Momentum Analyticity and Finiteness of
Compactified String Amplitudes

Part I: Tori

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ABSTRACT

We generalize to the case of compactified superstrings a construction given previously for critical superstrings of finite one loop amplitudes that are well-defined for all external momenta. The novel issues that arise for compactified strings are the appearance of infrared divergences from the propagation of massless strings in four dimensions and, in the case of orbifold schemes, the contribution of tachyons in partial amplitudes with given spin structure and twist sectors. Methods are presented for the resolution of both problems and expressions for finite amplitudes are given in terms of double and single dispersion relations, with explicit spectral densities.
1. Introduction

In previous papers [1,2,3], it was pointed out that one-loop four-point amplitudes in heterotic or Type II superstrings in their critical dimension are properly defined by the standard integral representation over moduli only for purely imaginary values of the Mandelstam variables \( s_{ij} \) [1,4]. Away from imaginary \( s_{ij} \), the integral representation is divergent and, for real (physical) values of \( s_{ij} \), it is formally real. Both the reality and the divergence of the one-loop amplitude are physically unacceptable. The imaginary part of the (forward) one loop amplitude is related – by unitarity – to the absolute value squared of the tree level four point function, which is of course known to be non-zero.

Both problems were solved in [1] for the critical string, by showing that there exists an analytic continuation of the integral representation to all \( s_{ij} \) which exhibits precisely the expected poles and branch cuts in \( s_{ij} \). An explicit construction of this analytic continuation was given in [1] for the case of the four-point function of states in the gravitational multiplet.

In the present paper, we shall extend the analysis used to deal with these problems to the case of superstrings propagating on four dimensional Minkowski space-time \( M_4 \) times a six dimensional compactified space \( K_6 \). A number of important novel features arise for compactified strings that did not present themselves for the critical superstring.

First, the presence of massless particles in four dimensions generically produces infrared (IR) divergences in the scattering amplitudes. IR divergences are familiar already from QED, where they signal the fact that physical charged states contain admixtures of soft photons. The Bloch-Nordsieck theorem guarantees that these IR divergences cancel out of the calculation of any physical cross section. In practice, an IR regulator may be introduced which produces finite but regulator dependent amplitudes. Physical cross sections should be finite once all the relevant contributions have been taken into account and they should be insensitive to the specific regulator used. If the theory contains charged massless particles, as in unbroken non-Abelian gauge theories, the IR divergences become even more severe, and great care must be used to make the IR regulator consistent with all symmetries. For gauge theories, dimensional regularization is in general a suitable regulator, while in supersymmetric theories, dimensional reduction is preferred.

Amplitudes derived from compactified superstring theories are also generally IR divergent, since they contain an unbroken non-Abelian gauge sector as well as gravity. Compared to the critical superstring, the IR divergences occurring for compactified strings present an extra source of infinities (independent of \( s_{ij} \)), not yet encountered in the analysis of [1,2,3]. It is a rather subtle issue to obtain a consistent IR regularization scheme in compactified superstring theory, where modular invariance is of crucial importance. Two regularization schemes that have already been discussed in the literature will be applied here, but both have different drawbacks that we shall point out. We shall show in subsequent work that both schemes lead to well-defined, finite analytic continuations of the compactified amplitudes, consistent with modular invariance.

Second, in the case of orbifold compactifications, extra divergences may arise from the presence of the tachyon in certain partial amplitudes that only involve the Neveu-Schwarz
(NS) sector, or that involve only one twisting. Such tachyon related divergences in fact already arise in the uncompactified critical superstring amplitudes, when partial amplitudes of definite spin structure are considered. For the four-point amplitudes, where spin structure sums are easily carried out explicitly, these tachyon divergences did not have to be dealt with explicitly. Also, for orbifold compactifications, the way the tachyon enters is more complicated than in the uncompactified superstring. By a suitable grouping together of expansion terms of the amplitudes, these tachyon pole and branch cut contributions are easily isolated and neutralized.

In this work we shall consider only the toroidal compactifications of either the IIB superstring or the Heterotic model and produce the analogous results for the case of orbifold models in a different study.

2. Amplitudes in Toroidal Compactifications

We shall consider only amplitudes for external states that have remained massless after the compactification: In the toroidal case, these states include the graviton, a number of Kaluza-Klein $U(1)$ gauge fields that arise from compactifying the metric $G_{\mu i}$ or the anti-symmetric tensor field $B_{\mu i}$ (including the axion). All these states arise from NS-NS states in the uncompactified string and are represented by vertex operators in which the external momenta for the compactified directions are set to zero. They are particularly simple and the calculation of amplitudes involving them is simplified by the fact that the dependence upon the external vertex operators is identical to that of the critical superstring, with non-trivial toroidal states propagating only inside the loop. Restriction to these states thus only requires further evaluation of the partition function for the zero modes of the strings, to be inserted into the integrand for the critical superstring.

The partition function (on a torus worldsheet with complex modulus $\tau$) evaluated for strings compactified on a torus, were given for both the Type II and heterotic strings as a sum over the momenta belonging to the lattice $\Gamma$ that characterizes the compactification:

$$Z(\Gamma) = (\text{Im } \tau)^{10-d} \tilde{Z}(\Gamma),$$

$$\tilde{Z}(\Gamma) = \sum_{(P_L, P_R) \in \Gamma} e^{i\pi \tau P_L \cdot P_L - i\pi \bar{\tau} P_R \cdot P_R}. \quad (2.1)$$

In (2.1), the momenta $(P_L, P_R)$ parameterize the Lorentzian lattice $\Gamma$ with signature $(p, q)$. Modular invariance of $Z(\Gamma)$ requires $\Gamma$ to be self-dual and even, in the sense that the quadratic form $(P_L^2 - P_R^2)/2$ is integer valued. Such lattices are known to exist only when $p - q$ is an integer multiple of 8 [5].

