Infinitely many solutions to a fractional nonlinear Schrödinger equation

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Abstract
This paper considers the fractional Schrödinger equation

\[ (-\Delta)^s u + V(|x|)u - u^p = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N) \]  

where \(0 < s < 1\), \(1 < p < \frac{N+2}{N-2}\), \(V(|x|)\) is a positive potential and \(N \geq 2\). We show that if \(V(|x|)\) has the following expansion:

\[ V(|x|) = V_0 + a \frac{1}{|x|^m} + o \left( \frac{1}{|x|^m} \right) \text{ as } |x| \to +\infty, \]

in which the constants are properly assumed, then (0.1) admits infinitely many non-radial solutions, whose energy can be made arbitrarily large. This is the first result for fractional Schrödinger equation. The \(s = 1\) case corresponds to the known result in Wei-Yan [28].

Key Words. Fractional Laplacian, fractional Schrödinger equation, Lyapunov-Schmidt

1 Introduction and main results

The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. The nonlinear fractional nonlinear Schrödinger equation is as follows:

\[ i\psi_t = (-\Delta)^s \psi + \tilde{V}(x)\psi - |\psi|^{p-1}\psi \]  

where \((-\Delta)^s\) \((0 < s < 1)\) denotes the classical fractional Laplacian, \(\tilde{V}\) is a bounded potential and \(p > 1\).

We are interested in finding standing wave solutions, which are solutions of the form \(\psi(x, t) = u(x)e^{i\lambda t}\) with the function \(u\) real-valued. Let \(V(x) = \tilde{V}(x) + \lambda\), then \(\psi\) is a solution of (1.3) if and only if \(u\) solves the following equation

\[ (-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \]  

A similar problem to (1.3) is the following fractional scalar field equation

\[ (-\Delta)^s u + u = Q(x)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N. \]  

It is also absorbing to study the singularly perturbed problem

\[ \varepsilon^{2s}(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N \]  

or

\[ \varepsilon^{2s}(-\Delta)^s u + u = Q(x)u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \]

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where $\varepsilon > 0$ is a small parameter. The natural place to look for solutions that decay at infinity is the space $H^{2s}(\mathbb{R}^N)$ of all functions $u \in L^2(\mathbb{R}^N)$ such that

$$
\int_{\mathbb{R}^N} (1 + |\xi|^{4s})|\hat{u}(\xi)|^2 d\xi < \infty,
$$

where $\hat{\cdot}$ denotes the Fourier transform. The fractional Laplacian $(-\Delta)^s u$ for $u \in H^{2s}(\mathbb{R}^N)$ is defined by

$$
(-\Delta)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi).
$$

For $\frac{\varepsilon}{s} = \frac{\varepsilon}{s}$, an interesting problem is to find solutions with a spike pattern concentrating around some points. As for the standard case $s = 1$ of $\frac{\varepsilon}{s} = \frac{\varepsilon}{s}$ or $\frac{\varepsilon}{s}$, this has been the topic of many works relating the concentration points with critical points of the potential, starting in 1986 from the pioneering work Floer-Weinstein $\frac{\varepsilon}{s}$. Later many works show that the number of the critical points of $V(x)$ (or $Q(x)$) (see for example $\frac{\varepsilon}{s}, \frac{\varepsilon}{s}$), the type of the critical points of $V(x)$ (or $Q(x)$) (see for example $\frac{\varepsilon}{s}, \frac{\varepsilon}{s}$), can effect the number of solutions of $\frac{\varepsilon}{s}$ (or $\frac{\varepsilon}{s}$). It is now known that when the parameter $\varepsilon$ goes to zero, the number of the solutions may tend to infinity. For the $s = 1$ case of $\frac{\varepsilon}{s}$ or $\frac{\varepsilon}{s}$, in 2010 Wei-Yan $\frac{\varepsilon}{s}$ get a multiplicity result under some symmetry assumption of $V(x)$ near the infinity. Recently we are told that del Pino-Wei-Yao $\frac{\varepsilon}{s}$ get the similar result with a weaker symmetry assumption on $V(x)$.

As to the fractional case $0 < s < 1$, very few is known. Recently Dávila-del Pino-Wei $\frac{\varepsilon}{s}$ obtained the first result of spike pattern for the fractional Schrödinger equation $\frac{\varepsilon}{s}$ with $1 < p < \frac{N}{N+2s}$. A natural question is can we get multiplicity result for $\frac{\varepsilon}{s}$ (or $\frac{\varepsilon}{s}$) with $0 < s < 1$? What is the situation in the fractional case? In this paper we will give an affirmative answer!

This paper is concerned about the following fractional Laplacian problem

$$
(-\Delta)^s u + V(|x|)u - |u|^{p-1}u = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N) \tag{1.6}
$$

where $0 < s < 1, 1 < p < \frac{N+2s}{N}$ and $N \geq 2$. We suppose that $V(x)$ satisfies the following assumption.

**Assumption $\mathcal{V}$**. $V$ is positive and radially symmetric, i.e. $V(x) = V(|x|) > 0$ and there are constants $a > 0$ and $V_0 > 0$ such that

$$
V(|x|) = V_0 + \frac{a}{|x|^m} + o\left(\frac{1}{|x|^m}\right), \quad \text{as } |x| \to +\infty, \tag{1.7}
$$

where

$$
\max\left\{0, (N+2s)\left[1 - (p-1)N - 2ps + \max\left\{s, p - \frac{N}{2}\right\}\right]\right\} < m < N + 2s. \tag{1.8}
$$

Without loss of generality, we may assume $V_0 = 1$ for the sake of simplicity.

It’s easy to see that

$$
1 - (p-1)N - 2ps + \max\left\{s, p - \frac{N}{2}\right\} < 1 \quad \text{for any } p > 1, \ s \in (0,1).
$$

By direct computations we find that in three dimension case, if

$$
1 + \frac{1 - s}{3 + 2s} < p < \frac{3 + 2s}{3 - 2s}, \quad \frac{1}{6} < s \leq \frac{1}{2},
$$

then we just need that $m \in (0, N + 2s)$.

The aim of this paper is to obtain infinitely many non-radial positive solutions to (1.6), whose energy may be arbitrarily large. Our main result in this paper is stated in the following theorem.

**Theorem 1.1.** If $V(|x|)$ satisfies the assumption $\mathcal{V}$, then the problem (1.6) admits infinitely many non-radial positive solutions. Moreover, the energy of these solutions may be arbitrarily large.

**Remark 1.1.** The condition on potential $V(x)$ is more general than that of $V$ in $\frac{\varepsilon}{s}$ for $s = 1$. The main reason is that in our case we can deduce the exact relationship between the radius and the number of spikes, while in $\frac{\varepsilon}{s}$, the authors can’t solve it exactly using the leading terms of energy.

We believe that the symmetry on $V$ is technical and then make the following conjecture.
Conjecture 1.1. Problem (1.6) has infinitely many solutions if there are constants $a > 0, m \in (0, N + 2s)$ and $V_0 > 0$, such that

$$V(x) = V_0 + \frac{a}{|x|^m} + o \left( \frac{1}{|x|^m} \right), \quad \text{as } |x| \to +\infty.$$  

Remark 1.2. Using the same argument, we can prove that if

$$Q(|x|) = Q_0 - \frac{a}{|x|^m} + o \left( \frac{1}{|x|^m} \right), \quad \text{as } |x| \to +\infty,$$

where the constants are similarly assumed, then problem (1.3) has infinitely many positive non-radial solutions.

Before close this introduction, let us outline the main idea in the proof of Theorem 1.1. Our aim is to construct solutions with a large number of bumps near the infinity. Since

$$\lim_{|x| \to +\infty} V(|x|) = 1,$$

we will use the solution of

$$(-\Delta)^s u + u - |u|^{p-1} u = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N) \quad (1.9)$$

to build up the approximate solution for problem (1.6). It is known (see for instance [20]) the existence of a positive, radial least energy solution $w(x)$, which gives the lowest possible value for the energy

$$J_1(v) = \frac{1}{2} \int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |v|^{p+1}$$

among all nontrivial solutions of (1.9). An important property, which has been proven recently by Frank-Lenzmann-Silvestre [20] (see also [4, 19]), is that there exists a radial least energy solution which is non-degenerate, in the sense that the space of solutions of the equation

$$(-\Delta)^s \phi + \phi - p a |\phi|^{p-1} \phi = 0, \quad \phi \in H^{2s}(\mathbb{R}^N) \quad (1.10)$$

consists exactly of the linear combinations of the translation-genersators $\frac{\partial w}{\partial x_j}, j = 1, \ldots, N$. Also we have the following behavior for $w(x)$ (1.11):

$$w'(|x|) < 0; \quad w(|x|) = \frac{A}{|x|^{N+2s}} (1 + o(1)), \quad A > 0, \quad \text{as } |x| \to +\infty.$$  

Let

$$q_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k,$$

where $0$ is the zero vector in $\mathbb{R}^{N-2}$, $r \in \left[ \frac{1}{C_0 k^{N+2s-m}}, C_0 k^{N+2s-m} \right]$ for large positive constant $C_0$ independent of $k$.

Set $x = (x', x'')$, $x \in \mathbb{R}^2$, $x'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \left\{ u \mid u \in H^{2s}(\mathbb{R}^N), \text{ u is even in } x_b \ (b = 2, \ldots, N) \text{ and} \right\}

u(r \cos \theta, r \sin \theta, x'') = u \left( r \cos \left( \theta + \frac{2\pi j}{k} \right), r \sin \left( \theta + \frac{2\pi j}{k} \right), x'' \right), \quad j = 1, \ldots, k-1.$$

Define

$$W(x) = \sum_{j=1}^{k} w(x - q_j),$$

then Theorem 1.1 is a direct consequence of the following result.

