Complex frequency characteristic and Green's function for cantilever beam under longitudinal oscillations

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Abstract. Relevant objective of this article is elaboration of methods of efficient determination of the dynamic characteristics of the materials both within the framework of well-known and classical models as well as within the advanced ones in the context of interaction with the environment under the impact of external dynamic loads. In the present work free and forced oscillations are studied with a help of spectral methods what allows to compare theoretical results with experimental data, obtained by other authors. The key point of the work is theoretical determination of the proper functions of the oscillatory system, its frequency characteristic using Green's function. We examine longitudinal oscillations, caused by the substitute load, which provokes oscillations of a sound spectrum. Univariate character of the task is determined by the linearity organization of the longitudinal oscillations of a beam. The oscillations are modeled with a help of the substitute load, which is a subject to Dirac impulse function, which belongs to a class of generalized functions. The obtained solution makes it possible to use the proposed approach in order to determine frequency characteristics in the acoustic spectrum, taking into account that the oscillations are included in a full frequency spectrum and can be used for an adequate valuation of the possibility of the appearance of resonance phenomena.

1. Introduction

This article deals with the theoretical approaches to the oscillations of constructions in acoustic spectrum. Dynamic loads of any kind, which create oscillations in building constructions, have particular characteristics. At the same time, other impacts cause the oscillations of a sound spectrum. The range of spectra of these frequencies is quite wide, but there can be a frequency coalescence, which leads to the appearance of a resonance. Different authors have done quite wide range of researches, dedicated to the longitudinal oscillations of beams with different boundary conditions, both in terms of setting goals and in terms of elaborating of the methods and approaches to solve the set tasks [1, 2]. While the issue, connected with oscillations, which involve force, is quite well-publicized [3-5], the question of acoustic oscillations is studied to a lesser extent, especially the questions of frequency coalescence, which leads to the resonance, what goes against the characteristics of a comfortable physical environment.
Within this work the authors are limited to the investigation of the longitudinal oscillations, caused by the substitute load, which excites the oscillations of a sound spectrum. In order to illustrate this load we use Dirac impulse function, which represents a generalized function of time and coordinates. In the case under consideration this function is considered to be univariate, resulting from the linearity of the task of determination of the longitudinal oscillations of the beam.

2. Methods
First of all we should specify that oscillations of a sound spectrum are modeled by a recurrent flow of impulses (by numerous blows of a conventional hammer or under the influence of a standard impact machine), in mathematical transformations Grin’s function [6] is applied that is why the module of elasticity of beam material is presented in a complex form.

External force $\vec{F}(x, t)$ is connected with its density by the proportion

$$\vec{F}(x, t) = \int_{x_1}^{x_2} \int_{t_1}^{t_2} \tilde{f}(x, t) dx dt,$$

where $\tilde{f}(x, t)$ – is a certain density of the external force, which represents spreading on the surface (surface loading) at a certain size and a time law at a certain distance (figure 1).

![Figure 1. Distribution of density of the external power.](image)

Examine a cantilever beam which is widely used in building practice and executes longitudinal oscillations (figure 2). Such oscillatory system has an infinite number of the degrees of freedom.

![Figure 2. Analytical model of a cantilever beam under longitudinal oscillations](image)

The longitudinal oscillations are illustrated by the following equation [7, 8]
\[
\frac{\partial^2 U(x,t)}{\partial t^2} - \frac{E}{\rho} \frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{\rho S} \tilde{f}(x,t),
\]
(2)

where \( U(x,t) \) – is the longitudinal displacement of the cross section of the beam with \( x \) - coordinate; \( E = E(1 + i \eta) \) – is a complex Young’s modulus; \( \rho \) – density of the material of the beam; \( S \) – a cross-sectional area of the beam.

Impose some signs: \( m = \rho S \) – mass of the unit of length of the beam; \( \tilde{c} \), where \( \tilde{c} \) – is the speed of longitudinal waves in the beam in a complex form. Therefore the equation of longitudinal oscillations of the beam can be presented in a following way:

\[
\frac{\partial^2 U(x,t)}{\partial t^2} - \tilde{c}^2 \frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{m} \tilde{f}(x,t).
\]
(3)

To get a solution it is necessary to establish initial and boundary conditions. The initial conditions are:

- \( U(x,0) \) – displacement of all the coordinates of the beam when \( t=0; \)
- \( \frac{\partial U(x,0)}{\partial t} \) – an initial speed of the cross-section of the beam.

