Measurable regularity of infinite-dimensional Lie groups based on Lusin measurability

Natalie Nikitin

Abstract

We discuss Lebesgue spaces \( L^p([a,b], E) \) of Lusin measurable vector-valued functions and the corresponding vector spaces \( AC_{L^p}([a,b], E) \) of absolutely continuous functions. These can be used to construct Lie groups \( AC_{L^p}([a,b], G) \) of absolutely continuous functions with values in an infinite-dimensional Lie group \( G \). We extend the notion of \( L^p \)-regularity of infinite-dimensional Lie groups introduced by Glöckner to this setting and adopt several results and tools.

Introduction

In [12], Milnor calls an infinite-dimensional Lie group \( G \) (with Lie algebra \( g \) and identity element \( e \)) regular if for every smooth curve \( \gamma: [0,1] \rightarrow g \) the initial value problem

\[
\eta' = \eta \cdot \gamma, \quad \eta(0) = e,
\]

has a (necessarily unique) solution \( \text{Evol}(\gamma): [0,1] \rightarrow G \) and the function

\[
\text{evol}: C^\infty([0,1], g) \rightarrow G, \gamma \mapsto \text{Evol}(\gamma)(1)
\]

so obtained is smooth.

Further, [10] and [13] deal with the concept of \( C^k \)-regularity, investigating whether the above initial value problem has a solution for every \( C^k \)-curve \( \gamma \) (the solution \( \text{Evol}(\gamma) \) being a \( C^{k+1} \)-curve then) and whether the function \( \text{evol}: C^k([0,1], g) \rightarrow G \) is smooth.

Generalizing this theory even more, in [9] Glöckner constructs Lebesgue spaces \( L^p_B([a,b], E) \) of Borel measurable functions \( \gamma: [a,b] \rightarrow E \) with values in Fréchet spaces \( E \) (for \( p \in [1, \infty) \)) and introduces spaces of certain absolutely continuous \( E \)-valued functions \( \eta: [a,b] \rightarrow E \) (denoted by \( AC_{L^p}([a,b], E) \)) with derivatives in \( L^p_B([a,b], E) \). Having a Lie group structure on the spaces \( AC_{L^p}([0,1], G) \) available, in [9] a Fréchet-Lie group \( G \) is called \( L^p \)-semiregular if the initial value problem [1] has a solution \( \text{Evol}(\gamma) \in AC_{L^p}([0,1], G) \) for every \( \gamma \in L^p_B([0,1], g) \), and \( G \) is called \( L^p \)-regular if it is \( L^p \)-semiregular and the map \( \text{Evol}: L^p_B([0,1], g) \rightarrow AC_{L^p}([0,1], G), \gamma \mapsto \text{Evol}(\gamma) \) is smooth.

Since the sum of two vector-valued Borel measurable functions may be not Borel measurable, certain assumptions need to be made to obtain a vector space structure on the space of the considerable maps. This implies that the concepts of \( L^p \)-regularity (mentioned above) only make sense for Fréchet-Lie groups (and some other classes of Lie groups described in [9]).
To loosen this limitation, we recall the notion of Lusin-measurable functions (Definition 1.1), which have the advantage that vector-valued Lusin-measurable functions always form a vector space, and define the corresponding Lebesgue spaces \( L^p([a,b], E) \) in Definition 2.8. Further, in Lemma 1.8 and Lemma 1.9 we recall that under certain conditions there is a close relation between Lusin and Borel measurable functions (known as the Lusin’s Theorem). This leads to the result that the Lebesgue spaces \( L^p_B([a,b], E) \) constructed in [9] coincide with our Lebesgue spaces \( L^p([a,b], E) \), due to the conditions needed for Borel measurable functions to form a vector space. (Note that Lebesgue spaces of Lusin measurable functions are also considered in [7], for example.)

We lean on the theory established in [9] and construct AC\(_L^p\)-spaces for sequentially complete locally convex spaces. In Definition 4.7 we define the notion of \( L^p \)-regularity for infinite-dimensional Lie groups modelled on such spaces and adopt several useful results from [9]. In particular:

**Theorem.** Let \( G \) be an \( L^p \)-semiregular Lie group. Then Evol: \( L^p_B([0,1], g) \to AC_{L^p}([0,1], G) \) is smooth if and only if Evol is smooth as a function to \( C([0,1], G) \).

As a consequence, we get:

**Theorem.** Let \( G \) be a Lie group modelled on a sequentially complete locally convex space and \( p,q \in [1,\infty] \) with \( q \geq p \). If \( G \) is \( L^p \)-regular, then \( G \) is \( L^q \)-regular. Furthermore, in this case \( G \) is \( C^0 \)-regular.

Furthermore, we show:

**Theorem.** Let \( G \) be a Lie group modelled on a sequentially complete locally convex space. Let \( \Omega \subseteq L^p([0,1], g) \) be an open \( 0 \)-neighbourhood. If for every \( \gamma \in \Omega \) there exists the corresponding Evol(\( \gamma \)) \( \in AC_{L^p}([0,1], G) \), then \( G \) is \( L^p \)-semiregular. If, in addition, the function Evol: \( \Omega \to AC_{L^p}([0,1], G) \) is smooth, then \( G \) is \( L^p \)-regular.

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**Notation** All the topological spaces are assumed Hausdorff (except for the \( L^p \)-spaces), all vector spaces are \( \mathbb{R} \)-vector spaces (and locally convex topological vector spaces are called "locally convex spaces" for short). Wherever we write \([a,b] \), we always mean an interval in \( \mathbb{R} \) with \( a < b \).

## 1 Measurable functions

Recall that the Borel \( \sigma \)-algebra \( B(X) \) on a topological space \( X \) is the \( \sigma \)-algebra generated by the open subsets of \( X \). A function \( \gamma: X \to Y \) between topological spaces is called Borel measurable if the preimage \( \gamma^{-1}(A) \) of every open (resp. closed) subset \( A \subseteq Y \) is in \( B(X) \). Further, for a measure \( \mu: B(X) \to [0,\infty] \) on \( X \), a subset \( N \subseteq X \) is called \( \mu \)-negligible if \( N \subseteq N' \) for some \( N' \in B(X) \) with
\( \mu(N') = 0 \). By saying that a property holds for \( \mu \)-almost every \( x \in X \) (or for almost every \( x \in X \), or for a.e. \( x \in X \)) resp. \( \mu \)-almost everywhere (or almost everywhere, or a.e.) we will always mean that there exists some \( \mu \)-negligible subset \( N \subseteq X \) such that the property holds for every \( x \notin N \).

**Definition 1.1.** Let \( X \) be a locally compact topological space, let \( Y \) be a topological space and let \( \mu : \mathcal{B}(X) \to [0, \infty] \) be a measure on \( X \). A function \( \gamma : X \to Y \) is called Lusin \( \mu \)-measurable (or \( \mu \)-measurable) if for each compact subset \( K \subseteq X \) there exists a sequence \((K_n)_{n \in \mathbb{N}}\) of compact subsets \( K_n \subseteq K \) such that each of the restrictions \( \gamma|_{K_n} \) is continuous and \( \mu(K \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0 \).

**Remark 1.2.** Clearly, every continuous function \( \gamma : X \to Y \) is \( \mu \)-measurable, more generally, if \( \gamma|_K \) is continuous for every compact subset \( K \subseteq X \), then \( \gamma \) is \( \mu \)-measurable. Furthermore, it is easy to see that if \( f : Y \to Z \) is a continuous function between topological spaces and \( \gamma : X \to Y \) is \( \mu \)-measurable, then the composition \( f \circ \gamma : X \to Z \) is \( \mu \)-measurable. Actually, it may happen that a composition \( f \circ \gamma \) with a continuous function \( f : X \to Y \) is not \( \mu \)-measurable (see [5 IV §5 No. 3]). However, a criterion for a composition of Lusin measurable functions to be Lusin measurable can be found, for example, in [16] Chapter I, Theorem 9]. The next lemma describes a special case.

**Lemma 1.3.** Let \( f : X_1 \to X_2 \) be a continuous function between locally compact spaces, let \( Y \) be a topological space. Let \( \mu_1, \mu_2 \) be measures on \( X_1, X_2 \), respectively, and assume that \( \mu_1(f^{-1}(N)) = 0 \), whenever \( \mu_2(N) = 0 \). Then for every \( \mu_2 \)-measurable function \( \gamma : X_2 \to Y \), the composition \( \gamma \circ f : X_1 \to Y \) is \( \mu_1 \)-measurable.

**Proof.** Given a compact subset \( K \subseteq X_1 \), the continuous image \( L := f(K) \subseteq X_2 \) is compact, hence there exists a sequence \((L_n)_{n \in \mathbb{N}}\) of compact subsets of \( L \) such that \( \gamma|_{L_n} \) is continuous for every \( n \) and \( \mu_2(L \setminus \bigcup_{n \in \mathbb{N}} L_n) = 0 \). Then each \( K_n := f^{-1}(L_n) \cap K \) is closed in \( K \), hence compact and \( \gamma \circ f|_{K_n} \) is continuous.

Furthermore,

\[
\mu_1(K \setminus \bigcup_{n \in \mathbb{N}} K_n) = \mu_1(K \setminus f^{-1}(\bigcup_{n \in \mathbb{N}} L_n)) \leq \mu_1(f^{-1}(L \setminus \bigcup_{n \in \mathbb{N}} L_n)) = 0,
\]

by assumption on \( f \), whence \( \gamma \circ f \) is \( \mu_1 \)-measurable. \( \square \)

We recall that a measure \( \mu : \mathcal{B}(X) \to [0, \infty] \) on a locally compact space \( X \) is called a Radon measure if \( \mu \) is inner regular and \( \mu(K) < \infty \) for every compact subset \( K \subseteq X \). An essential criterion for measurability with respect to Radon measures is recited below (and can be found, for example, in [4 IV §5 No. 1]).

**Lemma 1.4.** Let \( X \) be a locally compact topological space, \( Y \) be a topological space and let \( \mu : \mathcal{B}(X) \to [0, \infty] \) be a Radon measure. A function \( \gamma : X \to Y \) is \( \mu \)-measurable if and only if for each \( \varepsilon > 0 \) and each compact subset \( K \subseteq X \) there exists a compact subset \( K_{\varepsilon} \subseteq K \) such that \( \gamma|_{K_{\varepsilon}} \) is continuous and \( \mu(K \setminus K_{\varepsilon}) \leq \varepsilon \).

**Proof.** First, assume that \( \gamma \) is \( \mu \)-measurable, let \( K \subseteq X \) be a compact subset and fix \( \varepsilon > 0 \). Given a sequence \((K_n)_{n \in \mathbb{N}}\) of compact sets \( K_n \subseteq K \) as in Definition 1.1 the sequence \((K \setminus \bigcup_{m=1}^{n} K_m)_{n \in \mathbb{N}}\) is decreasing and \( \mu(K \setminus K_1) < \infty \). Hence, we have

\[
\lim_{n \to \infty} \mu(K \setminus \bigcup_{m=1}^{n} K_m) = \mu\left(\bigcap_{n \in \mathbb{N}} (K \setminus \bigcup_{m=1}^{n} K_m)\right) = \mu(K \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0,
\]

\[3\]
therefore, there exists some $N_\varepsilon \in \mathbb{N}$ such that $\mu(K \setminus \bigcup_{m=1}^{N_\varepsilon} K_m) \leq \varepsilon$. As
the union is finite, the set $K_\varepsilon := \bigcup_{m=1}^{N_\varepsilon} K_m$ is compact and $\gamma|_{K_\varepsilon}$ is continuous.

Now, we show that the converse is also true. To this end, for a compact set $K \subseteq X$ we construct a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of $K$ such that for each $n$ the restriction $\gamma|_{K_n}$ is continuous and $\mu(K \setminus \bigcup_{m=1}^{n} K_m) \leq 1/n$. Then

$$\mu(K \setminus \bigcup_{m=1}^{n} K_m) = \mu(\bigcap_{m=1}^{n} (K \setminus K_m)) = \lim_{n \to \infty} \mu(K \setminus K_n) \leq \lim_{n \to \infty} \frac{1}{n} = 0,$$

using the same arguments as in the first part of the proof, hence $\gamma$ will be $\mu$-measurable.

The construction of the sequence $(K_n)_{n \in \mathbb{N}}$ is made as follows. For $n = 1$, by assumption there is a compact subset $K_1 \subseteq K$ such that $\gamma|_{K_1}$ is continuous and $\mu(K \setminus K_1) \leq 1$ (taking $\varepsilon = 1$). Having constructed $K_1, \ldots, K_n$ with the required properties, we choose $\varepsilon = 1/(n+1)$, then the assumption yields a compact subset $K_{n+1} \subseteq K$ such that $\gamma|_{K_{n+1}}$ is continuous and $\mu(K \setminus K_{n+1}) \leq \varepsilon$. But by the inner regularity of $\mu$, for the measurable set $A := K_\varepsilon \setminus \bigcup_{m=1}^{n} K_m \subseteq K$ there is a compact subset $K_{n+1} \subseteq A$ such that $\mu(A \setminus K_{n+1}) \leq \varepsilon$ (since $\mu(A) < \infty$). As $K_{n+1} \subseteq K_\varepsilon$, the restriction $\gamma|_{K_{n+1}}$ is continuous and we have

$$\mu(K \setminus \bigcup_{m=1}^{n+1} K_m) = \mu(K \setminus K_\varepsilon) + \mu(A \setminus K_{n+1}) \leq \varepsilon = \frac{1}{n+1}.$$

\[\square\]

**Remark 1.5.** Given a Radon measure $\mu$ on $X$, for a $\mu$-measurable function $\gamma: X \to Y$ and every compact subset $K \subseteq X$ we can always construct a sequence $(K_n)_{n \in \mathbb{N}}$ (proceeding as in the second part of the previous proof) which is pairwise disjoint, this will be useful in Subsection 2.3 where we consider some integrals of $\mu$-measurable functions. Using the inner regularity of $\mu$, such a sequence can also be constructed for every Borel set $B \subseteq X$ with $\mu(B) < \infty$, so that every function $\bar{\gamma}: X \to Y$ such that $\bar{\gamma}(x) = \gamma(x)$ for $\mu$-almost every $x \in X$ is $\mu$-measurable.

Furthermore, if the locally compact space $X$ is $\sigma$-compact (in particular, if $X$ is compact), then there is a sequence $(K_n)_{n \in \mathbb{N}}$ such that $\mu(X \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0$.

In the following, the measure $\mu: \mathcal{B}(X) \to [0, \infty]$ on a locally compact space $X$ will be always assumed a Radon measure. The proof of the following lemma can be found (partially) in [4].

**Lemma 1.6.** Let $X$ be a locally compact topological space, and $(Y_n)_{n \in \mathbb{N}}$ be topological spaces. A map $\gamma := (\gamma_n)_{n \in \mathbb{N}}: X \to \prod_{n \in \mathbb{N}} Y_n$ is $\mu$-measurable if and only if each of the compositions $\gamma_n: X \to Y_n$ is $\mu$-measurable.

**Proof.** Assume that $\gamma$ is $\mu$-measurable. Each of the coordinate projections $p_n: \prod_{n \in \mathbb{N}} Y_n \to Y_n$ is continuous, thus by Remark 1.2 each of the compositions $p_n \circ \gamma = \gamma_n$ is $\mu$-measurable.

On the other hand, fix $\varepsilon > 0$ and a compact subset $K \subseteq X$. Using Lemma 1.4 for each $n \in \mathbb{N}$ we find a compact subset $K_n \subseteq K$ such that $\gamma_n|_{K_n}$ is continuous.
and \( \mu(K \setminus K_n) \leq \varepsilon/2^n \), as each \( \gamma_n \) is \( \mu \)-measurable. Then the intersection 

\[
K_\varepsilon := \bigcap_{n \in \mathbb{N}} K_n
\]

is a compact subset of \( K \) with

\[
\mu(K \setminus K_\varepsilon) = \mu(\bigcup_{n \in \mathbb{N}} (K \setminus K_n)) \leq \sum_{n=1}^{\infty} \mu(K \setminus K_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.
\]

Since \( K_\varepsilon \subseteq K_n \) for each \( n \in \mathbb{N} \), the restriction \( \gamma|_{K_\varepsilon} \) is continuous, thus \( \gamma \) is \( \mu \)-measurable by Lemma 1.3. \( \square \)

**Remark 1.7.** If \( E \) is a topological vector space, then for \( \mu \)-measurable functions \( \gamma, \eta: X \to E \) and every scalar \( \alpha \in \mathbb{R} \), the functions \( \gamma + \eta: X \to E \), \( \alpha \gamma: X \to E \) are \( \mu \)-measurable (by Remark 1.2 and Lemma 1.6). Thus, vector-valued \( \mu \)-measurable functions build a vector space without any assumptions on the range \( E \) (if \( \mu \) is a Radon measure). This allows us to construct locally convex vector spaces (the Lebesgue spaces) in Section 2 consisting of certain \( \mu \)-measurable functions with values in arbitrary locally convex spaces, expanding the theory established in [9], where the author deals with classical Borel measurable functions, which do not necessarily form a vector space without additional assumptions on \( E \).

The relation between \( \mu \)-measurable functions and Borel measurable functions is known as the Lusin’s Theorem and can be found in several versions in [2], [6], and others. Below, we recall the criterions which suffice for our purposes.

**Lemma 1.8.** Assume that \( X \) is a \( \sigma \)-compact locally compact space and let \( \gamma: X \to Y \) be a \( \mu \)-measurable function. Then there exists a Borel measurable function \( \bar{\gamma}: X \to Y \) such that \( \bar{\gamma}(x) = \gamma(x) \) for almost every \( x \in X \).

**Proof.** Given a sequence of compact subsets \( (K_n)_{n \in \mathbb{N}} \) as in Remark 1.5 we set

\[
X \setminus \bigcup_{n \in \mathbb{N}} K_n := N \in \mathcal{B}(X)
\]

and define the function \( \bar{\gamma}: X \to Y \) via

\[
\bar{\gamma}(x) := \gamma(x), \quad \text{if } x \in \bigcup_{n \in \mathbb{N}} K_n, \quad \bar{\gamma}(x) := y_0 \in Y, \quad \text{if } x \in N.
\]  

Then obviously \( \bar{\gamma}(x) = \gamma(x) \) a.e. Now we show that \( \bar{\gamma}^{-1}(A) \in \mathcal{B}(X) \) for every closed subset \( A \subseteq Y \). Consider

\[
\bar{\gamma}^{-1}(A) = (\bar{\gamma}^{-1}(A) \cap N) \cup (\bar{\gamma}^{-1}(A) \cap \bigcup_{n \in \mathbb{N}} K_n).
\]

If \( y_0 \in A \), then \( \bar{\gamma}^{-1}(A) \cap N = N \in \mathcal{B}(X) \), otherwise \( \bar{\gamma}^{-1}(A) \cap N = \emptyset \in \mathcal{B}(X) \). Further

\[
\bar{\gamma}^{-1}(A) \cap \bigcup_{n \in \mathbb{N}} K_n = \bigcup_{n \in \mathbb{N}} (\bar{\gamma}^{-1}(A) \cap K_n) = \bigcup_{n \in \mathbb{N}} (\gamma^{-1}(A) \cap K_n) \in \mathcal{B}(X),
\]

being a countable union of sets closed in \( X \) (being closed in \( K_n \)). \( \square \)

**Lemma 1.9.** Let \( Y \) be a topological space with countable base. Then each Borel measurable function \( \gamma: X \to Y \) is \( \mu \)-measurable.
Proof. We denote by \((V_n)_{n \in \mathbb{N}}\) the countable base of \(Y\). Fix a compact subset \(K \subseteq X\) and \(\varepsilon > 0\). Consider the sets

\[ B_n := K \cap \gamma^{-1}(V_n), \quad C_n := K \setminus \gamma^{-1}(V_n), \]

which are Borel sets with finite measure. Thus, the inner regularity of \(\mu\) yields compact subsets \(K_n \subseteq B_n\), \(K'_n \subseteq C_n\) such that \(\mu(B_n \setminus K_n)\), \(\mu(C_n \setminus K'_n) \leq \varepsilon/2^{n+1}\).

