Discretized Yang-Mills and Born-Infeld actions on finite group geometries

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Abstract

Discretized nonabelian gauge theories living on finite group spaces $G$ are defined by means of a geometric action $\int \text{Tr } F \wedge *F$. This technique is extended to obtain discrete versions of the Born-Infeld action. The discretizations are in 1-1 correspondence with differential calculi on finite groups.

A consistency condition for duality invariance of the discretized field equations is derived for discretized $U(1)$ actions $S[F]$ living on a 4-dimensional abelian $G$. Discretized electromagnetism satisfies this condition and therefore admits duality rotations.

Yang-Mills and Born-Infeld theories are also considered on product spaces $M^D \times G$, and we find the corresponding field theories on $M^D$ after Kaluza-Klein reduction on the $G$ discrete internal spaces. We examine in some detail the case $G = \mathbb{Z}_N$, and discuss the limit $N \to \infty$.

A self-contained review on the noncommutative differential geometry of finite groups is included.

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1 Introduction

The differential geometry of finite groups $G$ has proved to be a useful tool in constructing gauge and gravity theories on discrete spaces [1, 2, 3, 4, 5]. These spaces are the finite group “manifolds” associated to every differential calculus (DC) on $G$, in 1-1 correspondence with unions of its conjugation classes [1, 6, 2]. The construction of DC on finite $G$ is a particular instance of a general procedure yielding DC on Hopf algebras, first studied by Woronowicz [7] in the noncommutative context of quantum groups (see also [8] for a review with applications to field theory).

In this paper we extend the results of [2, 3, 4] and formulate discretized gauge theories by means of a geometric action $\int F \wedge *F$ on a finite group $G$. Using the same geometrical tools, we also construct a discretized version of Born-Infeld theory.

The continuum Born-Infeld theory [9], in its commutative and noncommutative settings, related by the Seiberg-Witten map, has become relevant in the description of $D$-brane dynamics, see for ex. [10]. Noncommutative structures ([11] for a review) in string/brane theory have emerged in the last years (see for ex. [12, 10] and ref.s therein), and are the object of intense research, see e.g. [13] and included ref.s. This motivates in part our investigation concerning a particularly simple form of noncommutative Born-Infeld (BI) theory: the one that arises by considering the BI action living on finite group spaces.

Here the noncommutativity is mild, in the sense that fields commute between themselves (in the classical theory), and only the commutations between fields and differentials, and of differentials between themselves are nontrivial. In this framework we obtain a discretized Yang-Mills and BI theory for every differential calculus on a finite group.

We study the issue of duality invariance of a nonlinear electromagnetic theory described by a generic action $S[F]$ on a 4-dimensional abelian finite group. A consistency condition is found, and generalizes the results of the continuum limit.

Considering then Yang-Mills and BI theories as living on the product space $(D$-Minkowski) $\times$ (finite group space) we obtain, after use of Kaluza-Klein reduction techniques, $D$-dimensional Yang-Mills and BI theories coupled to scalar fields. The harmonic analysis on the discrete internal spaces is trivial. It is tempting to interpret the product space $(D$-Minkowski) $\times$ (finite group space) as a bundle of $n$ $(D-1)$-dimensional branes evolving in time, $n$ being the dimension of the finite group.

The paper is organized as follows. In Section 2, after a short review of the differential geometry of finite groups, we show that the metric is essentially unique (for each real differential calculus) and then define the Hodge dual. Section 3 recalls previously obtained results on gauge theories living on finite group manifolds, with some new observations. In particular the Hodge dual is used to formulate discretized gauge theories on finite $G$ manifolds in purely geometrical terms, as in the continuum case. In Section 4 we address the question of duality invariance. In Section 5 we consider Yang-Mills theory on $M^D \times G$, and reduce it via Kaluza-Klein techniques to a continuum gauge theory in $M^D$ coupled to scalar fields. We then specialize our analysis in Section 6 to the case $G = Z_N$. As in the case $M^D \times Z_2$ (see for ex. [3]), a potential for the scalar field emerges in the $D$-dimensional action. In Section 7 a finite group lattice action for Born-Infeld theory is presented. Section 8 deals with Kaluza-Klein BI theory on $M^D \times G$, with some explicit
2 A brief review of differential calculus on finite groups

Let $G$ be a finite group of order $n$ with generic element $g$ and unit $e$. Consider $\text{Fun}(G)$, the set of complex functions on $G$. An element $f$ of $\text{Fun}(G)$ is specified by its values $f_g \equiv f(g)$ on the group elements $g$, and can be written as
\[
f = \sum_{g \in G} f_g x^g, \quad f_g \in \mathbb{C}
\]
where the functions $x^g$ are defined by
\[
x^g(g') = \delta_{g,g'}^g
\]
Thus $\text{Fun}(G)$ is a $n$-dimensional vector space, and the $n$ functions $x^g$ provide a basis.
$\text{Fun}(G)$ is also a commutative algebra, with the usual pointwise sum and product, and unit $I$ defined by $I(g) = 1, \forall g \in G$. In particular:
\[
x^g x^{g'} = \delta_{g,g'}^g x^g, \quad \sum_{g \in G} x^g = I
\]
The left and right actions of the group $G$ on itself
\[
L_g g' = g g' = R_{g'} g \quad \forall g, g' \in G,
\]
induce the left and right actions (pullbacks) $L_g, R_g$ on $\text{Fun}(G)$
\[
[L_g f](g') = f(g g') = [R_{g'} f](g) \quad \forall f \in \text{Fun}(G).
\]
For the basis functions we find easily:
\[
L_{g_1} x^g = x^{g -1} g, \quad R_{g_1} x^g = x^{g g_1 -1}
\]
Moreover:
\[
L_{g_1} L_{g_2} = L_{g_2 g_1}, \quad R_{g_1} R_{g_2} = R_{g_1 g_2}, \quad L_{g_1} R_{g_2} = R_{g_2} L_{g_1}
\]
The $G$ group structure induces a Hopf algebra structure on $\text{Fun}(G)$, and this allows the construction of differential calculi on $\text{Fun}(G)$, according to the techniques of ref. [7, 8]. We summarize here the main properties of these calculi. A detailed treatment can be found in [2], and Hopf algebraic formulas, allowing contact with the general method of [7, 8], are listed in the Appendix.

A (first-order) differential calculus on $\text{Fun}(G)$ is defined by a linear map $d: \text{Fun}(G) \to \Gamma$, satisfying the Leibniz rule $d(ab) = (da)b + a(db), \quad \forall a, b \in \text{Fun}(G)$. The “space of 1-forms” $\Gamma$ is an appropriate bimodule on $\text{Fun}(G)$, which essentially means that its elements can be multiplied on the left and on the right by elements of $\text{Fun}(G)$. From the Leibniz
rule \( da = d(Ia) = (dI)a + Ida \) we deduce \( dI = 0 \). Consider the differentials of the basis functions \( x^g \). From \( 0 = dI = d(\sum_{g \in G} x^g) = \sum_{g \in G} dx^g \) we see that in this calculus only \( n - 1 \) differentials are independent.

A differential calculus is left or right covariant if the left or right action of \( G \) (\( \mathcal{L}_g \) or \( \mathcal{R}_g \)) commutes with the exterior derivative \( d \). Requiring left and right covariance in fact defines the action of \( \mathcal{L}_g \) and \( \mathcal{R}_g \) on differentials: \( \mathcal{L}_g db \equiv d(\mathcal{L}_g b), \forall b \in \text{Fun}(G) \) and similarly for \( \mathcal{R}_g db \). More generally, on elements of \( \Gamma \) (one-forms) we define \( \mathcal{L}_g \) as:

\[
\mathcal{L}_g(adb) \equiv (\mathcal{L}_g a)\mathcal{L}_g db = (\mathcal{L}_g a)d(\mathcal{L}_g b) \tag{2.8}
\]

and similarly for \( \mathcal{R}_g \). A differential calculus is called bicovariant if it is both left and right covariant.

As in usual Lie group manifolds, we can introduce in \( \Gamma \) the left-invariant one-forms \( \theta^g \):

\[
\theta^g \equiv \sum_{h \in G} x^h g^{-1} dx^h = \sum_{h \in G} x^h dx^h, \tag{2.9}
\]

These \( \theta^g \) correspond to the \( \theta^g \) of ref.s [1, 2, 3, 4].

It is immediate to check that indeed \( \mathcal{L}_h \theta^g = \theta^g \). The right action of \( G \) on the elements \( \theta^g \) is given by:

\[
\mathcal{R}_h \theta^g = \theta^{ad(h)g}, \quad \forall h \in G \tag{2.10}
\]

where \( ad \) is the adjoint action of \( G \) on itself, i.e. \( ad(h)g \equiv hgh^{-1} \). Notice that \( \theta^e \) is biinvariant, i.e. both left and right invariant.

From \( \sum_{g \in G} dx^g = 0 \) one finds:

\[
\sum_{g \in G} \theta^g = \sum_{g,h \in G} x^h dx^h = \sum_{h \in G} x^h \sum_{g \in G} dx^g = 0 \tag{2.11}
\]

Therefore we can take as basis of the cotangent space \( \Gamma \) the \( n - 1 \) linearly independent left-invariant one-forms \( \theta^g \) with \( g \neq e \) (but smaller sets of \( \theta^g \) can be consistently chosen as basis, see later). Using (2.3) the relations (2.9) can be inverted:

\[
dx^h = \sum_{g \in G} x^h g^{-1} \theta^g = \sum_{g \neq e} (x^h g^{-1} - x^h)\theta^g \tag{2.12}
\]

Analogous results hold for right invariant one-forms \( \zeta^g \):

\[
\zeta^g = \sum_{h \in G} x^h \theta^h \tag{2.13}
\]

Using the definition of \( \theta^g \) (2.9), the commutations between \( x \) and \( \theta \) are easily obtained:

\[
x^h dx^g = x^h \theta^{h^{-1}g} = \theta^{h^{-1}g} x^g \quad (h \neq g) \quad \Rightarrow \theta^g x^h = x^h \theta^g \quad (g \neq e) \tag{2.14}
\]

and imply the general commutation rule between functions and left-invariant one-forms:

\[
\theta^g f = [\mathcal{R}_g f] \theta^g \quad (g \neq e) \tag{2.15}
\]
Thus functions do commute between themselves (i.e. $Fun(G)$ is a commutative algebra) but do not commute with the basis of one-forms $\theta^g$. In this sense the differential geometry of $Fun(G)$ is noncommutative.

The differential of an arbitrary function $f \in Fun(G)$ can be found with the help of (2.12):

$$
\begin{align*}
    df &= \sum_h f_h \, dx^h = \sum_{g,h} f_h \, x^h g^{-1} \theta^g = \sum_{g \neq e} \left( \sum_h f_h \, x^h g^{-1} - f \right) \theta^g \\
    &= \sum_{g \neq e} \left( [R_g f] - f \right) \theta^g = \sum_{g \neq e} (t_g f) \, \theta^g.
\end{align*}
$$

(2.16)

Here the finite difference operators $t_g = R_g - 1$ are the analogues of (left-invariant) tangent vectors. They satisfy the Leibniz rule:

$$
    t_g (f f') = (t_g f) f' + R_g (f) t_g f' = (t_g f) R_g f' + f t_g f'.
$$

(2.17)

and close on the fusion algebra:

$$
    t_g t_{g'} = (R_{gg'} - 1) - (R_g - 1) - (R_{g'} - 1) = \sum_h C^h_{g,g'} t_h,
$$

(2.18)

where the structure constants $C^h_{g,g'}$ are

$$
    C^h_{g,g'} = \delta^h_{gg'} - \delta^h_{g} - \delta^h_{g'},
$$

(2.19)

The commutation rule (2.15) allows to express the differential of a function $f \in Fun(G)$ as a commutator of $f$ with the biinvariant form $\sum_{g \neq e} \theta^g = -\theta^e$:

$$
    df = \left[ \sum_{g \neq e} \theta^g, f \right] = -[\theta^e, f].
$$

(2.20)

An exterior product, compatible with the left and right actions of $G$, can be defined as

$$
    \theta^g \wedge \theta^{g'} = \theta^g \otimes \theta^{g'} - \sum_{k, k'} \Lambda^{g g'}_{k k'} \theta^k \otimes \theta^{k'} = \theta^g \otimes \theta^{g'} - \theta^{g g'} g^{-1} \otimes \theta^g = \\
    \theta^g \otimes \theta^{g'} - [R_g \theta^{g'}] \otimes \theta^g, \quad (g, g' \neq e),
$$

