Accelerated Learning in the Presence of Time Varying Features with Applications to Machine Learning and Adaptive Control

Joseph E. Gaudio\textsuperscript{1}, Travis E. Gibson\textsuperscript{2}, Anuradha M. Annaswamy\textsuperscript{1}, and Michael A. Bolender\textsuperscript{3}

\textsuperscript{1}Massachusetts Institute of Technology
\textsuperscript{2}Brigham and Women’s Hospital and Harvard Medical School
\textsuperscript{3}Air Force Research Laboratory

March 11, 2019

Abstract
Features in machine learning problems are often time varying and may be related to outputs in an algebraic or dynamical manner. The dynamic nature of these machine learning problems renders current accelerated gradient descent methods unstable or weakens their convergence guarantees. This paper proposes algorithms for the case when time varying features are present, and demonstrates provable performance guarantees. We develop a variational perspective within a continuous time algorithm. This variational perspective includes, among other things, higher-order learning concepts and normalization, both of which stem from adaptive control, and allows stability to be established for dynamical machine learning problems. These higher-order algorithms are also examined for achieving accelerated learning in adaptive control. Simulations are provided to verify the theoretical results.

1 Introduction
As a field, machine learning has focused on both the processes by which computer systems automatically improve through experience, and on the underlying principles that govern learning systems \cite{14, 6, 31, 17}. A particularly useful approach for accomplishing this process of automatic improvement is to embody learning in the form of approximating a desired function and to employ optimization theory to reduce an approximation error at optimal rates as more data is observed. The field of adaptive control, on the other hand, has focused on the process of controlling engineering systems in order to accomplish regulation and tracking of critical variables of interest (e.g. speed in automotive systems, position and force in robotics, Mach number and altitude in aerospace systems, frequency and voltage in power systems) in the presence of uncertainties in the underlying system models, changes in the environment, and unforeseen variations in the overall infrastructure \cite{18, 1, 28, 41}. The approach used for accomplishing such regulation and tracking is to learn the underlying parameters through an online estimation algorithm. Stability theory is employed for enabling guarantees for the safe evolution of the critical variables, and convergence of the regulation and tracking errors to zero. In both machine learning and adaptive control the core algorithm is often inspired by gradient descent or gradient flow \cite{41}. As the scope of problems in both fields increases, the associated complexity and challenges increase as well, necessitating a better understanding of how the underlying algorithms can be designed to enhance learning and stability.
Modifications to standard gradient descent have been actively researched within the optimization community since computing began. The seminal accelerated gradient method proposed by [42] has not only received significant attention in the optimization community [43, 2, 8], but also in the neural network learning community [55, 52]. Nesterov’s original method, or a variant [13, 34, 58] are the standard methods for training deep neural networks. To gain insight into Nesterov’s method, which is a difference equation, [51] identified the second order ordinary differential equation (ODE) at the limit of zero step size. Still pushing further in the continuous time analysis of accelerated methods, several recent results have leveraged a variational approach showing that, at least in continuous time, there exists a broad class of accelerated methods where one can obtain an arbitrarily fast convergence rate [55, 57]. Converting back to discrete time to obtain an implementable algorithm with rates matching that of the differential equation is also an active area of research [5, 56]. Note in all the aforementioned work, while the parameter update is time varying, the features and output of the cost function are static.

The adaptive control community has also analyzed several modifications to gradient descent over the past 40 years. These modifications have been introduced to ensure provably safe and smooth learning in the presence of both structured parametric uncertainty and unstructured uncertainty due to: unmodeled dynamics, magnitude saturation, delays, and disturbances [27, 33, 3, 16]. A majority of these modifications are for first order gradient-like updates. One notable exception is the “high-order tuner” proposed by [39] which has been useful in providing stable algorithms for time-delay systems [15].

In this paper, we consider a general class of learning problems where features (regressors) are time varying. We also consider the case where those time varying features are inputs or states of an unknown dynamical system. Most accelerated methods in the machine learning literature are analyzed for the case where features are assumed to be constant [55]. Our approach is thus more comparable to the gradient descent methods analyzed in the field of online optimization [23, 21] and as previously alluded to, the field of adaptive control [46, 41]. Our framework, when applied to control generalizes the notion of a “higher order tuner” [39], illustrating their applicability beyond its first intended use of learning in systems with relative degree greater than one.\footnote{The relative degree of an output in a dynamical system is the number of times the output needs to be differentiated before an input appears.} Utilizing a common variational perspective that is inspired by the one adopted by [55], this paper will aim to realize two objectives. The first objective is the derivation of an accelerated learning algorithm with time varying features in machine learning problems. The second objective is to achieve accelerated learning in adaptive control problems.

We begin with a review of time varying features to demonstrate how they may be related to outputs in an algebraic manner as well as through the outputs of a dynamical system. We then propose a new class of online accelerated algorithms that are inspired by the “high order tuners” used in adaptive control [15] that take into account the time variation of the features and provide guarantees of stability and convergence. We propose the same “high order tuners” for adaptive control with accelerated convergence of model tracking errors. The derivation of these algorithms comes from a variational approach which relates the potential, kinetic, and damping characteristics of the algorithm. The proof of stability does not require an “ideal scaling condition” assumption as in [55], and allows for continuous time variation of the features. This paper is concluded with numerical experiments demonstrating the acceleration of the derived algorithm for time varying regression as well as adaptive control of dynamical systems.
2 Warmup: Time Varying Features and Model Reference Adaptive Control

This section provides a brief introduction into regression with time varying features as well as adaptive control.

2.1 Time Varying Regression

A time varying regression system may be expressed as:

\[ y(t) = \theta^* T \phi(t) \]

where \( \theta^*, \phi \in \mathbb{R}^N \) represent the unknown constant parameter and the known time varying feature respectively. The variable \( y \in \mathbb{R} \) represents the known time varying output. Given that \( \theta^* \) is unknown, we formulate an estimator:

\[ \hat{y}(t) = \theta^T(t) \phi(t) \]

where \( \hat{y} \in \mathbb{R} \) is the predicted output synthesized with an estimated parameter \( \theta \in \mathbb{R}^N \). Define the error between the actual output and the estimated output as:

\[ e_y(t) = y(t) - \hat{y}(t) = \tilde{\theta}(t) \phi(t) \] (1)

where \( \tilde{\theta} = \theta - \theta^* \) is the parameter estimation error. An overview of the time varying regression error model may be seen in Figure 1. The differential equation for the output error is then:

\[ \dot{e}_y(t) = \dot{\theta}^T(t) \phi(t) + \tilde{\theta}^T(t) \dot{\phi}(t) \] (2)

The goal is to design a rule to adjust \( \theta \) in a continuous manner using knowledge of \( \phi \) and \( e_y \) such that \( e_y \) tends towards zero. A continuous, gradient descent-like update is desired as the output of the regression system \( y \) may be corrupted by noise and feature dimensions may be large. To do so, consider the squared loss cost function: \( L = \frac{1}{2} e_y^2(t) \). The gradient of this function with respect to the parameters can be expressed as: \( \nabla_{\theta} L = \phi(t) e_y(t) \). The standard gradient flow update law (the continuous time limit of gradient descent) may be expressed as follows with user-designed gain parameter \( \gamma > 0 \) [41]:

\[ \dot{\theta}(t) = -\gamma \nabla_{\theta} L = -\gamma \phi(t) e_y(t). \] (3)

The parameter error model may then be stated as:

\[ \dot{\tilde{\theta}}(t) = -\gamma \phi(t) \phi^T(t) \tilde{\theta}(t) \] (4)

Stability analysis of the update law in (3) for the error model in (1) is provided in Appendix B.2.

