Analysis of new stabilized hp discontinuous Galerkin methods for elasticity problem✩

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Abstract

In the paper, we propose three new hp discontinuous Galerkin methods for the elasticity problem and make a comparison of the three numerical methods. And we prove the optimal order of convergence in energy norm and $L^2$-norm by the superpenalization technique. Finally, we give a numerical example to verify our theoretical results.

Keywords: Discontinuous Galerkin method; elasticity problem; error estimates; superpenalization.

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1. Introduction

Elasticity problem is an important branch of solid mechanics, which describes the changes of stress, strain, and displacement of the elastic medium by external factors. It is also the foundation of material mechanics, plastic mechanics and some interdisciplinary. Since the elasticity problem is very

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complicated, so it is a challenge due to the huge computation (see [7]). Many researchers studied the finite difference methods, finite element methods and some discontinuous Galerkin (DG) methods. From the view of physics, the DG method is very natural to handle with the elasticity problem because DG method is locally conservative, stable, high order accurate and easily adaptive. The first DG method was introduced in [11] by Reed and Hill for the hyperbolic equations, and then many DG methods were designed, one can refer [6]. The DG methods for linear elasticity was firstly studied by [8], afterwards, a local discontinuous Galerkin (LDG) method for linear elasticity problem is developed in [1, 5]. Cai and Ye presented a kind of mixed discontinuous finite element method in [4]. Besides, Rivière, Shaw and Wheeler introduced a standard DG scheme for linear elasticity in [12]. Houston and Schötzau gave an adaptive mixed DG method for nearly incompressible linear elasticity in [9]. However, the above mentioned methods mainly consider the order of error estimates depending on $h$ for the linear elasticity problem.

In the work, we propose absolutely stable $hp$ DG methods for the elasticity problem, which are different from the general DG methods, and we prove the optimal order of convergence in the energy norm and $L^2$-norm by the superpenalization technique.

The remaining parts of this paper are organized as follows. In Section 2, we introduce some notations and the model problem. In Section 3, we derive the new $hp$ DG methods for the elasticity problem and prove the stability of the methods. In Section 4, we prove the optimal order of convergence of our methods for the elasticity problem in the energy norm and $L^2$-norm. Finally, we give a numerical example to illustrate the performance of our theoretical
results.

2. Model problem and notations

In the paper, we consider the following elasticity problem:

\[- \nabla \cdot \sigma = f \quad \text{in } \Omega, \]
\[u = g_D \quad \text{on } \Gamma_D, \]
\[\sigma(u)n = g_N \quad \text{on } \Gamma_N, \]

(2.1)

where \( \Omega \subset \mathbb{R}^n \) (\( n = 2 \) or \( 3 \)) is a convex polygonal domain with \( \partial \Omega = \Gamma_D \cup \Gamma_N \),
and the stress tensor \( \sigma(u) = \lambda \nabla \cdot u I + 2\mu \varepsilon(u) \), \( I \) is the identity tensor,
\( \sigma_{ij} = C_{ijkl}(x)\varepsilon_{kl}(u) \quad \forall i, j, k, l = 1, 2, \ldots, n, \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T) \), and
\( C = (C_{ijkl}(x))_{1 \leq i, j, k, l \leq n} \) is a fourth-order tensor satisfying the symmetric
property: \( C_{ijkl}(x) = C_{jikl}(x), C_{ijkl}(x) = C_{ijlk}(x), C_{ijkl}(x) = C_{klij}(x) \).
\( f \) is the external force, and \( g_D \) and \( g_N \) are the given functions. In the paper,
we will omit the argument \( x \) in \( C \) and take the tensor \( C \) to be piecewise
constant in \( \Omega \).

Let \( \mathcal{T}_h \) be a nondegenerate quasiuniform subdivision of \( \Omega \) with elements
\( K \). And we denote \( h_K = \text{diam}(K), h = \max\{h_K\}_{K \in \mathcal{T}_h}, \Gamma = \bigcup_{K \in \mathcal{T}_h} \partial K \) and
\( \Gamma_h = \Gamma \setminus \partial \Omega \), where \( \partial K \) is the boundary of element \( K \). Also, we let \( e \) be the
dge (face in 3D) of element \( K \), and \( n \) be the unit outward vector normal to
\( \partial \Omega \).

