Minimum Enclosing Polytope in High Dimensions

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Abstract

We study the problem of covering a given set of \( n \) points in a high, \( d \)-dimensional space by the minimum enclosing polytope of a given arbitrary shape. We present algorithms that work for a large family of shapes, provided either only translations and no rotations are allowed, or only rotation about a fixed point is allowed; that is, one is allowed to only scale and translate a given shape, or scale and rotate the shape around a fixed point. Our algorithms start with a polytope guessed to be of optimal size and iteratively moves it based on a greedy principle: simply move the current polytope directly towards any outside point till it touches the surface. For computing the minimum enclosing ball, this gives a simple greedy algorithm with running time \( O(nd/\epsilon) \) producing a ball of radius \( 1 + \epsilon \) times the optimal. This simple principle generalizes to arbitrary convex shape when only translations are allowed, requiring at most \( O(1/\epsilon^2) \) iterations. Our algorithm implies that core-sets of size \( O(1/\epsilon^2) \) exist not only for minimum enclosing ball but also for any convex shape with a fixed orientation. A Core-Set is a small subset of \( \text{poly}(1/\epsilon) \) points whose minimum enclosing polytope is almost as large as that of the original points. When only rotation about a fixed point is allowed, for a certain class of convex bodies with an axis of symmetry that includes cylinders, cones and ellipsoids, we prove that our techniques work provided the problem is confined to a half space. Without the half-space restriction, we obtain an algorithm whose running time is exponential in \( 1/\epsilon^2 \), and corresponding core-sets. This automatically gives us a \( 2^{O(1/\epsilon^2)} \) \( nd \) time algorithm for the min-cylinder problem provided we are given a fixed point on the axis. Although we are unable to combine our techniques for translations and rotations for general shapes, for the min-cylinder problem, we give an algorithm similar to the one in [9], but with an improved running time of \( 2^{O(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon})} \) \( nd \). This generalizes to computing the minimum radius \( k \)-dimensional flat in time \( \text{exp}(\frac{O(k^2)}{\epsilon^2} \log \frac{1}{\epsilon}) \) \( nd \).

1 Introduction

Given a set \( S \) of \( n \) points in \( d \) dimensions, we study the problem of finding the minimum enclosing polytope of a given arbitrary shape when \( d \) is large. Being a fundamental problem in computational geometry with applications in data mining, learning, statistics and clustering ([5], [6], [8]), this problem has a rich history. Bădoiu et. al. [1] gave an algorithm that computes the minimum enclosing ball approximately, with radius at most \( 1 + \epsilon \) times the optimal radius in time \( O(nd/\epsilon^2 + (1/\epsilon)^{10}) \), independent of the number of dimensions, using convex programming. Their algorithm was based on the idea of Core-Sets, a small set of \( \text{poly}(1/\epsilon) \) points whose minimum enclosing ball is almost as large as that of all the \( n \) points. This was improved to \( O(nd/\epsilon + (1/\epsilon)^5) \) in [8] by finding

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smaller core-sets of size $\lceil 1/\epsilon \rceil$. They also provide a simple $O(nd/\epsilon^2)$ time algorithm for finding the minimum enclosing ball that does not require convex programming. Combining the two results gives a $O(nd/\epsilon + (1/\epsilon)^5)$ time algorithm for finding the minimum enclosing ball while eliminating the use of convex programming.

For the minimum enclosing cylinder problem, Har-Peled and Varadarajan [9] gave an algorithm with running time of $2^{O(1/\epsilon^2 \log 1/\epsilon)} nd$ that finds a cylinder with radius at most $1 + \epsilon$ times the optimal radius. They also generalized their algorithm to computing the minimum radius $k$-dimensional flat in time $exp\left(\frac{cO(k^2)}{\epsilon^2} \right) nd$, where the radius of a $k$-flat is the maximum distance of the given set of points from this $k$-flat.

