On a hydrodynamic description of waves propagating perpendicular to the magnetic field in relativistically hot plasmas

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The novel hydrodynamic model of plasmas with the relativistic temperatures consisted of four equations for the material fields: the concentration and the velocity field and the average reverse relativistic \( \gamma \) functor and the flux of the reverse relativistic \( \gamma \) functor is applied to study high-frequency part of spectrum of electromagnetic waves propagating perpendicular to the external magnetic field. The thermal effects considered for the temperatures close to the rest energy of electrons considerably change the dispersion equation in compare with the nonrelativistic temperatures. Analytical analysis of the changes is presented.

Keywords: relativistic plasmas, hydrodynamics, microscopic model, arbitrary temperatures

I. INTRODUCTION

Propagation of waves in the magnetized plasmas perpendicular to the magnetic field with the electric field perpendicular to the external magnetic field leads to the intermix of the longitudinal and transverse polarizations into well known extraordinary waves. It appears at the analysis of the plasmas in the nonrelativistic regime. If temperatures of plasmas increases up to the relativistic temperatures it leads to considerable changes of the hydrodynamic model and corresponding spectra of waves. The role of relativistic temperature effects in properties of waves propagating parallel to the external field is demonstrated in Ref. [1]. It is shown that the circularly polarized transverse waves are considerably modified by the thermal effect while there is no contribution of the thermal effects for the small nonrelativistic temperatures in the linear properties of these waves. It shows that the properties of relativistic plasmas considerably differ from the properties of the nonrelativistic plasmas. Therefore, the analysis of relativistic plasmas requires variety of models, particularly hydrodynamic models. Here, we present the further application of the recently suggested relativistic hydrodynamic model [2].

All macroscopic models must be derived from the corresponding microscopic model. Here we consider classic plasmas, where particles move up to the relativistic velocities getting close to the speed of light \( c \). The concentration of particles \( n(\mathbf{r}, t) \) in the arbitrary inertial frame [3], [4], [5] can be defined in the following form

\[
n(\mathbf{r}, t) = \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)).
\]  

(1)

The integral operator counts the number of particles in the vicinity of the point of space. hence, we have the number of particles in the volume \( \Delta \) around point of space \( \mathbf{r} \) in an arbitrary moment in time \( t \).

Other hydrodynamic functions are defined in the similar way [2] via operator

\[
\langle ... \rangle \equiv \frac{1}{\Delta} \int d\xi \sum_{i=1}^{N} \delta(\mathbf{r} + \xi - \mathbf{r}_i(t)),
\]

(2)

so we have \( \mathbf{j} = \langle \mathbf{v}_i(t) \rangle, \mathbf{v} = \mathbf{j}/n, \Gamma = \langle 1 \rangle, t^a = (\tau, \mathbf{r}), \eta^{ab} = (\eta^a_\xi \eta^b_\xi) - n n^a v^b, t^{ab} = (\eta^a_\xi v^b_\xi) - \Gamma v^a v^b - t^a v^b - t^a v^b, \) for \( M^{abcd} \) see equation (17) of Ref. [2], here \( \gamma_1 = 1/\sqrt{1 - \mathbf{v}_i(t)^2/c^2} \). The form of these functions is not chosen, but it appears via step by step derivation of the hydrodynamic equations as a continuous sequence. The energy-momentum density is absence here since it is not appear in the continuity, Maxwell or other hydrodynamic equations being under consideration. The microscopic dynamics based on the presentation of classic particles as the delta functions is considered in literature [6]. However, it contains no explicit transition to the macroscopic scale.

This paper is organized as follows. In Sec. II the relativistic hydrodynamic equations are presented and discussed. In Sec. III the spectrum of collective excitations is considered analytically. In Sec. IV a brief summary of obtained results is presented.