To propose the numerical methods, we need to introduce the following
broken Sobolev spaces:

\[ H^s(\mathcal{T}_h) = \{ v \in L^2(\Omega) : v|_K \in H^s(K), \quad \forall K \in \mathcal{T}_h \}, \]
\[ H^s(\mathcal{T}_h) = \{ v \in (L^2(\Omega))^n : v_i|_K \in H^s(K), \quad 1 \leq i \leq n \}. \]
The norm associated with space $H^s(\mathcal{T}_h)$ is defined by

$$||v||_{s,h} = \left( \sum_{K \in \mathcal{T}_h} ||v||_{s,K}^2 \right)^{1/2},$$

(2.2)

where $|| \cdot ||_{s,K}$ is the usual Sobolev norm on element $K$.

The finite element space $V_h \subset H^s(\mathcal{T}_h)$ is given by

$$V_h = \{ v : v|_K \in (\mathbb{P}_r(K))^n, \; \forall K \in \mathcal{T}_h \},$$

(2.3)

where $\mathbb{P}_r(K)$ is a space of polynomial of degree at most $r$ on $K$ for $r \geq 1$.

Also, we introduce the average, jump operators and some approximation properties. For each interior edge $e = \partial K^+ \cap \partial K^-$ or boundary edge $e = \partial K^+ \cap \partial \Omega$, we define

$$\{v\} := \begin{cases} (v^+ + v^-)/2 & \text{on } \partial K \cap \Gamma_h, \\ v^+ & \text{on } \partial K \cap \partial \Omega, \end{cases}$$

and

$$[v] := \begin{cases} v^+ - v^- & \text{on } \partial K \cap \Gamma_h, \\ v^+ & \text{on } \partial K \cap \partial \Omega, \end{cases}$$

where $v^\pm(x) = \lim_{\epsilon \to 0} v(x \pm \epsilon n)$.

It is well known that for $\phi \in H^s(K)$ there exists $z^h_r \in \mathbb{P}_r(K)$ satisfying the following properties (cf. [3]):

$$||\phi - z^h_r||_{q,K} \leq C \frac{h^{s-q}}{r^{s-q}} ||\phi||_{s,K} \quad s \geq 0,$$

(2.4)

$$||\phi - z^h_r||_{0,e} \leq C \frac{h^{s-1}}{r^{s-1/2}} ||\phi||_{s,K} \quad s > \frac{1}{2},$$

(2.5)

$$||\phi - z^h_r||_{1,e} \leq C \frac{h^{s-3/2}}{r^{s-3/2}} ||\phi||_{s,K} \quad s > \frac{3}{2},$$

(2.6)

where $\mu = \min(r + 1, s)$, $r = 1, 2, \ldots$ and $C$ is a constant depending on $s$ but independent of $\phi$, $h$, $r$. 

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Define the energy norm as follows:

\[ |||v|||^2 = \|v\|^2_{T_h} + \|v\|^2_{\partial T_h}, \]  

(2.7)

where

\[ |||v|||^2_{T_h} = \sum_{K \in T_h} \int_K \sigma(v) : \epsilon(v) dx, \]

\[ |||v|||^2_{\partial T_h} = \frac{\beta r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} ||[v]||^2_{0,e} + \frac{\gamma r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} ||[n \cdot v]||^2_{0,e} \]

and \( \beta \) and \( \gamma \) are the stabilized parameters.

3. Stabilized hp DG methods

Firstly, we give a variational problem of the problem (2.1) as follows:

Find \( w \in H^s(T_h) \) such that

\[ B_h(w, v) = L(v) \quad \forall v \in H^s(T_h), \]  

(3.1)

where

\[ B_h(w, v) = \sum_{K \in T_h} \int_K \sigma(w) : \epsilon(v) dx - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\sigma(w)n\} \cdot [v] d\ell \]

\[ + \alpha \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\sigma(v)n\} \cdot [w] d\ell + \frac{\beta r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [w] \cdot [v] d\ell \]

\[ + \frac{\gamma r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [n \cdot w] [n \cdot v] d\ell, \]  

(3.2)

and

\[ L(v) = \int_\Omega f \cdot v dx + \int_{\Gamma_N} g_N \cdot v d\ell + \alpha \int_{\Gamma_D} \sigma(v)n \cdot g_N d\ell \]

\[ + \frac{\beta r^2}{h} \int_{\Gamma_D} g_D \cdot v d\ell + \frac{\gamma r^2}{h} \int_{\Gamma_D} (n \cdot w)(n \cdot v) d\ell. \]  

(3.3)

As for the variational problem (3.1), we have the following result:
**Theorem 3.1.** Let \( s > \frac{3}{2} \). Suppose that the weak solution \( u \) of problem (2.1) belongs to \( H^s(\mathcal{T}_h) \), then \( u \) satisfies the variational formulation (3.1). The converse is also valid if \( u \) belongs to \( H^1(\Omega) \cap H^s(\mathcal{T}_h) \).

