COMPACT HERMITIAN MANIFOLDS WITH QUASI-NEGATIVE CURVATURE

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Abstract. In this work, we show that along a particular choice of Hermitian curvature flow, the non-positivity of Chern-Ricci curvature will be preserved if the initial metric has non-positive bisectional curvature. As an application, we show that the canonical line bundle of a compact Hermitian manifold with non-positive bisectional curvature and quasi-negative Chern-Ricci curvature is ample.

1. Introduction

Let \((M, g)\) be a compact Hermitian manifold with complex dimension \(n\). In hyperbolic geometry, a fundamental question is to determine the positivity of the canonical line bundle \(K_M\). In particular, a conjecture of Kobayashi asserts that the canonical line bundle is ample if the manifold is Kobayashi hyperbolic \([10, \text{p.370}]\).

In the viewpoint of differential geometry, the ampleness of \(K_M\) is related to the negativity of the Chern-Ricci curvature while the hyperbolicity can be assured by the negativity of holomorphic sectional curvature using Yau’s Schwarz Lemma \([23, 13]\). When \(M\) is in addition Kähler and projective, it was conjectured by Yau that if \(M\) has negative holomorphic sectional curvature, then \(K_M\) is ample. It was solved recently by Wu and Yau \([19]\). Soon after, the assumption on projectivity was removed by Tosatti and Yang \([17]\). More recently, Diverio and Trapani \([4]\) further generalized the result which only assume quasi-negative holomorphic sectional curvature. In \([20]\), Wu and Yau also gave a direct proof of this result. See also \([6, 7, 8]\) for related works.

In view of the conjecture by Kobayashi, it is natural to consider the case when \((M, g_0)\) is only assumed to be Hermitian with negative holomorphic sectional curvature. Along this direction, the Hermitian case is still open.

Conjecture 1.1. \([22]\) Conjecture 1.1] Let \((M, g)\) be a compact complex manifold.

(a) If \(M\) is Kobayashi hyperbolic, then its canonical line bundle \(K_M\) is ample.
If $M$ admits a Hermitian metric with quasi-negative holomorphic sectional curvature, then $K_M$ is ample.

If $M$ admits a Hermitian metric with negative holomorphic sectional curvature, then $K_M$ is ample.

Clearly, (b) implies (c) and (a) implies (c) by Yau’s Schwarz Lemma. Inspired by the proof in [19], Yang and Zheng introduced the concept of real bisectional curvature of a Hermitian metric which is slightly stronger than the holomorphic sectional curvature. Using the technique in [19], they further extended the solution by Diverio and Trapani to the case when $M$ is a Kähler manifold equipped with a Hermitian metric with quasi-negative real bisectional curvature. See also [9] for another curvature condition involving the torsion.

To the best of author’s knowledge, if the Kählerity is a priori unknown, then there are not much progress towards the conjecture even the slightly stronger curvature condition, real bisectional curvature. In this paper, we are interested in quasi-negative case. As an attempt, we consider the much stronger curvature assumption, namely the quasi-negative bisectional curvature. In particular, if the bisectional curvature is negative, then the canonical line bundle must be ample. In fact, when $(M, g)$ is Kähler, the curvature condition is related to another conjecture by Yau which was solved recently by Liu [12].

**Theorem 1.1.** [12] Let $M^n$ be a compact Kähler manifold with nonpositive holomorphic bisectional curvature. Then there exists a finite cover $M'$ of $M$ such that $M'$ is a holomorphic and metric fibre bundle over a compact Kähler manifold $N$ with non-positive bisectional curvature and $c_1(N) < 0$, and the fiber is a flat complex torus.

In [12], Liu observed that if the initial Kähler metric has nonpositive bisectional curvature, then the nonpositivity of Ricci curvature will be preserved for a short time along the Kähler-Ricci flow. Moreover, the full curvature tensor will be dominated by the Ricci curvature. By using the strong maximum principle of Kähler-Ricci flow, Liu was able to extend the result in [21] and confirm the full conjecture by Yau.

Inspired by the idea in [12], we attempt to deform the Hermitian metric so that its Chern-Ricci curvature behaves in a better way. Using the idea introduced by Ustinovskiy [18], we show that the Hermitian analogy of curvature conditions introduced by Liu [12] will be preserved for a short time by a particular choice of Hermitian curvature flow introduced by Streets and Tian [15].

**Theorem 1.2.** Suppose $(M, g_0)$ is a compact Hermitian manifold with non-positive bisectional curvature. Let $g(t), t \in [0, T]$ be a solution to the equation

\[
\begin{cases}
\frac{\partial}{\partial t} g_{ij} = -S_{ij}; \\
g(0) = g_0.
\end{cases}
\]
where $S$ is the second Ricci curvature with respect to the Chern connection. Then there is $\tau > 0$ such that the Chern-Ricci curvature $\text{Ric}(g(t))$ is nonpositive on $[0, \tau]$. 

For a more detailed result, we refer to Proposition 4.2. With the help of strong maximum principle of tensor, we have the following main result.

**Theorem 1.3.** Under the assumption in Theorem 1.2, if in addition $g_0$ has quasi-negative Chern-Ricci curvature, then the canonical line bundle is ample. In particular, there is a Kähler-Einstein metric $g = -\text{Ric}(g)$ on $M$.

In fact, in the non-Kähler Hermitian geometry, there are various types of Ricci curvature associated to the Chern-curvature tensor due to the presence of torsion. The Chern-Ricci curvature and the second Ricci curvature is particularly interesting. In view of the curvature assumptions in Conjecture 1.1 it seems to be more natural to use the Hermitian curvature flow introduced in [15] rather than the Chern-Ricci flow introduced by Gill [5]. Relatively, the Chern-Ricci flow seems to be more sensitive to the information from Bott-Chern class rather than differential geometric quantities. From Theorem 1.3 it is surprising that the Hermitian Ricci flow reflects the ampleness of the canonical line bundle. We believe that the Hermitian Ricci flow will be useful in studying the existence of Kähler-Einstein metric on complex manifolds with negative curvature.

