Periodic shadowing and $\Omega$-stability

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Abstract

We show that the following three properties of a diffeomorphism $f$ of a smooth closed manifold are equivalent: (i) $f$ belongs to the $C^1$-interior of the set of diffeomorphisms having periodic shadowing property; (ii) $f$ has Lipschitz periodic shadowing property; (iii) $f$ is $\Omega$-stable. Bibliography: 20 titles.

Mathematics Subject Classification: 37C50, 37D20
Keywords: periodic shadowing, hyperbolicity, $\Omega$-stability

1 Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well developed part of the global theory of dynamical systems (see, for example, the monographs [1, 2]).

This theory is closely related to the classical theory of structural stability. It is well known that a diffeomorphism has shadowing property in a neighborhood of a hyperbolic set [3, 4] and a structurally stable diffeomorphism has shadowing property on the whole manifold [5 – 7]. Analyzing the proofs of the first shadowing results by Anosov [3] and Bowen [4], it is easy to see that, in a neighborhood of a hyperbolic set, the shadowing property is Lipschitz (and the same holds in the case of a structurally stable diffeomorphism, see [1]).

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3 The research of the third author is supported by NSC (Taiwan) 98-2811-M-002-061
The shadowing property means that, near a sufficiently precise approximate trajectory of a dynamical system, there is an exact trajectory. One can pose a similar question replacing arbitrary approximate and exact trajectories by periodic ones (the corresponding property is called periodic shadowing property, see [8]).

In this paper, we study relations between periodic shadowing and structural stability (to be more precise, Ω-stability).

It is easy to give an example of a diffeomorphism that is not structurally stable but has shadowing property (see [9], for example). Similarly, there exist diffeomorphisms that are not Ω-stable but have periodic shadowing property.

Thus, structural stability is not equivalent to shadowing (and Ω-stability is not equivalent to periodic shadowing).

One of possible approaches in the study of relations between shadowing and structural stability is the passage to $C^1$-interiors. At present, it is known that the $C^1$-interior of the set of diffeomorphisms having shadowing property coincides with the set of structurally stable diffeomorphisms [10]. Later, a similar result was obtained for orbital shadowing property (see [11] for details).

In this paper, we show that the $C^1$-interior of the set of diffeomorphisms having periodic shadowing property coincides with the set of Ω-stable diffeomorphisms.

We are also interested in the study of the above-mentioned relations without the passage to $C^1$-interiors. Let us mention in this context that Abdenur and Diaz conjectured that a $C^1$-generic diffeomorphism with shadowing property is structurally stable; they have proved this conjecture for so-called tame diffeomorphisms [12]. Recently, it was proved that Lipschitz shadowing and the so-called variational shadowing are equivalent to structural stability [13, 9].

The second main result of this paper states that Lipschitz periodic shadowing property is equivalent to Ω-stability.

2 Main results

Let us pass to exact definitions and statements.

Let $f$ be a diffeomorphism of a smooth closed manifold $M$ with Riemannian metric dist. We denote by $Df(x)$ the differential of $f$ at a point
Denote by $T_x M$ the tangent space of $M$ at a point $x$; let $|v|, v \in T_x M,$ be the norm generated by the metric $dist$.

As usual, we say that a sequence $\xi = \{x_i \in M, i \in \mathbb{Z}\}$ is a $d$-pseudotrajectory of $f$ if

$$\text{dist}(f(x_i), x_{i+1}) < d, \quad i \in \mathbb{Z}. \quad (1)$$

**Definition 1.** We say that $f$ has periodic shadowing property if for any positive $\varepsilon$ there exists a positive $d$ such that if $\xi = \{x_i\}$ is a periodic $d$-pseudotrajectory, then there exists a periodic point $p$ such that

$$\text{dist}(f^i(p), x_i) < \varepsilon, \quad i \in \mathbb{Z}. \quad (2)$$

Denote by $\text{PerSh}$ the set of diffeomorphisms having periodic shadowing property.

**Definition 2.** We say that $f$ has Lipschitz periodic shadowing property if there exist positive constants $L, d_0$ such that if $\xi = \{x_i\}$ is a periodic $d$-pseudotrajectory with $d \leq d_0$, then there exists a periodic point $p$ such that

$$\text{dist}(f^i(p), x_i) \leq L d, \quad i \in \mathbb{Z}. \quad (3)$$

Denote by $\text{LipPerSh}$ the set of diffeomorphisms having Lipschitz periodic shadowing property.

Denote by $\Omega S$ the set of $\Omega$-stable diffeomorphisms (it is well known that $f \in \Omega S$ if and only if $f$ satisfies Axiom A and the no cycle condition, see, for example, [14]). Denote by $\text{Diff}^1(M)$ the space of diffeomorphisms of $M$ with the $C^1$ topology. For a set $P \subset \text{Diff}^1(M)$ we denote by $\text{Int}^1(P)$ its $C^1$-interior.

Let us state our main result.

**Theorem.** $\text{Int}^1(\text{PerSh}) = \text{LipPerSh} = \Omega S$.

The structure of the paper is as follows. In Sec. 3, we prove the inclusion $\Omega S \subset \text{LipPerSh}$. Of course, this inclusion implies that $\Omega S \subset \text{PerSh}$. Since the set $\Omega S$ is $C^1$-open, we conclude that $\Omega S \subset \text{Int}^1(\text{PerSh})$. In Sec. 4, we prove the inclusion $\text{Int}^1(\text{PerSh}) \subset \Omega S$. In Sec. 5, we prove the inclusion $\text{LipPerSh} \subset \Omega S$. 

