ON POLIGNAC’S CONJECTURE

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Abstract. A few years ago we identified a recursion that works directly with the gaps among the generators in each stage of Eratosthenes sieve. This recursion provides explicit enumerations of sequences of gaps among the generators, which are known as constellations.

As the recursion proceeds, adjacent gaps within longer constellations are added together to produce shorter constellations of the same sum. These additions or closures correspond to removing composite numbers that are divisible by the prime for that stage of Eratosthenes sieve. Although we don’t know where in the cycle of gaps a closure will occur, we can enumerate exactly how many copies of various constellations will survive each stage.

In this paper, we broaden our study of these systems of constellations of a fixed sum. By generalizing our methods, we are able to demonstrate that for every even number $2n$ the gap $g = 2n$ occurs infinitely often through the stages of Eratosthenes sieve. Moreover, we show that asymptotically the ratio of the number of gaps $g = 2n$ to the number of gaps $g = 2$ at each stage of Eratosthenes sieve converges to the estimates made for gaps among primes by Hardy and Littlewood in Conjecture B of their 1923 paper.

1. Introduction

We work with the prime numbers in ascending order, denoting the $k^{th}$ prime by $p_k$. Accompanying the sequence of primes is the sequence of gaps between consecutive primes. We denote the gap between $p_k$ and $p_{k+1}$ by $g_k = p_{k+1} - p_k$. These sequences begin

\[
P_1 = 2, \quad P_2 = 3, \quad P_3 = 5, \quad P_4 = 7, \quad P_5 = 11, \quad P_6 = 13, \quad \ldots
\]

\[
g_1 = 1, \quad g_2 = 2, \quad g_3 = 2, \quad g_4 = 4, \quad g_5 = 2, \quad g_6 = 4, \quad \ldots
\]

A number $d$ is the difference between prime numbers if there are two prime numbers, $p$ and $q$, such that $q - p = d$. There are already many interesting results and open questions about differences between prime numbers; a seminal and inspirational work about differences between primes is Hardy and Littlewood’s 1923 paper \cite{3}.
A number \( g \) is a gap between prime numbers if it is the difference between consecutive primes; that is, \( p = p_i \) and \( q = p_{i+1} \) and \( q - p = g \). Differences of 2 or 4 are also gaps; so open questions like the Twin Prime Conjecture, that there are an infinite number of gaps \( g_k = 2 \), can be formulated as questions about differences as well.

**Polignac’s conjecture.** In 1849 de Polignac conjectured that for every \( n > 0 \) the gap \( g = 2n \) occurs infinitely often among primes.

In this paper we provide supporting evidence for this conjecture by proving that the analogue for Eratosthenes sieve is true. We show that for any \( n > 0 \) the gap \( g = 2n \) occurs infinitely often in the stages of Eratosthenes sieve, and we show that the ratio of occurrences of \( g = 2n \) to \( g = 2 \), which ratio we denote by \( w_{2n,1} \), asymptotically approaches the ratio implicit in Hardy and Littlewood’s Conjecture B [3]:

\[
    w_{2n,1}(\infty) = \prod_{q>2, q|n} \frac{q-1}{q-2}.
\]

To accomplish this, we need to generalize the work in [4] and [6]. In those papers, we studied the cycle of gaps in each stage of Eratosthenes sieve, denoting the corresponding cycle of gaps \( G(p_k#) \). Here we study the cycle of gaps \( G(N) \) among the generators in \( \mathbb{Z} \) mod \( N \) for any \( N \).

A constellation among primes [7] is a sequence of consecutive gaps between prime numbers. Let \( s = c_1c_2 \cdots c_j \) be a sequence of \( j \) numbers. Then \( s \) is a constellation among primes if there exists a sequence of \( j + 1 \) consecutive prime numbers \( p_{i0}p_{i0+1} \cdots p_{i0+j} \) such that for each \( i = 1, \ldots, j \), we have the gap \( p_{i0+i} - p_{i0+i-1} = c_i \). In Eratosthenes sieve, \( s \) is a constellation if for some \( p_k \) and some \( i_0 \) and all \( i = 1, \ldots, j \), \( c_i = g_{i_0+i} \) in \( G(p_k#) \).

