We use 2-color QCD as a model to study the effects of simultaneous presence of the so-called $\theta$ parameter, chemical potentials for baryon number, $\mu_B$ and for isospin charge, $\mu_I$. We pay special attention to $\theta$, $\mu_B$, $\mu_I$ dependence of different vacuum condensates, including chiral and diquark condensates, as well as the gluon condensate, $\langle \frac{2g^2}{32\pi^2} G_{\mu\nu} G^{\mu\nu} \rangle$, and the topological susceptibility. We find that two phase transitions of the second order will occur when $\theta$ relaxes from $\theta = 2\pi$ to $\theta = 0$, if $\mu$ is of order of the pion mass, $m_\pi$. We demonstrate that the transition to the superfluid phase at $\theta = \pi$ occurs at a much lower chemical potential than at $\theta = 0$. We also show that the strong $\theta$ dependence present near $\theta = \pi$ in vacuum (Dashen’s phenomenon), becomes smoothed out in the superfluid phase. Finally, we comment on the relevance of this study for the real world with $N_c = 3$.

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I. MOTIVATION

In this paper we investigate the behavior of 2-color QCD under the influence of three parameters: $\theta$, $\mu_B$ and $\mu_I$. The main motivation for such a study is, of course, the attempt to understand the cosmological phase transition when $\theta$, being non-zero and large at the very beginning of the phase transition, slowly relaxes to zero, as the axion resolution of the strong CP problem suggests. Therefore, the universe may undergo many QCD phase transitions when $\theta$ relaxes to zero. Another motivation is the attempt to understand the complicated phase diagram of QCD as a function of external parameters $\theta$, $\mu_B$ and $\mu_I$. Finally, our study may be of interests for the lattice community – the determinant of the Dirac operator for $N_c = 2$ is real when $\theta = \pi$ in the presence of nonzero $\mu$. As we show, in this case the superfluid phase is realized at a much lower chemical potential than at $\theta = 0$. This gives a unique chance to study the superfluid phase on the lattice at a much smaller $\mu$ than would normally be required.

To study all these problems in real 3 color QCD at finite $\mu_B$ is, of course, a very difficult task. To get some insight into what might happen we shall use a controlled analytical method to study these questions in the non-physical (but nevertheless, very suggestive) $N_c = 2$ theory. We use the chiral effective Lagrangian approach to attack the problem. We shall determine the phase diagram in the $\mu_B, \mu_I, \theta$ planes, various condensates and lowest lying excitations. We expect that our approach is valid as long as all external parameters
and the quark mass, \( m_q \), are much smaller than \( \Lambda_{QCD} \). We perform most of our calculations for the case of two flavors \( N_f = 2 \) where the algebra simplifies considerably.

One exciting effect that we find is that \( \theta \) dependence of the theory at fixed \( \mu \) may become non-analytic. This is due to the fact that the critical chemical potential for transition to the superfluid phase varies with \( \theta \). Therefore, a change of \( \theta \) might trigger a second-order phase transition, accompanied by a discontinuity in the topological susceptibility \( \chi \). We also find that the strong \( \theta \) dependence, present near \( \theta = \pi \) in vacuum, is washed out in the superfluid phase. We expect that for equal quark masses a first order phase transition (Dashen’s phenomenon) will occur in the \( N_c = 2, N_f = 2 \) theory at \( \theta = \pi \), in the normal phase, but will disappear in the superfluid phase.

We also find some interesting results, which appear even at \( \theta = 0 \). Most importantly we compute the dependence of the gluon condensate, \( \langle \frac{b_g^2}{32\pi^2} G_{\mu\nu}^a G^{\mu\nu a} \rangle \), on the chemical potential. The gluon condensate decreases with density near the normal to superfluid phase transition, but, counter-intuitively, increases for \( m_\pi \ll \mu \ll \Lambda_{QCD} \).

We also evaluate novel vacuum expectation values which appear in the superfluid phase: \( \langle i\bar{u} \gamma_0 \gamma_5 \tau_2 d \rangle \) in the baryon breaking phase and \( \langle i\bar{u} \gamma_0 \gamma_5 d \rangle \) in the isospin breaking phase. These densities, being nonzero even at \( \theta = 0 \), nonetheless have never been discussed in the literature previously. These densities, themselves, break the baryon and isospin symmetries respectively, and so may be considered as additional order parameters.

The presentation of our results is organized as follows. In section II, we introduce our notations for the low energy effective Lagrangian. In section III, we introduce the \( \theta \) parameter into the effective Lagrangian description. In section IV, we discuss the phase diagram of our theory in detail, computing the spectrum of lowest lying excitations, characterizing the phases in terms of chiral condensates and densities and paying special attention to physics near the point \( \theta = \pi \). In section V, we check that our results satisfy known Ward Identities supporting the self consistency of our approach. In section VI, we study the gluon condensate \( \langle \frac{b_g^2}{32\pi^2} G_{\mu\nu}^a G^{\mu\nu a} \rangle \) as a function of \( \mu \) and \( \theta \). In Conclusion, we discuss the relevance of the obtained results for 3 color QCD and make some speculative remarks on evolution of the early universe during the QCD phase transition. In appendix I, we clarify some technical issues associated with global aspects of the goldstone manifold.

II. THE EFFECTIVE THEORY AT FINITE \( \mu_B \) AND \( \mu_I \)

Two color QCD at zero chemical potential is invariant under \( SU(2N_f) \) rotations in the chiral limit. This enhanced symmetry (as compared to the \( SU(N_f) \times SU(N_f) \times U(1) \) of three color QCD) is manifest in the Lagrangian if we choose to represent it in a basis of quarks \( \psi \) and conjugate quarks \( \bar{\psi} \). For \( N_f = 2 \) we use

\[
\Psi \equiv \begin{pmatrix} u \\ d \\ \bar{u} \\ \bar{d} \end{pmatrix} \equiv \begin{pmatrix} u_L \\ d_L \\ \sigma_3 \tau_2 (u_R)^* \\ \sigma_2 \tau_2 (d_R)^* \end{pmatrix}
\]

where the Pauli matrices \( \tau_2 \) and \( \sigma_2 \) act in colour and spin space respectively. We work in Euclidean space and use the definitions, \( \gamma_\nu = \begin{pmatrix} 0 & \sigma_\nu \\ \sigma_\nu & 0 \end{pmatrix} \), \( \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \sigma_\nu = (-i, \sigma_k) \).
The microscopic Lagrangian then reads,
\[ L = i\Psi^\dagger \sigma_\nu D_\nu \Psi \] (2)
and possesses a symmetry,
\[ \Psi \rightarrow U\Psi, \ U \in SU(4) \] (3)

The enhanced symmetry manifests itself in the low energy effective theory through the manifold of goldstone modes associated with spontaneous breaking of chiral symmetry, \( SU(2N_f) \rightarrow Sp(2N_f) \). In our case, \( N_f = 2 \), and the goldstone manifold is \( SU(4)/Sp(4) \), corresponding to the condensation of \( \Psi \Psi \rightarrow SU(4) \) flavor sextet. The fields on this manifold can be represented by a \( 4 \times 4 \) antisymmetric unitary matrix \( \Sigma \), with \( \det \Sigma = 1 \), that transforms under (3) as,
\[ \Sigma \rightarrow U\Sigma U^T \] (4)

We parameterize the vacuum manifold as,
\[ \Sigma = U\Sigma_c U^T, \ U \in SU(4), \ \Sigma_c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \] (5)

In what follows we use notations suggested in [3, 4, 5] for the description of baryonic as well as isospin chemical potentials. In these notations the baryon charge of the quark is \( 1/2 \), which comes from \( 1/N_c \), so that the baryon (diquark in \( N_c = 2 \)) has baryon charge 1. Thus, chemical potentials enter the microscopic Lagrangian as,
\[ L = \bar{\psi}\gamma_\mu D_\nu \psi - \frac{1}{2}\mu_B \bar{\psi}\gamma^0 \psi - \frac{1}{2}\mu_I \bar{\psi}\gamma^0 \sigma^3 \psi \] (6)

In the basis of SU(4) spinors [1, 2, 3, 4, 5], the baryon and isospin (third component) charge matrices in block-diagonal form are [1, 2, 3, 4, 5],
\[ B \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ I \equiv \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \] (7)
so that the Lagrangian reads,
\[ L = i\Psi^\dagger \sigma_\nu D_\nu \Psi - \Psi^\dagger (\mu_B B + \mu_I I) \Psi \] (8)

The effective Lagrangian for the field \( \Sigma \) of goldstone modes is determined by the symmetries inherited from the microscopic two-color QCD Lagrangian. To lowest order in derivatives and at zero quark mass the effective Lagrangian is [5],
\[ \mathcal{L} = \frac{F^2}{2} \text{Tr} \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger \] (9)

The \( \mu \)-dependence enters the effective Lagrangian through the covariant extension of the derivative,
\[ \partial_0 \Sigma \rightarrow \nabla_0 \Sigma = \partial_0 \Sigma - \left[ (\mu_B B + \mu_I I) \Sigma + \Sigma(\mu_B B + \mu_I I)^T \right], \ \nabla_i \Sigma = \partial_i \Sigma \]
\[ \partial_0 \Sigma^\dagger \rightarrow \nabla_0 \Sigma^\dagger = \partial_0 \Sigma^\dagger + \left[ (\mu_B B + \mu_I I) \Sigma + \Sigma(\mu_B B + \mu_I I)^T \right]^\dagger, \ \nabla_i \Sigma = \partial_i \Sigma^\dagger \] (10)
required by an extended local gauge symmetry [1]. Therefore, to this order in chiral perturbation theory, the Lagrangian at finite \( \mu \) does not require any extra phenomenological parameters beyond the pion decay constant, \( F \). This fact gives predictive power to chiral perturbation theory at finite \( \mu \). In using the effective Lagrangian constructed above we must, of course, assume that chiral symmetry for \( N_c = N_f = 2 \) QCD is spontaneously broken. Since we have regarded the hadronic modes as heavy, the theory is expected to be valid only up to the mass of the lightest non-goldstone hadron.
III. THE MASS TERM AND $\theta$ PARAMETER

