Isoperimetric deformations of curves on the Minkowski plane

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Abstract

We formulate an isoperimetric deformation of curves on the Minkowski plane, which is governed by the defocusing mKdV equation. Two classes of exact solutions to the defocusing mKdV equation are also presented in terms of the $\tau$ functions. By using one of these classes, we construct an explicit formula for the corresponding motion of curves on the Minkowski plane even though those solutions have singular points. Another class give regular solutions to the defocusing mKdV equation. Some pictures illustrating typical dynamics of the curves are presented.
1 Introduction

It is well-known that a certain class of integrable systems describes motions of plane and space curves in various settings. For instance, the nonlinear Schrödinger equation describes a motion of space curves which is a physical model of vortex filaments [1], and the modified Korteweg-de Vries (mKdV) equation describes motion of space and plane curves preserving the arc length [2, 3]. Curve motions have been studied not only in the Euclidean geometry but also in various Klein geometries [4, 5, 6, 7, 8]. For example, the KdV equation describes a motion of plane curves that preserves the areal velocity [9]. Moreover, in recent years, explicit formulas for these curve motions have been established in [10, 11, 12, 13] by using the theory of \( \tau \) functions.

In this paper, we consider the isoperimetric motions of curves on the Minkowski plane, namely, motions preserving the arc length, and show that the simplest nontrivial motion is described by the defocusing mKdV equation,

\[
\frac{\partial \kappa}{\partial t} = \frac{\partial^3 \kappa}{\partial x^3} - \frac{3}{2} \kappa^2 \frac{\partial \kappa}{\partial x}. \quad (1.1)
\]

The regular soliton type solutions of the defocusing mKdV equation show rather different behavior from other equations; it admits the solutions where the solitons run on a shock wave [14, 15]. On the other hand, the focusing mKdV equation

\[
\frac{\partial \hat{\kappa}}{\partial t} = \frac{\partial^3 \hat{\kappa}}{\partial x^3} + \frac{3}{2} \hat{\kappa}^2 \frac{\partial \hat{\kappa}}{\partial x}, \quad (1.2)
\]

which governs the curve motions preserving the arc length in the Euclidean plane describes the ordinary dynamics of solitons. Those two mKdV equations cannot be transformed to each other by scale change. It should be remarked that the defocusing mKdV equation has not been studied well because of less physical relevance compared to the focusing mKdV equation.

This paper is organized as follows. In Section 2, we review some basic notions of curves on the Minkowski plane. In Section 3, we formulate an integrable isoperimetric deformation of curves on the Minkowski plane. In Section 4.1, we first construct a class of exact solutions to the defocusing mKdV equation in terms of the \( \tau \) functions. Those solutions, however, give singular solutions. We also construct regular solutions to the defocusing mKdV equation in Section 4.2 by suitable choice of parameters and reductions on the \( \tau \) functions, following the idea given in [16]. We finally present the formulas for corresponding curve motions on the Minkowski plane.

2 Curves on the Minkowski plane

In this section, we state a curve theory on the Minkowski plane. We provide the vector space \( \mathbb{R}^2 \) with the Lorentzian inner product \( \langle \cdot, \cdot \rangle \)

\[
\langle v, w \rangle = v_1 w_1 - v_2 w_2
\]

for arbitrary vectors \( v = [v_1, v_2] \) and \( w = [w_1, w_2] \). We write \( \mathbb{R}^{1,1} \) for \( (\mathbb{R}^2, \langle \cdot, \cdot \rangle) \), and call it the Minkowski plane. We say that a vector \( v \in \mathbb{R}^{1,1} \) is spacelike if \( \langle v, v \rangle > 0 \) or \( v = 0 \), timelike if \( \langle v, v \rangle < 0 \), and lightlike if \( \langle v, v \rangle = 0 \) and \( v \neq 0 \). For a vector \( v \in \mathbb{R}^{1,1} \), its norm \( |v| \) is defined as

\[
|v| = \begin{cases} 
\langle v, v \rangle^{1/2} & \text{if } v \text{ is spacelike}, \\
(\langle v, v \rangle)^{1/2} & \text{if } v \text{ is timelike}, \\
0 & \text{if } v \text{ is lightlike}.
\end{cases}
\]
Therefore a spacelike (resp. timelike) vector \( v \in \mathbb{R}^{1,1} \) is of unit length if and only if \( v \in H_+^1 \) (resp. \( H_-^1 \)), where \( H_+^1 \) are the hyperbolas of two sheets

\[
H_+^1 = \{ v \in \mathbb{R}^{1,1} \mid \langle v, v \rangle = 1 \}, \quad H_-^1 = \{ v \in \mathbb{R}^{1,1} \mid \langle v, v \rangle = -1 \}.
\]

The Lorentz group

\[
O(1, 1) = \{ A \in GL(2, \mathbb{R}) \mid ^tA E' A = E' \}, \quad E' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

preserves the inner product. Indeed the equality \( \langle Av, Aw \rangle = \langle v, w \rangle \) holds for all \( v, w \in \mathbb{R}^{1,1} \) if and only if \( A \in O(1, 1) \). We also consider the subgroup \( SO(1, 1) = O(1, 1) \cap SL(2, \mathbb{R}) \), which has two connected components

\[
SO^+(1, 1) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(1, 1) \mid d > 0 \right\} = \left\{ \begin{bmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{bmatrix} \mid \phi \in \mathbb{R} \right\},
\]

\[
SO^-(1, 1) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(1, 1) \mid d < 0 \right\} = \left\{ \begin{bmatrix} -\cosh \phi & \sinh \phi \\ \sinh \phi & -\cosh \phi \end{bmatrix} \mid \phi \in \mathbb{R} \right\}.
\]