Theorem 1.2. Suppose $V(|x|)$ satisfies the assumption $V$. Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, Problem (1.6) has a solution $u_k$ of the form

$$u_k(x) = W(x) + \varphi(x),$$

where $\varphi(x) \in H_s$ and the energy at $u_k$ goes to infinity as $k$ goes to infinity.
Remark 1.3. Note that there is no parameter in the problem (1.4). Using the number of spikes as parameter, we get the first multiplicity result for fractional nonlinear Schrödinger equation, which seems a new phenomenon for fractional nonlinear Schrödinger equation.

Remark 1.4. Since the approximate solution has polynomial decay, we should deal with every term carefully in the calculus which makes our proof a little bit complicated. By the way, in [23], the approximation has exponential decay.

The paper is organized as follows. In Section 2 we introduce some preliminaries. In Section 3 the ansatz is established. In Section 4 we deal with the corresponding linearized problem. In Section 5 the nonlinear problem is considered and the proof of Theorem 1.2 is given. Finally some important estimates and the expansion of the energy are stated in Section 6.

Notations. In what follows, the symbol $C$ always denotes a various constant independent of $k$.

2 Preliminaries

In this section, we get a useful a-priori estimate for a related linear equation.

Let $0 < s < 1$. Various definitions of the fractional Laplacian $(-\Delta)^s$ of a function $\varphi$ defined in $\mathbb{R}^N$ are available, depending on its regularity and growth properties, see for example [0]. A useful (local) representation given by Caffarelli and Silvestre [3], is via the following boundary value problem in the half space $\mathbb{R}^{N+1}_+ = \{(x, y) \mid x \in \mathbb{R}^N, y > 0\}$:

$$\nabla \cdot (y^{1-2s} \nabla \tilde{\varphi}) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+, \quad \tilde{\varphi}(x, 0) = \varphi(x) \quad \text{on} \quad \mathbb{R}^N.$$  

Here $\tilde{\varphi}$ is the $s$-harmonic extension of $\varphi$, explicitly given as a convolution integral with the $s$-Poisson kernel $p_s(x, y)$,

$$\tilde{\varphi}(x, y) = \int_{\mathbb{R}^N} p_s(x - z, y) \varphi(z) dz,$$

where

$$p_s(x, y) = c_{N, s} \frac{y^{4s-1}}{|x|^2 + |y|^{2s}}$$

and $c_{N, s}$ achieves $\int_{\mathbb{R}^N} p_s(x, y) dx = 1$. Then under suitable regularity, $(-\Delta)^s \varphi$ is the Dirichlet-to-Neumann map for this problem, that is

$$(-\Delta)^s \varphi(x) = \lim_{y \to 0^+} y^{1-2s} \partial_y \tilde{\varphi}(x, y). \quad (2.1)$$

For $m > 0$ and $g \in L^2(\mathbb{R}^N)$, let us consider now the equation

$$(-\Delta)^s \varphi + m \varphi = g \quad \text{in} \quad \mathbb{R}^N.$$  

Then in terms of Fourier transform, for $\varphi \in L^2(\mathbb{R}^N)$, this problem reads

$$\left(|\xi|^{2s} + m\right) \hat{\varphi} = \hat{g}$$

and has a unique solution $\varphi \in H^{2s}(\mathbb{R}^N)$ given by the convolution

$$\varphi(x) = T_m(g) := \int_{\mathbb{R}^N} k(x - z) g(z) dz \quad (2.2)$$

where the Fourier transform of $k$ is

$$\hat{k}(\xi) = \frac{1}{|\xi|^{2s} + m}.$$  

Then we have the following main properties of the fundamental solution $k(x)$ (see for example [20, 18]): $k(x)$ is radially symmetric and positive, $k \in C^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying

(i) $|k(x)| + |x| |\nabla k(x)| \leq \frac{C}{|x|^{N-2s}}$ for all $|x| \leq 1$;

(ii) $\lim_{|x| \to \infty} k(x) |x|^{N+2s} = \alpha > 0$;

(iii) $|x| |\nabla k(x)| \leq \frac{C}{|x|^{N+2s}}$ for all $|x| \geq 1$. 

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or equivalent the closure of the set of all functions in $\mathcal{C}$ where $\tilde{c}$ is careful calculation, it is deduced that $\varphi$ we rewrite (2.5) as

\[ \rho \]  

Proof. Let $\mu = \mu N + 2s$. Then there exists a positive constant $C$ such that

\[ \| (1 + |x|)^{\mu} T_\mu g \|_{L^\infty(\mathbb{R}^N)} \leq C \| (1 + |x|)^{\rho} g \|_{L^\infty(\mathbb{R}^N)} \]  

Lemma 2.2. Let $g \in L^2(\mathbb{R}^N)$. Then the unique solution $\tilde{\varphi} \in H$ of the problem (2.3) is given by the $s$-harmonic extension of the function $\varphi = T_\mu g$.

Lemma 2.3. Assume that $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the following holds: if $\varphi = T_\mu g$ then there is a $C > 0$ such that

\[ \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\beta} \leq C \| g \|_{L^\infty(\mathbb{R}^N)} \]  

where $\beta = \min \{1, 2s\}$.

Lemma 2.4. Let $\varphi \in H^{2s}$ be the solution of

\[ (-\Delta)^s \varphi + W(x) \varphi = g \quad \text{in } \mathbb{R}^N \]  

with bounded potential $\varphi$. If $\inf_{x \in \mathbb{R}^N} W(x) =: m > 0$, $g \geq 0$. Then $\varphi \geq 0$ in $\mathbb{R}^N$.

Using these lemmas, we obtain an a-priori estimate for any solution $\varphi = T_\mu g$ (2.5).

Lemma 2.5. Let $W$ be a continuous function, and assume that for $k$ points $q_1, \ldots, q_k$, there is an $R > 0$ and $B = \bigcup_{j=1}^k B_R(q_j)$ such that

\[ \inf_{x \in \mathbb{R}^N \setminus B} W(x) =: m > 0. \]

Then given any number $\frac{\Delta}{2} < \mu < N + 2s$, there exists a uniform positive constant $C = C(\mu, R)$ independent of $k$ such that for any $\varphi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $g$ satisfying

\[ \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} < +\infty, \quad \text{where } \rho(x) = \sum_{j=1}^k \frac{1}{1 + |x - q_j|^\mu}, \]

we have the validity of the estimate

\[ \| \rho^{-1} \varphi \|_{L^\infty(\mathbb{R}^N)} \leq C \left( \| \varphi \|_{L^\infty(B)} + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \right). \]

Proof. We rewrite (2.5) as

\[ (-\Delta)^s \varphi + \tilde{W} \varphi = \tilde{g}, \]

where $\tilde{g} = (m - W) \chi_B \varphi + g$, $\tilde{W} = m \chi_B + W(1 - \chi_B)$ and $\chi_B$ is the characteristic function on $B$. By careful calculation, it is deduced that

\[ |\tilde{g}(x)| \leq C \| \varphi \|_{L^\infty(B)} + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \rho \leq M \rho \]

where

\[ M = C \| \varphi \|_{L^\infty(B)} \sup_{z \in B} \left( \sum_{j=1}^k \frac{1}{1 + |x - q_j|^\mu} \right)^{-1} + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \]

\[ \leq C \| \varphi \|_{L^\infty(B)} \max_{1 \leq j \leq k} \sup_{z \in B(q_j)} \left( \frac{1}{1 + |x - q_j|^\mu} \right)^{-1} + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \]

\[ \leq C \| \varphi \|_{L^\infty(B)} (1 + R^\alpha) + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \]

\[ \leq C(\mu, R) \left( \| \varphi \|_{L^\infty(B)} + \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \right). \]
From Lemma 2.3 with \( 0 < \mu < N + 2s \), the positive solution \( \varphi_0 \) to the problem

\[
(-\Delta)^s \varphi_0 + m \varphi_0 = \frac{1}{(1 + |x|)^\mu}
\]

satisfies \( \varphi_0 = O(|x|^{-\mu}) \) as \( |x| \to +\infty \). Since \( \inf_{x \in \mathbb{R}^N} \bar{W}(x) \geq m \) obviously, we have

\[
\left( (-\Delta)^s + \bar{W} \right) \bar{\varphi} \geq M \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu}
\]

where \( \bar{\varphi}(x) = M \sum_{j=1}^{k} \varphi_0(x - q_j) \). Setting \( \psi = \varphi - \bar{\varphi} \), one finds that

\[
(-\Delta)^s \psi + \bar{W} \psi = \bar{g} \leq 0.
\]

Using Lemma 2.4 we get that \( \varphi \leq \bar{\varphi} \). Arguing similarly for \(-\varphi \), we get that \( |\varphi| \leq \bar{\varphi} \). Then it holds that

\[
\| \rho^{-1} \varphi \|_{L^\infty(\mathbb{R}^N)} \leq \| \rho^{-1} \bar{\varphi} \|_{L^\infty(\mathbb{R}^N)} = M \left\| \left( \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu} \right)^{-1} \sum_{i=1}^{k} \varphi_0(x - q_i) \right\|_{L^\infty(\mathbb{R}^N)}
\]

\[
\leq CM \left\| \left( \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu} \right)^{-1} \sum_{i=1}^{k} \frac{1}{(1 + |x - q_i|)^\mu} \right\|_{L^\infty(\mathbb{R}^N)}
\]

\[
\leq CM.
\]

The desired estimate follows right now. \( \square \)

Examining the above proof, we can deduce the following immediately.

**Corollary 2.1.** Let \( \rho(x) \) be defined as in the previous lemma. Assume that \( \varphi \in H^{2s}(\mathbb{R}^N) \) satisfies the problem (2.3) and that

\[
\inf_{x \in \mathbb{R}^N} W(x) =: m > 0.
\]

Then we have that \( \varphi \in L^\infty(\mathbb{R}^N) \) and it satisfies

\[
\| \rho^{-1} \varphi \|_{L^\infty(\mathbb{R}^N)} \leq C \| \rho^{-1} g \|_{L^\infty(\mathbb{R}^N)} \quad (2.6)
\]

where \( C = C(\mu) \) independent of \( k \).

