ON RATIONAL EISENSTEIN PRIMES AND THE RATIONAL CUSPIDAL GROUPS OF MODULAR JACOBIAN VARIETIES

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Abstract. Let \( N \) be a non-square-free positive integer and let \( \ell \) be a prime such that \( \ell^2 \) does not divide \( 4N \). Consider the Hecke ring \( T(N) \) of weight 2 for \( \Gamma_0(N) \). Then, we define rational Eisenstein primes of \( T(N) \) containing \( \ell \). If \( m \) is such a rational Eisenstein prime, then we prove that \( m = (\ell, I_{DM,N}) \), where an ideal \( I_{DM,N} \) of \( T(N) \) is defined in \( \S 3 \). Furthermore, we prove that \( C(N)[m] \neq 0 \), where \( C(N) \) is the rational cuspidal group of \( J_0(N) \). To do this, we compute the precise order of the cuspidal divisor \( C_{DM,N} \), which is defined in \( \S 4 \), and the index of \( I_{DM,N} \) in \( T(N) \otimes \mathbb{Z}_\ell \).

1. Introduction

Let \( N \) be a positive integer. Consider the modular curve \( X_0(N) \) over \( \mathbb{Q} \). This curve is endowed with a Hecke correspondence \( T_p \) for each prime number \( p \). The correspondence \( T_p \) induces an endomorphism of the Jacobian variety \( J_0(N) := \text{Pic}^0(X_0(N)) \) of \( X_0(N) \), which is again denoted by \( T_p \). Let \( T(N) \) be the \( \mathbb{Z} \)-subalgebra of \( \text{End}(J_0(N)) \) which is generated by the family of endomorphisms \( T_n \) for all \( n \geq 1 \).

Suppose that \( m \) is a maximal ideal of \( T(N) \), and let \( \ell \) be the characteristic of \( T(N)/m \). Let

\[
\rho_m : \text{Gal} \left( \overline{\mathbb{Q}} / \mathbb{Q} \right) \rightarrow \text{GL}_2(T(N)/m)
\]

be the two-dimensional semisimple representation attached to \( m \) (cf. [15, \S 5]). Up to isomorphism, this representation is characterized by the fact that

- it is unramified outside \( \ell N \);
- for each prime number \( p \mid \ell N \), the characteristic polynomial of \( \rho_m(\text{Frob}_p) \) is

\[
X^2 - T_p (\text{mod } m) X + p,
\]

where \( \text{Frob}_p \) is a Frobenius element for the prime \( p \) in \( \text{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \).

If \( \rho_m \) is reducible, \( m \) is called an Eisenstein prime or Eisenstein.

To understand Eisenstein primes of \( T(N) \), we study some subgroups of \( J_0(N) \) annihilated by them. One is the cuspidal group \( C_N \), which is the group generated by the equivalence classes of degree 0 cuspidal divisors, and another is the rational torsion subgroup \( T(N) \), which is the group of all the elements of finite order in \( J_0(N)(\mathbb{Q}) \).

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Let $\mathcal{I}_0(N)$ denote the ideal generated by $T_p - p - 1$ for primes $p$ not dividing $N$, i.e.,
$$\mathcal{I}_0(N) := \langle T_p - p - 1 : \text{for primes } p \mid N \rangle \subseteq \mathbb{T}(N).$$
When $N$ is prime, Mazur proved the following theorem [9].

**Theorem 1.1.** Let $m \subseteq \mathbb{T}(N)$ be an Eisenstein prime containing a prime $\ell$. Then,

1. $m$ is of the form $(\ell, I_{N,N})$, where $I_{N,N} = (T_N - 1, \mathcal{I}_0(N))$.
2. $\mathcal{T}(N)/I_{N,N} \cong \mathbb{Z}/n\mathbb{Z}$, where $n$ is the numerator of $\frac{1}{12}(N - 1)$.
3. $C_N[m] \neq 0$.
4. $\dim J_0(N)[m] = 2$.
5. (Ogg’s conjecture) $\mathcal{T}(N) = C_N$.

Due to the existence of old forms at composite level $N$, the structures of subgroups of $J_0(N)$ annihilated by Eisenstein primes are more complicated. A natural generalization of the above theorem to the square-free level case is as follows.

**Theorem 1.2.** Let $m \subseteq \mathbb{T}(N)$ be an Eisenstein prime containing $\ell$, where $N = \prod_{i=1}^k p_i$ is square-free. Then,

1. $m$ is of the form $(\ell, T_{p_i} - 1, T_{p_j} - p_j, \mathcal{I}_0(N) : \text{for } 1 \leq i \leq s \text{ and } s + 1 \leq j \leq n) = (\ell, I_{M,N})$, where $M = \prod_{i=1}^s p_i$. If we take the largest such $s$ (and we will do the rest), then $s \geq 1$.
2. $\mathcal{T}(N)/I_{M,N} \cong \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}[1/2]$, where $n$ is the numerator of $\frac{1}{12}\prod_{i=1}^s (p_i - 1)\prod_{i=s+1}^t (p_i + 1)$.
3. If $m$ is new, then $p_j \equiv -1 (mod \ell)$ for all $s + 1 \leq j \leq n$.
4. $C_N[m] \neq 0$.
5. Assume that $m$ is new and $\ell$ does not divide $6N$. Let $s_0$ be the number of prime divisors $p_i$ of $N$ such that $p_i \equiv 1 (mod \ell)$. If either $s_0 \neq s$ or $s = t$, then
$$\dim J_0(N)[m] = 1 + s_0 + (t - s) + a,$$
where $a = s$ (resp. 0) if $s_0 = s = t$ (resp. otherwise).
6. (Generalized Ogg’s conjecture) $\mathcal{T}(N)[\ell^\infty] = C_N[\ell^\infty]$ for a prime $\ell \geq 5$, where $A[\ell^\infty]$ denotes the $\ell$-primary part of $A$. If $3$ does not divide $N$, then $\mathcal{T}(N)[3^\infty] = C_N[3^\infty]$.

The first three assertions are proved by the author [20, §2]. The fourth one is conjectured by Ribet, and proved by the author (loc. cit.). The fifth one is proved by Ribet and the author [16]. The last one is proved by Ohta [13]. (And if $N$ is the product of two distinct primes, then the result is slightly improved by the author [21].)

However, to do the same job in general, there are some discrepancy between the square-free level case and the other. First of all, there are some cuspidal divisors which are not rational. Therefore generalized Ogg’s conjecture must be modified. Next, there are some Eisenstein primes which do not contain $\mathcal{I}_0(N)$. (In fact, if $\ell$ does not divide $N$, $\rho_m$ is isomorphic to $\alpha \oplus \chi_\ell \cdot \alpha^{-1}$, where $\alpha$ is a character of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ not ramified outside $N$ and $\chi_\ell$ is the mod $\ell$ cyclotomic character, by Faltings-Jordan [4, Theorem 2.3].) The possible characters $\alpha$ can be studied by the method of Mazur [10, §5], and we can construct some of such non-trivial characters explicitly. In this direction, see also the paper of Stevens [17]. Finally, if $\ell^2$ divides $N$, then the geometry of the special fiber of the Néron model of $J_0(N)$ at $\ell$ is rather difficult.

One possible generalization of the above statement to the non-square-free level case is to impose the condition of “rationality”. More precisely, an Eisenstein prime $m$ of $\mathbb{T}(N)$ is called rational Eisenstein if $\rho_m \cong 1 \oplus \chi_\ell$, where $1$ is the trivial character, or equivalently it contains $\mathcal{I}_0(N)$. And the intersection $C_N \cap \mathcal{T}(N)$ is called the rational cuspidal subgroup of $J_0(N)$. It can also be defined as the subgroup generated by the equivalence classes of the degree 0 cuspidal divisors stable under the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. (By the theorem of Manin-Drinfeld [3, 8], the points on $J_0(N)$ defined by such classes are of finite order, and hence they belong to $\mathcal{T}(N)$.) Let us remark that if $\mathcal{T}(N)[m] \neq 0$, then $m$ is rational Eisenstein by Eichler-Shimura relation (and a standard argument using Brauer-Nesbitt Theorem and Chebotarev density theorem, cf. [9, chap. II, §14]).

The following conjecture is an optimistic generalization of the above theorem to non-square-free $N$.

**Conjecture 1.3.** Let $m \subseteq \mathbb{T}(N)$ be a rational Eisenstein prime containing $\ell$. Let $N = \prod p_i^{e(p)}$. 
(1) $m$ is of the form $(\ell, T_p - \epsilon(p), I_0(N) : \text{for primes } p \mid N) = (\ell, T_{M,N}^0)$, where $\epsilon(p)$ is either 0, 1 or $p$.

(2) $\mathbb{T}(N)/\mathbb{T}^0_{M,N} \cong \mathbb{Z}/n\mathbb{Z}$, where $n$ is the order of the cuspidal divisor $C_{M,N}^0$.

(3) If $m$ is new, then $D = 1$, and $p \equiv -1 \pmod{\ell}$ for all prime divisors $p$ of $N$ such that $p \nmid M$ and $p^2 \nmid N$.

(4) $\mathbb{C}(N)[m] \neq 0$.

(5) $\dim J_0(N)[m]$ can be computed (in most cases) if $m$ is new and $\ell$ does not divide $6N$.

(6) (Generalized Ogg’s conjecture) $T(N) = C(N)$.

(Here, $M$ denotes the product of the prime divisors of $N$ such that $T_p - 1 \in m$, and $D$ denotes the product of the primes $p$ such that $p^2 \mid N$ but $T_p \notin m$. For the definitions of $T_{M,N}^0$ and $C_{M,N}^0$, see §3 and §4, respectively.)

The fifth statement of this conjecture is proved in the upcoming paper by the author [22] (including the case where $\ell = 3$ but $N$ is still not divisible by $\ell$), but the last one is out of reach at present. In this article, we prove the rest under a mild assumption on $\ell$.

**Theorem 1.4 (Main theorem).** Let $m \subseteq \mathbb{T}(N)$ be a rational Eisenstein prime containing $\ell$. Suppose that $\ell^2$ does not divide $4N$.

1. $m$ is of the form $(\ell, T_{M,N}^0)$ for some divisor $D$ of $N$ and some divisor $M$ of $N$ such that $M(N^0/D) \neq 1$.

2. When $M(N^0/D) \neq 1$, we get $\mathbb{T}(N)/\mathbb{T}^0_{M,N} \otimes \mathbb{Z}_\ell \cong \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}_\ell$, where $n$ is the order of $C_{M,N}^0$.

3. If $m$ is new, then $D = 1$ and $p \equiv -1 \pmod{\ell}$ for all prime divisors $p$ of $N$ such that $p \nmid M$.

4. $\mathbb{C}(N)[m] \neq 0$.