(2.21)

where the tensor product between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes \rho a'$, $a(\rho \otimes \rho') = (a\rho) \otimes \rho'$ and $(\rho \otimes \rho') a = \rho \otimes (\rho' a)$. The braiding matrix $\Lambda$:

$$
    \Lambda^{g g'}_{k k'} = \delta^g_{k} \delta'^g_{k'} = \delta^g_{g'}, \quad (g, g' \neq e).
$$

(2.22)

satisfies the Yang-Baxter equation $\Lambda^{\alpha \beta}_{\gamma \delta} \Lambda^{\delta \gamma}_{\epsilon \alpha} \Lambda^{\epsilon \delta}_{\beta \gamma} = \Lambda^{\beta \gamma}_{\delta \epsilon} \Lambda^{\epsilon \delta}_{\gamma \beta} \Lambda^{\gamma \epsilon}_{\beta \delta}$. With this exterior product we find

$$
    \theta^g \wedge \theta^g = 0 \quad (\forall g), \quad \theta^g \wedge \theta^{g'} = -\theta^{g'} \wedge \theta^g \quad (\forall g, g' : \quad [g, g'] = 0, \ g \neq e).
$$

(2.23)
Moreover
\[ \theta^g \wedge \theta^{g'} f = (\mathcal{R}_{gg'} f) \theta^g \wedge \theta^{g'} \] (2.24)

Left and right actions on \( \Gamma \otimes \Gamma \) are simply defined by:
\[ \mathcal{L}_h (\rho \otimes \rho') = \mathcal{L}_h \rho \otimes \mathcal{L}_h \rho', \quad \mathcal{R}_h (\rho \otimes \rho') = \mathcal{R}_h \rho \otimes \mathcal{R}_h \rho' \] (2.25)

Compatibility of the exterior product with \( \mathcal{L} \) and \( \mathcal{R} \) means that
\[ \mathcal{L}(\theta^i \wedge \theta^j) = \mathcal{L}\theta^i \wedge \mathcal{L}\theta^j, \quad \mathcal{R}(\theta^i \wedge \theta^j) = \mathcal{R}\theta^i \wedge \mathcal{R}\theta^j \] (2.26)

Only the second relation is nontrivial and is verified upon use of the definition (2.21). We can generalize this definition to exterior products of left-invariant 1-forms as follows \[18\]:
\[ \theta_{i_1} \wedge ... \wedge \theta_{i_k} \equiv A_{i_1..i_k}^j \theta_j \otimes ... \otimes \theta_k \] (2.27)

or in short-hand notation (FRT matrix notations \[14\]):
\[ \theta^1 \wedge ... \wedge \theta^k = A_{1...k} \theta^1 \otimes ... \otimes \theta^k \] (2.28)

The labels 1...k in \( A \) refer to index couples, and \( A_{1...k} \) is the analogue of the antisymmetrizer of \( k \) spaces, defined by the recursion relation
\[ A_{1...k} = (1 - \Lambda_{k-1,k} + \Lambda_{k-2,k-1} - \ldots - (-1)^k \Lambda_{12} \Lambda_{23} \ldots \Lambda_{k-1,k}) A_{1...k-1}, \] (2.29)

where \( \Lambda_{12} = 1 - \Lambda_{12} \). The space of \( k \)-forms \( \Gamma^{\wedge k} \) is therefore defined as in the usual case but with the new permutation operator \( \Lambda \), and can be shown to be a bicovariant bimodule (see for ex. \[15\]), with left and right action defined as for \( \Gamma \otimes ... \otimes \Gamma \) with the tensor product replaced by the wedge product. The property (2.24) generalizes to:
\[ \theta^{i_1} \wedge ... \wedge \theta^{i_k} f = (\mathcal{R}_{i_1..i_k} f) \theta^{i_1} \wedge ... \wedge \theta^{i_k} \] (2.30)

The graded bimodule \( \Omega = \sum_k \Gamma^{\wedge k} \) is the exterior algebra of forms. As recalled in the Appendix, this algebra is also a Hopf algebra \[16, 17\].

The exterior derivative is defined as a linear map \( d : \Gamma^{\wedge k} \to \Gamma^{\wedge (k+1)} \) satisfying \( d^2 = 0 \) and the graded Leibniz rule
\[ d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^k \rho \wedge d\rho' \] (2.31)

where \( \rho \in \Gamma^{\wedge k}, \rho' \in \Gamma^{\wedge k'}, \Gamma^{\wedge 0} \equiv Fun(G) \). Left and right action is defined as usual:
\[ \mathcal{L}_g (d\rho) = d\mathcal{L}_g \rho, \quad \mathcal{R}_g (d\rho) = d\mathcal{R}_g \rho \] (2.32)

In view of relation (2.10), and (A.20), the algebra \( \Omega \) has natural quotients over the ideals \( H_g = \{ \theta^h g h^{-1}, \forall h \} \), corresponding to the various conjugacy classes of the elements \( g \) in \( G \). The different bicovariant calculi on \( Fun(G) \) are in 1-1 correspondence with different quotients of \( \Omega \) by any sum of the ideals \( H = \sum H_g \), cf. \[1, 6, 2\]. In practice one simply sets \( \theta^g = 0 \) for all \( g \neq e \) not belonging to the particular union \( G' \) of conjugacy classes
characterizing the differential calculus. The dimension of the space of independent 1-forms for each bicovariant calculus on $\text{Fun}(G)$ is therefore equal to the dimension of the subspace $\Gamma/H$.

The Cartan-Maurer equation for the differential forms $\theta^g$ (2.9) is obtained by direct calculation, using the definition (2.9), the expression (2.12) of $dx^h$ in terms of $\theta^i$s, and the commutations (2.14):

$$d\theta^g = - \sum_{\substack{h \neq e, h' \neq e}} \delta^g_{hh'} \theta^h \wedge \theta^{h'} + \sum_{k \neq e} \theta^k \wedge \theta^g + \sum_{k \neq e} \theta^g \wedge \theta^k = - \sum_{\substack{h \neq e, h' \neq e}} \sum_{k \neq e} C^g_{hh'} \theta^h \wedge \theta^{h'} , \quad (g \neq e)$$

where the structure constants $C^g_{hh'}$ are given in (2.19). Using the identity:

$$\sum_{\substack{h \neq e, h' \neq e}} \delta^k_{hh'} \theta^h \wedge \theta^{h'} = \sum_{\substack{h \neq e, h' \neq e}} \delta^k_{hh'} \left( \theta^h \otimes \theta^{h'} - \theta^{h'h} \wedge \theta^h \right) = 0$$

the Cartan-Maurer equation can be rewritten by means of the anticommutator of $\theta^g$ with the biinvariant form $\theta^e$:

$$d\theta^g = - \theta^e \wedge \theta^g - \theta^g \wedge \theta^e$$

cf. the case of 0-forms (2.20). Considering now a generic element $\rho = a \theta$ of $\Gamma$ it is easy to find that $d\rho = - \theta^e \wedge \rho - \rho \wedge \theta^e$. The general rule is

$$d\rho = [ - \theta^e, \rho ]_{\text{grad}} \equiv - \theta^e \wedge \rho + (-1)\deg(\rho) \rho \wedge \theta^e$$

valid for any $k$-form, where $[ - \theta^e, \rho ]_{\text{grad}}$ is the graded commutator. Thus $- \theta^e = \sum_{k \neq e} \theta^k$ is the $X$ generator of Woronowicz theory [7], or BRST operator [17]; in our case $X$ belongs to $\Gamma$.

There are two (Hopf algebra) conjugations on $\text{Fun}(G)$ [1, 4]

$$(x^g)^* = x^g , \quad (x^g)^* = x^{g^{-1}}$$

On one-forms

$$(\theta^g)^* = - \theta^{g^{-1}} , \quad (\theta^g)^* = \zeta^g ,$$

with $(f \theta^g h)^* = h^*(\theta^g)^* f^*$, $(f \theta^g h)^* = h^*(\theta^g)^* f^*$. These involutions can be extended to the whole tensor algebra of one-forms ([7]):

$$(\theta^1 \otimes \theta^2 \otimes \cdots \otimes \theta^k)^* = \Pi_{j_1 \cdots j_k}^{i_1 \cdots i_k} (\theta^{j_k})^* \otimes (\theta^{j_{k-1}})^* \otimes \cdots \otimes (\theta^{j_1})^* ,$$

where the linear operator $\Pi_{j_1 \cdots j_k}^{i_1 \cdots i_k}$ is the permutation $\begin{pmatrix} 1, 2, \ldots, k^{-1}, k \end{pmatrix}$ where nearest neighbour transpositions are replaced with the braid operator $\Lambda^{-1}$. A realization of $\Pi_{i_1 \cdots k}$ is $\Pi_{i_1 \cdots k} = \Lambda_1^{-1} \Lambda_2^{-1} \Lambda_1^{-1} \Lambda_2^{-1} \cdots \Lambda_{k-1}^{-1} \Lambda_k^{-1} \cdots \Lambda_2^{-1} \Lambda_1^{-1} \Lambda_2^{-1} \cdots \Lambda_{k-1}^{-1} \Lambda_k^{-1} \cdots \Lambda_2^{-1} \Lambda_1^{-1}$ [see (2.28),(2.29) for the notation]. Explicitly:

$$(\theta^{i_1} \otimes \theta^{i_2} \otimes \cdots \otimes \theta^{i_k})^* = (-1)^k \theta^{ad(i_2 \cdots i_k)} \otimes \theta^{ad(i_3 \cdots i_k)} \otimes \cdots \otimes \theta^{i_k}$$
In particular, this antilinear involution maps exterior forms onto exterior forms, and we have
\[(\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^k)^\ast = (-1)^{\frac{k(k+1)}{2}} \theta^{i_1} \wedge \cdots \wedge \theta^{i_{k-1}} \wedge \theta^{i_k}\] (2.41)
and more generally \( (\rho \wedge \rho')^\ast = (-1)^{\text{deg}(\rho) \text{deg}(\rho')} \rho^* \wedge \rho'^\ast \) with \( \rho, \rho' \) arbitrary forms. These same formulas hold also when we consider the \( \ast \)-conjugation. We’ll use the \( \ast \)-conjugation in the sequel. Consistency of this conjugation requires that if \( \theta^g \neq 0 \) then \( \theta^{g^{-1}} \neq 0 \) as well: we have to include in \( \Gamma/H \) at least the two ideals \( H_g \) and \( H_{g^{-1}} \) (if they do not coincide). We obtain thus a \( \ast \)-differential calculus, i.e. \( (df)^\ast = d(f^\ast) \).

The fact that both \( \theta^g \) and \( \theta^{g^{-1}} \) are included in the basis of left-invariant 1-forms characterizing the differential calculus also ensures the existence of a unique metric (up to a normalization).

