On the smoothness of the partition function for multiple Schramm-Loewner evolutions

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Abstract

We consider the measure on multiple chordal Schramm-Loewner evolution (SLE_κ) curves. We establish a derivative estimate and use it to give a direct proof that the partition function is $C^2$ if $κ < 4$.

1 Introduction

The (chordal) Schramm-Loewner evolution with parameter $κ > 0$ (SLE_κ) is a measure on curves connecting two distinct boundary points $z, w$ of a simply connected domain $D$. As originally defined by Schramm [6], this is a probability measure on paths. If $κ ≤ 4$, the measure is supported on simple curves that do not touch the boundary. Although Schramm [6] originally defined SLE_κ as a probability measure, if $κ ≤ 4$ and $z, w$ are locally analytic boundary points, it is natural to consider SLE_κ as a finite measure $μ(z, w)$ with partition function, that is, as a measure with total mass $Ψ_D(z, w) = H_D(z, w)^b$. Here $H$ denotes the boundary Poisson kernel normalized so that $H_H(0, x) = x^{-2}$ and $b = (6 - κ)/2κ$ is the boundary scaling exponent. If $f : D → f(D)$ is a conformal transformation, then

$$H_D(z, w) = |f'(z)||f'(w)|H_{f(D)}(f(z), f(w)).$$

There are several reasons for considering SLE_κ as a measure with a total mass. First, SLE_κ is known to be the scaling limit of various two-dimensional discrete models that are considered as measures with partition functions, and hence it is natural to consider the (appropriately normalized) partition function in the scaling limit. Second, the “restriction property” or “boundary perturbation” can be described more naturally for the nonprobability measures; see [11] below. This description leads to one way to define SLE_κ in multiply connected domains or, as is important for this paper, for multiple SLE_κ in a simply connected domain. See [3, 9] for more information. We write $μ_D^♯(z, w) = μ_D(z, w)/Ψ_D(z, w)$ for the probability measure which is well defined even for rough boundaries.

The definition of the measure on multiple SLE_κ paths immediately gives a partition function defined as the total mass of the measure. The measure on multiple SLE_κ paths

$$γ = (γ^1, ..., γ^n)$$

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has been constructed in [1, 2, 3]. Even though the definition in [2] is given for the so-called “rainbow” arrangement of the boundary points, it can be easily extended to the other arrangements [1, 10]. One can see that unlike SLE\(\kappa\) measure on single curves, conformal invariance and domain Markov property do not uniquely specify the measure when \(n \geq 2\). This definition makes it unique by requiring the measure to satisfy the restriction property, which is explained in Section 2.

Study of the multiple SLE\(\kappa\) measure involves characterizing the partition function. For \(n = 2\), the partition function is explicitly given in terms of the hypergeometric function. For \(n \geq 3\), the goal is to characterize the partition function by a particular second-order PDE. However, it does not directly follow from the definition that the partition function is \(C^2\). There are two main approaches to address this problem. One approach is to show that the PDE system has a solution and use it to describe the partition function. In [1], it is shown that a family of integrals taken on a specific set of cycles satisfy the required PDE system. In [8], conformal field theory and partial differential equation techniques such as Hörmander’s theorem are used to show that the partition function satisfies the PDE system. The other approach, which is the one we take in this work, is to directly prove that the partition function is \(C^2\). Then Itô’s formula can be used next to show that the partition function satisfies the PDEs.

The basic idea of our proof is to interchange derivatives and expectations in expressions for the partition function. This interchange needs justification and we prove an estimate about SLE\(\kappa\) to justify this.

Here we summarize the paper. We finish this introduction by reviewing examples of partition functions for SLE\(\kappa\). Definitions and properties of multiple SLE\(\kappa\) and the outline of the proof are given in Section 2. Section 3 includes an estimate for SLE\(\kappa\) using techniques similar to the ones in [4]. Proof of Lemma 2.3, which explains estimates for derivatives of the Poisson kernel is given in Section 4.

1.1 Examples

- SLE\(\kappa\) in a subset of \(\mathbb{H}\). Let \(\kappa \leq 4\) and suppose \(D \subset \mathbb{H}\) is a simply connected domain such that \(K = \mathbb{H} \setminus D\) is bounded and \(\text{dist}(0, K) > 0\). Also, assume that \(\gamma\) is parameterized with half-plane capacity. By the restriction property we have

\[
\frac{d\mu_{D}(0, \infty)}{d\mu_{\mathbb{H}}(0, \infty)}(\gamma) = 1\{\gamma \cap K = \emptyset\} \exp \left\{ \frac{c}{2} m_{\mathbb{H}}(\gamma, K) \right\},
\]

where \(m_{\mathbb{H}}(\gamma, D)\) denotes the Brownian loop measure of the loops that intersect both \(\gamma\) and \(K\) and

\[
c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}
\]

is the central charge. We normalize the partition functions, so that \(\Psi_{\mathbb{H}}(0, \infty) = 1\). For an initial segment of the curve \(\gamma_t\), let \(g_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}\) be the unique conformal transformation with \(g_t(z) = z + o(1)\) as \(z \to \infty\). Then

\[
\partial_t g_t(z) = \frac{a}{g_t(z) - U_t},
\]
where $a = 2/\kappa$ and $U_t$ is a standard Brownian motion. Suppose $\gamma_t \cap K = \emptyset$ and let $D_t = g_t(D \setminus \gamma_t)$. One can see that

$$m_{\Xi}(\gamma_t, K) = -\frac{a}{6} \int_0^t S \Phi_s(U_s) ds,$$

where $S$ denotes the Schwarzian derivative and $\Phi_s(U_s) = H_{D_s}(U_s, \infty)$. It follows from conditioning on $\gamma_t$ that

$$M_t = \exp \left\{ \frac{c}{2} m_{\Xi}(\gamma_t, K) \right\} \Psi_{D_t}(U_t, \infty)$$

is a martingale. We assume the function $V(t, x) = \Psi_{D_t}(x, \infty)$ is $C^2$ for a moment. Therefore, we can apply Itô’s formula and we get

$$-\frac{ac}{12} V(t, U_t) S \Phi_t(U_t) + \partial_t V(t, U_t) + \frac{1}{2} \partial_{xx} V(t, U_t) = 0.$$