If $\ell = 2$ then the same holds as long as 4 does not divide $N$ and $N^0D/M$ is an odd integer greater than 1.

(See §1.1 for unfamiliar notation.) To prove this theorem, we first study the image of $T_p$ in $\mathbb{T}(N)/m$ for each prime divisor $p$ of $N$. Next, we study the rational cuspidal divisor $C_{M,N}^0$, which is annihilated by $T_{M,N}^0$. More precisely, we compute the precise order of $C_{M,N}^0 \in J_0(N)(\mathbb{Q})$ and the Hecke action on it. This gives a lower bound of the index of $T_{M,N}^0$. Then, we define Eisenstein series $E_{M,N}^0$ and compute its residues at various cusps. This gives an upper bound. (In fact, we can prove a bit stronger results of the first and last statements of this theorem. For instance, see Theorem 3.3.)

As an application of this computation, we prove the following.

**Theorem 1.5.** Let $\alpha_p(N)^*$, $\beta_p(N)^*$ denote two degeneracy maps from $J_0(N)$ to $J_0(Np)$, and let $[N]_p^*$ denote the maps from $J_0(N)$ to $J_0(Np)$ defined by

- $[N]_p^+(x) := \alpha_p(N)^*(x) - \beta_p(N)^*(x)$
- $[N]_p^-(x) := p \cdot \alpha_p(N)^*(x) - \beta_p(N)^*(x)$
- $[N]_p^0(x) := \alpha_p(N)^*(x)$.

Let $\ell$ denote a prime not dividing $2N$.

- Assume that $p$ divides $M$. Then, the intersection of the kernel of $[N]_p^+$ and $\langle C_{M,N} \rangle$ is of order 2 (resp. trivial) if $M$ is a prime congruent to 1 modulo 8 and $N/M$ is a divisor of 2 (resp. otherwise). In any cases, $[N]_p^+\langle C_{M,N} \rangle = (p + 1) \cdot C_{M/p,Np}$ and $[N]_p^+\langle C_{M,N} \rangle = \langle C_{M/p,Np} \rangle [\ell]$.

- Assume that $p$ divides $N^0/M$. Then, the intersection of the kernel of $[N]_p^+$ and $\langle C_{M,N} \rangle$ is of order 2 (resp. trivial) if $M$ is a prime congruent to 1 modulo 8 and $N = 2M$ (resp. otherwise). Moreover, $[N]_p^+\langle C_{M,N} \rangle = C_{M,Np}$ and $[N]_p^+\langle C_{M,N} \rangle = \langle C_{M,Np} \rangle [\ell]$.

- Assume that $p^2$ divides $N$. Then, the intersection of the kernel of $[N]_p^+$ and $\langle C_{M,N} \rangle$ is trivial. Moreover, $[N]_p^+\langle C_{M,N} \rangle = p \cdot C_{M,Np}$ and $[N]_p^+\langle C_{M,N} \rangle = \langle C_{M,Np} \rangle [\ell]$.

Here, $C_{M,N}$ denotes $C_{M,N}^1$. For the proof, see §4.4. If $p$ does not divide $N$, then this is an easy consequence of the result proved by Ribet [14]. In fact, when $p \nmid N$, he determined the precise kernel of the map

$$\gamma_p(Np)^* : J_0(N) \times J_0(N) \to J_0(Np).$$

On the other hand, if $p$ divides $N$, then the kernel of $\gamma_p(Np)^*$ is not known in general. (Some cases are done by Ling [6].) The above theorem says that at least by the (proper) map $[N]_p^*$, the subgroup $\langle C_{M,N} \rangle [\ell] \subseteq J_0(N)[m]$
maps injectively into $J_0(Np)[m']$ (for the corresponding $m'$). This result is crucial to compute the dimension of $J_0(N)[m] := \{ x \in J_0(N)(\mathbb{Q}) : T x = 0 \text{ for all } T \in m \}$ for non-square-free $N$ by the (same) inductive argument used in [16]. (Note that in loc. cit., we have the result of Ribet, so the above theorem is not used.)

In general (without the condition of rationality), we expect that all Eisenstein maximal ideals of $\mathbb{T}(N)$ should have support at the cuspidal group $C_N$ of $J_0(N)$. In this direction, Jordan, Scholl and Ribet recently announced the following, which does not impose the “rationality” condition.

**Theorem 1.6.** Let $\mathbb{T}(N)$ be the ring of Hecke operators on the space of modular forms of weight 2 for $\Gamma_0(N)$. Let $\mathcal{I}$ be the ideal of those $t \in \mathbb{T}(N)$ that lift to an operator $\tilde{t} \in \mathbb{T}(N)$ such that $\tilde{t}$ vanishes on the space of Eisenstein series. Then, the annihilator of $C_N$ is locally equal to $\mathcal{I}$ at prime numbers that are prime to $6N$.

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1.1. **Notation.** Throughout this paper, $p$ and $N$ always denote a prime and a positive integer, respectively.

We denote by $p^n \mid | N$ if $p^n \mid N$ but $p^{n+1} \not| N$. In such a case, we say that $p^n$ exactly divides $N$, and denote by $\text{val}_p(N) := n$ the (normalized) valuation of $N$ with respect to $p$.

We denote by

$$\text{N}_{\text{ef}} := \prod_{\text{val}_p(N) = 1} p, \ N^\square := \prod_{\text{val}_p(N) \geq 2} p \text{ and } \text{N}^R := \prod_{p|N} p = \text{N}_{\text{ef}} \cdot N^\square.$$  

We denote by $\mathcal{H}$ the complex upper half plane, i.e., $\mathcal{H} := \{ z \in \mathbb{C} : \text{Im} z > 0 \}$.

2. **Background**

Throughout this section, we fix the notation, $r := \text{val}_p(N)$.

2.1. **The cusps of $X_0(N)$.** As in [12], the cusps of $X_0(N)$ can be regarded as the pairs $\left( \frac{a}{d}, \Gamma_0 \right)$, where $d$ is a divisor of $N$, $1 \leq x \leq d$, and $(x, d) = 1$ with $x$ taken modulo $(d, N/d)$. Such a cusp $\left( \frac{a}{d} \right)$ is called a cusp of level $d$. Thus, the number of cusps is $\sum_{d|N} \varphi((d, N/d))$. (For more detail, see [2, §3.8].)

We define the divisor $(P_d)$ as the sum of all the cusps of level $d$. Note that the degree of $(P_d)$ is $\varphi((d, N/d))$ and $(P_d)$ is invariant under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (cf. [7, §2]).

2.2. **The degeneracy maps.** Let $\alpha_p(N) : X_0(Np) \to X_0(N)$ be the degeneracy covering with the modular interpretation $(E, C) \mapsto (E, C[N])$, where $C$ denotes a cyclic subgroup of order $Np$ in an elliptic curve $E$.

Let $\beta_p(N)$ be the “other” degeneracy covering $X_0(Np) \to X_0(N)$; it has the modular interpretation $(E, C) \mapsto (E/C[p], C/C[p])$. If we identify the set of complex points on $X_0(Np)$ with $\Gamma_0(Np) \setminus \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, then for $\tau \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ we get

$$\begin{cases}
\alpha_p(N)(\tau) \equiv \tau \pmod{\Gamma_0(N)} \\
\beta_p(N)(\tau) \equiv p\tau \pmod{\Gamma_0(N)}.
\end{cases}$$

Hence on cusps, they have the following description (cf. [7, p. 42]):

Let $\left( \frac{a}{p^i d} \right)$ be a cusp of $X_0(Np)$, where $d$ is a divisor of $N/p^r$ and $0 \leq i \leq r + 1$. Let $y := (d, N/d)$. Then, $1 \leq x \leq p^id$ and $(x, p^id) = 1$ with $x$ taken modulo $p^i y$, where $a = \min\{i, r + 1 - i\}$. Moreover,

$$\alpha_p(N) \left( \begin{array}{c} x \\ p^i d \end{array} \right) = \left( \begin{array}{c} x \\ p^i d \end{array} \right) \quad \text{and} \quad \beta_p(N) \left( \begin{array}{c} x \\ p^i d \end{array} \right) = \left( \begin{array}{c} px \\ p^i d \end{array} \right).$$

By the property of the representatives of the cusps, we get

$$\alpha_p(N) \left( \begin{array}{c} x \\ p^i d \end{array} \right) = \alpha_p(N) \left( \begin{array}{c} x' \\ p^i d \end{array} \right) = \alpha_p(N) \left( \begin{array}{c} x' \\ p^i d \end{array} \right).$$
if and only if $x \equiv x' \pmod{p^b y}$, where $b = \min\{i, r - i\}$ for $0 \leq i \leq r$ and $b = 0$ for $i = r + 1$. Moreover,

$$\alpha_p(N) \left( \frac{x}{p^{r+1}d} \right) = \alpha_p(N) \left( \frac{x'}{p^r d} \right)$$

if and only if $x \equiv x' \pmod{p^r y}$. Thus, for $0 \leq i \leq r/2$, the points $\left( \frac{x^i}{p^i d} \right)$ are ramified of degree $p$. (This is true for $i = r = 0$ as well.) And the remaining points are unramified (cf. [17, p. 538]). Analogously, we get

$$\beta_p(N) \left( \frac{x}{p^r d} \right) = \beta_p(N) \left( \frac{x'}{p^r d} \right)$$

if and only if $x \equiv x' \pmod{p^r y}$, where $c = \min\{i - 1, r + 1 - i\}$ for $1 \leq i \leq r + 1$ and $c = 0$ for $i = 0$. Moreover,

$$\beta_p(N) \left( \frac{x}{d} \right) = \beta_p(N) \left( \frac{x'}{pd} \right)$$

if and only if $px \equiv x' \pmod{y}$. Thus, for $r/2 + 1 \leq i \leq r + 1$, the points $\left( \frac{x^i}{p^i d} \right)$ are ramified of degree $p$. (This is true for $i = 1$ and $r = 0$ as well.) And the remaining points are unramified.

The degeneracy coverings $\alpha_p(N), \beta_p(N)$ have degree $p$ if $r \geq 1$, and degree $p + 1$ if $r = 0$. They induce maps

$$\alpha_p(N), \beta_p(N) : J_0(N/p) \to J_0(N), \quad \alpha_p(N)^*, \beta_p(N)^* : J_0(N) \to J_0(N/p)$$

via the two functorialities of the Jacobian.

