FUNCTION SPACES AND EXTENSION RESULTS FOR NONLOCAL DIRICHLET PROBLEMS

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Abstract. We study function spaces and extension results in relation with Dirichlet problems involving integrodifferential operators. For such problems, data are prescribed on the complement of a given domain $\Omega \subset \mathbb{R}^d$. We introduce a function space that serves as a trace space for nonlocal Dirichlet problems and study related extension results.

1. Introduction

In this work, we study function spaces related to Dirichlet problems for a class of integrodifferential operators, which satisfy the maximum principle. We introduce a new function space, which can be understood as a nonlocal trace space. Let us illustrate our task with a very simple problem. Let $\Omega = B_1 \subset \mathbb{R}^d$ be the unit ball and assume $0 < s < 1$. We ask ourselves the question, for which functions $g: \mathbb{R}^d \setminus \Omega \to \mathbb{R}$, there is a function $u: \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon} \frac{u(x+h)-u(x)}{|h|^{d+2s}} \, dh = 0 \quad \text{for } x \in \Omega,$$

(1.1)

$$u(x) = g(x) \quad \text{for } x \in \mathbb{R}^d \setminus \Omega.$$  

(1.2)

Note that (1.1) is equivalent to $(-\Delta)^s u = 0$ in $\Omega$. In order to discuss the possible choices of data $g$, we need to specify the function space of possible solutions $u$. Moreover, we have to explain in which sense the above equation is to be understood. Since the validity of (1.1) for some $x \in \Omega$ involves values of $u$ on $\mathbb{R}^d \setminus \Omega$, where $u = g$ is imposed, there is a direct link between the function space for solutions $u$ and the function space for the data $g$.

The set-up of boundary value problems is well understood for differential operators, i.e., in the limit case $s = 1$. However, by considering our results for $s \to 1^-$, we will obtain a new extension result for classical Sobolev spaces, cf. Corollary 4 and Corollary 9.

Let us explain how to define a variational solution $u$ satisfying (1.1)-(1.2), cf. [3, 5]. Define two vector spaces by

$$V^s(\Omega|\mathbb{R}^d) = \{ v \in L^2_{\text{loc}}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(y) - v(x))^2}{|x-y|^{d+2s}} \, dx \, dy < \infty \}.$$ 

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Let us collect a few basic observations on these spaces.

1. \( V^s(\mathbb{R}^d) \) and \( V^s(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) equal the Sobolev-Slobodeckij space \( \dot{H}^s(\mathbb{R}^d) \) and \( H^s(\mathbb{R}^d) \), respectively.

2. \( H^s_{\Omega}(\mathbb{R}^d) \) is a Banach space together with the norm

\[
\|v\|_{H^s}^2 = \|v\|_{L^2}^2 + (1-s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(y) - v(x))^2}{|x-y|^{d+2s}} dy \, dx
\]

Let us define the notion of a variational solution.

**Definition 1** (cf. Definition 2.5 in [5]). Let \( \Omega \subset \mathbb{R}^d \) be open such that \( \Omega \) and \( \Omega^c \) both have positive measure. Let \( g \in V^s(\Omega|\mathbb{R}^d) \). Then \( u \in V^s(\Omega|\mathbb{R}^d) \) is called a variational solution to \((1.1)-(1.2)\), if \( u - g \in H^s_{\Omega}(\mathbb{R}^d) \) and for every \( \varphi \in H^s_{\Omega}(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(y) - u(x)) \varphi(y) - \varphi(x))}{|x-y|^{d+2s}} dy \, dx = 0.
\]

**Remark 2.** (i) The above definition implies that solutions \( u \) belong to \( L^2(\mathbb{R}^d, dm) \) with \( m(dx) = (1 + |x|)^{-d-2s} \, dx \). It would be possible to work under the assumption \( u \in L^1(\mathbb{R}^d, dm) \), but the presentation would be less transparent. (ii) In peridynamics, the definition of variational solutions to nonlocal boundary value problems looks similar, cf. [8]. However, it is rather different because of the usage of more restrictive function spaces. Regularity of \( u \) respectively \( g \) is required in regions, which are away from that region, where the nonlocal equation is considered. The above definition avoids such an assumption.

With the above definition at hand, we are now in the position to explain the main question addressed in this article. In order to apply **Definition 1** one needs to prescribe the data function \( g \) in the vector space \( V^s(\Omega|\mathbb{R}^d) \), i.e. in particular one needs to prescribe all values of \( g \) in \( \mathbb{R}^d \). This leads to the following question:

**Question:** For which Banach space of functions \( g : \Omega^c \to \mathbb{R} \)

(a) is there an extension operator \( g \mapsto \text{ext}(g) \in V^s(\Omega|\mathbb{R}^d) \), and

(b) is there a trace operator from \( V^s(\Omega|\mathbb{R}^d) \) into this space?

Extension and trace theorems are well known in the study of classical local Dirichlet problems. Thus, for the case of Sobolev spaces of integer order, these questions are classical and answers were given long time ago, cf. [9] for an early work and [1] for a general exposition. For a large class of domains \( \Omega \), functions in \( H^{1/2}(\partial \Omega) \) can be extended to elements of \( H^1(\Omega) \) and these themselves have a trace in \( H^{1/2}(\partial \Omega) \). A side result of our research on nonlocal quantities is that instead of \( H^1(\Omega) \) one could also consider the much larger space of all \( L^2(\Omega) \)-functions \( v \) with

\[
\int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{(|x-y| + \delta_x + \delta_y)^{d+2}} dy \, dx < \infty,
\]

where \( \delta_x = \text{dist}(z, \partial \Omega) \) for \( z \in \mathbb{R}^d \).
Trace and extension results have been established for various function spaces including Sobolev spaces with fractional order of differentiability. To our best knowledge, extensions from the complement of a domain to the whole space have not been dealt with so far. One reason for this might be that Dirichlet problems with prescribed data on the complement have not yet been studied intensively.

Let us formulate our main result, which answers the aforementioned question. We allow the domain \( \Omega \) to have a rather rough boundary, but we stress the fact that our results are new even for domains \( \Omega \) with a smooth boundary. See Section 3 for the definition of inner respectively exterior thickness of domains. Note that any bounded Lipschitz domain has these properties. The inner radius of an open set \( D \subset \mathbb{R}^d \) is defined as

\[
\text{inr}(D) = \frac{1}{2} \sup_{B \subset D} \text{diam}(B),
\]

where the supremum is taken over all balls \( B \subset D \).

**Theorem 3.** Assume \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^d \) be open, interior thick and exterior thick such that \( \partial \Omega \) has Lebesgue measure zero, and \( \text{inr}(\Omega) < \infty \) or \( \text{inr}(\Omega^c) = \infty \). Then the following is true:

(a) If \( f \in L^p_{\text{loc}}(\mathbb{R}^d), \ 1 \leq p < \infty \), satisfies

\[
\int_{\Omega} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dy \, dx < \infty,
\]

then

\[
\int_{\Omega^c} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx < \infty.
\]

(b) There exists a linear extension operator \( \text{ext} \), which maps \( L^p_{\text{loc}}(\Omega^c), \ 1 \leq p < \infty \), to measurable functions defined on \( \mathbb{R}^d \) such that

\[
(1 - s) \int_{\Omega} \int_{\mathbb{R}^d} \frac{|\text{ext}(f)(x) - \text{ext}(f)(y)|^p}{|x - y|^{d+sp}} \, dy \, dx \lesssim \int_{\Omega} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx,
\]

with constants that depend only on \( \inf s, p, d \) and \( \Omega \).

Theorem 3 follows directly from Theorem 5 and Theorem 8.

Considering the limit \( s \to 1^- \), Theorem 3 implies a new extension-type result for classical Sobolev spaces. We formulate this observation in the special case \( \Omega = B_1 \subset \mathbb{R}^d \) and refer to Corollary 9 for the general case and to Remark 10 for some related result.

**Corollary 4.** Given \( 1 < p < \infty \), there is a constant \( c = c(d, p) \geq 1 \) such that

\[
\int_{B_1} |\nabla \text{ext}(f)|^p \leq c \int \int \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dx \, dy
\]

for every \( f \in L^p(B_2 \setminus B_1) \) such that the right-hand side is finite.

