Qualitative properties for elliptic problems with CKN operators

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Abstract

The purpose of this paper is to study basic property of the operator

$$L_{\mu_1,\mu_2}u = -\Delta + \frac{\mu_1}{|x|^2}x \cdot \nabla + \frac{\mu_2}{|x|^2},$$

which generates at the origin due to the critical gradient and the Hardy term, where $\mu_1, \mu_2$ are free parameters. This operator arises from the critical Caffarelli-Kohn-Nirenberg inequality. We analyze the fundamental solutions in a weighted distributional identity and obtain the Liouville theorem for the Lane-Emden equation with that operator, by using the classification of isolated singular solutions of the related Poisson problem in a bounded domain $\Omega \subset \mathbb{R}^N (N \geq 2)$ containing the origin.

Keywords: Degenerate operator, fundamental solution, Liouville theorem, Lane-Emden problem.

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1 Introduction

The well-known Caffarelli-Kohn-Nirenberg inequality (CKN inequality for short) proposed in [7] states as following

$$\left( \int_{\mathbb{R}^N} |x|^{-b(p+1)}|u|^{p+1}dx \right)^{\frac{2}{p+1}} \leq C_{a,b,N} \int_{\mathbb{R}^N} |x|^{-2a}|
abla u|^2dx,$$

where $N \geq 2$,

$$-\infty < a < \frac{N - 2}{2}, \quad a \leq b \leq a + 1 \quad \text{and} \quad p = \frac{N + 2(1 + a - b)}{N - 2(1 + a - b)}.$$

A critical CKN inequality with $b = a + 1$ (see [8, 29]) is

$$\int_{\mathbb{R}^N} |x|^{-2a}|
abla u|^2dx \geq \left( \frac{N - 2 - 2a}{2} \right)^2 \int_{\mathbb{R}^N} |x|^{-2(a+1)}|u|^2dx,$$  \hspace{1cm} (1.1)

which, for $a = 0$ and $N \geq 3$, reduces to the classical Hardy inequality

$$\int_{\mathbb{R}^N} |
abla u|^2dx \geq \frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}dx.$$  \hspace{1cm} (1.2)

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It is known that the related elliptic operator arising form \( (1.1) \)
\[
\mathcal{L}_au = -\text{div}(|x|^{-2a} \nabla u) - \frac{(N - 2 - 2a)^2}{4} \frac{u}{|x|^{2(a+1)}}
\]
\[
= |x|^{-2a} \left( -\Delta u + \frac{2a}{|x|^2} x \cdot \nabla u - \frac{(N - 2 - 2a)^2}{4} \frac{u}{|x|^2} \right), \quad \forall u \in C^2_c(\mathbb{R}^N),
\]
where \( a < \frac{N-2}{2} \). If we take \( \mu_1, \mu_2 \) to replace \( 2a \) and \(-\frac{1}{4}(N - 2 - 2a)^2\) respectively in \( \mathcal{L}_a \), we propose a degenerate operator
\[
\mathcal{L}_{\mu_1, \mu_2} = -\Delta + \frac{\mu_1}{|x|^2} x \cdot \nabla + \frac{\mu_2}{|x|^2}
\]
(1.3)
with two free parameters \( \mu_1, \mu_2 \). Note that the operator \( \mathcal{L}_{\mu_1, \mu_2} \) degenerates at the origin both for the gradient term and the critical Hardy term, and here we call it the CKN operator. Our aim is to consider the qualitative properties of the solutions of the elliptic equations with that operator.

When \( \mu_2 = 0 \), \( \mathcal{L}_{\mu_1, 0} = -\Delta + \frac{\mu_1}{|x|^2} x \cdot \nabla \) is a type of degenerate elliptic operator, which together with its divergence form, plays an important role in the harmonic analysis, see [27], it attracts lots of attentions, the basic regularities of related equations [18], qualitative properties for equation with more general degenerate operators in divergence form [28].

When \( \mu_1 = 0 \), \( \mathcal{L}_{0, \mu_2} \) reduces to the Hardy-Leray operator, here we can write \( \mathcal{L}_\mu = -\Delta + \frac{\mu}{|x|^2} \), which is the prototype of the degenerate operators. The equations with Hardy-Leray operators has been studied extensively in the last decades. The authors in [24] initiated the analysis of isolated singular solutions of semilinear problems \( u \mapsto \mathcal{L}_\mu u + g(u) \) under the condition \( \mu \geq -\frac{(N-2)^2}{4} \), where \( \frac{(N-2)^2}{4} \) is the best constant of Hardy inequality [12], more related Hardy inequalities refer to [5, 13, 20]. Normally, the distributional solution of the Hardy problem \( \mathcal{L}_\mu u + g = 0 \) in \( \Omega \) would be proposed as
\[
\int_\Omega u \mathcal{L}_\mu \xi \, dx + \int_\Omega g \xi \, dx = 0, \quad \forall \xi \in C^\infty_c(\Omega),
\]
where \( g \) is a nonlinearity of \( x \) and \( u \). In this distributional sense, [4, 17, 19] show the existence for particular nonlinearity of \( u \) under some restriction that \( N \geq 3 \) and \( \mu \in [\mu_0, 0) \). Later on, Cîrstea at el in [14], Cîrstea in [15] classified the isolated singular classical solution of \( \mathcal{L}_\mu u + b(x) h(u) = 0 \) in \( \Omega \setminus \{0\} \), where both \( b \) and \( h \) consist of regularly varying and slowly varying parts (see their definitions in §1.2.2 of [15]). There a solution is considered as a \( C^1(\Omega \setminus \{0\}) \)-solution in the sense of distributions in \( \Omega \setminus \{0\} \), that is,
\[
\int_\Omega \nabla u \nabla \varphi dx - \int_\Omega \frac{\lambda}{|x|^2} u \varphi dx + \int_\Omega b(x) h(u) \varphi dx = 0, \quad \forall \varphi \in C^1_c(\Omega \setminus \{0\}).
\]
Thanks to a notion of weak solutions of \( \mathcal{L}_\mu u = 0 \) combined with a dual formulation of the equation introduced in [9] the equation
\[
\mathcal{L}_\mu u + g(u) = \nu \quad \text{in} \quad \Omega \quad \ u = 0 \quad \text{on} \ \partial \Omega,
\]
where \( \Omega \) is a bounded smooth domain, \( g \) is a continuous nondecreasing function and \( \nu \) is a Radon measure. When the pole of the Leray-Hardy potential is addressed on the boundary of the domain \( \Omega \), [11, 12] extend the approach to classify the boundary isolated singular solutions of Poisson problem
\[
\mathcal{L}_\mu u + g(u) = \nu_1 \quad \text{in} \quad \Omega, \quad \ u = \nu_2 \quad \text{on} \ \partial \Omega,
\]
where \( \nu_1, \nu_2 \) are bounded Radon measures respectively on \( \Omega \) and \( \partial \Omega \). Recently, the singularities of the Hardy problems have been also studied extensively [23, 21, 23, 26] and the references therein.