II. RELATIVISTIC HYDRODYNAMIC MODEL

Here we follow Refs. [1], [2] where a set of relativistic hydrodynamic equations is obtained and applied for plasmas with the relativistic temperatures. The model is composed of four equations. There are other hydrodynamic models of high-temperature relativistic plasmas, where the interaction of particles is considered in terms of the momentum evolution equation [7], [8], [9], [10], [11].

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First equation is the continuity equation

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0. \quad (3)$$

Next, the velocity field evolution equation is

$$n\partial_t v^a + n(\mathbf{v} \cdot \nabla)v^a + \frac{\partial^2 p}{m} = -\frac{e}{m} \Gamma E^a + \frac{e}{mc^2}abc(\Gamma v^b + t^b)B^c$$

and

$$\frac{e}{mc^2}(\Gamma v^a v^b + v^a t^b + v^b t^a)E^b - \frac{e}{mc^2}E^a,$$  \quad (4)

where $p$ is the flux of the thermal velocities.

The equation of evolution of the averaged reverse relativistic gamma factor is

$$\partial_t \Gamma + \partial_\mathbf{v}(\Gamma v^b + t^b) = -\frac{e}{mc^2}n\mathbf{v} \cdot \mathbf{E} \left( 1 - \frac{1}{c^2} \left( \frac{v^a + \frac{5p}{n} + 1}{c^2} \right) \right). \quad (5)$$

Function $\Gamma$ is also called the hydrodynamic Gamma function $[\Gamma].$

The final equation in this set of hydrodynamic equations is the equation of evolution for the thermal part of current of the reverse relativistic gamma factor (the hydrodynamic Theta function):

$$(\partial_t + \mathbf{v} \cdot \nabla)\Gamma + \partial_\mathbf{v}(\mathbf{v} \cdot \nabla)\Gamma + t^a(\nabla \cdot \mathbf{v})$$

$$+ \Gamma \frac{\partial}{\partial \mathbf{v}} \left( \frac{e}{m} nE^a \left[ 1 - \frac{v^2}{c^2} - \frac{3p}{mc^2} \right] \right)$$

$$+ \frac{e}{mc^2}abc n v^b B^c \left[ 1 - \frac{v^2}{c^2} - \frac{5p}{mc^2} \right] - \frac{2e}{mc^2}E^a p \left[ 1 - \frac{v^2}{c^2} \right]$$

$$+ \frac{e}{mc^2}n v^b B^c \left[ 1 - \frac{v^2}{c^2} - \frac{9p}{mc^2} \right] - \frac{10e}{3mc^2}M E^a. \quad (6)$$

Function $M$ is presented via the equation of state. All hydrodynamic equations are obtained in the mean-field approximation (the self-consistent field approximation).

The equations of electromagnetic field have the traditional form presented in the three-dimensional notations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = 4\pi(e_n - e_e),$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B}, \quad (7)$$

and

$$\nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E} + \frac{4\pi q_e}{c} n_e \mathbf{v}_e,$$ \quad (8)

where the ions exist as the motionless background.

**III. EXTRAORDINARY WAVES IN THE RELATIVISTIC MAGNETIZED PLASMAS**

We consider small amplitude collective excitations of the macroscopically motionless equilibrium state. This equilibrium state is described by the relativistic Maxwellian distribution. The equilibrium state is described within equilibrium concentration $n_0$. The velocity field $v_0$ in the equilibrium state is equal to zero. The equilibrium electric field $E_0$ is also equal to zero. The plasma is located in the constant and uniform external magnetic field $B_0 = B_0 \mathbf{e}_z$. Two second rank tensors and one fourth rank tensor are involved in the description of the thermal effects. The symmetric tensors $p^{ab}$ and $t^{ab}$ are assumed to be diagonal tensors: $p^{ab} = \delta^{ab}$ and $t^{ab} = i\delta^{ab}$. The "diagonal" form is assumed for the symmetric fourth rank tensor $M^{abcd}$ as well: $M^{abcd} = M_0(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})/3$. We consider propagation of perturbations in the direction perpendicular to the external magnetic field $k = (k_x, 0, 0)$.

