Non-Abrikosov vortices in liquid metallic hydrogen

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We consider non-Abrikosov vortex solutions in liquid metallic hydrogen (LMH) in the framework of two-component Ginzburg-Landau model. We have shown that there are three types of non-Abrikosov vortices depending on chosen boundary conditions at the core of vortices, namely, Neumann (N)-type, Dirichlet (D)-type and Gross-Pitaevskii (GP)-type vortices. The Neumann-type vortex has a non-vanishing condensation at the core, that is different from the ordinary vortex, and the magnetic flux could be reversed as well in LMH. Furthermore, we have obtained a new type of a neutral vortex which has no magnetic field. The presence of such a vortex is related to metallic superfluid state suggested by Babaev[1].

I. INTRODUCTION

The Abrikosov vortex has been intensively studied in one-component superconductor. Recently, the interest in non-Abrikosov vortices in two-component or multicomponent superconductor has been increased widely due to their unusual properties[1, 10, 12]. The non-Abrikosov vortex may have a fractional magnetic flux and non-vanishing condensation at the core of the vortex[1–3, 17]. In comparison with the Abrikosov vortex, the non-Abrikosov vortex may possesses a specific kind of interaction which is repulsive at short distances and attractive at larger scales[4]. details for for nonmonotonic vortex interaction in two-band superconductors were discussed in[6]. Besides, new interesting phenomena, like the presence of type-1.5 superconductivity and an abnormal external field response, have been found in two-component superconductor[4, 7, 12]. A liquid metallic hydrogen represents one of possible realizations of two-component superconductor medium where the existence of non-Abrikosov vortices and their characteristics can be verified in experiment. Another attractive feature of the LMH is that it could be an alternative candidate for the high temperature superconductor with coexistence phase of electron-electron and proton-proton Cooper pairs[15, 16]. Such a multi-component superconductor dresses novel features that do not appear in the normal superconductor.

In the Ginzburg-Landau (GL) theory of superconductivity, the GL parameter $\kappa$ defined as a ratio of the penetration length $\lambda$ to the coherence length $\xi$ divides the superconductor into two classes, type-I and type-II superconductor with the parameter values $\kappa < 1/\sqrt{2}$ and $\kappa > 1/\sqrt{2}$ respectively. Since the two-component superconductor has two independent GL parameters, $\kappa_{1,2}$, a new type of superconductivity appears in the regime $\kappa_1 < 1/\sqrt{2}$ and $\kappa_2 > 1/\sqrt{2}$ (assuming $\kappa_1 < \kappa_2$) which is coined as a type-1.5 superconductivity[4, 12]. Another interesting issue is that there are different types of topologically stable non-Abrikosov vortex solutions in LMH. Except the D-type and N-type vortices discussed in[2], a new GP-type vortex has been found in LMH. A main purpose of the present paper is to study these new vortex solutions and their properties in LMH. It has been shown that type-II and type-1.5 superconductivity may exist in LMH[7, 12], and for each type of superconductivity there are three types of non-Abrikosov vortex solutions. We consider the following properties of the vortices related to the presence of the fractional magnetic flux, non-vanishing condensate at the core of the vortex, magnetic flux inversion and a neutral vortex. An extended GL theory has been studied in[15] which has advantages in studying of electronic, magnetic, calorimetric, and other properties of twoband superconductors. And the GL theory for multiband superconductors from multiband BCS Hamiltonian is discussed in[14].

The paper is organized as follows. In Section 2 we introduce the two-component Ginzburg-Landau theory for LMH. In Section 3 we present numerical vortex solutions of LMH with two condensates having only one effective complex phase factor. The D-type and N-type vortices with unusual properties are described as well. Section 4 deals with the case of both condensates with non-zero complex phases. We demonstrate that there exist three types of vortex solutions, D-type, N-type and GP-type vortices, determined by different boundary conditions at the core. We have found a neutral vortex in LMH which has no magnetic field. Conclusions and discussions are given in the last section.

II. GINZBURG-LANDAU MODEL

Let us start with a free energy of LMH described by two-component Ginzburg-Landau model[1, 2, 19].