Straightforward calculation shows that $V(t, x) = H_{D_t}(x, \infty)^b$ is $C^2$ and satisfies this PDE. Here, $b$ is the boundary scaling exponent

$$b = \frac{6 - \kappa}{2\kappa}.$$

- **Other examples.** Similar ideas were used in [2] to describe the partition function of two $SLE_\kappa$ curves with a PDE. Differentiability of the partition function was justified using the explicit form of the solution in terms of the hypergeometric function. The PDE system in [9] characterizes the partition function of the annulus $SLE_\kappa$. That PDE is more complicated and one cannot find an explicit form for the solution. In fact, it is not easy to even show that the PDE has a solution. Instead, it was directly proved that the partition function is $C^2$ and Itô’s formula was used to derive the PDE.

## 2 Definitions and Preliminaries

We will consider the multiple $SLE_\kappa$ measure only for $\kappa \leq 4$ on simply connected domains $D$ and distinct locally analytic boundary points $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$. The measure is supported on $n$-tuples of curves

$$\gamma = (\gamma^1, \ldots, \gamma^n),$$

where $\gamma^j$ is a curve connecting $x_j$ to $y_j$ in $D$. If $n = 1$, then $\mu_D(x_1, y_1)$ is $SLE_\kappa$ from $x_1$ to $y_1$ in $D$ with total mass $H_D(x_1, y_1)^b$ whose corresponding probability measure $\mu^\#_D(x_1, y_1) = \mu_D(x_1, y_1)/H_D(x_1, y_1)^b$ is (a time change of) $SLE_\kappa$ from $x_1$ to $y_1$ as defined by Schramm.

**Definition** If $\kappa \leq 4$ and $n \geq 1$, then $\mu_D(x, y)$ is the measure absolutely continuous with respect to $\mu_D(x_1, y_1) \times \cdots \times \mu_D(x_n, y_n)$ with Radon-Nikodym derivative

$$Y(\gamma) := I(\gamma) \exp \left\{ \frac{c}{2} \sum_{j=2}^n m[K_j(\gamma)] \right\}.$$
Here $c = (6 - \kappa)(3\kappa - 8)/2\kappa$ is the central charge, $I(\gamma)$ is the indicator function of the event 
\[ \{\gamma^j \cap \gamma^k = \emptyset, 1 \leq j < k \leq n\}, \]
and $m[K_j(\gamma)]$ denotes the Brownian loop measure of loops that intersect at least $j$ of the paths $\gamma^1, \ldots, \gamma^n$.

Brownian loop measure is a measure on (continuous) curves $\eta : [0, t_\eta] \rightarrow \mathbb{C}$ with $\eta(0) = \eta(t_\eta)$. Let $\nu^\#(0, 0; 1)$ be the law of the Brownian bridge starting from 0 and returning to 0 at time 1. Brownian loop measure can be considered as the measure 
\[ m = \text{area} \times \left( \frac{1}{2\pi t^2} dt \right) \times \nu^\#(0, 0; 1) \]
on the triplets $(z, t_\eta, \eta\bar{}(t))$, where $\eta\bar{}(t) = t_\eta^{1/2} \eta(t/t_\eta)$ for $t \in [0, 1]$. For a domain $D \subset \mathbb{C}$, we denote the restriction of $m$ to the loops $\eta \subset D$ by $m_D$. One important property of $m_D$ is conformal invariance. More precisely, if $f : D \rightarrow f(D)$ is a conformal transformation, then 
\[ f \circ m_D = m_D \circ f, \]
where $f \circ m_D$ is the pushforward measure.

Note that if $\sigma$ is a permutation of $\{1, \ldots, n\}$ and $\gamma_\sigma = (\gamma^\sigma(1), \ldots, \gamma^\sigma(n))$, then $Y(\gamma) = Y(\gamma_\sigma)$. The partition function is the total mass of this measure 
\[ \Psi_D(x, y) = \|\mu_D(x, y)\|. \]
We also write 
\[ \tilde{\Psi}_D(x, y) = \frac{\Psi_D(x, y)}{\prod_{j=1}^n H_D(x_j, y_j)^b}, \]
which can also be written as 
\[ \tilde{\Psi}_D(x, y) = \mathbb{E}[Y], \]
where the expectation is with respect to the probability measure $\mu_D^\#(x_1, y_1) \times \cdots \times \mu_D^\#(x_n, y_n)$. Note that $\tilde{\Psi}_D(x, y)$ is a conformal invariant, 
\[ f \circ \tilde{\Psi}_D(x, y) = \tilde{\Psi}_D(f(x), f(y)), \]
and hence is well defined even if the boundaries are rough. Since $SLE_\kappa$ is reversible [7], interchanging $x_j$ and $y_j$ does not change the value.

To compute the partition function we use an alternative description of the measure $\mu_D(x, y)$. We will give a recursive definition.

- For $n = 1$, $\mu_D(x_1, y_1)$ is the usual $SLE_\kappa$ measure with total mass $H_D(x_1, y_1)^b$.
- Suppose the measure has been defined for all $n$-tuples of paths. Suppose $x = (x', x_{n+1}), y = (y', y_{n+1})$ are given and write an $(n+1)$-tuple of paths as $\gamma = (\gamma', \gamma^{(n+1)})$.
  - The marginal measure on $\gamma'$ induced by $\mu_D(x, y)$ is absolutely continuous with respect to $\mu_D(x', y')$ with Radon-Nikodym derivative $H_D(x_{n+1}, y_{n+1})^b$. Here $D$ is the component of $D \setminus \gamma'$ containing $x_{n+1}, y_{n+1}$ on its boundary. (If there is no such component, then we set $H_D(x_{n+1}, y_{n+1}) = 0$ and $\mu_D(x, y)$ is the zero measure.)
Given \( \gamma' \), the curve \( \gamma^{n+1} \) is chosen using the probability distribution \( \mu_D' \). One could try to use this description of the measure as the definition, but it is not obvious that it is consistent. However, one can see that the first definition satisfies this property using the following lemma.