Let \( \gamma : I \subset \mathbb{R} \rightarrow \mathbb{R}^{1,1}, \xi \rightarrow \gamma(\xi) \) be a regular curve on the Minkowski plane, parametrized by an arbitrary parameter \( \xi \). We say that \( \gamma \) is spacelike if its velocity \( (d/d\xi) \gamma \) is spacelike everywhere. A spacelike curve \( \gamma \) is said to be unit-speed if its velocity is of unit length everywhere. Therefore, the velocity \( T \) of a unit-speed spacelike curve \( \gamma \) moves along the hyperbola of two sheets \( H_+^1 \). Since \( T \) is a continuous vector field along \( \gamma \), \( T \) moves along one of the sheets of \( H_+^1 \). Similarly the notion of unit-speed timelike curve is defined. Hereafter, we consider a unit-speed spacelike curve

\[
\gamma(x) = \begin{bmatrix} \gamma_1(x) \\ \gamma_2(x) \end{bmatrix},
\]

and assume that \( \gamma'(x) > 0 \). We say that such a tangent vector field \( T = \gamma' \) is positive pointing. The positive pointing tangent vector field \( T \) is obviously expressed as

\[
T = \begin{bmatrix} \cosh \theta \\ \sinh \theta \end{bmatrix}
\]

with some function \( \theta \). We define the normal vector field by

\[
N = J'T = \begin{bmatrix} \sinh \theta \\ \cosh \theta \end{bmatrix}, \quad J' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

They satisfy \( |T| = |N| = 1 \) and \( \langle T, N \rangle = 0 \). Introducing a frame \( \Phi : I \rightarrow SO^+(1, 1) \) by \( \Phi = [T, N] \), we have the Frenet formula

\[
\Phi' = \Phi L, \quad L = \begin{bmatrix} 0 & \theta' \\ \theta' & 0 \end{bmatrix}.
\]

On the other hand, there exists a function \( \kappa \) called the curvature of \( \gamma \), such that \( T' = \kappa N \) because \( \langle T, T' \rangle = 0 \). Thus we have \( \kappa = \theta' \), from which \( \theta \) is sometimes referred to as the potential function. The discussion is summarized as the fundamental theorem of plane curves as follows.

**Proposition 2.1** For a given function \( \kappa = \kappa(x) \), there exists a unit-speed spacelike curve \( \gamma = \gamma(x) \) on \( \mathbb{R}^{1,1} \) with a positive pointing tangent vector field, such that \( \kappa \) is the curvature of \( \gamma \). In fact \( \gamma \) is given by the integral

\[
\gamma(x) = \int_{x_0}^x \frac{\cosh \theta}{\sinh \theta} \, dx, \quad \theta(x) = \int_{x_0}^x \kappa \, dx.
\]
Moreover, if two unit-speed spacelike curves $\gamma, \tilde{\gamma}$ have the same curvature, then they differ only by a Lorentzian motion, namely there exists a matrix $A \in \text{SO}(1,1)$ and a vector $v \in \mathbb{R}^{1,1}$ such that $\tilde{\gamma}(x) = A \gamma(x) + v$.

**Remark 2.2** The arclength function of a curve $\gamma(\xi)$ on the Minkowski plane is defined by

$$x(\xi, t) = \int_{\xi_0}^{\xi} |\gamma_\xi| \, d\xi,$$

so that the arclength parametrized curve $\gamma(x)$ is unit-speed.

**Remark 2.3** In terms of an arbitrary parameter $\xi$, we can rewrite Proposition 2.1 as follows. Let $k = k(\xi)$ be a function and $x = x(\xi)$ be a monotonously increasing function. Then, up to Lorentzian motions, there uniquely exists a spacelike curve $\tilde{\gamma} = \tilde{\gamma}(\xi)$ on $\mathbb{R}^{1,1}$ with a positive pointing tangent vector field, such that $k$ is the curvature of $\tilde{\gamma}$ and $x$ is the arclength of $\tilde{\gamma}$. In fact $\tilde{\gamma}$ is given by the integral

$$\tilde{\gamma}(\xi) = \int_{\xi_0}^{\xi} \left[ \frac{\cosh \theta}{\sinh \theta} \right] x' \, d\xi, \quad \tilde{\theta}(\xi) = \int_{\xi_0}^{\xi} k \, x' \, d\xi.$$

### 3 Deformation of curves on the Minkowski plane

We formulate an arclength preserving deformation of a unit speed spacelike curve $\gamma = \gamma(x)$ on $\mathbb{R}^{1,1}$ with positive pointing tangent vector field, and show that it can be governed by the defocusing mKdV equation. Introducing a deformation parameter $t$, we denote again by $\gamma = \gamma(\xi, t)$ the deformation, where $\gamma(\xi, 0)$ is the initial unit-speed curve $\gamma(x)$. We decompose $(\partial / \partial t) \gamma$ in the form

$$\gamma = f T + g N,$$

where $T = |\gamma_\xi|^{-1} \gamma_\xi$ and $N = J'T$. Here the subscripts mean differentiation with respect to the indicated variables.

**Proposition 3.1** Let $\gamma = \gamma(\xi, t)$ be a family of spacelike curves on $\mathbb{R}^{1,1}$ with positive pointing tangent vector fields, such that the initial curve $\gamma(\xi, 0)$ is unit speed. Then the arclength is independent of $t$ if and only if $\langle \gamma_\xi : \gamma_\xi \rangle = 0$.

