Two-dimensional Moist Stratified Turbulence

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We study moist stratified turbulence in a two-dimensional Boussinesq system influenced by condensation and evaporation. We work in a periodic setting with $\tau_a \gg \tau_c \approx \tau_e$, where $\tau_a$, $\tau_c$ and $\tau_e$ are the time-scales associated with advection, condensation and evaporation, respectively. Our numerical simulations demonstrate the emergence of ordered structures with a clear power-law type spectral scaling from initially spatially un-correlated vapor and liquid fields. With our simple evaporation and condensation schemes, wherein both the processes push a parcel towards saturation, an asymptotic analysis in the limit of rapid condensation and strong stratification shows that the equations support a straightforward state of balance characterized by a hydrostatic, saturated, vertically sheared horizontal flow (VSHF). Utilizing initial conditions which have sufficient total water content (i.e. it is possible to saturate the entire domain by advective re-arrangement), we see the formation of coherent structures. The spectral scaling of the vapor and liquid variance attain an invariant form. At long times, after a series of oscillations in the kinetic energy, the system transitions rather abruptly to a regime which is close to saturation and dominated by a strong VSHF. In essence, the system approaches the state of saturated balance prescribed above. On the other hand, for a broader class of initial conditions which are in-sufficient in the above mentioned sense, apart from similar scaling and structure formation, the flow persists in an oscillatory state such that the contributions of the vertical and horizontal components to the kinetic energy constitute a quasi or aperiodic oscillation. Even though we cannot achieve balance with these general initial conditions, the time-scale of this oscillation is much larger than that of condensation and evaporation indicating a distinct dominant slow component in this moist stratified two-dimensional turbulent system.

I. INTRODUCTION

Water, in its different phases, plays a significant role in shaping climate of the Earth \cite{1},\cite{2}. From a dynamical perspective, the energy release associated with changes in phase of water
substance has a profound effect on specific atmospheric phenomena as well as the large-scale general circulation of the tropical and extra-tropical troposphere [3],[4]. To develop an understanding of the interaction of advection and condensation/evaporation, in the present work, we study turbulence as generated by a moist stratified two-dimensional (2D) system which consists of nonlinear gravity wave-modes (including zero frequency vertically sheared horizontal flows - VSHF) [5] and a dynamically consistent heat source (due to condensation/evaporation) [10].

The stably stratified 2D Boussinesq equations (without the influence of water substance) have been studied as a model of thermal convection [12]. Numerical work on the scaling of kinetic and potential energy in this system is seen to be quite non-universal: in addition to the original Obhukov-Bogliano postulates, scaling laws for the temperature variance (or entropy) have been proposed and realized [13],[14],[16]. Unlike pure 2D turbulence, the stably stratified system does not conserve enstrophy (see below). In spite of this simulations suggest the possibility of an inverse transfer of kinetic energy, the formation of a VSHF and a forward cascade of temperature variance (or potential energy) [14],[15],[16]. On the other hand, noting that the 2D system involves the nonlinear interaction of gravity waves, the susceptibility of these waves to parametric subharmonic instability leads to the anticipation of a robust transfer of energy to smaller scales [17],[18],[19]. Numerical work in a geophysical context has documented this forward transfer of energy [20],[21],[22],[23]. Further, the possible development of singularities [24] and fronts [25],[23] as well as the global well-posedness of these equations has also attracted significant attention [26],[27],[28],[29]. One of the main issues we investigate in the present work is whether the moist stratified two-dimensional system is capable of exploiting the available potential energy in a systematic manner so as to spontaneously organize (initially) spatially un-correlated vapor and liquid fields.

Two-dimensional stratified models have been used quite extensively in atmospheric settings, for example the tendency to organize condensing convection by gravity waves in 2D has been noted by Huang [30]. Similarly moist 2D models have been used for studying tropical cyclone formation [31],[32], squall-line thunderstorms [33],[34], atmospheric density currents [35], unstable modes in moist convection [36] as well as in elucidating properties of precipitating convection [37]. In another vein, 2D models arise in the so-called super-parameterization schemes for general circulation modeling wherein a high resolution 2D model is embedded in a coarser three-dimensional (3D) model to help resolve features of geophysical systems which support an extended range of scales.
Quite naturally, as the 2D model feeds into the larger scale 3D flow, it is important to develop a feel for 2D moist turbulence in its own right.

In addition to the scaling and structure formation, we hope to address the issue of balance and "slowness" in this simplified moist system. Indeed, the notion of balance and the existence (or not) of a slow-manifold in geophysical systems has a rich history [41],[42],[43]. We refer the reader to the text by Daley [44] for an introduction and to Vanneste [45] (and the references therein) for recent progress in the subject. Loosely, given the wide range of temporal scales inherent in geophysical systems, it is natural to inquire whether physically relevant information is actually contained in an independent simpler reduced dynamical system that evolves on a comparatively slower time-scale. Usually such a simplified or reduced description is possible in certain limiting scenarios, the celebrated example being the emergence of quasigeostrophic dynamics in the limit of small Froude and Rossby numbers for a rotating and stratified fluid [7],[46]. In particular, for the system at hand, by imposing strong stratification (which implies the presence of high frequency gravity waves) and rapid condensation/evaporation (which implies rapid heating/cooling) we would like to see if it is possible to asymptotically extract an emergent state of balance or a reduced system which is able to capture the main features of the entire system. If in fact the analysis points to such a simplified description, it also of interest to numerically examine whether general initial conditions are capable of attaining this state.

