Polynomial degree vs. quantum query complexity

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Abstract

The degree of a polynomial representing (or approximating) a function \( f \) is a lower bound for the quantum query complexity of \( f \). This observation has been a source of many lower bounds on quantum algorithms. It has been an open problem whether this lower bound is tight.

We exhibit a function with polynomial degree \( M \) and quantum query complexity \( \Omega(M^{1/321\ldots}) \). This is the first superlinear separation between polynomial degree and quantum query complexity. The lower bound is shown by a generalized version of the quantum adversary method.

1 Introduction

Quantum computing provides speedups for factoring \[29\], search \[15\] and many related problems. These speedups can be quite surprising. For example, Grover’s search algorithm \[15\] solves an arbitrary exhaustive search problem with \( N \) possibilities in time \( O(\sqrt{N}) \). Classically, it is obvious that time \( \Omega(N) \) would be needed.

This makes lower bounds particularly important in the quantum world. If we can search in time \( O(\sqrt{N}) \), why can we not search in time \( O(\log^c N) \)? (Among other things, that would have meant \( NP \subseteq BQP \).) Lower bound of Bennett et al. \[10\] shows that this is not possible and Grover’s algorithm is exactly optimal.

Currently, we have good lower bounds on the quantum complexity of many problems. They mainly follow by two methods\(^1\): the hybrid/adversary method \[10\] \[4\] and the polynomials method \[9\]. The polynomials method is useful for proving lower bounds both in classical \[23\] and quantum complexity \[9\]. It is known that

\(^1\)Other approaches, such as reducing query complexity to communication complexity \[11\] are known, but have been less successful.
1. the number of queries $Q_E(f)$ needed to compute a Boolean function $f$ by an exact quantum algorithm exactly is at least $\frac{\deg(f)}{2}$, where $\deg(f)$ is the degree of the multilinear polynomial representing $f$.

2. the number of queries $Q_2(f)$ needed to compute $f$ by a quantum algorithm with two-sided error is at least $\frac{\deg(f)}{2}$, where $\tilde{\deg}(f)$ is the smallest degree of a multilinear polynomial approximating $f$.

This reduces proving lower bounds on quantum algorithms to proving lower bounds on degree of polynomials. This is a well-studied mathematical problem with methods from approximation theory [14] available. Quantum lower bounds shown by polynomials method include a $Q_2(f) = \Omega(\sqrt{D(f)})$ relation for any total Boolean function $f$ [9], lower bounds on finding mean and median [22], collisions and element distinctness [2, 18]. Polynomials method is also a key part of recent $\Omega(\sqrt{N})$ lower bound on set disjointness which resolved a longstanding open problem in quantum communication complexity [25].

Given the usefulness of polynomials method, it is an important question how tight is the polynomials lower bound. [9, 13] proved that, for all total Boolean functions, $Q_2(f) = O(\deg^6(f))$ and $Q_E(f) = O(\deg^4(f))$. The second result was recently improved to $Q_E(f) = O(\deg^3(f))$ [21]. Thus, the bound is tight up to polynomial factor.

Even stronger result would be $Q_E(f) = O(\deg(f))$ or $Q_2(f) = O(\tilde{\deg}(f))$. Then, determining the quantum complexity would be equivalent to determining the degree of a function as a polynomial. It has been an open problem to prove or disprove either of these two equalities [9, 13].

In this paper, we show the first provable gap between polynomial degree and quantum complexity: $\deg(f) = 2^d$ and $Q_2(f) = \Omega(2.5^d)$. Since $\deg(f) \geq \tilde{\deg}(f) \geq Q_2(f)$, this implies a separation both between $Q_E(f)$ and $\deg(f)$ and between $Q_2(f)$ and $\tilde{\deg}(f)$.

To prove the lower bound, we use the quantum adversary method of [4]. The quantum adversary method runs a quantum algorithm on different inputs from some set. If every input in this set can be changed in many different ways so that the value of the function changes, many queries are needed.

The previously known version of quantum adversary method gives a weaker lower bound of $Q_2(f) = \Omega(2.1213^{d})$. While this already gives some gap between polynomial degree and quantum complexity, we can achieve a larger gap by using a new, more general version of the method.

The new component is that we carry out this argument in a very general way. We assign individual weights to every pair of inputs and distribute each weight
among the two inputs in an arbitrary way. This allows us to obtain better bounds than with the previous versions of the quantum adversary method.

We apply the new lower bound theorem to three functions for which deterministic query complexity is significantly higher than polynomial degree. The result is that, for all of those functions, quantum query complexity is higher than polynomial degree. The biggest gap is polynomial degree $2^d = M$ and query complexity $\Omega(2^{0.5d}) = \Omega(M^{1.321\ldots})$.

Spalek and Szegedy [32] have recently shown that our method is equivalent to two other methods, the spectral method of [8] that was known prior to our work and the Kolmogorov complexity method of [19] that appeared after the conference version of our paper was published. Although all three methods are equivalent, they have different intuition. It appears to us that our method is the easiest to use for results in this paper.

2 Preliminaries

2.1 Quantum query algorithms

Let $[N]$ denote $\{1, \ldots, N\}$.

We consider computing a Boolean function $f(x_1, \ldots, x_N): \{0, 1\}^N \rightarrow \{0, 1\}$ in the quantum query model (for a survey on query model, see [6, 13]). In this model, the input bits can be accessed by queries to an oracle $X$ and the complexity of $f$ is the number of queries needed to compute $f$. A quantum computation with $T$ queries is just a sequence of unitary transformations $U_0 \rightarrow O \rightarrow U_1 \rightarrow O \rightarrow \ldots \rightarrow U_{T-1} \rightarrow O \rightarrow U_T$.

The $U_j$’s can be arbitrary unitary transformations that do not depend on the input bits $x_1, \ldots, x_N$. The $O$’s are query (oracle) transformations which depend on $x_1, \ldots, x_N$. To define $O$, we represent basis states as $|i, z\rangle$ where $i$ consists of $\lceil \log(N + 1) \rceil$ bits and $z$ consists of all other bits. Then, $O_x$ maps $|0, z\rangle$ to itself and $|i, z\rangle$ to $(-1)^{x_i}|i, z\rangle$ for $i \in \{1, \ldots, N\}$ (i.e., we change phase depending on $x_i$, unless $i = 0$ in which case we do nothing).

The computation starts with a state $|0\rangle$. Then, we apply $U_0, O_x, \ldots, O_x, U_T$ and measure the final state. The result of the computation is the rightmost bit of the state obtained by the measurement.

The quantum computation computes $f$ exactly if, for every $x = (x_1, \ldots, x_N)$, the rightmost bit of $U_T O_x \ldots O_x U_0 |0\rangle$ equals $f(x_1, \ldots, x_N)$ with certainty.

The quantum computation computes $f$ with bounded error if, for every $x = (x_1, \ldots, x_N)$, the probability that the rightmost bit of $U_T O_x U_{T-1} \ldots O_x U_0 |0\rangle$ equals $f(x_1, \ldots, x_N)$ is at least $1 - \epsilon$ for some fixed $\epsilon < 1/2$. 

