Unified entropic measures of quantum correlations induced by local measurements

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We introduce quantum correlations measures based on the minimal change in unified entropies induced by local rank-one projective measurements, divided by a factor that depends on the generalized purity of the system in the case of non-additive entropies. In this way, we overcome the issue of the artificial increasing of the value of quantum correlations measures based on non-additive entropies when an uncorrelated ancilla is appended to the system, without changing the computability of our entropic correlations measures with respect to the previous ones. Moreover, we recover as limiting cases the quantum correlations measures based on von Neumann and Rényi entropies (i.e., additive entropies), for which the adjustment factor becomes trivial. In addition, we distinguish between total and semiquantum correlations and obtain some relations between them. Finally, we obtain analytical expressions of the entropic correlations measures for typical quantum bipartite systems.

Keywords: Quantum correlations, Quantum unified entropies, Local projective measurements

I. INTRODUCTION

Quantum correlations lie at the heart of the difference between classical and quantum worlds. There are at least two paradigms to address this issue beyond the usual entangled-separable distinction \cite{1}. For instance, steering correlations have recently been formulated in a operational way in \cite{2}, although their origins can be found in the seminal works by Einstein, Podolski and Rosen \cite{3} and Schrödinger \cite{4}. These correlations intermediate between entanglement and nonlocality \cite{5} (i.e., a violation of Bell inequalities). On the other hand, it possible to identify quantum correlations induced by local rank-one projective measurements, divided by a factor that depends on the generalized purity only in the case of nonadditive entropies (this adjusting factor becomes trivial). In addition, we recover as limiting cases the quantum correlations measures based on von Neumann and Rényi entropies (i.e., additive entropies), for which the adjustment factor becomes trivial. Indeed, we propose as quantum correlations measures the minimal change in unified entropies (or total entropies), for which the adjustment factor becomes trivial. In \cite{12, 13}, we propose quantum correlations measures based on the minimal change in unified entropies induced by a local rank-one measurement, divided by a factor that depends on the generalized purity of the system in the case of nonadditive entropies (this adjusting factor becomes trivial for additive entropies). Several quantum correlations measures discussed in the literature, like \cite{16, 27}, among others, are particular cases of (or close to) our proposal (see \cite{28} for a recent review of quantum measurements). Indeed, the case of trace form entropies \cite{29}, which are nonadditive entropies (except the von Neumann case), has been dealt in \cite{19, 20, 24} and deserves a particular mention. These entropic quantum correlations measures artificially increase when an uncorrelated ancilla is appended to the system (the geometric discord \cite{22} has the same issue, as it has been pointed out in \cite{20}). The nonadditivity of trace form entropies is the cause of this problem. We solve this in the case of (q, s)-entropies by introducing a generalized purity factor, similarly to what has been done with the geometric discord, that is dividing it by the purity \cite{25}. In this way, we obtain a family of (q, s)-entropic measures of quantum correlations that are invariant under the addition of an uncorrelated ancilla, both in the cases of additive and non-additive entropies. In addition, the computability of our entropic quantum correlations measures remain equal to the previous ones \cite{19, 20, 24}, since the adjustment factor is simply the trace of a power of the density operator.

The outline of this work is as follows. Our proposal and main results are given in Sec. IIA. In IIA, we review the notion and some properties of (q, s)-entropies and majorization, and we introduce a family of entropic measures of disturbance due to a projective measurement. In IIB, we introduce the general entropic quantum correlations measures by quantifying disturbances due to local projective measurements, distinguishing between total and semiquantum correlations. Besides, we provide basic properties that justify our proposal. In IIC, we find a lower bound of the
II. ENTROPIC MEASURES OF QUANTUM CORRELATIONS