2.2 Model Reference Adaptive Control and Identification

In the previous subsection, the output was a linear combination of the elements of the feature. In a class of problems (including identification of dynamic systems and adaptive control) the features may be related to the errors of a dynamical system. To demonstrate this, features \( \phi \in \mathbb{R}^N \) may be related to a measurable state \( x \in \mathbb{R}^n \) through a dynamical system with unknown constant parameter \( \theta^* \in \mathbb{R}^N \) as:

\[ \dot{x}(t) = Ax(t) + b(u(t) + \theta^T \phi(t)) \]
where \( u \in \mathbb{R} \) is an input to the system. A single input system is considered here for notational simplicity, where \( A \in \mathbb{R}^{n \times n} \), and \( b \in \mathbb{R}^{n \times 1} \) are known dynamics and input matrices respectively. It can be noted that the results of this paper extend naturally to multiple input systems. Additionally, it should be noted that it is common in adaptive control for the feature to be a function of the state, i.e., \( \phi = \phi(x) \). This dynamical system is akin to a linearized recurrent neural network and is similar to the dynamical systems considered in [24, 22, 47, 11, 10, 12]. Similar to the linear regression case where an output predictor was created with the same form as the time varying system, but with an estimate of the unknown parameter, a state predictor with state \( x_m \in \mathbb{R}^n \) may be designed for this system as follows:

\[
\dot{x}_m(t) = Ax_m(t) + b(u(t) + \theta^T(t)\phi(t))
\]

Define the error between the considered dynamical system and predictor dynamical system as \( e = x_m - x \). The error model for identification and control schemes may then be stated as:

\[
\dot{e}(t) = Ae(t) + b\hat{\theta}^T(t)\phi(t)
\]

where the relation of the feature to the error can be seen to be through a differential equation, which is fundamentally different from (1). An overview of the dynamical error model may be seen in Figure 1 with transfer function \( W(s) := (sI - A)^{-1}b \). The dynamical nature of this error equation prohibits the use of an update law based solely on gradient descent of a loss function. Rather, the standard adaptive update may be chosen as follows, with a gain \( \gamma > 0 \) selected to adjust the learning rate [11]:

\[
\dot{\theta}(t) = -\gamma \phi(t)e^T(t)Pb
\]

where \( P = P^T \in \mathbb{R}^{n \times n} \) is a positive definite matrix computed for a stable matrix \( A \) as: \( A^TP + PA = -Q \), where \( Q = Q^T \in \mathbb{R}^{n \times n} \) is a user selected positive definite matrix (see Appendix B.1). Comparing (6) to (3), it can be noticed that the structure is similar with the multiplication of the feature by the error. The difference between them is through the inclusion of elements from the differential equation relating the parameter error to the model tracking error (5). Stability analysis of the update in (6) for the error model in (5) is provided in Appendix B.3.

### 3 Accelerated Algorithm Derivation

This section derives new accelerated parameter update algorithms for both the time varying regression, as well as the model reference adaptive control and identification problems. For the remainder of the paper, the notation of time dependence of variables will be omitted when it is clear from the context.

We begin with a common variational perspective in order to derive our accelerated parameter update algorithms. In particular the Bregman Lagrangian (see [55], Equation 1) is restated below as:

\[
\mathcal{L}(\theta, \dot{\theta}, t) = e^{\alpha t + \gamma t} \left( D_h(\theta + e^{-\alpha t} \dot{\theta}, \theta) - e^{\gamma t} L(\theta) \right)
\]

\(^2\)In adaptive control the input is usually designed as \( u(t) = -\theta^T(t)\phi(x) \).
where \( D_h \) is the Bregman divergence defined with a distance-generating function \( h \) as:

\[
D_h(y, x) = h(y) - h(x) - \langle \nabla h(x), y - x \rangle.
\]

This Lagrangian can be seen to weight the loss \( L(\theta) \) versus kinetic energy \( D_h(\theta + e^{-\alpha_t} \dot{\theta}, \theta) \) of an algorithm. The user defined time varying parameters \((\alpha_t, \beta_t, \gamma_t)\) will be defined in the following section.

### 3.1 Time Varying Regression

In order to make connections with adaptive control, we will use the squared Euclidean norm

\[
h(x) = \frac{1}{2} \| x \|^2
\]

in the Bregman divergence along with the squared loss \( L = \frac{1}{2} e_y^2 \) as was used in Section 2.1. The following are our choice of the time varying scaling parameters:

\[
\alpha_t = \ln (\beta N_t), \quad \beta_t = \ln \left( \frac{\gamma}{\beta N_t} \right), \quad \gamma_t = \int_{t_0}^t \beta N_t \, dx
\]

where \( \gamma, \beta > 0 \) are scalar design parameters and

\[
N_t \triangleq (1 + \mu \phi^T \phi)
\]

with scalar \( \mu > 0 \) is a function of the time varying feature, and is referred to as a normalizing signal.

It can be noticed that the second “ideal scaling condition” (Equation 2b, \( \dot{\gamma}_t = e^{\alpha_t} \)) of [55] holds but the first “ideal scaling condition” (Equation 2a, \( \dot{\beta}_t \leq e^{\alpha_t} \)) does not need to hold in general. In this sense, the results of this paper are applicable to a larger class of algorithms. With this choice of parameters, distance-generating function and loss function, the following non-autonomous Lagrangian results:

\[
\mathcal{L}(\theta, \dot{\theta}, t) = e^{\int_{t_0}^t \beta N_t \, dx} \frac{1}{\beta N_t} \left( \frac{1}{2} \dot{\theta}^T \dot{\theta} - \gamma \beta N_t \frac{1}{2} e_y^2 \right)
\]

This Lagrangian is a function of not only the parameter and its time derivative, but is also a function of time directly through normalizing signal \( N_t \). Using this Lagrangian, a functional may be defined as: \( J(\theta) = \int_T \mathcal{L}(\theta, \dot{\theta}, t) \, dt \), where \( T \) is a time interval. To minimize this functional, a necessary condition from the calculus of variations is that the Lagrangian solves the Euler-Lagrange equation [38]:

\[
d \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}}(\theta, \dot{\theta}, t) \right) = \frac{\partial \mathcal{L}}{\partial \theta}(\theta, \dot{\theta}, t) \tag{9}
\]

The second order differential equation resulting from the application of equation [9] is:

\[
\ddot{\theta} + \left( \beta N_t - \frac{\dot{N}_t}{N_t} \right) \dot{\theta} = -\gamma \beta N_t \phi e_y \tag{10}
\]

Here \( \beta \) can be seen to adjust “friction”. Taking \( \beta \to \infty \) (strong friction limit) results in the standard first order update law [3][3]. The second order differential equation in [10] may be implemented as the following two equations, similar to [15]:

\[
\dot{\theta} = -\gamma \nabla_\theta L = -\gamma \phi e_y
\]

\[
\dot{\theta} = -\beta (\theta - \dot{\theta}) N_t
\]

where it can be seen that the algorithm decomposes to the output of the first order update law [3] (a “gradient step”), passed through a filter normalized by the feature (a “mixing step”). The\footnote{This notion will be more rigorously shown in Section 4.}
normalization in the second equation is in fact required to establish stability. Similar in form to batch normalization [29] and the update law in ADAM [33], the normalization present in this update law is different in that it normalizes by the feature itself as opposed to estimated moments. A block diagram of this system of equations in (11) can be seen in Figure 2, where \( s = \frac{d}{dt} \) represents the differential operator. In this block diagram it can be seen that \( \beta \rightarrow \infty \) decreases the nominal time constant (for a given \( \phi \)) of the time varying filter. Thus the filter effect then disappears, and the first order update law (3) is recovered.

3.2 Model Reference Adaptive Control and Identification

In a similar manner to (8) the following non-autonomous Lagrangian is defined:

\[
\mathcal{L}(\theta, \dot{\theta}, t) = e^{\int_{\alpha}^t \beta N_t dx} \frac{1}{\beta N_t} \left( \frac{1}{2} \dot{\theta}^T \dot{\theta} - \gamma \beta N_t \left[ \frac{d}{dt} \left( \frac{e^T P e}{2} \right) + \frac{e^T Q e}{2} \right] \right)
\]

Comparing the Lagrangian in (12) to that in (8), it can be seen that they only differ by the term in the square brackets, representing the loss function considered. The extra terms in the square brackets account for energy storage in the error model dynamics in equation (5). This may be seen as:

\[
\frac{d}{dt} \left( \frac{e^T P e}{2} \right) + \frac{e^T Q e}{2} = e^T P (\dot{e} - Ae) = e^T P \dot{\theta}^T \phi,
\]

where the loss is only zero for this dynamical error model when both \( e \) and \( \dot{e} \) are zero. Using this Lagrangian, a functional may be defined as

\[
J(\theta) = \int \mathcal{L}(\theta, \dot{\theta}, t) dt.
\]

The minimization of this functional with the Euler-Lagrange equation (9) and error dynamics (5) results in the following second order differential equation:

\[
\ddot{\theta} + \left( \beta N_t - \frac{\dot{N}_t}{N_t} \right) \dot{\theta} = -\gamma \beta N_t \phi e^T P \dot{\theta}
\]

Again, \( \beta \) can be seen to represent “friction”, with \( \beta \rightarrow \infty \) resulting in the standard first order update law (6). The second order differential equation in (13) may be expressed as the following two differential equations, similar to (15):

\[
\begin{align*}
\dot{\theta} &= -\gamma \phi e^T P b \\
\dot{\theta} &= -\beta (\theta - \dot{\theta}) N_t
\end{align*}
\]

where once more, it can be seen that the algorithm decomposes to the output of the first order update law (6) being passed through a filter normalized by the feature.

