MARPA, A PRACTICAL GENERAL PARSER: THE RECOGNIZER

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ABSTRACT. The Marpa recognizer is described. Marpa is a practical and fully implemented algorithm for the recognition, parsing and evaluation of context-free grammars. The Marpa recognizer is the first to unite the improvements to Earley’s algorithm found in Joop Leo’s 1991 paper to those in Aycock and Horspool’s 2002 paper. Marpa tracks the full state of the parse, as it proceeds, in a form convenient for the application. This greatly improves error detection and enables event-driven parsing. One such technique is “Ruby Slippers” parsing, in which the input is altered in response to the parser’s expectations.

1. INTRODUCTION

Despite the promise of general context-free parsing, and the strong academic literature behind it, as of 2010 it had never been incorporated into a highly available tool like those that exist for LALR [9] or regular expressions. The MARPA project was intended to take the best results from the literature on Earley parsing off the pages of the journals and bring them to a wider audience. Marpa::XS [12], a stable version of this tool, was uploaded to the CPAN Perl archive on Solstice Day in 2011. This paper describes the algorithm of Marpa::R2 [11], a later version.

As implemented in [11], Marpa parses, without exception, all context-free grammars. Time bounds are the best of Leo [14] and Earley [4]. The Leo bound, $O(n)$ for LR-regular grammars, is especially relevant to Marpa’s goal of being a practical parser: If a grammar is in a class of grammar currently in practical use, Marpa parses it in linear time.

Error-detection properties are extremely important for practical parsing, but have been overlooked in the past. Marpa breaks new
ground in this respect. Marpa has the immediate error detection property, and goes well beyond that: it is fully aware of the state of the parse, and can report this to the user while tokens are being scanned.

Marpa allows the lexer to check its list of acceptable tokens before a token is scanned. Because rejection of tokens is easily and efficiently recoverable, the lexer is also free to take an event-driven approach. Error detection is no longer an act of desperation, but a parsing technique in its own right. If a token is rejected, the lexer is free to create a new token in the light of the parser’s expectations. This approach can be described as making the parser’s “wishes” come true, and we have called this “Ruby Slippers Parsing”.

One use of the Ruby Slippers technique is to parse with a clean but oversimplified grammar, programming the lexical analyzer to make up for the grammar’s short-comings on the fly. The author has implemented an HTML parser [10], based on a grammar that assumes that all start and end tags are present. Such an HTML grammar is too simple even to describe perfectly standard-conformant HTML, but the lexical analyzer is programmed to supply start and end tags as requested by the parser. The result is a very simply and cleanly designed parser that parses very liberal HTML and accepts all input files, in the worst case treating them as highly defective HTML.

Section 2 describes the notation and conventions of this paper. Section 3 deals with Marpa’s grammar rewrites. Sections 4, 5, and 6 introduce Earley’s algorithm. Section 7 describes Leo’s modification to Earley’s algorithm. Section 8 describes the modifications proposed by Aycock and Horspool. Section 9 presents the pseudocode for Marpa’s recognizer. Section 10 describes notation for, and other preliminaries to, the theoretical results. Section 11 contains a proof of Marpa’s correctness, while Section 12 contains its complexity results. Finally, Section 13 generalizes Marpa’s input model.

The nature of this paper is such that an adequate literature survey would be as large as the rest of this paper. Instead, we have placed a full, if somewhat informal, literature survey online [13].

2. Preliminaries

We assume familiarity with the theory of parsing, as well as Earley’s algorithm. We will use the type system of Farmer 2012 [6], without needing most of its apparatus. The notation $x : T$ indicates that the variable $x$ is of type $T$. More often, this paper will use subscripts to
indicate type. We will often designate particular sets as types, but any set can be a type.

\[
\begin{align*}
X &: T & \text{The variable } X \text{ of type } T \text{ (wide form)} \\
X_T &: T & \text{The variable } X \text{ of type } T \text{ (narrow form)} \\
\text{set-one}_T &: T^* & \text{The variable } \text{set-one} \text{ of type set of } T \\
\text{SYM} &: & \text{The type for a symbol} \\
\text{set-two}_{\text{SYM}} &: \text{SYM}^* & \text{The variable } \text{set-two} \text{, a set of symbols}
\end{align*}
\]

Subscripts may be omitted when the type is obvious from the context. Multi-character variable names will be common, and operations will never be implicit.

- Multiplication: \( a \times b \)
- Concatenation: \( a . b \)
- Subtraction: \( \text{symbol-count} - \text{terminal-count} \)

We often write “iff" for “if and only if”. We also often substitute the more prominent double colon (::) for the “mid” divider (|).

A useful feature of Farmer 2012 [6] is his notion of ill-definedness. For example, the value of partial functions may be ill-defined for some arguments in their domain. Farmer’s handling of ill-definedness is the traditional one, and was well-entrenched, but he was first to describe and formalize it.

We write \( \bot \) for ill-defined. A value is well-defined iff it is not ill-defined. We write \( x \downarrow \) to say that \( x \) is well-defined, and we write \( x \uparrow \) to say that \( x \) is ill-defined.

Traditionally, and in this paper, any formula with an ill-defined operand is false. This means that a equality both of whose operands are ill-defined is false, so that \( \neg (\bot = \bot) \). For cases where this is inconvenient, we introduce a new relation, \( \simeq \), such that

\[ a \simeq b \equiv a = b \lor (a \uparrow \land b \uparrow). \]

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1. In fact, a type in Farmer 2012 [6] can be much more than a set. Types in [6] are collections of classes (“superclasses”). But in this paper, every explicitly stated type will be a ZF set.

2. See Farmer 2004 [5]. Note that Farmer refers to ill-defined values as “undefined”. We found this problematic. For example, a partial function may not have a value for every argument in its domain. Saying that that the value of the partial function for these arguments is defined as “undefined” is confusing. In this paper we say that these values are defined, but ill-defined.
We define a tuple recursively:

- An ordered pair is a 2-tuple, or duple, for example \( \langle 7, 11 \rangle \). The first entry of a 2-tuple is its **head**. The second entry of a 2-tuple is its **tail**.

- The ordered pair of a set \( h \), and an \( n \)-tuple, call it \( \text{tupA} \), is an \((n + 1)\)-tuple, call it \( \text{tupB} \). \( h \) is the **head** of \( \text{tupB} \). \( \text{tupA} \) is the **tail** of \( \text{tupB} \).

We write tuples using angle brackets. The head of a tuple \( S \) can be written \( \text{hd}(S) \). The tail of a tuple \( S \) can be written \( \text{tl}(S) \). For example, where

\[
S = \langle 42, 1729, 42 \rangle,
\]

then \( S \) is a 3-tuple, or triple,

\[
\text{hd}(S) = 42,
\]

\[
\text{tl}(S) = \langle 1729, 42 \rangle,
\]

\[
\text{hd}(\text{tl}(S)) = 1729,
\]

\[
\text{tl}(\text{tl}(S)) = 42, \quad \text{and}
\]

\[
\text{tl}(\text{tl}(\text{tl}(S))) \uparrow.
\]

We define the natural numbers, \( \mathbb{N} \), to include zero. \( D \to C \) is the set of partial functions from domain \( D \) to codomain \( C \). \( D \to C \) is the set of total functions from domain \( D \) to codomain \( C \). It follows that \( \mathbb{N} \to C \) is the set of infinite sequences of terms from the set \( C \); and that \( 42 \to C \) is the set of sequences of length 42 of terms from the set \( C \). We say that

\[
\text{domSet} \to C \equiv \{ fn \mid (\exists D \in \text{domSet} \mid fn \in D \to C) \}
\]

so that \( \mathbb{N} \to C \) is the set of finite sequences of terms from the set \( C \).

We write \( |\text{seq}| \) for the cardinality, or length, of the sequence \( \text{seq} \). \( \text{seq}[i] \) is the \( i \)‘th term of the sequence \( \text{seq} \), and is well-defined when \( 0 \leq i < |\text{seq}| \). We write often specify a sequence by giving its terms inside brackets. For example,

\[
[ a, 42, []]
\]

is the sequence of length 3 whose terms are, in order, \( a \), the number 42, and the empty sequence.

The last index of the sequence \( \text{seq} \) is \( \#\text{seq} \), so that \( \text{seq}[\#\text{seq}] \) is the last term of \( \text{seq} \). \( \#\text{seq} \) is ill-defined for the empty sequence, that is, if \( |\text{seq}| = 0 \). If \( \text{seq} \) is not the empty sequence, then \( |\text{seq}| = \#\text{seq} + 1 \).
To avoid sub- and superscripts, we usually write summation as a unary operation on a sequence. For example, we write

\[ \sum [ix : N < 100 :: ix] \text{ for } \sum_{ix=0}^{100} ix. \]

Let \( \text{vocab} \) be a non-empty set of symbols. The string type, \( \text{STR} \), is the set of all finite sequences of symbols:

\[ \text{STR} = \mathbb{N} \rightarrow \text{vocab} \]

Where \( s : \text{STR} \) is a string, we write \( |s_{\text{STR}}| \) for the string length, counted in symbols. We write \( \text{STR}^+ \) for the set of all non-null strings:

\[ \text{STR}^+ = \{ x : \text{STR} \mid |x_{\text{STR}}| > 0 \} \]

In this paper we use, without loss of generality, the grammar \( g \), where \( g \) is the 3-tuple

\[ \langle \text{vocab}_{\text{SYM}}, \text{rules}, \text{accept}_{\text{SYM}} \rangle. \]

Here \( \text{accept}_{\text{SYM}} \in \text{vocab} \). Call the language of \( g \), \( L(g) : \text{STR}^* \).

\( \text{rules}_{\text{RULE}}^* \) is a set of rules (type \( \text{RULE} \)), where a rule is a duple of the form \( \langle \text{lhs}_{\text{SYM}} ::= \text{rhs}_{\text{STR}} \rangle \), such that

\[ \text{lhs}_{\text{SYM}} \in \text{vocab} \text{ and } \text{rhs}_{\text{STR}} \in \text{STR}^+. \]

\( \text{lhs}_{\text{SYM}} \) is referred to as the left hand side (LHS) of \( r_{\text{RULE}} \). \( \text{rhs}_{\text{STR}} \) is referred to as the right hand side (RHS) of \( r_{\text{RULE}} \). The LHS and RHS of \( r_{\text{RULE}} \) may also be referred to as \( \text{LHS}(r_{\text{RULE}}) \) and \( \text{RHS}(r_{\text{RULE}}) \), respectively. This definition follows [2], which departs from tradition by disallowing an empty RHS.

The rules imply the traditional rewriting system, in which

- \( x_{\text{STR}} \Rightarrow y_{\text{STR}} \) states that \( x \) derives \( y \) in exactly one step;
- \( x_{\text{STR}} \overset{\ast}{\Rightarrow} y_{\text{STR}} \) states that \( x \) derives \( y \) in one or more steps; and
- \( x_{\text{STR}} \overset{*}{\Rightarrow} y_{\text{STR}} \) states that \( x \) derives \( y \) in zero or more steps.

We call these rewrites derivation steps. A sequence of one or more derivation steps, in which the left hand side of all but the first is the right hand side of its predecessor, is a derivation. We say that the symbol \( x \) induces the string of length 1 whose only term is that symbol, that is, the string \( [x_{\text{SYM}}] \). Pedantically, the terms of derivations, and the arguments of concatenations like \( s_{\text{STR}} \cdot s_{\text{STR}} \cdot s_{\text{STR}} \) must be strings. But in concatenations and derivation steps we often write the symbol to represent the string it induces so that

\[ a_{\text{STR}} \cdot b_{\text{SYM}} \cdot c_{\text{STR}} = a_{\text{STR}} \cdot [b_{\text{SYM}}] \cdot c_{\text{STR}}. \]
We say that a string $x$ is **nullable**, $\text{Nullable}(x_{\text{STR}})$, iff the empty string can be derived from it:

$$\text{Nullable}(x_{\text{STR}}) \equiv x \xrightarrow{\ast} \epsilon.$$ 

We say that a string $x$ is **nulling**, $\text{Nulling}(x_{\text{STR}})$, iff it always eventually derives the null string:

$$\text{Nulling}(s_{\text{STR}}) \equiv \forall y_{\text{STR}} \mid x \xrightarrow{\ast} y \implies y \xrightarrow{\ast} \epsilon.$$ 

We say that symbols are nulling or nullable based on the string they induce:

$$\text{Nullable}(x_{\text{SYM}}) \equiv \text{Nullable}([x]).$$

$$\text{Nulling}(x_{\text{SYM}}) \equiv \text{Nulling}([x]).$$

A string or symbol is

- **non-nullable** iff it is not nullable;
- a **proper nullable** iff it is nullable, but not nulling; and
- **non-nulling** iff it is not nulling.

Following Aycock and Horspool [2], all nullable symbols in grammar $g$ are nulling – every symbol which can derive the null string always derives the null string. It is shown in [2] how to do this without losing generality or the ability to efficiently evaluate a semantics that is defined in terms of an original grammar that includes symbols which are both nullable and non-nulling, empty rules, etc.

Also without loss of generality, it is assumed that there is a dedicated acceptance rule,

$$\text{accept}_{\text{RULE}} = \langle \text{accept}_{\text{SYM}} ::= \text{start}_{\text{SYM}} \rangle,$$

where $\text{start}_{\text{SYM}} \in \text{vocab}$, and that the accept symbol, $\text{accept}_{\text{SYM}}$, is such that

$$\forall x \in \text{rules} \mid \not\exists \text{pre}_{\text{STR}}, \text{post}_{\text{STR}} \mid \text{pre.} \text{accept}_{\text{SYM}} \cdot \text{post} = \text{RHS}(x_{\text{RULE}})$$

and

$$\text{accept}_{\text{SYM}} = \text{LHS}(x) \implies \text{accept}_{\text{RULE}} = x.$$ 

Our definition of the rightmost non-nulling symbol of a string, $\text{Right-NN}(x_{\text{STR}})$, is
Right-NN(\text{STR}) \equiv \exists \text{rnn} : \text{SYM} :: \\
\exists \text{pre} : \text{STR}, \text{post} : \text{STR} :: \\
x_{\text{STR}} = \text{pre}.\text{rnn}.\text{post} \land \text{Nullable(\text{post})} \land \neg \text{Nulling(\text{rnn})}.

Our definition of the rightmost non-nulling symbol of a rule, Right-NN(\text{r}_{\text{RULE}}), is

Right-NN(\text{r}_{\text{RULE}}) \equiv \text{Right-NN(RHS(r))}.

A rule $x_{\text{RULE}}$ is directly right-recursive if and only if

$LHS(x_{\text{RULE}}) = \text{Right-NN}(x_{\text{RULE}})$.

