Stationary resonances of rapidly-rotating Kerr black holes

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The Klein-Gordon equation for a massive scalar field in the background of a rapidly-rotating Kerr black hole is studied analytically. In particular, we derive a simple formula for the stationary (marginally-stable) resonances of the field in the black-hole spacetime. The analytically derived formula is shown to agree with direct numerical computations of the resonances. Our results provide an upper bound on the instability regime of rapidly-rotating Kerr black holes to massive scalar perturbations.

I. INTRODUCTION

The ‘no-hair’ conjecture [1, 2], put forward by Wheeler more than four decades ago, asserts that stationary black-hole spacetimes should be described by the Kerr-Newman metric. This conjecture therefore suggests that stationary black holes can be characterized by only three externally observable parameters: mass, charge, and angular momentum.

According to the no-hair conjecture, it is expected that static fields (with the exception of the electric field which is associated with a globally conserved charge) cannot survive in the exterior of black holes [1–8]. In particular, such fields are expected to be radiated away to infinity or to be swallowed by the black hole itself [3, 5]. Massless test fields indeed follow this scenario: their relaxation phase in the exterior of black holes is characterized by ‘quasinormal ringing’, damped oscillations with a discrete spectrum [9, 10] (see also [11] and references therein). These characteristic oscillations are then followed by late-time decaying tails [12, 13].

However, it turns out that (non-static) massive scalar fields [14] can survive in the exterior of rotating black holes due to the well-known phenomena of superradiant scattering [15–27]: a bosonic field of the form $e^{im\phi}e^{-i\omega t}$ impinging on a rotating Kerr black hole can be amplified as it scatters off the hole if it satisfies the superradiant condition

$$\omega \leq m\Omega, \quad (1)$$

where

$$\Omega = \frac{a}{2Mr_+} \quad (2)$$

is the angular velocity of the black-hole horizon. Here $M$, $Ma$, and $r_+$ are the black-hole mass, angular momentum, and horizon-radius, respectively. If in addition the scalar field has a non-zero rest mass, then the mass term (the gravitational attraction between the black hole and the massive field) effectively works as a mirror, preventing the field from escaping to infinity.

In a seminal work, Detweiler [20] studied the Klein-Gordon equation for the black-hole-scalar-field system in the regime $M\mu \ll 1$, that is in the regime where the Compton wavelength of the field is much larger than the length-scale set by the black hole. (Here $\mu \equiv M\mathcal{G}/\hbar c$, where $\mathcal{M}$ is the mass of the field. We shall use natural units in which $G = c = 1$ [28].) Using Eqs. (18) and (26) of [20] one finds that marginally stable modes (that is, stationary modes which are characterized by $\Im\omega = 0$) of the massive scalar field exist for the marginal frequency

$$\omega = m\Omega \quad (3)$$

with the discrete spectrum

$$\mu = m\Omega \left[1 + \frac{1}{2} \left(\frac{mM\Omega}{l + 1 + n}\right)^2 + O((\mathcal{M}\Omega)^4)\right] \quad (4)$$

of the field-masses. Here $l$ is the spherical harmonic index of the mode, $m$ is the azimuthal harmonic index with $-l \leq m \leq l$, and $n$ is the resonance parameter which is a non-negative integer [20].

It should be emphasized that the formula (4) is only valid in the regime $M\mu \ll 1$ studied in [20]. Thus, the formula (4) for the field-masses of the stationary resonances is only valid for slowly-rotating black holes (that is, in the regime $M\Omega \ll 1$).
The main goal of the present study is to obtain an analytical formula for the field-masses of the stationary resonances in the regime of rapidly-rotating black holes with \( a \approx M \) (that is, for \( M \Omega \approx 1/2 \)). It is worth mentioning that we have recently obtained a simple upper bound on the field-masses of the stationary (marginally-stable) resonances [26]:

\[
\mu < \sqrt{2}m\Omega .
\]  

This upper bound is valid in the entire range \( 0 \leq a/M \leq 1 \) of the dimensionless black-hole spin. Note that the formula above (which is only valid in the \( a \ll M \) regime) conforms to this upper bound.

