ENDS OF RIEMANNIAN MANIFOLDS
WITH NONNEGATIVE RICCI CURVATURE
OUTSIDE A COMPACT SET

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ABSTRACT. We consider complete manifolds with Ricci curvature nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number.

1. INTRODUCTION

Toponogov [T] showed that in a complete manifold of nonnegative sectional curvature, a line splits off isometrically, i.e. any nonnegatively curved $M^n$ is isometric to a Riemannian product $N^k \times \mathbb{R}^{n-k}$, where $N^k$ does not contain a line. Later, Cheeger and Gromoll [CG] generalized this to manifolds of nonnegative Ricci curvature, known as the Cheeger-Gromoll splitting theorem. As a consequence, such a manifold has at most two ends (see §2 for the definition of an end). In [A], Abresch studied manifolds with asymptotically nonnegative sectional curvature. He showed that the number of ends of such a manifold is finite and can be estimated from above explicitly. In this note, we consider manifolds with Ricci curvature being nonnegative outside a compact set and prove that the number of ends of such a manifold is finite and in particular, we give an explicit upper bound for the number. That is, we prove the following theorem.

Theorem. Let $(M^n, o)$ be a Riemannian manifold with base point $o$. If the Ricci curvature is nonnegative outside the geodesic ball $B(o, a)$ of radius $a$ and is bounded from below on $B(o, a)$ by $-(n-1)\Lambda^2$ (for $\Lambda \geq 0$), then there exists a universal bound on the number of ends, e.g.

$$\text{the number of ends of } M^n \leq \frac{2n}{n-1} (\Lambda a)^{-n} \exp \left( \frac{17(n-1)}{2} \Lambda a \right).$$

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We learned that P. Li and L. F. Tam proved a similar theorem as an application of the theory of harmonic functions on a complete manifold. Our approach here is more geometrical. A previous version of the Theorem, under the additional condition of a lower bound on the sectional curvature, was proved by Z. Liu. After reading a preliminary version of our paper, Z. Liu informed us that he could also modify his proof, using ideas from this paper, to prove the same theorem as above (see [LT, L]).

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2. IDEA OF THE PROOF OF THE THEOREM

In what follows, we always let $M^n$ be a manifold as in the Theorem.

There are various (but equivalent) definitions of an end of a manifold (cf. [A]), for the sake of our argument, we use the following definition.

**Definition 2.1.** Two rays $\gamma_1$ and $\gamma_2$ starting at the base point $o$ are called cofinal if for any $r > 0$ and any $t \geq r$, $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same component of $M - B(o, r)$. An equivalence class of cofinal rays is called an end of $M$. We will use $[\gamma]$ to denote the class of the ray $\gamma$.

The following proposition is a key to the proof of the theorem.

**Proposition 2.2.** Let $M^n$ be as in the theorem, $[\gamma_1]$ and $[\gamma_2]$ be two different ends of $M^n$, then $d(\gamma_1(4a), \gamma_2(4a)) > 2a$.

Proposition 2.2 will be proved in §3. Assuming it, we now give a proof of the theorem.

**Proof of the theorem.** Let $k$ be an integer and $\gamma_1, \ldots, \gamma_k$ be rays from the base point $o$ going to $k$ different ends. We need to bound $k$ from above. Consider the sphere $S(o, 4a)$ of radius $4a$. Let $\{p_j\}$ be a maximal set of points on $S(o, 4a)$ such that the balls $B(p_j, \frac{1}{2}a)$ are disjoint. Clearly, the balls $B(p_j, a)$ cover $S(o, 4a)$, and since the set $\{\gamma_i(4a), i = 1, \ldots, k\}$ is contained in $S(o, 4a)$, each $\gamma_i(4a)$ is contained in some $B(p_j, a)$. But each ball $B(p_j, a)$ contains at most one $\gamma_i(4a)$ by the Proposition 2.2,
and hence the number of balls is not less than $k$. Thus it suffices to bound the number of balls $B(p_j, \frac{1}{2}a)$.

