GLOBAL SMALL SOLUTIONS TO THE CRITICAL RADIAL DIRAC EQUATION WITH POTENTIAL

FEDERICO CACCIAFESTA

Abstract. We solve globally a radial cubic Dirac equation perturbed with a small potential, with data of small critical norm $H^1$. The main tool are new endpoint estimates of the perturbed Dirac flow for a class of radial-type initial data.

1. Introduction

Consider a zero mass nonlinear Dirac equation
\[ iu_t - Du = F(u), \quad u(0, x) = f(x) \tag{1.1} \]
for the spinor field $u : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{C}_4$, where $D$ is the operator defined by
\[ D = i^{-1} \sum_{k=1}^3 \alpha_k \partial_k = -i(\alpha \cdot \nabla) \]
while the $4 \times 4$ Dirac matrices are defined as
\[ \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{1.2} \]
in terms of the Pauli matrices
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3} \]
We shall assume that the nonlinear term $F(u)$ is cubic of a very specific form, namely
\[ \text{either } F(u) = \langle \beta u, u \rangle u \quad \text{or} \quad F(u) = \langle u, u \rangle u, \tag{1.4} \]
where $\langle \cdot, \cdot \rangle$ denotes the standard hermitian product in $\mathbb{C}^4$ and $\beta$ is the $4 \times 4$ Dirac matrix
\[ \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}. \]
Notice that nonlinearities of the form (1.4) are the most interesting from a physical point of view (see e.g. [15]).

The typical approach to the Dirac equation is based on the identity
\[ (i\partial_t - D)(i\partial_t + D) \equiv (-\partial_t + \Delta)I_4 \tag{1.5} \]
which follows from the anticommuting relations
\[ \alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik} I_4, \quad i, k = 1, 2, 3, \tag{1.6} \]
\[ \alpha_i \beta + \beta \alpha_i = 0, \quad i = 1, 2, 3, \quad \beta^2 = I_4. \tag{1.7} \]
In many cases the identity allows to reduce problems for the massless Dirac equation to analogous ones for the wave equation, for which many effective tools are available.
The classical strategy to solve nonlinear dispersive equations is by the use of a fixed point argument in a suitable space, via the appropriate space-time Strichartz estimates. A huge literature is available on these estimates for several dispersive operators (wave, Schroedinger, Klein-Gordon, Dirac and others). For homogeneous nonlinear terms one typically finds a threshold regularity \( s_c \) in the scale of Sobolev spaces \( H^s \), such that for subcritical data with \( s > s_c \) solvability holds, while for supercritical data with \( s < s_c \) one has various degrees of ill-posedness. Data of critical regularity give rise to difficult questions which depend on the precise structure of the equation.

If we denote with \( e^{itD} \) the propagator for the Dirac operator \( D \), which can be defined via Fourier transform as the operator of symbol \( e^{it\xi} \), Strichartz estimates for the linear 3D Dirac equation can be written as follows (see \([9],[10]\), and, for the Dirac equation, \([2],[3]\)):

\[
\| [D^+]^{\frac{1}{2}} e^{itD} f \|_{L^p_t L^q_x} \lesssim \| f \|_{L^2},
\]

where the exponents \((p,q)\) are admissible, that is to say

\[
\frac{2}{p} + \frac{2}{q} = 1, \quad 2 < p \leq \infty, \quad 2 \leq q < \infty.
\]

Here and in the following we shall use the notation \( |D| = (-\Delta)^{1/2} \), the mixed space-time Strichartz space \( L^p_t L^q_x \) is \( L^p(\mathbb{R}^3; L^q(\mathbb{R}^3_x)) \), and \( H^s \) is the homogeneous Sobolev space with norm \( \| f \|_{H^s} = \| |D|^s f \|_{L^2} \). As it is well known, estimate (1.8) is false for the endpoint couple \((p,q) = (2,\infty)\), which would correspond to the estimate

\[
\| e^{itD} f \|_{L^2_t L^{\infty}_x} \lesssim \| f \|_{H^1}.
\]

The failure of (1.9) for generic data was first noticed for 3D the wave equation in \([11]\), and the corresponding statement for the Dirac equation follows easily from the relation connecting the Dirac and wave flows (see (2.2) below).

A clever use of the estimates (1.8) is sufficient for the study of (1.1) in the subcritical case. Local existence, and global existence for small data, was proved in \([7]\) for nonlinearities of the form \((\beta u, u)^{(p-1)/2} u\) with \( p > 3 \) and data in \( H^s \) with \( s > 3/2 - 1/(p - 1) \). The case of cubic nonlinearities and small data in \( H^s \), \( s > 1 \), was solved in \([13]\), where the result was generalized to all space dimensions. We also mention that systems involving the Dirac equation, like Dirac-Klein-Gordon and Maxwell-Dirac, have been the object of intense attention and the subcritical theory was completed only recently (see \([5],[4],[6]\)).

The failure of the endpoint estimate (1.8) means that above methods break down in the critical case of a cubic nonlinearity with \( H^1 \) data, and indeed the problem of global existence is still open in this case. In an attempt to overcome this limitation, in \([12]\) the following precised estimate was proved:

\[
\| e^{itD} f \|_{L^2_t L^{\infty}_x} \lesssim \| f \|_{H^1}, \quad \forall p < \infty.
\]

The norm at the left hand side distinguishes between the integrability in the radial and tangential directions. Using estimate (1.10), Machihara et al. were able to prove global well posedness for problem (1.1) for small \( H^1 \)-norm data with slight additional angular regularity, and in particular for all radial \( H^1 \) data. This is especially interesting since, as we shall see, radial data do not correspond to radial solution for the Dirac equation.

The main goal of the present paper is to study a special class of solutions to equation (1.1) and more generally to a potential perturbation of the form

\[
iu_t - Du + V(x)u = F(u), \quad u(0) = f(x),
\]

CRITICAL DIRAC EQUATION 2
which can be considered as a suitable generalization of the radial solutions to the wave equation in the context of the Dirac equation. We recall that, for small potentials \( V \) with suitable decay at infinity, the full range of Strichartz estimates (1.8) holds also for the perturbed flow \( e^{i(t(D+V))} \), as proved in [3]. Hence subcritical problems can be treated exactly as in the unperturbed case and one can extend the results of [7], [13] to (1.11) in a straightforward way. Here we focus on the more difficult case of critical \( H^1 \) data with an additional symmetry assumption like

\[
f = f_1 + Df_2 \quad \text{with} \quad f_1 \in \dot{H}^1, \ f_2 \in \dot{H}^2, \ \ f_1, f_2 \ \text{radial.}
\]

In addition, in order to preserve the symmetry of solutions, we need to assume that the potential \( V(x) \) is \textit{spherically symmetric} in the sense of [16].

