Algebras of infinite qubit systems

G. Sardanashvily

Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia
E-mail: sard@grav.phys.msu.su
URL: http://webcenter.ru/~sardan/

Abstract. The input and output algebras of an infinite qubit system and their representations are described.

Let $Q$ be the two-dimensional complex space $\mathbb{C}^2$ equipped with the standard positive non-degenerate Hermitian form $\langle \cdot | \cdot \rangle_2$. Let $M_2$ be the algebra of complex $2 \times 2$-matrices seen as a $\mathbb{C}^*$-algebra. A system of $m$ qubits is usually described by the Hilbert space $E_m = \otimes^m Q$ and the $\mathbb{C}^*$-algebra $A_m = \otimes^m M_2$, which coincides with the algebra $B(E_m)$ of bounded operators in $E_m$. We straightforwardly generalize this description to an infinite set $S$ of qubits by analogy with a spin lattice [1]. Its algebra $A_S$ admits non-equivalent irreducible representations. If $S = \mathbb{Z}^+$, there is one-to-one correspondence between the representations of $A_S$ and those of an algebra of canonical commutation relations.

Given a system of $m$ qubits, one also considers an algebra of complex functions on the set $m \times \mathbb{Z}_2$. It is regarded as the output algebra of a qubit system, while $A_m$ is the input one [2]. There is a natural monomorphism of this algebra to $A_m$. Using the technique of groupoids [3], we enlarge this algebra and generalize it to infinite qubit systems.

We start from the input algebra, and follow the construction of infinite tensor products of Hilbert spaces and $\mathbb{C}^*$-algebras in [1]. Let $\{Q_s, s \in S\}$ be a set of two-dimensional Hilbert spaces $Q_s = \mathbb{C}^2$. Let $\times S Q_s$ be the complex vector space whose elements are finite linear combinations of elements $\{q_s\}$ of the Cartesian product $\prod S Q_s$ of the sets $Q_s$. The tensor product $\otimes S Q_s$ of complex vector spaces $Q_s$ is the quotient of $\times S Q_s$ with respect to the vector subspace generated by the elements of the form:

- $\{q_s\} + \{q'_s\} - \{q''_s\}$, where $q_r + q'_r = q''_r$ for some element $r \in S$ and $q_s = q'_s = q''_s$ for all the others,

- $\{q_s\} - \lambda \{q'_s\}$, $\lambda \in \mathbb{C}$, where $q_r = \lambda q'_r$ for some element $r \in S$ and $q_s = q'_s$ for all the others.

Given an element $\theta = \{\theta_s\} \in \prod S Q_s$ such that all $\theta_s \neq 0$, let us denote $\otimes^\theta S Q_s$ the subspace of $\otimes S Q_s$ spanned by the vectors $\otimes q_s$ where $q_s \neq \theta_s$ only for a finite number of elements $s \in S$. 
It is called the \(\theta\)-tensor product of vector spaces \(Q_s, s \in S\). Let us choose a family \(\theta = \{\theta_s\}\) of normalized elements \(\theta_s \in Q_s\), i.e., all \(|\theta_s| = 1\). Then \(\otimes^\theta Q_s\) is a pre-Hilbert space with respect to the positive non-degenerate Hermitian form

\[
\langle \otimes^\theta q_s | \otimes^\theta q'_s \rangle := \prod_{s \in S} \langle q_s | q'_s \rangle_{2}.
\]

Its completion \(Q^\theta_S\) is a Hilbert space whose orthonormal basis consists of the elements \(e_{ir} = \otimes q_s, r \in S, i = 1, 2\), such that \(q_s \neq r = \theta_s\) and \(q_r = e_i\), where \(\{e_i\}\) is an orthonormal basis for \(Q\).

