Thermodynamics of charged topological dilaton black holes

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A class of \((n+1)\)-dimensional \((n \geq 3)\) topological black hole solutions in Einstein-Maxwell-
dilaton theory with Liouville-type potentials for the dilaton field is presented. In these space-
times, black hole horizon and cosmological horizon can be an \((n-1)\)-dimensional positive,
zero or negative constant curvature hypersurface. Because of the presence of the dilaton
field, these topological black holes are neither asymptotically flat nor (anti)-de Sitter. We
calculate the charge, mass, temperature, entropy and electric potential of these solutions.
We also analyze thermodynamics of these topological black holes and disclose the effect of
the dilaton field on the thermal stability of the solutions.

I. INTRODUCTION

It is generally believed that in asymptotically flat spacetime, the topology of the event horizon
of a stationary black hole in four dimensions, is uniquely determined to be the two-sphere \(S^2\) [1, 2].
The “topological censorship theorem” of Friedmann, Schleich and Witt is another indication of the
impossibility of non-spherical horizons [3, 4]. This theorem states that in a globally hyperbolic,
asymptotically flat spacetime satisfying the null energy condition, any two causal curves extending
from past to future null infinity are homotopic. A black hole with toroidal surface topology would
provide a possible violation of topological censorship theorem, as a light ray from past infinity
linking with the hole of the torus and then back to future infinity would not be deformable to a
light ray traveling from past to future outside the black hole. Thus the hole must quickly close up,
before a light ray can pass through [5]. Therefore, general relativity does not allow an observer to
probe the topology of spacetime, and any topological structure collapses too quickly to allow light
to traverse it.

However, when the asymptotic flatness and the four dimensional spacetime are given up, there
are no fundamental reasons to forbid the existence of static or stationary black holes with non-trivial topologies. For instance, for five-dimensional asymptotically flat stationary black holes, in
addition to the known \(S^3\) topology of event horizons, stationary black hole solutions with event
horizons of \(S^2 \times S^1\) topology (black rings) have been constructed [6]. It has been shown that for

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asymptotically anti-de Sitter (AdS) spacetime, in the four-dimensional Einstein-Maxwell theory, there exist black hole solutions whose event horizons may have zero or negative constant curvature and their topologies are no longer the two-sphere $S^2$. The properties of these black holes are quite different from those of black holes with usual spherical topology horizon, due to the different topological structures of the event horizons. Besides, the black hole thermodynamics is drastically affected by the topology of the event horizon. It was argued that the Hawking-Page phase transition [7] for the Schwarzschild-AdS black hole does not occur for locally AdS black holes whose horizons have vanishing or negative constant curvature, and they are thermally stable [8]. The studies on the topological black holes have been carried out extensively in many aspects [9, 10, 11, 12, 13, 14, 15, 16, 17].

On the other hand, it is quite possible that gravity is not given by the Einstein action, at least at sufficiently high energies. In string theory, gravity becomes scalar-tensor in nature. The low energy limit of the string theory leads to the Einstein gravity, coupled non-minimally to a scalar dilaton field [18]. When a dilaton is coupled to Einstein-Maxwell theory, it has profound consequences for the black hole solutions. Many attempts to construct exact solutions of Einstein-Maxwell-dilaton (EMd) gravity have been made in the literature. For example, exact solutions of EMd gravity in the absence of a dilaton potential have been constructed in [19, 20]. The dilaton changes the causal structure of the spacetime and leads to curvature singularities at finite radii. These black holes are asymptotically flat. In recent years, non-asymptotically flat black hole spacetimes have received a lot of interest. There are two motivations for exploring non-asymptotically flat nor (A)dS solutions of Einstein gravity. First, these solutions can shed some light on the possible extensions of AdS/CFT correspondence. Indeed, it has been speculated that the linear dilaton spacetimes, which arise as near-horizon limits of dilatonic black holes, might exhibit holography [21]. The second motivation comes from the fact that such solutions may be used to extend the range of validity of methods and tools originally developed for, and tested in the case of, asymptotically flat or asymptotically AdS black holes. Black hole spacetimes which are neither asymptotically flat nor (A)dS have been explored by many authors [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. Thermodynamics of $(n+1)$-dimensional black hole solutions with unusual asymptotics have also been explored [37, 38].