A. Unified entropies, majorization and \((q,s)\)-disturbances

Let a quantum system be described by a density operator \(\rho\), that is, a trace-one positive semidefinite operator acting on an \(N\)-dimensional Hilbert space, \(\mathcal{H}^N\). The quantum unified \((q,s)\)-entropies of the state are defined as \([14, 15]\),

\[
S_{(q,s)}(\rho) = \frac{\text{Tr} \rho^q}{1-q} - 1,
\]

for entropic indexes \(q > 0, q \neq 1\) and \(s \neq 0\). Notice that the quantum Tsallis entropies \([10]\) are obtained for \(s = 1\),

\[
S_{(q,1)}(\rho) \equiv S_T^q(\rho) = \frac{\text{Tr} \rho^q - 1}{1-q},
\]

being an interesting case \(q = 2\), \(S_{(2,1)}(\rho) \equiv S_2(\rho) = 1 - \text{Tr} \rho^2\), which is directly related to the purity of the state. On the other hand, von Neumann entropy \([8]\) is recovered in the limiting case \(q \to 1\),

\[
S_{(1,s)}(\rho) \equiv S(\rho) = -\text{Tr} \rho \ln \rho,
\]

whereas Rényi entropies \([9]\) are recovered in the limiting case \(s \to 0\),

\[
S_{(q,0)}(\rho) \equiv S_R^q(\rho) = \frac{\ln \text{Tr} \rho^q}{1-q}.
\]

A feature of \((q,s)\)-entropies is their nonadditive character \([14]\), which is reflected in the sum rule for product states \(\rho^A \otimes \rho^B\) acting on a Hilbert space \(\mathcal{H}^{N^A} \otimes \mathcal{H}^{N^B}\),

\[
S_{(q,s)}(\rho^A \otimes \rho^B) = S_{(q,s)}(\rho^A) + S_{(q,s)}(\rho^B) + (1-q)s S_{(q,s)}(\rho^A) S_{(q,s)}(\rho^B).
\]

Notice that in the cases \(q = 1\) or \(s = 0\), one recovers the additivity of von Neumann and Rényi entropies.

A closed related concept to entropy is majorization (see e.g. \([31]\)). Let us consider two density operators \(\rho\) and \(\sigma\), and the corresponding probability vectors \(p\) and \(q\) formed by the eigenvalues of \(\rho\) and \(\sigma\), respectively, sorted in decreasing order. Then, \(\rho\) is majorized by \(\sigma\), denoted as \(\rho \prec \sigma\), means that

\[
\sum_{i=1}^n p_i \leq \sum_{i=1}^n q_i \quad \text{for all} \quad n = 1, \ldots, N - 1, \quad \text{and} \quad \sum_{i=1}^N p_i = \sum_{i=1}^N q_i,
\]

where \(N = \max \{\text{rank } \rho, \text{rank } \sigma\}\) and rank denotes the rank of a density operator. Notice that if rank \(\rho \leq \text{rank } \sigma\) we complete the vector \(p\) with 0 entries to have the same length of \(q\), and vice versa. This has no impact in the value of unified entropies due to the expansibility property.

It can be shown that \((q,s)\)-entropies preserve the majorization relation (see e.g. \([15, 32]\)), that is,

\[
\text{if } \rho \prec \sigma \quad \text{then} \quad S_{(q,s)}(\rho) \geq S_{(q,s)}(\sigma),
\]

with equality if and only if \(\rho\) and \(\sigma\) have the same eigenvalues. We observe that the reciprocal does not hold in general, which means that majorization is stronger (as an order relation) than a single choice of the entropic indexes.

Now, using the Schur-concavity it is straightforward to show that \((q,s)\)-entropies are lower and upper bounded:

\[
0 \leq S_{(q,s)}(\rho) \leq \frac{N^{1-q}s - 1}{(1-q)s},
\]

where the first inequality is attained for pure states, whereas the second one for the maximally mixed state \(\rho^* = \frac{I}{N}\).
On the other hand, it can be shown that the eigenvalues of a density operator $\rho$ are invariant under arbitrary unitary transformations $U$, in other words $\rho$ and $U\rho U^\dagger$ have the same eigenvalues. Hence, we have that $(q, s)$-entropies are invariant under unitary transformations

$$S_{(q,s)}(\rho) = S_{(q,s)}(U\rho U^\dagger).$$

Moreover, we will see in the next subsection that the change of in entropy due to local measurements plays a key role in order to quantify quantum correlations. Before that, we recall the action of any bistochastic map over an arbitrary state. A bistochastic (or completely positive, trace-preserving unital) map $\mathcal{E}$ can be written in the Kraus form as $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ with both sets of positive operators $\{E_k\}$ (completely positive) and $\{E_k^\dagger\}$ (unital) summing the identity (see e.g. [33]). Notice that this map leaves invariant the maximally mixed state (i.e., terms of majorization) than the initial state $\rho$. We have $S(\rho) = S(U\rho U^\dagger)$. As projective measurements are particular cases of bistochastic maps, we have also an inequality similar to (11) for projective measurements. Thus, we propose to use the difference of $(q, s)$-entropies between the final and initial states (rescaled by a factor depending on the generalized purity) as a signature of the disturbance of the state of a system due to the measurement, that is

$$D_{(q,s)}^{\Pi}(\rho) = \frac{S_{(q,s)}(\Pi(\rho)) - S_{(q,s)}(\rho)}{\langle \text{Tr } \rho^s \rangle}.$$  

