Nordhaus-Gaddum-type theorem for total proper connection number of graphs

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Abstract

A graph is said to be total-colored if all the edges and the vertices of the graph are colored. A path $P$ in a total-colored graph $G$ is called a total-proper path if (i) any two adjacent edges of $P$ are assigned distinct colors; (ii) any two adjacent internal vertices of $P$ are assigned distinct colors; (iii) any internal vertex of $P$ is assigned a distinct color from its incident edges of $P$. The total-colored graph $G$ is total-proper connected if any two distinct vertices of $G$ are connected by a total-proper path. The total-proper connection number of a connected graph $G$, denoted by $tpc(G)$, is the minimum number of colors that are required to make $G$ total-proper connected. In this paper, we first characterize the graphs $G$ on $n$ vertices with $tpc(G) = n - 1$. Based on this, we obtain a Nordhaus-Gaddum-type result for total-proper connection number. We prove that if $G$ and $\overline{G}$ are connected complementary graphs on $n$ vertices, then $6 \leq tpc(G) + tpc(\overline{G}) \leq n + 2$. Examples are given to show that the lower bound is sharp for $n \geq 4$. The upper bound is reached for $n \geq 5$ if and only if $G$ or $\overline{G}$ is the tree with maximum degree $n - 2$.

Keywords: total-proper path, total-proper connection number, complementary graph, Nordhaus-Gaddum-type.

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1 Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow the terminology and notation of Bondy and Murty in [1] for those not defined here. If $G$

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is a graph and \( A \subseteq V(G) \), then \( G[A] \) denotes the subgraph of \( G \) induced by the vertex set \( A \), and \( G - A \) the graph \( G[V(G) \setminus A] \). If \( A = \{v\} \), then we write \( G - v \) for short. An edge \( xy \) is called a pendant edge if one of its end vertices, say \( x \), has degree one, and \( x \) is called a pendant vertex. For a vertex \( v \in V(G) \), we use \( N_G(v) \) to denote the neighborhood of \( v \) in \( G \) and use \( d_G(v) \) to denote the degree of \( v \) in \( G \), sometimes we simply write \( N(v) \) and \( d(v) \) if \( G \) is clear. For graphs \( X \) and \( G \), we write \( X \cong G \) if \( X \) is isomorphic to \( G \). Throughout this paper, \( \mathbb{N} \) denotes the set of all positive integers.

Let \( G \) be a nontrivial connected graph with an edge-coloring \( c : E(G) \to \{1, 2, \ldots, t\} \), \( t \in \mathbb{N} \), where adjacent edges may be colored with the same color. If adjacent edges of \( G \) receive different colors by \( c \), then \( c \) is a proper coloring. The minimum number of colors required in a proper coloring of \( G \) is referred as the chromatic index of \( G \) and denoted by \( \chi'(G) \). Meanwhile, a path in \( G \) is called a rainbow path if no two edges of the path are colored with the same color. The graph \( G \) is called rainbow connected if for any two distinct vertices of \( G \), there is a rainbow path connecting them. For a connected graph \( G \), the rainbow connection number of \( G \), denoted by \( rc(G) \), is defined as the minimum number of colors that are required to make \( G \) rainbow connected. These concepts were first introduced by Chartrand et al. in [3] and have been well-studied since then. For further details, we refer the reader to a book [9].

Motivated by rainbow connection coloring and proper coloring in graphs, Borozan et al. [2] introduced the concept of proper-path coloring. Let \( G \) be a nontrivial connected graph with an edge-coloring. A path in \( G \) is called a proper path if no two adjacent edges of the path are colored with the same color. The \( k \)-proper connection number of a connected graph \( G \), denoted by \( pc_k(G) \), is defined as the minimum number of colors that are required in an edge-coloring of \( G \) such that any two distinct vertices of \( G \) are connected by \( k \) internally pairwise vertex-disjoint proper paths. We write \( pc(G) \) for short when \( k = 1 \). For more details, we refer to a dynamic survey [8].

