A REMARK ON THE RAMSEY NUMBER OF THE HYPERCUBE

KONSTANTIN TIKHOMIROV

Abstract. A well known conjecture of Burr and Erdős asserts that the Ramsey number \( r(Q_n) \) of the hypercube \( Q_n \) on \( 2^n \) vertices is of the order \( O(2^n) \). In this paper, we show that \( r(Q_n) = O(2^n - cn) \) for a universal constant \( c > 0 \), improving upon the previous best known bound \( r(Q_n) = O(2^n) \), due to Conlon, Fox and Sudakov.

1. Introduction

The Ramsey number \( r(H) \) of a graph \( H \) is the smallest integer \( N \in \mathbb{N} \) such that any two-coloring of the complete graph on \( N \) vertices contains a monochromatic copy of \( H \). Given an integer \( n \), denote by \( Q_n \) the \( n \)-dimensional hypercube viewed as a graph (where the edge set is formed by the “geometric” edges of the hypercube). Burr and Erdős [3] conjectured that the Ramsey number of \( Q_n \) is of order \( O(2^n) \) (where the implicit constant is absolute). Improvements of the trivial bound \( r(Q_n) \leq r(K_{2^n}) \) were obtained by Beck [2], Graham, Rödl and Ruciński [9], Shi [12, 13], Fox and Sudakov [5], Conlon, Fox and Sudakov [4], as well as Lee [11]. Disregarding multiplicative constants, the best upper bound \( r(Q_n) = O(2^2n) \) prior to our work was obtained in [4] using the dependent random choice, and it applies to arbitrary bipartite graphs with a given maximum degree.

Theorem ([4, Theorem 4.1]). For every bipartite graph \( H \) on \( m \) vertices with maximum degree \( d \), one has \( r(H) \leq 2^{d+6}m \).

Let us remark at this point that, as was shown in [5], for every \( d \geq 2 \) and \( m \geq d+1 \) there exists a bipartite graph \( H \) on \( m \) vertices with the maximum degree at most \( d \) and such that \( r(H) \geq 2cdm \) for some constant \( c > 0 \). Therefore, a proof of the aforementioned conjecture of Burr and Erdős or even a weaker bound \( r(Q_n) = 2^{n+o(n)} \) should necessarily make use of properties of the hypercube other than the size of its vertex set and the vertex degrees.

The above theorem follows immediately as a corollary of an embedding theorem in the same work [4] (we provide a slightly simplified version here).

Theorem ([4, Theorem 4.7]). Let \( H \) be a bipartite graph on \( m \) vertices with maximum degree \( d \geq 2 \). If \( G \) is a bipartite graph with edge density \( \alpha \in (0,1) \) and at least \( 16d^{1/d}\alpha^{-d}m \) vertices in each part, then \( H \) is a subgraph of \( G \).

The dependent random choice is a well established technique in extremal combinatorics (see, among others, papers [1, 5, 7, 14] as well as survey [6]) which allows us to generate collections of graph vertices with many common neighbors. In the context of bipartite graph embeddings, the use of the dependent random choice can be roughly outlined as follows (the technical details which we omit here will be considered later in the paper with proper rigor).

(I) Given an ambient bipartite graph \( G = (V_G^{up}, V_G^{down}, E_G) \) with the vertex set \( V_G^{up} \sqcup V_G^{down} \), one constructs a collection \( S (|S| \geq m) \) of vertices in \( V_G^{up} \) such that for an \( 1 - o(1/m) \) fraction of \( d \)-tuples of vertices \( (v_1, \ldots, v_d) \) from \( S \), the number of common neighbors of \( (v_1, \ldots, v_d) \) in \( V_G^{down} \) is at least \( m \). The construction procedure for the set \( S \) with such properties is a variation of the first moment method.

The work is partially supported by the NSF Grant DMS 2054666.
Corollary 1.2. Let $Q_n$ be as in the last theorem. Then for every $n \geq n_0$, the Ramsey number of the hypercube $Q_n$ satisfies $r(Q_n) \leq 2^{2n+\epsilon(n)} + 2$. 

Remark. Our proof shows that one can take $c = 0.03656$ assuming that $n_0$ is sufficiently large. 

As an immediate corollary, we get
Theorem 1.1 utilizes “block” embeddings similar to the above (although more technically involved), and is based on the following alternatives. Given a bipartite graph $G$ on $2^{2n-c_n} + 2^{2n-c_n}$ vertices of density $1/2$, at least one of the following three conditions holds:

(a) The dependent random choice method from (I)–(II) succeeds in embedding the hypercube $Q_n$ into the ambient graph $G$ since the common neighborhoods of $n$–tuples of vertices in the set $S$ from (I) tend to have small overlaps.

(b) $G$ contains a large subgraph of a density significantly larger than $1/2$. In this case, we apply the dependent random choice to that subgraph of $G$ instead of $G$ itself to construct the hypercube embedding, again according to the scheme (I)–(II).

(c) $G$ contains a subgraph having a “block” structure similar to that of the random graph $\Gamma$ from the above example. In this case, we construct a special randomized facet-wise embedding of the hypercube.

The structural part of this trichotomy (not implementing an actual embedding of $Q_n$ and only concerned with the properties of the ambient graph $G$) is formally stated as Proposition 6.3. This proposition is then combined with appropriate embedding procedures to obtain the main result of the note.
Figure 1. An illustration of the “block” embedding of the cube into the bipartite graph $\Gamma$. The lower part of the vertex set of $\Gamma$ is partitioned into blocks, so that within each block every vertex has a same set of neighbors. The cube is embedded into $\Gamma$ so that each collection of vertices $T_b$, $b = \pm 1$ is mapped into a single block. In the picture, the vertices of the cube which belong to $T$ are enlarged, and the dotted ellipses mark the facets of the cube corresponding to the blocks of vertices. The edge density of $\Gamma$ in this illustration is made greater than $1/2$ in view of the small number of blocks (otherwise, no block embedding would be possible).

Below is the outline of the paper.

In Section 2, we revise the notation used in this paper, and recall a standard concentration inequality for independent Bernoulli random variables.

In Section 3, we provide a rigorous description of the dependent random choice technique as outlined in (I)–(II). The material for this section is taken, with some minor alterations, from [5, 6], and is applied to deal with the cases (a) and (b) above.

In Section 4, we introduce the notion of $(p, M)$–condensed common neighborhoods of tuples of graph vertices and show that existence of a non-condensed collection of neighborhoods (with appropriately chosen parameters) guarantees that the dependent random choice method produces an embedding of $Q_n$ into $G$. The main result of this section — Lemma 4.7 — is used to treat the case (a).

In Section 5 we define block-structured bipartite graphs and develop a randomized procedure for embedding the hypercube into graphs of that type. That embedding procedure is crucial for the case (c).

Finally, in Section 6 we state and prove the structural Proposition 6.3 and complete the proof of the main result of the paper.

Acknowledgements. The author would like to thank Han Huang for valuable discussions.

2. Notation and preliminaries

The notation $[m]$ will be used for a set of integers $\{1, 2, \ldots, m\}$. We will denote constants by $c, C$ etc. Sometimes we will add a subscript to a name of a constant to assign it to an appropriate statement within the paper. For example, the universal constant $c_{2.1} \in (0, 1]$ is taken from Lemma 2.1.
We will occasionally use standard asymptotic notations \( o(\cdot) \) and \( \Omega(\cdot) \), for example, for two functions \( f \) and \( g \) on positive integers, \( f(n) = o(g(n)) \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \) and \( f(n) = \Omega(g(n)) \) if \( \lim \inf_{n \to \infty} \frac{|f(n)|}{|g(n)|} > 0 \).

In this note, any bipartite graph \( G \) is viewed as a triple \((V^\text{up}_G, V^\text{down}_G, E_G)\), where \( V^\text{up}_G \) and \( V^\text{down}_G \) are sets of “upper” and “lower” vertices, and \( E_G \) is a collection of edges connecting vertices in \( V^\text{up}_G \) to those in \( V^\text{down}_G \).

Given a bipartite graph \( G = (V^\text{up}_G, V^\text{down}_G, E_G) \), its edge density is defined as the ratio

\[
\frac{|E_G|}{|V^\text{up}_G| |V^\text{down}_G|}.
\]

For a vertex \( v \) in a graph \( G \), the set of neighbors of \( v \) in \( G \) will be denoted by \( \mathcal{N}_G(v) \). Further, given a collection of vertices \( \{v^{(1)}, \ldots, v^{(r)}\} \), the set of their common neighbors will be denoted by \( \mathcal{CN}_G(v^{(1)}, \ldots, v^{(r)}) \).

The hypercube on \( 2^n \) vertices \( \{-1, 1\}^n \), viewed as a bipartite graph, will be denoted by \( Q_n \).

We will need the following standard concentration inequality for independent Bernoulli variables.

**Lemma 2.1** (Chernoff). Let \( b_1, \ldots, b_n \) be i.i.d. (independent and identically distributed) Bernoulli \((p)\) random variables (with \( p \in (0, 1) \)). Then

\[
\mathbb{P}\left\{ \left| \sum_{i=1}^{n} b_i - pn \right| \geq t \right\} \leq 2 \exp\left( - \frac{qt^2}{2pn} \right), \quad t \in (0, pn],
\]

where \( q_{\text{ch}} \in (0, 1) \) is a universal constant.

3. The dependent random choice

In this section, we revise the version of the dependent random choice method outlined in the introduction in (I)–(II). The material of this section, with some minor modifications, is taken from \([5, 6]\). We provide the proofs for completeness.

**Lemma 3.1.** Let \( G' = (V^\text{up}_{G'}, V^\text{down}_{G'}, E_{G'}) \) be a bipartite graph of a density \( \alpha \in (0, 1] \); let \( \beta \in (0, \alpha] \) and let \( r, s \in \mathbb{N} \) be arbitrary parameters. Let \( X_1, X_2, \ldots, X_s \) be i.i.d. uniform elements of \( V^\text{down}_{G'} \). Then

\[
\mathbb{E}\left| \mathcal{CN}_{G'}(X_1, X_2, \ldots, X_s) \right| \geq \alpha^s |V^\text{up}_{G'}|,
\]

and the expected number of ordered \( r \)-tuples of elements of \( \mathcal{CN}_{G'}(X_1, X_2, \ldots, X_s) \) with at most \( \beta^r |V^\text{down}_{G'}| \) common neighbors, is at most

\[
\beta^r s |V^\text{up}_{G'}|^r.
\]

**Proof.** Denote \( A := \mathcal{CN}_{G'}(X_1, X_2, \ldots, X_s) \). We have

\[
\mathbb{E}|A| = \sum_{v \in V^\text{up}_{G'}} \mathbb{P}\{v \in A\} = \sum_{v \in V^\text{up}_{G'}} \mathbb{P}\{X_1 \in \mathcal{N}_{G'}(v)\}^s
\]

\[
= |V^\text{up}_{G'}| \sum_{v \in V^\text{up}_{G'}} \frac{1}{|V^\text{up}_{G'}|} \left( \frac{|\mathcal{N}_{G'}(v)|}{|V^\text{down}_{G'}|} \right)^s
\]

\[
\geq |V^\text{up}_{G'}| \left( \sum_{v \in V^\text{up}_{G'}} \frac{|\mathcal{N}_{G'}(v)|}{|V^\text{up}_{G'}| |V^\text{down}_{G'}|} \right)^s
\]

\[
= |V^\text{up}_{G'}| \alpha^s.
\]
Further, for every \( r \)-tuple of distinct elements \( y_1, \ldots, y_r \) of \( V_{G'}^{\text{up}} \), the probability of the event \( \{ y_1, \ldots, y_r \} \subset A \) equals

\[
\mathbb{P}\{ y_1, \ldots, y_r \in N_{G'}(X_1) \}^s = \left( \frac{\text{set of common neighbors of } y_1, \ldots, y_r}{|V_{G'}^{\text{down}}|} \right)^s.
\]

Thus, the expected number of ordered \( r \)-tuples in \( A \) with at most \( \beta^r |V_{G'}^{\text{down}}| \) common neighbors, is at most

\[
\beta^r s |V_{G'}^{\text{up}}|^r.
\]

\[
\square
\]

As a corollary, we have

**Corollary 3.2.** For every \( \varepsilon \in (0, 1) \) there is \( r_{\varepsilon, \alpha} = r_{\varepsilon, \alpha}(\varepsilon) > 0 \) with the following property. Let \( G' = (V_{G'}^{\text{up}}, V_{G'}^{\text{down}}, E_{G'}) \) be a bipartite graph of density at least \( \alpha \in [\varepsilon, 1] \), let \( n \geq r_{\varepsilon, \alpha} \) and assume that either

\[
|V_{G'}^{\text{up}}| \geq 2^{n+\varepsilon n}, \quad |V_{G'}^{\text{down}}| \geq \frac{2^{n+\varepsilon n}}{\alpha^n}.
\]

or

\[
|V_{G'}^{\text{down}}| \geq 2^{n+\varepsilon n}, \quad |V_{G'}^{\text{up}}| \geq \frac{2^{n+\varepsilon n}}{\alpha^n}.
\]

Then the hypercube \( Q_n \) can be embedded into \( G' \).

