The Minimal Modal Interpretation of Quantum Theory

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We introduce a realist, unextravagant interpretation of quantum theory that builds on the existing physical structure of the theory and allows experiments to have definite outcomes, but leaves the theory’s basic dynamical content essentially intact. Much as classical systems have specific states that evolve along definite trajectories through configuration spaces, the traditional formulation of quantum theory asserts that closed quantum systems have specific states that evolve unitarily along definite trajectories through Hilbert spaces, and our interpretation extends this intuitive picture of states and Hilbert-space trajectories to the case of open quantum systems as well. We provide independent justification for the partial-trace operation for density matrices, reformulate wave-function collapse in terms of an underlying interpolating dynamics, derive the Born rule from deeper principles, resolve several open questions regarding ontological stability and dynamics, address a number of familiar no-go theorems, and argue that our interpretation is ultimately compatible with Lorentz invariance. Along the way, we also investigate a number of unexplored features of quantum theory, including an interesting geometrical structure—which we call subsystem space—that we believe merits further study. We include an appendix that briefly reviews the traditional Copenhagen interpretation and the measurement problem of quantum theory, as well as the instrumentalist approach and a collection of foundational theorems not otherwise discussed in the main text.
I. INTRODUCTION

A. Why Do We Need a New Interpretation?

1. The Copenhagen Interpretation

Any mathematical-physical theory like quantum theory\(^1\) requires an interpretation, by which we mean some asserted connection with the real world. The traditional Copenhagen interpretation, with its axiomatic Born rule for computing empirical outcome probabilities and its notion of wave-function collapse for establishing the persistence of measurement outcomes, works quite well in most practical circumstances.\(^2\) At least according to some surveys [254, 287], the Copenhagen interpretation is still the most popular interpretation today.

Unfortunately, the Copenhagen interpretation also suffers from a number of serious drawbacks. Most significantly, the definition of a measurement according to the Copenhagen interpretation relies on a questionable demarcation, known as the Heisenberg cut (Heisenbergscher Schnitt) [198, 304], between the large classical systems that carry out measurements and the small quantum systems that they measure; this ill-defined Heisenberg cut has never been identified in any experiment to date and must be put into the interpretation by hand. (See Figure 1.) An associated issue is the interpretation’s assumption of wave-function collapse—known more formally as the Von Neumann-Lüders projection postulate [214, 305]—by which we refer to the supposed instantaneous, discontinuous change in a quantum system immediately following a measurement by a classical system, in stark contrast to the smooth time evolution that governs dynamically closed systems.

The Copenhagen interpretation is also unclear as to the ultimate meaning of the state vector of a system: Does a system’s state vector merely represent the experimenter’s knowledge, is it some sort of objective probability distribution,\(^3\) or is it an irreducible ingredient of reality like the state of a classical system? For that matter, what constitutes an observer, and can we meaningfully talk about the state of an observer within the formalism of quantum theory? Given that no realistic system is ever perfectly free of quantum entanglements with other systems, and thus no realistic system can ever truly be assigned a specific state vector in the first place, what becomes of the popular depiction of quantum theory in which every particle is supposedly described by a specific wave function propagating in three-dimensional space? The Copenhagen interpretation leads to additional trouble when trying to make sense of thought experiments like Schrödinger’s cat, Wigner’s friend, and the quantum Zeno paradox.\(^4\)

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\(^1\) We use the term “quantum theory” here in its broadest sense of referring to the general theoretical framework consisting of Hilbert spaces and state vectors that encompasses models as diverse as nonrelativistic point particles and quantum field theories. We are not referring specifically to the nonrelativistic models of quantum-mechanical point particles that dominated the subject in its early days.

\(^2\) See Appendix 1 for a detailed definition of the Copenhagen interpretation, as well as a description of the famous measurement problem of quantum theory and a systematic classification of attempts to solve it according to the various prominent interpretations of the theory.

\(^3\) Recent work [35, 85, 247] casts considerable doubt on assertions that state vectors are nothing more than probability distributions over more fundamental ingredients of reality.

\(^4\) We will discuss all of these thought experiments in Section IV C.
2. An Ideal Interpretation

Physicists and philosophers have expended much effort over many decades on the search for an alternative interpretation of quantum theory that could resolve these problems. Ideally, such an interpretation would eliminate the need for an ad hoc Heisenberg cut, thereby demoting measurements to an ordinary kind of interaction and allowing quantum theory to be a complete theory that seamlessly encompasses all systems in Nature, including observers as physical systems with quantum states of their own. Moreover, an acceptable interpretation should fundamentally (even if not always superficially) be consistent with all experimental data and other reliably known features of Nature, including relativity, and should be general enough to accommodate the large variety of both presently known and hypothetical physical systems. Such an interpretation should also address the key no-go theorems developed over the years by researchers working on the foundations of quantum theory, should not depend on concepts or quantities whose definitions require a physically unrealistic measure-zero level of sharpness, and should be insensitive to potentially unknowable features of reality, such as whether we can sensibly define “the universe as a whole” as being a closed or open system.

3. Instrumentalism

In principle, an alternative to this project is always available: One could instead simply insist upon instrumentalism—known in some quarters as the “shut-up-and-calculate approach” [220, 222]—meaning that one should regard the mathematical formalism of quantum theory merely as a calculational recipe or algorithm for predicting measurement results and empirical outcome probabilities obtained by the kinds of physical systems (agents) that can self-consistently act as observers, and, furthermore, that one should refuse to make definitive metaphysical claims about any underlying reality.5 We provide a more extensive description of the instrumentalist approach in Appendix 1d.

On the other hand, if the physics community had uniformly accepted instrumentalism from the very beginning of the history of quantum theory, then we might have missed out on the many important spin-offs from the search for a better interpretation: Decoherence, quantum information, quantum computing, quantum cryptography, and the black-hole information paradox are just a few of the far-reaching ideas that ultimately owe their origin to people thinking seriously about the meaning of quantum theory.

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5 “Whereof one cannot speak, thereof one must be silent.” [326]
B. Our Interpretation

In this paper and in a brief companion letter [34], we present a realist interpretation of quantum theory that hews closely to the basic structure of the theory in its widely accepted current form. Our primary goal is to move beyond instrumentalism and describe an actual reality that lies behind the mathematical formalism of quantum theory. We also intend to provide new hope to those who find themselves disappointed with the pace of progress on making sense of the theory’s foundations [314, 317].

Our interpretation is fully quantum in nature. However, for purposes of motivation, consider the basic theoretical structure of classical physics: A classical system has a specific state that evolves in time through the system’s configuration space according to some dynamical rule that may or may not be stochastic, and this dynamical rule exists whether or not the system’s state lies beneath a nontrivial evolving probability distribution on the system’s configuration space; moreover, the dynamical rule for the system’s underlying state is consistent with the overall evolution of the system’s probability distribution, in the sense that if we consider a probabilistic ensemble over the system’s initial underlying state and apply the dynamical rule to each underlying state in the ensemble, then we correctly obtain the overall evolution of the system’s probability distribution as a whole.

In particular, note the insufficiency of specifying a dynamical rule solely for the evolution of the system’s overall probability distribution but not for the system’s underlying state itself, because then the system’s underlying state would be free to fluctuate wildly and discontinuously between macroscopically distinct configurations. For example, even if a classical system’s probability distribution describes constant probabilities $p_1$ and $p_2$ for the system to be in macroscopically distinct states $q_1$ or $q_2$, there would be nothing preventing the system’s state from hopping discontinuously between $q_1$ and $q_2$ with respective frequency ratios $p_1$ and $p_2$ over arbitrarily short time intervals. Essentially, by imposing a dynamical rule on the system’s underlying state, we can provide a “smoothness condition” for the system’s physical configuration over time and thus eliminate these kinds of instabilities.

In quantum theory, a system that is exactly closed and that is exactly in a pure state (both conditions that are unphysical idealizations) evolves along a well-defined trajectory through the system’s Hilbert space according to a well-known dynamical rule, namely, the Schrödinger equation. However, in traditional formulations of quantum theory, an open quantum system that must be described by a density matrix due to entanglements with other systems—a so-called improper mixture—does not have a specific underlying state vector, let alone a Hilbert-space trajectory or a dynamical rule governing the time evolution of such an underlying state vector and consistent with the overall evolution of the system’s density matrix. It is a chief goal of our interpretation of quantum theory to provide these missing ingredients.

C. Conceptual Summary

We present a technical summary of our interpretation in Section VIA. In short, for a quantum system in an improper mixture, our interpretation identifies the eigenstates of the system’s density matrix with the possible states of the system in reality and identifies the eigenvalues of that density matrix with the probabilities that one of those possible states is actually occupied, and introduces just enough minimal structure beyond this simple picture to provide a dynamical rule for underlying state vectors as they evolve along Hilbert-space trajectories and to evade criticisms made in the past regarding similar interpretations. This minimal additional structure consists of a single additional class of conditional probabilities amounting essentially to a series of smoothness conditions that kinematically relate the states of parent systems to the states of their subsystems, as well as dynamically relate the states of a single system to each other over time.

D. Comparison with Other Interpretations of Quantum Theory

Our interpretation, which builds on the work of many others, is general, model-independent, and encompasses relativistic systems, but is also conservative and unextravagant: It includes only metaphysical objects that are either already a standard part of quantum theory or that have counterparts in classical physics. We do not posit the existence of exotic “many worlds” [66, 91, 92, 94, 106–108, 307, 308, 318], physical “pilot waves” [57, 58, 61, 89], or any fundamental GRW-type dynamical-collapse or spontaneous-localization modifications to quantum theory [7, 37, 4].

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6 In the language of [314], our interpretation therefore comports “with the idea that the state of a physical system is described by a vector in [a] Hilbert space rather than by numerical values of the positions and momenta of all the particles in the system,” and is not an interpretation “with no description of physical states at all.” That is, our interpretation is not “only an algorithm for calculating probabilities.”
Indeed, our interpretation leaves the widely accepted mathematical structure of quantum theory essentially intact. At the same time, we will argue that our interpretation is ultimately compatible with Lorentz invariance and is nonlocal only in the mild sense familiar from the framework of classical gauge theories.

Furthermore, we make no assumptions about as-yet-unknown aspects of reality, such as the fundamental discreteness or continuity of time or the dimensionality of the ultimate Hilbert space of Nature. Nor does our interpretation rely in any crucial way upon the existence of a well-defined maximal parent system that encompasses all other systems and is dynamically closed in the sense of having a so-called cosmic pure state or universal wave function that precisely obeys the Schrödinger equation; by contrast, this sort of cosmic assumption is a necessarily exact ingredient in the traditional formulations of the de Broglie-Bohm pilot-wave interpretation [57, 58, 61, 89] and the Everett-DeWitt many-worlds interpretation [66, 91, 92, 94, 106–108, 307, 308, 318]. (Given their stature among interpretations of quantum theory, we will have more to say about the de Broglie-Bohm interpretation in Section III A 6 and the Everett-DeWitt interpretation in Sections III A 9 and VI D 6.)

Indeed, by considering merely the possibility that our observable universe is but a small region of an eternally inflating cosmos of indeterminate spatial size and age [16, 128, 130, 131, 155–157, 207, 208, 279, 300], it becomes clear that the idea of a biggest closed system (“the universe as a whole”) may not generally be a sensible or empirically verifiable concept to begin with, let alone an axiom on which a robust interpretation of quantum theory can safely rely. Our interpretation certainly allows for the existence of a biggest closed system, but is also fully able to accommodate the alternative circumstance that if we were to imagine gradually enlarging our scope to parent systems of increasing physical size, then we might well find that the hierarchical sequence never terminates at any maximal, dynamically closed system, but may instead lead to an unending “Russian-doll” succession of ever-more-open parent systems. 

E. Outline of this Paper

In Section II, we lay down the conceptual groundwork for our interpretation and review features of classical physics whose quantum counterparts will play an important role. In Section III, we define our interpretation of quantum theory in precise detail and compare it to some of the other prominent interpretations, as well as identify an important nontrivial geometrical structure, herein called subsystem space, that has been lurking in quantum theory all along.

Next, in Section IV, we describe how our interpretation makes sense of the measurement process, provide a first-principles derivation of the Born rule for computing empirical outcome probabilities, and discuss possible corrections to the naïve Born rule that are otherwise invisible in the traditional Copenhagen interpretation according to which, in contrast to our own interpretation, the Born rule is taken axiomatically to be an exact statement about reality. We also revisit several familiar “paradoxes” of quantum theory, including Schrödinger’s cat, Wigner’s friend, and the quantum Zeno paradox.

In Section V, we study issues of locality and Lorentz invariance and consider several well-known thought experiments and no-go theorems. Additionally, we show that our interpretation evades claims [54, 97, 101, 227, 228] that interpretations similar to our own are incompatible with Lorentz invariance and necessarily depend on the existence of a “preferred” inertial reference frame. We also address the question of nonlocality more generally, and argue that the picture of quantum theory that emerges from our interpretation is no more nonlocal than are classical gauge theories.

We conclude in Section VI, which contains a concise summary of our interpretation as well as a discussion of falsifiability, future research directions, and relevant philosophical issues. In our appendix, we present a brief summary of the Copenhagen interpretation and the measurement problem of quantum theory—including a systematic analysis of the various ways that the prominent interpretations have attempted to solve the measurement problem—as well as a description of the instrumentalist approach to quantum theory and a summary of several important foundational theorems not covered in the main text.

Moreover, our interpretation does not introduce any new violations of time-reversal symmetry, and gives no fundamental role to relative states [106]; a cosmic multiverse or self-locating uncertainty [8]; coarse-grained histories or decoherence functionals [133, 134, 164, 165]; decision theory [92, 306, 309]; Dutch-book coherence, SIC-POVMs, or urgleichungs [18, 75, 76, 118–124]; circular frequentist arguments involving unphysical “limits” of infinitely many copies of measurement devices [8, 70, 77, 109]; infinite imaginary ensembles [32, 33]; quantum reference systems or perspectivalism [17–52]; relational or non-global quantum states [54, 171, 173, 249, 267]; many-minds states [11]; mirror states [171, 173]; faux-Boolean algebras [95, 96, 298]; “atomic” subsystems [26, 27]; algebraic quantum field theory [101]; secret classical superdeterminism or fundamental information loss [281–283, 285]; cellular automata [284, 286]; classical matrix degrees of freedom or trace dynamics [5, 6]; or discrete Hilbert spaces or appeals to unknown Planck-scale physics [69, 70].
II. PRELIMINARY CONCEPTS

A. Ontology and Epistemology

Our aim is to present our interpretation of quantum theory in a language familiar to physicists. We make an allowance, however, for a small amount of widely used, model-independent philosophical terminology that will turn out to be very helpful.

For our purposes, we will use the term “ontology” (adjective “ontic”) to refer to a state of being or objective physical existence—that is, the way things, including ourselves as physical observers, really are. This language is crucial for being able to talk about realist interpretations of quantum theory such as our own.

Meanwhile, we will use the term “epistemology” (adjective “epistemic”) to refer to knowledge or information regarding a particular ontic piece of reality. We can further subdivide epistemology into subjective and objective parts: The subjective component refers to information that a particular observer possesses about an ontic piece of reality, whereas the objective component refers to the information an ontic piece of reality reveals about itself to the rest of the world beyond it—meaning the most complete information that any observer could possibly possess about that piece of reality.

All successful scientific theories and models make predictions about things we can directly or indirectly observe, but until we attach an interpretation, we can’t really talk of an underlying ontology or its associated epistemology. To say that one’s interpretation of a mathematical theory of physics adds an ontology and an associated epistemology to the theory is to say that one is establishing some sort of connection between, on the one hand, the mathematical objects of the theory, and, on the other hand, the ontic states of things as they really exist as well as what epistemic information those ontic ingredients of reality make known about themselves and can be known by observers.

We regard an interpretation of a mathematical theory of physics as being realist if it asserts an underlying ontology of some kind, and anti-realist if it asserts otherwise. We call an interpretation agnostic if it refrains from making any definitive claims about an ontology, either whether one exists at all or merely whether we can say anything specific about it. In the sense of these definitions, instrumentalism is agnostic, whereas the interpretation we introduce in this paper is realist.

B. Classical Theories

Before laying out our interpretation of quantum theory in detail, we consider the salient features of the classical story, presented as generally as possible and in a manner intended to clarify the conceptual parallels and distinctions with the key ingredients that we’ll need in the quantum case.

1. Classical Kinematics and Ontology

The kinematical structure of a classical theory is conceptually straightforward and admits an intuitively simple ontology: An ontic state of a classical system, meaning the state of the system as it could truly exist in reality, corresponds at each instant in time \( t \) to some element \( q \) of a configuration space \( \mathcal{C} \) whose elements are by definition mutually exclusive possibilities. A full sequence \( q(t) \) of such elements over time \( t \) constitutes an ontic-state trajectory for the system. (See Figure 2.)

2. Classical Epistemology

From the mutual exclusivity of all the system’s allowed ontic states, we also naturally obtain the associated epistemology of classical physics: In the language of probability theory, this mutual exclusivity allows us to regard the system’s configuration space as being a single sample space \( \mathcal{C} = \Omega \), meaning that if we don’t know the system’s actual ontic state precisely, then we are free to describe the system in terms of an epistemic state defined to be a probability distribution \( p : \mathcal{C} \to [0, 1] \) on \( \mathcal{C} = \Omega \), where \( p(q) \in [0, 1] \) is the epistemic probability that the actual ontic state is truly

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9 Bell coined the term “beables” (that is, “be-ables”) to refer to ontic states of being or reality, as opposed to the mere “observables” (“observe-ables”) that appear in experiments. As Bell wrote [43], “The beables of the theory are those elements which might correspond to elements of reality, to things which exist. Their existence does not depend on ‘observation.’” Today, the term “beables” is often used in the quantum foundations community to refer more specifically to a fixed and unchanging set of elements of reality, such as the coordinate basis in the original nonrelativistic formulation of the de Broglie-Bohm interpretation of quantum theory.
Figure 2. A schematic picture of a classical configuration space $\mathcal{C}$, with examples of allowed ontic states $q$ and an example of an ontic-state trajectory $q(t)$.

$q$ and definitely not any other allowed ontic state $q' \neq q$.\textsuperscript{10} Equivalently, we can regard the system’s epistemic state $p$ as the collection $\{(p(q), q)\}_{q \in \mathcal{C}}$ of ordered pairs that each identify one of the system’s possible ontic states $q \in \mathcal{C}$ together with the corresponding probability $p(q) \in [0, 1]$ with which $q$ is the system’s actual ontic state. In analogy with ontic-state trajectories $q(t)$, we will sometimes say that a full sequence $p(t)$ of a system’s epistemic states over time constitutes an epistemic-state trajectory for the system.

Note that epistemic states and epistemic probabilities describing kinematical configurations in classical physics are always ultimately subjective, in the sense that they depend on the particular observer in question and are nontrivial only due to prosaic reasons of ignorance on the part of that observer. For instance, the system’s ontic state might be determined by a hidden random number generator or by the roll of an unseen die, or the observer’s recording precision might be limited by technological constraints.

An optimal observer has no subjective uncertainty and knows the system’s present ontic state precisely; in this idealized case, the probability distribution $p(q)$ singles out that one ontic state with unit probability and we say that the system’s epistemic state is pure. In the more general and realistic case in which the observer’s knowledge is incomplete and the probability distribution $p(q)$ is nontrivial, we say that the system’s epistemic state is mixed.

3. Formal Epistemic States

The subjectivity of classical epistemic states means that, if we wish, we could work with formal epistemic probability distributions over possibilities that are not mutually exclusive. As an example, suppose that we fill a large container with marbles, 72% of which are known to come from a box containing only red marbles and 28% of which are known to come from a box containing marbles that are either red or blue according to an unknown red:blue ratio. If we single out one marble from the container at random, then we can formally describe the epistemic state of the marble in terms of the non-exclusive possible statements $x =$ “red” and $y =$ “red or blue” as $\{(0.72, x), (0.28, y)\}$. Similar reasoning would apply when describing the epistemic state of a dispenser that stochastically releases marbles in the state $x$ at a frequency of 72% and in the state $y$ at a frequency of 28%.

However, to avoid ambiguities in our discussion ahead—especially when employing the notion of entropy to quantify our level of uncertainty—we will generally assume that all our classical epistemic states always encode logically rigorous probability distributions involving only mutually exclusive possibilities.

\textsuperscript{10} We do not attempt to wade into the centuries-old philosophical debate regarding the ultimate meaning of probability in terms of anything more fundamental, as encapsulated wonderfully by Russell in 1929 (quoted in [179]): “Probability is the most important concept in modern science, especially as nobody has the slightest notion what it means.” Interestingly, this quotation parallels Feynman’s famous quip in 1961 that “I think I can safely say that nobody understands quantum mechanics.” [110] For our purposes, we treat probability as a primitive, irreducible concept, just as we treat concepts like ontology and epistemology.
Then the correlational entropy of the apparatus grows by at least an amount $\Delta S_{\text{apparatus}}$—that measures some previously unknown numerical property of another system to a precision of $n$ precision of a system’s information in terms of its correlational entropy. Consider a system—say, a measurement signals, and information transfer.

Figure 3. The subject system $Q$ and the measurement apparatus $A$ (a) before the measurement and (b) after the measurement.

### 4. Classical Surprisal and Entropy

Given a system’s epistemic state $p$, the smooth function

$$
\log \frac{1}{p(q)} \in [0, \infty],
$$

called the surprisal [292], captures our “surprise” at learning that the system’s actual ontic state happens to be $q$. Our “average level of surprise,” also called the (Shannon) entropy of the epistemic state $p$, is defined by

$$
S \equiv \left\langle \log \frac{1}{p} \right\rangle = -\sum_q p(q) \log p(q) \in [0, \log (#\text{states in } C)]
$$

and provides an overall measure of how much we currently do not yet know about the system’s ontology [180, 181, 262, 263]. That is, if we think of an epistemic state $p$ as characterizing our information about a system’s underlying ontic state, then the entropy $S$ defined in (2) provides us with a quantitative measure of how much information we lack and that is therefore still hiding in the system.

Indeed, in the limit of a pure epistemic state, meaning that we know the actual ontic state of the system with certainty and thus $p(q) \to 1$ for a single value of $q$ and $p(q') \to 0$ for all $q' \neq q$, the entropy goes to zero, $S \to 0$. On the other hand, if we know nothing about the actual ontic state of the system, so that all the probabilities $p(q)$ become essentially equal, then the entropy approaches its maximal value $S \to \log (#\text{states in } C)$.

### 5. Classical Measurements and Signals

Combining classical kinematics with the definition of entropy in (2) as a total measure of the information that we lack about a system’s underlying ontic state, we can give a precise meaning to the notions of classical measurements, signals, and information transfer.

Suppose that we wish to study a system $Q$ whose epistemic state is $p_Q$—that is, consisting of individual probabilities $p_Q(q)$ for each of the system’s possible ontic states $q$—and whose entropy $S_Q > 0$ represents the amount of information that we currently lack about the system’s actual ontic state. We proceed by sending over a measurement apparatus $A$ in an initially known ontic state “∅”, by which we mean that the dial on the apparatus initially reads “empty” and the initial entropy of the apparatus is $S_A = 0$. (See Figure 3a.)

When the apparatus $A$ comes into local contact with the subject system $Q$ and performs a measurement to determine the state of $Q$, the state of the measurement apparatus becomes correlated with the state of the subject system; for example, if the actual ontic state of the subject system $Q$ happens to be $q$, then the ontic state of the apparatus $A$ evolves from “∅” to “$q$”. As a consequence, the measurement apparatus, which is still far away from us, develops a nontrivial epistemic state $p_A'$ identical to that of the subject system—meaning that $p_A'(q') = p_Q(q')$—and thus the apparatus develops a nonzero final correlational entropy $S_A' = S_Q$. (See Figure 3b.)

Correlational entropy of this kind therefore indicates that there has been the transmission of a signal—that is, the exchange of observable information, which, as we know, is constrained by the vacuum speed of light $c$. In the present example, this signal is communicated from the subject system $Q$ to the measurement apparatus $A$. We therefore also obtain a criterion for declaring that no signal has passed between two systems, namely, when their respective epistemic states evolve independently of one another and thus neither system develops any correlational entropy.

With these concepts of information, signaling, and correlational entropy in hand, we can obtain a measure for the precision of a system’s information in terms of its correlational entropy. Consider a system—say, a measurement apparatus—that measures some previously unknown numerical property of another system to a precision of $n$ bits. Then the correlational entropy of the apparatus grows by at least an amount $\Delta S \sim \log n$, and so the minimum
measurement error of the apparatus satisfies the error-entropy bound

$$\text{minimum error } \sim 1/n \sim e^{-\Delta S} \geq e^{-S},$$

where $S \geq \Delta S$ is the total final entropy (2) of the apparatus. Hence, the precision with which a measurement apparatus can specify any numerical quantity is bounded from below by $\exp(-S)$. We will see that this same sort of error crops up in the analogous case of quantum measurements when we study them in Section IV A.\(^{11}\)

### 6. Classical Dynamics

A classical system has well-defined dynamics if final ontic states can be predicted (probabilistically at least) based on given data characterizing initial ontic states. More precisely, a system has dynamics if for any choice of final time $t'$ there exists a rule for picking prescribed initial times $t_1 \geq t_2 \geq \cdots$ and there exists a mapping $p(\cdot; t' | \cdot; t_1), \cdot; t_2, \ldots$ that takes arbitrary initial ontic states $q_1$ at $t_1$, $q_2$ at $t_2$, and yields corresponding conditional probabilities $p(q'; t' | q_1), q_2, \ldots \in [0, 1]$ that the system’s ontic state at $t'$ is $q'$:

$$p(\cdot; t' | \cdot; t_1), \cdot; t_2, \ldots : (q_1), (q_2), \ldots \rightarrow p(q'; t' | q_1), (q_2), \ldots .$$

The initial times $t_1, t_2, \ldots$ that are necessary to define the mapping could, in principle, be only infinitesimally separated, and their total number, which could be infinite, is called the order of the dynamics. (As an example, Markovian dynamics, to be defined shortly, requires initial data at only a single initial time, and is therefore first order.) Just as was the case for the epistemic probabilities characterizing an observer’s knowledge of a given system’s ontic state at a single moment in time, the conditional probabilities that define a classical system’s dynamics can be subjective in the sense that they are merely a matter of the observer’s prosaic ignorance regarding details of the given system. However, these conditional probabilities can also (or instead) be objective in the sense that they cannot be trivialized based solely on knowledge about the given system itself; for example, nontrivial conditional probabilities may arise from the observer’s ignorance about the properties or dynamics of a larger environment in which the given system resides—an environment that could, for instance, be infinitely big and old—or they may emerge in an effective sense from chaos, to be described later on.

We call the dynamics deterministic in the idealized case in which the conditional probabilities $p(q'; t' | q_1), q_2, \ldots$ specify a particular ontic state $q'$ at $t'$ with unit probability for each choice of initial data $(q_1), q_2, \ldots$. Otherwise, whether subjective, objective, or a combination of both, we say that the dynamics is stochastic.

By assumption, the conditional probabilities appearing in (4) are determined solely by the initial ontic data $(q_1), q_2, \ldots$ together with a specification of the final ontic state $q'$ at $t'$. In particular, the conditional probabilities are required to be independent of the system’s evolving epistemic state $p$, and thus $p(\cdot; t' | \cdot; t_1), \cdot; t_2, \ldots$ naturally lifts to a (multi-)linear dynamical mapping relating the system’s evolving epistemic state at each of the initial times $t_1 \geq t_2 \geq \cdots$ to the system’s epistemic state at the final time $t'$:

$$p(\cdot; t' | \cdot; t_1), \cdot; t_2, \ldots : p(q'; t' | q_1), q_2, \ldots \rightarrow p(\cdot; t' | q_1), q_2, \ldots p(q_1), p(q_2), \ldots ,$$

which we can think of as a kind of dynamical Bayesian propagation formula.

If a system has well-defined dynamics (4), then we see from (5) that the dynamics actually operates at two levels—namely, at the level of ontic states and at the level of epistemic states—and consistency requires that these two levels of dynamics must related by $p(\cdot; t' | \cdot; t_1), \cdot; t_2, \ldots$. Indeed, we can regard $p(\cdot; t' | \cdot; t_1), \cdot; t_2, \ldots$ equivalently as describing the ontic-level dynamics in the sense of a mapping (4) from initial ontic states to the conditional probabilities for final ontic states, or as describing epistemic-level dynamics in the sense of a multilinear mapping (5) between initial and final epistemic states. This equivalence means that not only does the existence of ontic-level dynamics naturally define epistemic-level dynamics, but any given epistemic-level dynamics also determines the system’s ontic-level dynamics because we can just read off the ontic-level dynamical mapping (4) from the matrix elements (5) of the multilinear dynamical mapping.

\(^{11}\) Curiously, $\sim \exp(-S)$ effects also seem to play an important role in the famous black-hole information paradox \cite{140, 141, 166, 167, 246}, because the mixed-state final density matrix computed semiclassically appears to differ by $\sim \exp(-S)$ off-diagonal entries \cite{30, 175} from that of the pure-state final density matrix state that would be required by information conservation. See also footnote 45.

\(^{12}\) We address the definition and status of “superdeterminism,” a stronger notion of determinism, in Section V TD 1.
Although it’s possible to conceive of an alternative world in which dynamics and trajectories are specified only at the epistemic level and do not determine ontic-level dynamics or trajectories, it’s important to recognize the inadequacy of such a description: If all we knew were the rules dictating how epistemic states \( q \) evolve in time, then there would be nothing to ensure that a system’s underlying ontic state should evolve in a sensible manner and avoid fluctuating wildly between macroscopically distinct configurations. Quantum theory, as traditionally formulated, finds itself in this predicament, because a quantum system’s dynamics (when it exists) is specified only in terms of density matrices and not directly as a multilinear dynamical mapping (5) acting on epistemic states themselves. As part of our interpretation of quantum theory, we fill in this missing ingredient by providing an explicit translation of matrix mappings (4) involving multiple initial times \( t \) and any additional times \( \tilde{t} \).

Returning to our classical story, the assumed independence of the conditional probabilities \( p(q'; t'| (q_1; t_1), \ldots) \) from the system’s evolving epistemic state \( q \) also implies that any conditional probabilities depending on ontic states at additional initial times \( \tilde{t}_1, \tilde{t}_2, \ldots \notin \{t_1, t_2, \ldots\} \) must (if they exist) be equal to \( p(q'; t'| (q_1; t_1), \ldots) \),

\[
p(q'; t'| (q_1; t_1), \ldots) = \sum_{\tilde{q}_1, \ldots} p(q'; t'| (q_1; t_1), \ldots, (\tilde{q}_1, \tilde{t}_1), \ldots) p(\tilde{q}_1; \tilde{t}_1) \ldots
\]

because otherwise the dynamical Bayesian propagation formula

\[
p(q'; t'| (q_1; t_1), \ldots) = \sum_{\tilde{q}_1, \ldots} p(q'; t'| (q_1; t_1), \ldots, (\tilde{q}_1, \tilde{t}_1), \ldots) p(\tilde{q}_1; \tilde{t}_1) \ldots
\]

would imply that \( p(q'; t'| (q_1; t_1), \ldots) \) has a disallowed dependence on the epistemic state of the system at the additional times \( \tilde{t}_1, \tilde{t}_2, \ldots \). As an example, in the case of a system whose dynamics is Markovian [217], meaning first order, the mapping \( p(\cdot; t'| t) \) defined in (4) requires the specification of just a single initial time \( t \), and any additional mappings (4) involving multiple initial times \( t \geq t_1 \geq t_2 \geq \ldots \) must be equal to the single-initial-time mapping \( p(\cdot; t'| t) \):

\[
p(\cdot; t'| (t'), (t_1), (t_2), \ldots) = p(\cdot; t'| t).
\]

In other words, a system whose dynamics is Markovian retains no “memory” of its ontic states prior to its most recent ontic state, apart from whatever memory is directly encoded in that most recent ontic state.

We call a system closed in the idealized case in which the system does not interact with or exchange information with its environment. Unless we allow for literal destruction of information, closed classical systems are always governed fundamentally by deterministic dynamics. (Subtleties arise for chaotic systems, to be discussed shortly.) More realistically, however, systems are never exactly closed and are instead said to be open, in which case information can “leak out.” The existence of dynamics for an open system is a delicate question because conditional probabilities inherited from an enclosing parent system may contain a nontrivial dependence on the parent system’s own epistemic state, but any such open-system dynamics, if it exists, is generically stochastic.

7. Classical Systems with Continuous Configuration Spaces

Many familiar classical systems have continuous configuration spaces that we can regard as manifolds of some dimension \( N \geq 1 \). When working with a classical system of this kind, we can parameterize the ontic states \( q = (q_1, \ldots, q_N) \) of the configuration space \( C \) in terms of \( N \) continuously valued degrees of freedom \( q_\alpha \) \((\alpha = 1, \ldots, N)\), which play the role of coordinates for the configuration-space manifold. It is then more natural to describe the system’s epistemic states in terms of probability densities \( \rho(q) \equiv \rho(q_1, \ldots, q_N) \equiv dq(q)/d^Nq \) rather than in terms of probabilities \( p(q) \) per se.

If we can treat time as being a continuous parameter in our description of a classical system having a continuous configuration space, then, purely at the level of kinematics, the ontic states \( q(t) = (q_1(t), \ldots, q_N(t)) \) of the system at infinitesimally separated times \( t, t + dt, t + 2dt, \ldots \) are independent kinematical variables, and thus the instantaneous coordinate values \( q_\alpha(\tilde{t}), \) instantaneous velocities \( \dot{q}_\alpha(\tilde{t}) \equiv dq_\alpha(\tilde{t})/dt \equiv (q_\alpha(t + dt) - q_\alpha(t))/dt, \) instantaneous accelerations \( \ddot{q}_\alpha(\tilde{t}) \equiv d^2q_\alpha(\tilde{t})/dt^2, \) and so forth, are all likewise independent kinematical variables. We are therefore free to extend the notion of the system’s epistemic state to describe the joint probabilities \( p(q, \dot{q}, \ddot{q}, \ldots) \) that the system’s ontic state is \( q \equiv (q_1, \ldots, q_N), \) that its instantaneous velocities have values given by \( \dot{q} \equiv (\dot{q}_1, \ldots, \dot{q}_N), \) and so forth.

This generalization is particularly convenient when examining a system whose dynamical mapping (4) is second order in time, meaning that it involves initial ontic states \( q \) at a pair of infinitesimally separated times \( t \) and \( t + dt. \) In that case, we can equivalently express the dynamics in terms of a mapping involving the system’s ontic state \( q = (q_1, \ldots, q_N) \) and its instantaneous velocities \( \dot{q} = (\dot{q}_1, \ldots, \dot{q}_N) \) at the single initial time \( t. \) Hence, if we now think of
the ordered pair \((q, \dot{q})\) as describing a point in a generalized kind of configuration space (mathematically speaking, the tangent bundle of the configuration-space manifold), then we can treat the dynamics as though it were Markovian—that is, depending only on data at a single initial time. When the dynamics is, moreover, deterministic, and we can express the dynamical mapping from initial data \((q, \dot{q}; t)\) to final data \((q', \dot{q}; t')\) as a collection of equations, known as the system’s equations of motion, then we enter the familiar terrain of textbook classical physics, with its language of Lagrangians \(L(q, \dot{q})\), action functionals \(S[q] \equiv \int dt \ L(q(t), \dot{q}(t); t)\), canonical momenta \(p_\alpha\), phase spaces \((q, p)\) (mathematically speaking, the cotangent bundle of the system’s configuration-space manifold), Hamiltonians \(H(q, p)\), and Poisson brackets

\[
\{f, g\}_{\text{PB}} \equiv \sum_\alpha \left[ \frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial q_\alpha} \frac{\partial f}{\partial p_\alpha} \right].
\]  

(8)

For classical systems of this type, we can formulate the second-order equations of motion as the so-called canonical equations of motion

\[
\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} = -\{H, q_\alpha\}_{\text{PB}}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} = -\{H, p_\alpha\}_{\text{PB}},
\]  

(9)

and, for consistency, the system’s generalized epistemic state \(\rho(q, p)\) on its phase space must then satisfy the classical Liouville equation

\[
\frac{\partial \rho}{\partial t} = \{H, \rho\}_{\text{PB}}.
\]  

(10)

A familiar corollary is the Liouville theorem: The total time derivative of the probability distribution \(\rho(q(t), p(t); t)\) vanishes on any trajectory that solves the equations of motion,

\[
\frac{d\rho}{dt} = 0,
\]  

(11)

meaning that the ensemble of phase-space trajectories described by \(\rho(q(t), p(t); t)\) behaves like an incompressible fluid.

8. Chaos

Classical systems with continuous configurations spaces can exhibit an important kind of dynamics known as chaos. While chaotic dynamics is formally deterministic according to the classification scheme described earlier—initial ontic states are mapped to unique final ontic states with unit probability—the predicted values of final ontic states are exponentially sensitive to minute changes in the initial data, and thus the decimal precision needed to describe the system’s final epistemic state is exponentially greater than the decimal precision needed to specify the system’s initial data. Indeed, our relative predictive power, which we could define as being the ratio of our output precision to our input precision, typically goes to zero as we try to approach the idealized limit of infinitely sharp input precision. Because any realistic recording device is limited to a fixed and finite decimal precision, chaotic dynamics is therefore effectively stochastic even if the system of interest is closed off from its larger environment.

C. Quantum Theories

1. Quantum Kinematics

The picture our interpretation paints for quantum theory is surprisingly similar to the classical story, the key kinematical difference being that a quantum system’s configuration space is “too large” to be a single sample space of mutually exclusive states. More precisely, our interpretation asserts that the configuration space of a quantum system corresponds (up to meaningless overall normalization factors) to a vector space \(\mathcal{H}\), called the system’s Hilbert space, \(^{13}\) It is not a goal of this paper to provide deeper principles underlying this fact, which we take on as an axiom; for a detailed discussion of the complex vector-space structure of Hilbert spaces in quantum theory, and attempts to derive it from more basic principles and generalize it in new directions, see Section VI C 1. It’s interesting to note the historical parallel between the way that quantum theory modifies classical ontology by replacing classical configuration spaces (even those that are discrete) with smoothly interpolating continuous vector spaces, and, similarly, the way that probability theory itself modified epistemology centuries ago by replacing binary true/false assignments with a smoothly interpolating continuum running from false (0) to true (1).
for which the notion of mutual exclusivity between states is defined by the vanishing of their inner product, much in the same way that nondegenerate eigenstates of the Hermitian operators representing observables in the traditional formulation of quantum theory are always orthogonal.

As a consequence, every quantum system exhibits a continuous infinity of distinct sample spaces, each corresponding to a particular orthonormal basis for the system’s Hilbert space. It follows that an epistemic state makes logically rigorous sense only as a probability distribution over an orthonormal basis, although, just as we saw in Section II B 3 for the classical case, it is sometimes useful to consider formal subjective epistemic states over possibilities that are not mutually exclusive.

