We discuss the relation between attractive singular potentials \(-\alpha_s/r^s, \ s \geq 2\) and inelasticity. We show that mentioned singular potentials can be regularized by infinitesimal imaginary addition to interaction constant \(\alpha_s = \alpha_s \pm i0\). Such a procedure enables unique definition of scattering observables and is equal to an absorption (creation) of particles in the origin. It is shown, that suggested regularization is an analytical continuation of the scattering amplitudes of repulsive singular potential as a function of interaction constant \(\alpha_s\). The nearthreshold properties of regularized in a mentioned way singular potential are examined. We obtain expressions for the scattering lengths, which turn to be complex even for infinitesimal imaginary part of interaction constant. The problem of perturbation of nearthreshold states of regular potential by a singular one is treated, the expressions for level shifts and widths are obtained. The physical sense of suggested regularization is that the scattering observables are insensitive to any details of the short range modification of singular potential, if there exists sufficiently strong inelastic short range interaction. In this case the scattering observables are determined by the solutions of Schrodinger equation with regularized potential \(-\langle\alpha_s \pm i0\rangle/r^s\). Possible application of developed formalism to systems with short range annihilation are discussed.

I. INTRODUCTION.

Singular potentials and related collapse problem is of special interest in many fields. Attractive singular potentials are concerned in the three-body problem in connection with famous Efimov and Thomas effect [1], in nuclear physics (low energy NN and N\bar{N} interaction [2,3], relativistic equations (Dirac equation with strong Coulomb potential and more singular potentials [4]), atomic and molecular physics (van der Waals interaction [5]) and other fields. The review of results, devoted to this one of the oldest quantum mechanical problem can be found in [6,7].

The collapse or ”fall to the center” [8], produced by singular potential, means that the energy spectrum is not bounded from below. The wave function has infinite number of zeroes in the vicinity of the origin, while two independent solutions of Schrodinger equation have the same order of singularity and there is no obvious way to choose their linear combination and define scattering observables.

The common way to deal with the pathological behavior of singular potentials is to impose that exact physical short range interaction is not that singular and the knowledge of the short-range parameters of interaction enables unique definition of scattering observables. Following this logic numerous methods of singular potentials regularization have been suggested [9,10]. The obtained results are usually very strongly depend on regularizing parameters (such as cut-off radius).

Another approach to the problem of collapse, which we will follow in this paper, is to exploit the relation between singular potentials and inelastic scattering [11,12]. The aim of the present study is to investigate the physical content of such a relation. The link between singular potentials and inelasticity arises from the fact that Hamiltonian with real attractive singular potential is not self-adjoint [13]. To make this link explicit, we allow interaction constant to be complex \(\alpha_s = \alpha_s \pm i\omega (\omega > 0)\). First we show that there is no collapse for complex values of interaction constant. This follows from the simple observation, that one of the solutions of Schrodinger equation sharply decreases, while the other solution increases near the origin inside the domain where complex potential acts [14]. Thus one can choose between independent solutions and calculate observables. Less trivial is that even in case there is an infinitesimal imaginary part of \(\alpha_s = \alpha_s \pm i0\), the collapse is eliminated and observables are uniquely defined. The non-self-adjointness of Hamiltonian for real positive values of interaction constant \(\alpha_s\) manifests itself in the fact that S-matrix is not unitary even for infinitesimal imaginary part of interaction constant. The S-matrix module is less than unit for \(\alpha_s = \alpha_s + i0\) (absorption) and greater than unit for \(\alpha_s = \alpha_s - i0\) (creation). In this sense the scattering on a singular potential can be ”redefined” as a scattering on an absorptive (creative) potential with interaction constant \(\alpha_s = \alpha_s \pm i0\). We will show that the physical meaning of such an approach is that the scattering observables can be uniquely defined and are independent on any details of short range modification of singular potential, if this short range interaction includes sufficiently strong inelasticity. We mention possible application of developed formalism to systems with short range annihilation, including nucleon-antineucleon and atom-antiatom systems.
A. Potentials more singular than $-\alpha/r^2$.