The metric is defined as a bimodule pairing, symmetric on left-invariant 1-forms. It maps couples of 1-forms \( \rho, \sigma \) into \( \text{Fun}(G) \), and satisfies the properties
\[< f \rho, \sigma h > = f < \rho, \sigma > h \ , \ < \rho f, \sigma > = < \rho, f \sigma > \] (2.42)
where \( f \) and \( h \) are arbitrary functions belonging to \( \text{Fun}(G) \). Up to a normalization the above properties determine the metric on the left-invariant 1-forms. Indeed from \( < \theta^g, f \theta^h > = < \theta^g, \theta^h > \mathcal{R}_{h^{-1}} f = \mathcal{R}_{g} f < \theta^g, \theta^h > \) one deduces:
\[g^{rs} \equiv < \theta^r, \theta^s > \equiv -\delta^{r}_{s-1} \]
(2.43)
Thus \( g^{rs} \) is symmetric and \( \theta^r \) has nonzero pairing only with \( \theta^{r^{-1}} \). The pairing is compatible with the \( \ast \)-conjugation
\[< \rho, \sigma > \ast = < \sigma^\ast, \rho^\ast > \] (2.44)
We can generalize \(< , > \) to tensor products of \( k \) left-invariant one-forms:
\[< \theta^{i_1} \otimes \cdots \otimes \theta^{i_k}, \theta^{j_1} \otimes \cdots \otimes \theta^{j_k} > \equiv < \theta^{i_1}, \theta^{j_1} > < \theta^{i_2}, \theta^{j_2} > \cdots < \theta^{i_k}, \theta^{j_k} > \]
(2.45)
where \( j_i' = (i_{t+1}i_{t+2}\ldots i_k)j_i(i_{t+1}i_{t+2}\ldots i_k)^{-1} \), i.e. \( \theta^{j_i'} = \mathcal{R}_{i_{t+1}i_{t+2}\ldots i_k} \theta^{j_i} \), and \( j_k' = j_k \). Using (2.40) the above definition is equivalent to
\[< (\theta^{i_1} \otimes \cdots \otimes \theta^{i_k})^\ast, \theta^{j_1} \otimes \cdots \otimes \theta^{j_k} > = \delta^{i_1j_1} \cdots \delta^{i_kj_k} \] (2.46)
From this formula we see that the pairing is symmetric on left-invariant tensors (tensor products of left-invariant one-forms). The pairing (2.45) is extended to all tensors by \(< f \rho, \sigma h > = f < \rho, \sigma > h \) where now \( \rho \) and \( \sigma \) are generic tensors of same order. Then we prove easily that \(< \rho f, \sigma > = < \rho, f \sigma > \) for any function \( f \) so that \(< , > \) is a bimodule pairing. Left and right invariance of \(< , > \) is straightforward. Moreover the pairing is compatible with the \( \ast \)-conjugation: formula (2.44) holds also when \( \rho \) and \( \sigma \) are arbitrary tensors.

In general for a differential calculus with \( m \) independent tangent vectors, there is an integer \( p \geq m \) such that the linear space of left-invariant \( p \)-forms is 1-dimensional, and \((p+1)\)-forms vanish identically \(^1\). This means that every product of \( p \) basis one-forms

\(^1\)with the exception of \( Z_2 \), see ref. [3]
\( \theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} \) is proportional to one of these products, that can be chosen to define the volume form \( \text{vol} \):

\[
\theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} = \epsilon^{g_1, g_2, ..., g_p} \text{vol}
\]

where \( \epsilon^{g_1, g_2, ..., g_p} \) is the proportionality constant, a real number since the braiding matrix \( \Lambda \) is real.

The volume \( p \)-form is obviously left invariant. It is also right invariant [2] (the proof is based on the \( \text{ad}(G) \) invariance of the \( \epsilon \) tensor: \( \epsilon^{\text{ad}(g) h_1, ..., \text{ad}(g) h_p} = \epsilon^{h_1, ..., h_p} \)).

Finally, if \( \text{vol} = \theta^{k_1} \wedge ... \wedge \theta^{k_p} \), then

\[
\text{vol}^* = (-1)^{\frac{p(p+1)}{2}} \epsilon^{k_1^{-1}...k_p^{-1}} \text{vol}
\]

so that \( \text{vol} \) is either real or imaginary. If \( \text{vol}^* = -\text{vol} \) we can always multiply it by \( i \) and obtain a real volume form. In that case comparing \( (\theta^{g_1} \wedge ... \wedge \theta^{g_p})^* = (-1)^{\frac{p(p+1)}{2}} \theta^{g_p^{-1}} \wedge ... \wedge \theta^{g_1^{-1}} = \epsilon^{g_p^{-1}...g_1^{-1}} (-1)^{\frac{p(p+1)}{2}} \text{vol} \) with \( (\theta^{g_1} \wedge ... \wedge \theta^{g_p})^* = \epsilon^{g_1, ..., g_p} \text{vol} \) yields

\[
\epsilon^{g_p^{-1}...g_1^{-1}} = (-1)^{\frac{p(p+1)}{2}} \epsilon^{g_1, ..., g_p}
\]

Computing the pairing of \( \text{vol} \) with itself yields:

\[
< \text{vol}, \text{vol} >= N, \quad N \in \mathbb{R}
\]

Clearly the pairing (2.45) or \( \text{vol} \) can be normalized so that \( < \text{vol}, \text{vol} >= \pm 1 \) but we’ll use (2.50) in the following.

Having identified the volume \( p \)-form it is natural to define the integral of the generic \( p \)-form \( f \) \( \text{vol} \) on \( G \)

\[
\int f \text{ vol} = \sum_{g \in G} f(g)
\]

the right-hand side being just the Haar measure of the function \( f \).

The Hodge dual, an important ingredient for gauge theories, can be defined as the unique map from \( k \)-forms \( \sigma \) to \( (p-k) \)-forms \( \ast \sigma \) such that (see [18] for a similar construction):

\[
\rho \wedge \ast \sigma = < \rho, \sigma > \text{vol} \quad \rho, \sigma \text{ n-forms}
\]

The Hodge dual is left linear; if \( \text{vol} \) is central it is also right linear:

\[
\ast (f \rho h) = f(\ast \rho) h
\]

with \( f, h \in \text{Fun}(G) \). Moreover

\[
\ast N = N \text{ vol} , \quad \ast \text{vol} = N .
\]

**Note:** the “group manifold” of a finite group is simply a collection of points corresponding to the group elements, linked together in various ways, each corresponding to a particular differential calculus on \( \text{Fun}(G) \) [1, 2]. The links are associated to the tangent vectors
\( R_h - 1 \) of the differential calculus, or equivalently to the right actions \( R_h \), where \( h \) belongs to the union \( G' \) of conjugacy classes characterizing the differential calculus. Two points \( x^g \) and \( x^{g'} \) are linked if \( x^{g'} = R_h x^g \), i.e. if \( g' = gh^{-1} \) for some \( h \) in \( G' \). The link is oriented from \( x^g \) to \( x^{g'} \) (unless \( h = h^{-1} \) in which case the link is unoriented): the resulting “manifold” is an oriented graph. From every point exactly \( m \) (= number of independent 1-forms) links originate. Some examples of finite group manifolds can be found in [2].

3 Gauge theories on finite groups

A natural question arises: is it possible to construct field theories on the discrete spaces provided by finite group manifolds? The answer is affirmative: exploiting the differential calculus on finite \( G \), gauge and gravity theories have been constructed in ref.s [1, 2, 3, 4, 5]. To prepare the ground for discretized Born-Infeld theory, we recall in this section how to construct \( G = U(N) \) gauge field theory on finite \( G \) group spaces and add some new observations. The discretized Yang-Mills action involves only geometric objects: the 2-form field strength \( F \), the \( * \)-Hodge operator and the invariant integral on \( G \).

The gauge field of a Yang-Mills theory on a finite group \( G \) is a matrix-valued one-form \( A(x) = A_h(x) \theta^h \). The components \( A_h \) are functions on \( G \); they can be considered functions of the “coordinates” \( x^g \), since any element of \( \text{Fun}(G) \) can be expanded on the basis functions \( x^g \). Moreover they are matrix-valued, i.e. \( A = (A_h)^\alpha_\beta \theta^h, \alpha, \beta = 1, \ldots N \). In the following matrix multiplication is implicit.

As in the usual case, \( G \) gauge transformations are defined as

\[
A' = -(dT)T^{-1} + TAT^{-1}
\]

(3.1)

where \( T(x) = T(x)^{\alpha_\beta} \) is an \( N \times N \) representation of a \( G \) group element; its matrix entries belong to \( \text{Fun}(G) \). In components:

\[
A'_h = -(t_h T)R_h T^{-1} + TA_h R_h T^{-1} = T [1 + A_h] [R_h T] - 1
\]

(3.2)

Matter fields \( \psi_\alpha(x) \) transform in a representation of \( G \) as \( \psi \rightarrow \psi' = T \psi \), and their covariant derivative, defined by

\[
D \psi = d \psi + A \psi
\]

(3.3)

transforms homogeneously: \( (D \psi)' = T(D \psi) \). The 2-form field strength \( F \) arises as usual in the square of the covariant derivative \( (D^2) \psi = F \psi \); it is given by the familiar expression

\[
F = dA + A \wedge A = d(A_k \theta^k) + A_h \theta^h \wedge A_k \theta^k
\]

\[
\begin{align*}
&= (t_h A_k) \theta^h \wedge \theta^k + A_k d \theta^k + A_h (R_h A_k) \theta^h \wedge \theta^k \\
&= [t_h A_k - A_j C^j_{hk} + A_h (R_h A_k)] \theta^h \wedge \theta^k
\end{align*}
\]

(3.4)

and satisfies the Bianchi identity:

\[
dF + A \wedge F - F \wedge A = 0
\]

(3.5)
Note that $A \wedge A \neq 0$ even if the gauge group $\mathcal{G}$ is abelian. Thus $U(1)$ gauge theory on a finite group space looks like a nonabelian theory, a situation occurring also in noncommutative $*$-deformed gauge theories.

Under gauge transformations (3.1) $F$ varies homogeneously:

$$F' = TFT^{-1}$$  \hspace{1cm} (3.6)

The gauge variation (3.2) suggests the definition of the link fields $U_h(g)$ and link 1-form $U(g)$:

$$U_h = 1 + A_h, \quad U = \sum_{h \neq e} U_h \theta^h = \sum_{h \neq e} \theta^h + A$$  \hspace{1cm} (3.7)

transforming homogeneously:

$$U'_g = TU_g \mathcal{R}_g T^{-1}, \quad U' = TU T^{-1}. \hspace{1cm} (3.8)$$

The field strength (3.4) can be expressed in terms of the link fields:

$$F = (U_h \mathcal{R}_h U_k) - U_g \delta^g_{hk} \theta^h \wedge \theta^k = U_h (\mathcal{R}_h U_k) \theta^h \wedge \theta^k = U \wedge U$$  \hspace{1cm} (3.9)

where we have used the identity (2.34).

Defining the components $f_{h,k}$ as:

$$F \equiv f_{h,k} \theta^h \otimes \theta^k$$  \hspace{1cm} (3.10)

eq. (3.9) yields:

$$f_{h,k} = U_h (\mathcal{R}_h U_k) - U_k (\mathcal{R}_k U^{-1}_{k-h_k}). \hspace{1cm} (3.11)$$

The Yang-Mills action is the geometrical action quadratic in $F$ given by

$$A_{YM} = - \int Tr(F \wedge * F) = - \int Tr < F, F > vol. \hspace{1cm} (3.12)$$

Recalling the pairing properties (2.42) the proof of gauge invariance of $Tr < F, F >$ is immediate:

$$Tr < F', F' > = Tr < TFT^{-1}, TFT^{-1} > = Tr T < F, F > T^{-1} = Tr < F, F >. \hspace{1cm}$$

The metric (2.43) is an euclidean metric (as is easily seen using a real basis of one-forms) and as usual we require (3.12) to be real and positive definite. This restricts the gauge group $\mathcal{G}$ and imposes reality conditions on the gauge potential $A$. Since (cf. (2.51))

$$\int Tr < F, F > vol = \sum_G Tr < F, F >,$$  \hspace{1cm}

positivity of (3.12) requires $-(Tr < F, F >) \geq 0$. Explicitly

$$< F, F > = F_{r,s} \theta^r \otimes \theta^s, F_{m,n} \theta^m \otimes \theta^n = F_{r,s} R_{rs} F_{m,n} < \theta^r \otimes \theta^s, \theta^m \otimes \theta^n >$$

$$= F_{r,s} R_{rs} F_{m,n} \delta^{-1}_{n} \delta^{-1}_{s} \varepsilon_{m,s^{-1}} = F_{r,s} R_{rs} F_{s^{-1} r^{-1} s^{-1}}, \varepsilon_{r,s,s^{-1}} \hspace{1cm} (3.13)$$

and

$$-Tr < F, F > = -(F_{r,s})_{\beta} (R_{rs} F_{s^{-1} r^{-1} s^{-1}})^{\beta}_{\alpha}. \hspace{1cm} (3.14)$$
We see that (3.14) is positive definite if 

\[-(\mathcal{R}_{rs} F_{s-1,r-1})^\beta_{\alpha} = (F_{r,s}^*)^\alpha_{\beta}, \text{ i.e. if} \]

\[ F_{r,s}^\dagger = -\mathcal{R}_{rs} F_{s-1,r-1} \tag{3.15} \]

The $\dagger$ conjugation by definition acts as hermitian conjugation on the matrix structure, and as the $^*$-conjugation (introduced in the previous Section) on the $Fun(G)$ entries of the matrix. The gauge action then can be rewritten (sum on the indices $h, k, \alpha, \beta$ understood)

\[ A_{YM} = \sum_G Tr(F_{h,k} F_{h,k}^\dagger) = \sum_G (F_{h,k})^\alpha_{\beta} (F_{h,k}^*)^\alpha_{\beta} \].