**Remark 2.1.** We build these results for any \( \frac{N}{2} < \mu < N + 2s \), but for our purpose, from now on we choose

\[
\mu = \frac{N}{2} - \frac{m}{N + 2s} + 1 + \sigma \in \left( \frac{N}{2}, N + 2s \right).
\]

Here \( \sigma > 0 \) is small enough.

Due to the symmetry, we define \( \Omega_j \) as follows

\[
\Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\{ \frac{y'}{|y'|}, \frac{q_j}{|q_j|} \right\} \geq \cos \frac{\pi}{k} \right\},
\]

and introduce the following estimate for later use. For any \( \beta \geq \frac{N + 2s - m}{N + 2s} \) and fixed \( \ell \), as \( k \to \infty \), it holds that

\[
\sum_{i \neq \ell} \frac{1}{|q_i - q_\ell|^\beta} \leq \frac{1}{2^\beta} \sum_{i \neq \ell} r_i^\beta \sin^\beta \frac{\pi r_i}{k} \leq \frac{C k^\beta}{r^\beta} \sum_{i=1}^{k} \frac{1}{r_i^\beta} \leq \begin{cases} \frac{C k^\beta}{r^\beta} = O(r^{-\frac{m}{N + 2s}}) & \beta > 1, \\ \frac{C k^\beta \ln k}{r^\beta} = O(r^{-\frac{m}{N + 2s} \ln r}) & \beta = 1, \\ \frac{C k^\beta}{r^\beta} = O(r^{-\left(\beta - \frac{N + 2s - m}{N + 2s}\right)}) & \beta < 1. \end{cases}
\]
Remark 2.2. It holds that
\[ \rho(x) \leq C + C \sum_{j=2}^{k} \frac{1}{|q_1 - q_j|^{\frac{1}{2} - \frac{1}{N+2\sigma} + 1 + \sigma}} \leq C + C \left( \frac{k}{r} \right)^{\frac{1}{2} - \frac{1}{N+2\sigma} + 1 + \sigma} \leq C. \]

according to Lemma 6.1. Also we easily have
\[ \int_{\Omega_1} \rho^2 \leq \int_{\Omega_1} \left( \frac{1}{(1 + |x - q_1|)^{\frac{1}{2} - \frac{1}{N+2\sigma} + 1 + \sigma}} + \frac{1}{(1 + |x - q_1|)^{\frac{1}{2} - \frac{1}{N+2\sigma} + \sigma}} \sum_{j=2}^{k} \frac{1}{|q_1 - q_j|^{1 - \frac{1}{N+2\sigma}}} \right)^2 \, dx \leq C. \]

In what follows, we use \( \|f\|_* \) to mean \( \|\rho^{-1}f\|_{L^\infty(\mathbb{R}^N)} \) for convenience, i.e.
\[ \|f\|_* = \|\rho^{-1}f\|_{L^\infty(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^{\frac{1}{2} - \frac{1}{N+2\sigma} + 1 + \sigma}} \right)^{-1} f(x). \]

A useful fact is that if \( f, g \in L^2(\mathbb{R}^N) \) and \( F = T_m(f), G = T_m(g) \), then the following holds
\[ \int_{\mathbb{R}^N} G(-\Delta)^s F - \int_{\mathbb{R}^N} F(-\Delta)^s G = -\int_{\mathbb{R}^N} T_m(f)g + \int_{\mathbb{R}^N} fT_m(g) = 0 \]
since the kernel \( k \) is radially symmetric.

3 Ansatz

In this section, we set up the approximation solution and estimate the corresponding error term. By a solution of the problem
\[ (-\Delta)^s u + Vu - u^p = 0 \quad \text{in } \mathbb{R}^N, \]
we mean a \( u \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) such that the above equation is satisfied. Let us observe that it suffices to solve
\[ (-\Delta)^s u + Vu - u^p = 0 \quad \text{in } \mathbb{R}^N \tag{3.1} \]
where \( u_+ = \max\{u, 0\} \) with the help of Lemma 2.4.

We look for a solution \( u \) of the form
\[ u = W + \varphi, \quad W = \sum_{j=1}^{k} W_j, \quad W_j = w(x - q_j) \]
where \( \varphi \in H_s \) is a small function, disappearing as \( k \to +\infty \). In terms of \( \varphi \), the equation (3.1) becomes
\[ (-\Delta)^s \varphi(x) + V(|x|)\varphi(x) - pW^{p-1}\varphi(x) = E + N(\varphi) \quad \text{in } \mathbb{R}^N, \tag{3.2} \]
where
\[ N(\varphi) = (W + \varphi)^+_p - W^p - pW^{p-1}\varphi, \]
\[ E = \sum_{j=1}^{k} (1 - V(|x|)) W_j + \left( \sum_{j=1}^{k} W_j \right)^p - \sum_{j=1}^{k} W_j^p. \]

Rather than solving the problem (3.2) directly, we shall first solve a projected version of it, precisely,
\[ \begin{cases} (-\Delta)^s \varphi(x) + V(|x|)\varphi(x) - pW^{p-1}\varphi(x) = E + N(\varphi) + c \sum_{j=1}^{k} Z_j & \text{in } \mathbb{R}^N, \\ \varphi \in H_s, \\ \int_{\mathbb{R}^N} Z_j \varphi = 0, & j = 1, \ldots, k, \end{cases} \tag{3.3} \]
for some pair \((\varphi, c)\) where \(\varphi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\), \(c\) is a constant,
\[
Z_j = \frac{\partial W_j}{\partial x} \quad \text{for} \ j = 1, \ldots, k, \quad \text{and} \quad |Z_j| \leq \frac{C}{(1 + |x - q_j|)^{N+2s}} \text{ obviously.}
\]

After the problem (3.3) solved, a variational process will carry out to find a suitable \(r\) and then make the constant \(c\) in (3.3) be zero, i.e. we solve the problem (3.2).

At the end of this section, we give the estimate of \(E\).

**Lemma 3.1.** It holds that
\[
\|E\|_s \leq C \left( \frac{k}{r} \right)^\min\{N+2s,(N+2s)p-\mu\} + C \frac{1}{r^{N+2s-\mu}} + C \frac{1}{r^m} = o \left( \frac{1}{r^{m/2}} \right). \tag{3.4}
\]

**Proof.** By symmetry, we just assume that \(x \in \Omega_1\) in the following proof. Obviously we know
\[
|x - q_j| \geq |x - q_1| \quad \text{for} \ j = 2, \ldots, k.
\]

If \(|x| \geq |q_1|/2 = r/2\), then
\[
V(|x|) - 1 = O \left( \frac{1}{|x|^m} \right) = O \left( \frac{1}{r^m} \right)
\]
and in this region
\[
\left| \rho^{-1} \sum_{j=1}^{k} (1 - V(|x|)) W_j \right| \leq \frac{C}{r^m} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_i|)^\mu} \sum_{j=1}^{k} W_j
\]
\[
\leq \frac{C}{r^m} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_i|)^\mu} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^{N+2s}}
\]
\[
\leq \frac{C}{r^m} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_i|)^\mu} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^{N+2s-\mu}}
\]
\[
\leq \frac{C}{r^m}.
\]

While for \(|x| \leq r/2\), then
\[
|x - q_1| \geq |q_1| - |x| \geq \frac{r}{2} \quad \text{and} \quad |x - q_j| \geq \frac{r}{2} \quad \text{for} \ j = 2, \ldots, k.
\]

Hence
\[
\left| \rho^{-1} \sum_{j=1}^{k} (1 - V(|x|)) W_j \right| \leq C \rho^{-1} \sum_{j=1}^{k} W_j \leq C \rho^{-1} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^{N+2s}}
\]
\[
\leq C \rho^{-1} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu} \cdot \frac{1}{(1 + |x - q_j|)^{N+2s-\mu}}
\]
\[
\leq C \left( \frac{1}{(1 + |x - q_j|)^\mu} \right)^{-1} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu} \cdot \frac{1}{r^{N+2s-\mu}}
\]
\[
\leq \frac{C}{r^{N+2s-\mu}}.
\]

For the other part in \(E\), we observe that
\[
\left| \left( \sum_{j=1}^{k} W_j \right)^p - \sum_{j=1}^{k} W_j^p \right| \leq CW_1^{p-1} \sum_{j=2}^{k} W_j + C \sum_{j=2}^{k} W_j^p + C \left( \sum_{j=1}^{k} W_j \right)^p.
\]
In the case of $\mu \leq (N + 2s)(p - 1)$,
\[
\rho^{-1} W^{p-1} \sum_{j=2}^{k} W_j \leq C \rho^{-1} \frac{1}{(1 + |x - q_1|)^{(N + 2s)(p-1)}} \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^{N + 2s}} \\
\leq C(1 + |x - q_1|)^{\mu} \frac{1}{(1 + |x - q_1|)^{(N + 2s)(p-1)}} \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^{N + 2s}} \\
\leq C \sum_{j=2}^{k} \frac{1}{|q_j - q_1|^{N + 2s}} \leq C \left( \frac{k}{r} \right)^{N + 2s};
\]
Otherwise if $\mu > (N + 2s)(p - 1)$, then
\[
\rho^{-1} W^{p-1} \sum_{j=2}^{k} W_j \leq C \rho^{-1} \frac{1}{(1 + |x - q_1|)^{(N + 2s)(p-1)}} \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^{N + 2s - \mu + (N + 2s)(p-1)}} \\
\leq C \rho^{-1} \frac{1}{(1 + |x - q_1|)^{\mu}} \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^{N + 2s - \mu}} \\
\leq C \sum_{j=2}^{k} \frac{1}{|q_j - q_1|^{(N + 2s)p - \mu}} \leq C \left( \frac{k}{r} \right)^{(N + 2s)p - \mu},
\]
where we used Lemma 5.1. It is easy to deduce that
\[
\rho^{-1} \sum_{j=2}^{k} W_j \leq C \rho^{-1} \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^{(N + 2s)p - \mu}} \frac{1}{(1 + |x - q_1|)^{\mu}} \\
\leq C \sum_{j=2}^{k} \frac{1}{|q_j - q_1|^{(N + 2s)p - \mu}} \leq C \left( \frac{k}{r} \right)^{(N + 2s)p - \mu}
\]
and
\[
\rho^{-1} \left( \sum_{j=2}^{k} W_j \right)^p \leq C \rho^{-1} \left( \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^{(N + 2s)p - \mu}} \frac{1}{(1 + |x - q_1|)^{\mu}} \right)^p \\
\leq C \left( \sum_{j=2}^{k} \frac{1}{|q_j - q_1|^{(N + 2s)p - \mu}} \right)^p \leq C \left( \frac{k}{r} \right)^{(N + 2s)p - \mu}.
\]
The condition of $m$ leads obviously to $N + 2s - \frac{\mu}{p} > 1$, $N + 2s - \mu > \frac{mp}{2}$ and $(N + 2s)p - \mu > \frac{N + 2s}{2}$. Thus we get the desired result by combining these above estimates.

