Positive controllability of networks under relative actuation

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Abstract

For arrays of identical linear systems coupled through relative actuation four problems are studied: controllability, positive controllability, pairwise controllability, and positive pairwise controllability. To this end, related to the eigenvalues of the system matrix, certain graphs with possibly vector-valued edge weights are constructed. It is shown that array controllability and graph connectivity are equivalent. Similar equivalences are established also between positive controllability & strong connectivity, pairwise controllability & pairwise connectivity, and positive pairwise controllability & strong pairwise connectivity.

1 Introduction

Probably since Huygens pointed out the synchronization of two pendulum clocks, it must have been self-evident that the collective behavior of a group of interacting systems should be determined by the connectivity of certain graph(s) representing (in some way) the interconnection between the individual units. What in general is not evident however is how to dig out the graphs whose connectivity determines what need be determined. For instance, consider the individual electrical oscillator in Fig. 1, where unit

![Figure 1: 10th order LC oscillator.](image1)

(1H) inductors are connected by unit (1F) capacitors as shown. Let us form two separate arrays, each containing three identical replicas of this oscillator coupled via unit (1Ω) resistors as in Fig. 2a and Fig. 2b, respectively. Although neither array look more connected than the other to the eye, there is a significant qualitative difference in their behaviors: starting from arbitrary initial conditions the oscillators in Fig. 2a always synchronize in the steady state, whereas those in Fig. 2b do not tend to oscillate in unison. This failure to synchronize can be traced back to the lack of connectivity of a certain graph.

![Figure 2: Arrays of coupled oscillators. The array (a) synchronizes; the array (b) does not.](image2)
Implicit in the above example is the importance of the role connectivity plays in network controllability. In fact, if the resistors in Fig. 2b are replaced by current sources (as our control inputs) the new array cannot be steered toward synchronization. The reason, not surprisingly, is the disconnectivity of the graph that was behind the failure of synchronization in the old array. To see the relation between connectivity and controllability explicitly, let us visit a simpler example where the graph is not hidden. Consider three identical water tanks (integrators) connected via water pumps as shown in Fig. 3. Letting $x_i$ denote the water volume ($m^3$) contained in the $i$th tank and $u_{\sigma}$ the flow rate ($m^3/h$) through the $\sigma$th pump we can write the dynamics as

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1 & 1 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}.$$

The pleasant (zero column sum) structure of the matrix $B$ is shared by the incidence matrix representing the graph $\Gamma$ in Fig. 4. Observe that $\Gamma$ is (weakly) connected yet not strongly connected. This has two apparent implications. First, because the graph $\Gamma$ is connected the array is (relatively) controllable by which we mean that the relative states $x_i - x_j$ can be adjusted arbitrarily. That is, with bidirectional pumps the relative water levels can be simultaneously steered to any desired values regardless of the initial distribution. Second, because the graph $\Gamma$ is not strongly connected the array is not positively controllable. This translates to that with unidirectional pumps ($u_{\sigma} \geq 0$) the water levels cannot in general be equalized. At least three pumps are needed for that since at least three edges are needed for a 3-node graph to be strongly connected.

The water tanks example clearly illustrates the link between network controllability and graph connectivity. Meanwhile, as the oscillator array example indicates, the graphs whose connectivity determines controllability may not be apparent and therefore revealing them may require some effort. This paper is a report on such effort. Our setup is an array of linear time-invariant (LTI) systems driven by relative actuators. Specifically, the $i$th individual system’s ($n$th order) dynamics reads $\dot{x}_i = Ax_i + \sum_{\sigma} B_{i\sigma} u_{\sigma}$ with $\sum_i B_{i\sigma} = 0$. For this setup we study, from the connectivity point of view, four problems: controllability, positive controllability, pairwise controllability, and positive pairwise controllability.

**Controllability.** The literature on network controllability has so far concentrated on a somewhat different problem concerning a different setup than ours given above. The generally adopted node dynamics are first order and there is coupling between nodes even when the inputs are zero. In addition, there is no relative actuation constraint. Namely, $\dot{x}_i = \sum_j a_{ij} x_j + \sum_{\sigma} b_{i\sigma} u_{\sigma}$ where $x_i \in \mathbb{R}$. Since the inputs are not relative, complete controllability is possible and that is what has been thoroughly investigated. Generally speaking, the problem that has received much attention concerns with the question of how to achieve controllability with as few inputs (or driver nodes) as possible; see, for instance, [12, 8, 19, 10]. In this particular direction a wealth of results has accumulated, e.g., [3, 2, 11, 16, 15], starting possibly with Lin’s work [6] on structural controllability. While these work dwell upon the “how?” for networks.
with first order node dynamics, we focus (in Section 4) on the simpler “yes/no?” for higher order dynamics. Namely, for an array with $q$ systems (nodes) and $p$ (relative) inputs, represented by the pair $[A, (B_{i,\sigma})_{i,\sigma=1}^{q,p}]$ we present a necessary and sufficient condition for controllability from the graph connectivity point of view. The result is based on tools from classical control theory. The presented connectivity condition can indeed be seen as a certain reformulation of Popov-Belevitch-Hautus (PBH) test exploiting the special structure of our setup.

Positive controllability. One of the earliest things we learn in life is how to steer a particular system (our body) with one-way actuators, for our muscles function that way. That is, a muscle can only pull or contract, but cannot (actively) push or extend. Another instance from biology of a one-way actuator is insulin, a key hormone in regulating the sugar level in blood. Insulin cannot undo what it does therefore pancreas employs another one-way agent, glucagon, to achieve proper regulation. Examples are not scarce outside biology; see, for instance, and references therein. The earliest work on controllability of LTI systems with positive controls (one-way actuators) is [14]. Later Brammer provides a general eigenvector test [1] which arguably is the most effective tool we have today on positive controllability of continuous-time LTI systems. Certain refinements/reformulations of Brammer’s test are reported in [5,18]. Among the very few works on positive controllability of networks is [7], where the authors study first order node dynamics. Here, for arrays with $n$th order node dynamics, we provide in Section 5 a necessary and sufficient strong connectivity condition for the positive controllability of an array. Just as our connectivity condition for controllability can be seen as a reformulation of PBH test, our strong connectivity condition for positive controllability is a natural extension of the refinements [18,7] of Brammer’s eigenvector test.

Pairwise controllability. For an uncontrollable array, while it is not possible to steer all relative states, it is of interest to determine the subarrays of states that can be controlled relatively. The problem, at least for primitive arrays, is closely related to determining the connected components of an unconnected graph, which can be studied by means of paths connecting pairs of nodes. This motivates us to analyze (from connectivity point of view) the so called pairwise controllability, roughly described as follows. For a given pair $(k, \ell)$ of indices, an array is $(k, \ell)$-controllable when the difference $x_k - x_\ell$ can be arbitrarily adjusted. (The actual definition is subtler; see Def. 2). The outcome of our analysis is presented in Section 6, where we provide necessary and sufficient connectivity conditions for $(k, \ell)$-controllability. From the geometric point of view what is done is in effect checking whether a certain subspace (corresponding to $(k, \ell)$-controllability of the array) is contained in the overall controllable subspace.

Positive pairwise controllability. Last in our sequence of problems is the characterization of pairwise controllability of an array with positive controls. The off-the-shelf tools (such as controllability matrix, PBH test, Brammer’s test) we use for the previous problems turn out not to be of much help here. Hence the analysis is of slightly different spirit and lengthier than before. However the end results (presented in Section 7) are of the same nature. In particular, positive pairwise controllability is interpreted in terms of strong connectivity of a pair of graph nodes.

To summarize, the contribution of this paper is intended to be in showing that the well-known close relation between controllability and connectivity for arrays with first order node dynamics naturally continue to exist for a class of arrays with higher order node dynamics. To this end, we study the above mentioned four facets of (relative) controllability. In particular, we establish connectivity characterizations of (pairwise) controllability as well as strong connectivity characterizations of positive (pairwise) controllability. With the possible exception of the contents of Section 7, the analysis methods employed in our work are not new; we borrow a great deal from the classical control theory toolbox. However, what we believe is fresh here is the perspective through which we tackle the problems at hand.

2 Array

A pair $[A, (B_{i,\sigma})_{i,\sigma=1}^{q,p}]$ is meant to represent an array of $q \geq 2$ LTI systems

$$\dot{x}_i = Ax_i + \sum_{\sigma=1}^{p} B_{i,\sigma} u_\sigma, \quad i = 1, 2, \ldots, q$$

As mentioned earlier, when we use the word controllable to indicate an array we mean that all relative states $x_i - x_j$ (as opposed to actual states $x_i$) can be controlled. Since the actuation is relative in our setup, this is the most that can be achieved in terms of controllability. For instance, the total amount of water in the tanks in Fig. 3 is constant and therefore independent of the control inputs driving the array.

The term “positive controllability” seems to have been coined by Yoshida and his coauthors in the paper [17].
where $x_i \in \mathbb{R}^n$ is the state of the $i$th system with $A \in \mathbb{R}^{n \times n}$. The $\sigma$th (scalar) input is denoted by $u_\sigma \in \mathbb{R}$. The input matrices $B_{i\sigma} \in \mathbb{R}^{n \times 1}$ are assumed to satisfy

$$\sum_{i=1}^q B_{i\sigma} = 0, \quad \sigma = 1, 2, \ldots, p.$$  \hfill (3)

The constraint (3) means that the actuation is relative. Hence the average of the states $x_{i\sigma} = q^{-1} \sum x_i$ evolves independently of the inputs driving the array, i.e., we have $\dot{x}_{i\sigma} = Ax_{i\sigma}$. The shorthand notation $(B_{i\sigma})$ represents the ordered collection $(B_{i\sigma})_{q, \sigma=1}^p$. Given some $d \times n$ matrix $M$, we write $(MB_{i\sigma})$ to mean the collection $(B_{i\sigma})_{q, \sigma=1}^p$ with $B_{i\sigma} = MB_{i\sigma}$. For an index set $I = \{\sigma_1, \sigma_2, \ldots, \sigma_r\} \subset \{1, 2, \ldots, p\}$ the subcollection $(B_{i\sigma})_{\sigma \in I}$ is denoted by $(B_{i\sigma})_{\sigma \in I}$. The corresponding incidence matrix is constructed as

$$\text{inc}(B_{i\sigma})_{\sigma \in I} = \begin{bmatrix} B_{1\sigma_1} & B_{1\sigma_2} & \cdots & B_{1\sigma_r} \\ B_{2\sigma_1} & B_{2\sigma_2} & \cdots & B_{2\sigma_r} \\ \vdots & \vdots & \ddots & \vdots \\ B_{q\sigma_1} & B_{q\sigma_2} & \cdots & B_{q\sigma_r} \end{bmatrix}.$$ 

Definition 1 An array $[A, (B_{i\sigma})]$ is said to be controllable if for each set of initial conditions $(x_1(0), x_2(0), \ldots, x_q(0))$ there exist a finite time $\tau > 0$ and input signals $u_\sigma : [0, \tau] \to \mathbb{R}$ such that $x_\sigma(\tau) = x_j(\tau)$ for all $(i, j)$. The array is said to be positively controllable if the input signals can be chosen to satisfy $u_\sigma : [0, \tau] \to \mathbb{R}_{\geq 0}$.

Definition 2 For a pair of distinct indices $k, \ell \in \{1, 2, \ldots, q\}$ an array $[A, (B_{i\sigma})]$ is said to be $(k, \ell)$-controllable if for each set of initial conditions $(x_1(0), x_2(0), \ldots, x_q(0))$ there exist a finite time $\tau > 0$ and input signals $u_\sigma : [0, \tau] \to \mathbb{R}$ such that $x_k(\tau) = x_j(\tau)$ and $x_\ell(\tau) = e^{A\tau}x_k(0)$ for all $i \neq k, \ell$. The array is said to be positively $(k, \ell)$-controllable if the input signals can be chosen to satisfy $u_\sigma : [0, \tau] \to \mathbb{R}_{\geq 0}$.

Our goal in this paper is to interpret the above definitions in terms of connectivity properties of certain graphs related to the array $[A, (B_{i\sigma})]$. In particular, we characterize (positive) controllability and (positive) $(k, \ell)$-controllability in terms of (strong) connectivity and (strong) $(k, \ell)$-connectivity, respectively. Since our analysis heavily depends on graphs it is worthwhile to recall the relevant basics of graph theory. This we do next.

3 Graph

The next few definitions are borrowed from [3]. The convex cone that is positively spanned by the vectors $g_1, g_2, \ldots, g_p \in \mathbb{R}^q$ is defined as

$$\text{cone}\{g_1, g_2, \ldots, g_p\} = \left\{ \zeta : \zeta = \sum_{\sigma=1}^p \alpha_\sigma g_\sigma, \alpha_\sigma \in \mathbb{R}_{\geq 0} \right\}.$$ 

In other words, $\text{cone}\{g_1, g_2, \ldots, g_p\}$ is the set of all positive combinations of $g_1, g_2, \ldots, g_p$. For $\alpha \in \mathbb{R}^p$ we write $\alpha \geq 0$ to mean $\alpha$ has no negative entry. Likewise, $\alpha \leq 0$ means $-\alpha \geq 0$. The convex cone spanned by the columns of a matrix $G$ is denoted by $\text{cone}G$. That is, cone $G = \{\zeta : \zeta = Ga, \alpha \geq 0\}$. The range and null spaces of $G$ are denoted by range $G$ and null $G$, respectively. The conjugate transpose of $G$ is denoted by $G^*$. (If $G$ is real then $G^*$ is simply the transpose of $G$.) The synchronization subspace is defined as $S_n = \text{range}[I_q \otimes I_n]$, where $I_q$ is the $q$-vector of all ones and $I_n$ is the $n \times n$ identity matrix. $S_n^\perp$ denotes the orthogonal complement of $S_n$. We say $G$ belongs to class-$G_n$ $(G \in G_n)$ if range $G \subset S_n^\perp$. We let $e_1$ be the unit $q$-vector with 1th entry one and the remaining entries zero.