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3 $\Omega S \subset \text{LipPerSh}$

First we introduce some basic notation. Denote by $\text{Per}(f)$ the set of periodic points of $f$ and by $\Omega(f)$ the nonwandering set of $f$. Let $N = \sup_{x \in M} ||Df(x)||$.

Let us formulate several auxiliary definitions and statements.

It is well known that if a diffeomorphism $f$ satisfies Axiom A, then its nonwandering set can be represented as a disjoint union of a finite number of compact sets:

$$\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_m,$$

where the sets $\Omega_i$ are so-called basic sets (hyperbolic sets each of which contains a dense positive semi-trajectory).

We say that a diffeomorphism $f$ has Lipschitz shadowing property on a set $U$ if there exist positive constants $L, d_0$ such that if $\xi = \{x_i, \ i \in \mathbb{Z}\} \subset U$ is a $d$-pseudotrajectory with $d \leq d_0$, then there exists a point $p \in U$ such that inequalities (3) hold.

We say that a diffeomorphism $f$ is expansive on a set $U$ if there exists a positive number $a$ (expansivity constant) such that if two trajectories $\{f^i(p) : i \in \mathbb{Z}\}$ and $\{f^i(q) : i \in \mathbb{Z}\}$ belong to $U$ and the inequalities

$$\text{dist}(f^i(p), f^i(q)) \leq a, \ i \in \mathbb{Z},$$

hold, then $p = q$.

The following statement is well known (see [1, 14], for example).

**Proposition.** If $\Lambda$ is a hyperbolic set, then there exists a neighborhood $U$ of $\Lambda$ such that $f$ has Lipschitz shadowing property on $U$ and is expansive on $U$.

We also need the following two lemmas (see [15]).

**Lemma 1.** Let $f$ be a homeomorphism of a compact metric space $(X, \text{dist})$. For any neighborhood $U$ of the nonwandering set $\Omega(f)$ there exist positive numbers $B, d_1$ such that if $\xi = \{x_i, \ i \in \mathbb{Z}\}$ is a $d$-pseudotrajectory of $f$ with $d \leq d_1$ and

$$x_k, x_{k+1}, \ldots, x_{k+l} \notin U$$

for some $l > 0$ and $k \in \mathbb{Z}$, then $l \leq B$.

Let $\Omega_1, \ldots, \Omega_m$ be the basic sets in decomposition (4) of the nonwandering set of an $\Omega$-stable diffeomorphism $f$. 

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Lemma 2. Let $U_1, \ldots, U_m$ be disjoint neighborhoods of the basic sets $\Omega_1, \ldots, \Omega_m$. There exist neighborhoods $V_j \subset U_j$ of the sets $\Omega_j$ and a number $d_2 > 0$ such that if $\xi = \{x_i, \ i \in \mathbb{Z}\}$ is a $d$-pseudotrajectory of $f$ with $d \leq d_2$ such that $x_0 \in V_j$ and $x_i \notin U_j$ for some $j \in \{1, \ldots, m\}$ and some $t > 0$, then $x_i \notin V_j$ for $i \geq t$.

Lemma 3. $\Omega S \subset \text{LipPerSh}$.

Proof. Apply the above proposition and find disjoint neighborhoods $W_1, \ldots, W_m$ of the basic sets $\Omega_1, \ldots, \Omega_m$ in decomposition (4) such that (i) $f$ has Lipschitz shadowing property on any of $W_j$ with the same constants $L, d_0^*$; (ii) $f$ is expansive on any of $W_j$ with the same expansivity constant $a$.

Find neighborhoods $V_j, U_j$ of $\Omega_j$ (and reduce $d_0^*$, if necessary) so that the following properties are fulfilled:

- $V_j \subset U_j \subset W_j$, $j = 1, \ldots, m$;
- the statement of Lemma 2 holds for $V_j$ and $U_j$ with some $d_2 > 0$;
- the $\mathcal{L}d_0^*$-neighborhoods of $U_j$ belong to $W_j$.

Apply Lemma 1 to find the corresponding constants $B, d_1$ for the neighborhood $V_1 \cup \cdots \cup V_m$ of $\Omega(f)$.

We claim that $f$ has the Lipschitz periodic shadowing property with constants $\mathcal{L}, d_0$, where

$$d_0 = \min \left( d_0^*, d_1, d_2, \frac{a}{2\mathcal{L}} \right).$$

Take a $\mu$-periodic $d$-pseudotrajectory $\xi = \{x_i, \ i \in \mathbb{Z}\}$ of $f$ with $d \leq d_0$. Lemma 1 implies that there exists a neighborhood $V_j$ such that $\xi \cap V_j \neq \emptyset$; shifting indices, we may assume that $x_0 \in V_j$.

In this case, $\xi \subset U_j$. Indeed, if $x_{i_0} \notin U_j$ for some $i_0$, then $x_{i_0+k\mu} \notin U_j$ for all $k$. It follows from Lemma 2 that if $i_0 + k\mu > 0$, then $x_i \notin V_j$ for $i \geq i_0 + k\mu$, and we get a contradiction with the periodicity of $\xi$ and the inclusion $x_0 \in V_j$.

