Gravitational instantons with quadratic volume growth

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Abstract
There are two known classes of gravitational instantons with quadratic volume growth at infinity, known as type ALG and ALG*. Gravitational instantons of type ALG were previously classified by Chen–Chen. In this paper, we prove a classification theorem for ALG* gravitational instantons. We determine the topology and prove existence of “uniform” coordinates at infinity for both ALG and ALG* gravitational instantons. We also prove a result regarding the relationship between ALG gravitational instantons of order n and those of order 2.

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1 INTRODUCTION

We begin with the following definitions.

**Definition 1.1.** A hyperkähler 4-manifold \((X, g, I, J, K)\) is a Riemannian 4-manifold \((X, g)\) with a triple of Kähler structures \((g, I), (g, J), (g, K)\) such that \(IJ = K\).

We denote by \(\omega = (\omega_1, \omega_2, \omega_3)\) the Kähler forms associated to \(I, J, K\), respectively. The form \(\Omega = \omega_2 + \sqrt{-1}\omega_3\) is a nonvanishing holomorphic 2-form with respect to the complex structure \(I\). These satisfy the condition

\[
\omega_1 \wedge \omega_1 = \frac{1}{2} \Omega \wedge \bar{\Omega}.
\]

Conversely, any two-dimensional Kähler manifold \((X, g, I, \omega_1)\) with a nonvanishing holomorphic 2-form \(\Omega\) that satisfies (1.1) is hyperkähler.

**Definition 1.2.** A gravitational instanton \((X, g, \omega)\) is a complete hyperkähler 4-manifold \(X\).

If \(g\) is any gravitational instanton satisfying a curvature decay condition

\[
| Rm | = O(s^{-2-\epsilon}),
\]

for some \(\epsilon > 0\) as \(s \to \infty\), where \(s(x) = d_g(x_0, x)\) for some \(x_0 \in X\), Chen–Chen proved that it must be of type ALE, ALF, ALG, or ALH; see [6, Theorem 1.1]. We refer the reader to [2, 5, 7, 17, 19, 21] and references therein for more background on gravitational instantons.

In this paper, we are interested in two classes of gravitational instantons known as type ALG and ALG\(^*\), whose (unique) tangent cones at infinity are two-dimensional flat cones. The ALG type satisfies the curvature decay (1.2), but the ALG\(^*\) case has curvature decay \(O(s^{-2}(\log s)^{-1})\), as \(s \to \infty\). Both the ALG and ALG\(^*\) types have quadratic volume growth

\[
c \cdot t^2 < \text{Vol}(B_t(x_0)) < C \cdot t^2
\]

for \(t\) sufficiently large.

For any rational elliptic surface \(S\), and any singular fiber \(D\) of finite monodromy, that is, of Kodaira type II, III, IV, II\(^*\), III\(^*\), IV\(^*\), or I\(^0\)\(^*\), Hein constructed in [17] a nice background Kähler form on \(S \setminus D\), which is \(\bar{\partial} \bar{\partial}\)-cohomologous to the restriction of a Kähler form from \(S\), and proved that the corresponding Tian–Yau metric [27, Theorem 1.1] is an ALG gravitational instanton on \(S \setminus D\). Conversely, it was shown in [7] that any ALG gravitational instanton can be compactified to a rational elliptic surface \(S\), and the Kähler form with respect to the elliptic complex structure must be \(\bar{\partial} \bar{\partial}\)-cohomologous to the restriction of a Kähler form from \(S\). In this paper, we prove...
a similar classification for ALG" gravitational instantons. For any rational elliptic surface $S$, and any singular fiber $D$ of Kodaira type $I^n_v$, $1 \leq v \leq 4$, in [17], Hein also constructed a nice background Kähler form on $S \setminus D$, which is $\bar{\partial}\bar{\partial}$-cohomologous to the restriction of a Kähler form from $S$, and proved that the corresponding Tian–Yau metric [27, Theorem 1.1] is asymptotic to a semiflat metric near the divisor. We will show that these metrics are actually ALG" gravitational instantons on $S \setminus D$ according to Definition 1.3; see Section 2. We also prove a converse; namely, that any ALG" gravitational instanton can be compactified to a rational elliptic surface $S$, and the Kähler form with respect to the elliptic complex structure must be $\bar{\partial}\bar{\partial}$-cohomologous to the restriction of a Kähler form from $S$; see Theorem A.

As a corollary of this classification result, we will determine the topology of ALG" gravitational instantons; see Theorem B. Using Chen–Chen’s classification of ALG gravitational instantons from [7], we will also determine the topology of ALG gravitational instantons; see Theorem D. In both cases, we will address the issue of the existence of “uniform” coordinates at infinity; see Theorems C and E. Finally, in Theorem F, we prove a result about the relationship between ALG gravitational instantons of order $n$ and those of order 2.

In the following subsections, we will outline the paper and state our main results in much more detail.

### 1.1 ALG" gravitational instantons

In Section 2, we will define the standard ALG" model space, which is denoted by

$$\mathfrak{M}_{2\nu}(R), \ g_{\mathfrak{M}_{2\nu}}, \ \omega_{\mathfrak{M}_{2\nu}}$$

and depends on parameters $\nu \in \mathbb{Z}_+, \kappa_0 \in \mathbb{R}$ and $R, L \in \mathbb{R}_+$. Here, we just note that the manifold $\mathfrak{M}_{2\nu}(R)$ is diffeomorphic to $(R, \infty) \times I^3_\nu$, where $I^3_\nu$ is an infranilmanifold, which is a circle bundle of degree $\nu$ over a Klein bottle. We will let $r$ denote the coordinate on $(R, \infty)$, and define $s = rV^{1/2}$, where $V = \kappa_0 + \frac{\nu}{\pi} \log r$. The hyperkähler structure is obtained via a Gibbons–Hawking ansatz, and explicit formulas for the metric and hyperkähler structure can be found in Section 2. We note that the paramter $L$ is just an overall scaling parameter.

**Definition 1.3 (ALG" gravitational instanton).** A complete hyperkähler 4-manifold $(X, g, \omega)$ is called an ALG" gravitational instanton of order $n > 0$ with parameters $\nu \in \mathbb{Z}_+, \kappa_0 \in \mathbb{R}$ and $L \in \mathbb{R}_+$ if there exists an ALG" model space $(\mathfrak{M}_{2\nu}(R), g_{\mathfrak{M}_{2\nu}}, \omega_{\mathfrak{M}_{2\nu}})$ with $R > 0$, a compact subset $X_R \subset X$, and a diffeomorphism $\Phi : \mathfrak{M}_{2\nu}(R) \to X \setminus X_R$ such that

$$|\nabla^k g_{\mathfrak{M}}(\Phi^* g - g_{\mathfrak{M}_{2\nu}})|_{g_{\mathfrak{M}}} = O(s^{-k-n}),$$

$$|\nabla^k g_{\mathfrak{M}}(\Phi^* \omega_i - \omega_{i,\mathfrak{M}_{2\nu}})|_{g_{\mathfrak{M}}} = O(s^{-k-n}), \ i = 1, 2, 3,$$

as $s \to \infty$ for any $k \in \mathbb{N}_0$.

**Remark 1.4.** We note that ALG" gravitational instantons satisfy the following properties.

1. $\text{Vol}(B_t(\kappa_0)) \sim t^2$ as $t \to \infty$.
2. The tangent cone at infinity is $\mathbb{R}^2/\{\pm 1\}$. 


(3) \( |\text{Rm}_g| = O(s^{-2} (\log s)^{-1}) \) as \( s \to \infty \).

(4) There exist ALG* coordinates on \( X \) so that the order satisfies \( n \geq 2 \); see [10, Theorem 1.10].

The following is our main classification theorem for ALG* gravitational instantons.

**Theorem A** (ALG* classification). We have the following relationship between ALG* gravitational instantons and rational elliptic surfaces.

1. Let \((X, g, \omega)\) be an ALG* gravitational instanton with parameters \( \nu, \kappa_0, \) and \( L \). Then, \( \nu \leq 4 \), and \( X \) can be compactified to a rational elliptic surface \( S \) with global section by adding a Kodaira singular fiber \( D \) of type \( \text{I}_\nu^* \) at infinity, with respect to the complex structure \( I \). The 2-form \( \Omega = \omega_2 + \sqrt{-1} \omega_3 \) is a rational 2-form on \( S \) with \( \text{div}(\Omega) = -D \). Furthermore, we can choose \( S \) so that there exists a Kähler form \( \omega \) on \( S \) and a smooth function \( \varphi : X \to \mathbb{R} \) such that

\[
\omega_1 = \omega + \sqrt{-1} \partial \bar{\partial} \varphi. \tag{1.7}
\]

2. Conversely, let \((S, I)\) be a rational elliptic surface with a type \( \text{I}_\nu^* \) fiber \( D \). For any \( \kappa_0 \in \mathbb{R} \), any Kähler form \( \omega \) on \( S \), and any rational 2-form \( \Omega = \omega_2 + \sqrt{-1} \omega_3 \) on \( S \) with \( \text{div}(\Omega) = -D \), there exist \( c > 0 \), \( L > 0 \), and a smooth function \( \varphi : X \to \mathbb{R} \), where \( X \equiv S \setminus D \), such that

\[
(X, g, \omega_1 = \omega + \sqrt{-1} \partial \bar{\partial} \varphi, c \cdot \omega_2, c \cdot \omega_3)
\]

is an ALG* gravitational instanton with parameters \( \nu, \kappa_0 \), and \( L \), where \( g \) is the metric determined by \( \omega_1 \) and the elliptic complex structure \( I \).

Part (1) solves a particular case of Yau’s conjecture in [28, page 246], and shows how these hyperkähler structures are related to the geometry of rational elliptic surfaces. We note that the construction in Part (2) is similar to the modification by Chen–Chen in [7, Theorem 4.3] of Hein’s construction [17] using Tian–Yau’s methods [27, Theorem 1.1]. Hein’s construction produces hyperkähler structures that are asymptotic to a semiflat hyperkähler structure. But, as mentioned above, our ALG* model space is defined via a Gibbons–Hawking construction. An important aspect of our proof is to show that the model semiflat metrics are indeed Gibbons–Hawking; see Subsection 2.2.

An application of the above classification is to show that we can view any ALG* gravitational instanton with \( 1 \leq \nu \leq 4 \) as living on a fixed manifold \( X_\nu \).

**Theorem B.** If \((X, g, \omega)\) and \((X', g', \omega')\) are any two ALG* gravitational instantons with \( \nu = \nu' \), then \( X \) is diffeomorphic to \( X' \).

Our next result is the following refinement of this, which takes into account the ALG* coordinates.

**Theorem C.** Let \((X, g, \omega)\) and \((X', g', \omega')\) be ALG* gravitational instantons of order 2 with the same \( \nu, \kappa_0, L \), with ALG* coordinates \( \Phi : \mathcal{M}_{2\nu}(R) \to X \setminus X_R \) and \( \Phi' : \mathcal{M}_{2\nu}(R) \to X' \setminus X'_R \). Then, there exists a mapping \( F : \mathcal{M}_{2\nu}(R) \to \mathcal{M}_{2\nu}(R) \) and a diffeomorphism \( \Psi' : X \to X' \) such that \( F \) preserves the model hyperkähler structure and \( \Psi' \circ \Phi = \Phi' \circ F \) when restricted to \( \mathcal{M}_{2\nu}(R) \) for a sufficiently large
TABLE 1 Invariants of ALG spaces.

| \( \infty \) | \( I^*_0 \) | II | II* | III | III* | IV | IV* |
|---|---|---|---|---|---|---|---|
| \( \beta \in (0,1] \) | \( \frac{1}{2} \) | \( \frac{1}{6} \) | \( \frac{5}{6} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{1}{3} \) | \( \frac{1}{3} \) |
| \( \tau \in \mathbb{H} \) | Any | \( e^{\sqrt{-1} \frac{\tau}{\pi}} \) | \( e^{\sqrt{-1} \frac{\tau}{\pi}} \) | \( -1 \) | \( -1 \) | \( e^{\sqrt{-1} \frac{\tau}{\pi}} \) | \( e^{\sqrt{-1} \frac{\tau}{\pi}} \) |

R. Consequently, the hyperkähler structure \((X, (\Psi')^* g', (\Psi')^* \omega')\) is ALG* of order 2 in the ALG* coordinates defined by \( \Phi \).

In other words, we can view any ALG* gravitational instanton of order 2 with parameters \( \nu, \chi_0 \), and \( L \) as a gravitational instanton on a fixed manifold \( X_\nu \) with respect to a fixed ALG* coordinate system \( \Phi_{X_\nu} \). Theorem B and Theorem C will be proved in Section 4.

1.2 ALG gravitational instantons

For background of analysis on ALG spaces, related classification results, and relations to moduli spaces of monopoles and Higgs bundles, we refer the reader to [1, 3, 7, 8, 11, 15, 17, 20] and also the references therein.

In Definition 2.5 below, we will define the standard ALG model space

\[
(C_{\beta, \tau, L}(R), g^C, \omega^C),
\]

for parameters \( L, R \in \mathbb{R}_+ \), \( \beta \in \{ \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \} \), and \( \tau \in \mathbb{H} \), where \( \mathbb{H} \subset \mathbb{C} \) is the upper half-space. Except for the case that \( \beta = 1/2 \), the parameter \( \tau \) is determined by \( \beta \); see Table 1. Here, we just note that \( C_{\beta, \tau, L}(R) \) is diffeomorphic to \((R, \infty) \times N^3_\beta \), where \( N^3_\beta \) is a flat 3-manifold that is a torus bundle over a circle, and the metric \( g^C \) is a flat metric; the explicit formulas are given in Subsection 2.3. We let \( r \) denote the coordinate on \((R, \infty) \).

**Definition 1.1** (ALG gravitational instanton). A complete hyperkähler 4-manifold \((X, g, \omega)\) is called an ALG gravitational instanton of order \( n > 0 \) with parameters \( (\beta, \tau) \) as in Table 1, and \( L \in \mathbb{R}_+ \) if there exist \( R > 0 \), a compact subset \( X_R \subset X \), and a diffeomorphism \( \Phi : C_{\beta, \tau, L}(R) \to X \setminus X_R \), such that

\[
\left| \nabla^k_{g^C} (\Phi^* g - g^C) \right|_{g^C} = O(r^{-k-n}),
\]

\[
\left| \nabla^k_{g^C} (\Phi^* \omega_i - \omega_i^C) \right|_{g^C} = O(r^{-k-n}), \quad i = 1, 2, 3,
\]

as \( r \to \infty \), for any \( k \in \mathbb{N} \).