Corollary 4 is a special case of Corollary 9.

The article is organized as follows. In Section 2 we present the setup of our work together with the main results, Theorem 5 and Theorem 8. Section 3 provides basic properties of the function spaces under consideration. In Section 4 we present the proof of Theorem 5. The proof of Theorem 8 is given in Section 5.
2. Setup and detailed results

Throughout the whole paper we assume that $\Omega \subset \mathbb{R}^d$ is an open set with the property that both, $\Omega$ and $\Omega^c = \mathbb{R}^d \setminus \Omega$, have positive Lebesgue measure. For our main result, we will assume some very mild additional assumption. We will use the symbol $g \lesssim h$ to denote that the inequality $g \leq ch$ holds with a positive constant $c$ that is independent of $g$ and $h$. We adopt the convention that $0^p = \infty$ for $a < 0$, in particular, $\frac{1}{0} = \infty$. We assume $0 < p < \infty$ and $0 < s \leq 1$.

In short, our main result answers the question from the previous section. It roughly says that the vector space of all functions $g \in L^2_{loc}(\Omega^c)$ with

\begin{equation}
\int_{\Omega^c} \int_{\Omega^c} \frac{|g(x) - g(y)|^2}{(|x - y| + \delta_x + \delta_y)^{d+2s}} \, dy \, dx < \infty
\end{equation}

has the desired properties, see Theorem 3. A special feature of our result is that the limit case $s = 1$ can be included. Thus we obtain a new extension result for $W^{1,2}(\Omega)$-functions, see below for details.

Let us now explain the set-up in detail. For $f \in L^p(\mathbb{R}^d)$, define

\begin{equation}
|f|^p_{W^{s,p}(\Omega^c)} := \int_{\Omega^c} \int_{\Omega^c} |f(x) - f(y)|^p \, dx \, dy,
\end{equation}

\begin{equation}
\|f\|^p_{W^{s,p}(\Omega^c)} := \|f\|^p_{L^p(\mathbb{R}^d)} + |f|^p_{W^{s,p}(\Omega^c)},
\end{equation}

and let $W^{s,p}(\Omega^c) = \{f \in L^p(\mathbb{R}^d) : \|f\|^p_{W^{s,p}(\Omega^c)} < \infty\}$. If $f \in L^p(\mathbb{R}^d)$, then $f \in W^{s,p}(\Omega^c)$, if $f$ satisfies some regularity condition across the boundary $\partial \Omega$, whereas the behavior of $f$ far from $\partial \Omega$ is not considered.

Example. Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, given by $f = 1_\Omega$. Then the function $f$ belongs to $W^{s,p}(\Omega^c)$ if and only if $s < \frac{1}{p}$.

Recall that the inner radius of an open set $D \subset \mathbb{R}^d$ is defined as $\text{inr}(D) = \frac{1}{2} \sup_{B \subset D} \text{diam}(B)$, where the supremum is taken over all balls $B \subset D$. For $x \in \mathbb{R}^d$, set $\delta_x = \text{dist}(x, \Omega)$. For $0 < \delta, \varepsilon \leq \infty$ set

$\Omega^\text{int}_\delta = \{x \in \Omega : \text{dist}(x, \Omega^c) \leq \delta\}, \quad \Omega^\text{ext}_\varepsilon = \{x \in \Omega^c : \text{dist}(x, \Omega) < \varepsilon\}.

$Note that $\Omega^\text{int}_\text{inr}(\Omega) = \Omega$. For a function $g$ let

$|g|^p_{A,B} := \int_A \int_B \frac{|g(x) - g(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx.$

The following result introduces a useful (semi)norm that is equivalent to $\|f\|^p_{W^{s,p}(\Omega^c)}$ respectively $|f|^p_{W^{s,p}(\Omega^c)}$. For the definition of interior thick domains we refer the reader to Subsection 3.2, here let us only mention that bounded Lipschitz domains are interior thick.

Theorem 5. Let $0 < p < \infty$ and $0 < s \leq 1$. Suppose that $\Omega \subset \mathbb{R}^d$ is an open interior thick set. Then there exists a constant $c = c(p, \Omega)$ not depending on $s$, such that

\begin{equation}
c^{-1} |f|^p_{W^{s,p}(\Omega^c)} \leq \int_{\Omega \cup \Omega^\text{ext}_\text{inr}(\Omega)} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx \leq \frac{c}{s} |f|^p_{W^{s,p}(\Omega^c)}
\end{equation}

for every $f \in L^p(\mathbb{R}^d)$. The following norms

$\left( \| \cdot \|^p_{L^p(\Omega,(1+|x|)^{-d-sp} \, dx)} + | \cdot |^p_{W^{s,p}(\Omega^c)} \right)^{1/p},$
Then both expressions, and the answer to the question posed earlier immediately follows, cf. (2.5) and (2.7). Theorem 3: 

\[
\left( \| \cdot \|_{L^p(\mathbb{R}^d,(1+|x|)^{-d-s} \, dx)}^p + \| \cdot \|_{W^{s,p}(\Omega; \mathbb{R}^d)}^p \right)^{1/p},
\]

and

\[
\left( \| f \|_{L^p(\mathbb{R}^d,(1+|x|)^{-d-s} \, dx)}^p + \int_{\mathbb{R}^d} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(|x-y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx \right)^{1/p}
\]

are comparable with constants depending only on \( p, \Omega \) and the lower bound for \( s \).

**Remark 6.** Note that, for the case \( s \to 1^- \), the different \( s \)-dependence on the two sides in (2.4) is not important.

**Example 7.** Define \( f : \mathbb{R}^d \to \mathbb{R} \) by \( f(x) = \sqrt{|x| - 1} \) for \( 1 < |x| < 2 \) and \( f(x) = 0 \) elsewhere. Assume \( \Omega = B_1 \subset \mathbb{R}^d \) as in Corollary 4. Then both expressions,

\[
\int_{\Omega \cup \Omega^c} \int_{\Omega^c} \frac{|f(x) - f(y)|^2}{(|x-y| + \delta_x + \delta_y)^{d+2s}} \, dy \, dx \quad \text{and} \quad |f|_{W^{s,2}(\Omega; \mathbb{R}^d)}^2
\]

diverge for \( s \to 1^- \). As observed in [13, Sec. 2.2.4], the expression \((1 - s)|f|_{W^{s,2}(\Omega; \mathbb{R}^d)}^2\) remains bounded.

The following theorem contains our main result.

**Theorem 8.** Let \( \Omega \subset \mathbb{R}^d \) be an open set which is exterior thick and such that \( \partial \Omega \) has Lebesgue measure zero, and \( \text{int}(\Omega) = 0 \) or \( \text{int}(\Omega^c) = \infty \). Then there exists a linear operator \( \text{ext} \) which maps \( L^1_{\text{loc}}(\Omega^c) \) to the space of measurable functions on \( \mathbb{R}^d \) with the following properties.

(a) For all \( f \in L^1_{\text{loc}}(\Omega^c) \), \( \text{ext}(f)|_{\Omega^c} = f \) and \( \text{ext}(f))|_{\Omega} \in C^\infty(\Omega) \). Furthermore, if \( z_0 \in \partial \Omega \) and the limit \( g = \lim_{x \to x_0} f(x) \) exists, then also the limit \( \lim_{x \to x_0} \text{ext}(f)(x) \) exists and equals \( g \).