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\[ h = \frac{\hbar^2}{2m_1} |(\nabla + ig_1 A)\phi_1|^2 + \frac{\hbar^2}{2m_2} |(\nabla - ig_2 A)\phi_2|^2 + \frac{1}{2} (\nabla \times A)^2 + V(\phi_1, \phi_2), \]  

(1)

where \( \phi_1 \) and \( \phi_2 \) are two complex fields corresponding to the order parameters of electron and proton condensates, and \( g_1 \) and \( g_2 \) are absolute values of the Cooper pair’s charge, \( g_1 = g_2 = 2e \), \( m_1 \), \( m_2 \) are the masses of electronic and protonic Cooper pairs, respectively.

In LMH the both, electron and proton, condensates are conserved independently since the electronic Cooper pairs cannot convert to protonic Cooper pairs. Therefore, there is no intrinsic Josephson interband interaction \( \eta(\phi_1^\dagger \phi_2 + h.c.) \) in LMH. Moreover, there is no interband interaction \( \lambda_{12}|\phi_1|^2|\phi_2|^2 \) in two-gap metallic superconductors. In this weak-coupling approximation the effective potential for LMH can be expressed as follows

\[ V(\phi_1, \phi_2) = \frac{1}{2} \lambda_{111}|\phi_1|^4 + \frac{1}{2} \lambda_{222}|\phi_2|^4 - \mu_1 |\phi_1|^2 - \mu_2 |\phi_2|^2, \]  

(2)

where \( \lambda_{111,22} \) are the quartic coupling constants and \( \mu_{1,2} \) are chemical potentials. One can simplify the Hamiltonian (1) with the normalized valuables \( \phi_{1,2} = \hbar^{1/2} \lambda_{11,22} \phi_{1,2} \).

The masses of the scalar fields and vector field can be expressed as follows \[21, 22\]

\[ m_{\phi_1} = \sqrt{2 \mu_1}, \]  

(7a)

\[ m_{\phi_2} = \sqrt{2 \mu_2}, \]  

(7b)

\[ m_A = \sqrt{2 g^2 (\langle |\phi_1|^2 \rangle + \langle |\phi_2|^2 \rangle)} \]  

(7c)

where \( m_{\phi_1} = 1/\xi_{1,2}, \xi_{1,2} \) is the characteristic length of \( \phi_{1,2}, m_A = 1/\lambda, \lambda \) is the penetration length. It is clear that there are two mass ratios due to the presence of two condensates in LMH

\[ \beta_1 = \frac{m_{\phi_1}}{m_A} = \sqrt{\frac{\lambda_{11} \lambda_{22} \mu_1}{(\lambda_{11} \mu_2 + \lambda_{22} \mu_1)g^2}}, \]  

(8a)

\[ \beta_2 = \frac{m_{\phi_2}}{m_A} = \sqrt{\frac{\lambda_{11} \lambda_{22} \mu_2}{(\lambda_{11} \mu_2 + \lambda_{22} \mu_1)g^2}}. \]  

(8b)

So that LMH is different from the one-component superconductor. In the case of \( \beta_1 > 1, \beta_2 > 1 \), it represents type-II superconductivity, while when \( \beta_1 < 1, \beta_2 < 1 \) it has type-I superconductivity. A new feature appears in the region \( \beta_1 < 1, \beta_2 > 1 \) or \( \beta_1 > 1, \beta_2 < 1 \), where a new, type-1.5, superconductivity has been arisen \[4, 12\].

A self-dual solution in two-component GL model may also be of interest, and it is considered in two-band superconductor in \[23\], where two condensates have a non-trivial interband interaction \( \lambda_{12} \neq 0 \). When \( \mu_1 = \mu_2, \beta_1 = \beta_2 = \lambda/g^2 = 1 \) the condensates satisfy Bogomol’nyi first-order equations which have self-dual vortices as solutions. As we mentioned above, in LMH the interband interaction can be neglected. Due to this the energy minimization does not imply first-order equations, so that self-dual vortex does not exist in LMH.