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$Q_E(f) \ (Q_2(f))$ denotes the minimum number $T$ of queries in a quantum algorithm that computes $f$ exactly (with bounded error). $D(f)$ denotes the minimum number of queries in a deterministic query algorithm computing $f$.

2.2 Polynomial degree and related quantities

For any Boolean function $f$, there is a unique multilinear polynomial $g$ such that $f(x_1, \ldots, x_N) = g(x_1, \ldots, x_N)$ for all $x_1, \ldots, x_N \in \{0, 1\}$. We say that $g$ represents $f$. Let $\deg(f)$ denote the degree of polynomial representing $f$.

A polynomial $g(x_1, \ldots, x_N)$ approximates $f$ if $1 - \varepsilon \leq g(x_1, \ldots, x_N) \leq 1$ whenever $f(x_1, \ldots, x_N) = 1$ and $0 \leq g(x_1, \ldots, x_N) \leq \varepsilon$ whenever $f(x_1, \ldots, x_N) = 0$. Let $\tilde{\deg}(f)$ denote the minimum degree of a polynomial approximating $f$. It is known that

**Theorem 1** [9]

1. $Q_E(f) = \Omega(\deg(f))$;
2. $Q_2(f) = \Omega(\tilde{\deg}(f))$;

This theorem has been a source of many lower bounds on quantum algorithms [9][22][2].

Two other relevant quantities are sensitivity and block sensitivity. The sensitivity of $f$ on input $x = (x_1, \ldots, x_N)$ is just the number of $i \in [N]$ such that changing the value of $x_i$ changes the value of $f$:

$$f(x_1, \ldots, x_N) \neq f(x_1, \ldots, x_{i-1}, 1-x_i, x_{i+1}, \ldots, x_N).$$

We denote it $s_x(f)$. The sensitivity of $f$ is the maximum of $s_x(f)$ over all $x \in \{0, 1\}^N$. We denote it $s(f)$.

The block sensitivity is a similar quantity in which we flip sets of variables instead of single variables. For $x = (x_1, \ldots, x_N)$ and $S \subseteq [N]$, let $x^{(S)}$ be the input $y$ in which $y_i = x_i$ if $i \notin S$ and $y_i = 1-x_i$ if $i \in S$. The block sensitivity of $f$ on an input $x$ (denoted $bs_x(f)$) is the maximum number $k$ of pairwise disjoint $S_1, \ldots, S_k$ such that $f(x^{(S_1)}) \neq f(x)$. The block sensitivity of $f$ is the maximum of $bs_x(f)$ over all $x \in \{0, 1\}^N$. We denote it $bs(f)$.

3 Main results

3.1 Overview

The **basis function.** $f(x)$ is equal to 1 iff $x = x_1x_2x_3x_4$ is one of the following values: 0011, 0100, 0101, 0111, 1000, 1010, 1011, 1100. This function has the
degree of 2, as witnessed by polynomial $f(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3x_4 - x_1x_4 - x_2x_3 - x_1x_2$ and the deterministic complexity $D(f) = 3$, as shown in section 4.3 where we discuss the function in more detail.

**Iterated function.** Define a sequence $f^1 = f, f^2, \ldots$ with $f^d$ being a function of $4^d$ variables by

$$
\begin{align*}
f^{d+1} &= f(f^d(x_1, \ldots, x_{4^d}), f^d(x_{4^d+1}, \ldots, x_{2.4^d}), \\
f^d(x_{2.4^d+1}, \ldots, x_{3.4^d}), f^d(x_{3.4^d+1}, \ldots, x_{4^d+1})).
\end{align*}
$$

Then, $\deg(f^d) = 2^d$, $D(f^d) = 3^d$ and, on every input $x$, $s_x(f^d) = 2^d$ and $bs_x(f^d) = 3^d$.

We will show

**Theorem 2** $Q_2(f^d) = \Omega(2.5^d)$.

Thus, the exact degree is $\deg(f^d) = 2^d$ but even the quantum complexity with 2-sided error $Q_2(f^d)$ is $\Omega(2.5^d) = \deg(f^d)^{1.321\ldots}$. This implies an $M$-vs-$\Omega(M^{1.321\ldots})$ gap both between exact degree and exact quantum complexity and between approximate degree and bounded-error quantum complexity.

The proof is by introducing a combinatorial quantity $Q'_2(f)$ with the following properties:

**Lemma 1** For any Boolean function $g$, $Q_2(g) = \Omega(Q'_2(g))$.

**Lemma 2** Let $g$ be an arbitrary Boolean function. If $g^1, g^2, \ldots$ is obtained by iterating $g$ as in equation (1), then

$$Q'_2(g^d) \geq (Q'_2(g))^d.$$

**Lemma 3** $Q'_2(f) \geq 2.5$.

Theorem 2 then follows from Lemmas 1-2-3.

### 3.2 Previous methods

Our approach is a generalization of the quantum adversary method [4].

**Theorem 3** Let $A \subset \{0,1\}^N$, $B \subset \{0,1\}^N$, $R \subset A \times B$ be such that $f(A) = 0, f(B) = 1$ and

- for every $x \in A$, there are at least $m$ inputs $y \in B$ such that $(x, y) \in R$,
- for every $y \in B$, there are at least $m'$ inputs $x \in A$ such that $(x, y) \in R$,
• for every \( x = (x_1 \ldots x_N) \in A \) and every \( i \in [N] \) there are at most \( l \) inputs \( y \in B \) such that \((x, y) \in R \) and \( x_i \neq y_i \).

• for every \( y = (y_1 \ldots y_N) \in B \) and every \( i \in [N] \), there are at most \( l' \) inputs \( x \in A \) such that \((x, y) \in R \) and \( x_i \neq y_i \).

Then, \( Q_2(f) = \Omega(\sqrt{mnl'}) \).

There are several ways to apply this theorem to \( f^d \) defined in the previous section. The best lower bound that can be obtained by it seems to be \( Q_2(f) = \Omega(2^{1.1213 \ldots d}) \) (cf. appendix A). This gives some separation between \( Q_2(f) \) and \( \deg(f) = 2^d \) but is weaker than our new method that we introduce next.

### 3.3 New method: weight schemes

We now formally define the combinatorial quantity \( Q'_2(f) \) that we use in Lemmas 1, 2 and 3.

**Definition 1** Let \( f : \{0, 1\}^N \to \{0, 1\}, A \subseteq f^{-1}(0), B \subseteq f^{-1}(1) \) and \( R \subseteq A \times B \). A weight scheme for \( A, B, R \) consists of numbers \( w(x, y) > 0, w'(x, y, i) > 0 \) for all \((x, y) \in R \) and \( i \in [N] \) satisfying \( x_i \neq y_i \), we have

\[
w'(x, y, i)w'(y, x, i) \geq w^2(x, y). \tag{2}
\]

**Definition 2** The weight of \( x \) is \( \text{wt}(x) = \sum_{y : (x, y) \in R} w(x, y) \), if \( x \in A \) and \( \text{wt}(x) = \sum_{y : (y, x) \in R} w(x, y) \) if \( x \in B \).

