p, q-Duality and Hamiltonian flows in the space of integrable systems or integrable systems as canonical transforms of the free ones

A. Mironov a,b, A. Morozov b

a Theory Department, Lebedev Physics Institute, Moscow 117924, Russia
b ITEP, Moscow 117259, Russia

Received 21 August 2001; accepted 20 October 2001

Abstract

Variation of coupling constants of integrable system can be considered as canonical transformation or, infinitesimally, a Hamiltonian flow in the space of such systems. Any function $T(\vec{p}, \vec{q})$ generates a one-parametric family of integrable systems in vicinity of a single system: this gives an idea of how many integrable systems there are in the space of coupling constants. Inverse flow is generated by a dual “Hamiltonian”, $\tilde{T}(\vec{p}, \vec{q})$ associated with the dual integrable system. In vicinity of a self-dual point the duality transformation just interchanges momenta and coordinates in such a “Hamiltonian”: $\tilde{T}(\vec{p}, \vec{q}) = T(\vec{q}, \vec{p})$. For integrable system with several coupling constants the corresponding “Hamiltonians” $T_i(\vec{p}, \vec{q})$ satisfy Whitham equations and after quantization (of the original system) become operators satisfying the zero-curvature condition in the space of coupling constants:

$$\frac{\partial}{\partial g_a} - \tilde{T}_a(\vec{p}, \vec{q}), \frac{\partial}{\partial g_b} - \tilde{T}_b(\vec{p}, \vec{q}) = 0.$$

Some explicit formulas are given for harmonic oscillator and for Calogero–Ruijsenaars–Dell system.

© 2002 Published by Elsevier Science B.V. Open access under CC BY license.

1. Introduction

The study of non-perturbative quantum phenomena revealed the real role of integrable systems in physics: τ-functions of integrable hierarchies appear to describe partition functions of quantum theories, with time-variables identified with coupling constants [1,2]. Broadening understanding of this relation, together with the growing interest to peculiarities of essential (non-perturbative) quantum physics, stimulates a new attention to the basics of integrability theory, which brings one back to the very foundations of classical Hamiltonian mechanics that were not sufficiently investigated in the past. The present Letter is devoted to one of such basic subjects, to the description of integrable systems in terms of the Hamiltonian evolution in the space of coupling constants. This approach can help one to describe the entire variety of integrable systems and to reveal the meaning of peculiar relations, like dualities, between different such systems.

2. Hamiltonian flow in the space of integrable systems

2.1. Variation of couplings as canonical transform—peculiar feature of integrable systems

Integrable system with $N$ coordinates $q_i$ and $N$ momenta $p_i$ is characterized by existence of $N$ Poisson-
commuting Hamiltonians \( H_i(\tilde{p}, \tilde{q}) \), \( \{ H_i, H_j \} = \frac{\partial H_i}{\partial \tilde{p}_j} \frac{\partial H_j}{\partial \tilde{q}_i} - \frac{\partial H_i}{\partial \tilde{q}_j} \frac{\partial H_j}{\partial \tilde{p}_i} = 0 \). For such a system one can consider a canonical transformation, treating these Hamiltonians as new momenta-like or coordinate-like variables. This transformation (of which the infinitesimal version is a certain Hamiltonian flow) will be the subject of the present Letter.

To make the problem precise, let us consider a one-parametric family of integrable models, parameterized by a single coupling constant \( g \) such that the model is free when \( g = 0 \). This means that at \( g = 0 \) the Hamiltonians \( H_i^{(0)}(\tilde{p}) = H_i(\tilde{p}, \tilde{q}|g = 0) \) are functions only of momenta \( \tilde{p} \), though, for conventional choices of Hamiltonians in particular applications, these functions can be non-trivial. The typical examples are: \( H_k^{(0)}(\tilde{p}) = \sum_{i=0}^{N} p_i^2 \) and \( H_k^{(0)}(\tilde{p}) = \sum_{i=1}^{N} e^{p_i} \) for \( p \)-rational and \( p \)-trigonometric models, respectively, and some elliptic functions of \( \tilde{p} \) for their elliptic generalizations.

The adequate definition of the new canonical variables \( \tilde{P}_x = \tilde{P}(\tilde{p}, \tilde{q}|g) \) and \( \tilde{Q}_x = \tilde{Q}(\tilde{p}, \tilde{q}|g) \) is [3]

\[
H_i^{(0)}(\tilde{P}) = H_i(\tilde{p}, \tilde{q}|g),
\]

\[
\tilde{H}_i^{(0)}(\tilde{Q}) = \tilde{H}_i(\tilde{p}, \tilde{q}|g),
\]

where \( \tilde{H}_i(\tilde{p}, \tilde{q}|g) \) define the Hamiltonians of the dual integrable system (for other discussions of this duality see [4] and [5]). Note that in (1) they depend on the “dressed” variables \( \tilde{P} \) and \( \tilde{Q} \) and, moreover, \( \tilde{P} \) and \( \tilde{Q} \) are interchanged. The shape of the dual Hamiltonians is dictated by the requirement that the new variables \( \tilde{P} \) and \( \tilde{Q} \) are canonical, i.e.,

\[
\sum_i dP_i \wedge dQ_i = \sum_i dp_i \wedge dq_i
\]

