An infinite family of prime knots with a certain property for the clasp number

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Abstract

The clasp number $c(K)$ of a knot $K$ is the minimum number of clasp singularities among all clasp disks bounded by $K$. It is known that the genus $g(K)$ and the unknotting number $u(K)$ are lower bounds of the clasp number, that is, $\max\{g(K), u(K)\} \leq c(K)$. Then it is natural to ask whether there exists a knot $K$ such that $\max\{g(K), u(K)\} < c(K)$. In this paper, we prove that there exists an infinite family of prime knots such that the question above is affirmative.

1 Introduction

It is known that every knot in $S^3$ bounds a singular disk in $S^3$ whose singular set consists of only clasp singularities as illustrated in Figure 1. We call such a singular disk a clasp disk of the knot. Let $K$ be a knot in $S^3$, and $D$ a clasp disk of $K$. Let $c(D)$ denote the number of clasp singularities in $D$. Then the clasp number of $K$ is $c(K) = \min\{c(D) \mid D \text{ is a clasp disk of } K\}$. We refer the reader to [4, 6, 7, 11, 12, 13, 14, 18] for related topics of the clasp number. In this paper, we suppose that every link is in $S^3$ and oriented, and the notation of prime knots follows Rolfsen’s book [16].

For a knot $K$, let $g(K)$ and $u(K)$ be the genus and the unknotting number of $K$ respectively. In [18], T. Shibuya proved that the genus and the unknotting number are lower bounds of the clasp number. In other words, for a knot $K$ we have $\max\{g(K), u(K)\} \leq c(K)$. Most of the prime knots with up to 10 crossings satisfy the equality above (cf. Appendix). Then it is natural to ask the following question.

Question 1.1. Does there exist a prime knot $K$ such that $\max\{g(K), u(K)\} < c(K)$?

For an integer $n$, let $K_n$ be the knot as illustrated in Figure 2. Here, an integer in the rectangle denotes the number of half twists. We note that $K_0 = 3_1 \# 4_1$ (Figure 3) and $K_1 = 10_{97}$ (Figure 4). Note that all the knots $K_n$ are distinct one another (Proposition 3.1). The main result of this paper is the following theorem. It is the affirmative answer of Question 1.1.
Theorem 1.2. If $n$ is odd, then the knots $K_n$ are prime and we have $\max\{g(K_n), u(K_n)\} < c(K_n)$.

In Appendix, we show a table for the clasp numbers of prime knots with up to 10 crossings. From the table, we raise the following question concerning an upper bound of the clasp number.

Question 1.3. Let $cr(K)$ be the crossing number of a knot $K$. For any non-trivial knot $K$, does the following inequality hold?

$$c(K) \leq \left\lfloor \frac{cr(K) - 1}{2} \right\rfloor,$$

where $[r]$ denotes the integer part of a rational number $r$.

Question 1.3 is affirmative for prime knots with up to 10 crossings (cf. Appendix) and the knots $K_n$ (Proposition 3.6). In Section 2 we calculate the Conway polynomial of a knot $K$ with $c(K) \leq 2$, giving an alternative proof of Morimoto’s result [14] (Lemma 2.1). (Since it seems to be hard to obtain a reference [14], it is also our purpose to expose it to the reader.) Moreover we give a sufficient condition for a knot $K$ to satisfy $c(K) \geq 3$. In Section 3 we investigate the knots $K_n$ and prove Theorem 1.2 and we also prove that Question 1.3 is affirmative for the knots $K_n$. 

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2 A sufficient condition for a knot $K$ to satisfy $c(K) \geq 3$

In [14], K. Morimoto calculated the Alexander module of a knot from a clasp disk. In this section, we calculate the Conway polynomial of a knot $K$ with $c(K) \leq 2$, giving an alternative proof of Morimoto’s result [14], and then we give a sufficient condition for a knot $K$ to satisfy $c(K) \geq 3$.

First, we prove the following lemma.

**Lemma 2.1.** (cf. [14]) Let $K$ be a knot with $c(K) \leq 2$. The Conway polynomial $\nabla_{K}(z)$ of $K$ is expressed as follows:

$$\nabla_{K}(z) = (b_{1}b_{2} + \varepsilon b_{3}(b_{3} + \delta))z^{4} + (b_{1} + b_{2} - \varepsilon\delta)z^{2} + 1$$

$(b_{1}, b_{2}, b_{3} \in \mathbb{Z}, \varepsilon \in \{\pm 1\}, \delta \in \{0, 1\})$.

