Optical tests of Bell’s inequalities not resting upon the absurd fair sampling assumption

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Abstract

A simple local hidden-variables model is exhibited which reproduces the results of all performed tests of Bell’s inequalities involving optical photon pairs. For the old atomic-cascade experiments, like Aspect’s, the model agrees with quantum mechanics even for ideal setups. For more recent experiments, using parametric down-converted photons, the agreement occurs only for actual experiments, involving low efficiency detectors. Arguments are given against the fair sampling assumption, currently combined with the results of the experiments in order to claim a contradiction with local realism. New tests are proposed which are able to discriminate between quantum mechanics and a restricted, but appealing, family of local hidden-variables models. Such tests require detectors with efficiencies just above 20%.

1

2 Introduction

Almost forty years after Bell’s work[1], no experiment has been performed showing a real (loophole-free) violation of a Bell inequality. Consequently the important question whether nature obeys local realism is still open. Furthermore a true refutation of local realism, that is a disproof of the whole family
of local hidden variables (LHV) theories, seems today more difficult than it was believed twenty years ago. The failure is concealed behind the assertion that there are loopholes in the performed tests of Bell’s inequalities, due to (allegedly unimportant) nonidealities of the measuring devices. This state of opinion is rather unfortunate because it discourages people from making the necessary effort to close an open question of fundamental relevance. Indeed it is important not only for pure science but also for the development of future technologies, like quantum computation. The truth is that none of the performed experiments has been able to discriminate between quantum mechanics and the whole family of LHV theories. Therefore the agreement of these experiments with quantum mechanics leaves untouched the question of local realism.

The attempt to prove the impossibility of hidden variables has been a recurrent endeavor since the early days of quantum mechanics. The reason has been the attempt to feel comfortable in spite of the failure to devise a picture of the world, consistent with the new theory and fitting in the tradition of (classical) physics, as supported e. g. by Einstein[2]. That is, a world view where physical systems have properties independently of any observation ("the moon is there when nobody looks"), actions propagate in space-time at a speed not greater than that of light (without "spooky actions at a distance") and probabilities appear due to ignorance, maybe unavoidable, rather than by an essential indeterminacy (that is "God does not play dice"). The mainstream of the scientific community concealed the failure with the statement that the purpose of physics is not to understand the world, but just to be able to predict the results of experiments. This lead to a pragmatic approach to quantum mechanics, supported mainly by Niels Bohr and known as the Copenhagen interpretation. But actually it consists of giving up any interpretation[3]. Attempts at a deeper understanding of nature (in particular, avoiding the essential indeterminacy) are commonly known as hidden variables theories. Thus, in order to feel comfortable with the absence of a fully satisfactory interpretation, it was necessary to prove that hidden variables cannot exist. This lead to a number of impossibility theorems, the most celebrated proven by John von Neumann’s in 1932 (the view of von Neumann differs from Bohr’s, but we shall not discuss this point here). All no-hidden-variables theorems previous to 1965 were shown to be useless by John Bell[4]. In turn he stated a new theorem[1] which apparently excluded the most interesting families of hidden variables, those which are local.
Most people are (quite correctly) fond of quantum mechanics and interpret (incorrectly) Bell’s theorem as proving the impossibility of a local realistic interpretation of nature, even without the need of specific experiments (e.g. tests of Bell’s inequalities). This is a theoretical prejudice which support the claim that local realism has been empirically refuted, in spite of the fact that there are LHV models compatible with all performed tests of Bell’s inequalities (see below, section 2.)

The failure to get an empirical disproof of local realism, after many years of attempts, shows that the task is extremely difficult, if not impossible. This contrasts with the perceived wisdom that quantum mechanics predicts highly nonlocal effects. (A good illustration of the difficulties involved is Fry experiment[5]. It is the most careful proposal for a loophole-free test of Bell’s inequalities, it was published in 1995 but not yet performed eight years later). As time elapses without a true refutation of local realism, more is reinforced the historical analogy with the difficulty of making a perpetuum mobile, which gave support to the principles of thermodynamics, or the difficulty of measuring the absolute velocity of Earth, which lead to relativity theory. That is, time is reinforcing the conjecture that there are fundamental principles preventing the violation of a Bell inequality[6],[7].

An apparently unsurmountable difficulty for the stated conjecture is that Bell’s theorem seems to imply that either quantum mechanics is wrong or local realism is untenable. But the difficulty is not so strong as is currently assumed. We must take into account that quantum mechanics contains two quite different ingredients: 1) the formalism (including the equations), and 2) the postulates of measurement (e. g. "the possible results of measuring an observable are the eigenvalues of the associated operator"). It is a fact that both ingredients are required for the proof of Bell’s theorem, but only the quantum formalism has been supported (beyond any reasonable doubt) by the experiments. In contrast most of the postulates of the measurement are unnecessary, if not absurd, a claim which I hope is shared by people who today reject any theory of measurement not derived from the quantum formalism itself.

