LOCALLY COMPACT HECKE PAIRS AND PROPERTY (RD)

VAHID SHIRBISHEH

Abstract. This paper has two primary purposes: The first one is to introduce an extended setting to study Hecke pairs \((G, H)\) which admit a regular representation on \(L^2(H \backslash G)\). As the result, many pairs of locally compact groups \((G, H)\) which had been studied in noncommutative harmonic analysis, Lie theory and representation theory are included in the theory of Hecke \(C^*\)-algebras. These Hecke pairs mainly consist of locally compact groups and their compact subgroups, or cocompact subgroups, or open Hecke subgroups. We clarify similarities, differences and relationships of our formulation with the discrete case. In the discrete case, using the Schlichting completion, we show that the left regular representations of associated Hecke algebras are bounded homomorphisms. Several criteria for relative unimodularity of discrete Hecke pairs are also discussed. The second purpose of this paper is to study property (RD) for Hecke pairs in our generalized setting. We discuss length functions on Hecke pairs and the growth of Hecke pairs. We establish an equivalence between property (RD) of locally compact groups and property (RD) of certain locally compact Hecke pairs. This allows us to transfer several important results concerning property (RD) of locally compact groups into our setting, and consequently identify many classes of examples of locally compact Hecke pairs with property (RD). As another corollary, we show that a reduced discrete Hecke pair \((G, H)\) has (RD) if and only if its Schlichting completion \(\overline{G}\) has (RD). Then it follows that the relative unimodularity is a necessary condition for a discrete Hecke pair to possess property (RD). Various aspects of discrete and locally compact Hecke pairs are discussed by means of numerous examples.

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1. Introduction

Hecke $C^*$-algebras were introduced by Jean-Benoît Bost and Alain Connes in [10], in order to study the class field theory of the field $\mathbb{Q}$ of rational numbers by means of quantum statistical mechanics. Since then, many examples of Hecke pairs $(G, H)$ have appeared in the related literature. In these Hecke pairs either $G$ is simply a discrete group or it is a locally compact group and $H$ is a compact open subgroup of $G$. In both cases the necessary and sufficient condition to define a regular representation and subsequently to associate a $C^*$-algebra to the pair $(G, H)$ is that $H$ should be commensurable with all its conjugates, see Remark 2.13. In this setting the homogeneous space $H\backslash G$ associated to the pair $(G, H)$ is always a discrete space, and so we refer to such pairs as discrete Hecke pairs. The discreteness of the homogeneous space $H\backslash G$ facilitates many constructions and results. However, this setting misses many interesting pairs of locally compact groups and their subgroups which have been studied in other fields, for instance $(SL_2(\mathbb{R}), SO_2(\mathbb{R}))$ and $(SL_2(\mathbb{R}), SL_2(\mathbb{Z}))$. We shall see that these pairs also admit $C^*$-algebraic representations similar to the Hecke $C^*$-algebras defined by Bost and Connes. On the other hand, even in the discrete case, it was observed in [54, 35] by Kroum Tzanev, S. Kaliszewski, Magnus B. Landstad, and John Quigg that the theory of locally compact groups provides us with a convenient and fruitful setting for studying discrete Hecke pairs and their associated Hecke $C^*$-algebras. Recently, this point of view has found more supporting evidences in the work of Claire Anantharaman-Delaroche, see [2]. Another motivation for studying locally compact Hecke pairs comes from our results in the present paper. Here we apply our generalized setting based on locally compact groups to study discrete Hecke pairs and their property (RD), see for instance Theorems 2.40, 3.18, and 3.20. Therefore our first aim in this paper is to extend the definition of Hecke $C^*$-algebras for several classes of pairs $(G, H)$ of locally compact groups whose homogeneous space $H\backslash G$ is not necessarily discrete.

In this paper, a pair $(G, H)$ consisting of a locally compact group $G$ and a closed subgroup $H$ of $G$ is simply called a pair. Regarding the Bost-Connes construction of Hecke $C^*$-algebras, we carefully examine various steps of associating a $C^*$-algebra to a “suitable” pair. Our primary concern is to determine classes of pairs such that for every pair $(G, H)$ in these classes, there exists a left regular representation from the Hecke algebra associated to $(G, H)$ into the $C^*$-algebra of bounded operators on the Hilbert space $L^2(H\backslash G)$. Besides discrete Hecke pairs, we show that there are at least two other important classes of pairs which admit such a construction, and therefore their Hecke algebras are suitable for $C^*$-algebraic formulation. The first class consists of pairs $(G, H)$ in which $H$ is a compact subgroup of $G$. In this case, not only we prove the existence of a well defined left regular representation, but we also show that it is a bounded homomorphism, see Proposition 2.28. Although this latter statement is not a necessary step to define Hecke $C^*$-algebra, it is a helpful feature in application, due to the the parallelism between reduced Hecke $C^*$-algebras and reduced group $C^*$-algebras. Furthermore, by applying a certain topologization process of discrete Hecke pairs named the Schlichting completion, this statement is applied to prove the same boundedness statement for the left regular representations associated with discrete Hecke pairs. This is particularly important in the study of property (RD) for discrete Hecke pairs, see for instance Remark 3.36. The second class of locally compact Hecke pairs capable of a $C^*$-algebraic representation consists of pairs $(G, H)$ in which $H$ is a cocompact (or at least cofinite) subgroup of $G$, see Proposition 2.41.
When the Hecke pair \((G, H)\) is discrete, it is known that a group homomorphism \(\Delta_{(G,H)} : G \to \mathbb{Q}^+\) is associated to the Hecke pair, which we name it “the relative modular function”. In a sense, it is a generalization of the modular function of a locally compact group. Therefore we define the involution on the Hecke algebra associated with \((G, H)\) using this modular function. Then it follows that the associated left regular representation is an involutive homomorphism if and only if the relative modular function is the constant function 1. Moreover, in Section 3, we prove that a necessary condition for a discrete Hecke pair \((G, H)\) to possess property (RD) is that the relative modular function must be the trivial homomorphism, i.e. \(\Delta_{(G,H)} = 1\), see Proposition 3.23. Motivated by these observations, we discuss various algebraic and analytic criteria to determine when the relative modular function is trivial. For instance, it is shown that if the Hecke algebra associated to \((G, H)\) satisfies a certain condition which is weaker than commutativity, then the relative modular function is trivial, see Proposition 2.19.

Following the works of G. Schlichting in [46, 47], K. Tzanev and other authors, in [54, 25, 35], constructed a certain topologization process for densely embedding a discrete (reduced) Hecke pair \((G, H)\) into a Hecke pair \((\overline{G}, \overline{H})\), where \(\overline{G}\) is a totally disconnected locally compact group and \(\overline{H}\) is a compact open subgroup of \(\overline{G}\). This technique is called the Schlichting completion of Hecke pairs and is our main tool to transfer several results from locally compact groups and Hecke pairs into the setting of discrete Hecke pairs. Therefore we briefly discuss the Schlichting completion and some of its examples, including those cases in which this process gives rise to discrete groups, see Example 2.37.

The second part of this paper is devoted to the study of property (RD) for locally compact Hecke pairs. We explained our motivation for studying property (RD) for Hecke pairs in [50]. Therefore, in the following, we only outline our main results concerning property (RD) in this paper. Our first major result in the second part of the paper asserts that when \(H\) is a compact subgroup of a locally compact group \(G\), the Hecke pair \((G, H)\) possesses property (RD) if and only if the group \(G\) does, see Theorem 3.18. An important consequence of Theorem 3.18 is that a discrete Hecke pair \((G, H)\) has (RD) if and only if the totally disconnected locally compact group \(\overline{G}\) appearing in its Schlichting completion has (RD), see Theorem 3.20. These theorems together with some complementary discussions allow us to apply several well known results concerning property (RD) of locally compact groups and Lie groups to identify various classes of locally compact or discrete Hecke pairs possessing property (RD). Using our results, we also discuss several examples of Hecke pairs which do not possess (RD), for instance the Hecke pair appeared in the work of Bost and Connes, see Example 3.24. Our study of property (RD) for Hecke pairs also includes a discussion about length functions on Hecke pairs and growth of Hecke pairs. For instance, it is observed that when a Hecke pair \((G, H)\) is amenable, property (RD) is equivalent to polynomial growth.

The works presented in this paper not only contribute to the theory of Hecke \(C^*\)-algebras and property (RD) for Hecke pairs, as explained in the above, it also serves as a prototype of ideas and methods which relates the study of Hecke pairs with the area of locally compact (and discrete) groups. Therefore it is expected that these ideas will be expanded to study other aspects related to these two areas. Some of these interactions have been explored in the literature, see for instance [2]. However, there are many other interesting subjects to be studied yet, such as the geometric theory of Hecke pairs, the role of Hecke pairs in induced representations, Constructing a \(C^*\)-algebraic framework to study
Gelfand pairs, etc. We hope that this paper provides sufficient preparations and clues for further developments in the study of these fascinating subjects.

2. Locally compact Hecke pairs \((G, H)\)

Our main aim in this section is to define an extended setting to study Hecke pairs \((G, H)\) which admit left regular representations on \(L^2(H \backslash G)\). We also study various conditions which imply that these representations are bounded and/or involutive homomorphisms. Here, our guiding principle in developing the theory of locally compact Hecke pairs is that it should coincide with the discrete case when the subgroup in the Hecke pair is an open Hecke subgroup. In this way, we will be able to transfer useful results and ideas from locally compact case into discrete case and vice versa.

2.1. Basic notions and definitions.

In order to define a general and consistent \(C^*\)-algebraic framework to study Hecke algebras, we have to define appropriate notions of Hecke algebras as well as involutions on these Hecke algebras. First we need to differ between discrete cases and non-discrete cases.

**Definition 2.1.**

(i) Let \((G, H)\) be a pair of locally compact groups. It is called a **discrete pair**, if \(H\) is open in \(G\), otherwise it is called **non-discrete**.

(ii) A discrete pair \((G, H)\) is called a **discrete Hecke pair** if every double coset of \(H\) in \(G\) is a finite union of finitely many left cosets of \(H\) in \(G\). In this case, we also say that \(H\) is a **Hecke subgroup of** \(G\).

(iii) Given a discrete Hecke pair \((G, H)\), the vector space of all finite support complex functions on the set \(G//H\) of double cosets of \(H\) in \(G\) is denoted by \(\mathcal{H}(G, H)\) and the above condition allows us to define a **convolution product** on \(\mathcal{H}(G, H)\) by

\[
(f_1 * f_2)(HxH) := \sum_{Hy \in H \backslash G} f_1(Hxy^{-1}H)f_2(HyH),
\]

for all \(f_1, f_2 \in \mathcal{H}(G, H)\) and \(x \in G\).

Given a discrete Hecke pair, the vector space \(\mathcal{H}(G, H)\) with the above product is a complex algebra, which is usually called the Hecke algebra associated with the pair \((G, H)\), see [27, 36] for detailed introductions to this type of Hecke algebras. The algebra \(\mathcal{H}(G, H)\) also admits an involution defined by

\[
f^\circ(HxH) := \overline{f(Hx^{-1}H)}, \quad \forall f \in \mathcal{H}(G, H), x \in G.
\]

However, due to some technical reasons explained in Remark 2.6, we have to consider a slightly different involution on \(\mathcal{H}(G, H)\), see Definition 2.5.

Using a convolution product similar to (2.1), \(\mathcal{H}(G, H)\) is endowed with a **left regular representation** as follows:

\[
\lambda : \mathcal{H}(G, H) \to B(L^2(H \backslash G)), \quad \lambda(f)(\xi)(Hx) := (f * \xi)(Hx),
\]

\[\text{Alternative names for Hecke subgroups appearing in the literature are “almost normal subgroups” and “commensurated subgroups”, see [10, 35, 49]. However, we follow [35]. See also Example 2.20 below for the origin of the name “Hecke subgroup”. Besides, we use the phrase “almost normal” for another notion, see Definition 2.24(iii) below.}\]
for all \( f \in \mathcal{H}(G, H), \xi \in \ell^2(H \setminus G) \) and \( Hx \in H \setminus G \). By definition, the reduced Hecke C*-algebra of the Hecke pair \((G, H)\) is the norm completion of the image of this representation. The class of these Hecke pairs also includes pairs \((G, H)\), where \( G \) is a locally compact group and \( H \) is a compact open subgroup of \( G \). These Hecke pairs and their associated Hecke C*-algebras have been studied extensively in the literature of Hecke C*-algebras, see for instance [17, 27, 35, 54].

Now we turn our attention to non-discrete pairs. As it is clear from the discrete case, we are not only interested in the Hecke algebra of a Hecke pair \((G, H)\), but also we need an explicit regular representation to embed this algebra inside the C*-algebra of bounded operators on the Hilbert space \( L^2(H \setminus G) \), where the homogeneous space \( H \setminus G \) is equipped with an appropriate measure. This requirement makes us to define the convolution product, \( L^1 \)-norm and \( L^2 \)-norm using integrations over the locally compact space of right cosets of \( H \) in \( G \), and therefore, we need to set up an appropriate measure theoretic framework. In what follows, we often follow the notations of Folland’s book [22] for measure theoretic considerations.

In this paper, \( \mu \) always denotes a right Haar measure on \( G \), \( \eta \) denotes a right Haar measure on \( H \), \( \Delta_G \) and \( \Delta_H \) denote the modular functions of \( G \) and \( H \), respectively. The vector space of all complex valued compact support continuous functions on a (locally compact) topological space \( X \) is denoted by \( C_c(X) \). There is an onto map \( P : C_c(G) \to C_c(H \setminus G) \) defined by

\[
Pf(Hx) := \int_H f(hx) \, d\eta(h), \quad \forall f \in C_c(G), x \in G.
\]

It is well known that we have

(2.4) \[ \Delta_G|_H = \Delta_H \]

if and only if there exists a right \( G \)-invariant Radon measure \( \nu \) on \( H \setminus G \). In this case, \( \nu \) is unique up to a positive multiple. By choosing the multiple suitably, we get Weil’s formula for the decomposition of an integral on \( G \) into a double integral on \( H \) and \( H \setminus G \) as follows:

(2.5) \[ \int_G f(x) \, d\mu(x) = \int_{H \setminus G} Pf(y) \, d\nu(y) = \int_{H \setminus G} \int_H f(hy) \, d\eta(h) \, d\nu(y), \]

for all \( f \in C_c(G) \). Then we briefly say that the triple \( (\eta, \mu, \nu) \) satisfies Weil’s formula. One notes that \( y \) in the above formula varies over a complete set of representatives of right cosets. In fact, we could use \( Hy \) instead of \( y \), and we sometimes use this latter notation in order to clarify our computations. One notes that invariant measures on homogeneous spaces are often discussed over the spaces of left cosets, but the reader can find a similar formulation for the spaces of right cosets in [24].

**Assumption 2.2.** We restrict our study to those subgroups of locally compact groups which satisfy the above condition, and therefore all homogeneous spaces \( H \setminus G \) possess a right \( G \)-invariant Radon measure \( \nu \) which satisfies Equality (2.5).

In fact, this condition often naturally holds in most interesting cases, see for instance Subsection 2.3. Inspired by the discrete case, for a given non-discrete pair \((G, H)\), we define

(2.6) \[ \mathcal{H}(G, H) := \{ f \in C_c(H \setminus G); f(Hxh) = f(Hx), \forall x \in G, h \in H \}. \]

Every element of \( \mathcal{H}(G, H) \) can be thought of as a function on the set of double cosets of \( H \) in \( G \). Therefore, for every \( f \in \mathcal{H}(G, H) \) and \( x \in G \), each of the expressions \( f(HxH) \),
For $f$ and $g$ as above, let $S_f$ and $S_g$ be the supports of $f$ and $g$, respectively. Let $\pi : G \to H \backslash G$ be the natural quotient map. By Lemma 2.47 of [22], there are compact subsets $A_f, A_g \subseteq G$ such that $\pi(A_f) = S_f$ and $\pi(A_g) = S_g$. Since $A_f A_g$ is a compact subset of $G$, $\pi(A_f A_g)$ is compact, and one easily observes that $\text{supp}(f \ast g) \subseteq \pi(A_f A_g)$. It is straightforward to check that $f \ast g$ is continuous. Finally, it follows from the right $G$-invariance of $\nu$ that $f \ast g \in \mathcal{H}(G, H)$. Thus the above product is well defined.

In order to define appropriate involutions on $\mathcal{H}(G, H)$ when $(G, H)$ is a discrete pair as well as when it is a non-discrete pair, again we have to treat each case separately. We proceed with some notations for discrete case. For every given discrete Hecke pair $(G, H)$, we define two functions $L, R : G \to \mathbb{N}$ by

$$L(x) := [H : H_x] = |H x H / H|, \quad R(x) := [H : H_{x^{-1}}] = |H \backslash H x H|,$$

for all $x \in G$, where $H_x := H \cap x H x^{-1}$. The integer $L(x)$ (resp. $R(x)$) is the number of distinct left (resp. right) cosets appearing in the double coset $H x H$, and so we have $R(x) = L(x^{-1})$.

