Gaussian multipartite bound information

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We demonstrate the existence of Gaussian multipartite bound information which is a classical analog of Gaussian multipartite bound entanglement. We construct a tripartite Gaussian distribution from which no secret key can be distilled, but which cannot be created by local operations and public communication. Further, we show that the presence of bound information is conditional on the presence of a part of the adversary’s information creatable only by private communication. Existence of this part of the adversary’s information is found to be a more generic feature of classical analogs of quantum phenomena obtained by mapping of non-classically correlated separable quantum states.

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Quantum entanglement and secret classical correlations are nonlocal information resources in the sense that they cannot be created by local operations and classical communication (LOCC) [1] and local operations and public communication (LOPC) [2, 3], respectively. The distillability of these resources is the key prerequisite for their utility in reliable communication. Indeed, imperfect entangled states can sometimes be distilled [4] by LOCC into a maximally entangled singlet state - a perfect resource for the faithful transmission of quantum states via teleportation [5]. Remarkably, there are entangled states which cannot be distilled to singlet form [6]. These states, fittingly said to be “bound entangled”, demonstrate a fundamental property of entanglement, namely its irreversibility under LOCC transformations [7]. More precisely, bound entanglement requires singlets for creation but no singlets can be distilled from it. Despite being nondistillable and hence unusable directly for quantum communication, bound entanglement can be activated [8, 9] and contain distillable entanglement when tensored [10] or mixed [11].

Moving to the secret correlations, they appear in the scenario in which two honest parties, Alice and Bob, and an adversary Eve, share independent realizations of three random variables A, B and E, characterized by the probability distribution P(A, B, E). As with entanglement, Alice and Bob can sometimes distill them by LOPC to a secret key, i.e., a common string of random bits about which Eve has practically no information. Inspired by the existence of bound entanglement, Gisin and Wolf posed a question [12] as to whether there are also nondistillable secret correlations, referred to as bound information (BI), that act as a classical analogue to bound entanglement. So far, a bipartite distribution has been found containing BI only in the asymptotic limit [2], but surprisingly, an example of tripartite BI was derived in [13]. A probability distribution P(A, B, C, E) shared by three honest parties Alice, Bob and Clare, and an adversary Eve, carries tripartite BI if [13]: (i) any pair of honest parties cannot distill a secret key, even if they can collaborate with the third party, and (ii) the distribution cannot be distributed by LOPC, i.e., the reduced probability distribution P(A, B, C) cannot be distributed among the honest parties if their public communication is constrained to contain at most information of the variable E.

The example of BI from Ref. [13] is discrete. It is obtained from a specific bound entangled state ˆρABC of three two-level systems by measuring its purification |ψ⟩ABC,E, ˆρABC = TrE(|ψ⟩⟨ψ|ABC,E), where E labels Eve’s purifying system, with measurements ˆPi, i = A, B, C, E in the computational basis as [14]

P(A, B, C, E) = Tr(ˆPA ⊗ ˆPB ⊗ ˆPC ⊗ ˆPE|ψ⟩⟨ψ|ABC,E). (1)

The distribution contains secret correlations across one splitting of honest parties into two groups which guarantees that both (i) and (ii) hold. The key step in the construction of BI [13], as well as of other classical analogs [15, 16] of quantum phenomena, is the proof of the absence of secret correlations across particular splittings. The proof is based on the concept of intrinsic information which is for the distribution P(A, B, E) defined as [17]

IABL|E = minE→E′ IABL|E′. (2)

where IABL|E is the conditional mutual information and the minimization is performed over all channels E → E′. Importantly, if IABL|E = 0 then Alice and Bob do not share secret correlations [3]. In the proofs of Refs. [13, 15, 16] a suitable channel E → Ŕ is found erasing a part of Eve’s information which nullifies the conditional mutual information. The intrinsic information then vanishes and there are no secret correlations across the considered splitting. The new variable Ŕ represents a publicly communicated message used by the honest parties for the LOPC preparation of the reduced distribution of their variables. However, in order to create the entire considered distribution including the original variable E, it is desirable to know whether Eve’s part of the information, which is erased by the channel E → Ŕ, can also
be established by LOPC or whether some other resources are needed.