The intersection \(K_\varepsilon := \bigcap_{n \in \mathbb{N}} (K_n \cup K'_n)\) is a compact subset of \(K\) and

\[ \mu(K \setminus K_\varepsilon) = \mu(K \setminus \bigcap_{n \in \mathbb{N}} (K_n \cup K'_n)) \leq \sum_{n=1}^{\infty} \mu(B_n \setminus K_n) + \mu(C_n \setminus K'_n) \leq \varepsilon/2^{n+1}, \]

It remains to show the continuity of \(\gamma|_{K_\varepsilon}\), then \(\gamma\) will be \(\mu\)-measurable by Lemma 1.4. Fix some \(x \in K_\varepsilon\) and let \(V_x \subseteq Y\) be a neighborhood of \(\gamma(x)\). Then \(\gamma(x) \subseteq V_m \subseteq V_x\) for some \(m\), thus \(x \in K_m\), that is \(x \in X \setminus K'_m\) which is an open subset in \(X\). Then \(U_x := K_\varepsilon \cap (X \setminus K'_m)\) is an open \(x\)-neighborhood in \(K_\varepsilon\) and \(U_x \subseteq K_m\), whence \(\gamma(U_x) \subseteq V_m \subseteq V_x\). Therefore, the restriction \(\gamma|_{K_\varepsilon}\) is continuous in \(x\), hence continuous, as \(x\) was arbitrary.

2 Lebesgue spaces

2.1 Definition and basic properties

First, we recall the definition of Lebesgue spaces of real-valued Borel measurable functions.

**Definition 2.1.** For \(p \in [1, \infty]\), we denote by \(L^p([a,b])\) the vector space of Borel measurable functions \(\gamma: [a,b] \to \mathbb{R}\) which are \(p\)-integrable with respect to the Lebesgue-Borel measure \(\lambda\) (that is \(\int_a^b |\gamma(t)|^p \, dt < \infty\)), endowed with the seminorm

\[ \|\gamma\|_{L^p} := \left( \int_a^b |\gamma(t)|^p \, dt \right)^{1/p}. \]

Further, \(L^\infty([a,b])\) denotes the vector space of Borel measurable essentially bounded functions \(\gamma: [a,b] \to \mathbb{R}\), endowed with the seminorm

\[ \|\gamma\|_{L^\infty} := \text{ess sup}_{t \in [a,b]} |\gamma(t)|. \]

Setting \(N_p := \{ \gamma \in L^p([a,b]) : \|\gamma\|_{L^p} = 0 \}\), we obtain normed vector spaces \(L^p([a,b]) := L^p([a,b])/N_p\).

In [9], the author defines Lebesgue spaces of Borel measurable functions with values in Fréchet spaces as follows.
Definition 2.2. Let $E$ be a Fréchet space. For $p \in [1, \infty]$, the space $\mathcal{L}^p_B([a, b], E)$ is the vector space of Borel measurable functions $\gamma: [a, b] \to E$ such that $\gamma([a, b])$ is separable and $q \circ \gamma \in \mathcal{L}^p([a, b])$ for each continuous seminorm $q$ on $E$. The locally convex topology on $\mathcal{L}^p_B([a, b], E)$ is defined by the (countable) family of seminorms

$$
\|\|_{\mathcal{L}^p_B([a, b], E)} \to [0, \infty[, \quad \|\|_{\mathcal{L}^p_B,[a, b], E} := \|q \circ \gamma\|_{\mathcal{L}^p} = \left( \int_a^b q(\gamma(t))^p \, dt \right)^{\frac{1}{p}}.
$$

Further, the vector space $\mathcal{L}^\infty_B([a, b], E)$ consists of Borel measurable functions $\gamma: [a, b] \to E$ such that $\gamma([a, b])$ is separable and bounded. The locally convex topology on $\mathcal{L}^\infty_B([a, b], E)$ is defined by the (countable) family of seminorms

$$
\|\|_{\mathcal{L}^\infty_B([a, b], E)} \to [0, \infty[, \quad \|\|_{\mathcal{L}^\infty_B,[a, b], E} := \|q \circ \gamma\|_{\mathcal{L}^\infty} = \text{ess sup}_{t \in [a, b]} q(\gamma(t)).
$$

Setting $N_p := \{ \gamma \in \mathcal{L}^p_B([a, b], E) : \gamma(t) = 0 \text{ for a.e. } t \in [a, b] \}$ we obtain a Hausdorff locally convex space

$$
\mathcal{L}^p_B([a, b], E) := \mathcal{L}^p_B([a, b], E)/N_p,
$$

consisting of equivalence classes

$$
[\gamma] := \{ \check{\gamma} \in \mathcal{L}^p_B([a, b], E) : \check{\gamma}(t) = \gamma(t) \text{ for a.e. } t \in [a, b] \},
$$

with seminorms

$$
\|\|_{\mathcal{L}^p_B,[a, b], E} := \|\|_{\mathcal{L}^p_B,[a, b], E}.
$$

Remark 2.3. For locally convex spaces $E$ having the property that every separable closed vector subspace $S \subseteq E$ can be written as a union $S = \bigcup_{n \in \mathbb{N}} F_n$ of vector subspaces $F_1 \subseteq F_2 \subseteq \cdots$ which are Fréchet spaces in the induced topology (called (FEP)-spaces in [9]), the spaces $\mathcal{L}^p_B([a, b], E)$ and $\mathcal{L}^\infty_B([a, b], E)$ are constructed in [9] in the same way.

Definition 2.4. If $E$ is an arbitrary locally convex space, then the vector space $\mathcal{L}^\infty_B([a, b], E)$ consists of Borel measurable functions $\gamma: [a, b] \to E$ such that $\gamma([a, b])$ is compact and metrizable. The seminorms $\|\|_{\mathcal{L}^\infty_B}$ (as in Definition 2.2) define the locally convex topology on $\mathcal{L}^\infty_B([a, b], E)$.

Setting $N_c := \{ \gamma \in \mathcal{L}^\infty_B([a, b], E) : \gamma(t) = 0 \text{ for a.e. } t \in [a, b] \}$ we obtain a Hausdorff locally convex space

$$
\mathcal{L}^\infty_B([a, b], E) := \mathcal{L}^\infty_B([a, b], E)/N_c,
$$

consisting of equivalence classes

$$
[\gamma] := \{ \check{\gamma} \in \mathcal{L}^\infty_B([a, b], E) : \check{\gamma}(t) = \gamma(t) \text{ for a.e. } t \in [a, b] \},
$$

with seminorms

$$
\|\|_{\mathcal{L}^\infty_B,[a, b], E} := \|\|_{\mathcal{L}^\infty_B,[a, b], E}.
$$

Remark 2.5. Note that in [9], the author constructs all of the above Lebesgue spaces even in a more general form, consisting of Borel measurable functions $\gamma: X \to E$ defined on arbitrary measure spaces $(X, \Sigma, \mu)$. 

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As we do not need additional assumptions on the range space $E$ or on the images of Lusin measurable functions for them to build a vector space, we may construct Lebesgue spaces for arbitrary locally convex spaces. From now on, the measure will always be the Lebesgue-Borel measure $\lambda$ and we will call Lusin $\lambda$-measurable functions just measurable.

**Definition 2.6.** Let $E$ be a locally convex space and $p \in [1, \infty]$. We denote by $L^p([a,b],E)$ the vector space of measurable functions $\gamma : [a,b] \to E$ such that for each continuous seminorm $q$ on $E$ we have $q \circ \gamma \in L^p([a,b])$. We endow each $L^p([a,b],E)$ with the locally convex topology defined by the family of seminorms

$$\|\cdot\|_{L^p,q} : L^p([a,b],E) \to [0, \infty[, \quad \|\gamma\|_{L^p,q} := \|q \circ \gamma\|_{L^p}.$$  

The next lemma can be found in [7].

**Lemma 2.7.** Let $E$ be a locally convex space, $\gamma : [a,b] \to E$ be a measurable function. Then the following assertions are equivalent:

1. $\gamma(t) = 0$ a.e.,
2. $\alpha(\gamma(t)) = 0$ a.e., for each continuous linear functional $\alpha$ on $E$,
3. $q(\gamma(t)) = 0$ a.e., for each continuous seminorm $q$ on $E$.

**Definition 2.8.** For $p \in [1, \infty]$, from the above lemma follows that

$$N_p := \{ \gamma \in L^p([a,b],E) : \gamma(t) = 0 \text{ a.e.} \} = \{0\},$$

thus we obtain Hausdorff locally convex spaces

$$L^p([a,b],E) := L^p([a,b],E)/N_p$$

consisting of equivalence classes

$$[\gamma] := \{ \bar{\gamma} \in L^p([a,b],E) : \bar{\gamma}(t) = \gamma(t) \text{ a.e.} \},$$

with seminorms

$$\|\cdot\|_{L^p,q} : L^p([a,b],E) \to [0, \infty[, \quad \|\gamma\|_{L^p,q} := \|\gamma\|_{L^p,q}.$$  

**Remark 2.9.** For $1 \leq p \leq r \leq \infty$ we have

$$C([a,b],E) \subseteq L^\infty([a,b],E) \subseteq L^r([a,b],E) \subseteq L^p([a,b],E) \subseteq L^1([a,b],E)$$

with continuous inclusion maps, as for a continuous seminorm $q$ on $E$ we have

$$\|\gamma\|_{L^p,q} \leq (b-a)^{\frac{1}{p} - \frac{1}{r}} \|\gamma\|_{L^r,q}.$$  

(Here, $C([a,b],E)$ is endowed with the topology of uniform convergence, with continuous seminorms $\|\gamma\|_{L^\infty,q} = \|\gamma\|_{L^\infty,q}$)

We show that the Lebesgue spaces $L^p_B([a,b],E)$ constructed in [7] coincide with the Lebesgue spaces $L^p([a,b],E)$ defined in Definition 2.8.

**Proposition 2.10.** If $E$ is a Fréchet space, then $L^p_B([a,b],E) \cong L^p([a,b],E)$ as topological vector spaces, for each $p \in [1, \infty]$. 

Proof. Let $[\gamma] \in L^p_B([a, b], E)$. Then $\gamma: [a, b] \to E$ is Borel measurable and $\text{im}(\gamma)$ is separable. Then the co-restriction $\gamma|\text{im}(\gamma): [a, b] \to \text{im}(\gamma)$ is Borel measurable, hence measurable by Lemma [1.9] because the range is separable and metrizable, thus has a countable base. Then $\gamma: [a, b] \to E$ is measurable. Furthermore, $q \circ \gamma \in L^p([a, b])$ for each continuous seminorm $q$ on $E$, hence $[\gamma] \in L^p([a, b], E)$.

To show that the linear injective function

$$\Phi: L^p_B([a, b], E) \to L^p([a, b], E), \quad [\gamma] \mapsto [\gamma].$$

is an isomorphism of topological vector spaces, it suffices to prove surjectivity, as the continuity of $\Phi$ and $\Phi^{-1}$ will be obvious then.

Let $[\gamma] \in L^p([a, b], E)$. If $p \in [1, \infty)$, then consider a partition $[a, b] = N \cup \bigcup_{n \in \mathbb{N}} K_n$ and define $\bar{\gamma}: [a, b] \to E$ via

$$\bar{\gamma}(t) := \gamma(t), \text{ if } t \in \bigcup_{n \in \mathbb{N}} K_n, \quad \bar{\gamma}(t) := 0, \text{ if } t \in N.
$$

Then $\bar{\gamma}$ is Borel measurable (see Lemma [1.8]), the image $\text{im}(\bar{\gamma})$ is separable and $q \circ \bar{\gamma} \in L^p([a, b])$, thus $\bar{\gamma} \in L^p_B([a, b], E)$. If $p = \infty$, then there is some null set $N' \subseteq [a, b]$ such that $\gamma([a, b] \setminus N')$ is bounded. Consider a partition $[a, b] = N' \cup N \cup \bigcup_{n \in \mathbb{N}} K_n$ and define $\bar{\gamma}: [a, b] \to E$ via

$$\bar{\gamma}(t) := \gamma(t), \text{ if } t \in \bigcup_{n \in \mathbb{N}} K_n, \quad \bar{\gamma}(t) := 0, \text{ if } t \in N' \cup N.
$$

Again, $\bar{\gamma}$ is Borel measurable and the image $\text{im}(\bar{\gamma})$ is separable and bounded. Thus $\bar{\gamma} \in L^\infty_B([a, b], E)$. In any case, $\Phi([\gamma]) = [\gamma]$, hence $\Phi$ is surjective.

Remark 2.11. If $E$ is an (FEP)-space, then also $L^p_B([a, b], E) \cong L^p([a, b], E)$ as topological vector spaces. To see this, we only need to show that every $\gamma \in L^p_B([a, b], E)$ is measurable, the rest of the proof is identical to the above.

Since $\text{im}(\gamma)$ is separable, the vector subspace $\text{span}(\text{im}(\gamma))$ is separable and closed, hence there is an ascending sequence $F_1 \subseteq F_2 \subseteq \cdots$ of vector subspaces such that

$$\text{span}(\text{im}(\gamma)) = \bigcup_{n \in \mathbb{N}} F_n$$

and each $F_n$ is a separable Fréchet space (see [8] Lemma 1.39). Consider the sets $B_1 := \gamma^{-1}(F_1)$, $B_n := \gamma^{-1}(F_n \setminus F_{n-1})$ for $n \geq 2$. Then $[a, b]$ is a disjoint union of $(B_n)_{n \in \mathbb{N}}$, each $B_n \in \mathcal{B}([a, b])$ and $\gamma|_{B_n}: B_n \to F_n$ is Borel measurable, hence measurable by Lemma [1.9]. Therefore, $\gamma: [a, b] \to E$ is measurable.

Remark 2.12. If $E$ is an arbitrary locally convex space, then $L^\infty_B([a, b], E) \subseteq L^\infty([a, b], E)$. Again, it suffices to prove that each $\gamma \in L^\infty_B([a, b], E)$ is measurable. This is true (by Lemma [1.9]), since the closure of the image of $\gamma$ is compact and metrizable, hence has a countable base.

2.2 Mappings between Lebesgue spaces

The following results can be found in [8] 1.34, 1.35.
Lemma 2.13. Let $E, F$ be locally convex spaces and $f : E \to F$ be continuous and linear. If $\gamma \in \mathcal{L}^p([a,b], E)$ for $p \in [1, \infty]$, then $f \circ \gamma \in \mathcal{L}^p([a,b], F)$ and the map

$$
\mathcal{L}^p([a,b], f) : \mathcal{L}^p([a,b], E) \to \mathcal{L}^p([a,b], F), \quad \gamma \mapsto f \circ \gamma
$$

is continuous and linear.

Proof. From Remark 1.2 follows that $f \circ \gamma$ is measurable. Further, for every continuous seminorm $q$ on $F$, the composition $q \circ f$ is a continuous seminorm on $E$, whence $q \circ (f \circ \gamma) \in \mathcal{L}^p([a,b])$. Therefore $f \circ \gamma \in \mathcal{L}^p([a,b], F)$.

Since

$$
\|f \circ \gamma\|_{\mathcal{L}^p,q} = \|\gamma\|_{\mathcal{L}^p,q \circ f},
$$

the linear function $\mathcal{L}^p([a,b], f)$ is continuous. 

Remark 2.14. From Lemma 2.13 we can easily conclude that for locally convex spaces $E$ and $F$ we have

$$
\mathcal{L}^p([a,b], E \times F) \cong \mathcal{L}^p([a,b], E) \times \mathcal{L}^p([a,b], F)
$$
as locally convex spaces. In fact, the function

$$
\mathcal{L}^p([a,b], E \times F) \to \mathcal{L}^p([a,b], E) \times \mathcal{L}^p([a,b], F), \quad \gamma \mapsto (\text{pr}_1 \circ \gamma, \text{pr}_2 \circ \gamma)
$$
is continuous linear (where $\text{pr}_1, \text{pr}_2$ are the projections on the first, resp., second component of $E \times F$) and is a linear bijection with the continuous inverse

$$
\mathcal{L}^p([a,b], E) \times \mathcal{L}^p([a,b], F) \to \mathcal{L}^p([a,b], E \times F), \quad (\gamma_1, \gamma_2) \mapsto \lambda_1 \circ \gamma_1 + \lambda_2 \circ \gamma_2,
$$
where $\lambda_1 : E \to E \times F, x \mapsto (x, 0)$ and $\lambda_2 : F \to E \times F, y \mapsto (0, y)$ are continuous and linear.

As in [9, Remark 3.7], the following holds:

Lemma 2.15. Let $E$ be a locally convex space, let $a \leq \alpha < \beta \leq b$ and

$$
f : [c, d] \to [a, b], \quad f(t) := \alpha + \frac{t - c}{d - c} (\beta - \alpha).
$$

If $\gamma \in \mathcal{L}^p([a,b], E)$ for $p \in [1, \infty]$, then $\gamma \circ f \in \mathcal{L}^p([c,d], E)$ and the function

$$
\mathcal{L}^p(f, E) : \mathcal{L}^p([a,b], E) \to \mathcal{L}^p([c,d], E), \quad \gamma \mapsto \gamma \circ f
$$
is continuous and linear.

Proof. Note that the composition $\gamma \circ f$ is measurable (by [14, Theorem 3], the function $f$ has the property required in Lemma 1.3).

Assume first $p < \infty$. By [2, Satz 19.4], the function $q^p \circ (\gamma \circ f)$ is $p$-integrable for each continuous seminorm $q$ on $E$, and

$$
\int_c^d q(\gamma(f(t)))^p dt = \frac{d - c}{\beta - \alpha} \int_{f(c)}^{f(d)} q(\gamma(t))^p dt < \infty,
$$
(3)
hence $\gamma \circ f \in L^p([c,d], E)$. Furthermore, we see that
\[
\|\gamma \circ f\|_{L^p,q} \leq \left(\frac{d-c}{\beta-\alpha}\right)^{1/p} \|\gamma\|_{L^p,q},
\]
whence the linear function $L^p(f, E)$ is continuous.

Now, assume $p = \infty$. Then for every continuous seminorm $q$ on $E$ we have
\[
\text{ess sup}_{t \in [c,d]} q(\gamma(f(t))) \leq \text{ess sup}_{t \in [a,b]} q(\gamma(t)) < \infty,
\]
that is, $\gamma \circ f \in L^\infty([c,d], E)$ and
\[
\|\gamma \circ f\|_{L^\infty,q} \leq \|\gamma\|_{L^\infty,q},
\]
hence the linear map $L^\infty(f, E)$ is continuous and the proof is finished.