Notice that

$$B(p_j, \frac{1}{2}a) \subset B(o, \frac{9}{2}a) \subset B(p_j, \frac{17}{2}a).$$

It follows from the Bishop-Gromov volume comparison theorem that

$$\text{vol} B(p_j, \frac{17}{2}a) \leq \frac{17a/2}{1} \frac{\sinh^{n-1} \Lambda t}{\sinh^{n-1} \Lambda t} \text{vol} B(p_j, \frac{1}{2}a).$$

Therefore, the number of balls $B(p_j, \frac{1}{2}a)$ is no more than

$$\frac{17}{2} \frac{\sinh^{n-1} \Lambda t}{\sinh^{n-1} \Lambda t}.$$

Since

$$\frac{\int_0^{17a/2} \sinh^{n-1} \Lambda t \, dt}{\int_0^{1a/2} \sinh^{n-1} \Lambda t \, dt} \leq \frac{2n}{n-1} \frac{17(n-1)}{2} \Lambda a,$$

the theorem follows.

**Remark 2.3.** The bound for the number of ends given here is far from being sharp. An improved bound can be obtained from a more general volume comparison theorem which we can state as follows (for definitions involved, one is referred to [AG]):

**A volume comparison theorem.** Let $M^n$ be an asymptotically non-negatively Ricci curved manifold. Then for any $p \in M^n$ and for every $0 \leq r \leq R$,

$$\frac{\text{vol} B(p, R)}{\text{vol} B(p, r)} \leq w_n \left( \frac{R}{r} \right)^n,$$

where $w_n = (1 + 2u(0)d(o, p))^{n-1} 2^{2n} \exp(6(n-1)C_1)$.

Moreover, if $0 \leq r \leq R \leq d(o, p)$ or $2d(o, p) \leq r \leq R$, $w_n$ can be chosen as $2^{2n} \exp(6(n-1)C_1)$ (see [AG] for the definitions of $u(0)$ and $C_1$).

The proof of this theorem will appear elsewhere.

**Proof of Proposition 2.2.** Let $M$ be a manifold as in the theorem.
For each ray $\gamma$, there is an associated function called the Busemann function, which is defined as follows:

$$b_\gamma(x) = \lim_{t \to \infty} (t - d(x, \gamma(t))).$$

For any given point $p$, let $\alpha_t$ be a minimizing geodesic from $p$ to $\gamma(t)$. As $t \to \infty$, $\alpha_t$ has a convergent subsequence which converges to a ray at $p$. Such a ray is called an asymptotic ray to $\gamma$ at $p$.

Let $\gamma$ be a line. We define $\gamma^+ : [0, \infty) \to M$ by $\gamma^+(t) = \gamma(t)$ and $\gamma^- : [0, \infty) \to M$ by $\gamma^-(t) = \gamma(-t)$.

Let $b_\gamma^+$ ( $b_\gamma^-$, resp.) be the associated Busemann function of $\gamma^+$ ( $\gamma^-$, resp.).

In [EH], J. Eschenburg and E. Heintze showed, under the assumption that the Ricci curvature is nonnegative everywhere, that $b_\gamma^+$ are smooth harmonic functions with $\text{Hess } b_\gamma^+ = 0$ and $b_\gamma^+ + b_\gamma^- = 0$. Applying their arguments locally, we can show the following lemma.

**Lemma 3.1.** Let $N$ be the $\delta$-tubular neighborhood of $\gamma$. Suppose that from every point $p$ in $N$, there is an asymptotic ray to $\gamma^+$ and an asymptotic ray to $\gamma^-$ such that the Ricci curvature is nonnegative on both asymptotic rays. Then through every point in $N$, there is a line $\alpha$ which, when parametrized properly, satisfies

$$b_\gamma^+(\alpha^+(t)) = t \quad \text{and} \quad b_\gamma^-(\alpha^-(t)) = t.$$