The mass term in the fundamental Lagrangian is defined as,

$$L_m = m_u \bar{u}u + m_d \bar{d}d$$

while the $\theta$ term in the fundamental Lagrangian is,

$$L_\theta = i \theta \cdot \frac{g^2 \tilde{G}G}{32\pi^2}$$

We keep $m_u \neq m_d$ on purpose: as is known $m_u = m_d$ is a very singular limit when one discusses $\theta$ dependence, see below. We would like to incorporate the $\theta$ dependence directly into the mass matrix. This can be achieved by performing a chiral rotation,

$$\psi \rightarrow e^{i \theta \gamma_5/2N_f} \psi$$

With this field redefinition, the topological $\theta$ term in the Lagrangian disappears, due to the axial anomaly, and the mass term becomes,

$$L_m = \bar{\psi} \left( \frac{1 + \gamma_5}{2} M^\dagger \psi + \bar{\psi} \frac{1 - \gamma_5}{2} M \psi \right)$$

where the mass matrix $M$ is,

$$M = e^{-i \theta/N_f} \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

In the basis of SU(4) spinors, the mass term becomes,

$$L_m = \frac{i}{2} \Psi^T \mathcal{M} \sigma_2 \tau_2 \Psi + h.c.$$}

where, in block-diagonal form,

$$\mathcal{M} = \begin{pmatrix} 0 & M^T \\ -M & 0 \end{pmatrix}$$

The transformation properties of $L_m$ under (3) imply that to lowest order, $\mathcal{M}$ enters the effective Lagrangian as,

$$\mathcal{L}_m = -g \text{Re} \text{Tr} (\mathcal{M} \Sigma),$$

where the coefficient $g$ is determined by the chiral condensate in the limit $m \rightarrow 0^+, \theta = 0, \mu_B = \mu_I = 0$ [2],

$$g = -\frac{\langle \bar{\psi} \psi \rangle_0}{2N_f}$$

as will be confirmed below. In our notations the chiral condensate includes the sum over all flavors, $\langle \bar{\psi} \psi \rangle = \sum_f \langle \bar{\psi}_f \psi_f \rangle$.

The chiral effective Lagrangian incorporating the effects of $\mu_B, \mu_I, \theta$ and non-zero quark masses, thus becomes,

$$\mathcal{L} = \frac{F^2}{2} \text{Tr} \nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger - g \text{Re} \text{Tr} (\mathcal{M} \Sigma)$$
We shall use the Lagrangian (20) for the rest of this work. So far our discussion easily generalizes to arbitrary $N_f$. However, significant algebraic simplification can be obtained by considering $N_f = 2$. Indeed, for $N_f = 2$, the effective Lagrangian (20) with $m_u \neq m_d$, $\theta \neq 0$, can be reduced to the same Lagrangian but with $m_u' = m_d'$ and $\theta' = 0$. This is achieved, by performing an $SU(4)$ (more specifically $SU(2)_A$) rotation,

$$ \Sigma = U_0 \tilde{\Sigma} U_0^T $$

with the particular choice of,

$$ U_0 = \begin{pmatrix} L & 0 \\ 0 & R^* \end{pmatrix}, \quad L = R^* = e^{i\alpha \sigma^3/2} \quad \cos \alpha = \frac{(m_u + m_d) \cos(\theta/2)}{\sqrt{(m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2)}} \quad \sin \alpha = \frac{(m_u - m_d) \sin(\theta/2)}{\sqrt{(m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2)}} $$

Our parameter $\alpha$ is related to the commonly used Witten’s variables $\varphi_u, \varphi_d$ [7], via,

$$ \varphi_u = \theta/2 - \alpha, \quad \varphi_d = \theta/2 + \alpha \quad \varphi_u + \varphi_d = \theta, \quad m_u \sin \varphi_u = m_d \sin \varphi_d $$

After such a transformation, the Lagrangian (20) takes the form,

$$ L = \frac{F^2}{2} \text{Tr} \nabla_\nu \tilde{\Sigma} \nabla_\nu \tilde{\Sigma} - g m(\theta) \text{Re} \text{Tr} (\mathcal{M}_0 \tilde{\Sigma}) $$

with,

$$ \mathcal{M}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m(\theta) = \frac{1}{2} \left( (m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2) \right) $$

The detailed explanation of this reduction, which along the way clarifies certain global properties of the vacuum manifold, is presented in appendix I. Technically, the simplification is due to pseudo-reality of $SU(N_f = 2)$ (see also section IVC for a more quantitative discussion).

**IV. PHASE DIAGRAM**

**A. Vacuum Alignment and Spectrum**

Our next step is to find the classical minimum of the effective Lagrangian (20) to determine the phase diagram, pattern of spontaneous symmetry breaking and, subsequently, the spectrum of excitations. For arbitrary $N_f$, quark masses, $\theta$ and chemical potentials this is a non-trivial algebraic problem. However, as was shown in the previous section, for $N_f = 2$, the effective Lagrangian reduces to the form (25), which was already analyzed in [5]. Thus, we may immediately read off all quantities of interest.
First, let’s study the phase diagram for fixed \( m_u, m_d, \theta \). To get acquainted with our theory, let’s begin with the trivial environment \( \mu_B = \mu_I = 0 \). The effective Lagrangian possesses an \( Sp(4) \) symmetry at this point. The classical minimum is given by,

\[
\langle \tilde{\Sigma} \rangle = \Sigma_c
\]  

(27)

The \( Sp(4) \) symmetry is unbroken. The low-lying excitations are a quintet of pseudo-goldstones (3 pions and 2 diquarks), with dispersions,

\[
E = \sqrt{p^2 + m^2_\pi(\theta)}
\]  

(28)

\[
m^2_\pi(\theta) = \frac{gm(\theta)}{F^2} = \frac{m(\theta) \langle \bar{\psi}\psi \rangle_0}{4F^2}
\]  

(29)

The pseudo-Goldstone mass \( m_\pi \) acquires a dependence on \( \theta \) through the effective quark mass parameter \( m(\theta) \) (26) (this \( \theta \) dependence is implicitly implied in all formulas below, unless otherwise stated). As we shall see, the whole phase diagram turns out to be determined by the parameter \( m_\pi(\theta) \). We note that \( m_\pi(\theta) \) reaches its maximum at \( \theta = 0 \) and minimum at \( \theta = \pi \). Moreover, for \( m_u = m_d, \theta = \pi, m_\pi \) vanishes to first order in \( M \).

We note that strong \( P \) and \( CP \) symmetries are explicitly broken in the system with \( \theta \neq 0 \). So at \( \theta \neq 0 \), the pions (diquarks) are no longer pure pseudoscalars (scalars). This will become particularly clear when we discuss Bose-condensates of our goldstones in the superfluid phase.

Now let’s turn on chemical potentials. For \( \mu_B \neq 0, \mu_I \neq 0 \), the symmetry of the problem is broken to \( U(1)_B \times U(1)_I \).\(^1\) We introduce the following notations to describe vacuum alignment of \( \Sigma \) at finite chemical potentials,

\[
\Sigma_B = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Sigma_I = i \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}
\]  

(30)

As is known\(^2\), there are 3 distinct phases in the \((\mu_B, \mu_I)\) plane,\(^2\)

I. Normal Phase (N): \( |\mu_B| < m_\pi(\theta), |\mu_I| < m_\pi(\theta) \)

\[
\langle \tilde{\Sigma} \rangle = \Sigma_c
\]  

(31)

Symmetry breaking: \( U(1)_B \times U(1)_I \rightarrow U(1)_B \times U(1)_I \)

Spectrum:

\[
\begin{align*}
q^\pm & \quad E = \sqrt{p^2 + m^2_\pi} \pm \mu_B \\
\pi^0 & \quad E = \sqrt{p^2 + m^2_\pi} \\
\pi^\pm & \quad E = \sqrt{p^2 + m^2_\pi} \pm \mu_I
\end{align*}
\]  

(32)

\(^1\) If only one of the chemical potentials is turned on, say \( \mu_I = 0, \mu_B \neq 0 \), then the symmetry is actually, \( SU(2)_V \times U(1)_B \).

\(^2\) In the original paper\(^3\), a certain physically reasonable ansatz was taken for the classical static minimum \( \langle \tilde{\Sigma} \rangle \) of \( 20 \). It was shown that this ansatz is, indeed, a local minimum, and the authors assumed that this minimum is also global. We note, that using the explicit parametrization of the vacuum manifold presented in Appendix I, it is possible to prove that the ansatz is, indeed, a global minimum.
II. Baryon Phase (B): \( |\mu_B| > m_\pi(\theta), |\mu_I| < |\mu_B| \)

\[
\langle \tilde{\Sigma} \rangle = \frac{m_\pi^2}{\mu_B^2} \Sigma_c + \left( 1 - \frac{m_\pi^4}{\mu_B^4} \right) ^{\frac{1}{2}} \Sigma_B \tag{33}
\]

Symmetry breaking: \( U(1)_B \times U(1)_I \rightarrow U(1)_I \)

Spectrum:

\[
\begin{align*}
\bar{q}^\pm & \quad E^2 = p^2 + \frac{1}{2} \mu_B^2 \left( 1 + 3 \frac{m_\pi^4}{\mu_B^4} \right) \pm \mu_B \left( 4 p^2 \frac{m_\pi^4}{\mu_B^4} + \frac{1}{4} \mu_B^2 \left( 1 + 3 \frac{m_\pi^4}{\mu_B^4} \right)^2 \right)^{\frac{1}{2}} \\
\pi^0 & \quad E = \sqrt{p^2 + \mu_B^2} \\
\pi^\pm & \quad E = \sqrt{p^2 + \mu_B^2} \pm \mu_I
\end{align*}
\]

III. Isospin Phase (I): \( |\mu_I| > m_\pi(\theta), |\mu_B| < |\mu_I| \)

\[
\langle \tilde{\Sigma} \rangle = \frac{m_\pi^2}{\mu_I^2} \Sigma_c + \left( 1 - \frac{m_\pi^4}{\mu_I^4} \right) ^{\frac{1}{2}} \Sigma_I \tag{35}
\]

Symmetry breaking: \( U(1)_B \times U(1)_I \rightarrow U(1)_B \)

Spectrum: Same as for B Phase, but with \( q^\pm \leftrightarrow \pi^\pm \) and \( \mu_B \leftrightarrow \mu_I \).

The phase transition between \( N \) phase and \( B \) phase, as well as \( N \) phase and \( I \) phase is second order, whereas the phase transition between \( B \) phase and \( I \) phase is first order. As noted in [5], the symmetry of the phase diagram/spectrum, with respect to \( \mu_B \leftrightarrow \mu_I \) is a direct consequence of the symmetry of the microscopic theory, \( d_L \leftrightarrow -\tilde{d}_R, d_R \leftrightarrow \tilde{d}_L \), \( \mu_B \leftrightarrow \mu_I \).