In the following we put $2M = 1$. Essential point is that interaction constant is considered to be complex $\alpha_s = \alpha_s \pm i\omega$. Near the origin we can neglect energy and all the nonsingular potentials, increasing at the origin slower than $1/r^2$, so the Schrödinger equation becomes:

$$-rac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2} - \frac{\alpha_s}{r^s} \Phi(r) = 0$$

The following variable substitution in the wave-function $\Phi(r)$:

$$\Phi(r) = \sqrt{r}K(z)$$  \hspace{1cm} (1)

$$z = \frac{2\sqrt{\alpha_s}}{s-2} r^{-(s-2)/2}$$  \hspace{1cm} (2)

yields in Bessel equation:

$$K'' + \frac{1}{z}K' + \left(1 - \frac{\mu^2}{z^2}\right)K = 0$$

$$\mu = \frac{\sqrt{(2l+1)^2 - 4\alpha_s^2}}{s-2}$$  \hspace{1cm} (3)

Its solution is:

$$K = CH^{(1)}_{\mu}(z) + DH^{(2)}_{\mu}(z)$$  \hspace{1cm} (4)

Here $H^{(1)}_{\mu}(z)$ and $H^{(2)}_{\mu}(z)$ are Hankel functions of order $\mu$ [15]. It is worth to mention that the new variable $z = 2p_{eff}r/(s-2)$ is semiclassical phase, and local momentum $p_{eff} = \sqrt{\alpha_s/r^s + \alpha_2/r^2 - l(l+1)/r^2}$ coincides with classical momentum $p_{eff} \approx \sqrt{\alpha_s/r^s}$ near origin.

Let us cut the singular potential at small distance $r_0$, and replace it with square well:

$$U(r) = \left\{ \begin{array}{ll}
-\alpha_s/r^s & \text{if } r > r_0 \\
-\alpha_s/r_0^s & \text{if } r \leq r_0 
\end{array} \right.$$  \hspace{1cm} (5)

The logarithmic derivative of the wave-function in the square well at point $r_0$ is:

$$\left. \frac{\Phi'}{\Phi} \right|_{r=r_0} = p_{eff} \cot(p_{eff}r_0)$$  \hspace{1cm} (6)

We can choose $r_0$ small enough to ensure:

$$|p_{eff}r_0| \approx \sqrt{\alpha_s/r^{(s-2)/2}} \gg 1$$

As far as $\alpha_s$ is complex (for distinctness we fix $\text{Im} \alpha_s > 0$) we have:

$$\text{Im} p_{eff}r_0 \gg 1$$

Then, taking into account that in this case $\cot(p_{eff}r_0) \to -i$, we get for the mentioned above logarithmic derivative

$$\left. \frac{\Phi'}{\Phi} \right|_{r=r_0} = -ip_{eff}$$

which corresponds to the condition of full absorption of particle with momentum $p_{eff}$ at distance $r_0$.

The logarithmic derivative of the wave-function in singular potential for $r > r_0$ is:

$$\frac{\Phi'}{\Phi} = -ip_{eff} \frac{\exp(iz - i\pi/4 - i\mu\pi/2) - S\exp(-iz - i\pi/4 - i\mu\pi/2)}{\exp(iz - i\pi/4 - i\mu\pi/2) + S\exp(-iz - i\pi/4 - i\mu\pi/2)}$$

We used here the asymptotic expansion of Hankel functions of large argument. Matching logarithmic derivatives we get for $S \equiv D/C$:  

2
Thus the solution near the origin is:

\[ \Phi = \sqrt{r} H_\mu^{(1)} \left( \frac{2 \sqrt{\alpha_s}}{s-2}, -\frac{s-2}{2} \right) \]  

(7)

which means that only "incoming" wave presents in the solution near origin (it is convenient to define "incoming" wave as a solution of Schrödinger equation, which logarithmic derivative at point \( r \) is equal to \(-ip_{eff}(r)\)). For \( \omega > 0 \) this wave-function rapidly decreases with \( r \to 0 \).

The above results are valid in the limit \( r_0 \to 0 \).

We can now go to the limit \( \omega \to +0 \), as far as \( \tilde{S} \) is independent on \( \omega \). We must ensure that \( \text{Im} \, z = 2/(s-2) \text{Im} \, pr_0 \gg 1 \) which is achieved by the order of performing limiting transitions. First we calculate \( \tilde{S} \) for nonzero \( \omega \) (and put \( r_0 \to 0 \)), and then put \( \omega \to +0 \). Let us note, that according to our cut-off procedure (5) the singular potential is real everywhere except infinitesimal vicinity of the origin.

It is interesting, that in spite \( \text{Im} \, \alpha \to +0 \) the scattering length has nonzero imaginary part. As we’ve mentioned above, this is the manifestation of the singular properties of potential which violates the self-adjointness of Hamiltonian.

For \( \omega = \text{Im} \, \alpha, \omega \to -0 \) one can easily get \( \tilde{S} = \infty \) and mentioned S-wave scattering length for creation is:

\[ a^\text{abs}_S = \exp(-i \pi/(s-2)) \left( \frac{\sqrt{\alpha_s}}{s-2} \right)^{2/(s-2)} \Gamma((s-3)/(s-2))/\Gamma((s-1)/(s-2)) \]  

(10)

Thus the scattering length has well-determined and conjugated values in the upper and lower complex halfplanes of \( \alpha_s \).

