NON-EQUILIBRIUM QUANTUM FIELD EVOLUTION
IN FRW COSMOLOGIES

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Based on a lecture at the NATO Advanced Research Workshop on ELECTROWEAK PHYSICS AND THE EARLY UNIVERSE held at Sintra, Portugal from March 22 to 27, 1994.
We derive the effective equations for the out of equilibrium time evolution of the order parameter and the fluctuations of a scalar field theory in spatially flat FRW cosmologies. After setting the problem in general we propose a non-perturbative, self-consistent Hartree approximation. The method consists of evolving an initial functional thermal density matrix in time and is suitable for studying phase transitions out of equilibrium. The renormalization aspects are studied in detail and we find that the counterterms depend on the initial state. We investigate the high temperature expansion and show that it breaks down at long times. The infinite time limit is computed for de Sitter spacetime. We obtain the time evolution of the initial Boltzmann distribution functions, and argue that in the Hartree approximation, the time evolved state is a “squeezed” state. We illustrate the departure from thermal equilibrium by studying the case of a free field in de Sitter and radiation dominated cosmologies. It is found that a suitably defined non-equilibrium entropy per mode increases linearly with comoving time in a de Sitter cosmology, whereas it is not a monotonically increasing function in the radiation dominated case. This work has been done in collaboration with D. Boyanovsky and R. Holman.

1 Introduction

The usual assumption in inflation is that the dynamics of the spatial zero mode of the (so-called) inflaton field is governed by some approximation by the effective
potential which incorporates the effects of quantum fluctuations of the field. Thus the equation of motion is usually an ordinary non-linear differential equation of the form:

$$\ddot{\phi} + 3 \dot{a} \dot{\phi} + V'_\text{eff}(\phi) = 0$$  \hspace{1cm} (1)

where $a(t)$ is the expansion factor in the metric:

$$ds^2 = dt^2 - a^2(t) \, d\vec{x}^2$$  \hspace{1cm} (2)

The problem here is that the effective potential is really only suited for analyzing static situations; it is the effective action evaluated for a field configuration that is constant in time. Thus, it is inconsistent to use the effective potential in a dynamical situation. Notice that such inconsistency appears for any inflationary scenario (old, new, chaotic, ...).

More generally, the standard methods of high temperature field theory are based on an equilibrium formalism; there is no time evolution in such a situation. Such techniques preclude us from treating non-equilibrium situations such as surely exist for very weakly coupled theories in the early universe.

More precisely, the effective potential develops an imaginary part around points $\phi_0$ where $V''(\phi_0) < 0$ [$\phi_0$: false vacuum]. This fact leads to long-wavelength modes with imaginary frequencies $\omega(k) = \sqrt{k^2 + V''(\phi_0)}$ for $0 \leq k < \sqrt{|V''(\phi_0)|}$. When the field $\phi$ starts localized around $\phi_0$, these long-wavelength unstable modes push the system to evolve on time away from $\phi_0$. The smaller is the non-linear coupling, the better is this small fluctuations approximations and the more important are these unstabilities.

In collaboration with Daniel Boyanovsky and Rich Holman we try to rectify this situation by addressing three issues: a) obtaining the evolution equations for the order parameter including the quantum fluctuations, b) studying departures from thermal equilibrium if the initial state is specified as a thermal ensemble, c) understanding the renormalization aspects and the validity of the high temperature expansion.

Our ultimate goal is to study the dynamics of phase transitions in the early universe, in particular, the formation and evolution of correlated domains and symmetry breaking in an expanding universe. From some of our previous studies on the dynamics of phase transitions in Minkowski space, we have learned that the familiar picture of “rolling” is drastically modified when the fluctuations are taken into account. As the phase transition proceeds fluctuations become large and correlated regions (domains) begin to grow. This enhancement of the fluctuations modifies substantially the evolution equation of the order parameter. Thus the time dependence of the order parameter is not enough to understand the dynamical aspects of the phase transition; it must be studied in conjunction with that of the fluctuations.

Our approach is to use the functional Schrödinger formulation, wherein we specify the initial wavefunctional $\Psi[\Phi(\vec{x}); t]$ (or more generally a density matrix $\rho[\Phi(\vec{x}), \dot{\Phi}(\vec{x}); t]$), and then use the Schrödinger equation to evolve this state in time. We can then use this state to compute all of the expectation values required in the construction of the effective equations of motion for the order parameter of the theory, as well as that for the fluctuations.

One advantage of this approach is that it is truly a dynamical one; we set up initial conditions at some time $t_o$ by specifying the initial state and then we follow
the evolution of the order parameter \( \phi(t) \equiv \langle \Phi(\vec{x}) \rangle \) and of the fluctuations as this state evolves in time. Another advantage is that it allows for departures from equilibrium. Thus, issues concerning the restoration of symmetries in the early universe can be addressed in a much more general setting.

Our analysis applies quite generally to any arbitrary spatially flat FRW cosmology. We also determine the time evolution of the initial (Boltzmann) distribution functions, relate the time evolution to “squeezed states” and perform a numerical integration in the case of free fields for de Sitter and radiation dominated cosmologies. We expect to provide a quantitative analysis of the evolution of the order parameter and the dynamics of phase transitions for interacting fields in a forthcoming article [8].