### 2.3. The Hecke ring

Let $\alpha_p(N)'$ and $\beta_p(N)'$ denote the transposes of $\alpha_p(N)$ and $\beta_p(N)$ viewed as correspondences, respectively. We define the Hecke correspondence $T_p$ on $X_0(N)$ by $\alpha_p(N) \circ \beta_p(N)'$:

$$\begin{align*}
X_0(N) & \xrightarrow{\beta_p(N)} X_0(N/p) \\
& \xrightarrow{\alpha_p(N)} X_0(N)
\end{align*}$$

By direct computation (via the modular interpretation), we get

(2.1) $\alpha_p(N) \circ \beta_p(N)' = \beta_p(N/p)' \circ \alpha_p(N/p)$ for $r \geq 2$;

$$\alpha_p(N) \circ \beta_p(N)' = \beta_p(N/p)' \circ \alpha_p(N/p) - w_p$$

for $r = 1$,

where $w_p$ is the Atkin-Lehner involution on $X_0(N/p)$ with respect to $p$.

**Definition 2.1.** The $p^{th}$ Hecke operator $T_p \in \text{End}(J_0(N))$ is the pullback of the correspondence $T_p$ to $J_0(N)$ (cf. [11, §13]). Namely,

$$T_p := \beta_p(N)^* \circ \alpha_p(N)^* : J_0(N) \xrightarrow{\alpha_p(N)^*} J_0(N/p) \xrightarrow{\beta_p(N)^*} J_0(N).$$

**The Hecke operator** $T_n$ are defined inductively as follows:

- $T_1 = 1$;
- $T_{pk} = T_p T_{pk-1} - p T_{pk-2}$ for $k \geq 2$;
- $T_{ab} = T_a T_b$ if $(a, b) = 1$.

**The Hecke ring** $\mathbb{T}(N)$ of level $N$ is the subring of $\text{End}(J_0(N))$ generated (over $\mathbb{Z}$) by $T_n$ for all integers $n \geq 1$. Sometimes (in §3 and 6), $\mathbb{T}(N)$ is regarded as the subring of the endomorphism ring of the space of cusp forms of weight 2 for $\Gamma_0(N)$ generated by the same named operator $T_n$ for all integers $n \geq 1$ (cf. [11, §1]).

Suppose that $r \geq 1$, and let $T_p$ (resp. $\tau_p$) denote the $p^{th}$ Hecke operator in $\mathbb{T}(N)$ (resp. $\mathbb{T}(N/p)$). Then by the formula (2.1), we get

(2.2) $T_p = \alpha_p(N/p)^* \circ \beta_p(N/p)^*$ for $r \geq 2$;

$$T_p + w_p = \alpha_p(N/p)^* \circ \beta_p(N/p)^*$$

for $r = 1$. 
Also, we get
\begin{align}
T_p \circ \alpha_p(N/p)^* &= \alpha_p(N/p)^* \circ \tau_p, \\
T_p \circ \beta_p(N/p)^* &= p \cdot \alpha_p(N/p)^* \quad \text{for } r \geq 2;
\end{align}
(2.3)
\begin{align}
T_p \circ \alpha_p(N/p)^* &= \alpha_p(N/p)^* \circ \tau_p - \beta_p(N/p)^*, \\
T_p \circ \beta_p(N/p)^* &= p \cdot \alpha_p(N/p)^* \quad \text{for } r = 1.
\end{align}

2.4. Old and new. Throughout this subsection, we assume that \( r \geq 1 \). We define the map:
\[
\gamma_p(N)^* : J_0(N/p) \times J_0(N/p) \to J_0(N)
\]
by the formula \( \gamma_p(N)^*(x, y) = \alpha_p(N/p)^*(x) + \beta_p(N/p)^*(y) \).

Let \( J := J_0(N) \) and \( \mathcal{T} := \mathcal{T}(N) \). The image of \( \gamma_p(N)^* \) is called the \( p \)-old subvariety of \( J \) and denoted by \( J_{p-old} \). The quotient of \( J \) by \( J_{p-old} \) is called the \( p \)-new quotient of \( J \) and denoted by \( J_{p-new} \). By the formula (2.3), (on \( J_{p-old} \)) we get the matrix relations (cf. [18, p. 499]):
\begin{align}
T_p &= \begin{pmatrix}
\tau_p & p \\
0 & 0
\end{pmatrix} \quad \text{for } r \geq 2 \quad \text{and} \quad T_p = \begin{pmatrix}
\tau_p & p \\
-1 & 0
\end{pmatrix} \quad \text{for } r = 1,
\end{align}
(2.4)
which make \( \gamma_p(N)^* \) to be Hecke-equivariant, where \( \tau_p \) is the \( p \)-th Hecke operator in \( \mathcal{T}(N/p) \). The image of \( \mathcal{T} \) in \( \text{End}(J_{p-old}) \) (resp. \( \text{End}(J_{p-new}) \)) is called the \( p \)-old (resp. \( p \)-new) quotient of \( \mathcal{T} \) and denoted by \( \mathcal{T}_{p-old} \) (resp. \( \mathcal{T}_{p-new} \)). A maximal ideal of \( \mathcal{T} \) is called \( p \)-old (resp. \( p \)-new) if its image in \( \mathcal{T}_{p-old} \) (resp. \( \mathcal{T}_{p-new} \)) is still maximal.

Note that a maximal ideal of \( \mathcal{T}(N) \) is either \( p \)-old or \( p \)-new (or both).

From the description above (and the Cayley-Hamilton theorem), in \( \mathcal{T}_{p-old} \) we get
\begin{align}
T_p^2 - \tau_p T_p &= 0 \quad \text{for } r \geq 2; \\
T_p^2 - \tau_p T_p + p &= 0 \quad \text{for } r = 1.
\end{align}
(2.5)

By the formula (2.2), the image of \( T_p \) (resp. \( T_p + w_p \)) is contained in the \( p \)-old subvariety and hence \( T_p \) (resp. \( T_p + w_p \)) maps to \( 0 \in \mathcal{T}_{p-new} \) if \( r \geq 2 \) (resp. \( r = 1 \)). Therefore we get the following.

Lemma 2.2. Let \( m \) be a maximal of \( \mathcal{T}(N) \).

1. Assume that \( m \) is \( p \)-new. If \( r = 1 \) then \( T_p \equiv \pm 1 \pmod{m} \), and if \( r \geq 2 \) then \( T_p \in m \).
2. Assume that \( r \geq 2 \). If \( T_p \not\in m \), then there is a maximal ideal \( n \) of \( \mathcal{T}(N/p^{r-1}) \) corresponding to \( m \). Also, if \( T_p - \kappa \in m \), then \( T_p - \kappa \in n \) as well.

Proof. The first claim is clear from the discussion above. For the second one, suppose that \( T_p - \kappa \in m \) and \( \kappa \not\in m \). Then \( m \) is not \( p \)-new and hence there is a maximal ideal \( \alpha \) of \( \mathcal{T}(N/p) \) corresponding to \( m \) (cf. [15, §7]). Since \( T_p^2 - \tau_p T_p = 0 \in \mathcal{T}(N/p^{r-1}) \) and \( \kappa \not\in m \),
\[
0 = \kappa^{-1}[\kappa(\kappa - \tau_p)] = 0 \in \mathcal{T}(N)/m \cong \mathcal{T}(N/p)/\alpha.
\]
Therefore \( \tau_p - \kappa \in \alpha \) and \( \kappa \not\in \alpha \). By doing the same process until finding a maximal ideal \( n \) of \( \mathcal{T}(N/p^{r-1}) \), the claim follows.

Remark 2.3. In the second case of the above lemma, the fact that \( T_p \not\in n \) does not guarantee that \( n \) is not \( p \)-new because \( p \) exactly divides \( N/p^{r-1} \).

2.5. The maps \([N]_p^* \). As before, let \( \alpha_p(N) \) and \( \beta_p(N) \) denote two degeneracy maps from \( X_0(N/p) \) to \( X_0(N) \).

Definition 2.4. We define the maps \([N]_p^* : J_0(N) \to J_0(Np)\) as follows: For \( x \in J_0(N) \),
\[
\begin{cases}
[N]_p^*(x) := \alpha_p(N)^*(x) - \beta_p(N)^*(x) \\
[N]_p(x) := p \cdot \alpha_p(N)^*(x) - \beta_p(N)^*(x) \\
[N]_p(x) := \alpha_p(N)^*(x).
\end{cases}
\]
We also define the maps $[N]_p^*$ from the space of modular forms of weight 2 for $\Gamma_0(N)$ to that for $\Gamma_0(Np)$ as follows: For a modular form $\mathcal{E}$ of weight 2 for $\Gamma_0(N)$,

\begin{align}
[N]_p^+(\mathcal{E})(z) := & \alpha_p(N)^* \mathcal{E}(z) - \beta_p(N)^* \mathcal{E}(z) = \mathcal{E}(z) - p \cdot \mathcal{E}(pz) \\
[N]_p^-(\mathcal{E})(z) := & \alpha_p(N)^* \mathcal{E}(z) - (1/p) \cdot \beta_p(N)^* \mathcal{E}(z) = \mathcal{E}(z) - \mathcal{E}(pz) \\
[N]_p(\mathcal{E})(z) := & \alpha_p(N)^* \mathcal{E}(z) = \mathcal{E}(z).
\end{align}

(cf. [19, Definition 2.5]). The merit of these definitions is that they preserve “modules” annihilated by Eisenstein ideals.

**Proposition 2.5.** Let $T_p$ and $\tau_p$ denote the $p^{th}$ Hecke operators in $\mathbb{T}(Np)$ and $\mathbb{T}(N)$, respectively. Let $x \in J_0(N)$.

1. If $\tau = 0$ and $(\tau_p - p - 1)x = 0$, then $(T_p - 1)[N]^+_p(x) = 0$ and $(T_p - p)[N]^−_p(x) = 0$.
2. Suppose that $\tau \geq 1$.
   - If $(\tau_p - 1)x = 0$, then $(T_p - 0)[N]^−_p(x) = 0$.
   - If $(\tau_p - p)x = 0$, then $(T_p - 0)[N]^+_p(x) = 0$.
   - If $(\tau_p - \kappa)x = 0$, then $(T_p - \kappa)[N]^−_p(x) = 0$.

If we denote by $T_p$ and $\tau_p$ the $p^{th}$ Hecke operators acting on the spaces of modular forms of weight 2 for $\Gamma_0(Np)$ and $\Gamma_0(N)$, respectively, then the same statement is true for a modular form $x = \mathcal{E}$ of weight 2 for $\Gamma_0(N)$.

**Proof.** This is clear by the discussions in §2.3. □

**Remark 2.6.** Since two degeneracy maps $\alpha_p(N)$ and $\beta_p(N)$ commute with the Hecke operators $T_n$ if $(n, p) = 1$, so do $[N]_p^*$. Thus, $[N]_p^*$ preserve eigenspaces of all the Hecke operators $T_n$ if $(n, p) = 1$.

**Remark 2.7.** The map $[N]_p^*$ is injective for any pair $(p, N)$. This follows from Ribet [14] if $p$ does not divide $N$. If $p$ divides $N$, then $\alpha_p(N)$ is totally ramified at the cusp $0 = (\frac{1}{1})$, which implies the claim.