As in [9, 3.15] (called the ”locality axiom” there) the following holds:

**Lemma 2.16.** For any $a = t_0 < t_1 < \ldots < t_n = b$, the function
\[
\Gamma_E : L^p([a,b], E) \to \prod_{j=1}^n L^p([t_{j-1}, t_j], E), \quad [\gamma] \mapsto ([\gamma|_{[t_{j-1}, t_j]}])_{j=1,\ldots,n}
\]
is an isomorphism of topological vector spaces.

**Proof.** The function $\Gamma_E$ is defined and continuous, by Lemma 2.15. For surjectivity, let $([\gamma_1], \ldots, [\gamma_n]) \in \prod_{j=1}^n L^p([t_{j-1}, t_j], E)$ and define $\gamma : [a,b] \to E$ via
\[
\gamma(t) := \gamma_j(t), \text{ if } t \in [t_{j-1}, t_j], \quad \gamma(t) := \gamma_n(t), \text{ if } t \in [t_{n-1}, t_n].
\]
The obtained map $\gamma$ is measurable (see Remark 1.5) and for Borel measurable $\bar{\gamma}$ as in Lemma 1.8 and $p < \infty$ we have
\[
\int_a^b q(\bar{\gamma}(t))^p \, dt = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} q(\gamma_j(t))^p \, dt < \infty.
\]
Further, for $p = \infty$ we have
\[
\text{ess sup}_{t \in [a,b]} q(\bar{\gamma}(t)) = \max_{j=1,\ldots,n} \text{ess sup}_{t \in [t_{j-1}, t_j]} q(\gamma_j(t)) < \infty.
\]
Thus, in any case $[\gamma] \in L^p([a,b], E)$. As $\Gamma_E([\gamma]) = ([\gamma_1], \ldots, [\gamma_n])$, the function $\Gamma_E$ is surjective. As it is obviously injective and linear, $\Gamma_E$ is a continuous isomorphism of vector spaces and the continuity of the inverse $\Gamma_E^{-1}$ follows easily from Equations (6), resp., (7).

Furthermore, the $L^p$-spaces have the subdivision property ([9, Lemma 5.26]).

**Lemma 2.17.** Let $E$ be a locally convex space, let $\gamma \in L^p([0,1], E)$. For $n \in \mathbb{N}$ and $k \in \{0, \ldots, n-1\}$ define
\[
\gamma_{n,k} : [0,1] \to E, \quad \gamma_{n,k}(t) := \frac{1}{n} \gamma\left(\frac{k+t}{n}\right).
\]
Then $\gamma_{n,k} \in \mathcal{L}^p([0,1], E)$ for every $n, k$ and

$$
\lim_{n \to \infty} \max_{k \in \{0, \ldots, n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^p,q} = 0
$$

for each continuous seminorm $q$ on $E$.

More generally, the same holds for $\gamma \in \mathcal{L}^p([a,b], E)$ and

$$
\gamma_{n,k} : [a,b] \to E, \quad \gamma_{n,k}(t) := \frac{1}{n} \left( a + \frac{k(b-a)+t-a}{n} \right).
$$

**Proof.** The functions $f_{n,k} : [0,1] \to [k/n, k+1/n]$, $f_{n,k}(t) := k+t/n$ are as in Lemma 2.15, hence $\gamma_{n,k} = 1/n (\gamma \circ f_{n,k}) \in \mathcal{L}^p([0,1], E)$.

Further, for fixed $n \in \mathbb{N}$ and $p = \infty$ we have

$$
\|\gamma_{n,k}\|_{\mathcal{L}^\infty,q} = \frac{1}{n} \|\gamma \circ f_{n,k}\|_{\mathcal{L}^\infty,q} \leq \frac{1}{n} \|\gamma\|_{\mathcal{L}^\infty,q}
$$

for every continuous seminorm $q$ on $E$ and every $k \in \{0, \ldots, n-1\}$, by (5). Hence

$$
\max_{k \in \{0, \ldots, n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^\infty,q} \leq \frac{1}{n} \|\gamma\|_{\mathcal{L}^\infty,q} \to 0 \text{ as } n \to \infty.
$$

Now, if $2 \leq p < \infty$, then for $n \in \mathbb{N}$ and a continuous seminorm $q$ on $E$ we have

$$
\|\gamma_{n,k}\|_{\mathcal{L}^p,q} = \frac{1}{n} \|\gamma \circ f_{n,k}\|_{\mathcal{L}^p,q} \leq \frac{n^p}{n^p} \|\gamma\|_{\mathcal{L}^p,q} = n^{p-1} \|\gamma\|_{\mathcal{L}^p,q}
$$

for each $k \in \{0, \ldots, n-1\}$, by (6). Hence

$$
\max_{k \in \{0, \ldots, n-1\}} \|\gamma_{n,k}\|_{\mathcal{L}^p,q} \leq n^{p-1} \|\gamma\|_{\mathcal{L}^p,q} \to 0 \text{ as } n \to \infty.
$$

Finally, let $p = 1$. Fix some $\varepsilon > 0$ and a continuous seminorm $q$ on $E$. Each of the sets

$$
A_m := \{ t \in [a,b] : q(\gamma(t)) > m \}
$$

are in $\mathcal{B}([0,1])$ and

$$
\lim_{m \to \infty} \int_{A_m} q(\gamma(t)) \, dt = \int_{\bigcap_{m \in \mathbb{N}} A_m} q(\gamma(t)) \, dt = 0,
$$

because $(A_m)_{m \in \mathbb{N}}$ is a decreasing sequence and $\bigcap_{m \in \mathbb{N}} A_m = \emptyset$. Therefore, for some $m \in \mathbb{N}$ we have

$$
\int_{A_m} q(\gamma(t)) \, dt < \frac{\varepsilon}{2}.
$$

We fix some $N \in \mathbb{N}$ such that $m/N < \varepsilon/2$ and for every $n \geq N$ we define

$$
A_{n,k} := \{ t \in [0,1] : f_{n,k}(t) \in A_m \}.
$$
Then
\[
\int_{A_{n,k}} q(\gamma_{n,k}(t)) \, dt = \frac{1}{n} \int_{A_{n,k}} q(\gamma(f_{n,k}(t))) \, dt = \int_{f_{n,k}(A_{n,k})} q(\gamma(t)) \, dt,
\]
by Equation (3). Since \(f_{n,k}(A_{n,k}) = A_m \cap [k/n, k+1/n]\), we obtain
\[
\int_{f_{n,k}(A_{n,k})} q(\gamma(t)) \, dt \leq \int_{A_m} q(f(t)) \, dt < \frac{\varepsilon}{2},
\]
by the choice of \(m\). Further
\[
\|\gamma_{n,k}\|_{L^1,q} = \int_0^1 q(\gamma_{n,k}(t)) \, dt = \int_{A_{n,k}} q(\gamma_{n,k}(t)) \, dt + \int_{[0,1]\setminus A_{n,k}} q(\gamma_{n,k}(t)) \, dt < \varepsilon,
\]
because \(q(\gamma_{n,k}(t)) = 1/qn(\gamma(f_{n,k}(t))) \leq \frac{m}{n} < \frac{\varepsilon}{2}\) for \(t \in [0,1]\setminus A_{n,k}\). Consequently,
\[
\max_{k \in \{0,...,n-1\}} \|\gamma_{n,k}\|_{L^1,q} < \varepsilon,
\]
in other words, \(\max_{k \in \{0,...,n-1\}} \|\gamma_{n,k}\|_{L^1,q} \to 0\) as \(n \to \infty\), as required.

The following result can be found in [3, Lemma 2.1] (for suitable vector spaces).

Lemma 2.18. Let \(X\) be a topological space, \(U \subseteq X\) be an open subset and \(E\), \(F\) be locally convex spaces. Let \(f : U \times E \to F\) be continuous and linear in the second argument. If \(\eta \in C([a,b], U)\) and \(\gamma \in L^p([a,b], E)\) for \(p \in [1, \infty]\), then \(f \circ (\eta, \gamma) \in L^p([a,b], F)\).

Proof. By Lemma 1.6 and Remark 1.2, the composition \(f \circ (\eta, \gamma)\) is a measurable function.

Now, consider the continuous function
\[
h_\eta : [a, b] \times E \to F; \quad h_\eta(t, v) := f(\eta(t), v).
\]
Let \(q\) be a continuous seminorm on \(E\). Then \(h_\eta([a, b] \times \{0\}) = \{0\} \subseteq B_1^1(0)\), thus \([a, b] \times \{0\} \subseteq V\), where \(V := h_\eta^{-1}(B_1^1(0))\) is an open subset of \([a,b] \times E\). Using the Wallace Lemma, we find an open subset \(W \subseteq E\) such that \([a, b] \times \{0\} \subseteq [a, b] \times W \subseteq V\). Then there is a continuous seminorm \(\pi\) on \(E\) such that
\[
[a, b] \times \{0\} \subseteq [a, b] \times B_1^1(0) \subseteq [a, b] \times W \subseteq V.
\]
We show that for each \((t, v) \in [a, b] \times E\) we have \(q(h_\eta(t, v)) \leq \pi(v)\). In fact, if \(\pi(v) > 0\), then (using the linearity of \(f\) in \(v\)) we have \((1/\pi(v))q(h_\eta(t, v)) = q(h_\eta(t, (1/\pi(v))v)) \leq 1\). If \(\pi(v) = 0\), then for each \(r > 0\) we have \(rv \in B_1^1(0)\), whence \(rq(h_\eta(t, v)) = q(h_\eta(t, rv)) \leq 1\), that is \(q(h_\eta(t, v)) \leq 1/r\), consequently \(q(h_\eta(t, v)) = 0 = \pi(v)\).

Now, if \(p < \infty\), then
\[
\int_a^b q(f(\eta(t), \gamma(t)))^p \, dt = \int_a^b q(h_\eta(t, \gamma(t)))^p \, dt \leq \int_a^b \pi(\gamma(t))^p \, dt < \infty,
\]
In other words, $\tilde{f}$ is continuous.

Proof. If $p = \infty$, then $q(f(\eta(t), \gamma(t))) \leq \pi(\gamma(t))$, whence

$$\text{ess sup}_{t \in [a, b]} q(f(\eta(t), \gamma(t))) \leq \text{ess sup}_{t \in [a, b]}(\pi(\gamma(t))) < \infty,$$

thus $q \circ (f \circ (\eta, \gamma)) \in \mathcal{L}^p([a, b])$.

The following lemma ([9] Lemma 2.4) will be used in the proof of Proposition 2.23.

**Lemma 2.19.** Let $E_1$, $E_2$, $E_3$ and $F$ be locally convex spaces, $U \subseteq E_1$, $V \subseteq E_2$ be open subsets and the function $f : U \times V \times E_3 \to F$ be a $C^1$-function and linear in the third argument. Then the function

$$\tilde{f} : U \times C([a, b], V) \times L^p([a, b], E_3) \to L^p([a, b], F),$$

$$(u, \eta, [\gamma]) \mapsto [f(u, \bullet) \circ (\eta, \gamma)]$$

is continuous. (Here $C([a, b], V)$ is endowed with the topology of uniform convergence.)

Proof. Fix some $(\tilde{u}, \tilde{\eta}, [\tilde{\gamma}]) \in U \times C([a, b], V) \times L^p([a, b], E_3)$ and let $q$ be a continuous seminorm on $F$. The subset $K := \{\tilde{u}\} \times \tilde{\eta}([a, b]) \subseteq U \times V$ is compact, hence from Lemma [9] Lemma 1.61 follows that there are seminorms $\pi$ on $E_1 \times E_2$ and $\pi_3$ on $E_3$ such that $K + B_1^\pi(0) \subseteq U \times V$ and

$$q(f(u, v, w) - f(u', v', w')) \leq \pi_3(w - w') + \pi(u - u', v - v')\pi_3(w')$$

for all $(u, v), (u', v') \in K + B_1^\pi(0)$, $w, w' \in E_3$. We may assume that $\pi(x, y) = \max\{\pi_1(x), \pi_2(y)\}$ for some continuous seminorms $\pi_1$ on $E_1$, $\pi_2$ on $E_2$. Then, setting

$$U_0 := B_1^{\pi_1}(\tilde{u}), \quad V_0 := \tilde{\eta}([a, b]) + B_1^{\pi_2}(0),$$

we define an open neighborhood

$$\Omega := U_0 \times C([a, b], V_0) \times L^p([a, b], E_3)$$

of $(\tilde{u}, \tilde{\eta}, [\tilde{\gamma}])$ and show that if $(u, \eta, [\gamma]) \to (\tilde{u}, \tilde{\eta}, [\tilde{\gamma}])$ in $\Omega$, then $\tilde{f}(u, \eta, [\gamma]) \to \tilde{f}(\tilde{u}, \tilde{\eta}, [\tilde{\gamma}])$ in $L^p([a, b], E_3)$, because

$$\|\tilde{f}(u, \eta, [\gamma]) - \tilde{f}(\tilde{u}, \tilde{\eta}, [\tilde{\gamma}])\|_{L^p, q} \leq \|(\gamma - \tilde{\gamma})\|_{L^p, \pi_3} + \max\{\pi_1(u - \tilde{u}), \|\eta - \tilde{\eta}\|_{\infty, \pi_2}\}\|\tilde{\gamma}\|_{L^p, \pi_3} \to 0.$$

In other words, $\tilde{f}$ is continuous in $(\tilde{u}, \tilde{\eta}, [\tilde{\gamma}])$.

Before investigating differentiable mappings between Lebesgue spaces, we recall some details concerning the concept of the differentiability on locally convex spaces. The concept we work with goes back to Bastiani [1] and is well known as Keller’s $C^1_k$-calculus [11]. The results below can be found, for example, in [8], [9] and many others.
Definition 2.20. Let $E$, $F$ be locally convex spaces, let $f : U \to F$ be a function defined on an open subset $U \subseteq E$. The map $f$ is called a $C^0$-map if it is continuous; it is called a $C^1$-map if it is $C^0$ and for every $x \in U$, $y \in E$ the directional derivative

$$df(x, y) := \lim_{h \to 0} \frac{f(x + hy) - f(x)}{h}$$

exists in $F$ and the obtained differential

$$df : U \times E \to F, \quad (x, y) \mapsto df(x, y)$$

is continuous. Further, for $k \geq 2$, a continuous map $f$ is called $C^k$ if it is $C^1$ and $df$ is $C^{k-1}$. Finally, a continuous function $f$ is called a $C^\infty$-map or smooth, if $f$ is $C^k$ for every $k \in \mathbb{N}$.

If $f : [a, b] \to F$ is a function defined on an interval $[a, b] \subseteq \mathbb{R}$, then $f$ is called $C^0$ if it is continuous, and it is called $C^1$ if for every $x \in [a, b]$ the (possibly one-sided) limit

$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists and the obtained derivative

$$f' : [a, b] \to F$$

is continuous.

Several properties of differentiable functions will be used repeatedly without further mention.

Remark 2.21. (i) Each continuous, linear function is smooth.

(ii) The differential $df : U \times E \to F$ of a $C^1$-map is linear in the second argument.

(iii) If $f : U \to F$, $g : F \to G$ are $C^k$-maps between locally convex spaces, then the composition $g \circ f : U \to G$ is $C^k$. Furthermore, if $g$ is a linear topological embedding such that $g(F)$ is closed in $G$ and if $g \circ f : U \to G$ is $C^k$, then $f$ is $C^k$.

(iv) A function $f : U \to \prod_{j \in J} F_j$ (where $F_j$ are locally convex spaces) is $C^k$ if and only if each of the components $\text{pr}_j \circ f : U \to F_j$ is $C^k$.

Remark 2.22. For an open subset $U \subseteq E$ and a $C^1$-map $f : U \to F$ define

the open subset

$$U^{[1]} := \{(x, y, h) \in U \times E \times \mathbb{R} : x + hy \in U\}$$

and the function $f^{[1]} : U^{[1]} \to F$ via

$$f^{[1]}(x, y, h) := \frac{f(x + hy) - f(x)}{h}, \quad \text{for } h \neq 0,$$
and
\[ f^{[1]}(x, y, 0) = df(x, y). \]

Then \( f^{[1]} \) is continuous.

Conversely, if for \( f: U \to F \) there is a continuous map \( f^{[1]}: U^{[1]} \to F \) such that \( f^{[1]}(x, y, h) = \frac{f(x + b y) - f(x)}{h} \) for \( h \neq 0 \), then \( f \) is a \( C^1 \)-function and \( df(x, y) = f^{[1]}(x, y, 0). \)

Returning to our theory, we show that as in [9 Proposition 2.3], the following holds:

**Proposition 2.23.** Let \( E_1, E_2, F \) be locally convex spaces, let \( V \subseteq E_1 \) be open and the function \( f: V \times E_2 \to F \) be \( C^{k+1} \) for \( k \in \mathbb{N} \cup \{0, \infty\} \) and linear in the second argument. Then for \( p \in [1, \infty) \) the function
\[ \Theta_f: C([a, b], V) \times L^p([a, b], E_2) \to L^p([a, b], F), \]
\[ (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)] \]
is \( C^k \).

**Proof.** For \( k = 0 \), the assertion holds by Lemma 2.19. Further, we may assume \( k < \infty \) and proceed by induction.

**Induction start:** \( k = 1 \). The map \( \Theta_f \) is continuous by the previous step; we show that for all \( (\eta, [\gamma]) \in C([a, b], V) \times L^p([a, b], E_2) \) and \( (\bar{\eta}, [\bar{\gamma}]) \in C([a, b], E_1) \times L^p([a, b], E_2) \) the directional derivative
\[ d(\Theta_f)(\eta, [\gamma], \bar{\eta}, [\bar{\gamma}]) := \lim_{h \to 0} \frac{\Theta_f(\eta + h\bar{\eta}, [\gamma + h\bar{\gamma}]) - \Theta_f(\eta, [\gamma])}{h} \]
exists in \( L^p([a, b], F) \) and equals \( df \circ (\eta, \gamma, \bar{\eta}, \bar{\gamma}) \).

Given \( \eta, [\gamma], \bar{\eta}, [\bar{\gamma}] \) as above, we note that \( \eta([a, b]) \) is a compact subset of the open subset \( V \subseteq E_1 \), thus there exists an open \( 0 \)-neighborhood \( U \subseteq E_1 \) such that \( \eta([a, b]) + U \subseteq V \). Further, there is some balanced \( 0 \)-neighborhood \( W \subseteq U \) such that \( W + W \subseteq U \). As \( \eta([a, b]) \) is bounded in \( E_1 \) (being compact), for some \( \varepsilon > 0 \) we have \( \bar{\eta}([a, b]) \subseteq \frac{1}{\varepsilon} W \). In this manner we obtain an open subset
\[ \Omega := ] - \varepsilon, \varepsilon [ \times (\eta([a, b]) + W) \times \frac{1}{\varepsilon} W \times E_2 \times E_2 \subseteq \mathbb{R} \times V \times E_1 \times E_2 \times E_2 \]
for which holds \( ] - \varepsilon, \varepsilon [ \times \eta([a, b]) \times \bar{\eta}([a, b]) \times \gamma([a, b]) \times \bar{\gamma}([a, b]) \subseteq \Omega \) and for all \( (t, w, \bar{w}, x, \bar{x}) \in \Omega \) we have \( (w + t\bar{w}, x + t\bar{x}) \in V \times E_2 \) (that is, \( \Omega \) is contained in \( V \times E_2 \)) constructed as in Remark 2.22.