The initial state we pick for the field \( \Phi(\vec{x}, t) \) is that corresponding to a thermal density matrix centered at \( \phi(t) \). It is then useful to try to understand the high temperature limit of our calculations. We are able to compute both the leading and subleading terms in the high \( T \) expansion of \( \langle \phi^2(t) \rangle \). From this we show that the high \( T \) expansion cannot be valid for all time, but breaks down in the large time limit. We then compute the limit of long times \( (t \rightarrow \infty) \) during the phase transition.

The time evolution of the Boltzmann distribution functions (initially the thermal equilibrium distribution functions) is obtained in ref.[7]. It is pointed out that to one-loop order and also in the Hartree approximation, the time evolved density matrix describes quantum “squeezed” states and time evolution corresponds to a Bogoliubov transformation.

To illustrate the departure of equilibrium, we have studied numerically in ref.[7] the case of a free massive scalar field in de Sitter and radiation dominated cosmologies. It was found that a suitably defined coarse-grained non-equilibrium entropy (per \( \vec{k} \) mode) grows linearly with time in the de Sitter case but it is not a monotonically increasing function of time in the radiation dominated case. This result may cast some doubt on the applicability of this definition of the non-equilibrium entropy. There still remain some (open) fundamental questions regarding the connection of this entropy and the thermodynamic entropy of the universe, in particular whether the amount of entropy produced is consistent with the current bounds.

This work sets the stage for a numerical study of the dynamics of phase transitions in cosmology fully incorporating the non-equilibrium aspects in the evolution of the order parameter and which at the same time can account for the dynamics of the fluctuations which will necessarily become very important during the phase transition.

We have applied similar methods to investigate the formation of disordered chiral condensates in high energy collisions [9].

\section{Evolution Equations}

We consider the inflaton scalar field \( \Phi(\vec{x}, t) \) in the spatially flat FRW cosmology (2) with action given by

\[ S = \int d^4x a^3(t) \left[ \frac{1}{2} \dot{\Phi}^2(\vec{x}, t) - \frac{1}{2} \frac{\left( \nabla \Phi(\vec{x}, t) \right)^2}{a(t)^2} - V(\Phi(\vec{x}, t)) \right] \]  

\[ V(\Phi) = \frac{1}{2} [m^2 + \xi \mathcal{R}] \Phi^2(\vec{x}, t) + \frac{\lambda}{4!} \Phi^4(\vec{x}, t) \]  

\[ \mathcal{R} = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \]
with $R$ the Ricci scalar and $\xi$ a non-minimal coupling constant.

In the Schrödinger representation (at an arbitrary fixed time $t$), the Hamiltonian becomes

$$H(t) = \int d^3 x \left\{ -\frac{\hbar^2}{2a^3(t)} \delta^2 \Phi(x) + \frac{a(t)}{2} (\nabla \Phi)^2 + a^3(t) V(\Phi) \right\}$$  \hspace{1cm} (6)

Since we consider a ‘thermal ensemble’, we work with a (functional) density matrix $\hat{\rho}$ with matrix elements in the Schrödinger representation $\rho[\Phi(\vec{x}), \tilde{\Phi}(\vec{x}), t]$. We will assume that the density matrix obeys the functional Liouville equation

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [H(t), \hat{\rho}]$$ \hspace{1cm} (7)

Normalizing the density matrix such that $Tr\hat{\rho} = 1$, we define as ‘order parameter’

$$\phi(t) = \frac{1}{\Omega} \int d^3 x \langle \Phi(\vec{x}, t) \rangle = \frac{1}{\Omega} \int d^3 x \ Tr [\hat{\rho}(t) \Phi(\vec{x})]$$ \hspace{1cm} (8)

where $\Omega$ is the comoving volume.

The evolution equations for the order parameter are as follows:

$$\frac{d^2 \phi(t)}{dt^2} + \frac{3}{a(t)} \frac{d a(t)}{dt} \frac{d \phi(t)}{dt} + \frac{1}{\Omega} \int d^3 x \left\langle \frac{\delta V(\Phi)}{\delta \Phi(x)} \right\rangle = 0$$ \hspace{1cm} (9)

It is now convenient to write the field in the Schrödinger picture as

$$\Phi(\vec{x}) = \phi(t) + \eta(\vec{x}, t)$$ \hspace{1cm} (10)

$$\langle \eta(\vec{x}, t) \rangle = 0$$  \hspace{1cm} (11)

Expanding the right hand side of (9) we find the effective equation of motion for the order parameter:

$$\frac{d^2 \phi(t)}{dt^2} + \frac{3}{a(t)} \frac{\dot{a}(t)}{a(t)} \frac{d \phi(t)}{dt} + V'(\phi(t)) + \frac{V''(\phi(t))}{2\Omega} \int d^3 x \left\langle \eta^2(\vec{x}, t) \right\rangle + \cdots = 0$$ \hspace{1cm} (12)

where primes stand for derivatives with respect to $\phi$.