In this paper, we present algorithms for computing the minimum enclosing polytope for a large family of shapes, provided either only translations and no rotations are allowed, or only rotation about a fixed point is allowed; that is, one is allowed to only scale and translate a given shape, or scale and rotate the shape around a fixed point. We hope that it may be possible to combine the techniques for translation and rotation to solve the problem without these restrictions. Our algorithms are based on a simple greedy principle applied iteratively: simply move the current polytope directly towards any outside point till it touches the surface.

For computing the minimum enclosing ball, this gives a simple greedy algorithm that repeatedly moves a ball directly towards the farthest uncovered point till it touches the surface. If we start with a ball of the optimal radius, we show that running $O(1/\epsilon)$ such steps gives the optimal position of the ball approximately, within a running time of $O(nd/\epsilon)$ (section 2). This simple principle generalizes to arbitrary convex shape when only translations are allowed, requiring at most $O(1/\epsilon^2)$ iterations (section 3). It also works if the shape can be expressed as a union of a few convex shapes – however, requiring a running time exponential in $1/\epsilon^2$. Our algorithm implies that core-sets of size $O(1/\epsilon^2)$ exist not only for the minimum enclosing ball but also for any convex shape with a fixed orientation.

Next we look at covering a set of points by a convex body while allowing only rotation about a fixed point (section 4). For a certain class of convex bodies with an axis of symmetry that includes cylinders, cones and ellipsoids, we prove that our techniques work provided the problem is confined to a half space bordering at the point of rotation. Without this restriction, we obtain an algorithm whose running time is exponential in $1/\epsilon^2$. This gives us an $2^{O(1/\epsilon^2)} nd$ time algorithm for the min-cylinder problem provided we are given a fixed point on the axis. This also implies that core-sets whose size depend only on $\epsilon$ exist for rotational problems as well. Although we are unable to combine our techniques for translations and rotations for general shapes, for the min-cylinder problem, we give an algorithm almost identical to the one in [9], but with an improved running time of $2^{\frac{1}{\epsilon^2} \log \frac{1}{\epsilon}} nd$ (section 5). This generalizes to computing the minimum radius $k$-dimensional flat in time $exp\left(\frac{cO(k^2)}{\epsilon^2} \log \frac{1}{\epsilon}\right) nd$.

2 Minimum Enclosing Ball

Given a set $S$ of $n$ points in $d$ dimensions, we provide an algorithm to compute the minimum enclosing ball with radius at most $1 + \epsilon$ times the optimal in time $O(nd/\epsilon)$.

We start with a simple algorithm $MEB$ (figure 1) that works as follows: the algorithm starts with an arbitrary ball of the optimal radius, and for any point at least $\epsilon$ outside this ball, moves the ball till the surface touches the outside point. This involves guessing the optimal radius of the minimum enclosing ball. Assume without loss of generality that the optimal ball is of unit radius.

Let $d(P, Q)$ denote the distance between two points $P$ and $Q$. Let $B(C, r)$ denote the ball of
Algorithm MEB
1. Start with a ball of optimal radius.
2. Repeat until every point is within \(1 + \epsilon\) of the current center \(C\).
3. Find the farthest point \(P\) from \(C\).
4. Move \(C\) towards the point \(P\) till \(P\) touches the unit sphere centered at \(C\).

Figure 1: A simple algorithm for finding the minimum enclosing ball

radius \(r\) centered at point \(C\).

