The $q$-cosine Fourier transform and the $q$-heat equation

Ahmed Fitouhi · Fethi Bouzeffour

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Abstract The aim of this work is to establish in great detail The $q$-Fourier analysis related to the $q$-cosine. The wise reader will note that the considered $q$-cosine coincides with the one given by T.H. Koornwinder and S.F. Swarttouw. Through the $q$-cosine product formula, we define and analyze the properties of the $q$-even translation and the $q$-convolution. Adopting the Titchmarsh approach, we study the $q$-cosine Fourier transform and its inverse formula.

The second theme of this paper is an application of the $q$-Fourier analysis developed earlier. We extend the heat representation theory inaugurated by P.C. Rosenbloom and D.V. Widder to the $q$-analogue. We construct the $q$-solution source, the $q$-heat polynomials and solve the $q$-analytic Cauchy problem.

Keywords Basic orthogonal polynomials and functions · Basic hypergeometric integrals

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1 Introduction

During the last years, an intensive work was founded about the so-called $q$-basic theory. Taking account of the well-known Ramanujan works shown at the beginning of this century by Jackson ([9, 10]), many authors such as Askey, Gasper, Ismail, Rogers, Andrew, Koornwinder, and others (see references) have recently developed this topic.

The present article is devoted to the study of the $q$-analogue of the Fourier transforms and to showing how it plays a central role in solving the $q$-heat equation associated to the second $q$-derivative operator. The method used here differs from those given by T.H. Koornwinder and R.F. Swarttouw, who discovered a $q$-analogue of Hankel’s Fourier–Bessel via some $q$-analogue orthogonality relations. We note that Ph. Feinsilver [4] gave a $q$-Harmonic Analysis for a $q$-Laplace transform with inversion formula.

Without entering into a dilemma through the analysis presented here, it seems that the point of view of T.H. Koornwinder and R.F. Swarttouw [12] is more suitable for harmonic analysis. We take as definition of the $q$-cosine the one given by the previous authors with a simple change and we prefer to write it as a series of functions denoted as $b_n(x; q^2)$. This $q$-cosine appears as an eigenfunction of the operator $\Delta_q$. Owing to a nice paper [12], we give a product formula written with the $q$-Jackson integral and we study the $q$-translation and the $q$-convolution. Next we define the $q$-analogue of the cosine Fourier transform with the purpose to find the transformation inverse. To this end, we prove the equivalent of the so-called Riemann–Lebesgue Lemma and discover that the Titchmarsh approach holds [15].

A motivation behind this work is to state some result about the $q$-heat equation associated to $\Delta_q$ operator. We attempt to extend the heat representation theory studied in many cases ([5, 7, 14], etc.). We define the $q$-heat polynomials and find that they are linked to the $q$-Hermite polynomials [13] and constitute with the $q$-associated functions a biorthogonal system. We conclude by solving the $q$-analytic Cauchy problem related to the $q$-heat equation.

2 Notations and preliminaries

We begin by recalling some $q$-elements of quantum analysis adapting the notation used in the book of Gasper and Rahman [6]. Let $a$ and $q$ be real numbers such that $0 < q < 1$, the $q$-shift factorial is defined by

\[(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \ldots, \infty.\] (1)

A basic hypergeometric series is

\[\varphi_r(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s, q; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k.\]
A function $f$ is $q$-regular at zero if $\lim_{n \to \infty} f(xq^n) = f(0)$ exists and is independent of $x$.

The $q$-derivative $D_q f$ of a function $f$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0. \quad (2)$$

The $q$-derivative at zero is defined by

$$D_q f(0) = \lim_{n \to \infty} \frac{f(xq^n) - f(0)}{xq^n},$$

if it exists and does not depend on $x$.

We introduce the set

$$\mathbb{R}_q = \{q^k; \ k \in \mathbb{Z}\}.$$  

The $q$-integral of Jackson is defined by

$$\int_0^a f(x) \, dq x = (1-q)a \sum_{k=0}^{\infty} f(aq^k)q^k,$$

$$\int_0^\infty f(x) \, dq x = (1-q) \sum_{k=-\infty}^{\infty} f(q^k)q^k.$$  

The $q$-integration by parts is given for suitable functions $f$ and $g$ by

$$\int_0^\infty f(x) D_q g(x) \, dq x = \left[ f(x)g(x) \right]_0^\infty - \int_0^\infty f(x) D_q g(q^{-1}x) \, dq x. \quad (3)$$

The $q$-analogue of the Gamma function is defined as

$$\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1-q)^{1-x}, \quad (4)$$

which tends to $\Gamma(x)$ when $q$ tends to $1^-$.

