THE RENORMALIZATION METHOD AND QUADRATIC-LIKE MAPS

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ABSTRACT. The renormalization of a quadratic-like map is studied. The three-dimensional Yoccoz puzzle for an infinitely renormalizable quadratic-like map is discussed. For an unbranched quadratic-like map having the a priori complex bounds, the local connectivity of its Julia set is proved by using the three-dimensional Yoccoz puzzle. The generalized version of Sullivan’s sector theorem is discussed and is used to prove his result that the Feigenbaum quadratic polynomial has the a priori complex bounds and is unbranched. A dense subset on the boundary of the Mandelbrot set is constructed so that for every point of the subset, the corresponding quadratic polynomial is unbranched and has the a priori complex bounds.

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1. Introduction

Let $P_c(z) = z^2 + c$ be a quadratic polynomial. A central problem in the study of the dynamics of $P_c$ is to understand the topology and geometry of the Julia set $J_c$ of $P_c$. The filled-in Julia set $K_c$ of $P_c$ is, by definition, the set of points not going to infinity under iterations of $P_c$. The Julia set $J_c$ of $P_c$ is, by definition, the boundary of $K_c$.

In order to have the more penetrating study of the dynamics of a quadratic polynomial $P_c$, Douady and Hubbard [DH3] introduced the concept of a quadratic-like map. In §1, we review the definition of a quadratic-like map and the work of Douady and Hubbard [DH3] which proves that a quadratic-like map with connected Julia set is hybrid equivalent to a unique quadratic polynomial. We also review the result due to Douady and Yoccoz (see [HUB, MI2]) about the landing of external rays at a repelling periodic point of a quadratic polynomial. In the same section, we also review some fundamental results about the Julia set of a quadratic-like map and some basic facts of hyperbolic geometry.

The Mandelbrot set $\mathcal{M}$ is the set of complex parameters $c$ such that the Julia set $J_c$ of $P_c$ is connected. The Julia set $J_c$ in the complement of $\mathcal{M}$ is a Cantor set. Douady and Hubbard [DH1] proved that $\mathcal{M}$ is connected. They further conjectured that $\mathcal{M}$ is locally connected. Many important research problems follow from this conjecture. For example, the hyperbolicity conjecture (explained below) would follow from this conjecture [DH2]. A quadratic polynomial is hyperbolic if it has an attractive or super-attractive periodic point in the complex plane. Let $\mathcal{HP}$ be the set of parameters $c$ such that $P_c$ is hyperbolic. The hyperbolicity conjecture says that $\mathcal{HP}$ is open and dense in $\mathcal{M}$. To study the local connectivity of the Mandelbrot set $\mathcal{M}$, it would be helpful to answer the question: for which $c$ in $\mathcal{M}$ is the Julia set $J_c$ locally connected? Yoccoz made substantial progress in this direction. He proved that if $P_c$ is non-renormalizable (or finitely renormalizable), then the Julia set $J_c$ is locally connected. We discuss his result in §3. (Using this result, Yoccoz further proved that $\mathcal{M}$ is locally connected at a finitely renormalizable point $c$ (see [HUB])). Also in §3, we discuss the two-dimensional Yoccoz puzzle of a quadratic-like map and its relation with the renormalizability of this quadratic-like map.

There remain many points in $\mathcal{M}$ which are infinitely renormalizable. In §2, we define infinitely renormalizable quadratic-like maps and define infinitely renormalizable folding mappings, and discuss the relation between the two definitions.

In §4, we prove one of our main results: for an unbranched infinitely renormalizable quadratic-like map having the a priori complex bounds, its Julia set is locally connected. We prove this result by using the three-dimensional Yoccoz puzzle of an infinitely renormalizable quadratic-like map. Also in §4, we prove that the filled-in Julia set of any renormalization of a renormalizable quadratic-like map, about the
period of the two-dimensional Yoccoz puzzle, does not depend on the choices of renormalization domains. In particular, the renormalized filled-in Julia set is the limiting component in two-dimensional Yoccoz puzzle containing the critical point.

In §5, we discuss Sullivan's sector theorem. We prove a generalized version. The proof repeatedly applies the sharpest version of Koebe's distortion theorem (see [BIE]), and uses Sullivan's idea about using hyperbolic contraction to trap points in a hyperbolic neighborhood.

Using the generalized version of Sullivan's sector theorem, we prove in §6 his result that the Feigenbaum quadratic polynomial has the a priori complex bounds and is unbranched. By combining this with the result in §4, we complete the proof of the result which was first announced in [JIH] and which says that the Julia set of the Feigenbaum quadratic polynomial is locally connected.

In §7, we construct a subset $\tilde{\Upsilon}$ of the Mandelbrot set which is dense on the boundary $\partial M$ of the Mandelbrot set $M$ such that for every point $c$ in this subset, the corresponding quadratic polynomial $P_c(z) = z^2 + c$ is unbranched and infinitely renormalizable and has the a priori complex bounds. Thus, for $c$ in $\tilde{\Upsilon}$, the Julia set $J_c$ of a quadratic polynomial $P_c(z) = z^2 + c$ is locally connected. A similar result concerning about the local connectivity of the Mandelbrot set at infinitely renormalizable points is proved in [JI4].

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2. Quadratic Polynomials, Quadratic-Like Maps, and Hyperbolic Geometry

Let $P_c(z) = z^2 + c$ be a quadratic polynomial. Let $C$ be the complex plane. Let $V = \{ z \in C \mid |z| < r \}$ be a disk in $C$. For $r$ large enough, $U = P_c^{-1}(V)$ is a simply connected domain, its closure is relatively compact in $V$, and $P_c : U \to V$ is a holomorphic, proper map of degree two (see Fig. 1). This is a model of an object defined by Douady and Hubbard [DH3].
Definition 1. A quadratic-like map is a triple \((U, V, f)\) where \(U\) and \(V\) are simply connected domains isomorphic to a disc with \(\overline{U} \subset V\), and where \(f : U \rightarrow V\) is a holomorphic, proper map of degree two (see Fig. 1).

Remark 1. A proper map \(f\) means that \(f^{-1}(K)\) is compact for every compact set \(K\). Douady and Hubbard [DH3] also defined a polynomial-like map.

Suppose \((U, V, f)\) is a quadratic-like map. We use 
\[
K_f = \cap_{n=0}^{\infty}f^{-n}(U)
\]
to denote the set of \(z\) in \(U\) such that images of \(z\) under iterations of \(f\) are all in \(U\). The set \(K_f\) is called the filled-in Julia set. It is a compact subset of \(U\). The Julia set \(J_f\) of \(f\) is the boundary of \(K_f\).

A quadratic-like map \((U, V, f)\) has only one branched point \(b\) at which the derivative \(f'(b)\) of \(f\) equals zero. We call \(b\) the critical point of \(f\) and, without loss of generality, we always assume that \(b = 0\). The following theorem is well-known.

Theorem 1. Suppose \((U, V, f)\) is a quadratic-like map. If the critical point 0 is not in \(K_f\), then \(K_f = J_f\) is a Cantor set of zero Lebesgue measure in \(\mathbb{C}\). And moreover, the set \(K_f\) (or \(J_f\)) is connected if and only if the critical point 0 is in \(K_f\).

Suppose that \(\Omega\) is a domain in \(\mathbb{C}\) and that \(f\) is a self-map of \(\Omega\). A point \(z\) in \(\Omega\) is called a periodic point of period \(k \geq 1\) if \(f^{ok}(z) = z\) and \(f^{oi}(z) \neq z\) for \(1 \leq i < k\). The number \(\lambda = (f^{ok})'(z)\) is called the multiplier (or eigenvalue) of \(f\) at \(z\). A periodic point of period 1 is called a fixed point. A periodic point \(z\) of \(f\) is said to be repelling, attractive, or neutral if \(|\lambda| > 1\), \(0 < |\lambda| < 1\), or \(|\lambda| = 1\). Moreover, a periodic point is called super-attractive if \(\lambda = 0\), and is called parabolic if \(\lambda = e^{2\pi ip/q}\) where \(p\) and \(q\) are integers. For an attractive or super-attractive or parabolic periodic point of period \(k \geq 1\) of a quadratic-like map \(f : U \rightarrow V\), let \(O(p) = \{f^{oi}(p)\}^{k-1}_{i=0}\) be the periodic orbit of \(f\). The set 
\[
B(p) = \{z \in V \mid f^{on}(p) \rightarrow O(p) \text{ as } n \rightarrow \infty\}
\]
is called the basin of \(O(p)\). Let \(CB(f^{oi}(p))\) be the connected component containing \(f^{oi}(p)\) of \(B(p)\). The set \(IB(p) = \cup_{i=0}^{k-1} CB(f^{oi}(p))\) is called the immediate basin of
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O(p). The proofs of the following two theorems can be found in Blanchard’s survey article [BLA].

**Theorem 2.** Suppose $z_0$ is a super-attractive periodic point of period $k$ of a holomorphic function $f$ defined on $\Omega$. Then there is a neighborhood $U$ of $z_0$, a unique holomorphic diffeomorphism $h : U \to h(U)$ with $h(z_0) = 0$ and $h'(z_0) = 1$, and a unique integer $n > 1$ such that

$$h \circ f^k \circ h^{-1}(z) = z^n$$

on $h(U)$.

**Theorem 3.** Let $(U, V, f)$ be a quadratic-like map and let $J_f = \partial K_f$ be its Julia set. Let $E_f$ be the set of all repelling periodic points of $f$. Then

1. $J_f$ is completely invariant, i.e., $f(J_f) = J_f$ and $f^{-1}(J_f) = J_f$;
2. $J_f$ is perfect, i.e., $J_f' = J_f$, where $J_f'$ means the set of limit points of $J_f$;
3. $E_f$ is dense in the Julia set $J_f$, i.e., $\overline{E_f} = J_f$;
4. for any $z$ in $V$, the limit set of $\{f^{-n}(z)\}_{n=0}^{\infty}$ is $J_f$;
5. $J_f$ has no interior point;
6. If $f$ has an attractive or super-attractive or parabolic periodic point $p$ in $V$, then the immediate basin $IB(p)$ contains the critical point 0 and the critical orbit $O(0) = \{f^{on}(0)\}_{n=0}^{\infty}$;
7. If $f$ has neither any attractive, nor any super-attractive, nor any neutral periodic points in $V$, then $K_f = J_f$.

Let $(U, V, f)$ and $(U', V', g)$ be two quadratic-like maps. They are topologically conjugate if there is a homeomorphism $h$ from a neighborhood $K_f \subset X \subset U$ to a neighborhood $K_g \subset Y \subset U'$ such that $h \circ f = g \circ h$ on $X$ where $K_f$ and $K_g$ are filled-in Julia sets. If $h$ is quasiconformal (see [AH1]) (respectively, holomorphic), then they are quasiconformally (respectively, holomorphically) conjugate. If $h$ can be chosen such that $h_\pi = 0$ a.e. on $K_f$, then they are hybrid equivalent. Let

$$I(f) = \{g \mid g \text{ is hybrid equivalent to } f\}$$

be the inner class of $f$. The following theorem is due to Douady and Hubbard.

**Theorem 4 [DH3].** If $(U, V, f)$ is a quadratic-like map such that $K_f$ is connected, then there is a unique quadratic polynomial $P(z) = z^2 + c_f$ in $I(f)$.

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. Then $\infty$ is a super-attractive fixed point of any quadratic polynomial $P_c(z) = z^2 + c$. The filled-in Julia set $K_c$ of $P_c$ is the set of all points not going to infinity under iterations of $P_c$. Let $D_r = \{z \in \mathbb{C} \mid |z| < r\}$. As we mentioned in the beginning of this section, for $r > 1$ large enough, $U = P_c^{-1}(D_r)$ is a simply connected domain and $\overline{U} \subset D_r$. 
Thus \((U, V, P_c)\) for \(V = D\) is a quadratic-like map. In particular, \((D_r, D_{1/2}, P_0)\) is a quadratic-like map for every \(r > 1\); the filled-in Julia set \(K_0\) of \(P_0\) is the closed unit disk \(D_1\). Applying Theorem 2, there is a holomorphic diffeomorphism \(h_1\) defined on a neighborhood \(\mathbb{C} \setminus D_r\) (for \(r > 1\) large) about \(\infty\) such that \(h_1(\infty) = \infty, h'_1(\infty) = 1\), and such that

\[ h_1^{-1} \circ P_c \circ h_1(z) = z^2 \]

on \(\mathbb{C} \setminus D_r\). Let \(B_1(\infty) = h_1(\mathbb{C} \setminus D_r)\) and let \(B_n(\infty) = P_{-n}(B_1(\infty))\). If the filled-in Julia set \(K_c\) of \(P_c\) is connected, then all

\[ P_c: B_n(\infty) \cap \mathbb{C} \to B_{n-1}(\infty) \cap \mathbb{C} \]

are unramified covering maps of degree two. We can inductively define holomorphic diffeomorphisms \(h_n\) on \(\mathbb{C} \setminus D_{r^{1/n}}\) such that

\[ h_n^{-1} \circ P_c \circ h_n(z) = z^2 \]

for \(z\) in \(\mathbb{C} \setminus D_{r^{1/n}}\) and for \(n > 1\). As \(n\) tends to infinity, we get a holomorphic diffeomorphism \(h_\infty\) defined on \(\mathbb{C} \setminus D_1\) such that

\[ h_\infty^{-1} \circ P_c \circ h_\infty(z) = z^2 \]  

(1)

for all \(z\) in \(\mathbb{C} \setminus D_1\). Therefore, \(B_c(\infty) = h_\infty(\mathbb{C} \setminus D_1)\) is the basin of \(\infty\) for \(P_c\) and \(K_c = \mathbb{C} \setminus B_c(\infty)\). Furthermore, for every \(r > 1\) and for \(U_r = h_\infty(D_r)\), \((U_r, U_{r^2}, P_c)\) is a quadratic-like map and its filled-in Julia set is always \(K_c\). Let \(S^R = \{z \in \mathbb{C} \mid |z| = R\}\) and let \(s_R = h_\infty(S^R)\) for \(R > 1\). Then

\[ P_c(s_R) = s_{R^2}. \]  

(2)

The topological circle \(s_R\) for every \(R > 1\) is called an equipotential curve of \(P_c\). A curve

\[ e_\theta = h_\infty(\{z \in \mathbb{C} \mid |z| > 1, \arg(z) = \theta\}) \]

for \(0 \leq \theta < 2\pi\) is called an external ray of \(P_c\). Then

\[ P_c(e_\theta) = e_{2\theta}. \]  

(3)

Remark 2. Let

\[ G(z) = \max\{0, \lim_{n \to \infty} \frac{1}{2n} \log |P_c^n(z)|\} \]

be the Green’s function of \(K_c\) in \(\mathbb{C}\). Then \(G(P_c(z)) = 2G(z)\). For any \(R > 1\), the equipotential curve \(s_R = G^{-1}(\log R)\) is a level curve of \(G\).

If the mapping \(h_\infty\) in Equation (1) can be extended continuously to the unit circle \(S^1\), then we have a unique continuous map \(H: \mathbb{C} \setminus D_1 \to \mathbb{C} \setminus K_f\) such that \(H|((\mathbb{C} \setminus D_1)) = h_\infty\). Using \(H\), we can define an equivalence relation on \(S^1\): \(z_1 \sim z_2\) if and only if \(H(z_1) = H(z_2)\). Let \([z]\) be the equivalent class of \(z\). Then \(P_0([z]) = [P_0(z)]\) defines a map of the quotient space \(X = S^1/\sim\), since \(z^2 \sim z^2\) if \(z_1 \sim z_2\). The
dynamical system \((\tilde{P}_0, X)\) is topologically conjugate to \((P_c, J_c)\) by \(\tilde{H}(z) = H(z)\). The question arises:

**Question 1.** For which \(c\) can \(h_\infty\) be extended continuously to the unit circle \(S^1\)?

A connected set \(X\) in \(\mathbb{C}\) is locally connected if for any point \(p\) in \(X\) and any neighborhood \(V\) about \(p\), there is another neighborhood \(U \subset V\) about \(p\) such that \(U \cap X\) is connected. The following classical theorem proved by Carathéodory in one complex variable gives a sufficient and necessary condition to extend \(h_\infty\) continuously to \(S^1\). The proof of this theorem can be found in [MI1].

**Theorem 5 [CAR].** Let \(h\) be a Riemann mapping from \(D_1\) onto a simply connected open domain \(\Omega\). Then \(h\) can be extended continuously to the unit circle \(S^1\) if and only if the boundary \(\partial \Omega\) (as well as \(\Omega\)) is locally connected.

**Remark 3.** If \(\partial \Omega\) is a Jordan curve, then \(h\) can be extended to a homeomorphism from \(D_1\) onto \(\overline{\Omega}\). Moreover, if \(\partial \Omega\) is made of finite number of analytic curves, then the extension restricted to the unit circle \(S^1\) has non-zero derivative at every point other than a corner (see [BIE]).

We have an equivalent question by just concerning the topology of a Julia set now:

**Question 2.** For which \(c\) is \(J_c\) locally connected?

An external ray \(e_\theta\) lands at \(J_c\) if \(e_\theta\) has only one limit point at \(J_c\). An external ray is periodic with period \(m\) if \(e_\theta \cap P^m_c(e_\theta) = \{\infty\}\) for \(1 \leq i < m\) and if \(P^m_c(e_\theta) = e_\theta\). The following theorem is proved by Douady and Yoccoz (refer to [MI1,MI2,HUB]).

**Theorem 6.** Let \(P_c(z) = z^2 + c\) be a quadratic polynomial with connected Julia set \(J_c\). Then every repelling periodic point of \(P_c\) is a landing point of finitely many periodic external rays with the same period.

Let \(S\) be a Riemann surface and let \((\tilde{S}, \pi)\) be the universal cover of \(S\), where \(\tilde{S}\) is a simple connected Riemann surface and \(\pi: \tilde{S} \to S\) is the universal covering map. From the Uniformization Theorem (see [AH2]), we can identify \(\tilde{S}\) with one of the extended complex plane \(\overline{\mathbb{C}}\), the complex plane \(\mathbb{C}\), or the open unit disk \(D_1\). A Riemann surface \(S\) is hyperbolic if \(\tilde{S} = D_1\).

Let \(D\) be the hyperbolic disk which is the open unit disk with the hyperbolic metric

\[
d_Hs = \frac{|dz|}{1-|z|^2}.
\]

Let \(d_H\) be the hyperbolic distance. Every holomorphic diffeomorphism \(h: D_1 \to D_1\), which is not a linear fractional transformation, strictly decreases the hyperbolic distance \(d_H\), i.e,

\[
d_H(h(z_1), h(z_2)) < d_H(z_1, z_2)
\]
for all \( z_1 \) and \( z_2 \) in \( D \). For a hyperbolic Riemann surface \( S \), one can define the hyperbolic distance \( d_{H,S} \) from \( d_H \) and \( \pi \). Any holomorphic map \( h : S \to S \), which is not an isometry with respect to \( d_{H,S} \), strictly decreases this hyperbolic distance \( d_{H,S} \), i.e., for any \( z_1 \) and \( z_2 \) in \( S \),

\[
d_{H,S}(h(z_1), h(z_2)) < d_{H,S}(z_1, z_2).
\]

A bounded domain \( \Omega \) in \( \mathbb{C} \) is a hyperbolic Riemann surface. An important family of hyperbolic Riemann surfaces is the family of bounded doubly connected domains in \( \mathbb{C} \). A bounded doubly connected domain \( \Omega \) is called an annulus. From complex analysis (see [BIE]), any annulus is holomorphically diffeomorphic to a unique round annulus \( A_r = \{ z \in \mathbb{C} \mid r < |z| < 1 \} \) for \( 0 < r < 1 \). The number

\[
\text{mod}(\Omega) = -\log r
\]

is called the modulus of \( \Omega \). It is a conformal invariant, i.e., \( \text{mod}(h(\Omega)) = \text{mod}(\Omega) \) whenever \( h \) is a conformal homeomorphism from \( \Omega \) onto \( h(\Omega) \).