A (directed) graph $\Gamma = (\mathcal{V}, \mathcal{E}, g)$ has a set of vertices (or nodes) $\mathcal{V} = \{v_1, v_2, \ldots, v_q\}$, a set of edges (or arcs) $\mathcal{E} = \{a_1, a_2, \ldots, a_p\}$, and a function $g : \mathcal{E} \to \mathcal{V} \times \mathcal{V}$ that maps each edge to an ordered pair $g(a_\sigma) = (v_i, v_j)$ for some $i \neq j$. We allow parallel edges, i.e., $g$ need not be injective. By slight abuse of notation we sometimes call $(v_i, v_j)$ an edge and write $(v_i, v_j) \in \mathcal{E}$ when some $a_\sigma \in \mathcal{E}$ exists satisfying $g(a_\sigma) = (v_i, v_j)$. Also, we write $-(v_i, v_j)$ to mean $(v_j, v_i)$. A directed path from $v_k$ to $v_\ell$ ($k \neq \ell$) is a sequence of pairs $((v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_r}, v_{i_{r+1}}))$ satisfying $i_1 = k$, $i_{r+1} = \ell$, and $(v_{i_j}, v_{i_{j+1}}) \in \mathcal{E}$ for all $j = 1, 2, \ldots, r$. An undirected path between $v_k$ and $v_\ell$ ($k \neq \ell$) is a sequence of pairs $((v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_r}, v_{i_{r+1}}))$ satisfying $i_1 = k$, $i_{r+1} = \ell$, and for each $j = 1, 2, \ldots, r$ either
(v_i, v_{i+1}) or (v_{i+1}, v_i) belongs to E. We adopt the convention that there is a (un)directed path from each vertex to itself despite that we allow no loop edges (v_i, v_i). For k ≠ ℓ the graph Γ is said to be strongly (k, ℓ)-connected if there exist two directed paths, one from v_k to v_ℓ, the other from v_ℓ to v_k. It is said to be (k, ℓ)-connected if there exists an undirected path between v_k and v_ℓ. It is said to be (strongly) connected if it is (strongly) (k, ℓ)-connected for all (k, ℓ). The incidence matrix [g_{σ}] = G ∈ ℝ^{q×p} of the graph Γ is such that the edge a_{σ} with g(a_{σ}) = (v_i, v_ℓ) is represented by the σth column of G in the following way: g_{σ} = 1, g_{σ} = -1, and the remaining entries of the column are zero. I.e., g(a_{σ}) = (v_i, v_ℓ) implies the column of G equals e_i - e_ℓ. Note that G ∈ G_1 since 1^{∗}G = 0. We now make the following simple observations.

**Proposition 1** Let G ∈ ℝ^{q×p} be the incidence matrix of some graph Γ = (V, E, g). We have the following.

1. The graph Γ is strongly (k, ℓ)-connected if and only if coneG ⊃ range[e_k - e_ℓ].

2. The graph Γ is (k, ℓ)-connected if and only if rangeG ⊃ range[e_k - e_ℓ].

3. The graph Γ is strongly connected if and only if coneG ⊃ S_{k}^{+}.

4. The graph Γ is connected if and only if rangeG ⊃ S_{k}^{+}.

**Proof.** If Γ is strongly (k, ℓ)-connected, a directed path (v_i, v_j, v_k, . . . , v_r) exists with i = k, i_{r+1} = ℓ, and (v_i, v_{i+1}) ∈ E for all j = 1, 2, . . . , r. This implies each e_i - e_{i+1} is a column of G for all j = 1, 2, . . . , r. Since we can write 

\[ (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \cdots + (e_{i+q} - e_{i+q+1}) = e_k - e_\ell \]

we have e_k - e_\ell ∈ cone G. Likewise, the existence of a directed path from v_\ell to v_k yields e_\ell - e_k ∈ cone G. Therefore coneG ⊃ {e_k - e_\ell, e_\ell - e_k}, which implies coneG ⊃ range[e_k - e_\ell].

Let us establish the other direction by contradiction. Suppose there does not exist a directed path from v_k to v_\ell while coneG ⊃ range[e_k - e_\ell]. Let V_k ⊂ V be the set of all vertices to which there is a directed path from v_k. Similarly, let V_\ell ⊂ V be the set of all vertices from which there is a directed path to v_\ell. (Note that v_k ∈ V_k and v_\ell ∈ V_\ell.) Since no directed path exists from v_k to v_\ell, the sets V_k and V_\ell are disjoint. Now define the edge sets E_k = {(v_i, v_j) ∈ E : v_i, v_j ∈ V_k} and E_\ell = {(v_i, v_j) ∈ E : v_i, v_j ∈ V_\ell}, which, too, are disjoint. Let G_k and G_\ell be the incidence matrices of the subgraphs Γ_k = (V_k, E_k, g) and Γ_\ell = (V_\ell, E_\ell, g), respectively. Since the vertices and edges can be relabeled, we can assume, without loss of generality, G has the following block structure.

\[
G = \begin{bmatrix}
G_k & M_- & 0 \\
0 & M_+ & 0 \\
0 & M_+ & G_\ell
\end{bmatrix}
\]

For the ease of discussion G is partitioned in various ways as shown above. Observe that the entries of the matrix M_- (if exists) are either 0 or -1. To see that suppose otherwise, i.e., M_- has an entry g_{σ} = 1. Let j be such that g_{σ} = -1. Since g_{σ} belongs to the center column partition, the edge a_{σ} (satisfying g(a_{σ}) = (v_i, v_\ell)) is not directed to E_k nor to E_\ell. Moreover, v_i ∈ V_k because g_{σ} belongs to the upper row partition. By definition, v_i ∈ V_k implies there is a directed path from v_k to v_i. The existence of the edge (v_i, v_\ell) implies that this path can be extended to v_j. Consequently v_j ∈ V_k. Since both vertices v_i, v_\ell belong to V_k we have to have a_{σ} ∈ E_k, but this we already ruled out. Therefore M_- can have only nonpositive entries.

Since coneG ⊃ range[e_k - e_\ell] we can find a p-vector α ≥ 0 satisfying Gα = e_k - e_\ell. Let q_\ell be the number of vertices in V_\ell and q_k the number of vertices in V_k. We can write

\[
Gα = \begin{bmatrix}
G_k & M_- & 0 \\
0 & M_+ & 0 \\
0 & M_+ & G_\ell
\end{bmatrix} \begin{bmatrix}
α_1 \\
α_2 \\
α_3
\end{bmatrix} = \begin{bmatrix}
f_+ \\
f_0 \\
f_-
\end{bmatrix}
\]

where f_+ ∈ ℝ^{q_\ell} and f_- ∈ ℝ^{q_k}. Since k ≤ q_\ell and ℓ ≥ q_k + 1 > q_k the vector f_+ = [G_k M_- 0]α contains exactly one +1 entry while its other entries are zero. (Likewise, f_- contains exactly one -1 entry while its other entries are zero and f_0 (if exists) is a vector of zeros.) In particular, 1^{∗}q f_+ = 1. Now define
\( \beta = [G_k M \ldots 0]^{\top} \mathbf{1}_{q_k} \). Using \( G_k^* \mathbf{1}_{q_k} = 0 \) (because \( G_k \) is an incidence matrix) and the fact that \( M \ldots \) has no positive entry we can write \( \beta = [G_k M \ldots 0]^{\top} \mathbf{1}_{q_k} = [0 M \ldots 0]^{\top} \mathbf{1}_{q_k} \leq 0 \). Combining \( \beta \leq 0 \) with \( \alpha \geq 0 \) yields \( \beta^* \alpha \leq 0 \). However this results in the following contradiction

\[
1 = \mathbf{1}_{q_k}^* f^* = \mathbf{1}_{q_k}^* [G_k M \ldots 0] \alpha = \beta^* \alpha \leq 0 .
\]

2. Given \( \Gamma = (\mathcal{V}, \mathcal{E}, g) \) with \( \mathcal{V} = \{v_1, v_2, \ldots, v_q\} \) and \( \mathcal{E} = \{a_1, a_2, \ldots, a_p\} \), define the mapping \( g_\alpha : \mathcal{E}_\alpha \rightarrow \mathcal{V} \times \mathcal{V} \) for the augmented edge set \( \mathcal{E}_\alpha = \{a_1, a_2, \ldots, a_{2p}\} \) as follows.

\[
g_\alpha(a_\sigma) = \begin{cases} 
g(a_\sigma) & \text{for } \sigma = 1, 2, \ldots, p \\
-g(a_\sigma) & \text{for } \sigma = p + 1, p + 2, \ldots, 2p .
\end{cases}
\]

Let \( \Gamma_\alpha = (\mathcal{V}, \mathcal{E}_\alpha, g_\alpha) \). It is not hard to see that \( \Gamma \) is \((k, \ell)\)-connected if and only if \( \Gamma_\alpha \) is strongly \((k, \ell)\)-connected. Moreover, the incidence matrix of \( \Gamma_\alpha \) reads \( G_\alpha = [G \ldots G_k] \). Therefore \( \text{cone } G_\alpha = \text{range } G \).

Using the first statement of the proposition we can now write

\[
\Gamma \ (k, \ell)\text{-connected} \iff \Gamma_\alpha \text{ strongly } (k, \ell)\text{-connected}
\]

\[
\iff \text{cone } G_\alpha \supset \text{range } [e_k - e_\ell]
\]

\[
\iff \text{range } G \supset \text{range } [e_k - e_\ell] .
\]

3. If \( \Gamma \) is strongly connected, then, by definition, \( \Gamma \) is strongly \((k, \ell)\)-connected for all pairs \((k, \ell)\). The first statement of the proposition then allows us to write

\[
\text{cone } G \supset \sum_{k, \ell} \text{range } [e_k - e_\ell] = S^+_1 .
\]

If \( \Gamma \) is not strongly connected, then there exists a pair \((k, \ell)\) for which \( \text{cone } G \not\supset \text{range } [e_k - e_\ell] . \) Since \( S^+_1 \supset \text{range } [e_k - e_\ell] \) we have to have \( \text{cone } G \not\supset S^+_1 \).  

4. The result follows from the second statement. The demonstration is similar to that of the third statement.  \[\blacksquare\]

**Proposition 4** motivates us for the following generalization.

**Definition 3** A **class-\(G_n\) matrix** \( G \in \mathbb{C}^{(q_n) \times p} \) is said to be:

- **strongly \((k, \ell)\)-connected** if \((G \text{ is real and})\) \( \text{cone } G \supset \text{range } [(e_k - e_\ell) \otimes I_n] \),
- **\((k, \ell)\)-connected** if \( \text{range } G \supset \text{range } [(e_k - e_\ell) \otimes I_n] \),
- **strongly connected** if \((G \text{ is real and})\) \( \text{cone } G \supset S^+_1 \),
- **connected** if \( \text{range } G \supset S^+_1 \).

A brief digression is in order here. **Definition 3** intends to extend connectivity, a central notion for graphs, to class-\(G_n\) matrices, which may be taken to represent (or be) generalized graphs. In this representation each column of \( G \in \mathcal{G}_n \) may be treated as an edge. Then a path between \( k \)th and \( \ell \)th vertices may be said to exist if for each \( \eta \in \mathbb{R}^n \) we can find some edges (columns) \( g_\sigma \) and some weights \( \alpha_\sigma \) that take us from vertex \( k \) to vertex \( \ell \) by satisfying \( e_k \otimes \eta + \alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_r g_r = e_\ell \otimes \eta . \) (For a directed path we would require the weights to be positive.) Depicting the endpoints \( e_k \otimes \eta \) and \( e_\ell \otimes \eta \) as dots (in space they belong to) and the vectors \( \alpha_\sigma g_\sigma \) as successive line segments connecting the two dots, a geometric interpretation can be obtained. Hence, although it would be difficult (provided it is possible/meaningful) to draw a generalized graph, the notion of connectivity seems to maintain to some degree its visual feature.

**Definition 3** has an interesting claim on hypergraphs\(^4\) which can be observed on a simple instance. Consider the array \((1)\) of water tanks under the constraint \( u_0 = 2u_1 \) which arises, say, because the voltages driving the water pumps are not independent. Then the dynamics reads \([\dot{x}_1 \dot{x}_2 \dot{x}_3]^* = [1 1 -2]^* u_1 \). Let now \( G = [1 1 -2]^* \in \mathcal{G}_1 \) represent the incidence matrix of a 3-vertex graph; call this graph \( \Gamma \). Since \( G \) has a single column, \( \Gamma \) has a single edge. This edge is incident to all the vertices, for the corresponding column has no zero entries. Therefore \( \Gamma \) is a 3-vertex hypergraph with a single (hyper)edge that is incident to all three vertices. According to the classical definition of connectivity for hypergraphs, \( \Gamma \)

\(^4\) A hypergraph is such that each edge is allowed to be incident to more than two vertices.
is connected because any two vertices are adjacent to one another through the only edge. According to Definition [3] however, G (therefore Γ) is not connected. In fact, no two vertices are connected since range\( G \not\supset [e_{\ell} - e_k] \) for all pairs \((k, \ell)\). To support Definition [3] against this discrepancy let us first obtain the Laplacian matrix \( L \) of \( \Gamma \) from the incidence matrix \( G \) as
\[
L = G G^* = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}.
\]
Then, treating \( L \) as the node admittance matrix of a resistive network, we obtain the conductances \( \gamma_{ij} \) between the nodes \( i \) and \( j \) through
\[
\begin{bmatrix} \gamma_{12} + \gamma_{31} & -\gamma_{12} & -\gamma_{31} \\ -\gamma_{12} & \gamma_{23} + \gamma_{12} & -\gamma_{23} \\ -\gamma_{31} & -\gamma_{23} & \gamma_{31} + \gamma_{23} \end{bmatrix} = L
\]
as \( \gamma_{12} = -1 \Omega, \gamma_{23} = 2 \Omega, \) and \( \gamma_{31} = 2 \Omega \). These values yield the simple delta network in Fig. [5]. Now, the connectivity of this network can be determined via the circuit theory tool effective conductance. Namely, the vertices \( k \) and \( \ell \) of the graph \( \Gamma \) is connected if the effective conductance between the nodes \( k \) and \( \ell \) of the corresponding resistive network in Fig. [5] is positive. A quick calculation shows that the effective conductance \( \) for any pair of nodes is zero for this network. Hence no two vertices of \( \Gamma \) is connected, just as Definition [3] predicates.

In the remainder of the paper we attempt to interpret different controllability aspects of the array (2) in terms of (strong) connectivity properties of certain class-\( G_n \) matrices.