Our first result is an endpoint estimate for the linear flow:

**Theorem 1.1.** Let \( V(x) \) be a \( 4 \times 4 \) matrix of the form

\[
V(x) = V_1(|x|)\mathbb{I}_4 + i\beta(\alpha \cdot \hat{x})V_2(|x|), \quad V_1, V_2 : \mathbb{R}^+ \to \mathbb{R}, \ x \in \mathbb{R}^3
\]

(where \( \hat{x} = x/|x| \)). Assume that for some \( \sigma > 1 \) and some sufficiently small \( \delta > 0 \)

\[
|V(x)| \leq \frac{\delta}{|x|^{1/2} \log |x|^{|\sigma|/2} + |x|^\sigma}.
\]

Then the following endpoint Strichartz estimate

\[
\|e^{i(t(D+V))}f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{\dot{H}^1}
\]

holds for all initial data \( f \in \dot{H}^1 \) where

\[
\dot{H}^1 = \{ f_1 + Df_2, \ f_1 \in \dot{H}^1(\mathbb{R}^3), \ f_2 \in \dot{H}^2(\mathbb{R}^3), \ f_1, f_2 \ \text{radial} \}.
\]

We recall that condition (1.13) is sufficient to ensure that the perturbed Dirac operator \( D + V \) is self-adjoint on the domain \( H^1(\mathbb{R}^3)^4 \). For this and many other properties of the Dirac equation, a comprehensive reference is [16].

The natural application of estimate (1.14) is to prove global well posedness for the critical equation (1.11). However the nonlinear term \( F(u) \) does not operate on the space \( \dot{H}^1 \) and additional restrictions on the algebraic structure of the data are necessary. More precisely, it is possible to decompose the space \( L^2(\mathbb{R}^3)^4 \) as a direct sum

\[
L^2(\mathbb{R}^3)^4 \simeq \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{m_j=\cdots=-j}^{j} \bigoplus_{k_j=\pm 1/2} L^2(0, +\infty; dr) \otimes H_{m_j,k_j},
\]

where the spaces \( H_{m_j,k_j} \) are two dimensional and are generated by spherical harmonics on the sphere \( S^2 \) (this is called a decomposition in \textit{partial wave subspaces}). When \( j = 1/2 \), we have four spaces

\[
L^2(0, +\infty; dr) \otimes H_{m_{1/2},k_{1/2}}
\]

corresponding to the four possible choices of indices

\[
(m_{1/2}, k_{1/2}) = (-1/2, -1), \ (-1/2, 1), \ (1/2, -1), \ (1/2, 1).
\]

Then we notice the mildly surprising fact that each of these four spaces is invariant not only for the Dirac operator but also for the action of cubic nonlinearities of the forms (1.4). In a sense, these spaces can be considered as a suitable generalization of radial functions adapted to the structure of the nonlinear problem (1.11). A detailed analysis of the partial wave decomposition is given in Section 5, with explicit forms for the functions in these spaces (see Lemma 5.5 and in particular (5.22)–(5.25)).

Thanks to this invariance, we can prove the following global existence result:
Theorem 1.2. Consider the equation on $\mathbb{R} \times \mathbb{R}^3$

$$iu_t - D u + V(x) u = F(u), \quad u(0, x) = f(x)$$  \hspace{1cm} (1.17)

where the potential has the form

$$V = V_1(|x|) \mathbb{I} + i \beta (\alpha \cdot \hat{x}) V_2(|x|), \quad \hat{x} = x/|x|$$

and satisfies assumption (1.13), while

$$F(u) = \langle \beta u, u \rangle \quad \text{or} \quad F(u) = \langle u, u \rangle.$$  

Assume the initial data $f$ belong to a space $\dot{H}^1((0, \infty), dr) \otimes H_{m_1/2, k_1/2}$ for one of the choices (1.16). Then if the $\dot{H}^1$ norm of the data is sufficiently small, problem (1.17) has a unique global solution in the class $C_t(\mathbb{R}, \dot{H}^1) \cap L_2^t(\mathbb{R}, L^\infty).$

The paper is organized as follows. Sections 2 contains an extension of the endpoint estimate for the free Dirac operator which is then adapted in Section 3 to a mixed endpoint-smoothing estimate with weights for the nonhomogeneous linear Dirac equation. Section 4 is devoted to the proof of Theorem (1.1). In Section 5 we recall the structure of the Dirac operator, the partial wave decomposition, and we investigate the interaction of the algebraic structure with the nonlinear term. Section 6 contains the proof of Theorem (1.2).

The author wishes to thank prof. Piero D’Ancona for his help and the referee for some useful suggestions.

2. THE HOMOGENEOUS ENDPOINT ESTIMATE

Consider the 3-dimensional, massless, free Dirac equation (1.1). As already observed, the Dirac flow does not preserve radiality so that we cannot hope to repeat here the simple argument used in [11] for the wave equation. However we can prove an endpoint estimate for suitable classes of function; to this end we need a deeper insight in the structure of the Dirac operator expressed in radial coordinates.

Let $u = e^{itD}f$ and notice that, thanks to identity (1.5), $u$ is (formally) a solution of

$$\begin{align*}
\Box u &= 0 \\
u(0) &= f(x) \\
u_t(0) &= iDf.
\end{align*}$$  \hspace{1cm} (2.1)

This gives the formula

$$u = e^{itD}f = \cos(t|D|)f + i \frac{\sin(t|D|)}{|D|} Df$$  \hspace{1cm} (2.2)

and we easily see that this representation is valid for generic distribution data.

We start by proving the following result:

Proposition 2.1. Let $f$ belong to the space $\dot{H}^1$ defined in (1.15). Then the following endpoint Strichartz estimate holds:

$$\|e^{itD}f\|_{L_t^3 L_x^\infty} \lesssim \|f\|_{\dot{H}^1}.$$  \hspace{1cm} (2.3)

By formula (2.2), we see that the proof is an immediate consequence of the following Lemma. Notice that the proof of the first estimate (2.4) is inspired by an argument of [8] which holds for all $n \geq 3$ without modification, while we fix $n = 3$ in the second estimate.
Lemma 2.2. Let $f$ be a radial function and let $e^{t|D|}$ be the linear propagator associated to the wave operator. Then the following estimates hold:

$$
\|e^{t|D|}f\|_{L^2_t L^\infty_x} \lesssim \|f\|_H \quad (n \geq 3) 
$$

(2.4)

$$
\left\| \frac{e^{t|D|}}{|D|} Df \right\|_{L^2_t L^\infty_x} \lesssim \|f\|_H \quad (n = 3). 
$$

(2.5)

Proof. Using Fourier transform in spherical coordinates and the radiality of $f$ and setting $\rho = |x|$, $|\xi| = \lambda$, $x \cdot \xi = \rho \lambda \cos \theta$, we have

$$
e^{t|D|}f = \int e^{i(x \cdot \xi + t|\xi|)} \hat{f}(\xi) d\xi = \int_0^\infty \int_0^\pi e^{i\lambda(t + \rho \cos \theta)} \hat{f}(\lambda)(\lambda^{n-1}(\sin \theta)^{n-2} d\theta d\lambda. 
$$

(2.6)

With the change of variable $y = \cos \theta$ in (2.6) we obtain

$$
= \int d\lambda e^{i\lambda y} g(\lambda) \int_1^1 e^{i\rho y(1 - y^2)^{\frac{n-2}{n}} dy
$$

with $g(\lambda) = \hat{f}(\lambda)\lambda^{n-1}H(\lambda)$ ($H$ represents the classical Heaviside function). Now changing the order of the integrals we obtain

$$
\int_1^1 dy (1 - y^2)^{\frac{n-2}{n}} \int_{-\infty}^{\infty} e^{i\rho y(1 - y^2)^{\frac{n-2}{n}}} g(\rho) d\rho = \int_1^1 dy (1 - y^2)^{\frac{n-2}{n}} ~\hat{g}(t + \lambda y
$$