For any choice of the entropic indexes this quantity is nonnegative and vanishes if only if the measurement does not disturb the state (i.e., $\Pi(\rho) = \rho$), which happens when measuring in the basis that diagonalises $\rho$. Notice that the rescaling factor plays no role for von Neumann and Rényi entropies (additive entropies), on the contrary it does for nonadditive entropies. In the next subsection, we will clarify the importance of the rescaling $\langle \text{Tr } \rho^s \rangle^s$ when dealing with quantum correlations measures based on nonadditive entropies.

Finally, notice that two interesting cases arise from the definition (12). The first one consists in considering the von Neumann entropy, in this case the disturbance can be recast as the quantum relative entropy (or quantum Kullback-Leibler divergence) between $\rho$ and $\Pi(\rho)$, that is

$$D_{(1,s)}^{\Pi}(\rho) \equiv D^{\Pi}(\rho) = S(\rho \| \Pi(\rho)),$$

where $S(\rho \| \sigma) = \text{Tr}(\rho \ln \rho - \ln \sigma)$ is the quantum relative entropy. The second one comes from evaluating (12) at Tsallis entropy with entropic index equal to 2, for which the disturbance expresses in terms of the Hilbert-Schmidt distance between $\rho$ and $\Pi(\rho)$ divided by the purity of $\rho$,

$$D_{(2,1)}^{\Pi}(\rho) \equiv D_{2}^{\Pi}(\rho) = \frac{\|\rho - \Pi(\rho)\|^2}{\text{Tr } \rho^2},$$

where $\|A\| = \sqrt{\text{Tr } A^\dagger A}$ is the Hilbert-Schmidt norm.

**B. Quantum correlations from disturbance due to a local projective measurement**

Let us consider a bipartite quantum system $AB$ with density operator $\rho^{AB}$ acting on a product finite dimensional Hilbert space, $\mathcal{H}^{N_{AB}} = \mathcal{H}^{N_A} \otimes \mathcal{H}^{N_B}$, where $N_{AB} = N_A N_B$. Following [10], we consider the local rank-one projective measurements (without postselection), $\Pi^{A} = \{P_i^A \otimes I^B\}$, $\Pi^{B} = \{I^A \otimes P_j^B\}$ and $\Pi^{AB} = \{P_i^A \otimes P_j^B\}$, where $\{P_i\}$
and \{P^B_i\} are a set of orthogonal rank-one projectors that sum to the identity, \(I^A\) and \(I^B\), respectively. Then, the resulting states after these measurements are

\[
\Pi^A(\rho^{AB}) = \sum_i P^A_i \otimes I^B \rho^{AB} P^A_i \otimes I^B = \sum_i p^A_i P^A_i \otimes \rho^{B|i},
\]

(15)

\[
\Pi^B(\rho^{AB}) = \sum_j I^A \otimes P^B_j \rho^{AB} I^A \otimes P^B_j = \sum_j p^B_j \rho^{A|j} \otimes P^B_j,
\]

(16)

\[
\Pi^{AB}(\rho^{AB}) = \Pi^A \circ \Pi^B(\rho^{AB}) = \Pi^B \circ \Pi^A(\rho^{AB})
\]

\[
= \sum_{ij} P^A_i \otimes P^B_j \rho^{AB} P^A_i \otimes P^B_j = \sum_{ij} \rho^{AB}_{ij} P^A_i \otimes P^B_j,
\]