We begin by introducing the basic equations and the schemes employed for condensation and evaporation. The physical problem at hand is to consider the fate of perturbations to a basic state which is saturated and in hydrostatic balance. A linear analysis in a periodic reveals that the most unstable modes have a columnar structure whose growth rate has a particularly simple form in the limit of rapid condensation (and evaporation). The emergence of columnar structures is realized in simulations starting from both single-scale and spatially un-correlated data. Another aspect of the linear analysis is the presence of a distinct slow mode, also given by an asymptotic expansion of the suitably non-dimensionalized nonlinear governing equations. In fact, this slow mode represents a state of balance which is characterized by a hydrostatic, saturated VSHF. Numerical simulations with moderate Froude numbers and rapid condensation (i.e. $\tau_c \to 0$) from so-called sufficient initial data (i.e. initial conditions that are capable of saturating the domain by incompressible advective re-arrangement) are shown to immediately attain this state of balance. We then progress to a more delicate regime where the Froude number tends to zero along with
rapid condensation, i.e. we consider a strongly stratified system influenced by rapid condensation. Utilizing initial conditions which have sufficient total water content, we document the formation of coherent structures, the spectral scaling of the vapor and liquid fields, and at long times, the transition to a regime which is close to saturation and dominated by a strong VSHF. For a broader class of initial conditions which are insuffi cient in the above mentioned sense, the flow exhibits similar scaling and structure formation. However, the contributions to kinetic energy of the vertical and VSHF components constitute aperiodic or quasiperiodic oscillations. Finally, a summary and discussion conclude the paper.

II. GOVERNING EQUATIONS

We consider the evolution of nonlinear periodic perturbations to a saturated Boussinesq fluid which is at rest and in hydrostatic balance. The moist Boussinesq framework has been the subject of numerous recent investigations: for example, in the scaling of moist turbulence [47], the modeling of precipitation fronts [48], moist Rayleigh Bernard convection [49] and in studying droplet growth in a 2D setting [50], tropical cyclone formation [32] and squall-line thunderstorms [34]. The governing equations are:

\[
\frac{D\vec{u}}{Dt} = -\nabla p' + \frac{g}{T_0} T' \hat{k},
\]

\[
\frac{DT'}{Dt} = -\lambda w + \frac{L}{C_p} S,
\]

\[
\frac{Dq_v}{Dt} = -S; \quad \frac{Dq_l}{Dt} = S
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.
\]

In the above \(\vec{u} = (u, w)\) is the 2D incompressible velocity field while \(T', p'\) are the temperature and pressure perturbations. \(L\) is the latent heat of condensation and \(C_p\) is the specific heat. We consider an equation of state for an ideal gas wherein \(p = \rho RT\). In effect these equations isolate the effect of condensation (evaporation) via dynamical heating (cooling). We have not included a rain-out velocity or the influence of water substance on the buoyancy and the equation of state. A discussion of these effects and their role in the moist framework can be found in Klein & Majda [51].

As per the usual Boussinesq setup, the buoyancy frequency is given by \((g\lambda/T_0)^{\frac{1}{2}}\) where \(g, T_0, \lambda\)
denote the gravitational acceleration, reference temperature and the background temperature gradient, respectively (i.e. $T = T_0 + \lambda z + T')$. In addition, $q_v(x, z, t)$ and $q_l(x, z, t)$ are the mixing ratios of water vapor and liquid water, and $q_s(x, z)$ is a fixed saturation mixing ratio profile. In essence, the mixing ratios are governed by advection-condensation-evaporation equations, and $q = q_v + q_l$ (the total water content) is materially conserved, i.e. $Dq/Dt = 0$.

Physically, $q_v$ is advected via the fluid flow and if it exceeds the saturation value at the new destination, the mixing ratio proceeds to relax back to its saturation value. Similarly $q_l$ is advected via the flow, and whenever $q_l > 0$ and $q_v < q_s$, the parcel experiences evaporation. These statements are quantified via

$$C = \frac{1}{\tau_c}(q_v(\vec{x}, t) - q_s(\vec{x})) \ H(q_v(\vec{x}, t) - q_s(\vec{x})) ; \quad E = \frac{1}{\tau_e}q_l \ H(q_s(\vec{x}) - q_v(\vec{x}, t))$$

and therefore $S = C - E$ in (1). Here $\tau_c, \tau_e$ are the time-scales associated with condensation and evaporation. $H(\cdot)$ denotes the Heaviside step function. Equation (2) encodes an essential simplification that at any given instant of time a parcel experiences either condensation or evaporation. In the limit of rapid condensation (and evaporation), i.e. $\tau_c, \tau_e \to 0$, we have $0 \leq q_v(\vec{x}, t) \leq q_s(\vec{x})$ and $0 \leq q_l(\vec{x}, t) \leq q_l(\vec{x}_0, 0) + q_s(\vec{x}_0)$ — where $(\vec{x}_0, 0)$ is the Lagrangian origin of the parcel at $(\vec{x}, t)$. One can see that the upper bound on $q_v$ is local while that on $q_l$ is spatio-temporally non-local.

