ON THE INTEGRAL KERNELS OF DERIVATIVES OF THE
ORNSTEIN-UHLENBECK SEMIGROUP

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Abstract. This paper presents a closed-form expression for the integral kernels associated with the derivatives of the Ornstein-Uhlenbeck semigroup $e^{tL}$. Our approach is to expand the Mehler kernel into Hermite polynomials and applying the powers $L^N$ of the Ornstein-Uhlenbeck operator to it, where we exploit the fact that the Hermite polynomials are eigenfunctions for $L$. As an application we give an alternative proof of the kernel estimates by [11], making all relevant quantities explicit.

1. Introduction

Much effort [2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13] has gone into developing the harmonic analysis of the Ornstein-Uhlenbeck operator

$$L := \frac{1}{2} \Delta - \langle x, \nabla \rangle.$$ (1)

On the space $L^2(\mathbb{R}^d, d\gamma)$, where $\gamma$ is the Gaussian measure

$$d\gamma(x) := \pi^{-d/2} e^{-|x|^2} dx,$$ (2)

this operator can be viewed as the Gaussian counterpart of the Laplace operator $\Delta$. Indeed, one has $L = -\nabla^* \nabla$, where $\nabla$ is the usual gradient and the $\nabla^*$ is its adjoint in $L^2(\mathbb{R}^d, d\gamma)$. It is a classical fact that the semigroup operators $e^{tL}$, $t > 0$, are integral operators of the form

$$e^{tL}u(\cdot) = \int_{\mathbb{R}^d} M_t(\cdot, y)u(y) \, d\gamma(y),$$

where $M_t$ is the so-called Mehler kernel [12]

$$M_t(x, y) = \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right) \frac{e^{t|y|^2}}{(1 - e^{-2t})^{d/2}}$$ (3)

(see [13] for a representation of $M_t$ which makes the symmetry in $x$ and $y$ explicit).

Developing a Hardy space theory for $L$ is the subject of active current research [11, 6]. In this theory the derivatives $(d^k/dt^k)e^{tL} = L^N e^{tL}$ play an important role. The aim of the present paper is to derive closed form expressions for the integral kernels of these derivatives, that is, to determine explicitly the kernels $M_t^N$ such that we have the identity

$$L^N e^{tL}u(\cdot) = \int_{\mathbb{R}^d} M_t^N(\cdot, y)u(y) \, d\gamma(y).$$ (4)

Direct application of the derivatives $d^N/dt^N$ to the Mehler kernel yields expressions which become intractible even for small values of $N$. Our approach will be to
expand the Mehler kernel in terms of the $L^2$-normalised Hermite polynomials and then to apply $L^N$ to it, thus exploiting the fact that the Hermite polynomials are eigenfunctions for $L$.

As an application of our main result, which is proved in section 5 after developing some preliminary material in the sections 2-4, we shall give a direct proof for the kernel bounds of [11] in section 6.

2. Preliminaries

In this preliminary section we collect some standard properties of Hermite polynomials and their relationship with the Ornstein-Uhlenbeck operator. Most of this material is classical and can be found in [12, 14].

2.1. Hermite polynomials. The Hermite polynomials $H_n$, $n \geq 0$, are defined by Rodrigues’s formula

$$H_n(x) := (-1)^n e^{x^2} \partial_x^n e^{-x^2}.$$  

Their $L^2$-normalizations,

$$h_n := \frac{H_n}{\sqrt{2^n n!}},$$

form an orthonormal basis for $L^2(\mathbb{R}, d\gamma)$. We shall use the fact that the generating function for the Hermite polynomials is given by