(b) Let \( 1 \leq p < \infty \). There exists a constant \( c = c(\Omega, p) \) such that the following inequalities hold for all \( f \in L^1_{\text{loc}}(\Omega^c) \) and \( 0 < \delta \leq \epsilon \leq \infty \)

\[
|\text{ext}(f)|_{\Omega_\delta,\Omega^c}^{s,p} \leq \frac{c}{s} |f|_{\Omega_\epsilon,\Omega^c}^{s,p}, \quad 0 < s \leq 1,
\]

\[
|\text{ext}(f)|_{\Omega_\delta,\Omega^c}^{s,p} \leq \frac{c}{s(1-s)} |f|_{\Omega_\epsilon,\Omega^c}^{s,p}, \quad 0 < s < 1.
\]

In particular,

\[
|\text{ext}(f)|_{\Omega_\delta,\Omega^c}^{s,p} \leq \frac{c}{s} |f|_{\Omega_\epsilon,\Omega^c}^{s,p}, \quad 0 < s \leq 1,
\]

\[
|\text{ext}(f)|_{\mathbb{R}^d,\mathbb{R}^d}^{s,p} \leq \frac{c}{s(1-s)} |f|_{\Omega_\epsilon,\Omega^c}^{s,p}, \quad 0 < s < 1.
\]

(c) Let \( 1 \leq p < \infty \), \( \beta \in \mathbb{R} \) or \( p = \infty \), \( \beta = 0 \). There exists a constant \( c = c(\Omega, \beta, p) \) such that the following inequality holds for all \( f \in L^1_{\text{loc}}(\Omega^c) \)

\[
\| \text{ext}(f) \|_{L^p(\Omega_\beta,\Omega_\beta^c)} \leq c \| f \|_{L^p(\Omega_\epsilon,\Omega_\epsilon^c)}.
\]

From Theorem 5 and Theorem 8, the answer to the question posed earlier immediately follows, cf. Theorem 3.
Corollary 9. Let $\Omega$ be a bounded Lipschitz-domain and $1 < p < \infty$. Then there exists a constant $c = c(\Omega,p)$ such that

$$|\text{ext}(f)|_{W^{1,p}(\Omega)} \leq c\|f\|^{1-p}_{\text{inr}(\Omega)^{1-p}} \|f\|^{p}_{\text{ext}(\Omega)},$$

where we take $|\text{ext}(f)|_{W^{1,p}(\Omega)} = \|\nabla \text{ext}(f)\|_{L^p(\Omega)}$, if $\text{ext}(f) \in W^{1,p}(\Omega)$, and $|\text{ext}(f)|_{W^{1,p}(\Omega)} = \infty$ otherwise.

Proof. We put $\delta = \varepsilon = \text{inr}(\Omega)$ in (2.6), multiply its both sides by $(1-s)$ and take $s \to 1^-$. If $|f|^{s,p}_{\text{inr}(\Omega)^{s,p}} < \infty$ for some $s$, then $f \in L^p(\Omega)$ and inequality (2.9) follows from [2, Theorem 2]. In the other case inequality (2.9) is trivial. \hfill \Box

Remark 10. In the case $p = 2$, a result related to Corollary 9 has recently been established in [4]. For the trace map $T$, the authors prove an estimate of the form

$$\|Tu\|_{H^{1/2}(\partial\Omega)} \leq C\|u\|_{\mathcal{S}(\Omega)},$$

where

$$\|u\|^2_{\mathcal{S}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \int_{\Omega} \int_{\Omega \cap B(x,\delta(x))} \frac{(u(y) - u(x))^2}{\delta(x)^{d+2}} \mathbf{1}_{B_1(|y-x|)} dy \, dx,$$

and $\delta$ denotes the distance function with respect to $\partial\Omega$. The authors of [4] are interested in models from peridynamics. It is interesting that our approach to nonlocal function spaces, in the limit case $s \to 1^-$, leads to a similar nonlocal trace theorem as their approach. Note that [4] does not contain extension results like Theorem 8.

3. Preliminary results

In this section, we prove basic properties of the function spaces $W^{s,p}(\Omega|\Omega^c)$ and collect several results on inner thick respectively exterior thick domains.

3.1. Basic properties of $W^{s,p}(\Omega|\Omega^c)$. Recall the definitions from (2.2) and (2.3).

Proposition 11. Let $\Omega \subset \mathbb{R}^d$ be an open set, $0 < s < 1$ and $p \geq 1$. Then the space $W^{s,p}(\Omega|\Omega^c)$ equipped with the norm $\|\cdot\|_{W^{s,p}(\Omega|\Omega^c)}$ is a Banach space.

Proof. The proof is straightforward. Let $(f_n)$ be a Cauchy sequence in $(W^{s,p}(\Omega|\Omega^c), \|\cdot\|_{W^{s,p}(\Omega|\Omega^c)})$. Then $(f_n)$ is a Cauchy sequence in $L^p(\mathbb{R}^d)$, hence there exists a function $f \in L^p(\mathbb{R}^d)$ such that $f_n \to f$ in $L^p(\mathbb{R}^d)$. Let $f_{k_n}$ be a subsequence convergent a.e. to $f$. By the Fatou lemma

$$\|f_{k_n} - f\|^p_{W^{s,p}(\Omega|\Omega^c)} = \int_{\Omega} \int_{\Omega} \liminf_{l \to \infty} \frac{|(f_{k_n} - f_l)(x) - (f_{k_n} - f_l)(y)|^p}{|x-y|^{d+sp}} \, dy \, dx \leq \liminf_{l \to \infty} \int_{\Omega} \int_{\Omega} \frac{|(f_{k_n} - f_l)(x) - (f_{k_n} - f_l)(y)|^p}{|x-y|^{d+sp}} \, dy \, dx \to 0 \text{ as } n \to \infty.$$

From the above calculation and triangle inequality we deduce that $f \in W^{s,p}(\Omega|\Omega^c)$. Since $(f_n)$ is a Cauchy sequence in $(W^{s,p}(\Omega|\Omega^c), \|\cdot\|_{W^{s,p}(\Omega|\Omega^c)})$ and its subsequence converges to $f$, the whole sequence converges to $f$. \hfill \Box

Remark 12. For $p \in (0,1)$ the space $W^{s,p}(\Omega|\Omega^c)$ equipped with a metric $\rho(f,g) := \|f-g\|^p_{W^{s,p}(\Omega|\Omega^c)}$ is complete. The proof is basically the same as above.
Proposition 13. If a measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) satisfies \( |f|_{W^{s,p}(\Omega^c)} < \infty \), then \( f \in L^p(\mathbb{R}^d, (1 + |x|)^{-d-sp} \, dx) \). Furthermore, the norms
\[
\left( \| \cdot \|_{L^p(\Omega,(1+|x|)^{-d-sp} \, dx)} + \| \cdot \|_{W^{s,p}(\Omega^c)} \right)^{1/p}
\]
are comparable.

Proof. Let \( R > 1 \) be large enough so that \( B(0,R) \) intersects both \( \Omega \) and \( \text{int} \Omega^c \). For a given \( f \) as in the proposition, let \( n \in \mathbb{N} \) be such that for \( E_n = \{ x \in \mathbb{R}^d : |f(x)| \leq n \} \) the intersections \( F_n = E_n \cap \Omega \cap B(0,R) \) and \( G_n = E_n \cap \Omega^c \cap B(0,R) \) are of positive Lebesgue measure. Note that
\[
|w - z| \leq R + |z| \leq R(1 + |z|), \quad \text{for } w \in B(0,R) \text{ and } z \in \mathbb{R}^d.
\]
Therefore
\[
2|f|_{W^{s,p}(\Omega^c)} \geq \int_{E_n} \int_{\Omega \setminus E_n} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dy \, dx + \int_{\Omega \setminus E_n} \int_{G_n} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dy \, dx
\]
\[
\geq \frac{2^{-p}}{R^{d+sp}} \left( \int_{E_n} \int_{\Omega \setminus E_n} \frac{|f(y)|^p}{1 + |y|^{d+sp}} \, dx \, dy + \int_{\Omega \setminus E_n} \int_{G_n} \frac{|f(x)|^p}{1 + |x|^{d+sp}} \, dx \, dy \right)
\]
\[
\geq \frac{2^{-p}(|F_n| \cap |G_n|)}{R^{d+sp}} \int_{E_n} \frac{|f(x)|^p}{1 + |x|^{d+sp}} \, dx.
\]
Choose \( n \in \mathbb{N} \) sufficiently large so that \( |F_n| \cap |G_n| \) is positive. Since obviously \( \int_{E_n} |f(x)|^p(1 + |x|)^{-d-sp} \, dx < \infty \), we conclude that \( f \in L^p(\mathbb{R}^d, (1 + |x|)^{-d-sp} \, dx) \).

