Cobordism independence of Grassmann manifolds

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Abstract. This note proves that, for \( F = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), the bordism classes of all non-bounding Grassmannian manifolds \( G_k(F^{n+k}) \), with \( k < n \) and having real dimension \( d \), constitute a linearly independent set in the unoriented bordism group \( \mathcal{N}_d \) regarded as a \( \mathbb{Z}_2 \)-vector space.

Keywords. Grassmannians; bordism; Stiefel–Whitney class.

1. Introduction

This paper is a continuation of the ongoing study of cobordism of Grassmann manifolds. Let \( F \) denote one of the division rings \( \mathbb{R} \) of reals, \( \mathbb{C} \) of complex numbers, or \( \mathbb{H} \) of quaternions. Let \( t = \dim \mathbb{R} F \). Then the Grassmannian manifold \( G_k(F^{n+k}) \) is defined to be the set of all \( k \)-dimensional (left) subspaces of \( F^{n+k} \). \( G_k(F^{n+k}) \) is a closed manifold of real dimension \( nkt \).

In [8], Sankaran has proved that, for \( F = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), the Grassmannian manifold \( G_k(F^{n+k}) \) bounds if and only if \( v(n+k) > v(k) \), where, given a positive integer \( m \), \( v(m) \) denotes the largest integer such that \( 2^{v(m)} \) divides \( m \).

Given a positive integer \( d \), let \( \mathcal{G}(d) \) denote the set of bordism classes of all non-bounding Grassmannian manifolds \( G_k(F^{n+k}) \) having real dimension \( d \) such that \( k < n \). The restriction \( k < n \) is imposed because \( G_k(F^{n+k}) \approx G_n(F^{n+k}) \) and, for \( k = n \), \( G_k(F^{n+k}) \) bounds. Thus, \( \mathcal{G}(d) = \{ [G_k(F^{n+k})] \in \mathcal{N}_d \mid nkt = d, k < n, \text{ and } v(n+k) \leq v(k) \} \subset \mathcal{N}_d \).

The purpose of this paper is to prove the following:

**Theorem 1.1.** \( \mathcal{G}(d) \) is a linearly independent set in the \( \mathbb{Z}_2 \)-vector space \( \mathcal{N}_d \).

Similar results for Dold and Milnor manifolds can be found in [6] and [1] respectively.

2. The real Grassmannians — a Brief review

The real Grassmannian manifold \( G_k(\mathbb{R}^{n+k}) \) is an \( nk \)-dimensional closed manifold of \( k \)-planes in \( \mathbb{R}^{n+k} \). It is well-known (see [3]) that the mod-2 cohomology of \( G_k(\mathbb{R}^{n+k}) \) is given by

\[ H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n]/\{ w.\bar{w} = 1 \}, \]
where \( w = 1 + w_1 + w_2 + \cdots + w_k \) and \( \tilde{w} = 1 + \tilde{w}_1 + \tilde{w}_2 + \cdots + \tilde{w}_n \) are the total Stiefel–Whitney classes of the universal \( k \)-plane bundle \( \gamma_k \) and the corresponding complementary bundle \( \gamma_k^* \), both over \( G_k(\mathbb{R}^{n+k}) \), respectively.

For computational convenience in this cohomology we use the flag manifold \( \text{Flag}(\mathbb{R}^{n+k}) \) consisting of all ordered \((n+k)\)-tuples \( (V_1, V_2, \ldots, V_{n+k}) \) of mutually orthogonal one-dimensional subspaces of \( \mathbb{R}^{n+k} \) with respect to the ‘standard’ inner product on \( \mathbb{R}^{n+k} \). It is standard (see [4]) that the mod-2 cohomology of \( \text{Flag}(\mathbb{R}^{n+k}) \) is given by

\[
H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2) \cong \mathbb{Z}_2[e_1, e_2, \ldots, e_{n+k}] / \left\{ \prod_{i=1}^{n+k} (1 + e_i) = 1 \right\},
\]

where \( e_1, e_2, \ldots, e_{n+k} \) are one-dimensional classes. In fact each \( e_i \) is the first Stiefel–Whitney class of the line bundle \( \lambda_i \) over \( \text{Flag}(\mathbb{R}^{n+k}) \) whose total space consists of pairs, a flag \( (V_1, V_2, \ldots, V_{n+k}) \) and a vector in \( V_1 \).