In this paper we demonstrate the existence of multipartite BI for random Gaussian continuous variables. We construct a tripartite Gaussian distribution containing secret correlations across only one splitting which thus carries tripartite Gaussian bound information (GBI). By using the Gaussian entanglement criteria [18, 19], we prove the presence of GBI without using the concept of intrinsic information. This approach also reveals that the creation of the part of Eve’s information, which is erased by the channel $E \rightarrow \bar{E}$, requires private communication between honest parties across the splittings where the distribution contains no secret correlations. Moreover, we find that this information is interconnected with the BI as if it is dropped the BI disappears. Finally, we give an example of a discrete distribution where creation of Eve’s information requires private communication and we argue that a necessary prerequisite for this property is the mapping of the type (1) of a non-classically correlated separable quantum state.

Gaussian secret correlations. We consider the set of Gaussian states and measurements associated with quantum systems with infinite-dimensional Hilbert state spaces. $N$ systems, which can be physically realized by $N$ light modes, are described by $N$ pairs of position and momentum quadrature operators, labeled by $\hat{x}_j$ and $\hat{p}_j$, respectively, $j = 1, 2, \ldots, N$, satisfying the canonical commutation relations $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$. $N$-mode quantum states can be represented by the Wigner function of $2N$ real variables and Gaussian states possess a Gaussian Wigner function. For example, distributions containing a tripartite Gaussian bound information (GBI). By constructing Gaussian bound information (GBI) from the state we need to find its purification.

A Gaussian measurement consists of a set of Gaussian operators obtained by all possible displacements of a fixed Gaussian state and normalized such that the completeness relation is satisfied. Incorporating Gaussian measurements and Gaussian states into the mapping (1), we can construct Gaussian distributions with unique cryptographic properties. In the case of a bipartite state $\hat{\rho}_{AB}$ the mapping can, e.g., yield a distribution $P(A, B, E)$ with distillable secret correlations. Necessary and sufficient conditions for secure-key correlations are not known, yet nevertheless one can show [20, 21], that a Gaussian distribution $P(A, B, E)$ is distillable if

$$\max(\Delta I_{DR}, \Delta I_{RR}) > 0,$$

where $\Delta I_{DR} = I_{AB} - I_{AE}$ and $\Delta I_{RR} = I_{AB} - I_{BE}$ are differences of mutual information [22], $I_{AB}$ between Alice and Bob and $I_{AE}$ ($I_{BE}$) between Alice (Bob) and Eve.

The mapping of entanglement onto distillable secret correlations is best exemplified by the two-mode squeezed vacuum state. It is a Gaussian entangled state $|\tau(m)\rangle_{AB}$ of two modes $A$ and $B$ described by the CM $\tau_{AB}(m) = \omega_{AB}(m) \oplus |\omega_{AB}(m)\rangle_{AB}$ with

$$[\omega_{AB}(m)]_{ij} = \sqrt{m^2 + \delta_{ij} - 1}, \quad i, j = 1, 2, 3,$$

where $m = \cosh(2r)$ ($r > 0$ is the squeezing parameter). By measuring the position quadratures on modes $A$ and $B$ (with outcomes $x_A$ and $x_B$), the state is mapped onto the Gaussian distribution

$$P_{AB}(x_A, x_B) = (2\pi)^{-1} e^{-(x_A^2 + x_B^2)}/\omega_{AB}(m) \langle \hat{r}_A \rangle,$$

for which $I_{AB} = \log_2 [\cosh(2r)]$. As the state itself is pure, Eve is completely uncorrelated with Alice and Bob. Consequently, $I_{AE} = I_{BE} = 0$ and the global distribution contains distillable secret correlations according to the criterion (3). A closer look at the distribution further shows that it plays the role of a continuous-variable analog of a basic unit of discrete secret correlations, a secret bit [14], defined as a probability distribution satisfying $P(A, B, E) = P(A, B)P(E)$ and $P(A = 1) = 1$. Indeed, not only does the distribution satisfy the first condition, but it also approaches the continuous-variable analog $P_{AB}(x_A, x_B) \propto \delta(x_A - x_B)$ of the second condition with increasing $r$.