If we turn to (10), we see, that the real part of S-wave scattering length is a fixed sign function for any positive value of \( \alpha_s \), which indicates there are no bound states in regularized singular potential \(-\alpha_s \pm i0)/r^s\) (we return to the spectrum problem of singular potential further on). Let us compare the scattering length (10) with that of the repulsive singular potential \( \alpha_s/r^s \). One can get:

\[ a^\text{rep}_S = \left( \frac{\sqrt{\alpha_s}}{s-2} \right)^{2/(s-2)} \Gamma((s-3)/(s-2))/\Gamma((s-1)/(s-2)) \]  

(11)

It is easy to see, that (10) can be obtained from (11) simply by choosing certain branch (corresponding to absorption or creation) of the function \( \left( \frac{\sqrt{\alpha_s}}{s-2} \right)^{2/(s-2)} \) when passing through the branching point \( \alpha_s = 0 \). The scattering length in singular potential is an analytical function of \( \alpha_s \) in the whole complex plane of \( \alpha_s \) with a cut along positive real axis. The jump of scattering length is:

\[ \Delta a \equiv a^\text{abs}_S - a^\text{rep}_S = -2i \sin\left( \frac{\pi}{s-2} \right) \left( \frac{\sqrt{\alpha_s}}{s-2} \right)^{2/(s-2)} \Gamma((s-3)/(s-2))/\Gamma((s-1)/(s-2)) \]  

It is also curious that the real part of regularized scattering length for \( 3 < s < 4 \) is negative, which corresponds to attraction, while for \( s > 4 \) it becomes positive, which corresponds to repulsion. One can say that strong absorption (creation) in the origin results in effective repulsion. The attractive effect of the potential tail dominates over repulsive effect of the origin only for \( s < 4 \). For more rapidly decreasing potentials with \( s > 4 \) the overall effect is repulsive. In case of \( s = 4 \) there is exact compensation of core repulsion and tail attraction, so \( \text{Re} \, a = 0 \).

It is worth to mention, that above results can be obtained by fixing \( \alpha_s \) real, but making infinitesimal complex rotation of \( r \)-axis:
\[ r = \rho \exp(i\varphi) \]

which yields in Schrodinger equation:

\[
\left[ -\frac{\partial^2}{\partial \rho^2} + \frac{i(l+1) - \alpha_2}{\rho^2} - \frac{\alpha_2(s-i(s-2)\varphi)}{\rho^s} \right] \Phi(\rho \exp(i\varphi)) = 0
\]

One can see that negative values of \( \varphi \) correspond to absorption, while positive values correspond to creation.

Summarizing, we can say that the scattering on attractive singular potential \(-\alpha_s/r^{-4}\) \((s > 2)\) can be defined as full absorption (creation) of the particle in such a potential in the origin. This procedure is supported by the existence of the limit for \( S \equiv D/C \) (which defines the linear combination of independent solutions) when interaction constant is approaching real axis from upper (lower) complex half-plane: \( \alpha_s = \alpha_s \pm i0 \). The physical sense of such a regularization is as follows. In case, when a singular interaction includes strong short range inelastic component (the restrictions on the strength of such a short range inelastic component would be discussed further on) the scattering observables are insensitive to any details of the short range physics and are determined by the solution (7) of Schrodinger equation with regularized singular potential \(-(\alpha_s \pm i0)/r^s\).

**B. Potential \(-\alpha/r^2\)**

Let us now turn to the very important case \(-\alpha/r^2\). The wave-function now is:

\[
\Phi = \sqrt{r} \left[ CJ_{\nu_+}(kr) + DJ_{\nu_-}(kr) \right]
\]

\[
\nu_{\pm} = \pm \sqrt{1/4 - \alpha}
\]

where \( k = \sqrt{|E|} \), and \( J_{\nu_{\pm}} \) are the Bessel functions \([15]\). We use the same cut-off procedure (5) and put \( \alpha = \alpha + i\omega \). Using Bessel function behavior at small \( r \) we get:

\[
\Phi \sim r^{\nu_+ + 1/2} + \bar{S} r^{\nu_- + 1/2}
\]

where \( \bar{S} \equiv D/C \). In the following we will be interested in the values of \( \alpha \) greater than critical \( \text{Re} \alpha > 1/4 \). We mention that for \( \nu_+ = \sqrt{1/4 - \alpha}, \text{Re} \nu > 0, \text{Im} \nu < 0 \), for small \( \omega \) we find \( \text{Re} \nu = \omega/2 \text{Re} \alpha - 1/4 \)