Since for $n \geq 3$ one has $(1 - y^2)^{\frac{n-2}{n}} \leq 1$, the change of variable $y \rightarrow y/r$ yields again

$$
\leq \frac{1}{r} \int_{-r}^r \hat{g}(t + y) dy = M(\hat{g})(t) 
$$

(2.7)

where $M$ denotes the standard maximal operator. Then we have for all $t$

$$
\|e^{t|D|}f\|_{L^\infty_x} \lesssim M(\hat{g})(t),
$$

(2.8)

and thus by the $L^p$-boundedness of maximal operator and Plancherel’s theorem

$$
\|e^{t|D|}f\|_{L^2_t L^\infty_x} \lesssim \|g\|_{L^2_x} = \left(\int_0^\infty (\lambda^{n-1}\hat{f}(\lambda))^2 d\lambda \right)^{\frac{1}{2}} = \left(\int_0^\infty \lambda^{\frac{n-1}{2}} \hat{f}(\lambda)^2 \lambda^{n-1} d\lambda \right)^{\frac{1}{2}} = \|\lambda^{\frac{n-1}{2}} \hat{f}\|_{L^2_x} = \|f\|_H \lesssim |\xi|
$$

(2.9)

which gives (2.4).

We now turn to estimate (2.5), for which the calculations are similar. Indeed we can write (here we are fixing $n = 3$)

$$
e^{t|D|} Df = \int e^{i(x \cdot \xi + t|\xi|)} (\alpha \cdot \xi) \hat{f}(\xi) d\xi = \int_0^\infty d\lambda \int_0^{2\pi} d\phi \int_0^\pi d\theta e^{i\lambda(t + \rho \cos \theta)} A(\theta, \phi) \hat{f}(\lambda) \lambda^2 \sin \theta
$$

(2.10)

where

$$
\alpha \cdot \xi = \sum_{k=1}^3 \frac{(\alpha_k \cdot \xi_k)}{|\xi|}. 
$$

Using spherical coordinates as before we have

$$
\int_0^\infty d\lambda \int_0^{2\pi} d\phi \int_0^\pi d\theta e^{i\lambda(t + \rho \cos \theta)} A(\theta, \phi) \hat{f}(\lambda) \lambda^2 \sin \theta = \alpha_1 \cos \theta + \alpha_2 \sin \theta \cos \phi + \alpha_3 \sin \theta \sin \phi
$$

with the operator $A(\theta, \phi) = \alpha_1 \cos \theta + \alpha_2 \sin \theta \cos \phi + \alpha_3 \sin \theta \sin \phi$. Observing that

$$
\int_0^{2\pi} \alpha_2 \sin \theta \cos \phi d\phi = \int_0^{2\pi} \alpha_3 \sin \theta \sin \phi d\phi = 0,
$$

(2.11)
we see that (2.11) is equal to
\[
\simeq \int_0^\infty d\lambda \int_0^\pi d\theta e^{i\lambda(t+\rho \cos \theta)} \alpha_1 \cos \theta \hat{f}(\lambda) \lambda^2 \sin \theta.
\]
Setting as before \(g(\lambda) = \hat{f}(\lambda) \lambda^2 \mathcal{H}(\lambda)\), changing variable \(\cos \theta \to y\) and then \(y \to y/r\) yield
\[
= \int_{-1}^1 dy (\alpha_1 \cdot y) \int_{-\infty}^{+\infty} dp \, e^{i\lambda(t+\rho y)} g(\rho) = \frac{1}{r} \int_{-r}^r (\alpha_1 \cdot \frac{y}{r}) \hat{g}(t+y) \, dy \simeq cM(\hat{g})(t)
\]  \hspace{1cm} (2.12)
since the term \((\alpha_1 \cdot \frac{y}{r})\) is bounded, and so we have the bound
\[
\left\| \frac{e^{itD}}{|D|} \mathcal{D} f \right\|_{L^2_x} \lesssim M(\hat{g})(t).
\]  \hspace{1cm} (2.13)
The \(L^p\)-boundedness of maximal operator and Plancherel Theorem yield as above
\[
\left\| \frac{e^{itD}}{|D|} \mathcal{D} f \right\|_{L^1_t L^2_x} \lesssim \|f\|_{H^1}
\]  \hspace{1cm} (2.14)
which gives the desired estimate (2.5).

Combining estimates (2.4) and (2.5) and using representation (2.2) for the solution of the free Dirac system we obtain estimate (2.3).

3. THE MIXED ENDPOINT-SMoothing ESTIMATE

We consider now the non homogeneous equation
\[ iu_t - \mathcal{D} u = F(t, x), \quad u(0, x) = 0. \]  \hspace{1cm} (3.1)
By Duhamel’s formula and the representation (2.2) we can write the solution \(u\) as
\[
u(t, x) = \int_0^t e^{i(t-s)\mathcal{D}} F(s, x) ds = \int_0^t \left( \cos((t-s)|D|) F(s, x) + i \frac{\sin((t-s)|D|)}{|D|} \mathcal{D} F(s, x) \right) ds.
\]  \hspace{1cm} (3.2)
Thus in order to estimate the solution \(u\) to (3.1) we can deal separately with the two integrals
\[
\int_0^t e^{i(t-s)|D|} F(s, x) ds \quad \text{and} \quad \int_0^t \frac{e^{i(t-s)|D|}}{|D|} \mathcal{D} F(s, x) ds.
\]
We prove the following:

**Proposition 3.1.** Let \(n = 3\) and assume \(F(t, x)\) has the structure
\[
F(t, x) = F_1(|x|) \mathcal{H}_4 + i \beta(\alpha \cdot \hat{x}) F_2(|x|).
\]  \hspace{1cm} (3.3)
Then the following estimate holds
\[
\left\| \int_0^t e^{i(t-s)|D|} F(s) ds \right\|_{L^1_t L^\infty_x} \lesssim \|\langle x \rangle^\frac{1}{2} + |D| F\|_{L^1_t L^2_x}.
\]  \hspace{1cm} (3.4)

The key step in the proof of (3.4) is the following non homogeneous estimate for the wave propagator with a radial term.