**Proof.** Fix any \( \varepsilon \in (0, 1) \). We will assume that \( n \) is large. Let \( G' \) be the graph satisfying the above assumptions. We can suppose without loss of generality that \( |V_{G'}^{\text{up}}| \geq 2^{n+\varepsilon n}, |V_{G'}^{\text{down}}| \geq \frac{2^{n+\varepsilon n}}{\alpha^n} \).

Set

\[
s := \left\lfloor \frac{\log(|V_{G'}^{\text{up}}|/2^n)}{\log(1/\alpha)} \right\rfloor, \quad \beta := \frac{2}{|V_{G'}^{\text{down}}|^{-1/n}}.
\]

Observe that \( \beta \leq 2^{-\varepsilon} \alpha \) and that

\[
\frac{|V_{G'}^{\text{up}}|}{2^n} \geq 2^{\varepsilon n} = \varepsilon^{-\varepsilon n/\log_2(1/\varepsilon)} \geq \frac{1}{\alpha^\varepsilon n/\log_2(1/\varepsilon)}
\]

implying

\[
s \geq \lfloor \varepsilon n/\log_2(1/\varepsilon) \rfloor.
\]

Let \( X_1, X_2, \ldots, X_s \) be i.i.d. uniform elements of \( V_{G'}^{\text{down}} \), and denote \( A := CN_{G'}(X_1, X_2, \ldots, X_s) \). In view of Lemma 3.1, \( \mathbb{E}|A| \geq \alpha^s |V_{G'}^{\text{up}}| \) whereas deterministically \( |A| \leq |V_{G'}^{\text{up}}| \). This implies that with probability at least \( \frac{1}{2} \alpha^s \),

\[
|A| \geq \frac{1}{2} \alpha^s |V_{G'}^{\text{up}}| \geq 2^{n-1}.
\]

Combining this with the second assertion of the lemma, we get that with a positive probability the set \( A \) satisfies (3), and the number of ordered \( n \)-tuples of elements of \( A \) with at most \( \beta^n |V_{G'}^{\text{down}}| \) common neighbors is at most \( 4 \alpha^{-s} \beta^{ns} |V_{G'}^{\text{up}}|^n \). We fix such realization of \( A \) for the rest of the proof.

Let \( (Y_1, \ldots, Y_n) \) be a uniform random ordered \( n \)-tuple of distinct elements in \( A \). In view of the above, the probability that \( (Y_1, \ldots, Y_n) \) have less than \( \beta^n |V_{G'}^{\text{down}}| \) common neighbors, is at most

\[
\frac{4 \alpha^{-s} \beta^{ns} |V_{G'}^{\text{up}}|^n}{|A| (|A| - 1) \cdots (|A| - n + 1)} \leq 4 \alpha^{-s} \beta^{ns} |V_{G'}^{\text{up}}|^n \left( \frac{1}{2} \alpha^s |V_{G'}^{\text{up}}| - n \right) \leq 8 \cdot 2^n \left( \frac{1}{\alpha} \left( \frac{\beta}{\alpha} \right)^n \right)^s,
\]
where in the last inequality we used (3) and our assumption that $n$ is sufficiently large. Further, we have $\frac{1}{\alpha} \left( \frac{\beta}{\alpha} \right)^n \leq \frac{1}{\varepsilon} 2^{-\varepsilon n} < 1$, and, in view of (2),

\begin{equation}
8 \cdot 2^n \left( \frac{1}{\alpha} \left( \frac{\beta}{\alpha} \right)^n \right)^s \leq 8 \cdot 2^n \left( \frac{1}{\varepsilon} 2^{-\varepsilon n} \right)^{\lfloor \varepsilon n / \log_2(1/\varepsilon) \rfloor} < 2^{-n+1},
\end{equation}

whenever $n$ is greater than a large constant multiple of $\varepsilon^{-2} \log_2(1/\varepsilon)$. Set

$\mathcal{T} := \{ v \in \{-1, 1\}^n : \text{vector } v \text{ has an odd number of } -1 \text{'s} \},$

and let $f(\mathcal{T})$ be the uniform random $2^{n-1}$-tuple of distinct elements in $A$. Then, by (4), with a positive probability for every $v \in \{-1, 1\}^n \setminus \mathcal{T}$, the set of vertices $f(N_{Q_n}(v))$ has at least $\beta^n |V_{G'}^{\downarrow}| \geq 2^{n-1}$ common neighbors in $G'$. It follows that with a positive probability the mapping $f : \mathcal{T} \to A$ can be extended to an embedding of $Q_n$ into $G'$.

The last statement immediately implies the bound $r(Q_n) \leq 2^{2n+o(n)}$ via the standard construction which we already mentioned in the introduction; see Remark 1.3 there.

4. Condensed common neighborhoods

The goal of this section is to investigate graph embeddings using dependent random choice in the setting when common neighborhoods of subsets of vertices of the host graph tend to have small overlaps.

Lemma 3.1 gives a probabilistic description of the set of common neighbors of i.i.d. uniform random vertices in a bipartite graph. We start by considering a de-randomization of the lemma similar to the one in the proof of Corollary 3.2. The de-randomization is accomplished by an application of Markov’s inequality and a union bound estimate. For convenience, we introduce a technical definition of a standard vertex pair which groups together the properties useful for us.

**Definition 4.1.** Let $G' = (V_{G'}^{\uparrow}, V_{G'}^{\downarrow}, E_{G'})$ be a bipartite graph of density at least $\alpha \in (0, 1)$, and let $\alpha_0 \in (0, \alpha)$, $\mu \in (0, \alpha_0/2]$, $r \in \mathbb{N}$, and $K > 0$. An ordered pair $(v_1, v_2)$ of [not necessarily distinct] vertices in $V_{G'}^{\downarrow}$ is $(\alpha_0, \alpha, \mu, r, K)$-standard if the following is true:

- The number of common neighbors of $v_1, v_2$ in $G'$ is at least $(1 - \mu)\alpha^2 |V_{G'}^{\uparrow}|$;
- For every $1 \leq k \leq |\mathcal{CN}_{G'}(v_1, v_2)|$, $m \geq 1$, and for every finite collection $(I_j)_{j=1}^m$ of subsets of $[k]$ satisfying $|I_j| = r$, $1 \leq j \leq m$, we have: if $(Y_i)_{i=1}^k$ is a random $k$-tuple of vertices in $\mathcal{CN}_{G'}(v_1, v_2)$ uniformly distributed on the set of $k$-tuples of distinct vertices in $\mathcal{CN}_{G'}(v_1, v_2)$ then with probability at least $1/2$,

$\left| \{1 \leq j \leq m : |\mathcal{CN}_{G'}(Y_i, i \in I_j)| \leq \beta r |V_{G'}^{\downarrow}| \} \right| \\
\leq m \cdot K^{2r} \alpha^{-2r}$ for every $\beta \in [\alpha_0, \alpha]$.

**Remark 4.2.** Regarding the first part of the above definition, note that, according to Lemma 3.1, the expected number of common neighbors for a random pair of vertices $v_1, v_2 \in V_{G'}^{\downarrow}$ is at least $\alpha^2 |V_{G'}^{\uparrow}|$. The extra multiple $1 - \mu$ controls acceptable deviation from the expected cardinality.

**Remark 4.3.** The second condition can be roughly interpreted as “the size of a common neighborhood $\mathcal{CN}_{G'}(y_1, \ldots, y_r)$, for $y_1, \ldots, y_r \in \mathcal{CN}_{G'}(v_1, v_2)$, is typically of order at least $\Omega(K^{-1/2} \alpha^r |V_{G'}^{\downarrow}|)$.” Again, according to Lemma 3.1, for a random $r$-tuple of elements in $V_{G'}^{\uparrow}$ their common neighborhood in $V_{G'}^{\downarrow}$ has expected cardinality at least $\alpha^r |V_{G'}^{\downarrow}|$. The rather complicated second part of the definition involving the collections $(I_j)_{j=1}^m$ will turn out convenient when dealing with a standard vertex pair in the proof of Lemma 4.7. Similarly to the first condition, we introduce a parameter $(K)$ to control the deviation of the actual set size from the expected cardinality in uniform random setting.
Lemma 4.4 (Existence of standard pairs of vertices). For every $\alpha_0 \in (0,1)$ and $\mu \in (0, \alpha_0/2]$ there is $C_G = C_G(\alpha_0, \mu) \geq 1$ with the following property. Let $\alpha \in (0,1]$, $r \geq 2$, and let $G' = (V_G^{up}, V_G^{down}, E_G')$ be a bipartite graph of density at least $\alpha$. Assume that

$$\alpha^2 |V_G^{up}| \geq r^2.$$ 
(5)

Then there exists an $(\alpha, \alpha, \mu, r, C_G^{(3)})$–standard ordered pair $(v_1, v_2)$ in $V_G^{down}$.

Proof. Set

$$\tilde{\delta} := \frac{\beta}{r}.$$ 

Let $X_1, X_2$ be i.i.d. uniform random elements of $V_G^{down}$, and set $A := \mathcal{CN}_G(X_1, X_2)$. According to Lemma 3.1, we have

$$\mathbb{E}|A| \geq \alpha^2 |V_G^{up}|,$$

whereas at the same time clearly $|A| \leq |V_G^{up}|$ deterministically. Thus,

$$|V_G^{up}| \mathbb{P}\{|A| \geq (1 - \tilde{\delta})\alpha^2 |V_G^{up}|\} + (1 - \tilde{\delta})\alpha^2 |V_G^{up}| (1 - \mathbb{P}\{|A| \geq (1 - \tilde{\delta})\alpha^2 |V_G^{up}|\}) \geq \alpha^2 |V_G^{up}|,$$

implying

$$\mathbb{P}\{|A| \geq (1 - \tilde{\delta})\alpha^2 |V_G^{up}|\} \geq \tilde{\delta}\alpha^2.$$ 

Further, let $\beta_{\ell} := \alpha \left(1 - \frac{\alpha - \alpha_0}{\alpha}\right)$, $\ell = 0, \ldots, r - 1$, and let $L := 2(\delta \alpha^2)^{-1}$. Denote by $\mathcal{E}$ the event that for every $\ell = 0, \ldots, r - 1$ the number of ordered $r$–tuples of elements of $A$ with at most $\beta_{\ell}^r |V_G^{down}|$ common neighbors, is at most

$$Lr\beta_{\ell}^{2r} |V_G^{up}| r.$$

Applying Lemma 3.1 together with Markov’s inequality, we get that the probability of the intersection of events $\mathcal{E} \cap \{|A| \geq (1 - \tilde{\delta})\alpha^2 |V_G^{up}|\}$ is at least $\delta \alpha^2 - \frac{1}{4} > 0$.