### 3 $q$-Trigonometric functions

We define the $q$-cosine as

$$\cos(x; q^2) = 1 \phi_1(0; q; q^2, (1-q)^2 x^2) = \sum_{n=0}^{\infty} (-1)^n b_n(x; q^2), \quad (5)$$

where we have put

$$b_n(x; q^2) = b_n(1; q^2)x^{2n} = q^{n(n-1)}(1-q)^{2n} \frac{(q; q)_{2n}}{x^{2n}}. \quad (6)$$
In the same way, the $q$-sine is given by

$$\sin(x; q^2) = (1 - q)x_1 \phi_1(0; q^3; q^2, (1 - q)^2 x^2) = \sum_{n=0}^{\infty} (-1)^n c_n(x; q^2),$$

with

$$c_n(x; q^2) = c_n(1; q^2) x^{2n+1} = \frac{q^{n(n-1)}(1 - q)^{2n+1}}{(q; q)_{2n+1}} x^{2n+1}.$$  

These $q$-trigonometric functions differ and should not be confused with the functions $\cos_q$ and $\sin_q$ considered in [6, p. 23]; but coincide with the one given in [12] and [15] with a minor change of variable. Furthermore, we have

**Proposition 3.1** The following statements hold:

1. $b_n(0, q^2) = \delta_{n,0}, \quad \Delta_q b_n(x; q^2) = b_{n-1}(x; q^2), \quad n \geq 1$;

2. $|b_n(x; q^2)| \leq \frac{x^{2n}}{(2n)!},$

where

$$\Delta_q u(x) = (D^2_{\Delta_q} u)(q^{-1}x). \quad (7)$$

**Proof** We only prove Part 2 since Part 1 is deduced from the definition of $\Delta_q$.

The coefficients $b_n(1; q^2)$, defined by (6), can be written as

$$b_n(1; q^2) = \prod_{j=0}^{n-1} \frac{q^j - q^{j+1}}{1-q^{2j+1}} \frac{q^j - q^{j+1}}{1-q^{2j+2}} = \prod_{j=0}^{n-1} \frac{e^{-jt} - e^{-(j+1)t}}{1 - e^{-2(j+1)t}} \frac{e^{-jt} - e^{-(j+1)t}}{1 - e^{-2(j+2)t}},$$

where we have put $q = e^{-t}, t > 0$.

Since the functions

$$f(t) = \frac{e^{-jt} - e^{-(j+1)t}}{1 - e^{-(j+1)t}} \quad \text{and} \quad g(t) = \frac{e^{-jt} - e^{-(j+1)t}}{1 - e^{-(2j+2)t}},$$

decrease on $]0, \infty[$, we obtain

$$b_n(1; q^2) \leq \frac{1}{(2n)!}. \quad \Box$$
As a consequence of the previous proposition, we can show that for \( \lambda \in \mathbb{C} \) the function

\[
\cos(\lambda x; q^2) = \sum_{n=0}^{\infty} (-1)^n b_n(x; q^2) \lambda^{2n},
\]

is the unique analytic solution of the \( q \)-differential equation

\[
\Delta_q u(x) = -\lambda^2 u(x), \quad (8)
\]

with

\[
u(0, q) = 1, \quad (D_q u)(0) = 0. \quad (9)
\]

**Proposition 3.2** For \( x \in \mathbb{R}_q \) and \( \frac{\log(1-q)}{\log(q)} \in \mathbb{Z} \), we have

1. \[
|\cos(x, q^2)| \leq \frac{1}{(q; q^2)_\infty};
\]
2. \[
\lim_{x \to \infty} \cos(x, q^2) = 0;
\]
3. \[
|\sin(x, q^2)| \leq \frac{1}{(q; q^2)_\infty};
\]
4. \[
\lim_{x \to \infty} \sin(x, q^2) = 0.
\]

**Proof** To prove Parts 1 and 2, we use the properties of \( \varphi_1 \) given in [12] and their connection to the \( q \)-cosine. We obtain

\[
|\cos(q^{1+n}x; q^2)| \leq \frac{1}{(q; q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0, \\ q^n & \text{if } n \leq 0. \end{cases} \quad (10)
\]

hence Parts 1 and 2 follow. A similar argument shows Parts 3 and 4. \( \square \)

Now we try to find a product formula for the \( q \)-cosine functions. We begin by proving the following result.

**Proposition 3.3** For reals \( x \) and \( y \), \( y \neq 0 \), we have

\[
\cos(x, q^2) \cos(y, q^2) = \sum_{k=0}^{\infty} q^k \left( \frac{x}{y} \right)^{2k} \sum_{s=-k}^{k} (-1)^{k-s} \frac{q^{(k-s)}}{(q; q)_{k-s}(q; q)_k} \cos(q^s y, q^2). \quad (11)
\]

*Note that this formula can be expressed in terms of \( \varphi_1 \) as follows*
\[
\cos(x, q^2) \cos(y, q^2) = \sum_{s=-\infty}^{\infty} q^s \left(\frac{x}{y}\right)^{2s} \frac{(q^{1+2s}; q)_{\infty}}{(q; q)_{\infty}} \times \varphi_1(0; q^{1+2s}; q, q \frac{x^2}{y^2}) \cos(q^s y, q^2). \tag{12}
\]

