$k$-planar Crossing Number of Random Graphs and Random Regular Graphs

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Abstract

We give an explicit extension of Spencer’s result on the biplanar crossing number of the Erdős-Rényi random graph $G(n, p)$. In particular, we show that the $k$-planar crossing number of $G(n, p)$ is almost surely $\Omega((n^2 p^2)^2)$. Along the same lines, we prove that for any fixed $k$, the $k$-planar crossing number of various models of random $d$-regular graphs is $\Omega((dn^2)^2)$ for $d > c_0$ for some constant $c_0 = c_0(k)$.

1 Introduction

Planar graphs have been heavily studied in the literature and their applications have sparked interdisciplinary work in a variety of fields, e.g., design problems for circuits, subways, and utility lines. The focus of this paper is a variation of the crossing number of a graph, which is itself a natural extension of planarity. The crossing number of a graph $G$, denoted $\text{cr}(G)$, is the minimum number of edge crossings in a drawing of $G$ in the plane. In particular, we will focus on the variation of crossing number known as the $k$-planar crossing number. The $k$-planar crossing number of $G$, denoted $\text{cr}_k(G)$, is defined as the minimum of $\text{cr}(G_1) + \cdots + \text{cr}(G_k)$ over all partitions of $G$ into $G_1 \cup \cdots \cup G_k$. The $k = 2$ case is commonly referred to as the biplanar crossing number.

In this paper, we investigate the $k$-planar crossing number of two models of random graphs: Erdős-Rényi random graphs and random $d$-regular graphs. Spencer, in [6], gave a lower bound on the biplanar crossing number of Erdős-Rényi random graph $G(n, p)$.

\textbf{Theorem 1.} [6] There are constants $c_0$ and $c_1$ such that for all $p \geq c_0/n$, the biplanar crossing number $\text{cr}_2(G)$, with $G \sim G(n, p)$, is with high probability at least $c_1(n^2 p^2)^2$.

Spencer remarked that the methods used in Theorem 1 allow one to show that for all $k$, when $p \geq c_k/n$ for some $c_k$, $\text{cr}_k(G) = \Omega((n^2 p^2)^2)$ where $G \sim G(n, p)$. However, a few people in the community were unable to extend Spencer’s proof, so in this paper, we give an explicit proof of the lower bound for the $k$-planar crossing number of $G(n, p)$ for arbitrary $k$. Throughout this paper, $o, O, \Omega$ are always for $n \to \infty$.

\textbf{Theorem 2.} For all integers $k \geq 1$, there are constants $c_0 = c_0(k)$ and $c_1 = c_1(k)$ such that for all $p \geq c_0/n$, the $k$-planar crossing number of $G(n, p)$ is with high probability at least $c_1(n^2 p^2)^2$.

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Along similar lines, we investigate the k-planar crossing number of several models of random d-regular graphs in Section 4. The key ingredients of the proof involve Friedman’s results on Alon’s second eigenvalue conjecture in [4]. In particular, we consider \( G_{n,d}, \mathcal{H}_{n,d}, \mathcal{I}_{n,d}, \mathcal{J}_{n,d} \) and some related models. Please refer to [4] for the definitions of \( G_{n,d}, \mathcal{H}_{n,d}, \mathcal{I}_{n,d}, \mathcal{J}_{n,d} \). For two families of probability spaces, \( (\Omega_n, \mathcal{F}_n, \mu_n)_{n=1,2,\ldots} \) and \( (\Omega_n, \mathcal{F}_n, \nu_n)_{n=1,2,\ldots} \) over the same sets \( \Omega_n \) and sigma-algebras \( \mathcal{F}_n \), denote \( \mu = \{ \mu_n \} \) and \( \nu = \{ \nu_n \} \). We say \( \mu \) dominates \( \nu \) if for any family of measurable events \( \{ E_n \} \), \( \mu_n(E_n) \rightarrow 0 \) implies \( \nu_n(E_n) \rightarrow 0 \). We say that \( \mu \) and \( \nu \) are contiguous if \( \mu \) dominates \( \nu \) and \( \nu \) dominates \( \mu \). Following Friedman’s notation, let \( \mathcal{L}_n \) be a family of probability spaces of d-regular graphs on \( n \) vertices that is dominated by \( G_{n,d}, \mathcal{H}_{n,d}, \mathcal{I}_{n,d} \) or \( \mathcal{J}_{n,d} \).