(17)

where \(\rho^{B|i} = \frac{\text{Tr}(P^A_i \otimes I^B \rho^{AB})}{p^A_i}\) with \(p^A_i = \text{Tr}(P^A_i \otimes I^B \rho^{AB})\), \(\rho^{A|j} = \frac{\text{Tr}(I^A \otimes P^B_j \rho^{AB})}{p^B_j}\) with \(p^B_j = \text{Tr}(I^A \otimes P^B_j \rho^{AB})\) and \(p^{AB}_{ij} = \text{Tr}(P^A_i \otimes P^B_j \rho^{AB})\). According to [16], these states are called classical-quantum (CQ), quantum-classical (QC) and classical-classical (CC) correlated states with respect to the local measurements \(\Pi^A\), \(\Pi^B\) and \(\Pi^{AB}\), respectively. A state is said CQ correlated if there is a local projective measurement over \(A\) that does not disturb it, i.e., \(\Pi^A(\rho^{AB}) = \rho^{AB}\) (analogously for QC and CC correlated states). All these states are separable (i.e., nonentangled), as they are convex combinations of product states [1], although not all separable states are of the forms (15)–(17).

The following properties justify our proposal (19) as measures of quantum correlations:

(i) nonnegativity: \(D^{K}_{(q,s)}(\rho^{AB}) \geq 0\) with equality if and only if \(\rho^{AB} \in \Omega^K\). Accordingly, \(D^{K}_{(q,s)}\) are semi-quantum correlations measures (with respect to \(H^K\)), whereas \(D^{AB}_{(q,s)}\) are total quantum correlation measures;

(ii) invariance under local unitary operators: \(D^{K}_{(q,s)}(U \otimes V \rho^{AB} U^\dagger \otimes V^\dagger) = D^{K}_{(q,s)}(\rho^{AB})\), where \(U\) and \(V\) are a unitary operations over \(A\) and \(B\) respectively; and

(iii) invariance when an uncorrelated ancilla is appended to the system: \(D^{K}_{(q,s)}(\rho^{AB} \otimes \rho^C) = D^{K}_{(q,s)}(\rho^{AB})\) for bipartitions \(A|BC\) or \(B|AC\) (for the bipartition \(AB|C\) the quantum correlations measures naturally vanish).

The first property is a direct consequence of majorization relation between the states after and before local projective measurements. The second one can be derived from the definition of our measure, Eq. (19), noting that \(\Pi^K(U \otimes V \rho^{AB} U^\dagger \otimes V^\dagger) = U \otimes V \Pi^K(\rho^{AB}) U^\dagger \otimes V^\dagger\), with \(\Pi^K = U^\dagger \otimes V^\dagger \Pi^K U \otimes V\), and recalling the invariance of \((q,s)\)-entropies under unitary transformations. The third property is more subtle and it is related to the sum rule (5) of the \((q,s)\)-entropies. Indeed, the generalized purity factor \((\text{Tr}(\rho^{AB})^q)^s\) plays a crucial role to fulfill this property in the case of nonadditive entropies, without affecting the complexity of computability of the measures. In general, this property has not been taking into account in the literature of nonadditive entropic measures of quantum correlations. For instance, entropic quantum correlations measures based on the difference of trace form entropies\(^1\) i.e., \(S_\phi(\rho) = \text{Tr} \phi(\rho)\)

\(^1\) Notice that \((q,s)\)-entropies reduce to a trace form only if \(s = 1\) (Tsallis entropies).
with $\phi$ concave and $\phi(0) = 0$ [29], have been dealt in Refs. [19] [21]. However, these measures are not invariant when an uncorrelated ancilla is appended to the system, except for the von Neumann case. This is direct consequence of nonadditivity of trace form entropies. For a more general discussion about necessary and reasonable conditions of quantum correlations measures, see [36]. Moreover, our semiquantum correlations measures can be also interpreted as a quantum deviation from the Bayes rule in a way similar to that discussed in [21].

We remark that our quantum correlations measures include some important cases already discussed in the literature. The first one consists in evaluating (19) for the von Neumann entropy. In this case we reobtain the so-called information deficit [18], which can be rewritten in terms of the minimal relative entropy over the sets $\Omega^K$ [21].

$$D^K(\rho^{AB}) = \min_{\Pi^K} S(\rho^{AB} \| \Pi^K(\rho^{AB})) = \min_{\chi^{AB} \in \Omega^K} S(\rho^{AB} \| \chi^{AB}).$$

(20)

The second one arises when evaluating (19) for the Tsallis entropy with entropic index equal to 2. This case is close to the geometric discord [22].