4 Controllability
Consider the array (2). By \( \mu_1, \mu_2, \ldots, \mu_m \) we denote the distinct eigenvalues of \( A^* \). Note that \( m \leq n \) and these eigenvalues are shared by \( A \). For each \( \kappa \in \{1, 2, \ldots, m\} \) we let \( V_\kappa \in \mathbb{C}^{n \times d_\kappa} \) denote a full column rank matrix satisfying range\( V_\kappa = \text{null}[A^* - \mu_\kappa I_n] \). Therefore \( d_\kappa \) is the geometric multiplicity of the eigenvalue \( \mu_\kappa \). We let \( V_\kappa \in \mathbb{R}^{n \times d_\kappa} \) when \( \mu_\kappa \in \mathbb{R} \). Note that the columns of \( V_\kappa \) are the linearly independent eigenvectors of \( A^* \) corresponding to the eigenvalue \( \mu_\kappa \). In particular, we have \( A^* V_\kappa = \mu_\kappa V_\kappa \). For notational convenience we sometimes represent the array (2) as a single big system
\[
\dot{x} = Ax + Bu
\]
where \( x = [x_1^* \quad x_2^* \quad \cdots \quad x_q^*]^* \) is the state and \( u = [u_1^* \quad u_2^* \quad \cdots \quad u_p^*]^* \) is the input. Clearly, we have
\[
A = [I_q \otimes A]
\]
while \( B \in \mathbb{R}^{(qn) \times p} \) has the structure
\[
B = \text{inc} \left( B_{\cdot \cdot} \right) = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ B_{q1} & B_{q2} & \cdots & B_{qp} \end{bmatrix}.
\]
Note that \( B \in G_n \) due to (3) and \( \text{inc} \left( V_\kappa^* B_{\cdot \cdot} \right) = [I_q \otimes V_\kappa^* B] \). The controllability matrix associated to the pair \( [A, B] \) reads \( [B \ AB \ \cdots \ A^{q-1}B] \). However, being equal to \( [I_q \otimes A] \), the matrix \( A \) satisfies

![Figure 5: The delta network with node admittance matrix L.](image-url)
the characteristic equation of $A$ (which is of order $n$) and therefore the controllability index for the pair $[A, B]$ is at most $n$. Hence we can treat

$$W = [B \ AB \ \cdots \ A^{n-1}B]$$

as the controllability matrix. Indeed, range $W$ is the controllable subspace for the pair $[A, B]$. Observe $W \in G_n$. Recall that the vector $1_q$ spans the synchronization subspace $S_1$. Let $S$ denote its normalized version, i.e., $S = 1_q/\sqrt{q}$ and hence $S^*S = 1$. Also, let $D \in \mathbb{R}^{q \times (q-1)}$ be some matrix whose columns make an orthonormal basis for $S^\perp$, sometimes called the disagreement subspace. Note that $D^*D = I_{q-1}$ and the columns of the matrix $[D \ S]$ make an orthonormal basis for $\mathbb{R}^n$. The following identities are easy to show and find frequent use in the sequel.

\begin{enumerate}
\item $DD^* + SS^* = I_q$.
\item range $[D \otimes I_n] = S_n^\perp$.
\item null $[D^* \otimes I_n] = S_n$.
\item $[S^* \otimes I_n]B = 0$.
\end{enumerate}

The distance of $x$ to $S_n$ we denote by $\|x\|_{S_n} = ||[D^* \otimes I_n]x||$. Finally, we define the reduced parameters

$$A_r = [I_{q-1} \otimes A]$$
$$B_r = [D^* \otimes I_n]B$$
$$W_r = [B_r \ A_r B_r \ \cdots \ A_r^{n-1}B_r].$$

The controllability index for the pair $[A_r, B_r]$ is at most $n$. Hence range $W_r$ equals the controllable subspace associated to the pair $[A_r, B_r]$.

**Lemma 1** The following are equivalent.

1. The array $[A, (B_r)]$ is controllable.
2. range $W \supset S_n^\perp$.
3. The pair $[A_r, B_r]$ is controllable.

**Proof.** $1 \implies 3$. Suppose $[A, (B_r)]$ is controllable. For the system (4) this means every initial condition $x(0)$ can be driven to $S_n$ in finite time. Consider the system $\dot{x}_r = A_r x_r + B_r u_r$. Choose an arbitrary initial condition $x_r(0) \in (\mathbb{R}^n)^{q-1}$. Now set the initial condition of the system (4) as $x(0) = [D \otimes I_n]x_r(0)$. Then let $u : [0, \tau] \rightarrow \mathbb{R}^p$ be an input signal that yields $\|x(\tau)\|_{S_n} = 0$ for some finite $\tau > 0$. Such input signal exists thanks to the controllability of the pair $[A, (B_r)]$. Let $u_r(t) = u(t)$ for $t \in [0, \tau]$. Recall that $\|x(\tau)\|_{S_n} = 0$ means $[D^* \otimes I_n]x(\tau) = 0$. We can write

$$x_r(\tau) = e^{A_r\tau} x_r(0) + \int_0^\tau e^{A_r(\tau-t)}B_r u_r(t)dt$$

$$= [I_{q-1} \otimes e^{A\tau}]x_r(0) + \int_0^\tau [I_{q-1} \otimes e^{A(\tau-t)}]B_r u_r(t)dt$$

$$= [D^* D \otimes e^{A\tau}]x_r(0) + \int_0^\tau [I_{q-1} \otimes e^{A(\tau-t)}][D^* \otimes I_n]B_r u_r(t)dt$$

$$= [D^* \otimes I_n]q \text{ and hence } 0 \text{ means } [D^* \otimes I_n]x(\tau) = 0. \text{ We can write}$$

$$x_r(\tau) = e^{A_r\tau} x_r(0) + \int_0^\tau e^{A_r(\tau-t)}B_r u_r(t)dt$$

$$= [D^* \otimes I_n]x_r(0) + [D^* \otimes I_n] \int_0^\tau [I_q \otimes e^{A(\tau-t)}]B u(t)dt$$

$$= [D^* \otimes I_n]e^{A\tau} x(0) + \int_0^\tau e^{A(\tau-t)}B u(t)dt$$

$$= [D^* \otimes I_n]e^{A\tau} x(0) + \int_0^\tau e^{A(\tau-t)}B u(t)dt$$

$$= 0.$$
3 $\implies$ 2. Suppose $[A_r, B_r]$ is controllable. Observe that we can write for all $k \in \{0, 1, \ldots, n - 1\}$

\[
A_r^k B_r = [I_{q-1} \otimes A]^k [D^* \otimes I_n]B \\
= [I_{q-1} \otimes A^k] [D^* \otimes I_n]B \\
= [D^* \otimes I_n] [I_q \otimes A^k]B \\
= [D^* \otimes I_n] [I_q \otimes A]^k B \\
= [D^* \otimes I_n] A^k B
\]

yielding $W_t = [D^* \otimes I_n] W$. Similarly, using $[S^* \otimes I_n]B = 0$ we can write

\[
[S^* \otimes I_n] A^k B = [S^* \otimes I_n] [I_q \otimes A^k]B \\
= A^k [S^* \otimes I_n] B \\
= 0
\]

yielding $[S^* \otimes I_n] W = 0$. Since $[A_r, B_r]$ is controllable, $W_t$ is full column rank, which allows us to write

\[
\text{range } W = \text{range } (D^* \otimes I_n) W_t = \text{range } (D \otimes I_n) W_t
\]

Let us now gather our recent findings and obtain

\[
\text{range } W = \text{range } (D \otimes I_n) W_t = S_n^\perp.
\]

Therefore $\text{range } W \supset S_n^\perp$.

2 $\implies$ 1. Suppose $\text{range } W \supset S_n^\perp$. Consider the system \([A, B]\). Let $x(0)$ be an arbitrary initial condition. Then let $\xi = (DD^* \otimes I_n)x(0)$. Note that $\xi \in S_n^\perp$. Therefore $\xi$ belongs to the controllable subspace $\text{range } W$ associated to the pair $[A, B]$. Consequently we can find an input signal $u : [0, \tau] \to \mathbb{R}^p$ with finite $\tau > 0$ that satisfies

\[
e^{A\tau} \xi + \int_0^\tau e^{A(t-t)} B u(t) dt = 0.
\]

Now using this control signal and the identity $D^* S = 0$ we can write

\[
[D^* \otimes I_n] x(t) = [D^* \otimes I_n] \left(e^{A\tau} x(0) + \int_0^\tau e^{A(t-t)} B u(t) dt\right) \\
\]

\[
= [D^* \otimes I_n] \left((DD^* \otimes S^*) \otimes e^{A\tau} x(0) + \int_0^\tau e^{A(t-t)} B u(t) dt\right) \\
\]

\[
= [D^* \otimes I_n] \left([I_q \otimes e^{A\tau}] [DD^* \otimes I_n] x(0) + \int_0^\tau e^{A(t-t)} B u(t) dt\right) \\
\]

\[
= [D^* \otimes I_n] \left(e^{A\tau} \xi + \int_0^\tau e^{A(t-t)} B u(t) dt\right) \\
\]

That is, $\|x(t)\|_{S_n} = 0$. The array $[A, (B_z)]$ then has to be controllable because $x(0)$ was arbitrary.

\[\blacksquare\]

**Theorem 1** The following are equivalent.

1. The array $[A, (B_z)]$ is controllable.

2. The controllability matrix $W$ is connected.

3. All the matrices $\text{inc}(V_1^* B_z), \text{inc}(V_2^* B_z), \ldots, \text{inc}(V_m^* B_z)$ are connected.
Proof. 1 $\iff$ 2. By definition $W$ connected means range $W \supset S_n^+$, which is equivalent to the controllability of the array $[A, (B_*)]$ by Lemma 1.

2 $\implies$ 3. Suppose for some $\kappa \in \{1, 2, \ldots, m\}$ the matrix $\text{inc}(V_*^*B_*)$ is not connected. Recall $\text{inc}(V_*^*B_*) = [I_q \otimes V_*]B$. Now, range $[I_q \otimes V_*]B \not\subset S_d^+$ implies null $B^*[I_q \otimes V_*] \not\subset S_{d_\kappa}$. Therefore there exists a vector $f \notin S_{d_\kappa}$ satisfying $[I_q \otimes V_*]f = 0$. Define $\xi = [I_q \otimes V_*]f$. We have $\xi \notin S_n$ because $f \notin S_{d_\kappa}$ and $V_\kappa$ is full column rank. Recall $A^*V_\kappa = \mu_\kappa V_\kappa$. This allows us to write $A^*[I_q \otimes V_*] = \mu_\kappa[I_q \otimes V_*]$. We can now proceed as

$$W^* \xi = \begin{bmatrix} B^* \\ B^* A^* \\ \vdots \\ B^* A^{(n-1)*} \end{bmatrix} [I_q \otimes V_*]f = \begin{bmatrix} B^*[I_q \otimes V_*]f \\ \mu_\kappa B^*[I_q \otimes V_*]f \\ \vdots \\ \mu_\kappa^{n-1} B^*[I_q \otimes V_*]f \end{bmatrix} = 0.$$ 

Since $\xi \notin S_n$ we deduce null $W^* \not\subset S_n$. Hence range $W \not\subset S_n^+$, i.e., $W$ is not connected. 

3 $\implies$ 1. Suppose the array $[A, (B_*)]$ is not connectable. Then by Lemma 1 the pair $[A_\kappa, B_\kappa]$ is not controllable. Thanks to PBH test this implies that there exists an eigenvector $\eta \in (C^n)^{r-1}$ of $A_\kappa^*$ satisfying $B_\kappa^* \eta = 0$. Note that $A_\kappa^*$ and $A^*$ share the same eigenvalues. Therefore for some $\kappa \in \{1, 2, \ldots, m\}$ we have $[A_\kappa^* - \mu_\kappa I_{(q-1)n}] \eta = 0$. Since null $[A_\kappa^* - \mu_\kappa I_{(q-1)n}] = \text{range}[I_{q-1} \otimes V_\kappa]$ there exists $h \in (C^{d_\kappa})^{r-1}$ satisfying $[I_{q-1} \otimes V_\kappa]h = \eta$. Note that $\eta$ is nonzero because it is an eigenvector. Therefore $h \neq 0$. Now define $g = [D \otimes I_{d_\kappa}]h$. We have $g \neq 0$ because $h$ is nonzero and $[D \otimes I_{d_\kappa}]$ is full column rank. Then the inclusion $g \in \text{range}[D \otimes I_{d_\kappa}] = S_{d_\kappa}^+$ allows us to write $g \notin S_{d_\kappa}$. Moreover,

$$B^*[I_q \otimes V_*]g = B^*[I_q \otimes V_*][D \otimes I_{d_\kappa}]h = B^*[D \otimes I_n][I_{q-1} \otimes V_\kappa]h = B^* \eta = 0.$$ 

Hence null $B^*[I_q \otimes V_*] \not\subset S_{d_\kappa}$, yielding range $[I_q \otimes V_*]B \not\subset S_{d_\kappa}^+$, i.e., $\text{inc}(V_*^*B_*)$ is not connected.

An example. Using Theorem 1 let us now study controllability of each of the two arrays of electrical oscillators shown in Fig. 6, where all inductances are 1H and all capacitances are 1F.

Figure 6: Arrays of electrical oscillators.

Let, for the $i$th oscillator, $y_i \in \mathbb{R}^5$ be the vector of inductor currents and $v_i \in \mathbb{R}^5$ be the vector of node voltages. We can then write

$$C\dot{v}_1 + y_1 = e_2 u_1 - e_k u_3, \quad \dot{C}v_1 = v_1$$
$$C\dot{v}_2 + y_2 = e_3 u_2 - e_2 u_1, \quad \dot{\dot{C}}v_2 = v_2$$
$$C\dot{v}_3 + y_3 = e_k u_3 - e_3 u_2, \quad \dot{\dot{C}}v_3 = v_3$$

where $e_2, e_3, e_k \in \mathbb{R}^5$, $e_k = e_5$ for the array in Fig. 6a, $e_k = e_4$ for the array in Fig. 6b, $L = I_5$, and

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$
Consider the system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

where \( A \) and \( B \) are matrices. Proposition 2 of the paper states that if there exists a matrix \( \mu \) such that

\[ \lambda \in \text{range}(A - \mu B) \]

then the system is controllable. Recall that

\[ \text{Positive controllability} \]

implies that there exists an input signal \( u(t) \) such that \( x(t) \) is controllable, but the array in Fig. 6b is not.