$x_{\text{RULE}}$ is right-recursive, Right-Recursive($x_{\text{RULE}}$), if and only if

$\exists y_{\text{STR}} \mid \text{Right-NN}(x_{\text{RULE}}) \Rightarrow y \land \text{Right-NN}(y) = LHS(x_{\text{RULE}})$.

Our definition of a grammar did not sharply distinguish terminals from non-terminals. The implementation of [11] allows terminals to be the LHS of rules, and every symbol except acceptSYM can be a terminal. [11] has options that allow the user to reinstate the traditional restrictions, in part or in whole. We note that, as a result of these definitions, sentential forms will be of type STR.

Let the input to the parse be $w : \text{STR}^+$. Locations in the input will be of type LOC. When we state our complexity results later, they will often be in terms of $n$, where $n = |w|$. $w[i]_{\text{SYM}}$ is character $i$ of the input, and is well-defined when $0 \leq i_{\text{LOC}} < |w|$.

We note that the previous definition of $w$ did not allow zero-length inputs. To simplify the mathematics, we exclude null parses and trivial grammars from consideration. In the implementation of [11], the MARPA parser deals with null parses and trivial grammars as special cases. (Trivial grammars are those that recognize only the null string.)

In this paper, EARLEY will refer to the Earley algorithm as it is presented in this paper — a simplified version of Earley 1970 [4] preceded and followed by a rewrite. LEO will refer to Leo’s revision of [4] as described in Leo 1991 [14]. AH will refer to the Aycock and Horspool’s revision of [4] as described in their 2002 paper [2]. MARPA will refer to the parser described in this paper, which combines features of EARLEY, LEO and AH. Where RECCE is a recognizer, $L(\text{RECCE}, g)$ will be the language accepted by RECCE when parsing $g$. 

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3. Rewriting the Grammar

We have already noted that no rules of \( g \) have a zero-length RHS, and that all symbols must be either nulling or non-nullable. These restrictions follow Aycock and Horspool [2]. The elimination of empty rules and proper nullables is done by rewriting the grammar. [2] shows how to do this without loss of generality.

Because MARPA claims to be a practical parser, it is important to emphasize that all grammar rewrites in this paper are done in such a way that the semantics of the original grammar can be reconstructed simply and efficiently at evaluation time. As one example, when a rewrite involves the introduction of new rule, semantics for the new rule can be defined to pass its operands up to a parent rule as a list. Where needed, the original semantics of a pre-existing parent rule can be “wrapped” to reassemble these lists into operands that are properly formed for that original semantics.

In the implementation of [11], the MARPA parser allows users to associate semantics with an original grammar that has none of the restrictions imposed on grammars in this paper. The user of a MARPA parser may specify any context-free grammar, including one with properly nullable symbols, empty rules, etc. The user specifies his semantics in terms of this original, “free-form”, grammar. [11] implements the rewrites, and performs evaluation, in such a way as to keep them invisible to the user. From the user’s point of view, the “free-form” of his grammar is the one being used for the parse, and the one to which his semantics are applied.

4. Earley’s Algorithm

This paper presents a specialized version of Earley’s algorithm. The version in this paper is simplified to take advantage of MARPA’s rewriting. Descriptions of the standard Earley’s algorithm are now plentiful.

A dotted rule (type \( DR \)) is a duple of rule and position in the rule.

\[
\begin{align*}
\text{Rule}(\text{d}r_{\text{DR}}) & \equiv \text{hd}(\text{d}r), \\
\text{Pos}(\text{d}r_{\text{DR}}) & \equiv \text{tl}(\text{d}r), \text{ and} \\
0 & \leq \text{Pos}(\text{d}r) \leq |\text{Rule}(\text{d}r)|.
\end{align*}
\]

The position of a dotted rule indicates the extent to which the rule has been recognized, and is represented with a large raised dot, so that

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3Focusing on the classic Earley-related literature, these include [1] pp. 320-321, [2], [9], [11], [17], and [13].
if
\[
\langle A_{\text{SYM}} ::= X_{\text{SYM}} \cdot Y_{\text{SYM}} \cdot Z_{\text{SYM}} \rangle
\]
is a rule,
\[
\langle A_{\text{SYM}} ::= X \cdot Y \cdot Z \rangle
\]
is the dotted rule with the dot at pos = 2, between \(Y_{\text{SYM}}\) and \(Z_{\text{SYM}}\).

Every rule concept, when applied to a dotted rule, is applied to the rule of the dotted rule. The following are examples:

\[
\begin{align*}
\text{LHS}(x_{\text{DR}}) & \equiv \text{LHS}(\text{Rule}(x)). \\
\text{RHS}(x_{\text{DR}}) & \equiv \text{RHS}(\text{Rule}(x)). \\
\text{Right-Recursive}(x_{\text{DR}}) & \equiv \text{Right-Recursive}(\text{Rule}(x)).
\end{align*}
\]

We also state the following:

\[
\begin{align*}
\text{Postdot}(x_{\text{DR}}) & \equiv \text{def} \begin{cases} 
\text{next}_{\text{SYM}}, & \text{if } \exists \text{pre}_{\text{STR}}, \text{post}_{\text{STR}}, A_{\text{SYM}} :: \not\perp, \text{otherwise.} \\
\end{cases} \\
\text{Next}(x_{\text{DR}}) & \equiv \text{def} \begin{cases} 
\langle A_{\text{SYM}} ::= \text{pre}_{\text{STR}} \cdot \text{next}_{\text{SYM}} \cdot \text{post}_{\text{STR}} \rangle : \text{DR}, \\
\text{if } x = \langle A ::= \text{pre} \cdot \text{next} \cdot \text{post} \rangle \\
\not\perp, & \text{otherwise.} \\
\end{cases} \\
\text{Penult}(x_{\text{DR}}) & \equiv \text{def} \begin{cases} 
\text{next}_{\text{SYM}}, & \text{if } \exists \text{pre}_{\text{STR}}, \text{post}_{\text{STR}}, A_{\text{SYM}} :: \\
\text{ Nullable}(\text{post}) \land \neg \text{Nulling}(\text{next}) \not\perp, & \text{otherwise} \\
\end{cases}
\end{align*}
\]

A **penult** is a dotted rule \(d_{\text{DR}}\) such that \(\text{Penult}(d) \downarrow\). We note that \(\text{Penult}(x_{\text{DR}})\) is never a nullable symbol. The **initial dotted rule** is

\[ (1) \quad \text{initial}_{\text{DR}} = \langle \text{accept}_{\text{SYM}} ::= \cdot \text{start}_{\text{SYM}} \rangle. \]

A **predicted dotted rule** is a dotted rule, other than the initial dotted rule, with a dot position of zero, for example,\n
\[ \text{predicted}_{\text{DR}} = \langle A_{\text{SYM}} ::= \cdot \text{alpha}_{\text{STR}} \rangle. \]

A **confirmed dotted rule** is the initial dotted rule, or a dotted rule with a dot position greater than zero. A **completed dotted rule** is a dotted rule with its dot position after the end of its RHS, for example,\n
\[ \text{completed}_{\text{DR}} = \langle A_{\text{SYM}} ::= \text{alpha}_{\text{STR}} \cdot \rangle. \]

Predicted, confirmed and completed dotted rules are also called, respectively, **predictions, confirmations** and **completions**.
A traditional Earley item (type EIMT) is a duple of dotted rule and origin.

\[ \text{DR}(x_{\text{EIMT}}) = \text{hd}(x). \]

\[ \text{Origin}(x_{\text{EIMT}}) = \text{tl}(x). \]

The origin is the location where recognition of the rule started. For convenience, the type ORIG will be a synonym for LOC, indicating that the variable designates the origin entry of an Earley item.

We find it convenient to apply dotted rule concepts to EIMT’s, so that the concept applied to the EIMT is the concept applied to the dotted rule of the EIMT. The following are examples:

\[ \text{LHS}(x_{\text{EIMT}}) \equiv \text{LHS}((\text{DR}(\text{dr})). \]

\[ \text{Postdot}(x_{\text{EIMT}}) \equiv \text{Postdot}((\text{DR}(\text{dr})). \]

\[ \text{Penult}(x_{\text{EIMT}}) \equiv \text{Penult}((\text{DR}(\text{dr})). \]

\[ \text{Next}(x_{\text{EIMT}}) \equiv \text{Next}((\text{DR}(\text{dr})). \]

\[ \text{Rule}(x_{\text{EIMT}}) \equiv \text{Rule}((\text{DR}(\text{dr})). \]

\[ \text{Right-Recursive}(x_{\text{EIMT}}) \equiv \text{Right-Recursive}((\text{DR}(\text{dr})). \]

An Earley parser builds a table of Earley sets,

\[ \text{table}[\text{EARLEY}, i], \text{ where } 0 \leq i_{\text{LOC}} \leq |w|. \]

Earley sets are of type ES. Earley sets are often named by their location: That is, the bijection between \( i_{\text{ES}} \) and \( i_{\text{LOC}} \) allows locations to be treated as the “names” of Earley sets. We often write \( i_{\text{ES}} \) to mean the Earley set at \( i_{\text{LOC}} \), and \( x_{\text{LOC}} \) to mean the location of Earley set \( x_{\text{ES}} \). The type designator ES is often omitted to avoid clutter, especially in cases where the Earley set is not named by location. Occasionally the naming location is an expression, so that

\[ (\text{Origin}(x_{\text{EIMT}}))_{\text{ES}} \]

is the Earley set at the origin of the EIMT \( x_{\text{EIMT}} \).

At points, we will need to compare the Earley sets produced by the different recognizers. \( \text{table}[\text{RECCE}, i] \) will be the Earley set at \( i_{\text{LOC}} \) in the table of Earley sets of the RECCE recognizer. For example, \( \text{table}[\text{MARPA}, j] \) will be Earley set \( j_{\text{LOC}} \) in MARPA’s table of Earley sets. In contexts where it is clear which recognizer is intended, \( \text{table}[k] \), or \( k_{\text{ES}} \), will symbolize Earley set \( k_{\text{LOC}} \) in that recognizer’s table of Earley sets. If \( \text{working}_{\text{ES}} \) is an Earley set, \( |\text{working}_{\text{ES}}| \) is the number of Earley items in \( \text{working}_{\text{ES}} \).
We often want the count of all the Earley items in a table, and we abbreviate the expression for this by omitting the quantifier, so that

\[
\sum \left[ \text{table}[\text{RECC}, i] \right] \equiv \sum \left[ i : \mathbb{N} \leq |w| :: \text{table}[\text{RECC}, i] \right].
\]

For example, \(\sum \left[ \text{table}[\text{MARPA}, i] \right]\) is the total number of Earley items in all the Earley sets of a MARPA parse.

Recall that there was a unique acceptance symbol, \textit{accept_{SYM}}, in \textit{g}. The input \(w\) is accepted if and only if,

\[
\langle \langle \text{accept}_{SYM} ::= \text{start}_{SYM} \cdot \rangle, 0 \rangle \in \text{table}[|w|]
\]

5. Confluences

An Earley item is also called a \textbf{parse item}. For MARPA, we will define another kind of parse item, a Leo item, later. Traditional parse items have type PIMT.

In MARPA, with each parse item is a set of confluences, which track the reasons the MARPA algorithm had for adding that parse to the Earley set. In an ambiguous parse, the MARPA algorithm may have more than one reason to add a parse item to an Earley set. Each reason is called a \textbf{confluence}.

A \textbf{confluence} is a duple, whose entries are called \textbf{inflows}. The first inflow of a confluence is the \textbf{mainstem}, and is either a parse item or ill-defined. The second inflow of a confluence is the \textbf{tributary}, and may be an Earley item, a token, or ill-defined. The hydrological terminology may seem ornate, but experience has shown that the overloading of more ordinary terms like ”predecessor”, ”cause”, and ”component”\(^4\) can be befuddling.

We hope this terminology is at least mildly intuitive. In hydrology, a confluence is a meeting of two upstream river channels to form a third, downstream, channel. Of the two upstream channels, one (usually the larger) is the mainstem, and the other is a tributary. For example, near Cairo, Illinois, there is a confluence of the Ohio and Mississippi Rivers, in which the the upstream Mississippi channel is the mainstem and the Ohio is a tributary.

Continuing the hydrological analogy, a sequence of parse items in which all but the first is the successor of its mainstem is called a \textbf{trunk}. The first term in the sequence is the \textbf{top} of the \textbf{trunk}, and the last term is the \textbf{bottom} of the trunk. Sequence terms which are neither top or bottom are \textbf{interior} terms of the trunk.

\(^4\)The term “component” was used in Irons \cite{Irons}.
Similarly, we can define a **tributary sequence** as a sequence in which all but the first is the successor of its tributary. Once again, the first term in the tributary sequence is the top of the trunk, and the last term is the bottom of the trunk. Tributary sequence terms which are neither top or bottom are interior terms of the trunk.

We sometimes refer to the confluences and their inflows as the **links** of parse items, reflecting that fact that they are typically implemented as pointers, or “links”. We also sometimes refer to confluences as **causations** of a parse item, because each confluence is the reason for the Earley algorithm to add the parse item to the Earley set.

### 6. Operations of the Earley Algorithm

For any Earley operation there is a current parse location, current\textsubscript{LOC}, and a current Earley set, current\textsubscript{ES}. Recall that we often write i\textsubscript{ES} for the Earley set at i\textsubscript{LOC}, and (exp)\textsubscript{ES} for the Earley set at the location given by the expression exp.

We write the set of confluences of a PIMT p\textsubscript{PIMT} in Earley set es as Confluences(p, es). Each Earley operation is shown in the form of an inference rule, the conclusion of which consists of

- a parse item, call it p\textsubscript{PIMT} of the Earley items to be added to current\textsubscript{ES}; and
- a confluence to be added to Confluences(p\textsubscript{PIMT}, current\textsubscript{ES}).

We note that when we said the confluence and parse item were “added” just now, that they are added to sets, and that an “add” is a no-op for an object that is already an element of the set. Implementations must take care not to allow duplicate confluences in confluence sets, and not to allow duplicate Earley items in Earley sets.

Each location starts with an empty Earley set. For the purposes of this description of EARLEY, the order of the Earley operations when building an Earley set is non-deterministic. When no more Earley items can be added, the Earley set is complete. In the MARPA implementation, the Earley sets are built in order from 0 to |w|.

