Polygamy relations of multipartite entanglement beyond qubits

Zhi-Xiang Jin\(^1\)\(^3\) and Shao-Ming Fei\(^1\)\(^,\)\(^2\)\(^,\)\(^3\)

\(^1\) School of Mathematical Sciences, Capital Normal University, Beijing 100048, People’s Republic of China
\(^2\) Max-Planck-Institute for Mathematics in the Sciences, Leipzig 04103, Germany

E-mail: jzxjinzhi@126.com and feishm@cnu.edu.cn

Received 21 September 2018, revised 4 March 2019
Accepted for publication 12 March 2019
Published 26 March 2019

Abstract

We investigate the polygamy relations related to the concurrence of assistance for any multipartite pure states. General polygamy inequalities given by the \(\alpha\)th \((0 \leq \alpha \leq 2)\) power of concurrence of assistance are first presented for multipartite pure states in arbitrary-dimensional quantum systems. We further show that the general polygamy inequalities can even be improved to be tighter inequalities under certain conditions on the assisted entanglement of bipartite subsystems. Based on the improved polygamy relations, lower bound for distribution of bipartite entanglement is provided in a multipartite system. Moreover, the \(\beta\)th \((0 \leq \beta \leq 1)\) power of polygamy inequalities are obtained for the entanglement of assistance as a by-product, which are shown to be tighter than the existing ones. A detailed example is presented.

Keywords: polygamy relations, multipartite entanglement, hamming weight, arbitrary-dimensional systems

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum entanglement \([1–8]\) has been extensively studied due to its importance in quantum communication and quantum information processing in recent years. The study of quantum entanglement from various viewpoints has been a very active area and has led to many impressive results. Monogamy of entanglement is one of the nonintuitive phenomena of quantum physics that distinguish quantum from classical physics. Different from the classical world, it is not possible to prepare three qubits in a way that any two qubits are maximally entangled. Qualitatively, monogamy of entanglement measures the shareability of entanglement in a
composite quantum system. Moreover, the monogamy property has emerged as the ingredient in the security analysis of quantum key distribution [9].

The monogamy relation was first quantified by Coffman, Kundu and Wootters [10] for three qubits, which satisfies $\mathcal{E}_{ABC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC}$. The CKW inequality shows the mutually exclusive nature of multipartite quantum entanglement in a quantitative way: more entanglement shared between two qubits (A and B) necessarily implies less entanglement between the other two qubits (A and C). CKW inequality was generalized for multiqubit systems [11] and also studied intensively in more general settings [12, 13]. However, the CKW inequality fails for higher-dimensional quantum systems. It is also not generally true for three-qubit systems with other entanglement measures like entanglement of formation [14]. Monogamy of multiqubit entanglement and some higher-dimensional quantum systems were later characterized in terms of various entanglement measures [15–17].

Whereas the monogamy of entanglement shows the restricted sharability of multipartite quantum entanglement, the distribution of entanglement in multipartite quantum systems was shown to have a dually monogamous property. Using concurrence of assistance [18] as the measure of distributed entanglement, the polygamy of entanglement provides a lower bound for distribution of bipartite entanglement in a multipartite system [19]. Polygamy of entanglement is characterized as a polygamy inequality, $\mathcal{E}_{A|BC} \leq \mathcal{E}_{A|B} + \mathcal{E}_{A|C}$ for a tripartite quantum state $p_{ABC}$, where $\mathcal{E}_{A|BC}$ is the assisted entanglement [20] between A and BC. Polygamy of entanglement was generalized to multiqudit systems [19] and arbitrary dimensional multipartite states [19, 21–23].

The study of quantum entanglement in higher-dimensional quantum systems is of importance in quantum information processing. Monogamy and polygamy of entanglement can restrict the possible correlations between the authorized users and the eavesdroppers, which tightens security bounds in quantum cryptography. And optimized monogamy and polygamy relations give rise to finer characterizations of the entanglement distributions. Furthermore, to optimize the efficiency of entanglement usage as a resource in quantum cryptography, higher-dimensional quantum systems rather than qubits are preferred in some physical systems for stronger security in quantum key distribution [24].

In this paper, we provide a tighter polygamy inequalities for arbitrary dimensional quantum systems. General polygamy inequalities given by the $\alpha$th ($0 \leqslant \alpha \leqslant 2$) power of concurrence of assistance are first presented for multipartite pure states in arbitrary-dimensional quantum systems. We further show that the general polygamy inequalities can even be improved to be tighter inequalities under certain conditions on the assisted entanglement of bipartite subsystems. Based on the improved polygamy relations, lower bound for distribution of bipartite entanglement is provided for multipartite systems. Moreover, the $\beta$th ($0 \leqslant \beta \leqslant 1$) power of polygamy inequalities are obtained for the entanglement of assistance, which are shown to be tighter than the existing ones.

