A CHARACTERIZATION OF TWO WEIGHT TRACE INEQUALITIES FOR POSITIVE DYADIC OPERATORS IN THE UPPER TRIANGLE CASE

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Abstract. Two weight trace inequalities for positive dyadic operators are characterized in terms of discrete Wolff’s potentials in the upper triangle case \(1 < q < p < \infty\).

1. Introduction

The purpose of this paper is to establish the two weight \(T1\) theorem for positive dyadic operators in the upper triangle case \(1 < q < p < \infty\). We first fix some notations. We will denote \(D\) by the family of all dyadic cubes \(Q = 2^{-i}(k + [0, 1)^n), i \in \mathbb{Z}, k \in \mathbb{Z}^n\). Let \(\sigma\) and \(\omega\) be nonnegative Radon measures on \(\mathbb{R}^n\) and let \(K : D \to [0, \infty)\) be a map. For an \(f \in L^1_{loc}(d\sigma)\) the positive dyadic operator \(T_K[f d\sigma]\) is defined by

\[
T_K[f d\sigma](x) := \sum_{Q \in D} K(Q) \int_Q f \, d\sigma 1_Q(x) \quad x \in \mathbb{R}^n.
\]

We will denote by \(K_\sigma(Q)(x)\) the function

\[
K_\sigma(Q)(x) := \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} K(Q') \sigma(Q') 1_{Q'}(x), \quad x \in Q \in D,
\]

and \(K_\sigma(Q)(x) = 0\) when \(\sigma(Q) = 0\). For \(s > 1\) discrete Wolff’s potential of \(\omega\) \(W^s_{K, \sigma}(\omega)(x)\) is defined by

\[
W^s_{K, \sigma}(\omega)(x) := \sum_{Q \in D} K(Q) \sigma(Q) \left(\int_Q K_\sigma(Q)(y) \, d\omega(y)\right)^{s-1} 1_Q(x), \quad x \in \mathbb{R}^n.
\]

The pair \((K, \sigma)\) is said to be satisfy the dyadic logarithmic bounded oscillation (DLBO) condition, if they fulfill

\[
\sup_{x \in Q} K_\sigma(Q)(x) \leq A \inf_{x \in Q} K_\sigma(Q)(x),
\]

where the constant \(A\) does not depend on \(Q \in D\). For each \(1 < p < \infty\), \(p'\) will denote the dual exponent of \(p\), i.e., \(p' = \frac{p}{p-1}\).

In their significant paper [2], Cascante, Ortega and Verbitsky established the following:

Proposition 1.1 ([2 Theorem A]). Let \(0 < q < p < \infty\) and \(1 < p < \infty\). Suppose that the pair \((K, \sigma)\) satisfy the DLBO condition. Then two weight trace inequality

\[
\|T_K[f d\sigma]\|_{L^q(d\omega)} \leq C_1 \|f\|_{L^p(d\sigma)}
\]

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holds if and only if
\[ \|W_{p,\sigma}^p[\omega]\|_{L^r(d\omega)}^{1/p'} \leq C_2 < \infty, \text{ where } \frac{1}{q} = \frac{1}{r} + \frac{1}{p}. \]

Moreover, the least possible \(C_1\) and \(C_2\) are equivalent.

In his elegant paper [10] Sergei Treil gives a simple proof of the following two weight \(T1\) theorem for positive dyadic operators in the lower triangle case.

**Proposition 1.2 (Theorem 2.1).** Let \(1 < p \leq q < \infty\). Then two weight trace inequality \((1.1)\) holds if and only if
\[
\begin{align*}
\sup_{Q \in \mathcal{D}} & \frac{1}{\omega(Q)} \left( \int_Q \left( \sum_{Q' \subset Q} K(Q')\omega(Q')1_{Q'} \right)^q d\omega \right)^{1/q} \leq C_2 < \infty, \\
\sup_{Q \in \mathcal{D}} & \frac{1}{\omega(Q)} \left( \int_Q \left( \sum_{Q' \subset Q} K(Q')\sigma(Q')1_{Q'} \right)^{p'} d\sigma \right)^{1/p'} \leq C_2 < \infty.
\end{align*}
\]

Moreover, the least possible \(C_1\) and \(C_2\) are equivalent.

Proposition 1.2 was first proved for \(p = 2\) in [6] by the Bellman function method. Later in [4] this was proved in full generality \(1 < p \leq q < \infty\). The checking condition in Proposition 1.2 is called “Sawyer type checking condition”, since this was first introduced by Eric T. Sawyer in [7, 8].