**Lemma 3.2.** Let \(n \geq 3\), \(F(t, \cdot)\) be a radial function. Then the following estimate holds
\[
\left\| \int_0^t e^{i(t-s)|D|} F(s) ds \right\|_{L^1_t L^\infty_x} \lesssim \|\langle x \rangle^\frac{1}{2} + |D| \frac{n-3}{2} F\|_{L^2_t L^2_x}.
\]  \hspace{1cm} (3.5)
Proof. We start with (3.5). Expanding \( u \) as in the homogeneous case (see formulas (2.6 and 2.7)), from the radiality of \( F \) we can estimate the \( L^\infty \) norm of the solution at fixed \( t \) as (here \( \tilde{G}(s, \lambda) = \lambda^{n-1} \tilde{F}(s, \lambda)H(\lambda) \) and \( H \) is the Heaviside function)

\[
\|u\|_{L^\infty} \lesssim \sup_r \frac{1}{r} \int_{-r}^{r} \left( \int_0^t \left| \tilde{G}(s, y + t - s) \right| ds \right) dy = \sup_r \frac{1}{r} \int_{-r}^{r} \left( \int_0^t \left| \tilde{G}(s, y + t - s) \right| (y + t - s)^{\frac{n}{2}+} (y + t - s)^{-\frac{n}{2}-} ds \right) dy \lesssim \sup_r \frac{1}{r} \int_{-r}^{r} dy \left( \left( \int_0^t \left| \tilde{G}_1(s, y + t - s) \right|^2 ds \right)^{\frac{1}{2}} \cdot \left( \int_0^t (y + t - s)^{-1} ds \right)^{\frac{1}{2}} \right)
\]

where in the last inequality we have used Cauchy-Schwarz inequality and \( \tilde{G}_1 \) is the function defined by

\[
\tilde{G}_1(s, y) = \tilde{G}(s, y)(y)^{\frac{n}{2}+}.
\]

Setting now \( h(z) := \left( \int_0^t \left| \tilde{G}_1(s, z - s) \right|^2 ds \right)^{\frac{1}{2}} \), we have

\[
\sup_r \frac{1}{r} \int_{-r}^{r} dy \left( \int_0^t \left| \tilde{G}_1(s, y + t - s) \right|^2 ds \right)^{\frac{1}{2}} = M(h)(t).
\]

The \( L^p \) boundedness of the maximal operator yields

\[
\|u\|_{L^p_{t,x}} \lesssim \|h(t)\|_{L^2} = \left( \int \int \left| \tilde{G}_1(s, t - s)^2 \right|^2 dsdt \right)^{\frac{1}{2}} = \|\tilde{G}_1\|_{L^2_{t,x}}.
\]

The last quantity is precisely

\[
\|\left| (\gamma)^{\frac{n}{2}+} \mathcal{F}_{\lambda \to \rho} \left( \lambda^{n-1} \tilde{F}(s, \lambda)H(\lambda) \right) \right| \|_{L^2_{t,x}} \lesssim \|\rho f\|_{L^2}
\]

and to conclude the proof we need to estimate it by

\[
\lesssim \|\langle x \rangle^{\frac{n}{2}+} |D|^{\frac{n-1}{2}} \mathcal{F}f\|_{L^2_{t,x}}.
\]

Since \( \mathcal{F}^{-1}(|D|^{\frac{n-1}{2}} f) = |\xi|^{\frac{n-1}{2}} \hat{f} \) we see that it is enough to prove the general inequality (we can neglect the dependence on time)

\[
\|\langle \rho \rangle^{\frac{n}{2}+} \mathcal{F}_{\lambda \to \rho} \left( \lambda^{\frac{n-1}{2}} \hat{f}(\lambda)H(\lambda) \right) \|_{L^2} \lesssim \|\langle x \rangle^{\frac{n}{2}} f\|_{L^2}
\]

for \( k = 1/2+ \).

We prove (3.6) by interpolation. The case \( k = 0 \) is trivial, since from Placherel’s Theorem we obviously have

\[
\|\mathcal{F}_{\lambda \to \rho} \left( \lambda^{\frac{n-1}{2}} \hat{f}(\lambda)H(\lambda) \right) \|_{L^2} \lesssim \|\lambda^{\frac{n-1}{2}} \hat{f}(\lambda)\|_{L^2} = \|f\|_{L^2}.
\]

The case \( k = 1 \) is just a little more complicated. Since of course \( \langle \rho \rangle \leq 1 + |\rho| \), we need only prove that

\[
\|\rho \mathcal{F}_{\lambda \to \rho} \left( \lambda^{\frac{n-1}{2}} \hat{f}(\lambda)H(\lambda) \right) \|_{L^2} \lesssim \|\langle x \rangle f\|_{L^2}
\]

or equivalently

\[
\|\partial_\lambda \left( \lambda^{\frac{n-1}{2}} \hat{f}(\lambda)H(\lambda) \right) \|_{L^2} \lesssim \|\langle x \rangle f\|_{L^2}.
\]

We write

\[
\|\partial_\lambda \left( \lambda^{\frac{n-1}{2}} \hat{f}(\lambda)\chi_{R^+}(\lambda) \right) \|_{L^2} \lesssim \|\lambda^{\frac{n-1}{2}} \partial_\lambda \hat{f}(\lambda)\|_{L^2} + \|\lambda^{\frac{n-2}{2}} \hat{f}(\lambda)\|_{L^2} = I_1 + I_2
\]

with the shorthand notation \( L^2_{\pm} = L^2(0, \infty) \). For \( I_1 \) we trivially have

\[
I_1 = \|\lambda^{\frac{n-1}{2}} (\rho f)\|_{L^2} \lesssim \|\langle x \rangle f\|_{L^2}.
\]
Lemma (Plancherel’s Theorem yields again for the first term of all, the solution of the equation form (where $\hat{F}$ estimate holds estimate ($\hat{F}$ arty function $i\beta\phi$). Let’s now turn to estimate ($\hat{F}$) we obtain estimate (3.5).

Lemma 3.3. Let $n = 3$ and $F(t, \cdot)$ be of the form (3.3). Then the following estimate holds

$$\left\| \int_0^t \frac{e^{i(t-s)|D|}}{|D|^{1-j}} A_j F_{\text{rad}}(s) \, ds \right\|_{L^2_t L^\infty_x} \lesssim \| \langle x \rangle \|_{L^2_t L^2_x}.$$ (3.9)

for $k = 0, 1$.

Proof. Since the operator $iD\beta(\alpha \cdot \dot{x})\phi$ applied to a radial function $\phi$ produces the radial function $i\beta\phi''$, Lemma (3.2) holds, and we only need to control terms of the form (where $F_{\text{rad}}$ denotes a radial function)

$$\left\| \int_0^t \frac{e^{i(t-s)|D|}}{|D|^{1-j}} A_j F_{\text{rad}}(s) \, ds \right\|_{L^2_t L^\infty_x}$$

with $A_j = i\beta^j(\alpha \cdot \dot{x})$, $j = 0, 1$. Recalling (2.10)-(2.12), since the quantities $A_j$ are obviously bounded, we can estimate in both cases with

$$\left\| \int_0^t \frac{e^{i(t-s)|D|}}{|D|^{1-j}} A_j F_{\text{rad}}(s) \, ds \right\|_{L^2_t L^\infty_x} \lesssim \int_0^\infty \int_{\mathbb{R}^3} |\hat{G}(s, y + t - s)| \, ds \, dy$$

where as before $\hat{G}(s, \lambda) = \lambda^2 \hat{F}_{\text{rad}}(s, \lambda) H(\lambda)$. Proceeding exactly as in the proof of Lemma (3.2) we obtain estimate (3.9). □

Estimate (3.4) is an immediate consequence of (3.5), (3.9) and representation (3.2)

4. Proof of Theorem (1.1)

We now turn to the proof of Theorem (1.1). This is based on a simple application of a smoothing estimate for a Dirac equation with potential proved in [3] (see also [2], [1] for related results):