4 Stability Analysis

This section proves the stability of the accelerated update algorithms derived in this paper. The class \( \mathcal{L}_p \) is described in Definition 1 of Appendix A. Unless otherwise specified, \( \| \cdot \| \) represents the 2-norm. Stability analysis using Lyapunov functions have been of increased use in recent years in state of the art machine learning approaches [57, 56]. A brief overview is given in Appendix [B.1].
4.1 Time Varying Regression

It can be noted that the Lyapunov function proposed in [55] cannot be used to demonstrate stability of the accelerated algorithm in (11). This can be seen in Appendix B.4. To show stability for the accelerated update law (11) for time varying regression (1), consider the following candidate Lyapunov function inspired by the higher order tuner approach in [15]:

\[ V = \frac{1}{\gamma} \| \vartheta - \vartheta^* \|^2 + \frac{1}{\gamma} \| \vartheta - \vartheta \|^2 \]  

(15)

which is a non-negative scalar quantity which represents squared error present in the algorithm. Choosing the normalization parameter \( \gamma \) in (7) as \( \mu = 2\gamma/\beta \) and using equations (1) and (11), the time derivative of the candidate Lyapunov function in (15) may be bounded as:

\[ \dot{V} \leq -\frac{2\beta}{\gamma} \| \vartheta - \vartheta \|^2 - \| e_y \|^2 - \| \vartheta - \vartheta \| \| \phi \|^2 \]  

Thus it can be concluded that \( V \) is a Lyapunov function with \( (\vartheta - \vartheta^*) \in L_\infty \) and \( (\vartheta - \vartheta) \in L_\infty \). By integrating \( \dot{V} \) from \( t_0 \) to \( t \infty \):

\[ \int_{t_0}^{t_\infty} \| e_y \|^2 dt \leq -\int_{t_0}^{t_\infty} \dot{V} dt = V(t_0) - V(\infty) < \infty, \]  

thus \( e_y \in L_2 \). Likewise, \( \int_{t_0}^{t_\infty} \frac{2\beta}{\gamma} \| \vartheta - \vartheta \|^2 dt \leq -\int_{t_0}^{t_\infty} \dot{V} dt = V(t_0) - V(\infty) < \infty \), thus \( \dot{\vartheta} - \vartheta \in L_2 \cap L_\infty \). Furthermore:

\[ \frac{\| \vartheta - \vartheta \|^2}{2\beta} \leq \frac{\gamma V(t_0)}{2\beta} \]

Here the effect of the parameter \( \beta \) is very apparent once again. As \( \beta \to \infty \), \( \| \vartheta - \vartheta \|^2 \to 0 \). If in addition \( \phi \in L_\infty \) (the magnitude of the features are bounded), then from equation (11) \( e_y \in L_2 \cap L_\infty \), and from equation (11) \( \dot{\vartheta}, \ddot{\vartheta} \in L_2 \cap L_\infty \). If the additional assumption is made that \( \dot{\phi} \in L_\infty \) (the time derivative of the features are bounded), then from equation (2), it can be seen that \( \dot{e}_y \in L_\infty \) and from equation (11) \( \dot{\vartheta}, \ddot{\vartheta} \in L_\infty \) and thus Corollary 1 in Appendix A:

\[ \lim_{t \to \infty} e_y = 0, \quad \lim_{t \to \infty} (\vartheta - \vartheta) = 0, \quad \lim_{t \to \infty} \dot{\vartheta} = 0, \quad \lim_{t \to \infty} \ddot{\vartheta} = 0 \]

which is to say that the prediction error goes to zero as time goes to infinity, and the parameter estimate and algorithm reach a steady state value. For the parameter estimation error to converge to zero \( (\dot{\vartheta} \to 0) \), persistence of excitation of the system regressor is needed (see Appendix A Definition 2).

4.2 Accelerated Model Reference Adaptive Control and Identification

To show stability for the accelerated update law (14) for the dynamical error model in (5), consider the following candidate Lyapunov function inspired by the higher order tuner approach for adaptive control in [15]:

\[ V = \frac{1}{\gamma} \| \vartheta - \vartheta^* \|^2 + \frac{1}{\gamma} \| \vartheta - \vartheta \|^2 + e^T P e \]  

(16)

which can be seen to be (15) with an additional term corresponding to the model tracking error. Choosing the normalization parameter \( \gamma \) in (7) as \( \mu = 2\gamma/\beta \) and the symmetric positive definite matrix \( P \) in the Lyapunov equation \( A^T P + PA = -Q \) from before as \( Q = 2I \) and using equations (5) and (14), the time derivative of the candidate Lyapunov function in (16) may be bounded as:

\[ \dot{V} \leq -\frac{2\beta}{\gamma} \| \vartheta - \vartheta \|^2 - \| e \|^2 - \| e \| - 2\| Pb \| \| \vartheta - \vartheta \| \| \phi \|^2 \]

\[ \text{Can be chosen without loss of generality.} \]
Thus it can be concluded that $V$ is a Lyapunov function with $e \in \mathcal{L}_\infty$, $(\dot{\vartheta} - \theta^*) \in \mathcal{L}_\infty$, and $(\theta - \vartheta) \in \mathcal{L}_\infty$. By integrating $\dot{V}$ from $t_0$ to $\infty$: $\int_{t_0}^{\infty} ||\dot{e}||^2 dt \leq -\int_{t_0}^{\infty} \dot{V} dt = V(t_0) - V(\infty) < \infty$, thus $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Likewise, $\int_{t_0}^{\infty} \frac{2\beta}{2\beta} ||\theta - \vartheta||^2 dt \leq -\int_{t_0}^{\infty} \dot{V} dt = V(t_0) - V(\infty) < \infty$, thus $(\theta - \vartheta) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

Furthermore, it can be concluded that again: $||\theta - \vartheta||_{\mathcal{L}_2} \leq \sqrt{\gamma V(t_0)}$, where as $\beta \to \infty$: $||\theta - \vartheta||_{\mathcal{L}_2} \to 0$.

If in addition $\phi \in \mathcal{L}_\infty$ then from equation (5) $\dot{e} \in \mathcal{L}_\infty$, and thus from Corollary 1 in Appendix A

$$\lim_{t \to \infty} e = 0$$

which is to say that the model tracking error goes to zero as time goes to infinity. It can be noted that compared to the stability analysis in Section 4.1 $\lim_{t \to \infty} \dot{e} = 0$ when $\phi \in \mathcal{L}_\infty$ without the additional requirement that $\dot{\varphi} \in \mathcal{L}_\infty$. Also, from equation (14) $\dot{\vartheta}, \dot{\tilde{\vartheta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. If the additional assumption is made that $\dot{\varphi} \in \mathcal{L}_\infty$ then from equation (14) $\ddot{\vartheta}, \ddot{\tilde{\vartheta}} \in \mathcal{L}_\infty$, and thus from Corollary 1 in Appendix A

$$\lim_{t \to \infty} (\theta - \vartheta) = 0, \quad \lim_{t \to \infty} \dot{\vartheta} = 0, \quad \lim_{t \to \infty} \dot{\tilde{\vartheta}} = 0$$

which states that the parameter estimate and algorithm reach a steady state value. For the parameter estimation error $\dot{\vartheta} \to 0$, persistence of excitation of the regressor of the system is needed (see Appendix A Definition 2).