It remains to check that, conditioned on $\mathcal{E} \cap \{|A| \geq (1 - \tilde{\delta})\alpha^2 |V_G^{up}|\}$, the pair $(X_1, X_2)$ is $(\alpha, \alpha, \mu, r, C' Lr^2)$–standard for some $C' = C'(\alpha_0) > 0$. For the rest of the proof, we fix a realization of $X_1, X_2$ from $\mathcal{E} \cap \{|A| \geq (1 - \tilde{\delta})\alpha^2 |V_G^{up}|\}$. Pick any $1 \leq k \leq |A|, m \geq 1$, and any finite collection $(I_j)_{j=1}^m$ of subsets of $[k]$ satisfying $|I_j| = r, 1 \leq j \leq m$. Further, let $(Y_i)_{i=1}^k$ be a random $k$–tuple of vertices in $A$ uniformly distributed on the set of $k$–tuples of distinct vertices in $A$. In view of our conditions on $X_1, X_2$, (5), and the definition of $\delta$, we have for every $j \leq m,$

$$\mathbb{P}\{\mathcal{CN}_{G'}(Y_i, i \in I_j) \leq \beta_{\ell}^r |V_G^{down}|\} \leq \frac{Lr\beta_{\ell}^{2r} |V_G^{up}| r}{|A| \cdot (|A| - 1) \cdots (|A| - r + 1)} \leq \tilde{C}r\beta_{\ell}^{2r} \alpha^{-2r}$$

for some universal constant $\tilde{C} > 0$, whence

$$\mathbb{E}\{1 \leq j \leq m : \mathcal{CN}_{G'}(Y_i, i \in I_j) \leq \beta_{\ell}^r |V_G^{down}|\} \leq \tilde{C} \cdot Lr\beta_{\ell}^{2r} \alpha^{-2r}, 0 \leq \ell \leq r - 1.$$ 

Applying Markov’s inequality (this time with respect to the randomness of $Y_i$’s), we get that with [conditional] probability at least $1/2$,

$$\{1 \leq j \leq m : \mathcal{CN}_{G'}(Y_i, i \in I_j) \leq \beta_{\ell}^r |V_G^{down}|\} \leq m \cdot 2\tilde{C}r Lr^2 \beta_{\ell}^{2r} \alpha^{-2r}, 0 \leq \ell \leq r - 1.$$ 

Finally, assuming that (6) holds, take any $\beta \in [\alpha_0, \alpha]$, and let $\ell \in \{0, \ldots, r - 1\}$ be the largest index such that $\beta_{\ell} \geq \beta$. Note that $\beta_{\ell} \leq \beta + \frac{\alpha - \alpha_0}{\alpha} \leq \left(1 + \frac{\alpha - \alpha_0}{\alpha}\right)\beta$. Then

$$\{1 \leq j \leq m : \mathcal{CN}_{G'}(Y_i, i \in I_j) \leq \beta_{\ell}^r |V_G^{down}|\} \leq \{1 \leq j \leq m : \mathcal{CN}_{G'}(Y_i, i \in I_j) \leq \beta_{\ell}^r |V_G^{down}|\} \leq m \cdot 2\tilde{C}r Lr^2 \beta_{\ell}^{2r} \alpha^{-2r} \leq m \cdot C' Lr^2 \beta_{\ell}^{2r} \alpha^{-2r},$$

for some $C' = C'(\alpha_0) > 0$. The result follows. ☐
Next, we discuss the main notion of this section.

**Definition 4.5.** Let $G' = (V_{G'}^\up, V_{G'}^\down, E_{G'})$ be a bipartite graph, and let $r \in \mathbb{N}$, $M > 0$, $p \in [0, 1]$ be parameters. Further, let $(v_1, v_2)$ be an ordered pair of vertices in $V_{G'}^\down$ with a non-empty set of common neighbors. We say that the collection \( \{CN_{G'}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G'}(v_1, v_2)^r \} \) is \((p, M)\)-condensed if, letting $Y_1, \ldots, Y_r$ be i.i.d. uniform random elements of $CN_{G'}(v_1, v_2)$, we have

\[
P\{ |CN_{G'}(Y_1, \ldots, Y_r) \cap CN_{G'}(\tilde{Y}_1, \ldots, \tilde{Y}_r) | \geq M \} \geq p.
\]

**Remark 4.6.** The above definition will be applied to standard pairs of vertices $(v_1, v_2)$, i.e. in the setting when a typical common neighborhood $CN_{G'}(y_1, \ldots, y_r)$ has size of order at least \( \Omega(K^{-1/2}\alpha^r|V_{G'}^\down|) \) (for an appropriate choice of the parameter $K$), where $\alpha$ is the edge density of $G'$. For $M$ much less than $\alpha^r|V_{G'}^\down|$ and for $p = o(1)$, the assertion that the collection \( \{CN_{G'}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G'}(v_1, v_2)^r \} \) is not \((p, M)\)-condensed implies that the neighborhoods $CN_{G'}(y_1, \ldots, y_r)$ typically have small overlaps.

**Lemma 4.7** (Embedding into a graph comprising a non-condensed set of common neighborhoods). Let $G' = (V_{G'}^\up, V_{G'}^\down, E_{G'})$ be a bipartite graph of density at least $\alpha \in (0, 1]$, and assume that parameters $0 < \alpha_0 < \alpha$, $\mu \in (0, \alpha_0/2]$, $\gamma \geq 3$ and $m \in \mathbb{N}$ satisfy $\alpha^2|V_{G'}^\up| \geq 2\max(m, 2\alpha^r|V_{G'}^\down|)$, $\alpha_0^2|V_{G'}^\down| \leq 1$, and

\[
\frac{1}{4 \max(m, 2\alpha^r|V_{G'}^\down|)} \cdot \frac{\alpha_0^\gamma}{\alpha_0^\gamma} \geq \alpha_0^r,
\]

where $G_{\alpha_0} = G_{\alpha_0}(\alpha_0, \mu)$ is taken from Lemma 4.4. Assume further that $(v_1, v_2)$ is an ordered pair of vertices in $V_{G'}^\down$ which is $(\alpha_0, \mu, r, G_{\alpha_0}^r)$-standard. Assume that the collection \( \{CN_{G'}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G'}(v_1, v_2)^r \} \) is not \((p, M)\)-condensed where the parameters $p \in [0, 1]$ and $M > 0$ satisfy

\[
r^2 \leq p \cdot \alpha^r|V_{G'}^\down|,
\]

\[
\sqrt{\beta} \leq \frac{\alpha_0^\gamma|V_{G'}^\down|^2}{16 \cdot 3^7G_{\alpha_0}^r \max(m^2, 4\alpha_0^r|V_{G'}^\down|)} \cdot 20 \log |V_{G'}^\down|
\]

and

\[
1 \leq M \leq \frac{2^9G_{\alpha_0}^2 \max(m^3, 8\alpha_0^r|V_{G'}^\down|^3)}{2^9G_{\alpha_0}^4 \max(m^3, 8\alpha_0^r|V_{G'}^\down|^3) \gamma^6 \cdot 400 \log^2 |V_{G'}^\down|},
\]

where the constant $\beta$ is taken from Lemma 2.4. Further, let $H = (V_H^\up, V_H^\down, E_H)$ be an $r$-regular bipartite graph on $m + m$ vertices. Then $H$ can be embedded into $G'$.

**Remark 4.8.** Observe that the lemma does not require any structural assumptions on the graph $H$ except for the regularity and bounds on the size of the vertex set. Moreover, one may consider versions of this lemma which operate under the only assumptions on the vertex set cardinality and the maximum degree, without the regularity requirement. We omit the discussion of such generalizations in order not to complicate the exposition further.

Before providing a proof of the lemma, let us discuss our construction of the graph embedding rather informally. Let $(v_1, v_2)$ be a standard pair of vertices in $V_{G'}^\down$ satisfying the above assumptions. For convenience, we label vertices of $H$ as $t^\up_1, \ldots, t^\up_m$, $t^\down_1, \ldots, t^\down_m$, and let $I_j$, $1 \leq j \leq m$, be $r$-subsets of $[m]$ corresponding to neighbors of vertices in $V_H^\down$, i.e. $N_H(t^\down_j) = \{t^\up_i, i \in I_j \}$ for every $j \leq m$. The embedding of $H$ is accomplished by mapping the vertices $V_H^\up$ into $V_{G'}^\up$ using the first moment argument, and then embedding $V_H^\down$ into $V_{G'}^\down$ one vertex at a time via a combination of deterministic and probabilistic reasoning.

The first moment method is applied to produce an $m$-tuple of distinct vertices $(y_1, y_2, \ldots, y_m)$ in $CN_{G'}(v_1, v_2)$, such that
• The number of indices \( j \leq m \) with \(|CN_{G'}(y_i, i \in I_j)|\) much smaller than \( \alpha r|V_{G'}^{down}|\) is much smaller than \( m \), and
• The pairwise intersections \( CN_{G'}(y_i, i \in I_{j_1}) \cap CN_{G'}(y_i, i \in I_{j_2}) \) have size less than \( M \) for a vast majority of pairs of indices \((j_1, j_2) \in [m]^2\).

We refer to the proof below for quantitative bounds. The vertex subset \( V_{H}^{up} \) is then mapped to \( y_1, y_2, \ldots, y_m \). This provides a satisfactory starting point for the embedding since, by the above conditions, the common neighborhoods of \( \{y_i, i \in I_j\}, j \leq m \), tend to have relatively large cardinalities and small pairwise overlaps. At this stage, the goal is to map \( V_{G'}^{down} \) into a collection of distinct vertices \( f(t_{down}^{j_1}), \ldots, f(t_{down}^{j_m}) \) from \( V_{G'}^{down} \) such that \( f(t_{down}^{j_i}) \) is contained in the common neighborhood of \( \{y_i, i \in I_j\} \) for every \( j \leq m \). We split the index set \([m]\) into three subsets \( Q, W, R \) according to properties of the common neighborhoods \( CN_{G'}(y_i, i \in I_j) \) as follows:

• \( Q \) is a set of indices \( j \) such that \(|CN_{G'}(y_i, i \in I_j)|\) is very small;
• \( W \) is the subset of all \( j \in [m] \setminus Q \) such that the common neighborhood \( CN_{G'}(y_i, i \in I_j) \) has large overlaps with many other sets \( CN_{G'}(y_i, i \in I_j') \), \( j' \neq j \);
• \( R \) is the complement of \( Q \) and \( W \) in \([m]\), the set of “regular” indices.

Both \( Q \) and \( W \) are small (and possibly empty), and \( t_{down}^{j_i}, j \in Q \cup W \), are embedded in the first place using a deterministic argument. Strong upper bounds on sizes of \( Q \) and \( W \) guarantee that the embedding does not fail at this point. The embedding of \( t_{down}^{j_i}, j \in R \), is then accomplished via a randomized construction.

We now turn to the rigorous argument.

**Proof of Lemma** [4,7] For better readability, we split the proof into blocks.

**An assumption on \( m \).** We claim that without loss of generality we can assume that the parameter \( m \) satisfies

\[
(10) \quad m \geq \alpha r|V_{G'}^{down}|.
\]

Indeed, suppose that the lemma is proved under the extra assumption (10). Let \( \tilde{m} < \alpha r|V_{G'}^{down}| \) be any positive integer, let \( \tilde{H} \) be a bipartite \( r \)-regular graph on \( \tilde{m} + \tilde{m} \) vertices, and suppose that the parameters \( \alpha, \alpha_0, \mu, r, p, M \) and an ordered pair \((v_1, v_2)\) all satisfy the assumptions of the lemma, with \( m \) replaced with \( \tilde{m} \) in (7), (8), and (9). We want to show that \( \tilde{H} \) can be embedded into \( G' \). Define \( m \) as the smallest integer multiple of \( \tilde{m} \) satisfying (10) and observe that \( m \leq 2\alpha r|V_{G'}^{down}| \) implying that \( \max(m, 2\alpha r|V_{G'}^{down}|) = \max(\tilde{m}, 2\alpha r|V_{G'}^{down}|) = 2\alpha r|V_{G'}^{down}| \) and hence \( m \) satisfies (7), (8), and (9). Define a bipartite graph \( H \) on \( m + m \) vertices as the disjoint union of \( m/\tilde{m} \) copies of the graph \( \tilde{H} \). By our assumption, \( H \) can be embedded into \( G' \), and therefore the same is true for \( \tilde{H} \). This proves the claim. For the rest of the proof, the parameter \( m \) is supposed to satisfy (10).

**Choosing an \( m \)-tuple of vertices in \( V_{H}^{up} \).** Let \((t_{up}^{1}, \ldots, t_{up}^{m})\) and \((t_{down}^{1}, \ldots, t_{down}^{m})\) be the vertices of \( H \) from \( V_{H}^{up} \) and \( V_{H}^{down} \), respectively, ordered arbitrarily. As in the proof outline above, for every \( 1 \leq j \leq m \), we let \( I_j \) be the collection of indices in \([m]\) such that \( N_H(t_{down}^{j_i}) = \{t_{up}^{i}, i \in I_j\} \) (we observe that, in view of \( r \)-regularity of \( H \), the number of all ordered pairs of indices \((j_1, j_2) \in [m]^2\) such that \( I_{j_1} \cap I_{j_2} = \emptyset \), is at least \( m \cdot (m - r^2) \)).

