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Radiative Transfer in a Fluid

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Abstract

We study the Radiative Transfer equations coupled with the time dependent temperature equation of a fluid: existence, uniqueness, a maximum principle are established. A short numerical section illustrates the pros and cons of the method.

Keywords: Radiative transfer, Temperature equation, Integral equation, Numerical analysis

Introduction

In fluid mechanics, \textit{Radiative Transfer} is an important subfield of \textit{Heat Transfer} with many applications to combustion, micro-wave ovens and climate models.

For the physics of radiative transfer for the atmosphere the reader is sent to Goody and Yung (1961), Bohren (2006), to the numerically oriented Zdunkowski and Trautmann (2003) and to the two mathematically oriented Chandrasekhar (1950) and Fowler (2011).

When Planck’s theory of black bodies is used the radiations have a continuum of frequencies governed by the temperature of the emitting body.

Even when the interactions with the fluid medium are neglected, the radiative transfer equations have 5 spatial dimensions. Hence the problem is
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numerically quite difficult. The stratified approximation is used when the radiation source is far, typically, the Sun. It is a two dimensional model with one spatial and one angular dimension to which the authors have contributed recently: see Bardos and Pironneau (2021) for stratified radiative transfer alone and Pironneau (2021), Golse and Pironneau (2022), Golse and Pironneau (2021) for stratified radiative transfer coupled with the temperature equation in the stationary case.

In this article the complete 5 dimensional radiative transfer model is studied when coupled with the time dependent temperature equation in the fluid.

Existence and uniqueness of a solution is well known when the physically constants do not depend on the frequency of the radiating source, the so called grey model (see Bardos et al (1988)) and Golse (1987)). In the non-grey case, some results have been obtained by Mercier (1987), Golse and Perthame (1986), et al. The present article extends these studies done in the eighties.

The radiative transfer system coupled to the Navier-Stokes equations has been studied by Pomraning (1973) and Ghattassi et al (2020) at least. In the later an existence theorem is given when the coefficients depend on the spatial variables but not on the frequencies of the source.

The paper begins with a statement of the radiative transfer equations in Section 1. In Section 2 an existence result is given. In Section 3, uniqueness is shown. The proof is complex and relies on an argument given by Mercier (1987) and Crandall and Tartar (1980). A maximum principle is also shown. Finally in Section 4 a numerical example is given.

1 Fundamental equations and approximations

To find the temperature $T$ in an incompressible fluid exposed to electromagnetic waves it is necessary to solve the Navier-Stokes equations coupled with the Radiative Transfer equations, as explained in Pomraning (1973). It is a complex partial differential system formulated in terms of a time dependent temperature field $T(x, t)$ function of the position $x$ in the physical domain $\Omega$ and a light intensity field $I_\nu(x, \omega, t)$ of frequency $\nu$ in each direction $\omega$:

Given $I_\nu, T, u, \rho$ at time zero, find $I_\nu, T, u, p, \rho$, such that for all $\{x, \omega, t, \nu\} \in \Omega \times S_2 \times (0, \bar{T}) \times \mathbb{R}^+$,

$$
\frac{1}{c} \partial_t I_\nu + \omega \cdot \nabla I_\nu + \rho \kappa_\nu a_\nu \left[ I_\nu - \frac{1}{4\pi} \int_{S^2} p(\omega, \omega') I_\nu(\omega') d\omega' \right] = \rho \kappa_\nu (1 - a_\nu) [B_\nu(T) - I_\nu],
$$

$$
\rho c V (\partial_t T + u \cdot \nabla T) - \nabla \cdot (\rho c P \kappa T \nabla T)
$$

$$
+ \frac{1}{c} \int_0^\infty \int_{S^2} I_\nu d\mu d\nu + \nabla \cdot \int_0^\infty \int_{S^2} I_\nu(\omega') \omega d\omega d\nu = 0
$$

$$
\partial_t u + u \cdot \nabla u - \frac{\mu F}{\rho} \Delta u + \frac{1}{\rho} \nabla p = g, \quad \nabla \cdot u = 0, \quad \partial_t \rho + \nabla \cdot (\rho u) = 0,
$$

(1)
where $\nabla, \Delta$ are with respect to $x$, $B_{\nu}(T) = \frac{2h\nu^3}{c^2[e^{\frac{h\nu}{kT}} - 1]}$, is the Planck function, $h, c, k$ are the Planck constant, the speed of light in the medium and the Boltzmann constant. The density of the medium is $\rho$, the pressure is $p$; $c_P, c_V$ are the compressibility of the fluid at constant pressure or volume; in large area these may be altitude/depth dependent. The absorption coefficient $\kappa_{\nu} := \rho\kappa_{\nu}$ comes from computations in atomic physics, but for our purpose it is seen as the percentage of light absorbed per unit length. The scattering albedo is $a_{\nu} \in (0, 1)$ and $\frac{1}{4\pi}p(\omega, \omega')$ is the probability that a ray in direction $\omega'$ scatters in direction $\omega$. The constants $\kappa_T$ and $\mu_F$ are the thermal and molecular diffusions; $g$ is the gravity.

Existence of solution for the fluid part of (1) has been established by Lions (1996).

### 1.1 The Mathematical Problem

Denote the angular average radiative intensity by $J_{\nu}(t, x) = \frac{1}{4\pi} \int_{S^2} I_{\nu}(\omega)d\omega$. If $\frac{1}{c}$ is neglected in (1), the following holds:

$$\nabla \cdot \int_0^\infty \int_{S^2} I_{\nu}(\omega')\omega d\omega dv = 4\pi \int_0^\infty \rho\kappa_{\nu}(1 - a_{\nu})(B_{\nu}(T) - J_{\nu}) d\nu. \quad (2)$$

Consequently we are led to study the well-posedness of the following system for $I_{\nu}, J_{\nu}, T$:

$$\begin{cases}
\omega \cdot \nabla I_{\nu} + \kappa_{\nu}I_{\nu} = \kappa_{\nu}(1 - a_{\nu})B_{\nu}(T) + \kappa_{\nu}a_{\nu}J_{\nu}, \quad J_{\nu} := \int_{S^2} I_{\nu} d\omega, \\
\partial_t T + u \cdot \nabla T - \lambda \Delta T = \int_0^\infty \kappa_{\nu}(1 - a_{\nu})(J_{\nu} - B_{\nu}(T)) dv, \\
I_{\nu}(x, \omega) = Q_{\nu}(x, \omega), \quad \omega \cdot n < 0, \quad x \in \partial \Omega, \\
\frac{\partial T}{\partial n} \bigg|_{\partial \Omega} = 0, \\
T \bigg|_{t=0} = T_{in}.
\end{cases} \quad (3)$$