**Conservation properties** : Apart from the material invariance of the total water substance $(Dq/Dt = 0)$, (1) also preserves the material conservation of moist static energy given by $D(T + \frac{L}{C_p}q_v)/Dt = 0$. With regard to an energy conservation law, (1) yields

$$\frac{\partial}{\partial t} \int \left[ u^2 + w^2 + \frac{g}{T_0\lambda} Tr^2 \right] = \frac{g}{T_0\lambda C_p} \int ST'$$

Therefore, the heating and temperature fluctuations have to be correlated for a growth in perturbation energy [3] — a detailed discussion of the conservation properties of moist systems that include bulk cloud microphysics can be found in Frierson, Paulius & Majda [48].

**Vorticity-stream formulation** : We introduce a streamfunction ($\psi$), with $u = -\partial_z \psi, w = \partial_x \psi$ and vorticity $\omega = -\Delta \psi$. Further we consider $q_s = q_s(z) = q_0 - \beta z$, i.e. the saturation mixing ratio is a linear function of $z$. Defining $q_v' = q_v - q_s$, (1) can be re-written as
\[
\frac{D\omega}{Dt} = -\left(\frac{g}{T_0}\right) \frac{\partial T'}{\partial x},
\]
\[
\frac{DT'}{Dt} = -\lambda \psi_x + \frac{L}{C_p} S,
\]
\[
\frac{Dq_v'}{Dt} = \beta \psi_x - S; \quad \frac{Dq_l}{Dt} = S
\]

(4)

Linearizing (4) about our base state of rest, hydrostatic balance, no liquid water and saturation (i.e. \(S = 0\)), we obtain a constant co-efficient system of partial differential equations. Substituting plane wave solutions of the form \(\exp\{i\vec{k} \cdot \vec{x} - \sigma t\}\), in addition to a distinct \(\sigma = 0\) mode, we get

\[
\sigma^3 + \sigma^2 \left(\frac{\delta_1}{\tau_c} + \frac{\delta_2}{\tau_e}\right) + \sigma \frac{k_x^2}{k_z^2} \left(\frac{\lambda g}{T_0}\right) + \frac{k_z^2}{k_x^2} \left(\frac{\lambda g}{T_0}\right) \left(\frac{\delta_1}{\tau_c} + \frac{\delta_2}{\tau_e}\right) - \frac{L}{C_p} \frac{\beta g}{T_0} \delta_1 = 0
\]

(5)

where \(\delta_1 = H(q_v - q_s)\) and \(\delta_2 = H(q_s - q_v)\). As expected, when \((\delta_1, \delta_2) = (1, 0)\) i.e. during condensation, \(\frac{L}{C_p} \beta > \lambda\) leads to an unstable situation. Quite interestingly, when \(\tau_c \to 0\) (rapid condensation) (5) gives \(\max(|\sigma|) = \sqrt{\frac{\delta g}{\delta_c}}\) where \(\delta = \frac{L}{C_p} \beta - \lambda\) and this maximal growth is attained for fields independent of \(z\) — in effective columnar or vertically coherent structures are the fastest growing modes in linear rapid condensation. On the other hand, the slow mode \(\sigma = 0\) is possible only when \(\delta_1 = \delta_2 = 0\), i.e. \(q_v = q_s\). In addition, this leads to the condition of hydrostatic balance with \(w = 0\) implying \(u = u(z)\). Hence, the slow dynamics of the linear system consists of a hydrostatically balanced, saturated VSHF.

A. Non-dimensionalization and asymptotic expansion

Non-dimensionalizing the momentum and energy equations from (1), we obtain (dropping primes)

\[
\frac{Du}{Dt} = -(Eu) \frac{\partial \phi}{\partial x},
\]
\[
\frac{ Dw}{Dt} = -(Eu) \frac{\partial \phi}{\partial z} + \left(\frac{\Gamma_\theta}{Fr^2}\right) T
\]
\[
\frac{ DT}{Dt} = -(\frac{1}{\Gamma_\theta})w + \left(\frac{\tau_a}{\tau_c}\right) S
\]
\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.
\]

(6)

Here \(Eu\) and \(Fr\) are the Euler and Froude numbers, and \(\phi = p/\rho_0\). \(\Gamma_\theta\) is a non-dimensional number defined by \(\Gamma_\theta = \Theta/\lambda L\) where the temperature is non-dimensionalized using the scale
parameter $\Theta$. Further, we have non-dimensionalized $LS/C_p$ by the same scale parameter as the temperature along with $\tau_c$ the time-scale associated with condensation (taken to be the same as that for evaporation). Finally, $\tau_a$ represents $U/L$ the advective time-scale. As we have only nondimensionalized the momentum and energy equations, for completeness $S$ in (6) is derived from the state of the vapor and liquid water fields.

Since our interest is in stratified systems, we set $Fr = \epsilon < 1$ in the following asymptotic analysis. The potential and kinetic energies in the Boussinesq system are defined by $u^2 + w^2$ and $(g/\lambda T_0) T^2$ respectively. Using an a priori equipartition of the potential and kinetic energies we see that $Fr \sim \Gamma_\theta$. Further, we choose $Eu = 1$. Clearly, in this environment, the influence of heating on the dynamics is likely to be significant when the time-scale associated with condensation (and evaporation) is much smaller than the advective time-scale, i.e. $\tau_a \tau_c \geq 1/\epsilon$. In this regime, we consider the two cases: (i) $\tau_a = 1/\epsilon$ and (ii) $\tau_a = 1/\epsilon^2$ with $\epsilon < 1$. In the first case, as $\epsilon \to 0$, the system is influenced by equally strong effects of stratification (via $Fr$) and heating (via $1/\tau_c$). In the second case, the system is stratified and in the presence of a dominant heat source.

