$C^{2,\alpha}$ estimates for nonlinear elliptic equations of twisted type

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Abstract We prove a priori interior $C^{2,\alpha}$ estimates for solutions of fully nonlinear elliptic equations of twisted type. For example, our estimates apply to equations of the type convex + concave. These results are particularly well suited to equations arising from elliptic regularization. As application, we obtain a new proof of an estimate of Streets and Warren on the twisted real Monge–Ampère equation.

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1 Introduction

In this paper we will derive an a priori interior $C^{2,\alpha}$ estimate for solutions of the nonlinear, uniformly elliptic equation

$$F(D^2 u) = 0$$ (1.1)

under the assumption that $F$ is of twisted type (see Definition 1.3, below). Higher order estimates for fully nonlinear elliptic equations are a subtle matter, and depend heavily on the structure of $F$. The fundamental result in this vein is the famous estimate of Evans [6] and Krylov [10, 11], which says that if $F$ is convex, or concave, then the $C^\alpha$ norm of $D^2 u$ on the interior is controlled by the $L^\infty$ norm of $D^2 u$. Nadirashvili and Vlăduţ [13] have produced counterexamples to Evans–Krylov type estimates for general fully nonlinear equations. As a result, it is an interesting and important problem to understand when Evans–Krylov type estimates hold for nonlinear equations which are neither convex nor concave.

Let us briefly recount the positive results currently available in the literature. Caffarelli and Cabré [3] proved an Evans–Krylov type estimate for functions $F$ which are the minimum
of convex and concave functions—in particular, their result applies to equations of Isaac’s type. Caffarelli and Yuan [4] proved $C^2,\alpha$ a priori estimates under the assumption that level set $\{F = 0\}$ has uniformly convex intersection with a family of planes. In essence, this allows one of the principle curvatures of $\{F = 0\}$ to be negative. Yuan [23] proved higher regularity for the 3 dimensional special Lagrangian equation. More recently, motivated by the pluriclosed flow introduced by Streets and Tian [16–19], Streets and Warren [20] exploited the partial Legendre transform to prove an Evans–Krylov type theorem for the real twisted Monge–Ampère equation

$$\log \det u_{xx} - \log \det(-u_{yy}) = 0,$$

(1.2)

where $u(x, y)$ is assumed to be uniformly convex in the $x$ variables, and uniformly concave in the $y$ variables. The authors also consider parabolic and complex analogs of this equation.

The concern of this note is to extend the Evans–Krylov estimate to a general class of nonconvex equations of twisted type. Let us give the precise setting for our result. Let $S^{n \times n} \subset M^{n \times n}$ be the set of symmetric $n \times n$ matrices with real entries.

**Definition 1.1** We say that $F\cap$ is weakly concave on $\Sigma_1$ if there exists an open subset $U \subset \mathbb{R}$ and a continuous function $G : U \to \mathbb{R}$ which is smooth on $U$, such that

(i) $F\cap(U) \subset \text{int}U$,

(ii) $G' > 0$, $G'' \leq 0$ and $G(F\cap(\cdot))$ is concave on $\Sigma_1$,

(iii) There exists a continuous function $Q(x) : \mathbb{R} \to \mathbb{R}_{>0}$ such that $G'(x) \geq Q(x)$ for $x \in U$.

We have endeavored to make Definition 1.1 as broad as possible, which unfortunately necessitates its somewhat technical appearance. In many cases, the conditions of Definition 1.1 are easily seen to be satisfied, as we will discuss in the examples below.

**Theorem 1.2** Suppose that $u$ is a smooth solution of

$$F(D^2u) = 0$$

(1.3)

on $B_1 \subset \mathbb{R}^n$, where $F := F\cup + F\cap$. Let $\mathcal{U} := D^2u(\overline{B_1})$, and let $\mathcal{V} \supset \mathcal{U}$ be an open convex set. Suppose that $F\cup$ is uniformly elliptic, convex and $C^2$ on $\mathcal{V}$, and $F\cap$ is $C^2$ on $\mathcal{V}$. Assume furthermore that $F\cap$ is degenerate elliptic and weakly concave on $\mathcal{U}$. Then, for every $0 < \alpha < 1$ we have an estimate

$$\|D^2u\|_{C^\alpha(B_{1/2})} \leq C (n, \Lambda, \alpha, \gamma, \Gamma, F\cup(0), \|D^2u\|_{L^\infty(B_1)}),$$

where

$$0 < \gamma := \inf_{x \in F\cup(D^2u)(B_1)} Q'(\cdot)$$

and

$$\Gamma := \text{Osc}_{B_1} G(-F\cup(D^2u))$$

depend only on $\Lambda$, $\|D^2u\|_{L^\infty(B_1)}$ and the functions $G$, $Q$ of Definition 1.1.