**Remark 1.2.** We note that ALG gravitational instantons satisfy the following properties.

1. \( \text{Vol}(B_t(x_0)) \sim t^2 \) as \( t \to \infty \).
2. The tangent cone at infinity is a two-dimensional cone with cone angle \( 2\pi \beta \).
3. The ALG order \( n \) is at least 2 in the \( I^*_0, \text{II, III, IV} \) cases and \( n = 2 - \frac{1}{\beta} \) in the \( \text{II}^*, \text{III}^*, \text{IV}^* \) cases; see [5, Theorem A].
As a corollary of Chen–Chen’s classification of ALG gravitational instantons in [7], our next result shows that we can view any ALG gravitational instanton with \( \beta \) as in Table 1 as living on a fixed space \( X_\beta \).

**Theorem D.** If \( (X, g, \omega) \) and \( (X', g', \omega') \) are any two ALG gravitational instantons with \( \beta = \beta' \), then \( X \) is diffeomorphic to \( X' \).

This will be proved in Section 4; see Theorem 4.1. Our next result is the following refinement of this, which takes into account the ALG coordinates.

**Theorem E.** Let \( (X, g, \omega) \) and \( (X', g', \omega') \) be ALG gravitational instantons of order \( n \) with the same \( \beta, \tau, L \), with ALG coordinates \( \Phi : C_{\beta, \tau, L}(R) \to X \setminus X_R \) and \( \Phi' : C_{\beta, \tau, L}(R) \to X' \setminus X'_R \). Then, there exists a mapping \( F : C_{\beta, \tau, L}(R) \to C_{\beta, \tau, L}(R) \) and a diffeomorphism \( \Psi : X \to X' \) such that \( F \) preserves the model hyperkähler structure and \( \Psi' \circ \Phi = \Phi' \circ F \) when restricted to \( C_{\beta, \tau, L}(R) \) for a sufficiently large \( R \). Consequently, the hyperkähler structure \( (X, (\Psi')^* g', (\Psi')^* \omega') \) is ALG of order \( n \) in the ALG coordinates defined by \( \Phi \).

In other words, we can view any ALG gravitational instanton of order \( n \) with parameters \( \beta \), \( \tau \), and \( L \) as a gravitational instanton on a fixed space \( X_\beta \) with respect a fixed ALG coordinate system \( \Phi_{X_\beta} \).

As mentioned above, when \( \beta > 1/2 \), the ALG order \( n \) can be improved to \( 2 - \frac{1}{\beta} \). Our next results examine in this case the relationship between ALG gravitational instantons of order \( 2 - \frac{1}{\beta} \) and those of order 2. Given an ALG gravitational instanton \( (X_\beta, g, \omega) \) with parameters \( (\beta, \tau) \) as in Table 1 and \( L > 0 \), we consider the periods
\[
[\omega] \in H^2_{dR}(X_\beta) \times H^2_{dR}(X_\beta) \times H^2_{dR}(X_\beta).
\]

(1.12)

In Section 5, we will prove the following.

**Theorem F.** Assume that \( \beta > 1/2 \), and let \( (X_\beta, g, \omega) \) be an ALG gravitational instanton with parameters \( (\beta, \tau) \) as in Table 1 and \( L > 0 \), which is of order \( n > 0 \) in the ALG coordinate system \( \Phi \). Then, there exists a diffeomorphism \( F : C_{\beta, \tau, L}(R) \to C_{\beta, \tau, L}(R) \) homotopic to the identity, and a 2-parameter family of ALG gravitational instantons \( (X_\beta, g_{a,b}, \omega_{a,b}) \) with \( (a, b) \in \mathbb{R}^2 \) containing \( (X_\beta, g, \omega) \), all with the same parameters \( \beta \), \( \tau \), and \( L \), and the same periods \( [\omega_{a,b}] = [\omega] \), which are ALG of order \( 2 - \frac{1}{\beta} \) in the \( \Phi \circ F \) coordinates when \( (a, b) \neq (0, 0) \), and ALG of order 2 in the \( \Phi \circ F \) coordinates when \( (a, b) = (0, 0) \).

This result is an improvement of [7, Theorem 1.7].

2 | THE MODEL HYPERKÄHLER STRUCTURES

In this section, we explain some properties of ALG* gravitational instantons in more detail. In Hein’s thesis [17], he proved that by choosing a nice background metric, the corresponding Tian–Yau metric [27, Theorem 1.1] is a gravitational instanton asymptotic to Greene–Shapere–Vafa-Yau’s semiflat ansatz in a neighborhood of a singular fiber of type I\( _0^* \) on a rational elliptic surface. One of
the main goals of this section is to show that Greene–Shapere–Vafa–Yau’s semiflat ansatz hyperkähler structure is asymptotic to a Gibbons–Hawking ansatz hyperkähler structure; thus, the $I_\nu$ metrics in [17] are $\text{ALG}^\ast$ according to Definition 1.3. We note that this was also remarked in [12, page 197].

2.1 | Gibbons–Hawking construction

In this subsection, we review the Gibbons–Hawking construction of the $\text{ALG}^\ast$ model metric. See [10, Section 2] for more details. Let $\nu$ be any positive integer. Recall that the Heisenberg nilmanifold $\text{Nil}^3_{2\nu}$ of degree $2\nu$ is the quotient of $\mathbb{R}^3$ by the following actions

\begin{align}
(\theta_1, \theta_2, \theta_3) &\mapsto (\theta_1 + 2\pi, \theta_2, \theta_3), \\
(\theta_1, \theta_2, \theta_3) &\mapsto (\theta_1, \theta_2 + 2\pi, \theta_3 + 2\pi \theta_1), \\
(\theta_1, \theta_2, \theta_3) &\mapsto (\theta_1, \theta_2, \theta_3 + 2\pi^2 \nu^{-1}).
\end{align}

We consider

\begin{align}
\Theta &\equiv \nu \pi (d\theta_3 - \theta_2 d\theta_1), \\
V &\equiv \kappa_0 + \frac{\nu}{\pi} \log r, \\
\kappa_0 &\in \mathbb{R}, \\
r &\in (R, \infty), \quad R > e^{\frac{\pi}{\nu}(1-\kappa_0)},
\end{align}

on the manifold

\begin{align}
S^1_{\theta_3} \to \widehat{\mathfrak{M}}_{2\nu}(R) \equiv (R, \infty) \times \text{Nil}^3_{2\nu} \to \widehat{\mathcal{U}} \equiv (\mathbb{R}^2 \setminus B_R(0)) \times S^1_{\theta_2},
\end{align}

where $(r, \theta_1)$ are polar coordinates on $\mathbb{R}^2$. The Gibbons–Hawking metric on $\widehat{\mathfrak{M}}_{2\nu}(R)$ is

\begin{align}
g_{\kappa_0} = V(dx^2 + dy^2 + d\theta_2^2) + V^{-1} \Theta^2 = V(dr^2 + r^2 d\theta_1^2 + d\theta_2^2) + V^{-1} \frac{\nu^2}{\pi^2} (d\theta_3 - \theta_2 d\theta_1)^2,
\end{align}

where $x + \sqrt{-1} y \equiv r \cdot e^{\sqrt{-1} \theta_1}$.

Consider an orthonormal basis

\begin{align}
\{ E^1, E^2, E^3, E^4 \} = \left\{ V^{1/2} dx, V^{1/2} dy, V^{1/2} d\theta_2, V^{-1/2} \Theta \right\}.
\end{align}

Then, the model hyperkähler forms on $\widehat{\mathfrak{M}}_{2\nu}(R)$ are given by

\begin{align}
\omega_I = \omega_{1,\kappa_0}^{\text{H}} = E^1 \wedge E^2 + E^3 \wedge E^4 = V dx \wedge dy + d\theta_2 \wedge \Theta, \\
\omega_I = \omega_{2,\kappa_0}^{\text{H}} = E^1 \wedge E^3 - E^2 \wedge E^4 = V dx \wedge d\theta_2 - dy \wedge \Theta, \\
\omega_K = \omega_{3,\kappa_0}^{\text{H}} = E^1 \wedge E^4 + E^2 \wedge E^3 = dx \wedge \Theta + V dy \wedge d\theta_2.
\end{align}

The $\mathbb{Z}_2$-action

\begin{align}
\iota(r, \theta_1, \theta_2, \theta_3) = (r, \theta_1 + \pi, -\theta_2, -\theta_3)
\end{align}
induces an automorphism of the hyperkähler structure, and we define the \( \text{ALG}^* \) model space as

\[
(\mathfrak{M}_{2\nu}(R), g_{\nu_0}, \omega_{\nu_0, x_0}, \omega_{\nu_0, x_1}, \omega_{\nu_0, x_2}, \omega_{\nu_0, x_3}) \equiv (\mathfrak{M}_{2\nu}(R), g_{\nu_0}, \omega_{1, x_0}, \omega_{2, x_0}, \omega_{3, x_0})/\iota.
\] (2.12)

For any scaling parameter \( L > 0 \), we define

\[
(\mathfrak{M}_{2\nu}(R), g_{\nu_0, L}, \omega_{\nu_0, L}, \omega_{\nu_0, L, x_0}, \omega_{\nu_0, L, x_1}, \omega_{\nu_0, L, x_2}, \omega_{\nu_0, L, x_3}) \equiv (\mathfrak{M}_{2\nu}(R), L^2 \cdot g_{\nu_0}, L^2 \cdot \omega_{\nu_0, L, x_0}, L^2 \cdot \omega_{\nu_0, L, x_1}, L^2 \cdot \omega_{\nu_0, L, x_2}, L^2 \cdot \omega_{\nu_0, L, x_3}).
\] (2.13)

### 2.2 Semiflat structure

In this subsection, we show that the \( \text{ALG}^* \) model space has a Greene–Shapere–Vafa–Yau semiflat structure [13, 17].

We first explain Kodaira’s model for an \( I^* \) fiber. Assume that we have a local elliptic fibration \( f : U \rightarrow B \xi \), where \( B \xi \) is a disc with coordinate \( \xi \) and the monodromy around the origin is of type \( I^* \). Let \( \pi : B_u \rightarrow B \xi \) denote the double cover branched over the origin given by \( \xi = u^2 \). Then, \( f \) pulls back to an elliptic fibration \( f' : U' \rightarrow B_u \) with a singular fiber of type \( I_{2\nu} \) over the origin. By [18], we can identify \( f' \) over the punctured disc with

\[
(B^*_u \times \mathbb{C}_v)/(\mathbb{Z} \oplus \mathbb{Z}),
\] (2.14)

where \( \mathbb{Z} \oplus \mathbb{Z} \) acts by

\[
(m, n) \cdot (u, v) = (u, v + m\tau_1(u) + n\tau_2(u)),
\] (2.15)

and \( \tau_1(u) = 1, \tau_2(u) = -\sqrt{-1} \nu \log u \). The \( \mathbb{Z}_2 \)-action on \( U' \) is given by \( (u, v) \mapsto (-u, -v) \).

**Remark 2.1.** If we use the holomorphic coordinates \( (\xi, w) = (u^2, uv) \) on the quotient, then the periods are

\[
\tau_{1,\text{model}}(\xi) = \xi^{1/2}, \quad \tau_{2,\text{model}}(\xi) = \frac{\nu}{2\pi \sqrt{-1}} \xi^{1/2} \log(\xi).
\] (2.16)

Furthermore, the monodromy matrix is given by

\[
A_{\nu} = \begin{pmatrix} -1 & -\nu \\ 0 & -1 \end{pmatrix}.
\] (2.17)

Let \( \Omega' = gdu \wedge dv \) be a meromorphic 2-form on \( U' \) away from the central fiber with \( g(u) = u^{-2k(u^2)} \), where \( k \) is a regular holomorphic function on \( B_\xi \) with \( k(0) \neq 0 \). Clearly \( \Omega' \) is invariant under the \( \mathbb{Z}_2 \)-action, so descends to a 2-form \( \Omega \) on \( U \).

The semiflat ansatz from [17, Equation 3.14] is given by

\[
\omega'_{sf, \nu} = \sqrt{-1} |k(u^2)|^2 \frac{\nu}{\pi \nu} \log |u| du \wedge d\bar{u} + \sqrt{-1} \frac{\nu}{2} \frac{\pi \nu}{\nu} \log |u| \frac{(dv - \Gamma du) \wedge (d\bar{v} - \overline{\Gamma} d\bar{u})}{2},
\] (2.18)

where \( \Gamma(u, v) \) is a function given by

\[
\Gamma(u, v) = \frac{1}{\sqrt{-1}} \frac{\Im(v)}{u \log |u|}.
\] (2.19)
Note that this Kähler form is defined upstairs, but descends to the quotient space.

We choose real coordinates \((v_1, v_2)\) defined by

\[
v = v_1 \tau_1 + v_2 \tau_2 = v_1 + v_2 \frac{\nu}{\pi \sqrt{-1}} \log u.
\]

(2.20)

There is an \(\mathbb{R}\)-action with period \(\sqrt{\epsilon}\) given by

\[
(u, v_1, v_2) \mapsto (u, v_1 + \epsilon^{-1/2} t, v_2),
\]

(2.21)

which leaves \(\omega'_s\epsilon\) and \(\Omega'\) invariant, and with associated vector field

\[
Y = \frac{1}{\sqrt{\epsilon}} \frac{\partial}{\partial v_1} = \frac{1}{\sqrt{\epsilon}} \left( \frac{\partial}{\partial v} + \frac{\partial}{\partial \overline{v}} \right).
\]

(2.22)

This action is Hamiltonian for each of the three Kähler forms

\[
\omega_1 = \omega'_s\epsilon, \quad \omega_2 = \text{Re}(\Omega'), \quad \omega_3 = \text{Im}(\Omega').
\]

(2.23)

We next compute the Hamiltonian functions \(H_i\), that is,

\[
\omega_i(Y, \cdot) = dH_i,
\]

(2.24)

for \(i = 1, 2, 3\).

