FRAME PHASE-RETRIEVABILITY AND EXACT PHASE-RETRIEVABLE FRAMES

DEGUANG HAN, TED JUSTE, YOUFA LI, AND WENCHANG SUN

Abstract. An exact phase-retrievable frame $\{f_i\}_{i=1}^N$ for an $n$-dimensional Hilbert space is a phase-retrievable frame that fails to be phase-retrievable if any one element is removed from the frame. Such a frame could have different lengths. We shall prove that for the real Hilbert space case, exact phase-retrievable frame of length $N$ exists for every $2n-1 \leq N \leq n(n+1)/2$. For arbitrary frames we introduce the concept of redundancy with respect to its phase-retrievability and the concept of frames with exact PR-redundancy. We investigate the phase-retrievability by studying its maximal phase-retrievable subspaces with respect to a given frame which is not necessarily phase-retrievable. These maximal PR-subspaces could have different dimensions. We are able to identify the one with the largest dimension, which can be considered as a generalization of the characterization for phase-retrievable frames. In the basis case, we prove that if $M$ is a $k$-dimensional PR-subspace, then $|\text{supp}(x)| \geq k$ for every nonzero vector $x \in M$. Moreover, if $1 \leq k < \lceil (n+1)/2 \rceil$, then a $k$-dimensional PR-subspace is maximal if and only if there exists a vector $x \in M$ such that $|\text{supp}(x)| = k$.

1. Introduction

A finite sequence $\mathcal{F} = \{f_i\}_{i=1}^N$ of vectors in an $n$-dimensional Hilbert space $H$ is called a frame for $H$ if there are two constants $0 < C_1 \leq C_2$ such that

$$C_1 \|f\|^2 \leq \sum_{i=1}^N |\langle f, f_i \rangle|^2 \leq C_2 \|f\|^2$$

holds for every $f \in H$. Equivalently, a finite sequence is a frame for $H$ if and only if it is a spanning set of $H$. Two frames $\{f_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ are called similar if there exists an invertible operator $T$ such that $g_i = T f_i$ for every $i$. For a given frame $\mathcal{F} = \{f_i\}_{i=1}^N$, the spark of $\mathcal{F}$ is the cardinality of the smallest linearly dependent subset of the frame. A full-spark frame is a frame whose spark is $n + 1$, i.e., every $n$-vectors in $\mathcal{F}$ are linearly independent.

In recent years, frames have been extensively studied in the context of the so-called phase-retrieval problem which arises in various fields of science and
engineering applications, such as X-ray crystallography, coherent diffractive imaging, optics and many more. The problem asks to recover a signal of interest from the magnitudes of its linear or nonlinear measurements. For the linear measurements with a frame \( \{ f_i \}_{i=1}^N \), one wants to reconstruct \( f \) from its intensity measurements \( \{ |\langle f, f_i \rangle| \}_{i=1}^N \). Clearly the intensity measurements are the same for both \( f \) and \( \lambda f \) for every unimodular scalar \( \lambda \). Therefore the phase retrieval problem asks to recover \( f \) up to an unimodular scalar.

We refer to [1]-[24] and the reference therein for some historic background of the problem and some recent developments on this topic.

**Definition 1.1.** A frame \( \{ f_i \}_{i=1}^N \) for a Hilbert space \( H \) is called phase-retrievable if the induced quotient map \( A : H/\mathbb{T} \rightarrow \mathbb{R}^N \) defined by \( A(f/\mathbb{T}) = \{ |\langle f, f_i \rangle| \}_{i=1}^N \) is injective, where \( \mathbb{T} = \{ \lambda \in \mathbb{R} : |\lambda| = 1 \} \).

There are few basic concepts when talking about frames or frame sequences: signal recoverability, redundancy and the exactness of frames. The signal recoverability of a sequence \( F = \{ f_i \} \) can be measured by the space spanned by \( F \), and the redundancy of a finite frame \( F = \{ f_i \}_{i=1}^N \) for an \( n \)-dimensional Hilbert space can be measured by \( N/n \). An exact frame for a Hilbert space \( H \) is a frame such that it fails to be a frame if we remove any one element from \( F \). So exact frames are precisely the bases or the frames with redundancy one. These concepts naturally lead us to the following questions when dealing with the phase-retrieval problem: Given a frame \( F = \{ f_i \}^N_{i=1} \) (which may not be phase-retrievable). How to measure its phase-retrievability? How to measure its redundancy with respect to the phase-retrievability and what can be said about those phase-retrievable frames that have the exact PR-redundancy?

Even a frame is not phase-retrievable, it is still possible that it can be used to perform phase retrieval for some subsets of the Hilbert space. So for the purpose of theory development and practical applications, it seems natural to investigate the subsets on which phase-retrieval can be performed with respect to a given frame (usually your favorite one but not phase-retrievable). In this paper we initiate the study on the maximal phase-retrievable subspaces for a given frame. This consideration naturally leads to the concept of frame redundancy with respect to the phase-retrievability and the notion of exact phase-retrievable frames. Unlike exact frames, exact phase-retrievable frames could have different lengths. This paper will be focused on the existence problem of exact phase-retrievable frames (or more generally, the frames with the exact PR-redundancy) with all the possible lengths, and the maximal phase-retrievable subspaces of all possible dimensions.

There are two well-known necessary and sufficient conditions for phase-retrievable frames (c.f. [2, 3, 4, 5, 6]). The first one is given in terms of the so-called “complement property”: A frame \( \{ f_i \}_{i=1}^N \) is said to have the complement property if for every \( \Omega \subseteq \{ 1, ..., N \} \) we have either \( \{ f_i \}_{i \in \Omega} \) or \( \{ f_i \}_{i \notin \Omega} \) spans \( H \).
Proposition 1.1. The complement property is necessary for a frame to be phase-retrievable. It is also sufficient for real Hilbert spaces.

The second condition is based on the rank-one operator lifting of the frame \( \{f_i\}_{i=1}^{N} \). For each \( f, g \in H \), let \( f \otimes g \) be the rank-one operator defined by \( (f \otimes g)x = \langle x, g \rangle f \) for every \( x \in H \). In what follows we use \( \langle A, B \rangle = tr(AB^*) \) to denote the Hilbert-Schmidt inner product on the space of \( n \times n \) matrices and \( S_2 \) be the set of all the Hermitian \( n \times n \) matrices with rank less than or equal to 2. Given a sequence \( \mathcal{F} = \{f_i\}_{i=1}^{N} \) in \( H \) and let \( \Theta_{L(\mathcal{F})} \) be the analysis operator of \( L(\mathcal{F}) := \{L(f_i)\} \), where \( L(x) := x \otimes x \). From the definition of phase-retrievable frames, it is easy to see get the following:

Proposition 1.2. A frame \( \{f_i\}_{i=1}^{N} \) is phase-retrievable if and only if

\[
\ker(\Theta_{L(\mathcal{F})}) \cap S_2 = \{0\}.
\]

The above characterization indicates that \( \ker(\Theta_{L(\mathcal{F})}) \cap S_2 \) seems to be a good candidate to measure the phase-retrievability for a frame \( \mathcal{F} \). This motivated us to introduce the following concept of redundancy with respect to the phase-retrievability (or PR-redundancy) and the concept of frames with the exact PR-redundancy property. Let \( \mathcal{F} = \{f_i\}_{i=1}^{N} \) be a frame for \( H \). For each subset \( \Lambda \) of \( \{1, ..., N\} \), let \( \mathcal{F}_\Lambda = \{f_i\}_{i \in \Lambda} \) and use \( |\Lambda| \) to denote the cardinality of \( \Lambda \).

Definition 1.2. Given a frame \( \mathcal{F} = \{f_i\}_{i=1}^{N} \) for \( H \). Let \( k \) be the smallest integer such that there exists a subset \( \Lambda \) of \( \{1, ..., N\} \) with the property that \( |\Lambda| = k \) and

\[
\ker(\Theta_{L(\mathcal{F}_\Lambda)}) \cap S_2 = \ker(\Theta_{L(\mathcal{F})}) \cap S_2.
\]

Then we call \( N/k \) the PR-redundancy of \( \mathcal{F} \). A frame \( \mathcal{F} = \{f_i\}_{i=1}^{N} \) for \( H \) is said to have the exact PR-redundancy property if its PR-redundancy is 1. A phase-retrievable frame with the exact PR-redundancy will be called an exact phase-retrievable frame.

