Linear Matrix Inequality Method for a Quadratic Performance Index Minimization Problem with a class of Bilinear Matrix Inequality Conditions

M Tanemura$^1$ and Y Chida$^2$

$^1$Interdisciplinary Graduate School of Science and Technology, Shinshu University, JP
$^2$Faculty of Engineering, Shinshu University, JP
E-mail: 15st208h@shinshu-u.ac.jp

Abstract. There are a lot of design problems of control system which are expressed as a performance index minimization under BMI conditions. However, a minimization problem expressed as LMIs can be easily solved because of the convex property of LMIs. Therefore, many researchers have been studying transforming a variety of control design problems into convex minimization problems expressed as LMIs. This paper proposes an LMI method for a quadratic performance index minimization problem with a class of BMI conditions. The minimization problem treated in this paper includes design problems of state-feedback gain for switched system and so on. The effectiveness of the proposed method is verified through a state-feedback gain design for switched systems and a numerical simulation using the designed feedback gains.

1. Introduction

Bilinear matrix inequalities (BMIs) are used as a way to express minimization problem in a lot of control design problems. However, it is difficult to obtain the global optimal solution of the problems because of the non-convex property of BMIs. In order to overcome the difficulty, the techniques of changing variables are often used to transform BMIs into linear matrix inequalities (LMIs). In the minimization of a linear function with LMIs, the global optimal solution can be obtained because of the convex property of LMIs. Therefore, many researchers have been studying transforming a variety of control design problems into convex minimization problems expressed as LMIs. This paper deals with a quadratic performance index minimization problem with a class of BMIs. It is assumed that the BMIs treated in this paper can be transformed into LMIs by the techniques of changing variables. This minimization problem, for instance, includes design problems of gain for switched linear systems [4, 5]. This paper transforms into a convex minimization problem with LMIs as follows: first, the BMIs are transformed into LMIs by the changing variables, that is, the changing variables linearize the non-convex BMI constraints. However, the changing variables transform a matrix inequality of minimization of the performance index into a non-convex constraint. The matrix inequality includes a second-order term of the new variables and therefore it becomes a non-convex constraint for the new variables. This paper proposes a sufficient LMI condition to satisfy the non-convex inequality of minimization of the performance index. The analysis of the conservative of the proposed condition is difficult; however, numerical simulations show the practicability of the condition.
The effectiveness of the proposed method is verified through a state-feedback gain design for switched systems and a numerical simulation using the designed feedback gains.

2. Problem statement
This paper deals with the following optimal problem:

$$\min_{x, X} J(x) \text{ s.t. } C_{BMI}(x, X),$$

where $x \in \mathbb{R}^{n_x \times 1}$ denotes the vector of decision variables; $J(x)$ is a performance index; $C_{BMI}(x, X)$ denotes BMI constraints for $x$ and $X$; $X \in \mathbb{R}^{n_x \times n_x}$ is a matrix variable. It is assumed that this optimal problem possesses the following two properties.

Property 1
The performance index, $J(x)$, consists of the following quadratic equation for $x$:

$$J(x) = x^T A x + b^T x + c; \quad A > 0,$$

$$A \in \mathbb{R}^{n_x \times n_x}, \quad b \in \mathbb{R}^{n_x \times 1}, \quad c \in \mathbb{R}. \tag{2}$$

Property 2

$C_{BMI}(x, X)$ can be transformed into $C_{LMI}(y, Y)$ for $y$ and $Y(X)$ by the changing variable,

$$y = Y(X)x, \quad Y(X) > 0,$$

where $y \in \mathbb{R}^{n_x \times 1}$ is the new variable vector; $Y(X) \in \mathbb{R}^{n_x \times n_x}$ is a positive definite matrix and an affine matrix for $X$. An example of $Y(X)$ is shown as follows:

$$Y(X) = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}. \tag{3}$$

The optimal problem (1) is hard to solve because of the non-convex property of $C_{BMI}(x, X)$. This paper proposes a sufficient condition to transform this problem into a convex problem.