When \( \mu_1, \mu_2 \neq 0 \), the CKN operator \( L_{\mu_1, \mu_2} \) defined in (1.3) degenerates at the origin thanks to both the gradient term and the Hardy potential. It is worth noting that the gradient term has the singularity \( |x|^{-1} \) indeed, which is also critical at the origin, compared with the Laplacian operator. The CNK operator in the diverging form \( L_a \) has been discussed in [18, 25]. So our aim is to characterize the roles of the two critical terms in the related fundamental solution and the related Poisson problems.

To consider the fundamental solution of \( L_{\mu_1, \mu_2} \), we provide the following setting of \( \mu_1, \mu_2 \) in this article is the following:

\[
\mu_1 \in \mathbb{R}, \quad \mu_2 \geq -\frac{(2 - N + \mu_1)^2}{4}.
\]

In this setting, direct computation shows that the homogeneous problem

\[
L_{\mu_1, \mu_2} u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}
\]

has two the radially symmetric solutions

\[
\Phi_{\mu_1, \mu_2}(x) = \begin{cases} 
|x|^\tau_{-(\mu_1, \mu_2)} & \text{if } \mu_2 > -\frac{(2 - N + \mu_1)^2}{4}, \\
-x|x|\ln |x| & \text{if } \mu_2 = -\frac{(2 - N + \mu_1)^2}{4}
\end{cases}
\]

and

\[
\Gamma_{\mu_1, \mu_2}(x) = |x|^\tau_{+(\mu_1, \mu_2)},
\]

where

\[
\tau_{-(\mu_1, \mu_2)} = \frac{(2 - N + \mu_1) - \sqrt{(2 - N + \mu_1)^2 + 4\mu_2}}{2}
\]

and

\[
\tau_{+(\mu_1, \mu_2)} = \frac{(2 - N + \mu_1) + \sqrt{(2 - N + \mu_1)^2 + 4\mu_2}}{2}.
\]

When \( \mu_2 = -\frac{(2 - N + \mu_1)^2}{4} \), we observe that

\[
\tau_{-(\mu_1, \mu_2)} = \tau_{+(\mu_1, \mu_2)} = \frac{2 - N + \mu_1}{2} := \tau_0(\mu_1)
\]

and

\[
\tau_0(\mu_1) < 0 \iff \mu_1 < N - 2.
\]

In the following, we use the notation of \( \tau_0 \) replaced by \( \tau_0(\mu_1, \mu_2) \), \( \tau_0 \) done by \( \tau_0(\mu_1) \) for simplicity if there is no confusion.

**Theorem 1.1** Assume that \( N \geq 2, \mu_1 \in \mathbb{R} \) and

\[
\mu_2 \geq -\frac{(2 - N + \mu_1)^2}{4}.
\]

Let \( d\gamma_{\mu_1, \mu_2} := |x|^\tau_{-\mu_1} dx \) and

\[
L^*_{\mu_1, \mu_2} = -\Delta + (-2\tau_1 + \mu_1) \frac{x}{|x|^2} \cdot \nabla,
\]
then we have
\[ \int_{\mathbb{R}^N} \Phi_{\mu_1,\mu_2} \mathcal{L}^*_{\mu_1,\mu_2}(\xi) d\gamma_{\mu_1,\mu_2} = c_{\mu_1,\mu_2} \xi(0), \quad \forall \xi \in C^2_c(\mathbb{R}^N), \tag{1.7} \]
where
\[ c_{\mu_1,\mu_2} = \begin{cases} \sqrt{(2 - N + \mu_1)^2 + 4\mu_2 |S^{N-1}|} & \text{if } \mu_2 > -\frac{(2-N+\mu_1)^2}{4}, \\ |S^{N-1}| & \text{if } \mu_2 = -\frac{(2-N+\mu_1)^2}{4}. \end{cases} \]

Remark 1.1 (i) The identity (1.7) means that
\[ \mathcal{L}^*_{\mu_1,\mu_2} \Phi_{\mu_1,\mu_2} = c_{\mu_1,\mu_2} \delta_0 \]
in the weighted distributional sense (1.7), where \( \delta_0 \) is Dirac mass concentrates at the origin.

(ii) When \( \mu_1 = 0 \), the fundamental solution expressed by Dirac mass in the weighted distributional is derived in [9] and singular solution of Lane-Emden equations with Hardy operators is classified in [10] in this framework.

(iii) When \( \mu_2 = 0 \), \( \mu_1 \leq N - 2 \), \( \Gamma_{\mu_1,\mu_2} = 1 \) and \( \Phi_{\mu_1,\mu_2}(x) = -|x|^{-\mu_1} \ln |x| \), which is the fundamental solution of \( \mathcal{L}_{\mu_1,0} \) in the weighted distributional identity (1.7), which coincides the normal distributional identity and \( \mathcal{L}^*_{\mu_1,0} = \mathcal{L}_{\mu_1,0} \).