**Theorem 1** Algorithm MEB terminates in \(O(1/\epsilon)\) iterations.

The basic idea behind this theorem is that in each iteration the center \(C\) in the algorithm moves closer to the optimal center \(C_{OPT}\). If \(d_i\) is the distance of \(C\) from \(C_{OPT}\) in the \(i^{th}\) iteration, then \(d_i\) decreases as follows.

\[
d_{i+1}^2 \leq d_i^2 - \epsilon^2
\]

This is because \(\angle C_i, C_{i+1}, C_{OPT}\) is obtuse (figure 2), as \(PC_{i+1}\) is not shorter than \(PC_{OPT}\) implying \(\angle PC_{i+1}C_{OPT}\) is acute. So, \(d_{i+1}^2 \leq d_i^2 - d(C_i, C_{i+1})^2 \leq d_i^2 - \epsilon^2\). Since the initial value of \(d_i^2\) is at most a constant, and since it decreases by \(\epsilon^2\) in each iteration, the algorithm must terminate in \(O(1/\epsilon^2)\) iterations. A tighter analysis based on the following lemma will show that it actually terminates in \(O(1/\epsilon)\) iterations.

**Lemma 2.1** If \(C\) and \(C_{OPT}\) are distance \(d\) apart, there must be a point that is at least \(d^2/4\) from the surface of the unit ball centered at \(C\).
Theorem 1.1. (Proof) Let \( \mathcal{A} \) be the algorithm that guesses the optimal radius. For a certain guess, if the algorithm terminates in \( \text{MEB}(\text{OPT}, \phi) \) iterations, \( \phi \) becomes less than 1/2, and the gap decreases by at least \( \epsilon \) in each iteration, and the correct radius can be ascertained approximately within a factor of 1 + \( \epsilon \).

Proof. Let \( \mathcal{A} \) be the algorithm that guesses the optimal radius. For a certain guess, if the algorithm terminates in \( \text{MEB}(\text{OPT}, \phi) \) iterations, \( \phi \) becomes less than 1/2, and the gap decreases by at least \( \epsilon \) in each iteration, and the correct radius can be ascertained approximately within a factor of 1 + \( \epsilon \).

Proof of theorem 1.1. Note that \( d_i^2 \leq d_i^2 - (1/2)^{i+1} \leq 1/2 \). In each iteration, \( \phi_i \) decreases by at least \( \epsilon \). After that, since it decreases by at least \( \epsilon \) in each iteration, there can be at most \( \epsilon \) further iterations.

Algorithm \( \text{MEB} \) requires guessing the optimal radius. The distance of the farthest point from the optimal radius is within factor 2 of the optimal radius. A binary search with at most \( O(\log(1/\epsilon)) \) tries can be used to ascertain the correct radius approximately within a factor of 1 + \( \epsilon \), and the gap decreases as the iterations proceed.

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2.1 Eliminating the Binary Search

To eliminate the binary search, in algorithm \( \text{MEB}^{\text{OPT}} \) (figure 3), we start with a ball of radius less than optimal and increase it in certain iterations. The radius of the ball is always less than the optimal radius but the gap decreases as the iterations proceed.

The algorithm maintains a lower bound, and an error \( \delta \), such that \( r \leq r_{\text{OPT}} \leq r + \delta \). Again as before we assume that the optimal radius is 1. In each iteration we reduce the error \( \delta \) by a factor of 3/4 based on the following lemma.

Lemma 2.2. After \( O(1/\delta) \) iterations of moving the ball to farthest outside point, every outside point must be within 3\( \delta \) from the surface.

Proof. Let \( h_i \) denote the distance of farthest point from surface of \( B(C_i, r_{\text{OPT}}) \), after \( i \) iterations. Then the farthest point is at a distance \( h_i - \delta \) from \( B(C_i, r_{\text{OPT}}) \). Just as in proof of theorem 1.1, it is easy to check that (only difference is that \( d(C_i, C_{i+1}) = h_i - \delta \) )

\[
d_{i+1}^2 \leq d_i^2 - (h_i - \delta)^2
\]

Now as long as \( h_i \geq 2\delta \), we have \( d_{i+1}^2 \leq d_i^2 - h_i^2/4 \). Again lemma 2.1 says \( h_i \geq d_i^2/4 \), implying that as long as \( h_i \geq 2\delta \), we have \( d_{i+1}^2 \leq d_i^2 - d_i^2/64 \). So as in proof of theorem 1.1 in \( O(1/\delta) \) iterations, either \( d_i \leq \delta \) or \( h_i \leq 2\delta \). In either case, distance of the farthest point from the center is at most \( r_{\text{OPT}} + 2\delta \leq r + 3\delta \).
**Algorithm MEBOPT**

