LAX REPRESENTATIONS VIA TWISTED EXTENSIONS OF
INFINITE-DIMENSIONAL LIE ALGEBRAS: SOME NEW
RESULTS

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Abstract. We find new integrable partial differential equations with Lax rep-
resentations generated by extensions of Lie algebras of the Kac–Moody type
as well as the Lie algebra of Hamiltonian vector fields on $\mathbb{R}^2$.

1. Introduction

Lax representations provide the basic construction that allows applications of a
number of techniques for studying nonlinear partial differential equations (PDEs),
whence they are considered as the key feature indicating integrability thereof, see
[36, 37, 34, 29, 12, 1, 20, 32, 5] and references therein. Therefore the problem of
finding intrinsic conditions that ensure existence of a Lax representation for a given
PDE is of great importance in the theory of integrable systems. In the recent papers
[22] – [26] we propose an approach to tackle this problem. We have shown there
that for a number of PDEs including the potential Khokhlov–Zabolotskaya equation,
the Boyer–Finley equation, the hyper-CR equation of Einstein–Weyl structures,
the reduced quasiclassical self-dual Yang–Mills equation, the 4D Martínez Alonso–
Shabat equation, the 4D universal hierarchy equation, and other equations, their
known Lax representations can be inferred from non-triviality of the second twisted
cohomology groups of the Lie algebras of contact symmetries of the PDEs. Moreover,
we have shown that the technique allows one to find new Lax representations of
some PDEs.

The aim of the present paper is to gain a better understanding of the relationship
between the structure theory of infinite-dimensional Lie algebras and the theory
of integrable systems. In particular, we construct new examples of deriving Lax
representations of PDEs from twisted extensions of some Lie algebras.

In a number of above-mentioned examples the symmetry algebras of the PDEs
have the form of the semi-direct sum $\mathfrak{s}_0 \ltimes \mathfrak{q}_{N,\varepsilon}$ of a finite-dimensional Lie algebra
$\mathfrak{s}_0$ and the infinite-dimensional Lie algebra $\mathfrak{q}_{N,\varepsilon}$ of the Kac–Moody type, [6], that
is, the deformation of the tensor product $\mathfrak{q}_{N,0} = \mathbb{R}_N[\mathfrak{s}] \otimes \mathfrak{w}$, where $\mathbb{R}_N[\mathfrak{s}]$ is the
commutative associative algebra of truncated polynomials of degree $N$ and $\mathfrak{w}$ is
the Lie algebra of vector fields on $\mathbb{R}$, see Section 3 for definition of $\mathfrak{q}_{N,\varepsilon}$. The
second twisted cohomology groups of the Lie algebras $\mathfrak{s}_0 \ltimes \mathfrak{q}_{N,\varepsilon}$ from our examples
turn out to be nontrivial, and the nontrivial twisted 2-cocycles generate twisted
extensions of these Lie algebras. Linear combinations of the Maurer–Cartan forms

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of the twisted extensions provide the Wahlquist–Estabrook forms that generate Lax representations of the PDEs.

In examples from [24]–[26] we have $N \geq 3$, and the natural question is whether there exist Lie algebras of the form $s_c \ltimes q_{1,c}$ with $N < 3$ and nontrivial second twisted cohomology groups whose twisted extensions generate Lax representations of some PDEs. In Sections 4, 5, and 6 we present three systems (4.3) - (4.4), (5.5) - (5.6), and (6.2) - (6.3) whose Lax representations are generated by extensions of the Lie algebras $q_{1,-1}, q_{1,-2},$ and $q_{2,-1}$, respectively. Equation (5.5) can be considered as a 3D generalization of the generalized 2D Hunter–Saxton equation [9, 10, 3, 27] with the special value of the parameter. We compare the structure of the symmetry algebras of the obtained systems with the structure of the Lie algebras that generate the Lax representations.

Paper [24] shows that for a number of known 4D integrable equations their Lax representations can be derived from the twisted extensions of the symmetry algebras, which turn out to be of the form $s_c \ltimes (R_N[s] \otimes R[t] \otimes \omega)$. In [26] we apply the technique to a Lie algebra of this form and construct a new 4D integrable equation. In the present paper we address an interesting question of finding new Hamiltonian vector fields on $\mathbb{R}^2$. We derive equation (7.4) from Lax representation defined by extensions of the Lie algebra $\mathfrak{h}$ of Hamiltonian vector fields on $\mathbb{R}^2$ as a subalgebra. In Section 7 we present such an example. We derive equation (7.4) from Lax representation defined by extensions of the Lie algebra $\mathfrak{h} \otimes \mathfrak{w}$. Other examples of integrable systems whose symmetry algebras include $\mathfrak{h}$ as ‘building blocks’ are given by the family of heavenly equations [33, 1, 7, 18] and their ‘symmetric deformations’ [19]. Hence we refer equation (7.4) to as the ‘degenerate heavenly equation’.

2. Preliminaries and notation

2.1. Symmetries and Lax representations. The presentation in this section closely follows [13, 16, 17] and [35]. All our considerations are local. Let $\pi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\alpha$, $\pi: (x^1, \ldots, x^n, u^1, \ldots, u^m) \mapsto (x^1, \ldots, x^n)$, be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of the infinite order. The local coordinates on $J^\infty(\pi)$ are $(x^j, u^\alpha, u^\alpha_t)$, where $I = (i_1, \ldots, i_n)$ are multi-indices with $i_k \geq 0$, and for every local section $f: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ of $\pi$ the corresponding infinite jet $j_\infty(f)$ of a section $j_\infty(f): \mathbb{R}^n \to J^\infty(\pi)$ such that $u^\alpha_t(j_\infty(f)) = \frac{\partial f^\alpha}{\partial x^j} = \frac{\partial f}{\partial x^j} = \frac{\partial f^\alpha}{\partial x^j} = \frac{\partial f^\alpha}{\partial x^j} = \frac{\partial f^\alpha}{\partial x^j}$.