**Proof.** Let $K$ be a knot with $c(K) \leq 2$. Then there exists a clasp disk $D$ of $K$ such that $c(D) = 2$. We may assume that $D$ is a surface which is the union of a disk $B_{0}$ and two clamping bands $B_{1}$ and $B_{2}$ (see Figure 5). Here, a clamping band means a pair of embedded 2-disks in $S^{3}$ with a clasp singularity as illustrated in Figure 6. The sign of a clamping band is defined as the linking number of the Hopf link bounding the clamping band. For each $i (i = 1, 2)$, let $c_{i}$ be the clasp singularity in $D$ corresponding to the clamping band $B_{i}$. We define the sign $\varepsilon_{i} \in \{\pm 1\}$ of the clasp singularity $c_{i}$ as the sign of $B_{i}$. For each $\varepsilon \in \{\pm 1\}$, an (\varepsilon)-Hopf band is an annulus in $S^{3}$ whose boundary is a Hopf link with linking number $\varepsilon$. Let $F$ be the oriented surface obtained from $D$ by the following operation:

$(*)$ For each $i (i = 1, 2)$, replacing the neighborhood of the clasp singularity $c_{i}$ with a plumbing of an ($\varepsilon_{i}$)-Hopf band, denoted by $H_{i}$, as illustrated in Figure 7.
Note that the surface $F$ is a genus two Seifert surface of $K$. For example, the Seifert surface in Figure 8 is obtained from the clasp disk in Figure 5 by the operation ($\ast$). We take a homological basis $\{[\alpha_1],[\alpha_2],[\beta_1],[\beta_2]\}$ of $H_1(F;\mathbb{Z})$ as follows (see Figure 8):

1. For each $i$ ($i = 1,2$), $\alpha_i$ is a loop corresponding to the core of the clasping band $B_i$ and $\beta_i$ is the core loop of the Hopf band $H_i$.

2. Orientations of $\alpha_i$ and $\beta_i$ are chosen so that the intersection number $[\beta_i] \cdot [\alpha_i]$ is 1.

![Figure 7: A construction of a Seifert surface $F$ from a clasp disk $D$](image)

![Figure 8: The Seifert surface obtained from the clasp disk in Figure 5 and a homological basis $\{[\alpha_1],[\alpha_2],[\beta_1],[\beta_2]\}$](image)

Let $a_{ij} = \text{lk}(\alpha_i, \alpha_j^\perp)$ be the linking number of $\alpha_i$ and $\alpha_j^\perp$, where $\alpha_j^\perp$ is a loop obtained by pushing $\alpha_j$ to the positive normal direction of $F$. The Seifert matrix $V$ of $K$ obtained from the homological basis $\{[\alpha_1],[\alpha_2],[\beta_1],[\beta_2]\}$ is calculated as follows:

$$V = \begin{pmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 \\
-1 & 0 & -\varepsilon_1 & 0 \\
0 & -1 & 0 & -\varepsilon_2
\end{pmatrix}.$$
Therefore the Alexander polynomial $\Delta_K(t)$ of $K$ is calculated as follows:

$$
\Delta_K(t) \doteq \det(tV - V^T)
$$

$$
= \det \begin{pmatrix}
  a_{11}(t-1) & a_{12}t - a_{21} & 1 & 0 \\
  a_{21}t - a_{12} & a_{22}(t-1) & 0 & 1 \\
  -t & 0 & -\varepsilon_1(t-1) & 0 \\
  0 & -t & 0 & -\varepsilon_2(t-1)
\end{pmatrix}
$$

$$
= \det \begin{pmatrix}
  \varepsilon_1a_{11}(t-1)^2 - t & \varepsilon_1(t-1)(a_{12}t - a_{21}) & 0 & 0 \\
  \varepsilon_2(t-1)(a_{21}t - a_{12}) & \varepsilon_2a_{22}(t-1)^2 - t & 0 & 0 \\
  0 & 0 & (\varepsilon_1 - \varepsilon_2)(t-1)^2 & 0 \\
  0 & 0 & 0 & (\varepsilon_1 - \varepsilon_2)(t-1)^2
\end{pmatrix}
$$

$$
= (\varepsilon_1a_{11}(t-1)^2 - t)(\varepsilon_2a_{22}(t-1)^2 - t) - \varepsilon_1\varepsilon_2(t-1)^2(a_{12}t - a_{21})(a_{21}t - a_{12})
$$

$$
= \varepsilon_1\varepsilon_2(a_{11}a_{22} - a_{12}a_{21})(t-1)^4 + (\varepsilon_1a_{11} - \varepsilon_2a_{22} + \varepsilon_1\varepsilon_2(a_{21} - a_{12})^2)t(t-1)^2 + t^2.
$$

