A CANONICAL MODEL OF THE ONE-DIMENSIONAL
DYNAMICAL DIRAC SYSTEM WITH BOUNDARY CONTROL

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Abstract. The one-dimensional Dirac dynamical system $\Sigma$ is

$$iu_t + \sigma_3 u_x + Vu = 0, \quad x, t > 0; \quad u_{|t=0} = 0, \quad u_{|x=0} = f, \quad t > 0,$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix; $V = \begin{pmatrix} 0 & p \\ \bar{p} & 0 \end{pmatrix}$ with $p = p(x)$ is a potential; $u = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}$ is the trajectory in $H = L^2(\mathbb{R}_+; \mathbb{C}^2)$; $f \in \mathcal{F} = L^2([0,\infty); \mathbb{C})$ is a boundary control. System $\Sigma$ is not controllable: the total reachable set $\mathcal{W} = \text{span}_{t \geq 0}\{u^f(\cdot,t) | f \in \mathcal{F}\}$ is not dense in $H$, but contains a controllable part $\Sigma_u$. We construct a dynamical system $\Sigma_a$, which is controllable in $L^2(\mathbb{R}_+; \mathbb{C}^2)$ and connected with $\Sigma_u$ via a unitary transform. The construction is based on geometrical optics relations: trajectories of $\Sigma_a$ are composed of jump amplitudes that arise as a result of projecting in $\mathcal{W}$ onto the reachable sets $\mathcal{W}^f = \{u^f(\cdot,t) | f \in \mathcal{F}\}$. System $\Sigma_a$, which we call the amplitude model of the original $\Sigma$, has the same input/output correspondence as system $\Sigma$. As such, $\Sigma_a$ provides a canonical completely reachable realization of the Dirac system.

1. Introduction. The Dirac operator $L$ acts in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+; \mathbb{C}^2)$ of $\mathbb{C}^2$-valued functions $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ of $x \geq 0$ by

$$Ly := i\sigma_3 \frac{dy}{dx} + Vy,$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & p \\ \bar{p} & 0 \end{pmatrix}$$

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with a smooth $^1$ $\mathbb{C}$-valued $p$. It is defined on $\text{Dom} \, L = \{y \mid y, Ly \in \mathcal{H}\}$.

With the operator $L$ one associates a dynamical system $\Sigma$ of the form
\begin{align*}
   iu_t + Lu &= 0, \quad x > 0, \; t > 0; \quad (2) \\
   u|_{t=0} &= 0, \quad x \geq 0; \quad (3) \\
   u_1|_{x=0} &= f, \quad t \geq 0, \quad (4)
\end{align*}
where $f \in \mathcal{F} := L_2([0, \infty); \mathbb{C})$ is the boundary control (a $\mathbb{C}$-valued function of time).

For controls of the class
\[ \mathcal{M} := \{f \in \mathcal{F} \mid f \text{ is smooth, supp } f \subset (0, \infty)\} \]
vanishing near $t = 0$, the system has a unique classical solution (wave) $u = u^f(x, t) = \left(\begin{array}{c} u_1^f(x, t) \\ u_2^f(x, t) \end{array}\right)$. The input/output correspondence in $\Sigma$ is realized by the map $R : f \mapsto u_2^f(0, \cdot)$, which acts in the space $\mathcal{F}$.

Associated with the system are the reachable sets $\mathcal{U}^t = \{u^f(\cdot, t) \mid f \in \mathcal{M}\}$ and the total reachable set $\mathcal{U} = \text{span}\{\mathcal{U}^t \mid t > 0\} \subset \mathcal{H}$. The relation $\mathcal{U} \neq \mathcal{H}$ holds, which is interpreted as a lack of controllability of the system (2)-(4) [4, 8].

Since $L$ does not depend on $t$ and $i \frac{d}{dt} \mathcal{M} = \mathcal{M}$ occurs, one has $Lu^f(\cdot, t) = -iu_1^f(\cdot, t) = u^{-i\mathcal{M}}(\cdot, t) \in \mathcal{U}$. Hence, $L\mathcal{U} \neq \mathcal{U}$ holds and, therefore, $L$ has a part
\[ L_u := L|_{\mathcal{U}}, \]
which is a closable operator acting in the subspace $\mathcal{U} \subset \mathcal{H}$. We call $L_u$ the wave part of $L$. It determines a dynamical system $\Sigma_u$ of the same form as (2)-(4), but with the replacement of $L$ by $L_u$ and $\mathcal{H}$ by $\mathcal{U}$.

- In the present paper, we construct an operator $L_a$, which is unitarily equivalent to $L_u$ and is called the amplitude model of $L_u$. The model acts in the Hilbert spaces $\mathcal{G} = L_2(\mathbb{R}_+; \mathbb{C})$ of functions $y = y(\tau), \tau \geq 0$; on compactly supported functions it has the form
\[ L_a y = i \frac{dy}{d\tau} + vy + Qy, \]
where $v$ is a real function and $Q$ is an operator, which acts locally as a Volterra operator with one-dimensional anti-Hermitian part $\frac{1}{2i}(Q - Q^*)$ [9].

Operator $L_a$ determines the model dynamical system $\Sigma_a$ of the form
\begin{align*}
   ia_t + L_a a &= 0, \quad \tau > 0, \; t > 0; \quad (5) \\
   a|_{t=0} &= 0, \quad \tau \geq 0; \quad (6) \\
   a|_{\tau=0} &= f, \quad t \geq 0, \quad (7)
\end{align*}
with the controls $f \in \mathcal{F}$, and $\mathbb{C}$-valued waves $a = a^f(\tau, t)$. In contrast to the original $\Sigma$, system $\Sigma_a$ is controllable: its reachable sets $\mathcal{V}^t = \{a^f(\cdot, t) \mid f \in \mathcal{M}\}$ and $\mathcal{V} = \text{span}\{\mathcal{V}^t \mid t > 0\}$ provide $\mathcal{V} = \mathcal{G}$. A fact of fundamental importance is that system $\Sigma_a$ is completely determined by the map $R$ and, moreover, is constructed from $R$ in an effective way. In fact, this construction is the main subject of the paper. In the terminology of system theory, the model $\Sigma_a$ provides a canonical controllable realization of the original $\Sigma$, which is relevant to the input/output map: see [10], Chapter 10, sec 10.6.

$^1$Everywhere smooth is $C^\infty$-smooth on $\mathbb{R}_+$. 


By \( P^\xi \) we denote the projectors in \( \mathcal{H} \) onto \( \mathcal{H}^\xi \), \( \xi > 0 \). When these projectors act on the waves \( u^f \), the discontinuities (jumps) of \( P^\xi u^f \) occur at points \( x = \xi \). The model waves \( u^f \) are composed from these jumps. Such a construction is based on the amplitude formula, which is a form of geometrical optics relations. These relations describe propagation of singularities of waves \( u^f \) initiated by singular controls \( f \). Formulas of this type are the main tool for solving inverse problems by the boundary control method \([2, 3]\).

From the point of view of operator theory, the passage from \( \Sigma \) to \( \Sigma_a \) diagonalizes the family \( \{P^\xi\}_{\xi \geq 0} \): each \( P^\xi \) is transferred to the projector \( Y^\xi \), which acts in \( G \) by cutting off functions on segment \( 0 \leq \tau \leq \xi \).

• Amplitude model is elaborated in the framework of a program outlined in \([1]\). The program aims to develop a new (the so-called wave) functional model for important class of symmetric operators: see \([1], [5]-[8]\). In fact, the amplitude model is an intermediate step towards the wave model. However, \( \Sigma_a \) is also connected with some general notions of the operator theory and, as such, is of certain independent interest.

2. Dynamical Dirac system. In this section, system (2)–(4) is endowed with standard attributes of control and system theory: spaces and operators. Throughout the paper we adhere to the agreement:

**Convention 1.** All time-dependent functions are extended to \( t < 0 \) by zero.

Waves. We start with the general well-known properties of solutions to problem (2)–(4). They are established by the same way that is used in the paper \([4]\) for the Dirac equation, which differs from (2) by a simple coordinate transformation.

• As mentioned in the Introduction, for \( f \in \mathcal{M} \), the problem has a unique classical solution \( u^f \). For it, the representation

\[
 u^f(x, t) = f(t - x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^t f(t - s) \begin{pmatrix} w_1(x, s) \\ w_2(x, s) \end{pmatrix} ds, \quad x \geq 0, \ t \geq 0
\]

holds with the kernels \( w_{1,2} \), which are smooth in \( 0 \leq x \leq t \leq T \) provided \( w_1(0, \cdot) = 0 \). Since the right-hand side makes sense for all \( f \in \mathcal{F} = L^2([0, \infty) ; \mathbb{C}) \), we consider it as a (generalized) solution to (2)–(4) for such \( f \) and denote by the same notation \( u^f \).