**Lemma 2.1.** Let \( \gamma \) denote a \((n + 1)\)-tuple of paths which we write as \( \gamma = (\gamma', \gamma^{(n+1)}) \), and let \( \tilde{D} \) be the connected component of \( D \setminus \gamma' \) containing the end points of \( \gamma^{(n+1)} \) on its boundary. Then

\[
\sum_{j=2}^{n+1} m[K_j(\gamma)] = \sum_{j=2}^{n} m[K_j(\gamma')] + m_D(\gamma^{(n+1)}, D \setminus \tilde{D}).
\]

**Proof.** Let \( K^1_j(\gamma) \) denote the set of loops in \( K_j(\gamma) \) that intersect \( \gamma^{(n+1)} \) and let \( K^2_j(\gamma) \) denote the set of loops that do not intersect \( \gamma^{(n+1)} \). Then

\[
m[K^2_j(\gamma)] = m_D(\gamma^{(n+1)}, D \setminus \tilde{D}).
\]  (2)

Note that \( K^1_j(\gamma) \) is equivalent to the set of loops in \( D \) that intersect \( \gamma^{(n+1)} \) and at least \( j - 1 \) paths of \( \gamma' \). Moreover, \( K^2_j(\gamma) \) is equivalent to the set of loops that intersect at least \( j \) paths of \( \gamma' \), but do not intersect \( \gamma^{(n+1)} \). Therefore,

\[
K_j(\gamma') = K^1_{j+1}(\gamma) \cup K^2_j(\gamma).
\]

Now the result follows from this, the fact that \( K^2_{n+1}(\gamma) = \emptyset \) and (2).

We can also take the marginals in a different order. For example, we could have defined the recursive step above as follows.

- The marginal measure on \( \gamma^{n+1} \) induced by \( \mu_D(x, y) \) is absolutely continuous with respect to \( \mu_D(x_{n+1}, y_{n+1}) \) with Radon-Nikodym derivative \( \Psi_{\tilde{D}}(x', y') \) where \( \tilde{D} = D \setminus \gamma \). (It is possible that \( \tilde{D} \) has two separate components in which case we multiply the partition functions on the two components.)

We will consider boundary points on the real line. We write just \( H, \Psi, \tilde{\Psi}, \mu, \mu^\# \) for \( H, \Psi, \tilde{\Psi}, \mu, \mu^\# \) and note that

\[
\tilde{\Psi}(x, y) = \mathbb{E}[Y] = \Psi(x, y) \prod_{j=1}^{n} |y_j - x_j|^{2b},
\]

where the expectation is with respect to the probability measure

\[
\mu^\#(x_1, y_1) \times \cdots \times \mu^\#(x_n, y_n).
\]

- If \( n = 1 \), then \( Y \equiv 1 \) and \( \tilde{\Psi}(x, y) = 1 \).
For $n = 2$ and $\gamma = (\gamma_1, \gamma_2)$, then
\[ \mathbb{E}[Y \mid \gamma_1] = \left[ \frac{H_{D \setminus \gamma_1}(x_2, y_2)}{H_D(x_2, y_2)} \right]^b. \]

The right-hand side is well defined even for non-smooth boundaries provided that $\gamma_1$ stays a positive distance from $x_2, y_2$. In particular,
\[ \mathbb{E}[Y] = \mathbb{E} [\mathbb{E}(Y \mid \gamma_1)] = \mathbb{E} \left[ \left( \frac{H_{D \setminus \gamma_1}(x_2, y_2)}{H_D(x_2, y_2)} \right)^b \right] \leq 1. \]

If $8/3 < \kappa \leq 4$, then $c > 0$ and $Y > 1$ on the event $I(\gamma)$ so the inequality $\mathbb{E}[Y] \leq 1$ is not obvious.

More generally, if $\gamma = (\gamma', \gamma_{n+1})$,
\[ \mathbb{E}[Y \mid \gamma'] = Y(\gamma') \left[ \frac{H_{D \setminus \gamma'}(x_{n+1}, y_{n+1})}{H_D(x_{n+1}, y_{n+1})} \right]^b \leq Y(\gamma'). \]

Using this we see that $\tilde{\Psi}_D(x, y) \leq 1$.

For $n = 2$, if $x_1 = 0, y_1 = \infty, y_2 = 1$ and $x_2 = x$ with $0 < x < 1$, we have (see, for example, [2, (3.7)])
\[ \tilde{\Psi}(x, y) = \phi(x) := \frac{\Gamma(2a) \Gamma(6a - 1)}{\Gamma(4a) \Gamma(4a - 1)} x^a F(2a, 1 - 2a, 4a; x), \] (3)
where $F = 2F_1$ denotes the hypergeometric function and $a = 2/\kappa$. This is computed by finding
\[ \mathbb{E} \left[ H_{\mathbb{H} \setminus \gamma_1}(x, 1)^b \right]. \]

In fact, this calculation is valid for $\kappa < 8$ if it is interpreted as
\[ \mathbb{E} \left[ H_{\mathbb{H} \setminus \gamma_1}(x, 1)^b; H_{\mathbb{H} \setminus \gamma_1}(x, 1) > 0 \right]. \]

It will be useful to write the conformal invariant $\tilde{\Psi}$ in a different way. If $V_1, V_2$ are two arcs of a domain $D$, let
\[ \mathcal{E}_D(V_1, V_2) = \int_{V_1} \int_{V_2} H_D(z, w) |dz| |dw|. \]