### 3. Rational Eisenstein primes

Let $m$ be a rational Eisenstein prime containing a prime $\ell$, i.e., $\rho_m \simeq I \oplus \chi_\ell$. Then it contains

$$I_0(N) := (T_p - p - 1 : \text{for primes } p \nmid N)$$

because

$$T_p \pmod{m} = \text{tr}(\rho_m(\text{Frob}_p)) = 1 + p$$

for a prime $p$ not dividing $\ell N$, and $T_\ell = 1 \equiv 1 + \ell \pmod{m}$ if $\ell$ does not divide $N$ by Ribet [20, §2]. (In fact, Ribet proved this one for $\ell \geq 3$. Nevertheless, the same argument works for $\ell = 2$.)

In this section, we classify all rational Eisenstein primes of $\mathbb{T}(N)$ containing a prime $\ell$. To understand them, it is necessary to compute the image of $T_p$ in the residue field $\mathbb{T}(N)/m$ for all prime divisors $p$ of $N$. Let $\epsilon(p)$ denote the image of $T_p$ in $\mathbb{T}(N)/m$.

**Lemma 3.1.** Let $p$ be a prime divisor of $N$. Then,

$$\epsilon(p) \in \begin{cases} 
\{1, p\} & \text{if } p \parallel N; \\
\{1, p, 0\} & \text{otherwise}.
\end{cases}$$

**Proof.** If $p \parallel N$, the result follows from the same argument as in [20, Lemma 2.1].

Let $r := \text{val}_p(N)$, and assume that $r \geq 2$. If $m$ is $p$-new, then $T_p \in m$ by Lemma 2.2. Suppose that $T_p \not\in m$. Then, again by Lemma 2.2, there is a maximal ideal $n$ of $\mathbb{T}(N/p^{r-1})$ corresponding to $m$. Since $n$ contains $I_0(N/p^{r-1})$, $\epsilon(p)$ is either 1 or $p$ (by the case above). This implies that

$$\epsilon(p) = 1 \text{ or } p \in \mathbb{T}(N/p^{r-1})/n \simeq \mathbb{T}(N)/m.$$  □
Definition 3.2. Let $D$ be a divisor of $N^\square$ and let $M$ be a divisor of $N^{sf}D$. Then, we define the ideal $I_{M,N}^D$ of $T(N)$ as follows:

$$I_{M,N}^D := (T_p - 1, T_q - q, T_r, I_0(N)) : \text{for primes } p \mid M, q \mid (N^{sf}D)/M, r \mid (N^\square/D)).$$

Here, $D$ denotes the product of primes $p$ such that $p^2 \mid N$ but $T_p$ does not belong to that ideal, and $M$ denotes the product of all prime divisors $p$ of $N$ such that $T_p - 1$ belongs to that ideal. If $D = 1$, then we simply denote it by $I_{M,N}$. If $q \equiv 1 \pmod{\ell}$ for a prime $q \mid (N^{sf}D)/M$, then $I_{M,N}^D = I_{M,q\cdot N,N}^D$. Thus, when we say that $m$ is of the form $(\ell, I_{M,N}^D)$, then we always assume that $q \not\equiv 1 \pmod{\ell}$ for all prime divisors $q$ of $(N^{sf}D)/M$.

In conclusion, we have the following theorem.

Theorem 3.3. Let $m$ be a rational Eisenstein prime of $T(N)$ containing a prime $\ell$. Then, $m$ is of the form $(\ell, I_{M,N}^D)$ for some divisor $D$ of $N^\square$ and some divisor $M$ of $N^{sf}D$ with $M(N^\square/D) \neq 1$. Furthermore if $m$ is new, then $D = 1$ and $p \equiv -1 \pmod{\ell}$ for all divisors $p$ of $N^{sf}/M$.

Proof. For the first statement, from the above discussion it suffices to show that $a := (\ell, I_{M,N}^D)$ cannot be maximal.

Now assume that $a$ is maximal. Note that from its definition, for a prime divisor $p$ of $N^\square$, $T_p \not\in a$ but $T_p - p \in a$. Thus in particular, $\ell$ cannot divide $N^\square$. First, assume that $\ell$ is odd. Since $T_p \not\in a$ for a prime divisor $p$ of $N^\square$, $a$ is not $p$-new by Lemma 2.2. Therefore we can find a maximal ideal $b = (\ell, I_{M,N^\square})$ of $T(N^\square)$ corresponding to $a$. However, this contradicts [20, Proposition 5.5]. Next, assume that $\ell = 2$. From its definition, $N^\square = 1$ and $N = 2$. Thus, this follows from the result by Mazur [9].

For the last claim, suppose that $m$ is new. Since $D$ is the product of prime divisors $p$ of $N^\square$ such that $T_p \not\in m$, by Lemma 2.2 we get $D = 1$. Now assume that $p$ divides $N^{sf}/M$. Then, $T_p - p \in m$. Since $m$ is new and $p$ exactly divides $N$, the eigenvalue of $T_p$ in $m$ is either 1 or $-1$. By our definition, $p \not\equiv 1 \pmod{\ell}$ and hence $p \equiv -1 \pmod{\ell}$.

4. The Cuspidal Divisor $C_{M,N}^D$ on $J_0(N)$

4.1. Definition of $C_{M,N}^D$. For a divisor $M$ of $N^{sf}$ and a divisor $D$ of $N^\square$ with $M(N^\square/D) \neq 1$, we define the point $C_{M,N}^D \in J_0(N)(\Q)$ as the equivalence class of the degree 0 divisor

$$C_{M,N}^D := \sum_{d|ML} (-1)^{\omega(d)} \times \varphi(L/(d,L))(P_d) \in \text{Div}^0(X_0(N)),$$

where $\omega(d)$ is the number of prime divisors of $d$, $L = N^\square/D$, and $\varphi$ is the Euler’s totient function, i.e., $\varphi(N) = N - \prod_{p \mid N} (1 - p^{-1})$. If $D = 1$, we simply denote it by $C_{M,N}$.

Note that since the degree of $(P_d)$ is $\varphi((d,L))$ for each divisor $d$ of $ML$, the degree of $C_{M,N}^D$ is 0 unless $M = L = 1$. In that case, $C_{M,N}^D = (P_1)$. Therefore we exclude this situation and only consider the situation with $M(N^\square/D) \neq 1$.

If $M$ is a divisor of $N^{sf}$, then we define the point $C_{M,N}^D \in J_0(N)(\Q)$ using the maps $[\ast]_p$ as follows.

- If $M = dp$ with $(d, N^\square) = 1$ and $r = \text{val}_p(N) \geq 2$, then
  $$C_{M,N}^D := [N/p]_p \ast \cdots \ast [N/p^{r-1}]_p(C_{M,N/p^{r-1}}^D/p) = \alpha_p(N/p)^\ast \cdots \ast \alpha_p(N/p^{r-1})^\ast(C_{M,N/p^{r-1}}^D/p),$$

  where $C_{M,N/p^{r-1}}^D$ is defined as above because $M$ divides $(N/p^{r-1})^{sf}$.

- If $M = dpq$ with $(d, N^\square) = 1$ and $r = \text{val}_q(N) \geq 2$, then
  $$C_{M,N}^D := [N/q]_q \ast \cdots \ast [N/q^{r-1}]_q(C_{M,N/q^{r-1}}^D/q) = \alpha_q(N/q)^\ast \cdots \ast \alpha_q(N/q^{r-1})^\ast(C_{M,N/q^{r-1}}^D/q),$$

  where $C_{M,N/q^{r-1}}^D \in J_0(N/q^{r-1})$ is defined as above. By doing analogously, we can define $C_{M,N}^D$ as long as $M$ is a divisor of $N^{sf}D$ with $M(N^\square/D) \neq 1$. 

4.2. The order of $C_{M,N}^D$. We first compute the order of $C_{M,N}^D$ when $M$ is a divisor of $N^{sf}$. The remaining cases follow from this because $\alpha_p(N / p^n) \ast$ is injective.

**Theorem 4.1.** Let $N := \prod_{i=1}^{s} p_i^{e(i)}$ with $e(i) = 1$ for $1 \leq i \leq t$ and $e(i) \geq 2$ for $t + 1 \leq i \leq u$. Let $M := \prod_{s+1}^{r} p_i$ for some $0 \leq s \leq t$ and let $D := \prod_{i=t+1}^{s} p_i$ for some $t \leq \tau \leq u$. Suppose that $\tau \neq u$ and $N$ is not a power of 2. Then, the order of $C_{M,N}^D$ is the numerator of

$$N_{M,N}^D := \frac{\prod_{i=1}^{u} (p_i-1) \cdot \prod_{j=s+1}^{u} (p_j^2-1) \cdot \prod_{k=t+1}^{\tau} p_k^{e(k)-1} \cdot \prod_{\tau+1}^{u} p_k^{e(k)-2}}{24}$$

(4.1)

If $N = 2^k$ with $k \geq 2$, then the order of $C_{1,N}$ is the numerator of $2^{k-4}$.

In the case where $t = u$, the order of $C_{M,N}$ is computed in [20, §3]. In the case where $s = t = 0$ and $u = 1$, it is computed by Ling (cf. [7, p. 41]). Before proving this theorem, we discuss its consequence.

**Corollary 4.2.** Let $N := \prod p_i^{e(i)}$. Let $A := (M,N,\square) \neq 1$ and let $B := \prod_{|A|} p_i^{e(-1)}$ for any divisor $D$ of $N^{\square}$, which is divisible by $A$, the order of $C_{M,N}^D$ is the order of $C_{M,N,B}^{D/A}$, which is the numerator of

$$N_{M,N,B}^{D/A} \cdot h = \frac{\prod_{p|M} (p-1) \cdot \prod p_i^{e(p)} \cdot \prod_{p|(N^{\square}/M)} (p-1) \cdot \prod p_i^{e(p)-2} \cdot (D/A) \cdot h}{24},$$

where $h \equiv 2$ (resp. $h \equiv 1$) if $M$ is prime congruent 1 modulo 8 and $N/B$ is either $M$ or $2M$ (resp. otherwise).

**Proof.** By the theorem above, the order of $C_{M,N,B}^{D/A}$ is the numerator of $N_{M,N,B}^{D/A} \cdot h$ because $M$ divides $(N/B)^{sf}$. (When $N/B$ is square-free, this follows from [20, Theorem 1.3].) Since $[\ast]_p$ is injective in each case, the order of $C_{M,N}^D$ is equal to that of $C_{M,N,B}^{D/A}$ from its definition, and hence the claim follows.