Here $\Omega$ is assumed to be a bounded open subset of $\mathbb{R}^3$ with $C^1$ boundary, and we denote by $n$ the outward unit normal field on $\partial \Omega$. We further assume that $\nu \mapsto \kappa_{\nu}$ and $\nu \mapsto a_{\nu}$ are measurable functions satisfying

$$0 \leq \kappa_m \leq \kappa_{\nu} \leq \kappa_M, \quad 0 \leq a_{\nu} \leq a_M < 1, \quad \nu > 0, \text{ a.e.},$$

for some positive constants $a_M$ and $\kappa_m < \kappa_M$. Finally, we assume that the fluid velocity field $(t, x) \mapsto u(t, x)$ is smooth on $[0, +\infty) \times \overline{\Omega}$ and satisfies

$$\nabla \cdot u(t, x) = 0 \text{ for } x \in \Omega, \quad u(t, x) = 0 \text{ for } x \in \partial \Omega, \quad t \geq 0.$$
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2 Existence

Given a passive parameter \( t \), consider the auxiliary problem

\[
\begin{cases}
\omega \cdot \nabla I_\nu(t, x, \omega) = \kappa_\nu(S_\nu(t, x) - I_\nu(t, x, \omega)), & x \in \Omega, \ |\omega| = 1, \\
I_\nu(t, x, \omega) = Q_\nu(x, \omega), & \omega \cdot n < 0,
\end{cases}
\]

where the source \( S_\nu \) is isotropic, i.e. not a function of \( \omega \). Define the exit time

\[
\tau_{x, \omega} = \sup \{ s > 0 \text{ s.t. } x - s\omega \in \Omega \}.
\]

By the method of characteristics

\[
I_\nu(t, x, \omega) = 1_{\tau_{x, \omega} < +\infty} Q_\nu(x - \tau_{x, \omega} \omega) e^{-\kappa_\nu \tau_{x, \omega}} + \int_0^{\tau_{x, \omega}} e^{-\kappa_\nu s} \kappa_\nu S_\nu(t, x - s\omega) ds.
\]

Averaging in \( \omega \), one finds

\[
J_\nu(t, x) = \mathcal{J}[S_\nu](t, x) := \frac{1}{4\pi} \int_{\mathbb{S}^2} 1_{\tau_{x, \omega} < +\infty} Q_\nu(x - \tau_{x, \omega} \omega) e^{-\kappa_\nu \tau_{x, \omega}} d\omega
+ \frac{1}{4\pi} \int_{\mathbb{S}^2} \int_0^{\tau_{x, \omega}} e^{-\kappa_\nu s} \kappa_\nu S_\nu(t, x - s\omega) ds d\omega.
\] (4)

Since \( \kappa_\nu > 0 \), the functional \( \mathcal{J} \) satisfies the following monotonicity property:

\[
S_\nu(t, x) \leq S'_\nu(t, x) \quad \text{for a.e. } x \in \Omega \text{ and } t > 0
\implies \mathcal{J}[S_\nu](t, x) \leq \mathcal{J}[S'_\nu](t, x) \quad \text{for a.e. } x \in \Omega \text{ and } t > 0.
\]

In particular,

\[
0 \leq Q_\nu(x, \omega), \ S_\nu(t, x) \leq B_\nu(T_M), \ x \in \Omega, \ |\omega| = 1, \ \nu, t > 0
\implies 0 \leq \mathcal{J}[S_\nu](t, x) \leq B_\nu(T_M), \ x \in \Omega, \ \nu, t > 0.
\]

That \( \mathcal{J}[S_\nu] \geq 0 \) is obvious. As for the upper bound, observe that

\[
\mathcal{J}[B_\nu(T_M)] = \frac{1}{4\pi} B_\nu(T_M) \int_{\mathbb{S}^2} e^{-\kappa_\nu \tau_{x, \omega}} d\omega + \frac{1}{4\pi} B_\nu(T_M) \int_{\mathbb{S}^2} \int_0^{\tau_{x, \omega}} e^{-\kappa_\nu s} \kappa_\nu ds d\omega
= \frac{1}{4\pi} B_\nu(T_M) \left[ \int_{\mathbb{S}^2} e^{-\kappa_\nu \tau_{x, \omega}} d\omega + \int_{\mathbb{S}^2} (1 - e^{-\kappa_\nu \tau_{x, \omega}}) d\omega \right] = B_\nu(T_M),
\]

so that the desired upper bound follows from the monotonicity of \( \mathcal{J} \).

In order to solve the system (3), we consider the iterative scheme detailed in Algorithm 1, where we have assumed that

\[
0 \leq T_{in}(x) \leq T_M, \quad 0 \leq Q_\nu(x, \omega) \leq B_\nu(T_M), \quad x \in \Omega, \ |\omega| = 1, \ \nu > 0.
\]
Algorithm 1 to solve (3).

1. Start from $T^0 \equiv 0$ and $J^0_{\nu} = J[0]$;
2. for $n = 0, 1, \ldots, N - 1$
   (a) for all $\nu \in (0, \infty)$ and all $\tau \in (0, Z)$, by knowing $T^n \equiv T^n(t, x)$ and $J^n_{\nu} \equiv J^n_{\nu}(t, x)$, define with (4)
   $$J^{n+1} = J[a_{\nu}J^n_{\nu} + (1 - a_{\nu})B_{\nu}(T^n)]$$;
   (b) Define $T^{n+1}$ to be the solution of the semilinear drift-diffusion equation
   $$\begin{cases}
   \partial_t T^{n+1} + \mathbf{u} \cdot \nabla T^{n+1} - \lambda \Delta T^{n+1} + B(T^{n+1}) = \int_0^\infty \kappa_{\nu}(1 - a_{\nu})J^{n+1}_{\nu} d\nu, \\
   T^{n+1}|_{t=0} = T_{in}, \\
   \frac{\partial T^{n+1}}{\partial n}|_{\partial \Omega} = 0, \quad x \in \Omega, \ t > 0,
   \end{cases}$$
   where $B(T) := \int_0^\infty \kappa_{\nu}(1 - a_{\nu})B_{\nu}(\min(T_+, T_M))d\nu$.