For a constellation \( s \), the length of \( s \) is the number of gaps in \( s \), denoted \(|s|\). A driving term for a gap \( g \) in \( G(p#) \) is a constellation whose gaps sum to \( g \). A driving term of length 1 is the gap itself.

The power of the recursion on the cycle of gaps is seen in the following theorem, which enables us to calculate the number of occurrences of a constellation \( s \) through successive stages of Eratosthenes sieve.

**Theorem 1.1.** (from [4, 5]) Given a gap \( g = 2n \), let \( n_{g,j}(p) \) be the number of driving terms for \( g \) in \( G(p#) \) of length \( j \). For every prime \( p_k \) such that \( g < 2p_k+1 \),

\[
    n_{g,j}(p_{k+1}) = (p_{k+1} - j - 1) \cdot n_{g,j}(p_k) + j \cdot n_{g,j+1}(p_k).
\]

The challenge in applying this approach to Polignac’s conjecture is that the condition \( g < 2p_k+1 \) in Theorem 1.1 requires us to go far into the stages of Eratosthenes sieve, before we can get exact counts for the driving terms of \( g \) of each length \( j \), the \( n_{g,j}(p_k) \) in Equation 1.
Following [9], once we have the exact counts \( n_{g,j}(p_k) \) for a prime \( p_k \) such that \( g < 2p_{k+1} \), we can set up a dynamic system representing the counts through all subsequent stages of Eratosthenes sieve. Since the gap \( g = 2 \) has no driving terms other than the gap itself, we take the ratios of the other gaps to the number of 2’s at each stage of the sieve. So instead of the raw counts of Equation 1, we use the ratios

\[
 w_{g,j}(p) = \frac{n_{g,j}(p)}{n_{2,1}(p)}:
\]

If we assemble the \( w_{g,j}(p_k) \) into a vector, we get a matrix equation with a remarkably simple eigenstructure, from which we derive the following corollary to Theorem 1.1.

**Figure 1.** This figure illustrates the dynamic system of Theorem 1.1 through stages of the recursion for \( G(p^#) \), using just the counts of gaps and their driving terms. The coefficients of the system at each stage of the recursion are independent of the specific gap and its driving terms. Below the diagram for the system, we record the initial conditions for a set of gaps at \( p_0 = 13 \). From this information we can derive the recursive count for primes \( q > p_0 \). Since the raw counts are superexponential, we take the ratio of the count for each constellation to the count for \( g = 2 \).
Corollary 1.2. (from [6]) For any gap $g$ with initial ratios $w_{g,j}(p_k)$, such that $g < 2p_{k+1}$, the ratio of occurrences of this gap $g$ to occurrences of the gap 2 in $G(p^#)$ as $p \to \infty$ converges to the sum of these initial ratios across all the driving terms for this gap:

$$w_{g,1}(\infty) = \sum_j w_{g,j}(p_k).$$

1.1. Approach toward Polignac’s conjecture. To establish an equivalent of Polignac’s conjecture for Eratosthenes sieve, we show that for every $n > 1$ the gap $g = 2n$ does occur at some stage of the sieve and that as $p \to \infty$

$$w_{g,1}(\infty) = \prod_{q > 2, q | n} \frac{q - 1}{q - 2}.$$

To obtain this result, we first consider $\mathbb{Z} \mod Q$ and its cycle of gaps $G(Q)$, in which $Q$ is the product of the prime divisors of $2n$. We then bring this back into Eratosthenes sieve by filling in the primes missing from $Q$ to obtain a primorial $p^#$.

Once we are working with $G(p^#)$, we are poised to apply Theorem 1.1 and Corollary 1.2. However, the condition $g < 2p_{k+1}$ could still require us to work with later stages of the sieve. We are able to show that the conclusion of Corollary 1.2 holds under the construction we are using.

2. The cycle of gaps among the generators in $\mathbb{Z} \mod N$

Let $G(N)$ denote the cycle of gaps among the generators in $\mathbb{Z} \mod N$, with the first gap being that between 1 and the next generator. There are $\phi(N)$ gaps in $G(N)$ that sum to $N$.

