Partial symmetry, reflection monoids and Coxeter groups

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Abstract. This is the first of a series of papers in which we initiate and develop the theory of reflection monoids, motivated by the theory of reflection groups. The main results identify a number of important inverse semigroups as reflection monoids, introduce new examples, and determine their orders.

Introduction

The symmetric group $\mathfrak{S}_n$ comes in many guises: as the permutation group of the set $\{1, \ldots, n\}$; as the group generated by reflections in the hyperplanes $x_i - x_j = 0$ of an $n$-dimensional Euclidean space; as the Weyl group of the reductive algebraic group $GL_n$, or (semi)simple group $SL_{n+1}$, or simple Lie algebra $sl_{n+1}$; as the Coxeter group associated to Artin’s braid group, …

If one thinks of $\mathfrak{S}_X$ as the group of (global) symmetries of $X$, then the partial symmetries naturally lead one to consider the symmetric inverse monoid $I_X$, whose elements are the partial bijections $Y \to Z$ ($Y, Z \subset X$). It too has many other faces. It arises in its incarnation as the “rook monoid” as the Renner monoid of the reductive algebraic monoid $M_n$. An associated Iwahori theory and representations have been worked out by Solomon [29,31]. There is a braid connection too, with $I_n$ naturally associated to the inverse monoid of “partial braids” defined recently in [6].

But what is missing is a realization of $I_n$ as a “partial reflection monoid”, or indeed, a definition and theory of partial mirror symmetry and the monoids generated by partial reflections that generalizes the theory of reflection groups.

Such is the purpose of the present paper. Reflection monoids are defined as monoids generated by certain partial linear isomorphisms $\alpha : X \to Y$ ($X, Y$ subspaces of $V$), that are the restrictions (to $X$) of reflections. Initially one is faced with many possibilities, with the challenge being to impose enough structure for a workable theory while still encompassing as many interesting examples as possible. It turns out that a solution is to consider monoids of partial linear isomorphisms whose domains form a $W$-invariant semilattice for some reflection group $W$ acting on $V$.

Two pieces of data will thus go into the definition of a reflection monoid: a reflection group and a collection of well behaved domain subspaces (see §1 for the precise definitions). What results is a theory of reflection monoids for which our main theorems in this paper determine their orders and identify the natural examples.

For instance, just as $\mathfrak{S}_n$ is the reflection group associated to the type $A$ root system, so now $I_n$ becomes the reflection monoid associated to the type $A$ root system, and where the domains form a $\mathfrak{S}_n$-invariant Boolean lattice (see [2]). This in fact turns out to be a common feature: if
the reflection group is $S_n$, and the domain subspaces are allowed to vary, one often gets families of monoids pre-existing in the literature. These families are then generalized by replacing $S_n$ by an arbitrary reflection group. Thus, the group of signed permutations of $\{1, \ldots, n\}$ is the Weyl group of type $B$, and the inverse monoid $I_{\pm n}$ of partial signed permutations is a reflection monoid of type $B$, with again a Boolean lattice of domain subspaces.

Another interesting class of examples arises from the theory of hyperplane arrangements. The reflection or Coxeter arrangement monoids have as their input data a reflection group $W$ and for the domains, the intersection lattice of the reflecting hyperplanes of $W$. These intersection lattices possess many beautiful combinatorial and algebraic properties (see [20]). Thus, the Coxeter arrangement monoids tie up reflection groups and the intersection lattices of their reflecting hyperplanes in one very natural algebraic object. Our main result here (Theorem 4 below) is that the orders of these reflection monoids are the sum of the indices of the parabolic subgroups of the original reflection group.

By the “rigidity of tori”, a maximal torus $T$ in a linear algebraic group $G$ has automorphisms a finite group, the Weyl group of $G$, and this is a reflection group in the space $\mathcal{X}(T) \otimes \mathbb{R}$, where $\mathcal{X}(T)$ is the character group of the torus. For a linear algebraic monoid $M$, our view is that there are two finite inverse monoids needed to play an equivalent role. One, the Renner monoid, is already well known for the important role it plays in the Bruhat decomposition of $M$. The other is a reflection monoid in $\mathcal{X}(T) \otimes \mathbb{R}$, where the extra piece of data, the semilattice of domain spaces, comes from the character semigroup $\mathcal{X}(T)$ of the Zariski closure of $T$. Our main result in this direction, Theorem 7, is that these two are pretty closely related.

The paper is organized as follows: §1 contains the basics and our first main result, on the orders of a large class of finite inverse monoids; §2 has a couple of examples based around the symmetric group. The idea is that “anyone” could read these sections, irrespective of whether they have an interest in reflection groups or semigroups. We fix our notation concerning reflection groups in §3 and this allows us to explore our first two families of examples in §§4 and 5: the Boolean and Coxeter arrangement reflection monoids. In §6, we give some fundamental abstract properties. The reflection monoid associated to a reductive algebraic monoid and its relation to the Renner monoid are the subject of §7. The last section, as the Bourbaki-ism in its title suggests, is a portmanteau of results of independent interest.

In the sequel to this paper, a general presentation is derived (among other things) for reflection monoids. This presentation is determined explicitly, and massaged a little more, for the Boolean and Coxeter arrangement monoids associated to the Weyl groups. The benchmark here is provided by a classical presentation [21] for the symmetric inverse monoid $I_n$, which we rederive in its new guise as the “Boolean reflection monoid of type $A$”.

1. Monoids of partial linear isomorphisms and reflection monoids

The symmetric group $S_X$ and the general linear group $GL(V)$ measure the symmetry of a set and a vector space. We start this section with two algebraic objects that measure instead partial symmetry. One is (reasonably) well known, the other less so, but nevertheless implicit in the area.

For a non-empty set $X$, a partial permutation is a bijection $Y \to Z$, where $Y, Z$ are subsets of $X$. We allow $Y$ and $Z$ to be empty, so that the empty function $0_X : \emptyset \to \emptyset$ is regarded as a partial permutation. The set of all partial permutations of $X$ is made into a monoid with zero $0_X$ using the usual rule for composition of partial functions: it is called the symmetric inverse monoid on $X$, and denoted by $I_X$. If $X = \{1, 2, \ldots, n\}$, we write $I_n$ for $I_X$. See [11] §§.

Now let $k$ be a field and $V$ a vector space over $k$. A partial linear isomorphism of $V$ is a vector space isomorphism $Y \to Z$, where $Y, Z$ are vector subspaces of $V$. The set of partial linear isomorphisms of $V$ is also made into a monoid using composition of partial functions (and with zero the linear isomorphism $0 \to 0$, from the zero subspace to itself). We call it the general linear monoid on $V$ and denote it by $ML(V)$.

It is possible to toggle back and forth between these two monoids, using the inclusions $ML(V) \subset I_V$ and $I_X \subset ML(V)$, for $V$ the $k$-space with basis $X$. Either can be taken as
the motivating example of an inverse monoid \cite{11,16}: a monoid \( M \) such that for all \( a \in M \) there is a unique \( b \in M \) such that \( aba = a \) and \( bab = b \). The element \( b \) is the inverse of \( a \) and is denoted by \( a^{-1} \). Intuitively, if \( a \) is the partial map \( Y \to Z \), then \( a^{-1} \) is the inverse partial map \( Z \to Y \), and it is precisely in order to capture this idea of “local inverses” that the notion of inverse monoid was formulated.

When \( X \) is finite, or \( V \) finite dimensional, any partial permutation/isomorphism \( Y \to Z \) can be obtained by restricting to \( Y \) a full permutation/isomorphism \( g : X \to X \). We will write \( g_Y \) for the partial map with domain \( Y \) and effect that of restricting \( g \) to \( Y \). Equivalently, \( g_Y = \varepsilon_Y g \) where \( \varepsilon : X \to X \) is the identity and \( \varepsilon_Y : Y \to Y \) its restriction to \( Y \), a partial identity. Thus every partial map is the product of an idempotent and a unit. One has to be careful with such representations: \( g_Y = h_Z \) if and only if \( Y = Z \) and \( g h^{-1} \) is in the isotropy group \( G_Y = \{ g \in G \mid v g = v \text{, for all } v \in Y \} \) of \( Y \). We have

\[
g_Y h_Z = (g h)_{Y \cap Z^{-1}},
\]

and \((g_Y)^{-1} = (g^{-1})_Y\). From now on, all our vector spaces will be finite dimensional.

Again, generalities are suggested by these primordial examples: a monoid \( M \) is factorizable if \( M = E G \) with \( E \) the idempotents and \( G \) the units of \( M \). The role of the isotropy group is played by the idempotent stabilizer \( G_e = \{ g \in G \mid e g = e \} \), and we have equality \( \varepsilon_1 g_1 = \varepsilon_2 g_2 \) if and only if \( e_1 = e_2 \) and \( g_2 g_1^{-1} \in G_{e_2} \). The units act on the idempotents: if \( e \in E \) and \( g \in G \) then \( g^{-1} e g \in E \) (with \( g^{-1} \varepsilon_Y g = \varepsilon_{Y g} \) in the examples above).

Looking a little more closely at the domain \( Y \cap Z g^{-1} \) of \( g_Y h_Z \) suggests the following:

**Definition 1.** Let \( V \) be a vector space and \( G \subset GL(V) \) a group. A collection \( S \) of subspaces of \( V \) is called a system in \( V \) for \( G \) if and only if

\begin{enumerate}
  \item[(S1).] \( V \in S \),
  \item[(S2).] \( S G = S \), ie: \( X g \in S \) for any \( X \in S \) and \( g \in G \), and
  \item[(S3).] if \( X, Y \in S \) then \( X \cap Y \in S \).
\end{enumerate}

If \( S_i (i \in I) \) is a family of systems for \( G \) then \( \bigcap S_i \) is too, and thus for any set \( \Omega \) of subspaces we write \( \langle \Omega \rangle_G \) for the intersection of all systems for \( G \) containing \( \Omega \), and call this the system for \( G \) generated by \( \Omega \).

Clearly one can always find trivial systems for \( G \)–just take \( V \) itself for instance–as well as plenty of examples, about which we can’t say a great deal: the system \( \langle \Omega \rangle_G \) for \( G \) generated by any set \( \Omega \) of subspaces. There is one system though that is intrinsic to \( G \), encoding some of its structure: for \( H \) a subgroup of \( G \) let \( \text{Fix}(H) = \{ v \in V \mid v g = v \text{ for all } g \in H \} \) be the fixed subspace of \( H \), and

\[ S = \{ \text{Fix}(H) \mid H \text{ a subgroup of } G \}. \]

Then \( \text{Fix}(H) g = \text{Fix}(g^{-1} H g) \), \( \text{Fix}(H_1) \cap \text{Fix}(H_2) = \text{Fix}(H_1, H_2) \), where \( \langle H_1, H_2 \rangle \) is the subgroup generated by the \( H_i \), and \( V \) is the fixed space of the trivial subgroup. We will return to this example in \cite{15}.

**Definition 2.** Let \( G \subset GL(V) \) be a group and \( S \) a system in \( V \) for \( G \). The monoid of partial linear isomorphisms given by \( G \) and \( S \) is the submonoid of \( ML(V) \) defined by

\[ M(G, S) := \{ g_X \mid g \in G, X \in S \}. \]

If \( G \) is a reflection group then \( M(G, S) \) is called a reflection monoid.

We will remind the reader of the definition of reflection group in \cite{13} and properly justify the terminology “reflection monoid” in \cite{15}. Observe that the monoid structure on \( M(G, S) \) is guaranteed by (1) and (S1)-(S3). If \( S \) a system for \( G \), \( X \in S \), and \( \varepsilon : V \to V \) is the identity isomorphism, then \( \varepsilon_X \in M(G, S) \), and thus every \( X \in S \) is the domain of some element of \( M(G, S) \). Conversely, by (S1)-(S3) and (1), every element of \( M(G, S) \) has domain some element of \( S \), so that \( S \) is precisely the set of domains of the partial isomorphisms in \( M(G, S) \).
If \( g_X \in M(G, S) \) then \((g_X)^{-1} = g_X^{-1} \in M(G, S)\), and we have an inverse monoid with units the \( g \in G \) and idempotents the partial identities \( \varepsilon_X \) for \( X \in S \). Moreover any \( g_X = \varepsilon_X g \), so \( M(G, S) \) is factorizable.

Even with these very modest preliminaries, it is possible to prove a result with non-trivial consequences:

**Theorem 1.** Let \( G \subset GL(V) \) be a finite group, \( S \) a finite system in \( V \) for \( G \), and \( M(G, S) \) the resulting monoid of partial linear isomorphisms. Then

\[
|M(G, S)| = \sum_{X \in S} |G : G_X|,
\]

where \( G_X \) is the isotropy group of \( X \in S \).

**Proof.** For \( X \subset V \) let \( M(X) \) be the set of \( \alpha \in M(G, S) \) with domain \( X \). Then \( M(G, S) \) is the disjoint union of the \( M(X) \), and as \( S \) is precisely the set of domains of the \( \alpha \in M(G, S) \), we have \( |M(G, S)| = \sum_{X \in S} |M(X)| \). The elements of \( M(X) \) are the partial isomorphisms obtained by restricting the elements of \( G \) to \( X \), with \( g_1, g_2 \in G \) the same partial isomorphism if and only if they lie in the same coset of the isotropy subgroup \( G_X \subset G \). Thus \( |M(X)| \) is the index \([G : G_X]\) and the result follows. \( \square \)

If \( X, Y \in S \) lie in the same orbit of the \( G \)-action (S2) on \( S \), then their isotropy groups \( G_X, G_Y \) are conjugate, and the sum in Theorem 1 becomes

\[
|M(G, S)| = |G| \sum_{X \in \Omega} \frac{n_X}{|G_X|},
\]

where \( \Omega \) is a set of orbit representatives, and \( n_X \) is the number of subspaces in the orbit containing \( X \). Most of our applications of Theorem 1 will use the form (2).

We end the section by recalling a result from semigroup theory. At several points in the paper we will want to identify a monoid of partial isomorphisms with some pre-existing monoid in the literature. This is possible if the group of units are the same, the idempotents are the same and the actions on the groups of idempotents are the same:

**Proposition 1.** Let \( M = EG \) and \( N = FH \) be factorizable inverse monoids, and \( \theta : G \to H, \varphi : E \to F \) homomorphisms such that

- \( \varphi \) is equivariant: \((geg^{-1})\varphi = (g\theta)(e\varphi)(g\theta)^{-1}\) for all \( g \in G \) and \( e \in E \), and
- \( \theta \) respects stabilizers: \( G_e\theta \subset H_{e\varphi} \) for all \( e \in E \).

Then the map \( \chi : M \to N \) given by \((eg)\chi = (e\varphi)(g\theta)\) is a homomorphism. Moreover, \( \chi \) is surjective if and only if \( \theta, \varphi \) are surjective, and \( \chi \) is an isomorphism if and only if \( \theta, \varphi \) are isomorphisms with \( G_e\theta = H_{e\varphi} \) for all \( e \in E \).