Construction of GBI. We construct GBI by mapping of a bound entangled Gaussian state of three modes $A$, $B$ and $C$ with the CM given in Eq. (17) of Ref. [22]. To construct GBI from the state we need to find its purification. The separability criterion [18] reveals that the state can be decomposed into the product $|\tau(r)\rangle_{AC}\langle 0|_B$ of the two-mode squeezed vacuum state in modes $A$ and $C$, and the vacuum state $\langle 0|_B$ in mode $B$, and displacements.

$$\hat{x}_\alpha \rightarrow \hat{x}_\alpha', \quad \hat{p}_\alpha \rightarrow \hat{p}_\alpha + \alpha q_x,$$

$$\alpha = A, B, C,$$

where $-A_x = C_x = A_p = C_p = -1/2$, $B_x = B_p = 1$. The classical displacements $q_x$ and $q_p$ obey a Gaussian distribution $P(q_x, q_p) = \exp[-(q_x^2 + q_p^2)/(4x^2)]/(4x^2)$, where $x = (e^{2r} - 1)/2$. Hence, we construct a suitable purification as

$$|\psi\rangle_{ABCE} = \int \sqrt{P(q_x, q_p)D_A(\delta_A)D_C(\delta_C)|\tau(m)\rangle_{AC}}$$

$$D_B(\delta_B)|0\rangle_B |q_x|^{(}\rangle_{E_1} |q_p|^{(}\rangle_{E_2} dq_x dq_p,$$

where $|\cdot\rangle_{(}\rangle$ is a position (momentum) eigenstate, $m = \cosh(2r)$ and $D_\alpha(\delta_\alpha) = \exp(\delta_\alpha \hat{a}_\alpha^\dagger - \delta_\alpha^* \hat{a}_\alpha)$ is the displacement operator with $\delta_\alpha = i(xq_x + ip_x)/\sqrt{2}$.

The purification (8) has the CM $\Gamma = X \otimes (X)^{-1}$, where

$$X = \begin{pmatrix}
  a & 2x & b & 2x \\
  2x & c & 2x & \frac{1}{2} e^{2r} \\
  b & -2x & a & 4x \\
  2x & 4x & -2x & -2x \\
  2x & -2x & e^{-2r} & \frac{1}{2} -2x \\
  \frac{1}{2} e^{2r} & \frac{1}{2} e^{2r} & -2x & x \\
  1 & -2x & e^{2r} & y
\end{pmatrix},$$

with $a = \cosh(2r) + x$, $b = \sinh(2r) - x$, $c = 1 + 4x$ and $y = e^{2r}(e^{2r} - 1)/[2(e^{2r} - 1)]$. By measuring the
position quadratures on all modes of the purification we get a Gaussian distribution

$$\Pi_{\eta} = (\pi)^{-\frac{1}{2}} e^{-\eta^T X^{-1} \eta},$$

(10)

where $$\eta = (x_{A}, x_{B}, x_{C}, x_{E_{1}}, x_{E_{2}})^T$$ is the vector of measurement outcomes and $$X$$ is the classical covariance matrix (CCM)\(^9\). The distribution contains secret correlations across just one splitting and therefore contains multipartite GBI.

First we prove the absence of secret correlations across the $$B-(AC)$$ splitting. Assume Alice and Clare privately draw two random variables $$z_{A}$$ and $$z_{C}$$ from a Gaussian distribution with CCM $$\omega_{AC} = \cosh(2r)$$, Eq. (4). They also draw a third variable $$x_{E_{1}}$$ from a Gaussian distribution with variance $$\langle x_{E_{1}}^{2} \rangle = 2\alpha$$ and send it through a public channel to Bob. He privately generates a random variable $$z_{B}$$ obeying a Gaussian distribution with variance $$\langle z_{B}^{2} \rangle = 1/2$$. Alice, Bob and Clare displace their variables as in Eq. (6), where we have performed the replacements $$\tilde{x}_{A}(\tilde{x}_{A}^{\prime}) \rightarrow z_{A}(x_{A})$$, $$q_{x} \rightarrow x_{E_{1}}$$. The variables $$x_{A}, x_{B}$$ and $$x_{C}$$ obey the reduced distribution $$\Pi(x_{A}, x_{B}, x_{C})$$. As the distribution was created by the public communication of $$x_{E_{1}}$$ across the $$B-(AC)$$ splitting, the distribution\(^9\) indeed contains no secret correlations across the splitting.