This is $\pi$ times the usual excursion measure between $V_1$ and $V_2$; the factor of $\pi$ comes from our choice of Poisson kernel. Note that
\[ \mathcal{E}_H((-\infty, 0], [x, 1]) = \int_x^1 \int_{-\infty}^0 \frac{dr \, ds}{(s - r)^2} = \int_x^1 \frac{dr}{r} = \log(1/x), \]
Hence we can write $\tilde{\Psi}$ as $\phi(\exp \{-\mathcal{E}_H((-\infty, 0], [x, 1])\})$. More generally, if $x_1 < y_1 < x_2 < y_2$,
\[ \tilde{\Psi}(x, y) = \phi(\exp \{-\mathcal{E}_H([x_1, y_1], [x_2, y_2])\}) = \phi \left( \exp \left\{ -\int_{x_1}^{y_1} \int_{x_2}^{y_2} \frac{dr \, ds}{(s - r)^2} \right\} \right), \]
and if $D$ is a simply connected subdomain of $\mathbb{H}$ containing $x_1, y_2, x_2, y_2$ on its boundary, then

$$\tilde{\Psi}_D(x, y) = \phi \left( \exp \left\{ -E_D([x_1, y_1], [x_2, y_2]) \right\} \right) = \phi \left( \exp \left\{ - \int_{x_1}^{y_1} \int_{x_2}^{y_2} H_D(r, s) \, dr \, ds \right\} \right).$$  \hspace{0.5cm} (4)

This expression is a little bulky but it allows for easy differentiation with respect to $x_1, x_2, y_1, y_2$.

At this point we can state the main proposition.

**Proposition 2.2.** $\Psi$ and $\tilde{\Psi}$ are $C^2$ functions.

It clearly suffices to prove this for $\tilde{\Psi}$. The idea is simple — we will write the partition function as an expectation and differentiate the expectation by interchanging the expectation and the derivatives. This interchange requires justification and this is the main work of this paper.

We will use the following fact which is an analogue of derivative estimates for positive harmonic functions. The proof is straightforward but we delay it to Section 4.

**Lemma 2.3.** There exists $c < \infty$ such that for every $x_1 < y_1 < x_2 < y_2$ the following holds.

1. Suppose $D \subset \mathbb{H}$ is a simply connected domain whose boundary contains an open real neighborhood of $[x_1, y_1]$ and suppose that

\[
\delta := \min \{|x_1 - y_1|, \text{dist } \{x_1, y_1\}, \mathbb{H} \setminus D\} > 0.
\]

Then if $z_1, z_2 \in \{x_1, y_1\}$,

\[
|\partial_{z_1} H_D(x_1, y_1)| \leq c \delta^{-1} H_D(x_1, y_1).
\]

\[
|\partial_{z_1 z_2} H_D(x_1, y_1)| \leq c \delta^{-2} H_D(x_1, y_1).
\]

2. Suppose $D \subset \mathbb{H}$ is a simply connected domain whose boundary contains open real neighborhoods of $[x_1, y_1]$ and $[x_2, y_2]$ and suppose that

\[
\delta := \min \{|w_1 - w_2|; w_1 \neq w_2 \text{ and } w_1, w_2 \in \{x_1, x_2, y_1, y_2\}\}, \text{dist } \{x_1, y_1, x_2, y_2\}, \mathbb{H} \setminus D\}.
\]

Then if $z_1 \in \{x_1, y_1\}, z_2 \in \{x_2, y_2\}$,

\[
|\partial_{z_1 z_2} \tilde{\Psi}_D(x, y)| \leq c \delta^{-2} \tilde{\Psi}_D(x, y).
\]

Moreover, the constant can be chosen uniformly in neighborhoods of $x_1, y_1, x_2, y_2$.

We will also need to show that expectations do not blow up when paths get close to starting points. We prove this lemma in Section 3.

Let

\[
\Delta_{j,k}(\gamma) = \text{dist } \{x_k, y_k\}, \gamma_j^i,
\]

\[
\Delta(\gamma) = \min_{j \neq k} \Delta_{j,k}(\gamma).
\]
Lemma 2.4. If $\kappa < 4$, then for every $n$ and every $(x,y)$, there exists $c < \infty$ such that for all $\epsilon > 0$, and all $j \neq k$,

$$
\mathbb{E}[Y; \Delta \leq \epsilon] \leq c \epsilon^{\frac{12 \kappa}{n} - 1}.
$$

In particular,

$$
\mathbb{E}[Y \Delta^{-2}] \leq \sum_{m=-\infty}^{\infty} 2^{-2m} \mathbb{E}[Y; 2^m \leq \Delta < 2^{m+1}] < \infty.
$$

Proof. It suffices to show that for each $j,k$,

$$
\mathbb{E}[Y; \Delta_{j,k} \leq \epsilon] \leq c \epsilon^{\frac{12 \kappa}{n} - 1},
$$

and by symmetry we may assume $j = 1, k = 2$. If we write $\gamma = (\gamma^1, \gamma^2, \gamma')$, then the event \{\(\Delta_{1,2} \leq \epsilon\)\} is measurable with respect to $\{\gamma^1, \gamma^2\}$ and

$$
\mathbb{E}[Y \mid \gamma^1, \gamma^2] \leq Y(\gamma^1, \gamma^2).
$$

Hence it suffices to prove the result when $n = 2$. This will be done in Section 3 in that section we consider $\kappa < 8$. \(\blacksquare\)

For $n = 1, 2$, it is clear that $\tilde{\Psi}$ is $C^\infty$ from the exact expression, so we will assume that $n \geq 3$. By invariance under permutation of indices, it suffices to consider second order derivatives involving only $x_1, x_2, y_1, y_2$. We will assume $x_j < y_j$ for $j = 1, 2$ and $x_1 < x_2$ (otherwise we just relabel the vertices). The configuration $x_1 < x_2 < y_1 < y_2$ is impossible for topological reasons. If $x_1 < x_2 < y_2 < y_1$, we can find a M"obius transformation taking a point $y' \in (y_2, y_1)$ to $\infty$ and then the images would satisfy $y'_1 < x'_1 < x'_2 < y'_2$ and this reduces to above. So we may assume that

$$
x_1 < y_1 < x_2 < y_2.
$$

Case I: Derivatives involving only $x_j, y_j$ for some $j$.