For the Type II string compactified on a symmetric lattice, we have $p = q = 10 - d$, which certainly satisfies this condition; furthermore, all lattices $\Gamma$ are obtained by taking $p$ copies of the two-dimensional lattice $\Gamma_2$ with signature $(1, 1)$. The geometric means of defining the lattice is through the background flat metric $g_{ij}$ and constant anti-symmetric tensor field $B_{ij}$ on the $10 - d$-dimensional torus: The lattice momenta are conveniently
parametrized in terms of these as follows:

\[ P^i_L = n^i + \frac{1}{2}g^{ij}m_j - g^{ij}B_{jk}n^k, \]
\[ P^i_R = n^i - \frac{1}{2}g^{ij}m_j + g^{ij}B_{jk}n^k, \]  

(2.2)

where \( g^{ij} \) is the inverse matrix of \( g_{ij} \) with \( n^i \) and \( m_j \) integers. (The metric \( g_{ij} \) may be viewed as the matrix of inner products of the basis vectors of the lattice \( \Gamma \).)

In the case of the heterotic string, one has \( p = 26 - d, q = 10 - d \), and for \( q > 0 \), all lattices are obtained as sums of two copies of the root lattice \( \Gamma_8 \) of \( E_8 \) and \( q \) copies of the two-dimensional lattice \( \Gamma_2 \) introduced above: \( \Gamma = 2\Gamma_8 \oplus q\Gamma_2 \). (For the uncompactified heterotic string with \( q = 0 \), there is also the lattice \( \Gamma_{16} \) associated with \( Spin(32)/\mathbb{Z}_2 \).)

The lattice momenta may now be parametrized in terms of the metric \( g_{ij} \) on the \( 10 - d \) dimensional torus, the anti-symmetric tensor field \( B_{ij} \) and the 16 Abelian gauge fields \( A^I_i \) which belong to the Cartan subalgebra of either \( Spin(32)/\mathbb{Z}_2 \) or \( E_8 \times E_8 \) and arise from commuting Wilson lines on the torus.

\[ P^i_L = n^i + \frac{1}{2}g^{ij}m_j - g^{ij}B_{jk}n^k + \cdots \]
\[ P^i_R = n^i - \frac{1}{2}g^{ij}m_j + g^{ij}B_{jk}n^k + \cdots \]  

(2.3)

With the help of these parametrizations, the partition function \( Z(\Gamma) \) can be evaluated in terms of a convergent sum over powers \( q^r \bar{q}^\bar{r} \) (where as usual \( q = e^{2i\pi\tau} \)) as was done for the simple case of the toroidal compactification.

One-loop amplitudes in the toroidally compactified theory with \( N \) external massless vertex operators (for states in the NS-NS sector), and evaluated at vanishing compactification momentum, are obtained simply by substituting the lattice partition function \( Z(\Gamma) \) into the integrals for the corresponding critical superstring amplitudes. We also must, of course, restrict the momenta of the external states to the uncompactified directions. Notice that the associated tree level amplitudes are unmodified for the massless vertex operators we consider. We shall concentrate on the 4-point function for the scattering of states contained in the massless NS-NS multiplet \([6,7,8,9]\), given as follows for the Type II superstrings

\[ A_{II}(k_i, \epsilon_i) = (2\pi)^d \delta(k) g^4 A_{II}(s, t, u) K_{\mu_1 \mu_2 \mu_3 \mu_4} K_{\bar{\mu}_1 \bar{\mu}_2 \bar{\mu}_3 \bar{\mu}_4} \prod_{i=1}^{4} \epsilon_i^{\mu_i} (k_i), \]  

(2.4)

where the factors \( K \) depend on the particular states involved and \( A_{II}(s, t, u) \) is a universal scalar function depending only on the kinematics.

For the heterotic string, we concentrate on the scattering amplitude for gauge bosons with root (weight) lattice \( K^I_i \), given by

\[ A_H(k_i, \epsilon_i, K_i) = (2\pi)^d \delta(k) g^4 A_H(s, t, u; S, T, U) K_{\mu_1 \mu_2 \mu_3 \mu_4} \prod_{i=1}^{4} \epsilon_i^{\mu_i} (k_i) \]  

(2.5)
Here, $g$ is the string coupling constant, $k = k_1 + k_2 + k_3 + k_4$, with $k_i^\mu$ massless momenta in the $d$ uncompactified dimensions, and external states characterized by the on-shell conditions:

$$k_i^\mu \varepsilon_i^\mu = 0$$

$$k_i^\mu \varepsilon_i^N = k_i^\nu \varepsilon_i^{M\nu} = 0 \quad N = 1, \cdots, 10 \quad (2.6)$$

The kinematical factor $K$ is a polynomial in momenta [5] and

$$s = s_{12} = s_{34}, \quad t = s_{23} = s_{14},$$

and $u = s_{13} = s_{24}$ are the familiar Mandelstam variables associated with the momenta $k_i$ with $s + t + u = 0$, while for the heterotic string $S$, $T$, $U$ are internal root (weight) lattice Mandelstam variables.

The Type IIb one-loop amplitude for a compactification scheme with lattice $\Gamma$ of signature $(10 - d, 10 - d)$ is given by ([6, 10, 5]),

$$A_{II}(s, t, u) = \int_F \frac{d^2 \tau}{\tau_2^2} \prod_{i=1}^4 \frac{d^2 z_i}{\tau_2} \exp\left\{ \frac{1}{2} s_{ij} G(z_i, z_j) \right\} Z_{\text{reg}}(\Gamma). \quad (2.7)$$

The corresponding one loop amplitude for the heterotic string for states labelled by $K_j$ is given by

$$A_H(s, t, u; S, T, U) = \int_F \frac{d^2 \tau}{\tau_2^2} \prod_{i=1}^4 \frac{d^2 z_i}{\tau_2} \exp\left\{ \frac{1}{2} s_{ij} G(z_i, z_j) \right\} Z_{\text{reg}}(K_i z_i, \tau, \Gamma), \quad (2.8)$$

where $\Gamma$ is a lattice of signature $(10 - d, 26 - d)$. The integration region $F$ is the fundamental moduli domain of the torus,

$$F = \{ \tau = \tau_1 + i \tau_2 : \tau_2 > 0, \quad -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \quad |\tau| \geq 1 \}, \quad (2.9)$$

and

$$G(z_i, z_j) = -\ln \left| \frac{\vartheta_1(z_i - z_j, \tau)}{\vartheta_1'(0, \tau)} \right|^2 - \frac{\pi}{2\tau_2} (z_i - \bar{z}_i - z_j + \bar{z}_j)^2, \quad (2.10)$$

is the scalar Green function on the torus. The theta function $\vartheta_1(z, \tau)$ is defined for even characteristics [11].