There is a one-to-one correspondence between generators of $\mathbb{Z} \mod N$ and the gaps in $G(N)$. Let

$$G(N) = g_1 g_2 \cdots g_{\phi(N)}.$$ 

Then for $k < \phi(N)$, $g_k$ corresponds to the generator $\gamma = 1 + \sum_{j=1}^k g_j$, and since $\sum_{j=1}^{\phi(N)} = N$, the generator 1 corresponds to $g_{\phi(N)}$. Moreover, since 1 and $N-1$ are always generators, $g_{\phi(N)} = 2$. For any generator $\gamma$, $N-\gamma$ is also a generator, which implies that except for the final 2, $G(N)$ is symmetric.

In our previous work [4, 5, 6], we focused on Eratosthenes sieve, in which $N = p^#$, the primorials. For reference we provide a few base cases, since we will develop $G(N)$ by building the cycle of gaps via one prime factor of $N$ at a time.
Remark 2.1. i) For any prime number \( p \),
\[
\mathcal{G}(p) = \underbrace{1 \cdots 1}_{p-2} \underbrace{2}_{p+1}
\]

ii) \( \mathcal{G}(5^#) = \mathcal{G}(30) = 64242462 \).

As a convention, we write the cycles with the first gap being from 1 to the next generator. For a prime \( p \), every number is a generator in \( \mathbb{Z} \mod p \) except \( p \) itself. The last gap of 2 is the gap from \( p - 1 \) past \( p \) (or 0) around to 1. With \( \mathcal{G}(p) \) as a starting point, we can build \( \mathcal{G}(N) \) for any \( N \) by introducing one prime factor at a time.

Lemma 2.2. Given \( \mathcal{G}(N) \), for a prime \( q \) we construct \( \mathcal{G}(qN) \) as follows:

a) if \( q \mid N \), then we concatenate \( q \) copies of \( N \),
\[
\mathcal{G}(qN) = \underbrace{\mathcal{G}(N) \cdots \mathcal{G}(N)}_{q \text{ copies}}
\]

b) if \( q \nmid N \), then we build \( \mathcal{G}(qN) \) in three steps:
R1 Concatenate \( q \) copies of \( \mathcal{G}(N) \);
R2 Close at \( q \);
R3 Close as indicated by the element-wise product \( q \ast \mathcal{G}(N) \).

Proof. A number \( \gamma \) in \( \mathbb{Z} \mod N \) is a generator iff \( \gcd(\gamma,N) = 1 \).

a) Assume \( q \mid N \). Since \( \gcd(\gamma,N) = 1 \), we know that \( q \nmid \gamma \). For \( j = 0,1,\ldots,q-1 \), we have
\[
\gcd(\gamma + jN,qN) = \gcd(\gamma,qN) = \gcd(\gamma,N) = 1.
\]
Thus \( \gcd(\gamma,N) = 1 \) iff \( \gcd(\gamma + jN,qN) = 1 \), and so the generators of \( \mathbb{Z} \mod qN \) have the form \( \gamma + jN \), and the gaps take the indicated form.

b) If \( q \nmid N \) then we first create a set of candidate generators for \( \mathbb{Z} \mod qN \), by considering the set
\[
\{ \gamma + jN : \gcd(\gamma,N) = 1, \ j = 0,\ldots,q-1 \}.
\]
For gaps, this is the equivalent of step R1, concatenating \( q \) copies of \( \mathcal{G}(N) \). The only prime divisor we have not accounted for is \( q \); if \( \gcd(\gamma + jN,q) = 1 \), then this candidate \( \gamma + jN \) is a generator of \( \mathbb{Z} \mod qN \). So we have to remove \( q \) and its multiples from among the candidates.

When we remove a multiple of \( q \) as a candidate, we add together the gaps on each side of this multiple. We call this closing the gaps at this multiple.

We first close the gaps at \( q \) itself. We index the gaps in the \( q \) concatenated copies of \( \mathcal{G}(N) \):
\[
g_1 g_2 \cdots g_{\phi(N)} \cdots g_{q\phi(N)}.
\]
Recalling that the first gap $g_1$ is the gap between the generator $1$ and the next smallest generator in $\mathbb{Z} \mod N$, the candidate generators are the running totals $\gamma_j = 1 + \sum_{i=1}^{j-1} g_i$. We take the $j$ for which $\gamma_j = q$, and removing $q$ from the list of candidate generators corresponds to replacing the gaps $g_{j-1}$ and $g_j$ with the sum $g_{j-1} + g_j$. This completes step R2 in the construction.