Now, defining the state of the \( i \)-th oscillator as \( x_i = [v_i^* \ y_i^*]^T \) we can rewrite the oscillator dynamics as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
B_{11} & B_{22} & 0 \\
0 & B_{12} & B_{23} \\
0 & 0 & B_{33}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

with

\[
A = \begin{bmatrix}
0 & -C^{-1} \\
L^{-1} & 0
\end{bmatrix}, \quad B_{11} = \begin{bmatrix}
C^{-1} e_2 \\
0
\end{bmatrix}, \quad B_{22} = \begin{bmatrix}
C^{-1} e_3 \\
0
\end{bmatrix}, \quad B_{33} = \begin{bmatrix}
C^{-1} e_k \\
0
\end{bmatrix}.
\]

The matrix \( A^* \in \mathbb{R}^{10 \times 10} \) has five conjugate pairs of eigenvalues: \( \mu_{1,2} = \pm j \sqrt{\tan(5\pi/12)} \), \( \mu_{3,4} = \pm j1 \), \( \mu_{5,6} = \pm j \sqrt{1/2} \), \( \mu_{7,8} = \pm j \sqrt{1/3} \), \( \mu_{9,10} = \pm j \sqrt{\tan(\pi/12)} \). Each eigenvalue \( \mu_\kappa \) admits an eigenvector \( V_\kappa \in \mathbb{C}^{10} \) and each eigenvector generates a class-\( G_1 \) matrix inc(\( V_\kappa B_{\cdot \cdot} \)) \( \in \mathbb{C}^{3 \times 3} \). Now, each inc(\( V_\kappa B_{\cdot \cdot} \)) is a (weighted\( \square \)) incidence matrix of a 3-vertex graph, which is connected when the matrix inc(\( V_\kappa B_{\cdot \cdot} \)) is connected. These graphs for each of the arrays in Fig. 6 are given in Table 1. Observe that all the graphs of the array in Fig. 6a are connected; whereas, for the array in Fig. 6b, the graph corresponding to the eigenvalue pair \( \mu_{5,6} = \pm j \sqrt{1/2} \) is not connected. By Theorem 1 therefore the array in Fig. 6a is controllable, but the array in Fig. 6b is not.

5 Positive controllability

Recall that \( A^* \) and \( A^t \) share the same eigenvalues and null[\( A^* - \mu \kappa I_{(q-1)n} \)] = range[\( I_{q-1} \otimes V_\kappa \)] for \( \kappa = 1, 2, \ldots, m \). The below result is \( \square \) Cor. 1 expressed in our notation.

**Proposition 2** Consider the system \( \dot{x}_t = A_t x_t + B_t u_t \). Suppose for each initial condition \( x_t(0) \) there exists an input signal \( u_t : [0, \tau] \rightarrow \mathbb{R}_{\geq 0}^d \) with some finite \( \tau > 0 \) that achieves \( x_t(\tau) = 0 \). Then and only then the following two conditions simultaneously hold.

1. The pair [\( A_t, B_t \)] is controllable.
2. cone[\( I_{q-1} \otimes V_\kappa \)]B_t = (\( \mathbb{R}^d \))^\kappa-1 for all \( \mu_\kappa \in \mathbb{R} \).

A pair [\( A_t, B_t \)] is said to be positively controllable if it satisfies the conditions in Proposition 2.

**Lemma 2** The following are equivalent.

1. The array [\( A_t, (B_{\cdot \cdot}) \)] is positively controllable.
2. The pair [\( A_t, B_t \)] is positively controllable.

\( \square \) That is, each column is of the form \( w_{ij}(e_i - e_j) \) with \( w_{ij} \in \mathbb{C} \).

| Array | inc(\( V_{1,2} B_{\cdot \cdot} \)) | inc(\( V_{3,4} B_{\cdot \cdot} \)) | inc(\( V_{5,6} B_{\cdot \cdot} \)) | inc(\( V_{7,8} B_{\cdot \cdot} \)) | inc(\( V_{9,10} B_{\cdot \cdot} \)) |
|-------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| (a)   | ![Diagram](a)                | ![Diagram](a)                | ![Diagram](a)                | ![Diagram](a)                | ![Diagram](a)                |
| (b)   | ![Diagram](b)                | ![Diagram](b)                | ![Diagram](b)                | ![Diagram](b)                | ![Diagram](b)                |

**Table 1**: The graphs associated to the arrays in Fig. 6.
Proof. 1 \implies 2. Suppose \([A, (B, \cdot)]\) is positively controllable. For the system (4) this means each initial condition \(x(0)\) can be driven to \(S_n\) in finite time with some nonnegative input signal. Consider the system \(x_t = A_t x_t + B_t u_t\). Choose an arbitrary initial condition \(x_t(0) \in (\mathbb{R}^n)^{q-1}\). Set the initial condition of the system (4) as \(x(0) = [D \otimes I_n]x_t(0)\). Then let \(u : [0, \tau] \to \mathbb{R}_{\geq 0}^d\) be an input signal that yields \(\|x(\tau)\|_{S_n} = 0\) for some finite \(\tau > 0\). In the proof of Lemma 1 we discovered that the input signal \(u_t(t) = u(t)\) for \(t \in [0, \tau]\) renders \(x_t(\tau) = 0\). Hence the pair \([A, B]\) must be positively controllable because \(x_t(0)\) was arbitrary.

2 \implies 1. Suppose \([A_t, B_t]\) is positively controllable. Consider the system (4). Let \(x(0)\) be an arbitrary initial condition. Then let \(x_t(0) = [D^* \otimes I_n]x(0)\) be the initial condition for the system \(x_t = A_t x_t + B_t u_t\). Thanks to positive controllability of \([A, B]\) we can find an input signal \(u_t : [0, \tau] \to \mathbb{R}_{\geq 0}^d\) with finite \(\tau > 0\) that satisfies

\[e^{A_t x_t(0)} + \int_0^\tau e^{A_t(t-t)}B_t u_t(t) dt = 0.\]

Using this control signal to drive the system (4), i.e., \(u(t) = u_t(t)\) for \(t \in [0, \tau]\), we can write

\[\|x(\tau)\|_{S_n} = 0.\]

That is, \(\|x(\tau)\|_{S_n} = 0\). Hence \([A, (B, \cdot)]\) has to be positively controllable because \(x(0)\) was arbitrary.

Lemma 3 Let \(\mu_\kappa \in \mathbb{R}\). The following are equivalent.

1. \(\text{inc}(V^*_\kappa B_{\cdot})\) is strongly connected.

2. \(\text{cone}[I_{q-1} \otimes V^*_\kappa B_t] = (\mathbb{R}^d)^{q-1}\).

Proof. 1 \implies 2. Suppose \(\text{inc}(V^*_\kappa B_{\cdot})\) is strongly connected. That is, \(\text{cone}[I_{q-1} \otimes V^*_\kappa B_t] \supset S^\perp_{dx}\). Let \(g \in (\mathbb{R}^d)^{q-1}\) be arbitrary. Define \(f = [D \otimes I_{dx}]g\). Note that \(f \in S^\perp_{dx}\). Hence we can find a nonnegative vector \(\alpha \in \mathbb{R}_{\geq 0}^{dx}\) satisfying \([I_{q-1} \otimes V^*_\kappa B_t]B_t \alpha = f\). We can now write

\[\|x(\tau)\|_{S_n} = 0.\]

Hence \(g \in \text{cone}[I_{q-1} \otimes V^*_\kappa B_t]\). Since \(g\) was arbitrary we have \(\text{cone}[I_{q-1} \otimes V^*_\kappa B_t] = (\mathbb{R}^d)^{q-1}\).

2 \implies 1. Suppose \(\text{cone}[I_{q-1} \otimes V^*_\kappa B_t] = (\mathbb{R}^d)^{q-1}\). Choose an arbitrary vector \(h\) belonging to \(\in S^\perp_{dx}\). Since \(h \in S^\perp_{dx}\) we can find \(b \in (\mathbb{R}^d)^{q-1}\) satisfying \([D \otimes I_{dx}]b = h\). Then we can find a nonnegative vector \(\beta \in \mathbb{R}_{\geq 0}^{dx}\) satisfying \([I_{q-1} \otimes V^*_\kappa]B_t \beta = b\). We can now write (recalling \([S^* \otimes I_n]B = 0\))

\[I_q \otimes V^*_\kappa B \beta = [I_q \otimes V^*_\kappa][(DD^* + SS^*) \otimes I_n]B \beta = [I_q \otimes V^*_\kappa][DD^* \otimes I_n]B \beta = [D \otimes I_{dx}]B \beta = h.\]

This shows that \(\text{cone}[I_q \otimes V^*_\kappa]B \supset S^\perp_{dx}\). ■

Proposition 2, Lemma 1, Lemma 2 and Lemma 3 yield:
Theorem 2 The array $[A, (B_\kappa)]$ is positively controllable if and only if the following two conditions simultaneously hold.

1. The array $[A, (B_\kappa)]$ is controllable.
2. $\text{inc}(V_\kappa^* B_\kappa)$ is strongly connected for all $\mu_\kappa \in \mathbb{R}$.

6 Pairwise controllability

Let the integers $n_1, n_2, \ldots, n_m$ be the algebraic multiplicities of the distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$, respectively. Hence the characteristic polynomial of $A^*$ reads $(s - \mu_1)^{n_1}(s - \mu_2)^{n_2} \cdots (s - \mu_m)^{n_m}$ with $n_1 + n_2 + \cdots + n_m = n$. For $\kappa = 1, 2, \ldots, m$ let $U_\kappa \in \mathbb{C}^{n \times n}$ be a full column rank matrix satisfying range$U_\kappa = \text{null}[A^* - \mu_\kappa I_n]^{n_\kappa}$. Without loss of generality we let $U_\kappa$ be real when the corresponding eigenvalue $\mu_\kappa$ is real. Since range$U_\kappa$ is invariant with respect to $A^*$, for each $\kappa$ there exists a square matrix $A_\kappa \in \mathbb{C}^{n \times n}$ satisfying $A^* U_\kappa = U_\kappa A_\kappa^*$. Note that each $A_\kappa^*$ has a single distinct eigenvalue. In other words, $(s - \mu_\kappa)^{n_\kappa}$ is the characteristic polynomial of $A_\kappa^*$. Define $B_{i\kappa}^* = U_\kappa B_{i\kappa}$ and construct the following controllability matrix

$$W_{i\kappa}^* = [B_{i\kappa}^* A_{i\kappa} B_{i\kappa}^* \cdots A_{i\kappa}^{n_{i\kappa}} B_{i\kappa}^*].$$

Lemma 4 We have $\text{null} [\text{inc}(W_{i\kappa}^*)]^* = \text{null} W_{i\kappa}^*[I_q \otimes U_\kappa]$ for all $\kappa = 1, 2, \ldots, m$.

Proof. Let $B_\kappa = [I_q \otimes U_\kappa]B$ and $A_\kappa = [I_q \otimes A_\kappa]$. Let $W_\kappa = [B_\kappa A_\kappa B_\kappa \cdots A_{i\kappa}^{n_{i\kappa}} B_\kappa]$ and define its augmented version as $\tilde{W}_\kappa = [B_\kappa A_\kappa B_\kappa \cdots A_{i\kappa}^{n_{i\kappa}} B_\kappa]$. Observe that $A^* U_\kappa = U_\kappa A_\kappa^*$ implies $A^* U_\kappa = U_\kappa A_\kappa^*$ for any nonnegative integer $r$. We can therefore write

$$A^{*r} [I_q \otimes U_\kappa] = [I_q \otimes A^{*r}] [I_q \otimes U_\kappa].$$

Whence follows

$$W_{i\kappa}^*[I_q \otimes U_\kappa] = \begin{bmatrix} B_{i\kappa}^* \\ B_{i\kappa}^* A_{i\kappa} \\ \vdots \\ B_{i\kappa}^* A_{i\kappa}^{(n_{i\kappa}-1)} \end{bmatrix} [I_q \otimes U_\kappa] = \begin{bmatrix} B_{i\kappa}^*[I_q \otimes U_\kappa] \\ B_{i\kappa}^*[I_q \otimes U_\kappa] A_{i\kappa}^{(n_{i\kappa})} \\ \vdots \\ B_{i\kappa}^*[I_q \otimes U_\kappa] A_{i\kappa}^{(n_{i\kappa}-1)} \\ B_{i\kappa}^*[I_q \otimes U_\kappa] A_{i\kappa}^{(n_{i\kappa})} \end{bmatrix}.$$

(5)

Since $A_{i\kappa}^*$ satisfies the characteristic equation of $A_{\kappa}^*$ (which is of order $n_\kappa \leq n$) we have

$$\text{null} \tilde{W}_{i\kappa}^* = \text{null} W_{i\kappa}^*.$$

Moreover, carrying out the multiplication explicitly one can obtain the identity $W_{\kappa} W_{\kappa}^* = \text{inc}(W_{\kappa}^*[\kappa]) \times [\text{inc}(W_{\kappa}^*[\kappa])]^*$ which yields

$$\text{null} W_{\kappa}^* = \text{null} [\text{inc}(W_{\kappa}^*[\kappa])]^*.$$

(7)

Combining (6), (7), and (8) yields the result.

Lemma 5 The following are equivalent.

1. The array $[A, (B_\kappa)]$ is $(k, \ell)$-controllable.
2. range$W \supset$ range$[(e_k - e_\ell) \otimes I_n]$. 


Proof. 1 \implies 2. Suppose \([A, (B_\ell)]\) is \((k, \ell)\)-controllable. Consider the system [4]. Choose an arbitrary initial condition \(x(0) = [x_1(0)^* x_2(0)^* \cdots x_q(0)^*]^T \in \text{range}[(e_k - e_\ell) \otimes I_n]\). Clearly, we have \(x_i(0) = 0\) for \(i \neq k, \ell\). Moreover, \(x_{av}(0) = 0\). Let now \(u : [0, \tau] \to \mathbb{R}^p\) be some control signal (with \(\tau > 0\) finite) that achieves \(x_k(\tau) - x_i(\tau) = 0\) and \(x_i(\tau) = e^{A\tau}x_i(0) = 0\) for \(i \neq k, \ell\). Such \(u\) exists because \([A, (B_\ell)]\) is \((k, \ell)\)-controllable. Recall that \(\dot{x}_{av} = Ax_{av}\) because the actuation is relative [3]. Hence \(x_{av}(\tau) = e^{A\tau}x_{av}(0) = 0\). We can therefore write
\[
x_k(\tau) + x_i(\tau) = x_k(\tau) + x_\ell(\tau) + \sum_{i \neq k, \ell} x_i(\tau)
\]
\[
= qx_{av}(\tau)
\]
\[
= 0.
\]
Recall \(x_k(\tau) - x_i(\tau) = 0\). Hence \(x_k(\tau) + x_\ell(\tau) = 0\) means \(x_k(\tau) = x_\ell(\tau) = 0\) yielding \(x(\tau) = 0\). This implies for the system [4] that any initial condition from the set range \([e_k - e_\ell] \otimes I_n\] can be driven to the origin in finite time. This set then must be contained in the controllable subspace. In other words, range \(W \supseteq \text{range}[(e_k - e_\ell) \otimes I_n]\).