Our second concern is to consider the nonexistence of positive solution for Lane-Emden equation with CKN operators
\[ \begin{cases} \mathcal{L}_{\mu_1,\mu_2} u \geq Qu^p & \text{in } \Omega \setminus \{0\}, \\ u \geq 0 & \text{in } \partial \Omega, \end{cases} \tag{1.8} \]
where \( p > 0 \), \( \Omega \) is a bounded domain containing the origin and the potential \( Q \in C^\beta_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \) with \( \beta \in (0,1) \) is a positive function such that for some \( \theta > -2 \),
\[ \liminf_{|x| \to 0^+} Q(x)|x|^{-\theta} > 0. \tag{1.9} \]

Before stating our main results, let us involve two critical exponents
\[ p^\#_{\mu_1,\mu_2,\theta} = 1 + \frac{2 + \theta}{-\tau_+} \quad \text{for } -\frac{(N-2-\mu_1)^2}{4} \leq \mu_2 < 0. \tag{1.10} \]
Here \( p^\#_{\mu_1,\mu_2,\theta} \) is a particular exponent appearing only for \( \tau_+ < 0 \), which is the essence for the nonexistence of positive solutions to (1.8). Note that any positive solution of (1.8) blows up at least at \( \Gamma_{\mu} \) at the origin and this singularity would be improved by the interact of the nonlinearity \( Qu^p \), which may lead to an unadmissible singularity in some weighted \( L^1 \) space. Inspired by this observation, [6,17] showed the nonexistence of positive solutions to (1.8), if \( p \geq p^\#_{0,\mu_2,0} \) when \( \mu_1 = 0 \), \( \mu_2 \in [-\frac{(N-2)^2}{4},0) \) and \( Q \equiv 1 \).

Our interest in this paper is to obtain the nonexistence of positive solutions of (1.8) with the parameters of \( \mu_1, \mu_2 \) in some suitable range. Here a function \( (u) \) is a positive solution of (1.8), if \( u \) satisfies the inequalities \( \mathcal{L}_{\mu_1,\mu_2} u(x) \geq Q(x)u^p(x) \) for any \( x \in \Omega \setminus \{0\} \).

**Theorem 1.2** Let \( N \geq 3 \),
\[ \mu_1 < N - 2, \quad -\frac{(N-2-\mu_1)^2}{4} \leq \mu_2 < 0, \]
potential \( Q \) verify (1.9) with \( \theta > -2 \) and \( p^\#_{\mu_1,\mu_2,\theta} \) be defined in (1.10).

Then for \( p \geq p^\#_{\mu_1,\mu_2,\theta} \), problem (1.8) has no positive solution.
This type of nonexistence is based on the classification of Poisson problem with the CKN operator
\[ L_{\mu_1,\mu_2} u = g \quad \text{in } \Omega \setminus \{0\}, \quad u = 0 \quad \text{on } \partial \Omega, \]
where \( \Omega \) is a bounded domain containing the origin. We provide sharp conditions of \( g \) for the existence and nonexistence of a positive solution to the Poisson problem. The precise results see Section 3 below.

The rest of this paper is organized as follows. In Section 2, we build a weighted distributional identity for \( \Phi_{\mu_1,\mu_2} \). Section 3 is devoted to classify the isolated singular solution of the related Poisson problem and some important estimates. Finally, we deal with the nonexistence of super solutions for semilinear problem (1.8) in Section 4.

2 Fundamental solutions

Here we first remark that the derivation of \( \tau_\pm = \frac{(2-N+\mu_1)\pm \sqrt{(2-N+\mu_1)^2+4\mu_2}}{2} \) is based on the calculation
\[ L_{\mu_1,\mu_2} |x|^\tau = \left(-\tau^2 + (2 - N + \mu_1)\tau + \mu_2\right)|x|^\tau - 2 \]
and \( \tau_\pm \) are zero points of \( \tau(N - 2 - \mu_1 + \tau) - \mu_2 = 0 \) and \( \tau_+ + \tau_- = 2 - N + \mu_1 \). Furthermore, we observe that
\[ L_{\mu_1,\mu_2} (|x|^\tau (-\ln |x|)) = \left(-\tau^2 + (2 - N + \mu_1)\tau + \mu_2\right)|x|^\tau - 2 (-\ln |x|) \]
which implies that for \( \mu_2 = -\frac{(2-N+\mu_1)^2}{4} \) and \( \tau_0 = \frac{2-N+\mu_1}{2} \),
\[ L_{\mu_1,\mu_2} (|x|^\tau (-\ln |x|)) = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \]

From the definition \( L_{\mu_1,\mu_2}^* \), direct computation shows that
\[ |L_{\mu_1,\mu_2}^* \xi(x)| \leq c \left( \|\xi\|_{C^2} + \frac{1}{|x|} \|\xi\|_{C^1} \right), \quad \forall \xi \in C^2_c(\mathbb{R}^N) \] (2.1)
for some \( c > 0 \) independent of \( \xi \).