(2)

and the Poisson brackets are

\[
\{.,.\} = \sum_i \left( \frac{\partial}{\partial P_i} \otimes \frac{\partial}{\partial Q_j} - \frac{\partial}{\partial Q_i} \otimes \frac{\partial}{\partial P_j} \right)
\]

\[
= \sum_i \left( \frac{\partial}{\partial P_i} \otimes \frac{\partial}{\partial Q_i} - \frac{\partial}{\partial Q_i} \otimes \frac{\partial}{\partial P_i} \right).
\]

(3)

2.2. Infinitesimal version of canonical transforms—the Hamiltonian flows

Relations (1) define \( \tilde{P}_x \) as functions of \( \tilde{P}, \tilde{Q} \) and the coupling constant \( g \). Of special interest and importance is the infinitesimal version of this canonical transformation, considered as a Hamiltonian flow along the \( g \) direction in the space of coupling constants. Such transformation is generated by a new Hamiltonian \( T(\tilde{p}, \tilde{q}|g) \), according to the rule:

\[
\frac{\partial \tilde{P}_x}{\partial g} = \{ T(\tilde{P}, \tilde{Q}|g), \tilde{P}_x \} = -\frac{\partial T}{\partial \tilde{Q}_x},
\]

\[
\frac{\partial \tilde{Q}_x}{\partial g} = \{ T(\tilde{P}, \tilde{Q}|g), \tilde{Q}_x \} = -\frac{\partial T}{\partial \tilde{P}_x}.
\]

(4)

This new Hamiltonian does not commute with the old ones, but it converts them into a new set of commuting Hamiltonians.

Moreover, instead of considering a pre-given family of integrable systems, one can use any function \( T(\tilde{P}, \tilde{Q}|g) \) to define a whole one-parametric family of integrable systems, though explicit construction of the corresponding Poisson-commuting Hamiltonians is rarely possible. Further, a multi-parametric family of integrable systems is similarly generated by any collection of functions \( T_{a}(\tilde{p}, \tilde{q}|g) \) satisfying the compatibility (Whitham) equations:

\[
\frac{\partial T_b}{\partial g_a} - \frac{\partial T_a}{\partial g_b} + \{ T_a, T_b \} = 0.
\]

(5)

The coupling constant variation for the dual integrable system is governed by the dual Hamiltonian \( \tilde{T}(\tilde{p}, \tilde{q}|g) \): for

\[
\tilde{H}_i^{(0)}(\tilde{P}) = \tilde{H}_i(\tilde{p}, \tilde{q}|g)
\]

(6)

\[\text{[As mentioned in the abstract to this Letter, after quantization of the original system these } T_{a}(\tilde{p}, \tilde{q}|g_{a}) \text{ become a family of operators } \tilde{T}_{a}(\tilde{p}, \tilde{q}|g_{a}) \text{ satisfying the zero-curvature condition in the space of coupling constants:]}
\]

\[
\left[ \frac{\partial}{\partial g_{a}}, \tilde{T}_{b}(\tilde{p}, \tilde{q}) \right] - \tilde{T}_{a}(\tilde{p}, \tilde{q}) = 0.
\]

This can serve as an explanation of emergency of classical integrability (the zero-curvature equations) in the study of quantum integrable systems.
in the coupling constant $g$ takes the expansion

$$T(\vec{P}, \vec{Q}|g) = \frac{\partial \vec{T}}{\partial \vec{Q}_g}, \frac{\partial \vec{T}}{\partial P_g}. \tag{7}$$

Relation between $\vec{T}(\vec{p}, \vec{q}|g)$ and $T(\vec{p}, \vec{q}|g)$ becomes especially simple at any self-dual point (where $H_1(\vec{p}, \vec{q}|g)$ is the same system with coupling constant $g$)

$$\vec{T}(\vec{p}, \vec{q}|g) = T(\vec{q}, \vec{p}|g). \tag{8}$$

As immediate consequence, any symmetric function $T(\vec{p}, \vec{q}|g)$ defines a one-parametric family of self-dual integrable systems.

Note that it is not that immediate calculation to restore the Hamiltonian starting from arbitrarily given $\Phi_i(p,q)$ that

$$T(p,q) = p + \sum_{i=1}^1 \Phi_i(p,q)g^i \text{ etc. and obtains that}$$

$$\Phi_1(p,q) = \Phi_1(q,p),$$

$$\Phi_2(p,q) = \Phi_2(q,p) + \cdots.$$  

The result ultimately is that the symmetric part of $\partial_p \Phi_1(p,q) = \partial_q \Phi_1(p,q) + \cdots$ is given independently at any order, while the antisymmetric parts are fixed by $\Phi_i$’s at lower orders:

$$\Phi'_1(p,q) - \Phi'_1(q,p) = 0,$$

$$\Phi'_2(p,q) - \Phi'_2(q,p) = \Phi'_1(p,q) \Phi_1(p,q) + (p \leftrightarrow q),$$

$$\Phi'_3(p,q) - \Phi'_3(q,p) = \Phi'_1(p,q) \Phi_2(p,q) + \Phi'_2(p,q) \Phi_1(p,q) - \frac{1}{2} \Phi'_1(p,q) \Phi_2(p,q) \Phi_1(p,q) - (p \leftrightarrow q).$$

$$\vdots$$

where all the derivatives are taken w.r.t. to the first variable. A trivial solution to these Eqs. (9) is $\Phi_1(p,q) = p^{n_1} q^{n_2}$ with arbitrary $\{n_i\}$. This implies that the Hamiltonian $H(p,q) = pf(pq)$ is self-dual with arbitrary function $f$. Indeed, one easily checks this is the case (note that for such a Hamiltonian $pq = P Q$). We discuss later in detail the rational Calogero Hamiltonian that gets to this class of Hamiltonians.