In his excellent survey of the \(A_2\) theorem [3] Tuomas P. Hytönen introduces another proof of Proposition 1.2 which uses the “parallel corona” decomposition from the recent work of Lacey, Sawyer, Shen and Uriarte-Tuero [5] on the two weight boundedness of the Hilbert transform.

Following Hytönen’s arguments and applying a basic lemma due to [1], we shall establish the following two weight \(T1\) theorem for positive dyadic operators in the upper triangle case.

**Theorem 1.3.** Let \(1 < q < p < \infty\). Then two weight trace inequality \((1.1)\) holds if and only if
\[
\begin{align*}
\left\| W_{p,\omega}^q[\sigma]^{1/q} \right\|_{L^r(d\sigma)} & \leq C_2 < \infty, \\
\left\| W_{p,\sigma}^p[\omega]^{1/p'} \right\|_{L^r(d\omega)} & \leq C_2 < \infty, \text{ where } \frac{1}{q} = \frac{1}{r} + \frac{1}{p}. 
\end{align*}
\]

Moreover, the least possible \(C_1\) and \(C_2\) are equivalent.

**Remark 1.4.** The DLBO condition is essential and quite useful. In [9], we develop a theory of weights for positive operators in a filtered measure space based upon this condition.

The letter \(C\) will be used for constants that may change from one occurrence to another. Constants with subscripts, such as \(C_1, C_2\), do not change in different occurrences.

2. **Proof of Theorem 1.3**

In what follows we shall prove Theorem 1.3. We need a basic lemma [1, Theorem 2.1]. For the sake of completeness, we will give the proof and will also check the constants.
Lemma 2.1. Let $\sigma$ be a Radon measure on $\mathbb{R}^n$. Let $1 < s < \infty$ and $\{\alpha_Q\}_{Q \in D} \subset [0, \infty)$. Define, for $Q_0 \in D$,

$$
A_1 := \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^s d\sigma,
$$

$$
A_2 := \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1},
$$

$$
A_3 := \int_{Q_0} \sup_{x \in Q \subset Q_0} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^s d\sigma(x).
$$

Then

$$
A_1 \leq c(s) A_2, \quad A_2 \leq c(s)^{\frac{s}{s-1}} A_3 \quad \text{and} \quad A_3 \leq (s')^s A_1.
$$

Here,

$$
c(s) := \begin{cases} 
    s, & 1 < s \leq 2, \\
    (s(s-1) \cdots (s-k))^{\frac{s-1}{s-k-1}}, & 2 < s < \infty,
\end{cases}
$$

where $k = \lceil s - 2 \rceil$ is the smallest integer greater than $s - 2$.

Proof. By a standard limiting argument, we may assume without loss of generality that there is only a finite number of $\alpha_Q \neq 0$.

(i) We prove $A_1 \leq c(s) A_2$. We use an elementary inequality

$$
\left( \sum_i a_i \right)^s \leq s \sum_i a_i \left( \sum_{j \geq i} a_j \right)^{s-1},
$$

where $\{a_i\}_{i \in \mathbb{Z}}$ is a sequence of summable nonnegative reals. First, we verify the simple case $1 < s \leq 2$. It follows from (2.1) that

$$
A_1 = \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^s d\sigma
\leq s \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} \int_{Q} \left( \sum_{Q' \subset Q} \frac{\alpha_{Q'}}{\sigma(Q')} 1_{Q'} \right)^{s-1} d\sigma
\leq s \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \int_{Q} \left( \sum_{Q' \subset Q} \frac{\alpha_{Q'}}{\sigma(Q')} 1_{Q'} \right) d\sigma \right)^{s-1}
= s \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} = s A_2,
$$

where we have used $s - 1 \leq 1$ and Hölder’s inequality. Next, we prove the case $s > 2$. Let $k = \lceil s - 2 \rceil$ be the smallest integer greater than $s - 2$. Applying (2.1) $(k+1)$-times, we have

$$
A_1 = s(s-1) \cdots (s-k)
\times \sum_{P_k \subset \cdots \subset P_1 \subset P_0 \subset Q_0} \frac{\alpha_{P_0}}{\sigma(P_0)} \frac{\alpha_{P_1}}{\sigma(P_1)} \cdots \frac{\alpha_{P_k}}{\sigma(P_k)} \int_{P_k} \left( \sum_{P \subset P_k} \frac{\alpha_p}{\sigma(P)} 1_P \right)^{s-k-1} d\sigma.
$$
H" older's inequality gives

\[ \frac{1}{\sigma(P_k)} \int_{P_k} \left( \sum_{P \subset P_k} \alpha_P \right)^{s-k-1} d\sigma \leq \left( \frac{1}{\sigma(P_k)} \sum_{P \subset P_k} \alpha_P \right)^{s-k-1}. \]