Proof of Theorem 1.1. Case 1: \( \mu_2 > -\frac{(2-N+\mu_1)^2}{4} \). Let \( u_0 = \Phi_{\mu_1,\mu_2} \), then we have that \( \xi \in C^2(\mathbb{R}^N) \)
\[ 0 = \int_{\mathbb{R}^N \setminus B_\varepsilon} (|x|^{\tau_+ - \mu_1} \xi) L_{\mu_1,\mu_2} u_0 \, dx \]
\[ = \int_{\mathbb{R}^N \setminus B_\varepsilon} (-\Delta) u_0 (|x|^{\tau_+ - \mu_1} \xi) \, dx + \int_{\mathbb{R}^N \setminus B_\varepsilon} \frac{\mu_1}{|x|^2} x \cdot \nabla u_0 (|x|^{\tau_+ - \mu_1} \xi) \, dx \]
\[ + \int_{\mathbb{R}^N \setminus B_\varepsilon} \frac{\mu_2}{|x|^2} |x|^{\tau_+ - \mu_1} \xi u_0 \, dx, \]
where \( B_r(z) \) is the ball with radius \( r > 0 \) centered at \( z \), particularly, \( B_r = B_r(0) \).
We note that
\[
\mu_1 \int_{\mathbb{R}^N \setminus B_r} \frac{1}{|x|^2} x \cdot \nabla u_0 \ |x|^{\tau_+ - \mu_1} \xi \, dx = -\mu_1 [N + (\tau_+ - \mu_1 - 2)] \int_{\mathbb{R}^N \setminus B_r} |x|^{-N} \xi \, dx
\]
\[= -\mu_1 \int_{\mathbb{R}^N \setminus B_r} |x|^{-N} x \cdot \nabla \xi \, dx - \mu_1 \int_{\partial B_r} |x|^{1-N} \xi \, d\omega \]
and
\[
\int_{\mathbb{R}^N \setminus B_r} \frac{\mu_2}{|x|^2} |x|^{\tau_+ - \mu_1} \xi u_0 \, dx = \mu_2 \int_{\mathbb{R}^N \setminus B_r} |x|^{-N} \xi \, dx.
\]
Direct computation shows
\[
\int_{\mathbb{R}^N \setminus B_r} (-\Delta) u_0 \ (|x|^{\tau_+ - \mu_1}) \xi \, dx = \int_{\mathbb{R}^N \setminus B_r} \nabla u_0 \nabla (|x|^{\tau_+ - \mu_1}) \xi \, dx + \int_{\partial B_r} x \cdot \nabla u_0 \ |x|^{\tau_+ - \mu_1} \xi \, d\omega
\]
\[= \int_{\mathbb{R}^N \setminus B_r} u_0(-\Delta)(|x|^{\tau_+ - \mu_1}) \xi \, dx + \int_{\partial B_r} x \cdot \nabla u_0 \ |x|^{\tau_+ - \mu_1} \xi \, d\omega
\]
\[= \int_{\partial B_r} x \cdot \nabla (|x|^{\tau_+ - \mu_1}) u_0 \, d\omega,
\]
where \(\nu = -\frac{x}{|x|}\) be the unit outside normal vector of \(\mathbb{R}^N \setminus B_r\) and \(d\omega\) be the Hausdorff measure of \(S^{N-1}\),
\[
\int_{\mathbb{R}^N \setminus B_r} u_0(-\Delta)(|x|^{\tau_+ - \mu_1}) \xi \, dx = \int_{\mathbb{R}^N \setminus B_r} u_0(-\Delta) \xi \, dx + 2(\tau_+ - \mu_1) \int_{\mathbb{R}^N \setminus B_r} u_0 \frac{x}{|x|^2} \nabla \xi \, dx
\]
\[= (-\mu_1)(N - 2 + \tau_+ - \mu_1) \int_{\mathbb{R}^N \setminus B_r} |x|^{-N} \xi \, dx,
\]
\[
\int_{\partial B_r} \frac{x \cdot \nabla u_0}{|x|} |x|^{\tau_+ - \mu_1} \xi \, d\omega = \tau_- \int_{\partial B_r} |x|^{1-N} \xi \, d\omega = \tau_- \int_{\partial B_r} |x|^{1-N} (\xi(0) + \nabla \xi(0) \cdot x) \, d\omega
\]
\[\to \tau_- |S^{N-1}| \xi(0) \quad \text{as} \quad \varepsilon \to 0^+
\]
and
\[
\int_{\partial B_r} \frac{x \cdot \nabla (|x|^{\tau_+ - \mu_1}) u_0}{|x|} \, d\omega = (\tau_+ - \mu_1) \int_{\partial B_r} |x|^{1-N} \xi \, d\omega + \int_{\partial B_r} |x|^{1-N} x \cdot \nabla \xi \, d\omega
\]
\[= (\tau_+ - \mu_1) \int_{\partial B_r} |x|^{1-N} (\xi(0) + \nabla \xi(0) \cdot x) \, d\omega + \int_{\partial B_r} |x|^{1-N} x \cdot \nabla \xi \, d\omega
\]
\[\to (\tau_+ - \mu_1) |S^{N-1}| \xi(0) \quad \text{as} \quad \varepsilon \to 0^+,
\]
where \(|\nabla \xi|\) is bounded.

Now we conclude that
\[
0 = \int_{\mathbb{R}^N \setminus B_r} (-\Delta) u_0 \ (|x|^{\tau_+ - \mu_1}) \xi \, dx + \int_{\mathbb{R}^N \setminus B_r} \frac{\mu_1}{|x|^2} x \cdot \nabla u_0 \ (|x|^{\tau_+ - \mu_1}) \xi \, dx
\]
\[+ \int_{\mathbb{R}^N \setminus B_r} \frac{\mu_2}{|x|^2} |x|^{\tau_+ - \mu_1} \xi u_0 \, dx
\]
\[= \int_{\mathbb{R}^N \setminus B_r} u_0(-\Delta) \xi \, dx + (-2\tau_+ + \mu_1) \int_{\mathbb{R}^N \setminus B_r} \frac{x}{|x|^2} \cdot \nabla \xi \, d\gamma_{\mu_1, \mu_2}
\]
Here we note that and thanks to (2.1) the solution

\[ \tau_+ - \tau_- = \sqrt{(2 - N + \mu_1)^2 + 4\mu_2} \]

and thanks to \((2.1)\)

\[ \int_{R^N \setminus B_\varepsilon} u_0 L_{\mu_1, \mu_2}^* \xi d\gamma_{\mu_1, \mu_2} \to \int_{R^N} u_0 L_{\mu_1, \mu_2}^* \xi d\gamma_{\mu_1, \mu_2} \quad \text{as} \quad \varepsilon \to 0^+ . \]

Here we note that

\[ \sqrt{(2 - N + \mu_1)^2 + 4\mu_2} > 0 \quad \iff \quad \mu_2 > -\frac{(2 - N + \mu_1)^2}{4} . \]

As a consequence, we obtain that

\[ \int_{R^N} u_0 L_{\mu_1, \mu_2}^* \xi d\gamma_{\mu_1, \mu_2} = c_{\mu_1, \mu_2} \xi(0) , \quad (2.2) \]

where

\[ c_{\mu_1, \mu_2} = \left( \sqrt{(2 - N + \mu_1)^2 + 4\mu_2} \right) |S^{N-1}| . \]

**Case 2:** \( \mu_2 = -\frac{(2-N+\mu_1)^2}{4} \). When

\[ \mu_1 \in \mathbb{R} \quad \text{and} \quad \mu_2 = -\frac{(N-2)^2 - 2(N-2)\mu_1}{4} , \]

the solution \( u_0 = \Phi_{\mu_1, \mu_2} \) has different form and then

\[ \int_{\partial B_\varepsilon} \frac{x \cdot \nabla u_0}{|x|} |x|^{\tau_+ - \mu_1} \xi d\omega = \tau_- \int_{\partial B_\varepsilon} |x|^{1-N}(-\ln |x|)|\xi d\omega - \int_{\partial B_\varepsilon} |x|^{1-N} \xi d\omega \]

and

\[ \int_{\partial B_\varepsilon} \frac{x \cdot \nabla (|x|^{\tau_+ - \mu_1} \xi)}{|x|} u_0 d\omega = (\tau_+ - \mu_1) \int_{\partial B_\varepsilon} |x|^{1-N}(-\ln |x|)|\xi d\omega \]

