Further improvements to incidence and Beck-type bounds over prime finite fields

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Abstract

We establish improved finite field Szemerédi-Trotter and Beck type theorems. First we show that if \( P \) and \( L \) are a set of points and lines respectively in the plane \( \mathbb{F}_{p}^2 \), with \(|P|, |L| \leq N \) and \( N < p \), then there are at most \( C_{1} N^{2 - \frac{3}{402} + o(1)} \) incidences between points in \( P \) and lines in \( L \). Here \( C_{1} \) is some absolute constant greater than 1. This improves on the previously best-known bound of \( C_{1} N^{2 - \frac{1}{806} + o(1)} \).

Second we show that if \( P \) is a set of points in \( \mathbb{F}_{p}^2 \) with \(|P| < p \) then either at least \( C_{2} |P|^{1-o(1)} \) points in \( P \) are contained in a single line, or \( P \) determines least \( C_{2} |P|^{1+\frac{1}{109} - o(1)} \) distinct lines. Here \( C_{2} \) is an absolute constant less than 1. This improves on previous results in two ways. Quantitatively, the exponent of \( 1 + \frac{1}{109} - o(1) \) is stronger than the previously best-known exponent of \( 1 + \frac{1}{267} \). And qualitatively, the result applies to all subsets of \( \mathbb{F}_{p}^2 \) satisfying the cardinality condition; the previously best-known result applies only when \( P \) is of the form \( P = A \times A \) for \( A \subseteq \mathbb{F}_{p} \).

1 Introduction

This paper proves new results concerning two types of incidence problem between points and lines in the plane determined by a finite field \( \mathbb{F}_{p} \) of prime order \( p \).

Throughout, we use \( Y = O(X) \), \( X = \Omega(Y) \) and \( Y \ll X \) all to mean that there is an absolute constant \( C \) with \( Y \leq CX \). We use \( Y \approx X \) or \( Y = \Theta(X) \) to mean \( Y \ll X \) and \( X \ll Y \). If the implicit constant \( C \) is dependent on a parameter \( \epsilon \) then we indicate this with a subscript, e.g. \( Y \ll_{\epsilon} X \). We also use \( Y \ll X \), \( Y = \tilde{O}(Y) \) etc. to mean that there is an absolute constant \( c \) with \( Y \ll \log(X)^{c}X \).

1.1 Incidence bounds over a prime finite field

The first problem considered is that of obtaining efficient bounds on the number \( I(P, L) \) of incidences between a set \( P \) of points and a set \( L \) of lines in the plane \( \mathbb{F}_{p}^2 \).
If \(|P|, |L| \leq N\) then we know by Cauchy-Schwarz that \(I(P, L) \ll N^{3/2}\), so non-trivial incidence bounds are of the form \(I(P, L) \ll N^{3/2-\epsilon}\) for \(\epsilon > 0\).

When working over the plane \(\mathbb{R}^2\), Szemerédi and Trotter \[9\] obtained the sharp bound \(\epsilon \geq 1/6\). This was extended to \(\mathbb{C}\) by Toth \[11\] and a near-sharp generalisation to higher dimensional points and varieties was recently given by Solymosi and Tao \[8\], subsequently made sharp in the \(\mathbb{R}^4\) case by Zhal \[13\].

In the finite field case considered here, one must impose nondegenerate conditions in order to prove nontrivial bounds, since in the case \(P = \mathbb{F}_p^2\) it possible to obtain \(I(P, L) \approx N^{3/2}\). When \(N < p^{2-\delta}\), Bourgain, Katz and Tao \[4\] proved the existence of an \(\epsilon > 0\), dependent only on \(\delta > 0\). Helfgott and Rudnev \[5\] then obtained \(\epsilon \geq 1/10,678\) in the ‘small \(N\)’ case \(\delta \geq 1\). This was subsequently improved by the present author to \(\epsilon \geq 1/806 - o(1)\). In the ‘large \(N\)’ case \(\delta < 1\), Vinh \[12\] has obtained an explicit lower bound on \(\epsilon\) in terms of \(\delta\).

In this paper we further improve the state of the art in the ‘small \(N\)’ case to \(\epsilon \geq 1/662 - o(1)\):

**Theorem 1.** If \(P\) and \(L\) are a set of points and lines over \(\mathbb{F}_p\) with \(|P|, |L| \leq N < p\) then \(I(P, L) \lessapprox N^{3/2-1/662}\).

### 1.2 Beck-type theorems over a prime finite field

The second problem considered is that of obtaining Beck-type results over \(\mathbb{F}_p\). Given a set \(P\) of points in the plane, write \(L(P)\) for the set of lines determined by pairs of points. Beck \[2\] proved that for any finite \(P \subseteq \mathbb{R}^2\) at least one of two things happens. Either \(\Omega(|P|)\) of the points are colinear, or \(|L(P)| \gg |P|^2\).