We assume $j = 1$. We will write $x = (x, x'), y = (y, y'), \gamma = (\gamma^1, \gamma')$, and let $D$ be the connected component of $\mathbb{H} \setminus \gamma'$ containing $x, y$ on the boundary. Then

$$
\mathbb{E}[Y \mid \gamma'] = Y(\gamma') \left[ \frac{H_D(x,y)}{H(x,y)} \right]^b = Y(\gamma') Q_D(x,y)^b,
$$

where $Q_D(x,y)$ is the probability that a (Brownian) excursion in $\mathbb{H}$ from $x$ to $y$ stays in $D$. Hence

$$
\tilde{\Psi}(x,y) = \mathbb{E} \left[ Y(\gamma') Q_D(x,y)^b \right].
$$

Let $\delta = \delta(\gamma') = \text{dist}\{\{x, y\}, \gamma'\}$. Using Lemma 2.3 we see that

$$
\left| \partial_x [Q_D(x,y)^b] \right| \leq c \delta^{-1} Q_D(x_1, y_1)^b,
$$

$$
\left| \partial_{xy} [Q_D(x,y)^b] \right| + \left| \partial_{xx} [Q_D(x,y)^b] \right| \leq c \delta^{-2} Q_D(x_1, y_1)^b.
$$

(Here $c$ may depend on $x, y$ but not on $D$). Hence

$$
\mathbb{E} \left[ Y(\gamma') \left| \partial_x [Q_D(x,y)^b] \right| \right] \leq c \mathbb{E} \left[ Y(\gamma') \delta(\gamma')^{-1} Q_D(x,y)^b \right],
$$
and if $z = x$ or $y$,

$$
\mathbb{E} \left[ Y(\gamma') \left| \partial_{xz} [Q_D(x,y)^b] \right. \right] \leq c \mathbb{E} \left[ Y(\gamma') \delta(\gamma')^{-2} Q_D(x,y)^b \right].
$$

Since

$$
\mathbb{E} \left[ Y(\gamma') \delta(\gamma')^{-2} Q_D(x,y)^b \right] = \mathbb{E} \left[ \mathbb{E} \left( Y \delta^{-2} | \gamma' \right) \right] = \mathbb{E}[Y \delta^{-2}] \leq \mathbb{E}[\Delta^{-2}] < \infty,
$$

the interchange of expectation and derivative is valid,

$$
\partial_x \tilde{\Psi}(x,y) = \mathbb{E} \left[ Y(\gamma') \partial_x [Q_D(x,y)^b] \right], \quad \partial_{xz} \tilde{\Psi}(x,y) = \mathbb{E} \left[ Y(\gamma') \partial_{xz} [Q_D(x,y)^b] \right].
$$

**Case 2:** The partial $\partial_{z_1 z_2}$ where $z_1 \in \{x_j, y_j\}$, $z_2 \in \{x_k, y_k\}$ with $j \neq k$.

We assume $j = 1, k = 2$. We will write $x = (x_1, x_2, x'), y = (y_1, y_2, y')$, $\gamma = (\gamma^1, \gamma^2, \gamma')$. We will write $D' = D \setminus \gamma'$ and let $D_1, D_2$ be the connected components of $D'$ containing $\{x_1, y_1\}$ and $\{x_2, y_2\}$ on the boundary. It is possible that $D_1 = D_2$ or $D_1 \neq D_2$.

- If $D_1 \neq D_2$, then
  $$
  \mathbb{E}[Y | \gamma'] = Y(\gamma') Q_{D_1}(x_1, y_1)^b Q_{D_2}(x_2, y_2)^b.
  $$

- If $D_1 = D_2 = D$, then
  $$
  \mathbb{E}[Y | \gamma'] = Y(\gamma') Q_{D_1}(x_1, y_1)^b Q_{D_2}(x_2, y_2)^b \tilde{\Psi}_D((x_1, x_2), (y_1, y_2)),
  $$
  where $\tilde{\Psi}_D$ is defined as in (4).

In either case we have written

$$
\mathbb{E}[Y | \gamma'] = Y(\gamma') \Phi(z; \gamma'),
$$

where $z = (x_1, y_1, x_2, y_2)$ and we can use Lemma 2.3 to see that

$$
|\partial_{z_1 z_2} \Phi(z; \gamma')| \leq c \Delta(\gamma, z)^{-2} \Phi(z, \gamma'), \quad \Delta(\gamma, z) = \text{dist}\{\gamma, \{x_1, y_1, x_2, y_2\}\}.
$$

As in the previous case, we can now interchange the derivatives and the expectation.

## 3 Estimate

In this section we will derive an estimate for $SLE_\kappa$, $\kappa < 8$. While the estimate is valid for all $\kappa < 8$, the result is only strong enough to prove our main result for $\kappa < 4$. We follow the ideas in [4] where careful analysis was made of the boundary exponent for $SLE$. Let $g_t$ denote the usual conformal transformation associated to the $SLE_\kappa$ path $\gamma$ from 0 to $\infty$ parametrized so that

$$
\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad (5)
$$

where $a = 2/\kappa$ and $U_t = -W_t$ is a standard Brownian motion. Throughout, we assume that $\kappa < 8$, so that $D = D_\infty = \mathbb{H} \setminus \gamma$ is a nonempty set. If $0 < x < y < \infty$, we let

$$
\Phi = \Phi(x, y) = \frac{H_D(x, y)}{H_\mathbb{H}(x, y)},
$$

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where $H$ denotes the boundary Poisson kernel. If $x$ and $y$ are on the boundary of different components of $D$ (which can only happen for $4 < \kappa < 8$), then $H_D(x, y) = 0$. As usual, we let
\[ b = \frac{6 - \kappa}{2\kappa} = \frac{3a - 1}{2}. \]
As a slight abuse of notation, we will write $\Phi^b$ for $\Phi^b 1\{\Phi > 0\}$ even if $b \leq 0$.