\[ + \int_{\partial B_\varepsilon} |x|^{1-N}(-\ln |x|)|x \cdot \nabla \xi d\omega , \]
then
\[
0 = \int_{\mathbb{R}^N \setminus B_\varepsilon} (-\Delta) u_0(|x|^{r_1 - \mu_1} \xi) dx + \int_{\mathbb{R}^N \setminus B_\varepsilon} \mu_1 \frac{|\nabla u_0|}{|x|^2} |x|^{r_1 - \mu_1} \xi dx + \int_{\mathbb{R}^N \setminus B_\varepsilon} \frac{\mu_2}{|x|^2} |x|^{r_2 - \mu_2} u dx
\]
\[
= \int_{\mathbb{R}^N \setminus B_\varepsilon} u_0 \mathcal{L}_{\mu_1, \mu_2} \xi d\gamma_{\mu_1, \mu_2} - (r_1 - r_2) \int_{\partial B_\varepsilon} |x|^{1-N} (-\ln |x|) \xi d\omega - \int_{\partial B_\varepsilon} |x|^{1-N} (\xi(0) + O(|x|)) d\omega
\]
\[
- \int_{\partial B_\varepsilon} |x|^{1-N} (\ln |x|) x \cdot \nabla \xi d\omega
\]
\[
\to \int_{\mathbb{R}^N} u_0 \mathcal{L}_{\mu_1, \mu_2} \xi d\gamma_{\mu_1, \mu_2} - |S^{N-1}| \xi(0) \quad \text{as} \quad \varepsilon \to 0^+,
\]
where
\[
c_{\mu_1, \mu_2} = |S^{N-1}| \quad \text{if} \quad \mu_2 = \frac{\mu_1^2}{4} - \frac{(2 - N + \mu_1)^2}{4}.
\]
We complete the proof. \(\square\)

3 Basic tools

An important tool is the comparison principle.

Lemma 3.1 Assume that \(\mu_1 \in \mathbb{R}, \mu_2 \geq -\frac{(N-2-\mu_1)^2}{4}, \ O \) is a bounded \(C^2\) domain containing the origin and \(u_i \) with \(i = 1, 2\) are classical solutions of

\[
\left\{ \begin{array}{ll}
\mathcal{L}_{\mu_1, \mu_2} u_i = f_i & \text{in} \ O \ \setminus \ {0}, \\
{u_i = 0} & \text{on} \ \partial O
\end{array} \right.
\]

and

\[
\limsup_{|x| \to 0^+} u_1(x) \Phi_{\mu_1, \mu_2}^{-1}(x) \leq \liminf_{|x| \to 0^+} u_2(x) \Phi_{\mu_1, \mu_2}^{-1}(x).
\]

If \(f_1 \leq f_2 \) in \(O \ \setminus \ {0}\), then

\[
u_1 \leq u_2 \ \text{in} \ O \ \setminus \ {0}.
\]

Proof. Case 1: \(\mu_2 > -\frac{(N-2-\mu_1)^2}{4}\). Let \(u = u_1 - u_2\) satisfy that

\[
\mathcal{L}_{\mu_1, \mu_2} u \leq 0 \ \text{in} \ O \ \setminus \ {0} \quad \text{and} \quad \limsup_{|x| \to 0^+} u(x) \Phi_{\mu_1, \mu_2}^{-1}(x) \leq 0,
\]

then for any \(\varepsilon > 0\), there exists \(r_\varepsilon > 0\) converging to zero as \(\varepsilon \to 0\) such that

\[
u \leq \varepsilon \Phi_{\mu_1, \mu_2} \ \text{in} \ \overline{B_{r_\varepsilon}} \ \setminus \ {0}.
\]

We see that

\[
u = 0 \ < \ v \Phi_{\mu_1, \mu_2} \ \text{on} \ \partial O,
\]

then by standard comparison principle for \(v \geq v_0\), see CKN inequality, we have that \(v \leq v \Phi_{\mu_1, \mu_2} \in O \ \setminus \ {0}\). By the arbitrary of \(\varepsilon\), we have that \(v \leq 0\) in \(O \ \setminus \ {0}\).

Case 2: \(\mu_2 = -\frac{(N-2-\mu_1)^2}{4}\). Replace \(\Phi_{\mu_1, \mu_2}\) in case 1 by

\[
w_{t_0} := \Phi_{\mu_1, \mu_2} + t_0 \Gamma_{\mu_1, \mu_2}.
\]
where \( t_0 \geq 0 \) be such that
\[
 w_{t_0} \geq 0 \quad \text{on } \partial O
\]
thanks to the boundedness of \( O \).

Repeat the above argument of the case 1 replacing \( \Phi_{\mu_1,\mu_2} \) by \( w_{t_0} \).

\( \square \)

### 3.1 Poisson problems

Our second purpose in this article is to classify the isolated singular solutions of Poisson problem with CKN operator

\[
\begin{align*}
\mathcal{L}_{\mu_1,\mu_2} u &= f \quad \text{in } \Omega \setminus \{0\}, \\
    u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( f : \Omega \rightarrow \mathbb{R} \) is a measurable function and \( \Omega \) is a bounded smooth domain contains the origin. For Poisson problem (3.2), we have the following results.

**Theorem 3.1** Let \( \mu_1 \in \mathbb{R} \) and
\[
\mu_2 \geq -\frac{(N - 2 - \mu_1)^2}{4}
\]
and \( f \) be a function in \( C^{\gamma}_{\text{loc}}(\Omega \setminus \{0\}) \) for some \( \gamma \in (0, 1) \).

(i) Assume that
\[
f \in L^1(\Omega, d\gamma_{\mu_1,\mu_2}) \quad \text{i.e.} \quad \int_{\Omega} |f| d\gamma_{\mu_1,\mu_2} < +\infty,
\]
then for any \( k \in \mathbb{R} \) problem (3.2) has a classical solution \( u_k \) with the asymptotic behavior
\[
\lim_{x \to 0} u_k(x)|x|^{-\tau} = k.
\]

(ii) Assume that \( f \geq 0 \) and
\[
\lim_{r \to 0^+} \int_{\Omega \setminus B_r(0)} f d\gamma_{\mu_1,\mu_2} = +\infty,
\]
then problem (3.2) has no nonnegative solutions.