This equation of motion is clearly very different from the one obtained by using the effective potential. It may be easily seen (by writing the effective action as the classical action plus the logarithm of the determinant of the quadratic fluctuation operator) that this is the equation of motion obtained by the variation of the one-loop effective action.

The static effective potential is clearly not the appropriate quantity to use to describe scalar field dynamics in an expanding universe. Although there may be some time regime in which the time evolution is slow and fluctuations rather small, this will certainly not be the case at the onset of a phase transition. As the phase transition takes place, fluctuations become dominant and grow in time signaling the onset of long range correlations[5, 6].

### 3 Hartree equations

Motivated by our previous studies in Minkowski space[5, 6] which showed that the growth of correlation and enhancement of fluctuations during a phase transition may
becomes leads to the self-consistent gap equation [3]. In this approximation the Hamiltonian (Minkowski) case this approximation sums up all the daisy (or cactus) diagrams and is the average using the time evolved density matrix. This average will only be a function of time. This approximation makes the Hamiltonian quadratic at the expense of a self-consistent condition. In the time independent case this approximation sums up all the daisy (or cactus) diagrams and leads to the self-consistent gap equation [3]. In this approximation the Hamiltonian becomes

\[ H = \int d^3x \left\{ -\frac{\hbar^2}{2a^3(t)} \frac{\delta^2}{\delta \eta^2} + \frac{a(t)}{2} \left( \nabla^2 \eta \right)^2 + a^3(t) \left( V(\phi) + V'(\phi) \eta + \frac{1}{2!} V''(\phi) \eta^2 \right) + \frac{1}{3!} \lambda \phi \eta^3 + \frac{1}{4!} \lambda \eta^4 \right\} \]  

The Hartree approximation is obtained by assuming the factorization

\[ \eta^3(\vec{x}, t) \rightarrow 3 \langle \eta^2(\vec{x}, t) \rangle \eta(\vec{x}, t) \]  (14)

\[ \eta^4(\vec{x}, t) \rightarrow 6 \langle \eta^2(\vec{x}, t) \rangle \eta^2(\vec{x}, t) - 3 \langle \eta^2(\vec{x}, t) \rangle^2 \]  (15)

where \( \langle \cdots \rangle \) is the average using the time evolved density matrix. This average will be determined self-consistently (see below). Translational invariance shows that \( \langle \eta^2(\vec{x}, t) \rangle \) can only be a function of time. This approximation makes the Hamiltonian quadratic at the expense of a self-consistent condition. In the time independent (Minkowski) case this approximation sums up all the daisy (or cactus) diagrams and leads to the self-consistent gap equation [3]. In this approximation the Hamiltonian becomes

\[ H = \Omega a^3(t) \mathcal{V}(\phi) + \int d^3x \left\{ -\frac{\hbar^2}{2a^3(t)} \frac{\delta^2}{\delta \eta^2} + \frac{a(t)}{2} \left( \nabla^2 \eta \right)^2 + a^3(t) \left( \mathcal{V}^{(1)}(\phi) \eta + \frac{1}{2} \mathcal{V}^{(2)}(\phi) \eta^2 \right) \right\} \]  

\[ \mathcal{V}(\phi) = V(\phi) - \frac{1}{8} \lambda \langle \eta^2 \rangle^2, \quad \mathcal{V}^{(1)}(\phi) = V'(\phi) + \frac{1}{2} \lambda \langle \eta^2 \rangle, \quad \mathcal{V}^{(2)}(\phi) = V''(\phi) + \frac{1}{2} \lambda \langle \eta^2 \rangle. \]

It is convenient to introduce the discrete Fourier transform of the fields in the comoving frame as

\[ \eta(\vec{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_k \eta_\vec{k}(t) e^{-i\vec{k} \cdot \vec{x}} \]  (17)

The Hamiltonian (16) then takes the form

\[ H = \Omega a^3(t) \mathcal{V}(\phi(t)) + \frac{\hbar^2}{2} \sum_\vec{k} \left\{ -\frac{\delta^2}{a^3(t) \delta \eta^2} + 2a^3(t) \mathcal{V}_\vec{k}(\phi(t)) \eta_{-\vec{k}} + \omega^2_\vec{k}(t) \eta_\vec{k} \eta_{-\vec{k}} \right\} \]  

(18)

where the time dependent frequencies \( (\omega^2_\vec{k}(t)) \) and the linear term in \( \eta \) have the values

\[ \omega^2_\vec{k}(t) = a(t) \vec{k}^2 + a^3(t) \mathcal{V}^{(2)}(\phi(t)) \]  (19)

\[ \mathcal{V}_\vec{k}^{(1)}(\phi(t)) = \mathcal{V}^{(1)}(\phi(t)) \sqrt{\Omega} \delta_{\vec{k},0} \]  (20)
We propose the following Gaussian ansatz for the functional density matrix elements in the Schrödinger representation

$$\rho[\Phi, \bar{\Phi}, t] = \prod_k N_k(t) \exp \left\{ - \left[ \frac{A_k(t)}{2\hbar} \eta_k(t)\eta_{-k}(t) + \frac{A_k^*(t)}{2\hbar} \bar{\eta}_k(t)\bar{\eta}_{-k}(t) + \frac{B_k(t)}{\hbar} \eta_k(t)\bar{\eta}_{-k}(t) \right] + \frac{i}{\hbar} \pi_k(t) \left( \eta_{-k}(t) - \bar{\eta}_{-k}(t) \right) \right\}$$