$$D^G_2(\rho^{AB}) = \min_{\chi^{AB} \in \Omega^K} \|\rho^{AB} - \chi^{AB}\|^2.$$  

(21)

Indeed, using the expression of $D^G_2$ in terms of local projective measurements given in [23], we obtain

$$D^G_2(\rho^{AB}) = \frac{\min_{\Pi^K} \|\rho^{AB} - \Pi^K(\rho^{AB})\|^2}{\text{Tr}(\rho^{AB})^2} = \frac{D^G_2(\rho^{AB})}{\text{Tr}(\rho^{AB})^2}.$$  

(22)

Notice that $D^G_2$ is not invariant when an uncorrelated ancilla is appended to the system [30]. The purity rescaled factor solves this issue [25], although there is not the unique way to do it (see e.g [25, 37]). Finally, notice that in the case of Rényi entropies, which has recently been introduce in [27], our measure fulfills the desired invariance property when appending an uncorrelated ancilla to the system.

**C. Lower bound and its relation with entanglement**

First, let us note that since QC, CQ and CC correlated states [15]–[17] are separable, they fulfill some general entropic separability inequalities (see e.g [32]),

$$S_{(q,s)}(\Pi^K(\rho^{AB})) \geq \max \{S_{(q,s)}(\text{Tr}_A \Pi^K(\rho^{AB})), S_{(q,s)}(\text{Tr}_B \Pi^K(\rho^{AB}))\}.$$  

(23)

On the other hand, the corresponding final reduced states are

$$\text{Tr}_A \Pi^A(\rho^{AB}) = \rho^B \text{ and } \text{Tr}_B \Pi^A(\rho^{AB}) = \text{Tr}_B \Pi^B(\rho^{AB}) = \sum_i P_i^A \rho_i^A = \rho_{\text{diag}}^A,$$

(24)

$$\text{Tr}_B \Pi^B(\rho^{AB}) = \rho^A \text{ and } \text{Tr}_B \Pi^B(\rho^{AB}) = \text{Tr}_A \Pi^B(\rho^{AB}) = \sum_j P_j^B \rho_j^B = \rho_{\text{diag}}^B,$$

(25)

where $\rho_{\text{diag}}^L$ denotes the diagonal of $\rho^L$ in the basis underlying by $\{P_i^L\}$. Since $\rho_{\text{diag}}^L < \rho^L$ [33] and due to the Schur-concavity of the $(q,s)$-entropies, inequality (23) reduces to

$$S_{(q,s)}(\Pi^K(\rho^{AB})) \geq \max \{S_{(q,s)}(\rho^A), S_{(q,s)}(\rho^B)\}.$$  

(26)

Thus, plugging (26) into (18) to lowerbound $D^K_{(q,s)}(\rho^{AB})$ and taking the minimum, we obtain that the quantum correlations measures are lower bounded, as follows

$$D^K_{(q,s)}(\rho^{AB}) \geq \max \left\{ \frac{S_{(q,s)}(\rho^A) - S_{(q,s)}(\rho^{AB})}{(\text{Tr}(\rho^{AB})^q)^s}, \frac{S_{(q,s)}(\rho^B) - S_{(q,s)}(\rho^{AB})}{(\text{Tr}(\rho^{AB})^q)^s} \right\}.$$  

(27)

Notice that this lower bound could be nontrivial only for entangled sates; indeed, the right hand side of (27) is negative for separable states. A similar result has already been obtained in the case of trace form entropies [19].

Now, let us consider a pure state $\rho^{AB} = |\Psi^{AB}\rangle \langle \Psi^{AB}|$. Let us suppose that

$$|\Psi^{AB}\rangle = \sum_{k=1}^n \sqrt{\lambda_k} |k^A\rangle \otimes |k^B\rangle$$  