Thus, there exists a point $p$ such that inequalities (3) hold. Let us show that $p \in \text{Per}(f)$. By the choice of $U_j$ and $W_j$, $f^i(p) \in W_j$ for all $i \in \mathbb{Z}$. Let $q = f^{i_0}(p)$. Inequalities (3) and the periodicity of $\xi$ imply that

$$\text{dist}(f^i(q), x_i) = \text{dist}(f^i(q), x_{i+i}) \leq \mathcal{L}d, \quad i \in \mathbb{Z}.$$ 

Thus,

$$\text{dist}(f^i(q), f^i(p)) \leq 2\mathcal{L}d \leq a, \quad i \in \mathbb{Z},$$

which implies that $f^{i_0}(p) = q = p$. This completes the proof.
Remark. Thus, we have shown that an $\Omega$-stable diffeomorphism has periodic shadowing property (and its Lipschitz variant). It must be noted that it was shown in [16] that there exist $\Omega$-stable diffeomorphisms that do not have weak shadowing property (hence, they do not have orbital and usual shadowing properties, see [11] for details).

4 $\text{Int}^1(\text{PerSh}) \subset \Omega S$

In the proof, we refer to the following well-known statement. Denote by $\text{HP}$ the set of diffeomorphisms $f$ such that every periodic point of $f$ is hyperbolic; let $F = \text{Int}^1(\text{HP})$. It is known (see [17, 18]) that the set $F$ coincides with the set $\Omega S$ of $\Omega$-stable diffeomorphisms.

Thus, it suffices for us to prove the following statement.

**Lemma 4.** $\text{Int}^1(\text{PerSh}) \subset F$.

**Proof.** In the proof of this lemma, as well as in some proofs below, we apply the usual linearization technique based on exponential mapping.

Let $\exp$ be the standard exponential mapping on the tangent bundle of $M$ and let $\exp_x$ be the corresponding mapping $T_x M \to M$.

Let $p$ be a periodic point of $f$; denote $p_i = f^i(p)$ and $A_i = Df(p_i)$. We introduce the mappings

$$F_i = \exp^{-1}_{p_{i+1}} \circ f \circ \exp_{p_i} : T_{p_i} M \to T_{p_{i+1}} M.$$  \hspace{1cm} (5)

It follows from the standard properties of the exponential mapping that $D \exp_x(0) = \text{Id}$; hence,

$$DF_i(0) = A_i.$$

We can represent

$$F_i(v) = A_i v + \phi_i(v),$$

where

$$\frac{|\phi_i(v)|}{|v|} \to 0 \text{ as } |v| \to 0.$$

Denote by $B(r, x)$ the ball in $M$ of radius $r$ centered at a point $x$ and by $B_T(r, x)$ the ball in $T_x M$ of radius $r$ centered at the origin.
There exists $r > 0$ such that, for any $x \in M$, $\exp_x$ is a diffeomorphism of $B_T(r, x)$ onto its image, and $\exp_x^{-1}$ is a diffeomorphism of $B(r, x)$ onto its image. In addition, we may assume that $r$ has the following property.

If $v, w \in B_T(r, x)$, then
\[
\frac{\text{dist}(\exp_x(v), \exp_x(w))}{|v - w|} \leq 2;
\]
if $y, z \in B(r, x)$, then
\[
\frac{|\exp_x^{-1}(y) - \exp_x^{-1}(z)|}{\text{dist}(y, z)} \leq 2.
\]

Every time, constructing periodic $d$-pseudotrajectories of $f$, we take $d$ so small that the considered points of our pseudotrajectories, points of shadowing trajectories, their “lifts” to tangent spaces, etc. belong to the corresponding balls $B(r, p_i)$ and $B_T(r, p_i)$ (and we do not repeat this condition on the smallness of $d$).

To prove Lemma 4, it is enough for us to show that $\text{Int}^1(\text{PerSh}) \subset \text{HP}$ and to note that the left-hand side of this inclusion is $C^1$-open.

To get a contradiction, let us assume that a diffeomorphism $f \in \text{Int}^1(\text{PerSh})$ has a nonhyperbolic periodic point $p$. Fix a $C^1$-neighborhood $\mathcal{N} \subset \text{PerSh}$ of $f$.

For simplicity, let us assume that $p$ is a fixed point and that the matrix $A_0 = Df(p)$ has an eigenvalue $\lambda = 1$ (the remaining cases are considered using a similar reasoning, see, for example, [19]).

In our case, an analog of mapping (5),
\[
F = \exp_p^{-1} \circ f \circ \exp_p : T_pM \to T_pM,
\]
has the form
\[
F(v) = A_0v + \phi(v).
\]
Clearly, we can find a number $a \in (0, r)$ (recall that the number $r$ was fixed above when properties of the exponential mapping were described), coordinates $v = (u, w)$ in $T_pM$ with one-dimensional $u$, and a diffeomorphism $h \in \mathcal{N}$ such that if
\[
H = \exp_p^{-1} \circ h \circ \exp_p
\]
and $|v| \leq a$, then
\[
H(v) = Av = (u, Bw),
\]
where $B$ is a matrix of size $(n - 1) \times (n - 1)$ (and $n$ is the dimension of $M$). For this purpose, we take a matrix $A$, close to $A_0$ and having an eigenvalue $\lambda = 1$ of multiplicity one, and “annihilate” the $C^1$-small term $(A_0 - A)v + \phi(v)$ in the small ball $B_T(a, p)$.