To compute the order of a cuspidal divisor $C$ we use the method of Ling [7, §2]. (For a moment, we do not assume that $N$ is of the form as in the theorem.) To emphasize their dependence on the level, let $S_1(N)$ and $S_2(N)$ denote two sets $S_1$ and $S_2$ on [7, p. 34], respectively, and let $\Lambda(N)$ denote the matrix $\Lambda$ on [7, p. 35]. Let $\varpi := \prod p_i (1 + \text{val}_p(N))$ denote the number of divisors of $N$ and $\{d_1, d_2, \cdots, d_\varpi\}$ the set of divisors of $N$. Then, the $i \times j$ component of $\Lambda(N)$ is as follows:

$$\Lambda(N)_{ij} = \frac{1}{24} \frac{N}{(d_i, N/d_j)} \frac{(d_i, d_j)^2}{d_i \times d_j}.$$

By the argument on [7, p. 36], we first compute $\Lambda(N)^{-1} C \in S_1(N)$. Then, we find the smallest positive integer $k$ such that the entries of $k \Lambda(N)^{-1} C$ satisfy all the conditions of Proposition 1 in loc. cit. The following lemma shows that the computation of $\Lambda(N)^{-1}$ can be reduced to the prime power level case, which is done by Ling.

**Lemma 4.3.** Let $q$ be a prime not dividing $N$, and let

$$b_{ij} = \frac{1}{q^e(q^2 - 1)} (q^{i-1}, q^{r+j-1}) \cdot \kappa_{ij},$$

where

$$\kappa_{ij} = \begin{cases} q^2 & \text{if } i = j = 1 \text{ or } r + 1; \\ q^2 + 1 & \text{if } 2 \leq i = j \leq r; \\ -q & \text{if } |i - j| = 1; \\ 0 & \text{if } |i - j| \geq 2. \end{cases}$$

Then, by change of basis we have

$$\Lambda(N q^r)^{-1} = \left( \begin{array}{cccc} M_{11} & M_{12} & \cdots & M_{1(r+1)} \\ M_{21} & M_{22} & \cdots & M_{2(r+1)} \\ \vdots & \vdots & \ddots & \vdots \\ M_{(r+1)1} & M_{(r+1)2} & \cdots & M_{(r+1)(r+1)} \end{array} \right).$$
where $M_{mn}$ are $\varpi \times \varpi$ matrices such that $M_{mn} = b_{mn} \times \Lambda(N)^{-1}$.

**Proof.** Let $d$ and $\delta$ denote divisors of $N$. By direct computation we get

$$
\frac{1}{24} N q^r (dq^i, \delta q^j)^2 \frac{(dq^i, \delta q^j)^2}{dq^i \cdot \delta q^j} = \frac{1}{24} (d, \delta)^2 \frac{(d, \delta)^2}{d \cdot \delta} \times \frac{q^r}{(q^r, q^j)^2} \cdot (q^r, q^j)^2.
$$

Note that the number of divisors of $Nq^r$ is $\varpi(r + 1)$. As above, let $d_1, \ldots, d_{\varpi}$ denote the divisors of $N$. Then, we can denote by $\{D_1, D_2, \ldots, D_{\varpi(r+1)}\}$ the set of divisors of $Nq^r$ as follows:

$$
D_n := d_i \times q^j,
$$

where $i \equiv n \pmod{\varpi}$ with $1 \leq i \leq \varpi$ and $j := \frac{n - i}{\varpi}$. With this index $\{D_k\}$ of the divisors of $Nq^r$, we get

$$
\Lambda(Nq^r) = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1(r+1)} \\
A_{21} & A_{22} & \cdots & A_{2(r+1)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{(r+1)1} & A_{(r+1)2} & \cdots & A_{(r+1)(r+1)}
\end{pmatrix},
$$

where $A_{mn} = \tau_{mn} \times \Lambda(N)$ and $\tau_{mn} = \frac{q^r}{(q^r, q^j)^2} \frac{(q^r, q^j)^2}{q^r}$. Since the product of two matrices

$$
(\tau_{mn})_{1 \leq m \leq r+1} \quad \text{and} \quad (b_{mn})_{1 \leq m \leq r+1}
$$

is equal to $\text{Id}_{(r+1) \times (r+1)}$ by Ling [7, Proposition 3], $(A_{mn}) \times (M_{mn}) = \text{Id}_{\varpi(r+1) \times \varpi(r+1)}$ and hence the result follows.

As above, let $q$ be a prime not dividing $N$. For a cuspidal divisor $C = \sum_{d|N} a(d)(P_d)$ of $X_0(N)$, we denote the divisors $[C]^1$ and $[C]^q$ of $X_0(Nq^r)$ as follows:

$$
[C]^1 := \sum_{d|N} a(d)(P_d) \quad \text{and} \quad [C]^q := \sum_{d|N} a(d)(P_{dq}).
$$

With this notation, we get

$$
C_{Mq,Nq}^D = [C_{M,N}^D]^1 - [C_{M,N}^D]^q \quad \text{and} \quad C_{M,Nq}^D = [C_{M,N}^D]^1.
$$

For $r \geq 2$,

$$
C_{M,Nq^r}^D = (q - 1)[C_{M,N}^D]^1 - [C_{M,N}^D]^q \quad \text{and} \quad C_{M,Nq^r}^D = [C_{M,N}^D]^1.
$$

Therefore (with respect to the chosen index of the divisors of $Nq^r$ as above) we get

$$
C_{Mq,Nq}^D = \begin{pmatrix}
C_{M,N}^D \\
-C_{M,N}^D
\end{pmatrix} \in \text{S}_2(Nq) \quad \text{and} \quad C_{M,Nq}^D = \begin{pmatrix}
C_{M,N}^D \\
0
\end{pmatrix} \in \text{S}_2(Nq).
$$

For $r \geq 2$,,

$$
C_{M,Nq^r}^D = \begin{pmatrix}
(q - 1)C_{M,N}^D \\
-C_{M,N}^D
\end{pmatrix} \in \text{S}_2(Nq^r) \quad \text{and} \quad C_{M,Nq^r}^D = \begin{pmatrix}
C_{M,N}^D \\
0
\end{pmatrix} \in \text{S}_2(Nq^r).
$$

By the above lemma together with the above discussion, we directly get the following.

**Corollary 4.4.** Let $R(M, N)^D := \Lambda(N)^{-1} C_{M,N}^D \in S_1(N)$. For a prime $q$ not dividing $N$, we get

$$
R(Mq, Nq)^D = \frac{1}{q - 1} \begin{pmatrix}
R(M, N)^D \\
-R(M, N)^D
\end{pmatrix} \quad \text{and} \quad R(M, Nq)^D = \frac{1}{q^2 - 1} \begin{pmatrix}
qR(M, N)^D \\
-qR(M, N)^D
\end{pmatrix}.
$$
Lemma 4.5. For a divisor $\delta$ of $N$, we set

$$x_n(\delta) := \begin{cases} 1 & \text{if } p_n \nmid \delta, \\ -1 & \text{if } p_n \mid \delta, \end{cases} \quad y_n(\delta) := \begin{cases} p_n & \text{if } p_n \mid \delta \\ -1 & \text{if } p_n \nmid \delta \\ 0 & \text{if } p_n^2 \mid \delta \end{cases} \quad \text{and} \quad z_n(\delta) := \begin{cases} p_n & \text{if } p_n \mid \delta \\ -(p_n + 1) & \text{if } p_n \nmid \delta \end{cases}$$

Then, we get

$$R(M, N)^D = \frac{1}{N_{D, M, N}^D} (R_\delta)_N \in S_1(N),$$

where $R_\delta = \prod_{i=1}^{s} x_i(\delta) \prod_{j=s+1}^{t} y_j(\delta) \prod_{k=t+1}^{u} z_k(\delta)$.

Finally, we can prove the Theorem 4.1.

Proof of Theorem 4.1. We check the conditions in Proposition 1 of [7].

- The condition (0) implies that the order of $e_{M, N}^D$ is of the form

$$N_{D, M, N}^D \times g$$

for some integer $g \geq 1$ because $R_\delta = (-1)^\tau$, where $d := \prod_{i=1}^{u} p_i \prod_{k=t+1}^{u} p_k^2$. So, we can think $r_\delta := \frac{1}{N_{D, M, N}^D} R_\delta \times (N_{D, M, N}^D \times g) = R_\delta \times g$ and check the remaining conditions.

- Suppose that $s \neq \tau$. Let $p = p_\tau$. Then,

$$\sum_{\delta | N} R_\delta \cdot \delta = \sum_{\delta | N/p^{\nu(\tau)}} (R_\delta \cdot \delta + R_{\delta p} \cdot \delta p) = 0.$$ 

Suppose that $\tau \neq u$. Let $p = p_u$ and $e(u) \geq 2$. Then,

$$\sum_{\delta | N} R_\delta \cdot \delta = \sum_{\delta | N/p^{\nu(u)}} (R_\delta \cdot \delta + R_{\delta p} \cdot \delta p + R_{\delta p^2} \cdot \delta p^2) = 0.$$ 

Suppose that $s = t = u$. Then,

$$\frac{1}{N_{D, M, N}^D} \sum_{\delta | N} R_\delta \cdot \delta = \frac{1}{N_{D, M, N}^D} \prod_{i=1}^{s} (1 - p_i) = (-1)^s \times 24.$$ 

Thus, the condition (1) always holds.

- Here, we denote by $(T_3)$ the column matrix in Lemma 4.5 at level $N/p^{\nu(n)}$.

Let $p = p_n$ for some $1 \leq n \leq s$. Then,

$$\sum_{\delta | N} R_\delta \cdot N/\delta = \sum_{\delta | N/p} (R_\delta \cdot N/\delta + R_{\delta p} \cdot N/\delta p) = (p - 1) \cdot \sum_{\delta | N/p} T_\delta \cdot (N/p)/\delta.$$ 

Let $p = p_n$ for some $s + 1 \leq n \leq \tau$. Then,

$$\sum_{\delta | N} R_\delta \cdot N/\delta = \sum_{\delta | N/p^{\nu(n)}} (R_\delta \cdot N/\delta + R_{\delta p} \cdot N/\delta p) = p^{\nu(n) - 1}(p - 1) \cdot \sum_{\delta | N/p^{\nu(n)}} T_\delta \cdot (N/p^{\nu(n)})/\delta.$$
Let $p = p_n$ for some $\tau + 1 \leq n \leq u$. Then, $\epsilon(n) \geq 2$ and
\[
\sum_{\delta \mid N} R_\delta \cdot N/\delta = \sum_{\delta \mid N/p^{\epsilon(n)}} (R_\delta \cdot N/\delta + R_{\delta p} \cdot N/\delta p + R_{\delta p^2} \cdot N/\delta p^2) = p^{\epsilon(n)-2}(p-1)(p^2-1) \cdot \sum_{\delta \mid N/p^{\epsilon(n)}} T_\delta \cdot (N/p^{\epsilon(n)})/\delta.
\]
Thus, we get
\[
\frac{1}{N^{\ell}_{M,N}} \sum_{\delta \mid N} R_\delta \cdot N/\delta = 24 \prod_{k=\tau+1}^u (p_k - 1)
\]
and the condition (2) always holds.