(The symbol $\doteq$ means “is equal to, up to multiplication by a unit of $\mathbb{Z}[t, t^{-1}]$”, and $V^T$ means the transpose of $V$.) Note that if $\alpha_1 \cap \alpha_2 \neq \emptyset$, then we may assume that orientations of $\alpha_1$ and $\alpha_2$ are chosen so that the intersection number $|\alpha_1| \cdot |\alpha_2|$ is 1 (see Figure 2). We put $b_1, b_2, b_3, \varepsilon$ and $\delta$ as follows:

$$
b_1 := -\varepsilon_1a_{11}, \ b_2 := -\varepsilon_2a_{22}, \ b_3 := a_{12},
$$

$$
\varepsilon := -\varepsilon_1\varepsilon_2 \text{ and } \delta := a_{21} - a_{12} = \begin{cases} 0 & (\alpha_1 \cap \alpha_2 = \emptyset), \\ 1 & (\alpha_1 \cap \alpha_2 \neq \emptyset).
\end{cases}
$$

By these substitutions and $\delta^2 = \delta$, we have the following:

$$
\Delta_K(t) \doteq (b_1b_2 + \varepsilon b_3(b_3 + \delta))(t-1)^4 + (b_1 + b_2 - \varepsilon\delta)t(t-1)^2 + t^2
$$

$$
(\forall b_1, b_2, b_3 \in \mathbb{Z}, \ \varepsilon \in \{\pm 1\}, \ \delta \in \{0, 1\}).
$$

For the Conway polynomial $\nabla_K(z)$ of $K$, since $\nabla_K(t^{\frac{1}{4}} - t^{-\frac{1}{4}}) = \det(t\frac{1}{2}V - t^{-\frac{1}{2}}V^T) = t^{-2}\det(tV - V^T)$ and $z^2 = \frac{(t-1)^2}{t}$, we obtain the following:

$$
\nabla_K(z) = (b_1b_2 + \varepsilon b_3(b_3 + \delta))z^4 + (b_1 + b_2 - \varepsilon\delta)z^2 + 1.
$$

\[\square\]

We mention that if a knot $K$ has $c(K) = 1$, then we obtain $\nabla_K(z) = b_1z^2 + 1$ by substitutions $b_2 = 0, b_3 = 0$ and $\delta = 0$.

By Lemma 2.1, we obtain the following.

**Proposition 2.2.** Suppose that a knot $K$ has $\nabla_K(z) = m_4z^4 + m_2z^2 + 1$. If $m_4 \equiv 3 \pmod{8}$ and $m_2 \equiv 2 \pmod{4}$, then $c(K) \geq 3$.

**Proof.** Suppose that a knot $K$ has $c(K) \leq 2$ and $\nabla_K(z) = m_4z^4 + m_2z^2 + 1$ with $m_4 \equiv 3 \pmod{8}$ and $m_2 \equiv 2 \pmod{4}$. By Lemma 2.1 the following holds:

$$
m_4 = b_1b_2 + \varepsilon b_3(b_3 + \delta) \text{ and } m_2 = b_1 + b_2 - \varepsilon\delta
$$
for some $b_1, b_2, b_3 \in \mathbb{Z}$, $\varepsilon \in \{\pm 1\}$ and $\delta \in \{0, 1\}$.

The case of $\delta = 0$: We obtain $m_4 = b_1 b_2 + \varepsilon b_3^2$ and $m_2 = b_1 + b_2$. Since $b_1 + b_2$ is even, $b_1 - b_2$ is also even. Therefore,

$$d := \left(\frac{b_1 - b_2}{2}\right)^2 = \frac{1}{4} m_2^2 - m_4 + \varepsilon b_3^2 \equiv 6 + \varepsilon b_3^2 \pmod{8}. $$

Then $d \equiv 2, 5, 6$ or $7 \pmod{8}$, and $d$ cannot be a square integer. This is a contradiction.

The case of $\delta = 1$: We obtain $m_4 = b_1 b_2 + \varepsilon b_3 (b_3 + 1)$ and $m_2 = b_1 + b_2 - \varepsilon$.

$$d' := (b_1 - b_2)^2 = (m_2 + \varepsilon)^2 - 4 (m_4 - \varepsilon b_3 (b_3 + 1))$$

$$\equiv 1 - 4 \pmod{8}$$

$$\equiv 5 \pmod{8}.$$ 

Then $d'$ cannot be a square integer. This is a contradiction. \qed

## 3 Proof of Theorem 1.2

In this section, first we calculate the Conway polynomial (Proposition 3.1) and the Jones polynomial (Proposition 3.2) of the knot $K_n$ by using the skein relation. Next, we provide a lemma (Lemma 3.3) and then we prove Theorem 1.2.

Let $J$ be the oriented link as illustrated in Figure 9. It is obtained from the knot $K_n$ by smoothing one of $|2n-1|$ crossings in the rectangle in Figure 2. We note that it is equivalent to $L9a41\{1\}$ in LinkInfo table 3.