We mention that this result also occurs in a preprint of D. Fitzgerald; as there, we leave the straightforward proof to the reader.

2. Two examples for the symmetric group \( \mathfrak{S}_n \)

The representation of the symmetric group \( \mathfrak{S}_n \) by permutation matrices leads to two interesting examples of monoids of partial isomorphisms—both of which turn out to be reflection monoids, and both of which can be identified with familiar monoids of partial permutations.
2. The Boolean monoids

Let $V$ be a Euclidean space with basis $\{x_1, \ldots, x_n\}$ and inner product $(x_i, x_j) = \delta_{ij}$, the Kronecker delta. Let the symmetric group act on $V$ via $x_i\pi = x_i\pi$ for $\pi \in S_n$; we will abuse notation and write $\mathfrak{S}_n \subset GL(V)$, identifying $\mathfrak{S}_n$ with its image under this representation.

For $J \subset I = \{1, \ldots, n\}$, let
$$X(J) = \bigoplus_{j \in J} \mathbb{R}x_j \subset V,$$
and let $\mathcal{B}$ be the collection of all such subspaces as $J$ ranges over the subsets of $I$, with $X(\emptyset) = 0$. We have $X(I) = V$, $X(J)\pi = X(J\pi)$ for all $\pi \in \mathfrak{S}_n$, and
$$X(J_1) \cap X(J_2) = X(J_1 \cap J_2).$$

Indeed, partially ordering $\mathcal{B}$ by inclusion, the map $X(J) \rightarrow J$ is a lattice isomorphism $\mathcal{B} \rightarrow 2^n$ to the Boolean lattice $2^n$ of all subsets of $I$.

The result is that $\mathcal{B}$ is a system in $\mathfrak{S}_n$, which in honour of the lattice isomorphism above we will call the Boolean system for $\mathfrak{S}_n$. We form the associated monoid $M(\mathfrak{S}_n, \mathcal{B})$ and call it the Boolean monoid.

Clearly $X(J)$ has isotropy group $\mathfrak{S}_{\Gamma \setminus J}$. Moreover, $\mathfrak{S}_{\Gamma}$ acts transitively on the subsets $J$ of a fixed size $k$, so that a set $\Omega$ of orbit representatives on $\mathcal{B}$ is given by the $X(1, \ldots, k)$, with each orbit having size the number of $k$ element subsets of $I$. Plugging all of this into Theorem 1 and its alternative version 2, gives
$$|M(\mathfrak{S}_n, \mathcal{B})| = \sum_{J \subset I} |\mathfrak{S}_{\Gamma \setminus J}| = |\mathfrak{S}_n| \frac{n!}{\sum_{k=0}^{n} \binom{n}{k} \frac{1}{\mathfrak{S}_{n-k}}} = \sum_{k=0}^{n} \binom{n}{k}^2 k!$$

Readers from semigroup theory will recognize the formula on the right as the order of the symmetric inverse monoid $\mathcal{I}_n$; readers from reflection groups will recognize the action of $\mathfrak{S}_n$ that led to it.

Neither is a coincidence: Let $\theta = \text{id} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ and $\varphi : \varepsilon_{X(J)} \mapsto \varepsilon_J$ the isomorphism on the idempotents induced by the lattice isomorphism $\mathcal{B} \rightarrow 2^n$—that $\varphi$ is a homorphism follows, for example, by (4). As $X(J)\pi = X(J\pi)$, we have $\pi^{-1}\varepsilon_{X(J)\pi} = \varepsilon_{X(J\pi)}$, and so
$$(\pi^{-1}\varepsilon_{X(J)\pi})\varphi = \varepsilon_{J\pi} = \pi^{-1}\varepsilon_J\pi = (\pi\theta)^{-1}(\varepsilon_{X(J)}\varphi)(\pi\theta),$$
giving the equivariance of $\varphi$. The stabilizer of the idempotent $e = \varepsilon_{X(J)}$ consists of those $\pi \in \mathfrak{S}_n$ such that $x_j\pi = x_j$ for all $j \in J$, whereas for $e\varphi = \varepsilon_J$ we require $j\pi = j$. Proposition 1 thus gives,

**Proposition 2.** The map $\pi_{X(J)} \mapsto \pi_J$ is an isomorphism $M(\mathfrak{S}_n, \mathcal{B}) \rightarrow \mathcal{I}_n$ from the Boolean monoid to the symmetric inverse monoid.

2.2. The Coxeter arrangement monoids

We keep the same $\mathfrak{S}_n \subset GL(V)$ and notation from 2.1 but switch to a more interesting system. For $1 \leq i \neq j \leq n$, let $\mathcal{A}$ be the collection of hyperplanes $H_{ij} = (x_i - x_j)^\perp \subset V$, and $\mathcal{H} = L(\mathcal{A})$ be the set of all possible intersections of elements of $\mathcal{A}$, with the null intersection taken to be $V$. This time we order $\mathcal{H}$ by reverse inclusion, via which it acquires the structure of a lattice, with the join of any two subspaces their intersection and meet, the subspace generated by them.

Just as with the Boolean system $\mathcal{B}$, we can identify $\mathcal{H}$ with a well known combinatorial lattice. Recall that a partition of $I = \{1, \ldots, n\}$ is a collection $A = \{A_1, \ldots, A_p\}$ of nonempty pairwise disjoint subsets $A_i \subset I$, or blocks, whose union is $I$. If $\lambda_i = |A_i|$ then $\lambda = |A| = (\lambda_1, \ldots, \lambda_p)$ is a partition of $n$, ie: the integers satisfy $\lambda_i \geq 1$ and $\sum \lambda_i = n$. Order the set $\Pi(n)$
of partitions of \( I \) by refinement: \( A \leq A' \) if and only if every block of \( A \) is contained in some block of \( A' \).

The result is the partition lattice. It is not hard to show (see eg: [20 Proposition 2.9]) that the map that sends the hyperplane \( H_{ij} \) to the partition with a single non-trivial block \( A_1 = \{i, j\} \), extends to a lattice isomorphism \( \mathcal{H} \to \Pi(n) \). Indeed, if \( X(A) \in \mathcal{H} \) is the subspace,

\[
X(A) = \bigcap_{\lambda \in A} \bigcap_{i,j \in A_k} H_{ij},
\]

then \( X(A) \to A \) is the isomorphism. In particular \( \sum \alpha_i x_i \in X(A) \) if and only if \( \alpha_i = \alpha_j \) when \( i, j \) lie in the same block. There is a faithful \( \mathfrak{S}_n \)-action on \( \Pi(n) \) given by \( \Lambda \pi = \{ A_1 \pi, \ldots, A_p \pi \} \), while the \( \mathfrak{S}_n \)-action on \( \mathcal{H} \) is given by \( X(A) \pi = X(A \pi) \).

As a consequence of this, and the fact that it is by definition closed under intersection, we have \( \mathcal{H} \) is a system in \( V \) for the symmetric group. By virtue of its description as the intersection lattice of the hyperplanes \( H_{ij} \), we call \( \mathcal{H} \) the Coxeter arrangement system for \( \mathfrak{S}_n \). We will properly remind the reader about hyperplane arrangements in \([5]\). We remark that the Boolean system \( B \) of \([2,1]\) is also an arrangement system, with \( B = L(\Lambda) \), where \( \Lambda \) consists of the coordinate hyperplanes \( x_i \). For reasons that will be made clearer in \([5]\) the \( H_{ij} \) are a more natural collection of hyperplanes to associate with the symmetric group than the \( x_i \), so we will reserve the arrangement terminology for this case.

We now apply Theorem \([1]\) By \((6)\) and the comments following it, we have \( x_\pi = x \) for all \( x \in X(A) \) if and only if \( A \pi = A_i \) for all \( i \). The isotropy group of the subspace \( X(A) \) is thus isomorphic to a product of symmetric groups \( \mathfrak{S}_{A_1} \times \cdots \times \mathfrak{S}_{A_p} \), called a Young subgroup of \( \mathfrak{S}_n \), and Theorem \([1]\) becomes a sum, over all partitions, of the indices of the resulting Young subgroups.

We can also be quite explicit: for a partition \( \Lambda \), let \( b_i > 0 \) be the number of \( \lambda_j \) equal to \( i \), and

\[
b_\Lambda = b_1! \ldots b_n! (n!)^{b_1} \ldots (n!)^{b_n} = b_1! \ldots b_n! \lambda_1! \ldots \lambda_p!.
\]

**Proposition 3 ([20] Proposition 6.72).** In the action of the symmetric group \( \mathfrak{S}_n \) on \( \mathcal{H} \), two subspaces \( X(A) \) and \( X(A') \) lie in the same orbit if and only if \( \|A\| = \|A'\| \). The cardinality of the orbit of the subspace \( X(A) \) is \( n! / b_\Lambda \).

Plugging everything into version \((2)\) of Theorem \([1]\) including a summary of the discussion above, gives

**Theorem 2.** Let \( \mathfrak{S}_n \subset GL(V) \) and \( \mathcal{H} = L(\Lambda) \) the intersection lattice of the hyperplanes \( H_{ij} \). Then the Coxeter arrangement monoid \( \mathcal{M}(\mathfrak{S}_n, \mathcal{H}) \) has order,

\[
|M(\mathfrak{S}_n, \mathcal{H})| = \sum_{\Lambda} |\mathfrak{S}_\Lambda : \mathfrak{S}_{A_1} \times \cdots \times \mathfrak{S}_{A_p}| = (n!)^2 \sum_\Lambda \frac{1}{b_\Lambda \lambda_1! \ldots \lambda_p!},
\]

the first sum over all partitions \( \Lambda \) of \( I \), and the second over all partitions \( \lambda = (\lambda_1, \ldots, \lambda_p) \) of \( n \), with \( b_\lambda \) given by \((7)\).

Theorem \([4]\) of \([5,1]\) will generalise this result, replacing \( \mathfrak{S}_n \) by an arbitrary finite reflection group.

The formula on the right hand side of Theorem \([2]\) may also ring a bell with the cognoscenti. A **uniform block permutation** is a bijection \( \pi : \Lambda \to \Gamma \) between two partitions of \( I \). Thus, it is a bijection \( \pi : I \to I \), where the image of each block of \( \Lambda \) is a block of \( \Gamma \). If \( \Lambda = \{ A_1, \ldots, A_p \} \), then up to a rearrangement of the blocks, \( \Gamma = \{ A_1 \pi, \ldots, A_p \pi \} \) and we write \( [\pi]_\Lambda \) for this uniform block permutation, noting that \( [\pi]_\Lambda = [\tau]_\Delta \) if and only if \( \Lambda = \Delta \) and \( A_i \pi = A_i \tau \) for all \( i \). We define an associative product \( [\pi]_\Lambda [\tau]_\Gamma = [\pi \tau]_\Delta \), where \( \Delta = \Lambda \vee \Gamma \) \( \pi^{-1} \) and \( \vee \) is the join in the partition lattice (compare this expression with the domain on the right hand side of \((10)\)). This turns out to be a factorizable inverse monoid, the monoid of uniform block permutations \( \mathcal{P}_n \) (see \([18,15]\)). Its group of units is \( \mathfrak{S}_n \) and the idempotents are the \( [\varepsilon]_\Lambda \) where \( \varepsilon : I \to I \) is the identity permutation. We have \( \pi^{-1} [\varepsilon]_\Lambda \pi = [\varepsilon]_\Lambda \).
Let \(\theta = \text{id} : \mathcal{S}_n \rightarrow \mathcal{S}_n\) and \(\varphi : \varepsilon_{X(A)} \mapsto [\varepsilon]_A\) the isomorphism on the idempotents induced by the lattice isomorphism \(\mathcal{H} \rightarrow \Pi(n)\): that \(\varphi\) is a homomorphism follows for example from \(X(A_1) \cap X(A_2) = X(A_1 \cup A_2)\); remember that \(\mathcal{H}\) is ordered by reverse inclusion. Equivariance follows from \(X(A)\pi = X(A\pi)\) much as in the discussion preceding Proposition 2 as does the condition on the idempotent stabilizers. Thus,

**Proposition 4.** The map \(\pi_{X(A)} \mapsto [\pi]_A\) is an isomorphism \(M(\mathcal{S}_n, \mathcal{H}) \rightarrow \mathcal{P}_n\) from the Coxeter arrangement monoid to the monoid of uniform block permutations.

### 3. Reflection groups

In this section we fix notation and leave the reader unfamiliar with reflection groups to consult one of the standard references \([3, 13, 14]\). We have generally followed \([13]\). For concreteness (as much as anything else) all the reflection groups in this paper will be finite and real, that is, subgroups \(W \subset GL(V)\) generated by linear reflections of a real vector space \(V\).

Any (finite real) reflection group has the form \(W(\Phi) = \langle s_v | v \in \Phi \rangle\), where \(\Phi \subset V\) is a root system and \(s_v\) the reflection in the hyperplane orthogonal to \(v\). The finite real reflection groups are, up to isomorphism, direct products of \(W(\Phi)\) for \(\Phi\) from a well known list of irreducible root systems. These \(\Phi\) fall into five infinite families of types \(A_{n-1}, B_n, C_n, D_n\) (the classical systems) and \(I_2(m)\), and six exceptional cases of types \(H_3, H_4, F_4, E_6, E_7\) and \(E_8\). Notable amongst these are the \(\Phi\) whose associated group \(W(\Phi)\) is a finite crystallographic reflection, or Weyl group: these are the \(W(\Phi) \subset GL(V)\) that leave invariant some \(\mathbb{Z}\)-lattice \(L \subset V\). The \(W(\Phi)\) for \(\Phi\) of type \(I_2(m)\) are just the dihedral groups.

Table 1 gives standard \(\Phi\) for the classical Weyl groups with \(\{x_1, \ldots, x_n\}\) an orthonormal basis for \(V\). The root systems of types \(B\) and \(C\) have the same symmetry, but different lengths of roots. The associated Weyl groups are thus identical, and as it is these that ultimately concern us, we have given just the type \(B\) system in Table 1 (type \(C\) has roots \(\pm 2x_i\) rather than the \(\pm x_i\)). For convenience in expressing some of the formulae of \([14]\) we extend the notation by adopting the additional conventions \(A_{-1} = A_0 = \emptyset, B_0 = \emptyset, B_1 = \{\pm x_1\}\), and \(D_0 = D_1 = \emptyset, D_n = \{\pm x_i, \pm x_j (1 \leq i < j \leq n)\}\) for \(n = 2, 3\). Table 2 gives the orders of the exceptional groups (where \(W(G_2)\) is the group of type \(I_2(6)\)). We will have no need for their root systems in this paper.