Let \( \Omega \) be a bounded doubly connected domain in \( \mathbb{C} \). Then the complement \( \mathbb{C} \setminus \Omega \) of \( \Omega \) in \( \mathbb{C} \) has two components. One is a connected and simply connected compact set \( E \) and the other is an unbounded set \( F \). From the Grötzsch argument (see [AH1]), the bounded component \( E \) is a single point if and only if \( \text{mod}(\Omega) = \infty \) (see [BRH]).

Let \( \Omega \) be an annulus. If \( \Omega_1 \subseteq \Omega \) is a subannulus, then

\[
\text{mod}(\Omega_1) \leq \text{mod}(\Omega).
\]  \hfill (4)

If \( \Omega_1, \Omega_2 \subseteq \Omega \) are two disjoint subannuli, then

\[
\text{mod}(\Omega_1) + \text{mod}(\Omega_2) \leq \text{mod}(\Omega).
\]  \hfill (5)

The proofs of these two inequalities can be found in the book of Ahlfors [AH1].

Let \( E \) be a connected and simply connected compact subset of the open unit disk \( D_1 \). Let

\[
\text{mod}(D_1, E) = \sup_{\Omega \subseteq D_1 \setminus E} \text{mod}(\Omega), \text{ where } \Omega \subseteq D_1 \setminus E \text{ is a round subannulus } \}
\]

Note that \( \text{mod}(D_1, E) = \infty \) if \( E \) is a single point.

Let \( \text{diam}_H(E) = \sup_{z_1, z_2 \in E} d_H(z_1, z_2) \) be the hyperbolic diameter of \( E \) in \( D_1 \). The following theorem can be found in the book of McMullen [MC1].

**Theorem 7.** The hyperbolic diameter \( d_H(E) \) and the \( \text{mod}(D_1, E) \) are inversely related:

\[
\text{diam}_H(E) \to 0 \iff \text{mod}(D_1, E) \to \infty
\]

and

\[
\text{diam}_H(E) \to \infty \iff \text{mod}(D_1, E) \to 0.
\]

More precisely, there is a constant \( C > 0 \) such that

\[
C^{-1}\text{diam}_H(E) \leq \exp(-\text{mod}(D_1, E)) \leq C\text{diam}_H(E)
\]
when $\text{diam}_H(E)$ is small, while

$$\frac{C}{\text{diam}_H(E)} \geq \text{mod}(D_1, E) \geq C^{-1} \exp(-\text{diam}_H(E))$$

when $\text{dim}_a H(E)$ is large.

3. Renormalizable Quadratic-Like Maps

Let $(U_0, V_0, f_0)$ be a quadratic-like map and suppose its filled-in Julia set $K_{f_0}$ is connected. The map $f_0 : U_0 \to V_0$ is renormalizable if there are an integer $n \geq 2$ and an open subdomain $U_1$ of $U_0$ such that $0 \in U_1$ and such that $(U_1, V_1, f_0^{\circ n})$ is a quadratic-like map with connected filled-in Julia set, where $V_1 = f_0^{\circ n}(U_1)$. Let $f_1 = f_0^{\circ n}|U_1$. The filled-in Julia set $K_{f_1} = K_{f_1}(n, U_1)$ is a priory dependent on the choice of $U_1$ (refer to Theorem 13). The domain $U_1$ is called a renormalization and $(U_0, V_0, f_0)$ is called renormalizable about $n$. Otherwise, $(U_0, V_0, f_0)$ is called non-renormalizable.

The quadratic-like map $(U_0, V_0, f_0)$ is infinitely renormalizable if there is a strictly increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $(U_0, V_0, f_0)$ is renormalizable about $n_k$ for $k = 1, 2, \ldots$. Otherwise, $(U_0, V_0, f_0)$ is called finitely renormalizable.

A quadratic-like map $(U_0, V_0, f_0)$ is called real if $f_0(U_0 \cap \mathbb{R}^1) \subset V_0 \cap \mathbb{R}^1$ and $g = f_0|U_0 \cap \mathbb{R}^1$ is a real folding map. Let $(U_0, V_0, f_0)$ be a real quadratic-like map and suppose its filled-in Julia set $K_{f_0}$ is connected. Suppose $g = f_0|U_0 \cap \mathbb{R}^1$ has a fixed point $p \in \mathbb{R}^1$ with positive multiplier $f_0'(p)$. Let $p' \neq p$ be another inverse image of $p$ under $g$, that is, $g(p') = p$. Conjugating by a linear fractional transformation, we may assume that $p = -1$ and $p' = 1$ and that

$$[-1, 1] = \bigcap_{n=0}^{\infty} g^{-n}(V_0 \cap \mathbb{R}^1) = K_f \cap \mathbb{R}^1.$$

Hence $g$ is a folding map with unique quadratic critical point 0.

Let $g$ be a folding map of $[-1, 1]$ such that $g(-1) = g(1) = -1$ and such that 0 is a unique quadratic critical point. We say $g$ is renormalizable about $n > 1$ if there is a subinterval $I$ of $[-1, 1]$ such that $0 \in I$, such that $g^m(I) \cap I = \emptyset$ for all $0 < i < n$, and such that $g^m(I) \subseteq I$. Otherwise, $g$ is non-renormalizable. We say $g$ is infinitely renormalizable if there is a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that $g$ is renormalizable about $n_k$ for all $k > 0$. Otherwise, $g$ is called finitely renormalizable. The next theorem shows that for a real quadratic-like map, both definitions of renormalization are essentially equivalent.

**Theorem 8.** Let $(U_0, V_0, f_0)$ be a real quadratic-like map and suppose its filled-in Julia set $K_{f_0}$ is connected. Suppose $f_0$ has neither neutral, nor attractive, nor superattractive periodic points. Then $f_0$ is renormalizable if and only if the folding map $g = f_0|[-1, 1]$ is renormalizable.
Theorem 6. Suppose the folding map \( g = f_0[-1, 1] \) is renormalizable. This means that there is a maximal closed subinterval \( I \) of \([-1, 1]\) and an integer \( n > 1 \) such that 0 is in \( \hat{I} \), such that \( g^i(I) \cap \hat{I} = \emptyset \) for \( 0 < i < n \), and such that \( g^n(I) \subseteq I \). One of the endpoints of \( I \) is fixed by \( g^n \). It is a repelling fixed point. Take a neighborhood \( T' \) of \( I \) such that \( f^n|\left(L \cup R \right) \) is expanding, where \( L \cup R = T' \setminus I \). Let \( I = g^i(I) \) for \( 0 \leq i \leq n \). The inverse \( h_n \) of \( g^{n-1} : I \to I_n \) is a diffeomorphism and can be extended to a diffeomorphism on an open interval \( T \supset \hat{I} \). Take \( T \subset T' \). Let \( M' = h_n(T) \) and \( \overline{M'} \) a subset of T. Because the critical orbit \( CO = \{ f_k^0(0) \}_{k=0}^\infty \) of \( f_0 \) is in the real line \( \mathbb{R}^1 \), \( h_n \) can be extended to \( V_1 = (V_0 \setminus \mathbb{R}^1) \cup \mathcal{T} \) analytically. The image \( V_1' \) of \( V_1 \) under this extension is contained in \( U' = (U_0 \setminus \mathbb{R}^1) \cup \mathcal{M}' \). Let \( U_1 = f_0^{-1}(V_1') \). Then \( U_1 \subset V_1 \) and \( (U_1, V_1, f_1) \) for \( f_1 = f_0^n|U_1 \) is a quadratic-like map. Since \( g^n(I) \subseteq I \), the filled-in Julia set \( K_{f_1} \) is connected. Therefore, \( (U_0, V_0, f_0) \) is renormalizable about \( n \) and \( U_1 \) is a renormalization.

Suppose \( (U_0, V_0, f_0) \) is renormalizable about \( n > 1 \). Let \( U_1 \) be a renormalization and set \( V_1 = f^n(U_1) \). Then \( (U_1, V_1, f_1) \) for \( f_1 = f_0^n|U_1 \) is a quadratic-like map with the connected filled-in Julia set \( K_{f_1} \). Let \( I = K_{f_1} \cap \mathbb{R}^1 \). Then \( I = \cap_{i=0}^\infty f_1^{-i}(V_1 \cap \mathbb{R}^1) \). For every \( 1 \leq i < n \), \( f_0^i(I) \cap \hat{I} = \emptyset \) else \( f^n \) would have at least three fixed points in \( I \cup f^i(I) \) with one of them either attractive or parabolic. Since \( f_1(0) \) is in \( I \), \( f_0^n(I) \subseteq I \). Therefore, \( g = f_0[-1, 1] \) is a renormalizable folding map.

4. Two-Dimensional Yoccoz Puzzles and Renormalizability

In this section, we discuss a technique in the study of non-renormalizable quadratic polynomials, due to Yoccoz, and some of its applications to renormalization.

Let \( P_c(z) = z^2 + c \) be a quadratic polynomial with connected filled-in Julia set \( K_c \). The external ray \( e_0 \) of \( P_c \) is the only one fixed by \( P_c \) (see Fig. 2). It lands either at a repelling or at a parabolic fixed point \( \beta \) of \( P_c \) (see [MI1]). Suppose \( \beta \) is repelling. Applying Theorem 6, we see that \( e_0 \) is the only external ray landing at \( \beta \). Thus \( K_c \setminus \{ \beta \} \) is connected. We call \( \beta \) the non-separate fixed point of \( P_c \). Let \( \alpha \neq \beta \) be the other fixed point of \( P_c \). If \( \alpha \) is either an attractive or a super-attractive fixed point, then \( J_c = K_c \setminus (\cup_{n=0}^\infty P_{c}^{-n}(D(\alpha))) \) for a small disk centered at \( \alpha \). The Julia set \( J_c \) is a Jordan curve; every external ray lands at a unique point in \( J_c \) (see Remark 3). If \( P \) is a repelling fixed point, there are at least two periodic external rays landing at \( \alpha \). We use \( R_0(\alpha) \) to denote the union of a cycle of periodic external rays of period \( q \) landing at \( \alpha \) (see Fig. 2). The set \( R_0(\alpha) \) cuts \( \mathbb{C} \) into finitely many simply connected domains \( \Omega_0, \Omega_1, \ldots, \Omega_{q-1} \). Each domain contains points in the Julia set \( J_c \). Thus \( K_c \setminus \{ \alpha \} \) is disconnected. We call \( \alpha \) the separate fixed point of \( P_c \).
Henceforth, we assume that the fixed points $\beta$ and $\alpha$ are both repelling. Let $s_r$ be a fixed equipotential curve of $P_c$ and let $U_r$ be the open domain bounded by $s_r$. Then $(U_r, U_r, P_c)$ is a quadratic-like map. The set $R_0(\alpha)$ cuts $U_r$ into finitely many simply connected domains. Let $C_0$ be the closure of the domain containing 0, and let $B_{0,i}$ be the closure of the domain containing $P_c^{i}(0)$ for $1 \leq i < q$. Since $R_0(\alpha)$ is forward invariant under $P_c$, the image under $P_c$ of $C_0 \cap K_c$ or $B_{0,i} \cap K_c$ for every $1 \leq i < q$, is the union of some of $C_0 \cap K_c$, $B_{0,1} \cap K_c$, $\ldots$, $B_{0,q-1} \cap K_c$. The set $\eta_0 = \{ C_0, B_{0,1}, \ldots, B_{0,q-1} \}$ is called the original partition. We note that it is not a Markov partition because $P_c | C_0$ is a proper, holomorphic map of degree two. (But $P_c | B_{0,i}$ is a holomorphic diffeomorphism for every $1 \leq i < q$.)

Let $\Gamma_n = P_c^{-n}(\alpha)$ and let $R_n(\alpha) = P_c^{-n}(R_0(\alpha))$ for $n \geq 0$. The set $R_n(\alpha)$ is the union of some external rays landing at points in $\Gamma_n$; it cuts the domain $U_r$ into a finite number of simply connected domains. Let $C_n$ be the closure of the domain containing 0 and let $B_{n,1}, \ldots, B_{n,k_n}$ be the closures of others. Since $P_c(\Gamma_n) = \Gamma_{n-1}$, the image of $C_n$ or $B_{n,i}$ under $P_c$ is one of $C_{n-1}$, $B_{n-1,1}$, $\ldots$, $B_{n-1,k_{n-1}}$, for $1 \leq i \leq k_n$ and $n \geq 1$. Then $P_c | C_n$ is holomorphic, proper branch covering map of degree two; all $P_c | B_{n,i}$ are holomorphic diffeomorphisms. The set $\eta_n = \{ C_n, B_{n,1}, \ldots, B_{n,k_n} \}$ is called the $n^{th}$-partition. The sequence $\eta = \{ \eta_n \}_{n=0}^{\infty}$ is called the two-dimensional Yoccoz puzzle for $P_c$. A similar puzzle for certain cubic polynomial is constructed by Branner and Hubbard [BRH]. Yoccoz used this puzzle while studying the local connectivity of a non-renormalizable quadratic polynomial as follows.

Let $\Gamma_\infty = \bigcup_{n=0}^{\infty} \Gamma_n$. For any $x$ in $K_c \setminus \Gamma_\infty$, there is one and only one sequence $\{ D_n(x) \}_{n=0}^{\infty}$ such that $x \in D_n(x) \in \eta_n$. For any $x$ in $\Gamma_\infty$, there are $q$ such sequences. We call such a sequence

$$x \in \ldots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \ldots \subseteq D_1(x) \subseteq D_0(x)$$
an $x$-end. In particular,

$$0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_2 \subseteq C_1 \subseteq C_0$$

is called the critical end.

Suppose $D_{n+1} \subseteq D_n$ are domains in $\eta_{n+1}$ and $\eta_n$ for $n \geq 0$. Define $A_n = D_n \setminus \hat{D}_{n+1}$. If $D_{n+1} \subset D_n$, then $A_n$ is a non-degenerate annulus and its modulus (denoted $\text{mod}(A_n)$) is greater than 0. Otherwise, $A_n$ is a degenerate annulus and its modulus $\text{mod}(A_n)$ is zero. The domain $D_n$ is critical if $D_n = C_n$ and $D_{n+1} = C_{n+1}$; it is semi-critical if $D_n = C_n$ but $D_{n+1} \neq C_{n+1}$; it is non-critical if $D_n \neq C_n$. If $D_n$ is non-critical, then

$$\text{mod}(A_n) = \text{mod}(P_c(A_n))$$

since $P_c : A_n \rightarrow P_c(A_n)$ is a conformal homeomorphism. If $D_n$ is critical, then

$$\text{mod}(A_n) = \frac{\text{mod}(P_c(A_n))}{2}$$

since $P_c : A_n \rightarrow P_c(A_n)$ is a proper, holomorphic, unramified covering map of degree two. If $D_n$ is semi-critical, then

$$\frac{\text{mod}(P_c(A_n))}{2} \leq \text{mod}(A_n).$$

For a point $x$ in $K_c$ and an $x$-end

$$x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x),$$

we define

$$D_{nm}(x) = P_{c}^{om}(D_{n+m}(x))(= D_n(P_{c}^{om}(x)))$$

and

$$A_{nm}(x) = D_{nm}(x) \setminus \hat{D}_{(n+1)m}$$

for $n, m \geq 0$. The tableau $T(x) = (a_{nm})_{n \geq 0, m \geq 0}$ is an $\infty \times \infty$-matrix defined as follows: $a_{nm} = 1$ if $D_{nm}(x) = C_n$, and $a_{nm} = 0$ if $D_{nm}(x) \neq C_n$. The tableau $T(0) = (a_{nm}^0)_{n \geq 0, m \geq 0}$ is called the critical tableau. Note that

$$P_{c}^{om}(x) \subseteq \cdots \subseteq D_{nm}(x) \subseteq D_{(n-1)m}(x) \subseteq \cdots \subseteq D_{1m}(x) \subseteq D_{0m}(x)$$

is a $P_{c}^{om}(x)$-end.

**Lemma 1.** The tableau $T(x)$ satisfies the following rules:

(T1) if $a_{nm} = 1$ for $n, m \geq 0$, then $a_{im} = 1$ for all $0 \leq i \leq n$,

(T2) if $a_{nm} = 1$ for $n, m \geq 0$, then $a_{(n-i)(m+j)} = a_{(n-i)j}^0$ for all $0 \leq i + j \leq n$,

(T3) if

i) $a_{nm} = 1$ and $a_{(n+1)m} = 0$ for $n, m \geq 0$ and if
ii) \( a_{(n-i)(m+i)} = 1 \) for \( 1 \leq i \leq n \) and \( a_{(n-j)(m+j)} = 0 \) for \( 0 < j < i \), then \( a^0_{(n-i+1)(m+i)} = 1 \) implies \( a_{(n-1)(m+i)} = 0 \).

**Proof.** Rule (T1) is valid because

\[
0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_1 \subseteq C_0.
\]

Rule (T2) is valid because if \( a_{nm} = 1 \), then \( D_{im}(x) = C_i \) for all \( 0 \leq i \leq n \). Thus \( D_{(n-i)(m+j)}(x) = f^{o_j}(C_{n-i}) \) for all \( 0 \leq i + j \leq n \).

We prove Rule (T3). Conditions i) and ii) imply that \( P^c_i : C_n \rightarrow C_{n-i} \) is a degree two holomorphic proper branch covering map. Condition i) and Rule (T2) imply \( a_{k(m+j)} = a_{kj}^0 \) for \( 0 \leq k + j \leq n \). Now \( a^0_{(n-i+1)i} = 1 \) and \( a^0_{ij} = 1 \) for all \( 0 \leq j < \infty \) imply that \( P^c_i : C_{n+1} \rightarrow C_{n-i+1} \) is a degree two holomorphic proper branch covering map. Thus \( C_{n+1} = P^c_i(C_{n-i+1}) \cap C_n \). Assume \( a_{(n-i+1)(m+i)} = 1 \). Then \( D_{(n+1)m}(x) = P^c_i(C_{n-i+1}) \cap C_n = C_{n+1} \). This contradicts \( a_{(n+1)m} = 0 \). ■

**Lemma 2.** For any domain \( D \) in \( \eta_n \), for \( n \geq 0 \), \( D \cap K_c \) is connected.

**Proof.** Since the domain \( D \) is bounded by finitely many external rays \( \Pi = \{ e_{r_i} \}_{i=1}^m \) and by some equipotential curve, then \( \partial D \cap K_c \) consists of a finite number of points \( \{ p_i \}_{i=1}^n \). Every \( p_i \) is a landing point of two external rays in \( \Pi \). Suppose \( D \cap K_c \) is not connected for some \( D \) in \( \eta_n \). Then there are two disjoint open sets \( U \) and \( V \) such that \( D \cap K_c = (D \cap K_c \cap U) \cup (D \cap K_c \cap V) \). Suppose that \( p_1, \ldots, p_k \) are in \( U \) and that \( p_{k+1}, \ldots, p_n \) are in \( V \). The two external rays in \( \Pi \) landing at \( p_i \) cut \( C \) into two open domains. Let \( W_i \) be the one which is disjoint from \( D \). Then \( U' = U \cup \bigcup_{i=1}^k W_i \) and \( V' = V \cup \bigcup_{i=k+1}^n W_i \) are two disjoint open sets and \( K_c = (U' \cap K_c) \cup (V' \cap K_c) \). This contradicts the fact that \( K_c \) is connected.

Lemma 2 implies that for any \( x \)-end

\[
x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x),
\]
the intersection \( L_x = \bigcap_{n=0}^\infty D_n(x) \) is a compact connected non-empty set containing \( x \). Let \( T(x) = (a_{nm})_{n \geq 0, m \geq 0} \) be the tableau of the \( x \)-end. It is non-recurrent if there is an integer \( N \geq 0 \) such that \( a_{nm} = 0 \) for all \( n \geq N \) and all \( m \geq 1 \). Otherwise, \( T(x) \) is recurrent.