**Proof.** Differentiating (2.4) by $t$, we have

$$x_t = \int_{\xi_0}^{\xi} \frac{\langle \gamma_\xi : \gamma_\xi \rangle}{|\gamma_\xi|} \, d\xi.$$

Therefore $x_t = 0$ for all $\xi$ if and only if $\langle \gamma_\xi : \gamma_\xi \rangle = 0$. \hfill $\Box$

Since we have from (2.5) that $T_\xi = x_\xi \kappa N$, where $\kappa$ is the curvature of $\gamma$ at each $t$, it follows from the expression (3.1) that

$$\gamma_\xi = (f_\xi + x_\xi \kappa g) T + (g_\xi + x_\xi \kappa f) N.$$  

Therefore the isoperimetric condition $\langle \gamma_\xi : \gamma_\xi \rangle = 0$ is equivalent to the equality $f_\xi + x_\xi \kappa g = 0$. In the followings, we consider an isoperimetric deformation of a unit-speed spacelike curve, and hence we can assume that $\xi$ itself is the arclength parameter. Thus, the isoperimetric condition becomes

$$f_\xi + \kappa g = 0,$$  

and the frame $\Phi = [T, N]$ is deformed as

$$\Phi_t = \Phi M, \quad M = \begin{bmatrix} 0 & g_\xi + \kappa f \\ g_\xi + \kappa f & 0 \end{bmatrix}.$$  

(3.3)
Under the isoperimetric condition (3.2), the compatibility condition between (2.2) and (3.3), \( L_t - M_x - LM + ML = 0 \), is
\[
\kappa_t = (g_x + \kappa f)_x = \Omega g,
\]
where \( \Omega = \partial_x^2 - \kappa^2 - \kappa_x \partial_x^{-1}(\kappa \cdot) \) is the recursion operator of the defocusing mKdV hierarchy. In view of this, it is reasonable to choose \( g \) as \( g = \kappa_x \) and hence \( f = -\kappa^2/2 \). Thus we have:

**Theorem 3.2 (defocusing mKdV flow)** Let \( \gamma = \gamma(x,t) \) be a family of unit speed spacelike curves on \( \mathbb{R}^{1,1} \) with positive pointing vector field, and \( \kappa \) the curvature of \( \gamma \) at each \( t \). Then \( \gamma \) is an arclength preserving deformation, and it varies according to the formula
\[
\gamma_t = -\frac{\kappa^2}{2} T + \kappa_i N
\]
if and only if \( \kappa \) satisfies the defocusing mKdV equation
\[
\kappa_t = \kappa_{xxx} - \frac{3}{2} \kappa^2 \kappa_x.
\]

**Proof.** The frame is deformed as
\[
\Phi_t = \Phi M, \quad M = \begin{bmatrix} 0 & \kappa_x - \frac{1}{2} \kappa^3 & \kappa_{xx} - \frac{1}{2} \kappa^3 \kappa_x \end{bmatrix}.
\]
The compatibility condition between (2.2) and (3.6) is (3.5).

We call the deformation (3.4) the defocusing mKdV flow. If \( \gamma \) is a defocusing mKdV flow, then the potential function \( \theta \) satisfies the potential defocusing mKdV equation
\[
\theta_t = \theta_{xxx} - \frac{1}{2} \theta_x^3.
\]

## 4 Solutions

We construct solutions to the defocusing mKdV equation in terms of \( \tau \) functions, and derive an explicit formula for the defocusing mKdV flow (3.4).

### 4.1 Explicit formula

Let \( \tau = \tau(x,t;y) \) and \( \overline{\tau} = \overline{\tau}(x,t;y) \) be real-valued functions, where \( y \) is an auxiliary variable. For a real constant \( c \), we consider the system of bilinear equations
\[
D_x D_y \tau \cdot \tau = -2 \tau^3, \quad (D_x^2 - c) \tau \cdot \overline{\tau} = 0, \quad D_t \tau \cdot \tau = -2 \tau^2, \quad (D_x^3 - D_t - 3c D_x) \tau \cdot \overline{\tau} = 0.
\]

Here \( D_x, D_y \) and \( D_t \) are Hirota’s bilinear differential operators [17], defined as
\[
D^i_x D^j_y f \cdot g = \left. \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^i \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^j f(x,y) g(x',y') \right|_{x' = x, y' = y}.
\]
Theorem 4.1 For a pair of solutions \( \tau, \varpi \) to (4.1)–(4.4), we define \( \theta \) and \( \gamma \) by
\[
\theta = 2 \log \frac{\tau}{\varpi}, \\
\gamma = \frac{1}{2} \frac{\partial}{\partial y} \left[ -\log \left( \frac{\tau}{\varpi} \right) \right].
\]

Then, for any \( y \), \( \theta \) and \( \gamma \) satisfy the potential defocusing mKdV equation (3.7) and the defocusing mKdV flow (3.4).

Proof. First we have from (4.1) and (4.2) that
\[
T = \gamma = \frac{1}{2} \left[ -\frac{\log \tau + \log \varpi}{\sqrt{\tau \varpi}} \right] = \frac{1}{4} \left[ -\tau^{-2}D_xD_x \tau \cdot \tau - \varpi^{-2}D_xD_x \varpi \cdot \varpi \right] \\
= \frac{1}{2} \left[ (\tau/\varpi)^2 + (\tau/\varpi)^{-2} \right] = \frac{1}{2} \left[ e^\theta + e^{-\theta} \right] = \left[ \frac{\cosh \theta}{\sinh \theta} \right],
\]
which yields that \( T \) is positive pointing, and we have the Frenet formula (2.2). Thus \( T_t = \theta_t N \), by which it is sufficient for (3.6) to show
\[
\theta_t = \kappa_{xx} - \kappa^3 \frac{3}{2}. \tag{4.5}
\]

We have
\[
\theta_t - \kappa_{xx} + \frac{1}{2} \kappa^3 = \frac{2}{\tau^2} \left( \frac{D_x \tau \cdot \tau}{\tau} \right)^2 = \frac{2}{\tau^2} \left( D_x \tau \cdot \tau - D_x^2 \tau \cdot \tau + 3 \tau \frac{D_x^2 \tau \cdot \tau}{\tau^2} \right).
\]

On the other hand, it immediately follows from the bilinear equations (4.3) and (4.4), hence we have (4.5). Equation (4.5) is the potential defocusing mKdV equation (3.7).