**Theorem 3.2** Assume that
\[
\mu_1 \in \mathbb{R}, \quad \mu_2 < -\frac{(N - 2 - \mu_1)^2}{4}
\]
and \( f \) be a nonnegative nonzero function, then problem (3.2) has no nonnegative solutions.

Our method is to transform the CKN Poisson problem into a Hardy Poisson problem and the existence and nonexistence could be derived by the results in [9].

We recall the Poisson problem with Hardy-Leray operator

\[
\begin{align*}
\mathcal{L}_\mu u &= f \quad \text{in } \Omega \setminus \{0\}, \\
    u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

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where \( f : \overline{\Omega} \setminus \{0\} \rightarrow \mathbb{R} \) is a Hölder continuous locally in \( \overline{\Omega} \setminus \{0\} \) and

\[
\mathcal{L}_\mu = -\Delta + \frac{\mu}{|x|^2}
\]

is the Hardy-Leray operator with \( \mu \geq -\frac{(N-2)^2}{4} \). [9] classifies the isolated singularities of solutions to (3.6) by building the connection with weak solutions of

\[
\begin{aligned}
\mathcal{L}_\mu u &= f + c_\mu k\delta_0 \quad \text{in} \; \Omega, \\
u &= 0 \quad \text{on} \; \partial \Omega
\end{aligned}
\tag{3.7}
\]

in the \(|x|^{\tau_+(\mu)}dx\)-distributional sense, that is, \( u \in L^1(\Omega, |x|^{\tau_+(\mu)}dx) \) and satisfying

\[
\int_\Omega u\mathcal{L}^*_\mu(\xi)|x|^{\tau_+(\mu)}dx = \int_\Omega f|\xi|^{\tau_+(\mu)}dx + c_\mu k\xi(0), \quad \forall \xi \in C^1_0(\Omega),
\tag{3.8}
\]

where \( k \in \mathbb{R} \) and \( \mathcal{L}^*_\mu u = -\Delta u - 2\tau_+(\mu)\frac{x}{|x|^2} \cdot \nabla u \) and

\[
\tau_\pm(\mu) = \frac{2 - N}{2} \pm \sqrt{\frac{(N-2)^2}{4} + \mu}.
\]

It is worth noting that \( \tau_\pm(\mu) = \tau_\pm(0, \mu) \) with \( \mu_1 = 0 \) and \( \mu_2 = \mu \).

The classification of isolated singular solutions of (3.6) states as following:

**Theorem 3.3** [9, Theorem 1.3] Let \( f \) be a function in \( C^\gamma_{\text{loc}}(\overline{\Omega} \setminus \{0\}) \) for some \( \gamma \in (0, 1) \).

(i) Assume that

\[
f \in L^1(\Omega, |x|^{\tau_+(\mu)}dx) \quad \text{i.e.} \quad \int_\Omega |f(x)| |x|^{\tau_+(\mu)}dx < +\infty,
\tag{3.9}
\]

then for any \( k \in \mathbb{R} \), problem (3.7) admits a unique weak solution \( u_k \), which is a classical solution of problem (3.6). Furthermore, if assume more that

\[
\lim_{{|x| \to 0^+}} f(x)|x|^{2-\tau_-(\mu)} = 0,
\tag{3.10}
\]

then we have the asymptotic behavior

\[
\lim_{{|x| \to 0^+}} u_k(x)|x|^{-\tau_-(\mu)}(x) = k.
\tag{3.11}
\]

(ii) Assume that \( f \) verifies (3.9) and \( u \) is a nonnegative solution of (3.6), then \( u \) is a weak solution of (3.7) for some \( k \geq 0 \).

(iii) Assume that \( f \geq 0 \) and

\[
\lim_{{r \to 0^+}} \int_{\Omega \setminus B_r(0)} f(x)|x|^{\tau_+(\mu)}dx = +\infty,
\tag{3.12}
\]

then problem (3.6) has no nonnegative solutions.

Now we are in a position to show the isolated singular solution of (3.2).

**Proof of Theorem 3.1.** Let \( \tau = \frac{N-2}{2} \) and \( u \in C^2(\Omega \setminus \{0\}) \) be a solution of (3.2) and

\[
u(x) = v(x)|x|^{\tau} \quad \text{in} \; \Omega \setminus \{0\},
\]

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then we have that
\[
\mathcal{L}_{\mu_1, \mu_2}(|x|^\tau v(x)) = |x|^\tau \left( -\Delta v + (\mu_1 - 2\tau) \frac{x}{|x|^2} \cdot \nabla v + (\mu_2 + \tau \mu_1 - \tau(N - 2 + \tau)) \frac{v}{|x|^2} \right)
\]
\[
= |x|^\tau \mathcal{L}_{\tilde{\mu}} v,
\]
where
\[
\mathcal{L}_{\tilde{\mu}} v = -\Delta v + \tilde{\mu} \frac{v}{|x|^2}
\]
with
\[
\tilde{\mu} = \mu_2 + \frac{\mu_1^2}{4} - \frac{\mu_1}{2} (N - 2).
\]
Hence \( v \) is a classical solution of
\[
\begin{cases}
\mathcal{L}_{\tilde{\mu}} v = |x|^{-\frac{\mu_1}{2}} f & \text{in } \Omega \setminus \{0\}, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{3.13}
\]
where we note that \( \tilde{\mu} \geq -\frac{(N-2)^2}{4} \) is equivalent to
\[
\mu_2 \geq -\frac{(N-2-\mu_1)^2}{4}.
\]
It is remarkable that
\[
\tau_\pm(\tilde{\mu}) = \frac{N - 2}{2} \pm \sqrt{\frac{(N - 2)^2}{4} + \mu_2 + \frac{\mu_1^2}{4} - \frac{\mu_1}{2} (N - 2)}
\]
\[
= \frac{2 - N \pm \sqrt{2 - N + \mu_1}^2 + 4\mu_2}{2}
\]
\[
= \tau_\pm(\mu_1, \mu_2) + \frac{\mu_1}{2},
\]
which implies that
\[
f \in L^1(\Omega, d\gamma_{\mu_1, \mu_2}) \iff |x|^{-\frac{\mu_1}{2}} f \in L^1(\Omega, |x|^{\tau_-(\tilde{\mu})} dx)
\]
and for \( k \in \mathbb{R} \)
\[
\lim_{|x| \to 0^+} v(x)|x|^{-\tau_-(\tilde{\mu})}(x) = k \iff \lim_{|x| \to 0^+} u(x)|x|^{-\tau_-(\tilde{\mu})-\frac{\mu_1}{2}} = \lim_{|x| \to 0^+} u(x)|x|^{-\tau_-(\mu_1, \mu_2)-\frac{\mu_1}{2}} = k.
\]
Therefore, the existence and nonexistence of problem \([3.2]\) follows Theorem \([3.3]\) part (i) and (iii). We complete the proof.