In summary, it is not obvious that quantum mechanics and local realism are incompatible, and the success of the former does not refute the latter. As a consequence new experimental tests of LHV theories are urgently needed.
3 Local hidden-variables model for optical photon experiments

Early tests of Bell’s inequalities consisted of measuring the polarization correlation of entangled photon pairs produced in atomic cascades\[8\]. The first experiment of this type was performed in 1972\[9\], and the most celebrated, and sophisticated, was made by Aspect et al.\[10\]. Recent tests have used photon pairs produced in non-linear crystals by the process of parametric down-conversion (PDC)\[11\]. In most experiments the quantity measured is the polarization correlation of the photons and we shall consider only these experiments here, although the argument that follows applies more generally. The set-up consists of a source of entangled photon pairs, two collecting lens systems, placed on the opposite sides of the source, plus a polarization analyzer and a detector on each side. (Experiments using two-channel polarizers and two detectors on each side will be considered later.) The quantum prediction for single rates, \( R_1 \) or \( R_2 \), and coincidence rates, \( R_{12} \), are\[8\], with some simplifications made for the sake of clarity,

\[
R_1 = R_2 = \frac{1}{2} \eta R_0,
\]

\[
R_{12} (\phi) = \frac{1}{4} \eta^2 R_0 [1 + V \cos (2\phi)],
\]

(1)

where \( R_0 \) is the production rate of photon pairs in the source, \( \eta \) is the overall detection efficiency (which takes into account the collection efficiency, the losses in the polarizers, lenses, etc, and the quantum efficiency of the photon detectors) and \( \phi \) the angle between the polarization planes of the analyzers.

In recent PDC experiments it is common to use two-channel polarizers with two detectors inserted behind each polarizer. For every detection in one beam (conventionally called the signal beam), a time window is open for detection in the other (idler) beam. The predictions of quantum mechanics for these experiments may be written

\[
R_{++} (\phi) = R_{--} (\phi) = \frac{1}{2} \eta R_1 [1 + V \cos (2\phi)], \quad R_{+-} (\phi) = R_{-+} (\phi) = R_{++} \left( \phi + \frac{\pi}{2} \right).
\]

(2)

For ideal set-up (in particular perfect polarizers and detectors) \( V = \eta = 1 \) but, due to the nonideality of polarizers, \( V \) usually does not surpass 0.97 and, due to nonideality of the idler detector, \( \eta \) is less than 0.20 (the efficiency of
the detector in the signal beam is irrelevant). Both quantities, \(\eta\) and \(V\), may be measured in independent experiments, although other nonidealities of the set-up turn down their values when used in (2). With reasonable estimates for these effects the results of the performed experiments confirm quantum theory.

A local hidden variables (LHV) model \(^{11}\) for the experiment consists of three functions, \(\rho(\lambda)\), \(P_1(\lambda, \phi_1)\) and \(P_2(\lambda, \phi_2)\), fulfilling the conditions

\[
\rho(\lambda) \geq 0, \quad P_j(\lambda, \phi_j) \geq 0, \quad j = 1, 2, \quad (3)
\]

\[
\int \rho(\lambda) d\lambda = 1, \quad P_j(\lambda, \phi_j) \leq 1, \quad (4)
\]

and such that

\[
\frac{R_1}{R_0} = \int \rho(\lambda) P_1(\lambda, \phi_1) d\lambda, \quad \frac{R_2}{R_0} = \int \rho(\lambda) P_2(\lambda, \phi_2) d\lambda,
\]

\[
\frac{R_{12}(\phi)}{R_0} = \int \rho(\lambda) P_1(\lambda, \phi_1) P_2(\lambda, \phi_2) d\lambda, \quad \phi_1 - \phi_2 = \phi. \quad (5)
\]

Here \(\lambda\) collectively labels a set of parameters known as “the hidden variables” and \(\phi_1\) (\(\phi_2\)) is the angle of the polarization plane of the first (second) analyzer with respect to the horizontal. For later convenience I have separated the homogeneous, (3), from the inhomogeneous, (4), constraints, the former (the latter) with the property that multiplying their left hand sides times an arbitrary real number, they remain (may not remain) true. (For experiments with two channel polarizers we may use two pairs of functions so that \(P_{j-}(\lambda, \phi_j) = P_{j+}(\lambda, \phi_j + \pi/2)\)).

In the early papers on the subject it was frequent to assert that LHV models for the experiments, although possible, would be necessarily contrived\(^{12},^{8}\). Today we know that this is not the case, as is shown by the following model. In quantum mechanics, a pair of photons entangled in polarization may be represented by the state vector