For convenience, we introduce the following terminology;

**Definition 1.3** Operators satisfying the structural assumptions of Theorem 1.2 will be said to be of twisted type.

Twisted type equations arise naturally in problems in differential geometry; let us describe some examples.
Example 1.4 Suppose $F_{\cup}$ is a smooth convex, uniformly elliptic operator, and $F_{\cap}$ is a smooth concave degenerate elliptic operator. Then the operator $F = F_{\cup} + F_{\cap}$ is of twisted type, with $G(x) = x$, $Q(x) = 1$. In particular, $U = \mathbb{R}$ in Definition 1.1, and hence the condition (i) in Definition 1.1 is vacuous.

Let us consider a slightly less trivial, and more explicit family of examples.

Example 1.5 For a symmetric matrix $M$, we let $\sigma_k(M)$ denote the $k$-th symmetric polynomial of the eigenvalues of $M$. It is well-known that $\sigma_k(D^2u)$ is a degenerate elliptic operator on the closure of the cone $\Gamma_k$ of $k$-convex functions. Furthermore, if we take $G(x) = x^{1/k}$ for $x \geq 0$, then $\sigma_k(D^2u)^{1/k}$ is concave, and $G$ satisfies properties (ii) and (iii) of Definition 1.1 where for item (iii) we can take

$$Q(x) = \min \left\{ 1, k|x|^{1/k} - 1 \right\}.$$ 

In particular, the equations

$$\Delta u + \sigma_k(D^2u) = 1$$

(1.4)

are uniformly elliptic for $D^2u \in \overline{\Gamma_k}$, and are all of twisted type. We obtain a $C^2$ estimate directly from the equation in this case, since

$$0 \leq \sigma_2 = \frac{(|\Delta u|^2 - |D^2u|^2)}{2},$$

and so $|D^2u|^2 \leq (\Delta u)^2 \leq 1$ since $\sigma_k(D^2u) \geq 0$. Theorem 1.2 implies $C^{2,\alpha}$ estimates provided $\sigma_k(D^2u(x)) > 0$ for all $x \in B_1$, which is equivalent to condition (i) in Definition 1.1.

In general, these operators do not have convex sublevel sets, and hence the Evans–Krylov theorem does not apply. Moreover, the result of Caffarelli and Yuan [4] does not apply as the convexity of the level sets $\{\sigma_k = t\}$ degenerate as $t \to 0$. In the special case when $n = k = 2$ Eq. (1.4) is equivalent to the Special Lagrangian equation with phase $\pi/4$, and the level sets are convex [24, Lemma 2.1]. On the other hand, in dimension 2, no convexity is required to obtain the Hölder continuity of the second derivatives, as shown by Nirenberg [14] (see also [4]).

Finally, let us make a few remarks about some of the assumptions in Theorem 1.2 when $k = n$ in (1.4). Condition (i) of Definition 1.1 can be dropped as the constant rank theorem of Bian and Guan [1] implies that either $\sigma_n > 0$ everywhere, or $\sigma_n \equiv 0$, and so $C^{2,\alpha}$ estimates hold in either case. It would be interesting to know if this behaviour holds more generally as (formally) suggested by combining the strong maximum principle with Lemma 2.1 below. More to this point, assuming that $\sigma_n > 0$, the function $G(x) := \log(x)$ defined on $(0, 2)$ satisfies conditions (i)–(iii) of Definition 1.1, but does not satisfy the assumption that $G$ extends continuously to $[0, 2]$. One could still run the proof of Theorem 1.2 using $G = \log(x)$, with the effect that the $C^{2,\alpha}$ estimate would depend on a lower bound for $\sigma_n$ on $B_1$. This illustrates the necessity of the assumptions on $G$ in Definition 1.1.