In this paper, we would like to explore thermodynamics of the topological dilaton black holes in higher dimensional spacetimes in the presence of Liouville-type potentials for the dilaton field. The motivation for studying higher dimensional solutions of Einstein gravity originates from string theory, which is a promising approach to quantum gravity. String theory predicts that spacetime
has more than four dimensions. For a while it was thought that the extra spatial dimensions would be of the order of the Planck scale, making a geometric description unreliable, but it has recently been realized that there is a way to make the extra dimensions relatively large and still be unobservable. This is if we live on a three dimensional surface (brane) in a higher dimensional spacetime (bulk) \[^{39, 40}\]. In such a scenario, all gravitational objects such as black holes are higher dimensional.

The outline of this paper is as follows: In Sec. II, we construct a new class of \((n+1)\)-dimensional topological black hole solutions in EMd theory with two liouville type potentials and general dilaton coupling constant, and investigate their properties. In Sec. III, we obtain the conserved and thermodynamics quantities of the \((n+1)\)-dimensional topological black hole solutions and show that these quantities satisfy the first law of thermodynamics. We also investigate the effect of the dilaton field on the thermal stability of the solutions in this section. The last section is devoted to summary and conclusions.

**II. FIELD EQUATIONS AND SOLUTIONS**

The action of \((n+1)\)-dimensional \((n \geq 3)\) Einstein-Maxwell-dilaton gravity can be written \[^{24}\]

\[
S = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left( \mathcal{R} - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) - e^{-4\alpha \Phi/(n-1)} F_{\mu\nu} F^{\mu\nu} \right),
\]

where \(\mathcal{R}\) is the Ricci scalar curvature, \(\Phi\) is the dilaton field and \(V(\Phi)\) is a potential for \(\Phi\). \(\alpha\) is a constant determining the strength of coupling of the scalar and electromagnetic field, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic field tensor and \(A_\mu\) is the electromagnetic potential. The equations of motion can be obtained by varying the action (1) with respect to the gravitational field \(g_{\mu\nu}\), the dilaton field \(\Phi\) and the gauge field \(A_\mu\) which yields the following field equations

\[
\mathcal{R}_{\mu\nu} = \frac{4}{n-1} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) + 2 e^{-4\alpha \Phi/(n-1)} \left( F_{\eta\mu} F^{\eta\nu} - \frac{1}{2(n-1)} g_{\mu\nu} F_{\lambda\eta} F^{\lambda\eta} \right),
\]

\[
\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{\alpha}{2} e^{-4\alpha \Phi/(n-1)} F_{\lambda\eta} F^{\lambda\eta},
\]

\[
\nabla_\mu \left( e^{-4\alpha \Phi/(n-1)} F^{\mu\nu} \right) = 0.
\]

We would like to find topological solutions of the above field equations. The most general such metric can be written in the form

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) h_{ij} dx^i dx^j,
\]
where \( f(r) \) and \( R(r) \) are functions of \( r \) which should be determined, and \( h_{ij} \) is a function of coordinates \( x_i \) which spanned an \((n - 1)\)-dimensional hypersurface with constant scalar curvature \((n - 1)(n - 2)k\). Here \( k \) is a constant and characterizes the hypersurface. Without loss of generality, one can take \( k = 0, 1, -1 \), such that the black hole horizon or cosmological horizon in (5) can be a zero (flat), positive (elliptic) or negative (hyperbolic) constant curvature hypersurface. The Maxwell equation (4) can be integrated immediately to give

\[
F_{tr} = \frac{qe^{2\alpha\Phi/(n-1)}}{(rR)^{n-1}},
\]

where \( q \) is an integration constant related to the electric charge of the black hole. Defining the electric charge via \( Q = \frac{1}{4\pi} \int \exp\left[-4\alpha\Phi/(n - 1)\right]^* F d\Omega \), we get

\[
Q = \frac{q\omega_{n-1}}{4\pi},
\]

where \( \omega_{n-1} \) represents the volume of constant curvature hypersurface described by \( h_{ij}dx^i dx^j \).