Given a frame \( \mathcal{F} = \{f_i\}_{i=1}^{N} \) for \( H \). From the above definition we have the following: (i) There exists a subset \( \Lambda \) of \( \{1, ..., N\} \) such that \( \mathcal{F}_\Lambda \) is a frame for \( H \) with the exact PR-redundancy property. (ii) \( \mathcal{F} \) has the exact PR-redundancy property if and only if for any proper subset \( \Lambda \) of \( \{1, ..., N\} \), there exist two vectors \( x, y \in H \) such that \( |\langle x, f_j \rangle| = |\langle y, f_j \rangle| \) for every \( j \in \Lambda \), but \( |\langle x, f_i \rangle| \neq |\langle y, f_i \rangle| \) for some \( i \in \Lambda^c \). (iii) If \( \mathcal{F} \) is phase-retrievable, then it is an exact phase-retrievable frame if and only if \( \mathcal{F}_\Lambda \) is no longer phase-retrievable for any proper subset \( \Lambda \) of \( \{1, ..., N\} \).

In what follows we always assume that \( H = \mathbb{R}^n \) and use \( \mathbb{H}_n \) to denote the space of all the \( n \times n \) Hermitian matrices.

Lemma 1.3. If a frame \( \mathcal{F} = \{f_i\}_{i=1}^{N} \) for \( \mathbb{R}^n \) has the exact PR-redundancy property, then \( \{L(f_i)\}_{i=1}^{N} \) is a linearly independent set (and hence \( N \leq \dim \mathbb{H}_n = n(n+1)/2 \)). The converse is false.
Definition 1.3. Let \( \mathcal{F} = \{ f_i \}_{i=1}^N \) be a frame for \( H \) and \( M \) is a subspace of \( H \). We say that \( M \) is a phase-retrievable subspace with respect to \( \mathcal{F} \) if \( \{ P_M f_i \}_{i=1}^N \) is a phase-retrievable frame for \( M \), where \( P_M \) is the orthogonal projection from \( H \) onto \( M \). A phase-retrievable subspace \( M \) is called maximal if it is not a proper subspace of any other phase-retrievable subspaces with respect to \( \mathcal{F} \).

We will use the abbreviation “\( \mathcal{F} \)-PR subspace” to denote a phase-retrievable subspace with respect to \( \mathcal{F} \). Given a frame \( \mathcal{F} \). Naturally we would like to know the answers to the following questions: What are possible dimensions
such that there exists a \( k \)-dimensional maximal \( \mathcal{F} \)-PR subspace? What is the largest (or the smallest) dimension for all the maximal \( \mathcal{F} \)-PR subspaces?

As an motivating example, we will show that if \( \mathcal{F} = \{ f_i \}_{i=1}^n \) is a basis for \( H \), then there exists a \( k \)-dimensional maximal \( \mathcal{F} \)-PR subspace if and only if

\[
1 \leq k \leq \lfloor (n+1)/2 \rfloor, \text{ where } \lfloor a \rfloor \text{ denotes the integer part of } a.
\]

For any general frame \( \mathcal{F} \), we will identify the largest \( k \) such that there exists a \( k \)-dimensional maximal \( \mathcal{F} \)-PR subspace. This leads to a generalization of Proposition 1.4.

In the case that \( \mathcal{F} = \{ f_i \}_{i=1}^n \) is an orthonormal basis, we show that if \( M \) is a \( \mathcal{F} \)-PR subspace, then the support \( supp(x) \) (with respect to the dual basis) of every nonzero vector \( x \) in \( M \) has the cardinality greater than or equal to \( k \). Moreover, we will prove that for any given vector \( x \) with \( |supp(x)| = k \), there exists a \( k \)-dimensional maximal \( \mathcal{F} \)-PR subspace \( M \) containing \( x \). This support condition is also necessary in the case that \( k < \lfloor (n+1)/2 \rfloor \), i.e., in this case we have that a \( k \)-dimensional \( \mathcal{F} \)-PR subspace \( M \) is maximal if and only if there exists an nonzero vector \( x \) in \( M \) whose support has the cardinality \( k \).

The following simple property will be needed in the rest of the paper.

**Lemma 1.5.** Suppose that \( H \) is the direct sum of two subspaces \( X \) and \( Y \). If \( \mathcal{F}_1 \) is a frame for \( X \) with the exact PR-redundancy property and \( \mathcal{F}_2 \) is a frame for \( Y \) with the exact PR-redundancy property, then \( \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \) is a frame for \( H \) with the exact PR-redundancy property.

**Proof.** By passing to a similar frame we can assume that \( Y = X^\perp \). Clearly \( \mathcal{F} \) is a frame for \( H \). Now assume that a vector \( f \) is removed from \( \mathcal{F}_1 \). Since \( \mathcal{F}_1 \) is a frame for \( X \) with the exact PR-redundancy property, there exists some nonzero operator \( A = u \otimes u - v \otimes v \) with \( u, v \in X \) such that \( A \in ker(\Theta_L(\mathcal{F}_1 \setminus \{ f \})) \) and \( A \notin ker(\Theta_L(\mathcal{F}_1)) \). Since \( Y \perp X \), we also have \( A \in ker(\Theta_L(\mathcal{F}_2)) \). This implies that \( A \in ker(\Theta_L(\mathcal{F} \setminus \{ f \})) \) and \( A \notin ker(\Theta_L(\mathcal{F})) \).

The same argument works if we remove one element from \( \mathcal{F}_2 \). Thus \( \mathcal{F} \) has the exact PR-redundancy property.

\[ \square \]

### 2. Exact Phase-retrievable Frames

In this section we prove the existence theorem for exact phase-retrievable frames of length \( N \) with \( 2n-1 \leq N \leq n(n+1)/2 \).

**Theorem 2.1.** For every integer \( N \) with \( 2n-1 \leq N \leq n(n+1)/2 \), there exists an exact phase-retrievable frame of length \( N \).

Before giving a proof for the above theorem, we introduce some preliminary results. We use the following notations for matrices: \( A(I, J) \) is the submatrix of \( A \) consisting of the entries with row indices in \( I \) and column indices in \( J \). \( A(\cdot, J) = A(\{1, \ldots, n\}, J) \) and \( A(i, \cdot) = A(\{i\}, \{j\}) \)

**Lemma 2.2.** Let \( f(x_1, \ldots, x_n) \) be a polynomial and \( a_i \) be independent continuous random variables. Then \( f(a_1, \ldots, a_n) \neq 0 \) almost surely.
Proof. The conclusion can be proved by induction on \(n\) and we omit the details. \(\square\)

**Lemma 2.3.** Let \(A\) be an \(n \times m\) random matrix such that \(\text{rank}(A) = r\) almost surely. Let \(B\) be an \((n+1)\times(m+1)\) matrix such that \(B(1..n, 1..m) = A\) and \(B(n+1, m+1)\) is a continuous random variable which is independent of the entries of \(A\). Then we have \(\text{rank}(B) \geq r + 1\) almost surely.

**Proof.** Let \(\Omega\) be the sample space. Since \(A\) has only finitely many submatrices and \(\text{rank}(A) = r\) almost surely, there is a partition \(\{\Omega_i\}_{i=1}^N\) of \(\Omega\) such that for each \(1 \leq i \leq N\), there is an \(r \times r\) submatrix \(A_i\) which is of rank \(r\) almost surely on \(\Omega_i\). Therefore, the submatrix of \(A\) consisting of rows and columns in \(A_i\) and the \((n+1)\)-th row and the \((m+1)\)-th column is of rank \(r + 1\) almost surely on \(\Omega_i\), thanks to Lemma 2.2. This completes the proof. \(\square\)

The following lemma can be proved similarly, which we leave to interested readers.

**Lemma 2.4.** Let \(A\) be an \(n \times m\) random matrix such that \(\text{rank}(A) = r \leq n - 1\) almost surely. Let \(a\) be an \(n\)-dimensional vector with entries consisting of continuous independent random variables, which are also independent of the entries of \(A\). Then we have \(\text{rank}((A a)) = r + 1\) almost surely.

We are ready to give a proof of Theorem 2.1.

**Proof of Theorem 2.1.** Since every full-spark frame of length \(2n - 1\) is an exact PR-frame, we only need to prove the theorem for \(2n \leq N \leq n(n+1)/2\). First, we show that for \(2n \leq N \leq n(n+1)/2\), there exist \(n \times N\) matrices \(A\) such that

1. \(A\) contains the \(n \times n\) identity matrix as a submatrix;
2. the rest \(N - n\) columns of \(A\) consisting of independent continuous random variables or zeros and each column contains at least one 0 and two non-zero entries;
3. there are exactly \(n\) non-zero entries in every row of \(A\);
4. for each \(1 \leq i \leq n\), there exist mutually different indices \(j_1, \ldots, j_n\) such that \(a_{i,j_l}, a_{i,j_l} \neq 0\);
5. columns of \(A\) form an exact PR frame with probability 1.

It is obvious that a phase-retrievable frame which satisfies (P3) is exact. Let us explain (P4) in more details.