### III. VORTEX SOLUTIONS WITH ONE-COMPONENT COMPLEX PHASE

We consider a straight vortex with translational symmetry along the \( z \) direction and rotational symmetry in the \((x, y)\) plane using two kinds of ansatz \[2, 23, 24\]. First, we choose an ansatz with two condensates having only one effective complex phase in cylinder coordinates \((r, \varphi, z)\)

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{\rho(r)}{\sqrt{2}} \begin{pmatrix} \cos \left( \frac{f(r)}{2} \right) \\ \exp \left( -i n \varphi \right) \end{pmatrix}, \]  

(9a)

\[ A_\mu = \frac{n}{g} A(r) \partial_\mu \varphi. \]  

(9b)
Substituting Eq. (9) into Eq. (3) yields the Hamiltonian
\[
\mathcal{H} = \frac{1}{2} \rho^2 + \frac{1}{8} \rho^2 (f^2 + \frac{n^2}{r^2} \sin^2 f) + \frac{n^2 \rho^2}{2 r^2} (\lambda - \cos f + \frac{1}{2})^2 \\
+ \frac{n^2}{2 g r^2} A^2 + \frac{\beta}{8} (\rho^2 - \frac{2 \mu}{\beta})^2 + \frac{\alpha}{\beta} (\rho^2 - \frac{4 \gamma}{\alpha}) \rho^2 \cos f \\
+ \rho^4 \cos^2 f - \frac{\mu^2}{2 \beta}.
\]

With this, the equations of motion become
\[
\dot{\rho} + \frac{1}{r} \rho - \left[ \frac{1}{4} (f^2 + \frac{n^2}{r^2} \sin^2 f) + \frac{n^2}{r^2} (\lambda - \cos f + \frac{1}{2})^2 \right] \rho = \frac{\beta}{2} (\rho^2 - \frac{2 \mu}{\beta}) + \frac{\alpha}{\beta} (\rho^2 - \frac{2 \gamma}{\alpha}) \cos f + \rho^2 \cos^2 f | \rho.
\]

\[
(10a)
\]

\[
\dot{f} + \frac{1}{r} \frac{2 \dot{\rho}}{\rho} f - 2 \frac{n^2}{r^2} (\lambda - \frac{1}{2}) \sin f = [2 \gamma - \left( \frac{\alpha}{\beta} + \beta \cos f \right) \rho^2] \sin f,
\]

\[
(10b)
\]

\[
\dot{A} - \frac{1}{r} \dot{A} = g^2 \rho^2 (\lambda - \cos f + \frac{1}{2}) = 0.
\]

\[
(10c)
\]

The corresponding electromagnetic current reads
\[
\dot{j}_\mu = n g \rho^2 \left( \lambda - \frac{\cos f + 1}{2} \right) \partial_\mu \phi.
\]

Boundary conditions at infinity can be fixed by the vacuum expectation of the order parameters
\[
\rho(\infty) = 2 \langle |\phi|^2 \rangle = 2 \sqrt{\frac{2 \beta \mu - \alpha \gamma}{4 \beta^2 - \alpha^2}},
\]

\[
\cos f(\infty) = \frac{2 \langle |\phi|^2 \rangle - \langle |\phi|^2 \rangle^2}{\rho^2(\infty)} = \frac{2 \beta \gamma - \alpha \mu}{2 \beta \mu - \alpha \gamma}.
\]

\[
(11)
\]

In particular, the electromagnetic current vanishes at infinity, i.e. \( j_\mu = 0 \), so that we have
\[
A(\infty) = \frac{\cos f(\infty) + 1}{2} = \frac{2 \beta (\gamma + \mu) - \alpha (\gamma + \mu)}{2 (2 \beta \mu - \alpha \gamma)}.
\]

\[
(12)
\]

On the other hand, boundary conditions at the core can be obtained by substituting Taylor expansions for the functions \( A, f, \rho \) into the equations of motion and imposing regularity conditions at the core \( \rho \). A-type vortex is defined by imposing Dirichlet boundary condition for the total condensate density \( \rho(r) \)
\[
\rho(0) = 0, \quad A(0) = -\frac{1}{n}, \quad f(0) = \pi.
\]

\[
(13)
\]

