A mixed finite element for weakly-symmetric elasticity

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Abstract

We develop a finite element discretization for the weakly symmetric equations of linear elasticity on tetrahedral meshes. The finite element combines, for \( r \geq 0 \), discontinuous polynomials of order \( r \) for the displacement, \( H(\text{div}) \)-conforming polynomials of order \( r + 1 \) for the stress, and \( H(\text{curl}) \)-conforming polynomials of order \( r + 1 \) for the vector representation of the multiplier. We prove that this triplet is stable and has optimal approximation properties. The lowest order case can be combined with inexact quadrature to eliminate the stress and multiplier variables, leaving a compact cell-centered finite volume scheme for the displacement.

1 Introduction

1.1 Symmetric linear elasticity

We consider the equations of linear elasticity in mixed first-order form,

\[
A\sigma = \text{sym grad } u, \quad \text{div } \sigma = g, \quad x \in \Omega,
\]

\[
u = 0, \quad x \in \partial \Omega. \tag{1}
\]

Here \( \sigma \) is the stress tensor, \( u \) is the displacement, \( A \) is the compliance tensor, and \( \text{sym } G := \frac{1}{2}(G + G^T) \). These equations have the well-posed variational form:

\[
\int_{\Omega} (A \sigma : \tau + u \cdot \text{div } \tau) + (\text{div } \sigma - g) \cdot v \; dx = 0, \quad \tau \in H(\text{div}; S), \quad v \in L^2(V). \tag{2}
\]

Here \( S \) is the symmetric subspace of \( \mathbb{M} \), the space of \( 3 \times 3 \) matrices, \( V \) is \( \mathbb{R}^3 \), and \( H(\text{div}; S) \) and \( L^2(V) \) have the usual meanings. We define the bilinear forms

\[
a(\sigma, \tau) := \int_{\Omega} A \sigma : \tau \; dx, \quad b(\tau, v) := \int_{\Omega} \text{div } \tau \cdot v \; dx.
\]

The well-posedness of (2) is guaranteed by the coercivity of \( a \) on the kernel of \( b \) and the fact that \( b \) satisfies the Ladyzhenskaya-Babuška-Brezzi (LBB) stability condition [14, 17, 18]. We will use \( \ker(f, X) \) to refer to the kernel of operator \( f \) on space \( X \), and \( \ker(g, X; Y) \) to refer to the kernel of the bilinear form \( g : X \times Y \to \mathbb{R} \) interpreted as an operator in \( \mathcal{L}(X; Y^*) \). We thus state the well-posedness of (2) as the existence of constants \( \alpha > 0 \) and \( \beta > 0 \) such that,

\[
\inf_{\tau \in \ker(b, H(\text{div}; S); L^2(V))} a(\tau, \tau) \geq \alpha \|\tau\|_{\text{div}}^2,
\]

\[
\inf_{v \in L^2(V)} \sup_{\tau \in H(\text{div}; S)} b(\tau, v) \geq \beta \|\text{div } v\|_0.
\]

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We use \( \| \cdot \|_{(k, \text{div}, \text{curl}) \Omega} \) for the \( H^k(\Omega) \), \( H(\text{div}; \Omega) \), and \( H(\text{curl}; \Omega) \) norms, dropping \( \Omega \) where it is clear from context, and \( | \cdot |_{(k, \text{div}, \text{curl}) \Omega} \) for their seminorms.

Equation \( (1) \) conserves both linear momentum \( (\text{div} \sigma = g) \) and angular momentum \( (\sigma = \sigma^T) \). Finite elements that conserve both pointwise have recently been developed \([1, 3, 4, 11, 12, 13]\). These discretizations are either high order \([1, 3, 12]\), nonconforming \([4, 11]\), or require bubble \( \sigma \) variational equations, in which the symmetry of \( K \) vec:

\[
\sigma \quad \text{is enforced by a Lagrange multiplier } \hat{G}. \quad \text{Henceforth we name the function spaces } \Sigma := H(\text{div}; M), \quad V := L^2(\Sigma) \quad \text{and } \quad Q := L^2(\mathbb{K}). \quad \text{We assume that the original compliance tensor } A : S \to S \text{ is extended continuously to } A : M \to M \text{ so that } a \text{ is coercive on } \ker(\text{div}, \Sigma). \quad \text{We define the bilinear form } c(\tau, q) := \int_\Omega \text{skw} \tau : q \; dx, \text{ where skw } G := \frac{1}{2}(G - G^T), \text{ which lets us rewrite } (3) \text{ as}
\]

\[
a(\sigma, \tau) + \{b(\tau, u) + c(\tau, p)\} + \{b(\sigma, v) + c(\sigma, q)\} = (g, v), \quad (\tau, v, q) \in \Sigma \times V \times Q.
\]

On the one hand, weakly-imposed symmetry is a simplification because \( b \) is now a vector-valued version of the familiar bilinear operator on \( H(\text{div}; V) \times L^2(\mathbb{R}) \) that originally motivated the development of the LBB condition, and we have well-established choices of discrete spaces that satisfy LBB stability for this operator. On the other hand, this is a complication because LBB stability must be satisfied with respect to a larger constraint space, \( V \times Q \).

**Proposition 1.1.** There exists \( \gamma > 0 \) such that
\[
\inf_{(v, q) \in V \times Q} \sup_{\tau \in \Sigma} b(\tau, v) + c(\tau, q) \geq \gamma \| \tau \|_{\text{div}} (\| v \|_0 + \| q \|_0). \quad \square
\]

It is simple to check that proposition 1.1 is equivalent to the following two proposition.

**Proposition 1.2.** There exists \( \hat{\gamma} > 0 \) such that \( \inf_{v \in V} \sup_{\tau \in \Sigma} b(\tau, v) \geq \hat{\gamma} \| \tau \|_{\text{div}} \| v \|_0. \quad \square
\]

**Proposition 1.3.** There exists \( \hat{\gamma} > 0 \) such that \( \inf_{q \in Q} \sup_{\tau \in \ker(b, \Sigma; V)} c(\tau, q) \geq \hat{\gamma} \| \tau \|_{\text{div}} \| q \|_0. \)

This split is a useful form of proposition 1.2 because, as mentioned above, proposition 1.2 is a vector-valued generalization of a well established result, and because we know curl\([H(\text{curl}; M)] \subseteq \ker(b, \Sigma; V)\). We provide a brief sketch of a proof of proposition 1.3 because it illustrates a method used to prove the stability of finite element approximations of (1). The proof uses the operators vec : \( K \to V \) and \( \Xi : M \to M \) defined by
\[
(\text{vec} \; A) \times b := A b, \quad \Xi A := A^T - \text{tr}(A)I.
\]
which have the following identities for sufficiently smooth fields:

\[ \Xi^{-1} A = A^T - \frac{1}{2} \text{tr}(A) I, \]
\[ \text{div skw } A = - \text{curl vec skw } A, \]  
\[ \text{div } \Xi \mu = 2 \text{vec skw } \mu. \]  

We state the following proposition about these two operators.

**Proposition 1.4.** \( \Xi \) induces a bounded linear bijection of \( H^1(\mathcal{M}) \) onto itself.

**Proposition 1.5.** \( \Xi \) induces a bounded linear map of \( H(\text{curl}; \mathcal{M}) \) into \( H(\text{div}; \mathcal{M}) \).