Fix some \(i\), say, \(i = 1\). By (P3), there exist mutually different indices \(j_1, \ldots, j_n\) such that \(a_{1,j_l} \neq 0\) for \(1 \leq l \leq n\). (P4) says that every row contains a non-zero entry in such columns and different rows correspond to different columns.

Consider the following example,

\[
(2.1) \quad A = \begin{pmatrix}
1 & 0 & 0 & a_{1,4} & a_{1,5} & 0 \\
0 & 1 & 0 & a_{2,4} & 0 & a_{2,3} \\
0 & 0 & 1 & 0 & a_{3,5} & a_{3,3}
\end{pmatrix},
\]
where \( a_{i,j} \) are independent continuous random variables. For \( i = 1 \), set \( \{j_1, j_2, j_3\} = \{1, 4, 5\} \). Then we have \( a_{1,j_l} a_{i,j_l} \neq 0 \) for \( 1 \leq l \leq 3 \).

It is easy to see that \( A \) satisfies (P1) \( \sim \) (P5). In other words, such matrix exists for \( n = 3 \).

Now we assume that such matrix \( A \) exists for some \( n \) and \( N \) with \( n \geq 3 \). Let us consider the case of \( n + 1 \). We prove the conclusion in the following four steps.

(I). **There is an** \( (n+1) \times (N+n+1) \) **matrix satisfying (P1) \( \sim \) (P5).**

Define the \( (n+1) \times (N+n) \) matrix \( B \) as follows,

\[
B = \begin{pmatrix}
    a_{1,N+1} & 0 & 0 & 0 \\
    0 & a_{2,N+2} & 0 & 0 \\
    0 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & a_{n,N+n} \\
    0 & a_{n+1,N+1} & a_{n+1,N+2} & a_{n+1,N+n+1} & 1
\end{pmatrix},
\]

where all the symbols \( a_{i,j} \) are independent continuous random variables. It is easy to see that \( B \) meets (P1) \( \sim \) (P4). It remains to prove that (P5) holds for \( B \).

Take some \( J \subset \{1, \ldots, N+n+1\} \). Set

\[
J^c = \{1 \leq j \leq N+n+1 : j \notin J\},
J|_N = \{j \in J : j \leq N\},
J^c|_N = \{j \in J^c : j \leq N\}.
\]

Without loss of generality, we assume that \( N+n+1 \in J^c \).

Suppose that rank \( (B(:,J^c)) < n+1 \) on some sample set \( \Omega' \) which is of positive probability. Since \( N+n+1 \in J^c \), we have rank \( (A(:,J^c|_N)) < n \) a.s. on \( \Omega' \). Consequently, rank \( (A(:,J^c|_N)) = n \) a.s. on \( \Omega' \).

On the other hand, Since \( N+n+1 \in J^c \), not all of \( N+1, \ldots, N+n \) are contained in \( J^c \). Otherwise, rank \( (B(:,J^c)) = n+1 \) a.s. on \( \Omega' \). Hence there is some \( 1 \leq i \leq n \) such that \( N+i \in J \). By Lemma 2.3, rank \( (B(:,J)) = n+1 \) a.s. on \( \Omega' \).

(II). **There is an** \( (n+1) \times (N+n) \) **matrix satisfying (P1) \( \sim \) (P5).**

Since \( A \) satisfies (P2), by rearranging columns of \( A \), we may assume that \( A(:,N) = (0,a_{2,N},\ldots)^t \), where at least two entries are non-zero. Define the \( (n+1) \times (N+n) \) matrix \( B \) as follows,

\[
B = \begin{pmatrix}
    0 & a_{1,N+1} & 0 & 0 & 0 \\
    a_{2,N} & a_{2,N+1} & 0 & 0 & 0 \\
    * & 0 & a_{3,N+2} & \cdots & 0 \\
    * & 0 & \cdots & \cdots & \cdots \\
    a_{n+1,N} & a_{n+1,N+1} & a_{n+1,N+2} & a_{n+1,N+n-1} & 1
\end{pmatrix}.
\]

Again, we only need to prove that (P5) holds for \( B \).
As in Step I, we take some \( J \subset \{1, \ldots, N + n\} \). We suppose that \( N + n \in J^c \) and that \( \text{rank}(B(:, J^c)) < n + 1 \) on some sample set \( \Omega' \) which is of positive probability. Then we have \( \text{rank}(A(:, J|_N)) = n \) a.s. on \( \Omega' \).

If there is some \( 1 \leq i \leq n \) such that \( N + i \in J \), then we have \( \text{rank}(B(:, J)) = n + 1 \) a.s. on \( \Omega' \), thanks to Lemma 2.3.

Next we assume that \( N + i \in J^c \) for \( 1 \leq i \leq n \). Since \( \text{rank}(B(:, J^c)) < n + 1 \) a.s. on \( \Omega' \), for any \( j \leq N \) with \( A(1, j) \neq 0 \), we have \( j \in J \), thanks to Lemma 2.2. Similarly we get that \( N \in J \).

By setting \( i = 1 \) in (P4), we get mutually different \( 1 \leq j_1, \ldots, j_n \leq N \) such that \( A(1, j_i), A(l, j_i) \neq 0 \). Hence \( j_1, \ldots, j_n \in J|_N \). Moreover, \( \text{rank}(A(:, \{j_1, \ldots, j_n\})) = n \) a.s. on \( \Omega' \), thanks to Lemma 2.2. Note that \( N \in J|_N \) and \( N \neq j_l \) for \( 1 \leq l \leq n \). By Lemma 2.3, we have

\[
\text{rank}(B(:, \{j_1, \ldots, j_n, N\})) = n + 1, \quad \text{a.s. on } \Omega'.
\]

Hence

\[
\text{rank}(B(:, J)) \geq \text{rank}(B(:, \{j_1, \ldots, j_n, N\})) = n + 1, \quad \text{a.s. on } \Omega'.
\]

(III). There is an \((n + 1) \times (N + 2)\) matrix satisfying (P1) \sim (P5).

By rearranging columns of \( A \), we may assume that

1. \( A(:, \{1, \ldots, n\}) \) is the \( n \times n \) identity matrix (P1),
2. \( A(n, N) = 0 \) and there are at least two non-zero entries in the \( N \)-th column (P2),
3. \( A(i, N - i), A(n, N - i) \neq 0 \) for \( 1 \leq i \leq n - 1 \) (P4).

Define the \((n + 1) \times (N + 2)\) matrix \( B \) as follows,

\[
B = \begin{pmatrix}
I_{n \times n} & \ast & \ast & a_{1,N-1} & \ast & a_{1,N+1} & 0 \\
\ast & a_{2,N-2} & \ast & \ast & a_{2,N+1} & 0 \\
\ast & \ast & \ast & \ast & a_{3,N+1} & 0 \\
\vdots & \vdots & \vdots & \ast & \ast & \ast & 0 \\
a_{n-1,N-n+1} & \ast & \ast & a_{n,N-2} & a_{n,N-1} & 0 & a_{n,N+1} \\
a_{n,N-n+1} & \ast & a_{n,N-2} & a_{n,N-1} & 0 & a_{n,N+1} & 0 \\
0 & 0 & a_{n+1,N-n+1} & a_{n+1,N-2} & a_{n+1,N-1} & a_{n+1,N} & 0 & 1
\end{pmatrix}.
\]

Again, we only need to prove that (P5) holds for \( B \).

As in Step I, take some \( J \subset \{1, \ldots, N + 2\} \) and suppose that \( N + 2 \in J^c \) and \( \text{rank}(B(:, J^c)) < n + 1 \) on some sample set \( \Omega' \) which is of positive probability. Then we have \( \text{rank}(A(:, J|_N)) = n \) a.s. on \( \Omega' \).

There are three cases.

(i). \( N + 1 \in J^c \)

In this case, we conclude that

(a) \( \text{rank}(B(1..n, J^c|_N)) \leq n - 2 \), a.s. on \( \Omega' \);
(b) there is some \( 1 \leq j_0 \leq n - 1 \) such that \( N - j_0 \in J \).

In fact, if there is some \( \Omega'' \subset \Omega' \) with positive probability such that \( \text{rank}(B(1..n, J^c|_N)) = n - 1 \) a.s. on \( \Omega'' \), then we see from Lemma 2.3 that \( \text{rank}(B(1..n, J^c|_N \cup \{N + 1\})) = n \) a.s. on \( \Omega'' \). By Lemma 2.3, we get
rank \( (B(:, J^c)) = n + 1 \) a.s. on \( \Omega'' \), which contradicts with the assumption. This proves (a).