**Proof of Theorem \([3.2]\)**. Observe that
\[
\mu_2 < -\frac{(N - 2 - \mu_1)^2}{4} \iff \tilde{\mu} < -\frac{(N - 2)^2}{4}
\]
and as shown in the proof Theorem \([3.1]\) the nonexistence of problem \([3.2]\) follows by the nonexistence of positive solution of \([3.13]\) with \( \tilde{\mu} < -\frac{(N-2)^2}{4} \) from \([9, \text{Proposition 5.2}]\). 

\( \square \)
3.2 Some estimates

In order to improve the blowing up rate at the origin or decay at infinity, we need the following estimates. Let

\[ c(\tau) = -\tau(N - 2 - \mu_1 + \tau) + \mu_2 > 0, \quad (3.14) \]

the map: \( \tau \mapsto c(\tau) \) is concave and \( \tau_+ \geq \tau_- \) are the two zero points of \( c(\tau) = 0 \). There holds

\[ L_{\mu_1, \mu_2}|\cdot|^{\tau} = c(\tau)|\cdot|^{\tau - 2} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \quad (3.15) \]

Lemma 3.2 Assume that \( \mu_1 \in \mathbb{R}, \mu_2 \geq -\frac{(N-2-\mu_1)^2}{4}, \) the nonnegative function \( g \in C^3_{\text{loc}}(\mathbb{B}_r \setminus \{0\}) \) for some \( \beta \in (0, 1) \) and there exists \( \tau \in (\tau_-, \tau_+) \), \( c > 0 \) and \( r_0 > 0 \) such that

\[ g(x) \geq c|x|^{\tau - 2} \quad \text{in} \quad \mathbb{B}_r \setminus \{0\}. \]

Let \( u_g \) be a positive solution of problem

\[ L_{\mu_1, \mu_2}u \geq g \quad \text{in} \quad \mathbb{B}_r \setminus \{0\}, \quad u \geq 0 \quad \text{on} \quad \partial \mathbb{B}_r, \quad (3.16) \]

then there exists \( c_1 > 0 \) such that

\[ u_g(x) \geq c_1|x|^{\tau} \quad \text{in} \quad \mathbb{B}_{2r}. \]

Proof. For \( \tau \in (\tau_-, \tau_+) \), we have that

\[ L_{\mu_1, \mu_2}|\cdot|^{\tau} = c(\tau)|\cdot|^{\tau - 2} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \]

For \( r > 0 \), let

\[ w(x) = |x|^{\tau} - r^{\tau - \tau_+}|x|^{\tau_+} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \]

which verifies that

\[ L_{\mu_1, \mu_2}w = c(\tau)|x|^{\tau - 2} \quad \text{in} \quad \mathbb{B}_r \setminus \{0\}, \quad w = 0 \quad \text{on} \quad \partial \mathbb{B}_r, \]

where \( c(\tau) > 0 \). Then our argument follows by Lemma 3.1. \( \square \)

4 Nonexistence in a punctured domain

We prove the nonexistence of positive solution of (1.8) in a punctured domain \( \Omega \setminus \{0\} \) by contradiction, i.e. (1.8) is assumed to have a positive solution \( u_0 \) and we will obtain a contradiction from Theorem 3.1 and Theorem 3.2. In this section, we can assume that

\[ B_2 \subset \Omega \]

and

\[ Q(x) \geq q_0|x|^\theta \quad \text{in} \quad \mathbb{B}_1 \setminus \{0\}. \]

Proof of Theorem 1.1. By contradiction, we assume that \( u_0 \) is a positive super solution of (1.8) in \( \Omega \setminus \{0\} \). Let

\[ v_0(x) = |x|^{\tau_+} - 1 \quad \text{for} \quad \mathbb{B}_1 \setminus \{0\}, \]

then

\[ L_{\mu_1, \mu_2}v_0(x) = \frac{\mu_2}{|x|^2} < 0 \]
for \( \mu_2 < 0 \). Here we recall that \( \tau_+ < 0 \) for \( -\frac{(N-2-\mu_1)^2}{4} \leq \mu_2 < 0 \).

Since \( B_2 \subset \Omega \) and there exists \( t_1 > 0 \) such that

\[
u_0 \geq t_1 \quad \text{for } |x| = 1.
\]

Therefore, for any \( \epsilon > 0 \), there holds

\[
\liminf_{x \to 0} (u_0 - t_1 v_0 + \epsilon \Phi_{\mu_1, \mu_2})(x) \Phi_{\mu_1, \mu_2}^{-1} \geq \epsilon,
\]

\[
(u_0 - t_1 v_0 + \epsilon \Phi_{\mu_1, \mu_2})(x) \geq 0 \quad \text{for } |x| = 1
\]

and

\[
\mathcal{L}_{\mu_1, \mu_2}(u_0 - t_1 v_0 + \epsilon \Phi_{\mu_1, \mu_2}) \geq 0 \quad \text{in } B_1 \setminus \{0\}
\]

It follows by Lemma 3.1 that

\[
u_0 - t_1 v_0 + \epsilon \Phi_{\mu_1, \mu_2} \geq 0 \quad \text{in } B_1 \setminus \{0\}
\]

and by the arbitrary of \( \epsilon > 0 \), we have that

\[
u_0 \geq t_1 v_0 \quad \text{in } B_1 \setminus \{0\}.
\]

Let

\[
u_{\mu_1, \mu_2, \theta}^\# = \frac{N + \theta}{-\tau_+} - 1.
\]

Then

\[
u_{\mu_1, \mu_2, \theta}^\# = p_{\mu_1, \mu_2, \theta}^\# \quad \text{for } \mu_2 = -\frac{(N-2-\mu_1)^2}{4}
\]

and

\[
u_{\mu_1, \mu_2, \theta}^\# > p_{\mu_1, \mu_2, \theta}^\# \quad \text{for } -\frac{(N-2-\mu_1)^2}{4} < \mu_2 < 0.
\]