**Proposition 1.6.** vec induces an isomorphism between \( H(\text{div}; \mathcal{K}) \) and \( H(\text{curl}; \mathcal{V}) \).

**Proof of proposition 1.4.** Let \( q \in Q \) be given. There exists \( \nu \in H^1(\mathcal{M}) \) and a constant \( C_1 \) independent of \( q \) such that \( \text{div } \nu = \text{vec } q \) such that \( \| \nu \|_1 \leq C_1 \| \text{vec } q \|_0 \). By proposition 1.3, \( \mu := \Xi^{-1} \nu \in H^1(\mathcal{M}) \subset H(\text{curl}; \mathcal{M}) \) and thus \( \text{curl } \mu \in \ker(\mathcal{b}, \Sigma; \mathcal{V}) \). Proposition 1.3 also implies there exists \( C_2 \) such that

\[ \| \text{curl } \mu \|_{\text{div}} = \| \text{curl } \mu \|_0 \leq C_2 \| \nu \|_1 \leq C_1 C_2 \| \text{vec } q \|_0. \]

By (3),

\[ c(\text{curl } \mu, q) = \int_{\Omega} \text{skw curl } \mu : q \, dx = \int_{\Omega} 2 \text{vec skw curl } \mu \cdot \text{vec } q \, dx = \int_{\Omega} \text{div } \mu \cdot \text{vec } q \, dx, \]

and thus

\[ c(\text{curl } \mu, q) = \| \text{vec } q \|_0^2 \geq \frac{1}{\sqrt{2}C_1 C_2} \| \text{curl } \mu \|_{\text{div}} \| q \|_0. \]

The relationship between vec and \( \Xi \) in (3) is fundamental to the embedding of the elasticity complex into the de Rham complex via the Bernstein-Gelfand-Gelfand (BGG) resolution of the rigid-body motions. We refer the reader interested in more background on this result to the works of Eastwood [9] and Arnold, Falk, and Winther [6]. The latter authors use the BGG resolution to design stable discretizations of (3) by choosing two finite element de Rham subcomplexes, one for \( \mathcal{V} \) and one for \( \mathcal{K} \), such that the steps in the proof of proposition 1.3 above can be repeated at the discrete level. In particular, their method requires that there exists an approximate operator \( \Xi_h^{-1} \) mapping from the \( H(\text{div}; \mathcal{M}) \) subspace in the complex for \( \mathcal{K} \) (the proxy for \( \mathcal{K} \)-valued 2-forms) into the \( H(\text{curl}; \mathcal{M}) \) subspace in the complex for \( \mathcal{V} \) (the proxy for \( \mathcal{V} \)-valued 1-forms) for which a discrete version of proposition 1.5 is true. Using this property, the authors prove the stability of the following triplet of discrete subspaces \( \Sigma_h, \mathcal{V}_h, \) and \( Q_h \) (alongside which we give the finite element exterior calculus notation [8] for the equivalent spaces of differential forms):

\[ \Sigma_h := \mathcal{P}_{r+1}(T_h; \mathcal{M}) \cap \Sigma \quad \sim \mathcal{P}_{r+1} \Lambda^2(T_h; \mathcal{V}), \]
\[ \mathcal{V}_h := \mathcal{P}_{r}(T_h; \mathcal{V}) \cap \mathcal{V} \quad \sim \mathcal{P}_{r} \Lambda^3(T_h; \mathcal{V}), \]
\[ Q_h := \mathcal{P}_{r}(T_h; \mathcal{K}) \cap \mathcal{Q} \quad \sim \mathcal{P}_{r} \Lambda^3(T_h; \mathcal{K}). \]

Here \( T_h \) is a tetrahedral mesh of the domain \( \Omega \), i.e. a simplicial complex. We will denote by \( \Delta_k(T_h) \) the \( k \)-dimensional simplices in \( T_h \), by \( h_S \) the diameter of simplex \( S \), and \( h := \max_{S \in T_h} h_S \).

\( \mathcal{P}_r(T_h; \mathcal{K}) \) is the space of functions that are equal to \( \mathcal{K} \)-valued polynomials of degree at most \( r \) when restricted to each \( T \in \Delta_3(T_h) \), and \( \mathcal{P}_r \Lambda^k(T_h; \mathcal{V}) \) are the same functions interpreted instead as \( \mathcal{V} \)-valued \( k \)-forms.

The triplet \( \Sigma_h \times \mathcal{V}_h \times Q_h \) is by no means the only stable discretization of (3); see [6, § 1] for a survey.
1.3 The proposed finite element

In light of the $H(\text{div}; \mathbb{K}) \sim H(\text{curl}; \mathbb{V})$ isomorphism, we propose the triplet $\Sigma_h \times V_h \times \tilde{Q}_h$ for the discretization of (3), where

$$\tilde{Q}_h := P_{r+1}(T_h; \mathbb{K}) \cap H(\text{div}; \mathbb{K}) \sim \text{vec}^{-1}[P_{r+1} \Lambda^1(T_h; \mathbb{R})].$$

The stress and velocity spaces are the same as in the element of Arnold, Falk, and Winther, but the multiplier space $\tilde{Q}_h$ is now $H(\text{div}; \mathbb{K})$-conforming and has the same order as $\Sigma_h$. The discrete version of proposition 1.3 is known to hold when restricted to $\Sigma_h \times V_h$, so to prove the stability of our triplet in discretizing (3), it only remains to prove a discrete version of proposition 1.3, which we do in section 2.

In the lowest order case, our finite element can be efficiently reduced to a generalized displacement method, in fact to a finite volume scheme with pointwise second-order convergence at cell centers. We discuss this aspect of the finite element in section 3.

2 Stability and convergence

In this section we prove the stability and convergence properties of $\Sigma_h \times V_h \times \tilde{Q}_h$ as a discrete setting for (3). Theorem 2.1 is the only component of the proof of stability that has not already been established.

**Theorem 2.1** (c is stable on the kernel of $b$ for $\Sigma_h \times V_h \times \tilde{Q}_h$). Let $T_h$ be a conformal tetrahedral mesh of $\Omega$ that is shape regular, i.e. there exists $C_{\text{mesh}}$ such that $h_T^3 \leq C_{\text{mesh}} \cdot \text{vol} T, T \in T_h$. Then there exists $\gamma(\Omega, C_{\text{mesh}}) > 0$ independent of $h$ such that

$$\inf_{q_h \in \tilde{Q}_h} \sup_{\tau_h \in \text{ker}(b, \Sigma_h; V_h)} c(\tau_h, q_h) \geq \gamma \|\tau_h\|_{\text{div}} \|q_h\|_0.$$  \hspace{1cm} (7)

The proof of theorem 2.1 is similar to the proof of proposition 1.3 in that we will use only a subspace of ker($b, \Sigma_h; V_h$) defined by the curl of another space,

$$M_h := P_{r+2}(T_h; \mathbb{M}) \cap H(\text{curl}; \mathbb{M}) \sim P_{r+2} \Lambda^1(T_h; \mathbb{V}).$$

As $M_h \xrightarrow{\text{curl}} \Sigma_h \xrightarrow{\text{div}} V_h \rightarrow 0$ is a subcomplex of the de Rham complex, curl($M_h$) $\subseteq$ ker($b, \Sigma_h; V_h$).