Just as a helpful visual analogy, we can loosely imagine all of a system’s allowed ontic states as corresponding to finite-size regions collectively covering an abstract Venn diagram. Then any orthonormal basis for the system’s Hilbert space corresponds to a partition of the Venn diagram into disjoint regions, whereas any two non-orthogonal ontic states describe overlapping regions. (See Figure 4.) Keep in mind, however, that this picture is entirely metaphorical: We regard quantum states as irreducible sample-space elements, and thus we are not literally identifying the Venn diagram’s individual points as elements of the system’s sample space.

This new feature of quantum theory opens up a possibility not available to classical systems, namely, that a system’s sample space is fundamentally contextual, changing over time from one orthonormal basis to another—that is, from one sample space to another sample space incompatible with the first but that nonetheless lies in the same configuration space—as the system interacts with other systems, such as measurement devices or the system’s larger environment. Hence, our interpretation of quantum theory builds in the notion of quantum contextuality [269–271] from the very beginning, in close keeping with the Kochen-Specker theorem [189] described in Appendix 2. Many interpretations of quantum theory, including our own, take advantage of this general feature of the theory in order to resolve the measurement problem through the well-known quantum phenomenon of decoherence [56, 65, 183, 184, 252, 253, 331].

2. Density Matrices

As we will review in Section III B, the formalism of quantum theory naturally furnishes a mathematical object \( \hat{\rho} \), called a density matrix [197, 303, 304], that our interpretation places in a central role for defining a system’s epistemic state and its underlying ontology, especially in the context of relating parent systems with their subsystems in the presence of quantum entanglement. A density matrix, which we regard as being related to but conceptually distinct from the system’s epistemic state, is a Hermitian operator on the system’s Hilbert space whose eigenvalues are all nonnegative and sum to unity.

For the case of a density matrix that arises entirely due to external quantum entanglements—that is, for a system in a totally improper mixture—our interpretation identifies the eigenvalues of the density matrix as collectively describing a probability distribution—the system’s epistemic state—over a set of possible ontic states that are represented by the eigenstates of the density matrix, where one of those possible ontic states is the system’s actual ontic state. (See
Figure 5. A schematic depiction of our postulated relationship between a system’s density matrix $\hat{\rho}$ and its associated epistemic state $\{(p_1, \Psi_1), (p_2, \Psi_2), (p_3, \Psi_3), \ldots\}$, the latter consisting of epistemic probabilities $p_1, p_2, p_3, \ldots$ (the eigenvalues of the density matrix) and possible ontic states $\Psi_1, \Psi_2, \Psi_3, \ldots$ (represented by the eigenstates of the density matrix), where one of those possible ontic states (in this example, $\Psi_2$) is the system’s actual ontic state.

Figure 5.) In the very rough sense in which observables in the traditional formulation of quantum theory correspond to Hermitian operators whose eigenvalues represent measurement outcomes and whose eigenstates represent their associated state vectors, our interpretation therefore identifies a system’s density matrix as the Hermitian operator corresponding to the system’s “probability observable.”

3. Quantum Dynamics

As is well known, density matrices describing closed systems evolve in time according to a quantum version of the Liouville equation, known as the Von Neumann equation, which bears a striking resemblance\(^{14}\) to the classical Liouville equation (10) and is given by

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right]. \quad (12)$$

This equation for the time evolution of density matrices has been generalized [65, 183, 206, 257] to cases involving open systems losing information to their larger environments, but a missing ingredient has been a general specification of the dynamics that governs the system at the level of its ontic and epistemic states.

Our interpretation supplies this key ingredient by translating the evolution law governing density matrices into a Markovian dynamical mapping in the sense of (4) that satisfies the key consistency requirement (5) relating the dynamics of ontic states with the dynamics of epistemic states. This mapping belongs to a new class of conditional probabilities that both dynamically relate the ontic states of a single quantum system at initial and final times and also define the kinematical relationship between the ontic states of parent systems with those of their subsystems. In addition, the mapping serves as a kind of smoothness condition that sews together ontic states in a sensible manner and avoids some of the ontic-level instabilities that arise in other interpretations of quantum mechanics.

III. THE MINIMAL MODAL INTERPRETATION

A. Basic Ingredients

Having presented the motivation, conceptual foundations, and rough outlines of our interpretation in the preceding sections, we now commence our precise exposition. The axiomatic content of our interpretation, which we summarize in detail in Section VIA, is intended to be minimal and consists of

1. definitions provided in this section of what we mean by ontic and epistemic states in quantum theory, as well as a dichotomy between subjective (“classical”) and objective epistemic states;

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\(^{14}\) This resemblance becomes even closer after an appropriate change to complex phase-space variables, as we discuss in Section VIC2.
2. a postulated relationship detailed in Section III B between objective epistemic states and density matrices, and, as a consequence, a relationship also between ontic states and density matrices;

3. a rule (the partial-trace operation) described in Section III C for calculating density matrices for subsystems given density matrices for their parent systems, and, as an immediate corollary, also a rule for relating the objective epistemic states of parent systems with those of their subsystems;

4. and, finally, a general formula (defining a class of quantum conditional probabilities) in Section III D both for \textit{kinematically} relating the ontic states of parent systems with those of their subsystems, as well as for \textit{dynamically} relating ontic states to each other over time and objective epistemic states to each other over time.

Note that apart from the references to density matrices, all these rules and metaphysical entities have necessary (if often implicit) counterparts in the description of classical systems detailed in Section II B: For classical systems as well as for quantum systems, we must define what we mean by ontic and epistemic states, establish relationships between epistemic states of parent systems and subsystems as well as between ontic states of parent systems and subsystems (the latter relationship being trivial in the classical case, but crucial for making sense of measurements in the quantum case), and dynamical relationships between ontic states over time and epistemic states over time.

There are two chief differences between the classical and quantum cases, the first difference being that in the quantum case, we need to invoke density matrices, ultimately because quantum entanglement leads to the existence of fundamentally objective epistemic states and makes density matrices a crucial link between parent systems and their subsystems. The second difference is that we can’t afford to take as many of the other ingredients for granted in the quantum case as we could in the classical case—quantum theory forces us to be much more explicit about our assumptions.

We claim that all the familiar features of quantum theory follow as consequences when we take all these basic principles their logical conclusions, including the Born rule for computing empirical outcome probabilities and also consistency with the various no-go theorems, as we will discuss starting in Section IV. We treat several additional no-go theorems in Appendix 2.

1. \textit{Ontic States}

Quantum physics first enters our story with the condition that the distinct ontic states $\Psi_i$ that make up our system’s configuration space are each represented by a particular unit vector $|\Psi_i\rangle$ (up to overall phase) in a complex inner-product vector space $H$, called the system’s Hilbert space:

$$\Psi_i \leftrightarrow |\Psi_i\rangle \in H \text{ (up to overall phase).}$$

That is, we identify the system’s configuration space as the quotient $H/\sim$, where $\sim$ represents equivalence of vectors in the Hilbert space $H$ up to overall normalization and phase. Note that we regard (13) as merely a \textit{correspondence} between an ontic object and a mathematical object, and so we will not use the terms “ontic state” and “state vector” synonymously in this paper, in contrast to the practice of some authors.

2. \textit{Epistemic States}

As part of the definition of our interpretation, and in close parallel with the classical case presented in Section II B, we postulate that a quantum system’s evolving epistemic state consists at each moment in time $t$ of a collection $\{(p_i, \Psi_i)\}_i$ of ordered pairs that for each $i$ identify one of the system’s \textit{possible} ontic states $\Psi_i = \Psi_i(t)$ together with the corresponding epistemic probability $p_i = p_i(t) \in [0, 1]$ for which $\Psi_i$ is the system’s \textit{actual} ontic state at the current time $t$.\footnote{Throughout this paper, we will always work in the so-called Schrödinger picture, in which a system’s time evolution is carried by states and not by operators. Indeed, because most of the systems that we consider in this paper aren’t dynamically closed and thus don’t evolve according to unitary dynamics—unitary dynamics usually being a good approximation only for systems that are microscopic and therefore easy to isolate from their environments—none of the other familiar pictures (such as the Heisenberg picture) will generally be well-defined anyway.}

As in the classical case, an epistemic state is called pure in the idealized case in which its probability distribution is trivial—that is, when one epistemic probability $p_i$ is equal to unity and all the others vanish—and is called mixed when the probability distribution is nontrivial.

Note that a system’s epistemic probabilities $p_i$ at a particular moment in time should not generally be confused with the empirical outcome probabilities that are used to quantify the results of experiments and that are computed via the
Born rule. Whereas epistemic probabilities describe a system’s present state of affairs, we will ultimately show that empirical outcome probabilities can be identified with a measurement device’s predicted future epistemic probabilities conditioned on the hypothetical assumption that the device will have performed a particular measurement on a given subject system.

3. Modal Interpretations and Minimalism

Our use of the modifiers “possible and “actual,” together known formally as modalities, identifies our interpretation of quantum theory as belonging to the general class of modal interpretations originally introduced by Krips in 1969 [192–194] and then independently developed by van Fraassen (whose early formulations involved the fusion of modal logic [84, 203, 322] with quantum logic [55, 139]), Dieks, Vermaas, and others [19, 26, 27, 68, 73, 211, 212, 295, 296, 298, 299].

The modal interpretations are now understood to encompass a very large set of interpretations of quantum theory, including most interpretations that fall between the “many worlds” of the Everett-DeWitt approach and the “no worlds” of the instrumentalist approaches. Generally speaking, in a modal interpretation, one singles out some preferred basis for each system’s Hilbert space and then regards the elements of that basis as the system’s possible ontic states—one of which is the system’s actual ontic state—much in keeping with how we think conceptually about classical probability distributions. For example, as we will explain more fully in Section III A 6, and as emphasized in [298], the de Broglie-Bohm pilot-wave interpretation can be regarded as a special kind of modal interpretation in which the preferred basis is permanently fixed for all systems at a universal choice. Other modal interpretations, such as our own, instead allow the preferred basis for a given system to change—in our case by choosing the preferred basis to be the evolving eigenbasis of that system’s density matrix. However, we claim that no existing modal interpretation captures the one that we introduce in this paper.

A central guiding principle of our interpretation is minimalism: By intention, we make no changes to the way quantum theory should be used in practice, and the only requirements we impose are those that are absolutely mandated by the need for our notion of ontology and epistemology to account for the observable predictions of quantum theory—no more, no less. We are also metaphysically conservative in the sense that we include only ingredients that are either already a part of the standard formalism of quantum theory or that have counterparts in classical physics. We therefore naturally call our interpretation the minimal modal interpretation of quantum theory.

Our reasons for this minimalism go beyond philosophically satisfying notions of axiomatic simplicity and parsimony. Based on an abundance of historical examples, we know that when trying to add an ontology to quantum theory, including (even implicitly) more features than is strictly necessary is not just a metaphysical extravagance, but also usually leads to trouble. This trouble may take the form of unacceptable ontological instabilities, conflicts with various no-go theorems, an uncontrollable profusion of “epicycles,” or just an overwhelming structural ornateness.

By contrast, we will show that our minimal modal interpretation evades a recent no-go theorem [54, 97, 101, 227, 228] asserting that other members of the class of modal interpretations are incompatible with Lorentz invariance at an ontological level. We will also argue that the same minimalism makes it possible to provide first-principles derivations of familiar aspects of quantum theory, including the Born rule (and possible corrections to it) for computing the empirical outcome probabilities that emerge from experiments.

4. Hidden Variables and the Irreducibility of Ontic States

To the extent that our interpretation of quantum theory involves hidden variables, the actual ontic states underlying the epistemic states of systems play that role. However, one could also argue that calling them hidden variables is just an issue of semantics because they are on the same metaphysical footing as both the traditional notion of quantum states as well as the actual ontic states of classical systems.

In any event, it is important to note that our interpretation includes no other hidden variables: Just as in the classical case, we regard ontic states as being irreducible objects, and, in keeping with this interpretation, we do not regard a system’s ontic state itself as being an epistemic probability distribution—much less a “pilot wave”—over a set of more basic hidden variables. In a rough sense, our interpretation unifies the de Broglie-Bohm interpretation’s pilot wave and hidden variables into a single ontological entity that we call an ontic state.

In particular, we do not attach an epistemic probability interpretation to the components of a vector representing a system’s ontic state, nor do we assume a priori the Born rule, which we will ultimately derive as a means of computing empirical outcome probabilities. Otherwise, we would need to introduce an unnecessary additional level of probabilities into our interpretation and thereby reduce its axiomatic parsimony and explanatory power.
Via the phenomenon of environmental decoherence, our interpretation ensures that the evolving ontic state of a sufficiently macroscopic system—with significant energy and in contact with a larger environment—is highly likely to be represented by a temporal sequence of state vectors that presumably approximate coherent states, as described in Section VI C 2, and whose labels evolve in time according to recognizable semiclassical equations of motion. For microscopic, isolated systems, by contrast, we simply accept that the ontic state vector may not always have an intuitively familiar classical description.

5. Constraints from the Kolmogorov Axioms

The logically rigid Kolmogorov axioms [190], which are satisfied by any well-defined epistemic probability distribution, require that our epistemic probabilities \( p_i = p_i(t) \) at any moment in time \( t \) can never sum to a value greater than unity. In the idealized limit in which the system cannot decay, we can require the stronger condition that the epistemic probabilities always sum exactly to unity, \( \sum_i p_i = 1 \). By contrast, for systems having \( \sum_i p_i < 1 \), we naturally interpret the discrepancy \( (1 - \sum_i p_i) \in [0,1] \) as the system’s probability of no longer existing, as we will explain in greater detail in Section III C 5.

In keeping with the Kolmogorov axioms, we additionally require that the system’s corresponding possible ontic states \( \Psi_i = \Psi_i(t) \) at any single moment in time \( t \) are mutually exclusive. We translate this requirement into quantum language as the condition that no member \( |\Psi_i\rangle \) of the system’s set of possible ontic state vectors at the time \( t \) can be expressed nontrivially as superposition involving any of the others; none is a “blend” involving any of the others. This condition is equivalent to requiring that the system’s possible ontic state vectors at the time \( t \) must all be mutually orthogonal,

\[
\langle \Psi_i | \Psi_j \rangle = 0 \quad \text{for} \quad i \neq j, \tag{14}
\]

much in the familiar way that eigenstates corresponding to nondegenerate eigenvalues of the Hermitian operators representing observables in the traditional formulation of quantum theory are always orthogonal. Hence, any orthonormal basis for that Hilbert space defines a distinct allowed sample space admitting logically rigorous probability distributions. However, sets of non-orthogonal state vectors (such as the system’s Hilbert space as a whole) do not, strictly speaking, admit logically rigorous probability distributions, although, in keeping with our discussion in Section II B 3, it is occasionally useful to define formal probability distributions even in those cases.

6. The de Broglie-Bohm Pilot-Wave Interpretation of Quantum Theory

In contrast to regarding any orthonormal basis as a potentially valid sample space, the so-called “fixed” modal interpretations [68, 298]—which include the well-known de Broglie-Bohm pilot-wave interpretation as a special case—require that the sample space remains permanently fixed forever at some universally preferred choice, which is then a matter for the interpretation to try to justify once and for all. A generic state vector is then regarded as both being an epistemic probability distribution over this fixed sample space of ontic states and also as being an ontological entity in its own right, namely, a physical “pilot wave” that guides the evolution of the system’s hidden ontic state through this fixed sample space while being conceptually distinct from that hidden ontic state.

In its original nonrelativistic formulation, the de Broglie-Bohm interpretation [57, 58, 61, 89] takes its fixed orthonormal basis to be the position eigenbasis of point particles, and offers what might appear to be a straightforward solution to questions concerning measurements in quantum theory [219]. Unfortunately, in the relativistic regime, spatial position ceases to exist as an orthonormal basis: The Hilbert-space inner product of two would-be position eigenstates is nonvanishing, although it is exponentially suppressed in the particle’s Compton wavelength \( \lambda = \frac{h}{mc} \) and therefore indeed goes to zero in the nonrelativistic limit \( c \to \infty \) [241]. (Preserving causality in the presence of this spontaneous superluminal propagation famously necessitates the existence of antiparticles [111, 241, 312].)

Attempts to recast the de Broglie-Bohm interpretation in a relativistic context require giving up the elegance and axiomatic parsimony of the interpretation’s nonrelativistic formulation and involve replacing the nonrelativistic position eigenbasis with bases that remain orthonormal in the relativistic regime, such as the field-amplitude eigenbasis for bosonic fields [58, 60]. However, fermionic fields still represent a serious problem, because their \textit{field amplitudes} are inherently non-classical, anticommuting, nilpotent Grassmann numbers; although Grassmann numbers provide a convenient \textit{formal} device for expressing fermionic scattering amplitudes in terms of Berezin path integrals [53, 241], taking Grassmann numbers seriously as physical ingredients in quantum-mechanical Hilbert spaces would lead to nonsense “probabilities” that are not ordinary numbers. Trying instead to choose the fermion-number eigenbasis [43] brings back the question of ill-definiteness of spatial position, leading some advocates to drop orthonormal bases.
altogether in favor of POVMs [275]. Relativity therefore implies that there’s no safe choice of fixed orthonormal basis to provide the de Broglie-Bohm interpretation with its foundation: At best, there’s no canonical choice of basis to fix once and for all, and, at worst, there is no choice that’s consistent or sensible.

Whether or not the de Broglie-Bohm interpretation’s proponents ultimately find a satisfactory fixed orthonormal basis to define their sample space, the interpretation still runs into other troubles as well, including its inability to accommodate the non-classical changes of particle spectrum that can arise in quantum field theories and the non-classical changes in configuration space that can emerge from quantum dualities, both of which play a central role in much of modern physics. The de Broglie-Bohm interpretation also suffers from a somewhat more metaphysical problem: Because a pilot wave has an ontological existence over and above that of the hidden ontic state of its corresponding physical system, we run into the well-known difficulty [66] of making sense of the ontological status of all its many branches. Indeed, the pilot wave’s branches behave precisely as the “many worlds” of the Everett-DeWitt interpretation of quantum theory in all their complexity, despite the fact that only one of those branches is supposedly “occupied” by the system’s hidden ontic state and the rest of the branches are “empty worlds” filled with ghostly people living out presumably ghostly lives.

7. Subjective Uncertainty and Proper Mixtures

As we explained in Section II B, we use nontrivial epistemic states \( \{ (p_i, q_j) \}_i \) in classical physics in order to account for subjective uncertainty about a classical system’s actual ontic state at a given moment in time. Similarly, one way that epistemic states \( \{ (p_i, \Psi_j) \}_i \) can arise in quantum theory is when we have subjective uncertainty about a quantum system’s actual ontic state at a particular moment in time, in which case we call the system’s epistemic state a proper mixture.

Strictly speaking, the requirements of a logically rigorous probability distribution require that the possible ontic states \( \Psi_i \) that make up a proper mixture must be mutually exclusive and hence correspond to mutually orthogonal state vectors \( |\Psi_i\rangle \) in accordance with (14). However, after deriving the Born rule later in this paper, we will describe in Section IV B how to accommodate contexts in which it is useful to relax this mutual exclusivity, just as we explained in Section II B3 how we could consider formal probability distributions over non-exclusive possibilities in classical physics.

There is no real controversy or dispute over the metaphysical meaning of proper mixtures, at least in the sense that there is wide acceptance for regarding a proper mixture as a prosaic, subjective probability distribution over a hidden underlying ontic state. We will therefore put proper mixtures entirely aside for most of this paper, returning to them in Section IV B only after deriving the Born rule.

8. Objective Uncertainty and Improper Mixtures

Quantum theory also features what our minimal modal interpretation regards as an objective kind of uncertainty that has no real classical counterpart, and in this case we will always insist upon logically rigorous epistemic states involving mutually exclusive possible ontic states. To see where this new form of uncertainty comes from, and to make clear the importance of the relationship between parent systems and subsystems in the context of our interpretation of quantum theory, we need to step back for a moment and compare the notions of parent systems and subsystems in classical physics and in quantum physics.

Classically, a system \( C \) with configuration space \( \mathcal{C}_C \) is said to be a parent or composite system consisting of two subsystems \( A \) and \( B \) with respective configuration spaces \( \mathcal{C}_A \) and \( \mathcal{C}_B \) if the configuration space of system \( C \) is expressible as the Cartesian product \( \mathcal{C}_C = \mathcal{C}_A \times \mathcal{C}_B \). Note, of course, that each element \( c = (a, b) \) of \( \mathcal{C}_C \) is an ordered pair that identifies a specific element \( a \) of \( \mathcal{C}_A \) and a specific element \( b \) of \( \mathcal{C}_B \). We then naturally denote the parent configuration \( C \) by \( A+B \), and, at least in the case in which all the configuration spaces in question have finitely many elements \( N_A, N_B, N_{A+B} < \infty \), we have the simple relation that \( N_{A+B} = N_A N_B \). Note, of course, that either subsystem \( A \) or \( B \) (or both) could well be a parent system to even more elementary subsystems.

\[ \text{Even for a nonrelativistic system, it’s not clear why the coordinate basis should be favored in the de Broglie-Bohm interpretation over an orthonormal basis approximating the system’s far more classical-looking coherent states [145, 255], as defined in Section V 1 C 2.} \]

\[ \text{Non-classical changes in particle spectrum in quantum field theories include prosaic examples like transitions from tachyonic particle modes to massive radial “Higgs” modes and massless Nambu-Goldstone modes after the spontaneous symmetry breaking of a continuous symmetry [148, 149, 229], as well as more exotic examples like Skyrmions [240, 264–266] and bosonization [86, 216]. Prominent examples of dualities include generalizations of electric-magnetic duality in certain supersymmetric gauge theories [226, 258], holographic dualities like the AdS/CFT correspondence between gauge theories and theories of quantum gravity in higher dimensions [9, 67, 153, 154, 215, 325], and various dualities that play key roles in string theory [144, 260, 273]. In particular, the AdS/CFT correspondence as well as string theory suggest that spacetime can undergo radical changes in structure [24, 323, 324] via intermediary non-geometric phases, and even perhaps that the “space” in “spacetime” is itself an emergent property of more primitive ingredients [103, 259].} \]
The quantum case is much more subtle because the Cartesian product of a pair of nontrivial Hilbert spaces is not another Hilbert space. We instead identify a given quantum system \( C \) having a Hilbert space \( \mathcal{H}_C \) as being the parent system of two quantum subsystems \( A \) and \( B \) with respective Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) if there exists an orthonormal basis for \( \mathcal{H}_C \) whose elements are of the tensor-product form \( |a, b\rangle = |a\rangle \otimes |b\rangle \), where the sets of vectors \( |a\rangle \) and \( |b\rangle \) respectively constitute orthonormal bases for \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Then, by construction, every vector in \( \mathcal{H}_C \) consists of some linear combination of the vectors \( |a, b\rangle \). We express this fact by writing the parent system’s Hilbert space as the tensor product \( \mathcal{H}_C = \mathcal{H}_A \otimes \mathcal{H}_B \), and we denote the parent system, as we did in the classical case, by \( C = A + B \).

Observe that the dimensions of these various Hilbert spaces (assuming they are all of finite dimension) then satisfy \( \dim \mathcal{H}_{A+B} = (\dim \mathcal{H}_A)(\dim \mathcal{H}_B) \).

Given this background, consider a parent quantum system \( A + B \) consisting of two subsystems \( A \) and \( B \) and whose ontic state \( \Psi_{A+B} \) is described by a so-called entangled state vector of the form

\[
|\Psi_{A+B}\rangle = \alpha |\Psi_{A,1}\rangle |\Psi_{B,1}\rangle + \beta |\Psi_{A,2}\rangle |\Psi_{B,2}\rangle, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{15}
\]

What is the ontic state of subsystem \( A \)? What is the ontic state of subsystem \( B \)?

In the Copenhagen interpretation of quantum theory, these two questions do not possess well-defined answers. Instrumentalist interpretations do not regard even the questions themselves as being sensible or meaningful.

9. The Everett-DeWitt Many-Worlds Interpretation of Quantum Theory

By contrast, according to the Everett-DeWitt many-worlds interpretation of quantum theory, (15) implies that there exist two simultaneous “worlds” or “realities” or “branches”: In one world, associated with a probability \( |\alpha|^2 \), the ontic state of \( A \) is \( \Psi_{A,1} \) and the ontic state of \( B \) is \( \Psi_{B,1} \), whereas in the other world, associated with a probability \( |\beta|^2 \), the ontic state of \( A \) is \( \Psi_{A,2} \) and the ontic state of \( B \) is \( \Psi_{B,2} \).

Of course, because we can always choose from a continuously infinite set of different orthonormal bases for the Hilbert space of the parent system \( A + B \), as we discussed in Section II C 1, these two “worlds” are radically non-unique even for a fixed parent-system state vector; moreover, the different possible world-bases are not generally related to one another in a manner that can be conceptualized classically, and preserving manifest locality in the many-worlds interpretation generically requires switching from one world-basis to another as a function of time. These issues, known collectively as the preferred-basis problem, imply a breakdown in the popular portrayal of the many-worlds interpretation as describing unfolding reality in terms of a well-defined forking structure in the manner of Borges’ Garden of Forking Paths [62] or Lewis’s modal realism [204, 205], and are unsolvable without introducing additional axiomatic ingredients into the interpretation. In particular, without additional postulates, one cannot evade the preferred-basis problem merely by appealing to environment-induced decoherence, because the many-worlds interpretation traditionally assumes that “the universe as a whole”—which determines the single branch-set shared by all systems in Nature—is described by an always-pure state that never undergoes decoherence and therefore doesn’t possess a canonical preferred basis.\(^{18}\)

Another conundrum of the many-worlds interpretation is the difficulty in making sense of the components \( \alpha \) and \( \beta \) in (15) in terms of probabilities when both worlds are deterministically and simultaneously realized,\(^{19}\) especially given the fundamental logical obstruction to deriving probabilistic conclusions from deterministic assumptions and in light of the preferred-basis problem and the continuously infinite non-uniqueness of the choice of world-basis. Putting aside the preferred-basis problem, a common approach [8, 109] is to study a pure state defined in terms of a “limit” of many identical copies of a given measurement set-up, and then argue that “maverick branches”—meaning terms in the final-state superposition whose outcome frequency ratios deviate significantly from the Born rule—have arbitrarily small Hilbert-space amplitudes and thus can safely be ignored. However, infinite limits are always rigorously defined in terms of increasing but finite sequences, and for any finite number of copies of a given measurement set-up, maverick branches have nonzero amplitudes and outnumber branches with better-behaved frequency ratios. Asserting that the smallness of their amplitudes makes maverick branches “unlikely” therefore implicitly assumes the very probability interpretation to be derived [70, 77], and is akin to the sort of circular reasoning inherent in all attempts to use the law of large numbers to turn frequentism into a rigorous notion of probability.

Even if one could somehow add enough additional axioms to justify interpreting state-vector components like \( \alpha \) and \( \beta \) in (15) as instantaneous, kinematical probabilities, the traditional many-worlds interpretation of quantum theory

\(^{18}\) For an explicit discussion of these points in the context of the EPR-Bohm thought experiment, including the issue of nonlocality, see Section VI D 6. There are additional reasons to be suspicious of attempts to give all the branches of the many-worlds interpretation an equal ontological meaning, as, for example, Aaronson describes in the context of quantum computing in [1]. We also emphasize that attaching a many-worlds ontology to mathematical vectors in the first place is itself a nontrivial axiom; indeed, the instrumentalist approach, which we describe in Appendix 1 d, does not postulate any ontological status for state vectors.

\(^{19}\) Maudlin eloquently captures this key shortcoming of the many-worlds interpretation in “Problem 2: The problem of statistics” in [218], and [92, 306, 309] attempt to suppress the problem by burying it under the elaborate axiomatic apparatus of decision theory.
lacks an explicit model describing how the experiential trajectory of an individual observer dynamically unfolds from moment to moment through the interpretation’s multitudinous (and ill-defined) branching worlds. That is, even if the many-worlds interpretation admits a sequence of static probability distributions at individual moments in time, it does not possess anything like the dynamical conditional probabilities (4) connecting one moment in time of an observer’s experiential trajectory to the next moment in time.

In particular, without adding on significantly more assumptions and metaphysical structure, the many-worlds interpretation is unable to ensure the ontological stability of such an observer’s experiential trajectory through time: Observers in the many-worlds interpretation of quantum theory are vulnerable to radical macroscopic instabilities in experience (and memory) that parallel the sorts of macroscopic instabilities that are possible in a hypothetical version of classical physics that lacks ontic-level dynamics, as we explained in Section II B 6 shortly after (5). We will detail a concrete quantum analogue of such dynamical ontological instabilities (eigenstate swaps) in Section III B 3, and we will eliminate them in the context of our own interpretation of quantum theory when we introduce a dynamical notion of quantum conditional probabilities in Section III D. Modal interpretations have been criticized in the past for lacking any such dynamical smoothing conditions on ontic-state trajectories to eliminate these sorts of metaphysical instabilities, but to the extent that such criticisms of the traditional modal interpretations are justified, the same criticisms apply equally to the many-worlds interpretation as well.

Finally, because all realistic systems exhibit a nonzero degree of quantum entanglement with other systems, the traditional many-worlds interpretation depends crucially upon the existence of a maximal closed system—again, “the universe as a whole”—that admits a description in terms of an exact cosmic pure state. Indeed, the many-worlds interpretation relies on the existence of this cosmic pure state in order to identify the branches on which the superposed copies of various subsystems reside, and with what associated probabilities. However, as we discussed in Section I D, there are reasons to be skeptical that any such maximal closed system is physically guaranteed to exist and be well-defined, thus opening up the real possibility that the many-worlds interpretation isn’t fundamentally well-defined either.

10. Our Interpretation

Answering our ontological questions about the state vector (15) in the context of our own interpretation of quantum theory will require that we first introduce a class of well-known mathematical objects, called density matrices, that have no counterpart in classical physics. For now, we will simply say that our uncertainty over each subsystem’s actual ontic state is captured by what we call an objective epistemic state for that subsystem. A quantum epistemic state that includes at least some objective uncertainty of this kind—possibly together with some subjective uncertainty as well—is called an improper mixture. Turning things around, we can then identify an objective epistemic state as an improper mixture in the extremal case of zero subjective uncertainty.

In contrast to proper mixtures (that is, wholly subjective epistemic states), improper mixtures do not have a widely accepted a priori meaning, and it is a central purpose of our minimal modal interpretation of quantum theory to provide one, starting with the requirement that an objective epistemic state must be a logically rigorous probability distribution and therefore always involve mutually exclusive (14) possible ontic states. In Section III B, we will define density matrices and relate them to objective epistemic states, and then, once we have developed a prescription for relating the density matrices of parent systems with those of their subsystems in Section III C, we will finally have in our hands a precise means of resolving the aforementioned conundrum about the ontic states of A and B.

B. The Correspondence Between Objective Epistemic States and Density Matrices

Even if we happen to know a system’s objective epistemic state \( \{(p_i(t), \Psi_i(t))\}_i \) at a given time \( t \), we have not yet presented a framework for determining the system’s objective epistemic state at any other time \( t' \neq t \), nor a prescription for relating ontic states to each other over time or for relating objective epistemic states or ontic states between parent systems and their subsystems. Our first step will be to define a correspondence between the objective epistemic states of quantum systems and a class of mathematical objects—density matrices—that will serve as the fulcrum of this framework.

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20 See “Problem 3: The problem of effect” in [218].
1. Density Matrices

Any objective epistemic state \( \{(p_t (t), \Psi_t (t))\}_t \) of the kind defined in Section III A 2—meaning, in particular, that it involves mutually exclusive possible ontic states \( \Psi_t (t) \)—can be identified uniquely\(^{21}\) with a unit-trace, positive semi-definite matrix \( \hat{\rho} (t) \) known as the system’s density matrix:

\[
\hat{\rho} (t) = \sum_i p_i (t) |\Psi_i (t)\rangle \langle \Psi_i (t)| , \quad \text{with} \quad (\hat{\rho} (t))^\dagger = \hat{\rho} (t) , \quad \text{Tr} [\hat{\rho} (t)] = 1 , \\
0 \leq p_i (t) \leq 1 , \quad \sum_i p_i (t) = 1 , \\
\langle \Psi_i (t)| \Psi_j (t) \rangle = \delta_{ij} .
\]

(16)

By “identified” here, we mean that there exists a precise correspondence\(^{22}\) between the objective epistemic state \( \{(p_t (t), \Psi_t (t))\}_t \) of the system and the eigenvalue-eigenvector pairs of the density matrix \( \hat{\rho} (t) \) at each moment in time \( t \):

\[
\{(p_t (t), \Psi_t (t))\}_t \leftrightarrow \hat{\rho} (t) = \sum_i p_i (t) |\Psi_i (t)\rangle \langle \Psi_i (t)| .
\]

(17)

Essentially, the correspondence (17) provides a means of encoding a list of non-negative real numbers—the system’s epistemic probabilities—and a list of state vectors—representing the system’s possible ontic states—in a basis-independent manner, namely, as the eigenvalue-eigenvector spectrum of the system’s density matrix.

Recalling the definition (2) of the entropy of a classical system’s epistemic state, the correspondence (17) immediately implies that we can express the entropy \(- \sum_i p_i \log p_i \) of a quantum objective epistemic state \( \{(p_t (t), \Psi_t (t))\}_t \) in terms of the corresponding density matrix \( \hat{\rho} \) according to the basis-independent Von Neumann entropy formula

\[
S \equiv - \text{Tr} [\hat{\rho} \log \hat{\rho}] .
\]

(18)

For recent work attempting to derive important aspects of statistical mechanics from fundamentally quantum reasoning, including various fluctuation theorems and the second law of thermodynamics, see, for example, [15, 105, 147, 186, 201, 209, 244, 245].

In light of the association (17) between probabilities and matrices, together with linear-algebraic expressions like the Von Neumann entropy formula (18), quantum theory has been called “a noncommutative generalization of classical probability theory” [202]. However, we regard (17) as merely a correspondence between an epistemic object and a mathematical object, and so, just as we do not use the terms “ontic state” and “state vector” synonymously in this paper even though the associated objects are related by (13), we will not use the terms “(objective) epistemic state” and “density matrix” synonymously either, again in contrast to the practice of some authors.

The correspondence (17), which lies at the core of our minimal modal interpretation of quantum theory, provides a natural context for emphasizing the following tautological but crucial point:

If our interpretation does not explicitly identify a given quantity as being a literal epistemic probability, then that quantity is not—or at least not yet—a literal epistemic probability.

(19)

In particular, in keeping with our earlier comment in Section III A 4 that, insofar as our interpretation involves hidden variables, ontic states play the role of those hidden variables, and also our comment that we do not regard the state vectors representing them as being epistemic probability distributions for anything else, we do not assume the Born rule Prob(\ldots) = \ldots^2 \text{ a priori}, nor do we correspondingly interpret the absolute-value-squared values of the complex components of state vectors as being literal probabilities for any of our hidden variables. According to our interpretation, and in accordance with the basic correspondence (17) above, we do not interpret the absolute-value-squared components of state vectors as describing literal probabilities for our hidden variables until some process—perhaps measurement- or environment-induced decoherence—turns them into the eigenvalues of some system’s density

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\(^{21}\) We address subtleties regarding degeneracies \( p_i (t) = p_j (t) (i \neq j) \) in Section III B 2.

\(^{22}\) In the traditional language of the modal interpretations [298], this correspondence is termed a (core) property ascription or an ontic ascription, our ontic states \( \Psi_t \) are called (core) properties or value states, and our objective epistemic states \( \{(p_t, \Psi_t)\}_t \) are called property sets, mathematical states, or dynamical states [299].
matrix. For these reasons, once we eventually do derive the Born rule in Section IV A 1, we will refer to the absolute-value-squared components of pre-measurement state vectors as empirical outcome probabilities, because it is only after a measurement device performs a specific measurement that they eventually become actualized as true epistemic probabilities for some system.

2. Probability Crossings and Degeneracies

One frequently cited anomaly in the purportedly one-to-one relationship (17) between objective epistemic states and density matrices concerns the issue of eigenvalue degeneracies. However, although one can easily picture a classical epistemic state evolving smoothly through a probability crossing

\[ p_i(t_c) = p_j(t_c) \]  

that occurs at some specific moment in time \( t = t_c \), eigenvalue crossings in density matrices are arrangements that require infinitely sharp—that is, measure-zero—fine-tuning and thus never realistically occur: They are co-dimension-three events in the abstract four-dimensional space of \( 2 \times 2 \) density-matrix blocks \([28, 172]\) and would therefore require that the off-diagonal entries vanish precisely when the diagonal elements exactly agree.

Just for purposes of illustration, consider the following static example: Although the two-particle, spin-singlet state vector

\[ |\Psi_{\text{EPR-Bohm}}\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]  

familiar from the EPR-Bohm thought experiment (to be discussed in detail in Section V B) would naïvely lead to degenerate \( 2 \times 2 \) reduced density matrices for each individual particle, setting up this state vector exactly would require unrealistically fine-tuning the total spin to \( S_{\text{tot}, z} = 0 \) to infinite precision—a measure-zero state of affairs. In any realistic scenario, there will unavoidably exist degeneracy-breaking deviations among the components of state vectors like \( |\Psi_{\text{EPR-Bohm}}\rangle \), so that, at best, the state vector actually takes the form

\[ |\Psi_{\text{EPR-Bohm}}\rangle = \frac{1}{\sqrt{2}} \left( (1 + \epsilon_1) |\uparrow\downarrow\rangle - (1 + \epsilon_2) |\downarrow\uparrow\rangle + \epsilon_3 |\uparrow\uparrow\rangle + \epsilon_4 |\downarrow\downarrow\rangle \right), \quad |\epsilon_1|, |\epsilon_2|, |\epsilon_3|, |\epsilon_4| \ll 1. \]  

Some have suggested \([10]\) that the problem with density matrices is how to interpret them when they exhibit exact degeneracies, which we have now seen are unphysical idealizations. However, as we will explain next, the actual danger for interpretations that are based on a correspondence like (17) (such as most modal interpretations) arises from the fact that density matrices can never realistically have exact degeneracies in the first place.