Comparability of the first two norms follows from the following inequalities
\[
\|f\|_{L^p(\Omega,(1+|x|)^{-d-sp} \, dx)} + |f|_{W^{s,p}(\Omega^c)} \geq \int_{\Omega \setminus B(0,R)} \left( |f(x)|^p + \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(1 + |y|)^{d+sp}} \, dy \right) \, dx
\]
\[
\geq \int_{\Omega \setminus B(0,R)} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(1 + |y|)^{d+sp}} \, dy \, dx
\]
\[
\geq \int_{\Omega^c} \frac{|f(y)|^p}{(1 + |y|)^{d+sp}} \, dy
\]
with constants depending only on \( \Omega, R, d, s, p \). \( \square \)

3.2. Whitney decomposition, thickness and plumpness. We recall several geometric notions needed in the sequel. They allow us to present our main results for rather general domains \( \Omega \subset \mathbb{R}^d \). Note that, however, Theorem 5 and Theorem 8 are new even for domains with a smooth boundary.

For a nonempty open set \( D \subset \mathbb{R}^d \), \( D \neq \mathbb{R}^d \), we fix a Whitney decomposition \( \mathcal{W}(D) \) [10, VI.1] and write \( \mathcal{W}_m(D) \) for the family of Whitney cubes with side length \( 2^{-m} \), \( m \in \mathbb{Z} \). If \( Q \in \mathcal{W}(D) \), then
\[
\text{diam}(Q) \leq \text{dist}(Q, \partial D) \leq 4 \text{diam}(Q).
\]
For any cube \( Q \), its side length is denoted by \( \ell(Q) \) and its center by \( x_Q \). By \( Q^* \) we denote a cube with the same center as \( Q \), but side length \( \ell(Q^*) = (1 + \varepsilon)\ell(Q) \), where \( 0 < \varepsilon < 1/4 \) is fixed once for
all. Such cubes have the property that

\[ 1_D \leq \sum_{Q \in W(D)} 1_{Q^c} \leq M 1_D \]

with some constant \( M \) depending only on \( d \).

The next two definitions are slightly modified versions of [11, Definition 3.1]. Our definitions and [11, Definition 3.1] coincide if \( D \) or \( D^c \) has finite inner radius. In the case when \( \text{inr}(D) = \text{inr}(D^c) = \infty \), if the domain \( D \) is \( I \)-thick in the sense of Definition 14, then it is also \( I \)-thick in the sense of [11, Definition 3.1].

**Definition 14.** An open set \( D \subset \mathbb{R}^d \) is called \( I \)-thick (interior thick), if for every \( M > 0 \) there exists a constant \( C \) such that for every cube \( Q \in W(\mathbb{R}^d \setminus \overline{D}) \) with \( \text{diam} Q < M \text{inr}(D) \) there exists a reflected cube \( \tilde{Q} \in W(D) \) satisfying

\[ C^{-1} \text{diam}(Q) \leq \text{diam}(\tilde{Q}) \leq C \text{diam}(Q) \quad \text{and} \quad \text{dist}(\tilde{Q}, Q) \leq C \text{dist}(Q, \partial D). \tag{3.2} \]

**Definition 15.** An open set \( D \subset \mathbb{R}^d \) is called \( E \)-thick (exterior thick), if for every \( M > 0 \) there exists a constant \( C \) such that for every cube \( Q \in W(D) \) with \( \text{diam} Q < M \text{inr}(D^c) \) there exists a reflected cube \( \tilde{Q} \in W(\mathbb{R}^d \setminus \overline{D}) \) satisfying

\[ C^{-1} \text{diam}(Q) \leq \text{diam}(\tilde{Q}) \leq C \text{diam}(Q) \quad \text{and} \quad \text{dist}(\tilde{Q}, Q) \leq C \text{dist}(Q, \partial D). \tag{3.3} \]

**Remark 16.** The definitions of \( I \)- and \( E \)-thickness do not depend on the choice of the families of Whitney cubes \( W(D) \) and \( W(\mathbb{R}^d \setminus \overline{D}) \).

**Remark 17.** Let \( \lambda > 0 \) be fixed. In Definition 14 we may additionally assume that the reflected cubes satisfy

\[ \text{diam} \tilde{Q} \leq \lambda \text{diam} Q. \tag{3.4} \]

Indeed, if the opposite inequality holds, then

\[ \text{dist}(\tilde{Q}, \partial D) \geq \text{diam}(\tilde{Q}) > \lambda \text{diam} Q, \]

so in the ball \( B(x_{\tilde{Q}}, 5 \text{diam} \tilde{Q}) \) there exists a point \( z \in D \) with \( \text{dist}(z, \partial D) = \lambda \text{diam} Q \). Take \( Q' \) to be a cube from \( W(D) \) containing \( z \). Then

\[ \frac{\lambda}{5} \text{diam} Q \leq \frac{1}{5} \text{dist}(z, \partial D) \leq \text{diam} Q' \leq \text{dist}(z, \partial D) = \lambda \text{diam} Q, \]

so \( \text{diam} Q \) and \( \text{diam} Q' \) are comparable. Moreover, since \( z \in B(x_{\tilde{Q}}, 5 \text{diam} \tilde{Q}) \cap Q' \), we obtain

\[ \text{dist}(Q', \tilde{Q}) \leq \text{dist}(z, x_{\tilde{Q}}) \leq 5 \text{diam} \tilde{Q}, \]

therefore,

\[ \text{dist}(Q', Q) \leq \text{diam} Q' + \text{dist}(Q', \tilde{Q}) + \text{diam} \tilde{Q} + \text{dist}(\tilde{Q}, Q) \lesssim \text{dist}(Q, \partial D) \]

with a constant depending only on \( \lambda \) and \( C \). Consequently, (3.2) holds also for \( Q' \) in place of \( \tilde{Q} \) (perhaps with an enlarged \( C \)). Hence by redefining reflected cubes both (3.2) and (3.4) hold. A similar remark applies to Definition 15.
Remark 18. In Definition 14 we may additionally assume that the reflected cubes satisfy

\[(3.5)\quad \tilde{Q} \subset \{ x \in D : \text{dist}(x, \partial D) < \text{inr}(D^c) \}.\]

Indeed, by taking \( \lambda \leq \frac{1}{5} \) in Remark 17 and perhaps redefining reflected cubes, we obtain \( \text{dist}(w, \partial D) \leq 5 \text{diam} \tilde{Q} \leq \text{diam} Q < \text{inr}(D^c) \) for \( w \in \tilde{Q} \), as desired.

A similar remark applies to Definition 15.

Remark 19. Let \( D \) be exterior thick. The family of all reflected cubes in the sense of the definition above, i.e., \( \mathcal{F} := \{ \tilde{Q} : Q \in \mathcal{W}(D), \text{diam} \tilde{Q} < N \text{inr}(D^c) \} \) has the bounded overlap property, i.e., there exists a constant \( N \) such that

\[ \sum_{Q \in \mathcal{W}(D)} 1_{\tilde{Q}} \leq N 1_{\mathbb{R}^d \setminus \mathcal{F}}. \]

This estimate holds true because the size of \( \tilde{Q} \) and its distance to \( Q \) are comparable to the size of \( Q \). An analogous property holds for interior thick sets \( D \).

From [11, Proposition 3.6] it follows that if \( D \) is a bounded (\( \epsilon, \delta \))-domain [11, Definition 3.1(i)], then \( D \) is I-thick and \( \partial D \) has Lebesgue measure zero. Bounded Lipschitz domains are both I-thick and E-thick [11, Proposition 3.8].

We will show that the assumption that \( D \) is an (\( \epsilon, \delta \))-domain may be replaced by a weaker one. To this end we need the following definition.

Definition 20. [12, 7] A set \( A \subset \mathbb{R}^d \) is \( \kappa \)-plump with \( \kappa \in (0, 1) \) (or simply plump) if, for each \( 0 < r < \text{diam}(A) \) and each \( x \in A \), there is \( z \in B(x, r) \) such that \( B(z, \kappa r) \subset A \).

Lemma 21. If \( D \subset \mathbb{R}^d \) is plump, then it is also I-thick and \( \partial D \) has d-dimensional Lebesgue measure zero.