The proof takes the form of the macroelement method of Stenberg [17 18]. The macroelement method was developed to prove the stability of $b$ on $(H^1_0(\mathbb{V}) \times L^2_0(\mathbb{R}))$-conforming finite elements, but the structure of the method requires few changes to apply to $M_h \times \tilde{Q}_h$.

First we define mesh dependent norms and seminorms:

$$\|q\|_{1,h}^2 := \sum_{T \in \Delta_3(T_h)} h_T^2 \|\text{vec} q\|_{0,T}^2 + \sum_{f \in \Delta_2(T)} h_f \|\text{vec} q\|_{0,f}^2,$$

$$\|\mu\|_{0,h}^2 := \sum_{T \in \Delta_3(T_h)} h_T^2 \|\text{vec} \mu\|_{0,T}^2 + \sum_{f \in \Delta_2(\Xi T_h)} h_f^{-1} \|\text{vec} \mu\|_{0,f}^2.$$

Here $\Xi := \omega^- - \omega^+$ is the jump of $\omega$ across $f = T^- \cap T^+$ where $n$ points from $T^-$ to $T^+$ (and $\omega^+ := 0$ if $f \subset \partial \Omega$).

**Proposition 2.1.** \( |c(\mu, q_h)| \leq \|\mu\|_{0,h} \|q_h\|_{1,h}, \mu \in H(\text{curl}; \mathbb{M}), q_h \in \tilde{Q}_h. \)

**Proof.** Using (5) and integration by parts,

$$c(\text{curl} \mu, q_h) = c(\text{div} \Xi \mu, \text{vec} q_h)$$

\[
= - \sum_{T \in \Delta_3(T_h)} (\Xi \mu, \text{grad} \text{vec} q_h)_T + \sum_{f \in \Delta_2(T_h)} \int_f [\text{vec} q_h]^T (\Xi \mu)n \, ds, \tag{8}
\]
so the bound is an application of the Cauchy-Schwarz inequality.

For each vertex \( v \in \Delta_0(T_h) \) we now define \( T_h^v \) to be the submesh of tetrahedra adjacent to \( v \). The shape regularity constant \( C_{\text{mesh}} \) implies the existence of an upper bound on the number of tetrahedra in \( T_h^v \), \( |\Delta_3(T_h^v)| \leq k(C_{\text{mesh}}) \), so the simplicial complex topology of \( T_h^v \) is the same as one of a finite dimensional set of reference macroelements, \( \{M_h\} \).

We divide the facets of \( T_h^v \) by whether they are adjacent to \( v \) or not,

\[
\Gamma_+^v := \{ f \in \Delta_2(T_h^v) : v \in f \}, \quad \Gamma_-^v := \Delta_2(T_h^v) \setminus \Gamma_+^v,
\]

and we associate with \( T_h^v \) a subspace of \( M_h \) that extends by zero away from it,

\[
\hat{M}_h^v := \{ \mu_h \in M_h : \mu_h(x) = 0, \ x \in \Omega \setminus T_h^v, (\Xi \mu_h)n = 0, f \in \Gamma_-^v \}.
\]

\( \hat{M}_h^v \) is so defined that for any \( q_h \in \hat{Q}_h \) the support of \( c(\mu_h, q_h) \) is just the interior of \( T_h^v \),

\[
c(\text{curl} \mu_h, q_h) = - \sum_{T \in \Delta_3(T_h^v)} \int_{T} \Xi \mu_h : \text{grad} \ q_h \ dx + \sum_{f \in \Gamma_-^v} \int_{f} [\text{vec} \ q_h]^T (\Xi \mu_h)n \ ds, \quad \mu_h \in \hat{M}_h^v.
\]

We restrict \( \| \cdot \|_{1,h}^2 \) to \( T_h^v \) using only the interior facets,

\[
\|q_h\|_{1,h,T_h^v}^2 := \sum_{T \in \Delta_3(T_h^v)} h_T^2 \| \text{vec} \ q_h \|_{H^2(T)}^2 + \sum_{f \in \Gamma_-^v} h_f \| [\text{vec} \ q_h]_{f} \|^2.
\]

**Proposition 2.2.** If \( \|q_h\|_{1,h,T_h^v} = 0 \), then \( q_h \) is constant in \( T_h^v \).

**Proposition 2.3.** There exists \( C \) such that \( \|q_h\|_{1,h}^2 \leq \sum_{v \in \Delta_0(T_h)} \|q_h\|_{1,h,T_h^v}^2 \leq C \|q_h\|_{1,h}^2 \).

Nothing thus far has involved details of the spaces \( M_h \) and \( \hat{Q}_h \). We will use their properties for one key lemma.

**Lemma 2.1.** If \( q_h \in \ker(c, \hat{Q}_h; \hat{M}_h^v) \), then \( q_h \) is constant in \( T_h^v \).

**Proof.** We will first show that if \( q_h \in \ker(c, \hat{Q}_h; \hat{M}_h^v) \) then \( q_h \) is constant in each tetrahedron in \( T_h^v \). Let \( e \) be an edge adjacent to \( v \), let \( v_e \) be the opposite vertex, and let \( t_e \) be the tangent vector of \( e \). We claim that \( \Xi^{-1}(\text{grad vec} \ q_h \cdot t_e \Gamma_e^h) \) is a matrix field that is tangentially continuous across each facet \( f \) adjacent to \( e \). To verify this claim apply a tangent vector \( t_f \) of \( f \) to the field:

\[
\Xi^{-1}(\text{grad vec} \ q_h \cdot t_e \Gamma_e^h)t = (t_e \Gamma_e^h)^T \text{grad vec} \ q_h - \frac{1}{2} \text{tr}(\text{grad vec} \ q_h \cdot t_e \Gamma_e^h) I t = (t_e \Gamma_e^h)^T \text{grad vec} \ q_h - \frac{1}{2} (t_e^T \text{grad vec} \ q_h \cdot t_e) I t = (\text{vec} \ q_h \cdot t_e) t_e - \frac{1}{2} (\text{vec} \ q_h \cdot t_e) I t.
\]

Because \( \text{vec} \ q_h \cdot t_e \) and \( \text{vec} \ q_h \cdot t_f \) are continuous across \( f \), and because the directional derivative \( \partial_{t_e} \) is tangent to \( f \), the resulting vector must be continuous across \( f \).