We put $u^\alpha = u^\alpha_{(0,\ldots,0)}$. Also, we will simplify notation in the following way: e.g., in the case of $n = 3, m = 1$ we denote $x^1 = t, x^2 = x + y, y, and u^1_{(i,j,k)} = u_{t,x,y}$ with $i$ times $t$, $j$ times $x$, and $k$ times $y$.

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{#I \geq 0} \sum_{\alpha=1}^m u^\alpha_{I+1k} \frac{\partial}{\partial u^\alpha_I}, \quad k \in \{1, \ldots, n\}$$

$(i_1, \ldots, k, \ldots, i_n) + 1_k = (i_1, \ldots, i_k + 1, \ldots, i_n)$, are called total derivatives. They commute everywhere on $J^\infty(\pi)$: $[D_{x^j}, D_{x^j}] = 0$.

The evolutionary vector field associated to an arbitrary vector-valued smooth function $\phi: J^\infty(\pi) \to \mathbb{R}^m$ is the vector field

$$E_\phi = \sum_{#I \geq 0} \sum_{\alpha=1}^m D_I(\phi^\alpha) \frac{\partial}{\partial u^\alpha_I}$$
with $D_I = D_{i_1 \ldots i_n} = D_{i_1}^{i_1} \circ \cdots \circ D_{i_n}^{i_n}$.

A system of PDEs $F_r(x^i, u^a_\ell) = 0$ of the order $s \geq 1$ with $\#I \leq s$, $r \in \{1, \ldots, R\}$ for some $R \geq 1$, defines the submanifold $E = \{(x^i, u^a_\ell) \in J^\infty(\pi) \mid D_K(F_r(x^i, u^a_\ell)) = 0, \ \#K \geq 0\}$ in $J^\infty(\pi)$.

A function $\varphi: J^\infty(\pi) \to \mathbb{R}^m$ is called a (generator of an infinitesimal) symmetry of equation $E$ when $E_{\varphi}(F) = 0$ on $E$. The symmetry $\varphi$ is a solution to the defining system
\[ (2.1) \quad \ell_E(\varphi) = 0, \]
where $\ell_E = \ell_F|_E$ with the matrix differential operator
\[ \ell_F = \left( \sum_{\#I \geq 0} \frac{\partial F_r}{\partial u^a_\ell} D_I \right). \]

The symmetry algebra $\text{Sym}(E)$ of equation $E$ is the linear space of solutions to (2.1) endowed with the structure of a Lie algebra over $\mathbb{R}$ by the Jacobi bracket $\{ \varphi, \psi \} = E_\varphi(\psi) - E_\psi(\varphi)$. The algebra of contact symmetries $\text{Sym}_0(E)$ is the Lie subalgebra of $\text{Sym}(E)$ defined as $\text{Sym}(E) \cap C^\infty(J^1(\pi))$. Symmetries with generators of the form $\varphi^a = \eta^a - \sum_{i} \xi_i u^a_i$, $\eta^a, \xi_i \in C^\infty(J^0(\pi))$, are referred to as point symmetries. They correspond to vector fields $\sum_i \xi_i \partial_{x^i} + \sum a \eta^a \partial_{u^a}$ on $J^0(\pi)$.

Let the linear space $W$ be either $\mathbb{R}^N$ for some $N \geq 1$ or $\mathbb{R}^\infty$ endowed with local coordinates $w^a$, $a \in \{1, \ldots, N\}$ or $a \in \mathbb{N}$, respectively. Variables $w^a$ are called pseudopotentials [36]. Locally, a differential covering of $E$ is a trivial bundle $\pi: J^\infty(\pi) \times W \to J^\infty(\pi)$ equipped with extended total derivatives
\[ \tilde{D}_{x^k} = D_{x^k} + \sum_a T^a_k(x^i, u^a_\ell, w^b) \frac{\partial}{\partial w^a} \]
such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ whenever $(x^i, u^a_\ell) \in E$. Define the partial derivatives of $w^a$ by $w^a_{x^k} = \tilde{D}_{x^k}(w^a)$. This yields the over-determined system of PDEs
\[ (2.2) \quad w^a_{x^k} = T^a_k(x^i, u^a_\ell, w^b) \]
which is compatible whenever $(x^i, u^a_\ell) \in E$. System (2.2) is referred to as the covering equations or the Lax representation of equation $E$.

Dually, the differential covering is defined by the Wahlquist–Estabrook forms
\[ (2.3) \quad r^a = dw^a - \sum_{k=1}^m T^a_k(x^i, u^a_\ell, w^b) \, dx^k \]
as follows: when $w^a$ and $u^a$ are considered to be functions of $x^1, \ldots, x^n$, forms (2.3) are equal to zero if and only if system (2.2) holds.

2.2. Twisted cohomology of Lie algebras. For a Lie algebra $g$ over $\mathbb{R}$, its representation $\rho: g \to \text{End}(V)$, and $k \geq 1$ let $C^k(g, V) = \text{Hom}(\Lambda^k(g), V)$ be the space of all $k$-linear skew-symmetric mappings from $g$ to $V$. Then the Chevalley–Eilenberg differential complex
\[ V = C^0(g, V) \xrightarrow{d} C^1(g, V) \xrightarrow{d} \cdots \xrightarrow{d} C^k(g, V) \xrightarrow{d} C^{k+1}(g, V) \xrightarrow{d} \cdots \]


is generated by the differential $d: \theta \mapsto d\theta$ such that
\[
d\theta(X_1, \ldots, X_{k+1}) = \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_q) (\theta(X_1, \ldots, \hat{X}_q, \ldots, X_{k+1}))
\]
\begin{equation}
(2.4) \quad + \sum_{1 \leq p < q \leq k+1} (-1)^{p+q+1} \theta([X_p, X_q], X_1, \ldots, \hat{X}_p, \ldots, \hat{X}_q, \ldots, X_{k+1}).
\end{equation}

The cohomology groups of the complex $(C^*(\mathfrak{g}, V), d)$ are referred to as the cohomology groups of the Lie algebra $\mathfrak{g}$ with coefficients in the representation $\rho$. For the trivial representation $\rho_0: \mathfrak{g} \to \mathbb{R}$, $\rho_0: X \mapsto 0$, the cohomology groups are denoted by $H^*(\mathfrak{g})$.

Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ with non-trivial first cohomology group $H^1(\mathfrak{g})$ and take 1-form $\alpha \neq 0$ on $\mathfrak{g}$ such that $d\alpha = 0$. Then for each $c \in \mathbb{R}$ define the twisted differential $d_{c\alpha}: C^k(\mathfrak{g}, \mathbb{R}) \to C^{k+1}(\mathfrak{g}, \mathbb{R})$ by the formula
\[
d_{c\alpha} \theta = d\theta - c\alpha \wedge \theta.
\]

From $d\alpha = 0$ it follows that $d_{c\alpha}^2 = 0$. The cohomology groups of the complex
\[
C^1(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{c\alpha}} \cdots \xrightarrow{d_{c\alpha}} C^k(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{c\alpha}} C^{k+1}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{c\alpha}} \cdots
\]
are referred to as the twisted cohomology groups [30,31] of $\mathfrak{g}$ and denoted by $H^*_{c\alpha}(\mathfrak{g})$.

3. Lie algebras of the Kac–Moody type and their extensions

Consider the Lie algebra $\mathfrak{q}_{N,0} = \mathbb{R}N[s] \otimes \mathfrak{w}$, where $\mathbb{R}N[s] = \mathbb{R}[s]/\langle s^{N+1} = 0 \rangle$ is the commutative unital algebra of truncated polynomials of variable $s$ of degree $N$, and $\mathfrak{w} = \langle V_k \mid k \geq 0 \rangle, V_k = \frac{1}{k!} t^k \partial_t$, is the Lie algebra of polynomial vector fields on $\mathbb{R}$ referred to as the (one-sided) Witt algebra. Algebra $\mathfrak{q}_{N,0}$ admits the deformation\footnote{For the full description of deformations of the Lie algebra $\mathfrak{q}_{N,0}$ see [32].} generated by cocycle $\Psi \in H^2(\mathfrak{q}_{N,0}, \mathfrak{q}_{N,0})$,
\[
\Psi(s^p \otimes V_m, s^q \otimes V_n) = \begin{cases} \frac{pn - qm}{m + n} \left( \begin{array}{c} m + n \\ m \end{array} \right) s^{p+q} \otimes V_{m+n-1}, & m + n \geq 1, p + q \leq N, \\
0, & \text{otherwise.}
\end{cases}
\]

For each $\varepsilon \neq 0$ this cocycle defines new bracket $[\cdot, \cdot]_\varepsilon = [\cdot, \cdot] + \varepsilon \Psi(\cdot, \cdot)$ on the linear space $(s^p \otimes V_m | p \leq N, m \geq 0)$. We denote the resulting Lie algebra as $\mathfrak{q}_{N,\varepsilon}$. In other words, the Lie algebra $\mathfrak{q}_{N,\varepsilon}$ is isomorphic to the linear space of functions $f(t, s) = f_0(t) + s f_1(t) + \cdots + s^N f_N(t), f_k \in \mathbb{R}[t]$, equipped with the bracket
\begin{equation}
(3.1) \quad [f, g]_\varepsilon = f g_t - g f_t + \varepsilon s (f_s g_t - g_s f_t)
\end{equation}
such that there holds $s^k = 0$ for $k > N$. Likewise to [6] it can be shown that $\mathfrak{q}_{N,\varepsilon} \subseteq \mathfrak{g}(A^{(1)}_M)$ for some $M \geq N$, see [11] for definition of the Lie algebra $\mathfrak{g}(A^{(1)}_M)$. Therefore $\mathfrak{q}_{N,\varepsilon}$ are referred to as Lie algebras of the Kac–Moody type.

Consider the dual 1-forms $\theta_{p,k}$ to the basis $s^p \otimes V_m$ of $\mathfrak{q}_{N,\varepsilon}$, that is, the linear mappings $\theta_{p,k}: \mathfrak{q}_{N,\varepsilon} \to \mathbb{R}$ such that $\theta_{p,k}(s^p \otimes V_m) = \delta_{p,q} \delta_{k,m}$. Define the formal series
\begin{equation}
(3.2) \quad \Theta = \sum_{k=0}^{N} \sum_{m=0}^{\infty} \frac{b_k h_m}{m!} \theta_{k,m}
\end{equation}
with formal parameters $h_0$ and $h_1$ such that $h_k^0 = 0$ when $k > N$ and $dh_0 = dh_1 = 0$. Then (2.4) and (3.1) entail the structure equations
\begin{equation}
(3.3) \quad d\Theta = \Theta h_1 \land (\Theta + \varepsilon h_0 \Theta_{h_0})
\end{equation}
of the Lie algebra $q_{N,\varepsilon}$. Here and below we use notation $\Theta_{h_1} = \partial_{h_1} \Theta$ for partial derivatives of the formal series of 1-forms with respect to the formal parameters $h_1$.

For each $N \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ the map $D_0: s^p \otimes v_k \mapsto p \ s^p \otimes v_k$ is an outer derivation of $q_{N,\varepsilon}$. Denote by $a_0 \ltimes q_{N,\varepsilon}$ the associated one-dimensional ‘right’ extension, [8 §1.4.4], of $q_{N,\varepsilon}$. As a vector space $a_0 \ltimes q_{N,\varepsilon} = (Z_0) \oplus q_{N,\varepsilon}$, and the bracket on $q_{N,\varepsilon}$ is extended to $Z_0$ by the formula $[Z_0, s^p \otimes v_k]_e = D_0(s^p \otimes v_k) = p \ s^p \otimes v_k$.