4 Linearized theory

This section is devoted to solve a projected linear problem.
We consider the linear problem of finding $\varphi \in H^{2s}(\mathbb{R}^N)$ such that for certain constant $c$, we have
\[
\begin{align*}
\left\{ \begin{array}{ll}
(-\Delta)^s \varphi + V(|x|)\varphi - pW^{p-1}\varphi = g + c \sum_{j=1}^{k} Z_j & \text{in } \mathbb{R}^N, \\
\varphi \in H_s, \\
\int_{\mathbb{R}^N} Z_j \varphi = 0, & j = 1, \ldots, k.
\end{array} \right.
\end{align*}
\]
The constant $c$ is uniquely determined in terms of $\varphi$ and $g$ when $k$ is sufficient large from the equation

$$
c \int_{\mathbb{R}^N} \sum_{j=1}^{k} Z_j Z_1 = \int_{\mathbb{R}^N} \left[ (-\Delta)^s \varphi + V(|x|) \varphi - p W^{p-1} \varphi \right] Z_1 - \int_{\mathbb{R}^N} g Z_1
$$

$$
= \int_{\mathbb{R}^N} \left[ (-\Delta)^s Z_1 + V(|x|) Z_1 - p W^{p-1} Z_1 \right] \varphi + \mathcal{O}(\|g\|_*) \int_{\mathbb{R}^N} \rho |Z_1|
$$

$$
= \int_{\mathbb{R}^N} \left[ (V - 1) + p(W^{p-1} - W^{p-1}) \right] Z_1 \varphi + \mathcal{O}(\|g\|_*),
$$

where we use Lemma 1.3 to obtain

$$
\int_{\mathbb{R}^N} \rho |Z_1| \leq C \left(1 + \sum_{j=2}^{k} \frac{1}{|q_1 - q_j|^\mu} \right) \int_{\mathbb{R}^N} \frac{1}{(1 + |x|)^{N+2s}} dx \leq C.
$$

By direct calculation, it is easy to see that

$$
\int_{\mathbb{R}^N} Z_1^2 = \int_{\mathbb{R}^N} \left( \frac{\partial w(x - q_1)}{\partial r} \right)^2 dx = \int_{\mathbb{R}^N} \left( \frac{\partial w(x - q_1)}{\partial x_1} \right)^2 dx = \frac{1}{N} \int_{\mathbb{R}^N} (w'(|x|))^2 dx,
$$

and

$$
\sum_{j=2}^{k} \int_{\mathbb{R}^N} Z_j Z_1 = \sum_{j=2}^{k} \int_{\mathbb{R}^N} w'(|x - q_1|) \frac{x - q_1}{|x - q_1|} \cdot \left( \frac{q_1}{r} \right) w'(|x - q_1|) \frac{x - q_1}{|x - q_1|} \cdot \left( \frac{q_1}{r} \right) dx
$$

$$
= \sum_{j=2}^{k} \int_{\mathbb{R}^N} w'(|x|) \frac{x^1}{|x|} w'(|x + q_1 - q_j|) \frac{x + q_1 - q_j}{|x + q_1 - q_j|} \cdot \left( \frac{q_1}{r} \right) dx
$$

$$
= \sum_{j=2}^{k} \left( \int_{\{|x| \leq \frac{1}{2}|q_1 - q_j|\}} + \int_{\{|x| \geq \frac{1}{2}|q_1 - q_j|\}} \right) w'(|x|) \frac{x^1}{|x|} w'(|x + q_1 - q_j|) \frac{x + q_1 - q_j}{|x + q_1 - q_j|} \cdot \left( \frac{q_1}{r} \right) dx
$$

$$
\leq C \sum_{j=2}^{k} \frac{1}{|q_1 - q_j|^{N+2s}} \int_{\mathbb{R}^N} |w'(|x|)| dx \leq C \sum_{j=2}^{k} \frac{1}{|q_1 - q_j|^{N+2s}} = O \left( \left( \frac{k}{r} \right)^{N+2s} \right),
$$

where $x = (x^1, \ldots, x^N)$. It implies that $\{Z_j\}_{j=1}^{k}$ is approximately orthogonal provided $k$ large enough because of the symmetry.

As to the first term in the right hand side of (1.3), we do the following analysis.

$$
\left| \int_{\Omega_1} (V(|x|) - 1) Z_1 \varphi \right| \leq C \|\varphi\|_* \left( \int_{\Omega_1} |V(|x|) - 1| \rho \frac{1}{(1 + |x - q_1|)^{N+2s}} dx \right)
$$

$$
\leq C \|\varphi\|_* \left( \int_{\{x \in \Omega_1 \mid |x| \geq |q_1|/2\}} + \int_{\{x \in \Omega_1 \mid |x| \leq |q_1|/2\}} \right) |V(|x|) - 1| \rho \frac{1}{(1 + |x - q_1|)^{N+2s}} dx
$$

$$
\leq C \|\varphi\|_* \left( \int_{\{x \in \Omega_1 \mid |x| \geq |q_1|/2\}} \frac{1}{|x|^m} \frac{1}{(1 + |x - q_1|)^{N+2s}} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu} dx + \frac{1}{r^{N+2s}} \sum_{j=1}^{k} \frac{1}{(1 + |x - q_j|)^\mu} dx \right)
$$

$$
\leq C \|\varphi\|_* \left( \frac{1}{r^{N+2s}} \left( \frac{1}{(1 + |x - q_1|)^{N+2s}} + \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^\mu} \right) dx + \frac{1}{r^{N+2s}} \left( \frac{1}{(1 + |x - q_1|)^{N+2s}} + \sum_{j=2}^{k} \frac{1}{(1 + |x - q_j|)^\mu} \right) dx \right)
$$

$$
\leq C \|\varphi\|_* \left( \frac{1}{r^{m}} + \frac{1}{r^{m}} \left( \frac{k}{r} \right)^\mu + \frac{1}{r^{N+2s}} + \frac{1}{r^{N+2s}} \left( \frac{k}{r} \right)^\mu \right) = o(\|\varphi\|_*), \quad \text{as } k \to +\infty.
$$

(4.5)
In addition, note that, for any $j \neq 1$, $\ell \neq 1$ and $j \neq \ell$,

\[
\int_{\Omega_1} \frac{dx}{(1 + |x - q_1|)^{N+2s}(1 + |x - q_j|)^\mu} 
\leq \frac{C}{|q_j - q_1|^{1-\frac{N+2s}{N+2\sigma}}} \int_{\Omega_1} \left[ \frac{1}{(1 + |x - q_1|)^{\frac{N+2s+\sigma}} + \frac{1}{(1 + |x - q_j|)^{\frac{N+2s+\sigma}}} } \right] dx 
\leq \frac{C}{|q_j - q_1|^{1-\frac{N+2s}{N+2\sigma}}} \left[ \frac{1}{|q_\ell - q_1|^{\frac{1}{N+2\sigma}}} + \frac{1}{|q_\ell - q_j|^{\frac{1}{N+2\sigma}}} \right],
\]

where Lemma 6.3 is used in the first inequality. It is checked that

\[
\left| \int_{\mathbb{R}^N \setminus \Omega_1} (V - 1) Z_1 \varphi \right| \leq C\|\varphi\|_* \sum_{\ell = 2}^k \int_{\Omega_\ell} |V(|x|) - 1| \frac{\rho(x)}{(1 + |x - q_1|)^{N+2s}} dx 
\leq C\|\varphi\|_* \sum_{\ell = 2}^k \int_{\Omega_\ell} \left( 1 + \frac{1}{(1 + |x - q_1|)^\mu} + \frac{1}{(1 + |x - q_2|)^\mu} \right) \frac{\sum_{j \neq \ell}^k (1 + |x - q_j|)^\mu}{(1 + |x - q_1|)^{N+2s}} \frac{1}{(1 + |x - q_2|)^{N+2s}} dx 
\leq C\|\varphi\|_* \left( \frac{k}{r} \right)^{N+2s} \tag{4.6}.
\]

Thus from (4.3) and (4.6) we get that

\[
\int_{\mathbb{R}^N} (V - 1) Z_1 \varphi = o(\|\varphi\|_*).
\]