We give a solution to the bilinear equations (4.1)–(4.4). For a positive integer \( N \) and an integer \( k \), we denote by \( \rho_N \) the determinant
\[
\rho_N (k) = \det \begin{bmatrix} f^{(1)}_k & f^{(1)}_{k+1} & \cdots & f^{(1)}_{k+N-1} \\ f^{(2)}_k & f^{(2)}_{k+1} & \cdots & f^{(2)}_{k+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ f^{(N)}_k & f^{(N)}_{k+1} & \cdots & f^{(N)}_{k+N-1} \end{bmatrix}, \tag{4.6}
\]
and set \( \rho_0 = 1 \). Here the entries \( f^{(i)}_{j-1} \) are functions in \( x, t \), and auxiliary variables \( y, z \).

Proposition 4.2 For a positive integer \( N \) and an integer \( k \), define the entries of (4.6) by
\[
f^{(i)}_n = \alpha_i p_i^n e^{\eta_i} + \beta_i \left( -p_i \right)^n e^{-\eta_i}, \\
\eta_i = p_i x + 4p_i^3 t + \frac{y}{p_i},
\]
where \( \alpha_i, \beta_i \in \mathbb{R} \) and \( p_i \in \mathbb{R}^\times \) are arbitrary constants, and set
\[
\tau_N (k) = \frac{e^{-\text{xy}}}{\prod_{i=1}^{N} p_i} \rho_N (k).
\]
Then the pair of functions $\tau = \tau_N(k)$ and $\tau = \tau_N(k+1)$ satisfy the bilinear equations (4.1)–(4.4). In particular,

$$\theta = 2\log \frac{\rho_N(k)}{\rho_N(k+1)} \tag{4.7}$$

satisfies the potential defocusing mKdV equation (3.7).

To prove this, we make use of the following well-known lemma.

**Lemma 4.3** Let $f_{n}^{(i)}$ ($i,n \in \mathbb{Z}$) be a sequence of functions in $x$, $t$, $y$ and $z$ which satisfy

$$\frac{\partial}{\partial x} f_{n}^{(i)} = f_{n+1}^{(i+1)}, \quad \frac{\partial}{\partial y} f_{n}^{(i)} = f_{n+2}^{(i)},$$

$$\frac{\partial}{\partial t} f_{n}^{(i)} = -4f_{n+3}^{(i)}, \quad \frac{\partial}{\partial y} f_{n}^{(i)} = f_{n-1}^{(i)}. \tag{4.8}$$

For a positive integer $N$ and an integer $k$, define a function $\sigma(k) = \sigma(x,t;y,z;k)$ by $\sigma(k) = e^{-3k} \rho_N(k)$. Then $\sigma(k)$ satisfies the bilinear equations

$$D_x D_y \sigma(k) \cdot \sigma(k) = -2\sigma(k+1) \sigma(k-1), \tag{4.9}$$

$$(D_x^2 - D_z) \sigma(k+1) \cdot \sigma(k) = 0, \tag{4.10}$$

$$(D_x + D_y + 3D_x D_z) \sigma(k+1) \cdot \sigma(k) = 0. \tag{4.11}$$

The system of bilinear equations (4.9)–(4.11) are included in the discrete two-dimensional Toda lattice hierarchy [12, 17, 18, 19, 20, 21, 22, 23, 24]. A typical example of $f_{n}^{(i)}$ satisfying the condition (4.8) is given by

$$f_{n}^{(i)} = \alpha_i p_i^n e^{p_i x + p_i^2 z - 4p_i t + p_i^{-1} y} + \beta_i q_i^n e^{q_i x + q_i^2 z - 4q_i t + q_i^{-1} y}, \tag{4.12}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ and $p_i, q_i \in \mathbb{R}^\times$ are arbitrary constants.

**Proof.** Let us prove Proposition 4.2. Fix $N$ and $k$. Let $\sigma(k)$ a function as introduced in Lemma 4.3 with entries (4.12). Imposing on it the reduction condition $q_i = -p_i$, we have for all integers $i$ and $n$ that

$$f_{n+2}^{(i)} = \frac{\partial}{\partial z} f_{n}^{(i)} = p_i^2 f_{n}^{(i)},$$

which yields

$$\sigma(k+2) = \sigma(k) \prod_{i=1}^{N} p_i^2, \quad \frac{\partial}{\partial z} \sigma(k) = \sigma(k) \sum_{i=1}^{N} p_i^2. \tag{4.13}$$

Because $\sigma(k) = \tau_N(k) \prod_{i=1}^{N} p_i^k$, equations (4.13) are rewritten in terms of $\tau_N(k)$ as

$$\tau_N(k+2) = \tau_N(k), \quad \frac{\partial}{\partial z} \tau_N(k) = \tau_N(k) \sum_{i=1}^{N} p_i^2. \tag{4.14}$$

Therefore the system of bilinear equations (4.1)–(4.4) with $c = 0$ immediately follows from (4.9)–(4.11) on writing $\tau = \tau_N(k)$ and $\tau = \tau_N(k+1)$. \qed
Figure 1: Profiles of the solution to the defocusing mKdV equation (3.5) \( \kappa = \theta_x \), where \( \theta \) is in (4.7). Parameters in Proposition 4.2 are \( N = 2 \), \( p_1 = 0.3 \), \( p_2 = 0.9 \), \( \alpha_1 = \beta_1 = \beta_2 = 1 \) and \( \alpha_2 = -1 \), and \( t = -7 \) (a), \( t = 0 \) (b), \( t = 7 \) (c).