The paper is organized as follows: In section 2 we will collect some useful information about the Chern connection. In section 3, we will derive evolution equations for the Hermitian Ricci flow. In section 4, we prove the preservation of non-positivity of $\text{Ric}(g(t))$. In section 5, we give a proof on Theorem 1.3.

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2. **Chern connection**

In this section, we collect some useful formulas for the Chern connection. Those materials can be found in [16] [14]. Let $(M, g)$ be a Hermitian manifold. The **Chern connection** of $g$ is defined as follows: In local holomorphic coordinates $z^i$, for a vector field $X_i \partial_i$, where $\partial_i := \frac{\partial}{\partial z^i}$, $\bar{\partial}_i = \frac{\partial}{\partial \bar{z}^i}$,

$$\nabla_i X^k = \partial_i X^k + \Gamma^k_{ij} X^j; \ 
abla_i X^k = \bar{\partial}_i X^k.$$

For a $(1, 0)$ form $a = a_i dz^i$,

$$\nabla_i a_j = \partial_i a_j - \Gamma^k_{ij} a_k; \ 
\nabla_i a_j = \bar{\partial}_i a_j.$$
Here \( \nabla_i := \nabla_{\partial_i} \), etc. \( \Gamma \) are the coefficients of \( \nabla \), with
\[
\Gamma^k_{ij} = g^{kl} \partial_l g_{jl}.
\]
Noted that Chern connection is a connection such that \( \nabla g = \nabla J = 0 \) and the torsion has no \( (1, 1) \) component. The torsion of \( g \) is defined to be
\[
T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.
\]
We remark that \( g \) is Kähler if and only if \( T = 0 \). Define the Chern curvature tensor of \( g \) to be
\[
R_{ij} = -\partial_j \Gamma^l_{ik}.
\]
We raise and lower indices by using metric \( g \). Direct computations show:
\[
R_{ijkl} = R_{jilk}.
\]

The Chern-Ricci curvature (first Ricci curvature) is defined by
\[
R_{ij} = g^{kl} R_{ijkl} = -\partial_i \partial_j \log \det g.
\]

And the second Ricci curvature is defined by \( S_{ij} = g^{kl} R_{klij} \). Note that if \( g \) is not Kähler, then \( R_{ij} \) is not necessarily equal to \( S_{ij} \).

**Lemma 2.1.** The commutation formulas for the Chern curvature are given by
\[
[\nabla_i, \nabla_j] X^l = R_{ijkl} X^k, \quad [\nabla_i, \nabla_j] a_k = -R_{ijk} a_l;
\]
\[
[\nabla_i, \nabla_j] a^l = -R_{ijkl} a^k, \quad [\nabla_i, \nabla_j] a_k = R_{ijkl} a^l.
\]

When \( g \) is not Kähler, the Bianchi identities maybe fail. The failure can be measured by the torsion tensor.

**Lemma 2.2.** In a holomorphic local coordinates, let \( T_{ijk} = g_{pk} T^p_{ij} \), we have
\[
R_{ijkl} - R_{klij} = -\nabla_j T_{ikl},
\]
\[
R_{ijkl} - R_{iklj} = -\nabla_i T_{jkl},
\]
\[
R_{ijkl} - R_{klij} = -\nabla_j T_{ikl} - \nabla_k T_{jli} = -\nabla_i T_{jkl} - \nabla_i T_{ikj},
\]
\[
\nabla_p R_{ijkl} - \nabla_i R_{pjk} = -T^r_{pi} R_{rjkl},
\]
\[
\nabla_q R_{ijkl} - \nabla_j R_{iqkl} = -T^s_{qj} R_{iskl}.
\]

It can be checked easily that for \( X, Y \in T^{1,0} M \), \( R(X, \bar{X}, Y, \bar{Y}) \) is real-valued. We consider the following curvature condition.

**Definition 2.1.** We say that \((M, g)\) has holomorphic bisectional curvature bounded above by a function \( \kappa(x) \) if for any \( x \in M \), \( X, Y \in T^{1,0}_x M \),
\[
R(X, X, Y, Y) \leq \kappa B(X, X, Y, Y)
\]
where \( B_{ijkl} = g_{ij} g_{kl} + g_{il} g_{kj} \).

Here we should remark that our notation of bisectional curvature is stronger than that in [11].
**Definition 2.2.** We say that \((M, g)\) has Chern-Ricci curvature bounded above by a function \(\kappa(x)\) if for any \(p \in M\), \(X \in T^1_p M\),
\[
\text{Ric}(X, \bar{X}) \leq \kappa(p)g(X, \bar{X}).
\]
If \(\kappa\) is non-positive and negative at some point \(z \in M\), then we say that \(g\) has quasi-negative Chern-Ricci curvature.

In this note, all the curvature tensor \(Rm\) will be referring to the curvature tensor with respect to Chern connection.

### 3. Evolution equation along the Hermitian Ricci flow

In this section, we will discuss a special type of Hermitian Ricci flow introduced by [15] with \(Q \equiv 0\):

\[
\tag{3.1}
\begin{cases}
\frac{\partial}{\partial t}g_{ij} = -S_{ij}; \\
g(0) = g_0.
\end{cases}
\]

Here \(S_{ij} = g^{kl}R_{klij}\) is the second Ricci curvature with respect to the Chern connection while the Chern-Ricci curvature (or first Ricci curvature) is defined by \(R_{ij} = g^{kl}R_{ijkl}\). When the metric is Kähler, the first and second Ricci curvature coincides with the Riemannian Ricci curvature. However they are all different in general.