On the other hand, if \( N - j \in J^c \) for any \( 1 \leq j \leq n - 1 \), then the expansion of the determinant of \( B(:, \{ N - n + 1, N - n + 2, \ldots, N - 1, N + 1, N + 2 \} ) \) contains the term \( A(n, N + 1) \cdot 1 \cdot \prod_{i=1}^{n-1} A(i, N - i) \), which is not zero a.s. By Lemma 2.2, rank \( (B(:, J^c)) = n + 1 \) a.s. on \( \Omega' \). Again, we get a contradiction with the assumption. Hence (b) holds.

We see from (a) and (b) that rank \( (B(1..n, J^c|N \cup \{N - j_0\})) \leq n - 1 \), a.s. on \( \Omega'. \) Since \( A \) is a PR frame a.s., we have rank \( (B(1..n, J|N \setminus \{N - j_0\})) = n \) a.s. Now we see from Lemma 2.3 that rank \( (B(:, J|N)) = n + 1 \) a.s. on \( \Omega' \).

(ii). \( N + 1 \in J \) and \( N - j_0 \in J \) for some \( 0 \leq j_0 \leq n - 1 \).

Since rank \( (A(:, J|N)) = n \) a.s. on \( \Omega' \), by Lemma 2.3,

\[
\text{rank} \left( B(\{1, \ldots, n\}, J|N \cup \{N + 1\} \setminus \{N - j_0\}) \right) = n, \quad \text{a.s. on } \Omega'.
\]

Using Lemma 2.3 again, we get

\[
\text{rank} \left( B(:, J|N \cup \{N + 1\}) \right) = n + 1, \quad \text{a.s. on } \Omega'.
\]

Hence

\[
\text{rank} \left( B(:, J) \right) = n + 1, \quad \text{a.s. on } \Omega'.
\]

(iii). \( N + 1 \in J \) and \( N - j \in J^c \) for any \( 0 \leq j \leq n - 1 \).

By (P2), there is some \( 1 \leq i_0 \leq n - 1 \) such that \( A(i_0, N) \neq 0 \). Hence the expansion of the determinant of \( B(:, \{ N - n + 1, N - n + 2, \ldots, N, N + 2 \} ) \) contains the term \( B(n+1, N+2)A(n, N-i_0)A(i_0, N) \prod_{1 \leq i \leq n-1, i \neq i_0} A(i, N - i) \), which is not zero a.s. By Lemma 2.2, rank \( (B(:, J^c)) = n + 1 \) a.s. on \( \Omega' \), which contradicts with the assumption.

(IV). For \( 2n \leq N \leq n(n+1)/2 \), there exist \( n \times N \) matrices satisfying (P1) \( \sim \) (P5).

Let \( K_n \) be the set of all integers \( k \) such that there exists an \( n \times k \) matrix \( A \) satisfying (P1) \( \sim \) (P5).

Since \( K_3 \supseteq \{6\} \), we see from the previous arguments that

\[
K_4 \supseteq \{8, 9, 10\},
\]

\[
K_5 \supseteq \{10, 11, 12, 13, 14, 15\}.
\]

Hence for \( 3 \leq n \leq 5 \),

\[
(2.2) \quad K_n \supseteq \{k : 2n \leq k \leq n(n+1)/2\}.
\]

Now suppose that (2.2) is true for some \( n \geq 5 \). Since \( 2n + (n+1) \leq n(n+1)/2 + 2 \) for \( n \geq 5 \), we have

\[
\{k + 2 : 2n \leq k \leq n(n+1)/2\} \cup \{k + n : 2n \leq k \leq n(n+1)/2\}
\]

\[
\cup \{k + n + 1 : 2n \leq k \leq n(n+1)/2\}
\]

\[
= \{k : 2(n+1) \leq k \leq (n+1)(n+2)/2\}.
\]

Hence \( K_{n+1} \supseteq \{k : 2(n+1) \leq k \leq (n+1)(n+2)/2\} \). By induction, (2.2) is true for \( n \geq 3 \).
Finally, since columns of a randomly generated \( n \times (2n - 1) \) matrix form an exact PR frame almost surely, we get the conclusion as desired. □

The following are some explicit examples for \( n = 5 \) and \( 10 \leq N \leq 15 \). In each case, column vectors of \( A \) form an exact PR frame. Moreover, such matrices correspond to exact PR frames almost surely if the non-zero entries are replaced with independent continuous random variables.

\[ (n, N) = (5,10): \]
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 6 & 4 & 2 & 11 & 0 \\
0 & 1 & 0 & 0 & 0 & 13 & 10 & 8 & 0 & 3 \\
0 & 0 & 1 & 0 & 0 & 7 & 7 & 0 & 9 & 8 \\
0 & 0 & 0 & 1 & 0 & 16 & 0 & 8 & 30 & 13 \\
0 & 0 & 0 & 0 & 1 & 0 & 4 & 12 & 14 & 18
\end{pmatrix}
\]

\[ (n, N) = (5,11): \]
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 5 & 0 & 3 & 35 & 7 & 0 \\
0 & 1 & 0 & 0 & 0 & 18 & 0 & 14 & 27 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 23 & 5 & 0 & 1 & 14 \\
0 & 0 & 0 & 1 & 0 & 0 & 8 & 0 & 14 & 7 & 14 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 30 & 3 & 14
\end{pmatrix}
\]

\[ (n, N) = (5,12): \]
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 7 & 0 & 10 & 10 & 11 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 4 & 0 & 7 & 16 & 0 & 15 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 16 & 2 & 0 & 2 & 3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 23 & 3 & 0 & 9 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 12 & 2 & 11 & 0 & 2
\end{pmatrix}
\]

\[ (n, N) = (5,13): \]
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 12 & 16 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 6 & 0 & 8 & 5 & 0 & 0 & 15 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 9 & 5 & 0 & 0 & 11 & 12 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 16 & 0 & 6 & 1 & 0 & 0 & 8 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 7 & 6 & 0 & 10 & 0 & 9
\end{pmatrix}
\]

\[ (n, N) = (5,14): \]
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 11 & 0 & 20 & 0 & 16 & 4 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 5 & 0 & 0 & 1 & 16 & 0 & 0 & 4 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 3 & 6 & 0 & 0 & 0 & 13 & 8 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 17 & 0 & 0 & 8 & 8 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 3
\end{pmatrix}
\]
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\[(n, N) = (5, 15):\]

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 12 & 0 & 4 & 0 & 7 & 0 & 13 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 17 & 0 & 0 & 3 & 0 & 10 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 8 & 0 & 0 & 0 & 12 & 17 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 15 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 13 & 18
\end{pmatrix}.
\]

3. Phase-retrievable subspaces

We first prove the following special case.

**Proposition 3.1.** Let \( F = \{f_1\}^{n}_{i=1} \) be a basis for \( H \). Then there exists a \( k \)-dimensional maximal \( F \)-PR subspace if and only if \( 1 \leq k \leq [(n + 1)/2] \).

**Proof.** Suppose that \( M \) is a \( k \)-dimensional \( F \)-PR subspace. Then we have that \( n \geq 2k - 1 \) and hence \( k \leq (n + 1)/2 \). For the other direction, note that for each invertible operator \( T \) on \( H \), \( M \) is an maximal \( F \)-PR subspace if and only if \( (T')^{-1}M \) is a maximal \( T^*F \)-PR subspace. So it suffices to show that for each \( k \)-dimensional subspace \( M \) with \( 1 \leq k \leq [(n + 1)/2] \) there exists a basis \( \{u_i\}^{n}_{i=1} \) such that \( M \) is an maximal PR subspace with respect to \( \{u_i\}^{n}_{i=1} \).

Let \( \{\varphi_j\}^{2k-1}_{j=1} \subset M \) be a PR-frame for \( M \). Without losing the generality we can assume that \( \{\varphi_1, ..., \varphi_k\} \) is an orthonormal basis for \( M \). Extend it to an orthonormal basis \( \{e_i\}^{n}_{i=1} \) for \( H \), where \( e_i = \varphi_i \) for \( i = 1, ..., k \). Define \( u_i \) by

\[
u_i = e_i \quad (i = 1, ..., k, 2k, ..., n) \quad \text{and} \quad u_i = e_i + \varphi_i \quad (i = k + 1, ..., 2k - 1).
\]