Let \((Y_i)_{i=1}^{m}\) be a random \( m \)-tuple of vertices in \( CN_{G'}(v_1, v_2) \) uniformly distributed on the set of \( m \)-tuples of distinct vertices in \( CN_{G'}(v_1, v_2) \) (the condition that \((v_1, v_2)\) is standard and our assumption on \( |V_{G'}^{up}| \) imply that \(|CN_{G'}(v_1, v_2)| \geq m \) so that \((Y_i)_{i=1}^{m}\) are well defined). In view of the definition of \((p, M)\)-condensation combined with the last observation, we have

\[
E \left| \left\{(j_1, j_2) \in [m]^2 : |CN_{G'}(Y_i, i \in I_{j_1}) \cap CN_{G'}(Y_i, i \in I_{j_2})| \geq M \right\} \right| \leq m \cdot r^2 + \frac{p \cdot m^2}{q},
\]
where $\theta \in (0,1)$ is the probability that $2r$ i.i.d. uniform random elements of $\mathcal{CN}_{G^r}(v_1, v_2)$ are all distinct. Since $|\mathcal{CN}_{G^r}(v_1, v_2)| \geq m \geq \sqrt{r}$ in view of the first inequality in (8) and (10), we have $m \cdot \sqrt{r} \leq 2 \cdot m^2$ and

$$\theta \geq \left(1 - \frac{2}{r}\right)^2 \geq 3^{-6}.$$ 

Markov’s inequality and the definition of a standard vertex pair then imply that with a positive probability the collection $(Y_i)_{i=1}^m$ satisfies all of the following:

(a) $\left|\{1 \leq j \leq m : |\mathcal{CN}_{G^r}(Y_i, i \in I_j)| \leq \beta^r|V_{G^r}|\right| \leq m \cdot \beta^r \alpha^{-2r}$ for every $\beta \in \left[0, \alpha\right]$;

(b) $|\mathcal{CN}_{G^r}(Y_i, i \in I_{j_1}) \cap \mathcal{CN}_{G^r}(Y_i, i \in I_{j_2})| \geq M$ for at most $3^7p \cdot m^2$ pairs of indices $(j_1, j_2) \in [m]^2$.

For the rest of the proof, we fix a realization $(y_1, \ldots, y_m)$ of $(Y_i)_{i=1}^m$ satisfying the above conditions. We shall construct an embedding $f$ of $H$ into $G$ which maps each $t_i$ into $y_i$, $1 \leq i \leq m$.

**Partitioning the set of indices $[m]$.** Define $\beta_0$ via the relation

$$\beta_0^r := \frac{1}{4} \frac{\alpha^{2r}|V_{G^r}|}{m \cdot C_3^r},$$

and observe that in view of (7) and (10), $\alpha_0 \leq \beta_0 \leq \alpha$, and hence, by the condition (a),

$$|\{1 \leq j \leq m : |\mathcal{CN}_{G^r}(y_i, i \in I_j)| \leq \beta_0^r|V_{G^r}|\}| \leq m \cdot C_3^r \beta_0^r \alpha^{-2r}.$$ 

Let $Q$ be the set of all indices $1 \leq j \leq m$ with $|\mathcal{CN}_{G^r}(y_i, i \in I_j)| \leq \beta_0^r|V_{G^r}|$, so that

$$|Q| \leq m \cdot C_3^r \beta_0^r \alpha^{-2r} = \frac{1}{16} \frac{\alpha^{2r}|V_{G^r}|^2}{m \cdot C_3^r} \leq \alpha^r |V_{G^r}|,$$

where the last inequality follows from (10). Assume that $(q_s)_{s=1}^{|Q|}$ is an ordering of $Q$ such that $|\mathcal{CN}_{G^r}(y_i, i \in I_{q_s})| \leq |\mathcal{CN}_{G^r}(y_i, i \in I_{q_{s+1}})|$, $1 \leq s < |Q|$.

Further, let $W$ be the set of all indices $1 \leq j \leq m$ such that

- $|\mathcal{CN}_{G^r}(y_i, i \in I_j)| > \beta_0^r|V_{G^r}|$ and
- $|\mathcal{CN}_{G^r}(y_i, i \in I_j) \cap \mathcal{CN}_{G^r}(y_i, i \in I_j)| \geq M$ for at least $\sqrt{p} \cdot m$ indices $\tilde{j} \in [m]$.

(note that in view of the condition (b) above, $|W| \leq 3^7 \sqrt{p} m$), and let $(w_s)_{s=1}^{|W|}$ be an arbitrary ordering of the vertices from $W$. Finally, we let $R := [m] \setminus (Q \cup W)$, i.e. $R$ is the set of indices $1 \leq j \leq m$ such that

- $|\mathcal{CN}_{G^r}(y_i, i \in I_j)| > \beta_0^r|V_{G^r}|$ and
- $|\mathcal{CN}_{G^r}(y_i, i \in I_j) \cap \mathcal{CN}_{G^r}(y_i, i \in I_j)| \geq M$ for less than $\sqrt{p} \cdot m$ indices $\tilde{j} \in [m]$.

Similarly, let $(r_s)_{s=1}^{|R|}$ be an arbitrary ordering of the vertices in $R$.

**A deterministic embedding of $t_j$, $j \in Q \cup W$.** We define $f(t_j)$, $j \in Q \cup W$ via a simple iterative procedure comprised of $|Q| + |W|$ steps. A $s$–th step,

- If $1 \leq s \leq |Q|$ then we let $f(t_{q_s})$ to be any point in $\mathcal{CN}_{G^r}(y_i, i \in I_{q_s}) \setminus \{f(t_{q_1}), \ldots, f(t_{q_{s-1}})\}$;
- If $|Q| + 1 \leq s \leq |Q| + |W|$ then we define $f(t_{w_{s-|Q|}})$ as an arbitrary point in $\mathcal{CN}_{G^r}(y_i, i \in I_{w_{s-|Q|}}) \setminus \{f(t_{q_1}), \ldots, f(t_{q_{|Q|}}); f(t_{w_1}), \ldots, f(t_{w_{s-|Q|}-1})\}$.
To make sure that the above process does not fail, we need to verify that at each step \( s, 1 \leq s \leq |Q| \), the set \( \mathcal{CN}_{G'}(y_i, i \in I_q) \setminus \{ f(t_{q_1}^{\text{down}}), \ldots, f(t_{q_{s-1}}^{\text{down}}) \} \) is necessarily non-empty, and similarly, \( \mathcal{CN}_{G'}(y_i, i \in I_{w_{s-1}(Q)}) \setminus \{ f(t_{q_1}^{\text{down}}), \ldots, f(t_{q_{s-1}}^{\text{down}}) \} \setminus \{ f(t_{q_s}^{\text{down}}) \} \neq \emptyset \) for every \( |Q| + 1 \leq s \leq |Q| + |W| \), regardless of the specific choices for \( f(t_j^{\text{down}}) \) at previous steps.

Observe that the condition
\[
\mathcal{CN}_{G'}(y_i, i \in I_q) \setminus \{ f(t_{q_1}^{\text{down}}), \ldots, f(t_{q_{s-1}}^{\text{down}}) \} = \emptyset \quad \text{for some} \ 1 \leq s \leq |Q|,
\]

Together with our choice of the ordering \( (q_s)_{s=1}^{|Q|} \), would imply that
\[
|\mathcal{CN}_{G'}(y_i, i \in I_j)| \leq s \quad \text{for at least} \ s \ \text{indices} \ j \in [m].
\]

For \( s < \alpha_0^r |V_G^{\text{down}}| \), this would imply that the set \( \{ 1 \leq j \leq m : |\mathcal{CN}_{G'}(Y_i, i \in I_j) \leq \alpha_0 |V_G^{\text{down}}| \} \) is non-empty which would contradict the condition (a) and the inequality
\[
m \cdot C_4 r^3 \alpha_0^2 r^2 \alpha^{-2r} \leq m \cdot C_4 r^3 \alpha_0 \frac{\alpha^{-2r} |V_G^{\text{down}}|}{|V_G^{\text{down}}|} < 1,
\]

Which follows from (7) and the assumption \( \alpha_0^r |V_G^{\text{down}}| \leq 1 \). On the other hand, for \( \alpha_0^r |V_G^{\text{down}}| \leq 1 \), combined with condition (a) and the cardinality estimate for \( Q \) implies \( s \leq m \cdot C_4 r^3 \frac{s^2}{|V_G^{\text{down}}|^2} \alpha^{-2r} \), which, together with (11), yields
\[
m \cdot C_4 r^3 \alpha_0 \frac{\alpha^{-2r} |V_G^{\text{down}}|}{|V_G^{\text{down}}|} \geq |Q| \geq \alpha_0^r |V_G^{\text{down}}|^2 \frac{s \alpha^{-2r}}{m \cdot C_4 r^3} \alpha^{-2r}.
\]

Again leading to contradiction in view of the definition of \( \beta_0 \). Thus, the process defined above cannot fail at any step \( 1 \leq s \leq |Q| \).

To verify that our embedding process does not fail at steps \( s \in \{ |Q| + 1, \ldots, |Q| + |W| \} \), we note that, in view of the definition of \( \beta_0 \) and the second inequality in (8),
\[
\beta_0^r |V_G^{\text{down}}| \geq 2m \cdot C_4 r^3 \beta_0 \alpha^{-2r} + 2 \cdot 3^2 \sqrt{p} m \geq 2|Q| + 2|W|.
\]

**A randomized embedding of vertices** \( t_j^{\text{down}}, j \in R \). To complete construction of our embedding \( f \), it remains to define \( f(t_{s-1}), s = 1, \ldots, |R| \). Set
\[
h := \left\lceil \frac{10}{\log |V_G^{\text{down}}|} \right\rceil.
\]

For every \( j \in R \), let \( Z_{j_1}, \ldots, Z_{j_h} \) be uniform random vertices in
\[
\mathcal{CN}_{G'}(y_i, i \in I_j) \setminus f(Q \cup W)
\]

(Note that in view of (13), the set difference is non-empty and, moreover, \( |\mathcal{CN}_{G'}(y_i, i \in I_j) \setminus f(Q \cup W)| \geq \frac{1}{2} |\mathcal{CN}_{G'}(y_i, i \in I_j)| \), and assume that \( Z_{j_1}, \ldots, Z_{j_h} \), \( j \in R \), are mutually independent.

We then define \( f(t_{s-1}), s = 1, \ldots, |R| \), as any \( |R| \)-tuple of distinct vertices in \( V_G^{\text{down}} \) satisfying \( f(t_{s-1}) \in \{ Z_{j_1}, \ldots, Z_{j_h} \} \) for each \( s = 1, \ldots, |R| \), whenever such vertex assignment is possible, and declare failure otherwise. To complete the proof, we must verify that this vertex assignment succeeds with a positive probability. Note that a sufficient condition of success is
\[
\{ Z_{s-1}, \ldots, Z_{s_h} \} \setminus \bigcup_{\tilde{s}=1}^{s-1} \{ Z_{\tilde{s}+1}, \ldots, Z_{\tilde{s}_h} \} \neq \emptyset, \ s = 1, \ldots, |R|.
\]

Pick any \( s \in \{ 1, \ldots, |R| \} \), and let \( L_s \) be the collection of all indices \( \tilde{s} \in \{ 1, \ldots, s - 1 \} \) such that \( |\mathcal{CN}_{G'}(y_i, i \in I_{t_{s-1}}) \cap \mathcal{CN}_{G'}(y_i, i \in I_{t_{s-1}})| \geq M \). In view of the definition of \( R \) as the complement of \( Q \cup W \), we have \( |L_s| < \sqrt{p} m \). For every \( \tilde{s} \in \{ 1, \ldots, s - 1 \} \setminus L_s \), let \( b_{\tilde{s}} \) be the indicator of the event
\[
\{ Z_{\tilde{s}+1}, \ldots, Z_{\tilde{s}_h} \} \cap \mathcal{CN}_{G'}(y_i, i \in I_{t_{s-1}}) \neq \emptyset.
\]
We have, in view of (13),

\[ P\{b_s = 1\} \leq h \cdot \frac{2M}{\beta_0^r |V_d|}, \]

whence, by Chernoff’s inequality (Lemma 2.1),

\[ P\left\{ \sum_{\tilde{s} \in \{1, \ldots, s-1\}\setminus L_s} b_{\tilde{s}} \geq 4hm \cdot M \right\} \leq 2 \exp \left( - \frac{2c(1)m \cdot M}{\beta_0^r |V_d|} \right). \]

We conclude that with probability at least \( 1 - 2 \exp \left( - \frac{2c(1)m \cdot M}{\beta_0^r |V_d|} \right) \), we have

\[ |CN_{G'}(y_i, i \in I_{r_s}) \cap \bigcup_{\tilde{s}=1}^{s-1} \{Z_{r_{\tilde{s}1}}, \ldots, Z_{r_{\tilde{s}h}}\}| \leq \frac{4h^2m \cdot M}{\beta_0^r |V_d|} + h \sqrt{p} m \leq \frac{1}{4} \beta_0^r |V_d|, \]

where the last inequality follows as a combination of (8) and (9). On the other hand, conditioned on the last estimate, we have that

\[ \{Z_{r_{11}}, \ldots, Z_{r_{k1}}\} \setminus \bigcup_{\tilde{s}=1}^{s-1} \{Z_{r_{\tilde{s}1}}, \ldots, Z_{r_{\tilde{s}h}}\} \neq \emptyset \]

with [conditional] probability at least \( 1 - (3/4)^h \). Thus, (14) holds with probability at least

\[ 1 - m \cdot \left( (3/4)^h + 2 \exp \left( - \frac{2c(1)m \cdot M}{\beta_0^r |V_d|} \right) \right) > 0, \]

where in the last inequality we used the definition of \( h, \beta_0, \) and (10). The proof is complete. \( \square \)

5. Embedding into block-structured graphs

In this section, we consider a special class of bipartite graphs whose structure is similar to the graph \( \Gamma \) from the introduction.