**II. DESCRIPTION OF THE SYSTEM**

The physical system we consider consists of a test scalar field \( \Psi \) coupled to a rotating Kerr black hole of mass \( M \) and angular-momentum per unit mass \( a \). In Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) the spacetime metric is given by [29, 30]

\[
\begin{align*}
ds^2 &= - \left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar^2\sin^2 \theta}{\rho^2}dt d\phi + \frac{\rho^2}{\Delta}d\theta^2 + \left( r^2 + a^2 + \frac{2Ma^2r\sin^2 \theta}{\rho^2}\right)\sin^2 \theta d\phi^2, \\
\end{align*}
\]

where \( \Delta \equiv r^2 - 2Mr + a^2 \) and \( \rho \equiv r^2 + a^2 \cos^2 \theta \). The black-hole (event and inner) horizons are located at the zeroes of \( \Delta \):

\[
r_\pm = M \pm (M^2 - a^2)^{1/2}.
\]

We shall henceforth assume that the black hole is rapidly-rotating (near-extremal) with \( a \approx M \).

The dynamics of a massive scalar field \( \Psi \) in the Kerr spacetime is governed by the Klein-Gordon equation [31]

\[
(\nabla^a \nabla_a - \mu^2) \Psi = 0.
\]

One may decompose the field as

\[
\Psi_{lm}(t, r, \theta, \phi) = e^{im\phi} S_{lm}(\theta; a\omega) R_{lm}(r; a\omega)e^{-i\omega t},
\]

where \( \omega \) is the (conserved) frequency of the mode. (We shall henceforth omit the indices \( l \) and \( m \) for brevity.) With the decomposition, \( R \) and \( S \) obey radial and angular equations both of confluent Heun type coupled by a separation constant \( K(a\omega) \) [31, 30].

The angular functions \( S(\theta; a\omega) \) are the spheroidal harmonics which are solutions of the angular equation [31, 30]

\[
\left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) \right) + \left[ K + a^2(\mu^2 - \omega^2) - a^2(\mu^2 - \omega^2) \cos^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S = 0 .
\]

The angular functions are required to be regular at the poles \( \theta = 0 \) and \( \theta = \pi \). These boundary conditions pick out a discrete set of eigenvalues \( \{ K_{lm} \} \) labeled by the integers \( l \) and \( m \). For \( a^2(\mu^2 - \omega^2) \leq m^2 \) one can treat \( a^2(\omega^2 - \mu^2) \cos^2 \theta \) in Eq. (10) as a perturbation term on the generalized Legendre equation and obtain the perturbation expansion [34]

\[
K_{lm} + a^2(\mu^2 - \omega^2) = l(l+1) + \sum_{k=1}^{\infty} c_k a^{2k}(\mu^2 - \omega^2)^k
\]

for the separation constants \( K_{lm} \). The expansion coefficients \( \{ c_k(l, m) \} \) are given in Ref. [34].

The radial Teukolsky equation is given by [31, 32]

\[
\Delta \frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + \left[ H^2 + \Delta [2m \omega - K - \mu^2(r^2 + a^2)] \right] R = 0 ,
\]

where \( H \equiv (r^2 + a^2) \omega - am \). We are interested in solutions of the radial equation with the physical boundary conditions of purely ingoing waves at the black-hole horizon (as measured by a comoving observer) and a bounded (decaying) solution at spatial infinity [18, 27]. That is,

\[
R \sim \begin{cases} 
e^{-\sqrt{\mu^2 - \omega^2}y} & \text{as } r \to \infty \ (y \to \infty) ; \\
e^{-i(\omega - m\Omega) y} & \text{as } r \to r_H \ (y \to -\infty) ,
\end{cases}
\]

where the “tortoise” radial coordinate \( y \) is defined by \( dy = [(r^2 + a^2)/\Delta]dr \).

Note that a bound state (a state decaying exponentially at spatial infinity) is characterized by \( \omega^2 < \mu^2 \). The boundary conditions [13] single out a discrete set of complex resonances \( \{ \omega_n(\mu) \} \) which correspond to the bound states of the massive field [18, 27, 34, 35]. The stationary (marginally-stable) resonances, which are the solutions we are interested in in this paper, are characterized by \( 3\omega = 0 \).
III. THE STATIONARY SCALAR RESONANCES

As we shall now show, the field (3) with the marginal frequency (3) describes a stationary resonance of the Klein-Gordon equation (3) in the black-hole spacetime. In particular, we shall now derive an analytical formula for the discrete spectrum \( \{M\mu(m, l, n)\} \) of field-masses which satisfy the stationary resonance condition \( \Im \omega = 0 \). To that end, it is convenient to define new dimensionless variables

\[
x \equiv \frac{r - r_+}{r_+}, \quad \tau \equiv 8\pi M T_{BH} = \frac{r_+ - r}{r_+}; \quad k \equiv 2m\Omega r_+; \quad \epsilon \equiv \sqrt{\mu^2 - (m\Omega)^2} r_+,
\]

in terms of which the radial equation (12) becomes

\[
x(x + \tau) \frac{d^2 R}{dx^2} + (2x + \tau) \frac{dR}{dx} + VR = 0,
\]

where \( V \equiv H^2/r_+^2 x(x + \tau) - K_{lm} + 2m^2 a \Omega - \mu^2 r_+^2 (x + 1)^2 + a^2 \) and \( H = kr_+ x(x + 1) \).