Applying Theorem 10.9 in Brezis (2011) (see also section 4.7.2 in Lions and Magenes (1972)) shows that, for each $q \in L^2(0, T; H^{-1}(\Omega))$, there exists a unique solution to the convection-diffusion problem

$$\begin{cases}
   \partial_t \theta + \mathbf{u} \cdot \nabla \theta - \lambda \Delta \theta = q, \quad x \in \Omega, \ t > 0, \\
   \theta|_{t=0} = T_{in}, \\
   \frac{\partial \theta}{\partial n}|_{\partial \Omega} = 0,
   \end{cases}$$

of the form

$$\theta(t, \cdot) = \Sigma(t, 0)T_{in} + \int_0^t \Sigma(t, s)q(s, \cdot)ds \in L^2(0, T; H^1(\Omega)) \cap C(0, T; L^2(\Omega)).$$

With

$$q := \int_0^\infty \kappa_{\nu}(1 - a_{\nu})J^{n+1}_{\nu} d\nu - B(T^{n+1}),$$

we see that $T^{n+1}$ is a fixed point of the map $F$ defined by

$$F(\theta)(t, \cdot) = \Sigma(t, 0)T_{in} + \int_0^t \Sigma(t, s) \left( \int_0^\infty \kappa_{\nu}(1 - a_{\nu})J^{n+1}_{\nu}(s, \cdot) d\nu - B(\theta(s, \cdot)) \right) ds.$$
Observe that $\mathcal{B}$ is Lipschitz continuous with
\[
\text{Lip}(\mathcal{B}) \leq \kappa_M (1 - a_M) \sup_{0 \leq \theta \leq T_M} \int_0^{\infty} B'_\nu(\theta) d\nu = 4\kappa_M (1 - a_M) T_M^3,
\]
so that, arguing as in the proof of Theorem 1.2 in chapter 6 of Pazy (1983) shows that $\mathcal{F}$ has a unique fixed point. This defines a unique solution $T^{n+1} \in C([0, +\infty); L^2(\Omega)) \cap L^2_{\text{loc}}(0, \infty; H^1(\Omega))$.

Next, we seek to compare the solutions $T$ and $T'$ of
\[
\begin{align*}
\partial_t T + u \cdot \nabla T - \lambda \Delta T + \mathcal{B}(T) &= R, \quad x \in \Omega, \ t > 0, \\
T|_{t=0} &= T_{\text{in}}, \quad \partial T/\partial n|_{\partial\Omega} = 0,
\end{align*}
\]
and
\[
\begin{align*}
\partial_t T' + u \cdot \nabla T' - \lambda \Delta T' + \mathcal{B}(T') &= R', \quad x \in \Omega, \ t > 0, \\
T'|_{t=0} &= T'_{\text{in}}, \quad \partial T'/\partial n|_{\partial\Omega} = 0,
\end{align*}
\]
under the assumption that $0 \leq R \leq R'$ on $(0, +\infty) \times \Omega$. Proceeding as in the proof of Lemma 6.2 of Golse and Pironneau (2021), we multiply both sides of the equality satisfied by the difference $T - T'$ by $(T - T')_+$:
\[
\partial_t \frac{1}{2} (T - T')_+^2 + u \cdot \nabla \frac{1}{2} (T - T')_+^2 - \lambda \Delta \frac{1}{2} (T - T')_+^2 + \lambda |\nabla (T - T')_+|^2 + (\mathcal{B}(T) - \mathcal{B}(T'))(T - T')_+ = (R - R')(T - T')_+ \leq 0.
\]

Integrating over $\Omega$, and taking into account the boundary conditions satisfied by $u$ and $(T - T')$ shows that
\[
\frac{d}{dt} \int_\Omega \frac{1}{2} (T - T')_+^2(t, x) dx + \lambda \int_\Omega |\nabla (T - T')_+|^2(t, x) dx + \int_\Omega (\mathcal{B}(T) - \mathcal{B}(T'))(T - T')_+ \leq 0,
\]
since
\[
\int_{\partial\Omega} \left( u \cdot n (T - T')_+^2 - \lambda \frac{\partial}{\partial n} (T - T')_+^2 \right) d\sigma = 0.
\]
Then, $T \mapsto \mathcal{B}(T)$ is nondecreasing on $\mathbb{R}$, since $\kappa_{\nu}(1 - a_{\nu}) \geq 0$ and $T \mapsto B_{\nu}(\min(T_+, T_M))$ is nondecreasing on $\mathbb{R}$ for each $\nu > 0$. Hence
\[
(\mathcal{B}(T) - \mathcal{B}(T'))(T - T')_+ \geq 0
\]
so that
\[
\int_\Omega \frac{1}{2} (T - T')_+^2(t, x) dx \leq \int_\Omega \frac{1}{2} (T_{\text{in}} - T'_{\text{in}})_+^2(x) dx = 0.
\]
Therefore
\[
T_{\text{in}} \leq T'_{\text{in}} \text{ on } \Omega \text{ and } R \leq R' \text{ on } (0, +\infty) \times \Omega \implies T \leq T' \text{ on } (0, +\infty) \times \Omega.
\]
This comparison argument shows that

\[ 0 \leq J_{\nu}^{n+1}(t, x) \leq B_{\nu}(T_M) \text{ on } (0, +\infty) \times \Omega \implies 0 \leq T^{n+1} \leq T_M \text{ on } (0, +\infty) \times \Omega. \]

By the same token,

\[ J_{\nu}^n \leq J_{\nu}^{n+1} \text{ on } (0, +\infty) \times \Omega \implies T^n \leq T^{n+1} \text{ on } (0, +\infty) \times \Omega. \]

On the other hand, the monotonicity property of the function \( J \) shows that

\[ T^{n-1} \leq T^n \text{ and } J_{\nu}^{n-1} \leq J_{\nu}^n \implies J_{\nu}^n = J[a_{\nu}J_{\nu}^{n-1} + (1 - a_{\nu})B_{\nu}(T^n)] \leq J[a_{\nu}J_{\nu}^n + (1 - a_{\nu})B_{\nu}(T^n)] = J_{\nu}^{n+1} \]

on \((0, +\infty) \times \Omega\). Summarising, we have essentially proved the following result.