3. Near-Degeneracies and Eigenstate Swaps

The closest density-matrix counterpart to a probability crossing (20) is a near-degeneracy in which two probability eigenvalues reach a point of closest approach

\[ |p_i(t) - p_j(t)| \sim \rho_0 \xi > 0 \]  

at some specific moment in time \( t_0 \) and then turn around again, while their associated orthogonal eigenstates \( |\Psi_i(t)\rangle \perp |\Psi_j(t)\rangle \) undergo an ultra-fast eigenstate swap

\[ |\Psi_i(t)\rangle \rightarrow |\Psi_i(t + \delta t_{\text{swap}})\rangle \approx |\Psi_j(t)\rangle \perp |\Psi_i(t)\rangle, \]  

\[ |\Psi_j(t)\rangle \rightarrow |\Psi_j(t + \delta t_{\text{swap}})\rangle \approx |\Psi_i(t)\rangle \perp |\Psi_j(t)\rangle \]  

over an ultra-short time scale of order

\[ \delta t_{\text{swap}} \sim \rho_0 \xi \tau. \]  

23 Eigenstate swaps are called core property instabilities or fluctuations in \([298]\) and crossovers in \([172]\).
The complex-valued dimensionless quantity
\[ \xi \sim \exp(-\#\text{degrees of freedom}) \] (26)
has exponentially small magnitude in the total number of degrees of freedom of the system itself and of all other systems that substantially interact and entangle with it, and characterizes the size of the off-diagonal elements in the relevant \( 2 \times 2 \) density-matrix block in a basis in which the diagonal elements become momentarily equal at precisely \( t = t_0 \). The real-valued quantity \( \rho_0 \) is the would-be-degenerate eigenvalue in the idealized but unphysical measure-zero case \( \xi = 0 \). Hence, near \( t = t_0 \), the relevant \( 2 \times 2 \) density-matrix block takes the approximate form
\[ \left( \begin{array}{cc} \rho_0 + (t - t_0)/\tau & \rho_0 \xi \\ \rho_0 \xi^* & \rho_0 - (t - t_0)/\tau \end{array} \right) \subset \hat{\rho}. \] (27)
Meanwhile, the real-valued quantity \( \tau \), which has units of time, is the characteristic time scale over which the probability eigenvalues \( p_i(t) \) and \( p_j(t) \) would be changing in the absence of eigenstate-swap effects. To the approximate extent that we can speak of well-defined energies \( E \) for a system that is open and thus whose dynamics is not strictly unitary, ordinary time evolution corresponds roughly to the \( \exp(-iEt/h) \) relative phase factors familiar from elementary quantum theory, and thereby implies that
\[ \tau \sim \hbar/E. \] (28)
Taken seriously, eigenstate swaps (24) could conceivably cause large systems to undergo frequent but sudden fluctuations between macroscopically distinct ontic states over arbitrarily short time scales, and thus have long been considered to be serious ontological instabilities inherent in density-matrix-centered interpretations of quantum theory.\(^{24} \) However, as we will explain in detail when we discuss the internal dynamics of systems in our minimal modal interpretation of quantum theory in Section III D, eigenstate swaps fortunately turn out to be a mirage.

4. The Fundamentally Unobservable Nature of Eigenstate Swaps

As a first hint that we should not take eigenstate swaps (24) seriously as physically real phenomena for macroscopic systems, notice that a macroscopic system’s eigenstate-swap time scale \( \delta t_{\text{swap}} \) in (25) is parametrically small in the exponentially tiny quantity \( \xi \) described in (26), and is therefore always exponentially smaller than the system’s ordinary characteristic time \( \tau \) described in (28). Thus, the hierarchical discrepancy between a macroscopic system’s characteristic time scale \( \tau \) and its corresponding eigenstate-swap time scale \( \delta t_{\text{swap}} \) actually gets worse if we could somehow arrange for the system to approach the idealized limit of an exact degeneracy, because that would just make \( \delta t_{\text{swap}} \sim \xi \) even smaller. Similarly, if we were to attempt to hook the system up to a macroscopic measuring device with the goal of trying to observe the eigenstate swap experimentally—perhaps introducing a lot more energy in order to achieve a very fine temporal measurement resolution—then we would decrease the overall system’s ordinary characteristic time scale \( \tau \), but simultaneously we would vastly decrease \( \xi \) and thereby end up pushing the eigenstate-swap time scale even farther out of reach.

Interestingly, because a macroscopic quantum system’s Hilbert space has dimension
\[ \dim \mathcal{H} \sim \prod_{\text{degrees of freedom}} \left( \text{range of each degree of freedom} \right) \sim \exp(\#\text{degrees of freedom}) \] (29)
and its maximum entropy (18) goes as
\[ S \sim -\sum_{\text{basis}} \frac{1}{\dim \mathcal{H}} \log \frac{1}{\dim \mathcal{H}} \sim \log \dim \mathcal{H}, \] (30)
we see that the parameter \( \xi \) from (26) loosely corresponds to our error-entropy bound (3):
\[ \xi \sim \exp(-\#\text{degrees of freedom}) \sim e^{-S} \sim \text{minimum error.} \] (31)

\(^{24} \) As Vermaas writes on p. 133 of [298] in critiquing his own modal interpretation of quantum theory: “If one [takes eigenstate swaps seriously], it follows that the set of eigenprojections of a state that comes arbitrarily close to a degeneracy can change maximally in an arbitrarily small time interval \( t_2 - t_1 \). [...] [T]he set of core properties can change rapidly, resulting in an unstable property ascription during a finite time interval.” Later, on p. 260, he writes: “If a state has a spectral resolution which is nearly degenerate, then an arbitrarily small change of that state (by an interaction with the environment or by internal dynamics) can maximally change the set of the possible core properties. Perhaps this instability is one of the more serious defects of modal interpretations because, firstly, it can have consequences for their ability to solve the measurement problem. For even when a modal interpretation ascribes readings to a pointer at a specific instant, a small fluctuation of the state may mean that at the next instant the pointer possesses properties which are radically different to readings.”
This result provides additional justification for regarding both the distance of closest approach \((23)\) between the near-degenerate probability eigenvalues, as well as the eigenstate-swap time scale \(\delta t_{\text{swap}}\) in \((25)\), as being unobservably small.

Eigenstate swaps for macroscopic systems therefore remain effectively decoupled from the observable predictions of quantum theory. This decoupling open up an important window of opportunity for trying to smooth these kinds of instabilities out of existence altogether by a suitable choice of internal dynamics for quantum ontic states, with the added benefit of making the decoupling more manifest. We put forward just such a proposal in Section III.D, where we show explicitly that our choice ensures that a macroscopic system’s time-evolving actual ontic state essentially never undergoes eigenstate swaps.

C. Parent Systems, Subsystems, and the Partial-Trace Operation

Up to now, our discussion has centered on the case in which we consider just one particular system of interest. Of course, any study of quantum theory must begin with some particular system, but there will generally be other systems that we can consider simultaneously, and each may naturally be interpreted as a parent system enclosing our original system, or as a subsystem, or as an adjacent system, or as something else entirely. In this section, we motivate and explain how our minimal modal interpretation of quantum theory defines the relationship between the objective epistemic states of parent systems and those of their subsystems, especially in situations like \((15)\) that feature quantum entanglement.

1. The Partial-Trace Operation

Recalling our discussion of classical parent systems and subsystems in Section III.A.8, and given an epistemic state \(p_{A+B}\) for a classical composite parent system \(A+B\), we can naturally obtain reduced (or marginal) epistemic states \(p_A\) and \(p_B\) for the respective subsystems \(A\) and \(B\) by the familiar partial-sum (or marginalization) operation:

\[
p_A (a) = \sum_b p_{A+B} (a,b), \quad p_B (b) = \sum_a p_{A+B} (a,b).
\]

Our next goal will be to motivate an analogous quantum operation, called the partial-trace operation, that relates the objective epistemic states of parent systems with those of their subsystems, and we will see that density matrices play a crucial role.

It is important to note that we will make no appeals to the Born rule or Born-rule-based averages in justifying the definition of the partial-trace operation. Indeed, we will ultimately find that we can derive the Born rule and its corollaries (as well as possible corrections that are invisible in the traditional Copenhagen interpretation) from the partial-trace operation’s deeper principles of logical self-consistency and its relationship with classical partial sums.

Central to our entire interpretation is the notion that every quantum system has an ontology and objective epistemology defined through a density matrix of its own.\(^{25}\) Given a quantum objective epistemic state for a parent system \(W = A + B + C + D + \cdots\) consisting of an arbitrary number of subsystems \(A, B, C, D, \ldots\) and associated to some density matrix \(\hat{\rho}_W\), we therefore require a universal prescription for nontrivially assigning unit-trace, positive semi-definite density matrices to all the various possible definable subsystems \(A, B, C, D, \ldots, A+B, A+C, A+D, \ldots, A+B+C, A+B+D, \ldots, A+B+C+D, \ldots\) in such a way that there is no dependence on our arbitrary choice of orthonormal basis for each of the individual Hilbert spaces of the subsystems, nor on the arbitrary order in which we could imagine defining a descending sequence of density matrices for the subsystems. We must, for instance, end up with the same final density matrix \(\hat{\rho}_A\) for subsystem \(A\) whether we choose to define intermediate density matrices according to the sequence \(W \mapsto A+B+C \mapsto A+B \mapsto A\) or according to the sequence \(W \mapsto A+B+D \mapsto A+D \mapsto A\). Equivalently, any arbitrarily complicated diagram typified by the following example must internally commute:

\[
\begin{array}{ccc}
\hat{\rho}_A, & \hat{\rho}_{A+B}, & \hat{\rho}_{A+B+C} \\
\hat{\rho}_C & \hat{\rho}_{B+C} & \\
\hat{\rho}_{A+B} & \hat{\rho}_B, & \\
\end{array}
\]

\(^{25}\) In particular, it is important to remember that in our interpretation of quantum theory, the density matrix or objective epistemic state of any other system, even that of a parent system, does not directly define the ontology or objective epistemology of a system of interest.
Furthermore, our final result for any one subsystem’s density matrix—say, $\hat{\rho}_A$, for subsystem $A$—cannot depend on our arbitrary choice among the continuously infinite different ways that we could have defined the rest of the subsystems $B, C, \ldots$. In particular, we must ultimately get the same answer $\hat{\rho}_A$ whether we decompose $\mathcal{H}_W$ as $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \cdots$ and then define $\hat{\rho}_A$ by the sequence $W \mapsto A + B \mapsto A$, or whether we instead decompose $\mathcal{H}_W$ as $\mathcal{H}_A \otimes \mathcal{H}_{B'} \otimes \mathcal{H}_{C'} \otimes \cdots$ for some different definitions $B', C', \ldots$ of the other subsystems and then define $\hat{\rho}_A$ by the sequence $W \mapsto A + B' \mapsto A$. That is, the continuous infinity of possible diagrams generalizing the following example must each internally commute:

$$\begin{align*}
\hat{\rho}_{A+B+C} &= \hat{\rho}_{A+B'+C'} \\
\hat{\rho}_{A+B} &\leftrightarrow \hat{\rho}_{A+B'} \\
&\cdots \\
\hat{\rho}_A \\
\end{align*}$$

(34)

It is remarkable that a universal prescription meeting all these nontrivial requirements of logical self-consistency exists at all, much less that it turns out to be so straightforward. To define this prescription explicitly and to ensure that it is indeed nontrivial and has the correct behavior in the classical regime, we require furthermore that it should reduce to classical partial sums (32) if the density matrix of a parent system $A + B$ happens to be diagonal in the tensor-product basis $|a, b\rangle = |a\rangle \otimes |b\rangle$ with probability eigenvalues $p_{A+B} (a, b)$—that is, if there is no quantum entanglement in the sense of (15) between the subsystems $A$ and $B$:

$$\begin{align*}
\hat{\rho}_{A+B} &= \sum_{a,b} p_{A+B} (a,b) \langle a \mid \otimes \langle b \mid \\
\implies \hat{\rho}_A &= \sum_a \left( \sum_b p_{A+B} (a,b) \right) |a\rangle \langle a|, \quad \hat{\rho}_B = \sum_b \left( \sum_a p_{A+B} (a,b) \right) |b\rangle \langle b|. \\
\end{align*}$$

(35)

Additionally, we require that the prescription should be linear over the space of operators on the parent system’s Hilbert space, in keeping with the linearity property that holds for classical partial sums over convex combinations $xp_{A+B} + yp'_{A+B}$ ($x, y > 0, x + y = 1$) of classical epistemic states $p_{A+B}$ and $p'_{A+B}$.

The unique resulting partial-trace operation is well known and surprisingly simple to describe. We begin by considering a parent system $C$ that we can regard as consisting of just two subsystems $A$ and $B$ (where either $A$ or $B$ could be parent systems encompassing subsystems of their own), and then we write the density matrix $\hat{\rho}_{A+B}$ for $C = A + B$ in the tensor-product basis $|a\rangle \otimes |b\rangle$, which won’t generally be its diagonalizing basis, namely, if $A$ and $B$ are entangled:

$$\hat{\rho}_{A+B} = \sum_{a,b,a',b'} \rho_{A+B} ((a,b), (a',b')) |a\rangle \otimes |b\rangle \langle a' \mid \otimes \langle b'|.$$  

(36)

Linearity and consistency with partial sums in the special case

$$\rho_{A+B} ((a,b), (a',b')) = p_{A+B} (a,b)$$

(37)

then immediately imply that the partial trace down to either subsystem $A$ or $B$ must be defined by formally “turning around” the bras and kets of the other subsystem and evaluating the resulting inner products, where the result then defines the reduced density matrix $\hat{\rho}_A$ or $\hat{\rho}_B$ of the subsystem $A$ or $B$, respectively:

$$\begin{align*}
\hat{\rho}_A &\equiv \text{Tr}_B [\hat{\rho}_{A+B}] = \sum_{a,b,a',b'} \rho_{A+B} ((a,b), (a',b')) |a\rangle \langle a' \mid \otimes |b\rangle \langle b'| \\
&= \sum_{a,a'} \left( \sum_b \rho_{A+B} ((a,b), (a',b)) \right) |a\rangle \langle a|, \\
\hat{\rho}_B &\equiv \text{Tr}_A [\hat{\rho}_{A+B}] = \sum_{a,b,a',b'} \rho_{A+B} ((a,b), (a',b')) |a\rangle \langle a' \mid \otimes |b\rangle \langle b'| \\
&= \sum_{b,b'} \left( \sum_a \rho_{A+B} ((a,b), (a',b')) \right) |b\rangle \langle b|. \\
\end{align*}$$

(38)
That is, the matrix elements of $\hat{\rho}_A$ and $\hat{\rho}_B$ are given in the respective orthonormal bases $|a\rangle$ for the Hilbert space $\mathcal{H}_A$ of subsystem $A$ and $|b\rangle$ for the Hilbert space $\mathcal{H}_B$ of subsystem $B$ by the natural matrix-generalizations of classical partial sums (32):

$$\rho_A(a,a') = \sum_b \rho_{A+B}((a,b),(a',b)); \quad \rho_B(b,b') = \sum_a \rho_{A+B}((a,b),(a',b)).$$

(39)

2. An Example

With the formula (38) for partial traces in hand, we are finally ready to answer the questions that we posed in Section III A 8 regarding the respective ontic states of the two subsystems $A$ and $B$ belonging to a composite parent system $A+B$ described by the entangled state vector (15),

$$|\Psi_{A+B}\rangle = \alpha |\Psi_{A,1}\rangle |\Psi_{B,1}\rangle + \beta |\Psi_{A,2}\rangle |\Psi_{B,2}\rangle,$$

where we assume just for simplicity that $|\Psi_{A,1}\rangle \perp |\Psi_{A,2}\rangle$ and $|\Psi_{B,1}\rangle \perp |\Psi_{B,2}\rangle$ in (15).

Given that $A+B$ has the definite ontic state $|\Psi_{A+B}\rangle$ defined by (15), the density matrix of $A+B$ is

$$\hat{\rho}_{A+B} = |\Psi_{A+B}\rangle\langle \Psi_{A+B}| = (\alpha |\Psi_{A,1}\rangle |\Psi_{B,1}\rangle + \beta |\Psi_{A,2}\rangle |\Psi_{B,2}\rangle)(\alpha^* \langle \Psi_{A,1}| \langle \Psi_{B,1}| + \beta^* \langle \Psi_{A,2}| \langle \Psi_{B,2}|),$$

thereby implying from the partial-trace prescription (38) that the respective reduced density matrices for $A$ and $B$ are

$$\hat{\rho}_A = |\alpha|^2 |\Psi_{A,1}\rangle\langle \Psi_{A,1}| + |\beta|^2 |\Psi_{A,2}\rangle\langle \Psi_{A,2}|,$n

$$\hat{\rho}_B = |\alpha|^2 |\Psi_{B,1}\rangle\langle \Psi_{B,1}| + |\beta|^2 |\Psi_{B,2}\rangle\langle \Psi_{B,2}|.$$ (41)

Hence, according to our interpretation of quantum theory, the possible ontic states of subsystem $A$ are $\Psi_{A,1}$ and $\Psi_{A,2}$, and the possible ontic states of subsystem $B$ are $\Psi_{B,1}$ and $\Psi_{B,2}$, with respective epistemic probabilities given by

$$p_A(\Psi_{A,1}) = |\alpha|^2, \quad p_A(\Psi_{A,2}) = |\beta|^2,$n

$$p_B(\Psi_{B,1}) = |\alpha|^2, \quad p_B(\Psi_{B,2}) = |\beta|^2.$$ (42)

3. Correlation and Entanglement

Given a parent quantum system $A+B$ consisting of two subsystems $A$ and $B$, we say that $A$ and $B$ are correlated if the parent system’s density matrix $\hat{\rho}_{A+B}$ is not a simple tensor product of the reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$:

$$A \text{ and } B \text{ correlated : } \hat{\rho}_{A+B} \neq \hat{\rho}_A \otimes \hat{\rho}_B.$$ (43)

Correlation is a property that exists even in classical physics and corresponds to the failure of the epistemic probabilities for the composite system $A+B$ to factorize: $p(a,b) \neq p(a)p(b)$.

But quantum systems are capable of an even stronger property known as entanglement, which we first encountered in (15) and now define more generally within the context of our interpretation of quantum theory as describing the case in which the parent system’s density matrix $\hat{\rho}_{A+B}$ is not diagonal in the tensor-product basis $|a,b\rangle = |a\rangle \otimes |b\rangle$ corresponding to the two subsystems:

$$A \text{ and } B \text{ entangled : } \text{eigenbasis of } \hat{\rho}_{A+B} \text{ is not } |a,b\rangle = |a\rangle \otimes |b\rangle.$$ (44)

It is precisely due to entanglement that the partial-trace operation (38) does not generically reduce to simple partial sums (32), and, indeed, is arguably the very reason why we need density matrices in quantum theory in the first place.26

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26 Note that the traditional formulation of quantum theory—unlike our own interpretation—does not single out the diagonalizing eigenbasis of a density matrix as being fundamentally preferred. Outside of the idealized case of a quantum system belonging to a parent system in an exactly pure state, entanglement (44) therefore ceases to be a more than a purely formal property of density matrices known as non-separability, and there exist various measures, such as quantum discord [168, 231], for characterizing the quantumness of the resulting correlations.
4. Proper Mixtures and Improper Mixtures

Recall from their definition in Section III A 7 that proper mixtures refer to subjective epistemic states, by which we mean epistemic states that merely encode subjective uncertainty about a system’s actual ontic state and make perfect sense even in the context of classical physics. By contrast, improper mixtures, as we defined them in Section III A 10, refer to epistemic states that encode objective uncertainty arising from quantum entanglements with other systems, in addition to any subjective uncertainty that may also be present.

In the traditional language of quantum theory, there exists a sharp conceptual distinction between proper mixtures and improper mixtures, whereas our minimal modal interpretation of quantum theory blurs that distinction to a significant degree. Indeed, we regard both kinds of mixtures as describing an epistemic state that hides the system’s actual ontic state.

However, looking back at our basic correspondence (17), note that we have so far introduced density matrices solely to describe objective epistemic states—that is, fully objective improper mixtures. (We will extend the use of density matrices to more general epistemic states in Section IV B only after deriving the Born rule.) Furthermore, it is important to keep in mind that it really matters whether or not a system’s actual ontic state is hiding behind a density matrix describing a truly nontrivial objective epistemic state: There can exist physical differences between the case of a system whose density matrix happens to be pure \( \hat{\rho} = |\Psi\rangle \langle \Psi| \) and a system whose density matrix is mixed but whose actual underlying ontic state nonetheless happens to be \( \Psi \).

For example, consider a composite system \( A + B \) whose density matrix has the pure form

\[
\hat{\rho}_{A+B} = |1\rangle |\Psi\rangle \langle 1| |\Psi\rangle, \tag{45}
\]

where 1 is an allowed state of \( A \) and \( \Psi \) is an allowed state of \( B \). We would naturally conclude not only that the parent system \( A + B \) has actual ontic state \( (1, \Psi) \) with unit epistemic probability, but, moreover, that the subsystems \( A \) and \( B \) themselves have respective actual ontic states 1 and \( \Psi \) each with unit epistemic probability, for the simple reason that their own reduced density matrices are respectively \( \hat{\rho}_A = |1\rangle \langle 1| \) and \( \hat{\rho}_B = |\Psi\rangle \langle \Psi| \).

But now suppose instead that the parent system’s density matrix has the mixed form

\[
\hat{\rho}_{A+B} = p_1 |1\rangle |\Psi\rangle \langle 1| |\Psi\rangle + p_2 |2\rangle |\Phi\rangle \langle 2| |\Phi\rangle, \tag{46}
\]

where we assume that \( |1\rangle \) is orthogonal to \( |2\rangle \)—that is, \( \langle 1|2 \rangle = 0 \)—but not necessarily that \( |\Psi\rangle \) is orthogonal to \( |\Phi\rangle \). (Note, however, that \( |1\rangle |\Psi\rangle \) is nonetheless orthogonal to \( |2\rangle |\Phi\rangle \).) Remember that in our interpretation of quantum theory, we determine a system’s possible ontic states and associated epistemic probabilities from the orthonormal spectrum of the system’s own density matrix. Hence, although it would be correct to say that the actual ontic state of the parent system could be \( (1, \Psi) \) with epistemic probability \( p_1 \) or \( (2, \Phi) \) with epistemic probability \( p_2 \), it would be incorrect to conclude that the actual ontic state of the subsystem \( B \) is \( \Psi \) with epistemic probability \( p_1 \) or \( \Phi \) with epistemic probability \( p_2 \), because \( \langle \Psi | \Phi \rangle \neq 0 \) means that \( \Psi \) and \( \Phi \) are not mutually exclusive possibilities and therefore cannot correspond to the mutually orthogonal eigenstates of the density matrix of subsystem \( B \). As a further consequence, it would also be incorrect to say that if the actual ontic state of the parent system happens to be, say, \( (1, \Psi) \), then the actual ontic state of subsystem \( B \) must be \( \Psi \).

The reason for all this trouble is, unsurprisingly, entanglement (44): The diagonalizing basis for \( \hat{\rho}_{A+B} \) given in (46) is not the “correct” tensor-product orthonormal basis that corresponds to our original tensor-product factorization \( \mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B \) of the parent system’s Hilbert space \( \mathcal{H}_{A+B} \) into the Hilbert spaces \( \mathcal{H}_A \) of subsystem \( A \) and \( \mathcal{H}_B \) of subsystem \( B \). Of course, if the inner product \( \langle \Psi | \Phi \rangle \) is very small in magnitude, then we are welcome to choose a slightly different tensor-product decomposition \( \mathcal{H}_{A+B} = \mathcal{H}_{A'} \otimes \mathcal{H}_{B'} \) in order to make the diagonalizing basis for \( \hat{\rho}_{A+B} \) the tensor-product basis \( |a', b'\rangle = |a'\rangle \otimes |b'\rangle \) corresponding to suitably redefined subsystems \( A' \) and \( B' \). In that case, we could immediately read off from the eigenvalues of \( \hat{\rho}_{A+B} \) the corresponding epistemic probabilities for the subsystems \( A' \) and \( B' \), and declare that if the actual ontic state of the composite system happens to be, say, \( (a', b') \), then the actual ontic states of \( A' \) and \( B' \) must respectively be \( a' \) and \( b' \).

However, if we insist on working with our original subsystems \( A \) and \( B \), then it might seem that all we can conclude about subsystem \( B \) from the density matrix (46) of the parent system \( A + B \) is that the reduced density matrix of \( B \) is given by the partial trace \( \hat{\rho}_B = \text{Tr}_A [\hat{\rho}_{A+B}] \), as defined in (38). In particular, the fact that the parent system \( A + B \) has a specific actual ontic state might not appear to imply anything about the actual ontic state of \( B \) alone.

There is a seemingly obvious connection between the actual ontic state of the parent system \( A + B = A' + B' \) and the actual ontic state of \( B' \), but does the actual ontic state of the parent system tell us anything about the actual ontic state of \( B \), which is presumably “just a slightly different version” of the same subsystem as \( B' \)? Without addressing this question, our interpretation of quantum theory would seem to be woefully inadequate, as our notion of an actual ontic state would be infinitely sensitive to arbitrarily small (and thus observationally meaningless) redefinitions of
subsystems.\textsuperscript{27} We will study and resolve this important issue in Section III D when we explicitly define the relationship between the ontic states of parent systems and the ontic states of their subsystems. In Section III E, we will introduce the relatively unexplored notion of “subsystem spaces” to describe the continuously infinite set of different ways of defining different versions of a particular subsystem.

5. Imperfect Tensor-Product Factorizations, Truncated Hilbert Spaces, Approximate Density Matrices, and Unstable Systems

Given a system’s Hilbert space, there will generally exist many possible tensor-product factorizations defining valid subsystems. But, going the other way, there may be cases in which an \textit{a priori} desired choice of subsystem cannot be realized as an \textit{exact} tensor-product factorization of a given parent system’s Hilbert space.

For example, the Hilbert space of Nature is presently unknown, and so we have no reason to believe that it admits a simple tensor-product factorization $\mathcal{H}_{\text{Nature}} = \mathcal{H}_{\text{system}} \otimes \mathcal{H}_{\text{other}}$ that includes the exact Hilbert space $\mathcal{H}_{\text{system}}$ of any of the kinds of systems known today. Nonetheless, we profitably employ such Hilbert spaces all the time. Indeed, even for the familiar example of a quantum field theory, we know that the full system’s Hilbert-Fock space $\mathcal{H}_{\text{Fock}} = \mathcal{H}_{0 \text{ particles}} \oplus \mathcal{H}_{1 \text{ particle}} \oplus \mathcal{H}_{2 \text{ particles}} \oplus \cdots$ does not neatly tensor-product-factorize as $\mathcal{H}_{\text{system}}$, where $\mathcal{H}_{1 \text{ particle}}$ is the Hilbert space of a single particle of a certain species, and yet we successfully make use of one-particle Hilbert spaces $\mathcal{H}_{1 \text{ particle}}$ whenever we wish to study nonrelativistic or semi-relativistic physics.

We clearly need a prescription for obtaining \textit{approximate} subsystem Hilbert spaces even when the full Hilbert space of the parent system is either unknown or does not exactly permit the desired tensor-product factorization. That prescription, which is implicit in all the aforementioned examples, consists of first \textit{truncating} the full Hilbert space—or, if the full Hilbert space is unknown, then regarding it as \textit{already} being appropriately truncated—in order to make the desired tensor-product factorization possible, and only then taking appropriate partial traces.

An immediate corollary is that the restricted density matrix on the parent system’s truncated Hilbert space, and thus the associated reduced density matrix of the subsystem as well, are only \textit{approximate} objects and, despite still being positive semi-definite, will no longer have exactly unit trace, corresponding to the statement that their probability eigenvalues no longer add up all the way to unity: $\sum_i p_i < 1$. The discrepancy $(1 - \sum_i p_i) \in [0, 1]$ arises from the absence of the states in which our subsystem does not exist (or no longer exists)—the discrepancy is directly related to our inability to capture the system’s actual ontic state among the possible ontic states $\Psi$, remaining in our description of the system—and so we naturally interpret the discrepancy as representing the probability that our system, which we now recognize as being unstable, has actually decayed.

D. Quantum Conditional Probabilities

We are now ready to say more about the time evolution and dynamics of ontic states and objective epistemic states, as well as address lingering questions regarding eigenstate swaps (24) that we first encountered in Section III B 4 and parent-subsystem discrepancies arising from nontrivial objective epistemic states that we first encountered in Section III A 8.

1. Classical Dynamics and (Multi-)Linear Dynamical Mappings

Recall from Section II B 6 that classical systems with well-defined dynamics are those that, to an acceptable level of approximation, possess an ontic-level dynamical mapping (4) that is independent of the system’s epistemic state and naturally lifts to a multilinear dynamical mapping (5) relating initial and final epistemic states as well. In the special case of Markovian dynamics (7)—that is, for a system whose dynamical mapping is first order, meaning that it requires the input of initial data at only a \textit{single} initial time—the ontic-level dynamics takes the general form

$$p(q; t') \rightarrow p(q'; t' | q; t),$$

\textbf{\textit{initial}} \textbf{data} \hspace{1cm} \textbf{\textit{final}} \textbf{data} \hspace{1cm} \textbf{\textit{conditional}} \textbf{probabilities}

\textsuperscript{27} Indeed, a potential instability somewhat analogous to eigenstate swaps (24) arises in the context of small changes in the definition of a subsystem, as explored in [28, 100]. As Vermaas writes on p. 134 of [298]: “A final remark concerns yet another source for incorrect property ascriptions. In this book I always assume that one can precisely identify the systems. Consequently, one can also precisely identify the composites of these systems. If, however, one proceeds the other way round and starts with a set of composites, one has to answer the question of how exactly to factor these composites into disjoint subsystems. In Bacchagullupi, Donald and Vermaas (1995, Example 7.3) it is proved that the property ascription to a subsystem can depend with high sensitivity on the precise identification of that subsystem.”
and the corresponding epistemic-level dynamics (5) becomes a simple linear mapping

$$ p (\cdot ; t' | ; t) : p (q'; t') = \sum_q p (q'; t | q) p (q; t), $$ \hspace{1cm} (48)

which we can regard as a kind of dynamical Bayesian propagation formula.

Consider now a classical system $Q$ with a configuration space $C_Q = \{q\}$, and suppose that $Q$ is an open subsystem of some larger classical system $W = Q + E$, where $E$ is the environment of $Q$ inside $W$ and has configuration space $C_E = \{e\}$ so that the configuration space of $W$ is the Cartesian product $C_W = C_Q \times C_E = \{w = (q, e) | q \in C_Q \text{ and } e \in C_E\}$. Then even if the parent system $W$ as a whole has well-defined dynamics in the sense of a linear mapping $p_W (\cdot ; t' | ; t)$ that is independent of the epistemic state of $W$ and that we assume for simplicity is of the Markovian form (48), the open subsystem $Q$ of $W$ will not necessarily have well-defined dynamics of its own.

Indeed, by expressing the time evolution for the epistemic state of $Q$ over the time interval from $t$ to $t'$ as

$$ p_Q (q'; t') = \sum_{e'} p_W (w' = (q', e') ; t') $$

$$ = \sum_{e', w} p_W (w' = (q', e') ; t' | w = (q, e) ; t) p_W (w = (q, e) ; t) $$

$$ = \sum_q \sum_{e', e} p_W (w' = (q', e') ; t' | w = (q, e) ; t) \left( \frac{p_W (w = (q, e) ; t)}{p_Q (q; t)} \right) p_Q (q; t), $$ \hspace{1cm} (49)

we see immediately that the failure of $Q$ to possess its own dynamics is characterized by the complicated, nonlinear dependence of the ratio in parentheses (both its numerator and denominator) on the epistemic state $p_Q (q; t)$ of $Q$. From Bayes' theorem, we readily identify this ratio as being the conditional probability that the environment $E$ is in the ontic state $e$ given that the subsystem $Q$ is in the ontic state $q$ at the same time $t$:

$$ \frac{p_W (w = (q, e) ; t)}{p_Q (q; t)} = p_{E|Q} (e; t | q; t). $$ \hspace{1cm} (50)

Notice that the open-subsystem evolution law (49) provides us with a coarse-grained or effective (albeit nonlinear) notion of dynamics $p_{Q \subset W} (\cdot ; t' | ; t)$ for $Q$,

$$ p_{Q \subset W} (\cdot ; t' ; t) : p_{Q \subset W} (q'; t') = \sum_q p_{Q \subset W} (q'; t | q) p_Q (q; t), $$ \hspace{1cm} (51)

where we have defined the coarse-grained or effective conditional probabilities $p_{Q \subset W} (q'; t | q)$ to be the factor appearing in brackets in (49)—that is, in accordance with Bayes' theorem, by multiplying each parent-system conditional probability $p_W (w; t')$ by the epistemic probability $p_W (w; t)$ for $W$ at the time $t$, marginalizing over the environment $E$ at both the initial and final times, and then conditioning on $Q$ at the time $t$:

$$ p_{Q \subset W} (q'; t' | q) = \frac{p_{Q \subset W} (q'; t \text{ and } q) p_Q (q; t)}{p_Q (q; t)} $$

$$ = \frac{1}{p_Q (q; t)} \sum_{e', e} p_W (w' = (q', e') ; t' \text{ and } w = (q, e) ; t) $$

$$ = \sum_{e', e} p_W (w' = (q', e') ; t' | w = (q, e) ; t) \left( \frac{p_W (w = (q, e) ; t)}{p_Q (q; t)} \right). $$ \hspace{1cm} (52)

In the case in which the correlations between $Q$ and its environment $E$ inside $W$ wash out over some short characteristic time scale $\delta t_Q \ll t' - t$—say, through irreversible thermal radiation into outer space—the epistemic probabilities for $W$ approximately factorize,

$$ p_W (w = (q, e)) \approx p_Q (q) p_E (e), $$ \hspace{1cm} (53)
and so the ratio (50) appearing in parentheses in the evolution equation (49) for \(Q\) reduces to

\[
p_{E|Q}(e; t|q; t) = \frac{p_{W}(w = (q, e); t)}{p_{Q}(q; t)} = \frac{p_{Q}(q; t)p_{E}(e; t)}{p_{Q}(q; t)} = p_{E}(e; t). \tag{54}
\]

Hence, the dynamical mapping (51) for \(Q\) now defines properly linear dynamics \(p_{Q \subseteq W}(\cdot; t'|\cdot; t) = p_{Q}(\cdot; t'|\cdot; t)\) for \(Q\) on its own,

\[
p_{Q}(\cdot; t'|\cdot; t) : p_{Q}(q'; t) = \sum_{q} p_{Q}(q; t|q') p_{Q}(q; t) \text{ for } t' - t \gg \delta t_{Q}, \tag{55}
\]

where (52) has reduced to

\[
p_{Q}(q'; t'|q; t) = \sum_{e', e} p_{W}(w' = (q', e'); t'|w = (q, e); t) p_{E}(e; t). \tag{56}
\]

Nonetheless, even if the dynamics of the parent system \(W\) is deterministic, so that its own dynamical conditional probabilities \(p_{W}(w'; t'|w; t)\) are trivial and relate each initial ontic state \(w\) to a unique final ontic state \(w'\) with unit probability, keep in mind that the presence of the environment’s instantaneous epistemic probabilities \(p_{E}(e; t)\) in the formula (56) for the conditional probabilities \(p_{Q}(q'; t'|q; t)\) generally implies that the dynamics of \(Q\) is stochastic.

Notice that the characteristic time scale \(\delta t_{Q}\) sets a natural short-time cutoff on the dynamics for our open subsystem \(Q\). Although \(Q\) is a sensible system at the level of kinematics over time scales shorter than \(\delta t_{Q}\), it does not possess truly well-defined dynamics on time scales shorter than \(\delta t_{Q}\).

2. Linear CPT Dynamical Mappings

Interestingly, the classical linear dynamical mapping (48) has a much-studied quantum counterpart that generalizes the “deterministic” unitary Schrödinger dynamics of closed quantum systems to a form of linear stochastic dynamics governing the time evolution of a large class of open quantum systems.\(^{28}\) This quantum-dynamical mapping consists of a linear function \(\mathcal{E}^{t'\leftarrow t}\) relating an open system’s initial and final density matrices at respectively initial and final times \(t \leq t'\):

\[
\hat{\rho}(t) \mapsto \hat{\rho}(t') = \mathcal{E}^{t'\leftarrow t}[\hat{\rho}(t)]. \tag{57}
\]

Notice that \(\mathcal{E}^{t'\leftarrow t}\) defines dynamics on density matrices, as opposed to classical-type dynamics (48) on objective epistemic states directly. Functions like \(\mathcal{E}^{t'\leftarrow t}\) that map operators to operators are called superoperators.

Intuitively, if (57) describes a mapping taking in an initial density matrix and producing a final density matrix, then it should, in particular, preserve the unit-trace of density matrices:

\[
\text{Tr}[\hat{\rho}(t)] = 1 \implies \text{Tr}[\mathcal{E}^{t'\leftarrow t}[\hat{\rho}(t)]] = 1. \tag{58}
\]

The assumed linearity of \(\mathcal{E}^{t'\leftarrow t}\), combined with its preservation (58) of the trace of unit-trace operators, means that it must in fact preserve the traces of all operators \(\hat{O}\), and thus must in general be a trace-preserving (“T” or “TP”) mapping:

\[
\mathcal{E}^{t'\leftarrow t} \text{ is T: } \text{Tr}[\mathcal{E}^{t'\leftarrow t}[\hat{O}]] = \text{Tr}[\hat{O}] \text{ for all } \hat{O}. \tag{59}
\]

Furthermore, (57) should be a positive mapping, meaning that it preserves the positive-semi-definiteness of density matrices:

\[
\hat{\rho}(t) \geq 0 \implies \mathcal{E}^{t'\leftarrow t}[\hat{\rho}(t')] \geq 0. \tag{60}
\]

---

\(^{28}\) See [80, 185, 278] for early work in this direction, and see [257] for a modern pedagogical review.
If we were to imagine introducing an arbitrary, causally disconnected ancillary system with trivial dynamics determined by the identity mapping \( \text{id} [\cdot] \) on operators, then it would be reasonable to impose the (nontrivial) requirement that the resulting composite dynamics \( \mathcal{E}^{t' \leftarrow t} [\cdot] \otimes \text{id} [\cdot] \) governing the pair of systems should likewise be a positive mapping, a condition on our original dynamical mapping \( \mathcal{E}^{t' \leftarrow t} [\cdot] \) called complete positivity (“CP”):

\[
\mathcal{E}^{t' \leftarrow t} [\cdot] \text{ is CP: } \mathcal{E}^{t' \leftarrow t} [\cdot] \otimes \text{id} [\cdot] \text{ is a positive mapping. (61)}
\]

We are therefore led to the study of linear completely-positive-trace-preserving (“CPT”) dynamical mappings of density matrices. In generalizing unitary dynamics to linear CPT dynamics in this manner, as is necessary in order to account for the crucial and non-reductive quantum relationships between parent systems and their subsystems, note that we are not proposing any fundamental modification to the dynamics of quantum theory, such as in GRW-type spontaneous-localization models [7, 37, 137, 235, 236, 315], but are simply accommodating the fact that generic mesoscopic and macroscopic quantum systems are typically open to their environments to some nonzero degree. Indeed, linear CPT dynamical mappings are widely used in quantum chemistry as well as in quantum information science, in which they are known as quantum operations; when specifically regarded as carriers of quantum information, they are usually called quantum channels.