These yield

\[ A_1 \leq s(s-1) \cdots (s-k) \]
\[ \times \int_{Q_o} \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^k \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-k-1} 1_Q \right) d\sigma. \]

H" older's inequality with exponent \( \frac{k}{s-1} + \frac{s-k-1}{s-1} = 1 \) gives

\[ \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-k-1} 1_Q \leq \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^{\frac{k}{s-1}} \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} 1_Q \right)^{\frac{s-k-1}{s-1}}, \]

and, hence,

\[ A_1 \leq s(s-1) \cdots (s-k) \]
\[ \times \int_{Q_o} \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^{\frac{k}{s-1}} \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} 1_Q \right)^{\frac{s-k-1}{s-1}} d\sigma. \]

H" older's inequality with the same exponent gives

\[ A_1 \leq s(s-1) \cdots (s-k) A_2^{\frac{k}{s-1}} A_3^{\frac{s-k-1}{s-1}}. \]

Thus, we obtain \( A_1 \leq c(s)A_2 \).

(ii) We prove \( A_2 \leq c(s)^{\frac{1}{s-1}} A_3 \). It follows that

\[ A_2 = \int_{Q_o} \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} 1_Q \right) d\sigma \leq \int_{Q_o} \left( \sum_{Q \subset Q_o} \frac{\alpha_Q}{\sigma(Q)} 1_Q(x) \left( \sup_{x \in Q \subset Q_o} \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} \right) d\sigma(x). \]

H" older's inequality gives

\[ A_2 \leq A_1^{\frac{1}{s-1}} A_3^{\frac{1}{s-1}}. \]

Since we have had \( A_1 \leq c(s)A_2 \), we obtain \( A_2 \leq c(s)^{\frac{1}{s-1}} A_3 \).
(iii) We prove \( A_3 \leq (s')^* A_1 \). It follows that

\[
A_3 = \int_{Q_0} \sup_{x \in Q \subset Q_0} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^s d\sigma(x)
\]

\[
\leq \int_{Q_0} M_\sigma \left[ \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q(x) \right]^s d\sigma(x)
\]

\[
\leq (s')^* A_1,
\]

where \( M_\sigma \) is the dyadic Hardy-Littlewood maximal operator and we have used the \( L^s(d\sigma) \)-boundedness of \( M_\sigma \). This completes the proof. \( \square \)

**Proof of Theorem 1.3 (Sufficiency):** We follow the arguments due to Hytönen in \[3\]. Let \( Q_0 \in \mathcal{D} \) be taken large enough and be fixed. We shall estimate the quantity

\[
\sum_{Q \subset Q_0} K(Q) \int_Q f d\sigma \int_Q g d\omega,
\]

where \( f \in L^p(d\sigma) \) and \( g \in L^{q'}(d\omega) \) are nonnegative and are supported in \( Q_0 \).

We define the collections of principal cubes \( \mathcal{F} \) for the pair \((f, \sigma)\) and \( \mathcal{G} \) for the pair \((g, \omega)\). Namely, analogously for \( \mathcal{G} \),

\[
\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k,
\]

where \( \mathcal{F}_0 := \{Q_0\} \),

\[
\mathcal{F}_{k+1} := \bigcup_{F \in \mathcal{F}_k} \text{ch}_F(F)
\]

and \( \text{ch}_F(F) \) is defined by the set of all maximal dyadic cubes \( Q \subset F \) such that

\[
\frac{1}{\sigma(Q)} \int_Q f d\sigma > \frac{2}{\sigma(F)} \int_F f d\sigma.
\]

Observe that

\[
\sum_{F' \in \text{ch}_F(F)} \sigma(F') \leq \left( \frac{2}{\sigma(F)} \int_F f d\sigma \right)^{-1} \sum_{F' \in \text{ch}_F(F)} \int_{F'} f d\sigma
\]

\[
\leq \left( \frac{2}{\sigma(F)} \int_F f d\sigma \right)^{-1} \int_F f d\sigma = \frac{\sigma(F)}{2},
\]

and, hence,

\[
(2.3) \quad \sigma(E_{\mathcal{F}}(F)) := \sigma \left( F \setminus \bigcup_{F' \in \text{ch}_F(F)} F' \right) \geq \frac{\sigma(F)}{2},
\]

where the sets \( E_{\mathcal{F}}(F) \) are pairwise disjoint.