This paper started with the monoids \(M(G, \mathcal{S})\) for \(G\) an arbitrary (linear) group. One of the reasons to focus on the case that \(G\) is a reflection group is that the isotropy groups \(G_X\) are very often also reflection groups, and this makes the calculation in \([2]\) do-able. A theorem of Steinberg \([34, \text{Theorem } 1.5]\) asserts that for \(G = W(\Phi)\) and \(X \subset V\) any subspace, the isotropy group \(W(\Phi)_X\) is generated by the reflections \(s_v\) for \(v \in \Phi \cap X^\perp\).

### 4. The Boolean reflection monoids

In \([2, 1]\) we considered the permutation action of \(\mathcal{S}_n\) on a Euclidean space \(V\) of dimension \(n\); we now know that this is nothing more than a well known realization of \(\mathcal{S}_n\) as the reflection group \(W(A_{n-1})\). Indeed the map \(s_{x_i-x_j} \mapsto (i, j)\) induces an isomorphism \(W(A_{n-1}) \rightarrow \mathcal{S}_n\). 

---

**Table 1.** Standard root systems \(\Phi \subset V\) for the classical Weyl groups \([13, \text{§2.10}]\).

| Type of \(\Phi\)       | Order of \(W(\Phi)\) | Root system \(\Phi\) |
|------------------------|----------------------|-----------------------|
| \(A_{n-1} (n \geq 2)\) | \(n!\)               | \(\{x_i - x_j (1 \leq i \neq j \leq n)\}\) |
| \(B_n (n \geq 2)\)    | \(2^n n!\)           | \(\{\pm x_i (1 \leq i \leq n), \pm x_i \pm x_j (1 \leq i < j \leq n)\}\) |
| \(D_n (n \geq 4)\)    | \(2^{n-1} n!\)       | \(\{\pm x_i \pm x_j (1 \leq i < j \leq n)\}\) |
which we write as $g(\pi) \mapsto \pi$. Moreover, the $W(A_{n-1})$-action on the subspaces $X(J) \in \mathcal{B}$ is just $X(J)g(\pi) = X(J\pi)$.

The Boolean monoid $M(\mathcal{S}_n, \mathcal{B})$ of [2.1] is thus a reflection monoid, which we will denote as $M(A_{n-1}, \mathcal{B})$ from now on, and the isomorphism $W(A_{n-1}) \cong \mathcal{S}_n$ extends to an isomorphism $M(A_{n-1}, \mathcal{B}) \cong I_n$.

The moral of this section is that the Boolean groups of types $A, B, D$, and the orders of the resulting reflection monoids can be determined in a nice uniform way. Moreover, these Weyl groups have well known alternative descriptions as certain groups of permutations, and so too the resulting reflection monoids, at least in types $A$ and $B$, have descriptions as naturally occurring monoids of permutations.

First, the alternative descriptions of the reflection groups in types $B$ and $D$. As usual $I$ is a set and $-I = \{-x \mid x \in I\}$ a set with the same cardinality. The group $\mathcal{S}_{\pm I}$ of signed permutations of $I$ is $\mathcal{S}_{\pm I} = \{\pi \in \mathcal{S}_{I,-I} \mid (-x)\pi = -(x\pi)\}$, where $\mathcal{S}_{\pm I}$ has the obvious meaning. For $x \in I$, let $|x| = \{x, -x\}$, and $|I| = \{|x| : x \in I\}$. If $\pi \in \mathcal{S}_{\pm I}$, define $|\pi| \in \mathcal{S}_{|I|}$ by

$$|x||\pi| = |y| \iff \{x\pi, -x\pi\} = \{y, -y\}.$$ 

Then the map $\pi \mapsto |\pi|$ is a surjective homomorphism $|\cdot| : \mathcal{S}_{\pm I} \to \mathcal{S}_{|I|} \cong I_I$.

A signed permutation $\pi$ is even if the number of $x \in I$ with $x\pi \in -I$ is even, and the even signed permutations form a subgroup $\mathcal{S}_{\pm I}^e$ of index two in $\mathcal{S}_{\pm I}$. Indeed, if $\tau_x$ is the signed transposition $(x, -x)$, then $\{1, \tau_x\}$ are coset representatives for $\mathcal{S}_{\pm I}^e$ in $\mathcal{S}_{\pm I}$. In particular, as $|\tau_x| = 1$, restriction gives a surjective homomorphism $|\cdot| : \mathcal{S}_{\pm I}^e \to I_I$.

There are isomorphisms $W(B_n) \to \mathcal{S}_{\pm n}$ induced by

$$s_{x_i - x_j} \mapsto (i, j)(-i, -j)$$
and $W(D_n) \to \mathcal{S}_{\pm n}^e$ induced by $s_{x_i - x_j} \mapsto (i, j)(-i, -j)$ and $s_{x_i + x_j} \mapsto (i, -j)(-i, j)$. As above, we write $g(\pi)$ for the element of $W(B_n)$ or $W(D_n)$ corresponding to $\pi \in \mathcal{S}_{\pm n}$ or $\mathcal{S}_{\pm n}^e$.

Now let $V$ be Euclidean with orthonormal basis $\{x_1, \ldots, x_n\}$, $I = \{1, \ldots, n\}$, and

$$\mathcal{B} = \{X(J) \mid J \subset I\},$$

the subspaces from [3]. Using these descriptions of the reflection groups $W(\Phi)$ for $\Phi = A_{n-1}, B_n$ and $D_n$, we have

$$X(J)g(\pi) = X(J|\pi|)$$

for $g(\pi) \in W(\Phi)$, as well as the $X(J) = V$ and $X(J_1) \cap X(J_2) = X(J_1 \cap J_2)$ that we had in [2.1]. Thus $\mathcal{B}$ is a system in $V$ for $W(\Phi)$, which we continue to call the Boolean system. We write $M(\Phi, \mathcal{B})$ instead of $M(W(\Phi), \mathcal{B})$, and call these the Boolean reflection monoids of types $A_{n-1}, B_n$ or $D_n$. Note that $\mathcal{B}$ is not a system for any of the exceptional $W(\Phi)$.

Now to their orders. Let $\Phi = \Phi_n$ be a root system of type $A_{n-1}, B_n$ or $D_n$ as in Table [1] For $J \subset I$, we write $\Phi_J$ for $\Phi \cap X(J)$. By Steinberg’s theorem the isotropy group of the subspace $X(J)$ is generated by the $s_v$ with $v \in \Phi \cap X(J)^\perp = \Phi_{I \setminus J}$. As the homomorphisms $|\cdot|$ maps $\mathcal{S}_n, \mathcal{S}_{\pm n}$ and $\mathcal{S}_{\pm n}^e$ onto $\mathcal{S}_n$, the action [9] of $W(\Phi)$ on $\mathcal{B}$ is transitive, for fixed $k$, on the $X(J)$ of dimension $k$. Thus:

| Type of $\Phi$ | Order of $W(\Phi)$ |
|----------------|---------------------|
| $G_2$          | $2^3 3$             |
| $F_4$          | $2^7 3^2$           |
| $E_6$          | $2^9 3^4 5$         |
| $E_7$          | $2^{10} 3^4 7$      |
| $E_8$          | $2^{14} 3^5 5^2 7$  |

Table 2: Exceptional Weyl groups and their orders.
Theorem 3. Let $\Phi_n$ be a root system of type $A_{n-1}, B_n$ or $D_n$ as in Table 1 and $\mathcal{B}$ the Boolean system (8) for $W(\Phi_n)$. Then the Boolean reflection monoids have orders,

$$|M(\Phi_n, \mathcal{B})| = \sum_{J \subset I} |W(\Phi) : W(\Phi_{I \setminus J})| = |W(\Phi_n)| \sum_{k=0}^{n} \frac{n!}{k!}.$$ 

Compare Theorem 3 with (9). By the conventions of (8) we have $|W(A_k)| = (k + 1)!$, $|W(B_k)| = 2^k k!$, $|W(D_k)| = 1$, and $|W(D_k)| = 2^{k-1}k!$ for $k > 1$, giving the explicit versions,

| $\Phi_n$ | $A_{n-1}$ | $B_n$ | $D_n$ |
|---|---|---|---|
| $|M(\Phi_n, \mathcal{B})|$ | $\sum_{k=0}^{n} \frac{n!}{k!}$ | $\sum_{k=0}^{n} 2^k \frac{n!}{k!}$ | $2^{n-1}n! + \sum_{k=1}^{n} 2^k \frac{n!}{k!}$ |

The dichotomy $W(A_{n-1}) \cong \mathfrak{S}_n, M(A_{n-1}, \mathcal{B}) \cong \mathcal{I}_n$ has a type $B$ version, for which we need an inverse monoid to play the role of $\mathfrak{S}_\pm n$. This is the **monoid of partial signed permutations** of $I$:

$$\mathcal{I}_{\pm I} := \{ \pi \in \mathcal{I}_{I \cup \neg I} \mid (-x)\pi = -(x\pi) \text{ and } x \in \text{dom } \pi \iff -x \in \text{dom } \pi \},$$

with $\mathcal{I}_{\pm n}$ having the obvious meaning. Every element of $\mathcal{I}_{\pm n}$ has the form $\pi_X, (X = J \cup \neg J)$ for some signed permutation $\pi$ and $J \subset I$. Thus $\mathcal{I}_{\pm n}$ has units $\mathfrak{S}_{\pm I}$ and idempotents the $\varepsilon_X, (X = J \cup \neg J)$ with $\varepsilon : I \rightarrow I$ the identity map.

Let $\theta : W(B_n) \rightarrow \mathfrak{S}_{\pm n}$ be the isomorphism $g(\pi) \mapsto \pi$ described above, and

$$\varphi : \varepsilon_{X(J)} \mapsto \varepsilon_X \ (X = J \cup \neg J)$$

the isomorphism on the idempotents induced by the lattice isomorphism $\mathcal{B} \rightarrow 2^n$. Observe that if $\pi \in \mathfrak{S}_{\pm n}$ then $\pi^{-1} \varepsilon_X \pi = \varepsilon_{X[\pi]} \ (X[\pi] = J[\pi] \cup \neg J[\pi])$, and the equivariance of $\varphi$ follows from this and $X(J)\pi = X(J[\pi])$. Thus, another application of Proposition 1 gives,

**Proposition 5.** The map $g(\pi)_{X(J)} \mapsto \pi_X, (X = J \cup \neg J)$ is an isomorphism $M(B_n, \mathcal{B}) \rightarrow \mathcal{I}_{\pm n}$ from the Boolean monoid of type $B$ to the monoid of partial signed permutations.

Thus we have the pair $W(B_n) \cong \mathfrak{S}_{\pm n}$ and $M(B_n, \mathcal{B}) \cong \mathcal{I}_{\pm n}$, to go with the one in type $A$. What about a pair $W(D_n) \cong \mathfrak{S}_{\pm n}$ and $M(D_n, \mathcal{B}) \cong \mathcal{I}_{\pm n}$, or some such? The problem is that one can show, by thinking in terms of partial signed permutations, that the non-units of $M(B_n, \mathcal{B})$ and $M(D_n, \mathcal{B})$ are the same (which is why the orders of these reflection monoids are identical except for the $k = 0$ terms). This makes a nice interpretation of $M(D_n, \mathcal{B})$ in terms of “even signed permutations” unlikely.

5. **The Coxeter arrangement monoids**

Just as (4) generalizes the Boolean monoid $M(\mathfrak{S}_n, \mathcal{B})$ of (2.1) replacing $\mathfrak{S}_n$ by a classical Weyl group, so now we generalize the Coxeter arrangement monoid $M(\mathfrak{S}_n, \mathcal{H})$ of (2.2) replacing $\mathfrak{S}_n$ by an arbitrary finite reflection group.

5.1. **Generalities**

Steinberg’s Theorem (8) provides a good reason to study reflection monoids, rather than just monoids of partial isomorphisms. Another reason is that the system

$$\mathcal{S} = \{ \text{Fix}(H) \mid H \text{ a subgroup of } G \},$$
of \([11]\) has a particularly nice combinatorial structure when \(G\) is a reflection group. These systems and the resulting reflection monoids, especially when \(G\) is a Weyl group, are the subject of this section. Recall that \(V\) is a finite dimensional real space and \(W \subset GL(V)\) a finite reflection group.

A hyperplane arrangement \(\mathcal{A}\) is a finite collection of hyperplanes in \(V\). General references are [20,38], where the hyperplanes can be affine, but we restrict ourselves to linear arrangements. An important combinatorial invariant for \(\mathcal{A}\) is the intersection lattice \(L(\mathcal{A})\)–the set of all possible intersections of elements of \(\mathcal{A}\), ordered by reverse inclusion, and with the null intersection taken to be the ambient space \(V\). What results is a lattice [20, §2.1], with unique minimal element the space \(V\). If the \(\mathcal{A}\) are reflecting hyperplanes of a reflection group \(W \subset GL(V)\), then we have a reflection or Coxeter arrangement.

The intersection lattice \(L(\mathcal{A})\) of a Coxeter arrangement is a system of subspaces for the associated reflection group \(W\); if \(X \in \mathcal{A}\) and \(s_X \in W\) is the reflection in \(X\), then for \(g \in W\) we have \(s_X g = g^{-1} s_X g\), and so \(Xg \in \mathcal{A}\). Thus \(AW = \mathcal{A}\), extending to \(L(\mathcal{A}) W = L(\mathcal{A})\). We will write \(\mathcal{H}\) for \(L(\mathcal{A})\), calling it the reflection or Coxeter arrangement system, and reserving \(\mathcal{B}\) for the Boolean system. We call the resulting \(M(W, \mathcal{H})\) the reflection or Coxeter arrangement monoid of \(\mathcal{A}\) (or \(W\)). If \(W = W(\Phi)\), we write \(\mathcal{H}(\Phi)\) for \(\mathcal{H}\), and \(M(\Phi, \mathcal{H})\) for \(M(W(\Phi), \mathcal{H})\), observing that \(M(A_n-1, \mathcal{H})\) is the monoid \(M(\mathcal{S}_n, \mathcal{H})\), or the monoid of uniform block permutations, of \([2,2]\).

**Lemma 1.** Let \(W \subset GL(V)\) be a finite reflection group with reflecting hyperplanes \(\mathcal{A}\) and Coxeter arrangement system \(\mathcal{H} = L(\mathcal{A})\). Then \(\mathcal{H} = \{\text{Fix}(H) \mid H \text{ a subgroup of } W\}\).