**Lemma 3.** If \( T(x) \) is non-recurrent, then \( L_x = \{ x \} \).

**Proof.** Suppose \( N \geq 0 \) is an integer such that \( a_{nm} = 0 \) for all \( n \geq N \) and all \( m \geq 1 \). Then, for \( n > N \), every

\[
P^c_{n-N-1} : D_{n1}(x) \rightarrow D_{N(n-N)}(x)
\]
is a holomorphic diffeomorphism. Thus \( \text{mod}(A_{n0}(x)) \) is greater than or equal to \( \text{mod}(A_{N(n-N)}(x))/2 \) for every \( n > N \). There are only finitely many different annuli in \( \{ A_{Nm}(x) \} \) because \( \eta_N \) has only finitely many domains. If there were infinitely
many non-degenerate annuli in \( \{ A_{Nn}(x) \}_{m=0}^{\infty} \), then there would be infinitely many non-degenerate annuli in \( \{ A_{n0}(x) \}_{n=0}^{\infty} \) whose moduli are the same. This would imply that
\[
\text{mod}(D_0(x) \setminus L_x) \geq \sum_{n=0}^{\infty} \text{mod}(A_{n0}(x)) = \infty.
\]

Therefore \( L_x = \{ x \} \).

If there are only finitely many non-degenerate annuli in \( \{ A_{Nn}(x) \}_{m=0}^{\infty} \), The proof uses results from hyperbolic geometry. Let \( B_{N1} \) be the domain in \( \eta_N \) containing the critical value \( P_c(0) \). Since \( P_c^0(x) \) do not enter \( C_{N+1} \) for all \( 0 < i < \infty \), then \( P_c^0(x) \) does not enter \( B_{N1} \) for all \( 2 \leq i < \infty \). Let us thicken \( B_{N_i} \) to an open simply connected domain \( \tilde{B}_{N_i} \) such that \( B_{N_i} \subset \tilde{B}_{N_i} \) and such that \( P_c(0) \) is not in \( \tilde{B}_{N_i} \) for \( 1 < i \leq k_N \).

The map \( P_c \) has two inverse branches \( g_{1 \alpha} \) and \( g_{2 \alpha} \) defined on \( \tilde{B}_{N_i} \) for every \( 1 < i \leq k_N \). We consider \( \tilde{B}_{N_i} \) to be a hyperbolic Riemann surface with hyperbolic distance \( d_{H,i} \), for every \( 1 < i \leq k_N \), where \( k_N \) is the number of elements in \( \eta_N \). Then if \( g_{i \alpha} \), for \( 1 < i \leq k_N \) and \( k = 1 \) or \( 2 \), sends \( B_{N_i} \) into \( B_{N_j} \) for some \( 1 < j \leq k_N \), then it strictly contracts these hyperbolic distances; more precisely, there is a constant \( 0 < \lambda < 1 \) such that \( d_{H,j}(g_{i \alpha}(x), g_{i \alpha}(y)) < \lambda d_{H,i}(x,y) \) for \( x \) and \( y \) in \( B_{N_i} \) and for \( k = 0 \) and \( 1 \). Therefore, there is a constant \( C > 0 \) such that for any \( D_{n1}(x) \) and for any \( n > N \),
\[
d(D_{n1}(x)) = \max_{y,z \in D_{n0}(x)} |y - z| \leq C \lambda^n - N - 1
\]
since \( D_{(n-i)\alpha}(x) \) is in one of \( B_{N_i}, \ldots, B_{N_k} \) for every \( 2 \leq i \leq n - N \). Thus \( d(D_{n0}(x)) \) tends to zero as \( n \) goes to infinity and \( L_x = \{ x \} \).

The critical end
\[
0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_2 \subseteq C_1 \subseteq C_0
\]
is important. Let \( A_{n0}(0) = C_n \setminus \tilde{C}_{n+1} \).

**Lemma 4.** If \( \sum_{n=0}^{\infty} \text{mod}(A_{n0}(0)) = \infty \), then for any \( x \in K_c \) and any \( x \)-end
\[
x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x),
\]

\( L_x = \{ x \} \).

**Proof.** Consider the tableau \( T(x) = (a_{nm})_{n \geq 0, m \geq 0} \). If \( T(x) \) is non-recurrent, the lemma follows from Lemma 3.

Suppose \( T(x) \) is recurrent. If there is a column which is entirely 1’s, then there are integers \( M \geq 0 \) and \( N \geq 0 \) such that \( a_{iM} = 1 \) for all \( i \geq 0 \) and \( a_{nm} = 0 \) for all \( n \geq N \) and \( 0 \leq m < M \). Thus \( P_c^M : D_{n0}(x) \to D_{(n-M)0} = C_{n-M} \) is a holomorphic diffeomorphism for every \( n \geq N \). This implies
\[
m(D_0(x) \setminus L_x) \geq \sum_{n=0}^{\infty} \text{mod}(A_{n0}(x)) \geq \sum_{n=N}^{\infty} \text{mod}(A_{n0}(x)) = \sum_{n=N-M}^{\infty} \text{mod}(A_{n0}(0)) = \infty.
\]

So \( L_x = \{ x \} \).
Suppose that there is no column which is entirely 1's. Let $N > 0$ be an integer such that $a_{n0} = 0$ for $n \geq N$. For any $n \geq N$, let $m_n > 0$ be the integer such that $a_{nm_n} = 1$ and $a_{ni} = 0$ for $0 \leq i < m_n$. Then $P_{c,n}^{0,m_n} : D_{(n+m_n-1)0} \to D_{(n-1)m_n} = \mathcal{C}_{n-1}$ is a holomorphic diffeomorphism. Remember that $A_{(n-1)0} = D_{(n-1)0} \setminus \mathcal{D}_{n0}$ and $A_{(n-1)0} = \mathcal{C}_{n-1} \setminus \mathcal{C}_n$. We have

$$\text{mod}(A_{(n+m_n-1)0}(x)) = \text{mod}(A_{(n-1)0}(0)).$$

Let $q_n = n + m_n - 1$. Then $q_N < q_{N+1} < \cdots < q_n < q_{n+1} < \cdots$. Thus

$$\text{mod}(D_0 \setminus L_x) \geq \sum_{n=N}^{\infty} \text{mod}(A_{n0}(x)) \geq \sum_{n=N}^{\infty} \text{mod}(A_q,0(x))$$

$$= \sum_{n=N}^{\infty} \text{mod}(A_{(n-1)0}(0)) = \infty.$$ 

This implies that $L_x = \{x\}$. 

The first column of the critical tableau $T(0) = (a_{nm})_{n \geq 0,m \geq 0}$ is entirely 1's. If $T(0)$ has another column which is entirely 1's, that is, if there is an integer $m > 0$ such that $a_{im} = 1$ for all $i \geq 0$, then we call $T(0)$ a periodic critical tableau.

**Theorem 9 [YOC].** The critical tableau $T(0)$ is periodic if and only if $P_c$ is renormalizable.

**Proof.** Suppose $T(0)$ is periodic. Let $n_1 > 0$ be the smallest integer such that $a_{in_1} = 1$ for all $i \geq 0$. Let $N \geq 0$ be the smallest integer such that $a_{ij} = 0$ for all $i \geq N$ and $0 < j < n_1$. For any $n \geq n_1 + N$, $P_{c,n}^{0,n_1} : \mathcal{C}_n \to \mathcal{C}_{n-n_1}$ is a degree two proper holomorphic branch covering map. Thus $\{P_{c,n_1}^{0,n_1}(0)\}_{n_1}^{\infty}$ is contained in $\mathcal{C}_{n_1+N}$. If $\mathcal{C}_{n_1+N} \subset \mathcal{C}_N$, then $P_{c,n}^{0,n_1} : \mathcal{C}_{n_1+N} \to \mathcal{C}_N$ is a quadratic-like map with connected filled-in Julia set and $\mathcal{C}_{n_1+N}$ is a renormalization about $n_1$. Thus $P_c$ is renormalizable.

In general, let us consider a small open disk $D(\alpha)$ centered at the separate fixed point $\alpha$ of $P_c$ such that

$$\overline{D}(\alpha) \subset D'(\alpha) = P_{c,n_1}^{0,n_1}(D(\alpha))$$

and such that

$$D'(\alpha) \cap \{P_{c,n_1}^{0,n_1}(0)\}_{n_1}^{\infty+N} = \emptyset.$$ 

Thicken $C_0$ and $B_0i$ for $1 \leq i \leq q - 1$ as follows. Suppose $C_0$ (respectively, $B_0i$) is bounded by two external rays $R_{\theta_1}$ and $R_{\theta_2}$ of angles $\theta_1$ and $\theta_2$. Let $\epsilon > 0$ be a small number such that the domains

$$U_1 = \cup_{\theta_1-\epsilon < \theta < \theta_1+\epsilon}(R_{\theta} \setminus D(\alpha))$$

and

$$U_2 = \cup_{\theta_2-\epsilon < \theta < \theta_2+\epsilon}(R_{\theta} \setminus D(\alpha))$$

are disjoint from $\{P_{c,n_1}^{0,n_1}(0)\}_{n_1}^{\infty+N}$. Let

$$\mathcal{C}_0 = (U_1 \cup C_0 \cup U_2 \cup D(\alpha)) \cap U_r$$

(respectively,

$$\mathcal{B}_0 = (U_1 \cup B_0 \cup U_2 \cup D(\alpha)) \cap U_r$$)
where $U_r$ is the domain bounded by the equipotential curve $s_r$. Let

$$\tilde{\eta}_0 = \{\tilde{C}_0, \tilde{B}_{01}, \ldots, \tilde{B}_{0(q-1)}\}$$

and let

$$\tilde{\eta}_n = P^{-n}(\tilde{\eta}_n) = \{\tilde{C}_n, \tilde{B}_{n1}, \ldots, \tilde{B}_{nk_n}\}$$

for $1 \leq n \leq n_1 + N$. The diffeomorphism $g = P_{n_1-1}^{-1} : C_N \to P_c(C_{n_1+N})$ can be extended to $\tilde{C}_N$. Let $B'$ be the image of $\tilde{C}_N$ under $g$. Then $\tilde{C}_{m+N} = P_{n_1-1}^{-1}(B')$. Let $\tilde{C}_n$ denote the interior of $\tilde{C}_n$ for $0 \leq n \leq n_1 + N$. Then $\tilde{C}_{n_1+N} \subset \tilde{C}_N$. Thus

$$P_{n_1}^n : \tilde{C}_{n_1+N} \to \tilde{C}_N$$

is a quadratic-like map and $\tilde{C}_{n_1+N}$ is a renormalization about $n_1$. This proves the “only if” part.

Now suppose $P_c$ is renormalizable. Let $U_1$ be a renormalization about $n_1$, that is, $(U_1, V_1, f_1)$ is a quadratic-like map with connected filled-in Julia set $K_{f_1}$, where $f_1 = P_{n_1}^n|U_1$ and $V_1 = f_1(U_1)$. The map $f_1$ has two fixed points $\beta_1$ and $\alpha_1$ in $U_1$. Let

$$\alpha_1 \in \cdots \subseteq D_n(\alpha_1) \subseteq D_{n-1}(\alpha_1) \subseteq \cdots \subseteq D_1(\alpha_1) \subseteq D_0(\alpha_1)$$

be an $\alpha_1$-end. There is a $D_k(\alpha_1)$ such that $K_{f_1} \subset D_k(\alpha_1)$ and $D_k(\alpha_1) = C_k$. Since

$$K_{f_1} \subseteq f_1^{-1}(U_1 \cap C_k) \subseteq U_1 \cap C_k,$$

then $P_{n_1}^n$ sends $C_{k+m}$ to $C_{k+(i-1)n_1}$ for all $i > 0$. Thus $T(0)$ is periodic. It is the “if” part. 

We define a function $\tau$ on the set $\mathbb{N}$ of natural numbers by using the critical tableau $T(0) = (a_{nm})_{n \geq 0, m \geq 0}$ as follows: $\tau(n) = m$ if $a_{(n-i)j} = 0$ for $0 < i < n - m$ and if $a_{m(n-m)} = 1$; if there is no such integer $m > 0$, then $\tau(n) = -1$.

If the critical tableau $T(0)$ is periodic, then there are integers $n_1 > 0$ and $N \geq 0$ such that $a_{im_1} = 1$ for all $i \geq 0$ and such that $a_{ij} = 0$ for $i \geq N$ and $0 < j < n_1$. Thus $\tau(n) = n - n_1$ for $n \geq N + n_1$.

If the critical tableau $T(0)$ is non-recurrent, then there is the smallest integer $N \geq 0$ such that $a_{nm} = 0$ for all $n \geq N$ and $m > 0$. Thus the image $\tau(\mathbb{N})$ is contained in the finite set $\{-1, 0, 1, \ldots, N - 1\}$.

If the critical tableau $T(0)$ is not periodic and is recurrent, then every row of $T(0)$ has infinitely many 1’s and every column (except for the 0th-column) has a 0. An integer $n \geq 0$ is noble if for every entry $a_{nk}$ such that $a_{n1} = 1$, we have $a_{(n+1)k} = 1$.

**Lemma 5.** If the critical tableau $T(0)$ is not periodic and is recurrent, then the function $\tau$ satisfies the following properties:

(i) For any integer $m \geq 0$, $\tau^{-1}(m)$ is not empty.

(ii) If $m \geq 0$ is noble, then $\tau^{-1}(m)$ contains at least two different integers.
(iii) If \( \tau(n) = m \) and if \( m \) is noble, then \( n \) is also noble.

(iv) If \( \tau^{-1}(m) \) contains only one integer \( n \), then \( n \) is also noble.

**Proof.** We prove (i) first. Consider any \( m \)-row in \( T(0) \) for \( m \geq 0 \). Let \( k > 0 \) be the integer such that \( a^0_{mi} = 0 \) for \( 0 < i < k \) and such that \( a^0_{mk} = 1 \). From (T1), \( a^0_{(m+k-i)i} = 0 \) for \( 0 < i < k \). Thus \( \tau(m+k) = m \).

To prove (ii), suppose \( m \geq 0 \) is noble. Let \( k \) be the same integer as that in the proof of (i). Let \( m_1 \) be the integer such that \( a^0_{m1k} = 1 \) and such that \( a^0_{ik} = 0 \) for all \( i > m_1 \). Consider \( a^0_{(m_1-k)(2k)} \), \( a^0_{(m_1-2k)(3k)} \), \ldots, and \( a^0_{(m_1-(i-1)k)(i)k} \) where \( m_1 - ik \leq m < m_1 - (i-1)k \). From the tableau rules (T1) and (T3), \( a^0_{(m_1-k+1)(2k)} = 0 \), \( a^0_{(m_1-2k+1)(3k)} = 0 \), \ldots, \( a^0_{(m_1-(i-1)k+1)(ik)} = 0 \). If \( m = m_1 - ik \), from the tableau rules (T1) and (T3), \( a^0_{m(ik)} = 1 \). Since \( m \) is noble, \( a^0_{(m+1)(ik)} = 1 \). But from the tableau rules (T1) and (T3), \( a^0_{(m+1)(ik)} = 0 \). The contradiction implies that \( m > m_1 - ik \).

Now from the tableau rule (T2), \( a^0_{m(k+m_1-m)} = 0 \). Let \( k_1 > k + m_1 - m \) be the integer such that \( a^0_{m1i} = 0 \) for \( k + m_1 - m < i < k_1 \) and such that \( a^0_{mk_1} = 1 \). Then \( a^0_{(m+k_1-i)i} = 0 \) for \( 0 < i < k_1 \) and \( a^0_{mk_1} = 1 \). This says that \( \tau(m+k_1) = m \).

To prove (iii), suppose \( \tau(n) = m \) where \( m \) is noble. For any \( a^0_{mk} = 1 \), since \( a^0_{(n-i)i} = 0 \) for \( 0 < i < n - m \) and \( a^0_{m(n-m)} = 1 \) and since the tableau rule (T1), we have \( a^0_{(n-i)(k+i)} = 0 \) for \( 0 < i < n - m \) and \( a^0_{m(k+n-m)} = 1 \). Since \( m \) is noble, \( a^0_{(m+1)(n-m)} = 1 \). Assume \( a^0_{(n+1)k} = 0 \). From the tableau rule (T3), \( a^0_{(m+1)(k+n-m)} = 0 \). This contradicts to that \( m \) is noble. Thus \( a^0_{(n+1)k} = 1 \). This means that \( n \) is noble.

Now we prove (iv). Suppose \( n > 0 \) is the only integer such that \( \tau(n) = m \). We first consider \( a^0_{(m+1)(n-m)} \). If \( a^0_{(m+1)(n-m)} = 0 \), then we would have an integer \( k > n - m \) such that \( a^0_{mi} = 0 \) for \( n - m < i < k \) and such that \( a^0_{mk} = 1 \). From the tableau rule (T1), \( a^0_{(m+k-i)i} = 0 \) for \( 0 < i < k \). This would imply that \( \tau(m+k) = m \), which contradicts the assumption. Thus, \( a^0_{(m+1)(n-m)} = 1 \). If there is an entry \( a^0_{nk_1} = 1 \) with \( a^0_{(n+1)k_1} = 0 \), then \( k_1 > n - m \) and, from the tableau rule (T3), \( a^0_{m(n-m+k_1)} = 1 \) and \( a^0_{(m+1)(n-m+k_1)} = 0 \). Consider the smallest integer \( k_2 > n - m + k_1 \) such that \( a^0_{mi} = 0 \) for \( k_1 + n - m < i < k_2 \) and \( a^0_{mk_2} = 1 \). From the tableau rule (T1), \( a^0_{(m+k_2-i)i} = 0 \) for \( k_1 + n - m < i < k_2 \). So we can find another integer \( n_0 = k_2 - k_1 + m > n \) such that \( \tau(n_0) = m \). This would contradict the assumption.

**Theorem 10 [YOC].** Suppose \( P_c(z) = z^2 + c \) is a recurrent quadratic polynomial. The critical tableau \( T(0) \) is periodic if and only if \( L_0 \) contains more than one point.

**Proof.** We use the same notation as in the proof of Theorem 9 and the proof of Lemma 5. Suppose \( T(0) \) is periodic. Then for \( n > N + n_1 \)

\[ P_c^{\circ n_1} : C_{n+1} \to C_{n-n_1+1} \]
is a degree two proper holomorphic branch covering map. Replacing $C_{n+1}$ by $\tilde{C}_{n+1}$ if it is necessary, we may assume that this map is a quadratic-like map. Since $L_0$ is the filled-in Julia set of this map, it contains more than one point. This is the “only if” part.

To prove the “if” part, suppose $T(0)$ is not periodic. We will prove that $L_0$ contains only one point. Since $P_c$ is recurrent, there are infinitely many 1’s in every row of $T(0)$, that is, $T(0)$ is recurrent. Consider the first partition

$$\eta_1 = \{C_1, B_{11}, \ldots, B_{1(q-1)}, B_{01}, \ldots B_{0(q-1)}\},$$

where $B_{0i} = B_{0,i}$ for $1 \leq i < q$ and where $B_{1i} \subseteq C_0$ and $P_c(B_{1i}) = B_{0,i}$ for $1 \leq i < q$. (Remember that $\eta_0 = \{C_0, B_{0,1}, \ldots, B_{0,q-1}\}$ is the original partition.) Let $c(n) = P_{c^n}(0)$. If the critical orbit $CO = \{c(n)\}_{n=0}^{\infty}$ is contained in the union

$$C_1 \cup B_{01} \cup \ldots \cup B_{0(q-1)},$$

then $T(0)$ is periodic of period $q$. Hence there must be one critical value $c(n)$ in $B_{11} \cup \cdots \cup B_{1(q-1)}$. Let $c(n)$ be in $B_{1i}$. The annulus $A_{0n}(0) = C_0 \setminus \tilde{B}_{1i}$ is non-degenerate. Pull back $A_{0n}(0)$ by $P_c$ along $A_{i(n-i)}(0)$ for $0 \leq i \leq n$; we get a non-degenerate annulus $A_{n0}(0)$.