Theorem 4.1 ([3]). Let $V(x) = V(x)^*$ be a $4 \times 4$ complex valued matrix and assume that for some $\sigma > 1$ and some sufficiently small $\delta > 0$ one has

$$|V(x)| \leq \frac{\delta}{w_\sigma(x)} \quad \text{where} \quad w_\sigma(x) = |x|(1 + |\log |x||)^\sigma$$ (4.1)

Then the following smoothing estimate holds:

$$\|w_\sigma^{-1/2} e^{i(t\mathcal{D} + V)} f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{L^2}.$$ (4.2)

It is not difficult to deduce the endpoint estimate (1.14) for the perturbed flow from the previous result and our mixed endpoint-smoothing estimate (3.4). First of all, the solution of the equation

$$iu_t = Du + Vu, \quad u(0, x) = f$$
can be written, regarding \( Vu \) as a right-hand member of the equation
\[
    u = e^{it(D+V)}f = e^{itD}f + i \int_0^t e^{i(t-s)D}(Vu)ds.
\]
Then we can write
\[
    \|D|^{-1}u\|_{L^2_tL^\infty_x} = \|D|^{-1}e^{it(D+V)}f\|_{L^2_tL^\infty_x} \leq \|D|^{-1}e^{itD}f\|_{L^2_tL^\infty_x} + \left\| \int_0^t e^{i(t-s)D}(V(s)e^{it(D+V)}f)ds \right\|_{L^2_tL^\infty_x}.
\] (4.3)
The first term can be estimated by (2.3) (notice that \( |D| \) commutes with \( D \) and hence with the flow). In order to apply estimate (3.4) to the second term we need the following

**Lemma 4.2.** If \( f \in \hat{H}^1 \), then \( e^{itD}f \in \hat{H}^1 \), and if \( V \) is of the form (1.12), then \( V e^{itD}f \) is of the form (3.3).

**Proof.** We write \( f = f_1 + Df_2 \) with \( f_1, f_2 \) radial functions. Then we have, from (2.2),
\[
    e^{itD}f = \left( \cos(t|D|) + \frac{\sin(t|D|)}{|D|} D \right) (f_1 + Df_2) = 
    \cos(t|D|)f_1 + \sin(t|D|)|D|f_2 + D \left( \cos(t|D|)f_2 + \frac{\sin(t|D|)}{|D|} f_1 \right) 
    = \tilde{f}_1 + D\tilde{f}_2,
\]
where \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are radial functions with the appropriate regularity, and this concludes the proof of the first statement. The proof of the second statement is trivial. \( \square \)

We thus can estimate (4.3) with (2.3) and (3.4) obtaining
\[
    \lesssim \|f\|_{L^2} + \|(x)^{1/2}Vu\|_{L^2_tL^2_x}
\]
Now multiplying and dividing by \( w_\sigma(x)^{1/2} \) in the second norm on the right hand side yields
\[
    \lesssim \|f\|_{L^2} + \|(x)^{1/2} + w_\sigma^{1/2}V\|_{L^\infty} \cdot \|w_\sigma^{-1/2}u\|_{L^2_tL^2_x}.
\]
Notice that the weighted norm of \( V \) at the right hand side is bounded as it follows from assumption (1.13). Moreover (1.13) implies also that the assumption of Theorem 4.1 is satisfied. Then using (4.2) we conclude
\[
    \|D|^{-1}u\|_{L^2_tL^\infty_x} \leq \left( 1 + \|(x)^{1/2} + w_\sigma^{1/2}V\|_{L^\infty} \right) \|f\|_{L^2} \tag{4.4}
\]
that gives, under hypothesis (1.13) on the potential, the desired Strichartz endpoint estimate.

### 5. Partial wave subspaces and radial Dirac operator

The purpose of this section is to construct, following [16], invariant subspaces for the Dirac operator with a potential having a special symmetry. To this end we use the classical decomposition of the space \( L^2(\mathbb{R}^3)^4 \) in the direct sum of 2-dimensional Hilbert spaces, the *partial wave subspaces*, which are invariant for the Dirac operator. We shall also check that the lowest order partial wave subspaces are invariant even for the cubic nonlinearities that we consider here.

We begin by recalling the basic facts on the decomposition, referring to [16] for more details. We shall use the standard notation for polar coordinates in \( \mathbb{R}^3 \)
\[
    x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta
\]
with the unit vectors in the directions of the polar coordinate lines given by
\[
\begin{align*}
\mathbf{e}_r &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = \frac{x}{|x|} = \hat{x} \\
\mathbf{e}_\theta &= (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta) = \frac{\partial \mathbf{e}_r}{\partial \theta} \\
\mathbf{e}_\phi &= (-\sin \phi, \cos \phi, 0) = \frac{1}{\sin \theta} \frac{\partial \mathbf{e}_r}{\partial \phi}
\end{align*}
\]
Then we write for a function \(\psi \in L^2(\mathbb{R}^3)\)
\[
\psi(r, \theta, \phi) = r \tilde{\psi}(x(r, \theta, \phi), y(r, \theta, \phi), z(r, \theta, \phi)).
\] (5.1)
Since the function \(\psi(r, \cdot, \cdot)\) of the angular variables is square integrable on the unit sphere \(L^2(S^2)\), the mapping \(\tilde{\psi} \rightarrow \psi\) defines a unitary isomorphism
\[
L^2(\mathbb{R}^3) \cong L^2((0, \infty), dr; L^2(S^2)) = L^2((0, \infty), dr) \otimes L^2(S^2).
\]
Applying the transformation (5.1) on each component of the (vector valued) wavefunction, we obtain the analogous decomposition
\[
L^2(\mathbb{R}^3)^4 \cong L^2((0, \infty), dr) \otimes L^2(S^2)^4.
\]
The decomposition of the Hilbert space into a "radial" and an "angular" part is very useful since the angular momentum operators
\[
\mathbf{L} = x \wedge (-i\nabla) \quad \text{orbital angular momentum}
\]
\[
\mathbf{J} = \mathbf{L} + \mathbf{S} \quad \text{total angular momentum}
\]
act only on the angular part \(L^2(S^2)^4\) in a nontrivial way; here
\[
\mathbf{S} = -\frac{1}{4}(\alpha \wedge \alpha)
\]
is the spin angular momentum operator. Recalling the expression of \(\nabla\) in polar coordinates
\[
\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \left( \mathbf{e}_\theta \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right)
\] (5.2)
we obtain that, since \(x = r \cdot \mathbf{e}_r\),
\[
\mathbf{L} = i \mathbf{e}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} - i \mathbf{e}_\phi \frac{\partial}{\partial \theta}
\] (5.3)
where the differentiation applies to each component of the wavefunction.