\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left( a_H^\dagger a_V^\dagger + a_V^\dagger a_H^\dagger \right) |\text{vacuum}\rangle, \quad (6)
\]

where \(H\) (\(V\)) labels horizontal (vertical) polarization and \(a^\dagger(b^\dagger)\) are creation operators of photons in the first (second) beam. If we search for a LHV model, it is natural to attach two-dimensional polarization vectors (in the
plane perpendicular to the wave-vectors) to the incoming light signals, such as
\[ \alpha \equiv (\alpha_H, \alpha_V), \beta \equiv (\beta_V, \beta_H). \]  
(7)
The scalar product of these two vectors is the model analog of the probability amplitude associated to (6), whence the probability should be the square of that scalar product. This suggests a probability function, \( \rho \) (see eq. (3)) of the form
\[ \rho \propto (\alpha \cdot \beta)^2 \propto \cos^2 (\lambda_1 - \lambda_2), \]  
(8)
where the hidden variables \( \lambda_1 \) and \( \lambda_2 \) are angles, say with the horizontal, defining the directions of the polarization vectors of the photons. It is easy to see that, after normalization, this gives
\[ \rho = \frac{1}{\pi^2} \left[ 1 + \cos (2\lambda_1 - 2\lambda_2) \right]. \]  
(9)
Now we shall choose the probabilities, \( P_j(\lambda, \phi_j) \), of detection of a photon after passage through the corresponding polarizer. In a classical theory an electromagnetic signal, polarized in the plane forming an angle \( \lambda_j \) with respect to the horizontal, would be divided in the polarizer according to Malus’ law, that is a large intensity would go to the upper channel if \( \lambda_j \) is close to the angle, \( \phi_j \), of the polarizer (and similarly to the lower channel if \( \lambda_j \) is close to \( (\phi_j + \pi/2) \)). Thus it is natural to assume that a detection event would be more probable if the intensity is large, which suggests that \( P_j(\lambda, \phi_j) \) should decrease with increasing value of the angle \( |\lambda_j - \phi_j| \). A simple expression which fits in these conditions is the following one (but the prediction of the model depends but slightly of the details of that expression, see below)
\[ P_j(\lambda_j, \phi_j) = \beta \text{ if } |\lambda_j - \phi_j| \leq \gamma \pmod{\pi} \]  
\[ 0 \text{ otherwise.} \]  
(10)
By the nature of polarizers the angle \( \phi_j \) is equivalent to \( \phi_j + \pi \), whence the periodicity \( \pi \). For the validity of the model it is crucial that \( \beta \leq 1 \) in order that conditions (4) are fulfilled. In this form we obtain a LHV model, not at all artificial, reproducing the quantum predictions, eqs. (1), for all performed experiments involving optical photons.
In fact, inserting eqs. (9) and (10) in (5), it is easy to see that the model leads to
\[ \frac{R_{12}}{R_1} = \frac{2\beta\gamma}{\pi} \left[ 1 + \frac{\sin^2(2\gamma)}{4\gamma^2} \cos (2\phi) \right], \]  
(11)
which reproduces the quantum formula (11) if we assume

\[ V = \frac{\sin^2(2\gamma)}{4\gamma^2}, \quad \eta = \frac{4\gamma\beta}{\pi}. \]  

(12)

(It may be realized that using functions \( P_j \) different from (10), but depending only on \( |\lambda_j - \phi_j| \), would change just the constant factor in front of \( \cos(2\phi) \) in (11)). The constraint \( \beta \leq 1 \) shows that the model may agree with the quantum predictions only if

\[ V \leq \frac{\sin^2(\pi\eta/2)}{(\pi\eta/2)^2} \simeq 1 - \frac{\pi^2\eta^2}{12}. \]  

(13)

Typical values in actual experiments are \( \eta \lesssim 0.2, \ V \lesssim 0.97 \) (if raw data are used, see next section), so that the LHV model is compatible with the empirical results. A detailed comparison with a recent experiment is made in section 5.

4 The meaning of Bell´s inequalities, homogeneous and inhomogeneous

Bell´s theorem has two parts. The first one is the derivation of some inequalities which hold true for any LHV model. The second part consists of providing an example of experiment where the quantum-mechanical predictions violate a Bell inequality. I have no criticism to the first part, but I claim that the second part is confused. In the current "proofs" of Bell´s theorem, the second part consists of exhibiting a vector and several projection operators in the Hilbert space of a two-particle system which, via the standard "postulates" of measurement, violate a Bell inequality. My claim is that a correct proof should not involve the postulates of measurement. It should consists of the detailed proposal of an experiment where the predictions of quantum theory violate a Bell inequality, without any possible loophole. Therefore it seems to me that the first paper close to being a real proof of the second part of Bell´s theorem is Fry proposed experiment[5]. However, to be sure that an experiment truly violates a Bell inequality it would be necessary to perform it. Consequently, in my view Bell´s 1964 work actually consisted of the derivation of inequalities fulfilled by the whole family of LHV theories, plus the suggestion that these inequalities might be
used to tests local realism (i. e. LHV theories) vs. quantum mechanics. (This is not an underestimation of Bell’s work, which I believe is one the greatest achievements in the history of quantum mechanics.)