(3.16)

From (3.11) or (3.4) we see that (3.15) holds if

\[ A_{h}^\dagger = R_{h} A_{h-1} \] or equivalently: \[ U_{h}^\dagger = R_{h} U_{h-1} \]

(3.17)

These relations simply state that the one forms $A$ and $U$ are antihermitian:

\[ A^\dagger = -A \quad , \quad U^\dagger = -U \tag{3.18} \]

hermitian conjugation on matrix valued one-forms $A$ (or $U$) being defined as:

\[ A^\dagger = (A_h \theta^h)^\dagger = (\theta^h)^* A_{h}^\dagger = -\theta^h A_{h}^\dagger = -(R_{h} A_{h-1}^\dagger) \theta^h \]

(3.19)

From (3.18), (3.9) and (2.41) we obtain that the 2-form $F$ is antihermitian $F = -F^\dagger$.

Finally gauge trasformations must preserve the hermiticity properties of $A$ and $F$ and this is the case if the representation $T$ of $G$ is unitary. We thus conclude that the action (3.12) has maximal gauge group $G = U(N)$.

Notice that writing $A = (A_h)^\alpha_{\beta} \theta^h$, where $\alpha, \beta = 1, ... N$, the reality condition $A^\dagger = -A$ is equivalent to $A_h (g)^\beta_{\alpha} = A_{h-1} (gh)^\alpha_{\beta}$ and is thus not local (not fiberwise). It follows that $A$ has values in $M_{N \times N}(C)$ and not in the Lie algebra of $U(N)$. Nevertheless $A^\dagger = -A$ is a good reality condition because it cuts by half the total number of components of $A$. This can be seen counting the real components of the antihermitian field $A$: they are $N^2 \times n \times d$, where $n$ is the number of points of $G$, and the “dimension” $d$ counts the number of independent left-invariant one-forms. In conclusion, when $G = U(N)$, we have a bona fide pure gauge action, where the number of components of $A$ is consistent with the dimension of the gauge group.

Note also from (3.2) that since $t_h T$ is a finite difference of group elements, then $A_h$ cannot belong to the $G$ Lie algebra.

The action (3.16) can be expressed in terms of the link fields $U_h$: substituting (3.11) into (3.16) leads to

\[ A_{YM} = 2 \sum_G Tr[U_k U_k^\dagger U_h U_h^\dagger - U_k^\dagger U_h (\mathcal{R}_h U_k)(\mathcal{R}_k U_{k-1}^\dagger)] \tag{3.20} \]

after use of the cyclic property of $Tr$, and of (3.17). Moreover we also needed the obvious properties

\[ \sum_G f = \sum_G \mathcal{R}_k f \quad \forall k, \forall f \in Fun(G) \tag{3.21} \]

\[ \sum_k f_h = \sum_k f_{k-1hk} \quad \forall k, \forall f_h \in Fun(G) \tag{3.22} \]
When the finite group $G$ is abelian, the action (3.20) reduces to a Wilson-like action if we require $U_h$ to be unitary: $U_h^\dagger = (U_h)^{-1}$. See also ref. [19] for the case $G = \mathbb{Z}_N \times \cdots \times \mathbb{Z}_N$.

**Note 1.** Here we sketch an equivalent presentation of the Yang-Mills action. We first define it to be the geometric action

$$A_{YM} = \int Tr(F \wedge * F^\dagger) = \int Tr <F, F^\dagger > vol . \quad (3.23)$$

so that $A_{YM}$ is trivially real and invariant under gauge transformations (3.6) provided that $T^\dagger = T^{-1}$. We then impose reality conditions on $A$, and therefore on $U$ and $F$, that are preserved under $U(N)$ gauge transformations. We thus obtain a pure gauge theory.

**Note 2.** One can construct the finite group lattice analogues of Wilson loops. Indeed, consider the exterior product $U^k$:

$$U^k = U_{h_1} \theta_{h_1} \wedge U_{h_2} \theta_{h_2} \wedge \ldots \wedge U_{h_k} \theta_{h_k} =$$

$$= U_{h_1}[\mathcal{R}_{h_1}] U_{h_2} \ldots [\mathcal{R}_{h_1 \ldots h_{k-1}} U_{h_k}] \theta_{h_1} \wedge \theta_{h_2} \wedge \ldots \wedge \theta_{h_k} \equiv$$

$$\equiv U_{h_1 \ldots h_k} \theta_{h_1} \wedge \ldots \wedge \theta_{h_k} \quad (3.24)$$

(no sum on the indices $h_i$) such that $h_1 h_2 \cdots h_k = 1$. Then, the trace $Tr$ of the component $U_{h_1 \ldots h_k}$ gives a gauge invariant object which can be interpreted as the finite group analog of a Wilson loop. When the volume form $vol = \theta^1 \wedge \cdots \wedge \theta^p$ is central ($[vol, f] = 0, \forall f \in Fun(G)$), then also the $p$-form $U^p$ is gauge invariant.

**Note 3.** The gauge invariant action (3.16) on the finite group lattice is easily generalized to the case of higher order field strengths:

$$B = B_{h_1 \ldots h_k} \theta_{h_1} \otimes \ldots \otimes \theta_{h_k} , \quad (3.25)$$

transforming as $B \rightarrow TB T^{-1}$, or

$$B_{h_1 \ldots h_k} \rightarrow T B_{h_1 \ldots h_k} \mathcal{R}_{h_1 \ldots h_k} T^{-1} , \quad (3.26)$$

Then

$$\sum B_{h_1 \ldots h_k} B_{h_1 \ldots h_k}^\dagger \rightarrow T \sum B_{h_1 \ldots h_k} B_{h_1 \ldots h_k}^\dagger T^{-1} , \quad (3.27)$$

If $B$ satisfies $B^\dagger = -B$, then $B_{h_1 \ldots h_k}^\dagger = (-1)^{k-1} \mathcal{R}_{h_1 \ldots h_k} B_{ad(h_2 \ldots h_k)}^{-1} h_1^{-1} , ad(h_3 \ldots h_k) h_2^{-1} , \ldots h_k^{-1}$ and [use (2.46)]

$$B_{h_1 \ldots h_k} B_{h_1 \ldots h_k}^\dagger = - < B, B > \quad (3.28)$$

The gauge invariant analogue of (3.16) is

$$A = \sum G \sum_{h_i \neq e} Tr \left( B_{h_1 \ldots h_k} B_{h_1 \ldots h_k}^\dagger \right) = - \sum G Tr < B, B > . \quad (3.29)$$

In particular, setting $B = U^k$, where $U$ is the link 1-form, and taking into account the condition (3.17), the gauge invariant action (3.29) is a sum over special Wilson loops as described in Note 2.
4 Duality Rotations

In this Section we consider an “abelian” $U(1)$ gauge theory on an abelian finite group space. We choose a four dimensional differential calculus on this finite group. The $U(1)$ gauge theory is “abelian” in the sense that the field strength satisfies the Bianchi identities $dF = 0$, in particular this holds if $F = dA$. In this case infinitesimal gauge transformations read $\delta A = d\lambda$, where $\lambda$ is a gauge parameter, see for ex. the second ref. in [5]. As in standard electromagnetism the field strength $F$ is therefore invariant under gauge transformations.

It is remarkable that also for this discrete (noncommutative) version of electromagnetism one can consider electric-magnetic duality rotations (for duality rotations in noncommutative geometry where the noncommutativity is given by a $*$-product we refer to [22] and references therein). Following [20], see also the nice review [21], in this section we obtain a consistency condition for an (in general nonlinear) electromagnetic theory to admit duality rotations; we also show that the equations of motions (EOM) of Maxwell theory admit electric-magnetic rotations. As far as we know this is the first example of duality rotations on a lattice.

Let us consider a 4-dimensional $*$-calculus on $G$ with one-forms $\theta^u, \theta^{u^{-1}}, \theta^v, \theta^{v^{-1}}$ where $u, u^{-1}, v, v^{-1}$ are the four different elements of $G$ that determine the calculus. It follows that the metric $g^{h,k} = \langle \theta^h, \theta^k \rangle = -\delta_{k^{-1}}^h$ has determinant $\det g = 1$. This is an Euclidean metric as one can check expressing it in a basis of real one-forms. We also notice that since the finite group is abelian, the epsilon tensor $\varepsilon_g$ is the usual completely antisymmetric tensor (with $\varepsilon^{uu^{-1}v v^{-1}} = 1$).

Consider an action $S$ that depends on the gauge field $A$ only through the field strength $F$. The EOM are obtained by varying $S$ with respect to $A_g$. We have

$$\delta A_g \equiv \sum_{p \in G} \frac{\delta S}{\delta A_g(p)} \delta A_g(p) = \frac{1}{2} \sum_{p, q \in G} \frac{\delta S}{\delta F_{h,h'}(q)} \delta F_{h,h'}(q) \frac{\delta S}{\delta A_g(p)} \delta A_g(p) = \sum_{q \in G} \frac{\delta S}{\delta F_{h,h'}(q)} t_h \delta A_{h'}(q)$$

$$= \sum_{q \in G} -t_h \frac{\delta S}{\delta F_{h,h'}(q)} R_h \delta A_{h'}(q) = \sum_{q \in G} -t_h (R_h^{-1} \frac{\delta S}{\delta F_{h,h'}(q)}) \delta A_{h'}(q)$$

where in the second equality we used that $F_{h,h'} = t_h A_{h'} - t_{h'} A_h$ is antisymmetric, while in the third we have integrated by parts via the Leibniz rule (2.17) $t_q(fh) = t_q f R_h h + f t_q h$ and then used Stokes’ theorem: $\sum_{q \in G} t_q(f)(q) = \sum_{q \in G} (R_q f(q) - f(q)) = 0$. The EOM are therefore

$$0 = \frac{\delta S}{\delta A_{h'}} = -t_h R_h^{-1} \frac{\delta S}{\delta F_{h,h'}} \quad \text{(no sum on } h')$$

(4.1)

or, applying $R_h^{-1}$ (so that the argument of $t_h$ is antisymmetric in the indices $h, h'$):

$$t_h (R_h^{-1} \frac{\delta S}{\delta F_{h,h'}}) = 0 \quad \text{(no sum on } h')$$

(4.2)

We then define

$$\tilde{G}^{h,h'} \equiv \frac{1}{2} \varepsilon^{h'h''g'} G_{g,g'} \equiv R^{-1}_{h'h'} \frac{\delta S[F]}{\delta F_{h,h'}}$$

(4.3)
where we have written $S[F]$ in order to stress that this definition does not depend on the relation $F = dA$. Using (4.3) the EOM and the Bianchi identities $dF = 0$ have the same structure

\begin{align*}
  t_h \tilde{G}^{h,h'} &= 0 , \\
  t_h \tilde{F}^{h,h'} &= 0 ,
\end{align*}

(4.4) (4.5)

where $\tilde{F}^{h,h'} \equiv \frac{1}{2} \varepsilon^{hh'gg'} F_{g,g'}$. In summary, starting with the action $S$ we obtained the EOM (4.4) with $F = dA$. We now relax the condition $F = dA$ and consider the more general theory given by the EOM (4.4) and (4.5) with $\tilde{G}$ given in (4.3) and where now $S[F]$ is seen just as a function of $F$, with $F$ an arbitrary antisymmetric tensor (not the differential of $A$). It is this theory that possibly admits duality rotations. More precisely we show that if $S[F]$ satisfies (4.16) then we have $SO(1,1)$ duality rotations. As in [20] ([21]) we observe that under the infinitesimal $GL(2)$ rotation $G \rightarrow G + \Delta G$, $F \rightarrow F + \Delta F$ given by

\begin{equation}
  \Delta \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}
\end{equation}

(4.6)

then, using (4.6) and (4.3), condition (4.7) reads

\begin{align*}
  AR_{gg'} \tilde{G}^{gg'} + BR_{gg'} \tilde{F}^{gg'} &= \frac{\delta (S[F + \Delta F] - S[F])}{\delta F_{gg'}} - \frac{1}{2} C \sum_{q \in G} \frac{\delta S[F]}{\delta F_{hh'}(q)} \frac{\delta G_{hh'}(q)}{\delta F_{gg'}} - D \frac{\delta S[F]}{\delta F_{gg'}} \\
  &= \delta (S[F + \Delta F] - S[F]) - \frac{1}{2} C \sum_{q \in G} \tilde{G}^{hh'}(q) R_{hh'}^{-1} \frac{\delta G_{hh'}(q)}{\delta F_{gg'}} - D \frac{\delta S[F]}{\delta F_{gg'}}
\end{align*}

(4.8)

where in the last line we used the invariance of the integral $\sum_{q \in G}$ under the translation $R_{hh'}^{-1}$. In order to further simplify this expression we observe that

\begin{equation}
  \frac{\delta}{\delta F_{gg}} \sum_{q \in G} F_{hh'}(q) R_{hh'} \tilde{F}^{hh'}(q) = 4 R_{gg'} \tilde{F}^{gg'}
\end{equation}

(4.9)
Proof: Use invariance of the integral \( \sum_{q \in G} \) under the translation \( R^{-1}_{hh'} \) and then notice that \( R^{-1}_{hh'} \varepsilon^{hh'gg'} = R_{gg'} \varepsilon^{hh'gg'} \) since \( h, h', g, g' \) run over the 4 group elements \( u, u^{-1}, v, v^{-1} \).

