On the Spectral Analysis of Direct Sums of Riemann-Liouville Operators in Sobolev Spaces of Vector Functions

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Abstract

Let $J^\alpha_k$ be a real power of the integration operator $J_k$ defined on Sobolev space $W^p_k[0,1]$. We investigate the spectral properties of the operator $A_k = \bigoplus_{j=1}^n \lambda_j J^\alpha_k$ defined on $\bigoplus_{j=1}^n W^p_k[0,1]$. Namely, we describe the commutant $\{A_k\}'$, the double commutant $\{A_k\}''$ and the algebra $\text{Alg} A_k$. Moreover, we describe the lattices $\text{Lat} A_k$ and $\text{HypLat} A_k$ of invariant and hyperinvariant subspaces of $A_k$, respectively. We also calculate the spectral multiplicity $\mu A_k$ of $A_k$ and describe the set $\text{Cyc} A_k$ of its cyclic subspaces. In passing, we present a simple counterexample for the implication

$$\text{HypLat}(A \oplus B) = \text{HypLat} A \oplus \text{HypLat} B \Rightarrow \text{Lat}(A \oplus B) = \text{Lat} A \oplus \text{Lat} B$$

to be valid.

1 Introduction

It is well known \cite{9, 20, 33, 36} that the Volterra integration operator $J : f(x) \rightarrow \int_0^x f(t) \, dt$ as well as its real powers $J^\alpha$ play an exceptional role in the spectral theory of nonselfadjoint operators in $L^2[0,1]$. The paper is devoted to the spectral analysis of direct sums of multiples of powers $J^\alpha$ of the integration operator $J$ in Sobolev spaces. To describe its content we first briefly recall basic facts on the integration operator.

It is well known \cite{9, 20, 33, 36} that $J$ is unicellular on $L^p[0,1]$ for $p \in [1, \infty)$ and the lattice $\text{Lat} J$ of its invariant subspaces is anti-isomorphic to the segment $[0,1]$. The same is also true (see \cite{20, 36}) for the simplest Volterra operators

$$J^\alpha : f(x) \rightarrow \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t) \, dt, \quad \alpha > 0,$$

being the positive powers of the integration operator $J$.

More precisely, it is known (see \cite{9, 20, 33, 36}) that

$$\text{Lat} J^\alpha = \text{HypLat} J^\alpha = \{ E_a : a \in [0,1] \},$$

$$E_a : = \{ f \in L^p[0,1] : f(x) = 0 \text{ for a.a. } x \in [0,a] \}. \quad (1.1)$$
Description (1.1) yields (and, in fact, is equivalent to) [9, 20, 36] the following description of cyclic vectors of \( J^\alpha \)

\[
f \text{is a cyclic vector for } J^\alpha \iff \int_0^\varepsilon |f(x)|^p \, dx > 0 \text{ for all } \varepsilon > 0. \tag{1.2}
\]

This condition is called the \( \varepsilon \)-condition.

Description (1.1) of \( \text{HypLat} J^\alpha \) is closely connected with the description of the commutant \( \{ J^\alpha \}' \). The commutant \( \{ J \}' \) of the operator \( J \) defined on \( L_2[0,1] \) as well as the (weakly closed) algebra \( \text{Alg} J \) generated by \( J \) and \( I \) were originally described by D. Sarason [44] (see also a simple proof in [18]). Another, description of \( \text{Alg} J \) for \( J \) acting in \( L_p[0,1] \) has also been obtained in [29, 30]. Namely, it was shown in [29, 30] that if \( J \) is defined on \( L_p[0,1] \) \( (1 < p < \infty) \), then \( \{ J^\alpha \}' = \text{Alg} J^\alpha \) and \( K \in \{ J^\alpha \}' \) if and only if it is bounded and admits a representation

\[
(K f)(x) = \frac{d}{dx} \int_0^x k(x-t) f(t) \, dt, \quad k \in L_{p'}[0,1], \tag{1.3}
\]

where \( p'^{-1} + p^{-1} = 1 \). Using a criterion of boundedness of \( K \) defined on \( L_2[0,1] \) (see [30, Proposition 3.1']) it can easily be shown that for \( p = 2 \) description (1.3) is equivalent to that obtained in [44].

Now, let \( A = J^\alpha \otimes B(= \bigoplus_{j=1}^n \lambda_j J^\alpha) \) be a tensor product of the operator \( J^\alpha \) defined on \( L_p[0,1] \) and the \( n \times n \) nonsingular diagonal matrix \( B = \text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n \times n} \). The investigation of such operators with \( B = B^* \) was initiated by G. Kalisch [24]. He has extended the known Livsic theorem (see [9, 20]) to the case of (abstract) Volterra operators with finite-dimensional real part and characterized those of them that are unitarily equivalent to \( A \) with \( B = B^* \) and \( \alpha = 1 \) (see also [9, 20]).

Later on, sufficient conditions for a Volterra operator \( K : f \to \int_0^x K(x,t) f(t) \, dt \) defined on \( L_p[0,1] \otimes \mathbb{C}^n \) to be similar to the operator \( A \) have been indicated in [32]. So, \( A \) may be treated as a similarity model for a wide class of Volterra operators. This result has been applied in [32] to the problem of unique recovery of a Dirac type system by its monodromy matrix (see also references therein).

Further, one of the authors [29, 31] described the lattices \( \text{Lat} A \) and \( \text{HypLat} A \) and the set \( \text{Cyc} A \) of cyclic subspaces of the operator \( A = J^\alpha \otimes B(= \bigoplus_{j=1}^n \lambda_j J^\alpha) \) defined on \( L_p[0,1] \otimes \mathbb{C}^n, \ p_1 (1, \infty), \). In particular, in [29, 31] necessary and sufficient conditions for a sequence \( \{ \lambda_i \}_{i=1}^n \) guaranteeing the splitting of each of the lattices \( \text{Lat} A \) and \( \text{HypLat} A \), as well as of the commutant \( \{ A \}' \) and double commutant \( \{ A \}'' \) of \( A \) were found. More precisely, it was proved in [29, 31] that each of the following relations

\[
\text{Lat} \bigoplus_{j=1}^n \lambda_j J^\alpha = \bigoplus_{j=1}^n \text{Lat} \lambda_j J^\alpha, \tag{1.4}
\]

\[
\text{HypLat} \bigoplus_{j=1}^n \lambda_j J^\alpha = \bigoplus_{j=1}^n \text{HypLat} \lambda_j J^\alpha, \tag{1.5}
\]

\[
\left\{ \bigoplus_{j=1}^n \lambda_j J^\alpha \right\}' = \bigoplus_{j=1}^n \left( \lambda_j J^\alpha \right)' = \left\{ \bigoplus_{j=1}^n \lambda_j J^\alpha \right\}'' = \bigoplus_{j=1}^n \left( \lambda_j J^\alpha \right)'' \tag{1.6}
\]
is equivalent to the condition
\[ \arg \lambda_i \neq \arg \lambda_j \pmod{2\pi} \quad 1 \leq i < j \leq n. \tag{1.7} \]

Some partial cases of the equivalence \((1.4) \iff (1.7)\) have been obtained earlier in [23, 39, 40] (see Remark 2.20).

It is easily seen that \((1.6)\) is equivalent to the following fact: for any \(\lambda \not\in (0, +\infty)\) an operator equation
\[ J^\alpha X = \lambda X J^\alpha \tag{1.8} \]
has only zero bounded solution \(X\). Moreover, in [29, 31] a description of all nonzero solutions \(X\) of \((1.8)\) with \(\lambda \in (0, +\infty)\) was obtained. Recently, equation \((1.8)\), and even more general ones with a bounded \(A\) in place of \(J^\alpha\), has attracted attention of several mathematicians (see, for instance, [5, 6, 26], and [8, 10, 45]). In particular, some results from [29] on equation \((1.8)\) were rediscovered in [5] and [26] (the case \(\alpha = 1\)) and in [6] (the case \(\alpha \in \mathbb{Z}_+ \setminus \{0\}\)). These authors treat any solution \(X\) of \(AX = \lambda X A\) as an extended eigenvector of \(A\) (see Remark 2.22 (2)).

Note also that if \((1.7)\) is not fulfilled then \(A\) is not cyclic. The set \(\text{Cyc} A\) of cyclic subspaces of \(A\) was described in [29, 31] by using a notion of \(*\)-determinant (see Definition 2.15). For example, vectors \(f_1 := (f_{11}, f_{12}), f_2 := (f_{21}, f_{22})\) generate a cyclic subspace of the operator \(A = J \oplus J\) defined on \(L_p[0, 1] \oplus L_p[0, 1]\) if and only if the function \(*\)-det \(\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} := f_{11} * f_{22} - f_{12} * f_{21}\) satisfies \(\varepsilon\)-condition \((1.2)\).

Passing to the case of the Sobolev space we should mention the pioneering work of E. Tsekanovskii [46]. More precisely, it is shown in [46] (see also [41]) that the integration operator \(J_k : f(x) \to \int_0^x f(t) \, dt\) defined on \(W_p^k[0, 1]\) is unicellular too and \(\text{Lat}_k\) consists of continuous part \(\text{Lat}^c J_k\) and discrete part \(\text{Lat}^d J_k\), \(\text{Lat}_k = \text{Lat}^c J_k \cup \text{Lat}^d J_k\). Here
\[ \text{Lat}^c J_k = \{ E^k_{a,0} : a \in (0, 1) \} \cup E_{0,0}, \]
\[ E^k_{a,0} := \{ f \in W_p^k[0, 1] : f(x) = 0 \text{ for } x \in [0, a] \}, \quad E_{0,0} := W_p^{k,0}[0, 1], \tag{1.9} \]
is a continuous chain and \(\text{Lat}^d J_k = \{ E^k_{l} \}_{l=0}^k\) with \(E^k_k := W_p^k[0, 1]\) and
\[ E^k_l = \{ f \in W_p^k[0, 1] : f(0) = \cdots = f^{(k-l-1)}(0) = 0 \}, \quad l \in \{1, \ldots, k-1\}, \quad \tag{1.10} \]
is a discrete chain. It is clear that, for \(0 \leq a_1 \leq a_2 \leq 1,\)
\[ \{0\} = E^k_{1,0} \subset E^k_{a_2,0} \subset E^k_{a_1,0} \subset E^k_{0,0} = W_p^{k,0}[0, 1], \]
\[ = W_p^{k,0}[0, 1] = E^k_{0} \subset E^k_{1} \subset \cdots \subset E^k_{k} = W_p^k[0, 1]. \]

In [16] we investigated the spectral properties of the complex powers \(J_k^\alpha\) of the integration operator \(J_k\) defined on Sobolev space \(W_p^k[0, 1]\). Namely, in [16] we described the lattices \(\text{Lat} J_k^\alpha\) and \(\text{HypLat} J_k^\alpha\), the set of cyclic subspaces \(\text{Cyc} J_k^\alpha\), the operator algebra \(\text{Alg} J_k^\alpha\), the commutant \(\{ J_k^\alpha \}'\), and the double commutant \(\{ J_k^\alpha \}''\). In
particular, it turns out that \( \{ J_k^\alpha \}^\prime = \{ J_k^\alpha \}^\prime\prime \) and \( \text{Alg} \ J_k^\alpha \) can be described as follows:

\[
R \in \{ J_k^\alpha \}^\prime \Leftrightarrow (Rf)(x) = cf(x) + \int_0^x r(x-t)f(t) \, dt, \quad r \in W_p^{k-1}[0,1], \quad (1.11)
\]

\[
R \in \text{Alg} \ J_k^\alpha \quad \Leftrightarrow \begin{cases} R \in \{ J_k^\alpha \}^\prime, \ r^{(l)}(0) = 0, \ l \neq m\alpha - 1, \ m \leq \frac{k-1}{\alpha}, & 1 \leq \alpha \leq k - 1, \quad (1.12) \\ R \in \{ J_k^\alpha \}^\prime, \ r \in W_p^{k-1}[0,1], & 2 \leq k \leq \alpha + \frac{1}{\alpha}. \end{cases}
\]

It was also shown in [16] that the operator \( J_k^\alpha \) is unicellular on \( W_p^k[0,1] \) if and only if either \( k = 1 \) or \( \alpha = 1 \). Moreover, the unicellularity of \( J_k^\alpha \) is equivalent to the validity of the "Neumann-Sarason" identity \( \text{Alg} \ J_k^\alpha = \{ J_k^\alpha \}^\prime\prime \).

In this paper we extend the main results from [16] and [31] to the case of the operator \( A_k := J_k^\alpha \otimes B \) defined on Sobolev space \( W_p^k[0,1] \otimes C^n \) of vector-functions. Moreover, we investigate the spectral properties of the operator \( A_k := \bigoplus_{j=1}^n \lambda_j J_{k,j}^\alpha \).

The paper is organized as follows. In Section 2, we collect some auxiliary results about invariant subspaces for \( C_0 \) contractions and accretive operators. Here we also present and complete some results from [31] for the operator \( A = \bigoplus_{j=1}^n \lambda_j J_{j}^\alpha \) defined on \( \bigoplus_{j=1}^n L_p^q[0,1] \).

In Section 3 it is shown that the operator \( A = \bigoplus_{j=1}^n \lambda_j J_{j}^\alpha \) defined on \( \bigoplus_{j=1}^n L_p^q[0,1] \) and the operator \( A_{k,0} = \bigoplus_{j=1}^n \lambda_j J_{k,j}^\alpha \) defined on \( \bigoplus_{j=1}^n W_p^{k,j}[0,1] \) are isometrically equivalent. Hence all results on the operator \( A \) presented in Section 2 are immediately extended to the case of the operator \( A_{k,0} \).

In Section 4 we provide a spectral analysis of the operator \( A_k := \bigoplus_{j=1}^n \lambda_j J_{k,j}^\alpha \) defined on \( \bigoplus_{j=1}^n W_p^{k,j}[0,1] \). A descriptions of the (weakly closed) algebra \( \text{Alg} \ A_k \), commutant \( \{ A_k \}^\prime \) and double commutant \( \{ A_k \}^\prime\prime \) is presented in Subsection 4.4 Subsection 4.2 and Subsection 4.3 respectively.

In Subsection 4.4, we obtain a description of the lattice \( \text{Lat} \ A_k \) assuming that \( A_k := \bigoplus_{j=1}^n \lambda_j J_{k,j}^\alpha \) satisfies condition (1.7). This description is essentially based on a description of \( \text{Lat} \ T \) (Theorem 2.1) for finite-dimensional operator \( T \) in \( \bigoplus_{j=1}^n C^k \).

In Subsection 4.5 a description of the lattice \( \text{HypLat} \ A_k \) is contained. We emphasize that \( \text{HypLat} \ A_{k,0} = \text{HypLat}^c \ A_k \) and the "continuous part" of \( \text{HypLat} \ A_k \) does not depend on \( \alpha \).

It turns out that under condition (1.7) \( \text{HypLat} \ A_k \) as well as the commutant \( \{ A_k \}^\prime \) of the operator \( A_k \) splits, that is, relations (1.3)-(1.6) remain valid with \( \text{HypLat} \ A \) and \( \{ A \}^\prime \) replaced by \( \text{HypLat} \ A_k \) and \( \{ A_k \}^\prime \), respectively. On the other hand, under condition (1.7) \( \text{Lat} \ A_k \) does not split for \( k \geq 1 \) in contrast to (1.4).