There is a map \( \pi_{n+k} : \text{Flag}(\mathbb{R}^{n+k}) \rightarrow G_k(\mathbb{R}^{n+k}) \) which assigns to \((V_1, V_2, \ldots, V_{n+k})\), the \( k \)-dimensional subspace \( V_1 \oplus V_2 \oplus \cdots \oplus V_k \). In the cohomology, \( \pi_{n+k}^* : H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \rightarrow H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2) \) is injective and is described by

\[
\pi_{n+k}^*(w) = \prod_{i=1}^k (1 + e_i), \quad \pi_{n+k}^*(\tilde{w}) = \prod_{i=k+1}^{n+k} (1 + e_i).
\]

In [9], Stong has observed, among others, the following facts:

**Fact 2.1.** The value of the class \( u \in H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \) on the fundamental class of \( G_k(\mathbb{R}^{n+k}) \) is the same as the value of

\[
\pi_{n+k}^*(u) e_1^{k-1} e_2^{k-2} \cdots e_{r-1}^{n-1} e_r^{n-2} \cdots e_{n+k-1}
\]

on the fundamental class of \( \text{Flag}(\mathbb{R}^{n+k}) \).

**Fact 2.2.** In \( H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2) \) one has

\[
e_i^{n+k-l} \cdots e_{r-1}^{n-1} e_r^{n-2} \cdots e_{n+k-1} = 0
\]

if \( 1 \leq r \leq n+k \) and the set \( \{i_1, i_2, \ldots, i_r\} \subset \{1, 2, \ldots, n+k\} \). In particular \( e_i^{n+k} = 0 \) for each \( i, 1 \leq i \leq n+k \).

**Fact 2.3.** In the top dimensional cohomology of \( \text{Flag}(\mathbb{R}^{n+k}) \), a monomial \( e_1^{i_1} e_2^{i_2} \cdots e_{n+k}^{i_{n+k}} \) represents the non-zero class if and only if the set \( \{i_1, i_2, \ldots, i_{n+k}\} = \{0, 1, \ldots, n+k-1\} \).

The tangent bundle \( \tau \) over \( G_k(\mathbb{R}^{n+k}) \) is given (see [5]) by

\[
\tau \oplus \gamma_k \oplus \gamma_k \cong (n+k)\gamma_k.
\]

In particular, the total Stiefel–Whitney class \( W(G_k(\mathbb{R}^{n+k})) \) of the tangent bundle over \( G_k(\mathbb{R}^{n+k}) \) maps under \( \pi_{n+k}^* \) to

\[
\prod_{1 \leq i \leq k} (1 + e_i)^{n+k} \cdot \prod_{1 \leq i < j \leq k} (1 + e_i + e_j)^{-2}.
\]
Choosing a positive integer \( \alpha \) such that \( 2^\alpha \geq n+k \), we have, using Fact 2.2

\[
\pi^*_{n+k}(W(G_k(\mathbb{R}^{n+k}))) = \prod_{1 \leq i < j \leq k} (1 + e_i)^{n+k} \prod_{1 \leq i \leq k} (1 + e_i + e_j)^{2^\alpha - 2}.
\]

Thus, the \( m \)th Stiefel–Whitney class \( W_m = W_m(G_k(\mathbb{R}^{n+k})) \) maps under \( \pi^*_{n+k} \) to the \( m \)th elementary symmetric polynomial in \( e_i, 1 \leq i \leq k \), each with multiplicity \( n+k \), and \( e_i + e_j, 1 \leq i < j \leq k \), each with multiplicity \( 2^\alpha - 2 \). Therefore, if \( S_p(\sigma_1, \sigma_2, \ldots, \sigma_p) \) denotes the expression of the power sum \( \sum_{m=1}^{q} y_m^p \) as a polynomial in elementary symmetric polynomials \( \sigma_m \)'s in \( q \) 'unknowns' \( y_1, y_2, \ldots, y_q, q \geq p \), we have (see [3])

\[
S_p(\pi^*_{n+k}(W_1), \pi^*_{n+k}(W_2), \ldots, \pi^*_{n+k}(W_p)) = \sum_{1 \leq i \leq k} (n+k)e_i^p.
\]