**Definition 3** Let \( i \in [N] \). The load of variable \( x_i \) in assignment \( x \) is

\[
v(x, i) = \sum_{y : (x, y) \in R, x_i \neq y_i } w'(x, y, i)
\]

if \( x \in A \) and

\[
v(x, i) = \sum_{y : (y, x) \in R, x_i \neq y_i } w'(x, y, i)
\]

if \( x \in B \).

We are interested in schemes in which the load of each variable is small compared to the weight of \( x \).

Let the maximum A-load be \( v_A = \max_{x \in A, i \in [N]} \frac{v(x, i)}{\text{wt}(x)} \). Let the maximum B-load be \( v_B = \max_{x \in B, i \in [N]} \frac{v(x, i)}{\text{wt}(x)} \). The maximum load of a weight scheme is \( v_{\text{max}} = \sqrt{v_A v_B} \).
Let $Q'_2(f)$ be the maximum of $\frac{1}{v_{\max}}$ over all choices of $A \subseteq \{0, 1\}^N$, $B \subseteq \{0, 1\}^N$, $R \subseteq A \times B$ and all weight schemes for $A, B, R$. We will show in Lemma 1 if we have a weight scheme with maximum load $v_{\max}$, the query complexity has to be $\Omega(\frac{1}{v_{\max}})$.

3.4 Relation to other methods

Theorem 3 follows from our new Lemma 1 if we set $w(x, y) = 1$ for all $(x, y) \in R$ and $w(x, y, i) = w(y, x, i) = 1$ for all $i \in [N]$. Then, the weight of $x$ is just the number of pairs $(x, y) \in R$. Therefore, $wt(x) \geq m$ for all $x \in A$ and $wt(y) \geq m'$ for all $y \in B$. The load of $i$ in $x$ is just the number of $(x, y) \in R$ such that $x_i \neq y_i$. That is, $v(x, i) \leq l$ and $v(y, i) \leq l'$. Therefore, $v_A \leq \frac{l}{m}$, $v_B \leq \frac{l'}{m'}$ and $v_{\max} \leq \sqrt{\frac{m l'}{m'}}$. This gives us the lower bound of Theorem 3.

There are several generalizations of Theorem 3 that have been proposed. Bar-num and Saks [7] have a generalization of Theorem 3 that they use to prove a $\Omega(\sqrt{N})$ lower bound for any read-once function on $N$ variables. This generalization can be shown to be a particular case of our Lemma 1 with a weight scheme constructed in a certain way.

Barnum, Saks and Szegedy [8] have a very general and promising approach. They reduce quantum query complexity to semidefinite programming and show that a $t$-query algorithm exists if and only if a certain semidefinite program does not have a solution. Spalek and Szegedy have recently shown [32] that our weighted scheme method is equivalent to Theorem 4 in [8] which is a special case of their general method. Our method is also equivalent [32] to Kolmogorov complexity method by Laplante and Magniez [19].

Hoyer, Neerbek and Shi [16] have shown lower bounds for ordered searching and sorting using a weighted version of the quantum adversary method, before both this paper and [8]. Their argument can be described as a weight scheme for those problems, but it is more natural to think about it in the spectral terminology of [8].

4 Proofs

4.1 Lemma 1

In terms of weight schemes, Lemma 1 becomes

Lemma 1 If a function $g$ has a weight scheme with maximum load $v_{\max}$, then $Q'_2(g) = \Omega(\frac{1}{v_{\max}})$.

Proof: We can assume that $v_A = v_B = v_{\max}$. Otherwise, we just multiply all $w'(x, y, i)$ by $\sqrt{v_B/v_A}$ and all $w'(y, x, i)$ by $\sqrt{v_A/v_B}$. Notice that this does not
affect the requirement (2). In the new scheme \( v_A \) is equal to the old \( v_A \sqrt{v_B/v_A} = \sqrt{v_A v_B} = v_{\text{max}} \) and \( v_B \) is equal to the old \( v_B \sqrt{v_A/v_B} = \sqrt{v_A v_B} = v_{\text{max}} \).

Let \( |\psi_x^t\rangle \) be the state of a quantum algorithm after \( t \) queries on input \( x \). We consider

\[
W_t = \sum_{(x,y) \in R} w(x,y) |\langle \psi_x^t | \psi_y^t \rangle|.
\]

For \( t = 0 \), \( W_0 = \sum_{(x,y) \in R} w(x,y) \). Furthermore, if an algorithm computes \( f \) in \( t \) queries with probability at least \( 1 - \epsilon \), \( W_t \leq 2\sqrt{\epsilon(1 - \epsilon)}W_0 \) \[^4\][^16]. To prove that \( T = \Omega(\frac{1}{v_{\text{max}}}) \), it suffices to show

**Lemma 4** \( |W_j - W_{j-1}| \leq 2v_{\text{max}}W_0 \).

**Proof:** Let \( |\phi_x^t\rangle \) be the state of the algorithm immediately before query \( t \). We write

\[
|\phi_x^t\rangle = \sum_{i=0}^N \alpha_{x,i}^t |\phi_{x,i}^t\rangle
\]

with \( |\phi_{x,i}^t\rangle \) being the state of qubits not involved in the query. The state after the query is

\[
|\psi_x^t\rangle = \alpha_{x,0}^t |0\rangle|\phi_{x,0}^t\rangle + \sum_{i=1}^N \alpha_{x,i}^t (-1)^{x_i} |i\rangle|\phi_{x,i}^t\rangle.
\]

Notice that all the terms in \( \langle \psi_x^t | \phi_y^t \rangle \) and \( \langle \psi_x^t | \psi_y^t \rangle \) are the same, except for those which have \( x_i \neq y_i \). Thus,

\[
\langle \psi_x^t | \psi_y^t \rangle - \langle \phi_x^t | \phi_y^t \rangle \leq 2 \sum_{i:x_i \neq y_i} |\alpha_{x,i}^t||\alpha_{y,i}^t|
\]

and

\[
|W_j - W_{j-1}| \leq 2 \sum_{(x,y) \in R} \sum_{i:x_i \neq y_i} w(x,y) |\alpha_{x,i}^t||\alpha_{y,i}^t|.
\]

By the inequality \( 2AB \leq A^2 + B^2 \),

\[
|W_j - W_{j-1}| \leq \sum_{(x,y) \in R} \sum_{i:x_i \neq y_i} (w'(x, y, i)|\alpha_{x,i}^t|^2 + w'(y, x, i)|\alpha_{y,i}^t|^2).
\]

We consider the sum of all first and all second terms separately. The sum of all first terms is

\[
\sum_{(x,y) \in R} \sum_{i:x_i \neq y_i} w'(x, y, i)|\alpha_{x,i}^t|^2 = \sum_{x \in \mathcal{A}, i \in [N]} |\alpha_{x,i}^t|^2 \left( \sum_{y:(x,y) \in R, x_i \neq y_i} w'(x, y, i) \right)
\]

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Lemma 2

In terms of weight schemes, we have to prove

\[ \sum_{x \in A, i \in [N]} |\alpha^i_{x,i}|^2 \alpha(x,i) \leq v_A \sum_{x \in A, i \in [N]} |\alpha^i_{x,i}|^2 \alpha(x) \]

\[ = v_A \sum_{x \in A} \alpha(x) \sum_{i \in [N]} |\alpha^i_{x,i}|^2 = v_A \sum_{x \in A} \alpha(x) = v_A W_0. \]

Similarly, the second sum is at most \( v_B W_0 \). Finally, \( v_A = v_B = v_{max} \) implies that \( |W_j - W_{j-1}| \leq 2v_{max} W_0 \).