Along similar lines as the proof of Theorem 2, we prove that the k-planar crossing number of the random d-regular graph \( G \) in \( \mathcal{L}_n \) is \( \Omega(n^2d^2) \), where \( n \) is the number of vertices of \( G \) and \( d \) is the degree of regularity.

**Theorem 3.** For all integers \( k \geq 1 \), there are constants \( c_0 = c_0(k) \) and \( c_1 = c_1(k) \) such that for all \( d \geq c_0 \), the k-planar crossing number \( CR_k(G) \), with \( G \) in \( \mathcal{L}_n \), is with high probability at least \( c_1(n^2d^2) \).

The proof of Theorem 3 hinges on the following result on the edge densities of random d-regular graphs.

**Theorem 4.** For every \( k \), there is a constant \( c_0(k) \) such that for \( d \geq c_0(k) \), the random d-regular graph \( G \) in \( \mathcal{L}_n \) has the following property with high probability: For every pair of disjoint sets \( X, Y \subseteq V(G) \), each of size at least \( \frac{n}{6} \), there are at least \( \frac{dn}{2(63k^2)^2} \) edges \( \{ x, y \} \in G \text{ with } x \in X, y \in Y \).

We will also need to use the notion of bisection width, a key tool used to set lower bounds for crossing numbers. The 1/3/2/3 bisection width of \( G \), denoted \( b(G) \), is defined as

\[
b(G) = \min_{V_1 \cup V_2 = V, |V_1| \geq n/3} \{ e(V_1, V_2) \},
\]

where \( e(V_1, V_2) \) is the number of edges between \( V_1 \) and \( V_2 \).

The bisection width can intuitively be thought of as the minimum number of edges of \( G \) which must be removed in order to disconnect \( G \) into two connected components of roughly equal size. An optimal 1/3/2/3 bisection is a partition realizing the 1/3/2/3 bisection width. This parameter on \( G \) is used in the following theorem to give a lower bound on the crossing number.

**Theorem 5.** [5] Let \( G \) be a graph of \( n \) vertices, whose degrees are \( d_1, d_2, \ldots, d_n \). Then

\[
b(G) \leq 10 \sqrt{CR(G)} + 2 \sqrt{\sum_{i=1}^{n} d_i^2}.
\]

## 2 k-planar crossing number of \( G(n, p) \)

The proof goes by a sequence of lemmas.

**Lemma 6.** [6] Let \( X \) be an \( m \)-element set and let \( X_{11} \cup X_{12} \) and \( X_{21} \cup X_{22} \) be any two bipartitions of \( X \) such that each of \( X_{11}, X_{12}, X_{21} \), and \( X_{22} \) has size at least \( m/3 \). Then there exist two subsets \( Y_1 \) and \( Y_2 \) of \( X \), each of size at least \( m/6 \), which lie on different sides of both bipartitions. That is, either \( Y_1 \subseteq X_{11} \cap X_{21}, Y_2 \subseteq X_{12} \cap X_{22} \) or \( Y_1 \subseteq X_{11} \cap X_{22}, Y_2 \subseteq X_{12} \cap X_{21} \).
The following lemma is a slight variation of Theorem 3 in [6].

**Lemma 7.** When $p = \Omega(1/n)$, the random graph $G(n, p)$ satisfies the following property w.h.p.: for any fixed $k$ and for every pair of disjoint vertex sets $X, Y$, each of size at least $t = n/(6 \cdot 3^k - 2)$, there are at least $\frac{t}{21(2)}p$ edges $\{x, y\} \in G$ with $x \in X, y \in Y$.