**Definition 2.3.** By definition, the relative modular function of a discrete Hecke pair $(G, H)$ is the function $\Delta_{(G,H)} : G \to \mathbb{Q}^+$ defined by

$$\Delta_{(G,H)}(x) := \frac{L(x)}{R(x)}, \quad \forall x \in G.$$

The discrete Hecke pair $(G, H)$ is called relatively unimodular if $\Delta_{(G,H)}(x) = 1$ for all $x \in G$. We also refer to this property as the relative unimodularity of the discrete Hecke pair $(G, H)$.

In fact, $\Delta_{(G,H)}$ is a group homomorphism whose kernel contains $H$, see Theorem 2.7 of [27]. It follows that $\Delta_{(G,H)}$ is always a continuous bi-$H$-invariant function on $G$. Moreover, when $H$ is a compact open subgroup of $G$, it was noted in [54, 35] that

$$\Delta_{(G,H)}(x) = \Delta_G,$$

see for instance Page 669 of [35] for a proof. We should also mention that the above equality also appeared in Lemma 1 of G. Schlichting’s paper [46].

It is important to note that the left regular representation $\lambda$, defined in (2.3), is an involutive homomorphism with respect to the involution defined in (2.2). However, this involution does not necessarily preserve the $\ell^1$-norm of elements of $\mathcal{H}(G, H)$. Even worse, one can see that this involution is not always continuous with respect to the $\ell^1$-norm. Therefore we are not able to show directly that the left regular representation $\lambda$ is continuous. The following example illustrates these points:

**Example 2.4.** Let $(G, H)$ be the Bost-Connes Hecke pair, that is

$$G = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} ; a \in \mathbb{Q}^+, b \in \mathbb{Q} \right\}, \quad \text{and} \quad H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z} \right\}.$$
For given \( g = \left( \begin{array}{cc} 1 & b \\ 0 & \frac{m}{n} \end{array} \right) \in G \), where \( m \) and \( n \) have no common prime factors, one easily computes \( L(g) = n \) and \( R(g) = m \), see 2.1.1.3 of [27]. Let \( \chi_g \) denote the characteristic function of the double coset \( HgH \) considered as an element of \( \mathcal{H}(G, H) \). Then \( \|\chi_g\|_1 = R(g) = m \) and \( \|\chi_g\|_1 = L(g) = n \). Therefore by replacing \( g \) with the elements of the sequence \( \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{n} \end{array} \right) \right\} \), we observe that the involution \( \circ \) on \( \mathcal{H}(G, H) \) is not continuous in \( \ell^1 \)-norm.

The problem is that the mapping \( Hx \mapsto Hx^{-1} \) is not even a well defined change of variable in \( H \setminus G \), of course, unless \( H \) is a normal subgroup of \( G \). However, the map \( HxH \mapsto Hx^{-1}H \) is a well defined bijection over the set of double cosets of \( H \) in \( G \) and so the involution \( \char^\circ \) is well defined. Other authors have also noticed the above issue. For example, in [54], the \( \ell^1 \)-norm on \( \mathcal{H}(G, H) \) was defined by

\[
\|f\|_1 = \sum_{Hx \in H \setminus G} |f(Hx)|\sqrt{\Delta_{(G, H)}(x)} = \sum_{HxH \in G \setminus H \setminus G} |f(HxH)|\sqrt{L(x)R(x)},
\]

for all \( f \in \mathcal{H}(G, H) \) and \( x \in G \). It is straightforward to see that this norm is preserved by the involution of \( \mathcal{H}(G, H) \). However, in order to get a norm preserving involution which can be generalized to non-discrete case, we use the definition given in [35]:

**Definition 2.5.** For a discrete Hecke pair \((G, H)\), the involution on the algebra \( \mathcal{H}(G, H) \) is defined by

\[
(2.9) \quad f^\circ(HxH) := \frac{\Delta_{(G, H)}(x^{-1})}{\sqrt{L(x)R(x)}} f(Hx^{-1}H), \quad \forall f \in \mathcal{H}(G, H), x \in G.
\]

**Remark 2.6.** Let \((G, H)\) be a discrete Hecke pair. Advantages (and a disadvantage) of the above involution in comparison with the involution \( \char^\circ \) are as follows:

(i) The left regular representation defined in (2.3), is an involutive homomorphism with respect to the involution \( \char^\circ \), while it is not involutive with respect to the involution \( \char^\ast \), unless the Hecke pair \((G, H)\) is relatively unimodular.

(ii) The involution \( \char^\ast \) preserves the \( \ell^1 \)-norm, while the involution \( \char^\circ \) does not.

(iii) The involution \( \char^\ast \) can be generalized to non-discrete cases such that it coincides with the usual involution in \( L^1(G) \) when \( H = \{e\} \).

**Problem 2.7.** When \( G \) is a locally compact group and \( H \) is a compact open subgroup of \( G \), the existing proof for the equality \( \Delta_{(G, H)} = \Delta_G \) (2.8), depends on both openness and compactness of \( H \). Therefore, it is desirable to investigate the validity of this equality when \( H \) is just an open and unimodular Hecke subgroup of \( G \).

In the following example, we observe that the same definition for the relative modular function is not possible for non-discrete pairs:

**Example 2.8.** Let \( G \) be the special linear group of degree 2, \( SL_2(\mathbb{R}) \), and let \( H \) be the compact subgroup \( SO_2(\mathbb{R}) = \left\{ \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) \right\} : -\pi < \theta \leq \pi \} \). For any given non-zero real number \( t \), set \( x := \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \). Then one computes that \( H_x = \{I, -I\} \), where \( I \) is the \( 2 \times 2 \) identity matrix. Therefore \([H : H_x] = \infty\). This shows that functions \( R, L \) and \( \Delta_{(G, H)} \) are not well defined for the pair \((G, H)\), but we shall see that the pair \((G, H)\) admits
When \((G, H)\) is a non-discrete pair, it is tempting to define \(R\) (and similarly \(L\)) using the integral

\[
R(x) := \int_{H \backslash G} \chi_x(Hy) d\nu(y), \quad \forall x \in G,
\]

where \(\chi_x\) is the characteristic function of the double coset \(HxH\) viewed as a function on \(H \backslash G\). But this approach does not work either, because in this case \(R(h) = L(h) = 0\) for all \(h \in H\), and so the relative modular function is not well defined on elements of \(H\).

Due to the lack of a well defined relative modular function, both the aforementioned tricks for defining a norm preserving involution fail for non-discrete Hecke pairs. However, Equality (2.8) suggests that we use the modular function of \(G\) in non-discrete cases in lieu of the relative modular function in discrete cases. In fact, this is also consistent with the traditionally defined involution on \(L^1(G)\). Therefore, in order to define a suitable involution on the Hecke algebras associated with a non-discrete pair, we need to impose another restriction:

**Assumption 2.9.** When \((G, H)\) is a non-discrete pair, we assume \(H\) is unimodular.

Thus, regarding the above assumption and Assumption 2.2, we require that \(\Delta_G|_H = \Delta_H = 1\). One notes that when \(H\) is compact, these conditions are automatically satisfied. Moreover, when there is no risk of confusion, we can ignore \(\Delta_H\) and denote the modular function of \(G\) simply by \(\Delta\). It follows from the above assumption that \(\Delta\) is a bi-\(H\)-invariant function on \(G\), so the following definition is allowed:

**Definition 2.10.** Assume \((G, H)\) is a non-discrete pair. The involution on the algebra \(\mathcal{H}(G, H)\) is defined by

\[
(f \ast g)(Hx) = \Delta(x^{-1})f(Hx^{-1}), \quad \forall f \in \mathcal{H}(G, H), \ x \in G.
\]

To see how the right \(G\)-invariance of \(\nu\) is used to prove that this map is an involution on \(\mathcal{H}(G, H)\), we show that it is an anti-homomorphism. The rest of axioms of involution are straightforward. For every \(f, g \in \mathcal{H}(G, H)\) and \(x \in G\), we compute

\[
(f \ast g)(Hx) = \Delta(x^{-1})f \ast g(Hx^{-1}) = \Delta(x^{-1})\int_{H \backslash G} f(x^{-1}y^{-1})g(y) d\nu(y)
\]

Set \(Hz := Hxy\). Due to the right \(G\)-invariance of \(\nu\), this is a measure preserving change of variable over \(H \backslash G\). Thus we have

\[
(f \ast g)(Hx) = \int_{H \backslash G} \Delta(x^{-1})f(z^{-1})g((xz^{-1})^{-1}) d\nu(z)
\]

\[
= \int_{H \backslash G} \Delta(x^{-1})\Delta(z)f^*(z)\Delta(xz^{-1})g^*(xz^{-1}) d\nu(z) = g^* \ast f^*(Hx).
\]

Now that we have defined appropriate involutions for both discrete and non-discrete cases, we are ready to define Hecke algebras:

**Definition 2.11.**

(i) When \((G, H)\) is a discrete Hecke pair, the algebra \(\mathcal{H}(G, H)\) of finite support complex function on the set of double cosets of \(H\) in \(G\) with the involution defined in (2.9) is called the Hecke algebra of the discrete Hecke pair \((G, H)\).
(ii) When \((G, H)\) is a non-discrete pair satisfying Assumptions 2.2 and 2.9, the involutive algebra \(\mathcal{H}(G, H)\) defined by (2.6), (2.7) and (2.10) is called the Hecke algebra of the pair \((G, H)\).

As it is evident from the above definition, we have not defined a Hecke pair in non-discrete cases yet. The reason is that unlike discrete cases, a non-discrete pair \((G, H)\) satisfying Assumptions 2.2 and 2.9, and without any extra conditions, gives rise to a Hecke realm of algebras inside a \(C^\ast\)-algebra. Unfortunately, we are not always able to define a left regular representation on the Hilbert space \(L^2(H \setminus G)\), as it is clear from pairs \((G, H)\) of discrete groups, see Remark 2.13. On the other hand, the requirement that every double coset has only finitely many left cosets is not necessary for a non-discrete pair \((G, H)\) to admit a left regular representation, see Example 2.8 and Proposition 2.28. Therefore we propose the following definition as the generalization of Definition 2.1(ii) for all discrete and non-discrete pairs \((G, H)\):

**Definition 2.12.** (i) Let \(G\) be a locally compact group and let \(H\) be a closed subgroup of \(G\) satisfying Assumptions 2.2 and 2.9. The pair \((G, H)\) is called a Hecke pair if the map \(\lambda : \mathcal{H}(G, H) \to B(L^2(H \setminus G))\) defined by the convolution product

\[ \lambda(f)(\xi)(Hx) := f \ast \xi \text{ for all } f \in \mathcal{H}(G, H) \text{ and } \xi \in L^2(H \setminus G) \]

is a homomorphism. In this case, this homomorphism is called the left regular representation of the Hecke pair \((G, H)\).

(ii) The norm completion of the image of \(\mathcal{H}(G, H)\) under \(\lambda\) in the \(C^\ast\)-algebra \(B(L^2(H \setminus G))\) is called the reduced Hecke \(C^\ast\)-algebra of the Hecke pair \((G, H)\) and is denoted by \(C^r_r(G, H)\).

In order to check a given pair \((G, H)\) is a Hecke pair, we must first prove that \(\lambda(f)\) is a bounded operator on the Hilbert space \(L^2(H \setminus G)\) for all \(f \in \mathcal{H}(G, H)\). Next, the usual argument, based on the Fubini theorem, shows that \(\lambda\) is a homomorphism, and finally it is clear that \(\lambda\) is always injective. Moreover, regarding the similarity between Hecke pairs and groups, one wishes to prove that \(\lambda\) is a bounded operator, that is to find a constant \(M > 0\) such that \(\lambda(f) \leq M \|f\|_1\). Although we are not able to prove this last statement for general Hecke pairs, we shall prove it in some important cases, see Proposition 2.28, Theorem 2.40 and Remark 2.42. One also notes that Assumption 2.9 is needed only if we are interested to have an involution on the Hecke algebra \(\mathcal{H}(G, H)\). However, since the image of \(\mathcal{H}(G, H)\) under \(\lambda\) is closed under the involution of the \(C^\ast\)-algebra \(B(L^2(H \setminus G))\), we consider this latter involution for the Hecke \(C^\ast\)-algebra associated with \((G, H)\).

**Remark 2.13.** Consider a discrete pair \((G, H)\). As it has already been studied in the literature, the pair \((G, H)\) is a Hecke pair in the sense of Definition 2.12(i) if and only if every double coset of \(H\) is a union of finitely many left cosets of \(H\). The “if” part was proved in Proposition 1.3.3 of [17], see also Theorem 2.40 below. To see the “only if” part, assume there is a double coset \(HxH\) which contains infinitely many distinct right cosets, say \(\{Hx_k\}_{k=1}^\infty\). For every \(k \in \mathbb{N}\), let \(\chi_k \in \mathcal{H}(G, H)\) be the characteristic function of the double coset \(Hx_kH\) and let \(\delta_e \in l^2(G)\) be the characteristic function of the right coset \(H\). Then one easily computes \(\chi_k \ast \delta_e(Hx_k) = 1\) for all \(k \in \mathbb{N}\), and so \(\|\chi_k \ast \delta_e\|_2 = \infty\). Therefore for a given discrete Hecke pair \((G, H)\), Definition 2.12(i) coincides with Definition 2.1(ii).
Proposition 2.14. Let \((G, H)\) be a pair and let \(N\) be a normal closed subgroup of \(G\) contained in \(H\). Set \(G' := \frac{G}{N}\) and \(H' := \frac{H}{N}\).

(i) If the pair \((G, H)\) satisfies Assumptions 2.2 and 2.9, then the pair \((G', H')\) satisfies them too.

(ii) If \(N\) is unimodular and \(H'\) is compact, then both pairs \((G', H')\) and \((G, H)\) satisfy Assumptions 2.2 and 2.9.

(iii) Assume that the pair \((G, H)\) satisfies Assumptions 2.2 and 2.9, then the pair \((G, H)\) is a Hecke pair if and only if the pair \((G', H')\) is a Hecke pair. In this case, the Hecke algebras \(\mathcal{H}(G, H)\) and \(\mathcal{H}(G', H')\) are isomorphic.

(iv) With the assumptions of item (iii), the left regular representation of \(\mathcal{H}(G, H)\) is bounded if and only if the left regular representation of \(\mathcal{H}(G', H')\) is bounded. Moreover, the \(C^*\)-algebras \(\mathcal{C}^*(G, H)\) and \(\mathcal{C}^*(G', H')\) are isomorphic.

Proof. We only deal with the case that \((G, H)\) is a non-discrete pair. The proof for discrete pairs is way easier.

(i) Let \((G, H)\) be a Hecke pair. Since \(H\) is unimodular and \(N\) is normal in \(H\), both \(N\) and \(H'\) are unimodular too, and so the assumption 2.9 holds for \((G', H')\).

To check Assumption 2.2, it is enough to show that \(H'\backslash G'\) possesses a right \(G'\)-invariant measure. In the following argument \(x\) is always an arbitrary element of \(G\). Let \(\pi : G \rightarrow G', x \mapsto \bar{x}\) be the quotient map. Define \(\pi' : H\backslash G \rightarrow H'\backslash G'\) by \(Hx \mapsto H'\bar{x}\). It is easy to see that \(\pi'\) is a well defined bijection. Consider the diagram

\begin{equation}
\begin{array}{ccc}
G & \xrightarrow{\pi} & G' \\
\downarrow & & \downarrow \\
H\backslash G & \xrightarrow{\pi'} & H'\backslash G'
\end{array}
\end{equation}

Since \(\pi\) and the vertical arrows are continuous open mappings, \(\pi'\) is a homeomorphism. Therefore it gives rise to a linear isomorphism

\[ \varphi : C_c(H'\backslash G) \rightarrow C_c(H'\backslash G'), \quad \varphi(f)(H'\bar{x}) := f(Hx), \]

for all \(f \in C_c(H\backslash G)\) and \(H'\bar{x} \in H'\backslash G'\). In the following \(g\) represents an arbitrary element of \(C_c(H'\backslash G')\). First we note that the inverse of \(\varphi\) is given by \(\varphi^{-1}(g) = g \circ \pi'\).

Next, we define a functional \(I : C_c(H'\backslash G') \rightarrow \mathbb{C}\) by

\[ I(g) := \int_{H'\backslash G'} g \circ \pi'(Hx) d\nu(x). \]

One checks that \(I\) is a positive linear functional on \(C_c(H'\backslash G')\). Furthermore, \(I(g) = 0\) if and only if \(g = 0\). It is straightforward to check that \(I\) is invariant by the action of \(G'\) on \(C_c(H'\backslash G')\) induced by the right action of \(G'\) on \(H'\backslash G'\), that is \(I(R_{\bar{y}}(g)) = I(g)\) for all \(\bar{y} \in G'\), where as usual, \(R_{\bar{y}}(g)(H\bar{x}) = g(H\bar{x}\bar{y})\) for all \(H\bar{x} \in H'\backslash G'\). Therefore \(I\) defines a right \(G'\)-invariant Radon measure \(\theta\) on \(H'\backslash G'\), which amounts to saying that \(\Delta_{G'}|_{H'} = \Delta_{H'}\).
(ii) Assumptions 2.2 and 2.9 for \( (G', H') \) easily follow from Proposition 2.27 of [18]. Thus we only need to prove these assumptions for the pair \( (G, H) \). Since \( \tilde{N} \) is unimodular, \( N \subseteq \text{Ker}\Delta_G \). Therefore there is a continuous group homomorphism \( \rho : G' \to \mathbb{R}^+ \) such that \( \rho \circ \pi = \Delta_G \). Since \( H' \) is a compact subgroup of \( G' \), \( \rho(H') \) is a compact subgroup of \( \mathbb{R}^+ \). But the only compact subgroup of \( \mathbb{R}^+ \) is the trivial subgroup \( \{1\} \). Hence \( \rho|_{H'} = 1 \) and this implies that \( \Delta_G|_H = 1 \). Similarly one shows that \( \Delta_H = 1 \).