**Proof.** Firstly, we prove that if the solution \( u \) of problem (2.1) belongs to \( H^s(\Omega) \), then it solves (3.1). To do this, multiplying the first equation of the problem (2.1) by \( v \in V_h \) and integrating by parts, we get

\[
\int_K \sigma(u) : \epsilon(v) dx - \int_{\partial K} \sigma(u)n \cdot v dl = \int_K f \cdot v dx. \tag{3.4}
\]

Using (3.4), we have

\[
\sum_{K \in \mathcal{T}_h} \int_K \sigma(u) : \epsilon(v) dx - \sum_{e \in \mathcal{E}_h} \int_e \{ \sigma(u)n \} \cdot [v] dl = \int_{\Omega} f \cdot v dx + \int_{\partial \Omega} \sigma(u)n \cdot v dl. \tag{3.5}
\]

Adding the term \( \alpha \int_{\Gamma_D} \sigma(v)n \cdot u dl \) to both sides of (3.5) and using the boundary conditions, we have

\[
\sum_{K \in \mathcal{T}_h} \int_K \sigma(u) : \epsilon(v) dx - \sum_{e \in \mathcal{E}_h \cup \Gamma_D} \int_e \{ \sigma(u)n \} \cdot [v] dl + \alpha \int_{\Gamma_D} \sigma(v)n \cdot u dl = \int_{\Omega} f \cdot v dx + \int_{\Gamma_N} g_N \cdot v dl + \alpha \int_{\Gamma_D} \sigma(v)n \cdot g_D dl. \tag{3.6}
\]

Using (3.7) and the fact of \([u] = 0\), we see that (3.1) holds.

Conversely, if \( u \in H^1(\Omega) \cap H^s(\mathcal{T}_h) \), then we have

\[
\sum_{K \in \mathcal{T}_h} \int_K \sigma(u) : \epsilon(v) dx = - \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \sigma(u) \cdot v dx + \sum_{e \in \mathcal{E}_h} \int_e \sigma(u)n \cdot [v] dl + \int_{\partial \Omega} \sigma(u)n \cdot v dl. \tag{3.8}
\]

Using (3.8) and (3.1), taking the suitable test functions \( v \), we know that \( u \) satisfies the problem (2.1). The proof is completed. \( \square \)
Next, we propose the hp discontinuous Galerkin methods for the problem (2.1): Find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h,$$

(3.9)

where $B_h(\cdot, \cdot)$ and $L(\cdot)$ are defined by (3.2) and (3.3), respectively.

Remarks 3.1. If the parameter $\alpha$ of (3.2) is chosen to be $\{-1, 0, 1\}$, the methods are called symmetric interior penalty Galerkin (SIPG) method ($\alpha = -1$) and incomplete interior penalty Galerkin (IIPG) method ($\alpha = 0$) and nonsymmetric interior penalty Galerkin (NIPG) method ($\alpha = 1$), respectively. We point out that the above methods are novel because the stabilized parameters are chosen dependently on the mesh size $h$ and the polynomial degree $r$, which are different from the general SIPG, IIPG and NIPG methods.

Remarks 3.2. Denote $\Sigma_n(u_h)$ by

$$\Sigma_n(u_h) = \begin{cases} 
\{\sigma(u_h)n_K\} - r^2 h^{-1}(\beta[u_h] - \gamma [n_K \cdot u_h]n_K) & \text{on } \Gamma_h, \\
\sigma(u_h)n_K - r^2 h^{-1}(\beta(u_h - g_D) - \gamma n_K \cdot (u_h - g_D)n_K) & \text{on } \Gamma_D, \\
g_N & \text{on } \Gamma_N,
\end{cases}$$

then we have

$$\int_{\partial K} \Sigma_n(u_h) + \int_K f = 0$$

(3.10)

for all $K \in \mathcal{T}_h$. That is, these schemes are local equilibrium in a weak sense for each element $K \in \mathcal{T}_h$.

Next, we give the stability of our hp DG methods.
Lemma 3.1. For all \((w, v) \in V_h \times V_h\), then there exists a positive constant C independent of \(h\) and \(r\) such that

\[
B_h(w, v) \leq C_b |||w||| |||v|||.
\] (3.11)

Proof. For all \((w, v) \in V_h \times V_h\), we have

\[
B_h(w, v) = \sum_{K \in T_h} \int_K \sigma(w) : \epsilon(v) dx - \sum_{e \in E_h \cup E_D} \int_e \{\sigma(w)n\} \cdot [v] d\ell
+ \alpha \sum_{e \in E_h \cup E_D} \int_e \{\sigma(v)n\} \cdot [w] d\ell + \frac{\beta r^2}{h} \sum_{e \in E_h \cup E_D} \int_e [w] \cdot [v] d\ell
+ \frac{\gamma r^2}{h} \sum_{e \in E_h \cup E_D} \int_e [n \cdot w] \cdot [n \cdot v] d\ell
= T_1 + T_2 + T_3 + T_4 + T_5.
\] (3.12)