**Definition 5.1.** Let \( G' = (V_{G'}^{up}, V_{G'}^{down}, E_{G'}) \) be a bipartite graph. We say that \( G' \) is block-structured with parameters \((\delta, \gamma, k, g)\) if there is a partition

\[ V_{G'}^{down} = \bigsqcup_{\ell=1}^{k} S_{\ell}^{down} \]

of \( V_{G'}^{down} \) into non-empty sets, and a collection of non-empty subsets \( S_{\ell}^{up} \subset V_{G'}^{up}, \ell = 1, \ldots, k, \) having the following properties:

- For each \( \ell = 1, \ldots, k, |S_{\ell}^{down}| = g; \)
- For each \( \ell = 1, \ldots, k, |S_{\ell}^{up}| \geq \gamma |V_{G'}^{up}|; \)
- For each \( \ell = 1, \ldots, k, \) vertices in \( S_{\ell}^{down} \) are not adjacent to any of the vertices in \( V_{G'}^{up} \setminus S_{\ell}^{up}; \)
- For each \( \ell = 1, \ldots, k, \) the density of the bipartite subgraph of \( G' \) induced by the vertex subset \( S_{\ell}^{up} \sqcup S_{\ell}^{down}, \) is at least \( 1 - \delta. \)

We will further say that collections of subsets \((S_{\ell}^{up})_{\ell=1}^{k}, (S_{\ell}^{down})_{\ell=1}^{k}\) satisfying the above properties are compatible with the block-structured graph \( G' \). Note that the compatible collections may not be uniquely defined; in what follows for every block-structured bipartite graph (with given parameters) we arbitrarily fix a pair of compatible collections \((S_{\ell}^{up})_{\ell=1}^{k}, (S_{\ell}^{down})_{\ell=1}^{k}\), and refer to them as the compatible subsets.

**Remark 5.2.** Note that from the above definition it follows that \( |V_{G'}^{down}| = k g. \)
Definition 5.3. Let $r, w, u \geq 1$, let $G' = (V^\text{up}_{G'}, V^\text{down}_{G'}, E_{G'})$ be a block-structured bipartite graph (with some parameters $(\delta, \gamma, k, g)$), and let $(S^\text{up}_{G'})_{\ell=1}^k, (S^\text{down}_{G'})_{\ell=1}^k$ be the corresponding compatible sequences of subsets of vertices of $G'$. Further, let $y_1, \ldots, y_u \in V^\text{up}_{G'}$ (some of the vertices may repeat). We define
\[ M_{G'}(r, w; y_1, \ldots, y_u) \]
as the collection of all ordered $r$–tuples $(x_1, \ldots, x_r)$ of elements of $V^\text{down}_{G'}$ satisfying the following conditions:
- The vertices $x_1, \ldots, x_r$ are distinct;
- There are distinct indices $1 \leq \ell_0, \ell_1, \ldots, \ell_w \leq k$ such that $x_1, \ldots, x_{r-w} \in S^\text{down}_{\ell_0}$ and for every $1 \leq a \leq w$, $x_{r-w+a} \in S^\text{down}_{\ell_a}$;
- $\{x_1, \ldots, x_r\} \subset CN_{G'}(y_1, \ldots, y_u)$.

Note that in the notation for $M_{G'}(\ldots)$ we omit the parameters $\delta, \gamma, k, g$ for brevity.

Lemma 5.4 (Dependent random choice for block-structured graphs). Let $G' = (V^\text{up}_{G'}, V^\text{down}_{G'}, E_{G'})$ be a block-structured bipartite graph with parameters $(\delta, \gamma, k, g)$, and let $r, w, u \geq 1$ with $g \geq r - w \geq 2$ and $k \geq w + 1$. Let $(S^\text{up}_{G'})_{\ell=1}^k, (S^\text{down}_{G'})_{\ell=1}^k$ be the corresponding compatible sequences of subsets of vertices of $G'$. Further, assume that $Y_1, \ldots, Y_u$ are i.i.d. uniform random elements of $V^\text{up}_{G'}$. Then
\[ \mathbb{P}\{|CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{G'}| \geq g(1 - \delta)^{u+1} \]
for at least $k \cdot \frac{2}{\delta} (\gamma(1 - \delta))^u$ indices $\ell$ \[ \geq \frac{\delta}{2} (\gamma(1 - \delta))^u, \]
and for any $s > 0$, the expected number of $r$–tuples $(x_1, \ldots, x_r) \in M_{G'}(r, w; Y_1, \ldots, Y_u)$ with $|CN_{G'}(x_1, \ldots, x_r)| \leq s$ is bounded above by
\[ \left( \frac{s}{|V^\text{up}_{G'}|} \right)^u k! g^w g! \cdot (k-w-1)! (g-r+w)! . \]

Proof. Fix for a moment any $1 \leq \ell \leq k$. We have
\[ \mathbb{P}\{Y_1, \ldots, Y_u \in S^\text{up}_{\ell}\} \geq \gamma^u. \]
On the other hand, conditioned on the event $\{Y_1, \ldots, Y_u \in S^\text{up}_{\ell}\}$, we get, by repeating the first part of the argument from the proof of Lemma 3.1
\[ \mathbb{E}\left[|CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{\ell}| \mid Y_1, \ldots, Y_u \in S^\text{up}_{\ell}\right] \geq g(1 - \delta)^u. \]
Since the size of the intersection $CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{\ell}$ is deterministically bounded above by $g$, we obtain
\[ g(1 - \delta)^u \leq \mathbb{E}\left[|CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{\ell}| \mid Y_1, \ldots, Y_u \in S^\text{up}_{\ell}\right] \]
\[ \leq g \mathbb{P}\{|CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{\ell}| \geq g(1 - \delta)^{u+1} \mid Y_1, \ldots, Y_u \in S^\text{up}_{\ell}\} \]
\[ + g(1 - \delta)^{u+1} (1 - \mathbb{P}\{|CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{\ell}| \geq g(1 - \delta)^{u+1} \mid Y_1, \ldots, Y_u \in S^\text{up}_{\ell}\}), \]
and hence
\[ \mathbb{P}\{|CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{\ell}| \geq g(1 - \delta)^{u+1} \mid Y_1, \ldots, Y_u \in S^\text{up}_{\ell}\} \geq \frac{(1 - \delta)^u - (1 - \delta)^{u+1}}{1 - (1 - \delta)^{u+1}} \]
\[ \geq \delta(1 - \delta)^u. \]
That, together with (15), implies
\[ \mathbb{P}\{|CN_{G'}(Y_1, \ldots, Y_u) \cap S^\text{down}_{\ell}| \geq g(1 - \delta)^{u+1}\} \geq \delta(1 - \delta)^u \cdot \gamma^u. \]
Since the last estimate is true for every $1 \leq \ell \leq k$, we get
\[
\left( k - k \cdot \frac{\delta}{2} \left( \gamma(1 - \delta) \right)^u \right) \mathbb{P}\{ |CN_{G'}(Y_1, \ldots, Y_u) \cap S^\downarrow_{\ell} | \geq g (1 - \delta)^{u+1} \} \geq k \left( \frac{\delta}{2} \right)^u \mathbb{P}\{ |CN_{G'}(Y_1, \ldots, Y_u) \cap S^\downarrow_{\ell} | \geq g (1 - \delta)^u \} \geq k \delta (1 - \delta)^u \cdot \gamma^u,
\]
and hence
\[
\mathbb{P}\{ |CN_{G'}(Y_1, \ldots, Y_u) \cap S^\downarrow_{\ell} | \geq g (1 - \delta)^{u+1} \} \geq \frac{\delta}{2} (\gamma(1 - \delta))^u.
\]
Further, take any ordered $r$–tuple $(x_1, \ldots, x_r)$ of elements of $V^\downarrow_{G'}$ satisfying the following conditions:
- The vertices $x_1, \ldots, x_r$ are distinct;
- There are distinct indices $1 \leq \ell_0, \ell_1, \ldots, \ell_w \leq k$ such that $x_1, \ldots, x_{r-w} \in S^\downarrow_{\ell_0}$ and for every $1 \leq a \leq w$, $x_{r-w+a} \in S^\downarrow_{\ell_a}$;
- $|CN_{G'}(x_1, \ldots, x_r)| \leq s$.

Clearly, the probability of the event $\{Y_1, \ldots, Y_u \in CN_{G'}(x_1, \ldots, x_r)\}$ can be bounded above by \(\left( \frac{s}{|V^\uparrow_{G'}|} \right)^u\). Thus, the expected number of all ordered $r$–tuples satisfying the above three conditions and contained in the common neighborhood of $Y_1, \ldots, Y_u$, is at most
\[
\left( \frac{s}{|V^\uparrow_{G'}|} \right)^u \frac{k! g^w g!}{(k-w-1)!(g-r+w)!}.
\]

Lemma 5.5 (Embedding into a block-structured graph). Let $G' = (V^\uparrow_{G'}, V^\downarrow_{G'}, E_{G'})$ be a block-structured bipartite graph with parameters $(\delta, \gamma, k, g)$, let $n, w, u \geq 1$ with $n - w \geq 2$, and assume that
\[
(16) \quad k \cdot \frac{\delta}{2} (\gamma(1 - \delta))^u \geq 2^w, \quad g (1 - \delta)^{u+1} \geq 2^{n-w}.
\]
and
\[
(17) \quad 64 \left( \frac{\delta}{4} (\gamma(1 - \delta))^u \right)^{-1} \left( \frac{2^{n-1}}{|V^\uparrow_{G'}|} \right)^u \left( (1 - \delta)^{u+1} \right)^{-n} \left( \frac{\delta}{2} (\gamma(1 - \delta))^u \right)^{-w-1} < 2^{-n+1}.
\]
Then the hypercube $\{1, -1\}^n$ can be embedded into $G'$.