Proof. Let us note that if \( D \) is plump, then its boundary \( \partial D \) is porous, i.e., there exists a constant \( \alpha \) with the following property: for every \( x \in \mathbb{R}^d \) and \( 0 < r \leq 1 \), there exists \( y \in B(x, r) \) such that \( B(y, \alpha r) \subset B(x, r) \setminus \partial D \). Therefore \( \partial D \) has Lebesgue measure zero, see e.g. [6].

Let \( M > 1 \). For each cube \( Q \in \mathcal{W} \left( \text{int} D^c \right) \) such that \( \text{diam} Q < M \text{inr}(D) \) we will associate a reflected cube \( \tilde{Q} \in \mathcal{W}(D) \) in the following way. Let \( y_Q \in \partial D \) be a fixed point satisfying \( |x_Q - y_Q| = \text{dist}(x_Q, \partial D) \). We consider a ball \( B(y_Q, \frac{\text{diam} Q}{M}) \). By plumpness condition, there exist a ball \( B \subset B(y_Q, \frac{\text{diam} Q}{M}) \cap D \) of radius \( \kappa \frac{\text{diam} Q}{M} \). Let \( z \) be its center; as \( \tilde{Q} \) we fix any of the Whitney cubes from \( \mathcal{W}(D) \) containing \( z \).

Let \( z \) be a point as above. Then

\[ \kappa \frac{\text{diam} Q}{M} \leq \text{dist}(z, \partial D) \leq \frac{\text{diam} Q}{M}, \]

and hence by properties of Whitney cubes

\[ \frac{\kappa}{5M} \text{diam} Q \leq \text{diam} \tilde{Q} \leq \frac{\text{diam} Q}{M}. \]

Furthermore, for \( x \in Q \) and \( w \in \tilde{Q} \)

\[ |x - w| \leq |x - x_Q| + |x_Q - y_Q| + |y_Q - z| + |z - w| \leq (1 + 5 + 1 + 1) \text{diam} Q \]

\[ \leq 8 \text{dist}(Q, \partial D). \]
To summarize, the five numbers $\text{diam}(Q)$, $\text{diam}(\tilde{Q})$, $\text{dist}(Q, \partial D)$, $\text{dist}(\tilde{Q}, \partial D)$, $\text{dist}(Q, \tilde{Q})$ are comparable with constants depending only on $\kappa$ and $M$. \hfill \blacksquare

[11, Remark 3.7] provides an example of an interior thick set $\Omega$ such that $|\partial \Omega| > 0$. It follows from Lemma 21 that such $\Omega$ is not plump. This example is however not completely satisfactory in our case, since in our results we assume that $|\partial \Omega| = 0$. Therefore we provide another example.

**Example.** Consider annuli $A_n = \{ x \in \mathbb{R}^2 : 2^{-n-1} \leq |x| < 2^{-n} \}$ and let $a_n = 2^{-n-1}/n$, where $n = 1, 2, \ldots$. Let $O_n \subset A_n$ be a maximal set such that balls centered at points from $O_n$ with radii $a_n$ are pairwise disjoint and contained in $A_n$. Clearly $O_n \neq \emptyset$. Set

$$\Omega = \bigcup_{n=1}^{\infty} \bigcup_{x \in O_n} B(x, \frac{a_n}{2}).$$

It is easy to observe that $\Omega$ is both interior and exterior thick. However, the largest ball that is contained in $B(0, 2^{-n})$, has a radius smaller than $3a_n/2$. Since $(3a_n/2)/(2^{-n}) = 3/(4n) \to 0$, the set $\Omega$ is not plump. Moreover, $|\partial \Omega| = 0$.

### 4. Proof of Theorem 5

Let $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Recall the definition $\delta_z = \text{dist}(z, \partial \Omega)$ for $z \in \mathbb{R}^d$. The first inequality in (2.4) follows from the fact that $|x - y| + \delta_x + \delta_y \leq 3|x - y|$ for $x \in \Omega$ and $y \in \Omega^c$. This implies

$$3^{-d-p} \int_{\Omega} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dy \, dx \leq \int_{\Omega} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx.$$

This estimate implies the desired inequality.

The remainder of this section is devoted to the proof of the second inequality. We observe that

$$\int_{\{x \in \mathbb{R}^d : \delta_x < \text{inr}(\Omega)\}} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx \leq |f|_{W^{s,p}(\Omega^c)}^p + \int_{\{x \in \Omega^c: \delta_x < \text{inr}(\Omega)\}} \int_{\Omega^c} \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx =: |f|_{W^{s,p}(\Omega^c)}^p + I.$$

We note that if $x \in \text{int } \Omega^c$ satisfies $\delta_x < \text{inr}(\Omega)$, then a Whitney cube $Q \in \mathcal{W}(\text{int } \Omega^c)$ containing $x$ satisfies $\text{diam } Q \leq \text{dist}(Q, \partial \Omega) < \text{inr}(\Omega)$. Moreover, since $\partial \Omega$ has Lebesgue measure zero, we obtain

$$(4.1) \quad I \leq \sum_{Q_1 \in \mathcal{W}^b} \sum_{Q_2 \in \mathcal{W}(\text{int } \Omega^c)} \int_{Q_1} \int_{Q_2} \frac{|f(x) - f(y)|^p}{(|x - y| + \delta_x + \delta_y)^{d+sp}} \, dy \, dx,$$

where

$$\mathcal{W}^b = \{ Q_1 \in \mathcal{W}(\text{int } \Omega^c) : \text{diam } Q_1 < \text{inr}(\Omega) \}.$$

We take $M = 1$ in the Definition 14 so that $\tilde{Q}$ exists for all cubes $Q \in \mathcal{W}^b$, and let $C$ be the corresponding constant. Let $Q_1 \in \mathcal{W}^b$ and $Q_2 \in \mathcal{W}(\text{int } \Omega^c)$. For $y \in Q_2$ and $w \in \tilde{Q}_1$

$$|y - w| \leq \text{dist}(y, Q_1) + \text{diam } Q_1 + \text{dist}(Q_1, \tilde{Q}_1) + \text{diam } \tilde{Q}_1$$

$$\leq \text{dist}(y, Q_1) + (5C + 1) \text{diam } Q_1.$$
Recall that for any cube $Q$, its side length is denoted by $\ell(Q)$ and its center by $x_Q$. For $x \in Q_1$ and $y \in Q_2$ we denote

$$w = w(x,y) = x_{Q_1} + \left(\frac{x - x_{Q_1}}{2\ell(Q_1)} + \frac{y - x_{Q_2}}{2\ell(Q_2)}\right)\ell(Q_1)$$

and observe that $w \in \bar{Q}_1$.

We come back to estimating the double integral in (4.1)

$$\int_{Q_1} \int_{Q_2} \frac{|f(x) - f(y)|^p}{(|x-y| + \delta_x + \delta_y)^{d+sp}} dy \, dx$$

$$\leq (2^{p-1} \vee 1) \int_{Q_1} \int_{Q_2} \frac{|f(x) - f(w(x,y))|^p}{(\text{dist}(Q_1, Q_2) + \text{dist}(Q_1, \partial\Omega))^{d+sp}} dy \, dx$$

$$+ (2^{p-1} \vee 1) \int_{Q_1} \int_{Q_2} \frac{|f(w(x,y)) - f(y)|^p}{(\text{dist}(y, Q_1) + \text{dist}(Q_1, \partial\Omega))^{d+sp}} dy \, dx$$

$$= : (2^{p-1} \vee 1)(I_1(Q_1, Q_2) + I_2(Q_1, Q_2)).$$

To estimate $I_1(Q_1, Q_2)$, we change the variable $y$ to $w = w(x,y)$ in the integral and obtain

$$I_1(Q_1, Q_2) \leq \frac{2^d \ell(Q_2)^d}{\ell(Q_1)^d} \int_{Q_1} \int_{Q_2} \frac{|f(x) - f(w(x,y))|^p}{(\text{dist}(Q_1, Q_2) + \text{dist}(Q_1, \partial\Omega))^{d+sp}} dw \, dx$$

$$\lesssim \int_{Q_1} \int_{Q_2} \frac{|f(x) - f(w(x,y))|^p}{|x-w|^{d+sp}} dw \, dx \cdot \int_{Q_2} \frac{1 + \frac{\text{dist}(Q_1, Q_2)}{\text{dist}(Q_1, \partial\Omega)}}{\ell(Q_1)^d} -d-sp \, dy$$