- If $s = 0$ and $\tau = u$, then $\sum_{\delta \mid N} R_\delta = \prod_{i=1}^u (p_i - 1)$. Otherwise, $\sum_{\delta \mid N} R_\delta = 0$. In fact, if $s = 0$ and $\tau = u$, then $C_{M,N}^D = (P_1)$, which is not a degree 0 divisor. Note that we exclude this case and hence the condition (3) always holds.

- Let $\prod_{\delta \mid N} \delta R_\delta = \prod_{k=1}^u p_k^{b(n)}$. Then,
\[
b(n) = \begin{cases}
\sum_{\delta \mid N/p^{\epsilon(n)}} R_{\delta p} = -\sum_{\delta \mid N/p^{\epsilon(n)}} T_\delta & \text{if } 1 \leq n \leq \tau \\
\sum_{\delta \mid N/p^{\epsilon(n)}} (R_{\delta p} + 2R_{\delta p^2}) = \sum_{\delta \mid N/p^{\epsilon(n)}} (1-p) \cdot T_\delta & \text{if } \tau + 1 \leq n \leq u.
\end{cases}
\]
Thus, $b(n) = 0$ unless one of the following holds.
- $n = s = 1$ and $\tau = u$.
- $n = u = \tau + 1$ and $s = 0$.

If $s = 1$ and $\tau = u$, then $b(1) = -\prod_{i=2}^u (p_i - 1)$. If $u = \tau + 1$ and $s = 0$, then $b(u) = -\prod_{i=1}^u (p_i - 1)$.

Hence $b(n)$ is always even unless one of the following holds.
- $s = \tau = u = 1$, i.e., $N = M$ is prime.
- $s = 1$, $\tau = u = 2$, and $p_2 = 2$, i.e., $N = 2M$ with $M$ prime.
- $s = \tau = 0$, $u = 1$, and $p_1 = 2$, i.e., $N = 2^k$ with $k \geq 2$.

The condition (4) holds if and only if $b(n)$’s are even for all $1 \leq n \leq u$. Thus, it suffices to check the case where $N = 2^k$ for some $k \geq 2$ because we assumed that $\tau \neq u$. In this case, $\prod_{\delta \mid N} \delta R_\delta = 2^{u-1}$. Thus, the condition (4) implies that $g$ must be even.

Therefore the order of $C_{M,N}^D$ is the smallest integer $N^{\ell}_{M,N} \times g$ with $g \geq 1$ and hence $g = \frac{24}{N^{\ell}_{M,N}}$ if $\tau \neq u$ and $N$ is not a power of 2. If $N = 2^k$ with $k \geq 2$, then the order of $C_{1,N}^D$ is the numerator of $2^{k-4}$.
\[\square\]

Remark 4.6. With this inductive argument, we can easily recover all the (strange) computations in [20, §3].

4.3. The Hecke action on $(C_{M,N}^D)$. As above, let $D$ denote a divisor of $N^{\square}$ and $M$ a divisor of $N^{\text{nd}}D$ with $M(N^{\square}/D) \neq 1$.

Theorem 4.7. The ideal $T_{M,N}^D$ annihilates $(C_{M,N}^D)$.

Proof. We first assume that $M$ divides $N^{\text{nd}}$. Let $r := \text{val}_p(N)$. By the property of degeneracy maps in §2.2, we get the following:

Let $d$ be a divisor of $N/p^r$, and let $y := (d, N/d)$. Let $A_d$ be the set of representatives of $x$ modulo $y$ such that $1 \leq x \leq d$ and $(x, d) = 1$. Similarly, let $A_{pd}$ (resp. $B_{pd}$) be the set of representatives of $x$ modulo $y$ (resp. $yp$) such that $1 \leq x \leq pd$ and $(x, pd) = 1$. Then, since $(p, d) = 1$,
\[
(P_x) = \sum_{x \in A_d} \left( \begin{array}{c} x \\ d \end{array} \right) = \sum_{x \in A_d} \left( \begin{array}{c} px \\ d \end{array} \right).
\]
And
\[
(P_{pd}) = \sum_{x \in A_{pd}} \left( \begin{array}{c} x \\ pd \end{array} \right) \quad \text{if } r = 1 \quad \text{and} \quad (P_{pd}) = \sum_{x \in B_{pd}} \left( \begin{array}{c} x \\ pd \end{array} \right) \quad \text{if } r \geq 2.
\]
Moreover, for each $x' \in B_{pd}$, we can find the unique $x \in A_{pd}$ such that $x' \equiv x \pmod y$. For each $x \in A_{pd}$, we denote by $B_x$ the set of $x' \in B_{pd}$ such that $x' \equiv x \pmod y$. Then for any $x \in A_{pd}$, $\#B_x = p - 1$. 

\[\square\]
Let $\Delta_p$ denote the map $\beta_p(N)_* \circ \alpha_p(N)^*: \text{Div}(X_0(N)) \to \text{Div}(X_0(N))$, which induces the map $T_p$ on $J_0(N)$.

- Case 1: $r = 0$. Then for $x \in \mathcal{A}_d$,
  $$\beta_p(N)_* \circ \alpha_p(N)^*\left(\frac{x}{d}\right) = \beta_p(N)_*\left(\frac{p}{d} \left(\frac{x}{d}\right) + \left(\frac{x}{pd}\right)\right) = p\left(\frac{px}{d}\right) + \left(\frac{x}{d}\right).$$
  Therefore $\Delta_p((P_d)) = (p + 1) \cdot (P_d)$ by (4.2).
- Case 2: $r = 1$. Then, for $x \in \mathcal{A}_d$,
  $$\beta_p(N)_* \circ \alpha_p(N)^*\left(\frac{x}{d}\right) = \beta_p(N)_*\left(\frac{p}{d} \left(\frac{x}{d}\right)\right) = p\left(\frac{px}{d}\right).$$
  And for $x \in \mathcal{A}_{pd}$,
  $$\beta_p(N)_* \circ \alpha_p(N)^*\left(\frac{x}{pd}\right) = \beta_p(N)_*\left(\sum_{x' \in B_x} \left(\frac{x'}{pd}\right) + \left(\frac{x}{p^2d}\right)\right) = (p - 1)\left(\frac{x}{d}\right) + \left(\frac{x}{pd}\right).$$
  Therefore $\Delta_p((P_d)) = p \cdot (P_d)$ and $\Delta_p((P_{pd})) = (p - 1) \cdot (P_d) + (P_{pd})$ by (4.2) and (4.3).
- Case 3: $r \geq 2$. Then, for $x \in \mathcal{A}_d$,
  $$\beta_p(N)_* \circ \alpha_p(N)^*\left(\frac{x}{d}\right) = \beta_p(N)_*\left(\frac{p}{d} \left(\frac{x}{d}\right)\right) = p\left(\frac{px}{d}\right).$$
  And for $x' \in B_x \subset B_{pd}$,
  $$\beta_p(N)_* \circ \alpha_p(N)^*\left(\frac{x'}{pd}\right) = \beta_p(N)_*\left(\frac{p}{d} \left(\frac{x'}{pd}\right)\right) = p\left(\frac{x}{d}\right).$$
  Therefore $\Delta_p((P_d)) = p \cdot (P_d)$ and $\Delta_p((P_{pd})) = p(p - 1) \cdot (P_d)$ by (4.2) and (4.3).

Thus,
- Case 1: $r = 0$. Since $\Delta_p((P_d)) = (p + 1) \cdot (P_d)$ for any $d \mid N$, we get
  $$T_p(C^D_{M,N}) = (p + 1) \cdot C^D_{M,N}.$$  
- Case 2: $r = 1$ and $p \mid M$. Since $\Delta_p((P_d) - (P_{dp})) = (P_d) - (P_{dp})$ for any $d \mid N/p$, we get
  $$T_p(C^D_{M,N}) = C^D_{M,N}.$$  
- Case 3: $r = 1$ and $p \nmid M$. Since $\Delta_p((P_d)) = p \cdot (P_d)$ for any $d \mid N/p$, we get
  $$T_p(C^D_{M,N}) = p \cdot C^D_{M,N}.$$  
- Case 4: $r \geq 2$ and $p \mid D$. Since $\Delta_p((P_d)) = p \cdot (P_d)$ for any $d \mid N/p'$, we get
  $$T_p(C^D_{M,N}) = p \cdot C^D_{M,N}.$$  
- Case 5: $r \geq 2$ and $p \nmid D$. Since $\Delta_p((p - 1)(P_d) - (P_{dp})) = 0$ for any $d \mid N/p'$, we get
  $$T_p(C^D_{M,N}) = 0.$$  

Therefore the claim follows in the case where $M$ divides $N^d$.

If $(M, N^\square) \neq 1$, then by Proposition 2.5 and Remark 2.6, we can reduce the above case, and hence the claim follows. \qed
4.4. The maps $[N]_p^*$ on $(\mathcal{C}_{M,N})$. In §2.5 we defined the maps $[N]_p^*$ from $J_0(N)$ to $J_0(Np)$. Let $p$ be a prime divisor of $N$. Then by the description in §2.2 and the definition of $\mathcal{C}_{M,N}$, we get

$$
\begin{cases}
[N]_p^*(\mathcal{C}_{M,N}) = (p+1) \cdot \mathcal{C}_{M/p,Np} & \text{if } p \mid M, \\
[N]_p^*(\mathcal{C}_{M,N}) = \mathcal{C}_{M,Np} & \text{if } p \nmid N^{st}/M, \\
[N]_p^*(\mathcal{C}_{M,N}) = p \cdot \mathcal{C}_{M,Np} & \text{if } p \mid N^{\square}.
\end{cases}
$$

(4.4)

Now, we can prove Theorem 1.5.

Proof of Theorem 1.5. Note that the order of $\mathcal{C}_{M,N}$ is the numerator of

$$
\left(\frac{1}{24} \prod_{p|N} (p-1) \cdot \prod_{p|\sqrt{\Delta}} (p^2 - 1) \cdot \prod_{p|N^{\square}} p^{s(p)-2}\right) \times h,
$$

where $h = 2$ if one of the following holds, and $h = 1$ otherwise.

- $M$ is prime and $N = M$.
- $M$ is prime and $N = 2M$.
- $M = 1$ and $N = 2^k$ for some $k \geq 2$.