**Proof.** Write \(\mathcal{F}\) for the system of fixed subspaces. It is not hard to show using Steinberg’s theorem and induction on the order of \(W\) (see, eg. [20, Theorem 6.27]), that \(\text{Fix}(g) \in \mathcal{H}\) for all \(g \in W\), where \(\text{Fix}(g)\) is the fixed subspace of the element \(g\). As \(\text{Fix}(H) = \bigcap_k \text{Fix}(h)\) and \(H\) is finite, we get \(\mathcal{F} \subset \mathcal{H}\). Moreover, if \(s\) is a reflection then \(\text{Fix}(s)\) is the reflecting hyperplane of \(s\), so that \(\mathcal{A} \subset \mathcal{F} \subset \mathcal{H}\). As every element of \(\mathcal{H}\) is an intersection of elements of \(\mathcal{A}\), and \(\mathcal{F}\) is closed under \(\bigcap\), we have \(\mathcal{F} = \mathcal{H}\). \(\square\)

We will see in \([5]\) that a monoid isomorphism \(M(G, \mathcal{S}) \rightarrow M(G', \mathcal{S}')\) induces a poset isomorphism \(\mathcal{S} \rightarrow \mathcal{S}'\), with the subspaces ordered by inclusion. A comparison of the number of \(k\)-dimensional subspaces in \(\mathcal{H}\) and \(\mathcal{B}\) shows that for a given \(W\), there can be no isomorphism between the Boolean and Coxeter arrangement monoids (see [20, §6.4] for the number of subspaces in \(\mathcal{H}\) given in terms of Stirling numbers of the second kind).

Now to orders: let \(W = W(\Phi)\) and \(\Delta \subset \Phi\) a simple system. If \(I \subset \Delta\), let \(W_I = \{s_x \mid x \in I\}\) be the resulting special parabolic subgroup, with a parabolic subgroup being any \(W\)-conjugate of a special parabolic (see [13, §1.10]). The parabolic subgroups are thus parametrised by the pairs \(I, w\) with \(I \subset \Delta\) and \(w\) a (right) coset representative for \(W_I\) in \(W\).

The parabolics in \(\mathcal{S}_I\) are just the Young subgroups \(\mathcal{S}_{A_1} \times \cdots \times \mathcal{S}_{A_p}\) for \(A = \{A_1, \ldots, A_p\}\) a partition of \(I\). Theorem \([2]\) gives the order of \(M(\mathcal{S}_n, \mathcal{H})\) as the sum of the indices of these, and indeed this is the case in general:

**Theorem 4.** Let \(W \subset GL(V)\) be a finite reflection group with Coxeter arrangement system \(\mathcal{H}\). Then the Coxeter arrangement monoid \(M(W, \mathcal{H})\) has order the sum of the indices of the parabolic subgroups of \(W\).

**Proof.** By [14, Theorem 5.2] the isotropy groups \(W_X\) are parabolic, so it suffices to show that every parabolic arises as an isotropy group \(W_X\) for some \(X \in \mathcal{H}\), and that distinct subspaces in \(\mathcal{H}\) have distinct isotropy groups.

The second of these requires only elementary arguments: if \(X, Y\) are any subspaces of \(V\) with \(W_X = W_Y = W_\alpha\), then \(W_\alpha\) also fixes \(X + Y\) pointwise. If \(X\) and \(Y\) are distinct, with one not contained in the other, then \(X\) say, is a proper subspace of \(X + Y\). Thus it suffices to show that \(X \not\subset Y\) have distinct isotropy groups for \(X, Y \in \mathcal{H}\). Suppose otherwise, and recalling that \(X, Y \in \mathcal{H}\) are intersections of reflecting hyperplanes of \(W\), write \(X = Y \cap H_1 \cap \cdots \cap H_k\).
with the \( H_i \) reflecting hyperplanes of \( W \) and \( Y \not\in H_i \) for any \( i \). In particular \( W_{H_i} = \langle s_i \rangle \) with \( s_i \) the reflection in \( H_i \), and \( s_i \not\in W_Y \) \cite{[13] Theorem 1.10}. But we also have \( \langle W_Y, s_1, \ldots, s_k \rangle \subset W_X = W_Y \), so that the \( s_i \in W_Y \), a contradiction.

Next we reduce to the case where the roots \( \Phi \) span \( V \). Let \( U = \sum_x \mathbb{R} x \subset V \) and decompose \( V = U \oplus U^\perp \). Then \( U \) is a \( W \)-invariant subspace of \( V \) with the complement \( U^\perp \) fixed pointwise, and \( W \) is a finite reflection group in \( GL(U) \) with reflecting hyperplanes the \( x^\perp \cap U \) for \( x \in \Phi \), and associated Coxeter arrangement system \( \mathcal{H}_U := \{ X \cap U \mid X \in \mathcal{H} \} \). The map \( X \mapsto X \cap U \) is a lattice isomorphism \( \mathcal{H} \to \mathcal{H}_U \) which induces an isomorphism between the idempotents of \( M(W, \mathcal{H}) \) and \( M(W, \mathcal{H}_U) \). An application of Proposition \( \ref{11} \) then gives an isomorphism between these two reflection monoids. Henceforth then, we will assume that \( \sum_x \mathbb{R} x = V \).

We now remind the reader of the Coxeter complex in \( V \), whose cells have isotropy groups easily identified with specific parabolics (see \cite{[13] 1.15]). If \( I \subset \Delta \) let \( C_I \) be the subset of \( V \) given by:

\[
C_I := \{ v \in V \mid (x, v) = 0 \text{ for } x \in I \text{ and } (x, v) > 0 \text{ for } x \in \Delta \setminus I \} = \bigcap_{x \in I} x^\perp \cap \bigcap_{x \in \Delta \setminus I} x^{>0},
\]

where \( x^{>0} \) is the open half space consisting of those \( v \) with \( (x, v) > 0 \). The Coxeter complex \( \Sigma \) has codimension \( k \) cells the subsets \( C_{Iw} \subset V \) where \( w \in W \) and \( |I| = k \), and the isotropy group of the cell \( C_{Iw} \) is the parabolic \( w^{-1}W_Iw \). If \( \Delta' \subset \Phi \) is some other simple system then there is a \( w \in W \) with \( \Delta' = \Delta w \). Hence if \( I' \subset \Delta' \), then

\[
C_{I'} = \bigcap_{x \in I'} x^\perp \cap \bigcap_{x \in \Delta' \setminus I'} x^{>0},
\]

is the cell \( C_{Iw} \in \Sigma \) for \( I = I'w^{-1} \), and every cell of \( \Sigma \) has this form. Thus the parabolic \( w^{-1}W_Iw \) is the isotropy group \( W_Y \) for the \( Y = C_{I'} = C_{Iw} \) of \( \ref{11} \). If \( X = \bigcap_{x \in I'} x^\perp \) then \( X \in \mathcal{H} \) and \( Y \subset X \) is an open subset. In particular, \( Y \) spans \( X \), so that \( W_Y = W_X \). Hence every parabolic arises as a \( W_X \) for some \( X \in \mathcal{H} \).

\( \square \)

5.2. Coxeter arrangement monoids of type \( B \)

Just as in Theorem \( \ref{2} \) we can more explicit about the orders of the Coxeter arrangement monoids. The material here and in \( \ref{5,3} \) is adapted from \cite{20} §6.4.

We build a combinatorial model for the Coxeter arrangement system \( \mathcal{H}(B_n) \), much as the partition lattice \( \Pi(n) \) models the system in type \( A \). If \( I = \{1, \ldots, n\} \), then a coupled partition \( A \) of \( I \) is a collection,

\[
A = \{ A_{11} + A_{12}, \ldots, A_{q1} + A_{q2}, A_1, \ldots, A_p \},
\]

of non-empty pairwise disjoint subsets whose union is \( I \). The \( A_{ij} \) and \( A_i \) are blocks, with \( A_{11} + A_{12} \) a coupled block. The + sign is purely formal, indicating that these two blocks have been coupled. Thus, a coupled partition is just a partition with some extra structure. The coupled partition is completely determined by the blocks and the couplings, so that reordering the blocks, the coupled blocks or even the blocks within a coupled block, gives the same coupled partition. If \( \lambda_{ij} = |A_{ij}| \) then let \( \lambda = ||A|| = (\Lambda_{11} + \Lambda_{12}, \ldots, \Lambda_{q1} + \Lambda_{q2}, \Lambda_1, \ldots, \Lambda_p) \) be the resulting partition of \( n \), where now + really does mean +.

Let \( T \) be the set of pairs \( (\Delta, A) \) where \( \Delta \subset I \) and \( A \) is a coupled partition of \( I \setminus \Delta \). Define a relation on \( T \) by \( (\Delta, A) \leq (\Delta', A') \) if and only if

- \( \Delta \subset \Delta' \);
- each (uncoupled) block \( A_i \in A \) is either contained in \( \Delta' \), or an (uncoupled) block \( A'_j \in A' \) or a block \( A'_{ij} \) of a couple \( A \); and
- each couple \( A_{i1} + A_{i2} \) is either contained in \( \Delta' \), or \( A_{i1} \subset A'_{j1} \) and \( A_{i2} \subset A'_{j2} \) for some couple \( A'_{j1} + A'_{j2} \in A' \).
Now let $V$ be Euclidean with orthonormal basis $\{x_1, \ldots, x_n\}$. For $(\Delta, A) \in T$, let $X(\Delta, A) \subset V$ be the subspace consisting of those $x = \sum \alpha_i x_i$ where $\alpha_i = 0$ for $i \in \Delta$, $\alpha_i = \alpha_j$ if $i, j$ lie in the same block of $A$ (either uncoupled or in a couple) and $\alpha_i = -\alpha_j$ if $i, j$ lie in different blocks of the same coupled block. By the results of [20 §6.4], the Coxeter arrangement system $\mathcal{H}(B_n)$ has elements the subspaces $X(\Delta, A)$ for $(\Delta, A) \in T$.

Proposition 6. The map $f : (\Delta, A) \mapsto X(\Delta, A)$ is a bijection from $T$ to $\mathcal{H}(B_n)$ with $(\Delta, A) \leq (\Delta', A')$ if and only if $X(\Delta', A') \subset X(\Delta, A)$. In particular, $T$ with the relation $\leq$ defined above is a lattice and $f$ is a lattice isomorphism $T \rightarrow \mathcal{H}(B_n)$.

Orlik and Terao [20 §6.4] parametrize the subspaces in $\mathcal{H}(B_n)$ using triples consisting of a subset $\Delta$, a partition $A$ of $I \setminus \Delta$ and a $\Gamma \subset I$, although these triples do not have a lattice structure.

Proof. That $f$ is a bijection is a straightforward notational translation of the results of [20 §6.4]. If $(\Delta, A) \leq (\Delta', A')$ and $x = \sum \alpha_i x_i \in X(\Delta', A')$, then it is easy to show that $x$ satisfies the conditions for being an element of $X(\Delta, A)$: we have $\alpha_i = 0$ for $i \in \Delta'$, so that $\alpha_i = 0$ when $i \in \Delta$, as $\Delta \subset \Delta'$, and so on. Conversely, by looking at the coordinates of the elements of $X(\Delta, A)$, it is easy to see that any subspace of the form $X(\Delta', A')$ must satisfy $(\Delta, A) \leq (\Delta', A')$. \Box

We saw in §4 that $W(B_n) \cong \mathfrak{S}_{\pm n}$. There is another alternative (and well known) description of $W(B_n)$ that is useful in the current context. Let $2^I$ be the subsets of $I$, but now an Abelian group under symmetric difference $S \triangle T := (S \cup T) \setminus (S \cap T)$. The symmetric group $\mathfrak{S}_I$ acts on $2^I$ via $T \mapsto T \pi$, $(\pi \in \mathfrak{S}_n$ and $T \subset I)$, and we form the semi-direct product $\mathfrak{S}_I \ltimes 2^I$, in which every element has a unique expression as a pair $(\pi, T)$ with $\pi \in \mathfrak{S}_I$, $T \subset I$. The map $s_{x_i - x_j} \mapsto (i, j) \in \mathfrak{S}_n$, $s_{x_i} \mapsto \{i\} \in 2^n$ induces an isomorphism

$$W(B_n) \rightarrow \mathfrak{S}_I \ltimes 2^n, \quad (12)$$

and we write $g(\pi, T) \in W(B_n)$ for the element mapping to $(\pi, T)$. If $J \subset I$, then $\mathfrak{S}_J \ltimes 2^J$ is naturally a subgroup of $\mathfrak{S}_I \ltimes 2^I$.

For $T \in 2^I$ and $J \subset I$, let $J^+ = J \cap T$ and $J^- = J \setminus T$, decomposing $J$ as a disjoint union $J = J^+ \cup J^-$. If $A$ is the coupled partition (11) and $\pi \in \mathfrak{S}_n$, then let

$$A\pi := \{\ldots, A_{i_1^+}\pi + A_{i_2^+}\pi, \ldots, A_{i_i^+}\pi, \ldots\},$$

and if $T \in 2^I$, then let

$$A^\top := \{\ldots, (A_{i_1^-}^\top \cup A_{i_2^-}^\top) + (A_{i_1^+}^- \cup A_{i_2^+}^-), \ldots, A_{i_i^-}^\top + A_{i_i^+}^-\},$$

with the convention $A + \emptyset = \emptyset + A = A$. Define an action of $\mathfrak{S}_I \ltimes 2^I$ on the lattice $T$ by

$$f(\Delta, A)(\pi, T) = (\Delta\pi, A^\top T\pi). \quad (13)$$

Just as $\mathfrak{S}_n \ltimes 2^n$ models $W(B_n)$ and $T$ models $\mathcal{H}(B_n)$, so (13) models the action of $W(B_n)$ on $\mathcal{H}(B_n)$: we have $X(\Delta, A) g(\pi, T) = X(\Delta\pi, A^\top T\pi)$. In particular, the lattice isomorphism of Proposition 6 is equivariant with respect to the $\mathfrak{S}_I \ltimes 2^I$ action on $T$ and the $W(B_n)$ action on $\mathcal{H}(B_n)$.

This observation allows us to give a combinatorial version of the Coxeter arrangement monoid $M(B_n, \mathcal{H})$. Its elements are “uniform block signed permutations” of the elements of $T$ with the action just described, and may be written in the form $[\pi, T]_{(\Delta, A)}$ where $(\pi, T) \in \mathfrak{S}_I \ltimes 2^I$ and $(\Delta, A) \in T$. We have $[\pi, T]_{(\Delta, A)} = [\pi', T']_{(\Delta', A')}$ if and only if $\Delta = \Delta'$, $A = A'$, $\Delta\pi = \Delta'\pi'$ and $A_i\pi = A_i'\pi'$ for all $i$. The product is defined by

$$[\pi, T]_{(\Delta, A)} [\pi', T']_{(\Delta', A')} = [(\pi, T)(\pi', T')]_{(\Gamma, T)},$$

where $(\Gamma, T) = (\Delta, A) \vee (\Delta', A')(\pi, T)^{-1}$, with $\vee$ the join in the lattice $T$. We leave the diligent reader to verify that this definition does indeed give a monoid isomorphic to $M(B_n, \mathcal{H})$. 

and content ourselves with the observation that this example illustrates the advantage of the geometric approach over the combinatorial one.

For a coupled block \( A_1 + A_2 \subset I \), let \( \mathcal{B}_{A_1+A_2} \subset \mathcal{S}_{A_1+A_2} \cong 2^{A_1+A_2} \) be the subgroup consisting of those \((\pi, T)\) in that leave each block of the coupled invariant under the restriction of the action \([\mathcal{T}]\). Precisely, we require \((A_1^+ \cup A_2^-)\pi = A_1\), from which it follows that \((A_1^+ \cup A_2^-)\pi = A_2\).