![Figure 9: The link $J = L9a41\{1\}$](image)

### 3.1 The Conway polynomial of the knot $K_n$

The Conway polynomial $\nabla_L(z) \in \mathbb{Z}[z]$ of an oriented link $L$ is characterized by the following skein relation:

(i) $\nabla_{\text{unknot}}(z) = 1$ \hspace{1cm} (ii) $\nabla_{L_+}(z) - \nabla_{L_-}(z) = z \nabla_{L_0}(z)$,

where three oriented links $L_+, L_-$ and $L_0$ coincide except in the neighborhood of a point as illustrated in Figure 10. We call $(L_+, L_-, L_0)$ a skein triple. We can calculate the Conway polynomial $\nabla_{K_n}(z)$ of the knot $K_n$ as follows.

**Proposition 3.1.** The Conway polynomial $\nabla_{K_n}(z)$ of the knot $K_n$ is as follows:

$$\nabla_{K_n}(z) = -(4n + 1)z^4 + 2nz^2 + 1.$$
Proof. The case of $n = 0$: Since $K_0 = 3_1 \# 4_1$,
\[ \nabla_{K_0}(z) = \nabla_{3_1}(z)\nabla_{4_1}(z) = (z^2 + 1)(-z^2 + 1) = -z^4 + 1. \]

The case of $n > 0$: The Conway polynomial of $J$ is $\nabla_J(z) = -4z^3 + 2z$. By considering a skein triple $(K_n, K_{n-1}, J)$, the Conway polynomial of $K_n$ can be calculated inductively as follows:
\[
\nabla_{K_n}(z) = \nabla_{K_{n-1}}(z) + z\nabla_J(z) \\
= (\nabla_{K_{n-2}}(z) + z\nabla_J(z)) + z\nabla_J(z) \\
\cdots \\
= \nabla_{K_0}(z) + nz\nabla_J(z) \\
= (-z^4 + 1) + nz(-4z^3 + 2z) = -(4n + 1)z^4 + 2nz^2 + 1.
\]

Similarly, we can calculate it in the case of $n < 0$. 

By Proposition 3.1, we see that all the knots $K_n$ are distinct one another.

3.2 The Jones polynomial of $K_n$

The Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ of an oriented link $L$ is characterized by the following skein relation:
\[
(i) \ V_{\text{unknot}}(t) = 1 \quad (ii) \ t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}(t)
\]
for any skein triple $(L_+, L_-, L_0)$. We can calculate the Jones polynomial $V_{K_n}(t)$ of the knot $K_n$ as follows.

Proposition 3.2. The Jones polynomial $V_{K_n}(t)$ of the knot $K_n$ is as follows:
\[
V_{K_n}(t) = \begin{cases} 
  t^{-1} - 1 + 2t - 3t^2 + 3t^3 - 2t^4 + 2t^5 - t^6 & (n = 0) \\
  t^{2n}V_{K_0}(t) + \sigma t^\sigma (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_J(t)(1 + t^{2\sigma} + \cdots + t^{2(n-\sigma)}) & (n \neq 0)
\end{cases}
\]
where $\sigma := \frac{n}{|n|} \in \{\pm 1\}$ for $n \neq 0$.

Proof. The case of $n = 0$: Since $K_0 = 3_1 \# 4_1$,
\[
V_{K_0}(t) = V_{3_1}(t)V_{4_1}(t) = (t + t^3 - t^4)(t^{-2} - t^{-1} + 1 - t + t^2) \\
= t^{-1} - 1 + 2t - 3t^2 + 3t^3 - 2t^4 + 2t^5 - t^6.
\]
The case of $n > 0$: By considering a skein triple $(K_n, K_{n-1}, J)$, the Jones polynomial of $K_n$ can be calculated inductively as follows:

$$V_{K_n}(t) = t^2 V_{K_{n-1}}(t) + t\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)V_J(t)$$

$$= t^2\left(t^2 V_{K_{n-2}}(t) + t\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)V_J(t)\right) + t\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)V_J(t)$$

$$\cdots$$

$$= t^{2n} V_{K_0}(t) + t\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)V_J(t)\left(1 + t^2 + \cdots + t^{2(n-1)}\right).$$

Similarly, we can calculate it in the case of $n < 0$.

\[\square\]

### 3.3 Proof of Theorem 1.2

First, we prove the following lemma.

**Lemma 3.3.** For the knot $K_n$ ($n \in \mathbb{Z}$), we have the following:

1. $g(K_n) = 2$.
2. $u(K_n) \leq 2$.
3. $2 \leq c(K_n) \leq 4$. In particular, $c(K_n) \geq 3$ for odd $n$.
4. $K_n$ is prime for $n \neq 0, -4$.