Now consider $\tau^{-k}(n)$. For each $m$ in $\tau^{-k}(n)$,

$$\text{mod}(A_{m0}(0)) \geq \frac{\text{mod}(A_{n0}(0))}{2^k}.$$ 

If the number of $\tau^{-k}(n)$ is greater than or equal to $2^k$ for every $k > 0$, then

$$\text{mod}(C_0 \setminus L_0) \geq \sum_{m=1}^{\infty} \text{mod}(A_{m0}(0)) \geq \sum_{k=1}^{\infty} \sum_{m \in \tau^{-k}(n)} \text{mod}(A_{m0}(0)) \geq \sum_{k=1}^{\infty} \text{mod}(A_{n0}(0)) = \left(\text{mod}(A_{n0}(0))\right) \cdot \sum_{k=1}^{\infty} 1 = \infty.$$ 

So $L_0 = \{0\}$.

If there is an integer $k > 0$ such that the number of $\tau^{-k}(n)$ is less than $2^k$, then there are pre-images $m > q$ of $n$ under iterates of $\tau$ such that $m$ is the only pre-image of $q$ under $\tau$. From (iii), $m$ is noble. Hence $\tau^{-k}(m)$ are all noble and contain at least $2^k$ different integers. Moreover

$$\text{mod}(A_{p0}(0)) = \frac{\text{mod}(A_{m0}(0))}{2^k}.$$
for every $p$ in $\tau^{-k}(m)$. Therefore,

$$\mod(C_0 \setminus L_0) \geq \sum_{k=1}^{\infty} \mod(A_{k0}(0)) \geq \sum_{k=1}^{\infty} \sum_{p \in \tau^{-k}(m)} \mod(A_{p0}(0)) \geq \sum_{k=1}^{\infty} \mod(A_{m0}(0))$$

$$= \left(\mod(A_{m0}(0))\right) \cdot \sum_{k=1}^{\infty} 1 = \infty.$$  

Again we have $L_0 = \{0\}$. This completes the “if” part.  

**Theorem 11 [YOC].** If $P_c(z) = z^2 + c$ is a non-recurrent or recurrent non-renormalizable quadratic polynomial, then its filled-in Julia set $K_c$ is locally connected.

**Proof.** Let $\alpha$ be the separate fixed point of $P_c$. Construct the two-dimensional Yoccoz puzzle for $P_c$. For any $x$ in $K_c$, let

$$x \in \cdots \subseteq D_n(x) \subseteq D_{n-1}(x) \subseteq \cdots \subseteq D_1(x) \subseteq D_0(x)$$

be an $x$-end. If $P_c$ is non-recurrent, then the critical tableau is non-recurrent. Lemma 3 and Lemma 4 imply that the diameter $d(D_n(x))$ tends to zero as $n$ goes to infinity. If $P_c$ is recurrent and non-renormalizable, then $T(0)$ is recurrent and is not periodic. Lemma 4 and Theorem 10 imply that the diameter $d(D_n(x))$ tends to zero as $n$ goes to infinity.

If $x$ is not a preimage of $\alpha$ under any iterate of $P_c$, then $x$ is an interior point of $D_n(x)$ for all $n \geq 0$. From Lemma 2, $\{D_n(x)\}$ is a basis of connected neighborhoods at $x$. If $x$ is a preimage of $\alpha$ under some iterate of $P_c$, then there are $q$ different $x$-ends,

$$x \in \cdots \subseteq D_{i,n}(x) \subseteq D_{i,(n-1)}(x) \subseteq \cdots \subseteq D_{i,1}(x) \subseteq D_{i,0}(x)$$

where $q$ is the period of the external rays landing at $\alpha$. Let $\tilde{D}_n(x) = \cup_{i=1}^{q} D_{i,n}(x)$. Then $x$ is an interior point of $\tilde{D}_n$. Since $K_c \cap D_{1,n}(x)$, $\ldots$, $K_c \cap D_{q,n}(x)$ have a common point $x$, from Lemma 2, $K_c \cap \tilde{D}_n(x)$ is connected. So $\{\tilde{D}_n(x)\}_{n=0}^{\infty}$ is a basis of connected neighborhoods at $x$. 

From Theorem 4, all arguments in this section apply to a quadratic-like map. Suppose that $(U, V, f)$ is a quadratic-like map and that its filled-in Julia set $K_f$ is connected. Suppose two fixed points $\beta$ and $\alpha$ of $f$ are repelling. Let $\beta$ be the non-separate fixed point of $f$, that is, $K_f \setminus \{\beta\}$ is connected, and let $\alpha$ be the separate fixed point of $f$, that is, $K_f \setminus \{\alpha\}$ is disconnected. Since $(U, V, f)$ is hybrid equivalent to a quadratic polynomial $P_c$, there is a quasiconformal homeomorphism $H$ defined on $V$ such that

$$H \circ f = P_c \circ H$$
on $U$. We call $e_{\theta,f} = H^{-1}(e_{\theta} \cap H(U))$ the external ray of angle $\theta$ of $f$ where $e_{\theta}$ is the external ray of $P_c$ of angle $\theta$. 


Two points $H(\beta)$ and $H(\alpha)$ are non-separate and separate fixed points of $P_c$, respectively. Suppose $\Gamma$ is the union of a cycle of periodic external rays landing at $H(\alpha)$. Let $\Gamma' = H^{-1}(\Gamma \cap H(U))$. The set $\Gamma'$ cuts the domain $U$ into $q$ domains. Each of them contains points in the filled-in Julia set $K_c$. Let $C_0$ be the domain containing 0 and let $B_{0,i}$ be the domain containing $f^{0i}(0)$ for $1 \leq i < q$. The partition

$$\eta_0 = \{C_0, B_{0,1}, \ldots, B_{0,q-1}\}$$

is called the original partition for $f$. Let $\Gamma'_n = f^{-n}(\Gamma')$ and $U_n = f^{-n}(U)$. Then $\Gamma'_n$ cuts $U_n$ into finitely many domains. Let $C_n$ be the domain containing 0 and $B_{n,i}$ for $1 \leq i \leq k_n$ be others. Then

$$\eta_n = \{C_n, B_{n,1}, \ldots, B_{n,k_n}\}$$

is called the $n^{th}$-partition for $f$. We use $f^{-n}(\eta_0)$ to denote $\eta_n$, i.e., $\eta_n = f^{-n}(\eta_0)$, for $1 \leq n < \infty$. We have that $f(C_n)$ and $f(B_{n,i})$ for $1 \leq i \leq k_n$ are in $\eta_{n-1}$ for $n > 0$ (set $k_0 = q - 1$). We call $\eta = \{\eta_n\}_{n=0}^{\infty}$ the two-dimensional Yoccoz puzzle of $f$. Let $\Lambda = \bigcup_{n=0}^{\infty} f^{-n}(\alpha)$. Let $L_0 = \cap_{n=0}^{\infty} C_n$ be the connected component of $K_f \setminus \Lambda$ containing 0. We state Theorems 10 and 11 in the following form.

**Theorem 12 [YOC].** Suppose $(U, V, f)$ is a recurrent quadratic-like map. Then $(U, V, f)$ is renormalizable if and only if $L_0$ contains more than one point. Moreover, if $(U, V, f)$ is non-renormalizable, then any connected component of $K_f \setminus \Lambda$ consists of only one point and $K_f$ is locally connected.

5. **Infinitely Renormalizable Quadratic Julia Sets and Three-Dimensional Yoccoz Puzzles**

Suppose $(U, V, f)$ is a renormalizable quadratic-like map with connected filled-in Julia set $K_f$. Let $\eta = \{\eta_n\}_{n=0}^{\infty}$ be the two-dimensional Yoccoz puzzle for $f$. From the previous section, the critical tableau $T(0) = (a_{mm}^0)_{n\geq 0, m\geq 0}$ is periodic of period $n_1$. Let

$$0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_1 \subseteq C_0$$

be the critical end. There is an integer $N > 0$ such that $a_{ij}^0 = 0$ for all $i \geq N$ and for all $0 < j < n_1$. Let $f_0 = f^{n_1}|\hat{C}_{N+n_1}$. Then $f_0 : \hat{C}_{N+n_1} \rightarrow \hat{C}_N$ is a proper, holomorphic branch cover of degree two. Assume $C_{N+n_1} \subseteq C_N$. (Otherwise, we can replace $C_n$ with $\hat{C}_n$ (see Theorem 9)). Then $(\hat{C}_{N+n_1}, \hat{C}_N, f_0)$ is a quadratic-like map and its filled-in Julia set is $L_0 = \cap_{n=0}^{\infty} C_n$.

**Theorem 13.** Suppose $(U, V, f)$ is a renormalizable quadratic-like map with connected filled-in Julia set $K_f$. For any renormalization $U_1$ about $n_1$, let $f_1 = f^{n_1}|U_1$ and let $V_1 = f_1(U_1)$. Then the filled-in Julia set $K_{f_1}$ (or the Julia set $J_{f_1}$) of $(U_1, V_1, f_1)$ is always $L_0$ (or $\partial L_0$).
Theorem 12. and is the limiting piece containing 0 in the two-dimensional Yoccoz puzzle from
set is actually canonical; it is independent of choices of domains in renormalization.
But the renormalized filled-in Julia set of \((U, U)\) depends on choices of domains in renormalization. Let \(\hat{C}_N\) be the filled-in Julia set of \((U, V)\). It is renormalizable if the corresponding quadratic-like map is renormalizable. Let \(\hat{C}_N\) be the filled-in Julia set (or \(\hat{C}_N\)) of \((U, V)\). Since a filled-in Julia set is completely invariant and since 0 is in \(V\), the two inverse images of 0 under \(f\) are in \(K_{f}\). But these two points are also inverse images of 0 under \(f_0\) and under \(f_1\). Therefore, they are both in \(L_0\) and in \(K_{f_1}\). Using this argument, the set \(\Xi\) of all inverse images of 0 under iterates of \(f\) is contained in \(K_{f_2}\) and is also contained in \(L_0\) and in \(K_{f_1}\). Therefore,

\[
K_{f_1} = K_{f_2} = L_0 \text{ (or } J_{f_1} = J_{f_2} = \partial L_0)\]

because each of \(\partial L_0\), \(J_{f_1} = \partial K_{f_1}\), and \(J_{f_2} = \partial K_{f_2}\) is the closure of \(\Xi\) (see Theorem 3).

As we saw in \(\S 2\), the definition of the filled-in Julia set of a renormalization about \(n_1\) depends on choices of domains in renormalization. But the renormalized filled-in Julia set is actually canonical; it is independent of choices of domains in renormalization and is the limiting piece containing 0 in the two-dimensional Yoccoz puzzle from Theorem 12.

Suppose \((U_1, V_1, f_1)\) is a recurrent renormalizable quadratic-like map with connected filled-in Julia set \(K_1\). We call \(K_1\) (or \(J_1\)) a quadratic filled-in Julia set (or quadratic Julia set). It is renormalizable if the corresponding quadratic-like map is renormalizable. Let \(\beta_1\) and \(\alpha_1\) be the non-separate and separate fixed points of \(f_1\), i.e., \(K_1 \setminus \{\beta_1\}\) is still connected and \(K_1 \setminus \{\alpha_1\}\) is disconnected. Let \(\Lambda_1 = \cup_{n=0}^{\infty} f_1^{-n}(\alpha_1)\). Let \(K_2 = L_0\) be the connected component of \(K_1 \setminus \Lambda_1\) containing 0. From Theorem 12, \(K_1\) is renormalizable if and only if \(K_2\) contains more than one point. The quadratic filled-in Julia set \(K_2\) is called the renormalization of \(K_1\).

Inductively, let \(K_i\) be the renormalization of \(K_{i-1}\). Let \(f_i = f_{i-1}^{n_{i-1}}\) for \(i \geq 2\), where \(n_{i-1}\) is the period of the critical tableau \(T^{n_{i-1}}(0) = (a_{nm}^i(i - 1))_{n \geq 0, m \geq 0}\) of the two-dimensional Yoccoz puzzle for \((U_{i-1}, V_{i-1}, f_{i-1})\). Let \(\beta_i\) and \(\alpha_i\) be the non-separate and the separate fixed points of \(f_i\), i.e., \(K_i \setminus \{\beta_i\}\) is still connected and \(K_i \setminus \{\alpha_i\}\) is disconnected. Let \(\Lambda_i = \cup_{n=0}^{\infty} f_i^{-n}(\alpha_i)\) and let \(K_{i+1}\) be the connected component of \(K_i \setminus \Lambda_i\) containing 0. Then \(K_i\) is renormalizable if and only if \(K_{i+1}\) contains more
than one point. Here $K_i$, for $i > 1$, is called the $i^{th}$-renormalization of $K_1$. Theorem 12 can be generalized as follows.

**Theorem 14 [YOC].** Suppose that $(U_1, V_1, f_1)$ is a recurrent quadratic-like map and that $K_1$ is its filled-in Julia set. The quadratic Julia set $K_1$ is finitely renormalizable if and only if there is an integer $m \geq 1$ such that $K_1, \ldots, K_m$ contains more than one point and such that $K_{m+1}$ contains only the point 0. Moreover, if $K_1$ is finitely renormalizable, then $K_1$ is locally connected.

**Proof.** The first part of the theorem follows directly from Theorem 12. We prove the second part. Let $\alpha_m$ be the separate fixed point of $f_m$. Let $\Gamma_m$ be a cycle of periodic external rays of $f_1$ landing at $\alpha_m$ (refer to the end of §3). Using $\Gamma_m$, we can construct the two-dimensional Yoccoz puzzle: let $\eta_0^m$ be the set consisting of the closures of the connected components of $V_1 \setminus \Gamma_m$. Let $\eta_n^m = f_1^{-n}(\eta_0^m)$ for $n \geq 1$. Let $C_n^m$ be the member of $\eta_n^m$ containing 0. Since $f_m$ is non-renormalizable, we use a proof similar to that of Theorem 10 to show that $\sum_{n=0}^{\infty} \mod(A_{n0}^m(0)) = \infty$, where $A_{n0}^m(0) = C_n^m \setminus \hat{C}_{n+1}^m$. Applying Lemma 4, for every $x$-end $x \in \cdots \subseteq D_n^m(x) \subseteq D_{n-1}^m(x) \subseteq \cdots \subseteq D_1^m(x) \subseteq D_0^m(x)$, $L_x = \cap_{n=0}^{\infty} D_n^m(x)$ contains only $x$. By using a similar argument to the proof of Theorem 11, we can now show that $K_1$ is locally connected.

Now let us consider an infinitely renormalizable quadratic-like map $(U_1, V_1, f_1)$. Let $K_1$ be the filled-in Julia set of $f_1$. Let $K_i$ be the $i^{th}$-renormalization of $K_1$. Then $\mathcal{K} = \{K_i\}_{i=1}^{\infty}$ is a sequence of renormalizations of $K_1$. Let $\{(U_i, V_i, f_i)\}_{i=1}^{\infty}$ be a sequence of renormalizations with filled-in Julia set $K_i$ where $f_i = f_{i-1}^{c_{n_i}}$ and where $n_{i-1}$ is the period of the critical tableau $T_i^{-1}(0)$ of the two-dimensional Yoccoz puzzle for $(U_{i-1}, V_{i-1}, f_{i-1})$, for $i \geq 2$. Suppose $U_i$ is a renormalization of $(U_{i-1}, V_{i-1}, f_{i-1})$. We describe $(U_i, V_i, f_i)$ as $(n_1, n_2, \ldots)$-infinitely renormalizable. The grid $\{(T_i(0))\}_{i=1}^{\infty}$ is called the three-dimensional critical tableau for $(U_1, V_1, f_1)$. Let $c(n) = f_1^{c_{n}}(0)$. The critical orbit of $f_1$ is $CO = \{c(n)\}_{n=0}^{\infty}$. Let $GCO = \cup_{k=0}^{\infty} \cup_{n=0}^{\infty} f_1^{-k}(c(n))$ be the grand critical orbit of $f_1$.

**Definition 2.** An infinitely renormalizable quadratic-like map $(U_1, V_1, f_1)$ has a priori complex bounds if there is a constant $\lambda > 0$ and a sequence of renormalizations $\{(U_k, V_k, f_k)\}_{k=1}^{\infty}$ of $f_1$ such that

$$\mod(V_k \setminus U_k) \geq \lambda$$

for all $k \geq 1$.

**Theorem 15.** Suppose $(U_1, V_1, f_1)$ is an infinitely renormalizable quadratic-like map having the a priori complex bounds. Its filled-in Julia set $K_1$ is locally connected at every point in $GCO$. 
Proof. Suppose, without loss of generality, that \( \{U_i, V_i, f_i\}_{i=1}^\infty \) is the sequence of renormalizations in Definition 2. Let \( \lambda > 0 \) be the constant in Definition 2. Then \( \{U_i\}_{i=1}^\infty \) is a sequence of nested domains containing 0. Consider the annulus \( A_i = \overline{U_i} \setminus U_{i+1} \) for \( i \geq 1 \). For each \( i \geq 1 \), let \( cv_i = f_i(0) \) and let \( \gamma_i \) be a curve in \( V_i \setminus f_i^2(U_{i+1}) \) connecting \( cv_i \) and a point on the boundary of \( V_i \). Let \( 0 \in U'_i \subseteq U_i \) be the connected component of the pre-image of \( V_i \setminus \gamma_i \) under \( f_i^{o2} \). Then \( f_i^{o2} : U'_i \to V_i \setminus \gamma_i \) is a degree two branch covering. Moreover, \( f_i : U'_i \to f_i(U'_i) \subseteq U_i \) is also a degree two branch covering map. Thus \( f_i : U_i \setminus U'_i \to V_i \setminus f(U'_i) \) is a degree two branch covering map. This implies that

\[
\text{mod}(U_i \setminus U'_i) = \frac{1}{2} \cdot \text{mod}(V_i \setminus f_i(U'_i)).
\]

But \( V_i \setminus U_i \) is a sub-annulus of \( V_i \setminus f_i(U'_i) \). So

\[
\text{mod}(U_i \setminus U'_i) \geq \frac{1}{2} \cdot \text{mod}(V_i \setminus U_i) > \frac{\lambda}{2}.
\]

Remember that \( U_{i+1} \) is the domain of the renormalization \( f_{i+1} = f_i^{n_i} \) where \( n_i \geq 2 \). We have \( U_{i+1} \subseteq U_i \). Hence \( U_i \setminus U'_i \) is a sub-annulus \( A_i \). So

\[
\text{mod}(A_i) > \frac{\lambda}{2}.
\]

Let \( A_\infty = \cap_{i=1}^\infty U_i \). Since \( U_1 \setminus A_\infty = \cup_{i=1}^\infty A_i \),

\[
\text{mod}(U_1 \setminus A_\infty) \geq \sum_{i=1}^\infty \text{mod}(A_i) = \infty.
\]

Thus, \( A_\infty = \{0\} \). This implies that the diameter \( d(U_i) \) tends to 0 as \( i \) goes to infinity.