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2tx-t^2}.$$  

The relationship with the Ornstein-Uhlenbeck operator is encoded in the eigenvalue identity $LH_n = -nH_n$, from which it follows that for all $t \geq 0$ we have $e^{tL}H_n = e^{-tn}H_n$. From this one quickly deduces that the Mehler kernel is given by

$$M_t(x, y) := \sum_{n=0}^{\infty} e^{-tn} h_n(x) h_n(y).$$

We will need two further identities for the Hermite polynomials which can be found, e.g., in [1] Chapter 18: the integral representation

$$H_n(x) = \frac{(-2i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \xi^n e^{2ix\xi} e^{-\xi^2} d\xi$$

and the “binomial” identity

$$H_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} (2y)^{n-k} H_k(x).$$

2.2. Hermite polynomials in several variables. The Hermite polynomials on $\mathbb{R}^d$ are defined, for multiindices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ by the tensor extensions

$$H_\alpha(x) := \prod_{n=1}^{d} h_{\alpha_n}(x_n),$$

for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. The normalized Hermite polynomials

$$h_\alpha := \frac{H_\alpha}{\sqrt{2^{\|\alpha\|} \alpha!}},$$

where $\alpha! = \alpha_1! \cdots \alpha_d!$.
form an orthonormal basis in $L^2(\mathbb{R}^d, d\gamma)$, and we have the eigenvalue identity

\begin{equation}
LH_\alpha = -|\alpha|H_\alpha, \quad \text{where } |\alpha| = \alpha_1 + \cdots + \alpha_d.
\end{equation}

If we consider the action of $L^N e^{tL}$ on a Hermite polynomial $h_\alpha$, through the multinomial theorem applied to $|\alpha|^k$ we get (writing $L_d$ for the operator $L$ in dimension $d$ and $L_1$ for the operator $L$ in dimension 1)

\begin{equation}
L^N_d e^{tL} h_\alpha(x) = |\alpha|^N e^{-t|\alpha|} h_{\alpha_1}(x_1) \cdots h_{\alpha_d}(x_d)
\end{equation}

This implies that we can reduce the question of computing the $d$-dimensional version of the integral kernel to the one-dimension one.

3. A combinatorial lemma

From now on we concentrate on the Ornstein-Uhlenbeck operator $L$ in one dimension, i.e., in $L^2(\mathbb{R}, d\gamma)$. We are going to follow the approach of [12]. Recalling the identity $Lh_n = -nh_n$, we will apply $L^N$ to the generating function of the Hermite polynomials (6). A problem which immediately occurs is that $\Delta$ and $\langle x, \nabla \rangle$ do not commute, and because of this we cannot use a standard binomial formula to evaluate $L^N$. Instead, we note that

\begin{equation}
L^N g = (-1)^k D_N g,
\end{equation}

where

\begin{equation}
D_N := t\partial_t \circ t\partial_t \circ \cdots \circ t\partial_t = (t\partial_t)^k.
\end{equation}

The following lemma will be very useful.

**Lemma 1.** We have

\begin{equation}
D_N = \sum_{n=0}^{N} \binom{N}{n} t^n \partial_t^n,
\end{equation}

where $\binom{N}{n}$ are the Stirling numbers of the second kind.

The Stirling numbers of the second kind are quite well-known combinatorial objects. For the sake of completeness we will state their definition and recall some relevant properties below. For more information we refer the reader to [15]. The related Stirling numbers of the first kind will not be needed here.

We begin by recalling the definition of falling factorial

\begin{equation}
(j)_n := j(j - 1) \cdots (j - n + 1) = \frac{j!}{(j - n)!},
\end{equation}

for non-negative integers $k \geq n$. 
Definition 1. For non-negative integers \( N \geq n \), the number Stirling number of the second kind \( \{N\}_{n} \) is defined as the number of partitions of an \( N \)-set into \( n \) non-empty subsets.

These numbers satisfy the recursion identity

\[
\{N\}_{n} = N\{N-1\}_{n} + \{N-1\}_{n-1}.
\]

For all non-negative integers \( j \) and \( k \) one has the generating function identity

\[
 j^N = \sum_{n=0}^{N} \{N\}_{n} (j)_n.
\]

4. Weyl Polynomials

Before turning to the proof of lemma 1, let us already mention that it only depends on the commutator identity \([t, \partial_t] = -1\). This brings us to the observation that Weyl polynomials provide the natural habitat for our expressions. Roughly speaking, a Weyl polynomial is a polynomial in two non-commuting variables \( x \) and \( y \) which satisfy the commutator identity \([x, y] = -1\). This is made more precise in the following definition.