2 \implies 1. Suppose range \([e_k - e_\ell] \otimes I_n\] is contained in range \(W\), the controllable subspace of the system [4]. Let us be given an arbitrary initial condition \(x(0) = [x_1(0)^* x_2(0)^* \cdots x_q(0)^*]^T\). Let \(z = (x_k(0) - x_i(0))/2\). Then let \(\tilde{u} : [0, \tau] \to \mathbb{R}^p\) be some control signal (with \(\tau > 0\) finite) that steers the initial condition \(\tilde{x}(0) = (e_k - e_\ell) \otimes z\) to the origin \(\tilde{x}(\tau) = 0\). Such \(\tilde{u}\) exists because \(\tilde{x}(0) \in \text{range}[(e_k - e_\ell) \otimes I_n]\) belongs to the controllable subspace. In particular, we can write
\[
\int_0^\tau e^{A(\tau-t)}\tilde{u}(t)dt = -e^{A\tau}\tilde{x}(0).
\]
Using the same input for the initial condition \(x(0)\) yields
\[
x(\tau) = e^{A\tau}x(0) + \int_0^\tau e^{A(\tau-t)}\tilde{u}(t)dt
\]
\[
= e^{A\tau}x(0) - e^{A\tau}\tilde{x}(0)
\]
\[
= e^{A\tau}(x(0) - \tilde{x}(0))
\]
\[
= [I_q \otimes e^{A\tau}](x(0) - \frac{1}{2}[(e_k - e_\ell) \otimes (x_k(0) - x_i(0))]).
\]
It is now easy to verify \(x_k(\tau) = x_\ell(\tau) = e^{A\tau}(x_k(0) + x_\ell(0))/2\) and \(x_i(\tau) = e^{A\tau}x_i(0)\) for \(i \neq k, \ell\). □

Theorem 3 The following are equivalent.
1. The array \([A, (B_\ell)]\) is \((k, \ell)\)-controllable.
2. The controllability matrix \(W\) is \((k, \ell)\)-connected.
3. All the matrices inc \((W_1, 1)\), inc \((W_2, 2)\), \ldots, inc \((W_m, m)\) are \((k, \ell)\)-connected.

Proof. 1 \iff 2. By Lemma [5]

2 \implies 3. Suppose for some \(\kappa \in \{1, 2, \ldots, m\}\) the matrix inc \((W_\kappa, \kappa)\) is not \((k, \ell)\)-connected. That is, \(\text{range}[\text{inc}(W_\kappa)] \not\supseteq \text{range}[(e_k - e_\ell) \otimes I_n]\). Then null \(W^*[I_q \otimes U_\kappa] \not\subseteq \text{null}[(e_k - e_\ell)^* \otimes I_n]\) by Lemma [4]. Therefore there exists a vector \(f \notin \text{null}[(e_k - e_\ell)^* \otimes I_n]\) satisfying \(W^*[I_q \otimes U_\kappa]f = 0\). Define \(\xi = [I_q \otimes U_\kappa]f\). We have \(f \notin \text{null}[(e_k - e_\ell)^* \otimes I_n]\) because \(f \notin \text{null}[(e_k - e_\ell)^* \otimes I_n]\) and \(U_\kappa\) is full column rank. Hence \(W^*\xi = W^*\xi = [I_q \otimes U_\kappa]f = 0\) implies null \(W^* \not\subseteq \text{null}[(e_k - e_\ell)^* \otimes I_n]\). Equivalently, range \(W \not\supseteq \text{range}[(e_k - e_\ell) \otimes I_n]\), i.e., \(W\) is not \((k, \ell)\)-connected.

3 \implies 2. Suppose \(\text{range} W \not\supseteq \text{range}[(e_k - e_\ell) \otimes I_n]\). Hence null \(W^* \not\subseteq \text{null}[(e_k - e_\ell)^* \otimes I_n]\) and we can find \(\zeta \notin \text{null}[(e_k - e_\ell)^* \otimes I_n]\) satisfying \(W^*\zeta = 0\). Observe that null \(W^*\) is invariant with respect to \(A^*\). This allows us to write \(W^*e^{A^*t}\zeta \equiv 0\). Note that the \(n \times n\) matrix \([U_1 0 0 \cdots 0\] is nonsingular. This means that the \(qn \times qn\) matrix \([(I_q \otimes U_1)(I_q \otimes U_2) \cdots (I_q \otimes U_m)]\) is nonsingular. Therefore we can find vectors \(f_k \in (\mathbb{C}^n)^q\) satisfying
\[
\zeta = \sum_{\kappa=1}^m [I_q \otimes U_\kappa]f_k.
\]
Since we can write
\[
\sum_{\kappa=1}^{m} [I_q \otimes U_\kappa][\{e_k - e_\ell\}^* \otimes I_{n_\kappa}] f_\kappa = [(e_k - e_\ell)^* \otimes I_n] \sum_{\kappa=1}^{m} [I_q \otimes U_\kappa] f_\kappa
= [(e_k - e_\ell)^* \otimes I_n] \zeta \\
\neq 0
\]
we have to have
\[
[(e_k - e_\ell)^* \otimes I_{n_\kappa}] f_\kappa \neq 0
\] (8)
for some \(\kappa\). Recall \(A^*_\kappa U_\kappa = U_\kappa A^{*\kappa}_\kappa\), which implies \(e^{A^{*\kappa}_\kappa t} U_\kappa = U_\kappa e^{A^{*\kappa}_\kappa t}\). We can therefore write
\[
W^* \sum_{\kappa=1}^{m} [I_q \otimes U_\kappa e^{A^{*\kappa}_\kappa t}] f_\kappa = W^* \sum_{\kappa=1}^{m} [I_q \otimes e^{A^{*\kappa}_\kappa t} U_\kappa] f_\kappa = W^* [I_q \otimes e^{A^{*\kappa}_\kappa t}] \sum_{\kappa=1}^{m} [I_q \otimes U_\kappa] f_\kappa = W^* e^{A^{*\kappa}_\kappa t} \zeta = 0.
\] (9)

Now, since no two matrices \(A^{*\kappa}_\kappa, A^{*\nu}_\nu (\kappa \neq \nu)\) share a common eigenvalue, the set of mappings \(\{t \mapsto [I_q \otimes U_\kappa e^{A^{*\kappa}_\kappa t}] f_\kappa : f_\kappa \neq 0\}\) is linearly independent. Hence (9) implies \(W^*[I_q \otimes U_\kappa e^{A^{*\kappa}_\kappa t}] f_\kappa = 0\), which in turn implies
\[
W^*[I_q \otimes U_\kappa] f_\kappa = 0
\] (10)
for all \(\kappa\). Combining (8) and (10) allows us to claim null \(W^*[I_q \otimes U_\kappa] \not\subset \text{null} [(e_k - e_\ell)^* \otimes I_{n_\kappa}]\) for some \(\kappa\). Then by Lemma 4 we have null \(\text{inc} (W^{[*\kappa]})^* \not\subset \text{null} [(e_k - e_\ell)^* \otimes I_{n_\kappa}]\). Therefore range \(\text{inc} (W^{[*\kappa]})\) \(\not\subset\) range \([(e_k - e_\ell)^* \otimes I_{n_\kappa}]\). That is, \(\text{inc} (W^{[*\kappa]})\) is not \((k, \ell)\)-connected.

Note that the eigenvector test for controllability stated in Theorem 1 cannot be extended to \((k, \ell)\)-controllability in general. In particular, an array may fail to be \((k, \ell)\)-controllable even if all the matrices inc \((V_1^{*}B_{\vdots})\), inc \((V_2^{*}B_{\vdots})\), \ldots, inc \((V_m^{*}B_{\vdots})\) are \((k, \ell)\)-connected. A counterexample is as follows. Consider an array of \(q = 3\) fourth order systems with
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\quad \text{and} \quad
\text{inc} (B_{\vdots}) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
1 & 0 & -1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]

The matrix \(A^*\) has no eigenvalue other than \(\mu_1 = 0\) for which
\[
V_1 = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

The matrix \(W\) is not \((2, 3)\)-connected for this array. Therefore by Theorem 3 the array is not \((2, 3)\)-controllable. Despite this lack of \((2, 3)\)-controllability, the matrix inc \((V_1^{*}B_{\vdots})\) however is \((2, 3)\)-connected.
7 Positive pairwise controllability

For the system (11) let \( R \subset (\mathbb{R}^n)^g \) be the set of points positively reachable (in finite time) from the origin, i.e.,
\[
R = \left\{ \xi : \xi = \int_0^\tau e^{A(t-\tau)}Bu(t)dt \text{ for some } u : [0, \tau] \to \mathbb{R}_+^p \text{ with } \tau \geq 0 \right\}.
\]
The set \( R \) is a convex cone. The polar of \( R \) is denoted by \( R^\circ \), which itself is a convex cone in \((\mathbb{R}^n)^g\), and defined as \( R^\circ = \{ \eta : \xi^* \eta \leq 0 \text{ for all } \xi \in R \} \). Note that we can write
\[
R^\circ = \{ \eta : B^*e^{A^*t} \eta \leq 0 \text{ for all } t \geq 0 \}.
\]
The cone \( R^\circ \) is closed and the polar \( R^\circ^\circ \) of \( R^\circ \) equals \( \text{cl} R \), the closure of \( R \) [13].

Let \( \Lambda_k = [A_k^* - \mu_k I_{n_k}]^* \) for \( k = 1, 2, \ldots, m \). Since \( \mu_k \) is the only eigenvalue of \( A_k^* \), the matrix \( \Lambda_k \) has a single distinct eigenvalue at the origin, i.e., it is nilpotent. In particular, \( \Lambda_k^{n_k} = 0 \). Recall \( B_{i\sigma} = U_{i\sigma}^* B_{i\sigma} \) and range \( U_{k} = \text{null} [A^* - \mu_k I_n]^{n_k} \). Let us now define
\[
Q_{i\sigma}^{[\kappa]} = [B_{i\sigma}^{[\kappa]} \Lambda_{i\sigma}^{[\kappa]} B_{2\sigma}^{[\kappa]} \cdots B_{n\sigma}^{[\kappa]}]^* \quad \text{and} \quad B_{i\sigma} = [B_{i\sigma}^* B_{2\sigma}^* \cdots B_{n\sigma}^*]^*.
\]
Without loss of generality we henceforth assume

- \( \text{Re} \mu_1 \geq \text{Re} \mu_2 \geq \cdots \geq \text{Re} \mu_m \) and
- \( \mu_k = \mu_\nu (k \neq \nu) \) implies \( k > \nu \),

where \( \text{Re} \mu_k \) denotes the real part of \( \mu_k \). For each \( \mu_k \in \mathbb{R} \), let \( Q_k \in (\mathbb{R}^{n_k})^g \) be the largest subspace contained in cone \( \text{inc}(Q_{i\sigma}^{[\kappa]})_{\sigma \in \mathcal{I}_k} \) where the index sets \( \mathcal{I}_k \subset \{1, 2, \ldots, p\} \) are constructed as follows. \( \mathcal{I}_1 = \{1, 2, \ldots, p\} \) and
\[
\mathcal{I}_{k+1} = \mathcal{I}_k \setminus \mathcal{I}_k^-
\]
with
\[
\mathcal{I}_k^- = \left\{ \sigma \in \mathcal{I}_k : [I_\sigma \otimes \Lambda_{\sigma}^{[\kappa]}] B_{i\sigma}^{[\kappa]} \notin Q_k \text{ for some } r \in \{0, 1, \ldots, n_k - 1\} \right\} \quad \text{if } \mu_k \in \mathbb{R}
\]
\[
\mathcal{I}_k^- = \emptyset \quad \text{if } \mu_k \notin \mathbb{R}.
\]

**Lemma 6** Let \( N \in \mathbb{R}^{n \times n} \) be a nilpotent matrix \((N^n = 0)\) and \( C \in \mathbb{R}^{p \times n} \). Define the convex cones \( \mathcal{N} = \{ \eta \in \mathbb{R}^n : C N^\tau \eta \leq 0 \text{ for all } \tau \geq 0 \} \) and \( \mathcal{M} = \{ \eta \in \mathbb{R}^n : C N^\tau \eta \leq 0 \text{ for all } \tau \in \{0, 1, \ldots, n - 1\} \} \). Let \( D \subset \mathbb{R}^n \) be the smallest subspace containing \( \mathcal{M} \). Then \( \mathcal{N} \subset D \).