3. Proposed method

3.1. Outline
First, a completing the square eliminates the first-order term of $J(x)$, that is, it modifies $J(x)$ to a performance index consisting of a second-order and constant terms. Second, the changing variable transforms $C_{BMI}(x, X)$ into $C_{LMI}(y, Y)$. Third, the changed variables are substituted for the performance index. Then, the Schur complement derives the matrix inequality of the minimization of the performance index. The matrix inequality is the non-convex constraint because it includes a second-order term of a variable matrix. Finally, the sufficient LMI condition to satisfy the matrix inequality of the minimization will be proposed.

3.2. Completing the square and changing variables

In order to eliminate the first-order term, $b^T x$, in (2), $J(x)$ is modified by completing the square. First, $x_{opt}$ minimizing $J(x)$ without considering $C_{BMI}(x, X)$ is determined. Equation (2) is differentiated with respect to $x$ as follows:

$$\frac{dJ(x)}{dx} = 2Ax + b. \tag{5}$$
The inverse matrix of $A$ exists because of the positive definiteness of $A$, thereby $x_{opt}$ is determined as
\[ x_{opt} = -\frac{1}{2}A^{-1}b \] (6)
from (5). Then, substituting $\Delta x = x - x_{opt}$, which is an error from $x$ to $x_{opt}$, for $J(x)$ derives the new performance index $J(\Delta x)$,
\[ J(\Delta x) = \Delta x^TA\Delta x - \frac{1}{4}b^TA^{-1}b + c, \] (7)
consisting of the second-order term of $\Delta x$ and the constant terms. Thereby $C_{BMI}(x, X)$ is transformed into a new constraints $C_{BMI}(\Delta x, X)$. Second, the changing variable, $y = Y(X)\Delta x$, $Y(X) > 0$, (8)
for $C_{BMI}(\Delta x, X)$ derives the LMI constraint, $C_{LMI}(y, Y)$; where, $Y(X)$ is the positive definite matrix and the affine matrix for $X$. The existence of $Y(X)$ linearizing $C_{BMI}(\Delta x, X)$ is guaranteed from property 2. Third, substituting the new variables, $y$ and $Y(X)$, for the performance index (7) derives
\[ J(y, Y) = y^TY^{-1}AY^{-1}y - \frac{1}{4}b^TA^{-1}b + c. \] (9)
In order to derive the matrix inequality of the minimization of $J(y, Y)$, $\gamma \in \mathbb{R}$ satisfying $\gamma > J(y, Y)$ is introduced. $\gamma > J(y, Y)$ is transformed into the following inequality,
\[ \begin{bmatrix} YA^{-1}Y & y \\ y^T & \gamma + \frac{1}{4}b^TA^{-1}b - c \end{bmatrix} > 0, \] (10)
by the Schur complement. The minimization of $\gamma$ in (10) equals the minimization of $J(y, Y)$. Finally, a different representation of the minimization problem (1),
\[ \min_{\gamma, y, Y} \gamma \quad \text{s.t. (10) and } C_{LMI}(y, Y), \] (11)
is obtained. Even though $C_{LMI}(y, Y)$ is the LMI condition; (10) is the non-convex condition because (10) includes the second-order term of $Y$ at the $(1,1)$ element. In the following section, the sufficient LMI condition to satisfy (10) will be proposed.

3.3. Sufficient LMI condition
The theorem about the sufficient condition of (10) is shown as follows. **Theorem**
If there exist $Y \in \mathbb{R}^{n_x \times n_x}$, $y \in \mathbb{R}^{n_x \times 1}$ and $\gamma \in \mathbb{R}$ satisfying the following LMIs:
\[ Y > \alpha A, \] (12)
\[ \begin{bmatrix} \alpha Y & y \\ y^T & \gamma + \frac{1}{4}b^TA^{-1}b - c \end{bmatrix} > 0, \] (13)
where $\alpha > 0$ is a positive constant, then $Y$, $y$ and $\gamma$ satisfy (10).
Proof. \( A^{-1} > \alpha Y^{-1} \) holds from (12); therefore,