Definition 2. The Weyl algebra over a field \( F \) of characteristic zero is the ring \( F\langle x, y\rangle \) of all polynomials of the form \( p(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} c_{mn}x^m y^n \) with coefficients \( c_{mn} \in F \) in two noncommuting variables \( x \) and \( y \) which satisfy the commutator identity

\([x, y] := xy - yx = -1\).

We now have the following abstract version of lemma 1:

Lemma 2. In the Weyl algebra \( F\langle x, y\rangle \) we have the identity

\[
(xy)^m = \sum_{i=1}^{m} \left\{ \frac{m}{i} \right\} x^i y^i,
\]

where \( \left\{ \frac{m}{i} \right\} \) are the Stirling numbers of the second kind.

As a preparation for the proof of lemma 2, we make a couple of easy computations. If we set \( D := xy \), then

\[
 Dx^m = x^m D + mx^m, \\
 Dy^m = y^m D - my^m.
\]

This can be shown by induction on \( m \). For instance, note that

\[
 Dx^m = x(D + 1)x^{m-1} = xDx^{m-1} + x^m.
\]

If we take this a bit further and have \( p \in F[D] \), then

\[
 p(D)x^m = x^m p(D + m), \\
 p(D)y^m = y^m p(D - m).
\]
The $m$-th powers, $m \geq 1$, of $x$ and $y$ satisfy

\begin{equation}
    x^m y^m = \prod_{i=0}^{m-1} (D - i),
\end{equation}

\begin{equation}
    y^m x^m = \prod_{i=1}^{m} (D + i).
\end{equation}

This can be seen using induction:

\[ x^{m+1} y^{m+1} = x^m D y^m x y^m (D - m) \]
\[ y^{m+1} x^{m+1} = y^m (D + 1) x^m = y^m D x^m + y^m x^m (D + (m + 1)) \]

**Definition 3.** The weighted degree of a monomial $x^n y^m \in F(x, y)$ is the integer $m - n$. A polynomial in $F(x, y)$ is said to be homogeneous of weighted degree $j$ if all its constituting monomials have weighted degree $j$.

Left multiplication by $xy$ is homogeneity preserving, i.e., for all $j \in \mathbb{Z}$ it maps the set of homogeneous monomials of weighted degree $j$ into itself. To prove this, first consider a monomial $x^m y^n$ of degree $j = m - n$. Then,

\[ (xy)x^m y^n = (x^m y^n + mx^m y^n) = x^m (xy) y^n + mx^m y^n = x^{m+1} y^{n+1} + mx^m y^n, \]

and we see that weighted degree of homogeneity is indeed preserved. The general case follows immediately. Through (24) we conclude that left multiplication $x^i y^j$ is homogeneity preserving as well, for all non-negative integers $k$. We claim that left multiplication by $x^i y^j$ is homogeneity preserving only if $i = j$. To see this note that

\[ yx^i y^j = x^i y^{i+j} + ix^{i-1} y^j \]

from which we can deduce that

\[ x^m y^M x^n y^N = x^{n+m} y^{N+M} + \text{lower order terms}. \]

From which the claim follows.

Finally, a polynomial is homogeneity preserving if and only if all of its constituting monomials have this property. If this were not to be the case we could look at the highest-order non-homogeneous term and note from above $x^m y^N x^n y^N$ would give terms of a lower order in the polynomial expansion which cannot cancel as they have different powers of $x$ or $y$.

It follows from these observations that

\begin{equation}
    F_0 := \left\{ \sum_{n=0}^{N} c_n x^n y^n \mid N \in \mathbb{N}, c_1, \ldots, c_N \in F \right\}
\end{equation}

is precisely the set of homogeneity preserving polynomials in $F(x, y)$.

Now everything is in place to give the proof of lemma 2.