Equations of the form considered in Example 1.5 arise naturally, both in geometry and analysis. For example, the $k = n$ case of (1.4), arose in the work of the author and G. Székelyhidi on the $J$-flow on toric Kähler manifolds [5]. In fact, in geometric applications it is often natural to require that solutions of (1.4) are convex, which is a more restrictive condition than $k$-convex. Previously, Fu and Yau [7] solved a twisted Monge–Ampère type equation similar to (but more complicated than) (1.4) in their study of the Strominger system. Fu–Yau succeeded in bypassing $C^{2,\alpha}$ estimates by estimating directly the $C^3$ norm by a difficult maximum principle argument.
One might argue that instead of Eq. (1.4), one should instead study the concave equation
\[
\Delta u + \sigma_k(D^2u)^{1/k} = 1. \tag{1.5}
\]
Let us provide a brief rebuttal to this point. First, note that (1.5) is not uniformly elliptic without a strict lower bound \(\sigma_k > \lambda > 0\), while Eq. (1.4) is uniformly elliptic without any further assumptions. For this reason the family of Eq. (1.4) arises as a natural “elliptic regularization” of the \(k\)-Hessian equation. Furthermore, on a Kähler manifold \(X\) of dimension \(n\) it is natural to consider equations of the form
\[
\omega^{n-1} \wedge (\alpha + i\partial\overline{\partial} \varphi) + \omega^{n-k} \wedge (\alpha + i\partial\overline{\partial} \varphi)^k = c\omega^n
\]
where \(\omega\) is a fixed Kähler form, \(\alpha\) is a real, smooth, closed \((1, 1)\)-form, and \(\varphi\) is unknown. In this case, the constant \(c\) is determined cohomologically by the classes \([\alpha], [\varphi] \in H^{1,1}(X, \mathbb{R})\).

In the toric case, this equation reduces to Eq. (1.4). On the other hand, for the concave equation whose local expression is (1.5), the right hand side is not determined cohomologically and one needs additional information, often unavailable in geometric problems, to specify the equation completely.

**Example 1.6** Let \(\{x_1, \ldots, x_k\}\) be standard coordinates on \(\mathbb{R}^k\), and let \(\{y_1, \ldots, y_{\ell}\}\) be standard coordinates on \(\mathbb{R}^{\ell}\), so that we may identify \(\mathbb{R}^{k+\ell} = \mathbb{R}^k \times \mathbb{R}^{\ell}\). Set \(n = k + \ell\). Our identification induces an decomposition of any matrix \(M \in S^{n \times n}\) as
\[
M = M_{kk} \otimes M_{\ell k} \oplus M_{kk}^T \otimes M_{\ell \ell}
\]
where \(M_{kk}\) is the symmetric \(k \times k\) matrix corresponding to the \(\mathbb{R}^k\) factor in the obvious way, and similarly for \(M_{\ell \ell}\). Define maps by
\[
\pi_k(M) = M_{kk}, \quad \pi_\ell(M) = M_{\ell \ell}.
\]
For \(\kappa, \lambda > 0\), we set
\[
\mathcal{E}_{\lambda, \kappa} := \left\{ M \in S^{n \times n} : \begin{array}{l}
(i) \lambda I_k \leq \pi_k(M) \leq \lambda^{-1} I_k, \\
(ii) \lambda I_\ell \leq -\pi_\ell(M) \leq \lambda^{-1} I_\ell, \\
(iii) \|M\| \leq \kappa
\end{array} \right\}
\]
where \(I_j\) denotes the \(j \times j\) identity matrix for \(j = k, \ell\). Note the \(\mathcal{E}_{\lambda, \kappa}\) is a compact, convex subset of \(S^{n \times n}\).

Consider the real twisted Monge–Ampère equation of Streets and Warren [20], given in Eq. (1.2). Let \(u\) be a smooth solution of (1.2) on \(B_1\) and suppose that \(\overline{D^2u(B_1)} \subset \mathcal{E}_{\lambda, \kappa}\).