**Proposition 2.2.** The moment map for the tri-Hamiltonian action associated to \(Y\)

\[
\mu : U' \setminus (f')^{-1}(0) \to \mathbb{R} \times \mathbb{C}
\]

is given by

\[
\mu = (H_1, H_2 + \sqrt{-1} H_3) = \left( \epsilon^{1/2} v_2, -\epsilon^{-1/2} \int u^{-2} k(u^2) du \right).
\]

(2.25)

(2.26)

**Proof.** We first compute

\[
\omega'_s\epsilon(Y, \cdot) = \frac{\sqrt{-1}}{2} \frac{\pi \sqrt{\epsilon}}{v|\log |u||} \left(d\overline{v} - \overline{\Gamma} d\overline{u} - dv + \Gamma du\right)
\]

(2.27)

\[
= \frac{\sqrt{-1}}{2} \frac{\pi \sqrt{\epsilon}}{v|\log |u||} \left(2\sqrt{-1} \text{Im}(-dv + \Gamma du)\right) = \frac{\pi \sqrt{\epsilon}}{v|\log |u||} (\text{Im}(dv - \Gamma du)).
\]

Differentiating (2.20) yields

\[
dv = dv_1 + \left( \frac{v}{\pi \sqrt{-1}} \log u \right) dv_2 + \left( v_2 \frac{v}{\pi \sqrt{-1} u} \right) du.
\]

(2.28)

Noting that

\[
\text{Im}(v) = \text{Im} \left( v_1 + v_2 \frac{v}{\pi \sqrt{-1}} \log u \right) = -v_2 \frac{v}{\pi} \log |u|,
\]

(2.29)
the term involving $\Gamma$ is given by

$$-\Gamma du = -\frac{1}{\sqrt{-1}} \frac{\text{Im}(v)}{|u| \log |u|} du = \sqrt{-1}v_2 \frac{v}{\pi u} du.$$  \tag{2.30}$$

Consequently,

$$\omega'_{sf,c}(Y, \cdot) = \frac{\pi \sqrt{\varepsilon}}{|v| \log |u|} \text{Im} \left( \frac{v}{\pi \sqrt{-1}} \log(u) dv_2 \right) = \sqrt{\varepsilon} dv_2.$$  \tag{2.31}$$

Therefore, we can choose $H_1 = \varepsilon^{1/2} v_2$.

Next, we compute

$$\Omega'(Y, \cdot) = u^{-2} k(u^2) du \wedge dv \frac{1}{\sqrt{\varepsilon}} \left( \frac{\partial}{\partial v} + \frac{\partial}{\partial \bar{v}}, \cdot \right)$$

$$= -\varepsilon^{-1/2} u^{-2} k(u^2) du = d \left( -\varepsilon^{-1/2} \int u^{-2} k(u^2) du \right),$$

so we can choose

$$H_2 + \sqrt{-1} H_3 = -\varepsilon^{-1/2} \int u^{-2} k(u^2) du,$$  \tag{2.33}$$

where $\int u^{-2} k(u^2) du$ is any $u$-antiderivative of $u^{-2} k(u^2)$.

\begin{proposition}
The semiflat metric is ALG* near infinity, as defined in Definition 1.3.
\end{proposition}

\textbf{Proof.} The function $k$ admits a power series expansion in $\xi = u^2$. It is clear that the higher order terms will yield polynomially decaying terms in the following computation, so we just need to consider the case that $k$ is a constant. Multiplying $\Omega'$ by $e^{\sqrt{-1}\theta}$ corresponds to a hyperkähler rotation of $J, K$, so without loss of generality, we may assume that $k = ik_0$, where $k_0 \in \mathbb{R}_+$. In this case, the moment map is

$$\mu = \left( \varepsilon^{1/2} v_2, \sqrt{-1} \varepsilon^{-1/2} k_0 u^{-1} \right).$$  \tag{2.34}$$

Letting $I$ denote the elliptic complex structure, we next compute

$$g(Y, Y) = \omega'_{sf,c}(Y, IY) = \sqrt{-1} \varepsilon^{-1} \omega'_{sf,c} \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial \bar{u}}, \frac{\partial}{\partial v} - \frac{\partial}{\partial \bar{v}} \right)$$

$$= -2 \sqrt{-1} \varepsilon^{-1} \omega'_{sf,c} \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial \bar{v}} \right) = -2 \sqrt{-1} \varepsilon^{-1} \sqrt{\frac{1}{2}} \frac{\pi \varepsilon}{v|\log |u||} = \frac{\pi}{v|\log |u||}.$$  \tag{2.35}$$

Next, define

$$V \equiv g(Y, Y)^{-1} = \frac{v}{\pi} |\log |u||,$$  \tag{2.36}$$

$$\alpha_0(Z) \equiv g(Y, Z).$$  \tag{2.37}$$
We compute that

$$K^*\alpha_0(Z) = \alpha_0(KZ) = g(Y, KZ) = -\text{Im} \Omega'(Y, Z) = \text{Im} \left\{ d \left( \varepsilon^{-1/2} \int u^{-2} \sqrt{-1}k_0 du \right) \right\}(Z),$$

(2.38)

$$J^*\alpha_0(Z) = \alpha_0(JZ) = g(Y, JZ) = -\text{Re} \Omega'(Y, Z) = \text{Re} \left\{ d \left( \varepsilon^{-1/2} \int u^{-2} \sqrt{-1}k_0 du \right) \right\}(Z),$$

(2.39)

$$I^*\alpha_0(Z) = \alpha_0(IZ) = g(Y, IZ) = -\omega_{sf,\varepsilon}'(Y, Z) = -\varepsilon^{1/2}d\nu_2(Z),$$

(2.40)

so we have

$$\alpha_1 \equiv \text{Re} \left( \varepsilon^{-1/2}k_0u^{-2}du \right) = K^*\alpha_0,$$

(2.41)

$$\alpha_2 \equiv \text{Im} \left( \varepsilon^{-1/2}k_0u^{-2}du \right) = -J^*\alpha_0,$$

(2.42)

$$\alpha_3 \equiv -\varepsilon^{1/2}d\nu_2 = I^*\alpha_0.$$  

(2.43)

Then $V^{1/2}\alpha_1, V^{1/2}\alpha_2, V^{1/2}\alpha_3, V^{1/2}\alpha_0$ form the dual basis of an orthonormal basis. Define

$$\Theta \equiv V\alpha_0.$$  

(2.44)

Then the hyperkähler metric is

$$g = V(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + V^{-1}\Theta^2,$$

(2.45)

and the Kähler forms are

$$\omega_I = V\alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \Theta,$$

(2.46)

$$\omega_J = V\alpha_1 \wedge \alpha_3 - \alpha_2 \wedge \Theta,$$

(2.47)

$$\omega_K = \alpha_1 \wedge \Theta + V\alpha_2 \wedge \alpha_3.$$  

(2.48)

We compute that

$$\alpha_0(Z) = \omega_{sf,\varepsilon}'(Y, IZ) = -\omega_{sf,\varepsilon}'(IY, Z) = -\sqrt{-1}\varepsilon^{-1/2}\omega_{sf,\varepsilon}' \left( \frac{\partial}{\partial \nu} - \frac{\partial}{\partial \nu}, Z \right).$$

(2.49)

Using (2.18), this yields

$$\alpha_0 = \frac{\pi \varepsilon^{1/2}}{v |\log |u||} \text{Re}(d\nu - \Gamma du).$$

(2.50)

Using (2.28) and (2.36), we arrive at

$$\Theta = V\alpha_0 = \varepsilon^{1/2} \left( d\nu_1 + \frac{v}{\pi} \text{arg}(u)d\nu_2 \right).$$

(2.51)
We also compute that
\[ \alpha_1^2 + \alpha_2^2 = \frac{|k_0|^2}{2\epsilon} |u|^{-4}(du \otimes d\bar{u} + d\bar{u} \otimes du), \] (2.52)
so we have
\[ g = V \left( \frac{|k_0|^2}{2\epsilon} |u|^{-4}(du \otimes d\bar{u} + d\bar{u} \otimes du) + \epsilon dv_2^2 \right) + V^{-1} \epsilon \left( dv_1 + \frac{\nu}{\pi} \arg(u) dv_2 \right)^2 \] (2.53)
\[ = V \left( \frac{|k_0|^2}{2\epsilon} |u|^{-4}(du \otimes d\bar{u} + d\bar{u} \otimes du) + \epsilon dv_2^2 \right) + \frac{\nu^2}{\pi^2} V^{-1} \epsilon \left( \frac{\pi}{\nu} dv_1 + \arg(u) dv_2 \right)^2. \] (2.54)

We consider the metric
\[ \tilde{g} = 4\pi^2 \epsilon^{-1} g = V \left( 4\pi^2 \frac{|k_0|^2}{2\epsilon^2} |u|^{-4}(du \otimes d\bar{u} + d\bar{u} \otimes du) + 4\pi^2 dv_2^2 \right) \] (2.55)
\[ + \frac{\nu^2}{\pi^2} V^{-1} \left( 2\pi^2 \frac{d(v_1)}{\nu} + \arg(u) d(2\pi v_2) \right)^2. \]

We next show that this is isometric to the model metric given in Definition 1.3. Observe that
\[ 4\pi^2 \frac{|k_0|^2}{2\epsilon^2} |u|^{-4}(du \otimes d\bar{u} + d\bar{u} \otimes du) \] (2.56)
is the Euclidean metric on \( \mathbb{R}^2 \) in the coordinates \( \zeta = 2\pi \sqrt{-1} k_0 \epsilon^{-1} u^{-1} = x + \sqrt{-1} y \), so we define coordinates
\[ r = |\zeta| = 2\pi k_0 \epsilon^{-1} |u|^{-1}, \] (2.57)
\[ \theta_1 = \arg(\zeta) = -\arg(u) + \pi, \] (2.58)
\[ \theta_2 = -2\pi v_2, \] (2.59)
\[ \theta_3 = \frac{2\pi^2}{\nu} v_1. \] (2.60)

Then, for \( r \) sufficiently large,
\[ V = \frac{\nu}{\pi} |\log |u|| = \frac{\nu}{\pi} (\log r - \log(2\pi k_0) + \log \epsilon) \equiv \frac{\nu}{\pi} \log r + \kappa_0, \] (2.61)
and the rescaled metric \( \tilde{g} \) is then
\[ \tilde{g} = V (dr^2 + r^2 d\theta_1^2 + d\theta_2^2) + \frac{\nu^2}{\pi^2} V^{-1} (d\theta_3 + (\theta_1 - \pi)d\theta_2)^2. \] (2.62)

The mapping
\[ \bar{\theta}_3 = \theta_3 + \theta_1 \theta_2 - \pi \theta_2 \] (2.63)
defines a coordinate change $H$ satisfying
\begin{equation}
H^* \Theta = d\tilde{\Theta}_3 - \partial_2 d\tilde{\Theta}_1 \equiv \Theta, \tag{2.64}
\end{equation}
so we have
\begin{equation}
H^* g = H^*(4\pi^2 \epsilon^{-1} g) = V(dr^2 + r^2 d\tilde{\Theta}_1^2 + d\tilde{\Theta}_2^2) + \frac{\nu^2}{\pi^2} V^{-1} \Theta^2, \tag{2.65}
\end{equation}
\begin{equation}
H^* \omega_I = H^*(4\pi^2 \epsilon^{-1} \omega_I) = Vdx \wedge dy + d\tilde{\Theta}_2 \wedge \Theta, \tag{2.66}
\end{equation}
\begin{equation}
H^* \omega_J = H^*(4\pi^2 \epsilon^{-1} \omega_J) = Vdx \wedge d\tilde{\Theta}_2 - dy \wedge \Theta, \tag{2.67}
\end{equation}
\begin{equation}
H^* \omega_K = H^*(4\pi^2 \epsilon^{-1} \omega_K) = dx \wedge \Theta + Vdy \wedge d\tilde{\Theta}_2, \tag{2.68}
\end{equation}
which are exactly (2.6), (2.8), (2.9), and (2.10).

Note that if we reverse the above construction, we see that the leading terms of any ALG* gravitational instanton admit a semiflat structure.

Remark 2.4. It is easy to see from the above proof that the parameters $\epsilon, ik_0$ in the semiflat metric are related to the parameters $\kappa_0, L$ in the Gibbons–Hawking construction by
\begin{equation}
4\pi L^2 = \epsilon, \quad \kappa_0 = \frac{\nu}{2\pi} \log \left( \frac{\epsilon}{2\pi k_0} \right). \tag{2.69}
\end{equation}

2.3 ALG model space

In the ALG case, we have the following definition of the model space.

Definition 2.5 (Standard ALG model). Let $\beta \in (0,1], \tau \in \mathbb{H} \equiv \{ \tau \in \mathbb{C} | \text{Im} \tau > 0 \}$ be the parameters in Table 1, and $L \in \mathbb{R}^+$. Let $C_{\beta,\tau,L}$ be the manifold obtained by identifying $(\mathcal{U}, \mathcal{V})$ with $(e^{\sqrt{-1}2\pi \beta} \mathcal{U}, e^{-\sqrt{-1}2\pi \beta} \mathcal{V})$ in the space
\begin{equation}
\{(\mathcal{U}, \mathcal{V}) | \arg \mathcal{U} \in [0, 2\pi \beta] \} \subset (\mathbb{C} \times \mathbb{C})/(\mathbb{Z} \oplus \mathbb{Z}), \tag{2.70}
\end{equation}
where $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{C} \times \mathbb{C}$ by
\begin{equation}
(m, n) \cdot (\mathcal{U}, \mathcal{V}) = (\mathcal{U}, \mathcal{V} + (m + n\tau) \cdot L), \quad (m, n) \in \mathbb{Z} \oplus \mathbb{Z}. \tag{2.71}
\end{equation}
Define
\begin{equation}
C_{\beta,\tau,L}(R) \equiv \{ |\mathcal{U}| > R \} \subset C_{\beta,\tau,L}. \tag{2.72}
\end{equation}
Then there is a flat hyperkähler metric
\begin{equation}
g_C = \frac{1}{2}(d\mathcal{U} \otimes d\mathcal{U} + d\tilde{\mathcal{U}} \otimes d\mathcal{U} + d\mathcal{V} \otimes d\mathcal{V} + d\tilde{\mathcal{V}} \otimes d\mathcal{V}) \tag{2.73}
\end{equation}
on $C_{\beta,\tau,L}(R)$ with Kähler form

$$\omega_1^C = \frac{\sqrt{-1}}{2} (d\mathcal{U} \wedge d\bar{\mathcal{U}} + d\mathcal{V} \wedge d\bar{\mathcal{V}}),$$

and holomorphic 2-form

$$\Omega^C = \omega_2^C + \sqrt{-1}\omega_3^C = d\mathcal{U} \wedge d\mathcal{V}.$$  \hfill (2.75)

Each flat space $(C_{\beta,\tau,L}(R), g^C, \omega^C)$ given as the above is called a standard ALG model.

Remark 2.6. We denote the complex structures acting on tangent vectors associated to $\omega_1^C, \omega_2^C, \omega_3^C$ by $I_C, J_C, K_C$, respectively. Let $I_C^*, J_C^*, K_C^*$ be the dual mappings acting on 1-forms. Then, we have

$$I_C^*(d\mathcal{U}) = \sqrt{-1}d\mathcal{U}, \quad I_C^*(d\mathcal{V}) = \sqrt{-1}d\mathcal{V},$$

$$J_C^*(d\mathcal{U}) = -d\bar{\mathcal{V}}, \quad J_C^*(d\mathcal{V}) = d\bar{\mathcal{U}},$$

$$K_C^*(d\mathcal{U}) = -\sqrt{-1}d\bar{\mathcal{V}}, \quad K_C^*(d\mathcal{V}) = \sqrt{-1}d\bar{\mathcal{U}}.$$ \hfill (2.78)

These will be used below in Section 5.