To remove the remaining multiples of $q$ from among the candidate generators, we note that any multiples of $q$ that share a prime factor with $N$ have already been removed. We need only consider multiples of $q$ that are relatively prime to $N$; that is, we only need to remove $q\gamma_j$ for each generator $\gamma_j$ of $\mathbb{Z} \mod N$ by closing the corresponding gaps.

We can perform these closures by working directly with the cycle of gaps $G(N)$. Since $q\gamma_i + 1 - q\gamma_i = qg_i$, we can go from one closure to the next by tallying the running sum from the current closure until that running sum equals $qg_i$. Technically, we create a series of indices beginning with $i_0 = j$ such that $\gamma_j = q$, and thereafter $i_k = j$ for which $\gamma_j - \gamma_{i_{k-1}} = q \cdot g_k$. To cover the cycle of gaps under construction, which consists initially of $q$ copies of $G(N)$, $k$ runs only from $0$ to $\phi(N)$. We note that the last interval wraps around the end of the cycle and back to $i_0$: $i_{\phi(N)} = i_0$.

\[ \square \]

**Theorem 2.3.** In step R3 of Lemma 2.2, each possible closure in $G(N)$ occurs exactly once in constructing $G(qN)$.

**Proof.** Consider each gap $g$ in $G(N)$. Since $q \nmid N \mod q \neq 0$. Under step R1 of the construction, $g$ has $q$ images. Let the generator corresponding to $g$ be $\gamma$. Then the generators corresponding to the images of $g$ under step R1 is the set:

$$\{ \gamma + jN : j = 0, \ldots, q-1 \}.$$ 

Since $N \mod q \neq 0$, there is exactly one $j$ for which $(\gamma + jN) \mod q = 0$. For this gap $g$, a closure in R2 and R3 occurs once and only once, at the image corresponding to the indicated value of $j$. \[ \square \]

**Corollary 2.4.** Let $g$ be a gap. If for the prime $q$, $q \nmid g$, then

$$\sum w_{g,j}(qN) = \sum w_{g,j}(N).$$

**Proof.** Consider a driving term $s$ for $g$, of length $j$ in $G(N)$. In constructing $G(qN)$, we initially create $q$ copies of $s$.

If $q \mid N$, then the construction is complete. For each driving term for $g$ in $G(N)$ we have $q$ copies, and so $n_{g,j}(qN) = q \cdot n_{g,j}(N)$. Similarly $n_{2,1}(qN) = q \cdot n_{2,1}(N)$, and $w_{g,j}(qN) = w_{g,j}(N)$. We have equality for each $j$ and so the result about the sum is immediate.
If \( q \mid N \), then in step R1 we create \( q \) copies of \( s \). In steps R2 and R3, each of the possible closures in \( s \) occurs once, distributed among the \( q \) copies of \( s \). The \( j - 1 \) closures interior to \( s \) don’t change the sum, and the result is still a driving term for \( g \). Only the two exterior closures, one at each end of \( s \), change the sum and thereby remove the copy from being a driving term for \( g \). Since \( q \mid g \), these two exterior closures occur in separate copies of \( s \).

If the condition \( g < 2p_{k+1} \) applies, then each of the closures occur in a separate copy of \( s \), and we can use the full dynamic system of Theorem 1.1.

For the current result we do not know that the closures necessarily occur in distinct copies of \( s \), and so we can’t be certain of the lengths of the resulting constellations.

However, we do know that of the \( q \) copies of \( s \), two are eliminated as driving terms and \( q - 2 \) remain as driving terms of various lengths.

\[
\sum_j n_{g,j}(qN) = (q - 2) \sum_j n_{g,j}(N).
\]

Since \( n_{2,1}(qN) = (q - 2)n_{2,1}(N) \), the ratios are preserved

\[
\sum_j w_{g,j}(qN) = \sum_j w_{g,j}(N).
\]

\( \square \)

**Corollary 2.5.** Let \( g = 2n \) be a gap, and let \( \bar{q} \) be the largest prime factor of \( g \). Then

\[
w_{g,1}(\infty) = \sum_j w_{g,j}(\bar{q}#).
\]

**Proof.** For Eratosthenes sieve, by Corollary 1.2 and the preceding corollary, for all primes \( p > \bar{q} \),

\[
\sum_j w_{g,j}(p#) = \sum_j w_{g,j}(\bar{q}#),
\]

and we have our result. \( \square \)

### 3. Polignac’s conjecture for Eratosthenes sieve

We establish an equivalent of Polignac’s conjecture for Eratosthenes sieve.