N-type vortex corresponds to imposed Neumann boundary condition for \( \rho(r) \)
\[
\rho(0) \neq 0, \quad \rho(0) = 0, \quad A(0) = 0, \quad f(0) = \pi.
\]

\[
(14)
\]

It is interesting to observe that the N-type solutions exhibit a non-vanishing concentration at the core of vortex.

\[
\Phi = \oint A_\mu dx^\mu = [A(\infty) - A(0)] \frac{2 \pi n}{g}.
\]

\[
(15)
\]
and $n(0.6 + \frac{1}{3})\Phi_0$ with $\gamma = 0.4$ and $\gamma = 0.2$, respectively. The fluxes are shown to be fractional multiple of the flux quanta as well.

## B. N-type Vortex Solutions

In this case, we choose the following boundary conditions

\[
\begin{align*}
\rho(0) &= 0, \quad f(0) = \pi, \quad A(0) = 0, \\
\rho(\infty) &= 2\sqrt{\frac{2\beta\mu - \alpha\gamma}{4\beta^2 - \alpha^2}}, \quad f(\infty) = \arccos \frac{2\beta\gamma - \alpha\mu}{2\beta\mu - \alpha\gamma}, \\
A(\infty) &= \frac{2\beta(\gamma + \mu) - \alpha(\gamma + \mu)}{2(2\beta\mu - \alpha\gamma)}.
\end{align*}
\]

(19)

Similarly, there are solutions with type-1.5 and type-II superconductivity. We choose the same parameter $\gamma$ as we did in the case of D-type solutions. Results of type-1.5 vortex with $\gamma = 0.8, 0.6$ are shown in Fig. 3 and type-II vortex solutions with $\gamma = 0.4, 0.2$ are shown in Fig. 4.

With Eq. (18) and Eq. (19), the total magnetic flux through the $x - y$ plane with $\gamma = 0.8, 0.6, 0.4, 0.2$ are $\Phi = 0.9n\Phi_0, 0.8n\Phi_0, 0.7n\Phi_0, 0.6n\Phi_0$, respectively. Then one can find that N-type vortex has a magnetic flux smaller than D-type vortex has with the same parameters. Although both two types of vortex carry infinite energy like the vortex of superfluid in GP theory, one can always make a natural cut-off with a real size of the superconductor. By this way one can obtain a finite energy vortex with fractional flux $[2, 17]$. Notice, the D-type vortex is topologically stable due to the presence of non-trivial homotopy group $\Pi_2(S^2)$, and the N-type vortex stability originates from the homotopy $\Pi_1(S^1)$.

The normal one-component superconductors have only integer flux, whereas two-gap superconductors can possess both, integer and fractional fluxes $[1, 2, 17]$. Furthermore, it can be found that the condensates concentration at core in Neumann type solution is non-zero $\rho(0) \neq 0$. The behavior of both two components $|\phi_1|, |\phi_2|$ near the core is described in Fig. 5 in $z - x$ plane. Clearly, configuration of the component $|\phi_1|$ looks the same as the Abrikosov vortex, while the behavior of $|\phi_2|$ is different, the non-vanishing condensation at the core makes it looks like the profile of $W$. One should notice that such a behavior is caused by a specific electromagnetic interaction of the two condensates irrespectively of interband interaction $[18]$. There is another unexpected result relates to the magnetic field properties depicted in Fig. 6. The magnetic

![FIG. 2: Solutions for $\rho$, $f$, $A$ with type-II superconductivity corresponding to Dirichlet boundary condition with $n = 1, \alpha = 0, \beta = 2, \mu = 1, g = 1$. Two solutions are shown for parameter values $\gamma = 0.4$ (solid lines) and $\gamma = 0.2$ (dotted lines).](image1)

![FIG. 3: Solutions for $\rho$, $f$, $A$ with type-1.5 superconductivity corresponding to Neumann boundary condition with $n = 1, \alpha = 0, \beta = 2, \mu = 1, g = 1$. Two solutions are shown for $\gamma = 0.8$ (solid lines), $\gamma = 0.6$ (dotted lines).](image2)

![FIG. 4: Solutions for $\rho$, $f$, $A$ with type-II superconductivity with Neumann boundary condition and with parameter values $n = 1, \alpha = 0, \beta = 2, \mu = 1, g = 1$. Two solutions are shown for $\gamma = 0.4$ (solid lines), $\gamma = 0.2$ (dotted lines).](image3)
field reverses around at $r = 7$ and then keeps this opposite direction with a long decaying tail till space infinity. According to numerical solution the inversion will be more clear with a higher phase winding number $n$.