Now we derive some evolution equation for \(R_{ijkl}\). This can be found in [15, Section 6]. We include the computation here for reader’s convenience.

**Lemma 3.1.** Along the Hermitian Ricci flow (3.1), we have
\[
\partial_t R_{ijkl} = \Delta R_{ijkl} + g^{rs} \left[ T^p_{ri} \nabla_s R_{pjkl} + T^q_{sj} \nabla_r R_{iqkl} + T^p_{ri} T^q_{sj} R_{pqkl} \\
+ R_{ijr}^p R_{pskl} + R_{rjk}^p R_{ispl} - R_{rjpl} R_{iskl}^p \right] \\
- \frac{1}{2} \left[ S_i^p R_{pjk}^l + S_k^p R_{ijpl} + S_j^p R_{iqkl} + S_i^q R_{ijkl} \right]
\]
where \(\Delta = \Delta_{g(t)}\).

**Proof.**
\[
\partial_t R_{ij}^l = -\partial_t \partial_j \Gamma_i^l_{ik} \\
= \nabla_j \nabla_i S_k^l \\
= g^{rs} \nabla_j \nabla_i R_{rsk}^l.
\]

By Lemma 2.2, we have
\[
\nabla_i R_{rsk}^l = \nabla_r R_{is\bar{k}}^l + T^p_{ri} R_{psk}^l.
\]
And hence,
\[
\nabla_j \nabla_i R_{\bar{s}\bar{k}}^l = \nabla_j \nabla_r R_{\bar{s}\bar{k}}^l + \nabla_j \left( T_{ri}^p R_{p\bar{s}\bar{k}}^l \right)
\]
\[
= \nabla_r \nabla_j R_{\bar{s}\bar{k}}^l + R_{r\bar{q}}^p R_{\bar{q}\bar{k}}^l + R_{r\bar{q}}^p R_{\bar{q}\bar{i}}^l - R_{r\bar{j}}^l R_{\bar{s}\bar{k}}^p
\]
\[
- R_{r\bar{j}}^l q_{s} R_{i\bar{k}}^l + \left( \partial_r T_{ri}^p \right) R_{psk}^l + T_{ri}^p \nabla_j R_{psk}^l
\]
\[
= \nabla_r \nabla_s R_{ij\bar{k}}^l + \left( \partial_r T_{sj}^q \right) R_{iq\bar{k}}^l + T_{sj}^q \nabla_i R_{iq\bar{k}}^l
\]
\[
+ R_{r\bar{j}}^l q_{s} R_{i\bar{k}}^l + R_{r\bar{j}}^l p R_{p\bar{s}}^l - R_{r\bar{j}}^l \bar{p} R_{i\bar{s}}^l
\]
\[
- R_{r\bar{s}}^l q_{j} R_{i\bar{k}}^l + R_{r\bar{j}}^l q_{i} R_{p\bar{s}}^l + R_{r\bar{j}}^l q R_{p\bar{s}}^l - R_{r\bar{j}}^l p R_{i\bar{s}}^l.
\]

By Lemma 2.1,
\[
\nabla_i \nabla_j R_{\bar{s}\bar{k}}^l = \nabla_j \nabla_i R_{\bar{s}\bar{k}}^l - R_{ijr}^l R_{p\bar{s}}^l - R_{ij\bar{k}}^l p R_{r\bar{s}}^l
\]
\[
+ R_{ijq}^l q_{s} R_{i\bar{k}}^l + R_{ijp}^l l R_{r\bar{s}}^l.
\]

The result follows by combining this with (3.3), (3.4) and (3.1). \qed

By tracing \( k \) and \( l \), we arrive at the evolution equation of the Chern-Ricci curvature (or first Ricci curvature).

**Lemma 3.2.** Along the Hermitian Ricci flow, we have the following evolution equation for the first Ricci curvature.

\[
\partial_t R_{ij} = \Delta R_{ij} + g^{rs} \left( T_{ri}^p \nabla_s R_{p\bar{j}} + T_{sj}^q \nabla_r R_{i\bar{q}} + T_{ri}^p T_{sj}^q R_{pq} \right)
\]
\[
+ R_{ijr}^l p R_{r}^k - \frac{1}{2} \left[ S_{i}^p R_{pj} + S_{j}^q R_{iq} \right].
\]

**Proof.**

\[
\partial_t R_{ij} = \partial_t (g^{kl} R_{ijkl})
\]
\[
= S^{kl} R_{ijkl} + g^{kl} \partial_t R_{ijkl}
\]
\[
= \Delta R_{ij} + g^{rs} \left[ T_{ri}^p \nabla_s R_{p\bar{j}} + T_{sj}^q \nabla_r R_{i\bar{q}} + T_{ri}^p T_{sj}^q R_{pq} \right]
\]
\[
+ R_{ijr}^l p R_{p\bar{s}} + g^{kl} R_{ij\bar{k}}^l p R_{i\bar{s}pl} - g^{kl} R_{r\bar{j}p} R_{i\bar{s}}^p
\]
\[
- \frac{1}{2} \left[ S_{i}^p R_{pj} + S_{j}^q R_{iq} \right]
\]
\[
= \Delta R_{ij} + g^{rs} \left( T_{ri}^p \nabla_s R_{p\bar{j}} + T_{sj}^q \nabla_r R_{i\bar{q}} + T_{ri}^p T_{sj}^q R_{pq} \right)
\]
\[
+ R_{ijk}^l p R_{r}^k - \frac{1}{2} \left[ S_{i}^p R_{pj} + S_{j}^q R_{iq} \right].
\]

\qed
4. Preservation of Curvature Conditions

In this section, we will adapt the idea by Liu in [12] to the Hermitian Ricci flow to show that the non-positivity of Chern-Ricci curvature will be preserved for a short time. Before we give a proof on the preservation of curvature conditions, we first recall the doubling-time estimate for the Hermitian Ricci flow in [15].