Unfortunately, (1.2) does not admit a natural decomposition into a twisted type operator. Instead, we write it in the form
\[
-\left(\det(-D^2_y u))^\frac{1}{n} + \varepsilon \text{Tr} (D^2_y u) + (\det(D^2_x u))^\frac{1}{n} - \varepsilon \text{Tr} (D^2_x u) = 0.
\]
We note that under the above assumptions
\[
F_{\cup}(M) := -\left(\det(-\pi_\ell M))^\frac{1}{n} + \varepsilon \text{Tr} (\pi_k M)
\]
is smooth and uniformly elliptic on \(\mathcal{E}_{\lambda, \kappa}\) for \(\varepsilon > 0\) and convex. Furthermore
\[
F_{\cap}(M) := (\det(\pi_k M))^\frac{1}{n} - \varepsilon \text{Tr} (\pi_k M)
\]
is degenerate elliptic for \(\varepsilon > 0\) depending only on \(k, \ell, \lambda\), and is concave. Furthermore, the set \(\mathcal{E}_{\lambda, \kappa}\) is a convex open set in \(S^{n \times n}\). In particular, the real twisted Monge–Ampère equation (1.2) is of twisted type in the sense of Definition 1.3, and so Theorem 1.2 gives a new proof of an estimate of Streets and Warren [20, Theorem 1.1].
The proof of Theorem 1.2 hinges on the observation that $G(F\cap(D^2u))$ is a supersolution of the linear equation $L\varphi = 0$, where $L$ denotes the linearization of (1.1); see Lemma 2.1 below. With this observation we can employ the machinery used in the proof of the Evans–Krylov theorem with some modifications to account for the fact that we must work with general elliptic operators rather than the Laplacian. We take this up in detail in Sect. 2. In Sect. 3 we provide a brief description of some applications and extensions.

2 Proof of Theorem 1.2

For ease of notation, whenever $F_\cup$, $F_\cap$ are $C^2$, we will set

$$F^{ij}_\alpha := \frac{\partial F_\alpha}{\partial a_{ij}}, \quad F^{ij,rs}_\alpha := \frac{\partial^2 F_\alpha}{\partial a_{ij}\partial a_{rs}} \quad \text{(2.1)}$$

for $\alpha = \cup, \cap$ and $1 \leq i, j, r, s \leq n$. We begin with some simple calculations. First, if $u$ solves Eq. (1.3), then since $F_\cup$, $F_\cap$ are $C^2$ on $\overline{\mathcal{U}} = D^2u(B_1)$ we may differentiate freely. We have

$$Lu_a := \left(F^{ij}_\cup + F^{ij}_\cap\right)u_{aij} = 0 \quad \text{(2.2)}$$

$$\left(F^{ij}_\cup + F^{ij}_\cap\right)u_{abij} + \left(F^{ij,rs}_\cup + F^{ij,rs}_\cap\right)u_{aij}u_{brs} = 0. \quad \text{(2.3)}$$

The main observation is contained in the following lemma.

**Lemma 2.1** Under the assumptions of Theorem 1.2, $G(F\cap(D^2u))$ is a supersolution of the linearized equation on $B_1$; that is,

$$L[G(F\cap(D^2u))] \leq 0.$$

**Proof** Since $F_\cup$, $F_\cap$ are $C^2$ in an open neighborhood of $D^2u(B_1)$ and by Definition 1.1 part (i), $F\cap(D^2u(B_1)) \subset \text{int}U$, the function $G(F\cap(D^2u))$ is $C^2$ in $B_1$. Hence, we can compute

$$\partial_a G(F\cap(D^2u)) = G'F^{ij}_\cap u_{aij}$$

$$\partial_b\partial_a G(F\cap(D^2u)) = G'F^{ij}_\cap u_{abij} + \left(G''F^{ij,rs}_\cap + G''F^{ij}_\cap F^{rs}_\cap\right)u_{aij}u_{brs}.$$

As a result, we obtain

$$LG(F\cap(D^2u)) = G'F^{ab}_\cap L(u_{ab}) + \left(G'F^{ij,rs}_\cap + G''F^{ij}_\cap F^{rs}_\cap\right)\left(F^{ab}_\cup + F^{ab}_\cap\right)u_{aij}u_{brs}$$

$$= \left[-G'F^{ab}_\cap \left(F^{ij,rs}_\cap + F^{ij,rs}_\cap\right)\right]u_{aij}u_{brs}$$

$$+ \left[G'F^{ij,rs}_\cap + G''F^{ij}_\cap F^{rs}_\cap\right]\left(F^{ab}_\cup + F^{ab}_\cap\right)u_{aij}u_{brs}$$

$$= F^{ab}_\cup \left(G'F^{ij,rs}_\cap + G''F^{ij}_\cap F^{rs}_\cap\right)u_{aij}u_{brs} + G''F^{ab}_\cap F^{ij,rs}_\cap u_{aij}u_{brs}$$

$$- G'F^{ij,rs}_\cap F^{ab}_\cap u_{aij}u_{brs}$$

Now, since $G(F\cap)$ is concave, and $F_\cup$ is elliptic, the first term is non-positive. Moreover, since $G'' \leq 0$, and $F_\cap$ is (degenerate) elliptic, the second term is also non-positive. Finally, since $G' > 0$, and $F_\cup$ is convex, the third term is non-positive as well. As a result, we have

$$LG(F\cap(D^2u)) \leq 0.$$
The next lemma is a general trick for extending convex elliptic operators, which allows us to assume that $F_\cup$ is uniformly elliptic on all of $S^{n\times n}$.