From eqs. (3) to (5) it is possible to derive inequalities necessarily fulfilled by every LHV model of the experiments. An example is the (CH) inequality derived by Clauser and Horne\[12\] which I shall write, for a particular choice of polarization orientations, in the form

\[
\frac{3R_{12}(\phi) - R_{12}(3\phi)}{R_1 + R_2} \leq 1.
\]  

This inequality is inhomogeneous, in the sense that it compares single rates with coincidence rates. In its derivation it is necessary to use both, the homogeneous basic inequalities (3) and the inhomogeneous ones (4). Indeed, it is a general result that every genuine Bell inequality, valid for the whole family of LHV theories, must be inhomogeneous\[6\]. However, inhomogeneous inequalities cannot be tested in optical experiments due to the low efficiency of optical photon counters. Indeed, the inequality (14) cannot be violated in any experiment where the efficiency is smaller than 82%. The proposed solution to this problem was to replace one of the inhomogeneous inequalities (4) by another one, homogeneous, which allows deriving testable homogeneous inequalities. A typical example is the no-enhancement assumption of Clauser and Horne\[12\]

\[
P_j(\lambda, \phi_j) \leq P_j(\lambda, \infty), \ j = 1, 2,
\]

where \(P_j(\lambda, \infty)\) is the detection probability with the polarizer removed. It is possible to prove that this condition, plus (3) and the second (4) leads to an inequality of the form of (14), \(R_j\) meaning now the coincidence rate with the corresponding polarized removed. Such inequality is homogeneous and may be violated in experiments with low efficiency detectors, but the violation does not disprove the whole family of LHV theories.

Most frequently the substitute for the second (4) is not written as an inequality, but expressed as a variant of the ”fair sampling” assumption. For instance, ”all photons incident in a detector have a probability of detection that is independent on whether or not the photon has passed through a polarizer”, which was used for the derivation of the inequality tested in the first atomic-cascade experiment\[9\]. Or, ”if a pair of photons emerges from the polarizers the probability of their joint detection is independent of the angles \(\phi_1\) and \(\phi_2\)”, used in the first experimental proposal\[13\]. Hence the authors
derived the most celebrated homogeneous inequality, known as CHSH, valid for experiments with two-channel polarizers

\[ S = |3E(\pi/8) - E(3\pi/8)| \leq 2, \quad E(\phi) = \frac{R_{++}(\phi) + R_{--}(\phi) - R_{+-}(\phi) - R_{-+}(\phi)}{R_{++}(\phi) + R_{--}(\phi) + R_{+-}(\phi) + R_{-+}(\phi)}, \]  

where \( R_{++}, R_{--}, R_{+-}, R_{-+} \), are coincidence rates in the four pairs of detectors, taking one detector on each side.

If the quantum prediction (2) is inserted in the CHSH inequality we obtain

\[ V \leq \sqrt{2}/2 \simeq 0.7071, \]  

which is the inequality actually tested, and violated, in the performed experiments. In sharp contrast, if we insert the quantum prediction in the genuine, inhomogeneous, Bell inequality (14) (taking, e.g. \( R_{++} \) for \( R_{12} \)) we get

\[ \eta \left( 1 + \sqrt{2}V \right) \leq 2, \]  

which could be violated only if both, \( \eta \) and \( V \), were close enough to unity.

The sharp contrast between the tested inequality (17) and the Bell inequality (18) urges me to stress that Bell’s inequalities refer to correlations (e.g. in polarization) between distant measurements (in the most strong, relativistic, form space-like should be substituted for distant). In a test of Bell’s inequalities, the correlation is quantified by the parameter \( V \), but the distance is related to the position measurement, which involves \( \eta \). Consequently I claim that replacing the test of the inequality (18) by the test of (17) amounts to losing an essential condition of the test.

The parameter \( V \), related to the polarization, is a wave property, whilst the detection efficiency \( \eta \), related to localization, is a particle property. Thus we might interpret the inequality (18) as forbidding that photons behave as (localized) particles and (delocalized) waves at the same time, whilst (17) just tests the wave property. Indeed, in optical photons the wave properties manifest most strongly and it is the ”particle” parameter \( \eta \) which is usually small. In contrast, it is known that in gamma rays, which behave more strongly as particles, it is the polarization which is difficult to measure accurately, whilst 100% efficient detection is achievable. Thus, the high values of \( V \) (close to unity) found in optical experiments tell us, simply, that optical photons are extended objects (waves) rather than localized objects (particles). A more quantitative statement should replace extended (localized) by bigger (smaller) than atoms.
5 Criticism of the fair sampling assumption

In order to see whether the "fair sampling" assumption is plausible, as is usually assumed, it is illustrative to return to the old atomic-cascade experiments, like Aspect’s one[10]. In these experiments the angular correlation of the photons emitted by the source is poor due to the three-body character of the atomic decay. Thus the quantum prediction eq.(11) may be written in the form

$$R_1 = R_2 = \frac{1}{2} \frac{\Omega}{4\pi} \eta' R_0,$$
$$R_{12}(\phi) = \frac{1}{4} \Omega^2 \left( \frac{\Omega}{4\pi} \right)^2 R_0 \left[ 1 + V' \frac{\cos(2\phi)}{V} \right],$$

where the new parameters $\eta'$ and $V'$ are related to nonidealities of detectors and polarizers, respectively, $\alpha$ is the half angle of the apertures (assumed equal) and $\Omega$ the solid angle covered by them, that is