We first recall monogamy and polygamy inequalities related to concurrence and concurrence of assistance. Let $\mathcal{H}_X$ denote a discrete finite-dimensional complex vector space associated with a quantum subsystem X. For a bipartite pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, the concurrence is given by [25–27], $C(|\psi\rangle_{AB}) = \sqrt{2 \left[1 - \text{Tr}(\rho_A^2)\right]}$, where $\rho_A$ is the reduced density matrix obtained by tracing over the subsystem B, $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence for a bipartite mixed state $\rho_{AB}$ is defined by the convex roof extension, $C(\rho_{AB}) = \min\{p_i C(|\psi_i\rangle)\} \sum_i p_i C(|\psi_i\rangle)$, where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, with $p_i \geq 0$, $\sum_i p_i = 1$ and $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance is defined by [18, 28],
\[ C_a(\psi_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{\rho_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \]

where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} = \text{Tr}_C(\psi_{ABC} |\psi\rangle) = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i| \). For pure states \( \rho_{AB} = |\psi\rangle_{AB} \langle \psi| \), one has \( C(|\psi\rangle_{AB}) = C_a(\rho_{AB}) \).

For an \( N \)-qubit state \( \rho_{AB_1 \cdots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_{N-1}} \), the concurrence is given by \( C(\rho_{AB_1 \cdots B_{N-1}}) = \sum |\tau_{ABC}^{(i)}|^2 \) [11, 29], with \( \tau_{ABC}^{(i)} \) defined in (2). Further improved monogamy relations are presented in [15] and [16].

The dual inequality in terms of the concurrence of assistance for \( N \)-qubit states have the form [19],

\[ C_a^2(\rho_{AB_1 \cdots B_{N-1}}) \leq \sum_{i=1}^{N-1} C_a^2(\rho_{AB_i}). \]

For a bipartite arbitrary dimensional pure state \( |\phi\rangle_{AB} = \sum_{i=1}^{d_1} \sum_{k=1}^{d_2} a_{ik} |i\rangle_{A} |k\rangle_{B} \) in \( C^{d_1} \otimes C^{d_2} \), the concurrence is given by [30]

\[ C^2(|\phi\rangle_{AB}) = 2(1 - \text{Tr}(\rho_{AB}^2)) = 4 \sum_{i<j} \sum_{k<l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2. \]

And for a mixed state \( \rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB} \langle \phi_i| \), from (3) its concurrence of assistance satisfies \( C_a(\rho_{AB}) = \max_{\{\rho_i, |\phi_i\rangle\}} \sum_i p_i C(|\phi_i\rangle) \leq \sum_{m=1}^{d_1} \sum_{n=1}^{d_2} \max_{\{|\phi\rangle\}} \left( \sum_i p_i |\langle L^m_A \otimes L^n_B |\phi\rangle|^2 \right) \]

where \( \rho_{AB} = \sum_{i,j} \sum_{k,l} a_{ik}a_{jl} |i\rangle_{A} |j\rangle_{B} \langle i| \langle j| \) and \( \tau_{AB}^{(i)} \) is defined in (2) [31].

A general polygamy inequality for any multipartite pure state \( |\phi\rangle_{A_1 \cdots A_n} \in C^{d_1} \otimes \cdots \otimes C^{d_n} \) was established in [31],

\[ \tau_{AB}^2(|\phi\rangle_{A_1 \cdots A_i}) \leq \sum_{i=2}^{n} \tau_{AB}^2(\rho_{A_i A_k}), \]

where \( \rho_{A_i A_k} \) is the reduced density matrix of \( |\phi\rangle_{A_1 \cdots A_n} \) associated with the subsystems \( A_i A_k \), \( k = 2, \cdots, n \).