2.3. Generating functions of canonical transforms

Along with the Hamiltonians $T(\vec{P}, \vec{Q}|g)$ one can also consider the generating functions of canonical transformations in question, like $S(\vec{Q}, \vec{q}|g)$ or its Legendre transform $F(\vec{P}, \vec{q}|g) = \vec{P} \vec{Q} - S(\vec{Q}, \vec{q}|g)$, such that

$$\frac{\partial S}{\partial Q} = \frac{\partial F}{\partial P}, \quad \frac{\partial S}{\partial q} = \frac{\partial F}{\partial Q}. \tag{10}$$

and

$$T = \frac{\partial S}{\partial g}. \tag{11}$$

Of course, for canonical transformation

$$\frac{\partial P_i}{\partial q_j} + \frac{\partial Q_j}{\partial P_i} = 0,$$  

(12) as implied by (10), but one should be more careful when drawing similar conclusion from (11): the second $g$ derivatives satisfy Eq. (5), because $g$-derivative is taken at constant $P_g$ and $Q_g$, which themselves depend on $g$.

Similarly,

$$\frac{\partial F}{\partial P} = \frac{\partial S}{\partial Q}, \quad \frac{\partial F}{\partial q}. \tag{13}$$

At self-dual points $F(\vec{P}, \vec{q}|gSD) = F(\vec{q}, \vec{P}|gSD)$ is a symmetric function.

Note that

$$\frac{\partial F}{\partial g} = T \tag{14}$$

(with $p$ and $Q$ constant), i.e., $T$ is, in a sense, a more invariant quantity than $S$ and $F$, which does not depend on the choice of independent variables.

Another way to see it is to consider a flow from an integrable system with coupling constant $g_1$ to the same system with coupling constant $g_2$. Then, $T$ depends only on $g_2$, but not on $g_1$, while the generating functions depend on both $g_1$ and $g_2$. 

A. Mironov, A. Morozov / Physics Letters B 524 (2002) 217–226
2.4. Generating functions, evolution operator and eigenfunctions

Basically, if re-expressed in terms of \( \tilde{Q} \) and \( \tilde{q} \) (instead of \( P \) and \( Q \)) the Hamiltonian \( T = \partial S / \partial g \). Therefore, \( e^{iS} \) can be considered as a kind of an evolution operator (kernel) in the space of coupling constants, which performs a canonical transformation from the free system to the integrable one. After quantization it can be symbolically represented as

\[
e^{iS}(\tilde{Q},\tilde{q}|g) = \oplus_{\lambda} |\psi_{\lambda}(\tilde{Q})\rangle c_{\lambda}(\tilde{q}|\tilde{q}),
\]

where \( |\psi_{\lambda}\rangle \) and \( |\psi_{\lambda}^{(0)}\rangle \) are eigenfunctions of the system and \( c_{\lambda} \) are some coefficients, depending on the spectral parameter \( \lambda \). The evolution operator satisfies the quantum version of Eq. (1),

\[
H^{(0)}(\partial / \partial \tilde{Q}) = e^{-i\tilde{S}(\tilde{Q},\tilde{q}|g)} H(\partial / \partial \tilde{q}, \tilde{q}|g) e^{i\tilde{S}(\tilde{Q},\tilde{q}|g)}. \tag{16}
\]

Since eigenfunctions \( |\psi_{\lambda}^{(0)}(Q)\rangle \) of a free system, are just exponents of the spectral parameter \( \tilde{\lambda} \), the dressed eigenfunctions \( |\psi_{\lambda}(Q)\rangle \) are basically Fourier transforms of the evolution operator \( e^{iS(Q,q)} \)

\[
|\psi_{\lambda}(Q)\rangle \sim \int e^{i\tilde{S}(Q,q)} e^{i\lambda Q} dQ. \tag{17}
\]

Similarly,

\[
|\psi_{\lambda}(Q)\rangle \sim e^{iF(Q,q)} \delta(\lambda - P) dP \sim e^{i\tilde{F}(\lambda,q)}. \tag{18}
\]

Note that there is a freedom in solutions of Eq. (2) to shift \( \{Q_{1}\} \) by an arbitrary function of \( \{P_{1}\} \). This shift is quite complicated in terms of the generating function \( S \), but in \( F \) it is just an addition of the term depending only on \( \{P_{1}\} \). In the quantum case, this ambiguity in the definition of \( F \) is just a matter of normalization of the eigenfunction.

One can also consider the quantum counterpart of \( T(p,q|g) \) which is a Hamiltonian that gives rise to a Schrödinger equation w.r.t. the coupling constant

\[
\frac{\partial \psi}{\partial g} = i\tilde{T}\psi. \tag{19}
\]

Therefore, the wavefunction \( \psi \) can be also realized as a path integral over the phase space variables \( P(g), Q(g) \).

The generating functions \( S(Q,q), F(P,q) \) and \( T(p,q) \) satisfy the quasiclassical versions of these relations (when multiple derivatives of \( S, F \) and \( T \) are neglected). Exact definition of quantum evolution operators, including effects of discrete spectra and identification of the spectra at different values of coupling constants, i.e., precise definition of the spectral parameter \( \lambda \) in such a way that Eq. (15) is diagonal in \( \lambda \), will be considered elsewhere.