Jiang et al. [7] introduced the analogous concept of total-proper connection of graphs. Let \( G \) be a nontrivial connected graph with a total-coloring \( c : E(G) \cup V(G) \to \{1, 2, \ldots, t\} \), \( t \in \mathbb{N} \). We use \( c(u), c(uv) \) to denote the colors assigned to the vertex \( u \in V(G) \) and the edge \( uv \in E(G) \), respectively. A path \( P \) is called a total-proper path if (i) any two adjacent edges of \( P \) are assigned distinct colors; (ii) any two adjacent internal vertices of \( P \) are assigned distinct colors; (iii) any internal vertex of \( P \) is assigned a distinct color from its incident edges of \( P \). A total-coloring \( c \) is a total-proper coloring of \( G \) if every pair of distinct vertices \( u, v \) of \( G \) is connected by a total-proper path in \( G \). A graph with a total-proper coloring is said to be total-proper connected. If \( k \) colors are used, then \( c \) is referred as a total-proper \( k \)-coloring. The total-proper connection number of a connected graph \( G \), denoted by \( tpc(G) \), is the minimum number of colors that are required to make \( G \) total-proper connected. For the total-proper connection number of graphs, the following observations are immediate.

**Proposition 1.** Let \( G \) be a connected graph on \( n \) vertices. Then

(i) \( tpc(G) = 1 \) if and only if \( G = K_n \);

(ii) \( tpc(G) \geq 3 \) if \( G \) is noncomplete.
A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name “Nordhaus-Gaddum-type” is given because Nordhaus and Gaddum [10] first established the following type of inequalities for chromatic number of graphs in 1956. They proved that if \( G \) and \( \overline{G} \) are complementary graphs on \( n \) vertices whose chromatic number are \( \chi(G) \) and \( \chi(\overline{G}) \), respectively, then \( 2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1 \). Since then, many analogous inequalities of other graph parameters have been considered, such as diameter [5], domination number [4], proper connection number [6], and so on. In this paper, we consider analogous inequalities concerning total-proper connection number of graphs. We prove that if both \( G \) and \( \overline{G} \) are connected, then

\[
6 \leq \text{tpc}(G) + \text{tpc}(\overline{G}) \leq n + 2.
\]

The rest of this paper is organized as follows: In Section 2, we list some useful known results on total-proper connection number. In Section 3, we first characterize the graphs \( G \) on \( n \) vertices with \( \text{tpc}(G) = n - 1 \). Based on this result, we give the upper bound and show that this bound is reached for \( n \geq 5 \) if and only if \( G \) or \( \overline{G} \) is the tree with maximum degree \( n - 2 \). In the final section, we give the lower bound and show that it is sharp for \( n \geq 4 \).

## 2 Preliminaries

In this section, we list some preliminary results and definitions on the total-proper coloring which can be found in [7].

**Proposition 2.** [7] If \( G \) is a nontrivial connected graph and \( H \) is a connected spanning subgraph of \( G \), then \( \text{tpc}(G) \leq \text{tpc}(H) \). In particular, \( \text{tpc}(G) \leq \text{tpc}(T) \) for every spanning tree \( T \) of \( G \).

**Proposition 3.** [7] Let \( G \) be a connected graph of order \( n \geq 3 \) that contains a bridge. If \( b \) is the maximum number of bridges incident with a single vertex in \( G \), then \( \text{tpc}(G) \geq b + 1 \).

In [7], the authors determined the total-proper connection numbers of trees and complete bipartite graphs.

**Theorem 1.** [7] If \( T \) is a tree of order \( n \geq 3 \), then \( \text{tpc}(T) = \Delta(T) + 1 \).

A Hamiltonian path in a graph \( G \) is a path containing every vertex of \( G \) and a graph having a Hamiltonian path is a traceable graph.