**Proposition 3.1.** For every $\kappa < 8$ and $\delta > 0$, there exists $0 < c < \infty$ such that for all $\delta \leq x < y \leq 1/\delta$ and all $0 < \epsilon < (y - x)/10$,
\[ E\left[ \Phi^b; \text{dist}(\{x, y\}, \gamma) < \epsilon \right] \leq c \epsilon^{6a-1}. \]

It is already known that
\[ P\{\text{dist}(\{x, y\}, \gamma) < \epsilon\} \asymp \epsilon^{4a-1}, \]
and hence we can view this as the estimate
\[ E\left[ \Phi^b \mid \text{dist}(\{x, y\}, \gamma) < \epsilon \right] \leq c \epsilon^{2a}. \]

Using reversibility \[5\,7\] and scaling of $SLE_\kappa$ we can see that to prove the proposition it suffices to show that for every $\delta > 0$ there exists $c = c_\delta$ such that if $\delta \leq x < 1$,
\[ E\left[ \Phi^b; \text{dist}(1, \gamma) < \epsilon \right] \leq c \epsilon^{6a-1}. \]

This is the result we will prove.

**Proposition 3.2.** If $\kappa < 8$, there exists $c < \infty$ such that if $\gamma$ is an $SLE_\kappa$ curve from $0$ to $\infty$, $0 < x < 1$, $\Phi = \Phi(x, 1)$, $0 < \epsilon \leq 1/2$,
\[ E\left[ \Phi^b; \text{dist}(\gamma, 1) < \epsilon (1 - x) \right] \leq c x^a (1 - x)^{4a-1} \epsilon^{6a-1}. \]

We will relate the distance to the curve to a conformal radius. In order to do this, we will need 1 to be an interior point of the domain. Let $D^*_t$ be the unbounded component of
\[ K_t = \mathbb{H} \setminus \{(-\infty, x] \cup \gamma_t \cup \{z : z \in \gamma_t\}\}, \]
and let $T = T_1 = \inf\{t : 1 \not\in D^*_t\}$. Then for $t < T$, the distance from 1 to $\partial D^*_t$ is the minimum of $1 - x$ and $\text{dist}(1, \gamma_t)$. In particular, if $t < T$ and $\epsilon < 1 - x$, then $\text{dist}(\gamma_t, 1) \leq \epsilon$ if and only if $\text{dist}(1, \partial D^*_t) < \epsilon$. We define $T$ to be $[4(1 - x)]^{-1}$ times the conformal radius of 1 with respect to $D^*_t$ and $\Upsilon = \Upsilon_\infty$. Note that $\Upsilon_0 = 1$, and if $\text{dist}(1, \partial D^*_t) \leq \epsilon(1 - x)$, then $\Upsilon \leq \epsilon$. It suffices for us to show that
\[ E\left[ \Phi^b; \Upsilon < \epsilon \right] \leq c \epsilon^{6a-1}. \]

We set up some notation. We fix $0 < x < 1$ and assume that $g_t$ satisfies \[5\]. Let
\[ X_t = g_t(1) - U_t, \quad Z_t = g_t(x) - U_t, \quad Y_t = X_t - Z_t, \quad K_t = \frac{Z_t}{X_t}. \]
and note that the scaling rule for conformal radius implies that
\[ \Upsilon_t = \frac{Y_t}{(1 - x) g'_t(1)}. \]
The Loewner equation implies that
\[ dX_t = \frac{a}{X_t} dt + dB_t, \quad dZ_t = \frac{a}{Z_t} dt + dB_t, \]
\[ \partial_t g'_t(1) = -\frac{a g'_t(1)}{X_t^2}, \quad \partial_t g'_t(x) = -\frac{a g'_t(x)}{Z_t^2}, \quad \partial_t Y_t = -\frac{a Y_t}{X_t Z_t}. \]
\[ \partial_t \Upsilon_t = \Upsilon_t \left[ \frac{a}{X_t^2} - \frac{a}{X_t Z_t} \right] = -a Y_t \frac{1 - K_t}{K_t}. \]
Let \( D_t \) be the unbounded component of \( \mathbb{H} \setminus \gamma_t \) and let
\[ \Phi_t = \frac{H_{D_t}(x,1)}{H_{D_0}(x,1)} = x^2 \frac{g'_t(x) g'_t(1)}{Y_t^2}, \]
where we set \( \Phi_t = 0 \) if \( x \) is not on the boundary of \( D_t \), that is, if \( x \) has been swallowed by the path (this is relevant only for \( 4 < \kappa < 8 \)). Note that \( \Phi = \Phi_\infty \) and
\[ \partial_t \Phi^b_t = \Phi_t^b \left[ -\frac{ab}{X_t^2} - \frac{ab}{Z_t^2} + \frac{2ab}{X_t Z_t} \right] = -ab \frac{\Phi_t^b}{X_t^2} \left( \frac{1 - K_t}{K_t} \right)^2, \]
\[ \Phi^b_t = \exp \left\{ -ab \int_0^t \frac{1}{X_s^2} \left( \frac{1 - K_s}{K_s} \right)^2 ds \right\}. \]
Itô’s formula implies that
\[ d \frac{1}{X_t} = -\frac{1}{X_t^2} dX_t + \frac{1}{X_t^3} d\langle X \rangle_t = \frac{1}{X_t} \left[ \frac{1 - a}{X_t^2} dt - \frac{1}{X_t} dW_t \right], \]
and the product rule gives
\[ d[1 - K_t] = [1 - K_t] \left[ \frac{1 - a}{X_t^2} dt - \frac{a}{X_t Z_t} dt - \frac{1}{X_t} dW_t \right] = \frac{1 - K_t}{X_t^2} \left[ (1 - a) - \frac{a}{K_t} \right] dt - \frac{1 - K_t}{X_t} dW_t. \]
which can be written as
\[ dK_t = \frac{1 - K_t}{X_t^2} \left[ a \frac{K_t}{K_t} + a - 1 \right] dt + \frac{1 - K_t}{X_t} dW_t. \]
As in [3], we consider the local martingale
\[ M^*_t = (1 - x)^{1 - 4a} X_t^{1 - 4a} g'_t(1)^{4a - 1} = (1 - x)^{1 - 4a} (1 - K_t)^{4a - 1} \Upsilon_t^{1 - 4a}, \]
which satisfies
\[ dM^*_t = \frac{1 - 4a}{X_t} M^*_t dW_t, \quad M^*_0 = 1 \]
If we use Girsanov and tilt by the local martingale, we see that
\[ dK_t = \frac{1 - K_t}{X_t^2} \left[ \frac{a}{K_t} - 3a \right] dt + \frac{1 - K_t}{X_t} dW_t^*. \]