$$\eta_k(t) = \Phi_k - \phi(t)\sqrt{\Omega} \delta_{k,0} \quad \bar{\eta}_k(t) = \bar{\Phi}_k - \phi(t)\sqrt{\Omega} \delta_{k,0} \tag{21}$$

where $$\phi(t) = \langle \Phi(x) \rangle$$ and $$\pi_k(t)$$ is the Fourier transform of $$\langle \Pi(x) \rangle$$. This form of the density matrix is dictated by the hermiticity condition $$\rho^\dagger[\Phi, \bar{\Phi}, t] = \rho^*[\Phi, \bar{\Phi}, t]$$; as a result of this, $$B_k(t)$$ is real. The kernel $$B_k(t)$$ determines the amount of “mixing” in the density matrix, since if $$B_k = 0$$, the density matrix corresponds to a pure state because it is a wave functional times its complex conjugate.

In order to solve for the time evolution of the density matrix (21) we need to specify the density matrix at some initial time $$t_o$$. It is at this point that we have to assume some physically motivated initial condition. We believe that this is a subtle point that has not received proper consideration in the literature. A system in thermal equilibrium has time-independent ensemble averages (as the evolution Hamiltonian commutes with the density matrix) and there is no memory of any initial state. However, in a time dependent background, the density matrix will evolve in time, departing from the equilibrium state and correlation functions or expectation values may depend on details of the initial state.

We will assume that at early times the initial density matrix is thermal for the modes that diagonalize the Hamiltonian at $$t_o$$ (we call these the adiabatic modes). The effective temperature for these modes is $$k_B T_o = 1/\beta_o$$. It is only in this initial state that the notion of “temperature” is meaningful. As the system departs from equilibrium one cannot define a thermodynamic temperature. Thus in this case the “temperature” refers to the temperature defined in the initial state.

Inserting the Gaussian Ansatz (21) into the Liouville equation yields upon using the hamiltonian (13) the following equations for the functions $$A_k(t)$$, $$B_k(t)$$ and $$N_k(t)$$:

$$i\frac{\dot{N}_k}{N_k} = \frac{1}{2a^3(t)}(A_k - A_k^*) \tag{22}$$

$$i\dot{A}_k = \left[ \frac{A_k^2 - B_k^2}{a(t)^3} - \omega^2_k(t) \right] \tag{23}$$

$$i\dot{B}_k = \frac{B_k}{a^3(t)}(A_k - A_k^*) \tag{24}$$

The equation for $$B_k(t)$$ reflects the fact that a pure state $$B_k = 0$$ remains pure under time evolution.

Writing $$A_k$$ in terms of its real and imaginary components $$A_k(t) = A_{Rk}(t) + iA_{Ik}(t)$$ (and because $$B_k$$ is real) we find that

$$\frac{B_k(t)}{A_{Rk}(t)} = \frac{B_k(t_o)}{A_{Rk}(t_o)} \tag{25}$$

and that the time evolution is unitary (as it should be), that is

$$\frac{N_k(t)}{\sqrt{(A_{Rk}(t) + B_k(t))}} = \text{constant} \tag{26}$$
For the chosen thermal initial conditions at \( t = t_0 \), we have

\[ A_k(t_0) = A_k^*(t_0) = W_k(t_0) a^3(t_0) \coth [\beta_0 \hbar W_k(t_0)] \]  
\[ B_k(t_0) = - \frac{W_k(t_0)a^3(t_0)}{\sinh [\beta_0 \hbar W_k(t_0)]} \]  
\[ N_k(t_0) = \left[ \frac{W_k(t_0)a^3(t_0)}{\pi \hbar} \tanh \left[ \frac{\beta_0 \hbar W_k(t_0)}{2} \right] \right]^\frac{1}{2} \]  

(27)  
(28)

where

\[ W_k(t_0) = \left[ \frac{\vec{k}^2 + m^2(T_o)}{a(t_o)} \right]^\frac{1}{2}, \quad \frac{m^2(T_o)}{a^2(t_o)} = \mathcal{V}^{(2)}(\phi(t_0)) \]  

(29)

The initial density matrix is normalized such that \( Tr\rho(t_0) = 1 \). Since time evolution is unitary such a normalization will be constant in time. For \( T_o = 0 \) the density matrix describes a pure state since \( B_k = 0 \).