Let \( \Omega^\alpha := [0, \alpha] \subset \mathbb{R}_+ \). By (8), the relation

\[
 \text{supp } u^f(\cdot, t) \subset \Omega^t, \quad t > 0
\]

holds for all \( f \in \mathcal{F} \). It shows that the signals in the Dirac system propagate along the half-line \( x \geq 0 \) with the unit speed. This is the reason why we call \( u^f \) waves. By the same reason, the waves depend on the potential locally: for times \( 0 < t \leq \alpha \) the solution \( u^f(\cdot, t)|_{\Omega^\alpha} \) is determined by the values of \( V \) on \( \Omega^\alpha \) (does not depend on \( V|_{x > \alpha} \)).

Fix \( \sigma > 0 \) and introduce the reduced shift operator \( S_\sigma \) that delays time-dependent functions in time by

\[
 (S_\sigma h(\cdot, \cdot))(t) := h(\cdot, t - \sigma), \quad t \geq 0,
\]

where \( (\cdot, \cdot) \) denotes other possible arguments of \( h \) (and recall Convention 1!). Since the operator \( L \) (the potential \( V \)) does not depend on time, one has

\[
 u^{S_\sigma f}(\cdot, t) = (S_\sigma u^f)(\cdot, t) = u^f(\cdot, t - \sigma), \quad t \geq 0,
\]
i.e., delay of control implies the same delay of the wave. By the same reason, for \( f \in \mathcal{M} \) one has
\[
L u^f \equiv (2) - i u^f_t = u^{-i \frac{df}{d\tau}}, \quad x > 0, \ t > 0; \quad S_\sigma L = L S_\sigma. \quad (11)
\]
For the operator of the form (1), the latter relation is equivalent to the independence of \( V \) from time.

- If \( f \) is piece-wise continuous (with possible jumps on \( t \geq 0 \)) then the wave \( u^f \) is also piece-wise continuous for \( x \geq 0 \), and the equality
  \[
u^f (\xi - 0, t) - u^f (\xi + 0, t) = [f(t - \xi + 0) - f(t - \xi - 0)] \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad 0 \leq \xi \leq t \quad (12)
\]easily follows from the continuity of the integral in (8) on \( x \). In fact, (12) is a simplest geometrical optics relation, which describes propagation of singularities in the Dirac system. It expresses the jump amplitude of the wave via the jump amplitude of the control, and we call it the amplitude formula.

**Spaces and operators.** Here we deal with the problem (2)–(4) restricted on a finite time interval \([0, T]\)

\[
i u_t + i \sigma_3 u_x + V u = 0, \quad x > 0, \ 0 < t < T; \quad (13)
\]
\[
u|_{t=0} = 0, \quad x \geq 0; \quad (14)
\]
\[
u|_{x=0} = f, \quad t \geq 0 \quad (15)
\]
with a smooth potential \( V \). It is referred to as system \( \Sigma^T \).

- The Hilbert space \( \mathcal{H}^T := L_2(\Omega^T; \mathbb{C}^2) \) is said to be the inner space of system \( \Sigma^T \).
- The waves \( u^f (\cdot, t) \) belong to \( \mathcal{H}^T \) and are interpreted as the states of the system (at the moment \( t \)). The inner space contains an increasing family of subspaces
  \[
\mathcal{H}^\xi := \{ y \in \mathcal{H} \mid \text{supp } y \subset \Omega^\xi \}, \quad 0 \leq \xi \leq T.
\]

- The space of controls \( \mathcal{F}^T := L_2([0, T]; \mathbb{C}) \) is called the outer space of the system.

It contains an increasing family of subspaces
\[
\mathcal{F}^{T, \xi} := \{ f \in \mathcal{F}^T \mid \text{supp } f \subset [T - \xi, T] \}, \quad 0 \leq \xi \leq T \quad (\mathcal{F}^{T, T} \equiv \mathcal{F}^T).
\]
Each subspace is formed by delayed controls: \( T - \xi \) is the delay, \( \xi \) is action time. By (9) and (10), one has
\[
u^{S_{T-\xi}}(\cdot, T) = u^f (\cdot, \xi) \quad \text{in } \Omega^T; \quad (16)
\]
therefore, \( f \in \mathcal{F}^{T, \xi} \) implies \( u^f (\cdot, \xi) \in \mathcal{H}^\xi \).

- Also, the class of controls
  \[
\mathcal{M}^T := \{ f \in \mathcal{F}^T \mid f \text{ is smooth, supp } f \subset (0, T) \} \subset \mathcal{F}^T
\]
is in use. It is dense in \( \mathcal{F}^T \) and relation \( \int_{\mathcal{M}^T} = \mathcal{F}^T \) holds.

- The input/state map \( W^T : \mathcal{F}^T \to \mathcal{H}^T, W^T f := u^f (\cdot, T) \) is called the control operator. The representation
  \[
(W^T f)(x) = f(T - x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^T f(T - s) \begin{pmatrix} u_1(x, s) \\ u_2(x, s) \end{pmatrix} \, ds, \quad x \in \Omega^T \quad (17)
\]
follows from (8) and shows that \( W^T \) is continuous and maps \( \mathcal{F}^T \) onto its image \( W^T \mathcal{F}^T \subset \mathcal{H}^T \) isomorphically. Moreover, for every \( 0 < \xi \leq T \), operator \( W^T \) is an isomorphism from \( \mathcal{F}^{T, \xi} \) onto the (closed) subspace \( W^T \mathcal{F}^{T, \xi} \subset \mathcal{H}^\xi \).
The control operator preserves smoothness: if \( f \in \mathcal{M}^T \) then \( u^f(\cdot, T) = W^T f \) is smooth in the segment \( \Omega^T \) and vanishes near its endpoint \( x = T \).

In accordance with (11), the relation
\[
W^T \left[ -i \frac{df}{dt} \right] = LW^T \quad \text{on } \mathcal{M}^T
\]
(18)
is just a way of writing the Dirac equation (13).

- In system \( \Sigma^T \), the input/output correspondence is realized by the response operator \( R^T : \mathcal{F}^T \to \mathcal{F}^T \), \( (R^T f)(t) := u^f_x(0, t), \ 0 \leq t \leq T \). By (8) one has
\[
(R^T f)(t) = \int_0^t r(t - s) f(s) \, ds, \quad 0 \leq t \leq T
\]
(19)
with the \( \mathbb{C} \)-valued function \( r(t) = w_2(0, t) \) called the response function.

- In subsequent considerations, the key role is played by the connecting operator \( C^T : \mathcal{F}^T \to \mathcal{F}^T \), \( C^T := (W^T)^* W^T \). Its definition yields
\[
(C^T f, g)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{M}^T} = (u^f(\cdot, T), u^g(\cdot, T))_{\mathcal{F}^T},
\]
(20)
so that \( C^T \) connects the metrics of the outer and inner spaces. Since \( W^T \) acts isomorphically, for any \( T > 0 \) operator \( C^T \) is a positive-definite isomorphism in \( \mathcal{F}^T \).

**Lemma 2.1.** The relation
\[
C^T = \mathbb{1} - (R^T)^* R^T
\]
(21)
holds and represents the connecting operator via the response operator.

**Proof.** Take \( f, g \in \mathcal{M}^T \). The Blagoveshchenskii function
\[
B(s, t) := \left( u^f(\cdot, s), u^g(\cdot, t) \right)_{\mathcal{F}^T}, \quad 0 \leq s, t \leq T
\]
satisfies
\[
B_t + B_s = \int_{\Omega^T} \left[ u^f(x, s)u^f_x(x, t) + u^f(x, s)u^g(x, t) \right] \, dx \quad \text{see (13)}
\]
\[
= \int_0^T \left\{ u^f(x, s)[-\sigma_xu^f_x(x, t) + iV(x)u^g(x, t)] +
\right. \
[-\sigma_xu^f_x(x, t) + iV(x)u^f(x, t)]u^g(x, t) \bigg\} \, dx =
\]
\[
= \int_0^T \left\{ [-u^f_1(x, s)u^f_1(x, t) + u^f_2(x, s)u^g_2(x, t)] +
\right. \
+ [-u^f_1(x, s)u^f_2(x, t) + u^f_2(x, s)u^g_1(x, t)] \bigg\} \, dx =
\]
\[
u^f(0, 0)u^f_1(0, t) - u^f_1(0, 0)u^g_2(0, t) = f(s)g(t) - (R^T f)(s)\overline{(R^T g)(t)} =: h(s, t).
\]

In the equality \( \equiv \) we use the relation \( \text{supp } u^f(\cdot, s), u^g(\cdot, t) \subset [0, T] \) by (9), so that \( u^f(T, s) = u^g(T, t) = 0 \).