**Corollary 1.** [7] If \( G \) is a traceable graph that is not complete, then \( \text{tpc}(G) = 3 \).

**Theorem 2.** [7] Let \( G = K_{s,t} \) denote a complete bipartite graph with \( s \geq t \geq 2 \). Then \( \text{tpc}(G) = 3 \).
Given a total-colored path \( P = v_1v_2 \ldots v_{s-1}v_s \) between any two vertices \( v_1 \) and \( v_s \), we denote by \( \text{start}_e(P) \) the color of the first edge in the path, i.e. \( c(v_1v_2) \), and by \( \text{end}_e(P) \) the last color, i.e. \( c(v_{s-1}v_s) \). Moreover, let \( \text{start}_v(P) \) the color of the first internal vertex in the path, i.e. \( c(v_2) \), and by \( \text{end}_v(P) \) the last color, i.e. \( c(v_{s-1}) \). If \( P \) is just the edge \( v_1v_s \), then \( \text{start}_e(P) = \text{end}_e(P) = c(v_1v_s) \), \( \text{start}_v(P) = c(v_s) \) and \( \text{end}_v(P) = c(v_1) \).

**Definition 1.** Let \( c \) be a total-coloring of a graph \( G \) that makes \( G \) total-proper connected. We say that \( G \) has the strong property if for any pair of vertices \( u, v \in V(G) \), there exist two total-proper paths \( P_1, P_2 \) between them (not necessarily disjoint) such that (1) \( c(u) \neq \text{start}_v(P_i) \) and \( c(v) \neq \text{end}_v(P_i) \) for \( i = 1, 2 \), and (2) both \( \{c(u), \text{start}_e(P_1), \text{start}_e(P_2)\} \) and \( \{c(v), \text{end}_e(P_1), \text{end}_e(P_2)\} \) are 3-sets.

The authors in [7] studied the total-proper connection number of 2-connected graphs and gave an upper bound.

**Theorem 3.** [7] Let \( G \) be a 2-connected graph. Then \( tpc(G) \leq 4 \) and there exists a total-coloring of \( G \) with 4 colors such that \( G \) has the strong property.

From Definition 1 and Theorem 3 we get the following.

**Corollary 2.** Let \( H = G \cup \{v\} \) such that \( H \) is connected. If there is a total-proper \( k \)-coloring \( c \) of \( G \) such that \( G \) has the strong property, then \( tpc(H) \leq k \).

We also study the total-proper connection number of \( H \) when \( G \) is a complete bipartite graph, and get the exact value of \( tpc(H) \).

**Lemma 1.** Let \( H = K_{s,t} \cup \{v\} \) such that \( H \) is connected, where \( s \geq t \geq 2 \). Then \( tpc(H) = 3 \). Moreover, \( tpc(H') = 3 \).

**Fig 1:** The graph \( H' \)

*Proof. Let \( U \) and \( W \) be the two partite sets of \( K_{s,t} \), where \( U = \{u_1, \ldots, u_s\} \) and \( W = \{w_1, \ldots, w_t\} \). Since \( H \) and \( H' \) are both noncomplete, we only need to prove \( tpc(H) \leq 3 \) and \( tpc(H') \leq 3 \), i.e., demonstrating a total-proper 3-coloring of \( H \) or \( H' \). We divide our discussion according to the value of \( t \).

**Case 1.** \( t = 2 \)
If $v$ is adjacent to $W$, say $vw_1 \in E(H)$, then set $c(w_1) = c(u_1w_2) = 1$, and $c(w_2) = c(u_1w_1) = 2$. Assign all the remaining vertices and edges with color 3. Thus, there is a total-proper path $u_iw_1u_2w_2u_j$ connecting $u_i$ and $u_j$, where $2 \leq i,j \leq s$. As for the rest of vertex pairs, we can always find a path contained in the path $vw_1u_1w_2u_i$ for some $2 \leq i \leq s$. If there is another vertex $v'$ adjacent to $w_2$, based on the above coloring, set $c(v') = c(v'w_2) = 3$, then we obtain a total-proper 3-coloring of $H'$, see Fig.1.

If $v$ is adjacent to $U$, say $vu_1 \in E(H)$, then set $c(w_1) = c(u_2) = c(u_1w_2) = 1$, and $c(w_2) = c(u_1w_1) = c(u_2w_1) = c(vu_1) = 2$. Assign all the remaining vertices and edges with color 3. Thus, there is a total-proper path, contained in the path $vu_1w_2u_2w_1$ or $vu_1w_2u_i$ for some $3 \leq i \leq s$, connecting $v$ and any other vertex in $H$. And for vertex pairs in $U \cup W$, there is a total-proper path contained in the path $u_iw_2u_1w_1u_j$ for some $2 \leq i < j \leq s$.