The integral expressions for $A_{II}$ and $A_H$ contain infra-red divergences; one sees immediately from the fact that there are propagating massless internal states. For this reason we have not denoted the partition functions in the above as $Z(\Gamma)$, given in eq. (2.1), but rather $Z_{\text{reg}}(\Gamma)$, stressing the fact that a well-defined amplitude is obtained only after suitable IR regularization. The corresponding regularized function $Z_{\text{reg}}(K_i z_i, \tau, \Gamma)$ contains the lattice sum required for the heterotic string, given by

$$\mathcal{L}(K_i, z_i, \tau, \Gamma) = \eta(\tau)^{-24} \prod_{i<j} \left( \frac{\vartheta_1(z_i - z_j|\tau)}{\vartheta_1'(0|\tau)} \right)^{-K_i.K_j} \vartheta_\Gamma(K_i z_i, \tau) \quad (2.11)$$

* Repeated Lorentz indices will be assumed to be summed over throughout, whereas repeated particle identification indices will not be assumed to be summed over.
The $\Gamma$ lattice $\vartheta$-function is now not holomorphic in $\tau$ and given by

$$\vartheta_{\Gamma}(K_i z_i | \tau) = \sum_{(P_L, P_R) \in \Gamma} \exp \left\{ i \pi \tau \left( \frac{1}{\tau} K_i z_i \right)^2 - i \pi \bar{\tau} P_R^2 \right\},$$

which arises from the $(p, q) = (26 - d, 10 - d)$ dimensional lattice described earlier.

3. IR Regularization

The amplitudes defined above through the formal integration over moduli space are not well defined due to the presence of infrared singularities and must be regularized. Infrared regularization of string theory was originally applied in the study of the zero slope limit of compactified theories [10]. It has also been widely used in string-inspired simplifications of quantum field theory Feynman rules [12]. Most of these IR regularization methods have been developed on a case by case basis, but no simple systematic treatment of them or prescription for them seems to be available for the string. We propose that the minimal consistency requirements for a good regulator be as follows:

1. The regulator must make all amplitudes finite and well-defined with the help of some regulator parameter, after suitable analytic continuation in the external momenta;
2. It must correspond to a local action on the worldsheet and preserve modular invariance;
3. The naive amplitude must be recovered in the limit of vanishing regularization parameter.

Before we enter into the IR regularization scheme based on continuing in the dimensions of the target torus, we shall briefly comment on another interesting IR regularization scheme that was proposed in [13]. This regulator satisfies criteria that are more restrictive than the ones we have proposed here. In particular, it is required that the entire string spectrum have a non-zero mass gap, which would certainly prohibit the entrance of infrared singularities. This criterion seems too restrictive since, in the field theory limit, would exclude the use of ordinary dimensional regularization for IR divergences. Also, the compactified string theory is considered in non-trivial four-dimensional backgrounds, which leads to rather complicated conformal field theories. Finally, the proposal appeals to the use of amplitudes of non-critical string theory that involve quantization of the Liouville field. While much progress has been made on this problem, it is certainly no straightforward task to calculate such Liouville amplitudes, even in the limit of vanishing regulator. Thus, while certainly in the cases considered in [13] this regulator is consistent and has been applied successfully, in the cases of toroidal and orbifold compactification that we consider here, it does not appear necessary to make use of this complicated scheme.

Instead, for both the toroidal and orbifold compactification schemes, a slightly altered version of dimensional regularization/dimensional reduction [14, 15] satisfies all the
requirements we have proposed above and will ultimately lead to well-defined scattering amplitudes. The method simply consists in analytically continuing the uncompactified dimension from \( d = 4 \) to \( d = 4 + 2\epsilon \) and the compactified dimensions from 6 to 6 − \( \epsilon \).

The analytic continuation of the toroidally or orbifold compactified dimensions poses no particular problem, as it may be formulated as the extension to fractional dimensions of the theory on a torus with variable constant background metric and anti-symmetric tensor fields. This regularization scheme applies to all loop orders through a worldsheet conformal field formulated on a flat torus analytically continued to 6 − \( \epsilon \) dimensions. We shall show shortly that all amplitudes are indeed IR regulated through this method, and that the naive limit corresponds to the original amplitudes.

One may be worried that all self-dual, even tori cannot be properly formulated in fractionalized dimensions. However, the formulation of tori by background fields reveals that this continuation is completely analogous to dimensional reduction and dimensional continuation of ordinary field theory. Also, the tori specified by lattices of signature \( (q + 8n, q) \), with \( n \) integer and \( q > 0 \), are isomorphic to tori specified by lattices \( \Gamma = n\Gamma_8 \oplus q\Gamma_2 \). While the \( E_8 \) lattice cannot be deformed, the number \( q \) of two-dimensional lattices can be varied continuously. Thus, there appears to be no problem in analytically continuing in the number of lattice dimensions.

Our regulator of choice is to multiply the internal modes by the lattice contribution from −2\( \epsilon \) scalars on the circle. For our purposes we regulate the infra-red divergences in the compactified four-dimensional model by introducing into the superstring a combination of 2\( \epsilon \)-scalars living on the line and −2\( \epsilon \) scalars on the circle, followed by ignoring the dependence on the vertex operators. This choice maintains the central charge of the matter at \( c = 10 \), thus no Weyl anomalies are naively introduced. The regulator should be thought of only as that, a regulator that maintains the symmetries of the amplitude, because −2\( \epsilon \) bosons on the world-sheet does not really define a pure model (in the geometric sense found by turning on background fields \( G_{ij} \) and \( B_{ij} \)). An alternative way of regulating the IR would be to curve the target space-time, or introducing additional fields but perhaps these regulators are too difficult to calculate in because of the possible introduction of the Liouville mode.