Figure 2: Profiles of the defocusing mKdV flow (3.4) given in Theorem 4.1. Parameters in Proposition 4.2 are \( N = 2 \), \( p_1 = 0.3 \), \( p_2 = 0.9 \), \( \alpha_1 = \beta_1 = \beta_2 = 1 \) and \( \alpha_2 = -1 \), and \( t = -7 \) (a), \( t = 0 \) (b), \( t = 7 \) (c).

### 4.2 Regular solutions

The formula in Theorem 4.1 together with Proposition 4.2 gives exact solutions to the potential defocusing mKdV equation (3.7) and the corresponding defocusing mKdV flow (3.4). However, as shown in Figures 1 and Figure 2, they have singular points since the \( \tau \) function \( \rho_N(k) \), \( \rho_N(k+1) \) have zeros in general. Actually, it seems difficult to choose parameters such that those \( \tau \) functions are positive valued.

In contrast to this, by using the \( \tau \) functions \( \rho_N(k) \), \( \rho_{N+1}(k) \) appropriately, we are able to give regular solutions to the potential defocusing mKdV equation (3.7) according to the idea suggested in [16]. With this choice of \( \tau \) functions, however, it is difficult to construct an explicit formula for the corresponding motion of curves on the Minkowski plane since functions \( \tau = \tau_N(k) \) and \( \tilde{\tau} = \tau_{N+1}(k) \), which are defined by the same way as that in Proposition 4.2, do not satisfy the bilinear equations (4.1) and (4.2). Therefore, we only present a regular solutions to the potential defocusing mKdV equation and observe the corresponding defocusing mKdV flow by using a numerical method. First we introduce the following lemma:
Lemma 4.4 Let \( g_n^{(i)} \) (\( i, n \in \mathbb{Z} \)) be a sequence of functions in \( u, v \) and \( w \), which satisfy

\[
\frac{\partial}{\partial u} g_n^{(i)} = g_{n+1}^{(i)}, \quad \frac{\partial}{\partial v} g_n^{(i)} = g_{n+2}^{(i)}, \quad \frac{\partial}{\partial w} g_n^{(i)} = g_{n+3}^{(i)}.
\]  

(4.15)

For a positive integer \( N \) and an integer \( k \), we denote by \( h_N \) the determinant

\[
h_N(k) = \det \begin{pmatrix}
g_k^{(1)} & g_{k+1}^{(1)} & \cdots & g_{k+N-1}^{(1)} \\
g_k^{(2)} & g_{k+1}^{(2)} & \cdots & g_{k+N-1}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
g_k^{(N)} & g_{k+1}^{(N)} & \cdots & g_{k+N-1}^{(N)}
\end{pmatrix},
\]

and set \( h_0 = 1 \). Then \( h_N \) satisfies the bilinear equations of the mKP hierarchy [18, 25, 26, 27, 28]

\[
\begin{align*}
(D_u^2 - D_v) h_{N+1}(k) \cdot h_N(k) &= 0, \\
(D_u^3 - 4D_w + 3D_u D_v) h_{N+1}(k) \cdot h_N(k) &= 0.
\end{align*}
\]

(4.16)

Here we note that, for integers \( n, i \) and parameters \( \alpha, \beta, p_i, q_i \), the function

\[
g_n^{(i)}(u, v, w) = \alpha_p^n e^{p_i u + p_i^2 v + p_i^3 w} + \beta q_i^n e^{q_i u + q_i^2 v + q_i^3 w}
\]

satisfies (4.15). Imposing on this a reduction condition \( q_i = -p_i \), it follows that the determinant function \( h_N \) satisfies \( (\partial / \partial v) h_N = h_N \sum_{i=1}^N p_i^2 \), and hence \( D_v h_{N+1} \cdot h_N = (p_{N+1})^2 h_{N+1} h_N \). Thus the bilinear equations (4.16) become

\[
\begin{align*}
(D_u^2 - p_{N+1}^2) h_{N+1} \cdot h_N &= 0, \\
(D_u^3 - 4D_w + 3p_{N+1}^2 D_v) h_{N+1} \cdot h_N &= 0.
\end{align*}
\]

(4.17)

Using Lemma 4.4 and the bilinear equations (4.17), we construct a class of regular solutions the potential defocusing mKdV equation (3.7) by applying the Galilean transformation and by imposing appropriate conditions on the parameters as follows. We denote by \( \text{sgn} \) the sign function:

\[
\text{sgn} x = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0.
\end{cases}
\]

Proposition 4.5 Fix an integer \( k \). For a positive integer \( N \), define the entries of (4.6) by

\[
\begin{align*}
\eta_i &= p_i x + 4p_i^3 \left(1 - \frac{3}{2} a p_i^2\right) t, \\
\theta &= 2 \text{log} \frac{\rho_N(k)}{\rho_{N+1}(k)}.
\end{align*}
\]

(4.18)

where \( a, \alpha, \beta, \) and \( p_i \) are arbitrary real constants for \( i = 1, 2, \ldots, N + 1 \). Then putting \( a = \frac{p_{N+1}^2}{2} \), the function

\[
\begin{align*}
\theta &= 2 \text{log} \frac{\rho_N(k)}{\rho_{N+1}(k)}.
\end{align*}
\]

(4.18)

satisfies the potential defocusing mKdV equation (3.7). Moreover, if we choose the parameters in such a way that

\[
\begin{align*}
0 < p_1 < \cdots < p_{N+1}, \\
\alpha_i &> 0, \quad \text{sgn} \beta_i = (-1)^{k+i-1}
\end{align*}
\]