(28)
is the Schmidt decomposition of $|\psi^{AB}\rangle$ ($n \leq \min\{N^A, N^B\}$ and $\{|k^L\rangle\}$ are a orthonormal set). Thus, it can be shown that the reduced states $\rho^A = \text{Tr}_B |\Psi^{AB}\rangle \langle \Psi^{AB}|$ and $\rho^B = \text{Tr}_A |\Psi^{AB}\rangle \langle \Psi^{AB}|$ have the same unified entropy and, as a consequence, the lower bound \cite{27} reduces to $S_{(q,s)}(\rho^A) = S_{(q,s)}(\rho^B)$ for pure states $\rho^{AB}$. Moreover, this bound is saturated when the local measurements are taking in the Schmidt basis. After these measurements, i.e., choosing the local projectors as $P^L_k = |k^L\rangle \langle k^L|$ (completed to obtain $N^L$ projector), the state is given by $\Pi^K(\rho^{AB}) = \sum_k \lambda_k P^A_k \otimes P^B_k$, with unified entropies $S_{(q,s)}(\Pi^K(\rho^{AB})) = S_{(q,s)}(\rho^A) = S_{(q,s)}(\rho^B)$. Therefore, we obtain that for pure states the entropic quantum correlation measures becomes a generalization of the entanglement entropy,

$$D^K_{(q,s)}(\rho^{AB}) = S_{(q,s)}(\rho^A) = S_{(q,s)}(\rho^B),$$

which for the von Neumann entropy reduces to the standard one \cite{38}.

D. Relationships between total and semiquantum correlations

It is possible to find some interesting relationships between total and semiquantum correlations when bilocal disturbances, $D^{\Pi^{AB}}_{(q,s)}(\rho^{AB})$, are rewritten in terms of unilocal disturbances,

$$D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) = D^{\Pi^A}_{(q,s)}(\rho^{AB}) + \pi^A_{(q,s)} D^{\Pi^B}_{(q,s)}(\Pi^A(\rho^{AB})),$$  \hspace{1cm} (30)

$$D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) = D^{\Pi^B}_{(q,s)}(\rho^{AB}) + \pi^B_{(q,s)} D^{\Pi^A}_{(q,s)}(\Pi^B(\rho^{AB})),$$  \hspace{1cm} (31)

where $\pi^A_{(q,s)} = \left( \frac{\text{Tr}(\Pi(\rho^{AB}))}{\text{Tr}(\rho^{AB})} \right)^s$ (for sake of brevity, we omit the dependence of this factor on the state). This quantity, $\pi^B_{(q,s)}$, is nonnegative but it can take values below or above 1, depending on the value of the entropic parameter $q$. As $\Pi(\rho) < \rho$, we have that $\Pi(\rho)^q < \rho^q$ if $q \geq 1$, whereas, $\rho < \Pi(\rho)$ holds if $0 \leq q < 1$. Thus, $\pi^B_{(q,s)} \in (0,1] if q \geq 1, else \pi^B_{(q,s)} \geq 1$. In particular, for Rényi entropies the factor is always equal to 1.

Now, let us consider two possible measurement scenarios:

- $\Pi^{AB}_0 = \Pi^A_0 \circ \Pi^B_0$ is a bilocal measurement that minimizes the total quantum correlation measure, i.e., $D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) = D^{\Pi^A}_{(q,s)}(\rho^{AB})$,

- $\Pi^{AB}_1 = \Pi^A_1 \circ \Pi^B_1$, where $\Pi^I_1$, optimize the unilocal disturbances, i.e., $D^{\Pi^I_{(q,s)}}_{(q,s)}(\rho^{AB}) = D^{\Pi^I_{((q,s)}}_{(q,s)}(\rho^{AB})$.

Applying Eqs. (30)–(31) to both scenarios, we obtain

$$D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) = D^{\Pi^A}_0(\rho^{AB}) + \pi^A_{(q,s)} D^{\Pi^B}_0(\Pi^A(\rho^{AB})) = D^{\Pi^B}_0(\rho^{AB}) + \pi^B_{(q,s)} D^{\Pi^A}_0(\Pi^B(\rho^{AB})),$$  \hspace{1cm} (32)

and

$$D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) = D^{\Pi^A}_1(\rho^{AB}) + \pi^A_{(q,s)} D^{\Pi^B}_1(\Pi^A(\rho^{AB})) = D^{\Pi^B}_1(\rho^{AB}) + \pi^B_{(q,s)} D^{\Pi^A}_1(\Pi^B(\rho^{AB})).$$  \hspace{1cm} (33)