**Theorem 1** Assume that

\[ 0 \leq \kappa_m \leq \kappa_{\nu} \leq \kappa_M, \quad 0 \leq a_{\nu} \leq a_M < 1, \quad \nu > 0, \quad \text{a.e.}, \]

and pick boundary and initial data satisfying, for some \(T_M\),

\[ 0 \leq T_{\text{in}}(x) \leq T_M, \quad 0 \leq Q_{\nu}(x, \omega) \leq B_{\nu}(T_M), \quad x \in \Omega, \quad |\omega| = 1, \quad \nu > 0. \]

Then

(1) the sequences \( T^n \) and \( J_{\nu}^n \) satisfy

\[ 0 = T^0 \leq T^1 \leq \ldots \leq T^n \leq T^{n+1} \leq \ldots \leq T_M \text{ on } (0, \infty) \times \Omega, \]

and

\[ 0 \leq J_{\nu}^0 \leq J_{\nu}^1 \leq \ldots \leq J_{\nu}^n \leq J_{\nu}^{n+1} \leq \ldots \leq B_{\nu}(T_M) \text{ on } (0, \infty) \times \Omega, \text{ for all } \nu > 0. \]

(2) There exist a temperature field \( T \in C([0, +\infty); L^2(\Omega)) \cap L^2_{\text{loc}}(0, \infty; H^1(\Omega)) \) and a radiative intensity \( I_{\nu} \in L^\infty((0, \infty) \times \Omega \times S^2 \times (0, +\infty)) \) satisfying (3) in the sense of weak solutions.

**Proof** Statement (1) is a consequence of the monotonicity properties established above. It implies statement (2) by the following elementary observations: first, one can pass to the limit by monotone convergence in the expression

\[ J_{\nu}^{n+1} = J[a_{\nu}J_{\nu}^n + (1 - a_{\nu})B_{\nu}(T^n)] \]

to find that

\[ J_{\nu} = J[(a_{\nu}J_{\nu} + (1 - a_{\nu})B_{\nu}(T)], \]

since \( B_{\nu} \) is an increasing function for each \( \nu > 0 \). By the method of characteristics, the formula

\[ I_{\nu}(t, x, \omega) = 1_{\tau_{x, \omega} < +\infty}Q_{\nu}(x - \tau_{x, \omega}\omega)e^{-\kappa_{\nu, x}\omega} \]
\[ + \int_0^{\tau_{x, \omega}} e^{-\kappa_{\nu, x}\kappa_{\nu}(a_{\nu}J_{\nu}(t, x - s\omega) + (1 - a_{\nu})B_{\nu}(T(t, x - s\omega)))}ds \]

defines a solution of the transfer equation in (3).
Finally, for each $\phi \in C_c([0, \infty); H^1(\Omega))$ such that $\partial_t \phi \in L^2((0, \infty) \times \Omega)$, one has
\[
\int_0^\infty \int_\Omega (\lambda \nabla T^n \cdot \nabla \phi - T^n(\partial_t \phi + u \cdot \nabla \phi))dxdt = \int_\Omega T_{in}(x)\phi(0, x)dx
\]
\[
\int_0^\infty \int_0^\infty \int_\Omega \kappa_\nu(1 - a_\nu)(J_\nu^n - B_\nu(T^n))\phi dx d\nu dt.
\]
One can pass to the limit by dominated convergence in all the terms except the one involving $\nabla T^n$. This last term is mastered by the energy balance for the convection-diffusion equation:
\[
\frac{1}{2} \int_\Omega T^n(t, x)^2 dx + \lambda \int_0^t \int_\Omega |\nabla T^n(t, x)|^2 dx = \frac{1}{2} \int_\Omega T_{in}(x)^2 dx
\]
\[
+ \int_0^t \int_\Omega \kappa_\nu(1 - a_\nu)(J_\nu^n - B_\nu(T^n))T^n d\nu ds \leq t|\Omega|B(T_M),
\]
which implies that $T^n$ is bounded in $L^2_{loc}(0, \infty; H^1(\Omega))$. Since we already know that $T^n \rightarrow T$ in $L^p((0, \tau) \times \Omega)$ for all $p \in [1, \infty)$ as $n \rightarrow \infty$, we conclude that $T^n \rightarrow T$ weakly in $L^2_{loc}(0, \infty; H^1(\Omega))$. With this last piece of information, we pass to the limit in the weak formulation of the convection-diffusion equation and conclude that
\[
\int_0^\infty \int_\Omega (\lambda \nabla T \cdot \nabla \phi - T(\partial_t \phi + u \cdot \nabla \phi))dxdt = \int_\Omega T_{in}(x)\phi(0, x)dx
\]
\[
\int_0^\infty \int_0^\infty \int_\Omega \kappa_\nu(1 - a_\nu)(J_\nu^n - B_\nu(T^n))\phi dx d\nu dt.
\]
In other words, $T$ is a weak solution of the second equation in (3). This concludes the proof.

\[\square\]

3 Uniqueness and maximum principle

For each $\epsilon > 0$ let $s_\epsilon \in C^\infty(\mathbb{R})$ be such that
\[
s_\epsilon(z) = 0 \text{ for } z \leq 0, \quad s_\epsilon(z) = 1 \text{ for } z \geq \epsilon, \quad 0 \leq \epsilon s_\epsilon'(z) \leq 2 \text{ for } z \in \mathbb{R},
\]
and let $s_+(z) = 1_{z > 0}$. Set
\[
S_\epsilon(y) = \int_0^y s_\epsilon(z)dz.
\]
Henceforth, we use the notation
\[
\langle \phi \rangle := \frac{1}{4\pi} \int_0^\infty \int_{S^2} \phi(\omega, \nu) d\omega d\nu
\]
Let $(I_\nu, T)$ and $(I'_\nu, T')$ be two solutions of the system above; then
\[
\nabla \cdot \langle \omega(I_\nu - I'_\nu)_+ \rangle + D_1 + D_2 = 0,
\]
where
\[ D_1 := \langle \kappa_\nu (1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))))s_+(I_\nu - I'_\nu) \rangle, \]
and
\[ D_2 := \langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (J_\nu - J'_\nu))s_+(I_\nu - I'_\nu) \rangle. \]
Since
\[ \int_{\mathbb{S}^2} ((I_\nu - I'_\nu) - (J_\nu - J'_\nu)) d\omega = 0, \]
one has
\[ D_2 := \langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (J_\nu - J'_\nu))(s_+(I_\nu - I'_\nu) - s_+(J_\nu - J'_\nu)) \rangle \geq 0, \]
since \( z \mapsto s_+(z) \) is nondecreasing, so that
\[ ((I_\nu - I'_\nu) - (J_\nu - J'_\nu))(s_+(I_\nu - I'_\nu) - s_+(J_\nu - J'_\nu)) \geq 0. \]