(4.19)

for all \( i \), then \( \theta \) gives a regular solution to (3.7).
Proof. First we only prove that $\theta$ is a solution to the potential defocusing mKdV equation (3.5), and afterward we shall verify regularity of $\theta$ on the condition (4.19). We change the independent variables from $(u,v,w)$ to $(x,t,z)$ by
\[ x = u + \frac{3}{2}r^2 w, \quad t = \frac{1}{4}w, \quad z = v, \]
where $r$ is a real constant, and define $\rho_N(x,t) = h_N(u,0,w)$. Then putting $r = p_N^{2} + 1$, we have
\[
\begin{aligned}
(D_2^2 - r^2) \rho_{N+1} \cdot \rho_N &= 0, \\
(D_3^2 - D_t - 3r^2 D_x) \rho_{N+1} \cdot \rho_N &= 0.
\end{aligned}
\]
Thus the pair $\rho_{N+1}$, $\rho_N$ gives a solution to the bilinear equations (4.3) and (4.4). Therefore (4.18) satisfies the potential defocusing mKdV equation (3.7).

\[\square\]

Figure 3: Profiles of the solution to the defocusing mKdV equation (3.5) $\kappa = \theta$, where $\theta$ is in (4.18). Parameters in Proposition 4.5 are $N = 2$, $p_1 = 0.5$, $p_2 = 0.7$, $p_3 = 0.9$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = 1$, $\beta_2 = -1$ and $\beta_3 = 0$, and $t = -30$ (a), $t = 0$ (b), $t = 30$ (c).

Figure 4: Profiles of the solution to the defocusing mKdV equation (3.5) $\kappa = \theta$, where $\theta$ is in (4.18). Parameters in Proposition 4.5 are $N = 2$, $p_1 = 0.5$, $p_2 = 0.7$, $p_3 = 0.9$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_3 = 1$ and $\beta_2 = -1$, and $t = -30$ (a), $t = 0$ (b), $t = 30$ (c).

Now we establish the regularity of $\theta$ by proving the positivity of $\rho_N$ in Proposition 4.2. The condition on the parameters (4.19) plays a crucial role.

Lemma 4.6 Fix an integer $k$. For a positive integer $N$, let $F$ be an $N \times N$ matrix
\[
F = \begin{bmatrix}
\alpha_1 p_1^{k} e^{\eta_1} + \beta_1 q_1^{k} e^{\xi_1} & \cdots & \alpha_1 p_1^{k + N - 1} e^{\eta_1} + \beta_1 q_1^{k + N - 1} e^{\xi_1} \\
\vdots & \ddots & \vdots \\
\alpha_N p_N^{k} e^{\eta_N} + \beta_N q_N^{k} e^{\xi_N} & \cdots & \alpha_N p_N^{k + N - 1} e^{\eta_N} + \beta_N q_N^{k + N - 1} e^{\xi_N}
\end{bmatrix},
\]
where \( \alpha_i, \beta_i, p_i, q_i, \eta_i \) and \( \xi_i \) are arbitrary real parameters for all \( i \). Choose these parameters as

\[
q_N < q_{N-1} < \cdots < q_1 < 0 < p_1 < p_2 < \cdots < p_N, \quad \alpha_i > 0, \quad sgn \beta_i = (-1)^{k+i-1} \quad (i = 1, 2, \ldots, N).
\]

(4.20)

Then \( \det F \) is positive.

**Proof.** The matrix \( F \) is expressed as a product of \( N \times 2N \) and \( 2N \times N \) matrices as

\[
F = \begin{bmatrix}
\alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \alpha_2 & \beta_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \alpha_N & \beta_N
\end{bmatrix}
\begin{bmatrix}
p_1 e^{\eta_1} & \cdots & p_1^{k+N-1} e^{\eta_1} \\
q_1 e^{\xi_1} & \cdots & q_1^{k+N-1} e^{\xi_1} \\
p_2 e^{\eta_2} & \cdots & p_2^{k+N-1} e^{\eta_2} \\
q_2 e^{\xi_2} & \cdots & q_2^{k+N-1} e^{\xi_2} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
p_N e^{\eta_N} & \cdots & p_N^{k+N-1} e^{\eta_N} \\
q_N e^{\xi_N} & \cdots & q_N^{k+N-1} e^{\xi_N}
\end{bmatrix}.
\]

Then by using the Cauchy-Binet formula, we have

\[
\det F = \sum_{\mu_1, \mu_2, \ldots, \mu_N} d(\mu_1, \mu_2, \ldots, \mu_N), \quad (4.21)
\]

where the summation in (4.21) is taken over all possible combinations of

\[
\mu_1 \in \{ \alpha_1, \beta_1 \}, \quad \mu_2 \in \{ \alpha_2, \beta_2 \}, \ldots, \quad \mu_N \in \{ \alpha_N, \beta_N \},
\]

\[
d(\mu_1, \mu_2, \ldots, \mu_N) = \det \begin{bmatrix}
\mu_1 & 0 & \cdots & 0 \\
0 & \mu_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mu_N
\end{bmatrix}
\begin{bmatrix}
v_1 e^{\omega_1} & \cdots & v_1^{k+N-1} e^{\omega_1} \\
v_2 e^{\omega_2} & \cdots & v_2^{k+N-1} e^{\omega_2} \\
\vdots & \vdots & \ddots & \vdots \\
v_N e^{\omega_N} & \cdots & v_N^{k+N-1} e^{\omega_N}
\end{bmatrix}
\]

\[
= \prod_{n=1}^{N} \mu_n v_n^k e^{\omega_n} \det \begin{bmatrix}
1 & v_1 & \cdots & v_1^{k+N-1} \\
1 & v_2 & \cdots & v_2^{k+N-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & v_N & \cdots & v_N^{k+N-1}
\end{bmatrix}
\]

\[
= \prod_{n=1}^{N} \mu_n v_n^k e^{\omega_n} \prod_{1 \leq i < j \leq N} (v_j - v_i),
\]

and each \((v_i, \omega_i)\) is accordingly given by

\[
(v_i, \omega_i) = \begin{cases} 
(p_i, \eta_i) & \text{if } \mu_i = \alpha_i \\
(q_i, \xi_i) & \text{if } \mu_i = \beta_i.
\end{cases}
\]