**Lemma 2.** Let \( \lambda_i = |A_i| \). Then the group \( \mathcal{B}_{A_1+A_2} \) has order \( \lambda_1! \lambda_2! c(\lambda_1 \lambda_2) \), where

\[
c(\lambda_1, \lambda_2) = \min\{\lambda_1, \lambda_2\} \sum_{j=0}^{\min\{\lambda_1, \lambda_2\}} \binom{\lambda_1}{j} \binom{\lambda_2}{j}.
\]

**Proof.** To have \((A_1^+ \cup A_2^-)\pi = A_1\), it is clearly necessary that \(A_1^+ \cup A_2^-\) and \(A_1\) have the same cardinality, and on the other hand, if this is so, then \(\pi\) can be a bijection extending any bijection \(A_1^+ \cup A_2^- \to A_1\). The \(T\) for which these two sets have the same size are precisely those with \(|T \cap A_1| = |T \cap A_2|\), of which there are \(c(\lambda_1, \lambda_2)\), and for each one there are \(\lambda_1!\) bijections \(A_1^+ \cup A_2^- \to A_1\), each one in turn extendable to \(\lambda_2!\) bijections \(\pi : A_1 + A_2 \to A_1 + A_2\). \(\square\)

Observe that if \( A_2 = \emptyset \), so we have a (uncoupled) block, then \( \mathcal{B}_{A_1+A_2} \) becomes the symmetric group \( S_{A_1} \). The following proposition summarizes all we need about the action of \( W(B_n) \) on \( \mathcal{H}(B_n) \):

**Proposition 7.** In the action of the Weyl group \( W(B_n) \) on \( \mathcal{H}(B_n) \), two subspaces \( X(\Delta, \Lambda) \) and \( X(\Delta', \Lambda') \) lie in the same orbit if and only if \(|\Delta| = |\Delta'|\) and \(||\Lambda|| = ||\Lambda'||\). The cardinality of the orbit of the subspace \( X(\Delta, \Lambda) \) is

\[
2^{n-m-p-q} \binom{n}{n-m} \frac{(n-m)!}{b_{\lambda}},
\]

where \( m = |\Delta| \), \( \Lambda \) has the form \( \{1\} \) and \( b_{\lambda} \) is given by \(7\). Moreover, if \( W_X \) is the isotropy group of \( X = X(\Delta, \Lambda) \) then

\[
W_X \cong S_{\pm \Delta} \times \mathcal{B}_{A_{11}+A_{12}} \times \cdots \times \mathcal{B}_{A_{q1}+A_{q2}} \times S_{A_1} \times \cdots \times S_{A_p} \quad (14)
\]

The groups \( [14] \) thus describe the parabolics in \( W(B_n) \), just as the Young subgroups do for \( W(\Lambda_{n-1}) \).

**Proof.** The orbit description and size is \([20] \) Proposition 6.75. For the isotropy group, we have \( x g(\pi, T) = x \) for all \( x \in X(\Delta, \Lambda) \) precisely when \((\pi, T) \in S_n \times 2^n \) leaves \( \Delta \) and each block of \( \Lambda \) invariant. The expression for \( W_X \) follows. \(\square\)

In the following we identify the group \( [14] \) with a subgroup of \( S_{\pm I} \). The proof uses Proposition \([7] \) and is another application of Theorem \([11] \) and \([2] \):

**Theorem 5.** The Coxeter arrangement monoid \( M(B_n, \mathcal{H}) \) has order

\[
|M(B_n, \mathcal{H})| = \sum_{(\Delta, \Lambda)} [S_{\pm \Delta} \times \mathcal{B}_{A_{11}+A_{12}} \times \cdots \times \mathcal{B}_{A_{q1}+A_{q2}} \times S_{A_1} \times \cdots \times S_{A_p}]
\]

\[
= 4^n (n!)^2 \sum_{(m, \lambda)} \frac{1}{4^m (m!)^2 d_{\lambda}},
\]

the first sum over all \((\Delta, \Lambda) \in \mathcal{T}\), and the second over all pairs \((m, \lambda)\) where \(0 \leq m \leq n\) is an integer, \( \lambda = (\lambda_1, \ldots, \lambda_p) \) is a partition of \( n - m \) and \( d_{\lambda} = 2^p b_{\lambda_1} \cdots b_{\lambda_p} \) with \( b_{\lambda} \) as in \([7] \).
5.3. Coxeter arrangement monoids of type D

We repeat [5,2] for type $D$, just running through the answers. Replace $T$ by the sublattice $T^\circ$ consisting of the $(\Delta, A)$ with $|\Delta| \neq 1$, the map $(\Delta, A) \mapsto X(\Delta, A)$ above restricting to an isomorphism $T^\circ \rightarrow \mathcal{H}(D_n)$.

Let $2^I \setminus 2 I$ be the subgroup consisting of those $T \subset I$ with $|T|$ even. Then the group isomorphism [12] restricts to an isomorphism $W(D_n) \rightarrow \mathcal{S}_I \times 2^I$. The action [13] of $\mathcal{S}_I \times 2^I$ on $T^\circ$ models the action of $W(D_n)$ on $\mathcal{H}(D_n)$ as before.

**Proposition 8.** If $X(\Delta, A)$ and $X(\Delta', A')$ lie in the same orbit of the action of $W(D_n)$ on $\mathcal{H}(D_n)$, then $|\Delta| = |\Delta'|$ and $|A| = |A'|$. Conversely, suppose that $|\Delta| = |\Delta'|$ and $|A| = |A'|$.

1. If $|\Delta| \geq 2$ then $X(\Delta, A)$ and $X(\Delta', A')$ lie in the same orbit, which has cardinality as in Proposition 7.

2. If $\Delta = \emptyset$, then the $W(B_n)$ orbit determined by $|A| = (\lambda_1 + \lambda_2, \ldots, \lambda_q + \lambda_{q+2}, \lambda_1, \ldots, \lambda_p)$ forms a single $W(D_n)$ orbit, except when each $\lambda_1 + \lambda_2$ and $\lambda_i$ are even, in which case it decomposes into two $W(D_n)$ orbits of size

$$\frac{2^{n-p-q+1}n!}{b_\lambda}.$$ 

3. If $X = X(\Delta, A) \in \mathcal{H}(D_n) \subset \mathcal{H}(B_n)$, then the isotropy groups $W(D_n)_X$ and $W(B_n)_X$ coincide when $\Delta = \emptyset$ and each $\lambda_1 + \lambda_2$ and $\lambda_i$ are even, otherwise, $W(D_n)_X$ has index 2 in $W(B_n)_X$.

**Proof.** The first two parts are just [20, Proposition 6.79]. For the third, the index of $W(D_n)_X$ in $W(B_n)_X$ is at most 2 as $W(D_n)_X = W(D_n) \cap W(B_n)_X$ with $W(D_n)$ of index two in $W(B_n)$. Thus either $W(D_n)_X$ has index 2 in $W(B_n)_X$ or the isotropy groups coincide, with the latter happening precisely when $Xg(\pi, T) = X$ for $g(\pi, T) \in W(B_n)$ implies that $g(\pi, T)$ is in $W(D_n)$, ie: that $|T|$ is even. It is easy to check that this happens if and only if $\Delta = \emptyset$ and each $\lambda_1 + \lambda_2$ and $\lambda_i$ is even.

**Theorem 6.** The Coxeter arrangement monoid $M(D_n, \mathcal{H})$ has order,

$$|M(D_n, \mathcal{H})| = 2^{2n-1}(n!)^2 \sum_{m, \lambda} \frac{\varepsilon_{m, \lambda}}{4^m(m!)^2 d_\lambda},$$

the sum over all pairs $(m, \lambda)$ where $0 \leq m \leq n$ is an integer $\neq 1$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a partition of $n - m$, with $\varepsilon_{m, \lambda} = 1$ if $m = 0$ and each $\lambda_i$ is even, and $\varepsilon_{m, \lambda} = 2$ otherwise.

5.4. Coxeter arrangement monoids of exceptional types

Finally, to the orders of the Coxeter arrangement monoids in the exceptional cases, where a combinatorial description of $\mathcal{H}$ is harder, but an enumeration of the orbits of the $W(\Phi)$-action on $\mathcal{H}$—and their sizes and common isotropy groups—suffices for our purposes. All this information is contained in [18,19] (see [20, Appendix C1]). For example, we can reproduce the essential information when $\Phi = F_4$ from [20, Table C.9, page 292] as

$$A_0, 12A_1, 12\tilde{A}_1, 72(A_1 \times \tilde{A}_1), 16A_2, 16\tilde{A}_2, 18B_2, 12C_3, 12B_3, 48(A_1 \times \tilde{A}_2), 48(\tilde{A}_1 \times A_2), F_4,$$

where each term $n\Phi$ indicates a $W(F_n)$-orbit on $\mathcal{H}$ of size $n$ and with isotropy group $W(\Phi)$. For our purposes the tildes can be ignored (so that $\tilde{A}_n = A_n$) and we also have $W(C_n) = W(B_n)$.

The data can then be plugged directly into (2), using the orders given in Tables [12] to get a calculation for the order of the Coxeter arrangement monoid of type $F_4$,

$$|M(F_4, \mathcal{H})| = 2^73^2 \left( 1 + \frac{12}{2} + \frac{12}{2} + \frac{72}{2} + \cdots + \frac{48}{2^23} + \frac{1}{2^73^2} \right) = 11 \cdot 4931.$$
Proposition 9. The orders of the exceptional Coxeter arrangement monoids are

| $|M(\Phi, \mathcal{H})|$   | $G_2$   | $F_4$   | $E_6$   | $E_7$   | $E_8$   |
|------------------------|---------|---------|---------|---------|---------|
|                        | $7^2$   | $11 \cdot 4931$ | $2^4 \cdot 5^2 \cdot 40543$ | $3 \cdot 113 \cdot 24667553$ | $11 \cdot 79 \cdot 55099865069$ |

What significance (if any) there is to these strange prime factorizations, we do not know.

6. Generalities on reflection monoids

We pause to explore some of the basic properties of reflection monoids. Among other things, we justify the terminology “reflection monoid”.

Recall the definition of an inverse monoid $M$ from [11] The sets of units and idempotents,

$$G = G(M) = \{a \in M \mid aa^{-1} = 1 = a^{-1}a\} \text{ and } E = E(M) = \{a \in M \mid a^2 = a\},$$

form a subgroup and commutative submonoid respectively. It is an elementary fact in semigroup theory that any commutative monoid of idempotents carries the structure of a meet semi-lattice, with order $e \leq f$ iff $ef = e$, and a unique maximal element. For this reason, $E$ is referred to as the semilattice of idempotents. An inverse submonoid of an inverse monoid $M$ is simply a subset $N$ that forms an inverse monoid under the same multiplicative and $^{-1}$ operations. More details can be found in [11][16].

We also observed in [11] that two primordial examples of inverse monoids are the symmetric inverse monoid $I_X$ and the general linear monoid $ML(V)$, which have units the symmetric group $\mathcal{S}X$ and general linear group $GL(V)$ respectively. The idempotents $E(I_X)$ consist of the partial identities $\varepsilon_Y$ for $Y \subset X$ Similarly the idempotents of $ML(V)$ are the partial identities on subspaces of $V$.

We shall be particularly interested in factorizable inverse monoids: monoids $M$ with $M = EG = GE$. Factorizability captures formally an idea used informally in §[11]2 and §[14] if $\alpha \in M$ where $M$ is an inverse submonoid of $I_X$, we have $\alpha \in EG$ if and only if $\alpha$ is a restriction of a unit of $M$. Similarly for $ML(V)$. In particular, $I_n$ and $ML(V)$, for $V$ finite dimensional, are factorizable, but $I_X$ for $X$ infinite is not (we remind the reader of our running assumption that $V$ is finite dimensional).

Of course this paper is about the monoids $M(G, S)$ of Definition [2] where $g_X$ is a unit precisely when $X = V$ and $g_X^2 = g_X$ precisely when the restriction of $g$ to $X$ is the identity on $X$. Thus $G$ is the group of units and the idempotents $E$ are the partial identities $\varepsilon_X$ for $X \in S$. If $S$ is ordered by inclusion, then $X \mapsto \varepsilon_X$ is an isomorphism of meet semi-lattices $S \rightarrow E$.

Proposition 10. Let $V$ be a finite dimensional vector space. Then $M \subset ML(V)$ is a factorizable inverse submonoid if and only if $M = M(G, S)$ for $G$ the group of units of $M$ and $S = \{\text{dom } \alpha \mid \alpha \in M\}$.

Proof. We observed in [11] that $M(G, S)$ is a factorizable inverse submonoid of $ML(V)$. Conversely, if $M$ is a factorizable inverse submonoid, let $G$ be its group of units and let $S = \{\text{dom } \alpha \mid \alpha \in M\}$. Then $V = \text{dom } \varepsilon$, for $\varepsilon : V \rightarrow V$ the identity map, $\text{dom } \alpha \cap \text{dom } \beta = \text{dom } (\alpha \alpha^{-1} \beta \beta^{-1})$, and $(\text{dom } \alpha) g = (\text{dom } g^{-1} \alpha)$ for $g \in G$. Thus $S$ is a system of subspaces for $G$ in $V$, allowing us to form the monoid $M(G, S)$ of partial isomorphisms. If $g \in G$ and $X = \text{dom } \alpha \in S$ then $g_X = \alpha \alpha^{-1} g \in M$, so that $M(G, S)$ is a factorizable inverse submonoid of $M$. Since $M(G, S)$ contains all the units and all the idempotents of $M$, it follows that $M(G, S) = M$. \[\]
Corollary 1. A submonoid \( M \subset ML(V) \) is a reflection monoid if and only if \( M \) is a factorizable inverse monoid generated by partial reflections.

Proof. If \( M = M(W, S) \) for \( W \) a reflection group, then \( M \) is a factorizable inverse submonoid of \( ML(V) \) with any element having the form \( \varepsilon_X g \) for some \( X \in S \) and \( g \in W \). Now \( g = s_1 \ldots s_k \) for some reflections \( s_1, \ldots, s_k \) and \( \varepsilon_X s_1 \) is a partial reflection, so \( \varepsilon_X g = (\varepsilon_X s_1) s_2 \ldots s_k \) is a product of partial reflections. Conversely, if \( M \) is a factorizable inverse submonoid of \( ML(V) \) then \( M = M(G, S) \) for \( G = G(M) \) by Proposition 10. It is easy to see that the non-units in \( M \) form a subsemigroup, and hence every unit of \( M \) must be a product of (full) reflections, so \( G \) is a reflection group. Indeed, if \( S \) is the set of generating partial reflections for \( M \), then \( G = \langle S' \rangle \), for \( S' \subset S \) the full reflections. \( \Box \)

If \( V_i (i = 1, 2) \) are vector spaces with \( G_i \subset GL(V_i) \) and \( S_i \) are systems in \( V_i \) for \( G_i \), then a monoid homomorphism \( M(G_1, S_1) \rightarrow M(G_2, S_2) \) induces a group homomorphism \( G_1 \rightarrow G_2 \) and a homomorphism \( E_1 \rightarrow E_2 \) of semilattices. By the comments immediately prior to Proposition 10 the latter is equivalent to a poset map \( S_1 \rightarrow S_2 \) between the two systems, ordered by inclusion. If the homomorphism \( M(G_1, S_1) \rightarrow M(G_2, S_2) \) is an isomorphism we have isomorphisms of groups \( G_1 \rightarrow G_2 \) and posets \( S_1 \rightarrow S_2 \).