**Proof.** Large deviation inequalities (see, e.g. Theorem A.1.13 of the Appendix of [2]) provides that $M$ independent trials, each with probability $p$, provides fewer than $M p/2$ successes with probability at most $e^{-Mp/8}$. Hence the number of edges between our sets is under $(1/2)p n^2/(6 \cdot 3^k - 2)^2$ with probability at most $e^{-p n^2/(288gk^2)}$. This bound, combined with an upper bound $4^n$ to the number of choices of $X, Y$, establishes the lemma. 

It is clear that the expected degree of each vertex is $(n-1)p$. The following bound on the maximum degree of $G(n, p)$ follows from the Chernoff bound for independent Bernoulli random variables.

**Lemma 8.** In $G(n, p)$ with $p > m/n$ for some constant $m$, with high probability the maximum degree in $G(n, p)$ is at most $(1 + \log n)p$.

**Proof.** If $X$ is the sum of $n-1$ independent Bernoulli random variables, which take value 1 with probability $p$, then for every $\delta > 0$ the Chernoff bound gives that

$$P[X > (1 + \delta)(n - 1)p] \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^{(n-1)p}.$$

Applying this bound with $\delta = \log n$ gives an upper bound of $n^{-\log \log n + 1}$ on the probability that a vertex has degree above $p n (1 + \log n)$, if $p > 2/n$. The union bound then gives that the expected number of vertices with degree above $p n (1 + \log n)$ is $n^{-\log \log n + 2} = o(1)$ which proves the assertion.

This condition on maximum degree is necessary in order to use Theorem 5 to provide the lower bound on the crossing number. In order for Theorem 5 to give $\text{cr}_k(G) = \Omega((n^2 p)^2)$, it must be the case that $\sqrt{\sum d^2} = o(b(G))$. If $\Delta$ denotes the maximum degree of a vertex in $G$, then it is sufficient that $\Delta \sqrt{n} = o(n^2 p)$, which is satisfied when $\Delta \leq (1 + \log n)p$ and $p > m/n$ for a sufficiently large constant $m$.

**Proof of Theorem 2.** Let $G_1 \cup G_2 \cup \cdots \cup G_k$ be the partition of the edges of our sample $G$ of $G(n, p)$ that realizes the $k$-planar crossing number of $G$. We assume without loss of generality that the sample satisfies the requirements in Lemmas 7 and 8.

Consider now the optimal 1/3-2/3 bisections of $G_1$ and $G_2$. By Lemma 6, we find two disjoint sets, each of the same size, at least $n/6$, which are separated from each other by both optimal 1/3-2/3 bisections. Call the union of these two sets $Y_2$, and observe $|Y_2| \geq n/3$.

Let $G_3|Y_2$ denote the restriction of $G_3$ to the vertices of $Y_2$ and consider now the optimal 1/3-2/3 bisection of $G_3|Y_2$, and an equipartition of $Y_2$ similar to that which we used to define $Y_2$. Lemma 6 applies again, resulting in two disjoint, equal sized subsets of $Y_2$, of size at least $n/(3 \cdot 6)$, which are subsets of different sides of all three partitions we have considered so far. Call the union of these two sets $Y_3$, and observe $|Y_3| \geq n/3^2$.

If for some $3 \leq i \leq k-1$ the set $Y_i$ is already defined, consider now the optimal 1/3-2/3 bisection of the graph $G_{i+1}|Y_i$, and the partition of $Y_i$ from which we defined $Y_i$. Lemma 6 applies again, resulting in two equal sized disjoint subsets of $Y_i$ of size at least $n/(3^{i-1} \cdot 6)$, which are subsets of different sides of all $i+1$ partitions considered so far. Call the union of these two sets $Y_{i+1}$, and observe $|Y_{i+1}| \geq n/3^i$. 