**Proposition 4.1.** Suppose $(M, g_0)$ is compact Hermitian manifold with
\[ \sup_M |Rm_{g_0}| + |T_{g_0}|^2 + |\nabla_{g_0} T_{g_0}| \leq K_0 \]
for some $K_0 > 0$. Then there is $c_1(n) > 0$ such that the Hermitian Ricci flow has a short-time solution on $M \times [0, c_1 K_0^{-1}]$ with initial metric $g_0$ and satisfies
\[ \sup_{M \times [0, c_1 K_0^{-1}]} |Rm_g| + |T|^2 + |\nabla T| \leq 2K_0 \]

**Proof.** By [15, Corollary 7.4], we have the short-time existence to (3.1). It remains to establish the doubling-time estimates. By [15, Lemma 7.2, Lemma 6.1], the function $F = |Rm|^2 + |T|^4 + |\nabla T|^2$ satisfies
\[ \left( \frac{\partial}{\partial t} - \Delta \right) F \leq 2c_0 F^3 \]
for some $c_0(n) > 0$. Apply maximum principle to $F(x, t)$, we may conclude that for all $t \in [0, C_0 K_0^{-1}] \cap [0, (c_0 K)^{-1}]$,
\[ F(x, t) \leq \left( \frac{1}{K_0^{-1} - c_0 t} \right)^2. \]
The assertion follows by choosing sufficiently small $c_1(n) > 0$. □

**Proposition 4.2.** Suppose $(M, g(t))$ is a solution to the Hermitian Ricci flow (3.1) such that $g_0$ has non-positive bisectional curvature. There is $t_0 > 0$, $K > 0$ such that for all $t \in [0, t_0]$, $Rm(g(t))$ satisfies the following conditions.

1. $\text{Ric} \leq 0$;
2. $|R_{\bar{u}x\bar{v}x}| \leq (1 + Kt)R_{\bar{u}\bar{v}}R_{\bar{u}\bar{v}}$ for all $x, u, v \in T^{1,0}M$, $|x| = 1$;

To begin with, we consider
\[ R_{ijkl}^\epsilon = R_{ijkl} - \epsilon B_{ijkl} \]
where $\epsilon$ is a small positive number. Then Proposition 4.2 is a direct consequence of the following Lemma by letting $\epsilon \to 0$.

**Lemma 4.1.** Under the assumption of Proposition 4.2, there is $\tau > 0$, $K > 0$ such that for any $\epsilon > 0$, $t \in [0, \tau]$, the followings hold.

1. $\text{Ric}^\epsilon < -\epsilon e^{-Kt}$;
2. $|R_{\bar{u}x\bar{v}x}^\epsilon| \leq (1 + Kt)R_{\bar{u}\bar{v}}^\epsilon R_{\bar{u}\bar{v}}^\epsilon$ for all $x, u, v \in T^{1,0}M$, $|x| = 1$;

Before we give a proof of Lemma 4.1, we first show that $g(0)$ satisfies the assumptions in the Lemma.
Lemma 4.2. Under the assumption of Proposition 4.2, \( Rm(0) \) satisfies

1. \( Ric^\epsilon \leq -\epsilon(n + 1) \);
2. \( |R_{u\bar{v}\bar{x}\bar{x}}^\epsilon|^2 < R_{u\bar{u}}^\epsilon R_{\bar{v}\bar{v}}^\epsilon \) for all \( x, u, v \in T^{1,0}_p \).

Proof. The conclusion on \( Ric^\epsilon \) follows immediately from \( Ric(g_0) \leq 0 \) and \( Ric(B) = n + 1 \).

To check (2), for a fixed \( x \in T^{1,0}_p \), \( R_{i\bar{q}x\bar{x}}^\epsilon \) is Hermitian form and hence we may choose eigenvectors \( \{e_i\}_{i=1}^n \) such that

\[
R_{ij\bar{x}\bar{x}}^\epsilon = \lambda_i \delta_{ij}
\]

where \( \lambda_i < 0 \) as \( BK(g_0) \leq 0 \). Therefore, for \( u = \sum_{i=1}^n u^i e_i \) and \( v = \sum_{i=1}^n v^i e_i \),

\[
|R_{u\bar{v}\bar{x}\bar{x}}^\epsilon|^2 = \left| \sum_{i=1}^n \lambda_i u^i v^i \right|^2
\]

\[
\leq \left[ \sum_{i=1}^n \lambda_i |u^i|^2 \right] \left[ \sum_{i=1}^n \lambda_i |v^i|^2 \right]
\]

\[
= R_{u\bar{u}}^\epsilon R_{v\bar{v}}^\epsilon
\]

\[
< R_{u\bar{u}}^\epsilon R_{v\bar{v}}^\epsilon
\]

since \( BK(g_0) < 0 \). \( \square \)

Proof of Lemma 4.4. Denote \( K_0 = \sup_{M \times \{0\}} |Rm| + |T|^2 + |\nabla T| \) and choose \( K >> K_0 \) and \( \tau = K^{-1} \). By proposition 4.1, the Hermitian Ricci flow \( g(t) \) exists on \( M \times [0, \tau] \) with

\[
\sup_{M \times [0, \tau]} |Rm^c| + |T|^2 + |\nabla T| \leq 2K_0.
\]

Suppose condition (a) and (b) are true on \([0, \tau]\), then we are done. Let \( t_0 \in (0, \tau] \) be the first time such that one of them fails.