We define a function \( \mu_h^v \) supported on each tetrahedron \( T \) adjacent to \( e \) by

\[
\mu_h^v|_T := \lambda_v \lambda_{v_e} \Xi^{-1}(\text{grad vec} \ q_h \cdot t_e \Gamma_e^h),
\]

where \( \lambda_{v} \) is the barycentric coordinate of vertex \( w \) in \( K \), which linearly interpolates between 1 at \( w \) and 0 at the facet opposite \( w \). Because \( q_h \) is a polynomial of degree \( r + 1 \), \( \mu_h^v \) is a polynomial of degree \( r + 2 \); by design, \( \mu_h^v \) has tangential continuity across the facets of \( T \) adjacent to \( e \), and vanishes on the others, so \( \mu_h^v \in M_h \); because \( e \) is adjacent to \( v \), every facet \( f \) adjacent to \( e \) is in \( \Gamma_+^v \), so \( \mu_h^v \) vanishes on \( \Gamma_0 \), and thus \( \mu_h^v \in \hat{M}_h^v \).
If \( f \in \Gamma^r_i \) is adjacent to \( e \), then by the fact that the scaling \( \lambda_v \) must commute with the pointwise linear algebra,
\[
\begin{align*}
[\text{vec } q_h]^T (\Xi \lambda_v \lambda_v \Xi^{-1} (\text{grad } q_h t_e t_e^T)) n &= \lambda_v \lambda_v [\text{vec } q_h]^T \text{grad } q_h t_e t_e^T n = 0.
\end{align*}
\]
Therefore, by (9),
\[
0 = c(\text{curl } \mu^e_h, q_h) = - \sum_{T \in \text{adj}(e)} \int_T \Xi \lambda_v \lambda_v \Xi^{-1} \text{grad } q_h t_e t_e^T \text{grad vec } q_h \ dx
= - \sum_{T \in \text{adj}(e)} \int_T \lambda_v \lambda_v (\text{grad vec } q_h t_e t_e^T) : (\text{grad vec } q_h)^T \ dx
= - \sum_{T \in \text{adj}(e)} \int_T \lambda_v \lambda_v |\partial_{ve} \text{vec } q_h|^2 \ dx.
\]
Because we can repeat this construction on every \( e \), we must conclude that \( \partial_{ve} \text{vec } q_h = 0 \) on every tetrahedron \( T \in \Delta_3(T^h) \) and for every edge \( e \) adjacent to \( T \) and \( v \). But every \( T \) must have three edges adjacent to \( v \), whose tangents are linearly independent. We conclude that \( q_h \) is constant on every cell.

If \( q_h \) is constant on every cell, then \([\text{vec } q_h] \in \mathcal{P}_0(f; V)\) for each \( f \in \Gamma^r_i \). Let \( \Upsilon : \Lambda^1(V) \to C^\infty(M) \) be the equivalence map between \( V \)-valued 1-forms and matrices. For each \( f \) there must be some \( \phi_f \in \mathcal{P}_0 \Lambda^1(f; V) \) such that
\[
\int_f [\text{vec } q_h]^T (\Xi \mu_h) n \ ds = \int_f \phi_f \wedge \text{Tr}_f (\Upsilon^{-1}(\mu_h)), \quad \mu_h \in M_h.
\]
The degrees of freedom for \( M_h \) must be equivalent to the canonical degrees of freedom for \( \mathcal{P}_{r+2} \Lambda^1(T; V) \), which include all moments over facets of the form
\[
\int_f \phi \wedge \text{Tr}_f (\Upsilon^{-1}(\mu_h)), \quad \phi \in \mathcal{P}_{r+1}^{-1} \Lambda^1(f; V).
\]
For every \( r \geq 0 \), \( \mathcal{P}_0 \Lambda^1(f; V) \subset \mathcal{P}_{r+1}^{-1} \Lambda^1(f; V) \). Therefore for every facet there is a function \( \mu^f_h \in M_h \) such that
\[
\int_f [\text{vec } q_h]^T (\Xi \mu^f_h) n \ ds = \int_f \phi_f \wedge \text{Tr}_f (\Upsilon^{-1}(\mu^f_h)) = ||[\text{vec } q_h]||,
\]
and such that \( \Upsilon^{-1}(\mu^f_h) \) is in the kernel of all other functionals that define \( \mathcal{P}_{r+2} \Lambda^1(T^h; V) \). This implies that \( (\Xi \mu^f_h)n = 0 \) on all facets \( g \neq f \), and so \( \mu^f_h \in M^f_h \). By (9), \( 0 = c(\text{curl } \mu^f_h, q_h) = ||[\text{vec } q_h]|| \). We conclude that \( q_h \) is constant in \( T^h \).

Lemma 2.2. \textbf{There exists } \beta(C_{\text{mesh}}) > 0 \text{ such that for each } v \in \Delta_0(T^h)
\[
\inf_{q_h \in Q_h} \sup_{\mu^f_h \in M^f_h} c(\text{curl } \mu^f_h, q_h) \geq \beta \left| \mu^f_h \right|_{\text{curl}} ||q_h||_{1, h, v}.
\]

\textbf{Proof.} (Cf. Stenberg [17, Lemma 3.1].)

Let \( N^h \subset Q_h \) be the space of constant functions over \( T^h \). Lemma 2.1 proves that
\[
\inf_{q_h \in Q_h \setminus N^h} \sup_{\mu^f_h \in M^f_h} \frac{c(\text{curl } \mu^f_h, q_h)}{||\mu^f_h||_{\text{curl}}} > 0.
\]
Therefore (10) defines a norm on $\tilde{Q}_h \setminus N^c_h$. By proposition 2.2, $|1, h, T^w_\nu$ also defines a norm on $\tilde{Q}_h \setminus N^c_h$, so by the equivalence of finite dimensional norms there is $\beta_w > 0$ such that

$$\inf_{q_h \in \tilde{Q}_h \setminus N^c_h} \sup_{\mu_h \in M^c_h} c(\text{curl} \mu_h, q_h) > \beta_w |\text{curl} q_h|_{1, h, T^w_\nu}.$$ 

All integrals in the definitions $c, |1, h, T^w_\nu$ depend continuously on the Jacobians and inverse Jacobians of the mappings of the elements onto their corresponding elements in the reference macroelement $M^c_h$, i.e. $\beta_w(\{J_\nu\} \cup \{J^{-1}_\nu\})$ is a continuous function. A scaling argument shows that $\beta_w$ is unchanged under a rescaling of coordinates, so we may assume $\det(J_\nu, I_0) = 1$ for every $v$. Because of the mesh regularity constant $C_{\text{mesh}}$, there is an upper bound $\sup_{w, T} \max \{|J_\nu|, |J^{-1}_\nu|\} \leq C(C_{\text{mesh}})$. Therefore there is a compact set containing $\{J_\nu, T\} \cup \{J^{-1}_\nu, T\}$ for all $w$ with the same macroelement as $v$, and so $\beta_w$ attains its minimum,

$$\beta_{M^c_h} := \min_{T^w_\nu \in M^c_h} \beta_w > 0.$$ 

Because there are finitely many macroelements, there must be a $\beta$ such that $\beta_w \geq \beta > 0$ for all $v$.

We are now ready to prove the main stability result.