\[
YA^{-1}Y > Y\alpha Y^{-1}Y
\]

\[
= \alpha Y
\]

holds. Then, \( Z \) is defined as \( YA^{-1}Y - \alpha Y =: Z \) \((> 0)\); as a result, (10) is expressed as follows:

\[
\begin{bmatrix}
YA^{-1}Y & y \\
y^T & \gamma + \frac{1}{4} b^T A^{-1} b - c
\end{bmatrix}
= \begin{bmatrix}
\alpha Y & y \\
y^T & \gamma + \frac{1}{4} b^T A^{-1} b - c
\end{bmatrix}
+ \begin{bmatrix}
Z & 0 \\
0 & 0
\end{bmatrix}.
\]

(16)

The first term on the right side of (16) is a positive definite matrix from (13) and the second term on the right side of (16) is a positive semidefinite matrix from \( Z > 0 \). Therefore, (16) is a positive definite matrix, therefore, (10) is satisfied.

From the above, the convex optimal problem,

\[
\min_{\gamma,Y} \gamma \quad \text{s.t. (12), (13) and } C_{LMI}(y,Y),
\]

(17)

is solved instead of (11), where it should be noted that \( Y(X) \) consists of \( X \). The global optimal solution, \( \gamma, \hat{y} \) and \( Y(\hat{X}) \) of (17) can easily be obtained because the optimal problem (17) possesses the convex property for the variables. With the obtained solutions of (17),

\[
\Delta \hat{x} := \hat{Y}(\hat{X})^{-1} \hat{y},
\]

(18)

is calculated. Then, an optimal solution \( \hat{x}_{opt} \) satisfying \( C_{BMI}(x,X) \) is obtained by

\[
\hat{x}_{opt} = x_{opt} + \Delta \hat{x}.
\]

(19)

4. Feedback gain design for switched systems

4.1. Plant and control system

The proposed method is applied to the state-feedback gain design for switched systems. This paper deals with \( N \) discrete-time single input systems,

\[
x_g[k+1] = A_{gi}x_g[k] + b_{gi}u_g[k], \quad i = 1, \cdots, N,
\]

(20)

\[
A_{gi} \in R^{n_g \times n_g}, \quad b_{gi} \in R^{n_g \times 1},
\]

with the state-feedback control,

\[
u_g[k] = -f_{gi}^T x_g[k], \quad i = 1, \cdots, N.
\]

(21)

In the following section, the design method of the feedback gains \( f_{gi} \) is mentioned.

4.2. State-feedback gain design

The sufficient condition of the stability of the switched systems (20) controlled by (21) is the existence of a matrix \( P \in R^{n_g \times n_g} \) such that [6]

\[
(A_{gi} - b_{gi}f_{gi}^T)^T P (A_{gi} - b_{gi}f_{gi}^T) - P < 0, \quad P > 0, \quad i = 1, \cdots, N.
\]

(22)

Designing \( f_{gi} \) satisfying (22) guarantees the stability of the closed system of (20) and (21). Furthermore, a performance index is introduced to design \( f_{gi} \) possessing the high control
This paper decides ideal gains $f_{gi}^*$, and then, designs $f_{gi}$ minimizing an error between $f_{gi}$ and $f_{gi}^*$. Accordingly, the quadratic performance index,

$$J(f_g) = (f_g - f_g^*)^T M (f_g - f_g^*)$$

$$= f_g^T M f_g - 2 f_g^T M f_g^* + f_g^* M f_g^*,$$  \hspace{1cm} (23)

is introduced, where $M > 0$ is a weighting matrix. From the above, solving the optimal problem,

$$\min_{f_g, P} J(f_g) \text{ s.t. } (22),$$  \hspace{1cm} (24)

designs $f_g$ that stabilizes the closed system of (20) and (21), and possesses the high control performance. However, the optimal problem (25) is the non-convex problem for variables $f_g$ and $P$. The optimal problem (25) resolves into a convex optimal problem with the proposed sufficient condition in the following section.