5 Comparison of Approaches

Table 1 shows a comparison of the Lagrangian functional and the resulting second order ODE from a given parameterization of the algorithm proposed by [55] to the results provided in this paper for regression. This parameterization was chosen to coincide with the notions used in this paper (squared loss and the squared Euclidean norm for the error in (1)) along with parameters chosen as in Equation 12 of [55]. It can be seen that both Lagrangians have an increasing function multiplying the kinetic and potential energies with an additional time varying term weighting the potential energy. Our approach however is a function of the feature as compared to an explicit function of time. This is a more natural parameterization because the resulting ODE shown for comparison purposes in Table 1 does not have a damping term that decays to zero with time. Therefore our algorithm does not change from an overdamped to underdamped system as time progresses as is commonly seen in accelerated methods [51]. Thus our approach provides for an algorithm capable of running continuously as features are processed. No restart is required as is often used in accelerated algorithms in the machine learning literature [44]. The more natural damping term shown in our second order ODE is an explicit function of both the feature and time derivative of the feature vector. It can be noted once more that the time derivative of the feature does not need to be known as this ODE may be implemented using (11), which allows for online processing of the features, without a priori knowledge of its future variation.

Normalization by the magnitude of the time varying feature (7) can be seen to be explicitly included in our algorithm. This normalization is in fact necessary in order to provide a proof of stability as was found by [15], due to the required feature dependent scaling. Table 2 shows the candidate Lyapunov function proposed by [55] applied to our algorithm, and the Lyapunov function considered in this paper. It can be seen that the candidate Lyapunov proposed by [55] represents a scaled kinetic plus potential energy, and results in a time derivative that cannot be guaranteed to

---

5 As is common in adaptive control, $\phi = x$. It was proved that $e \in \mathcal{L}_\infty$, with $x_m \in \mathcal{L}_\infty$ by design of a suitable input $u$. Thus with $x = x_m - e$, $\phi = x$ is bounded by construction and thus this is not a restrictive assumption.

6 This is again not a restrictive assumption in adaptive control with $\phi = x$, as $\dot{x} = \dot{x}_m - \dot{e}$ is bounded by construction.
be non-increasing for arbitrary initial conditions and time variations of the feature. Our Lyapunov function is fundamentally different in its construction and is indeed able to verify stability. It should be noted that the class of algorithms in [55] was not designed for time varying features and that the comparisons are due to its general form in continuous time, representing a large class of algorithms commonly used in machine learning. It can also be noted that the accelerated algorithms proposed in this paper are proven stable regardless of the initial condition of the algorithm (see Section 4). That is to say that an optimization problem-specific schedule on the parameters of the problem is not required to set in order to cope with the initial conditions of the algorithm, as is usually required for momentum methods commonly used in machine learning [52]. The accelerated algorithms proposed in this paper are proven to be stable and provide for a unified framework for convergence in output (11) (respectively model tracking (14)) error for time varying features with arbitrary initial conditions where the relation between feature and error may be algebraic (1) or dynamical (5).

6 Numerical Experiments

We conducted numerical experiments for the time varying regression and state feedback adaptive control problem. The implementation was carried out in Matlab and Simulink, in order to efficiently simulate continuous dynamical systems.

6.1 Time Varying Regression

A time varying regression system was simulated with the error model as in (1). The standard gradient flow update law (3) and accelerated update law (11) are compared in each simulation alongside a continuous parameterization of Nesterov’s accelerated method as shown in Table 1. A three dimensional problem was considered for the sake of clarity of presentation. For each simulation, the unknown parameter was set as $\theta^* = [1, -2, 5]^T$, with learning rate $\gamma = 0.1$. The accelerated time varying regression algorithm uses $\beta = 1$ and $\mu = 2\gamma/\beta$. For the accelerated method in [55], the following parameters were set to correspond to Nesterov acceleration and to have the same constant multiplying the gradient term, $\phi e_y$, in order to have a more direct comparison: $p = 2$, $C = \gamma\beta/p^2$.  

| Parameterization from [55] | Our Approach |
|----------------------------|--------------|
| $L(\theta, \dot{\theta}, t) = \frac{e^{t/2}}{p} \left( \frac{1}{2} \dot{\theta}^T \dot{\theta} - Cp^2 p^{-2} \frac{1}{2} \dot{e}_y^2 \right)$ | $L(\theta, \dot{\theta}, t) = e^{t/2} \frac{\beta N_t}{N_t} \left( \frac{1}{2} \dot{\theta}^T \dot{\theta} - \gamma \beta N_t \frac{1}{2} e_y^2 \right)$ |
| $\ddot{\theta} + \frac{p+1}{t} \dot{\theta} = -Cp^2 p^{-2} \phi e_y$ | $\ddot{\theta} + \left[ \beta N_t - \frac{\dot{\theta}}{N_t} \right] \dot{\theta} = -\gamma \beta N_t \phi e_y$ |

Table 2: Comparison of candidate Lyapunov functions for the accelerated algorithm in (10).

| Lyapunov Function in [55] Applied to [10] | Our Approach |
|------------------------------------------|--------------|
| $V = \frac{1}{2} \| \dot{\theta} + \frac{1}{\beta(1+\mu \phi^T \phi)} \dot{\theta} \|^2 + \frac{\gamma}{\beta(1+\mu \phi^T \phi)} \frac{1}{2} \dot{e}_y^2$ | $V = \frac{1}{2} \| \dot{\theta} - \theta^* \|^2 + \frac{1}{\gamma} \| \dot{\theta} - \theta \|^2$ |
| $\dot{V} = -\gamma \dot{e}_y^2 \left( 1 + \frac{1}{\beta(1+\mu \phi^T \phi)} \right) + \frac{\gamma}{\beta(1+\mu \phi^T \phi)} \dot{e}_y \dot{\theta}^T \phi$ | $\dot{V} \leq -\frac{2\beta}{\gamma} \| \dot{\theta} - \dot{\theta} \|^2 - \frac{1}{\gamma} \| e_y \|^2 - \left[ \| e_y \| - 2 \| \dot{\theta} - \theta \| \phi \|^2 \right] \leq 0$ |

be non-increasing for arbitrary initial conditions and time variations of the feature. Our Lyapunov function is fundamentally different in its construction and is indeed able to verify stability. It should be noted that the class of algorithms in [55] was not designed for time varying features and that the comparisons are due to its general form in continuous time, representing a large class of algorithms commonly used in machine learning. It can also be noted that the accelerated algorithms proposed in this paper are proven stable regardless of the initial condition of the algorithm (see Section 4). That is to say that an optimization problem-specific schedule on the parameters of the problem is not required to set in order to cope with the initial conditions of the algorithm, as is usually required for momentum methods commonly used in machine learning [52]. The accelerated algorithms proposed in this paper are proven to be stable and provide for a unified framework for convergence in output (11) (respectively model tracking (14)) error for time varying features with arbitrary initial conditions where the relation between feature and error may be algebraic (1) or dynamical (5).
For the first simulation shown in Figure 3, the feature vector was initially set equal to the zero with initial conditions of all algorithms initialized at zero (consistent with not knowing the feature variation and unknown parameter ahead of time). At time $t = 0.1$, the feature vector steps to a constant value of $\phi = [1, 1, 1]^T$. Consequently, the output $y$ steps to a value of 4. The gradient flow algorithm for regression (3) (denoted “reg”) as well as accelerated algorithms (denoted “A-reg” for the algorithm presented in this paper (11) and “W-reg” for the algorithm by [55], parameterized in Table 1) are seen to converge in output. The accelerated algorithm presented in this paper (11) can be seen to converge at a faster rate, and without significant oscillations present (as the algorithm does not become underdamped). Given that the feature steps to a constant value, and is held constant in this experiment, the presence of oscillations in the response modeling Nesterov acceleration seen here and explained in further detail by [51] is not desired. As a separate note,
given that the system does not have a persistently exciting regressor (Appendix A, Definition 2); the parameter $\theta$ does not converge to the true value. This demonstrates the non-convex formulation in the parameter space.

Figure 4 shows the response for features parameterized as: $\phi = [1, 1 + 3\sin(\omega t), 1 + 3\cos(\omega t)]^T$ (here with $\omega = 1$ resulting in $\dot{\phi}_{\text{max}} = 3$), which can be seen to consist of time varying functions added to components of the constant feature of Figure 3. Additional plots in Appendix C show a progression in the increase of the variable $\omega$ and thus the time variation of the feature vector (increase in $\dot{\phi}_{\text{max}}$). It can be seen that as the time variation of the feature increases, our algorithm remains stable, whereas the “W-reg” algorithm becomes unstable. A similar destabilizing effect can occur for many accelerated algorithms commonly employed in the machine learning community when features are time varying. It should also be noted that the feature profile variation considered here is persistently exciting. Thus in addition to output error tending towards zero, as proved in Section 4.1 the error in the parameter space can additionally be seen to tend towards zero (i.e. $\theta \rightarrow \theta^*$).