Case 1: Condition (a) is true on \([0, t_0]\) and fails at \( t = t_0 \). Then there is \( p \in M, X_0 \in T^{1,0}_p \) with \( |X_0| = 1 \) such that

\[
Ric^\epsilon(X_0, X_0) = -\epsilon e^{-Kt}.
\]

Moreover, for all \( z \in M, t \in [0, t_0], y, u, v \in T^{1,0}_z \) with \( |Y| = 1 \),

\[
Ric^\epsilon(Y, \bar{Y}) \leq -\epsilon e^{-Kt};
\]

\[
|R_{u\bar{y}y\bar{y}}^\epsilon|^2 < (1 + Kt)R_{u\bar{u}}^\epsilon R_{v\bar{v}}^\epsilon.
\]

As in [12] Page 1599], we may use polarization and (4.2) to infer that for sufficiently small \( \epsilon > 0 \), any \( e_k, e_l \in T^{1,0} \) with unit 1 and \( e_i, e_j \in T^{1,0} \),

\[
|R_{ijkl}^\epsilon|^2 \leq C_n R_{i\bar{i}}^\epsilon R_{j\bar{j}}^\epsilon; \quad |R_{ijkl}^\epsilon|^2 \leq C_n K_0 R_{i\bar{i}}^\epsilon.
\]

Consider the following tensor

\[
A_{ij} = R_{ij} + \epsilon \left[ e^{-Kt} - (n + 1) \right] g_{ij} = Ric^\epsilon_{ij} + \epsilon e^{-Kt} g_{ij}
\]
which satisfies \( A(X_0, \bar{X}_0) = 0 \) and \( A(Y, \bar{Y}) \leq 0 \) for all \( Y \in T_x^{1,0}M, \ x \in M \). We may assume \( |X_0|_{g(t_0)} = 1 \) by rescaling.

Extend \( X_0 \) locally to a vector field around \((p, t_0)\) such that at \((p, t_0)\),

\[
\begin{align*}
\nabla_q X^p &= 0; \\
\nabla_{\bar{p}} X^q &= T^q_{pl} X^l.
\end{align*}
\]  

(4.5)

Then \( A(X, \bar{X}) \) locally defined a function and satisfies

\[
\Box A(X, \bar{X}) \geq 0.
\]  

(4.6)

where we denote \( \left( \frac{\partial}{\partial t} - \Delta \right) \) by \( \Box \) for notational convenience.

Now we compute the evolution equation for \( A(X, \bar{X}) \). At \((p, t_0)\),

\[
\begin{align*}
\frac{\partial}{\partial t} A(X, \bar{X}) &= (\partial_t A_{ij}) X^i \bar{X}^j + A_{ij} \left( \partial_t X^i \bar{X}^j + X^i \partial_t \bar{X}^j \right) \\
&= (\partial_t R_{ij} - \epsilon [e^{-Kt} - (n + 1)] S_{ij} - \epsilon Ke^{-Kt} g_{ij}) X^i \bar{X}^j \\
&\quad + A_{ij} \left( \partial_t X^i \cdot \bar{X}^j + X^i \cdot \partial_t \bar{X}^j \right) \\
&\quad \leq (\partial_t R_{ij}) X^i \bar{X}^j - \frac{1}{2} \epsilon Ke^{-Kt}.
\end{align*}
\]  

(4.7)

Here we have used (4.2) and the fact that for any \( Y \in T_p^{1,0}M \),

\[
A_{X_0 Y} = 0
\]  

(4.8)

Now we compute the \( \Delta A(X, \bar{X}) \). We may in addition assume that at \((p, t_0)\),

\[ g_{ij} = \delta_{ij} \]. Then

\[
\begin{align*}
\Delta A(X, \bar{X}) &= \frac{1}{2} g^{rs} (\nabla_r \nabla_s + \nabla_s \nabla_r) \left( A_{ij} X^i X^j \right) \\
&= \Delta A_{ij} \cdot X^i \bar{X}^j + A_{ij} X^i \Delta \bar{X}^j + A_{ij} \bar{X}^j \Delta X^i \\
&\quad + A_{ij,r} X^i_{,r} \bar{X}^j + A_{ij,\bar{r}} X^i_{,\bar{r}} \bar{X}^j \\
&\quad + A_{ij,r} X^i_{,r} \bar{X}^j + A_{ij,\bar{r}} X^i_{,\bar{r}} \bar{X}^j \\
&\quad + A_{ij} X^i_{,r} X^j_{,r} + A_{ij} X^i_{,\bar{r}} X^j_{,\bar{r}} \\
&= \Delta R_{ij} \cdot X^i \bar{X}^j + R_{ij,r} X^i_{,r} \bar{X}^j + R_{ij,\bar{r}} X^i_{,\bar{r}} \bar{X}^j \\
&\quad + R_{ij,r} X^i_{,r} \bar{X}^j + R_{ij,\bar{r}} X^i_{,\bar{r}} \bar{X}^j \\
&\quad + A_{ij} X^i_{,r} X^j_{,r} + A_{ij} X^i_{,\bar{r}} X^j_{,\bar{r}}
\end{align*}
\]  

(4.9)

Here we have also used \( \nabla g = 0 \) and (4.8) on the terms involving \( A(\Delta X, \bar{X}) \) or its conjugate.
By rescaling, we may assume 
\[ 
\Box A(X, X) \leq X^i X^j \Box R_{ij} - \frac{1}{2} \epsilon Ke^{-Kt} 
\]
\[ 
- g^{rs} \left[ R_{ij;rt} X_i X^r X^j + R_{ij;st} X_i X^s X^j \right] 
\]
\[ 
+ R_{ij;st} X_i X^s X^j + R_{ij;st} X_i X^s X^j \right) 
\]
\[ 
- e \left[ e^{-Kt} - (n+1) \right] g^{rs} \left( X_i X^j + X_i X^j \right). 
\]

By (4.5) and (4.2), then it reduces to
\[ 
\Box A(X, X) \leq R_{X \bar{X} k} p R_{p k} - \frac{1}{2} \left( S_{X \bar{X} p} R_{p \bar{X}} + S_{X \bar{X} q} R_{X q} \right) - \frac{1}{4} \epsilon K. 
\]

As \( K \gg K_0 \), by (4.3), (4.8), (4.4) and (4.2), we have
\[ 
\Box A(X, \bar{X}) \leq R_{X \bar{X} k} p R_{p k} - \frac{1}{8} \epsilon K 
\]
\[ 
\leq - \frac{1}{10} \epsilon K 
\]
which contradicts with (4.6).