Let \( \alpha_i \) be the separate fixed point of \( f_i \). Let \( \Gamma_i \) be a cycle of periodic external rays of \( f_1 \) landing at \( \alpha_i \) (refer to the end of §3). Let \( \eta_0 \) be the set consisting of the closures of the connected components of \( U_1 \setminus \Gamma_i \) and let \( \eta_n = f_i^{-n}(\eta_0) \) for \( n \geq 1 \). Then \( \{\eta_n\}_{n=0}^\infty \) is a puzzle for \( f_1 \). Let \( C_n \) be the member of \( n_f \) containing 0. Consider the critical end

\[
0 \in \cdots \subseteq C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_1 \subseteq C_0
\]

and the critical tableau \( T^i(0) = (a_{nm}(i))_{n \geq 0, m \geq 0} \). Since \( (U_1, V_1, f_1) \) is \((n_1, n_2, \ldots, m_i)\)-renormalizable, \( T^i(0) \) is periodic of period \( m_i = \prod_{j=1}^i \eta_j \). Thus there is an integer \( N > 0 \) such that \( a_{nm}^i(i) = 0 \) for \( n \geq N \) and \( 0 < i < m_i \). Thus

\[
f_1^{\text{om}_i} = f_i^{\text{om}_i} : \hat{C}_{m_i+N} \to \hat{C}_N
\]

is a degree two proper holomorphic map. We may assume that \( C_{m_i+N} \subseteq \hat{C}_N \) (otherwise, we can modify \( C_n \) as the proof of Theorem 9). Therefore,

\[
f_{i+1} = f_1^{\text{om}_i} = f_i^{\text{om}_i} : \hat{C}_{m_i+N} \to \hat{C}_N
\]
is a quadratic-like map. Since
\[ K_{i+1} = \bigcap_{j=0}^{\infty} f_1^{-jm_i}(C^i_N) = \bigcap_{j=0}^{\infty} C^i_{jm_i+N} , \]
there is a \( C^i_{k(i)} \) contained in \( U_i \). The diameter \( d(C^i_{k(i)}) \) of \( C^i_{k(i)} \) tends to zero as \( i \) goes to infinity. From Lemma 2, \( C^i_{k(i)} \cap K_1 \) is connected. So \( \{C^i_{k(i)}\}_{i=1}^{\infty} \) is a basis of connected neighborhoods at 0.

For any \( x = f_1^{\circ n}(0) \), consider \( \{D_{i,n}(x) = f_1^{\circ n}(C^i_{k(i)})\}_{i=1}^{\infty} \). It is a basis of connected neighborhoods at \( x \). For any \( y \) in \( f_1^{-m}(x) \) (where 0 is not in \( f^{-n}(x) \) for \( 0 < n \leq m \)), there is an open neighborhood \( W \) of \( y \) such that \( f_1^{\circ m} : W \to f_1^{\circ m}(W) \) is a homeomorphism. Let \( g \) be its inverse. Then \( \{g(D_{i,n}(x))\}_{i=1}^{\infty} \) is a basis of connected neighborhoods at \( y \).

Suppose \((U_1, V_1, f_1)\) is an \((n_1, n_2, \ldots)\)-infinitely renormalizable quadratic-like map. We call the puzzle \( \{\{\eta^i_n\}_{n=0}^{\infty}\}_{i=1}^{\infty} \) constructed in Theorem 15, the three-dimensional Yoccoz puzzle for \((U_1, V_1, f_1)\). Let \( m_i = \prod_{j=1}^{i} n_j \) and \( \{K_i\}_{i=1}^{\infty} \) be the sequence of renormalizations of \( K_1 \). Let \( c(j) = f^{\circ j}(0) \) for \( j \geq 0 \), and let \( CO = \{c(j)\}_{j=0}^{\infty} \) be the critical orbit of \( f_1 \).

**Definition 3.** An infinitely renormalizable quadratic-like map \((U_1, V_1, f_1)\) is unbranched if there are a constant \( \lambda > 0 \) and a sequence of domains \( \{W_k\}_{k=1}^{\infty} \) such that \( W_k \supset K_{i_k} \), such that
\[
\text{mod}(W_k \setminus K_{i_k}) \geq \lambda,
\]
and such that \( W_k \cap CO = \{c(jm_{i_k})\}_{j=0}^{\infty} \) for all \( k \geq 1 \).

**Theorem 16.** Suppose \((U_1, V_1, f_1)\) is an infinitely renormalizable unbranched quadratic-like map having the a priori complex bounds. Then its filled-in Julia set \( K_1 \) is locally connected.

**Proof.** Suppose, without loss of generality, that \( k = i \) and \( i_k = i \) in Definition 3, and that \( \{U_1, V_1, f_1\}_{i=1}^{\infty} \) is a sequence of renormalizations in Definition 2. Let \( \lambda > 0 \) be a constant satisfying Definitions 2 and 3.

Let \( \{\{\eta^i_n\}_{n=0}^{\infty}\}_{i=1}^{\infty} \) be the three-dimensional Yoccoz puzzle for \((U_1, V_1, f_1)\). Let \( \{C^i_{k(i)}\}_{i=1}^{\infty} \) be the basis of connected neighborhoods constructed in Theorem 15. By choosing \( k(i) \) large enough, we can have
\[
\text{mod}(W_i \setminus C^i_{k(i)}) \geq \frac{\lambda}{2}
\]
for all \( i \geq 1 \).

If \( x = 0 \), Theorem 15 says that \( K_1 \) is locally connected at \( x \). For each \( x \neq 0 \) in \( K_1 \), there are two cases: either (1) \( x \) is non-recurrent, which means that there is an integer \( i \geq 1 \) such that \( \{f_1^{\circ n}(x)\}_{n=0}^{\infty} \cap C^i_{k(i)} = \emptyset \); or else (2) \( x \) is recurrent.

In case (1), we prove it by applying the results in hyperbolic geometry. Let \( x \) be a non-recurrent point. Then there is a \( C^i_{k(i)} \) such that the orbit \( O(x) = \{f_1^{\circ n}(x)\}_{n=0}^{\infty} \) is
disjoint from the interior of $C^i_{k(i)}$. Let $r$ be an external ray of $f_1$ cutting $U_1 \setminus f_1(C^i_{k(i)})$ into a simply connected domain $\Omega$. Consider two inverse branches $Q_1$ and $Q_2$ of $f_1|\Omega$. Let $\tilde{\Omega} = V_1 \setminus (f_1(C^j_{k(j)})) \cup r$ for some $j > i$ such that $\tilde{\Omega} \subset \Omega$. Let $Q_1$ and $Q_2$ be the inverses of $f_1|\tilde{\Omega}$ and let $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ be the images of $\tilde{\Omega}$ under $\tilde{Q}_1$ and $\tilde{Q}_2$, respectively. Consider $\Omega$, $\tilde{\Omega}_1$, and $\tilde{\Omega}_2$ as hyperbolic Riemann surfaces. Then $Q_1 : \Omega \to \Omega_1$ and $Q_2 : \Omega \to \Omega_2$ strictly contract the hyperbolic distances of $\tilde{\Omega}$, $\tilde{\Omega}_1$, and $\tilde{\Omega}_2$. Since $\tilde{\Omega}$ contains finite number of connected components of $K_1 \setminus f_1(C^i_{k(i)})$, we can cut $\Omega$ into finite number of simply connected domains $\Omega_1, \dotsc, \Omega_\ell$ such that $\Omega_k \cap K_1$ is connected for each $1 \leq k \leq \ell$. Since the orbit $O(x)$ is contained in $\tilde{\Omega}$, from the images of $\Omega_1, \dotsc, \Omega_\ell$ under the semi-group generated by $Q_1$ and $Q_2$, we can get a basis of connected neighborhoods at $x$. Therefore, $K_1$ is locally connected at $x$.

In case (2), $f_1^{\circ q_i}(x)$ enters $C^i_{k(i)}$ infinitely many times. For each $i \geq 1$, consider the puzzle $\eta^i = \{\eta^i_{n,j}\}_{n=0}^\infty$. For the $x$-end,

$$x \in \cdots \subseteq D^i_n(x) \subseteq D^i_{n-1}(x) \subseteq \cdots \subseteq D^i_1(x) \subseteq D^i_0(x),$$

let $T^i(x) = (a_{nm}(i))_{n\geq0,m\geq0}$ be the corresponding tableau. Let $q_i$ be the integer such that $a_{k(i)q} = 0$ for $0 \leq q < q_i$ and such that $a_{k(i)q_i} = 1$. Let $p_i = k(i) + q_i$. Then $g_{i,x} = f_1^{\circ q_i} : D^i_{p_i}(x) \to C^i_{k(i)}$ is a proper holomorphic diffeomorphism. Since there is no critical point in $W_i \setminus C^i_{k(i)}$, this diffeomorphism can be extended to a proper holomorphic diffeomorphism on $W_i$. Let $x_i = f_1^{\circ q_i}(x)$. We can modify $W_j$ such that

$$\frac{\lambda}{2} \leq \text{mod}(W_j \setminus C^j_{k(j)}) \leq 2\lambda.$$

From the previous theorem, the diameter $d(C^j_{k(j)})$ tends to zero as $j$ goes to infinity. We can find an integer $j > i$ such that $W_j \subset C^i_{k(i)}$. Let

$$x_j = f_1^{\circ q_i}(x) = f_1^{\circ (q_j - q_i)}(x_i).$$

Consider the puzzle $\eta^j = \{\eta^j_{n,j}\}_{j=1}^\infty$. Let

$$x_i \in \cdots \subseteq D^j_n(x_i) \subseteq D^j_{n-1}(x_i) \subseteq \cdots \subseteq D^j_1(x_i) \subseteq D^j_0(x_i)$$

be the $x_i$-end and let $T^j(x_i) = (b_{nm}(j))_{n\geq0,m\geq0}$ be the corresponding tableau. Then one can check that $f_1^{\circ (q_j - q_i)}$ is a proper holomorphic diffeomorphism from $D^j_{k(j)+q_j-q_i}(x_i) \to C^j_{k(j)}$.

In other words, $b_{k(j)m} = 0$ for $0 \leq m < q_j - q_i$, but $b_{k(j)(q_j - q_i)} = 1$. Let $g_{ij}$ be the inverse of $f_1^{\circ (q_j - q_i)} : D^j_{k(j)+q_j-q_i}(x_i) \to C^j_{k(j)}$. Then $g_{ij}$ can be extended to $W_j$. Since $C^i_{k(i)}$ is bounded by external rays landing at some pre-images of $\alpha_i$ under iterations of $P_c$ and is a part of an invariant net under $f_1$, $W_{ij} = g_{ij}(W_j)$ is contained in $C^i_{k(i)}$. 


Thus
\[ \text{mod}(W_i \setminus W_{ij}) \geq \frac{\lambda}{2}. \]
Consider \( X_i = g_{i,x}(W_i) \) and \( X_j = g_{j,x}(W_j) = g_{i,x}(W_{ij}) \). Then
\[ \text{mod}(X_i \setminus X_j) \geq \frac{\lambda}{2}. \]
Therefore, inductively, we can find a sequence of domains \( \{X_{is}\}_{s=1}^{\infty} \) such that
\[ \text{mod}(X_{is} \setminus X_{is+1}) \geq \frac{\lambda}{2} \]
for \( s \geq 1 \). Thus the diameter of \( X_{is} \) tends to zero as \( s \) goes to infinity. For each puzzle \( \eta^i = \{n^i_n\}_{n=0}^{\infty} \), consider both the \( x \)-end,
\[ x \in \cdots \subseteq D_n^{is}(x) \subseteq D_{n-1}^{is}(x) \subseteq \cdots \subseteq D_1^{is}(x) \subseteq D_0^{is}(x) \]
and the corresponding tableau \( T^{is}(x) = (a_{nm}(is))_{n,m \geq 0} \). For each \( k(is) \), there is an integer \( q_i \) such that \( a_{k(is)q}^i = 0 \) for \( 0 \leq q < q_i \) and such that \( a_{k(is)q_i}^i = 1 \). Then for \( p_is = k(is) + q_is \), \( j^{q_is} : D_{p_is}(x) \to C_{k(is)} \) is a proper holomorphic diffeomorphism.
This implies that
\[ D_{p_is}^{is}(x) \subseteq X_{is}. \]
So the diameter \( d(D_{p_is}^{is}(x)) \) tends to zero as \( s \) goes to infinity. From Lemma 2, \( \{D_{p_is}^{is}(x)\}_{s=1}^{\infty} \) forms a basis of connected neighborhoods at \( x \).

6. A Generalized Sullivan’s Sector Theorem

Let \( I = [0,1] \) be the closed unit interval. Let \( E_0 \) be the set of all functions \( G \) such that

1. \( G : I_G \supseteq I \to G(I_G) \) is a homeomorphism and \( G(0) = 0 \), \( G(1) = 1 \), and

2. \( G \) can be extended to be a schlicht function \( g \) on \( C_G = (C \setminus R^1) \cup \hat{I}_G \) preserving upper- and lower-half planes.

We assume \( I_G \) is the maximum interval satisfies (1) and (2) for each \( G \) in \( E_0 \) and call it the definition interval of \( G \). We will not distinguish \( g \) and \( G \) anymore. Take
\[ S_\gamma(z) = r^\hat{\gamma}e^{\hat{\gamma}z} : C \setminus \{x < 0\} \to C \]
as the standard \( \gamma \)-root where \( z = re^{\theta i} \) for \( r > 0 \), \( -\pi < \theta < \pi \), and \( \gamma > 1 \). For every \( a \leq 0 \), we call \( L_a(z) = ES_\gamma(z-a) + F \) a \( \gamma \)-root at \( a \) where \( E = 1/\left( (1-a)^\hat{\gamma} - (-a)^\hat{\gamma} \right) \)
and \( F = -(-a)^\hat{\gamma}E \) are determined by \( L_a(0) = 0 \) and \( L_a(1) = 1 \). Then \( L_a \) is an element in \( E_0 \) whose definition interval is \( [a, \infty) \).

Suppose that \( L_a \) is a \( \gamma \)-root at \( a \) and that \( G \) is an element in \( E_0 \) whose definition interval is \( I_G \). We say that \( L_a \) and \( G \) are compatible if \( [a,1] \subset G(I_G) \). For a
compatible pair \( L_a \) and \( G \), let \( a' = G^{-1}(a) \), \( J = [a', 1] \), \( L \cup R = I_G \setminus J \), and \( b = \min\{|L|, |R|\} \).

We consider several disks related to a compatible pair \( L_a \) and \( G \) (see Fig. 3). Set \( D^1 \) to be the closed disk centered at the middle point \((1 + a')/2\) of \([a', 1]\) with diameter \(1 + |a'| + 2b\). Set \( D^2 \) to be the closed disk centered at the middle point \((1 + a')/2\) of \([a', 1]\) with diameter \(k = \min\{1 + |a'| + b, 2(1 + |a'|)\}\). Set \( D^3 \) to be the maximum closed disk centered at \( a' \) and contained in \( D^2 \). Then the radius of \( D^3 \) is \( d = \min\{b/2, (1 + |a'|)/2\} \). We note that

\[ D^3 \subset D^2 \subset D^1. \]

The map \( G \) is a schlicht function on \( D^1 \). Let \( \mu = (1 + |a'|)/b \) and \( \nu = (2\mu + 3)^4 \). From Koebe’s distortion theorem (see [BIE]) for any \( \xi \) and \( \eta \) in \( D^2 \),

\[ \nu^{-1} \leq \frac{|G' (\xi)|}{|G' (\eta)|} \leq \nu. \]

Using the fact that \( G(I) = I \), there is at least one \( \eta \) in \( D^2 \) such that \( |G'(\eta)| = 1 \). Hence

\[ \nu^{-1} \leq |G'(\xi)| \leq \nu \]

for all \( \xi \) in \( D^2 \). This implies that

\[ \frac{|a'|}{\nu} \leq |a| \leq |a'| \cdot \nu. \]

Let \( \text{UH} = \{z = x + yi \in \mathbb{C} \mid y > 0\} \) be the upper-half plane. For any \( z \) in \( \text{UH} \), let \( \theta(z) = \arg(z) \).

**Theorem 17.** Suppose that \( L = L_a \) and \( G \) are a pair of compatible elements in \( \mathcal{E}_0 \). There is a constant \( 0 < \theta < \pi \) depending only on \( \mu \) such that the image \( L(G(\text{UH})) \)
of the upper-half plane under \( L \circ G \) contains an open triangle \( \Delta \) based on \([L(a), 0]\) whose angle at \( L(a) \) is \( \pi/\gamma \) and whose angle at 0 is \( \theta \).

**Proof.** The image \( G(D^3) \) of \( D^3 \) under \( G \) contains the closed disk \( D^1 \) centered at \( a \) with radius \( d/\nu \). Similarly, for any \( a' \leq x \leq 1 \), consider the closed disk \( D(x) \) centered at \( x \) with radius \( d \). Then \( D(x) \subset D^2 \) and \( G(D(x)) \) contains the closed disk centered at \( G(x) \) with radius \( d/\nu \). This implies that the convex-hull \( X \) of \( \{0\} \cup D^4 \) is contained in \( G(D^2) \). Either \( X = D^4 \) or \( X \cap \text{UH} \) has an angle at 0. In the later case, the angle \( \varphi \) of \( X \cap \text{UH} \) at 0 has \( \sin \varphi = d/|a|\nu \).

Since \( L \) is a \( \gamma \)-root for \( \gamma > 1 \), the convex set \( L(X) \) contains a triangle \( \Lambda \) based on \([L(a), 0]\) whose angle at \( L(a) \) is \( \pi/\gamma \) and whose angle \( \omega \) at 0 can be calculated from

\[
\frac{\sin \omega}{\sin(\frac{\pi}{\gamma} + \omega)} = \left( \frac{d}{|a|\nu} \right)^{\frac{1}{2}}
\]

through the law of sines. Because \( d = \min\{b/2, (1 + |a'|)/2\} \),

\[
\frac{d}{|a|} \geq \frac{d}{|a'|\nu} \geq \min\left\{ \frac{1}{2\mu\nu}, \frac{1}{2\nu} \right\}.
\]

Hence \( \Lambda \) contains a triangle \( \Delta \) based on \([L(a), 0]\) whose angle at \( L(a) \) is still \( \pi/\gamma \) and whose angle \( \theta \) at 0 is calculated from

\[
\frac{\sin \theta}{\sin(\frac{\pi}{\gamma} + \theta)} = \left( \min\left\{ \frac{1}{2\mu\nu}, \frac{1}{2\nu} \right\} \right)^{\frac{1}{\gamma}}.
\]

Suppose \( Q = (L \circ G)^{-1}(\Delta) \), where \( \Delta \) is the triangle obtained in Theorem 17. Then \( Q \subset D^2 \subset D^3 \).

Suppose \( \{(L_i, G_i)\}_{i=1}^n \) is a sequences of compatible pairs in \( \mathcal{E}_0 \) where \( L_i \) is a \( \gamma \)-root at \( a_i \). Let \( a'_i, b_i, k_i, d_i, \mu_i, \nu_i, \) and \( \theta_i \) be the numbers, and \( D^1_i, D^2_i, D^3_i, \Delta_i, \) and \( Q_i \) be the sets corresponding to each compatible pair \( L_i \) and \( G_i \). Let

\[
\mathcal{L} = L_n \circ G_n \circ \cdots \circ L_i \circ G_i \circ \cdots \circ L_0 \circ G_0.
\]

Then \( \mathcal{L} \) is a schlicht function defined on \( C_1 = (C \setminus \mathbb{R}^1) \cup \hat{I} \).

**Definition 4.** We call \( \mathcal{L} \) a root-like map if there are constants \( C > 0 \) and \( \lambda > 1 \) such that

1. \( a_0 = 0 \) and \( a_1 \geq 1/C \),
2. \( |a_j| \geq \max\{((\lambda^{j-1})/C) \cdot |a_i|, (1 + (\lambda - 1)/C) \cdot |a_i|\} \) for all \( 1 \leq i < j \leq n \), and
3. \( \mu_i < C \) for all \( 1 \leq i \leq n \).
Theorem 18. Suppose $L$ is a root-like map. There is a constant $\theta > 0$ depending only on $\lambda$ and $C$ such that the image of the upper-half plane under $L$ is contained in the sector

$$\text{Sec}_\theta = \{ z \in \mathbb{C} \mid 0 \leq \arg(z) \leq \pi - \theta \}.$$ 

Before we prove this theorem, we introduce some basic results in hyperbolic geometry. Let $C_I = (\mathbb{C} \setminus \mathbb{R}^1) \cup \tilde{I}$ be a plane domain. Then $q(z) = -z^2/(1 - z^2)$ is a diffeomorphism from the upper-half plane $\mathbb{U}H$ onto $C_I$. Consider $\mathbb{U}H$ to be a hyperbolic plane with Poincaré metric $d_{\mathbb{U}H} = |dz|/y$ for $z = x + yi$. This metric induces a hyperbolic metric $d_{C_I} = q^*(|dz|/y)$ on $C_I$. The plane domain $C_I$ under this metric is a hyperbolic Riemann surface. Let $d = d_{C_I}$ be the induced hyperbolic distance. We note that $q$ maps the positive imaginary line in $\mathbb{U}H$ onto the interval $I$ and maps the real line, which is the boundary of $\mathbb{U}H$, onto the set $\mathbb{R}^1 \setminus \tilde{I}$.