Case 2. $t \geq 3$

If $s = t = 3$, then $H$ is traceable so that $tpc(H) = 3$. If $s \geq 4$, we consider two subcases.

1) Assume there is a 6-cycle $C_6$ in $K_{s,t}$ such that $H - C_6$ is still connected. Without loss of generality, we suppose $C_6 = u_1w_1u_2w_2u_3w_3$. We color $C_6$ with the colors 1, 2, 3 by the sequence of vertices and edges on the cycle. That is, set $c(u_1) = c(w_2) = c(w_1) = c(u_3w_3) = 1$, $c(u_2) = c(w_3) = c(u_1w_1) = c(w_2u_3) = 2$, and $c(w_1) = c(u_3) = c(u_2w_2) = c(w_3u_1) = 3$. Let $i,j \geq 4$ be two integers. Assign $u_i$ and $u_3w_j$ (if any) with color 1, and assign $w_j$ and $w_1u_i$ with color 2. The remaining vertices and edges are all colored 3. Then we claim that this total-coloring makes $H$ total-proper connected. Any pair $(u_i,w_j) \in U \times W$ is connected by the edge $u_iw_j$. The total-proper path for the pairs from $U \times U$ is contained in the path $P = u_iw_1u_2w_2u_3w_3u_j$ for some $1 \leq i,j \leq s$. And the total-proper path for the pairs from $W \times W$ is contained in the path $P = w_1u_1w_2w_2u_3w_j$ for some $1 \leq i,j \leq t$. Now consider the pairs of $(v) \times (U \cup W)$. By the assumption, we know that $vu_\ell \in E(H)$ or $vw_\ell \in E(H)$ for $\ell \geq 4$. Without loss of generality, suppose $\ell = 4$. If $vu_4 \in E(H)$, then for pairs $(v,u_i)$ $(1 \leq i \leq s)$ there is a total-proper path contained in the path $P = vu_4w_1u_2w_2u_3w_3u_j$ for some $1 \leq j \leq s$, and for pairs $(v,w_j)$ $(1 \leq i \leq t)$ there is a total-proper path contained in the path $P = vu_4w_1u_2w_2u_3w_j$ for some $1 \leq j \leq t$. The case when $vw_4 \in E(H)$ is similar.

2) Assume there is not such a 6-cycle in subcase 1). As $s \geq 4$ we can deduce that $t = 3$ and $v$ is only adjacent to $W$, say $vw_2 \in E(H)$. Then we color $H$ as above. Then it is sufficient to check the pairs in $(v) \times (U \cup W)$. For pairs in $(v) \times U$, there is a total-proper path $P = vw_2u_3w_3u_i$ for some $1 \leq i \leq s$, and for pairs in $(v) \times W$, we can find a total-proper path contained in the path $P = vw_2u_3w_3u_iu_1w_1$.

The proof is complete. □
3 Upper bound on $tpc(G) + tpc(\overline{G})$

At the beginning of this section, we give total-proper connection numbers of four unicyclic graphs, which are useful to characterize the graphs on $n$ vertices that have total-proper connection number $n - 2$.

**Lemma 2.** Let $H_1, H_2, H_3$ and $H_4$ be the graphs on $n \geq 5$ vertices shown in the Fig. 2, respectively. Then $tpc(H_1) = n - 2$; $tpc(H_2) = n - 2$ if $n = 5$, $tpc(H_2) = n - 3$ if $n \geq 6$; and for $i = 3, 4$, $tpc(H_i) = n - 2$ if $n = 5$ or $6$, $tpc(H_i) = n - 3$ if $n \geq 7$.