Proof. Let $(S^\uparrow_{\ell})_{\ell=1}^k$, $(S^\downarrow_{\ell})_{\ell=1}^k$ be the corresponding compatible sequences of subsets of vertices of $G'$. Applying Lemma 5.4, we get that there are vertices $y_1, \ldots, y_u \in V^\uparrow_{G'}$ such that
- $|CN_{G'}(y_1, y_2, \ldots, y_u) \cap S^\downarrow_{\ell} | \geq g (1 - \delta)^{u+1}$ for at least $k \cdot \frac{\delta}{2} (\gamma(1 - \delta))^u$ indices $\ell$;
- the number of $n$–tuples $(x_1, \ldots, x_n) \in M_{G'}(n, w; y_1, \ldots, y_u)$ with $|CN_{G'}(x_1, \ldots, x_n)| \leq 2^{n-1}$ is bounded above by
\[
(18) \quad \left( \frac{\delta}{4} (\gamma(1 - \delta))^u \right)^{-1} \left( \frac{2^{n-1}}{|V^\uparrow_{G'}|} \right)^u \frac{k! g^w g!}{(k-w-1)!(g-n+w)!}.
\]
Further, let \( L \subset [k] \) be the subset of all indices \( \ell \) such that \( |\mathcal{CN}_{G'}(y_1, y_2, \ldots, y_u) \cap S^{\downarrow}_\ell | \geq g(1 - \delta)^{u+1} \), and for every \( \ell \in L \), let \( S^{\downarrow}_\ell \) be any subset of \( \mathcal{CN}_{G'}(y_1, y_2, \ldots, y_u) \cap S^{\downarrow}_\ell \) of cardinality \( g(1 - \delta)^{u+1} \). Observe that in view of (16) we have

\[
|L| \geq 2^u \quad \text{and} \quad |\mathcal{S}^{\downarrow}_\ell| \geq 2^{n-u} \quad \text{for all} \ \ell \in L.
\]

Now, we construct a random mapping

\[
f : \mathcal{T} = \{ v \in \{-1, 1\}^n : \text{vector } v \text{ has an odd number of } -1\text{'s} \} \to V_{G'}^{\downarrow}
\]

as follows. Let \((\mathcal{T}_b)_{b \in \{-1, 1\}^w}\) be a partition of \( \mathcal{T} \), where

\[
\mathcal{T}_b := \{ v \in \mathcal{T} : (v_{n-w+1}, \ldots, v_n) = b \}, \quad b \in \{-1, 1\}^w.
\]

Observe that \( |\mathcal{T}_b| = 2^{n-w-1} \). Let \((Z_b)_{b \in \{-1, 1\}^w}\) be the random 2\(^{n-w}\)-tuple uniformly distributed on the collection of all 2\(^w\)-tuples of distinct indices from \( L \). Then, conditioned on \((Z_b)_{b \in \{-1, 1\}^w}\), for every \( b \in \{-1, 1\}^w \) we let \( f((v : v \in \mathcal{T}_b)) \) be a random 2\(^{n-w-1}\)-tuple [conditionally] uniformly distributed on the collection of all 2\(^{n-w-1}\)-tuples of distinct elements of \( S^{\downarrow}_{Z_b} \), and we require that the random vectors \( f((v : v \in \mathcal{T}_b)) \), \( b \in \{-1, 1\}^w \) are [conditionally] mutually independent.

We recall at this point that the sets \( S^{\downarrow}_\ell \), \( \ell \in L \), are pairwise disjoint, and hence the mapping \( f \) injects everywhere on the probability space. Furthermore, the image of \( f \) is contained within \( \mathcal{CN}_{G'}(y_1, \ldots, y_u) \).

Pick any vertex \( v \in \{-1, 1\}^n \setminus \mathcal{T} \) and let \( v^{(1)}, \ldots, v^{(n)} \) be the neighbors of \( v \) in the hypercube ordered in such a way that

\[
(v^{(1)}_{n-w+1}, \ldots, v^{(1)}_n) = \ldots = (v^{(n-w)}_{n-w+1}, \ldots, v^{(n-w)}_n) = (v_{n-w+1}, \ldots, v_n),
\]

that is, for \( b = (v_{n-w+1}, \ldots, v_n) \in \{-1, 1\}^w \) we have \( v^{(1)}, \ldots, v^{(n-w)} \in \mathcal{T}_b \). Observe that the random \( n \)-tuple \( f(v^{(1)}), \ldots, f(v^{(n)}) \) is uniformly distributed on the set of all \( n \)-tuples of distinct vertices \((x_1, \ldots, x_n)\) such that there are distinct indices \( \ell_0, \ell_1, \ldots, \ell_w \in L \) with \( x_1, \ldots, x_{n-w} \in S^{\downarrow}_{Z_0} \) and \( x_{n-w+a} \in S^{\downarrow}_{Z_a} \) for every \( 1 \leq a \leq w \). Each \( n \)-tuple \((x_1, \ldots, x_n)\) satisfying the above conditions belongs to the set \( \mathcal{M}_{G'}(n, w; y_1, \ldots, y_u) \), and the total number of such \( n \)-tuples is estimated from below by

\[
\frac{q!}{(q-w-1)!(p-n+w)!} p^w p!
\]

where \( q := [k \cdot \frac{\delta}{2} (\gamma(1 - \delta))^u] \), and \( p := [g(1 - \delta)^{u+1}] \). In view of the bound (18) and the assumptions on the parameters, the probability that the number of common neighbors of \( f(v^{(1)}), \ldots, f(v^{(n)}) \) in \( G' \) is less than \( 2^{n-1} \), is less than

\[
\left( \frac{\delta}{4} (\gamma(1 - \delta))^u \right) \frac{2^{n-1}}{|V_{G'}^{\downarrow}|} \left( \left( \frac{k!}{(k-w-1)!(g-n+w)!} \right) \left( \frac{q!}{(q-w-1)!(p-n+w)!} \right) \right)^{w+1} \left( \left( \frac{w}{q-w} \right)^{w+1} \right)^{n-w}.
\]

In view of the first inequality in (16), we have

\[
\left( \frac{k-w}{q-w} \right)^{w+1} \leq \left( \frac{\delta}{2} (\gamma(1 - \delta))^u \right)^{-w+1} \left( 1 + \frac{w}{k \cdot \frac{\delta}{2} (\gamma(1 - \delta))^u - w} \right)^{w+1} \leq \left( \frac{\delta}{2} (\gamma(1 - \delta))^u \right)^{-w+1} \left( 1 + \frac{w}{2w - w} \right)^{w+1} \leq 8 \left( \frac{\delta}{2} (\gamma(1 - \delta))^u \right)^{-w+1},
\]
and, similarly, by the second inequality in (16),
\[
\left( \frac{g-n+w+1}{p-n+w+1} \right)^{n-w} \leq \left( (1-\delta)^{-u-1} \right)^{n-w} \left( \frac{g (1-\delta)^{u+1}}{g (1-\delta)^{u+1} - n + w + 1} \right)^{n-w} \\
\leq \left( (1-\delta)^{-u-1} \right)^{n-w} \left( \frac{2^{n-w}}{2^{n-w} - n + w + 1} \right)^{n-w} \\
\leq 8(1-\delta)^{-u-1})^{n-w}. 
\]
Hence,
\[
P\{|CN_{G'}(f(v^{(1)}), \ldots, f(v^{(n)}))| \leq 2^{n-1} \} \\
\leq 64 \left( \frac{\delta}{4(\gamma(1-\delta))^u} \right)^{u-1} \left( \frac{2^{n-1}}{|V_{G'_up}|} \right)^u \left( (1-\delta)^{u+1} \right)^{-n} \left( \frac{\delta}{2(\gamma(1-\delta))^u} \right)^{-u-1} < 2^{-n+1},
\]
where the last inequality follows from the assumption (17). Taking the union bound, we get that with a positive probability, for every \(v \in \{-1, 1\}^n \setminus T\), the set \(f(N(\{-1, 1\}^n)(v))\) has at least \(2^{n-1}\) common neighbors in \(G'\). A simple iterative procedure then produces an embedding of \(\{-1, 1\}^n\) into \(G'\), completing the proof.

6. Trichotomy

In this section, we show that any bipartite graph \(G\) has at least one of the following three properties: it either contains a large collection of non-condensed common neighborhoods of vertices, or a large subgraph of higher edge density, or a large block-structured subgraph as defined in Section 5. As a corollary of the trichotomy we obtain the main result of the paper.

**Lemma 6.1.** Let \(h, r > 0\) be parameters. Let \(G' = (V_{G'_up}, V_{G'_down}, E_{G'})\) be a bipartite graph, and let \(Y_1, \ldots, Y_r\) be i.i.d. uniform random vertices in \(V_{G'_up}\). Assume further that
\[
\mathbb{E} |CN_{G'}(Y_1, \ldots, Y_r)| \geq h.
\]
Then there is a subset \(S \subset V_{G'_down}^{\text{up}}\) of size at least \(h/2\) such that the induced subgraph of \(G'\) on \(V_{G'_up}^{\text{up}} \cup S\) has density at least \(\left( \frac{h}{2|V_{G'_down}|} \right)^{1/r}\).

**Proof.** For every vertex \(v \in V_{G'_down}^{\text{down}}\), let \(p_v \in [0, 1]\) be the probability of the event \(\{Y_1 \in N_{G'}(v)\}\), so that
\[
\sum_{v \in V_{G'_down}^{\text{down}}} p_v = \mathbb{E} |CN_{G'}(Y_1, \ldots, Y_r)| \geq h = |V_{G'_down}^{\text{down}}| \frac{h}{|V_{G'_down}|}.
\]
Let \(S\) be the collection of all \(v \in V_{G'_down}^{\text{down}}\) with \(p_v \geq \frac{h}{2|V_{G'_down}|}\), and note that
\[
|S| \geq \sum_{v \in S} p_v \geq \frac{h}{2}.
\]
The result follows.

**Lemma 6.2.** There is a universal constant \(\varepsilon \in (0, 1]\) with the following property. Let \(h, r, M > 0\) and \(p \in (0, 1]\) be parameters. Let \(G' = (V_{G'_up}, V_{G'_down}, E_{G'})\) be a bipartite graph, and assume that \((v_1, v_2)\) is a pair of vertices in \(V_{G'_down}^{\text{down}}\) such that the collection of common neighborhoods \(\{CN_{G'}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G'}(v_1, v_2)\}\) is \((p, M)\)-condensed. Further, assume that for i.i.d. uniform random vertices \(Y_1, \ldots, Y_r\) in \(CN_{G'}(v_1, v_2)\),
\[
\mathbb{E} |CN_{G'}(Y_1, \ldots, Y_r)| \leq h.
\]
Then there is a subset $S \subset V^{\downarrow}_{G'}$ of size at least $\frac{e^{-2}pM}{h \log_2(p/2)}$ such that the induced subgraph of $G'$ on $CN_{G'}(v_1, v_2) \cup S$ has density at least $\left( \frac{e^{-2}pM}{-h \log_2(p/2)} \right)^{1/r}$.

Proof. Let $\tilde{Y}_1, \ldots, \tilde{Y}_r$ be i.i.d. uniform random vertices in $CN_{G'}(v_1, v_2)$ mutually independent with $Y_1, \ldots, Y_r$. Further, for every $i \geq 0$ let $\mathcal{E}_i$ be the event (measurable with respect to collection of variables $(Y_1, \ldots, Y_r)$) that

$$P\{ |CN_{G'}(Y_1, \ldots, Y_r) \cap CN_{G'}(\tilde{Y}_1, \ldots, \tilde{Y}_r)| \geq M \mid Y_1, \ldots, Y_r \} \in (2^{-i-1}, 2^{-i}]$$

By the properties of the conditional probability and in view of the definition of a $(p, M)$-condensed collection, we have

$$E \ P\{ |CN_{G'}(Y_1, \ldots, Y_r) \cap CN_{G'}(\tilde{Y}_1, \ldots, \tilde{Y}_r)| \geq M \mid Y_1, \ldots, Y_r \} = P\{ |CN_{G'}(Y_1, \ldots, Y_r) \cap CN_{G'}(\tilde{Y}_1, \ldots, \tilde{Y}_r)| \geq M \} \geq p.$$ 

On the other hand, for every non-negative random variable $\xi$ we have

$$E \xi \leq \sum_{i \geq 0} 2^{-i} P\{ \xi \in (2^{-i-1}, 2^{-i}] \}.$$ 

It follows that

$$\sum_{i \geq 0} 2^{-i} P(\mathcal{E}_i) \geq p,$$

and hence there is an index $0 \leq i_0 \leq -2 \log_2(p/4)$ with

$$2^{-i_0} P(\mathcal{E}_{i_0}) \geq \frac{p}{-2 \log_2(p/4)}.$$ 

Using the assumption that $E |CN_{G'}(Y_1, \ldots, Y_r)| \leq h$, we get a bound on the conditional expectation

$$E \left[ |CN_{G'}(Y_1, \ldots, Y_r)| \mid \mathcal{E}_{i_0} \right] \leq h \cdot \frac{1}{P(\mathcal{E}_{i_0})} \leq h \cdot \frac{-2 \log_2(p/4)}{2^{i_0} p}.$$ 

Thus, there is a collection of [non-random] vertices $y_1, \ldots, y_r \in CN_{G'}(v_1, v_2)$ with

$$P\{ |CN_{G'}(y_1, \ldots, y_r) \cap CN_{G'}(\tilde{Y}_1, \ldots, \tilde{Y}_r)| \geq M \} \in (2^{-i_0-1}, 2^{-i_0}]$$

and such that $|CN_{G'}(y_1, \ldots, y_r)| \leq h \cdot \frac{-2 \log_2(p/4)}{2^{i_0} p}$.