We further define the stopping parents, for \( Q \in \mathcal{D} \),

\[
\begin{align*}
\pi_{\mathcal{F}}(Q) &:= \min \{ F \supset Q : F \in \mathcal{F} \}, \\
\pi_{\mathcal{G}}(Q) &:= \min \{ G \supset Q : G \in \mathcal{G} \}, \\
\pi(Q) &:= (\pi_{\mathcal{F}}(Q), \pi_{\mathcal{G}}(Q)).
\end{align*}
\]
Then we can rewrite the series in (2.2) as follows:

\[ \sum_{Q \subset Q_0} = \sum_{F \in F} \sum_{Q: \pi(Q) = (F,G)} \sum_{Q} + \sum_{G \subset F} \sum_{F \subset G} \sum_{Q: \pi(Q) = (F,G)} \cdot \]

where we have used the fact that if \( P, Q \in D \) then \( P \cap Q \in \{P, Q, \emptyset\} \). Since the proof can be done in completely symmetric way, we shall concentrate ourselves on the first case only.

It follows that, for \( F \in F \),

\[ \sum_{G \subset F} \sum_{Q: \pi(Q) = (F,G)} K(Q) \int \int g \, d\omega \]

\[ = \sum_{G \subset F} \sum_{Q: \pi(Q) = (F,G)} K(Q) \sigma(Q) \left( \frac{1}{\sigma(Q)} \int \int f \, d\sigma \right) \int \int g \, d\omega \]

\[ \leq \frac{2}{\sigma(F)} \int \int f \, d\sigma \sum_{G \subset F} \sum_{Q: \pi(Q) = (F,G)} K(Q) \sigma(Q) \int \int g \, d\omega. \]

We need the two observations. Suppose that \( \pi(Q) = (F,G) \) and \( G \subset F \). If \( F' \in ch_F(F) \) satisfies \( F' \subset Q \), then by definition of \( \pi_F \) we must have

\[ \pi_F (\pi_G(F')) = F. \]

By this observation we define

\[ ch(F) := \{ F' \in ch_F(F) : \pi_F (\pi_G(F')) = F \}. \]

We further observe that, when \( F' \in ch^*_F(F) \), we can regard \( g \) as a constant on \( F' \) in the above integrals. By these observations we see that, by use of Hölder’s inequality,

\[ \sum_{G \subset F} \sum_{Q: \pi(Q) = (F,G)} K(Q) \sigma(Q) \int \int g \, d\omega \]

\[ \leq \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q \, d\omega \right)^{1/q} \]

\[ \times \left( \int_{E_{ch}(F)} g^q \, d\omega + \sum_{F' \in ch^*_F(F)} \left( \frac{1}{\omega(F')} \int_{E_{ch}(F)} g \, d\omega \right)^{q'} \omega(F') \right)^{1/q'} \]

\[ =: \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q \, d\omega \right)^{1/q} \| g_F \|_{L^q(\omega)}. \]
Thus, we obtain
\[
\sum_{F \in \mathcal{F}} \sum_{G \subset F} \sum_{Q: \pi(Q) = (F,G)} K(Q) \int_Q f \, d\sigma \int_Q g \, d\omega \\
\leq \sum_{F \in \mathcal{F}} \frac{2}{\sigma(F)} \left( \int_F f \, d\sigma \left( \int_Q K(Q) \sigma(Q) \right)^q \, d\omega \right)^{1/q} g_{F|L^q(\sigma)} \\
\leq 2 \left( \sum_{F \in \mathcal{F}} \left( \frac{1}{\sigma(F)} \int_F f \, d\sigma \right)^p \sigma(F) \right)^{1/p} \\
\times \left( \sum_{F \in \mathcal{F}} \left( \frac{1}{\sigma(F)^{1/p}} \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) \right)^q \, d\omega \right)^{1/q} \right)^{p'} \right)^{1/p'} \|g_{F|L^{p'}(\sigma)}\|_{L^p(\sigma)}.
\]

= : 2I_1 \times I_2.