Matching of logarithmic derivatives at cut-off point \( r_0 \) gives for \( \bar{S} \):

\[
\bar{S} = r_0^{\nu_+ - \nu_-} \text{const} \sim r_0^{2\nu} = r_0^{\omega/2 \text{Re} \alpha - 1/4} r_0^{-2i \sqrt{\text{Re} \alpha - 1/4}}
\]

One can see, that due to the presence of the imaginary part of interaction constant \( \omega > 0 \) \( \bar{S} \to 0 \) when \( r_0 \to 0 \). Practically, this limit is achieved very slowly, i.e. if we want \( |\bar{S}| < \varepsilon \), one needs to get \( r_0 < \varepsilon \sqrt{\text{Re} \alpha - 1/4}/\omega \)

Thus, for \( \alpha = \alpha + i0 \) the wave-function is

\[
\Phi = C \sqrt{r} J_{\nu_+}(kr)
\]

\[
\nu_+ = + \sqrt{1/4 - \alpha}
\]

For large argument this function behaves like:

\[
\Phi \sim \cos(z - \nu_+ \pi/2 - \pi/4)
\]

The corresponding scattering phase is:

\[
\delta = i\pi/2 \sqrt{\alpha - 1/4 - \pi/4}
\]

One can see that the regularized wave-function and phase-shift are analytical functions of \( \alpha \) in the whole complex plane with a cut along real axis \( \alpha > 1/4 \). The jump of the phase-shift on the cut is:

\[
\Delta \delta = i\pi \sqrt{\alpha - 1/4}
\]

The mentioned procedure enables to choose between the independent solutions of Schrodinger equation. In case of absorption \((\alpha = \alpha + i0)\) we must choose the solution with negative imaginary index \( \nu_+ \), in case of creation- with positive imaginary index \( \nu_- \). We see, that scattering on potential \(-\alpha/r^2\) with \( \text{Re} \alpha > 1/4, \omega \to \pm i0 \) results in partial absorption (creation), characterized by imaginary scattering phase (16).
C. Regularization of real singular potential by complex less singular potential.

As it was mentioned above, the essence of suggested regularization is that the presence of strong enough inelastic interaction at short distance suppresses one of the independent solutions of Schrodinger equation and singles out another one. Now we would like to determine the minimum order of singularity of infinitesimal imaginary potential required for regularization of given singular potential. The potential of interest is a sum $-\alpha_{s_1}/r^{s_1} - \alpha_{s_2}/r^{s_2}$, $s_1 = s_2 + t$. Here we keep $\alpha_{s_1}$ real, but put $\omega = \text{Im} \alpha_{s_2} > 0$.

We again use the cut-off at some small $r_0$ and replace potential with square well. The logarithmic derivative of square well wave-function for small enough $r_0$ is:

$$\frac{\Phi'}{\Phi} |_{r=r_0} = p \cot(pr_0)$$
$$pr_0 = \frac{\sqrt{\alpha_{s_1}}}{r(s_1/2-1)} + \frac{i\omega \sqrt{1/\alpha_{s_1}}}{2pr(s_1/2-1-t)}$$

One can see, that if $t < s_1/2 - 1$ then for $r_0 \to 0 \text{ Im} pr_0 \to +\infty$ and $p \cot(pr_0) \to -ip$. Thus we return to previously examined case, where we found that $S \equiv D/C = 0$. It means, that we can put $r_0 = 0$ and than go to limit $\omega \to +0$ (in numerical calculations keeping $\omega \sqrt{1/\alpha_{s_1}/2}^{1/2} \ll 1$).

If $t = s_1/2 - 1$ for $r_0 \to 0 \text{ Im} pr_0 \to i\omega \sqrt{1/\alpha_{s_1}/2}$ and $p \cot(pr_0) \to p \cot(i\omega \sqrt{1/\alpha_{s_1}/2})$.

Using asymptotic behavior of Bessel functions, we get for $S$:

$$S = \frac{i \exp(i2pr_0/(s_2 - 1) - i\pi/4 - i\nu\pi/2) + \cot(pr_0) \exp(i2pr_0/(s_2 - 1) - i\pi/4 + i\nu\pi/2)}{\exp(i2pr_0/(s_2 - 1) - i\pi/4 - i\nu\pi/2) \cot(pr_0) - i \exp(i2pr_0/(s_2 - 1) - i\pi/4 + i\nu\pi/2)}$$

The value of $S$ oscillates with decreasing $r_0$ and there is no limit for $r_0 \to 0$, until $\text{Im} pr_0 \gg 1$ for which case we return to $S = 0$. So for $t = s_1/2 - 1$ the regularization is possible only for large, noninfinitesimal $\omega \gg 1/\alpha_{s_1}$.