Now, for \( f^{[1]}: (V \times E_2)^{[1]} \to F \) the function
\[ \Omega \to F, \quad (t, w, \bar{w}, x, \bar{x}) \mapsto f^{[1]}(w, x, \bar{w}, \bar{x}, t) \]
is \( C^1 \) and linear in \((x, \bar{x})\), thus from Lemma 2.19 follows (identifying the space \( L^p([a, b], E_2 \times E_2) \) with \( L^p([a, b], E_2) \times L^p([a, b], E_2) \), see Remark 2.14) that
\[ (t, \varphi, \bar{\varphi}, [\psi], [\bar{\psi}]) \mapsto [f^{[1]}(\bullet, t) \circ (\varphi, \psi, \bar{\varphi}, \bar{\psi})] \in L^p([a, b], F) \]
is continuous on
\[ ] - \varepsilon, \varepsilon [ \times C([a, b], \eta([a, b]) + W) \times C([a, b], 1/\varepsilon W) \times L^p([a, b], E_2) \times L^p([a, b], E_2). \]
Hence

\[ -\varepsilon, \varepsilon \to L^p([a, b], F), \quad t \mapsto [f^1(\bullet, t) \circ (\eta, \gamma, \bar{\eta}, \bar{\gamma})] \]

is continuous. It follows

\[
d(\Theta_f)(\eta, [\gamma], \bar{\eta}, [\bar{\gamma}]) = \lim_{h \to 0} \frac{1}{h} (\Theta_f(\eta + t\bar{\eta}, [\gamma + t\bar{\gamma}]) - \Theta_f(\eta, [\gamma]))
\]

\[
= \lim_{h \to 0} \frac{1}{h} ([f \circ (\eta + t\bar{\eta}, \gamma + t\bar{\gamma})] - [f \circ (\eta, \gamma)])
\]

\[
= \lim_{h \to 0} [f^1(\bullet, h) \circ (\eta, \gamma, \bar{\eta}, \bar{\gamma})]
\]

\[
= [f^1(\bullet, 0) \circ (\eta, \gamma, \bar{\eta}, \bar{\gamma})] = [df \circ (\eta, \gamma, \bar{\eta}, \bar{\gamma})]
\]

in \(L^p([a, b], F)\).

It remains to show that

\[
d(\Theta_f) : C([a, b], V) \times L^p([a, b], E_2) \times C([a, b], E_1) \times L^p([a, b], E_2) \to L^p([a, b], F)
\]

is continuous. But as the function

\[
V \times E_1 \times E_2 \times E_2 \to F, \quad (w, \bar{w}, x, \bar{x}) \mapsto df(w, x, \bar{w}, \bar{x}) \tag{9}
\]

is \(C^1\) and linear in \((x, \bar{x})\), by induction start

\[
C([a, b], V) \times C([a, b], E_1) \times L^p([a, b], E_2) \times L^p([a, b], E_2) \to L^p([a, b], F),
\]

\[
(\varphi, \bar{\varphi}, [\psi], [\bar{\psi}]) \mapsto [df \circ (\varphi, \bar{\varphi}, \psi, \bar{\psi})]
\]

is continuous (we identify the \(L^p\)-spaces again, as above), hence \(d(\Theta_f)\) is continuous. Therefore, \(\Theta_f\) is \(C^1\).

**Induction step:** Now, assume that \(f\) is \(C^{k+2}\). Then \(\Theta_f\) is \(C^1\) by induction start and \(df\) is \(C^{k+1}\). Then the map in (9) is \(C^{k+1}\) and linear in \((x, \bar{x})\), hence by induction hypothesis, the map \((\varphi, \bar{\varphi}, [\psi], [\bar{\psi}]) \mapsto [df \circ (\varphi, \bar{\varphi}, \psi, \bar{\psi})] = d(\Theta_f)(\varphi, [\psi], \bar{\varphi}, [\bar{\psi}])\) is \(C^k\). Hence \(\Theta_f\) is \(C^{k+1}\). □

### 2.3 Integrable \(L^p\)-functions

**Definition 2.24.** Let \(E\) be a locally convex space and let \(\gamma : [a, b] \to E\) be such that \(\alpha \circ \gamma \in L^1([a, b])\) for every continuous linear form \(\alpha \in E'\). If there exists some \(w \in E\) such that

\[
\alpha(w) = \int_a^b \alpha(\gamma(t)) \, dt
\]

for every \(\alpha\), then \(w\) is called the **weak integral of \(\gamma\) from \(a\) to \(b\)**, and we write \(\int_a^b \gamma(t) \, dt := w\). As the continuous linear forms separate the points on \(E\), the weak integral of a function \(\gamma\) is unique if it exists.

**Remark 2.25.** Since \(|\alpha|\) is a continuous seminorm on \(E\) for \(\alpha \in E'\), each \(\gamma \in L^1([a, b], E)\) satisfies the condition \(\alpha \circ \gamma \in L^1([a, b])\). Further, if \(\int_a^b \gamma(t) \, dt\) exists in \(E\), then for every continuous seminorm \(q\) on \(E\) we have

\[
q\left(\int_a^b \gamma(t) \, dt\right) \leq \int_a^b q(\gamma(t)) \, dt.
\]
In [3, Lemma 1.19, Lemma 1.23 and Lemma 1.43], the author proves that each $\gamma \in \mathcal{L}^1([a,b], E)$, resp. $\gamma \in \mathcal{L}^\infty_c([a,b], E)$, has a weak integral in $E$ for suitable spaces $E$. We show below that sequential completeness of $E$ suffices for each $\mathcal{L}^1$-function to be weak integrable.

**Proposition 2.26.** Let $E$ be a sequentially complete locally convex space. Then each $\gamma \in \mathcal{L}^1([a,b], E)$ has a weak integral $\int_a^b \gamma(t) \, dt \in E$.

Further, the function

$$\eta: [a,b] \to E, \quad \eta(t) := \int_a^t \gamma(s) \, ds$$

is continuous.

**Proof.** As $\gamma$ is measurable, pick a disjoint sequence $(K_n)_{n \in \mathbb{N}}$ of compact sets $K_n \subseteq [a,b]$ such that $\gamma|_{K_n}$ is continuous and $\lambda([a,b] \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0$. Then for each $n \in \mathbb{N}$ the weak integral $\int_{K_n} \gamma(t) \, dt$ exists in $E$ (by [15, 3.27 Theorem]). Then

$$\sum_{n=1}^\infty q \left( \int_{K_n} \gamma(t) \, dt \right) \leq \sum_{n=1}^\infty \int_{K_n} q(\gamma(t)) \, dt = \int_a^b q(\gamma(t)) \, dt < \infty,$$

that is, the series $\sum_{n=1}^\infty \int_{K_n} \gamma(t) \, dt$ is absolutely convergent in $E$, hence convergent, as $E$ is assumed sequentially complete.

Next, we see that $\sum_{n=1}^\infty \int_{K_n} \gamma(t) \, dt = \int_a^b \gamma(t) \, dt$, because for every continuous linear form $\alpha \in E'$ we have

$$\alpha \left( \sum_{n=1}^\infty \int_{K_n} \gamma(t) \, dt \right) = \sum_{n=1}^\infty \alpha \left( \int_{K_n} \gamma(t) \, dt \right) = \sum_{n=1}^\infty \int_{K_n} \alpha(\gamma(t)) \, dt = \int_a^b \alpha(\gamma(t)) \, dt.$$

To prove the continuity of $\eta$ in every $t \in [a,b]$, let $q$ be a continuous seminorm on $E$ and let $\varepsilon > 0$. Then there exists some $\delta > 0$ such that whenever $|t-r| < \delta$, we have $\int_r^t q(\gamma(s)) \, ds < \varepsilon$ (follows from the classical Fundamental Theorem of Calculus, see [3, VII. 4.14]). Therefore

$$q(\eta(t) - \eta(r)) = q \left( \int_a^t \gamma(s) \, ds - \int_a^r \gamma(s) \, ds \right) = q \left( \int_r^t \gamma(s) \, ds \right) \leq \int_r^t q(\gamma(s)) \, ds < \varepsilon,$$

whence $\eta$ is continuous in $t$.

The differentiability (almost everywhere) of $\eta$ defined in (10) is shown in [3, Lemma 1.28] in the Fréchet case. In the next proposition, we get (as in [4, §5]) a similar result for $E$ merely metrizable.

**Proposition 2.27.** Let $E$ be a metrizable locally convex space, let $\gamma \in \mathcal{L}^1([a,b], E)$. If the function

$$\eta: [a,b] \to E, \quad \eta(t) := \int_a^t \gamma(s) \, ds$$

is continuous.
is everywhere defined, then \( \eta \) is continuous and for almost every \( t \in [a, b] \) the derivative \( q'(t) \) exists and equals \( \gamma(t) \).

**Proof.** Note that the continuity of \( \gamma \) can be shown as in the proof of Proposition 2.26. We may assume that \( \gamma(t) = 0 \) for each \( t \notin \bigcup_{n \in \mathbb{N}} K_n \), where \( (K_n)_{n \in \mathbb{N}} \) is a sequence as in Definition 1.1. Our aim is to show that for almost every \( t \in [a, b] \) the difference quotient

\[
\frac{1}{r}(\eta(t + r) - \eta(t)) = \frac{1}{r} \left( \int_a^{t+r} \gamma(s) \, ds - \int_a^{t} \gamma(s) \, ds \right) = \frac{1}{r} \int_t^{t+r} \gamma(s) \, ds
\]

tends to \( \gamma(t) \) as \( r \to 0 \). That is, for every \( \varepsilon > 0 \) and continuous seminorm \( q_m \) on \( E \) we have

\[
q_m \left( \frac{1}{r} \int_t^{t+r} \gamma(s) \, ds - \gamma(t) \right) = q_m \left( \frac{1}{r} \int_t^{t+r} \gamma(s) - \gamma(t) \, ds \right) < \varepsilon
\]

for \( r \neq 0 \) small enough.

We fix some \( \varepsilon > 0 \) and some continuous seminorm \( q_m \). The set \( \gamma([a, b]) = \{0\} \cup \bigcup_{n \in \mathbb{N}} \gamma(K_n) \subseteq E \) is separable, say \( \gamma([a, b]) = \{a_k : k \in \mathbb{N} \} \). Thus for every \( t \in [a, b] \) we find some \( a_m(t) \) such that

\[
q_m(\gamma(t) - a_m(t)) < \frac{1}{3} \varepsilon,
\]

hence for every \( r \neq 0 \) small enough we have

\[
\frac{1}{r} \int_t^{t+r} q_m(\gamma(t) - a_m(t)) \, ds < \frac{1}{3} \varepsilon.
\]

Furthermore, each of the functions

\[
h_{k,m} : [a, b] \to \mathbb{R}, \quad h_{k,m}(t) := q_m(\gamma(t) - a_k)
\]

is in \( \mathcal{L}^1([a, b]) \), hence by the classical Fundamental Theorem of Calculus (see [6, VII, 4.14]) there exist some sets \( N_{k,m} \subseteq [a, b] \) such that \( \lambda(N_{k,m}) = 0 \) and for every \( t \notin N_{k,m} \) we have

\[
\left| \frac{1}{r} \int_t^{t+r} q_m(\gamma(s) - a_k) \, ds - q_m(\gamma(t) - a_k) \right| < \frac{1}{3} \varepsilon.
\]

for \( r \neq 0 \) small enough.

Consequently, for \( t \notin \bigcup_{m \in \mathbb{N}} N_{k,m} \) we have

\[
\left| \frac{1}{r} \int_t^{t+r} q_m(\gamma(s) - \gamma(t)) \, ds \right| \leq \left| \frac{1}{r} \int_t^{t+r} q_m(\gamma(s) - \gamma(t)) - q_m(\gamma(s) - a_m(t)) \, ds \right| + \left| \frac{1}{r} \int_t^{t+r} q_m(\gamma(s) - a_m(t)) \, ds \right|
\]

\[
+ \left| \frac{1}{r} \int_t^{t+r} q_m(\gamma(t) - a_m(t)) \, ds \right| + \left| \frac{1}{r} \int_t^{t+r} q_m(\gamma(s) - \gamma(t)) - q_m(\gamma(s) - a_m(t)) \, ds \right|
\]

\[
< \frac{1}{r} \int_t^{t+r} q_m(\gamma(s) - \gamma(t)) \, ds + \frac{1}{3} \varepsilon + \frac{1}{3} \varepsilon,
\]

\[\vdots\]
using the estimates in (13) and (12). Finally,

\[ \left| \frac{1}{r} \int_{t}^{t+r} q_m(\gamma(s) - \gamma(t)) - q_m(\gamma(s) - a_m(t)) \, ds \right| \leq \frac{1}{|r|} \int_{t}^{t+r} |q_m(\gamma(s) - \gamma(t)) - q_m(\gamma(s) - a_m(t))| \, ds \leq \frac{1}{|r|} \int_{t}^{t+r} q_m(\gamma(s) - \gamma(t) - \gamma(s) + a_m(t)) \, ds \]

\[ = \frac{1}{|r|} \int_{t}^{t+r} q_m(\gamma(t) - a_m(t)) \, ds < \frac{1}{3} \varepsilon, \]

using (12) again.

Altogether, we have

\[ q_m \left( \frac{1}{r} \int_{t}^{t+r} \gamma(s) - \gamma(t) \, ds \right) = \frac{1}{|r|} q_m \left( \int_{t}^{t+r} \gamma(s) - \gamma(t) \, ds \right) \leq \frac{1}{|r|} \int_{t}^{t+r} q_m(\gamma(s) - \gamma(t)) \, ds \leq \frac{1}{|r|} \int_{t}^{t+r} q_m(\gamma(s) - \gamma(t)) \, ds < \varepsilon \]

by the above. Thus the desired estimate (11) holds for each \( t \notin \bigcup_{m \in N} N_{k,m} \) and \( \lambda(\bigcup_{k,m \in N} N_{k,m}) = 0 \), whence the proof is finished.

Even if \( E \) is not metrizable (so that \( \eta' \) does not necessarily exist, not even almost everywhere), we show that (as in [9]) the equivalence class \([\gamma]\) is still uniquely determined by \( \eta \).

**Lemma 2.28.** Let \( E \) be a sequentially complete locally convex space and let \( \gamma \in L^1([a,b], E) \). If \( \int_{a}^{t} \gamma(s) \, ds = 0 \) for all \( t \in [a, b] \), then \( \gamma(s) = 0 \) for almost all \( s \in [a, b] \).

**Proof.** Let \( \alpha \) be a continuous linear functional on \( E \). Then we have

\[ \int_{a}^{t} (\alpha \circ \gamma)(s) \, ds = \alpha \left( \int_{a}^{t} \gamma(s) \, ds \right) = 0 \]

for every \( t \in [a, b] \). From the Fundamental Theorem of Calculus (see [3] VII. 4.14) follows that \( (\alpha \circ \gamma)(t) = 0 \) for almost every \( t \in [a, b] \). As \( \alpha \in E' \) was arbitrary, from Lemma 2.7 follows that \( \gamma(t) = 0 \) for almost every \( t \). \( \square \)

### 3 AC-functions

#### 3.1 Vector-valued AC-functions

The vector spaces \( AC_{L^p}([a, b], E) \) are defined similarly to [9] Definition 3.6.

**Definition 3.1.** Let \( E \) be a sequentially complete locally convex space. For \( p \in [1, \infty] \) we denote by \( AC_{L^p}([a, b], E) \) the vector space of continuous functions
Lemma 3.3. Let \( E \) be a sequentially complete locally convex space and endow \( AC_{L^p}([a, b], E) \) with the topology of uniform convergence. Then for \( p \in [1, \infty] \) the inclusion map

\[
\text{incl}: AC_{L^p}([a, b], E) \to C([a, b], E), \quad \eta \mapsto \eta
\]

is continuous.

**Proof.** Let \( \eta \in AC_{L^p}([a, b], E) \) and denote \( \eta' = [\gamma] \). For a continuous seminorm \( q \) on \( E \) and \( t \in [a, b] \) we have

\[
q(\eta(t)) = q\left(\eta(t_0) + \int_{t_0}^{t} \gamma(s) \, ds\right) \leq q(\eta(t_0)) + q\left(\int_{t_0}^{t} \gamma(s) \, ds\right)
\]

\[
\leq q(\eta(t_0)) + \int_{t_0}^{t} q(\gamma(s)) \, ds = q(\eta(t_0)) + \|\gamma\|_{L^1,q}
\]

\[
\leq q(\eta(t_0)) + (b - a)^{1 - \frac{1}{p}} \|\gamma\|_{L^p,q} = q(\eta(t_0)) + (b - a)^{1 - \frac{1}{p}} \|\eta'\|_{L^p,q},
\]

see Remark 2.9. Thus

\[
\|\eta\|_{\infty,q} \leq q(\eta(t_0)) + (b - a)^{1 - \frac{1}{p}} \|\eta'\|_{L^p,q},
\]

whence the (linear) inclusion map is continuous (recall that the topology on \( C([a, b], E) \) is defined by the family of seminorms \( \|\eta\|_{\infty,q} := \sup_{t \in [a, b]} q(\eta(t)) \) with continuous seminorms \( q \) on \( E \)).

**Remark 3.4.** For a subset \( V \subseteq E \), denote by \( AC_{L^p}([a, b], V) \) the set of functions \( \eta \in AC_{L^p}([a, b], E) \) such that \( \eta([a, b]) \subseteq V \). Then, as a consequence of Lemma 3.3, \( AC_{L^p}([a, b], V) = \text{incl}^{-1}(C([a, b], V)) \) is open in \( AC_{L^p}([a, b], E) \) if \( V \) is open in \( E \).
Remark 3.5. It is well known that the evaluation map \( C([a,b], E) \rightarrow E, \eta \mapsto \eta(\alpha) \) is continuous linear for \( \alpha \in [a,b] \). By Lemma 3.3, so is the inclusion map incl: \( AC_{L^p}([a,b], E) \rightarrow C([a,b], E) \), hence the evaluation map 
\[
ev_\alpha : AC_{L^p}([a,b], E) \rightarrow E, \quad \eta \mapsto \eta(\alpha)
\]
is continuous, linear.

The next lemma (which is not difficult to prove) will be used in the proof of Lemma 3.7 (which corresponds to 9 Lemma 3.18).

Lemma 3.6. Let \( E \) be a locally convex space, \( U \subseteq E \) be an open subset and \( f: U \rightarrow \mathbb{R} \) be continuous. Then for every compact subset \( K \subseteq U \) and every \( \varepsilon > 0 \) there exists some \( \varepsilon > 0 \) that an open neighborhood \( U \) of \( \eta \) incl:

\[
\eta = \eta(\alpha)
\]
is continuous, linear.