Lemma 6. A hyperbolic neighborhood $\Phi(r) = \{ z \in C_I \mid d(z, I) < r \}$ is the union of two Euclidean disks $D^+$ and $D^-$, symmetric to each other with respect to $I$, centered at $c^+$ and at $c^- = -c^+$, with the same radius $R^+ = R^-$. Moreover,

$$R^+ = \frac{1}{2\sin \beta} \quad \text{and} \quad c^+ = \frac{1}{2} + \frac{\cot \beta}{2} i,$$

where

$$\beta = 4 \cot^{-1}(e^r)$$

is the angle at 0 between $\partial D^+$ and the negative real line (and the angle at 1 between $\partial D^+$ and the ray $[1, \infty)$).

Proof. Consider the pre-image $\Phi' = q^{-1}(\Phi(r))$. It is a hyperbolic neighborhood in $\mathbb{U}H$ and consists of all points in $\mathbb{U}H$ whose hyperbolic distances to the half-line $l_+ = \{ z = yi \mid y > 0 \}$ are less than $r$. The boundary $\partial \Phi'$ consists of two rays starting from 0. Thus $\Phi'$ is a sector, symmetric with respect to the half-line $l_+$. Suppose $\beta/2$ is the outer angle of this sector (with respect to the real line). Since a geodesic in $\mathbb{U}H$ is a semi-circle or half-line perpendicular to the real line, it is easy to check that

$$\log \left( \cot \frac{\beta}{4} \right) = r.$$ 

Therefore, $\Phi(r)$ is the union of two disks $D^+$ and $D^-$ symmetric with respect to $I$. The angle between $\partial D^+$ (or $\partial D^-$) and the negative real line is $\beta$. Moreover, every point $z$ in $\partial D^+$ (or $\partial D^-$) views $I$ under the same angle $\beta$, that is, every triangle $\triangle(0z1)$ has the angle $\beta$ at $z$. Now consider the point $u$ such that the segment $\overline{Tu}$ is a diameter of $D^+$. The triangle $\triangle(u01)$ is a right triangle. We can calculate the length $2R^+$ of the segment $1u$ and length $|u|$ of the segment $0u$ as follows:

$$2R^+ = \frac{1}{\sin \beta}, \quad |u| = \cot \beta.$$
Therefore,

\[ R^+ = \frac{1}{2 \sin \beta} \quad \text{and} \quad c^+ = \frac{1}{2} + \frac{\cot \beta}{2} \cdot i. \]

Lemma 7. Let \( z \) be a point in \( C_1 \) and let \( \Phi(r) = D^+ \cup D^- \) be the smallest hyperbolic neighborhood containing \( z \). The Euclidean radius \( R^+ \) of \( D^+ \) (and \( D^- \)) is

\[ \frac{|z - 1|}{2 \sin(\arg(z))}. \]

Proof. Let \( u \) be the point in \( \partial D^+ \) such that the segment \( \overline{1u} \) is a diameter of \( D^+ \). The angle of the triangle \( \triangle(0z1) \) at \( z \) and the angle of the triangle \( \triangle(0u1) \) at \( u \) are both \( \beta \) (see the previous lemma). Now applying the law of sines,

\[ \frac{\sin(\arg(z))}{|1 - z|} = \sin \beta = \frac{1}{|1 - u|}. \]

Therefore,

\[ 2R^+ = \frac{|z - 1|}{\sin(\arg(z))}. \]

Suppose \( \mathcal{L} \) is a root-like map. Then there is a constant angle \( \sigma \) and a constant \( C_0 > 0 \) depending only on \( C \) such that \( \theta_i \geq \sigma \) and \( \nu_i \leq C_0 \) for all \( 0 \leq i \leq n \).

For each \( 1 \leq i \leq n - 1 \), let \( A_{i+1} = Q_{i+1} \setminus \Delta_i \). Let \( A_1 = Q_1 \setminus \{ z \in \mathbb{C}, 0 \leq \arg(z) < \pi - \pi/\gamma \} \). Suppose \( \Phi(r_i) = D^+_i \cup D^-_i \) is the smallest hyperbolic neighborhood in \( C_1 \) containing \( A_i \neq \emptyset \) and let \( D^0_i \) be the smallest disk centered at \( 1/2 \) containing \( \Phi(r_i) \).

Let \( R^+_i \) be the Euclidean radius of \( D^+_i \) and let \( R_i \) be the Euclidean radius of \( D^0_i \).

Since \( z \) and 1 are in \( D^2_i \), from Lemmas 6 and 7,

\[ R_i \leq 2 \cdot R^+_i \leq \frac{k_i}{\sin \theta_i} \leq \frac{2 \cdot (1 + |a'_i|)}{2 \cdot (1 + C) \cdot |a'_i|} = C_1 \cdot |a'_i| \]

where \( k_i = \min\{1 + |a'_i| + b_i, 2(1 + |a'_i|)\} \) is the diameter of \( D^2_i \) and where \( C_1 = 2 \cdot (1 + C)/\sin \sigma \) is a positive constant depending only on \( C \) (see Fig. 4).

For every \( 1 \leq i \leq n \), the definition interval of \( \phi_i = L_i \circ G_i \) is \( [a'_i, a''_i] = [a'_i, \infty) \cap I_{G_i} \).

According to (iii) of Definition 4, the right-endpoint \( a''_i \) satisfies

\[ a''_i \geq \frac{1 + |a'_i|}{C} + 1 \geq 1 + \frac{|a'_i|}{C}. \]

For every \( 1 \leq i < j \leq n \),

\[ |a'_j| \geq \frac{\lambda^{j-i}}{C C_0} |a'_i|. \]

Therefore, if \( \tau = \min\{1, 1/(C \cdot C_0)^2\} \), we have

\[ [a'_j, a''_j] \supset I_{i, \tau} = [\lambda^{j-i} \tau a'_i, \lambda^{j-i} \tau |a'_i| + 1]. \]
In other words, $\phi_j = L_j \circ G_j$ are schlicht functions on $C_{i \tau} = (C \setminus R^1) \cup I_{i \tau}$ for all $1 \leq i < j \leq n$.

**Fig. 4**

**Lemma 8.** There is a fixed integer $n_0 > 0$ depending only on $C$ and on $\lambda$ such that for any $0 \leq i < n$ and any $j \geq i + n_0$,

$$|a'_j| + \frac{1}{2} > \lambda^{j-i-n_0} R_i \quad \text{and} \quad a''_j - \frac{1}{2} > \lambda^{j-i-n_0} R_i.$$

**Proof.** Take $m_0$ as the biggest integer such that

$$m \leq \frac{\log(C_1)}{\log \lambda}.$$

From the above estimates, one can see that $n_0 = \max\{m_0 + 1, 0\}$ is the integer satisfying the lemma.

**Lemma 9.** There is a constant $C_2 > 0$ depending only on $\lambda$ such that for any $0 \leq i < n-n_0$, let $\Pi_i = L_n \circ G_n \circ \cdots \circ L_j \circ G_j \circ \cdots \circ L_{i+n_0+1} \circ G_{i+n_0+1}$, the distortion of $\Pi_i$ on $\Phi(r_i)$ is bounded by $C_2$, more precisely,

$$\left| \frac{\Pi'_i(\xi)}{\Pi'_i(\eta)} \right| \leq C_2$$

for all $\xi$ and $\eta$ in $\Phi(r_i)$.

**Proof.** Consider the disk $D_5^j$ centered at $1/2$ with radius $t_j = \min\{|a'_j| + 1/2, a''_j - 1/2\}$. The composition $\phi_j = L_j \circ G_j$ is a schlicht function on $D_5^j$. Let $r_{ij} = R_i/t_j$ be
the ratio of the radii of $D_i$ and $D_j$. Then

$$r_{ij} \leq \lambda^{i+n_0-j}$$

for $i + n_0 + 1 \leq j \leq n$. From Koebe’s distortion theorem (see [BIE]),

$$\left| \frac{\phi_j' (\xi)}{\phi_j' (\eta)} \right| \leq \left( \frac{1 + r_{ij}}{1 - r_{ij}} \right)^4 \leq \left( \frac{1 + \lambda^{-k}}{1 - \lambda^{-k}} \right)^4$$

for $0 < k = j - i - n_0 \leq n - i - n_0$ and for all $\xi$ and $\eta$ in $D_i$. Since $\phi_j$ is a schlicht function on $C$, it contracts the hyperbolic distance $d$ on $C$. Thus

$$\phi_j (\Phi (r_i)) \subset \Phi (r_i) \subset D_i.$$ 

Therefore, by the chain rule,

$$\left| \frac{\Pi_j' (\xi)}{\Pi_j' (\eta)} \right| = \prod_{j=n_0+1}^{n} \frac{|\phi_j' (\xi)|}{|\phi_j' (\eta)|} \leq C_2 = \left( \prod_{k=1}^{\infty} \frac{1 + \lambda^{-k}}{1 - \lambda^{-k}} \right)^4$$

for all $\xi$ and $\eta$ in $\Phi (r_i)$. \hfill \Box

**Lemma 10.** There is a constant $C_3 > 0$ depending only on $C$ such that for $0 \leq i \leq n$, for $i \leq j \leq i + n_0$, for $\phi_j = L_j \circ G_j$, and for all $\xi$ and $\eta$ in $D_i^+$ (or $D_i^-$),

$$\left| \frac{\phi_j' (\xi)}{\phi_j' (\eta)} \right| \leq C_3.$$ 

**Proof.** Suppose $x$ is a real number. Let $\alpha = \alpha (x) = \min \{|x - z| \mid z \in D_i^+ \cap \text{UH}\}$. Suppose that $c = c_i^+$ and that $R = R_i^+$ are the center and the radius of $D_i^+$. Suppose $h = h_i^+$ is the length of the segment $c \frac{1}{2}$ (the straight line connecting $c$ and $1/2$). Since the two triangles $\triangle (x \frac{1}{2} c)$ and $\triangle (0 \frac{1}{2} c)$ are both right triangles, then

$$(\alpha + R)^2 = \left( \frac{1}{2} - x \right)^2 + h^2$$

and

$$R^2 = \left( \frac{1}{2} \right)^2 + h^2.$$ 

Therefore,

$$\frac{\alpha + R}{R} = \sqrt{1 + \frac{x^2 - x}{R^2}}.$$ 

This implies that there is a constant $0 < C_4 < 1$ depending only on $C$ such that for $x = a_j$ or $x = a_j''$,

$$\frac{R}{\alpha + R} \leq C_4.$$
Now consider the largest disk \( D_j^6 \) centered at \( c \) such that \( \phi_j \) is a schlicht function on it. Then the radius of \( D_j^6 \) is greater than or equal to \( \min\{R + \alpha(a'_j), R + \alpha(a''_j)\} \).

From Koebe’s distortion theorem (see [BIE]),
\[
\frac{|\phi'_j(\xi)|}{|\phi'_j(\eta)|} \leq C_3 = \left( \frac{1 + C_4}{1 - C_4} \right)^4
\]
for any \( \xi \) and \( \eta \) in \( D_i^+ \).

Combining Lemmas 9 and 10, we obtained the following estimate:

**Lemma 11.** There is a constant \( C_5 > 0 \) depending on \( \lambda \) and on \( C \) such that for any \( 0 \leq i < n \), let \( \Sigma_i = L_n \circ G_n \circ \cdots \circ L_j \circ G_j \circ \cdots \circ L_i \circ G_i \), the distortion of \( \Sigma_i \) on \( D_i^+ \) (or \( D_i^- \)) is bounded by \( C_5 \), more precisely,
\[
\frac{\Sigma_i'(\xi)}{\Sigma_i'(\eta)} \leq C_5
\]
for all \( \xi \) and \( \eta \) in \( D_i^+ \) (or \( D_i^- \)).

**Proof.** Since each \( \phi_j \) is a schlicht function on \( C_I \), it contracts the hyperbolic distance \( d \) on \( C_I \). So
\[
\phi_j(D_i^+) \subset D_i^+
\]
for all \( i \leq j \leq n \). If \( n - i \leq n_0 \), then from Lemma 10 and the chain rule,
\[
\frac{|\Sigma_i'(\xi)|}{|\Sigma_i'(\eta)|} \leq C_3^{m_0}
\]
for all \( \xi \) and \( \eta \) in \( D_i^+ \).

Now we consider \( n - i > n_0 \) and write \( \Sigma_i = \Pi_i \circ \Theta_i \) where \( \Theta_i = L_{i+n_0} \circ G_{i+n_0} \circ \cdots \circ L_i \circ G_i \) and where \( \Pi_i = L_n \circ G_n \circ \cdots \circ L_{i+n_0+1} \circ G_{i+n_0+1} \). From Lemma 10,
\[
\frac{|\Theta_i'(\xi)|}{|\Theta_i'(\eta)|} \leq C_3^{m_0}
\]
for all \( \xi \) and \( \eta \) in \( D_i^+ \), and from Lemma 9,
\[
\frac{|\Pi_i'(\xi)|}{|\Pi_i'(\eta)|} \leq C_2
\]
for all \( \xi \) and \( \eta \) in \( D_i^+ \). Again, because \( \Theta_i \) is a schlicht function on \( C_I \) and contracts the hyperbolic distance \( d \) on \( C_I \), we have \( \Theta_i(D_i^+) \subset D_i^+ \). Therefore, from the chain rule,
\[
\frac{|\Sigma_i'(\xi)|}{|\Sigma_i'(\eta)|} \leq C_5 = C_2 \cdot C_3^{m_0}
\]
for all \( \xi \) and \( \eta \) in \( D_i^+ \).
Lemma 12. Suppose $G$ is in $\mathcal{E}_0$ and $D$ is a closed simply connected convex domain with $I \subset D \subset \mathbb{C}_G$. Then for all $z = x + yi$ with $y > 0$ in $D$

$$\sin \left( \arg(G(z)) \right) \geq \frac{\sin(\arg(z))}{N_0}$$

where $N_0 = \sup_{\xi, \eta \in D} |G'(\xi)/G'(\eta)|$ measures the distortion of $G$ on $D$.

Proof. Since $G$ maps $\mathbb{H}$ into itself, it contracts the hyperbolic metric $dz/y$ on $\mathbb{H}$. Suppose $z = x + yi$ with $y > 0$ and $G(z) = X + Yi$. Then $|G'(z)|y \leq Y$. Therefore,

$$\sin(\arg(z)) = \frac{y}{|z|} \leq \frac{Y}{|z| \cdot |G'(z)|} = \sin \left( \arg(G(z)) \right) \cdot \frac{|G(z)|}{|z| \cdot |G'(z)|}.$$ 

So

$$\frac{\sin(\arg(z))}{N_0} \leq \sin \left( \arg(G(z)) \right)$$

for all $z = x + yi$ with $y > 0$ in $D$. 

Now we complete the proof of Theorem 18 as follows.

Proof of Theorem 18. For any $z_0$ in $\mathbb{H}$, let $z_{i+1} = L_i(G_i(z_i))$ for $0 \leq i \leq n$. Since $0 \leq \arg(z_1) \leq \pi/\gamma$, the smallest positive integer $i$ such that $z_i$ lies in $\Delta_i$ must be either bigger than zero or not exist. If such a positive integer does not exist, then $0 \leq \arg(z_{n+1}) \leq \pi - \theta_n \leq \pi - \sigma$. Now let us suppose that this smallest number exists and is $i_0 + 1$. Then $z_{i_0}$ is in $A_{i_0} \neq \emptyset$ which is a subset of $D_{i_0}^+$. Since $0 < \arg(z_{i_0}) \leq \pi - \theta_i$, Lemmas 11 and 12 assure us there is a constant angle $0 < \theta \leq \sigma$ such that $0 < \arg(z_{n+1}) \leq \pi - \theta$.

7. FEIGENBAUM-LIKE QUADRATIC-LIKE MAPS

In this section, we discuss Feigenbaum-like quadratic-like maps and prove the result of Sullivan which says that a Feigenbaum-like quadratic-like map has the a priori complex bounds and is unbranched.

Let us first recall some facts about infinitely renormalizable folding mappings. Let $(U, V, f)$ be a real quadratic-like map, that is, $f(U \cap \mathbb{R}^1) \subseteq V \cap \mathbb{R}^1$. Conjugating by a linear fraction transformation, we may assume that $f(-1) = f(1) = -1$ and that $f|[-1, 1]$ is a folding map of $[-1, 1]$ with a unique quadratic critical point 0. For example, $P_t(z) = t - (1 + t)z^2$ for $0 \leq t \leq 1$ is a real quadratic-like map whenever it is restricted to any domain bounded by an equipotential curve. Furthermore, suppose $(U, V, f)$ is infinitely renormalizable. Then the filled-in Julia set $K_f$ is connected. Let $\beta_0 = -1$ and let $\alpha_0$ be the fixed point of $f$ in $(-1, 1)$. Then $\beta_0$ is the non-separate fixed point, $\alpha_0$ is the separate fixed point, and $K_f \cap \mathbb{R}^1 = [-1, 1]$. The mapping $f_0 = f|[-1, 1]$ is an infinitely $(n_1, n_2, \ldots, n_k, \ldots)$-renormalizable folding mapping where $\{n_k\}_{k=1}^\infty$ is the maximum sequence of integers such that $f_0$ is renormalizable.
about \( m_k \) for \( m_k = \prod_{i=1}^{k} n_i \). Let \( I_k = [-a_k, a_k] \) be the maximal interval containing 0 (set \( m_0 = 0 \) and \( I_0 = [-1, 1] \)) such that

(a) \( f_0^{m_k}(i) \) is monotone when restricted to \([-a_k, 0] \) and to \([0, a_k] \),

(b) \( f_0^{m_k}(I_k) \subset I_k \),

(c) \( I_k, f_0(I_k), \ldots, f_0^{(m_k-1)}(I_k) \) have pairwise disjoint interiors, and

(d) \( f_0^{m_k} \) has exactly two fixed points \( \beta_k \) and \( \alpha_k \) in \( I_k \) where \( a_k = |\beta_k| \).

There is a domain \( U_k \supseteq I_k \) such that \( (U_k, V_k, f_0^{m_k}) \) is a quadratic-like map with connected filled-in Julia set \( K_k \), where \( K_k \) is the \( k \)-th renormalization of \( K_1 \). Then \( \beta_k \) is the non-separate fixed point and \( \alpha_k \) is the separate fixed point of this quadratic-like map and then \( K_k \cap \mathbb{R}^1 = I_k \). An infinitely renormalizable real quadratic-like map is called Feigenbaum-like if \( n_k = 2 \) for all \( k \).

Suppose that \( \beta(i) = f_0^n(0) \) is the \( i \)-th critical value of \( f \) and that \( J_k(i) \) is the interval bounded by \( \beta(i) \) and \( \beta(m_k + i) \) for \( k \geq 0 \) and \( 0 \leq i < m_k \). We note that \( J_k(0) = J_k(m_k) \). Then \( f_0 : J_k(0) \to J_k(1) \) is folding for all \( k \geq 0 \) and \( f_0 : J_k(i) \to J_k(i + 1) \) is a homeomorphism for every \( k \geq 1 \) and \( 1 \leq i < m_k \). Let \( \zeta_k = \{J_k(i)\}_{0 \leq i < m_k} \) for \( k \geq 0 \). Let \( I_k(i) = f_0^i(I_k), 0 \leq i < m_k \), and let \( \xi_k = \{I_k(i)\}_{0 \leq i < m_k} \). Note that \( J_k(i) \subseteq I_k(i) \).