Obviously, for $t > s_1/2 - 1$ mentioned above regularization is impossible since $\text{Im} pr_0 \to 0$ with $r_0 \to 0$.

Summarizing the above results, we may say that the scattering is insensitive to any details of regularizing short range interaction in singular potential $-\alpha_{s_1}/r^{s_1}$ if the inelastic component of such an interaction behaves more singular than $-1/r^{(s_1/2+1)}$.

D. Boundary condition

As it follows from (7) the regularization of singular potential by means of infinitesimal imaginary addition to interaction constant is equal to the boundary condition at the origin. For $s > 2$:

$$\Phi \sim \sqrt{r} H_{\mu}^{(1)} \frac{2\sqrt{\alpha_{s_1}}}{s-2} r^{-(s-2)/2}$$
$$\mu = \frac{\sqrt{(2l+1) - 4\alpha_{s_2}}}{s-2}$$

or:

$$\left. \frac{\Phi'}{\Phi} \right|_{r=0} = -ip_{eff}$$

As one can see, this is the "full absorption" boundary condition.

For $s = 2$:

$$\Phi \sim \sqrt{r} J_{\nu_\pm}(kr)$$
$$\nu_\pm = \pm \sqrt{1/4 - \alpha}$$

or:

$$\left. \frac{\Phi'}{\Phi} \right|_{r=0} = \frac{(1/2 \mp i|\nu_\pm|)}{\sqrt{1/4 - \nu_\pm^2}} p_{eff}$$

This boundary condition corresponds to partial absorption (creation).
E. Singular potential and WKB approximation.

WKB approximation, consistent with the boundary condition (17) for $s > 2$ is:

$$\Phi = \frac{1}{\sqrt{p_{eff}}} \exp(i \int_{r}^{a} p_{eff} dr) \quad (23)$$

One can easily check that the above expression coincides with an asymptotic form of solution (7) for small $r$ (large $z$).

The WKB approximation holds if:

$$\left| \frac{\partial}{\partial r} \left( \frac{1}{p_{eff}} \right) \right| \ll 1$$

In case of zero-energy scattering on singular potential with $s > 2$ this condition holds for:

$$r \ll (2\sqrt{\alpha/s})^{(2/s-2)}$$

i.e. near origin.

For $s = 2$ the semiclassical approximation is valid only for $\alpha \gg 1$. One can see that in this case the boundary condition (22) becomes the condition of full absorption (creation) (19).

We can conclude, (for distinctness we will speak here of absorptive potentials with $\alpha = \alpha + i0$) in case WKB approximation is valid everywhere the Schrodinger equation solution includes incoming wave only. The corresponding S-matrix $S = 0$ within such an approximation.

On the other hand nonzero value of S-matrix means that an outgoing wave presents in the solution. The outgoing wave can appear in the solution only in the regions where (23) does not hold. For example, in the zero energy limit $E \to 0$ the S-matrix is nonzero:

$$S = \frac{1 - i\alpha S^{abs}}{1 + i\alpha S^{abs}} \quad (24)$$

with $S^{abs}$ from (10) and $k = \sqrt{E}$. One can show that the outgoing wave is reflected from those parts of potential which change sufficiently fast in comparison with effective wavelength:

$$\left| \frac{\partial}{\partial r} \left( \frac{1}{p_{eff}} \right) \right| \gtrsim 1$$

For zero energy scattering and $l = 0$ this holds for:

$$r \geq (2\sqrt{\alpha/s})^{(2/s-2)}$$

F. Remarks on the spectrum of singular potential.

1. Spectrum of potential more singular than $-\alpha_2/r^2$.

The "infinitely deep" bound states produced by singular potential is the central point of the collapse problem.

A simple qualitative picture of the energy spectrum of singular potential is as follows. One can have an estimation $E_{apr}$ from above for the ground state $E_0$ in a singular potential, replacing singular potential by a square well at small $r_0$. Such an estimation is a ground state energy in the mentioned square well with depth $\alpha_s/r_0^s$ and width $r_0$. For small enough $r_0$:

$$E_{apr} \simeq -\alpha_s/r_0^s + \pi^2/r_0^2 \quad (25)$$

$$E_{apr} \to -\infty \quad r_0 \to 0 \quad (26)$$

This ensures that exact level $E_0 < E_{apr}$ also tends to minus infinity with decreasing of $r_0$.  