**Proof.** Let us first find an explicit expression of the subspace \( D \) in terms of our parameters. To this end let \( C_\sigma \in \mathbb{R}^{1 \times n} \) denote the rows of \( C \), i.e., \([C_1^* C_2^* \cdots C_p^*]^* = C \). Hence we can write \( \mathcal{M} = \{ \eta : C_\sigma N^\tau \eta \leq 0 \text{ for all } \sigma, \tau \} \). Define the set of pairs \( \mathcal{P}^- = \{1, 2, \ldots, p\} \times \{0, 1, \ldots, n - 1\} \) as \( \mathcal{P}^- = \{ (\sigma, \tau) : \mathcal{M} \cap \{ \eta : C_\sigma N^\tau \eta < 0 \} \neq \emptyset \} \). Then let \( \mathcal{P}_0 = \mathcal{P} \setminus \mathcal{P}^- \) and \( \mathcal{M}_0 = \{ \eta : C_\sigma N^\tau \eta \leq 0 \text{ for all } \sigma, \tau \in \mathcal{P}_0 \} \). Define the subspace \( D_0 = \{ \eta : C_\sigma N^\tau \eta = 0 \text{ for all } \sigma, \tau \in \mathcal{P}_0 \} \). Clearly, \( D_0 \subset \mathcal{M}_0 \). We now claim
\[
D_0 \supset \mathcal{M}_0.
\]
(11)
The relation (11) trivially holds for the extreme possibilities \( \mathcal{P}_0 = \emptyset \) or \( \mathcal{P}^- = \emptyset \). For the case where neither \( \mathcal{P}_0 \) nor \( \mathcal{P}^- \) is empty, let us establish our claim by contradiction. Suppose \( D_0 \not\supset \mathcal{M}_0 \). Then we can find an \( \eta \in \mathcal{M}_0 \) satisfying \( C_\sigma N^\tau \eta < 0 \) for some \( (\sigma, \tau) \in \mathcal{P}_0 \). Now let \( F = \{ \eta_1, \eta_2, \ldots, \eta_l \} \subset \mathcal{M} \) be a finite collection of vectors with the property that for each pair \( (\sigma, \tau) \in \mathcal{P}^- \) there exists some \( \eta_i \in F \) satisfying \( C_\sigma N^\tau \eta_i < 0 \). Such \( F \) exists by how the set \( \mathcal{P}^- \) is defined. Let the scalars \( \alpha_1, \alpha_2, \ldots, \alpha_l \) satisfy \( \alpha_i > 0 \) for all \( i \) and \( \alpha_1 + \alpha_2 + \cdots + \alpha_l = 1 \). Then construct the convex combination of the vectors in \( F \) as \( \tilde{\eta} = \alpha_1 \eta_1 + \alpha_2 \eta_2 + \cdots + \alpha_l \eta_l \). Since \( \eta_i \in \mathcal{M} \) and \( \mathcal{M} \) is convex, the new vector \( \tilde{\eta} \) belongs to \( \mathcal{M} \). Moreover, since \( \alpha_i \) are strictly positive, we have \( C_\sigma N^\tau \tilde{\eta} < 0 \) for all \( (\sigma, \tau) \in \mathcal{P}^- \). Now define for \( \lambda \in (0, 1) \)
\[
\tilde{\eta} = \lambda \eta + (1 - \lambda) \hat{\eta}.
\]
(12)
Note that \( C_\sigma N^\tau \tilde{\eta} < 0 \) for all \( \lambda \in (0, 1) \). Also, it is easy to check that by choosing \( \lambda \) sufficiently small we can make \( \tilde{\eta} \) satisfy \( C_\sigma N^\tau \tilde{\eta} \leq 0 \) for all \( (\sigma, \tau) \), i.e., \( \tilde{\eta} \in \mathcal{M} \). Then \( C_\sigma N^\tau \tilde{\eta} < 0 \) implies \( (\bar{\sigma}, \bar{\tau}) \in \mathcal{P}^- \), which
contradicts \((\hat{\sigma}, \hat{r}) \in \mathcal{P}_0\). Hence (12) holds true. In particular, since we also have \(D_0 \subset M_0\), we can write \(D_0 = M_0\).

Next we show \(D = M_0\). Choose an arbitrary vector \(\eta\) that belongs to \(M_0\), i.e., \(C_{\sigma} N^r \eta \leq 0\) for all \((\sigma, r) \in P_0\). Let \(\hat{\eta} \in M\) be as before. That is, \(C_{\sigma} N^r \hat{\eta} < 0\) for all \((\sigma, r) \in P\) and \(C_{\sigma} N^r \tilde{\eta} \leq 0\) for all \((\sigma, r) \in P_0\). Consider (12). Choose \(\lambda \in (0, 1)\) small enough so that \(C_{\sigma} N^r \hat{\eta} \leq 0\) for all \((\sigma, r)\). Hence \(\tilde{\eta} \in M\). Let us now rewrite (12) as

\[
\eta = \lambda^{-1} \hat{\eta} + (\lambda - 1) \lambda^{-1} \tilde{\eta}.
\]

Since both \(\hat{\eta}\) and \(\tilde{\eta}\) belong to \(M\), we have \(\hat{\eta}, \tilde{\eta} \in D\). Therefore \(\eta \in D\). Since \(\eta\) was arbitrary we have to have \(D = M_0\), i.e., \(M_0\) (or \(D_0\)) is the smallest subspace containing \(M\).

So far we have not used the nilpotency of \(N\). An obvious implication of \(N^n = 0\) is \(N M \subset M\), i.e., the set \(M\) is invariant with respect to \(N\). This invariance imposes a special structure on \(P\). To see that let some pair \((\sigma, r)\) with \(r \geq 1\) belong to \(P\). That is, we can find some \(\eta \in M\) satisfying \(C_{\sigma} N^r \eta < 0\). Then \(N \eta \in M\) yields \((\sigma, r - 1) \in P\) because \(C_{\sigma} N^{r - 1} (N \eta) < 0\). Consequently, the complement set \(P_0\) enjoys (for \(r \neq n - 1\))

\[
(\sigma, r) \in P_0 \implies (\sigma, r + 1) \in P_0.
\]

We are now ready to establish \(N \subset D\). Suppose otherwise. Then we can find \(\eta \in N\) satisfying \(\eta \notin M_0\). Since \(N^n = 0\) the matrix exponential \(e^{N t}\) has finitely many terms. This allows us to write (for all \(\sigma\))

\[
C_{\sigma} \left( I_n + N t + N^2 t^2 \frac{2}{2} + \cdots + N^{n - 1} \frac{t^{n - 1}}{(n - 1)!} \right) \eta \leq 0 \quad \text{for all } t \geq 0.
\]

For each \(\sigma\) let \(r_\sigma \in \{0, 1, \ldots, n\}\) be the smallest index satisfying \(C_{\sigma} N^r \eta = 0\) for all \(r \geq r_\sigma\). Considering (14) as \(t \to \infty\) we deduce

\[
C_{\sigma} N^{r_\sigma - 1} \eta < 0 \quad \text{if } \phantom{\eta} r_\sigma \neq 0.
\]

Hence we can write

\[
C_{\sigma} N^r \eta \leq 0 \quad \text{for all } \phantom{\eta} \sigma, r \geq \max \{0, r_\sigma - 1\}.
\]

For each \(\sigma\) this time let \(\hat{r}_\sigma \in \{0, 1, \ldots, n\}\) be the smallest index satisfying \((\sigma, \hat{r}_\sigma) \in P_0\) if exists. Otherwise (i.e., in case \((\sigma, n - 1) \notin P_0\)) let \(\hat{r}_\sigma = n\). Now, due to the property (13) and \(\eta \notin M_0\) we have to have \(r_\sigma > \hat{r}_\sigma\) for some \(\sigma\). Therefore the integer \(p = \max_\sigma (r_\sigma - \hat{r}_\sigma)\) is positive. Define the vector \(\hat{\eta} = N^{p - 1} \eta\). Given some pair \((\sigma, \hat{r}) \in P_0\) define \(r = \hat{r} + \rho - 1\). Since \(\hat{r} \geq \hat{r}_\sigma\) we have \(r \geq \hat{r}_\sigma - 1\) and therefore \(C_{\sigma} N^{\hat{r}} \eta \leq 0\) by (16). Hence

\[
C_{\sigma} N^{\hat{r}} \hat{\eta} = C_{\sigma} N^{\hat{r} + p - 1} \eta = C_{\sigma} N^r \eta \leq 0
\]

meaning \(\hat{\eta} \in M_0\), for the pair \((\sigma, \hat{r}) \in P_0\) was arbitrary. Let now the index \(v \in \{1, 2, \ldots, p\}\) satisfy \(r_\nu - \hat{r}_v = \rho\). Since \(\rho\) is positive so is \(r_\nu\). As a result \(C_{\nu} N^{r_\nu - 1} \eta < 0\) by (14). Then

\[
C_{\nu} N^{\hat{r}_v} \hat{\eta} = C_{\nu} N^{\hat{r}_v + p - 1} \eta = C_{\nu} N^{\hat{r}_v} \hat{\eta} < 0
\]

meaning \(\hat{\eta} \notin D_0\) because \((v, \hat{r}_v) \in P_0\) by definition. But this contradicts \(\hat{\eta} \in M_0\) because \(D_0 = M_0\).

**Lemma 7** Let \(\eta \in (\mathbb{R}^n)^q\) satisfy \(\eta = \sum_{k=1}^n [I_q \otimes U_\kappa] f_\kappa\) with \(f_\kappa \in (\mathbb{C}^n)^q\) and \(B^* e^{A^* t} \eta \leq 0\) for all \(t \geq 0\). There exists \(\tau \geq 0\) such that for each \(\kappa\) we have \(\text{Re} \left( B^* \kappa [I_q \otimes e^{A^* t}] f_\kappa \right) \leq 0\) for all \(\sigma \in I_\kappa\) and \(t \geq \tau\).

**Proof.** Note that \(B^* e^{A^* t} \eta \leq 0\) for all \(t \geq 0\) means for all \(\sigma\) we have \(B^* \sigma e^{A^* t} \eta \leq 0\) for all \(t \geq 0\). We can
\[
\sum_{\kappa=1}^{m} \text{Re} \left( B_{\sigma}^{[\kappa]}[I_q \otimes e^{A^\kappa t}]f_\kappa \right) = \text{Re} \left( \sum_{\kappa=1}^{m} B_{\sigma}^* [I_q \otimes U_\kappa][I_q \otimes e^{A^\kappa t}]f_\kappa \right) \\
= \text{Re} \left( \sum_{\kappa=1}^{m} B_{\sigma}^* [I_q \otimes e^{A^\kappa t}]f_\kappa \right) \\
= \text{Re} \left( \sum_{\kappa=1}^{m} B_{\sigma}^* [I_q \otimes e^{A^\kappa t}] [I_q \otimes U_\kappa]f_\kappa \right) \\
= \text{Re} \left( \sum_{\kappa=1}^{m} B_{\sigma}^* [I_q \otimes e^{A^\kappa t}]f_\kappa \right) \\
= B_{\sigma}^* e^{A^\kappa t} \text{Re} \left( \sum_{\kappa=1}^{m} [I_q \otimes U_\kappa]f_\kappa \right) \\
\leq 0 
\] (17)

where \( U_\kappa e^{A^\kappa t} = e^{A^\kappa t} U_\kappa \) follows from \( U_\kappa A^\kappa = A^\kappa U_\kappa \). Recall our ordering \( \text{Re} \mu_1 \geq \text{Re} \mu_2 \geq \cdots \geq \text{Re} \mu_m \) with the extra condition: whenever two distinct indices \( \kappa, \nu \) satisfy \( \text{Re} \mu_\kappa = \mu_\nu \) we have \( \kappa > \nu \). Since each \( A^\kappa_\ast \) has a single distinct eigenvalue \( \mu_\kappa \) inequality (17) implies the existence of \( \tau \geq 0 \) such that for all \( t \geq \tau \) we can write

\[
\sum_{\kappa=1}^{\nu} \text{Re} \left( B_{\sigma}^{[\kappa]}[I_q \otimes e^{A^\kappa t}]f_\kappa \right) \leq 0 
\] (18)

for all \( \nu \in \{1, 2, \ldots, m\} \) and \( \sigma \). We now claim for all \( \nu \) and \( t \geq \tau \)

\[
B_{\sigma}^{[\kappa]}[I_q \otimes e^{A^\kappa t}]f_\kappa = 0 \quad \text{for all} \quad \kappa \leq \nu \quad \text{and} \quad \sigma \in I_\nu \setminus I_\nu^- .
\] (19)

Let us establish our claim by induction. Suppose (19) holds for some \( \nu \in \{1, 2, \ldots, m - 1\} \). Then (18) allows us to write for \( \sigma \in I_\nu \setminus I_\nu^- 
\[
\text{Re} \left( B_{\sigma}^{[\nu+1]}[I_q \otimes e^{A^\nu+1 t}]f_{\nu+1} \right) = \sum_{\kappa=1}^{\nu+1} \text{Re} \left( B_{\sigma}^{[\kappa]}[I_q \otimes e^{A^\kappa t}]f_\kappa \right) \\
\leq 0 .
\] (20)

If \( \mu_{\nu+1} \notin \mathbb{R} \) then (20) implies \( B_{\sigma}^{[\nu+1]}[I_q \otimes e^{A^{\nu+1} t}]f_{\nu+1} \equiv 0 \), for otherwise, the oscillations would not let the signal \( t \mapsto \text{Re} \left( B_{\sigma}^{[\nu+1]}[I_q \otimes e^{A^{\nu+1} t}]f_{\nu+1} \right) \) remain nonpositive indefinitely. Hence (19) holds for \( \nu + 1 \) because

\[
I_{\nu+1} \setminus I_{\nu+1}^- \subset I_{\nu+1} = I_\nu \setminus I_\nu^- .
\] (21)

Let us now consider the case \( \mu_{\nu+1} \in \mathbb{R} \). Since we can write \( e^{A^{\nu+1} t} = e^{\mu_{\nu+1} t} e^{A^\nu t} \) inequality (20) implies \( B_{\sigma}^{[\nu+1]}[I_q \otimes e^{A^\nu t}]f_{\nu+1} \leq 0 \) for all \( t \geq \tau \) and \( \sigma \in I_{\nu+1} \). This means the vector \( \tilde{f}_{\nu+1} = [I_q \otimes e^{A^\nu t}]f_{\nu+1} \) belongs to the convex cone \( \mathcal{N}_{\nu+1} = \{ \zeta : B_{\nu}^{[\nu+1]}[I_q \otimes e^{A^\nu t}]\zeta \leq 0 \quad \text{for all} \quad \sigma \in I_{\nu+1} \quad \text{and} \quad t \geq 0 \} \). Let \( \mathcal{M}_{\nu+1} = (\text{cone} [Q_{\nu+1} [Q_{\nu+1}^+[\nu+1] = 0 \quad \text{for all} \quad \sigma \in I_{\nu+1} \quad \text{and} \quad r] \}. \) Since \( Q_{\nu+1} \) is the largest subspace contained in \( \text{cone} [Q_{\nu+1}^+[\nu+1] \), the smallest subspace containing its polar \( \mathcal{M}_{\nu+1} \) is \( Q_{\nu+1}^- \). Since the matrix \( [I_q \otimes A^\nu_{\nu+1}] \) is nilpotent, we can invoke Lemma 6 which tells us that the subspace \( Q_{\nu+1}^- \) contains also \( \mathcal{N}_{\nu+1} \). Therefore we have \( \tilde{f}_{\nu+1} \in Q_{\nu+1}^- \). Then (since by definition \( [I_q \otimes A^\nu_{\nu+1}]B_{\nu+1}^{[\nu+1]} \in Q_{\nu+1} \) for all \( \sigma \in I_{\nu+1} \setminus I_{\nu+1}^- \) and \( r \) the structure (due to \( A^\nu_0 = 0 \))

\[
[I_q \otimes e^{A^\nu t}] = \sum_{r=0}^{n-1} [I_q \otimes A^\nu_{\kappa}] \frac{t^r}{r!}
\] (22)
allows us to write $B_{\sigma}^{[\nu+1]*}[I_q \otimes e^{A_{\nu+1}^*}]f_{\nu+1} = 0$ for all $\sigma \in I_{\nu+1} \setminus I_{\nu+1}$ and $t \geq 0$. Note that $e^{A_{\nu+1}^*} = e^{A_{\nu+1}^*}e^{A_{\nu+1}^*}$ and $f_{\nu+1} = [I_q \otimes e^{A_{\nu+1}^*}]f_{\nu+1}$. Hence we have (19) for $\nu+1$ for all $t \geq \tau$ thanks to (21). What now remains is that we show (19) for $\nu = 1$. This however follows from the previous arguments once we note that by (18) we have $\text{Re} (B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_{\nu}) \leq 0$ and that $I_1$ equals $\{1, 2, \ldots, p\}$ by definition.