When $1 < p \leq 2$, it holds that

\[
\left| \int_{\Omega_1} (W_1^{p-1} - W_1^{p-1}) Z_1 \varphi dx \right| \leq C\|\varphi\|_* \int_{\Omega_1} \left( \frac{k}{j = 2} \sum_{j = 2}^k W_j \right)^{p-1} \rho |Z_1| dx 
\leq C\|\varphi\|_* \left( \frac{k}{j = 2} \sum_{j = 2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1} \int_{\Omega_1} \left[ \frac{1}{(1 + |x - q_1|)^\mu} + \frac{1}{(1 + |x - q_2|)^\mu} \right] \frac{1}{(1 + |x - q_1|)^{N+2s}} dx 
\leq C\|\varphi\|_* \left( \frac{k}{r} \right)^{(N+2s)(p-1)} ,
\]

and, similar to (4.6),

\[
\left| \int_{\mathbb{R}^{N} \setminus \Omega_1} (W_1^{p-1} - W_1^{p-1}) Z_1 \varphi dx \right| \leq C\|\varphi\|_* \sum_{\ell = 2}^k \int_{\Omega_\ell} \left( \frac{k}{j = 2} \sum_{j = 2}^k W_j \right)^{p-1} \rho |Z_1| dx 
\leq C\|\varphi\|_* \sum_{\ell = 2}^k \int_{\Omega_\ell} \left[ \frac{1}{(1 + |x - q_1|)^{N+2s}(p-1)} + \left( \frac{k}{r} \right)^{(N+2s)(p-1)} \right] \frac{1}{(1 + |x - q_1|)^{N+2s}} dx 
\leq C\|\varphi\|_* \sum_{\ell = 2}^k \int_{\Omega_\ell} \left[ \frac{1}{(1 + |x - q_1|)^\mu} + \frac{1}{(1 + |x - q_2|)^\mu} + \frac{1}{(1 + |x - q_j|)^\mu} \right] \frac{dx}{(1 + |x - q_1|)^{N+2s}}.
\]
Thus we obtain, from the above two estimates, that
\[ \left| \int_{\mathbb{R}^N} (W^{p-1}_1 - W^{p-1}) Z_1 \varphi dx \right| \leq C \| \varphi \|_\ast \left( \frac{k}{r} \right)^{\frac{N}{p} + 2s}. \]

For the case \( p > 2 \), with Lemma 4.2,
\[ \left| \int_{\Omega_1} (W^{p-1}_1 - W^{p-1}) Z_1 \varphi dx \right| \leq C \| \varphi \|_\ast \int_{\Omega_1} \left( W^{p-2}_1 \sum_{j=2}^k W_j + \left( \sum_{j=2}^k W_j \right)^{p-1} \right) |Z_1| dx \]
\[ \leq C \| \varphi \|_\ast \left( \frac{k}{r} \right)^{\frac{N}{p} + 2s} \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} + \left( \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1} \leq C \| \varphi \|_\ast \left( \frac{k}{r} \right)^{\frac{N}{p} + 2s}, \] (4.7)

and, also similar to (4.6),
\[ \left| \int_{\mathbb{R}^N \setminus \Omega_1} (W^{p-1}_1 - W^{p-1}) Z_1 \varphi dx \right| \leq C \| \varphi \|_\ast \sum_{j=2}^k \int_{\Omega_1} \left( W^{p-2}_1 \sum_{j=2}^k W_j + \left( \sum_{j=2}^k W_j \right)^{p-1} \right) |Z_1| dx \]
\[ \leq C \| \varphi \|_\ast \left( \frac{k}{r} \right)^{\frac{N}{p} + 2s} \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} + \left( \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1} \leq C \| \varphi \|_\ast \left( \frac{k}{r} \right)^{\frac{N}{p} + 2s}, \] (4.8)
on account that, in \( \Omega_1 \),
\[ W^{p-2}_1 \sum_{j=2}^k W_j \leq \frac{C}{|q_1 - q_1|^{(N+2s)(p-2)}} \left( \frac{1}{(1 + |x - q_1|)^{N+2s}} + \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1} \]
\[ \left( \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1} \leq \frac{C}{(1 + |x - q_1|)^{N+2s}(p-1)} + C \left( \sum_{j=2}^k \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1}. \]

So it is concluded from (4.7) and (4.8) that
\[ \left| \int_{\mathbb{R}^N} (W^{p-1}_1 - W^{p-1}) Z_1 \varphi dx \right| \leq C \| \varphi \|_\ast \left( \frac{k}{r} \right)^{\frac{N}{p} + 2s}. \]

Combining the above inequalities leads to the following lemma right now.

Lemma 4.1. If \((\varphi, c)\) solves the problem (4.4), then
\[ c = o(\| \varphi \|_\ast) + O(\| g \|_\ast). \]

In the rest of this section we shall build a solution to the problem (4.1).

Proposition 4.1. Given \( k \) large enough, the exists a solution \( \varphi = T(g) \) to (4.1) which defines a linear operator of \( g \), provided that \( \| g \|_\ast < +\infty \). Moreover,
\[ \| \varphi \|_\ast \leq C \| g \|_\ast \quad \text{and} \quad c \leq C \| g \|_\ast, \]
where the positive constant \( C \) is independent of \( k \).

The key difference of the proof between this proposition and Proposition 4.1 in [10] is that now we should build an a priori estimate which is independent of \( k \), see the coming Lemma 4.2. Once we get such estimate, the remaining is just the same as that in [10].

Lemma 4.2. Under the assumptions of Proposition 4.1, there exists a positive constant \( C \) independent of \( k \) such that for any solution \( \varphi \) with \( \| \varphi \|_\ast < +\infty \), we have the following an a priori estimate
\[ \| \varphi \|_\ast \leq C \| g \|_\ast. \]
Proof. We argue by contradiction. Suppose that there are $g_k, r_k \in \left[ \frac{1}{C_0} k^{-\frac{N+2s}{2}}, C_0 k^{-\frac{N+2s}{2}} \right]$ and $\varphi_k$ solving \eqref{eq:1} for $g = g_k, r = r_k$ with $\|g_k\|_{\ast} \to 0$ and $\|\varphi_k\|_{\ast} \geq C' > 0$. We may assume that $\|\varphi_k\|_{\ast} = 1$.

For simplicity, we drop the subscript $k$.

From the conditions of potential $V$, obviously $\inf_{\mathbb{R}^N} V > 0$. On the other hand, in the equation of $\varphi$,

$$(-\Delta)^s \varphi + (V - pW^{p-1})\varphi = g + c \sum_{j=1}^{k} Z_j,$$

we find that

$$V(x) - pW^{p-1}(x) \geq V(x) - C \left( \frac{1}{(1 + |x - q_1|)^{N+2s}} + \sum_{j=2}^{k} \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p-1},$$

$$\geq V(x) - C \left( \frac{1}{(1 + |x - q_1|)^{N+2s}} + \left( \frac{1}{r} \right)^{N+2s} \right)^{p-1} \geq \frac{1}{2} V(x)$$

for any $x \in \Omega_1 \setminus B_R(q_1)$, which leads to

$$\inf_{\mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_R(q_j)} (V(x) - pW^{p-1}(x)) \geq \frac{1}{2} \inf_{\mathbb{R}^N} V(x) > 0.$$ 

Accordingly, by Lemma 2.5 and Lemma 4.1, it holds that

$$\|\varphi\|_{\ast} \leq C \left( \|\varphi\|_{L^\infty(\bigcup_{j=1}^{k} B_R(q_j))} + \|g\|_{\ast} + |c| \left\| \sum_{j=1}^{k} Z_j \right\|_{\ast} \right) \leq C\|\varphi\|_{L^\infty(\bigcup_{j=1}^{k} B_R(q_j))} + o(1),$$

from which we may assume that, up to a subsequence,

$$\|\varphi\|_{L^\infty(B_R(q_1))} \geq \gamma > 0.$$ (4.9)

Let us set $\tilde{\varphi}(x) = \varphi(x + q_1)$, then $\tilde{\varphi}$ satisfies

$$(-\Delta)^s \tilde{\varphi} + V(|x + q_1|)\tilde{\varphi} - pu^{p-1}(x)\tilde{\varphi} = \tilde{g}$$ (4.10)

where

$$\tilde{g}(x) = g(x + q_1) + c \left( Z_1(x + q_1) + \sum_{j=2}^{k} Z_j(x + q_1) \right) + p \left[ w(x) + \sum_{j=2}^{k} w(x + q_1 - q_j) \right]^{p-1} - w^{p-1}(x) \tilde{\varphi}.$$ (4.11)

For any point $x$ in an arbitrarily compact set of $\mathbb{R}^N$, we have, from Remark 2.2, that

$$|g(x + q_1)| \leq \|g\|_{\ast} o(x + q_1) \leq C\|g\|_{\ast} = o(1).$$

It is easy to see that $V(x + q_1) \to 1$,

$$c = o(\|\varphi\|_{\ast}) + O(\|g\|_{\ast}) \to 0$$

and

$$\left| Z_1(x + q_1) + \sum_{j=2}^{k} Z_j(x + q_1) \right| \leq \frac{C}{(1 + |x + q_1|)^{N+2s}} + C \sum_{j=2}^{k} \frac{1}{|x + q_1 - q_j|^{N+2s}}$$