6
Our regularizing procedure \( \alpha_s \to \alpha_0 \pm i\omega \) results in:

\[
E_{apr} \simeq -\frac{\alpha_0}{r_s^0} \pm i\omega/r_s^0 + \frac{\pi^2}{r_s^0}
\]

As far as we always keep \( \text{Im} p_{eff} r_0 = \pm \omega r_0^{(s-2)/2} \sqrt{1/\alpha_s} \to \infty \) the width of the “ground” state would be infinitely large even in case \( \omega \to \pm 0 \).

This infinitely large width corresponds to the full absorption (creation) of the particle in the origin. One can say that the collapse is eliminated because the particle disappears before it approaches close enough to the scattering center. Practically this means, that there are no bound states in the regularized in a mentioned above way singular potential. Absence of bound states is clear also from the already mentioned fact, that scattering amplitudes in regularized attractive singular potential can be obtained by an analytical continuation in \( \alpha_s \) of corresponding scattering amplitude in repulsive singular potential.

2. Spectrum of potential \(-\alpha_2/r^2\).

The S-matrix in potential \(-\alpha_2 \pm i0)/r^2\), according to (16) is energy independent:

\[
|S| = \exp(\mp \pi \sqrt{\alpha_2 - 1/4})
\]

Obviously, such an S-matrix has no singularities and there are no bound states in such a potential.

Meanwhile, if \(-\alpha_2 \pm i0)/r^2\) is combined with another potential, which can produce bound states itself, these bound states may obtain finite width. Let us treat, for example, the case of attractive coulomb potential, which is combined with potential \(-\alpha_2 + i0)/r^2\).

It is easy to see, that the energy is:

\[
E = -\frac{1}{2n^2} \quad n = n_r + \nu_+ + 1/2
\]

where radial quantum number \( n_r \) is positive integer or zero.

Finally, for \( \alpha > 1/4 \):

\[
E = \frac{1}{2} \frac{n_r + 1/2}{n_r^2 + n_r + \alpha} - \frac{i}{2} \frac{\sqrt{\alpha - 1/4}}{n_r^2 + n_r + \alpha}
\]

Further we will examine in more details the modification of spectrum of the regular potential by a singular one.

G. Nearthreshold scattering and perturbation theory

We will be interested in the modification of the low energy scattering amplitude of regular potential \( U(r) \), by a potential, which has singular behavior \(-\alpha_s \pm i0)/r^s - \alpha_2/r^2\) at short distances. We will also treat the modification of the spectrum of nearthreshold states in such a potential. If \( \alpha_s \) is small enough, there is a range where

\[
U(r) \ll \alpha_s/r^s \ll (l(l + 1) - \alpha_2)/r^2 \quad (27)
\]

Let us first treat the case, when regular potential is approximately constant in the mentioned range \( U(r) \approx p^2 \). Then from (27) we get:

\[
p \alpha_s^{1/(s-2)} \ll 1 \quad (28)
\]

For such values of \( r \) the wave function is:

\[
\Phi \sim \sqrt{r} (J_\mu(pr) - \tan(\delta_s Y_\mu(pr))) \quad (29)
\]

\[
Y_\mu = \frac{J_\mu \cos(\mu \pi) - J_{-\mu}}{\sin(\mu \pi)} \quad (30)
\]

\[
\mu = \sqrt{(l + 1/2)^2 - \alpha_2}
\]
here $\delta_s$ is a phase shift, produced by singular potential in the presence of regular potential. For small $r \ll \alpha_s^{1/(s-2)}$ the wave-function is determined by singular and centrifugal potential:

$$\Phi \sim \sqrt{r} H_n^s \left( \frac{2\sqrt{\alpha_s}}{s-2} r^{-(s-2)/2} \right)$$

$$\nu = 2\mu/(s-2)$$

Matching logarithmic derivatives and taking into account (28) we get for the phase shift $\delta_s$:

$$\delta_s = -\sin(\pi\mu) \left( \frac{p_a^{1/(s-2)}}{2(s-2)^{2/(s-2)}} \right)^{2\mu} \exp\left(-i\pi\nu\right) \frac{\Gamma(1-\mu) \Gamma(1-\nu)}{\Gamma(1+\mu) \Gamma(1+\nu)}$$

which for integer values of $2\mu = 2l + 1$ becomes:

$$\delta_s = (-1)^{l+1} \left( \frac{p_a^{1/(s-2)}}{2(s-2)^{2/(s-2)}} \right)^{2l+1} \sin(\pi\nu) \frac{\Gamma(1/2 - l) \Gamma(1 - \nu)}{\Gamma(3/2 + l) \Gamma(1 + \nu)}$$

Let us mention, that for nonzero $l$ the value of $\text{Re} \, \delta_s$ may become smaller, than the correction to expression (29), produced by the tail of singular potential, which is now small in comparison with $U(r)$. Such a correction depends on certain form of the tail of singular potential and can be calculated as a first order of distorted wave approximation.