2. Weighted polygamy relation for concurrence of assistance

Polygamy of entanglement states that if a multipartite state is maximally entangled with respect to a given kind of multipartite entanglement, then it must be pure [32]. This observation implies that all maximally entangled states are necessarily uncorrelated with any other systems. One can even propose this condition as another requisite for a good multipartite entanglement quantifier. Furthermore, it is also important to note that this polygamy holds for all kinds of entanglement, that is, whenever a system reaches a maximum amount of entanglement under any partitions, it becomes ‘free’ of its environment.
Therefore, for states that do not reach the maximum amount of entanglement of assistance under any partition, the polygamy inequality of entanglement provides a lower bound for the distribution of bipartite entanglement in a multipartite system. Meanwhile, the bipartite shareability of entanglement in a multipartite system gives an upper bound of the entanglement. Tighter polygamy inequalities give rise to finer characterization of the entanglement distributions, which are tightly related to the security of quantum cryptographic protocols based on entanglement [9] (it limits the amount of correlations that an eavesdropper can have with the honest parties). In the following, we give a class of polygamy inequalities that are tighter than existing ones. First, we give the definition of Hamming weight.

For any non-negative integer \( j \) and its binary expansion

\[
j = \sum_{i=0}^{n-1} j_i 2^i,
\]

with \( \log_2 j \leq n \) and \( j_i \in \{0, 1\} \), \( i = 0, 1, \cdots, n - 1 \), we can always define a unique binary vector \( j \) associated with \( j \).

\[
\vec{j} = (j_0, j_1, \cdots, j_{n-1}).
\]  

(5)

For the binary vector \( \vec{j} \) defined in (5), the Hamming weight \( w_H(\vec{j}) \) is defined by the number of 1’s in \( \{j_0, j_1, \cdots, j_{n-1}\} \) [1].

**Lemma 1.** For any real numbers \( x \) and \( t \), \( 0 \leq t \leq 1, \ 0 \leq x \leq 1 \), we have \((1 + t)^x \leq 1 + (2^x - 1) t^x\).

**Proof.** Let \( f(x, y) = (1 + y)^x - y^x \) with \( 0 \leq x \leq 1, \ y \geq 1 \). Then \( \frac{\partial f}{\partial y} = x[(1 + y)^{x-1} - y^{x-1}] \leq 0 \). Therefore, \( f(x, y) \) is an decreasing function of \( y \), i.e. \( f(x, y) \leq f(x, 1) = 2^x - 1 \). Set \( y = \frac{1}{t}, \ 0 < t \leq 1 \), we obtain \((1 + t)^x \leq 1 + (2^x - 1) t^x\). When \( t = 0 \), the inequality is trivial. \( \square \)

The following theorem provides states that a class of polygamy inequalities satisfied by the \( \alpha \)-power of \( \tau_a \). For convenience, we denote \( \tau_{a}(\rho_{AB}) = \tau_{a}^{AB} \) the concurrence of assistance \( \rho_{AB} \) and \( \tau_{a}^{(\rho_{A|B_0\cdots B_{n-1}})} = \tau_{a}^{A|B_0\cdots B_{n-1}} \)

**Theorem 1.** For any multiparty pure state \( \rho_{A|B_0\cdots B_{n-1}} \), we have

\[
\tau_{a}^{\alpha} A|B_0\cdots B_{n-1} \leq \sum_{j=0}^{N-1} (2^\frac{j}{\alpha} - 1)^{w_H(\vec{j})} \tau_{a}^{\alpha} A|B_j
\]

(6)

for \( 0 < \alpha \leq 2 \), where \( \vec{j} = (j_0, j_1, \cdots, j_{n-1}) \) is the vector from the binary representation of \( j \) and \( w_H(\vec{j}) \) is the Hamming weight of \( \vec{j} \).

**Proof.** Without loss of generality, we can always have

\[
\tau_{a} A|B_j \geq \tau_{a} A|B_{j+1} \geq 0,
\]

(7)

by relabeling the subsystems. From (4), it is sufficient to show that

\[
\left( \sum_{j=0}^{N-1} \tau_{a} A|B_j \right)^{\frac{1}{\alpha}} \leq \sum_{j=0}^{N-1} (2^\frac{j}{\alpha} - 1)^{w_H(\vec{j})} \tau_{a}^{\alpha} A|B_j
\]

(8)
We first prove the inequality (8) for the case that $N$ is a power of 2, $N = 2^n$, by mathematical induction. For $n = 1$, by using lemma 1 we have

$$\tau_{a|B}^\alpha \leq \tau_{a|B}^\alpha + (2^{2^2 - 1}) \tau_{a|B}^\alpha,$$

which is just the inequality (8) for $N = 2$.