**Proof** To show (11) and (12), we begin by expanding the \(q\)-cosines in series absolutely and uniformly convergent on every compact of \(\mathbb{R}\). From the product rule of series and the fact that
\[
\frac{1}{(q; q)_{2n-2k}} = (q^{2n-2k+1}; q)_{\infty} = 0, \quad k > n,
\]
we obtain for \(y \neq 0\)
\[
\cos(x; q^2) \cos(y; q^2) = \sum_{k=0}^{\infty} q^{2k} \left(\frac{x}{q; q} \right)_{2k} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2-n}}{(q; q)_{2n-2k}} q^{2nk} y^{2n}.
\]

On the other hand, we have
\[
\frac{1}{(q; q)_{2n-2k}} = q^{-k(2k-1)+2nk} \sum_{s=-k}^{s=k} (-1)^{k-s} \frac{q^{k-s}}{(q; q)_{k-s} (q; q)_{k+s}} q^{2ns}.
\]

We deduce (11) after the interchange of summation order. To prove (12), we write
\[
\cos(x; q^2) \cos(y; q^2) = I + J,
\]
with
\[
I = \sum_{s=0}^{\infty} \cos(q^s y; q^2) \sum_{k \geq s} q^k \left(\frac{x}{y}\right)^{2k} (-1)^{k-s} \frac{q^{(k-s)(k-s-1)}}{(q; q)_{k-s} (q; q)_{k+s}} \phi_1(0; q^{1+2s}; q, q \frac{x^2}{y^2}) \cos(q^s y, q^2).
\]

In \(I\), we make the change \(k - s \) into \(k\) and use the equality
\[
(q; q)_{k+2s} = (q; q)_{2s} (q^{1+2s}; q)_k,
\]
to obtain
\[
I = \sum_{s=0}^{\infty} q^s \left(\frac{x}{y}\right)^{2s} \frac{(q^{2s+1}; q)_{\infty}}{(q; q)_{\infty}} \phi_1(0; q^{1+2s}; q, q (q^2 / y^2)) \cos(q^s y, q^2).
\]

Now we make the change \(k + s \) into \(k\) in \(J\) and use the equalities
\[
(q; q)_{k-2s} = (q; q)_{-2s} (q^{1-2s}; q)_k, \quad -s \geq 1,
\]
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\[
\frac{(k - 2s)(k - 2s - 1)}{2} = \frac{(k - 2)(k - 3)}{2} - 2sk + 2s^2 - 1,
\]

and

\[
(q^{1-2s}; q)_\infty \phi_1(0; q^{1-2s}; q, q^{1-2s}x^2/y^2) = q^{s(2s-1)}q^{1-2s}(x^2/y^2)^{2s}(q^{1+2s}; q)_\infty \phi_1(0; q^{1+2s}; q, qx^2/y^2).
\]

This identity is easily deduced from [11]. Then we obtain

\[
J = \sum_{s=-\infty}^{-1} q^s (x^2/y^2)^{2s} \frac{(q^{1+2s}; q)_\infty}{(q; q)_\infty} \phi_1(0; q^{1+2s}; q, qx^2/y^2) \cos(q^s y; q^2).
\]

We add these sums to find that (12) holds.

\[\square\]

Remark 3.4 (1) If we replace $y$ by $qy$, $x$ by $qx$, and assume the proposition the hypothesis, we obtain from (12) that the following integral representation holds

\[
\cos(q^x; q^2) \cos(q^y; q^2) = \frac{(q^{2(x-y)+1}; q)_\infty}{(q; q)_\infty} \int_0^{\infty} u^{2(x-y)} \phi_1(0; u^{2(x-y)+1}; q, qu^2) \cos(q^y u; q^2) \, dq u.
\]

(2) The product formula (11) leads to

\[
\cos(x; q^2) \cos(y; q^2) = \sum_{n=0}^{\infty} b_n(x; q^2) \Delta_q^n \cos(y; q^2).
\]

4 $q$-Translation and $q$-convolution

We define, for $x$ and $y$ in $\mathbb{R}_q$, the measure

\[
d_q \mu_{(x,y)} = \sum_{s=-\infty}^{\infty} \mathcal{D}(x, y; q^s)q^s \delta_{yq^s},
\]

where $\delta_u$ denotes the unit mass supported at $u$, and

\[
\mathcal{D}(x, y; q^s) = \left( \frac{x}{y} \right)^{2s} \frac{(q(\frac{x}{y})^2; q)_\infty}{(q; q)_\infty} \phi_1(0; q(\frac{x}{y})^2; q, q^{1+2s}).
\]

Proposition 4.1 (1) For $x$ and $y$ in $\mathbb{R}_q$, we have

\[
d_q \mu_{(x,y)} = d_q \mu_{(y,x)}.
\]

(2) $d_q \mu_{(x,y)}$ is of bounded variation.
\[ \int dq \mu(x,y)(t) = 1. \]

**Proof**  For \( n, m \in \mathbb{Z} \), the relation (2.3) from [12] leads to

\[ D(q^n, q^m; q^s) = D(q^m, q^n; q^{s+m-n}). \]

We obtain Part 1 after the change \( s - n + m \) by \( s \).