For I_1, using \(\sigma(F) \leq 2\sigma(E_F(F))\),
\[
\frac{1}{\sigma(F)} \int_F f \, d\sigma \leq \inf_{y \in F} M_\sigma f(y)
\]
and the disjointness of the \(E_F(F)\), we have
\[
I_1 \leq 2^{1/p} \left( \sum_{F \in \mathcal{F}} \int_{E_F(F)} (M_\sigma f)_p \, d\sigma \right)^{1/p} \\
\leq 2^{1/p} \left( \int_{Q_0} (M_\sigma f)_p \, d\sigma \right)^{1/p} \leq 2^{1/p} \|f\|_{L^p(\sigma)}.
\]

Recall that \(\frac{1}{q} = \frac{1}{r} + \frac{1}{p} \) and let \(\theta := \frac{q'}{p'}\). Then we have \(\theta > 1\) and \(\theta'p' = r\). It follows from Hölder’s inequality with exponent \(\theta\) that
\[
I_2 \leq \left[ \sum_{F \in \mathcal{F}} \left( \frac{1}{\sigma(F)^{1/p}} \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) \right)^q \, d\omega \right)^{1/q} \right)^{p'} \right]^{1/p'} \\
\times \left( \sum_{F \in \mathcal{F}} \|g_{F|L^q(\sigma)}\|_{L^{p'}(\sigma)} \right)^{1/q} \\
=: I_{21} \times I_{22}.
\]
It follows by applying Lemma 2.4 that
\[
\int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) \right)^q \, d\omega \\
\leq c(q) \sum_{Q \subset F} K(Q) \sigma(Q) \omega(Q) \left( \int_{\omega(Q)} \sum_{Q' \subset Q} K(Q') \sigma(Q') \omega(Q') \right)^{q-1} \\
= c(q) \int_F \sum_{Q \subset F} K(Q) \omega(Q) \left( \int_{Q} \mathcal{K}_\omega(Q)(y) \, d\sigma(y) \right)^{q-1} \, 1_Q \, d\sigma.
\]
This implies
\[
\left\{ \frac{1}{\sigma(F)^1/p} \left( \int_F \left( \sum_{Q \subset F} K(Q) \omega(Q) 1_Q \right)^q \, d\omega \right)^{1/q} \right\}^r
\]
\[
\leq c(q)^{r/q} \left( \frac{1}{\sigma(F)} \int \sum_{Q \subset F} K(Q) \omega(Q) \left( \int Q R_Q(Q)(y) \, d\sigma(y) \right)^{q-1} 1_Q \, d\sigma \right)^{r/q} \sigma(F)
\]
\[
\leq 2c(q)^{r/q} \int_{E_F(F)} \left( M_{\sigma} \mathcal{W}^q_{K,\omega}[\sigma] \right)^{r/q} \, d\sigma,
\]
and hence,
\[
I_{21} \leq 2^{1/q} c(q)^{1/q} (r/q)^{1/q} \| \mathcal{W}^q_{K,\omega}[\sigma] \|_{L^r(d\sigma)}.
\]
It remains to estimate \( I_{22} \). It follows that
\[
I_{22}' = \sum_{F \in F} \int_{E_F(F)} \omega(F) \left( \int \sum_{Q \subset F} g^q \, d\omega \right)^{q'} \omega(F').
\]
By the pairwise disjointness of the set \( E_F(F) \), it is immediate that
\[
\sum_{F \in F} \int_{E_F(F)} g^q \, d\omega \leq \| g \|_{L^q(d\omega)}^{q'}.
\]
For the remaining double sum, we use the definition of \( ch^*_F(F) \) to reorganize:
\[
\sum_{F \in F} \sum_{Q \subset F} \left( \frac{1}{\omega(F')} \int q \, d\omega \right)^{q'} \omega(F')
\]
\[
= \sum_{F \in F} \sum_{G \subset F} \sum_{F' \in ch^*_F(F) : \pi_F(G) = F'} \left( \frac{1}{\omega(F')} \int q \, d\omega \right)^{q'} \omega(F')
\]
\[
\leq \sum_{G \subset F} \sum_{F' \in ch^*_F(F) : \pi_F(G) = F'} \left( \frac{2}{\omega(G)} \int q \, d\omega \right)^{q'} \omega(G)
\]
\[
\leq 2 \cdot 2^{q'} \| M_{\omega} g \|_{L^{q'}(d\omega)}^{q'} \leq 2 \cdot 2^{q'} \| g \|_{L^q(d\omega)}^{q'}.
\]
All together, we obtain
\[
\sum_{Q \subset Q_0} K(Q) \int_Q f \, d\sigma \int_Q g \, d\sigma \leq C \| \mathcal{W}^q_{K,\omega}[\sigma] \|_{L^r(d\sigma)} \| f \|_{L^p(d\sigma)} \| g \|_{L^q(d\omega)}^{q'}.
\]
This yields the sufficiency of Theorem 1.3.