In this connection we recall (see [14]) that for a direct sum \( T_1 \oplus T_2 \) of two operators on a Banach space the relations (1.3)-(1.6) are equivalent to each other and both are implied by (1.4). Thus, the operator \( A_k \) presents a simple counterexample to the validity of the implication

\[
\text{HypLat}(T_1 \oplus T_2) = \text{HypLat} T_1 \oplus \text{HypLat} T_2 \quad \Longrightarrow \quad \text{Lat}(T_1 \oplus T_2) = \text{Lat} T_1 \oplus \text{Lat} T_2.
\]

Other counterexamples can be found in [11].

In Subsection 4.6 we compute the spectral multiplicity and present a description of the cyclic subspaces \( \text{Cyc} \ A_k \) for the operator \( A_k \).
It should be emphasized that descriptions of the sets $\text{Cyc } A_k$ and $\text{Cyc } A_{k,0}$ essentially differ. Namely, the first description does not depend on a choice of a sequence $\{\lambda_j\}_n$, though the second one depends on $\{\arg \lambda_j\}_n$ and is similar to that obtained in \cite{31} for $\bigoplus_{j=1}^n L_p[0,1]$.

A description of the set of cyclic subspaces of the operator $A = \bigoplus_{j=1}^n \lambda_j J_k^\alpha \oplus \bigoplus_{j=m+1}^n \lambda_j J_{k,j,0}^\alpha$ acting in the mixed space $\bigoplus_{j=1}^m W_{r,k_j}^p [0,1] \oplus \bigoplus_{j=m+1}^n W_{r,k_j}^p [0,1]$ is presented too.

Main results of the paper have been announced (without proofs) in \cite{15}.

### 1.1 Notations and agreements

1. $X, X_1, X_2$ stand for Banach spaces;

2. $[X_1, X_2]$ is the space of bounded linear operators from $X_1$ to $X_2$; $[X] := [X, X]$;

3. $\mathbb{I}$ and $\mathbb{I}_k$ denote the identity operators on $X$ and on $\mathbb{C}^k$, respectively; $\mathbb{O} := 0 \cdot \mathbb{I}$, $\mathbb{O}_k := 0 \cdot \mathbb{I}_k$;

4. $J(0; k)$ denotes the Jordan nilpotent cell of order $k$;

5. $\ker T = \{x \in X : Tx = 0\}$ is the kernel of $T \in [X]$;

6. $\text{ran } T = \{Tx : x \in X\}$ is the range of $T \in [X]$;

7. $\text{Cyc } T$ denotes the set of cyclic subspaces of an operator $T \in [X]$ (see Definition \ref{Def2.12});

8. $\{T\}'$ and $\{T\}''$ denote the commutant and the double commutant (or bicommutant) of an operator $T \in [X]$, respectively;

9. $\text{Alg}\{T_1, \ldots, T_n\}$ stands for a weakly closed subalgebra of $[X]$ generated by $T_1, \ldots, T_n \in [X]$ and the identity $\mathbb{I}$;

10. $\text{Lat } \mathcal{A}$ denotes the lattice of invariant subspaces of the algebra $\mathcal{A}$;

11. $\text{Lat } T (:= \text{Lat}(\text{Alg } T))$ and $\text{HypLat } T (:= \text{Lat}(\{T\}')$ denote the lattices of invariant and hyperinvariant subspaces of $T \in [X]$, respectively;

12. $\text{ran } E$ is the closed linear span of the set $E \subset X$;

13. $r * f$ stands for the convolution of functions $r, f \in L_1[0,1] : (r * f)(x) := \int_0^x r(x - t)f(t) \, dt$;

14. $\mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\}$; $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

As usual, $W_p^{r,k}[0,1]$ ($p \in (1, \infty), k \in \mathbb{Z}_+ \setminus \{0\}$) stands for the Sobolev space consisting of functions $f$ having $k - 1$ absolutely continuous derivatives and $f^{(k)} \in L_p[0,1]$. $W_p^{r,k}[0,1]$ is a Banach space equipped with the norm

$$\|f\|_{W_p^{r,k}[0,1]} = \left[ \sum_{j=0}^{k-1} |f^{(j)}(0)|^p + \int_0^1 |f^{(k)}(t)|^p \, dt \right]^{1/p}.$$
\( W_{p,0}^k[0,1] := \{ f \in W_p^k[0,1] : f(0) = \cdots = f^{(k-1)}(0) = 0 \} \).

We set \( W_p^0[0,1] := L_p[0,1] \) and \( W_{p,0}^0[0,1] = L_p[0,1] \).

Let \( J_{k,0}^\alpha \) and \( J_k^\alpha \) stand for the operator \( J^\alpha \) defined on \( W_{p,0}^k[0,1] \) and \( W_p^k[0,1] \), respectively. The operator \( J_{k,0}^\alpha \) is well defined on \( W_{p,0}^k[0,1] \) for any \( \alpha > 0 \). The operator \( J_k^\alpha \) is well defined on \( W_p^k[0,1] \) if either \( \alpha \in \mathbb{Z}_+ \setminus \{0\} \) or \( \alpha > k - \frac{1}{p} \).

Therefore throughout the paper we assume that

1. the operator \( A := \oplus_{j=1}^n \lambda_j J^\alpha \) is defined on \( \bigoplus_{j=1}^n L_p[0,1] \) for \( \alpha > 0 \);
2. the operator \( A_{k,0} := \oplus_{j=1}^n \lambda_j J_{k,j,0}^\alpha \) is defined on \( \bigoplus_{j=1}^n W_{p,0}^{k_j}[0,1] \) with \( k_j \geq 0 \) and \( \alpha > 0 \);
3. the operator \( A_k := \oplus_{j=1}^n \lambda_j J_{k,j}^\alpha \) is defined on \( \bigoplus_{j=1}^n W_p^{k_j}[0,1] \) with \( k_j \geq 1 \) and for \( \alpha \in \mathbb{Z}_+ \setminus \{0\} \) or \( \alpha > \max_{1 \leq j \leq n} k_j - \frac{1}{p} \).

We will also assume that \( \lambda_j \neq 0 \) for \( j \in \{1, \ldots, n\} \).

### 2 Preliminaries

#### 2.1 Invariant subspaces of some operators

Here we present some known results on invariant subspaces of finite-dimensional nilpotent operators and \( C_0 \) contractions. We also recall a condition about splitting of \( \text{Alg}(A \oplus B) \), where \( A, B \in [X] \).

**Theorem 2.1.** [4, 21] If \( Q \) is nilpotent on a finite-dimensional vector space \( V \), then

\[
\text{Lat}(Q) = \bigcup_M \{ [M, Q^{-1}M] : M \in \text{Lat}(Q \mid QV) \},
\]

where \( [M, Q^{-1}M] \) is an interval in the lattice of all subspaces of \( V \). Each interval satisfies the equation

\[
\dim Q^{-1}M - \dim M = \dim \ker Q.
\]

The following result was first discovered by P. Halmos [22] for operators defined on finite-dimensional spaces. The generalization to \( C_0 \) contractions on Hilbert spaces belongs to H. Bercovici [2, Proposition 5.33], [3, Corollary 2.11] and P. Wu [48, Theorem 1.2], and [49, Theorem 5])(see also references therein).

**Theorem 2.2.** Let \( T \) be a \( C_0 \)-contraction defined on a separable Hilbert space. Then every invariant subspace of \( T \) is the closure of the range and the kernel of some bounded linear transformation that commutes with \( T \), that is,

\[
\text{Lat} T = \{ \ker C : C \in \{T\}' \} = \overline{\text{ran} C : C \in \{T\}'}.
\]

**Definition 2.3.** (see [33, 36]) Let \( A \) and \( B \) be bounded operators defined on a Banach space \( X_1 \) and \( X_2 \) respectively. \( A \) is said to be quasisimilar to \( B \) if there exist deformations \( K : X_1 \to X_2 \) and \( L : X_2 \to X_1 \) (i.e. \( \overline{\text{ran} K} = X_2, \ker K = \{0\}, \overline{\text{ran} L} = X_1, \ker L = \{0\} \)) such that \( AL = LB \) and \( KA = BK \).
Remark 2.4. (i) Standard manipulations with Cayley transform implies that Theorem 2.2 holds also for quasinilpotent accretive operators with finite-dimensional real part.

(ii) Let operator $A$ be defined on a Banach space. Let also $A$ be quasisimilar to a $C_0$ contraction $T$. Then, obviously the statement of Theorem 2.2 is true for $A$, that is, $\text{Lat } A = \{\ker C : C \in \{A\}'\} = \{\text{ran } C : C \in \{A\}'\}$.

Let $X$ be a Banach space and let $n$ be a positive integer. Then $X^{(n)}$ denotes the direct sum of $n$ copies of $X$. If $A$ is an operator on $X$, then $A^{(n)}$ denotes the direct sum of $n$ copies of $A$ (regarded as an operator on $X^{(n)}$).

The following theorem is implicitly contained in [43] (see also [42, Theorem 7.1, Theorem 7.2])

Theorem 2.5. Let $T_1, \ldots, T_r \in [X]$ and

$$\text{Lat}(T_1^{(n)} \oplus \cdots \oplus T_r^{(n)}) = \text{Lat } T_1^{(n)} \oplus \cdots \oplus \text{Lat } T_r^{(n)}, \quad n = 1, 2, \ldots$$

Then $\text{Alg}(T_1 \oplus \cdots \oplus T_r) = \text{Alg } T_1 \oplus \cdots \oplus \text{Alg } T_r$.

2.2 Spectral analysis of the operator $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ defined on $\bigoplus_{i=1}^n L_p[0, 1]$

Throughout this subsection $X$ stands for $L_p[0, 1]$, with $p \in (1, \infty)$. Here we present some results from [31] on spectral analysis of the operator $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ defined on $\bigoplus_{i=1}^n X$. Moreover, we obtain a description of $\text{Alg } A$ and investigate its properties.

We begin with the following simple statement.

Lemma 2.6. Let $A_i, M_i, N_i \in [X]$ for $i \in \{1, \ldots, n\}$ and $A = \bigoplus_{i=1}^n A_i$. Assume also that the following identities are satisfied

$$A_i^m = M_i A_1^m N_i, \quad m \in \mathbb{Z}_+, \quad i \in \{1, \ldots, n\}. \quad (2.3)$$

Then

$$\text{Alg } A = \left\{ \bigoplus_{i=1}^n R_i : R_1 \in \text{Alg } A_1, \ R_i = M_i R_1 N_i, \ i \in \{2, \ldots, n\} \right\}. \quad (2.4)$$

Proof. Let $M := \bigoplus_{i=1}^n M_i$ and $N := \bigoplus_{i=1}^n N_i$. Then for any polynomial $p(\cdot)$ identities (2.3) yield $p(A_i) = M_i p(A_1) N_i$. Hence,

$$p(A) = \bigoplus_{i=1}^n p(A_i) = \bigoplus_{i=1}^n M_i p(A_1) N_i = M \left( \bigoplus_{i=1}^n p(A_1) \right) N.$$ 

On the other hand, by definition of $\text{Alg } A$ polynomials $p(A)$ are dense in $\text{Alg } A$ in weak operator topology. Hence the last identities imply $\text{Alg } A = M \text{Alg}(\bigoplus_{i=1}^n A_1) N$. To complete the proof it remains to note that $\text{Alg}(\bigoplus_{i=1}^n A_1) = \bigoplus_{i=1}^n \text{Alg } A_1$. \qed
Next we apply Lemma 2.6 to describe $\text{Alg} \ A$ for the operator $A = \bigoplus_{i=1}^{n} \lambda_i J^\alpha$ with factors $\lambda_i$ having equal arguments,

$$\lambda_i = \lambda_1 / s_i^\alpha, \quad 1 = s_1 \leq s_2 \leq \ldots \leq s_n, \quad i \in \{1, \ldots, n\}. \quad (2.5)$$

**Theorem 2.7.** Let the operator $A = \bigoplus_{i=1}^{n} \lambda_i J^\alpha$ be defined on $\bigoplus_{i=1}^{n} X$ with $\lambda_i$ satisfying condition (2.5). Then $\text{Alg} \ A$ is

$$\text{Alg} \ A = \left\{ R = \text{diag}(R_1, \ldots, R_n) : (R_i f)(x) = \frac{d}{dx} \int_{0}^{x} r_i(x-t)f(t) \, dt; \quad r_1 \in L_p[0,1], \quad r_i(x) = r_1(s_i^{-1} x), \quad R_1 \in [L_p[0,1]] \right\}. \quad (2.6)$$

**Proof.** To apply Lemma 2.6 we introduce the operators $M_i$ and $N_i$ by setting

$$(M_i f)(x) := f(s_i^{-1} x), \quad (N_i f)(x) := \begin{cases} f(s_i x), & x \in [0, s_i^{-1}], \\ 0, & x \in [s_i^{-1}, 1]. \end{cases} \quad (2.7)$$

Clearly, $\ker N_i = \{0\}$, $\text{ran} \ N_i = \chi_{[0,s_i^{-1}]} L_p[0,1]$, $\ker M_i = \chi_{[s_i^{-1}, 1]} L_p[0,1]$ and $\text{ran} \ M_i = L_p[0,1]$. It can easily be checked that $M_i N_i = I_{L_p[0,1]}$ and, moreover,

$$(\lambda_i J^\alpha)^m = M_i (\lambda_1 J^\alpha)^m N_i, \quad m \in \mathbb{Z}_+, \quad i \in \{1, \ldots, n\}.$$ 

Setting $A_i := \lambda_i J^\alpha$ and applying Lemma 2.6 we obtain

$$\text{Alg} \ A = \left\{ R = \bigoplus_{i=1}^{n} R_i : R_i \in \text{Alg}(\lambda_1 J^\alpha), \quad R_i = M_i R_1 N_i, \quad i \in \{2, \ldots, n\} \right\}. \quad (2.8)$$

On the other hand, according to (1.3), any (bounded) $R_1 \in \text{Alg}(\lambda_1 J^\alpha)$ admits a representation

$$R_1 : f(x) \rightarrow \frac{d}{dx} \int_{0}^{x} r_1(x-t)f(t) \, dt, \quad r_1 \in L_p[0,1]. \quad (2.9)$$

Straightforward calculations show that that

$$(M_i R_1 N_i f)(x) = \frac{d}{dx} \int_{0}^{x} r_i(s_i^{-1}(x-t))f(t) \, dt, \quad i \in \{2, \ldots, n\}.$$ 

Combining the last equality with (2.8) we complete the proof. \hfill \Box

To state the results on $\{A\}'$ we need some additional notations. For any $a \in \mathbb{R}_+ \setminus \{0\}$ we define an operator $L_a : X \rightarrow X$ by

$$L_a : f(x) \rightarrow g(x) = \begin{cases} f(ax), & 0 < a \leq 1, \\ 0, & x \in [0, 1 - a^{-1}], \\ f(ax - a + 1), & x \in [1 - a^{-1}, 1], \end{cases} \quad (2.10)$$

We set also

$$L_a \{J^\alpha\}' := \{L_a K : K \in \{J^\alpha\}'\}, \quad \{J^\alpha\}' L_a := \{KL_a : K \in \{J^\alpha\}'\}.$$ 

It is easily checked that $L_a \{J^\alpha\}' = \{J^\alpha\}' L_a$. 