Now let us assume that the inequality (8) is true for $N = 2^{n-1}$ with $n \geq 2$, and consider the case that $N = 2^n$. For an $(N + 1)$-partite quantum state $\rho_{AB_0 \cdots B_{N-1}}$ and its bipartite reduced density matrices $\rho_{AB_j}$, $j = 0, 1, \cdots, N - 1$, we have

$$\left( \sum_{j=0}^{N-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2} = \left( \sum_{j=0}^{2^{n-1}-1} \tau_{a|B_j}^\alpha + \sum_{j=2^{n-1}}^{2^n-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2} = \left( \sum_{j=0}^{2^{n-1}-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2} \left( 1 + \frac{\sum_{j=2^{n-1}}^{2^n-1} \tau_{a|B_j}^\alpha}{\sum_{j=0}^{2^{n-1}-1} \tau_{a|B_j}^\alpha} \right)^\frac{\phi}{2}.$$

(9)

Due to (7) we have

$$\sum_{j=2^{n-1}}^{2^n-1} \tau_{a|B_j}^\alpha \leq \sum_{j=0}^{2^{n-1}-1} \tau_{a|B_j}^\alpha,$$

(10)

By using lemma 1 we get

$$\left( \sum_{j=0}^{N-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2} \leq \left( \sum_{j=0}^{2^{n-1}-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2} + (2^{2^2 - 1} - 1) \left( \sum_{j=2^{n-1}}^{2^n-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2}.$$

(11)

Here, the induction hypothesis assures that

$$\left( \sum_{j=0}^{2^{n-1}-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2} \leq \sum_{j=0}^{2^{n-1}-1} (2^{2^2 - 1} - 1)^{\phi n |j\rangle \langle j|} \tau_{a|B_j}^\alpha.$$

(12)

From above relations we obtain

$$\left( \sum_{j=2^{n-1}}^{2^n-1} \tau_{a|B_j}^\alpha \right)^\frac{\phi}{2} \leq \sum_{j=2^{n-1}}^{2^n-1} (2^{2^2 - 1} - 1)^{\phi n |j\rangle \langle j|} \tau_{a|B_j}^\alpha.$$

(13)

Taking into account (11)–(13) we have
\[
\left( \sum_{j=0}^{2^n-1} \tau_a^{\alpha_{ABj}} \right)^{\frac{2}{n}} \leq \sum_{j=0}^{2^n-1} (2^{\frac{2}{n}} - 1)^{\nu_{\alpha_{ABj}}} \tau_a^{\alpha_{ABj}},
\]

which proves the inequality (8) for \( N = 2^n \).

Now for an arbitrary positive integer \( N \), consider an \((N + 1)\)-partite state \( \rho_{AB_0 \ldots B_{N-1}} \). We can always assume that \( 0 \leq N \leq 2^n \) for some \( n \). Consider a \((2^n + 1)\)-partite quantum state

\[
\rho_{AB_0 \ldots B_{2^n-1}} = \rho_{AB_0 \ldots B_{N-1}} \otimes \delta_{B_N \ldots B_{2^n-1}},
\]

which is a product of \( \rho_{AB_0 \ldots B_{N-1}} \) and an arbitrary \((2^n - N)\)-partite quantum state \( \delta_{B_N \ldots B_{2^n-1}} \).

Because \( \rho_{AB_0 \ldots B_{2^n-1}} \) is a \((2^n + 1)\)-partite state, inequality (14) leads to

\[
\tau_a^{\alpha} (\rho_{AB_0 \ldots B_{2^n-1}}) \leq \sum_{j=0}^{2^n-1} (2^{\frac{2}{n}} - 1)^{\nu_{a}} \tau_a^{\alpha} (\sigma_{ABj}),
\]

where \( \sigma_{ABj} \) is the bipartite reduced density matrix of \( \rho_{AB_0 \ldots B_{2^n-1}} \) for \( j = 0, 1, \ldots, 2^n - 1 \). Since \( \rho_{AB_0 \ldots B_{2^n-1}} \) is a separable state with respect to the bipartition between \( AB_0 \ldots B_{N-1} \) and \( B_N \ldots B_{2^n-1} \), one has

\[
\tau_a^{\alpha} (\rho_{AB_0 \ldots B_{2^n-1}}) = \tau_a^{\alpha} (\rho_{AB}),
\]

and

\[
\tau_a^{\alpha} (\sigma_{AB}) = 0,
\]

for \( j = N, \ldots, 2^n - 1 \). Moreover, for \( j = 0, 1, \ldots, N - 1 \) one has

\[
\sigma_{ABj} = \rho_{ABj}.
\]