Having established (19), let us now be given any index $\gamma \in \{1, 2, \ldots, m\}$. If $\gamma = 1$ then by letting $\nu = 1$ in (18) we have $\text{Re} (B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_{\gamma}) \leq 0$ for all $\sigma \in I_\gamma$ and $t \geq \tau$. Consider now the case $\gamma \geq 2$. Invoking (19) with $\nu = \gamma - 1$ we obtain $B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_{\gamma} = 0$ for all $\kappa \leq \gamma - 1$ and $\sigma \in I_{\gamma-1} \setminus I_{\gamma-1} = I_\gamma$. Then by (18) we can write

\begin{align*}
\text{Re} (B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_{\gamma}) & = \text{Re} (B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_{\gamma}) + \sum_{\kappa=1}^{\gamma-1} \text{Re} (B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_{\kappa}) \\
& = \sum_{\kappa=1}^{\gamma} \text{Re} (B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_{\kappa}) \\
& \leq 0
\end{align*}

for all $\sigma \in I_\gamma$ and $t \geq \tau$. Hence the result. \hfill \blacksquare

**Assumption 1** If $\mu_\nu \notin \mathbb{R}$ satisfies $\text{Re} \mu_\nu = \mu_\nu$ for some $\nu \in \{1, 2, \ldots, m\}$ then $\Lambda_\kappa = 0$.

**Theorem 4** Under Assumption 1, the following two conditions are equivalent.

1. $\mathcal{R}^\circ \subset \text{null} \{(e_k - e_\ell)^* \otimes I_\nu\}$.

2. The below statements simultaneously hold.

   (a) $\text{inc} (Q_{\alpha}^{[\nu]} \sigma \in I_\nu)$ is strongly $(k, \ell)$-connected for all $\mu_\nu \in \mathbb{R}$.

   (b) $\text{inc} (Q_{\alpha}^{[\nu]} \sigma \in I_\nu)$ is $(k, \ell)$-connected for all $\mu_\nu \notin \mathbb{R}$.

**Proof.** 1 $\implies$ 2. Suppose the condition 2a fails. That is, cone $\text{inc} (Q_{\alpha}^{[\nu]} \sigma \in I_\nu) \notin \text{range} \{(e_k - e_\ell)^* \otimes I_\nu\}$ for some $\mu_\nu \in \mathbb{R}$. Without loss of generality assume that this $\mu_\nu$ is the largest real eigenvalue for which 2a fails. Absence of strong $(k, \ell)$-connectivity implies that the polar $(\text{cone} \{\text{inc} (Q_{\alpha}^{[\nu]} \sigma \in I_\nu)\}^o)$ is not contained in null $\{(e_k - e_\ell)^* \otimes I_\nu\}$. Hence we can find a vector $f_\nu \notin \text{null} \{(e_k - e_\ell)^* \otimes I_\nu\}$ satisfying $B_{\sigma}^{[\nu]*}[I_q \otimes \Lambda_\nu^*]f_\nu \leq 0$ for all $\nu \in I_\nu$ and $r \in \{0, 1, \ldots, n_\nu - 1\}$.

For $\kappa < \nu$ and $\mu_\nu \in \mathbb{R}$ let $\mathcal{M}_\kappa = (\text{cone} \{\text{inc} (Q_{\alpha}^{[\nu]} \sigma \in I_\nu)\}^o)$. That is, $\mathcal{M}_\kappa = \{f_\nu \in (\mathbb{R}^{n_\nu})^o : B_{\sigma}^{[\nu]*}[I_q \otimes \Lambda_\nu^*]f_\nu \leq 0 \text{ for all } \sigma \in I_\nu \text{ and } r \}$. Since $Q_\kappa$ is the largest subspace contained in cone $\text{inc} (Q_{\alpha}^{[\nu]} \sigma \in I_\nu)$, the smallest subspace containing its polar $\mathcal{M}_\kappa$ is $Q_\kappa^\perp$. Also, strong $(k, \ell)$-connectivity of inc $Q_{\alpha}^{[\nu]} \sigma \in I_\nu$ implies $Q_\kappa ^\perp \supset \text{range} \{(e_k - e_\ell)^* \otimes I_\nu\}$. Consequently, $Q_\kappa ^\perp \subset \text{null} \{(e_k - e_\ell)^* \otimes I_\nu\}$. Let $\mathcal{P}_\kappa = I_\kappa \times \{0, 1, \ldots, n_\kappa - 1\}$ and define $\mathcal{P}^-_\kappa = \{(\sigma, r) \in \mathcal{P}_\kappa : \mathcal{M}_\kappa \cap \{f_\nu : B_{\sigma}^{[\nu]*}[I_q \otimes \Lambda_\nu^*]f_\nu < 0\} \neq \emptyset\}$. Note that $(\sigma, r) \in \mathcal{P}^-_\kappa$ means there exist $f_\nu \in \mathcal{M}_\kappa \subset Q_\kappa^\perp$ satisfying $B_{\sigma}^{[\nu]*}[I_q \otimes \Lambda_\nu^*]f_\nu \neq 0$ yielding $[I_q \otimes \Lambda_\kappa^*]B_{\sigma}^{[\nu]} \notin \mathcal{Q}_\kappa$. Therefore $(\sigma, r) \in \mathcal{P}^-_\kappa$ implies $\sigma \in I_\kappa$. Furthermore, we can write $[I_q \otimes \Lambda_\kappa^*] = [I_q \otimes \Lambda_\kappa^*]^o$ and it is easy to see that the nilpotency of $\Lambda_\kappa$ is inherited by the matrix $[I_q \otimes \Lambda_\kappa^*]$. Hence

$$
(\sigma, r) \in \mathcal{P}^-_\kappa \implies (\sigma, r - 1) \in \mathcal{P}^-_\kappa
$$

for $r \neq 0$ (see the proof of Lemma 18). For each $\kappa < \nu$ with $\mu_\nu \in \mathbb{R}$ choose now $f_\nu \in \mathcal{M}_\kappa$ that satisfies $B_{\sigma}^{[\nu]*}[I_q \otimes \Lambda_\nu^*]f_\nu < 0$ for all $(\sigma, r) \in \mathcal{P}^-_\kappa$. Such choice exists thanks to convexity of $\mathcal{M}_\kappa$ (see the proof of Lemma 18). Then define $y_\nu^{[\nu]}(t) = B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_\nu$. The implication (22) together with the identity (22) ensure the existence of a scalar $\delta_\kappa > 0$ and a polynomial $\Delta_\kappa(t)$ of order $n_\kappa - 1$ satisfying

$$
y_\nu^{[\nu]}(t) \leq \begin{cases} -\delta_\kappa & \text{for } \sigma \in I_\kappa^- \\ 0 & \text{for } \sigma \in I_\kappa \setminus I_\kappa^- \\ \Delta_\kappa(t) & \text{for } \sigma \notin I_\kappa 
\end{cases}
$$

for all $t \geq 0$. As for $y_\nu^{[\nu]}(t) = B_{\sigma}^{[\nu]*}[I_q \otimes e^{A_{\nu}^*}]f_\nu$, note that we can write

$$
y_\nu^{[\nu]}(t) \leq \begin{cases} 0 & \text{for } \sigma \in I_\nu \\ \Delta_\nu(t) & \text{for } \sigma \notin I_\nu 
\end{cases}
$$

for all $t \geq 0$.
for some polynomial $\Delta_\nu(t)$ of order $n_\nu - 1$. Let us now construct the vector

$$
\eta = \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\nu[I_\nu \otimes U_\kappa]f_\kappa
$$

where $c_\nu = 1$ and the remaining scalars $c_\kappa > 0$ satisfy

$$
c_\kappa \delta_\kappa e^{\mu_\kappa t} \geq \sum_{\kappa < \gamma \leq \nu, \mu_\kappa \in \mathbb{R}} c_\gamma \Delta_\gamma(t)e^{\mu_\gamma t}.
$$

(25)

We can always find such scalars because (for real eigenvalues) we have $\mu_\kappa > \mu_\gamma$ when $\kappa < \gamma$.

Note that for $\kappa < \nu$ we have $f_\kappa \in \mathcal{M}_\kappa \subset \mathcal{Q}_\kappa^* \subset \text{null } [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]$ which allows us to write $[(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}][I_\nu \otimes U_\kappa]f_\kappa = U_\kappa [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]f_\kappa = 0$. Hence

$$
[(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]\eta = [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}] \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\mu U_\kappa [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]f_\kappa
$$

$$
= \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\kappa U_\kappa [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]f_\kappa
$$

$$
= \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\kappa U_\kappa f_\kappa 
$$

$$
\neq 0
$$

because $U_\nu$ is full column rank and $[(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]f_\nu \neq 0$. Hence $\eta \notin \text{null } [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]$. Let us now study the behavior of the entries of $B^* \mathbf{e}^{A^*t} \eta$. Recall that $B^* \sigma_\nu$ denotes the $\sigma$th row of $B^*$. We can write for $\sigma \in \{1, 2, \ldots, p\}$

$$
B^* \sigma_\nu \mathbf{e}^{A^*t} \eta = \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\kappa B^* \sigma_\nu[I_\nu \otimes e^{A^*t}][I_\nu \otimes U_\kappa]f_\kappa
$$

$$
= \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\kappa B^* \sigma_\nu[I_\nu \otimes e^{A^*t}]f_\kappa
$$

$$
= \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\kappa \Delta_\kappa(t)\sigma_\nu(t)
$$

(26)

where the summation is through the indices $\kappa \leq \nu$, $\mu_\kappa \in \mathbb{R}$ and we used the identities $e^{A^*t} U_\kappa = U_\kappa e^{A^*t}$ (because $A^* U_\kappa = U_\kappa A^*_\kappa$) and $e^{A^*t} = e^{\mu_\kappa t} e^{A^*_\kappa t}$ (because $A^*_\kappa = \mu_\kappa I_{n_\kappa} + A^*_\kappa$). Note that for each $\sigma \notin \mathcal{I}_\nu$ there is a unique $\kappa_\sigma < \nu$ satisfying $\sigma \in \mathcal{I}_{\kappa_\sigma}$ and $\sigma \notin \mathcal{I}_\kappa$ for $\kappa < \kappa_\sigma$. Hence for $\sigma \notin \mathcal{I}_\nu$ we can decompose (26) as

$$
B^* \sigma_\nu \mathbf{e}^{A^*t} \eta = \sum_{\kappa \leq \nu, \mu_\kappa \in \mathbb{R}} c_\kappa e^{\mu_\kappa t} y^{[\kappa]}_\sigma(t) + \sum_{\kappa > \kappa_\sigma} c_\kappa e^{\mu_\kappa t} y^{[\kappa]}_\sigma(t)
$$

$$
\leq -c_\kappa \delta_\kappa e^{\mu_\kappa t} + \sum_{\kappa > \kappa_\sigma} c_\kappa \Delta_\kappa(t) e^{\mu_\kappa t}
$$

$$
\leq 0
$$

where we used (23). If $\sigma \in \mathcal{I}_\nu$ then $\sigma \in \mathcal{I}_\kappa$ for all $\kappa \leq \nu$ and we have $y^{[\kappa]}_\sigma(t) \leq 0$ for all $\kappa \leq \nu$ yielding $B^* \sigma_\nu \mathbf{e}^{A^*t} \eta \leq 0$. Hence $B^* \mathbf{e}^{A^*t} \eta \leq 0$ for all $t \geq 0$, i.e., $\eta \in \mathcal{R}^\circ$. This implies however $\mathcal{R}^\circ \notin \text{null } [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]$ for we earlier obtained $\eta \notin \text{null } [(e_\kappa - e_\ell)^* \otimes I_{n_\kappa}]$.

