SPRINGERS WORK ON UNIPOTENT CLASSES AND WEYL GROUP REPRESENTATIONS

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1. This paper will discuss some of the contributions of T. A. Sprin-ger (1926-2011) to the theory of algebraic groups, with emphasis on his work on unipotent classes and representations of Weyl groups. Many aspects of Springer’s work such as his contributions to the theory of Jordan algebras will not be discussed here.

Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic $p \geq 0$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Many questions about the structure and representations of $G$ and its forms over non-algebraically closed fields involve in essential way the understanding of unipotent elements in $G$; these elements, which fall into finitely many conjugacy classes, have a complicated combinatorial structure. A very important breakthrough in understanding this combinatorial structure was achieved by Springer. This breakthrough introduced a remarkable connection between unipotent classes in $G$ and irreducible representations of the Weyl group. This connection has become an indispensable ingredient in current work in the representation theory of groups over $F_q, \mathbb{C}, \mathbb{R}$ and in the theory of character sheaves. A $q$-analog of it gives rise to representations of affine Hecke algebras hence of $p$-adic groups.

2. When $G = GL_n(\mathbb{C})$, the unipotent conjugacy classes were classified by Weierstrass (1867) and Jordan (1870). L. E. Dickson (1901,1904) classified all conjugacy classes (in particular unipotent ones) in $Sp_4(F_q), Sp_6(F_q)$. J. Williamson (1937, 1939) classified all conjugacy classes (in particular unipotent ones) in $Sp_{2n}(k)$ with $p = 0$. Springer (Ph.D. Thesis, Leiden 1951, under H. D. Kloosterman) classifies all conjugacy classes (in particular unipotent ones) in $Sp_{2n}(k)$, with $p \neq 2$; he also gives the structure of centralizers. He says: “The classification for the other classical linear groups (orthogonal and unitary) can be done with an analogous method” (this is a translation from the Dutch original).

3. In [St65] Steinberg proved the existence of regular unipotent elements in $G$. An independent proof in the case where $p$ is a good prime for $G$ (semisimple) is given by Springer in [S66] where he also proves the following beautiful result: the centralizer of a regular unipotent element equals the centre times a unipotent
group $U$ and $U$ is connected if and only if $p$ is a good prime for $G$. In this paper it is also shown that, if $p$ is not a good prime for $G$, a regular unipotent element is not contained in the identity component of $U$. This is based on a very delicate analysis of the structure constants of a $\mathbb{Z}$-form of the Lie algebra of $G$.

In [S66a], Springer shows that the centralizer of any element of $G$ (for example a unipotent one) contains a closed abelian subgroup of dimension equal to the rank of $G$.

4. Let $G^{un}$ (resp. $\mathfrak{g}^{nil}$) be the variety of unipotent (resp. nilpotent) elements in $G$ (resp. $\mathfrak{g}$). Assume that $p$ is a good prime for $G$. In [S69], Springer shows that there exists a bijective morphism $f : G^{un} \to \mathfrak{g}^{nil}$ which commutes with the $G$-action. This is defined as follows. Let $u$ be regular unipotent in $G$. Let $z(u)$ be the Lie algebra of the centralizer of $u$ in $G$. Pick a regular nilpotent element $X \in z(u)$. Define a map $f_0$ from the $G$-orbit of $u$ to the $G$-orbit of $X$ by $gug^{-1} \mapsto gXg^{-1}$. This is well defined and it extends to the required bijection $f$ (the “Springer bijection”). Now $f$ is not unique; it depends on a number of parameters equal to the rank of $G$. The proof uses [S66] which provides knowledge of centralizers of a regular unipotent/nilpotent element. (Later it was shown that $f$ is an isomorphism except in type A where one needs a stronger hypothesis on $p$.) For example, if $G = GL_n(k)$, then $1 + e \mapsto a_1e + a_2e^2 + \ldots + a_{n-1}e^{n-1}$ is a Springer bijection for any $(a_1, a_2, \ldots, a_n) \in k^* \times k \times \ldots \times k$. In 1999 Serre showed that the map on conjugacy classes induced by a Springer bijection is canonical.

5. In this subsection we assume that $k$ is an algebraic closure of a finite field $F_q$ and that $G$ has a fixed $F_q$-rational structure. In [G], Green found all irreducible characters of $G(F_q)$ when $G = GL_n$. In [Sr], Srinivasan found all irreducible characters of $G(F_q)$ when $G = Sp_4$ and $q$ is odd. In 1968-69 there was a seminar at the Institute for Advanced Study attended by Springer, Macdonald, Srinivasan, Carter and others. The papers of Green and Srinivasan were studied and Macdonald formulated his conjecture (mentioned in a paper by Springer for this seminar) which associates an irreducible character of $G(F_q)$ to a maximal torus of $G$ defined over $F_q$ and a sufficiently general character of it. In a 1968 manuscript Macdonald gave a conjectural formula for its character which involved some unknown quantities. The missing part was a function on the unipotent elements in $G(F_q)$ (or equivalently, via the Springer bijection, at least when $p$ is a good prime, a function on the nilpotent elements in $\mathfrak{g}(F_q)$) which gives the character values at unipotent elements (the “Green functions”). Then in [S70] Springer came up with a new idea (inspired by work of Harish-Chandra): he proposed a definition of the Green function associated to a maximal torus $T$ defined over $F_q$ (assuming that $p$ is large enough):

$$f_T(N) = (\text{constant}) \sum_A \psi(<N, A>)$$

where $N \in \mathfrak{g}(F_q)$ is nilpotent, $A$ runs over the rational points of a $G(F_q)$-orbit of a regular element defined over $F_q$ in the Lie algebra of $T$; $\psi$ is a non-trivial character.
Springer’s work on unipotent classes and Weyl group representations.

