ON THE SPECTRA OF HYPERBOLIC SURFACES
WITHOUT THIN HANDLES

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We obtain a sharp lower bound on the eigenvalues of the Laplace–Beltrami operator on a hyperbolic surface with injectivity radius bounded from below. Bibliography: 5 titles.

1. Introduction

Let $\Omega$ be a hyperbolic surface, that is, a Riemannian manifold of real dimension 2 with constant Gaussian curvature $-1$; we assume that $\Omega$ is compact and has no boundary. Denote by $g$ the genus of $\Omega$. Let $\Delta$ be the Laplace–Beltrami operator on $\Omega$; it has purely discrete spectrum, since $\Omega$ is compact. Denote by $\lambda_j = \lambda_j(\Omega)$ the $j$th eigenvalue of $-\Delta$ ($j = 0, 1, 2, \ldots$).

Our main result is the following theorem.

**Theorem 1.** Let $r > 0$. There exists a constant $c(r) > 0$ such that if the injectivity radius of $\Omega$ is greater than $r$, then $\lambda_{\lceil \varepsilon g \rceil} \geq c(r) \cdot \varepsilon^2$ for any $\varepsilon \leq 2$.

In what follows, we denote by $c(r)$ any positive constant depending only on $r$ (but not on $\varepsilon$, $g$, and $\Omega$).

Proposition 8 below shows that our estimate is order sharp.

A theorem by Otal and Rosas ([3]) says that $\lambda_{2g-2} > 1/4$ for any $\Omega$ of genus $g$. On the other hand, for given $\delta > 0$, $N \in \mathbb{N}$, and $g = 2, 3, \ldots$ there exists a hyperbolic surface $\Omega$ of genus $g$ with $\lambda_{2g-3} < \delta$ and $\lambda_{2g-2+N} < 1/4 + \delta$. The validity of these inequalities is related to the existence of thin handles on $\Omega$ (see [1]). In other words, the eigenvalues are small when the injectivity radius of $\Omega$ degenerates. Theorem 1 gives a lower bound on the eigenvalues under an assumption on this radius.

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2. Proof of Theorem 1

Our proof of Theorem 1 is a slight refinement of Buser’s argument leading to the estimate $\lambda_{2g-2} \geq 10^{-12}$ (see [1]) combined with a simple Lemma 7 on graphs.

We are going to apply the Dirichlet–Neumann bracketing technique. Recall that if $X \subset \Omega$ is a set of positive area with a piecewise smooth boundary, then its Cheeger constant is defined as

$$h(X) := \inf \frac{l(A)}{\min\{|B|, |B'|\}},$$

where $A$ ranges over the family of all finite unions of piecewise smooth curves on $X$ cutting $X$ into two disjoint subsets $B$ and $B'$. Here, $l(A)$ is the length of $A$ and $|\cdot|$ is the Riemannian volume on $\Omega$. A very standard combination of the geometric version of the minimax principle with the Cheeger–Yau isoperimetric inequality ([1], see also [4, 5]) leads to the following conclusion.

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Theorem 2. Let $k \in \mathbb{N}$, and assume that $\Omega$ is represented as the union of sets $X_1, \ldots, X_k$ with piecewise smooth boundaries and disjoint interiors. Then

$$\lambda_k(\Omega) \geq \min_{j=1, \ldots, k} \frac{h^2(X_j)}{4}.$$

An appropriate partition of $\Omega$ will be obtained via a triangulation of controlled size. For this, recall a result by Buser (Theorem 4.5.2 in [1], see also [2]).

Definition 3. A closed domain $D \subset \Omega$ is called a trigon if $D$ is of one of the following two types:

1. $D$ is a simply connected embedded geodesic triangle (an ordinary triangle);
2. $D$ is an embedded doubly connected domain bounded by a geodesic cycle and two geodesic arcs (a collar-type trigon).

The geodesic boundary components of such a trigon $D$ are called the sides of $D$.

Theorem 4 (Buser). The surface $\Omega$ can be triangulated into trigons with side lengths $\leq \log 4$ and areas between $0.19$ and $1.36$.