Using that $D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) \leq D^{\Pi^{AB}}_{(q,s)}(\rho^{AB})$ (and the analogous relations for the unilocal disturbances) on Eqs. (32)–(33) respectively, it can been shown that $D^{\Pi^{AB}}_{(q,s)}(\rho^{AB})$ is lower and upper bounded as follows,

$$D^{AB}_{(q,s)}(\rho^{AB}) \geq \max\{D^{A}_{(q,s)}(\rho^{AB}) + \pi^A_{(q,s)} D^{B}_{(q,s)}(\Pi^A(\rho^{AB})), D^{B}_{(q,s)}(\rho^{AB}) + \pi^B_{(q,s)} D^{A}_{(q,s)}(\Pi^B(\rho^{AB}))\},$$  \hspace{1cm} (34)

$$D^{AB}_{(q,s)}(\rho^{AB}) \leq \min\{D^{A}_{(q,s)}(\rho^{AB}) + \pi^A_{(q,s)} D^{B}_{(q,s)}(\Pi^A(\rho^{AB})), D^{B}_{(q,s)}(\rho^{AB}) + \pi^B_{(q,s)} D^{A}_{(q,s)}(\Pi^B(\rho^{AB}))\}. $$  \hspace{1cm} (35)

In particular, given that the nonoptimal unilocal disturbances in (34) are nonnegative, we naturally obtain that total quantum correlation are greater than or equal to the semiquantum ones,

$$D^{AB}_{(q,s)}(\rho^{AB}) \geq \max\{D^{A}_{(q,s)}(\rho^{AB}), D^{B}_{(q,s)}(\rho^{AB})\}. $$  \hspace{1cm} (36)

This result can be also obtained more directly from the fact that $S_{(q,s)}(\Pi^0_0(\rho^{AB})) \geq S_{(q,s)}(\Pi^I_1(\rho^{AB}))$. Notice that (36) is in accordance with the inclusion relations among the sets of CQ, QC and CC correlated states, i.e., $\Omega^{AB} = \Omega^A \cap \Omega^B \subset \Omega^L$. 


Moreover, noting that $2D_{(q,s)}^{AB}(\rho^{AB}) \geq D_{(q,s)}^{AB}(\rho^{AB}) + D_{(q,s)}^{AB}(\rho^{AB}) \geq 2D_{(q,s)}^{AB}(\rho^{AB})$ we can deduce from Eqs. (32)–(33) the following inequality for the sum of semiquantum correlations:

$$D_{(q,s)}^{AB}(\rho^{AB}) + \Delta_0 \geq D_{(q,s)}^{AB}(\rho^{AB}) + D_{(q,s)}^{B}(\rho^{AB}) \geq D_{(q,s)}^{AB}(\rho^{AB}) + \Delta_1,$$

(37)

where we defined the quantities $\Delta_i := D_{(q,s)}^{AB}(\rho^{AB}) - \pi^B_{i,s}D_{(q,s)}^{AB}(\rho^{AB}) - \pi^A_{i,s}D_{(q,s)}^{AB}(\rho^{AB})$, with $i = 0, 1$.

Notice that for CQ and QC correlated states, one has $\Delta_i = 0$. Therefore, from these observations together with (36), we obtain

- if $D_{(q,s)}^{A}(\rho^{AB}) = 0$, then $D_{(q,s)}^{AB}(\rho^{AB}) = D_{(q,s)}^{B}(\rho^{AB})$,
- if $D_{(q,s)}^{B}(\rho^{AB}) = 0$, then $D_{(q,s)}^{AB}(\rho^{AB}) = D_{(q,s)}^{A}(\rho^{AB})$,
- if $D_{(q,s)}^{AB}(\rho^{AB}) = 0$, then $D_{(q,s)}^{A}(\rho^{AB}) = D_{(q,s)}^{B}(\rho^{AB}) = 0$.

Furthermore, a triangle-like inequality between total and semiquantum correlations,

$$D_{(q,s)}^{A}(\rho^{AB}) + D_{(q,s)}^{B}(\rho^{AB}) \geq D_{(q,s)}^{AB}(\rho^{AB}),$$

(38)

is trivially satisfied for CQ, QC and CC correlated states. The validity of the triangle-like inequality (38) in the general case relies on the sign of $\Delta_1$. If $\Delta_1 \geq 0$, the inequality is generally true. On the contrary, if $\Delta_1 < 0$ for some $\rho^{AB}$ then it could be the case that the inequality does not hold for those states.