For every vertex $v \in CN_{G'}(y_1, \ldots, y_r)$, let $p_v$ be the probability of the event $\{ \tilde{Y}_1 \in N_{G'}(v) \}$. By our assumptions,

$$\sum_{v \in CN_{G'}(y_1, \ldots, y_r)} p_v \geq E \left[ |CN_{G'}(y_1, \ldots, y_r) \cap CN_{G'}(\tilde{Y}_1, \ldots, \tilde{Y}_r)| \geq 2^{-i_0-1} M. \right.$$ 

Let $S$ be the set of all vertices $v \in CN_{G'}(y_1, \ldots, y_r)$ such that

$$p_v \geq \frac{2^{-i_0-2} M}{h \cdot \frac{-2 \log_2(p/4)}{2^{i_0} p}} = \frac{pM}{-8 h \log_2(p/4)},$$ 

and note that

$$|S| \geq \sum_{v \in S} p_v \geq 2^{-i_0-2} M.$$ 

The induced subgraph of $G'$ on $CN_{G'}(v_1, v_2) \cup S$ has density at least

$$\left( \frac{pM}{-8 h \log_2(p/4)} \right)^{1/r},$$

completing the proof. \qed
Proposition 6.3 (The main structural proposition). Let \( G = (V^\text{up}_G, V^\text{down}_G, E_G) \) be a non-empty bipartite graph of density at least \( \alpha \in (0, 1) \). Further, let \( \alpha_0 \leq \alpha / 2, \mu \in (0, \alpha_0 / 2], M \geq 1, r \geq 2, \) and \( p \in (0, 1] \) be parameters and assume that \( ((1 - \mu)\alpha)^2 |V^\text{up}_G| \geq r^2 \). Then at least one of the following is true:

(a) There is a subgraph \( G' = (V^\text{up}_{G'}, V^\text{down}_{G'}, E_{G'}) \) of \( G \) of density at least \( (1 - \mu)\alpha \), with \( V^\text{up}_{G'} := V^\text{up}_G \) and \( V^\text{down}_{G'} := V^\text{down}_G \), and an \( (\alpha_0, (1 - \mu)\alpha, \mu, r, C\ell_4(\alpha_0, \mu, r^3))-\)standard vertex pair \( (v_1, v_2) \) in \( V^\text{down}_G \) such that the collection of neighborhoods

\[
\{ CN_{G'}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G'}(v_1, v_2) \}
\]

is not \( (p, M) \)-condensed.

(b) There is a subgraph \( G' = (V^\text{up}_{G'}, V^\text{down}_{G'}, E_{G'}) \) of \( G \) with \( |V^\text{up}_{G'}| \geq \frac{\alpha^2}{2} |V^\text{up}_G|, |V^\text{down}_{G'}| \geq h/2 \) of density at least \( \left( \frac{h}{2|V^\text{down}_G|} \right)^{1/r} \). If the collection of neighborhoods \( \{ CN_{G'}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G'}(v_1, v_2) \} \) is not \( (p, M) \)-condensed then we arrive at the option (a) and complete the proof.

(c) There is a block-structured subgraph \( G' = (V^\text{up}_{G'}, V^\text{down}_{G'}, E_{G'}) \) of \( G \) with \( V^\text{up}_{G'} := V^\text{up}_G \) and with parameters

\[
(1 - \left( \frac{\alpha_0^2 M}{h \log_2 (p/2)} \right))^{1/r}, (1 - \mu)^3 \alpha^2, \left[ \frac{2\mu |V^\text{down}_G|}{|V^\text{up}_G|} \right], \left[ \frac{p M}{ \alpha_0 M} \right].
\]

Proof. The proof is accomplished via an iterative process.

Set \( \ell := 1, G^{(1)} := G \), so that \( G^{(1)} \) has density at least \( \alpha \).

Beginning of cycle

At the start of the cycle, we assume that we have given a bipartite subgraph \( G^{(\ell)} \) of \( G \) having the same vertex set as \( G \), with the edge density

\[
\alpha^{(\ell)} \geq \alpha - \frac{(\ell - 1) \cdot \left( \frac{\alpha_0 M}{|V^\text{down}_G|} \right)}{\mu} \geq (1 - \mu)\alpha.
\]

Applying Lemma 4.4, we obtain an \( (\alpha_0, (1 - \mu)\alpha, \mu, r, C\ell_4(\alpha_0, \mu, r^3))-\)standard ordered pair \( (v_1^{(\ell)}, v_2^{(\ell)}) \) in \( V^\text{down}_{G^{(\ell)}} \). If the collection of neighborhoods \( \{ CN_{G^{(\ell)}}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G^{(\ell)}}(v_1^{(\ell)}, v_2^{(\ell)}) \} \) is not \( (p, M) \)-condensed then we arrive at the option (a) and complete the proof.

Otherwise, if the i.i.d. uniform random elements \( Y_1^{(\ell)}, \ldots, Y_r^{(\ell)} \) of \( CN_{G^{(\ell)}}(v_1^{(\ell)}, v_2^{(\ell)}) \) satisfy

\[
\mathbb{E} \left| CN_{G^{(\ell)}}(Y_1^{(\ell)}, \ldots, Y_r^{(\ell)}) \right| \geq h
\]

then, by Lemma 6.1, there is an induced subgraph \( G' = (V^\text{up}_{G'}, V^\text{down}_{G'}, E_{G'}) \) of \( G^{(\ell)} \) with \( V^\text{up}_{G'} := CN_{G^{(\ell)}}(v_1^{(\ell)}, v_2^{(\ell)}) \) and \( |V^\text{down}_{G'}| \geq h/2 \) of density at least \( \left( \frac{h}{2|V^\text{down}_{G^{(\ell)}}|} \right)^{1/r} \), implying (b) and completing the proof.

Otherwise, \( \{ CN_{G^{(\ell)}}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G^{(\ell)}}(v_1^{(\ell)}, v_2^{(\ell)}) \} \) is \( (p, M) \)-condensed and

\[
\mathbb{E} \left| CN_{G^{(\ell)}}(Y_1, \ldots, Y_r) \right| \leq h.
\]

Lemma 6.2 then implies that there is a subset \( S^{\text{down}}_G \subset V^\text{down}_{G^{(\ell)}} \) of non-isolated vertices of size \( \left[ \frac{\alpha_0 M}{\alpha_0} \right] \) such that the induced subgraph of \( G^{(\ell)} \) on \( CN_{G^{(\ell)}}(v_1^{(\ell)}, v_2^{(\ell)}) \cup S^{\text{down}}_G \) has density at least

\[
\left( \frac{\alpha_0 M}{h \log_2 (p/2)} \right)^{1/r}.
\]

Set \( S^{\text{up}} := CN_{G^{(\ell)}}(v_1^{(\ell)}, v_2^{(\ell)}) \) and note that \( |S^{\text{up}}| \geq (1 - \mu)((1 - \mu)\alpha)^2 |V^\text{up}_G| \). If \( \ell \geq \frac{\mu |V^\text{down}_{G^{(\ell)}}|}{|V^\text{up}_G|} \), then we exit the cycle and proceed with the rest of the proof. Otherwise, we let \( G^{(\ell+1)} := (V^\text{up}_{G^{(\ell+1)}}, V^\text{down}_{G^{(\ell+1)}}, E_{G^{(\ell+1)}}) \) be the subgraph of \( G^{(\ell)} \) obtained from \( G^{(\ell)} \) by removing all edges
adjacent to vertices in the set $S_{\ell}^{down}$. By our construction process, the edge density $\alpha^{(\ell+1)}$ of $G^{(\ell+1)}$ satisfies

$$\alpha^{(\ell+1)} \geq \alpha - \frac{\ell \cdot [\mu \alpha |V_G^{down}|]}{|V_G^{down}|} \geq (1 - \mu)\alpha.$$ 

Set

$$\ell := \ell + 1$$

and return to the beginning of the cycle.

**End of cycle**

Set

$$k := \left\lceil \frac{\mu \alpha |V_G^{down}|}{|\cup_{1}^{n} pM|} \right\rceil.$$ 

Upon completion of the cycle, we have two collections of subsets $(S_{\ell}^{up})_{\ell=1}^{k}$ and $(S_{\ell}^{down})_{\ell=1}^{k}$ such that for each $\ell$, $|S_{\ell}^{down}| = \left\lceil \frac{\mu \alpha |V_G^{down}|}{|\cup_{1}^{n} pM|} \right\rceil$, $|S_{\ell}^{up}| \geq (1 - \mu)^3 \alpha^2 |V_G^{up}|$, and the induced subgraph of $G$ on $S_{\ell}^{up} \sqcup S_{\ell}^{down}$ has density at least

$$\left(1 - \left(\frac{\mu \alpha |V_G^{down}|}{-h \log_2(p/2)}\right)^{1/r}, (1 - \mu)^3 \alpha^2, \left\lceil \frac{\mu \alpha |V_G^{down}|}{|\cup_{1}^{n} pM|} \right\rceil, [\cup_{1}^{n} pM] \right).$$

The result follows. □

**Proof of Theorem 1.1.** Let

$$\mu := 10^{-10}, \quad \alpha := \frac{1}{2}, \quad \alpha_0 := 0.1, \quad c := c' - 100\mu,$$

where $c' = 0.03657...$ is the positive solution of the quadratic equation $64(c')^2 + 25c' - 1 = 0$. We will assume that $n$ is sufficiently large and let $m := 2^{n-1}$. Let $G = (V_G^{up}, V_G^{down}, E_G)$ be a bipartite graph of density at least $\alpha = \frac{1}{2}$, where $|V_G^{up}| = |V_G^{down}| = \lfloor 2^{n-4cn} \rfloor$. Define

$$M := \left\lceil \frac{\mu \alpha |V_G^{down}|}{2^{9G_{\mu n}} \cdot m^3 n^6 \cdot 400 \log^2 |V_G^{down}|} \right\rceil = (1 - \mu)^4 n^2 - 4cn + o(n)$$

and

$$p := \left( \frac{\mu \alpha |V_G^{down}|}{16 \cdot 37G_{\mu n}^2 m^2 \cdot 20 \log |V_G^{down}|} \right)^2 = (1 - \mu)^4 n^2 - 4cn + o(n),$$

and note that with this definition and our assumption that $n$ is large, we have $M \geq 1$ and $n^2 \leq p \cdot ((1 - \mu)\alpha)^n |V_G^{down}|$. Further, set

$$u := \left\lfloor \sqrt{n} \right\rfloor,$$

define $w \in \mathbb{N}$ via the relation

$$n - w = \left\lfloor \log_2 \left( \left\lceil \frac{\mu \alpha |V_G^{down}|}{2^{1-u}} \right\rceil \right) \right\rfloor = n - 8cn + 8n \log_2 (1 - \mu) + o(n),$$

and define $h > 0$ via the formula

$$\left(\frac{\mu \alpha |V_G^{down}|}{-h \log_2(p/2)}\right)^{(n+w)/n} = 2^{2w+c-n}(1 - \mu)^{-3n},$$

so that

$$h = (1 - \mu)^{8n} 2^{n-8cn+o(n)} (2^{17cn-16n \log_2 (1 - \mu) - n} (1 - \mu)^{-3n})^{\frac{1}{1+8c-8 \log_2 (1 - \mu)}}.$$

To evaluate the magnitude of \( h \), note that the right hand side of the last identity can be crudely bounded from below by

\[
\exp(-100\mu n + o(n)) 2^{n-8cn}(2^{17cn})^{\frac{1}{1+8c}} = \exp(-100\mu n + o(n)) 2^{n+n \cdot \frac{1-25c-64c^2}{1+8c}},
\]

where, in view of the definition of \( c \), \( \frac{1-25c-64c^2}{1+8c} \geq \frac{25-100\mu}{1+8c} \geq 125\mu \), and hence \( h \geq 2^{n+1000\mu n + o(n)} \).

Applying Proposition 6.3 with \( r := n \), we get that at least one of the conditions (a), (b), (c) there holds. Below, we deal with each of the three possibilities.