(iii) Regarding the way we defined the measure \( \theta \) on \( H'\backslash G' \) in the above, we have
\[
\int_{H'\backslash G'} \varphi(f)(H'\bar{x})d\theta(H'\bar{x}) = \int_{H'\backslash G'} f(Hx)d\nu(Hx),
\]
for all \( f \in C_c(H\backslash G) \). Hence the isomorphism \( \varphi \) extends to an isometric isomorphism \( \varphi_2 \) between \( L^2(H\backslash G) \) and \( L^2(H'\backslash G') \). On the other hand, one easily checks that \( \varphi \) maps \( \mathcal{H}(G, H) \) onto \( \mathcal{H}(G', H') \). Let \( \varphi_H : \mathcal{H}(G, H) \to \mathcal{H}(G', H') \) denote the linear isomorphism obtained by restricting \( \varphi \) to \( \mathcal{H}(G, H) \). Then a straightforward computation shows that \( \varphi_H \) preserves the convolution, so it is an algebra isomorphism, and we have
\[
\varphi_H(f) \ast \varphi_2(\xi) = \varphi_2(f \ast \xi), \quad \forall f \in \mathcal{H}(G, H), \xi \in L^2(H\backslash G).
\]
This completes the proof.

(iv) Similar to the above item, \( \varphi \) extends to an isometric isomorphism between \( L^1(H\backslash G) \) and \( L^1(H'\backslash G') \). Then the desired statements follow immediately from (iii).

\[\square\]

Remark 2.15. With the notations of the above proposition, we note the followings:

(i) To see that the compactness of \( H' \) is necessary in Proposition 2.14(ii), consider the group \( H = \mathbb{R} \rtimes \mathbb{R}^x \) and its normal subgroup \( N = \mathbb{R} \). The locally compact group \( H \) is not unimodular, because it is isomorphic to the matrix group
\[
\left\{ \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \right\}, \quad x, y \in \mathbb{R}, y \neq 0.
\]
However, both \( N \) and \( \frac{H}{N} = \mathbb{R}^x \) are unimodular.

(ii) It is straightforward to see that the map \( \varphi_H \) preserves involution if and only if \( \Delta_G(x) = \Delta_{G'}(\bar{x}) \) for all \( x \in G \). However, the above example also shows that this formula is not generally true.

(iii) Fix a Haar measure \( \sigma \) on \( N \). Let \( \alpha \) and \( \beta \) be the right invariant Haar measures on \( G' \) and \( H' \), respectively, such that the triples \( (\sigma, \mu, \alpha) \) and \( (\sigma, \eta, \beta) \) satisfy Weil’s formula. Then one checks that the triple \( (\beta, \alpha, \theta) \) satisfies Weil’s formula.

There are also other ways of constructing new Hecke pairs from given ones. For example, direct product of finitely many Hecke pairs or considering extensions of groups by Hecke pairs, as it was done for discrete Hecke pairs in Section 3 of [51].

2.2. Unimodularity and relative unimodularity.

In this subsection, \( (G, H) \) is always a discrete Hecke pair. Two involutions \( \circledast \) and \( \ast \) defined on \( \mathcal{H}(G, H) \) in (2.2) and (2.9), respectively, agree if and only if the Hecke pair \( (G, H) \) is relatively unimodular. On the other hand, it is easy to check that this latter condition holds if and only if the involution \( \circledast \) preserves the \( \ell^1 \)-norm of \( H\backslash G \), see also Example 2.4. Moreover, when \( H \) is a compact open subgroup of a locally compact group
$G$, the equality $\Delta_{(G,H)}=1$ is equivalent to the unimodularity of $G$. Also, we will show that the relative unimodularity is a necessary condition for property (RD) of discrete Hecke pairs, see Corollary 3.23. Thus it is important to find various criteria implying this condition, which is the subject of the present subsection. We shall also briefly address the non-discrete pairs $(G, H)$ for which $G$ is unimodular. We begin our investigation with some easy observations.

In order to have $\|f\circ\|_1=\|f\|_1$, an obviously sufficient condition is that $|f(x)|=|f(x^{-1})|$ for all $x \in G$. This condition holds for all $f \in \mathcal{H}(G, H)$ whenever

$$\tag{2.12} HxH=Hx^{-1}H, \quad \forall x \in G.$$ 

In the following we give examples for which Equality (2.12) holds:

**Example 2.16.**

(i) Consider the Hecke pair $(\text{SL}_2(\mathbb{Q}_p), \text{SL}_2(\mathbb{Z}_p))$, where $p$ is a prime number. The group $\text{SL}_2(\mathbb{Q}_p)$ is a totally disconnected locally compact group and $\text{SL}_2(\mathbb{Z}_p)$ is a compact open subgroup of this group. Therefore every double coset of $\text{SL}_2(\mathbb{Z}_p)$ is a finite union of left cosets. An important feature of this Hecke pair is that Condition (2.12) holds for it, see Section 2.2.3.2 of [27].

(ii) Let $G$ be an abelian group. Consider the action $\theta$ of $\mathbb{Z}/2$ on $G$ by inversion, that is $\theta_1(g):=-g$ for all $g \in G$, where $\mathbb{Z}/2$ has been considered as the multiplicative group $\{1, -1\}$. Then $(G \rtimes \mathbb{Z}/2, \{0\} \rtimes \mathbb{Z}/2)$ is a Hecke pair. For every $g \in G$, we have two elements $(g, -1), (g, 1) \in G \rtimes \mathbb{Z}/2$. We observe that $(g, -1)^{-1}=(g, -1)$ and $(g, 1)^{-1}=(-g, 1)$. In the latter case, one computes

$$\mathbb{Z}/2 \langle g, 1 \rangle \mathbb{Z}/2 = \{(g, -1), (-g, -1), (g, 1), (-g, 1)\} = \mathbb{Z}/2 \langle g, 1 \rangle^{-1}\mathbb{Z}/2.$$

Therefore (2.12) holds for every Hecke pair as above.

Another important consequence of Condition (2.12) is that $\mathcal{H}(G, H)$ is a commutative algebra, see Section 2.2.3.2 of [27] for a proof. In the following example, we observe that the converse is not true in general. However, we can prove Proposition 2.19 below, which is a way more general condition implying the relative unimodularity.

**Example 2.17.** Consider the semi-crossed product group $G=(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$ defined by the action of $\mathbb{Z}/2$ on $\mathbb{Z}/2 \times \mathbb{Z}/2$ by flipping the components, that is $(-1)(x, y):=(y, x)$ for all $x, y \in \mathbb{Z}/2 = \{0, 1\}$, where again the acting copy of $\mathbb{Z}/2$ is considered to be the multiplicative group $\{1, -1\}$. The Hecke algebra associated to the Hecke pair $((\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2, \{(0, 0)\} \rtimes \mathbb{Z}/2)$ is commutative, see for instance Proposition 3.5 of [54]. However one can easily check that Condition (2.12) does not hold for this Hecke pair.

The following definition introduces an algebraic notion which implies the relative unimodularity of discrete Hecke pairs:

**Definition 2.18.** Let $(G, H)$ be a discrete Hecke pair. For every $g \in G$, let $\chi_g$ and $\chi_{g^{-1}}$ be the characteristic functions of double cosets $HgH$ and $Hg^{-1}H$, respectively. The Hecke pair $(G, H)$ as well as its Hecke algebra, $\mathcal{H}(G, H)$, and its reduced Hecke $C^*$-algebra, $C^*_r(G, H)$, are called *locally commutative* if $\chi_g \ast \chi_{g^{-1}} = \chi_{g^{-1}} \ast \chi_g$ for all $g \in G$.

**Proposition 2.19.** If $(G, H)$ is a locally commutative discrete Hecke pair, then it is relatively unimodular.
Proof. For given \( g \in G \), let \( \chi_g \) and \( \chi_{g^{-1}} \) be as in Definition 2.18. Assume that the double coset \( HgH \) is the disjoint union of right cosets \( Hx_i \) for \( i = 1, \ldots, R(g) \). We compute
\[
\chi_{g^{-1}} \ast \chi_g(H) = \sum_{Hx \in H \setminus G} \chi_{g^{-1}}(Hy \cdot H) \chi_g(Hy) = \sum_{i=1}^{R(g)} \chi_{g^{-1}}(Hx_i^{-1}H) = R(g).
\]
By replacing \( g \) with \( g^{-1} \), we get \( L(g) = \chi_g \ast \chi_{g^{-1}}(H) \). The desired statement follows from these equalities. \( \square \)

An immediate question is that whether is the converse of the above proposition also true? In other words, if \( \chi_{g^{-1}} \ast \chi_g(H) = \chi_g \ast \chi_{g^{-1}}(H) \) for all \( g \in G \), is it true that \( \chi_{g^{-1}} \ast \chi_g(HxH) = \chi_g \ast \chi_{g^{-1}}(HxH) \) for all \( g \in G \) and all double cosets \( HxH \in G/H \)?

Clearly every commutative Hecke algebra is locally commutative. Therefore the above proposition applies to Examples 2.16, 2.17 and the following example:

**Example 2.20.** Let \( G = GL(2, \mathbb{Q})^+ := \{ g \in GL(2, \mathbb{Q}); \det(g) > 0 \} \) and \( H = SL(2, \mathbb{Z}) \). Then \( H \) is a Hecke subgroup of \( G \), see Proposition 1.4.1 of [11]. Moreover, the Hecke algebra \( \mathcal{H}(G, H) \) is commutative, see Theorem 1.4.2 of [11]. It is worth mentioning that the terminologies “Hecke subgroup”, “Hecke pair” and “Hecke algebra” have been originated from this Hecke pair which was introduced for the first time in the realm of modular forms by E. Hecke, for more historical notes see [36].

Besides above examples, commutative Hecke algebras appear in numerous situations, for instance see Proposition 3.5 of [54].

**Definition 2.21.** In this paper, a (discrete or non-discrete) Hecke pair \((G, H)\) is called a **Gelfand pair**, if the Hecke algebra \( \mathcal{H}(G, H) \) is commutative.

Although there are also other characterizations for Gelfand pairs in the literature, see for example [19], we use the above definition to simplify our terminology. An immediate corollary of Equality (2.8) and Proposition 2.19 is:

**Corollary 2.22.** Let \( G \) be a locally compact group which possesses a compact open subgroup \( H \). If the discrete Hecke pair \((G, H)\) is locally commutative, then \( G \) is unimodular.

**Remark 2.23.** When \( H \) is a normal compact open subgroup of a locally compact group \( G \), the discrete Hecke pair \((G, H)\) is always locally commutative, and therefore \( G \) must be unimodular. On the other hand, in this case \( \mathcal{H}(G, H) \) is isomorphic to the complex group algebra of the quotient group \( G/H \), so it might be a noncommutative algebra. This phenomenon explains why the notion of locally commutative Hecke pairs is more general than the notion of Gelfand pairs.

One notes that the above corollary mostly applies to totally disconnected locally compact groups, because they have plenty of compact open subgroups. In the following we describe two more algebraic conditions implying the relative unimodularity. They come from the work of G.M. Bergman and H.W. Lenstra in [9]. First we need some definitions:

**Definition 2.24.** (i) Two subgroups \( H \) and \( K \) of a group \( G \) are called **commensurable** if \( H \cap K \) is a finite index subgroup of each one of them.
(ii) A subgroup $H$ of a group $G$ is called *nearly normal* if it is commensurable with a normal subgroup $N$ of $G$.

(iii) A subgroup $H$ of a group is called *almost normal* if it has only finitely many conjugates.

Condition (iii) in the above definition should not be confused with the definition of almost normal subgroups according to [10], see Definition 2.1(ii) and its footnote. One also observes that $H$ is an almost normal subgroup of $G$ if and only if the normalizer of $H$ in $G$, is a finite index subgroup of $G$.

**Proposition 2.25.** A discrete Hecke pair $(G, H)$ is relatively unimodular if one of the following conditions holds:

(i) The subgroup $H$ is nearly normal in $G$.

(ii) The subgroup $H$ is almost normal in $G$.

In particular, if a locally compact group $G$ possesses a compact open subgroup $H$ which is also almost normal in $G$, then $G$ is unimodular.

**Proof.** Since the notions defined in Definition 2.24 are algebraic, we do not have to worry about the topology in the following arguments:

(i) According to Theorem 3 of [9], $H$ is nearly normal if and only if there exists a natural number $n$ such that $1 \leq L(g) \leq n$ for all $g \in G$.\(^2\) Thus $\Delta_{(G,H)}(g) \leq n$ for all $g \in G$. On the other hand, since $\Delta_{(G,H)}$ is a group homomorphism from $G$ into the multiplicative group $\mathbb{Q}^+$, the boundedness of its image amounts to $\Delta_{(G,H)} = 1$.

(ii) It follows from (i) and the fact that if $H$ is an almost normal and a Hecke subgroup of $G$, then it is nearly normal.

\[ \square \]

In [42], B. H. Neumann determined the class of all finitely generated groups all whose subgroups are nearly normal, see also Theorem 2.8 of [51] for a summary. Also, in Example 2.10 of [51], we explained a method to construct nearly normal subgroups of free product of two discrete groups. In its simplest form, it is based on the fact that every finite subgroup of a finitely generated group gives rise to a nearly normal subgroup of a finitely generated free group.

The above discussion suggests a similar study about the unimodularity of locally compact groups. By Assumption 2.9, the subgroup $H$ in a non-discrete pair $(G, H)$ is always assumed to be unimodular, so we are interested in cases where both $G$ and $H$ are unimodular.

**Remark 2.26.** Besides Corollary 2.22 and elementary cases where $H$ is either a discrete, normal, or compact subgroup of a unimodular group $G$, see Corollary 1.5.4(a) of [18] and Corollary 2.8 of [22], we have the simultaneous unimodularity of $G$ and $H$ in the following cases:

(i) When $H$ is a lattice in $G$. A discrete subgroup $H$ of a locally compact group $G$ is called a *lattice in $G$* if the homogeneous space $H \backslash G$ has a $G$-invariant Radon measure $\nu$ such that $\nu(H \backslash G) < \infty$. In this case, by Theorem 9.1.6 of [18], $G$ is unimodular too.

\(^2\)We note that this statement also follows from the work of G. Schlichting in [47], see Proposition 1(i) of [7].
(ii) When $H$ is a unimodular and cocompact subgroup of a locally compact group $G$. In this case, by Proposition 9.1.2 of [18], $G$ is unimodular too. In fact the same conclusion is valid even when $H \backslash G$ has a finite relatively invariant measure. A measure $\nu$ on $H \backslash G$ is called relatively invariant if there exists a function $\chi : G \rightarrow [0, \infty]$ such that $\nu(Eg) = \chi(g)\nu(E)$ for all $g \in G$ and for all measurable subsets $E \subseteq H \backslash G$. In this case the function $\chi$ is a continuous homomorphism from $G$ into $\mathbb{R}^+$ and is called the character of $\nu$. When $H$ is unimodular, there is a relatively invariant measure on $H \backslash G$, which is unique up to a positive multiple, and moreover, if $\nu(H \backslash G) < \infty$, then $G$ is unimodular too, see Corollary B.1.8 of [8].

(iii) When $G$ is unimodular and $H$ is a nearly normal subgroup of $G$. Let $N$ be a normal subgroup of $G$ commensurable with $H$. If $N$ is not closed replace it with its closure, which is still commensurable with $H$. Then the statement follows from Lemma 2.27 below.

(iv) When $G$ is unimodular and $H$ is an open subgroup of $G$, in particular, when $H$ contains a compact open subgroup of $G$. In this case the restriction of a left (resp. right) Haar measure of $G$ to $H$ is a left (resp. right) Haar measure on $H$. Since every left Haar measure of a unimodular group is a right Haar measure too, the same is true for left and right Haar measures of $H$. This implies that $H$ is a unimodular group as well. One notes that $(G, H)$ is a discrete pair in this case.

The following elementary lemma might has already appeared in the literature, but we could not find it.

**Lemma 2.27.** Let $H$ and $K$ be two commensurable closed subgroups of a locally compact group $G$. Then $H$ is unimodular if and only if $K$ is unimodular.