Let \( P_M \) be the orthogonal projection onto \( M \). Clearly we have

\[
\{P_M u_i\}^{n}_{i=1} = \{\varphi_1, ..., \varphi_{2k-1}, 0, ..., 0\},
\]

and hence \( \{u_i\}^{n}_{i=1} \) is a phase-retrievable for \( M \). It is also easy to verify that \( \{u_i\}^{n}_{i=1} \) is a basis for \( H \). Now we show that \( M \) is an maximal PR subspace with respect to \( \{u_1, ..., u_n\} \). Let \( \hat{M} = \text{span} \{M, u\} \) with \( u = \sum_{j=k+1}^{n} a_j e_j \) in \( M^\perp \) and \( ||u|| = 1 \). Then \( P_M u_i = e_i \) for \( 1 \leq i \leq k \), \( P_M u_i = \varphi_i + a_i u \) for \( k + 1 \leq i \leq 2k - 1 \) and \( P_M u_i = a_i u \) for \( i \geq 2k - 1 \). If \( a_i = 0 \) for \( i = 2k, ..., n \), then \( \{P_M u_i\}^{n}_{i=1} \) is not phase-retrievable for \( \hat{M} \) since it only contains at most \( 2k - 1 \) nonzero elements. If \( a_{i_0} \neq 0 \) for some \( i_0 \geq 2k \), then clearly \( \{P_M u_i\}^{n}_{i=1} \) is phase-retrievable for \( \hat{M} \) if and only if \( \{P_M u_i\}^{2k-1}_{i=1} \cup \{a_i u\} \) is phase-retrievable for \( \hat{M} \). Thus \( \hat{M} \) is not a PR subspace with respect to \( \{u_1, ..., u_n\} \) since we need at least \( 2k + 1 \) number of elements in a phase-retrievable frame for the \( (k + 1) \)-dimensional space \( \hat{M} \). \( \square \)

Now lets consider the general frame case: Let \( F \) be a frame for \( H \). For each subset \( \Lambda \) of \( \{1, ..., N\} \), let

\[
d_\Lambda = \max \{\dim \text{span} (F_\Lambda), \dim \text{span} (F_{\Lambda^c})\}.
\]
Define
\[ d(\mathcal{F}) = \min\{d_\Lambda : \Lambda \subset \{1, \ldots, N\}\}. \]

**Theorem 3.2.** Let \( \mathcal{F} \) be a frame for \( H \). Then \( k \) is the largest integer such that there exists a \( k \)-dimensional maximal \( \mathcal{F} \)-PR subspace if and only if \( k = d(\mathcal{F}) \).

Clearly, \( d(\mathcal{F}) = n \) if and only if \( \mathcal{F} \) has the complement property. Thus the above theorem is a natural generalization of Proposition 1.1. We need to follow lemma for the proof of Theorem 3.2.

**Lemma 3.3.** Let \( T x = \sum_{i=1}^{k} \langle x, x_k \rangle x_k \) be a rank-\( k \) operator and \( M \) be a subspace of \( H \) such that \( \dim TM = k \), then \( \dim P(M) = k \), where \( P \) is the orthogonal projection onto \( \text{span}\{x_1, \ldots, x_k\}\).

**Proof.** Since \( \langle x, x_k \rangle = \langle Px, x_k \rangle \), we get that \( \text{range}(T|_M) = \text{range}(T|_{PM}) \). Thus \( \dim P(M) \geq k \) and hence \( \dim P(M) = k \). \( \Box \)

**Proof of Theorem 3.2** Clearly we only need to prove that if \( d(\mathcal{F}) = k \), then there exists a \( k \)-dimensional \( \mathcal{F} \)-PR subspace and every \((k+1)\)-dimensional subspace is not phase-retrievable with respect to \( \mathcal{F} \).

Suppose that \( M \) is a \((k+1)\)-dimensional subspace of \( H \) and it is also phase-retrievable with respect to \( \mathcal{F} \). Then, by Proposition 1.1 we get that \( d(P \mathcal{F}) = k+1 \), and hence \( d(\mathcal{F}) \geq d(P \mathcal{F}) \geq k+1 \), which leads to a contradiction. Therefore every \((k+1)\)-dimensional subspace is not phase-retrievable with respect to \( \mathcal{F} \).

Next we show that there exists a \( k \)-dimensional \( \mathcal{F} \)-PR subspace. Let \( \Omega \) be a subset of \( \{1, \ldots, N\} \) be such that \( \dim H_\Omega \geq k \), where \( H_\Omega = \text{span}\, \mathcal{F}_\Omega \). For \( X = (x_1, \ldots, x_k) \in H^{(k)} := H \oplus \ldots \oplus H \), define \( T_X(z) = \sum_{i=1}^{k} \langle z, x_k \rangle x_k \).

Consider the following set
\[ S_\Omega = \{(x_1, \ldots, x_k) \in H^{(k)} : \dim T_X(H_\Omega) = k\}. \]

Since \( \dim \text{span}\, \mathcal{F}_\Omega \geq k \), we get that there exists a linearly independent set \((f_{i_1}, \ldots, f_{i_k})\) in \( \mathcal{F}_\Omega \). This implies that \((f_{i_1}, \ldots, f_{i_k}) \in S_\Omega \) and hence \( S_\Omega \) is not empty.

Moreover, since \( \dim T_X(H_\Omega) = k \) if and only if there exists an \( k \times k \) submatrix of the \( n \times |\Omega| \) matrix \([T_X f_\omega]\) whose determinant is a nonzero polynomial of the input variables \( x_1, \ldots, x_k \), we obtain that \( S_\Omega \) is open dense in \( H^{(k)} \).

Now for each subset \( \Lambda \) in \( \{1, \ldots, N\} \). Let \( \Omega_\Lambda = \Lambda \) if \( d_\Lambda = \dim \text{span}\,(\mathcal{F}_\Lambda) \), and otherwise \( \Omega_\Lambda = \Lambda^c \). Thus we have \( \dim \text{span}\, \mathcal{F}_{\Omega_\Lambda} \geq k \) for every subset \( \Lambda \). Since each \( S_{\Omega_\Lambda} \) is open dense in \( H^{(k)} \), we get that
\[ S := \bigcap_{\Lambda \subset \{1, \ldots, N\}} S_{\Omega_\Lambda} \]
is open dense in \( H^{(k)} \). Let \( X = (x_1, \ldots, x_k) \in S \) and \( M = \text{span}\{x_1, \ldots, x_k\} \). Then by Lemma 3.3 we obtain that \( \dim P(H_{\Omega_\Lambda}) = k \). This implies that
From the proof of Theorem 3.2, we also have the following:

**Corollary 3.4.** Let $F$ be a frame for $H$. Then for almost all the vectors $(x_1, \ldots, x_\ell)$ in $H(\ell)$ (here $\ell \leq d(F)$), the subspace $\text{span}\{x_1, \ldots, x_\ell\}$ is phase-retrievable with respect to $F$. More precisely, for each $\ell \leq d(F)$, the following set

$$\{(x_1, \ldots, x_\ell) \in H(\ell) : \text{span}\{x_1, \ldots, x_\ell\} \text{ is phase retrievable with respect to } F\}$$

is open dense in $H(\ell)$.

The following lemma follows immediately from the definitions, and it tells us that it is enough to focus on maximal phase-retrievable subspaces for frames with the exact PR-redundancy property.

**Lemma 3.5.** Let $F = \{f_i\}_{i=1}^N$ be a frame for $H$, and $\Lambda \subset \{1, \ldots, N\}$. If $\ker(\Theta_{L(F)}) \cap \mathcal{S}_2 = \ker(\Theta_{L(F)}) \cap \mathcal{S}_2$, then $M$ is a $F$-PR subspace if and only if it is a $F_\Lambda$-PR subspace. Consequently, $M$ is an maximal $F$-PR subspace if and only if it is an maximal $F_\Lambda$-PR subspace.

Now we would like to know what are the possible values of $d(F)$. Since every frame contains a basis, we get by Proposition 3.1 that $k \geq \lceil n/2 \rceil$. The following theorem tells us that for every $k$ between $(n+1)/2$ and $n$, there is a frame $F$ with the exact PR-redundancy property such that $k = d(F)$.

**Theorem 3.6.** Let $H = \mathbb{R}^n$ and $k$ be an integer such that $n \geq k \geq \lceil n/2 \rceil$. Then for each $N$ between $2k-1$ and $k(k+1)/2 + (n-k)(n-k+1)/2$, there exists a frame $F$ of length $N$ such that it has the exact PR-redundancy property and $d(F) = k$, i.e., $k$ is the largest integer such that there exists a $k$-dimensional maximal $F$-PR subspace.

Before giving the proof we remark while the proof of the this theorem uses Theorem 2.1 it is also a generalization of Theorem 2.1 since it clearly recovers Theorem 2.1 if we let $n = k$.

**Proof.** Let $M$ be a $k$-dimensional subspace of $H$. For each $k$

We divide the proof into two cases.

Case (i). Assume that $2k-1 \leq N \leq k(k+1)/2$.