Integrating the equation \( B_t + B_s = h \) in the domain of definition of \( B \) with regard to \( B(s, 0) \equiv 0 \), we obtain
\[
B(s, t) = \int_{s-t}^s h(\xi, \xi - s + t) \, d\xi, \quad 0 \leq s, t \leq T.
\]
Thus, we arrive at the relations

$$
(C^T f, g)_{\mathcal{H}^T}^\text{see (20)} = (u^f(\cdot, T), u^g(\cdot, T)) = B(T, T) = \int_0^T h(\xi, \xi) \, d\xi = \\
= \int_0^T [f(t)g(t) - (R^T f)(t)(R^T g)(t)] \, dt = (f, g)_{\mathcal{H}^T} - (R^T f, R^T g)_{\mathcal{H}^T} = \\
= ([I - (R^T)^* R^T]f, g)_{\mathcal{H}^T}.
$$

With regard to the arbitrariness of $f, g$, density of $\mathcal{M}^T$ in $\mathcal{H}^T$, and boundedness of the operators, which we deal with, we arrive at (21).

\begin{corollary}
The relation

$$
(C^T f)(T) = f(T)
$$

holds for all $f \in \mathcal{M}^T$.

Indeed, we have

$$
[(R^T)^* R^T f](t) \stackrel{\text{see (19)}}{=} \int_0^T \int_{\max \{t, \eta\}}^T r(s - \eta)r(s - t) \, ds \, f(\eta) \, d\eta;
$$

hence $[(R^T)^* R^T f](T) = 0$, which leads to (22).

\begin{itemize}
  \item There are some properties of the connecting operator, which are used later.
  \end{itemize}

For $f, g \in \mathcal{M}^T$ one has

$$
\left(C^T \left[ \frac{df}{dt}, g \right] \right)_{\mathcal{H}^T} = \left(C^T f, \left[ \frac{dg}{dt} \right] \right)_{\mathcal{H}^T} = (\text{see (20), (11)}) = \\
= - (Lu^f(\cdot, T), u^g(\cdot, T))_{\mathcal{H}^T} + (u^f(\cdot, T), Lu^g(\cdot, T))_{\mathcal{H}^T} = \\
= - (i\sigma_3 \partial_x + V)u^f(\cdot, T), u^g(\cdot, T))_{\mathcal{H}^T} + (u^f(\cdot, T), [i\sigma_3 \partial_x + V]u^g(\cdot, T))_{\mathcal{H}^T} = \\
= (i\sigma_3 u^f(\cdot, T), u^g(\cdot, T))_{\mathcal{H}^T} + (u^f(\cdot, T), i\sigma_3 u^g(\cdot, T))_{\mathcal{H}^T} = \\
= -i \int_0^T [\sigma_3 u^f(x, T) \cdot \bar{u}^g(x, T)] \, dx = i \left[ u^f(0, T)\bar{u}^g(0, T) - u^g(0, T)\bar{u}^g(0, T) \right] = \\
= i \left[ f(T)\bar{g}(T) - (R^T f)(T)(R^T g)(T) \right].
$$

Thus, we arrive at the relations

$$
(Lu^f(\cdot, T), u^g(\cdot, T))_{\mathcal{H}^T} = (u^f(\cdot, T), Lu^g(\cdot, T))_{\mathcal{H}^T} = \\
= \left(C^T \left[ -i \frac{df}{dt}, g \right] \right)_{\mathcal{H}^T} = \left(C^T f, \left[ -i \frac{dg}{dt} \right] \right)_{\mathcal{H}^T} = \\
= -i \left[ f(T)\bar{g}(T) - (R^T f)(T)(R^T g)(T) \right].
$$

The class of smooth controls

$$
\mathcal{M}_0^T := \{ f \in \mathcal{M}^T \mid f(T) + (R^T f)(T) = 0 \}
$$

is dense in $\mathcal{H}^T$. Indeed, the operator $I + R^T$ is an isomorphism of $\mathcal{H}^T$, which satisfies $[I + R^T] \mathcal{M}^T = \mathcal{M}^T$, whereas $\mathcal{M}_0^T = [I + R^T]^{-1} \{ g \in \mathcal{M}^T \mid g(T) = 0 \}$ is the image of a dense set.

As a consequence of (23), we obtain

$$
C^T \left[ \frac{df}{dt} \right] = \left[ \frac{df}{dt} \right] C^T f, \quad f \in \mathcal{M}_0^T. \quad (24)
$$
The corresponding smooth waves, which belong to $W^T M^T$, satisfy $u^f(0, T) = f(T) = -(R^T f)(T) = -u^f_2(0, T)$, $u^f(T, T) = 0$, and form a dense set in $W^T$.

**Controllability.** The sets of waves

$$W^T := \{ u^f(\cdot, \xi) \mid f \in W^T \} \supseteq W^T [F^T, \xi \cap M^T] \supseteq H^{\xi}, \quad 0 \leq \xi \leq T,$$

are called reachable. With increasing $\xi$, the reachable sets expand.

As is seen from (17), the control operator $W^T$ maps $F^T, \xi$ onto $U^\xi$ isomorphically. Meanwhile, the specificity of the Dirac system is that relation $H^\xi \ominus U^\xi \neq \{0\}$ holds for all $\xi > 0$, i.e., the waves do not constitute $L_2$-complete systems in the segments $\Omega^\xi$, which they fill up. As an example, in the case of (13) with $V = 0$ one has

$$W^T = \{ y = \phi \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \mid \phi \in L_2(\Omega^\xi; \mathbb{C}) \}, \quad 0 \leq \xi \leq T$$

and, respectively,

$$H^\xi \ominus W^T = \{ y = \psi \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \mid \psi \in L_2(\Omega^\xi; \mathbb{C}) \}, \quad 0 \leq \xi \leq T$$

see [4, 8]. This effect is interpreted as a lack of controllability of the system (13)–(15); it distinguishes the Dirac system from the second-order dynamical system associated with the Sturm-Liouville operator [5].

For the ‘big’ system (2)–(4) we define the total reachable set

$$U := \text{span} \{ U^T \mid T > 0 \},$$

and also have a lack of controllability: $H \ominus W^T \neq \{0\}$ holds, where $H = L_2(\mathbb{R}_+; \mathbb{C}^2)$ [8].

- In view of the foregoing, it is $W^T$, not $H^T$, that is naturally regarded as the inner space of the system $\Sigma^T$. Such a view is consistent with the philosophy of system theory [10], and in the sequel we in fact deal with the ‘wave part’ of the system (13)–(15). In particular, we consider the control operator $W^T$ as a map from the outer space $F^T$ to the wave space $W^T$.

The elements of $W^T$ (waves) are characterized by the following property.

**Proposition 1.** An element $u \in H^T$ belongs to $W^T$ (is a wave) if and only if its components are connected by $u_2 = M^T u_1$ with the Volterra operator $M^T : L_2(\Omega^T; \mathbb{C}) \rightarrow L_2(\Omega^T; \mathbb{C})$ of the form

$$(M^T h)(x) = \int_x^T m(x, s) h(s) \, ds, \quad x \in \Omega^T$$

with a kernel $m$ smooth in $0 \leq x \leq s \leq T$.

Indeed, by (17), the components of the wave $u^f(\cdot, T) = W^T f \in W^T$ obey

$$u^f_1(x, T) = f(T - x) + \int_x^T w_1(x, s) f(T - s) \, ds, \quad x \in \Omega^T;$$

$$u^f_2(x, T) = \int_x^T w_2(x, s) f(T - s) \, ds, \quad x \in \Omega^T.$$

Solving the first relation as a Volterra equation of the second kind on $f$ and substituting the result to the second relation we easily arrive at (26).
As a consequence, \( u \in \mathcal{H}^\xi \) holds if and only if it obeys
\[
\text{supp } u \subset \Omega^\xi; \quad u|_{\Omega^\xi} = \begin{pmatrix} a \\ M^\xi a \end{pmatrix}, \quad a \in L_2(\Omega^\xi; \mathbb{C}) \quad (0 < \xi \leq T).
\] (27)
At the same time, it is easy to see that \( u \in \mathcal{H}^\xi \perp := \mathcal{H}^T \ominus \mathcal{H}^\xi \) if and only if
\[
u|_{\Omega^\xi} = \begin{pmatrix} -(M^\xi)^* b \\ b \end{pmatrix}, \quad b \in L_2(\Omega^\xi; \mathbb{C}).
\] (28)

Let \( P^\xi \) be the projector in \( \mathcal{H}^T \) on the subspace \( \mathcal{H}^\xi \); it is called the wave projector. As is seen from (25), in the case \( V = 0 \), projector \( P^\xi \) just cuts off waves onto \( \Omega^\xi \). However, in the general case, it acts in more complicated way. We omit the proof of the following result, which is easily derived from (26), (27), (28).