#### 6.1. Initialization.

\[
\begin{align*}
\text{current}_{\text{LOC}} = 0 \\
\langle \text{initial}_{\text{DR}}, 0 \rangle : \text{EIMT} \\
\langle \perp, \perp \rangle : \text{CFLU}
\end{align*}
\]

Here initial\textsubscript{DR} is from [1]. Earley **initialization** only takes place in Earley set 0, and always adds exactly one EIM, with exactly one confluence. The mainstem and tributary of the confluence are both ill-defined.
6.2. Scanning.

\[
\text{mainstem}_{\text{EIMT}} \in (\text{current}_{\text{LOC}} - 1)_{\text{ES}} \\
\text{Postdot}(\text{mainstem}_{\text{EIMT}}) = w[\text{current}_{\text{LOC}} - 1]
\]

\[
\langle \text{Next}(\text{mainstem}_{\text{EIMT}}), \text{Origin}(\text{mainstem}) \rangle : \text{EIMT} \\
\langle \text{mainstem}_{\text{EIM}}, (w[\text{current}_{\text{LOC}} - 1]_{\text{SYM}}) \rangle : \text{CFLU}
\]

In the confluence added by a scanning operation, \text{mainstem}_{\text{EIMT}} is the mainstem, and the symbol \(w[\text{current}_{\text{LOC}} - 1]\) is the tributary. In the context of a parse location, a symbol is often called a \text{token}. We also say that \text{Postdot}(\text{mainstem}_{\text{EIMT}}) is the transition symbol of the confluence, and of the scanning operation.

6.3. Reduction.

\[
\text{tributary}_{\text{EIMT}} \in \text{current}_{\text{ES}} \\
\text{mainstem}_{\text{EIMT}} \in (\text{Origin}(\text{tributary}_{\text{EIMT}}))_{\text{ES}} \\
\text{Postdot}(\text{mainstem}_{\text{EIMT}}) = \text{LHS}(\text{tributary}_{\text{EIMT}})
\]

\[
\langle \text{Next}(\text{mainstem}_{\text{EIMT}}), \text{Origin}(\text{mainstem}) \rangle : \text{EIMT} \\
\langle \text{mainstem}_{\text{EIM}}, \text{tributary}_{\text{EIM}} \rangle : \text{CFLU}
\]

\text{Postdot}(\text{tributary}_{\text{EIMT}}) is the transition symbol of the reduction operation.

6.4. Prediction.

\[
\text{mainstem}_{\text{EIMT}} \in \text{current}_{\text{ES}} \\
\text{Postdot}(\text{mainstem}) = \text{lhs}_{\text{SYM}} \\
\langle \text{lhs}_{\text{SYM}} ::= \text{rhs}_{\text{STR}} \rangle \in \text{rules}
\]

\[
\langle (\text{lhs}_{\text{SYM}} ::= \bullet \text{rhs}_{\text{STR}}), \text{current}_{\text{LOC}} \rangle : \text{EIMT} \\
\langle \text{mainstem}_{\text{EIMT}}, \bot \rangle : \text{CFLU}
\]

The \text{EIMT} added by a prediction operation can be the mainstem of other prediction operations at \text{current}_{\text{LOC}}, so that one prediction operation can trigger a long series of others. These prediction operations can add many Earley items to \text{current}_{\text{ES}}, but the items added will not depend on the location or the input — they will depend only on the postdot symbol of the mainstem. This means that a number of optimizations are possible.

The tributary of the operation is ill-defined.

7. The Leo algorithm

In [14], Joop Leo presented a method for dealing with right recursion in \(O(n)\) time. Leo showed that, with his modification, Earley’s algorithm is \(O(n)\) for all LR-regular grammars. (LR-regular is LR where
lookahead is infinite length, but restricted to distinguishing between regular expressions.)

Summarizing Leo's method, it consists of spotting potential right recursions and memoizing them. Leo restricts the memoization to situations where the right recursion is unambiguous. Potential right recursions are memoized by Earley set, using what Leo called "transitive items". In this paper Leo's "transitive items" will be called Leo items. Leo items in the form that MARPA uses will be type LIM. "Traditional" Leo items, that is, those of the form used in Leo's paper [14], will be type LIMT.

We illustrate the Leo method by examining an EARLEY parse of the input "xxxx" using the grammar

\[
\langle S ::= RR \rangle \\
\langle RR ::= x \rangle \\
\langle RR ::= x . RR \rangle.
\]

This grammar and input produce the Earley sets in the display which follows. Each line in the display shows one EIMT, and has four columns.

- The first column shows the EIMT. The notation @n indicates that the EIMT in ES n, so that @3 indicates that EIMT which precedes it is in ES 3.

Since our primary interest is in completions, the next three columns are shown only for completed EIMT's.

- The second column is a note. "Accept" indicates that the EIMT is an accept EIMT. "Bottom", "Interior" and "Top" indicate that the EIMT is in a corresponding position in a Leo stack. Leo stacks will be explained below.

- The third column shows the EIMT's confluence. Our grammar is unambiguous, so there is always exactly one confluence for each EIMT. Within the confluence, EIMT inflows are represented by their equation number.

- The fourth column indicates whether the EIM is actually used as part of the parse. Our grammar is unambiguous, so there is only one parse.

\[
\langle \langle S ::= \bullet RR \rangle , 0 \rangle @0 \\
\langle \langle RR ::= \bullet x . RR \rangle , 0 \rangle @0 \\
\langle \langle RR ::= \bullet x \rangle , 0 \rangle @0
\]
A glance at (4)-(32) shows that many of the completions are not used in the parse. This seems wasteful, and we wonder if this waste can be avoided.

Five of the unused completions are accept EIMT's, that is, EIMT’s with completed start rules. There is at most one of these per ES, so that the overhead is $O(n)$, and small. A mechanism for eliminating useless accept rules would gain us little, and would have to, itself, come at a very small cost to be justified. We therefore look elsewhere.

For our analysis of the other useless completions, we will need some new conceptual tools. We introduce the concepts of Leo uniqueness, Leo eligibility, and Leo stack.

We say that an EIMT $x_{EIMT}$ is **Leo unique** in the Earley set $es_{ES}$, $Leo-Unique(x, es)$, iff it is a penult in $es$, and its postdot symbol is unique in $es$. More precisely,

$$Leo-Unique(eim_{EIMT}, es_{ES}) \equiv_{def} eim = \exists eim2 : EIMT :: eim2 \in es \land Penult(eim2)$$

In (33) it is important to emphasize that $eim_{EIMT}$ ranges over all the EIMT’s, not just the penults. This means that if a penult $pen_{EIMT}$ shares a postdot symbol with an non-penult $np_{EIMT}$ in the same Earley set, then $pen_{EIMT}$ is not Leo unique.

If $eim_{EIMT}$ in (33) is Leo unique in an ES, then the symbol $Postdot(eim)$ is also said to be **Leo unique** in that ES. For each Leo unique symbol, call it $transition$, in an ES, call it es, there is exactly one dotted rule, call it $dr$, and exactly one rule, call it $r$. We call $dr_{DR}$, the dotted rule for $transition_{SYM}$ in $es_{ES}$. We call $r_{RULE}$, the rule for $transition_{SYM}$ in $es_{ES}$.

An EIMT $x_{EIMT}$ is **Leo eligible** in Earley set $es_{ES}$,

$$Leo-Eligible(x_{EIMT}, es_{ES})$$

iff it is right recursive and Leo unique in $es_{ES}$. More precisely,

$$Leo-Eligible(x_{EIMT}, es_{ES}) \equiv_{def} Right-Recursive(Rule(x_{EIMT})) \land Leo-Unique(x_{EIMT}, es_{ES}).$$

An EIMT is a **Leo completion** iff the mainstem of one of its confluences is Leo eligible.

An EIMT, call it $eim1$, is a **Leo tributary** of another EIMT, call it $eim2$, in the Earley set at $current_{LOC}$ iff $eim2$ has a confluence whose tributary is $eim1_{EIMT}$, and whose mainstem is Leo eligible at the origin
of \texttt{eim1}. More precisely, \texttt{eim1} is a Leo tributary of \texttt{eim2} iff
\[
\exists \text{main} : \text{EIMT} :: \\
\langle \text{main}, \text{eim1} \rangle \in \text{Confluences} (\text{eim2}, \text{current}) \\
\land \text{Leo-Eligible} (\text{main}, \text{Origin} (\text{eim1})).
\]

A \textbf{Leo tributary sequence} is a tributary sequence in which every term except the top is a tributary of its successor. Every term of a Leo tributary sequence is in the same Earley set, so that it is intuitive to visualize a Leo tributary sequence as a vertical stack.

A \textbf{Leo stack} is a Leo tributary sequence such that all of the following are true:
- The bottom of the sequence is not a Leo completion.
- The top of the sequence is not a Leo tributary of any \texttt{EIMT}.
- The top and bottom are distinct.

A Leo stack must have at least two terms, and all but the bottom term will be a Leo completion. Joop Leo’s technique for eliminating useless items from an Earley table was based on the following insight:

Every Leo completion in a Leo stack, except the top of the stack, can be deduced from the top of the stack.\(^5\)

This meant that all items in a Leo stack, except the top and bottom can be memoized during the parse, and ignored if they turn to be useless.

Where \(n\) is the size of the largest Leo stack in a parse using the grammar in (3), the number of “Leo memoizable” \texttt{EIMT}’s is
\[
\sum_{2 \leq i \leq n} i - 2
\]
of which
\[
\sum_{2 \leq i < n} i - 2
\]
can be ignored.

For the example parse of (4)-(32), the number of items which can be memoized is 3, but 2 of these need to be used for evaluation, so that we actually save the processing of only one \texttt{EIMT}. This is, frankly, unimpressive.

But (34) and (35) are both \(O(n^2)\), and for larger \(n\) the effect of quadratic growth takes over quickly. Continuing to use the grammar

\(^5\)For this reason, in his 1991 [14], Leo’s term for his concept analogous to our Leo stack was “deterministic reduction path”.

in [3], and to let \( n \) be the size of the largest Leo stack, when \( n \geq 4 \), the number of ignorable EIMT’s is

\[
\frac{(n - 3) \times (n - 2)}{2}
\]

and the number of other EIMT’s works out to \( 7 \times n \). For \( n = 19 \), over half of the EIMT’s can be ignored.

Right recursions are often very long, so Leo memoization, if the cost is \( O(1) \) with small constants, is a clear win for many grammars in practical use. Beginning in section (7.2), we shall outline such a low-cost method.

7.1. Differences with Leo 1991. Our version of Leo memoization is somewhat different from that in Leo 1991 [14].

- MARPA’s Leo memoization is eager, while that of [14] is lazy.
- MARPA only does Leo memoization for right recursive rules.

Omission of Leo memoization does not affect correctness, so these changes preserve the correctness as shown in [14]. And, later in this paper, we will show that these changes also leave the complexity results of [14] intact.

Looking more closely at our first difference, the algorithm of [14] in some cases delayed Leo memoization until after later ES’s were constructed. It was not clear to us that this produced any savings. It did make the algorithm more complex, and presented a real obstacle to MARPA’s on-the-fly features, such as event generation. For these reasons, in MARPA, Leo memoization is eager.

Our second difference was to consider Leo memoization only for right recursive EIMT’s. In [14], any penult was subject to Leo memoization, not just right recursions. (We recall that an EIMT \( eim_{\text{DR}} \) is a penult if \( \text{Penult}(eim) \downarrow \).)

By restricting Leo memoization to right-recursive rules, MARPA incurs the cost of Leo memoization only in cases where Leo sequences can be infinitely long. This more careful targeting of the memoization is for efficiency reasons. If all penults are memoized, memoizations will be performed where the longest Leo stack is finite, so that the payoff is limited, and often quite small. A possible future optimization would be to identify non-right-recursive rules which generate Leo stacks which are long enough to justify inclusion in the Leo memoizations. But research would be needed to show that such an optimization is worthwhile.
7.2. Leo items. A traditional Leo item (LIMT) is a triple consisting of a dotted rule, a symbol (called the transition symbol), and a location.

\[
\text{DR}(\text{limt}_{\text{LIMT}}) \equiv \text{hd}(\text{limt}).
\]

\[
\text{Transition}(\text{limt}_{\text{LIMT}}) \equiv \text{hd}(\text{tl}(\text{limt})).
\]

\[
\text{Origin}(\text{limt}_{\text{LIMT}}) \equiv \text{tl}(\text{tl}(\text{limt})).
\]

As will be explained in more detail later, the LIMT \( \text{limt}_{\text{LIMT}} \) indicates that

\[
(\text{DR}(\text{limt}_{\text{LIMT}}), \text{Origin}(\text{limt}_{\text{LIMT}})) : \text{EIMT}
\]

is to be added on Leo reductions over the symbol \( \text{Transition}(\text{limt}_{\text{LIMT}}) \).

Pedantically, LIMT’s are not members of Earley sets, so we introduce a partial function from pairs of location and symbol to LIMT’s:

\[
\text{LIMT-Map} : \mathbb{N} \times \text{vocab} \rightarrow \text{LIMT}.
\]

That LIMT-Map is a function implies that, in each Earley set, there is at most one Leo item per symbol.

In practice we will usually avoid direct reference to LIMT-Map, finding it convenient and natural to overload the notion of, and notation for, Earley set membership. Where an Earley set \( i_{\text{ES}} \) and a LIMT \( x_{\text{LIMT}} \) are such that

\[
\exists \text{postdot} \in \text{vocab} \mid x = \text{LIMT-Map}(i_{\text{LOC}}, \text{postdot}_{\text{SYM}}),
\]

then we will often say that \( x_{\text{LIMT}} \) is in the Earley set \( i_{\text{ES}} \), and write

\[
x_{\text{LIMT}} \in i_{\text{ES}}.
\]

Accordingly, we have spoken of parse items, which may be Leo items, being “added to the Earley set”, and we will continue to speak in this way. We note that in (36), LIMT-Map is a partial function, and therefore its value is not necessarily defined.

Implementing the Leo logic requires adding Leo reduction as a new basic operation, adding a new premise to the Earley reduction operation, and extending the Earley sets to memoize Earley items as LIMT’s.

7.3. LIMT mainstems. In Section 6.3, we defined the Earley mainstem of an EIMT. We define the LIMT mainstem of an EIMT by analogy. When say that \( \text{stem}_{\text{LIMT}} \) is the LIMT mainstem of \( x_{\text{EIMT}} \) if and only if
LIMT-Mainstem(stem\textsubscript{LIMT}, x\textsubscript{EIMT}) is true, where
\[
\text{LIMT-Mainstem}(\text{stem}_{LIMT}, x_{EIMT}) \equiv \text{def}\]
\[
\text{Transition}(\text{stem}_{LIMT}) = \text{LHS}(x_{EIMT})
\wedge \text{stem}_{LIMT} \in (\text{Origin}(x_{EIMT}))_{ES}.
\]

7.4. Leo reduction.
\[
\text{tributary}_{EIMT} \in \text{current}_{ES}
\]
\[
\text{LIMT-Mainstem}(\text{mainstem}_{LIMT}, \text{tributary})
\]
\[
\langle \text{DR}(\text{mainstem}_{LIMT}), \text{Origin}(\text{mainstem}) \rangle : \text{EIMT}
\]
\[
\langle \text{mainstem}_{LIMT}, \text{tributary}_{EIMT} \rangle : \text{CFLU}
\]

The new Leo reduction operation resembles the Earley reduction operation, except that it looks for a mainstem LIMT, instead of an EIMT. LHS(tributary\textsubscript{EIMT}) is the transition symbol of the Leo reduction. A confluence which results from Leo reduction is called a Leo confluence. The mainstem of a Leo confluence is always a LIMT.