3. Quantum Conditional Probabilities

To motivate our defining formula for the generalized quantum counterpart to the classical conditional probabilities appearing in (47), we begin by considering a collection of mutually disjoint quantum systems \( Q_1, \ldots, Q_n \) that we can identify as being subsystems of some parent system \( W = Q_1 + \cdots + Q_n \), where we include the possibility that \( Q_2 = \cdots = Q_n = \emptyset \) are all trivial so that \( W = Q_1 \). Suppose furthermore that the parent system \( W \) has dynamics over a given time interval \( \Delta t \equiv t' - t \geq 0 \), meaning that we can approximate the time evolution of the density matrix \( \hat{\rho}_W \) for \( W \) over the time interval \( \Delta t \) by a linear CPT dynamical mapping (57):

\[
\hat{\rho}_W (t) \mapsto \hat{\rho}_W (t') = \mathcal{E}^{t' \leftarrow t}_W [\hat{\rho}_W (t)].
\]

We can expand the density matrix \( \hat{\rho}_W (t) \) of the parent system \( W \) at the initial time \( t \) in terms of its probability eigenvalues \( p_w (w; t) \equiv p_{W,w} (t) \) and the projection operators (or eigenprojectors) \( \hat{P}_W (w; t) \equiv |\Psi_W (w; t)\rangle \langle \Psi_W (w; t)| \) onto its orthonormal eigenbasis:

\[
\hat{\rho}_W (t) = \sum_w p_w (w; t) |\Psi_W (w; t)\rangle \langle \Psi_W (w; t)| = \sum_w p_w (w; t) \hat{P}_W (w; t).
\]

Similarly, selecting subsystem \( Q_1 \) without loss of generality, we can expand its (reduced) density matrix \( \hat{\rho}_{Q_1} (t') = \text{Tr}_{Q_2 + \cdots + Q_n} [\hat{\rho}_W (t')] \) at the final time \( t' \) in terms of its own probability eigenvalues \( p_{Q_1, i_1} (i_1; t') \equiv p_{Q_1, i_1} (t') \) and its own eigenprojectors \( \hat{P}_{Q_1, i_1} (i_1; t') = |\Psi_{Q_1, i_1} (i_1; t')\rangle \langle \Psi_{Q_1, i_1} (i_1; t')| \):

\[
\hat{\rho}_{Q_1} (t') = \text{Tr}_{Q_2 + \cdots + Q_n} [\hat{\rho}_W (t')] = \sum_{i_1} p_{Q_1, i_1} (i_1; t') |\Psi_{Q_1, i_1} (i_1; t')\rangle \langle \Psi_{Q_1, i_1} (i_1; t')| = \sum_{i_1} p_{Q_1, i_1} (i_1; t') \hat{P}_{Q_1, i_1} (i_1; t').
\]

We can also expand each of the identity operators \( \hat{1}_{Q_2}, \ldots, \hat{1}_{Q_n} \) on the respective Hilbert spaces of the other mutually disjoint subsystem \( Q_2, \ldots, Q_n \) in terms of their own respective eigenprojectors \( \hat{P}_{Q_2, i_2} (i_2; t'), \ldots, \hat{P}_{Q_n, i_n} (i_n; t') \),

\[
\hat{1}_{Q_2} = \sum_{i_2} \hat{P}_{Q_2, i_2} (i_2; t'), \ldots, \hat{1}_{Q_n} = \sum_{i_n} \hat{P}_{Q_n, i_n} (i_n; t'),
\]

where we’ll see that symmetry with \( Q_1 \) and consistency with the linear CPT dynamical mapping (62) for the parent system \( W \) necessitates that these projection operators are all evaluated at the same final time \( t' \).

29 Despite the unfortunate but conventional acronym, “CPT” here should not be confused with the (C)harge-(P)arity-(T)ime-reversal transformations that are familiar from particle physics; some authors use the acronym “CPTP” instead.

30 Given the generically local interactions found in realistic fundamental physical models like quantum field theories and the resulting tendency of decoherence to reduce open systems to states of relatively well-defined spatial position [184] automatically, proponents of GRW-type spontaneous-localization models must justify why their modifications to quantum theory aren’t redundant [182] and how we would experimentally distinguish those claimed modifications from the more prosaic effects of decoherence.

31 Starting from a simple measure of distinguishability between density matrices that is non-increasing under linear CPT dynamics [251], one can argue [64, 196] that linear CPT dynamics implies the absence of any backward flow of information into the system from its environment. [71] strengthens this reasoning by proving that exact linear CPT dynamics exists for a given quantum system if and only if the system’s initial correlations with its environment satisfy a quantum data-processing inequality that prevents backward information flow.
From the spectral decompositions (63), (64), and (65), together with our formula (62) for the linear CPT dynamics of the parent system \( W \), we can trivially express the epistemic probability \( p_{Q_i, 1: i'} (i_1; t') \equiv p_{Q_i, 1: i'} (i_1; t') \) for subsystem \( Q_i \) to be in the ontic state \( \Psi_{i, 1: i'} \) at the final time \( t' \) as

\[
p_{Q_i, 1: i'} (i_1; t') = \text{Tr}_{Q_i} \left[ \hat{P}_{Q_i, 1: i'} (i_1; t') \hat{\rho}_{Q_i} (t') \right]
\]

\[
= \text{Tr}_W \left[ (\hat{P}_{Q_i, 1: i'} (i_1; t') \otimes \hat{1}_{Q_2} \otimes \cdots \otimes \hat{1}_{Q_n} \right] \hat{\rho}_W (t') \right]
\]

\[
= \text{Tr}_W \left[ (\hat{P}_{Q_i, 1: i'} (i_1; t') \otimes \hat{1}_{Q_2} \otimes \cdots \otimes \hat{1}_{Q_n} \right] \mathcal{E}^{t' - t}_W \left[ \hat{P}_W (w; t) \right] \right] p_W (w; t),
\]

\[
= \sum_{i_1, \ldots, i_n, W} \text{Tr}_W \left[ (\hat{P}_{Q_i, 1: i'} (i_1; t') \otimes \hat{P}_{Q_1, 1: i'} (i_1; t') \otimes \cdots \otimes \hat{P}_{Q_n, 1: i'} (i_1; t') \right] \mathcal{E}^{t' - t}_W \left[ \hat{P}_W (w; t) \right] \right] p_W (w; t).
\]

(66)

This last expression reduces to an intuitive Bayesian propagation formula with marginalization,

\[
p_{Q_i, 1: i'} (i_1; t') = \sum_{i_1, \ldots, i_n, W} p_{Q_1, 1: i, \ldots, Q_n, W} (i_1, \ldots, i_n; t' \mid w; t) p_W (w; t),
\]

(67)

provided that we interpret the trace over \( W \) appearing in the summation on \( w \) in (66) as the quantum conditional probability \( p_{Q_1, 1: i, \ldots, Q_n, W} (i_1, \ldots, i_n; t' \mid w; t) \) for each subsystem \( Q_n \) to be in the ontic state \( \Psi_{Q_n, 1: i_n, t'} = \Psi_{Q_n, 1: i_n, t'} \) at time \( t' \) for \( \alpha = 1, \ldots, n \) given that the parent system \( W \) was in the ontic state \( \Psi_W (w; t) = \Psi_W (w; t) \) at time \( t' \):

\[
\begin{aligned}
p_{Q_1, 1: i, \ldots, Q_n, W} (i_1, \ldots, i_n; t' \mid w; t) \equiv & \text{Tr}_W \left[ (\hat{P}_{Q_1, 1: i, \ldots, Q_n, W} (i_1, \ldots, i_n; t')) \mathcal{E}^{t' - t}_W \left[ \hat{P}_W (w; t) \right] \right] \\
\sim & \text{Tr} \left[ \hat{P}_{i_1} (t') \cdots \hat{P}_{i_n} (t') \mathcal{E} \left[ \hat{P}_w (t) \right] \right].
\end{aligned}
\]

(68)

As we'll see, these quantum conditional probabilities serve essentially as smoothness conditions that sew together ontologies in a natural way.

Note that our quantum conditional probabilities (68) are only defined in terms of the projection operators onto the orthonormal eigenstates of density matrices—that is, the eigenstates representing the system's possible ontic states in accordance with our general correspondence (17)—and not in terms of projection operators onto generic state vectors, in contrast to the Born rule for computing empirical outcome probabilities. Furthermore, observe that the linear CPT dynamical mapping \( \mathcal{E}^{t' - t}_W [\cdot] \) in our definition (68) plays the role of a parallel-transport superoperator that carries the parent-system projection operator \( \hat{P}_W (w; t) \) from \( t \) to \( t' \) before we compare it with the subsystem projection operators \( \hat{P}_{Q_1, 1; i_1, t'}, \ldots, \hat{P}_{Q_n, 1; i_n, t'} \); the Born rule involves the comparison of state vectors without any kind of parallel transport. Moreover, we emphasize that our definition (68), unlike the axiomatic wave-function collapse of the traditional Copenhagen interpretation of quantum theory, does not introduce any new fundamental violations of time-reversal symmetry: indeed, given a system that possesses well-defined reversed dynamics expressible in terms of a corresponding reversed linear CPT dynamical mapping that evolves the system’s density matrix backward in time, we are free to use that reversed linear CPT dynamical mapping in our definition (68) of the quantum conditional probabilities.

Observe also that the tensor-product operator \( \hat{P}_{Q_i, 1: i_1, t'} \otimes \cdots \otimes \hat{P}_{Q_n, 1: i_n, t'} \) and the time-evolved projection operator \( \mathcal{E}^{t' - t}_W \left[ \hat{P}_W (w; t) \right] \) are both positive semi-definite operators. Hence, each \( p_{Q_1, 1: i, \ldots, i_n, W} (i_1, \ldots, i_n; t' \mid w; t) \) defined according to (68) consists of a trace over a product of two positive semi-definite operators, and is therefore guaranteed to be non-negative.

\[
p_{Q_1, 1: i, \ldots, i_n, W} (i_1, \ldots, i_n; t' \mid w; t) \geq 0.
\]

(69)

From this starting point, we can show that the quantities \( p_{Q_1, 1: i, \ldots, i_n, W} (i_1, \ldots, i_n; t' \mid w; t) \) miraculously satisfy several important and nontrivial requirements that support their interpretation as a quantum class of conditional probabilities.

---

32 Proof: Let \( A \geq 0 \) and \( B \geq 0 \) be positive semi-definite matrices. Then \( \sqrt{A} \) and \( \sqrt{B} \) exist and are likewise positive semi-definite, and so, using the cyclic property of the trace, we have

\[
\text{Tr} [AB] = \text{Tr} \left[ \sqrt{A} \sqrt{B} \sqrt{B} \sqrt{A} \right] = \text{Tr} \left[ \sqrt{B} \sqrt{A} \sqrt{B} \sqrt{A} \right] = \text{Tr} \left[ (\sqrt{B} \sqrt{A})^\dagger \sqrt{A} \sqrt{B} \right] \geq 0.
\]

QED

Notice that this proof does not generalize to traces over products of more than two operators.
1. From the completeness of the subsystem eigenprojectors \( \hat{P}_{Q_i} (i_1; t') \), ..., \( \hat{P}_{Q_n} (i_n; t') \), the unit-trace of the parent-system eigenprojectors \( \hat{P}_W (w; t) \), and the trace-preserving property of \( \mathcal{E}_{W}^{t+t} [\cdot] \), the set of quantities \( p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) \) for any fixed value of \( w \) give unity when summed over the indices \( i_1, \ldots, i_n \) labeling all the possible ontic states of the mutually disjoint partitioning subsystems \( Q_1, \ldots, Q_n \):

\[
\sum_{i_1, \ldots, i_n} p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) = 1.
\] (70)

2. Combining the properties (69) and (70), we see immediately that the quantities \( p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) \) are each real numbers in the interval between 0 and 1:

\[
p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) \in [0,1].
\] (71)

3. Recalling our derivation of (66), and from the linearity property of \( \mathcal{E}_{W}^{t+t} [\cdot] \) together with the spectral decomposition (63) of the density matrix \( \hat{\rho}_W (t) \) of \( W \) in terms of its eigenprojectors \( \hat{P}_W (w; t) \) and the dynamical equation (62) for the time evolution of \( \hat{\rho}_W (t) \), we see that multiplying the quantities \( p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) \) by the epistemic probabilities \( p_W (w;t) \equiv p_{W,w} (t) \) for \( W \) at time \( t \) and summing over \( w \) and \( i_1, \ldots, i_n \) except for one subsystem index \( i_a \) gives the epistemic probabilities \( p_{Q_a} (i_a; t') \equiv p_{Q_a,i_a} (t') \) for \( Q_a \), in agreement with the rule (67) for Bayesian propagation and marginalization:

\[
\sum_{i_1, \ldots, (\text{no } i_a), \ldots, i_n} p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) p_W (w; t) = p_{Q_a} (i_a; t').
\] (72)

4. Due to the cyclic property of the trace, the quantities \( p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) \) are manifestly invariant under arbitrary unitary transformations. (Subtleties can arise for unitary transformations that involve time, such as for Lorentz transformations, as we explain in Section V D.)

5. In the idealized case in which \( W = Q_1 \equiv Q \) is a closed system undergoing “deterministic” unitary dynamics, so that

\[
\hat{\rho}_Q (t') = U_Q (t' \leftarrow t) \hat{\rho}_Q (t) U_Q^\dagger (t' \leftarrow t)
\]

for some unitary time-evolution operator \( U_Q (t' \leftarrow t) \), we have

\[
\hat{P}_Q (j; t') = U_Q (t' \leftarrow t) \hat{P}_Q (j; t) U_Q^\dagger (t' \leftarrow t)
\]

and thus, as expected, the quantities \( p_{Q|Q} (i; t'|j; t) \) trivialize to the deterministic formula

\[
p_{Q|Q} (i; t'|j; t) \equiv \text{Tr}_Q \left[ \hat{P}_Q (i; t') \hat{P}_Q (j; t') \right] = \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}
\] (73)

Considering the inflexibility of quantum theory and its famed disregard for the sorts of familiar concepts favored by human beings, it’s quite remarkable that the theory contains such a large class of quantities \( p_{Q_1,...,Q_n|W} (i_1, \ldots, i_n; t'|w; t) \) satisfying the properties (69)-(73) of conditional probabilities. Essentially, our interpretation of quantum theory takes this fact at face value by actually calling these quantities conditional probabilities.

4. Probabilistic and Non-Probabilistic Uncertainty

However, one should keep in mind that quantum theory does appear to limit what kinds of probabilities we can safely define. In particular, our derivation (66) does not extend to non-disjoint subsystems, nor to subsystems at multiple distinct final times, nor to multiple parent systems. Hence, certain kinds of hypothetical statements involving the ontic states of our interpretation of quantum theory turn out not to admit generally well-defined probabilities, despite lying behind a veil of uncertainty.

Physicists tend to use the terms “uncertainty” and “probability” almost synonymously, but the two concepts are distinct. Indeed, there is no rigorous \textit{a priori} reason to believe as a general truth about Nature that the frequency ratio of \textit{every} kind of repeatable event should “tend to” some specific value in the limit of many trials; the existence
of such a limit is actually a highly nontrivial and non-obvious constraint because we could easily imagine outcomes instead occurring in a completely unpredictable way that never “settles down.” Science would certainly be a far less successful predictive enterprise if observable phenomena did not generally obey such a constraint, but it makes no difference to the predictive power of science if hidden variables do not always comply.

In fact, economists have known for many years that although certain types of uncertainty, called probabilistic uncertainty or risk, could safely be described in terms of probabilities, other kinds of uncertainty, called non-probabilistic uncertainty, could not be. An example of this distinction from computer programming is the difference between the possibility of two pseudo-random numbers agreeing and the possibility of a user deciding to input two numbers that agree. Further examples closer to physics are examined in [280], which elegantly explains the nonexistence of a joint probability distribution \( p(x, y, z, \ldots) \) as the failure of its arguments \( x, y, z, \ldots \) to possess a well-defined joint limiting relative frequency, perhaps due to the need for proper subsets of the arguments \( x, y, z, \ldots \) to satisfy their own mandated limiting relative frequencies.

We can present a more concrete example [169] demonstrating that joint probabilities may not exist for certain sets of random variables even in classical probability theory. Consider a set of three classical random bits \( X, Y, Z \) with individual probabilities given by the chart

|       | +    | −    |
|-------|------|------|
| \( p_X(x) \) | 1/2  | 1/2  |
| \( p_Y(y) \)  | 1/2  | 1/2  |
| \( p_Z(z) \)  | 1/2  | 1/2  |

and pairwise-joint probabilities

|       | (+,+) | (+,−) | (−,+)| (−,−) |
|-------|-------|-------|-----|-------|
| \( p_{X,Y}(x,y) \) | \( \frac{1}{4}(1 + \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 + \frac{1}{\sqrt{2}}) \) |
| \( p_{X,Z}(x,z) \) | \( \frac{1}{4}(1 + \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 + \frac{1}{\sqrt{2}}) \) |
| \( p_{Y,Z}(y,z) \) | \( \frac{1}{4}(1 + \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) \) | \( \frac{1}{4}(1 + \frac{1}{\sqrt{2}}) \) |

satisfying the correct partial sums (32):

\[
\sum_y p_{X,Y}(x,y) = \sum_z p_{X,Z}(x,z) = \frac{1}{2} = p_X(x),
\]

\[
\sum_x p_{X,Y}(x,y) = \sum_z p_{Y,Z}(y,z) = \frac{1}{2} = p_Y(y),
\]

\[
\sum_x p_{X,Z}(x,z) = \sum_y p_{Y,Z}(y,z) = \frac{1}{2} = p_Z(z).
\]

Then it is impossible to define a joint probability distribution \( p_{X,Y,Z}(x,y,z) \) for all three random bits \( X, Y, Z \).

### 5. Hidden Ontic-Level Nonlocality

Note that the definition (68) of our quantum conditional probabilities is not manifestly local at the ontic level, as we’ll make clear explicitly when we discuss the EPR-Bohm thought experiment in Section V B in the context of our minimal modal interpretation of quantum theory. However, the causal structure of special relativity only places constraints on observable signals, and our quantum conditional probabilities are, by construction, compatible with standard density-matrix dynamics and thus constrained by the no-communication theorem [159, 239] to disallow superluminal observable signals, as we will explain in greater detail Section V.

The lack of manifest ontic-level locality in (68) is a feature, not a bug, because, as we’ll see when we analyze the EPR-Bohm thought experiment in Section V B and the GHZ-Mermin thought experiment in Section V C, there is no way to accommodate ontic hidden variables without hidden ontic-level nonlocality. Hence, if our formula (68) didn’t allow for benign ontic-level nonlocality, then we would clearly be doing something wrong.

---

33. In his light-hearted paper [3], Aaronson refers to the latter as “Knightian” uncertainty, in honor of economist Frank Knight’s seminal 1921 book [188] on the subject.

34. We can prove this claim by contradiction: Supposing to the contrary that we could indeed assign joint probabilities to \( (x, y, z) = (−, −, −) \) and \( (x, y, z) = (−, −, +) \), we see that \( p_{X,Y,Z}(−, −, −) \leq p_{Y,Z}(−, −) = 1/4 \) and \( p_{X,Y,Z}(−, −, +) \leq p_{Y,Z}(−, +) = 1/4 \) \( (1 - 1/\sqrt{2}) \), but then the partial-sum formula (32) breaks down because \( p_{X,Y,Z}(−, −, −) + p_{X,Y,Z}(−, −, +) \leq (1/4) \cdot (2 - 1/\sqrt{2}) < (1/4) \cdot (1 + 1/\sqrt{2}) = p_{X,Y}(−, −) \). QED

35. See Section V1D6 for a discussion of the status of locality in the Everett-DeWitt many-worlds interpretation.
Setting \( t' = t \) and suppressing time from our notation for clarity, the dynamical mapping \( \mathcal{E}^{t'' \rightarrow t} [\cdot] \) drops out and the quantum conditional probabilities (68) yield an explicit instantaneous kinematical (and generically probabilistic) relationship between the ontic states of the parent system \( W = Q_1 + \cdots + Q_n \) and the ontic states of the partitioning collection of mutually disjoint subsystems \( Q_1,\ldots,Q_n \):

\[
 p_{Q_1,\ldots,Q_n|W} (i_1,\ldots,i_n|w) = \text{Tr}_W \left[ \left( \hat{P}_{Q_1} (i_1) \otimes \cdots \otimes \hat{P}_{Q_n} (i_n) \right) \hat{P}_W (w) \right] \\
= \langle \Psi_{W,w} | \left( |\Psi_{Q_1,i_1}\rangle \langle \Psi_{Q_1,i_1} | \otimes \cdots \otimes |\Psi_{Q_n,i_n}\rangle \langle \Psi_{Q_n,i_n} | \right) |\Psi_{W,w}\rangle.
\]

(74)

In particular, this result leads to a simpler version of our formula (67) for Bayesian propagation and marginalization:

\[
 p_Q (i_1) = \sum_{i_2,\ldots,i_n} = p_{Q_1,\ldots,Q_n|W} (i_1,\ldots,i_n|w) p_W (w).
\]

(75)

Notice how our quantum conditional probabilities play the role of sewing together the ontologies of \( W \) and \( Q_1,\ldots,Q_n \).

In the simplest case, for which the actual ontic state \( |\Psi_{W,w}\rangle = |\Psi_{Q_1,i_1}\rangle \otimes \cdots \otimes |\Psi_{Q_n,i_n}\rangle \) of the parent system \( W \) involves approximately no entanglement between its mutually disjoint subsystems \( Q_1,\ldots,Q_n \), we obtain the classical-looking result \( p_{Q_1,\ldots,Q_n|W} (j_1,\ldots,j_n|w = (i_1,\ldots,i_n)) = \delta_{j_1 i_1} \cdots \delta_{j_n i_n} \), as expected. Because decoherence ensures that human-scale macroscopic systems exhibit negligible quantum entanglement with one another, we see immediately that the ontologies of typical macroscopic parent systems and their macroscopic subsystems fit together in a classically intuitive, reductionist manner. However, naive reductionism generically breaks down for microscopic systems like electrons, and thus demanding classically intuitive relationships\(^{30}\) between microscopic parent systems and subsystems would mean committing a fallacy of composition or division.\(^{37}\)

With the formula (74) in hand, we can easily generalize our earlier entangled example (40) to the case in which the composite parent system \( A + B \) itself has a nontrivial density matrix

\[
 \hat{\rho}_{A+B} = p_{\Phi} (|\Phi_{A+B}\rangle \langle \Phi_{A+B} |) + p_{\Psi} (|\Psi_{A+B}\rangle \langle \Phi_{A+B} |) + p_{\Phi} (|\Phi_{A+B}\rangle \langle \Psi_{A+B} |) + p_{\Phi} (|\Psi_{A+B}\rangle \langle \Psi_{A+B} |),
\]

where

\[
 |\Phi_{A+B}\rangle = \alpha |\Phi_{A,1}\rangle |\Phi_{B,1}\rangle + \beta |\Phi_{A,2}\rangle |\Phi_{B,2}\rangle, \quad |\Psi_{A+B}\rangle = \gamma |\Psi_{A,1}\rangle |\Phi_{B,1}\rangle + \delta |\Psi_{A,2}\rangle |\Phi_{B,2}\rangle,
\]

and where we assume for simplicity that the state vectors \( |\Phi_{A,i}\rangle \) for subsystem \( A \) are all mutually orthogonal and likewise that the state vectors \( |\Psi_{B,i}\rangle \) for subsystem \( B \) are all mutually orthogonal. The partial-trace prescription (38) then yields the reduced density matrix of \( A \),

\[
 \hat{\rho}_A = p_{\Psi} |\Psi_{A,1}\rangle \langle \Psi_{A,1} | + p_{\Psi} |\Psi_{A,2}\rangle \langle \Psi_{A,2} |,
\]

with a similar formula for \( B \), and the instantaneous kinematical relationship (74) yields the conditional probabilities

\[
 p (|\Psi_{A,B}\rangle = p_{\Psi} |\Psi_{A,1}\rangle \langle \Psi_{A,1} | + p_{\Psi} |\Psi_{A,2}\rangle \langle \Psi_{A,2} |,
\]

again with similar formulas for \( B \).

\(^{30}\) Vermaas [298] refers to the two logical directions underlying these classically reductionist relationships as the property of composition and the property of division.

\(^{37}\) Maudlin implicitly makes this kind of error in criterion 1.A of his three-part classification of interpretations of quantum theory in [218] when he assumes that the metaphysical completeness of state vectors implies that the state vector of a parent system completely determines the state vectors of all its subsystems even for the case of microscopic systems. It’s also worth mentioning that Maudlin’s criterion 1.B that state vectors always evolve according to linear dynamical equations is somewhat misleading, because all realistic systems are always at least slightly open and thus must be described by density matrices that do not evolve according to exactly linear dynamical equations.
The formula (74) also allows us to resolve an issue that we brought up in our discussion surrounding (46) in Section III C 4, where we saw that the existence of entanglement between a pair of systems $A$ and $B$ seemed to prevent us from relating the ontic state of subsystem $B$ to a given ontic state of its parent system $A + B$ even if there existed a slight redefinition of our subsystem decomposition $A + B = A' + B'$ for which the parent system’s ontic state had the simple non-entangled form $(a', b')$. In that case, the quantum conditional probability $p_{B|A+B} (b|w = (a', b')) \approx p_{B'|A'+B'} (b'|w = (a', b')) = 1$ would be very close to unity, thereby smoothing out the supposed discrepancy and, furthermore, eliminating the need to define subsystem $B$ with measure-zero sharpness.

7. Quantum Dynamics of Open Subsystems

Our discussion of dynamics in quantum theory begins with the assumption of a system $W$ being a composite parent system $W = Q + E$ consisting of a subsystem $Q$ and its larger environment $E$ inside $W$, then there is no guarantee that $Q$ itself has well-defined dynamics of its own. Indeed, the reduced density matrix of $Q$ at the final time $t'$ is generally given by

$$\hat{\rho}_Q (t') = \operatorname{Tr}_E [\hat{\rho}_W (t')] = \operatorname{Tr}_E [\mathcal{E}_{W}^{t' - t} [\hat{\rho}_W (t)]] ,$$

which is not generically linear (or even analytic) in the reduced density matrix $\hat{\rho}_Q (t)$ at the initial time $t$ because of the complicated way that $\hat{\rho}_Q (t)$ is related to $\hat{\rho}_W (t)$ through the partial trace (38).

We encountered a similar issue in Section III D 1 when we discussed the general relationship between classical parent-system and subsystem dynamics, where we saw in (49) that a classical open subsystem’s dynamical mapping is not generically linear. Mimicking our approach in that discussion, we begin by rewriting the subsystem evolution equation (76) as

$$\hat{\rho}_Q (t') = \operatorname{Tr}_E [\mathcal{E}_{W}^{t' - t} [A_{Q \subset W} [\hat{\rho}_Q (t)]]],$$

where we have introduced the nonlinear, manifestly positive-definite (but not generally trace-preserving) mapping

$$A_{Q \subset W} [\cdot] \equiv \hat{\rho}_W^{1/2} (t) \left( \hat{\rho}_{Q}^{-1/2} (t) \otimes \mathbb{1}_E \right) (\cdot) \otimes \mathbb{1}_E \left( \hat{\rho}_{Q}^{-1/2} (t) \otimes \mathbb{1}_E \right) \hat{\rho}_W^{1/2} (t) ,$$

which is known as an assignment mapping. The alternative version (77) of our original subsystem evolution equation (76) for $Q$ is the natural noncommutative quantum counterpart to the classical formula (49), and motivates introducing a coarse-grained or effective version $p_{Q \subset W} (\cdot; t'; t)$ of our original quantum conditional probabilities (68) for $Q_{1} \equiv Q$ inside $W$:

$$p_{Q \subset W} (j; t'|i; t) \equiv \operatorname{Tr}_W \left[ \left( \hat{P}_Q (j; t') \otimes \mathbb{1}_E \right) \mathcal{E}_{W}^{t' - t} [A_{Q \subset W} [\hat{P}_Q (i; t)]] \right].$$

Again in parallel with the classical case (53), if initial correlations between $Q$ and its environment $E$ approximately wash out

$$\hat{\rho}_W \approx \hat{\rho}_Q \otimes \hat{\rho}_E$$

over some short characteristic time scale

$$\delta t_Q \ll t' - t,$$

then

$$\hat{\rho}_W^{1/2} (t) \left( \hat{\rho}_Q^{-1/2} (t) \otimes \mathbb{1}_E \right) = \mathbb{1}_Q \otimes \hat{\rho}_E^{1/2} (t)$$

and thus the assignment mapping (78) trivializes,

$$A_{Q \subset W} [\hat{\rho}_Q (t)] = \hat{\rho}_Q (t) \otimes \mathbb{1}_E.$$
It follows as an immediate consequence that over time scales \( t' - t \gg \delta t_Q \), the subsystem evolution equation (77) reduces to linear CPT dynamics for \( Q \) on its own,
\[
\dot{\hat{\rho}}_Q (t') = \text{Tr}_E \left[ \mathcal{E}'_W \left[ \hat{\rho}_Q (t) \otimes \hat{1}_E \right] \right] = \sum_{\alpha} \hat{E}'_{\alpha} \hat{\rho}_Q (t) \hat{E}'_{\alpha \dagger} + \sum_{\alpha} \hat{E}'_{\alpha} \hat{\rho}_Q \hat{E}'_{\alpha \dagger} = \hat{1}_Q, \tag{82}
\]
where \( \{ \hat{E}'_{\alpha} \} \) is the usual set of Kraus operators for the dynamics [191]. Indeed, this line of reasoning, together with the additional assumptions that the linear CPT density-matrix dynamics for \( Q \) is Markovian and homogeneous in time, precisely leads to the well-known and highly effective Lindblad equation [206], which can be expressed in its diagonal form as [65, 183, 257]
\[
\frac{\partial\hat{\rho}_Q}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho}_Q \right] + \sum_{k=1}^{N^2-1} \gamma_k \left( \hat{A}_k \hat{\rho}_Q \hat{A}_k^\dagger - \frac{1}{2} \hat{A}_k^\dagger \hat{A}_k \hat{\rho}_Q - \frac{1}{2} \hat{\rho}_Q \hat{A}_k \hat{A}_k^\dagger \right) \tag{83}
\]
for a suitable collection of parameters \( \gamma_k \) and operators \( \hat{H}, \hat{A}_k \). Letting \( \mathcal{E}'_{Q} \) denote this reduced linear CPT dynamics for \( Q \) alone,
\[
\dot{\hat{\rho}}_Q (t) \rightarrow \hat{\rho}_Q (t') = \mathcal{E}'_{Q} \left[ \hat{\rho}_Q (t) \right], \tag{84}
\]
the coarse-grained conditional probabilities (79) now coincide with our exact general definition (68) for \( W = Q_1 \equiv Q \):
\[
p_{Q \subset W} (j; t'|i; t) \rightarrow p_{Q} (j; t'|i; t) \equiv \text{Tr}_Q \left[ \hat{P}_Q (j; t') \mathcal{E}'_{Q} \left[ \hat{P}_Q (i; t) \right] \right] \sim \text{Tr} \left[ \hat{P}_Q (t') \mathcal{E} \left[ \hat{P}_Q (i; t) \right] \right] \text{ for } t' - t \gg \delta t_Q. \tag{85}
\]
In other words, by coarse-graining in time on the scale \( \delta t_Q \) over which correlations between \( Q \) and its environment \( E \) wash out, we no longer need to coarse grain in the sense (79) of explicitly referring to the parent system \( W = Q + E \). As we remarked in the classical case, the characteristic time scale \( \delta t_Q \) determines a natural short-time cutoff on the existence of dynamics for our open subsystem \( Q \). Notice also that our interpretation of quantum theory can fully accommodate the possibility that the parent system \( W \) likewise has a nonzero characteristic time scale \( \delta t_W \neq 0 \) that sets the cutoff on the existence of its own dynamics as well; for all we know, there is no maximal parent system in Nature for which this temporal cutoff scale exactly vanishes. That is, it may well be that all linear CPT dynamics is ultimately only an approximate notion, although the same might well be true for density matrices themselves, as we explained in the context of discussing truncated Hilbert spaces in Section III C 5.

8. Dynamical Relationships Between Ontic States Over Time and Objective Epistemic States Over Time

Whether the dynamical quantum conditional probabilities \( p_{Q} (j; t'|i; t) \) are exact
\[
p_{Q} (j; t'|i; t) \equiv \text{Tr}_Q \left[ \hat{P}_Q (j; t') \mathcal{E}'_{Q} \left[ \hat{P}_Q (i; t) \right] \right] \sim \text{Tr} \left[ \hat{P}_Q (t') \mathcal{E} \left[ \hat{P}_Q (i; t) \right] \right], \tag{86}
\]
or coarse-grained either in the sense (79) of making explicit reference to a parent system or in the sense (85) of existing only over time scales exceeding some nonzero cutoff \( \delta t_Q \) as introduced in (81), they manifestly define dynamics for the ontic states of \( Q \) in a manner that parallels the classical case (47):38
\[
p_{Q} (j; t'|i; t) : \ (i; t), \ (j; t') \rightarrow p_{Q} (j; t'|i; t). \tag{87}
\]
The dynamical quantum conditional probabilities \( p_{Q} (j; t'|i; t) \) also provide an explicit dictionary that translates between the manifestly quantum linear CPT dynamics (84) of density matrices and the linear dynamical mapping of objective epistemic states familiar from the equation (48) for classical epistemic states:
\[
p_{Q} (j; t'|t; t) : \ p_{Q} (j; t') = \sum_{i} p_{Q} (j; t'|i; t) p_{Q} (i; t). \tag{88}
\]

---

38 These dynamical quantum conditional probabilities also supply an important ingredient that is missing from the traditional modal interpretations and that is identified in “Problem 3: The problem of effect” in [218], namely, the lack of a “detailed dynamics for the value [ontic] states.”
9. Entanglement and Imprecision in Coarse-Grained Quantum Conditional Probabilities

When working over time intervals so small that we cannot assume approximate factorization of the density matrix \( \hat{\rho}_W \neq \hat{\rho}_Q \otimes \hat{\rho}_E \) of the parent system \( W = Q + E \), we claim that quantum entanglement (44)—and not merely classical correlation (43)—between \( Q \) and its environment \( E \) inside \( W \) determines the size of our imprecision in using the coarse-grained quantum conditional probabilities \( p_{Q\subset W} (j; t'| i; t) \) defined in (79).

As evidence in support of this claim, suppose that the dynamics for the parent system \( W \) over the time interval \( t' - t \) exactly decouples into dynamics for subsystem \( Q \) alone and dynamics for the environment \( E \) alone, with no interactions between the two subsystems, so that the linear CPT dynamical mapping (62) factorizes:

\[
E^t_{W \rightarrow t} \left[ \right] = \left( E^t_{Q \rightarrow t} \otimes E^t_{E \rightarrow t} \right) \left[ \right].
\]

(89)

(This circumstance includes the trivial case \( t' = t \) in which \( E^t_{W \rightarrow t} \left[ \right] \) is just the identity mapping, as well as the case in which all these dynamical mappings describe unitary time evolution.) If \( Q \) and \( E \) are only classically correlated with each other at the initial time \( t \) but are not entangled in the sense of (44), then the density matrix \( \hat{\rho}_W (t) \) of the parent system takes the form

\[
\hat{\rho}_W (t) = \sum_{i,e} p_W ((i, e); t) \hat{P}_Q (i; t) \otimes \hat{P}_E (e; t),
\]

(90)

and a simple calculation shows that the coarse-grained quantum conditional probabilities \( p_{Q\subset W} (j; t'| i; t) \) defined in (79) reduce to the expected exact quantum conditional probabilities \( p_Q (j; t'| i; t) \) defined in (86) for \( Q \) alone:

\[
p_{Q\subset W} (j; t'| i; t) = \text{Tr}_Q \left[ \hat{P}_Q (j; t') E^t_{Q \rightarrow t} \left[ \hat{P}_Q (i; t) \right] \right] \equiv p_Q (j; t'| i; t).
\]

(91)

Within the scope of our assumption of decoupled dynamics (89), deviations from (91) can therefore arise only if \( Q \) is entangled with its environment \( E \). We interpret this result as implying that for general parent-system dynamics not necessarily factorizing (89) into independent dynamics for \( Q \) and \( E \), we should only ever trust the validity of coarse-grained conditional probabilities (79) up to an intrinsic error—a new kind of “uncertainty principle”—determined by the amount of entanglement between \( Q \) and \( E \).

10. Eigenstate Swaps

Recall that an eigenstate swap (24), as we have defined it in Section III B 3, describes an exchange between two orthogonal density-matrix eigenstates over a time scale \( \delta t_{\text{swap}} \) that is exponentially small in the total number of degrees of freedom of both the system itself and of all other systems that substantially interact and entangle with it, in keeping with (25).

It is now a simple matter to explain why actual ontic states avoid eigenstate swaps: If an eigenstate swap in the system’s density matrix takes place from \( t \) to \( t + \delta t_{\text{swap}} \), where \( \delta t_{\text{swap}} \) is assumed to be larger than the minimal time scale \( \delta t_Q \) over which the dynamics of our system \( Q \) actually exists, then, in accordance with (85), there is approximately zero conditional probability \( p_Q (i; t + \delta t_{\text{swap}} | i; t) \sim | \langle \Psi_{Q,i} (t + \delta t_{\text{swap}}) | \Psi_{Q,i} (t) \rangle |^2 \approx 0 \) for the system’s ontic state to be \( \Psi_{Q,i} (t + \delta t_{\text{swap}}) \approx \Psi_{Q,i} (t) \) at the time \( t + \delta t_{\text{swap}} \) given that it was \( \Psi_{Q,i} (t) \) at the earlier time \( t \). Indeed, the probability that the system’s actual ontic state will instead be \( \Psi_{Q,i} (t + \delta t_{\text{swap}}) \approx \Psi_{Q,i} (t) \) is approximately unity. Notice again the smoothing role played by our quantum conditional probabilities—they sew together the evolving ontology of \( Q \) in a manner that avoids eigenstate-swap instabilities.

These results may seem surprising given that \( \Psi_{Q,i} (t) \) is nearly certain to become \( \Psi_{Q,i} (t') \) over even shorter time scales \( t' - t \ll \delta t_{\text{swap}} \), but it is important to keep in mind that our objective quantum conditional probabilities do not naively compose: We cannot generically express \( p_Q (i; t + \delta t_{\text{swap}} | i; t) \), defined in accordance with (85), as a sum of products of quantum conditional probabilities of the form \( p_Q (i_m; t_m | i_{m-1}; t_{m-1}) \) over a sequence of many tiny intermediate time intervals \( t_m - t_{m-1} \ll \delta t_{\text{swap}} \) going from \( t \) to \( t + \delta t_{\text{swap}} \).

\[\text{Observe that the operation } E^t_{Q \rightarrow t} \left[ \hat{P}_Q (i; t) \right] \text{ appearing in (85) evolves ontic states as though they were not part of density matrices, and so is blind to the eigenstate swap. Note also that over long time scales } t' - t \gg \delta t_{\text{swap}} > \delta t_Q \text{, there can be appreciable conditional probabilities for the final ontic state of a system to end up being nearly orthogonal to its initial ontic state. That is, the present arguments eliminate only ultra-fast swaps to orthogonal ontic states, but allow for transitions that take place over reasonably long time intervals.}\]
11. Ergodicity Breaking, Classical States, and the Emergence of Statistical Mechanics

On short time scales, our dynamical quantum conditional probabilities \( \mathcal{E}_Q \) generically allow a macroscopic system’s ontic-level trajectory to explore ergodically a large set of ontic states differing in the values of only a few of the system’s degrees of freedom, while (super-)exponentially suppressing transitions to ontic states that differ in the values of large numbers of degrees of freedom. Hence, \( \mathcal{E}_Q \) leads emergently to a partitioning of the macroscopic system’s overall Hilbert space into distinct ergodic components that do not mix over short time intervals and that we can therefore identify as the system’s classical states, much in the way that a ferromagnet undergoes a phase transition from a single disordered ergodic component to a collection of distinct ordered ergodic components as we cool the system below its Curie temperature. For a treatment of ergodicity breaking and the related emergence of statistical mechanics from quantum considerations, see [173, 209, 244, 245].