**Lemma 2.2** Suppose $F_\cup$ is a $C^2$ uniformly elliptic operator with ellipticity constants $0 < \lambda < \Lambda < +\infty$. Suppose also that $F_\cup$ is convex on a compact convex set $V \subset S^{n\times n}$. There exists an extension $\hat{F}_\cup : S^{n\times n} \to \mathbb{R}$ which is continuous, uniformly elliptic with ellipticity constants $\lambda$, $\Lambda$, such that $\hat{F}_\cup|_V = F_\cup$.

**Proof** We use an envelope trick exploited by Wang [22], and Tosatti et al. [21] to extend $F_\cup$ to a uniformly elliptic convex operator outside of $V$. In order to simplify notation, let us suppress the subscript $\cup$ for the proof of the lemma. For $N \in S^{n\times n}$, and $M \in V$, we set

$$L_M N := DF(M) \cdot (N - M) + F(M).$$

That is, $L_M(N)$ is the value of the linear approximation to $F$ about the point $M \in V$. We then define $\tilde{F}_\cup$ to be the convex envelope of $F_\cup$; set

$$\tilde{F}(N) := \sup_{M \in V} L_M(N).$$

Since $F$ is convex over the convex, compact set $V$, we clearly have that $\tilde{F}$ is convex, and agrees with $F$ on $V$. We include the short proof of uniform ellipticity for the readers’ convenience. Fix $N, P \in S^{n\times n}$, with $P \geq 0$. We have

$$\tilde{F}(N + P) = DF(M_1) \cdot (N + P - M_1) + F(M_1)$$

$$\tilde{F}(N) = DF(M_2) \cdot (N - M_2) + F(M_2).$$

for some $M_1, M_2 \in V$. By maximality, we have

$$\tilde{F}(N + P) \geq DF(M_2) \cdot (N + P - M_2) + F(M_2)$$

$$\tilde{F}(N) \geq DF(M_1) \cdot (N - M_1) + F(M_1).$$

Combining these two equations yields

$$\tilde{F}(N + P) - \tilde{F}(N) \geq DF(M_2) \cdot (N + P - M_2) - DF(M_2) \cdot (N - M_2)$$

$$\geq DF(M_2) \cdot P$$

$$\geq \lambda \|P\|,$$

and similarly for the upper bound. \qed

By the preceding lemma, we will now assume that $F_\cup$ is defined on all of $S^{n\times n}$, keeping in mind that it is only continuous away from $V$. The following proposition is essential, and is based on the proof of the Evans–Krylov theorem (cf. [4, Proposition 1], [2]).

**Proposition 2.3** Let the assumptions of Theorem 1.2 be in force. Then, for any $\varepsilon > 0$, there exists a positive constant $\eta = \eta(n, \lambda, \Lambda, \varepsilon, \gamma, \Gamma, \|D^2u\|_{L^\infty(B_1)})$ and a quadratic polynomial $P$ so that, for all $x \in B_1$, we have

$$\left| \frac{1}{\eta^2}u(\eta x) - P(x) \right| \leq \varepsilon$$

$$F(D^2 P) = 0.$$
Proof Fix constants $\rho, \xi, \delta, k_0 > 0$ to be determined. We let $C$ denote a constant depending on the stated data which may change from line to line. Define

$$t_k := \sup_{B_{1/2k}} F_{\cup}(D^2 u), \quad 1 \leq k \leq k_0$$

$$s_k := \inf_{B_{1/2k}} G(F_{\cap}(D^2 u)), \quad 1 \leq k \leq k_0.$$  

Note that $s_k = G(-t_k)$ since $G$ is increasing.