6.2 State Feedback Adaptive Control

A state feedback model reference adaptive control (MRAC) problem was simulated with the error model in (5). The standard MRAC update law (6) and accelerated update law (14) were compared in a simulation of linearized longitudinal dynamics of an F-16 aircraft with integral command tracking. For this simulation, the unknown parameter was set as $\theta^* = [0.1965, -0.3835, -1]^T$, with learning rate $\gamma = 0.1$. The accelerated MRAC algorithm uses $\beta = 1$ and $\mu = \gamma \|Pb\|^2 / \beta$. More details regarding the simulation implementation can be found in Appendix D including the explanation for this choice unknown parameter and definitions of relevant variables of this physically motivated example in adaptive flight control.

The simulation results are shown in Figure 5. The initial conditions of the states of the dynamical systems and algorithm were set equal to zero along with the command into the system. At time $t = 5$, the command for the state $x_2$ to track changes to a value of 1. Consequently, the states of the system change in order to track the command. Both the standard MRAC (6) and accelerated MRAC (14) (denoted “A-MRAC”) algorithms are seen to converge in both command tracking and model tracking error to zero. The accelerated algorithm however, can be seen to converge at a faster rate. Additionally, the accelerated algorithm can be seen to result in fewer (albeit initially larger) oscillations which may be due to the presence of damping in the algorithm and the filtering effect as in (14). The rapid reduction in oscillations is desirable, particularly given that the system was provided a constant command to track.

7 Conclusions and Related Work

In this work we derived an accelerated gradient algorithm for optimization in time varying and dynamical machine learning problems. The variational approach taken connects the acceleration phenomena to energy concepts and provides a common method for analyzing both algebraic and dynamical error models. Our streaming accelerated algorithms were ultimately proven to be stable, with error for the algebraic and dynamical error models, $e_y$ and $e$ respectively, converging to zero asymptotically.

Learning for dynamical systems has been an active area of research within the machine learning community, especially within the area of reinforcement learning [4, 53, 54, 47]. There has also been a large increase in recent work studying learning and control for unknown linear dynamical systems: least squares [49], linear quadratic regulator, robust control [11, 10, 12], and spectral
filtering \cite{24, 22}. One major difference between these works and the one presented here is that our algorithm is streaming, with the exception of \cite{24, 22}. Control techniques have also been leveraged in the opposite direction, treating gradient descent explicitly as a dynamical system, and leveraging tools from robust control theory \cite{37, 7, 9, 25, 26}. It is an exciting time to be studying problems at the intersection of machine learning and control.

This work continues in the tradition of \cite{51} and \cite{55} whereby insight is gained into accelerated gradient descent methods through a continuous lens. Future work will be to obtain discrete time implementations of our algorithms with matching rates \cite{56, 5}, and to connect those back to classic discrete time adaptive control algorithms \cite{18, 19, 20}. Other fruitful directions forward would be to study these accelerated algorithms within the context of output feedback control, and to rigorously prove convergence rates when the regressor vectors are persistently exciting \cite{30, 40}.

References

[1] Karl J. Áström and Björn Wittenmark. *Adaptive Control: Second Edition*. Addison-Wesley Publishing Company, 1995.

[2] Amir Beck and Marc Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2(1):183–202, Jan 2009.

[3] Nikolaos Bekiaris-Liberis and Miroslav Krstic. Delay-adaptive feedback for linear feedforward systems. *Systems & Control Letters*, 59(5):277–283, May 2010.

[4] Dimitri P. Bertsekas. *Dynamic Programming and Optimal Control*, volume 1. Athena Scientific, 2017.

[5] Michael Betancourt, Michael I. Jordan, and Ashia C. Wilson. On symplectic optimization. *arXiv preprint arXiv:1802.03653*, 2018.

[6] Christopher M. Bishop. *Pattern Recognition and Machine Learning*. Springer, 2006.

[7] Ross Boczar, Laurent Lessard, Andrew Packard, and Benjamin Recht. Exponential stability analysis via integral quadratic constraints. *arXiv preprint arXiv:1706.01337*, 2017.

[8] Sébastien Bubeck. Convex optimization: Algorithms and complexity. *Foundations and Trends® in Machine Learning*, 8(3-4):231–357, 2015.

[9] Saman Cyrus, Bin Hu, Bryan Van Scoy, and Laurent Lessard. A robust accelerated optimization algorithm for strongly convex functions. *arXiv preprint arXiv:1710.04753*, 2017.

[10] Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. On the sample complexity of the linear quadratic regulator. *arXiv preprint arXiv:1710.01688*, 2018.

[11] Sarah Dean, Horia Mania, Nikolai Matni, Benjamin Recht, and Stephen Tu. Regret bounds for robust adaptive control of the linear quadratic regulator. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, *Advances in Neural Information Processing Systems 31*, pages 4192–4201. Curran Associates, Inc., 2018.

[12] Sarah Dean, Stephen Tu, Nikolai Matni, and Benjamin Recht. Safely learning to control the constrained linear quadratic regulator. *arXiv preprint arXiv:1809.10121*, 2018.
[13] John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of Machine Learning Research*, 12:2121–2159, July 2011.

[14] Richard O. Duda, Peter E. Hart, and David G. Stork. *Pattern Classification, 2nd Edition*. John Wiley & Sons, 2001.

[15] S. Evesque, A. M. Annaswamy, S. Niculescu, and A. P. Dowling. Adaptive control of a class of time-delay systems. *Journal of Dynamic Systems, Measurement, and Control*, 125(2):186, 2003.

[16] Travis E. Gibson, Amuradha M. Annaswamy, and Eugene Lavretsky. On adaptive control with closed-loop reference models: Transients, oscillations, and peaking. *IEEE Access*, 1:703–717, 2013.

[17] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. MIT Press, 2016.

[18] Graham C. Goodwin, Peter J. Ramadge, and Peter E. Caines. Discrete-time multivariable adaptive control. *IEEE Transactions on Automatic Control*, 25(3):449–456, jun 1980.

[19] Graham C. Goodwin, Peter J. Ramadge, and Peter E. Caines. Discrete time stochastic adaptive control. *SIAM Journal on Control and Optimization*, 19(6):829–853, nov 1981.

[20] Graham C Goodwin and Kwai Sang Sin. *Adaptive Filtering Prediction and Control*. Prentice Hall, 1984.

[21] Elad Hazan. Introduction to online convex optimization. *Foundations and Trends® in Optimization*, 2(3-4):157–325, 2016.

[22] Elad Hazan, Holden Lee, Karan Singh, Cyril Zhang, and Yi Zhang. Spectral filtering for general linear dynamical systems. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, *Advances in Neural Information Processing Systems 31*, pages 4639–4648. Curran Associates, Inc., 2018.

[23] Elad Hazan, Alexander Rakhlin, and Peter L. Bartlett. Adaptive online gradient descent. In J. C. Platt, D. Koller, Y. Singer, and S. T. Roweis, editors, *Advances in Neural Information Processing Systems 20*, pages 65–72. Curran Associates, Inc., 2008.

[24] Elad Hazan, Karan Singh, and Cyril Zhang. Learning linear dynamical systems via spectral filtering. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems 30*, pages 6702–6712. Curran Associates, Inc., 2017.

[25] Bin Hu and Laurent Lessard. Control interpretations for first-order optimization methods. In *2017 American Control Conference (ACC)*. IEEE, may 2017.

[26] Bin Hu and Laurent Lessard. Dissipativity theory for nesterov’s accelerated method. *arXiv preprint arXiv:1706.04381*, 2017.

[27] Petros A. Ioannou and Petar V. Kokotovic. Robust redesign of adaptive control. *IEEE Transactions on Automatic Control*, 29(3):202–211, mar 1984.

[28] Petros A Ioannou and Jing Sun. *Robust Adaptive Control*. PTR Prentice-Hall, 1996.