In Ref. [18] an unusual delocalization of the magnetic field has been found which is different from the normal Abrikosov vortex exponential localization. This effect can take place in the superconductive phase of LMH as well giving a new possible way to probe LMH at low temperature in experiment.

IV. VORTEX SOLUTIONS WITH TWO-COMPONENT COMPLEX PHASES

We set both component condensates, $\phi_1$ and $\phi_2$, to have nonzero complex phases. The ansatz is chosen as follows

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{\rho(r)}{\sqrt{2}} \begin{pmatrix} \cos \frac{f(r)}{2} \exp(-in_1 \phi) \\ \sin \frac{f(r)}{2} \exp(-in_2 \phi) \end{pmatrix},$$

(20a)

$$= \frac{\rho}{\sqrt{2}} \exp-ip\phi \begin{pmatrix} \cos \frac{f(r)}{2} \exp(-in_1 \phi) \\ \sin \frac{f(r)}{2} \exp(-in_2 \phi) \end{pmatrix},$$

(20b)

where $n = n_1 - n_2$ and $p = n_2$. With this, the Hamiltonian reads

$$\mathcal{H} = \frac{1}{2} \rho^2 + \frac{1}{8} \rho^2 (f^2 + \frac{n^2}{r^2} \sin^2 f)$$

$$+ \frac{\rho^2}{2r^2} (qA - n \cos f \frac{1}{2} - p)^2 + \frac{g^2}{2r^2} \lambda^2$$

$$+ \frac{\beta}{8} \left[ (\rho^2 - \frac{2\mu}{\beta})^2 + \frac{\alpha}{\beta} \rho^2 \right] \rho^2 \cos f + \rho^4 \cos^2 f$$

$$- \frac{\mu^2}{2\beta},$$

(21)

and equations of motion become

$$\ddot{\rho} + \frac{1}{r} \dot{\rho} - \frac{1}{4} (f^2 + \frac{n^2}{r^2} \sin^2 f)$$

$$+ \frac{1}{r^2} (qA - n \cos f \frac{1}{2} - p)^2 \rho = \frac{\beta}{2} (\rho^2 - \frac{2\mu}{\lambda}).$$

(22a)

$$\ddot{f} + \frac{1}{r} \dot{f} + 2 \dot{\phi} \frac{\dot{f}}{\rho} - 2 \frac{n}{r^2} (qA - \frac{n}{2} - p) \sin f$$

$$= [2\gamma - (\frac{\alpha}{2} + \beta \cos f) \rho^2] \sin f,$$

(22b)

$$\ddot{A} - \frac{1}{r} \dot{A} - g^2 \rho^2 [A - \frac{n}{2q} (\cos f + 1) - \frac{p}{q}] = 0.$$  

(22c)

The electromagnetic current is

$$j_\mu = g^2 \rho^2 (qA - n \cos f \frac{1}{2} - p) \dot{\phi}.$$

(23)

Similarly, magnetic flux through the $x - y$ plane can be fixed as

$$\Phi = \oint A_\mu dx^\mu = [A(\infty) - A(0)] \frac{2\pi q}{g}.$$

(24)
The boundary condition for D-type and N-type vortices can be obtained in a similar manner as in section 3. However, with two non-zero phase windings, there is a third type, GP-type superconductivity due to chosen boundary condition. In addition, one can find a neutral vortex which behaves as the vortex in the superfluid which is described by Gross-Pitaevskii theory.