Similarly one has \( \delta(\sum_{q \in G} G_{hh'}(q) R_{hh'} G_{hh'}(q)) = 2 \sum_{q \in G} \delta G_{hh'}(q) G_{hh'}(q) \) and therefore
\[
\frac{\delta}{\delta F_{gg'}} \sum_{q \in G} G_{hh'}(q) R_{hh'} \tilde{G}_{hh'}(q) = 2 \sum_{q \in G} \tilde{G}_{hh'}(q) R_{hh'}^\prime \frac{\delta G_{hh'}(q)}{\delta F_{gg'}} \quad (4.10)
\]

Next we substitute (4.9) and (4.10) in (4.8) and factorize out the functional derivative \( \frac{\delta}{\delta F_{gg'}} \). We thus arrive at the equivalent condition (the constant term \( (A + D) S[F = 0] \) is obtained by observing that when \( F = 0 \) also \( G = 0 \))
\[
S[F + \Delta F] - S[F] = \frac{B}{4} \sum_{G} F_{hh'} R_{hh'} \tilde{F}_{hh'} - \frac{C}{4} \sum_{G} G_{hh'} R_{hh'} \tilde{G}_{hh'} = (A + D)(S[F] - S[F = 0]) \quad (4.11)
\]

This expression can be further simplified. Use \( S[F + \Delta F] - S[F] = \frac{1}{2} \sum_{G} \frac{\delta S[F]}{\delta F_{hh'}} \Delta F_{hh'} = \frac{1}{2} C \sum_{G} G_{hh'} R_{hh'} \tilde{G}_{hh'} + \frac{1}{2} D \sum_{G} F_{hh'} R_{hh'} \tilde{G}_{hh'} \) in order to write
\[
\frac{C}{4} \sum_{G} G_{hh'} R_{hh'} \tilde{G}_{hh'} - \frac{B}{4} \sum_{G} F_{hh'} R_{hh'} \tilde{F}_{hh'} = (A + D)(S[F] - S[F = 0]) - \frac{D}{2} \sum_{G} F_{hh'} R_{hh'} \tilde{G}_{hh'} \quad (4.12)
\]

Then, assuming that the Lagrangian is parity even, the parity even and parity odd terms in (4.12) must separately vanish and we obtain
\[
C \sum_{G} G_{hh'} R_{hh'} \tilde{G}_{hh'} = B \sum_{G} F_{hh'} R_{hh'} \tilde{F}_{hh'} \quad (4.13)
\]
\[
(A + D)(S[F] - S[F = 0]) = \frac{D}{2} \sum_{G} F_{hh'} \frac{\delta S[F]}{\delta F_{hh'}} \quad (4.14)
\]

We now show that linear electromagnetism satisfies (4.13) and (4.14) with \( B = C \) and \( A = D \), i.e. that linear electromagnetism admits the duality rotation group \( SO(1,1) \) times the group of scale transformations (corresponding to \( A = D \neq 0 \)).

Let
\[
S_{EM} = -\frac{1}{4} \sum_{G} F_{hh'} F_{hh'}^\dagger = \frac{1}{4} \sum_{G} F_{hh'} R_{hh'} \tilde{F}_{hh'} F_{kk'} g^{hh'} g^{kk'},
\]
then \( \tilde{G}_{hh'} = F_{ll'} g^{hh'} g^{ll'} \equiv F_{hh'} \), and therefore \( G_{hh'} = \tilde{F}_{ll'} g_{hh'} g_{ll'} \equiv \tilde{F}_{hh'} \). Since \( R_{hh'} \varepsilon^{hh'gg'} = R_{gg'} \varepsilon^{hh'gg'} \), we have \( F_{hh'} R_{hh'} \tilde{F}_{hh'} = \tilde{F}_{hh'} R_{hh'} \tilde{F}_{hh'} \) and we conclude that (4.13) is satisfied iff \( C = B \). Similarly \( \frac{1}{2} D \sum_{G} F_{hh'} \frac{\delta S[F]}{\delta F_{hh'}} = 2 D S_{EM}[F] \) so that (4.13) is satisfied iff \( A = D \).

For a nonlinear theory \( S[F] \), that in the weak field limit reproduces \( S_{EM}[F] \) (i.e. \( S[F] = S[F = 0] + S_{EM}[F] + O(F^4) \)), condition (4.14) is equivalent to
\[
D(S[F] - S[F = 0] - \frac{1}{4} \sum_{G} F_{hh'} \frac{\delta S[F]}{\delta F_{hh'}}) = 0. \quad (4.15)
\]

---

\(^2\)We recall that \( \text{det} g = 1 \) so that the Hodge operator \( \ast \) squares to the identity and not to minus the identity as for Minkowski space. This is why in Euclidean space the duality group is \( SO(1,1) \) while in Minkowski space it is \( SO(2) \).
But $S[F]$ is not homogeneous in $F$ and therefore (4.14) is satisfied iff $A = D = 0$. We conclude that a nonlinear electromagnetic theory, that has definite parity and that in the weak field limit reproduces linear electromagnetism, admits an $SO(1, 1)$ duality rotation group if and only if

$$\sum_G G_{hh'} R_{hh'} G^{hh'} = \sum_G F_{hh'} R_{hh'} F^{hh'} .$$

(4.16)

**Note 1.** In Section 6 we present two Born-Infeld type actions on finite groups with four independent tangent vectors (four dimensions). The action (7.5) does not satisfy condition (4.16). Whether the action (7.17) in four dimensions admits $SO(1, 1)$ duality rotations remains to be determined.

## 5 Kaluza-Klein gauge theory on $M^D \times G$

We extend here the results of Section 3 to include spaces of the type $M^D \times$ finite $G$. This extension is straightforward (see also [3]), and allows us to apply Kaluza-Klein techniques when the internal space is a discrete finite $G$ “manifold”.

We’ll use the letter $x$ for the $M^D$ coordinates, and $y$ for the $G$ coordinate functions. A basis of 1-forms on $M^D \times G$ is given by $dx^A = \{dx^\mu, \theta^g\}$, with $x^{\mu*} = x^\mu$ and

$$dx^\mu \wedge \theta^g = -\theta^g \wedge dx^\mu$$  \hspace{1cm} (5.1)  

$$(dx^\mu \wedge \theta^g)^* = -(\theta^g)^* \wedge (dx^\mu)^* = \theta^{g-1} \wedge dx^\mu = -dx^\mu \wedge \theta^{g-1}$$  \hspace{1cm} (5.2)  

$$dx^{A_1} \wedge ... \wedge dx^{A_D + p} \equiv \epsilon^{A_1...A_D + p} vol(M^D \times G)$$  \hspace{1cm} (5.3)  

$$dx^{\mu_1} \wedge ... \wedge dx^{\mu_D} \wedge \theta^{g_1} \wedge ... \wedge \theta^{g_p} = \epsilon^{\mu_1...\mu_D} \epsilon^{g_1...g_p} vol(M^D) \wedge vol(G)$$  \hspace{1cm} (5.4)  

$$\int_{M^D \times G} f(x, y) d^D x \left( \int_G f(x, y) vol(G) \right)$$  \hspace{1cm} (5.5)  

$$< dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}, dx^{\nu_1} \wedge ... \wedge dx^{\nu_k} >= (k!)^2 \delta^{\mu_1...\mu_k}_{\nu_1...\nu_k},$$  \hspace{1cm} (5.6)  

$$< dx^\mu \wedge \theta^g, dx^\nu \wedge \theta^h >= -2\delta^{\mu}_\nu \delta^{g, h}_{\theta}$$  \hspace{1cm} (5.7)  

where $vol(M^D \times G) = vol(M^D) \wedge vol(G) = d^D x vol(G)$, $\epsilon^{\mu_1...\mu_D}$ is the usual Levi-Civita tensor and $\epsilon^{g_1...g_p}$ has been defined in (2.47). The normalization of exterior products is such that, for example, $dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$ and similar for $dx^\mu \wedge \theta^g$.

The gauge potential 1-form $A(x, y)$ is then expanded as:

$$A(x, y) = A_\mu(x, y) dx^\mu + A_g(x, y) \theta^g$$  \hspace{1cm} (5.8)  

The gauge variation $A' = -(dT)T^{-1} + TAT^{-1}$ becomes the usual one for the components $A_\mu$, and the one given in (3.2) for $A_g$, with $T$ depending on $x$ and $y$. Thus $A_\mu$ belongs to the Lie algebra of $G$, whereas $A_g$ belongs to the group algebra of $G$.

Defining as usual the field strength as $F = dA + A \wedge A$ and its components as

$$F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu + F_{\mu k} dx^\mu \wedge \theta^k + F_{h, k} \theta^h \otimes \theta^k$$  \hspace{1cm} (5.9)
we find
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu \quad (5.10) \]
\[ F_{\mu k} = \partial_\mu U_k + A_\mu U_k - U_k (R_k A_\mu) \equiv D_\mu U_k \quad (5.11) \]
\[ F_{h,k} = U_h (R_h U_k) - U_k (R_k U_k) \quad (5.12) \]

where the link field is defined as (cf. Section 3) \( U_g(x, y) = 1 + A_g(x, y) \). The gauge transformation \( F'(x, y) = T(x, y) F(x, y) T^{-1}(x, y) \) implies:
\[ F'_{\mu\nu}(x, y) = T(x, y) F_{\mu\nu}(x, y) T^{-1}(x, y) \quad (5.13) \]
\[ F'_{\mu k}(x, y) = T(x, y) F_{\mu k}(x, y) R_k T^{-1}(x, y) \quad (5.14) \]
\[ F'_{h,k}(x, y) = T(x, y) F_{h,k}(x, y) R_{hk} T^{-1}(x, y) \quad (5.15) \]

From the antihermiticity \( A^\dagger = -A \) (or \( U^\dagger = -U \)), one finds, as in Section 3, \( F^\dagger = -F \). In components:
\[ F_{\mu\nu}^\dagger = -F_{\mu\nu}, \quad F_{\mu k}^\dagger = R_k F_{\mu k}^{-1}, \quad F_{h,k}^\dagger = -R_{hk} F_{k^{-1} h^{-1}, k^{-1}} \quad (5.16) \]

Then the Yang-Mills action on \( M^D \times G \) can be expanded as follows:
\[ A_{YM} = -\int_{M^D \times G} \text{Tr} \ F \wedge *F = -\int_{M^D \times G} \text{Tr} \ <F, F > \text{vol}(M^D \times G) = \]
\[ = -\int_{M^D} d^D x \sum_{G} \text{Tr} \ [F_{\mu\nu} F_{\mu\nu} - 2F_{\mu k} F_{\mu k}^\dagger - F_{h,k} F_{h,k}^\dagger] = \]
\[ = \int_{M^D} d^D x \sum_{G} \text{Tr} \ [-F_{\mu\nu} F_{\mu\nu} + 2D_\mu U_k (D_\mu U_k)^\dagger + \]
\[ + 2U_k U_k^\dagger U_h U_h^\dagger - 2U_k^\dagger U_h (R_h U_k) (R_k U_k^\dagger) (R_{hk} U_{hk}^\dagger)] \quad (5.17) \]

Note that this action is real, and describes a Yang-Mills theory in \( D \) dimensions minimally coupled to the scalar fields \( U_g \).