Remark 2.7. The model space has the following properties. Letting $r = |\mathcal{U}|$, the cross-section $r = r_0$ is a flat 3-manifold. There is a holomorphic map $z_C : C_{\beta,\tau,L}(R) \to \mathbb{C}$ defined as $z_C = \mathcal{U}^{\frac{1}{2}}$, with torus fibers that have area $L^2 \cdot \text{Im} \tau$. The infinite end of the model space compactifies complex analytically by adding a singular fiber of the specified type in the first row of Table 1.

3 | CLASSIFICATION OF ALG* GRAVITATIONAL INSTANTONS

In this section, we will prove Theorem A. In Subsection 3.1, we will study the compactification of ALG* gravitational instantons. Then, in Subsection 3.2, we will classify ALG* gravitational instantons.

3.1 | Compactification of ALG* gravitational instantons

Proposition 3.1. Using the complex structure $I$, any ALG* gravitational instanton $(X, g, I, J, K)$ of order 2 must be biholomorphic to an elliptic surface $S$ minus an $I^*_v$-fiber.

Proof. Since $(re^{\sqrt{-1}\theta_1})^2$ is invariant under $t$, by [10, Theorem 1.3], there exists a harmonic function $z$ on $X$ such that $\Phi^*z$ is asymptotic to $(re^{\sqrt{-1}\theta_1})^2$ on $M_2\gamma(R)$. Since

$$I_M^* d((re^{\sqrt{-1}\theta_1})^2) = \sqrt{-1}d((re^{\sqrt{-1}\theta_1})^2),$$

(3.1)
we have
\[ \eta \equiv I^*_X dz - \sqrt{-1} dz = O(s^{-1+\mu}) \] (3.2)
for all \( \mu > 0 \), and \( \Delta_X \eta = 0 \). By Bochner’s formula and the maximum principle, \( \eta = 0 \) on \( X \), which implies that \( z \) is holomorphic on \( X \).

Since \( dz - d((\Phi^{-1})^*(re^{\sqrt{-1}\theta_1})^2) = O(s^{-1+\mu}) \), the fiber at \( z \) on \( X \) for \( |z| \) large is diffeomorphic to the fiber at \( (re^{\sqrt{-1}\theta_1})^2 \) on \( \mathfrak{M}_{2\nu}(R) \), which is the same as the fiber at \( re^{\sqrt{-1}\theta_1} \) on \( \hat{\mathfrak{M}}_{2\nu}(R) \) for \( r \) large. This implies that the fibration \( z : X \to \mathbb{C} \) is elliptic.

We next prove the existence of a holomorphic section \( \sigma_1 \) defined on a neighborhood of infinity using a similar argument as in [6, Section 4.7]. First, we define a \( \partial \)-operator acting on sections.

Above we have proved that there is a holomorphic function \( z : X \to \mathbb{C} \). Let \( U = \mathbb{C} \setminus \overline{B_{R^2}(0)} \) and let \( \sigma : U \to X \) be any smooth section, that is, \( \sigma \circ \sigma = \text{Id}_U \). The differential of \( \sigma : U \to X \) is
\[ \sigma_* : TU \to TX, \] (3.3)
where these are the real tangent bundles. If we complexify, then
\[ \sigma_* \otimes \mathbb{C} : T^{1,0}(U) \oplus T^{0,1}(U) \to T^{1,0}(X) \oplus T^{0,1}(X). \] (3.4)

Letting \( \Pi : T_cX \to T^{1,0}(X) \) denote the projection mapping, we have a mapping
\[ \Pi \circ (\sigma_* \otimes \mathbb{C}) : T^{0,1}(U) \to T^{1,0}(X). \] (3.5)

Given any point \( p \in U \), \( z \) gives a local holomorphic coordinate on the base, and we choose a local affine coordinate \( w \) on the fiber such that \( w = 0 \) is a locally defined holomorphic section, and on each fiber, \( w \) can be viewed as the coordinate on \( \mathbb{C} \) if we write the fiber as \( \mathbb{C}/\mathbb{Z}^2 \). Then locally, we can write \( \sigma : z \mapsto (z, s(z, \bar{z})) \) for a function \( z : U \to \mathbb{C} \), and we see that
\[ \Pi \circ (\sigma_* \otimes \mathbb{C}) \left( \frac{\partial}{\partial \bar{z}} \right) = \frac{\partial s}{\partial \bar{z}}(z, \bar{z}) \frac{\partial}{\partial w}. \] (3.6)

In other words, we have
\[ \Pi \circ (\sigma_* \otimes \mathbb{C}) : T^{0,1}(U) \to T^{1,0}(F), \] (3.7)
where \( T^{1,0}(F) \) is the \( (1,0) \) part of the vertical tangent bundle. This defines an operator from smooth sections over \( U \)
\[ \bar{\partial} : \Gamma(U) \to \Gamma(\Lambda^{0,1}(U) \otimes \Lambda^{-1}) \] (3.8)
such that \( \bar{\partial} \sigma = 0 \) if and only if \( \sigma \) is a holomorphic section, where \( \Lambda^{-1} \equiv \sigma^* T^{1,0}(F) \).

In coordinates, since each fiber is a compact Riemann surface, we have that the holomorphic 2-form is \( \Omega = f(z) dz \wedge dw \), where \( f : U \to \mathbb{C} \) is nonvanishing. By plugging in the \( T^{1,0}(F) \) component to \( \Omega \), we can define
\[ \Omega \circ \bar{\partial} : \Gamma(U) \to C^\infty(U, \mathbb{C}), \] (3.9)
since \( \Lambda^{1,0}(U) \otimes \Lambda^{0,1}(U) \cong \Lambda^{1,1}(U) \) is a trivial bundle. In coordinates, this mapping is given by

\[
\Omega \circ \bar{\sigma} = f(z) \frac{\partial s}{\partial \bar{z}}(z, \bar{z}).
\]  

(3.10)

Next, recall that on the model space, the zero section of the semiflat structure is holomorphic. Since \( z \) is asymptotic to \((\Phi^{-1})^* (r e^{-1/101})^2\), we see that the zero section intersects each far-enough fiber of \( z \) at exactly one point. This defines an “almost holomorphic” section \( \sigma_0 : U \to X \) near infinity. Locally, we can write \( \sigma_0(z) = (z, s_0(z, \bar{z})) \), and we define

\[
e(z, \bar{z}) = f(z) \frac{\partial s_0}{\partial \bar{z}}(z, \bar{z}).
\]  

(3.11)

Since \( U \) is biholomorphic to a punctured disc \( \Delta^* \) and \( H^1(\Delta^*, \Theta) = 0 \), there exists \( E : U \to \mathbb{C} \) solving

\[
\frac{\partial}{\partial \bar{z}} E(z, \bar{z}) = e(z, \bar{z}).
\]  

(3.12)

We then define

\[
s_1(z, \bar{z}) = s_0(z, \bar{z}) - \frac{E(z, \bar{z})}{f(z)}, \tag{3.13}
\]

and use this to define a section \( \sigma_1 \) over \( U \). We then have

\[
\frac{\partial}{\partial \bar{z}} (s_0(z, \bar{z}) - E(z, \bar{z})/f(z)) = \frac{\partial}{\partial \bar{z}} (s_0(z, \bar{z})) - \frac{\partial}{\partial \bar{z}} (E(z, \bar{z}))/f(z)
\]

\[
= \frac{\partial}{\partial \bar{z}} (s_0(z, \bar{z})) - (1/f(z)) f(z) \frac{\partial}{\partial \bar{z}} (s_0(z, \bar{z})) = 0.
\]  

(3.14)

Consequently, \( \sigma_1 \) is a holomorphic section over \( U \).

We then use the following fact from [18]: any elliptic surface with a section defined over a punctured disc \( \Delta^* \) is biholomorphic to \( (\Delta^* \times \mathbb{C})/(\mathbb{Z} \tau_1 \oplus \mathbb{Z} \tau_2) \), where \( \tau_1, \tau_2 \) are the periods. Recalling Remark 2.1, there exists a local coordinate \( u \) on the base such that the periods of the model space are given by \( \tau_{1,\text{model}}(u), \tau_{2,\text{model}}(u) \). On the \( ALG^* \) manifold, \( \tau_1 \) and \( \tau_2 \) are asymptotic to the model periods in \( ALG^* \) coordinates. Then, by a holomorphic coordinate transformation of the base, we can find coordinates \( \tilde{u} \) asymptotic to \( z^{-1} \) so that \( \tau_1(\tilde{u}) = \tau_{1,\text{model}}(\tilde{u}) \) and \( \tau_2(\tilde{u}) = \tau_{2,\text{model}}(\tilde{u}) \). This shows that the \( ALG^* \) manifold is biholomorphic to the model space near infinity. Finally, the argument of Case (v), Sub-case (2) in [18, Section 8] shows that we can add a central fiber of type \( I^*_\nu \) to compactify and obtain \( S \).

\[ \square \]

**Proposition 3.2.** *Using the complex structure \( I \), any \( ALG^* \) gravitational instanton \( (X, g, I, J, K) \) of order 2 is biholomorphic to a rational elliptic surface \( S \) with global section, minus an \( I^*_\nu \) fiber \( D \), with \( \nu \in \{1, 2, 3, 4\} \).*

**Proof.** We first show that the elliptic surface constructed in Proposition 3.1 is a rational elliptic surface. To see this, it follows from [10, Corollary 1.6] that the first betti number \( b_1(X) = 0 \). Since \( D \) is a configuration of 2-spheres corresponding to an extended Dynkin diagram of dihedral type, we have \( b_1(D) = 0 \). Let \( N \) be an open neighborhood of \( D \) which deformation retracts onto \( D \) and
such that $X \cap N$ is connected. The following portion of the reduced Mayer–Vietoris sequence

$$0 \longrightarrow H^1(S; \mathbb{R}) \longrightarrow H^1(X; \mathbb{R}) \oplus H^1(N; \mathbb{R})$$

(3.15)

implies that $b_1(S) = 0$. Next, it is straightforward to see that $\Omega = \omega_2 + \sqrt{-1}\omega_3$ defines a meromorphic 2-form on $S$ with a pole of order 1 along the divisor $D$ at infinity. Consequently, $\text{div}(\Omega) = -D$, which implies that $-K = [D]$. Similar to the proof of [7, Theorem 3.3], by Kodaira’s classification of complex surfaces and the Castelnuovo theorem, $S$ must be biholomorphic to $\mathbb{P}^2$ blown up at nine points, so there exists an exceptional curve $E$ satisfying $E^2 = -1$. By the adjunction formula, $E \cdot K = -1$, so $E \cdot D = 1$. Let $D'$ be any singular fiber in the interior of $X$. Then, $D'$ is homologous to $D$ in $S$. If $D'$ had multiplicity, say $D' = mC$, then $E \cdot C = 1/m$ which implies that $m = 1$, so that there are no multiple fibers. This implies that there exists a global holomorphic section of the elliptic fibration; see, for example, [16]. Finally, by the classification of singular fibers on rational elliptic surfaces due to Miranda–Persson, $\nu \in \{1, 2, 3, 4\}$; see [23].

3.2 Classification of $\text{ALG}^*$ gravitational instantons

In this subsection, we complete the proof of Theorem A. Recall that we have proved in Proposition 3.1 that the holomorphic function $z$ provides an elliptic fibration with associated holomorphic section $\sigma_1$ over $\{R^2 < |z| < \infty\}$ that provides a compactification. Choose an arbitrary Kähler form $\omega_S$ on the compactification such that the area of each fiber is the same as $\omega_{\text{ALG}^*}$. Denote this area by $\varepsilon$.

**Proposition 3.3.** There exist a holomorphic section $\sigma_2$ and a smooth real-valued function $\varphi_1$, both defined over $\{\infty > |z| > R^2\} \subset X$, such that

$$\omega_S = \omega_{sf,\varepsilon}[\sigma_2] + \sqrt{-1}\delta\bar{\delta}\varphi_1$$

(3.16)

on $\{\infty > |z| > R^2\}$, where $\omega_{sf,\varepsilon} [\sigma_2]$ is Greene–Shapere–Vafa–Yau’s semiflat metric [13] with $\sigma_2$ as the zero section, $\Omega$ as the holomorphic 2-form, and area of each fiber given by $\varepsilon$.

**Proof.** This is proved in Claim 1 on page 382 of [17], which is based on [14, Lemma 4.3].

Similarly, there exists a holomorphic section $\sigma_3$ and a smooth function $\varphi_2$, both defined over $\{\infty > |z| > R^2\}$, such that

$$\omega_{\text{ALG}^*} = \omega_{sf,\varepsilon} [\sigma_3] + \sqrt{-1}\delta\bar{\delta}\varphi_2$$

(3.17)

on $\{\infty > |z| > R^2\}$. Let $T$ be the translation by adding $\sigma_2 - \sigma_3$ on each fiber, which satisfies $T^* \Omega = \Omega$ because $\sigma_2$ and $\sigma_3$ are both holomorphic. Then

$$\omega_{\text{ALG}^*} = T^* (\omega_S - \sqrt{-1}\delta\bar{\delta}\varphi_1) + \sqrt{-1}\delta\bar{\delta}\varphi_2.$$  

(3.18)

If we use the local holomorphic section $\sigma_1 - \sigma_2 + \sigma_3$ to compactify $X$ into a rational elliptic surface $S'$, then

$$T^* (\omega_S) = \omega_{\text{ALG}^*} + \sqrt{-1}\delta\bar{\delta}\varphi_3.$$

(3.19)
can be extended to a smooth Kähler form on \( \{ \infty > |z| > R^2 \} \subset S' \), where

\[
\varphi_3 = T^* \varphi_1 - \varphi_2
\]  

(3.20)
is a smooth function. To obtain a Kähler form on \( S' \), we choose a cut-off function \( \chi \) on \( \mathbb{R} \) that is 1 on \((2, \infty)\) and is 0 on \((-\infty, 1)\). We also choose a smooth nonnegative function \( f \) on \( \mathbb{C} \) that is 1 on \( R^2 \leq |z| \leq 4R^2 \) and is 0 on \( |z| < R^2/4 \) and \( |z| > 16R^2 \). We can find a smooth real-valued function \( \psi \) on \( \mathbb{C} \) such that

\[
\sqrt{-1} \partial \bar{\partial} \psi = \sqrt{-1} f dz \wedge d\bar{z}.
\]  

(3.21)

Then,

\[
\omega = \omega_{\text{ALG}} + \sqrt{-1} \partial \bar{\partial} \left( \chi \left( \frac{|z|}{R} \right) \varphi_3 + C \cdot \psi(z) \right)
\]  

(3.22)

will be the required smooth Kähler form on \( S' \) for \( C \) sufficiently large. This finishes the proof of Part (1).