Recall that Green’s relation \( \mathcal{R} \) on a monoid \( M \) is defined by the rule that \( a \mathcal{R} b \) if and only if \( aM = bM \). The relation \( \mathcal{J} \) is the left-right dual of \( \mathcal{R} \); we define \( \mathcal{H} = \mathcal{R} \cap \mathcal{L} \) and \( \mathcal{D} = \mathcal{R} \lor \mathcal{L} \). In fact, \( \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} \). Finally, \( a \mathcal{J} b \) if and only if \( MaM = MbM \). In an inverse monoid, \( a \mathcal{D} b \) if and only if \( aa^{-1} = bb^{-1} \) and similarly, \( a \mathcal{L} b \) if and only if \( a^{-1}a = b^{-1}b \).

Proposition 11. Let \( \alpha, \beta \) be elements of the monoid \( M = M(G, S) \) of partial linear isomorphisms, with \( \alpha = g_X \) and \( \beta = h_Y \) where \( g, h \in G \) and \( X, Y \in S \). Then

(i). \( \alpha \mathcal{R} \beta \) if and only if \( X = Y \);
(ii). \( \alpha \mathcal{L} \beta \) if and only if \( Xg = Yh \);
(iii). \( \alpha \mathcal{D} \beta \) if and only if \( Y \in XG \);
(iv). if \( S \) consists of finite dimensional spaces, then \( \mathcal{J} = \mathcal{D} \).

Parts (i) and (ii) follow from [11] Proposition 2.4.2 and the well known characterization of \( \mathcal{R} \) and \( \mathcal{L} \) in \( ML(V) \). The rest is now a straightforward exercise for the reader.

7. Linear algebraic monoids and Renner monoids

Given a linear algebraic group \( G \) one can extract from it a finite group, the Weyl group, which turns out to play a number of roles. On one hand, it acts as a group of reflections of a space naturally associated to \( G \), and as such is a Weyl group in the sense of [3]. On another, there is the Bruhat decomposition of \( G \) with respect to a Borel subgroup, and the terms in the decomposition are parametrized by the elements of the Weyl group.

The theory of linear algebraic monoids was developed independently, and then subsequently collaboratively, by Mohan Putcha and Lex Renner, in the 1980’s. Among its chief achievements is the classification [26, 27] of the reductive monoids, and the formulation of a Bruhat decomposition [25], analogous to that for groups.

In this section we show that one can also extract, in a natural fashion, two finite monoids from a linear algebraic monoid \( M \). The first, which is new, is a reflection monoid in the same space that the Weyl group is a reflection group. The other, which is well known and coined the Renner monoid by Solomon [31], plays the same role as the Weyl group in the Bruhat decomposition of \( M \). In general it does not seem to be possible to find a single monoid to play all the roles that the Weyl group plays. Nevertheless our main result, Theorem 7 below, shows that this reflection monoid and the Renner monoid are very closely related.

The prerequisites for this section are more demanding than for earlier ones, and the reader who is unfamiliar with the theory of algebraic groups may find it helpful at first to think in terms of Example 1. In any case, standard references on algebraic groups are [2, 12, 32], and...
on algebraic monoids, the books of Putcha and Renner [22][24]. We particularly recommend the excellent survey of Solomon [30].

Throughout, $k$ is an algebraically closed field. An affine (or linear) algebraic monoid $M$ over $k$ is an affine algebraic variety together with a morphism $\varphi : M \times M \to M$ of varieties, such that the product $xy := \varphi(x, y)$ gives $M$ the structure of a monoid (ie: $\varphi$ is an associative morphism of varieties and there is a two-sided unit $1 \in M$ for $\varphi$). We will assume that the monoid $M$ is irreducible, that is, the underlying variety is irreducible, in which case the group $G = M$ (Zariski closure). Adjectives normally applied to $G$ are then transferred to $M$; thus we have reductive monoids, simply connected monoids, soluble monoids, and so on. We have, in analogy to the group case, that any affine algebraic monoid can be embedded as a closed submonoid in $M_n(k)$ for some $n$.

From now on, let $M$ be reductive. The key players, just as they are for algebraic groups, are the maximal tori $T \subset G$ and their closures $\overline{T} \subset M$. Let $\mathcal{X}(T)$ be the character group of all morphisms of algebraic groups $\chi : T \to G_m$ (with $G_m$ the multiplicative group of $k$) and $\mathcal{X}(\overline{T})$ similarly the commutative monoid of morphisms of $\overline{T}$ into $k$ as a multiplicative monoid. Then $\mathcal{X}(T)$ is a free $\mathbb{Z}$-module, and restriction (together with the denseness of $\mathcal{X}(T)$) embeds $\mathcal{X}(\overline{T}) \hookrightarrow \mathcal{X}(T)$.

The Weyl group $W_G = N_G(T)/T$ of automorphisms of $T$ acts faithfully on $\mathcal{X}(T)$ via $\chi^{g}(t) = \chi(g^{-1}tg)$, thus realizing an injection $W_G \hookrightarrow GL(V)$ for $V = \mathcal{X}(T) \otimes \mathbb{R}$. We will abuse notation and write $W_G$ for both the Weyl group and its image in $GL(V)$. The non-zero weights $\Phi := \Phi(G, T)$ of the adjoint representation $G \to GL(g)$ form a root system in $V$ with the Weyl group $W_G$ the reflection group $W(\Phi)$ associated to $\Phi$ (here, $g$ is the associated Lie algebra).

We now need a digression to review some basic facts about convex polyhedral cones, for which we follow [9, §1.2]. If $V$ is a real space and $v_1, \ldots, v_s$ a finite set of vectors, then the convex polyhedral cone with generators $\{v_i\}$ is the set $\sigma = \sum \lambda_i v_i$ with $\lambda_i \geq 0$. The dual cone $\sigma^\vee \subset V^*$ consists of those $u \in V^*$ taking non-negative values on $\sigma$. A face $\tau \subset \sigma$ is the intersection with $\sigma$ of the kernel $\mathfrak{k}^\perp$ of a $u \in \sigma^\vee$, and the faces form a meet semilattice $\mathcal{F}(\sigma)$ under inclusion. If $\tau \in \mathcal{F}(\sigma)$ and $\overline{\tau}$ is the $\mathbb{R}$-span in $V$ of $\tau$, then $\sigma \cap \tau = \tau$. In particular, if $\bigcap \tau_i = \bigcap \tau_j$ in $V$ then $\bigcap \tau_i = \bigcap \tau_j$ in $\mathcal{F}(\sigma)$.

If $\{\tau_j\} \subset \mathcal{F}(\sigma)$ are faces of $\sigma$ and $\tau = \bigcap \tau_j$, then we have $\overline{\tau} \subset \bigcap \overline{\tau}_j$. In general this inclusion is not an equality, a simple observation with non-trivial consequences. The reader interested in the source of the failure of the homomorphism of Theorem 4 to be an isomorphism can trace it back to here. The cone in Figure 1 has faces $\tau_1, \tau_2$ with $\overline{\tau}$ the zero subspace but $\bigcap \overline{\tau}_j$ 1-dimensional.

A cone is simplicial if it has a set $A = \{v_i\}$ of linearly independent generators. If $\tau_i$ is the cone on $\{v_1, \ldots, v_i, \ldots, v_s\}$, then $\tau_i = \sigma \cap u_i^\perp$, where $u_i$ is the vector corresponding to $v_i$ in the dual basis for $V^*$. Thus $\tau_i$ is a face of $\sigma$, and the face lattice $\mathcal{F}(\sigma)$ is isomorphic to the lattice of all subsets of $A$—or, if one prefers, to the Boolean lattice on the 1-dimensional faces $\mathbb{R}^+ v_i$ of $\sigma$. If $\tau \in \mathcal{F}(\sigma)$ corresponds to $A_\tau \subset A$ then $\tau_j \cap \tau_2$ corresponds to $A_{\tau_1} \cap A_{\tau_2}$, and $\overline{\tau}$ is the $\mathbb{R}$-span of $A_\tau$. In particular, the $\mathbb{R}$-span of $\bigcap A_\tau$ is the intersection of the $\mathbb{R}$-spans of the $A_{\tau_i}$.
and so if $\tau = \bigcap \tau_j$, we have $\mathcal{T} = \bigcap \mathcal{T}_j$ when $\sigma$ is simplicial. Finally, a cone is strongly convex if the dual $\sigma^\vee$ spans $V^*$. Simplicial cones are strongly convex. On the other hand, if $\dim V = 2$, then any strongly convex cone is simplicial [2] 1.2.13.

Returning to our algebraic monoid we may assume, by conjugating suitably, that the maximal torus $T$ is a subgroup of the group $\mathbb{T}_n$ of invertible diagonal matrices, where $n$ is the rank of $G$.

**Definition 3.** Let $\mathcal{M}$ be a reductive algebraic monoid. If $\chi_j$ is the restriction to $T$ of the $j$-th coordinate function on $\mathbb{T}_n$, then let $\sigma \subset \mathfrak{X}(T) \otimes \mathbb{R}$ be the cone given by $\sigma = \sum \mathbb{R}^+ \chi_i$.

This cone will turn out to have a number of nice properties, the first of which being that the character monoid $\mathfrak{X}(\mathcal{T}) = \sigma \cap \mathfrak{X}(T)$. Secondly, the Weyl group $W_G$, in its reflectional action on $V$, acts on $\sigma$, and this induces an action $\tau \mapsto \tau g$ of $W_G$ on the face lattice $\mathcal{F}(\sigma)$.

Finally, and most importantly, the face lattice $\mathcal{F}(\sigma)$ models idempotents: there is a lattice isomorphism $\mathcal{F}(\sigma) \to E(\mathcal{T})$, written $\tau \mapsto e_\tau$, that is $W_G$-equivariant with respect to the Weyl group actions on $\mathcal{F}(\sigma)$ and $E(\mathcal{T})$. In short, $e_\tau^g = e_{\tau g}$ for any $\tau \in \mathcal{F}(\sigma)$ and $g \in W_G$. Solomon [30 Corollary 5.5], working instead with the dual cone $\sigma^\vee$ in the group of 1-parameter subgroups of $T$, has a lattice anti-isomorphism $\mathcal{F}(\sigma^\vee) \to E(\mathcal{T})$.

**Example 1.** Let $\mathcal{M} = M_n(k)$, the monoid of $n \times n$ matrices over $k$, which can be naturally identified with $n^2$-dimensional affine space over $k$. The units are the general linear group $G = GL_n(k)$, a reductive group. The tori in $G$ are the conjugates of subgroups of diagonal matrices, and an example of a maximal torus $T$ is the subgroup of all invertible diagonal matrices,

$$\text{diag}(\lambda_1, \ldots, \lambda_n) := \begin{bmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix} \text{ with } \lambda_1 \ldots \lambda_n \neq 0.$$ 

Any other maximal torus is a conjugate of this one. The Zariski closure $\overline{T}$ consists of all the diagonal matrices with no restriction on the $\lambda_i$. The map $\chi_i : T \to \mathbb{G}_m$ sending $\text{diag}(\lambda_1, \ldots, \lambda_n)$ to $\lambda_i$ is a character of $T$, with an arbitrary character $\chi \in \mathfrak{X}(T)$ having the form $\chi = \chi_1^{t_1} \ldots \chi_n^{t_n}$ for some $t_i \in \mathbb{Z}$. Thus $\mathfrak{X}(T)$ is a free $\mathbb{Z}$-module with basis the $\chi_i$ and $V = \mathfrak{X}(T) \otimes \mathbb{R}$ is an $n$-dimensional space.

The normalizer $N_G(T)$ consists of the monomial matrices: those having precisely one non-zero entry in each row and column. The Weyl group $W_G = N_G(T)/T$ thus consists of the permutation matrices $A(\pi) = \sum E_{i_1 \pi i_n}$, where $E_{ij}$ is the matrix with $(i, j)$-th entry 1 and all other entries 0, and $\pi \in \mathfrak{S}_n$. The $W_G$-action $\chi^g(t) = \chi(g^{-1}tg)$ on $V$ becomes the $\chi_i^{A(\pi)} = \chi_{i\pi}$ of [2] with the weights $\Phi = \{\chi_i \chi_j^{-1} \mid 1 \leq i \neq j \leq n\}$ the root system $A_{n-1}$ of [3].

The idempotents $E(\mathcal{T})$ are those $\text{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \{0, 1\}$ for all $i$. Indeed, the idempotents are parametrized by the subsets of $\{1, \ldots, n\}$ according to the positions on the diagonal of the 1’s. Moreover, the order $e \leq f \iff ef = e$ corresponds precisely to the subset for $e$ being contained in the subset for $f$. Thus $E(\mathcal{T})$ is isomorphic as a lattice to the Boolean lattice of all subsets of $\{1, \ldots, n\}$ with unique minimal element the zero matrix and unique maximal element the identity matrix.

The cone $\sigma = \sum \mathbb{R}^+ \chi_i$ is the positive quadrant in $V$, hence simplicial, and with the integral points in $\sigma$ the characters in the monoid $\mathfrak{X}(\mathcal{T})$. The $W_G$-action on $\sigma$ permutes the vertices of the $(n - 1)$-simplex consisting of the points $\sum \mu_i \chi_i$ with $\sum \mu_i = 1$, and the face lattice $\mathcal{F}(\sigma)$ can be identified with the face lattice of this simplex. Figure 2 illustrates it all when $n = 3$ with the 2-simplex shaded and $\text{diag}(\lambda_1, \ldots, \lambda_n)$ represented by $\lambda_1 \ldots \lambda_n$.

Finally, an abstract simplex with vertices $X = \{\chi_1, \ldots, \chi_n\}$ has faces the subsets of $X$, with inclusion of faces corresponding to inclusion of subsets. Thus the face lattice $\mathcal{F}(\sigma)$ is the Boolean lattice of subsets of $X$ and the isomorphism $\mathcal{F}(\sigma) \to E(\mathcal{T})$ sends $\{\chi_{i_1}, \ldots, \chi_{i_k}\}$ to the diagonal matrix with 1’s in positions $\{i_1, \ldots, i_k\}$ and 0’s elsewhere.