**Proof of theorem 2.1.** (Cf. Stenberg [15, Theorem 2.1], Verfürth [19, Proposition 3.3].)

Let $q_h \in \tilde{Q}_h$ be given. By lemma 2.2 for each $v \in T_h$ we can choose $\mu^*_h \in M^c_h$ such that $c(\text{curl} \mu^*_h, q_h) \geq \frac{1}{2}\beta |q_h|_{1, h, T^w_\nu}$ and $|\mu^*_h|_{\text{curl}} \leq |q_h|_{1, h, T^w_\nu}$. Defining $\mu_h := \sum_{v \in \Delta_0(T_h)} \mu^*_h$, we have (using proposition 2.2)

$$c(\text{curl} \mu_h, q_h) \geq \sum_{v \in \Delta_0(T_h)} c(\text{curl} \mu^*_h, q_h) \geq \frac{1}{2}\beta \sum_{v \in \Delta_0(T_h)} |q_h|_{1, h, T^w_\nu} \geq \frac{1}{2}\beta \|q_h\|_{1, h}^2.$$ 

Therefore there is a constant $C_1(C_{\text{mesh}}) > 0$ such that

$$c(\text{curl} \mu_h, q_h) \geq C_1 |\mu_h|_{\text{curl}} \|q_h\|_{1, h} = C_1 \|q_h\|_{1, h}^2 |\mu_h|_{\text{curl}} \|q_h\|_0.$$ 

As a corollary to the proof of proposition 2.3 there exists $\mu^* \in H^1(M)$ and $C_2(\Omega)$ such that $c(\text{curl} \mu^*, q_h) \geq C_2_q |q_h|_{1, h}^2$ and $|\mu^*|_1 \leq |q_h|_0$. Let $\Pi_h^0 \mu^*$ be the smoothed projection [3, § 5.4] of $H^1(M)$ into $H^1(\Omega) \cap H^1(M) \subset M_h$, which satisfies

$${|\Pi_h^0 \mu^* - \mu^*|_{0, T} \leq C(C_{\text{mesh}})h_T \|\mu^*\|_{1, T}, \quad |\Pi_h^0 \mu^*|_{1, T} \leq C(C_{\text{mesh}})\|\mu^*\|_{1, T}}.$$ (11)

We have by proposition 2.2

$$c(\text{curl} \Pi_h^0 \mu^*, q_h) = c(\text{curl} \mu^*, q_h) + c(\text{curl}(\Pi_h^0 \mu^* - \mu^*), q_h) \geq C_2 \|\mu^*\|_0^2 - C\|\Pi_h^0 \mu^* - \mu^*\|_{0, h} \|q_h\|_{1, h}. \quad (12)$$

By the standard trace inequality for $H^1(T)$, $\|\Xi(n)|\|_{0, \partial T}^2 \leq C(C_{\text{mesh}})(h_T^{-1}\|\mu\|_{0, T}^2 + h_T^2|\mu|_{1, T}^2)$, and so by (11),

$$\|\Pi_h^0 \mu^* - \mu^*\|_{0, h}^2 = \sum_{T \in \Delta_0(T_h)} h_T^{-2} \|\Pi_h^0 \mu^* - \mu^*\|_{0, T}^2 + \sum_{f \in \Delta_2(T_h)} h_f^{-1} |\Xi(\Pi_h^0 \mu^* - \mu^*)|_{0, f}^2 \leq C \sum_{T \in \Delta_0(T_h)} h_T^{-2} \|\Pi_h^0 \mu^* - \mu^*\|_{0, T}^2 + \|\Pi_h^0 \mu^* - \mu^*\|_{1, T}^2 \leq C\|\mu^*\|_1^2.$$
Therefore there exist $C_3(\Omega, C_{\text{mesh}})$ and $C_4(C_{\text{mesh}})$ such that
\[ c(\nabla \Pi_1 \mu^*, \eta_h) \geq \|q_h\|_0 (C_2\|\mu^*\|_1 - C\|\mu^*\|_1\frac{\|q_h\|_1}{\|q_h\|_0}) \geq \|q_h\|_0 \|\Pi_1 \mu^*\|_{\text{curl}}(C_3 - C_4\frac{\|q_h\|_1}{\|q_h\|_0}). \]

Letting $t = \|q_h\|_1/\|q_h\|_0$ and combining these two bounds, we get
\[ \inf_{q_h \in Q_h} \sup_{\mu \in M_h} c(\nabla \mu, q_h) \geq \|\mu_h\|_{\text{curl}}\|q_h\|_0 \min_{t > 0} \{C_1 t, C_3 - C_4 t\} = \|\mu_h\|_{\text{curl}}\|q_h\|_0 \frac{C_1 C_3}{C_1 + C_4}. \]

Having established the discrete stability of our conforming mixed finite element approximation, the standard convergence estimates apply.

**Theorem 2.2.** (Cf. Arnold, Falk, and Winther [7, Theorem 7.2]) Suppose $(\sigma, u, p)$ is the solution of (10) and $(\sigma_h, u_h, p_h)$ is the Galerkin solution in $\Sigma_h \times V_h \times Q_h$. Let $P_h$ be the $L^2$ projection onto $V_h$. There exists $C$ independent of $h$ such that
\[ \|\sigma - \sigma_h\|_{\text{div}} + \|u - u_h\|_0 + \|p - p_h\|_0 \leq C \inf_{(\chi_h, v_h, q_h) \in \Sigma_h \times V_h \times Q_h} \|\sigma - \tau_h\|_{\text{div}} + \|u - v_h\|_0 + \|p - q_h\|_0, \]
\[ \|\sigma - \sigma_h\|_0 + \|p - p_h\|_0 + \|u_h - P_{V_h} u\|_0 \leq C \inf_{(\chi_h, v_h) \in \Sigma_h \times Q_h} \|\sigma - \tau_h\|_0 + \|p - q_h\|_0, \]
\[ |\sigma - \sigma_h|_{\text{div}} = \|(1 - P_{V_h})g\|_0. \]

If $u$ and $\sigma$ are sufficiently smooth, then
\[ \|\sigma - \sigma_h\|_0 + \|p - p_h\|_0 + \|u_h - P_{V_h} u_h\|_0 \leq Ch^{r+2} \|u\|_{r+3}, \]
\[ \|u - u_h\|_0 \leq Ch^{r+1} \|u\|_{r+2}, \]
\[ |\sigma - \sigma_h|_{\text{div}} \leq Ch^{r+1} \|\sigma\|_{r+1}. \]

Because $\tilde{Q}_h$ has the same approximation order as $\Sigma_h$, an improved estimate is achieved compared to the $\Sigma_h \times V_h \times Q_h$ element.

### 3 Reduction to finite volume method

Let $A$, $B$, and $C$ be the assembled matrices for the bilinear forms $a$, $b$, and $c$ in $\Sigma_h \times V_h \times \tilde{Q}_h$, let $g$ be the assembled right-hand side, and let $\{s; u; p\}$ be the corresponding vector of degrees of freedom for $(\sigma_h, u_h, p_h)$. The matrix equation solved by $(\sigma_h, u_h, p_h)$ is
\[ \begin{bmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ u \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}. \quad (13) \]

Defining $Z := A^{-1} + A^{-1}C^T(-CA^{-1}C^T)^{-1}CA^{-1}$, $u$ solves $-BZ^{-1}B^Tu = g$. It is not practical to form this double Schur complement for general discretizations. For our lowest order $\Sigma_h \times \tilde{Q}_h$ space, however, a sparse approximation to $Z^{-1}$ can be constructed with nodal degrees of freedom and quadrature. For the remainder of this section $r = 0$.

(Our approach is inspired by the Multipoint Flux Mixed Finite Element (MFMFE) method of Wheeler and Yotov [20], and can be seen as a vector-valued extension of it.)
Because $\Sigma_h \times \tilde{Q}_h$ restricts to $\mathcal{P}_1(T; \mathbb{M} \times \mathbb{V})$ for each tetrahedron $T$, we can choose a nodal basis for $\Sigma_h \times \tilde{Q}_h$: instead of the moments on facets and edges, we take values at the functions at the boundary vertices of those facets and edges.