**Proof.** Without loss of generality, we can assume that $K$ is a finite index subgroup of $H$. By replacing $K$ with $N := \bigcap_{h \in H} hKh^{-1}$ (which is a finite index closed normal subgroup of both $K$ and $H$), we can even assume $K$ is normal. Now if $H$ is unimodular $K$ is unimodular too. When $K$ is unimodular, $H$ is unimodular as well by Remark 2.26(ii). □

### 2.3. When $H$ is compact.

In this subsection we show that when $H$ is a compact subgroup of a locally compact group $G$, the pair $(G, H)$ is a Hecke pair. We also discuss briefly some classes of examples of such pairs. The special case where $H$ is a compact and open subgroup of $G$ will be treated with more details in the next subsection, because it is better studied within the class of Hecke pairs with open Hecke subgroups.

Define a map

$$ \iota : C_c(H \backslash G) \rightarrow C_c(G), $$

$$ \iota(f)(x) = \tilde{f}(x) := f(Hx), \quad \forall f \in C_c(H \backslash G), \ x \in G. $$

Since $H$ is compact, it is well defined. We also consider extensions of this map from $L^2(H \backslash G)$ into $L^2(G)$ and from $L^1(H \backslash G)$ into $L^1(G)$, and denote them with the same notation. Since $\tilde{f}$ is left $H$-invariant, one checks that

$$ \int_H |\tilde{f}(hx)|^2 d\eta(h) = |\tilde{f}(x)|^2 \int_H d\eta(h) = \eta(H)|f(Hx)|^2, \quad \forall x \in G. $$
Thus it follows from Weil’s formula, (2.5), that \( \| \tilde{f} \|_2^2 = \eta(H) \| f \|_2^2 \) for all \( f \in L^2(H \backslash G) \) where the \( L^2 \)-norms of \( \tilde{f} \) and \( f \) are computed in \( L^2(G) \) and \( L^2(H \backslash G) \), respectively. Similarly, we have \( \| \tilde{f} \|_1 = \eta(H) \| f \|_1 \). One also notes that for every \( f \in \mathcal{H}(G, H) \), \( \tilde{f} \) is a bi-\( H \)-invariant function on \( G \), and so \( f(Hx) = \tilde{f}(xh) \) for all \( x \in G \) and \( h \in H \). Therefore for every \( f \in \mathcal{H}(G, H) \), \( g \in L^2(H \backslash G) \) and \( x \in G \), we can compute

\[
\tilde{f} \ast g(x) = f \ast g(Hx) = \int_{H \backslash G} f(Hxy^{-1})g(Hy) d\nu(y) \\
= \frac{1}{\eta(H)} \int_{H \backslash G} \tilde{f}(xy^{-1}) \tilde{g}(y) \left( \int_H d\eta(h) \right) d\nu(y) \\
= \frac{1}{\eta(H)} \int_{H \backslash G} \left( \int_H \tilde{f}(xy^{-1}h^{-1}) \tilde{g}(hy) d\eta(h) \right) d\nu(y) \\
= \frac{1}{\eta(H)} \int_G \tilde{f}(xy^{-1}) \tilde{g}(y) d\mu(y) \\
= \frac{1}{\eta(H)} \tilde{f} \ast \tilde{g}(x).
\]

Hence

\[
\| f \ast g \|^2_2 = \frac{1}{\eta(H)} \| \tilde{f} \ast \tilde{g} \|^2_2 = \frac{1}{\eta(H)^2} \| \tilde{f} \ast \tilde{g} \|^2_2 \\
\leq \frac{1}{\eta(H)^2} \| \tilde{f} \|^2_2 \| \tilde{g} \|^2_2 = \| f \|^2_1 \| g \|^2_2.
\]

We summarize the above computations in the following proposition:

**Proposition 2.28.** Let \( H \) be a compact subgroup of a locally compact group \( G \). Then the pair \((G, H)\) is a Hecke pair. Furthermore, the left regular representation \( \lambda : \mathcal{H}(G, H) \to B(L^2(H \backslash G)) \) is bounded and we have \( \| \lambda(f) \| \leq \| f \|_1 \).

**Remark 2.29.** The continuity of \( \lambda \), when \( H \) is compact, allows us to define \( L^1(G//H) \) as the involutive Banach algebra consisting of functions in \( L^1(H \backslash G) \) that are almost everywhere right \( H \)-invariant. Whenever we want to prove some relations involving functions in \( L^1(G//H) \), we can prove it for functions in \( \mathcal{H}(G, H) \) and then extend it to \( L^1(G//H) \) using continuity.

Now we discuss several examples of Hecke pairs \((G, H)\) satisfying the assumption of Proposition 2.28. These examples have already studied in other branches of mathematics such as noncommutative harmonic analysis, representations theory, Lie theory, geometric group theory and random walks. Therefore not only we have no shortage of examples for the above proposition, but we also have plenty of opportunities to find new notable applications for our extended \( C^* \)-algebraic formulation of Hecke algebras.

When \( H \) is a finite subgroup of \( G \), no matter \( G \) is discrete or not, one can apply Proposition 2.28 to show that the left regular representation is a bounded operator, see Examples 2.16(ii) and 2.17.

Compact Lie subgroups of Lie groups give rise to another class of examples. We are particularly interested in two important subclasses of these Hecke pairs. The first one consists of pairs \((G, H)\), where \( H \) is a maximal compact subgroup of a Lie group \( G \), see
Another subclass consists of those Gelfand pairs \((G, H)\) which arise in Lie theory and representation theory.

**Example 2.30.** In the following we give some examples of the Hecke pairs \((G, H)\) described in the above.

(i) We have already mentioned the Hecke pair \((SL_2(\mathbb{R}), SO_2(\mathbb{R}))\) in Example 2.8. In fact, this Hecke pair is a Gelfand pair too, see Theorem 11.2.3 of [18]. Furthermore, \(SO_2(\mathbb{R})\) is a maximal compact subgroup of \(SL_2(\mathbb{R})\), for a quick reference and without using Lie theory see Theorem 11.1.1 of [18]. More generally, for every \(n \geq 2\), \(SO_n(\mathbb{R})\) is a maximal compact subgroup of \(SL_n(\mathbb{R})\).

(ii) Every closed subgroup of a compact Lie group is an example too. For instance, consider a maximal connected abelian subgroup \(H\) of a compact Lie group \(G\). In this case, \(H\) is called a maximal torus of \(G\) and is isomorphic to \(\mathbb{T}^k\) for some natural number \(1 \leq k \leq \dim(G)\), see [29, 48] for their existence, various properties and concrete examples. One notes that these Hecke pairs are also examples for Hecke pairs that will be discussed in Section 2.5, particularly in Remark 2.42(ii).

Maximal compact subgroups of general locally compact groups provide us with examples of Hecke pairs too. The homogeneous space associated with these Hecke pairs are homeomorphic to \(\mathbb{R}^k\) for some natural numbers \(k\), see [1, 3]. We can extend the above classes of Hecke pairs further by considering pro-Lie groups. A locally compact group \(G\) is called pro-Lie group if there exists a small compact normal subgroup \(K\) of \(G\) such that \(G/K\) is a Lie group, see [4, 30] for equivalent definitions. Then every (maximal) compact subgroup of \(G/K\) gives rise to a (maximal) compact subgroup \(H\) of \(G\) and it follows from Proposition 2.14 that the Hecke algebra of \(\mathcal{H}(G, H)\) is isomorphic to the Hecke algebra \((G/K, H/K)\). One notes that \(K\) does not need to be compact here, as long as it is normal in \(G\) and \(G/K\) is a Lie group. One can find more similar situations in the literature, for instance, see Theorem 2 of [40] and Theorem 5 of [43].

Another class of examples of Hecke pairs satisfying the assumptions of Proposition 2.28 consists of Hecke pairs \((G, H)\), where \(G\) is a totally disconnected locally compact group and \(H\) is a compact (and in most interesting cases, open) subgroup of \(G\). Many concrete examples of this type arise naturally as the automorphism groups of graphs (especially trees) and compact subgroups of them. Here we content ourself with examples of compact subgroups of the automorphism groups of homogeneous trees. To some extent similar compact subgroups can be realized inside of the automorphism group of a vertex transitive graph too. We refer the interested reader to [55] to explore this idea further. We follow the book [21] for basics about automorphism groups of homogeneous trees and the interested reader can refer to this book for further details and proofs.

**Example 2.31.** Let \(\mathfrak{X}\) be a homogeneous tree of degree \(q + 1\), i.e. a (connected) tree whose every vertex has \(q + 1\) neighbors. Let the same notation \(\mathfrak{X}\) denote the set of vertices of \(\mathfrak{X}\) and let \(\mathfrak{E}\) denote the set of edges of \(\mathfrak{X}\). Note that we consider \(\mathfrak{X}\) as an undirected graph, so \(\{x, y\} = \{y, x\}\) for all \(\{x, y\} \in \mathfrak{E}\). An automorphism of \(\mathfrak{X}\) is a bijective map \(\sigma : \mathfrak{X} \to \mathfrak{X}\) which preserves the edges of \(\mathfrak{X}\), i.e. \(\{\sigma(x), \sigma(y)\} \in \mathfrak{E}\) if and only if \(\{x, y\} \in \mathfrak{E}\). The set \(Aut(\mathfrak{X})\) of all automorphisms of \(\mathfrak{X}\) is a subgroup of the group \(S(\mathfrak{X})\) of all permutations of \(\mathfrak{X}\) and it is called the automorphism group of \(\mathfrak{X}\). Since \(\mathfrak{X}\) is connected, by giving \(\mathfrak{X}\) the natural metric structure in which every edge is isometric to the unit interval \([0, 1]\), we can also interpret \(Aut(\mathfrak{X})\) as the group of all isometries of \(\mathfrak{X}\). It is endowed with the
permutation topology, namely the topology whose basis of neighborhoods at the neutral element consists of (pointwise) stabilizer subgroups of finite subsets of $\mathfrak{X}$. The group $\text{Aut}(\mathfrak{X})$ with this topology is a locally compact group. It is well known that $\text{Aut}(\mathfrak{X})$ possesses plenty of lattices, see for instance [5]. Thus, by Remark 2.26(i), $\text{Aut}(\mathfrak{X})$ is unimodular, and consequently, for every compact open subgroup $H$ of $\text{Aut}(\mathfrak{X})$, the discrete Hecke pair $(\text{Aut}(\mathfrak{X}), H)$ is relatively unimodular.

Harmonic analysis on this group and its various subgroups was studied in [21]. For every $x \in \mathfrak{X}$ (resp. $\{x, y\} \in \mathfrak{E}$), let $K_x$ (resp. $K_{\{x,y\}}$) denote the subgroup of $\text{Aut}(\mathfrak{X})$ which stabilizes the vertex $x$ (resp. the edge $\{x,y\}$). There are certain characterizations and examples of compact subgroups of $\text{Aut}(\mathfrak{X})$ as follows.

(i) For given $x, y \in \mathfrak{X}$, the subgroups $K_x$ and $K_{\{x,y\}}$ are compact open subgroups of $\text{Aut}(\mathfrak{X})$. In fact, by Theorem 5.2 of [21], subgroups of these forms are maximal compact subgroups of $\mathfrak{X}$.

(ii) Given a closed subgroup $K$ of $\text{Aut}(\mathfrak{X})$, $K$ is compact if and only if $K(x)$, the orbit of $x$ under the action of $K$, is finite for all $x \in \mathfrak{X}$, see Proposition 5.1 of [21].

(iii) Let $\gamma = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$ be a geodesic in $\mathfrak{X}$. Define $M_\gamma := \bigcap_{i=-\infty}^{\infty} K_{x_i}$. Then $M_\gamma$ is a compact subgroup of $\text{Aut}(\mathfrak{X})$. It is not open in $\text{Aut}(\mathfrak{X})$, because otherwise regarding the sequence of compact open subgroups $M_\gamma \subsetneq \cdots \subsetneq \bigcap_{i=-2}^{1} K_{x_i} \subsetneq K_{x_0}$, the compact open subgroup $K_{x_0}$ would have infinite measure (with respect to a Haar measure) which is a contradiction.

The above examples of Hecke pairs are particularly important to find examples of Hecke pairs with property (RD), see Théorème 2(3) of [34] and Example 3.27.

Yet there is another important class of Hecke pairs of the type described in Proposition 2.28. This class consists of the Schlichting completions of reduced discrete Hecke pairs $(G, H)$. Since the subgroup in a Schlichting completion is open, we study these examples in the next subsection.

2.4. When $H$ is open.

In this subsection $(G, H)$ is always a discrete Hecke pair. Using the Schlichting completion of $(G, H)$, we reduce the study of this type of Hecke pairs to the case that the subgroup in the Hecke pair is compact and open.

**Definition 2.32.** Given a discrete Hecke pair $(G, H)$, the normal subgroup of $G$ defined by $K_{(G,H)} := \bigcap_{x \in G} xHx^{-1}$ is called the core of the Hecke pair $(G, H)$. The Hecke pair $(G, H)$ is called reduced if its core is the trivial subgroup. The pair $(\frac{G}{K_{(G,H)}}, \frac{H}{K_{(G,H)}})$ is a reduced discrete Hecke pair which is called the reduction of $(G, H)$ or the reduced Hecke pair associated with $(G, H)$ and is denoted by $(G_r, H_r)$.

The reduced Hecke pair $(G_r, H_r)$ associated with $(G, H)$ has a similar properties as $(G, H)$ in a number of situations, see for instance Lemmas 3.21 and 3.31. In fact, we have already seen, in Proposition 2.14, that the Hecke algebras of Hecke pairs $(G, H)$ and $(G_r, H_r)$ are naturally isomorphic. Therefore one can often assume that discrete Hecke pairs are reduced.
The Schlichting completion is a process to associate a totally disconnected locally compact group $\overline{G}$ with a given reduced discrete Hecke pair $(G, H)$ such that $G$ is dense in $\overline{G}$, and more importantly, the closure of $H$ in $\overline{G}$, denoted by $\overline{H}$, is a compact open subgroup of $\overline{G}$. Then the discrete Hecke pair $(\overline{G}, \overline{H})$ is called the Schlichting completion of $(G, H)$. We briefly recall basic definitions of the Schlichting completion. One notes that the construction can be applied to the cases in which $H$ is an open Hecke subgroup of a locally compact group $G$, but it might change the underlying topology of $G$. The Schlichting completion of a Hecke pair was studied by K. Tzanev in [54], based on the works of G. Schlichting in [46, 47]. However, we follow the paper [35] by S. Kaliszewski, M. B. Landstad and J. Quigg for notations, definitions and basic results. We should also mention that a similar construction was studied in [25] by H. Glöckner and G.A. Willis.

Let $X$ be a set (with the discrete topology). Denote the set of all maps from $X$ into $X$ by $\text{Map}(X)$ and equip it with the pointwise convergence topology. This topology is equivalent with the product topology on $\text{Map}(X) = \prod_{x \in X} X$ and the permutation topology discussed in Example 2.31, so for every $\phi \in \text{Map}(X)$ a basis of neighborhoods of $\phi$ consists of sets of the form

$$U_{\phi,F} := \{\tau \in \text{Map}(X); \tau(x) = \phi(x), \forall x \in F\},$$

where $F$ varies in the collection $\mathcal{F}$ of all finite subsets of $X$. One notes that the set $\text{Per}(X)$ of all permutations on $X$ is a subset of $\text{Map}(X)$. The group $\text{Per}(X)$ with the induced topology is a locally compact (and Hausdorff) group. However, it is not closed in $\text{Map}(X)$ when $X$ is infinite, see Example 3.4 of [35]. Therefore, for a given subgroup $\Gamma$ of $\text{Per}(X)$, it is important to have a criterion to determine whether $\overline{\Gamma}$, the closure of $\Gamma$ in $\text{Map}(X)$, is contained in $\text{Per}(X)$, because if it is the case, then $\overline{\Gamma}$ will be a group too. The desired criterion is that every suborbit of the action of $\Gamma$ on $X$ must be finite, i.e. $|\Gamma_{x}(y)| < \infty$ for all $x, y \in X$. We recall that a suborbit of the action of $\Gamma$ on $X$ is an orbit of a stabilizer subgroup. One notes that a suborbit $\Gamma_{x}(y)$ is finite if and only if $\Gamma_{x} \cap \Gamma_{y}$ is finite index in $\Gamma_{x}$. A subgroup $\Gamma$ of $\text{Per}(X)$ with this property is called a Hecke group on $X$.

**Proposition 2.33.** ([35], Proposition 3.6) If $\Gamma$ is a Hecke group on $X$, then $\overline{\Gamma}$ is a totally disconnected locally compact subgroup of $\text{Per}(X)$. Moreover, for every $x \in X$, $\overline{\Gamma}_{x} = \overline{\Gamma_{x}}$ is a compact open subgroup of $\overline{\Gamma}$.

Now, we employ the above construction in the setting of Hecke pairs. Given a discrete pair $(G, H)$, the group $G$ acts on the set $G/H$ by translation from left, i.e. $\theta_{x}(yH) := xyH$ for all $x, y \in G$. It is easy to see that the map $\theta : G \to \text{Per}(G/H)$ is an injective homomorphism if and only if $K_{(G,H)}$ is the trivial subgroup $\{e\}$. Thus when the pair $(G, H)$ is reduced, $G$ can be realized as a subgroup of $\text{Per}(G/H)$. Let $\theta : G \to \text{Per}(G/H)$ denote the map $x \mapsto \theta_{x}$. When $\theta$ is injective, we often drop it from our notations.