1. Initialize $C$ = any arbitrary point of the $n$ points
2. $r = 1/2$ of distance of farthest point from any one point
3. $\delta = 1/2$ of distance of farthest point from any one point
4. Repeat until $\delta \leq \epsilon$
5. For $O(1/\delta)$ iterations
6. Find farthest point $P$ from $C$
7. Move $B(C, r)$ till its surface touches $P$.
8. Let $s = $ distance of the farthest point from the surface of current ball $B$
9. If $s \leq 3\delta/4$
10. $\delta = 3\delta/4$
11. else $r = r + \delta/4$
12. $\delta = 3\delta/4$

Figure 3: An algorithm that does not require binary search

Using the above lemma, we can test if the current estimate $r$ in fact has an error of at most $\delta/4$ in $O(4/\delta)$ iterations. If after so many iterations, the farthest point is more than $3\delta/4$ away from the surface, then we can conclude that the error was more than $\delta/4$ and so $r$ can be increased to $r + \delta/4$. Otherwise, since every point is within distance $3\delta/4$ outside the surface, we can conclude that the error in $r$ is at most $3\delta/4$. Since each iteration in step 4 runs in time $O(nd/\delta)$, and $\delta$ decreases geometrically to $\epsilon$, we have proved the following theorem.

**Theorem 2** Algorithm MEBOPT finds an approximate minimum enclosing ball in time $O(nd/\epsilon)$

### 3 Generalizing to Convex Polytopes

The simple algorithm of moving towards the outside point works not only for finding the minimum enclosing ball but also for minimum enclosing polytope of any given convex shape with a fixed orientation. That is, one is only allowed to translate and scale the given shape but is not allowed to rotate it. Again, for ease of exposition, we assume that the maximum inter-point distance is at most 1. We present algorithm MINCON (figure 4), similar to MEB, that finds an approximate optimal solution in $1/\epsilon^2$ iterations. Again as in algorithm MEB, we guess the optimal size of the given shape but do not know its position to begin with. We repeatedly find an outside point at least $\epsilon$ away from the surface and move the current polytope by the shortest distance till the point touches the surface.

**Theorem 3** Algorithm MINCON terminates in $1/\epsilon^2$ iterations. That is, after so many iterations no point will be more than $\epsilon$ outside the surface.

**Proof** The proof is very similar to that of theorem 1. Let $Q$ be the point on the current polytope closest to $P$ (figure 5). Also, we know that $P$ is in the optimal polytope. Let $P'$ denote the corresponding point in the current polytope. That is, the vector $\overrightarrow{PP'}$ is the displacement of the optimal polytope from the current polytope. We will prove that $\angle P'QP$ is obtuse. If not, there
Algorithm MINCON —

1. Start with a polytope guessed to be of optimal size positioned anywhere.
2. Repeat until done
3. Find any point $P$ that is at least $\epsilon$ away from the surface of the current polytope
4. Find the point $Q$ on the polytope that is closest to $P$.
5. Move the polytope so that $Q$ coincides with $P$.

Figure 4: Algorithm for finding Minimum Enclosing Convex Polytope

Figure 5: The polytope keeps moving closer to the optimal position in each iteration
is a point on the segment \( P'Q \) that is closer to \( P \) than \( Q \). And since both \( P' \) and \( Q \) are in the current polytope, that point must also be within the current polytope. So \( Q \) cannot be the closest point to \( P \), which is a contradiction. If \( d_i \) denotes the distance of the current polytope of the optimal one in this \( i \)th iteration, then \( d_i = d(P', P) \). After the displacement by the vector \( QP \), the polytope will be off from the optimal position by the vector \( P'P - QP = P'Q \). So \( d_{i+1}^2 = d(P', Q)^2 \leq d(P', P)^2 - d(Q, P)^2 \leq d_i^2 - \epsilon^2 \). This proves that the algorithm terminates in \( 1/\epsilon^2 \) iterations.