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Theorem 2.8. [34] Proposition 4.6] Suppose $A = \bigoplus_{i=1}^{n} \lambda_i J^\alpha$ is defined on $\bigoplus_{i=1}^{n} X$ and $\lambda_i$ satisfy condition (2.5). Set also $a_{ij} := s_i^{-1} s_j$ for $i, j \in \{1, \ldots, n\}$. Then the commutant $\{A\}'$ is of the form

$\{A\}' = \{K : K = (K_{ij})_{i,j=1}^{n}, \ K_{ij} \in L_{a_{ij}} \{J^\alpha\}'\}.$

Next we complete Theorem 2.7 by establishing the Neumann type identity, $\{A\}'' = \text{Alg} A$. Note, that for the case $p = 2$ and $\alpha = 1$ it follows from a general result of B.S.-Nagy and C. Foias [34] on a dissipative operator with finite dimensional imaginary part.

Theorem 2.9. Suppose $A = \bigoplus_{i=1}^{n} \lambda_i J^\alpha$ is defined on $\bigoplus_{i=1}^{n} X$ and $\lambda_i$ satisfy condition (2.5). Then $\{A\}'' = \text{Alg} A$.

Proof. It is known (and easily seen) that if $T_1$ and $T_2$ are bounded operators on a Banach space $Y$, then $\{T_1 \oplus T_2\}'' \subset \{T_1\}'' \oplus \{T_2\}''$. Hence $\{A\}'' = \bigoplus_{i=1}^{n} \{\lambda_i J^\alpha\}'' \subset \bigoplus_{i=1}^{n} \{\lambda_i J^\alpha\}'$. It follows that any $R \in \{A\}''$ admits a direct sum decomposition $R = \bigoplus_{i=1}^{n} R_i$ with $R_i \in \{\lambda_i J^\alpha\}'' = \{\lambda_i J^\alpha\}'$, $i \in \{1, \ldots, n\}$. According to [13], $R_i$ admits a representation $(R_if)(x) = \frac{d}{dx} \int_{0}^{x} r_i(x-t)f(t) dt$, where $r_i \in L_{\mu}(0,1)$ and it is such that $R_i \in [X]$.

Further, let $K = (K_{ij})_{i,j=1}^{n}$ be an operator matrix with entries $K_{ij} = L_{a_{ij}}$ for $i > j$ and $K_{ij} = \emptyset$ for $i \leq j$. Let also $a_{ij} := s_i^{-1}s_j$ for $i, j \in \{1, \ldots, n\}$. Then, by Theorem 2.8, $K \in \{A\}'$. Clearly, relation $RK = KR$ yields

$$R_i L_{a_{i1}} = L_{a_{i1}} R_i, \quad i \in \{2, \ldots, n\}. \quad (2.11)$$

It is easily seen that

$$(R_i L_{a_{i1}} f)(x) = \frac{d}{dx} \int_{0}^{x} r_i(x-t)f(s_i^{-1}t) dt, \quad i \in \{2, \ldots, n\}. \quad (2.12)$$

On the other hand,

$$(L_{a_{i1}} R_i f)(x) = \frac{d}{dx_1} \int_{0}^{x_1} r_1(x_1-t)f(t) dt \bigg|_{x_1=s_i^{-1}x} = s_i \frac{d}{dx} \int_{0}^{s_i^{-1}x} r_1(s_i^{-1}x-t)f(t) dt = \frac{d}{dx} \int_{0}^{x} r_1(s_i^{-1}(x-t))f(s_i^{-1}t) dt.$$

Comparing this relation with (2.12) and taking into account (2.11) and the obvious relation $\text{ran}(L_{a_{i1}}) = X$, we obtain $r_i(x) = r_i(s_i^{-1}x)$, $i \in \{2, \ldots, n\}$. By Theorem 2.7, this means that $R \in \text{Alg} A$, that is $\{A\}'' \subset \text{Alg} A$. Since the inclusion $\{A\}'' \supset \text{Alg} A$ is obvious, we get $\{A\}'' = \text{Alg} A$. \hfill $\square$

In the following theorem we obtain a description of $\text{Lat} A$ similar to that of $\text{Lat} T$ for $C_0$-contractions $T$ described in Theorem 2.2. It is interesting to note that though a description is completely the same, the operator $A$ in not accretive in $L_2[0,1]$ for $\alpha > 1$ (cf. Remark 2.4 (i)).

Theorem 2.10. Let $A = \bigoplus_{i=1}^{n} \lambda_i J^\alpha$ be defined on $\bigoplus_{i=1}^{n} X$ and $\lambda_i$ satisfy conditions (2.5). Then every invariant subspace of $A$ is the closure of the range (the kernel) of a bounded linear transformation that commutes with $A$. 

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Proof. Alongside the operator $A$ we consider the operator $A_1 := \bigoplus_{i=1}^n \lambda_1 s_i^{-1} J$. By Theorem 2.7, $\text{Alg} A = \text{Alg} (\bigoplus_{i=1}^n \lambda_1 s_i^{-\alpha} J^\alpha) = \text{Alg} (\bigoplus_{i=1}^n \lambda_1 s_i^{-1} J) = \text{Alg} A_1$. Hence $\text{Lat} A = \text{Lat} A_1$ and $\{A\}' = \{A_1\}'$. So we can assume that $\lambda_1 = 1$ and $\alpha = 1$. We put

$$K := \bigoplus_{i=1}^n J \in \left[ \bigoplus_{i=1}^n L_p[0, 1], \bigoplus_{i=1}^n L_2[0, 1] \right],$$

$$L := \bigoplus_{i=1}^n J \in \left[ \bigoplus_{i=1}^n L_2[0, 1], \bigoplus_{i=1}^n L_p[0, 1] \right],$$

$$B := \bigoplus_{i=1}^n s_i J \in \left[ \bigoplus_{i=1}^n L_2[0, 1] \right].$$

It is clear that $\ker K = \{0\}, \ker L = \{0\}, \text{ran} K = \bigoplus_{i=1}^n L_2[0, 1], \text{ran} L = \bigoplus_{i=1}^n L_p[0, 1], KA_1 = BK$ and $A_1L = LB$. Hence $A_1$ is quasisimilar to $B$. So, we can assume that $A_1$ is defined on $\bigoplus_{i=1}^n L_2[0, 1]$. Note that $A_1$ is accretive, since $s_i > 0$ for $i \in \{1, ..., n\}$. Now the assertions of the theorem follow from Theorem 2.2 (see also Remark 2.4 (i)).

Next, we recall a description of $\text{HypLat} A$.

**Theorem 2.11.** [31, Proposition 4.8] Suppose $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ is defined on $\bigoplus_{i=1}^n X$ and $\lambda_i$ satisfy condition (2.5). Then the lattice $\text{HypLat} A$ is of the form

$$\text{HypLat} A = \left\{ \bigoplus_{i=1}^n E_{a_i} : (a_1, \ldots, a_n) \in P(s_1, \ldots, s_n) \right\},$$

where $P(s_1, \ldots, s_n) := \left\{ (a_1, \ldots, a_n) \in [0, 1]^n : s_i a_{i+1} \leq s_{i+1} a_i \leq s_i a_{i+1}, 1 \leq i \leq n-1 \right\}$.

**Definition 2.12.** (cf. [36])

(1) A subspace $E$ of a Banach space $X_1$ is called a cyclic subspace for an operator $T \in [X_1]$ if $\text{span}\{T^n E : n \geq 0\} = X_1$;

(2) a vector $f(\in X_1)$ is called cyclic for $T$ if $\text{span}\{T^n f : n \geq 0\} = X_1$;

(3) the set of all cyclic subspaces of an operator $T$ is denoted by $\text{Cyc} T$.

**Definition 2.13.** (1) The number

$$\mu_T := \inf_E \{ \dim E : E \text{ is a cyclic subspace of the operator } T \text{ on } X_1 \}$$

is called the spectral multiplicity of an operator $T$ on $X_1$;

(2) operator $T$ is called cyclic if $\mu_T = 1$. 

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It is well known that the concept of spectral multiplicity plays an important role in control theory (see for instance [47]). Investigating some other problems of control theory, N.K. Nikol’skii and V.I. Vasjunin [38] introduced one more "cyclic" characteristic of an operator.

**Definition 2.14.** [38] Let \( T \in [X] \). Then

\[
\text{disc} \ T := \sup_{E \in \text{Cyc} \ T} \min \{ \dim E' : E' \subset E, E' \in \text{Cyc} \ T \}.
\]

\text{disc} \ T \ is called a disc-characteristic of an operator \( T \). ("disc" is the abbreviation of "Dimension of the Input Subspace of Control".)

Clearly, \( \text{disc} \ T \geq \mu_T \).

To present a description of \( \text{Cyc} \ A \) we recall the following definition.

**Definition 2.15.** [29, 31, 35] The determinant of a functional matrix \( F(x) = (f_{ij}(x))_{i,j=1}^n \) calculated with respect to the convolution product

\[
(f \ast g)(x) = \int_0^x f(x-t)g(t) \, dt = \int_0^x g(x-t)f(t) \, dt = (g \ast f)(x)
\]

is called *-determinant and is denoted by \( \ast - \text{det} F(x) \). Similarly, *-minors of \( F(x) \) are the minors calculated with respect to the convolution product. *-rank \( F(x) \) will is the highest order of *-minors of \( F(x) \) satisfying \( \varepsilon \)-condition (1.2).

Next we complete [31, Theorem 2.3] by computing disc \( A \).

**Theorem 2.16.** Suppose \( A = \bigoplus_{i=1}^n \lambda_i J^\alpha \) is defined on \( \bigoplus_{i=1}^n X \) and \( \lambda_i \) satisfy condition (2.5). Then the system \( \{ f_{i1} \}_{i=1}^N \) of vectors

\[
f_l = f_{i1} \oplus \cdots \oplus f_{in} \in \bigoplus_{i=1}^n X, \quad l \in \{1, \ldots, N\}, \quad i \in \{1, \ldots, n\}
\]

generates a cyclic subspace for the operator \( A \) if and only if

(i) \( N \geq n \);

(ii) the matrix

\[
F_n(x) = \begin{pmatrix}
  f_{i1}(s_1x) & f_{i2}(s_2x) & \cdots & f_{in}(s_nx) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{N1}(s_1x) & f_{N2}(s_2x) & \cdots & f_{Nn}(s_nx)
\end{pmatrix}
\]

is of maximal *-rank, namely, *-rank \( F_n(x) = n \);

(iii) \( \text{disc} A = \mu_A = n \).
Proof. (i), (ii) and the equality \( \mu_A = n \) were proved in [31] Theorem 2.3 (see also [14 Proposition 3.2] for another proof).

(iii) Let us prove that disc \( A = n \). Let \( E = \text{span}\{f_1, \ldots, f_N\} \) be an \( N \)-dimensional subspace cyclic for the operator \( A \). It is necessary to show that this space contains an \( n \)-dimensional subspace which is also cyclic for the operator \( A \). Since \( \ast \)-rank \( F_n(x) = n \), it follows that there exists an \( n \times n \) submatrix \( G_n(x) \) of \( F_n(x) \) such that \( \ast \)-rank \( G_n(x) = n \). Hence we can choose \( n \)-vectors \( f_{i_1}, \ldots, f_{i_n} (i_1, \ldots, i_n \in \{1, \ldots, N\}) \) such that \( \text{span}\{f_{i_1}, \ldots, f_{i_n}\} \) is a cyclic subspace for \( A \).

Corollary 2.17. Let \( K \in \{J^\alpha\}' \) and \( K_n = \bigoplus_{i=1}^n K \) be defined on \( \bigoplus_{i=1}^n L_2[0,1] \). Then \( \mu_{K_n} \geq n \).

Proof. It follows from Theorem 2.7 that \( K_n \in \text{Alg} A \), where \( A = \bigoplus_{i=1}^n J \) is defined \( \bigoplus_{i=1}^n L_2[0,1] \). Hence, by Theorem 2.16 \( \mu_{K_n} \geq \mu_A = n \).

Remark 2.18. In the recent paper [4 Proposition 7.6] Corollary 2.17 was proved for the case \( n = 2 \).

Next we recall the following notation. Let \( T_j \in [X_j] \) (\( j = 1, 2 \)) and \( R \in \text{Cyc}(T_1 \oplus T_2) \). It is clear that \( P_j R \in \text{Cyc} T_j \), where \( P_j \) is the projection from \( X_1 \oplus X_2 \) onto \( X_j \), \( j \in \{1, 2\} \). Following [38], we write

\[
\text{Cyc}(T_1 \oplus T_2) = \text{Cyc} T_1 \lor \text{Cyc} T_2
\]

if \( P_j R \in \text{Cyc} T_j \) (\( j = 1, 2 \)) yields \( R \in \text{Cyc}(T_1 \oplus T_2) \) for every \( R \subset X_1 \oplus X_2 \). In particular, if \( \text{Lat}(T_1 \oplus T_2) = \text{Lat} T_1 \oplus \text{Lat} T_2 \) then \( \text{Cyc}(T_1 \oplus T_2) = \text{Cyc} T_1 \lor \text{Cyc} T_2 \).

Next we complete [31 Proposition 4.1, Proposition 4.2].

Theorem 2.19. Suppose \( A = \bigoplus_{j=1}^r \lambda_j J^\alpha \) is defined on \( \bigoplus_{j=1}^r X \) and

\[
\arg \lambda_i \neq \arg \lambda_j \pmod{2\pi}, \quad 1 \leq i < j \leq r.
\]

Then

\[
\text{Alg} A = \{A\}' = \{A\}'' = \bigoplus_{j=1}^r \text{Alg} J^\alpha = \bigoplus_{j=1}^r \{J^\alpha\}' = \bigoplus_{j=1}^r \{J^\alpha\}'',
\]

\[
\text{Lat} A = \text{HypLat} A = \bigoplus_{j=1}^r \text{Lat} J^\alpha = \bigoplus_{j=1}^r \text{HypLat} J^\alpha,
\]

\[
\text{Cyc} A = \bigvee_{j=1}^r \text{Cyc} J^\alpha,
\]

\[
\text{disc} A = \mu_A = 1.
\]

Proof. (2.15)-(2.17) and the splitting of \( \{A\}' \) and \( \{A\}'' \) were proved in [29], [31]. We present two different proofs of the splitting of \( \text{Alg} A \) due to the first and to the second author, respectively.