In the same time $\delta_s$ has positive imaginary part according to absorbing character of singular potential.

$$\text{Im} \, \delta_s = (-1)^{l+1} \left( \frac{p_a^{1/(s-2)}}{2(s-2)^{2/(s-2)}} \right)^{2l+1} \sin(\pi\nu) \frac{\Gamma(1/2 - l) \Gamma(1 - \nu)}{\Gamma(3/2 + l) \Gamma(1 + \nu)}$$

(31)

In case of coulomb potential $U(r) = -\beta/r$, the solution in singular potential should be matched with zero-energy coulomb wave-function:

$$\Phi_c \sim \sqrt{r} (J_\eta(\sqrt{8\beta}r) - \tan(\delta_s) Y_\eta(\sqrt{8\beta}r))$$

$$\eta = 2\sqrt{(l+1/2)^2 - \alpha_2}$$

The expression for the phase shift now is:

$$\delta_s = -\sin(\pi\eta) \left( \frac{8\beta \alpha_s^{1/(s-2)}}{2(s-2)^{2/(s-2)}} \right)^{\eta} \exp\left(-i\pi\eta\right) \frac{\Gamma(1-\eta)}{\Gamma(1+\eta)}$$

In the upper expression $\eta$ is noninteger. For the validity of the above expression it is required that $\beta \alpha_s^{1/(s-2)} \ll 1$.

From the above results we can immediately get the low energy phase shift and scattering volume $a_l$ produced by pure singular potential $-(\alpha_s + i0)/r^s$, putting $U(r) = k^2 = E$.

$$\delta_s = (-1)^{l+1} \left( \frac{k \alpha_s^{1/(s-2)}}{2(s-2)^{2/(s-2)}} \right)^{2l+1} \exp\left(-i\pi\nu\right) \frac{\Gamma(1/2 - l) \Gamma(1 - \nu)}{\Gamma(3/2 + l) \Gamma(1 + \nu)}$$

$$a_l = (-1)^{l} \left( \frac{\alpha_s^{1/(s-2)}}{2(s-2)^{2/(s-2)}} \right)^{2l+1} \exp\left(-i\pi\nu\right) \frac{\Gamma(1/2 - l) \Gamma(1 - \nu)}{\Gamma(3/2 + l) \Gamma(1 + \nu)}$$

The nearthreshold states produced by regular potential $U(r)$ are perturbed by the short range singular potential. In particular they get the widths, which in our case of small $\delta_s$ are proportional to $\text{Im} \, \delta_s$.

If the nearthreshold states spectrum in $U(r)$ has semiclassical character, than from the quantization rule:

$$\int \sqrt{E_n + \delta E_n - U(r)} dr + \delta_s = \text{const}$$

one gets:

$$\delta E_n = -\delta_s \omega_n$$

(32)

where $\omega_n$ is semiclassical frequency.
\[ \omega_n = (\int (E_n - U(r))^{-1/2} dr)^{-1} \]

Taking into account (31) we get for the width of the state:

\[ \Gamma_n/2 = (-1)^{l+1} \left( \frac{\rho \alpha_s^{1/(s-2)}}{2(s-2)^2(s-2)} \right)^{2l+1} \frac{\sin(\pi \nu)}{\Gamma(3/2 + l)\Gamma(1 + \nu)} \omega_n \]

We see that singular potential modification of nearthreshold scattering in regular potential is small when the characteristic wave-length of the particle in the regular potential \(1/p\) is much greater than scattering length in singular potential, so that \(\rho \alpha_s^{1/(s-2)} \ll 1\).

**H. Physical examples.**

1. **Nearthreshold nucleon-antinucleon scattering.**

The meson-exchange inspired models of nucleon-antinucleon low energy interaction [2,3,16–18] encounter attractive potentials that behave like \(1/r^3\) near origin. These models require short-range regularization of singular attractive potentials. Such a regularization is usually done by means of cut-off radius. Another important ingredient of mentioned models is a short-range imaginary potential, which describes nucleon-antinucleon annihilation. We will demonstrate the regularization of attractive singular potentials by means of imaginary addition to the strength parameter of singular potential. The short-range annihilation is automatically taken into account in our approach due to mentioned above full absorption of the particle in the vicinity of the scattering center. In the same time no cut-off radius is needed. We calculate the scattering volume in the state with quantum numbers \(J = 0, S = 1, L = 1, T = 0\), where singular \(1/r^3\) terms are attractive. It is known that this particular scattering state has a nearthreshold resonance, so the scattering volume is very much enhanced. The reproduction of such an enhancement can be a test for the model. We use the version of real OBEP potential \(W_{OBEP}\), used in Kohno-Weise model [16]. This potential includes singular terms at short distance, which are now regularized by introducing imaginary singular potential of the form \(W_I = -i\omega/r^3 \exp(-r/\tau)\) with \(\omega \to 0\):

\[ W = W_{OBEP} - i\omega/r^3 \exp(-r/\tau) \]

The scattering volume \(3P_0 T = 0\) calculated in the limit \(\omega \to 0\) turns to be:

\[ a_r = -7.66 - i4.87 \text{ fm}^3 \]

The result is independent on diffuseness \(\tau\), as far as \(\omega\) is infinitesimal.