[29] Sergey Ioffe and Christian Szegedy. Batch normalization: Accelerating deep network training by reducing internal covariate shift. *arXiv preprint arXiv:1502.03167*, 2015.
[30] Benjamin M. Jenkins, Anuradha M. Annaswamy, Eugene Lavretsky, and Travis E. Gibson. Convergence properties of adaptive systems and the definition of exponential stability. *SIAM Journal on Control and Optimization*, 56(4):2463–2484, jan 2018.

[31] M. I. Jordan and T. M. Mitchell. Machine learning: Trends, perspectives, and prospects. *Science*, 349(6245):255–260, jul 2015.

[32] R. E. Kalman and J. E. Bertram. Control system analysis and design via the “second method” of lyapunov: I—continuous-time systems. *Journal of Basic Engineering*, 82(2):371, 1960.

[33] S. P. Karason and A. M. Annaswamy. Adaptive control in the presence of input constraints. *IEEE Transactions on Automatic Control*, 39(11):2325–2330, 1994.

[34] Diederik P. Kingma and Jimmy L. Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2017.

[35] Alex Krizhevsky, Ilya Sutskever, and Geoffrey E. Hinton. Imagenet classification with deep convolutional neural networks. In F. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 25*, pages 1097–1105. Curran Associates, Inc., 2012.

[36] Eugene Lavretsky and Kevin A. Wise. *Robust and Adaptive Control with Aerospace Applications*. Springer London, 2013. ch. 11.

[37] Laurent Lessard, Benjamin Recht, and Andrew Packard. Analysis and design of optimization algorithms via integral quadratic constraints. *SIAM Journal on Optimization*, 26(1):57–95, jan 2016.

[38] David G. Luenberger. *Optimization by Vector Space Methods*. John Wiley & Sons, 1969.

[39] A. S. Morse. High-order parameter tuners for the adaptive control of linear and nonlinear systems. In *Systems, Models and Feedback: Theory and Applications*, pages 339–364. Birkhäuser Boston, 1992.

[40] Kumpati S. Narendra and Anuradha M. Annaswamy. Persistent excitation in adaptive systems. *International Journal of Control*, 45(1):127–160, jan 1987.

[41] Kumpati S. Narendra and Anuradha M. Annaswamy. *Stable Adaptive Systems*. Dover, 2005.

[42] Yurii Nesterov. A method of solving a convex programming problem with convergence rate $o(1/k^2)$. *Soviet Mathematics Doklady*, 27:372–376, 1983.

[43] Yurii Nesterov. *Introductory Lectures on Convex Optimization*. Springer, 2004.

[44] Brendan O’Donoghue and Emmanuel Candès. Adaptive restart for accelerated gradient schemes. *Foundations of Computational Mathematics*, 15(3):715–732, jul 2013.

[45] V. M. Popov. *Hyperstability of Control Systems*. Springer-Verlag, 1973.

[46] Maxim Raginsky, Alexander Rakhlin, and Serdar Yuksel. Online convex programming and regularization in adaptive control. In *49th IEEE Conference on Decision and Control (CDC)*. IEEE, 2010.
[47] Benjamin Recht. A tour of reinforcement learning: The view from continuous control. *arXiv preprint arXiv:1806.09460*, 2018.

[48] Shankar Sastry and Marc Bodson. *Adaptive Control: Stability, Convergence and Robustness*. Prentice-Hall, 1989.

[49] Max Simchowitz, Horia Mania, Stephen Tu, Michael I. Jordan, and Benjamin Recht. Learning without mixing: Towards a sharp analysis of linear system identification. In Sébastien Bubeck, Vianney Perchet, and Philippe Rigollet, editors, *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pages 439–473. PMLR, July 2018.

[50] Brian L. Stevens and Frank L. Lewis. *Aircraft Control and Simulation*. Wiley, 2003.

[51] Weijie Su, Stephen Boyd, and Emmanuel J. Candès. A differential equation for modeling nesterov’s accelerated gradient method: Theory and insights. *Journal of Machine Learning Research*, 17(153):1–43, 2016.

[52] Ilya Sutskever, James Martens, George Dahl, and Geoffrey Hinton. On the importance of initialization and momentum in deep learning. In Sanjoy Dasgupta and David McAllester, editors, *Proceedings of the 30th International Conference on Machine Learning*, volume 28 of *Proceedings of Machine Learning Research*, pages 1139–1147. PMLR, 2013.

[53] Richard S. Sutton and Andrew G. Barto. *Reinforcement Learning: An Introduction*. MIT Press, 2018.

[54] Stephen Tu and Benjamin Recht. Least-squares temporal difference learning for the linear quadratic regulator. In Jennifer Dy and Andreas Krause, editors, *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 5005–5014, Stockholmsmässan, Stockholm Sweden, July 2018. PMLR.

[55] Andre Wibisono, Ashia C. Wilson, and Michael I. Jordan. A variational perspective on accelerated methods in optimization. *Proceedings of the National Academy of Sciences*, 113(47):E7351–E7358, nov 2016.

[56] Ashia Wilson. *Lyapunov Arguments in Optimization*. PhD thesis, University of California, Berkeley, 2018.

[57] Ashia C. Wilson, Benjamin Recht, and Michael I. Jordan. A lyapunov analysis of momentum methods in optimization. *arXiv preprint arXiv:1611.02635*, 2016.

[58] Ashia C. Wilson, Rebecca Roelofs, Mitchell Stern, Nathan Srebro, and Benjamin Recht. The marginal value of adaptive gradient methods in machine learning. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems 30*, pages 4148–4158. Curran Associates, Inc., 2017.
Appendix

Organization of the Appendix

Mathematical definitions and Barbalat’s lemma are provided in Appendix A. Lyapunov stability definitions and stability analysis of some of the algorithms presented in this paper is provided in Appendix B. Additional plots demonstrating the effects of an increase in the time variation of a feature on the stability of regression algorithms in this paper is presented in Appendix C. Appendix D provides details regarding the model reference adaptive control simulation implementation.

A Definitions and Barbalat’s Lemma

This section details useful definitions regarding signals which are used throughout this paper.

The following definition regarding signals is modified from [41]:

Definition 1 For any fixed \( p \in [1, \infty) \), \( f : \mathbb{R}^{+} \to \mathbb{R} \) is defined to belong to \( \mathcal{L}_p \) if \( f \) is locally integrable and
\[
\| f(t) \|_{\mathcal{L}_p} \triangleq \left( \lim_{t \to \infty} \int_0^t \| f(\tau) \|^p d\tau \right)^{\frac{1}{p}} < \infty.
\]
When \( p = \infty \), \( f \in \mathcal{L}_\infty \), if,
\[
\| f \|_{\mathcal{L}_\infty} \triangleq \sup_{t \geq 0} \| f(t) \| < \infty.
\]

The notion of persistence of excitation has a long history in adaptive systems. It is commonly used to denote the condition that all states in the adaptive system are excited to allow for perfect system identification. Here it refers the the condition of all of the states of a system (collectively the regressor) being excited such that parameter convergence occurs. The following definition is regarding persistence of excitation:

Definition 2 ([30]) Let \( \omega \in [t_0, \infty) \to \mathbb{R}^p \) be a time varying parameter with initial condition defined as \( \omega_0 = \omega(t_0) \); then the parameterized function of time \( y(t, \omega) : [t_0, \infty) \times \mathbb{R}^p \to \mathbb{R}^m \) is persistently exciting if there exists \( T > 0 \) and \( \alpha > 0 \) such that
\[
\int_{t}^{t+T} y(\tau, \omega) y^T(\tau, \omega) d\tau \succeq \alpha I
\]
for all \( t \geq t_0 \) and \( \omega_0 \in \mathbb{R}^p \).

The notation \( X \succeq Y \) denotes that \( X - Y \) is positive semidefinite for square matrices \( X, Y \) of the same dimension.

