Properties of a New Subclass of Analytic Functions With Negative Coefficients Defined by Using the Q-Derivative

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Abstract

In this paper we define a new class of analytic functions with negative coefficients involving the q-differential operator. Our main purpose is to determine coefficient inequalities and distortion theorems for functions belonging to this class. Connections with previous results are pointed out.

Keywords: analytic functions, Janowski functions, negative coefficients, q-calculus, q-differential operator.

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1 Introduction

The quantum calculus or $q$-calculus has called the attention of many great researchers from Geometric function theory field due to its numerous applications in mathematics and physics. In 1999 was defined the first $q$-analogue of a starlike function by Ismail et al. [1]. In fact, many results obtained for univalent functions can be extended by using q-analogues functions. Recently, analytic functions with negative coefficient were studied in papers [5, 8–10].

Let $U = \{z : |z| < 1\}$ be the open unit disk of the complex plane and let $\mathcal{A}$ represent the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U.$$  

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For $0 < q < 1$, the q-derivative of a function $f \in \mathcal{A}$ is defined by (see [2])

$$d_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z}, \quad (z \neq 0),$$

with $d_q f(0) = f'(0)$.

Motivated by the aforementioned works, we define the following class of functions associated with Janowski functions:

**Definition 1.1.** For $0 \leq \mu \leq 1$, $k \geq 0$ and $-1 \leq B < A \leq 1$, we let $\mathcal{ST}(q, \mu, k; A, B)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the condition

$$\Re \left\{ \frac{(B - 1)zd_q f(z)}/[(1 - \mu)z + \mu f(z)] - (A - 1) \right\} > 0, \quad (B + 1)zd_q f(z)/[(1 - \mu)z + \mu f(z)] - (A + 1) \right\} \bigg|_{z \in U}.$$  (1.3)

Let $\mathcal{T}$ denote the subclass of analytic functions $f \in \mathcal{A}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0.$$  (1.5)

Further, let $\mathcal{ST}(q, \mu, k; A, B) \equiv \mathcal{ST}(q, \mu, k; A, B) \cap \mathcal{T}$.

Also, we remark that $\mathcal{ST}(q, \mu, k; A, B)$ reduces to the following known classes:

(i) In case $A = 1 - 2\alpha$, $0 \leq \alpha < 1$, $B = -1$, $\mu = 1$ and $q \to 1^-$, we obtain the class $S_p(k, \alpha)$ of $k$–uniformly starlike functions of order $\alpha$ (see [4]);

(ii) In case $A = 1, B = -1, \mu = 1, k = 1$ and $q \to 1^-$, we get the class $S_p$ of uniformly starlike functions (see [6]);

(iii) In case $\mu = 1$ and $q \to 1^-$, we obtain the class $k - \mathcal{ST}[A, B]$ which was introduced and studied by Noor and Malik (see [3]);

(iv) In case $A = 1 - 2\alpha$, $0 \leq \alpha < 1$, $B = -1$, $\mu = 1, k = 0$ and $q \to 1^-$, we get the class of starlike functions of order $\alpha$ denoted $S'(\alpha)$;

(v) In case $A = 1, B = -1, \mu = 0, k = 0$ and $q \to 1^-$, we obtain the class $R$ of functions whose derivative has positive real part.

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \mu \leq 1, k \geq 0$ and $-1 \leq B < A \leq 1$.

2 Coefficient Estimates

We begin with a result that provides coefficient inequalities for functions in the class $\mathcal{ST}(q, \mu, k; A, B)$.

**Theorem 2.1.** A function $f \in \mathcal{A}$ of the form (1.1) is in the class $\mathcal{ST}(q, \mu, k; A, B)$ if

$$\sum_{n=2}^{\infty} \left\{ (2k + 1)(|n|q - \mu) + ||n||_{q}(B + 1) - \mu (A + 1) \right\} |a_n| \leq |B - A|. \quad (2.1)$$

**Proof.** It is suffices to prove that

$$k|g(z) - 1| - \Re[g(z) - 1] < 1, \quad z \in U,$$  (2.2)

where the function $g$ is defined by

$$g(z) := \frac{(B - 1)zd_q f(z)/[(1 - \mu)z + \mu f(z)] - (A - 1)}{(B + 1)zd_q f(z)/[(1 - \mu)z + \mu f(z)] - (A + 1)}, \quad z \in U.$$  (2.3)
First of all,

\[ k|g(z) - 1| - \text{Re}[g(z) - 1] \leq (k + 1)|g(z) - 1|, \quad z \in U. \quad (2.4) \]