6 Kaluza-Klein gauge theory on \( M^D \times Z_N \)

In this section we first study the geometry of \( Z_N \) equipped with a bicovariant calculus and a *-conjugation, and consider the limit \( N \to \infty \) i.e. \( Z_N \to S_1 \). It is then easy to generalize the results to \( M^D \times Z_N \), consider the Yang-Mills action on this space and understand its \( N \to \infty \) limit.

6.1 *-bicovariant calculus on \( Z_N \)

Let \( u: w^j u^k = w^{j+k}, \ u^N = u^0 = e \) be the generator of the cyclic group \( Z_N \). A basis of functions on \( Z_N \) is given by \( x^u = \{x^e, x^u, x^{u^2}, ..., x^{u^{N-1}}\} \). It is convenient to use a basis of functions that reproduce the algebra of the \( Z_n \) elements \( w^j \). This basis is given by [1]
\[ y^j \equiv \sum_{k=0}^{N-1} q_j^k u^k, \quad \text{where } q \equiv e^{\frac{2\pi i}{N}}. \] Thus \[ y^j y^k = y^{j+k}, \quad y^0 = I. \] For example \( y^1 = y \) is given by

\[ y = x^e + qx^w + q^2 x^{w^2} + ... q^{n-1} x^{w^{n-1}} \]

(6.1)

having the values:

\[ y(u^k) = q^k = e^{\frac{2\pi i k}{N}} \]

(6.2)
on the \( N \) points of \( \mathbb{Z}_N \). Using \( \sum_{j=0}^{N-1} q_{kj} = N \delta_{k,0} \) one finds the inverse transformation:

\[ x^{u^j} = \frac{1}{N} \sum_{k=0}^{N-1} q^{-jk} y^k. \]

The cyclic group \( \mathbb{Z}_N \) can be seen as a discrete approximation of the circle \( S_1 \) of radius \( R \). Let \( 0 \leq x \leq 2\pi R \) be the \( S_1 \) coordinate on the circle (not to be confused with the basis functions \( x^{u^j} \) of \( \mathbb{Z}_N \) : then the points of \( \mathbb{Z}_N \) have coordinates \( x_k = x(u^k) = 2\pi R k/N \) on the circle, and their corresponding \( y \) values are \( y(x_k) = e^{rac{i \pi k}{N}} \). In the limit \( N \to \infty \) these points fill the whole circle, and the \( x_k \) become the continuous values of the \( S_1 \) coordinate \( x \).

Among the many differential calculi on \( \mathbb{Z}_N \) the most discussed in the literature [1, 2, 4] is the one where only \( \theta^u = -\theta^w \neq 0 \). This calculus is not (for \( N \neq 2 \)) a \( * \)-calculus: \((df)^* \neq d(f^*)\). The compatibility of a differential calculus with the \( * \)-product is essential if we want to consider the calculus on \( \mathbb{Z}_N \) as a discretization of the real differential calculus on \( S_1 \). Indeed on a circle of radius \( R \) we have \((df)^* = d(f^*)\), where \( * \) denotes complex conjugation, and \( f = \sum_n f_n y^n(x) = \sum_n f_n e^{in \pi x/R} \), \( f_n \in \mathbb{C} \), \( x^* = x \), i.e. \( y^*(x) = y^{-1}(x) \).

In the following we therefore choose the \( * \)-bicovariant calculus generated by the 1-forms \( \theta^u \) and \( \theta^{u^{-1}} \). This is the minimal \( * \)-bicovariant calculus with \( \theta^u \) nonzero. In fact, from \( y^* = y^{-1} \) and \((h df)^* = d(f^*) h^* \) we obtain \( \theta^{u*} = -\theta^{u^{-1}} \).

In order to gain a better insight about the geometry of \( \mathbb{Z}_N \) it is convenient to consider the real and closed forms

\[ \alpha = -iR y^{-1} dy, \quad \beta = iR y dy^{-1}; \]

in the \( N \to \infty \) limit, requiring that the exterior differential \( d \) becomes the commutative one, we find \( \alpha = \beta = dx, 0 \leq x \leq 2\pi R \). For finite \( N, \alpha \) and \( \beta \) are linearly independent, and recalling (2.12) and (6.1) we obtain

\[ \alpha = iR(\zeta - \zeta^*) , \quad \beta = iR(q^{-1} \zeta - q \zeta^*) \]

(6.3)

where we have defined

\[ \zeta = (1 - q) \theta^u, \quad \zeta^* = (q^{-1} - 1) \theta^{u^{-1}}. \]

(6.4)

From (6.3) we see that \( \zeta, \zeta^* \neq 0 \) in the \( N \to \infty \) limit and therefore \( \theta^u \) and \( \theta^{u^{-1}} \) are ill defined in this limit.

**Metric**

\(^3\)If \( N \) is odd there is no 1-dimensional \( * \)-bicovariant calculus and the 2-dimensional \( * \)-bicovariant calculus defined by \( \theta^u \) and \( \theta^{u^{-1}} \) is the most natural. If \( N \) is even on the contrary there exist a unique 1-dimensional \( * \)-bicovariant calculus, it is generated by the 1-form \( \theta^{u^{N/2}} \) that is pure imaginary \( \theta^{u^{N/2}} = -\theta^{u^{-N/2}} = -\theta^{u^{-N/2}} \); for \( N \to \infty \) however this calculus does not lead to the standard calculus on \( S_1 \).
Using \( y^u = q^{-1} \theta^u y \), \( y^{u-1} = \theta^u y \) it is easy to see that, up to normalization, there exists a unique metric (bimodule pairing) on the space of 1-forms such that on left-invariant 1-forms it is symmetric, and satisfies the properties in (2.42). The normalization, compatible with the \( N \to \infty \) limit, is naturally fixed requiring \( < \alpha, \alpha > = 1 \). Explicitly we have

\[
\begin{align*}
&g^{\zeta \zeta^*} = < \zeta, \zeta^* > = \frac{1}{2R^2}, \\
g^{\zeta \zeta} = = < \zeta, \zeta > = 0, \\
g^{\zeta \zeta^*} = = < \zeta^*, \zeta^* > = 0
\end{align*}
\] (6.5)

We then also have \( < \beta, \beta > = 1 \). As in (2.44), this pairing is compatible with the \(^*\)-conjugation \( < \rho, \sigma >^* = < \sigma^*, \rho^* > \). Having the metric \( g \), we can find the 1-form \( \gamma \) that is orthogonal to \( \alpha \) and has unit length, \( < \alpha, \gamma > = 0, \ < \gamma, \gamma > = 1 \). This form is given by

\[
\gamma = -R(\zeta + \zeta^*) = \frac{q + q^{-1}}{q - q^{-1}} \alpha - \frac{2i}{q - q^{-1}} \beta
\] (6.7)

Notice that \( \gamma^* = \gamma \). In the \( N \to \infty \) limit \( \alpha, \beta, \gamma \) are well defined (\( \alpha \) and \( \beta \) become \( dx \), and \( \gamma \) can be checked from (6.7) or from \( < \gamma, \gamma > = 1 \) to remain finite). Then from (6.7) and (6.3) we see that both \( \zeta \) and \( \zeta^* \) are well defined (besides being \( \neq 0 \)) in the \( N \to \infty \) limit, while \( \theta^u \) and \( \theta^{u-1} \) diverge as \( 1/(q^{-1} - 1) \). One also has \( tu/(q^{-1} - 1) \to iR\partial/\partial x \) for \( N \to \infty \).

The pairing \( < \ , \ > \) can be generalized to act on the space of 2-forms. The space of left-invariant 2-forms is one dimensional because the wedge product for abelian groups is the commutative one (e.g. \( \zeta \wedge \zeta^* = \zeta \otimes \zeta^* - \zeta^* \otimes \zeta \)). Any 2-form can be written as \( f \, vol \) where \( vol \) is the volume form associated to the metric:

\[
vol = \alpha \wedge \gamma = -2iR^2 \zeta \wedge \zeta^*.
\]

Up to a normalization the (\(^*\)-bimodule) pairing \( < \ , \ > \) is uniquely defined. As in the commutative (continuous) case we choose here the normalization such that

\[
< vol, vol > = 1.
\]

Notice that \( vol \) is central, therefore the space of 2-forms has no noncommutativity. Notice also that \( vol^* = (-1)^{\vert \alpha \vert \vert h \vert} \gamma^* \wedge \alpha^* = vol \).

**\(^*\)-Hodge operator**

The \(^*\)-Hodge operator is defined as in (2.52). We have

\[
* \zeta = i\zeta, \quad * \zeta^* = -i \zeta^*.
\] (6.8)

**Integral**

The Haar measure \( h \) on \( \mathbb{Z}_N \) is given by

\[
h(f) = \sum_{k=0}^{N-1} f(u^k)
\] (6.9)
Integration on 2-forms is then simply given by
\[ \int f \text{vol} = \frac{2\pi R}{N} h(f) \] (6.10)
the normalization is chosen such that in the \( N \to \infty \) limit we obtain the usual integral over the circle \( S_1 \) of radius \( R \). It is not difficult to check that the integral (6.10) is cyclic, that it satisfies Stokes’ theorem and that it is real. For any \( p \)- and \( (2-p) \)-form \( \omega^{(p)} \), \( \omega^{(2-p)} \), we have
\[ \int \omega^{(p)} \wedge \omega^{(2-p)} = (-1)^{p(2-p)} \int \omega^{(2-p)} \wedge \omega^{(p)} \quad \text{(cyclicity)} \] (6.11)
\[ \int d\omega^{(1)} = 0 \] (6.12)
\[ \left( \int \omega^{(2)} \right)^{*} = \int \omega^{(2)*} \] (6.13)

6.2 \( \ast \)-bicovariant calculus on \( \mathcal{M}^{D} \times \mathcal{Z}_N \)

It is straightforward to generalize the results of the previous subsection to the \( \mathcal{M}^{D} \times \mathcal{Z}_N \) case. A basis of 1-forms is given by \( \{ dx^A \} = \{ dx^\mu, \alpha, \gamma \} = \{ dx^\mu \} \). The metric \( g^{AB} \) is block diagonal
\[ g^{\mu\nu} \neq 0, \quad g^{\mu\alpha} = g^\mu^\gamma = g^\alpha^\gamma = 0, \quad g^{\alpha\alpha} = g^{\gamma\gamma} = 1. \]
The volume element is
\[ \text{vol} = \sqrt{\det g_{AB}} \ dx^0 \wedge dx^1 \wedge \ldots dx^{D-1} \wedge \alpha \wedge \gamma, \]
where \( g^{\alpha C} g_{CB} = \delta^\alpha_B \). In the \( \{ dx^A \} \) basis the epsilon tensor, defined by
\[ dx^A_1 \wedge dx^A_2 \wedge \ldots dx^{A_{D+2}} = \varepsilon^{A_1 A_2 \ldots A_{D+2}} \text{vol} \]
is the classical one. The integral
\[ \int_{\mathcal{M}^{D} \times \mathcal{Z}_N} f(x^\mu, y) \text{vol} \equiv \int_{\mathcal{M}^{D}} \sqrt{|\det g_{\mu\nu}|} \ d^D x \left( \int_{\mathcal{Z}_N} f(x^\mu, y) \alpha \wedge \gamma \right) \]
is again cyclic, real and satisfies Stokes’ theorem.

The \( \ast \)-Hodge operator is still given by (2.52), and satisfies the left- and right-linearity property (2.53) and the compatibility with the \( \ast \)-conjugation (2.54). The normalization of the pairing can be chosen such that the Hodge operator squares to \( \pm id \).