**Proposition 2.** For \( u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y \\ M^Ty \end{pmatrix} \in \mathcal{H}^T \), the representation
\[
P^\xi u = \begin{cases} \begin{pmatrix} a \\ M^\xi a \end{pmatrix} & \text{in } \Omega^\xi \\ 0 & \text{in } \Omega^\xi \ominus \Omega^\xi \end{cases}, \quad a = \left[1 + (M^\xi)^* M^\xi \right]^{-1} \left[\theta^\xi y + (M^\xi)^* \theta^\xi M^Ty \right] (29)
\]
holds, where \( \theta^\xi \) restricts functions given on \( \Omega^T \) to \( \Omega^\xi \). For continuous \( u \) the equalities
\[
(P^\xi u)_1(0) = u_1(0), \quad (P^\xi_2 u)(\xi) = u_2(\xi)
\]
hold, where \( P^\xi_\perp = \mathbb{1} - P^\xi \).

**Operator \( L^T_u \).** Let \( A \) be an operator in a Hilbert space \( \mathcal{F} \) and \( \mathcal{F} \subset \mathcal{F} \) a (closed) subspace. We say that \( A \) has a part in \( \mathcal{F} \) if \( \mathcal{F} \ominus \text{Dom } A = \mathcal{F} \) and \( A \mid_{\mathcal{F} \ominus \text{Dom } A} \subset \mathcal{F} \) holds\(^2\). If \( A \) is closed then its part \( A \mid_{\mathcal{F} \ominus \text{Dom } A} \) is also a closed operator in \( \mathcal{F} \).

By (11) and with regard to \( i \frac{d}{dt} \mathcal{M}^T = \mathcal{M}^{T^*} \), for \( f \in \mathcal{M}^T \) one has \( \mathcal{L}u(\cdot, T) = u - i \frac{d}{dt} \mathcal{M}^{T^*}(\cdot, T) \subset \mathcal{H}^T \). As a result, one can establish the following fact. Recall that the ‘maximal’ Dirac operator \( L \), which acts in \( \mathcal{H} \), is defined on \( \{ y \mid y, Ly \in \mathcal{H} \} \).

**Proposition 3.** For any \( T > 0 \), the operator \( L \) has a part in \( \mathcal{H}^T \).

We denote it by \( L^T_u \) and call the wave part of \( L \). In fact, it is this part, which governs the evolution of system (13)–(15): one can replace there \( L \) by \( L^T_u \).

Speaking in advance, we will construct a functional model \( L^T_u \) of \( L^T_u \). The model is unitarily equivalent to \( L^T_u \) and acts in a space of \( \mathbb{C} \)-valued functions. As we will see, the dynamical system \( \Sigma_a \) governed by this model is controllable.

**Dual projectors.** • Recall that \( P^\xi \) projects (orthogonally) in \( \mathcal{H}^T \) onto \( \mathcal{H}^\xi \). The projectors
\[
P^{T, \xi} := (W^T)^{-1} P^\xi W^T : \mathcal{T} \rightarrow \mathcal{T}, \quad 0 \leq \xi \leq T
\] (30)
are called dual to the wave projectors.

**Lemma 2.2.** The dual projector \( P^{T, \xi} \) is the skew projector in \( \mathcal{T} \) onto \( \mathcal{T}^{T, \xi} \) in parallel to the subspace \( (C^T)^{-1} [\mathcal{T} \ominus \mathcal{T}^{T, \xi}] \). The relation \( (P^{T, \xi})^* C^T = C^T P^{T, \xi} \) is valid.

\(^2\) See a discussion on terms in [8].
Proof. 1. We verify the characteristic properties of the skew projector in \(\mathcal{F}_T\) parallel to \((C^T)^{-1} [\mathcal{F}_T \ominus \mathcal{F}_T,\xi]\).

The definition of \(\mathcal{P}^{T,\xi}\) implies \((\mathcal{P}^{T,\xi})^2 = \mathcal{P}^{T,\xi}\). Since \(W_T,\mathcal{F}_T,\xi = \mathcal{W}_T\), for \(f \in \mathcal{F}_T,\xi\) one has

\[
\mathcal{P}^{T,\xi}f = (W_T)^{-1}P^T(W_T) = (W_T)^{-1}P^Tw(f,\cdot) = (W_T)^{-1}w(f,\cdot) = f. \tag{31}
\]

Thus, \(\mathcal{P}^{T,\xi}\) acts on \(\mathcal{F}_T,\xi\) identically. Meanwhile, \(W_T,\mathcal{F}_T,\xi = \mathcal{W}_T\) implies \((W_T)^* [\mathcal{W}_T \ominus \mathcal{W}_T] = \mathcal{F}_T \ominus \mathcal{F}_T,\xi\), from which it follows that

\[
\mathcal{P}^{T,\xi}(C^T)^{-1} [\mathcal{F}_T \ominus \mathcal{F}_T,\xi] = (W_T)^{-1}P^T(W_T)^{-1}[W_T^*]^{-1} [\mathcal{F}_T \ominus \mathcal{F}_T,\xi] =
\]

\[
(W_T)^{-1}P^T\mathcal{W}_T = \{0\}. \tag{32}
\]

Thus, \(\mathcal{P}^{T,\xi}\) annihilates \((C^T)^{-1} [\mathcal{F}_T \ominus \mathcal{F}_T,\xi]\).

The relations

\[
\mathcal{F}_T = (W_T)^{-1}\mathcal{W}_T = (W_T)^{-1}\mathcal{W}_T + (W_T)^{-1}[\mathcal{W}_T \ominus \mathcal{W}_T] =
\]

\[
\mathcal{F}_T,\xi + (C^T)^{-1}(W_T)^* [\mathcal{W}_T \ominus \mathcal{W}_T] = \mathcal{F}_T,\xi + (C^T)^{-1} [\mathcal{F}_T \ominus \mathcal{F}_T,\xi]
\]

show that \(\mathcal{F}_T\) is a direct sum of the subspaces mentioned in the statement of the Lemma. In the meantime, (31) and (32) certify the characteristic properties of \(\mathcal{P}^{T,\xi}\) as the projector in \(\mathcal{F}_T\) onto \(\mathcal{F}_T,\xi\) in parallel to \((C^T)^{-1} [\mathcal{F}_T \ominus \mathcal{F}_T,\xi]\).

2. By \(W_T^*\mathcal{P}^{T,\xi} = P^T\mathcal{W}_T\) one has \((W_T)^*W_T\mathcal{P}^{T,\xi} = C^T\mathcal{P}^{T,\xi} = (W_T)^*P^T\mathcal{W}_T\). Since \((P^\xi)^* = P^\xi\), we have \((\mathcal{P}^{T,\xi})^*(W_T)^* = (W_T)^*\mathcal{P}^{T,\xi}\). The latter leads to \((\mathcal{P}^{T,\xi})^*C^T = (W_T)^*\mathcal{P}^{T,\xi}\mathcal{W}_T\), and we arrive at \((\mathcal{P}^{T,\xi})^*C^T = C^T\mathcal{P}^{T,\xi}\). \(\square\)

- Here a representation of the dual projectors is provided.

Along with the system \(\Sigma_T\) of the form (13)–(15), we deal with a family of ‘shortened’ systems \(\{\Sigma^\xi\}_{0 < \xi \leq T}\) of the same type but with the final moments \(\xi\) instead of \(T\). By \(\mathcal{F}^\xi, W^\xi, R^\xi, C^\xi, \ldots\), we denote the corresponding objects related to these systems. Recall that each \(C^\xi\) is a positive-definite isomorphism of \(\mathcal{F}^\xi\).

Introduce the map \(e^{T,\xi} : \mathcal{F}^\xi \rightarrow \mathcal{F}_T\),

\[(e^{T,\xi}g)(t) := g(t - (T - \xi)), \quad 0 \leq t \leq T\]

(recall Convention 1) that embeds the shortened system into original one and provides \(e^{T,\xi}\mathcal{F}^\xi = \mathcal{F}_T,\xi\). Its adjoint is \((e^{T,\xi})^* : \mathcal{F}_T \rightarrow \mathcal{F}^\xi\),

\[
((e^{T,\xi})^*f)(t) = f(t + (T - \xi)), \quad 0 \leq t \leq \xi
\]

and relations

\[
(e^{T,\xi})^*e^{T,\xi} = \mathbb{I}_{\mathcal{F}^\xi}, \quad e^{T,\xi}(e^{T,\xi})^* = X^{T,\xi} \tag{33}
\]

are valid, where

\[
X^{T,\xi}(t) := \begin{cases}0, & 0 \leq t < T - \xi; \\f(t), & T - \xi \leq t \leq T
\end{cases}
\]

is the (orthogonal) projector in \(\mathcal{F}_T\) onto \(\mathcal{F}_T,\xi\).