3. Some conceptual questions for further examination

The previous section explained possible needs to study integrable systems from the non-conventional perspective: in terms of the auxiliary Hamiltonians \( T(\tilde{p},\tilde{q}|g) \), describing the associated Hamiltonian flows in the space of coupling constants. Some immediate problems were also mentioned, like (i) criteria, distinguishing the Hamiltonians \( T \) associated with conventional integrable systems at the entire space of such Hamiltonians; (ii) structure of algebra, generated by all the Hamiltonians \( H_{a} \) and \( T_{a} \), in which the \( H_{a} \)'s generate a commutative (Cartan) subalgebra; (iii) duality relation between \( T \) and \( T \); (iv) relation between integrable systems and flat connections over the space of coupling constants encoded in (5).

One can also add to this list: (v) identification of the flows and Hamiltonians \( T_{a} \) in terms of Seiberg–Witten theory [2,6]; (vi) treatment of \( T \)'s in terms of the theta-function formalism of Refs. [3,5]; (vii) the “gauge-equivalence” of the Hamiltonians \( T_{a} \), corresponding to different choices of the constant-\( \tilde{p}, \tilde{q} \) sections over the moduli space (i.e., with different quantities kept constant when the couplings \( g_{a} \) are varied); (viii) the role of the distinguished “flat-modulus” \( \tilde{a} \) [6], for which the WDVV equations [7] hold; (ix) further development of the notion of generalized \( \tau \)-function, making use of the evolution operator \( e^{S} \), introduced in the previous section (see also the concluding Section 5).

Since the subject is relatively new, we do not think it is time now to get into further discussions of these (and many other) problems. Instead in the remaining part of this Letter we consider some simple examples: the
Hamiltonians \( T(\tilde{p}, \tilde{q}; g) \), associated with the flows in the coupling constant space for some well-known integrable systems like harmonic oscillator and Calogero–Ruijsenaars-double-elliptic (DELL) family.

4. Particular examples of Hamiltonian flows

4.1. Harmonic oscillator

Perhaps surprisingly, even for this simple model one obtains rather sophisticated formulas.

For harmonic oscillator \( H(p, q|\omega) = \frac{1}{2}(p^2 + \omega^2 q^2) \).

Let the frequency \( \omega \) play the role of the coupling constant so that \( H^{(0)}(p) = \frac{1}{2}p^2 \). Then

\[
P = \sqrt{p^2 + \omega^2 q^2},
\]

\[
Q = \frac{\omega q}{p} \arctan \frac{\omega q}{p}.
\]

(20)

Note that in this case the \( \omega \)-evolution possesses “the conservation law”

\[
\tan \frac{\omega Q}{P} = \frac{\omega q}{p}.
\]

(21)

The inverse transformation looks like

\[
p = P \cos \frac{\omega Q}{P},
\]

\[
q = \frac{P}{\omega} \sin \frac{\omega Q}{P},
\]

(22)
i.e., the Hamiltonian of the dual flow is

\[
\tilde{H}(\tilde{p}, \tilde{q}|\omega) = \frac{\tilde{q}}{\omega} \sin \frac{\omega \tilde{p}}{\tilde{q}}.
\]

(23)

(Note that with this definition \( \tilde{H}^{(0)}(p) = p \) and also note that \( p, q \)-duality does not respect conventional dimensions of \( p \) and \( q \) so that it should not be a surprise that \( \omega q/p \) in \( H \) got substituted by \( \omega \tilde{p}/\tilde{q} \) in \( \tilde{H} \).

Of course, \( \tilde{p} \) and \( \tilde{q} \) are nothing but \( Q_\omega \) and \( P_\omega \).)

The generator of \( \omega \)-evolution such that

\[
\frac{\partial P}{\partial \omega} \bigg|_{p,q=\text{const}} = \frac{\omega q^2}{\sqrt{p^2 + \omega^2 q^2}} = \frac{P}{\omega} \sin^2 \frac{\omega Q}{P} = \frac{\partial T}{\partial Q}.
\]

(24)

\[3\] In fact, the treatment of the DELL case is very sensitive to the choices mentioned in (vii) above, and we postpone serious discussion of the DELL case to another occasion, in order not to draw attention away from the simple ideas of the present Letter.
where \( \frac{\partial^2}{\partial x^2} = v + \frac{1}{2} \) and the Hermite polynomials, satisfying

\[
( -\partial_x^2 + x^2 ) e^{-x^2/2} H_v(x) = (2v + 1) e^{-x^2/2} H_v(x)
\]  

are given by inverse Laplace transform of the generating function \( e^{-x^2/4 + tx} \).

Taking this integral in the saddle point approximation, one gets

\[
H_v(x) \sim \int \frac{dt}{t^{v+1}} e^{-t^2/4 + tx}.
\]  

which, at large \( v \), is in accordance with Eq. (30).

### 4.2. Generic Schrödinger-like systems

For a single particle any dynamics is integrable, and any Hamiltonian is canonically equivalent to the free one. Therefore, it may make sense to look at the corollaries of our formalism in such context. This can help to see the difference between models which have and have not simple integrable generalizations to the multiparticle case.