For the term \(|T_1|\), using the Cauchy-Schwarz inequality, we get

\[
|T_1| \leq \sum_{K \in T_h} \left( \int_K C_{ijkl} \epsilon_{kl}(w) \epsilon_{ij}(w) dx \right) \left( \int_K C_{ijkl} \epsilon_{kl}(v) \epsilon_{ij}(v) dx \right)^{\frac{1}{2}}
\leq |||w||| |||v|||.
\] (3.13)

In order to bound the terms \(|T_2|\) and \(|T_3|\), we recall the following inverse estimate (cf. [10, 12, 14])

\[
||C^{1/2} \epsilon(v)||_{0,e} \leq C_0 h^{-\frac{1}{2}} r \|C^{1/2} \epsilon(v)||_{0,K} \quad \forall v \in V_h,
\] (3.14)

where \(C_0\) is a positive constant independent of \(h\) and \(r\).

Assume \(e \subset \partial K_1 \cap \partial K_2\), by Cauchy-Schwarz and triangle inequality, we have

\[
\int_e \{C_{ijkl} \epsilon_{kl} n_j\} [v_i] d\ell \leq \||C_{ijkl} \epsilon_{kl} n_j\||_{0,e} \||[v_i]||_{0,e}
\leq \frac{1}{2} (||C_{ijkl} \epsilon_{kl} n_j ||_{0,e} + ||C_{ijkl} \epsilon_{kl} n_j ||_{0,e} ) \||[v_i]||_{0,e}
\leq C_0 r h^{-1/2} r \left( ||C^{1/2} \epsilon(v)||_{0,K_1 \cup K_2} \right) \||[v]||_{0,e},
\] (3.15)
where $\rho$ is a positive constant with respect to $C$. Furthermore, summing over all internal edges, we have

$$|T_2| \leq C_0\rho \sqrt{\frac{h}{\beta}} \left( \sum_{K \in T_h} \|C^{1/2}\varepsilon(v)\|^2_{0,K} \right)^{\frac{1}{2}} \left( \frac{\beta r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \|v\|^2_{1,e} \right)^{\frac{1}{2}}$$

$$\leq C_2 \|w\| \|v\|, \quad (3.16)$$

where the parameter $n_0$ is denoted by the maximum number of neighboring element.

As for the term $T_3$, taking the above argument, we get

$$|T_3| \leq C_3 \|w\| \|v\|. \quad (3.17)$$

As for the terms $|T_4 + T_5|$, we have

$$|T_4 + T_5| \leq \left| \frac{\beta r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [w] \cdot [v] d\ell \right| + \left| \frac{\gamma r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [n \cdot w] [n \cdot v] d\ell \right|$$

$$\leq \left| \left( \frac{\beta r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [w]^2 d\ell \right)^{\frac{1}{2}} \left( \frac{\beta r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [v]^2 d\ell \right)^{\frac{1}{2}} \right|$$

$$+ \left| \left( \frac{\gamma r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [n \cdot w]^2 d\ell \right)^{\frac{1}{2}} \left( \frac{\gamma r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [n \cdot v]^2 d\ell \right)^{\frac{1}{2}} \right|$$

$$\leq C_4 \|w\| \|v\|. \quad (3.18)$$

Combining with all the bounds together and taking $C = \max\{C_4, C_2, C_3\}$, we see that (3.11) holds. This completes the proof. \qed

**Lemma 3.2.** For all $w \in V_h$, then there exists a positive constant $C_s$ independent of $h$ and $r$ such that

$$B_h(w, w) \geq C_s \|w\|^2. \quad (3.19)$$
Proof. Using (3.20) and letting $\alpha = 1$, we have

$$B_h(w, w) = |||w|||, \forall w \in V_h. \quad (3.20)$$

Using the Cauchy-Schwarz inequality, the inverse inequality and Young’s inequality, we get

$$\sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{C_{ijkl} \epsilon_{kl}(w) n_j\}[w_i]d\ell \leq \left( \frac{C_0^2 n_0}{\beta} \sum_{K \in T_h} |||C^{1/2} \epsilon(w)|||^2_{0,K} \right)^{\frac{1}{2}} \left( \frac{\beta r^2 \rho^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} |||w|||^2_{0,e} \right)^{\frac{1}{2}}$$

$$\leq \frac{C_0^2 n_0 \eta}{2\beta} \sum_{K \in T_h} \int_K \sigma(w) : \epsilon(w) dx + \frac{\beta r^2 \rho^2}{2\eta h} \sum_{e \in \Gamma_h \cup \Gamma_D} |||w|||^2_{0,e}. \quad (3.21)$$