7.5. Revised Earley reduction. The Earley reduction of 6.3 still applies, with an additional premise:
\[
\text{tributary}_{EIMT} \in \text{current}_{ES}
\]
\[
\not\exists x: \text{LIMT} :: \text{LIMT-Mainstem}(x, \text{tributary})
\]
\[
\text{mainstem}_{EIMT} \in (\text{Origin}(\text{tributary}_{EIMT}))_{ES}
\]
\[
\text{Postdot}(\text{mainstem}_{EIMT}) = \text{LHS}(\text{tributary}_{EIMT})
\]
\[
\langle \text{Next}(\text{mainstem}_{EIMT}), \text{Origin}(\text{mainstem}) \rangle : \text{EIMT}
\]
\[
\langle \text{mainstem}_{EIMT}, \text{tributary}_{EIMT} \rangle : \text{CFLU}
\]

The additional premise prevents Earley reduction from being applied where there is an LIMT with LHS(tributary\textsubscript{EIMT}) as its transition symbol. This reflects the fact that Leo reduction replaces Earley reduction if and only if there is a Leo memoization.

7.6. Leo memoization. We are now ready to define the inference rules for Leo memoization. We define one rule that holds if a LIMT mainstem can be found for the tributary EIMT in current\textsubscript{ES}.
\[
\text{LIMT-Mainstem}(\text{stem}_{LIMT}, \text{trib}_{EIMT})
\]
\[
\text{Leo-Eligible} (\text{trib}_{EIMT}, \text{current}_{ES})
\]
\[
\langle \text{DR}(\text{stem}_{LIMT}), \text{Penult}(\text{trib}_{EIMT}), \text{Origin}(\text{stem}) \rangle : \text{LIMT}
\]
\[
\langle \text{stem}_{LIMT}, \text{trib}_{EIMT} \rangle : \text{CFLU}
\]
and another inference rule that holds if $\text{trib}_{\text{EIMT}}$ has no mainstem $\text{LIMT}$,

$$\forall \text{stem} : \text{LIMT} :: \neg \text{LIMT-Mainstem} (\text{stem, trib}_{\text{EIMT}})$$

Leo-Eligible ($\text{trib}_{\text{EIMT}}, \text{current}_{\text{ES}}$)

$$\langle \text{DR(trib}_{\text{EIMT}}), \text{Penult(trib}), \text{Origin(trib)} \rangle : \text{LIMT}$$

$$\langle \bot, \text{trib}_{\text{EIMT}} \rangle : \text{CFLU}.$$

We note that when the confluence of a $\text{LIMT}$ has no mainstem, its mainstem is ill-defined.

**Algorithm 1** Leo stack expansion

1. **procedure** $\text{LEOEXPAND(top : EIMT, confl : CFLU, current : ES)}$
   $\triangleright$ Expand Leo stack for $\text{confl} \in \text{Confluences(top, current)}$. top may have more than one Leo confluence and this algorithm must be run for each of them.
2.  $\langle \text{trunkLIM}_{\text{LIMT}}, \text{previousCompletion}_{\text{EIMT}} \rangle \leftarrow \text{confl}$
3.  **while** $\text{trunkLIM}_{\text{LIMT}} \downarrow$ **do**
4.      $\langle \text{nextTrunkLIM}_{\text{LIMT}}, \text{penult}_{\text{EIMT}} \rangle : \text{CFLU}$
5.      $\leftarrow \forall c :: c \in \text{Confluences(trunkLIM, current)}$
   $\triangleright$ $\text{LIMT}$’s always have exactly one confluence.
6.      $\text{completion}_{\text{EIMT}} \leftarrow \langle \text{Next(\text{DR(penult}_{\text{EIMT}}))}, \text{Origin(penult)} \rangle$
7.      Add $\text{completion}_{\text{EIMT}}$ to $\text{current}_{\text{ES}}$.
8.      Add $\langle \text{penult}_{\text{EIMT}}, \text{previousCompletion} \rangle : \text{CFLU}$
9.      to $\text{Confluences(completion, current}_{\text{ES}})$
   $\triangleright$ ”Adds” are to sets — elements already in the set must not be duplicated.
10. $\text{trunkLIM}_{\text{LIMT}} \leftarrow \text{nextTrunkLIM}_{\text{LIMT}}$
11. $\text{previousCompletion}_{\text{EIMT}} \leftarrow \text{completion}_{\text{EIMT}}$

12. **end while**
13. **end procedure**

7.7. *Leo evaluation.* MARPA evaluation deals with Leo-memoized $\text{EIMT}$’s by expanding those Leo stacks which contain memoized $\text{EIMT}$’s that are useful. For each right recursion, there will be only one such Leo stack. In the example parse of [1]-[32], that Leo stack is in $\text{ES 4}$. Algorithm [1] on page [21] recreates the Leo stack for a confluence of a top $\text{EIMT}$. We note that top of a Leo stack may have more than one confluence, and that the others may be either Leo or ordinary confluences. Algorithm [1] must be run for each of the Leo confluences.
8. The Aycock-Horspool Finite Automaton

In this paper a “split LR(0) ε-DFA” as described by Aycock and Horspool [2], will be called an Aycock-Horspool Finite Automaton, or AHFA. This section will summarize the ideas from [2] that are central to MARPA.

Aycock and Horspool based their AHFA’s on a few observations.

- In practice, Earley items sharing the same origin, but having different dotted rules, often appear together in the same Earley set.
- There is in the literature a method for associating groups of dotted rules that often appear together when parsing. This method is the LR(0) DFA used in the much-studied LALR and LR parsers.
- The LR(0) items that are the components of LR(0) states are, exactly, dotted rules.
- By taking into account symbols that derive the null string, the LR(0) DFA could be turned into an LR(0) ε-DFA, which would be even more effective at grouping dotted rules that often occur together into a single DFA state.

AHFA states are, in effect, a shorthand for groups of dotted rules that occur together frequently. Aycock and Horspool realized that, by changing Earley items to track AHFA states instead of individual dotted rules, the size of Earley sets could be reduced, and conjectured that this would make Earley’s algorithm faster in practice.

As a reminder, the original Earley items (EIMTs) were duples, \( \langle x_{\text{DR}}, x_{\text{ORIG}} \rangle \), where \( x_{\text{DR}} \) is a dotted rule. An Aycock-Horspool Earley item is a duple \( \langle y_{\text{AH}}, y_{\text{ORIG}} \rangle \), where \( y_{\text{AH}} \) is an AHFA state.

MARPA uses Earley items of the form created by Aycock and Horspool. A MARPA Earley item has type EIM, and a MARPA Earley item is often referred to as an EIM.

Aycock and Horspool did not consider Leo’s modifications, but MARPA incorporates them, and MARPA also changes its Leo items to use AHFA states. Marpa’s Leo items (LIMs) are triples of the form \( \langle \text{top}_{\text{AH}}, \text{transition}_{\text{SYM}}, \text{top}_{\text{ORIG}} \rangle \), where \( \text{transition}_{\text{SYM}} \) and \( \text{top}_{\text{ORIG}} \) are as in the traditional Leo items, and \( \text{top}_{\text{AH}} \) is an AHFA state. A MARPA Leo item has type LIM.
[2] also defines a partial transition function for pairs of AHFA state and symbol,

\[ \text{GOTO} : fa, (\epsilon \cup \text{vocab}) \mapsto fa. \]

\[ \text{GOTO}(\text{from}_{AH}, \epsilon) \] is a null transition. (AHFA’s are not fully deterministic.) If \( \text{predicted}_{AH} \) is the result of a null transition, it is called a predicted AHFA state. If an AHFA state is not a predicted AHFA state, it is called a confirmed AHFA state. The initial AHFA state is a confirmed AHFA state.

The states of an AHFA are not a partition of the dotted rules – a single dotted rule can occur in more than one AHFA state. In combining the improvements of Leo [14] and Aycock and Horspool [2], the following theorem is crucial.

**Theorem 8.1.** If a MARPA Earley item (EIM) is the result of a Leo reduction, then its AHFA state contains only one dotted rule.

**Proof.** Let the EIM that is the result of the Leo reduction be

\[ \text{result}_{EIM} = \langle \text{result}_{AH}, \text{result}_{ORIG} \rangle. \]

Let the Earley set that contains \( \text{result}_{EIM} \) be \( \text{iES} \). Since \( \text{result}_{EIM} \) is the result of a Leo reduction we know, from the definition of a Leo reduction, that

\[ \text{complete}_{DR} \in \text{result}_{AH} \]

where \( \text{complete}_{DR} \) is a completed rule. Let

\[ c_{RULE} = \text{Rule}(\text{complete}_{DR}) \text{and } cp = \text{Pos}(\text{complete}), \]

so that \( \text{complete}_{DR} = \langle c_{RULE}, cp \rangle \).

We note that \( cp > 0 \) because, in MARPA grammars, completions are never predictions.

Suppose, for a reduction to absurdity, that the AHFA state contains another dotted rule, \( \text{other}_{DR} \), that is, that

\[ \text{other}_{DR} \in \text{result}_{AH}, \]

where \( \text{complete}_{DR} \neq \text{other}_{DR} \). Let \( o_{RULE} \) be the rule of \( \text{other}_{DR} \), and \( op \) its dot position,

\[ \text{other}_{DR} = \langle o_{RULE}, op \rangle. \]

AHFA construction never places a prediction in the same AHFA state as a completion, so \( \text{other}_{DR} \) is not a prediction. Therefore, \( op > 0 \). To show this outer reduction to absurdity, we first prove by a first inner reductio that \( c_{RULE} \neq o_{RULE} \), then by a second inner reductio that \( c_{RULE} = o_{RULE} \).

\[ ^6 \text{In [2] confirmed states are called "kernel states", and predicted states are called "non-kernel states".} \]
Assume, for the first inner reductio, that \(c_{\text{RULE}} = o_{\text{RULE}}\). By the construction of an AHFA state, both \(\text{complete}_{\text{DR}}\) and \(\text{other}_{\text{DR}}\) resulted from the same series of transitions. But the same series of transitions over the same rule would result in the same dot position, \(cp = op\), so that if \(c_{\text{RULE}} = o_{\text{RULE}}\), \(\text{complete}_{\text{DR}} = \text{other}_{\text{DR}}\), which is contrary to the assumption for the outer reductio. This shows the first inner reductio.

Next, we assume for the second inner reductio that \(c_{\text{RULE}} \neq o_{\text{RULE}}\). Since both \(\text{complete}_{\text{DR}}\) and \(\text{other}_{\text{DR}}\) are in the same EIM and neither is a prediction, both must result from transitions, and their transitions must have been from the same Earley set. Since they are in the same AHFA state, by the AHFA construction, that transition must have been over the same transition symbol, call it \(\text{transition}_{\text{SYM}}\). But Leo uniqueness applies to \(\text{complete}_{\text{DR}}\), and requires that the transition over \(\text{transition}_{\text{SYM}}\) be unique in \(i_{\text{ES}}\).

But if \(c_{\text{RULE}} \neq o_{\text{RULE}}\), \(\text{transition}_{\text{SYM}}\) was the transition symbol of two different dotted rules, and the Leo uniqueness requirement does not hold. The conclusion that the Leo uniqueness requirement both does and does not hold is a contradiction, which shows the second inner reductio. Since the assumption for the second inner reductio was that \(c_{\text{RULE}} \neq o_{\text{RULE}}\), we conclude that \(c_{\text{RULE}} = o_{\text{RULE}}\).

By the two inner reductio’s, we have both

\[c_{\text{RULE}} \neq o_{\text{RULE}} \text{ and } c_{\text{RULE}} = o_{\text{RULE}},\]

which completes the outer reduction to absurdity. For the outer reductio, we assumed that \(\text{other}_{\text{DR}}\) was a second dotted rule in \(\text{result}_{\text{AH}}\), such that \(\text{other}_{\text{DR}} \neq \text{complete}_{\text{DR}}\). We can therefore conclude that

\[\text{other}_{\text{DR}} \in \text{result}_{\text{AH}} \implies \text{other}_{\text{DR}} = \text{complete}_{\text{DR}}.\]

If \(\text{complete}_{\text{DR}}\) is a dotted rule in the AHFA state of a Leo reduction EIM, then it must be the only dotted rule in that AHFA state. □

9. The Marpa Recognizer

9.1. Complexity. Alongside the pseudocode of this section are observations about its space and time complexity. In what follows, we will charge all time and space resources to Earley items, or to attempts to add Earley items. We will show that, to each Earley item actually added, or to each attempt to add a duplicate Earley item, we can charge amortized \(\mathcal{O}(1)\) time and space.

At points, it will not be immediately convenient to speak of charging a resource to an Earley item or to an attempt to add a duplicate Earley item. In those circumstances, we speak of charging time and space

- to the parse; or
• to the Earley set; or
• to the current procedure’s caller.

We can charge time and space to the parse itself, as long as the total time and space charged is $O(1)$. Afterwards, this resource can be re-charged to the initial Earley item, which is present in all parses. Soft and hard failures of the recognizer use worst-case $O(1)$ resource, and are charged to the parse.

We can charge resources to the Earley set, as long as the time or space is $O(1)$. Afterwards, the resource charged to the Earley set can be re-charged to an arbitrary member of the Earley set, for example, the first. If an Earley set is empty, the parse must fail, and the resource can be charged to the parse.

In a procedure, resource can be “caller-included”. Caller-included resource is not accounted for in the current procedure, but passed upward to the procedure’s caller, to be accounted for there. A procedure to which caller-included resource is passed will sometimes pass the resource upward to its own caller, although of course the top-level procedure does not do this.

For each procedure, we will state whether the time and space we are charging is inclusive or exclusive. The exclusive time or space of a procedure is that which it uses directly, ignoring resource charges passed up from called procedures. Inclusive time or space includes resource passed upward to the current procedure from called procedures.

Recall that Earley sets may be represented by $i_{ES}$, where $i$ is the Earley set’s location $i_{LOC}$. The two notations should be regarded as interchangeable. The actual implementation of either should be the equivalent of a pointer to a data structure containing, at a minimum, the Earley items, a memoization of the Earley set’s location as an integer, and a per-set-list. Per-set-lists will be described in Section 9.12.