First proof. We will derive the splitting of \( \text{Alg} A \) from the splitting of \( \text{Cyc} A \).
By (2.16) \( g := \frac{x^{\alpha - 1}}{\Gamma(\alpha)} \oplus \cdots \oplus \frac{x^{\alpha - 1}}{\Gamma(\alpha)} \in \text{Cyc} A \). Hence, there exists a sequence \( \{P_n(x)\}_{n=1}^\infty \) such that \( \text{s-lim}_{n \to \infty} P_n(A)g = 0 \oplus \cdots \oplus 0 \oplus \frac{x^{\alpha - 1}}{\Gamma(\alpha)} \). We claim that
\[
\text{s-lim}_{n \to \infty} AP_n(A) = \emptyset \oplus \cdots \oplus \emptyset \oplus \lambda_J J^\alpha. \tag{2.18}
\]
Indeed, for any \( f = f_1 \oplus \cdots \oplus f_r \in \bigoplus_{j=1}^r X \) one has
\[
\text{s-lim}_{n \to \infty} AP_n(A)f = \text{s-lim}_{n \to \infty} (\lambda_1 J^\alpha P_n(\lambda_1 J^\alpha)f_1 \oplus \cdots \oplus \lambda_r J^\alpha P_n(\lambda_r J^\alpha)f_r)
\]
\[
= \text{s-lim}_{n \to \infty} \left( \lambda_1 \frac{x^{\alpha - 1}}{\Gamma(\alpha)} \ast (P_n(\lambda_1 J^\alpha)f_1)(x) \oplus \cdots \oplus \lambda_r \frac{x^{\alpha - 1}}{\Gamma(\alpha)} \ast (P_n(\lambda_r J^\alpha)f_r)(x) \right)
\]
\[
= \text{s-lim}_{n \to \infty} \left( \lambda_1 f_1 \ast (P_n(\lambda_1 J^\alpha)\frac{x^{\alpha - 1}}{\Gamma(\alpha)}) \oplus \cdots \oplus \lambda_r f_r \ast (P_n(\lambda_r J^\alpha)\frac{x^{\alpha - 1}}{\Gamma(\alpha)}) \right)
\]
\[
= \lambda_1 f_1 \ast 0 \oplus \lambda_2 f_2 \ast 0 \oplus \cdots \oplus \lambda_r f_r \ast \frac{x^{\alpha - 1}}{\Gamma(\alpha)} = \text{diag}(\emptyset, \ldots, \emptyset, \lambda_r J^\alpha)f.
\]
So (2.18) is proved. A similar argument shows that for any \( j \in \{1, \ldots, r\} \) there exists a sequence of polynomials \( \{P_{j,n}\}_{n=1}^\infty \) such that
\[
\text{s-lim}_{n \to \infty} AP_{j,n}(A) = \emptyset \oplus \cdots \oplus \emptyset \oplus \lambda_J J^\alpha \oplus \emptyset \oplus \cdots \oplus \emptyset.
\]
Hence the splitting of \( \text{Alg} A \) is proved.

Second proof. Keeping in mind notations of Theorem 2.24 (see below), for any \( j \in \{1, \ldots, r\} \) we let \( n_j = n \) and \( \lambda_{j_1} := \cdots := \lambda_{j_n} := \lambda_j \). Then setting \( A(j) := \bigoplus_{i=1}^{n_j} \lambda_{j_i} J^\alpha \) we rewrite \( A(j) \) and \( A \) as
\[
A(j) = \bigoplus_{i=1}^{n_j} \lambda_{j_i} J^\alpha = (\lambda_j J^\alpha)^{(n)} \quad \text{and} \quad A = \bigoplus_{j=1}^{r} A(j) = \bigoplus_{j=1}^{r} (\lambda_j J^\alpha)^{(n)},
\]
where the factors \( \lambda_j \) have different arguments, \( \lambda_j \neq \lambda_k \) for \( j \neq k \). Therefore by Theorem 2.24 the lattice \( \text{Lat}(\bigoplus_{j=1}^{r} (\lambda_j J^\alpha)^{(n)}) \) splits, \( \text{Lat}(\bigoplus_{j=1}^{r} (\lambda_j J^\alpha)^{(n)}) = \bigoplus_{j=1}^{r} \text{Lat}(\lambda_j J^\alpha)^{(n)} \). One completes the proof by applying Theorem 2.5 with \( T_j = \lambda_j J^\alpha, \ j \in \{1, \ldots, n\} \).

\[\Box\]

Remark 2.20. Some particular statements of Theorem 2.19 were obtained in [1, 23, 39, 40] for the case \( p = 2 \).

Namely, A. Atzmon [1] proved that for every integer \( k \geq 2 \), the operator \( iJ^{1-1/k} \oplus e^{\pi i} J^{1-1/k} \) is cyclic.

In [39, 40] B.P. Osilenker and V.S. Shulman proved that (2.13) implies the splitting of \( \text{Lat}(\bigoplus_{j=1}^{r} \lambda_j J) \). Their proof cannot be extended to the case \( \alpha \neq 1 \).

L.T. Hill [23] showed that if \( \alpha \in (0, 1) \) and \( \lambda \) is a nonzero complex number, then \( \text{Lat}(J^\alpha \oplus \lambda J^\alpha) \) splits if and only if \( \lambda \) is not positive. His proof cannot be extended neither to the case of \( \alpha > 1 \) nor to the number of summands \( n > 2 \).

The following result is easily implied by combining Theorems 2.8 and 2.19.

Corollary 2.21. [28, 27] Let \( c \in \mathbb{C} \) and let \( R \in [X] \) be a solution of the equation \( R J^\alpha = cJ^\alpha R \). Then the following statements hold.
(i) if \( c \not\in \mathbb{R}_+ \), then \( R = \emptyset \);

(ii) if \( c = a^\alpha > 0, \ a > 0 \), then \( R \in L_a\{J^\alpha\}' \), where \( L_a \) is defined by (2.10).

**Remark 2.22.** (i) It was shown in [19] that the operators \( J \) and \( cJ \) are similar if and only if \( c = 1 \). Corollary 2.21 implies that operators \( J^\alpha \) and \( cJ^\alpha \) are not even quasisimilar for any \( c \neq 1 \).

(ii) In particular cases Corollary 2.21 (i) was recently reproved by another method in [5], [26] (the case \( \alpha = 1, \ p = 2 \)) and in [6] (the case \( \alpha \in \mathbb{Z}_+ \setminus \{0\}, \ p = 2 \). Some solutions \( R \) of the equation \( RJ^\alpha = cJ^\alpha R \) in the case \( c > 0, \ \alpha \in \mathbb{Z}_+ \) were also indicated in [5], [6], [26].

**Lemma 2.23.** Suppose that \( A \in [X_1] \) is quasisimilar to \( B \in [X_2] \) with intertwining deformations \( L \) and \( K \). That is, \( AL = LB \) and \( KA = BK \). Let also \( KL = A^2 \) and \( KL = B^2 \). Then

(i) \( E \in \text{Cyc} \ A \Rightarrow KE \in \text{Cyc} \ B \);

(ii) \( F \in \text{Cyc} \ B \Rightarrow LF \in \text{Cyc} \ A \);

(iii) \( \text{disc} \ A = \text{disc} \ B \).

**Proof.** The proof is left for the reader. \( \square \)

Now we can consider the case of any diagonal nonsingular matrix \( B \).

Next we complete [31, Proposition 3.2, Theorem 3.4, Corollary 3.5, Theorem 4.10, Theorem 4.11].

**Theorem 2.24.** Suppose \( A(j) := \bigoplus_{i=1}^{n_j} \lambda_{ji}J^\alpha \) is defined on \( \bigoplus_{i=1}^{n_j} X, \ j \in \{1, \ldots, r\} \) and \( A := \bigoplus_{j=1}^r A(j) \) is defined on \( \bigoplus_{j=1}^r (\bigoplus_{i=1}^{n_j} X) \). Let also

\[
\arg \lambda_{j1} = \arg \lambda_{ji} \pmod{2\pi}, \quad 1 \leq j \leq r, \ 1 \leq i \leq n_j, \\
\arg \lambda_{i1} \neq \arg \lambda_{i1} \pmod{2\pi}, \quad 1 \leq i < j \leq r.
\]

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Then

\[
\text{Alg} A = \bigoplus_{j=1}^{r} \text{Alg} A(j), \quad (2.19)
\]

\[
\{A\}' = \bigoplus_{j=1}^{r} \{A(j)\}', \quad (2.20)
\]

\[
\{A\}'' = \bigoplus_{j=1}^{r} \{A(j)\}'', \quad (2.21)
\]

\[
\text{Lat} A = \bigoplus_{j=1}^{r} \text{Lat} A(j), \quad (2.22)
\]

\[
\text{HypLat} A = \bigoplus_{j=1}^{r} \text{HypLat} A(j), \quad (2.23)
\]

\[
\text{Cyc} A = \bigvee_{j=1}^{r} \text{Cyc} A(j), \quad (2.24)
\]

\[
\text{disc} A = \mu_A = \max_{1 \leq j \leq r} \mu_{A(j)}.
\]

Proof. Relations (2.20)-(2.24) and the equality \( \mu_A = \max_{1 \leq j \leq r} \mu_{A(j)} \) were proved in [31]. Let us prove (2.19). By Theorem 2.19, for any \( j \in \{1, \ldots, r\} \)

\[
\bigoplus \cdots \bigoplus \bigoplus \lambda j_1 J^\alpha \bigoplus \bigoplus \cdots \bigoplus \bigoplus \in \text{Alg}(\lambda_{11} J^\alpha \bigoplus \cdots \bigoplus \lambda_{r1} J^\alpha).
\]

Thus, by Theorem 2.7 we have that

\[
\bigoplus \cdots \bigoplus \bigoplus A(j) \bigoplus \bigoplus \bigoplus \bigoplus \bigoplus \in \text{Alg} A,
\]

and hence (2.19) is proved.

Let us prove that \( \text{disc} A = \mu_A \). Assume that \( A_1 := A \) is defined on

\[
\bigoplus_{j=1}^{r} (\bigoplus_{i=1}^{n_j} L_2[0,1]). \quad \text{Then [37, Statement 1.13] and [38, Corollary 13] imply the equality disc} A_1 = \mu_{A_1}. \quad \text{We define}
\]

\[
K := \bigoplus_{j=1}^{r} \bigoplus_{i=1}^{n_j} \lambda_{ji} J^\alpha \in \left[ \bigoplus_{j=1}^{r} \left( \bigoplus_{i=1}^{n_j} L_2[0,1] \right) \right],
\]

\[
L := \bigoplus_{j=1}^{r} \bigoplus_{i=1}^{n_j} \lambda_{ji} J^\alpha \in \left[ \bigoplus_{j=1}^{r} \left( \bigoplus_{i=1}^{n_j} L_2[0,1] \right) \right],
\]

and \( A_2 := A \). It is clear that \( K \) and \( L \) are deformations and \( A_1 L = LA_2, KA_1 = A_2 K \). Now application of Lemma 2.23 completes the proof. \( \square \)

3 The operator \( A_{k,0} \)

Let \( J_{k,l}^\alpha \) stand for the operator \( J_k^\alpha \) acting on the subspace \( E_l^k \) of \( W^k_p[0,1] \) defined by (1.10) \( (l \leq k - 1) \) and \( E_k^k := W^k_p[0,1] \).

Next we establish isometric equivalence of \( J_{k,0}^\alpha \) and \( J^\alpha \).
Lemma 3.1. The operator $J_{k,i}^\alpha$ defined on $E_k^i$ is isometrically equivalent to the operator $J_i^\alpha$ defined on $W_p^i[0,1]$. In particular, the operator $J_{k,0}^\alpha$, defined on $W_p^k[0,1]$ is isometrically equivalent to the operator $J_0^\alpha := J^\alpha$ defined on $W_p[0,1]$.

Proof. It is clear that the operator $U = \frac{d^{k-1}}{dx^{k-1}} : E_k^i \to W_p^i[0,1]$ isometrically maps $E_k^i$ on $W_p^i[0,1]$. Moreover,

$$U^{-1} = U^* = J^{k-1} : W_p^i[0,1] \to E_k^i.$$ 

The assertion follows now from the identity $J_{k,i}^\alpha = U^{-1}J_0^\alpha U$. \hfill \Box

Corollary 3.2. The operator $A_{k,0} := \bigoplus_{i=1}^n \lambda_i J_{k,0}^\alpha$ defined on $\bigoplus_{i=1}^n W_p^k[0,1]$ is isometrically equivalent to the operator $A := \bigoplus_{i=1}^n \lambda_i J^\alpha$ defined on $\bigoplus_{i=1}^n L_p[0,1]$. Corollary 3.2 makes it possible to translate all results on the operator $A$ defined on $\bigoplus_{i=1}^n L_p[0,1]$ to the results on operator $A_{k,0}$ defined on $\bigoplus_{i=1}^n W_p^k[0,1]$. For instance, Theorem 2.10 takes the following form

Theorem 3.3. Let $A_{k,0} := \bigoplus_{i=1}^n \lambda_i J_{k,0}^\alpha$ be defined on $\bigoplus_{i=1}^n W_p^k[0,1]$ and $\lambda_i$ satisfy condition (2.5). Then every invariant subspace of $A_{k,0}$ is the closure of the range (the kernel) of a bounded linear transformation that commutes with $A_{k,0}$.

4 The operator $A_k$

This section contains the main results of the paper. Namely, we described the spectral properties of the operator $A_k := \bigoplus_{i=1}^n \lambda_i J_k^\alpha$ defined on $X^{\alpha} = \bigoplus_1^n X$ where $X = W_p^k[0,1]$.

4.1 The algebra $\text{Alg} A_k$

Theorem 4.1. Suppose $A_k := \bigoplus_{i=1}^n \lambda_i J_k^\alpha$ is defined on $\bigoplus_{i=1}^n W_p^k[0,1]$ and

$$\lambda_i = \lambda_1/s_i^\alpha, \quad 1 = s_1 \leq s_2 \leq \ldots \leq s_n, \quad i \in \{1, \ldots, n\}. \quad (4.1)$$

Let also

$$R := \bigoplus_{i=1}^n R_i \in \left[ \bigoplus_{i=1}^n W_p^k[0,1] \right], \quad (R_i f)(\cdot) = c_i f(\cdot) + (r_i * f)(\cdot), \quad i \in \{1, \ldots, n\}. \quad (4.2)$$

Then the following is true:

(1) if $1 \leq \alpha \leq k - 1$, then

$$\text{Alg} A_k = \left\{ R : c_1 = \cdots = c_n \in \mathbb{C}; \ r_1 \in W_p^{k-1}[0,1]; \ r_i(x) = s_i^{-1} r_1(s_i^{-1} x), \quad 1 \leq i \leq n; \ r_i^{(l)}(0) = 0, \quad l \neq m\alpha - 1, \quad 1 \leq m \leq [(k - 1)/\alpha] \right\}; \quad (4.3)$$
(2) if \(2 \leq k \leq \alpha + \frac{1}{p}\), then
\[
\text{Alg } A_k = \{ R : c_1 = \cdots = c_n \in \mathbb{C}; \ r_1 \in W_{p,0}^{k-1}[0,1]; \ r_i(x) := s_i^{-1}r_1(s_i^{-1}x), \quad 1 \leq i \leq n \}. \tag{4.4}
\]

**Proof.** Let
\[
(M_i f)(x) := f(s_i^{-1}x), \quad (N_i f)(x) := \begin{cases} f(s_i x), & x \in [0, s_i^{-1}], \\ \sum_{m=0}^{k-1} \frac{(xs_i^{-1})^m}{m!} f(m), & x \in [s_i^{-1}, 1]. \end{cases}
\]

It can easily be checked that
\[
(\lambda_i J^\alpha_k)^m = M_i(\lambda_1 J^\alpha_k)^m N_i, \quad m \in \mathbb{Z}_+, \quad i \in \{1, \ldots, n\}.
\]

Setting \(A_i := \lambda_i J^\alpha_k\) and applying Lemma 2.4 we obtain
\[
\text{Alg } A = \left\{ R = \bigoplus_{i=1}^n R_i : R_1 \in \text{Alg}(\lambda_1 J^\alpha_k), \ R_i = M_i R_1 N_i, \quad i \in \{2, \ldots, n\} \right\}. \tag{4.5}
\]

Next we confine ourselves to the case \(1 \leq \alpha \leq k - 1\). The case \(2 \leq k \leq \alpha + \frac{1}{p}\) is considered similarly. By (1.12), \(R_1 \in \text{Alg}(\lambda_1 J^\alpha_k)\) if and only if
\[
R_1 : f(x) \rightarrow c_1 f(x) + \int_0^x r_1(x-t)f(t) \, dt, \quad c_1 \in \mathbb{C}, \quad r_1 \in W_{p}^{k-1}[0,1], \quad r_1^{(l)}(0) = 0, \quad l \neq m\alpha - 1, \quad 1 \leq m \leq [(k-1)/\alpha]. \tag{4.6}
\]

Straightforward calculations show that
\[
(M_i R_1 N_i f)(x) = c_1 + \int_0^x s_i^{-1} r_1(s_i^{-1}(x-t)) f(t) \, dt, \quad i \in \{2, \ldots, n\}.
\]

Combining the last relations with (4.5) we arrive at the required description. \(\square\)

In the proof of the following theorem we need a concept of the weak operator topology in the algebra \([X]\). Recall the following definition.