### 4.3. Linearization with proposed method

From (5) and (6), the new variable $\Delta f_g$ is calculated by

$$\Delta f_g = f_g - f_g^* =: \begin{bmatrix} \Delta f_{g1} \\ \vdots \\ \Delta f_{gN} \end{bmatrix}. \hspace{1cm} (26)$$

Then, substituting $\Delta f_g$ for $J(f_g)$ derives the new performance index,

$$J(\Delta f_g) = \Delta f_g^T M \Delta f_g,$$  \hspace{1cm} (27)

consisting of the second-order term of $\Delta f_g$, and substituting $\Delta f_g$ for (22) derives the new condition,

$$(A_{gi}^* - b_{gi} \Delta f_{gi})^T P (A_{gi}^* - b_{gi} \Delta f_{gi}) - P < 0, \hspace{0.5cm} P > 0, \hspace{0.5cm} i = 1, \cdots, N,$$  \hspace{1cm} (28)

$$A_{gi}^* := A_{gi} - b_{gi} f_{gi}^*.$$  \hspace{1cm} (29)

In order to transform (28) into LMI conditions, the changing variable,

$$y_i = X \Delta f_{gi}, \hspace{0.5cm} X := P^{-1},$$  \hspace{1cm} (30)

is calculated and (28) is transformed into

$$\begin{bmatrix} X \\ X A_{gi}^* - y_i b_{gi}^T \\ X \end{bmatrix} > 0, \hspace{0.5cm} i = 1, \cdots, N$$

by the Schur complement. From the changed variable (29), the performance index (27) is reduced to

$$J(y, Y) = y^T Y^{-1} M Y^{-1} y,$$  \hspace{1cm} (31)

$$Y(X) := \begin{bmatrix} X & X & 0 \\ 0 & \cdots & X \\ 0 & \cdots & X \end{bmatrix}, \hspace{0.5cm} y := \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}. \hspace{1cm} (32)$$
Then, $\gamma$ satisfying $\gamma > J(y, Y)$ is introduced and $\gamma > J(y, Y)$ is transformed into
\[
\begin{bmatrix}
YM^{-1}Y & y \\
y^T & \gamma
\end{bmatrix} > 0
\]  \hspace{1cm} (33)
by the Schur complement. Minimizing $\gamma$ of (33) minimizes the performance index $J(y, Y)$; however, (33) is the non-convex condition because (33) has the product of $Y$. The proposed sufficient condition transforms the inequality (33) into LMI conditions,
\[
\begin{bmatrix}
\alpha Y & y \\
y^T & \gamma
\end{bmatrix} > 0,
\]  \hspace{1cm} (35)
where $\alpha > 0$ is a positive constant. From the above, the feedback gains are obtained by solving the convex optimal problem,
\[
\min_{\gamma, y, X, Y(X)} \gamma \quad \text{s.t.} \ (34), (35) \text{ and } (30),
\]  \hspace{1cm} (36)
where $Y(X)$ and $y$ are defined by (32). With $\hat{Y}(\hat{X})$ and $\hat{y}$, which are obtained by solving (36),
\[
\Delta \hat{f}_g := \hat{Y}(\hat{X})^{-1}\hat{y},
\]  \hspace{1cm} (37)
is calculated and the optimal solution $\hat{f}_g =: [\hat{f}_g^1 \cdots \hat{f}_g^{N}]^T$ is obtained by
\[
\hat{f}_g = \hat{f}_g^* + \Delta \hat{f}_g.
\]  \hspace{1cm} (38)