4.2 Lemma 2

In terms of weight schemes, we have to prove

Lemma 2 Let \( g \) be a function with a weight scheme with maximum load \( v_1 \). Then, the function \( g^d \) obtained by iterating \( g \) as in equation (1) has a weight scheme with maximum load \( v_1^d \).

The lemma follows by inductively applying

Lemma 5 If \( g \) has a weight scheme with maximum load \( v_1 \) and \( g^{d-1} \) has a weight scheme with maximum load \( v_{d-1} \), then \( g^d \) has a weight scheme with maximum load \( v_1 v_{d-1} \).

Proof: Similarly to lemma 1, assume that the schemes for \( g \) and \( g^{d-1} \) have \( v_A = v_B = v_{max} \).

Let \( n \) be the number of variables for the base function \( g(x_1, \ldots, x_n) \). We subdivide the \( n^d \) variables \( x_1, \ldots, x_{n^d} \) of the function \( g^d \) into \( n \) blocks of \( n^{d-1} \) variables. Let \( x^j = (x_{j-1}n^{d-1} + 1, \ldots, x_{j+n^{d-1}}) \) be the \( j \)th block. Furthermore, let \( \tilde{x} \) be the vector \( (g^{d-1}(x^1), g^{d-1}(x^2), \ldots, g^{d-1}(x^n)) \).

Then, \( g^d(x) = g(\tilde{x}) \).

We start by defining \( A, B \) and \( R \). Let \( A_1, B_1, R_1 (A_{d-1}, B_{d-1}, R_{d-1}) \) be \( A, B, R \) in the weight scheme for \( g \) (\( g^{d-1} \), respectively). \( x \in A \) (\( B \), respectively) if

- \( \tilde{x} \in A_1 \) (\( B_1 \), respectively), and
- for every \( j \in [n] \), \( x^j \in A_{d-1} \) if \( \tilde{x}_j = 0 \) and \( x^j \in B_{d-1} \) if \( \tilde{x}_j = 1 \).

\((x, y) \in R \) if \( (\tilde{x}, \tilde{y}) \in R_1 \) and, for every \( j \in [n] \),

- \( x^j = y^j \) if \( \tilde{x}_j = \tilde{y}_j \).
- \((x^j, y^j) \in R_{d-1} \) if \( \tilde{x}_j = 0, \tilde{y}_j = 1 \).
- \((y^j, x^j) \in R_{d-1} \) if \( \tilde{x}_j = 1, \tilde{y}_j = 0 \).
Let $w_1(x, y)$ denote the weights in the scheme for $g$ and $w_{d-1}(x, y)$ the weights in the scheme for $g^{d-1}$. We define the weights for $g^d$ as

$$w_d(x, y) = w_1(\bar{x}, y) \prod_{j: \bar{x}_j = y_j} wt_{d-1}(x^j) \prod_{j: \bar{x}_j \neq y_j} wt_{d-1}(x^j, y^j)$$

where $wt_{d-1}$ is the weight of $x^j$ in the scheme for $g^{d-1}$.

For $i \in [n^d]$, let $i_1 = \lceil \frac{i}{n^{d-1}} \rceil$ be the index of the block containing $i$ and $i_2 = (i - 1) \mod n^{d-1} + 1$ be the index of $i$ within this block. Define

$$w'_d(x, y, i) = w_d(x, y) \sqrt{\frac{w'_1(\bar{x}, \bar{y}, i_1)}{w'_1(\bar{y}, \bar{x}, i_1)}} \sqrt{\frac{w'_{d-1}(x^{i_1}, y^{i_1}, i_2)}{w'_{d-1}(y^{i_1}, x^{i_1}, i_2)}}.$$  \(1\)

The requirement (2) is obviously satisfied. It remains to show that the maximum load is at most $v_1 v_{d-1}$. We start by calculating the total weight $wt_d(x)$. First, split the sum of all $w_d(x, y)$ into sums of $w_d(x, y)$ over $y$ with a fixed $z = \bar{y}$.

Claim 1

$$\sum_{y \in \{0,1\}^{n^d}: \bar{y} = z} w_d(x, y) = w_1(\bar{x}, z) \prod_{j=1}^{n} wt_{d-1}(x^j).$$

Proof: Let $y$ be such that $\bar{y} = z$. Then,

$$w_d(x, y) = w_1(\bar{x}, z) \prod_{j: \bar{x}_j = z_j} wt_{d-1}(x^j) \prod_{j: \bar{x}_j \neq z_j} wt_{d-1}(x^j, y^j)$$

When $\bar{x}_j \neq z_j$, $y^j$ can be equal to any $y' \in \{0,1\}^{n^d-1}$ such that $g^{d-1}(y') = z_j$. Therefore, the sum of all $w_d(x, y)$, $\bar{y} = z$ is

$$w_1(\bar{x}, z) \prod_{j: \bar{x}_j = z_j} wt_{d-1}(x^j) \prod_{j: \bar{x}_j \neq z_j} \left( \sum_{y' \in \{0,1\}^{n^d-1}, g^{d-1}(y') = z_j} wt_{d-1}(x^j, y') \right). \ (3)$$

Each of sum in brackets is equal to $wt_{d-1}(x^j)$. Therefore, (3) equals

$$w_1(\bar{x}, z) \prod_{j=1}^{n} wt_{d-1}(x^j).$$

Corollary 1

$$wt_d(x) = w_1(\bar{x}) \prod_{j=1}^{n} wt_{d-1}(x^j). \quad (4)$$
Claim 2

Next, we calculate the load

\[ v(x, i) = \sum_{y \in \{0,1\}^m} w_d'(x, y, i) \]

in a similar way. We start by fixing \( z = \tilde{y} \) and all variables in \( y \) outside the \( i_1 \)th block. Let \( W \) be the sum of \( w_d(x, y) \) and \( V \) be the sum of \( w_d'(x, y, i) \), over \( y \) that have \( \tilde{y} = z \) and the given values of variables outside \( y^{i_1} \).