In the last passage we have used (3.7) with $D := \Omega$, $Q := Q_1$ and the inequality $s \leq 1$ (although any upper bound for $s$ would suffice). We obtain

$$\sum_{Q_1 \in W^h} \sum_{Q_2 \in W(\text{int} \Omega^c)} I_1(Q_1, Q_2)$$

$$\lesssim \sum_{Q_1 \in W^h} \int_{Q_1} \int_{Q_1} \frac{|f(x) - f(w(x,y))|^p}{|x-w|^{d+sp}} dw \, dx \cdot \sum_{Q_2} \int_{Q_2} \frac{1 + \frac{\text{dist}(Q_1, Q_2)}{\text{dist}(Q_1, \partial\Omega)}}{\ell(Q_1)^d} -d-sp \, dy.$$

By properties of Whitney cubes,

$$\sum_{Q_2} \int_{Q_2} \frac{1 + \frac{\text{dist}(Q_1, Q_2)}{\text{dist}(Q_1, \partial\Omega)}}{\ell(Q_1)^d} -d-sp \, dy$$

$$\leq \ell(\bar{Q}_1)^{-d} c(d) \int_{\mathbb{R}^d} \left(1 + \frac{|y - x_{Q_1}|}{\text{dist}(Q_1, \partial\Omega)}\right)^{-d-sp} \, dy$$

$$= \left(\frac{\text{dist}(Q_1, \partial\Omega)}{\ell(Q_1)}\right)^d c(d) \int_{\mathbb{R}^d} (1 + |z|)^{-d-sp} \, dz \leq \frac{c(d, C)}{s},$$

where the constant $c(d, C)$ depends only on $d$ and $C$, but not on the cube $Q_1$. Thus by Remark 19

$$\sum_{Q_1 \in W^h} \sum_{Q_2 \in W(\text{int} \Omega^c)} I_1(Q_1, Q_2) \leq \frac{c(d, p, C)}{s} |f|_{W^{s,p}(\Omega^c)}^p.$$
We are left with estimating $I_2(Q_1, Q_2)$. We interchange the order of integration and change the variable $x$ to $w = w(x, y)$. By (4.2), this gives us

$$I_2(Q_1, Q_2) \leq \frac{2d \ell(Q_1)^d}{\ell(Q_1)^d} \int_{Q_2} \int_{Q_1} \frac{|f(w) - f(y)|^p}{\left(\text{dist}(y, Q_1) + \text{dist}(Q_1, \partial \Omega)\right)^{d+sp} dw dy}$$

$$\leq c(d, p, C) \int_{Q_2} \int_{Q_1} \frac{|f(w) - f(y)|^p}{|w - y|^{d+sp}} dw dy.$$ 

By Remark 19, we get an estimate of the form (4.3) for $I_2(Q_1, Q_2)$ instead of $I_1(Q_1, Q_2)$. The proof is complete.

Note that the constant in Theorem 5 depends on $\Omega$ only through $d$ and the constant $C$ from Definition 14 taken for $M = 1$.

5. Proof of Theorem 8

We may assume that $\Omega \neq \emptyset$. We fix $M = 1$ if $\text{inr}(\Omega^c) = \infty$ and $M = \frac{2\sqrt{d} \text{inr}(\Omega)}{\text{inr}(\Omega^c)\ell(\Omega)}$ if $\text{inr}(\Omega^c) < \infty$, and we fix $\lambda = 1/125$. We take reflected cubes and the constant $C$ as in Definition 14 and Remark 17 for these particular choices of $M$ and $\lambda$. By Remark 18, the reflected cubes satisfy (3.5) with $D = \Omega^c$.

5.1. Definition of the extension. Let $Q_0 = [0, 1]^d$. We fix a function $\psi_0 \in C_c^\infty(Q_0^c)$ such that $\psi_0 = 1$ on $Q_0$ and $0 \leq \psi_0 \leq 1$. We shift and rescale this function to other cubes, i.e., we let

$$\psi_Q(x) = \psi_0\left(\frac{x - x_Q}{\ell(Q)} + x_Q_0\right), \quad x \in \mathbb{R}^d.$$ 

Recall from Subsection 3.2 that for $Q \in \mathcal{W}(D)$ we have $\text{diam}(Q) \leq \text{dist}(Q, \partial D) \leq 4\text{diam}(Q)$. For any cube $Q$, its side length is denoted by $\ell(Q)$ and its center by $x_Q$. By $Q^*$ we denote a cube with the same center as $Q$, but side length $\ell(Q^*) = (1 + \varepsilon)\ell(Q)$, where $0 < \varepsilon < 1/4$ is fixed as above.

We consider the following family of functions

$$\phi_Q(x) = \frac{\psi_Q(x)}{\sum_{R \in \mathcal{W}(\Omega)} \psi_R(x)}, \quad x \in \mathbb{R}^d.$$ 

Thus $\phi_Q \geq 0$ and $\sum_{Q \in \mathcal{W}(\Omega)} \phi_Q = 1\Omega$. Let $f \in L^1_{\text{loc}}(\Omega^c)$. Let

$$a_Q = \frac{1}{|Q|} \int_Q f(x) dx, \quad Q \in \mathcal{W}(\Omega).$$ 

Note that the reflected cube $\tilde{Q}$ is well defined thanks to the choice of $M$. We extend a given function $f \in L^1_{\text{loc}}(\Omega^c)$ from $\Omega^c$ to $\mathbb{R}^d$ by defining $\text{ext}(f)$ as follows:

$$\text{ext}(f)(x) = \begin{cases} \sum_{Q \in \mathcal{W}(\Omega)} a_Q \phi_Q(x) & \text{if } x \in \Omega, \\ f(x) & \text{if } x \in \Omega^c. \end{cases}$$ 

Let $\mathcal{N}_\Omega(Q) = \{R \in \mathcal{W}(\Omega) : R \cap Q^* \neq \emptyset\}$ be the collection of Whitney cubes intersecting $Q$. Observe that for $x \in Q_1 \in \mathcal{W}(\Omega)$ and any $t \in \mathbb{R}$

$$\text{(5.1)} \quad \text{ext}(f)(x) = \sum_{Q \in \mathcal{N}_\Omega(Q_1)} a_Q \phi_Q(x) = t + \sum_{Q \in \mathcal{N}_\Omega(Q_1)} (a_Q - t) \phi_Q(x).$$
5.2. A remark on reflected cubes. Let
\[ W^{<\delta}(\Omega) = \{ Q : W(\Omega) : \text{dist}(Q, \partial\Omega) < \delta \}. \]

Let us note that if \( Q_1 \in W^{<\delta}(\Omega), Q_2 \in \mathcal{N}_\delta(Q_1) \) and \( Q_3 \in \mathcal{N}_\delta(Q_2) \), then
\[ \text{diam } Q_3 \leq 5 \text{ diam } Q_2 \leq 25 \text{ diam } Q_1. \]

Therefore, if \( z \in \tilde{Q}_3 \), then
\[ \text{dist}(z, \partial\Omega) \leq 5 \text{ diam } \tilde{Q}_3 \leq 5\lambda \text{ diam } Q_3 \leq 125\lambda \text{ diam } Q_1 \leq 125\lambda \text{ dist}(Q_1, \partial\Omega) < \delta, \]
that is, \( \tilde{Q}_3 \subset \Omega^\text{ext}_\delta \). In particular, \( \tilde{Q}_1, \tilde{Q}_2 \subset \Omega^\text{ext}_\delta \), because we may take \( Q_3 = Q_2 \) or \( Q_3 = Q_1 \).