This technique also extends to shapes that are a union of a small number of convex-shapes.

**Theorem 4** Given a shape and orientation that can be expressed as a union of \( c \) convex-shapes, the smallest enclosing polytope with that shape and orientation can be computed within \( \epsilon \) approximation in time \( O(1/\epsilon^2) \).

**Proof** The algorithm is identical to MINCON, except that at each step we guess one of the convex bodies that contains the outside point and move the polytope till that convex body touches the point.

This in fact proves that Core-Sets exist not only for minimum enclosing ball but also for any convex shape with a given orientation.

**Definition 1** Given a set of points, \( S \), and convex shape and orientation, we say that a subset \( T \) of \( S \) forms a Core-Set if the minimum enclosing polytope of \( T \) has every point of \( S \) within distance at most \( \epsilon \) outside its surface.

**Theorem 5** For a set of points with maximum inter-point distance 1, and for a given convex shape and orientation, there is a Core-Set of size \( O(1/\epsilon^2) \).

**Proof** Instead of starting with a polytope of the optimal size, we start with one just small enough so that it can never be positioned to have every outside point within distance \( \epsilon \) from its surface. Now we know that if we start with this size, then algorithm MINCON would never terminate in the \( O(1/\epsilon^2) \) iterations it otherwise would have. We let the algorithm run for one more than \( O(1/\epsilon^2) \) iterations and look at the \( 1 + O(1/\epsilon^2) \) points that are visited. We will prove that these points form the required Core-Set. The minimum enclosing polytope of these \( 1 + O(1/\epsilon^2) \) points must be larger than the one we started with as otherwise, by theorem 4 algorithm MINCON would not require more than \( 1 + O(1/\epsilon^2) \) iterations on these points. Since the initial polytope can be chosen so that every outside point is within distance arbitrarily close to \( \epsilon \) from the surface of this initial polytope, we have proved the theorem.

### 4 Allowing Rotations

So far we did not allow the convex polytope to be rotated and only allowed translations. In this section we prove that our techniques work if only rotation about a fixed point and no translations are allowed, provided certain conditions are met.

Given a polytope that has an axis of symmetry (that is, every cross section along the axis is hyper-sphere of dimension \( d - 1 \)) with the axis passing through the origin, and a set of points \( S \),
Algorithm MINROT

1. Start with the axis as any ray in the given half-space shooting from the origin.
2. Iteratively find any outside point and rotate the axis by the smallest angle till the surface of the polytope touches the outside point.
   We assume that the distance between the outside point and the point on the surface it touches is at least $\epsilon$, as otherwise we are done.

Figure 6: Algorithm for rotational problem in a half-space

our goal is to rotate the axis till the points in $S$, are covered by the polytope. We will also assume that the following conditions are satisfied

- All these points and the optimal polytope lie in a half-space with the bounding hyper-plane passing through the origin.
- Any $d$-dimensional hyper-sphere centered at the origin intersects the polytope in a single hyper-sphere of dimension $d - 1$. This $d - 1$-dimensional hyper-sphere divides the original $d$-dimensional hyper-sphere into two disjoint regions. We also assume that the interior of the polytope intersects the $d$-dimensional hyper-sphere in the smaller of these two regions. (this is similar to the convexity requirement in algorithm MINCON. In three dimensions this would mean that every sphere passing through the origin cuts the polytope in at most one circle. Also the interior of the polytope intersects the sphere in the smaller of the two regions on the sphere formed by the circle).

Examples that satisfy these conditions are cylinders, half-cones, ellipsoids lying in a half-space with axis passing through the origin. Again, for ease of exposition, we assume that all points are at most at unit distance from the origin. Our algorithm MINROT (figure 6) repeatedly rotates the polytope by the smallest angle so as to touch an uncovered point at least $\epsilon$ outside the surface.