Because

\[
(k + 1)|g(z) - 1| = \frac{2(k + 1)\sum_{n=2}^{\infty} |n|q \cdot a_n z^{n-1}}{|(B - A) + \sum_{n=2}^{\infty} (B + 1) |n|q - \mu(A + 1)\cdot a_n z^{n-1}|} \leq \frac{2(k + 1)\sum_{n=2}^{\infty} |n|q \cdot a_n z^{n-1}}{|B - A| - \sum_{n=2}^{\infty} (B + 1) |n|q - \mu(A + 1)|\cdot a_n|} \quad (2.5)
\]

and (2.1) take place, then \((k + 1)|g(z) - 1|\) is bounded above by 1. Hence \(f \in ST(q, \mu, k; A, B)\) and the theorem is proved.

**Corollary 2.1.** (see [3], Theorem 2.1) Let function \(f \in A\) be of the form (1.1). If

\[
\sum_{n=2}^{\infty} \{2(k + 1)(n - 1) + |n(B + 1) - (A + 1)|\} \cdot |a_n| \leq |B - A|,
\]

then \(f \in k - ST[A, B]\).

**Corollary 2.2.** (see [7], Theorem 1) Let function \(f \in A\) be of the form (1.1). If

\[
\sum_{n=2}^{\infty} (n - \alpha) \cdot |a_n| \leq 1 - \alpha,
\]

then function \(f\) is in the class \(S^*(\alpha)\).

For \(f \in TST(q, \mu, k; A, B)\) the converse of Theorem 2.1 is also true.

**Theorem 2.2.** A function \(f \in T\) given by (1.5) is in the class \(TST(q, \mu, k; A, B)\) if and only if

\[
\sum_{n=2}^{\infty} \{2(k + 1)(|n|q - \mu) + |n|q(B + 1) - \mu(A + 1)|\} \cdot |a_n| \leq |B - A|.
\]

The result is sharp with the extremal function \(f\) given by

\[
f(z) = z - \frac{|B - A|}{2(k + 1)(|n|q - \mu) + |n|q(B + 1) - \mu(A + 1)}, \quad z \in U. \quad (2.9)
\]

**Proof.** In view of Theorem 2.1, we need only to prove that (2.8) holds if \(f \in TST(q, \mu, k; A, B)\). Conversely, assuming that \(f \in TST(q, \mu, k; A, B)\) and \(z\) is real, we find that \(g(z) \geq k|g(z) - 1|\), where \(g\) is given in (2.3). Therefore,

\[
\frac{(B - A) - \sum_{n=2}^{\infty} (B + 1) |n|q - \mu(A + 1)\cdot a_n z^{n-1}}{(B - A) - \sum_{n=2}^{\infty} (B + 1) |n|q - \mu(A + 1)\cdot a_n z^{n-1}} \geq 2k \left| \frac{\sum_{n=2}^{\infty} (|n|q - \mu) a_n z^{n-1}}{(B - A) - \sum_{n=2}^{\infty} (B + 1) |n|q - \mu(A + 1)\cdot a_n z^{n-1}} \right|.
\]

Next, letting \(z \to 1^-\) through real values, we obtain the inequality (2.8). Also, the equality in (2.8) take place for the function \(f\) given in (2.9).

**Corollary 2.3.** Let function \(f\) be of the form (1.5). Then \(f \in k - ST[A, B] \cap T\) if and only if

\[
\sum_{n=2}^{\infty} \{2(k + 1)(n - 1) + |n(B + 1) - (A + 1)|\} \cdot |a_n| \leq |B - A|.
\]

**Corollary 2.4.** (see [7], Theorem 2) Let function \(f \in T\) be of the form (1.5). Then \(f\) is a starlike function of order \(\alpha\) if and only if

\[
\sum_{n=2}^{\infty} (n - \alpha) \cdot |a_n| \leq 1 - \alpha.
\]
3 Distortion Theorems

Theorem 3.1. Let the function $f$ of the form (1.5) be in the class $\mathcal{T}ST(q, \mu, k; A, B)$. Then, for $|z| = r$,

$$|f(z)| \leq r + \frac{|B - A|}{2(k + 1)(|2|_q - \mu) + |2|_q(B + 1) - \mu(A + 1)|} r^2$$

and

$$|f(z)| \geq r - \frac{|B - A|}{2(k + 1)(|2|_q - \mu) + |2|_q(B + 1) - \mu(A + 1)|} r^2,$$

with equality for

$$f(z) = z - \frac{|B - A|}{2(k + 1)(|2|_q - \mu) + |2|_q(B + 1) - \mu(A + 1)|} z^2.$$  

Proof. By using the hypothesis and the triangle inequality, we find that

$$r - r^2 \sum_{n=2}^{\infty} a_n \leq |f(z)| \leq r + r^2 \sum_{n=2}^{\infty} a_n,$$

which, in conjunction with (2.8) gives us the inequalities (3.1) and (3.2).