$$\Omega = 2\pi \left( 1 - \cos \alpha \right).$$

(For the sake of clarity we have ignored a factor close to unity in front of the right hand side of the second eq.(19), but this does not affects the argument that follows). The function $F(\alpha)$ takes into account that the polarization correlation of a photon pair decreases when the angle between their wave-vectors departs from $\pi$. In the most frequent case of a 0-1-0 cascade (that is, both the initial and final states of the atom having zero spin) the form of the function is

$$F(\alpha) = 1 - \frac{2}{3} \left( 1 - \cos \alpha \right)^2.$$

It is easy to see that the LHV model of section 2 is compatible with the quantum predictions (19) even for ideal set-up, that is $\eta' = V' = 1$.

The point that I want to stress is that in atomic-cascade experiments the "fair sampling assumption", that the ensemble of detected photon pairs is representative of the whole ensemble of pairs emitted from the source, contradicts quantum mechanics. In fact, quantum theory predicts that the polarization correlation of a photon pair decreases when the angle between their wave-vectors departs from $\pi$. As a consequence the correlation is higher for the photons actually detected than for the average pairs produced in the source, a fact giving rise to the factor $F(\alpha)$ in eq.(11). It may be said that fair
sampling refers to the ensemble of pairs such that both enter the apertures, and this might be applied to both atomic-cascade and PDC experiments. But such ensemble is nonsense according to quantum mechanics, in the same way as the "ensemble of photons passing the upper slit" is nonsense in a two-slit experiment\cite{14}. In summary, fair sampling is either false or nonsense in quantum mechanics.

In the context of LHV theories the fair sampling assumption is, simply, absurd. In fact, the starting point of any hidden variables theory is the hypothesis that quantum mechanics is not complete, which essentially means that states which are considered identical in quantum theory may not be really identical. For instance if two atoms, whose excited states are represented by the same wave-function, decay at different times, in quantum mechanics this fact may be attributed to an "essential indeterminacy", meaning that identical causes (identical atoms) may produce different effects (different decay times). In contrast, the aim of introducing hidden variables would be to explain the different effects as due to the atomic states not being really identical, only our information (encapsuled in the wave-function) being the same for both atoms. That is, the essential purpose of hidden variables is to attribute differences to states which quantum mechanics may consider identical. Therefore it is absurd to use the fair sampling assumption -which rests upon the identity of all photon pairs- in the test of LHV theories, because that assumption excludes hidden variables a priori.

For similar arguments it is not allowed to subtract accidental coincidences, but the raw data of the experiments should be used. In fact, what is considered accidental in the quantum interpretation of an experiment might be essential in a hidden variables theory. Actually any interpretation of quantum theory in terms of hidden variables should take into account the existence of a fundamental noise (quantum fluctuations). The quantum formalism has been developed containing (very clever) rules which allow an efficient removal of that noise in the calculations, e. g. the use of normal ordering in quantum field theory. But a hidden variables theory might relate noise in measurements (e. g. dark rate in detection) to quantum fluctuations, so attributing a fundamental character to a part, at least, of the experimental noise. This is the reason why uncorrected (raw) data should be used in the interpretation of experiments testing LHV theories.
New test of local hidden variables with optical photon experiments

In experiments with optical photons the main loophole for a disproof of local realism derives from the low efficiency (and/or high dark rate) of available photon counters. It is well known that efficiencies higher than about 80% (with no noise) are required for a loophole-free experiment. Apparently such photon counters will not be available in the near future and, consequently, it seems appropriate to perform new experiments which might provide some indication about the results to be expected when the efficiency of detectors increases. As we have remarked, making new experiments, whose interpretation requires the fair sampling assumption, adds very little to our knowledge. The alternative is to test quantum mechanics against restricted families of LHV theories and I propose models of the type presented in the previous section.

Actually, the proposal of testing restricted families of LHV theories is not new, because introducing assumptions, e. g. (15), in addition to eqs. (3) and (4) in order to get testable inequalities, amounts to restricting the family of LHV theories to be tested. Such additional assumptions, allegedly plausible, have been used for the interpretation of all performed tests, as was commented in section 3. However, as plausibility is not a scientific criterion, it seems to me that a proposal where it is clearly exhibited the restricted family of LHV theories to be tested, is less misleading than using the word ”plausible”. Such a proposal is what is made in the following.

We consider a family of LHV models with two angular hidden variables \{\lambda_1, \lambda_2\}, a probability density, \rho, of the photon pairs created in the source depending only on \(|\lambda_1 - \lambda_2|\) and probability functions, \(P_j\), depending on \(|\lambda_j - \phi_j|\). It would be desirable to define a criterion of closeness with quantum predictions and, after that, to optimize the functions \rho and \(P_j\) in order to get the best LHV model of the said family, but this program will be left for the future. Here we shall consider two particular models appropriate for low an high efficiencies, respectively. More specifically for situations where the inequality (13) is either slightly of strongly violated. In both case we shall use functions \(P_j\) of the form (10).