From (16)–(19), we have

\[
\tau_a^{\alpha} (\rho_{ABj}) = \tau_a^{\alpha} (\rho_{ABj}) \leq \sum_{j=0}^{2^n-1} (2^{\frac{2}{n}} - 1)^{\nu_{a}} \tau_a^{\alpha} (\sigma_{ABj})
\]

and

\[
\tau_a^{\alpha} (\rho_{AB}) = \tau_a^{\alpha} (\rho_{AB}) \leq \sum_{j=0}^{N-1} (2^{\frac{2}{n}} - 1)^{\nu_{a}} \tau_a^{\alpha} (\rho_{ABj}).
\]

This completes the proof. \( \square \)

We have obtained the general polygamy inequality of the \( \alpha \)th \((0 \leq \alpha \leq 2)\) power of concurrence of assistance for arbitrary-dimensional quantum systems. In fact, (4) is a special case of (6) for \( \alpha = 2 \). Besides, based on the improved polygamy relations, we get a new upper bound for bipartite entanglement in multiparticle systems for \( 0 \leq \alpha < 2 \), which is better than (4). To illustrate the advantage of (6), we give an example as follows.

Let us consider the three-qubit state \( \rho = |\psi\rangle \langle \psi| \) in the generalized Schmidt decomposition form, where

\[
|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle,
\]

with
\( \lambda_i \geq 0, \ i = 0, 1, 2, 3, 4 \) and \( \sum_{i=0}^4 \lambda_i^2 = 1 \). We have \( \tau_{\alpha A|B_1\cdots B_{N-1}} = 2\lambda_0 \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2} \), \( \tau_{\alpha AB} = 2\lambda_0 \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2} \), and \( \tau_{\alpha AC} = 2\lambda_0 \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2} \). Take \( \lambda_0 = \lambda_1 = \frac{1}{2} \).

Corollary 1. For any multiparty pure state \( \rho_{A|B_1\cdots B_{N-1}} \), we have

\[
\tau_{\alpha A|B_1\cdots B_{N-1}} \leq \sum_{j=0}^{N-1} (2^{\frac{\alpha}{2}} - 1)^{\nu N(\bar{\alpha})} \tau_{\alpha A^j} \tag{21}
\]

for any multiparticle quantum state \( \rho_{A|B_1\cdots B_{N-1}} \). Thus, we have the following corollary.

The class of weighted polygamy inequalities in theorem 1 can be further tightened under some condition on bipartite quantum relations.

Theorem 2. For any multiparty pure state \( \rho_{A|B_1\cdots B_{N-1}} \), if

\[
\tau_{\alpha A|B_i}^2 \geq \sum_{j=i+1}^{N-1} \tau_{\alpha A|B_j}^2 \tag{22}
\]

for \( i = 0, 1, \cdots N-2 \), we have

\[
\tau_{\alpha A|B_1\cdots B_{N-1}} \leq \sum_{j=0}^{N-1} (2^{\frac{\alpha}{2}} - 1)^{\nu N(\bar{\alpha})} \tau_{\alpha A^j} \tag{23}
\]

for \( 0 \leq \alpha \leq 2 \).

Proof. From lemma 1, we have

\[
\tau_{\alpha A|B_1\cdots B_{N-1}} \leq \tau_{\alpha A^0} \leq \tau_{\alpha A^0} + (2^{\frac{\alpha}{2}} - 1) \left( \sum_{j=1}^{N-1} \tau_{\alpha A^j}^2 \right)^{\frac{\alpha}{2}}
\]

\[
\leq \tau_{\alpha A^0} + (2^{\frac{\alpha}{2}} - 1) \tau_{\alpha A^1} + (2^{\frac{\alpha}{2}} - 1)^2 \left( \sum_{j=2}^{N-1} \tau_{\alpha A^j}^2 \right)^{\frac{\alpha}{2}}
\]

\[
\leq \cdots
\]

\[
\leq \tau_{\alpha A^0} + (2^{\frac{\alpha}{2}} - 1) \tau_{\alpha A^1} + \cdots + (2^{\frac{\alpha}{2}} - 1)^{N-1} \tau_{\alpha A^{N-1}}.
\]