Thus we have a polynomial

\[
S_p(G_k(\mathbb{R}^{n+k})) = S_p(W_1, W_2, \ldots, W_p) \in H^p(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)
\]

of Stiefel–Whitney classes of \( G_k(\mathbb{R}^{n+k}) \) such that

\[
\pi^*_{n+k}(S_p(G_k(\mathbb{R}^{n+k}))) = \begin{cases} 
\sum_{1 \leq i \leq k} e_i^p, & \text{if } n+k \text{ is odd and } p < n+k \\
0, & \text{otherwise.}
\end{cases} \tag{2.4}
\]

### 3. Proof of Theorem 1.1

It is shown in [2] that

\[
[G_{2k}(\mathbb{R}^{2n+2k})] = [G_k(\mathbb{R}^{n+k})]^4 \quad \text{in } \mathfrak{M}_{4nk}.
\]

From this, we have, in particular,

\[
[G_k(F^{n+k})] = [G_k(\mathbb{R}^{n+k})]^{2^\beta} \quad \text{in } \mathfrak{M}_{nk2^\beta}, \quad \beta \geq 0.
\]

For this one has to simply observe that the mod-2 cohomology of the \( F \)-Grassmannian is isomorphic as ring to that of the corresponding real Grassmannian by an obvious isomorphism that multiplies the degree by \( t \). On the other hand, since \( \mathfrak{M}_* \) is a polynomial ring over the field \( \mathbb{Z}_2 \), we have the following:

**Remark 3.1.** A set \( \{[M_1], [M_2], \ldots, [M_m]\} \) is linearly independent in \( \mathfrak{M}_d \) if and only if the set \( \{[M_1]^{2^\beta}, [M_2]^{2^\beta}, \ldots, [M_m]^{2^\beta}\} \) is linearly independent in \( \mathfrak{M}_{d,2^\beta}, \beta \geq 0 \).

Therefore, noting that \( t = 1, 2, \) or \( 4 \), it is enough to prove Theorem 1.1 for real Grassmannians only. Thus, from now onwards, we shall take

\[
\mathcal{G}(d) = \{ [G_k(\mathbb{R}^{n+k})] \mid nk = d, k < n, \text{ and } \nu(n+k) \leq \nu(k) \}.
\]

If \( G_k(\mathbb{R}^{n+k}) \) is an odd-dimensional real Grassmannian manifold then both \( n \) and \( k \) must be odd, and so \( \nu(n+k) > \nu(k) \). This means that \( G_k(\mathbb{R}^{n+k}) \) bounds and so it follows that \( \mathcal{G}(d) = \emptyset \) if \( d \) is odd. Therefore we assume that \( d \) is even.
Lemma 3.2. In $H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2)$ one has, for $1 \leq j \leq n$,
\[
\left( \sum_{1 \leq i \leq k} e_i^{n+k-(2j-1)} \right) e_1^{k-1} e_2^{k-2} \cdots e_j^{j-1} e_{k-(j-1)}^{n+k-(j-1)} \cdots e_k^{n+k-1} = e_1^{k-1} e_2^{k-2} \cdots e_{k-j}^{j-1} e_{k-(j-1)}^{n+k-(j-1)} \cdots e_k^{n+k-1}.
\]

Proof. Note that
(a) if $i \neq k - (j - 1)$ then the exponent of $e_i$ in the product
\[
e_1^{k-1} e_2^{k-2} \cdots e_{k-j}^{j-1} e_{k-(j-1)}^{n+k-(j-1)} \cdots e_k^{n+k-1}
\]
is greater than or equal to $j$, and
(b) $\{n+k-(2j-1)\} + j = n+k-(j-1)$.

Therefore, invoking Fact 2.2, the lemma follows.

PROPOSITION 3.3.
Let $\mathcal{O}(d) = \{ [G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) \mid n+k \text{ is odd} \}$. Then $\mathcal{O}(d)$ is linearly independent in $\mathcal{R}_d$.