**Proof.** (1) Since we can obtain a genus two Seifert surface of $K_n$ as illustrated in Figure 11, we have $g(K_n) \leq 2$. Since the degree of $\nabla_{K_n}(z)$ is four (Proposition 3.1), we have $g(K_n) \geq 2$. Therefore we obtain $g(K_n) = 2$.

(2) Since we can unknot the knot $K_n$ by crossing changes at dotted circles in Figure 2, we have $u(K_n) \leq 2$.

(3) By (1) we have $c(K_n) \geq 2$, and by Figure 12 we have $c(K_n) \leq 4$. Suppose that $n$ is odd. Since $\nabla_{K_n}(z) = -(4n + 1)z^4 + 2nz^2 + 1$, it implies that $-(4n + 1) \equiv 3 \pmod{8}$ and $2n \equiv 2 \pmod{4}$. Therefore by Proposition 2.2 we obtain $c(K_n) \geq 3$.

(4) Suppose that $K_n$ is a composite knot. By (1), $K_n$ is decomposed into two genus one knots. Since the degree of the Conway polynomial $\nabla_{K_n}(z)$ of $K_n$ is four, $\nabla_{K_n}(z)$ is decomposed into two Conway polynomials with degree two, that is, $\nabla_{K_n}(z) = (pz^2 + 1)(qz^2 + 1)$ for some integers $p$ and $q$. Therefore we have

$$pq = -(4n + 1) \quad \text{and} \quad p + q = 2n.$$

Then $p$ and $q$ are integral roots of the equation $x^2 - 2nx - (4n + 1) = 0$, and hence the discriminant of the equation is a square integer. Hence there is a non-negative integer $s$ such that $n^2 + 4n + 1 = s^2$. Since it implies that $(n + 2 + s)(n + 2 - s) = 3$, we have $s = 1$, and $n = 0$ and $-4$. $\square$
Remark 3.4. (1) On $u(K_n)$, by using the Nakanishi index (cf. [5]) reduced modulo 2, we can see that $u(K_n) = 2$ for even $n$. The authors conjecture that $u(K_n) = 2$ for all $n$.

(2) On $c(K_n)$, we see that $c(K_0) = 2$ and $c(K_1) = 3$. The authors conjecture that $c(K_n) = 4$ for any integer $n$ except 0 and 1.

(3) On the primeness of $K_n$, the authors conjecture that $K_{-4}$ is also prime.

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that $n$ is odd. By Lemma 3.3, the knot $K_n$ is prime and we have $\max\{g(K_n), u(K_n)\} < c(K_n)$.

We show that Question 1.3 is affirmative for the knots $K_n$.

Lemma 3.5. For the knots $K_n (n \in \mathbb{Z})$, $cr(K_n) = 2n + 8$ for positive $n$, $cr(K_0) = 7$, and $-2n + 7 \leq cr(K_n) \leq -2n + 10$ for negative $n$.

Proof. The case of $n > 0$: Since we can see that $K_n$ has a reduced alternating diagram (see Figure 2), by [15, Theorem 13.5] we have $cr(K_n) = 2n + 8$.

The case of $n = 0$: Since $K_0 = 3_1 \# 4_1$, by [15, Theorem 13.5] we have $cr(K_0) = 7$.

The case of $n < 0$: It is known that the degree of the Jones polynomial of a knot is a lower bound of the crossing number of the knot. The Jones polynomial of $K_n$ is as follows:

$$V_{K_n}(t) = t^{2n}V_{K_0}(t) - t^{-1}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{J}(t)(1 + t^{-2} + \cdots + t^{2(n+1)}),$$

where

$$V_{K_0}(t) = t^{-1} - 1 + 2t - 3t^2 - 3t^3 - 2t^4 + 2t^5 - t^6$$

and

$$V_{J}(t) = -t^{-\frac{1}{2}} + 2t^{-\frac{1}{2}} - 4t^\frac{1}{2} + 6t^2 - 6t^3 + 5t^4 - 6t^5 + 3t^6 - 2t^7 + t^8.$$ 

Therefore we have $cr(K_n) \geq 7 - 2n$. On the other hand, it is easy to see that $cr(K_n) \leq -2n + 10$. 

By Lemma 3.3 (3) and Lemma 3.5 we have the following.

