \hookrightarrow L^\infty$), we get

$$\|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} \quad \text{for all } t \in [0, T].$$

In consequence, the energy estimate (3.24) ensures that

$$\|(\rho, \omega)(t)\|_{H^s} + \int_0^t \|\nabla_h \omega(\tau)\|_{H^s}^2 \, d\tau \leq \|(\rho_0, \omega_0)\|_{H^s}^2 e^{Ct} e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \quad (4.7)$$

On the other hand, by virtue of Proposition 3.1, Proposition 3.5 and Lemma 3.6, we can deduce that

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \leq C + \sup_{2 \leq q < \infty} \int_0^t \frac{\|S_q \nabla u(\tau)\|_{L^\infty} \, d\tau}{\sqrt{q}} \left( \log \left( e + \|\omega\|_{L^1(\omega)} \right) \right)^{\frac{1}{2}}$$

$$\leq C + \left( \sup_{2 \leq p < \infty} \int_0^t \frac{\|\nabla u(\tau)\|_{L^p} \, d\tau}{\sqrt{p}} \right)^2 + \log \left( e + \|\omega\|_{L^1(\omega)} \right) \quad (4.8)$$

Inserting (4.8) in (4.7) gives

$$\|(\rho, \omega)(t)\|_{H^s} + \int_0^t \|\nabla_h \omega(\tau)\|_{H^s}^2 \, d\tau \leq \|(\rho_0, \omega_0)\|_{H^s}^2 e^{Ct} e^{\int_0^t \|\nabla u(\tau)\|_{H^s} \, d\tau},$$

which together with the Gronwall inequality and Proposition 3.7 gives the desired result (4.6). This completes the proof of Proposition 4.1.

Now, let us turn to prove Theorem 1.1. We first construct the following approximate scheme:

$$\begin{align*}
(t_0 + u \cdot \nabla) u^n - \Delta_h u^n + \nabla \Pi^n &= \rho^n e_3, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
(t_0 + u \cdot \nabla) \rho^n &= 0, \\
\text{div} u^n &= 0, \\
(u^n, \rho^n)|_{t=0} &= (S_{n+1} u_0, S_{n+1} \rho_0).
\end{align*} \quad (4.9)$$

Since $u^n, \rho^n \in H^\infty := \bigcap_{s > 0} H^s$, we know that (4.9) has a unique global solution $(\rho^n, u^n)$ by taking advantage of Proposition 4.1. Thus, Proposition 3.1, Proposition 3.5 and Proposition 3.8 ensure

$$\rho^n \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1 \cap L^\infty), \quad u^n \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1), \quad \nabla_h u^n \in L^2_{\text{loc}}(\mathbb{R}^+; H^1),$$

$$\partial_t \omega^n \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2), \quad \nabla_h \partial_t \omega^n \in L^2_{\text{loc}}(\mathbb{R}^+; L^2) \quad \text{and} \quad \nabla u \in L^1_{\text{loc}}(\mathbb{R}^+; L^\infty).$$

Mimicking the compactness argument used for proving Proposition 4.1, one can conclude that there exists a solution $(\rho, u)$ such that (deduced from the Fatou lemma)

$$\rho \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1 \cap L^\infty), \quad u \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1), \quad \nabla_h u \in L^2_{\text{loc}}(\mathbb{R}^+; H^1),$$

$$\partial_t \omega \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2), \quad \nabla_h \partial_t \omega \in L^2_{\text{loc}}(\mathbb{R}^+; L^2) \quad \text{and} \quad \nabla u \in L^1_{\text{loc}}(\mathbb{R}^+; L^\infty).$$

From the above estimates, we eventually obtain the time continuity by the same argument as used in [27 Proposition F.4].

It remains for us to show the uniqueness statement. Let $(\rho, u, \Pi)$ and $(\overline{\rho}, \overline{u}, \overline{\Pi})$ be two solutions of the system (1.1) with the same initial data $(u_0, \rho_0)$ such that

$$\rho, \overline{\rho} \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1 \cap L^\infty), \quad u, \overline{u} \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1), \quad \nabla_h u, \nabla_h \overline{u} \in L^2_{\text{loc}}(\mathbb{R}^+; H^1),$$

$$\partial_t \omega, \partial_t \overline{\omega} \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^2), \quad \nabla_h \partial_t \omega, \nabla_h \partial_t \overline{\omega} \in L^2_{\text{loc}}(\mathbb{R}^+; L^2) \quad \text{and} \quad \nabla u, \nabla \overline{u} \in L^1_{\text{loc}}(\mathbb{R}^+; L^\infty). \quad (4.10)$$
Then the difference \((\delta \rho, \delta u, \delta \Pi)\) between two solutions \((\rho, u, \Pi)\) and \((\tilde{\rho}, \tilde{u}, \tilde{\Pi})\) satisfies
\[
\begin{cases}
\partial_t \delta u + u \cdot \nabla \delta u - \Delta_h \delta u + \nabla \delta \Pi = \delta \rho e_3 + \delta u \cdot \nabla \tilde{u}, \\
\partial_t \delta \rho + u \cdot \nabla \delta \rho = -\delta u \cdot \nabla \tilde{\rho}.
\end{cases}
\]