One observes that the stabilizer of a given left cost $xH \in G/H$ under the action $\theta$ is the subgroup $xHx^{-1}$. Therefore when $(G, H)$ is a reduced discrete Hecke pair, the intersection of every two stabilizer subgroups is finite index in each one of them, and so every suborbit of the action of $G$ on $G/H$ is finite. In other words, $G$ is a Hecke group on $G/H$ whenever $(G, H)$ is a reduced discrete Hecke pair, see also Lemma 4.1 of [35]. By the above proposition, $\overline{G}$, the closure of $G$ in $\text{Map}(G/H)$, is a totally disconnected locally compact group and $\overline{H}$, the closure of $H$ in $\overline{G}$, is a compact open subgroup of $\overline{G}$.

**Definition 2.34.** Let $(G, H)$ be a reduced discrete Hecke pair.
(i) The Hecke pair \((G, H)\), associated with \((G, H)\) in the above, is called the Schlichting completion of \((G, H)\).

(ii) The topology on \(G\) induced by the inclusion \(\theta\) is called the Hecke topology of the pair \((G, H)\). It is the group topology generated by the collection \(\{xHx^{-1}, x \in G\}\).

(iii) If \(H\) is a compact open subgroup of \(G\) in the Hecke topology, then the pair \((G, H)\) is called a Schlichting pair.

The Schlichting completion possesses the following characterization:

**Theorem 2.35.** ([35], Theorem 4.8) Let \((G, H)\) be a reduced Hecke pair with the Schlichting completion \((\overline{G}, \overline{H})\). Assume \((L, K)\) is a Schlichting pair. For every homomorphism \(\varphi : G \to L\) such that \(\varphi(G)\) is dense in \(L\) and \(\varphi(H) \subseteq K\), there exists a unique continuous homomorphism \(\overline{\varphi} : \overline{G} \to L\) which extends \(\varphi\), in other words, \(\overline{\varphi}\theta = \varphi\).

Furthermore, if \(H = \varphi^{-1}(K)\), then \(\overline{\varphi}\) is a topological group isomorphism from \(\overline{G}\) onto \(L\) and maps \(\overline{H}\) onto \(K\).

One notes that the surprising part of the above theorem is the injectivity of \(\overline{\varphi}\) in the last statement.

**Remark 2.36.** All the above constructions for defining the Schlichting completion are algebraic, and so they are applicable even when \(H\) is an open Hecke subgroup of a locally compact group \(G\). Therefore, when \((G, H)\) is a discrete Hecke pair with respect to two different topologies on \(G\) (and \(H\)), we obtain (algebraically) the same reduced Hecke pair \((G_r, H_r)\) and (algebraically and topologically) the same Schlichting completion.

**Example 2.37.** Let \((G, H)\) be a discrete Hecke pair.

(i) Assume that \((G, H)\) is reduced. Then it follows that the Schlichting completion of \((G, H)\) is the same as \((G, H)\) if and only if \(H\) is of finite group. In this case, \(G\) has to be a discrete group.

(ii) Let \(H\) be a nearly normal subgroup of \(G\). If \(H\) is finitely generated, then the subgroup \(H_r\) in the reduced Hecke pair \((G_r, H_r)\) associated to \((G, H)\) is finite, and consequently we have \((\overline{G_r}, \overline{H_r}) = (G_r, H_r)\). To prove this, it is enough to show that \(H\) has a finite index subgroup \(L\) which is normal in \(G\). Let \(N\) be a normal subgroup of \(G\) which is commensurable with \(H\). Since \(H\) is finitely generated, \(N\) is finitely generated too, and therefore there are only finitely many subgroups of index equal to \([N : H \cap N]\) in \(N\). Set

\[
L := \cap_{x \in G} x(H \cap N)x^{-1}.
\]

Then \(L\) is a finite index subgroup of \(H\) which is also normal in \(G\).\(^3\) This proves the above claim.

When \(H\) is not finitely generated, the Hecke pair \((G, H)\) might even be reduced, and consequently its Schlichting completion is different than itself. To see an example, let \(N\) be an infinite product of \(\mathbb{Z}/2\mathbb{Z}\) and let \(A\) be the full automorphism group of \(N\). Set \(G := N \rtimes A\) and let \(H\) be a subgroup of \(N\) of index 2. Then since \(A\) acts transitively on nontrivial elements of \(N\), \(H\) does not contain any non-trivial normal subgroup of \(G\).\(^4\)

\(^3\)I learnt this argument from Henry Wilton over a Q&A in mathoverflow.net.

\(^4\)I learnt this example from Jeremy Rickard over a Q&A in mathoverflow.net.
One notes that when \( H \) is a finite index subgroup of \( G \), the condition of \( H \) being finitely generated is unnecessary. Because, in any case, the subgroup \( K_{(G,H)} \) is a finite index subgroup of \( H \). Therefore we have \((\overline{G}, \overline{H}) = (G_r, H_r)\).

(iii) Let \((G, H)\) be a the Bost-Connes Hecke pair discussed in Example 2.4. One checks that it is a reduced Hecke pair and it is shown in Example 11.4 of [35] that its Schlichting completion \((\overline{G}, \overline{H})\) is as follows:

\[
\overline{G} = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} : a \in \mathbb{Q}^+, b \in \mathcal{A} \right\}, \quad \text{and} \quad \overline{H} = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbb{Z} \right\},
\]

where \( \mathcal{A} \) and \( \mathbb{Z} \) are the ring of finite adeles on \( \mathbb{Q} \) and its the maximal compact subring, respectively. Moreover, the Hecke topology of the above Schlichting completion coincides with the topology coming from the topology of \( \mathcal{A} \), the restricted product topology of \( p \)-adic fields.

(iv) Given a prime number \( p \), consider the Hecke pair \((SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))\). By looking at intersections of the form \( SL_2(\mathbb{Z}) \cap x_nSL_2(\mathbb{Z})x_n^{-1} \), where \( x_n = \begin{pmatrix} 0 & p^{-n} \\ -p^n & 0 \end{pmatrix} \) for \( n \in \mathbb{N} \), one easily observes that \( K_{(SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \).

Therefore the pair \((PSL_2(\mathbb{Z}[1/p]), PSL_2(\mathbb{Z}))\) is the reduction of the Hecke pair \((SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))\). Then it is shown in Example 11.8 of [35] that the Schlichting completion of \((PSL_2(\mathbb{Z}[1/p]), PSL_2(\mathbb{Z}))\) is \((PSL_2(\mathbb{Q}_p), PSL_2(\mathbb{Z}_p))\), which is the reduction of the Hecke pair appearing in Example 2.16(i). Similar computations show that \((PSL_2(\mathbb{A}), PSL_2(\mathbb{Z}))\) is the Schlichting completion of \((PSL_2(\mathbb{Q}), PSL_2(\mathbb{Z}))\). One notes that the Hecke topology of these Schlichting completions again coincide with the conventional totally disconnected locally compact topologies of the above groups coming from \( p \)-adic valuations and the restricted product of \( p \)-adic fields.

The following lemma simply states that passing to the Schlichting completion does not change the algebraic and analytic aspects of reduced discrete Hecke pairs and their associated Hecke algebras.

**Lemma 2.38.** [Proposition 4.9 of [35]] Let \((G, H)\) be a reduced discrete Hecke pair and let \((\overline{G}, \overline{H})\) be its Schlichting completion. Then the following statements hold:

(i) The mapping \( \alpha : H \backslash G \to \overline{H} \backslash \overline{G} \) (resp. \( \alpha' : G / H \to \overline{G} / \overline{H} \)), defined by \( Hg \mapsto \overline{H}g \) (resp. \( gH \mapsto g\overline{H} \)) for all \( g \in G \) is a \( G \)-equivariant bijection. In particular, \( \alpha \) induces an isometric isomorphism between Hilbert spaces \( L^2(H \backslash G) \) and \( L^2(\overline{H} \backslash \overline{G}) \).

(ii) The mapping \( \beta : G / H \to \overline{G} / \overline{H} \), defined by \( HgH \mapsto \overline{H}g\overline{H} \) for all \( g \in G \) is a bijection.

(iii) The mapping \( \beta' \) commutes with the convolution product, and therefore it induces an isometric isomorphism between the Hecke algebras \( \mathcal{H}(G, H) \) and \( \mathcal{H}(\overline{G}, \overline{H}) \), with respect to the corresponding \( L^1 \)-norms.

In the next section, we shall need a variation of the above lemma as follows:

**Remark 2.39.** Consider the conventions and notations of the above lemma. Let \( \Gamma \) be a closed subgroup of \( G \) containing \( H \) and let \( \overline{\Gamma}^{\overline{G}} \) be its closure in \( \overline{G} \). Then the same conclusions as above hold for mappings \( \alpha' : H \backslash \Gamma \to \overline{H} \backslash \overline{\Gamma}^{\overline{G}} \) and \( \beta' : G / H \to \overline{\Gamma}^{\overline{G}} / \overline{H} \) which are defined by restricting \( \alpha \) and \( \beta \), respectively.
An immediate application of the Schlichting completion is the next theorem, for further applications see Proposition 5.1 of [54], the recent work of C. Anantharaman-Delaroche in [2] as well as several results in Subsections 3.2 and 3.3.

**Theorem 2.40.** Let \((G, H)\) be a discrete Hecke pair. Then the left regular representation \(\lambda : \mathcal{H}(G, H) \to B(\ell^2(H \backslash G))\) is bounded. In fact, we have \(\|\lambda(f)\| \leq \|f\|_1\) for all \(f \in \mathcal{H}(G, H)\).

**Proof.** By Proposition 2.14, without loss of generality, we can assume \((G, H)\) is a reduced discrete Hecke pair. Let \(\alpha\) and \(\beta\) be as Lemma 2.38. Since both Hecke pairs \((G, H)\) and \((\overline{G}, \overline{H})\) are discrete, the above mappings induce the following isometric isomorphisms:

\[
\alpha^* : \ell^2(\overline{H} \backslash \overline{G}) \to \ell^2(H \backslash G), \quad \alpha^*(\xi) := \xi \circ \alpha, \quad \forall \xi \in \ell^2(\overline{H} \backslash \overline{G}),
\]

\[
\beta^* : \mathcal{H}(\overline{G}, \overline{H}) \to \mathcal{H}(G, H), \quad \beta^*(f) := f \circ \beta, \quad \forall f \in \mathcal{H}(\overline{G}, \overline{H}),
\]

where the norm on Hecke algebras \(\mathcal{H}(G, H)\) and \(\mathcal{H}(\overline{G}, \overline{H})\) are the corresponding \(\ell^1\)-norms. One notes that a similar isometry as \(\alpha^*\) exists from \(\ell^1(\overline{H} \backslash \overline{G})\) onto \(\ell^1(H \backslash G)\). It is straightforward to check that these isometries commute with the convolution products defining the left regular representations of \(\mathcal{H}(G, H)\) and \(\mathcal{H}(\overline{G}, \overline{H})\). Since \(\overline{H}\) is compact, by applying Proposition 2.28 and the above discussion for every \(f \in \mathcal{H}(G, H)\) and \(\xi \in \ell^2(H \backslash G)\), we have

\[
\|f \ast \xi\|_2 = \|\alpha^*((\beta^* f) \ast (\alpha^* \xi))\|_2
= \|\beta^* f \ast (\alpha^* \xi)\|_2
\leq \|\beta^* f\|_1 \|\alpha^* \xi\|_2
= \|f\|_1 \|\xi\|_2.
\]

It worths mentioning that the above theorem is stronger than Proposition 1.3.3 of [17], and more importantly, it is a necessary step to show that Hecke pairs of polynomial growth possess property (RD), see Proposition 3.37.

### 2.5. When \(H \backslash G\) has a finite relatively invariant measure.

In this subsection, we assume that \(H\) is a unimodular closed subgroup of \(G\) such that the homogeneous space \(H \backslash G\) has a finite relatively invariant measure \(\nu\). Therefore by Remark 2.26(ii), \(G\) is unimodular too, and so both Assumptions 2.2 and 2.9 hold for the pair \((G, H)\).

**Proposition 2.41.** For \((G, H)\) as above, the pair \((G, H)\) is a Hecke pair.

**Proof.** For a given \(f \in \mathcal{H}(G, H)\), we set \(M_f := \max\{\|f(Hx)\|; Hx \in H \backslash G\}\). Then for every \(\xi \in L^2(H \backslash G)\), by applying Minkowski’s inequality for integrals (Theorem 6.19 of [23]), we
obtain
\[ \| f \ast \xi \|_2 = \left( \int_{\mathcal{H} \setminus G} \left( \int_{\mathcal{H} \setminus G} |f(\mathcal{H}xy^{-1})\xi(\mathcal{H}y)|^2 d\nu(\mathcal{H}y) \right)^2 d\nu(\mathcal{H}x) \right)^{1/2} \]
\[ \leq \int_{\mathcal{H} \setminus G} \left( \int_{\mathcal{H} \setminus G} |f(\mathcal{H}xy^{-1})|^2 |\xi(\mathcal{H}y)|^2 d\nu(\mathcal{H}y) \right)^{1/2} d\nu(\mathcal{H}x) \]
\[ \leq M_f \int_{\mathcal{H} \setminus G} \| \xi \|_2 d\nu(\mathcal{H}x) \]
\[ = M_f \nu(\mathcal{H} \setminus G) \| \xi \|_2. \]

This shows that \( f \ast \xi \in L^2(\mathcal{H} \setminus G) \), and therefore the left regular representation \( \lambda : \mathcal{H}(G, H) \to B(L^2(\mathcal{H} \setminus G)) \) is a well defined homomorphism. \( \square \)

In the following we describe two cases of Hecke pairs \((G, H)\) with \( H \) being cocompact in \( G \) and for which the left regular representation \( \lambda \) is bounded.

**Remark 2.42.**

(i) If \( H \) is a finite index closed subgroup of \( G \), then \( H \) is also open, and so the Hecke pair \((G, H)\) is a discrete Hecke pair. Thus by Theorem 2.40, \( \lambda \) is bounded.

(ii) When \( H \) contains a normal cocompact subgroup \( K \) of \( G \). In this case the quotient groups \( G/K \) and \( H/K \) are both compact. Therefore by Propositions 2.14 and 2.28, \( \lambda \) is bounded.

In Proposition 3.13, we shall show how the boundedness of \( \lambda \) in the above cases implies property (RD).

### 3. Property (RD) for Locally Compact Hecke Pairs

Property (RD) for Hecke pairs \((G, H)\), where \( G \) is a discrete group, was studied in [50], where we also explained our motivation for studying property (RD) for Hecke pairs. More or less, the same theory applies to all discrete Hecke pairs. We only need to clarify a few things about length functions, growth rates and applications of the Schlichting completion in the study of property (RD) for discrete Hecke pairs. These topics will be dealt with after defining and studying the basic features of property (RD) for general (not necessarily discrete) Hecke pairs. One will immediately observe that these two settings coincide with each other when they are considered on discrete Hecke pairs.

#### 3.1. Length functions on locally compact groups and Hecke pairs.

The definition of property (RD) is based on certain geometric and analytic notions defined on groups. The geometric aspects of property (RD) rely on the notion of length functions on groups (and Hecke pairs). In this subsection we collect basic definitions and lemmas concerning length functions which are necessary for our study of property (RD).

A length function on a locally compact group \( G \) is a Borel function \( l : G \to [0, \infty[ \) such that for all \( g, h \in G \), we have

- \( l(e) = 0 \),
\[ l(g) = l(g^{-1}), \text{ and} \]
\[ l(gh) \leq l(g) + l(h). \]

Then the set
\[ N_l := \{ g \in G ; l(g) = 0 \} \]
is a subgroup of \( G \) and is called the \textit{kernel of} \( l \). The length function \( l \) is called \textit{closed} if its kernel is a closed subgroup of \( G \). A \textit{length function on a Hecke pair} \((G, H)\) is a length function on \( G \) such that \( H \subseteq N_l \). In this case, \( l \) is a bi-\( H \)-invariant function on \( G \). It follows that when the Hecke pair \((G, H)\) is discrete, \( l \) is a continuous function, and so a closed length function.

In [33], a length function is assumed to be continuous. Following [32, 13], we weaken this assumption by assuming \( l \) to be only a Borel function in order to include word length functions associated with compact generating subsets of \( G \). Since our aim is to study property (RD), we also need to consider other notions related to length functions:

**Definition 3.1.** In the following \( G \) is a locally compact group and \((G, H)\) is a Hecke pair.

(i) A length function \( l \) on \( G \) (resp. \((G, H)\)) is called \textit{locally bounded} if \( l \) is bounded on every compact subset of \( G \) (resp. \( H \setminus G \)).

(ii) A length function \( l \) on \( G \) (resp. \((G, H)\)) is called \textit{proper} if \( l^{-1}([0,n]) \) is relatively compact in \( G \) (resp. \( H \setminus G \)).