Our aim here is to construct exact, \((n + 1)\)-dimensional topological solutions of the EMd gravity with an arbitrary dilaton coupling parameter \( \alpha \). The case in which we find topological solutions of physically interest is to take the dilaton potential of the form

\[
V(\Phi) = 2\Lambda_0 e^{2\zeta_0 \Phi} + 2\Lambda e^{2\zeta \Phi},
\]

where \( \Lambda_0, \Lambda, \zeta_0 \) and \( \zeta \) are constants. This kind of potential was previously investigated by a number of authors both in the context of Friedmann-Robertson-Walker (FRW) scalar field cosmologies \[41\] and EMd black holes (see e.g \[24, 35, 36, 37\]). In order to solve the system of equations (2) and (3) for three unknown functions \( f(r), R(r) \) and \( \Phi(r) \), we make the ansatz

\[
R(r) = e^{2\alpha\Phi/(n-1)}.
\]

Using (9), the Maxwell fields (6) and the metric (5), one can easily show that equations (2) and (3) have solutions of the form

\[
f(r) = -\frac{k(n - 2)(\alpha^2 + 1)^2 b^{-2\gamma} r^{2\gamma}}{(\alpha^2 - 1)(\alpha^2 + n - 2)} - \frac{m}{r^{(n-1)(1-\gamma)-1}} + \frac{2q^2(\alpha^2 + 1)^2 b^{-2(n-2)\gamma}}{(n - 1)(\alpha^2 + n - 2)} r^{2(n-2)(\gamma-1)}
\]  

\[
+ \frac{2\Lambda(\alpha^2 + 1)^2 b^{2\gamma}}{(n - 1)(\alpha^2 - n)} r^{2(1-\gamma)},
\]

\[
\Phi(r) = \frac{(n - 1)\alpha}{2(1 + \alpha^2)} \ln\left(\frac{b}{r}\right),
\]

where \( b \) is an arbitrary constant and \( \gamma = \alpha^2/(\alpha^2 + 1) \). In the above expression, \( m \) appears as an integration constant and is related to the ADM (Arnowitt-Deser-Misner) mass of the black hole.
According to the definition of mass due to Abbott and Deser \cite{42} (see also \cite{43}), the mass of the solution (10) is

\[ M = \frac{b^{(n-1)\gamma}(n-1)\omega_{n-1}}{16\pi(a^2 + 1)}m. \]  

(12)

In order to fully satisfy the system of equations, we must have

\[ \zeta_0 = \frac{2}{\alpha(n-1)}, \quad \zeta = \frac{2\alpha}{n-1}, \quad \Lambda_0 = \frac{k(n-1)(n-2)a^2}{2b^2(a^2 - 1)}. \]  

(13)

Notice that here \( \Lambda \) is a free parameter which plays the role of the cosmological constant. For later convenience, we redefine it as \( \Lambda = -\frac{n(n-1)}{2l^2} \), where \( l \) is a constant with dimension of length. One may note that in the absence of a non-trivial dilaton (\( \alpha = \gamma = 0 \)), the solution (10) reduces to

\[ f(r) = k - \frac{m}{r^{n-2}} + \frac{2\gamma^2}{(n-1)(n-2)r^{2(n-2)}} - \frac{2\Lambda}{n(n-1)}r^2, \]  

(14)

which describes an \((n+1)\)-dimensional asymptotically AdS topological black hole with a positive, zero or negative constant curvature hypersurface (see for example \cite{11,12}).