We define a corner quadrature rule for approximating the integrals in $a$ and $c$:

$$a_h(\sigma_h, \tau_h) := \sum_{T \in \mathcal{D}_h} \frac{|\text{vol} T|}{4} \sum_{v \in \mathcal{D}_0(T_h); v \in \mathcal{T}} \{ (A|_{T}(v) \tau_h|_{T}(v)) : \sigma_h|_{T}(v) \},$$

$$c_h(\tau_h, q_h) := \sum_{T \in \mathcal{D}_h} \frac{|\text{vol} T|}{4} \sum_{v \in \mathcal{D}_0(T_h); v \in \mathcal{T}} \tau_h|_{T}(v) : q_h|_{T}(v).$$

The approximate matrices assembled from $a_h$ and $c_h$ with our nodal degrees of freedom, $\tilde{A}$ and $\tilde{C}$, are block diagonal, with one block for each vertex $v$ in $\mathcal{T}_h$. The blocks will not have uniform size: the size of each block will depend on how many facets and edges are adjacent to $v$. This block structure extends to the approximation $\tilde{Z}$ of $Z$, so that computing $\tilde{Z}^{-1}$ involves only the solution of small, independent saddle-point systems at each vertex, and $-\mathcal{B}\tilde{Z}^{-1}\mathcal{B}^T$ is sparse, connecting cell-centered $u$ degrees of freedom to vertex-adjacent neighbors.

Because corner quadrature exactly integrates linear polynomials, we can combine the Bramble-Hilbert lemma with bounded projections to bound the loss of accuracy due to quadrature error.

**Proposition 3.1.** Given $\xi \in \Sigma \times Q$, let $\xi_\Sigma$ and $\xi_Q$ be its components, and let $\Pi_h \xi := (\Pi_h^{\Sigma}, \Pi_h \xi_Q)$, where $\Pi_h^{\Sigma}$ is the smoothed 2-form projection onto $\Sigma_h$ from $\mathbb{L}$ and $\Pi_h$ is the L$^2$ projection on $\tilde{Q}_h$. Let $f(\xi, \zeta) := a(\xi_\Sigma, \zeta_\Sigma) + c(\zeta_\Sigma, \xi_Q) + c(\xi_\Sigma, \zeta_Q)$ and $f_h(\xi_h, \zeta_h) := a(\xi_h, \zeta_h) + c(\xi_h, \zeta_h) + c(\xi_h, \zeta_h) + c(\xi_h, \xi_h, \zeta_h)$. Let $E(A, \xi_h, \zeta_h) := f(\xi_h, \zeta_h) - f_h(\xi_h, \zeta_h)$, where $A$ is the compliance field, and let $E_c(\tau_h, q_h) := c(\tau_h, q_h) - c_h(\tau_h, q_h)$. There exists $C$ independent of $h$ such that

$$|E(A, \Pi_h \xi, \zeta_h)| \leq Ch\|A\|_{1,\infty} \|\xi\|_1 \|\zeta_h\|_h, \quad \xi \in H^1(\mathbb{M} \times \mathbb{V}), \zeta_h \in \Sigma_h \times \tilde{Q}_h, \quad (14)$$

$$|E(A, \Pi_h \xi, \Pi_h \zeta)| \leq Ch^2 \|A\|_{2,\infty} \|\xi\|_1 \|\zeta\|_h, \quad \xi, \zeta \in H^1(\mathbb{M} \times \mathbb{V}), \quad (15)$$

$$|E_c(\tau_h, q_h)| \leq C\|\tau_h\|_0 \|q_h\|_1, \quad \tau \in H^1(\mathbb{M}), q_h \in \tilde{Q}_h. \quad (16)$$

These norms are the norm in $W^{k,\infty}(\Omega)$.

**Proof.** (Cf. Wheeler and Yotov [20, Lemma 3.5, Lemma 4.2].)

Let $P_h$ be the $L^2$ projection onto $\mathcal{P}_0(T_h; \mathbb{X})$ for each field $\mathbb{X}$. We have

$$E(A, \Pi_h \xi, \zeta_h) = E((I - P_h)A, \Pi_h \xi, \zeta_h) + E(P_h A, (I - P_h)\Pi_h \xi, \zeta_h) + E(P_h A, P_h \Pi_h \xi, \zeta_h).$$

The last error term is zero because the product is linear. Therefore

$$E(A, \xi, \zeta_h) \leq C\|\zeta_h\|_0 (\|(I - P_h)A\|_{0,\infty}\|\xi\|_0 + \|P_h A\|_{0,\infty}\|(I - P_h)\Pi_h \xi\|_0).$$

We have $\|(I - P_h)A\|_{0,\infty} \leq Ch\|A\|_{1,\infty}$ and

$$
\|(I - P_h)\Pi_h \xi\|_0 \leq \|(I - \Pi_h)\xi\|_0 + \|(I - P_h)\xi\|_0 + \|P_h (I - \Pi_h) \xi\|_0 \leq Ch\|\xi\|_1,
$$

which proves (14).

To prove (15), we expand the error as

$$E(A, \Pi_h \xi, \Pi_h \zeta) = E(A, P_h \Pi_h \xi, P_h \Pi_h \zeta) +$$

$$E((I - P_h)A, (I - P_h)\Pi_h \xi, \Pi_h \zeta) + E((I - P_h)A, P_h \Pi_h \xi, (I - P_h)\Pi_h \zeta) +$$

$$E(P_h A, (I - P_h)\Pi_h \xi, (I - P_h)\Pi_h \zeta) +$$

$$E(P_h A, P_h \Pi_h \xi, (I - P_h)\Pi_h \zeta) + E(P_h A, (I - P_h)\Pi_h \xi, P_h \Pi_h \zeta).$$


The first term is exact for $A \in P_1(T_h)$, and so is $o(h^2\|A\|_{2,\infty})$; the last two error terms are zero because their products are linear. Therefore

$$E(A, \Pi_h \xi, \Pi_h \zeta) \leq C h^2 \left(\|A\|_{2,\infty}\|\Pi_h \xi\|_0\|\Pi_h \zeta\|_0 + \|A\|_{1,\infty}(\|\xi\|_1\|\Pi_h \zeta\|_0 + \|\Pi_h \xi\|_1\|\zeta\|_1) + \|A\|_0\|\xi\|_1\|\zeta\|_1\right).$$

The last bound follows from $E_c(\tau_h, q_h) = E_c(\tau_h, (I - P_h)q_h).$

Unlike the MFME method, inexact quadrature could potentially affect the stability of the discretization. We must prove that this is not the case.

**Proposition 3.2.** Theorem 2.1 holds for $c_h$ (though with a different bound $\gamma$).

**Proof.** Given $q_h$, let $\mu^*$ be as in the proof of theorem 2.1. We can modify (12) for $c_h$ using the bound (16):

$$c_h(\Pi_h \mu^*, q_h) = c(\Pi_h \mu^*, q_h) - E_c(\Pi_h \mu^*, q_h)$$

$$\geq \|q_h\|_0\|\Pi_h \mu^*\|_{\text{curl}}(C_3 - C_4 \frac{\|q_h\|_{1,0}}{\|q_h\|_0}) - |E_c(\Pi_h \mu^*, q_h)|$$

$$\geq \|q_h\|_0\|\Pi_h \mu^*\|_{\text{curl}}(C_3 - (C_4 + C) \frac{\|q_h\|_{1,0}}{\|q_h\|_0}).$$

Therefore to complete the proof using the macroelement technique, all that needs to be done is to verify that lemma 2.1 holds for $c_h$: that if $q_h \in \ker(c_h, \tilde{Q}_h; \tilde{M}_h^1)$, then $q_h$ is constant in $T_h^v$.