The following equilibrium functions $\Gamma_0$, $\delta_0$, $p_0$, $t_0$, $q_0$, $M_0$ are involved in the description of the equilibrium state $p^{ab} = c^2\delta^{ab}\tilde{Z}f_1(\beta)/3$, $p^{ab} = c^2\delta^{ab}\tilde{Z}f_2(\beta)/3$ $M^{abcd} = c^4(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})\tilde{Z}f_3(\beta)/15$, $\Gamma_0 = n_0 K_1(\beta)/K_2(\beta)$, and $q = 0$, where $\beta = mc^2/T$, $\tilde{Z} = 4\pi Z(mc)^3 = n_0 K_1^{-1}(\beta)$, $f_1(\beta) = \int_1^{+\infty} \frac{dx}{x(x^2 - 1)^{3/2}} e^{-x\beta}$, $f_2(\beta) = \int_1^{+\infty} \frac{dx}{x^5(x^2 - 1)^{3/2}} e^{-x\beta}$, and $f_3(\beta) = \int_1^{+\infty} \frac{dx}{x^3(x^2 - 1)^{5/2}} e^{-x\beta}$.

Functions $f_1(\beta)$, $f_2(\beta)$ and $f_3(\beta)$ are calculated numerically below for the chosen values of temperatures. We introduce three characteristic velocities $\delta p = U^2_0\delta n$, $\delta t = U^2_0\delta n$, and $\delta M = U^4_0\delta n$.

Let us present the linearized equations for the plane wave excitations. For instance for the concentration we have $\delta n = N_0 e^{-i\omega t - ik_x x}$, where $\omega$ is the frequency, $N_0$ is the constant amplitude. We start with the linearized continuity equation $[\delta]$:

$$-i\omega \delta n + n_0 q_e k_x \delta v_x = 0. \quad (12)$$

Next, we show the linearized equations for the evolution of the three projections of velocity field

$$-i\omega n_0 \delta v_x + i k_x \frac{\delta p}{m} = \frac{q_e}{m} \left( \Gamma_0 - \frac{\ell_0}{c^2} \right) \delta E_x + \Omega_e(\Gamma_0 \delta v_y + \delta t_y), \quad (13)$$

$$-i\omega n_0 \delta v_y = \frac{q_e}{m} \left( \Gamma_0 - \frac{\ell_0}{c^2} \right) \delta E_y - \Omega_e(\Gamma_0 \delta v_x + \delta t_x). \quad (14)$$
and

\[-\omega n_0 \delta v_z = \frac{q_e}{m} \Gamma_0 \delta E_z - \frac{q_e}{mc^2} \Gamma_0 \delta E_z, \quad (15)\]

where \( \Omega_e = q_e B_0 / mc \) is the cyclotron frequency.

The Maxwell equations lead to

\[\omega^2 \delta E_x + 4\pi q_e \omega n_0 \delta v_x = 0, \quad (16)\]

\[(\omega^2 - k_x^2 c^2) \delta E_y + 4\pi q_e \omega n_0 \delta v_y = 0, \quad (17)\]

and

\[(\omega^2 - k_x^2 c^2) \delta E_z + 4\pi q_e \omega n_0 \delta v_z = 0. \quad (18)\]

To get a closed set of equations including the x- and y-projections of the velocity field and the x- and y-projections of the electric field we need to include the evolution equations for the x- and y-projections of the flux of the reverse relativistic factor:

\[-\omega \delta t_x - \omega \Gamma_0 \delta v_x + i k_x \delta \bar{t} = n_0 \left( 1 - \frac{5 q_e}{n_0 c^2} \right) \left[ \frac{q_e}{m} \delta E_y - \Omega_e \delta v_y \right] - \frac{10 q_e}{3 m c^4} M_0 \delta E_x, \quad (19)\]

\[\xi_{NR} = \begin{pmatrix} \xi_{xx,NR} & \xi_{xy,NR} \\ \xi_{yx,NR} & \xi_{yy,NR} \end{pmatrix} = \begin{pmatrix} -1 + \frac{\omega^2}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2} & \frac{\omega^2}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2} \\ -\frac{\omega^2}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2} & 1 \end{pmatrix}, \quad (22)\]

Let us to point out that this matrix is not symmetric. Moreover, it is not Hermitian.