9.2. Top-level code. The top-level code is Algorithm 2 on page 26. Exclusive time and space for the loop over the Earley sets is charged to the Earley sets. Inclusive time and space for the final loop to check for $\text{accept}_{br}$ is charged to the Earley items at location $|w|$. Overhead is charged to the parse. All these resource charges are obviously $O(1)$.

9.3. Ruby Slippers parsing. The top-level code of Algorithm 2 (p. 26) represents a significant change from AH [2]. Scan pass and Reduction pass are separated. As a result, when the scanning of tokens that start at location $i_{LOC}$ begins, the Earley sets for all locations prior to $i_{LOC}$ are complete. This means that the scanning operation has
Algorithm 2 Marpa Top-level

1: procedure Main
2:   Initial
3:   for i, 0 ≤ i ≤ |w| do
4:     \[\text{\texttt{Scan pass}}(i, w[i − 1])\]
5:     if \(|i_{ES}| = 0\) then
6:       reject \(w\) and return
7:     end if
8:   end for
9:   for every \(\langle x_{AH}, 0 \rangle: \text{EIM} \in \text{table}(|w|)\) do
10:    if accept_{DR} \(\in x_{AH}\) then
11:      accept \(w\) and return
12:    end if
13:   end for
14:   reject \(w\)
15: end procedure

available, in the Earley sets, full information about the current state of the parse, including which tokens are acceptable during the scanning phase.

Algorithm 3 Initialization

1: procedure Initial
2:   Add EIM pair((0)_{ES}, start_{AH}, 0)
3: end procedure

9.4. Initialization. The initialization code is Algorithm 3 on page 26. Inclusive time and space is \(O(1)\) and is charged to the parse.

9.5. Scan pass. The code for the scan pass is Algorithm 4 on page 27. transitions is a set of tables, one per Earley set. The tables in the set are indexed by symbol. Symbol indexing is \(O(1)\), since the number of symbols is a constant, but since the number of Earley sets grows with the length of the parse, it cannot be assumed that Earley sets can be indexed by location in \(O(1)\) time. For the operation transitions(1_{LOC}, s_{SYM}) to be in \(O(1)\) time, 1_{LOC} must represent a link directly to the Earley set. In the case of scanning, the lookup is always in the previous Earley set, which can easily be tracked in \(O(1)\) space
Algorithm 4 Marpa Scan pass

1: procedure \textsc{Scan pass}(i_{\text{LOC}}, a_{\text{SYM}})
2: \hspace{1em} for each mainstem_{EIM} \in \text{transitions}((i - 1), a) do \hfill ▷ Each pass through this loop is an EIM attempt.
3: \hspace{1em} \langle \text{from}_{\text{AH}}, \text{origin}_{\text{LOC}} \rangle \leftarrow \text{mainstem}_{EIM}
4: \hspace{1em} \text{to}_{\text{AH}} \leftarrow \text{GOTO} (\text{from}_{\text{AH}}, a_{\text{SYM}})
5: \hspace{1em} \text{ADD EIM PAIR}(i_{\text{ES}}, \text{to}_{\text{AH}}, \text{origin}_{\text{LOC}})
6: \hspace{1em} end for
7: end procedure

and retrieved in $O(1)$ time. Inclusive time and space can be charged to the mainstem_{EIM}. Overhead is charged to the Earley set at $i_{\text{LOC}}$.

Algorithm 5 Reduction pass

1: procedure \textsc{Reduction pass}(i_{\text{LOC}})
2: \hspace{1em} ▷ \text{table}[i] may include EIM’s added by by \textsc{Reduce one LHS} and the loop must traverse these.
3: \hspace{1em} for each Earley item work_{EIM} \in \text{table}[i] do
4: \hspace{1em} \langle \text{work}_{\text{AH}}, \text{origin}_{\text{LOC}} \rangle \leftarrow \text{work}_{EIM}
5: \hspace{1em} \text{lh-sides}_{\text{SYM}*} \leftarrow \text{a set containing the LHS}
6: \hspace{1em} \text{for each } \text{lh-sides}_{\text{SYM}*} \text{ do}
7: \hspace{1em} \hspace{1em} \text{\textsc{Reduce one LHS}}(i_{\text{LOC}}, \text{origin}_{\text{LOC}}, \text{lh-sym})
8: \hspace{1em} \text{end for}
9: \hspace{1em} end for
10: \textsc{Memoize transitions}(i_{\text{LOC}})
11: end procedure

9.6. Reduction pass. The code for the reduction pass is Algorithm 5 on page 27. The loop over \text{table}[i] must also include any items added by \textsc{Reduce one LHS}. This can be done by implementing \text{table}[i] as a list and adding new items at the end.

Exclusive time is clearly $O(1)$ per work_{EIM}, and is charged to the work_{EIM}. Additionally, some of the time required by \textsc{Reduce one LHS} is caller-included, and therefore charged to this procedure. Inclusive time from \textsc{Reduce one LHS} is $O(1)$ per call, as will be seen in Section 9.8 and is charged to the work_{EIM} that is current during that call to \textsc{Reduce one LHS}. Overhead may be charged to the Earley set at $i_{\text{LOC}}$. 
Algorithm 6 Memoize transitions

1: procedure Memoize transitions (i_{LOC})
2: for every postdot_{SYM}, a postdot symbol of i_{ES} do
3: if Leo-Eligible (postdot_{SYM}, i_{ES}) then
4: Set transitions (i_{LOC}, postdot_{SYM}) to a LIM
5: else
6: Set transitions (i_{LOC}, postdot_{SYM})
   to the set of eim : EIM such that
   eim ∈ i_{ES} ∧ Postdot (eim) = postdot_{SYM}
7: end if
8: end for
9: end procedure

9.7. Memoize transitions. The code for the memoization of transitions is Algorithm 6 on page 28. The transitions table for i_{ES} is built once all EIMs have been added to i_{ES}. We first look at the resource, excluding the processing of Leo items. The non-Leo processing can be done in a single pass over i_{ES}, in \( O(1) \) time per EIM. Inclusive time and space are charged to the Earley items being examined. Overhead is charged to i_{ES}.

We now look at the resource used in the Leo processing. A transition symbol transition_{SYM} is Leo eligible if it is Leo unique and its rule is right recursive. (If transition_{SYM} is Leo unique in i_{ES}, it will be the postdot symbol of only one rule in i_{ES}.) All but one of the determinations needed to decide if transition_{SYM} is Leo eligible can be precomputed from the grammar, and the resource to do this is charged to the parse. The precomputation, for example, for every rule, determines if it is right recursive.

One part of the test for Leo eligibility cannot be done as a precomputation. This is the determination whether there is only one EIM in i_{ES} whose postdot symbol is transition_{SYM}. This can be done in a single pass over the EIM's of i_{ES} that notes the postdot symbols as they are encountered and whether any is encountered twice. The time and space, including that for the creation of a LIM if necessary, will be \( O(1) \) time per EIM examined, and can be charged to EIM being examined.

9.8. Reduce one LHS. The code to reduce a single LHS symbol is Algorithm 7 on page 29. To show that

\[ \text{transitions}(\text{origin}_{LOC}, \text{lhs}_{SYM}) \]

can be traversed in \( O(1) \) time, we note that the number of symbols is a constant and assume that origin_{LOC} is implemented as a link back
Algorithm 7 Reduce one LHS symbol

1: procedure Reduce one LHS(i_{LOC}, origin_{LOC}, lhs_{SYM})
\hspace{1em} \triangleright \text{Each pass through the following loop is an EIM attempt}
2: \hspace{1em} for each pim \in transitions(origin_{LOC}, lhs_{SYM}) do
\hspace{2em} \triangleright \text{pim is a “postdot item”, either a LIM or an EIM}
3: \hspace{2em} if pim is a LIM, pim_{LIM} then
4: \hspace{3em} Leo reduction(i_{LOC}, pim_{LIM})
5: \hspace{2em} else
6: \hspace{3em} Earley reduction(i_{LOC}, pim_{EIM}, lhs_{SYM})
7: \hspace{2em} end if
8: \hspace{1em} end for
9: end procedure

to the Earley set, rather than as an integer index. This requires that
work_{EIM} in Reduction pass carry a link back to its origin. As imple-
mented in [11], Marpa’s Earley items have such links.

Inclusive time for the loop over the EIM attempts is charged to each
EIM attempt. Overhead is \(O(1)\) and caller-included.

Algorithm 8 Earley reduction

1: procedure Earley reduction(i_{LOC}, from_{EIM}, trans_{SYM})
2: \langle from_{AH}, origin_{LOC} \rangle \leftarrow from_{EIM}
3: to_{AH} \leftarrow \text{GOTO}(from_{AH}, trans_{SYM})
4: Add EIM pair(i_{ES}, to_{AH}, origin_{LOC})
5: end procedure

9.9. Earley reduction operation. The code that performs Earley
reduction is Algorithm 8 on page 29. Exclusive time and space is
clearly \(O(1)\). Earley reduction is always called as part of an EIM
attempt, and inclusive time and space is charged to the EIM attempt.

Algorithm 9 Leo reduction

1: procedure Leo reduction(i_{LOC}, from_{LIM})
2: \langle from_{AH}, trans_{SYM}, origin_{LOC} \rangle \leftarrow from_{LIM}
3: to_{AH} \leftarrow \text{GOTO}(from_{AH}, trans_{SYM})
4: Add EIM pair(i_{ES}, to_{AH}, origin_{LOC})
5: end procedure
9.10. **Leo reduction operation.** The code that performs Leo reduction is Algorithm 9 on page 29. Exclusive time and space is clearly $O(1)$. **Leo reduction** is always called as part of an **EIM** attempt, and inclusive time and space is charged to the **EIM** attempt.

**Algorithm 10 Add EIM pair**

1: procedure **ADD EIM PAIR**($i_{ES}$, $confirmed_{AH}$, $origin_{LOC}$)
2:   $confirmed_{EIM}$ ← $\langle confirmed_{AH}, origin_{LOC} \rangle$
3:   $predicted_{AH}$ ← GOTO($confirmed_{AH}, \epsilon$)
4:   if $confirmed_{EIM}$ is new then
5:     Add $confirmed_{EIM}$ to table[$i$]
6:   end if
7:   if $predicted_{AH} \neq \Lambda$ then
8:     $predicted_{EIM}$ ← $\langle predicted_{AH}, i_{LOC} \rangle$
9:     if $predicted_{EIM}$ is new then
10:    Add $predicted_{EIM}$ to table[$i$]
11:   end if
12: end if
13: end procedure

9.11. **Adding a pair of Earley items.** The code in Algorithm 10 on page 30 attempts to add a pair of Earley items (**EIMs**), one confirmed and the other a prediction. Algorithm 10 first attempts to add a confirmed **EIM**. Then Algorithm 10 checks for the existence of $predicted_{EIM}$, the **EIM** for the null-transition of the confirmed **EIM**. If $predicted_{EIM}$ exists, Algorithm 10 attempts to add $predicted_{EIM}$.

Inclusive time and space is charged to the calling procedure. Trivially, the space is $O(1)$ per call.

We show that time is also $O(1)$ by singling out the two non-trivial cases: checking that an Earley item is new, and adding it to the Earley set. **MARPA** checks whether an Earley item is new in $O(1)$ time by using a data structure called a PSL. PSL’s are the subject of Section 9.12. An Earley item can be added to the current set in $O(1)$ time if Earley set is seen as a linked list, to the head of which the new Earley item is added.

The resource used by **ADD EIM PAIR** is always caller-included. No time or space is ever charged to a predicted Earley item. At most one attempt to add a $predicted_{EIM}$ will be made per attempt to add a $confirmed_{EIM}$, so that the total resource charged remains $O(1)$. 
9.12. **Per-set lists.** In the general case, where \( x \) is an arbitrary datum, it is not possible to use duple \( \langle i_{ES}, x \rangle \) as a search key and expect the search to use \( O(1) \) time. Within MARPA, however, there are specific cases where it is desirable to do exactly that. This is accomplished by taking advantage of special properties of the search.

If it can be arranged that there is a link direct to the Earley set \( i_{ES} \), and that \( 0 \leq x < c \), where \( c \) is a constant of reasonable size, then a search can be made in \( O(1) \) time, using a data structure called a PSL. Data structures identical to or very similar to PSL’s are briefly outlined in both [4, p. 97] and [1, Vol. 1, pages 326-327]. But neither source gives them a name. The term PSL (“per-Earley set list”) is new with this paper.

A PSL is a fixed-length array of integers, indexed by an integer, and kept as part of each Earley set. While MARPA is building a new Earley set, \( j_{ES} \), the PSL for every previous Earley set, \( i_{LOC} \), tracks the Earley items in \( j_{ES} \) that have \( i_{LOC} \) as their origin. The maximum number of Earley items that must be tracked in each PSL is the number of AHFA states, \( |fa| \), which is a constant of reasonable size that depends on \( g \).

It would take more than \( O(1) \) time to clear and rebuild the PSL’s each time that a new Earley set is started. This overhead is avoided by “time-stamping” each PSL entry with the Earley set that was current when that PSL entry was last updated.

As before, where \( i_{ES} \) is an Earley set, let \( i_{LOC} \) be its location, and vice versa. \( i_{LOC} \) is an integer which is assigned as Earley sets are created. Let \( ID(x_{AH}) \) be the integer ID of an AHFA state. Numbering the AHFA states from 0 on up as they are created is an easy way to create \( ID(x_{AH}) \). Let \( PSL[i_{ES}][y] \) be the entry for integer \( y \) in the PSL in the Earley set at \( i_{LOC} \).

Consider the case where MARPA is building \( j_{ES} \) and wants to check whether Earley item \( x_{EIM} = \langle x_{AH}, x_{ORIG} \rangle \) is new. MARPA looks at

\[
\text{time-stamp}_{LOC} = PSL[i_{ES}][ID(x_{AH})],
\]

and proceeds as follows:

- PSL entries are initially undefined. If \( \text{time-stamp}_{LOC} \) is undefined, then the entry has never been used, and \( x_{EIM} \) is new. \( x_{EIM} \) will be added to \( j_{ES} \) and the time stamp will be reset.
- If \( \text{time-stamp} = j_{LOC} \), then \( x_{EIM} \) is not new, and will not be added to \( j_{ES} \). The time stamp is left as it is.
- If \( \text{time-stamp} \neq j_{LOC} \), then \( x_{EIM} \) is new. \( x_{EIM} \) will be added to \( j_{ES} \) and the time stamp will be reset.
Resetting the time stamp is done as follows:

$$\text{PSL}[^{x_{ES}}] [^{\text{ID}}(^{x_{AH}})] \leftarrow J_{\text{LOC}}.$$ 

9.13. **Complexity summary.** For convenience, we collect and summarize here some of the observations of this section.