Next we study the physical properties of these solutions. To do this, we first look for the curvature singularities. In the presence of dilaton field, the Kretschmann scalar \( R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} \) diverges at \( r = 0 \), it is finite for \( r \neq 0 \) and goes to zero as \( r \to \infty \). Thus, there is an essential singularity located at \( r = 0 \). It is notable to mention that in the \( k = \pm 1 \) cases these solutions does not exist for the string case where \( \alpha = 1 \). As one can see from Eq. (10), the solution is ill-defined for \( \alpha = \sqrt{n} \). The cases with \( \alpha < \sqrt{n} \) and \( \alpha > \sqrt{n} \) should be considered separately. In the first case where \( \alpha < \sqrt{n} \), there exist a cosmological horizon for \( \Lambda > 0 \), while there is no cosmological horizons if \( \Lambda < 0 \) (see fig. \( \Pi \)). Indeed, in the latter case (\( \alpha < \sqrt{n} \) and \( \Lambda < 0 \)) the spacetimes associated with the solution (10) exhibit a variety of possible casual structures depending on the values of the metric parameters \( \alpha, m, q \) and \( k \) (see figs. 2-3). For simplicity in these figures, we kept fixed the other parameters \( l = b = q = 1 \). These figures show that our solutions can represent topological black hole, with inner and outer event horizons, an extreme topological black hole or a naked singularity provided the parameters of the solutions are chosen suitably. In the second case where \( \alpha > \sqrt{n} \), the spacetime has a cosmological horizon for \( \Lambda < 0 \) despite the value of curvature constant \( k \), while for \( \Lambda > 0 \) we have cosmological horizon in the case \( k = 1 \) and naked singularity for \( k = 0, -1 \). One can obtain the casual structure by finding the roots of \( f(r) = 0 \). Unfortunately, because of the nature of the exponent in (10), it is not possible to find analytically the location of the horizons. To have further understanding on the nature of the horizons, we plot in figures 4-7, the mass parameter \( m \) as a function of the horizon radius for different value of dilaton coupling.
FIG. 1: The function $f(r)$ versus $r$ for $k = -1$, $n = 4$, $\alpha = 0.5$ and $m = 2$. $\Lambda = +6$ (bold line), $\Lambda = -6$ (dashed line).

FIG. 2: The function $f(r)$ versus $r$ for $\Lambda = -6$, $n = 4$, $\alpha = 0.67$ and $m = 2$. $k = 1$ (bold line), $k = 0$ (continuous line) and $k = -1$ (dashed line).

constant $\alpha$ and curvature constant $k$. Again, we have fixed $l = b = q = 1$, for simplicity. It is easy to show that the mass parameter $m$ of the topological black hole can be expressed in terms of the horizon radius $r_h$ as

$$m(r_h) = k(n-2)(\alpha^2+1)^2b^{-2\gamma}\frac{r_h^{n-2+\gamma(3-n)}}{(\alpha^2-1)(n+\alpha^2-2)} + \frac{2\Lambda(\alpha^2+1)^2b^{2\gamma}}{(n-1)(\alpha^2-n)}r_h^{n(1-\gamma)-\gamma} + \frac{2q^2(\alpha^2+1)^2b^{-2\gamma(n-2)}}{(n-1)(n+\alpha^2-2)}r_h^{(n-3)(\gamma-1)-1}.$$  \hfill (15)

These figures show that for a given value of $\alpha$, the number of horizons depend on the choice of the value of the mass parameter $m$. We see that, up to a certain value of the mass parameter $m$, there are two horizons, and as we decrease the $m$ further, the two horizons meet. In this case we get extremal black hole with mass $m_{ext}$. These figures also show that for $k = 0, 1$, with increasing $\alpha$, the $m_{ext}$ also increases, while for $k = -1$ it decreases as $\alpha$ increases. Besides figure[7] shows that
FIG. 3: The function $f(r)$ versus $r$ for $\Lambda = -6$, $n = 4$, $m = 2$ and $k = 0$. $\alpha = 0.4$ (bold line), $\alpha = 0.67$ (continuous line) and $\alpha = 0.8$ (dashed line).

FIG. 4: The function $m(r_h)$ versus $r_h$ for $\Lambda = -6$, $n = 4$ and $k = 0$. $\alpha = 0$ (bold line), $\alpha = 0.5$ (continuous line) and $\alpha = 0.6$ (dashed line).

FIG. 5: The function $m(r_h)$ versus $r_h$ for $\Lambda = -6$, $n = 4$ and $k = 1$. $\alpha = 0$ (bold line), $\alpha = 0.5$ (continuous line) and $\alpha = 0.6$ (dashed line).
FIG. 6: The function $m(r_h)$ versus $r_h$ for $\Lambda = -6$, $n = 4$ and $k = -1$. $\alpha = 0$ (bold line), $\alpha = 0.5$ (continuous line) and $\alpha = 0.6$ (dashed line).