Lemma 3.7. Let \( E, F \) be sequentially complete locally convex spaces, \( V \subseteq E \) be an open subset and \( p \in [1, \infty] \). If \( f: V \rightarrow F \) is a \( C^1 \)-function then

\[
f \circ \eta \in AC_{L^p}([a,b], F)
\]
for every \( \eta \in AC_{L^p}([a,b], V) \).

Proof. The composition \( f \circ \eta \) is continuous and the differential \( df: V \times E \rightarrow F \) is continuous linear in the second argument, thus \( df \circ (\eta, \gamma) \) is continuous linear. In other words, the function

\[
\zeta: [a,b] \rightarrow F, \quad \zeta(t) := f(\eta(a)) + \int_a^t df(\eta(s), \gamma(s)) \, ds
\]
(16)
is in \( AC_{L^p}([a,b], F) \).

We claim that for each continuous linear form \( \alpha \in E^\prime \), the composition \( \alpha \circ f \circ \eta \) is in \( AC_{L^p}([a,b], \mathbb{R}) \) (hence almost everywhere differentiable) and that

\[
(\alpha \circ f \circ \eta)'(t) = \alpha(\eta(\alpha)) + \int_a^t df(\eta(s), \gamma(s)) \, ds
\]
for almost every \( t \in [a,b] \). From this will follow that

\[
\alpha(f(\eta(t))) = (\alpha \circ f \circ \eta)(a) + \int_a^t \alpha(df(\eta(s), \gamma(s))) \, ds
\]

\[
= \alpha(\zeta(t)) = \alpha(f(\eta(a)) + \int_a^t df(\eta(s), \gamma(s)) \, ds)
\]

for each \( \alpha \in E^\prime \) and \( t \in [a,b] \), therefore \( f \circ \eta = \zeta \in AC_{L^p}([a,b], F) \), as \( E^\prime \) separates points on \( E \).

To prove the claim, we may assume that \( F = \mathbb{R} \) and we show that the composition \( f \circ \eta: [a,b] \rightarrow \mathbb{R} \) is absolutely continuous in the sense that for each \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that

\[
\sum_{j=1}^n |f(\eta(b_j)) - f(\eta(a_j))| < \varepsilon \quad \text{whenever} \quad a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b \quad \text{with} \quad \sum_{j=1}^n |b_j - a_j| < \delta.
\]

Now, as \( \eta([a,b]) \) is a compact subset of the open subset \( V \), there is some open neighborhood \( U \subseteq V \) of \( \eta([a,b]) \) and some continuous seminorm \( q \) on \( E \) such that

\[
|f(u) - f(\tilde{u})| \leq q(u - \tilde{u})
\]
(17)
for all \( u, \bar{u} \in U \), by [9] Lemma 1.60. Given \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that \( \sum_{j=1}^{n} |\sigma(b_j) - \sigma(a_j)| < \varepsilon \) whenever \( a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots < a_n < b_n \leq b \) and \( \sum_{j=1}^{n} |b_j - a_j| < \delta \), because the function

\[
\sigma: [a, b] \to \mathbb{R}, \quad \sigma(t) := \int_{a}^{t} q(\gamma(s)) \, ds
\]

is absolutely continuous. Therefore, we have

\[
\sum_{j=1}^{n} |f(\eta(b_j)) - f(\eta(a_j))| \leq \sum_{j=1}^{n} q(\eta(b_j) - \eta(a_j)) = \sum_{j=1}^{n} \left( \int_{a_j}^{b_j} \gamma(s) \, ds \right)
\leq \sum_{j=1}^{n} \int_{a_j}^{b_j} q(\gamma(s)) \, ds < \delta,
\]

where we used (17) in the first step. Hence \( f \circ \eta \) is absolutely continuous, thus, by [6, VII. 4.14], there is some \( \varphi \in L^1([a, b]) \) such that

\[
f(\eta(t)) = f(\eta(a)) + \int_{a}^{t} \varphi(s) \, ds,
\]

in other words

\[f \circ \eta \in AC_{L^1}([a, b], \mathbb{R}) \quad \text{and} \quad (f \circ \eta)'(t) = \varphi(t) \text{ for a.e. } t \in [a, b].\]

Now, we want to show that \( \varphi(t) = df(\eta(t)), \gamma(t)) \) for almost every \( t \in [a, b], \) that is, \( \varphi \in L^p([a, b]) \). To this end, we may assume that there exists a sequence \( (K_n)_{n \in \mathbb{N}} \) of compact subsets of \( [a, b] \) such that for every \( n \in \mathbb{N} \) the restriction \( \gamma|_{K_n} \) is continuous, \( \lambda([a, b] \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0 \) and \( \gamma(t) = 0 \) for every \( t \notin \bigcup_{n \in \mathbb{N}} K_n \).

Each of the sets

\[
L_n := \eta([a, b]) \cup \bigcup_{m=1}^{n} \gamma(K_m)
\]

is compact and metrizable, hence by [9] Lemma 1.11, there exists a locally convex topology \( T_{X_n} \) on each vector subspace

\[
X_n := \text{span}(L_n),
\]

which is metrizable, separable and coarser than the induced topology \( \mathcal{O}_{X_n} \).

Then on each \( X_n \), there is a countable family \( \Lambda_n \) of continuous (with respect to \( T_{X_n} \)) linear functionals separating the points (see [Chapter II, Prop. 4]Schwartz). Consequently, the countable family \( \Lambda := \bigcup_{n \in \mathbb{N}} \Lambda_n \) separates the points on the vector space \( X := \bigcup_{n \in \mathbb{N}} X_n \), which enables to define a metrizable locally convex topology \( T_X \) coarser than the induced topology \( \mathcal{O}_X \). On the other hand, each of the \( m \)-fold sums

\[
L_{m,n} := [-m, m]L_n + \cdots + [-m, m]L_n
\]

is compact (with respect to \( T_{X_n} \) and \( \mathcal{O}_{X_n} \), and \( X_n = \bigcup_{m \in \mathbb{N}} L_{m,n}, \) thus

\[
X = \bigcup_{m,n \in \mathbb{N}} L_{m,n}
\]

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is $\sigma$-compact.

Then, the space $X \times X \times \mathbb{R}$ has a locally convex metrizable $\sigma$-compact topology $T$, say, $X \times X \times \mathbb{R} = \bigcup_{n\in\mathbb{N}} C_n$. Then $O_{C_n} = T_{C_n}$, where $O_{C_n}, T_{C_n}$ are the topologies on $C_n$ induced by $E \times E \times \mathbb{R}$ and $X \times X \times \mathbb{R}$, respectively. Hence $V^{[1]} \cap C_n \in T_{C_n}$ and $T_{C_n}$ is compact and metrizable, hence second countable. Therefore, $V^{[1]} \cap C_n$ is $\sigma$-compact (being locally compact with countable base), that is, $V^{[1]} \cap C_n$ is a countable union of compact subsets, hence so is $(V \cap X)^{[1]} = V^{[1]} \cap (X \times X \times \mathbb{R}) = \bigcup_{n\in\mathbb{N}} (V^{[1]} \cap C_n)$, so we may write $(V \cap X)^{[1]} = \bigcup_{n\in\mathbb{N}} A_n$ with compact subsets $A_n$.

Next, we will construct a metrizable locally convex topology on $X$ such that $\eta \in AC_{L^p}([a, b], V \cap X)$ with $\eta' = [\gamma] \in L^p([a, b], V \cap X)$ and such that $f^{[1]}|_{(V \cap X)^{[1]}}$ remains continuous. To this end, fix some $(x, y, t) \in (V \cap X)^{[1]}$. Then there exists $n \in \mathbb{N}$ such that $(x, y, t) \in A_n$. Further, for every $k \in \mathbb{N}$ there exists a continuous seminorm $q_{n, k}$ on $E \times E \times \mathbb{R}$ such that

$$|f^{[1]}(x, y, t) - f^{[1]}(v, w, s)| < \frac{1}{k} \quad \forall (v, w, s) \in B^{q_{n, k}}_1(x, y, t)$$

(see Lemma 4.6). Consequently, there is a continuous seminorm $\pi_{n, k}$ on $E$ and $\delta > 0$ such that

$$|f^{[1]}(x, y, t) - f^{[1]}(v, w, s)| < \frac{1}{k} \quad \forall (v, w, s) \in B_{\pi_{n, k}}^1(x) \times B_{\pi_{n, k}}^1(y) \times [t - \delta, t + \delta] \subseteq B^{q_{n, k}}_1(x, y, t).$$

We endow $X$ with the metrizable locally convex topology $T$ defined by the countable family $\{\pi_{n, k} : n, k \in \mathbb{N}\}$. This topology is coarser than the induced topology, hence $\eta': [a, b] \to V \cap X$ remains continuous and $\eta(t) - \eta(a) = \int_a^t \gamma(s) \, ds$ is the weak integral of $\gamma$ in $X$ for every $t \in [a, b]$. To see this, let $\alpha$ be a continuous linear functional on $(X, T)$. Then $\alpha$ is continuous with respect to the induced topology on $X$ (which is finer than $T$) hence there is some continuous linear extension $\mathcal{A} \in E'$ of $\alpha$. Thus

$$\alpha(\eta(t) - \eta(a)) = \mathcal{A}(\eta(t) - \eta(a)) = \int_a^t \mathcal{A}(\gamma(s)) \, ds = \int_a^t \alpha(\gamma(s)) \, ds.$$ 

Therefore, $\eta \in AC_{L^p}([a, b], V \cap X)$ with $\eta' = [\gamma]$ and, by the construction of the topology, the map $f^{[1]}$ is continuous in every $(x, y, t) \in (V \cap X)^{[1]}$ with respect to the obtained topology on $X \times X \times \mathbb{R}$. As $T$ is metrizable, the map $\eta': [a, b] \to V \cap X$ is differentiable in almost every $t \in [a, b]$ with $\eta'(t) = \gamma(t)$ (see Proposition 2.27), so in every such $t$ we have

$$\frac{1}{h}(f(\eta(t + h) - f(\eta(t))) = \frac{1}{h}(f(\eta(t)) + \frac{\eta(t + h) - \eta(t)}{h} - f(\eta(t)))$$

$$= f^{[1]}(\eta(t), \frac{\eta(t + h) - \eta(t)}{h}, h) \to df(\eta(t), \gamma(t))$$

as $h \to 0$. That means, for almost every $t \in [a, b]$ we have

$$\varphi(t) = (f \circ \eta)'(t) = df(\eta(t), \gamma(t)),$$

whence $\varphi \in L^p([a, b])$ and $f \circ \eta \in AC_{L^p}([a, b], \mathbb{R})$. 

$\square$
Also the following (corresponding to [9, Lemma 3.28]) holds:

**Proposition 3.8.** Let $E$, $F$ be sequentially complete locally convex spaces, let $V \subseteq E$ be an open subset and $p \in [1, \infty]$. If $f : V \to F$ is a $C^{k+2}$ function (for $k \in \mathbb{N} \cup \{0, \infty\}$), then the map

$$AC_{L^p}([a,b],f) : AC_{L^p}([a,b],V) \to AC_{L^p}([a,b], F), \quad \eta \mapsto f \circ \eta$$

is $C^k$.

**Proof.** The map $AC_{L^p}([a,b],f)$ is defined by Lemma 3.7 by definition of the topology on $AC_{L^p}([a,b],F)$ (see Definition 3.1). $AC_{L^p}([a,b],f)$ will be $C^k$ if each of the components of

$$AC_{L^p}([a,b],V) \to F \times L^p([a,b], F), \quad \eta \mapsto (f(\eta(a)),(f \circ \eta)') \quad (19)$$

is $C^k$. The first component

$$AC_{L^p}([a,b],V) \to F, \quad \eta \mapsto (f \circ \text{pr}_1 \circ \Phi)(\eta) = f(\eta(a))$$

is indeed $C^k$, where $\Phi$ is as in Definition 3.1 and $\text{pr}_1$ is the projection on the first component. Further, for $\eta' = [\gamma] \in L^p([a,b],E)$ we have $(f \circ \eta)' = [df \circ (\eta, \gamma)]$ by (18) and

$$C([a,b],V) \times L^p([a,b], E) \to L^p([a,b], F), \quad (\eta, [\gamma]) \mapsto [df \circ (\eta, \gamma)]$$

is $C^k$, the derivative $df : V \times E \to F$ being $C^{k+1}$ and linear in the second argument (see Proposition 2.22c). Hence, the second component of (19) is $C^k$, as required.

**Remark 3.9.** Since any continuous linear function $f : E \to F$ is smooth, we conclude from Proposition 3.8 that

$$AC_{L^p}([a,b],E \times F) \cong AC_{L^p}([a,b],E) \times AC_{L^p}([a,b], F)$$

as locally convex spaces (proceeding as in Remark 2.14).

The next result will be helpful.

**Lemma 3.10.** Let $E$ be a sequentially complete locally convex space, $p \in [1, \infty]$ and $a = t_0 < t_1 < \ldots < t_n = b$. Then the function

$$\Psi : AC_{L^p}([a,b], E) \to \prod_{j=1}^n AC_{L^p}([t_{j-1}, t_j], E), \quad \eta \mapsto (\eta|_{[t_{j-1}, t_j]})_{j=1,...,n} \quad (20)$$

is a linear topological embedding with closed image.

**Proof.** Clearly, for $\eta \in AC_{L^p}([a,b], E)$ with $\eta' = [\gamma]$ and every $j \in \{1, \ldots, n\}$ we have $\eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], E)$ with $(\eta|_{[t_{j-1}, t_j]})' = [\gamma|_{[t_{j-1}, t_j]}]$ by Lemma 2.16, that is, the function $\Psi$ is defined. Also the linearity and injectivity are clear.

We show that each of the components

$$AC_{L^p}([a,b], E) \to AC_{L^p}([t_{j-1}, t_j], E), \quad \eta \mapsto \eta|_{[t_{j-1}, t_j]}$$


is continuous, which will be the case if each

\[ AC_{L^p}([a, b], E) \to E \times L^p([t_{j-1}, t_j], E), \quad \eta \mapsto (\eta(t_{j-1}), [\gamma]_{[t_{j-1}, t_j]}) \]

is continuous (using the isomorphism as in Definition 5.1). But the first component is the continuous evaluation map on \( AC_{L^p}([a, b], E) \), see Remark 3.9 and the second component is a composition of the continuous maps \( AC_{L^p}([a, b], E) \to L^p([a, b], E), \eta \mapsto [\gamma] \) and \( L^p([a, b], E) \to L^p([t_{j-1}, t_j], E), [\gamma] \mapsto [\gamma]_{[t_{j-1}, t_j]} \), see Definition 5.1 and Lemma 2.10. Therefore, \( \Psi \) is continuous.

Note that \( \eta_1, \ldots, \eta_n \in \text{im}(\Psi) \subseteq \prod_{j=1}^n AC_{L^p}([t_{j-1}, t_j], E) \) if \( \eta_j(t_j) = \eta_{j+1}(t_j) \) for all \( j \in \{1, \ldots, n-1\} \), thus the map

\[ \Gamma(\eta_1, \ldots, \eta_n) : [a, b] \to E, \quad t \mapsto \eta_j(t) \text{ for } t \in [t_{j-1}, t_j] \]

is continuous and it is easy to show that \( \Gamma(\eta_1, \ldots, \eta_n) \in AC_{L^p}([a, b], E) \) and that

\[ \Gamma : \text{im}(\Psi) \to AC_{L^p}([a, b], E) \]

is the inverse of \( \Psi^{\text{im}(\Psi)} \). The continuity of \( \Gamma \) follows from the continuity of

\[ \text{im}(\Psi) \to E \times L^p([a, b], E), \]

\( (\eta_1, \ldots, \eta_n) \mapsto (\eta_1(a), \Gamma(\eta_1, \ldots, \eta_n)') = (\eta_1(a), \Gamma^{-1}_E(\eta_1', \ldots, \eta_n')) \),

where \( \Gamma_E \) is the isomorphism from Lemma 2.10. Hence \( \Psi \) is a topological embedding.

Finally, let \( (\eta_{1, \alpha}, \ldots, \eta_{n, \alpha})_{\alpha \in \mathcal{A}} \) be a convergent net in \( \text{im}(\Psi) \) with limit point \( (\eta_1, \ldots, \eta_n) \in \prod_{j=1}^n AC_{L^p}([t_{j-1}, t_j], E) \), thus for every \( j \in \{1, \ldots, n-1\} \) we have

\[ \eta_j(t_j) = \lim_{\alpha \in \mathcal{A}} \eta_{j, \alpha}(t_j) = \lim_{\alpha \in \mathcal{A}} \eta_{j+1, \alpha}(t_j) = \eta_{j+1}(t_j), \]

therefore \( (\eta_1, \ldots, \eta_n) \in \text{im}(\Psi) \). \( \square \)

**Remark 3.11.** Lemma 3.10 immediately implies that whenever \( \eta \in C([a, b], E) \) and \( \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], E) \) for some \( a = t_0 < t_1 < \ldots < t_n = b \), we have \( \eta \in AC_{L^p}([a, b], E) \).

**Lemma 3.12.** Let \( E \) be a sequentially complete locally convex space, \( p \in [1, \infty] \). Let \( \eta \in AC_{L^p}([a, b], E) \) and for \( a \leq \alpha < \beta \leq b \) define

\[ f : [c, d] \to [a, b], \quad f(t) := \alpha + \frac{t - c}{d - c} (\beta - \alpha). \]

Then \( \eta \circ f \in AC_{L^p}([c, d], E) \) and

\[ (\eta \circ f)' = \frac{\beta - \alpha}{d - c} [\gamma \circ f], \]

where \( [\gamma] = \eta' \).

Furthermore, the function

\[ AC_{L^p}(f, E) : AC_{L^p}([a, b], E) \to AC_{L^p}([c, d], E), \quad \eta \mapsto \eta \circ f \]

is continuous and linear.
Proof. We know that \( \gamma \circ f \in L^p([c, d], E) \) (Lemma 2.15) and for \( t \in [c, d] \) we have
\[
\eta(f(t)) - \eta(f(c)) = \int_{f(c)}^{f(t)} \gamma(s) \, ds,
\]
since \([\gamma] = \eta'\). Then for any continuous linear functional \( A \) on \( E \) we have
\[
\int_{f(c)}^{f(t)} A(\gamma(s)) \, ds = \frac{\beta - \alpha}{d - c} \int_c^t A(\gamma(f(s))) \, ds
\]
(see [2 19.4 Satz]), whence
\[
\eta(f(t)) - \eta(f(c)) = \frac{\beta - \alpha}{d - c} \int_c^t \gamma(f(s)) \, ds,
\]
in other words, \( \eta \circ f \in AC_{L^p}([c, d], E) \) with \( (\eta \circ f)' = \frac{\beta - \alpha}{d - c} [\gamma \circ f] \).