Let $\alpha_0: a_0 \ltimes q_{N,\varepsilon} \to \mathbb{R}$ be the dual form to $-Z_0$, that is, $\alpha_0(Z_0) = -1$ and $\alpha_0(s^p \otimes v_k) = 0$. Then the structure equations for $a_0 \ltimes q_{N,\varepsilon}$ read
\[
\begin{cases}
d\Theta = \Theta h_1 \land (\Theta + \varepsilon h_0 \Theta_{h_0}) + h_0 \alpha_0 \land \Theta_{h_0},
d\alpha_0 = 0.
\end{cases}
\]

For some values of $N$ and $\varepsilon$ the Lie algebras $a_0 \ltimes q_{N,\varepsilon}$ admit further right extensions. In [22 [23 [24 [25] we have shown examples of integrable PDEs whose Lax representations can be inferred from such extensions when $N \geq 3$. In the next three sections we consider integrable PDEs that are related to extensions of $a_0 \ltimes q_{N,\varepsilon}$, $a_0 \ltimes q_{N,\varepsilon}$, and $a_0 \ltimes q_{N,\varepsilon}$.

As it was shown in [23, for $N \in \mathbb{N}$ and $\varepsilon = -r^{-1}$ with $r \in \{1, \ldots, N\}$ the Lie algebra $a_0 \ltimes q_{N,\varepsilon}$ admits the right extension $a_1 \ltimes q_{N,\varepsilon}$ generated by the outer derivation $D_1: a_0 \ltimes q_{N,\varepsilon} \to a_0 \ltimes q_{N,\varepsilon}$ with
\[D_1(s^p \otimes v_k) = \begin{cases} k s^{p+r} \otimes v_{k-1}, & p + r \leq N, k \geq 1, \\
0, & \text{otherwise.}
\end{cases}\]
We have $[D_0, D_1] = D_0 \circ D_1 - D_1 \circ D_0 = r D_1$. Then $a_1 \ltimes q_{N,\varepsilon-1/r}$ as a vector space is $(Z_1) \oplus (a_0 \ltimes q_{N,\varepsilon-1/r})$, with the extension of the bracket of $q_{N,\varepsilon-1/r}$ given by $[Z_0, Z_1]_{-1/r} = -r Z_1$ and $[Z_1, s^p \otimes v_k]_{-1/r} = D_1(s^p \otimes v_k)$. Consider the dual form $\alpha_1$ to the vector $-Z_1$ that, put $\alpha_1(Z_0) = -1$, $\alpha_1(Z_0) = \alpha_1(s^p \otimes v_k) = 0$. Then the structure equations for the Lie algebra $a_1 \ltimes q_{N,\varepsilon-1/r}$ get the form
\begin{equation}
(3.4) \quad \begin{cases}
d\Theta = \Theta h_1 \land \left(\Theta - \frac{h_0}{r} \Theta h_0 - h_0^r \alpha_1\right) + h_0 \alpha_0 \land \Theta_{h_0}, \\
da_{0} = 0, \\
da_{1} = r \alpha_0 \land \alpha_1.
\end{cases}
\end{equation}
These equations yield $H^1(a_1 \ltimes q_{N,\varepsilon-1/r}) = (a_0)$ and $[a_0 \land \alpha_1] \in H^2_{r a_0}(a_1 \ltimes q_{N,\varepsilon-1/r})$, therefore the Lie algebra $a_1 \ltimes q_{N,\varepsilon-1/r}$ admits the twisted extension $a_2 \ltimes q_{N,\varepsilon-1/r}$ with the structure equations obtained by appending equation
\begin{equation}
(3.5) \quad da_{2} = r \alpha_0 \land \alpha_2 + \alpha_0 \land \alpha_1
\end{equation}
to system (3.3).

Furthermore, we have $[\alpha_1 \land \alpha_2] \in H^2_{r a_0}(a_2 \ltimes q_{N,\varepsilon-1/r})$, therefore the Lie algebra $a_2 \ltimes q_{N,\varepsilon-1/r}$ admits the twisted extension $a_3 \ltimes q_{N,\varepsilon-1/r}$ whose structure equations are obtained by appending equation
\begin{equation}
(3.6) \quad da_{3} = 2r \alpha_0 \land \alpha_3 + \alpha_1 \land \alpha_2
\end{equation}
The restriction of form \( r q a d u \) (4.2). We do not need explicit expressions for the other forms \( \pi q w \). Then we consider the linear combination \( v \theta q w \rightarrow R \theta q w \rightarrow R \). In other words, the extended Lie algebra \( a_k \times q N_{-1/r} \) has the structure equations

\[
\begin{align*}
d\Theta &= \Theta h_1 \wedge \left( \Theta - \frac{h_0}{r} \Theta h_0 - h_0 a_1 \right) + h_0 a_0 \wedge \Theta h_0, \\
d\alpha_0 &= 0, \\
d\alpha_1 &= r a_0 \wedge \alpha_1, \\
d\alpha_{m+1} &= m r a_0 \wedge \alpha_{m+1} + a_1 \wedge \alpha_m, \quad m \in \{1, \ldots, k\}.
\end{align*}
\]

(3.7)

4. Integrable equation associated to \( a_2 \times q 1_{-1} \)

Consider the Lie algebra \( a_3 \times q 1_{-1} \) defined by the structure equations (3.7) with \( r = 1 \) and \( k = 2 \), that is, by system

\[
\begin{align*}
d\Theta &= \Theta h_1 \wedge \left( \Theta - h_0 \Theta h_0 - h_0 a_1 \right) + h_0 a_0 \wedge \Theta h_0, \\
d\alpha_0 &= 0, \\
d\alpha_1 &= a_0 \wedge \alpha_1, \\
d\alpha_2 &= a_0 \wedge a_2 + a_0 \wedge \alpha_1, \\
d\alpha_3 &= 2 a_0 \wedge a_3 + a_0 \wedge a_2.
\end{align*}
\]

(4.1)

Frobenius’ theorem allows one to integrate equations (4.1) step by step. In particular, we have

\[
\begin{align*}
\alpha_0 &= \frac{dq}{q}, \quad \alpha_1 = q dy, \quad \alpha_2 = q (dw + \ln q dy), \quad \alpha_3 = q^2 (dv - w dy), \\
\theta_{0,0} &= a dt, \quad \theta_{1,0} = q (dx - \ln a dy + u dt), \quad \theta_{1,1} = q a^{-1} (du - b dy - p dt),
\end{align*}
\]

where \( q \neq 0, \ a \neq 0, \ t, x, y, u, v, w \) are free parameters ('constants of integration'). We do not need explicit expressions for the other forms \( \theta_{i,j} \) in what follows.