For low efficiency we argue as follows. The probability density \rho may be
written as a Fourier series

\[
\rho = \frac{1}{\pi^2} \left[ 1 + \sum_{n=1}^{\infty} a_n \cos (2n\lambda_1 - 2n\lambda_2) \right], \tag{22}
\]

normalized in the interval \([0, \pi]\). The simple model of section 2 corresponds to \(a_1 = 1\), \(a_n = 0\) for \(n > 1\). This model agrees with quantum predictions only if the inequality \((13)\) holds true. If the inequality is violated, we need a function \((22)\) with \(a_1 > 1\) which, by the positivity of \(\rho\), necessarily requires that some \(a_n\) with \(n > 1\) are not zero. The most simple choice is

\[
\rho = \frac{1}{\pi^2} \left[ 1 + (1 + \varepsilon) \cos (2\lambda_1 - 2\lambda_2) + \varepsilon \cos (4\lambda_1 - 4\lambda_2) \right]. \tag{23}
\]

which involves an adjustable parameter, \(\varepsilon \in [0, \frac{1}{3}]\) (compare with eq.(12)).

The proposed density is non-negative definite, being equivalent to

\[
\rho = \frac{2}{\pi^2} \left[ (1 - 3\varepsilon) \cos^2 (\lambda_1 - \lambda_2) + 4\varepsilon \cos^4 (\lambda_1 - \lambda_2) \right]. \tag{24}
\]

Putting eqs.(10) and (23) into eqs.(5) we get

\[
\frac{R_{12}}{R_1} = \frac{2}{\pi} \frac{\beta \gamma}{\pi} \left[ 1 + (1 + \varepsilon) \sin^2 (2\gamma) \cos (2\phi) + \varepsilon \frac{\sin^2 (4\gamma)}{16\gamma^2} \cos (4\phi) \right]. \tag{25}
\]

This model prediction generalizes \((11)\) (see section 3) and agrees with it if \(\varepsilon = 0\).

Tests of the LHV model against quantum mechanics require new experiments with higher values of the parameters \(\eta\) and/or \(V\), e. g. using parametric down converted photons. This means not too small quantum efficiencies of the photon counters, but also high collecting apertures for the idler photon in each pair. In contrast the apertures for the signal photon may be arbitrarily small, with the limitation of not reducing too much the single counting rate in this beam. For such experiments our LHV model predicts deviations of the function \(R_{12}(\phi)\) from the quantum prediction \((11)\), because the parameter \(\varepsilon\) cannot be zero (see \((25)\)). In fact, the condition \(\beta \leq 1\) gives in this case (using the second eq.(12))

\[
V \leq (1 + \varepsilon) \frac{\sin^2 (\pi\eta/2)}{\pi^2\eta^2/4}, \tag{26}
\]
so that $\varepsilon > 0$ whenever the inequality (13) is violated. Thus the LHV model predicts deviations from the quantum formula (1), which may be tested experimentally, consisting in the presence of a $\cos(4\phi)$ term in the fit of the empirical curve $R_{12}(\phi)$.

We define the deviation between the quantum and model predictions by

$$
\delta \equiv \left< \left[ \frac{R_{12}^Q(\phi)}{R_{12}^Q(\phi)} - \frac{R_{12}^{LHV}(\phi)}{R_{12}^{LHV}(\phi)} \right]^2 \right>^{1/2},
$$

(27)

where $<>$ means average over angles. Taking (25) and (26) into account, we get

$$
\delta \geq \sqrt{2\varepsilon \sin^2(\pi \eta)} \geq \sqrt{2} \cos^2(\pi \eta/2) \left[ V - \frac{\sin^2(\pi \eta/2)}{(\pi \eta/2)^2} \right].
$$

(28)

This is our proposed Bell inequality. In the empirical test, the parameter $V$ may be obtained from a fit of the empirical data to the quantum prediction eq.(1). The inequality requires only the measurements of coincidence rates and detection efficiency.

A related inequality may be obtained as follows. We define two parameters $V_A$ and $V_B$ :

$$
V_A = \frac{R_{12}(0) - R_{12}(\pi/2)}{R_{12}(0) + R_{12}(\pi/2)}, \quad V_B = \sqrt{2 \frac{R_{12}(\pi/8) - R_{12}(3\pi/8)}{R_{12}(\pi/8) + R_{12}(3\pi/8)}},
$$

(29)

which according to the quantum prediction are both equal to the parameter $V$ of eq.(2). In contrast, in our model (with $\varepsilon \neq 0$) we get from (25)

$$
V_B = V = (1 + \varepsilon) \frac{\sin^2(\pi \eta/2\beta)}{(\pi \eta/2\beta)^2}, \quad V_A = V_B \left[ 1 + \varepsilon \frac{\sin^2(\pi \eta/\beta)}{(\pi \eta/\beta)^2} \right]^{-1}.
$$

(30)

Taking the constraint $\beta \leq 1$ into account, this implies

$$
\frac{V_B}{V_A} \geq 1 + \cos^2(\pi \eta/2) \left[ V_B - \frac{\sin^2(\pi \eta/2)}{(\pi \eta/2)^2} \right],
$$

(31)

which may be tested by measuring the three quantities $V_A, V_B$ (or $V$) and $\eta$. Quantum predictions should violate the inequality for high enough $V$ and $\eta$. This inequality has the advantage that the parameters $V_A$ and $V_B$ have been measured in most experiments, which allows tests using the results
already reported (see below). However, this inequality is only useful for relatively small detection efficiencies, where the model resting upon eq. (33) is appropriate. But the inequality does not hold true for the model described below (see eq. (34)), appropriate for high efficiencies.