**Proof of lemma**. As $(xy)^k$ is homogeneity preserving, we infer that there are coefficients $a^k_i$ in $F$ such that

\begin{equation}
    (xy)^k = \sum_{i=0}^{k} a^k_i x^i y^j.
\end{equation}
We will apply $x^j$ to the right on both sides of (27) and derive an expression for the $a_i^k$. First note that (22) gives
\[(xy)^k x^j = x^j (xy + j)^k ,\]
and (24) together with (22) gives
\[x^j y^i x^j = \prod_{\ell=0}^{i-1} (xy - \ell)x^j = x^j \prod_{\ell=0}^{i-1}(xy - \ell + j).\]
Hence, to find the coefficients $a_i^k$ it is sufficient to consider
\[(xy + j)^k = \sum_{i=0}^{k} a_k^i \prod_{\ell=0}^{i-1}(xy - \ell + j).\]
Comparing the constant terms on both the left-hand side and right-hand side, we find
\[j^k = \sum_{i=0}^{k} a_k^i \prod_{\ell=0}^{i-1}(j - \ell) = \sum_{i=0}^{k} a_k^i (j)_i,\]
where $(j)_i$ is the falling factorial as in (19). Comparing (28) with the generating function of the Stirling numbers of the second kind \(\{ k \}_i\) as given in (18), we see that $a_i^k = \{ k \}_i$. This concludes the proof of lemma 2. \(\square\)

5. The integral kernel of $L^N e^{tL}$

As mentioned before, as a first step we would like to apply $D_N$ to the generating function $g(x, t) := e^{-2tx + t^2} = e^{-(x-t)^2 + x^2}$ for the Hermite polynomials \(\{ H_N \}\). We first compute the action of $\partial_t^N$ on the generating function.

**Lemma 3.** We have
\[(29) \quad \partial_t^N e^{-(x-t)^2 + x^2} = e^{-(x-t)^2 + x^2} H_N(x - t).\]

**Proof.** We first note that,
\[\partial_t e^{-(x-t)^2} = 2(x - t)e^{-(x-t)^2} = -\partial_x e^{-(x-t)^2}.\]
Using this we get
\[\partial_t^N e^{-(x-t)^2 + x^2} = e^{x^2} \partial_t^N e^{-(x-t)^2} = e^{x^2} \partial_t \partial^N e^{-(x-t)^2} = e^{x^2} \partial_t \partial^N e^{-(x-t)^2} = \ldots \]
\[= (-1)^N e^{x^2} \partial_x^N e^{-(x-t)^2} \]
By a change of variables,
\[\partial_t^N e^{-(x-t)^2 + x^2} = (-1)^N e^{-(x-t)^2 + x^2} e^{(x-t)^2} \partial_y^N e^{-y^2} \bigg|_{y = x - t} .\]
Hence, by (5),
\[\partial_t^N e^{-(x-t)^2 + x^2} = e^{-(x-t)^2 + x^2} H_N(x - t) . \]
\(\square\)
Lemma 4. For all \( x \in \mathbb{R} \) and \( t > 0 \) we have

\[
L^N e^{-(x-t)^2+x^2} = (-1)^N e^{-(x-t)^2+x^2} \sum_{n=0}^{N} \binom{N}{n} t^n H_n(x-t).
\]

Proof. This is now easy to prove. Recalling that \( L = -t\partial_t \) and using (30), we get

\[
L^N e^{-(x-t)^2+x^2} = (-1)^N \sum_{n=0}^{N} \binom{N}{n} t^n \partial_t^n e^{-(x-t)^2+x^2}
\]

\[
= (-1)^N \sum_{n=0}^{N} \binom{N}{n} t^n H_n(x-t).
\]

Our next theorem is the main result of this paper and provides an explicit expression for the integral kernel of \( L^N e^{tL} \).