where \( W_t^* \) is a standard Brownian motion in the new measure \( \mathbb{P}^* \). We reparametrize so that \( \log \Upsilon_t \) decays linearly. More precisely, we let \( \sigma(t) = \inf \{ t : \Upsilon_t = e^{-at} \} \) and define \( \hat{X}_t = X_{\sigma(t)}^*, \hat{Y}_t = Y_{\sigma(t)}^* \), etc. Since \( \hat{Y}_t := \Upsilon_{\sigma(t)} = e^{-at} \), and
\[ -a \hat{Y}_t = \partial_t \hat{Y}_t = -a \hat{X}_t \frac{1 - \hat{K}_t}{K_t} \hat{K}_t \]
we see that
\[ \hat{\sigma}(t) = \frac{\hat{X}_t^2 \hat{K}_t}{1 - K_t}, \]
Therefore,
\[ d\hat{K}_t = \left[ a - 3a\hat{K}_t \right] dt + \sqrt{\hat{K}_t (1 - \hat{K}_t)} dB_t^*. \]

for a standard Brownian motion \( B_t^* \) (in the measure \( \mathbb{P}^* \)).

Let \( \lambda = 2a^2 \), and
\[ N_t = e^{\lambda t} \hat{\Phi}_t^b \hat{K}_t^a = \exp \left\{ -ab \int_0^t \frac{1 - \hat{K}_s}{\hat{K}_s} ds \right\} \exp \left\{ -ab \int_0^t \frac{1}{\hat{K}_s} ds \right\} \hat{K}_t^a. \]

Itô’s formula shows that \( N_t \) is a local \( \mathbb{P}^*-\)martingale satisfying
\[ dN_t = N_t a \sqrt{\frac{1 - \hat{K}_t}{\hat{K}_t}} dB_t^* , \quad N_0 = x^a \]

One can show it is a martingale by using Girsanov to see that
\[ d\hat{K}_t = \left[ 2a - 4a\hat{K}_t \right] dt + \sqrt{\hat{K}_t (1 - \hat{K}_t)} d\hat{B}_t, \]
where \( \hat{B}_t \) is a Brownian motion in the new measure \( \hat{\mathbb{P}} \). By comparison with a Bessel process, we see that the solution exists for all time. Equivalently, we can say that
\[ \hat{M}_t := \hat{M}_t^a N_t, \]
is a $\mathbb{P}$-martingale with $\hat{M}_0 = x^a$. (Although $M^*_t$ is only a local martingale, the time-changed version $\hat{M}_t := M^*_t(\hat{t})$ is a martingale.)

Using [3] we see that $E[\Phi^b_t | \gamma_{\sigma(t)}] \leq c \hat{K}^a_t \Phi^b_t$. If $\epsilon = e^{-at}$, then

$$E[\Phi^b; \sigma(t) < \infty] = c E[E(\Phi^b I\{\sigma(t) < \infty\} | \gamma_{\sigma(t)})] \leq c E[\hat{K}^a_t \Phi^b; \sigma(t) < \infty] = c e^{-\lambda t} e^{(1-4a)at} (1-x)^{4a-1} \hat{M}_0^{-1} E[\hat{M}_t (1-\hat{K}_t)^{1-4a}; \sigma(t) < \infty]$$

$$= c e^{a(1-6a)t} x^a (1-x)^{4a-1} \hat{E}[(1-\hat{K}_t)^{1-4a}]$$

So the result follows once we show that

$$\hat{E}[(1-\hat{K}_t)^{1-4a}] < \infty$$

is uniformly bounded for $t \geq t_0$. The argument for this proceeds as in [4]. If we do the change of variables $\hat{K}_t = [1 - \cos \Theta_t]/2$, then Itô’s formula shows that

$$d\Theta_t = \left(4a - \frac{1}{2}\right) \cot \Theta_t dt + dB_t.$$ 

This is a radial Bessel process that never reaches the boundary. It is known that the invariant distribution is proportional to $\sin^{8a-1} \theta$ and that it approaches the invariant distribution exponentially fast. One then computes that the invariant distribution for $\hat{K}_t$ is proportional to $x^{4a-1} (1-x)^{4a-1}$. In particular, $(1-\hat{K}_t)^{1-4a}$ is integrable with respect to the invariant distribution.

### 4 Proof of Lemma 2.3

We prove the first part of Lemma 2.3 for $x_1 = 0, y_1 = 1$. Other cases follow from this and a Möbius transformation sending $x_1, y_1$ to 0, 1.

**Lemma 4.1.** There exists $c < \infty$ such that if $D$ is a simply connected subdomain of $\mathbb{H}$ containing 0, 1 on its boundary, then

$$|\partial_x H_D(0,1)| + |\partial_y H_D(0,1)| \leq c \delta^{-1} H_D(0,1),$$

$$|\partial_{xx} H_D(0,1)| + |\partial_{xy} H_D(0,1)| + |\partial_{yy} H_D(0,1)| \leq c \delta^{-2} H_D(0,1),$$

where $\delta = \text{dist}([0,1], \partial D \cap \mathbb{H})$.