**Proof:** Fixing \( z \) and the variables outside \( y^{i_1} \) fixes all terms in \( w_d(x, y) \), except \( w_{d-1}(x^{i_1}, y^{i_1}) \). Therefore, \( w_d(x, y) = Cw_{d-1}(x^{i_1}, y^{i_1}) \) where \( C \) is fixed. This means \( W = Cw_{d-1}(x^{i_1}) \). Also,

\[ w_d'(x, y, i) = Cw_{d-1}(x^{i_1}, y^{i_1}) \cdot \left( \frac{w_{d-1}'(x^{i_1}, y^{i_1}, i_2)}{w_{d-1}'(y^{i_1}, x^{i_1}, i_2)} \right) \left( \frac{w_1'(\tilde{x}, \tilde{y}, i_1)}{w_1'(\tilde{y}, \tilde{x}, i_1)} \right). \]

Property \( \mathbb{P} \) of the scheme for \( (A_{d-1}, B_{d-1}, R_{d-1}) \) implies

\[ w_{d-1}(x^{i_1}, y^{i_1}) \cdot \left( \frac{w_{d-1}'(x^{i_1}, y^{i_1}, i_2)}{w_{d-1}'(y^{i_1}, x^{i_1}, i_2)} \right) \leq w_{d-1}'(x^{i_1}, y^{i_1}, i_2), \]

\[ w_d'(x, y, i) \leq Cw_{d-1}'(x^{i_1}, y^{i_1}, i_2) \left( \frac{w_1'(\tilde{x}, \tilde{y}, i_1)}{w_1'(\tilde{y}, \tilde{x}, i_1)} \right). \]

If we sum over all possible \( y^{i_1} \in \{0,1\}^{n_{d-1}} \), we get

\[ V \leq C v_{d-1}(x^{i_1}, i_2) \left( \frac{w_1'(\tilde{x}, \tilde{y}, i_1)}{w_1'(\tilde{y}, \tilde{x}, i_1)} \right). \]

Since \( v_{d-1}(x^{i_1}, i_2) \leq v_{d-1} wt_{d-1}(x^{i_1}) \), we have

\[ V \leq C v_{d-1} wt_{d-1}(x^{i_1}) \left( \frac{w_1'(\tilde{x}, \tilde{y}, i_1)}{w_1'(\tilde{y}, \tilde{x}, i_1)} \right) \left( \frac{w_1'(\tilde{x}, \tilde{y}, i_1)}{w_1'(\tilde{y}, \tilde{x}, i_1)} \right) W. \]
We now consider the part of $v(x, i)$ generated by $w'_d(x, y, i)$ with a fixed $\tilde{y}$. By the argument above, it is at most $v_{d-1} \sqrt{\frac{w'_d(\tilde{x}, \tilde{y}, i_1)}{w'_1(\tilde{y}, \tilde{x}, i_1)}}$ times the sum of corresponding $w_d(x, y)$. By Claim 1, this sum is $w_1(\tilde{x}, z) \prod_{j=1}^{n} w_{t_d-1}(x^j)$. By summing over all $\tilde{y}$, we get

$$v(x, i) \leq \sum_{z \in \{0,1\}^n} v_{d-1} \sqrt{\frac{w'_d(\tilde{x}, z, i_1)}{w'_1(z, \tilde{x}, i_1)}} w_1(\tilde{x}, z) \prod_{j=1}^{n} w_{t_d-1}(x^j)$$

$$= v_{d-1} \prod_{j=1}^{n} w_{t_d-1}(x^j) \sum_{z \in \{0,1\}^n} \sqrt{\frac{w'_d(\tilde{x}, z, i_1)}{w'_1(z, \tilde{x}, i_1)}} w_1(\tilde{x}, z)$$

(5)

By property 2, $\sqrt{\frac{w'_d(\tilde{x}, z, i_1)}{w'_1(z, \tilde{x}, i_1)}} w_1(\tilde{x}, z) \leq w'_1(\tilde{x}, z, i_1)$. Therefore,

$$\sum_{z \in \{0,1\}^n} \sqrt{\frac{w'_d(\tilde{x}, z, i_1)}{w'_1(z, \tilde{x}, i_1)}} w_1(\tilde{x}, z) \leq \sum_{z \in \{0,1\}^n} w'_1(\tilde{x}, z, i_1) = v(\tilde{x}, i_1) \leq v_1 wt(\tilde{x})$$

and (5) is at most

$$v_{d-1} \prod_{j=1}^{n} w_{t_d-1}(x^j) v_1 wt(\tilde{x}) = v_1 v_{d-1} wt_d(x)$$

By induction, $v_d \leq (v_1)^d$. This proves lemma 2.

4.3 Lemma 3

We now look at the base function $f$ in more detail. The function $f$ is shown in Figure 1. The vertices of the two cubes correspond to $(x_1, x_2, x_3, x_4) \in \{0,1\}^4$. Black circles indicate that $f(x_1, x_2, x_3, x_4) = 1$. Thick lines connect pairs of black vertices that are adjacent (i.e., $x_1x_2x_3x_4$ and $y_1y_2y_3y_4$ differing in exactly one variable with $f(x_1, x_2, x_3, x_4) = 1$ and $f(y_1, y_2, y_3, y_4) = 1$).

From the figure, we can observe several properties. Each black vertex ($f = 1$) has exactly two black neighbors and two white neighbors. Each white vertex ($f = 0$) also has two white and two black neighbors. Thus, for every $x \in \{0,1\}^4$, there are two variables $x_i$ such that changing $x_i$ changes $f(x)$. We call these two sensitive variables and the other two insensitive. From Figure 1 we also see that, for any $x \in \{0,1\}^4$, flipping both sensitive variables changes $f(x)$ and flipping both insensitive variables also changes $f(x)$. 

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Thus, the sensitivity of $f$ is 2 on every input. The block sensitivity is 3 on every input, with each of the two sensitive variables being one block and the two insensitive variables together forming the third block.

Finally, $D(f) = 3$. The algorithm queries $x_1$ and $x_3$. After both of those are known, the function depends only on one of $x_2$ and $x_4$ and only one more query is needed. The lower bound follows from $bs(f) = 3$.

We now proceed to proving the lemma. In terms of weight schemes, the lemma is

**Lemma 3.** The function $f$ has a weight scheme with the maximum load of 2.5.

**Proof:** Let $A = f^{-1}(0)$, $B = f^{-1}(1)$. $R$ consists of all $(x, y)$ where $x \in A$ and $y$ differs from $x$ in exactly

- one of the sensitive variables or
- both sensitive variables or
- both insensitive variables.

Thus, for every $x \in A$, there are four inputs $y \in B$ such that $(x, y) \in R$. Also, for every $y \in B$, there are four inputs $x \in A$ such that $(x, y) \in R$ and again, these are $x$ differing from $y$ in one sensitive variable, both sensitive variables or both insensitive variables. Notice that, if $y$ differs from $x$ in both variables that are insensitive for $x$, then those variables are sensitive for $y$ and conversely. (By flipping one of them in $y$, we get to an input $z$ which differs from $x$ in the other variable insensitive to $x$. Since the variable is insensitive for $x$, $f(x) = f(z)$. Together with $f(x) \neq f(y)$, this implies $f(y) \neq f(z)$. )
Let $w(x, y) = 1$ for $(x, y) \in R$ with $x, y$ differing in one variable and $w(x, y) = 2/3$ if $x, y$ differ in two variables. Thus, $wt(x) = 2 \cdot 1 + 2 \cdot \frac{2}{3} = \frac{10}{3}$ for all $x$. $w'(x, y, i)$ is

- $1$ if $x$ and $y$ differ in one variable,
- $\frac{1}{3}$ if they differ in both variables sensitive for $x$,
- $\frac{4}{3}$ if they differ in both variables insensitive for $x$.