Therefore all claims easily follow from the discussion above. \qed

5. Eisenstein series $\mathcal{E}_{M,N}^D$.

5.1. Definition. Let $\mathcal{E}_{p,p}$ denote the unique Eisenstein series of weight 2 for $\Gamma_0(p)$ whose $q$-expansion is the power series $c'$ on $[9$, p. 78$]$, and for $k \geq 2$ let $\mathcal{E}_{1,p^k}$ denote an Eisenstein series of weight 2 for $\Gamma_0(p^k)$ defined by

$$
\mathcal{E}_{1,p^k}(z) := \left[p^{k-1}\right]_p \circ \cdots \circ \left[p^2\right]_p \circ \left[p\right]_p (\mathcal{E}_{p,p}(z) - \mathcal{E}_{p,p}(pz)).
$$

As before, let $D$ denote a divisor of $N^{\square}$ and $M$ a divisor of $N^{st}D$ with $M(N^{\square}/D) \neq 1$.

Definition 5.1. We inductively define Eisenstein series $\mathcal{E}_{M,N}^D$ of weight 2 for $\Gamma_0(N)$ as follows.

1. If $N = p^k$, then $\mathcal{E}_{1,p^k}^1 := \mathcal{E}_{1,p^k}$ and $\mathcal{E}_{p,p^k}^p(z) := \mathcal{E}_{p,p^k}(z)$, where $a = \min\{1, k-1\}$.
2. Let $p$ denote a prime not dividing $N$ and assume that we have defined $\mathcal{E}_{M,N}^D$. Then for $r \geq 2$,

$$
\begin{cases}
\mathcal{E}_{M,N}^D(pz) := [N]_p^r (\mathcal{E}_{M,N}^D(z) - p \cdot \mathcal{E}_{M,N}^D(pz)) \\
\mathcal{E}_{M,N}^D(z) := [N]_p^r (\mathcal{E}_{M,N}^D(z) - \mathcal{E}_{M,N}^D(pz)) \\
\mathcal{E}_{M,N}^D(Nz) := \mathcal{E}_{M,N}^D(z) - (p+1) \cdot \mathcal{E}_{M,N}^D(pz) + p \cdot \mathcal{E}_{M,N}^D(p^2z) (= [Np^{r-1}]_p (\mathcal{E}_{M,N}^{p,D}(p) \mathcal{E}_{M,N}^{p,D}(pz)) \text{ if } r \geq 3) \\
\mathcal{E}_{M,N}^D(z) := [Np^{r-1}]_p (\mathcal{E}_{M,N}^{p,D}(p) \mathcal{E}_{M,N}^{p,D}(pz)) \text{ if } r = 2.
\end{cases}
$$

(5.1)

Here $a = \min\{1, r - 2\}$ is the minimum of 1 and $r - 2$.

Remark 5.2. If we denote by $\mathcal{E}$, the Eisenstein series of weight 2 and level 1 whose $q$-expansion is the power series $e$ on $[9$, p. 78$]$, and we set $\mathcal{E}_{1,1}^1(z) := \mathcal{E}(z)$. Then, the above formula (5.1) is “compatible” when $N = 1$. For instance, $\mathcal{E}_{1,1}^1(pz) = \mathcal{E}_{1,1}^1(z) - (p+1) \cdot \mathcal{E}_{1,1}^1(pz) + p \cdot \mathcal{E}_{1,1}^1(p^2z)$ for $k \geq 2$. Nevertheless, we split the definition as above not to make a confusion because $\mathcal{E}_{1,1}^1(z)$ is not a genuine modular form (cf. [19, Definition 2.5]).

5.2. The residue of $\mathcal{E}_{M,N}^D$ at various cusps. Let $D$ denote a divisor of $N^{\square}$ and $M$ a divisor of $N^{st}D$ with $M(N^{\square}/D) \neq 1$, as always. We compute the residues of $\mathcal{E}_{M,N}^D$ at cusps as meromorphic differentials (cf. [9, p. 86–87] or [20, Proposition 4.2]).

Theorem 5.3. If $M \neq N^{\mathcal{R}}$ (resp. $M = N^{\mathcal{R}}$), then the residue of $\mathcal{E}_{M,N}^D$ at $\infty = \left(\frac{1}{N}\right)$ is 0 (resp. $\prod_{p|N}(1 - p)$).

Also, the residue of $\mathcal{E}_{M,N}^D$ at any cusp of level $ML$ is

$$
(-1)^{(\varepsilon_ML)} \prod_{p|ML} (p-1) \cdot \prod_{p|\sqrt{\Delta}M} (p^2 - 1) \cdot \prod_{p|N^{\square}} p^{\varepsilon_p(N)-2} \prod_{p|(N^{st}/M, N^{st})} L \cdot P.
$$
where \( L := (N^2/D) \)

In the theorem, we only compute the residues at some chosen cusps. However, we can easily compute the residues at all the cusps of \( \mathcal{X}_0(N) \), and it will be clear later how to do it systematically. We start with the following well-known lemma in the theory of compact Riemann surfaces.

**Lemma 5.4.** Let \( \phi \) be a non-constant map between compact Riemann surfaces \( C_1 \) and \( C_2 \). Let \( \omega \) be a meromorphic differential on \( C_2 \). Then, the residue of \( \phi^* (\omega) \) at \( x \) is the product of the ramification index of \( \phi \) at \( x \) and the residue of \( \omega \) at \( \phi(x) \).

**Proof.** This is a local computation. On a local chart of \( x \), \( \phi \) sends \( z \) to \( z^e \), where \( e \) is the ramification index of \( \phi \) at \( x \). If we write \( \omega = f(z)dz \) on a local chart of \( \phi(x) \), then \( \phi^* (\omega) = f(z^e)dz^e = e z^{e-1} f(z^e)dz \). Thus, if \( f(z) = \sum a_n z^n \), then \( e z^{e-1} f(z^e) = e \sum a_n z^n (e + 1) \). Therefore the residue of \( \phi^* (\omega) \) at \( x \) is \( e a_{-1} \) and the claim follows because the residue of \( \omega \) at \( \phi(x) \) is \( -a_{-1} \). \( \square \)

As a corollary, we can compute the residues of \( \alpha_p(N)^*(\mathcal{E}_{M,N}^D) \) and \( \beta_p(N)^*(\mathcal{E}_{M,N}^D) \) at various cusps, regarded as meromorphic differentials. Combining the computation with the definition of \( \mathcal{E}_{M,N}^D \), we get the following.

**Corollary 5.5.** Let \( p \) denote a prime not dividing \( N \) and let \( r \geq 2 \). Suppose that for any divisor \( d \) of \( N \), the residues of \( \mathcal{E} := \mathcal{E}_{M,N}^D \) at any cusps of level \( d \) are the same, and denoted by \( A_d \). Then, the residues of \( \mathcal{E}_{M,p^a,N}^D \) at cusps of the same level are the same.

Let \( \text{Res}_a(\mathcal{E}_{M,p^a,N}^D) \) denote the residue of \( \mathcal{E}_{M,p^a,N}^D \) at a cusp of level \( n \). Then, we get the following.

1. For \( 0 \leq a \leq 1 \), \( \text{Res}_{p^a}(\mathcal{E}_{M,p,N}^D) = (-1)^{a} (p-1) A_d \).
2. For \( 0 \leq a \leq 1 \), \( \text{Res}_{p^a}(\mathcal{E}_{M,p^a,N}^D) = (1 - a)^2 \frac{p-1}{p} A_d \).
3. For simplicity, let \( X^r \), \( Y^r \) and \( Z^r \) denote Eisenstein series \( \mathcal{E}_{M,p^a,N}^D \), \( \mathcal{E}_{M,p^a,N}^D \), and \( \mathcal{E}_{M,N}^D \), respectively. Then,
   - \( \text{Res}_d(X^2) = p(p-1) A_d := \chi_0 \), \( \text{Res}_d(X^2) = (1-p) A_d := \chi_1 \) and \( \text{Res}_p(X^2) = \chi_1 \).
   - \( \text{Res}_d(Y^2) = (p^2 - 1) A_d := \gamma_0 \), \( \text{Res}_d(Y^2) = 0 \) and \( \text{Res}_p(Y^2) = 0 \).
   - \( \text{Res}_d(Z^2) = (p^2 - 1) \frac{p-1}{p} A_d := \zeta_0 \), \( \text{Res}_d(Z^2) = \frac{1-p^2}{p} A_d := \zeta_1 \) and \( \text{Res}_p(Z^2) = 0 \).

**Proof.** This is clear from the above lemma and the discussion in \( \S 2.2 \). \( \square \)

Now, we get the proof of Theorem 5.3 by induction due to the above corollary and the following lemma. Note that from our assumption that \( M(N^2/D) \neq 1 \), there does exist an initial prime \( p \) to apply the induction.

**Lemma 5.6.** For \( r \geq 2 \), let \( X^r \) and \( Z^r \) denote Eisenstein series \( \mathcal{E}_{p^a,N}^D \) and \( \mathcal{E}_{p^a,N}^D \), respectively. Then for each \( a \) with \( 0 \leq a \leq r \), the residues of \( X^r \) (resp. \( Z^r \)) at any cusps of level \( p^a \) are the same, and denoted by \( \text{Res}_{p^a}(X^r) \) (resp. \( \text{Res}_{p^a}(Z^r) \)). Moreover for each \( a \) with \( 2 \leq a \leq r \), we get

1. \( \text{Res}_{p^a}(X^r) = p^r - (p-1) \), \( \text{Res}_{p^a}(X^r) = p^{r-2} (1-p) \) and \( \text{Res}_{p^a}(X^r) = p^{r-2a} (1-p) \).
2. \( \text{Res}_{p^a}(Z^r) = p^{r-3}(p^2-1) \), \( \text{Res}_{p^a}(Z^r) = p^{r-3}(1-p^2) \) and \( \text{Res}_{p^a}(Z^r) = 0 \).

**Proof.** By Mazur [9, chap II, §5], the residues of \( \mathcal{E}_{p^a} \) at the cusps 0 and \( \infty \) are \( p - 1 \) and \( 1 - p \), respectively. By Definition 5.1 and Lemma 5.4, the claims follow. \( \square \)

6. The Index of \( \mathcal{T}_{M,N}^D \)

In this section, we prove the following theorem.

**Theorem 6.1.** Let \( \ell \) be a prime such that \( \ell^2 \) does not divide \( N \). If \( \ell \geq 3 \), then
\[
\mathbb{T}(N)/\mathcal{T}_{M,N}^D \otimes \mathbb{Z}_{\ell} \cong (\mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Z}_{\ell},
\]
where \( n \) is the order of \( C_{M,N} \). If \( \ell = 2 \), we suppose further that \( N^R/M \) is an odd integer greater than 1. Then,
\[
\mathcal{T}(N)/\mathcal{T}_{M,N}^D \otimes \mathbb{Z}/2 = (\mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Z}/2.
\]

**Proof.** Note that if \( N \) is square-free, this is done by [20, Theorem 1.4]. Thus, we assume that \( N = 1 \). We basically follow the argument in §5 of loc. cit.