Take a positive $\varepsilon$ such that $8 \varepsilon < a$. Since $h \in \mathcal{N}$, there exists a corresponding $d \in (0, \varepsilon)$ from the definition of periodic shadowing (for the diffeomorphism $h$). Take a natural number $K$ such that $Kd > 8 \varepsilon$. Reducing $d$, if necessary, we may assume that

$$8 \varepsilon < Kd < 2a. \tag{6}$$

Let us construct a sequence $y_k \in T_p M$, $k \in \mathbb{Z}$, as follows:

$$y_0 = 0, \quad y_{k+1} = Ay_k + \left( \frac{d}{2}, 0 \right), \quad 0 \leq k \leq K - 1,$$

$$y_{k+1} = Ay_k - \left( \frac{d}{2}, 0 \right), \quad K \leq k \leq 2K - 1,$$

and $y_{k+2K} = y_k$, $k \in \mathbb{Z}$. Clearly,

$$y_K = \left( \frac{Kd}{2}, 0 \right). \tag{7}$$

Let

$$x_k = \exp_p(y_k).$$

Since

$$\exp_p^{-1}(h(x_k)) = H(y_k) = Ay_k$$

and

$$|y_{k+1} - Ay_k| = \frac{d}{2},$$

the sequence $\xi = \{x_k\}$ is a $2K$-periodic $d$-pseudotrajectory of $h$.

By our assumption, there exists a periodic point $p_0$ of $h$ such that

$$\text{dist}(p_k, x_k) < \varepsilon, \quad k \in \mathbb{Z},$$

where $p_k = h^k(p_0)$. Let

$$p_k = \exp_p(q_k), \quad k \in \mathbb{Z},$$

and
where \( q_k = (U_k, W_k) \), and let \( y_k = (u_k, w_k) \); then
\[
|U_k - u_k| \leq |q_k - y_k| < 2\varepsilon, \quad k \in \mathbb{Z},
\]
which implies that
\[
|U_0| \leq |q_0| < 2\varepsilon.
\]
Since \( q_{k+1} = H(q_k), U_k = U_0 \) for all \( k \) due to the structure of \( H \). We conclude that \( |U_K| < 2\varepsilon \) and get a contradiction with the inequalities \( |U_K - u_K| < 2\varepsilon \), (6), and (7). The lemma is proved.

5 LipPerSh \( \subset \Omega S \)

In this section, we assume that \( f \in \text{LipPerSh} \) (with constants \( \mathcal{L} \geq 1, d_0 > 0 \)). Clearly, in this case \( f^{-1} \in \text{LipPerSh} \) as well (and we assume that the constants \( \mathcal{L}, d_0 \) are the same for \( f \) and \( f^{-1} \)).

In the construction of pseudotrajectories, we apply the same linearization technique as in the previous section.

**Lemma 5.** Every point \( p \in \text{Per}(f) \) is hyperbolic.

**Proof.** To get a contradiction, let us assume that \( f \) has a nonhyperbolic periodic point \( p \) (to simplify notation, we assume that \( p \) is a fixed point; literally the same reasoning can be applied to a periodic point of period \( m > 1 \)).

In this case, mapping (5) takes the form
\[
F(v) = \exp_p^{-1} \circ f \circ \exp_p(v) = Av + \phi(v),
\]
where \( A \) is a nonhyperbolic matrix. The following two cases are possible:

(Case 1): \( A \) has a real eigenvalue \( \lambda \) with \( |\lambda| = 1 \);

(Case 2): \( A \) has a complex eigenvalue \( \lambda \) with \( |\lambda| = 1 \).

We treat in detail only Case 1; we give a comment concerning Case 2. To simplify presentation, we assume that 1 is an eigenvalue of \( A \); the case of eigenvalue \(-1\) is treated similarly.

We can find coordinates \( v \) in \( T_p M \) such that, with respect to this coordinate, the matrix \( A \) has block-diagonal form,
\[
A = \text{diag}(B, P),
\]
(8)
where $B$ is a Jordan block of size $l \times l$:

$$B = \begin{pmatrix}
    1 & 1 & 0 & \ldots & 0 \\
    0 & 1 & 1 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & 1
\end{pmatrix}.$$  

Of course, introducing new coordinates, we have to change the constants $\mathcal{L}, d_0, N$; we denote the new constants by the same symbols. In addition, we assume that $\mathcal{L}$ is integer.

We start considering the case $l = 2$; in this case,

$$B = \begin{pmatrix}
    1 & 1 \\
    0 & 1
\end{pmatrix}.$$  

Let

$$e_1 = (1, 0, 0, \ldots, 0) \text{ and } e_2 = (0, 1, 0, \ldots, 0)$$

be the first two vectors of the standard orthonormal basis.

Let $K = 25\mathcal{L}$.

Take a small $d > 0$ and construct a finite sequence $y_0, \ldots, y_Q$ in $T_p M$ (where $Q$ is determined later) as follows: $y_0 = 0$ and

$$y_{k+1} = Ay_k + de_2, \quad k = 0, \ldots, K - 1.$$  

Then

$$y_K = (Z_1(K)d, Kd, 0, \ldots, 0),$$

where the natural number $Z_1(K)$ is determined by $K$ (we do not write $Z_1(K)$ explicitly). Now we set

$$y_{k+1} = Ay_k - de_1, \quad k = K, \ldots, 2K - 1.$$  

Then

$$y_{2K} = (Z_2(K)d, 0, 0, \ldots, 0),$$

where the natural number $Z_2(K)$ is determined by $K$ as well. Take $Q = 2K + Z_2(K)$; if we set

$$y_{k+1} = Ay_k - de_1, \quad k = 2K, \ldots, Q - 1,$$  

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then \( y_Q = 0 \). Let us note that both numbers \( Q \) and
\[
Y := \frac{\max_{0 \leq k \leq Q-1} |y_k|}{d}
\]
are determined by \( K \) (and hence, by \( L \)).

Now we construct a \( Q \)-periodic sequence \( y_k, k \in \mathbb{Z} \), that coincides with the above sequence for \( k = 0, \ldots, Q \).