Thus, the phase diagram in the \( (\mu_B, \mu_I) \) plane looks the same at \( \theta \neq 0 \) as at \( \theta = 0 \), with the important replacement, \( m_\pi^2 \rightarrow m_\pi^2(\theta) \). This is a very natural conclusion. Indeed, at \( \theta \neq 0 \) diquarks (pions) still carry baryon (isospin) number. Hence, their energy is lowered at finite baryon (isospin) chemical potential. As soon as \( \mu_B (\mu_I) \) reaches the vacuum diquark (pion) mass \( m_\pi(\theta) \), Bose-condensation occurs leading to spontaneous breaking of \( U(1)_B (U(1)_I) \) symmetry.

Quantitatively, the \( \theta \) dependence of the Goldstone mass \( m_\pi(\theta) \) implies that the transition to superfluid phase is shifted to a smaller chemical potential \( \mu_B, \mu_I \), compared to \( \theta = 0 \). In the limiting case, when \( m_u = m_d \) and \( \theta = \pi \), the transition occurs in the vicinity of \( \mu = 0 \) (see Section IVC for a more precise discussion). For physical values, \( m_d = 7MeV, m_u = 4MeV \), the transition at \( \theta = \pi \) occurs at \( \mu = \left( \frac{m_d-m_u}{m_d+m_u} \right) ^{\frac{1}{2}} m_\pi(0) \sim 70MeV \).

B. Chiral Condensates and Densities

In section IVA, we have established the phase diagram of \( N_c = 2, N_f = 2 \) QCD at finite \( \mu_B, \mu_I \) and \( \theta \). In this section, we wish to characterize this phase diagram in terms of
chiral condensates and densities. Similar computations have been performed\cite{2,5} at $\theta = 0$. However, we evaluate a wider range of expectation values and find some condensates that are non-zero even at $\theta = 0$, which have not been discussed in the original papers. As expected, we also find new condensates at $\theta \neq 0$.

We follow the standard procedure for computing microscopic condensates from the effective Lagrangian. We start from a slightly generalized version of the microscopic Lagrangian\cite{8} together with the mass term (14),

$$L = i\Psi^\dagger \sigma_\nu D_\nu \Psi - \Psi^\dagger T \Psi + i/2 \left( \Psi^T J_2 \tau_2 \Psi - \Psi^\dagger J_2 \tau_2 \Psi^* \right)$$  \quad (36)

Here the hermitian, traceless matrix $T$ incorporates the chemical potentials for all 15 charges associated with the $SU(4)$ symmetry, and the chiral condensate source $J$ is an arbitrary, antisymmetric matrix (12 real components). We may express $T$ and $J$ in terms of a basis,

$$T = t_A \lambda_A, \quad J = j_a X_a, \quad \lambda_A = \lambda_A^\dagger, \quad \text{Tr} (\lambda_A) = 0, \quad X_a^T = -X_a, \quad t_A, j_a \in \mathbb{R}$$  \quad (37)

Differentiating the vacuum free energy density $F$, we obtain our condensates and charge densities:

$$\frac{\partial F}{\partial t_A} = -\langle \Psi^\dagger \lambda_A \Psi \rangle$$
$$\frac{\partial F}{\partial j_a} = i/2 \langle \Psi^T X_a \sigma_2 \tau_2 \Psi - \Psi^\dagger X_a \sigma_2 \tau_2 \Psi^* \rangle$$  \quad (38)

The relations\cite{38} hold for any $T$, $J$, however, we will apply them when the derivatives and expectation values are evaluated at physical parameters, $T = T_0 = \mu_B B + \mu_I I$ and $J = M$.

In the effective theory, the sources $T$ and $J$ are incorporated by replacing $T_0 \rightarrow T$ in the covariant derivative (10), and $M \rightarrow J$ in the mass term (18). The condensates\cite{38}, thus, become,

$$\frac{\partial F}{\partial t_A} = F^2 \langle \text{Tr} (\Sigma^\dagger \lambda_A \nabla_0 \Sigma - \nabla_0 \Sigma^\dagger \lambda_A \Sigma) \rangle$$
$$\frac{\partial F}{\partial j_a} = -g \langle \text{Re} \text{Tr} (X_a \Sigma) \rangle$$  \quad (39)

It remains to evaluate the expressions\cite{39}, with $\Sigma$ given by the time-independent, classical minimum of the effective Lagrangian\cite{20}. We must remember that to simplify algebra we expressed, $\Sigma = U_0 \Sigma U_0^T$, with $U_0$ given by\cite{22}. We should also remember that we incorporated $\theta$ dependence into the mass-matrix by a chiral rotation\cite{13} of the quark fields. The condensates and densities, expressed in terms of the original quark fields, are listed in Tables 1,2. We define the charge conjugation matrix $C = \gamma_0 \gamma_2 \gamma_5$. We also introduce the parameter $\lambda(\theta)$ in Tables 1,2,

$$\lambda(\theta) = \begin{cases} 1 & \text{Normal Phase} \\ \frac{m_2^2(\theta)}{\mu^2_2} & \text{Baryon Phase} \\ \frac{m_2^2(\theta)}{\mu^2_1} & \text{Isospin Phase} \end{cases}$$  \quad (40)

which obtains its $\theta$ dependence through $m_2^2(\theta)$. 
### TABLE I: Chiral condensates in $N_c = N_f = 2$ QCD at finite $\theta$

| Condensate $/ \langle \bar{\psi} \psi \rangle_0$ | N Phase ($\lambda = 1$) | B Phase ($\lambda = \frac{m^2(\theta)}{\mu_B^2}$) | I Phase ($\lambda = \frac{m^2(\theta)}{\mu_I^2}$) |
|--------------------------------------------------|-------------------------|---------------------------------|---------------------|
| $iu^T C\gamma_5 \tau_2 d$                         | 0                       | $-\frac{1}{2} \cos(\frac{\theta}{2}) (1 - \lambda^2)^\frac{1}{2}$ | 0                   |
| $u^T C\tau_2 d$                                  | 0                       | $-\frac{1}{2} \sin(\frac{\theta}{2}) (1 - \lambda^2)^\frac{1}{2}$ | 0                   |
| $i\bar{u}\gamma_5 d$                            | 0                       | 0                               | $-\frac{1}{2} \cos(\frac{\theta}{2}) (1 - \lambda^2)^\frac{1}{2}$ |
| $\bar{u}d$                                       | 0                       | 0                               | $-\frac{1}{2} \sin(\frac{\theta}{2}) (1 - \lambda^2)^\frac{1}{2}$ |
| $\bar{u}u$                                       |                          | $\frac{1}{2} \lambda \cos(\frac{\theta}{2} - \alpha)$ |                     |
| $i\bar{u}\gamma_5 u$                            |                          | $-\frac{1}{2} \lambda \sin(\frac{\theta}{2} - \alpha)$ |                     |
| $\bar{d}d$                                       |                          | $\frac{1}{2} \lambda \cos(\frac{\theta}{2} + \alpha)$ |                     |
| $i\bar{d}\gamma_5 d$                            |                          | $-\frac{1}{2} \lambda \sin(\frac{\theta}{2} + \alpha)$ |                     |

### TABLE II: Densities in $N_c = N_f = 2$ QCD at finite $\theta$

| Density            | N Phase ($\lambda = 1$) | B Phase ($\lambda = \frac{m^2(\theta)}{\mu_B^2}$) | I Phase ($\lambda = \frac{m^2(\theta)}{\mu_I^2}$) |
|--------------------|-------------------------|---------------------------------|---------------------|
| $\frac{1}{2} \bar{\psi}\gamma_0 \psi$       | 0                       | $4F^2 \mu_B (1 - \lambda^2)$    | 0                   |
| $iu^T \gamma_0 C\gamma_5 \tau_2 d$           | 0                       | $-4F^2 \mu_B \lambda (1 - \lambda^2)^\frac{1}{2} \cos(\alpha)$ | 0                   |
| $u^T \gamma_0 C\tau_2 d$                     | 0                       | $-4F^2 \mu_B \lambda (1 - \lambda^2)^\frac{1}{2} \sin(\alpha)$ | 0                   |
| $\frac{1}{2} \bar{\psi}\gamma_0 \sigma_3 \psi$ | 0                       | 0                               | $4F^2 \mu_I (1 - \lambda^2)$ |
| $i\bar{u}\gamma_0 \gamma_5 d$                | 0                       | 0                               | $4F^2 \mu_I \lambda (1 - \lambda^2)^\frac{1}{2} \cos(\alpha)$ |
| $\bar{u}\gamma_0 d$                          | 0                       | 0                               | $4F^2 \mu_I \lambda (1 - \lambda^2)^\frac{1}{2} \sin(\alpha)$ |
| $u^T \gamma_0 C\tau_2 u$                     | 0                       | 0                               | 0                   |
| $d^T \gamma_0 C\tau_2 d$                     | 0                       | 0                               | 0                   |
We can now see, how our phase diagram is described in terms of condensates and charge densities. First, let’s gauge our intuition by considering the Normal phase. At $\theta = 0$, the parameter $\alpha$ of eq. (22) is 0, and the only condensates (Table 1) are $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$. At non-zero $\theta$, we also get condensates $\langle i\bar{u}\gamma^5u \rangle$, $\langle i\bar{d}\gamma_5d \rangle$, while $\langle \bar{u}u \rangle$, $\langle \bar{d}d \rangle$ get depleted. The appearance of $P$ and $CP$ odd condensates $\langle i\bar{u}\gamma_5u \rangle$, $\langle i\bar{d}\gamma_5d \rangle$ is a direct consequence of explicit $P$ and $CP$ breaking by the $\theta$ term. Finally, for $\theta \neq 0$, $m_u \neq m_d$, the parameter $\alpha \neq 0$, and we see explicit effects of isospin symmetry breaking: $\langle \bar{u}u \rangle \neq \langle \bar{d}d \rangle$ (correspondingly for $P$ odd condensates). Such effects are absent to lowest order in $M$ at $\theta = 0$. As expected, all charge densities (Table 2) in the Normal phase vanish.

Let’s now see what happens in superfluid phases. At $\theta = 0$, the Baryon phase is characterized by a scalar diquark condensate $\langle i\bar{u}^T\gamma_5\tau_2d \rangle$, which breaks the $U(1)_B$ symmetry. The Isospin phase is characterized by a pseudo-scalar pion condensate $\langle i\bar{u}\gamma_5d \rangle$, which breaks the $U(1)_I$ symmetry. These condensates appear at the expense of depleting $\langle \bar{\psi}\psi \rangle$. As expected, at finite $\theta$, $U(1)_B$ and $U(1)_I$ violating condensates of opposite parity also appear: $\langle u^T\gamma_5\tau_2d \rangle$ in $B$ phase and $\langle \bar{u}d \rangle$ in $I$ phase.