To prove the continuity of the linear function \( AC_{L^p}(f, E) \), we show that
\[
AC_{L^p}([a, b], E) \to E \times L^p([c, d], E), \quad \eta \mapsto (\eta(f(c)), (\eta \circ f)')
\]
is continuous (where we used the isomorphism from Definition 3.1). The first component
\[
AC_{L^p}([a, b], E) \to E, \quad \eta \mapsto \text{ev}_{f(c)}(\eta)
\]
is continuous, by Remark 3.1. Further, the map
\[
\Psi : L^p([a, b], E) \to L^p([c, d], E), \quad [\gamma] \mapsto \frac{\beta - \alpha}{d - c} [\gamma \circ f]
\]
is continuous, by Lemma 2.15; hence the second component
\[
AC_{L^p}([a, b], E) \to L^p([c, d], E), \quad \eta \mapsto \Psi(\eta)' = (\eta \circ f)'
\]
is continuous, and the proof is finished. \( \square \)

3.2 Manifold-valued AC-functions

Let \( M \) be a smooth manifold modelled on a locally convex space \( E \), that is, \( M \) be a Hausdorff topological space together with an atlas of charts \( \varphi : U_\varphi \to V_\varphi \) (homeomorphisms between open subsets of \( M \) and \( E \)) such that the transition maps \( \psi \circ \varphi^{-1} \) are smooth functions (as in [3]). The definition of the tangent space \( T_xM \), the tangent manifold \( TM \), differentiable functions between manifolds and tangent maps between tangent spaces are defined in the usual way.

The properties of the spaces \( AC_{L^p}([a, b], E) \), proved in the preceding, enable us to define spaces of absolutely continuous functions with values in manifolds \( M \) modelled on sequentially complete locally convex spaces.

The corresponding definition can be found in [2 3.2.10].

Definition 3.13. Let \( M \) be a smooth manifold modelled on a sequentially complete locally convex space \( E \). For \( p \in [1, \infty] \), denote by \( AC_{L^p}([a, b], M) \) the space of continuous functions \( \eta : [a, b] \to M \) such that there exists some partition \( a = t_0 < t_1 < \ldots < t_n = b \) with
\[
\varphi_j \circ \eta|[t_{j-1}, t_j] \in AC_{L^p}([t_{j-1}, t_j], E)
\]
for some charts \( \varphi_j : U_j \to V_j \) such that \( \eta([t_{j-1}, t_j]) \subseteq U_j \) for \( j = 1, \ldots, n \).
As in [9] Lemma 3.21, the construction of $AC_{L^p}([a,b], M)$ does not depend on the choice of the partition $a = t_0 < t_1 < \ldots < t_n = b$ or of the charts $\varphi_j$.

**Lemma 3.14.** Let $\eta \in AC_{L^p}([a,b], M)$, let $[\alpha, \beta] \subseteq [a,b]$ and $\varphi : U \to V$ be any chart for $M$ such that $\eta([\alpha, \beta]) \subseteq U$. Then

$$\varphi \circ \eta|_{[\alpha, \beta]} \in AC_{L^p}([\alpha, \beta], E).$$

**Proof.** We have $\alpha \in [t_k, t_{k+1}]$ and $\beta \in [t_{l-1}, t_l]$ for some $k, l$, for simplicity we may assume $\alpha = t_k$, $\beta = t_l$. For $j \in \{k+1, \ldots, l\}$ we have

$$\varphi \circ \eta|_{[t_{j-1}, t_j]} = (\varphi \circ \varphi_j^{-1}) \circ (\varphi_j \circ \eta|_{[t_{j-1}, t_j]}).$$

Since $\varphi \circ \varphi_j^{-1}$ is a smooth function and $\varphi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], E)$, the above composition is in $AC_{L^p}([t_{j-1}, t_j], E)$ by Lemma 3.7. From Remark 3.11 follows $\varphi \circ \eta|_{[\alpha, \beta]} \in AC_{L^p}([\alpha, \beta], E)$.

**Remark 3.15.** From the above lemma follows that if the smooth manifold $M$ is a locally convex space (with global chart $\text{id}_M$), then the space $AC_{L^p}([a,b], M)$ defined in Definition 3.13 is the same as the space defined in Definition 3.3.

The next results correspond to [9] Lemma 3.24] and [9] Lemma 3.30.

**Lemma 3.16.** Let $M, N$ be smooth manifolds modelled on sequentially complete locally convex spaces $E$ and $F$, respectively. If $f : M \to N$ is a $C^1$-map, then $f \circ \eta \in AC_{L^p}([a,b], N)$ for each $\eta \in AC_{L^p}([a,b], M)$ and $p \in [1, \infty]$.

**Proof.** Consider a partition $a = t_0 < t_1 < \ldots < t_n = b$ and charts $\varphi_j : U_j \to V_j$ for $M$ such that $\eta([t_{j-1}, t_j]) \subseteq U_j$ and $\varphi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], E)$ for each $j \in \{1, \ldots, n\}$. Since $f \circ \eta|_{[t_{j-1}, t_j]}$ is continuous, we find a partition $t_{j-1} = s_0 < s_1 < \cdots < s_m = t_j$ and charts $\psi_i : P_i \to Q_i$ for $N$ such that $f(\eta([s_{i-1}, s_i])) \subseteq P_i$ for each $i \in \{1, \ldots, m\}$. Then

$$\psi_i \circ f \circ \eta|_{[s_{i-1}, s_i]} = (\psi_i \circ f \circ \varphi_j^{-1}) \circ (\varphi_j \circ \eta|_{[s_{i-1}, s_i]}) \in AC_{L^p}([s_{i-1}, s_i], F),$$

by Remark 3.11 and Lemma 3.7. Hence $f \circ \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], N)$ for each $j \in \{1, \ldots, n\}$, whence $f \circ \eta \in AC_{L^p}([a,b], N)$.

**Lemma 3.17.** Let $E_1$, $E_2$ and $F$ be sequentially complete locally convex spaces. Let $M$ be a smooth manifold modelled on $E_1$ and $V \subseteq E_2$ be an open subset. If $f : M \times V \to F$ is a $C^{k+2}$-map and $\zeta \in AC_{L^p}([a,b], M)$ for $p \in [1, \infty]$, then

$$AC_{L^p}([a,b], V) \to AC_{L^p}([a,b], F), \quad \eta \mapsto f \circ (\zeta, \eta)$$

is a $C^k$-map.

**Proof.** Since $(\zeta, \eta) \in AC_{L^p}([a,b], M \times V)$, the above map is defined by Lemma 3.16; it will be $C^k$ if for a partition $a = t_0 < t_1 < \ldots < t_n = b$ for $\zeta$ (as in Definition 3.13) the function

$$AC_{L^p}([a,b], V) \to \prod_{j=1}^n AC_{L^p}([t_{j-1}, t_j], F), \quad \eta \mapsto (f \circ (\zeta, \eta)|_{[t_{j-1}, t_j]})_{j=1, \ldots, n}$$

is $C^k$. 

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is $C^k$ (where we use the topological embedding with closed image on the space $AC_{L^p}([a, b], F)$ as in Lemma 3.10). This will hold if every component

$$AC_{L^p}([a, b], V) \to AC_{L^p}([t_{j-1}, t_j], F), \quad \eta \mapsto f \circ (\zeta, \eta)|_{[t_{j-1}, t_j]}$$

is $C^k$.

Now, given charts $\varphi: U_j \to V_j$ for $M$ with $\zeta([t_{j-1}, t_j]) \subseteq U_j$, for every $j \in \{1, \ldots, n\}$, the function

$$AC_{L^p}([a, b], V) \to AC_{L^p}([t_{j-1}, t_j], V_j \times V), \quad \eta \mapsto \big(\varphi_j \circ \zeta|_{[t_{j-1}, t_j]}, \eta|_{[t_{j-1}, t_j]}\big)$$

is smooth by Lemma 3.10 (identifying $AC_{L^p}([t_{j-1}, t_j], V_j \times V)$ with the product $AC_{L^p}([t_{j-1}, t_j], V_j) \times AC_{L^p}([t_{j-1}, t_j], V)$, see Remark 3.9). As the composition $f \circ (\varphi_j^{-1} \times \text{id}_V): V_j \times V \to F$ is $C^{k+2}$, by Proposition 3.8 the function

$$AC_{L^p}([a, b], V) \to AC_{L^p}([t_{j-1}, t_j], F),$$

$$\eta \mapsto \big(f \circ (\varphi_j^{-1} \times \text{id}_V) \circ (\varphi_j \circ \zeta|_{[t_{j-1}, t_j]}, \eta|_{[t_{j-1}, t_j]}\big)$$

$$= f \circ (\zeta, \eta)|_{[t_{j-1}, t_j]}$$

is $C^k$. Therefore, the function in (22) is $C^k$ and the proof is finished. \qed

### 3.3 Lie group-valued $AC$-functions

We consider smooth Lie groups $G$ modelled on locally convex spaces $E$, that is, $G$ is a group endowed with a smooth manifold structure modelled on $E$ such that the group multiplication $m_G: G \times G \to G$ and the inversion $j_G: G$ are smooth functions. We will always denote the identity element of $G$ by $e_G$ and the Lie group by $g := T_{e_G}G \cong E$.

Similar to [14 Proposition 4.2], we endow $AC_{L^p}([a, b], G)$ with a Lie group structure for any Lie group $G$ modelled on a sequentially complete locally convex space, after recalling the following fact.

**Remark 3.18.** Let $G$ be a group, $U \subseteq G$ be a symmetric subset containing the identity element of $G$. Assume that $U$ is endowed with a smooth manifold structure modelled on a locally convex space $E$ such that the inversion $U \to U, x \mapsto x$ on $U$ is smooth, the subset $U_m := \{(x, y) \in U \times U : xy \in U\}$ is open in $U \times U$ and the multiplication $U_m \to U, (x, y) \mapsto xy$ is smooth on $U_m$. Further, assume that for each $g \in G$, there exists an open identity neighborhood $W \subseteq U$ such that $gWg^{-1} \subseteq U$ and $W \to U, x \mapsto gxg^{-1}$ is smooth. Then $G$ can be endowed with a unique smooth manifold structure modelled on $E$ such that $G$ becomes a smooth Lie group and $U$ with the given manifold structure becomes an open smooth submanifold.

**Proposition 3.19.** Let $G$ be a smooth Lie group modelled on a sequentially complete locally convex space $E$, let $p \in [1, \infty]$. Then there exists a unique $C^k$-structure on $AC_{L^p}([a, b], G)$ such that for each open symmetric $e_G$-neighborhood $U \subseteq G$ the subset $AC_{L^p}([a, b], U)$ is open in $AC_{L^p}([a, b], G)$ and such that each

$$AC_{L^p}([a, b], \varphi): AC_{L^p}([a, b], U) \to AC_{L^p}([a, b], V), \quad \eta \mapsto \varphi \circ \eta$$

is a smooth diffeomorphism for every chart $\varphi: U \to V$ for $G$.  

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Proof. Step 1: $AC_{L^p}([a,b], G)$ is a group.

As $m_G$ and $j_G$ are smooth, we have $m_G \circ (\eta, \xi), j_G \circ \eta \in AC_{L^p}([a,b], G)$ for all $\eta, \xi \in AC_{L^p}([a,b], G)$, by Lemma 3.16 (identifying $AC_{L^p}([a,b], G \times G)$ with $AC_{L^p}([a,b], G) \times AC_{L^p}([a,b], G)$). Then $\hat{G} := AC_{L^p}([a,b], \hat{G})$ is a group with multiplication

$$m_G := AC_{L^p}([a,b], m_G) : \hat{G} \times \hat{G} \to \hat{G}, \quad (\eta, \xi) \mapsto m_G \circ (\eta, \xi) := \eta \cdot \xi,$$

inversion

$$j_G := AC_{L^p}([a,b], j_G) : \hat{G} \to \hat{G}, \quad \eta \mapsto j_G \circ \eta =: \eta^{-1}$$

and identity element $e_{\hat{G}} : t \mapsto e_G$.

Step 2: Existence of a Lie group structure on $AC_{L^p}([a,b], G)$.

Consider an open symmetric $e_G$-neighbourhood $U \subseteq G$ and a chart $\varphi : U \to V$. As $\tilde{V} := AC_{L^p}([a,b], V)$ is open in $AC_{L^p}([a,b], E)$ (see Remark 3.4), we endow the symmetric subset $\tilde{U} := AC_{L^p}([a,b], U) := \{ \eta \in AC_{L^p}([a,b], G) : \eta([a,b]) \subseteq U \}$ with the $C^\infty$-manifold structure turning the bijection

$$\tilde{\varphi} := AC_{L^p}([a,b], \varphi) : \tilde{U} \to \tilde{V}, \quad \eta \mapsto \varphi \circ \eta$$

into a global chart (the map is defined by Lemma 3.16). Obviously, $e_{\tilde{G}} \in \tilde{U}$.

Further, by Lemma 3.7 the function

$$AC_{L^p}([a,b], \varphi \circ j_G|U \circ \varphi^{-1}) : \tilde{V} \to \tilde{V}, \quad \eta \mapsto (\varphi \circ j_G|U \circ \varphi^{-1}) \circ \eta$$

is smooth. Thus, writing

$$\tilde{U} \to \tilde{U}, \quad \eta \mapsto (\varphi^{-1} \circ AC_{L^p}([a,b], \varphi \circ j_G|U \circ \varphi^{-1}) \circ \varphi)(\eta)$$

$$= \varphi^{-1} \circ \varphi \circ j_G|U \circ \varphi^{-1} \circ \varphi \circ \eta$$

$$= j_G \circ \eta,$$

we see that the inversion on $\tilde{U}$ is smooth.

Now, consider the open subset $U_m := \{ (x, y) \in U : xy \notin U \}$ of $U \times U$. As $V_m := (\varphi \times \varphi)(U_m)$ is open in $E \times E$, the set $\tilde{V}_m := AC_{L^p}([a,b], V_m)$ is open in $AC_{L^p}([a,b], E) \times AC_{L^p}([a,b], E)$, whence $\tilde{U}_m := (\varphi^{-1} \times \varphi^{-1})(\tilde{V}_m)$ is open in $\tilde{U} \times \tilde{U}$. Again, by Lemma 3.7 the function

$$AC_{L^p}([a,b], \varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|\tilde{V}_m) : \tilde{V}_m \to \tilde{V}, \quad \eta \mapsto (\varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|\tilde{V}_m) \circ \eta$$

is smooth. Therefore

$$\tilde{U}_m \to \tilde{U},$$

$$(\eta, \xi) \mapsto ((\varphi^{-1} \circ AC_{L^p}([a,b], \varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|\tilde{V}_m) \circ (\varphi \times \varphi))(\eta, \xi)$$

$$= \varphi^{-1} \circ \varphi \circ m_G \circ (\varphi^{-1} \times \varphi^{-1})|\tilde{V}_m \circ (\varphi \times \varphi) \circ (\eta, \xi)$$

$$= m_G \circ (\eta, \xi),$$

which is the multiplication on $\tilde{U}_m$, is smooth.

Finally, fix some $\eta \in \tilde{G}$ and write $K := \text{im}(\eta) \subseteq G$. As the function

$$h : G \times G \to G, (x, y) \mapsto xyx^{-1}$$

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is smooth and \( h(K \times \{e_G\}) = \{e_G\} \subseteq U \), the compact set \( K \times \{e_G\} \) is a subset of the open set \( h^{-1}(U) \subseteq G \times G \). By the Wallace Lemma, there are open subsets \( W_K, \ W \) of \( G \) such that \( K \times \{e_G\} \subseteq W_K \times W \subseteq h^{-1}(U) \). We may assume \( W \subseteq U \), then we see that \( \tilde{W} := AC_L^p([a, b], W) \) is open in \( \tilde{U} \) and for each \( \xi \in \tilde{W} \) we have

\[
\eta \cdot \xi \cdot \eta^{-1} = h \circ (\eta, \xi) \in \tilde{U}
\]

by Lemma 3.16. Using Lemma 3.17, we see that \( \text{id} : \tilde{W} \to \tilde{U} \) is smooth,

\[
AC_L^p([a, b], \varphi(W)) \to \tilde{V},
\]

\[
\xi \mapsto (\varphi \circ h \circ (\text{id}_{W_K} \times \varphi^{-1}|_{\varphi(W)}) \circ (\eta, \xi)
\]

is smooth, whence

\[
\tilde{W} \to \tilde{U}, \quad \xi \mapsto \varphi^{-1} \circ (\varphi \circ h \circ (\text{id}_{W_K} \times \varphi^{-1}|_{\varphi(W)}) \circ (\eta, \varphi \circ \xi) = h \circ (\eta, \xi)
\]

\[
= \eta \cdot \xi \cdot \eta^{-1}
\]

is smooth.

Consequently, by Remark 3.12, there exists a unique Lie group structure on \( \tilde{G} \) turning \( \tilde{U} \) into a smooth open submanifold and \( \tilde{\varphi} \), a \( G \)-chart around \( e_{\tilde{G}} \).

Step 3: Uniqueness of the Lie group structure on \( AC_L^p([a, b], G) \).

Let \( U' \subseteq G \) be an open symmetric \( e_G \)-neighborhood and \( \varphi' : U' \to V' \) be a \( G \)-chart around \( e_G \). Denote by \( \tilde{G'} \) the group \( AC_L^p([a, b], G) \) endowed with the Lie group structure turning \( \tilde{U'} := AC_L^p([a, b], U') \) into an open submanifold and \( \tilde{\varphi}' : \tilde{U'} \to AC_L^p([a, b], V') \) into a chart (constructed as in Step 2). We show that both identity maps \( \text{id} : \tilde{G'} \to \tilde{G} \) and \( \text{id} : \tilde{G} \to \tilde{G'} \) are continuous, that is, both Lie group structures coincide.

The subset \( U' \cap U \) is open in \( U' \), therefore \( \varphi'(U' \cap U) \) is open in \( V' \), thus \( AC_L^p([a, b], \varphi'(U' \cap U)) \) is open in \( AC_L^p([a, b], V') \), and consequently \( \tilde{U'} \cap \tilde{U} = \tilde{\varphi}'^{-1}(AC_L^p([a, b], \varphi'(U' \cap U))) \) is open in \( \tilde{G}' \). Writing \( \text{id}_{\tilde{U}' \cap \tilde{U}} = \tilde{\varphi}'^{-1} \circ AC_L^p([a, b], \varphi \circ \varphi'^{-1}|_{U' \cap U}) \circ \tilde{\varphi}'|_{\tilde{U}' \cap \tilde{U}} : \tilde{U}' \cap \tilde{U} \to \tilde{G} \) and using Lemma 3.4, we see that \( \text{id} : \tilde{G'} \to \tilde{G} \) is continuous on the open identity neighborhood \( \tilde{U'} \cap \tilde{U} \), hence continuous. In the same way, we show that also \( \text{id} : \tilde{G} \to \tilde{G'} \) is continuous, as required.

See [9] Remark 4.3] for the next result.

**Lemma 3.20.** The inclusion map

\[
\text{incl} : AC_L^p([a, b], G) \to C([a, b], G), \quad \eta \mapsto \eta
\]

is a smooth homomorphism.

**Proof.** Let \( U \subseteq G \) be an open identity neighborhood, \( \varphi : U \to V \) be a chart for \( G \). Then \( C([a, b], \varphi) : C([a, b], U) \to C([a, b], V), \eta \mapsto \varphi \circ \eta \) is a chart for \( C([a, b], G) \) and \( AC_L^p([a, b], \varphi) : AC_L^p([a, b], U) \to AC_L^p([a, b], V), \eta \mapsto \varphi \circ \eta \) is a chart for \( AC_L^p([a, b], G) \). The function

\[
AC_L^p([a, b], V) \to C([a, b], V), \quad \eta \mapsto (C([a, b], \varphi) \circ \text{incl} \circ AC_L^p([a, b], \varphi)^{-1})(\eta) = \eta
\]
is smooth, being a restriction of the smooth inclusion map from Lemma 3.3. Hence the group homomorphism incl is smooth.