**Definition 4.2.** Let \(\{f_i\}_{i=1}^N\) and \(\{g_i\}_{i=1}^N\) be the sets of unit vectors in \(X\) and \(X^*\), respectively, and let \(\varepsilon\) be a positive number. For any \(R \in B[X]\) define \(\mathcal{V} := \mathcal{V}(\varepsilon; \{f_i, g_i\}_{i=1}^N)\) to be the set of all operators \(T\) satisfying
\[
|\langle T - R, f_i, g_i \rangle| < \varepsilon, \quad i \in \{1, \ldots, N\}.
\]

Then \(\mathcal{V}\) is a weak neighborhood of \(R\) and the family of all such sets \(\mathcal{V}\) is a base of weak neighborhoods of \(R\).
Theorem 4.3. Suppose \( A_k = \bigoplus_{j=1}^r \lambda_j J^n_k \) is defined on \( \bigoplus_{j=1}^r W_p^k [0, 1] \) and
\[
\arg \lambda_i \neq \arg \lambda_j \pmod{2\pi}, \quad 1 \leq i < j \leq r.
\]

Let also
\[
R := \bigoplus_{j=1}^r R_j \in \left[ \bigoplus_{j=1}^r W_p^k [0, 1] \right], \quad (R_j f)(\cdot) = c_j f(\cdot) + (r_j * f)(\cdot), \quad j \in \{1, \ldots, r\}.
\]

Then the following are true:

(1) if \( 1 \leq \alpha \leq k - 1 \), then
\[
\text{Alg} \ A_k = \left\{ R : c_1 = \cdots = c_r \in \mathbb{C}; \ r_j \in W_p^{k-1} [0, 1], \quad r_j^{(am-1)}(0) = (\lambda_j \lambda_1^{-1})^m r_1^{(am-1)}(0), \quad m \leq \left[ \frac{k-1}{\alpha} \right], \quad 1 \leq j \leq r; \right. \\
\left. r_j^{(l)}(0) = 0, \quad l \neq am - 1, \quad m \leq \left[ (k-1)/\alpha \right], \quad 1 \leq j \leq r \right\};
\] (4.7)

(2) if \( 2 \leq k \leq \alpha + \frac{1}{p} \), then
\[
\text{Alg} \ A_k = \left\{ R : c_1 = \cdots = c_r \in \mathbb{C}; \ r_j \in W_p^{k-1} [0, 1], \quad 1 \leq j \leq r \right\}.
\]

Proof. (i) Theorem 2.19 and Corollary 3.2 imply that \( \mathcal{O} \oplus \cdots \oplus \mathcal{O} \oplus \lambda_j J^\alpha_{k,0} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \in \text{Alg} \left( \bigoplus_{j=1}^r \lambda_j J^\alpha_{k,0} \right) \) for any \( j \in \{1, \ldots, r\} \). It easily implies that \( M_j := \mathcal{O} \oplus \cdots \oplus \mathcal{O} \oplus (\lambda_j J^\alpha_{k,0})^{k+1} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \in A^k \text{Alg} \ A_k \). Thus \( M_j \in \text{Alg} \ A_k \) and (1.12) implies that if either \( \alpha \in \mathbb{Z}_+ \setminus \{0\} \) or \( \alpha > k - \frac{1}{p} \), then
\[
\text{Alg} \ A_k \supset \left\{ R : c_1 = \cdots = c_r \in \mathbb{C}; \ r_j \in W_p^{k-1} [0, 1], \quad 1 \leq j \leq r \right\};
\] (4.8)

(ii) Let \( 2 \leq k \leq \alpha + \frac{1}{p} \). Then combining the obvious inclusion \( \text{Alg} \ A_k \subset \bigoplus_{j=1}^r \text{Alg} \lambda_j J^\alpha_{k} \) with (1.12) we arrive at opposite inclusion in (4.8). Thus, (2) is proved.

(iii) Let us prove the inclusion ”\( \subset \)” in (4.7). Description (1.12) and inclusion \( \text{Alg} \ A_k \subset \bigoplus_{j=1}^r \text{Alg} \lambda_j J^\alpha_{k} \) imply that
\[
\text{Alg} \ A_k \subset \left\{ R : c_j \in \mathbb{C}; \ r_j \in W_p^{k-1} [0, 1], \quad r_j^{(l)}(0) = 0, \quad l \neq am - 1, \quad m \leq \left[ (k-1)/\alpha \right], \quad 1 \leq j \leq r \right\}.
\]

For \( j \in \{1, \ldots, r\} \) and \( m \in \{1, \ldots, \left[ \frac{k-1}{\alpha} \right] \} \) by definition, put:
\[
x_{jm} := 0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0, \quad y_{jm} := 0 \oplus \cdots \oplus 0 \oplus \frac{x^{am}}{\Gamma(a)} \oplus 0 \oplus \cdots \oplus 0.
\]
Let $R := \bigoplus_{j=1}^{r} R_j \in \text{Alg} A_k$. Choose $\varepsilon_1 > 0$ and put

$$
\varepsilon := \min \{ 2^{-1} \lambda_j^m \varepsilon_1 : 1 \leq j \leq r, \ 0 \leq m \leq k_1 \} \quad \text{and} \quad k_1 := \left\lceil \frac{k-1}{\alpha} \right\rceil.
$$

Next, choose vectors $\{x_{jm}\}_{j,m=1}^{r,k_1}$ and $\{y_{jm}\}_{j,m=1}^{r,k_1}$ belonging to $\mathcal{W}_p^k[0,1]$ and $(\mathcal{W}_p^k[0,1])^* = \mathcal{W}_p^k[0,1]$, respectively and define a weak neighborhood

$$
\mathcal{V} := \mathcal{V}(\varepsilon : \{x_{jm}\}_{j,m=1}^{r,k_1}, \{y_{jm}\}_{j,m=1}^{r,k_1}) \text{ of } R \text{ according to Definition 4.2. Then by definition of } \text{Alg} A_k \text{ there exists a polynomial } p(x) := \sum_{i=0}^{N} a_i x^l \text{ such that } p(A_k) \text{ belongs to the weak neighborhood } \mathcal{V} \text{ of } R, \ p(A_k) \in \mathcal{V}, \text{ that is}
$$

$$
|(R - p(A_k)) x_{jm}, y_{jm}| < \varepsilon, \quad j \in \{1, \ldots, r\}, \quad m \in \{0, \ldots, k_1\}. \quad (4.9)
$$

It is clear that (4.9) is equivalent to the following system

$$
\left| \left( (R_j - p(\lambda_j J^m_k)) \right) \mathbf{1} \frac{x^m}{\Gamma(\alpha m)} \right| < \varepsilon, \quad j \in \{1, \ldots, r\}, \quad m \in \{0, \ldots, k_1\}.
$$

After simple computations this systems reduces to the following one

$$
|c_j - a_0| < \varepsilon, \quad \left| \frac{r_j^{(\alpha m - 1)}(0)}{\lambda_j^m} - a_m \right| < \frac{\varepsilon}{\lambda_j^m}, \quad j \in \{1, \ldots, r\}, \quad m \in \{0, \ldots, k_1\}.
$$

Finally, triangle inequality implies that

$$
|c_1 - c_j| < 2\varepsilon \leq \varepsilon_1, \quad \left| \frac{r_j^{(\alpha m - 1)}(0)}{\lambda_j^m} - \frac{r_j^{(\alpha m - 1)}(0)}{\lambda_j^m} \right| < \frac{2\varepsilon}{\lambda_j^m} \leq \varepsilon_1, \quad m \in \{0, \ldots, k_1\}.
$$

Hence,

$$
c_j = c_1, \quad r_j^{(\alpha m - 1)}(0) = (\lambda_j \lambda_1^{-1})^m r_1^{(\alpha m - 1)}(0), \quad m \in \{1, \ldots, k_1\}, \quad j \in \{1, \ldots, r\}.
$$

Thus, the inclusion "$c$" in (4.7) is proved.

(iii) Let $R$ belongs to the algebra defined by the right side of (4.7). Since $r_j \in \mathcal{W}_p^{k-1}[0,1]$, it follows that

$$
r_j(x) = r_{j,0} + r_{j,k-2} := \left( r_j(x) - \sum_{i=0}^{k-2} r_j^{(i)}(0) \frac{x^i}{i!} \right) + \sum_{i=0}^{k-2} r_j^{(i)}(0) \frac{x^i}{i!}, \quad j \in \{1, \ldots, r\}.
$$

According to this decomposition we can write $R = R_0 + R_{k-2}$, where

$$
R_0 = R_{1,0} \oplus \cdots \oplus R_{r,0}, \quad R_{k-2} = R_{1,k-2} \oplus \cdots \oplus R_{r,k-2},
$$

and

$$
(R_{j,0} f)(\cdot) := (r_{j,0} \ast f)(\cdot), \quad (R_{j,k-2} f)(\cdot) := (r_{j,k-2} \ast f)(\cdot), \quad j \in \{1, \ldots, r\}.
$$

Furthermore, $R_0 \in \text{Alg} A_k$ by (4.8) and $R_{k-2} \in \text{Alg} A_k$ by (iii) . Thus (1) is proved.
Theorem 4.4. Suppose $A_k(j) := \bigoplus_{i=1}^{n_j} \lambda_{ji} A_k^i$ is defined on $\bigoplus_{i=1}^{n_j} W_p^k[0,1], \ j \in \{1, \ldots, r\}$ and $A_k := \bigoplus_{j=1}^{r} A(j)$ is defined on $\bigoplus_{j=1}^{r} \bigoplus_{i=1}^{n_j} W_p^k[0,1]$. Let also

$$\lambda_{ji} = \frac{\lambda}{s_{ji}}, \ 1 = s_{j1} \leq s_{j2} \leq \ldots \leq s_{jn},$$

$$\text{arg} \lambda_{i1} \neq \text{arg} \lambda_{j1} \pmod{2\pi}, \quad 1 \leq i < j \leq r.$$

Let also

$$R := \bigoplus_{j=1}^{r} \bigoplus_{i=1}^{n_j} R_{ji} \in \left[ \bigoplus_{j=1}^{r} \bigoplus_{i=1}^{n_j} W_p^k[0,1] \right], \quad (R_{ji} f)(\cdot) = (r_{ji} * f)(\cdot), \quad 1 \leq j \leq r.$$

Then the following are true:

1. if $1 \leq \alpha \leq k - 1$, then

$$\text{Alg} A_k = \left\{ c I + R : \ c \in \mathbb{C}; \ r_{j1} \in W_p^{k-1}[0,1], \ 1 \leq j \leq r; \ r_{ji}(x) = s_{ji}^{-1} r_{j1}(s_{ji}^{-1} x), \ 1 \leq j \leq r, \ 1 \leq i \leq n_j; \ r_{j1}^{(\alpha m-1)}(0) = (\lambda_{j1} \lambda_{i1})^{m} r_{j1}^{(\alpha m-1)}(0), \ m \leq \left\lfloor \frac{k-1}{\alpha} \right\rfloor, \ 1 \leq j \leq r; \ r_{j1}^{(l)}(0) = 0, \ l \neq \alpha m - 1, \ m \leq \left\lfloor \frac{k-1}{\alpha} \right\rfloor, \ 1 \leq j \leq r \right\};$$

2. if $2 \leq k \leq \alpha + \frac{1}{p}$, then

$$\text{Alg} A_k = \left\{ c I + R : \ c \in \mathbb{C}; \ r_{j1} \in W_p^{k-1}[0,1], \ 1 \leq j \leq r; \ r_{ji}(x) = s_{ji}^{-1} r_{j1}(s_{ji}^{-1} x), \ 1 \leq j \leq r, \ 1 \leq i \leq n_j \right\}.$$

Remark 4.5. In this paper we do not consider questions about the reflexivity of the operator $A_k$. Such results are contained in [17].

4.2 The commutant $\{A_k\}'$

As in Section 2 we define operator $L_a \in [W_p^{k}[0,1]]$ for $a \in (0,1)$ and $L_a \in [W_p^{k}_0[0,1], W_p^{k}[0,1]]$ for $a \in (1, \infty)$ by

$$L_a : f(x) \rightarrow g(x) = \begin{cases} f(ax) & 0 < a \leq 1, \\ 0, & x \in [0, 1 - a^{-1}], \\ f(ax - a + 1), & x \in [1 - a^{-1}, 1], \end{cases} \quad (4.10)$$

Next we investigate solvability of the equation

$$R J_k^c = c J_k^c R \quad (4.11)$$

in the space $X = W_p^{k}[0,1]$ and describe the set of its solutions. The following proposition plays a crucial role in the sequel. Its proof is based on Corollary 2.21 and use some ideas from [16].
Proposition 4.6. Let \( c \in \mathbb{C} \) and let \( R \in [X] \) be a solution of equation \((4.11)\) where \( X = W^k_p[0,1] \). Then

1. If \( c \notin \mathbb{R}_+ \), then \( R = 0 \);
2. If \( 0 < c = a^\alpha \leq 1 \), \( a > 0 \), then \( R \in L_a \{ J^\alpha_k \}' = \{ J^\alpha_k \}'L_a \), that is,
\[
(Rf)(x) = \frac{d}{dx} \int_0^x r(x-t)f(at)\,dt, \quad r \in W^k_p[0,1];
\]
3. If \( 1 < c = a^\alpha \), \( a > 0 \), then \( R \in L_a \{ J^\alpha_k \}' \), that is,
\[
(Rf)(x) = \left( L_a \frac{d}{dx} (r \ast f) \right)(x)
\]
\[
= \begin{cases} 
0, & x \in [0,1-a^{-1}], \\
\alpha^{-1} \frac{d}{dx} \int_0^x r(ax-a+1-t)f(t)\,dt, & r \in W^k_p[0,1], \quad x \in [1-a^{-1}, 1].
\end{cases}
\]

Proof. Let \( c \in \mathbb{C} \) and \( RJ^\alpha_k = cJ^\alpha_kR \). Consider the block matrix representations of the operators \( J^\alpha_k \) and \( R \) with respect to the direct sum decomposition \( W^k_p[0,1] = W^k_{p,0}[0,1] + X_k \), where \( X_k := \text{span}\{1, x, \ldots, x^{k-1}\} \). Since \( W^k_{p,0}[0,1] \in \text{Lat} J^\alpha_k \), one has
\[
J^\alpha_k = \begin{pmatrix} J^\alpha_{11} & J^\alpha_{12} \\ \varnothing & J^\alpha_{22} \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.
\]
Now the equality \( RJ^\alpha_k = cJ^\alpha_kR \) splits into
\[
R_{11}J^\alpha_{11} = cJ^\alpha_{11}R_{11} + cJ^\alpha_{12}R_{21}, \quad (4.12)
\]
\[
R_{21}J^\alpha_{11} = cJ^\alpha_{22}R_{21}, \quad (4.13)
\]
\[
R_{21}J^\alpha_{12} + R_{22}J^\alpha_{22} = cJ^\alpha_{22}R_{22},
\]
\[
R_{12}J^\alpha_{12} + R_{12}J^\alpha_{22} = cJ^\alpha_{11}R_{12} + cJ^\alpha_{12}R_{22}.
\]

It is clear that \( J^\alpha_{22} \) is a nilpotent operator on \( X_k \) and consequently \( J^\alpha_{22} = 0 \). Therefore one derives from \((4.13)\) that \( R_{21}J^\alpha_{11} = cJ^\alpha_{22}R_{21} = \varnothing \). It follows that \( R_{21} = \varnothing \) since \( \text{ran} J^\alpha_{11} \) is dense in \( W^k_{p,0}[0,1] \). Now equation \((4.12)\) takes the form \( R_{11}J^\alpha_{11} = cJ^\alpha_{11}R_{11} \), that is, \( R_{11} \) intertwines the operators \( J^\alpha_{11} \) and \( cJ^\alpha_{11} \).