Let us consider the following boundary conditions:

The Dirichlet boundary conditions

\[ \rho(0) = 0, \; f(0) = \pi, \; A(0) = \frac{p \pm 1}{q}, \]  

(25)

The Neumann boundary conditions

\[ \dot{\rho}(0) = 0, \; f(0) = \pi, \; A(0) = \frac{p}{q}, \]  

(26)

The Gross-Pitaevskii (GP) type boundary conditions for \( n = 2 \)

\[ \rho(0) = 0, \; f(0) = 0, \; A(0) = \frac{2p + n}{2q}. \]  

(27)

GP-type boundary conditions can be imposed also in the case of one component phase winding for \( n = 2 \), however, the corresponding vortex solution is not stable. This is why we did not discuss it in section 3. However, we can choose \( n_1 = 1, n_2 = -1 \), and \( n = 2 \), then the composite vortices are not only topologically but also thermodynamically stable\(^{[1]}\). One can find that a neutral type vortex exists not only with the Gross-Pitaevskii boundary condition, but also with the Neumann boundary when \( n = 0 \).

At infinity, with Eq. (6), Eq. (20) and Eq. (23) one can obtain

\[
\rho(\infty) = 2\sqrt{\frac{2\beta\mu - \alpha\gamma}{4\beta^2 - \alpha^2}}, \\
\cos f(\infty) = \frac{2\beta\gamma - \alpha\mu}{2\beta\mu - \alpha\gamma}, \\
A(\infty) = \frac{(2n + 4p)\beta\mu - (n + 2p)\alpha\gamma + 2n\beta\gamma - n\alpha\mu}{2q(2\beta\mu - \alpha\gamma)}.
\]

(28)

Then the total magnetic flux of the three types of vortex is

\[
\Phi = \begin{cases} 
\frac{n(\mu + \gamma)(2\beta - \alpha) + 2(2\beta\mu - \alpha\gamma)2\pi q}{g} & \text{D-type} \\
\frac{n(\mu + \gamma)(2\beta - \alpha)2\pi q}{g} & \text{N-type} \\
\frac{2q(2\beta\mu - \alpha\gamma)2\pi q}{g} & \text{GP-type} \\
\frac{2n\beta\gamma - \alpha\mu}{2q(2\beta\mu - \alpha\gamma)2\pi q} & \text{LMH with GP-type boundary condition and parameters (i.e. \( n = 2, \; p = -1, \; q = 1 \). D-type vortex and N-type vortex solutions are shown in Fig. 7 and Fig. 8. Moreover, there is a GP-type vortex with non-zero \( f(0) \) in LMH, non-trivial solutions are shown in Fig. 9 with \( \gamma = 0.8, 0.2 \) and \( \alpha = 0, \; \beta = 2, \; \mu = 1, \; g = 1 \). In Fig. 7 and Fig. 8 one can find that \( f \) (dashed lines) decline monotonically from \( \pi \), while solid lines in these plots firstly decrease from \( \pi \) to a minimum then increase to the vacuum expectation. More interestingly, in Fig. 9 lines \( f \) of GP-type vortex increase monotonically near the core which shows an opposite behavior comparing to the N-type and D-type vortex. With the same parameters we choose for the three types of vortex, one can find GP-type vortex carries the smallest flux. So that, with an appropriate cut off, one can more easily find GP-type vortex than N-type and D-type vortex in LMH in experiments.

Moreover, it is noticed that there is a type of neutral solution with GP-type boundary conditions in Eq. (22)

\[ A = 0, \; f = \pi/2, \; \rho(0) = 0 \]  

(30)

We show it in Fig. 10 a neutral type vortex exists in LMH with GP-type boundary condition and parameters (i.e. \( n = 2, \; p = -1, \; q = 1, \; \alpha = 0, \; \beta = 2, \; \mu = 1, \; g = 1, \) and \( \gamma = 0 \)). Without magnetic flux, this neutral vortex looks like a vortex in superfluid described by the Gross-Pitaevskii theory.