Using (3.21) and (3.2), we obtain

$$B_h(w, w) \geq \left( 1 - \frac{C_0^2 n_0 \eta |1 - \alpha|}{2\beta} \right) \sum_{K \in T_h} \int_K \sigma(w) : \epsilon(w) dx + \left( 1 - \frac{|1 - \alpha| \rho^2}{2\eta} \right) \frac{\beta r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} |||w|||^2_{0,e} + \frac{\gamma r^2}{h} \sum_{e \in \Gamma_h \cup \Gamma_D} |||n \cdot w|||^2_{0,e}. \quad (3.22)$$

Choosing $\eta$ such that $1 - \frac{C_0^2 n_0 \eta |1 - \alpha|}{2\beta} > 0$ and $1 - \frac{|1 - \alpha| \rho^2}{2\eta} > 0$, and taking $C_s = \min \left\{ 1 - \frac{C_0^2 n_0 \eta |1 - \alpha|}{2\beta}, 1 - \frac{|1 - \alpha| \rho^2}{2\eta} \right\}$, using (3.20) and (3.23), we see that (3.19) holds. The proof is completed.

Theorem 3.2. There is a unique solution $u_h$ to the variational problem (3.1).

Proof. Suppose $u_h^1$ and $u_h^2$ are two different solutions of (3.1), then we have

$$B_h(u_h^1 - u_h^2, v) = 0 \quad \forall v \in V_h.$$
Choosing \( v = u_h^1 - u_h^2 \) and using Lemma 3.2, we have

\[ \|\| u_h^1 - u_h^2 \|\| = 0, \]

which implies that \( u_h^1 = u_h^2 \).

Using Lemma 3.1 and Lemma 3.2, we can easily prove the existence of the numerical solution \( u_h \) by Lax-Milgram theorem and Resize theorem for symmetric schemes and nonsymmetric schemes, respectively. And we omit the details of the proof. The proof is completed.

\[ \square \]

4. Error estimates

In this section, we will prove the optimal convergence rate in terms of \( h \) and \( r \) but suboptimal with respect to \( r \) if \( \mathbf{u}_I \) is discontinuous for all the above methods, where \( \mathbf{u}_I \) is the interpolation of \( \mathbf{u} \).

**Lemma 4.1.** Let \( \mathbf{u} \in H^2(\Omega_h) \). If \( \mathbf{u}_I \in C(\overline{\Omega}) \cap V_h \), then we have

\[ |B_h(\mathbf{u} - \mathbf{u}_I, \mathbf{v})| \leq C \frac{h^{\mu-1}}{r^{s-1}} \|\| \mathbf{u} \|\|_s \|\| \mathbf{v} \|\| \quad \forall \mathbf{v} \in V_h; \]  

(4.1)

If \( \mathbf{u}_I \notin C(\overline{\Omega}) \), then we get

\[ |B_h(\mathbf{u} - \mathbf{u}_I, \mathbf{v})| \leq C \frac{h^{\mu-1}}{r^{s-3/2}} \|\| \mathbf{u} \|\|_s \|\| \mathbf{v} \|\| \quad \forall \mathbf{v} \in V_h, \]  

(4.2)

where \( s \geq 2 \), and \( C \) is a positive constant independent of \( h \) and \( r \).

**Proof.** Using (3.2), we have

\[ B_h(\mathbf{u} - \mathbf{u}_I, \mathbf{v}) = \sum_{K \in \Omega_h} \int_K \mathbf{\sigma}(\mathbf{u} - \mathbf{u}_I) : \mathbf{\epsilon}(\mathbf{v}) \, dx - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \mathbf{\sigma}(\mathbf{u} - \mathbf{u}_I) \mathbf{n} \} \cdot [\mathbf{v}] \, d\ell \]
\[ + \sum_{e \in \Gamma_h} \int_{e} \{ \sigma(v)n \} \cdot [u - u_I] d\ell + \frac{\beta r^2}{h} \sum_{e \in \Gamma_h} \int_{e} [u - u_I] \cdot [v] d\ell \]
\[ + \frac{\gamma r^2}{h} \sum_{e \in \Gamma_h} \int_{e} [n \cdot (u - u_I)] \cdot [n \cdot v] d\ell \]
\[ \leq Q_1 + Q_2 + Q_3 + Q_4 + Q_5. \] (4.3)