Fix such a triangulation; denote by $\mathcal{T}_c$ and $\mathcal{T}_t$ the sets of its collar-type trigons and ordinary triangles, respectively. Also, denote by $\mathcal{S}_c$ and $\mathcal{S}_t$ the sets of sides of our triangulation that are cycles and geodesic arcs, respectively. Let $\mathcal{N}$ be the set of vertices of the triangulation. The proof of Theorem 4 from [1] shows that trigons from $\mathcal{T}_c$ have symmetries: namely, the $\mathcal{S}_t$-sides of such a trigon have equal lengths. From this we derive that the lengths of arcs from $\mathcal{S}_a$ are bounded from below by an absolute constant; also, the angles of the triangulation are also bounded from below by an absolute constant. (For $\mathcal{T}_c$-trigons, these statements are obvious due to the restrictions on the area, whereas for segments and angles in the boundaries of collars, the computation is done in [1].)

Lemma 5. If $a_1, a_2$ are two sides of the triangulation with no common vertex, then $\text{dist}_\Omega(a_1, a_2)$ is bounded from below by an absolute constant $d_0 > 0$.

Proof. First, note that the distance from any side $c \in \mathcal{S}_c$ to any other side is bounded from below by a universal constant, since otherwise the area of some trigon from $\mathcal{T}_c$ degenerates.

Next, we claim that the distances between vertices of the triangulation are bounded from below by a universal constant. Indeed, let $U$ be a metric ball on $\Omega$ centered at some point $v \in \mathcal{N}$. If the radius of $U$ is small enough, then for every $\tau \in \mathcal{T}_c \cup \mathcal{T}_t$ the intersection $U \cap \tau$ cannot intersect any side of the triangulation except for those that start at $v$; this can be easily checked for both types of trigons, and this leads to our claim.

Now, assume that $\gamma$ is a geodesic arc of small length joining $a_1$ and $a_2$; it cannot intersect a side from $\mathcal{S}_c$, since such sides are far enough from all the other sides. Assume that $\gamma$ passes through some trigon $\tau \in \mathcal{T}_c \cup \mathcal{T}_t$. Then it lies close enough to some vertex $v \in \mathcal{N}$ (because the angles of trigons are bounded from below). Since the vertices are separated, the whole curve $\gamma$ lies close enough to some vertex $v \in \mathcal{N}$, but in this case $\gamma$ can join only sides starting at $v$. The proof is finished. \qed

Now we estimate the Cheeger constants.

Lemma 6. Let $X \subset \Omega$ be the union of $N$ distinct trigons from our triangulation ($N = 1, 2, \ldots$). Assume that $X$ is “connected,” in the sense that two trigons are considered to be adjacent if they share a common side, and not just a common vertex. (More formally, we may say that the interior of $X$ is connected.)
Then, under the assumption on the injectivity radius of $\Omega$, we have

$$h(X) \geq \frac{c(r)}{N}.$$  \hfill (1)

**Proof.** Let $A, B, B'$ be the sets from the definition of the Cheeger constant for $X$; we have $A \neq \emptyset$, since $X$ is connected. By Yau’s lemma ([1, Lemma 8.3.6], see also [5]), we may assume that $B, B'$ are connected. If $l(A) \geq r$, then note that $\min\{|B|, |B'|\} \leq 1.36 \cdot 1/2 \cdot N$, and this leads to (1). Next, assume that $A$ contains a cycle $\gamma$. Then $\gamma$ is homotopic to identity in $\Omega$ (since $l(A) < r$ and by the condition on the injectivity radius). The cycle $\gamma$ must bound in $\Omega$ a component of area $\leq l(\gamma)/h(\mathbb{H}) = l(\gamma)$ (it is known that the Cheeger constant of the whole Lobachevsky plane $\mathbb{H}$ is 1), and this also gives (1). So, assume that $A$ contains no cycles.