For Part (2), let \( z : S \to \mathbb{P}^1 \) be the elliptic projection such that \( D \) lies over \([0,1]\). Then we can view \( z : X \to \mathbb{C} \), where \( X = S \setminus D \). By Remark 2.4, there exists \( c > 0 \) and \( L > 0 \) such that for any holomorphic section \( \sigma \) defined over \( \{ R^2 < |z| < \infty \} \), the Greene–Shapere–Vafa–Yau’s semiflat metric \( \omega_{\text{SF},[\sigma]}[13] \) using \( \sigma \) as the zero section, \( c \cdot \Omega \) as the holomorphic 2-form, and using the same area \( \epsilon \) on each fiber as \( \omega \), has an ALG" end with parameters \( \nu, \kappa_0, \) and \( L \). The function \( \Phi^\nu z \) is asymptotic to \( c'(re^{\sqrt{-1}0})^2 \), where \( c' \in \mathbb{C} \setminus \{0\} \). Similar to Proposition 3.3, there exists a holomorphic section \( \sigma \) and a smooth function \( \varphi_4 \), both defined over \( \{ R^2 < |z| < \infty \} \), such that

\[
\omega = \omega_{\text{SF},[\sigma]} + \sqrt{-1} \partial \bar{\partial} \varphi_4.
\]  

(3.23)

Then, we use the following modification as in [7, Theorem 4.3] instead of the original construction in [17]. Choose \( t \) sufficiently large so that

\[
\omega_t \equiv \omega - \sqrt{-1} \partial \bar{\partial} \left( \chi \left( \frac{|z|}{R} \right) \varphi_4 - t \cdot \psi(z) \right)
\]  

(3.24)
is positive, and the integral

\[
\int_X \left( \omega_t^2 - \frac{c^2}{2} \Omega \wedge \bar{\Omega} \right)
\]  

(3.25)
is positive. The \( dz \wedge d\bar{z} \) component of \( \omega_t \) is asymptotic to

\[
\frac{\sqrt{-1}}{8|c'|} \left( \kappa_0 + \frac{\nu}{2\pi} \log \frac{|z|}{|c'|} \right) |z|^{-1} dz \wedge d\bar{z}.
\]  

(3.26)

If we fix a large enough \( R \), then

\[
\omega_{t,t'} \equiv \omega_t - \chi \left( \frac{|z|}{R} - 100 \right) \left( 1 - \chi \left( \frac{|z|}{R} - t' \right) \right) \frac{\sqrt{-1}}{10000|c'|} \left( \kappa_0 + \frac{\nu}{2\pi} \log \frac{|z|}{|c'|} \right) |z|^{-1} dz \wedge d\bar{z}
\]  

(3.27)
is still positive, but

$$\int_X \left( \omega_{t,t'}^2 - \frac{c^2}{2} \Omega \wedge \hat{\Omega} \right) \to -\infty$$

(3.28)
as \( t' \to \infty \). Since this integral is positive for \( t' \) sufficiently small, we see that there exists a value \( t' \) such that the integral becomes 0. For this value of \( t' \), there exists a smooth bounded function \( \varphi_5 : X \to \mathbb{R} \) such that

$$\omega_{\text{ALG}^*} \equiv \omega_{t,t'} + \sqrt{-1} \partial \bar{\partial} \varphi_5$$

(3.29)
is hyperkähler; see [27, Theorem 1.1]. By Hein’s estimates for higher order derivatives of \( \varphi_5 \) [17, Proposition 2.9], \( \omega_{\text{ALG}^*} \) is asymptotic to \( \omega_{t,t'} \), which is the same as \( \omega_{sf,\varepsilon} [\sigma] \) near infinity. By Subsection 2.2, the hyperkähler structure is \( \text{ALG}^* \) according to Definition 1.3. Finally, the order can be improved to 2 by [10, Theorem 1.10].

## 4  |  TOPOLOGY AND COORDINATES AT INFINITY

In this section, we will determine the topology of \( \text{ALG} \) and \( \text{ALG}^* \) gravitational instantons, and also address the issue of existence of “uniform” coordinates at infinity.

### 4.1 | Topology of gravitational instantons

In this subsection, we will prove that any two gravitational instantons of type \( \text{ALG} \) or \( \text{ALG}^* \) with the same \( \beta \) or \( \nu \) are diffeomorphic, by a diffeomorphism with nice properties. Recall that a rational elliptic surface admits a section if and only if there are no multiple fibers; see, for example, [16].

**Theorem 4.1.** Let \( z : S \to \mathbb{P}^1 \) and \( z' : S' \to \mathbb{P}^1 \) be rational elliptic surfaces without multiple fibers. Let \( \Sigma, \Sigma' \) be sections of \( z, z' \), and assume that \( D, D' \) are singular fibers of the same Kodaira type \( (I_0^*, \ II^*, \ II, \ III^*, \ III, \ IV^*, \ IV, \ I_1^*, \ I_2^*, \ I_3^*, \ \text{or} \ I_4^*) \) in \( S, S' \), respectively, both over the point \( p_{\infty} \equiv [0,1] \in \mathbb{P}^1 \). Then there exists a diffeomorphism

$$\Psi : S \setminus D \to S' \setminus D'$$

(4.1)
such that near infinity, \( z = z' \circ \Psi \), the section \( \Sigma \) is mapped to \( \Sigma' \), and any point \( w_1 \tau_1 + w_2 \tau_2 \) on the fiber is mapped to \( w_1 \tau'_1 + w_2 \tau'_2 \) on the fiber, where \( \Sigma \) and \( \Sigma' \) are viewed as the zero section \( w_1 = w_2 = 0, \tau_1, \tau_2 \) are the periods of the fiber on \( S \), and \( \tau'_1, \tau'_2 \) are the periods of the fiber on \( S' \).

To prove this, we will use the Weierstraß representation of a rational elliptic surface, which we describe next. Let \( z : S \to \mathbb{P}^1 \) be any rational elliptic surface with a section \( \Sigma \subset S \). The section hits some multiplicity 1 component of each singular fiber. The components in a singular fiber that do not hit this component (if there are any) form an ADE-configuration of \((-2)\)-curves, which can be blown down to a surface \( \hat{S} \) with only rational double point (RDP) singularities.
### Table 2

| Type | $I_0$ | $I_N$ | $I_0^*$ | $I_N^*$ | II | III | IV | IV* | III* | II* |
|------|-------|-------|---------|---------|----|-----|----|-----|------|-----|
| $e_p$ | 0     | $N$   | 6       | $N+6$   | 2  | 3   | 4  | 8   | 9    | 10  |
| $a_p$ | $a \geq 0$ | 0     | $a \geq 2$ | 2     | $a \geq 1$ | 1  | $a \geq 2$ | 3  | $a \geq 3$ | 3   |
| $b_p$ | $b \geq 0$ | 0     | $b \geq 3$ | 3     | 1   | $b \geq 2$ | 2  | 4   | $b \geq 5$ | 5   |
| $\delta_p$ | 0     | $N$   | 6       | $N+6$   | 2  | 3   | 4  | 8   | 9    | 10  |

**Theorem 4.2** Lecture II of [22]. There exist a section $A$ of $\mathcal{O}_{p,1}(4)$ and $B$ of $\mathcal{O}_{p,1}(6)$ such that the surface $\hat{S}$ is biholomorphic to

$$S_{(A,B)} = \{Z_2^2Z_0 = Z_1^3 + AZ_1Z_0^2 + BZ_0^3\} \subset \mathbb{P}(\mathcal{O}_{p,1} \oplus \mathcal{O}_{p,1}(2) \oplus \mathcal{O}_{p,1}(3))$$

(4.2)

by a biholomorphism that takes $z$ onto $\pi_{(A,B)} : S_{(A,B)} \to \mathbb{P}^1$, which is the restriction to $S_{(A,B)}$ of the projection

$$\pi : \mathbb{P}(\mathcal{O}_{p,1} \oplus \mathcal{O}_{p,1}(2) \oplus \mathcal{O}_{p,1}(3)) \to \mathbb{P}^1.$$ 

(4.3)

The biholomorphism can be chosen to map the given section $\Sigma$ onto the zero section $\{Z_1 = 0, Z_0 = 0\}$. Finally, this biholomorphism lifts to a biholomorphism of $S$ with the minimal resolution of $S_{(A,B)}$.

The pair $(A, B)$ is called **Weierstraß data**. The discriminant is $\Delta = 4A^3 + 27B^2$. We let

$$a_p = \text{the order of vanishing of } A \text{ at } p,$$

(4.4)

$$b_p = \text{the order of vanishing of } B \text{ at } p,$$

(4.5)

$$\delta_p = \text{the order of vanishing of } \Delta \text{ at } p,$$

(4.6)

$$e_p = \text{the Euler characteristic of the singular fiber}.$$ 

(4.7)

In [25], Tate showed that the vanishing orders $a_p, b_p, \delta_p$ completely determine the fiber type; see Table 2. Note that we only consider Weierstraß data with the following constraint: there is no point $p \in \mathbb{P}^1$ where $a_p \geq 4$ and $b_p \geq 6$ [22, Proposition III.3.2].

We can assume that the divisor $D = D_\infty$ is over the point $p_\infty \equiv [0, 1] \in \mathbb{P}^1$. We will next write out the constraints on $A, B$ in each case and prove that the space of Weierstraß data with a fixed Kodaira fiber type $D_\infty$ is connected in the following cases that arise in the compactification of ALG and ALG* gravitational instantons.

**Cases of $I_p^*$**: In these cases,

$$A = a_0Z_1^4 + a_1Z_1^3Z_2 + a_2Z_1^2Z_2^2,$$

(4.8)

$$B = b_0Z_1^6 + b_1Z_1^5Z_2 + b_2Z_1^4Z_2^2 + b_3Z_1^3Z_2^3,$$

(4.9)
and

\[
\Delta = (4a_2^3 + 27b_3^2)z_1^6z_2^6 + (12a_1a_2^2 + 54b_2b_3)z_1^7z_2^5 \\
+ (12a_1^2a_2 + 12a_0a_2^2 + 54b_1b_3 + 27b_2^2)z_1^8z_2^4 \\
+ (4a_1^3 + 24a_0a_1a_2 + 54b_0b_3 + 54b_1b_2)z_1^9z_2^3 \\
+ (12a_0^2a_2 + 12a_0a_1^2 + 27b_1^2 + 54b_0b_2)z_1^{10}z_2^2 \\
+ (12a_0^2a_1 + 54b_0b_1)z_1^{11}z_2 + (4a_0^3 + 27b_0^2)z_1^{12}.
\]

(4.10)

In the \(I_0^*\) case, we need to show that the set

\[
\{4a_2^3 + 27b_3^2 \neq 0\} \subset \{(a_0, a_1, a_2, b_0, b_1, b_2, b_3) \in \mathbb{C}^7\}
\]

is connected. This is trivially true because the inequalities do not affect the connectedness.

In the \(I_1^*\) case, the set is

\[
\{a_2 \neq 0, b_3 \neq 0, 4a_2^3 + 27b_3^2 = 0, 12a_1a_2^2 + 54b_2b_3 \neq 0\}.
\]

(4.11)

It is connected because the condition \(4a_2^3 + 27b_3^2 = 0\) defines an elliptic curve in \(\mathbb{C}^2\), which is connected. The inequalities also do not affect the connectedness.

In the \(I_2^*\) case, the set is

\[
\{a_2 \neq 0, b_3 \neq 0, 4a_2^3 + 27b_3^2 = 12a_1a_2^2 + 54b_2b_3 = 0, 12a_0a_2^2 + 12a_0a_1^2 + 54b_1b_3 + 27b_2^2 \neq 0\}.
\]

(4.12)

The condition \(12a_1a_2^2 + 54b_2b_3 = 0\) is the same as

\[
b_2 = \frac{12a_1a_2^2}{54b_3}.
\]

(4.13)

If we ignore the inequalities, we get a graph over the product of the elliptic curve \(4a_2^3 + 27b_3^2 = 0\) with \(\{(a_0, a_1, b_0, b_1) \in \mathbb{C}^4\}\), so this set is also connected.

In the \(I_3^*\) case, the set is

\[
\{a_2 \neq 0, b_3 \neq 0, 4a_2^3 + 27b_3^2 = 12a_1a_2^2 + 54b_2b_3 = 12a_0a_2^2 + 12a_0a_1^2 + 54b_1b_3 + 27b_2^2 = 0, \\
4a_1^3 + 24a_0a_1a_2 + 54b_0b_3 + 54b_1b_2 \neq 0\}.
\]

(4.14)

The condition

\[
12a_1^2a_2 + 12a_0a_2^2 + 54b_1b_3 + 27b_2^2 = 0
\]

(4.15)

is the same as

\[
b_1 = \frac{12a_1^2a_2 + 12a_0a_2^2 + 27b_2^2}{54b_3}.
\]

(4.16)
If we ignore the inequalities, we get a graph of a map from the product of the elliptic curve \(4a_2^3 + 27b_2^2 = 0\) with \(\{(a_0, a_1, b_0) \in \mathbb{C}^3\}\) to \(\{(b_1, b_2) \in \mathbb{C}^2\}\). To define this map, we compute \(b_2\) first and then compute \(b_1\), so this set is also connected.

In the I* case, the set is
\[
\{a_2 \neq 0, b_3 \neq 0, 4a_2^3 + 27b_2^2 = 12a_1a_2^2 + 54b_2b_3 = 12a_1^2a_2 + 12a_0a_2^2 + 54b_1b_3 + 27b_2^2 = 0, \quad 4a_1^2 + 24a_0a_1a_2 + 54b_0b_3 + 54b_1b_2 = 0, \quad 12a_0^2a_2^2 + 12a_0a_2^2 + 27b_2^2 + 54b_0b_2 \neq 0\}.
\] (4.18)

The condition
\[
4a_1^2 + 24a_0a_1a_2 + 54b_0b_3 + 54b_1b_2 = 0
\] (4.19)
is the same as
\[
b_0 = -\frac{4a_1^3 + 24a_0a_1a_2 + 54b_1b_2}{54b_3}
\] (4.20)

If we ignore the inequalities, we get a graph of a map from the product of the elliptic curve \(4a_2^3 + 27b_2^2 = 0\) with \(\{(a_0, a_1) \in \mathbb{C}^2\}\) to \(\{(b_0, b_1, b_2) \in \mathbb{C}^3\}\). To define this map, we compute \(b_2\) first, \(b_1\) second, and then compute \(b_0\), so this set is also connected.