**Proof.** Let $g : D \to \mathbb{H}$ be a conformal transformation with $g(0) = 0, g(1) = 1, g'(0) = 1$. Then if $|x| < \delta, |y - 1| < \delta$,

$$H_D(x,y) = \frac{g'(x) g'(y)}{|g(y) - g(x)|^2}. \quad (6)$$
In particular \( g'(0) g'(1) = H_D(0, 1) \leq H_H(0, 1) = 1 \) and hence \( g'(1) \leq 1 \). Using Schwartz reflection we can extend \( g \) to be a conformal transformations of disks of radius \( \delta \) about 0 and 1. By the distortion estimates (the fact that \( |a_2| \leq 2, |a_3| \leq 3 \) for schlicht functions) we have

\[
|g''(0)| \leq 4 \delta^{-1} g'(0) \leq 4 \delta^{-1}, \quad |g''(0)| \leq 18 \delta^{-2} g'(0) \leq 18 \delta^{-2},
\]

and similarly \( |g''(1)| \leq 4 \delta^{-1} g'(1) \) and \( |g''(1)| \leq 18 \delta^{-1} g'(1) \). By direct differentiation of the right-hand side of (6) we get the result. \( \square \)

**Lemma 4.2.** There exists \( c < \infty \) such that if \( x_1 < y_1 \leq 0 < 1 \leq x_2 < y_2 \), \( \tilde{\Psi}_D(x, y) \) is as in (6), and \( z_1 \in \{x_1, y_1\}, z_2 \in \{x_2, y_2\} \), then

\[
|\partial_{z_1} \tilde{\Psi}_D(x, y) + |\partial_{z_2} \tilde{\Psi}_D(x, y)| \leq c \delta^{-1} \Psi_D(x, y),
\]

\[
|\partial_{z_1 z_2} \tilde{\Psi}_D(x, y)| \leq c \delta^{-2} \tilde{\Psi}_D(x, y),
\]

where

\[
\delta := \min \{\{|w_1 - w_2|; w_1 \neq w_2 \text{ and } w_1, w_2 \in \{x_1, x_2, y_1, y_2\}\}, \text{dist } \{(x_1, y_1, x_2, y_2), \mathbb{H} \setminus D\}\}.
\]

**Proof.** Let

\[
\tilde{\Psi}_D(x, y) = \phi(u_D(x, y)).
\]

where

\[
u_D(x, y) = e^{-\mathcal{E}_D(x, y)}, \quad \mathcal{E}_D(x, y) = \int_{x_1}^{y_1} \int_{x_2}^{y_2} H_D(r, s) \, dr \, ds.
\]

Using the Harnack inequality we can see that for \( j = 1, 2 \),

\[
H_D(x, s) \asymp H_D(x_j, s), \quad H_D(r, y) \asymp H_D(r, y_j)
\]

if \( |x - x_j| \leq \delta/2, |y - y_j| \leq \delta/2 \). From this we see that

\[
\int_{x_2}^{y_2} H_D(z_1, s) \, ds + \int_{x_1}^{y_1} H_D(r, z_2) \, ds \leq c \delta^{-1} \mathcal{E}_D(x, y),
\]

\[
H_D(z_1, z_2) \leq c \delta^{-2} \mathcal{E}_D(x, y).
\]

Let \( z_1 \) be \( x_1 \) or \( y_1 \) and let \( z_2 \) be \( x_2 \) or \( y_2 \). Then,

\[
\partial_{z_1} \tilde{\Psi}_D(x, y) = \phi'(u_D(x, y)) \partial_{z_1} u_D(x, y)
\]

\[
\partial_{z_1 z_2} \tilde{\Psi}_D(x, y) = \phi''(u_D(x, y)) \left[ \partial_{z_1} u_D(x, y) \right] \left[ \partial_{z_2} u_D(x, y) \right] + \phi'(u_D(x, y)) \partial_{z_1 z_2} u_D(x, y).
\]

\[
\partial_{z_1} u_D(x, y) = \begin{pmatrix} \pm \int_{x_2}^{y_2} H_D(z_1, s) \, ds \end{pmatrix} u_D(x, y).
\]

\[
\partial_{z_2} u_D(x, y) = \begin{pmatrix} \pm \int_{x_1}^{y_1} H_D(r, z_2) \, ds \end{pmatrix} u_D(x, y).
\]
\[ \partial_{zz_1} u_D(x, y) = \left[ \pm \int_{x_1}^{y_1} \int_{x_2}^{y_2} H_D(r, z_2) \, dr \, H_D(z_1, s) \, ds \pm H_D(z_1, z_2) \right] u_D(x, y) \]

This gives
\[
|\partial_{z_1} u_D(x, y)| + |\partial_{z_2} u_D(x, y)| \leq c \delta^{-1} E_D(x, y) u_D(x, y),
\]
\[
|\partial_{zz_1} u_D(x, y)| \leq c \delta^{-2} E_D(x, y) u_D(x, y).
\]

The result will follow if we show that
\[
\frac{(1-x) |\phi''(1-x)|}{\phi(x)}, \quad \frac{\phi'(x)}{\phi(x)},
\]

are uniformly bounded for \( x > x_0 \).

Recall that \( u(x) = c x^\alpha F(x) \) where \( F(x) = {}_2F_1(2a, 1 - 2a, 4a; x) \). We recall that \( F \) is analytic in the unit disk with power series expansion
\[
F(x) = 1 + \sum_{n=1}^\infty b_n x^n,
\]

where the coefficients \( b_j \) satisfy
\[
b_n = C n^{4a-2} [1 + O(n^{-1})].
\]

We therefore get asymptotic expansions for the coefficients of the derivatives of \( F \). The important thing for us is that if \( \kappa < 8 \), then \( 4a - 1 > 1 \) and we have as \( x \downarrow 1 \)
\[
F(1-x) = O(1), \quad F'(x) = o(x^{-1}), \quad F''(x) = o(x^{-2}).
\]

In other words, the quantities
\[
F(x), \quad \frac{(1-x) F'(x)}{F(x)}, \quad \frac{(1-x)^2 F''(x)}{F(x)},
\]

are uniformly bounded for \( 0 \leq x < 1 \). If \( g(x) = x^\alpha F(x) \), then
\[
g'(x) = g(x) \left[ \frac{a}{x} + \frac{F'(x)}{F(x)} \right],
\]
\[
g''(x) = g(x) \left[ \left( \frac{a}{x} + \frac{F'(x)}{F(x)} \right)^2 - \frac{a}{x^2} + \frac{F''(x)}{F(x)} - \frac{F'(x)^2}{F(x)^2} \right].
\]

Therefore, for every \( x_0 > 0 \), the quantities
\[
\phi(x), \quad \frac{(1-x) \phi'(x)}{\phi(x)}, \quad \frac{(1-x)^2 \phi''(x)}{\phi(x)},
\]

are uniformly bounded for \( x_0 < x < 1 \). \qed
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