FIG. 7: The function $m(r_h)$ versus $r_h$ for $\Lambda = -6$, $\alpha = 0.5$ and $n = 4$. $k = 1$ (bold line), $k = 0$ (continuous line) and $k = -1$ (dashed line).

for fixed value of the other parameters, $m_{\text{ext}}$ decreases with decreasing the constant curvature $k$. Numerical calculations show that when we have extremal topological black hole, the temperature of the black hole vanishes. The Hawking temperature of the topological black hole on the outer horizon $r_+$ can be calculated using the relation

$$T_+ = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi},$$

where $\kappa$ is the surface gravity. Then, one can easily show that

$$T_+ = -\frac{(\alpha^2 + 1)}{2\pi(n - 1)} \left( \frac{k(n - 2)(n - 1)b^{-2\gamma}}{2(\alpha^2 - 1)} r_+^{2\gamma - 1} + \Lambda b^{2\gamma} r_+^{1 - 2\gamma} + q^2 b^{-2(n - 2)\gamma} r_+^{(2n - 3)(\gamma - 1) - \gamma} \right)$$

$$= -\frac{k(n - 2)(\alpha^2 + 1)b^{-2\gamma}}{2\pi(\alpha^2 + n - 2)} r_+^{2\gamma - 1} + \frac{(n - \alpha^2)m}{4\pi(\alpha^2 + 1)} r_+^{(n - 1)(\gamma - 1)}$$

$$- \frac{q^2(\alpha^2 + 1)b^{-2(n - 2)\gamma}}{\pi(\alpha^2 + n - 2)} r_+^{(2n - 3)(\gamma - 1) - \gamma}. $$

(17)
Equation (17) shows that when \( k = 0 \), the temperature is negative for the two cases of (i) \( \alpha > \sqrt{n} \) despite the sign of \( \Lambda \), and (ii) positive \( \Lambda \) despite the value of \( \alpha \). As we argued above in these two cases we encounter with cosmological horizons, and therefore the cosmological horizons have negative temperature. Numerical calculations show that the temperature of the event horizon goes to zero as the black hole approaches the extreme case. It is a matter of calculation to show that

\[
m_{\text{ext}} = \frac{2k(n - 2)(\alpha^2 + 1)^2b^{-2\gamma}}{(n - 2\alpha^2 + n - 2)(2-n)(\gamma-1)+\gamma} + \frac{4q^2(\alpha^2 + 1)^2b^{2(2-n)\gamma}}{(n - 2\alpha^2 + n - 2)(2-n)(1-\gamma)-1}.
\]

(18)

In summary, the metric of Eqs. (5) and (10) can represent a topological black hole with inner and outer event horizons located at \( r_- \) and \( r_+ \), provided \( m > m_{\text{ext}} \), an extreme topological black hole in the case of \( m = m_{\text{ext}} \), and a naked singularity if \( m < m_{\text{ext}} \). It is worth noting that in the absence of a non-trivial dilaton field \( \alpha = \gamma = 0 \), expressions (17) and (18) reduce to that of an \( (n + 1) \)-dimensional asymptotically AdS topological black hole [11, 12].

III. THERMODYNAMICS OF TOPOLOGICAL BLACK HOLE

In this section we are going to explore thermodynamics of the topological dilaton black hole we have just found. The entropy of the topological black hole typically satisfies the so called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area [44]. This near universal law applies to almost all kinds of black holes, including dilaton black holes, in Einstein gravity [45]. It is a matter of calculation to show that the entropy of the topological black hole is

\[
S = \frac{b^{(n-1)\gamma} \omega_{n-1} r_+^{(n-1)(1-\gamma)}}{4}.
\]

(19)

The electric potential \( U \), measured at infinity with respect to the horizon, is defined by

\[
U = A_\mu \chi^\mu \Big|_{r=\infty} - A_\mu \chi^\mu \big|_{r=r_+},
\]

(20)

where \( \chi = \partial_t \) is the null generator of the horizon. One can easily show that the gauge potential \( A_t \) corresponding to the electromagnetic field (6) can be written as

\[
A_t = \frac{q b^{(3-n)\gamma}}{\Upsilon r^\gamma},
\]

(21)

where \( \Upsilon = (n - 3)(1 - \gamma) + 1 \). Therefore, the electric potential may be obtained as

\[
U = \frac{q b^{(3-n)\gamma}}{\Upsilon r_+^\gamma}.
\]