$$\leq C + C \sum_{j=2}^{k} \frac{1}{|q_1 - q_j|^{N+2s}} \leq C.$$
For the last term in (4.11), as $1 < p \leq 2$,
\[
\left\| \left( w(x) + \sum_{j=2}^{k} w(x + q_j - q_j) \right)^{p-1} \right\| - w^{p-1}(x) \left\| \phi \right\|
\]
\[
\leq C \left( \sum_{j=2}^{k} w(x + q_j - q_j) \right)^{p-1} \leq C \left( \sum_{j=2}^{k} \frac{1}{|x + q_j - q_j|^{N+2s}} \right)^{p-1}
\]
\[
\leq C \left( \sum_{j=2}^{k} \frac{1}{|q_j - q_j|^{N+2s}} \right)^{p-1} \leq C \left( \frac{1}{r} \right)^{(N+2s)(p-1)}.
\]
while for $p > 2$,
\[
\left\| \left( w(x) + \sum_{j=2}^{k} w(x + q_j - q_j) \right)^{p-1} \right\| - w^{p-1}(x) \left\| \phi \right\|
\]
\[
\leq C w^{p-2} \sum_{j=2}^{k} w(x + q_j - q_j) + C \left( \sum_{j=2}^{k} w(x + q_j - q_j) \right)^{p-1}
\]
\[
\leq \sum_{j=2}^{k} \frac{C}{|x + q_j - q_j|^{N+2s}} + C \left( \sum_{j=2}^{k} \frac{1}{|x + q_j - q_j|^{N+2s}} \right)^{p-1}
\]
\[
\leq C \left( \frac{k}{r} \right)^{N+2s} + C \left( \frac{k}{r} \right)^{(N+2s)(p-1)} \leq C \left( \frac{k}{r} \right)^{N+2s}.
\]
Hence $\tilde{g} \to 0$ uniformly on any compact set of $\mathbb{R}^N$ as $k \to \infty$. Meanwhile, from
\[
(\Delta)^s \tilde{\phi} + \tilde{\phi} = (1 - V(x + q_j)) \tilde{\phi} + pw^{p-1} \tilde{\phi} + \tilde{g},
\]
and Lemma 2.3 we obtain that
\[
\sup_{x \neq y} \frac{\left| \tilde{\phi}(x) - \tilde{\phi}(y) \right|}{|x - y|^{\beta}} \leq C \left( \left\| (1 - V) \tilde{\phi} \right\|_{L^\infty} + \left\| w^{p-1} \tilde{\phi} \right\|_{L^\infty} + \left\| \tilde{g} \right\|_{L^\infty} \right) \leq C (\left\| \phi \right\|_{s} + \left\| \tilde{g} \right\|_{L^\infty} \leq C
\]
where $\beta = \min\{1, 2s\}$. Hence up to a subsequence, we may assume that $\tilde{\phi} \to \varphi_0$ uniformly on any compact set. It is easy to observe that $\varphi_0$ satisfies
\[
\begin{cases}
(\Delta)^s \varphi_0 + \varphi_0 - pw^{p-1} \varphi_0 = 0 & \text{in } \mathbb{R}^N, \\
\varphi_0 \in H_s, \\
\int_{\mathbb{R}^N} \frac{\partial w}{\partial x^j} \varphi_0 = 0,
\end{cases}
\]
where $x = (x^1, \ldots, x^N)$. Besides, we know, from Remark 2.2 that
\[
\int_{B_n(0)} \varphi_0^2 \leq \int_{B_n(0)} \varphi_0^2 = \int_{B_n(q_i)} \varphi_k^2 \leq \left\| \varphi_k \right\|^2_{L^\infty} \int_{B_n(q_i)} \rho^2 \leq C,
\]
which means that $\varphi_0 \in L^2(\mathbb{R}^N)$. Then the non-degeneracy result in [20] implies that $\varphi_0$ must be a linear combination of the partial derivatives $\frac{\partial \varphi_0}{\partial x^i}, i = 1, \ldots, N$. But the symmetry and orthogonality condition yield that $\varphi_0 \equiv 0$, which is a contradiction to (4.9). The lemma is then proved.
5 The variational reduction and the proof of Theorem 1.2

In this section we first solve the intermediate nonlinear problem (3.3), i.e.
\[
\begin{cases}
(-\Delta)^s \varphi(x) + V(|x|)\varphi(x) - pW^{p-1}\varphi(x) = E + N(\varphi) + c \sum_{j=1}^k Z_j & \text{in } \mathbb{R}^N, \\
\varphi \in H_s, \\
\int_{\mathbb{R}^N} Z_j \varphi = 0 & \text{for any } j = 1, \ldots, k.
\end{cases}
\]

Then we solve the final nonlinear problem (3.2) variationally.

Proposition 5.1. Assume that $k$ is large enough, for any $r \in \left[\frac{1}{C_0} k^{-\frac{N+2s}{2}}, C_0 k^{-\frac{N+2s}{2}} \right]$, the problem (3.2) has a unique small solution $\varphi = \Phi(r)$ with
\[
\|\varphi\| \leq C \left( \frac{k}{r} \right)^{\min\{N+2s,(N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m} = o \left( \frac{1}{r^{m/2}} \right).
\]
Furthermore, the map $r \rightarrow \Phi(r)$ is of class $C^1$, and
\[
\|\Phi'(r)\| \leq C \left( \frac{k}{r} \right)^{\min\{N+2s,(N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m}.
\]

Proof. Problem (3.3) can be written as the fixed point problem
\[
\varphi = T(E + N(\varphi)) =: A(\varphi) \quad \text{for } \varphi \in H_s.
\]

Let
\[
\mathfrak{S} = \{ \varphi \in H_s | \|\varphi\| \leq s_0 \},
\]
where $s_0 > 0$ is a small number determined later.

If $\varphi \in \mathfrak{S}$, either $1 < p \leq 2$,
\[
\|N(\varphi)\| \leq C\|\varphi^p\| \leq C\|\varphi\|^2\rho|\varphi|^{p-1}\| E_{\infty}(\mathbb{R}^N) \leq C\|\varphi\|^p;
\]
or $p > 2$,
\[
\|N(\varphi)\| \leq C\|\varphi^p\| \leq C\|\varphi\|^2W^{p-2}\| + C\|\varphi^p\| \leq C\|\varphi\|^2W|E_{\infty}(\mathbb{R}^N) + C\|\varphi\|^p\|\varphi|^{p-1}\|E_{\infty}(\mathbb{R}^N) \leq C\|\varphi\|^2.
\]

By Proposition 4.1 and Lemma 3.1,
\[
\|A(\varphi)\| \leq C (\|E\| + \|N(\varphi)\|) \leq C\|E\| + C(\|\varphi\| + \|\varphi\|^p) \|\varphi\| \leq s_0
\]
if we choose $C(s_0 + s_0^{p-1}) \leq \frac{1}{2}$ and $k$ large enough such that
\[
C \left( \frac{k}{r} \right)^{\min\{N+2s,(N+2s)p-\mu\}} + \frac{C}{r^{N+2s-\mu}} + \frac{C}{r^m} \leq \frac{1}{2}s_0.
\]

On the other hand, for any $\varphi_i \in H_s, i = 1, 2$,
\[
|N(\varphi_1) - N(\varphi_2)| = |N'(t)(\varphi_1 - \varphi_2)|
\]
where $t$ lies between $\varphi_1$ and $\varphi_2$.

For $1 < p \leq 2$, $|N'(t)| \leq C|t|^{p-1} \leq C(|\varphi_1|^{p-1} + |\varphi_2|^{p-1})$ which tells us that
\[
\|N(\varphi_1) - N(\varphi_2)\| \leq C\|\varphi_1 - \varphi_2\| + C(|\varphi_1|^{p-1} + |\varphi_2|^{p-1}) \|\varphi_1|^{p-1}\| E_{\infty}(\mathbb{R}^N) + C(|\varphi_1|^{p-1} + |\varphi_2|^{p-1}) \|\varphi_2|^{p-1}\| E_{\infty}(\mathbb{R}^N) \leq C\|\varphi_1 - \varphi_2\| + C\|\varphi_1|^{p-1}\| E_{\infty}(\mathbb{R}^N) + C\|\varphi_2|^{p-1}\| E_{\infty}(\mathbb{R}^N) \leq C s_0^{p-1}\|\varphi_1 - \varphi_2\| \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|.
\]
provided $s_0$ small enough. And for $p > 2$, $|N'(t)| \leq C(W^{p-2}|t| + |t|^{p-1})$, from which we can deduce that

$$
\|N(\varphi_1) - N(\varphi_2)\|_* \leq C\|\varphi_1 - \varphi_2\|_* \left(\|\rho W\|_{L^{\infty}(\mathbb{R}^N)}(\|\varphi_1\|_* + \|\varphi_2\|_*) + (\|\varphi_1\|_*^{p-1} + \|\varphi_2\|_*^{p-1})\|\rho\|_{L^{\infty}(\mathbb{R}^N)}\right)
$$

$\leq C\|\varphi_1 - \varphi_2\|_* (\|\varphi_1\|_* + \|\varphi_2\|_*) + (\|\varphi_1\|_*^{p-1} + \|\varphi_2\|_*^{p-1})$

$\leq C(s_0 + s_0^{p-1})\|\varphi_1 - \varphi_2\|_* \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_*$

with $s_0$ small enough.

Thus we obtain that $A$ is a contraction mapping and the problem (3.3) has a unique solution $\varphi$. Obviously according to Lemma 3.1,

$$
\|\varphi\|_* \leq C\left(\frac{k}{r}\right)^{\min\{N+2s,(N+2s)p-\mu\}} + \frac{C}{p^{N+2s-\mu}} + \frac{C}{p^m}.
$$

For the proof of $\Phi(r) \in C^1$, please refer to [10]. Here we don’t repeat it.

Next, we will use the above introduced ingredients to find existence results for the nonlinear problem (3.2), i.e. the equation

$$
(-\Delta)^s u + V(x)u - u_p^* = 0.
$$

(5.1)

Set the following energy functional

$$
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} u(-\Delta)^s u + V(x)u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} u_{p+1}^{p+1}
$$

(5.2)

whose nontrivial critical points are solutions to (5.1).

We want to find a solution of (5.1) with the form $U = W + \varphi$ where $\varphi = \Phi(r)$ is found in Proposition 5.1. Then it is easy to observe that

$$
(-\Delta)^s U + VU - U_p^* = c \sum_{j=1}^{k} Z_j.
$$

Hence we need to find suitable $r$ such that the coefficient $c = 0$. The problem can be formulated variationally as follows.