Helfgott and Rudnev \[5\] proved a form of Beck’s theorem for finite fields. In particular case of the direct product \(P = A \times A\) they showed that \(|L(P)| \gg |P|^{1+1/267}\) so long as \(|P| < p\). As with the problem of counting incidences, a nondegeneracy condition of this kind is required for results of this type because \(|L(P)| \approx p^2\) when \(P = \mathbb{F}_p^2\). In fact, Iosevich, Rudnev and Zhai \[1\] recently showed that \(|L(P)| \approx p^2\) whenever \(|P| > p \log p\).

We improve on the Helfgott-Rudnev bound in two respects. First, we improve the exponent \(1/267\) to \(1/109 - o(1)\). Second, we establish the result for general \(P \subseteq \mathbb{F}_p^2\) rather than simply those of the form \(P = A \times A\):

**Theorem 2.** If \(P \subseteq \mathbb{F}_p^2\) and \(|P| < p\) then at least one of the following must occur:

1. At least \(\tilde{\Omega}(|P|)\) points from \(P\) are contained in a single line.
2. \(|L(P)| \gtrapprox |P|^{1+1/109}\.\)
1.3 Structure of the paper

The proofs of Theorems 1 and 2 develop the method in [6], and readers are referred to that paper for a sketch of the approach.

The structure of this paper is as follows. Section 2 shows how to refine a setup of points and lines to a particular configuration. Section 3 interprets this configuration as a partial sum-product problem, for which Section 4 obtains efficient bounds. Finally, Section 5 uses the results from the previous three sections to prove Theorems 1 and 2.

We remark that the results of Sections 2 and 3 are general and apply when working over any field. Sections 4 and 5 are specific to the $\mathbb{F}_p$ setting.

2 Refining points and lines

This section shows that if there exist many points that are each incident to many lines then a large set of points must lie in a certain kind of configuration. In Section 3 we will interpret this configuration as corresponding to a partial sum-product problem. We begin with a definition:

**Definition 3.** Let $p$ be a point in the plane, and $P$ be a set of points in the plane. We say that the pair $(P,p)$ is $K$-good if $P$ is supported over at most $K$ lines through $p$.

The main results of this section are the following two propositions.

**Proposition 4.** Let $P$ and $L$ be a set of points and lines respectively over a plane such that every point in $P$ is incident to $\Theta(K)$ lines in $L$. Suppose that $|P| \gg 1$ and $|P|K^2 \gg |L|$.

Then there exist distinct points $p_1, p_2 \in P$, and a point-set $Q \subseteq P$ with $|Q| \approx \frac{K^4|P|}{|L|^2}$ such that $(Q, p_1)$ and $(Q, p_2)$ are both $O(K)$-good, and the $O(K)$ lines supporting $Q$ are in each case elements of $L$.

**Proposition 5.** Let $Q, p_1, p_2$ be as provided by Proposition 4. Suppose in addition that each of the $O(K)$ supporting lines through each of $p_1, p_2$ is incident to at most $\frac{|P|}{K}$ points in $Q$ and that $|Q| \gg |L|/K, |P|/K, K$.

Then there exist distinct points $p_3, p_4 \in Q$ such that $p_2, p_3, p_4$ are colinear along a line $l \in L$ that is not incident to $p_1$, and there exists a point-set $R \subseteq Q$ with $|R| \approx \frac{|P|K^8}{|L|^4}$ such that $(R, p_3)$ and $(R, p_4)$ are both $O(K)$-good.

The rest of Section 2 is concerned with proving these propositions. Section 2.1 establishes some preliminary incidence lemmata. Section 2.2 then gives the proof of Proposition 4 and Section 2.3 gives the proof of Proposition 5.
2.1 Lemmata

We first record two standard results.

**Lemma 6.** Let $P_1$ be the set of points in $P$ incident to at least $\frac{I(P, L)}{2|P|}$ lines in $L$. Then $I(P_1, L) \approx I(P, L)$. Similarly, if $L_1$ is the set of lines in $L$ incident to at least $\frac{I(P, L)}{2|L|}$ points in $P$ then $I(P, L_1) \approx I(P, L)$.