**Proof.** Let $p$ be any point in $N$. Applying arguments as in the proof of Lemma 3 in [EH], we find that at $p$, $b_\gamma^+ + b_\gamma^- = 0$, and $b_\gamma^\pm$ are $C^1$ smooth with $||\text{grad } b_\gamma^\pm|| = 1$. Hence the asymptotes to $\gamma^\pm$ are uniquely determined at $p$ and fit together to a line, say, $\gamma_p$. Arguments as in the proof of Lemma 2 together with the concluding remarks in [EH] imply that $b_\gamma^+$ ( $b_\gamma^-$, resp.) is actually $C^\infty$ smooth with $\text{Hess } b_\gamma^\pm = 0$ on $\gamma_p$. Thus the restriction of $b_\gamma^\pm$ to $\gamma_p$ must be a linear function with derivative 1. After a reparametrization of $\gamma_p$, Lemma 3.1 then follows.

**Remark 3.2.** The same argument as in [EH] of course also implies a local splitting for the metric in $N$, under the assumptions of Lemma 3.1.

**Lemma 3.3.** $M^n$ cannot admit a line $\gamma$ with the following property:

(I) \[ d(\gamma(t), B(o, a)) \geq |t| + 2a \quad \text{for all } t. \]
Proof. Suppose there were such a line $\gamma$. Consider the $a$-tubular neighborhood of $\gamma$. We claim that from any point $p$ in this neighborhood, all its asymptotic rays to $\gamma^+$ (or $\gamma^-$) are away from $B(o, a)$, in particular, the Ricci curvature is nonnegative on such a ray. In fact, let $s$ be such that $d(p, \gamma(s)) < a$, then,

$$d(p, \gamma^\pm(t)) \leq d(p, \gamma(s)) + d(\gamma(s), \gamma^\pm(t)) = d(p, \gamma(s)) + d(\gamma(s), \gamma(\pm t)) \leq a + \vert s \vert + t$$

but any curve from $p$ to $\gamma^\pm(t)$ passing through $B(o, a)$ has length

$$l \geq d(p, B(o, a)) + d(\gamma^\pm(t), B(o, a)) \geq d(\gamma(s), B(o, a)) + d(\gamma(\pm t), B(o, a)) - a \geq \vert s \vert + t + 3a$$

the last inequality follows from the property (I). Clearly, this implies that any minimizing geodesic, say, $\alpha_t$ from $p$ to $\gamma^\pm(t)$ does not pass through $B(o, a)$. Hence any convergent subsequence of $\alpha_t$ will converge to a ray which is away from $B(o, a)$. This proves the claim.

Next, we claim that through every point of the $\alpha$-tubular neighborhood of $\gamma$, there exists a line with the property (I). Indeed, it follows from the above claim and Lemma 3.1 that through every point of the $\alpha$-tubular neighborhood of $\gamma$, there is a line $\beta$ such that

$$b^-_\gamma(\beta^+(t)) = t \quad \text{and} \quad b^-_\gamma(\beta^-(t)) = t.$$ We need to show that $\beta$ also has the property (I), i.e.

$$d(\beta(t), B(o, a)) \geq \vert t \vert + 2a \quad \text{for all} \quad t.$$ By symmetry, we may assume that $t \geq 0$. Then for any $r \geq 0$,

$$d(\beta(t), B(o, a)) \geq d(\gamma(r), B(o, a)) - d(\beta(t), \gamma(r)) \geq r - d(\beta(t), \gamma(r)) + 2a$$

(here we used the property (I) for $\gamma$). Letting $r \rightarrow \infty$ in the above inequality, we have

$$d(\beta(t), B(o, a)) \geq b^+_\gamma(\beta(t)) + 2a = t + 2a.$$ Now let $\alpha(t) : [0, d] \rightarrow M$ be a minimizing geodesic from $\gamma(0)$ to $o$, then there is a partition of the interval $[0, d]$: $t_0 = 0 < t_1 < \cdots < t_k = d$ such that $d(\alpha(t_i), \alpha(t_{i+1})) < a$.
The last claim implies that there is a line through $\alpha(t_1)$ with the property (I). Continuing this process inductively, we would find a line with the property (I) through $\alpha(t_k)$, the base point $o$, which is absurd.