It is convenient to introduce the complex function:

\[ A_k(t) = \tanh [\beta_0 \hbar W_k(t_o)] \Re A_k(t) + i \Im A_k(t) \]  

(30)

Then \( A_k(t) \) obeys a Riccati equation

\[ i\dot{A}_k(t) = \frac{1}{a^3(t)} \left[ A_k^2(t) - \omega_k^2(t) a^3(t) \right] \]  

with the initial conditions:

\[ A_k(t_o) = W_k(t_o)a^3(t_o) \]  

(32)

Eq. (31) can be easily linearized by introducing the functions \( \varphi^H_k(t) \) as

\[ \dot{A}_k(t) = -i a^3(t) \frac{\dot{\varphi}^H_k(t)}{\varphi^H_k(t)} \]  

(33)

The equal time two-point function thus becomes

\[ \langle \eta^2(\vec{x}, t) \rangle = \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \left| \varphi^H_k(t) \right|^2 \coth [\beta_0 \hbar W_k(t_o)/2], \]  

(34)

which leads to the following set of self-consistent time dependent Hartree equations:

\[ \ddot{\phi} + 3 \frac{\dot{a}}{a}\phi + V'(\phi) + \lambda \phi \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \left| \varphi^H_k(t) \right|^2 \coth [\beta_0 \hbar W_k(t_o)/2] = 0 \]  

(35)

\[ \left[ \frac{d^2}{dt^2} + 3 \frac{\dot{a}(t)}{a(t)} \frac{d}{dt} + \frac{\vec{k}^2}{a^2(t)} + V''(\phi(t)) + \lambda \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \left| \varphi^H_k(t) \right|^2 \coth [\beta_0 \hbar W_k(t_o)/2] \right] \varphi^H_k(t) = 0 \]  

(36)

\[ \varphi^H_k(t_o) = \frac{1}{\sqrt{a^3(t_o)W_k(t_o)}}, \quad \varphi^H_k(t)|_{t_o} = i \sqrt{\frac{W_k(t_o)}{a^3(t_o)}} \]  

(37)

That is, we are faced with two complicated coupled non-linear and non-local equations for \( \phi(t) \) and \( \varphi_k(t) \).
The Hartree self-consistent equations (35) need regularization and renormalization since the momentum integrals of the mode functions (34) diverge.

Because the Bose-Einstein distribution functions are exponentially suppressed at large momenta, the finite temperature contribution will always be convergent and we need only address the zero temperature contribution.

A WKB analysis of eq. (35) provides the large k behaviour of the mode functions \( \varphi_k(t) \) [7]:

\[
\frac{|\varphi_k^H(t)|^2}{2} \overset{k \to \infty}{=} \frac{1}{2a^2(t)k^4} + \frac{1}{4k^3} \left[ \frac{\dot{a}^2(t_o)}{a^2(t)} - \left( -\frac{\mathcal{R}}{6} + V''(\phi) + \frac{\lambda}{2} \langle \eta^2(\vec{x},t) \rangle \right) \right] + \mathcal{O}(1/k^4) + \cdots
\]

Introducing an upper momentum cut-off \( \Lambda \) we obtain

\[
\langle \eta^2(\vec{x},t) \rangle = \frac{\hbar}{8\pi^2} \frac{\Lambda^2}{a^2(t)} + \frac{\hbar}{8\pi^2} \ln \left( \frac{\Lambda}{K} \right) \left[ \frac{\dot{a}^2(t_o)}{a^2(t)} - \left( -\frac{\mathcal{R}}{6} + V''(\phi(t)) + \frac{\lambda}{2} \langle \eta^2(\vec{x},t) \rangle \right) \right] + \text{finite}
\]

where we have introduced a renormalization point \( K \), and the finite part depends on time, temperature and \( K \).

There are several physically important features of the divergent structure obtained above. First, the quadratically divergent term reflects the fact that the physical momentum cut-off is being red-shifted by the expansion. (This term will not appear in dimensional regularization).

Secondly, the logarithmic divergence contains a term that reflects the initial condition (the derivative of the expansion factor at the initial time \( t_o \)). The initial condition breaks any remnant symmetry. For example, in de Sitter space there is still invariance under the de Sitter group, but this is also broken by the initial condition at an arbitrary time \( t_o \). Thus this term is not forbidden, and its appearance does not come as a surprise. As a consequence of this term, we need a time dependent term in the bare mass proportional to \( 1/a^2(t) \).

We are now in a position to present the renormalization prescription within the Hartree approximation. The differential equation for the mode functions (36) must be finite. Thus the renormalization conditions are obtained from

\[
m_B^2(t) + \frac{\lambda_B}{2} \phi^2(t) + \xi_B \mathcal{R} + \frac{\lambda_B}{2} \langle \eta^2 \rangle_B = m_R^2 + \frac{\lambda_R}{2} \phi^2(t) + \xi_R \mathcal{R} + \frac{\lambda_R}{2} \langle \eta^2 \rangle_R
\]

where the subscripts \( B, R \) refer to bare and renormalized quantities respectively and \( \langle \eta^2 \rangle_B \) is read from (39)

\[
\langle \eta^2 \rangle_B = \frac{\hbar}{8\pi^2} \frac{\Lambda^2}{a^2(t)} + \frac{\hbar}{8\pi^2} \ln \left( \frac{\Lambda}{K} \right) \left[ \frac{\dot{a}^2(t_o)}{a^2(t)} - \left( -\frac{\mathcal{R}}{6} + m_R^2 + \frac{\lambda_R}{2} \phi^2(t) + \xi_R \mathcal{R} + \frac{\lambda_R}{2} \langle \eta^2 \rangle_R \right) \right] + \text{finite}
\]