1. Let \( c \notin \mathbb{R}_+ \). Then Corollary \(2.21\) (i) yields \( R_{11} = \varnothing \). Furthermore, since \( J^\alpha_{0,k}x^m \in W^k_{p,0}[0,1], m \in \{0, \ldots, k-1\} \), one has
\[
0 = R_{11}J^\alpha_{0,k}x^m = RJ^\alpha_{0,k}x^m = cJ^\alpha_{0,k}Rx^m.
\]

It follows that \( Rx^m = 0 \) for \( m \in \{0, \ldots, k-1\} \), hence \( R = \varnothing \).

2. Let \( 0 < c = a^\alpha \leq 1 \) for some \( a > 0 \). Then Corollary \(2.21\) (ii) yields
\[
(R_{11}f)(x) = \frac{d}{dx} \int_0^x r(x-t)f(at)\,dt, \quad r \in L^p[0,1].
\]
Let us prove that \( r \in W^k_p[0,1] \). We have
\[
a^\alpha(J^\alpha_k R1)(x) = (RJ^\alpha_k 1)(x) = (R_{11}J^\alpha_k 1)(x)
\]
\[
= \frac{d}{dx} \int_0^x r(x-t) \frac{(at)^\alpha}{\Gamma(\alpha k + 1)} \,dt = a^\alpha(J^\alpha R)(x).
\]
Hence $r = R1 \in W_p^k[0, 1]$.

So, the operator $R_{11}$ defined on $W_{p,0}^k[0, 1]$ admits a continuation $T$ as an operator defined on $W_p^k[0, 1]$ by

$$
T : W_p^k[0, 1] \to W_p^k[0, 1],
T : f(x) \to \frac{d}{dx} \int_0^x r(x-t)f(at) \, dt.
$$

Since $T \upharpoonright W_{p,0}^k[0, 1] = R \upharpoonright W_{p,0}^k[0, 1] = R_{11}$ and $J_{k}^{\alpha m} \in W_{p,0}^k[0, 1]$ for $m \in \{0, \ldots, k-1\}$, we obtain

$$
J_{k}^{\alpha m} T x^m = a^{\alpha m} T J_{k}^{\alpha m} x^m = a^{\alpha m} R J_{k}^{\alpha m} x^m = J_{k}^{\alpha m} R x^m.
$$

It follows that $T x^m = R x^m$ for $m \in \{0, \ldots, k-1\}$. Thus $R = T$.

(3) Since $c = a^\alpha > 1$, Corollary 2.21 (ii) yields

$$
(R_{11} f)(x) = \left( L_a \frac{d}{dx} (r * f) \right)(x)
= \begin{cases}
0, & x \in [0, 1 - a^{-1}], \\
\frac{a^{-1} d}{dx} \int_0^x r(ax - a + 1 - t)f(t) \, dt, & x \in [1 - a^{-1}, 1],
\end{cases}
$$

where $r \in L_p'[0, 1]$. Let us prove that $r \in W_{p,0}^k[0, 1]$.

$$
a^{\alpha m}(J_{k}^{\alpha} R1)(x) = (R J_{k}^{\alpha})1(x) = (R_{11} J_{k}^{\alpha})1(x)
= \begin{cases}
0, & x \in [0, 1 - a^{-1}], \\
\frac{a^{-1} d}{dx} \int_0^x r(ax - a + 1 - t)J_{k}^{\alpha}(ax - a + 1) \, dt, & x \in [1 - a^{-1}, 1],
\end{cases}
$$

$$
= \begin{cases}
0, & x \in [0, 1 - a^{-1}], \\
J_{k}^{\alpha + 1}(x), & x \in [1 - a^{-1}, 1],
\end{cases}
$$

Hence

$$
(R1)(x) = \begin{cases}
0, & x \in [0, 1 - a^{-1}], \\
r(ax - a + 1), & x \in [1 - a^{-1}, 1].
\end{cases}
$$

Since $R1 \in W_p^k[0, 1]$, it follows that $r \in W_{p,0}^k[0, 1]$.

So, the operator $R_{11}$ defined on $W_{p,0}^k[0, 1]$ admits a continuation $T$ on $W_p^k[0, 1]$ defined by

$$
(T f)(x) = \begin{cases}
0, & x \in [0, 1 - a^{-1}], \\
\frac{a^{-1} d}{dx} \int_0^x r(ax - a + 1 - t)f(t) \, dt, & x \in [1 - a^{-1}, 1].
\end{cases}
$$

Since $T \upharpoonright W_{p,0}^k[0, 1] = R \upharpoonright W_{p,0}^k[0, 1] = R_{11}$ and $J_{k}^{\alpha m} \in W_{p,0}^k[0, 1]$ for $m \in \{0, \ldots, k-1\}$, one deduces

$$
J_{k}^{\alpha m} T x^m = a^{\alpha m} T J_{k}^{\alpha m} x^m = a^{\alpha m} R J_{k}^{\alpha m} x^m = J_{k}^{\alpha m} R x^m.
$$

It follows that $T x^m = R x^m$ for $m \in \{0, \ldots, k-1\}$. Thus $R = T$. □
Suppose \( \lambda = \lambda_1/\lambda_i \), \( 1 = s_1 \leq s_2 \leq \ldots \leq s_n \), \( a_{ij} = s_i^{-1}s_j \), \( 1 \leq i, j \leq n \).

Then the commutant \( \{A_k\}' \) is of the form
\[
\{A_k\}' = \{ R : \ R = (R_{ij})_{i,j=1}^n, \ R_{ij} = L_{a_{ij}}K_{ij} \},
\]
where
\[
(K_{ij}f)(x) = \frac{d}{dx} \int_0^x k_{ij}(x-t)f(t)\,dt, \quad k_{ij} \in \begin{cases} \mathbb{W}_p^k[0,1], & a_{ij} \leq 1, \\ \mathbb{W}_p^{k,0}[0,1], & a_{ij} > 1. \end{cases}
\]

**Proof.** Let \( R = (R_{ij})_{i,j=1}^n \) be the block matrix partition of the operator \( R \) with respect to the direct sum decomposition \( X^{(n)} = \bigoplus_{i=1}^n \mathbb{W}_p^k[0,1] \). Then the equality \( RA_k = A_kR \) is equivalent to the following system
\[
R_{ij}J_k^\alpha = \lambda_i \lambda_j^{-1}J_k^\alpha R_{ij} = (s_i^{-1}s_j)^\alpha J_k^\alpha R_{ij} = a_{ij}^\alpha J_k^\alpha R_{ij}, \quad 1 \leq i, j \leq n.
\]
To complete the proof it remains to apply Proposition 4.8.

**Theorem 4.9.** Suppose \( A_k = \bigoplus_{j=1}^r \lambda_j J_k^\alpha \) is defined on \( \bigoplus_{j=1}^r \mathbb{W}_p^k[0,1] \) and \( \arg \lambda_i \neq \arg \lambda_j \ (\text{mod } 2\pi) \) for \( 1 \leq i < j \leq r \). Then the commutant \( \{A_k\}' \) splits, that is,
\[
\{A_k\}' = \bigoplus_{j=1}^r \{\lambda_j J_k^\alpha\}'.
\]

**Proof.** Following the proof of Theorem 4.8 one arrives at the relations
\[
R_{ij}J_k^\alpha = \lambda_i \lambda_j^{-1}J_k^\alpha R_{ij}, \quad 1 \leq i, j \leq r.
\]
(4.14)
The latter results with \( i = j \) yield \( R_{ii} \in \{J_k^\alpha\}' \) for \( i \in \{1, \ldots, r\} \), hence by Proposition 4.6 (2)
\[
R_{ii} : f \rightarrow \frac{d}{dx} \int_0^x p_{ii}(x-t)f(t)\,dt, \quad r_{ii} \in \mathbb{W}_p^k[0,1], \quad i \in \{1, \ldots, r\}.
\]
Since \( \arg \lambda_i \neq \arg \lambda_j \ (\text{mod } 2\pi) \) (\( 1 \leq i < j \leq r \)), it follows that \( \lambda_i \lambda_j^{-1} \notin \mathbb{R}_+ \), hence by Proposition 4.6 (1) \( R_{ii} = 0 \) (\( 1 \leq i \neq j \leq r \)). This completes the proof.

Combining Theorems 4.8 and 4.9 we arrive at

**Theorem 4.10.** Suppose \( A_k(j) := \bigoplus_{i=1}^{n_j} \lambda_{ij} J_k^\alpha \) is defined on \( \bigoplus_{i=1}^{n_j} \mathbb{W}_p^k[0,1] \) \( j \in \{1, \ldots, r\} \) and \( A_k := \bigoplus_{j=1}^r A(j) \) is defined on \( W = \bigoplus_{j=1}^r \bigoplus_{i=1}^{n_j} \mathbb{W}_p^k[0,1] \). Let also
\[
\arg \lambda_{ij} = \arg \lambda_{ji} \ (\text{mod } 2\pi), \quad 1 \leq j \leq r, \quad 1 \leq i \leq n_j,
\]
\[
\arg \lambda_i \neq \arg \lambda_{ij} \ (\text{mod } 2\pi), \quad 1 \leq i < j \leq r.
\]
Then
\[
\{A_k\}' = \bigoplus_{j=1}^r \{A_k(j)\}'.
\]
where the algebras \( \{A_k(j)\}' \) are described in Theorem 4.8.
4.3 The double commutant \( \{A_k\}'' \)

**Theorem 4.11.** Suppose \( A_k = \bigoplus_{i=1}^{n} \lambda_i J_k^\alpha \) is defined on \( W = \bigoplus_{i=1}^{n} W_p^k[0,1] \) and

\[
\lambda_i = \frac{\lambda_1}{s_i^\alpha}, \quad 1 = s_1 \leq s_2 \leq \ldots \leq s_n, \quad a_{ij} = s_i^{-1}s_j, \quad 1 \leq i, j \leq n.
\]

Then

(1) \[
\{A_k\}'' = \{cI + R : c \in \mathbb{C}, \ R = \text{diag}(R_1, \ldots, R_n), \ (R_it)(\cdot) = (r_i * f)(\cdot), \ r_i(x) = s_i^{-1}r_1(s_i^{-1}x), \ r_i \in W_p^{k-1}[0,1], \ 1 \leq i \leq n\}.
\]

(2) The dimension \( d_{k,\alpha} \) of the quotient space \( \{A_k\}''/\text{Alg} A_k \) is \( k - 1 - [(k - 1)/\alpha] \). In particular, \( \text{Alg} A_k = \{A_k\}'' \) if and only if either \( \alpha = 1 \) or \( k = 1 \).

**Proof.** Let us set

\[
e_i := (0, \ldots, 0, 1, 0, \ldots, 0), \quad E_{ij} := e_i^T e_j, \quad 1 \leq i, j \leq n.
\]

Then Theorem 4.8 implies

\[
\{A_k\}' = \text{Alg} \{J_k \otimes E_{ii}, \ 1 \leq i \leq n; \ L_{aij} \otimes E_{ij}, \ 1 \leq j \leq i \leq n; \ L_{aij} J_k^\alpha \otimes E_{ij}, \ 1 \leq i < j \leq n\}.
\]

Since \( \bigoplus_{i=1}^{n} \lambda_i J_k^\alpha \subset \bigoplus_{i=1}^{n} \lambda_i J_k^\alpha \), it follows from (1.11)

\[
\{A_k\}'' \subset \{T := (c_1 I + R_1) \oplus \cdots \oplus (c_n I + R_n) : c_i \in \mathbb{C}, \ (R_i f)(\cdot) = (r_i * f)(\cdot), \ r_i \in W_p^{k-1}[0,1]\}.
\]

It is clear that \( T(J_k \otimes E_{ii}) = (J_k \otimes E_{ii})T \) for \( i \in \{1, \ldots, n\} \). It can easily be checked that

\[
T(L_{aij} \otimes E_{ij}) = (L_{aij} \otimes E_{ij})T, \quad 1 \leq j \leq i \leq n, \quad (4.15)
\]

\[
T(L_{aij} J_k^\alpha \otimes E_{ij}) = (L_{aij} J_k^\alpha \otimes E_{ij})T, \quad 1 \leq i < j \leq n \quad (4.16)
\]

if and only if \( c_1 = \cdots = c_n \) and \( r_i(x) = s_i^{-1}r_1(s_i^{-1}x) \) for \( 1 \leq i \leq n \). Indeed, (4.15) and (4.16) are equivalent to the first and the second of the following relations

\[
(c_j I + R_j)L_{aij} f = L_{aij} (c_j I + R_j)f, \quad f \in W_p^k[0,1], \quad 1 \leq j \leq i \leq n,
\]

\[
(c_j I + R_j)L_{aij} J_k^\alpha f = L_{aij} J_k^\alpha (c_j I + R_j)f, \quad f \in W_p^k[0,1], \quad 1 \leq i < j \leq n,
\]

respectively. According to the definition of \( L_{aij} \) (see (4.10)), we obtain

\[
c_j f(a_{ij}x) + \int_0^x r_j(x-t)f(a_{ij}t) dt = c_i f(a_{ij}x) + \int_0^{a_{ij}x} r_i(a_{ij}x-t)f(t) dt \quad (4.17)
\]
for \( f \in W^k_p[0, 1], x \in [0, 1] \) and \( 1 \leq j \leq i \leq n \), and
\[
c_j(J_k^f)(a_{ij}x - a_{ij} + 1) + \int_{1-a_{ij}^{-1}}^{x} r_j(x - t)(J_k^f)(a_{ij}t - a_{ij} + 1) \, dt
\]
(4.18)
\begin{align*}
&= c_i(J_k^f)(a_{ij}x - a_{ij} + 1) + \int_0^x r_i(a_{ij}x - a_{ij} + 1)(J_k^f)(t) \, dt \\
\end{align*}
for \( f \in W^k_p[0, 1], x \in [1 - a_{ij}^{-1}, 1] \) and \( 1 \leq j < i \leq n \).