The Baryon and Isospin phases also carry non-vanishing $SU(4)$ charge densities. The I phase, is characterized by the isospin density,

$$n_I = \frac{1}{2} \langle \bar{\psi}\gamma_0\sigma_3\psi \rangle = 4F^2\mu_I \left( 1 - \frac{m_\pi^4(\theta)}{\mu_I^4} \right)$$

This is precisely the density, which one expects to induce by applying an isospin chemical potential $\mu_I$. At $\theta = 0$ it coincides with the previous results [4, 5]. In addition, we also obtain the following axial charge density,

$$n_A = \langle i\bar{u}\gamma_0\gamma_5d \rangle = 4F^2\mu_I m_\pi^2(\theta) \frac{m_\pi^2}{\mu_I^2} \left( 1 - \frac{m_\pi^4(\theta)}{\mu_I^4} \right) \frac{1}{2} \cos \alpha(\theta)$$

which has not been discussed previously in the literature even at $\theta = 0$. This is the axial charge density, corresponding to off-diagonal generators of the $SU(2)_A$ group, which is both spontaneously and explicitly broken. Note that the axial charge density [12] does not vanish already at $\theta = 0$. Thus, we for now concentrate on $\theta = 0$, and hence $\alpha(\theta = 0) = 0$, to better understand the physical nature of this new density [12]. For simplicity, we take $|\mu_B| < m_\pi$.

The density $n_A$ spontaneously breaks the $U(1)_I$ symmetry and, hence, may be considered as an order parameter alongside the pion condensate,

$$\langle \pi^- \rangle = \langle i\bar{u}\gamma_5d \rangle = -\frac{1}{2} \langle \bar{\psi}\psi \rangle_0 \left( 1 - \frac{m_\pi^4}{\mu_I^4} \right) \frac{1}{2}$$

There was no explicit chemical potential conjugate to $n_A$ in the Lagrangian - once $U(1)_I$ is already spontaneously broken by $\langle \pi \rangle$, $n_A$ is induced automatically. The quantitative behaviour of these two order parameters is somewhat different. The pion condensate monotonically increases with $\mu_I$ after the Normal to Isospin phase transition, and $\langle \pi \rangle \rightarrow -\frac{1}{2} \langle \bar{\psi}\psi \rangle_0$ for $\mu_I \gg m_\pi$. On the other hand, the new charge density $n_A$ first increases after the phase transition, reaches a peak at $\mu_I = 3^{1/4}m_\pi$, and then decreases to 0 for $\mu \gg m_\pi$. Of course, we always consider only $\mu_I, \mu_B \ll \Lambda_{QCD}$.

One can understand the appearance of a new condensate $n_A = \langle i\bar{u}\gamma_0\gamma_5d \rangle$ in the following simple way. We are in the phase where the isospin density, $n_I \sim \langle \bar{u}\gamma_0u \rangle - \langle \bar{d}\gamma_0d \rangle$, as well as the condensate, $\langle \pi^- \rangle \sim \langle i\bar{u}\gamma_5d \rangle$, do not vanish. This implies that our ground state can
be understood as a coherent superposition of an infinitely large number of \( \pi^- \) mesons. We expect that we do not disturb the ground state of the system by adding one of these \( \pi^- \) mesons. On the other hand, we can relate the matrix element with an extra \( \pi^- \) meson to the matrix element without the \( \pi^- \) using the standard PCAC technique, \( \langle A|O|B\pi \rangle \sim i\langle A|[O,Q^5]|B\rangle \). The coefficient of proportionality would not be precisely \( 1/F \) in the present case because our pions are not in a trivial vacuum, but rather in the \( \langle \pi^- \rangle \) condensed phase. However, we expect that the general algebraic structure of the vacuum expectation value will be obtained correctly using this approach. Indeed, taking \( O = \bar{u}\gamma_0u - \bar{d}\gamma_0d \), as the isospin density and calculating the commutator \([O,Q^5]\), where \( Q^5 = \int d^3x \bar{u}\gamma_0\gamma_5d = \int d^3x u^\dagger\gamma_5d \) is the axial charge, one obtains the structure \( \langle \bar{iu}\gamma_0\gamma_5d \rangle \) entering the eq. \([42]\). Therefore, if we expect the vacuum expectation value of \( \langle O \rangle = \langle \bar{u}\gamma_0u - \bar{d}\gamma_0d \rangle \) to be nonzero, we should also expect a nonzero value for the axial density \( \langle \bar{iu}\gamma_0\gamma_5d \rangle \). This logic is definitely supported by the explicit calculations \([42]\).

One can also test formula \([42]\) at small isospin density \( n_I \). In this case our system may be understood as a dilute Bose-Condensate of non-relativistic \( \pi^- \) particles\([2]\). How is \( n_A \) manifested in this terminology? We shall work at fixed isospin number density (instead of at fixed \( \mu_I \)). Moreover, we will temporarily work in Minkowski space. In the Isospin phase, the diquarks are not important as we saw, so we parameterize \( \Sigma \) as

\[
\Sigma = \begin{pmatrix} 0 & -U \\ UT & 0 \end{pmatrix}, \quad U \in SU(2)
\]

The field \( U \) transforms as \( U \to LUR^\dagger \) under \( SU(2)_L \times SU(2)_R \) and the effective Lagrangian for \( U \) reads,

\[
\mathcal{L} = F^2 \left( \text{Tr} \partial_\mu U \partial^\mu U^\dagger + 2m_\pi^2 \text{Re} \text{Tr} U \right)
\]

Thus, we see that the Lagrangian describing the pion sector of \( N_c = N_f = 2 \) QCD is exactly the same as the one describing \( N_c = 3, N_f = 2 \) QCD. We express, \( U = \exp \left( \frac{i\sigma^a}{2F} \right) \).

Similarly to eq. \([38],[39]\), we identify,

\[
\bar{\psi}\gamma^\mu \frac{\sigma^a}{2} \psi = 2iF^2 \text{Tr} \left( [\partial^\mu U, U^\dagger] \frac{\sigma^a}{2} \right) \approx -\epsilon^{abc} \partial^\mu \pi^b \pi^c \]

\[
\bar{\psi}\gamma^5 \frac{\sigma^a}{2} \psi = -2iF^2 \text{Tr} \left( [\partial^\mu U, U^\dagger] \frac{\sigma^a}{2} \right) \approx 2F \partial^\mu \pi^a \]

\[
i\bar{\psi}\gamma^5 \frac{\sigma^a}{2} \psi = ig \text{Tr} \left( (U - U^\dagger) \frac{\sigma^a}{2} \right) \approx -g \frac{\pi^a}{F} \]

where we have expanded the corresponding currents to leading order in \( \pi \) fields. We also expand the Lagrangian to fourth order in \( \pi \) fields,

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \pi \partial^\mu \pi - \frac{1}{2} m_\pi^2 \pi^2 - \frac{1}{24F^2} \pi^2 \partial_\mu \pi \partial^\mu \pi + \frac{1}{96F^2} \partial_\mu (\pi^2) \partial^\mu (\pi^2) + \frac{1}{96F^2} m_\pi^2(\pi^2)^2
\]

We can ignore the \( \pi^0 \) particles as they are irrelevant for \( \pi^- \) condensation. It is useful to combine \( \pi = \frac{1}{\sqrt{2}}(\pi^1 + i\pi^2) \). To describe non-relativistic physics involving \( \pi^- \) particles, we can replace \( \partial_0 \pi \to im_\pi \pi, \partial_\pi \pi \to 0 \), in the quartic terms of Lagrangian \([19]\). Finally, we adopt a non-relativistic normalization of our \( \pi^- \) field, by introducing, a canonical, non-relativistic Bose field (of dimension \( 3/2 \)),

\[
\phi^\dagger = \sqrt{2m_\pi} \pi
\]
The Hamiltonian density in terms of the $\phi$ field reads,
\[ H = \frac{1}{2m_\pi} \partial_i \phi^\dagger \partial_i \phi + m_\pi \phi^\dagger \phi + \frac{1}{32F^2} (\phi^\dagger \phi)^2 \] (51)
while the condensates and densities become,
\[ n_I = \frac{1}{2} \bar{\psi} \sigma_3 \psi = \phi^\dagger \phi \] (52)
\[ n_A = i \bar{u} \gamma_0 \gamma_5 d = -2F \sqrt{m_\pi} \phi^\dagger \] (53)
\[ i \bar{u} \gamma_5 d = -\frac{g}{F \sqrt{m_\pi}} \phi^\dagger \] (54)
In this language, we see that both of our $U(1)_I$ order parameters, $n_A$ and $i \bar{u} \gamma_5 d$ are expressed in terms of the same non-relativistic Bose field $\phi^\dagger$.

The energy density of a spatially uniform Bose-Condensate as a function of isospin density is,
\[ \epsilon = m_\pi n_I + \frac{1}{32F^2} n_I^2 \] (55)
Therefore, the isospin chemical potential,
\[ \mu_I = \frac{\partial \epsilon}{\partial n_I} = m_\pi + \frac{1}{16F^2} n_I \] (56)
One can check that (55) agrees to first order in $n_I$ with the result (41) obtained in the grand-canonical ensemble treatment. Re-expressing the order parameters in terms of isospin density, we obtain (up to $U(1)_I$ phase),
\[ n_A = 2F (m_\pi n_I)^{\frac{1}{2}}, \quad \langle i \bar{u} \gamma_5 d \rangle = -\frac{1}{2} \langle \bar{\psi} \psi \rangle_0 \left( \frac{n_I}{4F^2 m_\pi} \right)^{\frac{1}{2}} \] (57)
in agreement to leading order with previous result [12], [13].

Thus, the appearance of the second order parameter $n_A$ is quite natural. Finally, we remark that the situation in the $B$ phase is the mirror image of the above discussion. The new $U(1)_B$ breaking density is,
\[ \langle i u^T \gamma_0 C \gamma_5 \tau_2 d \rangle = -4F^2 \mu_B \lambda \left( 1 - \lambda^2 \right)^{\frac{1}{2}} \cos(\alpha) \] (58)

C. $\theta$ Dependence

So far we have been mostly investigating the phase diagram in the $(\mu_B, \mu_I)$ plane at fixed $\theta$. In this section we would like to focus more on the $\theta$ dependence, drawing the phase diagram in the $(\theta, \mu)$ plane. This trivial exercise leads to rather interesting consequences, namely, the $\theta$ dependence at fixed $\mu$ becomes non-analytic. We further characterize the phase diagram in terms of the $\langle GG \rangle$ correlator and the topological susceptibility $\chi$. Finally, we confirm our calculations by checking the validity of low energy theorems.