We use \( L_{I_k}(i) \) and \( R_{I_k}(i) \) to denote the intervals in \( \xi_k \) adjacent to \( I_k(i) \) and on the left and right sides of \( I_k(i) \), respectively (there is only \( L_{I_k}(1) \) or \( R_{I_k}(2) \) in \( \xi_k \)). Let \( L_{I_k}^+(i) \) be the smallest interval containing \( L_{I_k}(i) \) and the left-end point of \( I_k(i) \) and let \( R_{I_k}^+(i) \) be the smallest interval containing \( R_{I_k}(i) \) and the right end-point of \( I_k(i) \), for \( i = 0 \) or \( 3 \leq i < m_k \). Let \( L_{I_k}^+(2) = [-1, c(2)] \) and \( R_{I_k}^+(1) = [c(1), 1] \). Similarly, we can define \( L_{J_k}(i) \) and \( R_{J_k}(i) \) and \( L_{J_k}^+(i) \) and \( R_{J_k}^+(i) \) for \( 0 \leq i < m_k \). The following theorem is due to Sullivan.

**Theorem 19 [SU2].** There is a constant \( C > 0 \) such that

\[
\min\{|L_{I_k}^+(i)|, |R_{I_k}^+(i)|\} \geq C \cdot |I_k(i)|,
\]

and such that

\[
\min\{|L_{J_k}^+(i)|, |R_{J_k}^+(i)|\} \geq C \cdot |J_k(i)|.
\]

for all \( k \geq 0 \) and \( 0 \leq i < m_k \).

Consider the slit domain \( V_0 = V \setminus [c(1), \infty) \). The map \( f|V_0 \) has two inverse branches (see Fig. 5)

\[
g_0 : V_0 \to U_{0,0} = U \cap \{z = x + yi \in \mathbb{C} \mid x < 0\}
\]

and

\[
g_1 : V_0 \to U_{0,1} = U \cap \{z = x + yi \in \mathbb{C} \mid x > 0\}.
\]
For each $k \geq 1$, let $g_{s(i)} : J_k(i + 1) \to J_k(i)$ for $1 \leq i < m_k$ where $s(i) = 0$ if $J_k(i)$ is contained in the negative half of the real line, and otherwise $s(1) = 1$. Then

$$A_k = g_{s(1)} \circ g_{s(2)} \circ \cdots \circ g_{s(m_k)} : J_k(0) \to J_k(1)$$

is a homeomorphism and can be extended homeomorphically to the maximum closed interval

$$T_k(m_k) \supseteq LJ_k^+(m_k) \cup J_k(m_k) \cup RJ_k^+(m_k).$$

Furthermore, $A_k$ can be extended analytically to $V_k = (V \setminus \mathbb{R}^1) \cup T_k(m_k)$. Let us continue to use $A_k$ to denote this extension. Let $U_k' = A_k(V_k)$, and let $U_k = f^{-1}(U_k')$ be the pre-image of $U_k'$ under $f$. Since $f$ has no attractive and no parabolic periodic point, then $g_0\left(A_k(T_k(m_k))\right)$ and $g_1\left(A_k(T_k(m_k))\right)$ are contained strictly in $T_k(m_k)$. Thus, $U_k \subset V_k$ and

$$f^{\circ m_k} : U_k \to V_k$$

is a quadratic-like map (see Fig. 6).
Let $T_k(i) = g_{s(i)}(T_k(i + 1))$ for $i = m_k - 1, m_k - 2, \ldots, 1$. Let $T_k(0) = f_0^{-1}(T_k(1))$. Then

$$J_k(0) = J_k(m_k) \subseteq I_k \subseteq T_k(0) \subset T_k(m_k).$$

The interval $T_k(m_k)$ is bounded by two critical values $c(q(k))$ and $c(r(k))$ of $f$; one of them is a maximum value and the other is a minimum value of $f^{\circ m_k}$. Suppose $T_k(1) = [d_1, e_1]$ where $d_1 < c(1) < e_1$. Let $d_i = f^{\circ i}(d_1)$ and let $e_i = f^{\circ i}(e_1)$ for $1 \leq i \leq m_k$. Note that $d_{m_k}$ and $e_{m_k}$ are $c(r(k))$ and $c(q(k))$. We normalize $T_k(i)$ into the unit interval $[0, 1]$: let $l_i : T_k(i) \to [0, 1]$ be the linear map such that $l_i(d_i) = 0$ and $l_i(e_i) = 1$. We define the linear map $g_s(i)$ as

$$g_s(i) = l_i \circ g_{s(i)} \circ l_i^{-1}$$

for $i = m_k - 1, \ldots, 2, 1$. The map $g_{s(i)}$ fixes 0 and 1 and is a univalent function defined on $C_{[0,1], V} = \left( (C \setminus R^1) \cup (0,1) \right) \cap V$.

The restriction $g_{s(i)}|[0,1]$ is a homeomorphism of $[0,1]$. Let

$$L_k = g_{s(1)} \circ g_{s(2)} \circ \cdots \circ g_{s(m_k-1)}$$

for $k \geq 1$.

**Lemma 13.** Suppose $(U, V, f)$ is a Feigenbaum-like quadratic-like map. Then

$$\{L_k\}_{k=3}^{\infty}$$

are uniform root-like maps.

**Proof.** Since $f$ is a Feigenbaum-like map, we have $m_k = 2^k$ and $T_k(m_k) = J_{k-2}(0)$ for $3 \leq k < \infty$. For any $k \geq 3$, consider the homeomorphism $f^{\circ (2^k - 1)} : J_k(2^k - 1) \to J_k(2^k + 1)$. Its inverse $g_{s(2^k - 1 + 1)} \circ \cdots \circ g_{s(2^k - 1)} : J_k(2^k) \to J_k(2^k - 1 + 1)$ can be extended to $T_{k-1}(2^k - 1)$, that is, we can consider

$$g_{s(2^k - 1 + 1)} \circ \cdots \circ g_{s(2^k - 1)} : T_{k-1}(2^k - 1) \to T_{k-1}(1).$$

Let

$$G_0 = g_{s(2^k - 1 + 1)} \circ \cdots \circ g_{s(2^k - 1)}$$

and let

$$L_{a_0} = g_{s(2^k - 1)}.$$

Then $G_0$ is a univalent map (or holomorphic embedding) of $C_{[0,1], V}$ such that $G_0(0) = 0$ and $G_0(1) = 1$ and such that $G_0([-1,1]$ is a homeomorphism of $[0,1]$. The map $L_{a_0}$ is a square root at 0 (see Fig. 7).

Consider the homeomorphism $f^{\circ (2^k - 2)} : J_k(2^k - 2) \to J_k(2^k - 1)$. Its inverse $g_{s(2^k - 2 + 1)} \circ \cdots \circ g_{s(2^k - 2)} : J_k(2^k - 1) \to J_k(2^k - 2 + 1)$ can be extended to $T_{k-2}(2^k - 2)$, that is, we can consider

$$G'_0 = g_{s(2^k - 2 + 1)} \circ \cdots \circ g_{s(2^k - 2)} : T_{k-2}(2^k - 2) \to T_{k-2}(1).$$
Let
\[ G_1 = \tilde{g}_{s(2^{k-2}+1)} \circ \cdots \circ \tilde{g}_{s(2^{k-1}-1)} \]
and let
\[ L_{a_1} = \tilde{g}_{s(2^{k-2})}. \]
The map \( G_1 \) is a univalent map of \( C[0,1], V \) and \( G_1[-1,1] \) is a homeomorphism of \([0,1]\). The map \( L_{a_1} \) is a square root at \( a_1 \). The preimage of \( c(1) \) under \( G_1' \) is \( c(2^{k-2}) \). One of the end-points of \( T_k(2^{k-1}) \) is 0; the other one is between \( J_{k-1}(0) \) and one of \( L.J_{k-1}(0) \) or \( R.J_{k-1}(0) \). From Theorem 19, two components of \( T_{k-2}(2^{k-2}) \backslash T_k(2^{k-1}) \) have lengths greater than a constant \( C \) (obtained from Theorem 19) times the length of \( T_k(2^{k-1}) \). Thus \((L_{a_1}, G_1)\) is a compatible pair and satisfies (i) and (iii) of Definition 4 (see Fig. 7).

Next we consider the homeomorphism \( f^0(2^{3-k} - 1) : J_k(2^{k-3} + 1) \to J_k(2^{k-2}) \). Its inverse \( g_{s(2^{k-3}+1)} \circ \cdots \circ g_{s(2^{k-2}-1)} : J_k(2^{k-2}) \to J_k(2^{k-3} + 1) \) can be extended to \( T_{k-3}(2^{k-2}) \); that is, we can consider
\[ G'_2 = g_{s(2^{k-3}+1)} \circ \cdots \circ g_{s(2^{k-2}-1)} : T_{k-3}(2^{k-2}) \to T_{k-3}(1). \]
Let
\[ G_2 = \tilde{g}_{s(2^{k-3}+1)} \circ \cdots \circ \tilde{g}_{s(2^{k-2}-1)} \]
and let
\[ L_{a_2} = \tilde{g}_{s(2^{k-3})}. \]
The map \( G_2 \) is a univalent map of \( C[0,1], V \) and \( G_2[-1,1] \) is a homeomorphism of \([0,1]\). The map \( L_{a_2} \) is a square root at \( a_2 \). The preimage of \( c(1) \) under \( G_2' \) is \( c(2^{k-3}) \). The interval \( T_k(2^{k-2}) \) is contained in \( T_{k-1}(2^{k-1}) \). From Theorem 19, two components of \( T_{k-3}(2^{k-3}) \backslash T_k(2^{k-2}) \) have lengths greater than a constant \( C \) (obtained from Theorem 19) times the length of \( T_k(2^{k-2}) \). Thus \((L_{a_2}, G_2)\) is a compatible pair and satisfies (iii) of Definition 4 (see Fig. 7).

In general, for \( 3 < i \leq k - 1 \), consider the homeomorphism \( f^0(2^{k-i-1}) : J_k(2^{k-i} + 1) \to J_k(2^{k-i+1}) \). Its inverse \( g_{s(2^{k-i}+1)} \circ \cdots \circ g_{s(2^{k-i+1}-1)} : J_k(2^{k-i+1}) \to J_k(2^{k-i} + 1) \) can be extended to \( T_{k-i}(2^{k-i}) \); that is, we can consider
\[ G'_{i+1} = g_{s(2^{k-i}+1)} \circ \cdots \circ g_{s(2^{k-i+1}-1)} : T_{k-i}(2^{k-i}) \to T_{k-i}(1). \]
Let
\[ G_{i+1} = \tilde{g}_{s(2^{k-i}+1)} \circ \cdots \circ \tilde{g}_{s(2^{k-i+1}-1)} \]
and let
\[ L_{a_{i+1}} = \tilde{g}_{s(2^{k-i})}. \]
The map \( G_{i+1} \) is a univalent map of \( C[0,1], V \) and \( G_{i+1}[0,1] \) is a homeomorphism. The map \( L_{a_{i+1}} \) is a square root at \( a_{i+1} \). The preimage of \( c(1) \) under \( G_{i+1}' \) is \( c(2^{k-i}) \). The interval \( T_k(2^{k-i+1}) \) is contained in \( T_{k-i+2}(2^{k-i+2}) \). From Theorem 19, two components \( T_{k-i}(2^{k-i}) \backslash T_k(2^{k-i+1}) \) have lengths greater than a constant \( C \) times the length of \( T_k(2^{k-i+1}) \). Thus \((L_{a_{i+1}}, G_{i+1})\) is a compatible pair and satisfies (iii) of Definition 4.
One of the endpoints of $T_k(2^{k-i}+1)$ is to the left of $c(2^{k-i}+1)$; the other is in the interval in $\zeta_k$ which is adjacent to $J_k(2^{k-i}+1)$. The branch point of $g_{s(2^{k-i})}$ is always $c(1)$. From Theorem 19, we have a constant $\lambda > 1$ such that $\{a_i\}_{i=1}^k$ satisfies (ii) of Definition 4 for $C = 1$. 

Fig. 7
For all $k \geq 3$, we therefore decompose
\[ L_k = L_{a_k} \circ G_k \circ L_{a_{k-1}} \circ G_{k-1} \circ \cdots \circ L_{a_1} \circ G_1 \circ L_{a_0} \circ G_0. \]
From the construction above, $L_k$, $3 \leq k < \infty$, is uniform root-like map.

Let $(U, V, f)$ be a Feigenbaum-like quadratic-like map. Let $m_k = 2^k$. Consider the renormalizations
\[ f^{\circ m_k} : U_k \to V_k \]
for $1 \leq k < \infty$ where $V_k = (V \setminus \mathbb{R}^1) \cup \mathbb{T}k(m_k)$. Let $U_k \cap \mathbb{R}^1 = [-o_k, o_k]$. Let $w_{k,\theta}$ be the the ray starting at $o_k$ with slope $\tan \theta$ for $0 < \theta < \pi/2$. Let $0 \in R_{k,\theta}$ be the domain bounded by $w_{k,\theta}, -w_{k,\theta}, \overline{w}_{k,\theta}$, and $-\overline{w}_{k,\theta}$.

**Lemma 14.** Suppose $(U, V, f)$ is a Feigenbaum-like quadratic-like map. Then there is a constant angle $\theta_0 > 0$ such that
\[ U_k \subseteq R_{k,\theta_0} \]
for all $k \geq 0$ (see Fig. 8).

**Proof.** Consider $f^{\circ (m_k-1)} : J_k(1) \to J_k(0)$. Its inverse has the maximum extension
\[ \mathcal{A}_k = g_{s(1)} \circ g_{s(2)} \circ \cdots \circ g_{s(m_k-1)} : V_k = (V \setminus \mathbb{R}^1) \cup \mathbb{T}k(m_k) \to U_k'. \]
Remember that
\[ L_k = l_1 \circ \mathcal{A}_k \circ l_{m_k}^{-1} \]
where $l_{m_k}$ and $l_1$ are the linear maps normalizing $T_{m_k}$ and $T_k(1)$, respectively, to $[0, 1]$. Let
\[ w'_{k,\theta} = \{ z \in \mathbb{C} \mid \arg(z - d_1) = \theta, \Im(z) > 0 \} \]
be the ray starting at $d_1$ with angle $0 \leq \theta \leq \pi$ where $U'_k \cap \mathbb{R}^1 = T_k(1) = [d_1, e_1]$ with $d_1 < c(1) < e_1$. Let $R'_{k, \theta}$ be the sector containing $c(1)$ bounded by $w'_k$ and $-w'_k$. Applying Theorem 18, there is a constant angle $0 < \theta_1 \leq \pi$ such that $U'_k$ is contained in a sector domain $R'_{k, \theta_1}$ (see Fig. 8). Since $f : U \rightarrow V$ is a quadratic-like map, it is comparable with $z \mapsto z^2$ near 0. So there is a constant angle $0 < \theta_0 < \pi/2$ depending on $\theta_1$ such that $U_k = f^{-1}(U'_k)$ is contained in $R_{k, \theta_0}$ (see Fig. 8).

Take $I_0 = (-1, 1)$. Let $C_{I_0} = (C \setminus \mathbb{R}^1) \cup I_0$. Let $d = d_{H_C, I_0}$ be the hyperbolic distance on $C_{I_0}$. Let $\Omega_r = \{z \in C_{I_0} \mid d(z, I_0) < r\}$ be a hyperbolic neighborhood. From Lemma 6, $\Omega_r$ is the union of two disks $D^+_\beta$ and $D^-\beta$ centered at $c^+_\beta = i \cot \beta$ and $c^-\beta = -i \cot \beta$ with radii $R^+_\beta = R^\beta = 1/\sin \beta$ where $\beta$ is the angle between $\partial D^+_\beta$ and the line $[1, \infty)$ at 1 (see Fig. 9). Using the law of cosines for the triangle $\Delta(c_0z)$, for any $z = re^{i\phi}$ in $\partial D^+_\beta$,

$$r = \cot \beta \sin \phi + \sqrt{\csc^2 \beta - \cot^2 \beta \cos^2 \phi}.$$

Let $q(z) = \sqrt{z}$ be the square root from $C \setminus \{x < 0\}$ to the right half-plane $\mathbb{R}^H$. Let $\Pi^+_\beta = q(D^+_\beta)$. For $0 < \beta < \pi/4$, consider $\Omega_{r'} = D^+_{2\beta} \cup D^-_{2\beta}$. Let $z_0 = re^{i\theta} \neq 1$ be the intersection point of $\partial D^+_\beta$ and $\Pi^+_\beta$. Then $\tau$ is the unique non-zero solution of the equation

$$\sqrt{\cot \beta \sin(2\phi) + \sqrt{\csc^2 \beta - \cot^2 \beta \cos^2(2\phi)}} = \cot(2\beta) \sin \phi + \sqrt{\csc^2(2\beta) - \cot^2(2\beta) \cos^2 \phi}.$$

Thus $\tau = \tau(\beta)$ is a strictly increasing function and $\tau \rightarrow 0$ as $\beta \rightarrow 0$. Let $z_0 = 1 = re^{i\theta}$. Then $\theta = \theta(\tau)$ is a strictly increasing function and $\theta \rightarrow 0$ as $\tau \rightarrow 0$. Let $\theta = \theta \circ \tau(\beta)$. It is a strictly decreasing function and $\theta \rightarrow 0$ as $\beta \rightarrow 0$. Let $\beta = \beta(\theta)$ be its inverse function.
For any $0 < \theta_0 < \pi/2$, let $0 < \beta_0 = \beta(\theta_0) < \pi/2$ and let

$$Q_{\theta_0} = \{z \in \Pi_{\beta_0} \mid \theta_0 \leq \arg(z - 1) \leq \frac{\pi}{2}\}.$$ 

Then $Q_{\theta_0} \subseteq D^{+}_{2\beta_0}$. Let $\overline{Q}_{\theta_0} = \{z \mid z \in Q_{\theta_0}\}$ and let $S_{\theta_0} = Q_{\theta_0} \cup (-Q_{\theta_0}) \cup \overline{Q}_{\theta_0} \cup (-\overline{Q}_{\theta_0})$. For a number $0 < \nu_0 < 1$, let

$$\nu_0 \cdot S_{\theta_0} = \{w = \nu_0 \cdot z \mid z \in S_{\theta_0}\}.$$

Let $A_{\theta_0} = \Omega_r \setminus (\nu_0 S_{\theta_0})$. From the above calculation, we have

**Lemma 15.** There is a constant $C = C(\theta_0, \nu_0) > 0$ depending on $\theta_0$ and on $\nu_0$ such that the modulus $\text{mod}(A_{\theta_0})$ of the annulus $A_{\theta_0}$ is greater than $C$.

**Proof.** Let $a = \text{diam}(\nu_0 S_{\theta_0})$ be the diameter of $\nu_0 S_{\theta_0}$ and let $b = d(\partial(\nu_0 S_{\theta_0}), \partial \Omega_r)$ be the distance between $\partial(\nu_0 S_{\theta_0})$ and $\partial \Omega_r$. Then $a/b$ is bounded from above by a constant depending only on $\theta_0$ and on $\nu_0$. This implies the lemma (by using Grötzsch argument (refer to [AH1])).

Suppose $0 < C_0 < 1$ is a constant. If we use $q_a(z) = \sqrt{z - a}/\sqrt{1 - a}$ for $|a| < C_0$ to replace $q(z)$ in the above calculation, then Lemma 15 has a generalized version.

**Lemma 15’.** There is a constant $C = C(\theta_0, \nu_0, C_0) > 0$ depending on $\theta_0$, $\nu_0$, and $C_0$ such that the modulus $\text{mod}(A_{\theta_0})$ of the annulus $A_{\theta_0}$ is greater than $C$. 
**Theorem 20 [SU2].** A Feigenbaum-like quadratic-like map \((U, V, f)\) has the a priori complex bounds and is unbranched.