**Lemma 5.1.** Let $F(r) = J(U) = J(W + \Phi(r))$, then $c = 0$ if and only if $F'(r) = 0$.

**Proof.** Assume that $\bar{U}$ is the unique $s$-harmonic extension of $U = W + \Phi(r)$, then the well-known computation by Caffarelli and Silvestre [8] shows that

$$
F(r) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} |\nabla \bar{U}|^2 y^{1-2s} \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} V(|x|)U^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} U_p^{p+1}.
$$

So with $\partial_r U = \partial_r W + \Phi'(r) = \sum_{i=1}^{k} Z_i + \Phi'(r)$, (4.3), (4.4) and Proposition 5.3,

$$
F'(r) = \int_{\mathbb{R}^{N+1}} \nabla \bar{U} \cdot \nabla (\partial_r \bar{U}) y^{1-2s} + \int_{\mathbb{R}^N} V(|x|)U \partial_r U - \int_{\mathbb{R}^N} U_p^{p+1} \partial_r U
$$

$$
= \int_{\mathbb{R}^N} \left((-\Delta)^s U + V(|x|)U - U_p^*\right) \partial_r U = c \sum_{j=1}^{k} \int_{\mathbb{R}^N} Z_j \partial_r U
$$

$$
= c \sum_{i,j=1}^{k} \int_{\mathbb{R}^N} Z_i Z_j + c \sum_{j=1}^{k} \int_{\mathbb{R}^N} Z_j \Phi'(r)
$$

$$
= c \left(\frac{k}{N} \int_{\mathbb{R}^N} (w'(|x|))^2 \, dx + kO((\frac{k}{r})^{N+2s}) + O(\|\Phi'(r)\|_* \sum_{j=1}^{k} \int_{\mathbb{R}^N} |Z_j| \rho)\right)
$$

$$
= ck \left(\frac{1}{N} \int_{\mathbb{R}^N} (w'(|x|))^2 \, dx + o(1)\right),
$$

with $k$ large enough. The proof is finished.
Now our task is to find a critical of the functional $F(r)$. We have the following expansion of $F(r)$.

**Proposition 5.2.** There exists $k_0$ such that for any $k \geq k_0, r \in I_0$, the following expansion holds

$$F(r) = k \left[ A_1 + B_1 \frac{1}{r^m} - B_2 k^{N+2s} \frac{1}{r^{N+2s}} + o \left( r^{-m} \right) \right]$$

(5.3)

where $A_1, B_1, B_2$ are universal positive constants defined in Proposition 6.1 and the interval $I_0$ is given by

$$I_0 = \left[ \frac{1}{C_0} k \frac{N+2s}{N+2s}, C_0 k \frac{N+2s}{N+2s} \right].$$

**Proof.** Since $U = W + \varphi$, let us expand $J(U)$ at $W$ and get that

$$J(U) = \frac{1}{2} \int_{\mathbb{R}^N} U(-\Delta)^s U + VU^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} U^p + 1
$$

$$= J(W) + \int_{\mathbb{R}^N} [(-\Delta)^s U + VU - U^p] \varphi - \frac{1}{2} \int_{\mathbb{R}^N} \varphi (-\Delta)^s \varphi + V\varphi^2 - pW^{p-1} \varphi^2
$$

$$- \frac{1}{p+1} \int_{\mathbb{R}^N} \left( (W + \varphi)^{p+1} - W^{p+1} - (p + 1)W^p \varphi - \frac{p(p+1)}{2} W^{p-1} \varphi^2 \right)
$$

$$+ \int_{\mathbb{R}^N} (U^p - W^p - pW^{p-1} \varphi) \varphi. $$

Since $\int_{\mathbb{R}^N} \varphi Z_j = 0$ for all $j = 1, \ldots, k$, the second term disappears. From Remark 2.2, we have

$$\left| \int_{\mathbb{R}^N} (\varphi (-\Delta)^s \varphi + V\varphi^2 - pW^{p-1} \varphi^2) \right|
$$

$$= \int_{\mathbb{R}^N} |E + N(\varphi)| |\varphi| \leq C k (\|E\|_* + \|N(\varphi)\|_*) \|\varphi\| \int_{\Omega_1} \rho^2
$$

$$\leq C k (\|E\|_* + \|N(\varphi)\|_*) \|\varphi\|_* = ko(r^{-m}).$$

Similarly, it is easy to see that

$$\left| \int_{\mathbb{R}^N} \left( (W + \varphi)^{p+1} - W^{p+1} - (p + 1)W^p \varphi - \frac{p(p+1)}{2} W^{p-1} \varphi^2 \right) \right|
$$

$$\leq C \int_{\mathbb{R}^N} |\varphi|^{\min\{p+1,3\}} \leq C k \|\varphi\|_{\max\{p+1,3\}} \int_{\Omega_1} \rho^2 = ko \left( r^{-m} \right),$$

and

$$\int_{\mathbb{R}^N} (U_p - W^p - pW^{p-1} \varphi) \varphi = O \left( \int_{\mathbb{R}^N} W^{p-1} \varphi^2 \right) = kO (\|\varphi\|^2) = ko(r^{-m}).$$

The proof is completed. 

**Proof of Theorem 1.2.** Now consider the following problem $\max_{r \in I_0} F(r)$. We want to verify that the maximum points lie in the interior of the interval $I_0$. For this, let

$$r_0 = \left( \frac{(N + 2s) B_2}{m B_1} \right) \frac{1}{k^{N+2s-m}} \in I_0$$

for large positive constant $C_0$. It is easy to show that for large $k$,

$$F(r_0) = k A_1 + k^{1 - \frac{N+2s}{N+2s-m}} B_1 \left( \frac{m B_1}{(N+2s) B_2} \right) \frac{m}{N+2s-m} \frac{N+2s-m}{N+2s} \frac{1}{N+2s} + o \left( k^{1 - \frac{N+2s}{N+2s-m}} \right).$$

On the other hand, it always holds that

$$F \left( \frac{1}{C_0} k \frac{N+2s}{N+2s-m} \right) = k A_1 + k^{1 - \frac{N+2s}{N+2s-m}} (B_1 C_0 - B_2 C_0^{N+2s}) + o \left( k^{1 - \frac{N+2s}{N+2s-m}} \right)
$$

$$< k A_1 + k^{1 - \frac{N+2s}{N+2s-m}} B_1 \left( \frac{m B_1}{(N+2s) B_2} \right) \frac{m}{N+2s-m} \frac{N+2s-m}{N+2s} \frac{1}{N+2s}$$

$$= F(r_0).$$
Recall that Proposition 6.1. It holds that

\[ F(C_0k^{\frac{N+2s}{N+2m}}) = kA_1 + k^{1-\frac{(N+2s)m}{N+2m}} \left( B_1 \frac{B_0}{C_0^m} - \frac{B_2}{C_0^{N+2s}} \right) + \omega \left( k^{1-\frac{(N+2s)m}{N+2m}} \right) \]

\[ < kA_1 + k^{1-\frac{(N+2s)m}{N+2m}} B_1 \frac{B_0}{C_0^m} + \omega \left( k^{1-\frac{(N+2s)m}{N+2m}} \right) \]

\[ < kA_1 + k^{1-\frac{(N+2s)m}{N+2m}} B_1 \left( \frac{mB_1}{(N+2s)B_2} \right)^{\frac{m}{N+2s-m}} \frac{N+2s-m}{2(N+2s)} , \]

if we choose \( C_0 \) large enough such that

\[ B_1C_0^m - B_2C_0^{N+2s} < 0 , \quad \frac{B_1}{C_0^m} < \frac{mB_1}{(N+2s)B_2} \left( \frac{N+2s-m}{2(N+2s)} \right) \]

which can be done because of \( 0 < m < N + 2s \). If we let \( F(r) = \max_{r \in I_0} F(r) \), then \( r_1 \) is an interior point of \( I_0 \) and thus \( F'(r_1) = 0 \), which gives a critical point of \( F(r) \).

Therefore Lemma 6.1 implies Theorem 1.2.

6 Appendix: Energy expansion

In this section, the important expansion of the energy at \( W \) is given. First we list the following lemmas, whose proofs can be found in [23].

**Lemma 6.1.** For any \( \alpha > 0 \),

\[ \sum_{j=1}^{k} \frac{1}{|x_j - x|^{\alpha}} \leq C + C \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{\alpha}} , \quad \forall \ x \in \mathbb{R}^N \]

where \( C > 0 \) is a constant independent of \( k \).

**Lemma 6.2.** For any constant \( 0 < \sigma < N - 2 \), there is a constant \( C > 0 \), such that

\[ \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} \mathrm{d}z \leq \frac{C}{(1+|y|)^\sigma} . \]

The proof of this lemma, a more general one actually, can also be found in [22, 28].

**Lemma 6.3.** For any constant \( 0 \leq \sigma \leq \min\{\alpha, \beta\} \), there is a constant \( C > 0 \) such that, for any \( i \neq j \),

\[ \frac{1}{(1+|x-q_i|)^{\alpha}} \leq \frac{1}{(1+|x-q_j|)^{\beta}} \left[ \frac{1}{(1+|x-q_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|x-q_j|)^{\alpha+\beta-\sigma}} \right] . \]

The proof of the above lemma may be found in [23].

Next we focus on the expansion of energy at \( W \). Recall the positive least energy solution \( w \) to (1.4).

**Proposition 6.1.** It holds that

\[ J(W) = k \left[ A_1 + \frac{B_1}{r^m} - \frac{B_2k^{N+2s}}{r^{N+2s}} + o \left( \frac{k}{r} \right)^{N+2s} \right] , \tag{6.1} \]

where

\[ A_1 = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} \omega^{p+1} \mathrm{d}x , \quad B_1 = \frac{a}{2} \int_{\mathbb{R}^N} \omega^2 \mathrm{d}x \]

and \( B_2 \) are all positive numbers.

**Proof.** Recall that

\[ q_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right) , \quad j = 1, \ldots, k , \]

where \( 0 \) is the zero vector in \( \mathbb{R}^N \), \( r \in \left[ \frac{1}{C_0} k^{\frac{N+2s}{N+2m}} , C_0 k^{\frac{N+2s}{N+2m}} \right] \) for a large positive constant \( C_0 \). By direct calculus, we get that

\[ |q_1 - q_j| = 2r \sin \frac{(j-1)\pi}{k} , \quad 0 < c' \leq \frac{\sin \frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}} \leq c'' , \]

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from which we can find that for any $\ell > 1$,
\[
\sum_{j=2}^{k} \frac{1}{|q_j - q_1|^{\ell}} = \frac{1}{(2r)^{\ell}} \sum_{j=2}^{k} \left( \frac{1}{\sin (\frac{\pi}{2} \frac{j-1}{k})} \right)^{\ell} = C_\ell \left( \frac{k}{r} \right)^{\ell} + o \left( \left( \frac{k}{r} \right)^{\ell} \right),
\]  
where $C_\ell > 0$.