To prove Part 2, we suppose \(|\frac{x}{y}| \leq 1\); from the formulas (2.4) in [12] we have

\[ |dq \mu(x,y)|_{\text{var}} \leq \left( \frac{|y|^2 + q|x|^2}{|y|^2 - q|x|^2} \right) \frac{(q|\frac{x}{y}|^2; q, q)_{\infty}}{(q, q)_{\infty}}. \]  \hspace{1cm} (16)

Finally, from (2.8) in [12], we can show that Part 3 is true. \( \square \)

We introduce the \( q \)-translation which generalizes the even translation given by \( \frac{1}{2}(\delta_{x+y} + \delta_{x-y}) \).

Let \( f \) be a function with support in \( \mathbb{R}_q \), the \( q \)-translation is defined for \( x \) and \( y \) in \( \mathbb{R}_q \) by

\[ T_{x,q} f(y) = \int_0^\infty f(t) dq \mu(x,y)(t). \]  \hspace{1cm} (17)

From the previous proposition and the \( q \)-product formula (12), we have

**Proposition 4.2**  Let \( f \) be a function with compact support in \( \mathbb{R}_q \). We have

(i)

\[ T_{q,y} \cos(x; q^2) = \cos(x; q^2) \cos(y; q^2). \]

(ii)

\[ T_{q,y} f(x) = T_{q,x} f(y), \]

\[ T_{q,0} f = f. \]

(iii)

\[ \Delta_q T_{q,y} f = T_{q,y} \Delta_q f, \]

\[ \Delta_{q,y} T_{q,y} f = T_{q,y} \Delta_{q,y} f. \]

(iv) **The function** \( u(x, y) = T_{q,y} f(x) \) **is a solution of the problem**

\[ \Delta_{q,x} u(x, y) = \Delta_{q,y} u(x, y), \]

\[ u(x, 0) = f(x). \]
From the relation
\[
\Delta^n_q(f)(x) = \sum_{k=0}^{n} \frac{(2n)_{n-k}}{(1-q)^{2n}} \sum_{k=-n}^{n} (-1)^{n-k} (q;q)_{n-k}(q;q)_{n+k} \cdot f(q^k x),
\]
we can write the \(q\)-translation of a function \(f\) as
\[
T_{q,x} f(x) = \sum_{n=0}^{\infty} b_n(y, q^2) \Delta^n_{q,x} f(x),
\]
and have in the limit when \(q\) tends to \(1^-\) the classical even translation cited before.

Now we denote by \(L^1_q(\mathbb{R}_q)\) the space of functions \(f\) defined on \(\mathbb{R}_q\) such that
\[
\|f\|_{1,q} = \int_{-\infty}^{\infty} |f(t)| \, d_q t < \infty.
\]
Then we are able to define the \(q\)-convolution by
\[
f_q \ast g(x) = \left(1 + q^{-1}\right)^{1/2} \frac{\Gamma_q(1/2)}{\Gamma_q^2(1/2)} \int_0^\infty T_{x,q} f(y) g(y) \, d_q y,
\]
where \(f\) and \(g\) are two functions in \(L^1_q(\mathbb{R}_q)\). We can show that this space is an algebra.

### 5 \(q\)-Analogue of Fourier-cosine

In this section, we suppose \(\frac{\log(1-q)}{\log(q)} \in \mathbb{Z}\). The \(q\)-analogue of Fourier transform is defined for \(\lambda \in \mathbb{R}_q\) by
\[
\mathcal{F}(f)(\lambda) = \left(1 + q^{-1}\right)^{1/2} \frac{\Gamma_q(1/2)}{\Gamma_q^2(1/2)} \int_0^\infty f(t) \cos(\lambda t; q^2) \, d_q t,
\]
where \(f\) is a function in \(L^1_q(\mathbb{R}_q)\).

This definition is the same (after a minor change) as that given by T.H. Koornwinder and R.F. Swarttouw (see [12]).

**Proposition 5.1** For \(f, g \in L^1_q(\mathbb{R}_q)\), the following properties hold:

1. \[
|\mathcal{F}_q(f)(\lambda)| \leq \frac{1}{[q(1-q)]^{1/2}} \|f\|_{1,q}, \quad \lambda \in \mathbb{R}_q;
\]

2. \[
\mathcal{F}_q(T_{q,x} f)(\lambda) = \cos(\lambda x; q^2) \mathcal{F}_q(f)(\lambda), \quad \lambda \in \mathbb{R}_q;
\]

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\( F_q(f \ast_g g) = F_q(f)F_q(g). \)

**Proof** Part 1. The inequality (21) follows from Proposition 3.2 and the identity

\[
(q; q^2)_\infty (q^2; q^2)_\infty = (q; q)_\infty.
\]

Part 2 is a direct consequence of the \( q \)-product formula (12).

Part 3 is obtained after the exchange of the integration order and taking into account the invariability of the \( q \)-integral by the \( q \)-translation. \( \square \)

Now we focus our attention on the inversion of the linear map \( F_q \). We proceed by looking at the \( q \)-analogue of the Riemman–Lebesgue Lemma, the localization theorem, and we show that the Titchmarsh approach holds in the \( q \)-theory.