The lattice contribution in eq.(2.1), \( Z(\Gamma) \), defined by the \( g = 1 \) contribution for the lattice \( \Gamma \), is

\[
Z(\Gamma) = \sum_{P_L, P_R \in \Gamma} e^{i\pi P_L \cdot P_L - i\pi P_R \cdot P_R}, \tag{3.1}
\]

where the lattice momenta are given in general through (2.2). The expression in eq.(3.1) may be regularized by subtracting \( \epsilon \)-bosons lying on the same or on a different lattice. For the example of an orthogonal lattice (discussed extensively in [10]), we have the simple form for the lattice contribution from a single \( U(1) \) factor,

\[
Z_o = (\tau_2)^{-\frac{1}{2}} \sum_{m,n} e^{-2\pi mn\tau_1 - \pi \tau_2 (m^2 a^2 + n^2 a^2)} \tag{3.2}
\]

However, as discussed previously, we may regularize by analytically continuing the lattice metric, anti-symmetric tensor field, and gauge fields for the heterotic string to fractional
dimensions. We shall continue to denote this regularization away from the integer dimensions by $Z_0$; the case in which $Z_0$ denotes an orthogonal lattice in $\epsilon$ dimensions being a special case of the construction.

Putting the above ingredients together, we have the following defined regularized partition function of the internal modes

$$Z_{\text{reg}}(\Gamma) = Z(\Gamma) Z_0^{-\epsilon}. \quad (3.3)$$

It is important to note that whichever the regulator, the behavior for large $\tau_2$ is the same, as can be seen from the following definition

$$Z_{\text{reg}}(\Gamma) = (\tau_2)^{10-d-\epsilon} \tilde{Z}_{\text{reg}}(\Gamma) \quad (3.4)$$

Here, and elsewhere, the partition functions labelled with a tilde represent the lattice sums without the $\tau_2$ prefactor. For example, $\tilde{Z}(\Gamma)$ is given by the lattice sum

$$\tilde{Z}(\Gamma) = \sum_{m,n} |q|^N_{m,n} e^{2\pi i \tau_1 \phi_{m,n}}, \quad (3.5)$$

with $m$ and $n$ ranging over sets of integers that parametrize $P_L$ and $P_R$, and

$$N_{m,n} = \frac{1}{2}(P_L \cdot P_L + P_R \cdot P_R) \quad \phi_{m,n} = \frac{1}{2}(P_L \cdot P_L - P_R \cdot P_R) \quad (3.6)$$

where the left and right lattice momenta are defined through eq. (2.2). This discussion is easily extended to the case of the heterotic string, where the $Z$-factor is $z_i$-dependent.

4. Analytic Structure

The convergence properties of the amplitudes in eqs. (2.4) and (2.5) are closely related to those for the corresponding uncompactified one. We shall in this section concentrate on the integral representations in eqs. (2.7) and (2.8). The lattice contributions are easily seen to not modify the exponential behavior in the large $\tau_2$ region (each term gives additional factors of $q$ and $\bar{q}$ which suppress the large $\tau_2$ behavior). Following the analysis of [1], we see that absolute convergence holds only in the domain

$$\text{Re} \ s_{ij} = 0, \quad (4.1)$$

for fixed non-zero $\epsilon > 0$. Infrared convergence results from the extra $\epsilon$-dependence in the $\tau_2$ factors in $Z_{\text{reg}}$ for $\epsilon > 0$.

Given a convergent definition of the amplitude at the point $\text{Re}(s_{ij}) = 0$, we now proceed to construct a unique analytic continuation that will extend to all momenta. We shall modify some of the arguments given for the critical string in [1] for the dimensionally
We begin by setting $z_4 = 0$ and separating the integration over the three remaining
locations of the vertex operators into the regions,

$$0 \leq \text{Im } z_1 \leq \text{Im } z_2 \leq \text{Im } z_3 \leq \tau_2 ,$$

and the permutations of $(1, 2, 3)$. The different regions, labelled by the orderings of the
three locations of the vertex operators, give rise to a decomposition of the integral repre-
sentation in eq.(2.7) into

$$A(s, t, u) = 2A(s, t) + 2A(t, u) + 2A(u, s)$$

where $u = -s - t$ and $s = -2k_1 \cdot k_2$, $t = -2k_2 \cdot k_3$. We only need to work with $A(s, t)$ since
the remaining two subamplitudes are found by permutations of the momenta.

With the ordering in (4.2) we introduce new variables $w_{ij}$ satisfying $|w_{ij}| \leq 1,$ defined
by

$$w_{ij} = e^{2\pi i (z_i - z_j)} \quad \text{Im } z_{ij} > 0$$

$$= q e^{2\pi i (z_i - z_j)} \quad \text{Im } z_{ij} < 0$$

with which we shall express the integral in (2.7). We shall also make use of the standard
parametrization of the vertex insertion points in terms of the real variables $\alpha_i$ and $u_i$

$$z_i - z_{i-1} = \frac{\alpha_i}{2\pi} + i \tau u_i , \quad i = 1, \ldots, 4,$$

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi \tau_1$ and $u_1 + u_2 + u_3 + u_4 = 1$.

In product expansion form, we may rewrite the prime form neglecting the $z$ independent factors which will vanish as a result of momentum conservation once we insert it into
the expression for the four-point functions: $z_{ij} \equiv z_i - z_j$,

$$E(z_i, z_j) = \frac{\theta_1[z_{ij}; \tau]}{\theta_1'[0; \tau]} = \prod_{n=0}^{\infty} (1 - q^n e^{2\pi i z_{ij}}) (1 - q^{n+1} e^{-2\pi i z_{ij}}) .$$

As usual we have defined $q \equiv e^{2\pi i \tau}$. Next we define,

$$\mathcal{R}(w_{ij}) = \prod_{i<j} |E(z_i, z_j)|^{2k_i \cdot k_j} = \prod_{i \neq j} \prod_{n=0}^{\infty} |1 - w_{ij} q^n|^{-s_{ij}} .$$

which is the component $e^{2\pi i G(z_i, z_j)}$ of the amplitude, without including the zero mode subtraction in the Greens function in eq. (2.10).