We proceed by imposing the requirement for the linear combination \( \theta_{1,1} - \theta_{1,0} = q a^{-1} (du - a dx - (b - a \ln a) dy - (p + a u) dt) \) to be a multiple of the contact form \( du - u_x dt - u_x dx - u_y dy \) on the bundle of jets of sections of the bundle \( \pi: R^4 \rightarrow R^3 \), \( \pi: (t, x, y, u) \mapsto (t, x, y) \), that is, we take \( a = u_x, \ b = u_y + u_x \ln u_x, \ p = u_t - u u_x \). Then we consider the linear combination \( \tau = \alpha_3 - \theta_{1,0} = q^2 (dv - q^{-1} (u dt + dx + (q w - \ln u_x) dy) \) and put \( q = v_x^{-1} \). This yields

\[
\tau = v_x^2 (dv - u v_x dt - v_x dx - (w - v_x \ln u_x) dy).
\]

The restriction of form \( \tau \) on the bundle of sections of the bundle \( R^6 \rightarrow R^3 \), \( (t, x, y, u, v, w) \mapsto (t, x, y) \), gives the over-determined system of PDEs

\[
\begin{align*}
v_t &= u v_x, \\
v_y &= w - v_x \ln u_x.
\end{align*}
\]

(4.2)

The integrability condition \( (v_t)_y = (v_y)_t \) thereof gives two equations

\[
(4.3) \quad u_{tx} = u u_{xx} - u_x^2 \ln u_x - u_x u_y
\]
and

\begin{equation}
(4.4) \quad w_t = u w_x.
\end{equation}

Thus system (4.2) provides a Lax representation of system (4.3), (4.4). We can find the Lax representation of equation (4.3) by proceeding as follows. Equation (4.4) is a copy of the first equation in (4.2). We consider \(w\) as a new pseudopotential and include equation (4.4) into the Lax representation for equation (4.3) by adding the copy \(w_t = u w_x \ln u_x\) of the second equation from (4.2) with additional function \(w_1\), and then repeat this process. In other words, we rename \(v_0 = v, w_1 = v_1\), then add the sequence of functions \(v_k, k \geq 2\), and consider the infinite system

\begin{equation}
(4.5) \quad \begin{cases}
v_{k,t} = u v_{k,x}, \\
v_{k,y} = v_{k+1} - v_{k,x} \ln u_x.
\end{cases} \quad k \geq 0.
\end{equation}

The compatibility conditions of this system coincides with equation (4.3). System (4.5) can be written in the finite form by introducing the function

\begin{equation}
(4.6) \quad r = e^{-\lambda y} \sum_{k=0}^{\infty} \lambda^k v_k
\end{equation}

where \(\sum_{k=0}^{\infty} \lambda^k v_k\) is a formal series with respect to a formal parameter \(\lambda\). Then we have

\begin{equation}
(4.7) \quad \begin{cases}
r_t = u r_x, \\
r_y = -r_x \ln u_x.
\end{cases}
\end{equation}

This system provides a Lax representation for equation (4.3).

Direct computations give the following statement:

**Proposition 1.** The contact symmetry algebra of equation (4.3) is generated by the vector fields

\begin{equation}
(4.8) \quad A_0 \partial_t - A'_0 y \partial_y - (A'_0 u - A''_0 y) \partial_u, \quad A_1 \partial_t - A'_1 \partial_u, \quad x \partial_x + y \partial_y + u \partial_u, \quad \partial_y,
\end{equation}

where \(A_i = A_i(t)\) are arbitrary smooth functions of \(t\). Restricting these functions to polynomials gives the Lie algebra isomorphic to \(a_2 \ltimes q_{1,-1}\). The contact symmetry algebra of system (4.3), (4.4) is obtained by appending the vector field \(B \partial_v\) with arbitrary smooth function \(B = B(y,v)\) to the vector fields (4.8). \(\square\)

5. 3D Generalized Hunter–Saxton Equation

The Lie algebra \(a_0 \ltimes q_{1,-1/2}\) admits the outer derivation \(D_2(s^p f(t)) = s^{p+1} f''(t)\), that is,

\[
D_2(s^p \otimes V_k) = \begin{cases} 
  k(k-1) s \otimes V_{k-2}, & p = 0 \text{ and } k \geq 2, \\
  0, & p = 1 \text{ or } k \in \{0,1\}.
\end{cases}
\]

This derivation produces the right extension \(b_1 \ltimes q_{1,-1/2} = (Z_1) \ltimes (a_0 \ltimes q_{1,-1/2})\) of \(a_0 \ltimes q_{1,-1/2}\), where \([Z_1, V_{k}]_{-1/2} = D_2(V_k)\) and \([Z_0, Z_1]_{-1/2} = Z_1\). Denote by \(\alpha_1\)

\footnote{We carried out computations of generators of contact symmetries in the Jets software [2].}
the dual form to \(-Z_1\), that is, put \(\alpha_1(Z_1) = -1, \alpha_1(Z_0) = 0, \alpha_1(V_k) = 0\). Then
the structure equations for \(b_1 \ltimes q_1_{-1/2}\) acquire the form
\[
\begin{cases}
  d\Theta &= \Theta_{h_1} \wedge (\Theta - 2 h_0 \Theta_{h_0}) + h_0 \alpha_0 \wedge \Theta_{h_0} - h_0 \alpha_1 \wedge \Theta_{h_1 h_1}, \\
  da_0 &= 0, \\
  da_1 &= \alpha_0 \wedge \alpha_1.
\end{cases}
\] (5.1)