One should notice, these results are consistent with results found in\(^{[1]}\). It has been claimed that if the composite vortices (\( \Delta \phi_1 = 2\pi, \; \Delta \phi_2 = -2\pi \)) in LMH were not yet ionized into two separated vortices, the both, superconductive superfluid phase and metallic superfluid phase, can appear in LMH. Obviously, it is our case \( n_1 = 1, \; n_2 = -1 \). We have demonstrated exactly that

\[ n_1 = -n_2 \]
there is a super phase (metallic superfluid) with only neutral vortex which is topologically and thermodynamically stable according to Babaev’s arguments. Solutions in Section 3 represent the case of $\Delta \phi_1 = 2\pi$, $\Delta \phi_2 = 0$ in which the magnetic field cannot appear in the vortex in this metallic superconductor. It indicates that there is a phase corresponding to the magnetic superfluid state in which protonic Cooper pairs co-exist with the electronic Cooper pairs.

**V. CONCLUSION**

In this paper we have considered vortex solutions with different topologies in various type superconductors. We have found D-type, N-type and GP-type non-Abrikosov vortices according to imposed different boundary conditions at the core. We have shown that GP-type boundary conditions can easily degenerate to the normal Abrikosov vortex in one component superconductor with the condition $f \equiv 0$, $A(0) = 0$, $\rho(0) = 0$.

**B. The case of $n_1 = n_2$**

Unfortunately, this case is energetically forbidden, but this kind of vortex can be induced by the vortex ($\Delta \phi_1 = 2\pi$, $\Delta \phi_2 = 2\pi$) imposed in external field [17]. With $n_1 = n_2$, naively, it seems that this two-component system is identical to the ordinary one-component case, since these two components lost their relative phase windings,

$$\phi = \frac{\rho(r)}{\sqrt{2}} \left( \frac{\cos f(r)}{\sin f(r)} \right) \exp (-in_1 \varphi).$$  \hspace{1cm} (31)

This view is only partly correct with the GP-type boundary condition. One can check that in such case it can be easily degenerated to the normal Abrikosov vortex in one component superconductor with the condition

$$f \equiv 0, \quad A(0) = 0, \quad \rho(0) = 0.$$  \hspace{1cm} (32)

There is a more trivial solution

$$f(r) \equiv \text{const.}, \quad A(r) \equiv 1, \quad \rho(0) = 0.$$  \hspace{1cm} (33)

which is nothing but the neutral vortex in Gross-Pitaevskii theory.

Besides the above neutral vortex, we have found a non-trivial neutral vortex with Neumann boundary condition. The neutral N-type vortex solutions with null magnetic flux are shown in Fig. [11]. The magnetic field cannot appear in the vortex in this metallic superconductor. It indicates that there is a phase corresponding to the magnetic superfluid state in which protonic Cooper pairs co-exist with the electronic Cooper pairs.
vortices carry a smaller magnetic flux than N-type and D-type vortices. The D-type vortex has no concentration of the condensate at the core, whereas N-type vortex has a non-trivial profile of the condensate at the core. In general, unlike the Abrikosov vortex, the condensate $\phi_2$ in the N-type vortex in Fig. 5 has a local maximum at the core which makes configuration looks like $W$, while $\phi_1$ has the ordinary configuration as the Abrikosov vortex.

Furthermore, the magnetic flux in LMH can be integer only in a special parameter limit, the fractional flux has been shown to be more general in LMH. Another important property of non-Abrikosov vortex is the magnetic field inversion effect which can be observed in the obtained delocalized solution in LMH. It shows that vortex carries a positive magnetic field along $Z$ axis near the center, while at a certain distance magnetic field flips to the negative direction. This effect may also exist in the case when both two condensates have non-zero complex phases as it is shown in Fig. 9 where the magnetic field is reversed clearly.

Moreover, there is a type of neutral vortex solution with two non-zero complex phases. In this case, magnetic flux cannot exist in the vortex, there is no super electromagnetic current in LMH. This is similar to the vortex of superfluid in Gross-Pitaevskii theory. Neutral vortex in LMH gives an important indication that there is a new ordered state in LMH, the metallic superfluid[1]. That new quantum ordered state cannot be purely categorized as a superconductor or superfluid and deserves further study.

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