By Theorem 2.1 there exists an exact PR-frame $\mathcal{G} = \{g_i\}_{i=1}^N$ for $M$. Without losing the generality we can also assume that $\{g_1, \ldots, g_k\}$ is an orthonormal basis for $M$. Extend it to an orthonormal basis $\{e_i\}_{i=1}^n$ with $e_1 = g_1, \ldots, e_k = g_k$. Let

$$F = \{f_i\}_{i=1}^N = \{e_1, \ldots, e_k, g_{k+1} + e_{k+1}, \ldots, g_n + e_n, g_{n+1}, \ldots, g_N\}.$$ 

Then it is a frame for $H$. Consider the subset $\Lambda = \{1, \ldots, k, n+1, \ldots, N\}$ of $\{1, \ldots, N\}$. We have $\text{dim span}\mathcal{F}_\Lambda = \dim M = k$, and $\text{dim span}\mathcal{F}_\Lambda \leq n - k$. Note that from $k \geq \lceil n/2 \rceil$ we get that $n - k \leq k$. Thus we have
\[ d(F) \leq \max\{n - k, k\} = k. \] On the other hand, it is easy to prove that \( d(F) \geq d(P_M F) = d(G) = k \), where \( P_M \) is the orthogonal projection onto \( M \). Therefore we have \( d(F) = k \).

Now we show that \( F \) has the exact PR-redundancy property. If fact, if \( \Lambda \) is a proper subset of \( \{1, \ldots, N\} \), then \( P_M F_\Lambda \) is not a PR frame for \( M \) since \( P_M F = G \) is an exact PR-frame for \( M \). Therefore, there exists \( x \) and \( y \) in \( M \) such that \( \langle x, P_M f_i \rangle = \langle y, P_M f_i \rangle \) for all \( i \in \Lambda \) and \( A = x \otimes x - y \otimes y \neq 0 \).

Since \( P_M F \) is a PR-frame for \( M \), we obtain that \( \langle x, P_M f_i \rangle \neq \langle y, P_M f_i \rangle \) for some \( i \in \Lambda^c \). Note that \( \langle z, f_i \rangle = \langle z, P_M f_i \rangle \) for every \( z \in M \). Therefore, we have that \( A \in \ker \Theta_{L(F_\Lambda)} \cap S_2 \) but \( A \notin \ker \Theta_{L(F)} \cap S_2 \), and hence \( \ker \Theta_{L(F_\Lambda)} \cap S_2 \neq \ker \Theta_{L(F)} \cap S_2 \) for any proper subset \( \Lambda \). So \( F \) has the exact PR-redundancy property.

Case (ii): Assume that \( k(k+1)/2 < N \leq k(k+1)/2 + (n-k)(n-k+1)/2 \). Since \( k \geq [(n+1)/2] \geq n/2 \), it is easy to verify that

\[
 k(k+1)/2 \geq (2k-1) + 2(n-k) - 1 = 2n - 2.
\]

Then we can write \( N = N_1 + N_2 \) such that

\[
 2k-1 \leq N_1 \leq k(k+1)/2 \quad \text{and} \quad 2(n-k) - 1 \leq N_2 \leq (n-k)(n-k+1)/2.
\]

By Theorem 2.1, there exist an exact PR-frame \( F_1 \) of length \( N_1 \) for \( M \) and an exact PR-frame \( F_2 \) of length \( N_2 \) for the \( M^\perp \). By Lemma 1.5, we know that \( F = F_1 \cup F_2 \) is a frame of length \( N \) with the exact PR-redundancy property. Clearly \( d(F) \leq k \) since

\[
 \max\{\dim \text{span} F_1, \dim \text{span} F_2\} = k.
\]

On the other hand, since \( F \) has a \( k \)-dimensional PR-subspace \( M \), we get from Theorem 3.2 that \( d(F) \geq k \). Thus we have \( d(F) = k \).

The following example shows that \( k(k+1)/2 + (n-k)(n-k+1)/2 \) is not necessarily the upper bound of \( N \) such that there exists a frame \( F \) of length \( N \) with the exact PR-redundancy property and \( d(F) = k \).

**Example 3.1.** Let \( \{e_1, e_2, e_3\} \) be an orthonormal basis for \( \mathbb{R}^3 \). Consider the following frame

\[
 F = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_2 + e_3\}.
\]

Then \( k = d(F) = 2 \) and \( 5 > k(k+1)/2 + (3-k)(3-k+1)/2 = 4 \). We can check that \( F \) has the exact PR-redundancy property. Let \( G \) be the frame after removing an element \( f \) from \( F \). Based on the following five cases, we can easily construct \( A = x \otimes x - y \otimes y \) such that \( A \neq 0 \), \( A \in \ker \Theta_{L(G)} \) but \( A \notin \ker \Theta_{L(F)} \):

(i) \( f = e_1 \): Let \( x = 2e_1 = e_2 \) and \( y = 4e_1 - e_2 \).

(ii) \( f = e_2 \): Let \( x = 2e_2 + e_1 \) and \( y = 4e_2 - e_1 \).

(iii) \( f = e_3 \): Let \( x = e_1 + e_3 \) and \( y = e_1 - 3e_3 \).

(iv) \( f = e_1 + e_2 \): Let \( x = e_1 + (e_2 + e_3) \) and \( y = e_1 - (e_2 + e_3) \).

(v) \( f = e_1 + e_2 + e_3 \): Let \( x = e_1 + e_3 \) and \( y = e_1 - e_3 \).
Proposition 3.7. Let \( H = \mathbb{R}^n \). Suppose that a frame \( F \) of length \( N \) has the exact PR-redundancy property and \( d(F) < n \). Then \( N < n(n + 1)/2 \).

Proof. Since \( F \) has the exact PR-redundancy, we get that \( N \leq n(n + 1)/2 \). If \( N = n(n + 1)/2 \), then, by Lemma 3.3, \( \{f_i \otimes f_i\}_N \) is linearly independent and hence a basis for \( \mathbb{H}_n \). This implies that \( F \) is phase-retrievable and so \( d(F) = n \). This contradiction shows that \( N < n(n + 1)/2 \). \( \square \)

Question. Give an integer \( k \) such that \( n > k \geq \lceil \frac{n+1}{2} \rceil \). What is the least upper bound \( N \) such that there exists a frame \( F \) of length \( N \) which has the exact PR-redundancy property and \( d(F) = k \)?

4. Maximal Phase-Retrievable Subspaces with respect to bases

Given a basis \( F = \{f_1, \ldots, f_n\} \). We would like to have a better understanding about the maximal phase-retrievable subspaces with respect to \( F \). We will first focus on orthonormal bases and then use the similarity to pass to general bases.

Now we assume that \( \mathcal{E} = \{e_1, \ldots, e_n\} \) is an orthonormal basis for \( \mathbb{R}^n \). By Proposition 3.1 we know that there exists a \( k \)-dimensional maximal \( \mathcal{E} \)-PR subspace for every integer \( k \) with \( 1 \leq k \leq \lceil \frac{n+1}{2} \rceil \). What more can we say about these \( k \)-dimensional maximal \( \mathcal{E} \)-PR subspaces? We explore its connections with the support property of vectors in these subspaces. Recall that for a vector \( x = \sum_{i=1}^{n} \alpha_i e_i \in \mathbb{R}^n \), the support of \( x \) is defined by \( \text{supp}_\mathcal{E}(x) := \{i | \alpha_i \neq 0\} \). We will also use \( \text{supp}(x) \) to denote \( \text{supp}_\mathcal{E}(x) \) if \( \mathcal{E} \) is well understood in the statements, and use \( |\Lambda| \) to denote the cardinality of any set \( \Lambda \).

Proposition 4.1. Suppose that \( M \) is a \( k \)-dimensional \( \mathcal{E} \)-PR subspace. Then for any nonzero vector \( x \in M \), we have \( |\text{supp}(x)| \geq k \).

Proof. Assume to the contrary that there exists a nonzero \( x \in M \) with \( |\text{supp}(x)| = j < k \). We may assume that \( \|x\| = 1 \) and that \( \text{supp}(x) = \{1, 2, \ldots, j\} \). Pick vectors \( y_1, \ldots, y_{k-1} \) in \( M \) such that the set \( \{x, y_1, \ldots, y_{k-1}\} \) is an orthonormal basis for \( M \). Then we have,

\[
\begin{align*}

P_M(e_1) &= \langle e_1, x \rangle x + \langle e_1, y_1 \rangle y_1 + \cdots + \langle e_1, y_{k-1} \rangle y_{k-1} \\
P_M(e_2) &= \langle e_2, x \rangle x + \langle e_2, y_1 \rangle y_1 + \cdots + \langle e_2, y_{k-1} \rangle y_{k-1} \\
&\vdots \\
P_M(e_j) &= \langle e_j, x \rangle x + \langle e_j, y_1 \rangle y_1 + \cdots + \langle e_j, y_{k-1} \rangle y_{k-1} \\
P_M(e_{j+1}) &= \langle e_{j+1}, y_1 \rangle y_1 + \cdots + \langle e_{j+1}, y_{k-1} \rangle y_{k-1} \\
&\vdots \\
P_M(e_n) &= \langle e_n, y_1 \rangle y_1 + \cdots + \langle e_n, y_{k-1} \rangle y_{k-1}.
\end{align*}
\]
The partition \( \{ P_{M}(e_1), ..., P_{M}(e_j) \} \) and \( \{ P_{M}(e_{j+1}), ..., P_{M}(e_n) \} \) does not have the complement property since the first set contains less than \( k \) elements and the members of the second set are all contained in the \((k-1)\)-dimensional subspace \( \text{span} \{ y_1, ..., y_{k-1} \} \). Thus \( M \) is not a \( E-PR \) subspace, which leads to a contradiction.