We are now in the position to prove Proposition 2.2.

**Proof of Proposition 2.2.** Suppose the contrary. That is, $d(\gamma_1(4a), \gamma_2(4a)) \leq 2a$. Since $[\gamma_1]$ and $[\gamma_2]$ are different ends, there exists an $A > 4a$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are in different unbounded components of $M - B(o, A)$ for all $t > A$. Let $C_t (t > A)$ be a minimizing geodesic joining $\gamma_1(t)$ and $\gamma_2(t)$. Then $C_t$ must pass through $B(o, A)$. In addition, we claim that the middle point $m_t$ of $C_t$ is in the ball $B(o, 2A)$. As a matter of fact, let $p$ be a point in $C_t \cap B(o, A)$ and without loss of generality we may assume that $d(p, \gamma_1(t)) \leq d(p, \gamma_2(t))$, then

$$d(o, m_t) \leq d(o, p) + d(p, m_t)$$

$$\leq A + \frac{1}{2} \rho_t - d(p, \gamma_1(t))$$

$$\leq A + \frac{1}{2} \rho_t - (t - A)$$

where $\rho_t$ is the length of $C_t$. Notice that

$$\rho_t = d(\gamma_1(t), \gamma_2(t))$$

$$\leq d(\gamma_1(t), \gamma_1(4a)) + d(\gamma_1(4a), \gamma_2(4a)) + d(\gamma_2(4a), \gamma_2(t))$$

$$\leq 2(t - 4a) + 2a = 2t - 6a.$$ 

Hence,

$$d(o, m_t) \leq A + \frac{1}{2}(2t - 6a) - (t - A)$$

$$= 2A - 3a.$$ 

This shows that $m_t$ is in the ball $B(o, 2A)$.

Now we reparametrize $C_t$ by translating the origin and with abuse of notation we still denote it by $C_t$ such that

$$C_t(-\frac{1}{2} \rho_t) = \gamma_1(t), \quad C_t(0) = m_t, \quad C_t(\frac{1}{2} \rho_t) = \gamma_2(t).$$

We claim that $C_t(s)$ satisfies property (I) for $-\frac{1}{2} \rho_t \leq s \leq \frac{1}{2} \rho_t$.

In fact, for any $s$ (we may assume $s \geq 0$),

$$d(C_t(s), B(o, a)) \geq d(C_t(\frac{1}{2} \rho_t), B(o, a)) - (\frac{1}{2} \rho_t - s)$$

$$\geq (t - a) - (t - 3a) + s$$

$$= s + 2a$$

where we used the fact $\rho_t \leq 2t - 6a$. Since $C_t(0) \in B(o, 2A)$ for all $t \geq A$, when $t \to \infty$, a subsequence of $C_t$ converges to a line $\gamma(s)$ with the property (I) for all $s$. (Notice that $\rho_t \to \infty$, as $t \to \infty$). This is a contradiction by Lemma 3.3.
REFERENCES

[A] U. Abresch, Lower curvature bounds, Toponogov's Theorem and bounded topology, Ann. Sci. ÉcoleNorm. Sup. Paris 28 (1985), 665–670.

[AG] U. Abresch and D. Gromoll, On complete manifolds with nonnegative Ricci curvature, J. Amer. Math. Soc. (to appear).

[CG] J. Cheeger and D. Gromoll, The Splitting Theorem for manifolds of non-negative Ricci curvature, J. Differential Geom. 6 (1971), 119–128.

[EH] J. Eschenburg and E. Heintze, An elementary proof of the Cheeger-Gromoll Splitting Theorem, Ann. Global Anal. Geom. 2 (1984), 141–151.

[L] Z. Liu., Ball covering on manifolds with nonnegative Ricci curvature near infinity, SUNY at Stony Brook, preprint, 1990.

[LT] P. Li and L. F. Tam, Harmonic functions and the structure of complete manifolds, University of Arizona, preprint, 1990.

[T] V. A. Toponogov, Riemannian spaces which contain straight lines, Amer. Math. Soc. Transl. (2) 37 (1964), 287–290.

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