The standard energy argument together with the fact from the Plancherel theorem that
\[
\int_{\mathbb{R}^3} \delta \rho e_3 \delta u \, dx = \int_{\mathbb{R}^3} \delta \rho \delta u_z \, dx = \int_{\mathbb{R}^3} (\Delta)^{-1} \delta \rho \Delta \delta u_z \, dx
\]
\[
= \int_{\mathbb{R}^3} (\Delta)^{-1} \delta \rho \Delta_h \delta u_z \, dx - \sum_{i=1}^2 \int_{\mathbb{R}^3} (\Delta)^{-1} \delta \rho \partial_i \delta u_i \, dx
\]
\[
\leq C \| \delta \rho \|_{H^{-1}} \| \nabla_h \delta u \|_{L^2}
\]
enables us to infer that
\[
\frac{1}{2} \frac{d}{dt} \| \delta u(t) \|_{L^2}^2 + \| \nabla_h \delta u(t) \|_{L^2}^2 \leq C \| \delta \rho \|_{H^{-1}} \| \nabla_h \delta u \|_{L^2} + \| \nabla \tilde{u} \|_{L^\infty} \| \delta u \|_{L^2}^2
\]
which implies
\[
\frac{d}{dt} \| \delta u(t) \|_{L^2} \leq C \| \delta \rho \|_{H^{-1}} + \| \nabla \tilde{u} \|_{L^\infty} \| \delta u \|_{L^2}.
\] (4.11)

And the Fourier localization technique and the classical commutator estimate (see for example [25]) allow us to conclude that
\[
\frac{d}{dt} \| \delta \rho(t) \|_{H^{-1}} \leq C \| \nabla u \|_{L^\infty} \| \delta \rho \|_{H^{-1}} + \| \delta u \|_{L^2} \| \tilde{\rho} \|_{L^\infty}.
\] (4.12)

Putting (4.11) together with (4.10) and (4.12), and using the Gronwall inequality entails \((\delta \rho, \delta u) \equiv 0\).

This completes the proof of Theorem 1.1.

A Appendix

In this section, we shall give two useful lemmas which have been used throughout the paper.

**Lemma E.1** ([II]). There exists the positive constant \(C\) such that
\[
\int_{\mathbb{R}^3} fghdx_1dx_2dx_3 \leq C \| f \|_{L^2}^\frac{1}{2} \| \partial_{x_3} f \|_{L^2}^\frac{1}{2} \| g \|_{L^2}^\frac{1}{2} \| \nabla_h g \|_{L^2}^\frac{1}{2} \| h \|_{L^2}^\frac{1}{2} \| \nabla_h h \|_{L^2}^\frac{1}{2}. \] (5.1)

**Lemma E.2.** Let \(s > -1\) and \(1 \leq p \leq \infty\). Assume that \(u\) be a divergence free vector fields over \(\mathbb{R}^d\). There exists a positive constant \(C\) such that for all \(q \geq -1\)
\[
2^{qs} \| R_q(u, u) \|_{L^2} \leq c_q \| \nabla u \|_{L^\infty} \| u \|_{B^q_{p,r}}, \quad \text{with} \quad c_q \in \ell^2,
\] (5.2)
and
\[
\| R_q(u, \rho) \|_{L^2} \leq C \left( \| \nabla u \|_{L^\infty} \sum_{q' \geq q-4} 2^{q-q'} \| \Delta_{q'} \rho \|_{L^2} + \| \rho \|_{L^\infty} \sum_{|q-q'| \leq 4} \| \Delta_{q'} \nabla u \|_{L^2} \right), \] (5.3)
where \(R_q(u, v) := S_{q+1} u \cdot \nabla \Delta_{q'} v - \Delta_{q'} (u \cdot \nabla v)\).
Proof. We just give the proof of \([5.3]\), since the proof of \([5.2]\) is standard. First of all, one decomposes \(R_q(u, \rho)\) as follows:

\[
R_q(u, \rho) = u \cdot \nabla \Delta_q \rho - \Delta_q (u \cdot \nabla \rho) - \Delta_q ((I_d - S_{q+1})u \cdot \nabla \rho)
\]

where \(\bar{u} = (I_d - S_1)u\).