The amplitude $A(s, t)$ may then be rewritten as

$$A(s, t) = \int_F d^2 \tau \frac{1}{\tau_{-1+\epsilon}} \int_0^{2\pi} \prod_{i=1}^4 \frac{d \alpha_i}{2\pi} \delta(2\pi \tau - \sum \alpha_j)$$

$$\times \int_1^4 \prod_{i=1}^4 du_i \delta(1 - \sum u_j) |q|^{(-s_{u_1 u_3} - (u_2 u_4))} \mathcal{R}(w_{ij}) \tilde{Z}(\Gamma) \tilde{Z}_c^{-2\epsilon} .$$
The function $\mathcal{R}$ is defined from the product expansion of the $\vartheta$-functions and may be expanded in an infinite series expansion as follows:

$$
\mathcal{R}(w_{ij}) = \prod_{i \neq j} \prod_{n=0}^{\infty} |1 - w_{ij} q^n|^{-s_{ij}} \\
= \prod_{i=1}^{4} |1 - e^{i\alpha_i} |q|^{nu_i}|^{-s_{i}} \sum_{n_i=0}^{\infty} \sum_{|\nu_i| \leq n_i} P^{(4)}_{\{n_i,\nu_i\}}(s, t) \prod_{i=1}^{4} |q|^{nu_i} e^{iu_i \alpha_i}
$$

(4.9)

Here, $s_i = s$ for $i$ even, $s_i = t$ for $i$ odd, and $P^{(4)}_{\{n_i,\nu_i\}}(s, t)$ are polynomials in $s$ and $t$, which have previously been generated recursively in Appendix C of [1].

The explicit analytic continuation of the toroidal one-loop amplitude for the four point function and the singularities in the external momenta are then described by the following statement:

**Toroidally Compactified Partial Amplitudes**

For any integer $N$, we can break the original amplitude in eqs. (2.4) and (2.5) into the sum

$$
A(s, t; \epsilon) = \sum_{\{n_i, m, n\} \in D_N} \sum_{|\nu_i| \leq n_i} P^{(4)}_{\{n_i, \nu_i\}}(s, t) A_{\{r_i, \eta_i\}}(s, t) + M_N(s, t; \epsilon)
$$

(4.10)

with

$$
r_i = n_i + N_{m,n} \quad \eta_i = \nu_i + \phi_{m,n}.
$$

(4.11)

Here, $M_N(s, t)$ is a meromorphic function of $s$ and $t$ in the region of $\text{Re}(s) > N$ or $\text{Re}(t) < N$, and the numbers $D_N$ range through $D_N = \{m, n; n_i \text{ such that } n_1 + n_2 + n_3 + n_4 + N_{m,n} \leq 4N\}$. The partial amplitudes $A_{\{n_i, \nu_i\}}(s, t)$ are almost identical to those encountered for the critical superstring (together with the Theorem 1 of [1] describing their analytic behavior) and are obtained by adjusting only the power of $\tau_2$:

$$
A_{\{r_i, \eta_i\}} = \int_{1}^{\infty} \frac{d\tau_2}{\tau_2} e^{i\alpha_j} \int_{0}^{2\pi} \int_{0}^{4} \prod_{i=1}^{4} \frac{d\alpha_i du_i}{2\pi} \delta(1 - \sum_{j} u_j) |q|^{-su_1 u_3 - tu_2 u_4} \\
\times \prod_{j=1}^{4} |1 - e^{i\alpha_j} |q|^{u_j}|^{-s_{j,j+1}} |q|^{ru_i} e^{i\eta_i \alpha_i}.
$$

(4.12)

The above reduces the analytic continuation of the amplitudes in eq. (2.7) and eq. (2.8) to the analytic continuation of the much simpler amplitudes $A_{r_i, \eta_i}$, where there are no longer any infinite products, and where all $\alpha_i$ integrations are decoupled. The proof is
completely analogous to that of Theorem 1 in [1] and it will not be given here. In fact, the polynomials \( P^{(4)}_{\nu}(s, t) \) defined through the representation in eq. (4.9) are the same as those that appear in the analysis of the four-point function (of states in the gravitational multiplet) of the critical string. The only modifications from the critical string are thus the modified power of \( \tau_2 \), due to the altered number of uncompactified dimensions, and the shifts in the spectrum resulting from the addition of \( N_{m,n} \) and \( \phi_{m,n} \) in the definition of \( r_i \) and \( \eta_i \) arising from the compactification. These modifications are exactly the expected ones from the masses of the states in the toroidally compactified theory.

The structure of the branch cuts and the poles on top of branch cuts of the four-point amplitude arise in the individual amplitudes \( A_{\{r, \eta\}}(s, t) \), which we now examine.

### 5. Dispersion Relations

In order to manifest the cut structure without an \( i\epsilon \)-prescription, we need to resort to old-fashioned dispersion theory. In this section we complete the analysis of the construction of the dispersion relations necessary to describe the partial amplitudes \( A_{\{r, \eta\}}(s, t) \) in eq. (4.12). The analysis involves finding the dimensionally regularized versions of the spectral representation of the box diagram and its applications to study the analytic form of these partial amplitudes. For ease of presentation, we break the analysis into three subsections.

The crucial ingredient in studying the analytic form of the partial amplitudes, as found in [1], is the inverse Laplace transform \( \varphi_{\eta\nu} \) of the hypergeometric function. This is defined as follows

\[
\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{i\alpha|1 - xe^{i\alpha}|^{-s}} = C_{|\eta|}(s)x^{|\eta|}F\left(\frac{s}{2}, \frac{s}{2} + |\eta|; |\eta| + 1; x^2\right)
\]

\[
= x^{-r} \int_0^\infty d\beta x^{\beta} \varphi_{\eta\nu}(s; \beta),
\]

(5.1)

where \( \varphi_{\eta\nu}(s; \beta) = 0 \) for \( \beta < 0 \).