The boundary conditions are the following:

- \( U(0,t) \) – displacement of the cross-section \( x=0 \) at the moment of time \( t; \)
- \( \frac{\partial U(0,t)}{\partial x} \) – relative elongation at a point \( x=0 \) at the moment of time \( t. \)

To determine proper functions we examine free oscillations of the cantilever beam. The corresponding equation has the following form:

\[
\frac{\partial^2 U(x,t)}{\partial t^2} = \tilde{c}^2 \frac{\partial^2 U(x,t)}{\partial x^2}.
\]
(4)

Firstly we will search for a partial solution of this equation. We suppose that it is presented in a following form \( \tilde{U}_n(x,t) = T(t)X(x) \) and represents as usual the composition of two functions, one of which depends only on \( t \), while another depends only on dimension \( x \). Applying it to the equation (4) we will get

\[
\frac{\partial^2 T(t)}{\partial t^2} X(x) = \tilde{c}^2 T(t) \frac{\partial^2 X(x)}{\partial x^2}
\]
or

\[
\frac{\partial^2 T(t)}{\partial t^2} \cdot \tilde{c}^2 T(t) = \frac{\partial^2 X(x)}{\partial x^2} \cdot X(x) = -k^2,
\]

where \( k \) – is a separating constant of the variables \( x \) and \( t. \)

Herefrom we get two equations:

\[
\frac{\partial^2 T(t)}{\partial t^2} + k^2 \tilde{c}^2 T(t) = 0; \quad \frac{\partial^2 X(x)}{\partial x^2} + k^2 X(x) = 0.
\]
(5)
When $k \neq 0$ the general solution of these equations can be presented in a following way:

$$X(x) = C \cos(kx) + D \sin(kx);$$
$$T(t) = A \cos(\xi kt) + B \sin(\xi kt).$$

(6)

Consider the first equation. As far as the displacement in a bonding is equal to 0, then $U(0,t)=0$ and for the value $x=0$ we have $C\cdot I + D \cdot 0 = 0$, from which $C=0$. There remains the expression $D \sin(kx) = 0$. As there are no external forces, we can present this expression in a following way $(D \sin(kx))’_x = 0$, from what we get $Dk \cos(kx) = 0$. When $x=l$ we have $Dk \cos(kl) = 0$. The separating constant $k \neq 0$. If we take $D=0$, we will get zero (trivial) solution, which is of no interest.

That is why we accept that $D \neq 0$. In such a case, it is necessary to suppose that $\cos(kl) = 0$. It is possible when $kl = n \frac{\pi}{2}$, where $n=1, 3, 5, \ldots$ or $kl = \frac{\pi}{2} (2n-1)$. Here $n=1, 2, 3, \ldots$. Proper functions of our task will be

$$\varphi_n(x) = \sin\left[\frac{\pi x}{2l} (2n-1)\right],$$

(7)

where $n = 1, 2, 3, \ldots$.

The obtained proper functions can be further used in combination with strength dynamic calculations. Hereafter we are going to determine a complex frequency characteristic.

3. Results

For the determination of the complex frequency characteristic of the cantilever beam we will force on its cross section with coordinate $\xi$, at the moment of time $t=t_0$ by the load, which is mathematically marked by Dirac impulse function [9, 10]. Following on from (4) we have an equation:

$$\frac{\partial^2 U(x,t)}{\partial t^2} - \xi^2 \frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{m} \delta(t-t_0) \delta(x-\xi)$$

(8)

here $m$ is the mass, normalized through the density of load.

We find the solution in a form

$$\bar{U}(x,t) = \sum_{n=1}^{\infty} \bar{f}_n(t) \varphi_n(x)$$

(9)

where the function $\bar{f}$ no longer depends on $x$.