**Corollary 4.2.** If \( M \) is a \( k \)-dimensional \( E-PR \) subspace and there exists \( x \in M \) such that \( |\text{supp}(x)| = k \), then \( M \) is maximal.

Now suppose that \( k \leq [(n+1)/2] \). Let \( x \in H \) be a vector of norm one and \( |\text{supp}(x)| = k \). We show that \( x \) can be extended to an orthonormal set \( \{ x, u_1, ..., u_{k-1} \} \) such that \( M = \text{span} \{ x, u_1, ..., u_{k-1} \} \) is a \( k \)-dimensional \( E-PR \) subspace.

**Theorem 4.3.** Let \( u_1 \in \mathbb{R}^n \) be a unit vector such that \( |\text{supp}(u_1)| = k \) and \( k \leq [(n+1)/2] \). Then \( u_1 \) can be extended to an orthonormal set \( \{ u_1, ..., u_k \} \) such that \( M = \text{span} \{ u_1, ..., u_k \} \) is a \( k \)-dimensional maximal \( E-PR \) subspace.

**Proof.** We can assume that \( \{ e_1, ..., e_n \} \) is the standard orthonormal basis for \( \mathbb{R}^n \) and \( u_1 = \sum_{i=1}^{k} \alpha_i u_i \) such that \( \alpha_i \neq 0 \) for every \( 1 \leq i \leq k \).

It is easy to observe the following fact: Let \( m : 1 \leq m \leq k \). Suppose that \( \{ u_1, ..., u_m \} \) is an orthonormal set extension of \( u_1 \) and

\[
A(u_1, ..., u_m) = [u_1, ..., u_m].
\]

Let \( A_{\Lambda}(u_1, ..., u_m) \) be the matrix consisting of row vectors corresponding to \( \Lambda \). If \( A_{\Lambda}(u_1, ..., u_m) \) is invertible for every subset \( \Lambda \) of \( \{1, ..., n\} \) of cardinality \( k \) with the property that \( \Lambda \cap \{1, ..., k\} \neq \emptyset \), then the row vectors of \( A(u_1, ..., u_m) \) form a frame for \( \mathbb{R}^m \) that has the complement property.

Now we use the induction to show that such an matrix \( A(u_1, ..., u_m) \) exists for every \( m \in \{1, ..., k\} \). Clearly, the \( n \times 1 \) matrix \( A(u_1) \) satisfies the requirement. Now assume that such an \( n \times m \) matrix \( A(u_1, ..., u_m) \) has been constructed and \( m < k \). We want to prove that there exists a unit vector \( u_{m+1} \perp u_i (1 \leq i \leq m) \) such that \( A(u_1, ..., u_m, u_{m+1}) \) has the required property.

Let \( U = \text{span} \{ u_1, ..., u_m \} \) and let \( \Lambda \) be a subset of \( \{1, ..., n\} \) such that \( |\Lambda| = m + 1 \) and \( \Lambda \cap \{1, ..., k\} \neq \emptyset \). Define

\[
\Omega_{\Lambda} = \{ u \in U : A_{\Lambda}(u_1, ..., u_m, u) \text{ is invertible} \}.
\]

We claim that \( \Omega_{\Lambda} \) is an open dense subset of \( U \).

Using the fact that the set of invertible matrices form an open set in the space of all matrices, it is clear that \( \Omega_{\Lambda} \) is open in \( U \).

Now we show that \( \Omega_{\Lambda} \neq \emptyset \). Let \( \Lambda' \) be a subset of \( \Lambda \) with cardinality \( m \) and \( \Lambda' \cap \{1, ..., k\} \neq \emptyset \). Then, by our induction assumption, we have that \( A_{\Lambda'}(u_1, ..., u_m) \) is invertible, which implies that the \( m \) column vectors of \( A_{\Lambda}(u_1, ..., u_m) \) form a linearly independent set in the \( m+1 \) dimensional space \( \mathbb{R}^A = \Pi_{i \in \Lambda} \mathbb{R} \). Let \( z \in \mathbb{R}^{m+1} \) be a nonzero vector such that it is orthogonal to all the column vectors of \( A_{\Lambda}(u_1, ..., u_m) \). Define \( u = (u_1, ..., u_m)^T \in \mathbb{R}^n \) by
letting \( u_i = z_i \) for \( i \in \Lambda \), and 0 otherwise. Then \( u \in U \) and hence \( u \in \Omega_\Lambda \). Therefore we get that \( \Omega_\Lambda \neq \emptyset \).

For the density of \( \Omega_U \), let \( y \in U \) be an arbitrary vector and pick a vector \( u \in \Omega_\Lambda \). Consider the vector \( u_t = tu + (1 - t)y \in U \) for \( t \in \mathbb{R} \). Since \( A_\Lambda(u_1, \ldots, u_m, u) \) is invertible, we have that \( \det(A_\Lambda(u_1, \ldots, u_m, u_t)) \) is a nonzero polynomial of \( t \), and hence it is finitely many zeros. This implies that there exists a sequence \( \{t_j\} \) such that \( u_{t_j} \in \Omega_\Lambda \) and \( \lim_{j \to \infty} t_j = 0 \). Hence \( u_{t_j} \to y \) and therefore \( \Omega_U \) is dense in \( U \).

By the Baire Category theorem we obtain that the intersection \( \Omega \) of all such \( \Omega_\Lambda \) is open dense in \( Y \). Pick any \( u_{m+1} \in \Omega \), then \( A(u_1, \ldots, u_m, u_{m+1}) \) has the required property. This completes the induction proof for the existence of such an matrix \( A = [u_1, \ldots, u_k] \), where \( \{u_1, \ldots, u_k\} \) is an orthonormal set extending the given vector \( u_1 \).

Write \( u_j = (a_{1j}, a_{2j}, \ldots, a_{nj})^T \) for \( 1 \leq j \leq k \). Let \( M = \operatorname{span}\{u_1, \ldots, u_k\} \) and \( P \) be the orthogonal projection onto \( M \). Then

\[
P e_i = \sum_{j=1}^{k} <e_i, u_j> u_j = \sum_{j=1}^{n} a_{ij} u_j
\]

for all \( 1 \leq i \leq n \). For every subset \( \Lambda \) of \( \{1, \ldots, n\} \), since \( \{u_1, \ldots, u_k\} \) is an orthonormal set, we have that \( \{P e_j : j \in \Lambda\} \) are linearly independent if and only if \( A_\Lambda \) is invertible. Thus, \( \{P e_i\}_{i=1}^{n} \) has the complement property since the set of row vectors of \( A \) has the complement property.

**Remark 4.1.** Note that from the proof it is easy to see that the existence of such an matrix \( A(u_1, \ldots, u_k) \) does not require the condition \( k \leq [(n + 1)/2] \). However, the complement property of the row vectors for \( \mathbb{R}^k \) does require this condition.

We know that if \( M \) is a \( k \)-dimensional PR-subspace with respect to an orthonormal basis \( E \), then the condition \( \min\{|\text{supp}(x)| : 0 \neq x \in M\} = k \) is sufficient for \( M \) to be maximal. The following example show that this condition is not necessary in general. However, we will prove in Theorem 4.4 that it is indeed also necessary if \( k < [\frac{n+1}{2}] \).

**Example 4.1.** There exists a 2-dimensional maximal PR-subspace \( M \) in \( \mathbb{R}^4 \) such that \( |\text{supp}(x)| = 3 \) for every nonzero \( x \in M \). Indeed, let \( \{e_1, e_2, e_3, e_4\} \) and orthonormal basis for \( \mathbb{R}^4 \) and be \( M = \operatorname{span}\{e_1 + e_2 + e_3, e_1 - e_2 + e_4\} \). Then it can be easily verified that \( M \) is a PR-subspace and \( |\text{supp}(x)| = 3 \) for every nonzero \( x \in M \). It is clear that \( M \) is maximal since there is no 3-dimensional PR-subspace with respect to \( \{e_1, e_2, e_3, e_4\} \) in \( \mathbb{R}^4 \). It is interesting to note that if we view \( M \) as a subspace in \( \mathbb{R}^n \) with \( n \geq 5 \), then \( M \) is no longer maximal anymore.