\( \square \)
In theorem 2, the condition (22) are always satisfied by some states. Let us consider a four-qubit state $\rho = |W\rangle_{ABCD}|W\rangle$, where $|W\rangle_{ABCD} = a|1000\rangle + b|0100\rangle + c|0010\rangle + d|0001\rangle$, and $a^2 + b^2 + c^2 + d^2 = 1$. We have $\tau_{a}(\rho_{A|BCD}) = 2a\sqrt{1 - d^2}$, $\tau_{a}(\rho_{AB}) = 2ab$, $\tau_{a}(\rho_{MC}) = 2ac$, $\tau_{a}(\rho_{AD}) = 2ad$. The condition (22) is satisfied as long as $b^2 \geq c^2 + d^2$. For example, we set $b = \frac{1}{\sqrt{2}}$, $a = c = d = \frac{1}{\sqrt{2}}$. Then the state $\rho = |W\rangle_{ABCD}|W\rangle$ satisfies the condition (22). On the other hand, if $b^2 \leq c^2 + d^2$, e.g. $c = \frac{1}{\sqrt{2}}$ and $a = b = d = \frac{1}{\sqrt{6}}$, then $\rho$ does not satisfy the condition (22).

**Remark 1.** For any non-negative integer $j$ and the corresponding binary vector $\vec{f}$ in inequality (5), the Hamming weight $w_{H}(\vec{f})$ is upper bounded by $\log_{2}j$, which implies that $\tau_{a}^{|\vec{f}|_{A|BR_{1}\cdots R_{n-1}}} \leq \sum_{j=0}^{N-1}(2^{j} - 1)^{\frac{n-1}{2}}\tau_{a}^{\vec{f}}\rho_{A|BR_{1}\cdots R_{n-1}}^{|\vec{f}|}$, for $0 \leq \alpha \leq 2$. In other words, inequality (23) in theorem 2 is tighter than the inequality (6) in theorem 1 for states satisfying the condition (22).

### 3. Polygamy relations for entanglement of assistance

Now we study the polygamy relations for entanglement of assistance. For polygamy inequality beyond qubits, it was shown that the von Neumann entropy can be used to establish a polygamy inequality of tripartite quantum systems [33]. For any arbitrary dimensional tripartite pure state $|\psi\rangle_{ABC}$, one has $E(|\psi\rangle_{A|BC}) \leq E_{\alpha}(\rho_{AB}) + E_{\alpha}(\rho_{AC})$, where $E(|\psi\rangle_{A|BC}) = S(\rho_{A})$ is the entropy of entanglement between $A$ and $BC$ in terms of the von Neumann entropy $S(\rho) = -\text{Tr}\rho\ln\rho$, and $E_{\alpha}(\rho_{AB}) = \max_{\sum_{i}p_{i}|\psi_{i}\rangle_{A}}\sum_{i}p_{i}E(|\psi_{i}\rangle_{AB})$, with the maximization taking over all possible pure state decompositions of $\rho_{AB} = \sum_{i}p_{i}|\psi_{i}\rangle_{A}|\psi_{i}\rangle_{B}$. Later, a general polygamy inequality for any multipartite state $\rho_{A_{1}|A_{2}\cdots A_{n}}$ was established [34],

$$E_{\alpha}(\rho_{A_{1}|A_{2}\cdots A_{n}}) \leq \sum_{i=2}^{n}E_{\alpha}(\rho_{A_{i}|A_{1}}).$$

Recently, another class of multipartite polygamy inequalities in terms of the $\beta$th power of entanglement of assistance (EOA) has been introduced [23]. For any multipartite state $\rho_{A_{1}|B_{1}\cdots B_{n-1}}$, and $0 \leq \beta \leq 1$,

$$E_{\beta}^{\alpha}(\rho_{A_{1}|B_{1}\cdots B_{n-1}}) \leq \sum_{j=0}^{N-1}\beta^{2j}E_{\beta}^{\alpha}(\rho_{A_{1}|B_{1}\cdots B_{n-1}})$$

if $E_{\alpha}^{\beta}(\rho_{AB_{1}}) \geq E_{\alpha}^{\beta}(\rho_{AB_{1}}^{'})$ for $i = 0, 1, \cdots, N - 2$; and

$$E_{\beta}^{\alpha}(\rho_{A_{1}|B_{1}\cdots B_{n-1}}) \leq \sum_{j=0}^{N-1}\beta^{j}E_{\beta}^{\alpha}(\rho_{AB_{1}}),$$

if $E_{\alpha}^{\beta}(\rho_{AB_{1}}) \geq \sum_{j=i+1}^{N-1}E_{\alpha}^{\beta}(\rho_{AB_{1}})$ for $i = 0, 1, \cdots, N - 2$. With a similar consideration to $\tau_{A_{1}|B_{1}\cdots B_{n-1}}$, we have the following result for EOA.