We set \( x_k = \exp_p(y_k) \) and claim that if \( d \) is small enough, then \( \xi = \{x_k\} \) is a \( 4d \)-pseudotrajectory of \( f \) (and this pseudotrajectory is \( Q \)-periodic by construction).

Indeed, we know that \( |y_k| \leq Yd \) for \( k \in \mathbb{Z} \). Since \( \phi(v) = o(|v|) \) as \( |v| \to 0 \),
\[
|\phi(y_k)| < d, \quad k \in \mathbb{Z}, \tag{10}
\]
if \( d \) is small enough.

The definition of \( \{y_k\} \) implies that
\[
|y_{k+1} - Ay_k| = d, \quad k \in \mathbb{Z}. \tag{11}
\]

Note that
\[
\exp_p^{-1}(f(x_k)) = F(y_k) = Ay_k + \phi(y_k);
\]
thus, it follows from (10) and (11) that
\[
|y_{k+1} - \exp_p^{-1}(f(x_k))| \leq |y_{k+1} - Ay_k| + |\phi(y_k)| < 2d,
\]
which implies that \( \xi = \{x_k\} \) is a \( 4d \)-pseudotrajectory of \( f \) if \( d \) is small enough.

Now we estimate the distances between points of trajectories of the mapping \( F \) and its linearization.

Let us take a vector \( q_0 \in T_pM \) and assume that the sequence \( q_k = F^k(q_0) \) belongs to the ball \( |v| \leq (Y + 8L)d \) for \( 0 \leq k \leq K \). Let \( r_k = A^k q_0 \) (we impose no conditions on \( r_k \) since below we estimate \( \phi \) at points \( q_k \) only).

Take a small number \( \mu \in (0, 1) \) (to be chosen later) and assume that \( d \) is small enough, so that the inequality
\[
|\phi(v)| \leq \mu |v|
\]
holds for \( |v| \leq (Y + 8L)d \).

Then
\[
|q_1| \leq |Aq_0| + |\phi(q_0)| \leq (N+1)|q_0|, \ldots, |q_k| \leq |Aq_{k-1}| + |\phi(q_{k-1})| \leq (N+1)^k|q_0|
\]
for $1 \leq k \leq K$, and
\[
|q_1 - r_1| = |Aq_0 + \phi(q_0) - Aq_0| \leq \mu|q_0|,
\]
\[
|q_2 - r_2| = |Aq_1 + \phi(q_1) - Ar_1| \leq N|q_1 - r_1| + \mu|q_1| \leq \mu(2N + 1)|q_0|,
\]
\[
|q_3 - r_3| \leq N|q_2 - r_2| + \mu|q_2| \leq \mu(N(2N + 1) + (N + 1)^2)|q_0|,
\]
and so on.
Thus, there exists a number $\nu = \nu(K, N)$ such that
\[
|q_k - r_k| \leq \mu\nu|q_0|, \quad 0 \leq k \leq K.
\]
We take $\mu = 1/\nu$, note that $\mu = \mu(K, N)$, and get the inequalities
\[
|q_k - r_k| \leq |q_0|, \quad 0 \leq k \leq K, \quad (12)
\]
for $d$ small enough.
Since $f \in \text{LipPerSh}$, for $d$ small enough, the $Q$-periodic $4d$-pseudotrajectory $\xi$ is $4\mathcal{L}d$-shadowed by a periodic trajectory. Let $p_0$ be a point of this trajectory such that
\[
\text{dist}(p_k, x_k) \leq 4\mathcal{L}d, \quad k \in \mathbb{Z},
\]
(13)
where $p_k = f^k(p_0)$. Let $q_k = \exp_p^{-1}(p_k)$.
The inequalities $|y_k| \leq Yd$ and (13) imply that
\[
|q_k| \leq |y_k| + 2\text{dist}(p_k, x_k) \leq (Y + 8\mathcal{L})d, \quad k \in \mathbb{Z}. \quad (14)
\]
Note that $|q_0| \leq 8\mathcal{L}d$.
Set $r_k = A^kq_0$; we deduce from estimate (12) that if $d$ is small enough, then
\[
|q_K - r_K| \leq |q_0| \leq 8\mathcal{L}d. \quad (15)
\]
Denote by $v^{(2)}$ the second coordinate of a vector $v \in T_pM$.
It follows from the structure of the matrix $A$ that
\[
|r^{(2)}_K| = |q^{(2)}_0| \leq 8\mathcal{L}d. \quad (16)
\]
The relations
\[
|y^{(2)}_K| = Kd \text{ and } |q_K - y_K| \leq 8\mathcal{L}d
\]
imply that
\[ |q_{K}^{(2)}| \geq Kd - 8Ld = 17Ld \] (17)
(recall that \( K = 25L \)).

Estimates (15)–(17) are contradictory. Our lemma is proved in Case 1 for \( l = 2 \).

If \( l = 1 \), then the proof is simpler; the first coordinate of \( A^kv \) equals the first coordinate of \( v \), and we construct the periodic pseudotrajectory perturbing the first coordinate only.

If \( l > 2 \), the reasoning is parallel to that above; we first perturb the \( l \)th coordinate to make it \( Kd \), and then produce a periodic sequence consequently making zero the \( l \)th coordinate, the \((l-1)\)st coordinate, and so on.

If \( \lambda \) is a complex eigenvalue, \( \lambda = a + bi \), we take a real \( 2 \times 2 \) matrix
\[
R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\]
and assume that in representation (8), \( B \) is a real \( 2l \times 2l \) Jordan block:
\[
B = \begin{pmatrix} R & E_2 & 0 & \cdots & 0 \\ 0 & R & E_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R \end{pmatrix},
\]
where \( E_2 \) is the \( 2 \times 2 \) unit matrix.