Case I: With the aforementioned substitutions, (6) reads

$$
\begin{align*}
\frac{Du}{Dt} &= -\frac{\partial \phi}{\partial x} \\
\frac{Dw}{Dt} &= -\frac{\partial \phi}{\partial z} + \frac{1}{\epsilon} T \\
\frac{DT}{Dt} &= -\frac{1}{\epsilon} w + \frac{1}{\epsilon} S \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.
\end{align*}
$$

Note that as $\epsilon \to \infty$, (7) reduces to the incompressible 2D Euler equations with temperature being passively advected by the flow. For small $\epsilon$, plugging in an asymptotic expansion of the form $f = f^0 + \epsilon f^1 + ...$ for all fields, at $O(1/\epsilon)$ we obtain

$$
T^0 = 0; \quad w^0 = S^0
$$

while at $O(1)$ we have
\[ \frac{D^0}{Dt} u^0 = -\frac{\partial \phi^0}{\partial x} \]
\[ \frac{D^0}{Dt} w^0 = -\frac{\partial \phi^0}{\partial z} + T^1 \]
\[ w^1 = S^1 ; \frac{\partial u^0}{\partial x} + \frac{\partial w^0}{\partial z} = 0. \] (9)

Therefore it is clear that if \( S^0 = 0 \), \( q_v = q_s \) then \( w^0 = 0 \) and \( u^0 = u^0(z,t) \). In essence we have a linear saturated VSHF in hydrostatic balance (between \( \phi^0 \) and \( T^1 \)) which for steady conditions is the previously encountered \( \sigma = 0 \) mode. We call this a state of balance for the moist stratified problem. In general, for \( S^0 \neq 0 \), (9) is a 2D version of the VSHF with sources (or VSHFS) model proposed by Majda et al. [52]. In fact, a three dimensional balanced model with a structure similar to (9) has recently been developed to understand the formation and evolution of hurricane embryos [32].

It is worth asking when (9) can achieve a state of saturation. Quite evidently, if \( q(x,z,t = 0) \geq q_s(x,z) \) then a final state with \( q_v(x,z,t \rightarrow \infty) = q_s(x,z) \) is possible, but this is too strong a condition. In the absence of diffusion, we define an initial condition as sufficient in water substance as one which can saturate the domain by incompressible advective re-arrangement. In the numerical investigation, we will examine this special sufficient class of initial conditions as well as the more general class that is in-sufficient in the above mentioned sense. Our aim is to examine the long-time fate of such initial conditions, and in particular to test the accessibility of the balanced state in (9).

**Case II :** With \( \frac{\tau_a}{\tau_c} = \frac{1}{\epsilon^2} \), we obtain the same system as (7) except for a factor of \( \frac{1}{\epsilon^2} \) preceding \( S \) in the temperature equation. Now the leading order balance is at \( O(1/\epsilon^2) \) and reads : \( S^0 = 0 \). At \( O(1/\epsilon) \) we have \( T^0 = 0, w^0 = S^1 \) and at \( O(1) \) we recover (9) except for the condition that \( w^1 = S^2 \). Therefore, in this case even though we obtain a VSHFS system, the leading order balance requires the heating to vanish. In the numerical section of the paper, we examine the difference between this and the preceding limiting scenario.

**III. EVOLUTION OF THE SYSTEM**

Numerically, we solve the system in the vorticity-stream form using a pseudo-spectral method in a periodic domain of size \( L_d \). The time stepping is via a fourth order Runge Kutta scheme. Dissi-
vation is of the form of a hyperviscous damping given by \((-1)^{n+1} \nu_n \Delta^n f\) (for all fields \(f\)). In particular, we adopt the formulation employed by Maltrud and Vallis [53] wherein \(\nu_n = f_{\text{rms}}/(k_m^{2n-2})\), where \(k_m\) is the maximum resolved wavenumber. In all simulations the hyperviscous order is eight. All short-term simulations are performed at a resolution of \(512 \times 512\) while the long-term simulations are at a resolution \(350 \times 350\). The equations solved are given by (4), and we set \(g/T_0 = \lambda = \beta = 1/\epsilon\). The linear saturation profile is specified as \(q_s = (L_d/\epsilon)[1 - z/L_d]\). As \(\beta = \lambda\), the condition for linear instability reduces to \(L/C_p > 1\) and in all the simulations we set this ratio to 4 (different values were employed for shorter runs to verify the qualitative similarity of solutions).

**A. Short-time evolution**

We begin with a set of short-time linear and non-linear simulations. Based on the \(\tau_c \to 0\) linear analysis, we expect vertically coherent structures to be the most unstable modes. As the maximal growth rate is independent of wavenumber, we examine the growth of spatially un-correlated initial data with power at all scales as well as spatially correlated spectrally localized (single-scale) initial conditions. In both cases, \(\tau_c = \tau_e = \epsilon = 0.1\) and the resolution is \(512 \times 512\). Note that with this choice of parameters the system is influenced by equally strong stratification and heating. Figs. (1) and (2) show the initial conditions and their fate after linear and nonlinear evolution for a short period of time (specifically, 10 non-dimensional \(t/\tau_c\) units). Quite clearly, vertically coherent structures emerge most rapidly, these persist in the linear run while they begin to distort towards the end of the non-linear simulation. Indeed, we expect this distortion to proceed as will be seen in detail in the next section. Repeating the linear calculations from un-correlated initial conditions for varying \(\epsilon (= \tau_c = \tau_e)\), in Fig. (3) we examine the growth of the total dry energy (LHS of (3)). As is seen in a plot of the log of the dry energy vs \(t/\epsilon\), all the curves approximately collapse on to each other and the growth rate scales as \(1/\epsilon\). This is precisely the estimate from our linear \(\tau_c \to 0\) analysis after substituting \(g/T_0 = \lambda = \beta = 1/\epsilon\). Note further that we have better agreement at smaller \(\epsilon\). In fact, the growth rate appears to converge to \(\sqrt{3}\) as would be expected for our choice of \(L/C_p = 4\).
FIG. 1: The first panel is the single-scale initial condition. The second and third panels show the emergent perturbed vapor field \((q'_v)\) for \(\tau_c = \tau_e = \epsilon = 0.1\) in the linear and nonlinear simulations, respectively.