Suppose $q_h \in \ker(c_h, \tilde{Q}_h; \tilde{M}_h^1)$. Now as before we define $\mu_h^v$ by

$$\mu_h^v|_T := \lambda_v \nu_v \Xi^{-1}(\text{grad vec } q_h t e_T^v),$$

and observe

$$\text{div}(\Xi \mu_h^v) = \text{div}(\lambda_v \nu_v (\text{grad vec } q_h t e_T^v))$$

$$= \lambda_v \nu_v \text{div}(\text{grad vec } q_h t e_T^v) + (\text{grad vec } q_h t e_T^v)(\lambda_v \text{ grad } \nu_v + \nu_v \text{ grad } \lambda_v)$$

$$= \lambda_v \nu_v \text{div}(\text{grad vec } q_h t e_T^v) + (\lambda_v \partial_{x_v} \nu_v + \nu_v \partial_{x_v} \lambda_v) \partial_{x_v} \text{ vec } q_h.$$
Now $\lambda_v - \lambda_{v_e}$ is linear and is equal to 1 at $v$ and $-1$ at $v_e$, so $\partial_{v_e} (\lambda_v - \lambda_{v_e}) = 2/h_e$; likewise, $\partial_{v_e} (\lambda_v + \lambda_{v_e}) = 0$. Therefore

$$\partial_{v_e} \vec{q}_h \cdot (\vec{v} + \vec{q}_h ((v + v_e)/2) + \frac{h_e}{2} (\partial_{v_e} (\lambda_v - \lambda_{v_e}) \partial_{v_e} \vec{q}_h)) = ||\partial_{v_e} \vec{q}_h||^2.$$

As before, we conclude that $q_h$ must be constant on each tetrahedron. But $c_h(\mu_h, q_h) = c(\mu_h, q_h)$ in this case, so $q_h$ is constant on $T_h^e$. □

Having proven discrete stability with inexact quadrature, the bounds on the quadrature error let us prove optimal convergence of $u_h$ to the true solution $u$, as well as an improved estimate for $||u_h - P_h u||_0$ via a duality argument, which implies second-order pointwise convergence at the centroids of the elements.

**Theorem 3.1.** Let $(\sigma, u, p)$ be the solution of (8), and let $(\sigma_h, u_h, p_h)$ be the solution of

$$a_h(\sigma_h, \tau_h) + c_h(\sigma_h, q_h) + c_h(\tau_h, p_h) + b(\sigma_h, v_h) + b(\tau_h, u_h) = (g, v_h), \quad (\tau_h, v_h, q_h) \in \Sigma_h \times V_h \times Q_h.$$

Then assuming $\sigma, u,$ and $p$ are smooth enough, there exists $C_1$ independent of $h$ such that

$$\|\sigma - \sigma_h\|_0 + ||p - p_h||_0 \leq C_1 h \|A\|_{1,\infty} (\|\sigma\|_1 + ||p||_1), \quad (17)$$

$$|\sigma - \sigma_h|_{\text{div}} \leq C_1 h \|\text{div} \sigma\|_1, \quad (18)$$

$$\|u - u_h\|_0 \leq C_1 h \|A\|_{1,\infty} (\|\sigma\|_1 + ||p||_1 + ||u||_1). \quad (19)$$

If (9) is sufficiently regular that there exists $C_2$ independent of $g$ such that

$$||\sigma||_1 + ||p||_1 + ||u||_1 \leq C_2 \|g\|_0, \quad (20)$$

then there exists $C_3$ independent of $h$ such that

$$\|u_h - P_h u\|_0 \leq C_3 h^2 \|A\|_{2,\infty} (||\text{div} \sigma||_1 + ||\sigma||_1 + ||p||_1).$$

**Proof.** (Cf. Wheeler and Yotov [20, Theorems 3.4, 4.1, 4.3].)

Let $\xi := (\sigma, p), \, \xi_h := (\sigma_h, p_h), \, K_h := \ker (b, \Sigma_h, V_h) \times \tilde{Q}_h = \ker (\text{div}, \Sigma_h) \times \tilde{Q}_h$, and let $f, f_h, E$ and $\Pi_h$ be as in proposition 3.4.

Proposition 3.2 proves that there exists $\beta > 0$ independent of $h$ such that

$$\inf_{\xi_h \in K_h} \sup_{\zeta_h \in K_h} f_h(\xi_h, \zeta_h) \geq \beta \|\xi\|_0 \|\zeta\|_0.$$

Therefore, because $\text{div} \Pi_h^\text{tv} \sigma = P_h g = \text{div} \sigma_h$, we have $\Pi_h \xi - \xi_h \in K_h$, and so by (14)

$$\|\Pi_h \xi - \xi_h\|_0 \leq \frac{1}{\beta} \sup_{\zeta_h \in K_h} \frac{f_h(\Pi_h \xi - \xi_h, \zeta_h)}{||\zeta_h||_0} \leq \frac{1}{\beta} \sup_{\zeta_h \in K_h} \frac{f_h(\Pi_h \xi - \xi_h, \zeta_h)}{||\zeta_h||_0} + b(\zeta_h, u_h) \leq \frac{1}{\beta} \sup_{\zeta_h \in K_h} \frac{f(\xi, \zeta_h) - E(A, \xi, \zeta_h)}{||\zeta_h||_0} \leq C h \|A\|_{1,\infty} ||\xi||_1.$$

The bound 19 follows by the triangle inequality and the approximation properties of $\Pi_h^\text{tv}$.  

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The bound (18) is a standard result of the commuting property of $\Pi_h^{\text{div}}$ and is unchanged by inexact quadrature.

Due to the stability of $b$ on $\Sigma_h \times V_h$ and (13),

$$\| u_h - P_h u \|_0 \leq C \sup_{\tau_h \in \Sigma_h} b(\tau_h, u_h - P_h u) \| \tau_h \|_{\text{div}}$$

$$= C \sup_{\tau_h \in \Sigma_h} \frac{f(\xi, (\tau_h, 0)) - f_h(\xi, (\tau_h, 0))}{\| \tau_h \|_0}$$

$$= C \sup_{\tau_h \in \Sigma_h} \frac{f(\xi - \Pi_h \xi, (\tau_h, 0)) - f_h(\xi - \Pi_h \xi, (\tau_h, 0)) + E(A, \Pi_h \xi, (\tau_h, 0))}{\| \tau_h \|_0}$$

$$\leq Ch\| A \|_{1, \infty} \| \xi \|_1.$$  

The bound (18) follows by the triangle inequality and the approximations properties of $P_h$.

Now let us assume that (20) holds, and let $(\tau, v, q)$ be the solution of (6) with $g = P_h u - u_h$.