12. Connections to Other Work

In a spirit reminiscent of our minimal modal interpretation of quantum theory, a series of approaches [72, 142] to studying open quantum systems involve “unraveling” the dynamical equation for a given subsystem’s reduced density matrix as an ensemble average over a collection of stochastically evolving state vectors. Such techniques have been applied to quantum optics [72, 88], entanglement [74], decoherence [160], non-Markovian dynamics [98, 99, 126, 274], and geometrical phases [38].

Similarly, in [105], Esposito and Mukamel make use of a notion of “quantum trajectories” not unlike the evolving ontic-state trajectories in our interpretation of quantum theory. Esposito and Mukamel likewise note that defining these quantum trajectories for a given system \( Q \) requires working with the time-dependent eigenbasis that instantaneously diagonalizes the system’s density matrix \( \hat{\rho}_Q (t) \), in contrast to the fixed configuration space of a classical system. Moreover, defining a differential linear CPT dynamical mapping \( \mathcal{K}_Q' \) in terms of its finite-time counterpart \( \mathcal{E}_Q^{t,t'} [\cdot] \) from (84) according to

\[
\mathcal{K}_Q' [\cdot] = \mathcal{E}_Q^{t+\delta t_Q,t-t'} [\cdot] - \text{id} [\cdot],
\]

where \( \delta t_Q \) is the minimal time scale (81) over which our system’s dynamics (85) exists, we can naturally re-express our dynamical quantum conditional probabilities (86),

\[
p_Q (j; t' ; t ; i) \equiv \text{Tr}_Q \left[ \hat{P}_Q (j; t') \mathcal{E}_Q^{t,t'} \left[ \hat{P}_Q (i; t) \right] \right],
\]

in terms of a set of quantum transition rates

\[
W_Q ((j|i); t) = \frac{p_Q (j; t + \delta t_Q | i; t ) - p_Q (j; t | i; t)}{\delta t_Q} = \text{Tr}_Q \left[ \hat{P}_Q (j; t') \mathcal{K}_Q' \left[ \hat{P}_Q (i; t) \right] \right]
\]

that coincide in the formal limit \( \delta t_Q \to 0 \) with Esposito and Mukamel’s own definition of quantum transition rates. Esposito, Mukamel, and others [186, 201] use these quantum trajectories and quantum transition rates to study quantum definitions of work and heat, entropy production, fluctuation theorems, and other fundamental questions in statistical mechanics and thermodynamics.

Our quantum conditional probabilities, in their dynamical manifestation (86) for a system \( Q \) having well-defined dynamics, are also closely related to the causal quantum conditional states of Leifer and Spekkens [202]. To introduce the Leifer-Spekkens construction and to explain this purported connection to our minimal modal interpretation of quantum theory, we begin by considering an arbitrary linear CPT dynamical mapping \( \mathcal{E}_Q^{t,t'} [\cdot] \) of the form (84) for a system \( Q \),

\[
\hat{\rho}_Q (t) \mapsto \hat{\rho}_Q (t') = \mathcal{E}_Q^{t,t'} [\hat{\rho}_Q (t)].
\]

In particular, ergodicity breaking is crucial for explaining the complications that the authors of [209] encounter with their assumption that an open subsystem’s final equilibrium state should satisfy “subsystem state independence”—that is, that the open subsystem’s final equilibrium state should be independent of the subsystem’s initial state. This initial-state sensitivity also plays an important role in obtaining a robust resolution of the measurement problem for realistic measurement devices, an issue that arose in discussions with the author of [173] in the context of a similar modal interpretation of quantum theory developed concurrently with our own modal interpretation; we compare these two modal interpretations in Section VI E.
Then, formally introducing a tensor-product Hilbert space $\mathcal{H}_{Q(t')} \otimes \mathcal{H}_{Q(t)}$ consisting of two time-separated copies of the Hilbert space of $Q$, the Choi-Jamiołkowski isomorphism \cite{Choi1972, Jamiołkowski1972, Spekkens08}, also known as the channel-state duality, implies the existence of a unique operator $\hat{\rho}_{Q(t')}|Q(t)\rangle\langle Q(t')|$ on $\mathcal{H}_{Q(t')} \otimes \mathcal{H}_{Q(t)}$ that implements precisely the same time evolution for $Q$ through a quantum generalization of the classical Bayesian propagation rule $p_Q(j'; t) = \sum_i p_Q(j; t'| i; t) p_Q(i; t)$, namely,

$$\hat{\rho}_Q(t') = \text{Tr}_{Q(t)} \left[ \hat{\rho}_{Q(t')}|Q(t)\rangle\langle Q(t')| \left( \hat{1}_{Q(t')} \otimes \hat{\rho}_Q(t) \right) \right], \quad (94)$$

where $\hat{1}_{Q(t')}$ is the identity operator on $\mathcal{H}_{Q(t')}$. Leifer and Spekkens regard the operator $\hat{\rho}_{Q(t')}|Q(t)\rangle\langle Q(t')|$ as the Leifer-Spekkens quantum conditional state, as being a noncommutative generalization of classical causal quantum conditional state, as being a noncommutative generalization of classical conditional probabilities, in much the same way that density matrices $\hat{\rho}_Q$ themselves serve as noncommutative generalizations of classical probabilities $p_Q$.

Our dynamical quantum conditional probabilities \cite{Leifer07} are then precisely the diagonal matrix elements of the Leifer-Spekkens quantum conditional state $\hat{\rho}_{Q(t')}|Q(t)\rangle\langle Q(t')|$ in the tensor-product basis $|\Psi_Q(t')\rangle \otimes |\Psi_Q(t)\rangle$ for $\mathcal{H}_{Q(t')} \otimes \mathcal{H}_{Q(t)}$ constructed out of the eigenstates $|\Psi_Q(t')\rangle$ of $\hat{\rho}_Q(t')$ and the eigenstates $|\Psi_Q(t)\rangle$ of $\hat{\rho}_Q(t)$:

$$p_Q(j; t'| i; t) = \text{Tr}_{Q(t')} \left[ \hat{P}_Q(j; t') \hat{1}_{Q(t')} \otimes \hat{\rho}_Q(t) \right]$$

$$= \text{Tr}_{Q(t')} \left[ \hat{\rho}_{Q(t')}|Q(t)\rangle\langle Q(t')| \left( \hat{1}_{Q(t')} \otimes \hat{\rho}_Q(t) \right) \right]$$

$$\quad = \left( \langle \Psi_Q,j(t') | \otimes | \Psi_Q,i(t) \rangle \right) \left| \Psi_Q(t') \right\rangle \left\langle \Psi_Q(t) \right| \left( | \Psi_Q,j(t') \rangle \otimes | \Psi_Q,i(t) \rangle \right). \quad (95)$$

This result mirrors the way that our interpretation of quantum theory identifies the diagonal matrix elements of a system’s density matrix $\hat{\rho}_Q$ in its own eigenbasis as being the system’s epistemic probabilities $p_Q$.

These connections go deeper. For a system $Q$ without well-defined dynamics of its own but belonging to a parent system $W = Q + E$ with a linear CPT dynamical mapping $\mathcal{E}_{W}^{t'\rightarrow t} [\cdot]$ as in \cite{Spekkens08},

$$\hat{\rho}_W(t) \mapsto \hat{\rho}_W(t') = \mathcal{E}_{W}^{t'\rightarrow t} [\hat{\rho}_W(t)],$$

we introduced the coarse-grained quantum conditional probabilities $p_{Q\subset W}(j; t'| i; t)$ according to \cite{Leifer07},

$$p_{Q\subset W}(j; t'| i; t) \equiv \text{Tr}_{W} \left[ \left( \hat{1}_{Q(t')} \otimes \hat{1}_E \right) \mathcal{E}_{W}^{t'\rightarrow t} \left[ \hat{\rho}_{Q\subset W}(i; t) \right] \right],$$

where we defined the nonlinear assignment mapping $\mathcal{A}_{Q\subset W}^[\cdot] \equiv [\cdot]_{\otimes}^{1/2} \left( \hat{\rho}_Q^{1/2} \otimes 1_E \right) \left( [\cdot] \otimes 1_E \right) \hat{\rho}_Q^{1/2}$ in \cite{Leifer07},

$$\mathcal{A}_{Q\subset W}^[\cdot] \equiv \mathcal{A}_{Q\subset W}^{1/2} [\cdot] \left( \hat{\rho}_Q^{1/2} \otimes 1_E \right) \left( [\cdot] \otimes 1_E \right) \hat{\rho}_Q^{1/2}.$$

It follows from a straightforward computation that we can equivalently express our coarse-grained quantum conditional probabilities $p_{Q\subset W}(j; t'| i; t)$ as the diagonal matrix elements of a corresponding coarse-grained quantum conditional state $\hat{\rho}_{Q\subset W}^{Q(t')}|Q(t)\rangle\langle Q(t')|$, \cite{Leifer07},

$$p_{Q\subset W}(j; t'| i; t) = \text{Tr}_{Q(t')} \left[ \hat{\rho}_{Q(t')}|Q(t)\rangle\langle Q(t')| \left( \hat{1}_{Q(t')} \otimes \hat{\rho}_Q(t) \right) \right], \quad (96)$$

where $\hat{\rho}_{Q\subset W}^{Q(t')}|Q(t)\rangle\langle Q(t')|$ is an operator on the formal tensor-product Hilbert space $\mathcal{H}_{Q(t')} \otimes \mathcal{H}_{Q(t)}$ that we have defined by starting with the exact Leifer-Spekkens quantum conditional state $\hat{\rho}_{W(t')}|W(t)\rangle\langle W(t')|$ of the parent system $W$, multiplying by the density matrix $\hat{\rho}_W(t)$ of $W$ at the initial time $t$ to obtain a natural causal quantum joint state $\hat{\rho}_{W(t')+W(t)}^W$ for $W$, partial tracing over the environment $E$ at both the initial time $t$ and the final time $t'$ to obtain a coarse-grained causal quantum joint state $\hat{\rho}_{Q(t')+Q(t)}^Q$ for our original subsystem $Q$, and then “marginalizing” on $Q$ at the initial time $t$ by multiplying by $\hat{\rho}_Q^{-1}(t)$:

$$\hat{\rho}_{Q\subset W}^{Q(t')}|Q(t)\rangle\langle Q(t')| = \left( \hat{1}_{Q(t')} \otimes \hat{\rho}_Q^{-1/2}(t) \right) \text{Tr}_{E(t')} \left[ \hat{\rho}_E(t') \right] \left( \left( \hat{1}_{W(t')} \otimes \hat{\rho}_W^{-1/2}(t) \right) \hat{\rho}_{W(t')}|W(t)\rangle\langle W(t')| \left( \hat{1}_{W(t')} \otimes \hat{\rho}_W^{-1/2}(t) \right) \right] \left( \hat{1}_{Q(t')} \otimes \hat{\rho}_Q^{-1/2}(t) \right). \quad (97)$$

As expected, this coarse-grained quantum conditional state exactly satisfies a propagation rule of the form \cite{Leifer07}:

$$\hat{\rho}_Q(t') = \text{Tr}_{Q(t)} \left[ \hat{\rho}_{Q\subset W}^{Q(t')}|Q(t)\rangle\langle Q(t')| \left( \hat{1}_{Q(t')} \otimes \hat{\rho}_Q(t) \right) \right]. \quad (98)$$
E. Subsystem Spaces

An interesting geometrical structure emerges if we consider the continuously infinite set of different ways that we can define a particular subsystem Q with its own Hilbert space \( \mathcal{H}_Q \) by tensor-product-factorizing the Hilbert space \( \mathcal{H}_W = \mathcal{H}_Q \otimes \mathcal{H}_E \) of a given parent system \( W = Q + E \) that includes an environment \( E \). This geometrical structure, which we call the subsystem space for \( Q \) and which encompasses all the various possible versions of \( \mathcal{H}_Q \), exists even for a parent system \( W \) with as few as four mutually exclusive states (dim \( \mathcal{H}_W = 4 \)) and has no obvious counterpart in classical physics.

1. Formal Construction

We parameterize the smooth family of choices of bipartite tensor-product factorization of the parent system’s Hilbert space \( \mathcal{H}_W \) using an \( n \)-tuple of complex numbers \( \alpha = (\alpha_1, \ldots, \alpha_n) \) for some integer \( n \), writing \( \mathcal{H}_W = \mathcal{H}_{Q(\alpha)} \otimes \mathcal{H}_{E(\alpha)} \). We then obtain a complex vector bundle—namely, the subsystem space for our subsystem \( Q \) of interest—that consists of an identical Hilbert space \( \mathcal{H}_{Q(\alpha)} \cong \mathcal{H}_Q \) of some fixed dimension \( \text{dim} \mathcal{H}_Q \) attached to each point with coordinates \( \alpha \) on a base manifold of complex dimension \( n \). Each specific choice of tensor-product factorization \( \mathcal{H}_W = \mathcal{H}_{Q(\alpha)} \otimes \mathcal{H}_{E(\alpha)} \) and partial-trace down to \( \mathcal{H}_{Q(\alpha)} \cong \mathcal{H}_Q \) then corresponds to just one version of our subsystem \( Q \) of interest, or, equivalently, to one “slice” of this complex vector bundle—that is, to one slice of the subsystem space for our subsystem of interest.\(^{41}\)

The parent system’s density matrix \( \hat{\rho}_W \) on \( \mathcal{H}_W = \mathcal{H}_{Q(\alpha)} \otimes \mathcal{H}_{E(\alpha)} \) then defines a natural Hermitian inner product between any pair of vectors \( |\psi_{Q(\alpha)}\rangle \in \mathcal{H}_{Q(\alpha)} \) and \( |\chi_{Q(\alpha')}\rangle \in \mathcal{H}_{(\alpha')} \) living in two slices respectively located at \( \alpha \) and \( \alpha' \):

\[
\langle \psi_{Q(\alpha)} | \chi_{Q(\alpha')} \rangle \equiv \text{Tr}_W \left[ \hat{\rho}_W \left( |\psi_{Q(\alpha)}\rangle \langle \psi_{Q(\alpha)}| \otimes \hat{1}_E(\alpha) \right) \left( |\chi_{Q(\alpha')}\rangle \langle \chi_{Q(\alpha')}| \otimes \hat{1}_E(\alpha') \right) \right] = \langle \chi_{Q(\alpha')} | \psi_{Q(\alpha)} \rangle^*.
\]

Here \( \hat{1}_E(\alpha) \) and \( \hat{1}_E(\alpha') \) are respectively the identity operators on the tensor-product factors \( \mathcal{H}_{E(\alpha)} \) and \( \mathcal{H}_{E(\alpha')} \).

2. Actual Ontic Layer of Reality

The smooth family of reduced density matrices \( \hat{\rho}_{Q(\alpha)} \) that we can obtain by partial-tracing \( \hat{\rho}_W \) for different choices of \( \alpha \) defines a set of \( N = \text{dim} \mathcal{H}_Q \) smooth sections \( |\Psi_{Q(\alpha),i}\rangle \) (\( i = 1, \ldots, N = \text{dim} \mathcal{H}_Q \)) on this complex vector bundle. The notion of the subsystem’s actual ontic state therefore extends to a whole “actual ontic layer of reality” as we move through the different values of \( \alpha \), but we cannot generically identify this actual ontic layer of reality as a whole with just a single section \( |\Psi_{Q(\alpha),i}\rangle \) for fixed \( i \), for the simple reason that the epistemic probabilities \( p_{Q(\alpha),i} \) across this section may vary considerably from one value of \( \alpha \) to another. However, our kinematical quantum conditional probabilities (74) ensure that if the actual ontic state of \( W \) is of the non-entangled form \( |\Psi_{Q(\alpha),\epsilon}\rangle, |\Psi_{E(\alpha),\epsilon}\rangle \) for a particular choice of \( \alpha \), then all versions of our subsystem \( Q \) near this value of \( \alpha \) will be nearly certain to be in ontic states very similar to \( |\Psi_{Q(\alpha),i}\rangle \) in the Hilbert-space sense.

IV. THE MEASUREMENT PROCESS

A. Measurements and Decoherence

It is not our aim in this paper to investigate the measurement process and decoherence in any great depth, or to study realistic examples. For pedagogical treatments, see [56, 65, 183]. For additional detailed discussions, including a careful study of the extremely rapid rate of decoherence for typical systems in contact with macroscopic measurement devices or environments, see, for example, [184, 252, 253, 330, 331]. For a discussion of how realistic measuring devices with finite spatial and temporal resolution solve various delocalization [234] and instability [12–14] issues,

\(^{41}\) After this work was substantially complete, we noticed that a similar idea appears in [175], where the authors parameterize their subsystem space using a coordinate label \( \theta \) and employ it to study \( \sim \exp (−S) \) breakdowns in locality in the presence of a black hole of entropy \( S \).
Figure 6. The schematic set-up for a Von Neumann experiment, consisting of a subject system $Q$ together with a measurement apparatus $A$ and a larger environment $E$.

and, in particular, a treatment of the important role played both by environmental interactions and by ergodicity breaking in making sense of realistic measurement processes, see, for example, [29, 172, 173, 298].

1. Von Neumann Measurements

For the purposes of establishing how our minimal modal interpretation makes sense of measurements, how decoherence turns the environment into a “many-dimensional chisel” that rapidly sculpts the ontic states of systems into their precise shapes, and how the Born rule naturally emerges to an excellent approximation, we consider the idealized example of a so-called Von Neumann measurement. Along the way, we will address the status of both the measurement problem generally and the notion of wave-function collapse specifically in the context of our interpretation of quantum theory. Ultimately, we’ll find that our interpretation solves the measurement problem by replacing instantaneous axiomatic wave-function collapse with an interpolating ontic-level dynamics, and thereby eliminates any need for an ad hoc Heisenberg cut.

A Von Neumann measurement involves a subject system $Q$ with a complete basis of mutually exclusive ontic states $Q(i)$ (with $\langle Q(i)|Q(j) \rangle = \delta_{ij}$), a macroscopic measurement apparatus $A$ in an initial actual ontic state $A(\emptyset)$ with its display dial set to “empty,” and an even more macroscopic environment $E$ in an initial actual ontic state $E(\emptyset)$ in which it sees the dial on the apparatus $A$ as being empty. (See Figure 6.) We suppose furthermore that in the special case in which the actual ontic state of the subject system $Q$ is precisely one of the definite ontic states $Q(i)$, the resulting time evolution is unitary (although linear CPT evolution would be more realistic) and proceeds according to the following idealized two-step sequence:

1. The apparatus $A$ transitions to a new actual ontic state $A(i)$ in which its dial now displays the value “the subject system $Q$ appears to be in the state $Q(i)$,” and then

2. the environment $E$ subsequently changes to a new actual ontic state $E(i)$ in which its own configuration has been noticeably perturbed by the change in the apparatus $A$, such as through unavoidable and irreversible thermal radiation [158].

Written directly in terms of the appropriate state vectors, this two-step time evolution takes the form

$$|Q(i)\rangle|A(\emptyset)\rangle|E(\emptyset)\rangle \mapsto |Q(i)\rangle|A(i)\rangle|E(\emptyset)\rangle$$

$$\mapsto |Q(i)\rangle|A(i)\rangle|E(i)\rangle.$$ (100)

The assumption that the temporal sequence (100) represents unitary time evolution presents an obstruction to assuming that the different final ontic state vectors $|A(i)\rangle$ of the apparatus are exactly mutually orthogonal for different values of $i$, and similarly for the different final ontic state vectors $|E(i)\rangle$ of the environment $E$. However,

42 We discuss the modal interpretation of [172, 173], which developed concurrently with our own interpretation of quantum theory, in Section VI E.

43 The way that decoherence sculpts ontic states into shape is reminiscent of the way that the external pressure of air molecules above a basin of water maintains the water in its liquid phase.
our assumption that the apparatus and the environment are macroscopic systems with large respective Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_E \) ensures that inner products of the form \( \langle A ("i") | A ("j") \rangle \) and \( \langle E ("i") | E ("j") \rangle \) for \( i \neq j \) are both exponentially small in their respective numbers \( \gg 10^{23} \) of degrees of freedom:

\[
\langle A ("i") | A ("j") \rangle \sim \exp (- (\text{#apparatus degrees of freedom}) \times \Delta t / \tau_A), \quad (101)
\]

\[
\langle E ("i") | E ("j") \rangle \sim \exp (- (\text{#environment degrees of freedom}) \times \Delta t / \tau_E). \quad (102)
\]

Here \( \Delta t \) is the time duration of the measurement process (100), \( \tau_A \) is the characteristic time scale for the evolution of each individual degree of freedom of the apparatus \( A \), and \( \tau_E \) is similarly the characteristic time scale for the evolution of each individual degree of freedom of the environment \( E \).

If we now instead arrange for the initial actual ontic state \( \Psi \) of the subject system \( Q \) to be a general superposition of the complete orthonormal basis of ontic states \( Q (i) \),

\[
|\Psi\rangle = \sum_i \alpha_i |Q (i)\rangle = \alpha_1 |Q (1)\rangle + \alpha_2 |Q (2)\rangle + \cdots , \quad \langle \Psi | \Psi \rangle = 1, \quad (103)
\]

then the linearity of the time evolution (100) implies the temporal sequence

\[
|\Psi\rangle \langle A ("\psi") | E ("\psi") \rangle = \left( \sum_i \alpha_i |Q (i)\rangle \right) \langle A ("\psi") | E ("\psi") \rangle \rightarrow \left( \sum_i \alpha_i |Q (i)\rangle \langle A ("\psi") | E ("\psi") \rangle \right) \rightarrow \sum_i \alpha_i |Q (i)\rangle \langle A ("\psi") | E ("\psi") \rangle.
\]

(104)

According to our interpretation of quantum theory, the final density matrices of the subject system \( Q \) and of the measurement apparatus \( A \), obtained by appropriately partial-tracing (38) the composite final state appearing in (104), are respectively given in their diagonal form by

\[
\hat{\rho}_Q = \sum_i |\alpha_i^\prime|^2 |Q (i)\rangle \langle Q (i)\rangle, \quad (105)
\]

\[
\hat{\rho}_A = \sum_i |\alpha_i^\prime|^2 |A ("i")\rangle \langle A ("i")\rangle + \text{(small \perp terms)}, \quad (106)
\]

where the primes represent the \textit{very} tiny changes in basis needed to eliminate exponentially small off-diagonal corrections of order the product of (101) with (102),

\[
\text{off-diagonal corrections} \sim \exp \left(- \left( \text{#apparatus + environment degrees of freedom} \times \Delta t / \tau \right) \right). \quad (107)
\]

where \( \tau \sim \min (\tau_A, \tau_E) \), and where the \textit{double} primes appearing on the eigenvalues of \( \hat{\rho}_A \) account for the possibility that they might not precisely agree with the eigenvalues of \( \hat{\rho}_Q \), again due to discrepancies of order (107). No realistic ontic basis is infinitely sharp anyway in the context of our interpretation of quantum theory, and certainly no sharper than these sorts of corrections.

The exponential speed with which the corrections (107) become negligibly small—a phenomenon called decoherence of the originally coherent superposition (103)—is perfectly in keeping with the notion that a large apparatus conducting a measurement while sitting in an even larger environment leads to the appearance of the “instantaneous” wave-function collapse of the traditional Copenhagen interpretation, and indeed \textit{justifies} the use of wave-function collapse as a heuristic shortcut in elementary pedagogical treatments of the measurement process in quantum theory.44

In keeping with the basic correspondence (17) between objective epistemic states and density matrices at the heart of our interpretation of quantum theory, we can then conclude that the final actual ontic state of the subject system \( Q \) is one of the possibilities \( Q (i)\rangle \approx Q (i) \) with epistemic probability given by the corresponding density-matrix eigenvalue \( p_{Q,i} = |\alpha_i|^2 \approx |\langle Q (i) | \Psi \rangle|^2 \), up to exponentially small corrections (107). Similarly, the final actual ontic

---

44 Typical decoherence time scales for many familiar systems are presented in [184, 252, 253, 331].
state of the measurement apparatus $A$ is one of the possibilities $A ("i\,\prime") \approx A ("i")$ with approximately the same epistemic probability $p_{A \rightarrow \ell} = |\alpha_i'|^2 \approx |\langle Q (i)| \Psi \rangle|^2$, again up to exponentially small corrections (107). We therefore see that our interpretation of quantum theory solves the measurement problem by replacing axiomatic wave-function collapse with an underlying interpolating evolution for ontic states.

It is widely accepted that measurement-induced decoherence destroys quantum interference between distinct measurement outcomes and produces final objective density matrices for both subject systems and apparatuses that closely resemble the subjective mixtures that result from a so-called Von Neumann-Lüders projection [214, 305] without post-selection. (See Appendix 1 b.) Because the latter operation constitutes a linear CPT dynamical mapping, we see immediately that well-executed measurements by macroscopic apparatuses can be described to great accuracy by linear CPT dynamical mappings as well. Hence, to the extent that the measurement process experienced by the subject system $Q$ can be captured to an acceptable level of approximation by a linear CPT dynamical mapping $\mathcal{E}_Q : \Delta t^{-1} \to [\cdot]$ for $Q$ over the relevant time scale $\Delta t$ of the measurement, we directly find that the stochastic dynamical conditional probability (86) for $Q$ to end up in the final ontic state $Q (i') \approx Q (i)$ at time $t + \Delta t$ given that it was initially in the ontic state $\Psi$ at time $t$ is $p_{Q \rightarrow i} (i; t + \Delta t| \Psi; t) \approx |\langle Q (i)| \Psi \rangle|^2$, as expected. Similarly, if we can approximate the evolution of the apparatus $A$ as a linear CPT dynamical mapping, then $p_{A \rightarrow i'} (i'; t + \Delta t| \Psi'; t) \approx |\langle Q (i)| \Psi \rangle|^2$.

By any of these approaches, the famous Born rule

$$\text{Prob} (\ldots) = |\ldots|^2 \quad (108)$$

for computing empirical outcome probabilities therefore emerges automatically to an excellent approximation without having to be assumed a priori.

Notice that in line with the central correspondence (17) between objective epistemic states and density matrices, and the boxed statement (19) below it, the quantities $|\alpha_i|^2$ do not have an interpretation as literal epistemic probabilities for the underlying actual ontic states until they (or some approximate versions of them) have become actualized as eigenvalues of density matrices—that is, not until after the measurement is actually complete. Again, the only quantities that our minimal modal interpretation of quantum theory regards as being literal epistemic probabilities are those that our interpretation’s axioms explicitly identify as such.

We can also go beyond the Born rule for the individual subsystems and, importantly, say something jointly about their actual ontic states. The post-measurement reduced density matrix $\hat{\rho}_{Q \rightarrow A}$ of the composite system $Q + A$ consisting of the subject system together with the apparatus, computed according to the standard partial-trace operation (38), is not generically diagonal in exactly the “correct” tensor-product basis $|Q (j)'\rangle, A ("i'\prime") \rangle \equiv (|Q (j)'\rangle \otimes |A ("i'\prime")\rangle)$ with our original definitions of the subject system $Q$ and the apparatus $A$ as subsystems. Nonetheless, the diagonalizing basis for $\hat{\rho}_{Q \rightarrow A}$ is exponentially close to this tensor-product basis,

$$\hat{\rho}_{Q \rightarrow A} = \sum_{i,j} \alpha_i \alpha_j^* \left( \langle E ("i") | E ("j") \rangle \right) |Q (i)\rangle \langle Q (i)| \langle A ("i") | A ("j") \rangle,$$  \quad (109)

discrepancies and ensures with near-certainty that the final actual ontic states of the subsystems $Q$ and $A$ have correctly correlated values of the label $i$. We can therefore safely conclude that if the final actual ontic state of the apparatus is $A ("i'\prime") \approx A ("i")$, then the final actual ontic state of the subject system must with near-certainty be $Q (i') \approx Q (i)$ for the same value of $i$, and vice versa.

Looking back at (31), we also see that the off-diagonal corrections (107) to the final density matrix imply that the Born rule (108) is itself only ever accurate up to corrections $\sim \exp (-S \times \Delta t/\tau)$, where $S = S_{A + E}$ is the combined entropy of the measurement apparatus $A$ and environment $E$. As a consequence, all statistical quantities derived from the Born rule—including all expectation values, final-outcome state vectors, semiclassical observables, scattering cross sections, tunneling probabilities, and decay rates computed in quantum-mechanical theories and quantum field theories—likewise suffer from imprecisions $\sim \exp (-S \times \Delta t/\tau)$.

These results are nicely in keeping with the notion of our error-entropy bound (3), but are completely invisible in the traditional Copenhagen interpretation of quantum theory: In contrast to our own interpretation, the Copenhagen interpretation derives the partial-trace prescription (38) from the a priori assumption of the exact Born rule and features a Von Neumann-Lüders projection postulate [214, 305] that axiomatically sets the off-diagonal entries of the final density matrix precisely to zero and converts it into a proper mixture, as we explain in detail in Appendix 1.

On the other hand, for ordinary macroscopic measurement devices, our claimed corrections to the Born rule rapidly become far too small to notice, so our minimal modal interpretation of quantum theory is consistent with the predictions of the Copenhagen interpretation in most practical circumstances. Note that exponentially small corrections
of this kind should, in principle, also generically arise in any other interpretations of quantum theory (such as the Everett-DeWitt many-worlds interpretation) that attempt to derive the Born rule from decoherence.\footnote{These corrections to the Born rule have interesting consequences for gravitational physics: If every region of physical space has a bounded maximum entropy—given, say, in terms of the radius \( R \) and energy \( E \) of the region by the Bekenstein bound \( S_{\text{max}} = 2\pi k_B E R / \hbar c \) \cite{Bekenstein}}

Finally, it is easy to show within the scope of our analysis that if the dial on our apparatus \( A \) could display \emph{sequences} of outcomes arising from repeated measurements conducted in rapid succession, then the final density matrix of the apparatus would have non-negligible probability eigenvalues \emph{only} for possible ontic states of the form \( A ("1"\,\text{and}\,"1"\,\text{and}\,"1"\,\ldots) \) and \( A ("2"\,\text{and}\,"2"\,\text{and}\,"2"\,\ldots) \), but not for, say, \( A ("1"\,\text{and}\,"2"\,\text{and}\,"1"\,\ldots) \). Thus, realistic measurements, at least of the simplified Von Neumann type, produce robust, persistent, repeatable outcomes, provided that we are working over sufficiently short time scales that uncontrollable overall dynamics does not have time to alter our subject system \( Q \) appreciably. This observation provides yet another reason why our own interpretation of quantum theory, in contrast to the Copenhagen interpretation, does not need to assume wave-function collapse as a distinct axiomatic postulate.

\section*{B. Subjective Density Matrices and Proper Mixtures}

Having derived the Born rule (108) at last, we can now safely introduce the conventional practice of employing density matrices to describe \emph{subjective} epistemic states for quantum systems: Starting from the Born rule and given a subjective probability distribution \( p_\alpha \) over objective epistemic states described by a collection of density matrices \( \hat{\rho}_\alpha \) for a given system, where we allow that \( p_\alpha \) may be only a \emph{formal} probability distribution in the spirit of Section II B 3 and thus do not insist on mutual exclusivity, standard textbook arguments show that we can compute empirical expectation values in terms of the subjective density matrix

\begin{equation}
\hat{\rho} = \sum_\alpha p_\alpha \hat{\rho}_\alpha. \tag{110}
\end{equation}

Note, of course, that the mapping from formal subjective epistemic states to subjective density matrices is many-to-one because the mapping does not invariantly encode the original choice of objective density matrices \( \hat{\rho}_\alpha \)—the same subjective density matrix \( \hat{\rho} \) can be expressed in terms of infinitely many different choices of coefficients \( p_\alpha \) and their corresponding objective density matrices \( \hat{\rho}_\alpha \).

In the special case of a proper mixture, meaning an epistemic state that is \emph{wholly} subjective in nature, each objective density matrix \( \hat{\rho}_\alpha \) describes a single pure state \( \Psi_\alpha \) of the given system and thus the subjective density matrix (110) of the system takes the simpler form

\begin{equation}
\hat{\rho} = \sum_\alpha p_\alpha |\Psi_\alpha \rangle \langle \Psi_\alpha |. \tag{111}
\end{equation}

Note, however, that the von Neumann entropy formula \( S = -\text{Tr} \left[ \hat{\rho} \log \hat{\rho} \right] \) from (18) does not generically yield \(- \sum_\alpha p_\alpha \log p_\alpha \) for the proper mixture if \( p_\alpha \) is merely a \emph{formal} probability distribution over non-exclusive possible ontic states; the von Neumann entropy formula gives \(- \sum_\alpha p_\alpha \log p_\alpha \) only if \( p_\alpha \) is a \emph{logically rigorous} probability distribution involving \emph{mutually exclusive} possibilities \( \Psi_\alpha \) in the sense that the associated state vectors \( |\Psi_\alpha \rangle \) are mutually orthogonal—see (14)—in which case the quantities \( p_\alpha \) are the eigenvalues of \( \hat{\rho} \).

It is important to note that a proper mixture (111) fundamentally describes classical uncertainty over the state vector \( |\Psi_\alpha \rangle \) of a single system. For example, if a source generates a sequence of \( N \gg 1 \) physical copies of an elementary subsystem, with classical frequency ratios \( p_\alpha \in [0,1] \) for the emitted elementary subsystems to be described by definite corresponding state vectors \( |\Psi_\alpha \rangle \) in the elementary-subsystem Hilbert space \( \mathcal{H} \), then we can employ a subjective density matrix of the form (111) to provide an approximate description of the resulting proper mixture for a single randomly chosen copy of the elementary subsystem. However, in the idealized limit in which we can neglect any entanglements with the larger environment, the state of the parent system consisting of the \emph{entire physical sequence} of copies is not described by a nontrivial density matrix at all, but instead by a tensor-product state vector of the form

\begin{equation}
|\Psi \rangle = |\Psi_{\alpha_1} \rangle \otimes |\Psi_{\alpha_2} \rangle \otimes \cdots \otimes |\Psi_{\alpha_N} \rangle \tag{112}
\end{equation}

belonging to the tensor-product Hilbert space \( \mathcal{H}_{\text{sequence}} = \mathcal{H} \otimes \cdots \otimes \mathcal{H} \) (\( N \) factors) of the parent system. In essence, formally replacing the state vector (112) of the full parent system with the subjective density matrix (111) of an
arbitrary elementary subsystem—as is often done merely for practical convenience in order to simplify calculations—implicitly assumes that we have erased all information about the original ordering of the physical sequence. By contrast, if we do wish to retain and make use of information about the original ordering of the sequence—such as if we wish to examine or select particular subsequences of interest—then we must be careful not to confuse the full sequence’s own state vector (112) with the subjective density matrix (111) of an arbitrarily chosen member of the sequence, lest we run into apparent contradictions of the sort uncovered in [223].

C. Paradoxes of Quantum Theory Revisited

1. Schrödinger’s Cat, Wigner’s Friend, and the Local Nature of Ontology

In his famous 1935 thought experiment arguing against the Copenhagen interpretation [256], Schrödinger considered the hypothetical case of an unfortunate cat trapped in a perfectly sealed box together with a killing mechanism triggered by the decay of an unstable atom. If the experiment begins at time \( t = 0 \), and if the atom has a 50-50 empirical outcome probability of quantum-mechanically decaying over a nonzero time interval \( \Delta t \), then the atom’s state vector at exactly the time \( t = \Delta t \) is the quantum superposition

\[
|\text{atom}\rangle = \frac{1}{\sqrt{2}} (|\text{not decayed}\rangle + |\text{decayed}\rangle).
\]

(113)

Immediately thereafter, the linearity of time evolution would seem to imply that the final state vector of the composite parent system

\[
(\text{parent system}) \equiv (\text{atom}) + (\text{killing mechanism}) + (\text{cat}) + (\text{air in box}) + (\text{box})
\]

(114)

then has the highly entangled, quantum-superposed form

\[
|\text{parent system}\rangle = \frac{1}{\sqrt{2}} (|\text{not decayed}\rangle |\text{not triggered}\rangle |\text{alive}\rangle |\text{air in box}\rangle |\text{box}\rangle \\
+ |\text{decayed}\rangle |\text{triggered}\rangle |\text{dead}\rangle |(\text{air in box}')\rangle |(\text{box}')\rangle).
\]

(115)

Does this expression imply that the cat’s state of existence is smeared out into a quantum superposition of alive and dead, and remains in that superposed condition until a human experimenter opens the box to perform a measurement on the cat’s final state and thereby collapses the cat’s wave function?

The Everett-DeWitt many-worlds interpretation claims to resolve this paradoxical state of affairs by dropping the assumption that the final measurement carried out by the human experimenter has a definite outcome in any familiar sense. Instead, according to the many-worlds interpretation, the human experimenter merely “joins” the superposition upon opening the box to look inside, thereby splitting into two clones that each observe a different outcome. As always, beyond whatever metaphysical discomfort accompanies the notion that the world “unzips” into multiple copies with each passing moment in time (and, after all, such discomfort may be nothing more than human philosophical prejudice), there remains the tricky issue of making sense of probability itself when all possible outcomes of an experiment simultaneously occur, as well as understanding why the numerical coefficients \( 1/\sqrt{2} \) appearing in the superposed state vector (115) have the physical meaning of probabilities in the first place.

By contrast, our minimal modal interpretation of quantum theory resolves the problem much less extravagantly: We begin by noting that the moment the cat interacts with the killing mechanism (and inevitably also with the air in the box and with the walls of the box itself), the cat’s own density matrix rapidly decoheres to the approximate form

\[
\hat{\rho}_{\text{cat}} = \frac{1}{2} |\text{alive}\rangle \langle \text{alive}| + \frac{1}{2} |\text{dead}\rangle \langle \text{dead}|,
\]

up to small but unavoidable degeneracy-breaking corrections similar to those appearing in (22). According to our interpretation, the cat’s actual ontic state is therefore definitely alive or dead almost immediately after the

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46 The authors of [223] elide these subtle distinctions in order to generate a paradox in which the proper mixture describing a carefully post-selected subsequence of non-entangled two-qubit systems can also seemingly be represented as an entangled pure state, with the ultimate goal of casting doubt on interpretations of quantum theory (such as our own) that attempt to ascribe an ontological meaning to state vectors.