FIG. 2: The first panel is the random spatially un-correlated initial condition. The second and third panels show the emergent perturbed vapor field \((q'_v)\) for \(\tau_c = \tau_e = \epsilon = 0.1\) in the linear and nonlinear simulations.

FIG. 3: A plot of the log of the LHS of (3) Vs \(t/\epsilon\) for spatially un-correlated initial conditions (smaller \(\epsilon\) runs are carried out to progressively longer times). For varying \(\epsilon(= \tau_c = \tau_e)\) (from 1 to 0.001), the curves approximately fall on each other. Further, as \(\epsilon\) decreases the growth rate appears to converge to \(\sqrt{3}\) as would be anticipated from the \(\tau_c \to 0\) limit with \(L/C_p = 4\).
B. Scaling and structure formation

We now continue with the nonlinear simulations to detect and quantify the emergence of scaling and coherence from initially un-correlated conditions. Setting $\tau_c = \tau_e = \epsilon = 0.1$, Fig. (4) shows the evolution of the power spectrum of the vapor variance (i.e. $(q'_v)^2$) for un-correlated uniformly distributed initial data (first panel of Fig. (2)). The initial condition is such that the entire domain is super-saturated. Hence at small times we have condensation over the whole domain which manifests itself in a drop in the total vapor variance. Subsequently, as seen in the sequentially numbered curves, the spectrum evolves via a systematic growth of variance at all scales. From the second panel of Fig. (4) the initial growth of vapor variance is very rapid, followed by a regime of slower almost exponential increase. Quite interestingly, this initial rapid growth lasts for a longer time for progressively larger scales, hence in a short time the spectrum attains a characteristic negative slope. Finally, at longer times, the spectrum attains a fairly invariant form (with fluctuations at the largest scales). This invariant form, averaged over a number of time-steps is also shown in the first panel of Fig. (4). For reference, we have plotted a line with a slope of $-5/3$ in this figure. It is worth emphasizing that even though we obtain an invariant spectrum which scales approximately as $-5/3$, the mechanism of generation of this spectrum is quite different from a traditional 2D inverse cascade. Specifically, in the present unstable setting, the vapor field experiences a growth of energy at all scales and the spectral form emerges due to the comparatively larger growth at large scales. Along with this spectral evolution, we also notice the emergence of ordered structures from the initially spatially un-correlated data. Indeed, Fig. (5) shows the $q'_v$ field at times corresponding to the time slices labelled 6, 15 and in the spectrally invariant state.

C. Long time behavior

We now proceed to the long-time evolution of the system. As before $g/T_0 = \lambda = \beta = 1/\epsilon$, so $Fr = \frac{\epsilon}{\tau_a}$ and the two cases of interest are : $\frac{\tau_a}{\tau_c} = \frac{\tau_a}{\epsilon}$ and $\frac{\tau_a}{\tau_c} = \frac{\tau_a}{\epsilon^2}$. In the first case $\frac{\tau_a}{\tau_c} \sim \frac{1}{\tau_a Fr} \sim \frac{1}{\epsilon}$ i.e. as $\epsilon \to 0$ we have a system that is strongly stratified and operates in the limit of rapid condensation/evaporation. While in the second case, for $\epsilon \to 0$, in terms of orders of magnitude the system is moderately stratified when compared to the presence of a dominant heat source.
FIG. 4: First panel: A plot of the log of the vapor variance as it evolves in time. The numbers on the plots indicate the time ordering (note that the spacing between these time slices is not equal). The flat thick line is the initial power spectrum, which as prescribed is uniform in wavenumber. The other dark dotted line is the mean spectrum, averaged over 100 time steps towards the end of the run when the spectrum has attained a fairly invariant form. For reference we have also drawn a power-law with $-5/3$ slope. Second panel: The growth of variance in time for selected wavenumbers.

FIG. 5: Evolution of a spatially un-correlated initial condition. The three panels show the emergent perturbed vapor field ($q_v'$) for at time labels 6, 15 and 100 (i.e. in the spectrally invariant regime) as referred to in Fig. (4).