Lemma 2.3. The representations

\[
\mathcal{P}^{T,\xi} = e^{T,\xi}(C^\xi)^{-1}(e^{T,\xi})^*C^T = X^{T,\xi} + K^{T,\xi}, \quad 0 < \xi \leq T \tag{34}
\]

hold, where \(K^{T,\xi} : \mathcal{F}_T \rightarrow \mathcal{F}_T\) is a compact integral operator of the form

\[
(K^{T,\xi}f)(t) = \int_0^{T-\xi} k^{T,\xi}(t,\eta)f(\eta) \, d\eta, \quad 0 \leq t \leq T
\]
with the kernel \( k^{T,\xi} \) supported and smooth in \([T - \xi, T] \times [0, T - \xi] \). The relation
\[
\lim_{\xi \to +0} (\mathcal{P}^{T,\xi} f)(T - \xi + 0) = f(T)
\]
holds for \( f \in \mathcal{M}^T \).

Proof. (Sketch)

1. Let \( Q = e^{T,\xi} (C^\xi)^{-1}(e^{T,\xi})^* C^T \) be the right hand side of the first equality in (34). Representation (21) easily implies \( C^\xi = (e^{T,\xi})^* C^T e^{T,\xi} \) that leads to \( (Q)^2 = Q \).

Using the fact that \( C^T \) and \( C^\xi \) are isomorphisms, it is easy to verify that \( Q \) acts on the elements of \( \mathcal{F}^{T,\xi} \) identically.

Since \((e^{T,\xi})^* [\mathcal{F}^T \cap \mathcal{F}^{T,\xi}] = \{0\} \) one has \( Q([C^T]^{-1} [\mathcal{F}^T \cap \mathcal{F}^{T,\xi}] = \{0\} \). Thus, \( Q \) is a projector, which possesses the same characteristic properties as \( \mathcal{P}^{T,\xi} \). Hence, \( Q = \mathcal{P}^{T,\xi} \) holds.

2. In accordance with (21) one represents
\[
((C^\xi)^{-1} g)(t) = g(t) + \int_0^\xi \xi(t, \eta) g(\eta) d\eta, \quad 0 \leq t \leq \xi \quad (\xi \leq T) \tag{36}
\]
with a kernel \( \xi \), which smoothly depends on the parameter \( \xi \). Then a simple analysis of the structure of \( e^{T,\xi} [C^\xi]^{-1}(e^{T,\xi})^* C^T \) with regard to (21), (36) and (33) follows to the second equality in (34). The upper limit \( T - \xi \) of the integral corresponds to the fact that \( f|_{0 \leq t \leq T - \xi} = 0 \) (i.e., \( f \in \mathcal{F}^{T,\xi} \)) must imply \( \mathcal{P}^{T,\xi} f = X^{T,\xi} f = f \), i.e., \( K^{T,\xi} f = 0 \).

3. It is easy to bring the following considerations to rigorous proof.

By (22), if \( \xi \) is small then \( (C^T f)_{|T - \xi \leq t \leq T} \approx f(T) \). In the meantime, when \( \xi \) tends to zero, the first term in (36) becomes dominant owing to smallness of the interval of integration, which leads to \([e^{T,\xi} (C^\xi)^{-1}(e^{T,\xi})^* C^T f]_{|T - \xi \leq t \leq T} \approx (C^T f)(t) \approx f(T) \). Summarizing, we arrive at (35). \( \square \)

Space \( \Phi^T \). • The following view on dual projectors is also natural and helpful. Recall that \( \mathcal{F}^T = L_2([0, T]; \mathbb{C}) \), and \( C^T \) is a positive-definite isomorphism of \( \mathcal{F}^T \).

Endow \( \mathcal{F}^T \) with the new inner product
\[
(f, g)_{\Phi^T} := (C^T f, g)_{\mathcal{F}^T} \tag{20}
\]
and denote the resulting Hilbert space by \( \Phi^T \). The new metric is equivalent to the original one, so \( \mathcal{F}^T \) and \( \Phi^T \) consist of the same functions. It is convenient to accept the following agreement:

Convention 2. Simplifying the notation, we omit the embedding operator that maps \( f \) as element of \( \mathcal{F}^T \) to \( f \) as element of \( \Phi^T \). By \( \Phi^{T,\xi} \) we denote the image of \( \mathcal{F}^{T,\xi} \) in \( \Phi^T \) via this embedding. For operator \( A \) acting from \( \mathcal{F}^T \) to a Hilbert space \( \mathcal{G} \), its adjoint is denoted by \( A^* \). For \( A \) as an operator from \( \Phi^T \) to \( \mathcal{G} \), its adjoint is denoted by \( A^\dagger \).

The relations
\[
(A f, y)_{\mathcal{G}} = (f, A^* y)_{\mathcal{G}^T} = (C^T f, [C^T]^{-1} A^* y)_{\mathcal{G}^T} = (f, [C^T]^{-1} A^* y)_{\mathcal{G}}
\]
show that
\[
A^\dagger = [C^T]^{-1} A^*
\]
holds.
By (20) we have $(f,g)_{\Phi^T} = (W^T f, W^T g)_{\mathcal{W}^T}$. In the meantime, $W^T \Phi^T, \xi = W^T, \mathcal{F}^T, \xi = \mathcal{W}^T \xi$ holds for all $\xi$. Therefore, the operator $W^T : \Phi^T \to \mathcal{W}^T$ is unitary. It satisfies
\[(W^T)^{-1} = (W^T)^{-1}\]
and maps the subspaces $\Phi^T, \xi$ onto $\mathcal{W}^T$ isometrically.

As is easy to recognize, the result of Lemma 2.3 means that $\mathcal{P}^T, \xi$ is the orthogonal projector in $\Phi^T$ onto $\Phi^T, \xi$. Indeed, we have $(\mathcal{P}^T, \xi)^2 = \mathcal{P}^T, \xi$ and
\[
(\mathcal{P}^T, \xi, f, g)_{\Phi^T} \overset{\text{see } (34)}{=} (C^T e^{T, \xi} [C^T]^{-1} (e^{T, \xi})^* C^T f, g)_{\mathcal{W}^T} = (C^T f, e^{T, \xi} [C^T]^{-1} (e^{T, \xi})^* C^T g)_{\mathcal{W}^T} = (f, \mathcal{P}^T, \xi, g)_{\Phi^T},
\]
so $\mathcal{P}^T, \xi$ is a self-adjoint idempotent in $\Phi^T$. In the meantime, one has $\text{Ran} \, \mathcal{P}^T, \xi = \mathcal{F}^T, \xi = \Phi^T, \xi$ and $\text{Ker} \, \mathcal{P}^T, \xi = (C^T)^{-1}[\mathcal{F}^T \cap \mathcal{F}^T, \xi] = \Phi^T \cap \Phi^T, \xi$. Hence, $\mathcal{P}^T, \xi$ does possess all the characteristic properties of the projector mentioned above.

- Let us provide few more facts about the space $\Phi^T$. Recall that $\mathcal{M}_0^T = \{ f \in \mathcal{M}^T \mid f(T) + (R^T f)(T) = 0 \}$, $\mathcal{T}^T = W^T, \mathcal{M}_0^T$, and denote $\mathcal{U}_0^T := W^T, \mathcal{M}_0^T = \{ u \in W^T \mid u_1(0) + u_2(0) = 0 \}$.

Since $W^T : \Phi^T \to \mathcal{W}^T$ is a unitary operator, the relation $W^T \left[ -i \frac{d}{dt} \right] = L^T W^T$ shows that the operators $-\frac{d}{dt} |_{\mathcal{T}^T}$ and $L^T |_{\mathcal{U}_0^T}$ are unitarily equivalent. This implies
\[
(L^T u^f(\cdot, T), u^g(\cdot, T))_{\mathcal{W}^T} - (u^f(\cdot, T), L^T u^g(\cdot, T))_{\mathcal{W}^T} =
\]
\[
= \left( \left[ -i \frac{d}{dt} \right], g \right)_{\Phi^T} - \left( f, \left[ -i \frac{d}{dt} \right], g \right)_{\Phi^T} =
\]
\[
= \left( C^T \left[ -i \frac{d}{dt} \right], g \right)_{\mathcal{F}^T} - \left( C^T f, \left[ -i \frac{d}{dt} \right] \right)_{\mathcal{F}^T} =
\]
\[
\overset{\text{see } (23)}{=} -i \left[ f(T) g(T) - (R^T f)(T) (R^T g)(T) \right].
\]
Taking $f, g \in \mathcal{M}_0^T$ we see that the operators $L^T u^f |_{\mathcal{U}_0^T}$ and $\left[ -i \frac{d}{dt} \right] |_{\mathcal{U}_0^T}$ are densely defined and symmetric in $\mathcal{W}^T$ and $\Phi^T$ respectively, and unitarily equivalent (via the map $W^T : \Phi^T \to \mathcal{W}^T$).

**Amplitude transform.** Here we deal with the space $\mathcal{G} := L_2(\mathbb{R}_+; \mathbb{C})$ of C-valued functions of variable $\tau \geq 0$ and denote $\Theta^\gamma := [0, \gamma], \mathcal{G}^\gamma := \{ y \in \mathcal{G} \mid \text{supp} \, y \subset \Theta^\gamma \}$. By $Y^\gamma$ we denote the projector in $\mathcal{G}$ onto $\mathcal{G}^\gamma$ that cuts off functions on $\Theta^\gamma$ and extends them by zero to $\mathbb{R}_+ \setminus \Theta^\gamma$.