After simple computations with (4.17)-(4.18), we get
\[
\int_0^x \left[ r_j(x - t) - a_{ij}r_i(a_{ij}(x - t)) \right] f(a_{ij}t) \, dt = (c_i - c_j)f(a_{ij}x),
\]
\[
\int_0^x \left[ r_i(x - t) - a_{ij}^{-1}r_j(a_{ij}^{-1}(x - t)) \right] (J_k^f)(t) \, dt = (c_j - c_i)(J_k^f)(x).
\]

Now it is easy to see that any of the latter equations is equivalent to \( c_1 = \cdots = c_n \) and \( r_i(x) = s_i^{-1}r_i(s_i^{-1}x) \) for \( i \in \{1, \ldots, n\} \). Thus, (1) is proved.

(2) It is clear that \( W^{k-1}_p[0, 1] \approx W^{k-1}_p[0, 1] + \text{span}\{\frac{x^l}{l!} : l = 1, \ldots, k-2\} \). Hence (1) implies that
\[
\{A_k\}'' \approx \mathbb{C} + W^{k-1}_p[0, 1] \approx \mathbb{C} + W^{k-1}_p[0, 1] + \text{span}\{\frac{x^l}{l!} : l = 0, \ldots, k-2\}. \tag{4.19}
\]

Further, Theorem 4.12 yields \( A_k \) is isomorphic
\[
\text{Alg } A_k \approx \mathbb{C} + W^{k-1}_p[0, 1] + \text{span}\left\{\frac{x^{am-1}}{(am-1)!} : 1 \leq m \leq \left\lfloor \frac{k-1}{\alpha} \right\rfloor \right\}. \tag{4.20}
\]

Combining (4.19) with (4.20) we easily arrive at (2). \( \square \)

**Theorem 4.12.** Suppose \( A_k = \bigoplus_{j=1}^r \lambda_j J_k^\alpha \) is defined on \( \bigoplus_{j=1}^r W_p^k[0, 1] \) and \( \arg \lambda_i \neq \arg \lambda_j \pmod{2\pi} \) for \( 1 \leq i < j \leq r \). Then
\begin{enumerate}
\item \( \{A_k\}'' = \bigoplus_{j=1}^r \{J_k^\alpha\}'' \).
\item The dimension \( d_{k, \alpha} \) of the quotient space \( \{A_k\}'' / \text{Alg } A_k \) is \( d_{k, \alpha} = rk - 1 - [(k - 1)/\alpha] \). In particular, \( \text{Alg } A_k = \{A_k\}'' \) if and only if either
  \begin{enumerate}
  \item \( r = 1 \) and \( \alpha = 1 \), or
  \item \( r = 1 \) and \( k = 1 \).
  \end{enumerate}
\end{enumerate}
Proof. (1) is implied by Theorem 4.9. Furthermore, (1) and Theorem (4.3) imply that

\[ \{A_k\}'' \approx \bigoplus_{j=1}^{r} (\mathbb{C}^1 \oplus W_{p}^{k-1}[0,1]) \]

\[ \approx \mathbb{C}^r + \bigoplus_{j=1}^{r} W_{p,0}^{k-1}[0,1] + \bigoplus_{j=1}^{r} \text{span}\{ \frac{x_l}{l!} : l = 0, \ldots, k-2 \} \]

\[ \text{Alg} A_k \approx \mathbb{C}^1 + \bigoplus_{j=1}^{r} W_{p,0}^{k-1}[0,1] + \text{span}\left\{ \frac{x_{\alpha m - l}}{(\alpha m - l)!} : 1 \leq m \leq \left\lfloor \frac{k-1}{\alpha} \right\rfloor \right\}. \]

Now it is easy to see that

\[ d_{k,\alpha} = r + r(k-1) - 1 - \left\lfloor \frac{k-1}{\alpha} \right\rfloor = rk - 1 - \left\lfloor \frac{k-1}{\alpha} \right\rfloor. \]
Thus (2) is proved. \qed

Combining Theorems 4.11 and 4.12, we obtain

**Theorem 4.13.** Under the conditions of Theorem 4.10, we have

\[ \{A_k\}'' = \bigoplus_{j=1}^{r} \{A_k(j)\}'', \]

where the algebras \( \{A_k(j)\}'' \) are described in Theorem 4.11.

**Remark 4.14.** Recall that according to celebrated von Neumann theorem \( \{T\}'' = \text{Alg} T \) whenever \( T \) is a normal operator. B. Sz.-Nagy and C. Foiaș [33]-[34] generalized this result to the wide class of accretive (dissipative) operators. In particular, this result holds for the accretive operator \( A = J \otimes B \) defined on \( L_2[0,1] \otimes \mathbb{C}^n \), where \( B \) is a diagonal positive matrix, \( B = B^* > 0 \). By Theorem 4.1 this result remains also valid for non-accretive operator \( T := A_k = J_{\alpha} \otimes B \) defined on \( \bigoplus_{j=1}^{n} W_{k}^{2}[0,1] \), with the same \( B \).

### 4.4 Invariant subspaces

In [16] we proved that every subspace invariant under \( J_{\alpha}^k \) belongs either to the "continuous chain" \( \text{Lat}^c J_{\alpha}^k \) or to the "discrete chain" \( \text{Lat}^d J_{\alpha}^k \). It turns out that \( \text{Lat}^c J_{\alpha}^k \) does not depend on \( \alpha \): \( \text{Lat}^c J_{\alpha}^k = \text{Lat}^c J_k \) (see (1.9)). We proved also that the description of \( \text{Lat}^d J_{\alpha}^k \) easily follows from that of \( \text{Lat} J(0,k)'' \). This description is extracted from Theorem 2.1.

In this section we prove that every \( A_k \)-invariant subspace can be decomposed into a direct sum of two invariant subspaces: the first one belongs to the "continuous part" of \( \text{Lat} A_k \) and the second one belongs to the "discrete part" of \( \text{Lat} A_k \). We show also, that "continuous part" does not depend on \( \alpha \). Moreover, a description of the "discrete part" is deduced from Theorem 2.1.

Let \( \chi_s \) stand for the characteristic function of an arbitrary nonempty subset \( S \subset \mathbb{Z}_n := \{1, \ldots, n\} \). We denote by \( P_S \) and \( \check{P}_S \) the canonical projections from
\[ \bigoplus_{j=1}^{n} W_{p}^{k_{j}}[0, 1] \] and from \( \bigoplus_{j=1}^{n} C^{k_{j}} \) onto \( \bigoplus_{j=1}^{n} \chi_{s}(j)W_{p}^{k_{j}}[0, 1] \) and onto \( \bigoplus_{j=1}^{n} \chi_{s}(j)C^{k_{j}} \), respectively. Next we let

\[ A_{k,S} := \bigoplus_{j=1}^{n} \chi_{s}(j)\lambda_{j}J_{k_{j}}^{\alpha} \upharpoonright \text{ran } P_{S}, \quad \tilde{A}_{k,S} := \bigoplus_{j=1}^{n} \chi_{s}(j)\lambda_{j}J(0; k_{j})^{\alpha} \upharpoonright \text{ran } \tilde{P}_{S} \]

and denote by \( \pi_{S} \) the quotient mapping from \( \text{ran } P_{S} \) onto \( \text{ran } \tilde{P}_{S} \).

**Theorem 4.15.** Suppose \( A_{k} = \bigoplus_{j=1}^{n} \lambda_{j}J_{k_{j}}^{\alpha} \) is defined on \( \bigoplus_{j=1}^{n} W_{p}^{k_{j}}[0, 1] \) and \( \arg \lambda_{i} \neq \arg \lambda_{j} \pmod{2\pi} \) for \( 1 \leq i < j \leq n \). Then \( E \in \text{Lat } A_{k} \) if and only if there exists \( S \subset \mathbb{Z}_{+} \) and \( a_{1}, \ldots, a_{n} \in [0, 1] \) such that

\[ E = \text{Lat } A_{k,S} \bigoplus_{j=1}^{n} \chi_{s}(j)E_{a_{j}, 0}^{k_{j}}, \]

where

\[ \text{Lat } A_{k,S} = \bigcup_{M} \pi_{S}^{-1} \left\{ [M, (\tilde{A}_{k,S})^{-1}M] : M \in \text{Lat } \tilde{A}_{k,S} \upharpoonright \tilde{A}_{k,S}M \right\} \quad (4.21) \]

and \( S^{c} \) is the complement for \( S \) in \( \mathbb{Z}_{+} \) \( (S \cup S^{c} = \mathbb{Z}_{+}) \). Here \( [M, (\tilde{A}_{k,S})^{-1}M] \) is a closed interval in the lattice of all subspaces of \( \text{ran } \tilde{P}_{S} \). Each interval satisfies the equation

\[ \dim(\tilde{A}_{k,S})^{-1}M - \dim M = \sum_{j \in S} \min \{-[-\alpha], k_{j}\}. \quad (4.22) \]

**Proof.** For every \( E \in \text{Lat } A_{k} \), we put \( j \) in \( S := S_{E} \) if \( P_{j}E \not\subset W_{p,0}^{k_{j}}[0, 1] \) and put \( j \) in \( S^{c} \) otherwise. Next we introduce the subspaces \( E_{S} := \text{span}\{A_{k,S}^{m}P_{S}E : m \geq 0\} \) and \( E_{S^{c}} := \text{span}\{A_{k,S}^{m}P_{S^{c}}E : m \geq 0\} \subset \bigoplus_{j=1}^{n} \chi_{s}(j)W_{p,0}^{k_{j}}[0, 1] \). It is clear that \( E \subset E_{S} \oplus E_{S^{c}} \).

Let \( M = \max_{1 \leq j \leq n} k_{j} \). Then the subspace \( F := \overline{A_{k}^{M}E} \) is invariant for the operator \( A_{k,0} := A_{k} \upharpoonright \bigoplus_{j=1}^{n} W_{p,0}^{k_{j}}[0, 1] \) and, by Theorem 2.19, \( F = \bigoplus_{j=1}^{n} E_{a_{j}, 0}^{k_{j}} \) for some \( a_{j} \in [0, 1] \). By the construction of \( S \), it is clear that \( a_{j} = 0 \) for \( j \in S \) and hence

\[ F = \left( \bigoplus_{j=1}^{n} \chi_{s}(j)W_{p,0}^{k_{j}}[0, 1] \right) \cup \left( \bigoplus_{j=1}^{n} \chi_{s}(j)E_{a_{j}, 0}^{k_{j}} \right). \quad (4.23) \]

It is clear that \( E \supset F \supset E_{S^{c}} \). Hence \( E \supset P_{S^{c}}E \) and, therefore, \( E \supset P_{S}E \). The latter inclusion yields \( E \supset E_{S} \) and consequently \( E \) splits: \( E = E_{S} \oplus E_{S^{c}} \).

In turn, by Theorem 2.19, \( E_{S^{c}} \) splits: \( E_{S^{c}} = \bigoplus_{j=1}^{n} \chi_{s}(j)W_{p,0}^{k_{j}}[0, 1] \). On the other hand, combining \( (4.23) \) with the relations \( E = E_{S} \oplus E_{S^{c}} \supset F \), one gets \( E \supset \bigoplus_{j=1}^{n} \chi_{s}(j)W_{p,0}^{k_{j}}[0, 1] \). Therefore, \( \pi_{S}(E_{S}) \in \text{Lat } \tilde{A}_{S} \). Since the quotient map \( \pi_{S} \) establishes a bijective correspondence between \( E_{S} \in \text{Lat } A_{S} \) with \( E_{S} \supset \bigoplus_{j=1}^{n} \chi_{s}(j)W_{p,0}^{k_{j}}[0, 1] \) and \( \pi_{S}(E_{S}) \), one derives \( E_{S} = \pi_{S}^{-1}(\pi_{S}E_{S}) \). One completes the proof by applying Theorem 2.19. Furthermore, relations \( (4.21) \) and \( (4.22) \) are implied by the relations \( (2.1) \) and \( (2.2) \), respectively. \( \square \)
Corollary 4.16. [10] Let \( \pi \) be the quotient map

\[
\pi : W^p_k[0, 1] \to X_k := W^p_k[0, 1]/W^p_{p,0}[0, 1]
\]

and \( \hat{J}_k^\alpha \) be the quotient operator on \( X_k \). Then \( \text{Lat} J_k^\alpha = \text{Lat}^c J_k^\alpha \cup \text{Lat}^d J_k^\alpha \), where

(a)

\[
\text{Lat}^c J_k^\alpha = \{ E^k_{a,0} : 0 \leq a \leq 1 \}, \quad E^k_{a,0} := \{ f \in W^p_{p,0}[0, 1] : f(x) = 0, \ x \in [0, a] \}
\]

is the "continuous part" of \( \text{Lat} J_k^\alpha \);

(b)

\[
\text{Lat}^d J_k^\alpha = \pi^{-1}(\text{Lat} \hat{J}_k^\alpha) = \bigcup_M \pi^{-1}\left\{ [M, (\hat{J}_k^\alpha)^{-1}M] : M \in \text{Lat}(\hat{J}_k^\alpha \upharpoonright \hat{J}_k^\alpha M) \right\}
\]

is the "discrete part" of \( \text{Lat} J_k^\alpha \).

Here \( [M, (\hat{J}_k^\alpha)^{-1}M] \) is a closed interval in the lattice of all subspaces of \( X_k \). Each interval satisfies the equation

\[
\dim(\hat{J}_k^\alpha)^{-1}M - \dim M = d,
\]

where \( d = \min\{-[-\alpha], k\} \).

Corollary 4.17. [10] Operator \( J_k^\alpha \) is unicellular if and only if either \( \alpha = 1 \) or \( k = 1 \).