After that, almost the same reasoning works; we note that \( |Rv| = |v| \) for any 2-dimensional vector \( v \) and construct periodic pseudotrajectories replacing, for example, formulas (9) by the formulas
\[
y_{k+1} = Ay_k + dw_k, \quad k = 0, \ldots, K - 1,
\]
where \( j \)th coordinates of the vector \( w_k \) are zero for \( j = 1, \ldots, 2l - 2, 2l + 1, \ldots, n \), while the 2-dimensional vector corresponding to \((2l-1)\)st and \(2l\)th coordinates has the form \( R^kw \) with \(|w| = 1\), and so on. We leave details to the reader. The lemma is proved.

**Lemma 6.** There exist constants \( C > 0 \) and \( \lambda \in (0, 1) \) depending only on \( N \) and \( L \) and such that, for any point \( p \in \text{Per}(f) \), there exist complementary subspaces \( S(p) \) and \( U(p) \) of the tangent space \( T_pM \) that are \( Df \)-invariant, i.e.,
(H1) \( Df(p)S(p) = S(f(p)) \) and \( Df(p)U(p) = U(f(p)) \), and the inequalities
\[
(H2.1) \ |Df^j(p)v| \leq C\lambda^j|v|, \quad v \in S(p), j \geq 0,
\]
and
\[
(H2.2) \ |Df^{-j}(p)v| \leq C\lambda^j|v|, \quad v \in U(p), j \geq 0,
\]
hold.

**Remark.** Lemma 6 means that the set \( \text{Per}(f) \) has all the standard properties of a hyperbolic set, with the exception of compactness.

**Proof.** Take a periodic point \( p \in \text{Per}(f) \); let \( m \) be the minimal period of \( p \).

Denote \( p_i = f^i(p) \), \( A_i = Df(p_i) \), and \( B = Df^m(p) \). It follows from Lemma 5 that the matrix \( B \) is hyperbolic. Denote by \( S(p) \) and \( U(p) \) the invariant subspaces of \( B \) corresponding to parts of its spectrum inside and outside the unit disk, respectively. Clearly, \( S(p) \) and \( U(p) \) are invariant with respect to \( Df \), \( T_pM = S(p) \oplus U(p) \), and the following relations hold:

\[
\lim_{n \to +\infty} B^n v_s = \lim_{n \to +\infty} B^{-n} v_u = 0, \quad v_s \in S(p), v_u \in U(p). \quad (18)
\]

We prove that inequalities (H2.2) hold with \( C = 16L \) and \( \lambda = 1 + 1/(8L) \) (inequalities (H2.1) are established by similar reasoning applied to \( f^{-1} \) instead of \( f \)).

Consider an arbitrary nonzero vector \( v_u \in U(p) \) and an integer \( j \geq 0 \). Define sequences \( v_i, e_i \in T_pM \) and \( \lambda_i > 0 \) for \( i \geq 0 \) as follows:

\[
v_0 = v_u, \quad v_{i+1} = A_i v_i, \quad e_i = \frac{v_i}{|v_i|}, \quad \lambda_i = \frac{|v_{i+1}|}{|v_i|} = |A_i e_i|.
\]

Let
\[
\tau = \frac{\lambda_{m-1} \cdots \lambda_1 + \lambda_{m-1} \cdots \lambda_2 + \cdots + \lambda_{m-1} + 1}{\lambda_{m-1} \cdots \lambda_0}.
\]

Consider the sequence \( \{a_i \in \mathbb{R}, \ i \geq 0\} \) defined by the following formulas:

\[
a_0 = \tau, \quad a_{i+1} = \lambda_i a_i - 1. \quad (19)
\]

Note that
\[
a_m = 0 \quad \text{and} \quad a_i > 0, \quad i \in [0, m-1]. \quad (20)
\]

Indeed, if \( a_i \leq 0 \) for some \( i \in [0, m-1] \), then \( a_k < 0 \) for \( k \in [i + 1, m] \).
It follows from (18) that there exists \( n > 0 \) such that
\[
|B^{-n}\tau e_0| < 1. \tag{21}
\]

Consider the finite sequence \( \{w_i \in T_p M, \ i \in [0, m(n + 1)]\} \) defined as follows:
\[
\begin{cases}
    w_i = a_i e_i, & i \in [0, m - 1], \\
    w_m = B^{-n}\tau e_0, \\
    w_{m+1+i} = A_i w_{m+i}, & i \in [0, mn - 1].
\end{cases}
\]

Clearly,
\[
w_{km} = B^{k-1-n}\tau e_0, \quad k \in [1, n + 1],
\]
which means that we can consider \( \{w_i\} \) as an \( m(n + 1) \)-periodic sequence defined for \( i \in \mathbb{Z} \).

Let us note that
\[
A_i w_i = a_i A_i e_i = a_i \frac{v_{i+1}}{|v_i|}, \quad i \in [0, m - 2],
\]
\[
w_{i+1} = (\lambda_i a_i - 1) \frac{v_{i+1}}{|v_{i+1}|} = a_i \frac{v_{i+1}}{|v_i|} - e_i + 1, \quad i \in [0, m - 2],
\]
and
\[
A_{m-1} w_{m-1} = a_{m-1} \frac{v_m}{|v_{m-1}|} = \frac{v_m}{\lambda_{m-1}|v_{m-1}|} = e_m
\]
(in the last relation we take into account that \( a_{m-1}\lambda_{m-1} = 1 \) since \( a_m = 0 \)).