As for every term of (4.3), using the Cauchy-Schwarz inequality and (2.4)-(2.6), we have

\[ |Q_1| \leq \sum_{K \in T_h} \left( \int_{K} C_{ijkl} \epsilon_{kl}(u - u_I) \epsilon_{ij}(u - u_I) dx \right)^{\frac{1}{2}} \left( \int_{K} C_{ijkl} \epsilon_{kl}(v) \epsilon_{ij}(v) dx \right)^{\frac{1}{2}} \]
\[ \leq C \frac{h^{\frac{\mu-1}{r}}} {r^s} ||u||_s ||v||. \] (4.4)

\[ |Q_2| \leq \frac{h}{\beta r^2} \sum_{e \in \Gamma_h} \left( ||C_{ijkl} \epsilon_{kl}(u - u_I) n_j||_{0,e} \right)^2 \left( \frac{\beta r^2}{h} \sum_{e \in \Gamma_h} \left| |v_i||_{0,e} \right|^2 \right)^{\frac{1}{2}} \]
\[ \leq C \frac{h^{1/2}} {r} \left( \sum_{e \in \Gamma_h} \left| |u - u_I||^2_{3,e} \right) \right)^{1/2} ||v|| \]
\[ \leq C \frac{h^{\mu-1}} {r^{s-1/2}} ||u||_s ||v|| \]
\[ \leq C \frac{h^{\mu-1}} {r^{s-1}} ||u||_s ||v||. \] (4.5)

where we use the inequality \(1 \leq 1/r^{1-\frac{1}{r}}\).

Since the terms \(Q_3, Q_4\) and \(Q_5\) vanish if the interpolation \(u_I \in C(\bar{\Omega}) \cap V_h\) due to the jump \([u - u_I] = 0\) on each edge, so we see that (4.1) holds.

As for the case of \(u_I \notin C(\bar{\Omega})\), using the Cauchy-Schwarz inequality, the
inverse estimate and (2.5), we have

\[
|Q_3| \leq Ch^{-1/2}r \left( \sum_{K \in T_h} ||C^{1/2}e(v)||^2_{0, K} \right)^{1/2} \left( \sum_{e \in \Gamma_h \cup \Gamma_D} ||[u - u_I]||^2_{0, e} \right)^{1/2}
\]

\[
\leq C \frac{h^{\mu - 1}}{r^{s-3/2}} ||u||_s ||v||,
\]

(4.6)

\[
|Q_4 + Q_5| \leq \sum_{e \in \Gamma_h \cup \Gamma_D} \left( \frac{\beta r^2 h}{h} \int_e [u - u_I]^2 d\ell \right)^{1/2} \left( \frac{\beta r^2 h}{h} \int_e [v]^2 d\ell \right)^{1/2}
\]

\[
+ \sum_{e \in \Gamma_h \cup \Gamma_D} \left( \frac{\gamma r^2 h}{h} \int_e \nabla (u - u_I)^2 d\ell \right)^{1/2} \left( \frac{\gamma r^2 h}{h} \int_e \nabla v^2 d\ell \right)^{1/2}
\]

\[
\leq Ch^{-1/2}r \left( \sum_{e \in \Gamma_h \cup \Gamma_D} ||[u - u_I]||^2_{0, e} \right)^{1/2} ||v||
\]

\[
\leq C \frac{h^{\mu - 1}}{r^{s-3/2}} ||u||_s ||v||.
\]

(4.7)

Using (4.4), (4.5), (4.6), (4.7), we see that (4.2) holds. This completes the proof.

Next, we give the main result as follows:

**Theorem 4.1.** Under the assumption of Lemma 4.1, there is a positive constant independent of \(h\) and \(r\) such that

\[
|||u - u_h||| \leq C \frac{h^{\mu - 1}}{r^{s-1}} ||u||_s.
\]

(4.8)

If the interpolation \(u_I \notin C(\bar{\Omega})\), then

\[
|||u - u_h||| \leq C \frac{h^{\mu - 1}}{r^{s-3/2}} ||u||_s
\]

(4.9)

holds for \(s \geq 2\).
Proof. Using (3.19), we have
\[ C_s \| u - u_I \|^2 \leq B_h(u - u_I, u - u_I) \]
\[ \leq C \frac{h^{\mu-1}}{r^\tau} \| u \|_s \| u - u_I \|, \quad (4.10) \]
\[ C_s \| u_h - u_I \|^2 \leq B_h(u_h - u_I, u_h - u_I) \]
\[ = B_h(u - u_I, u_h - u_I) - B_h(u - u_h, u_h - u_I) \]
\[ = B_h(u - u_I, u_h - u_I) \]
\[ \leq C \frac{h^{\mu-1}}{r^\tau} \| u \|_s \| u_h - u_I \|, \quad (4.11) \]
where \( \tau = s - 1 \) if \( u_I \) is continuous, otherwise \( \tau = s - 3/2 \).