Proposition 3.6. For the knots $K_n (n \in \mathbb{Z})$, the following holds.

$$c(K_n) \leq \left\lfloor \frac{cr(K_n) - 1}{2} \right\rfloor.$$ 

This is a supporting evidence that the question is affirmative.
Appendix

In the table below, $g$ is the genus, $u$ is the unknotting number, $c$ is the clasp number, and $X = 2$ or 3 (cf. [8, 9, 16]). The clasp numbers of torus knots ([5, 13]) and doubled knots (cf. [10, 17]) were determined. We refer the newest information of the unknotting numbers mainly from KnotInfo [2]. In the table, we are using the notation of prime knots in Rolfsen's table [16]. (We note that (1) $10_{161}$ is equivalent to $10_{162}$ in Rolfsen’s table and Kawauchi’s table [8, 9], (2) in Kawauchi’s table, $10_{83}$ and $10_{86}$ are interchanged from Rolfsen’s table, and (3) Knot Atlas [1] and KnotInfo follow basically Rolfsen’s table, but they remove $10_{162}$ in Rolfsen’s table and denote $10_n$ ($163 \leq n \leq 166$) in Rolfsen’s table by $10_{n-1}$.)

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| knot | $g$ | $u$ | $c$ |
|------|-----|-----|-----|
| $3_1$ | 1   | 1   | 1   |
| $4_1$ | 1   | 1   | 1   |
| $5_1$ | 2   | 2   | 2   |
| $5_2$ | 1   | 1   | 1   |
| $6_1$ | 1   | 1   | 1   |
| $6_2$ | 2   | 1   | 2   |
| $6_3$ | 2   | 1   | 2   |
| $7_1$ | 3   | 3   | 3   |
| $7_2$ | 1   | 1   | 1   |
| $7_3$ | 2   | 2   | 2   |
| $7_4$ | 1   | 2   | 2   |
| $7_5$ | 2   | 2   | 2   |
| $7_6$ | 2   | 1   | 2   |
| $7_7$ | 2   | 1   | 2   |
| $8_1$ | 1   | 1   | 1   |
| $8_2$ | 3   | 2   | 3   |
| $8_3$ | 1   | 2   | 2   |
| $8_4$ | 2   | 2   | 2   |
| $8_5$ | 3   | 2   | 3   |
| $8_6$ | 2   | 2   | 2   |
| $8_7$ | 3   | 1   | 3   |
| $8_8$ | 2   | 2   | 2   |
| $8_9$ | 3   | 1   | 3   |
| $8_{10}$ | 3   | 2   | 3   |
| $8_{11}$ | 2   | 1   | 2   |
| $8_{12}$ | 2   | 2   | 2   |
| $8_{13}$ | 2   | 1   | 2   |
| $8_{14}$ | 2   | 1   | 2   |
| $8_{15}$ | 2   | 2   | 2   |
| $8_{16}$ | 3   | 2   | 3   |
| $8_{17}$ | 3   | 1   | 3   |
| $8_{18}$ | 3   | 2   | 3   |
| $8_{19}$ | 3   | 3   | 3   |
| $8_{20}$ | 2   | 1   | 2   |
| $8_{21}$ | 2   | 1   | 2   |
| $9_1$ | 4   | 4   | 4   |
| $9_2$ | 1   | 1   | 1   |
| $9_3$ | 3   | 3   | 3   |
| $9_4$ | 2   | 2   | 2   |
| $9_5$ | 1   | 2   | 2   |
| $9_6$ | 3   | 3   | 3   |
| $9_7$ | 2   | 2   | 2   |
| $9_8$ | 2   | 2   | 2   |
| $9_9$ | 3   | 3   | 3   |
| $9_{10}$ | 2   | 3   | 3   |
| $9_{11}$ | 3   | 2   | 3   |
| $9_{12}$ | 2   | 1   | 2   |
| $9_{13}$ | 2   | 3   | 3   |
| $9_{14}$ | 2   | 1   | 2   |
| $9_{15}$ | 2   | 2   | 2   |
| $9_{16}$ | 3   | 3   | 3   |
| $9_{17}$ | 3   | 2   | 3   |
| $9_{18}$ | 2   | 2   | 2   |
| $9_{19}$ | 2   | 1   | 2   |
| $9_{20}$ | 3   | 2   | 3   |
| $9_{21}$ | 2   | 1   | 2   |
| $9_{22}$ | 3   | 1   | 3   |