To simplify the discussion we shall take $\mu_I = 0$ and focus on $\theta$ dependence in the Normal and Baryon phases. The situation in the Isospin phase is again just the mirror image, as can be explicitly checked.
We begin by considering $\theta$ dependence at $\mu = 0$. The story is exactly the same as in the well-studied case $N_c = 3, N_f = 2$. The vacuum energy density $F(\theta)$ is,

$$F(\theta, \mu = 0) = -4F^2 m_\pi^2(\theta)$$  \hfill(59)

where,

$$m_\pi^2(\theta) = \frac{m(\theta)|\langle \bar{\psi}\psi\rangle_0|}{4F^2}$$  \hfill(60)

$$m(\theta) = \frac{1}{2} \left( (m_u + m_d)^2 \cos^2(\theta/2) + (m_u - m_d)^2 \sin^2(\theta/2) \right)^{1/2}$$  \hfill(61)

By differentiating $F(\theta)$ we can compute correlation functions of $G\tilde{G}$,

$$\frac{\partial F}{\partial \theta} = \langle i\frac{g^2G\tilde{G}}{32\pi^2} \rangle$$  \hfill(62)

$$-\frac{\partial^2 F}{\partial \theta^2} = \chi = -\int d^4x \langle T\frac{g^2G\tilde{G}}{32\pi^2}(x)\frac{g^2G\tilde{G}}{32\pi^2}(0) \rangle_{\text{conn}}$$  \hfill(63)

At $\mu = 0$ we find,

$$\langle i\frac{g^2G\tilde{G}}{32\pi^2} \rangle_{\mu = 0} = -\frac{1}{4} \frac{m_u m_d}{m(\theta)} \sin(\theta) \langle \bar{\psi}\psi \rangle_0$$

$$\chi(\mu = 0) = \frac{1}{4} \frac{m_u m_d}{m(\theta)} \left( \cos(\theta) + \frac{m_u m_d}{4m(\theta)^2} \sin^2(\theta) \right) \langle \bar{\psi}\psi \rangle_0$$  \hfill(64)

The expressions (64) reflect the well-known strong $\theta$ dependence in the region $m_u \approx m_d = m_q, \theta \approx \pi$. Let’s introduce the asymmetry parameter,

$$\epsilon = \frac{|m_u - m_d|}{m_u + m_d}$$  \hfill(65)

and assume $\epsilon \ll 1$.

The $CP$ odd order parameter $\langle iG\tilde{G} \rangle$ (see Fig. 1a) starts out at 0 when $\theta = 0$ and increases smoothly with $\theta$, reaching its maximum just before $\theta = \pi$ at,

$$\langle i\frac{g^2G\tilde{G}}{32\pi^2} \rangle_{\theta = \pi^-} \approx -\frac{m_q}{2} \langle \bar{\psi}\psi \rangle_0$$  \hfill(66)

Afterwards, the order parameter $\langle iG\tilde{G} \rangle$ experiences a steep crossover, dropping to its minimum of,

$$\langle i\frac{g^2G\tilde{G}}{32\pi^2} \rangle_{\theta = \pi^+} \approx +\frac{m_q}{2} \langle \bar{\psi}\psi \rangle_0$$  \hfill(67)

The crossover occurs in the region $|\theta - \pi| \sim \epsilon$ and hence, the topological susceptibility $\chi$ has a sharp peak around $\theta = \pi$ of width $\Delta \theta \sim \epsilon$ and height, $\chi(\pi)/|\chi(0)| = 1/\epsilon$ (see Fig. 1b).

Such behaviour of the $CP$ odd order parameter $\langle iG\tilde{G} \rangle$ strongly suggests that for $m_u = m_d$, spontaneous breaking of $CP$ symmetry occurs at $\theta = \pi$. This situation, known as Dashen’s phenomenon, has been extensively studied in $N_c = 3$ QCD with $N_f = 3$ and $N_f = 2$. For $N_f = 3$ with $m_s \gg m_u, m_d$ it is believed that spontaneous
\( CP \) breaking occurs at \( \theta = \pi \) for \( |m_u - m_d|m_s < m_u m_d \). For \( N_f = 2 \), \( CP \) violation occurs at \( \theta = \pi \), \( m_u = m_d \) and possibly in a small window of \( |m_u - m_d| \neq 0 \) \[1\].

However, it is important to note that Dashen’s phenomenon is not under complete theoretical control in our effective Lagrangian \[20\]. Indeed, for a moment, we fix \( m_u = m_d \). Then, for general \( \theta \), the mass term explicitly breaks the symmetry of the effective Lagrangian \[20\] from \( SU(4) \) to \( Sp(4) \). However, for \( \theta = \pi \), the mass term in the effective Lagrangian vanishes, restoring the symmetry to \( SU(4) \) and giving rise to apparently massless goldstones: \( m^2_\pi(\theta = \pi) = 0 \). Yet, no such symmetry restoration occurs in the fundamental microscopic QCD Lagrangian at \( \theta = \pi \). This contradiction is resolved by including higher order (quadratic) mass terms in the effective Lagrangian, which would explicitly break \( SU(4) \) even at \( \theta = \pi \) \[10\]. It is precisely these terms, which control the physics of Dashen’s phenomenon, and which are not included in the present work.

We do not wish to consider such higher order mass terms in this paper. For any fixed \( |m_u - m_d| m_s \neq 0 \) these terms can be neglected by considering sufficiently small \( m_q \). If the higher order mass terms are largely saturated by a third quark of mass \( m_{u,d} < m_s \ll \Lambda_{QCD} \), then we require,

\[
\frac{|m_u - m_d|}{m_u + m_d} \gg \frac{m_{u,d}}{m_s} \sim \frac{m^2_\pi(\theta = 0)}{M^2_\eta} \tag{68}
\]

This condition is, indeed, realized in the true physical world. If, on the other hand, the higher order terms are controlled by a light \( \eta' \) (as motivated by \( N_c \to \infty \)), we consider,

\[
\frac{|m_u - m_d|}{m_u + m_d} \gg \frac{m_0(\bar{\psi}\psi)}{F^2_\pi M^2_\eta'} \sim \frac{m^2_\pi(\theta = 0)}{M^2_\eta'} \tag{69}
\]

Of course, by imposing restrictions \[68\], \[69\] we automatically exclude the regions of parameter space where Dashen’s transition is realized, and we may discuss only the quantitatively steep crossover in the Normal phase. However, we shall see in a moment that by considering the system at finite \( \mu \), the rapid changes in the vicinity of \( \theta \approx \pi \) observed in the Normal phase will be washed out.

Let us now turn on finite \( \mu_B \). Once conditions \[68\], \[69\] are met, all the results of previous sections hold for any \( \theta \). In particular, the transition to the Baryon phase occurs at \( \mu = m_\pi(\theta) \) (see Fig. 2). As explained above, we can consider arbitrarily small ratio
A rapid crossover occurs in the Normal phase at $\theta = \pi$, which is conjectured to become a first order phase transition, when $m_u = m_d$.

In the Baryon phase, the free energy density reads,

$$ F(\theta) = -2F^2\mu_B^2 \left( 1 + \frac{m^4_\pi(\theta)}{\mu_B^4} \right) $$

Clearly, the $\theta$ dependence in the superfluid phase is different from that in the Normal phase. This is most clearly seen by computing,

$$ \langle iG\tilde{G} \rangle = \frac{m_u m_d}{16F^2\mu_B^2} \langle \bar{\psi}\psi \rangle_0^2 \sin(\theta), $$

$$ \chi = -\frac{m_u m_d}{16F^2\mu_B^2} \langle \bar{\psi}\psi \rangle_0^2 \cos(\theta) $$

We have to remember that expressions (71) hold for all $\theta$ only once we are entirely in the Baryon phase: $\mu_B > m_\pi(\theta = 0)$. On the other hand, if $m_\pi(\theta = \pi) < \mu_B < m_\pi(\theta = 0)$, then we use expression (64), for $\theta$ where the Normal phase is realized, and expression (71), for $\theta$ where the Baryon phase exists. Focusing for a moment on $\mu_B > m_\pi(\theta = 0)$, we see that the $\theta$ dependence is very smooth: there is no sign of rapid crossover in $\langle iG\tilde{G} \rangle$ near $\theta = \pi$ and the
large peak in the susceptibility $\chi$ disappears. Moreover, as $\mu_B$ increases, the $\theta$ dependence is suppressed, as expected. This smooth $\theta$ dependence at $\theta \sim \pi$ in the superfluid phase should be contrasted with sharp behavior in the Normal phase discussed above, see Figs 3a,b.

Now we would like to understand, how the strong $\theta$ dependence at $\mu = 0$ gets smoothed out as the chemical potential $\mu_B$ increases. For, $0 < \mu_B < m_\pi(\theta = \pi)$, $\theta$ dependence is the same as at $\mu = 0$. The key region is $m_\pi(\theta = \pi) < \mu_B < m_\pi(\theta = 0)$, where at fixed $\mu_B$, the

\[
\frac{\chi(\theta^+) - \chi(\theta^-)}{\chi(0)} = \frac{a_m m_d m_\pi^2(0)(\bar{\psi}\psi)_0^2}{64 F^4 \mu_B^6} \sin^2(\theta_c)
\]  

(72)

As the chemical potential increases slightly past $m_\pi(\theta = \pi)$, a narrow region of Baryon phase appears around $\theta = \pi$, deep inside the crossover region shown on Fig. 1. In terms of susceptibility $\chi$, this affects only the very top of the peak of $\chi(\theta)$ by introducing small discontinuities at $\theta = \theta_c$ and $\theta = 2\pi - \theta_c$ (see Fig. 3a). As $\mu$ further increases, the range of $\theta$ where the B phase exists starts growing. This is accompanied by growth in discontinuities of $\chi$ at the transition points. Eventually, for $m_\pi(\theta = \pi) \lesssim \mu \ll m_\pi(\theta = 0)$, the original peak in $\chi$ associated with the $CP$ crossover, is entirely replaced by discontinuities associated

FIG. 3: a) $\theta$ dependence of the topological susceptibility $\chi$, in $N_c = 2, N_f = 2$ QCD at $\mu = 1.02 m_\pi(\theta = \pi)$ (broken curve) together with $\chi$ at $\mu = 0$ (unbroken curve). Here, $\epsilon = 0.01$ b) the same with $\mu = 1.3 m_\pi(\theta = \pi)$; c) the same with $\mu = 0.7 m_\pi(\theta = 0)$; d) the same with $\mu = m_\pi(\theta = 0)$
with the second order Normal to Baryon phase transition (see Fig. 3b). Once this occurs, the magnitude of the jump $\chi(\theta^+) - \chi(\theta^-)$ starts to decrease (Fig. 3c), until finally at $\mu = m_\pi(\theta = 0)$, $\chi$ becomes continuous again and we are entirely in the Baryon phase (Fig. 3d). The washout of the sharp $\theta$ dependence near $\theta = \pi$ has been realized!