Using the triangle inequality and (4.11), we have
\[ \| u - u_h \| \leq \| u - u_I \| + \| u_h - u_I \| \]
\[ \leq C \frac{h^{\mu-1}}{r^\tau} \| u \|_s, \quad (4.12) \]
which completes the proof.

Remarks 4.1. From (4.8), we know that the error estimate is optimal in terms of both \( h \)-convergence and \( r \)-convergence, however, (4.9) shows that the error estimate is optimal in terms of \( h \)-convergence but suboptimal with respect to the polynomial degree \( r \).

Next, we prove the error estimates in \( L^2 \)-norm. As for SIPG method, we easily achieve the optimal order convergence in \( L^2 \)-norm by Aubin-Nitsche technique because the method satisfies the following adjoint consistency condition
\[ B_h(v, u) = \int_\Omega v \cdot f \, dx \quad \forall v \in V_h. \quad (4.13) \]
However, the argument fails for IIPG method and NIPG method which are adjoint inconsistent, so we display the superpenalization term and show that the optimal order convergence in $L^2$-norm, our main idea mainly comes from [2] and [13]. As for IIPG method and NIPG method, we choose the superpenalization terms as follows:

$$
\mathcal{K}(w, v) = \frac{\beta r^2}{h^d} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [w] \cdot [v] d\ell + \frac{\gamma r^2}{h^d} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e [n \cdot w] [n \cdot v] d\ell. \quad (4.14)
$$

Define the new energy norm as

$$
|||v||| = \left(|||v|||_{0}^2 + \mathcal{K}(v, v)\right)^{\frac{1}{2}}. \quad (4.15)
$$

It is easy to check that the boundedness, stability and Theorem (4.1) still hold with respect to the new energy norm (4.15).

Now, we give the following main result:

**Theorem 4.2.** For SIPG method, there exists a positive constant $C$ independent of $h$ such that

$$
||u - u_h||_0 \leq Ch^\mu ||u||_s. \quad (4.16)
$$

For IIPG method and NIPG method, the optimal error estimate also can be achieved if $d \geq 3$ under the assumptions of (4.14) and (4.15).

**Proof.** As for SIPG method, we consider the dual problem:

$$
- \nabla \cdot \sigma(\varphi) = u - u_h \quad \text{in } \Omega, \quad \sigma(\varphi)n = 0 \quad \text{on } \partial \Omega. \quad (4.17)
$$
Taking $v = u - u_h$, we have

$$
||u - u_h||_0^2 = B_h(u - u_h, \varphi)
= B_h(u - u_h, \varphi - \varphi_I) + B_h(u - u_h, \varphi_I)
\leq C_b|||\varphi - \varphi_I||| ||u - u_h||
\leq C ||\varphi||_2 ||u - u_h||,
$$

where $\varphi_I$ is the interpolation of $\varphi_I$.

Due to the elliptic regularity, we have

$$
||\varphi||_2 \leq C||u - u_h||_0.
$$

(4.20)

Using Theorem 4.1 and (4.20), we get

$$
||u - u_h||_0 \leq C||u - u_h|| \leq C h^\mu ||u||_s.
$$

(4.21)

As for IIPG method and NIPG method, we have

$$
||u - u_h||_0^2 = B_h(u - u_h, \varphi)
- \theta \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{1}{e} \{\sigma(\varphi)n\} \cdot [u - u_h]d\ell.
$$

(4.22)
Using the Cauchy-Schwarz inequality and the inverse estimate, we obtain

\[
\sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \sigma(w) n \right\} \cdot [v] d\ell
\leq \sum_{e \in \Gamma_h \cup \Gamma_D} \left( \frac{h^d}{\beta r^2} \int_e \left\| C_{ijkl}(w)n_j \right\|^2 d\ell \right)^{\frac{1}{2}} \left( \frac{\beta r^2}{h^d} \int_e \left\| v_i \right\|^2 d\ell \right)^{\frac{1}{2}}
\leq \left( \frac{h^d}{\beta r^2} \sum_{e \in \Gamma_h \cup \Gamma_D} \left\| C_{ijkl}(w)n_j \right\|_{0,e}^2 \right)^{\frac{1}{2}} \left( \frac{\beta r^2}{h^d} \sum_{e \in \Gamma_h \cup \Gamma_D} \left\| v_i \right\|_{0,e}^2 \right)^{\frac{1}{2}}
\leq Ch^{\frac{d-1}{2}} \left( \sum_{K \in T_h} \left\| C^{1/2} \epsilon(w) \right\|_{0,K}^2 \right)^{\frac{1}{2}} \left( \frac{\beta r^2}{h^d} \sum_{e \in \Gamma_h \cup \Gamma_D} \left\| v_i \right\|_{0,e}^2 \right)^{\frac{1}{2}}
\leq Ch^{\frac{d-1}{2}} \left\| w \right\|_2 \left\| v \right\|.
\]  