1. $\tau_a/\tau_c = 1/\epsilon^2$: Simple Balance

We first investigate the second regime considered in the asymptotic analysis of Case II in Section A. The resolution is $350 \times 350$ and we start from spatially un-correlated uniformly distributed data such that $q_v' \geq 0$. The simulation was carried out for $\epsilon = 0.1, 0.05$ and $0.01$. In all cases the results were qualitatively similar and we present results from the $\epsilon = 0.01$ run. Note that the saturation profile is scaled such that $\max (q_s) = L_d/\epsilon$. Therefore, for smaller $\epsilon$ we have progressively larger values permissible for the vapor and liquid fields. Examining the power spectrum of the vapor field (not shown), as before we see the growth of energy at all scales in a manner similar to that
FIG. 6: The first two panels show snapshots of the vapor field (i.e. \( q_v = q_v' + q_s \)) as it evolves in time for the case \( \frac{\tau_a}{\tau_c} = \frac{1}{\epsilon^2} \) and \( \frac{1}{\tau_a Fr} \sim \frac{1}{\epsilon} \) (with \( \epsilon = 0.01 \)). The fields correspond to approximately 4 and 10 non-dimensional time units respectively. The third panel shows the total water substance (i.e. \( q_l + q_v \)) at 10 non-dimensional time units.

FIG. 7: The first two panels show snapshots of the horizontal velocity field that correspond to the snapshots shown in Fig. (6). The third panel shows the mean horizontal flow as a function of the non-dimensional time (\( \frac{t}{\epsilon} \)) : once a VSHF is formed it persists for the duration of the simulation.

shown in Fig. (4). Indeed, Fig. (6) shows snapshots of the vapor field (i.e. \( q_v = q_v' + q_s \)), after the development of ordered structures from the initially un-correlated state. One sees that the domain tends to saturation with \( q_v \to q_s \). Correspondingly, the first two panels of Fig. (7) show the horizontal velocity field associated with Fig. (6), and as anticipated we observe the emergence of a VSHF as \( q_v \to q_s \). In addition, the last panel of Fig. (7) shows the mean horizontal flow with time, indicating the permanence of the VSHF once it is formed. Further, as the domain tends to saturation, the heating vanishes and the total water substance is passively advected by the VSHF. This shear flow advection can been seen in the third panel of Fig. (6) which shows the total water substance in the domain.
2. \( \tau_a/\tau_c = 1/\epsilon \): Aperiodic VSHF regime

We now proceed to the regime wherein the stratification strength is comparable to that of the heating due to condensation/evaporation, i.e. \( \frac{\tau_a}{\tau_c} \sim \frac{1}{\tau_c Fr} \sim \frac{1}{\epsilon} \). The simulations are carried out at a resolution of 350 \times 350 for \( \epsilon = 0.01, 0.001, 0.0001 \). In all cases, the evolution was similar and we focus on the case \( \epsilon = 0.001 \) but comparative plots from other runs are also presented.

a. Initial data with sufficient water substance: We first consider sufficient initial data in the sense defined earlier, i.e. the initial water substance is sufficient to saturate the domain via advective re-arrangement. To satisfy this condition we start with supersaturation everywhere and \( q_v \) is a uniformly distributed spatially un-correlated field. As expected the spectrum of the vapor field evolves from its initially flat form — much like in Fig. (4) — and coherent structures develop with this spectral evolution. In fact, the early behavior of the system is the same as in Section III B.

As the spectra of the vapor and liquid fields attain an invariant form, we examine the state of the flow. In particular, Fig. (8) shows the horizontal and vertical components of the kinetic energy as functions of non-dimensional time for \( \epsilon = 0.01, 0.001, 0.0001 \). In each case, at early times the system oscillates with comparable contributions to the kinetic energy from both the components. At a critical time \( (\tau^{*}_e) \) the system experiences a rather abrupt transition beyond which the horizontal component comes to dominate the kinetic energy budget. Focussing on \( \epsilon = 0.001 \), the first and second panels of Fig. (9) show the horizontal flow before and after \( \frac{t}{\epsilon} = \tau^{*}_e \approx 50 \): as is evident, at long times we observe the development of a dominant VSHF. In fact, this VSHF is a persistent feature which can be seen in the third panel of Fig. (9). Shifting our attention to the vapor field, the first panel of Fig. (10) shows the power spectrum of the vapor variance through the system’s evolution and the other two panels show the total vapor field at times corresponding the flow snapshots in Fig. (9). As can be seen from the power spectrum, even though the scaling is roughly maintained, the shift in the spectra with time indicates that the magnitude of the variance in the vapor field is slowly decaying, and as per the third panel of Fig. (9), the domain does tend to saturation. Though not shown, the development of a VSHF and the tendency to saturation shown for \( \epsilon = 0.001 \) is also seen in corresponding plots for the other two values of \( \epsilon \) we have considered. Gathering these pieces together we argue that at long times the system — with sufficient initial water substance — approaches a hydrostatic saturated VSHF as prescribed in asymptotic analysis for the rapidly condensing, \( Fr \to 0 \) limit.
b. Initial data with in-sufficient water substance  We now proceed to initial data that is in-sufficient in the sense defined earlier, i.e. the initial water substance cannot saturate the domain via advective re-arrangement. To satisfy this condition we supersaturate the domain in a localized Gaussian blob, while undersaturating it in a similar manner in the rest of the domain. We ensure that $q_\epsilon(x, z, t = 0) \geq 0$, the total initial water substance is less than the amount required for the domain to be saturated and that $q_\epsilon'$ is a spatially periodic function. The supersaturation ensures that the system evolves without external forcing.