Theorem 1. Let \( L \) be the Ornstein-Uhlenbeck operator on \( L^2(\mathbb{R}^d, d\gamma) \), let \( t > 0 \), and let \( N \geq 0 \) be an integer. The integral kernel \( M^{N}_t \) of \( L^N e^{tL} \) is given by

\[
M^{N}_t(x,y) = M_t(x,y) \sum_{|n|=N} \binom{N}{n_1, \ldots, n_d} \prod_{i=1}^d \sum_{m_i=0}^{n_i} 2^{-m_i} \binom{n_i}{m_i} \binom{m_i}{\ell_i} \left( -\frac{e^{-t} - e^{-2t}}{\sqrt{1 - e^{-2t}}} \right)^{2m_i - \ell_i} H_{\ell_i}(x_i) H_{2m_i - \ell_i}(y_i) e^{-t} - y_i \sqrt{1 - e^{-2t}}.
\]

Proof. We first prove the result for \( d = 1 \). Pulling \( L^N \) through the integral expression (14) for \( e^{-tL} \) involving the Mehler kernel, we must find a suitable expression for the kernel \( M^{N}_t(-x, y) = L^N M_t(-x, y) \). Using (7) and the normalization of \( H_m \) in (11), we get

\[
M^{N}_t(x, y) = L^N \sum_{m=0}^{\infty} \frac{e^{-tm}}{m!} \frac{1}{2m} H_m(x) H_m(y)
\]

\[
\leq L^N \frac{e^{y^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{2iy\xi} H_m(x) (-2i)^m e^{\gamma^2} \int_{-\infty}^{\infty} e^{-\xi^2} e^{2iy\xi} \xi e^{2iy\xi} d\xi
\]

\[
= L^N \frac{e^{y^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{2iy\xi} \xi e^{2iy\xi} (-x+i(\xi e^{-t}))^2 + x^2 d\xi.
\]

The operator \( L^N \) is applied with respect to \( x \) here, so by lemma 4 we get

\[
M^{N}_t(x, y) = \frac{e^{y^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{2iy\xi} \xi L^N e^{-(x+i(\xi e^{-t}))^2 + x^2} d\xi
\]

\[
= (-1)^N \sum_{m=0}^{\infty} \binom{N}{m} \int_{-\infty}^{\infty} e^{2iy\xi} L^N e^{-(x+i(\xi e^{-t}))^2 + x^2} d\xi
\]

\[
= (-1)^N \sum_{m=0}^{N} \binom{N}{m} \int_{-\infty}^{\infty} e^{2iy\xi} e^{-(x+i(\xi e^{-t}))^2 + x^2} d\xi
\]

\[
= (-1)^N \frac{e^{y^2} + \sum_{m=0}^{N} \binom{N}{m} \int_{-\infty}^{\infty} e^{2iy\xi} e^{-(x+i(\xi e^{-t}))^2 + x^2} d\xi}
\]
where in last line we have used the analytic continuation of the algebraic identity (30). Similarly we can expand \( H_m(y + i\xi e^{-t}) \) using (9). This gives

\[
H_m(y + i\xi e^{-t}) = \sum_{\ell=0}^{\infty} \left( \frac{m}{\ell} \right) H_\ell(y)(2i\xi e^{-t})^{m-\ell},
\]

so that \( M_t^{N} \) can be written as

\[
(-1)^N \frac{e^{x^2+y^2}}{\sqrt{\pi}} \sum_{m=0}^{N} \sum_{\ell=0}^{m} \left\{ \begin{array}{c} N \\ m \end{array} \right\} \left( \frac{m}{\ell} \right) H_\ell(y) 2^{m-\ell} \int_{-\infty}^{\infty} e^{2i\xi \zeta - \xi^2} e^{-(x+ix\xi^{-1})^2} (i\xi e^{-t})^{2m-\ell} \, d\xi.
\]

If we set \( M = 2m - \ell \), this reduces our task to computing the integral

\[
\frac{e^{x^2+y^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{2i\xi \zeta - \xi^2} e^{-(x+ix\xi^{-1})^2} (i\xi e^{-t})^{M} \, d\xi
\]

(33)

To make the computation less convolved, let us set

\[
\alpha_t := \sqrt{1 - e^{-2t}}, \text{ and, } \beta_t(x, y) := \frac{xe^{-t} - y}{\sqrt{1 - e^{-2t}}}.
\]