The most exciting result of this section is that for $m_u \neq m_d$ the “Dashen’s crossover” first splits into two second order Normal to superfluid phase transitions and for $\mu > m_\pi(0)$ gets entirely washed out. We would like now to provide some speculations regarding the degenerate case $m_u = m_d$, $\theta = \pi$. This point might be of importance for lattice fermions[9, 15], as it is equivalent to a theory where one quark mass is negative and $\theta$ parameter is not explicitly present. In principle, it is possible to analyze this situation rigourously by going to higher order Chiral Perturbation Theory. However, since the algebra becomes rather involved even at $\mu = 0$, we confine ourselves to a conjecture based on the above results and common $\theta = \pi$ lore. For $N_c = 2, N_f = 2$, at $\mu = 0$, we expect that Dashen’s phenomenon will occur along the same lines[10], as for $N_c = 3$. Spontaneous $CP$ violation will happen at $\theta = \pi$, however, no continuous symmetries will be broken and Goldstones will have a small, but finite mass $m_\pi(\theta = \pi) > 0$. At finite $\mu$, we expect a line of first order phase transitions at $\theta = \pi$ to extend to $\mu = m_\pi(\theta = \pi)$, where it splits into two second order phase transition lines (see Fig. 4). We remark that in the Baryon phase, P-parity is still spontaneously broken at $\theta = \pi$, while in the Isospin phase, P-parity is broken at $\theta = 0$, but not a $\theta = \pi$.

![Phase Diagram](image)

**FIG. 4**: Conjectured form of the Phase diagram of $N_c = 2, N_f = 2$ QCD for $m_u = m_d$. Solid line indicates a first order phase transition, while dashed lines indicate second order phase transitions. The region near $\theta = \pi$ is not to scale.

## V. WARD IDENTITIES

In this section we check the validity of Ward Identities (WI) [8, 12, 13, 14] at nonzero $\mu, \theta$. We anticipate that WI must remain untouched when external parameters such as $\mu,
θ or temperature \( T \) are introduced. Indeed, the anomaly (chiral and/or conformal) is a short distance (UV) phenomenon, which is not affected by medium effects (density \( \mu \neq 0 \), \( \theta \) and/or temperature \( T \)). This fact was implicitly used when we constructed the effective Lagrangian \((20)\). However, we are in a position to calculate each term entering the WI explicitly. Therefore, the check of the WI is a nontrivial test of self consistency of our results.

The first identity that we consider, relates the two \( CP \) odd order parameters,

\[
\langle i g^2 G \tilde{G} \rangle = \frac{1}{N_f} \langle i \psi \gamma_5 M \psi \rangle
\]

\((73)\)

The identity \((73)\) reflects the well known fact that there is no \( \theta \) dependence when \( m \to 0 \).

By consulting Table 1 and eqs. \((64), (71)\), we can explicitly check that our results satisfy the identity \((73)\) both in Normal and superfluid phases.

The next WI we would like to discuss is,

\[
\chi = -\int d^4 x \langle T \frac{g^2 G \tilde{G}}{32 \pi^2} (x) \frac{g^2 G \tilde{G}}{32 \pi^2} (0) \rangle_{\text{conn}} = \frac{1}{N_f} \langle \bar{\psi} M \psi \rangle + O(M^2)
\]

\((74)\)

The \( O(M^2) \) term in \((74)\) is usually dropped in the chiral limit, \( SU(N_f)_V \) symmetric limit at \( \theta = 0 \) assuming the resolution of the \( U(1) \) problem when flavor singlet \( \eta' \) is a heavy state.

Indeed, in this case Table 1 and eqs. \((64), (71)\) imply that the WI \((74)\) holds both in Normal and superfluid phases. An important remark is that both sides of \((73)\) and \((74)\) depend on \( \mu \) in a very nontrivial way. Nevertheless, the identities are preserved as expected.

Now, we would like to see what happens with \((74)\) when we relax the requirement of the \( SU(N_f)_V \) symmetric limit and also consider \( \theta \neq 0 \). In this case it is important to keep the \( O(M^2) \) term,

\[
O(M^2) = -\frac{1}{N_f^2} \int d^4 x \langle T \bar{\psi} \gamma_5 M \psi (x) \bar{\psi} \gamma_5 M \psi (0) \rangle_{\text{conn}}
\]

\((75)\)

We begin in the Normal phase and evaluate,

\[
\chi - \frac{1}{N_f^2} \langle \bar{\psi} M \psi \rangle = -\frac{1}{64} \left( \frac{m_u^2 - m_d^2}{m(\theta)^3} \right) \langle \bar{\psi} \psi \rangle_0
\]

\((76)\)

The above result implies that \( m_q^{-1} \) singularities develop in the \( O(M^2) \) term of eq. \((74)\), due to \( \eta' \)/goldstone mixing, which occurs for \( m_u \neq m_d \). However, the singularities disappear when \( m_u = m_d \), so that the \( O(M^2) \) term can be neglected in the chiral, \( SU(N_f)_V \) limit, as long as we are sufficiently far from \( \theta = \pi \). This is the physically expected result.

What happens with \((74)\) in the superfluid phase, when \( \theta \neq 0 \)? We can immediately see that the \( O(M^2) \) term can no longer be neglected even when \( m_u = m_d \). As has been discussed above, the Normal to superfluid transition is second order, so that the topological susceptibility \( \chi \) generically experiences a jump across the phase boundary \((72)\), while the chiral condensate \( \langle \bar{\psi} M \psi \rangle \) (Table 1) is continuous. Thus, for the WI \((74)\) to hold, the \( O(M^2) \) term must jump across the phase boundary, accounting for the discontinuity in \( \chi \) and, thus, contributing on the same footing as \( \langle \bar{\psi} M \psi \rangle \) to the righthand side of \((74)\). It is not surprising that the \( O(M^2) \) term becomes important. Indeed, from Table 1 and \((71)\) we see that both \( \chi \) and \( \langle \bar{\psi} M \psi \rangle \) are of order \( M^2/\mu_B^2 \) in the superfluid phase. The fact that \( \chi \) becomes \( O(M^2) \) rather than \( O(M) \) is part of smoothing out of \( \theta \) dependence in the superfluid phase. The
contact between $O(M)$ dependence in the Normal phase and $O(M^2)$ dependence in the superfluid phase is provided by the fact that $\mu^2 \sim O(M)$ at the Normal to superfluid phase transition. Thus, all correlators in eq. (74) develop $\mu^{-2}$ singularities in the superfluid phase, which are due to the modification of the goldstone spectrum (34).

Finally, from (74), to leading order in $M$, in the superfluid phase,

$$\int d^4x \langle T \bar{\psi} \gamma_5 M \psi(x) \bar{\psi} \gamma_5 M \psi(0) \rangle_{\text{conn}} = -\frac{1}{16 F^2 \mu_B^2} (m_u^2 + m_d^2 - 2m_u m_d \cos(\theta)) \langle \bar{\psi} \psi \rangle_0^2$$

which vanishes only if $m_u = m_d$, $\theta = 0$.

VI. GLUON CONDENSATE

Having determined the $\theta$ and $\mu$ dependence of different condensates and densities containing the quark degrees of freedom (Tables 1, 2), one can wonder if similar results can be derived for the gluon condensate $\langle G_{\mu \nu}^2 \rangle$, which describes the gluon degrees of freedom. As is known, the gluon condensate represents the vacuum energy of the ground state in the limit $m_q = 0$, $\mu = 0$ and plays a crucial role in such models as the MIT Bag model, where a phenomenological “bag constant” $B$ describes the non-perturbative vacuum energy of the system. The question we want to answer: how will the gluon condensate $\langle G_{\mu \nu}^2 \rangle$ (bag constant $B$) depend on $\mu, \theta$ if the system is placed into dense matter? This question is relevant for a number of different studies such as the equation of state in the interior of neutron stars, see e.g. [16], or stability of dense strangelets [17]. Of course, it is difficult to answer this question in full 3 color QCD at finite $\mu_B$, however, the answer can be easily obtained in 2 color QCD for $\mu \ll \Lambda_{QCD}$, which is the subject of the present work. We limit ourselves to considering only the Normal and Baryon phases, the results in the Isospin phase, as always, are obtained by replacing $\mu_B \rightarrow \mu_I$. We work in Minkowski space in this section.

We start from the equation for the conformal anomaly,

$$\Theta^\mu_\mu = -\frac{b g^2}{32 \pi^2} G^a_{\mu \nu} G^{a \mu \nu} + \bar{\psi} M \psi$$

where we have taken the standard 1 loop expression for the $\beta$ function and $b = \frac{11}{3} N_c - \frac{2}{3} N_f = 6$ for $N_c = N_f = 2$. As usual, a perturbative constant is subtracted in expression (78). For massless quarks and in the absence of chemical potential, eq. (78) implies that the QCD vacuum carries a negative non-perturbative vacuum energy due to the gluon condensate.