![Fig 2: The graphs $H_1, H_2, H_3$ and $H_4$.](image)

**Proof.** By Proposition 3 we get $tpc(H_1) \geq n - 2$ and $tpc(H_i) \geq n - 3$ for $i \in \{2, 3, 4\}$.

For $i = 1, 2, 3$, let $uvw$ be the triangle in $H_i$ and let $e_1$, $e_2$, . . . , and $e_{n-3}$ denote the bridges in $H_i$. Assume that $e = e_{n-3}$ in the graphs $H_2$ and $H_3$, and the edge $e$ is incident with the vertex $x$ and adjacent to the bridge $e_1$ in $H_2$, and $e$ is incident with the vertex $v$ in $H_3$. We first consider the graph $H_1$ and demonstrate a total-coloring of it with $n - 2$ colors. Let $c(u) = c(vw) = 1$, $c(e_j) = j + 1$ for $1 \leq j \leq n - 3$, $c(uv) = c(w) = 2$ and $c(v) = c(wu) = 3$. The remaining vertices are all colored 1. It is easy to check this total-coloring makes $H_1$ total-proper connected. Hence, we have $tpc(H_1) = n - 2$ when $n \geq 5$.

We should point out that for $i = 2, 3, 4$, the graph $H_i$ is traceable when $n = 5$, hence $tpc(H_i) = 3$ by Corollary 1. So we assume $n \geq 6$. Consider the graph $H_2$. Color as $H_1$ only with the exception that $c(e_{n-3}) = 1$ and $c(x) = 3$. It is easy to check that under this total-coloring, $H_2$ is total-proper connected. Hence, we have $tpc(H_2) = n - 2$ when $n = 5$ and $tpc(H_2) = n - 3$ when $n \geq 6$.

Consider the graph $H_3$. When $n = 6$, we claim that $tpc(H_3) = 4$. From Proposition 2, we get that $tpc(H_3) \leq 4$. If we use 3 colors to total-color $H_3$, no matter how we color it, there always exist two pendant vertices not being connected by a total-proper path. When $n \geq 7$, it can be easily checked that the total-coloring of $H_2$, only with the exception that $c(e) = 4$, makes $H_3$ total-proper connected. Hence, we have $tpc(H_3) = n - 2$ when $n = 5, 6$ and $tpc(H_3) = n - 3$ when $n \geq 7$.

Now we consider the graph $H_4$. We use $e_1$, $e_2$, . . . , and $e_{n-4}$ to denote the bridges incident with $u$, respectively, and use $uvwu$ to denote the quadrangle in $H_4$. First, we consider the case $n \geq 7$. We demonstrate a total-coloring of $H_4$ with $n - 3$ colors. Let $c(e_j) = j$ for $1 \leq j \leq n - 4$, $c(u) = n - 3$, $c(v) = c(x) = 2$, $c(vw) = c(xu) = 3$...
and \( c(w) = 4 \). The remaining edges and vertices are all colored 1. It is easy to check that under this total-coloring, \( H_4 \) is total-proper connected. When \( n = 6 \), we claim that \( tpc(H_4) = 4 \). From Proposition 2 we get that \( tpc(H_4) \leq 4 \). If we use 3 colors to total-color \( H_4 \), no matter how we color it, there always exists a vertex pair not being connected by a total-proper path. Hence, we have \( tpc(H_4) = n - 2 \) when \( n = 5, 6 \) and \( tpc(H_4) = n - 3 \) when \( n \geq 7 \).}

We use \( C_n \) and \( S_n \) to denote the cycle and the star on \( n \) vertices, respectively, and use \( T(a, b) \) to denote the double star that is obtained by adding an edge between the center vertices of \( S_a \) and \( S_b \). Given a cycle \( C_r = v_1v_2 \ldots v_r \), let \( C_r(T_1, T_2, \ldots, T_r) \) be the graph obtained from \( C_r \) and rooted trees \( T_i \) by identifying the root, say \( r_i \), of \( T_i \) with \( v_i \) on \( C_r \), \( i = 1, 2, \ldots, r \). We assume that \( |T_i| = n_i, n_i \geq 1, i = 1, 2, \ldots, r \). Then \( |C_r(T_1, T_2, \ldots, T_r)| = \sum_{i=1}^{r} |T_i| \). In particular, if \( |T_i| = 1 \) for each \( i \in \{1, 2, \ldots, r\} \), the graph \( C_r(T_1, T_2, \ldots, T_r) \) is just the cycle \( C_r \). For a nontrivial graph \( G \) such that \( G + uv \cong G + xy \) for every two pairs \((u, v), (x, y)\) of nonadjacent vertices of \( G \), we use \( G + e \) to denote the graph obtained from \( G \) by joining two nonadjacent vertices of \( G \).