It suffices for the positivity of \( \det F \) to show that \( d(\mu_1, \mu_2, \ldots, \mu_N) > 0 \) for each \((\mu_1, \mu_2, \ldots, \mu_N)\). We arbitrarily fix a choice \((\mu_1, \mu_2, \ldots, \mu_N)\). If \( \mu_i = \alpha_i \) for all \( i \), then \( d(\mu_1, \mu_2, \ldots, \mu_N) \) is obviously positive. If \( \mu_i = \beta_i \) for all \( i \), then \( d(\mu_1, \mu_2, \ldots, \mu_N) \) is positive, because

\[
\text{sgn} \prod_{n=1}^{N} \beta_n q_n^k = (-1)^{N(N-1)/2}, \quad \text{sgn} \prod_{1 \leq i < j \leq N} (q_j - q_i) = (-1)^{N(N-1)/2}.
\]
For the remaining cases, we divide the set \( \mu = \{ \mu_1, \mu_2, \ldots, \mu_N \} \) into \( \mu = \mu^\alpha \cup \mu^\beta \), where

\[
\mu^\alpha = \{ \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_{N-r}} \}, \quad \mu^\beta = \{ \beta_{i_1}, \beta_{i_2}, \ldots, \beta_i \}
\]

with some \( r \in \{1, 2, \ldots, N-1\} \). Here the indices are sorted in ascending order, namely \( j_1 < j_2 < \cdots < j_{N-r} \) and \( i_1 < i_2 < \cdots < i_r \). Then we have

\[
\text{sgn} \prod_{n=1}^{N} \mu_n v_n^k = (-1)^{i_1 + \cdots + i_r}, \quad (4.22)
\]

because

\[
\prod_{n=1}^{N} \mu_n v_n^k = \mu_1 \mu_2 \cdots \mu_N (v_1 v_2 \cdots v_N)^k
\]

\[
= \alpha_{j_1} \cdots \alpha_{j_{N-r}} \beta_{i_1} \cdots \beta_i (p_{j_1} \cdots p_{j_{N-r}} q_{i_1} \cdots q_i)^k
\]

\[
= \alpha_{j_1} \cdots \alpha_{j_{N-r}}(-1)^{i_1 + \cdots + i_r + r-k} [\beta_{i_1} \cdots \beta_i] (-1)^k (p_{j_1} \cdots p_{j_{N-r}} q_{i_1} \cdots q_i)^k
\]

\[
= (-1)^{i_1 + \cdots + i_r-r} \prod_{n=1}^{N} |\mu_n v_n^k|.
\]

Similarly we have

\[
\text{sgn} \prod_{1 \leq i < j \leq N} (v_j - v_i) = (-1)^{i_1 + \cdots + i_r-r}. \quad (4.23)
\]

In fact, for each \( j \in \{2, 3, \ldots, N\} \), it follows for all \( i = 1, 2, \ldots, j-1 \) that

\[
\text{sgn}(v_j - v_i) = \begin{cases} 1 & \text{if } v_j = p_j \\ -1 & \text{if } v_j = q_j, \end{cases}
\]

and consequently

\[
\text{sgn} \prod_{i=1}^{j-1} (v_j - v_i) = \begin{cases} 1 & \text{if } j \in \{j_1, \ldots, j_{N-r}\} \\ (-1)^{j-1} & \text{if } j \in \{i_1, \ldots, i_r\}. \end{cases}
\]

Thus we readily have (4.23) if \( i_1 \neq 1 \). We note that, if \( i_1 = 1 \), we have

\[
\text{sgn} \prod_{1 \leq i < j \leq N} (v_j - v_i) = (-1)^{i_2 + \cdots + i_r - (r-1)} = (-1)^{i_1 + i_2 + \cdots + i_r-r}.
\]

Therefore it follows from (4.22) and (4.23) that

\[
\text{sgn} d(\mu_1, \mu_2, \ldots, \mu_N) = \text{sgn} \left( \prod_{n=1}^{N} \mu_n v_n^k e_{\alpha} \prod_{1 \leq i < j \leq N} (v_j - v_i) \right) = 1.
\]

Thus every \( d(\mu_1, \mu_2, \ldots, \mu_N) \) is positive, and hence \( \det F \) is positive. \( \square \)

The positivity of \( \rho_{\alpha}(k) \) and thus the regularity of \( \theta \) follow immediately from Lemma 4.6 by putting \( q_i = -p_i \) and \( \xi_i = -\eta_i \) for \( i = 1, 2, \ldots, N \). We illustrate some spacelike curves by using the representation formula (2.3) and the regular solution in Proposition 4.5. By applying a numerical integration in (2.3) with \( \theta \) in (4.18), we have the following figures.

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Figure 5: Profiles of the defocusing mKdV flow (3.4) given as (2.3) with $\theta$ in (4.18). Parameters in Proposition 4.5 are $N = 2$, $p_1 = 0.5$, $p_2 = 0.7$, $p_3 = 0.9$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = 1$, $\beta_2 = -1$ and $\beta_3 = 0$, and $t = -18$ (a), $t = -5$ (b), $t = 8$ (c).

Figure 6: Profiles of the defocusing mKdV flow (3.4) given as (2.3) with $\theta$ in (4.18). Parameters in Proposition 4.5 are $N = 2$, $p_1 = 0.5$, $p_2 = 0.7$, $p_3 = 0.9$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_3 = 1$ and $\beta_2 = -1$, and $t = -15$ (a), $t = 0$ (b), $t = 15$ (c).