We first rewrite \( A_{\{r, \eta\}}(s, t) \) using the inverse Laplace transform of the hypergeometric function,

\[
A_{\{r, \eta\}}(s, t) = \int_0^\infty \prod_{j=1}^4 d\beta_j \varphi_{\{r_j, \eta_j\}}(s, t; \beta_i) \int_1^\infty \frac{d\tau_2}{\tau_2^{1+\epsilon}} \int_0^1 \prod_{i=1}^4 du_i \delta(1 - \sum_j u_j)
\]

\[
\times e^{2\pi \tau_2 (su_1 u_3 + tu_2 u_4 - \sum_{i=1}^4 u_i \beta_i)}. \]

(5.2)

A number of changes of variables, identical to those used in [1], leads to
\[ A_{\{r_i, \eta_i\}} = \int_0^\infty \prod_{j=1}^4 d\beta_j \Psi_{\{r_j, \eta_j\}}(s; \beta_i) \int_0^1 du_1 \int_0^{1-u_1} du_2 \left(1 - u_1 - u_2\right) \]
\[ \times \int_1^\infty d\mu \int_0^1 d\alpha \frac{1}{\mu^{-1+\epsilon} e^{-2\pi \mu \kappa}}, \] (5.3)

with the definitions

\[ \kappa \equiv u_1 \beta_1 + u_2 \beta_2 + (1 - u_1 - u_2) \left\{ \alpha (\beta_3 - s u_1) + (1 - \alpha)(\beta_4 - t u_2) \right\}, \] (5.4)

and

\[ \Psi_{\{r_j, \eta_j\}} = \prod_{j=1}^4 \phi_{\{r_j, \eta_j\}}(s; \beta_j). \] (5.5)

The \( \mu \)-integral ranges from 0 to 1,

\[ \int_0^1 \frac{d\mu}{\mu^{-1+\epsilon} e^{-2\pi \mu \kappa}} = E(\kappa). \] (5.6)

Keeping the dimension \( \epsilon < 2 \) gives \( E(\kappa) \) an entire function in all its variables. Note that this region would correspond to the ultraviolet portion of the box integral, which is divergent for \( d \geq 8 \). Thus far we have obtained the analog in the toroidal compactified case to Lemma 2 in [1], which is that the contribution of \( E(\kappa) \) to the partial amplitude is globally meromorphic in both \( s \) and \( t \) and may also be absorbed into the meromorphic function \( M_N(s, t; \epsilon) \).

Since we may absorb any meromorphic contribution from \( A_{\{r_i, \eta_i\}} \) into the function \( M_N(s, t; \epsilon) \) we may extend the lower limit of the integration region of \( \mu \) in eq. (5.3) from one down to zero, at the cost of changing \( M_N(s, t; \epsilon) \). Then we have after performing the \( \mu \)-integral explicitly,

\[ A_{\{r_i, \eta_i\}} = \int_0^\infty \prod_{j=1}^4 d\beta_j \Psi_{\{r_j, \eta_j\}}(s; \beta_i) \int_0^1 du_1 \int_0^{1-u_1} du_2 \left(1 - u_1 - u_2\right) \]
\[ \times \Gamma(2 - \epsilon) \int_0^1 d\alpha (2\pi \kappa)^{-2+\epsilon}. \] (5.7)

We see that the \( \beta \)-integral in eq. (5.7) is over the regulated box integral functions \( B(\beta_i; s, t; \epsilon) \) possessing masses \( \beta_j \), defined by

\[ B(\beta_i; s, t; \epsilon) = \Gamma(2 - \epsilon) \int_0^1 du_1 \int_0^{1-u_1} du_2 \left(1 - u_1 - u_2\right) \int_0^1 d\alpha (2\pi \kappa)^{-2+\epsilon}. \] (5.8)
The box integral in eq. (5.8), although in a non-standard form, may be recast in the form of a dispersion relation. (Given an \( i\epsilon \) prescription, we may actually find the complete expression for the string amplitude because all of the dimensionally regularized box integrals are known explicitly, at least in terms of dilogarithms [16].)

5.1 Dispersive Form of Regularized Box

In this section we modify the integral form of the dimensionally regularized box diagram in (5.8) into a form suitable for expressing the amplitudes in a dispersive form. To find the box dispersion relation, we first perform the \( \alpha \) integral in eq. (5.8), leading to

\[
B(\beta_i; s, t; \epsilon) = \int_0^1 d\alpha (2\pi \kappa)^{-2+\epsilon} = -\frac{\Gamma(1 - \epsilon)}{\Gamma(2 - \epsilon)} [2\pi (1 - u_1 - u_2)]^{-2+\epsilon} \\
\times \frac{1}{y} \left\{ (x_o + \beta_4 - t u_2 + y)^{-1+\epsilon} - (x_o + \beta_4 - t u_2)^{-1+\epsilon} \right\},
\]

where \( y = \beta_3 - s u_1 - \beta_4 + t u_2 \). Then we use the identity

\[
\int_0^\infty dx \frac{x^\mu}{(x + a)(x + b)} = \frac{\Gamma(1 + \mu)\Gamma(-\mu)}{b - a} \left( b^\mu - a^\mu \right),
\]

to rewrite the \( \alpha \) integral into

\[
\Gamma(2 - \epsilon) \int_0^1 (2\pi \kappa)^{-2+\epsilon} = \frac{(2\pi)^{-2+\epsilon}}{\Gamma(\epsilon)} (1 - u_1 - u_2)^{-2+\epsilon} \\
\times \int_0^\infty dx \frac{x^{-1+\epsilon}}{(x + x_o + \beta_3 - s u_1)(x + x_o + \beta_4 - t u_2)}.
\]

We represent the two linear factors in the denominator by spectral integrals over the real variables \( \sigma \) and \( \tau \):

\[
\frac{1}{x + x_o + \beta_3 - s u_1} = \int_0^\infty d\sigma \frac{1}{\sigma - s} \delta(x + x_o + \beta_3 - \sigma u_1),
\]

and

\[
\frac{1}{x + x_o + \beta_4 - t u_2} = \int_0^\infty d\tau \frac{1}{\tau - t} \delta(x + x_o + \beta_4 - \tau u_2).
\]