The Lie algebra \(b_1 \ltimes q_1_{-1/2}\) admits the sequence of twisted extensions \(b_k \ltimes q_1_{-1/2}\)
\(k \geq 2\), as it was described when constructing system (3.7). In this section we need
the second extension from this series. Specifically, we have \([\alpha_0 \wedge \alpha_1] \in H^2_{\alpha_0}(b_1 \ltimes q_1_{-1/2})\), hence we append equation
\[
(5.2) \quad da_2 = \alpha_0 \wedge \alpha_2 + \alpha_0 \wedge \alpha_1,
\]
to system (5.1) and get the structure equations of \(b_2 \ltimes q_1_{-1/2}\). Then \([\alpha_1 \wedge \alpha_2] \in H^3_{2\alpha_0}(b_2 \ltimes q_1_{-1/2})\), and the structure equations of \(b_3 \ltimes q_1_{-1/2}\) are obtained by
adding equation
\[
(5.3) \quad da_3 = 2 \alpha_0 \wedge \alpha_3 + \alpha_1 \wedge \alpha_2.
\]

Successively integrating equations (5.1), (5.2), and (5.3) by applying Frobenius’
theorem, we get
\[
\alpha_0 = \frac{dq}{q}, \quad \alpha_1 = q \, dy, \quad \alpha_2 = q \, (dw + \ln q \, dy), \quad \alpha_3 = q^2 \, (dv - w \, dy),
\]
\[
\theta_{0,0} = a \, dt, \quad \theta_{0,1} = \frac{da}{a} + p_1 \, dt, \quad \theta_{0,2} = \frac{dp_1}{a} + p_2 \, dt,
\]
\[
\theta_{1,0} = qa^{-1} \, (dx + p_1 \, dt + u \, dt), \quad \theta_{1,1} = qa^{-2} \, (du - p_1 \, dx + (a \, p_2 - p_1^2) \, dy + p_3 \, dt),
\]
where \(q \neq 0, a \neq 0, t, x, y, u, v, w, p_1, p_2, p_3\) are free parameters. By altering
notation as \(p_1 = u_x, p_2 = a^{-1} (u_x^2 - u_y),\) and \(p_3 = -u_t\) we obtain \(\theta_{1,1} = qa^{-2} \, (du -
\end{equation}

\begin{equation}
\end{equation}

\begin{equation}
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\begin{equation}
\end{equation}
with the special value \( \beta = -1 \) of parameter \( \beta \). Hence \([5.5]\) can be considered as a three-dimensional generalization of the particular case \( u_{tx} = u u_{xx} - u_x^2 \) of equation \([5.7]\).

Likewise to Section [3] we can find the Lax representation of equation \([5.5]\) by renaming \( v = v_0 \), \( w = v_1 \) and including equation \([5.6]\) in the infinite system

\[
\begin{align*}
v_{k,t} &= u v_{k,x}, \\
v_{k,y} &= v_{k+1} + u_x v_{k,x},
\end{align*}
\]

System \([5.8]\) is compatible by virtue of equation \([5.5]\). Series \([4.6]\) allows one to write \([5.8]\) in the form

\[
\begin{align*}
r_t &= u r_x, \\
r_y &= u_x r_x.
\end{align*}
\]

We have

**Proposition 2.** The contact symmetry algebra of equation \([5.5]\) admits generating vector fields

\[
A_0 \partial_t + (A_0' y - A_0' x) \partial_x - (2 A_0' u - A_0'' x + A_0'' y) \partial_u, \quad A_1 \partial_x - A_1' \partial_u,
\]

\[
x \partial_x + y \partial_y + u \partial_u, \quad \partial_y,
\]

where \( A_i = A_i(t) \) are arbitrary smooth functions of \( t \). Restriction of these functions to \( \mathbb{R}[t] \) gives the Lie algebra isomorphic to \( \mathfrak{v}_2 \ltimes \mathfrak{q}_{1, -1/2} \). The contact symmetry algebra of system \([5.5], [5.6]\) has additional generating vector field \( B \partial_v \) with arbitrary smooth function \( B = B(y, v) \).

---

**6. Integrable Equation Associated to \( a_2 \ltimes \mathfrak{q}_{2, -1} \)**

The structure equations of the Lie algebra \( a_3 \ltimes \mathfrak{q}_{2, -1} \) have the form \([4.4]\), where now the formal series \( \Theta \) is given by formula \([3.2]\) with \( N = 2 \). We take the same forms \( \alpha_0, ..., \alpha_3 \), and \( \theta_{0,0} \) as in Section [3] and put

\[
\theta_{1,0} = q (db - \ln a dy - p_1 dt), \quad \theta_{2,0} = -q^2 a^{-1} (dx + p_1 dy - p_2 dt).
\]

Then the linear combination

\[
\theta_{1,0} + \theta_{2,0} + \alpha_2 = q \left( db + dw + \frac{p_1 + q p_2}{a} dt - \frac{q}{a} dx - a \frac{\ln a - \ln q + q p_1}{a} dy \right)
\]

after altering notation \( b = u - w, q = a u_x, p_1 = (u_y + \ln u_x) u^{-1}_x \), and \( p_2 = -(u_t u_x + u_y + \ln u_x) u^{-2}_x \) acquires the form \( \theta_{1,0} + \theta_{2,0} + \alpha_2 = q (du - u_t dt - u_x dx - u_y dy) \), while for the form

\[
\tau = \alpha_3 + \theta_{2,0} = a^2 u_x^2 \left( dv - \frac{1}{a} \left( dx + \frac{u_t u_x + u_y + \ln u_x}{u_x} dt + a \frac{u_x w + u_y + \ln u_x}{u_x} dy \right) \right)
\]