Consider the set

$$E_k := \{x \in B_{1/2k} | F_{\cup}(D^2 u) \leq t_k - \xi\}.$$  

Suppose that

$$\exists 1 \leq \ell \leq k_0 \text{ so that } |E_\ell| \leq \delta |B_{1/2\ell}|.$$  

(2.4)

Let $w_\ell(x) = 2^{2\ell} u \left( \frac{x}{2^{2\ell}} \right)$, and let $\upsilon$ solve the equation

$$\begin{cases}
F_{\cup}(D^2 \upsilon(x)) = t_\ell & x \in B_1 \\
\upsilon(x) = w_\ell(x) & x \in \partial B_1.
\end{cases}$$

Note that $\upsilon$ exists by standard elliptic theory (see, e.g. [2, Chapter 9]). Since $F_{\cup}$ is uniformly elliptic, the Alexandroff–Bakelman–Pucci (see, for instance, [9, Theorem 2.21], [2, Theorem 3.2]) estimate implies

$$\|\upsilon - w_\ell\|_{L^\infty(B_1)} \leq C \|F_{\cup}(D^2 \upsilon) - F_{\cup}(D^2 w_\ell)\|_{L^n(B_1)} \leq C(n)(\xi^n + \delta)^{1/n}. \quad (2.5)$$

Set $\hat{\upsilon} = \upsilon - w_\ell(0) - \langle \nabla w_\ell(0), x \rangle$, and observe that $F_{\cup}(D^2 \hat{\upsilon}) = t_\ell$. Since $F_{\cup}$ is uniformly elliptic and convex we can apply the usual Evans–Krylov Theorem [6,10,11], [2, Theorem 6.6], [8], to find a constant $\beta = \beta(\lambda, \Lambda, n) \in (0, 1)$ depending only on universal data such that

$$|\hat{\upsilon}|_{C^{2,\beta}(B_{1/2})} \leq C \left( \|\upsilon\|_{L^\infty(B_1)} + |F_{\cup}(0) - t_\ell| \right). \quad (2.6)$$

By the uniform ellipticity of $F_{\cup}$, we have

$$|t_\ell - F_{\cup}(0)| \leq \Lambda \|D^2 u\|_{L^\infty(B_1)}. \quad (2.7)$$

Furthermore, applying the Alexandroff–Bakelman–Pucci estimate [2, Theorem 3.2] to the uniformly elliptic equation $F_{\cup}(D^2 \hat{\upsilon}) = t_\ell$ we obtain

$$\|\hat{\upsilon}\|_{L^\infty(B_1)} \leq C \left[ \sup_{x \in \partial B_1} |\hat{\upsilon}(x)| + |t_\ell| \right] \leq C \left[ \sup_{x \in \partial B_1} |w_\ell(x) - w_\ell(0) - \langle \nabla w_\ell(0), x \rangle| + |t_\ell| \right] \leq C \left( \|D^2 w_\ell\|_{L^\infty(B_1)} + |t_\ell| \right).$$

(2.8)

Combining estimates (2.6), (2.7) and (2.8)

$$\|D^2 \upsilon\|_{C^{\beta}(B_{1/2})} = \|D^2 \hat{\upsilon}\|_{C^{\beta}(B_{1/2})} \leq C \left( \|D^2 u\|_{L^\infty(B_1)} + |F_{\cup}(0)| \right).$$
Let $P$ denote the quadratic part of $\nu$ at the origin. Then, for $x \in B_{1/2}$ we have
\[
|w_\ell - P|(x) \leq |w_\ell - \nu|(x) + |\nu - P|(x) \\
\leq C(\xi^n + \delta)^{1/n} + \|D^2\nu\|_{C(\beta(B_{1/2}))}|x|^{2+\beta} \\
\leq C(\xi^n + \delta)^{1/n} + C|x|^{2+\beta}.
\]  
(2.9)

Before proceeding, let us address the case when the assumption in (2.4) does not hold. Namely, assume that
\[
\forall 1 \leq \ell \leq k_0, \quad |E_\ell| \geq \delta|B_{1/2\ell}|.
\]  
(2.10)

Define $w_k(x) = 2^{2k}u \left( \frac{x}{2^k} \right)$. By Lemma 2.1, we have
\[
L(G(F_\Gamma(D^2w_k)) - s_k) \leq 0.
\]

We now apply the weak Harnack inequality repeatedly to show that $G(F_\Gamma(D^2w_k))$ must concentrate in measure near a level set, for $k_0$ sufficiently large, depending only on $\xi, \delta, \Gamma$. This will imply that assumption (2.4) always holds.