Using these, we have the final representation of the regularized box graphs,

\[
B(\beta_i; s, t; \epsilon) = \int_0^\infty d\sigma \int_0^\infty d\tau \frac{\rho_B(\beta_i; s, t; \sigma, \tau; \epsilon)}{(\sigma - s)(\tau - t)},
\]

where the dimensionally regularized double spectral density \( \rho_B \) is given by
\[
\rho_B(\beta_i; s, t; \sigma, \tau; \epsilon) = \frac{(2\pi)^{-2+\epsilon}}{\Gamma(\epsilon)} \int_0^1 du_1 \int_0^{1-u_1} du_2 \ (1-u_1-u_2)^{-1+\epsilon} \\
\times \int_0^\infty dx \ x^{-1+\epsilon} \delta(x+x_o+\beta_3-\sigma u_1) \delta(x+x_o+\beta_4-\tau u_2)
\]

The integrals in eq. (5.15) may be evaluated to give

\[
\rho_B(\beta_i; s, t; \sigma, \tau; \epsilon) = \frac{(2\pi)^{-2+\epsilon}}{\Gamma(\epsilon)} \frac{1}{\sigma \tau} \left( \frac{\sigma + \tau}{\sigma \tau} \right)^{-1+\epsilon} \vartheta(A^2 - B^2)^{-1+\epsilon} + \frac{1}{2 \vartheta} \vartheta(A^2 - B^2) \vartheta(A)
\]

where \(A\) and \(B\) are defined as,

\[
A = \frac{\sigma \tau - (\beta_1 + \beta_3) \tau - (\beta_2 + \beta_4) \sigma}{2(\sigma + \tau)} \quad \quad B^2 = \frac{\beta_1 \beta_3 \tau + \beta_2 \beta_4 \sigma}{\sigma + \tau}.
\]  

(5.16)

The domain of support for the spectral density of the dimensionally regularized box diagram with masses \(\beta_j\) \((j = 1, \ldots, 4)\) is found by solving the three constraints imposed by the theta functions on \(A, A^2 - B^2\) in eq. (5.16) and from the fact that \(\sigma, \tau \geq 0\).

In the following we parameterize the solution to these three conditions and the allowed values of \(\sigma, \tau\). The first constraint \(A^2 - B^2\) becomes, after rewriting \(B^2\) which involves an arbitrary mass scale \(m^2\),

\[
\left[ 1 - \frac{\beta_1 + \beta_3}{\sigma} - \frac{\beta_2 + \beta_4}{\tau} \right]^2 + \left[ \frac{\beta_1 \beta_3 - m^4}{\sigma m^2} + \frac{\beta_2 \beta_4 - m^4}{\tau m^2} \right]^2 \\
\geq \left[ \frac{\beta_1 \beta_3 + m^4}{\sigma m^2} + \frac{\beta_2 \beta_4 + m^4}{\tau m^2} \right]^2.
\]

(5.18)

Next we define the variables \(x\) and \(y\),

\[
\left[ \frac{\beta_1 \beta_3 - m^4}{\sigma m^2} + \frac{\beta_2 \beta_4 - m^4}{\tau m^2} \right] = x \left[ \frac{\beta_1 \beta_3 + m^4}{\sigma m^2} + \frac{\beta_2 \beta_4 + m^4}{\tau m^2} \right]
\]

(5.19)

\[
\left[ 1 - \frac{\beta_1 + \beta_3}{\sigma} - \frac{\beta_2 + \beta_4}{\tau} \right] = y \left[ \frac{\beta_1 \beta_3 + m^4}{\sigma m^2} + \frac{\beta_2 \beta_4 + m^4}{\tau m^2} \right]
\]

(5.20)

so that eq. (5.18) becomes the hyperbolic equation:

\[
x^2 + y^2 \geq 1.
\]

(5.21)

The next constraint \(\sigma, \tau \geq 0\) then becomes a bound for \(x\), so that its allowed range is \(x^- \leq x \leq x^+\) with

\[
x^\pm = \left[ \frac{\beta_1 \beta_3 - m^4}{\beta_1 \beta_3 + m^4}, \frac{\beta_2 \beta_4 - m^4}{\beta_2 \beta_4 + m^4} \right].
\]

(5.22)
The last constraint $A \geq 0$ becomes the statement $y \geq 0$, since the inequality has the form,

$$A = \frac{\sigma \tau}{2(\sigma + \tau)} \left( \frac{\beta_1 \beta_3 + m^4}{\sigma m^2} + \frac{\beta_2 \beta_4 + m^4}{\tau m^2} \right) y.$$  \hfill (5.23)

The parameterization of the region above in $\sigma$ and $\tau$ in terms of $x$ and $y$ and the masses $\beta_j$ has the solution to $\sigma$ and $\tau$: \hfill (5.24)

$$\frac{1}{\sigma} = \frac{A_2}{\lambda}, \quad \frac{1}{\tau} = -\frac{A_1}{\lambda},$$

with

$$A_1 = \beta_1 \beta_3 - m^4 - x(\beta_1 \beta_3 + m^4),$$
$$A_2 = \beta_2 \beta_4 - m^4 - x(\beta_2 \beta_4 + m^4),$$
$$\lambda = (\beta_1 + \beta_3)A_2 - (\beta_2 + \beta_4)A_1 - 2y D m^2,$$
$$D = \beta_1 \beta_3 - \beta_2 \beta_4.$$ \hfill (5.25)

For a given mass configuration $\beta_i$, the spectral density vanishes if $\sigma$ and $\tau$ are outside the region specified by the allowed space of $x$ and $y$ above: The minimum values for $\sigma$ and $\tau$ are found to be,

$$(\beta_1 + \beta_3)^2 \quad (\beta_2 + \beta_4)^2.$$ \hfill (5.26)

The box diagram in eq. (5.8) possesses cuts at values of $s$ and $t$ beginning at the values in eq.(5.26), which is the expected two-particle threshold condition.