**Theorem 3.1.** For every \( n > 0 \), the gap \( g = 2n \) occurs infinitely often in Eratosthenes sieve, and the ratio of the number of occurrences of \( g = 2n \) to the number of 2’s converges asymptotically to

\[
w_{2n,1}(\infty) = \prod_{q > 2, q \mid n} \frac{q - 1}{q - 2}.
\]
We establish this result in two steps. First we find a stage of Eratosthenes sieve in which the gap \( g = 2n \) has driving terms. Once we can enumerate the driving terms for \( g \) in this initial stage of Eratosthenes sieve, we can establish the asymptotic ratio of gaps \( g = 2n \) to the gaps \( g = 2 \) as the sieve continues.

**Lemma 3.2.** Let \( g = 2n \) be given. Let \( Q \) be the product of the primes dividing \( 2n \), including \( 2 \),

\[
Q = \prod_{q \mid 2n} q, \quad \text{and} \quad n_1 = 2n / Q.
\]

Finally, let \( \bar{q} \) be the largest prime factor in \( Q \).

Then in \( G(\bar{q}^#) \) the gap \( g \) has driving terms, the total number of which satisfies

\[
\sum_j n_{g,j}(\bar{q}^#) = \phi(Q) \cdot \prod_{p < \bar{q}, p \mid Q} (p - 2).
\]

**Proof.** By Lemma 2.2 the cycle of gaps \( G(2n) \) consists of \( n_1 \) concatenated copies of \( G(Q) \). In \( G(Q) \), there are \( \phi(Q) \) driving terms for the gap \( g = 2n \). To see this, start at any gap in \( G(Q) \) and proceed through the cycle \( n_1 \) times. The length of each of these driving terms is initially \( n_1 \cdot \phi(Q) \).

We now want to bring this back into Eratosthenes sieve.

Let \( Q_0 = Q \), and let \( p_1, \ldots, p_k \) be the prime factors of \( \bar{q}^#/Q \). For \( i = 1, \ldots, k \), let \( Q_i = p_i \cdot Q_{i-1} \). In forming \( G(Q_i) \) from \( G(Q_{i-1}) \), we apply Corollary 2.4. Although we don’t have enough information about the lengths of the driving terms to apply the dynamic system of Theorem 1.1 for each length \( j \), we do know that

\[
\sum_{j=1}^{J} n_{2n,j}(Q_i) = (p_i - 2) \cdot \sum_{j=1}^{J} n_{2n,j}(Q_{i-1})
\]

Thus at \( p_k \) we have

\[
\sum_{j=1}^{J} n_{2n,j}(Q_k) = (p_k - 2) \cdot \sum_{j=1}^{J} n_{2n,j}(Q_{k-1})
\]

\[
= \left( \prod_{i=1}^{k} (p_i - 2) \right) \sum_{j=1}^{J} n_{2n,j}Q_0 = \left( \prod_{i=1}^{k} (p_i - 2) \right) \phi(Q)
\]

**Proof of Theorem 3.1.** Let \( g = 2n \) be given. Let \( Q \) be the product of the prime factors dividing \( g \) and let \( \bar{q} \) be the largest prime factor of \( g \). By
Lemma 3.2 we know that in $G(q#)$ there occur driving terms for $g$ if not the gap $g$ itself. We know the total number of these driving terms is

$$\sum_j n_{g,j}(q#) = \phi(Q) \cdot \prod_{p \leq q, p \not| Q} (p - 2).$$

The number of gaps 2 in $G(q#)$ is $n_{2,1}(q#) = \prod_{2 < p \leq q} (p - 2)$. So for the ratios we have

$$\sum_j w_{g,j}(q#) = \sum_j n_{g,j}(q#)/n_{2,1}(q#)$$

$$= \phi(Q) / \prod_{p|Q, p > 2} (p - 2) = \prod_{p|Q, p > 2} (p - 1)/(p - 2).$$

By Corollary 2.4 and Corollary 2.5 we have the result

$$w_{2n,1}(\infty) = \prod_{p|2n, p > 2} \left(\frac{p - 1}{p - 2}\right).$$

□

This establishes a strong analogue of Polignac’s conjecture for Eratosthenes sieve. Not only do all even numbers appear as gaps in later stages of the sieve, but they do so in proportions that converge to specific ratios. Using the gap $g = 2$ as the reference point since it has no driving terms other than the gap itself, the gaps for other even numbers appear in ratios to $g = 2$ implicit in the work of Hardy and Littlewood [3].