We may assume from the beginning that $r < d_0$ where $d_0$ is the constant from Lemma 5. The set $A$ is a union of curves; take any component $\gamma$ of $A$. Then $\gamma$ necessarily has endpoints (since $A$ contains no cycles), and these endpoints lie on $\partial X$. Take two of such endpoints, $p_1, p_2$, and a curve $\gamma_1 \subset \gamma$ joining them. By Lemma 5, the points $p_1$ and $p_2$ lie either on the same side of the triangulation or on two distinct sides starting at their common vertex; this side (or these sides) lie(s) on $\partial X$. But if $Y$ is an angle in $\mathbb{H}$ or a half-plane of $\mathbb{H}$, then $h(Y) = 1$ (see, e.g., the proof of Theorem 8.1.2 in [1]). This, together with the condition on the injectivity radius, say at $p_1$, leads to (1). \hfill $\Box$

Now, to obtain an appropriate partition of $\Omega$ via our triangulation, we give a simple lemma on graphs.

**Lemma 7.** Let $G$ be a finite connected nonoriented graph with degrees of vertices $\leq 3$. Let $k \in \mathbb{N}$. The set of vertices of $G$ can be represented as a disjoint union $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_\alpha \sqcup V'$ (for some $\alpha = 0, 1, 2, \ldots$) so that

(i) the graphs induced by $G$ on each of the sets $V_1, V_2, \ldots, V_\alpha, V'$ are connected;

(ii) $2^k \leq |V_1|, |V_2|, \ldots, |V_\alpha| \leq 2^{k+1} - 1$ and $0 \leq |V'| \leq 2^k$.

**Proof.** We argue by induction on the number of vertices in $G$; for the empty graph, the statement is obvious. We may assume that $G$ is a tree. Pick a leaf of $G$ and call it the root. Arrange the graph by levels according to the distance from the root. A vertex $v$ from some level is adjacent to $\leq 2$ vertices from the next level, which we call the children of $v$. Also, any vertex except for the root has a unique parent in this arrangement.

Let us construct a sequence of vertices $v_0, v_1, \ldots, v_\beta$ of $G$ (where $\beta$ will be some nonnegative integer). Take the root as $v_0$. Assume that $v_j$ is constructed and that $v_j$ and $v_r$ are its children. Without loss of generality, the total number of descendants of $v_j$ is greater than or equal to that of $v_r$. Then put $v_{j+1} := v_j$. If $v_j$ has only one child, then take it as $v_{j+1}$. Finally, if $v_j$ has no children, then stop our process and put $\beta := j$; this must necessarily happen at some step. Thus, we have constructed a sequence of vertices.

Now traverse this sequence in the reverse order (from $v_\beta$ to $v_0$) and keep track of the total number of descendants of vertices. If $v_{j+1}$ has $x$ descendants (including itself), then $v_j$ has $\leq 2x + 1$ descendants including itself. Then we have two cases:

(1) There exists a vertex $v_j$ with $\geq 2^k$ and $\leq 2^{k+1} - 1$ descendants including itself. Then, for $V_j$ we take the set consisting of $v_j$ and all its descendants. Cut them out of $G$ and apply the induction hypothesis to $G$ without $V_j$.

(2) $|G| < 2^k$. Then take $V'$ to be the whole set of vertices of $G$.

The lemma is proved. \hfill $\Box$
Proof of Theorem 1. First, assume that \( \varepsilon g \leq 1 \). Then we must prove that \( \lambda_1 > c(r) / g^2 \), but, by Theorem 2, it suffices to prove that \( h(\Omega) > c(r) / g \). Taking \( A \) from the definition of \( h(\Omega) \), we see that \( A \) must contain a cycle; in this case, we argue as in the corresponding case in the proof of Lemma 6 and easily obtain the desired estimate (recall that \( |\Omega| = 2\pi(2g - 2) \)).