**Proof of Theorem 1.3 (Necessity):** This fact was verified in [1, Theorem B (i)]. But, for reader’s convenience the full proof is given here. We assume that the trace inequality (1.1) holds. Then, by Lemma 2.1, there holds
\[
(2.4) \quad \sum_{Q \subset D} K(Q) \omega(Q) \int_Q f \, d\sigma \left( \frac{1}{\omega(Q)} \sum_{Q' \subset Q} K(Q') \omega(Q') \int_{Q'} f \, d\sigma \right)^{q-1} \leq C C^q \| f \|_{L^p(d\sigma)}^{q'}.
\]
where $f \in L^p(d\sigma)$ is nonnegative. For $g \geq 0$ we have
\[
\int_{\mathbb{R}^n} g(x)W_{K,\omega}^q[x]\,d\sigma(x)
\]
\[
= \sum_{Q \in D} K(Q)\omega(Q) \int_Q g\,d\sigma \left( \frac{1}{\omega(Q)} \sum_{Q' \subset Q} K(Q')\omega(Q')\sigma(Q') \right)^{q-1}
\]
\[
= \sum_{Q \in D} K(Q)\omega(Q)\sigma(Q) \left( \int_Q g\,d\sigma \right)^{1/q} \left( \frac{1}{\omega(Q)} \int_Q g\,d\sigma \right)^{1/q} \sum_{Q' \subset Q} K(Q')\omega(Q')\sigma(Q')^{q-1}
\]
\[
\leq \sum_{Q \in D} K(Q)\omega(Q) \left( \int_Q (M_\sigma g)^{1/q}\,d\sigma \right)^{1/q} \left( \frac{1}{\omega(Q)} \sum_{Q' \subset Q} K(Q')\omega(Q') \int_{Q'} (M_\sigma g)^{1/q}\,d\sigma \right)^{q-1}
\]
\[
\leq C C_1^1 \| (M_\sigma g)^{1/q} \|_{L^p(d\sigma)}^q
\]
\[
\leq C C_1^1 \| g \|_{L^{p/q}(d\sigma)}^q,
\]
where we have used (2.4) and the $L^{p/q}(d\sigma)$-boundedness of $M_\sigma$. This implies by duality
\[
\| W_{K,\omega}^q[s]^{1/q} \|_{L^{r}(d\sigma)} \leq C C_1 < \infty.
\]
To verify
\[
\| W_{K,\omega}^{p'/q}[\omega]^{1/p'} \|_{L^{r}(d\omega)} \leq C C_1 < \infty,
\]
we merely use the dual inequality of (1.1).

References

[1] Cascante C., Ortega J. and Verbitsky I., Nonlinear potentials and two weight trace inequalities for general dyadic and radial kernels, Indiana Univ. Math. J., 53 (2004), no. 3, 845–882.
[2] ______, On $L^p-L^q$ trace inequalities, J. London Math. Soc. (2), 74 (2006), no. 2, 497–511.
[3] Hytönen T., The $A_2$ theorem: Remarks and complements, arXiv:1212.3840 (2012).
[4] Lacey M., Sawyer E. and Uriarte-Tuero I., Two weight inequalities for discrete positive operators, arXiv:0911.3437 (2009).
[5] Lacey M., Sawyer E., Shen C.-Y. and Uriarte-Tuero I., Two weight inequality for the Hilbert transform: A real variable characterization, arXiv:1201.4319 (2012).
[6] Nazarov F., Treil S. and Volberg A., The Bellman functions and two-weight inequalities for Haar multipliers, J. of Amer. Math. Soc., 12 (1999), no. 4, 909–928.
[7] Sawyer E., A characterization of a two-weight norm inequality for maximal operators, Studia Math., 75 (1982), no. 1, 1–11.
[8] ______, A characterization of two weight norm inequalities for fractional and Poisson integrals, Trans. Amer. Math. Soc., 308 (1988), no. 2, 533–545.
[9] Tanaka H. and Terasawa Y., Positive operators and maximal operators in a filtered measure space, J. Funct. Anal., 264 (2013), no.4, 920–946.
[10] Treil S., A remark on two weight estimates for positive dyadic operators, arXiv:1201.1455 (2012).

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