4. Data & Observations

To anchor the above results in data, we exhibit a few tables of data extracted from stages of Eratosthenes sieve. In each table, the rows are indexed by the size of the gap, and the columns are indexed by the length of the driving terms. So if the table is for the cycle of gaps $G(p#)$, the $ij$th entry is the number of driving terms for the gap $g_i$ of length $j$ in $G(p#)$.

We have calculated the tables for $G(37#)$, for gaps from 2 to 3528 and for lengths $j$ of driving terms from 1 to 500. These tables are quite large. We present samples from the larger tables, which are posted on www.primegaps.com.

Our first table shows the table of nonzero entries for $G(13#)$, for gaps $g = 2, \ldots, 32$. This is the range of gaps for which the condition $g < 2p_{k+1}$ holds and thus for which the full dynamic system of Theorem 1.1 applies.
These results may lend some insight into the Jacobsthal function \([1]\). The Jacobsthal function \(g(N)\) is defined as the least integer such that for any \(g(N)\) consecutive integers there is at least one which is relatively prime to \(N\). We observe that this is equivalent to defining \(g(N)\) to be the maximum gap in \(G(N)\), and by Lemma 2.2 \(g(N) = g(Q)\), in which \(Q\) is the product of the prime factors of \(N\). From Lemma 2.2 and Theorem 2.3, letting \(\bar{q}\) be the maximum prime in \(Q\), we know that \(g(Q) \leq g(\bar{q}^\#)\).

From our tabulated data, it appears that the maximum gap that actually occurs in \(G(p_k^\#)\) is roughly \(2p_{k-1}\). We know from previous work \([4]\) that the gap \(g = 2p_{k-1}\) always occurs in \(G(p_k^\#)\). Although this gap is sometimes exceeded as the maximum gap, the tables suggest that this value is often the maximum gap.

| p  | \(p\) max \(g\) | p  | \(p\) max \(g\) | p  | \(p\) max \(g\) | p  | \(p\) max \(g\) |
|----|----------------|----|----------------|----|----------------|----|----------------|
| 3  | 4              | 5  | 6              | 7  | 10             | 9  | 11             |
| 4  | 11             | 12 | 13             | 14 | 17             | 19 | 22             |
|    | 14             | 15 | 16             | 17 | 26             | 19 | 23             |
|    | 19             | 20 | 21             | 20 | 29             | 21 | 23             |
|    | 23             | 24 | 24             | 24 | 31             | 25 | 41             |
|    | 24             | 25 | 25             | 25 | 25             | 25 | 37             |
|    | 25             | 26 | 26             | 26 | 37             | 26 | 46             |
|    | 26             | 27 | 27             | 27 | 37             | 27 | 46             |
|    | 27             | 28 | 28             | 28 | 37             | 28 | 46             |
|    | 28             | 29 | 29             | 29 | 37             | 29 | 46             |
|    | 29             | 30 | 30             | 30 | 37             | 30 | 46             |
|    | 30             | 31 | 31             | 31 | 37             | 31 | 46             |
|    | 31             | 32 | 32             | 32 | 37             | 32 | 46             |
|    | 32             | 33 | 33             | 33 | 37             | 33 | 46             |
|    | 33             | 34 | 34             | 34 | 37             | 34 | 46             |
|    | 34             | 35 | 35             | 35 | 37             | 35 | 46             |
|    | 35             | 36 | 36             | 36 | 37             | 36 | 46             |

In the next table we exhibit the part of the table for \(G(31^\#)\) at which the driving terms through length 9 are running out. In this part of the table we observe interesting patterns for the maximum gap associated with driving terms of a given length. The driving terms of length 4 have sums up to 90 but none of sums 82, 86, or 88. Interestingly, although the gap 128 is a power of 2, in \(G(31^\#)\) its driving terms span the lengths from 11 to 27; yet the gaps \(g = 126\) and \(g = 132\) already have driving terms of length 9.