5. Verification

5.1. Numerical example

This section deals with the three systems of (20) defined by the following matrices:
\[
A_{g1} = \begin{bmatrix}
0 & 1 \\
-1 & -1
\end{bmatrix}, \quad A_{g2} = \begin{bmatrix}
0 & 1 \\
1 & -1.5
\end{bmatrix}, \quad A_{g3} = \begin{bmatrix}
0 & 1 \\
-1 & 5
\end{bmatrix},
\]  \hspace{1cm} (39)
\[
b_{g1} = \begin{bmatrix}
1 \\
0
\end{bmatrix}, \quad b_{g2} = \begin{bmatrix}
0 & 0.5
\end{bmatrix}, \quad b_{g1} = \begin{bmatrix}
0 & -20
\end{bmatrix}.
\]  \hspace{1cm} (40)
The weighting matrices,
\[
Q_1 = \text{diag}[10, 10], \quad r_1 = 1,
\]  \hspace{1cm} (41)
\[
Q_2 = \text{diag}[10, 1], \quad r_2 = 1,
\]  \hspace{1cm} (42)
\[
Q_3 = \text{diag}[1, 10], \quad r_3 = 1,
\]  \hspace{1cm} (43)
were defined for each system and the ideal feedback gains,
\[
\hat{f}_{g1}^* = \begin{bmatrix}
-0.6180 \\
0.3820
\end{bmatrix}, \quad \hat{f}_{g2}^* = \begin{bmatrix}
1.9993 \\
-3.0000
\end{bmatrix}, \quad \hat{f}_{g3}^* = \begin{bmatrix}
0.0500 \\
-0.2500
\end{bmatrix},
\]  \hspace{1cm} (44)
were designed based on the optimal regulator. It is noted that (22) for $\hat{f}_{g1}^*$ is not feasible. The other parameters were decided as
\[
M = \begin{bmatrix}
Q_1^{-1} & 0 & 0 \\
0 & Q_2^{-1} & 0 \\
0 & 0 & Q_3^{-1}
\end{bmatrix}, \quad \alpha = 1.
\]  \hspace{1cm} (45)
Under the above parameters, solving the convex optimal problem (36) determined the feedback gains,

$$
\hat{f}_{g1} = \begin{bmatrix} -0.6555 \\ 0.3445 \end{bmatrix}, \quad \hat{f}_{g2} = \begin{bmatrix} 1.9993 \\ -2.9257 \end{bmatrix}, \quad \hat{f}_{g3} = \begin{bmatrix} 0.0500 \\ -0.2519 \end{bmatrix}.
$$

(47)

\( \hat{\gamma} \) of the sufficient condition is \( \hat{\gamma} = 0.0124 \) and the performance index \( J(\hat{f}_{g}) = J(\hat{f}_{g}) = 0.0058 \). For this reason, the conservative of the sufficient condition is small in this example.

5.2. Comparison with guaranteed cost control

This section designs the feedback gains based on the guaranteed cost control [7, 8, 9] in order to compare their control performances. From Ref. [9], if positive definite matrices \( S_i \in \mathbb{R}^{n_g \times n_g} \), vectors \( v_i \in \mathbb{R}^{1 \times n_g} \) and matrices \( G_i \in \mathbb{R}^{n_g \times n_g} \) satisfy

$$
\begin{bmatrix}
G_i + G_i^T - S_i & G_iA_i^T - v_i b_i & G_i^T v_i \\
A_g G_i - b_i v_i^T & S_j & 0 \\
G_i & 0 & Q_i^{-1} \\
v_i^T & 0 & 0 & r_i^{-1}
\end{bmatrix} > 0, \quad \forall (i, j) \in \{1, \cdots, N\} \times \{1, \cdots, N\},
$$

(48)

the state-feedback control (21) with \( f_{gi} = G_i^{-T} v_i^T \) guarantees the stability of the closed system and guarantees

$$
\sum_{k=0}^{\infty} \{x_g[k]TQ_i x_g[k] + u_g[k]r_i u_g[k]\} < x_g[0]^T S_r^{-1} x_g[0],
$$

(49)

where \( r_0 \) denotes the initial subsystem. Therefore, when the initial subsystem \( r_0 \) and the initial value \( x_g[0] \) are known, solving the optimal problem,

$$
\min_{\tau, S_i, v_i, G_i} \tau \text{ s.t. } \begin{bmatrix} -\tau & x_g[0]^T \\
-x_g[0] & -S_r \end{bmatrix} < 0 \text{ and (48)},
$$

(50)

determines \( f_{gi} = G_i^{-T} v_i^T \) minimizing (49).