*Observation 9.1.* The time and space charged to an Earley item which is actually added to the Earley sets is $O(1)$.

*Observation 9.2.* The time charged to an attempt to add a duplicate Earley item to the Earley sets is $O(1)$.

For evaluation purposes, MARPA adds a confluence to each EIM for every attempt to add that EIM, even if that EIM is a duplicate. Traditionally, complexity results treat parsers as recognizers, and such costs are ignored. This will be an issue when the space complexity for unambiguous grammars is considered.

*Observation 9.3.* The space charged to an attempt to add a duplicate Earley item to the Earley sets is $O(1)$ if the confluences are included, zero otherwise.

As noted in Section 9.11, the time and space used by predicted Earley items and attempts to add them is charged elsewhere.

*Observation 9.4.* No space or time is charged to predicted Earley items, or to attempts to add predicted Earley items.

10. **Preliminaries to the theoretical results**

10.1. **Nulling symbols.** Recall that MARPA grammars, without loss of generality, contain neither empty rules nor properly nullable symbols. This corresponds directly to a grammar rewrite in the implementation of [11], and its reversal during MARPA’s evaluation phase. For the correctness and complexity proofs in this paper, we assume an additional rewrite, this time to eliminate nulling symbols.

Elimination of nulling symbols is also without loss of generality, as can be seen if we assume that a history of the rewrite is kept, and that the rewrite is reversed after the parse. Clearly, whether a grammar $g$ accepts an input $w$ will not depend on the nulling symbols in its rules.

In [11], MARPA does not directly rewrite the grammar to eliminate nulling symbols. But nulling symbols are ignored in creating the AHFA states, and must be restored during MARPA’s evaluation phase, so that the implementation of [11] and this simplification for theory purposes track each other closely.
10.2. Comparing Earley items.

**Definition.** A MARPA Earley item **corresponds** to a traditional Earley item \( x_{EIMT} = \langle x_{DR}, x_{ORIG} \rangle \) if and only if the MARPA Earley item is a \( y_{EIM} = \langle y_{AH}, x_{ORIG} \rangle \) such that \( x_{DR} \in y_{AH} \). A traditional Earley item, \( x_{EIMT} \), corresponds to a Marpa Earley item, \( y_{EIM} \), if and only if \( y_{EIM} \) corresponds to \( x_{EIMT} \).

**Definition.** A set of EIM’s is **consistent** with respect to a set of EIMT’s, if and only if each of the EIM’s in the first set corresponds to at least one of the EIMT’s in the second set. A MARPA Earley set table\[Marpa, i\] is **consistent** if and only if all of its EIM's correspond to EIMT's in table\[Leo, i\].

**Definition.** A set of EIM’s is **complete** with respect to a set of EIMT’s, if and only if for every EIMT in the second set, there is a corresponding EIM in the first set. A MARPA Earley set table\[Marpa, i\] is **complete** if and only if, for every traditional Earley item in table\[Leo, i\], there is a corresponding Earley item in table\[Marpa, i\].

**Definition.** A MARPA Earley set is **correct** if and only that MARPA Earley set is complete and consistent.

10.3. About AHFA states. Several facts from [2] will be heavily used in the following proofs. For convenience, they are restated here.

**Observation 10.1.** Every dotted rule is an element of one or more AHFA states, that is,

\[
\forall x_{DR} \exists y_{AH} \mid x_{DR} \in y_{AH}.
\]

**Observation 10.2.** AHFA confirmation is consistent with respect to the dotted rules. That is, for all \( from_{AH}, t_{SYM}, to_{AH}, to_{DR} \) such that

\[
\text{GOTO}(from_{AH}, t_{SYM}) = to_{AH} \\
\land to_{DR} \in to_{AH},
\]

there exists \( from_{DR} \) such that

\[
\text{from}_{DR} \in from_{AH} \\
\land t_{SYM} = \text{Postdot}(from_{DR}) \\
\land \text{Next}(from_{DR}) = to_{DR}.
\]
Observation 10.3. AHFA confirmation is complete with respect to the dotted rules. That is, for all \( \text{from}_{AH}, \text{t}_{SYM}, \text{from}_{DR}, \text{to}_{DR} \) if
\[
\text{from}_{DR} \in \text{from}_{AH} \land \text{Postdot}(\text{from}_{DR}) = \text{t}_{SYM} \land \text{Next}(\text{from}_{DR}) = \text{to}_{DR}
\]
then there exists \( \text{to}_{AH} \) such that
\[
\text{GOTO}(\text{from}_{AH}, \text{t}_{SYM}) = \text{to}_{AH} \land \text{to}_{DR} \in \text{to}_{AH}.
\]

Observation 10.4. AHFA prediction is consistent with respect to the dotted rules. That is, for all \( \text{from}_{AH}, \text{to}_{AH}, \text{to}_{DR} \) such that
\[
\text{GOTO}(\text{from}_{AH}, \epsilon) = \text{to}_{AH} \land \text{to}_{DR} \in \text{to}_{AH},
\]
there exists \( \text{from}_{DR} \) such that
\[
\text{from}_{DR} \in \text{from}_{AH} \land \text{to}_{DR} \in \text{Predict}(\text{from}_{DR}).
\]

Observation 10.5. AHFA prediction is complete with respect to the dotted rules. That is, for all \( \text{from}_{AH}, \text{from}_{DR}, \text{to}_{DR} \), if
\[
\text{from}_{DR} \in \text{from}_{AH} \land \text{to}_{DR} \in \text{Predict}(\text{from}_{DR}),
\]
then there exists \( \text{to}_{AH} \) such that
\[
\text{to}_{DR} \in \text{to}_{AH} \land \text{GOTO}(\text{from}_{AH}, \epsilon) = \text{to}_{AH}
\]

11. Marpa is correct

11.1. Marpa’s Earley sets grow at worst linearly.

Theorem 11.1. For a context-free grammar, and a parse location \( i_{LOC} \),
\[
|\text{table}[\text{MARPA}, i]| = O(i).
\]

Proof. EIM’s have the form \( \langle x_{AH}, x_{ORIG} \rangle \). \( x_{ORIG} \) is the origin of the EIM, which in MARPA cannot be after the current Earley set at \( i_{LOC} \), so that
\[
0 \leq x_{ORIG} \leq i_{LOC}.
\]
The possibilities for \( x_{AH} \) are finite, since the number of AHFA states is a constant, \(|fa|\), which depends on \( g \). Since duplicate EIM’s are never added to an Earley set, the maximum size of Earley set \( i_{LOC} \) is therefore
\[
i_{LOC} \cdot |fa| = O(i_{LOC}). \qed
\]
11.2. Marpa’s Earley sets are correct.

**Theorem 11.2.** Marpa’s Earley sets are correct.

The proof is by triple induction, that is, induction with a depth down to 3 levels. We number the levels of induction 0, 1 and 2, starting with the outermost. The level 0 induction is usually called the outer induction. The level 1 induction is usually called the inner induction. Level 2 induction is referred to by number.

The outer induction is on the Earley sets. The induction variable is $i_{\text{LOC}}$, and the outer induction hypothesis is that every Earley set $\text{table}[\text{MARPA}, j]$, where $j_{\text{LOC}} < i_{\text{LOC}}$, is complete and consistent, and therefore correct. For the outer induction step we need to show that every Earley set $\text{table}[\text{MARPA}, k]$, $k_{\text{LOC}} \leq i_{\text{LOC}}$, is complete and consistent.

Since we have correctness for the Earley sets at locations less than $i_{\text{LOC}}$ by the outer induction hypothesis, all we need to show for the step of the outer induction is that the Earley set $\text{table}[\text{MARPA}, i]$ is correct. We leave it as an exercise to show, as the basis of the outer induction, that $\text{table}[\text{MARPA}, 0]$ is complete and consistent.

To show the outer induction step, we show first consistency, then completeness. We show consistency by an inner induction on the Marpa operations. The inner induction hypothesis is that $\text{table}[\text{MARPA}, i]$, as so far built, is consistent with respect to $\text{table}[\text{LEO}, i]$.

As the basis of the inner induction, an empty Marpa Earley set is consistent, trivially. We show the step of the inner induction by cases:

- Marpa scanning operations;
- Marpa reductions when there are no Leo reductions; and
- Marpa’s Leo reductions

11.2.1. *Marpa scanning is consistent.* For Marpa’s scanning operation, we know that the mainstem $\text{EIM}$ is correct by the outer induction hypothesis, and that the token is correct by the definitions in the preliminaries. We know, from Section 9.5, that at most two $\text{EIM}$’s will be added. We now examine them in detail.

Let

$$\text{confirmed}_{\text{AH}} = \text{GOTO}(\text{mainstem}_{\text{AH}}, \text{token}_{\text{SYN}})$$

If $\text{confirmed}_{\text{AH}} = \Lambda$, the pseudocode of Section 9.5 shows that we do nothing. If we do nothing, since $\text{table}[\text{MARPA}, i]$ is consistent by the inner induction hypothesis, it remains consistent, trivially.
Otherwise, let \( \text{confirmed}_{\text{EIM}} = \langle \text{confirmed}_{\text{AH}}, \text{i}_{\text{LOC}} \rangle \). We see that \( \text{confirmed}_{\text{EIM}} \) is consistent with respect to \( \text{table}[\text{LEO}, i] \), by the definition of Earley scanning (Section 6.2) and Observation 10.2. Consistency is invariant under union, and since \( \text{table}[\text{MARPA}, i] \) is consistent by the inner induction, \( \text{table}[\text{MARPA}, i] \) remains consistent after \( \text{confirmed}_{\text{EIM}} \) is added.

For predictions, if \( \text{confirmed}_{\text{AH}} \neq \Lambda \), let

\[
\text{predicted}_{\text{AH}} = \text{GOTO}(\text{confirmed}_{\text{AH}}, \epsilon)
\]

If \( \text{predicted}_{\text{AH}} = \Lambda \), the pseudocode of Section 9.11 shows that we do nothing. If we do nothing, since \( \text{table}[\text{MARPA}, i] \) is consistent by the inner induction hypothesis, it remains consistent, trivially. Otherwise, let

\[
\text{predicted}_{\text{EIM}} = \langle \text{predicted}_{\text{AH}}, \text{i}_{\text{LOC}} \rangle.
\]

We see that \( \text{predicted}_{\text{EIM}} \) is consistent with respect to \( \text{table}[\text{LEO}, i] \), by the definition of Earley prediction (Section 6.4) and Observation 10.4. Consistency is invariant under union and, since \( \text{table}[\text{MARPA}, i] \) is consistent by the inner induction, \( \text{table}[\text{MARPA}, i] \) remains consistent after \( \text{predicted}_{\text{EIM}} \) is added.

11.2.2. \textit{Earley reduction is consistent.} Next, we show that MARPA’s reduction operation is consistent, in the case where there is no Leo reduction. The reduction will be the result of the two EIM’s of a confluence, call them \( \text{mainstem}_{\text{EIM}} \) and \( \text{tributary}_{\text{EIM}} \). \( \text{mainstem}_{\text{EIM}} \) will be correct by the outer induction hypothesis and \( \text{tributary}_{\text{EIM}} \) will be consistent by the inner induction hypothesis. From \( \text{tributary}_{\text{EIM}} \), we will find zero or more transition symbols, \( \text{lhs}_{\text{SYM}} \). From this point, the argument is very similar to that for the case of the scanning operation.

Let

\[
\text{confirmed}_{\text{AH}} = \text{GOTO}(\text{mainstem}_{\text{AH}}, \text{lhs}_{\text{SYM}})
\]

If \( \text{confirmed}_{\text{AH}} = \Lambda \), we do nothing, and \( \text{table}[\text{MARPA}, i] \) remains consistent, trivially. Otherwise, let

\[
\text{confirmed}_{\text{EIM}} = \langle \text{confirmed}_{\text{AH}}, \text{i}_{\text{LOC}} \rangle.
\]

We see that \( \text{confirmed}_{\text{EIM}} \) is consistent with respect to \( \text{table}[\text{LEO}, i] \) by the definition of Earley reduction (Section 6.3), and Observation 10.2. By the invariance of consistency under union, \( \text{table}[\text{MARPA}, i] \) remains consistent after \( \text{confirmed}_{\text{EIM}} \) is added.

For predictions, the argument exactly repeats that of Section 11.2.1. \( \text{table}[\text{MARPA}, i] \) remains consistent, whether or not a \( \text{predicted}_{\text{EIM}} \) is added.
11.2.3. *Leo reduction is consistent.* We now show consistency for MARPA’s reduction operation, in the case where there is a Leo reduction. If there is a Leo reduction, it is signaled by the presence of $\text{mainstem}_{\text{LIM}}$,

$$\text{mainstem}_{\text{LIM}} = \langle \text{top}_{\text{AH}}, \text{lhs}_{\text{SYM}}, \text{top}_{\text{ORIG}} \rangle$$

in the Earley set where we would look for the $\text{mainstem}_{\text{EIM}}$. We treat the logic to create $\text{mainstem}_{\text{LIM}}$ as a matter of memoization of the previous Earley sets, and its correctness follows from the outer induction hypothesis.

As the result of a Leo reduction, LEO will add $\langle \text{top}_{\text{DR}}, \text{top}_{\text{ORIG}} \rangle$ to $\text{table}[\text{Leo}, j]$. Because the MARPA LIM is correct, using Observations 10.2 and 10.3 and Theorem 8.1 we see that $\text{top}_{\text{AH}}$ is the singleton set $\{\text{top}_{\text{DR}}\}$. From Section 9.10 we see that, as the result of the Leo reduction, MARPA will add

$$\text{leo}_{\text{EIM}} = \langle \text{top}_{\text{AH}}, \text{top}_{\text{ORIG}} \rangle$$

to $\text{table}[\text{MARPA}, i]$. The consistency of $\text{leo}_{\text{EIM}}$ follows from the definition of EIM consistency. The consistency of $\text{table}[\text{MARPA}, i]$, once $\text{leo}_{\text{EIM}}$ is added, follows by the invariance of consistency under union.

11.2.4. *Marpa’s Earley sets are consistent.* Sections 11.2.1, 11.2.2 and 11.2.3 show the cases for the step of the inner induction, which shows the induction. It was the purpose of the inner induction to show that consistency of $\text{table}[\text{MARPA}, i]$ is invariant under MARPA’s operations.