**Proposition 5.2** Let \( f \) be a function in \( L^1_q(\mathbb{R}_q) \), then

\[
\lim_{\lambda \to \infty} F_q(f)(\lambda) = 0, \quad \lambda \in \mathbb{R}_q.
\]

**Proof** To prove this, first we have from Proposition 3.2

\[
\left| f(x) \cos(\lambda x; q^2) \right| \leq \frac{1}{(q; q^2)_\infty^2} \left| f(x) \right| \in L^1_q(\mathbb{R}_q), \quad x, \lambda \in \mathbb{R}_q.
\]

And for \( \lambda \in \mathbb{R}_q \) we have

\[
\lim_{\lambda \to \infty} f(x) \cos(\lambda x; q^2) = 0, \quad \lambda \in \mathbb{R}_q,
\]

so the result is true. \( \square \)

**Proposition 5.3** We have the identity

\[
\int_0^\infty \frac{\sin(x; q^2)}{x} dqx = \frac{\Gamma^2_q(\frac{1}{2})}{1 + q^{-1}}.
\]

**Proof** This is a consequence of (2.8) in [12]. \( \square \)

**Proposition 5.4** Let \( f : (0, \infty) \to \mathbb{C} \) satisfy the conditions:

1. \( f \in L^1_q(\mathbb{R}_q) \),
2. For \( a \in \mathbb{R}_q \), there exists \( C(a) > 0 \) such that

\[
\left| f(aq^k) - f(0) \right| \leq C(a)q^k, \quad k = 0, 1, 2, \ldots.
\]

Then

\[
\lim_{\lambda \to +\infty} \int_0^\infty f(x) \frac{\sin(\lambda x; q^2)}{x} dqx = \frac{\Gamma^2_q(\frac{1}{2})}{1 + q^{-1}} f(0).
\]
Proof Indeed, the first hypothesis shows that for an arbitrary \( \varepsilon > 0 \) we have for large \( q^{-N}, N = 0, 1, \ldots, \) that
\[
\int_{q^{-N}}^{\infty} \left| \frac{f(x)}{x} \right| dq x \leq \frac{\varepsilon}{2} (q, q^2)^2_{\infty}
\]
and
\[
\left| \int_0^{\infty} f(x) \frac{\sin(\lambda x; q^2)}{x} dq x - f(0) \int_0^{q^{-N}} f(x) \frac{\sin(\lambda x; q^2)}{x} dq x \right|
\leq \frac{\varepsilon}{2} + \int_0^{q^{-N}} \frac{f(x) - f(0)}{x} \sin(\lambda x; q^2) dq x.
\]

The second hypothesis and Proposition 3.2 show that
\[
\left| f(q^{k-N}) - f(0) \frac{\sin(\lambda q^{k-N}; q^2)}{q^{k-N}} \right| \leq \frac{C(N)}{q^{-N} (q, q^2)^2_{\infty}}.
\]
Since from Proposition 3.2 we have that \( \sin(\lambda x; q^2) \) tends to zero as \( \lambda \) tends to \( \infty \), the proposition is then a direct consequence. \( \square \)

**Theorem 5.5** (The \( q \)-cosine Fourier integral theorem) If \( f \in L^1_q(\mathbb{R}_q) \) is such that for \( a \in \mathbb{R}_q \) there exist positive constants \( C(a) \) such that
\[
|T_{x, q} f(a q^k) - f(q^k)| \leq C(a) q^k, \quad k = 0, 1, \ldots,
\]
then
\[
\frac{(1 + q)^{1/2}}{\Gamma_q(1/2)} \int_0^{\infty} dq \xi \int_0^{\infty} f(t) \cos(\xi t; q^2) \cos(\xi x; q^2) dq t = f(x),
\]
\( x \in L^1_q(\mathbb{R}_q) \).

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### 6 q-Heat equation and q-heat polynomials

In this section, the two \( q \)-analogues of the elementary exponential functions are crucial and they are defined by
\[
E(x; q^2) = (-1 - q^2)x, q^2_{\infty}
\]
\[
= \sum_{n=0}^{\infty} \frac{(1 - q^2)^n}{(q^2; q^2)_{\infty}} q^{n(n-1)} x^n, \quad x \in \mathbb{R},
\]
and
\[
e(x; q^2) = \frac{1}{((1 - q^2)x; q^2)_{\infty}} = \sum_{n=0}^{\infty} \frac{(1 - q^2)^n}{(q^2; q^2)_{n}} x^n, \quad |x| < \frac{1}{1 - q^2}.
\]
These functions satisfy the identity
\[ e(x; q^2) E(-x; q^2) = 1, \]
and have as limit, when \( q \) tends to \( 1^- \), the classical exponential function.

Now we purpose to give the \( q \)-analogue of the heat equation associated to the second derivative operator (even in \( x \))
\[
\frac{\delta^2 u}{\delta x^2} = \frac{\delta u}{\delta t}, \quad x \in \mathbb{R}, \ t > 0.
\] (27)

We consider as \( q \)-heat equation associated to the second \( q \)-derivative operator the partial \( q \)-difference equation
\[
(\Delta_{q,x} u)(x, t) = (D_{q^2,t} u)(x, t).
\] (28)

We take as the initial condition
\[
u(x, 0) = f(x), \quad f \in L^1_q(\mathbb{R}_q). \] (29)

6.1 \( q \)-Solution source

To find the solution source related to the \( q \)-heat equation, we apply the Fourier method with the adapted \( q \)-Fourier cosine studied before.