The Dirac operator can be written in polar coordinates as follows. Combining (5.2) and (5.3) yields
\[
-i \nabla = -i \mathbf{e}_r \frac{\partial}{\partial r} - \frac{1}{r} (\mathbf{e}_r \wedge \mathbf{L})
\]
and thus
\[
- i \alpha \cdot \nabla = -i(\alpha \cdot \mathbf{e}_r) \frac{\partial}{\partial r} - \frac{1}{r} \alpha \cdot (\mathbf{e}_r \wedge \mathbf{L}).
\] (5.4)
By the basic property of the Dirac matrices,
\[
(\alpha \cdot A)(\alpha \cdot B) = A \cdot B + 2\mathbf{S} \cdot (A \wedge B);
\]
which holds for any matrix-valued vector fields \(A = (A^1, A^2, A^3)\), \(B = (B^1, B^2, B^3)\) with \(F^i, G^i \in \mathcal{M}_{4 \times 4}(\mathbb{C})\), and
\[
\gamma_5 \alpha = 2\mathbf{S},
\]
where \(\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), we obtain
\[
(\alpha \cdot A)(2\mathbf{S} \cdot B) = i\gamma_5 A \cdot B - i\alpha \cdot (A \wedge B),
\]
thus equation (5.4) is equal to
\[
= -i(\alpha \cdot \mathbf{e}_r) \frac{\partial}{\partial r} + \frac{i}{r}(\alpha \cdot \mathbf{e}_r)(2\mathbf{S} \cdot \mathbf{L}).
\]
Finally, introducing the spin orbit operator
\[ K = \beta(2\mathbf{S} \cdot \mathbf{L} + 1) \equiv \beta(J^2 - L^2 + 1/4) \] (5.5)
where we used the identity \( J^2 = (L + S)^2 = L^2 + 2\mathbf{S} \cdot \mathbf{L} + 3/4 \), we arrive at the following representation:

**Proposition 5.1.** The 3-dimensional Dirac operator can be written as
\[ \mathcal{D} = -i(\alpha \cdot \hat{x}) \left( \frac{\partial}{\partial r} + \frac{1}{r} - \frac{1}{r} \beta K \right) \] (6.6)
where \( K \) is the spin orbit operator defined in (5.5).

The key step to construct the invariant spaces is the following:

**Proposition 5.2.** For each choice \((j, m_j, k_j)\) with \( j = 1/2, 3/2, 5/2, \ldots \), \( m_j = -j, -j + 1, \ldots, j \), \( k_j = -(j + 1/2), (j + 1/2) \), there exist precisely two eigenfunctions \( \Phi_{m_j, k_j}^\pm \in C^\infty(S^2)^4 \) satisfying the following relations:
\[ J^2 \Phi_{m_j, k_j} = j(j + 1) \Phi_{m_j, k_j}, \]
\[ J_3 \Phi_{m_j, k_j} = m_j \Phi_{m_j, k_j}, \]
\[ K \Phi_{m_j, k_j} = -k_j \Phi_{m_j, k_j}. \]
The family \( \Phi_{m_j, k_j}^\pm \) forms an orthonormal basis of \( L^2(S^2)^4 \).

The functions \( \Phi_{m_j, k_j} \) can be written explicitly using spherical harmonics. We first recall the following representation of 3-dimensional spherical harmonics
\[ Y_l^m(\theta, \phi) = \frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} e^{im\phi} P_l^m(\cos \theta) \quad \forall -l \leq m \leq l, \] (5.7)
where \( P_l^m \) are the Legendre polynomials
\[ P_l^m(x) = \frac{(-1)^m}{2^{l+1}} (1 - x^2)^{m/2} \frac{d^{m+l}}{dx^{m+l}} (x^2 - 1)^l. \] (5.8)
As it is well known, the spherical harmonics form a complete orthonormal set in \( L^2(S^2) \), i.e. every function \( f \in L^2(S^2) \) can be written as
\[ f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_l^m Y_l^m(\theta, \phi) \]
for some constants \( f_l^m \); moreover, they are eigenfunctions of both the operators \( L^2 \) and \( L_3 \), i.e.
\[ L^2 Y_l^m = l(l + 1) Y_l^m, \quad L_3 Y_l^m = m Y_l^m. \] (5.9, 5.10)
We now define for \( j = 1/2, 3/2, 5/2, \ldots, m_j = -j, -j + 1, \ldots, +j \) the functions \( \Psi_{j+1/2}^{m_j} \in L^2(S^2)^2 \):
\[ \Psi_{j+1/2}^{m_j} = \frac{1}{\sqrt{2j + 1}} \begin{pmatrix} \sqrt{j + m_j} & Y_{j+1/2}^{m_j-1/2} \\ \sqrt{j - m_j} & Y_{j+1/2}^{m_j+1/2} \end{pmatrix}, \] (5.11)
\[ \Psi_{j+1/2}^{m_j} = \frac{1}{\sqrt{2j + 2}} \begin{pmatrix} \sqrt{j + 1 - m_j} & Y_{j+1/2}^{m_j+1/2} \\ -\sqrt{j + 1 + m_j} & Y_{j+1/2}^{m_j-1/2} \end{pmatrix}. \] (5.12)
These functions are, as it is easily seen, eigenfunctions of both the operators \( L^2 \) and \( J^2 = L^2 + \sigma \cdot L + 3/4 \) with eigenvalues \( l(l + 1) \) and \( j(j + 1) \) respectively. So we conclude, in view of (5.5), that the functions in Proposition (5.2) are given by
\[ \Phi_{m_j, \mp(j+1/2)}^+ = \begin{pmatrix} i \Psi_{j+1/2}^{m_j} \\ 0 \end{pmatrix}, \quad \Phi_{m_j, \mp(j+1/2)}^- = \begin{pmatrix} \Psi_{j+1/2}^{m_j} \\ 0 \end{pmatrix}. \] (5.13)
Thus the Hilbert space $L^2(S^2)^4$ is the orthogonal direct sum of 2-dimensional Hilbert spaces $\mathcal{H}_{m_j, k_j}$, which are spanned by simultaneous eigenfunctions $\Phi_{m_j, k_j}$ of $J^2$ and $K$:

$$L^2(S^2)^4 = \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm (j+\frac{1}{2})}^{\infty} \mathcal{H}_{m_j, k_j}$$

(5.14)

Easy calculations show that the functions $\Psi_{m_j, k_j}$ satisfy

$$(\sigma \cdot \hat{x}) \Psi_{j\pm1/2} = \Psi_{j\pm1/2}$$

and hence

$$i(\alpha \cdot \hat{x}) \Phi_{m_j, k_j} = \mp \Phi_{m_j, k_j}$$

(5.15)

This proves the following:

**Lemma 5.3.** The subspaces $\mathcal{H}_{m_j, k_j}$ are left invariant by the operators $\beta$ and $\alpha \cdot \hat{x}$. With respect to the basis $\{\Phi_{m_j, k_j}^+, \Phi_{m_j, k_j}^-\}$ defined above, these operators are represented by the $2 \times 2$ matrices

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -i\alpha \cdot \hat{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

(5.16)

The decomposition just shown obviously implies a similar one of $L^2(\mathbb{R}^3)^4$, in which each partial wave subspace $L^2((0, \infty), dr) \otimes \mathcal{H}_{m_j, k_j}$ is isomorphic to $L^2((0, \infty), dr)^2$ if we choose the basis $\{\Phi_{m_j, k_j}^+, \Phi_{m_j, k_j}^-\}$. There is in fact a unitary isomorphism between the Hilbert spaces:

$$L^2(\mathbb{R}^3)^4 \cong \bigoplus L^2((0, \infty), dr) \otimes \mathcal{H}_{m_j, k_j}.$$  

(5.17)

This decomposition and (5.6) allow us to easily calculate the action of the Dirac operator (at least on differentiable states) even in the presence of a suitable potential.