In each stage of Eratosthenes sieve, some copies of the driving terms of length \(j\) will have at least one interior closure, resulting in shorter driving terms at the next stage. For this part of the table, \(g \geq 2p_{k+1}\) and so more
than one closure could occur within a single copy of a driving term. Letting $31 = p_1$, we therefore know that a gap $g = 2n$ will occur as a gap in $G(p_{k#})$ for $k \leq \min j$, the length of the shortest driving term for $g$ in $G(31#)$.

| gap | $n_{g,j}(31)$: driving terms of length $j$ in $G(31#)$ | $\sum w_{g,j} \ w_{g,1}(\infty)$ |
|-----|---------------------------------------------------|---------------------------------|
| 74  | 1 1206 70194 1550662 17523160 11349768 445136490 | 1 1.02857 |
| 76  | 602 32194 765488 9470176 68041280 302507798 | 1.0588 1.0588 |
| 78  | 292 26060 826426 12166908 99284264 48904926 | 2.1818 2.1818 |
| 80  | 2 2876 139926 2656274 26634332 159280176 | 1.3333 1.3333 |
| 82  | 747 46878 1066848 12378176 83484438 | 1 1.0256 |
| 84  | 2 1012 58216 1485176 135450260 | 2.4 2.4 |
| 86  | 74 4726 147779 2453256 23265260 | 1 1.0244 |
| 88  | 2 2190 107182 20259108 | 1.1111 1.1111 |
| 90  | 8 300 9360 195708 18772184 | 2.6667 2.6667 |
| 92  | 20 860 26854 488854 5364068 | 1.0476 1.0476 |
| 94  | 16 740 19740 333162 3693782 | 1.0222 |
| 96  | 4 242 9636 249610 3693782 | 2 2 |
| 98  | 28 1482 53288 968210 | 1.2 1.2 |
| 100 | 8 672 26428 567560 | 1.3333 1.3333 |
| 102 | 78 7042 249300 | 2.153 2.153 |
| 104 | 182 6086 129016 | 1.0909 1.0909 |
| 106 | 16 1168 37144 | 1.0196 |
| 108 | 8 1244 44334 | 2 2 |
| 110 | 142 7686 | 1.4815 1.4815 |
| 112 | 68 5294 | 1.2 1.2 |
| 114 | 22 2388 | 2.1176 2.1176 |
| 116 | 224 4716 | 1.0370 1.0370 |
| 118 | 72 | 1 1.0175 |
| 120 | 1012 | 2.6667 2.6667 |
| 122 | 70 | 1 1.0169 |
| 124 | 28 | 1.0345 1.0345 |
| 126 | 4 2.4 2.4 |
| 128 | | 1 1 |
| 130 | | 1.4545 1.4545 |
| 132 | 2 2.2222 2.2222 |

From the tabled values for $G(31#)$, we see that the driving term of length 3 for $g = 74$ will advance into an actual gap in two more stages of the sieve. Thus the maximum gap in $G(41#)$ is at least 74, and the maximum gap for $G(43#)$ is at least 90.

For $g = 74, 82, 86, 94, 106, 118, 122$, note that in the table for $G(31#)$

$$\sum w_{g,j}(31#) \neq w_{g,1}(\infty).$$

Up through $G(31#)$ the ratio is 1; but for each gap, we know that this ratio will jump to equal $w_{g,1}(\infty)$ in the respective $G(q#)$. How does the ratio transition from 1 to the asymptotic value? If we look further in the data for $G(31#)$, we see that for the gap $g = 222$, $\sum_j w_{222,j}(31#) = 2$ but the asymptotic value is $w_{222,1}(\infty) = 72/35$. 
These gaps \( g = 2n \) have maximum prime divisor \( \bar{q} \) greater than the prime \( p \) for the current stage of the sieve \( \mathcal{G}(p^\#) \). From Corollary 2.4 and the approach to proving Lemma 3.2, we are able to establish the following.