(a) In this case, we are given a subgraph \( G' = (V_{up}^{G'}, V_{down}^{G'}, E_{G'}) \) of \( G \) of density at least \((1-\mu)\alpha\), with \( V_{up}^{G'} := V_{up}^{G} \) and \( V_{down}^{G'} := V_{down}^{G} \), and an \((\alpha_0, (1-\mu)\alpha, \mu, r, C_\alpha (\alpha, \mu, r^3)\)–standard vertex pair \((v_1, v_2)\) in \( V_{down}^{G'} \) such that the collection of neighborhoods

\[
\{CN_{G'}(y_1, \ldots, y_r) : (y_1, \ldots, y_r) \in CN_{G'}(v_1, v_2)^r\}
\]

is not \((p, M)\)–condensed. Observe that the assumptions of Lemma 4.7 (with \( \alpha \) replaced with \((1-\mu)\alpha\)) hold. Applying Lemma 4.7, we get that \( Q_n \) can be embedded into \( G \).

(b) In this case, there is a subgraph \( G' = (V_{up}^{G'}, V_{down}^{G'}, E_{G'}) \) of \( G \) with \(|V_{up}^{G'}| \geq \frac{1}{8}|V_{up}^{G}|\), \(|V_{down}^{G'}| \geq \frac{h}{2}|V_{down}^{G}|\), of density at least \((\frac{1}{2}|V_{down}^{G}|)^{1/n}\). Since \( h \geq 2^{n+1000\mu n + o(n)} \) and assuming \( n \) is large, we have

\[
|V_{up}^{G'}| \geq 2^{n+\tilde{c}n} \cdot \frac{2|V_{down}^{G'}|}{h}, \quad |V_{down}^{G'}| \geq 2^{n+\tilde{c}n}, \quad \left( \frac{h}{2|V_{down}^{G'}|} \right)^{1/n} \geq \tilde{c},
\]

for a universal constant \( \tilde{c} > 0 \). Applying Corollary 3.2 we obtain an embedding of \( Q_n \) into \( G \).

(c) In this case, \( G \) contains a block-structured subgraph \( G' = (V_{up}^{G'}, V_{down}^{G'}, E_{G'}) \) with \( V_{up}^{G'} = V_{up}^{G} \) and with parameters

\[
(\delta, \gamma, k, g) := \left( 1 - \left( \frac{c_0 \beta M}{-h \log_2(p/2)} \right)^{1/n}, (1-\mu)^3 \alpha^2, \left\lfloor \frac{\mu |V_{down}^{G'}|}{cn^{2/3}pM} \right\rfloor, \left\lceil \frac{\mu |E_{down}^{G'}|}{cn^{2/3}pM} \right\rceil \right).
\]

In view of our definition of \( u, w, h \) and assuming that \( n \) is sufficiently large, we have

\[
k \cdot \frac{\delta}{2} (\gamma(1-\delta))^u \geq 2^u, \quad g (1-\delta)^{u+1} \geq 2^{n-w},
\]

and

\[
64 \left( \frac{\delta}{4} (\gamma(1-\delta)) \right)^{-1} \left( \frac{2^{n-1}}{|V_{up}^{G'}|} \right)^u (1-\delta)^{u+1} - n \left( \frac{\delta}{2} (\gamma(1-\delta)) \right)^{u+1} < 2^{-n+1}.
\]

Hence, applying Lemma 5.3, we obtain an embedding of \( Q_n \) into \( G \). The proof is complete.

\[\square\]

**Remark 6.4** (Algorithmic perspective). It should be expected that our construction of the hypercube embedding can be turned into a randomized algorithm which runs in time polynomial in the number of vertices of the ambient graph. We leave this as an open question.
Appendix A. The random graph $\Gamma_{\varepsilon,n}$

The goal of this section is to show that for every $\varepsilon > 0$ and any sufficiently large $n$, there exists a bipartite graph on $\Theta(2^{2n-\varepsilon n})$ vertices and with the edge density $1/2$ which does not admit embedding of the hypercube $Q_n$ via the randomized procedure (I)--(II) from the introduction.

Fix a small parameter $\varepsilon > 0$, an integer $n \in \mathbb{N}$. Let $V_{\varepsilon,n}^{up}$ and $V_{\varepsilon,n}^{down}$ be two disjoint sets with $|V_{\varepsilon,n}^{up}| = |V_{\varepsilon,n}^{down}| = 2 \cdot \lceil 2^{n-\varepsilon n/2} \rceil^2$, and assume that the set $V_{\varepsilon,n}^{down}$ is partitioned into $2 \cdot \lceil 2^{n-\varepsilon n/2} \rceil$ subsets $V_{\varepsilon,n}^{down}(i)$, $1 \leq i \leq 2 \cdot \lceil 2^{n-\varepsilon n/2} \rceil$, of size $\lceil 2^{n-\varepsilon n/2} \rceil$ each. Further, for each $v \in V_{\varepsilon,n}^{up}$, let $I_v$ be a uniform random $\lceil 2^{n-\varepsilon n/2} \rceil$-subset of $\{1, \ldots, 2 \cdot \lceil 2^{n-\varepsilon n/2} \rceil\}$, and assume that the sets $I_v, v \in V_{\varepsilon,n}^{up}$, are mutually independent. Consider a random bipartite graph $\Gamma_{\varepsilon,n} = (V_{\varepsilon,n}^{up}, V_{\varepsilon,n}^{down}, E_{\varepsilon,n})$, where the edge set $E_{\varepsilon,n}$ is comprised of all unordered pairs of vertices $\{v, w\}, v \in V_{\varepsilon,n}^{up}, w \in \bigcup_{i \in I_v} V_{\varepsilon,n}^{down}(i)$. Note that the edge density of $\Gamma_{\varepsilon,n}$ is $1/2$ everywhere on the probability space. Our goal is to prove

**Proposition A.1.** For every $\varepsilon \in (0, 1]$ there is $n_\varepsilon \in \mathbb{N}$ such that given $n \geq n_\varepsilon$ and the random bipartite graph $\Gamma_{\varepsilon,n} = (V_{\varepsilon,n}^{up}, V_{\varepsilon,n}^{down}, E_{\varepsilon,n})$, with a positive probability $\Gamma_{\varepsilon,n}$ has the following property. For every subset $S$ of $V_{\varepsilon,n}^{up}$ with $|S| \geq 2^{n-1}$

$$|T(S) \subset V_{\varepsilon,n}^{down} \with |T| \leq 2^{n-\varepsilon n/4}$$

(19)

such that for at least half of $n$-tuples of vertices in $S$,

the common neighborhood of the $n$-tuple is contained in $T$.

**Proof.** Fix any $\varepsilon \in (0, 1)$. We will assume that $n$ is large. Fix for a moment any subset $S$ of $V_{\varepsilon,n}^{up}$ with $|S| \geq 2^{n-1}$. We will estimate the probability of the event $E_S$ that $S$ does not satisfy (19).
For each \( i \in \{1, \ldots, 2 \cdot \lceil 2^{n-\varepsilon/2} \rceil \} \), let \( \delta_i \in [0, 1] \) be the [random] number defined by
\[
\delta_i = \frac{|\{v \in S : v \text{ is adjacent to } V_{\varepsilon,n}^{\downarrow}(i)\}|}{|S|}.
\]
Note that \( \delta_i^n \) can be viewed as the proportion of ordered \( n \)-tuples of vertices in \( S \) (with repetitions allowed) comprising \( V_{\varepsilon,n}^{\downarrow}(i) \) in their common neighborhood. Define the random set \( U \) as
\[
U := \left\{ i \in \{1, \ldots, 2 \cdot \lceil 2^{n-\varepsilon/2} \rceil \} : \delta_i^n \geq \frac{1}{4 \cdot \lceil 2^{n-\varepsilon/2} \rceil} \right\}.
\]
Condition for a moment on any realization of \( \Gamma_{\varepsilon,n} \) such that \( |U| \leq \frac{2^{n-\varepsilon/4}}{2^{n-\varepsilon/2}} \). Since
\[
\sum_{i \in \{\lceil 2^{n-\varepsilon/2} \rceil / 4 \}} \delta_i^n \leq \frac{1}{2},
\]
for at least half of the ordered \( n \)-tuples \((v_1, \ldots, v_n)\) of vertices in \( S \) the set difference
\[
CN_{\Gamma_{\varepsilon,n}}(v_1, \ldots, v_n) \setminus \bigcup_{i \in U} V_{\varepsilon,n}^{\downarrow}(i)
\]
is empty, implying that \( S \) satisfies the property (19) with \( T := \bigcup_{i \in U} V_{\varepsilon,n}^{\downarrow}(i) \). Thus, necessarily
\[
\{U = L\} \leq \mathbb{P}\left\{ \left| \left\{ v \in S : i \in I_v \right\} \right| \geq \frac{1}{(4 \cdot \lceil 2^{n-\varepsilon/2} \rceil)^{1/n}} \text{ for every } i \in L \right\}.
\]
Let \((b_{v,i})\), \( v \in S \), \( i \in L \), be a collection of i.i.d. Bernoulli(1/2) random variables. Then, in view of the definition of the random sets \( I_v \), we can write
\[
\mathbb{P}\left\{ \frac{\left| \left\{ v \in S : i \in I_v \right\} \right|}{|S|} \geq \frac{1}{(4 \cdot \lceil 2^{n-\varepsilon/2} \rceil)^{1/n}} \text{ for every } i \in L \right\} \leq \mathbb{P}\left\{ \sum_{v \in S} b_{v,i} \geq |S| \frac{1}{(4 \cdot \lceil 2^{n-\varepsilon/2} \rceil)^{1/n}} \text{ for every } i \in L \right\} \cdot \mathbb{P}\left\{ \sum_{i=1}^{2 \cdot \lceil 2^{n-\varepsilon/2} \rceil} b_{v,i} = \lceil 2^{n-\varepsilon/2} \rceil \text{ for every } v \in S \right\}^{-1}.
\]
A crude estimate
\[
\mathbb{P}\left\{ \sum_{i=1}^{2 \cdot \lceil 2^{n-\varepsilon/2} \rceil} b_{v,i} = \lceil 2^{n-\varepsilon/2} \rceil \text{ for every } v \in S \right\}^{-1} \leq 2^{n|S|}
\]
will be sufficient for our purposes. Further, the Hoeffding inequality implies
\[
\mathbb{P}\left\{ \sum_{v \in S} b_{v,i} \geq |S| \frac{1}{(4 \cdot \lceil 2^{n-\varepsilon/2} \rceil)^{1/n}} \text{ for every } i \in L \right\} \leq \exp \left( -c|S| \cdot |L| \right)
\]
for some \( c = c(\varepsilon) > 0 \). Hence,
\[
\mathbb{P}\{U = L\} \leq 2^{n|S|} \exp \left( -c|S| \cdot |L| \right) \leq \exp \left( -c|S| \cdot |L|/2 \right),
\]
where in the last inequality we assumed that \( n \) is sufficiently large. A union bound estimate gives
\[
\mathbb{P}\left\{ |U| \geq \frac{2^{n-\varepsilon n/4}}{2^{n-\varepsilon n/2}} \right\} = \sum_{m \geq 2^{n-\varepsilon n/4}/[2^{n-\varepsilon n/2}]} \mathbb{P}\{ |U| = m \}
\leq \sum_{m \geq 2^{n-\varepsilon n/4}/[2^{n-\varepsilon n/2}]} \exp \left( -c|S| \cdot m/2 \right) \left( \frac{2e [2^{n-\varepsilon n/2}]}{m} \right)^m
\leq \exp \left( -c|S| \cdot 2^{en/4} / 4 \right),
\]
again under the assumption of large \( n \). We conclude that the probability that the set \( S \) does not satisfy (19), is bounded above by \( \exp \left( -c'|S| \cdot 2^{en/4} \right) \), for some \( c' = c'(\varepsilon) > 0 \). Another union bound estimate implies
\[
\mathbb{P}\{ S \text{ does not satisfy (19)} \text{ for some } S \subset V_{\varepsilon,n}^{up} \text{ of size at least } 2^{n-1} \}
\leq \sum_{m \geq 2^{n-1}} \exp \left( -c'm \cdot 2^{en/4} \right) \left( \frac{e \cdot 2^n}{m} \right)^m < 1.
\]
Hence, there is a realization of \( \Gamma_{\varepsilon,n} \) with the required properties.  

School of Mathematics, Georgia Institute of Technology, 686 Cherry street, Atlanta, GA 30332, and

Department of Mathematical Sciences, Carnegie Mellon University, Wean Hall 6113, Pittsburgh, PA 15213, e-mail: ktikhomi@andrew.cmu.edu