**Case 2**: Suppose condition (b) is not true at \( t = t_0 \). Then there is \( p \in M \), \( x_0, u_0, v_0 \in T^{1,0}_p M \) with \( |x_0|_{t_0} = 1 \) such that
\[ 
|R^e_{u_0 v_0 x_0}|^2 = (1 + Kt) R^e_{u_0 v_0} R^e_{v_0 v_0}. 
\]

By rescaling, we may assume \( |u_0|_{t_0} = |v_0|_{t_0} = 1 \). Moreover for all \((z, t) \in M \times [0, t_0]\), \( x, u, v \in T^{1,0}_z M \) with \( |x|_z = 1 \),
\[ 
R^e_{x \bar{x}} \leq -\epsilon e^{-Kt}; 
\]
\[ 
|R^e_{u_0 x \bar{x}}|^2 \leq (1 + Kt) R^e_{u_0} R^e_{v_0}. 
\]

Argue as in (1.3), we know that for any \( e_k, e_l \in T^{1,0} \) with unit 1,
\[ 
|R^e_{ijkl}|^2 \leq C_n R^e_{ik} R^e_{lj}, \quad |R^e_{ijk}\|^2 \leq C_n K_0 |R^e_{ij}|. 
\]

As in case 1, we extend \( x_0, u_0 \) to a local vector field \( X, U \) around \((p, t_0)\). We extend \( x_0 \) so that along each geodesics \( \gamma \) emanating from \( p, \nabla_\gamma X = 0 \) at \( t = t_0 \) and constant in \( t \). On the other hand, we extend \( u_0, v_0 \) to \( U \) and \( V \) such that at \((p, t_0)\),
\[ 
\nabla_x U^r = 0, \quad \nabla_p U^r = T^r_{pq} U^q, \quad \Box U^r = \frac{1}{2} S^r_{p q} U^p; 
\]
\[ 
\nabla_x V^r = 0, \quad \nabla_p V^r = T^r_{pq} V^q, \quad \Box V^r = \frac{1}{2} S^r_{p q} V^p. 
\]
In particular, $|X|_{t_0} = 1$ around $(p, t_0)$. Hence the function

$$F(x, t) = g_X^{-2}|R^e_{UVVX}|^2 - (1 + Kt)R^e_{UU}R^e_{VV}$$

attains its local maximum at $(p, t_0)$ and therefore satisfies

$$\square F \bigg|_{(p, t_0)} \geq 0. \tag{4.16}$$

We now differentiate each of them carefully. Using (4.15) and Lemma 3.2, a similar calculation as in case 1 yields

$$\square R^e_{UU} = \square R^e_{ij} \cdot U^i U^j + R^e_{ij} U^i \square U^j + R^e_{ij} \square U^i \cdot U^j
- g^{rs} R^e_{ij} U^i U^j - g^{rs} R^e_{ij} U^i U^j - g^{rs} R^e_{ij} U^i U^j
= \square R^e_{ij} \cdot U^i U^j + \epsilon(n + 1)S_{UU}
+ R^e_{ij} \square U^j + R^e_{ij} \square U^i + \epsilon(n + 1)g^{rs} g_{ij} T^i_{rp} T^j_{sq} U^p U^q
- g^{rs} R^e_{ij} T^i_{rp} T^j_{sq} U^p U^q - g^{rs} R^e_{ij} T^i_{rp} U^p U^q
= R^e_{UU} R^k_p - \epsilon(n + 1)S_{UU} + \epsilon(n + 1)g^{rs} g_{ij} T^i_{rp} T^j_{sq}. \tag{4.17}$$

Similarly,

$$\square R^e_{VV} = R^e_{VVK} R^k_p - \epsilon(n + 1)S_{VV} + \epsilon(n + 1)g^{rs} g_{ij} T^i_{rp} T^j_{sq}. \tag{4.18}$$

By combining (4.17), (4.18) with (4.13), (4.14), (4.2) and using the fact that $K > K_0$ and $\tau = K^{-1}$, we arrive at the following inequality.

$$\square [(1 + Kt)R^e_{UU} R^e_{VV}]
\geq (1 + Kt)R^e_{UU} (R^e_{VVK} R^k_p) + (1 + Kt)R^e_{VV} (R^e_{UU} R^k_p)
- 2(1 + Kt) \text{Re} (g^{rs} \nabla_r R^e_{UU} \cdot \nabla_s R^e_{VV}) + \frac{1}{2} K R^e_{UU} R^e_{VV}
\geq -2(1 + Kt) \text{Re} (g^{rs} \nabla_r R^e_{UU} \cdot \nabla_s R^e_{VV}) + \frac{1}{4} K R^e_{UU} R^e_{VV} \tag{4.19}$$