6.3 Yang-Mills action

The Yang-Mills action on \( \mathcal{M}^{D} \times \mathcal{Z}_N \) with gauge group \( \mathcal{G} = U(M) \) is given by
\[ A_{YM} = -\int_{\mathcal{M}^{D} \times \mathcal{Z}_N} Tr(F \wedge *F) \] (6.14)
where the trace is over \( M \times M \) matrices. Positivity and gauge invariance of this action are shown in Section 4. In order to write the Yang-Mills action in terms of \( F \) components we first obtain the commutation relations
\[ \alpha y^n = \frac{q^n + q^{-n}}{2} y^n \alpha - i \frac{q^n - q^{-n}}{2} y^n \gamma = \bar{y}_n \alpha - i \bar{y}_n \gamma \] (6.15)
\[ \gamma y^n = \frac{q^n + q^{-n}}{2} y^n \gamma + i \frac{q^n - q^{-n}}{2} y^n \alpha = \bar{y}_n \gamma + i \bar{y}_n \alpha \] (6.16)
we then expand $F_{AB}(y)$ as

$$F_{AB}(y) = F_{AB}(k)\hat{y}_k \frac{1}{\sqrt{2\pi R}}$$

(6.17)

and define

$$F_{AB}(\hat{y}) \equiv F_{AB}(k)\hat{y}_k \frac{1}{\sqrt{2\pi R}}, \quad F_{AB}(\check{y}) \equiv F_{AB}(k)\check{y}_k \frac{1}{\sqrt{2\pi R}}.$$  

(6.18)

Finally the action reads

$$A_{YM} = \int_{M^D \times Z_N} Tr\left[ F_{\mu\nu} F_{\mu\nu} + F_{\alpha\gamma} F_{\alpha\gamma} + 2F_{\alpha\mu}(y) F^{\alpha\mu}(\hat{y}) + 2F_{\gamma\mu}(y) F^{\gamma\mu}(\hat{y}) - 2iF_{\alpha\mu}(y) F^{\gamma\mu}(\check{y}) + 2iF_{\gamma\mu}(y) F^{\alpha\mu}(\check{y}) \right] vol$$

$$= \sum_k \int_{M^D} \sqrt{|det g_{\mu\nu}|} \left[ Tr\left[ F_{\mu\nu}(k) F_{\mu\nu}(-k) + F_{\alpha\gamma}(k) F_{\alpha\gamma}(-k) + 2q^k F_{\alpha\mu}(k) F^{\alpha\mu}(-k) + 2q^k F_{\gamma\mu}(k) F^{\gamma\mu}(-k) - 2i(q^k - q^{-k}) F_{\alpha\mu}(-k) F^{\gamma\mu}(k) \right] \right]$$

(6.19)

where in the second equality we have integrated on $Z_N$ using (6.10) and $\sum_{j=0}^{N-1} q^{kj} = N \delta_{k,0}$.

In the $N \to \infty$ limit $F_{AB}(k)$ becomes the $k-th$ Fourier mode, and the action (6.19) becomes the dimensional reduction (in the direction $\gamma$) of the usual Yang-Mills action on $M^D \times S_1 \times S_1$, where the first $S_1$ is in the $\alpha$ direction and the second $S_1$ is in the $\gamma$ direction. It is therefore Yang-Mills theory on $M^D \times S_1$ coupled to the adjoint scalar $\phi = A_{\gamma}$.

The interesting feature of this action arises for finite $N$: in this case we have a nontrivial scalar potential term. Using the link fields $U$ it is given in (5.17). These variables are not convenient in the present section because they are ill defined in the $N \to \infty$ limit.

## 7 Born - Infeld Theory on finite group spaces

We recall the continuum $D$-dimensional Born - Infeld action for non-linear electrodynamics [9] in flat space

$$A_{BI} = \int_{M^D} d^Dx \sqrt{det(\delta_{\mu\nu} + F_{\mu\nu})}$$  

(7.1)

When $D = 4$ this action takes the form (after explicitly computing the determinant):

$$A_{BI} = \int_{M^4} d^4x \sqrt{1 + \frac{1}{2} \bar{F} \cdot F + \left( \frac{1}{4} \bar{F} \cdot \bar{F} \right)^2},$$  

(7.2)

where $\bar{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$, the dot in products means complete index contraction (for ex. $F \cdot F \equiv F_{\mu\nu} F^{\mu\nu}$), and we consider the euclidean theory.

The action (7.1) can be generalized to the nonabelian case: then $F_{\mu\nu}$ is $G$ Lie algebra valued, and the determinant in (7.1) is not a number any more but belongs to the universal enveloping algebra of $G$. We can define its “absolute value” $| det |$ (still belonging to the
enveloping algebra) as the positive square root in \( \sqrt{\det} \). Any square root of \( |\det| \) is hermitian, so that an overall trace produces a real action:

\[
A_{BI} = \int_{M^D} d^Dx \, Tr \sqrt{|\det(\delta_{\mu\nu} + F_{\mu\nu})|}
\]  

(7.3)

This is a gauge invariant action for any choice of the square root. The trace can be symmetrized so to fix ordering ambiguities in products of the Lie valued \( F_{\mu\nu} \) elements (see for ex. [23] and included references).

We address now the problem of formulating BI theory (for a nonabelian gauge group \( G \)) on finite group spaces.

We first consider the special \( D = 4 \) case (7.2) on finite groups: since the indices of \( F \) run here on 4 values, we take finite groups with four independent tangent vectors (for ex. \( Z_N \times Z_N \) with the *-bicovariant calculus of Section 6.1). Later we present a generalization of the action (7.1) on any finite group \( G \).

As we have done for the Yang-Mills action, we replace the term \( F \cdot F \) by \( <F,F> = -F_{rt}F^r_t\). Using the bimodule pairing (2.42), also the quartic term \((F \cdot \tilde{F})^2\) can be replaced by the gauge covariant expression

\[
<F^2,F^2> = <F \wedge F, F \wedge F> 
\]

(7.4)

transforming under gauge variations as \( <F^2,F^2> \rightarrow T <F^2,F^2> \rightarrow T^{-1} \). Then the whole action:

\[
A_{BI} = \sum_G Tr \sqrt{1 + \frac{1}{2} <F,F> + \frac{1}{16} <F^2,F^2>}
\]

(7.5)

is gauge invariant, and reproduces in the continuum case the Born-Infeld action for non-abelian gauge fields.

Let us explore in more detail the structure of \( <F^2,F^2> \), and its expression in terms of the link field \( U_h \). The differential 4-form \( F^2 \)

\[
F^2 = U^4 = U_{h_1} [R_{h_1} U_{h_2} [R_{h_1} U_{h_3} [R_{h_1} U_{h_4} A_{h_1,h_2,h_3,h_4} \theta^{h_1} \otimes \theta^{h_2} \otimes \theta^{h_3} \otimes \theta^{h_4} =
\]

(7.6)

transforms under gauge variations as:

\[
F^2 \rightarrow T F^2 T^{-1} , \quad (F^2)_{h_1,h_2,h_3,h_4} \rightarrow T (F^2)_{h_1,h_2,h_3,h_4} [R_{h_1} h_2 h_3 h_4 T^{-1}].
\]

(7.7)

Consider then the quartic term (sum on the indices \( h_i \) understood)

\[
(F^2)_{h_1,...,h_4} (F^2)_{h_1,...,h_4}^\dagger
\]

(7.8)

This term is gauge covariant, i.e. it transforms as \( T \cdots T^{-1} \), and in fact coincides with \( <F^2,F^2> \) [use (2.46)].

**Born-Infeld action on arbitrary finite groups**
The analogue of $\delta_{\mu\nu} + F_{\mu\nu}$ becomes simply $E_{g,h} \equiv -\delta_{g,h^{-1}} + F_{g,h}$, cf. (2.43), and transforms under gauge variations in the same way as $F_{g,h}$:

$$E_{g,h}' = -\delta_{g,h^{-1}} + TF_{g,h}R_{gh}T^{-1} = T(-\delta_{g,h^{-1}} + F_{g,h})R_{gh}T^{-1} = TE_{g,h}R_{gh}T^{-1} \quad (7.9)$$

We need now a gauge covariant definition of determinant for a matrix transforming as in (7.9), and that possibly reduces to the usual determinant in some limit that recovers the continuum case. This limit exists for $N \to \infty$ in the case $G = Z_N \times Z_N \times \ldots \times Z_N$, and indeed the definition we propose in the following has this property.

**Lemma:**

$$\epsilon^{g_1 \cdots g_p} e^{h_1 \cdots h_p} R_{g_1 \cdots g_p} R_{h_1 \cdots h_p} = \epsilon^{g_1 \cdots g_p} e^{h_1 \cdots h_p} id \quad \text{(no sums on } g, h) \quad (7.10)$$

**Proof:** for any function $f$:

$$\epsilon^{g_1 \cdots g_p} e^{h_1 \cdots h_p} f \mathcal{N} = \langle \theta^{g_1} \wedge \ldots \wedge \theta^{g_p}, \theta^{h_1} \wedge \ldots \wedge \theta^{h_p} > f =$$

$$= \langle \theta^{g_1} \wedge \ldots \wedge \theta^{g_p}, (R_{h_1 \cdots h_p} f) \theta^{h_1} \wedge \ldots \wedge \theta^{h_p} >=$$

$$= \langle R_{g_1 \cdots g_p} R_{h_1 \cdots h_p} f, \theta^{g_1} \wedge \ldots \wedge \theta^{g_p}, \theta^{h_1} \wedge \ldots \wedge \theta^{h_p} >=$$

$$\epsilon^{g_1 \cdots g_p} e^{h_1 \cdots h_p} (R_{g_1 \cdots g_p} R_{h_1 \cdots h_p} f) \mathcal{N} \quad (7.11)$$

proving the Lemma ($\mathcal{N}$ is defined as $<\text{vol, vol}>$, cf. (2.50)).

**Proposition:** the determinant of the matrix $E_{g,h}$ defined as:

$$\det_G E_{g,h} =$$

$$\epsilon^{g_1 \cdots g_p} E_{g_1, h_1} (R_{g_1 h_1 g_2 h_2}) (R_{g_1 h_1 g_2 h_2 g_3 h_3}) \ldots (R_{g_1 h_1 g_2 h_2 \ldots g_{p-1} h_{p-1}} E_{g_p, h_p}) \epsilon^{h_1' \cdots h_p'} \equiv$$

$$\epsilon^{g_1 \cdots g_p} E_{g_1, h_1 g_2, h_2, \ldots g_p, h_p} e^{h_1' \cdots h_p'} \quad (7.12)$$

where $h_n' = (g_{n+1}g_{n+2} \ldots g_p)^{-1} h_n (g_{n+1}g_{n+2} \ldots g_p)$, transforms covariantly:

$$\det_G E_{g,h}' = T \det_G E_{g,h} T^{-1} \quad (7.13)$$

**Proof:** the quantity defined in the last line of (7.12) transforms as:

$$E_{g_1, h_1 g_2, h_2, \ldots g_p, h_p}' = TE_{g_1, h_1 g_2, h_2, \ldots g_p, h_p} (R_{g_1 h_1 g_2 h_2 \ldots g_p h_p} T^{-1}) =$$

$$T E_{g_1, h_1 g_2, h_2, \ldots g_p, h_p} (R_{g_1 g_2 \ldots g_p h_1' h_2' \ldots h_p'}) T^{-1} \quad (7.14)$$

where we used the definition of the $h'$ indices. Then recalling $R_{g_1 g_2 \ldots g_p h_1' h_2' \ldots h_p'} = R_{g_1 \ldots g_p} R_{h_1' \ldots h_p'}$ the gauge variation of $\det_G E_{g,h}$ reads

$$\det_G E_{g,h}' = T \epsilon^{g_1 \cdots g_p} E_{g_1, h_1 g_2, h_2, \ldots g_p, h_p} (R_{g_1 \ldots g_p} R_{h_1' \ldots h_p'} T^{-1}) \epsilon^{h_1' \cdots h_p'} = T \det_G E_{g,h} T^{-1} \quad (7.15)$$

after using the Lemma (7.10) in the last equality.

For abelian $G$ and constant matrices this determinant coincides with the usual determinant, multiplied by $p!$ since we have defined it by means of two $\epsilon$ tensors. We could
equally well define \( \det_G \) by fixing the order of \( g_1...g_p \) to be the one that defines the volume form, i.e. \( \text{vol} = \theta^{g_1} \wedge ... \wedge \theta^{g_p} \) so that \( \epsilon^{g_1...g_p} = 1 \) disappears from the formula and the indices \( g_1...g_p \) are not summed any more.

A covariant hermitian “absolute value” of the determinant can be defined by choosing the positive square root in:

\[
|\det_G(E_{g,h})| \equiv \sqrt{\det_G E_{g,h} (\det_G E_{g,h})^\dagger} \tag{7.16}
\]

and can be used to construct a real \( G \)-gauge invariant Born-Infeld action on the finite group \( G \):

\[
A_{BI}^G = \int_G \text{Tr} \sqrt{|\det_G (-\delta_{g,h} - 1 + F_{g,h})| \text{vol}(G)} = \sum_G \text{Tr} \sqrt{|\det_G (-\delta_{g,h} - 1 + F_{g,h})|} \tag{7.17}
\]

for any choice of square root; a positive definite action is obtained by choosing the positive square root. Note that for finite groups the order of the \( F \) factors in the definition of \( \det_G \) is fixed by (7.12).