The value of the same scattering volume, obtained within Kohno-Weise model with a cut-off \(r_c = 1\) fm is:

\[ a_{KW} = -8.83 - i4.45 \text{ fm}^3 \]

As one can see, both scattering volumes are rather close, reproducing the strong P-wave enhancement. In the same time suggested approach is free from any uncertainty related to the cut-off radius. The above result demonstrates that condition of full absorption, incorporated in our model, is rather realistic in case of \(NN\) interaction [19].

2. **Hydrogen-antihydrogen interaction.**

It is known, that long range interaction between atoms is dominated by attractive van der Waals potential \(-C_6/r^6\). The very simple model of low energy (fraction of eV) hydrogen-antihydrogen interaction is the absorption model, which is based on the assumption that the particles are fully absorbed on the sphere of radius \(r_c\), while at \(r > r_c\) the atoms interact via \(-C_6/r^6\) potential. The critical radius is close to hydrogen Bohr radius \(r_c \approx r_B\).

The scattering observables in such a model can be obtained within suggested regularization of attractive singular potential by imaginary addition to the interaction constant \(C_6 = C_6(1 + i\omega)\). In the limit \(\omega \to 0\) (which corresponds to \(r_c \to 0\)) we get for S-wave scattering length:

\[ a_S = (MC_6)^{1/4} \frac{\Gamma(3/4)}{2\sqrt{2\Gamma(5/4)}}(1 - i) \]
Here $M$ is a proton mass.

We would not discuss here the physics of hydrogen-antihydrogen interaction [20,21] and limitation of absorption model. We only mention, that strong absorption model seems to be more realistic for the interaction of $1S$ hydrogen with excited $nS$ antihydrogen, where transitions to a lot of inelastic channels are energetically allowed even in zero energy limit. If we take into account that $C_6 \sim n^4$, we get a useful law:

$$a_S \sim n$$

Let us summarize that in mentioned above physical examples the strong absorption makes the system insensitive to any details of short range physics. This radically simplifies the problem and makes useful the developed approach of singular potentials regularization by means of imaginary addition to interaction constant.

I. Conclusion.

We have found that the scattering amplitude in singular potential $-\alpha_s/r^s (s \geq 2)$ has well-defined values if interaction constant is complex $\alpha_s = \alpha_s \pm i\omega (\omega > 0)$. This is true even in the limiting case $\omega \to +0$. The corresponding S-matrix is nonunitary ($|S| < 0$ for $\omega > 0$ and $|S| > 0$ for $\omega < 0$), has conjugated values in the complex halfplanes of $\alpha_s$ and makes a jump on the real positive axis of $\alpha_s$. This property is a manifestation of non-self-adjointness of Hamiltonian with singular potential and can be used for "redefinition" of singular potential as absorptive (creative) potential by adding an infinitesimal imaginary part to the interaction constant $\alpha_s = \alpha_s \pm i0$. Such a procedure is equal to boundary condition of absorption (creation) of the particle in the origin. This procedure eliminates the collapse of the system, as far as the particle promptly disappears in the scattering center, rather than forms infinitely deep bound states. There are no bound states in such a regularized singular potential. The scattering observables in such a potential can be obtained from those of repulsive singular potential by analytical continuation and choosing certain branch in passing through the branching point $\alpha_s = 0$.

It was shown, that scattering length perturbation theory can be used when we are interested how the nearthreshold spectrum and scattering phase of regular potential is modified by a singular one. The perturbation parameter is the ratio of the scattering length in singular potential to the effective wavelength in the regular potential near origin. We have obtained an expression for the widths and shifts of mentioned states.

The physical sense of suggested regularization is that the scattering observables are insensitive to any details of short range modification of singular potential, if there is sufficiently strong short range inelastic interaction. In this case the scattering amplitude can be calculated by solving Schrodinger equation with the regularized singular potential $-(\alpha_s \pm i0)/r^s$. The developed formalism can be useful for examining of particle-antiparticle systems with singular interaction and short range annihilation.

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