Similarly we can show that there are no secret correlations across the $$C-(AB)$$ splitting. For this we again use the separability criterion\(^1\) and decompose the underlying state into the product $$| \tau(m) \rangle_{AB}|0 \rangle_{C}$$, where

$$m = \frac{1 + 2(e^{4r} - e^{2r})}{2e^{2r} - 1},$$

(11)

and the displacements\(^5\) and\(^7\) with $$A_{x} = -A_{p} = 1/(2y)$$, $$B_{y} = B_{p} = -e^{2r}/y$$, $$C_{x} = C_{p} = (1 - e^{2r})$$, and the displacements $$q_{x}$$ and $$q_{p}$$ obeying the Gaussian distribution $$\mathcal{P}(q_{x}, q_{p}) = \exp[-((q_{x}^{2} + q_{p}^{2})/y)/\pi y]$$. The decomposition tells us how to establish the reduced distribution $$\Pi(x_{A}, x_{B}, x_{C})$$ by LOPC with respect to the $$C-(AB)$$ splitting. Initially, Alice and Bob draw privately two random variables $$z_{A}$$ and $$z_{B}$$ from a Gaussian distribution with CCM $$\omega_{AB}(m)$$, Eq. (1), where $$m$$ is given in Eq. (11), and Clare privately generates a random variable $$z_{C}$$ obeying a Gaussian distribution with variance $$\langle z_{C}^{2} \rangle = 1/2$$. Alice and Bob also draw a third variable $$x_{E_{2}}$$ from a Gaussian distribution with variance $$\langle x_{E_{2}}^{2} \rangle = y/2$$ and send it through a public channel. All the participants then perform displacements\(^5\) where we have performed the replacements $$\tilde{x}_{A}(\tilde{x}_{A}^{\prime}) \rightarrow z_{A}(x_{A})$$, $$q_{x} \rightarrow x_{E_{1}}$$, and the coefficients $$\alpha_{x}$$ are given below Eq. (11). The variables $$x_{A}, x_{B}$$ and $$x_{C}$$ follow the reduced distribution $$\Pi(x_{A}, x_{B}, x_{C})$$, which we distributed by the public communication of $$x_{E_{2}}$$ across the $$C-(AB)$$ splitting. Consequently, the distribution\(^9\) contains no secret correlations across the splitting.

**Activating GBI.** As the distribution\(^9\) contains no secret correlations across the $$B-(AC)$$ and $$C-(AB)$$ splittings, no two honest parties can establish a secret key even with the help of the third one. The distribution, however, cannot be created by LOPC as it contains secret correlations across the $$A-(BC)$$ splitting. They are not detected by the criterion\(^9\) (see dotted and dashed curves in Fig. 1) but can be “activated” by allowing Bob and Clare to perform the joint operation $$x_{B}, x_{C} \rightarrow (x_{B} \pm x_{C})/\sqrt{2}$$. The presence of secret correlations can be seen from the reduced distribution of Alice’s, Bob’s and Eve’s variables. The corresponding information difference $$\Delta I_{RR}$$ arising in the condition\(^3\) can then be expressed as

$$\log_{2} \sqrt{\frac{4 \ - \ e^{-2r} \ - \ 11e^{2r} + 20e^{4r} - 20e^{6r} + 16e^{8r}}{2 \ - \ 8e^{2r} + 10e^{6r} + 4e^{8r}}}$$

(12)

and it is plotted against the squeezing parameter $$r$$ in Fig. 1. The figure and numerics reveal, that $$\Delta I_{RR} > 0$$ for $$r > r_{\Delta I_{RR}=0} \approx 0.166$$. Therefore Alice and Bob can distill a secret key using the reverse reconciliation protocol\(^2\). This would be impossible without the distribution\(^9\) to have secret correlations across the $$A-(BC)$$ splitting which accomplishes the construction of GBI.

**Specific adversary features.** The absence of secret correlations could be alternatively proved using the intrinsic information\(^2\). If Eve uses the channel $$(x_{E_{1}}, x_{E_{2}}) \rightarrow x_{E_{1}}(x_{E_{2}})$$, we get $$I_{B(AC)|E_{1}} = 0$$ ($$I_{C(AB)|E_{2}} = 0$$). Then $$I_{B(AC)|E_{1}E_{2}} = 0$$ ($$I_{C(AB)|E_{1}E_{2}} = 0$$) follows and the distribution\(^9\) contains no secret correlations across $$B-(AC)$$ ($$C-(AB)$$) splitting. It is, however, advantageous to use the formalism of our previous proof as it unveils the nonlocal nature of the part of Eve’s information which is erased by the channel, and its interconnection with the BI carried by the distribution\(^9\). More precisely, it shows first, that for $$r > \tilde{r} \approx 0.284$$ both of Eve’s variables are indispensable for the presence of the BI as dropping either of them results in its disappearance. Indeed, the dropping of $$x_{E_{1}}$$ causes the distribution $$\Pi(x_{A}, x_{B}, x_{E_{2}})$$ to give $$\Delta I_{DR} > 0$$ for $$r > 0.156$$ and the secret key can be distilled between Alice and Bob. Likewise, if $$x_{E_{2}}$$ is dropped and $$r > \tilde{r}$$, Alice and Clare can distill a secret key with the help of Bob (See