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Let $A$ and $B$ denote the two disjoint, equal sized sets, of size at least $n/(3^{k-2} \cdot 6)$, whose union defined $Y_k$. The following inequality follows from our construction:

$$e_G(A, B) \leq b(G_k|_{Y_{k-1}}) + b(G_{k-1}|_{Y_{k-2}}) + \ldots + b(G_3|_{Y_2}) + b(G_2) + b(G_1).$$

Observe that $e_G(A, B)$ is large by Lemma 7. Therefore, at least one of the $1/3 - 2/3$ bisection widths of the $k$ graphs on the right-hand side must be large; that is, at least $\frac{n^2 p}{k - 2(6 \cdot 3^{k-2})}$. (Note that the additional factor of $1/k$ comes from the fact that there are $k$ total summands on the right hand side.) In particular, it is at least a constant fraction of $n^2 p$. By Theorem 5, the bisection width of this graph is large enough to prove Theorem 1.

\[\square\]

3 Proof of Theorem 4

For any $X, Y \subseteq V(G)$, let $E(X, Y)$ be the set of edges with one vertex in $X$ and the other in $Y$ and denote the order of $E(X, Y)$ by $e(X, Y)$.

The following variant of Expanders Mixing Lemma is a slight extension by Beigel, Margulis and Spielman [3] of a bound originally proven by Alon and Chung [1].

**Theorem 9.** [3] Let $G$ be a $d$-regular graph such that every eigenvalue except the largest has absolute value at most $\mu$. Let $X, Y \subseteq V$ have sizes $\alpha n$ and $\beta n$, respectively. Then

$$|e(X, Y) - \alpha \beta dn| \leq \mu n \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}$$

The following theorem of Friedman [4] gives a bound on $\mu$ for random regular graphs.

**Theorem 10.** [4] Fix a real $\varepsilon > 0$ and a positive integer $d \geq 2$. Let $\lambda_i$ denote the $i^{th}$ eigenvalue of the adjacency matrix of $G$. Then there exists a constant $c$ such that for a random $d$-regular graph $G$ in $L_n$, we have with probability $1 - o(1)$ (as $n \to \infty$) that for all $i > 1$,

$$|\lambda_i(G)| \leq 2\sqrt{d - 1} + \varepsilon$$

*Proof of Theorem 4.* Theorem 10 gives that the $\mu$ in Theorem 9 is at most $2\sqrt{d - 1} + \varepsilon$ with high probability when $n$ is large. As in Theorem 4, let $\alpha = \beta = \frac{1}{2(6 \cdot 3^{k-2})}$. This gives that with high probability,

$$|e(X, Y) - \frac{dn}{(6 \cdot 3^{k-2})^2}| \leq 2n \sqrt{d - 1} \sqrt{\left(\frac{1}{6 \cdot 3^{k-2}} - \left(\frac{1}{6 \cdot 3^{k-2}}\right)^2\right)^2}$$

The radical on the right hand side is a constant for fixed $k$. Let twice this constant be defined as $c_k$. Then with high probability, $e(X, Y)$ differs from $\frac{dn}{(6 \cdot 3^{k-2})^2}$ by at most $c_k n \sqrt{d - 1}$, where $c_k < 1$ for all $k \geq 2$. Straightforward computation shows that when $d \geq (4 \cdot 6 \cdot 3^{k-2})^2 =: c_0(k)$, it will be the case that the right hand side of the inequality at most $\frac{dn}{2(6 \cdot 3^{k-2})^2}$ which completes the proof. \[\square\]

4 Proof of Theorem 3

In this section we will prove Theorem 3: Let $G$ be a random $d$-regular graph in $L_n$. Then

$$\text{cr}_k(G) = \Theta((dn)^2).$$
Proof of Theorem 3. Looking closely at the proof of Theorem 1, we realize any graph $G$ that satisfies the conditions in Theorem 4 (about edge density between two sufficiently large vertex sets) and Lemma 8 (about maximum degree) will have large $k$-planar crossing numbers. In our case, we can replace $p$ by $d/n$, the density of our $d$-regular graph $G$. Clearly $G$ has maximum degree $d$ which satisfies Lemma 8. Therefore Theorem 3 holds.

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