Corollary 3.1. Let the function $f$ given by (1.5) be in the class $k - \mathcal{T}ST[A, B] \cap \mathcal{T}$. Then, for $|z| = r$,

$$r - \frac{|B - A|}{2(k + 1) + 2B - A + 1} r^2 \leq |f(z)| \leq r + \frac{|B - A|}{2(k + 1) + 2B - A + 1} r^2,$$

with equality for

$$f(z) = z - \frac{|B - A|}{2(k + 1) + 2B - A + 1} z^2.$$  

Corollary 3.2. (see [7], Theorem 4) Let the function $f \in \mathcal{T}$ given by (1.5) be in the class $S^*(\alpha) \cap \mathcal{T}$. Then, for $|z| = r$,

$$r - \frac{1 - \alpha}{2 - \alpha} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{2 - \alpha} r^2,$$

with equality for

$$f(z) = z - \frac{1 - \alpha}{2 - \alpha} z^2.$$  

The next theorem can be proven by employing similar techniques as in the demonstration of Theorem 3.1, so we will omit the details of our proof.

Theorem 3.2. Let the function $f$ of the form (1.5) be in the class $\mathcal{T}ST(q, \mu, k; A, B)$. Then, for $|z| = r$,

$$|f'(z)| \leq 1 + \frac{2|B - A|}{2(k + 1)(|2|_q - \mu) + |2|_q(B + 1) - \mu(A + 1)|} r$$

and

$$|f'(z)| \geq 1 - \frac{2|B - A|}{2(k + 1)(|2|_q - \mu) + |2|_q(B + 1) - \mu(A + 1)|} r,$$

with equality for $f$ given by (3.3).
Corollary 3.3. Let the function \( f \) given by (1.5) be in the class \( k-ST[A, B] \cap \mathcal{J} \). Then, for \(|z| = r\),

\[
1 - \frac{2|B - A|}{2(k + 1) + |2B - A + 1|} r \leq |f'(z)| \leq 1 + \frac{2|B - A|}{2(k + 1) + |2B - A + 1|} r,
\]

with equality for \( f \) given by (3.5).

Corollary 3.4. (see [7], Theorem 4) Let the function \( f \in \mathcal{J} \) given by (1.5) be in the class \( S^*\alpha \cap \mathcal{J} \). Then, for \(|z| = r\),

\[
1 - \frac{2(1 - \alpha)}{2 - \alpha} r \leq |f'(z)| \leq 1 + \frac{2(1 - \alpha)}{2 - \alpha} r,
\]

with equality for \( f \) given by (3.6).

References

[1] M.E.H Ismail, E. Merkes, D. Styer, A generalization of starlike functions, Complex Var. Theory Appl., vol. 14, 77-84, 1999.
[2] F.H. Jackson, On q-functions and certain difference operator, Trans. R. Soc. Edinburgh, vol. 46, 253-281, 1908.
[3] K. L. Noor and S. N. Malik, On coefficient inequalities of functions associated with conic domains, Comput. Math. Appl. 62, 2209-2219, 2011.
[4] S. Owa, On uniformly convex functions, Math. Japon, 48, 377-384, 1998.
[5] Akhter Rasheed, Saqib Hussain, Muhammad Asad Zaighum and Maslina Darus, Class of Analytic functions related with uniformly convex and Janowski functions, Journal of Function Spaces, Vol. 2018, Article ID 4679857, 1-6, 2018.
[6] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc., vol. 18, 189-196, 1993.
[7] Herb Silverman, Univalent functions with negative coefficients, Proceedings of the American Mathematical Society, vol. 51, no. 1, 1975.
[8] H.M. Srivastava, R.M. El-Ashwah, N. Breaz, A certain Subclass of Multivalent Functions Involving Higher-Order Derivatives, Filomat, vol. 30, no.1, 113-124, 2016.
[9] H.M. Srivastava, M. Tahir, B.Khan, Q.Z. Ahmad and N. Khan, Some General Classes of q-Starlike Functions Associated with the Janowski Functions, Symmetry, vol. 292, no.11, 1-14, 2019.
[10] K. Vijaya, G. Murugusundaramoorthy, S. Yalcin, Certain Class of Analytic Functions Involving Salagean Type q-Difference Operator, Konuralp Journal of Mathematics, vol.6, no.2, 264-271, 2018.