Returning to generalities, the point of the cone $\sigma$ is that it gives a system of subspaces in $V$ for the Weyl group:
Definition 4. Let $\mathbb{M}$ be a reductive algebraic monoid with $\sigma$ the cone of Definition 3 and $\mathcal{F}(\sigma)$ its face lattice. Define the system $\mathcal{S}_\mathbb{M}$ for $W_G$ in $V = \mathcal{X}(T) \otimes \mathbb{R}$ to be the set of all intersections $\bigcap \mathcal{T}_j$, for $\tau_j \in \mathcal{F}(\sigma)$, and with the empty intersection taken to be $V$.

The terminology presupposes that $\mathcal{S}_\mathbb{M}$ is indeed a system for $W_G$, but this is immediate as $W_G$ acts on the face lattice $\mathcal{F}(\sigma)$. One can also describe $\mathcal{S}_\mathbb{M}$ as the system for $W_G$ generated by the subspaces $\mathcal{T}$.

Revisiting Example 1 the system $\mathcal{S}_\mathbb{M}$ is just the Boolean system of Example 2 consisting of all $X(J) = \sum_{j \in J} \mathbb{R}X_j$ for $J \subset \{1, \ldots, n\}$.

Definition 5. Let $\mathbb{M}$ be a reductive algebraic monoid with group of units $G$. The reflection monoid associated to $\mathbb{M}$ is the monoid $M(W_G, \mathcal{S}_\mathbb{M})$ where $W_G$ is the Weyl group of $G$ and $\mathcal{S}_\mathbb{M}$ is the system for $W_G$ given in Definition 4.

Wrapping up Example 1, the Weyl group $W_G$ acts on $V$ as the symmetric group $\mathfrak{S}_n$ permuting coordinates, and the system $\mathcal{S}_\mathbb{M}$ is the Boolean one: the reflection monoid associated to $\mathbb{M}$ is thus our first example $M(\mathfrak{S}_n, B)$ from Example 2—the symmetric inverse monoid $\mathcal{I}_n$ or the Boolean reflection monoid $M(A_{n-1}, B)$.

We now introduce a second intermediate monoid associated with $\mathbb{M}$ that is constructed via the cone $\sigma$ and uses the idea of a system of subsets for $W_G$; this idea is defined in general in Example 8.1. For now, we observe that $\sigma \in \mathcal{F}(\sigma)$, the lattice $\mathcal{F}(\sigma)$ is closed under intersection and $W_G$ acts on $\mathcal{F}(\sigma)$. These facts allow us to form an inverse monoid $M(W_G, \mathcal{F}(\sigma))$ in the same way that $M(G, \mathcal{S})$ is constructed when $G$ is a system of subspaces for a group $G$. The elements of $M(W_G, \mathcal{F}(\sigma))$ have the form $w_\tau$ ($w \in W_G, \tau \in \mathcal{F}(\sigma)$), where $w_\tau$ is the restriction of $w$ to $\tau$. Multiplication is given by $w_\tau w'_\tau = (ww')_\mu$ where $\mu = \tau \cap \tau'w^{-1}$.

The minimum face of $\sigma$ is $\sigma \cap -\sigma$, which is the largest subspace of $V$ contained in $\sigma$. Write $Z$ for this subspace and let $\pi : V \to V/Z$ be the canonical homomorphism. Then $\pi(\sigma)$ is a strongly convex polyhedral cone in $V/Z$, and the face lattices $\mathcal{F}(\sigma)$ and $\mathcal{F}(\pi(\sigma))$ are isomorphic.

Not suprisingly, there is a close connection between the reflection monoid $M(W_G, \mathcal{S}_\mathbb{M})$ and $M(W_G, \mathcal{F}(\sigma))$. First, note that idempotents in $M(W_G, \mathcal{S})$ are products $e_X = \prod \mathcal{E}_j$ where $X = \bigcap \mathcal{T}_j \in \mathcal{S}_\mathbb{M}$ and $\mathcal{E}_j$ is the partial identity on $\mathcal{T}_j$. Factorizability (Proposition 10) thus gives that any element of the reflection monoid has the form $e_X w = (\prod \mathcal{E}_j)w$ for $w \in W_G$.

Define $\theta : M(W_G, \mathcal{S}_\mathbb{M}) \to M(W_G, \mathcal{F}(\sigma))$ by

$$(\prod \mathcal{E}_j)w = (\prod \mathcal{E}_j)w,$$

where $e_j$ is the partial identity $e_{\mathcal{T}_j}$ on $\mathcal{T}_j$.

Proposition 12. The map $\theta : M(W_G, \mathcal{S}_\mathbb{M}) \to M(W_G, \mathcal{F}(\sigma))$ is a surjective homomorphism, and is an isomorphism if and only if the cone $\pi(\sigma)$ is simplicial.

Proof. The identity map of $W_G$ is an isomorphism between the groups of units of the two monoids. Let $\varphi$ be the map between the respective idempotents given by $\prod \mathcal{E}_j \varphi = \prod \mathcal{E}_j$. 

![Fig. 2. The simplicial cone $\sigma \subset \mathcal{X}(T) \otimes \mathbb{R}$ and the isomorphism $\mathcal{F}(\sigma) \to E(T)$ for $\mathcal{M} = M_n$k.](image-url)
As \( \bigcap \tau_i = \bigcap \tau_j \) implies \( \forall \tau_i = \bigcap \mu_j \) in \( F(\sigma) \), the map \( \varphi \) is well defined. It is clearly a homomorphism and surjective. Next, note that \( \tau_j w = \tau_j w (w \in W_G) \), and so \( w \varepsilon_X w^{-1} = \prod \varepsilon_{\tau_j}, \ (\tau_j = \tau_j w) \), giving \( (w \varepsilon_X w^{-1}) \varphi = w^{-1} \left( \prod \varepsilon_{\tau_j} \right) w = w^{-1} \left( \prod \varepsilon_j \right) \varphi w. \)

Finally, \( (\prod \varepsilon_j) w = \prod \varepsilon_j \) if and only if \( w \) leaves \( \bigcap \tau_j \) fixed pointwise, and this implies that \( w \) leaves \( \bigcap \tau_j \) fixed pointwise, so \( (\prod \varepsilon_j) w = \prod \varepsilon_j \).

We thus have all the ingredients needed to apply Proposition 1 and get \( \theta \) a surjective homomorphism.

Now suppose that \( \pi(\sigma) \) is not simplicial. Then \( \dim \pi(\sigma) > 2 \) so \( \dim \sigma > 2 + \dim Z \).

There are maximal faces in \( F(\pi(\sigma)) \) which intersect in \{\( Z \)\} and corresponding to maximal faces \( \tau_1, \tau_2 \) in \( F(\sigma) \) with \( \tau_1 \cap \tau_2 = Z \). As the spaces \( \tau_i (i = 1, 2) \) are hyperplanes in \( \tau \), the intersection \( \tau_1 \cap \tau_2 \) has codimension two in \( \tau \) and hence \( Z \) is strictly contained in \( \tau_1 \cap \tau_2 \). This translates into \( \varepsilon_1 \varepsilon_2 0 \neq 0 \) in \( M(W_G, S_M) \) but \( \varepsilon_1 \varepsilon_2 = 0 \) in \( M(W_G, F(\sigma)) \), where, as usual, \( \varepsilon_i \) is the partial identity on \( \tau_i \) and \( \varepsilon_i \) is the partial identity on \( \tau_i \). Thus, \( \theta \) fails to be injective even on the idempotents of \( M(W_G, S_M) \), and so on \( M(W_G, S_M) \) itself.

On the other hand suppose that \( \pi(\sigma) \) is simplicial. Let \( X, Y \in S_M \) with \( \varepsilon_X \varphi = \varepsilon_Y \varphi \). If \( X = \bigcap \tau_j \) and \( Y = \bigcap \tau_j \), then \( \varepsilon_X = \prod \varepsilon_j \) and \( \varepsilon_Y = \prod' \varepsilon_j \) the partial identity on \( \tau_i \). Thus \( \prod \varepsilon_j = \prod' \varepsilon_j \) where \( \varepsilon_i \) is the partial identity on \( \mu_i \). This is equivalent to \( \bigcap \tau_j = \bigcap \mu_i \) and so in \( \pi(\sigma) \) we have \( \bigcap \pi(\tau_j) = \bigcap \pi(\mu_i) \). Since \( \pi(\sigma) \) is simplicial, this gives that \( \bigcap \pi(\tau_j) = \bigcap \pi(\tau_j) = \bigcap \pi(\mu_i) \).

Now \( Z \subset \tau_j \) for all \( j \), so \( \bigcap \pi(\tau_j) = \pi(\tau_j) \) and similarly \( \bigcap \pi(\mu_i) = \pi(\mu_i) \), whence \( \bigcap \tau_j = \bigcap \mu_i \), and so \( \varepsilon_X = \varepsilon_Y \). Thus \( \varphi \) is an isomorphism.

To apply Proposition 1 we also need to show that \( (\prod \varepsilon_j) w = \prod \varepsilon_j \) implies \( (\prod \varepsilon_j) w = \prod \varepsilon_j \). The assumption is equivalent to \( w \) leaving \( \bigcap \tau_j \) fixed pointwise, and hence \( w \) leaves the \( \mathbb{R} \)-span of \( \bigcap \tau_j \) fixed pointwise. But this subspace is equal to \( \bigcap \tau_j \) as \( \sigma \) is simplicial, and so \( (\prod \varepsilon_j) w = \prod \varepsilon_j \) follows. Thus \( \theta \) is an isomorphism by Proposition 1.

Now to monoid number three, where we can be briefer. The Renner monoid \( R_M \) of \( M \) is defined to be \( R_M = \overline{NG(T)}/T. \) See [23] or [22, Chapter 11].

Just as \( L_n \) is the archetypal inverse monoid, and as \( M(A_{n-1}, B) \) with \( B \) the Boolean system it is the archetypal reflection monoid, so its incarnation as the rook monoid it is the standard example of a Renner monoid, namely for \( M = M_n(k) \) in Example 1 above. The elements of the rook monoid are the \( n \times n \) matrices having 0, 1 entries with at most one non-zero entry in each row and column. The etymology of “rook monoid” is that each element represents an \( n \times n \) chessboard with the 0 squares empty, the 1 squares containing rooks and the rooks mutually non-attacking. The Renner monoids have also been explicitly described in some other cases, for example when \( M \) is the “symplectic monoid” \( MSp_n(k) = \overline{Sp_n(k)} \subset M_n(k) [40]. \)

In general, we have \( E(R_M) = E(T) \) and \( R_M \) is a (factorizable) inverse monoid. Consider, for \( E = E(R_M) \) the fundamental representation \( \alpha : R_M \to L_E \), written \( axa = \alpha_a \), where \( \alpha_a : Eaa^{-1} \to Ea^{-1}a \) is defined by \( x\alpha_a = a^{-1}xa \) for all \( x \in Eaa^{-1} \). We shall describe the fundamental representation in a more general context, in particular being more precise about the location of its image, in [8.2] For now, we record that \( \alpha \) is a homomorphism 11 Theorem 5.4.4].

In general, \( \alpha \) is not an isomorphism (see [24, Proposition 8.3]), but if the reductive monoid \( M \) has a 0, then it follows from [22, Proposition 11.1] and [11, Theorem 5.4.4] that \( \alpha \) is an isomorphism. Further, by [22, Theorem 6.20], the length of a maximal chain in \( E(T) \) is equal to \( \dim T \), where \( \dim T = m \) when \( T = T_m \). From [30], \( \dim T = \text{rank} \mathcal{X}(T) = \dim V \) for \( V = \mathcal{X}(T) \otimes \mathbb{R} \). Recall that there is an isomorphism \( F(\sigma) \to E(T) \) which sends \( \tau \) to an
idempotent $e_\tau$. We conclude that $\mathcal{F}(\sigma)$ has a chain of length $\dim V$, and since $\mathfrak{e} = V$, it follows that the least member of such a chain must be the trivial subspace $0$. Hence $\sigma$ is strongly convex.

Restricting $\alpha$ to $W_G$ gives an isomorphism $\alpha : W_G \to W_G \alpha$. Let $E = E(\mathfrak{e})$ and $E'$ be the partial identities $\{\varepsilon_\tau : \tau \in \mathcal{F}(\sigma)\}$. Then we have an isomorphism $\beta : E' \to E\alpha$ given by

$$\varepsilon_\tau \beta = e_\tau \alpha,$$

where in turn $e_\tau \alpha = e_\tau \alpha$ is the identity on $E e_\tau$. Now define $\varphi : M(W_G, \mathcal{F}(\sigma)) \to R_M$ by $(e_\tau \varphi)(\alpha) = (e_\tau \beta)(\alpha \alpha)$. 

**Proposition 13.** Let $M$ be a reductive monoid with $0$. Then the map $\varphi : M(W_G, \mathcal{F}(\sigma)) \to R_M$ is an isomorphism.

**Proof.** We show first that $\beta$ is equivariant. For $w \in W_G$, $\tau \in \mathcal{F}(\sigma)$:

$$\left( w^{-1} \varepsilon_\tau w \right) \beta = (w^{-1} w_\tau) \beta = (w^{-1} w_\tau)(\varepsilon_\tau \beta) = e_\tau \alpha \beta = (w^{-1} \varepsilon_\tau w) \alpha = (w \alpha)^{-1} (e_\tau \alpha)(w \alpha) = (w \alpha)^{-1} (\varepsilon_\tau \beta)(w \alpha),$$

as required. If $\varepsilon_\tau w = \varepsilon_\tau$, then $w$ leaves $\tau$ fixed pointwise, so that $\kappa \wedge w = \kappa$ for all $\kappa \leq \tau$. Now $\varepsilon_\tau \beta)(\alpha \alpha) = \id_{E e_\tau \alpha} \alpha \alpha = \alpha \alpha | E e_\tau$. Also, $e \in E e_\tau$ if and only if $e = e_\kappa$ for some $\kappa \leq \tau$. Thus, for all $\kappa \leq \tau$, $e_\kappa \alpha \alpha = w^{-1} e_\kappa w = e_\kappa = e_\kappa$, so $\alpha \alpha | E e_\tau$ is the identity on $E e_\tau$ as required. Thus Proposition 1 gives $\varphi$ is a surjective homomorphism. To see that $\varphi$ is an isomorphism, all that remains is to show that $\alpha \alpha | E e_\tau = \id_{E e_\tau}$ implies $\varepsilon_\tau w = \varepsilon_\tau$, that is, $w$ leaves $\tau$ fixed pointwise. Since $E\alpha$ is strongly convex, $\tau$ contains one dimensional faces. Let $\kappa$ be one such. Then $\kappa$ is a ray with $e_\kappa \leq e_\tau$, so $e_\kappa \in E e_\tau$ and hence $e_\kappa \alpha \alpha = e_\kappa$, that is, $e_\kappa \wedge w = e_\kappa$. Thus $\kappa \wedge w = \kappa$. As $w$ acts on $V$ as a reflection, and so is orthogonal, it leaves $\kappa \wedge w$ fixed pointwise. Since this is so for all one dimensional faces contained in $\tau$, it follows that $w$ leaves $\tau$ fixed pointwise and $\varphi$ is an isomorphism.