47 Schrödinger himself coined the term “entanglement” (Verschränkung) in this 1935 paper.

48 We will discuss possible connections between the many-worlds interpretation and our own interpretation in greater depth in Section V E.

49 Contrary to some accounts [151], at no time during the experiment is the cat alone ever represented by a pure-state superposition of the form \( (1/\sqrt{2}) (|\text{alive}\rangle + |\text{dead}\rangle) \).
As we explained in Section I D, there exists a very real possibility that there is no well-defined maximal closed system in a pure state

examine that system’s

own

where by “local” we simply mean system-centric: In order to establish the ontology of a given system, one must

expected.

conditional probability, the ontic state of the cat is alive and the ontic state of Wigner’s friend is WF (“alive”), as

composite system (cat) + (Wigner’s friend) has, say, the ontic state (alive

Hence, our instantaneous kinematical parent-subsystem quantum conditional probabilities (74) ensure that if the

matrix through the partial-trace operation (38).

their subsystems in Section III D 6.

the context of characterizing the precise kinematical relationship (74) between the ontic states of parent systems and

quantum-superposed existence would mean committing the same kind of fallacy of division that we first described in

velocity (contrary to colloquial assertions that galaxies sufficiently far from our own are “receding from our own galaxy

Just as there’s no well-defined sense in which widely separated galaxies in an expanding universe have a sensible relative

In particular, given a system whose ontic state involves a highly quantum superposition, naïvely assuming (as one
does in the Everett-DeWitt many-worlds interpretation) that each of the system’s subsystems likewise has a highly

superposed existence would mean committing the same kind of fallacy of division that we first described in

the context of characterizing the precise kinematical relationship (74) between the ontic states of parent systems and

their subsystems in Section III D 6.

50 As we explained in Section 1D, there exists a very real possibility that there is no well-defined maximal closed system in a pure state

and enclosing all other systems. In that case, we are forced to approach notions of ontology in a system-centric way in any realist

interpretation of quantum theory, including in the de Broglie-Bohm and Everett-DeWitt interpretations. Indeed, if we eventually have

to stop going up the never-ending hierarchy of open parent systems at some particular system’s reduced density matrix, then we have

no choice but to try to make sense of that reduced density matrix on its own terms.
Schrödinger intended his thought experiment to be a proof of principle against the Copenhagen interpretation, but it is worth asking whether his experiment could ever practically be realized (barring obvious ethical questions). Microscopic versions of his experiment involving small assemblages of particles, known as “Schrödinger kittens,” are performed all the time [20, 21, 136, 233, 268], and these experiments have even been scaled up beyond microscopic size in recent years [116, 232, 294, 321].

However, the systems involved in these experiments are far from what we would consider complex living organisms. The trouble is that maintaining a large system in a non-negligible superposition of macroscopically distinct states requires completely decoupling the system of interest from the sort of warm, hospitable environment that life as we know it seems to require. When we recognize moreover that all large living creatures constantly give off thermal radiation themselves, it becomes untenable to imagine that partial-tracing down to just the degrees of freedom that we associate with a living cat could ever yield an actual ontic state for the cat that remains in such a distinctly non-classical superposition for more than an infinitesimal instant in time. One can certainly imagine some day having the technological ability to construct a quantum superposition involving a frozen-solid cat at a temperature near absolute zero in an evacuated chamber, but such an experiment would hardly be in keeping with the original spirit of Schrödinger’s “paradox.”

2. The Quantum Zeno Paradox

The quantum Zeno paradox [33, 78, 90, 151, 225] demonstrates yet another limitation of the traditional Copenhagen interpretation, and so it is important to explain why the problem is avoided in decoherence-based interpretations of quantum theory such as the one that we present in this paper.

The paradox concerns the behavior of systems that experience rapid sequences of measurements but that otherwise undergo unitary time evolution according to some Hamiltonian $\hat{H}$. For time intervals $T$ that are sufficiently large compared to the inverse-energy-differences between the system’s energy eigenstates, but small enough that time-dependent perturbation theory is valid, the probability of observing a transition away from an unstable state $\Psi_{\text{initial}}$ is approximately linear in $T$,

$$p_{\text{initial} \rightarrow \text{anything} \neq \text{initial}} (T) \simeq \alpha T, \quad (117)$$

where $\alpha \simeq \text{const}$ is just the total transition rate away from $\Psi_{\text{initial}}$ and where we assume for consistency with perturbation theory that $\alpha$ is much smaller than $1/T$. Hence, the probability that a measurement after a time $T$ will find the system still in its original state $\Psi_{\text{initial}}$ is

$$p_{\text{initial}} (T) \simeq 1 - \alpha T. \quad (118)$$

It follows that if we perform $N \gg 1$ sequential measurements separated by time intervals $T = t/N$ over which (118) holds, then the probability of finding the system still in the initial state $\Psi_{\text{initial}}$ at the final time $t = N \times (t/N)$ is given by the familiar exponential decay law

$$p_{\text{initial}} (t) \simeq e^{-\alpha t}. \quad (119)$$

This argument helps explain why exponential laws are so ubiquitous among systems exhibiting quantum decay.

By contrast, for extremely tiny time intervals $\Delta t \rightarrow 0$, the Born rule (108) implies that the probability that a measurement after a time $\Delta t$ will find the system still in its original state $\Psi_{\text{initial}}$ depends quadratically on $\Delta t$,

$$p_{\text{initial}} (\Delta t) = \left| \langle \Psi_{\text{initial}} | e^{-i\hat{H}\Delta t/\hbar} | \Psi_{\text{initial}} \rangle \right|^2 = 1 - \beta^2 \Delta t^2, \quad (120)$$

where

$$\beta = \sqrt{\left< (\hat{H} - \left< \hat{H} \right>)^2 \right>/\hbar} = \frac{\Delta E}{\hbar} \sim \frac{1}{\tau} \quad (121)$$

is just the variance of the system’s energy in the state $\Psi_{\text{initial}}$ in units of $\hbar$, and is related to the system’s characteristic time scale $\tau$ through the energy-time uncertainty principle $\Delta E \times \tau \sim \hbar$. If we carry out $N \gg 1$ sequential measurements separated by time intervals $\Delta t = t/N \ll \beta$, then the probability that we will see the system still in its initial state $\Psi_{\text{initial}}$ at the final time $t = N \times (t/N)$ seemingly asymptotes to unity,

$$p_{\text{initial}} (t) \rightarrow 1, \quad (122)$$
apparently implying that the quantum state never transitions. This result, known as the quantum Zeno paradox (or
the watched-pot paradox), would appear to suggest that a continuously observed unstable system can never decay.

As with many of the supposed “paradoxes” of quantum physics, the quantum Zeno paradox can be resolved by
examining the measurement process more carefully. To begin, any interpretation of quantum theory (unlike the
Copenhagen interpretation) that regards measurements as physical processes implies that they occur over a nonzero
time interval, and therefore implies that $\Delta t$ cannot, in fact, be taken literally to zero. The notion of a continuously
observed system is therefore a fiction, and so we can safely rule out the quantum Zeno paradox in its strongest sense,
namely, in its suggestion that frequently observed systems should never decay.

But we can go further and explain why even a weaker version of the quantum Zeno effect—that is, a noticeable delay
in the system’s decay rate or a deviation in the exponential decay law (119)—is extremely difficult to produce and
observe. By momentarily connecting our unstable system to a measurement apparatus $A$ with its own temporal reso-

lution scale $\Delta t_A \sim \hbar/\Delta E_A$—meaning that we replace our initial state vector $|\psi_{\text{initial}}\rangle$ with $|A\rangle$—the resulting
interactions risk enlarging the system’s total energy variance and thereby increasing the coefficient $\beta$ appearing in the
decay probability (120) [138]. The quantum Zeno effect is therefore far from a generic process for systems exposed
to environmental interactions, and successfully producing the effect requires very careful experiments [176, 327].

3. Particle-Field Duality

Disputes crop up occasionally over whether particles or fields represent the more fundamental feature of reality.\textsuperscript{51}
Such questions are perfectly valid to ask in the context of certain interpretations of quantum theory, including the
so-called “fixed” modal interpretations—the de Broglie-Bohm pilot-wave interpretation being an example—in which
a particular Hilbert-space basis is singled out permanently as defining the system’s ontology.

From the standpoint of our own interpretation, however, these questions are much less meaningful. A quantum
field is a quantum system having some specific dynamics and a Hilbert space that is usefully described at low energies
and weak coupling as a Fock space.\textsuperscript{52} Some of the states in this Hilbert space look particle-like, in the sense of
being eigenstates of a particle-number operator, whereas—at least in the case of a bosonic system—other states in
the Hilbert space look more like classical fields, in the sense of being coherent states [145, 255] with a well-defined
classical field amplitude.\textsuperscript{53}

According to our interpretation of quantum theory, the ontic basis of the system is contextual: Experiments that
count quanta decohere the system to an ontic basis of particle-like states, and experiments that measure forces on test
bodies decohere the system to an ontic basis of field-like states. Hence, neither of these two possible classes of states
represents the “fundamental” basis for the system in any permanent or universal sense. Indeed, to say one of these
two classes of states is more fundamental than the other would be precisely like saying that the energy eigenstates
of a harmonic oscillator are more fundamental than its coherent states or vice versa, or like saying that a harmonic
oscillator “is really” made up of energy eigenstates or that it “is really” made up of coherent states.

V. LORENTZ INVARiance AND LOCALity

A. Special Relativity and the No-Communication Theorem

Special relativity links locality with causality, but only to the extent of forbidding observable signals from propa-
gating superluminally. Hidden variables that exhibit nonlocal dynamics are therefore not necessarily a problem:

The no-communication theorem [159, 239] ensures that quantum systems with local density-matrix dynamics do not
transmit superluminal observable signals, which could otherwise spell trouble for causality.

Indeed, consider the simplified example of a dynamically closed system with a Hamiltonian $\hat{H}$ having a separa-
decomposition of the form $\hat{H} = \hat{H}_A \otimes 1 + 1 \otimes \hat{H}_B$ for a pair of subsystems $A$ and $B$—as is necessarily the case in
practice if the subsystems $A$ and $B$ are well-separated from each other in space. Then it is straightforward to show
[36, 304] directly by taking the partial trace (38) of the unitary Liouville-Von-Neumann equation (12),

$$\frac{\partial \hat{\rho}(t)}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho}(t) \right],$$

\textsuperscript{51} For a recent instance, see [170].

\textsuperscript{52} Note that we use the term “quantum field” here to refer to the physical system itself, and not to mathematical field operators.

\textsuperscript{53} As we explained in Section III A 6, the reliance of the de Broglie-Bohm pilot-wave interpretation on a universal, permanent ontic basis
leads to serious trouble dealing with relativistic systems: States of particles with definite positions are unavailable in general, because
relativistic systems do not admit a basis of sharp orthonormal position eigenstates, and neither coherent states nor field eigenstates
exist for fermionic systems [275].
that the dynamics of the reduced density matrix $\hat{\rho}_A(t)$ of subsystem $A$ has no influence whatsoever on the reduced density matrix $\hat{\rho}_B(t)$ of subsystem $B$, and vice versa.

But a more subtle issue arises when we consider whether an interpretation’s hidden variables are well-defined under arbitrary choices of inertial reference frame, or, equivalently, under arbitrary choices of foliation of four-dimensional spacetime into the three-dimensional spacelike hyperplanes that define slices of constant time. We explore this and related questions in subsequent sections, including the well-known EPR-Bohm and GHZ-Mermin thought experiments, and ultimately conclude that our minimal modal interpretation of quantum theory is indeed consistent with Lorentz invariance and that its nonlocality is no more severe than is the case for classical gauge theories.

B. The EPR-Bohm Thought Experiment and Bell’s Theorem

In 1964, Bell [41] employed a thought experiment due to Bohm [56, 59] and based originally on a 1935 paper by Einstein, Podolsky, and Rosen [102] to prove that no realist interpretation of quantum theory in which experiments have definite outcomes could be based on dynamically local hidden variables.\[54\]

Bell’s theorem has exerted a powerful influence on the foundations of quantum theory in all the years since, to the extent that all interpretations today drop at least one of Bell’s stated assumptions—either the existence of hidden variables, realism, dynamical locality, or the assertion that experiments have definite outcomes. For example, the de Broglie-Bohm pilot-wave interpretation [57, 58, 61, 89] involves a hidden level of manifestly nonlocal dynamics, as do all of the modal interpretations [26, 27, 68, 73, 192–194, 295, 296, 298, 299] including our own. The Everett-DeWitt many-worlds interpretation [66, 91, 92, 94, 106–108, 307, 308, 318] does away with the assumption that experiments have definite outcomes, and instead asserts that all outcomes occur simultaneously and are thus equally real. (We argue in Section VI D 6 that the many-worlds interpretation, as traditionally formulated, still suffers from some residual nonlocality.)

It is interesting to recapitulate Bell’s arguments carefully in order to understand precisely how they interface with our own interpretation of quantum theory. Along the way, we will spot a subtle additional loophole in his assumptions, although our interpretation does not ultimately exploit this loophole and thus does not void Bell’s conclusions about nonlocality.\[55\]

1. The Basic Set-Up of the EPR-Bohm Thought Experiment

The EPR-Bohm set-up begins with a pair of distinguishable spin-1/2 particles—denoted particle 1 and particle 2—that are initially prepared in an entangled pure state of total spin zero, so that the ontic state $\Psi$ of the composite two-particle system 1 + 2 is described by the state vector (21):

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \quad \hat{S}_{1+2,z} |\Psi\rangle = 0. \quad (123)$$

We then consider two spatially separated spin detectors $A$ and $B$, where $A$ locally measures the spin of particle 1 and $B$ locally measures the spin of particle 2. We suppose that $A$ measures the component $S_{1,\vec{a}} = \vec{a} \cdot \vec{S}_1$ of the spin $\vec{S}_1$ of particle 1 along the direction of a three-dimensional spatial unit vector $\vec{a}$ and that $B$ measures the component $S_{2,\vec{b}} = \vec{b} \cdot \vec{S}_2$ of the spin $\vec{S}_2$ of particle 2 along the direction of a three-dimensional spatial unit vector $\vec{b}$. (See Figure 7.)

---

54 It is important to note again that, despite Bell’s theorem, any interpretation of quantum theory consistent with the theory’s standard observable rules does not permit superluminal observable signals, as ensured by the no-communication theorem [159, 239].

55 As we will discuss shortly, the same loophole also exists in the assumptions of similar no-go theorems, such as the CHSH theorem of [42, 81, 82, 200], although not in the GHZ-Mermin arguments to be described in Section V C.
2. Spin Correlation Predicted by Quantum Theory

After many repetitions of the entire experiment—each time preparing a composite two-particle system $1 + 2$ in the ontic state (123) and then allowing the spin detectors $A$ and $B$ to perform local measurements on the spins of their respective individual particles—we can ask for the average value of the product $S_{1,\vec{a}}S_{2,\vec{b}}$, which is just the correlation function for the pair of measured spins. Working in units of $\hbar/2$ for clarity (that is, setting $\hbar/2 \equiv 1$), quantum theory predicts that this expectation value should be given by

\[
\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle_{QM} = -\vec{a} \cdot \vec{b},
\]

a result that follows directly from linearity and rotation invariance.\(^{56}\)

3. The Bell Inequality

Bell argued that no dynamically local hidden variables could account for the result (124). To prove this claim, he derived a "triangle inequality"

\[
\left| \langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle_{LHV} - \langle S_{1,\vec{a}}S_{2,\vec{c}} \rangle_{LHV} \right| \leq 1 + \langle S_{1,\vec{b}}S_{2,\vec{c}} \rangle_{LHV}
\]

(125)

that all interpretations based on local hidden variables (LHV) would necessarily have to satisfy, where $\vec{c}$ is a unit vector specifying the alignment of a third detector $C$. The correct quantum result (124) can easily violate this inequality as well as various generalizations, as has been confirmed repeatedly in experiments \([17, 22, 23, 114, 143, 152, 250, 290, 291, 311]^{57}\).

To derive the inequality (125), Bell assumed that the hidden variables $\lambda$ obeyed a probability distribution $p(\lambda)$ satisfying the standard Kolmogorov conditions \([190]:\)

\[
0 \leq p(\lambda) \leq 1, \quad \int d\lambda \, p(\lambda) = 1.
\]

(126)

The proof of the Bell inequality (125), which we present in Appendix 2a, then rests entirely on manipulations of averages weighted by $p(\lambda)$.

4. The Additional Loophole

Bell’s theorem and all the various generalizations that have arisen in subsequent papers by Bell and others, including the CHSH inequalities \([42, 81, 82]\) and Legget’s inequality for “crypto-nonlocal” interpretations \([200]\), implicitly assume that the hidden variables $\lambda$ admit a sensible joint epistemic probability distribution $p(\lambda)$ as described in (126). As we made clear in Section III D 4, this assertion is nontrivial, but none of these papers attempt to justify its validity. However, this assumption represents a significant loophole, one that appears to have been noticed first by Bene \([17–52]\) in the context of his “perspectival” interpretation of quantum theory, but then also discovered independently and examined in a small number of additional papers \([169, 293, 329]\).\(^{58}\) Interesting though this loophole may be, we do not exploit it in our own interpretation.

---

\(^{56}\) Proof: First, observe that \[
\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle = \sum_i \sum_j a_i b_j \langle S_{1,i}S_{2,j} \rangle = \sum_i \sum_j a_i b_j F_{ij},
\]

where $F_{ij} = \langle S_{1,i}S_{2,j} \rangle$ is a $3 \times 3$ matrix that is manifestly independent of $\vec{a}$ and $\vec{b}$. Because our final result for $\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle$ must be real and invariant under global rotations, and should obviously have the specific value $-1$ if $\vec{a} = \vec{b}$, we must have $F_{ij} = -\delta_{ij}$. Hence, \[
\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle = \sum_i \sum_j a_i b_j (-\delta_{ij}) = -\vec{a} \cdot \vec{b}. \text{ QED}
\]

\(^{57}\) For example, if we take $\vec{a}$ and $\vec{b}$ to be orthogonal and for $\vec{c}$ to be $45$ degrees between both of them, then the quantum formula (124) for the spin correlation function yields $\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle = 0$, $\langle S_{1,\vec{a}}S_{2,\vec{c}} \rangle \simeq -0.707$, and $\langle S_{1,\vec{b}}S_{2,\vec{c}} \rangle \simeq -0.707$, in which case the Bell inequality (125) is expressly violated: $\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle - \langle S_{1,\vec{a}}S_{2,\vec{c}} \rangle \simeq 0.707 \leq 0.293 \simeq 1 + \langle S_{1,\vec{b}}S_{2,\vec{c}} \rangle$. For a discussion of potential shortcomings and loopholes in these Bell-test experiments, see, for example, \([125, 127, 135, 199]\).

\(^{58}\) Fine \([112, 113]\) appears to have worked on similar ideas in 1982. He studied whether one could assume the existence of the particular joint probability $p(A, B, 1, 2)$, but he did not regard the states of $A$, $B$, 1, and 2 as hidden variables themselves. Indeed, he assumed a deeper level of hidden variables $\lambda$ with their own probability distribution $p(\lambda)$, and he never considered that $p(\lambda)$ might not exist. Some interpretations of quantum theory, such as Bene’s, essentially identify the states of $A$, $B$, 1, 2 as hidden variables and question the existence of $p(\lambda)$ itself.
5. Analysis of the EPR-Bohm Thought Experiment

To the extent that our interpretation of quantum theory involves hidden variables (see Section III A 4), they are just the actual ontic states \( \lambda = \{ \Psi, \Psi_1, \Psi_2 \} \) of the two-particle parent system and of the individual one-particle subsystems, where \( \Psi \) is the entangled state defined in (123) and where \( \Psi_1 = \uparrow \) or \( \downarrow \). Before any measurements take place, the epistemic probabilities for the composite two-particle parent system \( 1 + 2 \) are

\[
\text{Before: } p_{1+2}(\Psi) = 1, \quad p_{1+2}(\text{states } \perp \Psi) = 0, \quad (127)
\]

the epistemic probabilities for the individual one-particle subsystems 1 and 2 are\(^{59}\)

\[
\text{Before: } p_1(\uparrow) = p_1(\downarrow) = \frac{1}{2}, \quad p_2(\uparrow) = p_2(\downarrow) = \frac{1}{2}, \quad (128)
\]

and the epistemic probabilities for the individual spin detectors A and B are

\[
\text{Before: } p_A(\"\Psi\") = 1, \quad p_B(\"\Psi\") = 1, \quad (129)
\]

with all epistemic probabilities for orthogonal possible ontic states being zero. Importantly, in accordance with our instantaneous kinematical conditional probabilities (74) (but contrary to the hypothetical loophole that we described in Section V B 4), the two particles 1 and 2 also have the initial joint epistemic probabilities

\[
\text{Before: } \begin{align*}
p_{1,2|1+2}(\uparrow, \downarrow | \Psi) &= \frac{1}{2}, \quad p_{1,2|1+2}(\downarrow, \uparrow | \Psi) = \frac{1}{2}, \\
p_{1,2|1+2}(\uparrow, \uparrow | \Psi) &= 0, \quad p_{1,2|1+2}(\downarrow, \downarrow | \Psi) = 0,
\end{align*} \quad (130)
\]

which ensure that the individual ontic states of the two particles are initially anti-correlated. If we introduce a macroscopic measurement apparatus \( M \) (unavoidably coupled to an even larger environment \( E \), at the very least through irreversible thermal radiation [158]) that will locally record and compare the two readings on the spin detectors after their pair of measurements, then before any measurements have taken place, its only nonzero epistemic probability is

\[
\text{Before: } p_M(\"\Psi\") = 1. \quad (131)
\]

It is easy to check that after a local spin measurement by either one of the spin detectors A or B aligned along the \( z \) direction, and the consequent rapid decoherence, the objective epistemic state of the composite two-particle system \( 1 + 2 \) becomes

\[
\text{After } A \text{ or } B: \quad p_{1+2}(\uparrow\downarrow') = p_{1+2}(\downarrow\uparrow') = \frac{1}{2}, \quad p_{1+2}(\uparrow\uparrow') = p_{1+2}(\downarrow\downarrow') = 0, \quad (132)
\]

where the primes denote corrections (107)—exponentially small in the number of (spin-detector) + (environment) degrees of freedom—to the two-particle basis states that are necessary to diagonalize the final density matrix \( \hat{\rho}_{1+2} \) exactly. Given that the ontic state of the composite system 1 + 2 after the measurement is \( (\uparrow\downarrow') \), a simple application of the kinematical parent-subsystem formula (74) shows that there is nearly unit probability that particle 1 is in the ontic state \( \uparrow \) and particle 2 is in the ontic state \( \downarrow \), whereas given that the ontic state of 1 + 2 is \( (\downarrow\uparrow') \) after the measurement, there is nearly unit probability that particle 1 is in the ontic state \( \downarrow \) and particle 2 is in the ontic state \( \uparrow \). These results are nicely consistent with our initial joint epistemic probabilities (130) for the two particles.

Taking partial traces (38) of the density matrix of the full system after both spin detectors A and B have performed their measurements, we find that the final epistemic probabilities for the individual one-particle subsystems 1 and 2 are given to very high numerical accuracy by

\[
\text{After: } \begin{align*}
p_1(\uparrow') &= p_1(\downarrow') = \frac{1}{2}, \quad p_2(\uparrow') &= p_2(\downarrow') = \frac{1}{2},
\end{align*} \quad (133)
\]

in approximate agreement with (128) before the measurements took place, where, again, the primes denote exponentially tiny corrections (107) to the definitions of the ontic basis states. Likewise, the final joint epistemic probabilities

---

\(^{59}\) As we explained in our discussion surrounding (22), there will unavoidably exist tiny degeneracy-breaking parameters in the entangled two-particle parent system’s ontic state \( \Psi \) that—by an appropriate choice of our coordinate system—pick out the spin-\( z \) basis for each one-particle subsystem.
for the two particles are given by

\[
\begin{align*}
\text{After:} & & p_{1,2|1+2}(\uparrow', \downarrow' | (\uparrow\downarrow)' &= 1, \\
& & p_{1,2|1+2}(\downarrow', \uparrow' | (\uparrow\downarrow)' &= 0, \\
& & p_{1,2|1+2}(\uparrow', \downarrow' | (\uparrow\downarrow)' &= 0, \\
& & p_{1,2|1+2}(\downarrow', \uparrow' | (\uparrow\downarrow)' &= 0, \\
& & p_{1,2|1+2}(\uparrow', \uparrow' | (\uparrow\downarrow)' &= 0, \\
& & p_{1,2|1+2}(\downarrow', \downarrow' | (\uparrow\downarrow)' &= 0.
\end{align*}
\]

(134)

As for the individual spin detectors \( A \) and \( B \), we find the final epistemic probabilities

\[
\begin{align*}
\text{After:} & & p_A ("\uparrow\uparrow") &= \frac{1}{2}, \\
p_A ("\downarrow\downarrow") &= \frac{1}{2}, \\
p_B ("\uparrow\downarrow") &= \frac{1}{2}, \\
p_B ("\downarrow\uparrow") &= \frac{1}{2},
\end{align*}
\]

(135)

and for the apparatus \( M \) that locally reads off the pair of results,

\[
\begin{align*}
\text{After:} & & p_M ("\text{found } A ("\uparrow\uparrow") \text{ and } B ("\downarrow\downarrow")") &= \frac{1}{2}, \\
p_M ("\text{found } A ("\downarrow\downarrow") \text{ and } B ("\uparrow\uparrow")") &= \frac{1}{2}.
\end{align*}
\]

(136)

Notice that if our apparatus \( M \) could immediately check the spin detectors again and record their dials a second time, then its final epistemic probabilities would be

\[
\begin{align*}
\text{After:} & & p_M ("\text{found } A ("\uparrow\uparrow") \text{ and } B ("\downarrow\downarrow") \text{ twice")} &= \frac{1}{2}, \\
p_M ("\text{found } A ("\downarrow\downarrow") \text{ and } B ("\uparrow\uparrow") \text{ twice")} &= \frac{1}{2},
\end{align*}
\]

(137)

thereby implying that the measurement results are robust and persistent, as expected.

The spin detectors \( A \) and \( B \) play the role of macroscopic, highly classical intermediaries between the apparatus \( M \) and the individual particles 1 and 2. As a consequence, we would face troubling metaphysical issues if we were unable to assert a strong connection between the final actual ontic states of the spin detectors \( A \) and \( B \) and the final actual ontic state of the apparatus \( M \) that looks at them. Fortunately, one can easily show that after environmental decoherence, the composite system \( W = 1 + 2 + A + B + M \) has the final epistemic state

\[
\begin{align*}
\text{After:} & & p_W ("\text{found } A ("\uparrow\uparrow") \text{ and } B ("\downarrow\downarrow") \text{ twice")}, A ("\uparrow\uparrow") , B ("\downarrow\downarrow") , \uparrow, \downarrow)' &= \frac{1}{2}, \\
p_W ("\text{found } A ("\downarrow\downarrow") \text{ and } B ("\uparrow\uparrow") \text{ twice")}, A ("\downarrow\downarrow") , B ("\uparrow\uparrow") , \down, \up)' &= \frac{1}{2},
\end{align*}
\]

(138)

where, as usual, primes denote exponentially suppressed corrections \((107)\) in the definitions of the ontic states that are necessary to diagonalize the density matrix of \( W \) exactly. Hence, invoking again our kinematical parent-subsystem probabilistic smoothness condition \((74)\), we can say with near-certainty at the level of conditional probabilities that if, say, the final actual ontic state of the apparatus \( M \) happens to be "\text{found } A ("\uparrow\uparrow") \text{ and } B ("\downarrow\downarrow") \text{ twice")", then the final actual ontic states of the two spin detectors are respectively \( A ("\uparrow\uparrow") \text{ and } B ("\downarrow\downarrow") \) and, furthermore, that the final actual ontic states of the individual particles are really \( \uparrow \) for 1 and \( \downarrow \) for 2. These results are obviously all consistent with our expectations for the post-measurement state of affairs.

6. More General Alignments and Nonlocality

Generating a violation of the Bell inequality \((125)\) necessitates choosing nonparallel detector alignments, so it is no surprise that our investigation so far has not turned up any clear evidence of nonlocality. Hence, let us now suppose that the alignment \( \vec{u} \) of spin detector \( A \) is along the \( x \) direction. For clarity, we will suppress the primes on ontic states, remembering always that decoherence is an imperfect process and that we should expect exponentially small corrections to all our results.

If spin detector \( A \) performs its measurement on particle 1 first, then after the resulting measurement-induced decoherence, a simple calculation shows that particle 1 now has possible ontic spin states \( \leftarrow (\text{spin } x \text{ left}) \) and \( \rightarrow (\text{spin} \)
This nonlocal influence on the final measurement result obtained by the system of three distinguishable spin-1/2 particles 1, 2, and 3 in an initial pure state represented by the entangled assumption of joint probability distributions for the hidden variables, and can be captured by talking about just a local hidden variables. Unlike the arguments employed in Bell’s theorem, the GHZ result does not depend on the Bohm thought experiment involving three or more particles with spin could not be described in terms of dynamically our quantum conditional probabilities safely allow for a hidden level of nonlocality.

On the other hand, suppose that before performing its own local spin measurement, spin detector $B$ changes its alignment $\vec{b}$ to point along the $x$ direction just like the alignment $\vec{a}$ of spin detector $A$. In that case, if we use the local dynamics of particle 2 and spin detector $B$ in our dynamical conditional probabilities (86), then we find that regardless of whether the initial actual ontic state of particle 2 was $\uparrow$ or $\downarrow$, the conditional probability for $B$ to obtain either “$\leftarrow$” or “$\rightarrow$” for particle 2 is 1/2, independent of the result “$\leftarrow$” or “$\rightarrow$” obtained by spin detector $A$ for particle 1; this result is unsurprising because by tracing out spin detector $A$ and particle 1, we have chosen to ignore crucial information about the necessary final anti-correlation between the spins of the two particles. But if we instead condition on either the initial or final actual ontic state of the composite parent two-particle system 1 + 2, which has evolved nonlocally during the experiment, then a simple calculation yields with unit probability that the final measurement outcome for the spin of particle 2 by spin detector $B$ obtained by $B$ is in line with our claim in Section III D 5 that our quantum conditional probabilities safely allow for a hidden level of nonlocality.

C. The GHZ-Mermin Thought Experiment

In a 1989 paper [150], Greenberger, Horne, and Zeilinger (GHZ) showed that certain generalizations of the EPR-Bohm thought experiment involving three or more particles with spin could not be described in terms of dynamically local hidden variables. Unlike the arguments employed in Bell’s theorem, the GHZ result does not depend on the assumption of joint probability distributions for the hidden variables, and can be captured by talking about just a single measurement outcome. Following the simplest version of the GHZ argument, described in detail by Mermin in 1990 [221], we consider a system of three distinguishable spin-1/2 particles 1, 2, and 3 in an initial pure state represented by the entangled state vector

$$|\Psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right),$$  \hspace{1cm} (139)

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are one-particle eigenstates of the spin-$z$ operator $\hat{S}_z$ and where again we work in units for which $\hbar/2 \equiv 1$:

$$\hat{S}_z |\uparrow\rangle = + |\uparrow\rangle, \quad \hat{S}_z |\downarrow\rangle = - |\downarrow\rangle.$$  \hspace{1cm} (140)

It is then straightforward to show that the GHZ state vector described by (139) is a definite eigenstate of the three operators $\hat{S}_{1,x}\hat{S}_{2,y}\hat{S}_{3,y}$, $\hat{S}_{1,y}\hat{S}_{2,x}\hat{S}_{3,x}$, and $\hat{S}_{1,y}\hat{S}_{2,y}\hat{S}_{3,x}$ with eigenvalue +1,

$$\begin{align*}
\hat{S}_{1,x}\hat{S}_{2,y}\hat{S}_{3,y} |\Psi_{\text{GHZ}}\rangle &= + |\Psi_{\text{GHZ}}\rangle, \\
\hat{S}_{1,y}\hat{S}_{2,x}\hat{S}_{3,y} |\Psi_{\text{GHZ}}\rangle &= + |\Psi_{\text{GHZ}}\rangle, \\
\hat{S}_{1,y}\hat{S}_{2,y}\hat{S}_{3,x} |\Psi_{\text{GHZ}}\rangle &= + |\Psi_{\text{GHZ}}\rangle,
\end{align*}$$  \hspace{1cm} (141)

but is also a definite eigenstate of the operator

$$\hat{S}_{1,x}\hat{S}_{2,x}\hat{S}_{3,x} = - \left(\hat{S}_{1,x}\hat{S}_{2,y}\hat{S}_{3,y}\right) \left(\hat{S}_{1,y}\hat{S}_{2,x}\hat{S}_{3,y}\right) \left(\hat{S}_{1,y}\hat{S}_{2,y}\hat{S}_{3,x}\right)$$

with eigenvalue $-1$,

$$\hat{S}_{1,x}\hat{S}_{2,y}\hat{S}_{3,x} |\Psi_{\text{GHZ}}\rangle = - |\Psi_{\text{GHZ}}\rangle.$$  \hspace{1cm} (142)

One can carefully list all the possible “local instructions” that we could imagine somehow packaging along with each of the individual particles to ensure agreement with the outcomes required by the first three eigenvalue equations (141),

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60 Hardy [161, 162] has found certain two-particle variants of the GHZ construction that likewise do not depend on assumptions regarding joint probability distributions.
but not one of those sets of local instructions could then accommodate the fourth eigenvalue equation (142). The consequence is that any hidden variables accounting for all these possible measurement results must change nonlocally during the course of the experiment in order to ensure agreement with the necessary eigenvalue equations.

Lest one suppose that this issue involves only subatomic particles, we could consider placing macroscopic spin detectors $A$, $B$, and $C$ respectively near each of the three particles (see Figure 8), and include also a larger measurement apparatus $M$ that will ultimately visit each spin detector at the end of the full experiment and then compare their final readings, so that the initial state vector of the full parent system $W = 1 + 2 + 3 + A + B + M$ is

$$|\Psi_W\rangle = \frac{1}{\sqrt{2}} (|↑↑↑\rangle - |↓↓↓\rangle) |A (“∅”\rangle) |B (“∅”\rangle) |C (“∅”\rangle) |M (“∅”\rangle).$$  \hspace{1cm} (143)

If the spin detectors $A$, $B$, and $C$ perform their various local measurements on each particle’s spin-$x$ or spin-$y$ component, then the hidden ontic states of these macroscopic spin detectors would need to interact nonlocally in order to ensure that when the apparatus $M$ finally comes along to look at them, it would find agreement with the four eigenvalue equations in (141) and (142).

### D. The Myrvold No-Go Theorem

Building on a paper by Dickson and Clifton [97] and employing arguments similar to Hardy [161], Myrvold [227] argued that modal interpretations are fundamentally inconsistent with Lorentz invariance at a deeper level than mere unobservable nonlocality, leading to much additional work [54, 101, 228] in subsequent years to determine the implications of his result. Specifically, Myrvold argued that one could not safely assume that a quantum system, regardless of its size or complexity, always possesses a specific actual ontic state beneath its epistemic state, because any such actual ontic state could seemingly change under Lorentz transformations more radically than, say, a four-vector does—for example, binary digits 1 and 0 could switch, words written on a paper could change, and true could become false. The only way to avoid this conclusion would then be to break Lorentz symmetry in a fundamental way by asserting the existence of a “preferred” Lorentz frame in which all ontic-state assignments must be made.

Myrvold’s claims, and those of Dickson and Clifton as well as Hardy, rest on several invalid assumptions. Dickson and Clifton [97] assume that the hidden ontic states admit certain joint epistemic probabilities that are conditioned on multiple disjoint systems at an initial time, and such probabilities are not a part of our interpretation of quantum theory, as we explained in Section III D 4.61 Similarly, one of Myrvold’s arguments hinges on the assumed existence of joint epistemic probabilities for two disjoint systems at two separate times, and again such probabilities are not present in our interpretation. As Dickson and Clifton point out in Appendix B of their paper, Hardy’s argument also relies on several faulty assumptions about ontic property assignments in modal interpretations.

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61 The same implicit assumption occurs in Section 9.2 of [288].
A final argument of Myrvold—and repeated by Berkovitz and Henno [54]—is based on inadmissible assumptions about the proper way to implement Lorentz transformations in quantum theory and the relationship between density matrices and ontic states in our minimal modal interpretation. Myrvold’s mistake is subtle, so we go through his argument in detail.

1. Measurements Performed at Different Times and in Different Inertial Reference Frames

Myrvold begins by considering the general case of a composite parent system \( W = 1 + A + 2 + B \) consisting of a pair of well-localized, isolated subsystems 1 and 2 at spacelike separation and with respective observables \( X \) and \( Y \), together with a macroscopic detector \( A \) localized near subsystem 1 and a macroscopic detector \( B \) localized near subsystem 2. Letting \( P_X (x) \) for outcome \( x \) denote an arbitrary eigenprojector of \( X \), and, similarly, letting \( P_Y (y) \) for outcome \( y \) denote an arbitrary eigenprojector for \( Y \), Myrvold argues that the probability that \( B \) obtains \( y \) in a measurement of \( Y \), and then, at a later time, \( A \) obtains \( x \) in a measurement of \( X \), is given by

\[
\text{Tr}_W \left[ \left( \hat{U}_1^\dagger \hat{P}_X (x) \hat{U}_1 \right) \otimes \hat{P}_Y (y) \hat{\rho}_W \right],
\]

where \( \hat{U}_1 \) is the unitary time-evolution operator for subsystem 1 over the time interval between the measurements. This formula follows directly from a double application of the Born rule—once for \( \hat{Y} \) and then again for \( \hat{X} \)—with possibly different Lorentz-boosted matrices and ontic states in our minimal modal interpretation. Myrvold’s mistake is subtle, so we go through his argument in detail.

Because the entire experiment could have been described from a safe distance by an observer \( M \) who was moving at relativistic speed relative to \( W \) and for whom the measurements were simultaneous, Myrvold then asserts on empirical grounds that (144) must agree with

\[
\text{Tr}_W \left[ \hat{P}_X (x) \otimes \hat{P}_Y^\dagger (y) \hat{\rho}_W \right],
\]

where the primes refer to the fact that the operators inside the trace have been appropriately Lorentz-boosted compared to their unprimed counterparts. Extracting the unitary operators \( \hat{U}_{\Lambda,1} \) and \( \hat{U}_{\Lambda,2} \) that respectively implement the Lorentz boost on the two eigenprojectors,

\[
\hat{P}_X (x) = \hat{U}_{\Lambda,1} \hat{P}_X (x) \hat{U}_{\Lambda,1}^\dagger, \quad \hat{P}_Y (y) = \hat{U}_{\Lambda,2} \hat{P}_Y (y) \hat{U}_{\Lambda,2}^\dagger,
\]

Myrvold then uses the arbitrariness of the choice of eigenprojectors to show that

\[
\hat{\rho}_W = \left( \hat{U}_{\Lambda,1} \otimes \hat{U}_{\Lambda,2} \right) \left( \hat{U}_1^\dagger \otimes \hat{1}_2 \right) \hat{\rho}_W \left( \hat{U}_1^\dagger \otimes \hat{1}_2 \right) \left( \hat{U}_{\Lambda,1}^\dagger \otimes \hat{U}_{\Lambda,2}^\dagger \right).
\]

In other words, Lorentz transformations act on density matrices in a more subtle manner than other kinds of unitary transformations, because they also involve changes in time—hence the factor of \( \left( \hat{U}_1^\dagger \otimes \hat{1}_2 \right) \) and its adjoint.