Let us present corresponding matrix for the relativistically hot plasmas obtained from the linearized equations presented above:

\[\hat{\xi} = \begin{pmatrix} \frac{\omega^2}{\omega^2 - \Omega_e^2 (1 - \frac{5 u_p^2}{c^2})} \left[ -1 + \frac{\omega^2}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2} \right] & \frac{\omega^2}{\omega^2 - \Omega_e^2 (1 - \frac{5 u_p^2}{c^2})} \\ -\frac{\omega^2}{\omega^2 - \Omega_e^2 (1 - \frac{5 u_p^2}{c^2})} & -1 + \frac{\omega^2}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2} \left[ \frac{\omega^2}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2} - \frac{\omega^2}{\omega^2 - \Omega_e^2 (1 - \frac{5 u_p^2}{c^2})} \right] \end{pmatrix}, \quad (23)\]

where

\[\Sigma \equiv \frac{(\Gamma_0 - \frac{u_t^2}{c^2})}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2 (1 - \frac{5 u_p^2}{c^2})}, \quad (24)\]

and

\[\Xi \equiv \frac{(\Omega_e (1 - \frac{5 u_p^2}{c^2}) - \frac{10 u_p^4}{c^2})}{\omega^2 - k_x^2 u_p^2 - \Omega_e^2 (1 - \frac{5 u_p^2}{c^2})}. \quad (25)\]

Characteristic thermal velocities \( u_p \) and \( u_t \) reduce to the traditional nonrelativistic thermal velocity \( u \) appearing from the pressure (see equations (21) and (22)).

Element \( \xi_{xx,NR} \) at the transition to the relativistic case (23) has two modifications. First, we have \( \omega^2 / \omega^2 - \Omega_e^2 (1 - \frac{5 u_p^2}{c^2}) \) instead of \( \omega^2 / \omega^2 - \Omega_e^2 \) existing in the nonrelativistic regime. Such modification of the Langmuir frequency is not universal. In other elements of the matrix we have different representation of the Langmuir frequency. Second, the square of the cyclotron frequency \( \Omega_e^2 \) being in the denominator is replaced by the thermally modified version \( \Omega^2_e \left[ (1 - \frac{5 u_p^2}{c^2}) \right] \). It leads to the effective decrease of the cyclotron frequency by the thermal motion. This modification of the cyclotron frequency square \( \Omega^2_e \) being in the numerator has different transformation.

Element \( \xi_{xy,NR} \) transforms in two ways. First, the square of the cyclotron frequency \( \Omega_e^2 \) being in the de-
numerator is replaced by the thermally modified version
\( \Omega_e^2(1 - 5u^2/c^2) \). Second transformation is related to the
representation of the cyclotron frequency \( \Omega_e \) in the
numerator, where \( \Omega_e \) transforms into (1 - \( \frac{5u^2}{c^2} \)).

We see that combination \( \frac{\Omega}{\omega} \) multiplies by (1 - \( \frac{5u^2}{c^2} \)), but
the additional term proportional to \( uM \) appears.