5 Concluding remarks

In this paper, we formulated the motion of spacelike curves on the Minkowski plane preserving the arc length which is governed by the defocusing mKdV equation. Then we constructed two classes of exact solutions to the defocusing mKdV equation in terms of the $\tau$ functions. Especially, one of them can be used to construct the explicit formula for the corresponding defocusing mKdV flow in terms of the same $\tau$ functions. However, those solutions contain singular points where the $\tau$ functions have zeros. Since we are usually interested in regular solutions, we have also presented the regular solutions to the defocusing mKdV equation by using different types of the $\tau$ functions and choosing suitable parameters. These solutions describe the solitons running on a shock wave, including the dark solitons as special cases, whose behavior is different from solutions of other soliton equations. On the other hand, it seems that this class of solutions does not allow the similar explicit formula to the solutions mentioned above. Therefore, we used a numerical integration to observe the dynamics of the corresponding mKdV flow on the Minkowski plane curves. It may be an interesting and important problem to extend the motion of curves in this paper to those of discrete curves.
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References

[1] H. Hasimoto, A soliton on a vortex filament, J. Fluid Mech. 51 (1972), 477–485.
[2] R. E. Goldstein and D. M. Petrich, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, Phys. Rev. Lett. 67 (1991), 3203–3206.
[3] G. L. Lamb, Jr., Solitons and the motion of helical curves, Phys. Rev. Lett. 37 (1976), 235–237.
[4] K. -S. Chou and C. -Z. Qu, Integrable equations arising from motions of plane curves, Phys. D 162 (2002), 9–33.
[5] K. -S. Chou and C. -Z. Qu, Integrable equations arising from motions of plane curves. II, J. Nonlinear Sci. 13 (2003), 487–517.
[6] K. -S. Chou and C. -Z. Qu, Motions of curves in similarity geometries and Burgers-mKdV hierarchies, Chaos Solitons Fractals 19 (2004), 47–53.
[7] A. Fujioka and T. Kurose, Hamiltonian formalism for the higher KdV flows on the space of closed complex equicentroaffine curves, Int. J. Geom. Methods Mod. Phys. 7 (2010), 165–175.
[8] K. Kajiwara, T. Kuroda and N. Matsuura, Isogonal deformation of discrete plane curves and discrete Burgers hierarchy, Pac. J. Math. Ind. 8 (2016), 14.
[9] U. Pinkall, Hamiltonian flows on the space of star-shaped curves, Results Math. 27 (1995), 328–332.
[10] S. Hirose, J. Inoguchi, K. Kajiwara, N. Matsuura and Y. Ohta, Discrete local induction equation, arXiv:1708.01704 (2017).
[11] J. Inoguchi, K. Kajiwara, N. Matsuura and Y. Ohta, Explicit solutions to the semi-discrete modified KdV equation and motion of discrete plane curves, J. phys. A 45 (2012), 045206.
[12] J. Inoguchi, K. Kajiwara, N. Matsuura and Y. Ohta, Motion and Bäcklund transformations of discrete plane curves, Kyushu J. Math. 66 (2012), 303–324.
[13] H. Park, K. Kajiwara, T. Kurose and N. Matsuura, Defocusing mKdV flow on centroaffine plane curves, submitted to JSIAM Letters.
[14] T. L. Perelman, A. K. Fridman and M. M. El’yashevich, Modified korteweg-de vries equation in electrohydrodynamics, Sov. Phys. JETP 39 (1974), 643–646.
[15] T. L. Perelman, A. K. Fridman and M. M. El’yashevich, On the relationship between the n-soliton solution of the modified korteweg-de vries equation and the kdv equation solution, Phys. Lett. 47A (1974), 321–323.
[16] Y. Ohta, Wronskian solutions to soliton equations, RIMS Kokyuroku 684 (1989) 1–17.
[17] R. Hirota, *The direct method in soliton theory* (Cambridge University Press, Cambridge, 2004).
[18] M. Jimbo and T. Miwa, Solitons and infinite-dimensional Lie algebras, *Publ. Res. Inst. Math. Sci.* **19** (1983), 943–1001.
[19] K. Maruno, K. Kajiwara and M. Oikawa, Casorati determinant solution for the discrete-time relativistic Toda lattice equation, *Phys. Lett. A* **241** (1998), 335–343.
[20] K. Maruno and Y. Ohta, Casorati determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation, *J. Phys. Soc. Japan* **75** (2006), 054002.
[21] Y. Ohta, R. Hirota, S. Tsujimoto and T. Imai, Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy, *J. Phys. Soc. Japan* **62** (1993), 1872–1886.
[22] Y. Ohta, K. Kajiwara, J. Matsukidaira and J. Satsuma, Casorati determinant solution for the relativistic Toda lattice equation, *J. Math. Phys.* **34** (1993), 5190–5204.
[23] S. Tsujimoto, On a discrete analogue of the two-dimensional Toda lattice hierarchy, *Publ. Res. Inst. Math. Sci.* **38** (2002), 113–133.
[24] K. Ueno, and K. Takasaki, Toda lattice hierarchy, in: Group representations and systems of differential equations, (Tokyo, 1982), *Adv. Stud. Pure Math.* **4** (1984), 1–95.
[25] N. C. Freeman and J. J. C. Nimmo, Soliton solutions of the Korteweg-de Vries and the Kadomtsev-Petviashvili equations: the Wronskian technique, *Proc. Roy. Soc. London Ser. A* **389** (1983), 319–329.
[26] F. Gesztesy and W. Schweiger, Rational KP and mKP-solutions in Wronskian form, *Rep. Math. Phys.* **30** (1991), 205-222.
[27] R. Hirota, Y. Ohta and J. Satsuma, Solutions of the Kadomtsev-Petviashvili equation and the two-dimensional Toda equations, *J. Phys. Soc. Japan* **57** (1988), 1901–1904.
[28] R. Hirota, Y. Ohta and J. Satsuma, Wronskian structures of solutions for soliton equations, *Progr. Theoret. Phys. Suppl.* **94** (1988), 59–72.