(iii) A subset \( S \) of \( G \) is called a \textit{generating set of the group} \( G \) (resp. \textit{the Hecke pair} \((G, H)\)) if \( G = \bigcup_{n \in \mathbb{N}} \hat{S}^n \) (resp. \( H \setminus G = \bigcup_{n \in \mathbb{N}} H\hat{S}^n \)), where \( \hat{S} := S \cup S^{-1} \cup \{ e \} \).

The group \( G \) (resp. \textit{the Hecke pair} \((G, H)\)) is called \textit{compactly generated} if it has a compact generating set.

(iv) Let \( S \) be a generating set of \( G \). The function
\[ l_S(g) := \begin{cases} 
\min\{n ; g \in \hat{S}^n \} & e \neq g \in G \\
0 & g = e
\end{cases} \]
is called the \textit{word length function on} \( G \) \textit{associated with} \( S \). We sometimes drop the subscript \( S \) for the sake of brevity.

(v) Let \( l_1 \) and \( l_2 \) be two length functions on \( G \) (resp. \((G, H)\)). We say that \( l_1 \) \textit{dominates} \( l_2 \) if there are constants \( c_0, c_1 \geq 0 \) such that \( l_2(g) \leq c_1 l_1(g) + c_0 \) for all \( g \in G \). If \( l_1 \) and \( l_2 \) dominate each other, we call them \textit{equivalent}.

For basic properties and characterizations of compactly generated groups, we refer to Section 2.C of [16] or Theorem 6.11 and Corollary 6.12 of [53]. For instance, every almost connected locally compact group is compactly generated. We recall that a locally compact group \( G \) is called \textit{almost connected}, if \( G/G_0 \) is a compact group, where \( G_0 \) is the connected component of the identity.

**Remark 3.2.** Given an arbitrary Hecke pair \((G, H)\) with a generating set \( S \), we cannot imitate Item (iv) of the above definition to associate a word length function on \((G, H)\). Thus generally, we have to use length functions defined geometrically, as it is explained in the next Remark. When the subgroup \( H \) in the Hecke pair \((G, H)\) is compact, there is an alternative method to define a length function on \((G, H)\), using a given length function on \( G \), see Lemma 3.8. Finally, in Example 3.9(ii), for a given relatively unimodular discrete Hecke pair \((G, H)\), we define a length function on \((G, H)\) using its algebraic structure.
Let $l$ be a length function on a locally compact group $G$. Define a function $d_l : G \times G \to [0, \infty]$ by setting $d_l(x, y) := l(x^{-1}y)$ for all $x, y \in G$. It is straightforward to check that $d_l$ is a pseudo-metric on $G$, that is for every $x, y, z \in G$, we have

(i) $d_l(x, x) = 0$,
(ii) $d_l(x, y) = d_l(y, x)$,
(iii) $d_l(x, z) \leq d_l(x, y) + d_l(y, z)$.

This function is also left $G$-invariant;

(iv) $d_l(gx, gy) = d_l(x, y)$, for all $x, y, g \in G$.

If we assume that the kernel of $l$ is trivial, i.e. $N_l = \{e\}$, then $d_l$ is a metric on $G$, that is in addition to Items (i), (ii) and (iii), for every $x, y \in G$, we have

(v) $d_l(x, y) = 0 \iff x = y$.

This last condition is equivalent to saying that the topology defined by $d_l$ is Hausdorff.

Conversely, given a left $G$-invariant metric $d$ on $G$, we can define a length function $l_d : G \to [0, \infty]$ by specifying an element $x_0$ of $G$ and setting

$$l_d(g) := d(x_0, gx_0), \quad \forall g \in G.$$ 

Since the action of $G$ on itself is transitive and $d$ is left invariant, $l_d$ does not depend on $x_0$. The kernel of $l_d$ is always trivial. Moreover, we always have $d_{l_d} = d$, and if $N_l = \{e\}$, then we have $l_{d_l} = l$. In the following remark, we see that the correspondence between length functions on $G$ with trivial kernel and metrics on $G$ can be generalized to arbitrary length functions (including length functions on Hecke pairs) and pseudo-metrics on $G$.

**Remark 3.3.** (i) Assume $(X, d)$ is a metric space and $G$ acts continuously on $X$ from left by isometries, so $d(gx, gy) = d(x, y)$ for all $g \in G$ and $x, y \in X$. Fix a point $x_0 \in X$ and define $l_{d,x_0}(g) := d(x_0, gx_0)$ for all $g \in G$. Then $l_{d,x_0}$ is a length function on $G$. One notes that $l_{d,x_0}$ depends on $x_0$, and its orbit. The kernel of $l_{d,x_0}$ is the stabilizer of $x_0$, i.e. the subgroup $G_{x_0}$. Therefore it is always a closed subgroup of $G$, because the action is continuous.

If the action of $G$ on $X$ is transitive and free, then $l_{d,x_0}$ does not depend on $x_0$. In this case, there is a bijection between $G$ and $X$, which is defined by fixing a point $x_0 \in X$, and we can define a metric $d'$ on $G$ using $d$ by setting $d'(g_1, g_2) := d(g_1x_0, g_2x_0)$ and one checks that $l_{d,x_0} = l_{d'}$, or equivalently $d' = d_{d,x_0}$.

(ii) The parallelism between the notions of length functions and left invariant pseudo-metric structures on $G$ (or generally any set with an action of $G$) motivates the following definitions: Given a topological space $X$, we call a pseudo-metric $d$ on $X$ locally bounded if for every $x_0 \in X$ there exists a neighborhood of $x_0$ with bounded diameter, and we call $d$ proper if for every $x_0 \in X$ the map $X \to [0, \infty[$ defined by $x \mapsto d(x_0, x)$ is proper. Then one checks that every locally bounded and proper length function on a group gives rise to a left invariant, locally bounded and proper pseudo-metric on $G$, and vice versa. Following the terminology of [16], we call a pseudo-metric on $G$ adapted if it is locally bounded and proper.

The above method is applicable for any Hecke pair $(G, H)$ where $G$ is a group of isometries of some metric space $X$ and $H$ is contained in the stabilizer of some point of $X$. The following example describes an instance coming from Lie theory. For examples of discrete Hecke pairs coming from groups acting on trees, we refer the reader to [6], see also Example 2.31.
Example 3.4. In Chapter VI of [28], it was shown that every connected semisimple Lie group $G$ with finite center has a maximal compact subgroup $K$ such that for a suitable Riemannian structure on $G/K$, the natural action of $G$ on the homogeneous space $G/K$ (by left multiplication) identifies $G$ with $I_0(G/K)$, the identity component of the Lie group of all isometries of $G/K$. Moreover, $K$ is the stabilizer subgroup of some point $o \in G/K$. Thus by applying Remark 3.3(i), we can define a length function on the Hecke pair $(G, K)$. It is also known that every compact subgroup of $G$ is contained in a maximal compact subgroup of $G$ and every maximal compact subgroup of $G$ is a conjugate of $K$. Therefore this construction applies to all Hecke pairs $(G, H)$, where $H$ is a compact subgroup of $G$.

The study of groups by means of their actions on geometric structures (mostly metric spaces) is a vivid trend in group theory which lies in the domain of geometric group theory. Here, we content ourselves with some definitions and lemmas which are needed for our study and postpone the study of Hecke pairs by means of geometric notions to another time. The interested reader can find most of these materials as well as more comprehensive discussion of geometric group theory for locally compact groups in [16].

Definition 3.5. Let $(X_1, d_1)$ and $(X_2, d_2)$ be two pseudo-metric spaces and let $\varphi : X_1 \to X_2$ be a mapping.

(i) The mapping $\varphi$ is called large scale Lipschitz if there are constants $c_0 \geq 0$ and $c_1 > 0$ such that

$$d_2(\varphi(x), \varphi(y)) \leq c_1 d_1(x, y) + c_0, \quad \forall x, y \in X_1.$$ 

It is called Lipschitz if the above inequality holds for $c_0 = 0$.

(ii) The mapping $\varphi$ is called large scale bilipschitz if there are constants $c_0, c'_0 \geq 0$, $c_1, c'_1 > 0$ such that

$$c'_1 d_1(x, y) - c'_0 \leq d_2(\varphi(x), \varphi(y)) \leq c_1 d_1(x, y) + c_0, \quad \forall x, y \in X_1.$$ 

It is called bilipschitz if the above inequalities hold for $c_0, c'_0 = 0$.

(iii) The mapping $\varphi$ is called large scale bilipschitz equivalence (resp. bilipschitz equivalence) if it is bijective and large scale bilipschitz (resp. bilipschitz).

Lemma 3.6. Let $l_1$, $l_2$ be two length functions on $G$ and let $d_1$, $d_2$ be their associated pseudo-metrics, respectively.

(i) The length function $l_1$ dominates $l_2$ if and only if the identity map $id : (G, d_2) \to (G, d_1)$ is a large scale Lipschitz map. Therefore the length functions $l_1$ and $l_2$ are equivalent if and only if the identity map, $id$, is a large scale bilipschitz equivalence.

(ii) Let $l_1$ be the length function associated with a compact generating set $S_1$ of $G$ and let $l_2$ be a locally bounded length function on $G$. Then $l_1$ dominates $l_2$.

(iii) Let $l_1$ and $l_2$ be length functions associated with two compact generating sets $S_1$ and $S_2$ of $G$, respectively. Then they are equivalent.

(iv) Let $l_1$, $l_2$, $S_1$ and $S_2$ be as described in (iii). Then $id : (G, d_2) \to (G, d_1)$ is a bilipschitz equivalence.

Proof. (i) It follows from the correspondence between length functions on $G$ and pseudo-metrics on $G$.

(ii) Since $l_2$ is locally bounded and $S_1$ is compact, there is some $c \geq 0$ such that $c \geq l_2(s)$ for all $s \in S_1 \cup S_1^{-1}$. By definition, for every $e \neq g \in G$, there is $n \in \mathbb{N}$ such that
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\(l_1(g) = n\) and there are some \(s_1, \ldots, s_n \in (S_1 \cup S_1^{-1}) - \{e\}\) such that \(g = s_1 \cdots s_n\). For every \(1 \leq i \leq n\), we have \(l_1(s_i) = 1\). Thus we have

\[
\begin{align*}
    l_2(g) &= l_2(s_1 \cdots s_n) \leq l_2(s_1) + \cdots + l_2(s_n) \\
    &\leq cn = c l_1(g).
\end{align*}
\]

(iii) It follows from (ii) and the fact that every length function associated with a compact generating set is locally bounded.

(iv) It follows from the proof of (ii).

□

Remark 3.7. Let \(l_1\) and \(l_2\) be two length functions on \(G\).

(i) If \(l_1\) and \(l_2\) are equivalent, then \(l_1\) is proper (resp. locally bounded) if and only if \(l_2\) is proper (resp. locally bounded).

(ii) If \(G\) is compactly generated, then it possesses a compact generating set which is also a symmetric neighborhood of \(\{e\}\), for the proof see Page 514 of [13].

(iii) It follows immediately that every length function associated with a compact generating set is proper and locally bounded.

Whenever \(H\) is a compact subgroup of a locally compact group \(G\), the following lemma is a useful tool to define length functions on the Hecke pair \((G, H)\) using length functions defined on \(G\). Its proof is a modification of the proof of Lemma 2.1.3 of [33].

Lemma 3.8. Let \(H\) be a compact subgroup of a locally compact group \(G\). If \(l\) is a length function on \(G\) which is bounded on \(H\), then there exists a length function \(l'\) on \(G\) such that the following statements hold:

(i) The length functions \(l\) and \(l'\) are equivalent.

(ii) The kernel of \(l'\) contains \(H\), i.e. \(l'\) is a length function on the Hecke pair \((G, H)\).

(iii) If the length function \(l\) is locally bounded (resp. proper), then \(l'\) is locally bounded (resp. proper), both as a length function on \(G\) and as a length function on the Hecke pair \((G, H)\).

Proof. Define \(l_1 : G \to [0, \infty[\) by

\[
l_1(g) := \int_H l(hgh^{-1})d\eta(h), \quad \forall g \in G.
\]

One observes that \(l_1\) is a length function on \(G\) such that

\[
l_1(hg) = l_1(gh), \quad \forall g \in G, h \in H.
\]

Also, \(l_1\) is equivalent to \(l\). This follows from the boundedness of \(l\) on \(H\) and the following inequalities:

\[
l_1(g) \leq \eta(H)l(g) + 2 \int_H l(h)d\eta(h), \quad \forall g \in G,
\]

\[
l(g) \leq \frac{1}{\eta(H)} l_1(g) + \frac{2}{\eta(H)} \int_H l(h)d\eta(h), \quad \forall g \in G.
\]

It follows from (3.2) that \(l_1\) is bounded on \(H\) too. Now, define \(l' : G \to [0, \infty[\) by

\[
l'(g) := \inf_{h, h' \in H} l_1(hgh'), \quad \forall g \in G.
\]
Using (3.1), one observes that $l'(g) = \inf_{h \in H} l_1(hg) = \inf_{h \in H} l_1(gh)$ for all $g \in G$. Therefore $l'$ is a length function on $G$. Moreover, $l'(g) \leq l_1(g)$ and $l_1(g) \leq l_1(gh) + l_1(h)$ for all $g \in G$ and $h \in H$. By boundedness of $l_1$ on $H$ and these inequalities, $l_1$ and $l'$ are equivalent. Assertions (i), (ii) and (iii) follow from the above discussion. □

Besides the zero length function and constructions discussed in Remark 3.3 and Lemma 3.8, the following examples are useful too:

**Example 3.9.**

(i) Let $(G, H)$ be a Hecke pair. Define $l: G \to \{0, 1\}$ by

$$l(g) := \begin{cases} 0 & g \in H \\ 1 & \text{otherwise} \end{cases}$$

This is a length function on $G$ which is proper if and only if $H$ is cocompact in $G$. However, it is always locally bounded.

(ii) Let $(G, H)$ be a relatively unimodular discrete Hecke pair. Let $L$ be as defined in Subsection 2.1, i.e. $L(g)$ be the number of left cosets in $HgH$ for all $g \in G$. Define $l_c: G \to [0, \infty]$ by setting $l_c(g) := \log(L(g))$. One checks that $l_c(g) = 0$ if and only if $g$ belongs to the normalizer of $H$ in $G$, i.e. $g \in N_G(H)$. Due to relative unimodularity of $(G, H)$, one also checks that $l_c(g) = l_c(g^{-1})$ for all $g \in G$. For given $x, y \in G$, let $m = L(x)$, $n = L(y)$, $HxH = \cup_{i=1}^m x_iH$ and $HyH = \cup_{j=1}^n y_jH$. Then we have

$$HxyH \subseteq HxHHyH = \bigcup_{i=1}^m \bigcup_{j=1}^n x_i y_j H.$$ 

This shows that $L(xy) \leq L(x)L(y)$, and so $l_c(xy) \leq l_c(x) + l_c(y)$. Therefore $l_c$ is a length function on the Hecke pair $(G, H)$. We call it the characteristic length function of $(G, H)$.

By using $L(g)R(g)$ instead of $L(g)$ in the definition of $l_c$, one can drop the relative unimodularity condition of $(G, H)$. However, since we are dealing with property (RD) of Hecke pairs and relative unimodularity is a necessary condition for (RD), see Corollary 3.23, this condition is not an important restriction.

One observes that $l_c$ is always locally bounded. Moreover, it follows from Theorem 3 of [9] that $l_c$ is bounded if and only if $H$ is a nearly normal subgroup of $G$, see also the proof of Proposition 2.25(i). Therefore it is an interesting problem to find a condition which is equivalent to (or at least implies that) $l_c$ is a proper length function. Surprisingly, one notes that such a condition has already appeared in a different context in the literature, see Proposition 2 of [7].

While dealing with length functions on discrete Hecke pairs, it is necessary to pay attention to certain easy but important details.

**Remark 3.10.**

(i) Let $(G, H)$ be a discrete Hecke pair and let $(G_r, H_r)$ be its reduction. For every length function $l$ on $(G, H)$, we define a length function $l_r$ on the Hecke pair $(G_r, H_r)$ by $l_r(gK_{(G,H)}) := l(g)$ for all $gK_{(G,H)} \in G/K_{(G,H)}$. Then the mapping $l \mapsto l_r$ is a natural bijective correspondence between the set of length functions on $(G, H)$ and the ones on $(G_r, H_r)$. It is clear that this correspondence preserves local boundedness, properness and equivalence of length functions. Moreover, if $S$ is a generating set for $(G, H)$, then $S' := \{sK_{(G,H)}, s \in S\} \subseteq G_r$ is a generating set for $(G_r, H_r)$. One also notes that the same comment applies to any other normal subgroup $K$ of $G$ which is contained in $H$.
(ii) Let $(G, H)$ be a reduced Hecke pair and let $(\overline{G}, \overline{H})$ be its Schlichting completion. If $\overline{l} : \overline{G} \to [0, \infty]$ is a length function on the Hecke pair $(\overline{G}, \overline{H})$, then its restriction to $G$, say $l$, is clearly an algebraic length function on the Hecke pair $(G, H)$. We only need to check that $l$ is a Borel function. But this is an immediate consequence of the fact that $H$ is open in $G$ and $l$ is constant on every double coset of $H$ in $G$.