Denote
\[
W_j(x) = w(x - q_j), \quad j = 1, \ldots, k, \quad W(x) = \sum_{j=1}^{k} W_j(x).
\]

Then we have
\[
J(W_1) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ w(x - q_1) (-\Delta)^s w(x - q_1) + V(|x|) w^2(x - q_1) \right] dx - \frac{1}{p+1} \int_{\mathbb{R}^N} w^{p+1}(x - q_1) dx
\]
\[
= J_1(w) + \frac{1}{2} \int_{\mathbb{R}^N} (V(|x|) - 1) w^2(x - q_1) dx
\]
\[
= \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N} w^{p+1} dx + \frac{1}{2} \int_{\mathbb{R}^N} (V(|x - q_1|) - 1) w^2(x) dx.
\]

For any $\alpha > 0$, $x \in B_{r/2}(0)$, since
\[
\frac{1}{|x - q_1|^{\alpha}} = \frac{1}{|q_1|^{\alpha}} \left[ 1 + O \left( \frac{|x|}{|q_1|} \right) \right],
\]
we deduce that
\[
\frac{1}{2} \int_{\mathbb{R}^N} (V(|x - q_1|) - 1) w^2(x) dx = \left( \int_{\{x ||x|<\frac{q_1}{2}\}} + \int_{\{x ||x|\geq\frac{q_1}{2}\}} \right) (V(|x - q_1|) - 1) w^2(x) dx
\]
\[
= \frac{1}{2} \int_{\{||x||<\frac{q_1}{2}\}} w^2(x) dx + O \left( \int_{\{||x||\geq\frac{q_1}{2}\}} w^2(x) dx \right)
\]
\[
= \frac{1}{2} \frac{a}{|q_1|^{m}} \int_{\{||x||<\frac{q_1}{2}\}} w^2(x) dx + O \left( r^{-(m+1)} \int_{\{||x||<\frac{q_1}{2}\}} |x| w^2(x) dx + r^{-(N+4s)} \right) + o(r^{-m})
\]
\[
= B_1 + o(r^{-m}) + o(r^{-m}) = B_1 + o(r^{-m}),
\]
where the positive constant $B_1 = \frac{a}{2} \int_{\mathbb{R}^N} w^2(x) dx$. Recall that
\[
\Omega_j = \left\{ y = (y', y'' \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \frac{y'}{|y'|}, y'' \geq \frac{\pi}{k} \right\}.
\]

By symmetry, we can deduce that
\[
J(W) = \frac{1}{2} \int_{\mathbb{R}^N} W(-\Delta)^s W + V(|x|) W^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} W^{p+1}
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} W((-\Delta)^s W + W) + \frac{1}{2} \int_{\mathbb{R}^N} (V(|x|) - 1) W^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} W^{p+1}
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} \sum_{j=1}^{k} W_j^{p+1} + \frac{k}{2} \int_{\Omega} (V(|x|) - 1) W^2 - \frac{k}{p+1} \int_{\Omega} W^{p+1}
\]
\[
= \frac{k}{2} \int_{\mathbb{R}^N} W_j^{p+1} + \frac{k}{2} \sum_{j=1}^{k} \int_{\mathbb{R}^N} W_j W_j + \frac{k}{2} \int_{\Omega} (V(|x|) - 1) W^2 - \frac{k}{p+1} \int_{\Omega} W^{p+1}.
\]

Now let us do the computations term by term.

With (6.3) and (1.11) at hand, we find that
\[
\sum_{j=2}^{k} \int_{\mathbb{R}^N} W_j^{p+1} = \sum_{j=2}^{k} \int_{\mathbb{R}^N} w^p(|x - q_1|) w(|x - q_j|) dx
\]
\[
\begin{align*}
&= \sum_{j=2}^{k} \int_{\mathbb{R}^N} w^p(|x|)w(|x + q_j|)dx \\
&= \sum_{j=2}^{k} \int_{|x||x| \leq |q_j - q_j|/2} w^p(|x|) \left[ \frac{A}{|x + q_j - q_j|^{N-2s}} + o\left(\frac{1}{|x + q_j - q_j|^{N-2s}}\right) \right] dx \\
&\quad + \sum_{j=2}^{k} \int_{|x||x| \geq |q_j - q_j|/2} O\left(\frac{1}{|q_j - q_j|^{(N-2s)\alpha}}\right) w(|x + q_j|)dx \\
&= \sum_{j=2}^{k} \frac{A}{|q_j - q_j|^{N-2s}} \int_{\mathbb{R}^N} w^p(x)dx + o\left(\sum_{j=2}^{k} \frac{1}{|q_j - q_j|^{N-2s}}\right) + O\left(\sum_{j=2}^{k} \frac{1}{|q_j - q_j|^{(N-2s)\alpha}}\right) \\
&= \sum_{j=2}^{k} \frac{\tilde{B}_2}{|q_j - q_j|^{N-2s}} + o\left(\frac{1}{r^{N-2s}}\right) (6.5)
\end{align*}
\]

where the positive constant \( \tilde{B}_2 = A \int_{\mathbb{R}^N} w^p. \)

For any \( x \in \Omega_1 \), it is obvious that \( |x - q_j| \geq |x - q_1| \) and \( |x - q_j| \geq |q_j - q_1|/2 \) for \( j = 2, \ldots, k \). Then for any \( 0 \leq \alpha \leq N - 2s \),
\[
W_j(x) \leq \frac{C}{(1 + |x - q_j|)^{N+2s}} \leq \frac{C}{(1 + |x - q_1|)^{N+2s}} |q_j - q_1|^{N+2s-\alpha}.
\]

Hence for any \( 0 \leq \alpha < N + 2s - 1 \),
\[
\sum_{j=2}^{k} W_j(x) = O\left(\frac{1}{(1 + |x - q_1|)^{N+2s}} \sum_{j=2}^{k} \frac{1}{|q_j - q_j|^{N+2s-\alpha}}\right) = O\left(\frac{1}{(1 + |x - q_1|)^{N+2s}} \alpha \left(\frac{1}{r^{N+2s}}\right)\right). (6.6)
\]

Now we can deduce that
\[
\frac{1}{2} \int_{\Omega_1} (V(|x|) - 1) W^2 = \frac{1}{2} \int_{\Omega_1} (V(|x|) - 1) \left( W_1 + \sum_{j=2}^{k} W_j \right)^2 \\
= \frac{1}{2} \int_{\Omega_1} (V(|x|) - 1) W_1^2 + O\left(\int_{\Omega_1} |V(|x|) - 1| W_1 \sum_{j=2}^{k} W_j + \left(\frac{k}{r}\right)^{2N+4s-2\alpha} \int_{\Omega_1} \frac{1}{(1 + |x - q_1|)^{2s}}\right) \\
= B_1 \frac{\rho^m}{r^m} + o(r^{-m}) + O\left(\left(\frac{k}{r}\right)^{N+2s} \int_{\Omega_1} |V(|x|) - 1| W_1 + \left(\frac{k}{r}\right)^{N+3s}\right) \\
= B_1 \frac{\rho^m}{r^m} + o\left(\frac{r^{-m}}{r^{N+2s}} + \left(\frac{k}{r}\right)^{N+2s} O\left(\int_{\Omega_1} |V(|x|) - 1| w(x - q_1)\right)\right) \\
= B_1 \frac{\rho^m}{r^m} + o\left(\frac{r^{-m}}{r^{N+2s}} + \left(\frac{k}{r}\right)^{N+2s} O\left(\int_{\Omega_1} |V(|x|) - 1| w(x - q_1)\right)\right) \\
= B_1 \frac{\rho^m}{r^m} + o\left(\frac{r^{-m}}{r^{N+2s}} + \left(\frac{k}{r}\right)^{N+2s} O\left(\frac{1}{r^m} + \frac{1}{r^m}\right)\right), (6.7)
\]

where we choose \( \alpha = \frac{N+2s}{2}. \)
For the last term in the energy $J(W)$, it is not difficult to check that

$$
\frac{1}{p+1} \int_{\Omega_1} W^{p+1} = \frac{1}{p+1} \int_{\Omega_1} \left( W_1 + \sum_{j=2}^{k} W_j \right)^{p+1}
$$

$$
= \frac{1}{p+1} \int_{\Omega_1} W_1^{p+1} + \int_{\Omega_1} W_1^{p} \sum_{j=2}^{k} W_j^{p} + O \left( \int_{\Omega_1} W_1^{p-1} \left( \sum_{j=2}^{k} W_j \right)^2 \right) + O \left( \int_{\Omega_1} \left( \sum_{j=2}^{k} W_j \right)^{p+1} \right)
$$

$$
= \frac{1}{p+1} \int_{\Omega_1} W_1^{p+1} + \int_{\Omega_1} W_1^{p} \sum_{j=2}^{k} W_j^{p} + O \left( \sum_{j=2}^{k} \frac{1}{|q_j - q_1|^{N+2s}} \right)^{p+1} \tag{6.8}
$$

$$
+ O \left( \frac{1}{|q_1 - q_1|^{N+2s} - \frac{N+2s}{p+1}} \right)^{p+1} + O \left( \frac{1}{r^{N+4s}} \right).
$$

Combining (6.5), (6.7) and (6.8), we get the desired expansion of energy

$$
J(W) = k \left[ A_1 + \frac{B_1}{r^m} - \frac{1}{2} \sum_{j=2}^{k} \frac{\tilde{B}_2}{|q_1 - q_j|^{N+2s}} + O \left( r^{-m} + \left( \frac{k}{r} \right)^{N+2s} \right) \right], \tag{6.9}
$$

where

$$
A_1 = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^n} w^{p+1}(x)dx, \quad B_1 = \frac{a}{2} \int_{\mathbb{R}^n} w^2(x)dx, \quad \tilde{B}_2 = A \int_{\mathbb{R}^n} w^p.
$$

With (6.2) at hand, we finished the proof. \qed

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