Now, we can use the effective Lagrangian (20) to calculate the change in the trace of the energy-momentum tensor $\langle \theta^\mu_\mu \rangle$ due to a finite chemical potential $\mu_B \ll \Lambda_{QCD}$. The energy density $\epsilon$ and pressure $p$ are obtained from the free energy density $\mathcal{F}$,

$$\epsilon = \mathcal{F} + \mu_B n_B$$

$$p = -\mathcal{F}$$

Therefore, the conformal anomaly implies,

$$\langle \frac{b g^2}{32 \pi^2} G^a_{\mu \nu} G^{a \mu \nu} \rangle_{\mu, m, \theta} - \langle \frac{b g^2}{32 \pi^2} G^a_{\mu \nu} G^{a \mu \nu} \rangle_0 = -4 (\mathcal{F}(\mu, m, \theta) - \mathcal{F}_0) - \mu_B n_B(\mu, m, \theta) + \langle \bar{\psi} M \psi \rangle_{\mu, m, \theta}$$
Here, the subscript 0 on an expectation value means that it is evaluated at $\mu = m = 0, \theta = 0$. The good news is that we have already calculated all quantities on the right-hand side of eq. (81) - see expressions (59), (70) and Tables 1, 2. Thus, in the Normal phase we obtain,

$$\langle \frac{b g^2}{32\pi^2} C_{\mu\nu}^a G^{\mu\nu a} \rangle_{\mu, m, \theta} - \langle \frac{b g^2}{32\pi^2} G_{\mu\nu}^a G^{\mu\nu a} \rangle_0 = -3m(\theta) \langle \bar{\psi} \psi \rangle_0$$

When $\theta = 0$, (82) reduces to the standard result [14], which was derived in a different manner. As expected, $\langle G_{\mu\nu}^2 \rangle$ does not depend on $\mu$ in the Normal phase. The Baryon phase is more exciting,

$$\langle \frac{b g^2}{32\pi^2} C_{\mu\nu}^a G^{\mu\nu a} \rangle_{\mu, m, \theta} - \langle \frac{b g^2}{32\pi^2} G_{\mu\nu}^a G^{\mu\nu a} \rangle_{\mu=0, m, \theta} = 4F^2(\mu_B^2 - m_\pi^2) \left( 1 - \frac{2m_\pi^2}{\mu_B^2} \right)$$

which makes contact with the fact that in the Normal phase, when $\mu_B \leq m_\pi(\theta)$, the gluon condensate does not vary with $\mu_B$. However, for $\mu_B \geq m_\pi(\theta)$, the dependence of the gluon condensate $\langle G_{\mu\nu}^2 \rangle$ on $\mu_B$ in the Baryon phase becomes rather interesting. The condensate decreases with $\mu_B$ for $m_\pi < \mu_B < 2^{1/4}m_\pi$ and increases afterwards. The qualitative difference in the behaviour of the gluon condensate for $\mu_B \approx m_\pi$ and for $m_\pi \ll \mu_B \ll \Lambda_{QCD}$ can be explained as follows. Right after the Normal to Baryon phase transition occurs, the baryon density $n_B$ is small and our system can be understood as a weakly interacting gas of diquarks. The pressure of such a gas is negligible compared to the energy density, which comes mostly from diquark rest mass. Thus, $\langle G_{\mu\nu}^2 \rangle$ increases with $n_B$ and, according to the anomaly equation (78), $\langle G_{\mu\nu}^2 \rangle$ decreases. A similar decrease in $\langle G_{\mu\nu}^2 \rangle$ with baryon density is expected to occur in “dilute” nuclear matter (see [13] and review [19]). On the other hand, for $\mu_B \gg m_\pi$, energy density is approximately equal to pressure, and both are mostly due to self-interactions of the diquark condensate. Luckily, the effective Chiral Lagrangian (20) gives us control over these self-interactions as long as $\mu_B \approx \Lambda_{QCD}$. Such control is largely absent in corresponding calculations of $\langle G_{\mu\nu}^2 \rangle$ in nuclear matter. As $\Delta \epsilon \sim \Delta p$, the trace $\langle \Theta_{\mu\nu}^a \rangle$ decreases and the gluon condensate increases with baryon density. Such behaviour of $\langle G_{\mu\nu}^2 \rangle$ is quite unusual, as finite baryon density, on general grounds, is expected to suppress the gluons.

At small baryon density, we can also use a slight variant of the above method for calculation of the gluon condensate, originally developed in the context of nuclear matter. As long as our Bose-condensate of diquarks is dilute, we may neglect interactions between diquarks, and approximate the change in $\langle G_{\mu\nu}^2 \rangle$ as just the expectation value of $G_{\mu\nu}^2$ in each diquark times their number,

$$\langle \frac{b g^2}{32\pi^2} C_{\mu\nu}^a G^{\mu\nu a} \rangle_{\mu, m, \theta} - \langle \frac{b g^2}{32\pi^2} G_{\mu\nu}^a G^{\mu\nu a} \rangle_{0, m, \theta} = \frac{n_B}{2m_\pi} \langle q^- \rangle \langle \frac{b g^2}{32\pi^2} C_{\mu\nu}^a G^{\mu\nu a} \rangle_{\mu, m, \theta}$$

Here $q^-$ denotes a diquark state relativistically normalized to $\langle q^-(p)|q^-(p')\rangle = 2E_p(2\pi)^3\delta^3(p - p')$, giving rise to the factor $\frac{1}{2m_\pi}$ in (83). It remains to calculate the matrix.
element, \( \langle q^- | G^2_{\mu\nu} | q^- \rangle \). This can be done by sandwiching the anomaly equation (78) between two diquark states. As \( \langle q^- | \Theta^\mu | q^- \rangle = 2m^2_\pi \),

\[
2m^2_\pi = - \langle q^- | \frac{b\gamma^2}{32\pi^2} G^a_{\mu\nu} G^{\mu\nu a} | q^- \rangle + \langle q^- | \bar{\psi} M \psi | q^- \rangle
\]

We are used to the fact that the goldstone mass comes entirely from the symmetry breaking term \( \bar{\psi} M \psi \), so we might, naively, expect from eq. (86) that \( G^2_{\mu\nu} \) vanishes in a diquark. However, the diquark mass, to first order in \( M \) is given by,

\[
m^2_\pi = \langle q^- | \bar{\psi} M \psi | q^- \rangle
\]

therefore,

\[
\langle q^- | \frac{b\gamma^2}{32\pi^2} G^a_{\mu\nu} G^{\mu\nu a} | q^- \rangle = -m^2_\pi
\]

so that \( G^2_{\mu\nu} \) and \( \bar{\psi} M \psi \) contribute equally to the goldstone mass in eq. (86). Now from eq. (85),

\[
\langle \frac{b\gamma^2}{32\pi^2} G^a_{\mu\nu} G^{\mu\nu a} \rangle_{\mu,m,\theta} - \langle \frac{b\gamma^2}{32\pi^2} G^a_{\mu\nu} G^{\mu\nu a} \rangle_{0,m,\theta} = \frac{1}{2} n_B m_\pi
\]

This is in agreement with our full result (88), (84) to leading order in \( n_B \).

Finally, we note that by differentiating (82), (83) with respect to \( \theta \) we can obtain correlation functions of \( G^2_{\mu\nu} \) with \( G \bar{G} \), in Normal and superfluid phases.

VII. CONCLUSION. SPECULATIONS.

We conclude this paper with the following speculative remarks:

1. Naively, one could say that we studied in the present paper a purely academic question by considering \( N_c = 2 \) rather than the realized in nature QCD with \( N_c = 3 \). We should comment on this as follows. First, for \( N_c = 2 \), the so-called, diquarks become well-defined gauge invariant objects. However, diquarks, as has been argued in a number of papers, (see, e.g. recent papers on the subject [20, 21, 22, 23, 24, 25]) may play an important role in 3-color QCD dynamics. If the passage from \( SU(2)_{\text{color}} \) to \( SU(3)_{\text{color}} \) does not lead to dramatic disturbances of these diquarks, these predictions based on \( SU(2)_{\text{color}} \) remain qualitatively valid in real QCD! Arguments supporting the conjecture of smoothness of the transition \( SU(2)_{\text{color}} \) to \( SU(3)_{\text{color}} \) are presented in [24]. We also note that there is some similarity between the proposal of [25] and the present work to study the diquark dynamics. In the proposal [25] the idea is to introduce a color source to study the diquark dynamics, while in our paper with \( N_c = 2 \) the diquarks automatically become gauge invariant objects, and no source is required to study them.

2. It has been suggested that the \( \theta \)-induced \( CP \) odd state can be formed in heavy ion collisions at RHIC, see original papers [26] and a recent review [27]. Our analysis could be quite relevant for the study of the decays of a \( CP \) odd configuration, if it is formed.

3. It has been known for quite some time that violation of parity and \( CP \) parity (which is the case when \( \theta \neq 0 \)) may completely change the phase structure of a theory. Some lattice calculations, for example, suggest that the behavior of the system could be very nontrivial when \( \theta \neq 0 \), and some singular behavior and even phase transitions are expected, see e.g. [28].
In an environment where $C$, $P$, and $CP$ are strongly violated, the interaction of quarks and anti-quarks is not identical, as it is usually assumed, but rather, could be very different. Under such conditions the QCD phase transition in the early universe could have a much more complicated history than it is typically assumed. In particular, one can imagine that some very nontrivial objects, such as Witten’s nuggets\[29\], which behave as dark matter particles, can be formed. Moreover, due to the differences in interactions between quarks and anti-quarks in the presence of $\theta$, local separation of baryon charges may take place, and chunks of quarks or anti-quarks in condensed color superconducting phase may form during the QCD phase transition, serving as dark matter.\[30\] This scenario is based on the idea that while the universe is globally symmetric, the anti-baryon charge can be stored in chunks of dense color superconducting antimatter. In this case, instead of baryogenesis, one should discuss the separation of baryon charges. Such a global picture of our universe, is definitely, not in contradiction with the present observational constraints.\[30\] Rather, it may give a natural explanation for some global parameters such as $\Omega_{DM}/\Omega_{B}$\[30\], or even can naturally explain the famous 511KeV line from the bulge of our galaxy\[31\].

Typical relaxation time for $\theta$ is much larger than $\Lambda_{QCD}^{-1}$, therefore, one can neglect the dynamics of $\theta$ for studying the possible phases for each given $\theta$. This was the main reason for us to study $\theta$ dependence of the QCD phase diagram. We find in the present analysis that, indeed, two phase transitions of the second order will take place when $\theta$ relaxes from $\theta = 2\pi$ to $\theta = 0$. These phase transitions will occur under very generic conditions when $\mu_I$ is smaller than $m_\pi$, but of the same order of magnitude as $m_\pi$. The physical consequences of these phase transitions are still to be explored.

4. Aside from these far reaching speculations, we would like to mention here a few much more terrestrial consequences of the present study, which may have some impact on the lattice simulations. First of all, 2 color QCD is a nice laboratory to study a variety of different very deep problems of gauge theories at nonzero temperature and density, see e.g.\[32, 33, 34, 35\]. New elements, which were not studied previously and which are the subject of the present work, are related to $\theta$ dependence of different condensates. There are a few interesting observations which deserve to be mentioned here:

a) At $\theta = \pi$, when the determinant of the Dirac operator is real, the superfluid phase is realized at much lower critical chemical potential $\mu_c$ than at $\theta = 0$. In the limit $m_u = m_d$, we expect $\frac{\mu_c(\theta = \pi)}{\mu_c(\theta = 0)} \sim \left(\frac{m}{\Lambda_{QCD}}\right)^{\frac{1}{2}}$. It gives a unique chance to study the superfluid phase on the lattice at a much smaller $\mu$ than would normally be required. We hope that our conjecture for the disappearance of Dashen’s transition in the superfluid phase can be explicitly tested on the lattice.

b) Knowledge of $\theta$ dependence of different condensates allows one to calculate the topological susceptibility and other interesting correlation functions as a function of $\mu$. Corresponding Ward Identities at nonzero $\mu$ can be tested on the lattice.

c) Physics of gluon degrees of freedom and $\mu$ dependence of the gluon condensate can also be tested on the lattice. The behavior of the gluon condensate as a function of $\mu$ is very nontrivial, as has been explained in the text. Nevertheless, our prediction is robust in a sense that it is based exclusively on the chiral dynamics and no additional assumptions have been made to derive the corresponding expression.