**Case of II:** In this case,
\[
A = a_0z_1^4 + a_1z_1^2z_2 + a_2z_2^2 + a_3z_1z_2^3,
\] (4.21)
\[
B = b_0z_1^6 + b_1z_1^5z_2 + b_2z_1^4z_2^2 + b_3z_1^3z_2^3 + b_4z_1^2z_2^4 + b_5z_1z_2^5,
\] (4.22)
\[
\Delta = 27b_5^2z_1^2z_2^{10} + ...
\] (4.23)

The set
\[
\{b_5 \neq 0, 27b_5^2 \neq 0\} \subset \{(a_0, ..., a_3, b_0, ..., b_5) \in \mathbb{C}^{10}\}
\] (4.24)
is obviously connected.

**Case of III:** In this case,
\[
A = a_0z_1^4 + a_1z_1^2z_2 + a_2z_2^2 + a_3z_1z_2^3,
\] (4.25)
\[
B = b_0z_1^6 + b_1z_1^5z_2 + b_2z_1^4z_2^2 + b_3z_1^3z_2^3 + b_4z_1^2z_2^4,
\] (4.26)
\[
\Delta = 4a_3^3z_1^3z_2^9 + ...
\] (4.27)

The set
\[
\{a_3 \neq 0, 4a_3^3 \neq 0\} \subset \{(a_0, ..., a_3, b_0, ..., b_4) \in \mathbb{C}^9\}
\] (4.28)
is obviously connected.

**Case of IV:** In this case,
\[
A = a_0z_1^4 + a_1z_1^2z_2 + a_2z_2^2z_2^2,
\] (4.29)
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\[ B = b_0 z_1^6 + b_1 z_1^5 z_2 + b_2 z_1^4 z_2^2 + b_3 z_1^3 z_2^3 + b_4 z_1^2 z_2^4, \]  
\[ \Delta = 27 b_4^2 z_1^8 z_2^8 + \ldots \]  
(4.30)

The set
\[ \{b_4 \neq 0, 27 b_4^2 \neq 0\} \subset \{(a_0, \ldots, a_2, b_0, \ldots, b_4) \in \mathbb{C}^8\} \]  
(4.32)
is obviously connected.

**Case of IV**: In this case,
\[ A = a_0 z_1^4 + a_1 z_1^3 z_2, \quad B = b_0 z_1^6 + b_1 z_1^5 z_2 + b_2 z_1^4 z_2^2, \quad \Delta = 27 b_2^2 z_1^8 z_2^4 + \ldots \]  
(4.33)
The set
\[ \{b_2 \neq 0, 27 b_2^2 \neq 0\} \subset \{(a_0, a_1, b_0, b_1, b_2) \in \mathbb{C}^5\} \]  
(4.34)
is obviously connected.

**Case of III**: In this case,
\[ A = a_0 z_1^4 + a_1 z_1^3 z_2, \quad B = b_0 z_1^6 + b_1 z_1^5 z_2, \quad \Delta = 4 a_1^2 z_1^9 z_2^3 + \ldots \]  
(4.35)
The set
\[ \{a_1 \neq 0, 4a_1^3 \neq 0\} \subset \{(a_0, a_1, b_0, b_1) \in \mathbb{C}^4\} \]  
(4.36)
is obviously connected.

**Case of II**: In this case,
\[ A = a_0 z_1^4, \quad B = b_0 z_1^6 + b_1 z_1^5 z_2, \quad \Delta = 27 b_1^2 z_1^6 z_2^2 + \ldots \]  
(4.37)
The set
\[ \{b_1 \neq 0, 27 b_1^2 \neq 0\} \subset \{(a_0, b_0, b_1) \in \mathbb{C}^3\} \]  
(4.38)
is obviously connected.

**Definition 4.3.** The Weierstraß data satisfying the above conditions in each case will be called *allowable*. Allowable Weierstraß data \((A, B)\) are called *good* if \(\Delta\) has no multiple root on \(\mathbb{P}^1 \setminus \{[0, 1]\}\). Otherwise, it is called *bad*.

**Proof of Theorem 4.1.** We first prove the theorem assuming that \(z : S \to \mathbb{P}^1\) and \(z' : S' \to \mathbb{P}^1\) both correspond to good Weierstraß data. Note that good Weierstraß data are equivalent to all singular fibers being of type \(I_1\) (except for \(D_{\infty}\)). Thus, the Weierstraß models are smooth away from a single RDP singularity over \(P_{\infty}\). The good property is characterized by nonvanishing of the resultant \(R(\Delta, \Delta')\) on \(\mathbb{P}^1 \setminus \{[0, 1]\}\). This resultant is not identically zero on the space of allowable Weierstraß data, since otherwise it would mean that there is no rational elliptic surface with fiber \(D_{\infty}\) and all other fibers of type \(I_1\). This is a contradiction because such surfaces definitely exist; see [23]. Consequently, the bad condition defines a proper subvariety of the space of allowable Weierstraß data,
so does not affect the connectedness. Consequently, there is a holomorphic family corresponding to good Weierstraß data

\[
S_{(A_i, B_i)} \to S \to B
\]  
(4.39)

over a Zariski-open base space \( B \). To prove the theorem, clearly we can reduce to the case that the base \( B = B_2(0) \subset \mathbb{C} \) is a disc, \( S_{(A, B)} = \pi^{-1}(0) \), and \( S_{(A', B')} = \pi^{-1}(1) \). Since the \( D_\infty \) fibers of the Weierstraß models are all the same cuspidal cubic, by taking a fixed sequence of blow-ups of the ambient space \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(2) + \mathcal{O}_{\mathbb{P}^1}(3)) \) at the singular point of the cuspidal cubic, and then taking the corresponding sequences of proper transforms of \( S_{(A_i, B_i)} \), we can resolve this family to a family with smooth total space \( \tilde{S} \), with compact fiber over \( t \) given by \( \tilde{S}_{(A_i, B_i)} \), which is the minimal resolution of \( S_{(A_i, B_i)} \). Let \( z_t : \tilde{S}_{(A_i, B_i)} \to \mathbb{P}^1 \) denote the elliptic projection. By choosing some Riemannian metric on \( \tilde{S} \), we can therefore find a vector field \( Y_1 \) on \( \tilde{S} \), defined on a subset of \( \tilde{S} \) containing a neighborhood of \( \tilde{D}_\infty \) of each fiber of the family, whose vertical part vanishes on \( \tilde{D}_\infty \) and on each section \( \Sigma_0 \), and whose horizontal part projects to \( \partial/\partial t \), and which satisfies \( (z_t)_* (Y_1) = 0 \) for each \( t \in B \). We can also find a vector field \( Y_2 \) on \( \tilde{S} \), defined on a subset of \( \tilde{S} \) containing a large compact subset of each fiber of the family whose horizontal part projects to \( \partial/\partial t \). Choosing a cutoff function \( \chi \) that is 0 near infinity and 1 on the compact region of each fiber, the vector field \( Y = \chi Y_2 + (1 - \chi) Y_1 \) will then give a globally defined vector field on \( \tilde{S} \). Similar to [24, Theorem 4.1], the flow of this vector field will then provide a smooth family of diffeomorphisms \( F_t : \tilde{S}_{(A_i, B_i)} \to \tilde{S}_{(A_i, B_i_0)} \) satisfying \( z_t \circ F_t = z_0 \) and preserving the zero section near infinity.

We next modify \( F_t \) so that it has the affine property near infinity. Fixing a point \( p \in \mathbb{P}^1 \) near infinity, we can use \( F_t \) to identify \( H^1(\tilde{z}^{-1}(p); \mathbb{Z}) \) with \( H^1(\tilde{z}^{-1}(p); \mathbb{Z}) \), and we can make a smooth choice of basis

\[
[y_1(p, t)], [y_2(p, t)] \in H^1(\tilde{z}^{-1}(p); \mathbb{Z}).
\]  
(4.40)

Next, the diffeomorphisms \( F_t \) give an identification of the homological invariants

\[
R^1(z_t)_* \mathbb{Z} \cong R^1(z_0)_* \mathbb{Z}
\]  
(4.41)

near infinity. Given any other point \( q \) near infinity, we choose paths \( h(z, t) \) from \( p \) to \( q \), and then use the homological invariants to identify \( H^1(\tilde{z}^{-1}(p); \mathbb{Z}) \) with \( H^1(\tilde{z}^{-1}(q); \mathbb{Z}) \) to define

\[
[y_1(q, t)], [y_2(q, t)] \in H^1(\tilde{z}^{-1}(q); \mathbb{Z}).
\]  
(4.42)

These depend upon the path, but are well defined up to monodromy. After using \( F_t \) to identify \( H^1(\tilde{z}^{-1}(q); \mathbb{Z}) \) with \( H^1(\tilde{z}_0^{-1}(q); \mathbb{Z}) \), we see that this choice varies smoothly in \( t \).

Next, letting \( x = Z_1/Z_0, y = Z_2/Z_0 \), we see that \( \alpha_1 = dx/y \) is a smooth holomorphic 1-form on the total space of the Weierstraß moduli space (for the same reason that \( dx/y \) is a globally defined holomorphic 1-form on the elliptic curve \( y^2 = x^2 + ax + b \)). We then define multivalued period functions by

\[
\tau_1(q, t) = \int_{y_1(q, t)} \alpha_1, \quad \tau_2(q, t) = \int_{y_2(q, t)} \alpha_1,
\]  
(4.43)

which are smooth in both \( q \) and \( t \).
Since each $z_t : \tilde{S}(A_t, B_t) \to \mathbb{P}^1$ is an elliptic surface, using [18], for each $t$, there exists a neighborhood $U_{\infty,t}$ of $\tilde{D}_{\infty}$ such that we can identify $U_{\infty,t} \setminus \tilde{D}_{\infty}$ with a standard model
\[
\{\xi \in \mathbb{C} \mid 0 < |\xi| < R^{-1}\} \times \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}),
\]
where $\mathbb{Z} \oplus \mathbb{Z}$ acts by $(m, n) \cdot (\xi, w) = (\xi, w + m\tau_1(\xi^{-1}, t) + n\tau_2(\xi^{-1}, t))$, such that $\xi = z^{-1}$ and the given sections $\Sigma, \Sigma'$ are mapped to $\Sigma_0 = \{(\xi, 0) \mid 0 < |\xi| < R^{-1}\}$ in these coordinates. We can define a diffeomorphism $\Phi_t$ near infinity from $S_t \setminus D_t$ to $S_0 \setminus D_0$ such that $z_t = z_0 \circ \Phi_t$, the section $\Sigma_t$ is mapped to $\Sigma_0$, and any point $w_1\tau_1(q, t) + w_2\tau_2(q, t)$ on the fiber is mapped to $w_1\tau_1(q, 0) + w_2\tau_2(q, 0)$ on each fiber. Then the differential of $\Phi_t$ determines a vector field $Y_3$ on $\tilde{S}$, defined on a subset of $\tilde{S}$ containing a neighborhood of $\tilde{D}_{\infty}$ of each fiber of the family, whose horizontal part projects to $\partial/\partial t$, and which is affine near infinity. If we use $Y_3$ instead of $Y_1$ in the definition of $F_t$, then we get the required diffeomorphism that is affine near infinity.

Next, we consider the case that $z$ arises from bad Weierstraß data $(A, B)$. Above we have shown that the space of allowable Weierstraß data in each case is connected, and the bad Weierstraß data are a proper subvariety of the good Weierstraß data. We can therefore find a holomorphic family
\[
S_{(A_t, B_t)} \to S \to B
\]
over the base $B$ a disc, with $(A_0, B_0) = (A, B)$ and $(A_t, B_t)$ good for $t \neq 0$. Arguing as in the above case, since the $D_{\infty}$ fibers of the Weierstraß models are all the same cuspidal cubic, we can resolve the family $S$ to a family $\tilde{S}$ with smooth fibers, except for the central fiber, which will have some nontrivial RDP singularities away from $D_{\infty} \subset \hat{S}_{(A_0, B_0)}$, since we assumed that the Weierstraß data $(A, B)$ are bad. Consequently, the family of surfaces $\tilde{S}$ is then a smoothing of all the RDP singularities of $\hat{S}_{(A_0, B_0)}$. From the existence of simultaneous resolutions for RDP singularities [4, 26], after a base change, we may lift this family to a smooth family, with central fiber the minimal resolution of $S_{(A_0, B_0)}$. The proof then proceeds as in the previous case to show that we can find a diffeomorphism from any rational elliptic surface with bad Weierstraß data to a rational elliptic surface with good Weierstraß data that is standard near infinity.

Finally, we can obtain a diffeomorphism between any two surfaces with bad Weierstraß data by first finding a diffeomorphism from each surface to a surface with good Weierstraß data as in the previous paragraph, and then finding a diffeomorphism between these good surfaces as in the first part of the above proof.

Remark 4.4. In the case that $D$ is a regular fiber, Theorem 4.1 is equivalent to [7, Theorem 3.4]. Theorem 4.1 implies the following. In the case that $D$ is an $I^*_0$, $\Pi^*$, $\Pi$, $\Pi^*$, $\Pi^*$, $IV^*$, or $IV$ fiber, we have that any two ALG gravitational instantons with the same tangent cone at infinity are diffeomorphic to each other, which is Theorem D. In the case that $D$ is an $I^*_\nu$ fiber for $\nu = 1, 2, 3, 4$, we have that any two ALG* gravitational instantons with the same $\nu$ are diffeomorphic to each other, which is Theorem B. In the following subsection, we will make an improvement to this that takes into account the ALG or ALG* structure near infinity.

4.2 Coordinates at infinity

In this subsection, we prove Theorem C and Theorem E from the introduction, regarding the existence of “uniform” coordinate systems for ALG* and ALG gravitational instantons.
Proof of Theorem E. In the following arguments, the constant $R$ may increase in each step, but for simplicity of notation, we will just use a single constant $R$. Let

$$ (z_C, w_C) \equiv (\mathcal{U}^{1/\beta}, \mathcal{U}^{1-1/\beta} \mathcal{Y}) \quad (4.46) $$

be the coordinates on $C_{\beta, \tau, L}(R)$ and $z$ be the holomorphic function on $X$ asymptotic to $z_C$ that was proved to exist in [5, Theorem 4.14]. Then for large $R$, we can define a smooth map

$$ f_1 : \mathbb{C} \setminus \overline{B}_{2R}^{1/\beta}(0) \to \mathbb{C} \setminus \overline{B}_{R}^{1/\beta}(0) \quad (4.47) $$

by the following. For any number $t \in \mathbb{C}$, we define $p$ as the point with $z_C = t$ and $w_C = 0$. Then $f_1(t, \bar{t})$ is defined as $z(\Phi(p))$. Since $g$ is ALG of order $n$ in the $\Phi$ coordinates, it follows that

$$ |f_1(t, \bar{t}) - t| = O\left(|t|^{1+\beta(-n+\varepsilon)}\right) \quad (4.48) $$

and

$$ |\partial_t f_1(t, \bar{t}) - 1| + |\partial_{\bar{t}} f_1(t, \bar{t})| = O\left(|t|^{\beta(-n+\varepsilon)}\right) \quad (4.49) $$

for any $\varepsilon > 0$.