Let now \(\{A_s, s \in S\}\) be a set of unital \(C^*\)-algebras \(A_s = M_2\). These algebras are provided with the operator norms

\[
\|a\| = (\lambda_0 \lambda_0 + \lambda_1 \lambda_1 + \lambda_2 \lambda_2 + \lambda_3 \lambda_3)^{1/2}, \quad a = i\lambda_0 1 + \sum_{i=1,2,3} \lambda_i \sigma^i,
\]

where \(\sigma^i\) are the Pauli matrices. Given the family \(\{1_s\}\), let us construct the \(\{1_s\}\)-tensor product \(\otimes A_s\) of vector spaces \(A_s\). It is a normed involutive algebra with respect to the operations

\[
(\otimes a_s)(\otimes a'_s) = \otimes (a_s a'_s), \quad (\otimes a_s)^* = \otimes a_s^*\]

and the norm

\[
\| \otimes a_s \| = \prod_{s \in S} \|a_s\|.
\]

Its completion \(A_S\) is a \(C^*\)-algebra. Then the following holds [1].

**Proposition 1.** Given a family \(\theta = \{\theta_s\}\) of normalized elements \(\theta_s \in Q_s\), the natural representation of the involutive algebra \(\otimes A_s\) in the pre-Hilbert space \(\otimes^\theta Q_s\) is extended to the representation of the \(C^*\)-algebra \(A_S\) in the Hilbert space \(Q^\theta_S\) such that \(A_S = B(Q^\theta_S)\) is the algebra of all bounded operators in \(Q^\theta_S\).

**Proposition 2.** Given two families \(\theta = \{\theta_s\}\) and \(\theta' = \{\theta'_s\}\) of normalized elements, the representations of the \(C^*\)-algebra \(A_S\) in the Hilbert spaces \(Q^\theta_S\) and \(Q'^\theta_S\) are equivalent iff

\[
\sum_{s \in S} |\langle \theta_s | \theta'_s \rangle - 1| < \infty.
\]

For instance, if \(S = \mathbb{Z}^k\), we come to a spin lattice.

If \(S\) is a countable set, let us consider its bijection onto \(\mathbb{Z}^+ = \{1, 2, \ldots\}\). Let us denote by \(\sigma^i_r, r \in \mathbb{Z}^+\), the element \(\otimes a_s \in A_S\) such that \(a_r = \sigma^i, a_s \neq r = 1\). Then the elements

\[
a^\pm_s = \frac{1}{2}\sigma_1^3 \cdots \sigma_{s-1}^3 (\sigma_1^1 \pm i\sigma_2^2)
\]
make up the algebra $G_S$ of canonical anticommutation relations. The following holds [1].

**Proposition 3.** There is one-to-one correspondence between the representations $\pi$ of the $C^*$-algebra $A_S$ of a countable qubit system and those $\pi^o$ of the CAR algebra $G_S$, where $\pi^o$ is the restriction of $\pi$ to $G(S) \subset A_S$. Furthermore, a representation $\pi$ is cyclic (resp. irreducible) iff $\pi^o$ is cyclic (resp. irreducible). Representations $\pi$ and $\pi'$ of $A_S$ are equivalent iff its restrictions $\pi^o$ and $\pi'^o$ to $G_S$ are equivalent representations of $G_S$.

Turn now to the output algebra. One can associate to a system of qubits $\{Q_s, s \in S\}$ the following groupoid. Let $Z_2 = \{1, p : p^2 = 1\}$ be the smallest Coxeter group. Let us consider the set $X = Z_2^S$ of $Z_2$-valued functions on $S$. It is a set $2^S$ of all subsets of $S$, and it can be brought into a Boolean algebra. Let $G \subset X$ be a subset of functions which equal $p \in Z_2$ at most finitely many points of $S$. Both $X$ and $G$ are commutative Coxeter groups with respect to the pointwise multiplication. Given the action of $G$ on $X$ on the right, the product $\mathfrak{G} = X \times G$ can be brought into the action groupoid [3], where:

- a pair $((x, g), (x', g'))$ is composable iff $x' = xg$,
- the inversion $(x, g)^{-1} := (xg, g^{-1})$,
- the partial multiplication $(x, g)(xg, g') = (x, gg')$,
- the range $r : (x, g) \mapsto (x, g)(x, g)^{-1} = (x, 1)$,
- the domain $l : (x, g) \mapsto (x, g)^{-1}(x, g) = (xg, 1)$.