Note that

\[
[\Delta_q, S_{q+1}u] \cdot \nabla \rho = \Delta_q (T_{q+1} \partial_i \rho) + \Delta_q (T_{q+1} \partial_i \rho) - \Delta_q (T_{q+1} \partial_i \rho)
\]

\[
= R^1_q(u, \rho) + R^2_q(u, \rho) + R^3_q(u, \rho) + R^4_q(u, \rho) + R^5_q(u, \rho),
\]

and

\[
\Delta_q ((I_d - S_{q+1})u) \cdot \nabla \rho = \Delta_q (T_{q+1} \partial_i \rho) + \Delta_q (T_{q+1} \partial_i \rho) - \Delta_q (T_{q+1} \partial_i \rho)
\]

\[
= R^1_q(u, \rho) + R^2_q(u, \rho) + R^3_q(u, \rho) + R^4_q(u, \rho) + R^5_q(u, \rho).
\]

From above, it is clear to find that the only term \([\Delta_q, S_{q}u] \cdot \nabla \rho\) involves low frequencies of \(u\).

First of all, we observe that

\[
[S_{q'-1} S_{q+1} u, \Delta_q] \partial_i \Delta_q \rho
\]

\[
= 2^q \int_{\mathbb{R}^d} (S_{q'-1} S_{q+1} u(x) - S_{q'-1} S_{q+1} u(x-y)) \varphi(2^q (x-y)) \partial_i \Delta_q \rho(y) \, d y
\]

\[
= -2^q \int_{\mathbb{R}^d} \int_0^1 \partial_k S_{q'-1} S_{q+1} u(x + (1-\tau)(x-y)) \, d \tau \partial_k \varphi(2^q (x-y)) \partial_i \Delta_q \rho(y) \, d y
\]

\[
= -2^q \int_{\mathbb{R}^d} \int_0^1 \partial_k S_{q'-1} S_{q+1} u(x + (1-\tau)(x-y)) \, d \tau \partial_k \varphi(2^q (x-y)) \partial_i \Delta_q \rho(y) \, d y,
\]

where used the relation \(\Delta_q f = 2^q \int_{\mathbb{R}^d} \varphi(2^q (x-y)) f(y) \, d y\).

Therefore, we immediately get that

\[
\|R^1_q(u, \rho)\|_{L^p} \leq C \sum_{|q'-q| \leq 4} 2^{-(q-q')} \|\partial_k S_{q'-1} u_i\|_{L^\infty} \|\partial_i \Delta_q \rho\|_{L^p} \int_{\mathbb{R}^d} |x \varphi(x)| \, d x
\]

\[
\leq C \sum_{|q'-q| \leq 4} 2^{-(q-q')} \|\partial_k S_{q'-1} u_i\|_{L^\infty} \|\Delta_q \rho\|_{L^p}
\]

\[
\leq C \|\nabla u\|_{L^\infty} \sum_{|q'-q| \leq 4} 2^{-(q-q')} \|\Delta_q \rho\|_{L^p}. \tag{5.4}
\]

In a similar fashion as for proving \(R^2_q(u, \rho)\), we can bounded \([\Delta_q, S_{q}u] \cdot \nabla \rho\) as follows:

\[
\|[\Delta_q, S_{q}u] \cdot \nabla \rho\|_{L^p} \leq C \sum_{|q'-q| \leq 4} 2^{-(q-q')} \|\partial_k S_{q'-1} u_i\|_{L^\infty} \|\Delta_q \rho\|_{L^p}
\]

\[
\leq C \|\nabla u\|_{L^\infty} \sum_{|q'-q| \leq 4} 2^{-(q-q')} \|\Delta_q \rho\|_{L^p}. \tag{5.5}
\]

For the second term \(R^2_q(u, \rho)\), the Hölder inequality allows us to conclude that

\[
\|R^2_q(u, \rho)\|_{L^p} \leq C \sum_{|q'-q| \leq 4} \|\Delta_q \bar{u}_i\|_{L^p} \|S_{q'-1} \partial_i \rho\|_{L^\infty}
\]

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\[ \leq C \| \rho \|_{L^\infty} \sum_{|q' - q| \leq 4} \| \Delta_{q'}(\nabla u) \|_{L^p} \] (5.6)