**Proof.** We prove the result for points, leaving that for lines as an exercise. Let $P_2$ be the set of points in $P$ incident to at most $\frac{I(P, L)}{2|P|}$ lines in $L$. Then

$$I(P_2, L) = \sum_{p \in P_2} \# \{l \in L \text{ incident to } p\} \leq |P_2| \frac{I(P, L)}{2|P|} \leq \frac{I(P, L)}{2}.$$ 

Since $I(P, L) = I(P_1, L) + I(P_2, L)$ we obtain $I(P_1, L) \geq \frac{I(P, L)}{2}$ as required.

**Lemma 7.** Let $P_1$ be the set of points in $P$ incident to no more than $\max \left\{4, \frac{4|L|^2}{I(P, L)} \right\}$ lines in $L$. Then $I(P_1, L) \approx I(P, L)$. Similarly, if $L_1$ is the set of lines in $L$ incident to no more than $\max \left\{4, \frac{4|P|^2}{I(P, L)} \right\}$ points in $P$, then $I(P, L_1) \approx I(P, L)$.

**Proof.** We prove the result for points, leaving that for lines as an exercise. Let $\lambda = \max \left\{4, \frac{4|L|^2}{I(P, L)} \right\}$ and let $P_2$ be the set of points incident to at least $\lambda$ lines in $L$. Then we have

$$I(P_2, L) = \sum_{p \in P_2} \sum_{l \in L} \delta_{pl} \leq \frac{1}{\lambda} \sum_{p \in P_2} \sum_{l_1, l_2 \in L} \delta_{pl} \leq \frac{1}{\lambda} \left(I(P, L) + |L|^2\right) \leq \frac{I(P, L)}{2}.$$ 

Since $I(P, L) = I(P_1, L) + I(P_2, L)$ we obtain $I(P_1, L) \geq \frac{I(P, L)}{2}$ as required.

For points $p, q$ in the plane, let $l_{pq}$ be the line determined by $p$ and $q$. We use Lemma 6 to establish the following useful result.

**Lemma 8.** Let $P$ be a set of points and $L$ a set of lines, such that every point is incident to $\Theta(K)$ lines in $L$. For each $p \in P$, define $P_p = \{q \in P : l_{pq} \in L\}$. Then there exists $P_1 \subseteq P$ with $|P_1| \approx |P|$ such that $|P_p| \gg \frac{K^2|P|}{|L|}$ for each $p \in P_1$. 

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Proof. Since every point in \( P \) is incident to \( \Theta(K) \) lines in \( L \) we may write

\[
I(P, L) \approx K|P|.
\]

Let \( L_1 \) be the set of lines in \( L \) incident to \( \Omega\left(\frac{I(P, L)}{|L|}\right) = \Omega\left(\frac{K|P|}{|L|}\right) \) points in \( P \). By Lemma 6 we have \( I(P, L_1) \approx I(P, L) \approx K|P| \). Now let \( P_1 \) be the set of points in \( P \) incident to \( \Omega\left(\frac{I(P, L_1)}{|P|}\right) = \Omega(K) \) lines in \( L_1 \). By Lemma 6 we have

\[
I(P_1, L_1) \approx I(P, L_1) \approx K|P|.
\]  

(1)

Since \( P_1 \subseteq P \) and \( L_1 \subseteq L \) we know that each point in \( P_1 \) is incident to at most \( O(K) \) lines in \( L_1 \). So we have

\[
I(P_1, L_1) \ll K|P_1|.
\]  

(2)

Comparing (1) and (2) we see that \( |P_1| \gg |P| \), and so \( |P_1| \approx |P| \) since \( P_1 \) is a subset of \( P \). Now, each \( p \in P_1 \) is incident to \( \Omega(K) \) lines in \( L_1 \). And each line in \( L_1 \) is incident to \( \Omega\left(\frac{K|P|}{|L|}\right) \) points in \( P \). So we have \( |P_p| \gg \frac{K^2|P|}{|L|} \) for each \( p \in P_1 \).

\[
|P_1 \cap P_2| \gg \frac{K^2|P_1|}{|L|} \gg \frac{K^4|P|}{|L|}.
\]

Note that \( P_{p_1} \) is the set of points incident to lines that are themselves incident to \( p_1 \). Since \( p_1 \) is incident to only \( O(K) \) lines in \( L \) this means that \( (P_{p_1}, p_1) \) is \( O(K) \)-good. Similarly, \( (P_{p_2}, p_2) \) is \( O(K) \)-good. So \( (P_{p_1} \cap P_{p_2}, p_1) \) and \( (P_{p_1} \cap P_{p_2}, p_2) \) are both \( O(K) \)-good. We take \( Q \) to be an appropriately-sized subset of \( P_{p_1} \cap P_{p_2} \).