On the other hand
\[ D_1 = D_3^\varepsilon + s_\epsilon(T - T')\langle \kappa_\nu (1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) \rangle \]
where
\[ D_3^\varepsilon = \langle \kappa_\nu (1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_\epsilon(T - T')) \rangle, \]
while
\[
\begin{align*}
\partial_t s_\epsilon(T - T') + u \cdot \nabla s_\epsilon(T - T') - \lambda \Delta (T - T') s_\epsilon(T - T') \\
= s_\epsilon(T - T') \int_0^\infty \kappa_\nu (1 - a_\nu)((J_\nu - J'_\nu) - (B_\nu(T) - B_\nu(T'))) d\nu \\
= 4\pi s_\epsilon(T - T') \langle \kappa_\nu (1 - a_\nu)((J_\nu - J'_\nu) - (B_\nu(T) - B_\nu(T'))) \rangle.
\end{align*}
\]
Thus
\[
4\pi \nabla \cdot \langle \omega (I_\nu - I'_\nu)_+ \rangle + \partial_t s_\epsilon(T - T') + u \cdot \nabla s_\epsilon(T - T') - \lambda \Delta (T - T') s_\epsilon(T - T') \\
+ 4\pi (D_3^\varepsilon + D_2) = 0
\]
Then we integrate both sides of this equality on \( \Omega \):
\[
\frac{d}{dt} \int_{\Omega} S_\epsilon(T - T') d\mathbf{x} + 4\pi \int_{\partial\Omega} \langle \omega \cdot \mathbf{n}(I_\nu - I'_\nu)_+ \rangle d\sigma(\mathbf{x}) + \int_{\partial\Omega} S_\epsilon(T - T') \mathbf{u} \cdot nd\sigma(\mathbf{x}) \\
+ \lambda \int_{\Omega} |\nabla(T - T')|^2 s_\epsilon(T - T') d\mathbf{x} - \lambda \int_{\partial\Omega} s_\epsilon(T - T') \frac{\partial(T - T')}{\partial n} d\sigma(\mathbf{x}) \\
= -4\pi \int_{\Omega} (D_3^\varepsilon + D_2) d\mathbf{x}.
\]
Using the boundary conditions, specifically that
\[ I_\nu(x, \omega) = Q_\nu(x, \omega) \text{ and } I'_\nu(x, \omega) = Q'_\nu(x, \omega), \quad \omega \cdot n < 0, \]
with
\[ (Q_\nu - Q'_\nu)(x, \omega) \leq 0, \quad \omega \cdot n < 0, \]
implies
\[ \int_{\partial \Omega} \left\langle \omega \cdot n (I_\nu - I'_\nu) \right\rangle d\sigma(x) = \int_{\partial \Omega} \left\langle \omega \cdot n (I_\nu - I'_\nu) \right\rangle d\sigma(x) \geq 0, \]
\[ \int_{\partial \Omega} S_\epsilon(T - T') u \cdot n d\sigma(x) = \int_{\partial \Omega} s_\epsilon(T - T') \partial(T - T') \partial n d\sigma(x) = 0. \]
Hence
\[ \int_{\Omega} S_\epsilon(T - T')(t, x)dx + 4\pi \int_0^t \int_{\partial \Omega} \left\langle \omega \cdot n (I_\nu - I'_\nu) \right\rangle(\tau, x)d\sigma(x)d\tau + \lambda \int_0^t \int_{\Omega} |\nabla(T - T')(\tau, x)|^2 s_\epsilon(T - T')(\tau, x)dx d\tau + 4\pi \int_0^t \int_{\Omega} (D_3 + D_2)(\tau, x)dx d\tau \]
\[ = \int_{\Omega} S_\epsilon(T - T')(0, x)dx = 0 \]
under the assumption that
\[ T \big|_{t=0} = T_{in} \quad \text{and} \quad T' \big|_{t=0} = T'_{in} \quad \text{with} \quad T_{in} \leq T'_{in}. \]

Assume that
\[ \kappa_\nu(1 - a_\nu)(I_\nu + I'_\nu + B_\nu(T) + B_\nu(T')) \in L^1([0, t] \times \Omega \times S^2 \times (0, +\infty)); \]
by dominated convergence
\[ \int_0^t \int_{\Omega} D_3(\tau, x)dx d\tau \to \int_0^t \int_{\Omega} D_3(\tau, x)dx d\tau \]
where
\[ D_3 = \left\langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_+(T - T')) \right\rangle \geq 0 \]
since \( z \mapsto s_+(z) \) is nondecreasing and
\[ s_+(T - T') = s_+(B_\nu(T) - b_\nu(T')) \]
because \( B_\nu \) is increasing for each \( \nu > 0 \), so that
\[ ((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_+(T - T')) \geq 0. \]
By Fatou’s lemma
\[
\int_{\Omega} S_\epsilon(T - T')(t, x) \, dx \to \int_{\Omega} (T - T')_+(t, x) \, dx
\]
for a.e. \( t \geq 0 \), so that
\[
\int_{\Omega} (T - T')_+(t, x) \, dx + 4\pi \int_{\partial \Omega} \langle (\omega \cdot n)_+(I_\nu - I'_\nu)_+ \rangle(t, x) \, d\sigma(x) \, d\tau + 4\pi \int_{0}^{t} \int_{\Omega} (D_3 + D_2)(\tau, x) \, dx \, d\tau \leq 0 ,
\]
since
\[
\lim_{\epsilon \to 0} \int_{0}^{t} \int_{\Omega} |\nabla(T - T')(\tau, x)|^2 s'_\nu(T - T')(\tau, x) \, dx \, d\tau \geq 0 .
\]
Since all the terms on the left hand side of the previous equality are nonnegative, one has
\[
\int_{\Omega} (T - T')_+(t, x) \, dx = 4\pi \int_{0}^{t} \int_{\partial \Omega} \langle (\omega \cdot n)_+(I_\nu - I'_\nu)_+ \rangle(t, x) \, d\sigma(x) \, d\tau = 4\pi \int_{0}^{t} \int_{\Omega} (D_3 + D_2)(\tau, x) \, dx \, d\tau = 0 \quad \text{for a.e. } t > 0 .
\]
Once it is known that \( T \leq T' \) a.e. on \((0, +\infty) \times \Omega\), one has
\[
\int_{\Omega} \kappa_\nu (1 - a_\nu)(I_\nu - I'_\nu)_+ \, dx + \int_{\partial \Omega} \langle (\omega \cdot n)_+(I_\nu - I'_\nu)_+ \rangle(t, x) \, d\sigma(x) + \int_{\Omega} D_2 \, dx = \int_{\Omega} \langle \kappa_\nu (1 - a_\nu)(B_\nu(T) - B_\nu(T'))(t, x)s_+(I_\nu - I'_\nu) \rangle \, dx \leq 0
\]
since \( B_\nu(T) - B_\nu(T') \leq 0 \) while \( s_+(I_\nu - I'_\nu) \geq 0 \), so that \( I_\nu \leq I'_\nu \) a.e. on \((0, +\infty) \times \Omega \times S^2 \times (0, +\infty)\).