| $9_{23}$ | 2   | 2   | 2   |
| $9_{24}$ | 3   | 1   | 3   |
| $9_{25}$ | 2   | 2   | 2   |
| $9_{26}$ | 3   | 1   | 3   |
| $9_{27}$ | 3   | 1   | 3   |
| $9_{28}$ | 3   | 1   | 3   |
| $9_{29}$ | 3   | 2   | 3   |
| $9_{30}$ | 3   | 1   | 3   |
| $9_{31}$ | 3   | 2   | 3   |
| $9_{32}$ | 3   | 2   | 3   |
| $9_{33}$ | 3   | 1   | 3   |
| $9_{34}$ | 3   | 1   | 3   |
| $9_{35}$ | 1   | 3   | 3   |
| $9_{36}$ | 3   | 2   | 3   |
| $9_{37}$ | 2   | 2   | 2   |
| $9_{38}$ | 2   | 3   | 3   |
| $9_{39}$ | 2   | 1   | $X$ |
| $9_{40}$ | 3   | 2   | 3   |
| $9_{41}$ | 2   | 2   | $X$ |
| $9_{42}$ | 2   | 1   | 2   |
| $9_{43}$ | 3   | 2   | 3   |
| $9_{44}$ | 2   | 1   | 2   |
| $9_{45}$ | 2   | 1   | 2   |
| $9_{46}$ | 1   | 2   | 2   |
| $9_{47}$ | 3   | 2   | 3   |
| $9_{48}$ | 2   | 2   | 2   |
| $9_{49}$ | 2   | 3   | 3   |
| $10_1$ | 1   | 1   | 1   |
| $10_2$ | 4   | 3   | 4   |
| $10_3$ | 1   | 2   | 2   |
| $10_4$ | 2   | 2   | 2   |
| $10_5$ | 4   | 2   | 4   |
| $10_6$ | 3   | 3   | 3   |
| $10_7$ | 2   | 1   | 2   |
| $10_8$ | 3   | 2   | 3   |
| $10_9$ | 4   | 1   | 4   |
| $10_{10}$ | 2   | 1   | 2   |
| $10_{11}$ | 2   | $X$ | $X$ |
| $10_{12}$ | 3   | 2   | 3   |
| $10_{13}$ | 2   | 2   | 2   |
| $10_{14}$ | 3   | 2   | 3   |
| $10_{15}$ | 3   | 2   | 3   |
| $10_{16}$ | 2   | 2   | $X$ |
| $10_{17}$ | 4   | 1   | 4   |
| $10_{18}$ | 2   | 1   | 2   |
| $10_{19}$ | 3   | 2   | 3   |
| $10_{20}$ | 2   | 2   | 2   |
| $10_{21}$ | 3   | 2   | 3   |
| $10_{22}$ | 3   | 2   | 3   |
| $10_{23}$ | 3   | 1   | 3   |
| $10_{24}$ | 2   | 2   | 2   |
| $10_{25}$ | 3   | 2   | 3   |
| $10_{26}$ | 3   | 1   | 3   |
| $10_{27}$ | 3   | 1   | 3   |
| $10_{28}$ | 2   | 2   | $X$ |
| $10_{29}$ | 3   | 2   | 3   |
| $10_{30}$ | 2   | 1   | $X$ |
| $10_{31}$ | 2   | 1   | 2   |
| $10_{32}$ | 3   | 1   | 3   |
| $10_{33}$ | 2   | 1   | $X$ |
| $10_{34}$ | 2   | 2   | 2   |
| $10_{35}$ | 2   | 2   | 2   |
| $10_{36}$ | 2   | 2   | 2   |
| $10_{37}$ | 2   | 2   | 2   |
| $10_{38}$ | 2   | 2   | 2   |
| $10_{39}$ | 3   | 2   | 3   |
| $10_{40}$ | 3   | 2   | 3   |
| $10_{41}$ | 3   | 2   | 3   |
| $10_{42}$ | 3   | 1   | 3   |
| knot | g  | u  | c  |
|------|----|----|----|
| 10_43 | 3  | 2  | 3  |
| 10_44 | 3  | 1  | 3  |
| 10_45 | 3  | 2  | 3  |
| 10_46 | 4  | 3  | 4  |
| 10_47 | 4  | X  | 4  |
| 10_48 | 4  | 2  | 4  |
| 10_49 | 3  | 3  | 3  |
| 10_50 | 3  | 2  | 3  |
| 10_51 | 3  | X  | 3  |
| 10_52 | 3  | 2  | 3  |
| 10_53 | 2  | 3  | 3  |
| 10_54 | 3  | X  | 3  |
| 10_55 | 2  | 2  | 2  |
| 10_56 | 3  | 2  | 3  |
| 10_57 | 3  | 2  | 3  |
| 10_58 | 2  | 2  | 2  |
| 10_59 | 3  | 1  | 3  |
| 10_60 | 3  | 1  | 3  |
| 10_61 | 3  | X  | 3  |
| 10_62 | 4  | 2  | 4  |
| 10_63 | 2  | 2  | 2  |
| 10_64 | 4  | 2  | 4  |