The following lemma was attributed to Barbalat in [45] and has found significant use in the field of adaptive control and nonlinear control. The version from [41] is stated below with an associated corollary:

Lemma 1 ([41]) If \( f : \mathbb{R}^+ \to \mathbb{R} \) is uniformly continuous for \( t \geq 0 \), and if the limit of the integral
\[
\lim_{t \to \infty} \int_0^t |f(\tau)| d\tau
\]
exists and is finite, then
\[
\lim_{t \to \infty} f(t) = 0.
\]

Corollary 1 If \( f \in \mathcal{L}_2 \cap \mathcal{L}_\infty \), and \( \dot{f} \in \mathcal{L}_\infty \), then \( \lim_{t \to \infty} f(t) = 0 \).
B Stability Analysis

An overview of Lyapunov functions and their use in stability analysis is presented in Section B.1. Stability analysis for the first order update law of Section 2.1 is presented in Section B.2. Section B.3 then presents stability analysis of the first order update law of Section 2.2. Stability analysis for the Lyapunov function by [55] in Table 2 for time varying regression is presented in Section B.4.

B.1 Lyapunov Functions

This section provides a primer on Lyapunov functions and some of their common uses. While Lyapunov functions are ubiquitous in control theory and many similar definitions exist, this section was adapted from the definitions by [41]. Consider a general nonlinear dynamical system of the form:

\[ \dot{x} = f(x, t), \quad x(t_0) = x_0 \]  

(17)

where \( f(0, t) = 0 \ \forall \ t > 0 \). Lyapunov functions are often used to determine whether the equilibrium state of the dynamical system in (17) is stable, without explicitly finding the solution of (17). This is due to the potential difficulty in finding a solution of the nonlinear differential equation in (17). The method follows from finding a scalar function \( V(x, t) \) of the states \( x \) of a system and time. The time derivative \( \dot{V}(x, t) \) is then analyzed for all trajectories of the system in (17). The notion of a Lyapunov function comes from a energy perspective in which energy in a purely dissipative system is always positive and the time derivative is non-positive. It can be noted that even though the results of this paper rely on Lyapunov functions that are autonomous (i.e., \( V(x, t) = V(x) \)), some of what will be provided is additionally applicable to non-autonomous systems.

The following theorem establishes uniform asymptotic stability of the nonlinear dynamical system in (17), with proof available in [32].

**Theorem 1 (Lyapunov’s Direct Method)** The equilibrium state of (17) is uniformly asymptotically stable in the large if a scalar function \( V(x, t) \) with continuous first partial derivatives with respect to \( x \) and \( t \) exists such that \( V(0, t) = 0 \) and if the following conditions are satisfied:

1. \( V(x, t) \) is positive definite, i.e. there exists a continuous non-decreasing scalar function \( \alpha \) such that \( \alpha(0) = 0 \) and, for all \( t \) and all \( x \neq 0 \):

\[ 0 < \alpha(||x||) \leq V(x, t) \]

2. There exists a continuous non-decreasing scalar function \( \gamma \) s.t. \( \gamma(0) = 0 \) and the derivative \( \dot{V} \) of \( V \) along all system directions is negative-definite; that is that \( \dot{V} \) satisfies for all \( t \):

\[ \dot{V} = \frac{\partial V}{\partial t} + (\nabla V)^T f(x, t) \leq -\gamma(||x||) < 0, \quad \forall x \neq 0 \]

3. \( V(x, t) \) is decreascent, that is, there exists a continuous non-decreasing scalar function \( \beta \), such that \( \beta(0) = 0 \) and for all \( t \):

\[ V(x, t) \leq \beta(||x||) \]

4. \( V(x, t) \) is radially unbounded, that is:

\[ \lim_{||x|| \to \infty} \alpha(||x||) = \infty \]
It should be noted that in general, all of the conditions of Theorem 1 may not hold, in particular the condition 2 of Theorem 1, which requires \( \dot{V} \) being negative-definite along all system directions may be difficult to satisfy. In particular, \( \dot{V} < 0 \) may never hold for the entire state space of adaptive systems as the unknown parameter would have to show up in the expression for \( \dot{V} < 0 \). The parameter being unknown would restrict this condition from holding. However, it is common that \( \dot{V} \leq 0 \), that is, that the time derivative of \( V \) is negative semi-definite. The following proposition is used throughout this paper:

**Proposition 1** If \( V(x,t) \) in Theorem 1 is positive definite (condition 1) and \( \dot{V}(x,t) \leq 0 \) then the origin of (17) is stable; if in addition, condition 3 of Theorem 1 is satisfied, then uniform stability follows, \( x \) is bounded for all time, and \( V(x,t) \) is called a Lyapunov function.

In particular, linear time invariant (LTI) systems are often considered in this paper. The following theorem establishes stability for LTI systems and gives a connection to what is known as the Lyapunov equation:

**Theorem 2 ([41])** The equilibrium state \( x = 0 \) of the linear time invariant system

\[
\dot{x} = Ax
\]  

is asymptotically stable if, and only if, given any symmetric positive-definite matrix \( Q \), there exists a symmetric positive-definite matrix \( P \), which is the unique solution of the set of \( n(n+1)/2 \) linear equations (called the Lyapunov equation):

\[
A^T P + PA = -Q.
\]

Therefore, \( V(x) = x^T Px \) is a Lyapunov function for equation (18).

**B.2 Time Varying Regression**

To show stability of the first order update law (3) for the time varying regression error model (1), consider the following Lyapunov function candidate:

\[
V(\tilde{\theta}(t)) = \frac{1}{2\gamma} \tilde{\theta}^T(t)\tilde{\theta}(t)
\]  

which is a non-negative scalar quantity. The time derivative of this Lyapunov function candidate is:

\[
\dot{V}(\tilde{\theta}(t)) = \frac{1}{\gamma} \dot{\theta}^T(t)\dot{\theta}(t)
\]

Employing the equation for output error (1), as well as the first order update (3), the time derivative of the Lyapunov function may be expressed as:

\[
\dot{V}(\tilde{\theta}(t)) = -e_y^2(t) \leq 0
\]

From this, it can be concluded that \( V(\tilde{\theta}) \) is a Lyapunov function and \( \tilde{\theta} \in L_\infty \). By integrating \( \dot{V} \) from \( t_0 \) to \( \infty \):

\[
\int_{t_0}^{\infty} e_y^2(t)dt = -\int_{t_0}^{\infty} \dot{V}dt = V(\tilde{\theta}(t_0)) - V(\tilde{\theta}(\infty)) < \infty,
\]

thus \( e_y \in L_2 \). If in addition \( \phi \in L_\infty \) (the magnitude of the features are bounded), then from equation (1) it can be seen that \( e_y \in L_2 \cap L_\infty \) and from equation (4) \( \dot{\theta} \in L_\infty \). Also from (3), given that \( e_y \in L_2 \), it can be seen that \( \dot{\theta} \in L_2 \cap L_\infty \). If the additional assumption is made that \( \phi \in L_\infty \) (the time derivative of the features...
are bounded), then from (2), \( \dot{e}_y \in L_\infty \). Additionally from (3), it can then be seen that \( \ddot{\theta} \in L_\infty \). Then from Corollary 1 in Appendix A:

\[
\lim_{t \to \infty} e_y(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \dot{\theta}(t) = 0
\]

which is to say that the prediction error goes to zero as time goes to infinity, and the parameter estimate reaches a constant steady state value. For the parameter estimation error \( \dot{\theta} \to 0 \), persistence of excitation is needed (see Appendix A, Definition 2). This condition is similar in machine learning problems, where the objective function is defined based on a prediction error, but parameter convergence is not guaranteed without sufficient richness of data.

### B.3 Model Reference Adaptive Control and Identification

To demonstrate stability of the first order update (6) for the model reference adaptive control and identification error model in (5), consider the following Lyapunov function candidate:

\[
V(e(t), \tilde{\theta}(t)) = e^T(t)Pe(t) + \frac{1}{\gamma} \tilde{\theta}^T(t)\dot{\theta}(t)
\]

The time derivative of the Lyapunov function candidate may be expressed as:

\[
\dot{V}(e(t), \tilde{\theta}(t)) = 2e^T(t)P\dot{e}(t) + \frac{2}{\gamma} \tilde{\theta}^T(t)\dot{\theta}(t)
\]

Employing the Lyapunov equation \((A^TP + PA = -Q)\), the equation for model tracking error model (5), as well as the first order update (6), the time derivative of the Lyapunov function may be expressed as:

\[
\dot{V}(e(t), \tilde{\theta}(t)) = -e^T(t)Qe(t) \leq 0
\]

Thus it can be concluded that \( V(e(t), \tilde{\theta}(t)) \) is a Lyapunov function with \( e \in L_\infty \) and \( \tilde{\theta} \in L_\infty \). By integrating \( \dot{V} \) from \( t_0 \) to \( \infty \):

\[
\int_{t_0}^{\infty} e^T(t)Qe(t)dt = -\int_{t_0}^{\infty} \dot{V}dt = V(t_0) - V(\infty) < \infty, \text{ thus } e \in L_2.
\]

If in addition \( \phi \in L_\infty \) then from equation (5) \( \dot{e} \in L_\infty \) and from Corollary 1 in Appendix A:

\[
\lim_{t \to \infty} e(t) = 0
\]

which is to say that the model tracking error goes to zero as time goes to infinity. Once again, for the parameter estimation error \( \dot{\theta} \to 0 \), persistence of excitation of the regressor of the system is needed (see Appendix A, Definition 2). It can be noted that compared to the stability analysis for time varying regression in Appendix B.2, \( \lim_{t \to \infty} 0 e(t) = 0 \) when \( \phi \in L_\infty \) without the additional requirement that \( \phi \in L_\infty \).