![Figure 1: Information differences $\Delta I_{RR}$ (solid curve) in Eq. (12); $\Delta I_{RR}$ (dotted curve) and $\Delta I_{DR}$ (dashed curve) for CCM\(^9\) with respect to $A-(BC)$ splitting.](image-url)
Appendix [A] for the proof. In the remaining interval \( \tilde{r} \geq r > r_{\Delta I_{DR}=0} \) the condition (3) is not fulfilled if \( x_{E_2} \) is discarded and we cannot decide about the presence of BI. Nevertheless, Alice and Bob can create a new variable \( x_{A} \neq x_{B}/2 \) which, together with the variables \( x_{C} \) and \( x_{E_1} \), obeys a distribution satisfying \( \Delta I_{DR} > 0 \) for \( r > 0 \). Consequently, after discarding \( x_{E_2} \) from the distribution (10), secret correlations across both the \( A - (BC) \) and \( C - (AB) \) splittings are present and the property (i) of BI is no longer guaranteed.

Let us now focus on the remaining variable \( x_{E_2} (x_{E_1}) \) which is erased in the proof of the absence of secret correlations across the \( B - (AC) \) (\( C - (AB) \)) splitting. Can honest parties also publicly announce this variable by using only LOPC across the splitting? If this was the case then it would be possible to create the distribution with GBI (10) just from secret correlations with respect to the \( A - C \) or \( A - B \) splitting in the form of the distribution [5]. Surprisingly, we answer the question in the negative. Namely, the variable can be expressed as

\[
x_{E_1} = -A_p z_A - B_p z_B - C_p z_C + e x_{E_k} + \chi_{E_j}, \tag{13}
\]

where \( j = 2, k = 1 \) \((j = 1, k = 2)\) and \( e = (-1/2)(A_p A_p + B_p B_p + C_p C_p)\). The parameters \( A_{x,p}, B_{x,p}, C_{x,p} \) are given below Eq. (7) (Eq. (11)), and the random variable \( \chi_{E_2} (\chi_{E_1}) \) is uncorrelated with the other variables and obeys a Gaussian distribution with variance \( \langle \chi_{E_2}^2 \rangle = 1/(8\pi) \) \( \langle \chi_{E_1}^2 \rangle = 1/(2\pi) \). The variable (13) contains the term \( A_p z_A + B_p z_B + C_p z_C \) involving variables belonging to both parts of the \( B - (AC) \) (\( C - (AB) \)) splitting. Assume, that the variable \( x_{E_2} (x_{E_1}) \) is known publicly and the parties share the reduced distribution \( \Pi(x_A, x_B, x_C, x_{E_1}) \) \( \Pi(x_A, x_B, x_C, x_{E_2}) \), which can be completely created by LOPC across the splitting. The honest parties can then turn the “nonlocal” term into (distillable) secret correlations by combining locally with respect to the \( B - (AC) \) (\( C - (AB) \)) splitting the variable \( x_{E_2} (x_{E_1}) \) with their variables (See Appendix [3] for the proof). As it is impossible to create secret correlations by LOPC the variable \( x_{E_2} (x_{E_1}) \) cannot be announced publicly by LOPC and private communication across the \( B - (AC) \) (\( C - (AB) \)) splitting is needed for this task.

We have shown for the distribution (10), that although it does not contain secret correlations across a certain splitting, the creation of the whole of Eve’s information requires private communication across the splitting. This property obviously cannot be obtained by mapping of the type [11] of pure states or classically correlated separable states, which are diagonal in the local product basis [26] and product in the Gaussian scenario [27], but one can get it by mapping of some non-classically correlated separable states which is the case of the state used for construction of GBI. Interestingly, one can find the property also for some other probability distributions constructed via the mapping of such a state. For example, for the discrete probability distribution (8) from Ref. [13] derived from a fully separable state [28] exhibiting non-classical correlations [29], Alice and Bob do not share secret correlations. However, to create the distribution Alice and Bob have to reveal to Eve, i.e., publicly announce, the results of privately tossed, independent fair coins, but only if the results differ. In those cases for which Alice and Bob do not share their results with Eve, i.e., their results are the same, a shared secret bit is established between them. This is clearly impossible with LOPC and a private channel is required to decide when to share their results with Eve.