**Proposition 5.4.** The Dirac operator (5.6) with the potential

$$V(x) = V_1(|x|) \|_4 + i \beta (\alpha \cdot \hat{x}) V_2(|x|)$$

(5.18)

leaves the partial wave subspaces $C^0(0, \infty) \otimes \mathcal{H}_{m_j, k_j}$ invariant. With respect to the basis $\{\Phi_{m_j, k_j}^+, \Phi_{m_j, k_j}^-\}$ the Dirac operator on each subspace can be represented by the operator

$$d_{m_j, k_j} = \begin{pmatrix} V_1(|x|) & -\frac{d}{dx} + \frac{k_j}{\tau} + V_2(|x|) \\ \frac{d}{dx} + \frac{k_j}{\tau} + V_2(|x|) & V_1(|x|) \end{pmatrix}.$$  

(5.19)

which is well defined over $C^0_0(0, \infty)^2 \subset L^2((0, \infty), dr)^2$. Moreover, the Dirac operator $\mathcal{D}$ on $C^0_0(\mathbb{R}^3)^4$ is unitary equivalent to the direct sum of the partial wave Dirac operators $d_{m_j, k_j}$,

$$\mathcal{D} \cong \bigoplus_{j=\frac{1}{2}}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm (j+\frac{1}{2})}^{\infty} d_{m_j, k_j}$$

(5.20)

**Remark 5.1.** Proposition 5.4 holds for slightly more general potentials (see [16]), but we shall not need this fact here.

**Remark 5.2.** The operator in (5.19) is also known as radial Dirac operator. It can be proved that $d_{m_j, k_j}$ is essentially self-adjoint (for every $j$) on $C_0^0(0, \infty)$ if and only if $\mathcal{D} + V$ is essentially self-adjoint on $C_0^0(\mathbb{R}^3 \setminus \{0\})$. 

Thus using spherical coordinates it is possible to construct invariant spaces for the perturbed Dirac operator. What may come as a surprise is that for $j = 1/2$ the partial wave subspaces are also invariant for the cubic nonlinearity, and this fact is obviously crucial for the nonlinear application we shall prove in the next section.

**Lemma 5.5.** Let $j = 1/2$ and let $(m_{1/2}, k_{1/2})$ be one of the couples $(-1/2, -1)$, $(-1/2, 1)$, $(1/2, -1)$, $(1/2, 1)$. Then the partial wave subspaces $C_0^\infty((0, \infty), dr) \otimes \mathcal{H}_{m_{1/2}, k_{1/2}}$ are invariant for the cubic nonlinearities $F_1(u) = \langle u, u \rangle u$ and $F_2(u) = \langle \beta u, u \rangle u$, i.e.

$$u \in C_0^\infty((0, \infty), dr) \otimes \mathcal{H}_{m_{1/2}, k_{1/2}} \Rightarrow F_1(u), F_2(u) \in C_0^\infty((0, \infty), dr) \otimes \mathcal{H}_{m_{1/2}, k_{1/2}}.$$  

**Proof.** We explicitly write down the functions $\Phi^+, \Phi^-$ in the four cases: a straightforward calculation using formulas (5.7), (5.8), (5.11), (5.12) and (5.13) yields

$$\Phi^+_{-1/2, -1} = \begin{pmatrix} 0 \\ \frac{i}{2 \sqrt{\pi}} e^{i \phi} \sin \theta \\ 0 \\ 0 \end{pmatrix}, \quad \Phi^-_{-1/2, -1} = \begin{pmatrix} 0 \\ \frac{i}{2 \sqrt{\pi}} e^{i \phi} \cos \theta \\ 0 \\ -\frac{1}{2 \sqrt{\pi}} \end{pmatrix},$$

$$\Phi^+_{1/2, 1} = \begin{pmatrix} \frac{i}{2 \sqrt{\pi}} e^{i \phi} \cos \theta \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Phi^-_{1/2, 1} = \begin{pmatrix} 0 \\ \frac{i}{2 \sqrt{\pi}} e^{i \phi} \sin \theta \\ 0 \\ \frac{1}{2 \sqrt{\pi}} \end{pmatrix},$$

$$\Phi^+_{1/2, -1} = \begin{pmatrix} \frac{i}{2 \sqrt{\pi}} \cos \theta \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Phi^-_{1/2, -1} = \begin{pmatrix} 0 \\ \frac{i}{2 \sqrt{\pi}} \sin \theta \\ 0 \\ \frac{1}{2 \sqrt{\pi}} \end{pmatrix}.$$  

We prove Lemma (5.5) for the couple $(1/2, 1)$, i.e. for functions of the form (5.25), being the proof for the other cases completely analogous.

The generic function $u \in L^2((0, \infty), dr) \otimes \mathcal{H}_{1/2, 1}$ can be written as

$$u(r, \theta, \phi) = u^+(r) \Phi^+_{1/2, 1}(\theta, \phi) + u^-(r) \Phi^-_{1/2, 1}(\theta, \phi)$$

for some radial functions $u^+, u^-$. So $f$ takes the vectorial form

$$u = \begin{pmatrix} u^+(r) \frac{i}{2 \sqrt{\pi}} \cos \theta \\ u^+(r) \frac{i}{2 \sqrt{\pi}} \sin \theta \\ u^-(r) \frac{i}{2 \sqrt{\pi}} \cos \theta \\ u^-(r) \frac{i}{2 \sqrt{\pi}} \sin \theta \end{pmatrix}.$$  

(5.26)
Thus the Hermitian product \( \langle u, u \rangle \) yields
\[
\langle u, u \rangle = -\frac{1}{4\pi} \cos^2 \theta \ u^+(r)^2 - \frac{1}{4\pi} \sin^2 \theta \ u^+(r)^2 + \frac{1}{4\pi} u^-(r)^2 = -\frac{1}{4\pi} \left( u^+(r)^2 - u^-(r)^2 \right)
\]
that has no angular components. This proves that if \( u \in C_0^\infty((0, \infty), dr) \odot \mathcal{H}_{m_{1/2}, k_{1/2}} \) then \( F_1(u) \in C_0^\infty((0, \infty), dr) \odot \mathcal{H}_{m_{1/2}, k_{1/2}} \).

Minor modifications yield the same result also for the nonlinearity \( F_2(u) \). In fact we know from Lemma (5.21) that the operator \( \beta \) acts on the partial wave subspaces in a very simple way with respect to the basis \( \{ \Phi^+, \Phi^- \} \): if in fact we associate to the function \( u \) its coordinates \( (u^+(r), u^-(r)) \) with respect to such a basis we have
\[
\langle \beta u, u \rangle = -\frac{1}{4\pi} \cos^2 \theta \ u^+(r)^2 - \frac{1}{4\pi} \sin^2 \theta \ u^+(r)^2 + \frac{1}{4\pi} u^-(r)^2 = -\frac{1}{4\pi} \left( u^+(r)^2 + u^-(r)^2 \right)
\]
that again has no angular components, and this shows that if \( u \in C_0^\infty((0, \infty), dr) \odot \mathcal{H}_{m_{1/2}, k_{1/2}} \) then \( F_2(u) \in C_0^\infty((0, \infty), dr) \odot \mathcal{H}_{m_{1/2}, k_{1/2}} \). \( \square \)

6. Global existence for the nonlinear equation

As an application of the results we have presented in the previous section we can now prove global existence for problem (1.17) with small initial data in one of the four partial wave subspaces \( \mathcal{H}^1((0, \infty), dr) \odot \mathcal{H}_{m_{1/2}, k_{1/2}} \). Our goal is to prove

**Theorem 6.1.** Consider the Cauchy problem for the 3-dimensional nonlinear Dirac equation
\[(6.1)\]
\[iu_t - D_u + V(x)u = F(u), \quad u(0, x) = f(x)\]
where the potential \( V \) is of the form
\[V = V_2(|x|)\mathbb{I}_x + \mathbb{I}_x (\alpha \cdot \dot{x}) V_2(|x|)\]
and satisfies assumption (1.13), while the nonlinear term \( F(u) \) is either of the form \( \langle \beta u, u \rangle \) or \( \langle u, u \rangle \).