**Theorem 4.** Let \( G \) be a connected graph of order \( n \geq 3 \). Then \( tpc(G) = n - 1 \) if and only if \( G \in \{T(2, n - 2), C_4, C_4 + e, S_4 + e\} \).

**Proof.** By Theorem \( 1 \) and Corollary \( 1 \) we can easily check that \( tpc(G) = n - 1 \) if \( G \) is one of the above four graphs. So we concentrate on the verification of the converse of the theorem. Suppose that \( tpc(G) = n - 1 \). Then \( G \) cannot be complete, so \( tpc(G) \geq 3 \). If \( G \) is a tree, then by Theorem \( 1 \) we have \( \Delta(G) = n - 2 \), thus \( G \cong T(2, n - 2) \). Now, we consider the case that \( G \) contains cycles. Pick a longest cycle \( C_k = v_1v_2 \ldots v_k \) of \( G \), where \( k \geq 3 \). If \( k = n \), then \( 3 = tpc(C_k) = tpc(G) = n - 1 \). So \( n = 4 \). Thus \( G \cong C_4 \) or \( C_4 + e \). If \( k < n \), consider a unicyclic spanning subgraph \( H \) of \( G \) containing the cycle \( C_k \). Then \( H \) can be written as \( C_k(T_1, T_2, \ldots, T_k) \). Set \( r = \max\{\Delta(T_i) : 1 \leq i \leq k\} \) and let \( T_i \) be a tree with \( \Delta(T_i) = r \). Notice that \( \Delta(T_i) \leq |T_i| - 1 \leq n - k \), so \( r \leq n - k \). Then delete an edge \( e \) of \( H \), which is incident with \( v_i \) in \( C_k \), and denote the obtained graph as \( H' \), so \( H' \) is a spanning tree of \( G \) and \( \Delta(H') \leq n - k + 1 \), and the equality holds if and only if there is only one nontrivial subtree \( T_i = S_{n-k+1} \) in \( H \) whose center is \( v_i \) or there are exactly two pendant edges attaching to \( C_k \). Thus \( n - 1 = tpc(G) \leq tpc(H') = \Delta(H') + 1 \leq n - k + 2 \), therefore we have \( k \leq 3 \). So \( k = 3 \) and all the equalities must hold. Hence, there is only one nontrivial subtree in \( H \) and \( \Delta(H) = n - 1 \) or \( H \) is traceable on 5 vertices, the latter contradicting the condition \( tpc(G) = n - 1 \). So we can identify \( H \) as \( S_n + e \), and when \( n \geq 5 \), the graph \( H \) is just the graph \( H_1 \) in Fig. 2. By Lemma \( 3 \) and Proposition \( 2 \), we have \( tpc(G) \leq tpc(H_1) = n - 2 \), a contradiction. So \( n = 4 \) and \( G \cong S_4 + e \) since \( C_3 \) is a longest cycle of \( G \).
Lemma 3. Let $G$ be a graph on 5 vertices. If both $G$ and $\overline{G}$ are connected, then we have

$$tpc(G) + tpc(\overline{G}) = \begin{cases} 
7 & \text{if } G \cong T(2,3) \text{ or } \overline{G} \cong T(2,3); \\
6 & \text{otherwise.}
\end{cases}$$

Proof. If $G \cong T(2,3)$ or $\overline{G} \cong T(2,3)$, then from Theorem 4 we can easily get that $tpc(G) + tpc(\overline{G}) = 7$. Otherwise, we have $tpc(G) \leq n - 2 = 3$ and $tpc(\overline{G}) \leq n - 2 = 3$. Combining with Proposition 1 we get $tpc(G) + tpc(\overline{G}) = 3 + 3 = 6$ if $G \not\cong T(2,3)$ and $\overline{G} \not\cong T(2,3)$.