Proof. Arrange the members of $\mathcal{O}(d)$ in descending order of the values of $n+k$, so that
\[
\mathcal{O}(d) = \{ [G_{k_1}(\mathbb{R}^{n_1+k_1})], [G_{k_2}(\mathbb{R}^{n_2+k_2})], \ldots, [G_{k_s}(\mathbb{R}^{n_s+k_s})] \},
\]
where $n_1 + k_1 > n_2 + k_2 > \cdots > n_s + k_s$. Note that $n_1 = d$ and $k_1 = 1$.

For a $d$-dimensional Grassmannian manifold $G_k(\mathbb{R}^{n+k})$, consider the polynomials
\[
f_\ell(G_k(\mathbb{R}^{n+k})) = \prod_{1 \leq j \leq k} S_{n_j+k_j-(2j-1)}(G_k(\mathbb{R}^{n+k})) \in H^d(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)
\]
of Stiefel–Whitney classes of $G_k(\mathbb{R}^{n+k})$, where $1 \leq \ell \leq s$. Then, for each $\ell$, $1 \leq \ell \leq s$, we have, using 2.4,
\[
\pi_{n_1+k_1}(f_\ell(G_k(\mathbb{R}^{n+k}))) e_1^{k_1-1} e_2^{k_2-2} \cdots e_{k_{\ell-1}}^{n_{\ell-1}} e_{k_{\ell}}^{n_{\ell}} \cdots e_{n_\ell+k_\ell-1} = \left( \prod_{1 \leq j \leq k} \left( \sum_{1 \leq \ell \leq k} e_\ell^{n_\ell+k_\ell-(2j-1)} \right) \right) e_1^{k_1-1} e_2^{k_2-2} \cdots e_{k_{\ell-1}}^{n_{\ell-1}} e_{k_{\ell}}^{n_{\ell}} \cdots e_{n_\ell+k_\ell-1},
\]
applying Lemma 3.2 repeatedly for successive values of $j$.

Thus, in view of Facts 2.1 and 2.3 the Stiefel–Whitney number
\[
\langle f_\ell(G_k_1(\mathbb{R}^{n+k})), [G_{k_\ell}(\mathbb{R}^{n+k})]) \rangle \neq 0
\]
for each $\ell$, $1 \leq \ell \leq s$. On the other hand, using 2.4, it is clear that
\[
\langle f_\ell(G_k_1(\mathbb{R}^{n+k})), [G_{k_\ell}(\mathbb{R}^{n+k})]) \rangle = 0
\]
for each $h > \ell$, since $n_\ell + k_\ell - 1 \geq n_h + k_h$. Therefore, it follows that the $s \times s$ matrix
\[
\begin{bmatrix}
[f_\ell(G_k(\mathbb{R}^{n_k+k})), G_k(\mathbb{R}^{n_k+k})] \mid 1 \leq \ell, 1 \leq k \leq s
\end{bmatrix}
\]
is non-singular; being lower triangular with 1’s in the diagonal. This completes the proof.

Now we shall complete the proof of Theorem 1.1 using induction on $d$. First note that
\[
\mathcal{G}(2) = \{ [G_1(\mathbb{R}^{2+1})] \} = \{ [\mathbb{R}^2] \},
\]
\[
\mathcal{G}(4) = \{ [G_1(\mathbb{R}^{4+1})] \} = \{ [\mathbb{R}^4] \},
\]
and so both are linearly independent in $\mathcal{N}_2$, $\mathcal{N}_4$ respectively. Assume that the theorem holds for all dimensions less than $d$.

We have $\mathcal{G}(d) = \mathcal{E}(d) \cup \mathcal{O}(d)$, where
\[
\mathcal{E}(d) = \{ [G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) \mid n + k \text{ is even} \}
\]
and
\[
\mathcal{O}(d) = \{ [G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) \mid n + k \text{ is odd} \}.
\]
Observe that if $[G_k(\mathbb{R}^{n+k})] \in \mathcal{E}(d)$ then both $n$ and $k$ are even with $v(k) \neq v(n)$. On the other hand, $[G_2(\mathbb{R}^{\frac{d}{2}+2})] \in \mathcal{E}(d)$ if $d \equiv 0 \pmod{8}$. Thus, $\mathcal{E}(d) \neq \emptyset$ if and only if $d \equiv 0 \pmod{8}$.