Using the renormalization conditions (30) we obtain

\[
m_B^2(t) + \frac{\lambda_B \hbar}{16\pi^2} a^2(t) + \frac{\lambda_B \hbar}{16\pi^2} \ln \left( \frac{\Lambda}{K} \right) \frac{\dot{a}^2(t_o)}{a^2(t)} = m_R^2 \left[ 1 + \frac{\lambda_B \hbar}{16\pi^2} \ln \left( \frac{\Lambda}{K} \right) \right]
\]

\[
\lambda_B = \frac{\lambda_R}{1 - \frac{\lambda_B \hbar}{16\pi^2} \ln \left( \frac{\Lambda}{K} \right)}
\]

\[
\xi_B = \xi_R + \frac{\lambda_B \hbar}{16\pi^2} \ln \left( \frac{\Lambda}{K} \right) \left( \xi_R - \frac{1}{6} \right)
\]

\[
\langle \eta^2 \rangle_R = I_R + J
\]
where

\[ I_R = \hbar \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{|\varphi_k^H(t)|^2}{2} - \frac{1}{2ka^2(t)} + \frac{\theta(k - K)}{4k^3} \left[ -\frac{R}{6} - \frac{\dot{a}^2(t_o)}{a^2(t)} + m_R^2 \frac{\phi^2(t)}{2} + \frac{\lambda_R}{2}\langle\eta^2\rangle_R \right] \right\} \quad (46) \]

\[ J = \hbar \int \frac{d^3k}{(2\pi)^3} \exp \beta_0\hbar \mathcal{V}_k(t_o) - 1 \quad (47) \]

The conformal coupling \( \xi = 1/6 \) is a fixed point under renormalization\[10\]. In dimensional regularization the terms involving \( \Lambda^2 \) are absent and \( \ln \Lambda \) is replaced by a simple pole at the physical dimension. Even in such a regularization scheme, however, a time dependent bare mass is needed. The presence of this new renormalization allows us to introduce a new renormalized mass term of the form

\[ \frac{\Sigma}{a^2(t)} \]

This counterterm may be interpreted as a squared mass red-shifted by the expansion of the universe. However, we shall set \( \Sigma = 0 \) for simplicity.

Notice that there is a weak cut-off dependence on the effective equation of motion for the order parameter.

For fixed \( \lambda_R \), as the cutoff \( \Lambda \to \infty \)

\[ \lambda_B \approx -\frac{(4\pi)^2}{\ln \left( \frac{\Lambda}{R} \right)} \cdot \xi_B = \frac{1}{6} + O \left( \frac{1}{\ln \Lambda} \right) \], \( m_B^2(t) = \frac{1}{a^2(t)} \left[ \frac{\Lambda^2}{\ln \left( \frac{\Lambda}{R} \right)} + \dot{a}^2(t_o) \right] + O \left( \frac{1}{\ln \Lambda} \right) \quad (48) \]

This approach to \( 0^- \) of the bare coupling as the cutoff is removed translates into an instability in the bare theory. This is a consequence of the fact that the \( N \)-component \( \Phi^4 \) theory for \( N \to \infty \) is asymptotically free (see ref.\[11\]), which is not relieved in curved space-time. Clearly this theory is sensible only as a low-energy cut-off effective theory, and it is in this restricted sense that we will ignore the weak cut-off dependence and neglect the term proportional to the bare coupling in (49).

The renormalized self-consistent Hartree equations thus become after letting \( \Lambda = \infty \):

\[ \ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + m_R^2 \phi + \xi_R \quad R \quad \frac{\lambda_R}{2} \phi^3 + \frac{\lambda_R}{2} \frac{\phi^2}{2} \quad \langle\eta^2\rangle_R = 0 \quad (49) \]

\[ \left[ \frac{d^2}{dt^2} + 3 \frac{\dot{a}(t)}{a(t)} \frac{d}{dt} + \frac{\tilde{k}^2}{a(t)} \right] + m_R^2 + \xi_R \quad R \quad \frac{\lambda_R}{2} \quad \langle\eta^2\rangle_R \quad \varphi_k^H(t) = 0 \]

where \( \langle\eta^2\rangle_R \) is given by equations (46), (47).

4 High Temperature Limit

One of the payoffs of understanding the large-\( k \) behavior of the mode functions (as obtained in the previous section via the WKB method) is that it permits the evaluation of the high temperature limit. We shall perform our analysis of the high temperature expansion for the Hartree approximation.

The finite temperature contribution is determined by the integral

...
\[ J = \hbar \int \frac{d^3k}{(2\pi)^3} \frac{\left| \varphi_k^H(t) \right|^2}{e^{\beta_0 k T_o} - 1} \]  

(50)

For large temperature, only momenta \( k \geq T_o \) contribute. Thus the leading contribution is determined by the first term in eq. (38). We find

\[ J = \frac{1}{12\hbar} \left[ \frac{k_B T_o a(t_o)}{a(t)} \right]^2 \left[ 1 + \mathcal{O}(1/T_o) + \cdots \right] \]  

(51)

Thus we see that the leading high temperature behavior results in an effective time dependent temperature

\[ T_{\text{eff}}(t) = T_o \left[ \frac{a(t)}{a(t_o)} \right] \]

This expression corresponds to what would be obtained for an adiabatic (isentropic) expansion for radiation (massless particles) evolving in the cosmological background.