Theorem 5. Let $G$ be a graph of order $n \geq 5$. If both $G$ and $\overline{G}$ are connected, then we have $tpc(G) + tpc(\overline{G}) \leq n + 2$, and the equality holds if and only if $G \cong T(2, n - 2)$ or $\overline{G} \cong T(2, n - 2)$.

Proof. It follows from Lemma 3 that the result holds for $n = 5$. So we assume that $n \geq 6$. If $G \cong T(2, n - 2)$, then $\overline{G}$ contains a spanning subgraph $H$ that is obtained by attaching a pendant edge to the complete bipartite graph $K_{2,n-3}$. So we have $tpc(G) = 3$ by Lemma 1. Combining with Theorem 4, the result is clear. Similarly, we get that $tpc(G) + tpc(\overline{G}) = n + 2$ if $\overline{G} \cong T(2, n - 2)$. In the following, we prove that $tpc(G) + tpc(\overline{G}) < n + 2$ when $G \not\cong T(2, n - 2)$ and $\overline{G} \not\cong T(2, n - 2)$. Under this assumption, we have $3 \leq tpc(G) \leq n - 2$ and $3 \leq tpc(\overline{G}) \leq n - 2$ by Proposition 1 and Theorem 4.

We first consider the case that both $G$ and $\overline{G}$ are 2-connected. When $n = 6$, we claim that $tpc(G) = 3$. Suppose that the circumference of $G$ is $k$. If $k = 6$, then $tpc(G) \leq tpc(C_6) = 3$. If $k = 4$, then $G$ contains a spanning $K_{2,4}$, contradicting the fact that $\overline{G}$ is connected. Next, we assume that $G$ contains a 5-cycle $C = v_1v_2v_3v_4v_5$. Then $G$ is traceable, so $tpc(G) = 3$ by Corollary 1. Thus, we have $tpc(G) + tpc(\overline{G}) \leq 3 + n - 2 < n + 2$. For $n \geq 7$, we have $tpc(G) \leq 4$ and $tpc(\overline{G}) \leq 4$ by Theorem 3. Hence, we get $tpc(G) + tpc(\overline{G}) \leq 4 + 4 < n + 2$.

Now, we consider the case that at least one of $G$ and $\overline{G}$ has cut vertices. Without loss of generality, we suppose that $G$ has cut vertices. Let $u$ be a cut vertex of $G$, let $G_1, G_2, \ldots, G_k$ be the components of $G - u$, and let $n_i$ be the number of vertices in $G_i$ for $1 \leq i \leq k$ with $n_1 \leq \cdots \leq n_k$. We consider the following two cases.

Case 1. There exists a cut vertex $u$ of $G$ such that $n - 1 - n_k \geq 2$. Since $\Delta(G) \leq n - 2$, we have $n_k \geq 2$. We know that $\overline{G} - u$ contains a spanning complete bipartite graph $K_{n-1-n_k,n_k}$. Hence, it follows from Lemma 1 that $tpc(\overline{G}) = 3$. Combining with the fact that $tpc(G) \leq n - 2$, we get that $tpc(G) + tpc(\overline{G}) < n + 2$.

Case 2. Every cut vertex $u$ of $G$ satisfies that $n - 1 - n_k = 1$.

First, we suppose that $G$ has at least two cut vertices, say $u_1$ and $u_2$. Let $u_1v_1$ and $u_2v_2$ be two pendant edges of $G$. Obviously, the edges $u_1v_1$ and $u_2v_2$ are disjoint. So $u_1v_2, u_2v_1 \in E(\overline{G})$, and $\overline{G} - \{u_1, u_2\}$ contains a spanning complete bipartite graph $K_{2,n-4}$ with two partitions $U = \{v_1, v_2\}$ and $W = V(G) \setminus \{u_1, u_1, v_1, v_2\}$. By Lemma 1 we have that $tpc(\overline{G}) = 3$. Together with the fact that $tpc(G) \leq n - 2$, we get that $tpc(G) + tpc(\overline{G}) < n + 2$. 