Dirac impulse function expands in Fourier series according to the functions of the cantilever beam

$$\delta(x-\xi) = \sum_{n=1}^{\infty} b_n \sin\left[\frac{\pi x}{2l} (2n-1)\right];$$

$$b_n = \frac{2}{l} \int_{0}^{l} \delta(x-\xi) \sin\left[\frac{\pi x}{2l} (2n-1)\right] dx = \frac{2}{l} \sin\left[\frac{\pi \xi}{2l} (2n-1)\right] \sin\left[\frac{\pi \xi}{2l} (2n-1)\right];$$

(10)

where

$$\delta(x-\xi) = \frac{2}{l} \sum_{n=1}^{\infty} \sin\left[\frac{\pi \xi}{2l} (2n-1)\right] \sin\left[\frac{\pi \xi}{2l} (2n-1)\right].$$

Substituting the expressions for $\bar{U}(x,t)$ (9) and Dirac impulse function (10) into the equation (8) we will get
\[
\frac{\partial^2 \tilde{f}_n(t)}{\partial t^2} + \bar{p}_n \tilde{f}_n(t) = \frac{2}{m} \left[ \frac{\pi \xi}{2l} (2n-1) \right] \delta(t-t_0),
\]
(11)

where \( \bar{p}_n = \frac{\pi \xi}{2l} (2n-1) \).

We apply the integral Laplace transform of \( t [6, 11] \) to the obtained equation (11). In optical space we have

\[
p^2 F_n(p) - f_n(0)p - f_n'(0) + p^2 F_n(0) = \frac{2}{ml} \left[ \frac{\pi \xi}{2l} (2n-1) \right].
\]

As far as \( f_n(0)=0 \) and \( f_n'(0)=0 \), then

\[
F_n(p) = \frac{2}{ml} \left[ \frac{\pi \xi}{2l} (2n-1) \right].
\]

In optical space we get complex frequency characteristic of the cantilever beam

\[
\tilde{W}(x, \xi, p) = \sum_{n=1}^{\infty} \frac{1}{p^2 + \bar{p}_n^2} \sin \left[ \frac{\pi x}{2l} (2n-1) \right] \sin \left[ \frac{\pi \xi}{2l} (2n-1) \right],
\]
(12)

where \( p=io \).

This expression is similar to the result, given in the work [12].

Passing to the space of real variables we find Grin’s function

\[
\tilde{G}(x, \xi, t) = \frac{4}{m \pi c} \sum_{n=1}^{\infty} \sin \left[ \frac{\pi x}{2l} (2n-1) \right] \sin \left[ \frac{\pi \xi}{2l} (2n-1) \right] \sin \left[ \frac{\pi \delta_1}{2l} (2n-1) \right],
\]
(13)

where \( \delta_1 = t - \tau \).

The expression (13) is similar to the result which are given in the work [12], if we suppose that \( m=1 \).

Module of a complex frequency characteristic of the cantilever beam under longitudinal oscillations can be presented in a following way:

\[
|\tilde{W}(x, \xi, f)| = \frac{8l}{m \pi c^2} \sum_{n=1}^{\infty} \left[ \frac{(2n-1)^3 - a^2}{(2n-1)^2 - a^2} - i \eta (2n-1)^2 \right] \sin \left[ \frac{\pi x}{2l} (2n-1) \right] \sin \left[ \frac{\pi \xi}{2l} (2n-1) \right],
\]
(14)

where \( a = \frac{4lf}{c} \).

4. Discussion

Substitution of the specific values of variables into (14) gives a numerical value, which shows the correlation between the polar angle and the frequency.

As an example we will give a graph, showing the module of the frequency characteristic of the end of the 6 meters long cantilever beam, made of steel concrete, with cross-sectional dimension \( 0.4 \times 0.4 \) meters under the influence of the external force in the cross-section with the coordinate \( \xi = 4.2 \) meters (figure 3).
Figure. 3. Module of a complex frequency characteristic of the end of the cantilever beam, made of steel concrete with cross-sectional dimensions 0.4x0.4 m: l=6 m; m=400 kg/m; c=3000 m/s; η=0.05; x=6 m; ξ=4.2 m

The first resonance is observed at a frequency \( ω \approx 785 \) hertz, the second one – at a frequency \( ω \approx 2343 \) hertz. These frequencies are slightly lower than those ones, which are obtained using proper functions. The period of the first resonating frequency is approximately 0.008 seconds.

5. Conclusion
The given solution shows the possibility to use the proposed approach in order to determine frequency characteristics in acoustic spectrum. The important thing is that these oscillations are included in the whole spectrum of the construction frequencies and can be used for adequate evaluation of the possibility of the appearing of resonance phenomena.

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