This behavior only appears at leading order in the high temperature expansion. There are subleading terms that we computed subsequently [7]. To avoid cluttering of notation, we will set \( k_B = \hbar = 1 \) in what follows.

We define

\[ m^2(T_o) \equiv m_R^2 + \xi_R \mathcal{R}(t_o) + \frac{\lambda_R}{2} \phi^2(t_o) + \frac{\lambda_R}{2} \langle \eta^2(t_o) \rangle_R \]  

(52)

and we will assume that \( m^2(T_o) \ll T_o^2 \). Since we are interested in the description of a phase transition, we will write

\[ m_R^2 + \xi_R \mathcal{R}(t_o) + \frac{\lambda_R}{2} \phi^2(t_o) = -\frac{\lambda_R T_o^2}{24}; \quad T_o^2 > 0 \]  

(53)

Thus, to leading order in \( T_o \)

\[ m^2(T_o) = \frac{\lambda_R}{24} (T_o^2 - T_c^2) \]  

(54)

Our high temperature expansion will assume fixed \( m(T_o) \) and \( m(T_o)/T_o \ll 1 \).

It becomes convenient to define the variable

\[ x^2 = \frac{k^2}{T_o^2 a^2(t_o)} + \frac{m^2(T_o)}{T_o^2} \]

\[ \left| \varphi_k^H(t) \right|^2 = \left| \varphi^H(a(t_o), \sqrt{x^2 T_o^2 - m^2(T_o); t}) \right|^2 \]

Recall from our WKB analysis that the leading behavior for \( k \to \infty \) is (see equation (38))

\[ \left| \varphi_k^H(t) \right|^2 \to \frac{1}{2a^2(t)k} \]

adding and subtracting this leading term in the integral \( J \) and performing the above change of variables, we have

\[ J = J_1 + J_2 \]
\begin{align*}
J_1 &= \left[ \frac{a(t_o)}{a(t)} \right]^3 \left( \frac{T_o}{\pi} \right)^2 \int_{m(T_o)/T_o}^{\infty} \frac{dx}{e^x - 1} \left[ a^3(t) \sqrt{x^2T_o^2 - m^2(T_o)} \right] \frac{|\varphi^H(t)|^2}{2} - \frac{a(t)}{2a(t_o)} \\
J_2 &= \frac{T_o^2}{2\pi^2} \left[ \frac{a(t_o)}{a(t)} \right]^2 \int_{m(T_o)/T_o}^{\infty} \frac{dx}{e^x - 1} \\
&= \frac{1}{2} T_o^2 \left[ \frac{a(t_o)}{a(t)} \right]^2 \left[ \frac{1}{12} - \frac{m(T_o)}{2\pi^2 T_o} + \frac{m^2(T_o)}{8\pi^2 T_o^2} + \mathcal{O} \left( \frac{m^3(T_o)}{T_o^3} \right) + \cdots \right] \quad (55)
\end{align*}

We now must study the high temperature expansion of \( J_1 \). We restrict ourselves to the determination of the linear and logarithmic dependence on \( T_o \). For this purpose, it becomes convenient to introduce yet another change of variables \( x = \frac{m(T_o)}{T_o} z \) and use the fact that in the limit \( T_o \gg m(T_o) \),

\[
\frac{z}{e^{\frac{m(T_o)}{T_o} z} - 1} \approx \frac{T_o}{m(T_o)} \left[ 1 - \frac{m(T_o)}{2T_o} z + \cdots \right]
\]

This yields the following linear and logarithmic terms in \( T_o \):

\[
J_{1\text{lin}} = \left[ \frac{a(t_o)}{a(t)} \right]^3 \frac{T_o m(T_o)}{\pi^2} \int_1^{\infty} dz \left\{ a^3(t) m(T_o) \sqrt{z^2 - 1} \frac{|\varphi^H(t)|^2}{2} - \frac{a(t)}{2a(t_o)} \right\} \quad (56)
\]

Note that the above integral is finite.

The logarithmic contribution is obtained by keeping the \( \mathcal{O}(1/k^3) \) in the large momentum expansion of \( |\varphi^H_k(t)|^2 \) given by equation (38) (in terms of the new variable \( z \)). We obtain after some straightforward algebra:

\[
J_{1\text{log}} = -\ln \left[ \frac{m(T_o)}{T_o} \right] \frac{R}{8\pi^2} \left\{ -\frac{\dot{a}^2(t)}{a^2(t)} + \left[ \frac{a(t_o)}{a(t)} \right]^2 \left[ m^2(T_o) + \frac{\lambda_R T_c^2}{24} \right] - \frac{\lambda_R T_c^2}{24} \right\} \quad (57)
\]

That is, in the limit \( T_o \gg m(T_o) \), \( J_1 = J_{1\text{lin}} + J_{1\text{log}} + O((T_o)^0) \).

Comparing the \( \mathcal{O}(T_o^2, T_o, \ln T_o) \) contributions it becomes clear that they have very different time dependences through the scale factor \( a(t) \). Thus the high temperature expansion as presented will not remain accurate at large times since the term quadratic in \( T_o \) may become of the same order or smaller than the linear or logarithmic terms. The high temperature expansion and the long time limit are thus not interchangeable, and any high temperature expansion is thus bound to be valid only within some time regime that depends on the initial value of the temperature and the initial conditions.