A snapshot of the initial vapor field, and its subsequent evolution for $\epsilon = 0.001$ is shown in Fig. (11). As expected we do not approach saturation. In fact, the first panel of Fig. (12) shows the vertical and horizontal contribution to the kinetic energy, and quite clearly we do not observe...
FIG. 10: The first panel shows the mean vapor variance spectrum for different periods of time. The curves labelled 1, 2 and 3 correspond to $20 \leq \frac{t}{\epsilon} \leq 40$, $80 \leq \frac{t}{\epsilon} \leq 100$ and $120 \leq \frac{t}{\epsilon} \leq 140$ respectively. The second and third panels show the total vapor field before and after the transition at $\tau^*$, in particular the second panel corresponds to a non-dimensional time of 35 units while the third panel is a snapshot of the vapor field at 190 time units.

any transition to horizontal flow dominance as was noted in the previous section. Shifting our attention to the heating, the second panel of Fig. (12) shows a time-series of the domain-averaged heating. From the main plot as well as the inset it is seen that the heating evolves on multiple scales — there is an obvious large scale oscillation as seen in the main plot and also a fair amount of fluctuation at finer time-scales as indicated by the inset figure. To quantify the characteristics of the domain-averaged heating time-series, in Fig. (13) we show its power spectrum. Clearly, there is a large amount of power at small frequencies or large time-scales. Further, to characterize the fluctuations that occur within one large-scale oscillation, we consider the spectrum of an individual oscillation (specifically, the one highlighted in red in Fig. (12)). As shown in the right-corner inset, this exhibits close to red scaling. The physical picture that emerges is that the domain-averaged heating consists of a series of slow temporal oscillations which account for the bulk of the power in the heating spectrum, while the finer scale fluctuations within each of these oscillations behave as a random walk and exhibit a red-noise scaling. In order to see the departure from red behavior, the left inset of Fig. (13) shows the red-compensated spectrum. As expected, the peak at low frequencies stands out as a systematic deviation. Noting the behavior of the domain-averaged heating, in Fig. (14) we examine the red-compensated spectrum of the horizontal and vertical contributions to the kinetic energy obtained from all the three of the runs for $\epsilon = 0.01, 0.001$ and 0.0001, i.e. of time-series akin to the one shown in the first panel of Fig. (12). On comparing with the time-scale associated with condensation/evaporation displayed in the plot, in each case we note that for high frequencies the spectra follow a red scaling while at low frequencies we observe a series of distinct peaks corresponding to a slower scale oscillation. Further, as $\epsilon$ decreases we note
FIG. 11: The first panel shows the initial vapor field, while the inset shows a cross-section of $q_v$ and $q_s$ through the center of the domain. The initial condition is chosen so as to maintain supersaturation in a small portion, while under-saturating it to an extent that the total initial water substance is not enough to saturate the domain. The second panel shows the vapor ($q_v$) field at $\frac{t}{\epsilon} \approx 70$.

the clearer scale separation between $\tau_c (= \epsilon)$ and the low-frequency peaks as well as an amplitude separation between the red and low-frequency portions of the spectrum.

Following the asymptotic VSHFS description, we expect that in the limit $\epsilon \to 0$, the moist system with insufficient water substance will continue to evolve in the described aperiodic or quasi-periodic fashion whose dynamics are naturally significantly slower than the imposed rapid condensation/evaporation. Note that this is a rich regime in that the heating never vanishes, and the water substance continues to be a dynamically active scalar field.

IV. CONCLUSION AND DISCUSSION

We have considered a 2D moist Boussinesq system in an idealistic periodic setting with simple condensation and evaporation schemes. In particular, these schemes are designed to push individual parcels towards a saturated state. This allows possibly the simplest scenario wherein we can study the interaction of advection with evaporation and condensation in a dynamically consistent manner. Linearizing the problem about a state of saturation, with no liquid water and hydrostatic balance showed that the most unstable modes were vertically coherent. In fact, in the limit of rapid condensation, $\tau_c \to 0$, the form of the growth rate of the unstable modes took a particularly simple form, namely $\max(\sigma) = \sqrt{\frac{\delta}{L}}$ where $\delta = \frac{L}{c_p} \beta - \lambda$. A suite of small-time linear simulations verified these predictions. In particular, Fig. (3) shows the agreement of the observed growth rate with the linear prediction for small values of $\tau_c$. 
FIG. 12: The first panel shows the contribution of the horizontal (blue) and vertical (red) components to the kinetic energy of the flow as a function of $\frac{t}{\epsilon}$. The second panel shows the domain-averaged heating (normalized by its maximum value) as a function of $\frac{t}{\epsilon}$. There are approximately twenty large oscillations over the time-period shown (one such oscillation is highlighted in red). As is seen in the inset of the second panel — which zooms into a small time span as denoted by the star (around $\frac{t}{\epsilon} = 24.5$) on the larger plot — there are fluctuations in the average heating at a much finer time-scale.