**Theorem 3.** For any multipartite state $\rho_{A_{1}|B_{1}\cdots B_{n-1}}$, we have

$$E_{\beta}^{\alpha}(\rho_{A_{1}|B_{1}\cdots B_{n-1}}) \leq \sum_{j=0}^{N-1}(2^{\beta} - 1)^{2j}E_{\beta}^{\alpha}(\rho_{AB_{1}}),$$

for $0 \leq \beta \leq 1$. 

To illustrate the tightness of the inequality (26) compared with the inequality (25) in [23], we consider the three-qubit state $|W⟩_{ABC} = |W⟩_{AB}|W⟩_{C}$, where $|W⟩_{AB} = \frac{1}{\sqrt{3}}(|100⟩ + |010⟩ + |001⟩)$.

We have $E_a(\rho_{AB}) = S(\rho_A) = \log_2 3 - \frac{3}{2}$ and $E_a(\rho_{ABC}) = E_a(\rho_{AC}) = \frac{3}{2}$. Thus the marginal EOA from inequality (25) is $E^\beta_a(\rho_{AB}) + \beta E^\beta_a(\rho_{AC}) - E_a(\rho_{ABC}) = (1 + \beta)\left(\frac{3}{2}\right)^\beta + \frac{3}{2} - \log_2 3$. The marginal EOA from inequality (26) is $E^\beta_a(\rho_{AB}) + (2\beta - 1)E^\beta_a(\rho_{AC}) - E_a(\rho_{ABC}) = 2\beta\left(\frac{3}{2}\right)^\beta + \frac{3}{2} - \log_2 3$. Figure 1 shows that our inequality gives a smaller upper bound than (25) in [23], namely, our marginal EOA is smaller than inequality (25) in [23] for $0 < \beta < 1$.

Since $0 \leq (2\beta - 1)^{w(\beta)} \leq 1$ for any $0 \leq \beta \leq 1$, we have

$$E^\beta_{AB|B_1\cdots B_{k-1}} \leq \sum_{j=0}^{N-1} (2\beta - 1)^{w(\beta)} E^\beta_{ABj}$$

for any multipartite quantum state $\rho_{AB|B_1\cdots B_{k-1}}$. Thus, we have the following corollary.

**Corollary 2.** For any multipartite pure state $\rho_{AB_0\cdots B_{k-1}}$, we have

$$E^\beta_{AB|B_1\cdots B_{k-1}} \leq \sum_{j=0}^{N-1} E^\beta_{ABj}$$

for $0 \leq \beta \leq 1$.

With a similar consideration to theorem 2, we can tighten the class of weighted polygamy inequalities in theorem 3 under certain conditions on bipartite quantum correlations.

**Theorem 4.** For any multipartite state $\rho_{AB_0\cdots B_{k-1}}$, we have

$$E^\beta_{AB|B_1\cdots B_{k-1}} \leq \sum_{j=0}^{N-1} (2\beta - 1)^j E^\beta_{ABj}$$

conditioned that

$$E^2_{ABj} \geq \sum_{j=i+1}^{N-1} E^2_{ABj}$$

for $i = 0, 1, \cdots, N - 2$, $0 \leq \beta \leq 1$.

**Remark 2.** For any non-negative integer $j$, since $w_H(\overrightarrow{j}) \leq \log_2 j \leq j$, one has $E^\beta_{AB|B_1\cdots B_{k-1}} \leq \sum_{j=0}^{N-1} (2\beta - 1)^j E^\beta_{ABj} \leq \sum_{j=0}^{N-1} (2\beta - 1)^{w_H(\overrightarrow{j})} E^\beta_{ABj}$ for $0 \leq \beta \leq 1$. Therefore, inequality (27) in theorem 4 is tighter than the inequality (26) in theorem 3 for states satisfying the conditions $E^2_{ABj} \geq \sum_{j=i+1}^{N-1} E^2_{ABj}$, $i = 0, \cdots, N - 2$.

In particular, (27) reduces to (24) in [34] for $\beta = 1$. For $0 < \beta < 1$, (27) is a tighter polygamy inequality compared with (24). Since $w_H(\overrightarrow{j}) \leq j$, (27) in theorem 4 is in general tighter than the (26) in theorem 3. From the example shown in figure 1, one can see that (26) is generally tighter than the result in [23]. Hence our weighted polygamy relations give finer
characterizations of the entanglement distributions among the subsystems, and help better security analysis of quantum key distribution [9] in quantum information processing.