This allows us to write the exponential in the integral (33) as

\[
e^{2i(xe^{-t} - y)e^{-1-e^{-2t}}t^2} = e^{2i\alpha_t \beta_t(x, y)\xi e^{-\alpha_t^2 t^2}}.
\]

This reduces the problem, after the substitution \( \alpha_t \xi \to \xi \), to computing the integral

\[
\frac{e^{y^2}}{\sqrt{\pi}} \frac{i^M e^{-Mt}}{\alpha_t^{M+1}} \int_{-\infty}^{\infty} e^{2i\beta_t(x, y)\xi} e^{-\xi^2} \xi^M \, d\xi.
\]

The final integral is an integral expression for the Hermite polynomials (8), so

\[
\frac{e^{y^2}}{\sqrt{\pi}} \frac{i^M e^{-Mt}}{\alpha_t^{M+1}} \int_{-\infty}^{\infty} e^{2i\beta_t(x, y)\xi} e^{-\xi^2} \xi^M \, d\xi
\]

(8) \( e^{y^2-\beta_t(x, y)^2} \frac{1}{\alpha_t^{M+1}} \frac{(-1)^{M} e^{-Mt}}{2^M} H_M(\beta_t(x, y)).
\]

Next we note that \( \exp(y^2 - \beta_t(x, y)^2)\alpha_t^{-1} = M_t \), the Mehler kernel from (1). Hence,

\[
M_t^{N}(x, y) = M_t(x, y) \sum_{m=0}^{N} \sum_{\ell=0}^{m} \left( \frac{m}{\ell} \right) \left\{ \begin{array}{c} N \\ m \end{array} \right\} \left( -\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \right)^{2m-\ell} 2^{-m} \times H_\ell(x) H_{2m-\ell} \left( \frac{xe^{-t} - y}{\sqrt{1 - e^{-2t}}} \right).
\]

Applying (13) we get the result in \( d \) dimensions. □
6. An application

As an application of our main result, in this section we give an alternative proof of the bounds on the kernels $K$ and $\tilde{K}$ of [11] (see the definition below), making the dependence on the parameters more explicit. These kernels play a central role in the study of the Hardy space $H^1(\mathbb{R}^d, d\gamma)$ in [11], where the standard Calderón reproducing formula is replaced by

$$u = C \int_0^\infty (t^2 L)^{N+1} e^{\frac{1}{2} t^2 L} u \frac{dt}{t} + \int_{\mathbb{R}^d} u \, d\gamma,$$

where $C$ is a suitable constant only depending on $N$ and $\alpha$ (this can be seen by letting $u$ be a Hermite polynomial). The kernels $K$ and $\tilde{K}$ then occur in several decompositions, and the estimates below allow them to be related to classical results about the Mehler kernel.

**Definition 4.** We define the kernels $K$ and $\tilde{K}$ by

$$\int_{\mathbb{R}^d} K_{t^2, N, \alpha}(x, y) u(y) \, d\gamma(y) = (t^2 L)^N e^{\frac{1}{2} t^2 L} u(x),$$

$$\int_{\mathbb{R}^d} \tilde{K}_{t^2, N, \alpha, j}(x, y) u(y) \, d\gamma(y) = (t^2 L)^N e^{\frac{1}{2} t^2 L} t \partial_x^j u(x).$$

Note that the operators on the right-hand sides are indeed given by integral kernels: the first is a scaled version of the operator we have already been studying, and a duality argument implies that the second is given by the integral kernel $\tilde{K}_{t^2, N, \alpha, j}(x, y) = t \partial_x^j K_{t^2, N, \alpha}(x, y)$.

Thus, both kernels are given as appropriate derivatives of the Mehler kernel.

We begin with a technical lemma which is a rephrased version of [11, Lemma 3.4]. One should take note that we define the kernels with respect to the Gaussian measure whereas, [11] defines these with respect to the Lebesgue measure.