Since $\frac{1}{3} \cdot \frac{4}{3} = \left(\frac{2}{3}\right)^2$, this is a correct weight scheme.

We now calculate the load of $i$. There are two cases.

1. $x$ is insensitive to flipping $x_i$. Then, the only input $y$ such that $(x, y) \in R$ and $x_i \neq y_i$ is obtained by flipping both insensitive variables. It contributes $\frac{4}{3}$ to $v(x, i)$.

2. $x$ is sensitive to flipping $x_i$. Then, there are two inputs $y$: one obtained by flipping just this variable and one obtained by flipping both sensitive variables. The load is $v(x, i) = 1 + \frac{1}{3} = \frac{4}{3}$.

Thus, we get $\frac{wt(x)}{v(x, i)} = \frac{10}{4} = 2.5$ for all $x, i$.

4.4 Theorem 2

Theorem 2 now follows from Lemmas 1, 2, 3. By Lemma 3, the function $f$ has a weight scheme with the maximum load of $2.5$. Together with Lemma 2, this implies that $f^d$ has a weight scheme with the maximum load of $2.5^d$. By Lemma 1, this means that $Q_2(f) = \Omega(2.5^d)$.

5 Other base functions

Iterated functions similar to ours have been studied before. Nisan and Wigderson [24] used them to show a gap between communication complexity and log rank (an algebraic quantity that provides a lower bound on communication complexity). Buhrman and de Wolf [13] proposed to study the functions from [24] to find out if polynomial degree of a function characterizes its quantum complexity. However, the base functions that [24, 13] considered are different from ours.

We now consider the functions from [24, 13]. Our method shows the gaps between $\text{deg}(f)$ and $Q_2(f)$ for those functions as well but those gaps are considerably smaller than for our new base function.
**Function 1** \[23\] \[24\]. \( g(x_1, x_2, x_3) \) is 0 iff all variables are equal. We have \( \deg(g) = 2 \) (as witnessed by \( g = x_1 + x_2 + x_3 - x_1 x_2 - x_1 x_3 - x_2 x_3 \)), and \( D(g) = 3 \).

**Lemma 6** \( g \) has a weight scheme with max load \( \sqrt{2}/3 \).

**Proof:** Let \( A = g^{-1}(0), B = g^{-1}(1), R = A \times B \). We set \( w(x, y) = 2 \) if \( x, y \) differ in one variable and \( w(x, y) = 1 \) if \( x \) and \( y \) differ into two variables. (Notice that \( x \) and \( y \) cannot differ in all three variables because that would imply \( g(x) = g(y) \).)

The total weight \( wt(x) \) is

1. \( 3 \cdot 2 + 3 \cdot 1 = 9 \) for \( x \in A \) (since there are three ways to choose one variable and three ways to choose two variables and every way of flipping one or two variables changes the value).

2. \( 2 + 1 = 3 \) for \( x \in B \). (Each such \( x \) has two variables equal and third different. It is involved in \( w(y, x) \) with \( y \) obtained by flipping either the different variable or both equal variables.)

Let \( x \in A, y \in B \). If \( x, y \) differ in one variable \( x_i \), we define \( w'(x, y, i) = 2\sqrt{2} \) and \( w'(y, x, i) = \sqrt{2} \). If \( x, y \) differ in two variables, \( w'(x, y, i) = \sqrt{2}/2 \) and \( w'(y, x, i) = \sqrt{2} \) for each of those variables.

The load of \( i \) in \( x \) is:

1. \( g(x) = 0 \).

   We have to add up \( w'(x, y, i) \) with \( y \) differing from \( x \) either in \( x_i \) only or in \( x_i \) and one of other two variables. We get \( 2\sqrt{2} + 2 \cdot (\sqrt{2}/2) = 3\sqrt{2} \).

2. \( g(x) = 1 \).

   Then, there is only one input \( y \). It can differ in just \( x_i \) or \( x_i \) and one more variable. In both cases, \( w'(x, y, i) = \sqrt{2} \).

We have \( v_A = \frac{3\sqrt{2}}{9} = \frac{\sqrt{2}}{3} \) and \( v_B = \frac{\sqrt{2}}{3} \). Therefore, \( v_{max} = \frac{\sqrt{2}}{3} \).

This means that \( Q_2(g^d) = \Omega(\frac{3}{\sqrt{2}}^d) = \Omega(2.12^d) \).

**Function 2** (Kushilevitz, quoted in \[24\]). The function \( h(x) \) of 6 variables is defined by

1. \( h(x) = 0 \) if the number of \( x_i = 1 \) is 0, 4 or 5,
2. \( h(x) = 1 \) if the number of \( x_i = 1 \) is 1, 2 or 6,
if the number of \( x_i = 1 \) is 3, \( h(x) = 0 \) in the following cases: \( x_1 = x_2 = x_3 = 1, x_2 = x_3 = x_4 = 1, x_3 = x_4 = x_5 = 1, x_4 = x_5 = x_6 = 1, x_5 = x_1 = 1, x_1 = x_2 = 1, x_1 = x_3 = x_6 = 1, x_1 = x_4 = x_6 = 1, x_2 = x_4 = x_6 = 1, x_2 = x_5 = x_6 = 1, x_3 = x_5 = x_6 = 1 \) and 1 otherwise.

We have \( \deg(h) = 3 \) and \( D(h) = 6 \).

**Lemma 7** \( h \) has a weight scheme with max load \( 4/\sqrt{39} \).

**Proof:** We choose \( A \) to consist of inputs \( x \) with all \( x_i = 0 \) and those inputs \( x \) with three variables \( x_i = 1 \) which have \( h(x) = 0 \). \( B \) consists of all inputs \( x \) with exactly one variable equal to 1. \( R \) consists of \( (x, y) \) such that \( y \) can be obtained from \( x \) by flipping one variable if \( x = 0^6 \) and two variables if \( x \) contains three \( x_i \).

If \( x = 0^6 \) and \( y \in B \), we set \( w(x, y) = w'(x, y, i) = w'(y, x, i) = 1 \).

If \( x \) has three variables \( x_i = 1 \) and \( y \) is obtained by switching two of those to 0, we set \( w(x, y) = 1/8, w'(x, y, i) = 1/32 \) and \( w'(y, x, i) = 1/2 \).

To calculate the maximum loads, we consider three cases:

1. \( x = 0^6 \).

\( wt(x) = 6 \) and \( v(x, i) = 1 \) for all \( i \).

2. \( x \) has three variables \( x_i = 1 \).

Then, there are three pairs of variables that we can flip to get to \( y \in B \). Thus, \( wt(x) = 3/8 \). Each \( x_i = 1 \) gets flipped in two of those pairs. Therefore, its load is \( v(x, i) = 2 \cdot 1/32 = 1/16 \). The ratio \( \frac{wt(x)}{v(x, i)} \) is 6.

3. \( y \) has 1 variable \( y_i = 1 \).