Indeed, a density matrix is not a local object, but is more properly associated with an entire three-dimensional spacelike slice of constant time. And so, when we perform a Lorentz transformation, we have to be careful to account for the attendant change in time-slice, with possibly different consequences for different well-localized subsystems. In the present case, we account for this effect using the time-evolution operator \( \hat{U}_1 \). An important corollary for our interpretation of quantum theory is that the ontic states of a spatially extended composite system can change nontrivially under Lorentz transformations that shift some of their subsystems across large temporal displacements, although the same is obviously true even in classical physics; indeed, under a sufficiently large Lorentz transformation, we can significantly alter the “current” state of far-away galaxies.

For “fixed” modal interpretations like [68], for which there exists a universal ontic basis independent of density matrices or arbitrary choices of orthonormal basis, we are free to drop the Lorentz-transformation operator \( \hat{U}_{\Lambda,1} \otimes \hat{U}_{\Lambda,2} \) and its adjoint from (147), because it merely represents a physically meaningless unitary change of basis.\(^{62}\) However, in our modal interpretation, the full right-hand-side of (147) is necessary for yielding the correct density matrix \( \hat{\rho}_W \)—and the correct ontic basis—for the parent system \( W \) in the Lorentz-boosted frame. That is, without including the Lorentz-transformation operator \( \hat{U}_{\Lambda,1} \otimes \hat{U}_{\Lambda,2} \), the left-hand-side simply wouldn’t be the density matrix of any relevant physical system, and so its eigenstates would have no ontological meaning. Myrvold neglects this important fact completely, and we will see that it invalidates the applicability of his no-go theorem to our interpretation of quantum theory.

---

\(^{62}\) As Myrvold writes on p. 1776 of [227]: “Now, the Lorentz boost operators \( \Lambda, \Lambda' \) merely effect a transformation from a state given with respect to one reference frame’s coordinates to one given with respect to another reference frame’s coordinates. In what follows, it will be more convenient to utilize the coordinate basis of one reference frame \( \Sigma \) for all states, even those on hypersurfaces that are not equal-time hyperplanes for \( \Sigma \).”
Following Myrvold [227], we next examine a scenario inspired by the EPR-Bohm thought experiment of Section V B but seen from several different inertial reference frames, each corresponding to a different foliation of spacetime into three-dimensional slices of constant time.

Consider a composite four-state system \( W = 1 + A + 2 + B \) consisting of two localized qubits 1 and 2 that each have orthonormal basis states \( |+\rangle \) and \( |-\rangle \) and that lie very far apart in space at sharply defined positions. As in the later part of Myrvold’s papers, as well as in Berkovitz and Hemmo’s treatment [54], consider also a macroscopic qubit detector \( A \) located near qubit 1 and a similar macroscopic qubit detector \( B \) located near qubit 2, where the qubit detector \( A \) has ontic states \( A (\Psi) \), \( A (\Psi^\prime) \), and \( A (\Psi^\prime\prime) \) respectively describing its initial state and its allowed measurement outcomes for qubit 1, and, similarly, the qubit detector \( B \) has ontic states \( B (\Psi) \), \( B (\Psi^\prime) \), and \( B (\Psi^\prime\prime) \) respectively describing its initial state and its allowed measurement outcomes for qubit 2. We imagine that the local detectors measure their respective qubits approximately continuously, so that they always show the correct readings on their dials.

As with the spin detectors in the EPR-Bohm thought experiment analyzed in Section V B, and in parallel with our discussion surrounding (138), the qubit detectors \( A \) and \( B \) will play an important role as macroscopic, highly classical intermediaries between observers and the quantum qubits themselves. In particular, the presence of \( A \) and \( B \) makes clear our eventual metaphysical need for joint statements about the final actual ontic states of the various subsystems at the end of the experiment. At the very least, the final results recorded by observers should be correlated with the final actual ontic states of the detectors \( A \) and \( B \).

Suppose that on one particular three-dimensional constant-time slice, which we will denote by \( \alpha \), the parent system \( W = 1 + A + 2 + B \) is in a pure state represented by the post-measurement state vector

\[
|\Psi_W (\alpha)\rangle = + \frac{1}{\sqrt{12}} |+\rangle |A (\Psi)\rangle |+\rangle |B (\Psi)\rangle - \frac{1}{\sqrt{12}} |+\rangle |A (\Psi^\prime)\rangle |-\rangle |B (\Psi^\prime)\rangle - \frac{1}{\sqrt{12}} |-\rangle |A (\Psi^\prime\prime)\rangle |+\rangle |B (\Psi^\prime\prime)\rangle - \sqrt{\frac{9}{12}} |-\rangle |A (\Psi^\prime\prime)\rangle |-\rangle |B (\Psi^\prime\prime)\rangle .
\]

(See Figure 9.) Taking the partial trace (38) to obtain the corresponding reduced density matrix \( \hat{\rho}_{1+2} (\alpha) \) for the composite two-qubit subsystem 1 + 2, we find that it describes the possible ontic states \( (+_{1+2}), (-_{1+2}), (-_{1+2}), \) and \( (+_{1-2}) \), each with significant nonzero epistemic probabilities given respectively by \( 1/12, 1/12, 1/12, 9/12 \); that is,

\[
\hat{\rho}_{1+2} (\alpha) = \frac{1}{12} |+\rangle |+\rangle \langle +| + \frac{1}{12} |+\rangle |1\rangle \langle 2| + \frac{1}{12} |1\rangle \langle -2| + \frac{1}{12} |1\rangle \langle 1|2\rangle + \frac{9}{12} |1\rangle \langle 1| -2\rangle - \frac{1}{12} |1\rangle \langle 1| -2\rangle ,
\]

(149)

where, as explained in our discussion surrounding (22), we implicitly assume small degeneracy-breaking corrections to avoid the measure-zero case of any exactly equal probability eigenvalues. According to our interpretation of quantum theory, the actual ontic state of the composite two-qubit subsystem 1 + 2 on the constant-time slice \( \alpha \) could be any one of the four possible ontic states \( (+_{1+2}), (+_{1-2}), (-_{1+2}), (-_{1-2}) \).

Next, Myrvold supposes that the system undergoes unitary time evolution governed by a local time-evolution operator \( \hat{U}_{1+A} \otimes \hat{U}_{2+B} \) given by the following Hadamard transformations:

\[
\begin{align*}
\hat{U}_{1+A} |+\rangle |A (\Psi)\rangle & = \frac{1}{\sqrt{2}} (|+\rangle |A (\Psi)\rangle + |-\rangle |A (\Psi^\prime)\rangle), \\
\hat{U}_{1+A} |-\rangle |A (\Psi^\prime)\rangle & = \frac{1}{\sqrt{2}} (|+\rangle |A (\Psi^\prime)\rangle - |-\rangle |A (\Psi^\prime)\rangle), \\
\hat{U}_{2+B} |+\rangle |B (\Psi)\rangle & = \frac{1}{\sqrt{2}} (|+\rangle |B (\Psi)\rangle + |-\rangle |B (\Psi^\prime)\rangle), \\
\hat{U}_{2+B} |-\rangle |B (\Psi^\prime)\rangle & = \frac{1}{\sqrt{2}} (|+\rangle |B (\Psi^\prime)\rangle + |-\rangle |B (\Psi^\prime)\rangle).
\end{align*}
\]

(150)

\footnote{This label \( \alpha \) should not be confused with the coordinate parameterization we used for subsystem spaces in Section III E.}
(Notice that the entanglement between each qubit and its corresponding detector implies that the evolution of the qubit alone is not unitary, although it can be captured by a suitable linear CPT dynamical mapping.) The pure state of the parent system \( W = 1 + A + 2 + B \) then ends up represented by the following state vector on the later constant-time slice \( \beta \):

\[
\begin{aligned}
|\Psi_W (\beta)\rangle &= -\frac{1}{\sqrt{3}} |+1\rangle |A ("+" )\rangle |+2\rangle |B ("+" )\rangle \\
&\quad + \frac{1}{\sqrt{3}} |+1\rangle |A ("+" )\rangle |-2\rangle |B ("-" )\rangle \\
&\quad + \frac{1}{\sqrt{3}} |-1\rangle |A ("-" )\rangle |+2\rangle |B ("+" )\rangle .
\end{aligned}
\]

The possible ontic states of the composite two-qubit subsystem \( 1 + 2 \) are the same as before, but now with epistemic probability zero for \( (-1-2) \):

\[
\hat{\rho}_{1+2} (\beta) = \frac{1}{3} |+1\rangle |+2\rangle \langle +1| (+2| + \frac{1}{3} |+1\rangle |-2\rangle \langle +1| (-2| + \frac{1}{3} |-1\rangle |+2\rangle \langle -1| |+2\rangle .
\]

Now Myrvold considers a Lorentz-boosted (and therefore “tilted”) spacelike constant-time slice \( \gamma \) intersecting the localized composite detector-qubit subsystem \( 1 + A \) at the final time and intersecting the detector-qubit subsystem \( 2 + B \) at the initial time. Then, looking back at the unitary transformation law (147), we see that we can obtain the state vector representing the parent system’s pure state on the constant-time slice \( \gamma \) by starting with the pure state (148) on the constant-time slice \( \alpha \), carrying out the local unitary time evolution just on \( 1 + A \), and then acting with the Lorentz-boost operator \( \left( \hat{U}_{A,1+A} \otimes \hat{U}_{A,2+B} \right) \), with the result being

\[
\begin{aligned}
|\Psi_W (\gamma)\rangle &= -\sqrt{2} \frac{1}{3} |\hat{U}_{A,1+A} (|+1\rangle |A ("+" )\rangle )\rangle \langle \hat{U}_{A,2+B} (-2| |B ("-" )\rangle )| \\
&\quad + \frac{1}{\sqrt{3}} |\hat{U}_{A,1+A} (|-1\rangle |A ("-" )\rangle )\rangle \langle \hat{U}_{A,2+B} (+2| |B ("+" )\rangle )| \\
&\quad + \frac{1}{\sqrt{3}} |\hat{U}_{A,1+A} (|-1\rangle |A ("-" )\rangle )\rangle \langle \hat{U}_{A,2+B} (-2| |B ("-" )\rangle )|
\end{aligned}
\]

\[
\text{Crucially, Myrvold drops the Lorentz-transformation operators} \hat{U}_{A,1+A} \text{ and} \hat{U}_{A,2+B} \text{ from his corresponding expression for (153), despite that fact that, at the very least,} \hat{U}_{A,1+A} \text{ has a significant temporal effect on the composite detector-qubit subsystem} 1 + A, \text{ which is located far from the spacetime-center of the Lorentz boost. Moreover, because} \hat{U}_{A,1+A} \text{ implements a significant time shift on} 1 + A, \text{ and in light of the entanglement-inducing Hadamard time evolution (150),} \hat{U}_{A,1+A} \text{ is certainly not factorizable into separate transformation operators for qubit} 1 \text{ and detector} A, \text{ meaning that qubit} 1 \text{ will necessarily change according to a non-unitary dynamical mapping. It is therefore impossible to arrive at Myrvold’s subsequent conclusion that the reduced density matrix of the composite two-qubit system} 1 + 2 \text{ on this new constant-time slice} \gamma \text{ is given by}
\]

\[
\hat{\rho}_{1+2} (\gamma) = \frac{2}{3} |++\rangle \langle ++| + \frac{1}{6} |+-\rangle \langle +-| + \frac{1}{6} |-+\rangle \langle -+|,
\]

which has zero epistemic probability for \( (+1+2) \). The same erroneous reasoning applies to Myrvold’s second choice of Lorentz-boosted constant-time slice \( \delta \) intersecting the localized composite detector-qubit subsystem \( 1 + A \) at the initial time and the detector-qubit subsystem \( 2 + B \) at the final time.

Had these results held up, Myrvold would have been able to compare ontic states on each of the constant-time slices \( \alpha, \delta, \gamma, \text{ and} \beta \) and thereby arrive at the conclusion that the composite two-qubit system must be in the final ontic state \( (-1-2) \) on the constant-time slice \( \beta \), in seeming contradiction with the epistemic probabilities encoded in the density matrix (152). However, without these central results, Myrvold’s no-go theorem breaks down.

E. Quantum Theory and Classical Gauge Theories

Suppose that we were to imagine reifying all of the possible ontic states defined by each system’s density matrix as simultaneous actual ontic states in the sense of the many-worlds interpretation. Then because every density matrix

\footnote{Observe that the eigenbasis of each system’s density matrix therefore defines a preferred basis for that system alone. We do not assume the sort of universe-spanning preferred basis shared by all systems that is featured in the traditional many-worlds interpretation; such a universe-spanning preferred basis would lead to new forms of nonlocality, as we describe in Section V D 6.}
as a whole evolves locally, no nonlocal dynamics between the actual ontic states is necessary and our interpretation of quantum theory becomes manifestly dynamically local: For example, each spin detector in the EPR-Bohm or GHZ-Mermin thought experiments possesses all its possible results in actuality, and the larger measurement apparatus locally “splits” into all the various possibilities when it visits each spin detector and looks at the detector’s final reading.

To help make sense of this step of adding unphysical actual ontic states into our interpretation of quantum theory, recall the story of classical gauge theories, and specifically the example of the Maxwell theory of electromagnetism. The physical states of the theory involve only two possible polarizations, corresponding fundamentally to the two physical spin states of the underlying species of massless spin-1 gauge boson that we call the photon.

In a unitary gauge, meaning a choice of gauge in which we describe the theory using just two polarization states for the gauge field $A^\mu$, the theory appears to be nonlocal and not Lorentz covariant. For instance, in Weyl-Coulomb gauge, we impose the two manifestly non-covariant gauge conditions $A^0 = 0$ and $\nabla \cdot \vec{A} = 0$, which together eliminate the timelike and longitudinal polarizations of the gauge field $A^\mu$ and thereby leave intact just the two physical transverse states orthogonal to the direction of wave propagation. In this “ontologically correct” choice of gauge, the vector potential $\vec{A}$ becomes a nonlocal action-at-a-distance function of the distribution of currents over all of three-dimensional space [177], and is no longer part of a true Lorentz four-vector [313].

However, if we formally add an additional unphysical polarization state by replacing the two Weyl-Coulomb gauge conditions $A^0 = 0$ and $\nabla \cdot \vec{A} = 0$ with the weaker but Lorentz-covariant single condition $\partial_\mu A^\mu = (1/c) \partial A^0 / \partial t - \nabla \cdot \vec{A} = 0$ that defines Lorenz gauge, then we obtain a manifestly local, Lorentz-covariant description of the Maxwell theory in which all the gauge potentials become true Lorentz four-vectors given by causal integrals over appropriately time-delayed distributions of charges and currents. Because the Maxwell theory therefore has a mathematical description in which the nonlocality of the ontologically correct unitary gauge disappears, we see that apparent nonlocality of the Maxwell theory is totally harmless.

Nonetheless, switching to Lorenz gauge does not imply that the extra unphysical polarization state that we have formally added to the description achieves a true ontological status, and so we take precisely the same view toward the addition of extra unphysical “actual” ontic states that would provide a manifestly local mathematical description of our own interpretation of quantum theory. Just as different choices of gauge for a given classical gauge theory make different calculations or properties of the theory more or less manifest—each choice of gauge inevitably involves trade-offs—we see that switching from the “unitary gauge” corresponding to our interpretation of quantum theory to the “Lorenz gauge” in which it looks more like a density-matrix-centered version of the many-worlds interpretation makes the locality and Lorentz covariance of the interpretation more manifest at the cost of obscuring the interpretation’s underlying ontology and the meaning of probability.

In this analogy, gauge potentials $A^\mu$ correspond to ontic states $\Psi_i$, which can similarly undergo nonlocal changes. Gauge transformations $A^\mu \rightarrow A^\mu + \partial^\mu \lambda$ describe an unobservable change in the gauge theory’s ontology, and are analogous in our interpretation of quantum theory to carrying out a simultaneous but fundamentally unobservable

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**Figure 9.** A spacetime diagram representing the Myrvold thought experiment, consisting of particles 1 and 2 together with qubit detectors $A$ and $B$. The constant-time slices $\alpha$, $\beta$, $\gamma$, and $\delta$ are also shown.
shift in the hidden actual ontic states (and thus also the memories) of all our systems, such as by reassigning each system’s hidden (“private”) actual ontic state to one of the other possible ontic states defined by the system’s density matrix. Just as all observable predictions of electromagnetism can be expressed in terms of gauge-invariant, dynamically local quantities like electric fields \( \vec{E} \) and magnetic fields \( \vec{B} \), all outwardly observable statistical predictions of quantum theory ultimately derive from density matrices \( \hat{\rho} \), which are insensitive to our identification of each system’s hidden actual ontic state from among its possible ontic states and which always evolve locally in accordance with the no-communication theorem \([159, 239]\).

Seen from this perspective, we can also better understand why it is so challenging \([174]\) to make sense of a many-worlds-type interpretation as an ontologically and epistemologically reasonable interpretation of quantum theory: Attempting to do so leads to as much metaphysical difficulty as trying to make sense of the Lorenz gauge of Maxwell electromagnetism as an “ontologically correct interpretation” of the Maxwell theory.\(^{65}\) Hence, taking a lesson from classical gauge theories, we propose instead regarding many-worlds-type interpretations as merely a convenient mathematical tool—a particular “gauge choice”—for establishing definitively that a given “unitary-gauge” interpretation of quantum theory like our own is ultimately consistent with locality and Lorentz invariance.

VI. CONCLUSION

A. Summary

In this paper, we have introduced what we call the minimal modal interpretation of quantum theory. Our interpretation consists of several parsimonious ingredients:

1. In (13), we define ontic states \( \Psi_i \), meaning the states of the given system as it could actually exist in reality, in terms of arbitrary (unit-norm) state vectors \( |\Psi_i\rangle \) in the system’s Hilbert space \( \mathcal{H} \),

\[
\Psi_i \leftrightarrow |\Psi_i\rangle \in \mathcal{H} \text{ (up to overall phase)},
\]

and we define epistemic states \( \{(p_i, \Psi_i)\}_i \) as probability distributions over sets of possible ontic states,

\[
\{(p_i, \Psi_i)\}_i, \quad p_i \in [0,1],
\]

where these definitions parallel the corresponding notions from classical physics. We translate logical mutual exclusivity of ontic states \( \Psi_i \) as mutual orthogonality of state vectors \( |\Psi_i\rangle \), and we make a distinction between subjective epistemic states (proper mixtures) and objective epistemic states (improper mixtures): The former arise from classical ignorance and are uncontroversial, whereas the latter arise from quantum entanglements to other systems and do not have a widely accepted \textit{a priori} meaning outside of our interpretation of quantum theory. Indeed, the problem of interpreting objective epistemic states may well be unavoidable: Essentially all realistic systems are entangled to other systems to a nonzero degree and thus cannot be described exactly by pure states or by purely subjective epistemic states.\(^{66}\)

2. We posit a correspondence (17) between objective epistemic states \( \{(p_i, \Psi_i)\}_i \) and density matrices \( \hat{\rho} \):

\[
\{(p_i, \Psi_i)\}_i \leftrightarrow \hat{\rho} = \sum_i p_i |\Psi_i\rangle \langle \Psi_i|.
\]

The relationship between subjective epistemic states and density matrices is not as strict, as we explain in Section IVB.

3. We invoke the partial-trace operation \( \hat{\rho}_Q \equiv \text{Tr}_E[\hat{\rho}_{Q+E}] \), motivated and defined in (38) without appeals to the Born rule or Born-rule-based averages, to relate the density matrix (and thus the epistemic state) of any subsystem \( Q \) to that of any parent system \( W = Q + E \).

\(^{65}\) Indeed, in large part for this reason, some textbooks \([313]\) develop quantum electrodynamics fundamentally from the perspective of Weyl-Coulomb gauge.

\(^{66}\) As we explain in Section 1D, there are reasons to be skeptical of the common assumption that one can always assign an exactly pure state or purely subjective epistemic state to “the universe as a whole.”
4. We introduce a general class of quantum conditional probabilities (68),

\[ p_{Q_1,\ldots,Q_n|W}(i_1,\ldots,i_n; t') = \text{Tr}_W \left[ \left( \hat{P}_{Q_1}(i_1) \otimes \cdots \otimes \hat{P}_{Q_n}(i_n) \right) \hat{P}_W(w) \right] \]

relating the possible ontic states of any partitioning collection of mutually disjoint subsystems \(Q_1,\ldots,Q_n\) to the possible ontic states of a corresponding parent system \(W = Q_1 + \cdots + Q_n\) whose own dynamics is governed by a linear completely-positive-trace-preserving ("CPT") dynamical mapping \(\mathcal{E}_{W}^{t'\rightarrow t}\) over the given time interval \(t' - t\). Here \(\hat{P}_W(w; t)\) denotes the projection operator onto the eigenspace \(|\Psi_W(w; t)\rangle\) of the density matrix \(\hat{\rho}_W(t)\) of the parent system \(W\) at the initial time \(t\), and, similarly, \(\hat{P}_{Q_\alpha}(i; t')\) denotes the projection operator onto the eigenspace \(|\Psi_{Q_\alpha}(i; t')\rangle\) of the density matrix \(\hat{\rho}_{Q_\alpha}(t')\) of the subsystem \(Q_\alpha\) at the final time \(t'\) for \(\alpha = 1,\ldots,n\). In a rough sense, the dynamical mapping \(\mathcal{E}_{W}^{t'\rightarrow t}\) acts as a parallel-transport superoperator that moves the parent-system projection operator \(\hat{P}_W(w; t)\) from \(t\) to \(t'\) before we compare it with the subsystem projection operators \(\hat{P}_{Q_\alpha}(i; t')\). As special cases, these quantum conditional probabilities provide a kinematical smoothing relationship (74),

\[ p_{Q_1,\ldots,Q_n|W}(i_1,\ldots,i_n; w) = \text{Tr}_W \left[ \left( \hat{P}_{Q_1}(i_1) \otimes \cdots \otimes \hat{P}_{Q_n}(i_n) \right) \hat{P}_W(w) \right] = \langle \Psi_{W,w} | \left( \langle \Psi_{Q_1,i_1} | \langle \Psi_{Q_1,i_1} | \cdots \langle \Psi_{Q_n,i_n} | \langle \Psi_{Q_n,i_n} | \right) | \Psi_{W,w} \rangle , \]

between the possible ontic states of any partitioning collection of mutually disjoint subsystems \(Q_1,\ldots,Q_n\) and the possible ontic states of the corresponding parent system \(W = Q_1 + \cdots + Q_n\) at any single moment in time, and, taking \(Q = Q_1 = W\), also provide a dynamical smoothing relationship (86),

\[ p_Q(j; t'|i; t) = \text{Tr}_Q \left[ \hat{P}_Q(j) \otimes \mathcal{E}_{Q}^{t'\rightarrow t} \left[ \hat{P}_Q(i; t) \right] \right] \sim \text{Tr}_Q \left[ \hat{P}_j(t') \mathcal{E} \left[ \hat{P}_i(t) \right] \right] , \]

between the possible ontic states of a system \(Q\) over time and also between the objective epistemic states of a system \(Q\) over time.

Essentially, 1 establishes a linkage between ontic states and epistemic states, 2 establishes a linkage between (objective) epistemic states and density matrices, 3 establishes a linkage between parent-system density matrices and subsystem density matrices, and 4 establishes a linkage between parent-system ontic states and subsystem ontic states as well as between parent-system epistemic states and subsystem epistemic states, either at the same time or at different times.

After verifying that our quantum conditional probabilities satisfy a number of consistency requirements (69)-(73), we showed that they allow us to avoid ontological instabilities that have presented problems for other modal interpretations, analyzed the measurement process, studied various familiar "paradoxes" and thought experiments, and examined the status of Lorentz invariance and locality in our interpretation of quantum theory. In particular, our interpretation accommodates the nonlocality implied by the EPR-Bohm and GHZ-Mermin thought experiments without leading to superluminal signaling, and evades claims by Myrvold purporting to show that interpretations like our own lead to unacceptable contradictions with Lorentz invariance. As a consequence of its compatibility with Lorentz invariance, we claim that our interpretation is capable of encompassing all the familiar quantum models of physical systems widely in use today, from nonrelativistic point particles to quantum field theories and even string theory. We also compared our interpretation to some of the other prominent interpretations of quantum theory, such as the de Broglie-Bohm pilot-wave interpretation and the Everett-DeWitt many-worlds interpretation, and concluded that we could view the latter interpretation as being a local "Lorentz gauge" of our own interpretation.

### B. Falsifiability and the Role of Decoherence

As some other interpretations do, our own interpretation puts decoherence in the central role of transforming the Born rule from an axiomatic postulate into a derived consequence and thereby solving the measurement problem of quantum theory. We regard it as a positive feature of our interpretation of quantum theory that falsification of the capacity for decoherence to manage this responsibility would mean falsification of our interpretation. We therefore also take great interest in the ongoing arms race between proponents and critics of decoherence, in which critics offer up examples of decoherence coming up short [10, 12–14, 234] and thereby push proponents to argue that increasingly realistic measurement set-ups involving non-negligible environmental interactions resolve the claimed inconsistencies [29, 172, 173, 298].
C. Future Directions

1. Understanding and Generalizing the Hilbert-Space Structure of Quantum Theory

The Hilbert-space structure underlying quantum theory is closely related to the principle of linear superposition. Interesting recent work has examined whether this Hilbert-space structure can be motivated from more primitive ideas \[163\], perhaps by a “purification postulate” that all mixed states should have a correspondence to pure states in some formally defined larger system, together with the known linear structure of classical spaces of mixed epistemic states.

Note that requiring the vector space \( \mathcal{H} \) to be complex is necessary for the existence of energy as an observable, as well as for the existence of states having definite energy, at least for systems that are dynamically closed and thus possess a well-defined Hamiltonian in the first place; however there exist subtle ways to get around these requirements \[276, 277, 328\]. And because the Born rule \( (108) \) involves absolute-value-squares of inner products, one could also, in principle, explore dropping the requirement that the Hilbert space’s inner product must be positive definite, although then avoiding the dynamical appearance of “null” state vectors having vanishing norm (and thus vanishing probability) requires a delicate choice of Hamiltonian. \[67\] One could also try to alter the definition of the inner product to involve a PT transformation \[44–46\]. These and other approaches \[1\] to modifying the Hilbert-space structure of quantum theory may have interesting implications for our interpretation of quantum theory that would be worth exploring.

2. Coherent States

For a system with continuously valued degrees of freedom, the similarity between the classical Liouville equation \( (10) \) and the quantum Liouville equation \( (12) \) becomes much closer if we re-express the \( 2^N \)-dimensional classical phase space \( (q,p) \) and the Poisson brackets \( (8) \) in terms of the \( N \) dimensionless complex variables

\[
z_{\alpha} \equiv \frac{1}{\sqrt{2}} \left( \frac{1}{\ell_{\alpha}} q_{\alpha} + i \frac{\ell_{\alpha}}{\hbar} p_{\alpha} \right) \quad (155)
\]

and their complex conjugates \( z_{\alpha}^* \), where \( \ell_{\alpha} \) are characteristic length scales in the system \[272\]. In that case, introducing the complexified Poisson brackets

\[
\{f,g\}_{z,z^*} \equiv \sum_{\alpha} \left[ \frac{\partial f}{\partial z_{\alpha}} \frac{\partial g}{\partial z_{\alpha}^*} - \frac{\partial g}{\partial z_{\alpha}} \frac{\partial f}{\partial z_{\alpha}^*} \right], \quad (156)
\]

the classical Liouville equation takes a much more quantum-looking form, complete with the familiar prefactor of \( -i/\hbar \) on the right-hand side:

\[
\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} \{H, \rho\}_{z,z^*}. \quad (157)
\]

The complex variables \( z_{\alpha} \) are natural for another reason, namely, because they label the corresponding quantum system’s coherent states \( |z\rangle \equiv |\{z_{\alpha}\}_{\alpha} \rangle \quad [145, 255] \), which are defined to be the solutions to the eigenvalue equations \( \hat{z}_{\alpha} |z\rangle = z_{\alpha} |z\rangle \), with \( q_{\alpha} \equiv \sqrt{2\ell_{\alpha}} \text{Re} z_{\alpha} \) and \( p_{\alpha} \equiv \sqrt{2\hbar \text{Im} z_{\alpha}/\ell_{\alpha}} \) the expectation values of the operators \( \hat{q}_{\alpha} \) and \( \hat{p}_{\alpha} \). Coherent states are the closest quantum analogues to classical states:

- They saturate the Heisenberg uncertainty bounds \( \Delta q_{\alpha} \Delta p_{\alpha} \geq \hbar/2 \);
- they have Gaussian wave functions in both coordinate space and momentum space that both approach delta functions in the limit \( \hbar \to 0 \);
- they become orthogonal in the limit of large coordinate separation \( \sum_{\alpha} |z_{\alpha} - z'_{\alpha}|^2 \gg 1 \);
- they each occupy an \( N \)-dimensional disc of approximate volume \( (2\pi\hbar)^N = \hbar^N \) in \( N \)-dimensional phase space \( (q,p) \) and thereby nicely account for phase-space quantization;

\[67\] Unphysical null and negative-norm (“ghost”) states also arise when formally enlarging the Hilbert space of gauge theories in order to make their symmetries more manifest, but in the present context we are imagining that we treat positive- and negative-norm states as both being physical.
• they satisfy the overcompleteness relation \( \int (1/\pi^N) \, d^2 z^N \, |z\rangle \langle z| \), with a measure \( d^2 z^N / \pi^N = dq^N \, dp^N / \hbar^N \) that exactly replicates the familiar phase-space measure from semiclassical statistical mechanics;

• and, for coupled systems with small interactions between them, the rate at which coherent states become mutually entangled is very low, so that they remain uncorrelated even in the macroscopic limit.\(^{68}\)

Moreover, we can regard delta-function coordinate-basis eigenstates as coherent states in the limit \( \ell_\alpha \to 0 \), and delta-function momentum-basis eigenstates as coherent states in the limit \( \ell_\alpha \to \infty \). If we instead choose the length scales \( \ell_\alpha \) so that the terms in the Hamiltonian quadratic in coordinates and momenta are proportional to \( \sum_\alpha z^\dagger_\alpha z_\alpha \), then the corresponding coherent states remain coherent states under unitary time evolution over short time intervals, in which case the expectation values \( q_\alpha(t) \) and \( p_\alpha(t) \) approximately follow the same dynamical equations as would be expected if the system were classically and governed by second-order dynamics; this last fact, in particular, helps explain the ubiquity of second-order dynamics among classical systems with continuous degrees of freedom.

Because there is no real observable whose Hermitian operator’s eigenstates are coherent states, systems don’t end up in coherent states as a consequence of a Von Neumann measurement. Instead, sufficiently large systems with continuously valued degrees of freedom end up approximately in coherent states due to messy environmental perturbations because coherent states are so robust [289, 332]. A set of coherent states farther apart in phase space than the Planck constant \( h \) form an approximately orthogonal set, and we can imagine extending such a set to an orthonormal basis suitable for the spectrum of a macroscopic system’s density matrix by including additional state vectors as needed; for small \( h \), coherent states are very, very close to being orthogonal, so this approximation becomes better and better in the formal classical limit \( h \to 0 \). It would be worthwhile to investigate this story in greater detail as it relates to our interpretation of quantum theory.

3. The First-Quantized Formalism for Quantum Theories, Quantum Gravity, and Cosmology

In describing first-quantized closed systems of particles or strings [39, 242, 243], as well as in canonical methods for describing quantum gravity [93, 224], it is often useful to work with a generalized Hilbert space consisting of infinitely many copies of the given system’s physical Hilbert space, each copy corresponding to one instant along a suitable temporal parameterization. Unitary dynamics is then expressed as a Hamiltonian constraint equation of the Wheeler-DeWitt form \( \hat{H}|\Psi\rangle = 0 \). It would be an intriguing exercise to study how to re-express the formalism of our interpretation of quantum theory in this alternative framework, and, indeed, how to accommodate open systems exhibiting more general linear CPT dynamics. More broadly, we look forward to exploring quantum features of black holes and cosmology within the context of our interpretation, including the measure problem of eternal inflation [115, 129, 132, 156, 301].

D. Relevant Metaphysical Speculations

1. The Status of Superdeterminism

Throughout this paper, we have repeatedly emphasized the importance and nontriviality of the existence of dynamics for a classical or quantum system. In particular, the existence of dynamics for a system is a much stronger property than the mere possession of a particular kinematical trajectory by the system. Indeed, one could easily imagine a “superdeterministic” universe in which every system has some specific trajectory—written down at the beginning of time on some mystical “cosmic ledger,” say—but has no dynamics (not even deterministic dynamics) in the sense of the existence of a mapping (4) for arbitrary values of initial ontic states and that allows us to compute hypothetical alternative trajectories.\(^{69}\)

It is therefore a remarkable fact that so many systems in Nature—the Standard Model of particle physics being an example par excellence—are well described by dynamics simple enough that we can write them down on a sheet of paper. The time evolution of most systems apparently encodes far less information than their complicated trajectories might naively suggest—that is, the information encoded in their trajectories is highly compressible—and certainly far less information than would be expected for systems belonging to a superdeterministic universe.\(^{70}\)

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\(^{68}\) As explained in [184], “This argument may explain the dominance of the field aspect over the particle aspect for boson[ic] fields.”

\(^{69}\) It’s important to realize that the existence of dynamics—even deterministic dynamics—still leaves open a great deal of flexibility via initial conditions. Building on this idea, Aaronson [3] suggests that if our universe is not superdeterministic but is instead governed by “merely” deterministic or probabilistically stochastic dynamical laws, then un-cloneable quantum details (“freebits”) of our observable universe’s initial conditions could allow for the kind of non-probabilistic uncertainty that we discussed in Section 111D4 and thus make room for notions of free will, much as the low entropy of our observable universe’s initial conditions makes room for a thermodynamic arrow of time.

\(^{70}\) Following Aaronson’s classification of philosophical problems in [3], the Q question “Is the universe superdeterministic?” may be
A perennial debate in metaphysics concerns the fundamental nature of time itself: Presentism is the philosophical proposition that the present moment in time is an ontologically real concept, whereas an obvious alternative is that our notion of “the moving now” is merely an illusion experience by beings who actually live in a “block-time” universe whose extent into the past and future are equally real. In the classical deterministic case, we can always depict spacetime in block form, and we can even accommodate nontrivial stochastic dynamics by considering appropriate ensembles of block-time universes [3].

However, according to our interpretation of quantum theory, naïve notions of reductionism generically break down at the microscale, as we explained immediately following (74), and thus epistemic states for microscopic quantum systems each have their own time evolution and don’t sew together in a classically intuitive manner; indeed, the same may well be true on length scales that exceed the size of our cosmic horizon [63]. Hence, the notion of a block-time universe, which is, in a certain sense, a “many-times” interpretation of physics, may turn out to be just as untenable as the many-worlds interpretation of quantum theory.

3. Sentient Quantum Computers

In Section III A 4 and elsewhere in this paper, we have repeatedly emphasized that our interpretation of quantum theory regards ontic states as being irreducible objects, rather than as being epistemic probability distributions over a more basic set of preferred basis states or hidden variables. Because human brains are warm systems in constant contact with a messy larger environment, their reduced density matrices and thus their ontic states are guaranteed by decoherence to look classical and not to involve macroscopic quantum superpositions [288]; putting a human brain into an overall quantum superposition would therefore seem to require completely isolating the brain from its environment (including its blood supply) and lowering its temperature to nearly absolute zero, in which case no human awareness is conceivable. We can therefore sidestep metaphysical questions about the subjective first-person experiences of human observers who temporarily exist in quantum superpositions of mutually exclusive, classical-looking state vectors.

However, one might imagine someday building a sentient quantum computer capable of human-level intelligence. In principle, and in contrast to human beings, such a machine would be perfectly functional even at a temperature close to absolute zero and without any need for continual interactions with a larger environment. Could such a machine achieve something like subjective first-person experiences, and, if so, how would the machine experience existing in a quantum superposition of mutually exclusive state vectors? Or is there something about the existence of subjective first-person experiences that fundamentally requires continual decoherence and information exchange with a larger environment?

A recurring problem in the history of interpretations of quantum theory is the tendency for the subject to become mixed up with other thorny problems in philosophy. Just as our interpretation deliberately aims to be model independent and thereby attempts to avoid getting tangled up in the philosophical debate over the rigorous meaning of probability and its many schools of interpretation (from Laplacianism to frequentism to Bayesianism to decision theory), we also intentionally avoid making any definitive statements about a preferred philosophy of mind and the important metaphysical problem of understanding the connection between the physical third-person reality of atoms, planets, and galaxies and the subjective first-person reality of colors, thoughts, and emotions. By design, our interpretation concerns itself solely with physical third-person reality, and does not favor any of the schools of thought on the reality of first-person experiences (from dualism to eliminativism to functionalism to panpsychism).

4. The Wigner Representation Theorem

As we mentioned in Section 1D, we have no evidence for the existence of a closed maximal parent system described by a cosmic pure state or universal wave function—there may only be a succession of increasingly large parent systems that are all in nontrivial improper mixtures—so there may be “no place to stand” to say that we wish to perform a global physical transformation.

But looking again at the Wigner representation theorem [313, 319], we can now provide a new definition of physical transformations that does not depend on the existence of such cosmic pure-state systems: We can simply say that

fundamentally unanswerable, but our present discussion suggests the more meaningful and well-defined Q’ question “How compressible is trajectory information for the systems that make up the universe?”

71 We thank Scott Aaronson (private communication) for suggesting these points.
our system admits a particular class of physical transformations if there exists some sort of active way of altering
the physical state of the external observer (which we now understand is equivalent to using the phrase “passive
transformation”) that leaves the external observer’s own calculated empirical outcome probabilities invariant. That
statement is still perfectly true, and one can express it at the level of mathematics in terms of unitary or anti-unitary
transformations formally acting on the subject system, provided that we keep in mind that the transformation of the
external observer is not truly changing the subject system.