**Lemma 5.** For all $\alpha > 1$ and all $t > 0$ and $x, y$ in $\mathbb{R}^d$ we have

$$\frac{|e^{-\frac{1}{2} t} x - y|^2}{1 - e^{-2 t}} \geq \frac{\alpha}{2} e^{-2 t} \frac{|e^{-t} x - y|^2}{1 - e^{-2 t}} - \frac{t^2 \min(|x|^2, |y|^2)}{1 - e^{-2 t}}.\quad (36)$$

Additionally, we have

$$\alpha e^{-2 t} \leq \frac{1 - e^{-2 t}}{1 - e^{-2 t}} \leq \alpha.\quad (37)$$

**Theorem 2.** Let $N$ be a positive integer, $0 < t < T$. The for all large enough $\alpha > 1$ we have

1. If $t|x| \leq C$, then

$$|K_{t^2, N}(x, y)| \lesssim_{T, N} \alpha \exp\left(\frac{\alpha}{2} C^2\right) M_{t^2}(x, y) \exp\left(-\frac{\alpha}{8e^{2T}} \frac{|e^{-t} x - y|^2}{1 - e^{-2t}}\right).$$

2. If $t|x| \leq C$, then

$$|\tilde{K}_{t^2, N, \alpha, j}(x, y)| \lesssim_{T, N} \alpha \exp\left(\frac{\alpha}{2} C^2\right) M_{t^2}(x, y) \exp\left(-\frac{\alpha}{8e^{2T}} \frac{|e^{-t} x - y|^2}{1 - e^{-2t}}\right).$$
Proof. For $K_{t^2, \alpha, N}$, we use Theorem 1 to obtain, after taking absolute values,

$$|K_{t^2, \alpha, N}(x, y)| \leq M_{t^2}(x, y) \sum_{|k| = N} \left( N \right)_{m_1, \ldots, m_d} \prod_{i=1}^{d} e^{2k_i} \sum_{\ell_i = 0}^{m_i} \sum_{m_i = 0}^{n_i} 2^{-m_i} \left( \ell_i \right)_{m_i} \left( n_i \right)_{m_i} \times \left( \frac{e^{-t^2 x_i^2}}{1 - e^{-2t^2 \alpha^2}} \right)^{2m_i - \ell_i} |H_{\ell_i}(x_i)| \left| H_{2m_i - \ell_i} \left( \frac{x_i e^{-t^2 y_i^2}}{1 - e^{-2t^2 \alpha^2}} \right) \right|.$$ 

Recalling that $\ell_1 + \cdots + \ell_d \leq N$, using the assumptions $t \leq T$ and $t|x| \leq C$ we can bound $t^{2k_i}|H_{\ell_i}(x)|$ by considering the highest order term to obtain

$$t^{2k_i}|H_{\ell_i}(x)| \lesssim_{C, T} 1.$$ 

Using (37) we proceed by looking at

$$M_{t^2}(x, y) = M_{t^2}(x, y) \left( \frac{1 - e^{-2t^2 \alpha^2}}{1 - e^{-2t^2}} \right) \left( \frac{1 - e^{-t^2 x^2}}{1 - e^{-2t^2}} \right)^{1/2} \exp \left( \frac{|e^{-t^2 x^2} - y_i^2|^2}{1 - e^{-2t^2 \alpha^2}} \right) \exp \left( \frac{|e^{-t^2 x^2} - y_i^2|^2}{1 - e^{-2t^2 \alpha^2}} \right) \leq \alpha M_{t^2}(x, y) \exp \left( \frac{-1}{2} \frac{|e^{-t^2 x^2} - y_i^2|^2}{1 - e^{-2t^2 \alpha^2}} \right)^2.$$ 

We can now bound the final Hermite polynomial in the expression of the kernel. Setting $M_i = 2m_i - \ell_i$ we get

$$\left( \frac{e^{-t^2 x_i^2}}{1 - e^{-2t^2 \alpha^2}} \right)^{M_i} \left| H_{M_i} \left( \frac{e^{-t^2 x_i^2} - y_i}{1 - e^{-2t^2 \alpha^2}} \right) \right| \lesssim_N \left( \frac{e^{-t^2 x_i^2}}{1 - e^{-2t^2 \alpha^2}} \right)^{M_i} \left( \frac{e^{-t^2 x_i^2} - y_i}{1 - e^{-2t^2 \alpha^2}} \right)^{M_i} \leq \left( \frac{e^{-t^2 x_i^2} - y_i}{1 - e^{-2t^2 \alpha^2}} \right)^{M_i}.$$ 