In view of Proposition 3.3, we may assume without any loss that $\mathcal{E}(d) \neq \emptyset$. Then, by the above observation and by Theorem 2.2 of [8] every member of $\mathcal{E}(d)$ is of the form $[G_2(\mathbb{R}^{\frac{d}{2}+2})]$, where $[G_2(\mathbb{R}^{\frac{d}{2}+2})] \in \mathcal{G}(\frac{d}{2})$. By induction hypothesis, $\mathcal{G}(\frac{d}{2})$ is linearly independent in $\mathcal{N}_{\frac{d}{2}}$.

So, by Remark 3.1
\[
\mathcal{E}(d) \text{ is linearly independent in } \mathcal{N}_d. \tag{3.4}
\]

Again note that if $[G_k(\mathbb{R}^{n+k})] \in \mathcal{E}(d)$, then, by 2.24, the polynomial $S_p(G_k(\mathbb{R}^{n+k})) = 0$, $\forall p \geq 1$. So, for each of the polynomials $f_\ell$, $1 \leq \ell \leq s$, considered in Proposition 3.3 we have
\[
\langle f_\ell(G_k(\mathbb{R}^{n+k})) \rangle, [G_k(\mathbb{R}^{n+k})] \rangle = 0.
\]
Therefore, writing
\[
\mathcal{E}(d) = \{ [G_{k+1}(\mathbb{R}^{n_{k+1}+k_{k+1}})], [G_{k+2}(\mathbb{R}^{n_{k+2}+k_{k+2}})], \ldots, [G_{k+s}(\mathbb{R}^{n_{k+s}+k_{k+s}})] \},
\]
where $n_{s+1} + k_{s+1} > n_{s+2} + k_{s+2} > \cdots > n_{s+q} + k_{s+q}$, we see that the $s \times (s + q)$ matrix
\[
\begin{bmatrix}
[f_\ell(G_k(\mathbb{R}^{n_k+k})), G_k(\mathbb{R}^{n_k+k})] \mid 1 \leq \ell, 1 \leq k \leq s + q
\end{bmatrix}
\]
is of the form
\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
* & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
* & * & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}, \tag{3.5}
\]
Thus, no non-trivial linear combination of members of $\mathcal{O}(d)$ can be expressed as a linear combination of the members of $\mathcal{E}(d)$. This, together with (3.4) and Proposition 3.3, proves that the set $\mathcal{O}(d) = \mathcal{E}(d) \cup \mathcal{O}(d)$ is linearly independent in $\mathcal{R}_d$. Hence, by induction, Theorem 1.1 is completely proved.

**Remark 3.6.** Using the decomposition of the members of $\mathcal{E}(d)$, and the polynomials $f_l$, in the lower dimensions together with the *doubling homomorphism* defined by Milnor [7], one can obtain a set of polynomials of Stiefel–Whitney classes which yield, as in Proposition 3.3, a lower triangular matrix for $\mathcal{E}(d)$ with 1’s in the diagonal. Thus using (3.5) we have a lower triangular matrix, with 1’s in the diagonal, for the whole set $\mathcal{O}(d)$.

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**References**

[1] Dutta S and Khare S S, Independence of bordism classes of Milnor manifolds, *J. Indian Math. Soc.* 68(1–4) (2001) 1–16

[2] Floyd E E, Steifel–Whitney numbers of quaternionic and related manifolds, *Trans. Am. Math. Soc.* 155 (1971) 77–94

[3] Hiller H L, On the cohomology of real Grassmannians, *Trans. Am. Math. Soc.* 257 (1980) 521–533

[4] Hirzebruch F, Topological Methods in Algebraic Geometry (New York: Springer-Verlag) (1966)

[5] Hsiang W-C and Szczarba R H, On the tangent bundle for the Grassmann manifold, *Am. J. Math.* 86 (1964) 698–704

[6] Khare S S, On Dold manifolds, *Topology Appl.* 33 (1989) 297–307

[7] Milnor J W, On the Steifel–Whitney numbers of complex manifolds and of spin manifolds, *Topology* 3 (1965) 223–230

[8] Sankaran P, Determination of Grassmann manifolds which are boundaries, *Canad. Math. Bull.* 34 (1991) 119–122

[9] Stong R E, Cup products in Grassmannians, *Topology Appl.* 13 (1982) 103–113