Then, we can either flip this variable or one of 5 pairs of \( y_i = 0 \) variables to get to \( x \in A \). The weight is \( wt(y) = 1 + 5 \cdot 1/8 = 13/8 \). If \( y_i = 1 \), then the only input \( x \in A, (x, y) \in R \) with \( x_i \neq y_i \) is \( x = 0^6 \) with \( w'(y, x, i) = 1 \). Thus, \( v(y, i) = 1 \). If \( y_i = 0 \), then exactly two of 5 pairs of variables \( j : y_j = 0 \) include the \( i \)th variable. Therefore, \( v(y, i) = 2 \cdot 1/2 = 1 \).

Thus, \( v_A = 1/6, v_B = 8/13 \) and \( v_{\text{max}} = 2/\sqrt{39} \).

This gives a \( 3^d \) vs. \( \Omega((\sqrt{39}/2)^d) = \Omega(3.12...^d) \) gap between polynomial degree and quantum complexity.

### 6 Conclusion

An immediate open problem is to improve our quantum lower bounds or to find quantum algorithms for our iterated functions that are better than classical by more than a constant factor. Some other related open problems are:
1. **AND-OR tree.** Let

\[ f(x_1, \ldots, x_4) = (x_1 \land x_2) \lor (x_3 \land x_4). \]

We then iterate \( f \) and obtain a function of \( N = 4^n \) variables that can be described by a complete binary tree of depth \( \log_2 N = 2n \). The leaves of this tree correspond to variables. At each non-leaf node, we take the AND of two values at its two children nodes at even levels and OR of two values at odd levels. The value of the function is the value that we get at the root. Classically, any deterministic algorithm has to query all \( N = 4^n \) variables.

For probabilistic algorithms, \( N^{0.753\ldots} = (1 + \sqrt[4]{33})^2n \) queries are sufficient and necessary \([26, 27, 30]\). What is the quantum complexity of this problem? No quantum algorithm that uses less than \( N^{0.753\ldots} = (1 + \sqrt[4]{33})^2n \) queries is known but the best quantum lower bound is just \( \Omega(N^{0.5}) = \Omega(2^n) \).

A related problem that has been recently resolved concerns AND-OR trees of constant depth. There, we have a similar \( N^{1/d} \)-ary tree of depth \( d \). Then, \( \Theta(\sqrt{N}) \) quantum queries are sufficient \([11, 17]\) and necessary \([4, 7]\). The big-O constant depends on \( d \) and the number of queries in the quantum algorithm is no longer \( O(\sqrt{N}) \) if the number of levels is non-constant. Curiously, it is not known whether the polynomial degree is \( \Theta(\sqrt{N}) \), even for \( d = 2 \) \([28]\).

2. **Certificate complexity barrier.** Let \( C_0(f) \) and \( C_1(f) \) be 0-certificate and 1-certificate complexity of \( f \) (cf. \([13]\) for definition). Any lower bound following from theorems of \([4]\) or weight schemes of the present paper is \( O(\sqrt{C_0(f)C_1(f)}) \) for total functions and \( O(\sqrt{\min(C_0(f), C_1(f))N}) \) for partial functions\(^2\) \([19, 33]\).

This has been sufficient to prove tight bounds for many functions. However, in some cases quantum complexity is (or seems to be) higher. For example, the binary AND-OR tree described above has \( C_0(f) = C_1(f) = 2^n \). Thus, improving the known \( \Omega(2^n) \) lower bound requires going above \( \sqrt{C_0(f)C_1(f)} \).

To our knowledge, there is only one known lower bound for a total function which is better than \( \sqrt{C_0(f)C_1(f)} \) (and no lower bounds for partial functions better than \( \sqrt{\min(C_0(f), C_1(f))N} \). This is the \( \Omega(N^{2/3}) \) lower bound\(^2\).

\(^2\)The distinction between partial and total functions is essential here. The methods of \([4]\) and the present paper can be used to prove lower bounds for partial functions that are more than \( \sqrt{C_0(f)C_1(f)} \) but \( O(\sqrt{\min(C_0(f), C_1(f))N}) \). Examples are inverting a permutation \([4]\) and local search \([1]\).
of Shi [2, 18, 5] for element distinctness, a problem which has $C_0(f) = 2$, $C_1(f) = N$ and $\sqrt{C_0(f)C_1(f)} = \Theta(\sqrt{N})$. It uses methods quite specific to the particular problem and cannot be easily applied to other problems. It would be very interesting to develop more methods of proving quantum lower bounds higher than $O(\sqrt{C_0(f)C_1(f)})$ for total functions or higher than $O(\sqrt{\min(C_0(f), C_1(f))N})$ for partial functions.

3. Finding triangles. A very simple problem for which its true quantum complexity seems to exceed the $\Omega(\sqrt{C_0(f)C_1(f)})$ lower bound is as follows. We have $n^2$ variables describing adjacency matrix of a graph. We would like to know if the graph contains a triangle. The best quantum algorithm needs $O(n^{1.3})$ queries [31, 20] and an $\Omega(n)$ lower bound follows by a reduction from the lower bound on Grover’s search [12] or lower bound theorem of [4]. We have $C_0(f) = O(n^2)$ but $C_1(f) = 3$ (if there is a triangle, its three edges form a 1-certificate), thus $\Omega(n)$ is the best lower bound that follows from theorems in [4]. We believe that the quantum complexity of this problem is more than $\Theta(n)$. Proving that could produce new methods applicable to other problems where quantum complexity is more than $O(\sqrt{C_0(f)C_1(f)})$ as well.

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A Appendix: bounds using previous method

In this section, we look at what bounds can be obtained for $Q_2(f^d)$ for $f^d$ defined in section 3.1 using the previously known lower bound Theorem 3.

It can be verified that the block sensitivity of $f$ is 3 on every input. By induction, we can show that this implies $bs_x(f^d) = 3^d$ for every input $x \in \{0, 1\}^{4^d}$. This makes it tempting to guess that we can achieve $m = m' = 3^d$ and $l = l' = 1$ which would give a lower bound of $\Omega(3^d)$.

This is not the case. If we would like to use Theorem 3 with $l = l' = 1$, we need two requirements simultaneously:

1. For every $x \in A$, denote by $y_1, \ldots, y_{3^d}$ the elements of $B$ for which $(x, y_i) \in R$. Then, the sets of variables where $(x, y_i)$ and $(x, y_j)$ differ must be disjoint for all $i, j, i \neq j$.

2. For every $y \in B$, denote by $x_1, \ldots, x_{3^d}$ the elements of $A$ for which $(x_i, y) \in R$. Then, the sets of variables where $(x_i, y)$ and $(x_j, y)$ differ must be disjoint for all $i, j, i \neq j$.

If block sensitivity is $3^d$ on every input, we can guarantee the first requirement (by starting with $x \in A$ constructing disjoint $S_1, \ldots, S_{3^d}$ and putting $(x, x^{S_i})$ into $R$). But, if the set $A$ only contains one $x$, then $m' = 1$ and the lower bound is $\Omega(\sqrt{3^d})$ which is even worse than the previous one.