8
Now, we consider the subcase that \( G \) has only one cut vertex \( u \) and let \( uw \) be the pendant edge of \( G \). Then \( G - v \) is 2-connected. By Theorem 3 and Corollary 2, we have \( \text{tpc}(G) \leq 4 \), thus \( \text{tpc}(G) + \text{tpc}(\overline{G}) \leq n + 2 \). Now, we prove that the equality cannot hold. Note that \( d_{\overline{G}}(v) = n - 2 \). Let \( N_{\overline{G}}(v) = \{ w_1, w_2, \ldots, w_{n-2} \} \). Since \( \Delta(G) \leq n - 2 \), there exists a vertex \( w_i \) (\( 1 \leq i \leq n - 2 \)) not adjacent to \( u \) in \( G \), say \( uw_i \notin E(G) \). Then \( uw_i \in E(\overline{G}) \). If there is a vertex \( w_j \) (\( 2 \leq j \leq n - 2 \)) adjacent to \( u \) in \( \overline{G} \), then \( \overline{G} \) contains \( H_3 \) in Fig. 2 as a spanning subgraph, so \( \text{tpc}(\overline{G}) \leq \max\{4, n - 3\} \). If there is a vertex \( w_j \) (\( 2 \leq j \leq n - 2 \)) adjacent to \( u \) in \( \overline{G} \), then \( \overline{G} \) contains \( H_4 \) in Fig. 2 as a spanning subgraph, so \( \text{tpc}(\overline{G}) \leq \max\{4, n - 3\} \). If there are two vertices \( w_j, w_k \) (\( 2 \leq j \neq k \leq n - 2 \)) adjacent in \( \overline{G} \), then \( \overline{G} \) contains \( H_2 \) in Fig. 2 as a spanning subgraph, so \( \text{tpc}(\overline{G}) \leq n - 3 \). We conclude that \( \text{tpc}(\overline{G}) \leq \max\{4, n - 3\} \) if \( G - v \) is 2-connected. For \( n \geq 7 \), we get the result \( \text{tpc}(G) + \text{tpc}(\overline{G}) \leq n + 1 < n + 2 \). For \( n = 6 \), since \( G - v \) is a 2-connected graph on 5 vertices, \( G - v \) contains a spanning 5-cycle or a spanning \( K_{2,3} \), implying that \( \text{tpc}(G) = 3 \) by Corollary 1 and Lemma 1. Thus, we have \( \text{tpc}(G) + \text{tpc}(\overline{G}) \leq 3 + 4 = 7 < 8 \).

4 Lower bound on \( \text{tpc}(G) + \text{tpc}(\overline{G}) \)

As we have noted that \( \text{tpc}(G) = 1 \) if and only if \( G \) is a complete graph. In this case, the graph \( \overline{G} \) is not connected. So, if \( G \) and \( \overline{G} \) are both connected, then \( \text{tpc}(G) \geq 3 \). Similarly, we have \( \text{tpc}(\overline{G}) \geq 3 \). Hence, we obtain that \( \text{tpc}(G) + \text{tpc}(\overline{G}) \geq 6 \).

**Theorem 6.** Let \( G \) be a graph of order \( n \geq 4 \). If both \( G \) and \( \overline{G} \) are connected, then we have \( \text{tpc}(G) + \text{tpc}(\overline{G}) \geq 6 \), and the lower bound is sharp.

**Proof.** We only need to prove that there are graphs \( G \) and \( \overline{G} \) on \( n \geq 4 \) vertices such that \( \text{tpc}(G) = \text{tpc}(\overline{G}) = 3 \).

Let \( G \) be the graph with vertex set \( \{v\} \cup U \cup W \), where \( U = \{u_1, \ldots, u_{\frac{n}{2}}\} \) and \( W = \{w_1, \ldots, w_{\frac{n}{2}+1}\} \), such that \( N(v) = U \) and \( U \) is an independent set and \( G[W] \) is a clique, and for each vertex \( u_i, u_i \) is adjacent to \( w_i, w_i+1, \ldots, w_{i+\lceil\frac{n-1}{2}\rceil} \) where the subscripts are taken modulo \( \lceil\frac{n-1}{2}\rceil \). Obviously, the graphs \( G \) and \( \overline{G} \) are both traceable. It follows from Corollary 1 that \( \text{tpc}(G) = \text{tpc}(\overline{G}) = 3 \).

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