This general line of reasoning makes possible a very concise proof of the Wigner representation theorem. Consider a
subject system $Q$ together with an external observer $O$ in the presence of a larger environment $E$, and suppose that the
composite system $Q + O$ goes through a process in which $O$ performs a measurement on $Q$ and then the environment
$E$ immediately causes decoherence of the final density matrix of $Q + O$ to a diagonalizing eigenbasis whose individual
eigenstates correspond to the expected measurement outcomes. We require that the resulting probability eigenvalues
of the final post-measurement density matrix of the composite system $Q + O$ must be the same regardless of whether or
not we actively transformed the observer $O$ before the experiment. This condition is equivalent to the usual definition
of a physical transformation for the purposes of the Wigner representation theorem, as it translates directly into an
invariance constraint on Born-rule probabilities. Note that the physical transformation is to be performed before the
experiment, and thus before we have a density matrix with our final probability eigenvalues.

We can now employ the following trick: If we know in advance what observable the observer $O$ is planning to
measure, then, even before the experiment has been performed, we can pretend that the subject system $Q$ was already
in a mixed state whose density matrix is diagonal in the eigenbasis of that observable’s Hermitian operator, with
the Born-rule probabilities appearing as the eigenvalues of this mixed-state density matrix. Taking this approach
ends up producing the same final density matrix for the composite system $Q + O$ at the end of the experiment
and with the correct final probability eigenvalues, and so we can safely substitute our mixed-state density matrix
for the subject system $Q$ into the Born-rule formulas in advance. Then, to the extent that we can always pretend
inside our Born-rule formulas that a passive transformation—really an active transformation of just the external
observer $O$—is mathematically equivalent to an (inverse) transformation of the subject system $Q$, we can demand
that the probability eigenvalues of the mixed-state density matrix of $Q$ should remain unchanged under the physical
transformation. Because the only transformations that leave the eigenvalues of a general Hermitian matrix unchanged
are unitary or anti-unitary transformations, the theorem is proved. QED

5. Dynamical Symmetries and the Absence of Preferred Perspectives

As an aside, it is worth noting that a physical transformation in the present context coincides with the notion of an
(unbroken) dynamical symmetry when it maps solutions of the dynamics to new solutions of the same dynamics—
for example, at the classical level, because the physical transformation leaves the given system’s action functional
invariant up to possible boundary terms. Not all physical transformations are dynamical symmetries in this sense;
for example, although a change in the length of all our rulers may be a physical transformation in the sense of
Wigner’s representation theorem, it is not a dynamical symmetry of Nature because there exist plenty of well-known
physical systems whose dynamical equations involve dimensionful length scales and are therefore not scale-invariant.
By contrast, translations in space, as well as Lorentz transformations, appear to be dynamical symmetries of Nature.

In particular, if a given physical transformation happens to correspond to a dynamical symmetry, then there
cannot exist a fundamentally “preferred” perspective with respect to that symmetry transformation. For example,
because spatial translations are a dynamical symmetry of Nature, there cannot exist a fundamentally preferred
location in space, and because Lorentz transformations are also a dynamical symmetry of Nature, there cannot exist a
fundamentally preferred inertial reference frame, at least when considering sufficiently small spacetime intervals that
general relativity reduces to special relativity.

6. Nonlocality in the Everett-DeWitt Many-Worlds Interpretation

Although often claimed to be manifestly local, the Everett-DeWitt many-worlds interpretation of quantum theory
can only maintain this manifest locality by abandoning any sharply defined probabilistic branching structure that
spans all systems or assertions that it can solve the preferred-basis problem, as we mentioned in Section III A 9.
To see explicitly how this problem arises, consider again the standard EPR-Bohm experiment that we originally
examined in Section V B. The initial state vector of the composite system $1 + 2 + A + B$ is

$$|\Psi_{1+2+A+B}\rangle = \frac{1}{\sqrt{2}} (|↑\downarrow\rangle - |↓\uparrow\rangle) |A ("0")\rangle |B ("0")\rangle.$$
How should one regard this state vector in terms of branches (“worlds”) and their associated probabilities according to the many-worlds interpretation?

If the claim is that there exists just one branch with unit probability and on which the state vector of the two-particle system 1 + 2 is really |Ψ_{1+2}⟩ = (1/√2) (|↑↓⟩ − |↓↑⟩), then one runs into trouble with nonlocality, because after the spin detector A performs its local measurement on particle 1, two branches instantaneously emerge on which particles 1 and 2 are suddenly classically correlated (↑↓ in one branch and ↓↑ in the other branch) despite not having been classically correlated before.

An alternative approach is to argue that there were actually two branches with respective probabilities of 1/2 all along, meaning that one should regard the initial state vector of the composite system 1 + 2 + A as really being

\[ |Ψ_{1+2+A+B}⟩ = \frac{1}{√2} |↑↓⟩ |A ("ψ")⟩ |B ("ψ")⟩ − \frac{1}{√2} |↓↑⟩ |A ("ψ")⟩ |B ("ψ")⟩ \).

In that case, the two particles were really classically correlated on each branch even before the experiment and thus no “new” classical correlation suddenly appears nonlocally after the spin detector A carries out its local measurement on particle 1. But then one runs into a serious problem making the notion of the branches and their probabilities well-defined, because there is nothing that privileges splitting up the branches in the spin-z basis for the two-particle system 1 + 2; indeed, one could just as well have split up the branches in the spin-x basis (|←⟩ and |→⟩) instead, in which case the initial state vector of the composite system 1 + 2 + A + B would be

\[ \frac{1}{√2} |←→⟩ |A ("ψ")⟩ |B ("ψ")⟩ − \frac{1}{√2} |→←⟩ |A ("ψ")⟩ |B ("ψ")⟩ \).

One is therefore confronted directly with the preferred-basis problem, and decoherence cannot help resolve the paradox because the spin detectors A and B haven’t performed their measurements yet and thus all the various possible bases are on an equal footing.\(^{72}\)

So what are the branches? Which basis is the “correct” one for deciding? As we have seen in this section, if we pick one preferred definition for the branches that span the systems under consideration—and their associated probabilities—then we immediately run into the nonlocality issue again; for example, if we pick, say, the spin-x branches, and then A performs a spin-z measurement, then again the branches need to change in a nonlocal way in order to ensure the correct final correlation. If we declare that we must be noncommittal about assigning entangled states to branches, then we run into the trouble that entanglement is a ubiquitous phenomenon afflicting all systems to a nonzero degree, and thus we don’t obtain sharp definitions of what we mean by branches. Because there is no fixed choice of branch-set compatible with manifest locality, the many-worlds interpretation isn’t manifestly local unless we give up any notion of a preferred branch set that span the systems, but then we lose any hope of making sense of probabilities in the interpretation.

E. Comparison with the Hollowood Modal Interpretation

Our minimal modal interpretation of quantum theory differs in several key aspects from the recently introduced “emergent Copenhagen interpretation” by Hollowood [171–173], with whom we have collaborated over the past year. Specifically, we employ a different, manifestly non-negative, more general formula for our quantum conditional probabilities that doesn’t depend fundamentally on a temporal cut-off time scale; we use these quantum conditional probabilities not only for dynamical purposes but also to interpolate between the ontologies of parent systems and their subsystems; our quantum conditional probabilities allow ontic states of disjoint systems to influence each other; joint ontic-state and epistemic-state assignments exist for mutually disjoint systems; and we accept the inevitability of ontic-level nonlocality implied by the EPR-Bell and GHZ-Mermin thought experiments.

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\(^{72}\) Note that these conclusions cannot be evaded by the assumption of small degeneracy-breaking effects of the form (22), which would merely have the effect of increasing the total number of branches.
organizers Andrew Friedman and Elizabeth Petrik, as well as with Scott Aaronson and Ned Hall. Both authors are indebted to Steven Weinberg, who has generously exchanged relevant ideas and whose own recent paper [316] also explores the idea of reformulating quantum theory in terms of density matrices. The work of Pieter Vermaas has also been a continuing source of inspiration for both authors, as have many conversations with Allan Blaer.

APPENDIX

In this appendix, we summarize the traditional Copenhagen interpretation, in part just to establish our notation and terminology. With this background established, we then define the measurement problem and systematically analyze attempts to solve it according to the various prominent interpretations of quantum theory, including the instrumentalist approach. Finally, we describe several important theorems that have been developed over the years to constrain candidate interpretations of quantum theory.

1. The Copenhagen Interpretation and the Measurement Problem

a. A Review of the Copenhagen Interpretation

The Copenhagen interpretation asserts that every isolated quantum system is completely described by a particular unit-norm state vector \(|\Psi\rangle\) in an associated Hilbert space \(H\). Furthermore, according to the Copenhagen interpretation, every observable property \(\Lambda\) of the system corresponds to a Hermitian operator \(\hat{\Lambda} = \hat{\Lambda}^\dagger\) necessarily having a complete orthonormal basis of eigenstates \(|a\rangle\) with corresponding real eigenvalues \(\lambda_a\):

\[
\hat{\Lambda} |a\rangle = \lambda_a |a\rangle, \quad \lambda_a \in \mathbb{R}, \quad \langle a | a'\rangle = \delta_{aa'}, \quad \sum_{a} |a\rangle \langle a| = \hat{1}.
\] (158)

The eigenvalues \(\lambda_a\) represent the possible measurement outcomes for the random variable \(\Lambda\), and the empirical outcome probability \(p(\lambda)\) with which a particular eigenvalue \(\lambda = \lambda_a\) is obtained is given by the Born rule,

\[
p(\lambda) = \sum_{\langle a, \lambda = \lambda_a\rangle} |\langle a| \Psi\rangle|^2,
\] (159)

where, in order to accommodate the case of degeneracies in the eigenvalue spectrum of \(\hat{\Lambda}\), the sum is over all values of the label \(a\) for which the eigenvalue \(\lambda_a\) of the eigenstate \(|a\rangle\) is equal to the outcome eigenvalue \(\lambda\). When degeneracy is absent, so that we can uniquely label the eigenstates of \(\hat{\Lambda}\) by \(\lambda\), the Born rule (159) reduces to the simpler expression

\[
p(\lambda) = |\langle \lambda | \Psi\rangle|^2.
\] (160)

Either way, we can then express expectation values of observables \(\Lambda\) in terms of the system’s state vector \(|\Psi\rangle\) in the following way:

\[
\langle \Lambda \rangle = \langle \Psi | \hat{\Lambda} | \Psi \rangle.
\] (161)

In particular, letting \(\hat{P}_\lambda\) denote the Hermitian projection operator onto eigenstates of \(\hat{\Lambda}\) having the eigenvalue \(\lambda\),

\[
\hat{P}_\lambda \equiv \sum_{\langle a, \lambda = \lambda_a\rangle} |a\rangle \langle a|,
\] (162)

we can use (161) to express the Born rule (159) in the alternative form

\[
p(\lambda) = \langle P_\lambda \rangle = \langle \Psi | \hat{P}_\lambda | \Psi \rangle.
\] (163)

These statements all naturally extend to density matrices \(\hat{\rho}\) according to the formulas

\[
p(\lambda) = \text{Tr} \left[ \hat{\rho} \hat{P}_\lambda \right], \quad \langle \Lambda \rangle = \text{Tr} \left[ \hat{\rho} \hat{\Lambda} \right],
\] (164)
which we can identify as noncommutative generalizations of the respective classical formulas

\[ p(\lambda) = \sum_a p_a P_{\lambda,a}, \quad \langle \Lambda \rangle = \sum_a p_a \lambda_a, \quad (165) \]

where the quantities \( p_a \in [0,1] \) constitute a classical probability distribution over elementary outcomes \( a \) and

\[ P_{\lambda,a} = \begin{cases} 1 & \text{for } \lambda_a = \lambda, \\ 0 & \text{for } \lambda_a \neq \lambda \end{cases} \quad (166) \]

is the characteristic (or indicator) function for the set of elementary outcomes \( a \) corresponding to the generalized outcome \( \lambda \).

Systems that are dynamically closed undergo smooth, linear time evolution according to a unitary time-evolution operator \( \hat{U}(t) \),

\[ |\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle \quad \left[ \hat{U}(t)^\dagger = \hat{U}(t)^{-1}, \quad \langle \Psi(t)|\Psi(t)\rangle = \langle \Psi(0)|\Psi(0)\rangle = 1 \right]. \quad (167) \]

If we can express the time-evolution operator in terms of a Hermitian, time-independent Hamiltonian operator \( \hat{H} \) according to

\[ \hat{U}(t) = e^{-i\hat{H}t/\hbar}, \quad (168) \]

then the system’s state vector obeys the famous Schrödinger equation:

\[ i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle. \quad (169) \]

At the level of the system’s density matrix \( \hat{\rho}(t) \), these integral and differential dynamical equations respectively become

\[ \dot{\hat{\rho}}(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}(t)^\dagger, \quad \frac{\partial}{\partial t} \hat{\rho}(t) = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho}(t) \right]. \quad (170) \]

The Copenhagen interpretation axiomatically regards the Born rule (159) as an exact postulate, and defines the partial-trace prescription together with reduced density matrices for subsystems precisely to ensure that the Born-rule-based expectation value (164) obtained for any observable \( \hat{\Lambda} \) of any subsystem agrees with the expectation value of the corresponding observable \( \hat{\Lambda} \otimes \hat{1} \) of any parent system:

\[ \text{Tr}_{\text{subsystem}} \left[ \hat{\rho}_{\text{subsystem}\hat{\Lambda}} \right] = \frac{\text{Tr}_{\text{parent}} \left[ \hat{\rho}_{\text{parent}\hat{\Lambda}} \otimes \hat{1} \right]}{\text{Tr}_{\text{subsystem}} \text{Tr}_{\text{other}}}, \]

\[ \Longrightarrow \hat{\rho}_{\text{subsystem}} = \text{Tr}_{\text{other}} \left[ \hat{\rho}_{\text{parent}} \right]. \quad (171) \]

Consequently, the dynamics governing a general (open) subsystem of a dynamically closed parent system is determined by an equation that is generically non-unitary:

\[ \frac{\partial}{\partial t} \hat{\rho}(t) = -\frac{i}{\hbar} \text{Tr}_{\text{parent}} \left[ \left[ \hat{H}_{\text{parent}}, \hat{\rho}_{\text{parent}} \right] \right]. \quad (172) \]

The final axiom of the Copenhagen interpretation stipulates that if an external observer measures an observable \( \lambda \) of the system and obtains an outcome \( \lambda \) corresponding to the projection operator \( \hat{P}_\lambda \) defined in (162), then the system instantaneously “collapses” according to a non-unitary rule called the Von Neumann-Lüders projection postulate [214, 305]: Regardless of the system’s initial density matrix \( \hat{\rho}_{\text{initial}} \), which may well be pure \( |\Psi_{\text{initial}}\rangle \langle \Psi_{\text{initial}}| \) or may instead describe an improper mixture—meaning that the nontriviality of the density matrix arises at least in part from quantum entanglements with other systems—the system’s final density matrix \( \hat{\rho}_{\text{final},\lambda} \) is given by

\[ \hat{\rho}_{\text{final},\lambda} = \frac{\hat{P}_\lambda \hat{\rho}_{\text{initial}} \hat{P}_\lambda}{\text{Tr} \left[ \hat{P}_\lambda \hat{\rho}_{\text{initial}} \right]} \quad (173) \]

In keeping with (164), the denominator is the probability \( p(\lambda) \) of obtaining the outcome \( \lambda \) and ensures that the final density matrix \( \hat{\rho}_{\text{final},\lambda} \) has unit trace. In the special case in which the eigenvalue spectrum of the operator \( \hat{\Lambda} \) representing the observable \( \lambda \) has no degeneracies, so that the projection operator \( \hat{P}_\lambda \) reduces to the simple form \( |\lambda\rangle \langle \lambda| \) for a single eigenstate \( |\lambda\rangle \) of \( \Lambda \), the Von Neumann-Lüders projection postulate (173) reduces to

\[ \hat{\rho}_{\text{final},\lambda} = |\lambda\rangle \langle \lambda|, \quad (174) \]

so that the system ends up in a pure state represented by \( |\lambda\rangle \).
The Von Neumann-Lüders projection postulate (173)—known informally as wave-function collapse—both accounts for the statistical features of the post-measurement state of affairs and also ensures that definite measurement outcomes persist under identical repeated experiments performed over sufficiently short time intervals. However, (173) represents a discontinuous departure from the smooth time evolution determined by the Schrödinger equation (169), a discrepancy known as the measurement problem of quantum theory.

As a first step toward better characterizing the measurement problem, notice that we can gather together the different possible post-measurement density matrices \( \hat{\rho}_{\text{final},\lambda} \) appearing in (173) into a subjective probability distribution of the form

\[
\{ p(\lambda) \Rightarrow \hat{\rho}_{\text{final},\lambda} \}.
\]  

(175)

We can obtain the same statistical predictions from the block-diagonal “subjective” density matrix

\[
\hat{\rho}_{\text{final}} = \sum_\lambda p(\lambda) \hat{\rho}_{\text{final},\lambda} = \sum_\lambda \hat{P}_\lambda \hat{\rho}_{\text{initial}} \hat{P}_\lambda,
\]  

(176)

which describes the post-measurement system in the absence of post-selecting or conditioning on the actually observed value \( \lambda \). In the special case (174) in which the set of possible measurement outcomes exhibit no degeneracies, (176) reduces to a proper mixture, meaning that the nontriviality of the density matrix arises solely from subjective uncertainty over the true underlying state vector of the system:

\[
\hat{\rho}_{\text{final}} = \sum_\lambda p(\lambda) |\lambda\rangle \langle \lambda|.
\]  

(177)

More generally, note that the specific decomposition appearing on the right-hand-side of (176) is preferred among all possible decompositions of \( \hat{\rho}_{\text{final}} \) because we know that the system’s true density matrix is really one of the possibilities \( \hat{\rho}_{\text{final},\lambda} \) obtained from the Von Neumann-Lüders projection postulate (173). That is, our system is fundamentally described by the subjective probability distribution (175), and we have introduced the subjective density matrix (177) merely for mathematical convenience.

Remarkably, provided that the measurement in question is performed by a sufficiently macroscopic observer or measuring device, the quantum phenomenon of decoherence [56, 65, 183, 184, 252, 253, 331] naturally produces reduced (“objective”) density matrices describing improper mixtures that look formally just like the subjective density matrix (177) up to tiny corrections, with eigenstates that exhibit negligible quantum interference with one another under further time evolution. However, the subjective density matrix (177), unlike a decoherence-generated density matrix, is not a reduced density matrix arising from external quantum entanglements—it is ultimately just a formal stand-in for the subjective probability distribution (175) over persistent measurement outcomes, and, in the simplest case (177), is a proper mixture. Moreover, the Copenhagen interpretation provides no canonical recipe for assigning preferred decompositions to generic reduced density matrices. Hence, solving the measurement problem requires either additional ingredients or a new interpretation altogether.

c. A Variety of Approaches

Within the framework of the Copenhagen interpretation, one declares that observers or measurement devices that are “sufficiently classical”—meaning that they are on the classical side of the so-called Heisenberg cut—cause decoherence-generated density matrices to cease being reduced density matrices by breaking their quantum entanglements with any external systems, and, furthermore, cause them to develop the necessary preferred decomposition appearing on the right-hand-side of (177); the overall effect is therefore to convert decoherence-generated density matrices into subjective probability distributions of the form (175). Equivalently, one can phrase the Heisenberg cut as a threshold on the amount of quantum interference between the eigenstates of a decoherence-generated density matrix: If the amount of quantum interference falls below that threshold, then we can treat those eigenstates as describing classical possibilities in a subjective probability distribution (175). Unfortunately, according to either formulation, the Heisenberg cut remains ill-defined and has not been identified in any experiment so far.

An alternative approach is to postulate that all reduced density matrices gradually evolve into corresponding subjective probability distributions (175), with macroscopic systems evolving in this way more rapidly than microscopic systems. Achieving these effects requires altering the basic dynamical structure of quantum theory along the lines of GRW dynamical-collapse or spontaneous-localization constructions [7, 37, 137, 235, 236, 315].
The Everett-DeWitt many-worlds interpretation [66, 91, 92, 94, 106–108, 307, 308, 318] attempts to solve the measurement problem by reifying all the members of a suitably chosen basis as simultaneous “worlds” or “branches” while somehow also regarding them as the elements of an appropriate subjective probability distribution. Apart from making sense of probabilities when all outcomes are simultaneously realized, one key trouble with this interpretation is deciding which basis to choose, a quandary known as the preferred-basis problem and that we describe in Sections III A 9 and VI D 6.

The modal interpretations instead reify just one member of a suitable basis. In “fixed” modal interpretations—of which the de Broglie-Bohm interpretation [57, 58, 61, 89] is the most well-known example—this preferred basis is fixed for all systems, leading to problems that we detail in Section III A 6. By contrast, in density-matrix-centered modal interpretations such as the one that we introduce in this paper, one chooses the basis to be the diagonalizing eigenbasis of the density matrix of whatever system is currently under consideration. Essentially, the central idea of our own minimal modal interpretation is a conservative one: We identify improper density matrices as closely as possible with subjective probability distributions, and add the minimal axiomatic ingredients that are necessary to make this identification viable.

d. The Instrumentalist Approach

A final prominent option is the instrumentalist approach, in which one formally accepts the basic axioms of the Copenhagen interpretation without taking a definitive stand on the ontological meaning of state vectors or density matrices, or on any reality that underlies the formalism of quantum theory more generally. In that case, we can regard the Von Neumann-Lüders projection postulate (173) as being the natural noncommutative generalization of the classical post-measurement probability-update formula

$$p_{\text{initial},a} \rightarrow p_{\text{final},\lambda,a} = \frac{P_{\lambda,a}p_a}{\sum_{a'} P_{\lambda,a'}p_{a'}} ,$$

(178)

where we defined $P_{\lambda,a}$ in (166) as the characteristic function for the generalized outcome $\lambda$.

We can then replace the Heisenberg cut with a self-consistency condition on the definition of valid observers, called agents in this context. Specifically, in order for a system such as a creature or a measurement device to count as an agent, we require that whenever it performs generic measurements, the observable $Q$ representing the true-or-false question “Did the agent obtain a set of frequency ratios in agreement with the Born formula?” and the observable $Q'$ representing the true-or-false question “Did the agent find a persistent measurement outcome that appears to be in keeping with the Von Neumann-Lüders projection postulate?” each have respective Born-rule probabilities (164) $p(Q = \text{"true"})$ and $p(Q' = \text{"true"})$ acceptably close to unity—say, 99.9%. Provided that the agent is sufficiently macroscopic, decoherence will generally guarantee that these conditions hold.

It is important to keep in mind, however, that the Born rule (164) and the Von Neumann-Lüders projection postulate (173) are still nontrivial axioms and cannot be dropped within the instrumentalist approach: Without these axioms, we cannot conclude solely from $p(Q = \text{"true"}) \approx 1$ and $p(Q' = \text{"true"}) \approx 1$ that sufficiently macroscopic measurement devices (such as human beings) will experience the Born rule and the Von Neumann-Lüders projection postulate, the reason being that neither of the associated Hermitian operators $Q$ and $Q'$ actually correspond to unique questions; indeed, each of these operators generically has two highly degenerate eigenvalues 1 (“true”) and 0 (“false”), meaning that neither $Q$ nor $Q'$ can uniquely pick out an orthonormal basis of states describing classically sensible realities. Thus, at best, the conditions $p(Q = \text{"true"}) \approx 1$ and $p(Q' = \text{"true"}) \approx 1$ merely supply us with a self-consistency check—a necessary but not sufficient condition—on our definition of agents and a quantitative criterion for determining how macroscopic a valid agent must be, rather than making possible an ab initio derivation of the Born rule or the Von Neumann-Lüders projection postulate.

2. Foundational Theorems

We discuss the Bell theorem in Section V B of the main text, and the Myrvold theorem in Section V D. Here we present a brief proof of the Bell theorem and describe several other important theorems that put strong constraints on candidate interpretations of quantum theory.
As we explained in Section V B, the Bell theorem involves a set of three spin detectors $A$, $B$, and $C$ aligned respectively along three unit vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ and that make measurements on pairs of spin-$1/2$ particles governed by local hidden variables $\lambda$. Granting the fact that spin is quantized in units of $\hbar/2$ (one can account for this condition in classical language by insisting that the particles automatically line themselves up along the local detector alignments as they are being measured), Bell assumed that the respective results $A(\vec{a}, \lambda) = \pm 1$, $B(\vec{b}, \lambda) = \pm 1$, and $C(\vec{c}, \lambda) = \pm 1$ (in units of $\hbar/2$) of the three detectors depend only on data local to each detector. In keeping with the supposed anti-correlated state (123) of each pair of spin-$1/2$ particles, Bell also required that if any two detectors are aligned, then they must always measure opposite spins:

$$A(\vec{a}, \lambda) = -B(\vec{b} = \vec{a}, \lambda) = -C(\vec{c} = \vec{a}, \lambda),$$

$$B(\vec{b}, \lambda) = -A(\vec{a} = \vec{b}, \lambda) = -C(\vec{c} = \vec{b}, \lambda),$$

$$C(\vec{c}, \lambda) = -A(\vec{a} = \vec{c}, \lambda) = -B(\vec{b} = \vec{c}, \lambda).$$

Finally, Bell posited the existence of a probability distribution (126) for the hidden variables $\lambda$ themselves:

$$0 \leq p(\lambda) \leq 1, \quad \int d\lambda p(\lambda) = 1.$$

The Bell theorem then asserts that the three average spin correlations

$$\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle_{\text{LHV}} = \int d\lambda p(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda),$$

$$\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle_{\text{LHV}} = \int d\lambda p(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda),$$

$$\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle_{\text{LHV}} = \int d\lambda p(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda)$$

must satisfy the inequality (125),

$$|\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle_{\text{LHV}} - \langle S_{1,\vec{a}}S_{2,\vec{c}} \rangle_{\text{LHV}}| \leq 1 + \langle S_{1,\vec{b}}S_{2,\vec{c}} \rangle_{\text{LHV}},$$

as follows from a straightforward computation:

$$|\langle S_{1,\vec{a}}S_{2,\vec{b}} \rangle_{\text{LHV}} - \langle S_{1,\vec{a}}S_{2,\vec{c}} \rangle_{\text{LHV}}| = \int d\lambda p(\lambda) \left| \frac{1}{A(\vec{a}, \lambda)} \left( A(\vec{a}, \lambda) B(\vec{b}, \lambda) - \frac{1}{A(\vec{b}, \lambda)^2} C(\vec{c}, \lambda) \right) \right|$$

$$\leq \int d\lambda p(\lambda) \left| \frac{1}{A(\vec{a}, \lambda)} A(\vec{b}, \lambda) - \frac{C(\vec{c}, \lambda)}{A(\vec{b}, \lambda)} \right|$$

$$= \int d\lambda p(\lambda) \left( 1 + A(\vec{b}, \lambda) C(\vec{c}, \lambda) \right) = 1 + \langle S_{1,\vec{b}}S_{2,\vec{c}} \rangle_{\text{LHV}}.$$  

QED

b. Gleason’s Theorem

Gleason’s theorem [146] asserts that the only consistent probability measures for closed subspaces of a Hilbert space of dimension $\geq 3$ must be given by $p_i = \text{Tr} \left[ \hat{\rho} P_i \right]$, which generalizes the state-vector version of the Born rule. Here $\hat{\rho}$ is a unit-trace, positive semi-definite operator that we interpret as the system’s density matrix and $\hat{P}_i = |\Psi_i\rangle \langle \Psi_i|$ is a projection operator onto some state vector $|\Psi_i\rangle$.

All the well-known interpretations of quantum theory satisfy Gleason’s theorem. In particular, the basic correspondence (17) at the heart of our own minimal modal interpretation is consistent with the theorem.
It might seem plausible for an interpretation of quantum theory to claim that when we measure observables $A_1, A_2, \ldots$ belonging to a quantum system and obtain some set of outcome values $\lambda_1, \lambda_2, \ldots$, we are merely revealing that those observables secretly possessed all those values $\lambda_1, \lambda_2, \ldots$ simultaneously even before the measurement took place. The Kochen-Specker theorem [189] presents an obstruction to such claims. More precisely, the theorem rules out hidden-variables interpretations that assert that all observables that could be measured have simultaneous, sharply defined values before they are measured—values that depend only on the observables themselves and on the system to which they belong—and that measurements merely reveal those supposedly preexisting values. For viable hidden-variables interpretations of quantum theory, an immediate consequence of the theorem is that the properties of a quantum system must generically be contextual, in the sense that they can depend on the kinds of measurements performed on the system. Here we present one of the simplest versions of the theorem, due to Peres [238].

We begin by considering a four-state quantum system, which we can regard as consisting of a pair of two-state spin-1/2 subsystems and therefore having a Hilbert space $\mathcal{H}$ of the form

$$\mathcal{H} = \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}, \quad \dim \mathcal{H} = 2 \times 2 = 4.$$  \hspace{1cm} (179)

We consider also the following nine Hermitian matrices on this four-dimensional Hilbert space:

$$\begin{align*}
P_{2z} &\equiv (1 - \sigma_{2z})/2, \\
P_{1z} &\equiv (1 - \sigma_{1z})/2, \\
P_{1x,2z} &\equiv (1 - \sigma_{1x}\sigma_{2z})/2, \\
P_{1x,2z} &\equiv (1 - \sigma_{1z}\sigma_{2z})/2, \\
P_{1x} &\equiv (1 - \sigma_{1x})/2, \\
P_{2x} &\equiv (1 - \sigma_{2x})/2, \\
P_{1x,2x} &\equiv (1 - \sigma_{1z}\sigma_{2x})/2, \\
P_{1y,2y} &\equiv (1 - \sigma_{1y}\sigma_{2y})/2.
\end{align*}$$  \hspace{1cm} (180)

In our notation here, $1 \equiv 1_{2 \times 2} \otimes 1_{2 \times 2}$ is the $4 \times 4$ identity matrix, and $\sigma_{1i} \equiv \sigma_i \otimes 1_{2 \times 2}$ and $\sigma_{2i} \equiv 1_{2 \times 2} \otimes \sigma_i$ are respectively the Pauli sigma matrices for the first and second spin-1/2 subsystems.

Each of the $4 \times 4$ matrices in (180) has two +1 eigenvalues and two −1 eigenvalues; in the usual language of quantum theory, the associated observables therefore have possible measured values +1 and −1. Furthermore, all three matrices in each column of (180) commute with each other, and, likewise, all three matrices in each row commute with each other; hence, for each column or row, the three corresponding observables are mutually compatible and thus can be made simultaneously sharply defined by an appropriate preparation of the state of our system.

We next define three new $4 \times 4$ matrices $A_1, A_2, A_3$ by respectively summing each column of (180),

$$\begin{align*}
A_1 &\equiv P_{2z} + P_{1x} + P_{1x,2z} &\text{(eigenvalues +2, +2, 0, 0)}, \\
A_2 &\equiv P_{1z} + P_{2x} + P_{1z,2x} &\text{(eigenvalues +2, +2, 0, 0)}, \\
A_3 &\equiv P_{1x,2z} + P_{1x,2x} + P_{1y,2y} &\text{(eigenvalues +3, +3, +1, +1)}.
\end{align*}$$  \hspace{1cm} (181)

and three $4 \times 4$ matrices $B_1, B_2, B_3$ by respectively summing each row of (180),

$$\begin{align*}
B_1 &\equiv P_{2z} + P_{1z} + P_{1z,2z} &\text{(eigenvalues +2, +2, 0, 0)}, \\
B_2 &\equiv P_{1x} + P_{2x} + P_{1x,2x} &\text{(eigenvalues +2, +2, 0, 0)}, \\
B_3 &\equiv P_{1x,2z} + P_{1x,2x} + P_{1y,2y} &\text{(eigenvalues +2, +2, 0, 0)}.
\end{align*}$$  \hspace{1cm} (182)

Finally, we introduce an observable $\Sigma$ defined to be the sum (times two) of the nine original observables defined in (180), or, equivalently, defined to be the sum of the six observables $A_1, A_2, A_3, B_1, B_2, B_3$:

$$\Sigma \equiv 2P_{2z} + 2P_{1x} + 2P_{1x,2z} + 2P_{1z} + 2P_{2x} + 2P_{1z,2x} + 2P_{1y,2y}$$
\hspace{1cm} (183)
\hspace{1cm} = A_1 + A_2 + A_3 + B_1 + B_2 + B_3.

Each of the nine observables defined in (180) has permissible values +1 or −1, and so, from the first expression for $\Sigma$ in (183), we see that the existence of simultaneous pre-measurement values for these nine observables would imply that $\Sigma$ has an even pre-measurement value:

$$\Sigma = 2 \times (+1 \text{ or } -1) + 2 \times (+1 \text{ or } -1) + 2 \times (+1 \text{ or } -1)$$
$$= \text{even.}$$  \hspace{1cm} (184)

\hspace{1cm}
However, if we assume that the six observables $A_1, A_2, A_3, B_1, B_2, B_3$ likewise have simultaneous pre-measurement values, then we find instead that $\Sigma$ has an odd pre-measurement value, a contradiction:

$$\Sigma = (0 \text{ or } 2) + (0 \text{ or } 2) + (1 \text{ or } 3) + (0 \text{ or } 2) + (0 \text{ or } 2) + (0 \text{ or } 2)$$

$$= \text{odd.}$$

(185)

We are therefore forced to give up our assumption that a quantum system can always have simultaneous sharply defined pre-measurement values for all of its observables. QED

Neither the traditional Copenhagen interpretation nor the minimal modal interpretation we introduce in this paper asserts that all observables have well-defined values before measurements—indeed, our own interpretation makes manifest that a system’s set of possible ontic states can change contextually from one orthonormal basis of the system’s Hilbert space to another orthonormal basis in the course of interactions with other systems—and so are both consistent with the Kochen-Specker theorem. (For macroscopic systems, however, decoherence ensures that all observables simultaneously develop approximate pre-measurement values.) The de Broglie-Bohm pilot-wave interpretation evades this theorem as well, because although the interpretation assumes systems have hidden well-defined values of both canonical coordinates and canonical momenta at all times, the canonical momenta are not identified with the observable momenta that actually show up in measurements.

d. The Pusey-Barrett-Rudolph (PBR) Theorem

The Pusey-Barret-Rudolph (PBR) theorem [35, 85, 247] rules out so-called psi-epistemic interpretations of quantum theory that directly regard state vectors (as opposed to density matrices) as merely being epistemic probability distributions for hidden variables whose configurations determine the outcomes of measurements. Essentially, the theorem derives a one-to-one correspondence between each specific configuration of hidden variables and each state vector (up to trivial overall phase factor), thus implying that each configuration singles out a unique state vector. It is therefore impossible to regard each state vector as being an epistemic probability distribution over a nontrivial collection of different configurations of hidden variables.

To illustrate the theorem in a simple case, the authors consider a two-state system with an orthonormal basis $|0\rangle, |1\rangle$ and a second orthonormal basis defined by

$$|+\rangle \equiv \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad |-\rangle \equiv \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle).$$

(186)

If the two state vectors $|0\rangle$ and $|+\rangle$ merely describe probability distributions over configurations of hidden variables, and those probability distributions are allowed to overlap on the sample space of configurations of hidden variables, then there exists some nonzero probability $q > 0$ that a particular configuration $\lambda$ of hidden variables will reside in both probability distributions. Hence, if the system’s hidden variables have the configuration $\lambda$, then the corresponding probability distribution could be either $|0\rangle$ or $|+\rangle$.

If we now set up a pair of independent such systems as a composite system, then, with probability $q^2$, the configurations $\lambda_1$ and $\lambda_2$ of the two subsystems would permit being jointly described by the probability distributions arising from any of the possible tensor-product state vectors $|0\rangle |0\rangle, |0\rangle |+\rangle, |+\rangle |0\rangle, |+\rangle |+\rangle$. But then a measurement of an observable whose corresponding basis of orthonormal eigenstates is

$$\left\{ \begin{array}{l}
|\xi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle |1\rangle + |1\rangle |0\rangle), \\
|\xi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle |-\rangle + |1\rangle |+\rangle), \\
|\xi_3\rangle = \frac{1}{\sqrt{2}} (|+\rangle |1\rangle + |-\rangle |0\rangle), \\
|\xi_4\rangle = \frac{1}{\sqrt{2}} (|+\rangle |-\rangle + |-\rangle |+\rangle)
\end{array} \right\}$$

(187)

would have zero probability of yielding $|\xi_1\rangle$ if the composite system’s state vector happened to be $|0\rangle |0\rangle$, zero probability of yielding $|\xi_2\rangle$ if the composite system’s state vector happened to be $|0\rangle |+\rangle$, zero probability of yielding $|\xi_3\rangle$ if the composite system’s state vector happened to be $|+\rangle |0\rangle$, and zero probability of yielding $|\xi_4\rangle$ if the composite system’s state vector happened to be $|+\rangle |+\rangle$. Hence, whichever result is found by the measurement, there exists a contradiction with the notion that the original simultaneous configurations $\lambda_1$ and $\lambda_2$ of the conjoined subsystems
were really compatible with all four possible state vectors $|0\rangle$, $|0\rangle|+,\rangle$, $|+\rangle|0\rangle$, and $|+\rangle|+\rangle$. The authors then extend this general argument to a much larger class of possibilities beyond the particular pair $|0\rangle$ and $|+\rangle$ to argue that no two state vectors could ever describe overlapping probability distributions over hidden variables.

The Copenhagen interpretation and our own interpretation satisfy the PBR theorem, because they both regard state vectors as irreducible features of reality rather than as mere epistemic probability distributions over a deeper layer of hidden variables. The de Broglie-Bohm pilot-wave interpretation also evades the theorem because, in that interpretation, the state vector plays both the role of an epistemic probability distribution as well as a physical pilot wave that (nonlocally if necessary) guides the hidden variables during measurements to values that are always consistent with final measurement outcomes.

e. The Fine and Vermaas No-Go Theorems

Finally, there exist no-go theorems due to Fine [112, 113] and Vermaas [297] suggesting that axiomatically imposing joint probability distributions for properties of non-disjoint subsystems of a larger parent system leads to contradictions with axiomatic impositions of joint probability distributions for properties of disjoint subsystems. In our minimal modal interpretation of quantum theory, we do not postulate joint probability distributions for non-disjoint subsystems of a given parent system, and so our interpretation evades both theorems.
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a development of classical gauge theories from the perspective of Weyl-Coulomb gauge, in which the gauge potential is not a proper Lorentz four-vector, see Section 5.9.

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[317] Steven Weinberg and Jermy N. A. (interviewer) Matthews. “Questions and answers with Steven Weinberg”. *Physics Today*, July 2013. In particular, “… Some very good theorists seem to be happy with an interpretation of quantum mechanics in which the wavefunction only serves to allow us to calculate the results of measurements. But the measuring apparatus and the physicist are presumably also governed by quantum mechanics, so ultimately we need interpretive postulates that do not distinguish apparatus or physicists from the rest of the world, and from which the usual postulates like the Born rule can be deduced. This effort seems to lead to something like a ‘many worlds’ interpretation, which I find repellent. Alternatively, one can try to modify quantum mechanics so that the wavefunction does describe reality, and collapses stochastically and nonlinearly, but this seems to open up the possibility of instantaneous communication”. URL: http://scitation.aip.org/content/aip/magazine/physicstoday/news/10.1083/PT.4.2502, doi:10.1083/PT.4.2502.

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