4. Conclusion

Entanglement monogamy and polygamy are fundamental properties of multipartite entanglement. We have investigated the polygamy relations related to the concurrence of assistance. General polygamy inequalities given by the $\alpha$th ($0 \leq \alpha \leq 2$) power of concurrence of assistance have been presented for multipartite states in arbitrary-dimensional quantum systems. We have further shown that the general polygamy inequalities can even be improved to be tighter ones under certain conditions on the assisted entanglement of bipartite subsystems. Based on the improved polygamy relations, lower bound for distribution of bipartite entanglement has been provided for multipartite systems. Moreover, the $\beta$th ($0 \leq \beta \leq 1$) power of polygamy inequalities have been obtained for the entanglement of assistance as a by-product, which are shown to be tighter than the existing ones.

The higher-dimensional quantum systems are the key resources in various quantum information and communication processing tasks. For instance, the qudit ($d > 2$) systems are preferred in some quantum key distributions, where the use of qudits increases the coding density and provides stronger security compared to qubits [24]. Our results apply to general polygamy relations of multipartite entanglement in arbitrary higher-dimensional quantum systems. Moreover, our polygamy inequalities provide tighter constraints and finer characterizations of the entanglement distributions among the multipartite systems. These results may highlight future works on the study of multipartite quantum entanglement.

Acknowledgments

This work is supported by the Natural Science Foundation of China (NSFC) under Grants No. 11847209 and No.11675113; Key Project of Beijing Municipal Commission of Education under Grant No. KZ201810028042.
ORCID iDs

Zhi-Xiang Jin @ https://orcid.org/0000-0001-9217-1349

References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[3] Mintert F, Kuś M and Buchleitner A 2004 Phys. Rev. Lett. 92 167902
[4] Chen K, Albeverio S and Fei S M 2005 Phys. Rev. Lett. 95 040504
[5] Breuer H P 2006 J. Phys. A: Math. Gen. 39 11847
[6] Breuer H P 2006 Phys. Rev. Lett. 97 080501
[7] de Vicente J I 2007 Phys. Rev. A 75 052320
[8] Zhang C J, Zhang Y S, Zhang S and Guo G C 2007 Phys. Rev. A 76 012334
[9] Pawłowski M 2010 Phys. Rev. A 82 032313
[10] Koashi M and Winter A 2004 Phys. Rev. A 69 022309
[11] Osborne T J and Verstraete F 2006 Phys. Rev. Lett. 96 220503
[12] Gour G and Guo Y 2018 Quantum 2 81
[13] Guo Y 2018 Quantum Inf. Process. 17 222
[14] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[15] Zhu X N and Fei S M 2014 Phys. Rev. A 90 024304
[16] Jin Z X and Fei S M 2017 Quantum Inf. Process. 16 77
[17] Jin Z X, Li J, Li T and Fei S M 2018 Phys. Rev. A 97 032336
[18] Laustsen T, Verstraete F and van Enk S J 2003 Quantum Inf. Comput. 3 64
[19] Gour G, Bandyopadhyay S and Sanders B C 2007 J. Math. Phys. 48 012108
[20] Gour G, Meyer D A and Sanders B C 2005 Phys. Rev. A 72 042329
[21] Kim J S 2010 Phys. Rev. A 81 062328
[22] Kim J S and Sanders B C 2011 J. Phys. A: Math. Theor. 44 295303
[23] Kim J S 2018 Phys. Rev. A 97 042332
[24] Groblacher S, Jennewein T, Vaziri A, Weihs G and Zeilinger A 2006 New J. Phys. 8 75
[25] Uhlmann A 2000 Phys. Rev. A 62 032307
[26] Rungta P, Bužek V, Caves C M, Hillery M and Milburn G J 2001 Phys. Rev. A 64 042315
[27] Albeverio S and Fei S M 2001 J. Opt. B: Quantum Semiclass. Opt. 3 223–7
[28] Yu C S and Song H S 2008 Phys. Rev. A 77 032329
[29] Bai Y K, Ye M Y and Wang Z D 2009 Phys. Rev. A 80 044301
[30] Akhtarshenas S J 2005 J. Phys. A: Math. Gen. 38 6777
[31] Kim J S 2009 Phys. Rev. A 80 022302
[32] Cavalcanti D and Brand F G S L 2005 Phys. Rev. A 72 040303
[33] Buscemi F, Gour G and Kim J S 2009 Phys. Rev. A 80 012324
[34] Kim J S 2012 Phys. Rev. A 85 062302