11.2.5. *The inner induction for completeness.* It remains to show that, when MARPA’s operations are run as described in the pseudocode of Section 9 that $\text{table}[\text{MARPA}, i]$ is complete. To do this, we show that at least one EIM in $\text{table}[\text{MARPA}, i]$ corresponds to every EIMT in $\text{table}[\text{Leo}, i]$. We will proceed by cases, where the cases are LEO operations. For every operation that LEO would perform, we show that MARPA performs an operation that produces a corresponding Earley item. Our cases for the operations of LEO are Earley scanning operations; Earley reductions; Leo reductions; and Earley predictions.

11.2.6. *Scanning is complete.* For scanning, the MARPA pseudocode (Algorithm 4 on page 27) shows that a scan is attempted for every pair

$$\langle \text{mainstem}_{\text{EIM}}, \text{token}_{\text{SYM}} \rangle,$$

where $\text{mainstem}_{\text{EIM}}$ is an EIM in the previous Earley set, and $\text{token}_{\text{SYM}}$ is the token scanned at $i_{\text{LOC}}$. (Algorithm 4 actually finds $\text{mainstem}_{\text{EIM}}$ in
a set returned by $\text{transitions()}$. This is a memoization for efficiency and we will ignore it.

By the preliminary definitions, we know that $\text{tok}_{\text{sym}}$ is the same in both Earley and LEO. By the outer induction hypothesis we know that, for every traditional Earley item in the previous Earley set, there is at least one corresponding Marpa Earley item. Therefore, Marpa performs its scan operation on a complete set of mainstems.

Comparing the Marpa pseudocode (Section 9.5), with the Earley scanning operation (Section 6.2) and using Observations 10.3 and 10.5, we see that an Earley item will be added to $\text{table}[\text{Marpa, i}]$ corresponding to every scanned Earley item of $\text{table}[\text{Leo, i}]$. We also see, from the pseudocode of Section 9.11, that the Marpa scanning operation will add to $\text{table}[\text{Marpa, i}]$ an Earley item for every prediction that results from a scanned Earley item in $\text{table}[\text{Leo, i}]$.

11.2.7. Earley reduction is complete. We now examine Earley reduction, under the assumption that there is no LEO transition. The Marpa pseudocode shows that the Earley items in $\text{table}[\text{Marpa, i}]$ are traversed in a single pass for reduction.

To show that we traverse a complete and consistent series of tributary Earley items, we stipulate that the Earley set is an ordered set, and that new Earley items are added at the end. From Theorem 11.1, we know that the number of Earley items is finite, so a traversal of them must terminate.

Consider, for the purposes of the level 2 induction, the reductions of LEO to occur in generations. Let the scanned Earley items be generation 0. An EIMT produced by a reduction is generation $n + 1$ if its tributary Earley item was in generation $n$. Predicted Earley items do not need to be assigned generations. In Marpa grammars they can never contain completions, and therefore can never act as the tributary of a reduction.

The level 2 induction is on generations. In Section 11.2.6 we showed that generation 0 is complete – it contains Earley items corresponding to all of the generation 0 EIMT’s of LEO. This is the basis of the level 2 induction.

The generation variable for the level 2 induction is $g$. The induction hypothesis for the step of level 2 induction is that for some $g$, the Earley items of $\text{table}[\text{Marpa, i}]$ for the generations prior to $g$ are correct (that is, complete and consistent). For the step we need to show that the Earley items of $\text{table}[\text{Marpa, i}]$ for the generation up to $g$ are correct. Since we have the correctness of the generation prior to $g$ by the induction hypothesis, all that we need to show for the step
will be that the Earley items of \texttt{table}[\texttt{MARPA}, i] for generation \texttt{g} are correct.

From Section 11.2.4 we know that all Earley items in \texttt{MARPA}'s sets are consistent. Therefore, to show correctness, we have only to show completeness.

Since we stipulated that \texttt{MARPA} adds Earley items at the end of each set, we know that they occur in generation order. Therefore \texttt{MARPA}, when creating Earley items of generation \texttt{n + 1} while traversing \texttt{table}[\texttt{MARPA}, i], can rely on the level 2 induction hypothesis for the completeness of Earley items in generation \texttt{n}.

Let \texttt{working}_{\text{EIM}} \in \mathbf{i}_{\mathbf{ES}} be the Earley item currently being considered as a potential tributary for an Earley reduction operation. From the pseudocode, we see that reductions are attempted for every pair \texttt{mainstem}_{\text{EIM}}, \texttt{working}_{\text{EIM}}. (Again, \texttt{transitions}() is ignored as a memoization.) By the outer induction hypothesis we know that, for every traditional Earley item in the previous Earley set, there is at least one corresponding \texttt{MARPA} Earley item. We see from the pseudocode, therefore, that for each \texttt{working}_{\text{EIM}} that \texttt{MARPA} performs its reduction operation on a complete set of correct mainstems. Therefore \texttt{MARPA} performs its reduction operations on a complete set of confluences.

Comparing the \texttt{MARPA} pseudocode (Section 9.9) with the Earley reduction operation (Section 6.3) and using Observations 10.3 and 10.5, we see that a Earley reduction result of generation \texttt{n + 1} will be added to \texttt{table}[\texttt{MARPA}, i] corresponding to every Earley reduction result in generation \texttt{n + 1} of \texttt{table}[\texttt{LEO}, i], as well as one corresponding to every prediction that results from an Earley reduction result of generation \texttt{n + 1} in \texttt{table}[\texttt{LEO}, i]. This shows the level 2 induction and the case of reduction completeness.

11.2.8. \textit{Leo reduction is complete}. We now show completeness for \texttt{MARPA}'s reduction operation, in the case where there is a \texttt{Leo} reduction. In Section 11.2.3, we found that where \texttt{LEO} would create the EIMT

\[ \langle \texttt{top}_{\text{DR}}, \texttt{top}_{\text{ORIG}} \rangle, \]

\texttt{MARPA} adds

\[ \langle \texttt{top}_{\text{AH}}, \texttt{top}_{\text{ORIG}} \rangle \]
such that $\text{top}_\text{DR} \in \text{top}_\text{AH}$. Since $\text{top}_\text{DR}$ is a completed rule, there are no predictions. This shows the case immediately, by the definition of completeness.

11.2.9. *Prediction is complete.* Predictions result only from items in the same Earley set. In Sections 11.2.6, 11.2.7, and 11.2.8 we showed that, for every prediction that would result from an item added to $\text{table}[\text{LEO},i]$, a corresponding prediction was added to $\text{table}[\text{MARPA},i]$.

11.2.10. *Finishing the proof.* Having shown the cases in Sections 11.2.6, 11.2.7, 11.2.8 and 11.2.9 we know that Earley set $\text{table}[\text{MARPA},i]$ is complete. In Section 11.2.4 we showed that $\text{table}[\text{MARPA},i]$ is consistent. It follows that $\text{table}[\text{MARPA},i]$ is correct, which is the step of the outer induction. Having shown its step, we have the outer induction, and the theorem. □

11.3. **MARPA is correct.** We are now in a position to show that MARPA is correct.

**Theorem 11.3.** $L(\text{MARPA}, g) = L(g)$

*Proof.* From Theorem 11.2 we know that

$$\langle \text{accept}_\text{DR}, 0 \rangle \in \text{table}[\text{LEO}, |w|]$$

if and only there is a

$$\langle \text{accept}_\text{AH}, 0 \rangle \in \text{table}[\text{MARPA}, |w|]$$

such that $\text{accept}_\text{DR} \in \text{accept}_\text{AH}$. From the acceptance criteria in the LEO definitions and the MARPA pseudocode, it follows that

$$L(\text{MARPA}, g) = L(\text{LEO}, g).$$

By Theorem 4.1 in [14], we know that

$$L(\text{LEO}, g) = L(g).$$

The theorem follows from the previous two equalities. □

12. **MARPA recognizer complexity**

12.1. *Complexity of each Earley item.* For the complexity proofs, we consider only MARPA grammars without nulling symbols. We showed that this rewrite is without loss of generality in Section 10.1 when we examined correctness. For complexity we must also show that the rewrite and its reversal can be done in amortized $O(1)$ time and space per Earley item.
Lemma 12.1. All time and space required to rewrite the grammar to eliminate nulling symbols, and to restore those rules afterwards in the Earley sets, can be allocated to the Earley items in such a way that each Earley item requires $O(1)$ time and space.

Proof. The time and space used in the rewrite is a constant that depends on the grammar, and is charged to the parse. The reversal of the rewrite can be done in a loop over the Earley items, which will have time and space costs per Earley item, plus a fixed overhead. The fixed overhead is $O(1)$ and is charged to the parse. The time and space per Earley item is $O(1)$ because the number of rules into which another rule must be rewritten, and therefore the number of Earley items into which another Earley item must be rewritten, is a constant that depends on the grammar. □

Theorem 12.2. All time in Marpa can be allocated to the Earley items, in such a way that each Earley item, and each attempt to add a duplicate Earley item, requires $O(1)$ time.

Theorem 12.3. All space in Marpa can be allocated to the Earley items, in such a way that each Earley item requires $O(1)$ space and, if confluences are not considered, each attempt to add a duplicate Earley item adds no additional space.

Theorem 12.4. If confluences are considered, all space in Marpa can be allocated to the Earley items in such a way that each Earley item and each attempt to add a duplicate Earley item requires $O(1)$ space.

Proof of Theorems 12.2, 12.3, and 12.4. These theorems follows from the observations in Section 9 and from Lemma 12.1. □

12.2. Duplicate dotted rules. The same complexity results apply to Marpa as to Leo, and the proofs are very similar. Leo’s complexity results [14] are based on charging resource to Earley items, as were the results in Earley’s paper [4]. But both assume that there is one dotted rule per Earley item, which is not the case with Marpa.

Marpa’s Earley items group dotted rules into AHFA states, but this is not a partitioning in the strict sense – dotted rules can fall into more than one AHFA state. This is an optimization, in that it allows dotted rules, if they often occur together, to be grouped together aggressively. But it opens up the possibility that, in cases where Earley and Leo disposed of a dotted rule once and for all, Marpa might have to deal with it multiple times. Marpa’s duplicate rules do not change the complexity results, although showing this requires some additional theoretical apparatus, which this section contains.
Theorem 12.5.
\[ \sum |\text{table}[\text{MARPA}, i]| < c \sum |\text{table}[\text{LEO}, i]|, \]
where \( c \) is a constant that depends on the grammar.

Proof. We know from Theorem 11.2 that every MARPA Earley item corresponds to one of LEO’s traditional Earley items. If an EIM corresponds to an EIM, the AHFA state of the EIM contains the EIM’s dotted rule, while their origins are identical. Even in the worst case, a dotted rule cannot appear in every AHFA state, so that the number of MARPA items corresponding to a single traditional Earley item must be less than \(|fa|\). Therefore,
\[ \sum |\text{table}[\text{MARPA}, i]| < |fa| \sum |\text{table}[\text{LEO}, i]| \]
\[ \square \]

Earley [4] shows that, for unambiguous grammars, every attempt to add an Earley item will actually add one. In other words, there will be no attempts to add duplicate Earley items. Earley’s proof shows that for each attempt to add a duplicate, the causation must be different – that the confluenes causing the attempt will differ in either their mainstem or their tributary. Multiple confluenes for an Earley item would mean multiple derivations for the sentential form that it represents. That in turn would mean that the grammar is ambiguous, contrary to assumption.

In MARPA, there is an slight complication. A dotted rule can occur in more than one AHFA state. Because of that, it is possible that two of MARPA’s operations to add an EIM will represent identical Earley confluenes, and therefore will be consistent with an unambiguous grammar. Dealing with this complication requires us to prove a result that is weaker than that of [4], but that is still sufficient to produce the same complexity results.

Theorem 12.6. For an unambiguous grammar, the number of attempts to add Earley items will be less than or equal to
\[ c \sum |\text{table}[\text{MARPA}, i]|, \]
where \( c \) is a constant that depends on the grammar.

Proof. Let initial-tries be the number of attempts to add the initial item to the Earley sets. For Earley set 0, it is clear from the pseudocode that there will be no attempts to add duplicate EIM’s:
\[ \text{initial-tries} = |\text{table}[0]| \]
Let \textit{leo-tries} be the number of attempted Leo reductions in Earley set \( j_{\text{LOC}} \). For Leo reduction, we note that by its definition, duplicate attempts at Leo reduction cannot occur. Let \textit{max-AHFA} be the maximum number of dotted rules in any AHFA state. From the pseudo-code of Sections 9.8 and 9.10, we know there will be at most one Leo reduction for each each dotted rule in the current Earley set, \( j_{\text{LOC}} \).

\[
\text{leo-tries} \leq \text{max-AHFA} \times |\text{table}[j]|
\]

Let \textit{scan-tries} be the number of attempted scan operations in Earley set \( j_{\text{LOC}} \). Marpa attempts a scan operation, in the worst case, once for every EIM in the Earley set at \( j_{\text{LOC}} - 1 \). Therefore, the number of attempts to add scans must be less than equal to \(|\text{table}[j - 1]|\), the number of actual Earley items at \( j_{\text{LOC}} - 1 \).

\[
\text{scan-tries} \leq |\text{table}[j - 1]|
\]

Let \textit{predict-tries} be the number of attempted predictions in Earley set \( j_{\text{LOC}} \). MARPA includes prediction in its scan and reduction operations, and the number of attempts to add duplicate predicted EIM’s must be less than or equal to the number of attempts to add duplicate confirmed EIM’s in the scan and reduction operations.

\[
\text{predict-tries} \leq \text{reduction-tries} + \text{scan-tries}
\]

The final and most complicated case is Earley reduction. Recall that \( j_{\text{ES}} \) is the current Earley set. Consider the number of reductions attempted. MARPA attempts to add an Earley reduction result once for every triple

\[
\langle \text{mainstem}_{\text{EIM}}, \text{transition}_{\text{SYM}}, \text{tributary}_{\text{EIM}} \rangle.
\]

where

\[
\text{tributary}_{\text{EIM}} = \langle \text{tributary}_{\text{AH}}, \text{tributary-origin}_{\text{LOC}} \rangle
\]

\[\wedge\text{tributary}_{\text{DR}} \in \text{tributary}_{\text{AH}}\]

\[\wedge\text{transition}_{\text{SYM}} = \text{LHS}(\text{tributary}_{\text{DR}}).
\]

We now put an upper bound on number of possible values of this triple. The number of possibilities for \text{transition}_{\text{SYM}} is clearly at most \(|\text{symbols}|\), the number of symbols in \( g \). We have \text{tributary}_{\text{EIM}} \in j_{\text{ES}}, and therefore there are at most \(|\text{table}[j]|\) choices for \text{tributary}_{\text{EIM}}.