Example. Suppose that the operator \( A = \lambda_1 J_{k_1}^\alpha \oplus \lambda_2 J_{k_2}^\alpha \) (arg \( \lambda_1 \neq \arg \lambda_2 \) (mod 2\( \pi \)) is defined on \( W^p_{k_1}[0, 1] \oplus W^p_{k_2}[0, 1] \). By Theorem 4.13 one has the following description of its lattice of invariant subspaces:

\[
\text{Lat} A = \bigcup_{[a_1, a_2] \in [0,1] \times [0,1]} (E^{k_1}_{a_1,0} \oplus E^{k_2}_{a_2,0}) \cup \bigcup_{a \in [0,1]} \pi^{-1}_{\{1\}}(\text{Lat} \hat{A}_{\{1\}}) \oplus E^k_{a,0}
\]

\[
\cup \bigcup_{a \in [0,1]} E^k_{a,0} \oplus \pi^{-1}_{\{2\}}(\text{Lat} \hat{A}_{\{2\}}) \cup \bigcup_{a \in [0,1]} \pi^{-1}_{\{1,2\}}(\text{Lat} \hat{A}_{\{1,2\}}),
\]

where lattices \( \pi^{-1}_{\{1\}}(\text{Lat} A_{\{1\}}) = \text{Lat}^d J_{k_1}^\alpha \) and \( \pi^{-1}_{\{2\}}(\text{Lat} A_{\{2\}}) = \text{Lat}^d J_{k_2}^\alpha \) are described in Corollary 4.10. For example, if \( k_1 = 1, \ k_2 = 2, \ \lambda_1 = i, \ \lambda_2 = 1 \) and \( \alpha = 1 \), one has \( \pi^{-1}_{\{1\}}(\text{Lat} A_{\{1\}}) = \text{Lat}^d J_1^1 = W^p_{p,0}[0, 1] \oplus W^p_{p,0}[0, 1], \ \pi^{-1}_{\{2\}}(\text{Lat} A_{\{2\}}) = \text{Lat}^d J_2^1 = W^p_{p,0}[0, 1] \cup E^2_{0} \cup W^p_{p,0}[0, 1] \). It is easily seen that \( \hat{A}_{\{1,2\}} = 0 \oplus J(0; 2) \), hence, \( \text{Lat} \hat{A}_{\{1,2\}} \upharpoonright \text{ran}(A_{\{1,2\}}) : e_3 \to 0 \) (here \( \{e_1, e_2, e_3\} \) is the standard basis in \( C^3 \)). Thus, by Theorem 2.1

\[
\text{Lat} A_{\{1,2\}} = \bigcup_{M \subset \{e_3\}} [M, (\hat{A}_{\{1,2\}})^{-1}M] = [0, \{e_1, e_3\}] \cup \{\{e_3\}, \{e_1, e_2, e_3\}\}
\]

\[
= \{0\} \cup \bigcup_{\alpha, \beta \in C} \{\alpha e_1 + \beta e_3\} \cup \bigcup_{\alpha, \beta \in C} \{\alpha e_1 + \beta e_2, e_3\} \cup \{e_1, e_2, e_3\}
\]

\[
\approx \{0\} \cup \bigcup_{\alpha, \beta \in C} \{(\alpha, \beta)\} \cup \bigcup_{\alpha, \beta \in C} \{(\alpha, \beta), (0, x)\} \cup \{(1, 0), (0, 1), (0, x)\}.
\]
Hence
\[
\pi_{1,2}^{-1}(\text{Lat} A_{1,2}) = (W_{p,0}[0,1] \oplus W_{p,0}[0,1]) \\
\cup \bigcup_{\alpha,\beta \in C} \{\{f_1, f_2\} : f_1 \in W_{p}[0,1], f_2 \in E_1, \alpha f_1(0) + \beta f_2(0) = 0\} \\
\cup \bigcup_{\alpha,\beta \in C} \{\{f_1, f_2\} : f_1 \in W_{p}[0,1], f_2 \in W_{p}[0,1], \alpha f_1(0) + \beta f_2(0) = 0\} \\
\cup (W_{p}[0,1] \oplus W_{p}[0,1]).
\]

Remark 4.18. (i) An alternative description of \text{Lat}^d J^k might be obtained from the Halmos description of \text{Lat} T for \(T \in [\mathbb{C}^n]\) (see Theorem 2.2).

(ii) A quite different proof of the description of \text{Lat} J_k has been originally obtained by E. Tsekanovskii [46].

4.5 Hyperinvariant subspaces

To present a description of HypLat A_k we keep the notation from Subsection 4.4.

Theorem 4.19. Let the conditions of Theorem 4.8 hold. Then
\[
\text{HypLat} A_k = \bigcup_{S \subset \mathbb{Z}_n} \{E_{S^c} \oplus E_S\}.
\]

Here

(a) "the continuous part" \(E_{S^c}\) is of the form
\[
E_{S^c} = \left\{ \bigoplus_{j=1}^{n} \chi_{S^c}(j) E_{a_j,0}^k : a = \{a_j\}_{j \in S^c} \in P(\{s_j\}_{j \in S^c}) \right\},
\]
where
\[
P(\{s_i\}_{i \in S^c}) := P(s_{n_1}, \ldots, s_{n_{|S^c|}}) = \left\{(a_{n_1}, \ldots, a_{n_{|S^c|}}) \in \square_{|S^c|} : s_{n_j}a_{n_{j+1}} \leq s_{n_{j+1}}a_{n_j} \leq s_{n_{j+1}} + s_{n_j}a_{n_{j+1}}, 1 \leq j \leq |S^c| - 1 \right\}.
\]

(b) "the discrete part" \(E_S\) is of the form \(E_S = \bigoplus_{j=1}^{n} \chi S(j) E_{l_j}^k\), where \(1 \leq l_j \leq k-1\) and \(l_j \leq l_i\) if \(s_j \leq s_i\) for \(1 \leq i, j \leq n\);

In particular, if \(\lambda_1 = \cdots = \lambda_n\), then
\[
\text{HypLat} A_k = \bigcup_{S \subset \mathbb{Z}_n, a \in [0,1], 1 \leq l \leq k-1} \left\{ \bigoplus_{j=1}^{n} \chi_S(j) E_{l_j}^k \bigoplus_{i=j}^{n} \chi_{S^c}(j) E_{a_i,0}^k \right\}.
\]
Remark 4.21. It is well known (see [11]) that for two bounded operators \( T_1 \) and \( T_2 \) the splitting of \( \text{Lat}(T_1 \oplus T_2) \) implies the splitting of \( \text{HypLat}(T_1 \oplus T_2) \). In other words, the relation \( \text{Lat}(T_1 \oplus T_2) = \text{Lat}T_1 \oplus \text{Lat}T_2 \) yields the relation \( \text{HypLat}(T_1 \oplus T_2) = \text{HypLat}T_1 \oplus \text{HypLat}T_2 \). Theorem 4.20 demonstrates that the converse implication is not true. Nevertheless the converse implication is true for \( C_0 \) contractions \( T_1 \) and \( T_2 \) defined on Hilbert space ([11]).

Summing up Theorems 4.19 and 4.20, we obtain

**Theorem 4.22.** Under the conditions of Theorem 4.10, we have

\[
\text{HypLat} A_k = \bigoplus_{j=1}^r \text{HypLat} \lambda_j J_k^\alpha = \bigoplus_{j=1}^r \text{Lat} J_k.
\]

where the lattices \( \text{HypLat} A_k(j) \) are described in Corollary 4.19.

4.6 Cyclic subspaces

Some results of this subsection were announced in [13]. First, we present the following simple

**Lemma 4.23.** Let \( A \in [\mathbb{C}^k], \sigma(A) = \{0\} \) and \( P_{\ker A^*} \) be the orthoprojection from \( \mathbb{C}^k \) onto \( \ker A^* \). Then

1. \( \mu_A = \text{disc} A = \dim(\ker A^*) = \dim(\ker A); \)

2. \( E \in \text{Cyc} A \) if and only if \( PE = \ker A^* \).
Applying the operator $A$.

Hence

It means $E$

Proof. Necessity. Note that $\text{span}\{E, \text{ran} A\} \supset \text{span}\{A^j E : j \geq 0\}$ and $(\mathbb{I}_k - P_{\text{ker} A^*})E = P_{\text{ran} A}E \subset \text{ran} A$. Therefore, since $E \in \text{Cyc} A$, we have

$$
\mathbb{C}^k = \text{span}\{A^j E : j = 0, 1, \ldots, k - 1\} \subset \text{span}\{P_{\text{ker} A^*}E, (\mathbb{I}_k - P_{\text{ker} A^*})E, \text{ran} A\}
$$

$$
= \text{span}\{P_{\text{ker} A^*}E, \text{ran} A\} \subset \text{span}\{\ker A^*, \text{ran} A\} = \ker A^* \oplus \text{ran} A = \mathbb{C}^k.
$$

Hence $P_{\text{ker} A^*}E = \ker A^*$.

Sufficiency. Let $PE = \ker A^*$. Then

$$
\mathbb{C}^k = \text{span}\{P_{\text{ker} A^*}E, \text{ran} A\} \subset \text{span}\{E, (\mathbb{I}_k - P_{\text{ker} A^*})E, \text{ran} A\} = \text{span}\{E, \text{ran} A\}.
$$

Applying the operator $A^j$, we obtain $\text{ran} A^j = \text{span}\{A^j E, \text{ran} A^{j+1}\} (1 \leq j \leq k - 1)$. Hence

$$
\mathbb{C}^k = \text{span}\{E, \text{ran} A\} = \text{span}\{E, AE, \text{ran} A^2\} = \cdots = \text{span}\{E, \ldots, A^{k-1} E\}.
$$

It means $E \in \text{Cyc} A$. 

For every system $\phi = \{\overrightarrow{\phi_1}\}_1^N$, $\overrightarrow{\phi} \in \mathbb{C}^n$, we denote by $W(\phi)$ the $n \times N$ matrix consisting of the columns $\overrightarrow{\phi} : W(\phi) = (\overrightarrow{\phi}_1, \ldots, \overrightarrow{\phi}_N)$.

Corollary 4.24. Suppose that $A = \bigoplus_{j=1}^n \lambda_j J(0; k_j)\alpha$ is defined on $\bigoplus_{j=1}^n \mathbb{C}^{k_j}$ and $m_j := \min(-[-\alpha], k_j)$ for $1 \leq j \leq n$. Then

(1) $\mu_A = \text{disc} A = \sum_{j=1}^n m_j$;

(2) the following system

$$
\overrightarrow{\phi_l} = \text{col}(\phi_{l11}, \ldots, \phi_{l1k_1}, \phi_{l21}, \ldots, \phi_{l2k_2}, \ldots, \phi_{ln1}, \ldots, \phi_{lnk_n}), \quad 1 \leq l \leq N
$$

generates a cyclic subspace for the operator $A$ if and only if

(1) $N \geq \sum_{j=1}^n m_j$;

(2) the matrix $W_0 = P_{\text{ker} A^*}W(\phi)$ is of maximal rank, that is, $\text{rank} W_0 = \sum_{j=1}^n m_j$.

Theorem 4.25. Suppose $A_k = \bigoplus_{j=1}^n \lambda_j J_{k_j}^\alpha$ is defined on $\bigoplus_{j=1}^n W_{p_j}^{k_j}[0, 1]$ and $m_j := \min(-[-\alpha], k_j)$ for $1 \leq j \leq n$. Then

(1) $\mu_{A_k} = \text{disc} A_k = \sum_{j=1}^n m_j$;

(2) the system $\{f_i(x)\}_1^N$ of vectors $f_i(x) = \{f_{i1}(x), \ldots, f_{in}(x)\}$ generates a cyclic subspace for $A_k$ if and only if the following conditions hold

(i) $N \geq \sum_{j=1}^n m_j$;

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(ii) the matrix

\[
W(0) = \begin{pmatrix}
  f_{11}(0) & f_{21}(0) & \cdots & f_{N1}(0) \\
  f'_{11}(0) & f'_{21}(0) & \cdots & f'_{N1}(0) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{1n}(0) & f_{2n}(0) & \cdots & f_{Nn}(0) \\
  f'_{1n}(0) & f'_{2n}(0) & \cdots & f'_{Nn}(0) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{1n-m}(0) & f_{2n-m}(0) & \cdots & f_{Nn-m}(0)
\end{pmatrix}
\]

is of maximal rank, i.e., \( \text{rank } W(0) = \sum_{j=1}^{n} m_j \).

Proof. It is clear that \( E \in \text{Cyc } A_k \) implies \( \pi E \in \text{Cyc } \hat{A}_k \). To prove the converse assertion we choose a subspace \( E \subset \bigoplus_{j=1}^{n} W_{p,0}^{k_j}[0,1] \) such that \( \pi E \in \text{Cyc } \hat{A}_k \) and denote by \( F := \text{span}\{ A^j E : j \geq 0 \} \). Since \( \pi F = \bigoplus_{j=1}^{n} \mathbb{C}^{k_j} \), one gets that \( F \supset \bigoplus_{j=1}^{n} W_{p,0}^{k_j}[0,1] \). Therefore, just in the same way as in Theorem 4.15, we obtain that \( F = \pi^{-1}(\pi F) = \pi^{-1}(\bigoplus_{j=1}^{n} \mathbb{C}^{k_j}) = \bigoplus_{j=1}^{n} W_{p}^{k_j}[0,1] \), that is, \( E \in \text{Cyc } A_k \). To complete the proof it suffices to apply Corollary 4.24. \( \square \)

Remark 4.26. For \( \alpha = 1 \) and \( k_1 = \cdots = k_n = k \geq 1 \), that is, for the operator \( A_k = \bigoplus_{j=1}^{n} \lambda_j J_{k_j} \), Theorem 4.25 has been established in [12] by another method.

We emphasize that the description of the set \( \text{Cyc } A_{k,0} \) essentially differs from that of \( \text{Cyc } A_k \). Namely, in contrast to the operator \( A_{k,0} \), the description of the set \( \text{Cyc } A_k \) does not depend on the choice of \( \lambda_j \).

Summing up, we obtain a description of the cyclic subspaces for the operator \( A = \bigoplus_{j=1}^{m} \lambda_j J_{k_j} \oplus \bigoplus_{j=m+1}^{n} \lambda_j J_{k_j,0}^\alpha \) acting on the mixed space \( \bigoplus_{j=1}^{m} W_{p}^{k_j}[0,1] \oplus \bigoplus_{j=m+1}^{n} W_{p,0}^{k_j}[0,1] \).

Theorem 4.27. Suppose that the operators

\[ A_k(1) := \bigoplus_{j=1}^{m} \lambda_j J_{k_j}, \quad A_{k,0}(1) := \bigoplus_{j=1}^{m} \lambda_j J_{k_j,0}^\alpha \]

and

\[ A_{k,0}(2) := \bigoplus_{j=m+1}^{n} \lambda_j J_{k_j,0}^\alpha, \quad \text{and} \quad A := A_k(1) \oplus A_{k,0}(2) \]

are defined on

\[ X(1) := \bigoplus_{j=1}^{m} W_{p}^{k_j}[0,1], \quad X_{0}(1) := \bigoplus_{j=1}^{m} W_{p,0}^{k_j}[0,1] \]

and

\[ X_{0}(2) := \bigoplus_{j=m+1}^{n} W_{p,0}^{k_j}[0,1], \quad \text{and} \quad X := X(1) \oplus X_{0}(2) \]

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respectively. Furthermore, let $P(1)$ be the canonical projection from $X = X(1) \oplus X_0(2)$ onto $X(1)$. Then

(1) $\mu_A = \max\{\mu_{A_k(1)}, \mu_{A_{k,0}(1) \oplus A_{k,0}(2)}\}$;

(2) $E \in \text{Cyc} \, A$ if and only if

(a) $P(1)E \in \text{Cyc} \, A_k(1)$,
(b) $A^M E \in \text{Cyc} (A_{k,0}(1) \oplus A_{k,0}(2))$, where $M := \max_{1 \leq j \leq m} k_j$. Furthermore, the set $\text{Cyc} (A_{k,0}(1) \oplus A_{k,0}(2))$ is described in Theorems 2.24 and 2.16 and the set $\text{Cyc} A_k(1)$ is described in Theorem 4.25.

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