Summarising, we have proved the following

**Theorem 2** Let \((I_\nu, T)\) and \((I'_\nu, T')\) be two solutions of (3) such that
\[
\kappa_\nu (1 - a_\nu)(I_\nu + I'_\nu + B_\nu(T) + B_\nu(T')) \in L^1([0, t] \times \Omega \times S^2 \times (0, +\infty))
\]
for all \( t > 0 \). Assume that
\[
T \bigg|_{t=0} = T_{in} \quad \text{and} \quad T' \bigg|_{t=0} = T'_{in} \quad \text{with} \quad T_{in} \leq T'_{in} ,
\]
and that, when \( x \in \partial \Omega\),
\[
I'_\nu(x, \omega) = Q'_\nu(x, \omega) \quad \text{and} \quad I_\nu(x, \omega) = Q_\nu(x, \omega) \leq Q'_\nu(x, \omega) , \quad \omega \cdot n < 0 .
\]
Then
\[
I_\nu \leq I'_\nu \quad \text{and} \quad T \leq T' .
\]
If \( T'_{in} = T_{in} \) and \( Q_\nu = Q'_\nu \), exchanging the roles of \((I_\nu, T)\) and \((I'_\nu, T')\) in the theorem above leads to the following uniqueness result.
Corollary 1 There is at most one solution \((I_\nu, T)\) of (3) such that
\[
\kappa_\nu (1 - a_\nu)(I_\nu + B_\nu(T)) \in L^1([0, t] \times \Omega \times S^2 \times (0, +\infty)) \quad \text{for all } t > 0.
\]

In the case where \((I_\nu', T)\) is a Planck equilibrium, i.e. \(I_\nu' = B_\nu(T')\) with \(T'\) =constant, one obtains Mercier’s maximum principle:

Corollary 2 If \(0 \leq Q_\nu \leq B_\nu(T_M)\) and \(0 \leq T_{in} \leq T_M\) and \(\Omega\) has finite volume, the solution \((I_\nu, T)\) of (3) such that
\[
\kappa_\nu (1 - a_\nu)(I_\nu + B_\nu(T)) \in L^1([0, t] \times \Omega \times S^2 \times (0, +\infty))
\]
satisfies
\[
0 \leq I_\nu(t, x, \omega) \leq B_\nu(T_M) \quad \text{and} \quad 0 \leq T(t, x) \leq T_M
\]
for a.e. \((t, x, \omega, \nu)\) in \((0, +\infty) \times \Omega \times S^2 \times (0, +\infty)\).

4 A Numerical Scheme

We begin with an important observation for the numerical implementation:

Proposition 3 Equation (4) can be written as
\[
\tilde{J}_\nu(x) = Y_{\kappa_\nu}(x) * \tilde{S}_\nu(x) + \tilde{S}^E_\nu(x), \quad \text{with } Y_{\kappa_\nu}(x) = \kappa_\nu \frac{e^{-\kappa_\nu|x|}}{|2x|^{d-1}}, \quad d = 2, 3. \tag{5}
\]

where \(*\) denotes a convolution and tildes indicate an extension by zero outside \(\Omega\) and
\[
\tilde{S}^E_\nu(x) = \frac{1}{2^{d-1}\pi} \int_{|\omega|=1} 1_{\{\tau_{\nu,\omega}<+\infty\}} Q_\nu(x - s\omega) e^{-\kappa_\nu \tau_{\nu,\omega}} d\omega \tag{6}
\]

Proof This is because, by integration in spherical coordinates with \(|x| = s, \]
\[
\int_{|\omega|=1} \int_0^\infty \kappa_\nu \tilde{S}_\nu(x - s\omega)e^{-\kappa_\nu s} dsd\omega = \int_{\mathbb{R}^d} \kappa_\nu \tilde{S}_\nu(x - x')e^{-\kappa_\nu |x'|} \frac{dx'}{|x'|^{d-1}}
\]

Notice that the Fourier transform of \(Y_{\kappa_\nu}\) satisfies
\[
\mathcal{F}Y_{\kappa_\nu}(\xi) = \mathcal{F}Y_1 \left( \frac{\xi}{\kappa_\nu} \right) = \frac{|\xi|}{2\pi \kappa_\nu} \arctan \frac{|\xi|}{\kappa_\nu}.
\]

The numerical implementation is detailed in Algorithm 2.
Algorithm 2 \textbf{To solve (5) with } S_\nu = a_\nu J_\nu + (1 - a_\nu) B_\nu(T) \textbf{ for each } \nu > 0,

1. Compute \( \mathbf{x} \mapsto \tilde{S}_E^E(\mathbf{x}) \) by (6) and \( \mathcal{F}Y_{\kappa_\nu} = \frac{\kappa_\nu}{2\pi|\xi|} \arctan(\kappa_\nu|\xi|) \).

2. \textbf{for } \mathbf{n}=0,1,\ldots,N
   \hspace{1em} (a) Compute the Fourier transforms \( \mathcal{F}\tilde{S}_\nu \).
   \hspace{1em} (b) Compute \( Y_{\kappa_\nu} * S_\nu = \mathcal{F}^{-1}(\mathcal{F}Y_{\kappa_\nu} \cdot \mathcal{F}\tilde{S}_\nu) \).
   \hspace{1em} (c) Set \( J^{n+1}_\nu(\mathbf{x}) = \tilde{S}_E^E(\mathbf{x}) + \mathcal{F}^{-1}(\mathcal{F}Y_{\kappa_\nu} \cdot \mathcal{F}S_\nu) \).
   \hspace{1em} (d) Compute \( T^{n+1} \) solution of
   \hspace{1em} \begin{align*}
   \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T) d\nu &= \int_0^\infty \kappa_\nu (1 - a_\nu) J^{n+1}_\nu d\nu.
   \end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The light source, in the far right, sends lightwaves to the planet; it is assumed that the light is unaffected by the atmosphere. Hence point \( P \) in the atmosphere receives only the radiations emitted by the planet. A cross section of the planet and its atmosphere is shown in the plane defined by the axis \( Ox \) and the point \( P \). The light intensity in the direction \( \omega \) is a function of the light intensity at \( P_E \), the intersection of \( \omega \) and the circle \( |\mathbf{x}| = R \).}
\end{figure}

4.1 A Bidimensional Example

The geometry of the problem is shown on Figure 1 and the data are:

\[ \Omega = \{ x : |x| \in (R, R + H) \} , \quad Q_\nu(\mathbf{x}_E, \omega) = Q^0 B_\nu(T_s) \frac{x_E^+}{R} . \]