The following weak Harnack inequality extends the lower bound to an integral up to the boundary. The inequality can be derived from [12] (see also [2, Theorem 4.8], [4]), and implies that, for $x \in B_{1/2}
\[
G(F_\Gamma(D^2w_k))(x) - s_k \geq c(n, \lambda)\|G(F_\Gamma(D^2w_k)) - s_k\|_{L^p_0(B_1)},
\]

where $p_0 = p_0(n, \lambda, \Lambda)$. Since $G$ is concave and increasing, the definition of weak convexity implies that
\[
G(F_\Gamma(D^2w_k)) \geq s_k + \gamma \xi \quad \forall x \in E_k.
\]

In particular, thanks to (2.10) we obtain
\[
s_{k+1} \geq s_k + c(n, \lambda)\gamma \xi^{1/p_0} =: s_k + \theta.
\]

It follows immediately that after
\[
k_0 = \frac{\text{Osc}_{B_1} G(F_\Gamma(D^2u))}{\theta} =: \frac{\Gamma}{\theta}
\]
it must hold that $|E_{k_0}| \leq \delta|B_{1/2k_0}|$, for otherwise
\[
s_{k_0+1} \geq \sup_{B_1} G(F_\Gamma(D^2u))
\]
which is absurd. In particular, assumption (2.4) always holds provided we take $k_0$ large enough, depending on $\xi, \delta, \gamma, \Gamma$.

In conclusion, by (2.9), there is an $1 \leq \ell \leq k_0$, and a polynomial $P$ such that
\[
|w_\ell - P|(x) \leq C(\xi^n + \delta)^{1/n} + C|x|^{2+\beta}.
\]

Set $x = \rho y$, and $\hat{P}(y) = \rho^{-2}P(\rho y)$, then we have
\[
\left| \frac{1}{\rho^2} w_\ell(\rho y) - \hat{P}(y) \right| \leq C \frac{(\xi^n + \delta)^{1/n}}{\rho^2} + C\rho^\beta
\]
for $|y| \leq 1$. We may now apply [4, Lemma 2] to obtain a polynomial $\tilde{P}$ such that $F(D^2\tilde{P}) = 0$, and
\[
\left| \frac{1}{\rho^2} w_\ell(\rho y) - \tilde{P}(y) \right| \leq C \frac{(\xi^n + \delta)^{1/n}}{\rho^2} + C\rho^\beta.
\]
We now choose \( \rho \), then \( \xi, \delta \) (which determine \( k_0 \)), depending on \( n, \lambda, \Lambda, \gamma, \varepsilon \) such that
\[
\left| \frac{1}{\eta^2} u(\eta y) - \tilde{P}(y) \right| \leq \varepsilon
\]
where \( \eta = \eta(n, \lambda, \Lambda, \gamma, \varepsilon) = \rho/2^\ell \), and \( 1 \leq \ell \leq k_0 \).

The conclusion of Theorem 1.2 follows immediately from Proposition 2.3, either by appealing directly to Savin’s small perturbations theorem [15, Theorem 1.3], or by following the argument in Caffarelli and Yuan [4].

### 3 Applications and extensions

In this section we indicate a few applications of Theorem 1.2. The first application is a rigidity result (Lemma 3.1) for solutions of twisted type equations on \( \mathbb{R}^n \). As an application of this rigidity, we extend Theorem 1.2 to a more general class of equations in Theorem 3.2, which is the main result of this section. Theorem 3.2 may be of interest to geometers.

**Lemma 3.1** Suppose \( u : \mathbb{R}^n \to \mathbb{R} \) is a smooth function with \( |D^2 u|_{L^\infty(\mathbb{R}^n)} \leq K < \infty \). Suppose that \( u \) solves the twisted type equation \( F(D^2 u) = 0 \) on \( \mathbb{R}^n \). Then \( u \) is a quadratic polynomial.

**Proof** Without loss of generality we may assume that \( u(0) = 0, \nabla u(0) = 0 \). Since \( |D^2 u| \leq K \), we have the estimate \( |\nabla u(x)| \leq K |x| \). For \( R > 0 \) we define
\[
v_R(x) := R^{-2} u(Rx).
\]
Then \( v_R(x) \) satisfies \( F(D^2 v_R) = 0 \) on \( B_1 \), together with the bounds
\[
|\nabla v_R(x)|_{L^\infty(\mathbb{B}_1)} \leq \Lambda, \quad |D^2 v_R|_{L^\infty(\mathbb{B}_1)} \leq K.
\]
By Theorem 1.2 we have an estimate \( |D^2 v_R|_{C^\alpha(B_{1/2})} \leq C \) for a constant \( C \) independent of \( R \). Writing this in terms of \( u \) gives
\[
R^\alpha |D^2 u|_{C^\alpha(B_{R/2})} \leq C.
\]
It follows that \( D^2 u \) is a constant, and hence \( u \) is a quadratic polynomial. \( \square \)

We can now prove an extension of Theorem 1.2.