2.2 Proof of Proposition 4

By Lemma 7 there exists \( P_1 \subseteq P \) with \( |P_1| \approx |P| \) such that \( |P_p| \gg \frac{K^2|P|}{|L|} \gg 1 \) for each \( p \in P_1 \). In particular we can find a single \( p_1 \in P \) such that \( |P_{p_1}| \gg \frac{K^2|P|}{|L|} \). Applying Lemma 7 again, this time to \( P_{p_1} \) and \( L \), we can find \( p_2 \in P_{p_1} \) such that

\[
|P_{p_1} \cap P_{p_2}| \gg \frac{K^2|P_{p_1}|}{|L|} \gg \frac{K^4|P|}{|L|}.
\]

2.3 Proof of Proposition 5

Since \( Q \subseteq P \) we know that every point in \( Q \) is incident to \( \Theta(K) \) lines in \( L \). So by Lemma 7 there exists \( Q_1 \subseteq Q \) with \( |Q_1| \approx |Q| \) such that \( |Q_p| \gg \frac{K^2|Q|}{|L|} \) for each \( p \in Q_1 \). Now, \( Q_1 \) is contained in \( Q \) and so \( (Q_1, p_2) \) is \( O(K) \)-good, i.e. \( Q_1 \) is supported over \( O(K) \) lines in \( L \) that are incident to \( p_2 \). Let \( J \subseteq L \) be this set of \( O(K) \) supporting
lines. We have \( I(Q_1,J) = |Q_1| \) and \(|J| \ll K\). Let \( J_1 \) be the set of \( l \in J \) incident to at least \( \Omega \left( \frac{|Q_1|}{K} \right) \) points in \( Q_1 \). By Lemma 6 we know that

\[
I(Q_1,J_1) \approx I(Q_1,J) \approx |Q_1|.
\]  
(3)

But since \( Q_1 \subseteq P \) and \( J_1 \subseteq L \), we know that each line in \( J_1 \) is incident to at most \( O \left( \frac{|P|}{K} \right) \) points in \( Q_1 \). So we have

\[
I(Q_1,J_1) \ll \frac{|J_1|P}{K}.
\]  
(4)

Comparing (3) and (4) gives \(|J_1| \gg |Q_1|K|P|\). Since \(|Q_1| \gg |P|/K\) by hypothesis we know, by appropriate choice of constants in the statement of the theorem, that \(|J_1| \geq 2\). Since \(|J_1| \geq 2\) and all lines in \( J_1 \) are incident to \( p_2 \) we know that there is at least one line in \( J_1 \) that is not incident to \( p_1 \). Fix this line \( l^* \), so that \(|Q \cap l^*| \gg \frac{|Q|}{K} \), and in particular \(|Q \cap l^*| \) has at least, say, 100 elements since \(|Q| \gg K\). We have

\[
\frac{K^2|Q|}{|L|} \left| Q \cap l^* \right| \ll \sum_{p \in Q \cap l^*} |Q_p|.
\]  
(5)

On the other hand by Cauchy-Schwarz we have

\[
\sum_{p \in Q \cap l^*} |Q_p| \leq |Q|^{1/2} \left( \sum_{p_3,p_4 \in Q \cap l^*} |Q_{p_3} \cap Q_{p_4}| \right)^{1/2} = |Q|^{1/2} \left( \sum_{p \in Q \cap l^*} |Q_p| + \sum_{p_3 \neq p_4 \in Q \cap l^*} |Q_{p_3} \cap Q_{p_4}| \right)^{1/2}
\]

If the first summation on the right dominates then we have

\[
\sum_{p \in Q \cap l^*} |Q_p| \ll |Q|.
\]

Since \(|Q \cap l^*| \gg \frac{|Q|}{K} \) and \(|Q_p| \gg \frac{K^2|Q|}{|L|} \) we then obtain \(|Q|K \gg |L| \) by comparison with (5), contradicting the hypothesis \(|Q| \ll |L|/K\). So the second summation on the right dominates and we have instead

\[
\frac{K^4|Q|}{|L|} \left| Q \cap l^* \right|^2 \ll \sum_{p_3 \neq p_4 \in Q \cap l^*} |Q_{p_3} \cap Q_{p_4}|
\]
so there exist distinct \( p_3, p_4 \in Q \cap l^* \) such that

\[
|Q_{p_3} \cap Q_{p_4}| \gg \frac{K^4 |Q|}{|L|^2} \gg \frac{K^8 |P|}{|L|^3}.
\]

We take \( R \) to be an appropriately-sized subset of \( Q_{p_3} \cap Q_{p_4} \). Since \( p_2, p_3, p_4 \in l^* \) and \( p_1 \notin l^* \) we have the required result.