### 5.2 Dispersive Form of Partial Amplitudes

For the string partial amplitude in eq.(4.12) the inverse Laplace transform of the hypergeometric function leads to the factors $\Psi_{\{r, \eta_i\}}$, which can be interpreted as an infinite sum of Dirac point masses:

$$\Phi_{r, \eta_i} = \sum_{k_i=0}^\infty C_{k_i}(s_i)C_{k_i+|\eta_i|}(s_i)\delta(2k_i + r_i + |\eta_i| - \beta_i),$$ \hfill (5.27)

and

$$\Psi_{\{r, \eta_i\}} = \prod_{j=1}^4 \Phi_{r_j \eta_j}.$$ \hfill (5.28)

The functions $C_{k_i}(s_i)$ are defined through the inverse Laplace transform in eq. (5.1). The total spectral density can be factorized into, after performing the $\beta$ integration, the following form:

$$\rho(s, t; \sigma, \tau; \epsilon) = \sum_{k_i=0}^\infty \prod_{j=1}^4 C_{k_j}(s_j)C_{k_j+|\eta_j|}(s_j) \rho_B(\beta_j; \sigma, \tau; s, t; \epsilon).$$ \hfill (5.29)
The line, |, in eq. (5.29) denotes that the $\beta_i$ are equal to $\beta_i = 2k_i + r_i + |\eta_i|$.

Combining the above results, we arrive at the following theorem:

**Dispersive Form of Toroidally Compactified Partial Amplitudes**

The partial amplitudes $A_{r_i\eta_i}(s, t)$ can be expressed as

$$A_{\{r_i\eta_i\}}(s, t) = M_{\{r_i\eta_i\}}(s, t) + \int_0^\infty d\sigma \int_0^\infty d\tau \frac{\rho_{\{r_i\eta_i\}}(s, t; \sigma, \tau)}{(s - \sigma)(t - \tau)},$$  \hspace{2cm} (5.30)

where the double spectral density is given by

$$\rho_{\{r_i\eta_i\}}(s, t) = \int_0^\infty \prod_{j=1}^4 d\beta_j \varphi_{\{r_j\eta_j\}}(s, t; \beta_j) \rho_B(\beta_j; s, t; \sigma, \tau; \epsilon).$$ \hspace{2cm} (5.31)

It is assumed that we maintain the number of uncompactified dimensions $d < 8$. In this case no ultra-violet divergences arise within the box graph: We do not need to perform any subtractions in the double dispersion representation. The double spectral density in eq. (5.31) has been regularized and leads to an infra-red regularized expression for the partial amplitudes $A_{\{r_i\eta_i\}}(s, t)$.

The term $M_{\{r_i\eta_i\}}(s, t)$ in eq. (5.30) is a globally meromorphic function of $s$ and $t$, similar to the contribution $M_N(s, t)$ in eq. (4.10); for example, it receives contributions from changing the lower limit of integration in eq. (5.3), producing the meromorphic result in eq. (5.6).

We may denote the contribution of each box diagram that arises from the integral over the spectral density in eq. (5.31) by the notation $A_{\{r_i\eta_i\}}^{m_i^2}(s, t)$. The internal masses $m_i^2$ on the lines of the box depend on the value of the particular $\beta_i$ pulled out by the integration over $\sigma$ and $\tau$ in eq. (5.30). The support for the box diagrams has been found above, and the branch cuts in $s$ and $t$ appearing from the integration in eq. (5.30) begin at $\beta_i = 0$; further ones appear at the discrete values of $\beta_j$ extracted from the integration over the Dirac point masses in eq. (5.27).

The condition on the masses $\beta_j = 0$ correspond to the minimum mass values,

$$m_j^2|_{\text{min}} = r_j + |\eta_j|.$$ \hspace{2cm} (5.32)

Thus the integral representation on the right hand side of eq. (5.30) defines a holomorphic expression in $s$ and $t$ in the cut plane $s, t \in \mathbb{C} - \mathbb{R}_+$:

$$(s, t) \in \left( \mathbb{C} \backslash [(M_1^2 + M_3^2)^2, \infty) \right) \times \left( \mathbb{C} \backslash [(M_2^2 + M_4^2)^2, \infty) \right).$$ \hspace{2cm} (5.33)

with the mass parameters $M_i^2 = 2k_i + r_i + |\eta_i|$. The lattice momenta contribute to the locations of the branch cuts through $N_{m,n}$, defined in eq. (3.6),

$$N_{m,n} = \frac{1}{2}(P_L \cdot P_L + P_R \cdot P_R), \quad \phi_{m,n} = \frac{1}{2}(P_L \cdot P_L - P_R \cdot P_R),$$ \hspace{2cm} (5.34)
and enters through the definitions $r_i = n_i + N_{m,n}$ and $\eta_i = \nu_i + \phi_{m,n}$ respectively, from eq. (4.11).

There are several features of the dispersive form of the amplitude worth pointing out. First, as already noted above we have isolated the locations of the origins of the branch cuts in the values of $(s, t)$; these points define the masses for particle production in intermediate string states. Second, the one-loop amplitude is genuinely unitary in the double dispersive representation, and may be used for example to check against the optical theorem. Last, we may use the integral representation to construct an effective $i\epsilon$-prescription, as in [1].

6. Discussion

In this work we have extended the construction initially used to define the four-point function in critical string theory to the cases of a toroidal compactification scheme in both the IIb and Heterotic models. The pole structure and unitarity of the compactified string amplitudes are found through defining them via single and double dispersion relation. Furthermore, we have provided an regularization scheme which is used to isolate the infrared singularities in the amplitudes. The scattering amplitudes we consider are those of the scattering of states in the Neveu-Schwarz/Neveu-Schwarz sector; however, more general amplitudes at four-point are certainly amenable to these techniques.

In the future it would be quite interesting to explore the application of these methods to amplitudes containing external fermion creation operators, in which case possible ambiguities in the integration over supermoduli space might appear [17]. Additionally, the construction should also be applied to the more general case of higher-point amplitudes in both the critical and non-critical case. However, in the former case the spectral representation of the amplitudes does not admit a unique analytic continuation through the construction given here [18]. In part II in of this work we shall present the analogous construction for amplitudes within orbifold compactification schemes [19].

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