Now we derive the evolution equation of $|R^e_{UUXX}|^2$. Similar to the computation
of $\Box R_{U\bar{U}}$, using (4.15) and Lemma 3.2, we have
\[
\Box R_{U\bar{V}X\bar{X}} = \Box R_{ijkl} \cdot U^i\bar{V}^j X^k X^l - \epsilon \Box B_{ijkl} \cdot U^i\bar{V}^j X^k X^l \\
+ R_{ijkl}^e (\Box \bar{U}^i) \bar{V}^j X^k X^l + R_{ijkl}^e (\Box \bar{V}^j) X^k X^l \\
+ R_{ijkl}^e U^i \bar{V}^j \left( \Box X^k X^l + X^k \Box X^l \right) \\
- g^{rs} R_{ijkl;r} \left( U^i \bar{V}^j X^k X^l + U^i \bar{V}^j X^k X^l \right) \\
- g^{rs} R_{ijkl;s} \left( U^i \bar{V}^j X^k X^l + U^i \bar{V}^j X^k X^l \right) \\
- R_{ijkl}^e \left( U^i \bar{V}^j X^k X^l + U^i \bar{V}^j X^k X^l \right) \\
+ U^i \bar{V}^j X^k X^l \right) \\
= g^{rs} \left[ R_{UV \bar{r}} p R_{p \bar{s} XX} + R_{r \bar{V} X} p R_{U \bar{s} p \bar{X}} - R_{r \bar{V} p \bar{X}} R_{U \bar{s} X} p \right] \\
- \frac{1}{2} \left[ S_X^p R_{UV p \bar{X}} + S_X^q R_{UV q} \right] \\
+ \epsilon \left( S_{UV} g_{XX} + g_{UV} S_{XX} + S_{UV} g_{XX} + g_{UV} S_{XX} \right) \\
+ \epsilon B_{ijkl} T^a_r T^j \bar{S}_U \frac{1}{2} \left( B_{iv \bar{X} X} S_U^i + B_{U \bar{j} X X} \right).
\]

Similarly,
\[
\Box R_{V\bar{U}X\bar{X}} = g^{rs} \left[ R_{V\bar{U} \bar{r}} p R_{p \bar{s} \bar{X}} + R_{r \bar{U} X} p R_{V \bar{s} p \bar{X}} - R_{r \bar{U} p \bar{X}} R_{V \bar{s} X} p \right] \\
- \frac{1}{2} \left[ S_X^p R_{V \bar{U} p \bar{X}} + S_X^q R_{V \bar{U} q} \right] \\
+ \epsilon \left( S_{V \bar{U}} g_{XX} + g_{V \bar{U}} S_{XX} + S_{V \bar{U}} g_{XX} + g_{V \bar{U}} S_{XX} \right) \\
+ \epsilon B_{ijkl} T^a_r T^j \bar{S}_U \frac{1}{2} \left( B_{i \bar{U} \bar{X} X} S_V^i + B_{V \bar{j} X X} \right).
\]

We need to emphasize that $U$ and $V$ only appear in the first two entry of the Chern-curvature tensor. Therefore by combining (4.20), (4.21) and [4.13], we can show that
\[
(4.22) \quad \Box (R_{U\bar{V}X\bar{X}} R_{V\bar{U}X\bar{X}}) \leq -|\nabla R_{U\bar{V}X\bar{X}}|^2 - |\nabla R_{U\bar{V}X\bar{X}}|^2 + C_n K_0 R_{U\bar{U}} R_{V\bar{V}}.
\]

And hence at $(p, t_0)$,
\[
(4.23) \quad \left( \frac{\partial}{\partial t} - \Delta \right) F \leq 2(1 + K t) \text{Re} \left( g^{rs} \nabla_r R_{U\bar{U}} \cdot \nabla_s R_{V\bar{V}} \right) \\
- |\nabla R_{U\bar{V}X\bar{X}}|^2 - |\nabla R_{U\bar{V}X\bar{X}}|^2 \\
- \frac{1}{8} K R_{U\bar{U}} R_{V\bar{V}} + 2 S_{XX} |R_{U\bar{V}X\bar{X}}|^2.
\]

By using the fact that $\nabla F = 0$ and $F = 0$ at $(p, t_0)$, one can conclude that
\[
2(1 + K t) \text{Re} \left( g^{rs} \nabla_r R_{U\bar{U}} \cdot \nabla_s R_{V\bar{V}} \right) \leq |\nabla R_{U\bar{V}X\bar{X}}|^2 + |\nabla R_{U\bar{V}X\bar{X}}|^2.
\]
Using (4.2) and $F(p, t_0) = 0$ again, we deduce that
\[2S_{X\bar{X}}|R_{U\bar{V}X\bar{X}}|^2 \leq C_n K_0 R_{U\bar{U}} R_{V\bar{V}}^c\]
and hence at $(p, t_0)$,
\[
\left(\frac{\partial}{\partial t} - \Delta\right) F < -\frac{1}{16} K R_{U\bar{U}} R_{V\bar{V}}^c
\]
which contradicts with (4.16) provided that $K > \tilde{C}_n K_0$ for some $\tilde{C}_n >> 1$. □

5. Strong Maximum Principle

In this section, we will show that the Chern-Ricci curvature will become strictly negative shortly after the Hermitian Ricci flow evolves. We will adapt the strong maximum principle in [3] to the Hermitian Ricci flow setting. One may also consider the Kernel of the Chern-Ricci curvature in the Hermitian setting, see [18, Theorem 5.2] for related works.

Theorem 5.1. Suppose $(M, g(t))$ is a solution to the Hermitian Ricci flow on $M \times [0, T]$ with initial metric $g_0$. If $g_0$ has non-positive bisectional curvature and its Chern-Ricci curvature is negative at some $p \in M$, then there is $\tau > 0$ such that $Ric(g(t)) < 0$ on $(0, \tau]$.