**Theorem 4.4.** Assume that \( M = \operatorname{span}\{u_1, \ldots, u_k\} \) is a \( k \)-dimensional maximal PR-subspace with respect to \( \{e_1, \ldots, e_n\} \) and \( k < [\frac{n+1}{2}] \). Then \( k = \min\{|\text{supp}(x)| : 0 \neq x \in M\} \).
Proof. By Proposition 4.1, it suffices to show there is a nonzero vector \( x \in M \) such that \( |\text{supp}(x)| \leq k \).

Let \( \{u_1, ..., u_k\} \) be an orthonormal basis for \( M \). We adopt the notation used in the proof of Theorem 4.3. For every subset \( \Lambda \) of \( \{1, ..., n\} \), let \( A_\Lambda(u_1, ..., u_k) \) be the matrix consisting of row vectors of \([u_1, ..., u_k]\) corresponding to the row index in \( \Lambda \). It is obvious that if there is a subset \( \Lambda \) with \(|\Lambda| = n - k \) such that \( \text{rank} A_\Lambda(u_1, ..., u_k) < k \), then there is a nonzero vector \( x \in M \) such that \( \text{supp}(x) \subseteq \Lambda^c \) and hence \( |\text{supp}(x)| \leq k \). We will prove that such a subset \( \Lambda \) exists.

Assume, to the contrary, that \( \text{rank} A_\Lambda(u_1, ..., u_k) = k \) for any subset \( \Lambda \) with \(|\Lambda| = n - k \). Thus we have \( \text{rank} A_\Lambda(u_1, ..., u_k) = k \) for any subset \( \Lambda \) with \(|\Lambda| \leq n - k \).

For each subset \( \Lambda \), since \( k < \frac{n+1}{2} \), we only have three possible cases:

(i) \(|\Lambda| \geq n - k \) and \(|\Lambda^c| < n - k \).
(ii) \(|\Lambda^c| \geq n - k \) and \(|\Lambda| < n - k \).
(iii) \(|\Lambda| < n - k \) and \(|\Lambda^c| < n - k \).

Note that case (iii) implies that \(|\Lambda| > k \) and \(|\Lambda^c| > k \). Now we assign each \( \Lambda \) to a subset \( S(\Lambda) \) by the following rule: Set \( S(\Lambda) \) to be \( \Lambda \) or \( \Lambda^c \) depending case (i) or case (ii). Suppose that \( \Lambda \) satisfies (iii). Since the row vectors of \([u_1, ..., u_k]\) has the complement property, we have that either \( \text{rank} A_\Lambda(u_1, ..., u_k) = k \) or \( \text{rank} A_{\Lambda^c}(u_1, ..., u_k) = k \). In this case we set \( S(\Lambda) = \Lambda \) if \( \text{rank} A_\Lambda(u_1, ..., u_k) = k \), and otherwise set \( S(\Lambda) = \Lambda^c \). Let

\[
S = \{ S(\Lambda) : \Lambda \subseteq \{1, ..., n\} \}.
\]

Then for each \( \Lambda \) we have either \( S(\Lambda) = \Lambda \) or \( S(\Lambda) = \Lambda^c \), \( \text{rank} A_{S(\Lambda)}(u_1, ..., u_k) = k \) and \(|S(\Lambda)| \geq k + 1 \).

Let \( U = \text{span}\{u_1, ..., u_k\}^\perp \) and \( \Omega_\Lambda = \{ u \in U : \text{rank} A_{S(\Lambda)}(u_1, ..., u_k, u) = k + 1 \} \).

Then by the exact same argument as in the proof of Theorem 4.3, we get that \( \Omega_\Lambda \) is open dense in \( U \). The Baire-Category theorem implies that there exists unit vector \( u_{k+1} \in U \) such that \( \text{rank} A_{S(\Lambda)}(u_1, ..., u_k, u_{k+1}) = k + 1 \) for every subset \( \Lambda \subseteq \{1, ..., n\} \). This shows that the row vectors of the matrix \([u_1, ..., u_k, u_{k+1}]\) has the complementary property, and hence \( \text{span}\{u_1, ..., u_k, u_{k+1}\} \) is a PR-subspace with respect to the orthonormal basis \( \{e_1, ..., e_n\} \), which contradicts the maximality of \( M \).

Example 4.2. Let \( F = \{e_1, ..., e_n\} \) be an orthonormal basis for \( \mathbb{R}^n \). Then \( M = \text{span}\{x\} \) be a one-dimensional maximal \( F \)-PR subspace if and only if \(|\text{supp}(x)| = 1 \).

Example 4.3. Let \( x \in \mathbb{R}^n \) be a unit vector such that \(|\text{supp}(x)| = 2 \) and \( M \) be a 2-dimensional subspace containing \( x \). Then \( M \) is maximal \( F \)-PR subspace if and only if there exists an orthonormal basis \( \{x, y\} \) for \( M \) such that \( y = y_1 + y_2 \) with \( 0 \neq y_1 \in \text{span}\{e_i : i \in \text{supp}(x)\} \) and \( 0 \neq y_2 \in \text{span}\{e_i : i \notin \text{supp}(x)\} \). Indeed, by Corollary 4.2, it suffices to show that
\( M \) is a \( F - PR \) subspace. We may assume that \( \text{supp}(x) = \{1, 2\} \). Then we have

\[
\begin{align*}
P_M(e_1) &= \langle e_1, x \rangle x + \langle e_1, y_1 \rangle y \\
P_M(e_2) &= \langle e_2, x \rangle x + \langle e_2, y_1 \rangle y \\
P_M(e_3) &= \langle e_3, y_2 \rangle y \\
&\quad \vdots \\
P_M(e_n) &= \langle e_n, y_2 \rangle y
\end{align*}
\]

Then it is easy to check that \( \{P_M e_i\} \) has the complement property if and only if \( \{P_M e_1, P_M e_2\} \) are linearly independent, and \( \langle e_i, y_2 \rangle \neq 0 \) for some \( 3 \leq i \leq n \). This is in turn equivalent to the conditions that \( y_1 \neq 0 \) and \( y_2 \neq 0 \).

Finally, let examine the general basis case. Let \( F = \{f_1, \ldots, f_n\} \) be a basis for \( \mathbb{R}^n \), and \( F^* = \{f_1^*, \ldots, f_n^*\} \) be its dual basis. Let \( T \) be the invertible matrix such that \( f_i = T e_i \) for all \( i \), where \( E = \{e_1, \ldots, e_n\} \) be the standard orthonormal basis for \( \mathbb{R}^n \). We observe the following facts:

(i) \( M \) is a PR-subspace with respect to \( F \) if and only if \( T^t M \) is a PR-subspace with respect to \( E \).

(ii) The dual basis \( F^* = \{(T^{-1})^t T^{-1} e_i\}_{i=1}^n \), i.e., \( f_i^* = (T^{-1})^t T^{-1} e_i \).

(iii) The coordinate vector of \( x \) with respect to the basis \( F^* \) is the same as the coordinate vector of \( T^t x \) with respect to the basis \( E \).

Based on the above observations we summarize the main results of this section in the following theorem:

**Theorem 4.5.** Let \( F = \{f_1, \ldots, f_n\} \) be a basis for \( \mathbb{R}^n \), and \( F^* = \{f_1^*, \ldots, f_n^*\} \) be its dual basis. Then we have

(i) If \( M \) is a \( k \)-dimensional PR-subspace with respect to \( F \), then \( |\text{supp}_{F^*}(x)| \geq k \) for any nonzero vector \( x \in M \). Consequently, \( M \) is maximal if there exists a vector \( x \in M \) such that \( |\text{supp}_{F^*}(x)| = k \).

(ii) For any vector \( x \in \mathbb{R}^n \) such that \( |\text{supp}_{F^*}(x)| = k \), there exists a \( k \)-dimensional maximal PR-subspace \( M \) with respect to \( F \) such that \( x \in M \).

(iii) If \( k < \lceil (n+1)/2 \rceil \) and \( M \) is a \( k \)-dimensional PR-subspace with respect to \( F \), then \( M \) is maximal if and only if there exists a vector \( x \in M \) such that \( |\text{supp}_{F^*}(x)| = k \).

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Department of Mathematics, University of Central Florida, Orlando, FL 32816
E-mail address: deguang.han@ucf.edu

Department of Mathematics, University of Central Florida, Orlando, FL 32816
E-mail address: tjuste@knights.ucf.edu

College of Mathematics and Information Science, Guangxi University, Nanjing, China
E-mail address: youfalee@hotmail.com

School of Mathematical Sciences, Nankai University, Tianjin, China
E-mail address: sunwch@nankai.edu.cn