Suppose now the condition 2b fails for some $\mu_\nu \notin \mathbb{R}$. Without loss of generality assume that the condition 2a is satisfied and no eigenvalue with real part strictly larger than $\text{Re} \mu_\nu$ violates 2b. That inc $(Q^{[\nu]}\sigma_{\kappa_\nu})_{\nu \in \mathcal{I}_\nu}$ is not $(k, \ell)$-connected implies the existence of a (complex) vector $f_\nu \neq \text{null } [(e_\kappa - e_\ell)^* \otimes I_{n_\nu}]$ satisfying $B^* \sigma_{\kappa_\nu}[I_\nu \otimes \Lambda^*_\nu]f_\nu = 0$ for all $\sigma \in \mathcal{I}_\nu$ and $\nu \in \{0, 1, \ldots, n_\nu - 1\}$. Without loss of generality we can assume $\text{Re} (U_\nu [(e_\kappa - e_\ell)^* \otimes I_{n_\nu}]f_\nu) \neq 0$, for $U_\nu$ is full column rank and we can always replace $f_\nu$ with $\sqrt{-1} f_\nu$. Let $\mathcal{M}_\kappa$ and $\mathcal{P}_\kappa^\circ$ be as before. For each $\kappa < \nu$ with $\mu_\kappa \in \mathbb{R}$ choose $f_\kappa \in \mathcal{M}_\kappa$ that satisfies $B^* \sigma_{\kappa_\nu}[I_\nu \otimes \Lambda^*_\nu]f_\kappa \leq 0$ for all $(\sigma, r) \in \mathcal{P}_\kappa^\circ$. Recall $y^{[\kappa]}(t) = B^* \sigma_{\kappa_\nu}[I_\nu \otimes e^{A^*_\kappa t}]f_\nu$. Now (23) and (22) allows us to find scalars $\delta_\kappa > 0$ and polynomials $\Delta_\kappa(t)$ of order $n_\kappa - 1$ satisfying (24) for all $t \geq 0$ and all $\kappa < \nu$ with $\mu_\kappa \in \mathbb{R}$. As for the index $\nu$ note that $y^{[\nu]}_\sigma(t)$ is complex and we can write

$$
|y^{[\nu]}_\sigma(t)| \leq \begin{cases} 
0 & \text{for } \sigma \in \mathcal{I}_\nu \\
\Delta_\nu(t) & \text{for } \sigma \notin \mathcal{I}_\nu
\end{cases}
$$
for some polynomial $\Delta_\nu(t)$ of order (at most) $n_\nu - 1$. If $\Lambda_\nu = 0$ we let $\Delta_\nu(t)$ to be of order zero, i.e., $\Delta_\nu(t) \equiv \Delta_\nu(0) \geq 0$. (This we can do thanks to \[22\].) Let us now construct the vector

$$
\eta = \text{Re} \left( [I_q \otimes U_\nu] f_\nu \right) + \sum_{\kappa < \nu, \mu_\kappa \in \mathbb{R}} c_\kappa [I_q \otimes U_\kappa] f_\kappa
$$

where the real scalars $c_\kappa > 0$ satisfy

$$
c_\kappa \delta_\kappa e^{\mu_\kappa t} \geq \Delta_\nu(t)|e^{\mu_\kappa t}| + \sum_{\kappa < \gamma, \mu_\kappa, \mu_\gamma \in \mathbb{R}} c_\gamma \Delta_\gamma(t) e^{\mu_\gamma t}.
$$

We can always find such scalars thanks to two reasons. First, for $\mu_\kappa, \mu_\gamma \in \mathbb{R}$ we have $\mu_\kappa > \mu_\gamma$ when $\kappa < \gamma$. Second, if there is $\mu_\kappa \in \mathbb{R}$ satisfying $\text{Re} \mu_\kappa = \mu_\nu$ then by Assumption\[1\] we have $\Delta_\nu(t) \equiv \Delta_\nu(0)$.

As before for $\kappa < \nu$ we have $[(e_\kappa - e_\ell)^* \otimes I_\nu][I_q \otimes U_\kappa] f_\kappa = 0.$ Hence

$$
[(e_\kappa - e_\ell)^* \otimes I_\nu] f_\kappa = 0.
$$

Hence $\eta \notin \text{null } [(e_\kappa - e_\ell)^* \otimes I_\nu]$. Let us now study the behavior of the entries of $B^* e^{A^\tau t} \eta$. We can write for $\sigma \in \{1, 2, \ldots, p\}$

$$
B^* e^{A^\tau t} \eta = \sum_{\kappa < \nu, \mu_\kappa \in \mathbb{R}} c_\kappa e^{\mu_\kappa t} y_{\sigma}^\nu(t) + \sum_{\kappa < \nu, \mu_\kappa \in \mathbb{R}} c_\kappa e^{\mu_\kappa t} y_{\sigma}^\kappa(t) + \sum_{\kappa < \nu} c_\kappa \Delta_\kappa(t) e^{\mu_\kappa t} + \Delta_\nu(t)|e^{\mu_\kappa t}|
$$

where the summation is through the indices $\kappa < \nu, \mu_\kappa \in \mathbb{R}$. For $\sigma \notin \mathcal{I}_\nu$, we can decompose \[28\] as

$$
B^* e^{A^\tau t} \eta = \sum_{\kappa < \nu, \mu_\kappa \in \mathbb{R}} c_\kappa e^{\mu_\kappa t} y_{\sigma}^\nu(t) + \sum_{\kappa < \nu} c_\kappa \Delta_\kappa(t) e^{\mu_\kappa t} + \Delta_\nu(t)|e^{\mu_\kappa t}|
$$

where ($\kappa_\sigma$ is defined earlier and) we used \[27\]. If $\sigma \in \mathcal{I}_\nu$ then $\sigma \in \mathcal{I}_\kappa$ for all $\kappa \leq \nu$ and we have $y_{\sigma}^\kappa(t) \leq 0$ for all $\kappa < \nu$ as well as $y_{\sigma}^\nu(t) \equiv 0$. This yields $B^* e^{A^\tau t} \eta \leq 0$. Hence $B^* e^{A^\tau t} \eta \leq 0$ for all $t \geq 0$, i.e., $\eta \in \mathcal{R}^3$. This implies however $\mathcal{R}^3 \notin \text{null } [(e_\kappa - e_\ell)^* \otimes I_\kappa]$. For we already obtained $\eta \notin \text{null } [(e_\kappa - e_\ell)^* \otimes I_\nu]$.

\[2 \implies 1\]. Suppose the condition 1 fails. Then we can find a vector $\eta \in \mathcal{R}^3$ that satisfies $[(e_\kappa - e_\ell)^* \otimes I_\nu] \eta \neq 0$. Let vectors $f_\kappa \in (\mathbb{C}^{n_\kappa})^q$ be such that $\eta = \sum_{\kappa_1=1}^m [I_q \otimes U_\kappa] f_\kappa$. We can write

$$
0 \neq [(e_\kappa - e_\ell)^* \otimes I_\nu] \eta = [(e_\kappa - e_\ell)^* \otimes I_\nu] \sum_{\kappa_1=1}^m [I_q \otimes U_\kappa] f_\kappa = \sum_{\kappa_1=1}^m U_\kappa [(e_\kappa - e_\ell)^* \otimes I_\kappa] f_\kappa
$$

meaning for some $\kappa$ we have to have $U_\kappa [(e_\kappa - e_\ell)^* \otimes I_\kappa] f_\kappa \neq 0$. This implies $[(e_\kappa - e_\ell)^* \otimes I_\kappa] f_\kappa \neq 0$ because $U_\kappa$ is full column rank.

Suppose now $\mu_\kappa \in \mathbb{R}$. By Lemma\[7\] there exists $\tau \geq 0$ such that $B^* e^{A^\tau t} \eta \leq 0$ for all $\sigma \in \mathcal{I}_\kappa$ and $t \geq \tau$. Define $\tilde{f}_\kappa = [I_q \otimes e^{A^\tau t}] f_\kappa$. We can write

$$
[(e_\kappa - e_\ell)^* \otimes I_\nu] \tilde{f}_\kappa = [(e_\kappa - e_\ell)^* \otimes I_\nu] [I_q \otimes e^{A^\tau t}] f_\kappa = e^{A^\tau t} [(e_\kappa - e_\ell)^* \otimes I_\kappa] f_\kappa
$$

because $[(e_\kappa - e_\ell)^* \otimes I_\nu] f_\kappa \neq 0$ and $e^{A^\tau t}$ is nonsingular. Since we can write $e^{A^\tau t} = e^{\mu_\ell t} e^{A^\tau t}$ and $e^{\mu_\ell t}$ is always positive, $\tilde{f}_\kappa$ belongs to the cone $\mathcal{N}_\kappa = \{ \zeta : B^* e^{A^\tau t} \zeta \leq 0 \text{ for all } \sigma \in \mathcal{I}_\kappa \text{ and } t \geq \tau \}$. Let
$M_{\kappa} = \left( \text{cone} \left[ \text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}} \right] \right)^{\circ}$. That is, $M_{\kappa} = \left\{ \zeta : B_{[\kappa]}^{[\sigma]} \left[ I_{\varphi} \otimes \Lambda_{[\kappa]}^{[\sigma]} \right] \zeta \leq 0 \text{ for all } \sigma \in \mathcal{I}_{\kappa} \text{ and } r \right\}$. Since $Q_{\kappa}$ is the largest subspace contained in $\text{cone} \left[ \text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}} \right]$, the smallest subspace containing its polar $Q_{\kappa}^{\circ}$ is $Q_{\kappa}^{\circ}$. By Lemma 6 the subspace $Q_{\kappa}^{\circ}$ contains also $N_{\kappa}$. Therefore we have $f_{\kappa} \in Q_{\kappa}^{\circ}$. Then (29) yields $Q_{\kappa}^{\circ} \nsubseteq \text{null} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n_{\kappa}} \right]$. Consequently, $Q_{\kappa} \nsubseteq \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n_{\kappa}} \right]$. This allows us to write $\text{cone} \left[ \text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}} \right] \nsubseteq \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n_{\kappa}} \right]$ because $\text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n_{\kappa}} \right]$ is a subspace and $Q_{\kappa}$ is the largest subspace that cone inc $Q_{[\kappa]}^{[\sigma]} \in \mathcal{I}_{\kappa}$ contains. Then $\text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}}$ is not strongly $(k, \ell)$-connected by definition.

We now consider the other possibility: $\mu_{k} \notin \mathbb{R}$. By Lemma 7 there exists $\tau \geq 0$ such that $\text{Re} \left( B_{[\kappa]}^{[\sigma]} \left[ I_{\varphi} \otimes e^{A_{\kappa}^{\rho}} \right] \right) f_{\kappa} \leq 0$ for all $\sigma \in \mathcal{I}_{\kappa}$ and $t \geq \tau$. This implies $B_{[\kappa]}^{[\sigma]} \left[ I_{\varphi} \otimes e^{A_{\kappa}^{\rho}} \right] f_{\kappa} = 0$ for all $\sigma \in \mathcal{I}_{\kappa}$ because $\mu_{k}$ is the single distinct eigenvalue of $A_{\kappa}^{\rho}$ and it has non-zero imaginary part. (Otherwise, the oscillations would not let the signal $t \mapsto \text{Re} \left( B_{[\kappa]}^{[\sigma]} \left[ I_{\varphi} \otimes e^{A_{\kappa}^{\rho}} \right] f_{\kappa} \right)$ stay non-positive indefinitely.) Then the identity $e^{A_{\kappa}^{\rho} t} = e^{\mu_{k} t} e^{A_{\kappa}^{\rho} t}$ implies $B_{[\kappa]}^{[\sigma]} \left[ I_{\varphi} \otimes e^{A_{\kappa}^{\rho} t} \right] f_{\kappa} = 0$ for all $\sigma \in \mathcal{I}_{\kappa}$. By differentiating $B_{[\kappa]}^{[\sigma]} \left[ I_{\varphi} \otimes e^{A_{\kappa}^{\rho} t} \right] f_{\kappa} = 0$ (with respect to $t$) sufficiently many times and evaluating the derivatives at $t = 0$ we at once see that $f_{\kappa}$ belongs to null $\text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}}^{*}$. Since $\left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n_{\kappa}} \right] f_{\kappa} \neq 0$, this means null $\text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}}^{*} \nsubseteq \text{null} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n_{\kappa}} \right]$. Hence $\text{range} \left[ \text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}} \right] \nsubseteq \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n_{\kappa}} \right]$, i.e., $\text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}}$ is not $(k, \ell)$-connected.

The proof of the below result is similar to that of Lemma 5.

**Lemma 8** **The following are equivalent.**

1. The array $[A, (B_{\cdot})]$ is positively $(k, \ell)$-controllable.
2. $\mathcal{R} \supset \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$.

**Assumption 2** **If $\text{cl} \mathcal{R} \supset \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$ then $\mathcal{R} \supset \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$.

At the time of writing this paper we do not know whether it is possible that Assumption 2 is violated. For the benchmark example of network of double integrators it is not difficult to see that it holds. In general, Assumption 2 can be shown to hold for any array $[A, (B_{\cdot})]$ of chain of integrators with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $\text{inc} \left( B_{\cdot} \right) = G \otimes \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

where $G \in \mathbb{R}^{q \times p}$ is an incidence matrix with columns of the form $e_{i} - e_{j}$.

**Theorem 5** **Under Assumptions 7 and 8 the following two conditions are equivalent.**

1. The array $[A, (B_{\cdot})]$ is positively $(k, \ell)$-controllable.
2. The below statements simultaneously hold.
   (a) $\text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}}$ is strongly $(k, \ell)$-connected for all $\mu_{k} \in \mathbb{R}$.
   (b) $\text{inc} \left( Q_{[\kappa]}^{[\sigma]} \right)_{\sigma \in \mathcal{I}_{\kappa}}$ is $(k, \ell)$-connected for all $\mu_{k} \notin \mathbb{R}$.

**Proof.** 1 $\implies$ 2. Suppose the array $[A, (B_{\cdot})]$ is positively $(k, \ell)$-controllable. By Lemma 8 we have $\mathcal{R} \supset \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$. This implies $\mathcal{R}^{\circ} \subset \text{null} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$. Then by Assumption 1 and Theorem 4 the second condition follows.

2 $\implies$ 1. Suppose the second condition holds. By Assumption 1 and Theorem 4 we have $\mathcal{R}^{\circ} \subset \text{null} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$. This implies $\mathcal{R}^{\circ} \supset \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$. Since $\mathcal{R}^{\circ} = \text{cl} \mathcal{R}$ Assumption 2 yields $\mathcal{R} \supset \text{range} \left[ \left( e_{k} - e_{\ell} \right)^{*} \otimes I_{n} \right]$. Then by Lemma 8 the array $[A, (B_{\cdot})]$ is strongly $(k, \ell)$-controllable.
8 Conclusion

For networks of relatively actuated LTI systems we established in this paper that certain controllability properties of an array and certain connectivity properties of a set of graphs obtained from the array are equivalent. The main findings rested on four theorems. First, in Theorem 1 we presented the equivalence between array controllability and graph connectivity. Then in Theorem 2 we stated that an array can be steered by positive controls if the constructed graphs are strongly connected. Those two theorems in the first half of the paper were related to the overall controllability of the array. In the second half we focused on the problem of controlling the difference of the states of a particular pair of systems in the array. To this end, in Theorem 3 we obtained that this pairwise controllability can be understood through pairwise connectivity of certain graphs. Finally, in Theorem 4 we showed that positive pairwise controllability is closely related to strong pairwise connectivity of the graphs associated to the array.

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