For \( H = \sqrt{p^2 + g^2 V(q)} \) the Hamiltonian \( T \) can be found in the form of a perturbation expansion

\[
T(p,q|g) = \frac{T_0(q)}{p} + g^2 \frac{T_1(q)}{p^2} + O(g^3),
\]  

where

\[
\frac{\partial T_0(q)}{\partial q} = -V(q), \quad \frac{\partial T_1(q)}{\partial q} = \frac{1}{2} T_0 \frac{\partial^2 T_0}{\partial q^2}.
\]  

There is no way to integrate explicitly these equations in the general case. However, for harmonic oscillator exact answer exists for the entire expansion, Eq. (25). Similarly, exact answers are available for potentials of the rational and trigonometric Calogero models (see below). However, for elliptic Calogero model with \( V(q) \sim \text{sn}^{-2}(q|k) \) Eq. (35) implies that the generator \( T(p,q|g) \) involves the integral elliptic tangent (i.e., the Weierstrass zeta-function), \( T_n(q|k) = \int \frac{d\xi}{\text{sn}^2(\xi|k)} \sim \xi(\xi|\tau). \)  

Higher orders of the \( g \) expansion contain multiple integrals of \( T_n(q) \). This reflects a more sophisticated group structure of elliptic models.

The simplest way to derive formulas (34), (35) is to construct the Hamiltonian \( q = \hat{H}(P,Q) \) dual to \( \sqrt{p^2 + g^2 V(q)} \). The canonicity condition then gives

\[
\frac{\partial \hat{H}(P,Q)}{\partial Q} = \frac{p(P,Q)}{P}.
\]  

Therefore, the dual Hamiltonian can be obtained via solving the equation

\[
Q = \int \frac{d\xi}{\sqrt{1 - g^2 V(\xi) P^2}},
\]  

w.r.t. \( q \). Now taking the derivative of \( P = \sqrt{p^2 + g^2 V(q)} \) w.r.t. \( g \), one obtains

\[
-\frac{\partial T}{\partial Q} = \frac{\partial}{\partial Q}(q(P,Q)).
\]  

where \( q(P,Q) \) is given by (37). One easily obtains the expansion of these two formulas in powers of \( g \), in particular, reproducing (34), (35).

### 4.3. SU(2) rational and trigonometric Calogero–Ruijsenaars models

#### 4.3.1. The rational–rational case (rational Calogero model)

In this case the single Hamiltonian is \( H = \frac{1}{2}(p^2 + g^2 q^2) \), thus \( H^0(p) = \frac{1}{2} p^2 \), and

\[
P^2 = p^2 - \frac{g^2}{q^2},
\]

\[
q^2 = Q^2 - \frac{g^2}{Q^2}.
\]  

The rational Calogero model is self-dual.

\footnote{However, the conservation laws of the \( g \)-flow involve the ordinary (algebraic) elliptic tangents, \( T_n(q|k) = \text{sn}(q|k) \): say, in the SU(2) elliptic-rational model (the dual of the elliptic Calogero system) [3] \( \text{cn}(P|k) = \alpha(q) \text{cn}(\beta(q) p|\gamma(q)) \), \( q = \sqrt{Q^2 - \frac{\beta^2}{\text{sn}^2(P|k)}} \) : \( \alpha^2(q) = 1 - \frac{\beta^2}{\gamma^2} \), \( \beta^2(q) = k^2 q^2 \), \( \gamma(q) = \sqrt{1 - \frac{\beta^2(q)}{\gamma^2(q)}} (1 - k^2) \), the counterpart of (21) is \( Q \text{cn}(P|k) = q \text{cn}(\beta p|\gamma) \). The difference between the algebraic and integral tangents is that \( \text{cn}(q|k) = \frac{\text{dn}(q|k)}{\text{cn}(q|k)} \) while \( \text{cn}(q|k) = \frac{1}{\text{cn}(q|k)} \).}
One can express the dressed variables $P = P_g$, $Q = Q_g$, through the bare ones, $p$, $q$:

\[ P = \frac{1}{q} \sqrt{p^2 q^2 - g^2}, \]
\[ Q = \frac{p q^2}{\sqrt{p^2 q^2 - g^2}}. \]

The symmetricity (integrability) condition (12) is true:

\[ \partial S/\partial g = \frac{2}{q} \log \frac{Q - \sqrt{Q^2 - q^2}}{q}. \]

Similarly,

\[ F(P, q) = \sqrt{p^2 q^2 + g^2} + \frac{g}{2} \log \frac{\sqrt{P^2 q^2 + g^2} - g}{\sqrt{P^2 q^2 + g^2} + g}. \]

One can also consider the transformation from the system given at $(g_1, Q_1, P_1)$ to that given at $(g_2, Q_2, P_2)$. The corresponding generating function

\[ S(Q_1, Q_2, g_1, g_2) \]

\[ = g_1 \log \frac{g_1 \sqrt{Q_1^2 - Q_2^2} + \sqrt{s_1^2 Q_1^2 - s_2^2 Q_2^2}}{Q_2} \]
\[ + g_2 \log \frac{g_2 \sqrt{Q_2^2 - Q_1^2} + \sqrt{s_2^2 Q_2^2 - s_1^2 Q_1^2}}{Q_1} \]

is the function of both $g_1$ and $g_2$, while

\[ T = \frac{\partial S}{\partial g_2} = \frac{1}{2} \log \frac{P_2 Q_2 - g_2}{P_2 Q_2 + g_2} \]

is the function of only $g_2$.