**Proof.** We use the same notation as in the previous lemmas. Suppose \((U, V, f)\) is a Feigenbaum-like quadratic-like map. Let \(m_k = 2^k\). Consider the renormalizations

\[ f^{m_k} : U_k \to V_k \]

for \(1 \leq k < \infty \) where \(V_k = (V \setminus R^1) \cup T_k(m_k) \) and where \(U_k \cap R^1 = [-o_k, o_k] \). From Lemma 14, there is a constant angle \(0 < \theta_0 < \pi/2 \) such that

\[ U_k \subseteq R_k \theta_0 \]

for \(k > 0\). Let \(\beta_0 = \beta(\theta_0)\).

Consider \(f^{o(m_k^{-1})} : J_k(1) \to J_k(0)\). Its inverse has the maximum extension

\[ \mathcal{A}_k = g_{s(1)} \circ g_{s(2)} \circ \cdots \circ g_{s(m_k - 1)} : V_k \to U'_k \]

where \(U'_k \cap R^1 = T_k(1) = [d_1, e_1] \) with \(d_1 < c(1) < e_1\). From Theorem 19 (and the bounded geometry property of the attractive Cantor set), there is a constant \(C_0 > 0\) such that \(C_0^{-1} \leq |c(1) - d_1|/|e_1 - c(1)| < C_0\) for all \(k > 0\). We normalize \(\hat{T}_k(m_k) = (d_{m_k}, e_{m_k}) \) to \((-1, 1)\) by the linear map \(s_1\) such that \(s_1(d_{m_k}) = 1\) and \(s_1(e_{m_k}) = -1\). We normalize \((0, o_k)\) to \((0, 1)\) by the linear map \(s_2\) such that \(s_2(0) = 0\) and \(s_2(o_k) = 1\). We normalize \(\hat{T}_k(1) = (d_1, e_1)\) to \((-1, 1)\) by the linear map \(s_3\) such that \(s_3(e_1) = -1\) and \(s_3(d_1) = 1\). Let \(a = s_3(c(1))\). Then there is a constant we still denote it as \(0 < C_0 < 1\) such that \(|a| < C_0\) for all \(k > 0\). There is an integer \(n_0 > 0\) such that for any \(k > n_0\), \(\hat{\Omega}_r = D_{\beta_0}^+ \cup D_{\beta_0}^-\) is contained in \(s_1(V)\). Let \(B_k = s_3 \circ \mathcal{A}_k \circ s_3^{-1}\) and let \(q_0(z) = s_2 \circ g_{s_{m_k}} \circ s_3^{-1}\). Then \(q_0\) is comparable with \(\sqrt{z - a}/\sqrt{1 - a}\). Since \(B_k\) contracts the hyperbolic distance \(d_{H, C_{\iota_0}}\), then \(B_k(\hat{\Omega}_r) \subseteq \hat{\Omega}_r\). From Lemma 14,

\[ X'_k = q_0(B_k(\hat{\Omega}_r)) \subseteq S_{\theta_0} = Q_{\theta_0} \cup (-Q_{\theta_0}) \cup \overline{Q_{\theta_0}} \cup (-\overline{Q_{\theta_0}}) \]

Let \(X''_k = s_1 \circ s_3^{-1}(X'_k)\). Then \(X''_k \subseteq \nu_0 S_{\theta_0}\) for all \(k > 0\) where \(\nu_0 > 0\) is a constant obtained from Lemma 4.2. Let \(Y''_k = \hat{\Omega}_r\). Then from Lemma 15',

\[ \text{mod}(Y''_k \setminus X''_k) > C \]

for all \(k > n_0\) where \(C > 0\) is a constant.

Now let \(X_k = s_1^{-1}(X''_k)\) and let \(Y_k = s_1^{-1}(Y''_k)\). Then

\[ f^{m_k} : X_k \to Y_k \]

is quadratic-like map and \(\text{mod}(Y_k \setminus X_k) > C\) for all \(k > n_0\). This means that \((U, V, f)\) has the a priori complex bounds.

Let \(W_k = Y_k \setminus (LJ_k(0) \cup RJ_k(0))\). Applying Theorem 19 and the above argument, there is a constant \(C' > 0\) such that \(\text{mod}(W_k \setminus K_k) > C'\) for all \(k > n_0\) where \(K_k\) is the filled-in Julia set of \(f^{m_k} : X_k \to Y_k\). But \(W_k \cap CO = \{c(jm_k)\}_{j=0}^\infty\). So \((U, V, f)\) is unbranched. This completes the proof. \(\blacksquare\)
Theorem 20 and Theorem 16 now give us that

**Corollary 1.** The filled-in Julia set $K_f$ of a Feigenbaum-like quadratic-like map $(U, V, f)$ is locally connected.

Sullivan [SU2] (see also [MEV]) also proved that any real infinitely renormalizable quadratic-like map $f : U \to V$ of bounded type has the a priori complex bounds. Thus it is unbranched and its filled-in Julia set is locally connected from Theorem 16.

8. **The Local Connectivity of Certain Infinitely Renormalizable Quadratic Julia Sets**

We prove in this section the following result.

**Theorem 21.** There is a subset $\tilde{\Upsilon}$ in $M$ such that

1. $\tilde{\Upsilon}$ is dense in the boundary $\partial M$ of the Mandelbrot set $M$,
2. for every $c$ in $\tilde{\Upsilon}$, $P_c$ is unbranched, infinitely renormalizable and has the a priori complex bounds.

From Theorem 16, we have

**Corollary 2.** The filled-in Julia set $K_c$ of $P_c$ is locally connected for every $c$ in $\tilde{\Upsilon}$.

**Proof of Theorem 21.** Suppose $c_0$ is a Misiurewicz point in $M$. Then there is an integer $m > 1$ such that $p = P_{\circ c_0}^m(0)$ is a repelling periodic point of period $k \geq 1$. Let $\alpha$ be the separate fixed point of $P_{c_0}$. Without loss of generality, we assume that $P_{c_0}$ is non-renormalizable. (If $P_{c_0}$ is renormalizable, it must be finitely renormalizable. We would then take $\alpha$ as the separate fixed point of the last renormalization of $P_{c_0}$ (see §4)). Let $\Gamma$ be the union of a cycle of external rays landing at $\alpha$. Let $\gamma$ be a fixed equipotential curve of $P_{c_0}$. Using $\Gamma$ and $\gamma$, we construct the two-dimensional Yoccoz puzzle as follows (see §3). Let $C_{-1}$ be the domain bounded by $\gamma$. The set $\Gamma$ cuts $C_{-1}$ into a finite number of closed domains. Let $\eta_0$ denote the set of these domains. Let $\eta_n = P_{\circ c_0}^{-n}(\eta_0)$. Let $C_n$ be the member of $\eta_n$ containing 0.

Let

\[ p \in \cdots \subseteq D_n(p) \subseteq D_{n-1}(p) \subseteq \cdots \subseteq D_1(p) \subseteq D_0(p) \]

be a $p$-end, where $D_n(p) \in \eta_n$. Let

\[ c_0 \in \cdots \subseteq E_n(c_0) \subseteq E_{n-1}(c_0) \subseteq \cdots \subseteq E_1(c_0) \subseteq E_0(c_0) \]

be a $c_0$-end, where $E_n(c_0) \in \eta_n$. We have $P_{\circ c_0}^{m-1}(E_{n+m-1}(c_0)) = D_n(p)$.

Since the diameter $\text{diam}(D_n(p))$ tends to zero as $n \to \infty$ and since $p$ is a repelling periodic point, we can find an integer $l \geq m$ such that $|(P_{\circ c_0}^k)'(x)| \geq \lambda > 1$ for all $x \in D_l(p)$ and such that

\[ P_{\circ c_0}^{m-1} : E_{l+m-1}(c_0) \to D_l(p) \]
is a homeomorphism. Let \( q > 0 \) be the smallest integer such that \( P_{c_0}^{oq}(D_l(p)) \) contains 0, i.e., it is \( C_r \) in \( \eta_r \) where \( r_0 \geq 0 \). Then
\[
f = P_{c_0}^{oq} : D_l(p) \to C_{r_0}
\]
is a homeomorphism. Let \( r > r_0 \) be an integer such that \( B_0 = f^{-1}(C_r) \subseteq D_l(p) \) does not contain \( p \) where \( C_r \) is the member of \( \eta_r \) containing 0. Then
\[
P_{c_0}^{oq} : B_0 \to C_r
\]
is a homeomorphism. The domain \( B_0 \) is a member of \( \eta_{r+q} \). Let \( B_n \subseteq D_{l+nk} \) be the pre-image of \( B_0 \) under \( P_{c_0}^{onk}|D_{l+nk}(p) \) for \( n \geq 1 \). The domain \( B_n \) is a member of \( \eta_{r+q+nk} \) and
\[
P_{c_0}^{o(q+nk)} : B_n \to C_r
\]
is a homeomorphism.

From the structural stability theorem (see [PRZ,SHU]), the points \( \alpha \) and \( p \) and the sets \( \Gamma, \ c, \ D_r \) and \( B_n \) for \( n \geq 1 \) are all preserved by a small perturbation \( c \) of \( c_0 \) (refer to [JI4]). Therefore they can be constructed for \( P_c \) as long as \( c \) near \( c_0 \). Let \( U_0 \) be a neighborhood about \( c_0 \) such that the corresponding points \( \alpha(c) \) and \( p(c) \) and the corresponding sets \( \Gamma(c), \ C_r(c), \ D_r(c) \), and \( B_n(c) \) for \( n \geq 1 \) are all preserved for \( c \in U_0 \). Moreover, as \( n \) goes to infinity, the diameter \( \text{diam}(B_n(c)) \) tends to zero and the set \( B_n(c) \) approaches to \( p(c) \) uniformly on \( U_0 \). Let
\[
W_n = W_n(c_0) = \{ c \in C \mid P_{c_0}^m(0) \in B_n(c) \}.
\]
Then from the result in [JI4], \( W_n \subseteq U_0 \) for \( n \) large enough.

For any \( c \in W_n \), Let \( R_n(c) \) be the preimage of \( B_n(c) \) under the map \( P_{c_0}^{o(m-1)} : E_{l+m-1}(c) \to D_l(p,c) \) and let \( C_{r+q+nk+m}(c) = P_{c_0}^{-1}(R_n(c)). \) The domain \( C_{r+q+nk+m}(c) \) is the member containing 0 in \( \eta_{r+q+nk+m} \) and
\[
F_{n,c} = P_{c_0}^{o(q+nk+m)} : X_n(c) = \hat{C}_{r+q+nk+m}(c) \to Y_n(c) = \hat{C}_r(c)
\]
is a quadratic-like map. Let \( A_n(c) = \hat{C}_r(c) \setminus C_{r+q+nk+m}(c) \). Since the diameter \( \text{diam}(C_{r+q+nk+m}(c)) \) tends to zero as \( n \) goes to infinity uniformly in \( U_0 \), there is an integer \( N_0 > 0 \) such that
\[
\text{mod}(A_n(c)) \geq 1
\]
for all \( n \geq N_0 \) and \( c \in W_n \). Since
\[
\{ F_{n,c} : X_n \to Y_n \mid c \in W_n \}
\]
is a full family of quadratic-like maps, \( W_n \) contains a copy \( \mathcal{M}_n = \mathcal{M}_n(c_0) \) of the Mandelbrot set \( \mathcal{M} \) (see [DH3]). For any \( c \in \mathcal{M}_n, \ P_c \) is once renormalizable and
\[
\text{CO}(c) \cap C_{r+q+nk+m}(c) = \{ c(j(q + nk + m)) \}_{j=0}^{\infty}
\]
where \( \text{CO}(c) = \{ c(i) = P_{c_0}^{o(i)}(0) \}_{i=0}^{\infty}. \) Let
\[
\tilde{\mathcal{Y}}_1(c_0) = \bigcup_{n=N_0}^{\infty} \mathcal{M}_n.
\]
We use the induction to complete the construction of the subset \( \tilde{\Upsilon}(c_0) \) around \( c_0 \). Suppose we have constructed \( W_w \) where \( w = i_0i_1 \ldots i_{k-1} \) and \( i_0 \geq N_0, i_1 \geq N_{i_1}, \ldots, i_{k-1} \geq N_{i_0i_1 \ldots i_{k-2}} \). There is a parameter \( c \in M_w \) such that

\[
F_w = F_{w,c_w} : X_w = X_w(c_w) \to Y_w = Y_w(c_w)
\]

is hybrid equivalent (see §1) to \( P(z) = z^2 - 2 \). For \( F_w \), let \( \beta_w \) and \( \alpha_w \) be its non-separate and separate fixed points. Let \( \tilde{\beta}_w \) be another preimage of \( \beta_w \) under \( F_w \). Let \( \Gamma_w \) be the external rays of \( P_c \) landing at \( \alpha_w \). Let \( \tilde{Y}_{w0} \) be the domain containing 0 and bounded by \( \partial X_w \) and \( F_w^{-1}(\Gamma_w) \). Let \( \tilde{\beta}_w \in E_{w0} \) and \( \beta_w \in E_{w1} \) be the components of the closure of \( X_w \setminus Y_{w0} \). Let \( G_{w0} \) and \( G_{w1} \) be the inverses of \( F_w|E_{w0} \) and \( F_w|E_{w1} \). Let

\[
D_{wn} = G_{w1}^n(D_{w0})
\]

and

\[
B_{wn} = G_{w0}(D_{w(n-1)})
\]

for \( n \geq 1 \). From the structural stability theorem (see [PRZ,SHU]), the points \( \beta_w \) and \( \alpha_w \) and the sets \( \Gamma_w \) are all preserved by a small perturbation \( c \) of \( c_w \). Therefore we can find a small neighborhood \( U_w \) about \( c_w \) such that the corresponding domains \( D_{wn}(c) \) and \( B_{wn}(c) \) can be constructed for \( P_c, c \in U_w \) (refer to [J14]). Let

\[
W_{wn} = \{ c \in C \mid F_{w,c}(0) \in B_{wn}(c) \}.
\]

The diameter \( \text{diam}(B_{wn}(c)) \to 0 \) as \( n \to \infty \) uniformly on \( U_w \). From the result in [J14], \( W_{wn} \subseteq U_w \) for \( n \) large.

For each \( c \) in \( W_{wn} \), \( n \geq N_w \), let \( X_{wn}(c) = F_{w,c}^{-1}(B_{wn}(c)) \) and \( Y_{wn}(c) = \tilde{Y}_{w0}(c) \). Then

\[
F_{wn,c} = F_{w,c}^{n+1} : X_{wn}(c) \to Y_{wn}(c)
\]

is a quadratic-like map. Let

\[
A_{wn}(c) = X_{wn}(c) \setminus Y_{wn}(c).
\]

Since the diameter \( \text{diam}(Y_{wn}(c)) \) tends to zero as \( n \) goes to infinity uniformly in \( U_w \). There is an integer \( N_w > 0 \) such that

\[
\text{mod}(A_{wn}(c)) \geq 1
\]

for all \( n \geq N_w \) and \( c \in W_{wn} \). Since

\[
\{ F_{wn,c} : X_{wn} \to Y_{wn} \mid c \in W_{wn} \}
\]

is a full family of quadratic-like maps, \( W_{wn} \) contains a copy \( M_{wn} = M_{wn}(c_0) \) of the Mandelbrot set \( M \) (see [DH3]). For any \( c \in M_{wn} \), \( P_c \) is \( k \)-times renormalizable and \( A_{wn}(c) \) contains no critical values of \( P_c \). Let

\[
\tilde{\Upsilon}_k(c_0) = \bigcup_w \bigcup_{n=N_w}^{\infty} M_{wn}
\]

where \( w \) runs over all sequences of integers of length \( k \) in the induction.
We have thus constructed a subset \( \bar{\Upsilon}(c_0) = \bigcap_{k=1}^\infty \bar{\Upsilon}_k(c_0) \) such that every \( c \in \bar{\Upsilon}(c_0) \) is infinitely renormalizable and such that \( c_0 \) is a limit point of \( \bar{\Upsilon}(c_0) \). From the above construction, for every \( c \in \bar{\Upsilon}(c_0) \), \( P_c \) is unbranched and has the a priori complex bounds.

Let \( \bar{\Upsilon} = \bigcup_{c_0} \bar{\Upsilon}(c_0) \) where \( c_0 \) runs over all Misiurewicz points in \( M \). Then for every \( c \in \bar{\Upsilon} \), \( P_c \) is unbranched infinitely renormalizable and has the a priori complex bounds. Since the set of Misiurewicz points is dense in \( \partial M \) (see [CAG]), the set \( \bar{\Upsilon} \) is dense in \( \partial M \). It completes the proof of the theorem.

**Remark 3.** Douady (see [MI2]) constructed an example of an infinitely renormalizable quadratic polynomial whose filled-in Julia set is not locally connected.

**References**

[AH1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*. D. Van Nostrand-Reinhold Company, Inc., Princeton, New Jersey, 1966.

[AH2] L. V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*. McGraw-Hill, New York, 1973.

[BIE] L. Bieberbach, *Conformal Mapping*. Chelsea Publishing Company, New York, 1953.

[BLA] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*. Bull. of the American Mathematical Society 11, 1984, pp. 85-141.

[BRH] B. Branner and J. Hubbard, *The iteration of cubic polynomials, Part I: The global topology of parameter space & Part II: Patterns and parapatterns*. Acta Math 160, 1988, pp. 143-206 & Acta Math, to appear.

[CAR] C. Carathéodory, "Über die Begrenzung einfach zusammenhängender Gebiete." Math. Ann. 73, 1913, pp. 323-370. (Gesam. Math. Schr., v. 4).

[CAG] L. Carleson and T. Gamelin, *Complex Dynamics*. Springer-Verlag, Berlin, Heidelberg, 1993.

[DH1] A. Douady and J. H. Hubbard, *Itération des polynômes quadratiques complexes*. C.R. Acad. Sci. Paris, 294, 1982, pp. 123-126.

[DH2] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes I & II*. Publ. Math. d’Orsay, 1984 & 1985.

[DH3] A. Douady and J. H. Hubbard, *On the dynamics of polynomial-like maps*. Ann. Sci. Éc. Norm. Sup., 18, 1985, pp. 287-344.

[HUB] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J. -C. Yoccoz*. Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor’s Sixtieth Birthday, 1993.

[J1] Y. Jiang, *Infinitely renormalizable quadratic Julia sets*. Preprint of FIM, 1994, ETH, Zürich.

[J2] Y. Jiang, *On Sullivan’s sector theorem*. Preprint.

[J3] Y. Jiang, *Renormalization on one-dimensional folding maps*. Proceedings of the International Conference on Dynamical Systems and Chaos, Volum 1, World Scientific Publishing Co. Pte. Ltd., 1995, pp. 116-125.

[J4] Y. Jiang, *Local connectivity of the Mandelbrot set at certain infinitely renormalizable points*. MSRI Preprint # 63/1995.

[MC1] C. McMullen, *Complex Dynamics and Renormalization*. Ann. of Math. Stud., vol 135, Princeton Univ. Press, Princeton, NJ, 1994.

[MC2] C. McMullen, *Renormalization and 3-manifolds which fiber over the circle*. Preprint.
[MEV] W. de Melo and S. van Strien, One-Dimensional Dynamics. Springer-Verlag, Berlin, Heidelberg, 1993.

[MI1] J. Milnor, Dynamics in one complex variable: Introductory lectures. IMS preprint 1990/5, Stony Brook.

[MI2] J. Milnor, Local connectivity of Julia sets: expository lectures. IMS preprint 1992/11, Stony Brook.

[PRZ] F. Przytycki, On $U$-stability and structural stability of endomorphisms satisfying Axiom A. Studia-Math., 60, no. 1, 1977, pp. 61–77.

[RUE] D. Ruelle, Repellers for real analytic maps. Ergod. Th. & Dynam. Sys., 2, 1982, pp. 99-107.

[SHU] M. Shub, Endomorphisms of compact differentiable manifolds. Amer. J. Math., 91, 1969.

[SU1] D. Sullivan, Seminar on conformal and hyperbolic geometry. IHES Preprint, March/1982.

[SU2] D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures. American Mathematical Society Centennial Publications, Volume 2: Mathematics into the Twenty-First Century, AMS, Providence, RI, 1991.

[YOC] J. C. Yoccoz, unpublished work.

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