### B.4 Stability Using Lyapunov Function in [55] for Accelerated Time Varying Regression

The candidate Lyapunov function proposed for demonstrating stability in [55] Equation 8 is restated as:

\[
V = D_h(\theta^*, \theta + e^{-\alpha t}\dot{\theta}) + e^\beta_t(L(\theta) - L(\theta^*))
\]

\(^7\)As is common in adaptive control, \( \phi = x \). It was proved that \( e \in L_\infty \), with \( x_m \in L_\infty \) by design of a suitable input \( u \). Thus with \( x = x_m - e \), \( \phi = x \) is bounded by construction and thus this is not a restrictive assumption.
where the Bregman divergence \( D_h(y, x) = h(y) - h(x) - \langle \nabla h(x), y - x \rangle \) may be expanded with the same squared Euclidean norm \( h(x) = \frac{1}{2} \| x \|^2 \) and squared loss \( L = \frac{1}{2} e_y^2 \) considered in Section 3.1 as:

\[
V = \frac{1}{2} \left\| \dot{\theta} + \frac{1}{\beta(1 + \mu \phi^T \phi)} \dot{\phi} \right\|^2 + \frac{\gamma}{\beta(1 + \mu \phi^T \phi)} \frac{1}{2} e^2_y
\]

Evaluating the time derivative of this candidate Lyapunov function using the time varying regression error model (1), its time derivative (2) and accelerated algorithm (10):

\[
\dot{V} = -\gamma e^2_y \left( 1 + \frac{\mu \phi^T \dot{\phi}}{\beta(1 + \mu \phi^T \phi)^2} \right) + \frac{\gamma}{\beta(1 + \mu \phi^T \phi)} e_y \tilde{\theta}^T \dot{\phi}
\]

which can be seen to be sign indeterminate. There can exist time derivatives of the feature \( \dot{\phi} \) for which \( \dot{V} \) is positive and thus global stability cannot be established for arbitrary feature time variations. It can be noted that if the feature is constant, as is assumed implicitly by [55] (i.e., \( \dot{\phi} = 0 \)), then stability can be established.
C Time Varying Regression Increase in Feature Time Variation - Plot Progression

The following plots show a progression in the increase of the frequency of the features of Section 6.1.

Figure 6: (to be viewed in color) T.V. Regression - Step Response.

Figure 7: (to be viewed in color) T.V. Regression - One Period ($\omega = \frac{2\pi}{36}$).

Figure 8: (to be viewed in color) T.V. Regression - Near Instability of W-reg Algorithm ($\omega = \frac{1}{4}$).

Figure 9: (to be viewed in color) T.V. Regression - Persistently Exciting Feature Response ($\omega = 1$).
D State Feedback Adaptive Control Implementation Details

This section provides implementation details for the state feedback adaptive control simulation in Section 6.2. The F-16 model used in this paper is from [50]. A trim point for this nonlinear F-16 vehicle model was obtained at a straight and level flying condition at a velocity of 500 ft/s with an altitude of 15,000 ft. The model was linearized about this trim point in order to obtain linear dynamics for control design and simulation. The short period linearized longitudinal dynamics of the aircraft are considered in this paper, as is typical for inner loop flight control [36]. The longitudinal short period variables are:

\[ x_p = [\alpha \ q]^T, \quad u = \delta_e, \quad z_p = q \]

where the longitudinal state \( x_p \) is composed of the vehicle’s angle of attack \( \alpha \) (degrees) and pitch rate \( q \) (degrees per second). The pitch rate is a regulated variable \( z_p \). The elevator deflection \( \delta_e \) (degrees) is an input to the dynamics. The linearized dynamics and input matrices are:

\[
A_p = \begin{bmatrix}
-0.6398 & 0.9378 \\
-1.5679 & -0.8791
\end{bmatrix}, \quad b_p = \begin{bmatrix}
-0.0777 \\
-6.5121
\end{bmatrix}
\]

The goal is to design the control input \( u \) so that \( z_p \) tracks a bounded command \( z_{cmd} \) with zero error. To ensure a zero tracking error, an integral error \( x_e \) state is generated as:

\[
\dot{x}_e(t) = z_p(t) - z_{cmd}(t)
\]

where the integral error state in this paper represents the integral of the pitch rate command tracking error. The complete plant model augments the plant dynamics with the integral of the tracking error and is written as:

\[
\begin{bmatrix}
\dot{x}_p(t) \\
\dot{x}_e(t) \\
\dot{x}(t)
\end{bmatrix} =
\begin{bmatrix}
A_p & 0_{2\times1} & 0_{2\times1} \\
0_{1\times3} & 0 & 0_{1\times3} \\
0_{3\times1} & 0_{1\times3}
\end{bmatrix}
\begin{bmatrix}
x_p(t) \\
x_e(t) \\
x(t)
\end{bmatrix} +
\begin{bmatrix}
b_p \\
0_{2\times1} \\
0_{3\times1}
\end{bmatrix} u(t) +
\begin{bmatrix}
0_{2\times1} \\
-1_{1\times3}
\end{bmatrix} z_{cmd}(t)
\]

This can be expressed more compactly as: \( \dot{x}(t) = Ax(t) + bu(t) + b_z z_{cmd}(t) \), where \( A \in \mathbb{R}^{3\times3}, \ b \in \mathbb{R}^{3\times1}, \ b_z \in \mathbb{R}^{3\times1} \) are the known matrices provided above. A state feedback gain \( \theta^* \) may be designed with linear quadratic regulator (LQR) methods in order to stabilize this system. The following cost matrices were employed to penalize the integral of the pitch rate command tracking error. The complete plant model augments the plant dynamics with the integral of the tracking error and is written as:

\[
Q_{LQR} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad R_{LQR} = 1
\]

The Matlab command \( \theta^* = lqr(A, b, Q_{LQR}, R_{LQR})' \) resulted in the following gain:

\[
\theta^* = \begin{bmatrix}
0.1965 \\
-0.3835 \\
-1.0000
\end{bmatrix}^T
\]

A stable closed loop matrix \( A_m \) may then be formulated as:

\[
A_m \triangleq A - b\theta^T
\]

---

8 Model downloaded from: [http://www.aem.umn.edu/~balas/darpa_sec/SEC.Software.html](http://www.aem.umn.edu/~balas/darpa_sec/SEC.Software.html)
The plant model may then be expressed in a similar manner as Section 2.2 with the closed loop matrix as:

\[ \dot{x}(t) = A_m x(t) + b(u(t) + \theta^* T x(t)) + b_z z_{cmd}(t) \]

A set of desired dynamics, known as the reference model may then be stated with the closed loop matrix as:

\[ \dot{x}_m(t) = A_m x_m(t) + b(u + \bar{\theta}^T x(t)) + b_z z_{cmd}(t) \]

In order to track the reference model in an adaptive control formulation, the control input is set as:

\[ u(t) = -\bar{\theta}^T x(t) \]

where the adaptive parameter \( \theta \) may be adjusted according to the nominal MRAC [6] and accelerated MRAC [14] update laws. The model tracking error may be stated as \( e = x_m - x \). The error model may then be stated as:

\[ \dot{e}(t) = A_m e(t) + b\bar{\theta}^T x(t) \]

where \( \bar{\theta} = \theta - \theta^* \), and can be seen to have a similar representation to error model 2 in equation (5), with \( \phi = x \).