Putting
\[
U(\lambda, t) = \mathcal{F}(u(x, t))(\lambda),
\]
Eq. (28) becomes
\[
D_{q^2,t} U(\lambda, qt) = -\lambda^2 U(\lambda, t),
\]
and, taking into account conditions (29), we obtain
\[
U(\lambda, t) = \mathcal{F}(f)(\lambda)e(-\lambda^2 t; q^2).
\]

The problem consists in finding the function which has \( e(-\lambda^2 t; q^2) \) as its \( q \)-Fourier-cosine transform. For this end, we need the following lemma.

**Lemma 6.1** For \( n = 0, 1, 2, \ldots \) and \( t > 0 \), we have
\[
\int_0^\infty e\left(-\frac{\lambda^2}{qt(1+q)^2}, q^2\right) b_n(\lambda; q^2) d_q \lambda.
\]
\[
= (1 - q) \left( q^2, -\frac{1+q}{1-q} q^{2t} \right) (1-q^2)^n (\frac{1}{q^2} q^{2t})^n.
\]

**Proof** From (26) we find
\[
\int_0^\infty e\left(-\frac{\lambda^2}{qt(1+q)^2}, q^2\right) \lambda^{2n} d_q \lambda = (1 - q) \sum_{-\infty}^{\infty} \frac{q^{2n+1} k}{(-\frac{1}{1+q} q^2 t, q^2)}.
\]
Secondly, the use of the well-known Ramanujan [8] identity
\[
\sum_{k=-\infty}^{\infty} \frac{z^k}{(bq^k, q)_\infty} = \frac{(bz, q/bz, q, q)_\infty}{(b, z, q/b, q)_\infty}, \quad b \neq 0,
\]
leads to the result after minor computation. □

**Proposition 6.2**
\[
\frac{(1 + q^{-1})^{1/2}}{T_{q^2}(1/2)} \int_0^\infty e\left(-\frac{\lambda^2}{qt(1 + q^2)}, q^2\right) \cos(\lambda x, q^2) \, dq \lambda = A(t, q^2)e(-tx^2, q^2),
\]
where
\[
A(t, q^2) = \left[(1 - q)q^{-1}\right]^{1/2} \frac{(-\frac{1+q}{1-q} q^2 t, -\frac{1-q}{1+q} \frac{1}{t}, q^2)_\infty}{(-\frac{1-q}{1+q} \frac{1}{t}, -\frac{1+q}{1-q} q^3 t; q^2)_\infty}. \quad (30)
\]

As an immediate consequence we are now able to define the \(q\)-source solution associated to the \(q\)-heat equation (28) by
\[
G(x, t, q^2) = (A(t, q^2))^{-1} e\left(-\frac{x^2}{qt(1 + q^2)}; q^2\right). \quad (31)
\]

In the same manner as in the classical heat equation theory, we put
\[
G(x, y, t; q^2) = T_{y,q} G(x, t; q^2), \quad (32)
\]
with \(T_{y,q}\) being the \(q\)-translation studied in Sect. 4.

Through this approach we show that the solution of the \(q\)-Cauchy problem (28) and (29) can be written in the form of
\[
u(x, t) = (G(\cdot, t; q^2) *_q f)(x) = \int_0^\infty G(x, y, t; q^2) f(y) \, dq y. \quad (33)
\]

It is natural to ask how other properties such as the positivity of \(G(x, t; q^2)\) and the existence of the \(q\)-semigroup can be established.

### 6.2 \(q\)-Heat polynomials

**Proposition 6.3** It is easy to see that, for \(x \in \mathbb{R}\) and \(t > 0\), the analytic function
\[
\lambda \rightarrow e(-\lambda^2 t; q^2) \cos(\lambda x; q^2),
\]
is a solution of (28) and it has the expansion
\[
e(-\lambda^2 t, q^2) \cos(\lambda x, q^2) = \sum_{n=0}^{\infty} (-1)^n v_{2n}(x, t, q) \lambda^{2n},
\]
where
\[ v_{2n}(x, t, q) = \sum_{k=0}^{n} b_k(x, q^2) \frac{(1 - q^2)^{n-k}}{(q^2; q^2)_{n-k}} t^{n-k}, \tag{34} \]

with the functions \( b_n \) being given by (6).