The unit space $\mathfrak{G}^0 = r(\mathfrak{G}) = l(\mathfrak{G})$ of this groupoid is naturally identified with $X$. Given elements $x, y \in \mathfrak{G}^0$, let us denote the $r$-fibre $\mathfrak{G}^x = r^{-1}(x)$, the $l$-fibre $\mathfrak{G}^y = l^{-1}(y)$, and $\mathfrak{G}^y_x = \mathfrak{G}^y \cap \mathfrak{G}^x$.

Since $G$ acts freely on $X$, the action groupoid $\mathfrak{G}$ is principal, i.e., the map $(r, l) : \mathfrak{G} \to \mathfrak{G}^0 \times \mathfrak{G}^0$ is an injection. Provided with the discrete topology, $X$ is a locally compact space. Then $\mathfrak{G}$ is a locally compact groupoid. Since $\mathfrak{G}^0 \subset \mathfrak{G}$ is obviously an open subset, $\mathfrak{G}$ is an $r$-discrete groupoid.

Note that the action groupoid $\mathfrak{G}$ is isomorphic to the following one. Let us say that functions $x, y \in X$ are equivalent ($x \sim y$) iff they differ at most finitely many points of $S$. Let $\mathfrak{G}' \subset X \times X$ be the graphic of this equivalence relation, i.e., it consists of pairs $(x, y)$ of equivalent functions. One can provide $\mathfrak{G}'$ with the following groupoid structure [3]:

- a pair $((x, y), (y', z))$ is composable iff $y = y'$,
- $(x, y)^{-1} := (y, x)$,
- $(x, y)(y, z) = (x, z)$,
• \( r : (x, y) \mapsto (x, x) \), \( l : (x, y) \mapsto (y, y) \).

The unit space \( \mathcal{G}^0 \) of this groupoid is naturally identified with \( X \). The isomorphism of groupoids \( \mathcal{G} \) and \( \mathcal{G}' \) is given by the assignment \( \mathcal{G} \ni (x, g) \mapsto (x, xg) \in \mathcal{G}' \). In particular, \( \mathcal{G}_y \neq \emptyset \) iff \( x \sim y \).

Let \( \mathcal{K}(\mathcal{G}, \mathbb{C}) \) be the space of complex functions on \( \mathcal{G} \) of compact support provided with the inductive limit topology. Since \( \mathcal{G} \) is a discrete space, any function on \( \mathcal{G} \) is continuous.

A left Haar system for the groupoid \( \mathcal{G} \) exists. It is given by the measures \( \mu_x = \delta_x \times \mu_G \), where \( \delta_x \) is the Dirac measure on \( X \) with support at \( x \in X \) and \( \mu_G \) is the left Haar measure on the locally compact group \( G \). We have

\[
\int f((y, g)) d\mu_x((y, g)) = \int f((x, gg')) d\mu((y, g')).
\]

A left Haar system for the action groupoid \( \mathcal{G} \) is a family of measures \( \{ \mu_x, x \in X \} \) on \( \mathcal{G} \) indexed by points of the unit space \( \mathcal{G}^0 = X \) such that:

• the support of the measure \( \mu_x \) is \( \mathcal{G}^x \),

• for any \( (x, g) \in \mathcal{G} \) and any \( f \in \mathcal{K}(\mathcal{G}, \mathbb{C}) \), we have

\[
\int f((x, g)(y, g')) d\mu_x((y, g')) = \int f((y, g')) d\mu_x((y, g')).
\]
Thus, $A_G$ is exactly the group algebra of the locally compact group $G$ provided with the norm

$$\|f\| = \sum_g |f(g)|.$$ 

There is the monomorphism of this algebra to the algebra $\otimes A_s$ as follows. Let us assign to each element $g \in G$ the element $\hat{g} = \otimes a_s \in \otimes A_s$, where $a_s = 1$ if $g(s) = 1$ and $a_s = \sigma^1$ if $g(s) = p$. Then the above mentioned monomorphism is given by the association

$$f(g) \mapsto \sum_g f(g) \hat{g}.$$ 

Thus, one can think of $A_G$ and, consequently, $\mathcal{K}(\mathfrak{G}, \mathbb{C})$ as being a generalization of the output algebra in [2].