Introduce the **amplitude transform** $A^T : \mathcal{F}^T \to \mathcal{G}^T$,
\[
(A^T f) (\tau) := (\mathcal{P}^T, \tau f) (T - \tau + 0) \overset{\text{see } (34)}{=} f(T - \tau) + \int_\tau^T w(\tau, \eta) f(T - \eta) \, d\eta, \quad \tau \in \Theta^T
\]
with the kernel $w(\tau, \eta) = k^{T, \tau}(T - \tau, T - \eta)$, which obeys $w(0, \eta) = 0$ in view of (35). Such a transform is well defined and bounded on the dense set of smooth controls $f \in \mathcal{M}^T$ and then extended by continuity up to isomorphism from $\mathcal{F}^T$ onto $\mathcal{G}^T$.

As is easy to verify, its inverse takes the form
\[
((A^T)^{-1} y) (t) = y(T - t) + \int_0^t z(t, \tau) y(T - \tau) \, d\tau, \quad 0 \leq t \leq T
\]
with a kernel $z$ smooth in $0 \leq \tau \leq t \leq T$ obeying $z(T, \tau) = 0$.

Also, the definition implies

$$A^T\mathcal{F}_T^\xi = \mathcal{G}_\xi, \quad 0 \leq \xi \leq T. \quad (41)$$

**Lemma 2.4.** The relation

$$A^T\mathcal{P}_T^\xi = Y^\xi A^T, \quad 0 \leq \xi \leq T \quad (42)$$

holds. Transform $A^T : \Phi^T \to \mathcal{G}^T$ is a unitary operator.

**Proof.** 1. Projectors $\mathcal{P}_T^\tau : \Phi^T$ constitute an increasing family of projectors in $\Phi^T$; hence, one has $\mathcal{P}_T^\tau \mathcal{P}_T^\sigma = \mathcal{P}_T^\min\{\tau,\sigma\}$. For smooth $f$ we have $(\mathcal{P}_T^\tau f)(t)|_{t=T-\xi} = 0$. Therefore,

$$\mathcal{P}_T^\tau = \begin{cases} \mathcal{P}_T^\tau, & 0 < \tau < \xi \\ \mathcal{P}_T^\min\{\tau,\xi\}, \xi < \tau < T \\ 0, \quad \xi < \tau < T \end{cases}$$

and we get (42) on $\mathcal{M}^T$. Then one extends this equality to $\Phi^T$ by continuity.

2. Here we consider $\mathcal{P}_T^\tau : \Phi^T$ as an operator in $\Phi^T$, and $A^T$ as a map from $\Phi^T$ onto $\mathcal{G}^T$. By (42) we have $A^T\mathcal{P}_T^\tau (A^T)^\dagger = Y^\xi A^T (A^T)^\dagger$. Meanwhile, passing to adjoint operators in (42), we obtain $\mathcal{P}_T^\tau (A^T)^\dagger = (A^T)^\dagger Y^\xi$, which yields $A^T\mathcal{P}_T^\tau (A^T)^\dagger = A^T(A^T)^\dagger Y^\xi$. Comparing the results, we see that $Y^\xi$ commutes with $A^T(A^T)^\dagger$.

3. By (39), we easily represent

$$(A^T)^* y(t) = y(T - t) + \int_t^T \frac{w(T - \tau, T - t)}{w(T - \tau, T - t)} y(T - \tau) \, d\tau.$$  

Then, combining it with representations (21), (37) and (39), one easily obtains

$$A^T(A^T)^\dagger = A^T(C^T)^{-1}(A^T)^* = \mathbb{I}_{\mathcal{G}^T} + B^T$$

with a compact self-adjoint integral operator $B^T$.

Since $A^T(A^T)^\dagger$ commutes with $Y^\xi$, one has $B^TY^\xi = Y^\xi B^T$ for all $\xi$. By well-known arguments of the spectral theory of self-adjoint operators, the latter means that $B^T$ is multiplication by a real bounded function. Meanwhile, such an operator can be compact if and only if $B^T = \mathbb{O}_{\mathcal{G}^T}$. So, we arrive at $A^T(A^T)^\dagger = \mathbb{I}_{\mathcal{G}^T}$ and prove the second statement of the Lemma.  

\[ \blacksquare \]

- The meaning and background of the transform $A^T$ may be commented on as follows.

  Return to the Dirac system (13)–(15). By the definition (30), we have $W^T \mathcal{P}_T^\tau = \mathcal{P}_T^u \mathcal{F}_T^\tau$. Therefore, the projection $\mathcal{P}_T^u \mathcal{F}_T^\tau$ is also a wave initiated by the control $\mathcal{P}_T^\tau f$, i.e., $\mathcal{P}_T^u \mathcal{F}_T^\tau = u^{\mathcal{P}_T^\tau \mathcal{F}_T^\tau}$. The latter wave is supported on $\Omega^*$; so $u^{\mathcal{P}_T^\tau \mathcal{F}_T^\tau}(\tau + 0, T) = 0$ holds. In the
meantime, the geometrical optics (amplitude formula) provides

\[ [P^T u^f(\cdot, T)]_1 \big|_{x=\tau-0} = u_1^{P^T, \tau} f(\tau - 0, T) = \]

\[ = u_1^{P^T, \tau} f(\tau - 0, T) - u_1^{P^T, \tau} f(\tau + 0, T) \text{ see (12)} \]

\[ = (P^T, \tau f) (T - \tau + 0) - (P^T, \tau f) (T - \tau - 0) = \]

\[ = (P^T, \tau f) (T - \tau + 0) = (A^T f) (\tau). \]

More generally, by (10) we have

\[ (A^T S_{T-t} f)(\tau) = [P^T u^f(\cdot, t)]_1 \big|_{x=\tau-0}, \quad 0 < \tau \in \Theta, \ 0 \leq t \leq T. \]

Thus, the \( A^T \)-image of controls is formed by amplitudes of jumps that arise when projecting the reachable set \( \mathcal{W}^T \) onto 'shortened' sets \( \mathcal{W}^T \). The transforms of this type provide one of basic tools for solving inverse problems by the boundary control method \([2, 3, 4]\).

One more view is as follows. Since the operators \( W^T : \Phi^T \to \mathcal{W}^T \) and \( A^T : \Phi^T \to \mathcal{G}^T \) are unitary, the map

\[ U^T := A^T (W^T)^{-1} : \mathcal{W}^T \to \mathcal{G}^T \]

is also a unitary operator. The relations

\[ U^T P^\xi = A^T (W^T)^{-1} P^\xi = A^T P^\xi (W^T)^{-1} \text{ see (42)} \]

\[ Y^\xi A^T (W^T)^{-1} = Y^\xi U^T \]

show that \( U^T \) diagonalizes the family of wave projectors \( \{P^\xi\}_{0 \leq \xi \leq T} \) in the sense of the spectral theory of self-adjoint operators.

It is also worth noting that there is a connection of the amplitude transform with the classical problem of triangular factorization of linear operators \([9]\). Introduce the maps \( I^T : \Phi^T \to \Phi^T, (I^T y)(t) := y(T-t) \) and \( \tilde{A}^T := I^T A^T : \Phi^T \to \Phi^T \). As one can see, the second map satisfies \( \tilde{A}^T \Phi^T, \xi \subset \Phi^T, \xi \), i.e., it is triangular with respect to the family of subspaces \( \{\Phi^T, \xi\}_{0 \leq \xi \leq T} \). In the meantime, one has

\[ C^T = (W^T)^* W^T = (A^T)^* [(U^T)^{-1}]^* (U^T)^{-1} A^T = (A^T)^* A^T = (I^T A^T)^* (I^T A^T) = \]

\[ = (\tilde{A}^T)^* \tilde{A}^T. \]

Thus, \( \tilde{A}^T \) provides triangular factorization of the connecting operator with respect to the family \( \{\Phi^T, \xi\}_{0 \leq \xi \leq T} \). By the way, such a factorization is canonical in the sense of \([9]\).

**Operator \( L_a \).** As can be easily seen from the definition (39), transform \( A^T \) preserves smoothness: the set

\[ \gamma^T := A^T \mathcal{M}^T = U^T \mathcal{M}^T \]

consists of smooth functions supported in \( \Theta^T \) and vanishing near \( \tau = T \), whereas \( \mathcal{F}^T = A^T \mathcal{G}^T = \mathcal{G}^T \) holds.