The main result of the section now follows from the preceding two Propositions:

**Theorem 7.** Let $M$ be a reductive algebraic monoid with $0$.

- Let $G$ be its group of units with $T \subset G$ a maximal torus, $X(T)$ the character group and $W_G$ the Weyl group;
- Let $\sigma \subset X(T) \otimes \mathbb{R}$ be the polyhedral cone of Definition 3, $\mathcal{F}(\sigma)$ its face lattice, and $S_M$ the system for $W_G$ of Definition 4;
- Let $M(W_G, S_M)$ be the reflection monoid associated to $M$ and $M(W_G, \mathcal{F}(\sigma))$ the monoid given by the system of subsets $\mathcal{F}(\sigma)$;
- Finally, let $R_M$ be the Renner monoid of $M$.

Then $R_M \cong M(W_G, \mathcal{F}(\sigma))$ and there is a surjective homomorphism $M(W_G, S_M) \to R_M$ which is an isomorphism if and only if $\sigma$ is a simplicial cone.

**Example 2.** As an illustration of the lack of injectivity of $f$, let $M$ be the (normalization of) the closure of $\text{Ad}(G)k^\alpha$ for $G$ the adjoint simple group of type $B_2$. Then $\text{dim}(X(T) \otimes \mathbb{R}) = 3$ with $\mathfrak{e}$ a cone on a square (see Figure 6) or Figure 1). If $\tau_i$, $(i = 1, 2)$ are the cones on opposite, non-intersecting faces of the square, then $\tau_1 \cap \tau_2 = \{0\}$, whereas $\tau_1 \cap \tau_2$ is a 1-dimensional subspace.

Figure 3 gives the lattice of idempotents of the reflection monoid associated to $M$ (left) with a pair $\varepsilon_1 \varepsilon_2 \not= 0$ marked, mapping via $f$ to $e_1 \wedge e_2 = 0$ (right).

**Example 3.** Not only does the homomorphism $f$ fail to be injective in Example 2 but we can also show quite easily that $R_M$ cannot be isomorphic to a reflection monoid. For, suppose that $R_M \cong M(W, S)$ where $S$ is a system of subspaces of a Euclidean space $V$ on which $W$ acts as a reflection group. Since $W$ must be isomorphic to the group of units of $R_M$, we have $W = W(B_2)$. Hence four of the elements of order 2 in $W$ must be reflections. Also, the lattice $S$ must...
be isomorphic to the lattice shown on the right in Figure 3. Moreover, if the unique minimal element of $S$ is a non-zero subspace, we can factor it out to obtain a lattice of subspaces with minimal element $\{0\}$.

Reading from left to right, let the elements of $S$ indicated in Figure 3 be $U_0, U_1, U_2, U_3$, and $X_0, X_1, X_2, X_3$ respectively. The intersection of any two $U_i$’s is zero, as is the intersection of $X_0$ and $X_2$. Hence for any choice of non-zero vectors $u_i \in U_i$ ($i = 0, 1, 2, 3$), the set $\{u_0, \ldots, u_3\}$ is linearly independent.

The group of units of $R_M$ is the automorphism group of $E(R_M)$ where the action is by conjugation. Hence $W$ acting by conjugation on $\{\varepsilon_Y \mid Y \in S\}$ gives all automorphisms of $E(M(W,S))$ and since $\varepsilon_Y g = g^{-1} \varepsilon_Y g$ for all $Y \in S$ and $g \in W$, the $W$-action on $S$ gives all the automorphisms of $S$.

Now, automorphisms of $S$ are determined by their effect on the $U_i$. Let $g, g' \in W$ be such that their actions give rise to the automorphisms determined by interchanging $U_0$ with $U_3$ and $U_1$ with $U_2$, and interchanging $U_0$ with $U_1$ and $U_2$ with $U_3$ respectively. Choose $u_i \in U_i$ for $i = 0, 1$; then $u_0 g \in U_3$ and $u_1 g \in U_2$, so that $\{u_0, u_1, u_0 g, u_1 g\}$ is a basis for the subspace it spans, say $U$. It is readily verified that $-1$ is an eigenvalue of $g|_U$ of multiplicity 2, so that $-1$ cannot be a simple eigenvalue of $g$ itself. Thus $g$ (which has order 2) is not a reflection. Similarly, $g' \neq g$ is not a reflection. This is a contradiction since there is only one element of order 2 in $W$ which is not a reflection.

We conclude the subsection by mentioning that several authors have calculated the orders of certain Renner monoids. The most general results (which include all earlier ones) are in [39]. We will analyse in more detail the connection between reflection monoids and linear algebraic monoids in a future paper.

8. Complements

In this final section we elaborate on a number of miscellaneous issues thrown up in earlier sections, but not strictly part of the flow of the paper.

8.1. Factorizable inverse monoids

We first met factorizable inverse monoids in [6] where we characterized the factorizable inverse submonoids of $ML(V)$ as being the monoids $M(G,S)$ that form the main characters in our story. The results of that section suggest that, in a suitably “de-linearized” form, they can be used to provide a description of all factorizable inverse monoids.

We take our cue from group theory, where the “Cayley” representation embeds a group $G$ in the symmetric group $S_G$. The equivalent for an inverse monoid $M$ is the Vagner-Preston representation [11][16], which is a faithful representation $M \hookrightarrow I_M$ given by partial right multiplication. Thus any characterization of inverse monoids (up to isomorphism) can be restricted to the inverse submonoids of the symmetric inverse monoid.
Throughout this section, let $X$ be an arbitrary set. We observe that if $M$ is an inverse submonoid of $\mathcal{I}_X$, then $E = E(M) = M \cap E(\mathcal{I}_X) = \{ \varepsilon_Y | Y = \text{dom } \alpha \text{ for some } \alpha \in M \}$. Equally, $E = \{ \varepsilon_Y | Y = \text{im } \alpha \text{ for some } \alpha \in M \}$ since $\text{im } \alpha = \text{dom } \alpha^{-1}$ for all $\alpha \in M$. Putting $S = \{ \text{dom } \alpha | \alpha \in M \}$, we see that $S$ is a meet semilattice isomorphic to $E$. Moreover, $X \in S$ since $M$ is a submonoid, and finally, if $Y \in S$ and $g \in G = G(M)$, then $Y g = \text{im } (\varepsilon_Y g) \in S$. Thus $S$ provides an example of a system of subsets in $X$ for $G$: a collection $S \subset 2^X$ such that $X \in S$, $SG = S$ and $X \cap Y \in S$ for all $X, Y \in S$. If $G \subset \mathcal{S}_X$ is a group and $S$ a system in $X$ for $G$ then we form the monoid of partial permutations $M(G, S) := \{ g_Y | Y \in S \} \subseteq \mathcal{I}_X$.

Note that if $g_Y, h_Z \in M(G, S)$, then $(g_Y)^{-1} = (g^{-1})_Y g \in M(G, S)$ and $g_Y h_Z = (g h)_T$ with $T = Y \cap Z g^{-1}$, so that $M(G, S)$ is an inverse submonoid of $\mathcal{I}_X$. Clearly, $G$ is the group of units, and the idempotents are $E = \{ \varepsilon_Y | Y \in S \}$. Moreover, every element is by definition a restriction of a unit, so $M(G, S)$ is factorizable. Here is the promised characterization:

**Proposition 14.** $M$ is a factorizable inverse monoid if and only if there is a set $X$, a group $G \subset \mathcal{S}_X$, and a system $S$ in $X$ for $G$, with $M$ isomorphic to $M(G, S)$.

We have already seen that monoids of the form $M(G, S)$ are factorizable inverse submonoids of $\mathcal{I}_X$. For the converse, it suffices, by the Vagner-Preston representation, to assume $M \subset \mathcal{I}_X$ for some $X$. Let $G$ be its group of units, $S = \{ \text{dom } \sigma | \sigma \in M \}$ the system above, and form $M(G, S)$. Now proceed as in the proof of Proposition 10.

We also have:

**Theorem 8.** Let $G \subset \mathcal{S}_X$ be finite and $S$ a finite system in $X$ or $G$. Then $|M(G, S)| = \sum_{Y \in S} |G : G_Y|$ with $G_Y = \{ g \in G | yg = y \text{ for all } y \in Y \}$.

The proof is identical to Theorem 1.

### 8.2. Fundamental inverse monoids

We extend the themes of the previous section to describe another abstract class of inverse monoids of interest: the fundamental inverse monoids. On any inverse monoid $M$, define the relation $\mu$ by the rule:

$$a \mu b \text{ if and only if } a^{-1} e a = b^{-1} e b \text{ for all } e \in E.$$  

It is easy to see that $\mu$ is a congruence on $M$; it is idempotent-separating in the sense that distinct idempotents in $M$ are not related by $\mu$, and, in fact, it is the greatest idempotent-separating congruence on $M$. We say that $M$ is **fundamental** if $\mu$ is the equality relation; in general, $M/\mu$ is fundamental.

The Munn semigroup $\mathcal{S}[\mathcal{I}_E]$ of a semilattice $E$ is defined to be the set of all isomorphisms $E e \rightarrow E f$ where $e, f \in E$ with $E e \cong E f$. We have $\mathcal{I}_E$ an inverse submonoid of $\mathcal{I}_E$ whose semilattice of idempotents is isomorphic to $E$ (see [11] Theorem 5.4.4 or [16] Theorem 5.2.7).

Given any inverse monoid $M$ and $a \in M$, define an element $\alpha_a \in \mathcal{T}_{E(M)}$ as follows. The domain of $\alpha_a$ is $E a a^{-1}$ and $x \alpha_a = a^{-1} x a$ for $x \in E a a^{-1}$. Note that $\text{im } \alpha_a = E a a^{-1} a$. The main results (see [11] Theorems 5.4.4 and 5.4.5 or [16] Theorems 5.2.8 and 5.2.9) are that the mapping $\alpha : M \rightarrow \mathcal{S}[\mathcal{I}_E(M)]$ given by $\alpha a = \alpha_a$ is a homomorphism onto a full inverse submonoid of $\mathcal{T}_{E(M)}$ such that $a a = b a$ if and only if $a \mu b$. Moreover, an inverse monoid $M$ is fundamental if and only if $M$ is isomorphic to a full inverse submonoid of $\mathcal{T}_{E(M)}$.

The homomorphism $\alpha : M \rightarrow \mathcal{T}_{E(M)}$ of is called the **fundamental** or **Munn representation** of $M$. Note that $M$ is fundamental if and only if $\alpha$ is one-one.

It is well known that the symmetric inverse monoid $\mathcal{I}_X$ is fundamental for any set $X$—see, for example, [11] Chapter 5, Exercise 22. In contrast, for any nonempty set $X$, it is easy to see that...
the monoid of partial signed permutations $\mathcal{J}_X$ is not fundamental: a simple calculation shows that the identity of $\mathcal{J}_X$ and the transposition $(x, -x)$ are $\mu$-related. In Proposition 5 we saw that $\mathcal{J}_n$ is a reflection monoid, so there certainly are non-fundamental reflection monoids.

We now describe fundamental factorizable inverse monoids in terms of semilattices and their automorphism groups. We remark that the principal ideals of a semilattice $E$ regarded as a monoid are precisely the principal order ideals of $E$ regarded as a partially ordered set. It will be convenient to write $\varepsilon_x$ for the partial identity with domain $Ex$.

**Proposition 15.** (i) If $E$ is a semilattice with unique maximal element and $G$ is a subgroup of the automorphism group $\text{Aut}(E)$, then the collection $\mathcal{S} = \{Ex \mid x \in E\}$ of all principal ideals of $E$ forms a system in $E$ for $G$, and the resulting $M(G, S) \cong \langle G, E \rangle \subset T_E$.

(ii) If $M$ is a fundamental factorizable inverse monoid with group of units $G$ and idempotents $E = E(\hat{T})$, then $R_M \cong \langle W_G, E \rangle \subset T_E \subset I_E$.

**Proof.** Given $E$ and $G$ we observe that $\mathcal{S}$ does form a system in $E$ for $G$ since $E = E\hat{1}$, for the maximal element, $Ex \cap Ey = Exy$ and the image under $g \in G$ of $Ex$ is $E(xg)$. We can thus define the factorizable inverse monoid $M(G, S) \subset I_E$ as above. As $G$ is a subgroup of $\text{Aut}(E)$, it is a subgroup of the group of units of $T_E$, and hence if $\varepsilon_x \in M(G, S)$ with $g \in G$, then $\varepsilon_x \varepsilon g \in T_E$. Thus $M(G, S) \subset T_E$; in fact, it is clearly a full inverse submonoid of $T_E$ and so it is fundamental. Identifying $E(T_E)$ with $E$, it is also clear that $M(G, S)$ is generated as a submonoid by $G$ and $E$.

For part (ii), $M$ is isomorphic to a full submonoid of $T_E$ by the injectivity of the Munn representation, and we identify this submonoid with $M$. The group $G$ is a subgroup of the group of units of $T_E$, that is, of $\text{Aut}(E)$. As above $\mathcal{S} = \{\text{dom } \alpha \mid \alpha \in M\}$ is a system in $E$ for $G$, and since $M$ is factorizable we have $M = M(G, S)$. Thus $M$ is generated by $G$ and $E$ (identifying $E$ with $E(T_E)$). \qed

We finish by returning to reflection monoids and giving an example of a non-fundamental reflection monoid in which the restriction of the Munn representation to the group of units is one-one. (We have seen that $\mathcal{J}_X$ is not fundamental, but in this case there are distinct units which are $\mu$-related.) First, note that if $M = M(W, S)$ is a reflection monoid, and $\alpha \in M$ has domain $X$, then for any $Y \in S$ we have

$$\alpha^{-1}\varepsilon_Y \alpha = \varepsilon_{(Y \cap X)\alpha}. \tag{15}$$

Now let $V = \mathbb{R}^2$ and $W$ the reflection group of either of the two triangles shown (so $W \cong \mathbb{S}_3$). The $\mathbb{R}$-spans of the vectors shown, together with $V$ and 0, form a system (of subspaces) for $W$. Let $\rho \in W$ be the rotation through $2\pi/3$ and $\tau \in W$ the reflection in the $y$-axis. The $\mu$-class of the identity $\varepsilon_V$ is a normal subgroup of $W$ and so to show that $\mu$ is trivial on $W$, it is enough to show that $\rho$ and $\varepsilon_V$ are not $\mu$-related. This is clear from (15) using any of the six lines for $Y$. On the other hand, letting $X$ be the $x$-axis, we see that $\tau_X$ and $\varepsilon_X$ are distinct but $\mu$-related.

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