6. Let $W$ be the Weyl group of $G$. Let $v$ be an indeterminate and let $H$ be the Hecke algebra over $C(v)$ associated to $W$. Let $\{T_w; w \in W\}$ be the standard basis of $H$; if $s \in W$ is a simple reflection then $(T_s + v^2)(T_s - 1) = 0$. Let $E$ be a simple $H$-module over an algebraic closure of $C(v)$. In [BC], Benson and Curtis showed that $E$ can almost always be defined over $C(v^2)$ (note that $H$ is obtained by extension of scalars from $C(v^2)$ to $C(v)$ from a $C(v^2)$-algebra.) Around 1972 Springer discovered an example when this rationality property actually fails. I have heard about it from Springer during his visit to the University of Warwick in 1973. I will explain below Springer’s argument. Assume that $G$ is of type $E_7$ and that $E$ is one of the two simple $H$-module of dimension 512, namely the one which for $v = 1$ becomes the irreducible representation $E$ of $W$ which comes from the Steinberg representation of $W/\pm 1 = SP_6(F_2)$. Now the trace of a simple reflection $s_i$ of $W$ on $E$ is 0. Hence the dimensions of the $+1, -1$ eigenspaces of $s_i$ on $E$ are 256, 256 and the action of $T_{s_i} \in H$ on $E$ has 256 eigenvalues $v^2$ and 256 eigenvalues $-1$, so that $\det(T_{s_i}, E) = v^{512}(-1)^{256} = v^{512}$. Let $w_0$ be the longest element of $W$; it is a product of 63 simple reflections and $T_{w_0} \in H$ is a product of 63 factors of the form $T_{s_i}$. Hence $\det(T_{w_0}, E) = v^{63 \times 512}$. Now $T_{w_0}$ is central in $H$ hence it acts on $E$ as a scalar $\lambda$. We have $\det(T_{w_0}, E) = \lambda^{512}$. Hence $\lambda^{512} = v^{63 \times 512}$ and $\lambda$ equals $v^{63}$ times a root of 1. In particular, $\lambda \notin C(v^2)$. This provides the required example.

7. In [S76], Springer discovered a connection between unipotent elements in $G$ and representations of $W$, with very important consequences for representation theory. There is a precursor for this: in [G], Green remarks that if one takes leading coefficients in the Green polynomials of $GL_n(F_q)$ (character at a unipotent element indexed by a partition of $n$ of a series of representations corresponding to a conjugacy class in the symmetric group $S_n$ indexed by another partition of $n$) one obtains the character table of $S_n$. In particular, the number of irreducible components of top degree of a ”Springer fibre” in $GL_n$ is the degree of an irreducible representation of $S_n$. As I wrote in my Math. Review of Springer’s paper, “this observation [of Green] finds a beautiful explanation in the paper under review”.

Let $B$ be the variety of Borel subgroups of $G$. For any $u \in G^{un}$ let $B_u = \{B \in B; u \in B\}$ (a ”Springer fibre”). This is a nonempty variety of dimension, say $d_u$. Assume that $p$ is a sufficiently large prime number. (This assumption is made so that Fourier transform on $g$ can be used.) Springer shows that $W$ acts naturally on the $l$-adic cohomology spaces $H^i(B_u)$. (Here $i \in \mathbb{Z}$ and $l$ is a prime $\neq p$.) Springer shows that any irreducible representation of $W$ appears in $H^{2d_u}(B_u)$ for some unipotent $u \in G$ which is in fact unique up to conjugation.
This defines a surjective map from the set of irreducible representations of $W$ (up to conjugation) to the set of unipotent conjugacy classes in $G$. This is essentially Springer’s correspondence. Springer shows that his Green functions (trigonometric sums) can be expressed in terms of the Springer representations (in all degrees).

In a later paper [S78] Springer constructs his representations of $W$ in the case $p = 0$; as a corollary he deduces that the irreducible representations of $W$ can be defined over the rational numbers (this was known earlier by a less transparent argument).

Springer conjectured that his representations of $W$ can be defined without restriction on $p$; this was proved in [L], using intersection cohomology methods.

8. Let $u \in G^{un}, i \in \mathbb{Z}$. We consider the setup of no.5. In [S84] Springer shows that, when $p$ is a sufficiently large prime, $H^i(B_u)$ is pure of weight $i$. The proof is based on the idea that $H^i(B_u)$ is the cohomology of a projective variety and at the same time it can be interpreted in terms of a Slodowy slice (which is smooth) and then using estimates of Deligne for the eigenvalues of Frobenius. Combined with earlier results by Shoji and Beynon-Spaltenstein, this implies that $H^i(B_u) = 0$ for $i$ odd. Springer conjectured that when $k = \mathbb{C}$, the integral homology of $B_u$ has no torsion and is zero in odd degrees. This conjecture was proved in [DLP]. When $p$ is a bad prime for $G$, it is not known whether $H^i(B_u) = 0$ for $i$ odd.

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