The above relations and condition (21) imply that
\[
|w_{i+1} - A_i w_i| < 2, \quad i \in \mathbb{Z}. \tag{22}
\]

Now we take a small \( d > 0 \) and consider the \( m(n + 1) \)-periodic sequence \( \xi = \{x_i = \exp_{p_i}(dw_i), \ i \in \mathbb{Z}\} \).

We claim that if \( d \) is small enough, then \( \xi \) is a \( 4d \)-pseudotrajectory of \( f \).

Denote
\[
\zeta_{i+1} = \exp_{p_{i+1}}^{-1}(f(x_i)) \quad \text{and} \quad \zeta'_{i+1} = \exp_{p_{i+1}}^{-1}(x_{i+1}).
\]

Then
\[
\zeta_{i+1} = \exp_{p_{i+1}}^{-1} f(\exp_{p_i}(dw_i)) = F_i(dw_i) = A_i dw_i + \phi_i(dw_i),
\]

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where the mapping $F_i$ is defined in (5) and $\phi_i(v) = o(|v|)$, and

$$\zeta'_{i+1} = \exp_{p_{i+1}}^{-1}(x_{i+1}) = dw_{i+1}.$$ 

It follows from estimates (22) that

$$|\zeta'_{i+1} - \zeta_{i+1}| \leq 2d$$

for small $d$, and

$$\text{dist}(f(x_i), x_{i+1}) \leq 4d.$$

By Lemma 5, the $m$-periodic trajectory $\{p_i\}$ is hyperbolic; hence, $\{p_i\}$ has a neighborhood in which $\{p_i\}$ is a unique periodic trajectory. It follows that if $d$ is small enough, then the pseudotrajectory $\{x_i\}$ is $4d$-shadowed by $\{p_i\}$.

The inequalities $\text{dist}(x_i, p_i) \leq 4d$ imply that $|a_i| = |w_i| \leq 8L$ for $0 \leq i \leq m - 1$.

Now the equalities $\lambda_i = (a_{i+1} + 1)/a_i$ imply that if $0 \leq i \leq m - 1$, then

$$\lambda_0 \cdots \lambda_{i-1} = \frac{a_1 + 1}{a_0} \frac{a_2 + 1}{a_1} \cdots \frac{a_i + 1}{a_{i-1}} =
\frac{a_i + 1}{a_0} \left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_{i-1}}\right) \geq
\frac{1}{8L} \left(1 + \frac{1}{8L}\right)^i \geq \frac{1}{16L} \left(1 + \frac{1}{8L}\right)^i$$

(we take into account that $1 + 1/(8L) < 2$ since $L \geq 1$).

It remains to note that

$$|Df^i(p)v_u| = \lambda_{i-1} \cdots \lambda_0 |v_u|, \quad 0 \leq i \leq m - 1,$$

and that we started with an arbitrary vector $v_u \in U(p)$.

This proves our statement for $j \leq m - 1$. If $j \geq m$, we take an integer $k > 0$ such that $km > j$ and repeat the above reasoning for the periodic trajectory $p_0, \ldots, p_{km-1}$ (note that we have not used the condition that $m$ is the minimal period). Lemma 6 is proved.

**Lemma 7.** If $f \in \text{LipPerSh}$, then $f$ satisfies Axiom A.
Proof. Denote by $P_l$ the set of points $p \in \text{Per}(f)$ of index $l$ (as usual, the index of a hyperbolic periodic point is the dimension of its unstable manifold).

Let $R_l$ be the closure of $P_l$. Clearly, $R_l$ is a compact $f$-invariant set. We claim that any $R_l$ is a hyperbolic set. Let $n = \dim M$.

Consider a point $q \in R_l$ and fix a sequence of points $p_m \in P_l$ such that $p_m \to q$ as $m \to \infty$. By Lemma 6, there exist complementary subspaces $S(p_m)$ and $U(p_m)$ of $T_{p_m}M$ (of dimensions $n - l$ and $l$, respectively) for which estimates (H2.1) and (H2.2) hold.

Standard reasoning shows that, introducing local coordinates in a neighborhood of $(q, T_q M)$ in the tangent bundle of $M$, we can select a subsequence $p_{m_k}$ for which the sequences $S(p_{m_k})$ and $U(p_{m_k})$ converge (in the Grassmann topology) to subspaces of $T_q M$ (let $S_0$ and $U_0$ be the corresponding limit subspaces).

The limit subspaces $S_0$ and $U_0$ are complementary in $T_q M$. Indeed, consider the “angle” $\beta_{m_k}$ between the subspaces $S(p_{m_k})$ and $U(p_{m_k})$ which is defined (with respect to the introduced local coordinates in a neighborhood of $(q, T_q M)$) as follows:

$$\beta_{m_k} = \min |v^s - v^u|,$$

where the minimum is taken over all possible pairs of unit vectors $v^s \in S(p_{m_k})$ and $v^u \in U(p_{m_k})$.

It is shown in [16, Lemma 12.1] that the values $\beta_{m_k}$ are estimated from below by a positive constant $\alpha = \alpha(C, \lambda, N)$. Clearly, this implies that the subspaces $S_0$ and $U_0$ are complementary.

It is easy to show that the limit subspaces $S_0$ and $U_0$ are unique (which means, of course, that the sequences $S(p_m)$ and $U(p_m)$ converge). For the convenience of the reader, we prove this statement (our reasoning is close to that of [16]).