Similarly, we can conclude that
\[ \| R^6_q(u, \rho) \|_{L^p} \leq C \| \nabla u \|_{L^\infty} \sum_{-1 \leq q' \leq q + 2} 2^{q' - q} \| \Delta_{q'} \rho \|_{L^p}. \] (5.7)

The reminder term \( R^3_q(u, \rho) \) can be bounded by
\[
\| \partial_i \Delta_q(R(S_{q+1} \bar{u}, \rho)) \|_{L^p} \leq C \sum_{q' \geq q - 2} 2^q \| \Delta_{q'} S_{q+1} \rho \|_{L^p} \| \Delta_{q'} \bar{u}_i \|_{L^\infty} \\
\leq C \sum_{q' \geq q - 2} 2^{q - q'} \| \Delta_{q'} S_{q+1} \rho \|_{L^p} \| \Delta_{q'} \nabla u \|_{L^\infty} \\
\leq C \| \nabla u \|_{L^\infty} \sum_{q' \geq q - 2} 2^{q - q'} \| \Delta_{q'} \rho \|_{L^p}. \] (5.8)

In a similar way, one has
\[
\| R^5_q(u, \rho) \|_{L^p} \leq C \| \nabla u \|_{L^\infty} \sum_{q' \geq q - 2} 2^{q - q'} \| \Delta_{q'} \rho \|_{L^p}. \] (5.9)

It remain for us to bound the last three terms \( R^6_q(u, \rho), R^7_q(u, \rho) \) and \( R^8_q(u, \rho) \). Thanks to the property of support and the Hölder inequality, one has
\[
\| R^6_q(u, \rho) \|_{L^p} \leq C \sum_{|q' - q| \leq 4} \| S_{q'-1} (I_d - S_{q+1}) u_i \|_{L^\infty} \| \Delta_{q'} \partial_{q'} \rho \|_{L^p} \\
\leq C \sum_{|q' - q| \leq 4} 2^{-q} \| S_{q'-1} (I_d - S_{q+1}) \nabla u_i \|_{L^\infty} \| \Delta_{q'} \partial_{q'} \rho \|_{L^p} \\
\leq C \sum_{|q' - q| \leq 4} 2^{q - q'} \| S_{q'-1} \nabla u_i \|_{L^\infty} \| \Delta_{q'} \rho \|_{L^p} \\
\leq C \| \nabla u \|_{L^\infty} \sum_{|q' - q| \leq 4} 2^{q - q'} \| \Delta_{q'} \rho \|_{L^p}. \] (5.10)

For the term \( R^7_q(u, \rho) \), by the Hölder inequality, we obtain
\[
\| R^7_q(u, \rho) \|_{L^p} \leq C \sum_{|q' - q| \leq 4} \| S_{q'-1} \partial_{q'} \rho \|_{L^\infty} \| \Delta_{q'} (I_d - S_{q+1}) u_i \|_{L^p} \\
\leq C \sum_{|q' - q| \leq 4} 2^{q - q'} \| S_{q'-1} \rho \|_{L^\infty} \| \Delta_{q'} (I_d - S_{q+1}) \nabla u_i \|_{L^p} \\
\leq C \| \rho \|_{L^\infty} \sum_{|q' - q| \leq 4} 2^{q - q'} \| \Delta_{q'} (I_d - S_{q+1}) \nabla u \|_{L^p}. \] (5.11)

As for the last term \( R^8_q(u, \rho) \), by the Hölder inequality, we obtain
\[
\| R^8_q(u, \rho) \|_{L^p} \leq C \| \partial_i \Delta_q R((I_d - S_{q+1}) u_i, \rho) \|_{L^p} \\
\leq C \sum_{q' \geq q - 2} 2^q \| \Delta_{q'} \rho \|_{L^p} \| \Delta_{q'} (I_d - S_{q+1}) u_i \|_{L^\infty} \]
\[ \leq C \sum_{q' \geq q-2} 2^q \| \Delta_q \rho \|_{L^p} \| \tilde{\Delta}_q u_i \|_{L^\infty} \]
\[ \leq C \sum_{q' \geq q-2} 2^{q-q'} \| \Delta_q \rho \|_{L^p} \| \tilde{\Delta}_q \nabla u_i \|_{L^\infty} \]
\[ \leq C \| \nabla u \|_{L^\infty} \sum_{q' \geq q-2} 2^{q-q'} \| \Delta_q \rho \|_{L^p}. \quad (5.12) \]

Combining these estimates yields the desired result (5.3). \[ \square \]

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