Lemmas 3.21 - 3.23 can be found in [9, Lemma 4.8].

**Lemma 3.21.** For any \( \alpha \in [a, b] \), the evaluation map

\[
\text{ev}_\alpha : AC_{L^p}([a, b], G) \to G, \quad \eta \mapsto \eta(\alpha)
\]

is a smooth homomorphism.

*Proof.* The function is a composition of the smooth inclusion map from Lemma 3.20 and the smooth evaluation map on \( C([a, b], G) \), hence smooth.

**Lemma 3.22.** Let \( \eta \in AC_{L^p}([a, b], G) \) and for \( a \leq \alpha < \beta \leq b \) define

\[
f : [c, d] \to [a, b], \quad f(t) := \alpha + \frac{t - c}{d - c} (\beta - \alpha).
\]

Then \( \eta \circ f \in AC_{L^p}([c, d], G) \).

Furthermore, the function

\[
AC_{L^p}(f, G) : AC_{L^p}([a, b], G) \to AC_{L^p}([c, d], G), \quad \eta \mapsto \eta \circ f
\]

is a smooth homomorphism.

*Proof.* As \( \eta \circ f \) is a continuous curve, there exists a partition \( c = t_0 < t_1 < \ldots < t_n = d \) and for every \( j \in \{1, \ldots, n\} \) there is a chart \( \varphi_j : U_j \to V_j \) for \( G \) such that \( \eta(f([t_{j-1}, t_j])) \subseteq U_j \). But \( f([t_{j-1}, t_j]) = [f(t_{j-1}), f(t_j)] \) is an interval and from Lemma 3.14 follows that

\[
\varphi_j \circ \eta|_{[f(t_{j-1}), f(t_j)]} \in AC_{L^p}([f(t_{j-1}), f(t_j)], V_j).
\]

By Lemma 3.12 we have

\[
\varphi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], V_j),
\]

that is, \( \eta \circ f \in AC_{L^p}([c, d], G) \).

Finally, for any open identity neighborhood \( U \subseteq G \) and any chart \( \varphi : U \to V \) for \( G \) the function

\[
AC_{L^p}([a, b], V) \to AC_{L^p}([c, d], V),
\]

\[
\zeta \mapsto (AC_{L^p}([c, d], \varphi) \circ AC_{L^p}(f, G) \circ AC_{L^p}([a, b], \varphi)^{-1})(\zeta) = \zeta \circ f
\]

is smooth (see Lemma 3.12), hence the group homomorphism \( AC_{L^p}(f, G) \) is smooth.

**Lemma 3.23.** Let \( G \) be a Lie group modelled on a sequentially complete locally convex space \( E \), let \( p \in [1, \infty] \). Then the function

\[
\Gamma_G : AC_{L^p}([a, b], G) \to \prod_{j=1}^n AC_{L^p}([t_{j-1}, t_j], G), \quad \eta \mapsto (\eta|_{[t_{j-1}, t_j]})_{j=1, \ldots, n}
\]

is a smooth homomorphism and a smooth diffeomorphism onto a Lie subgroup of \( \prod_{j=1}^n AC_{L^p}([t_{j-1}, t_j], G) \).
Proof. First of all we introduce some notations. For \( j = 1, \ldots, n \) we denote \( G_j := AC_{L^p}([t_{j-1}, t_j], G) \), and for an open identity neighborhood \( U \subseteq G \) and a chart \( \varphi: U \to V \) we write \( U_j := AC_{L^p}([t_{j-1}, t_j], U) \), \( V_j := AC_{L^p}([t_{j-1}, t_j], V) \) and \( \varphi_j: U_j \to V_j, \zeta \mapsto \varphi \circ \zeta \).

Clearly, the map \( \Gamma_G \) is a group homomorphism and

\[
\text{im}(\Gamma_G) = \{ (\eta_j)_{j=1}^{n} \in \prod_{j=1}^{n} G_j : \eta_{j-1}(t_j) = \eta_j(t_j) \text{ for all } j \in \{2, \ldots, n\} \}
\]

is a subgroup of \( \prod_{j=1}^{n} G_j \). Moreover, the function

\[
\psi := \prod_{j=1}^{n} \varphi_j: \prod_{j=1}^{n} U_j \to \prod_{j=1}^{n} V_j, \quad (\zeta_j, \ldots, \zeta_n) \mapsto (\varphi \circ \zeta_1, \ldots, \varphi \circ \zeta_n)
\]

is a chart for \( \prod_{j=1}^{n} G_j \) and \( \psi(\text{im}(\Gamma_G) \cap \prod_{j=1}^{n} U_j) = \text{im}(\Gamma_E) \cap \prod_{j=1}^{n} V_j \), where \( \Gamma_E \) is the linear topological embedding with closed image from Lemma 3.10.

Therefore, \( \text{im}(\Gamma_G) \) is a Lie subgroup modelled on the closed vector subspace \( \text{im}(\Gamma_E) \) of \( \prod_{j=1}^{n} AC_{L^p}([t_{j-1}, t_j], E) \).

Finally, both compositions

\[
\psi \circ \Gamma_G \circ AC_{L^p}([a, b], \varphi)^{-1}: AC_{L^p}([a, b], V) \to \prod_{j=1}^{n} V_j, \quad \eta \mapsto \Gamma_E(\eta)
\]

and

\[
AC_{L^p}([a, b], \varphi) \circ \Gamma_G^{-1} \circ (\psi|_{\text{im}(\Gamma_G)})^{-1}: \text{im}(\Gamma_E) \cap \prod_{j=1}^{n} V_j \to AC_{L^p}([a, b], V), \quad \eta \mapsto \Gamma_E^{-1}(\eta)
\]

are smooth maps, thus we conclude that \( \Gamma_G \) is a smooth diffeomorphism onto its image. \( \square \)

See \cite{9} Lemma 5.10 for the next result.

**Proposition 3.24.** Let \( G \) be a smooth Lie group, let \( E, F \) be locally convex spaces and \( f: G \times E \to F \) be a \( C^{k+1} \)-function and linear in the second argument. Then for \( p \in [1, \infty] \) the function

\[
C([a, b], G) \times L^p([a, b], E) \to L^p([a, b], F), \quad (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)] \quad (23)
\]

is \( C^k \).

**Proof.** The function is defined by Lemma 2.18. We fix some \( \bar{\eta} \in C([a, b], G) \) and some open identity neighborhood \( U \subseteq G \). Then \( U \) contains some open identity neighborhood \( W \) such that \( WW \subseteq U \). The function in (23) will be \( C^k \) if the restriction

\[
Q \times L^p([a, b], E) \to L^p([a, b], F), \quad (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)]
\]

(24)

is \( C^k \), where \( Q := \{ \zeta \in C([a, b], G) : \bar{\eta}^{-1} \cdot \zeta \in C([a, b], W) \} \) is an open neighborhood of \( \bar{\eta} \).
Consider a partition \( a = \tau_0 < \tau_1 < \ldots < \tau_n = b \) such that
\[
\eta(\tau_{j-1})^{-1}\eta([\tau_{j-1}, \tau_j]) \subseteq W.
\]

From Lemma 2.16 follows that the above function will be \( C^k \) if
\[
Q \times L^p([a, b], E) \rightarrow \prod_{j=1}^n L^p([\tau_{j-1}, \tau_j], F),
\]
\[
(\eta, [\gamma]) \mapsto ([f \circ (\eta, \gamma)|_{[\tau_{j-1}, \tau_j]}])_{j=1,\ldots,n}
\]
is \( C^k \), which will be the case if each component
\[
Q \times L^p([a, b], E) \rightarrow L^p([\tau_{j-1}, \tau_j], F), \quad (\eta, [\gamma]) \mapsto [f \circ (\eta, \gamma)|_{[\tau_{j-1}, \tau_j]}]
\]
is \( C^k \).

Now, by 2.16 the function
\[
Q \times L^p([a, b], E) \rightarrow C([\tau_{j-1}, \tau_j], G) \times L^p([\tau_{j-1}, \tau_j], E),
\]
\[
(\eta, [\gamma]) \mapsto (\eta(\tau_{j-1})^{-1}\eta|_{[\tau_{j-1}, \tau_j]}, [\gamma]|_{[\tau_{j-1}, \tau_j]})
\]
is smooth; for \( \eta \in Q \) and \( t \in [\tau_{j-1}, \tau_j] \) we have
\[
\eta(\tau_{j-1})^{-1}\eta(t) = \eta(\tau_{j-1})^{-1}\eta(t)^{-1}\eta(t) \in WW \subseteq U.
\]

Thus
\[
Q \times L^p([a, b], E) \rightarrow C([\tau_{j-1}, \tau_j], V) \times L^p([\tau_{j-1}, \tau_j], E),
\]
\[
(\eta, [\gamma]) \mapsto (\varphi \circ \eta(\tau_{j-1})^{-1}\eta|_{[\tau_{j-1}, \tau_j]}, [\gamma]|_{[\tau_{j-1}, \tau_j]})
\]
is smooth, where \( \varphi: U \rightarrow V \) is a chart for \( G \). We define
\[
g: V \times E \rightarrow F, \quad (x, y) \mapsto f(\eta(\tau_{j-1}))^{-1}(x), y,
\]
which is \( C^{k+1} \) and linear in the second argument, and we use Proposition 2.23 to obtain a \( C^k \)-map
\[
Q \times L^p([a, b], E) \rightarrow L^p([\tau_{j-1}, \tau_j], F),
\]
\[
(\eta, [\gamma]) \mapsto [g \circ (\varphi \circ \eta(\tau_{j-1})^{-1}\eta|_{[\tau_{j-1}, \tau_j]}, [\gamma]|_{[\tau_{j-1}, \tau_j]}) = [f \circ (\eta, \gamma)|_{[\tau_{j-1}, \tau_j]}],
\]
which is exactly the required function from (25).

\[ \square \]

4 Measurable regularity of Lie groups

4.1 Logarithmic derivatives of \( AC \)-functions

The following definition is taken from [9, Definition 5.1].

Definition 4.1. Let \( M \) be a smooth manifold modelled on a sequentially complete locally convex space \( E \), let \( p \in [1, \infty] \). Consider \( \eta \in AC_L^p([a, b], M) \), a partition \( a = \tau_0 < \tau_1 < \ldots < \tau_n = b \) and charts \( \varphi_j: U_j \rightarrow V_j \) for \( M \) such that
\[
\eta([\tau_{j-1}, \tau_j]) \subseteq U_j \quad \text{for all } j
\]
\[
\eta_j := \varphi_j \circ \eta|_{[\tau_{j-1}, \tau_j]} \in AC_L^p([\tau_{j-1}, \tau_j], E).
\]
Denote \( \eta'_j := [\gamma_j] \in L^p([t_{j-1}, t_j], E) \) and set
\[
\gamma(t) := T\varphi_j^{-1}(\eta_j(t), \gamma_j(t))
\]
for \( t \in [t_{j-1}, t_j] \), and
\[
\gamma(b) := T\varphi_n^{-1}(\eta_n(b), \gamma_n(b)).
\]
The constructed function \( \gamma : [a, b] \to TM \) is measurable and we write \( \dot{\gamma} := [\gamma] \).

**Remark 4.2.** Note that the definition of \( \dot{\gamma} \) is independent of the choice of the partition \( a = t_0 < t_1 < \ldots < t_n = b \) and the charts \( \varphi_j \).

Recall that on the tangent bundle \( TG \) of a smooth Lie group \( G \), there is a smooth Lie group structure with multiplication \( Tm_G : TG \times TG \to TG \) and inversion \( Tj_G : TG \to TG \). For \( g, h \in G \), \( v \in T_gG \) and \( w \in T_hG \) write
\[
v.h := T_g \rho_h(v) \in T_{gh}G, \quad g.w := T_h \lambda_g(w) \in T_{gh}G,
\]
(where \( \lambda_g : x \mapsto gx \), \( \rho_h : x \mapsto xh \) are the left and the right translations on \( G \)) then
\[
Tm_G(v, w) = g.w + v.h.
\]

See [9, Lemma 5.4] for the following computations.

**Lemma 4.3.** Let \( G \) be a Lie group modelled on a sequentially complete locally convex space, let \( p \in [1, \infty] \). For \( \eta, \zeta \in AC_{L^p}([a, b], G) \) with \( \dot{\eta} = [\gamma] \), \( \dot{\zeta} = [\xi] \) we have
\[
(\eta \cdot \zeta) = [t \mapsto \gamma(t) \cdot \zeta(t) + \eta(t) \cdot \xi(t)]
\]
and
\[
(\eta^{-1}) = [t \mapsto -\eta(t)^{-1} \cdot \gamma(t) \cdot \eta(t)^{-1}].
\]
Further, if \( f : G \to H \) is a smooth function between Lie groups modelled on sequentially complete locally convex spaces, then
\[
(f \circ \eta) = [Tf \circ \gamma].
\]

**Proof.** We prove the last equation [25] first. Consider a partition \( a = t_0 < t_1 < \ldots < t_n = b \) and charts \( \varphi_j : U_j \to V_j \), \( \psi_j : P_j \to Q_j \) for \( G \) and \( H \), respectively, such that
\[
\varphi_j \circ \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], E),
\]
\[
\psi_j \circ f \circ \eta|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], F),
\]
where \( E \) and \( F \) are the model spaces of \( G \) and \( H \). Denote
\[
[\gamma_j] := (\varphi_j \circ \eta|_{[t_{j-1}, t_j]})' \in L^p([t_{j-1}, t_j], E),
\]
\[
[\xi_j] := (\psi_j \circ f \circ \eta|_{[t_{j-1}, t_j]})' \in L^p([t_{j-1}, t_j], F).
\]
Then (using (138)) we have
\[ [\xi_j] = ((\psi_j \circ f \circ \varphi_j^{-1}) \circ (\varphi_j \circ \eta_{[t_{j-1}, t_j]}))(d(\psi_j \circ f \circ \varphi_j^{-1})(\varphi_j \circ \eta_{[t_{j-1}, t_j]}, \gamma_j))].\]

Therefore, for \( [\delta] := (f \circ \eta) \) and \( t \in [t_{j-1}, t_j] \) we have
\[
\delta(t) = T\psi_j^{-1}((\psi_j \circ f \circ \eta)(t), d(\psi_j \circ f \circ \varphi_j^{-1})(\varphi_j \circ \eta)(t), \gamma_j(t))
\]
\[
= T\psi_j^{-1}((\psi_j \circ f \circ \varphi_j^{-1} \circ \varphi_j \circ \eta)(t), d(\psi_j \circ f \circ \varphi_j^{-1})(\varphi_j \circ \eta)(t), \gamma_j(t))
\]
\[
= (T\psi_j^{-1} \circ T(\psi_j \circ f \circ \varphi_j^{-1}))(\varphi_j \circ \eta)(t), \gamma_j(t))
\]
\[
= (Tf \circ T\varphi_j^{-1})(\varphi_j \circ \eta)(t), \gamma_j(t))
\]
\[
= (Tf \circ \gamma)(t),
\]
and analogously for \( t = b \).

Now,
\[
(\eta \cdot \zeta)^{-1} = (mG \circ (\eta, \zeta)) = [TmG \circ (\gamma, \zeta)] = [t \mapsto \eta(t), \zeta(t) + \gamma(t), \zeta(t)]
\]
and
\[
(\eta^{-1}) = (jG \circ \eta)^{-1} = [TjG \circ \gamma] = [t \mapsto -\eta(t)^{-1}, \gamma(t), \eta(t)^{-1}].
\]

The logarithmic derivative of \( \eta \in AC_{L^p}([a, b], G) \) is defined as follows (see Definition 5.6).

**Definition 4.4.** For \( \eta \in AC_{L^p}([a, b], G) \) define the left logarithmic derivative of \( \eta \) via
\[
\delta(\eta) := [\omega_l \circ \gamma],
\]
where \( [\gamma] = \hat{\eta} \) and \( \omega_l: TG \to \mathfrak{g}, v \mapsto \pi_{TG}(v)^{-1}. v \) with the bundle projection \( \pi_{TG} : TG \to G \).

As in [9] Lemma 5.11, we prove:

**Lemma 4.5.** Let \( G \) be a Lie group modelled on a sequentially complete locally convex space \( E \) and \( p \in [1, \infty] \). If \( \eta \in AC_{L^p}([a, b], G) \), then \( \delta(\eta) \in L^p([a, b], \mathfrak{g}) \).

**Proof.** By definition, there exists a partition \( a = t_0 < t_1 < \ldots < t_n = b \) and there exist charts \( \varphi_j : U_j \to V_j \) for \( G \) such that \( \eta([t_{j-1}, t_j]) \subseteq U_j \) and \( \eta_j := \varphi_j \circ \eta_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], E) \) for every \( j \in \{1, \ldots, n\} \). We denote \( [\gamma_j] := \eta_j^* \) and \( [\gamma] := \hat{\eta} \) and see that
\[
\omega_l \circ \gamma_{[t_{j-1}, t_j]} = \omega_l \circ T\varphi_j^{-1} \circ (\eta_j, \gamma_j) \in L^p([t_{j-1}, t_j], \mathfrak{g})
\]
by Lemma 2.13 since \( \omega_l \circ T\varphi_j^{-1}: V_j \times E \to \mathfrak{g} \) is continuous and linear in the second argument. From Lemma 2.10 follows \( \delta(\eta) = [\omega_l \circ \gamma] \in L^p([a, b], \mathfrak{g}) \).

Some of the following properties can be found in [9] Lemma 5.12.

**Lemma 4.6.** Let \( \eta, \zeta \in AC_{L^p}([a, b], G) \) and denote \( \delta(\eta) = [\gamma], \hat{\eta} = [\gamma], \delta(\zeta) = [\xi], \hat{\zeta} = [\xi] \). Then the following holds.
(i) We have
\[
\delta(\eta \cdot \zeta) = [t \mapsto \zeta(t)^{-1} \cdot \gamma(t) \cdot \zeta(t) + \xi(t)],
\]
and
\[
\delta(\eta^{-1}) = [t \mapsto -\gamma(t) \cdot \eta(t)^{-1}].
\]

(ii) We have \(\delta(\eta) = 0\) if and only if \(\eta\) is constant.

(iii) We have \(\delta(\eta) = \delta(\zeta)\) if and only if \(\eta = g \zeta\) for some \(g \in G\).