To get a contradiction, assume that there is a subsequence $p_{m_i}$ for which the sequences $S(p_{m_i})$ and $U(p_{m_i})$ converge to complementary subspaces $S_1$ and $U_1$ different from $S_0$ and $U_0$ (for definiteness, we assume that $S_0 \setminus S_1 \neq \emptyset$).

Due to the continuity of $Df$, the inequalities

$$|Df^j(q)v| \leq C \lambda^j |v|, \quad v \in S_0 \cup S_1,$$

and

$$|Df^j(q)v| \geq C^{-1} \lambda^{-j} |v|, \quad v \in U_0 \cup U_1,$$

hold for $j \geq 0$. 

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Since 
\[ T_q M = S_0 \oplus U_0 = S_1 \oplus U_1, \]
our assumption implies that there is a vector \( v \in S_0 \) such that
\[ v = v^s + v^u, \quad v^s \in S_1, v^u \in U_1, v^u \neq 0. \]

Then
\[ |Df^j(q)v| \leq C\lambda^j|v| \to 0, \quad j \to \infty, \]
and
\[ |Df^j(q)v| \geq C^{-1}\lambda^{-j}|v^u| - C\lambda^j|v^s| \to \infty, \quad j \to \infty, \]
and we get the desired contradiction.

It follows that there are uniquely defined complementary subspaces \( S(q) \) and \( U(q) \) for \( q \in R_l \) with proper hyperbolicity estimates; the \( Df \)-invariance of these subspaces is obvious. We have shown that each \( R_l \) is a hyperbolic set with \( \dim S(q) = n - l \) and \( \dim U(q) = l \) for \( q \in R_l \).

If \( r \in \Omega(f) \), then there exists a sequence of points \( r_m \to r \) as \( m \to \infty \) and a sequence of indices \( k_m \to \infty \) as \( m \to \infty \) such that \( f^{k_m}(r_m) \to r \).

Clearly, if we continue the sequence
\[ r_m, f(r_m), \ldots, f^{k_m-1}(r_m) \]
periodically with period \( k_m \), we get a periodic \( d_m \)-pseudotrajectory of \( f \) with \( d_m \to 0 \) as \( m \to \infty \).

Since \( f \in \text{LipPerSh} \), for large \( m \) there exist periodic points \( p_m \) such that \( \text{dist}(p_m, r_m) \to 0 \) as \( m \to \infty \). Thus, periodic points are dense in \( \Omega(f) \).

Since hyperbolic sets with different dimensions of the subspaces \( U(q) \) are disjoint, we get the equality
\[ \Omega(f) = R_0 \cup \cdots \cup R_n, \]
which implies that \( \Omega(f) \) is hyperbolic. The lemma is proved.

It was mentioned above that if a diffeomorphism \( f \) satisfies Axiom A, then its nonwandering set can be represented as a disjoint union of a finite number of basic sets (see representation [14]).

The basic sets \( \Omega_i \) have stable and unstable “manifolds”:
\[ W^s(\Omega_i) = \{ x \in M : \text{dist}(f^k(x), \Omega_i) \to 0, \quad k \to \infty \} \]
and
\[ W^u(\Omega_i) = \{ x \in M : \text{dist}(f^k(x), \Omega_i) \to 0, \ k \to -\infty \}. \]

If \( \Omega_i \) and \( \Omega_j \) are basic sets, we write \( \Omega_i \to \Omega_j \) if the intersection
\[ W^u(\Omega_i) \cap W^s(\Omega_j) \]
contains a wandering point.

We say that \( f \) has a 1-cycle if there is a basic set \( \Omega_i \) such that \( \Omega_i \to \Omega_i \).

We say that \( f \) has a \( t \)-cycle if there are \( t > 1 \) basic sets \( \Omega_{i_1}, \ldots, \Omega_{i_t} \) such that
\[ \Omega_{i_1} \to \cdots \to \Omega_{i_t} \to \Omega_{i_1}. \]

**Lemma 8.** If \( f \in \text{LipPerSh} \), then \( f \) has no cycles.

**Proof.** To simplify presentation, we prove that \( f \) has no 1-cycles (in the general case, the idea is literally the same, but the notation is heavy).

To get a contradiction, assume that \( p \in (W^u(\Omega_i) \cap W^s(\Omega_i)) \setminus \Omega(f) \).

In this case, there are sequences of indices \( j_m, k_m \to \infty \) as \( m \to \infty \) such that
\[ f^{-j_m}(p), f^{k_m}(p) \to \Omega_i, \ m \to \infty. \]

Since the set \( \Omega_i \) is compact, we may assume that
\[ f^{-j_m}(p) \to q \in \Omega_i \text{ and } f^{k_m}(p) \to r \in \Omega_i. \]

Since \( \Omega_i \) contains a dense positive semi-trajectory, there exist points \( s_m \to r \) and indices \( l_m > 0 \) such that \( f^{l_m}(s_m) \to q \) as \( m \to \infty \).

Clearly, if we continue the sequence
\[ p, f(p), \ldots, f^{k_m-1}(p), s_m, \ldots, f^{l_m-1}(s_m), f^{-j_m}(p), \ldots, f^{-1}(p) \]
periodically with period \( k_m + l_m + j_m \), we get a periodic \( d_m \)-pseudotrajectory of \( f \) with \( d_m \to 0 \) as \( m \to \infty \).

Since \( f \in \text{LipPerSh} \), there exist periodic points \( p_m \) (for \( m \) large enough) such that \( p_m \to p \) as \( m \to \infty \), and we get the desired contradiction with the assumption that \( p \notin \Omega(f) \). The lemma is proved.

Lemmas 5 – 8 show that \( \text{LipPerSh} \subset \Omega S \).
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