(22)
Then, we consider the first law of thermodynamics for the topological black hole. In order to do this, we obtain the mass $M$ as a function of extensive quantities $S$ and $Q$. Using the expression for the charge, the mass and the entropy given in Eqs. (7), (12) and (19) and the fact that $f(r_+) = 0$, one can obtain a Smarr-type formula as

$$M(S, Q) = -\frac{k(n-1)(n-2)(\alpha^2 + 1)b^{-\alpha^2}}{16\pi(\alpha^2 - 1)(\alpha^2 + n - 2)} (4S)^{\frac{\alpha^2 + n - 2}{n-1}} + \frac{\Lambda}{8\pi} \frac{(\alpha^2 + 1)b^{\alpha^2}}{(\alpha^2 - n)(4S)^{\frac{n-\alpha^2}{n-1}}}$$

$$+ \frac{2\pi Q^2(\alpha^2 + 1)b^{\alpha^2}}{\alpha^2 + n - 2} (4S)^{\frac{\alpha^2 + n - 2}{1-n}}.$$ 

(23)

One may then regard the parameters $S$, and $Q$ as a complete set of extensive parameters for the mass $M(S, Q)$ and define the intensive parameters conjugate to $S$ and $Q$. These quantities are the temperature and the electric potential

$$T = \left( \frac{\partial M}{\partial S} \right)_Q, \quad U = \left( \frac{\partial M}{\partial Q} \right)_S.$$ 

(24)

Numerical calculations show that the intensive quantities calculated by Eq. (24) coincide with Eqs. (17) and (22). Thus, these thermodynamics quantities satisfy the first law of thermodynamics

$$dM = TdS + UdQ.$$ 

(25)

Finally, we study thermal stability of the topological dilaton black hole. The stability of a thermodynamic system with respect to small variations of the thermodynamic coordinates is usually performed by analyzing the behavior of the entropy $S(M, Q)$ around the equilibrium. The local stability in any ensemble requires that $S(M, Q)$ be a convex function of the extensive variables or its Legendre transformation must be a concave function of the intensive variables. The stability
FIG. 9: $(\partial^2 M/\partial S^2)_{Q}$ versus $q$ for $l = b = 1$, $r_+ = 0.8$, $n = 5$ and $k = 1$. $\alpha = 0.8$ (bold line), $\alpha = 1.2$ (continuous line) and $\alpha = \sqrt{2}$ (dashed line).

FIG. 10: $(\partial^2 M/\partial S^2)_{Q}$ versus $\alpha$ for $l = b = 1$, $r_+ = 0.8$, $n = 5$, and $k = 0$. $q = 0.5$ (bold line), $q = 1$ (continuous line), and $q = 1.5$ (dashed line).

FIG. 11: $(\partial^2 M/\partial S^2)_{Q}$ versus $q$ for $l = b = 1$, $r_+ = 0.8$, $n = 5$ and $k = 0$. $\alpha = 0.8$ (bold line), $\alpha = 1.2$ (continuous line) and $\alpha = \sqrt{2}$ (dashed line).
FIG. 12: $(\partial^2 M/\partial S^2)_Q$ versus $\alpha$ for $l = b = 1$, $r_+ = 0.8$, $n = 5$ and $k = -1$. $q = 0.5$ (bold line), $q = 1$ (continuous line), and $q = 1.5$ (dashed line).

FIG. 13: $(\partial^2 M/\partial S^2)_Q$ versus $q$ for $l = b = 1$, $r_+ = 0.8$, $n = 5$ and $k = -1$. $\alpha = 0.8$ (bold line), $\alpha = 1.2$ (continuous line) and $\alpha = \sqrt{2}$ (dashed line).