Our model may be extended to experiments with two-channel polarizers using two pairs of functions such that $P_j^-(\lambda, \phi_j) = P_j^+(\lambda, \phi_j + \pi/2)$, which leads to the prediction

$$R_{+ -} (\pi/2 + \phi) = R_{- +} (\pi/2 + \phi) = R_{+ +} (\phi) = R_{- -} (\phi),$$  \hspace{1cm} (32)

the latter given by eq. (25). It is not difficult to see that, with this choice, $V_A$ coincides with the visibility of any of the functions (32) (if they are equal, or their average if they are not) and $V_B$ coincides with $\sqrt{2}/4$ times the quantity $S$ defined by eq. (16), usually measured in the tests of Bell’s inequality. This allows testing (31) in experiments with two-channel polarizers and some performed experiments are quite close to achieving a test of the inequality. For instance the experiment of Kurtsiefer et al. \cite{15} reports a value $S = 2.6979 \pm 0.0034$, leading to $V_B = 0.9539 \pm 0.00012$, with $\eta = 0.214$. Thus the inequality (18) (with $V_B$ substituted for $V$) is fulfilled, which means that the experiment is unable to discriminate between quantum mechanics and our model. However, an increase of $S$ by only 1% could provide a valid test of eq. (31).

Another possibility, in experiments with two-channel polarizers, would be to see whether the sum

$$R_{\text{tot}}(\phi) = R_{+ +} (\phi) + R_{- -} (\phi) + R_{+ -} (\phi) + R_{- +} (\phi)$$  \hspace{1cm} (33)

is independent of $\phi$. In our model, using eq. (32), the quantity $R_{\text{tot}}$ is proportional to $1 + \varepsilon \cos (4\phi)$, so that deviations from rotational invariance are expected of order $\varepsilon$ ($\varepsilon$ may be related to the measurable parameters $\eta, V_A$ and $V_B$ (or $V$) by the second eq. (30)). Measurements of the deviation from rotational invariance of the quantity $R_{\text{tot}}$ have been recently proposed as a test of the fair sampling assumption \cite{16}.

It might be believed that models of the type presented in this section would depart strongly from the quantum predictions when the detection efficiency increases, for instance above 80%, but this is not the case. For high values of $\eta$ and/or $V$ a good LHV model, of the same type, consists of using the probability function

$$\rho = \frac{1}{\sqrt{2\pi^3\sigma}} \exp \left( \frac{\lambda^2}{2\sigma^2} \right) (\mod \pi), \lambda = \lambda_1 - \lambda_2.$$  \hspace{1cm} (34)
This is a periodic function which may be expanded in Fourier series to give

\[ \rho = \frac{1}{\pi^2} \left[ 1 + 2 \sum_{n=1}^{\infty} \exp \left( -2n^2 \sigma^2 \right) \cos (2n\lambda) \right], \quad (35) \]

provided that \( \sigma \) is small enough so that the gaussian function in eq. (34) is negligible for \( \lambda = \pi/2 \). Using again (9) for the detection probabilities we get a prediction (for \( \beta = 1 \))

\[ \frac{R_{12}}{R_1} = \frac{2\gamma}{\pi} \left[ 1 + 2 \sum_{n=1}^{\infty} \exp \left( -2n^2 \sigma^2 \right) \frac{\sin^2(2n\gamma)}{(2n\gamma)^2} \cos (2n\phi) \right] \quad (36) \]

This is very close to the quantum prediction even for quite high detection efficiency, in particular using \( \sigma = \pi/18 \), as recently proposed [16]. For this choice it is possible to get \( V = 1 \) with efficiencies up to \( \eta = 0.848 \) and still the departures from the quantum prediction are only of a few percent. In fact we get

\[ V_A = 0.980, \quad V_B = 0.957, \quad \delta = 0.047. \]

Even closer is the prediction for the function \( E(\phi) \), see eq. (16), where the discrepancy corresponds to a term in \( \cos(6\phi) \) with a coefficient of order 0.02.