**Corollary 4.1.** Let \( g = 2n \), and let \( Q = q_1 q_2 \cdots q_k \) be the product of the distinct prime factors of \( g \), with \( q_1 < q_2 < \cdots < q_k \). Then for \( \mathcal{G}(p^\#) \),

\[
\sum_j w_{g,j}(p^\#) = \prod_{2 < q_i \leq p} \left( \frac{q_i - 1}{q_i - 2} \right).
\]

**Proof.** Let \( p = q_j \) for one of the prime factors in \( Q \). By Corollary 2.4, these are the only values of \( p \) at which the sum of the ratios \( \sum_j w_{g,j}(p) \) changes.

Let \( Q_j = q_1 q_2 \cdots q_j \). In \( \mathcal{G}(q_j^\#) \), \( g \) behaves like a multiple of \( Q_j \). As in the proof of Lemma 3.2 in \( \mathcal{G}(Q_j) \) each generator begins a driving term of sum \( 2n \), consisting of \( 2n/Q_j \) complete cycles. There are \( \phi(Q_j) \) such driving terms.

We complete \( \mathcal{G}(q_j^\#) \) as before by introducing the missing prime factors. The other prime factors do not divide \( 2n \), and so by Corollary 2.4 the sum of the ratios is unchanged by these factors. We have our result:

\[
\sum_j w_{g,j}(q_j^\#) = \prod_{2 < q_i \leq q_j} \left( \frac{q_i - 1}{q_i - 2} \right).
\]

\[\square\]

Once the gap \( g = 2n \) finally occurs in \( \mathcal{G}(p^\#) \), from the description of the dynamic system in [6], we know that the ratio \( w_{g,1}(p^\#) \) converges to its asymptotic value as quickly as

\[
a_2^k = \prod_{q=p_1}^{p_k} \frac{q - 3}{q - 2}
\]

converges to 0. This convergence is very slow; for \( p_1 = 17 \) and \( p_k \approx 3.01 \times 10^{15} \), \( a_2^k \) is still around 0.079138.

5. Conclusion

By identifying structure among the gaps in each stage of Eratosthenes sieve, we have been able, for a handful of conjectures about gaps between primes, to resolve the equivalent conjectures for Eratosthenes sieve. These results provide evidence toward the original conjectures, to the extent that gaps in stages of Eratosthenes sieve are indicative of gaps among primes themselves.

In [4] we established that across the stages of Eratosthenes sieve:
• **Spikes - liminf.** In the \( k \)th stage of the sieve, there are consecutive gaps \( g_k \) and \( g_{k+1} \) such that as \( k \to \infty \), \( \liminf \frac{g_{k+1}}{g_k} = 0 \).

• **Spikes - limsup.** In the \( k \)th stage of the sieve, there are consecutive gaps \( g_k \) and \( g_{k+1} \) such that as \( k \to \infty \), \( \limsup \frac{g_{k+1}}{g_k} = \infty \).

• **Superlinear growth.** For any \( n > 2 \), there exists a stage \( k_n \) of the sieve, such that for all stages \( k \geq k_n \), there exists a sequence of \( n \) consecutive gaps \( g_{k,i+1}, \ldots, g_{k,i+n} \)

\[
g_{k,i+1} < g_{k,i+2} < \cdots < g_{k,i+n}.
\]

• **Superlinear decay.** For every \( n > 2 \), there exists a stage \( k_n \) of the sieve, such that for all stages \( k \geq k_n \), there exists a sequence of \( n \) consecutive gaps \( g_{k,i+1}, \ldots, g_{k,i+n} \)

\[
g_{k,i+1} > g_{k,i+2} > \cdots > g_{k,i+n}.
\]

These results provide examples that persist through all subsequent stages of Eratosthenes sieve and thereby provide evidence to resolve conjectures by Erdős and Turán [2].

In this paper we have generalized the approach we have used in [4] [5] [6] in order to establish for Eratosthenes sieve the analogue of Polignac’s conjecture. We have shown that for every \( n > 0 \), there is a stage \( k_n \) such that for every stage \( k \geq k_n \) of the sieve there exist gaps of size \( g = 2n \).

Moreover, we have shown that the ratio of the number of gaps \( g = 2n \) to the number of gaps 2 in the \( k \)th stage of the sieve, which ratio is denoted \( w_{2n,1}(p_k) \), asymptotically approaches the ratio suggested by Hardy and Littlewood [3]:

\[
w_{2n,1}(\infty) = \prod_{q > 2, q|n} \frac{q - 1}{q - 2}.
\]

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