**Theorem 3.2** Suppose \( u \) is a smooth solution of the uniformly elliptic equation
\[
F(D^2 u, x) = 0, \quad x \in B_2.
\]
Assume that \( F(\cdot, \cdot) \) is smooth in both slots, and that \( F(\cdot, x) \) is of twisted type for each \( x \in B_2 \). Then for every \( 0 < \alpha < 1 \) we have the estimate
\[
|D^2 u|_{C^\alpha(B_{1/2})} \leq C
\]
where \( C \) depends on \( F, \alpha, |D^2 u|_{L^\infty(B_2)} \).

**Proof** Set
\[
N_u = \sup_{B_1} d_x |D^3 u(x)|, \quad d_x = d(x, \partial B_1),
\]
and assume that the supremum is achieved at some point $x_0 \in B_1$. Define
\[ \tilde{u}(z) = d_{x_0}^{-2} N_u^2 u (x_0 + d_{x_0} N_u^{-1} z) - A - A_i z_i. \]  
where $A, A_i$ are chose so that $\tilde{u}(0) = 0$ and $\nabla \tilde{u}(0) = 0$. The function $\tilde{u}(z)$ is defined on $B_{N_u}(0)$, and satisfies
\[ D^2 \tilde{u} = D^2 u, \quad \|D^3 \tilde{u}\|_{L^\infty(B_{2^{-1}N_u}(0))} \leq 2|D^3 \tilde{u}(0)| = 2. \]
Furthermore, $\tilde{u}$ solves the equation
\[ F(D^2 \tilde{u}, x_0 + d_{x_0} N_u^{-1} z) = 0, \]  
on $B_{N_u}(0)$.

In order to prove the theorem, we use a contradiction argument, together with this rescaling. Suppose there exists a sequence $u_n$ of functions on $B_2$ solving (3.1), and having a uniform bound with $\|D^2 u_n\|_{L^\infty(B_2)} \leq K$. For the sake of obtaining a contradiction, we assume that the sequence $\{N_n\}$ increases to $+\infty$. Let $x_n \in B_1$ be the point where $N_n$ is achieved. By passing to a subsequence (not relabeled) we may assume that $\{x_n\}$ converges to a point $x_\infty \in \overline{B}_1$.

Using the above rescaling we get a sequence $\tilde{u}_n$ solving the equation
\[ F(D^2 \tilde{u}_n, x_n + d_{x_n} N_n^{-1} z) = 0, \]  
on $B_{N_n}(0)$. Furthermore, for every fixed $k$, the functions $u_n$, $n \geq k$ are defined on $B_{N_k}(0)$ and uniformly bounded in $C^3(B_{2^{-1}N_k}(0))$. Since $F$ is smooth in both slots, and uniformly elliptic, it follows from the Schauder theory that $\{\tilde{u}_n\}_{n \geq k}$ are uniformly bounded in $C^{3,\alpha}(B_{2^{-1}N_k}(0))$.

Since $N_n \to \infty$ by assumption, we may find a function $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ and a diagonal subsequence (not relabeled) so that $\{u_n\}$ converges uniformly to $\tilde{u}$ in $C^{3,\alpha/2}$ on compact subsets of $\mathbb{R}^n$. In particular, we have $|D^3 \tilde{u}|(0) = 1$. Furthermore, $\tilde{u}$ satisfies
\[ F(D^2 \tilde{u}, x_\infty) = 0. \]

Applying the Schauder theory again, we conclude that $\tilde{u} \in C^\infty(\mathbb{R}^n)$. But, since $F(\cdot, x_\infty)$ is of twisted type, Lemma 3.1 implies that $\tilde{u}$ is a quadratic polynomial, which contradicts $|D^3 \tilde{u}|(0) = 1$.

We conclude that $u$ must have $N_u < C$, for some $C$ depending on $F$, and $|D^2 u|_{L^\infty(B_2)}$. The theorem follows immediately. \qed

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