In accordance with (17)

\[ \psi_2(q) \sim \int e^{iS(q, Q|x)} e^{iQx} dQ \sim q J_{i \lambda}(i \lambda q), \]

where $J_{i \lambda}(x)$ is the Bessel function. The exact quantum wavefunction, solving the equation $\left(-\partial^2_x + \frac{\lambda^2}{x^2}\right) \psi_2(x) = \lambda^2 \psi_2(x)$, is $\sqrt{\lambda} J_{i \lambda}(i \lambda x)$, $\nu^2 = -g^2 + 1/4$. It coincides with (50) in quasiclassical approximation.

For other members of the Calogero–Ruijsenaars family we just list some relevant formulas, parallel to those in the rational–rational case.
\[ p^2 = \frac{g^2 \sinh^2 Q \cosh^2 q}{\sinh^2 q (\sinh^2 Q - \sinh^2 q)}, \]
\[ a = \frac{\cosh Q}{\sqrt{\sinh^2 Q - \sinh^2 q}} \]
\[ \frac{\partial q}{\partial \sinh Q \cosh q} \sqrt{\sinh^2 Q - \sinh^2 q} = 0. \] (51)

In this case \( S \) is still a simple linear function of \( g \)
\[ S = g \arcsinh \left( \frac{\sinh Q}{\sinh q} \right). \] (52)

4.3.3. The trigonometric-rational case (rational Ruijsenaars model)
\[ \cosh^2 P = \cosh^2 p \left( 1 - \frac{\sinh^2 \epsilon}{q^2} \right), \]
\[ q^2 = Q^2 - \frac{\sinh^2 \epsilon}{\sinh^2 p}. \]
\[ \text{Tran} P = q \tanh p, \]
\[ T = \frac{1}{2} \log \frac{\tanh P - \tanh \epsilon}{\tanh P + \tanh \epsilon}, \]
\[ = \frac{1}{2} \log \frac{q \tanh p - \tanh \epsilon}{q \tanh p + \tanh \epsilon}; \]
\[ \sinh^2 P = \frac{\sinh^2 \epsilon}{Q^2 - q^2}. \]
\[ \sinh^2 P = \frac{\sinh^2 \epsilon \cosh^2 Q}{(Q^2 - q^2) (q^2 - \sinh^2 \epsilon)}. \]
\[ \frac{\partial}{\partial q} \arcsinh \frac{g}{\sqrt{Q^2 - q^2}} \]
\[ + \frac{\partial}{\partial Q} \arcsinh \frac{g \cdot Q}{\sqrt{(Q^2 - q^2) (q^2 - \sinh^2 \epsilon)}} = 0. \] (53)

In this case, \( S \) is no longer a simple linear function of the coupling constant \( \epsilon \).

4.3.4. The trigonometric–trigonometric case (trigonometric Ruijsenaars model)
\[ \cosh^2 P = \cosh^2 p \left( 1 - \frac{\sinh^2 \epsilon}{\sinh^2 q} \right), \]
\[ \cosh^2 q = \cosh^2 Q \left( 1 - \frac{\sinh^2 \epsilon}{\sinh^2 P} \right). \]
\[ \tan P \tanh Q = \tan p \tanh q, \]
\[ T = \frac{1}{2} \log \frac{\tanh P \tanh Q - \tanh \epsilon}{\tanh P \tanh Q + \tanh \epsilon}, \]
\[ = \frac{1}{2} \log \frac{\tanh p \tanh q - \tanh \epsilon}{\tanh p \tanh q + \tanh \epsilon}; \]
\[ \sinh^2 P = \frac{\sinh^2 \epsilon \cosh^2 Q}{\sinh^2 Q - \sinh^2 q}, \]
\[ \sinh^2 P = \frac{\sinh^2 \epsilon \sinh^2 Q \cosh^2 q}{(\sinh^2 Q - \sinh^2 q) (\sinh^2 q - \sinh^2 \epsilon)}; \]
\[ \frac{\partial}{\partial q} \arcsinh \frac{\sinh \epsilon \cosh Q}{\sqrt{\sinh^2 Q - \sinh^2 q}} \]
\[ \times \arcsinh \frac{\sinh \epsilon \sinh Q \cosh q}{\sqrt{(\sinh^2 Q - \sinh^2 q) (\sinh^2 q - \sinh^2 \epsilon)}} = 0. \] (54)

4.4. SU(\( N \)) rational Calogero model

To give an illustration of what happens in the multiparticle case we start here with the simplest case of the rational Calogero model. More profound examples will be considered elsewhere.

The Hamiltonians \( H_i(\vec{p}, \vec{q}) = \text{Tr} L^i \) are defined as traces of powers of the Lax matrix \( L_{ij} = p_i \delta_{ij} + g \frac{1}{\varphi_{ij}} \).

Accordingly \( H^0_i(\vec{p}) = \sum_{j} p^i_j \). The Hamiltonian flow within the family of rational Calogero models (along the \( g \) direction) is governed by
\[ T(\vec{p}, \vec{q}; g) = -g \sum_{ij} \frac{1}{p_{ij} q_{ij}} + O(g^3). \] (55)

Technically, higher-order terms can be done in the following way. As the first step, one needs to find dual Hamiltonians. Note that dual Hamiltonians technically each time emerge as necessary ingredient in constructing \( T \). Moreover, with these intermediate objects, constructing \( T \) becomes well-defined although often complicated procedure and reduces to finding solutions of some partial differential equations. Indeed, first of all the dual Hamiltonians emerge as solutions to partial differential canonicity equations and then \( T \) is obtained by integration the following set of equations:
\[ \sum_{i} \left( \frac{\partial \tilde{H}_k}{\partial \tilde{Q}_i} \frac{\partial T}{\partial \tilde{p}_i} - \frac{\partial \tilde{H}_k}{\partial \tilde{Q}_i} \frac{\partial T}{\partial \tilde{p}_i} \right) + \frac{\partial \tilde{H}_k}{\partial g} = 0. \] (56)
These equations just follow from the fact that the dual Hamiltonians \( \vec{H} \) are nothing but independent of the coupling constant \( g \) coordinates \( (\varphi_l) \).