Also,

$$\left( \frac{|e^{-t^2 x_i^2} - y_i|^2}{1 - e^{-2t^2 \alpha^2}} \right)^{M_i} \exp \left( -\frac{1}{2} \frac{|e^{-t^2 x_i^2} - y_i|^2}{1 - e^{-2t^2 \alpha^2}} \right) \lesssim 1.$$ 

Putting these estimates together, using Lemma 5 and taking $\alpha > 1$ so large that

$$1 - \frac{\alpha}{4e^{2T}} \leq -\frac{\alpha}{8e^{2T}}$$

$$1 - e^{-2t^2 \alpha^2} \geq \frac{t^2}{\alpha},$$

we obtain

$$|K_{t^2, \alpha, N}(x, y)| \lesssim_{T, N} \alpha M_{t^2}(x, y) \exp \left( \frac{|e^{-t^2 x^2} - y_i|^2}{1 - e^{-2t^2 \alpha^2}} \right) \exp \left( -\frac{1}{2} \frac{|e^{-t^2 x^2} - y_i|^2}{1 - e^{-2t^2 \alpha^2}} \right) \exp \left( \frac{1}{2} \frac{t^4 |x|^2}{1 - e^{-2t^2 \alpha^2}} \right) \leq \alpha M_{t^2}(x, y) \exp \left( \left( 1 - \frac{\alpha}{4e^{2T}} \right) \frac{|e^{-t^2 x^2} - y_i|^2}{1 - e^{-2t^2 \alpha^2}} \right) \exp \left( \frac{1}{2} \frac{t^4 |x|^2}{1 - e^{-2t^2 \alpha^2}} \right) \leq \alpha M_{t^2}(x, y) \exp \left( -\frac{\alpha}{8e^{2T}} \frac{|e^{-t^2 x^2} - y_i|^2}{1 - e^{-2t^2 \alpha^2}} \right) \exp \left( \frac{\alpha}{2} t^2 |x|^2 \right) \leq \alpha \exp \left( \frac{\alpha}{2} C \right) M_{t^2}(x, y) \exp \left( -\frac{\alpha}{8e^{2T}} \frac{|e^{-t^2 x^2} - y_i|^2}{1 - e^{-2t^2 \alpha^2}} \right).$$
using \( t|\varepsilon| \leq C \) in the last step.

For the bound on \( \tilde{K} \) we consider

\[
t \partial_x \left[ H_{\varepsilon}(\varepsilon^t \frac{\varepsilon^{-t} - \varepsilon}{\sqrt{1 - e^{-2t}}} \right] = t H_{\varepsilon}(\varepsilon^t \frac{\varepsilon^{-t} - \varepsilon}{\sqrt{1 - e^{-2t}}} \right) \partial_x H_{\varepsilon}(\varepsilon^t \frac{\varepsilon^{-t} - \varepsilon}{\sqrt{1 - e^{-2t}}} \right)
\]

So, as the first term on the right-hand side just decreases in degree we look at

\[
t \partial_x \left[ \frac{x_i e^{-t} - y_i}{\sqrt{1 - e^{-2t}}} \right] = m_i \frac{x_i e^{-t} - y_i}{\sqrt{1 - e^{-2t}}} \partial_x \left[ \frac{x_i e^{-t} - y_i}{\sqrt{1 - e^{-2t}}} \right] + H_{\varepsilon}(\varepsilon^t \frac{\varepsilon^{-t} - \varepsilon}{\sqrt{1 - e^{-2t}}} \right).
\]

The last term is bounded as \( t \downarrow 0 \), and the rest of the proof is as before. \( \square \)

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