Proof. Let $\tau$ be the constant obtained from Proposition 4.2. We adopt the argument in [3] to Hermitian Ricci flow setting. Let $y \in M$ be a point at which the Chern-Ricci curvature is negative. Let $\phi_0$ be a smooth nonnegative function such that $\phi_0(y) > 0$, $\phi_0 = 0$ outside a neighbour of $y$ and
\[Ric(g_0) + \phi_0 g_0 \leq 0\]
on $M$. Let $\phi(z, t)$ be the solution to the heat equation
\[
\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \phi(x, t) = 0, \quad \text{on } M \times [0, \tau];
\]
\[\phi(x, 0) = \phi_0.\]
It then follows by strong maximum principle that $\phi(x, t) > 0$ on $M \times (0, \tau]$. We may assume that $\phi(x, t) \leq 1$ by rescaling.

Let $k = c_n K_0$ with $c_n >> 1$. For any $\epsilon > 0$, consider the tensor
\[A^\epsilon = Ric_{g(t)} + e^{-kt} \phi^2 g(t) - \epsilon e^{Bt} g(t)\]
where $B$ is some large constant to be specified later. We claim that $A^\epsilon \leq 0$ on $M \times [0, \tau]$. Then the result follows by letting $\epsilon \to 0$. We omit the index $\epsilon$ for notational convenience. Noted that $A(0) < 0$ on $M$. Suppose not, there is $t_0 \in (0, \tau]$ such that for all $(z, t) \in M \times [0, t_0]$, $u \in T_z^{1,0} M$,
\[A_{u\bar{u}}(z, t) \leq 0.\]
And there is $x_0 \in M$, $v \in T^1_{x_0} M$ so that $A_{v \psi}(x_0, t_0) = 0$. We may further assume that $|v|_{t_0} = 1$ by rescaling. As in the proof of Lemma 4.1, we extend $v$ around $(x_0, t_0)$ locally such that

$$\nabla_q v^p = 0, \quad \nabla_q v^p = T^p_{q} v^l.$$  

Therefore, the function $A_{v \psi}$ attains local maximum at $(x_0, t_0)$ and therefore obeys

$$\left( \frac{\partial}{\partial t} - \Delta \right) \bigg|_{(x_0, t_0)} A_{v \psi} \geq 0.$$  

On the other hand, if we choose coordinate at $x_0$ such that $g_{ij} = \delta_{ij}$, then

$$\left( \frac{\partial}{\partial t} - \Delta \right) A_{v \psi} = \square A_{ij} \cdot v^i v^j + A_{ij} \left( v^i \square v^j + v^j \square v^i \right)$$

$$- g^{rs} \left( A_{ijs} v^i v^j + A_{ijr} v^i v^j + A_{ijr} v^i \right)$$

$$= \square A_{ij} v^i v^j - A_{ij} T^i_{r} T^i_{r} - A_{ijr} T^i_{r} v^j - A_{ijr} T^i_{r} v^j.$$  

where we have used (5.3). Moreover, the first bracket will be vanished due to the fact that

$$A_{v \psi} = 0$$

for all $u \in T^1_{x_0}$. This fact can be seen by considering the first variation of functions, $A(v + tu, \tilde{v} + \tilde{t}u)$ and $A(v + t\sqrt{-1}u, \tilde{v} - t\sqrt{-1}u)$ at $t = 0$.

Moreover, we use Lemma 3.2, 4.1 and 5.1 to deduce that

$$\square A_{ij} \cdot v^i v^j = R_{ij} T^i_{r} T^j_{r} + R_{ijr} T^i_{r} v^j + R_{ijr} T^j_{r} v^i$$

$$+ R_{vpq} R^k_{p} - \frac{1}{2} \left( S^p_{v} R_{vp} + S^q_{v} R_{vq} \right) - ke^{-kt} \phi^2 - 2|\nabla \phi|^2 e^{-kt} - \phi^2 e^{-kt} S_{v \psi}$$

$$- eB e^{Bt} + e^{Bt} S_{v \psi}.$$  

Combines with (5.5) together with (4.2), (5.6), Theorem 4.2 and the fact that $A_{v \psi}(x_0, t_0) = 0$, it gives

$$\left( \frac{\partial}{\partial t} - \Delta \right) A_{v \psi} = \left( \phi^2 e^{-kt} + e^{Bt} \right) g_{ij} T^i_{r} T^j_{r} - 4\phi e^{-kt} \text{Re} \left( \phi T_{v \psi} \right)$$

$$+ R_{vpq} R^p_{v} - ke^{-kt} \phi^2 - 2e^{-kt} |\nabla \phi|^2 - eB e^{Bt}$$

$$\leq -k + C_n K_0 \phi^2 e^{-kt} + e^{Bt} (-B + C_n K_0).$$  

Hence, if we choose $k$ and $B$ sufficiently large, then it contradicts with (5.4). In conclusion, we have shown that there is $k, B > 0$ such that for all $\epsilon > 0$, $(x, t) \in M \times [0, \tau]$,

$$\text{Ric}(g(t)) \leq \left( -e^{-kt} \phi(x, t) + e^{Bt} \right) g.$$  

In particular, by letting $\epsilon \to 0$, we have $\text{Ric}(g(t)) < 0$ when $t \in (0, \tau)$. \qed
By using Theorem 5.1, the main theorem is immediate.

Proof of Theorem 1.3. By the short time existence result in [15], there is a short time solution to the Hermitian Ricci flow (3.1) starting from g₀. By Theorem 4.2 and Theorem 5.1, there is τ > 0 such that Ric(g(τ)) < 0 on M. Therefore, c₁(K_M) > 0 and hence Kähler. The existence of Kähler-Einstein follows by a fundamental result of Aubin [1] and Yau [24], see also [16] for a parabolic proof using the Chern-Ricci flow starting from any smooth Hermitian metric. □

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