Now we choose a function $\chi_1$ that is 1 if $|t|^\beta \geq 20R$ and is 0 if $|t|^\beta \leq 10R$. Define

$$ f_2(t, \bar{t}) = \chi_1(t, \bar{t})(f_1(t, \bar{t}) - t) + t. \quad (4.50) $$

Then $|\partial_t f_2(t, \bar{t})|$ is very close to 1 and $|\partial_{\bar{t}} f_2(t, \bar{t})|$ is very close to 0, so $f_2$ is a diffeomorphism that is $f_1$ if $|t|^\beta \geq 20R$ and is the identity map if $|t|^\beta \leq 10R$. Then we define a map

$$ F_1 : C_{\beta, \tau, L}(3R) \to X \setminus X_R \quad (4.51) $$

by the following way: for each point $(z_C, w_C) \in C_{\beta, \tau, L}(3R)$, we write the fiber $\{z = f_2(z_C)\} \subset X \setminus X_R$ as $\mathbb{C}/(\mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2)$ such that the zero point is $\Phi(f_1^{-1}(f_2(z_C)), 0)$. Then,

$$ F_1(z_C, w_C) = w_1 \tau_1 + w_2 \tau_2 \quad (4.52) $$

on $\{z = f_2(z_C)\}$ if

$$ w_C = w_1 \tau_{1, \text{model}} + w_2 \tau_{2, \text{model}}, \quad (4.53) $$

where

$$ \tau_{1, \text{model}} = \mathcal{U}^{1-1/\beta} \cdot L \quad (4.54) $$

and

$$ \tau_{2, \text{model}} = \mathcal{U}^{1-1/\beta} \cdot L \cdot \tau. \quad (4.55) $$

Since $g$ is ALG of order $n$ in the $\Phi$ coordinates, we also have the estimates

$$ |\tau_1 - \tau_{1, \text{model}}| \leq C \cdot |\tau_{1, \text{model}}| \cdot |\mathcal{U}|^{-n+\varepsilon}, \quad |\tau_2 - \tau_{2, \text{model}}| = C \cdot |\tau_{2, \text{model}}| \cdot |\mathcal{U}|^{-n+\varepsilon}, \quad (4.56) $$
for any $\epsilon > 0$, with similar estimates on their derivatives. Therefore we can choose multivalued functions $\tau_1$ and $\tau_2$ so that $F_1$ is a well-defined single-valued smooth mapping. The point $F_1(p)$ is very close to $\Phi(p)$ if $|\mathcal{U}(p)| \geq 20R$, since the distance between $F_1(p)$ and $\Phi(p)$ is bounded by a constant multiple of the diameter of the fiber times $|\mathcal{U}|^{-n+\epsilon}$ for any $\epsilon > 0$. Since $F_1(p)$ is within the injectivity radius at $\Phi(p)$, we let $Y(p)$ be the tangent vector at $\Phi(p)$ corresponding to the shortest geodesic connecting these points. Then, $F_1(p) = \exp_{\Phi(p)}(Y(p))$. Note that if we parallel transport through this shortest geodesic, then the differential of $\Phi$ is very close to the differential of $F_1$.

Choose another cut-off function $\chi_2$ that is 0 if $|\mathcal{U}| \geq 200R$ and is 1 if $|\mathcal{U}| \leq 100R$. Then the map

$$F_2(p) \equiv \exp_{\Phi(p)}(\chi_2(p) \cdot Y(p))$$

is a diffeomorphism that is $\Phi$ if $|\mathcal{U}| \geq 200R$ and is $F_1$ if $3R \leq |\mathcal{U}| \leq 100R$. We can define the map

$$F' : C_{\beta, \tau, L}(3R) \to X' \setminus X'_R$$

in a similar way.

By Theorem 4.1, there exists a diffeomorphism $\Psi$ from $X$ to $X'$ such that when $|z|^\beta > R$, $z = z' \circ \Psi$. Moreover, $\Psi$ preserves the affine structure on each fiber near infinity. Then, we need to study the map $F_3 \equiv (F'_2)^{-1} \circ \Psi \circ F_2$. The zero section over the circle $\{|z|^\beta = 4R\}$ is mapped by $F_3$ to another smooth section over the circle $\{|z|^\beta = 4R\}$. It is a section of the $T^2$-fibration. If it is liftable to a smooth section of the line bundle $L^{-1}$ (the line bundle whose fiber at each point is the universal cover of the $T^2$-fiber), then $F_3$ also maps the zero section over $\{5R \geq |z|^\beta \geq 4R\}$ to the projection of a section $s$ of $L^{-1}$. Choose a cut-off function $\chi_3$ that is 0 if $|z|^\beta \geq 5R$ and is 1 when $|z|^\beta \leq 4R$. Then for

$$F_4(z, w_1\tau_1 + w_2\tau_2) \equiv (z, \chi_3 \cdot s(z, \bar{z}) + w_1\tau_1 + w_2\tau_2),$$

the map

$$F_5(p) \equiv \begin{cases} F' \circ F_4 \circ (F_2)^{-1}(p) & |z(p)| > R \\ \Psi(p) & |z(p)| \leq R \text{ or } p \in X'_R \end{cases}$$

will be $\Psi$ in the interior and $\Phi' \circ (\Phi)^{-1}$ near infinity, which is the required mapping.

However, in general, this section may not be liftable to a section of $L^{-1}$. To solve this problem, if we write $\mathcal{U} = z^\beta$ as $re^{i\theta}$, then we can rewrite the torus fibration over $\{|z| \geq R\}$ as the trivial product of $(r, \theta) \in [R, \infty) \times [0, 2\pi\beta]$ with $T^2$ after gluing the $T^2$ for $\theta = 0$ with the $T^2$ for $\theta = 2\pi\beta$ using the monodromy group. Therefore, the periods $\tau_1$ and $\tau_2$ on $\theta = 0$ can be identified with $a_{11}\tau_1 + a_{12}\tau_2$ and $a_{21}\tau_1 + a_{22}\tau_2$ using the monodromy group. The difference of two local sections on this $T^2$-fibration can be viewed as a section of an $\mathbb{R}^2$-fibration

$$\sigma = x_1(r, \theta)\tau_1 + x_2(r, \theta)\tau_2$$

on $[R, \infty) \times [0, 2\pi\beta]$, with the gluing condition that there exist integers $m, n$ such that

$$x_1(r, 0)(a_{11}\tau_1 + a_{12}\tau_2) + x_2(r, 0)(a_{21}\tau_1 + a_{22}\tau_2) - x_1(r, 2\pi\beta)\tau_1 - x_2(r, 2\pi\beta)\tau_2 = m\tau_1 + n\tau_2.$$ 

In general, $(m, n)$ may not be $(0, 0)$. Fortunately, since the matrix

$$\begin{pmatrix} a_{11} - 1 & a_{12} \\ a_{21} & a_{22} - 1 \end{pmatrix}$$

is a diffeomorphism that is $\Phi$ if $|\mathcal{U}| \geq 200R$ and is $F_1$ if $3R \leq |\mathcal{U}| \leq 100R$. We can define the map $F' : C_{\beta, \tau, L}(3R) \to X' \setminus X'_R$ in a similar way.
is invertible in every ALG case, there exists constants $C_1, C_2$ such that

$$C_1(a_{11}\tau_1 + a_{12}\tau_2) + C_2(a_{21}\tau_1 + a_{22}\tau_2) - C_1\tau_1 - C_2\tau_2 = m\tau_1 + n\tau_2. \quad (4.63)$$

Therefore, the map $F : C_{\beta, r, L}(R) \rightarrow C_{\beta, r, L}(R)$ defined by

$$F : (z, w_1\tau_1(z) + w_2\tau_2(z)) \mapsto (z, (w_1 + C_1)\tau_1(z) + (w_2 + C_2)\tau_2(z)) \quad (4.64)$$

is well defined and $F^{-1} \circ F_3$ now maps the zero section to the projection of a section of $L^{-1}$. From the definition of Greene–Shapere–Vafa–Yau’s semiflat metric, the map $F$ preserves the hyperkähler structure on the model space, and we are done.

Proof of Theorem C. The proof is similar to the proof of Theorem E with the following minor modifications. In the first part, we use the holomorphic function $z$ studied in the proof of Proposition 3.1. We define $f_1$ similarly, and since $g$ is ALG* of order 2 in the $\Phi$ coordinates, we obtain an estimate

$$|f_1(t, \bar{t}) - t| = O\left(|t|^\frac{\varepsilon}{2}\right) \quad (4.65)$$

for any $\varepsilon > 0$. We also have an estimate analogous to (4.49). Since $g$ is ALG* of order 2 in the $\Phi$ coordinates, we have the estimates

$$|	au_1 - \tau_{1,\text{model}}| \leq C \cdot |	au_{1,\text{model}}| \cdot r^{-2+\varepsilon}, \quad |	au_2 - \tau_{2,\text{model}}| = C \cdot |	au_{2,\text{model}}| \cdot r^{-2+\varepsilon}, \quad (4.66)$$

for any $\varepsilon > 0$, with similar estimates on their derivatives, where $\tau_{1,\text{model}}$ and $\tau_{2,\text{model}}$ were defined in (2.16), so we can define the mapping $F_1$ in the same way. The distance between $F_1(p)$ and $\Phi(p)$ is bounded by a constant multiple of the diameter of the fiber times $r^{-2+\varepsilon}$ for any $\varepsilon > 0$. Since the diameter of the fiber is proportional to $\sqrt{\log r}$, and the injectivity radius is proportional to $1/\sqrt{\log r}$, the construction of the mapping $F_2$ is also similar. In the last part, we only used the fact that the monodromy generator minus the identity matrix is invertible, which from (2.17) obviously holds in this case.

5 | REMARKS ON THE ORDER OF ALG GRAVITATIONAL INSTANTONS

In this section, we will prove Theorem F. Let $(X_\beta, g, \omega)$ be an ALG gravitational instanton of order $n > 0$, and let $s : X \rightarrow [1, \infty)$ be a smooth extension of $r$ via the diffeomorphism $\Phi : C_{\beta, r, L}(R) \rightarrow X \setminus X_R$, where $\Phi$ and $r$ are defined as in Definition 1.1.

Definition 5.1. Let $(X, g, \omega)$ be an ALG gravitational instanton of order $n > 0$. For any fixed $\delta \in \mathbb{R}$, the weight function $\hat{\omega}_\delta$ on $X$ is defined by $\hat{\omega}_\delta \equiv s^{-\delta - 1}$, and we define the weighted Sobolev norms as follows:

$$\|\omega\|_{L^2(X)} \equiv \left(\int_X |\omega| \cdot \hat{\omega}_\delta^2 \text{dvol}_X\right)^{\frac{1}{2}}, \quad \|\omega\|_{W^{k,2}(X, \hat{\omega}_\delta)} \equiv \left(\sum_{m=0}^k \|\nabla^m \omega\|_{L^2(X)}^2 \right)^{\frac{1}{2}}.$$
We remark that this convention differs from the convention in [6, Definition 4.1] because of the following.

**Lemma 5.2.** Let \((X, g, \omega)\) be an ALG gravitational instanton of order \(n > 0\). For any \(\delta \in \mathbb{R}\) and \(k \in \mathbb{N}_0\), there exists a constant \(C_{k, \delta} > 0\) so that

\[
\sum_{m=0}^{k} \sup_{x \in X} |(s(x))^{m-\delta} \nabla^m \omega(x)| \leq C \|\omega\|_{W^{k+3,2}_\delta(X)}
\]

for all \(\omega \in W^{k+3,2}_\delta(X)\).

**Proof.** The proof is a standard rescaling argument; see, for example, [9, Proposition 6.16]. □

In [5, Theorem A], Chen–Chen proved that there exists a diffeomorphism \(F_1 : C_{\beta, \tau, L}(R) \to C_{\beta, \tau, L}(R)\) homotopic to the identity such that when the cone angle \(\beta \leq 1/2\), the ALG order \(n\) must be at least 2 in the \(\Phi \circ F_1\) coordinates. On the other hand, when \(1/2 < \beta < 1\), the leading term of \((\Phi \circ F_1)\ast \omega - \omega^C\) must be a linear combination of closed anti-self-dual forms, which are \(\text{Re} \mathcal{U}^{\frac{1}{\beta}-2} \mathcal{U} \wedge d\overline{\mathcal{V}}\) and \(\text{Im} \mathcal{U}^{\frac{1}{\beta}-2} \mathcal{U} \wedge d\overline{\mathcal{V}}\). Using Remark 2.6, the following formulas are straightforward to verify:

\[
\mathcal{L}_Y \omega^C_1 = -d(I^c_1 \eta) = d \left( \text{Im} \mathcal{U}^{\frac{1}{\beta}-1} d\mathcal{U} \right) = 0, \quad (5.2)
\]

\[
\mathcal{L}_Y \omega^C_2 = -d(J^c_1 \eta) = d \left( \text{Re} \mathcal{U}^{\frac{1}{\beta}-1} d\overline{\mathcal{V}} \right) = \left( \frac{1}{\beta} - 1 \right) \text{Re} \left( \mathcal{U}^{\frac{1}{\beta}-2} \mathcal{U} \wedge d\overline{\mathcal{V}} \right), \quad (5.3)
\]

\[
\mathcal{L}_Y \omega^C_3 = -d(K^c_1 \eta) = -d \left( \text{Im} \mathcal{U}^{\frac{1}{\beta}-1} d\overline{\mathcal{V}} \right) = - \left( \frac{1}{\beta} - 1 \right) \text{Im} \left( \mathcal{U}^{\frac{1}{\beta}-2} \mathcal{U} \wedge d\overline{\mathcal{V}} \right), \quad (5.4)
\]

where \(\mathcal{L}\) denotes the Lie derivative and \(Y\) is the \(g^C\)-metric dual to \(\eta \equiv \text{Re} \mathcal{U}^{\frac{1}{\beta}-1} d\mathcal{U}\). Furthermore, we have

\[
\mathcal{L}_{I_1^c} \omega^C_1 = -d \eta = 0, \quad \mathcal{L}_{I_1^c} \omega^C_2 = d(K^c_1 \eta), \quad \mathcal{L}_{I_1^c} \omega^C_3 = -d(J^c_1 \eta), \quad (5.5)
\]

\[
\mathcal{L}_{J_1^c} \omega^C_1 = -d(K^c_1 \eta), \quad \mathcal{L}_{J_1^c} \omega^C_2 = -d \eta = 0, \quad \mathcal{L}_{J_1^c} \omega^C_3 = d(I^c_1 \eta) = 0, \quad (5.6)
\]

\[
\mathcal{L}_{K_1^c} \omega^C_1 = d(J^c_1 \eta), \quad \mathcal{L}_{K_1^c} \omega^C_2 = -d(I^c_1 \eta) = 0, \quad \mathcal{L}_{K_1^c} \omega^C_3 = -d \eta = 0. \quad (5.7)
\]