Therefore, we have to take larger set $A$. This can break the second requirement. Let $x, z \in A$ and $y \in B$. Then, we could have $(x, y) \in R$ and $(z, y) \in R$. $x$ and $y$ would differ in a set of variables $S_i$ which is one of $3^d$ disjoint blocks for $x$. Similarly, $z$ and $y$ would differ in a set $T_j$ which is one of $3^d$ disjoint blocks for $z$. Now, there is no reason why $S_i$ and $T_j$ have to be disjoint! Block sensitivity guarantees that $S_i \cap S_j = \emptyset$ for every fixed $x$ but it gives no guarantees about blocks for $x$ being disjoint from blocks for $z$.

Similarly, if we start with $y \in B$, we can ensure the second requirement but not the first.
The best that we could achieve with this approach was $m = m' = 3^d$, $l = 1$, $l' = 2^d$, as follows. Let $A = f^{-1}(0)$, $B = f^{-1}(1)$. We inductively construct two sets of $3^d$ disjoint perfect matchings between inputs in $A$ and inputs in $B$.

The first set of matchings consists of ordered pairs $(x, y)$, $x \in A$, $y \in B$. For $d = 1$, the first two matchings match each input $x \in A$ to the two inputs $y \in B$ that differ in exactly one variable. The first matching is $(0011, 0001)$, $(0101, 1101)$, $(1100, 1110)$, $(1010, 0010)$, $(0100, 1100)$, $(1000, 0000)$, $(0111, 1111)$, $(1011, 1001)$. The second matching matches each $x \in A$ to the other $y \in B$ which differs in exactly one variable. The third matching matches each $x \in A$ to $y \in B$ which differs from $x$ in both variables that are sensitive for $x$. This is the first set of 3 matchings.

The second set of matchings consists of ordered pairs $(y, x)$, $y \in B$, $x \in A$. The first two matchings are the same as in the first set. The third matching matches each $x \in A$ to $y \in B$ which differs from $x$ in both variables that are sensitive for $y$.

For $d > 1$, we introduce notation $x^1$, $x^2$, $x^3$, $x^4$ and $\tilde{x}$ similarly to section 4.2. The first $3^{d-1}$ matchings are constructed as follows. For each $x$, we find $\tilde{x}$. Then, we find $\tilde{y}$ such that $(\tilde{x}, \tilde{y})$ belongs to the first matching in the first set. Let $i$ be the variable for which $\tilde{x}_i \neq \tilde{y}_i$. In the $k^{th}$ matching $(1 \leq k \leq 3^{d-1})$, we match each $x \in A$ to $y \in B$ which is defined as follows:

- If $j \neq i$, then $x^j = y^j$.
- $x^i$ is such that $(x^i, y^i)$ belongs to the $k^{th}$ matching for $d - 1$ levels (taking matchings from the first set if $f(x^i) = 0$ and the second set if $f(y^i) = 1$).

The second $3^{d-1}$ matchings are constructed similarly, except that we use $\tilde{y}$ for which $(\tilde{x}, \tilde{y})$ belongs to the second matching of the first set.

To construct the last $3^{d-1}$ matchings, we take $\tilde{y}$ for which $(\tilde{x}, \tilde{y})$ belongs to the third matching. In $2 \times 3^d + k^{th}$ matching, we match $x$ with $y$ defined as follows.

- if $\tilde{x}_i \neq \tilde{y}_i$, then $y^i$ is the input of length $x^i$ for which $(x^i, y^i)$ belongs to the $k^{th}$ matching for $d - 1$ levels.
- if $\tilde{x}_i = \tilde{y}_i$, then $y^i = x^i$.

We then define $R$ as the set of $(x, y)$ which belong to one of the $3^d$ matchings we constructed. By induction, we show

**Lemma 8** For the first set of $3^d$ matchings, $m = m' = 3^d$, $l = 1$, $l' = 2^d$. For the second set of $3^d$ matchings, $m = m' = 3^d$, $l = 1$, $l' = 2^d$.

**Proof:** First, we prove $m = m' = 3^d$. In the base case, we can just check that the matchings are distinct and, thus, every $x \in A$ or $y \in B$ is matched to 3 distinct
elements of the other set. In the inductive case, consider an element \( x \in A \) (or \( y \in B \)) and two elements \( y_1 \in B \) and \( y_2 \in B \) to which it is matched. If \((x, y_1)\) and \((x, y_2)\) belong to two matchings in the same group of \( 3^{d-1} \) matchings, then, by the inductive assumption \( y'_1 \neq y'_2 \) and, hence, \( y_1 \neq y_2 \). If \((x, y_1)\) and \((x, y_2)\) belong to two matchings in different groups, then \( y'_1 \neq y'_2 \) implies \( y_1 \neq y_2 \).

To prove \( l = 1 \) (or \( l' = 1 \) for the second set), we first observe that this is true in the base case. For the inductive case, we again have two cases. If \((x, y_1)\) and \((x, y_2)\) belong to different sets of \( 3^{d-1} \) matchings, then, for each \( i \in \{1, 2, 3, 4\} \), either \( x_i = y'_1 \) or \( x_i = y'_2 \). This means that only one of \( y_1 \) and \( y_2 \) can differ from \( x \) in a variable belonging to \( x^1 \). If \((x, y_1)\) and \((x, y_2)\), we apply the inductive assumption to \((x^1, y'_1)\) and \((x^1, y'_2)\).

To prove \( l' = 2 \) in the base case, we notice that, if \((x_1, y)\) and \((x_2, y)\) belong to the first and the second matching, then the pairs \((x_1, y)\) and \((x_2, y)\) cannot differ in the same variable. In the inductive case, for every \( i \in \{1, 2, 3, 4\} \), either \( x_{1i} = y_i \) or \( x_{2i} = y_i \). If we have a variable \( j \) such that \( j \in \{(i - 1)4^{d-1} + 1, (i - 1)4^{d-1} + 2, \ldots, i \times 4^{d-1}\} \) and \( x_{1j} = y_i \), then \((x, y) \in R \) and \( x_j \neq y_j \) means that \((x, y)\) belongs to either one of the second \( 3^{d-1} \) matchings or one of the last \( 3^{d-1} \) matchings. By applying the inductive assumption, there are at most \( 2^{d-1} \) such \((x, y)\) in each of the two sets of \( 3^{d-1} \) matchings. This gives a total of at most \( 2 \times 2^{d-1} = 2^d \) such pairs \((x, y)\).

The weakness of Theorem 3 that we see here is that all variables get treated essentially in the same way. For each \( y \in B \), different variables \( y_i \) might have different number of \( x \in A \) such that \((x, y) \in R \), \( x_i \neq y_i \). Theorem 3 just takes the worst case of all of those (the maximum number). Our weight schemes allow to allocate weights so that some of load gets moved from variables \( i \) which have lots of \( x \in A \): \((x, y) \in R \), \( x_i \neq y_i \) to those which have smaller number of such \( x \in A \). This results in better bounds.

For the function of section 3.1, we get \( \Omega(2.12^d) \) by old method and \( \Omega(2.5^d) \) by the new method. For the two functions in section 5 the old method only gives bounds that are lower than polynomial degree while the new method shows that \( Q_2(f) \) is higher than \( \deg(f) \) for those functions as well.