**Theorem 6**  Algorithm MINROT terminates in $1/\epsilon^2$ iterations. That is, after so many iterations no point will be more than $\epsilon$ outside the surface.

**Proof** The proof is very similar to that of theorem 1. Let $\theta$ be the angle between the current axis and the optimal one. We will argue that the distance between corresponding points on the two axes on the units sphere centered at the origin decreases in each iteration. This distance $d = 2 \sin(\theta/2)$.

Let $P$ be the outside point chosen in a certain iteration and $Q$ be the closest point on the current polytope in terms of rotation required to move $Q$ to $P$. Look at the sphere centered at the origin passing through $P$ (figure 7). This sphere intersects the current polytope in a hyper-circle, $C$, passing through $Q$. The point $P$ lies within the optimal polytope. Let $P'$ be the corresponding point in the current polytope. $P'$ must be on the sphere inside the hyper-circular region $C$. Look at the great circle on the sphere passing through $P$ and $Q$. Project all points onto the two dimensional space containing this great circle. Under this projection the hyper-circle $C$ will become a segment $QQ'$. Since the points $P, Q, Q'$ lie on a half circle $\angle PQQ'$ is obtuse. Since, under the projection, $P'$ lies in the minor segment formed by $QQ'$, $\angle PQP'$ is also obtuse. Scale the distances so that the great circle is of unit radius. So $d_{i+1}^2 = d(P', Q)^2 \leq d(P', P)^2 - d(Q, P)^2 \leq d_{i}^2 - \epsilon^2$

Algorithm MINROT assumes that the polytope lies in a given half-space. We now provide an alternate algorithm for polytopes with an axis of symmetry passing through the origin without
Algorithm FULLROT

1. Start with the axis as any ray from the origin.
2. For each outside point $P$ look at the sphere centered at the origin.
3. The sphere intersects the polytope in at most two circles, $C_1$ and $C_2$ on different sides of the origin.
4. Now the axis could be rotated to either touch $C_1$ or $C_2$ to $P$.
   Guess one of them and rotate the axis by the smallest angle till $P$ touches the chosen circle.

Theorem 7
Algorithm FULLROT terminates in $1/\epsilon^2$ iterations. That is, after so many iterations no point will be more than $\epsilon$ outside the surface. The deterministic version of this algorithm runs in time $2^{O(1/\epsilon^2)}nd$. For ease of exposition, this algorithm, FULLROT (figure 8), is described for three dimensions.

Proof As in the proof of theorem 4 in any iteration there would be two points $Q_1$ and $Q_2$ on the circles $C_1$ and $C_2$ closest to the outside point $P$. As before $P'$ would lie in one of the minor segment formed by one of $C_1$ and $C_2$. We guess the correct one, say $C_1$. Again project all points
to the plane containing the great circle passing through $P$ and $Q_1$. Since the angle $Q_2Q_1Q'_1$ is $90$ deg, the angle $PQ_1Q'_1$ is obtuse. The rest of the proof is same as that of theorem 6.

Again, as before, we can extend our techniques to shapes that are a union of a small number of bodies that satisfy the conditions required by algorithm FULLROT.

**Theorem 8** Given a shape and orientation that can be expressed as a union of $c$ shapes, each satisfying the conditions required by algorithm FULLROT, we can find the smallest enclosing polytope with that shape and orientation in time $(2^c)O(1/\varepsilon^2)n^d$

Just as in section 3, we can derive core-sets for rotational problems.

**Theorem 9** For rotational problems with shapes satisfying conditions for algorithms MINROT and FULLROT, core-sets of sizes $O(1/\varepsilon^2)$ and $2^{O(1/\varepsilon^2)}$ exist, respectively.