Conclusions. We derived a distribution carrying tripartite GBI. The entire distribution cannot be constructed by LOPC solely from secret correlations shared by a certain pair of honest parties because the creation of a part of Eve’s information requires private communication with the third honest party. Moreover, this part of Eve’s information is found to be inextricably interconnected with the presence of the BI. As a similar nonlocal part of Eve’s information can also be traced for some other classical analogs of quantum phenomena, an open question arises about to which extent the interconnection between this information and the presence of the phenomenon is general. Our results reveal the nontrivial nature of the classical information resources needed for formation of some classical analogs of quantum phenomena and show that they are linked to the quantum resources not only in the form of entanglement but also in the form of separable non-classical correlations.

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Appendix A: Indispensability of the variable \( x_{E_2} \) for the presence of bound information

In this appendix we show that if the variable \( x_{E_2} \) is dropped from the distribution (10) and \( r > \tilde{r} \approx 0.284 \), Alice and Clare can distill a secret key with the help of Bob. Indeed, if Bob publicly communicates his variable \( x_B \) to Alice and Clare, they can create new variables \( x_A + g x_B \) and \( x_C + h x_B \). The distribution of these variables and the variables \( x_B \) and \( x_{E_1} \) held by Eve, where \( g \) and \( h \) are chosen such that \( \Delta I_{DR} \) is maximized, gives \( \Delta I_{DR} > 0 \) for \( r > \tilde{r} \approx 0.284 \). Consequently, Alice and Clare can distill a secret key with the help of Bob.

Appendix B: Secret correlations appear if the erased adversary’s information is known publicly

In this appendix we prove that if the parties share the reduced distribution \( \Pi(x_A, x_B, x_C, x_{E_1}) \) \( \Pi(x_A, x_B, x_C, x_{E_2}) \) and the variable \( x_{E_2} (x_{E_1}) \) is publicly known, then the honest parties share secret correlations across the \( B - (AC) \) (\( C - (AB) \)) splitting.
Assume first, that all the participants hold the publicly known variable \( x_{E_2} \) and they also hold the reduced distribution \( \Pi(x_A, x_B, x_C, x_{E_1}) \). Then Alice and Clare can create a new variable \( x'_A = (x_A + x_C)/2 - x_{E_2} \). The joint distribution of the variables \( x'_A, x_B, x_{E_1} \) and \( x_{E_2} \) then satisfies \( \Delta I_{DR} > 0 \) if \( r > r' \approx 0.38 \) and therefore contains (distillable) secret correlations with respect to the other hand, the honest parties create new variables \( x''_A = (x_A + x_C)/2 \) and \( x''_B = x_B + x_{E_2} \), the distribution of the variables \( x''_A, x''_B, x_{E_1} \) and \( x_{E_2} \) yields \( \Delta I_{DR} > 0 \) for \( r < r'' \approx 0.549 \) and consequently there are (distillable) secret correlations with respect to the same splitting also for \( r' \leq r'' \). As the latter secret correlations have been established by local operations with respect to the \( B - (AC) \) splitting which cannot create secret correlations, we have to conclude, that if the parties share the distribution \( \Pi(x_A, x_B, x_C, x_{E_1}) \) and know the variable \( x_{E_2} \), then the honest parties share secret correlations across the \( B - (AC) \) splitting.

Assume now, that the parties share the reduced distribution \( \Pi(x_A, x_B, x_C, x_{E_2}) \) and the publicly known variable \( x_{E_1} \). They can then prepare new variables \( \bar{x}_A = e^{2r}x_A + x_B/2 \) and \( \bar{x}_C = x_C + ye^{-2r}x_{E_1} \) which together with the variables \( x_{E_1} \) and \( x_{E_2} \) obey a distribution satisfying \( \Delta I_{DR} > 0 \) for \( r > 0 \) across the \( A - C \) splitting. Because the latter secret correlations have been created by local operations with respect to \( C - (AB) \) splitting which cannot create secret correlations, there are secret correlations across the \( C - (AB) \) splitting if the parties share the distribution \( \Pi(x_A, x_B, x_C, x_{E_2}) \) and know the variable \( x_{E_1} \).

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