As an example consider \( G = Z_N \) with the 2-dimensional differential calculus involving the two left-invariant 1-forms \( \theta^u \) and \( \theta^{u^{-1}} \). The matrix \( E_{g,h} \) becomes:

\[
\begin{pmatrix}
0 & -1 + F_{+-} \\
-1 - F_{+-} & 0
\end{pmatrix}, \tag{7.18}
\]

where \( F_{+-} = F_{u,u^{-1}} \). The determinant (7.12) in this case coincides with the usual determinant

\[
\det_G E_{g,h} = \det E_{g,h} = -1 + F_{+-}^2. \tag{7.19}
\]

A real gauge invariant BI action is then obtained as:

\[
A_{BI} = \sum_{Z_N} \text{Tr} \sqrt{|-1 + F_{+-}^2|} \tag{7.20}
\]

the positive square root yielding a positive definite action.

8 Born - Infeld Theory on \( M^D \times G \)

It is convenient to extract the \( E = g + F \) matrix from the following expansion of \( F \):

\[
F \equiv \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu + F_{\mu g} \, dx^\mu \wedge \theta^g + F_{g,h} \, \theta^g \otimes \theta^h =
\]

\[
= dx^\mu \otimes F_{\mu\nu} \, dx^\nu + dx^\mu \otimes F_{\mu h} \, \theta^h - \theta^g \otimes \mathcal{R}_{g^{-1}} F_{\nu g} \, dx^\nu + \theta^g \otimes \mathcal{R}_{g^{-1}} F_{g,h} \, \theta^h =
\]

\[
= (dx^\mu, \theta^g) \otimes \begin{pmatrix} F_{\mu\nu} & F_{\mu h} \\ -\mathcal{R}_{g^{-1}} F_{\nu g} & \mathcal{R}_{g^{-1}} F_{g,h} \end{pmatrix} \begin{pmatrix} dx^\nu \\ \theta^h \end{pmatrix} \tag{8.1}
\]
where we used
\[ dx^\mu \wedge \theta^g = dx^\mu \otimes \theta^g - \theta^g \otimes dx^\mu \]

Then the \( E_{A,B} \) (\( A = \mu, g \), \( B = \nu, h \)) matrix is given by:

\[
E_{A,B} = \begin{pmatrix}
g_{\mu\nu} + F_{\mu\nu} & F_{\mu h} \\
- \mathcal{R}_{g^{-1}} F_{\nu g} & -\delta_{g,h^{-1}} + \mathcal{R}_{g^{-1}} F_{g,h}
\end{pmatrix}
\tag{8.2}
\]

Next we have to define the determinant of a matrix with mixed indices \( \mu, g \). Again we find a definition that ensures gauge invariance, and a correct continuum limit when it exists. Consider the algebraic identity

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & BD^{-1} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
A - BD^{-1}C & 0 \\
0 & D
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
D^{-1} C & 1
\end{pmatrix}
\tag{8.3}
\]

With the usual definition of determinant this implies:

\[
\det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \det (A - BD^{-1}C)(\det D)
\tag{8.4}
\]

For our purposes we define the modified determinant \( \text{Det} \):

\[
\left[ \text{Det} \begin{pmatrix}
A_{\mu\nu} & B_{\mu h} \\
C_{\nu} & D_{g,h}
\end{pmatrix} \right]^2 = [\det_G (\mathcal{R}_g D_{g,h})]^\dagger (\det M_{\mu\nu})^\dagger \det M_{\mu\nu} \det_G (\mathcal{R}_g D_{g,h})
\tag{8.5}
\]

with \( M_{\mu\nu} \equiv A_{\mu\nu} - B_{\mu h}(D^{-1})^{h,g} C_{g,\nu} \) and

\[
\det M_{\mu\nu} = \epsilon^{\mu_1...\mu_D \nu_1...\nu_D} M_{\mu_1 \nu_1} ... M_{\mu_D \nu_D}
\tag{8.6}
\]

For the matrix \( E_{A,B} \) the modified determinant reads

\[
(\text{Det } E_{A,B})^2 = [\det_G (-\delta_{g,h^{-1}} + F_{g,h})]^\dagger (\det M_{\mu\nu})^\dagger \det M_{\mu\nu} \det_G (-\delta_{g,h^{-1}} + F_{g,h})
\tag{8.7}
\]

where

\[
M_{\mu\nu} \equiv g_{\mu\nu} + F_{\mu\nu} + F_{\mu h}(H^{-1})^{h,g} \mathcal{R}_{g^{-1}} F_{\nu g}
\tag{8.8}
\]

\[
H_{g,h} \equiv -\delta_{g,h^{-1}} + \mathcal{R}_{g^{-1}} F_{g,h}
\tag{8.9}
\]

Next we prove that \((\text{Det } E_{A,B})^2\) transforms covariantly. The matrix \( H \) and its inverse \( H^{-1} \) transform as

\[
H'_{g,h} = (\mathcal{R}_{g^{-1}} T) H_{g,h} (\mathcal{R}_h T^{-1}) \Rightarrow (H^{-1})^{h,g} = (\mathcal{R}_h T) (H^{-1})^{h,g} (\mathcal{R}_{g^{-1}} T^{-1})
\tag{8.10}
\]

so that \( M_{\mu\nu} \) transforms as \( M'_{\mu\nu} = T M_{\mu\nu} T^{-1} \). As a consequence \((\text{Det } E_{A,B})^2\) transforms covariantly

\[
(\text{Det } E'_{A,B})^2 = T (\text{Det } E_{A,B})^2 T^{-1}
\tag{8.11}
\]
A real gauge invariant BI action on \( M^D \times G \) can be constructed by taking twice the square root of \((\text{Det } E_{AB})^2:\)

\[
A_{BI} = \int d^Dx \sum_G Tr \sqrt{|\text{Det } E_{AB}|} \tag{8.12}
\]

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A Hopf algebraic formulas for the differential calculus on Fun(G)

The \( G \) group structure induces a Hopf algebra structure on \( \text{Fun}(G) \), with coproduct \( \Delta \), coinverse \( \kappa \) and counit \( \varepsilon \) defined by group multiplication, inverse and unit as:

\[
\Delta(f)(g,g') = f(gg'), \quad \Delta : \text{Fun}(G) \to \text{Fun}(G) \otimes \text{Fun}(G) \tag{A.1}
\]

\[
\kappa(f)(g) = f(g^{-1}), \quad \kappa : \text{Fun}(G) \to \text{Fun}(G) \tag{A.2}
\]

\[
\varepsilon(f) = f(e), \quad \varepsilon : \text{Fun}(G) \to \mathbb{C} \tag{A.3}
\]

In the first line we have used \( \text{Fun}(G \times G) \approx \text{Fun}(G) \otimes \text{Fun}(G) \) [indeed a basis for functions on \( G \times G \) is given by \( x^{g_1} \otimes x^{g_2}, g_1, g_2 \in G \)]. On the basis functions \( x^g \) the costructures take the form:

\[
\Delta(x^g) = \sum_{h \in G} x^h \otimes x^{h^{-1}g} \equiv x^g_{(1)} \otimes x^g_{(2)}, \quad \kappa(x^g) = x^{g^{-1}}, \quad \varepsilon(x^g) = \delta^g_e \tag{A.4}
\]

Left and right coactions on \( \Gamma \)

\[
\Delta_L(ab) = a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}, \quad \Delta_R(ab) = a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} \tag{A.5}
\]

where \( \Delta(a) \equiv a_{(1)} \otimes a_{(2)} \).

Left and right invariant forms

\[
\theta^g = \kappa(x^g_{(1)}) dx^g_{(2)} = \sum_h x^h d x^{hg} = \sum_h x^{h^{-1}g} d x^h \tag{A.6}
\]

\[
\zeta^g = d(x^g_{(1)}) \kappa(x^g_{(2)}) = \sum_h (dx^{gh}) x^h = - \sum_h x^{gh} d x^h \tag{A.7}
\]
Bimodule relations and $f^g_{g'}$ functionals

Definition of the $f^g_{g'}$ functionals:

$$\theta^g a = (f^g_{g'} * a) \theta^{g'} \equiv (id \otimes f^g_{g'}) \Delta(a) \theta^{g'}$$ \hspace{1cm} (A.8)

Applying the rule $\theta^g a = (\mathcal{R}^g a) \theta^g$ (2.15) to the basis functions $x^g$ yields

$$f^g_{g'} = \delta^g_{g'} g, \ g \neq e, \ g' \neq e$$ \hspace{1cm} (A.9)

where we denote as $g$ the functional on $Fun(G)$ dual to the basis function $x^g$, i.e. $g(x^g') = \delta^g_{g'}$

Left and right coaction on $\Gamma$

Left and right coaction of $G$ on $\Gamma$ are given by the mappings $\Delta_L : \Gamma \to Fun(G) \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes Fun(G)$ that encode the information about all left or right translations:

$$\Delta_L(ab) = \Delta(a)\Delta_L(b), \ \Delta_L(db) = (id \otimes d)\Delta(b) \ \forall a, b \in Fun(G), \ \rho \in \Gamma$$ \hspace{1cm} (A.10)

$$\Delta_R(ab) = \Delta(a)\Delta_R(b), \ \Delta_R(db) = (d \otimes id)\Delta(b) \ \forall a, b \in Fun(G), \ \rho \in \Gamma$$ \hspace{1cm} (A.11)

For example their action on the basic terms $x^{g_1}dx^{g_2} \in \Gamma$ is:

$$\Delta_L(x^{g_1}dx^{g_2}) = \Delta(x^{g_1})(id \otimes d)\Delta(x^{g_2}) = \sum_{h \in G} x^h \otimes x^{h^{-1}g_1}dx^{h^{-1}g_2}$$ \hspace{1cm} (A.12)

$$\Delta_R(x^{g_1}dx^{g_2}) = \Delta(x^{g_1})(d \otimes id)\Delta(x^{g_2}) = \sum_{h \in G} x^{g_1h}dx^{g_2h} \otimes x^{h^{-1}}$$ \hspace{1cm} (A.13)

Computing $\Delta_L$ and $\Delta_R$ on the basic differentials yields:

$$\Delta_L(dx^{g_1}) \equiv (id \otimes d)(\Delta x^{g_1}) = \sum_{h \in G} x^h \otimes dx^{h^{-1}g_1}$$ \hspace{1cm} (A.14)

$$\Delta_R(dx^{g_1}) \equiv (d \otimes id)(\Delta x^{g_1}) = \sum_{h \in G} dx^h \otimes x^{h^{-1}g_1}$$ \hspace{1cm} (A.15)

Adjoint representation

The adjoint representation matrix $M^g_{g'}$ is defined by:

$$\Delta_R(\theta^g) = \theta^{g'} \otimes M^g_{g'}$$ \hspace{1cm} (A.16)

On the other hand computing the right coaction on $\theta^g$ according to the formulas of the previous paragraph yields:

$$\Delta_R(\theta^g) = \sum_h \theta^{gh^{-1}} \otimes x^h = \sum_k \theta^k \otimes \sum_h \delta^{gh^{-1}}_k x^h$$ \hspace{1cm} (A.17)
so that

\[ M_{g', g} = \sum_h \delta_{g'}^{hgh^{-1}} x^h \quad \text{(A.18)} \]

Braiding matrix \( \Lambda \)

\[ \Lambda^{g'g}_{k'k} = f^{g}_{k}(M_{k', g'}) = \delta^{g'}_{k'k} \delta^{gg'}_{g}^{-1}, \quad g \neq e, g' \neq e \quad \text{(A.19)} \]

Hopf algebra structure of \( \Omega \)

\[ \Delta(\theta^g) \equiv \Delta_L(\theta^g) + \Delta_R(\theta^g) = 1 \otimes \theta^g + \sum_h \theta^{hgh^{-1}} \otimes x^h, \]

\[ \epsilon(\theta^g) = 0, \quad \kappa(\theta^g) = -\sum_h \theta^{h^{-1}gh} x^h. \quad \text{(A.20)} \]

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