after renaming \( a = v^{-1}_x \) we have

\[
\tau = \frac{u_x^2}{v_x} \left( dv - v_x \left( dx + \frac{u_t u_x + u_y + \ln u_x}{u_x} dt \right) + \frac{u_x w + (u_y + \ln u_x) v_x}{u_x} dy \right).
\]
Restricting this onto sections of the bundle \((t, x, y, u, v, w) \mapsto (t, x, y)\) produces system

\[
\begin{cases}
v_t &= \frac{u_t u_x + u_y + \ln u_x}{u_x^2} v_x, \\
v_y &= \frac{u_y + \ln u_x}{u_x} v_x + w,
\end{cases}
\]

which is compatible by virtue of equations

\[
(6.2) \quad u_{yy} = u_{tx} - \left(\frac{u_y + \ln u_x}{u_x}\right)^2 + u_t u_x u_{xx} + 2 \left(\frac{u_y + \ln u_x}{u_x}\right) - 1 \frac{u_{xy}}{u_x}
\]

and

\[
(6.3) \quad w_t = \frac{u_t u_x + u_y + \ln u_x}{u_x^2} w_x.
\]

The last equation is a copy of the first equation in \((6.1)\). Therefore we can take \(w\) as a new pseudopotential, rename \(v = v_0, w = v_1\), add the infinite sequence of pseudopotentials \(v_j, j \geq 2\), and consider infinite system

\[
(6.4) \quad \begin{cases}
v_{k,t} &= \frac{u_t u_x + u_y + \ln u_x}{u_x^2} v_{k,x}, \\
v_{k,y} &= \frac{u_y + \ln u_x}{u_x} v_{k,x} + v_{k+1},
\end{cases} \quad k \geq 0.
\]

This system is compatible by virtue of equation \((6.2)\) and thus defines a Lax representation thereof. Introducing series \((4.6)\), we rewrite \((6.4)\) in the form of another Lax representation

\[
(6.5) \quad \begin{cases}
r_t &= \frac{u_t u_x + u_y + \ln u_x}{u_x^2} r_x, \\
r_y &= \frac{u_y + \ln u_x}{u_x} r_x
\end{cases}
\]

for equation \((6.2)\).

**Proposition 3.** The following vector fields

\[
A_0 \partial_t - \left(\frac{A'_0 x + \frac{1}{2} A''_0 y^2}{2 x} \right) \partial_x - A'_0 y \partial_u, \quad A'_1 y \partial_x + A_1 \partial_u, \quad A_2 \partial_x,
\]

\[
2 x \partial_x + y \partial_y + (u + y) \partial_u, \quad \partial_y,
\]

with arbitrary smooth functions \(A_i = A_i(t)\) are generators for the contact symmetry algebra of equation \((6.2)\). Restricting these functions to polynomials gives the Lie algebra isomorphic to \(a_2 \ltimes q_{2,-1}\). The contact symmetry algebra of system \((6.2), (6.3)\) is obtained by appending the vector field \(B \partial_v\) with arbitrary smooth function \(B = B(y, v)\). \(\square\)
7. The degenerate heavenly equation