Because $g = P_h g$ in this case,

$$\| P_h u - u_h \|_0^2 = b(\tau, P_h u - u_h) = b(\Pi_h \tau, P_h u - u_h).$$

Defining $\zeta := (\tau, q)$, we have

$$b(\Pi_h \tau, P_h u - u_h) = f_h(\xi_h, \Pi_h \zeta) - f(\xi, \Pi_h \zeta)$$

$$= f(\xi_h - \xi, \Pi_h \zeta) - E(A, \xi_h, \Pi_h \zeta)$$

$$= f(\xi_h - \xi, \zeta) + f(\xi_h - \xi, \Pi_h \zeta - \zeta) - E(A, \xi_h, \Pi_h \zeta)$$

$$= b(\sigma - \sigma_h, v) + f(\xi_h - \xi, \Pi_h \zeta - \zeta) - E(A, \xi_h, \Pi_h \zeta)$$

$$= b(\sigma - \sigma_h, v) + f(\xi_h - \xi, \Pi_h \zeta - \zeta) - E(A, \Pi_h \xi, \Pi_h \zeta) - E(A, \xi_h - \Pi_h \xi, \Pi_h \zeta).$$

Because $\text{div}(\sigma) - \text{div}(\sigma_h) = (I - P_h) \text{div} \sigma$, it is orthogonal to $V_h$ and

$$|b(\sigma - \sigma_h, v)| = |b(\sigma - \sigma_h, v - P_h v)| \leq Ch^2 \| \text{div} \sigma \|_1 \| v \|_1.$$  

By this result, the already established bound for $\| \xi_h - \xi \|_0$, the approximation properties of $\Pi_h = (\Pi_h^{\text{div}}, P_h)$, and the quadrature bounds (12) and (13), we have

$$\| P_h u - u_h \|_0^2 \leq Ch^2 (\| \text{div} \sigma \|_1 \| v \|_1 + (\| A \|_{1, \infty} + \| A \|_{2, \infty}) (\| \sigma \|_1 + \| p \|_1 \| (\| \tau \|_1 + \| q \|_1)$$

$$\leq Ch^2 \| A \|_{2, \infty} (\| \text{div} \sigma \|_1 + \| \sigma \|_1 + \| p \|_1) (\| v \|_1 + \| \tau \|_1 + \| q \|_1).$$

Invoking (21) completes the proof. \qed

4 Discussion

Although we have proved optimal and improved convergence estimates of the $\Sigma_h \times V_h \times \bar{Q}_h$ element for arbitrary order, the existence of high order discretizations of linear elasticity whose stress is pointwise symmetric (1, 2, 12) means that our element may only have practical advantages for low orders. The limit of strongly-symmetric finite elements composed of “pure” polynomial elements (i.e. without bubble functions) appears to be $r = 3$ (12). It is worthwhile to compare our element to other low order offerings.

- Arnold, Falk, and Winther (6) introduced $\Sigma_h \times V_h \times Q_h$, against which our element has already been compared. We can estimate the sizes of $|Q_h|$ and $|\bar{Q}_h|$ using the Euler characteristic of $\Omega$ and the heuristics $|\Delta_2| \approx 2|\Delta_3|$ and $|\Delta_3| \approx 7|\Delta_0|$ (16), giving us an
approximation $|\Delta_1| \approx 7/6|\Delta_3|$. The lowest order $Q_h$ space has three degrees of freedom per cell, while $\tilde{Q}_h$ has two degrees of freedom per edge, which predicts $|\tilde{Q}_h| \approx 7/9|Q_h|$. The same paper, however, introduced a reduced element with a stress space $|\Sigma_h^+| = 2/3|\Sigma_h|$. Neither $\Sigma_h \times Q_h$ nor $\Sigma_h^+ \times Q_h$ admits a jointly nodal discretization, so reduction to just the displacement variables is not a local operation. $\Sigma_h$ alone, however, can be locally eliminated with corner quadrature, leaving a finite volume discretization of $V \times Q$ (in keeping with the interpretation of the elasticity complex as a resolution of the rigid-body motions).

- Hu and Zhang [13] developed a pointwise conservative and symmetric discretization $\hat{\Sigma}^+ \times V_{1,h}$, where $V_{1,h}$ is the discontinuous approximation space of rigid-body motions on each tetrahedron, and $\hat{\Sigma}^+ \times V_{1,h}$ is the union of $\mathcal{P}_1(\mathbb{T}^3; \mathbb{S}) \cap C^0$ with div-conforming bubble functions on each facet. This space has the advantage of being strongly-symmetric, and its size is $6(|\Delta_1| + |\Delta_2| + |\Delta_3|) \approx 18.9|\Delta_0|$, compared to the lowest order element introduced here, $|\Sigma_h \times V_h \times \tilde{Q}_h| = 2|\Delta_1| + 9|\Delta_2| + 3|\Delta_3| \approx 23.3|\Delta_3|$. The stress space $\hat{\Sigma}^+ \times V_{1,h}$ would seem to admit a nodal discretization and quadrature, meaning that their element could also be reduced to a first-order finite volume scheme, though with six degrees of freedom per cell instead of three.

Suppose we wished to solve a problem with a nonlinear stress-strain relationship, such that the stress degrees of freedom could not be eliminated a priori. The block-diagonal matrix for the approximation $c_h$ would still allow one to locally project $Q_h$ out of $\Sigma_h$ a priori, leaving a system of the size $|\Sigma_h| + |V_h| - |\tilde{Q}_h| \approx 18.7|\Delta_3|$, making our element more competitive. Our element also has the practical advantage of being composed of function spaces that are already widely implemented.

- Ambartsumyan, Khattatov, and Yotov [2] recently proposed a finite element for linear elasticity $\Sigma_h := \mathcal{P}_{r+1}(\mathbb{T}^3; \mathbb{S}) \cap C^0(\mathbb{K})$, i.e. the multiplier field is fully continuous. Their approach also reduces to a finite volume method, and is a mixed finite element presentation of the finite volume method of Nordbotten [15]. The similarity of our element to theirs is not intentional, but not surprising, as both approaches are generalization of the MFMFME method of Wheeler and Yotov [20]. Though unpublished, the proof of stability they describe in [2] suggests that it is the same as proposition 8.29. As both elements appear to have the same approximation properties, the notable difference is the size of the multiplier spaces ($\tilde{Q}_h \subset \tilde{Q}_h$): $|\tilde{Q}_h| = 3|\Delta_0| \approx 1/2|\Delta_3|$ vs. $|\tilde{Q}_h| = 2|\Delta_1| \approx 7/3|\Delta_3|$. Their element thus requires less work to compute the displacement Schur complement, while ours results in a more nearly symmetric stress, and a smaller approximation space when only the multipliers are projected out.

- Gopalakrishnan and Guzmán [11] and Arnold, Awanou, and Winther [4] have introduced strongly-symmetric but $H(\text{div})$-nonconforming discretizations. The reduced element in Arnold, Awanou, and Winther [4] appears to be the smallest: its size is $9|\Delta_2| + 6|\Delta_3| \approx 24|\Delta_3|$, and its convergence is first-order in the $L^2$ norm of displacement and stress in general, and the convergence in displacement is second-order with full elliptic regularity. These $H(\text{div})$-nonconforming methods require careful selection of the stress degrees of freedom to ensure stability, so a lack of nodal discretization would appear to rule out local elimination of the stress degrees of freedom. However, because all degrees of freedom are in cells and on facets, the methods are hybridizable, reducing to symmetric positive definite systems in Lagrange multipliers on the facets. The lowest order cases require three multipliers on each facet, meaning their reduced size is $\approx 6|\Delta_3|$.
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