(4.23)

Using (4.23) and (4.22), we have

\[
-\theta \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \left\{ \sigma(\varphi) n \right\} \cdot [u - u_h] d\ell
\leq Ch^{\frac{d-1}{2}} \left\| \varphi \right\|_2 \left\| u - u_h \right\|.
\]  

(4.24)

Using (4.20), (4.22), (4.21) and (4.24), we have

\[
\left\| u - u_h \right\|_0 \leq Ch \left\| u - u_h \right\| + Ch^{\frac{d-1}{2}} \left\| u - u_h \right\|
\leq Ch \cdot h^{\mu-1} \left\| u \right\|_s + Ch^{\frac{d-1}{2}} \cdot h^{\mu-1} \left\| u \right\|_s
= Ch^\mu \left\| u \right\|_s + Ch^{\mu + \frac{d-1}{2}} \left\| u \right\|_s,
\]  

(4.25)

which completes the proof.

5. Numerical tests

In this section, we present a 2-D numerical example in \( \Omega = (-1, 1) \times (-1, 1) \) with homogeneous Dirichlet boundary condition and empty Neumann boundary.
Let $\lambda = 0.03$, $\mu = 0.035$ and
\[
f(x, y) = \lambda \left( \frac{\pi^2}{4} \zeta_1, \frac{\pi^2}{4} \zeta_1 \right)^T + 2\mu \left( \frac{\pi^2}{8} \zeta_2 + \frac{\pi^2}{8} \zeta_1, \frac{\pi^2}{8} \zeta_2 + \frac{\pi^2}{8} \zeta_1 \right)^T
\]
with $\zeta_1 = \cos(\frac{\pi}{2} x + \frac{\pi}{2} y)$ and $\zeta_2 = \cos(\frac{\pi}{2} x) \cos(\frac{\pi}{2} y)$.

It is easy to check that the exact solution is
\[
u(x, y) = \left( \cos\left( \frac{\pi}{2} x \right) \cos\left( \frac{\pi}{2} y \right), \cos\left( \frac{\pi}{2} x \right) \cos\left( \frac{\pi}{2} y \right) \right)^T.
\]

In the computation, we set $\beta = 125$. For the adjoint inconsistent methods, we use superpenalization and choose $d = 3$. The numerical results of errors in $L^2$-norm and the energy norm are displayed in Table 1 as follows.

| Method | $k,d$ | $h = 2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ |
|--------|-------|-------------|---------|---------|---------|---------|
| SIPG   | $k = 1$ | $||u - u_h||_0$ | 0.12213 | 0.03113 | 0.00745 | 0.00150 | 0.00038 |
|        |       | $||u - u_h||$  | 0.20320 | 0.10402 | 0.05375 | 0.02985 | 0.01982 |
| IIPG   | $k = 1$ | $||u - u_h||_0$ | 0.12256 | 0.03161 | 0.00796 | 0.00199 | 0.00049 |
|        | $d = 3$ | $||u - u_h||$  | 0.20305 | 0.10333 | 0.05190 | 0.02598 | 0.01299 |
| NIPG   | $k = 1$ | $||u - u_h||_0$ | 0.12275 | 0.03171 | 0.00799 | 0.00200 | 0.00050 |
|        | $d = 3$ | $||u - u_h||$  | 0.20306 | 0.10333 | 0.05190 | 0.02598 | 0.01299 |

The comparisons of $||u - u_h||_0$, $||u - u_h||$ in ln-ln scale for all three methods are displayed in Figure 1 and Figure 2.
Figure 1: $||u - u_h||_0$ in ln-ln scale for the three methods

Figure 2: $|||u - u_h|||_0$ in ln-ln scale for the three methods

From the above figures and Table 1, we find that the optimal convergence rate in the energy norm is got for the three methods, and the optimal con-
vergence rate in $L^2$-norm is achieved for SIPG method, and are obtained for both IIPG method and NIPG method when $d = 3$, which conform with the theoretical results of Theorem 4.1 and Theorem 4.2.

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