In the space \( \mathcal{G}^T \) define the operator

\[ L_a^T := A^T \left[-i \frac{d}{dt}\right] (A^T)^{-1} = U^T L_a^T (U^T)^{-1}, \quad \text{Dom} \ L_a^T = \gamma^T. \]

As a consequence of the definitions, we conclude that the operators

\[ L_a^T |_{\mathcal{M}^T} \text{ in } \mathcal{M}^T, \quad \left[-i \frac{d}{dt}\right] |_{\mathcal{M}^T} \text{ in } \Phi^T, \quad L_a^T |_{\gamma^T} \text{ in } \mathcal{G}^T \]
are pairwise unitarily equivalent.

The following is the main result of the paper. Denote
\[ \lambda = \lambda(\tau) := r(\tau) + \int_0^\tau z(T - \eta, T - \tau) r(\eta) \, d\eta, \quad \tau \in \Theta^T, \]
where \( r \) is the kernel of \( R^T \), \( z \) is taken from (40).

**Theorem 2.5.** The representation
\[ L_a^T = \left[ i \frac{d}{d\tau} + v + Q \right] \]
holds, where \( v \) is a real smooth function, \( Q \) is a Volterra operator of the form
\[ (Qy)(\tau) = \int_\tau^T q(\tau, s) y(s) \, ds, \quad \tau \in \Theta^T \]
with a kernel \( q \) smooth in \( 0 \leq \tau \leq s \leq T \) and such that
\[ Q - Q^* = i (\cdot, \lambda)_{\Theta^T} \lambda. \]

**Proof.** 1. Taking \( y \in \mathcal{V}^T \) and substituting (39), (40) to \( A^T \left[ -i \frac{d}{dt} \right] (A^T)^{-1} y \), as a result of integration by parts and changing the order of integration, we arrive at the representation
\[ (L_a^T y)(\tau) = \left[ i \frac{dy}{d\tau} \right](\tau) + v(\tau) y(\tau) + \int_\tau^T q(\tau, s) y(s) \, ds, \quad \tau \in \Theta^T \]
with a smooth function \( v \) and kernel \( q \).

2. Let \( f, g \in \mathcal{M}^T \), \( a, b \in \mathcal{V}^T \) and \( A^T f = a \), \( A^T g = b \). By (46), we have
\[ \left( L_a^T a, b \right)_{\Theta^T} - (a, L_a^T b)_{\Theta^T} = \left( \left[ -i \frac{df}{dt} \right], g \right)_{\Theta^T} - \left( f, \left[ -i \frac{dg}{dt} \right] g \right)_{\Theta^T} = \]
\[ = -i \left[ f(T) g(T) - (R^T f)(T) (R^T g)(T) \right], \quad f, g \in \mathcal{M}^T. \]

In the meantime, the relations
\[ f(T) = ((A^T)^{-1} a)(0) \quad (40) \quad a(0) = b(0); \]
\[ (R^T f)(T) = (R^T (A^T)^{-1} a)(0) \quad (40) \quad (a, \lambda)_{\Theta^T}; \quad (R^T g)(T) = (b, \lambda)_{\Theta^T} \]
hold, whereas (50) provides
\[ (L_a^T a, b)_{\Theta^T} - (a, L_a^T b)_{\Theta^T} = -i \left[ a(0) b(0) - (a, \lambda)_{\Theta^T} (b, \lambda)_{\Theta^T} \right] \]
for \( a, b \in \mathcal{V}^T \).

3. Recall that \( \mathcal{M}_0^T = \{ f \in \mathcal{M}^T | f(T) + (R^T f)(T) = 0 \} \) and denote
\[ \mathcal{V}_0^T := A^T \mathcal{M}_0^T = U^T \mathcal{V}_0^T. \]
As it follows from (51), an element \( a \in \mathcal{V}^T \) belongs to \( \mathcal{V}_0^T \) if and only if it obeys \( a(0) + (a, \lambda)_{\Theta^T} = 0 \). As a result of the definitions and (52), we conclude that the operators
\[ L_a^T \big|_{\mathcal{V}_0^T}, \quad \left[ -i \frac{d}{dt} \right] \big|_{\mathcal{M}_0^T}, \quad L_a^T \big|_{\mathcal{V}^T}, \quad L_a^T \big|_{\mathcal{M}^T} \]
are symmetric densely defined and pairwise unitarily equivalent operators.
4. Consider which conditions the symmetry of $L^T_a|_{\mathcal{Y}_0^T}$ imposes on $v$ and $Q$ in (47). Taking $a, b \in \mathcal{Y}_0^T$, we have

\[ 0 = (L^T_a a, b)_{\mathcal{Y}^T} - (a, L^T_a b)_{\mathcal{Y}^T} \text{ int. by parts, (47)} \]

\[ = -i a(0) b(0) + (a, [v - \bar{v}] b + [Q - Q^*] b)_{\mathcal{Y}^T}. \]

Let $a(0) = 0$, so that $(a, \lambda)_{\mathcal{Y}^T} = 0$ holds and implies

\[ (a, [v - \bar{v}] b + [Q - Q^*] b)_{\mathcal{Y}^T} = 0. \]

Since the set of such $a$ is dense in $\mathcal{Y}^T \ominus \text{span} \{\lambda\}$, the latter leads to

\[ [v - \bar{v}] b + [Q - Q^*] b = i \kappa (b, \lambda)_{\mathcal{Y}^T} \lambda \quad \text{in} \ \Omega^T \] (53)

for all $b \in \mathcal{Y}^T_0$ with a real constant $\kappa$. All operators here are bounded, whereas the relation (53) is valid for the dense set of $b \in \mathcal{Y}_0^T$. Therefore it holds for all $b \in \mathcal{Y}^T$. However, operators $Q - Q^*$ and $i \kappa (\cdot, \lambda)_{\mathcal{Y}^T} \lambda$ are compact, whereas the multiplication by $v - \bar{v}$ is compact if and only if $v - \bar{v} \equiv 0$. Hence, we have $3v = 0$ and arrive at $Q - Q^* = i \kappa (\cdot, \lambda)_{\mathcal{Y}^T} \lambda$.

5. It remains to determine $\kappa$. By the aforesaid, for $a, b \in \mathcal{M}^T$ one has

\[ (L^T_a a, b)_{\mathcal{Y}^T} - (a, L^T_a b)_{\mathcal{Y}^T} \text{ int. by parts, (47)} \]

\[ = -i a(0) b(0) + (a, [v - \bar{v}] b + [Q - Q^*] b)_{\mathcal{Y}^T} = -i a(0) b(0) + i \kappa (a, \lambda)_{\mathcal{Y}^T} (b, \lambda)_{\mathcal{Y}^T}. \]

Comparing with (52), we get $\kappa = 1$. So, (47) is proved. □

System $\Sigma_{\kappa}^T$. • Define a function

\[ a(\tau, t) := (U^T u^f(\cdot, t))(\tau) = (A^T S_{T-t} f)(\tau) \text{ see (43)} \]

\[ = [P^T u^f(\cdot, t)]|_{x=\tau-0}, \quad 0 < \tau \leq T, \ 0 \leq t \leq T; \] (54)

and extend it to $\tau = 0$ by

\[ a(0, t) = (A^T S_{T-t} f)(0) \text{ see (39)} = (S_{T-t} f)(T) = f(t), \quad 0 \leq t \leq T. \]

Such a function satisfies

\[ -ia_t = U^T [-iu^f] = U^T L^T u^f = U^T L^T (U^T)^{-1} U^T u^f = L^T_a a. \]

As a result, we conclude that (54) defines a trajectory $a^f$ of a dynamical system $\Sigma_{\kappa}^T$ of the form

\[ \begin{align*}
  ia_t + ia_r + va + Qa &= 0, \quad 0 < \tau < T, \ 0 < t < T; \\
  a|_{t=0} &= 0, \quad \tau \in \Theta^T; \\
  a|_{\tau=0} &= f, \quad 0 \leq t \leq T.
\end{align*} \]

(55) \hspace{1cm} (56) \hspace{1cm} (57)

Thus, system $\Sigma_{\kappa}^T$ is governed by the same boundary control $f$ as the Dirac system $\Sigma^T$. The relation $u^f|_{x>t} \text{ see (9)} = 0$ easily implies $a^f|_{\tau=t} = 0$, i.e., signals in this system travel with unit speed. Its reachable sets are $\mathcal{X}^\kappa = U^T \mathcal{Y}^\kappa$. 