Now, having \( \vec{H} \) calculated, one can easily construct a recursive procedure of calculating \( T \) as series in \( g \):
\[
T = \sum g^{2k+1} T_k.
\]
Indeed, note that, in the leading order in \( g \), \( \vec{H} \) are just simple functions of \( Q_i \), say, just \( Q_k \) which do not depend on \( g \). This means that plugging the power expansion of \( T \) into (56) gives a system of \( N \) linear algebraic equations for the derivatives \( D_i^{(1)} \equiv \partial T_i / \partial P_i \). Solving these equations, one should just integrate \( D_i^{(1)} \), finally obtaining \( T_1 \). Then one can use this calculated \( T_1 \) in order to find \( T_2 \), etc. Note that each time there emerge non-trivial integrability conditions
\[
\partial D_i / \partial P_j = \partial D_j / \partial P_i .
\]

Thus, the most difficult part of the calculation is to find dual Hamiltonians. In the rational Calogero system, when the system is self-dual, the calculation is very immediate. Similarly, the calculation can be easily done for any member of the Calogero–Ruijsenaars–Dell family.

5. Conclusion

A peculiar feature of integrable systems is that interaction is introduced by a canonical transformation of a free system. In other words, the corresponding flows in the space of coupling constants are Hamiltonian ones. This observation should provide new insights into the structure of exact (non-perturbative) partition functions, which are always (generalized) \( \tau \)-functions [1,2].

Let us remind that the partition function, arising after functional integration over fields,
\[
\int D\varphi \exp \left( \mathcal{A} (\varphi_0 + \varphi) + i \sum_{\{k\}} t_{\{k\}} V_{\{k\}} (\varphi_0 + \varphi) \right)
\]
\[
= \tau (t_{\{k\}} | \varphi_0) = \left( \vec{G} (\varphi_0) \right) P \exp \left( i \sum_{\{k\}} \int \vec{H}_{\{k\}} dt_{\{k\}} \right)
\]
\[
= \langle \vec{G} e^{i\vec{S}} \rangle
\]
is essentially insensitive to the choice of original action \( \mathcal{A} (\varphi) \), and the equivalence classes (of actions) can be represented by integrable systems, i.e., by the systems where the partition function is a matrix element of a group element \( \vec{G} \) of some group [8]. Our evolution operator in the space of coupling constants, \( e^{i\vec{S}} = P \exp (i \sum_{\{k\}} \int \vec{H}_{\{k\}} dt_{\{k\}}) \), which performs a canonical transformation of integrable system into the free one, as discussed in Section 2 should also be a group element. The moduli \( \varphi_0 \), parameterizing the boundary conditions and vacuum states of the original theory are related to the (eigen)values of the Hamiltonians \( H_{\{k\}} \) (i.e., in our simple examples, the dressed momenta \( P \)). The couplings (perturbations) \( t_{\{k\}} \) (i.e., our \( g \)) become time-variables of the \( \tau \)-function. The fact that the space of relevant Hamiltonians (the moduli space) is an orbit of the canonical transformation group, which can be alternatively considered as changing the couplings can further clarifies the relation between the times- and moduli-dependencies of the partition functions [9] and bring us closer to the long-awaited group-theoretical treatment of entire \( \tau \)-function, its time-dependencies being associated with the commutative (Cartanian) part of the algebra (generated by the Hamiltonians \( H_{\{k\}} \)), while moduli-dependencies—with its non-Cartanian part (generated by \( T_a \) and their commutators among themselves and with \( H_{\{k\}} \)). See Ref. [10] for related approaches to this problem.

Acknowledgements

We acknowledge the hospitality of the University of British Columbia, where the work was completed, and the support of the Nato Collaborative Linkage Grant SA (PST.CLG.977361)5941. Our work is also supported by grants: CRDF #6531, INTAS 00-561 and also by the RFBR-01-02-17682a (A. Mironov) and RFBR-00-02-16477 (A. Morozov), the Grant of Support for the Scientific Schools 96-15-96798 (A. Mironov) and the Russian President’s Grant 00-15-99296 (A. Morozov).