From Proposition 3.1 we deduce immediately the following properties:
\[ \Delta_{q,x} v_{2n}(x, t, q) = D_{q^2,t} v_{2n}(x, t, q), \quad n \geq 0, \]
\[ v_{2n}(x, 0, q) = b_n(x, q^2), \]
\[ v_{2n}(x, t, q) \geq 0, \quad \text{if } t \geq 0. \]

We note that formula (34) can be inverted:
\[ b_n(x; q^2) = \sum_{k=0}^{n} (-1)^{n-k} v_{2k}(x, t; q) q^{(n-k)(n-k-1)/2} \frac{(1 - q^2)^{n-k}}{(q^2; q^2)_{n-k}} t^{n-k}. \tag{35} \]

**Proposition 6.4** The \( q \)-heat polynomials (34) possess the \( q \)-integral representation
\[ (1) \quad v_{2n}(x, t; q) = \int_{0}^{\infty} G(x, y, t, q^2) b_n(y; q^2) \, dq_y. \tag{36} \]
\[ (2) \quad b_n(x; q^2) = \int_{0}^{\infty} G(x, y, t, q^2) v_{2n}(q^{-1/2} y, t; q^{-1}) \, dq_y. \tag{37} \]

**Proof** We have
\[ \int_{0}^{\infty} G(x, y, t, q^2) b_n(y; q^2) \, dq_y = \int_{0}^{\infty} T_{q,x} G(y, t, q^2) b_n(y; q^2) \, dq_y \]
\[ = \int_{0}^{\infty} G(y, t, q^2) T_{q,x} b_n(y; q^2) \, dq_y \]
\[ = \sum_{k=0}^{n} b_k(x; q^2) \int_{0}^{\infty} G(y, t, q^2) b_{n-k}(y; q^2) \, dq_y \]
\[ = v_{2n}(x, t; q) \]

and
\[ \int_{0}^{\infty} G(x, y, t, q^2) v_{2n}(q^{-1/2} y, -t; q^{-1}) \, dq_y \]
\[ = \sum_{k=0}^{n} (-1)^{n-k} q^{(n-k)(n-k-1)/2} \frac{(1 - q^2)^{n-k}}{(q^2; q^2)_{n-k}} t^{n-k} \int_{0}^{\infty} G(x, y, t, q^2) b_k(y, q^2) \, dq_y \]
The $q$-cosine Fourier transform and the $q$-heat equation

\[
= \sum_{k=0}^{n} (-1)^{n-k} q^{(n-k)(n-k-1)} \frac{1 - q^2)^{n-k}}{(q^2; q^2)_{n-k}} r^{n-k} v_{2k}(x, t; q)
= b_n(x; q^2).
\]

In [14], the authors defined the so-called associated functions by the Appell transform. We extend this notion by defining for $t > 0$ the $q$-associated functions of $v_{2n}$ by

\[
w_{2n}(x, t; q) = (-1)^n \Delta^n G(x, y, t; q^2) \big|_{y=0}.
\]

It is easy to see that

\[
w_{2n}(x, t; q) = \frac{(1 + q^{-1})^{1/2}}{\Gamma_q(1/2)} \int_{0}^{\infty} e(-tq^{1/2}y, t; q^2) \lambda^{2n} \cos(\lambda x, q^2) d_q \lambda.
\]

**Proposition 6.5** (Biorthogonality) For $t > 0$ and $n, m \in \mathbb{N}$, we have

\[
\int_{0}^{\infty} w_{2m}(x, t; q)v_{2n}(q^{1/2}x, -t; q) d_q x = (-1)^m \delta_{n,m}.
\]

**Proof** By (37), we have

\[
\Delta^n b_n(x; q^2) = \int_{0}^{\infty} \Delta^n G(x, y, t; q^2) v_{2n}(q^{-1/2}y, t; q^{-1}) d_q y.
\]

Putting $x = 0$, we obtain

\[
\int_{0}^{\infty} w_{2m}(y, t; q)v_{2n}(q^{-1/2}y, t; q^{-1}) d_q y = (-1)^m \delta_{n,m}.
\]

6.3 Convergence of $\sum_{n \geq 0} \alpha_n v_{2n}(x, t; q)$

Now we establish the following estimates that will be needed later

**Lemma 6.6** For $n = 0, 1, \ldots$ and $0 < \frac{x^2}{t_0} < +\infty$, we have

\[
|v_{2n}(x_0, t_0, q)| \geq \frac{(1 - q^2)^n}{(q^2; q^2)_n} |t_0|^n \geq \frac{|t_0|^n}{n!}.
\]

**Proof** Indeed, the first inequality is a consequence of $b_0(1; q^2) = 1$ and the hypothesis, and the second follows from

\[
\frac{1}{n!} \leq \frac{(1 - q^2)^n}{(q^2; q^2)_n}.
\]

\[\Box \text{ Springer}\]
Corollary 6.7 For $n = 0, 1, \ldots$ and $0 < \frac{x_0^2}{t_0} < +\infty$, we have

$$|v_{2n}(x_0, t_0, q)| \geq C n^{-\frac{1}{2}} \left( \frac{|t_0| e}{n} \right)^n,$$

where $C$ is a constant depending on $x_0$ and $t_0$.