Then for every initial data \( f \in \mathcal{H}^1((0, \infty), dr) \odot \mathcal{H}_{m_{1/2}, k_{1/2}} \), with sufficiently small \( \mathcal{H}^1 \) norm, there exists a unique global solution \( u(t, x) \) to problem (1.17) in the class \( C_t(\mathbb{R}, \mathcal{H}^1) \cap L^2_t(\mathbb{R}, L^\infty) \).

**Proof.** The proof is identical for both choices of the form of the nonlinear term. We rewrite (1.17) in integral form
\[(6.2)\]
\[u = e^{itD} f + i \int_0^t e^{i(t-s)D} (V(s)u(s) + F(u(s))) \, ds = e^{itD} f + i \int_0^t e^{i(t-s)D} (V(s)u(s)) \, ds + i \int_0^t e^{i(t-s)D} (F(u(s))) \, ds = e^{itD + V} f + i \int_0^t e^{i(t-s)D} (F(u(s))) \, ds \equiv \Phi(u) \equiv I_1 + I_2\]
we denote by \( \Phi(u) \) the RHS of (6.2) and we check that the map \( \Phi \) is a contraction on the function space
\[X = L^2_t L^\infty_x \cap L^\infty_t \mathcal{H}^1_x.\]
In order to estimate the first term \( I_1 \) we use our endpoint Strichartz estimate (1.14), observing that if \( f \in \mathcal{H}^1((0, \infty), dr) \odot \mathcal{H}_{m_{1/2}, k_{1/2}} \) then in particular \( f \) is of
On the other hand, in view of Lemma (1.15) we have
\[ \|I_1\|_X \lesssim \|f\|_{\dot{H}^1} \] (6.3)
Now we need to handle the nonlinear term $I_2$. By Minkowski inequality
\[ \left\| \int_0^t e^{i(t-s)D} F(u(s)) ds \right\|_X \leq \int_0^\infty \| e^{itD} e^{-isD} F(u(s)) \|_X ds. \]
By energy conservation we have
\[ \| e^{itD} e^{-isD} F(u(s)) \|_{L^2_x H^1} = \| F(u(s)) \|_{L^2_x H^1}. \]
On the other hand, in view of Lemma (5.5), we can use estimate (2.3) and we have
\[ \| e^{itD} e^{-isD} F(u(s)) \|_{L^2_x L^\infty_x} = \| F(u(s)) \|_{\dot{H}^1} \lesssim \| F(u(s)) \|_{L^\infty_x H^1}. \]
Thus
\[ \|I_2\|_X \lesssim \| F(u(s)) \|_{L^\infty_x H^1}. \]
Then by Hölder inequality in $t, x$ we obtain
\[ \| F(u) \|_{L^1_t L^2_x H^1_x} \leq \| u \|_{L^2_t L^\infty_x L^2_x} \| F(u) \|_{L^\infty_x H^1_x} \]
which implies
\[ \| \Phi(u) \|_X \lesssim \| f \|_{\dot{H}^1} + \| u \|_X^3. \]
An analogous computation gives
\[ \| \Phi(u) - \Phi(v) \|_X \lesssim (\| u \|_X^2 + \| v \|_X^2) \| u - v \|_X. \] (6.4)
Therefore if the data belong to a sufficiently small ball in $\dot{H}^1$, $\Phi$ is a contraction on that ball, and its unique fixed point is the unique global solutions to problem (1.17) in the space $X$. \qed

References

[1] Piero D’Ancona and Luca Fanelli. $L^p$-boundedness of the wave operator for the one dimensional Schrödinger operator. Comm. Math. Phys., 268(2):415–438, 2006.
[2] Piero D’Ancona and Luca Fanelli. Decay estimates for the wave and Dirac equations with a magnetic potential. Comm. Pure Appl. Math., 60(3):357–392, 2007.
[3] Piero D’Ancona and Luca Fanelli. Strichartz and smoothing estimates of dispersive equations with magnetic potentials. Comm. Partial Differential Equations, 33(1-6):1028–1112, 2008.
[4] Piero D’Ancona, Damiano Foschi, and Sigmund Selberg. Local well-posedness below the charge norm for the Dirac-Klein-Gordon system in two space dimensions. J. Hyperbolic Differ. Equ., 4(2):295–330, 2007.
[5] Piero D’Ancona, Damiano Foschi, and Sigmund Selberg. Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system. J. Eur. Math. Soc. (JEMS), 9(4):877–899, 2007.
[6] Piero D’Ancona, Damiano Foschi, and Sigmund Selberg. Null structure and almost optimal local well-posedness of the Maxwell-Dirac system. Amer. J. Math., 132(3):771–839, 2010.
[7] Miguel Escobedo and Luis Vega. A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s > 1$. SIAM J. Math. Anal., 28(2):338–362, 1997.
[8] Daoyuan Fang and Chengbo Wang. Weighted Strichartz estimates with angular regularity and their applications. Forum Math. 23 (2011), 181-205.
[9] Jean Ginibre and Giorgio Velo. Generalized Strichartz inequalities for the wave equation. J. Funct. Anal., 133(1):50–68, 1995.
[10] Markus Keel and Terence Tao. Endpoint Strichartz estimates. Amer. J. Math., 120(5):955–980, 1998.
[11] Sergiu Klainerman and Matei Machedon. Space-time estimates for null forms and the local existence theorem. Comm. Pure Appl. Math., 46(9):1221–1268, 1993.
[12] Shuji Machihara, Makoto Nakamura, Kenji Nakanishi, and Tohru Ozawa. Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation. J. Funct. Anal., 219(1):1–20, 2005.
[13] Shuji Machihara, Makoto Nakamura, and Tohru Ozawa. Small global solutions for nonlinear Dirac equations. Differential Integral Equations, 17(5-6):623–636, 2004.
[14] Shuji Machihara, Kenji Nakanishi and Tohru Ozawa. Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation. Rev. Mat. Iberoamericana, 19 (2003), 179-194.

[15] Antonio F. Rañada. On nonlinear classical Dirac fields and quantum physics. In Old and new questions in physics, cosmology, philosophy, and theoretical biology, pages 363–376. Plenum, New York, 1983.

[16] Bernd Thaller. The Dirac equation. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

Federico Cacciafesta: SAPIENZA — Università di Roma, Dipartimento di Matematica, Piazzale A. Moro 2, I-00185 Roma, Italy
E-mail address: cacciafe@mat.uniroma1.it