We first consider the radial equation (15) in the far region \( x \gg \tau \). Then Eq. (15) is well approximated by

\[
x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + V_{\text{far}} R = 0,
\]

where \( V_{\text{far}} = - (\epsilon x)^2 + k^2 x / 2 + [-K_{lm} + 2m^2 a \Omega + k^2 - \mu^2 (r_+^2 + a^2)] \). A solution of Eq. (16) that satisfies the boundary condition (12) can be expressed in terms of the confluent hypergeometric functions \( M(a, b; z) \) [34, 39]

\[
R = C_1 (2\epsilon)^{1/2 + \beta} x^{-1/2 + \beta} e^{-\epsilon x} M\left(\frac{1}{2} + \beta - \kappa, 1 + 2\beta, 2\epsilon x\right) + C_2 (\beta \to -\beta),
\]

where \( C_1 \) and \( C_2 \) are constants. Here

\[
\beta^2 \equiv K_{lm} + \frac{1}{4} + \mu^2 (r_+^2 + a^2) - k^2 - 2m^2 a \Omega
\]

and

\[
\kappa \equiv \frac{1}{4} k^2 - \epsilon^2.
\]

The notation \( \beta \to -\beta \) means “replace \( \beta \) by \(-\beta\) in the preceding term.”

We next consider the near horizon region \( x \ll 1 \). The radial equation is given by Eq. (15) with \( V \to V_{\text{near}} \equiv -K_{lm} + 2m^2 a \Omega - \mu^2 (r_+^2 + a^2) + k^2 x / (x + \tau) \). The physical solution obeying the ingoing boundary condition at the horizon is given by [34, 39]

\[
R = \left(\frac{x}{\tau} + 1\right)^{ik} 2F_1\left(\frac{1}{2} + \beta - ik, \frac{1}{2} - \beta - ik; 1; -x/\tau\right),
\]

where \( 2F_1(a; b; c; z) \) is the hypergeometric function.

The solutions (17) and (20) can be matched in the overlap region \( \tau \ll x \ll 1 \). The \( x \ll 1 \) limit of Eq. (17) yields [34, 39]

\[
R \to C_1 (2\epsilon)^{1/2 + \beta} x^{-1/2 + \beta} + C_2 (\beta \to -\beta).
\]

The \( x \gg \tau \) limit of Eq. (20) yields [34, 39]

\[
R \to \tau^{1/2 - \beta} \frac{\Gamma(2\beta)}{\Gamma(\frac{1}{2} + \beta - ik)\Gamma(\frac{1}{2} + \beta + ik)} x^{-1/2 + \beta} + (\beta \to -\beta).
\]

By matching the two solutions in the overlap region one finds

\[
C_1 = \tau^{1/2 - \beta} \frac{\Gamma(2\beta)}{\Gamma(\frac{1}{2} + \beta - ik)\Gamma(\frac{1}{2} + \beta + ik)} (2\epsilon)^{-1/2 - \beta},
\]

and

\[
C_2 = \tau^{1/2 + \beta} \frac{\Gamma(-2\beta)}{\Gamma(\frac{1}{2} - \beta - ik)\Gamma(\frac{1}{2} - \beta + ik)} (2\epsilon)^{-1/2 + \beta}.
\]
Approximating Eq. (17) for \( x \to \infty \) one gets
\[ R \to \left[ C_1(2\epsilon)^{-\kappa} \frac{\Gamma(1 + 2\beta)}{\Gamma(1 + \beta - \kappa)} x^{-1 - \kappa} + C_2(\beta \to -\beta) \right] e^{\epsilon x} \]
\[ + \left[ C_1(2\epsilon)^{\kappa} \frac{\Gamma(1 + 2\beta)}{\Gamma(1 + \beta + \kappa)} x^{-1 + \kappa} (1)^{-1/2 - \beta + \kappa} + C_2(\beta \to -\beta) \right] e^{-\epsilon x} . \]  
\( \text{(25)} \)