(iv) If \(f : G \to H\) is a Lie group homomorphism, then
\[
\delta(f \circ \eta) = [L(f) \circ \gamma].
\]

(v) For \(\alpha \leq \beta \leq b\) and \(f : [c, d] \to [a, b]\), \(f(t) := \alpha + \frac{t-c}{d-c}(\beta - \alpha)\) we have
\[
\delta(\eta \circ f) = \frac{\beta - \alpha}{d-c}[\gamma \circ f].
\]

Proof. (i) Using Equations (26) and (27), we get
\[
\begin{align*}
\delta(\eta \cdot \zeta) & = [t \mapsto (\eta(t) \cdot \zeta(t))^{-1} \cdot (\gamma(t) \cdot \zeta(t) + \eta(t) \cdot \xi(t))] \\
& = [t \mapsto (\zeta(t)^{-1} \cdot \eta(t)^{-1}) \cdot \gamma(t) \cdot \zeta(t) + (\zeta(t)^{-1} \cdot \eta(t)^{-1}) \cdot \eta(t) \cdot \xi(t)] \\
& = [t \mapsto \zeta(t)^{-1} \gamma(t) \cdot \zeta(t) + \eta(t)],
\end{align*}
\]
and
\[
\delta(\eta^{-1}) = [t \mapsto \eta(t) \cdot \gamma(t)^{-1} \cdot \gamma(t) \cdot \eta(t)^{-1}] = [t \mapsto -\gamma(t) \cdot \eta(t)^{-1}].
\]

(ii) Now, we assume that \(\delta(\eta) = 0\), that is, \(\eta \circ \eta^{-1} \cdot \gamma(t) = 0 \in L^p([a, b], g)\). In other words, \(\eta(t)^{-1} \gamma(t) = 0 \in g\) for a.e. \(t \in [a, b]\). Let \(a = t_0 < t_1 < \ldots < t_n = b\), charts \(\varphi_j\) and \([\gamma_j]\) be as in Definition 3.13. Then for \(\gamma(t) \in T_{\eta(t)}G\) we have \(d\varphi_j(\gamma(t)) = 0 \in E\) for a.e. \(t \in [t_{j-1}, t_j]\). On the other hand, we have \(d\varphi_j(\gamma(t)) = \gamma_j(t)\) for a.e. \(t \in [t_{j-1}, t_j]\), thus \([\gamma_j] = 0 \in L^p([t_{j-1}, t_j], E)\). That means, that \(\varphi_j \circ \eta \circ [t_{j-1}, t_j]\) is constant, whence \(\eta_{|_{[t_{j-1}, t_j]}}\) is constant, being \(\eta\) is constant, being continuous.

Conversely, assume \(\eta(t) = g \in G\) for all \(t \in [a, b]\). Then for some chart \(\varphi\) around \(g\) we have
\[
\varphi(g) = \varphi(\eta(t)) = \varphi(g) + \int_a^t \gamma_\eta(s) \, ds,
\]
for every \(t \in [a, b]\), thus \(\gamma_\eta(s) = 0\) for a.e. \(s \in [a, b]\), by Lemma 2.28 in (29) in other words, \((\varphi \circ \eta)' = 0 \in L^p([a, b], E)\). Therefore,
\[
\gamma(t) = T\varphi^{-1}(\varphi(\eta(t)), 0) = T\varphi^{-1}(\varphi(g), 0)
\]
a.e., whence
\[
\delta(\eta) = [t \mapsto \eta(t)^{-1} \cdot T\varphi^{-1}(\varphi(g), 0)] = [t \mapsto g^{-1} \cdot T\varphi^{-1}(\varphi(g), 0)] = 0 \in L^p([a, b], g).
\]

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Identifying equivalence classes with functions, we obtain
\[
\delta(\eta \cdot \zeta^{-1}) = [t \mapsto \zeta(t), \zeta(t)^{-1} - \xi(t), \zeta(t)^{-1}] = [t \mapsto \xi(t), \zeta(t)^{-1} - \xi(t), \zeta(t)^{-1}] = [t \mapsto \xi(t), \zeta(t)^{-1} - \xi(t), \zeta(t)^{-1}] = 0 \in L^p([a, b], \mathfrak{g}).
\]

Then, by the above, the curve \(\eta \cdot \zeta^{-1}\) is constant, say \(\eta \cdot \zeta^{-1} = g \in G\), thus \(\eta = g\zeta\).

Conversely, assume \(\eta = g\zeta\). We define \(\eta_g : [a, b] \to G, t \mapsto g\) in \(AC_{L^p}([a, b], G)\), then \([\gamma_g] = \delta(\eta_g) = 0 \in L^p([a, b], \mathfrak{g})\) (by the above), whence
\[
\delta(\eta) = \delta(\eta_g \cdot \zeta) = [t \mapsto \xi(t)] = \delta(\zeta),
\]
using Equation (29).

(iv) By (28), we have

\[
\delta(f \circ \eta) = [t \mapsto f(\eta(t))^{-1} \cdot T f(\gamma(t))]
\]
\[
= [t \mapsto (T \lambda_{f(\eta(t))}^H \circ T f)(\gamma(t))]
\]
\[
= [t \mapsto T(\lambda_{f(\eta(t))}^H \circ f)(\gamma(t))]
\]
\[
= [t \mapsto T(f \circ \lambda_{\eta(t)^{-1}}^G)(\gamma(t))]
\]
\[
= [t \mapsto T f(\eta(t))^{-1} \cdot \bar{\gamma}(t)]
\]
\[
= [t \mapsto L(f)(\gamma(t))] = [L(f) \circ \gamma].
\]

Note that we used \(\lambda_{f(\eta(t))}^H \circ f = f \circ \lambda_{\eta(t)^{-1}}^G\) as \(f\) is a group homomorphism.

(v) Consider a partition \(c = t_0 < t_1 < \ldots < t_n = d\) and charts \(\varphi : U_j \to V_j\) with \(\eta(f([t_{j-1}, t_j])) \subseteq U_j\). Write
\[
f_j := f|_{[t_{j-1}, t_j]}, \quad \eta_j := \eta|_{[f(t_{j-1}), f(t_j)]}.
\]

Identifying equivalence classes with functions, we obtain
\[
\delta(\eta \circ f)|_{[t_{j-1}, t_j]} = \omega_t \circ T \varphi_j^{-1} \circ (\varphi_j \circ \eta \circ f_j, (\varphi_j \circ \eta \circ f_j)^r)
\]
\[
= \omega_t \circ T \varphi_j^{-1} \circ (\varphi_j \circ \eta \circ f_j, \beta - \alpha \frac{\beta - \alpha}{d - c} (\varphi_j \circ \eta_j)^r \circ f_j)
\]
\[
= \omega_t \circ T \varphi_j^{-1} \circ (\varphi \circ \eta_j, \beta - \alpha \frac{\beta - \alpha}{d - c} (\varphi_j \circ \eta_j)^r \circ f_j)
\]
\[
= \frac{\beta - \alpha}{d - c} (\delta(\eta) \circ f|_{[t_{j-1}, t_j]})
\]

using the formula in Lemma 3.12 and the linearity of \(\omega_t \circ T \varphi_j^{-1}\) in its second argument.

\[\square\]

4.2 \(L^p\)-regularity

The \(L^p\)-semiregularity and \(L^p\)-regularity of Lie groups modelled on sequentially complete locally convex spaces is defined as in [9], Definition 5.16.
**Definition 4.7.** Let $G$ be a smooth Lie group modelled on a sequentially complete locally convex space. For $p \in [1, \infty]$, the Lie group $G$ is called $L^p$-semiregular if for every $\gamma \in L^p([0,1], g)$ the initial value problem
\[
\delta(\eta) = \gamma, \quad \eta(0) = e
\]
has a solution $\eta_\gamma \in AC_{L^p}([a,b], G)$ (which is unique, by Lemma 4.6).

An $L^p$-semiregular Lie group $G$ is called $L^p$-regular if the obtained function
\[
\text{Evol}: L^p([0,1], g) \to AC_{L^p}([a,b], G), \quad \gamma \mapsto \eta_\gamma
\]
is smooth.

**Remark 4.8.** As in [9, Remark 5.18], we note that if a Lie group $G$ is $L^p$-regular, then the function
\[
ev_1: AC_{L^p}([0,1], G) \to G, \eta \mapsto \eta(1)
\]
is smooth, since so is the evaluation map $\text{ev}_1: AC_{L^p}([0,1], G) \to G, \eta \mapsto \eta(1)$ (see Lemma 3.21).

We prove the result from [9, Proposition 5.20].

**Proposition 4.9.** Let $G$ be an $L^p$-semiregular Lie group. Then the function $\text{Evol}$ is smooth if and only if $\text{Evol}$ is smooth as a function to $C([0,1], G)$.

**Proof.** First assume that $\text{Evol}: L^p([0,1], g) \to AC_{L^p}([0,1], G)$ is smooth. As the inclusion map $\text{incl}: AC_{L^p}([0,1], G) \to C([0,1], G)$ is smooth (see Lemma 3.20), the composition $\text{incl} \circ \text{Evol}: L^p([0,1], g) \to C([0,1], G)$ is smooth.

Conversely, assume that $\text{Evol}: L^p([0,1], g) \to C([0,1], G)$ is smooth; for some fixed $\bar{\gamma} \in L^p([0,1], g)$ we are going to find some open neighborhood $P$ of $\bar{\gamma}$ such that $\text{Evol}|_P: P \to AC_{L^p}([0,1], G)$ is smooth.

To this end, let $U \subseteq G$ be an open identity neighborhood and $\varphi: U \to V$ be a chart. Then $U$ contains some open identity neighborhood $W$ such that $WW \subseteq U$. For $\eta_{\bar{\gamma}} := \text{Evol}(\bar{\gamma})$, the subset
\[
Q := \{ \zeta \in C([0,1], G) : \eta_{\bar{\gamma}}^{-1} \cdot \zeta \in C([0,1], W) \}
\]
is an open neighborhood of $\eta_{\bar{\gamma}}$. Set
\[
P := \text{Evol}^{-1}(Q).
\]
Now, we want to show that the function
\[
P \to AC_{L^p}([0,1], G), \quad \gamma \mapsto \eta_\gamma := \text{Evol}(\gamma)
\]
is smooth.

As $\eta_\gamma$ is continuous, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\eta_{\gamma}(t_j)^{-1} \eta_{\gamma}([t_{j-1}, t_j]) \subseteq W$ for each $j \in \{1, \ldots, n\}$. Using the function $\Gamma_G$ from Lemma 3.23 the map in (35) will be smooth if
\[
P \mapsto \prod_{j=1}^n AC_{L^p}([t_{j-1}, t_j], G), \quad \gamma \mapsto (\eta_\gamma|_{[t_{j-1}, t_j]})_{j=1,\ldots,n}
\]
is smooth, which will be the case if each of the components
\[ P \mapsto AC_{L^p}([t_{j-1}, t_j], G), \quad \gamma \mapsto \eta_\gamma|_{[t_{j-1}, t_j]} \]  
(37)
is smooth. As left translations on the Lie group \( AC_{L^p}([t_{j-1}, t_j], G) \) are smooth diffeomorphisms, the function in (37) will be smooth if
\[ P \mapsto AC_{L^p}([t_{j-1}, t_j], G), \quad \gamma \mapsto \eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]} \]  
(38)
is a smooth map.

Now, for every \( t \in [t_{j-1}, t_j] \) we have
\[ \eta_\gamma(t_j)^{-1} \eta_\gamma(t) = \eta_\gamma(t_j)^{-1} \eta_\gamma(t) \eta_\gamma(t_j)^{-1} \eta_\gamma(t) \in WW \subseteq U, \]
in other words, \( \eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]} \in AC_{L^p}([t_{j-1}, t_j], U) \). Thus the smoothness of (38) will follow from the smoothness of (37).

Using the definition of the topology on \( AC_{L^p}([t_{j-1}, t_j], E) \) (see Definition 3.1), we will show that
\[ P \mapsto E \times L^p([t_{j-1}, t_j], E), \quad \gamma \mapsto (\varphi(\eta_\gamma(t_j)^{-1} \eta_\gamma(t_{j-1})), (\varphi \circ \eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]}))^t \]
is smooth.

Using the assumed smoothness of \( P \mapsto C([0, 1], G), \gamma \mapsto \eta_\gamma \), we see that the first component of the above function is smooth. Therefore, it remains to show that
\[ P \mapsto L^p([t_{j-1}, t_j], E), \quad \gamma \mapsto (\varphi \circ \eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]})^t \]  
(40)
is smooth.

Identifying equivalence classes with functions, we have
\[ (\varphi \circ \eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]})^t = d\varphi \circ (\eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]}). \]
Consider the smooth function
\[ \sigma : G \times g \to TG, \quad (g, v) \mapsto g.v. \]
We have
\[ d\varphi \circ (\eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]})^t = d\varphi \circ \sigma \circ \eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]} \cdot \delta(\eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]}), \]
\[ = d\varphi \circ \sigma \circ (\eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]} \cdot \delta(\eta_\gamma|_{[t_{j-1}, t_j]})), \]
\[ = d\varphi \circ \sigma \circ (\eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]} \cdot \gamma|_{[t_{j-1}, t_j]}), \]
using (iii) from Lemma 4.6. Hence the map in (40) will be smooth if
\[ P \mapsto L^p([t_{j-1}, t_j], E), \quad \gamma \mapsto d\varphi \circ \sigma \circ (\eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]} \cdot \gamma|_{[t_{j-1}, t_j]}) \]  
(41)
is smooth. But this is true, the function being a composition of the smooth functions
\[ P \mapsto C([t_{j-1}, t_j], U) \times L^p([t_{j-1}, t_j], E), \quad \gamma \mapsto (\eta_\gamma(t_j)^{-1} \eta_\gamma|_{[t_{j-1}, t_j]} \cdot \gamma|_{[t_{j-1}, t_j]}) \]
\[ , \]  
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and
\[ C([t_{j-1}, t_j], U) \times L^p([t_{j-1}, t_j], g) \to L^p([t_{j-1}, t_j], g), \quad (\eta, \gamma) \mapsto d\varphi \circ \sigma \circ (\eta, \gamma), \]

(the smoothness of the last function holds by Proposition 3.24 as the composition \(d\varphi \circ \sigma: G \times g \to E\) is linear in the second argument).

Analogously to [9, Corollary 5.21] we get the following:

**Corollary 4.10.** Let \( G \) be a Lie group and \( p, q \in [1, \infty] \) with \( q \geq p \). If \( G \) is \( L^p \)-regular, then \( G \) is \( L^q \)-regular. Furthermore, in this case \( G \) is \( C^0 \)-regular.

**Proof.** Assume that \( G \) is \( L^p \)-regular and \( q \geq p \). Since \( L^q([0, 1], g) \subseteq L^p([0, 1], g) \) with a smooth inclusion map (Remark 2.4), the Lie group \( G \) is \( L^q \)-semiregular and \( L^q([0, 1], g) \to C([0, 1], G), \gamma \mapsto \text{Evol}(\gamma) \) is smooth. From Proposition 4.9 follows, that \( L^q([0, 1], g) \to AC_L^{L^q}([0, 1], G), \gamma \mapsto \text{Evol}(\gamma) \) is smooth, whence \( G \) is \( L^q \)-regular.

Further, since \( C([0, 1], g) \subseteq L^p([0, 1], g) \), the Lie group is \( C^0 \)-regular. Since the inclusion map \( \text{incl}: C([0, 1], g) \to L^p([0, 1], g) \) is smooth, as well as the evaluation map \( \text{ev}: C([0, 1], G) \to G \), the composition \( C([0, 1], g) \to G, \gamma \mapsto \text{Evol}(\gamma)(1) \) is smooth, whence \( G \) is \( C^0 \)-regular.

The next Proposition shows that it suffices for a Lie group \( G \) to be \( L^p \)-regular, if it is merely locally \( L^q \)-regular (see [9, Definition 5.19, Proposition 5.25]).

**Proposition 4.11.** Let \( G \) be a Lie group modelled on a sequentially complete locally convex space \( E \), let \( g \) denote the Lie algebra of \( G \). Let \( \Omega \subseteq L^p([0, 1], g) \) be an open 0-neighbourhood. If for every \( \gamma \in \Omega \) the initial value problem \( (33) \) has a (necessarily unique) solution \( \eta_{\gamma} \in AC_L^{L^p}([0, 1], G) \), then \( G \) is \( L^p \)-semiregular. If, in addition, the function \( \text{Evol}: \Omega \to AC_L^{L^p}([0, 1], G), \gamma \mapsto \eta_{\gamma} \) is smooth, then \( G \) is \( L^p \)-regular.

**Proof.** First, fix some \( \gamma \in L^p([0, 1], g) \) and for \( n \in \mathbb{N}, k \in \{0, \ldots, n-1\} \) define \( \gamma_{n,k} \in L^p([0, 1], g) \) as in (3). Let \( Q \) be a continuous seminorm on \( L^p([0, 1], g) \) such that \( E^Q_v(0) \subseteq \Omega \). By Lemma 2.17 there exists some \( n \in \mathbb{N} \) such that \( \gamma_{n,k} \in \Omega \) for \( k \in \{0, \ldots, n-1\} \). We set \( \eta_{n,k} := \text{Evol}(\gamma_{n,k}) \in AC_L^{L^p}([0, 1], G) \) and define \( \eta_{n}: [0, 1] \to G \) via
\[ \eta_{n}(t) := (\eta_{n,0} \circ f_{n,0})(t), \quad \text{for } t \in [0, 1/n], \]
and
\[ \eta_{n}(t) := \eta_{n,0}(1) \cdots \eta_{n,k-1}(1)(\eta_{n,k} \circ f_{n,k})(t), \quad \text{for } t \in [k/n, k+1/n], \]
where
\[ f_{n,k}: [k/n, k+1/n] \to [0, 1], \quad f_{n,k}(t) := nt - k. \]

Then we easily verify that the function \( \eta_{n} \) is continuous and from Lemma 3.22 follows that \( \eta_{n}|_{[k/n, k+1/n]} \in AC_L^{L^p}([k/n, k+1/n], G) \), whence \( \eta_{n} \in AC_L^{L^p}([0, 1], G) \). Furthermore, \( \eta_{n}(0) = e \) and \( \delta(\eta_{n}) = \gamma_{n} \), see Lemma 3.15. Consequently, \( \text{Evol}(\gamma) := \eta_{n} \) solves the initial value problem in (33) for \( \gamma \), whence \( G \) is \( L^p \)-semiregular.
Now, assume that Evol: $\Omega \to AC_{L^p}([0,1], G)$ is smooth; we will show the smoothness of Evol on some open neighborhood of $\gamma$. From the continuity of each
\[
\pi_{n,k}: L^p([0,1], g) \to L^p([0,1], g), \quad \xi \mapsto \xi_{n,k},
\]
(see Lemma 2.15) follows that there exists an open neighborhood $W \subseteq L^p([0,1], g)$ of $\gamma$ such that $\pi_{n,k}(W) \subseteq \Omega$ for every $k \in \{0, \ldots, n-1\}$. Then
\[
\text{Evol} : W \to AC_{L^p}([0,1], G), \quad \xi \mapsto \eta_{\xi}
\]
is defined, where $\eta_{\xi}$ is as in (42) and (43). It will be smooth if we show (using Lemma 3.23) that each
\[
W \to AC_{L^p}([k/n, k+1/n], G), \quad \xi \mapsto \eta_{\xi}|_{[k/n, k+1/n]} \tag{44}
\]
is smooth. But, by construction, we have
\[
\eta_{\xi}|_{[0, 1/n]} = \text{Evol}(\xi_{n,0}) \circ f_{n,0}
\]
and
\[
\eta_{\xi}|_{[k/n, k+1/n]} = \text{evol}(\xi_{n,0}) \cdots \text{evol}(\xi_{n,k-1}) \text{Evol}(\xi_{n,k}) \circ f_{n,k},
\]
so the smoothness of (44) follows from Lemma 3.22 and Remark 4.8.

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