For higher efficiencies the model cannot reproduce the (ideal) quantum prediction \( V = 1 \). In particular, for 100% efficiency, i.e. \( \eta = 1 \Rightarrow \gamma = \pi/4 \), we get \( V \leq 8/\pi^2 \), whatever is the value of \( \sigma \). But, even in this limit, the model does not depart too much from quantum mechanics. For instance, with the same choice \( \sigma = \pi/18 \), we get

\[ V = 0.7627, \quad V_A = 0.8225, \quad V_B = 0.7043, \]

in comparison with the ideal quantum prediction \( V = V_A = V_B = 1 \). It may be realized that the value given by that model for the left side of the CH inequality [14] is 0.996, very close to the Bell limit, 1, although not so close to the quantum prediction with ideal polarizers, 1.207. It is remarkable that the function \( R_{\text{tot}}(\phi) \), eq. (33), defined with the prescription eq. (32), is unity for any angle \( \phi \), in exact agreement with the quantum prediction. It is also interesting that the model gives \( V_A > V_B \), in contrast with what happens with the model of the previous section, leading to the inequality (31). Also our proposed Bell inequality (28), which was derived for low detection efficiencies, is here useless its right hand side being zero. This shows that this inequality is only appropriate for relatively low detection efficiencies.
In summary, the proposed family of models agrees exactly with quantum predictions for low enough detection efficiency, and departs when the efficiency increases. The departure manifests in that the curve $R_{12}(\phi)$, when Fourier analyzed, either contains terms in $\cos(2n\phi)$ with $n > 1$ or has a lower coefficient in the $\cos(2\phi)$, or both. The deviations slowly increase with the detection efficiency up to a few percent for 85% efficiency. Then more rapidly up to about 20% for 100% efficiency. But neither the inequality (31) is necessarily violated by the models nor $R_{tot}(\phi)$, eq.(33), necessarily exhibits lack of rotational symmetry. In any case, for not too large detection efficiency (say below 30%) the inequality (28) provides a good test of an appealing family of local hidden variables models vs. quantum mechanics.

7 Discussion

Current optical tests of Bell’s inequalities - consisting of measuring a coincidence counting rate as a function of some angular parameter and confirming that the curve is a cosine one with visibility, $V$, above 0.71 - are useless for the purpose of refuting local realism. In this type of test it is therefore worthless to improve the experiment by increasing the distance between the detectors or increasing the statistics in order to provide violations with more standard deviations.

I propose experiments, easily feasible, which would test a particular, but interesting, family of local hidden variables theories. The test may be performed using detectors with not too high efficiency, just above 20%. On the other hand the test requires only the measurement of coincidence rates and it is, therefore, immune to many sources of noise. It is true that the refutation of that family would not rule out local realism, because other kinds of models exist in perfect agreement with quantum predictions up to about 64% efficiency[17], but those models are more artificial.

In the proposed experiments there exists the possibility that the standard quantum predictions are violated, which I claim would not contradict the hard core of quantum mechanics, that is the formalism. It would disprove only the standard theory of measurement. It is generally believed that departures from ideal behaviour of polarizers, detectors, etc. are just practical problems, which should be solved with the progress of the technology. But there may be nonidealities of a fundamental character. For instance, a detailed (quantum) theory of photon counters might show, at high effi-
ciencies, a nonlinear relation between the detection rate and the intensity of the incoming light beam, or an unavoidable increase of the dark rate. The models studied in the present paper show that relatively small nonidealities might be sufficient to make compatible with LHV theories the experiments involving entangled photon pairs.

References

[1] J. S. Bell, Physics, 1, 195 (1964.) This article is reproduced in the collection of articles on foundations of quantum mechanics by J. S. Bell, Speakable and unspeakable in quantum mechanics, Cambridge University, 1987.

[2] A. Einstein, in Albert Einstein: Philosopher-scientist, P. A. Schilpp, ed., Open Court, Evanston, 1949.

[3] C. Fuchs and A. Peres, ”Quantum mechanics needs no interpretation”, Physics Today, 53(3), 70-71 (2000) and 53(9),14,90(2000).

[4] J. S. Bell, Rev. Mod. Phys. 38, 447 (196). Reproduced in the book of Ref.1.

[5] E. Fry, T. Walther and S. Li, Phys. Rev. A 52, 4381 (1995).

[6] E. Santos, Phys. Rev. A 46, 3646 (1992).

[7] I. Percival, quant-ph/0008097.

[8] J. F. Clauser and A. Shimony, Rep. Prog. Phys. 41, 1881 (1978).

[9] S. J. Freedman and J. F. Clauser, Phys. Rev. Lett. 28, 938 (1972).

[10] A. Aspect, J. Dalibard and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).

[11] M. Horne, A. Shimony and A. Zeilinger, in J. S. Anandan, Ed., Quantum coherence, World Scientific, 1990.

[12] J. F. Clauser and M. Horne, Phys. Rev. D 10, 526 (1974).

[13] J. F. Clauser, M. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[14] E. Santos, *Phys. Rev. Lett.* **68**, 2700 (1992).

[15] C. Kurtsiefer, M. Oberparleiter and H. Weinfurter, *Phys. Rev. A* **64**, 023802 (2001).

[16] G. Adenier and A. Yu. Khrennikov, quant-ph/0306045.

[17] S. Caser, *Phys. Lett.* **102A**, 152 (1984).