Set \( g(x) = Q(x) u_0(x)^p \), then for some \( d_0 > 0 \)

\[
u_0(x) \geq d_0 |x|^{\tau_+} \quad \text{in } B_{\frac{1}{\tau_+}} \setminus \{0\}.
\]

**Part 1: Nonexistence for \(-\frac{(N-2-\mu_1)^2}{4} \leq \mu_2 < 0\) and \( p \geq q_{\mu_1, \mu_2, \theta}^\#.\)** Note that

\[
\mathcal{L}_{\mu_1, \mu_2} u_0(x) \geq g(x) \geq d_0^p |x|^\theta + \tau_+ \quad \text{in } B_{\frac{1}{\tau_+}} \setminus \{0\},
\]

where \( \theta + \tau_+ p + \tau_+ \leq -N \) and

\[
\lim_{\tau \to 0^+} \int_{B_{\tau}(0) \setminus B_r(0)} Q(x) |x|^{\tau_+ p} \Gamma_{\mu_1, \mu_2} dx = +\infty
\]

by the fact that \( p \geq q_{\mu_1, \mu_2, \theta}^\#.\) As a consequence, we see that \( u_0 \) is a solution of

\[
\mathcal{L}_{\mu_1, \mu_2} u_0 \geq g \quad \text{in } \Omega \setminus \{0\}, \quad u_0 \geq 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]

Then there is a contradiction from Theorem 3.1 part (ii).

**Part 2: Nonexistence for \(-\frac{(N-2-\mu_1)^2}{4} < \mu_2 < 0\) and \( p \in (p_{\mu_1, \mu_2, \theta}^\#, q_{\mu_1, \mu_2, \theta}^\#).\)** Let \( \tau_0 = \tau_+ < 0 \), then

\[
\mathcal{L}_{\mu_1, \mu_2} u_0(x) \geq q_0 d_0^p |x|^\theta + \tau_0 \geq q_0 d_0^p |x|^{-2} \quad \text{in } B_{\frac{1}{\tau_+}} \setminus \{0\},
\]

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where
\[ \tau_1 := p\tau_0 + \theta + 2. \]

By Lemma 3.2, we have that
\[ u_0(x) \geq d_1|x|^\tau_1 \text{ in } B_{\frac{1}{2}} \setminus \{0\}. \]

Iteratively, we recall that
\[ \tau_j := p\tau_{j-1} + \theta + 2, \quad j = 1, 2, \ldots. \]

Note that
\[ \tau_1 - \tau_0 = (p - 1)\tau_0 + \theta + 2 < 0 \]
for \( p \in (p_{\theta, \mu}^#, q_{\theta, \mu}^#). \)

If \( \tau_1 p + \theta + 2 \leq -N, \) then \( \tau_{j+1} p + \theta < -N \) and
\[ \mathcal{L}_{\mu_1, \mu_2} u_0(x) \geq g(x) \geq q_0 d_1^p |x|^\theta + \tau_1 p \text{ in } B_{\frac{1}{2}} \setminus \{0\} \]
and a contradiction comes from Theorem 3.1 part (ii).

If not, we iterate above procedure. If
\[ \tau_{j+1} := \tau_j p + \theta + 2 \in (\tau_-, \tau_+), \]

it following by Lemma 3.2 that
\[ u_0(x) \geq d_{j+1} |x|^{\tau_{j+1}} \]
where
\[ \tau_{j+1} = p\tau_j + 2 + \theta < \tau_j. \]

If \( \tau_{j+1} p + \tau_+ + \theta \leq -N, \) then
\[ \mathcal{L}_{\mu_1, \mu_2} u_0(x) \geq q_0 d_{j+1}^p |x|^\theta + \tau_{j+1} p \text{ in } B_{\frac{1}{2}} \setminus \{0\} \]
then we have a contradiction and we are done.

Furthermore, this iteration must stop by finite times since
\[ \tau_j - \tau_{j-1} = p(\tau_{j-1} - \tau_{j-2}) = p^{j-1}(\tau_1 - \tau_0) \rightarrow -\infty \text{ as } j \rightarrow +\infty. \]

As a consequence, we obtain the nonexistence for the case \( p \in (p_{\mu_1, \mu_2, \theta}^#, q_{\mu_1, \mu_2, \theta}^#). \)

Part 3: Nonexistence for \(- \frac{(N-2-\mu_1^2)}{4} < \mu_2 < 0 \) and \( p = p_{\mu_1, \mu_2, \theta}^# > 1. \) Note that
\[ \mathcal{L}_{\mu_1, \mu_2} u_0(x) \geq q_0 d_0^p |x|^\theta + \tau_+ p \text{ in } B_{\frac{1}{2}} \setminus \{0\}, \]
where in this case
\[ \theta + \tau_+ p + 2 = \tau_. \]

Here we see that for some \( \sigma_0 > 0 \)
\[ \frac{1}{2} Q u_0^{p-1}(x) \geq \sigma_0 |x|^{-2} \text{ in } B_{\frac{1}{2}} \setminus \{0\}. \]

If \( \mu_2 - \sigma_0 < - \frac{(N-2-\mu_1^2)}{4}, \) then we have that
\[ \mathcal{L}_{\mu_1, \mu_2 - \sigma_0} u_0(x) = f \geq 0, \]
which has no positive solution by Theorem 3.2.

If \( \mu_2 - \sigma_0 \geq -\frac{(N-2-\mu_1)^2}{4} \), we can write problem (1.8) as following

\[
\mathcal{L}_{\mu_1, \mu_2 - \sigma_0} u_0 \geq \frac{1}{2} Q(x) u_0^p \quad \text{in} \quad B_{\frac{2}{5}} \setminus \{0\},
\] (4.1)

which the critical exponent

\[
p^\#_{\mu_1, \mu_2 - \sigma_0, \theta} = 1 + \frac{2 + \theta}{-\tau_+(\mu_1, \mu_2 - \sigma_0)} < 1 + \frac{2 + \theta}{-\tau_+(\mu_1, \mu_2)} = p^\#_{\mu_1, \mu_2},
\]

since \( t \mapsto \tau_+(\mu_1, t) \) is decreasing. Thus it reduces to the case: part 2 for (4.1) for \( p = p^\#_{\mu_1, \mu_2, \theta} \) is supercritical and we obtain the nonexistence.

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