One of the main questions we wished to address in this work was the development of initially un-correlated vapor fields. Keeping in mind that the 2D Boussinesq system does not conserve enstrophy, and that there is significant theoretical and numerical evidence showing that the system supports a transfer of energy from large to small scales [17],[18],[19],[20],[21],[22],[23], it was not a priori evident to us that initially un-correlated fields would develop into coherent structures, or that one would see a growth of energy at large scales. Pursuing our nonlinear simulations we observe that coherent structures do emerge spontaneously from spatially un-correlated initial conditions. Further, as is shown in Fig. (4), the scaling of the vapor variance attains an invariant form and follows a power-law. Note that the scaling fluctuates at the very largest scales, and this is reflected in the changing snapshots of the vapor field. Quite interestingly, the growth of energy at large scales cannot be attributed to an inverse transfer kind of mechanism seen in pure 2D turbulence wherein energy is transferred from small to large scales. In fact, in an unstable setting, as per Fig. (4), the variance grows very rapidly at all scales and this rapid growth persists for a longer time at larger scales leading to a characteristic negatively sloped spectrum. Given that the emergence of order in the vapor field does not depend on the conservation of enstrophy, we are optimistic that a similar development of multiscale ordered structures from initially uncorrelated fields will be possible in a three-dimensional moist Boussinesq system.
FIG. 13: Power spectrum of the heating time-series seen in Fig. (12). Specifically, we consider two long segments, each encompassing multiple oscillations seen in the signal. The main plot in panel one shows the power spectrum when averaged over two such windows. The inset on the upper-right shows the spectrum of a single oscillation. In particular, the spectrum shown corresponds to the highlighted (red) region in the main plot of the second panel of Fig. (12). The second inset, on the lower-left corner shows the red-compensated spectrum, i.e. spectrum after a multiplication by the square of the frequency.

FIG. 14: The red-compensated power spectrum of the temporal contribution of the horizontal and vertical components to the kinetic energy for $\epsilon = 0.01, 0.001$ and 0.0001 (the $\epsilon = 0.001$ time series is shown in the first panel of Fig. (12)). The vertical bars denote the time-scale corresponding to $\tau_{\epsilon}(=\epsilon)$ for the three cases respectively ($\tau_{\epsilon_1}$ corresponds to $\epsilon = 0.01$ etc.).
Our other goal was to examine the notion of balance and slowness in physically relevant limits of this idealized moist system. Imposing strong stratification (implying the presence of high frequency gravity waves) along with rapid condensation (implying rapid heating/cooling via condensation/evaporation) we looked for a reduced description of the dynamics, or a state of balance towards which the system may strive to evolve at long times. Formally, non-dimensionalizing the equations and considering an asymptotic expansion in the limit \((Fr, \tau_c) = \epsilon \to 0\) we found that, if the leading order heating vanishes then the system tends to a hydrostatic and saturated VSHF (a state of moist balance). In general, for non-zero heating the reduced description consists of the so-called VSHF with sources model derived previously by Majda et al. [52].

We also considered the case with strong stratification in the presence of dominant heating, i.e., \(Fr = \epsilon\) while \(\tau_c = \epsilon^2\) for \(\epsilon \to 0\). As shown, the reduced description remains the same as previously discussed, though the vanishing of the leading order heating is imposed in a natural manner.

We began our numerical investigation by considering the second limit described above, i.e., strong stratification in the presence of dominant heating. As expected, coherent structures emerged from spatially un-correlated initial conditions. Further, in a short time we noted the tendency of the domain to achieve saturation along with the development of a persistent VSHF. In all, as per asymptotic expectations the system immediately tended to a hydrostatic, saturated VSHF.

Proceeding to the second limit, i.e. \((Fr, \tau_c) = \epsilon \to 0\), we distinguish between two classes of initial conditions. The first are special initial conditions and have sufficient water substance, i.e. the initial water substance is enough to saturate the domain via advective re-arrangement. Employing these initial conditions, we performed simulations for \(\epsilon \ll 1\). Once again, the fields evolved as per expectations with coherent structures and scaling emerging from initially un-correlated conditions. Focussing on the kinetic energy, we traced the contribution of the horizontal and vertical components to the kinetic energy as a function of non-dimensional time \((t/\epsilon)\). At small times, as the scaling evolved and as coherent structures emerged we noted oscillations in this plot with equal contribution from either component at different times. However, at \(t/\epsilon = \tau_c^*\) we observed a rather abrupt transition beyond which the horizontal component came to dominate the kinetic energy budget. Examining the state of the flow after \(t/\epsilon = \tau_c^*\), we saw the formation of a persistent and dominant VSHF. In addition, the vapor field tended to saturation for \(t/\epsilon \gg \tau_c^*\). In essence, we argue that the system transitions to the asymptotic reduced state of moist balance.

Continuing with our examination of the \(Fr = \tau_c = \epsilon \to 0\) limit, we proceeded to the more
general class of initial conditions which are insufficient in initial water substance. The now generic behavior involving coherent structures and scaling was observed, but in contrast to the previously considered special initial conditions the domain never tended to saturation. Examining the domain-averaged heating, we noted a wide range of temporal scales. The time-series of the heating as well as its spectrum revealed dominant slow oscillations and a fine temporal structure (characterized by a red spectrum) within each such oscillation. Comparing the contributions of the horizontal and vertical components to the total kinetic energy, we noted that there is no transition as seen before (to horizontal flow dominance), and in fact the time-series of the two components formed an aperiodic or quasi-periodic oscillation. The oscillations seen in the domain-averaged heating and the kinetic energy series were of comparable time-periods. Of course, there was no single period associated with these oscillations indicating their quasi-periodic or aperiodic nature. An important feature of these oscillations was that their time-scale is much larger than that of the imposed condensation/evaporation time-scale. In fact, for decreasing $\epsilon = \tau_c = Fr$ we observed a greater scale separation between the imposed condensation/evaporation time-scale and the low-frequency peaks associated with the aforementioned oscillation. In addition, the amplitude difference between these peaks and the higher frequency red portion of the heating time-series spectrum was also seen to increase. Therefore, even though the system did not achieve a state of moist balance, its long time evolution was dominated by series of slow oscillations.

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