5.3. An estimate of \( |a_{Q_1} - a_Q|^p \). We claim that, for \( Q_1, Q \in W(\Omega) \),
\[ |a_{Q_1} - a_Q|^p \lesssim \left( \frac{(\text{dist}(Q_1, Q) + \ell(Q) + \ell(Q))^d + sp}{|Q_1||Q|} \right) |f|_{Q_1, \tilde{Q}}^{s,p}. \] (5.2)

Indeed,
\[ |a_{Q_1} - a_Q| = \frac{1}{|Q_1||Q|} \left| \int_{\tilde{Q}_1} f(y) dy - |\tilde{Q}_1| \int_{\tilde{Q}} f(x) dx \right| \]
\[ \leq \frac{1}{|Q_1||Q|} \int_{\tilde{Q}_1} \int_{\tilde{Q}} |f(y) - f(x)| dy dx. \]

From \( |Q_j| \lesssim |\tilde{Q}_j| \), and Jensen inequality we deduce
\[ |a_{Q_1} - a_Q|^p \lesssim \frac{1}{|Q_1||Q|} \int_{\tilde{Q}_1} \int_{\tilde{Q}} |f(y) - f(x)|^p dy dx, \]
and the claim follows.

5.4. An estimate of \( |\phi_Q|_{s,p}^{s,p} \) for \( s \leq 1 \) and arbitrary cubes \( Q, Q_1, Q_2 \). It is easy to check that \( |\nabla \phi_Q| \lesssim \ell(Q)^{-1} \). Therefore \( |\phi_Q(x) - \phi_Q(y)| \lesssim \ell(Q)^{-1} |x - y| \wedge 1 \) for all \( x, y \). As a result, we obtain
\[ |\phi_Q|_{Q_1, Q_2}^{s,p} \lesssim \int_{Q_1} \int_{Q_2} \frac{\ell(Q)^{-p} |x - y|^p \wedge 1}{|x - y|^{d + sp}} dy dx \]
\[ \lesssim |Q_1||Q_2| \left( \frac{\ell(Q_1)^{p - sp - d} \ell(Q)^{-p}}{1 - s} \wedge \frac{\ell(Q_2)^{p - sp - d} \ell(Q)^{-p}}{1 - s} \wedge \text{dist}(Q_1, Q_2)^{-d - sp} \right). \] (5.3)

We note that the above inequality for \( s = 1 \) is nontrivial only if \( \text{dist}(Q_1, Q_2) > 0 \).

5.5. Proof of part (b), formula (2.6). It holds
\[ |\text{ext}(f)|_{s,p}^{s,p}_{\Omega^{\text{int}}_\delta, \Omega^{\text{int}}_\delta} \leq \sum_{Q_1 \in W^{<\delta}(\Omega)} \sum_{Q_2 \in W^{<\delta}(\Omega)} |\text{ext}(f)|_{Q_1, Q_2}^{s,p}. \] (5.4)

For \( Q_1, Q_2 \in W^{<\delta}(\Omega) \), we use (5.1) twice with \( t = a_{Q_1} \) and obtain
\[ |\text{ext}(f)|_{Q_1, Q_2}^{s,p} \lesssim \sum_{Q \in \mathcal{N}_\delta(Q_1) \cup \mathcal{N}_\delta(Q_2)} |a_Q - a_{Q_1}|^p |\phi_Q|_{Q_1, Q_2}^{s,p}. \]
If additionally \( Q_2 \in \mathcal{N}_\Omega(Q_1) \), then for \( Q \in \mathcal{N}_\Omega(Q_1) \cup \mathcal{N}_\Omega(Q_2) \)

\[
|a_Q - a_{Q_1}|^p |\phi_Q|^{s,p}_{Q_1, Q_2} \lesssim \frac{(\text{dist}(Q_1, Q) + \ell(Q_1) + \ell(Q))^{d+sp}}{|Q_1||Q|} \cdot f_{Q_1, Q}^{s,p} \cdot \frac{|Q_2| \ell(Q_1)^{-sp-d}}{1 - s} \lesssim \frac{|f|^{s,p}_{Q_1, \tilde{Q}}}{1 - s}.
\]

Otherwise, if \( Q_2 \in \mathcal{W}^{<\delta}(\Omega) \setminus \mathcal{N}_\Omega(Q_1) \), then \( \text{dist}(Q_1, Q_2) > 0 \) and consequently \( \ell(Q_1), \ell(Q_2) \lesssim \text{dist}(Q_1, Q_2) \). Then for \( Q \in \mathcal{N}_\Omega(Q_1) \)

\[
|a_Q - a_{Q_1}|^p |\phi_Q|^{s,p}_{Q_1, Q_2} \lesssim \frac{(\ell(Q_1) + \ell(Q))^{d+sp}}{|Q_1||Q|} \cdot f_{Q_1, \tilde{Q}}^{s,p} \cdot \frac{|Q_2|}{\text{dist}(Q_1, Q_2)^{d+sp}} \lesssim \frac{\ell(Q_1)^{sp}|Q_2|}{\text{dist}(Q_1, Q_2)^{d+sp}} |f|^{s,p}_{Q_1, \tilde{Q}}.
\]

On the other hand, for \( Q \in \mathcal{N}_\Omega(Q_2) \)

\[
|a_Q - a_{Q_1}|^p |\phi_Q|^{s,p}_{Q_1, Q_2} \lesssim \frac{(\text{dist}(Q_1, Q) + \ell(Q_1) + \ell(Q))^{d+sp}}{|Q_1||Q|} \cdot f_{Q_1, \tilde{Q}}^{s,p} \cdot \frac{|Q_2|}{\text{dist}(Q_1, Q_2)^{d+sp}} \lesssim |f|^{s,p}_{Q_1, \tilde{Q}}.
\]

Let \( \mathcal{W}^{<\delta}(\Omega) = \{ Q \in \mathcal{W}(\Omega) : \mathcal{N}_\Omega(Q) \cap \mathcal{W}^{<\delta}(\Omega) \neq \emptyset \} \). Combining the above inequalities yields

\[
|\text{ext}(f)|^{s,p}_{\Omega^{\text{ext}}_\delta, \Omega^{\text{ext}}_\delta} \lesssim \sum_{Q_1 \in \mathcal{W}^{<\delta}(\Omega)} \left( \sum_{Q_2 \in \mathcal{N}_\Omega(Q_1)} \sum_{Q \in \mathcal{N}_\Omega(Q_1) \cup \mathcal{N}_\Omega(Q_2)} \frac{|f|^{s,p}_{Q_1, \tilde{Q}}}{1 - s} \right) + \sum_{Q_2 \in \mathcal{W}^{<\delta}(\Omega) \setminus \mathcal{N}_\Omega(Q_1)} \sum_{Q \in \mathcal{N}_\Omega(Q_1)} \frac{\ell(Q_1)^{sp}|Q_2| |f|^{s,p}_{Q_1, \tilde{Q}}}{\text{dist}(Q_1, Q_2)^{d+sp}}
\]

\[
\lesssim \sum_{Q_1 \in \mathcal{W}^{<\delta}(\Omega)} \left( \sum_{Q \in \mathcal{N}_\Omega(Q_1)} \frac{|f|^{s,p}_{Q_1, \tilde{Q}}}{1 - s} \right) + \sum_{Q \in \mathcal{N}_\Omega(Q_1)} |f|^{s,p}_{Q_1, \tilde{Q}} \int_{\Omega \cup \mathcal{N}_\Omega(Q_1)} \frac{\ell(Q_1)^{sp} dx}{\text{dist}(x, Q_1)^{d+sp}}
\]

\[
+ \sum_{Q_2 \in \mathcal{W}^{<\delta}(\Omega) \setminus \{Q_1\}} |f|^{s,p}_{Q_1, \tilde{Q}}.
\]

(5.5)

The fact that in the last expression there are sets \( \Omega^{\text{ext}}_\delta \) follows from a remark in Subsection 5.2.
5.6. Proof of part (b), formula (2.5). It holds

\[
|\text{ext}(f)|_{\Omega_{\text{int}}^\text{ext},\Omega_{\text{ext}}}^{s,p} \leq \sum_{Q_1 \in \mathcal{W}^{<\delta}(\Omega)} \sum_{Q_2 \in \mathcal{W}^{c<\varepsilon}(\Omega)} |\text{ext}(f)|_{Q_1, Q_2 \cap \Omega_{\text{ext}}}^{s,p}. \tag{5.6}
\]