Conversely, let $l : G \to [0, \infty]$ be a length function on $(G, H)$. By Lemma 2.38(i), we can define $\overline{l} : \overline{G} \to [0, \infty]$ by $\overline{l}(x) := l(g_x)$ for all $x \in G$, where $g_x$ is an element of $G$ such that $\overline{H}x = \overline{H}g_x$. Using Lemma 2.38(i) and due to the fact that $l$ is a bi-$\overline{H}$-invariant function on $G$, $\overline{l}$ is a well defined bi-$\overline{H}$-invariant function on $\overline{G}$ such that $\overline{l}(x) = 0$ for all $x \in \overline{H}$. It follows that $\overline{l}$ is a continuous function, and so a Borel function. Since $HgH \mapsto Hg^{-1}H$ is a bijection from $G//H$ onto $G//H$ and using Lemma 2.38(ii), we observe that $\overline{l}(x) = \overline{l}(x^{-1})$ for all $x \in \overline{G}$. For given $x, y \in \overline{G}$, pick $g_x, g_y^{-1} \in G$ such that $\overline{H}x = \overline{H}g_x$ and $\overline{H}y^{-1} = \overline{H}g_y$. Thus $\overline{H}xy\overline{H} = \overline{H}g_xg_y^{-1}\overline{H}$, and therefore

$$\overline{l}(xy) = l(g_xg_y^{-1}) \leq l(g_x) + l(g_y^{-1}) = \overline{l}(x) + \overline{l}(y^{-1}) = \overline{l}(x) + \overline{l}(y).$$

This shows that $\overline{l}$ is a length function on the Hecke pair $(\overline{G}, \overline{H})$. Therefore there is a natural bijection between the sets of length functions on the Hecke pairs $(G, H)$ and $(\overline{G}, \overline{H})$ as above.

Finally, one notes that if $S$ is a finite generating set for $(G, H)$, then $\overline{H}S$ is a compact generating set for $\overline{G}$. Conversely, if $\overline{G}$ is compactly generated, then the discrete Hecke pair $(G, H)$ is finitely generated.

### 3.2. Main theorems concerning property (RD) for locally compact Hecke pairs.

In this subsection we state and prove our main theorems concerning property (RD) and their consequences.

**Definition 3.11.** Let $(G, H)$ be a Hecke pair.

(i) Given $s > 0$, every locally bounded length function $l$ on $(G, H)$ defines a weighted $L^2$-norm on $\mathcal{H}(G, H)$ as follows:

$$\|f\|_{s,l} := \left( \int_{H \backslash G} |f(x)|^2 (1 + l(x))^{2s} \right)^{1/2}, \quad \forall f \in \mathcal{H}(G, H),$$

(ii) We say that $(G, H)$ has property (RD) if there exist a locally bounded length function $l$ on $(G, H)$ and real numbers $s, c > 0$ such that

$$\|\lambda(f)\| \leq c\|f\|_{s,l}, \quad \forall f \in \mathcal{H}(G, H),$$

where $\|\lambda(f)\|$ is the convolution norm of $f$ defined using the left regular representation of $(G, H)$.

In the following remark we explain why we usually consider locally bounded and proper length functions in our study of property (RD).

**Remark 3.12.**

(i) The locally boundedness of a length function $l$ is required in order to insure that the weighted $L^2$-norm $\|f\|_{s,l}$ is well defined for all $f \in \mathcal{H}(G, H)$. 
(ii) Boundedness of length functions imposes a substantial restriction on groups having property (RD). In fact, if a locally compact group \( G \) has (RD) with respect to a bounded length function \( l \), then the space \( L^2(G) \) is closed under the convolution product, and so it becomes an algebra. However, when \( G \) is an abelian locally compact group, \( L^2(G) \) equipped with the convolution product is an algebra if and only if \( G \) is compact, see [45]. This shows that for the purpose of studying property (RD) for non-compact groups, we have to consider non-bounded length functions. One way to avoid bounded length functions is to consider proper length functions. The reason is that if \( l \) is a bounded and proper length function on a locally compact group \( G \), then \( G \) has to be compact, see also [15] for other clues that boundedness and properness of length functions are opposite notions in some sense.

The boundedness of the left regular representation is a necessary condition to study various features of property (RD) for Hecke pairs in analogy with property (RD) for groups, see the following proposition and Proposition 3.16.

**Proposition 3.13.** Let \( (G, H) \) be a Hecke pair such that \( H \backslash G \) possesses a finite relatively invariant measure \( \nu \). Then \( (G, H) \) has (RD) whenever the left regular representation \( \lambda : \mathcal{H}(G, H) \to B(L^2(H \backslash G)) \) is bounded.

See Remark 2.42 for two classes of Hecke pairs described satisfying the above condition.

**Proof.** Let \( C > 0 \) be the norm of \( \lambda \) and let \( l \) be the zero length function. Then for every \( f \in H(G, H) \) and every \( s > 0 \), we have

\[
\|\lambda(f)\| \leq C \|f\|_1 \leq C \nu(H \backslash G)^{1/2} \|f\|_2 = C \nu(H \backslash G)^{1/2} \|f\|_{s,l},
\]

where the last inequality follows from Proposition 6.12 of [23]. \( \square \)

**Example 3.14.** The Hecke pairs discussed in Example 2.30(ii) satisfy the assumptions of the above proposition, see also Remark 2.42. Therefore they have property (RD).

There are several equivalent definitions for property (RD) which are easier to work with. In order to discuss them, we need some observations and notations.

**Remark 3.15.** Let \( l_1 \) and \( l_2 \) be two length functions on a Hecke pair \( (G, H) \). If \( l_1 \) dominates \( l_2 \) and \( (G, H) \) has (RD) with respect to \( l_2 \), then it has (RD) with respect to \( l_1 \) too. Therefore when \( l_1 \) and \( l_2 \) are equivalent, \( (G, H) \) has (RD) with respect to \( l_1 \) if and only if it has (RD) with respect to \( l_2 \).

Assume \( l \) is a length function on a Hecke pair \( (G, H) \). For every \( r \geq 0 \), we set

\[
B_{r,l}(G, H) := \{xH \in H \backslash G; l(x) \leq r\},
\]

\[
C_{r,l}(G, H) := \{xH \in H \backslash G; r \leq l(x) < r + 1\}.
\]

When \( H \) is the trivial subgroup, we simply denote the above sets by \( B_{r,l}(G) \) and \( C_{r,l}(G) \), respectively. In the rest of this paper, the subscript \(+\) in \( \mathcal{H}_+(G, H) \), \( L^2_+(H \backslash G) \), etc., means that we are considering only nonnegative real functions.

**Proposition 3.16.** Let \( l \) be a locally bounded length function on a Hecke pair \( (G, H) \). If the left regular representation \( \lambda : \mathcal{H}(G, H) \to B(L^2(H \backslash G)) \) is bounded, then the following conditions are equivalent:

(i) The Hecke pair \( (G, H) \) has property (RD) with respect to \( l \).
(ii) There exists a polynomial \( P \) such that for every \( r > 0 \), if the support of a function \( f \in \mathcal{H}_+(G, H) \) is contained in \( B_{r,l}(G, H) \), then we have
\[
\|\lambda(f)\| \leq P(r)\|f\|_2.
\]

(iii) There exists a polynomial \( P \) such that for every \( r > 0 \), if the support of a function \( f \in \mathcal{H}_+(G, H) \) is contained in \( B_{r,l}(G, H) \) and \( \xi \in L^2_+(H \backslash G) \), then we have
\[
\|f * \xi\|_2 \leq P(r)\|f\|_2\|\xi\|_2.
\]

**Proof.** The proof of the above proposition is the same as Proposition 2.10 of [50]. We only prove the implication “(ii) \(\Rightarrow\) (i)” to explain why the conditions of the locally boundedness of \( l \) and the boundedness of the left regular representation are necessary.

Assume (ii) holds for some polynomial \( P \). Without loss of generality, it is enough to prove (i) for any given \( f \in \mathcal{H}_+(G, H) \). Since the support of \( f \) is compact and \( l \) is locally bounded, there is some \( r > 0 \) such that \( \text{supp}(f) \subseteq B_{r,l}(G, H) \). Find two positive numbers \( C, s \) such that \( P(n) \leq Cn^{s-1} \) for all \( n \in \mathbb{N} \). For every \( n \in \mathbb{N} \), let \( \chi_n \) be the characteristic function of \( C_{n-1,l}(G, H) \) which is a right \( H \)-invariant function. Since \( \lambda \) is continuous, (ii) holds for every function in the closure of \( \mathcal{H}_+(G, H) \) with respect to the \( L^1 \)-norm. Therefore since \( f \chi_n \in L^1(G, H) \) for all \( n \in \mathbb{N} \), we have
\[
\|\lambda(f \chi_n)\| \leq P(n)\|f \chi_n\|_2 \leq Cn^{s-1}\|f \chi_n\|_2.
\]
Thus we get
\[
\|\lambda(f)\| = \left\| \sum_{n=1}^{\infty} \lambda(f \chi_n) \right\| \leq C \sum_{n=1}^{\infty} n^{-s}\|f \chi_n\|_2.
\]
By the Cauchy-Schwartz inequality, we obtain
\[
\|\lambda(f)\| \leq C' \left( \sum_{n=1}^{\infty} n^{2s}\|f \chi_n\|_2^2 \right)^{1/2},
\]
where \( C' = C \left( \sum_{n=1}^{\infty} n^{-2} \right)^{1/2} \). On the other hand, for every \( n \in \mathbb{N} \) and \( Hg \in C_{n-1,l}(G, H) \), we have
\[
\|f \chi_n\|_2^2 = \int_{C_{n-1,l}(G, H)} |f(Hg)|^2 \delta \nu(Hg),
\]
and also \( n \leq l(g) + 1 \). Thus we obtain
\[
\|\lambda(f)\| \leq C' \left( \sum_{n=1}^{\infty} \int_{C_{n-1,l}(G, H)} |f(Hg)|^2 (l(g) + 1)^{2s} \delta \nu(Hg) \right)^{1/2} = C'\|f\|_{s,l}.
\]

One notes that when \((G, H)\) is a discrete Hecke pair, the above proof works even without using the fact that \( \lambda \) is bounded. That is why the boundedness of \( \lambda \) was not assumed in Proposition 2.10 of [50]. The above proposition had been appeared in the literature in the setting of discrete and locally compact groups before [50], see [12, 13, 14, 38]. For the next theorem, we need to borrow Lemma 3.5 of [13]. However, we have to change its statement slightly according to the applications that we have in mind. We also do not assume that \( \eta \) is a normalized measure, so various powers of \( \eta(H) \) appear in our formulas.
Lemma 3.17. Let $H$ be a compact subgroup of a locally compact group $G$. For every $\xi \in L^2(G)$ and $f \in C_c(G)$, define $H\xi \in L^2(G)$ and $Hf_H \in C_c(G)$ by

$$H\xi(x) := \left( \int_H |\xi(hx)|^2 d\eta(h) \right)^{1/2},$$

$$Hf_H(x) := \left( \int_H \int_H |f(hxk)|^2 d\eta(h) d\eta(k) \right)^{1/2},$$

for all $x \in G$. Then we have the following statements:

(i) $\|H\xi\|_2 = \eta(H)^{1/2}\|\xi\|_2$.

(ii) If $f$ is bi-$H$-invariant, then $|f \ast \xi(x)| \leq |f| \ast H\xi(x)$ for all $x \in G$.

(iii) $\|Hf_H\|_2 = \eta(H)\|f\|_2$.

(iv) $\|\lambda(f)\| \leq \eta(H)\|\lambda(Hf_H)\|$, where $\lambda$ is the left regular representation of the locally compact group $G$.

One notes that the item (ii) of the above lemma is proved by modifying a calculation in the proof of Lemma 3.5 of [13]. The following theorem is a crucial step to study property (RD) of Hecke pairs $(G, H)$ when $H$ is a compact subgroup of $G$:

Theorem 3.18. Let $H$ be a compact subgroup of a locally compact group $G$. Then the group $G$ has (RD) if and only if the Hecke pair $(G, H)$ has (RD).

Proof. Suppose $G$ has (RD) with respect to a locally bounded length function $l$ on $G$. By Lemma 3.8 and Remark 3.15, without loss of generality, we can assume that the kernel of $l$ contains $H$, and so $l$ is a locally bounded length function on the Hecke pair $(G, H)$ as well. Hence $B_{r,l}(G, H) = H \setminus B_{r,l}(G)$ for all $r \geq 0$. Let $P$ be the polynomial mentioned in Proposition 3.16(iii), coming from property (RD) for $G$. Given $r \geq 0$, for every function $f \in \mathcal{H}_+(G, H)$ such that $\text{ supp}(f) \subseteq B_{r,l}(G, H)$ and every function $\xi \in L^2(G \setminus H)$, we define $\hat{f} \in C_c(G)$ and $\hat{\xi} \in L^2(G)$ as in Subsection 2.3. Then $\text{ supp}(\hat{f}) \subseteq B_{r,l}(G)$ and we have

$$\|f \ast \xi\|_2^2 = \frac{1}{\eta(H)^3} \|\hat{f} \ast \hat{\xi}\|_2^2 \leq \frac{P(r)^2}{\eta(H)^3} \|\hat{f}\|_2^2 \|\hat{\xi}\|_2^2 = \frac{P(r)^2}{\eta(H)} \|f\|_2^2 \|\xi\|_2^2.$$

This proves that the Hecke pair $(G, H)$ has (RD) with respect to $l$.

Conversely, assume that the Hecke pair $(G, H)$ possesses (RD) with respect to a locally bounded length function $l$ and let $P$ be the corresponding polynomial. Given $r \geq 0$, assume that a function $f \in C_{c+}(G)$ subject to the condition $\text{ supp}(f) \subseteq B_{r,l}(G)$ and a function $\xi \in L^2(G)$ are given. In the first step, we assume that $f$ is bi-$H$-invariant and $\xi$ is left $H$-invariant. We define $\hat{f} \in \mathcal{H}(G, H)$ and $\hat{\xi} \in L^2(H \setminus G)$ by $\hat{f}(Hx) := f(x)$ and $\hat{\xi}(Hx) := \xi(x)$ for all $hx \in H \setminus G$, respectively. It is straightforward to check that

$$\|\hat{f}\|_2 = \frac{1}{\eta(H)} \|f\|_2, \quad \|\hat{\xi}\|_2 = \frac{1}{\eta(H)} \|\xi\|_2.$$
For every $x \in G$, we compute
\[
\hat{f} \ast \hat{\xi}(x) = \int_{H \setminus G} \hat{f}(Hxy^{-1})\hat{\xi}(Hy)d\nu(Hy)
\]
\[
= \int_{H \setminus G} f(xy^{-1})\xi(y)d\nu(Hy)
\]
\[
= \frac{1}{\eta(H)} \int_{H \setminus G} \left( \int_{H} f(xy^{-1})\xi(y)d\eta(h) \right) d\nu(Hy)
\]
\[
= \frac{1}{\eta(H)} \int_{H \setminus G} \left( \int_{H} f(xy^{-1}h^{-1})\xi(hy)d\eta(h) \right) d\nu(Hy)
\]
\[
= \frac{1}{\eta(H)} \int_{G} f(xy^{-1})\xi(y)d\mu(y)
\]
\[
= \frac{1}{\eta(H)} f \ast \xi(x).
\]
Since $\text{supp}(\hat{f}) \subseteq B_{r,l}(G, H)$, we have
\[
\| f \ast \xi \|_2^2 = \eta(H)^2 \| \hat{f} \ast \hat{\xi} \|_2^2
\]
\[
= \eta(H)^3 \| \hat{f} \ast \hat{\xi} \|_2^2
\]
\[
\leq \eta(H)^3 P(r)^2 \| \hat{f} \|_2^2 \| \hat{\xi} \|_2^2
\]
\[
= \eta(H) P(r)^2 \| f \|_2^2 \| \xi \|_2^2.
\]
Therefore the condition of Proposition 3.16(iii) with the polynomial $\sqrt{\eta(H)} P(r)$ holds in this case. Now, let $f$ and $\xi$ be arbitrary (no $H$-invariance conditions are assumed). Using Lemma 3.17 and the above calculation, we have
\[
\| Hf \ast H\xi \|_2^2 \leq \| Hf \ast H\xi \|_2^2
\]
\[
\leq \eta(H) P(r)^2 \| Hf \|_2^2 \| H\xi \|_2^2
\]
\[
= \eta(H)^2 P(r)^2 \| f \|_2^2 \| \xi \|_2^2.
\]
Hence,
\[
\| \lambda(Hf) \| \leq \eta(H) P(r) \| Hf \|_2 = \eta(H)^2 P(r) \| f \|_2.
\]
By Lemma 3.17(iv), this implies $\| \lambda(f) \| \leq \eta(H)^3 P(r) \| f \|_2$. Therefore, by Proposition 3.16(ii), $G$ has (RD).