Lemma 6.8 For $n = 0, 1, \ldots, \delta > 0$, and $\frac{x^2}{\delta(1+q)} < 1$, we have

$$\frac{(1-q^2)^n}{(q^2; q^2)_n} |v_{2n}(|x|, |t|, q)| \leq q^{-n(n-1)} \frac{\delta + |t|}{n!} e\left( \frac{x^2}{\delta(1+q)} ; q \right). \quad (40)$$

Proof To show (40), we note that

$$(q; q)_{2k} = (q, q^2; q^2)_k,$$

and

$$(q; q^2)_k \geq (q; q)_k.$$  

For $\delta > 0$, and by using the fact that

$$\frac{(1-q)^k}{(q; q)_k} \frac{|x|^2}{(\delta(1+q))^k} \leq q^{-\left(\frac{\delta}{\delta} \right)} \exp\left( \frac{|x|^2}{\delta(1+q)} \right),$$

we obtain

$$v_{2n}(|x|, |t|; q) \leq \frac{(1-q^2)^n}{(q^2; q^2)_n} \sum_{k=0}^{\infty} q^{k(k-1)} \binom{n}{k} \frac{(1-q)^k}{(q; q)_k} \frac{|x|^{2k}}{(1+q)^k} |t|^{n-k}$$

$$\leq q^{-\left(\frac{\delta}{\delta} \right)} n! \left\{ -\frac{|t|}{\delta}; q^2 \right\}_n e\left( \frac{|x|^2}{\delta(1+q)} ; q \right).$$

The inequalities

$$\left( -\frac{|t|}{\delta}; q^2 \right)_n \leq \left( \frac{|t|}{\delta} + 1 \right)^n,$$

and

$$q^{\left(\frac{\delta}{\delta} \right)} n! \leq \frac{(q; q)_n}{(1-q)^n} \leq n!,$$

give the result.

By the Stirling formula, we obtain

Corollary 6.9 For $n = 0, 1, \ldots, \delta > 0$, and $\frac{x^2}{\delta(1+q)} < 1$, we have
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\begin{equation}
v_{2n}(|x|, |t|, q) \leq K q^{-n(n-1)} \left( (\delta + |t|) \frac{n}{e} \right)^n, \tag{41}
\end{equation}

where $K$ is a constant depending on $\delta$.

**Theorem 6.10** Let $(\alpha_n)$ be a sequence of real or complex numbers such that

\[ \lim_{n \to \infty} n^{-1} q^{-2(n-1)} |\alpha_n|^{1/n} = \frac{1}{\sigma} < +\infty. \]

Then the series

\[ \sum_{n \geq 0} \alpha_n v_{2n}(x, t; q), \]

converges in the strip

\[ S_{\sigma} = \{(x, t), x \in \mathbb{R}, |t| < \sigma\}, \tag{42} \]

and converges uniformly in any region of this strip.

To prove the theorem, we adopt the same approach as in [14] by taking account of the $q$-equivalent estimation (41).

**Remark** If we write $u(x, t)$ as the sum of the previous series, then this function satisfies the $q$-heat equation (28) and

\[ u(x, 0) = \sum_{n=0}^{\infty} \alpha_n b_n(x; q^2), \]

where the $b_n(x; q^2)$ is given by (6).

6.4 Analytic Cauchy problem related to the $q$-heat equation

**Lemma 6.11** Under the hypothesis of Theorem 6.10 and putting

\[ u(x, t) = \sum_{n \geq 0} \alpha_n v_{2n}(x, t; q), \tag{43} \]

$u(x; t)$ is an analytic function of two variables $x$ and $t$ in the strip $S_{\sigma}$ given by (42) and satisfies the $q$-heat equation (28). Furthermore, the coefficients $\alpha_n$ are given by

\[ \alpha_n = \Delta_q^n u(x, t) \big|_{(x, t)=(0,0)}. \tag{44} \]

**Proof** To show this, we note that the theorem gives that $u(x, t)$ is analytic in the whole strip $S_{\sigma}$. Now for a fixed integer $p$ the series

\[ \sum_{n \geq 0} \alpha_{n+p} v_{2n}(x, t; q) \]
converges uniformly in any compact region of $S_\sigma$. To prove \((44)\), it suffices to see that for integers $n$ and $p$ we have
\[
\left. \left( \Delta_{q,x}^n v_{2p}(x, t; q) \right) \right|_{(0,0)} = \delta_{n,p},
\]
where $\delta_{n,p}$ is the Kronecker symbol.

Finally the following statement is established.

**Theorem 6.12** Under the hypothesis of Lemma 6.11, the function $u(x, t)$ given by \((43)\) has the $q$-Maclaurin expansion
\[
u(x, t) = \sum_{m,p \geq 0} \beta_{m,p} \frac{(1 - q^2)^m}{(q^2; q^2)_m} x^{2p} t^m,
\]
where
\[
\beta_{m,p} = \alpha_{m+p} b_p (1, q^2).
\] (45)

If for $x \in \mathbb{R}$ and $|t| < \sigma$ then function
\[
u(x, t) = \sum_{m,p} \beta_{m,p} \frac{(1 - q^2)^m}{(q^2; q^2)_m} x^{2p} t^m,
\]
satisfies the $q$-heat equation \((28)\) with the coefficients $\beta_{m,p}$ given by \((44)\), then $u(x, t)$ can be extended to an analytic function in the strip $S_\sigma$ and we have
\[
u(x, t) = \sum_{n \geq 0} \alpha_n v_{2n}(x, t; q).
\]

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