### 3 Interpretation as partial sum-products

If \( A \) and \( B \) are subsets of a field, then we write

\[
A + B = \{ a + b : a \in A, b \in B \}.
\]

If \( G \subseteq A \times B \) then we write

\[
A^G + B = \{ a + b : (a, b) \in G \}.
\]

We extend these definitions to the operations of multiplication, subtraction and division. This section shows that the configuration of points described in Proposition 9 implies the existence of sets \( A, B \) and \( G \subseteq A \times B \) with \( |A^G - B|, |A/B| \ll |A|, |B| \).

**Proposition 9.** Suppose \( P \) is a set of points in the plane over a field \( F \) and \( p_1, p_2, p_3, p_4 \notin P \) are points in the plane such that \((P, p_i)\) is \( K_i \)-good for each \( i \). Suppose moreover that \( p_2, p_3 \) and \( p_4 \) are colinear on a line \( l \) that is not incident to \( p_1 \) nor to any of the points in \( P \).

Then there exist sets \( A, B \subseteq F \) with \( |A| \leq K_3, |B| \leq K_4 \), and \( G \subseteq A \times B \) with \( |G| = |P| \) such that \( |A^G - B| \leq K_2 \) and \( |A/B| \leq K_1 \).

**Proof.** Let \( \tau \) be a projective transformation of the plane that sends \( p_3 \) and \( p_4 \) to the line at infinity, so that lines through \( \tau(p_3) \) are parallel to the vertical axis and lines through \( \tau(p_4) \) are parallel to the horizontal axis, and sends \( p_1 \) to the origin. Define \( G = \tau(P) \subseteq F^2 \).

The set \( G \) is supported over \( K_3 \) vertical lines \( K_4 \) horizontal lines. Let \( A \) be the set of \( x \)-intercepts of the vertical lines and \( B \) be the set of \( y \)-intercepts of the horizontal lines, so that \( G \subseteq A \times B \) and \( |A| \leq K_3, |B| \leq K_4 \).

Furthermore, \( G \) is supported over \( K_1 \) lines through the origin. We note that these are identified by their gradient, and that a point \((a, b) \in F^2 \) is incident to the line with gradient \( \xi \) if and only if \( \frac{a}{b} = \xi \). We therefore have \( |A/B| \leq K_1 \).
Finally, $G$ is supported over $K_2$ lines through $\tau(p_2)$. Since $\tau$ sends $p_3$ and $p_4$ to the line at infinity, and $p_2$ is collinear with these points, we know that $\tau(p_2)$ lies on the line at infinity. Say that all lines incident to $\tau(p_2)$ have gradient $\lambda \in F$. These are identified by the intercept, and a point $(a, b) \in F^2$ is incident to the line with gradient $\rho$ if and only if $a + \lambda b = \rho$. We therefore have $|A - \lambda B| \leq K_2$.

We now let $B' = \lambda B$ and $G' = \{(a, \lambda b) : (a, b) \in G\}$ to obtain $|G'| = |G| = |P|$, $|A| \leq K_3$, $|B'| = |B| \leq K_4$, $|A - B'| \leq K_1$ and $|A/B'| \leq K_2$. \hfill $\square$

\section{Bounding partial sum-products}

From Sections 2 and 3 we know that if too many points in $P$ are incident to too many lines in $L$ then there exist $A, B \subseteq F_p$ and a large $G \subseteq A \times B$ such that $|A^G_B|, |A/B|$ are both small relative to $|A|$ and $|B|$.

In this section, we will show that this is not possible, in the following sense:

\textbf{Proposition 10.} If $A, B \subseteq F_p$ and $G \subseteq A \times B$ with $|G| \leq p |B|$ then we have

\begin{equation*}
|G|^{55} \ll |A|^{36} |B|^{37} |A - B|^{28} |A/B|^8.
\end{equation*}

Finite field sum-product estimates are the driving force behind Proposition 10. These assert that max $\{|A + A|, |A \cdot A|\}$ must always be large relative to the cardinality of $A$. We will make use of the most-recent finite field sum-product estimate, due to Rudnev [7], who showed that max $\{|A + A|, |A \cdot A|\} \gtrsim |A|^{\frac{12}{11}}$. As Rudnev notes, the result makes use of the multiplicative energy $E_x(A)$, which is the number of solutions to $ab = cd$ with $a, b, c, d \in A$ rather than the product set itself, and extends perfectly well to difference sets rather than sumsets. We will adopt the following formulation of Rudnev’s result.

\textbf{Lemma 11 (Rudnev).} If $A \subseteq F_p$ and $|A| < p^{1/2}$ then $E_x(A)^4 \lesssim |A - A|^4 |A|^4$.