In the nonrelativistic limit element \( \varepsilon_{yx,NR} \) has non-
trivial difference in compare with element \( \varepsilon_{xy,NR} \). The difference
between elements \( \varepsilon_{yx} \) and \( \varepsilon_{xy} \) is more
pronounced. Element \( \varepsilon_{yx,NR} \) is proportional to the sum
of two terms \(-i\frac{\Omega}{\omega} \) in two terms \(-i\frac{\Omega}{\omega} \)
and \(-i\frac{\Omega}{\omega} \). The first of them \(-i\frac{\Omega}{\omega} \) represents in the way similar to element \( \varepsilon_{xy,NR} \):

\[
-\frac{\Omega}{\omega} (1 - \frac{5u^2}{c^2}) - \frac{10u^4}{3c^2}. 
\]

The second of them \(-i\frac{\Omega}{\omega} \) is proportional to \( k^2u^2 \). While it has
following transformation \( k^2u^2 \rightarrow k^2u^2(\Gamma_0/n_0 - u^2/c^2) \). Moreover, the square of the cyclotron frequency \( \Omega_e^2 \) being in the denominator transforms in the way described above \( \Omega_e^2 \rightarrow \Omega_e^2(1 - 5u^2/c^2) \).

Element \( \varepsilon_{yy,NR} \) composed of the electromagnetic part
\( \frac{k^2u^2}{2c^2} \), which does not change, and the matter part
\( \frac{k^2u^2}{2c^2} \). The singular square of the Langmuir
frequency modifies in the way similar to element \( \varepsilon_{xx} \):

\[
\omega_{Le}^2 \rightarrow \omega_{Le}^2(\Gamma_0/n_0 - u^2/c^2). 
\]

Moreover, the spectrum of the extraordinary waves in the magnetized
plasmas has been illustrated analytically. To this end the explicit form of the dielectric permeability tensor has
been found. To illustrate the relativistic-thermal effects this
tensor is compared with the corresponding nonrelativistic
tensor.

Considerable modification of the dielectric permeability
tensor has been obtained. It demonstrates the background for the further analysis of the linear and nonlinear wave phenomena in the relativistically hot plasmas within the presented hydrodynamic model based on the
dynamics of four material fields: the concentration and the
velocity field and the average reverse relativistic \( \gamma \)
functor and the flux of the reverse relativistic \( \gamma \) functor.

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VI. DATA AVAILABILITY

Data sharing is not applicable to this article as no new
data were created or analyzed in this study, which is a
purely theoretical one.

Appendix A: Appendix: Numerical estimation of the
relativistic-thermal parameters

Three temperature regimes are chosen \( T_1 = 0.1mc^2 \),
\( T_2 = mc^2 \), and \( T_3 = 10mc^2 \). It gives the following values of the
dimensionless reverse temperature \( \beta = mc^2/T \):

\( \beta_1 = 10, \beta_2 = 1, \beta_3 = 0.1. \)

For the relatively small relativistic temperature \( \beta_1 = 10 \), we calculate \( K_1/K_2 = 0.91, U_0^2/c^2 = 0.07, U_0^2/c^2 = 0.08 \), and \( U_0^2/c^2 = 0.02 \), where \( K_1(10) = 2 \times 10^{-5}, K_2(10) = 2.2 \times 10^{-5} \), \( f_1(10) = 5 \times 10^{-7}, f_2(10) = 4.2 \times 10^{-7} \), \( f_3(10) = 1.7 \times 10^{-7} \).

For \( \beta_2 = 1 \), we get \( K_1/K_2 = 0.38, U_0^2/c^2 = 0.1, U_0^2/c^2 = 0.28 \), and \( U_0^2/c^2 = 0.15 \), where \( K_1(1) = 0.6, K_2(1) = 1.6, f_1(1) = 1.35, f_2(1) = 0.46, f_3(1) = 1.17 \).

For \( \beta_3 = 0.1 \), we have \( K_1/K_2 = 0.05, U_0^2/c^2 = 0.02, U_0^2/c^2 = 0.33, \) and \( U_0^2/c^2 = 0.2 \), where \( K_1(0.1) = 10, K_2(0.1) = 200, f_1(0.1) = 2 \times 10^3, f_2(0.1) = 100, f_3(0.1) = 2 \times 10^3 \).

IV. CONCLUSION

Contribution of the thermal-relativistic effects in the
spectrum of the extraordinary waves in the magnetized

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