Note that an infinite cylinder with its axis passing through the origin satisfies the assumptions of algorithm FULLROT. So we have:

**Corollary 1** For the minimum radius cylinder problem if we are given a point on the axis of the optimal cylinder, algorithm FULLROT runs in time $2^{O(1/\varepsilon^2)}n^d$

Note that in the min-cylinder problem, the maximum distance between all points may not be 1 as assumed. This can be easily overcome by setting the initial position of the axis to pass through the farthest point from the origin - we omit the details here.

## 5 Minimum Radius Cylinder

Although we do not have any general results for a combination of rotation and translation for different shapes, we provide an algorithm for the min-cylinder problem without restrictions that runs in time $2^{O(1/\varepsilon^2)}n^d$. The algorithm is similar to the one mentioned in [9] with a running time of $2^{O((\frac{1}{\varepsilon} \log^2 \frac{1}{\varepsilon}))}n^d$. Our algorithm can be viewed as following the greedy principle underlying the other algorithms of this paper: In each iteration it moves the axis of the cylinder along the plane containing the axis and an outside point by guessing its optimal position in that plane approximately.

Without loss of generality assume that the optimal radius is 1. We will show later how this optimal radius can be computed approximately using a binary search. We start with a certain initial position of the axis that will be specified latter. Let $l_{OPT}$ be the axis of the optimal cylinder. Let $U$ and $V$ be the farthest two points on $l_{OPT}$ that are projections of points in set $S$ on $l_{OPT}$.

**Algorithm MINCYN** -

1. We iteratively compute an estimate $l_i$ of $l_{OPT}$ and points $U_i$ and $V_i$ on $l_i$ that are close to projections of $U$ and $V$ on $l_i$.
2. We maintain the following invariant: $d(U_i, U) \leq 5$ and $d(V_i, V) \leq 5$
3. $l_{i+1}, U_{i+1}$ and $V_{i+1}$ are computed from $l_i, U_i$ and $V_i$ as follows: Find any point $P$ that is at distance more than $1 + \epsilon$ from $l_i$. Look at the plane $h$ containing $l_i$ and $P$. We will try to
set $U_{i+1}$ and $V_{i+1}$ close to $U_h = \text{proj}(U, h)$ and $V_h = \text{proj}(V, h)$ respectively, by the following process.

From the invariant, we have $d(U_i, U_h) \leq 5$. So $U_h$ lies in a circle in $h$ of radius 5 centered at $U_i$. Create a mesh, where each element has side $\epsilon/8$, so that $U_i$ itself is a mesh point, and guess the mesh point closest to $U_h$ and at a distance at most 5 from $U_i$. We need to guess one out of $\frac{\pi(5)^2}{\epsilon^2}$ points and set this point to $U_{i+1}$. Clearly this point is at most $\epsilon/4$ from $U_h$.

Similarly we guess $V_{i+1}$ out of at most $O(1/\epsilon^2)$ points.

We will prove convergence by arguing that the potential function, $\Phi = d(U, U_h)^2 + d(V, V_h)^2$, decreases significantly in each iteration.

**Lemma 5.1** We maintain the invariant, $d(U_i, U) \leq 5$ and $d(V_i, V) \leq 5$, during each iteration of algorithm MINCYN.

**Proof** We will show that $d(U_i, U)$ only keeps decreasing and if the invariant is true to start with, it always remains true. Now for any point $X$ on the plane $h$, $d(X, U_h)^2 = d(X, U_h)^2 + d(U_h, U)^2$. Since we choose $U_{i+1}$ to be the mesh point closest to $U_h$, among mesh points including $U_i$, $d(U_{i+1}, U_h) \leq d(U_i, U_h)$. So the invariant 2 follows.