Lemma 11 concerns the complete difference set $A - A$, but we have control of only the partial difference set $A^G - B$. We prove the following lemma, which yields a Balog-Szemerédi-Gowers type estimate for sumsets and a more efficient bound for multiplicative energy.

\textbf{Lemma 12.} If $A, B \subseteq F_p$ and $G \subseteq A \times B$ then there exists $A' \subseteq A$ with $|A'| \gg \frac{|G|}{|B|}$ such that

1. $|A' - A'| \ll \frac{|A^G - B|^{1/3} |A|^2 |B|/|G|^{1/3}}{|A|}$
2. \( E_x(A') \gg \frac{|G|^6}{|B|^3|A|^2|A'/B|^2} \).

Implicitly the proof of the bound on \( A' - A' \) in Lemma \([\text{12}]\) is the same as that of the Balog-Szemerédi-Gowers type result due to Bourgain and Garaev \([\text{3}]\), although it is presented below slightly differently.

Combining Lemma \([\text{11}]\) and Lemma \([\text{12}]\) gives the statement of Proposition \([\text{10}]\). The remainder of this section is therefore concerned with the proof of Lemma \([\text{12}]\).

### 4.1 Proof of Lemma \([\text{12}]\)

We begin with some preliminary lemmata. Given \( G \subseteq A \times B \) we define \( N(a) \) for each \( a \in A \) as \( N(a) = \{ b \in B : (a, b) \in G \} \). The following result can be found in \([\text{10}]\) and is repeated in \([\text{3}]\).

**Lemma 13.** For sets \( G \subseteq A \times B \) and \( \epsilon > 0 \) there exists \( A' \subseteq A \) with \( |A'| \gg \frac{|G|}{|B|} \) such that for \((1 - \epsilon)|A'|^2\) pairs \((a_1, a_2) \in A' \times A'\) we have

\[
|N(a_1) \cap N(a_2)| \gg \frac{|G|^2}{|A|^2|B|}.
\]

**Lemma 14.** For sets \( A, B \) and \( G \subseteq A \times B \) there exists \( A' \subseteq A \) and \( H \subseteq A' \times A' \) with \( |A'| \gg \frac{|G|}{|B|} \) and \( |H| \geq (1 - \epsilon)|A'|^2 \) such that

1. \( |A' - A'| \ll \epsilon \frac{|A - B|^2|A|^2|B|}{|G|^2} \),
2. \( |A'/A'| \ll \epsilon \frac{|A/B|^2|A|^2|B|}{|G|^2} \).

**Proof.** Apply Lemma \([\text{13}]\). Let \( H \) be the set of \((a_1, a_2) \in A' \times A'\) for which \( |N(a_1) \cap N(a_2)| \) so that \( |H| \geq (1 - \epsilon)|A'|^2 \). We prove the result for \( |A' - A'| \), leaving that for \( |A'/A'| \) as an exercise.

For each \( x \in A' - A' \) fix \((a_1(x), a_2(x)) \in H\) such that \( a_1(x) - a_2(x) = x \). Now let \( Y = \{ (x, b) : x \in A' - A', b \in N(a_1(x)) \cap N(a_2(x)) \} \). It is clear that \( |Y| \gg |A' - A'| \frac{|G|^2}{|A|^2|B|} \).

On the other hand the injection

\[
f : Y \to (A - B) \times (A - B)
\]

\( f : (x, b) \mapsto (a_1(x) - b, a_2(x) - b) \)

shows that \( |Y| \leq |A - B|^2 \). Comparing the upper and lower bounds on \( |Y| \) gives the result. \(\square\)
The following result can also be found in [10], where it is set as exercise 2.5.4. It is not clear that a proof can usually be found in the literature, and so we give one here.

**Lemma 15.** Let \(0 < \epsilon < 1/4\) and let \(G \subseteq A \times B\) and \(H \subseteq B \times C\), such that \(|G| \geq (1 - \epsilon)|A||B|\) and \(|H| \geq (1 - \epsilon)|B||C|\). Then there exist \(A' \subseteq A\) and \(C' \subseteq C\) with \(|A'| \geq (1 - \sqrt{\epsilon})|A|\) and \(|C'| \geq (1 - \sqrt{\epsilon})|C|\) such that

\[
|A' - C'| \ll _{\epsilon} \frac{|A - B||B - C|}{|B|}.
\]