In this section we construct a Lie algebra that includes the algebra of polynomial Hamiltonian vector fields
\[
\frak{h} = \langle W_{m,n} = n t^m x^{n-1} \partial_t - m t^{m-1} x^n \partial_x \mid m, n \in \mathbb{N}_0, m^2 + n^2 \neq 0 \rangle
\]
on \mathbb{R}^2 as a subalgebra and has a twisted extension that generates an integrable PDE. To find such a Lie algebra we proceed as follows. The Lie algebra \( \frak{h} \) admits the grading \( \text{gr}(W_{m,n}) = m + n - 2 \) that defines the outer derivation \( D: W_{m,n} \mapsto (m+n-2)W_{m,n} \) and the right extension \( \langle Z \rangle \triangleleft \frak{h} \) with \( [Z,W_{m,n}] = D(W_{m,n}) \). Denote the dual forms to \( W_{p,q} \) and \(-Z \) as \( \theta_{i,j} \) and \( \alpha \), so \( \theta_{i,j}(W_{m,n}) = \delta_{i,m} \delta_{j,n}, \theta_{i,j}(Z) = 0, \alpha(W_{m,n}) = 0, \alpha(Z) = -1 \). Further, we take \( \frak{w} = \langle V_k = \frac{1}{k!} y^k \partial_y \mid k \geq 0 \rangle \) and consider the direct sum of Lie algebras
\[
\frak{h} = \langle (Z) \triangleleft \frak{h} \rangle \oplus \frak{w}.
\]
Denote by \( \omega_n \) the dual forms for \( V_k \), so \( \omega_m(V_k) = \delta_{m,k}, \omega_m(W_{k,n}) = \omega_m(Z) = 0, \) and \( \theta_{m,n}(V_k) = \alpha(V_k) = 0 \). Put
\[
\Theta = \sum_{m \geq 0, n \geq 0, m^2 + n^2 \neq 0} \frac{h^m_i h^n_j}{m! n!} \theta_{m,n}, \quad \Omega = \sum_{k \geq 0} \frac{h^k_j}{k!} \omega_k,
\]
where \( h_i \) are formal parameters such that \( dh_i = 0 \). Then the structure equations for the Lie algebra \( \frak{h} \) acquire the form
\[
\begin{align*}
\frac{d\Theta}{dx} &= -\Theta h_1 \wedge \Theta h_2 - \alpha \wedge (h_1 \Theta h_1 + h_2 \Theta h_2 - 2 \Theta), \\
\frac{d\Omega}{dx} &= \Omega h_3 \wedge \Omega, \\
\frac{d\alpha}{dx} &= 0.
\end{align*}
\]
This system entails \( H^1(\frak{h}) = \langle \alpha \rangle \) and \( [\theta_{1,0} \wedge \theta_{0,1}] \in H^2_{\text{fr}}(\frak{h}) \). Hence the Lie algebra \( \frak{h} \) admits the twisted extension defined by appending equation
\[
(7.2) \quad d\sigma = 2 \alpha \wedge \sigma + \theta_{1,0} \wedge \theta_{0,1}
\]
to system (7.1).
Applying Frobenius' theorem and integrating equations (7.2), we step by step obtain
\[
\theta_{1,0} = a_{11} dt + a_{12} dx, \quad \theta_{0,1} = a_{21} dt + a_{22} dx, \\
\alpha = \frac{1}{2} \frac{dq}{q}, \quad q = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
\]
\[
\theta_{2,0} = \frac{1}{q} (a_{12} da_{11} - a_{11} da_{12}) + b_1 dt + b_2 dx,
\]
\[
\theta_{0,2} = \frac{1}{q} (a_{22} da_{21} - a_{21} da_{22}) + b_3 dt + b_4 dx,
\]
\[
\theta_{1,1} = \frac{1}{2q} (a_{22} da_{11} - a_{11} da_{22} + a_{12} da_{21} - a_{21} da_{12}) + b_5 dt + b_6 dx,
\]
\[
\omega_0 = pdy, \quad \sigma = q (dv - x dt),
\]
where \( a_{i,j}, b_1, \ldots, b_4, p, v \) are free parameters such that \( q \neq 0, p \neq 0, \) and
\[
b_5 = \frac{1}{q} (-a_{21} a_{22} b_1 + a_{21}^2 b_2 + a_{11} a_{12} b_3 - a_{11}^2 b_4),
\]
\[
b_6 = \frac{1}{q} (-a_{22} b_1 + a_{21} a_{22} b_2 + a_{12}^2 b_3 - a_{11} a_{12} b_4),
\]
Put \( a_{21} = u a_{22} \), \( b_3 = -a_{22}^2 q^{-1} u_x \), \( b_4 = -a_{22}^2 q^{-1} u_y \), \( p = -a_{22}^2 q^{-1} u_y \). This yields \( \theta_{t,0} - \omega_0 = \frac{a_{22}}{q^2} (du - u_x dt - u_x dx - u_y dy) \). Consider the linear combination 
\[
\tau = \sigma + \theta_{t,0} - \frac{1}{2} \omega_0 
\]
\[
= a_{22} (a_{11} - u a_{12}) \left( dv + \frac{dx}{a_{11} - u a_{12}} - \frac{(x (a_{11} - u a_{12}) - u)}{2 (a_{11} - u a_{12})^2} dt - \frac{u_y dy}{2 (a_{11} - u a_{12})^2} \right).
\]
By altering notation \( a_{11} = u a_{12} - v_x^{-1} \) we get
\[
\tau = -\frac{a_{22}}{v_x} \left( dv - v_x dx - (x + u v_x) dt - \frac{1}{2} u_y v_x^2 dy \right).
\]
Then upon restricting form \( \tau \) to the sections of the bundle \( \mathbb{R}^5 \to \mathbb{R}^3 \), \((t, x, y, u, v) \mapsto (t, x, y)\), we obtain the over-determined system
\[
(7.3) \quad \begin{cases}
  v_t = x + u v_x, \\
v_y = \frac{1}{2} u_y v_x^2.
\end{cases}
\]
The compatibility condition of this system \((v_t)_y = (v_y)_t\) holds if and only
\[
(7.4) \quad u_{txy} = u u_{xxy} - 2 u_x u_y.
\]
Therefore system \((7.3)\) defines a Lax representation for equation \((7.4)\).

We notice that equation \((7.4)\) is invariant with respect to translations along \(x\), while its Lax representation \((7.3)\) does not admit such translations, cf \([21, \text{Th. 4}]\) and \([23, \text{Example 7}]\).

Direct computations give

**Proposition 4.** The contact symmetry algebra of equation \((7.4)\) has generators
\[
A_x \partial_t - A_t \partial_x + (A_{tt} - 2 u A_{tx} + u^2 A_{xx}) \partial_u, \quad B \partial_y, \quad t \partial_t + x \partial_x,
\]
where \(A = A(t, x)\) and \(B = B(y)\) are arbitrary smooth functions of their arguments. Restriction of these functions to polynomials \(A = t^m x^n, B = y^k, m, n, k \in \mathbb{N}_0, m^2 + n^2 \neq 0\), produces the Lie algebra isomorphic to \(\mathfrak{b}^\circ\).

8. Concluding remarks

In the present paper we have used the method of twisted extensions to derive new integrable PDEs related to some infinite-dimensional Lie algebras. In the obtained examples as well as in some examples in \([22, 20]\) the integrable equations were constructed starting from certain extensions of the Lie algebras of the Kac–Moody type or the Lie algebra of Hamiltonian vector fields on \(\mathbb{R}^2\). In examples of Sections 4, 5, and 6 the symmetry algebras of the obtained integrable systems turn out to be wider than initial infinite-dimensional Lie algebras used in the construction, while the specific form of equations \((4.4), (5.6)\), and \((6.3)\) allowed us to find infinite-component Lax representations \((4.5), (5.8)\), and \((6.1)\) as well as one-component Lax representations \((4.7), (5.9)\), and \((6.5)\) \(\mapsto\) \((4.3), (5.5)\), and \((6.2)\), respectively. The polynomial parts of symmetry algebras of equations \((4.3), (5.5)\), and \((6.2)\) coincide with the infinite-dimensional Lie algebras whose twisted extensions were used to construct systems \((4.2), (5.4), \) and \((6.1)\).

We hope further examples will enlighten relations between structure theory of Lie algebras and integrable PDEs. In particular, it is important to address the following issues in the future research:

- to find other examples of integrable systems related with the Lie algebras of the Kac–Moody type with small values of \(N\),
• to consider extensions of general deformations of the Lie algebras $R_N[s] \otimes \mathfrak{w}$, cf. [38],
• to construct other integrable systems whose symmetry algebras are extensions of the Lie algebras of Hamiltonian vector fields,
• to generalize the technique used in the present paper on the Lie–Rinehart algebras, cf. [28].

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