We can show that the number of possible choices of \text{mainstem}_{\text{EIM}} is at most the number of AHFA states, \(|\text{fa}|\), by a reductio. Suppose, for the reductio, there were more than \(|\text{fa}|\) possible choices of \text{mainstem}_{\text{EIM}}. Then there are two possible choices of \text{mainstem}_{\text{EIM}} with the same AHFA state. Call these \text{choice1}_{\text{EIM}} and \text{choice2}_{\text{EIM}}. We know, by the definition of Earley reduction, that \text{mainstem}_{\text{EIM}} \in j_{\text{ES}},
and therefore we have $\text{choice}_1 \in j_{ES}$ and $\text{choice}_2 \in j_{ES}$. Since all EIM's in an Earley set must differ, and $\text{choice}_1$ and $\text{choice}_2$ both have the same AHFA state, they must differ in their origin. But two different origins would produce two different derivations for the reduction, which would mean that the parse was ambiguous. This is contrary to the assumption for the theorem that the grammar is unambiguous. This shows the reductio and that the number of choices for mainstem EIM, compatible with tributary_{ORIG}, is at most $|fa|$.

Collecting the results, we see that the number of possible choices for each tributary EIM is

$$|fa| \cdot |\text{symbols}| \cdot |\text{table}[j]|,$$

where $|fa|$ is the number of possible choices for mainstem EIM, $|\text{symbols}|$ is the number of possible choices for transition EIM, and $|\text{table}[j]|$ is the number of possible choices for tributary EIM.

The number of reduction attempts will therefore be at most

$$\text{reduction-tries} \leq |fa| \cdot |\text{symbols}| \cdot |\text{table}[j]|.$$

Summing

$$\text{tries} = \text{scan-tries} + \text{leo-tries} + \text{predict-tries} + \text{reduction-tries} + \text{initial-tries},$$

we have, where $n = |w|$, the size of the input,

$$|\text{table}[0]| + \sum_{j=0}^{n} \max-\text{AHFA} \cdot |\text{table}[j]| + 2 \sum_{j=1}^{n} |\text{table}[j-1]| + 2 \sum_{j=0}^{n} |fa| \cdot |\text{symbols}| \cdot |\text{table}[j]|$$

initial EIM's

LIM's

scanned EIM's

reduction EIM's.

In this summation, prediction-tries was accounted for by counting the scanned and predicted EIM attempts twice. Since max-AHFA and $|\text{symbols}|$ are both constants that depend only on $g$, if we collect the terms of the summation, we will find a constant $c$ such that, where $c$
is a constant that depends on $g$,

$$\text{tries} \leq c \times \sum_{j=0}^{n} |\text{table}[\text{Marpa}, j]|.$$  

Changing the index from $j$ to $i$, and abbreviating the count of all Earley items according to the convention of (2) on page 11, we have

$$\text{tries} \leq c \times \sum |\text{table}[\text{Marpa}, i]|. \quad \square$$

As a reminder, we follow tradition by stating complexity results in terms of $n$, setting $n = |w|$, the length of the input.

**Theorem 12.7.** For a context-free grammar,

$$\sum |\text{table}[\text{Marpa}, i]| = \mathcal{O}(n^2).$$

*Proof.* By Theorem 11.4, the size of the Earley set at $i_{\text{loc}}$ is $\mathcal{O}(i)$. Summing over the length of the input, $|w| = n$, the number of EIM’s in all of Marpa’s Earley sets is

$$\sum_{i_{\text{loc}}=0}^{n} \mathcal{O}(i) = \mathcal{O}(n^2). \quad \square$$

**Theorem 12.8.** For a context-free grammar, the number of attempts to add Earley items is $\mathcal{O}(n^3)$.

*Proof.* Reexamining the proof of Theorem 12.6, we see that the only bound that required the assumption that $g$ was unambiguous was reduction-tries, the count of the number of attempts to add Earley reductions. Let other-tries be attempts to add EIM’s other than as the result of Earley reductions. By Theorem 12.7,

$$\sum |\text{table}[\text{Marpa}, i]| = \mathcal{O}(n^2),$$

and by Theorem 12.6,

$$\text{other-tries} \leq c \times \sum |\text{table}[\text{Marpa}, i]|,$$

so that $\text{other-tries} = \mathcal{O}(n^2)$.

Looking again at reduction-tries for the case of ambiguous grammars, we need to look again at the triple

$$(\text{mainstem}_{\text{EIM}}, \text{transition}_{\text{SYM}}, \text{tributary}_{\text{EIM}}).$$

We did not use the fact that the grammar was unambiguous in counting the possibilities for transition$_{\text{SYM}}$ or tributary$_{\text{EIM}}$, but we did make
use of it in determining the count of possibilities for $\text{mainstem}_{\text{EIM}}$. We still know that

$$\text{mainstem}_{\text{EIM}} \in \text{tributary-origin}_{\text{ES}},$$

where $\text{tributary-origin}_{\text{LOC}}$ is the origin of $\text{tributary}_{\text{EIM}}$. Worst case, every

$$\text{EIM} \in \text{tributary-origin}_{\text{ES}}$$

is a possible match, so that the number of possibilities for $\text{mainstem}_{\text{EIM}}$ now grows to $|\text{table}[\text{tributary-origin}]|$, and

$$\text{reduction-tries} = |\text{table}[\text{tributary-origin}]| \times |\text{symbols}| \times |\text{table}[j]|.$$

We know that $\text{tributary-origin} \leq j$, so that by Theorem 11.1

$$|\text{table}[\text{tributary-origin}]| \times |\text{table}[j]| = \mathcal{O}(j^2).$$

Adding other-tries and summing over the Earley sets, we have

$$\mathcal{O}(n^2) + \sum_{j_{\text{loc}}=0}^{n} \mathcal{O}(j^2) = \mathcal{O}(n^3).$$

□

**Theorem 12.9.** Either a right derivation has a step that uses a right recursive rule, or it has length is at most $c$, where $c$ is a constant which depends on the grammar.

**Proof.** Let the constant $c$ be the number of symbols. Assume, for a reductio, that a right derivation expands to a Leo sequence of length $c + 1$, but that none of its steps uses a right recursive rule.

Because it is of length $c + 1$, the same symbol must appear twice as the rightmost symbol of a derivation step. (Since for the purposes of these complexity results we ignore nulling symbols, the rightmost symbol of a string will also be its rightmost non-nulling symbol.) So part of the rightmost derivation must take the form

$$\text{earlier-prefix}_{\text{STR}} \cdot A_{\text{SYM}} \xrightarrow{+} \text{later-prefix}_{\text{STR}} \cdot A_{\text{SYM}}.$$

But the first step of this derivation sequence must use a rule of the form

$$A_{\text{SYM}} ::= \text{rhs-prefix}_{\text{STR}} \cdot \text{rightmost}_{\text{SYM}},$$

where $\text{rightmost}_{\text{SYM}} \xrightarrow{+} A_{\text{SYM}}$. Such a rule is right recursive by definition. This is contrary to the assumption for the reductio. We therefore conclude that the length of a right derivation must be less than or equal to $c$, unless at least one step of that derivation uses a right recursive rule. □
12.3. **The complexity results.** We are now in a position to show specific time and space complexity results.

**Theorem 12.10.** For every LR-regular grammar, Marpa runs in $O(n)$ time and space.

*Proof.* By Theorem 4.6 in [14, p. 173], the number of traditional Earley items produced by Leo when parsing input $w$ with an LR-regular grammar $g$ is

$$O(|w|) = O(n).$$

Marpa may produce more Earley items than Leo for two reasons: First, Marpa does not apply Leo memoization to Leo sequences which do not contain right recursion. Second, Marpa’s Earley items group dotted rules into states and this has the potential to increase the number of Earley items.

By theorem 8.1, the definition of an EIMT, and the construction of a Leo sequence, it can be seen that a Leo sequence corresponds step-for-step with a right derivation. It can therefore be seen that the number of EIMT’s in the Leo sequence and the number of right derivation steps in its corresponding right derivation will be the same.

Consider one EIMT that is memoized in Leo. By theorem 8.1 it corresponds to a single dotted rule, and therefore a single rule. If not memoized because it is not a right recursion, this EIMT will be expanded to a sequence of EIMT’s. How long will this sequence of non-memoized EIMT’s be, if we still continue to memoize EIMT’s which correspond to right recursive rules? The EIMT sequence, which was formerly a memoized Leo sequence, will correspond to a right derivation that does not include any steps that use right recursive rules. By Theorem 12.9, such a right derivation can be of length at most $c_1$, where $c_1$ is a constant that depends on $g$. As noted, this right derivation has the same length as its corresponding EIMT sequence, so that each EIMT not memoized in Marpa will expand to at most $c_1$ EIMT’s.

By Theorem 12.5, when EIMT’s are replaced with EIM’s, the number of EIM’s Marpa requires is at worst, $c_2$ times the number of EIMT’s, where $c_2$ is a constant that depends on $g$. Therefore the number of EIM’s per Earley set for an LR-regular grammar in a Marpa parse is less than

$$c_1 \times c_2 \times O(n) = O(n).$$

LR-regular grammar are unambiguous, so that by Theorem 12.6, the number of attempts that Marpa will make to add EIM’s is less than or equal to $c_3$ times the number of EIM’s, where $c_3$ is a constant that depends on $g$. Therefore, by Theorems 12.2 and 12.4 the time and
space complexity of MARPA for LR-regular grammars is
\[ c3 \times O(n) = O(n). \]

**Theorem 12.11.** For every unambiguous grammar, MARPA runs in \( O(n^2) \) time and space.

*Proof.* Let \( g \) be an unambiguous grammar. By Theorem 12.6 and Theorem 12.7 the number of attempts that MARPA will make to add EIM’s is
\[ c \times O(n^2), \]
where \( c \) is a constant that depends on \( g \). Therefore, by Theorems 12.2 and 12.4 the time and space complexity of MARPA for unambiguous grammars is \( O(n^2) \). □

**Theorem 12.12.** For every context-free grammar, MARPA runs in \( O(n^3) \) time.

*Proof.* By Theorem 12.2 and Theorem 12.8 □

**Theorem 12.13.** For every context-free grammar, MARPA runs in \( O(n^2) \) space, if it does not track confluences.

*Proof.* By Theorem 12.3 and Theorem 12.7 □

Traditionally only the space result stated for a parsing algorithm is that without confluences, as in 12.13. This is sufficiently relevant if the parser is only used as a recognizer. In practice, however, algorithms like MARPA are typically used in anticipation of an evaluation phase, for which confluences are necessary.

**Theorem 12.14.** For every context-free grammar, MARPA runs in \( O(n^3) \) space, including the space for tracking confluences.

*Proof.* By Theorem 12.4 and Theorem 12.8 □

13. **The MARPA input model**

In this paper, up to this point, the traditional input stream model has been assumed. As implemented in [11], MARPA generalizes the idea of input streams beyond the traditional model.

MARPA’s generalized input model replaces the input \( w \) with a set of tokens, *tokens*, whose elements are triples of symbol, start location and length:
\[ \langle t_{SYM}, \text{start}_{LOC}, \text{length} \rangle \]
such that \( \text{length} \geq 1 \) and \( \text{start}_{LOC} \geq 0 \). The size of the input, \( |w| \), is the maximum over *tokens* of \( \text{start}_{LOC} + \text{length} \).
Multiple tokens can start at a single location. (This is how MARPA supports ambiguous tokens.) The variable-length, ambiguous and overlapping tokens of MARPA bend the conceptual framework of "parse location" beyond its breaking point, and a new term for parse location is needed. Start and end of tokens are described in terms of earleme locations, or simply earlemes. Token length is also measured in earlemes.

Like standard parse locations, earlemes start at 0, and run up to $|\mathbf{w}|$. Unlike standard parse locations, there is not necessarily a token "at" any particular earleme. (A token is considered to be "at an earleme" if it ends there, so that there is never a token "at" earleme 0.) In fact, there may be earlemes at which no token either starts or ends, although for the parse to succeed, such an earleme would have to be properly inside at least one token. Here "properly inside" means after the token's start earleme and before the token's end earleme.

In the MARPA input stream, tokens may interweave and overlap freely, but gaps are not allowed. That is, for all $i_{\text{LOC}}$ such that $0 \leq i_{\text{LOC}} < |\mathbf{w}|$, there must exist

$$\text{token} = \langle \text{SYM}, \text{start}_{\text{LOC}}, \text{length} \rangle$$

such that

$$\text{token} \in \text{tokens} \quad \text{and} \quad \text{start}_{\text{LOC}} \leq i_{\text{LOC}} < \text{start}_{\text{LOC}} + \text{length}.$$ 

The intent of MARPA’s generalized input model is to allow users to define alternative input models for special applications. An example that arises in current practice is natural language, features of which are most naturally expressed with ambiguous tokens. The traditional input stream can be seen as the special case of the MARPA input model where for all $x_{\text{SYM}}$, $y_{\text{SYM}}$, $x_{\text{LOC}}$, $y_{\text{LOC}}$, $x_{\text{length}}$, $y_{\text{length}}$, if we have both of

$$\langle x_{\text{SYM}}, x_{\text{LOC}}, x_{\text{length}} \rangle \in \text{tokens} \quad \text{and} \quad \langle y_{\text{SYM}}, y_{\text{LOC}}, y_{\text{length}} \rangle \in \text{tokens},$$

then we have both of

$$x_{\text{length}} = y_{\text{length}} = 1 \quad \text{and} \quad x_{\text{LOC}} = y_{\text{LOC}} \implies x_{\text{SYM}} = y_{\text{SYM}}.$$ 

The correctness results hold for MARPA input streams, but to preserve the time complexity bounds, restrictions must be imposed. In stating them, let it be understood that

$$\langle x_{\text{SYM}}, x_{\text{LOC}}, \text{length} \rangle_{\text{TOK}} \in \text{tokens}$$
We require that, for some constant $c_1$, possibly dependent on the grammar $g$, that

$$\forall \text{sym} : \text{SYM}, x : \text{LOC}, \text{length} : \mathbb{N} ::$$

$$\langle \text{sym}, j, \text{length} \rangle_{\text{TOK}} \in \text{tokens} \implies \text{length} < c_1.$$

We also require that the cardinality of the set of tokens starting at any one location be less than a constant, call it $c_2$:

$$\forall j : \text{LOC} ::$$

$$\left| \left\{ t : \text{TOK} :: \exists \text{sym} : \text{SYM}, \text{length} : \mathbb{N} :: t = \langle \text{sym}, j, \text{length} \rangle \land t \in \text{tokens} \right\} \right| < c_2$$

Because the number of symbols is a constant depending on the grammar, (38) follows from (37). Restrictions 37 and 38 impose little or no obstacle to the practical use of MARPA’s generalized input model. And with them, the complexity results for MARPA stand.

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