**Proof.** Let \(A'\) be the set of \(a \in A\) with a \(G\)-degree of at least \((1 - \sqrt{\epsilon})|B|\), and \(C'\) be the set of \(c \in C\) with an \(H\)-degree of at least \((1 - \sqrt{\epsilon})|B|\). Since \(|G| \geq (1 - \epsilon)|A||B|\) and \(|H| \geq (1 - \epsilon)|B||C|\) we know that \(|A'| \geq (1 - \sqrt{\epsilon})|A|\) and \(|C'| \geq (1 - \sqrt{\epsilon})|C|\). Now if \(a \in A'\) and \(c \in C'\) then there are at least \((1 - 2 \sqrt{\epsilon})|B|\) elements \(b \in B\) such that \((a, b) \in G\) and \((b, c) \in H\). Denote the set of such elements by \(B_{ac}\). For each \(x \in A' - C'\) fix a single \(a(x) \in A', c(x) \in C'\) such that \(a(x) - c(x) = x\). Let \(Y = \{(x, b) : x \in A' - C', b \in B_{a(x)c(x)}\}\). It is clear that \(|Y| \gg _{\epsilon} |A' - C'||B|\) On the other hand the injection

\[
f : Y \to (A - B) \times (B - C)
\]

\[
f : (x, b) \mapsto (a(x) - b, b - c(x))
\]

shows that \(|Y| \leq |A - B||B - C|\). Comparing the upper and lower bounds on \(|Y|\) gives the result. \(\square\)

We record a particular consequence of Lemma 15 in the case where \(A = B = C\) and \(G = H\):

**Corollary 16.** Let \(0 < \epsilon < 1/4\) and let \(G \subseteq A \times A\) with \(|G| \geq (1 - \epsilon)|A|^2\). Then there exists \(A' \subseteq A\) with \(|A'| \geq (1 - 2 \sqrt{\epsilon})|A|\) such that

\[
|A' - A'| \ll _{\epsilon} \frac{|A - A|^2}{|A|}.
\]

**Proof.** By Lemma 15 there exist \(A_1, A_2 \subseteq A\) with \(|A_1|, |A_2| \geq (1 - \sqrt{\epsilon})|A|\) such that \(|A_1 - A_2| \ll _{\epsilon} \frac{|G - A|^2}{|A|}\). We then let \(A = A_1 \cap A_2\). \(\square\)

We now prove Lemma 12. Let \(\epsilon > 0\) be sufficiently small. By Lemma 14 there exists there exists \(A' \subseteq A\) and \(H \subseteq B \times A'\) with \(|A'| \gg _{\epsilon} \frac{|G|}{|B|}\) and \(|H| \geq (1 - \epsilon)|A'|^2\) such that

\[
|A' - A'| \ll _{\epsilon} \frac{|A - B|^2|A|^2}{|G|^2} \quad \text{and} \quad |A'/A'| \ll _{\epsilon} \frac{|A/B|^2|A|^2|B|}{|G|^2}.
\]

Apply Corollary 16 to obtain \(A'' \subseteq A'\) with \(|A''| \geq (1 - \epsilon)|A'|\) such that
\[ |A'' - A''| \ll \frac{|A' - A'|^2}{|A'|} \ll \frac{|A - B|^4 |A|^4 |B|^3}{|G|^5}. \]

Now let \( H' = H \cap (A'' \times A'') \). Since both \( H \) and \( A'' \times A'' \) are of cardinality at least \((1 - \epsilon)|A'|^2\) we have \(|H'| \gg |A'|^2\). So by Cauchy-Schwarz we have

\[
E \times (A'') \geq \frac{|H'|^2}{|A'' / A''|} \gg \frac{|A'|^4}{|A' / A'|} \gg \frac{|G|^6}{|B|^5 |A|^2 |A - B|^2}
\]

which completes the proof. \(\square\)

5 Proving Theorems 1 and 2

5.1 Proof of Theorem 1

Suppose that \( I(P, L) \gtrsim N^{3/2 - \epsilon} \). We shall show that \( \epsilon \geq 1/662 \). By Lemma 7 we may assume that every line in \( L \) is incident to at most \( O \left( N^{1/2 + \epsilon} \right) \) points in \( P \). By a dyadic pigeonholing we may find a subset \( P_1 \subseteq P \) and an integer \( K \) with

\[
|P_1| K \gtrsim N^{3/2 - \epsilon} \tag{6}
\]

such that every point in \( P \) is incident to \( \Theta(K) \) lines in \( L \). Note moreover that since \( |P_1| \leq N \) we have

\[
K \gtrsim N^{1/2 - \epsilon} \tag{7}
\]

Applying Proposition 4 and then 5 we know that at least one of the following is true:

1. \( |P_1| K^2 \ll |L| \).
2. \( |P_1| K^3 \ll |L|^2 \).
3. \( |P_1| K^5 \ll |L|^3 \).
4. \( K^5 \ll |L|^2 \).
5. There exists \( R \subseteq P_1 \) with \( |R| \approx \frac{|P_1| K^8}{|L|} \) and points \( p_1, p_2, p_3, p_4 \) such that \( (R, p_i) \) is \( O(K) \)-good for each \( i \). Moreover, the points \( p_2, p_3, p_4 \) are colinear along a line \( l \) in \( L \) that is not incident to \( p_1 \).
We quickly dispense with the first four cases. If $|P_1|K^2 \ll |L|$ then applying (6), (7) and the fact that $|L| \leq N$ we get $\epsilon \geq 1/2 - o(1)$. By the same arguments, the other three cases yield $\epsilon \geq 1/6 - o(1)$, $\epsilon \geq 1/10 - o(1)$ and $\epsilon \geq 1/10 - o(1)$ respectively.

We are left with the fifth case. Since the line $l$ that is incident to $p_2, p_3, p_4$ is an element of $L$, we know that it is incident to at most $O\left(N^{1/2+\epsilon}\right)$ points in $R$. We may assume that $|R \setminus \{l\}| \approx |R|$ or we are already done.

Apply Proposition 5 to $R \setminus \{l\}$ to obtain $A, B \subseteq \mathbb{F}_p$ with $|A|, |B| \ll K$ and $G \subseteq A \times B$ with $|G| \gg \frac{|P_1|K^8}{|L|}$ such that $|A - B|, |A/B| \ll K$.

By Proposition 10 we have $|G|^{55} \ll K^{109}$ and so $|P_1|^{55}K^{331} \ll |L|^{220}$. Applying (6) and (7) as before we get $\epsilon \geq 1/662 - o(1)$.

5.2 Proof of Theorem 2

For $l \in L(P)$, write $\mu(l)$ for the number of points in $P$ incident to $l$. It is clear that

$$|P|^2 \approx \sum_{l \in L(P)} \mu(l)^2.$$

By a dyadic pigeonholing there exists $L_1 \subseteq L(P)$ and an integer $k$ such that $\mu(l) \approx k$ for all $l \in L_1$ and

$$|L_1|k^2 \approx |P|^2.$$  \hfill (8)

We have $I(P, L_1) \approx |L_1|k$. So there exists $P_1 \subseteq P$ and an integer $K$ such that every point in $P_1$ is incident to $\Theta(K)$ lines in $L_1$ and $|P_1|K \gtrsim |L_1|k$. We therefore have

$$K \gtrsim \frac{|L_1|k}{|P_1|}.$$  \hfill (9)

Applying Proposition 4 and then Proposition 5 to $P_1, L_1$ and $K$ we know that at least one of the following is true

1. $|P_1|K^2 \ll |L_1|$.
2. $|P_1|K^3 \ll |L_1|^2$
3. $|P_1|K^5 \ll |L_1|^3$.
4. $K^5 \ll |L_1|^2$.
5. There exists $R \subseteq P_1$ with $|R| \approx \frac{|P_1|K^8}{|L_1|}$ and points $p_1, p_2, p_3, p_4$ such that $(R, p_i)$ is $O(K)$-good for each $i$. Moreover, the points $p_2, p_3, p_4$ are colinear along a line $l \in L_1$ that is not incident to $p_1$.  \hfill □
In the first case, we apply (9) to obtain $|P_1| \lesssim 1$. The second, third and fourth cases yield respectively $k \lesssim 1$, $k \lesssim 1$ and $k \gtrsim |P|$. If $k \lesssim 1$ then we have $|L(P)| \gtrsim |P|^2$, and if $k \gtrsim |P|$ then there are $\tilde{\Omega}(|P|)$ collinear points in $P$.

So we are left with the fifth case. Since $l \in L_1$ it is incident to at most $O(k)$ points in $P$, and so we may assume $|R \setminus \{l\}| \approx |R|$ or we are already done. Apply Proposition 9 to obtain $A, B \subseteq \mathbb{F}_p$ with $|A|, |B| \ll K$ and $G \subseteq A \times B$ with $|G| \gg \frac{|P_1|^8}{|L_1|^4}$ such that $|A \cap B|, |A/B| \ll K$.

By Proposition 10 we have $|G|^{55} \ll K^{109}$ and so $|P_1|^{55}K^{331} \ll |L_1|^{220}$. By (9) this gives $|L_1|^{111}K^{331} \lesssim |P_1|^{276}$. By (8) and the fact that $|P_1| \leq |P|$ we then get $k^{109} \lesssim |P|^{54}$ and so by (8) again $|P|^{110} \lesssim |L_1|^{109}$. We conclude therefore that $|L(P)| \geq |L_1| \gtrsim |P|^{109}$. \hfill \qed

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