As an illustration, we calculated \( J_{1\text{lin}} \) explicitly in the case of de Sitter space [7].

5 Large Time Limit

The expansion factor \( a(t) \) tends to infinity in the limit \( t \to \infty \). As we have just seen, the high temperature expansion breaks down in this limit. Physical momenta are given by

\[
l = \frac{k}{a(t)} \quad (58)
\]

For large \( a(t) \), only comoving momenta \( k = l a(t) \to \infty \) will be relevant. Thus, we can again use the WKB method to evaluate the mode functions \( \varphi^H_k(t) \) in this regime. Let
us consider for example the de Sitter universe \((a(t) = a_0 e^{Ht})\). We find from eq.(\[8\])
\[
|\varphi^H_{la(t)}(t)|^2 \overset{a(t) \to \infty}{=} \frac{H^2 + l^2}{2l^2} \left[ 1 + \frac{1}{2l^2} \left( \frac{m(T_0)^2}{a(t)^2} + \frac{\dot{a}(t)^2}{a(t)^2} \right) + O\left( \frac{1}{[a(t)]^4} \right) \right]
\]
for \(m/H << 1\). Using the asymptotic behaviour \((59)\) in eqs.\((45)-(47)\) leads to \((58)\)
\[
\langle \eta^2 \rangle_R = \frac{1}{(2\pi)^2} \left\{ H^2 Z\left( \frac{K}{m(T_0)} \right) + Y(T_0) \left[ \frac{a(t_0)}{a(t)} \right]^2 + O(a(t)^{-4}) \right\}
\]
where
\[
Z\left( \frac{K}{m(T_0)} \right) = \frac{1}{2} (x-1) \left( 1 + \frac{\dot{a}(t_0)^2}{m^2} \right) + \log \left( \frac{2}{1+x} \right) \text{ with } x \equiv \sqrt{1 + \frac{m^2}{K^2}}
\]
\[
Y(T) = \int_0^\infty \frac{k \, dk}{\sqrt{k^2 + m^2}} \frac{1}{1 + \frac{m^4}{4K^2(1+x)^2}} + \frac{\dot{a}(t_0)^2}{2} \log \left( \frac{2}{1+x} \right)
\]
It should be noticed that the integral in \(Y(T)\) is the mean value of \(\frac{1}{2}\) for a free Bose gas at temperature \(T_0\). Notice that both \(Y(T)\) and \(Z\left( \frac{K}{m(T_0)} \right)\) depend on the renormalization point \(K\) and that \(Z\left( \frac{K}{m(T_0)} \right)\) is positive for all \(\bar{K}\).

Assuming that \(\phi(t) \to \phi_\infty\) (a constant) for \(t \to \infty\), the Hartree equations \((59)\) yield for \(t \to \infty\),
\[
\phi^2_\infty = -\frac{2}{\lambda_r} \left[ m_r^2 + 12H^2\xi_r \right] - H^2 Z\left( \frac{K}{m(T_0)} \right)
\]
Since \(Z\left( \frac{K}{m(T_0)} \right) > 0\) and we consider \(\lambda_r << 1\), this equation has a solution provided
\[
m_r^2 + 12H^2\xi_r < 0
\]
The physical interpretation is as follows, in the symmetry breaking case where eq.\((14)\) holds, the order parameter \(\phi(t)\) for large times \(t\) (when the universe cools down fast) tends to the non-zero value \(\phi_\infty\) given by eq.\((33)\) independent of the initial value \(\phi(t_0)\).

The equal time two-point field correlator is given in the Hartree approximation by
\[
\Delta(\vec{x}, t) = \langle \eta(\vec{x}, t) \eta(0, t) \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} \left| \varphi^H_k(t) \right|^2 \coth \left[ \beta_0 \hbar \mathcal{W}_k(t_0)/2 \right],
\]
For long times [ \(a(t) >> 1\)] we find using the previous WKB results for \(\left| \varphi^H_k(t) \right|^2\) \((38)\),
\[
\Delta(\vec{x}, t) \overset{a(t) \to \infty}{=} \left( \frac{H}{2\pi} \right)^2 \left( \frac{A(r)}{r} + \frac{B(r)}{[2\pi r a(t)]^2} \right), \text{ where } A(r) = \frac{\pi}{2m} - \int_r^\infty dr \, K_0(mr) \quad B(r) = m r K_1(mr) \quad r \equiv |\vec{x}|
\]
We find in terms of the physical (redshifted) length \(R \equiv r a(t)\) when \(t \to \infty\) and \(R\) is kept fixed:
\[
\Delta(\vec{x}, t) \overset{a(t) \to \infty}{=} \frac{1}{(2\pi)^2} + \left( \frac{H}{2\pi} \right)^2 \left[ Ht - \log(2mR) + 1 - \gamma \right] + O(\log a(t)/[a(t)]^2)
\]
Notice that \(m = m(T_0)\) fixes the scale of the correlations.
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