These formulas imply that there exists another diffeomorphism \(F_2 : C_{\beta, \tau, L}(R) \to C_{\beta, \tau, L}(R)\) homotopic to the identity such that

\[
(\Phi \circ F_2)\ast \omega_1 = \omega^C_1 + O \left( r^{\frac{1}{\beta}-2-\epsilon} \right), \quad (5.8)
\]

\[
(\Phi \circ F_2)\ast \omega_2 = \omega^C_2 + O \left( r^{\frac{1}{\beta}-2-\epsilon} \right), \quad (5.9)
\]
as \( r \to \infty \), for some \( \varepsilon > 0 \). To see this, for a vector field \( Z = O(r^{\frac{1}{\beta} - 1}) \), let \( \chi \) be a cut-off function that is 1 on \( C_{\beta, \tau, L}(2R) \) and is supported on \( C_{\beta, \tau, L}(R) \), and then define \( \Phi_{Z,t} : C_{\beta, \tau, L}(R) \to C_{\beta, \tau, L}(R) \) by \( \Phi_{Z,t}(p) = \exp_{g_{c,p}}(t \cdot \chi \cdot Z_p) \). If \( 0 \leq t \leq 1 \), since \( \frac{1}{\beta} - 1 < 1 \), the vector \( t \chi Z_p \) has norm much smaller than the conjugate radius at \( p \) (which is comparable to \( r(p) \)), so \( \Phi_{Z,t} \) is a diffeomorphism for \( R \) sufficiently large. Then it is straightforward to see that we have the expansions
\[
\Phi_{Z,t}^* J_{\omega^1, Y, 1} = \omega^1 + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right), \\
\Phi_{Z,t}^* K_{\omega^2, Y, 1} = \omega^2 + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right), \\
\Phi_{Z,t}^* \omega^3 = \omega^3 + (\Phi_{Z,t}^* \omega^3_{\text{pert}}) + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right),
\]
as \( r = |\mathcal{U}| \to \infty \), so we can arrange that (5.8) is satisfied after pulling back by an appropriate diffeomorphism. Then (5.9) is proved similarly, using \( Y \) and \( I_C Y \), and this does not affect (5.8).

Note that we have the expansion
\[
(\Phi \circ F_2)^* \omega_3 = \omega_3^C + \frac{1}{\beta} \left( \frac{1}{\beta} - 1 \right)(a_0 \text{Re} + b_0 \text{Im}) \left( \mathcal{U}^{\frac{1}{\beta}} - 2 d\mathcal{U} \wedge d\overline{\mathcal{U}} \right) + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right)
\]
\[
= \omega_3^C + d(K^C d(-a_0 \text{Im} + b_0 \text{Re}) \mathcal{U}^{\frac{1}{\beta}}) + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right)
\]
\[
= \omega_3^C + (\Phi \circ F_2)^* \left( 2 \sqrt{-1} \partial_K \overline{\partial_K} (a_0 y - b_0 x) \right) + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right),
\]
as \( r \to \infty \), where \( \varepsilon > 0 \), \( z = x + \sqrt{-1} y \) is the \( I \)-holomorphic function on \( X \) asymptotic to \( z_{c_o(\Phi \circ F_2)^{-1}} = \mathcal{U}^{\frac{1}{\beta}} o(\Phi \circ F_2)^{-1} \), and \( \partial_K \) is the \( \partial \) operator using the complex structure \( K \). Estimates for all higher derivatives analogous to (5.8), (5.9), and (5.12) follow from standard arguments using the hyperkähler condition and elliptic regularity.

Next, we have the following key proposition.

**Proposition 5.3.** If \( \beta > 1/2 \), then for any ALG gravitational instanton \((X, g, \omega)\) satisfying
\[
(\Phi \circ F_2)^* \omega_1 = \omega_1^C + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right), \quad (\Phi \circ F_2)^* \omega_2 = \omega_2^C + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right),
\]
and
\[
(\Phi \circ F_2)^* \omega_3 = \omega_3^C + (\Phi \circ F_2)^* \left( 2 \sqrt{-1} \partial_K \overline{\partial_K} (a_0 y - b_0 x) \right) + O\left( r^{\frac{1}{\beta} - 2 - \varepsilon} \right),
\]
for any \((a, b) \in \mathbb{R}^2\), there exists a smooth function
\[
\varphi = 2 \cdot (a y - a_0 y - b x + b_0 x) + O\left( r^{\frac{1}{\beta} - \varepsilon} \right),
\]
as \( r \to \infty \), unique up to adding a constant, such that \((X, g_{a,b}, \omega_{a,b})\) is an ALG gravitational instanton with the same ALG coordinate system \( \Phi \circ F_2 \) and the same \( K \), where
\[
(\omega_{1,a,b}, \omega_{2,a,b}, \omega_{3,a,b}) = (\omega_1, \omega_2, \omega_3 + \sqrt{-1} \partial_K \overline{\partial_K} \varphi),
\]
and \( g_{a,b} \) is defined by \( K \) and \( \omega_{3,a,b} \).
Proof. We use $K$ as the complex structure and consider

$$\omega_{3,1} \equiv \omega_{3} + 2\sqrt{-1} \partial_{K} \tilde{\partial}_{K} (ay - a_{0}y - bx + b_{0}x). \quad (5.17)$$

The form $\omega_{3,1}$ is not positive in general. However, it must be positive on the region $\{ r \geq R_{1} \}$ for sufficiently large $R_{1}$. By the surjectivity in [5, Theorem 4.4], there exists a smooth function $\varphi_{1} \in W^{k+2,2}_{1/101/100}(\{ r \geq R_{1} \})$ such that

$$2\omega_{3,1} \wedge \sqrt{-1} \partial_{K} \tilde{\partial}_{K} \varphi_{1} = \omega_{1}^{2} - \omega_{3,1}^{2}, \quad (5.18)$$

because

$$\omega_{1}^{2} - \omega_{3,1}^{2} = \omega_{3}^{2} - \omega_{3,1}^{2} = -4(\sqrt{-1} \partial_{K} \tilde{\partial}_{K} (ay - a_{0}y - bx + b_{0}x))^{2} \in W^{k,2}_{-99/100}(\{ r \geq R_{1} \}) \quad (5.19)$$

for any $k \geq 0$, using the fact that $x, y$ are harmonic with respect to the hyperkähler structure $(g, \omega_{1}, \omega_{2}, \omega_{3})$ and $2 \cdot (\frac{1}{3} - 2) \leq -1$ by Table 1. Then, for large enough $R_{2} > R_{1}$, the form

$$\omega_{3,2} \equiv \omega_{3,1} + \sqrt{-1} \partial_{K} \tilde{\partial}_{K} \varphi_{1} \quad (5.20)$$

is smooth and positive on the region $\{ r \geq R_{2} \}$. Using Lemma 5.2, it follows that

$$(\partial_{K} \tilde{\partial}_{K} \varphi_{1})^{2} \in W^{k,2}_{-99/50}(\{ r \geq R_{2} \}) \quad (5.21)$$

for any $k \geq 0$. So, by the surjectivity in [5, Theorem 4.4], we can find a smooth $\varphi_{2} \in W^{k+2,2}_{1/50}(\{ r \geq R_{2} \})$ such that

$$2\omega_{3,2} \wedge \sqrt{-1} \partial_{K} \tilde{\partial}_{K} \varphi_{2} = \omega_{1}^{2} - \omega_{3,2}^{2} = -(\sqrt{-1} \partial_{K} \tilde{\partial}_{K} \varphi_{1})^{2} \in W^{k,2}_{-99/50}(\{ r \geq R_{2} \}) \quad (5.22)$$

for any $k \geq 0$. Then the form

$$\omega_{3,3} \equiv \omega_{3,2} + \sqrt{-1} \partial_{K} \tilde{\partial}_{K} \varphi_{2} = \omega_{3,1} + \sqrt{-1} \partial_{K} \tilde{\partial}_{K} (2 \cdot (ay - a_{0}y - bx + b_{0}x) + \varphi_{1} + \varphi_{2}) \quad (5.23)$$

is smooth and positive on the region $\{ r \geq R_{3} \}$ for large enough $R_{3} > R_{2}$. Using Lemma 5.2, it follows that

$$(\partial_{K} \tilde{\partial}_{K} \varphi_{2})^{2} \in W^{k,2}_{-99/25}(\{ r > R_{3} \}) \quad (5.24)$$

for any $k \geq 0$. Then

$$\omega_{3}^{2} - \omega_{3,3}^{2} = -(\sqrt{-1} \partial_{K} \tilde{\partial}_{K} \varphi_{2})^{2} \in W^{k,2}_{-99/25}(\{ r \geq R_{3} \}) \quad (5.25)$$

for any $k \geq 0$.

Now we take a cut-off function $\chi$ that is 1 on $(2, \infty)$ and is 0 on $(-\infty, 1)$. Then as long as $R_{4} > R_{3}$ is large enough, the form

$$\omega_{3,4} \equiv \omega_{3,1} + \sqrt{-1} \partial_{K} \tilde{\partial}_{K} \left( \chi \left( \frac{r}{R_{4}} \right) \cdot (2 \cdot (ay - a_{0}y - bx + b_{0}x) + \varphi_{1} + \varphi_{2}) \right) \quad (5.26)$$
is smooth and positive on the whole manifold $X$. Moreover, by (5.25),

$$\int_X (\omega_1^2 - \omega_{3,4}^2) < \infty.$$  (5.27)

Since $\log |z|$ is harmonic with respect to $\omega_3$, the integral

$$\int_X \omega_3 \wedge \sqrt{-1} \partial K \bar{\partial} K \left( \chi \left( \frac{r}{R_5} \right) \cdot \log |z| \right)$$  (5.28)

is a nonzero number independent of $R_5$. Define $t \in \mathbb{R}$ such that

$$2 \cdot t \cdot \int_X \omega_3 \wedge \sqrt{-1} \partial K \bar{\partial} K \left( \chi \left( \frac{r}{R_5} \right) \cdot \log |z| \right) = \int_X (\omega_1^2 - \omega_{3,4}^2),$$  (5.29)

and define

$$\omega_{3,5} = \omega_{3,4} + t \sqrt{-1} \partial K \bar{\partial} K \left( \chi \left( \frac{r}{R_5} \right) \cdot \log |z| \right).$$  (5.30)

Then $\omega_{3,5}$ is smooth and still positive for large enough $R_5 > R_4$. Moreover,

$$\int_X (\omega_{3,5}^2 - \omega_1^2) = \int_X (\omega_{3,4}^2 - \omega_1^2) + t \cdot \int_X (\omega_{3,4} + \omega_{3,5}) \wedge \sqrt{-1} \partial K \bar{\partial} K \left( \chi \left( \frac{r}{R_5} \right) \cdot \log |z| \right).$$  (5.31)

Using Stokes’ theorem,

$$\int_X \omega_{3,4} \wedge \sqrt{-1} \partial K \bar{\partial} K \left( \chi \left( \frac{r}{R_5} \right) \cdot \log |z| \right) = \int_X \omega_{3,5} \wedge \sqrt{-1} \partial K \bar{\partial} K \left( \chi \left( \frac{r}{R_5} \right) \cdot \log |z| \right)$$

$$= \int_X \omega_3 \wedge \sqrt{-1} \partial K \bar{\partial} K \left( \chi \left( \frac{r}{R_5} \right) \cdot \log |z| \right).$$  (5.32)

Therefore,

$$\int_X (\omega_{3,5}^2 - \omega_1^2) = 0.$$  (5.33)

Finally, we apply Tian–Yau’s theorem [27, Theorem 1.1] to find a bounded smooth function $\varphi_3 : X \rightarrow \mathbb{R}$ such that the form

$$\omega_{3,6} \equiv \omega_{3,5} + \sqrt{-1} \partial K \bar{\partial} K \varphi_3$$  (5.34)

satisfies

$$\omega_{3,6}^2 = \omega_1^2 = \omega_2^2.$$  (5.35)

Hein proved the higher order estimates for $\varphi_3$ [17, Proposition 2.9]. Taking $\omega_{3,a,b} = \omega_{3,6}$ completes the existence proof.
To show the uniqueness: if $\varphi_4$ and $\varphi_5$ are two potentials, then

$$2 \cdot \left( \omega_3 + \sqrt{-1} \partial_K \bar{\partial}_K \frac{\varphi_4 + \varphi_5}{2} \right) \wedge \sqrt{-1} \partial_K \bar{\partial}_K (\varphi_4 - \varphi_5)$$

$$= (\omega_3 + \sqrt{-1} \partial_K \bar{\partial}_K \varphi_4)^2 - (\omega_3 + \sqrt{-1} \partial_K \bar{\partial}_K \varphi_5)^2$$

(5.36)

$$= \omega_1^2 - \omega_1^2 = 0.$$

So $\varphi_4 - \varphi_5 = O(r^{1/\beta - \epsilon})$ is harmonic. By [15, Proposition 7], the leading term of $\varphi_4 - \varphi_5$ is a linear combination of $\log |z|$ and 1. However, the coefficient of $\log |z|$ must be 0 by Stokes’ theorem and the fact that

$$\int_X \left( \omega_3 + \sqrt{-1} \partial_K \bar{\partial}_K \frac{\varphi_4 + \varphi_5}{2} \right) \wedge \sqrt{-1} \partial_K \bar{\partial}_K (\varphi_4 - \varphi_5) = 0.$$  

(5.37)

Therefore, $\varphi_4 - \varphi_5$ is a constant plus a decaying harmonic function. By the maximum principle, there is no nonzero decaying harmonic function, so $\varphi_4 - \varphi_5$ is a constant. □

Finally, to finish the proof of Theorem F using Proposition 5.3, we apply an argument similar to the one used in [5, Theorem A] to $\omega_{0,0}$ to find another diffeomorphism $F_3 : C_{\beta, \tau, L}(R) \to C_{\beta, \tau, L}(R)$ homotopic to the identity such that $\omega_{0,0}$ is an ALG gravitational instanton with order 2 in the $\Phi \circ F_3$ coordinates. Actually, the diffeomorphism $F_2^{-1} \circ F_3$ can be chosen to be close to the identity map so that

$$(\Phi \circ F_2) \ast \omega_{a,b} = (\Phi \circ F_3) \ast \omega_{a,b} + O(r^{\frac{1}{\beta} - 2 - \epsilon'})$$

(5.38)

for another $\epsilon' > 0$ slightly smaller than $\epsilon > 0$.

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