| 10_65 | 3  | 2  | 3  |
| 10_66 | 3  | 3  | 3  |
| 10_67 | 2  | 2  | 2  |
| 10_68 | 2  | 2  | X  |
| 10_69 | 3  | 2  | 3  |
| 10_70 | 3  | 2  | 3  |
| 10_71 | 3  | 1  | 3  |
| 10_72 | 3  | 2  | 3  |
| 10_73 | 3  | 1  | 3  |
| 10_74 | 2  | 2  | X  |
| 10_75 | 3  | 2  | 3  |
| 10_76 | 3  | X  | 3  |
| 10_77 | 3  | X  | 3  |
| 10_78 | 3  | 2  | 3  |
| 10_79 | 4  | X  | 4  |
| 10_80 | 3  | 3  | 3  |
| 10_81 | 3  | 2  | 3  |
| 10_82 | 4  | 1  | 4  |
| 10_83 | 3  | 2  | 3  |
| 10_84 | 3  | 1  | 3  |
| knot | g  | u  | c  |
|------|----|----|----|
| 10_85 | 4  | 2  | 4  |
| 10_86 | 3  | 2  | 3  |
| 10_87 | 3  | 2  | 3  |
| 10_88 | 3  | 1  | 3  |
| 10_89 | 3  | 2  | 3  |
| 10_90 | 3  | 2  | 3  |
| 10_91 | 4  | 1  | 4  |
| 10_92 | 3  | 2  | 3  |
| 10_93 | 3  | 2  | 3  |
| 10_94 | 4  | 2  | 4  |
| 10_95 | 3  | 1  | 3  |
| 10_96 | 3  | 2  | 3  |
| 10_97 | 2  | 2  | 3  |
| 10_98 | 3  | 2  | 3  |
| 10_99 | 4  | 2  | 4  |
| 10_100 | 4  | X  | 4  |
| 10_101 | 2  | 3  | 3  |
| 10_102 | 3  | 1  | 3  |
| 10_103 | 3  | 3  | 3  |
| 10_104 | 4  | 1  | 4  |
| 10_105 | 3  | 2  | 3  |
| 10_106 | 4  | 2  | 4  |
| 10_107 | 3  | 1  | 3  |
| 10_108 | 3  | 2  | 3  |
| 10_109 | 4  | 2  | 4  |
| 10_110 | 3  | 2  | 3  |
| 10_111 | 3  | 2  | 3  |
| 10_112 | 4  | 2  | 4  |
| 10_113 | 3  | 1  | 3  |
| 10_114 | 3  | 1  | 3  |
| 10_115 | 3  | 2  | 3  |
| 10_116 | 4  | 2  | 4  |
| 10_117 | 3  | 2  | 3  |
| 10_118 | 4  | 1  | 4  |
| 10_119 | 3  | 1  | 3  |
| 10_120 | 2  | 3  | 3  |
| 10_121 | 3  | 2  | 3  |
| 10_122 | 3  | 2  | 3  |
| 10_123 | 4  | 2  | 4  |
| 10_124 | 4  | 4  | 4  |
| 10_125 | 3  | 2  | 3  |
| 10_126 | 3  | 2  | 3  |
| knot | g  | u  | c  |
|------|----|----|----|
| 10_127 | 3  | 2  | 3  |
| 10_128 | 3  | 3  | 3  |
| 10_129 | 2  | 1  | X  |
| 10_130 | 2  | 2  | X  |
| 10_131 | 2  | 1  | X  |
| 10_132 | 2  | 1  | 2  |
| 10_133 | 2  | 1  | 2  |
| 10_134 | 3  | 3  | 3  |
| 10_135 | 2  | 2  | 2  |
| 10_136 | 2  | 1  | 2  |
| 10_137 | 2  | 1  | 2  |
| 10_138 | 3  | 2  | 3  |
| 10_139 | 4  | 4  | 4  |
| 10_140 | 2  | 2  | 2  |
| 10_141 | 3  | 1  | 3  |
| 10_142 | 3  | 3  | 3  |
| 10_143 | 3  | 1  | 3  |
| 10_144 | 2  | 2  | 2  |
| 10_145 | 2  | 2  | 2  |
| 10_146 | 2  | 1  | 2  |
| 10_147 | 2  | 1  | 2  |
| 10_148 | 3  | 2  | 3  |
| 10_149 | 3  | 2  | 3  |
| 10_150 | 3  | 2  | 3  |
| 10_151 | 3  | 2  | 3  |
| 10_152 | 4  | 4  | 4  |
| 10_153 | 3  | 2  | 3  |
| 10_154 | 3  | 3  | 3  |
| 10_155 | 3  | 2  | 3  |
| 10_156 | 3  | 1  | 3  |
| 10_157 | 3  | 2  | 3  |
| 10_158 | 3  | 2  | 3  |
| 10_159 | 3  | 1  | 3  |
| 10_160 | 3  | 2  | 3  |
| 10_161 | 3  | 3  | 3  |
| 10_162 | 3  | 3  | 3  |
| 10_163 | 2  | 2  | X  |
| 10_164 | 3  | 2  | 3  |
| 10_165 | 2  | 1  | X  |
| 10_166 | 2  | 2  | X  |

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