By the definitions accepted above, for two trajectories $a^f$ and $a^g$ we derive
\begin{align*}
&-i [f(t)g(t) - (R^T f(t))(R^T g(t))] = (L^T_u u^f(\cdot, t), w^g(\cdot, t))_{\mathcal{M}^T} - (w^f(\cdot, t), L^T_u u^g(\cdot, t))_{\mathcal{M}^T} = \\
&= (L^T_u a^f(\cdot, t), v^g(\cdot, t))_{\mathcal{M}^T} - (a^f(\cdot, t), L^T_u v^g(\cdot, t))_{\mathcal{M}^T} = \\
&= -i [a^f(0, t) a^g(0, t) - (a^f(\cdot, t), \lambda)_{\mathcal{M}^T} (a^g(\cdot, t), \lambda)_{\mathcal{M}^T}] = \\
&= -i [f(t) g(t) - (a^f(\cdot, t), \lambda)_{\mathcal{M}^T} (a^g(\cdot, t), \lambda)_{\mathcal{M}^T}]
\end{align*}
for $a, b \in \mathcal{V}$. Such a result shows that, firstly, the relevant input/output map of the system $\Sigma_a^T$ is
\begin{align*}
R_a^T : \mathcal{V}^T \to \mathcal{V}, \quad (R_a^T f)(t) := (a^f(\cdot, t), \lambda)_{\mathcal{M}^T}, \quad 0 \leq t \leq T,
\end{align*}
and, secondly, the equality
\begin{align*}
R_a^T = R^T
\end{align*}
holds.

Emphasizing the role of geometrical optics (amplitude formula) in construction of $\Sigma_a^T$, we call this system the amplitude model of the Dirac system $\Sigma^T$. The principal fact is that, by (58), the original system and its model respond on the action of controls identically.

- Strictly speaking, in (47) we should denote $v$ and $Q$ by $v^T$ and $Q^T$. The following explains the reason why the superscript can be omitted.

Consider two systems $\Sigma_a^T$ and $\Sigma_a'^T$ with $0 < T < T'$. Let the controls $f \in \mathcal{V}^T$ and $f' \in \mathcal{V}^{T'}$ satisfy $f = f'|_{0 \leq t \leq T}$. Then, by stationarity of the system (13)-(15), we have $u^f(\cdot, t) = u^{f'}(\cdot, t)$ for $0 \leq t \leq T$. By the same stationarity, in both systems the reachable sets $\mathcal{V}^\xi$ are the same for $0 \leq \xi \leq T$. Hence, the projections $P^\xi u^f(\cdot, t)$ and $P^\xi u^{f'}(\cdot, t)$ coincide for all $0 \leq t, \xi \leq T$. As a result, we obtain
\begin{align*}
 a^f(\tau, t) = (P^T u^f(\cdot, t))(\tau - 0) = (P^T u^{f'}(\cdot, t))(\tau - 0) = a^{f'}(\tau, t)
\end{align*}
for all $0 \leq \tau, t \leq T$. Therefore, determining the potential $v$ and the kernel $q$ of $Q$ from the equation (55) via its solutions $a$, we arrive at $v^T = v^{T'}|_{0 \leq \tau \leq T}$ and $q^T = q^{T'}|_{0 \leq \tau \leq s \leq T}$ that motivates to write $v^T = v$, $q^T = q$ for all $T > 0$.

The same analysis shows that $v$ and $Q$ are connected with the original Dirac potential $V$ locally: the values of $v|_{\Omega T}$ and $q|_{0 \leq \tau \leq s \leq T}$ are determined by $V|_{\Omega T}$. However, this connection is of rather complicated implicit character.

- By the aforesaid, considering arbitrarily large $T$ (i.e., extending $\Theta^T$ to the semi-axis $\tau \geq 0$), we get a smooth real $v|_{\tau \geq 0}$ and smooth $q|_{0 \leq \tau \leq s \leq \infty}$, which provide $v|_{\Omega T} = v^T$ and $q|_{0 \leq \tau \leq s \leq T} = q^T$ for all $T > 0$, and depend on $V$ locally. Function $v$ and kernel $q$ determine the operator $L_a$ that is the closure of the operator
\begin{align*}
 i \frac{d}{d\tau} + v + Q, \quad \text{Dom} \ L_a = \mathcal{V} := \text{span} \{ \mathcal{V}^T \mid T > 0 \},
\end{align*}
and is called the amplitude model of the Dirac operator $L$. Let $U : \overline{\mathcal{M}}^T \to \mathcal{V}$ be a unitary operator defined by the condition $U u^f(\cdot, T) = U^T u^f(\cdot, T) = a^f(\cdot, T)$ for all $f \in \mathcal{M}^T$, $T > 0$. Recall that $L_a$ is the wave part of the Dirac operator $L$. Just resuming the previous results, we have $U L_u = L_a U$. 

System $\Sigma_a$. Let us summarize our considerations.

• The model operator $L_a$, the existence of which is established above, determines dynamical system $\Sigma_a$ of the form

$$
\begin{align*}
ia_t + ia_x + va + Qa = 0, & \quad \tau > 0, t > 0; \\
|a|_{t=0} = 0, & \quad \tau \geq 0; \\
|a|_{\tau=0} = f, & \quad t \geq 0.
\end{align*}
$$

which is referred to as the amplitude model of the Dirac system (2)–(4). Its trajectory $a = a^f(\tau, t)$ is connected with waves $u^f$ by the relation $a^f(\cdot, t) = U u^f(\cdot, t)$, $t \geq 0$. Respectively, its reachable sets are $\mathcal{V}^T = U^T \mathcal{Y}$ and $\mathcal{Y} = \text{span} \{ \mathcal{V}^T | T > 0 \}$, the relation $\mathcal{F} = \mathcal{G}$ holds. By the latter, system $\Sigma_a$ is controllable. It is a unitary copy of the wave subsystem $\Sigma_u$ and, as such, it provides the canonical controllable (reachable) realization of the system $\Sigma$ [10].

• The remarkable fact is that the model $\Sigma_a$ is fully determined by the input/output map of system $\Sigma$. Indeed, the relevant response operator of the system (2)–(4) is $R : \mathcal{F} \to \mathcal{F}$, $(Rf)(t) := u^f(0, t)$, $t \geq 0$. Obviously, it determines the family $\{ R^T \}_{T > 0}$ of ‘truncated’ response operators (19). Therefore, if we know $R$ then we are given all the operators $C^T$, $T > 0$: see (21). Hence, we can construct the projectors $\mathcal{P}^T, \xi$ by (34), determine the transforms $A^T$, $T > 0$ by (39) and then visualize the ‘model waves’ (states) of system (59)–(61) by

$$
a^f(\tau, T) = (A^T f)(\tau), \quad 0 \leq \tau \leq T
$$

for all $T > 0$. Moreover, knowing $a^f$ for rich enough set of $f$, one can find $v$ and $q$, for instance, from equation (59).

A bit of philosophy. • The results obtained perfectly illustrate the general principles of system theory: see [10], Chapter 10.6, Abstract Realization theory. These principles are worth of noting; we cite them with minor changes in the notation and terms.

1. If a system $\Sigma$ is tested via causal experiments and its (causal) input/output map $R$ determined from the equation of motion, then $R$ depends only on a subsystem $\Sigma_0$ of $\Sigma$ which is both completely reachable and completely observable. The other parts of $\Sigma$ have no effect on $R$ and may be chosen completely arbitrarily without altering $R$.

In our situation, $\Sigma_0$ is $\Sigma_u$ and ‘reachable’ is controllable.

2. If any two systems, $\Sigma$ and $\hat{\Sigma}$, are both completely reachable and completely observable and have the same input/output map $R$, then they differ only in the coordinatization of their state space.

In our case, the state space of $\Sigma_u$ is $\mathcal{Y}$, the state space of $\Sigma_a$ is $\mathcal{G}$, and these spaces are connected via the isometry $U : \mathcal{F} \to \mathcal{G}$, which realizes coordinatization of states.

3. If a realization $\Sigma$ of $R$ is not completely reachable or not completely observable, then $\Sigma$ contains certain part which has no relation to the experimental data represented by $R$ but arise in a completely arbitrary way determined only by the particular algorithm used to construct $\Sigma$.

In our case, such a ‘not completely reachable’ part of $\Sigma$ corresponds to the subspace $\mathcal{H} \ominus \mathcal{F}$.
4. If a realization $\Sigma$ of $R$ is completely reachable and completely observable, then it is essentially uniquely determined by $R$, since the coordinatization of states can never be inferred from input/output experiments. In any case the coordinatization of states is irrelevant, since the labeling of internal states in $\Sigma$ has no intrinsic physical significance.

In addition to this indisputable thesis, we note that among all possible realizations, there may exist those that are canonically distinguished by the physical properties of the system. For the Dirac system, such is the amplitude model $\Sigma_a$, adequate to the principle of finiteness of the domain of influence (hyperbolicity of system $\Sigma$) and the nature of the propagation of singularities (geometric optics).

- In the future we plan to introduce another model of system $\Sigma$, the so-called wave model $\Sigma_w$: see [5, 6, 7]. It will also be completely reachable (controllable), and, hence, by thesis 4, systems $\Sigma_a$ and $\Sigma_w$ have to be isometric. Both of them provide different realizations of the physical system $\Sigma_u$.

The Nature laws 1-4 constitute the ‘ideological’ background of the boundary control method [2, 3] that is an approach to inverse problems of mathematical physics, which is based upon their relations to control and system theory.

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