Finally, we should emphasize that all results presented above are valid only for very small chemical potentials $\mu_B, \mu_I \ll \Lambda_{QCD}$ when the chiral effective theory is justified. For larger chemical potentials we expect a transition to a deconfined phase at $\mu_B(\mu_I) \simeq 7\Lambda_{QCD}$\[36\]. We should also add that all results presented above can be easily generalized to $N_c = 3$.
QCD with $\mu_I \neq 0$, $\mu_B = 0$.

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APPENDIX A: PARAMETRIZATION OF THE VACUUM MANIFOLD

The purpose of this appendix is to clarify some global aspects of the manifold of goldstone fields in $N_c = 2$, $N_f = 2$ QCD. Once the vacuum manifold is parameterized, we show that to first order in quark mass, all effects of the $\theta$ parameter can be incorporated into a common real quark mass.

We begin with the assumption that the chiral symmetry breaking pattern of $N_c = 2$, $N_f = 2$ QCD is, exactly, $SU(4) \rightarrow Sp(4)$, so that the vacuum manifold $\mathcal{X} = SU(4)/Sp(4)$. We represent the manifold as,

$$\mathcal{X} = \{U \Sigma_c U^T, U \in SU(4)\}$$

(A1)

with $\Sigma_c$ given by eq. (5). Note that $\mathcal{X} \subset \mathcal{A}$, where $\mathcal{A}$ is the set of all $4 \times 4$ antisymmetric, unitary matrices of determinant 1. The original work [2] had implicitly assumed that $\mathcal{X} = \mathcal{A}$. As we shall show, this is almost, but not quite true. In fact, $\mathcal{A} = \mathcal{X} \cup i\mathcal{X}$, i.e $\mathcal{A}$ is a disjoint union of two pieces, both of which are homeomorphic to $SU(4)/Sp(4)$.

Even though such technical details do not affect the analysis of [2], they become important, once the $\theta$-parameter is introduced. In particular, if we minimized the effective Lagrangian (20) over $\Sigma \in \mathcal{A}$, we would obtain very different $\theta$ dependence. In fact, the theory (20) with $\Sigma \in \mathcal{X}$ corresponds to the theory with $\Sigma \in \mathcal{X}$ with the redefinition $\theta \rightarrow \theta + \pi$. As long as we represent our vacuum manifold by any one of the two pieces $\mathcal{X}$ or $i\mathcal{X}$, we obtain the same physics, if we define $\theta$ appropriately. However, if we enlarge the vacuum manifold to contain both pieces, the physics changes: we obtain cusps in $\theta$ dependence at $\theta = \pm \frac{\pi}{2}$, instead of Dashen’s phenomenon at $\theta = \pi$. Since we find no evidence of an additional spontaneously broken discrete symmetry that would connect the two disjoint pieces of $\mathcal{A}$, we insist on our original assumption that the vacuum manifold is $SU(4)/Sp(4) = \mathcal{X}$.

Now, let us demonstrate the above claims. We begin by writing any $\Sigma \in \mathcal{A}$ as,

$$\Sigma = \begin{pmatrix} a \sigma_2 & C \\ -C^T & b \sigma_2 \end{pmatrix}$$

(A2)

Here, $a, b$ are complex numbers and we have used the fact that $\Sigma$ is antisymmetric. The requirement, $\text{det} \Sigma = 1$, implies,

$$(\text{det}(C) + ab)^2 = 1$$

(A3)

We call, $\mathcal{A}_+ = \{\Sigma \in \mathcal{A}, \text{det}(C) + ab = \pm 1\}$. Then $\mathcal{A}$ is the disjoint union, $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$. We shall show, $\mathcal{A}_+ = \mathcal{X}$, $\mathcal{A}_- = i\mathcal{X}$. We begin with the observation that $\mathcal{X} \subset \mathcal{A}_+$. Indeed, $SU(4)$ is connected, so $\mathcal{X}$ is connected. But, $\Sigma_c \in \mathcal{X}$, and $\text{det}(C) + ab = 1$ for $\Sigma_c$. Therefore, $\text{det}(C) + ab = 1$ for any $\Sigma \in \mathcal{X}$, so $\mathcal{X} \subset \mathcal{A}_+$. 


Now, take $\Sigma \in \mathcal{A}_+$. The condition $\Sigma \Sigma^\dagger = 1$ implies,

\begin{align}
CC^\dagger + |a|^2 &= 1 \\
C^\dagger C + |b|^2 &= 1 \\
a^* \sigma_2 C^T \sigma_2 &= b C^\dagger
\end{align}

(A4) (A5) (A6)

The remaining condition for $\Sigma \in \mathcal{A}_+$ is,

$$det(C) + ab = 1$$

(A7)

Equation (A4) implies $|a| \leq 1$, and $C$ can be written (non-uniquely) in the form,

$$C = \sqrt{1 - |a|^2} e^{i\phi_c} S, \quad S \in SU(2)$$

(A8)

Substituting (A8) into (A5), produces, $|a| = |b|$. We write, $a = |a| e^{i\phi_a}$, $b = |b| e^{i\phi_b}$. If $a = 0$, the condition (A6) is satisfied automatically, and (A7) implies, $e^{2i\phi_c} = 1$. If $|a| = 1$, the condition (A6) is again satisfied, and (A8) gives, $e^{i(\phi_a + \phi_b)} = 1$. Finally, if $0 < a < 1$, (A6) becomes,

$$e^{2i\phi_c} \sigma_2 S^T \sigma_2 = e^{i(\phi_a + \phi_b)} S^\dagger$$

(A9)

But $SU(2)$ is pseudo-real: $\sigma_2 S^T \sigma_2 = S^\dagger$, so $e^{2i\phi_c} = e^{i(\phi_a + \phi_b)}$. Substitution into (A7) gives

$$e^{2i\phi_c} = e^{i(\phi_a + \phi_b)} = 1$$

(A10)

We note that if $e^{i\phi_c} = -1$, we can always reabsorb the negative sign into the definition of $S \in SU(2)$. Thus, $\Sigma \in \mathcal{A}_+$ if and only if it may be written as,

$$\Sigma = \begin{pmatrix}
|a| e^{i\phi_c} \sigma_2 \\
\sqrt{1 - |a|^2} S \\
-\sqrt{1 - |a|^2} S^T \\
|a| e^{-i\phi_a} \sigma_2
\end{pmatrix}, \quad |a| < 1, \quad S \in SU(2)$$

(A11)

We now show that any $\Sigma$ of form (A11) is in $\mathcal{X}$. We let, $e^{i\xi} = \sqrt{1 - |a|^2 + i|a|}$. Define matrices, $U_1, U_2, U_3, U \in SU(4)$ as,

\begin{align}
U_1 &= \exp \left( i \frac{\xi}{2} \begin{pmatrix}
0 & -i \sigma_2 \\
n i \sigma_2 & 0
\end{pmatrix} \right) \\
U_2 &= \begin{pmatrix}
-S & 0 \\
0 & 1
\end{pmatrix} \\
U_3 &= \exp(i\phi_a B/2) \\
U &= U_3 U_2 U_1
\end{align}

(A12) (A13) (A14) (A15) (A16)

One can now check that $U \Sigma_c U^T = \Sigma$. This concludes the proof of the fact that $\mathcal{X} = \mathcal{A}_+$. It is now trivial to show that $\mathcal{A}_- = i\mathcal{A}_+ = i\mathcal{X}$.

The most practically useful result of the above discussion is the explicit parametrization (A11) of the vacuum manifold $\mathcal{X}$. For instance, this parametrization allows one to prove that the local minimum of the effective Lagrangian (20) at finite $\mu_B$ and $\mu_I$, originally constructed in [5], is, actually, global. Moreover, we can now use the form (A11) to study the topology of the vacuum manifold $\mathcal{X}$. Indeed, the matrix $\Sigma \in SU(2)$ can be uniquely written as
$S = X_0 + iX_i \sigma_i$, $X_0 X_0 + X_i X_i = 1$. Also, $e^{i\phi a} = X_4 + iX_5$, $X_0^2 + X_5^2 = 1$. Clearly, $\mathcal{X}$ is in one to one correspondence with the set of vectors in $\mathbb{R}^6$, $\sqrt{1 - |a|^2}(X_0, X_1, X_2, X_3, 0, 0) + |a|(0, 0, 0, 0, X_4, X_5)$, $0 \leq |a| \leq 1$. But this is just a parametrization of $S^5$. Hence, $\mathcal{X} = SU(4)/Sp(4) \cong S^5$.

Finally, let us discuss the $SU(4)$ transformation (21). Since, $U_0 B U_0^\dagger = B$, $U_0 I U_0^\dagger = I$, the kinetic part of the effective Lagrangian (20) remains invariant. Therefore, we have to discuss only the mass term:

$$\mathcal{L}_m = -gR e \text{Tr}(\mathcal{M}\Sigma) = -gR e \text{Tr}(\mathcal{M}U_0 \tilde{\Sigma} U_0^T)$$ (A17)

where $\tilde{\Sigma} \in \mathcal{X}$ and, therefore, can be written in the form (A11), as,

$$\tilde{\Sigma} = \left( \begin{array}{cc} a \sigma_2 & C \\ -C^T & a^* \sigma_2 \end{array} \right), \quad C = \sqrt{1 - |a|^2} S, \quad S \in SU(2)$$ (A18)

Expanding,

$$\mathcal{L}_m = \frac{1}{2} g \left( e^{-i\theta/2} \text{Tr}(e^{i\alpha_3} \{M, C\}) + e^{i\theta/2} \text{Tr}(e^{-i\alpha_3} \{M, C^*\}) \right)$$ (A19)

At this point, we again use the pseudo-reality of $SU(2)$, $C^* = \sigma_2 C \sigma_2$, obtaining after some algebra,

$$\mathcal{L}_m = 2gm(\theta) \text{Tr}(C) = -gm(\theta) \text{Tr}(M_0 \tilde{\Sigma})$$ (A20)

Thus, to first order in $M$, all $\theta$ and $m_u, m_d$ dependence can be incorporated into the quark mass $m(\theta)$ (26).

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