Now, assume that \( \varepsilon g > 1 \). Pick \( k \in \mathbb{N} \) with \( 2^k \geq \frac{8\pi}{0.19\cdot\varepsilon} \geq 2^{k-1} \); this can be done because \( \varepsilon \leq 2 \). Let \( G \) be the graph of the triangulation obtained in Theorem 4: namely, the set of vertices of \( G \) is \( T_1 \cup T_c \), and two trigons are adjacent if they share a common side. Apply Lemma 7 to \( G \), take the partition of the set of vertices of \( G \) obtained by this lemma, and consider the corresponding partition of the surface: \( X_1 \cup X_2 \cup \cdots \cup X_n \cup X' \) for some \( \alpha = 0, 1, 2, \ldots \). Since the trigons have area \( \geq 0.19 \), we have \( |X_j| \geq 2^k \cdot 0.19 \) for all \( j = 1, 2, \ldots, \alpha \). Then

\[
\alpha \leq \frac{|\Omega|}{2^k \cdot 0.19} < \frac{4\pi g}{2^k \cdot 0.19} \leq \frac{\varepsilon g}{2} \leq \lfloor \varepsilon g \rfloor - 1.
\]

Hence, \( \alpha + 1 \leq \lfloor \varepsilon g \rfloor \). Now, by Lemma 6, we have \( h(X_j), h(X') \geq c(r) / 2^k \) for all \( j \). This, together with Theorem 2, leads to the desired estimate. \( \square \)

Finally, let us demonstrate the sharpness of our estimate (we may assume that \( \varepsilon = 1/k \)).

**Proposition 8.** For any \( k, l \in \mathbb{N} \) there exists a hyperbolic surface \( \Omega \) of genus \( kl + 1 \) with injectivity radius bounded from below by a universal constant such that \( \lambda_{l-1}(\Omega) \leq C / k^2 \), where \( C < +\infty \) is a universal constant.

**Proof.** Let \( P \) be a fixed pair of hyperbolic pants bounded by geodesic cycles of length, say, 1 (the existence and uniqueness of such a pair of pants is a well-known fact). Let \( P_1, \ldots, P_{2k} \) be copies of \( P \). For \( j = 1, 2, \ldots, 2k \), denote by \( \gamma_1(P_j), \gamma_2(P_j), \gamma_3(P_j) \) the boundary components of \( P_j \). For \( j = 1, 2, \ldots, k \), glue \( \gamma_2(P_{2j-1}) \) to \( \gamma_2(P_{2j}) \) and \( \gamma_3(P_{2j-1}) \) to \( \gamma_3(P_{2j}) \). Also, for \( j = 1, 2, \ldots, k - 1 \), glue \( \gamma_1(P_{2j}) \) to \( \gamma_1(P_{2j+1}) \). Denote by \( Q \) the surface obtained in this way; it is a hyperbolic surface with two geodesic boundary components of length 1. There exists a Sobolev function \( f: Q \rightarrow \mathbb{R} \) with the following properties: first, \( f = 0 \) on \( \partial Q \); second, \( f \) takes values in \( [j - 1, j] \) on \( P_j \) and on \( P_{2k-j+1} \) for \( j = 1, 2, \ldots, k \); third, \( |\operatorname{grad} f| \) does not exceed some absolute constant, where \( \operatorname{grad} \) stands for the metric gradient. (To construct such a function, just let it be equal to appropriate constants on the boundary components of pants and interpolate it into the interiors of pants.) Now, take \( l \) copies of \( Q \) and glue them together in cyclic order to obtain a hyperbolic surface \( \Omega \) without boundary. Then the genus of \( \Omega \) is \( kl + 1 \). Moreover, one can find Sobolev functions \( f_1, f_2, \ldots, f_l: \Omega \to \mathbb{R} \) with disjoint supports such that \( \int_{\Omega} f_j^2 \geq c_1 \cdot k^3, \int_{\Omega} |\operatorname{grad} f_j|^2 \leq c_2 \cdot k \) (the constants \( c_1, c_2 \) are absolute). By the geometric version of the minimax principle (that is, by the upper estimate from the Dirichlet–Neumann bracketing), this leads to the desired estimate on the eigenvalues. The injectivity radius of \( \Omega \) is bounded from below, since this is true for any pair of pants. \( \square \)

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