FIG. 14: $(\partial^2 M/\partial S^2)_Q$ versus $\alpha$ for $l = b = 1$, $r_+ = 0.8$, $q = 0.4$ and $n = 5$. $k = 1$ (bold line), $k = 0$ (continuous line), and $k = -1$ (dashed line).
FIG. 15: \( (\partial^2 M/\partial S^2)_Q \) versus \( q \) for \( l = b = 1, r_+ = 0.8, \alpha = \sqrt{2} \) and \( n = 5 \). \( k = 1 \) (bold line), \( k = 0 \) (continuous line), and \( k = -1 \) (dashed line).

can also be studied by the behavior of the energy \( M(S, Q) \) which should be a convex function of its extensive variable. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of \( M(S, Q) \) with respect to its extensive variables \([46, 47]\). In our case the mass \( M \) is a function of entropy and charge. The number of thermodynamic variables depends on the ensemble that is used. In the canonical ensemble, the charge is a fixed parameter and therefore the positivity of the \( (\partial^2 M/\partial S^2)_Q \) is sufficient to ensure local stability. Numerical calculations show that the topological black hole solutions are stable independent of the value of the charge and curvature constant parameters \( q \) and \( k \) in any dimensions if \( \alpha < \alpha_{\text{max}} \), while for \( \alpha > \alpha_{\text{max}} \) the system has an unstable phase. It is notable to mention that for \( k = 0, -1 \) we have \( \alpha_{\text{max}} \geq 1 \). On the other hand, there is always a lower limit for the electric charge, \( q_{\text{min}} \), for which the system is thermally stable provided \( q > q_{\text{min}} \) (see figs. 8-13). It is worth noting that \( \alpha_{\text{max}} \) and \( q_{\text{min}} \) depend on the dimensionality of the spacetime and the metric parameters \( m \) and \( l \). In figures 14 and 15 we plot \( (\partial^2 M/\partial S^2)_Q \) versus dilaton coupling constant \( \alpha \) and charge parameter \( q \) for different value of the curvature constant \( k \). These figures show that for fixed value of the other parameters, as we decrease the constant curvature \( k \), the value of \( \alpha_{\text{max}} \) increases while in contrast, \( q_{\text{min}} \) decreases.

IV. SUMMARY AND CONCLUSIONS

In \((n + 1)\)-dimensional spacetime, when the \((n - 1)\)-sphere of black hole event horizons is replaced by an \((n - 1)\)-dimensional hypersurface with positive, zero or negative constant curvature, the black hole is referred to as a topological black hole. In this paper, first we obtained a new
class of \((n + 1)\)-dimensional \((n \geq 3)\) topological black hole solutions in Einstein-Maxwell-dilaton gravity with Liouville-type potentials for the dilaton field. Then, we explored thermodynamics of these topological dilaton black holes and disclosed the effect of the dilaton field on the stability of the solutions. In contrast to the topological black holes in the Einstein-Maxwell theory, which are asymptotically AdS, the topological dilaton black holes we found here, are neither asymptotically flat nor (A)dS. Indeed, the Liouville-type potentials (the negative effective cosmological constant) plays a crucial role in the existence of these black hole solutions, as the negative cosmological constant does in the Einstein-Maxwell theory. In the \(k = \pm 1\) cases, these solutions does not exist for the string case where \(\alpha = 1\). In the presence of Liouville-type potential, we obtained exact solutions provided \(\alpha \neq \sqrt{n}\). In the absence of a dilaton field (\(\alpha = \gamma = 0\)), our solutions reduce to the \((n+1)\)-dimensional topological black hole solutions presented in [12]. We showed that our solutions can represent topological black hole with inner and outer event horizons, an extreme topological black hole or a naked singularity provided the parameters of the solutions are chosen suitably. We also computed the charge, mass, temperature, entropy and electric potential of the topological dilaton black holes and found that these quantities satisfy the first law of thermodynamics. We analyzed the thermal stability of the solutions in the canonical ensemble by finding a Smarr-type formula and considering \((\partial^2 M/\partial S^2)_Q\) for the charged topological dilaton black hole solutions in \((n + 1)\) dimensions. We showed that there is no Hawking-Page phase transition in spite of charge of the topological black hole provided \(\alpha \leq \alpha_{\text{max}}\), while the solutions have an unstable phase for \(\alpha > \alpha_{\text{max}}\). It is worth noting that for \(k = 0, -1\) we have \(\alpha_{\text{max}} \geq 1\). We found that there is always a low limit for the electric charge, \(q_{\text{min}}\), for which the solutions are stable provided \(q > q_{\text{min}}\).

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