Now let $Q_1 \in \mathcal{W}^{<\delta}(\Omega)$ and $Q_2 \in \mathcal{W}^{c<\varepsilon}(\Omega)$. We again use (5.1) with $t = aQ_1$,

\[
|\text{ext}(f)|_{Q_1, Q_2 \cap \Omega_{\text{ext}}}^{s,p} = \int_{Q_1} \int_{Q_2 \cap \Omega_{\text{ext}}} \left| \sum_{Q \in \mathcal{N}(Q_1)} (a_Q - a_{Q_1}) (\phi_Q(x) + a_{Q_1} - f(y))^p \right| \frac{1}{(|x-y| + \delta_x + \delta_y)^{d+sp}} dy \, dx \\
\leq \sum_{Q \in \mathcal{N}(Q_1)} \int_{Q_1} \int_{Q_2 \cap \Omega_{\text{ext}}} \left| (a_Q - a_{Q_1}) (\phi_Q(x) - \phi_Q(y))^p \right| \frac{1}{(|x-y| + \delta_x + \delta_y)^{d+sp}} dy \, dx \\
+ \text{dist}(Q_1, Q_2)^{-d-sp} \int_{Q_1} \int_{Q_2 \cap \Omega_{\text{ext}}} \left| a_{Q_1} - f(y) \right|^p dy \, dx =: A + \frac{B}{\text{dist}(Q_1, Q_2)^{d+sp}}.
\]

The first term above is estimated using (5.2) and (5.3),

\[
A \lesssim \sum_{Q \in \mathcal{N}(Q_1)} \frac{\text{dist}(Q_1, Q) + \ell(Q_1) + \ell(Q)}{|Q_1||Q|} \frac{|Q_1||Q_2|}{\text{dist}(Q_1, Q_2)^{d+sp}} |f|_{\mathcal{Q}^1, \mathcal{Q}}^{s,p}.
\]

For the second term,

\[
B = |Q_1| \int_{Q_2 \cap \Omega_{\text{ext}}} \left| \frac{1}{|Q_1|} \int_{Q_2 \cap \Omega_{\text{ext}}} (f(x) - f(y)) \, dx \right|^p dy \lesssim \int_{Q_2 \cap \Omega_{\text{ext}}} \int_{Q_1} \left| f(x) - f(y) \right|^p dx \, dy \\
\lesssim \text{dist}(Q_1, Q_2)^{d+sp} \int_{Q_2 \cap \Omega_{\text{ext}}} \int_{Q_1} \left| (x - y) \right|^p \frac{\ell(Q_1)^{sp} dx}{\text{dist}(x, Q_1)^{d+sp}} dy = \text{dist}(Q_1, Q_2)^{d+sp} |f|_{Q_2 \cap \Omega_{\text{ext}}, Q_1}^{s,p}.
\]

Inequalities obtained for $A$ and $B$ together with (5.6) yield

\[
|\text{ext}(f)|_{\Omega_{\text{int}}^\text{ext},\Omega_{\text{ext}}}^{s,p} \lesssim \sum_{Q_1 \in \mathcal{W}^{<\delta}(\Omega)} \left( \sum_{Q \in \mathcal{N}(Q_1)} |f|_{\mathcal{Q}^1, \mathcal{Q}}^{s,p} \int_{\Omega_{\text{ext}}} \frac{\ell(Q_1)^{sp} dx}{\text{dist}(x, Q_1)^{d+sp}} + \sum_{Q_2 \in \mathcal{W}^{c<\varepsilon}(\Omega)} |f|_{Q_2 \cap \Omega_{\text{ext}}, Q_1}^{s,p} \right) \\
\lesssim \frac{1}{s} |f|_{\mathcal{W}_2^{d}, \Omega_{\text{ext}}}^{s,p}, \tag{5.7}
\]

since by Subsection 5.2, the cubes $\tilde{Q}_1$ and $\tilde{Q}$ above are contained in $\Omega_{\text{ext}}^\delta \subset \Omega_{\text{ext}}^\varepsilon$.

5.7. Proof of part (b), formulas (2.7) and (2.8). Formula (2.7) follows directly by taking $\delta = \text{int} \Omega$ and $\varepsilon = \infty$ in (2.5) and enlarging the right hand side; alternatively, one may also apply (2.5) to $\delta = \varepsilon = \infty$.

To prove (2.8) we proceed as follows,

\[
|\text{ext}(f)|_{\mathbb{R}^d, \Omega}^{s,p} = |\text{ext}(f)|_{\Omega, \Omega}^{s,p} + 2 |\text{ext}(f)|_{\Omega, \Omega^c}^{s,p} + |f|_{\Omega^c, \Omega^c}^{s,p} \leq \frac{c}{s(1-s)} |f|_{\Omega^c, \Omega^c}^{s,p},
\]

by (2.5) and (2.6) applied to $\delta = \varepsilon = \infty$.

We note that the constants depend on $\Omega$ only through $d$, $M$ and $C$. 
5.8. **Proof of part (a).** The smoothness of $\text{ext}(f)$ on $\Omega$ follows directly from the definition. The proof of the second part is omitted as it is straightforward, it is based on the fact that if the cubes $Q \in \mathcal{W}(\Omega)$ approach $z \in \partial \Omega$, then so do the reflected cubes $\tilde{Q}$.

5.9. **Proof of part (c).** If $p = \infty$, then

$$
\| \text{ext}(f) \|_{L^\infty(\Omega)} = \sup_{Q \in \mathcal{W}(\Omega)} \sup_{x \in Q} \left| \sum_{Q \in N^\Omega_1(1)} a_Q \phi_Q(x) \right| 
\leq \sup_{Q \in \mathcal{W}(\Omega)} \# N^\Omega_1(1) \cdot \| f \|_{L^\infty(\Omega^{\text{ext}})} \lesssim \| f \|_{L^\infty(\Omega^c)}.
$$

Now let $p < \infty$. We first observe that

$$
|x| + 1 \asymp |x| + 1
$$

whenever $x \in Q \in \mathcal{W}(\Omega)$ and $\tilde{x} \in \tilde{Q} \in \mathcal{W}(\text{int} \Omega^c)$, with constants dependent only on the domain $\Omega$. Indeed, let $R = \text{dist}(0, \partial \Omega)$, then

$$
|\tilde{x}| \leq |x - x| + |x| \lesssim \text{diam } Q + |x|,
$$

and

$$
\text{diam } Q \leq \text{dist}(Q, \partial \Omega) \leq |x - 0| + \text{dist}(0, \partial \Omega) = |x| + R,
$$

so $|\tilde{x}| + 1 \lesssim |x| + 1$, as claimed. The proof of the opposite estimate is similar and omitted.

Let $\omega(x) = (1 + |x|)^p$. Since the numbers $\# N^\Omega_1(1)$ for $Q_1 \in \mathcal{W}(\Omega)$ are bounded from above by a constant depending only on the domain $\Omega$, we obtain

$$
\| f \|_{L^p(\Omega, \omega(x) dx)}^p = \sum_{Q \in \mathcal{W}(\Omega)} \int_{Q_1} \left| \sum_{Q \in N^\Omega_1(1)} a_Q \phi_Q(x) \right|^p \omega(x) dx 
\lesssim \sum_{Q \in \mathcal{W}(\Omega)} \sum_{Q \in N^\Omega_1(1)} |a_Q|^p \int_{Q_1} |\phi_Q(x)|^p \omega(x) dx
$$

By Jensen inequality, comparability of the sizes of cubes $Q$, $\tilde{Q}$ and $Q_1$ as in the sum above, and (5.8) we can estimate each summand as follows

$$
|a_Q|^p \int_{Q_1} |\phi_Q(x)|^p \omega(x) dx \leq \frac{1}{|Q|} \int_{\tilde{Q}} |f(x)|^p dx \cdot |Q_1| \cdot \sup_{Q} \omega \lesssim \int_{Q} |f(x)|^p \omega(x) dx.
$$

Using boundedness of $\# N^\Omega_1(1)$, and Remark 19 we obtain from the estimate (5.9) the following estimate:

$$
\| f \|_{L^p(\Omega, \omega(x) dx)}^p \lesssim \sum_{Q \in \mathcal{W}(\Omega)} \sum_{Q \in N^\Omega_1(1)} \int_{Q} |f(x)|^p \omega(x) dx \lesssim \sum_{Q \in \mathcal{W}(\Omega)} \int_{Q} |f(x)|^p \omega(x) dx
$$

This completes the proof of part (c) and thus the proof of Theorem 8.

□
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