**Lemma 5.2** In each iteration of algorithm MINCYN, the potential function $\Phi$ decreases by at least $\epsilon^2/2$.

**Proof** Note that $UU_h$ is perpendicular to the plane containing $U_h, U_i$ and $U_{i+1}$. So,

\[
d(U_i, U_{i+1})^2 = d(U_i, U_h)^2 + d(U_h, U_{i+1})^2 \\
= d(U_i, U_h)^2 - d(U_i, U_h)^2 + d(U_h, U_{i+1})^2 \\
\leq d(U_i, U_h)^2 - d(U_i, U_h)^2 + \epsilon^2/4
\]

Similar inequality holds for $d(V, V_{i+1})^2$. Adding the two we get, $\Phi_{i+1} \leq \Phi_i - d(U_i, U_h)^2 - d(V_i, V_h)^2 + \epsilon^2/2$.

We will prove that at least one of $d(U_i, U_h)$ and $d(V_i, V_h)$ is more than $\epsilon$. For if not then we will show that $P$ cannot be within distance 1 of any point in the segment $U_hV_h$, which is a contradiction because $U_hV_h$ is the projection of $UV$.

Let $l_p$ be the line in plane $h$ passing through $P$ and perpendicular to $l_i$, meeting $l_i$ at $P'$. Then, $d(P, P') \geq 1 + \epsilon$. Project all points to $l_p$. The segment $U_hV_h$ projects down to a segment $U_pV_p$. Since there is a point on $U_hV_h$ that is at most distance 1 from $P$ there must also be such a point, $Q$, on $U_pV_p$. Since $d(P', Q) \geq \epsilon$, at least one of $U_p$ and $V_p$ must be at least $\epsilon$ away from $P'$. Since distances only decrease under projections, at least one of $d(U_i, U_h)$ and $d(V_i, V_h)$ must be $\geq \epsilon$.

So, we get $\Phi_{i+1} \leq \Phi_i - \epsilon^2 + \epsilon^2/4 \leq \Phi_i - \epsilon^2/2$.

Finally we need to prove that we can choose an initial line $l_0$ and points on it $U_0$ and $V_0$ that satisfy the invariant. Look at any point $X$ in the set $S$. Let $Y$ be the farthest point from $X$ in $S$. Set $l_0$ to the line passing through $X$ and $Y$. It is easy to verify that every point in $S$ must be within distance 4 from $l_0$ (See lemma 5.2 in [9] for a more general statement). Look at the projections of points in $S$ on $l_0$ and $l$. For any point $Z$ in $S$, let $Z_0$ denote its projection on $l_0$ and $Z_l$ denote its projection on $l$. Then $d(Z, Z_0) \leq 4$ and $d(Z, Z_l) \leq 1$. So $d(Z_0, Z_l) \leq 5$. 

11
Set $U_0$ and $V_0$ to the farthest two points among projections of points of $S$ on $l_0$. Clearly these points are at most at distance 5 from $U$ and $V$ respectively.

So we have proved the following theorem

**Theorem 10** Algorithm $MINCYN$ terminates in $O(1/\epsilon^2)$ iterations. That is, after so many iterations no point will be more than $\epsilon$ outside the surface of the cylinder. A deterministic version of this algorithm runs in time $2^{O(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon})} nd$.

The deterministic version follows by simply eliminating the guess. Each guess requires guessing twice from $O(1/\epsilon^2)$ choices, this guessing happens at most $O(1/\epsilon^2)$ times. We also need to clarify how the optimal radius required by the algorithm can be determined. The distance of the farthest point from the initial position of the axis $l_0$ is within a constant factor of the optimal radius. The algorithm $MINCYN$ terminates only if the radius used is larger than the optimal radius and does not terminate if the radius used is too small. A binary search involving $\log(1/\epsilon)$ trials will result in a value that is within $1 + \epsilon$ of the optimal radius.

Using the techniques in [9] this algorithm generalizes to computing the min-radius $k$-dimensional flat - we omit the details here.

**Theorem 11** The minimum radius $k$-dimensional flat can be computed in time $2^{O(k^2/\epsilon^2) \log \frac{1}{\epsilon}} nd$.

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