First, by the same argument as in [9, chap II, Proposition 9.7], \( \mathcal{T}(N)/\mathcal{T}_{M,N}^D \simeq \mathbb{Z}/m\mathbb{Z} \) for some \( m \geq 1 \).

Next, since \( \langle C_{M,N} \rangle \) is annihilated by \( \mathcal{T}_{M,N}^D \), there is a natural surjection
\[
Z/m\mathbb{Z} \twoheadrightarrow \mathcal{T}(N)/\mathcal{T}_{M,N}^D = \text{End}(\langle C_{M,N} \rangle) \simeq \mathbb{Z}/n\mathbb{Z}.
\]

Therefore \( n \) divides \( m \).

Let \( \ell a(\ell) \) and \( \ell b(\ell) \) denote the exact powers of \( \ell \) dividing \( m \) and \( n \), respectively. Then, it suffices to prove that \( a(\ell) \leq b(\ell) \) for all primes \( \ell \) satisfying given assumptions. Let \( \ell \) denote a prime satisfying all the above assumptions. (Therefore, in particular, if \( \ell = 2 \), we exclude the case where \( N = 2^k \) with \( k \geq 2 \).) If \( a(\ell) = 0 \), then nothing to prove and hence we further assume that \( a(\ell) \geq 1 \). Let \( f \) denote a cusp form of weight 2 for \( \Gamma_0(N) \) over \( \mathbb{T}(N)/\mathcal{I} \simeq \mathbb{Z}/\ell a(\ell)\mathbb{Z} \) whose q-expansion is
\[
\sum_{n \geq 1} (T_n \mod \mathcal{I}) \cdot q^n,
\]
where \( \mathcal{I} := \langle \ell a(\ell), \mathcal{T}_{M,N}^D \rangle \).

- **Case 1:** Assume that \( M \neq N^R \) and \( \ell \) does not divide \( N^R/M \). Then the q-expansion of
\[
g = 24f + \mathcal{E}_{M,N}^D \quad (\text{mod } 24\ell a(\ell))
\]
is zero. Therefore by the q-expansion principle [5, §1.6], \( g \) is identically zero on the connected component of \( X_0(N)/\mathcal{F} \) containing the cusp \( \infty = (1,1) \). If \( \ell \) does not divide \( N \), then \( X_0(N)/\mathcal{F}/\ell \) is connected. However if \( \ell \) exactly divides \( N \), then \( X_0(N)/\mathcal{F}/\ell \) has two components, and all the cusps of level \( d\ell \) with \( (d, \ell) = 1 \) belong to the same component (cf. [11]). In particular, in both cases the cusps \( (1,1) \) and \( (1,1) \) belong to the same component, where \( L = N^D/D \) since we assume that \( \ell \) does not divide \( N^R/M \). Thus, the residue of \( \mathcal{E}_{M,N}^D \) at the cusp \( (1,1) \) is divisible by \( 24\ell a(\ell) \) because \( 24f \) is a cusp form. By Theorem 5.3, \( 24\ell a(\ell) \) divides \( 24N/\mathcal{I} \). In other words, \( \ell a(\ell) \) divides (the numerator of) \( N/\mathcal{I} \). This implies that \( a(\ell) \leq b(\ell) \) because the order of \( C_{M,N} \) is the numerator of \( N/\mathcal{I} \) unless \( N = 2^k \) with \( k \geq 2 \).

- **Case 2:** Assume that \( \ell \) divides \( N^d \) but \( \ell \) does not divide \( 2M \). Note that \( \mathcal{I}/\mathcal{I}/\ell = \mathcal{I} \) and \( \ell \) exactly divides \( N \). Thus, \( a := (\ell, \mathcal{I}) \) is not \( \ell \)-new by Lemma 2.2. By the same argument as in the proof of Theorem 5.2 in [20], we get
\[
\mathcal{T}(N)/\mathcal{I} \simeq \mathcal{T}(N)/\ell \mathcal{T}_{M,N}/\ell \simeq \mathcal{T}(N)/\ell \mathcal{T}_{M,N}/\ell = \mathcal{T}(N)/\ell \mathcal{T}_{M,N}/\ell.
\]
Since \( \ell \mid (\ell^2 - 1) \), we get \( \mathcal{T}(N)/\ell \mathcal{T}_{M,N}/\ell \otimes \mathbb{Z}/\ell \simeq \mathbb{Z}/\ell b(\ell)\mathbb{Z} \), and therefore \( a(\ell) \leq b(\ell) \).

- **Case 3:** Assume that \( M = N^R \) and \( \ell \) is an odd prime. Applying the following lemma inductively, we get
\[
\mathcal{T}(N)/\ell \mathcal{T}_{M,N}^D \otimes \mathbb{Z}/\ell \simeq \mathcal{T}(M)/\ell \mathcal{T}_{M,M}/\ell \otimes \mathbb{Z}/\ell.
\]
Note that the order of \( C_{M,N}^D \) is equal to the order of \( C_{M,N} \) by Corollary 4.2. Therefore \( \mathcal{T}(M)/\ell \mathcal{T}_{M,M}/\ell \otimes \mathbb{Z}/\ell \simeq \mathbb{Z}/\ell b(\ell)\mathbb{Z} \) by [20, Theorem 1.4]. This implies that \( a(\ell) = b(\ell) \) as desired.

\[
\square
\]

**Lemma 6.2.** Let \( p \) be a prime and \( M \) a square-free integer prime to \( p \). For a prime \( \ell \) different from \( p \) and for \( r \geq 2 \), we get an isomorphism
\[
\mathcal{T}(Mp^r)/\mathcal{T}_{Mp,p^r}^D \otimes \mathbb{Z}/\ell \simeq \mathcal{T}(M)/\mathcal{T}_{Mp,p}^D \otimes \mathbb{Z}/\ell.
\]

**Proof.** For ease of notation, let \( \mathcal{T} := \mathcal{T}(M)/\mathcal{T}_{Mp,p}^D \) and \( \mathcal{J} := \mathcal{T}_{Mp,p}^D \). Since \( \mathcal{J} \) is the completion of \( \mathcal{T} \) at \( a/\ell \), we have
\[
\mathcal{T} / \mathcal{J} \simeq \mathbb{Z}/\ell a^n.
\]

Therefore, \( \mathcal{T} / \mathcal{J} \) is a profinite \( \mathbb{Z}/\ell a^n \) group.

\[
\square
\]
First, let $R$ be the common subring of $\mathbb{T}^0$ and $\mathbb{T}^1$, which is generated (over $\mathbb{Z}$) by all $T_n$ with $(n,p) = 1$ (cf. [15, §7]). Then, $\mathbb{T}^0 = R[T_p]$ and $\mathbb{T}^1 = R[\tau_p]$; moreover, $T_p$ and $\tau_p$ are connected by the equation

$$T_p(T_p - \tau_p) = 0$$

(cf. §2.4). In fact by the matrix relation (2.4), the first component $\alpha_p(Mp^{r-1})^*(J_0(Mp^{r-1}))$ is a $\mathbb{T}^0$-submodule of $J_0(Mp^{r-1})$. Since $\alpha_p(Mp^{r-1})^*$ is injective, we get a natural surjection $\mathbb{T}^0 \to \mathbb{T}^1$. Since $T_p \not\in m$, we have $\mathbb{T}_m^0 \simeq \mathbb{T}_m^1$.

Then, since $T_\ell$ and $\mathbb{T}_m^1$ are semi-local, they decompose into local rings:

$$\mathbb{T}_\ell \simeq \prod_{\ell \in a} T_a$$

and

$$\mathbb{T}_m^1 \simeq \prod_{\ell \in b} T_b.$$

Therefore, we get

$$\mathbb{T}_\ell/I \simeq \prod_{\ell \in a \text{ and } I \subseteq a} T_a/I = \mathbb{T}_m/I \text{ and } \mathbb{T}_m^1/J \simeq \prod_{\ell \in b \text{ and } J \subseteq b} T_b/J = \mathbb{T}_n^1/J$$

due to $\mathbb{T}_\ell$ being a maximal ideal in $\mathbb{T}_\ell/I$ and $\mathbb{T}_m^1/J$.

Again since $T_p \not\in m$, $m$ is not $p$-new by Lemma 2.2. Thus, $\mathbb{T}_m \simeq \mathbb{T}_m^0$, and the latter is isomorphic to $\mathbb{T}_m^1$. By taking quotients, we get the following.

$$\mathbb{T}(Mp^r)/\mathbb{T}_\ell Mp, Mp^r \otimes \mathbb{Z}_\ell \simeq \mathbb{T}_\ell/I \simeq \mathbb{T}_m/I \simeq \mathbb{T}_m^0/I \simeq \mathbb{T}_m^1/I \simeq \mathbb{T}_n^1/J \simeq \mathbb{T}(Mp^{r-1})/\mathbb{T}_m^1Mp, Mp^r-1 \otimes \mathbb{Z}_\ell.$$

Since this is true for any $r \geq 2$, we get

$$\mathbb{T}(Mp^r)/\mathbb{T}_\ell Mp, Mp^r \otimes \mathbb{Z}_\ell \simeq \mathbb{T}(Mp^{r-1})/\mathbb{T}_m^1Mp, Mp^r-1 \otimes \mathbb{Z}_\ell \simeq \cdots \simeq \mathbb{T}(M)/\mathbb{T}_m^1Mp, Mp \otimes \mathbb{Z}_\ell.$$

\[ \square \]

7. Proof of the main theorem

**Theorem 7.1** (Main theorem). Let $m \subseteq \mathbb{T}(N)$ be a rational Eisenstein prime containing $\ell$. Suppose that $\ell^2$ does not divide $4N$.

1. $m$ is of the form $(\ell, I^D_{M,N})$ for some divisor $D$ of $N$ and some divisor $M$ of $N^{sf}D$ with $M(N^{\square}/D) \neq 1$.
2. When $M(N^{\square}/D) \neq 1$, we get $\mathbb{T}(N)/\mathbb{T}_M^{D_{M,N}} \otimes \mathbb{Z}_\ell \simeq \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}_\ell$, where $n$ is the order of $C_{M,N}^D$.
3. If $m$ is new, then $D = 1$ and $p \equiv -1 (\text{mod } \ell)$ for all prime divisors $p$ of $N^{sf}/M$.
4. $C(N)[m] \neq 0$.

If $\ell = 2$ then the same holds as long as 4 does not divide $N$ and $N^{sf}D/M$ is an odd integer greater than 1.

**Proof.** The first and third assertions follow from Theorem 3.3. The second one follows from Theorem 6.1. And the last one is clear from the second one because $C_{M,N}^D$ is annihilated by $I^D_{M,N}$ (Theorem 4.7).

\[ \square \]

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