These data are used with \( \mathbf{x}_E = (x_E, y_E) \), the intersection of the line \( \{ \mathbf{x} - t\omega \}_{t>0} \) with the circle \( \{ \mathbf{x} : |\mathbf{x}| = R \} \). As \( |\mathbf{x} - t\omega| = R \) requires \( t^2 - 2(\mathbf{x} \cdot \omega) + |\mathbf{x}|^2 - R^2 = 0 \), we have

\[ \tau_{x,\omega} = \mathbf{x} \cdot \omega - \sqrt{(\mathbf{x} \cdot \omega)^2 - |\mathbf{x}|^2 + R^2} . \]

As explained in Bardos and Pironneau (2021), for numerical convenience the problem can be rescaled so that the Planck function is \( B_\nu(T) = \frac{\nu^3}{e^{\frac{\nu}{T}} - 1} \) with \( T \) in Kelvin divided by 4780. The Stefan Boltzmann formula becomes

\[ \int_0^\infty B_\nu(T) d\nu = \sigma T^4 \text{ with } \sigma = \frac{\pi^4}{15} . \]

All cases are without scattering \( a = 0 \).
4.1.1 The Grey Case

In the grey case ($\kappa_\nu$ independent of $\nu$), the upper bar, as in $\bar{J}$, denotes the mean in $\nu$. Then it is easy to see that we need to solve iteratively the integral equation:

$$\bar{J}(x) = S^E(x) + \sigma Y_\kappa \tilde{T}^4, \quad S^E(x) = \frac{Q^0 \sigma T_s^4}{2\pi} \int_0^{2\pi} (x - \tau_x \omega \cos \theta) e^{-\kappa \tau_x \omega} d\theta \quad (7)$$

with $Y_\kappa = \frac{\kappa}{2\pi|x|} e^{-\kappa |x|}$. In absence of thermal diffusion, the temperature field is given by

$$\kappa \sigma T^4(x) = \kappa \bar{J}(x), \quad x \in \Omega. \quad (8)$$

Figure 2 shows a numerical result obtained with $\kappa = 0.5$, $N = 10$ iterations, starting from $T^0 = 0.01$. The monotone behaviour of $\bar{J}^n$ is clearly seen (but not displayed here).

![Fig. 2](image1.png)  
**Fig. 2** Temperature map in the atmosphere of the planet which receives light from the right.

![Fig. 3](image2.png)  
**Fig. 3** Temperature map in the atmosphere of the planet which receives light from the right and has thermal diffusion.

The grid used for the FFT is $64 \times 64$. The mesh for the ring is $36 \times 120$ approximately uniform in polar coordinates. For $S^E$ there are 60 integration points in $\theta$. The computing time is 1 second per iteration on a core i9 MacBook 2020; convergence is reached after 5 iterations.

4.1.2 Non-Zero Thermal Diffusion

Let $\kappa_T$ be the thermal diffusion and let $T_E$ be the temperature of the planet. Then (8) must be replaced by

$$-\kappa_T \Delta T + \sigma T^4(x) = \bar{J}(x), \quad x \in \Omega, \quad T_{\partial\Omega} = T_E. \quad (9)$$
It is discretized with triangular finite elements of degree 1 and solved iteratively by a fixed point method whereby $T^4$ is replaced by $T_m^3T_{m+1}$. Figure 3 shows a result with the same data used for Figure 2 and $\kappa_T = 0.01\sigma$. The temperature on the planet is fixed at 0.06, i.e. 13.8 °C.

### 4.1.3 The Frequency Dependent Case

When $\kappa_\nu$ is not constant the problem is numerically expensive because one Fourier transform is needed at each integration point in the integrals in $\nu$.

Recall that, when $a_\nu = 0$, we have to solve

$$\int_0^\infty \kappa_\nu B_\nu(T(x))d\nu = \int_0^\infty \kappa_\nu Y_{\kappa_\nu} \ast \tilde{B}_\nu(T)d\nu + S^E(x)$$  \hspace{1cm} (10)

with

$$S^E(x) = \frac{Q_0}{2\pi} \int_0^{2\pi} \left( (x - \tau_x, \omega \cos \theta)^+ \int_0^\infty B_\nu(T_s(\kappa_\nu e^{-\kappa_\nu \tau_x, \omega}) d\nu \right) d\theta$$  \hspace{1cm} (11)

Extracting $x \mapsto T(x)$ from (10), with a known right hand side, with a $\nu \mapsto \kappa_\nu$ given by values, is doable but expensive (see Golse and Pironneau (2021)). For a simple numerical example we may expand $\kappa_\nu$ in powers of $\nu$:

$$\kappa_\nu \approx \kappa_0 + \kappa_1 \nu + \kappa_2 \nu^2 + \kappa_3 \nu^3 + \kappa_4 \nu^4 + \cdots \implies \int_0^\infty \kappa_\nu B_\nu(T) = \sigma \kappa_0 T^4 + 24.886 \kappa_1 T^5 + 122.081 \kappa_2 T^6 + 726.012 \kappa_3 T^7 + 5060.55 \kappa_4 T^8 + \cdots$$  \hspace{1cm} (12)

These numerical values are evaluations of polynomials of $\pi$ and $\zeta$ function numbers computed with Maple.

For the numerical test we chose $\kappa_\nu = \kappa_0 + \kappa_1 \nu := 0.5 \pm 0.03\nu$, $\nu \in (0, 15)$. Then we have to solve iteratively

$$\sigma \kappa_0 T^4(x) + 24.886 \kappa_1 T^5(x) = Y_1 \ast B_\nu(x) + S^E(x), \quad x \in \Omega$$  \hspace{1cm} (13)

Figures 3, 4 and 5 illustrate the influence of a varying $\kappa_\nu$ on the temperature. There were 60 points for the integrations in $\nu$, 60 points for the integrations in $\theta$ and $64 \times 64$ for the Discrete Fourier Transforms.

All programs were written with the high level PDE solver freefem++ (see Hecht (2012)). The program for a non constant $\kappa$ is evidently much slower and took 580 seconds per case.

### 5 Conclusion

By using a technique developed in Golse and Pironneau (2021) for the stratified radiative transfer problem, we have proved existence, uniqueness and a
maximum principle for a rather general form of the multidimensional Radiative Transfer system coupled with the time dependent temperature equation with drift.

The proofs are constructive and yield a robust and fairly fast numerical numerical algorithm, at least in the grey case, which encapsulate the exact solution between a lower and a higher numerical one obtained by starting from a guessed temperature field below (resp. above) the exact temperature field.