The monomorphism $A_G \to \otimes A_s$ provides the representations of the involutive algebra $A_G$ in Hilbert spaces $Q^0_S$. One can construct representations of the whole algebra $\mathcal{K}(\mathfrak{G}, \mathbb{C})$ as follows.

Given the Hilbert space $Q^0_S$, let us consider the product $\mathfrak{E} = X \times Q^0_S$ seen as a group bundle in the Abelian groups $Q^0_S$ over $X$. It is a groupoid such that

- a pair $((x, v), (x', v'))$, $x, x' \in X$, $v, v' \in Q^0_S$, is composable iff $x' = x$,
- $(x, v)^{-1} := (x, -v)$,
- $(x, v)(x, v') = (x, v + v')$,
- $r((x, v)) = l((x, v)) = x$.

Its unit space is $X$, and its fibres $\mathfrak{E}_x$ are isomorphic to $Q^0_S$. Let $\text{Iso} \mathfrak{E}$ denote the set of all isomorphisms $\gamma^x : \mathfrak{E}_y \to \mathfrak{E}_x$, $x, y \in \mathfrak{G}^0$. This set possesses the natural groupoid structure such that:

- a pair $(\gamma^x, \gamma^{x'})$ is composable iff $x' = y$,
- $(\gamma^x)^{-1}$ is the inverse of $\gamma^x$,
- $\gamma^x \gamma^y = \gamma^y \circ \gamma^y$,
- $r(\gamma^x) = \text{Id} \mathfrak{E}_x$ and $l(\gamma^x) = \text{Id} \mathfrak{E}_y$.

The unit space $\mathfrak{E}^0$ of this groupoid is naturally identified with $X$ by the association $x \mapsto \text{Id} \mathfrak{E}_x$. The assignment

$$(x, g) \mapsto \gamma^x_g = \hat{g}$$
provides a homomorphism of the action groupoid $\mathcal{G}$ to $\text{Iso} \mathcal{E}$ and, consequently, a representation of $\mathcal{G}$ in $\mathcal{E}$.

Let the discrete topological space $X$ be provided with the measure $\mu_X$ such that

$$\int f(x) d\mu_X(x) = \sum_x f(x), \quad f \in \mathcal{K}(X, \mathbb{C}).$$

Let $L^2(\mathcal{E}, \mu_X)$ denote the Hilbert space of square $\mu_X$-integrable sections $\phi$ of the group bundle $\mathcal{E} \to X$. Then we have a desired representation $\pi$ of the algebra $\mathcal{K}(\mathcal{G}, \mathbb{C})$ in $L^2(\mathcal{E}, \mu_X)$ by the formula

$$(\pi(f)\phi)(x) = \sum_g f((xg^{-1}, g)) \tilde{g}\phi(xg^{-1}).$$

The algebra $\mathcal{K}(\mathcal{G}, \mathbb{C})$ can be provided with the operator norm $\|f\| = \|\pi(f)\|$ with respect to this representation. Its completion relative to this norm is the $C^*$-algebra of the action groupoid $\mathcal{G}$.

References

[1] G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory* (Wiley–Interscience, New York, 1972).

[2] M. Keyl, Fundamentals of quantum information theory, *Phys. Rep.* 369 (2002) 431.

[3] J. Renault, *A Groupoid Approach to $C^*$-algebras*, Lect. Notes Math. 793 (Springer-Verlag, Berlin, 1980).