The above theorem can be thought of as the continuous version of Theorem 2.11 of [50]. Since Theorem 2.2 of [51] is also a generalization of the latter theorem, it would be interesting to state and prove the generalization of the above theorem for compactly commensurable subgroups, i.e. two closed subgroups whose intersection is a cocompact subgroup of both. So far, we have only considered compact and cocompact (similarly cofinite) subgroups of locally compact groups to define non-discrete Hecke pairs. On the other hand, a compactly commensurable subgroup of a compact (resp. cocompact) subgroup is again a compact (resp. cocompact) subgroup. Hence, before generalizing the above theorem for compactly commensurable subgroups, we first need to define and study other non-discrete Hecke pairs besides the cases studied in this paper.

A version of the following corollary also appeared in [13] as Lemma 3.4.
Corollary 3.19. Let $K$ be a compact normal subgroup of a locally compact group $G$. Then the group $G$ has (RD) if and only if the quotient group $G/K$ has (RD).

As another application of Theorem 3.18, we establish an equivalence between the property (RD) of a reduced discrete Hecke pair and the property (RD) of the totally disconnected locally compact group appearing in its Schlichting completion.

Theorem 3.20. Let $(G, H)$ be a reduced discrete Hecke pair and let $(\overline{G}, \overline{H})$ be its Schlichting completion. Then the followings are equivalent:

(i) The Hecke pair $(G, H)$ has (RD).
(ii) The Hecke pair $(\overline{G}, \overline{H})$ has (RD).
(iii) The totally disconnected locally compact group $\overline{G}$ has (RD).

Proof. Equivalence of (i) and (ii) follows from Remark 3.10(ii), and Lemma 2.38(i, iii). Equivalence of (ii) and (iii) follows from Theorem 3.18. □

For a similar result concerning the amenability of Hecke pairs and their Schlichting completion see Proposition 5.1 of [54]. The following lemma which completes the above theorem follows from Proposition 2.14 and Remark 3.10(i):

Lemma 3.21. Let $(G, H)$ be a Hecke pair and let $N$, $G'$ and $H'$ be as considered in Proposition 2.14. Then the Hecke pair $(G, H)$ has (RD) if and only if the Hecke pair $(G', H')$ has (RD).

In particular, a discrete Hecke pair $(G, H)$ has (RD) if and only if its associated reduced discrete Hecke pair $(G_r, H_r)$ has (RD).

Remark 3.22. The only known obstruction for a discrete group $G$ to possess property (RD) is that when it is amenable, property (RD) is equivalent to polynomial growth, see Corollary 3.1.8 of [33] and see Proposition 3.37 for the same result in the framework of Hecke pairs. In addition to this obstruction, unimodularity is another necessary condition for a locally compact group to possess property (RD), see Theorem 2.2 of [32]. The following corollary shows that the similar notion of relative unimodularity is necessary for a discrete Hecke pair to possess property (RD).

This suggests that the theory of locally compact groups can provide a better understanding of Hecke pairs. This have been the main point of several papers which use the Schlichting completion to analyze Hecke pairs and Hecke $C^*$-algebras, see for example [54, 35, 2] and our recent work [52]. For another application of the Schlichting completion in the present paper see Proposition 3.25.

Corollary 3.23. If a discrete Hecke pair $(G, H)$ has (RD), then it must be relatively unimodular.

Proof. Using Lemma 3.21, without loss of generality, we can restrict our attention to the case that $(G, H)$ is a reduced discrete Hecke pair. If $\Delta_{(G, H)} \neq 1$, then the locally compact group $\overline{G}$ in the Schlichting completion of $(G, H)$ is not unimodular and therefore cannot have (RD) by Theorem 2.2 of [32]. It follows from this and Theorem 3.20 that the discrete Hecke pair $(G, H)$ cannot possess (RD). □

Example 3.24. The Bost-Connes Hecke pair, see Example 2.4, does not have property (RD), because it is not relatively unimodular. The same statement holds for all Hecke pairs appearing in the context of quantum dynamical systems associated with Hecke pairs, see
As another application of Theorem 3.20, we improve Corollary 3.7 of [51] as follows:

**Proposition 3.25.** Let \((G, H)\) be a discrete Hecke pair and let \(\Gamma\) be a group containing \(G\) as a closed finite index subgroup such that \((\Gamma, H)\) is a discrete Hecke pair too. Then the Hecke pair \((G, H)\) has (RD) if and only if the Hecke pair \((\Gamma, H)\) has (RD).

**Proof.** If the Hecke pair \((\Gamma, H)\) has (RD), then by Proposition 2.11 of [51], the Hecke pair \((G, H)\) has (RD) too.

Conversely, suppose that the Hecke pair \((G, H)\) has (RD). By Lemma 3.21, without loss of generality, we can assume that the Hecke pair \((\Gamma, H)\) is reduced. Let \(\overline{\Gamma}\) be the Schlichting completion of \((\Gamma, H)\) and let \(\overline{G}^\Gamma\) be the closure of \(G\) in \(\overline{\Gamma}\). By the same argument as the proof of Theorem 3.20 and using Remark 2.39, we deduce that the Hecke pair \((\overline{G}^\Gamma, \overline{H})\) has (RD). Since \(\overline{H}\) is a compact subgroup of \(\overline{G}^\Gamma\), by Theorem 3.18, the locally compact group \(\overline{G}^\Gamma\) has (RD). Since \(\overline{G}^\Gamma\) is a closed finite index subgroup of \(\overline{\Gamma}\), by Lemma 3.3 of [13], the locally compact group \(\overline{\Gamma}\) has (RD). Therefore by Theorem 3.20, the Hecke pair \((\overline{\Gamma}, H)\) has (RD). \(\Box\)

There are two points about the above proposition: First, Lemma 3.3 of [13] is still true without assuming that \(G\) is a compactly generated group. Secondly, it is necessary to assume that \((\Gamma, H)\) is a Hecke pair, see the following example:

**Example 3.26.** Consider the direct product \(G = H \times K\) of two copies of \(\mathbb{Z}\) and assume that \(t\) and \(s\) are generators of \(H\) and \(K\), respectively. Let \(\Gamma\) be a torsion version of HNN extension of \(G\) as follows:

\[
\Gamma := \langle t, s, u; utu^{-1} = s, u^2 = e, st = ts \rangle.
\]

Clearly, \((G, H)\) is a Hecke pair and \(\Gamma\) contains \(G\) as a finite index subgroup, but one easily observes that \(H\) is not a Hecke subgroup of \(\Gamma\).

Now, we are ready to illustrate our results in proving property (RD) for specific discrete Hecke pairs.

**Example 3.27.** It is proved in Théorème 2(3) of [34] that if \(G\) is a unimodular locally compact group acting properly on a locally finite tree \(X\) with finite quotients, then \(G\) has property (RD). A famous example for this type of groups is the group \(SL_2(\mathbb{Q}_p)\). Thus by theorem 3.18 and Lemma 3.21, the Hecke pair \((PSL_2(\mathbb{Q}_p), PSL_2(\mathbb{Z}_p))\) has property (RD). Then it follows from Theorem 3.20 that the Hecke pair \((PSL_2(\mathbb{Z}[1/p]), PSL_2(\mathbb{Z}))\) has (RD).

In order to investigate more examples of Hecke pairs possessing property (RD), we mainly use Theorems 3.18 and 3.20 and the main results of I. Chatterji, C. Pittet and L. Saloff-Coste in [13] and S. Mustapha in [41]. These latter results are summarized in the following remark:

**Remark 3.28.** (i) Let \(G\) be a connected Lie group. Let \(\tilde{G}\) and \(\mathfrak{g}\) denote the universal cover of \(G\) and the Lie algebra of \(G\), respectively. Then the main result (Theorem 0.1) of [13] asserts that the following statements are equivalent:

(a) The group \(G\) has (RD).
(b) The Lie algebra $\mathfrak{g}$ is a direct product of a semisimple Lie algebra and a Lie algebra of type $R$.
(c) The Lie group $\tilde{G}$ is a direct product of a connected semisimple Lie group and a Lie group of polynomial growth.

We recall that a Lie algebra is called of type $R$ if all the weights of its adjoint representation are purely imaginary. Then it is known that a Lie algebra is of type $R$ if and only if its associated Lie group is of polynomial growth, see [26, 31]. This explains the equivalence of Conditions (b) and (c) in the above. Also, for the similar statement about polynomial growth of $p$-adic Lie groups and several other equivalent conditions see [44].

(ii) Let $F$ be a local field of characteristic 0 and let $G$ be an algebraic group over $F$. Assume $G$ and its radical are both compactly generated. Then $G$ has property (RD) if and only if $G$ is a reductive group. For a more general statement see Theorem 1 of [41]. We note that it was also shown in Theorem 4.5 of [13] that every semisimple linear algebraic group on a local field has property (RD).

Example 3.29. The above discussion applies to the semisimple Lie group $SL_2(\mathbb{R})$, and therefore the Hecke pair $(SL_2(\mathbb{R}), SO_2(\mathbb{R}))$ discussed in Example 2.30(i) has property (RD).

Using Proposition 2.14 and Theorem 3.18, a locally compact Hecke pair $(G, H)$ has property (RD) provided that there exists a normal subgroup $K$ of $G$ such that $K \subseteq H$, $H/K$ is compact and $G/K$ is of one of the forms described in the above remark. This is particularly useful to find pro-Lie groups with property (RD). In the next subsection, we expand on the notion of growth of a Hecke pair.

3.3. Growth, amenability and property (RD).

In addition to compactly (finitely) generated groups, we might also deal with not necessarily compactly (possibly infinitely) generated groups and Hecke pairs associated with them. Thus we need to extend some notions from the setting of compactly generated groups to this class of groups and Hecke pairs. To this end, we have to study the growth rates of groups with respect to arbitrary locally bounded length functions (not just length functions associated with compact generating sets). Another motivation for our general approach is that word length functions are not well defined on compactly generated Hecke pairs.

Definition 3.30. Let $(G, H)$ be a Hecke pair and let $l$ be a locally bounded length function on $(G, H)$.

(i) The growth function associated with $l$ is the function $\mathcal{G}_l : [0, \infty] \to [0, \infty]$ defined by $\mathcal{G}_l(r) := \nu(B_{r,l}(G, H))$ for all $r \geq 0$.
(ii) We say that $(G, H)$ has infinite growth with respect to $l$ if $\mathcal{G}_l(r) = \infty$ for some $r \geq 0$, otherwise we say that $(G, H)$ has a finite growth with respect to $l$.
(iii) We say that $(G, H)$ is of polynomial growth with respect to $l$ if there are two positive constants $c, \alpha$ such that $\mathcal{G}_l(r) \leq cr^\alpha$ for all large enough real numbers $r$. In this case we also say that the degree of the growth of $(G, H)$ is at most $\alpha$.
(iv) We say that $(G, H)$ is of superpolynomial growth with respect to $l$ if its growth is faster than any polynomial, more precisely, if the limit $\lim_{r \to \infty} \frac{\ln \mathcal{G}_l(r)}{\ln r}$ exists and equals $\infty$. 
(v) We say that \((G, H)\) is of exponential growth with respect to \(l\) if it has a finite growth and if there are two positive constants \(d, \beta\) such that \(G_l(r) \geq d\beta^r\) for all sufficiently large real numbers \(r\).

(vi) We say that \((G, H)\) is of subexponential growth with respect to \(l\) if it has a finite growth with respect to \(l\), but it is not of exponential growth with respect to \(l\). It is equivalent to the condition that \(\lim_{r \to \infty} \frac{\log G_l(r)}{r} = 0\).

(vii) We say that \((G, H)\) is of intermediate growth with respect to \(l\) if it is of superpolynomial and subexponential growth with respect to \(l\), simultaneously.

There are several points regarding the above definition: When a Hecke pair \((G, H)\) is of polynomial (resp. intermediate or exponential) growth with respect to some length function \(l\), we may also say that \((G, H)\) has polynomial (resp. intermediate or exponential) growth rate with respect to \(l\). When \((G, H)\) is a discrete Hecke pair \(\nu\) is a multiple of the counting measure. Assuming \(H = \{e\}\), the above definitions reduce to the well known definitions of various growth rates for locally compact groups. It is clear that equivalent length functions give rise to the same growth rate. Therefore, when a group \(G\) is a compactly (or finitely) generated group, we usually consider the growth rate of \(G\) with respect to the word length function associated with a compact generating set and do not mention the length function.

Similar to amenability and property (RD), (see [54, 52] for amenability and see Remark 3.10, Theorem 3.20 and Lemma 3.21 for property (RD)), certain operations on Hecke pairs do not change the growth rate. Here, we only mention them for later references and skip the proofs.

**Lemma 3.31.** Let \((G, H)\) be a Hecke pair equipped with a length function \(l\). Let \(N, G'\) and \(H'\) be as in Proposition 2.14 and let \(l'\) be the length function defined on the Hecke pair \((G', H')\) using \(l\), see Remark 3.10(i). Then the Hecke pairs \((G, H)\) and \((G', H')\) have the same growth rate with respect to \(l\) and \(l'\), respectively.

The special case of the above lemma that \((G, H)\) is a discrete Hecke pair and \(N = K_{(G, H)}\) is particularly important in the study of growth rate and amenability of discrete Hecke pairs.

**Proposition 3.32.** Let \((G, H)\) be a reduced discrete Hecke pair. Let \(l\) be a length function on \((G, H)\) and let \(\bar{l}\) be the corresponding length function on its Schlichting completion \((\overline{G, H})\) defined using \(l\), see Remark 3.10(ii). Then the discrete Hecke pairs \((G, H)\) and \((\overline{G, H})\) have the same growth rate with respect to the length functions \(l\) and \(\bar{l}\), respectively.

Using the above results, we obtain a criterion for amenability of discrete Hecke pairs in the next proposition. First we recall the definition of amenability of pairs \((G, H)\) based on [20], see also [54, 2, 52] for more details and results concerning amenability of Hecke pairs.

**Definition 3.33.** Let \(H\) be a closed subgroup of a locally compact group \(G\). The pair \((G, H)\) is called amenable if it possesses the fixed point property, that is whenever \(G\) acts continuously on a compact convex subset \(Q\) of a locally convex topological vector space by affine transformations and the restriction of this action to \(H\) has a fixed point, the action of \(G\) has a fixed point too.

**Proposition 3.34.** Let \((G, H)\) be a discrete Hecke pair which has a finite generating set \(S\). If the Hecke pair \((G, H)\) is of subexponential growth, then it is amenable.

**Proof.** Assume that \((G, H)\) is of subexponential growth, then so is its reduction \((G_r, H_r)\). Thus the Schlichting completion \((\overline{G_r, H_r})\) of the latter Hecke pair is of subexponential
growth. Since $\overline{\mathcal{H}}_r$ is compact, it follows that $\mathcal{G}_r$ is of subexponential growth with respect to a length function defined by a compact generating set. It is a well known fact that compactly generated locally compact groups of subexponential growth are amenable, see [26]. Therefore $\mathcal{G}_r$ is amenable. Using Proposition 5.1 of [54], this implies that the Hecke pair $(G, H)$ is amenable, see also Remark 4 of [52].

Due to the fact that every compactly generated locally compact group of subexponential growth is unimodular, see Lemma 1.3 of [26], and by applying a similar argument as the above proof, one can prove the following proposition:

**Proposition 3.35.** Let $(G, H)$ be a discrete Hecke pair which has a finite generating set $S$. If the Hecke pair $(G, H)$ is of subexponential growth, then it is relatively unimodular.

Since, in the setting of finitely generated discrete Hecke pairs, polynomial growth implies property (RD) and property (RD) requires relative unimodularity, the above conclusion can be reached using the next proposition as well. First, we need to recall and use a characterization of amenability.

**Remark 3.36.** A locally compact group $G$ is amenable if and only if for every non-negative real function $f \in L^1(G)$, we have $\|f\|_1 = \|\lambda(f)\|$, see [39]. Using this, one can show that every compactly generated amenable group with property (RD) is of polynomial growth. Conversely, the continuity of the left regular representation implies that every compactly generated locally compact group of polynomial growth possesses property (RD). For the proof of these statements see Theorem 1.5 of [14].

Using the above remark, Theorem 3.20, Proposition 3.32, and Remark 4(iv) of [52], we obtain the following proposition:

**Proposition 3.37.** Let $(G, H)$ be a finitely generated discrete Hecke pair.

(i) If $(G, H)$ is of polynomial growth, then it has (RD).

(ii) If $(G, H)$ is amenable and possesses property (RD), it is of polynomial growth.

One notes that using Theorem 3.18 and Proposition 2.28, the above proposition is still valid for non-discrete Hecke pairs $(G, H)$, where $G$ is a compactly generated locally compact group and $H$ is a compact subgroup of $G$. When $G$ is a unimodular locally compact group and it has a decomposition of the form $G = PK$, where $K$ is compact and $P$ is amenable, a notably stronger result is also available, see Theorem 4.4 and Proposition 4.2 of [13].

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E-mail address: shirbisheh@gmail.com