The 5 dimensional PDE is thus replaced by an iteration involving a three dimensional integral and a convolution integral easily computed with an FFT and which constitutes a tremendous gain in computing time over more classical finite element discretization as in LeHardy et al (2017).

Most remarkable is that there are essentially no constraint, besides positivity, on the absorption $\kappa_\nu$ and the scattering $a_\nu$. If these depend on $x$, a change of variable needs to be applied to return to the case $\kappa_\nu$ independent of $x$. However if $\kappa_\nu$ depends on $T$ the method does not work, except by adding an iteration loop, sending this dependency on the right hand side of the equation of $I_\nu$.

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6 Appendix: Code documentation

The following may not appear in the published paper.

The following \texttt{freefem++} script \texttt{RT2Dfull2.edp} works for \( \kappa_\nu = \kappa_0 + \kappa_1 \nu \), \( \nu \in (\nu_{\text{min}},\nu_{\text{max}}) \). It recognizes the case \( \kappa \) constant (i.e. \( \kappa_1 = 0 \)) and by avoiding the integrals in \( \nu \) is then much faster in that case.

The data are:

```plaintext
1 // RT2Dfull2.edp. radiative transfer with no approximation
2 load "dfft"
3
4 int n=1, Niter=5,
5 nx=n+32, ny=n+32, NN=nx*ny;
6 real Q0=2*sqrt(2)*2.03e-5, Tsun=1.209, sigma=(pi^4)/15;
7
8 mesh Th=square(nx−1,ny−1,[−1+2*(nx−1)*x/nx,−1+2*(ny−1)*y/ny]);
9 // warning the numbering of vertices (x,y) i i = x/nx + nx*y/ny
10
11 real R=0.4, H=0.3;
12 real dtheta=pi/30; // controls the integration on the unit circle
13 real kappa0=0.5, kappa1=0., kappaT=0.01; // if zero no heat equation
14 real numin=0.01, numax=15, dnu=(numax−numin)/100;
15
16 real source=Q0, R2=R*R; // auxiliaries
```

We need two domains, the square for the dFFT and the ring for the physics:

```plaintext
1 mesh Th=square(nx−1,ny−1,[−1+2*(nx−1)*x/nx,−1+2*(ny−1)*y/ny]);
2 // warning the numbering of vertices (x,y) i i = x/nx + nx*y/ny
3 border R1(t=0,2*pi){x=R*cos(t); y=R*sin(t);}
4 border RH(t=0,2*pi){x=(R+H)*cos(t); y=(R+H)*sin(t);}
5 mesh Rh=buildmesh(RH(120)+R1(−120));
```

The finite element spaces and the functions are defined by

```plaintext
1 fespace Vh(Th,P1);
2 fespace Wh(Rh,P1);
3 Vh<complex> u,v,w,F; // u, v, w for FFT and J J for J (x, y)
4 Wh Tc=0.01, Tch, Tca; // Tc is T(x, y) and Tch and Tca are auxiliaries
```

We need a function to define \( \kappa_\nu \), one to define the Planck function \( B_\nu(T) \) and one to compute \( \tau_{x,\omega} \). The parameter \texttt{scal} in \texttt{txw} is here to save time and prevent recomputing the scalar product in \texttt{getSe}.

```plaintext
1 func real kappa(real nu){ return kappa0+kappa1*nu; }
2 func real txw(real X,real Y,real scal){
3 real aux=sqr(scal)+R2−sqr(X)−sqr(Y);
4 if(aux>0) return scal−sqr(aux);
5 else return scal+sqr(aux);
6 else return −1;
7 }
8
9 func real getSe(real X, real Y){
10 real aux = X*X+Y*Y;
11 real Jxy=0;
12 real Jyx=0;
13 if(aux>R2 & aux<(R+H)*(R+H))
14 for(real theta=0; theta<2*pi; theta+=dtheta){
15 real x=cos(theta), y=sin(theta);
16 real scal = X*x+y*y;  
17 real r = txw(X,Y,scal); 
18 real Bke = 0;
19 if(t>0) {
20 if(kappa1==0) Bke = kappa0*exp(−t*kappa0)*sigma*Tsun^4;
```

Now \texttt{getSe} implements (11) or (7) when applicable.
Now let us compute $Y_1$ and its Fourier transform $v$. We could have used its analytical values but then would have had to struggle with the correspondence between the Fourier modes and the grid points. To avoid the singularity at $x = 0$ we truncate it at $|x| > R/4$. The FEM function $u$ is needed to build an array of values at the grid points.

```
func real Yxy(real X, real Y, real kappa)
{
    real aux = sqrt(X*X + Y*Y);
    if ( aux>R/4 ) return kappa * exp(-aux*kappa) / (2*pi*aux);
    else return 0.0;
}
```

The computation of the right hand side of (13) is done as follows

```
F=0;
if ( kappa1==0)
{
    v[] = dfft (u[], ny,-1);
    w[] = dfft (u[], ny,-1);
    F=v'*w/sqr (NN);
}
else
{
    for ( real nu=numin; nu<numax; nu+=dnu )
    {
        u= Yxy(x,y,kappa (nu) ) ;
        v[] = dfft (u[], ny,-1);
        u = Bnu(Tc(x,y),nu) ;
        w[] = dfft (u[], ny,-1);
        u=
        v=w/sqr (NN);
        F=F+u*kappa (nu) *dnu ;
    }
    u[] = dfft (F[], ny,1)
    u= u + Jsource;
}
```

Finally the temperature is computed and converted into Celsius degree by the last formula.

```
Tca=sqrt(sqr (real (u) / ( sigma*kappa0 + 24.886*kappa1*Tc ))) ;
heat;
u = sqrt(Tc*Tc) * ( sigma*kappa0 + 24.886*kappa1*Tc ) ; // mean light intensity
Tca=Tc*4780-273; // temperature in Celsius
plot (Tca, ps="planettempdifffull2.ps", value=1, fill=1);
```

where `heat` is finite element solver for the temperature equation implemented as (notice how $T$ is prescribed on the planet at 0.06, which is 13.8 Celsius.  

```
problem heat(Tc,Tch) = int2d(Rh) ( kappaT*(dx(Tc)*dx(Tch)) + dy(Tc)*dy(Tch) ) + Tc*Tch ) - int2d(Rh) ( Tca*Tch ) + on ( R1, Tc=0.06 );
```

These next to last 2 blocks must be encapsulated into a iteration loop

```
for(int niter=0; niter<Niter; niter++){
    // the blocks here
}
```