\( Q_i \) and \( r_i \) in (48) were set as (41), (42) and (43); the initial subsystem \( r_0 \) was set as \( r_0 = 2 \); the initial value \( x_g[0] \) was set as \( x_g[0] = [1 \ 0.5]^T \), and then, the feedback gains \( f_{cost} \),

$$
\hat{f}_{g1}^{cost} = \begin{bmatrix} -0.7175 \\ 0.2825 \end{bmatrix}, \quad \hat{f}_{g2}^{cost} = \begin{bmatrix} 2.065 \\ -1.737 \end{bmatrix}, \quad \hat{f}_{g3}^{cost} = \begin{bmatrix} 0.0500 \\ -0.2809 \end{bmatrix}
$$

(51)

were obtained by solving the optimal problem (50). The following section compares the control performance of the gains (47) and the gains (51) through a numerical simulation.

5.3. Numerical simulation

The numerical simulation is carried out under the following conditions: a sampling period \( T_s \) is \( T_s = 0.01 \text{ s} \), the initial value of \( x_g[k] \) is \( x_g[0] = [1 \ 0.5]^T \), and a switching signal is random as shown in figure 1. Simulation results are shown in figure 2, 3, and 4. Figure 2 shows the time histories of \( u_g[k] \), and figure 3 and 4 show the time responses of \( x_{g1}[k] \) and \( x_{g2}[k] \), respectively. The solid line is the case of which controllers are \( \hat{f}_{gi} \) of (47) designed by the proposed method, and the broken line is the case of which controllers are \( \hat{f}_{gi}^{cost} \) of (51) by the guaranteed cost. Figure 3 and 4 show that the states of the proposed method rapidly converge to zero compared with that of the guaranteed cost. Accordingly, this numerical simulation shows that the feedback gains possessing the high performance are designed by solving the optimal problem using the proposed sufficient condition.
6. Conclusion
This paper deals with the minimization problem of the quadratic performance index with the BMIs that can be transformed into LMIs by the changing variables. This paper proposes the sufficient condition to linearize the optimal problem. The effectiveness of the proposed method is verified through the state-feedback gain design for switched systems and the numerical simulation using the designed feedback gains. The numerical simulation shows that solving the optimal problem using the proposed sufficient condition designs the feedback gains possessing the high performance compared with that of the guaranteed cost.

References
[1] Scherer C, Gahinet P and Chilali M 1997 IEEE Trans. Automat. Contr. 42 896-911
[2] Ikeda M, Lee T W and Uezato E 2000 Decision and Control Proc. Conf. (Sydney) 1 601-4
[3] Montagner V.F, Leite V.J.S, Oliveira R.C.L.F. and Peres P.L.D. 2006 Journal of computational and applied mathematics 194 192-206
[4] Kumata S, Tanemura M and Chida Y 2015 the Japan Joint Automatic Control Proc. Conf. (Hyogo) (in Japanese)
[5] Nakamura Y, Hirata K, Sugimoto K and Kogiso K 2007 Intelligent Control Symp. Int. Conf. (Singapore) 184-9
[6] Hai L and Antsaklis P J 2009 IEEE Trans. Automat. Contr. 54 308-22
[7] Yang G-H, Wang J L and Soh Y C 2000 Linear algebra and its applications 312 161-80
[8] Mukaidani H, Ishii Y, Nan B and Tsuji T 2004 IEEE Decision and Control 1 809-14
[9] Ying Z, Guangren D and Yannming F. 2006 IEEE Intelligent Control and Automation Proc. Cong. (Dalian) 1 1309-15