References

[1] See, for example, 
A. Morozov, Sov. Phys. Usp. 35 (1992) 671; 
A. Morozov, Uspekhi Fiz. Nauk 164 (1994) 3, hep-th/9303139; 

[2] A. Mironov, A. Morozov / Physics Letters B 524 (2002) 217–226 225
A. Mironov, A. Morozov / Physics Letters B 524 (2002) 217–226

[2] For particular applications of this idea, see, Quantum integrable systems:
M. Jimbo, T. Miwa, M. Sato, in: Holonomic Quantum Fields, I, Publ. Res. Inst. Math. Sci., Vol. 14, Kyoto University, 1978, p. 223;
M. Jimbo, T. Miwa, M. Sato, in: Holonomic Quantum Fields, II–V, Publ. Res. Inst. Math. Sci., Vol. 15, Kyoto University, 1979, pp. 201, 577, 871, 1531;
A.R. Its, A.G. Izergin, V.E. Korepin, N.A. Slavnov, Int. J. Mod. Phys. B 4 (1990) 1003;
A.G. Izergin, D.A. Coker, V.E. Korepin, J. Phys.: Math. Gen. A 25 (1992) 4315–4334;
2d gravity and matrix models:
M. Fukuma, H. Kawai, R. Nakayama, Int. J. Mod. Phys. A 6 (1991) 1383;
R. Dijkgraaf, E. Verlinde, H. Verlinde, Nucl. Phys. B 348 (1991) 435;
A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov, A. Oriol, Nucl. Phys. B 357 (1991) 565;
S. Kerchver, A. Marshakov, A. Mironov, A. Morozov, A. Zabrodin, Phys. Lett. B 275 (1992) 311;
S. Kerchver, A. Marshakov, A. Mironov, A. Morozov, A. Zabrodin, Nucl. Phys. B 380 (1992) 181;
Nonperturbative SUSY Yang–Mills theories (Seiberg–Witten theory):
A. Gorsky, I. Krichever, A. Marshakov, A. Mironov, A. Morozov, Phys. Lett. B 355 (1995) 466–474, hep-th/9505035;
H. Itouyama, A. Mironov, Nucl. Phys. B 477 (1996) 855–877, hep-th/9511126;
H. Itouyama, A. Mironov, Nucl. Phys. B 491 (1997) 529–573, hep-th/9511261;
H. Itouyama, A. Mironov, hep-th/9601168;
H. Braden, A. Marshakov, A. Mironov, A. Morozov, Nucl. Phys. B 558 (1999) 371–390, hep-th/9902205;
A. Gorsky, A. Marshakov, A. Mironov, A. Morozov, Nucl. Phys. B 527 (1999) 690–716, hep-th/9802007;
A. Mironov, hep-th/9903087;
A. Gorsky, A. Mironov, hep-th/0011197;
Chern–Simons theory and AdS/CFT correspondence:
A. Mironov, hep-th/0010031.

[3] H. Braden, A. Marshakov, A. Mironov, A. Morozov, Nucl. Phys. B 573 (2000) 553, hep-th/9906240.

[4] S.N. Ruijsenaars, Commun. Math. Phys. 115 (1988) 127–165;
O. Babelon, D. Bernard, Phys. Lett. B 317 (1993) 363–368;
V. Fock, in: P. Pyatov, S. Solodukhin (Eds.), Geometry and Integrable Models, World Scientific, 1995, p. 20;
V. Fock, A. Rosly, math.qa/9802054;
V. Fock, A. Gorsky, N. Nekrasov, V. Roubtsov, JHEP 0007 (2000) 028;
A. Gorsky, V. Rubtsov, hep-th/0103004.

[5] A. Mironov, A. Morozov, Phys. Lett. B 475 (2000) 71–76, hep-th/9912088;
A. Mironov, A. Morozov, hep-th/0001168;
A. Gorsky, A. Mironov, hep-th/0011197;
A. Mironov, hep-th/0010423.

[6] N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19, hep-th/9407087;
N. Seiberg, E. Witten, Nucl. Phys. B 431 (1994) 484, hep-th/9408099;
H.W. Braden, I.M. Krichever (Eds.), Integrability: The Seiberg–Witten and Whitham Equations, Gordon & Beach, 2000.

[7] A. Marshakov, A. Mironov, A. Morozov, Phys. Lett. B 389 (1996) 43–52, hep-th/9607109;
A. Marshakov, A. Mironov, A. Morozov, Mod. Phys. Lett. A 12 (1997) 773–788, hep-th/9701014;
A. Marshakov, A. Mironov, A. Morozov, Int. J. Mod. Phys. A 15 (2000) 1157–1206, hep-th/9701123.

[8] For the notion of group element and its relation to generalized τ-functions see, for example,
A. Mironov, A. Morozov, L. Vinet, Theor. Math. Phys. 100 (1995) 890–899, hep-th/9312213;
A. Gerasimov, S. Khoroshkin, D. Lebedev, A. Mironov, A. Morozov, Int. J. Mod. Phys. A 10 (1995) 2589–2614, hep-th/9405011;
A. Mironov, L. Vinet, Int. J. Mod. Phys. A 13 (1998) 1651–1708, hep-th/9409093;
S. Kerchver, A. Mironov, A. Morozov, hep-th/9501013;
A. Gerasimov, S. Kerchver, A. Marshakov, A. Mironov, A. Morozov, M. Olshanetsky, Int. J. Mod. Phys. A 12 (1997) 2523–2584, hep-th/9601161;
A. Mironov, Theor. Math. Phys. 114 (1998) 127, q-alg/9711006.

[9] S. Kerchver, A. Marshakov, A. Mironov, A. Morozov, Mod. Phys. Lett. A 8 (1993) 1047–1061, hep-th/9208046;
See also the second review in Ref. [1].

[10] A. Gerasimov, A. Morozov, K. Selivanov, Int. J. Mod. Phys. A 16 (2001) 1531–1558, hep-th/0005053;
A. Mironov, A. Morozov, Phys. Lett. B 490 (2000) 173–179, hep-th/0003290.