A bound state is characterized by a decaying field at spatial infinity. The coefficient of the growing exponent \( e^{\epsilon x} \) in Eq. (25) should therefore vanish. Taking cognizance of Eqs. (23)–(25), one finds the characteristic equation
\[ \frac{1}{\Gamma(1/2 + \beta - \kappa)} = \left[ \frac{\Gamma(-2\beta)}{\Gamma(2\beta)} \right]^2 \frac{\Gamma(1/2 + \beta - ik)\Gamma(1/2 + \beta + ik)}{\Gamma(1/2 - \beta - ik)\Gamma(1/2 - \beta + ik)} (2\epsilon)^{2\beta} \]  
\( \text{(26)} \)

for the stationary bound states of the massive scalar field. Note that the r.h.s. of Eq. (26) is of order \( O(\tau^{2\beta}) \ll 1. \)

Thus, using the well-known pole structure of the Gamma functions \( \text{[34]} \), one finds that the resonance condition (26) can be written as
\[ \frac{1}{2} + \beta - \kappa = -n + O(\tau^{2\beta}) , \]  
\( \text{(27)} \)
where \( n \geq 0 \) is a non-negative integer.

We shall henceforth assume that \( 2\beta > 1 \text{ [40]} \) and expand all quantities to first order in the small parameter \( \tau \). Taking cognizance of Eqs. (11), (14), (18), and (19), one finds
\[ \beta^2 = (l + \frac{1}{2})^2 - \frac{3}{2} m^2 + \frac{1}{4} m^2 \tau + O(\tau^2, \epsilon^2) \]  
\( \text{(28)} \)
and
\[ \kappa = \frac{m^2}{4\epsilon} - \epsilon + O(\tau^2) . \]  
\( \text{(29)} \)

Substituting Eqs. (28)–(29) into (27), one finds that the resonance condition for the stationary modes can be expressed as a polynomial equation for the dimensionless variable \( \epsilon \):
\[ 4[(2l + 1)^2 - m^2(4 - \tau) - (2n + 1)^2] \epsilon^2 + 4m^2(2n + 1) \epsilon - m^4 + O(\tau^{2\alpha}, \epsilon^3) = 0 , \]  
\( \text{(30)} \)
where \( \alpha \equiv \min\{1, \beta\} \). This simple equation can easily be solved to yield
\[ \bar{\epsilon}(l, m, n) \equiv \frac{\epsilon}{m} \frac{m}{2(\ell + 1 + 2n)} - \frac{m^3}{4\ell(\ell + 1 + 2n)^2 \tau} + O(\tau^{2\alpha}, \bar{\epsilon}^3) , \]  
\( \text{(31)} \)
where \( \ell \equiv \sqrt{(2l + 1)^2 - 4m^2} \text{ [42]} \). The field-masses of the stationary resonances are given by \( \mu = \sqrt{(m\Omega)^2 + (\epsilon/\tau)^2} \) [see Eq. (14)], which implies
\[ \mu = m\Omega\left[1 + 2\bar{\epsilon}^2 + O(\tau^2, \bar{\epsilon}^4)\right] . \]  
\( \text{(32)} \)

**IV. NUMERICAL CONFIRMATION**

We shall now test the accuracy of the analytically derived formula (32) for the field-masses of the stationary resonances. The stationary resonances can be computed using standard numerical techniques, see [23, 27] for details. In Table I we present a comparison between the *analytically* derived field-masses of the stationary resonances (32), and the *numerically* computed field-masses [23, 27] for the physically most interesting mode [18–27], \( l = m = 1 \) with \( n = 0 \). We find an almost perfect agreement between the two in the \( \tau \ll 1 \) \((a/M \gtrsim 0.99)\) regime. In fact, one finds that the agreement between the numerical data and the analytical formula (32) is quite good already at \( a/M = 0.9 \). This is quite surprising since the assumption \( \tau \ll 1 \) breaks down for this value of the dimensionless spin parameter.
TABLE I: Stationary resonances of a massive scalar field in the background of a rapidly-rotating Kerr black hole. The data shown is for the fundamental mode $l = m = 1$ with $n = 0$, see also [23, 27]. We display the ratio between the analytically derived field-mass, $\mu_{\text{ana}}$, and the numerically computed values, $\mu_{\text{num}}$. The agreement between the numerical data and the analytical formula (32) is better than 3% in the $a/M \gtrsim 0.9$ regime. (In fact, the agreement becomes much better than 1% in the $a/M \gtrsim 0.99$ regime).

| $a/M$ | 0.9  | 0.95 | 0.99 | 0.995 | 0.999 | 1.0  |
|-------|------|------|------|-------|-------|------|
| $\mu_{\text{ana}}/\mu_{\text{num}}$ | 1.029 | 1.024 | 1.007 | 1.006 | 1.004 | 1.003 |

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This statement seems to hold true for all bosonic fields.