Although the most general conditions for the validity of the triangle-like inequality (38) are hard to analyze, we can link the validity of (38) with a kind of local contractivity property of the unilocal disturbances. Specifically, let us assume as valid the following inequalities:

$$\pi^B_{(q,s)} D_{(q,s)}^{AB}(\rho^{AB}) \leq D_{(q,s)}^{AB}(\rho^{AB}),$$

(39)

$$\pi^A_{(q,s)} D_{(q,s)}^{AB}(\rho^{AB}) \leq D_{(q,s)}^{AB}(\rho^{AB}).$$

(40)

Then, we have

$$\pi^B_{(q,s)} D_{(q,s)}^{AB}(\rho^{AB}) \leq D_{(q,s)}^{AB}(\rho^{AB}) = D_{(q,s)}^{A}(\rho^{AB}),$$

(41)

$$\pi^A_{(q,s)} D_{(q,s)}^{AB}(\rho^{AB}) \leq D_{(q,s)}^{AB}(\rho^{AB}) = D_{(q,s)}^{B}(\rho^{AB}),$$

(42)

and, replacing any of these relations in (33), we obtain

$$D_{(q,s)}^{AB}(\rho^{AB}) \leq D_{(q,s)}^{A}(\rho^{AB}) + D_{(q,s)}^{B}(\rho^{AB}).$$

(43)

Finally, recalling that $D_{(q,s)}^{AB}(\rho^{AB}) \leq D_{(q,s)}^{AB}(\rho^{AB})$, it follows the triangle-like inequality (38).

Thus, we are able to link the validity of the triangle-like inequality, for all states and any entropic indexes, with the assumption of contractivity of unilocal disturbances under local projective measurements. In the case of von Neumann entropy, inequalities (39)–(40) are particular cases of the contractivity of the quantum relative entropy under trace-preserving completely positive maps [39]. Otherwise, for Tsallis entropy of entropic index 2, inequalities (39)–(40) are particular cases of the contractivity of the Hilbert-Schmidt distance under projective measurements [10]. Therefore, in both cases the triangle-like inequality is satisfied (notice that for the latter, this result has been proved in alternative way [26]). Unfortunately, the local contractivity is not valid for general entropic functionals. Indeed, we show that is the case for a wide range of the entropic index of the Rényi and Tsallis entropies in Fig. 1.

### III. EXAMPLES

#### A. Mixtures of a pure state and the maximally mixed one

An interesting example where the computations can be carried out analytically involves the family of pseudopure states, given by mixtures of an arbitrary pure state, $|\psi^{AB}\rangle \in H^{N_A} \otimes H^{N_B}$, with the maximally mixed state, yielding

$$\rho_p^{AB} = (1 - p) I^{AB} / N^{AB} + p |\psi^{AB}\rangle \langle \psi^{AB}|,$$

(44)

...
with $0 \leq p \leq 1$ (remind that $N^{AB} = N^A N^B$). The spectrum of $\rho^{AB}$ is given by the eigenvalue $(1-p)/N^{AB} + p$, with both the unilocal and the bilocal quantifiers are unique (do not depend on the entropic form) and are given by the local Schmidt basis $\{|1\rangle\rangle_{AB}\}$. This entropic-independent measurement fact is not a universal property, but depends on the particular states. In this case, measuring in the Schmidt basis yields a final spectrum that is majorized by any other spectrum corresponding to any other measurement, implying the entropic-independent optimization. After the collapse of the semiquantum and total quantifiers, the triangle-like inequality (38) holds for the most general measurement, the spectrum is given by the eigenvalue $\lambda_k$ the square of Schmidt coefficients (28). Using Eq. (19), we obtain

$$D^{(q,s)}_K(\rho^{AB}) = \frac{1}{(1-q)s} \left[ \left( \frac{(N^{AB} - n)(1-p)^q + \sum_{k=1}^n \left[ 1 + \left( \frac{N^{AB} \lambda_k - 1 \right) p \right] q^s }{(N^{AB} - 1)(1-p)^q + \left[ 1 + (N^{AB} - 1) p \right] q^s } \right) - 1 \right]$$

for the generalized quantum correlations of pseudopure states. It is remarkable that, in this particular case and given the collapse of the semiquantum and total quantifiers, the triangle-like inequality (38) holds for the most general $(q,s)$-entropic forms.

In particular, when $|\psi^{AB}\rangle$ is a maximally entangled state, with $N^A = N^B = N$, states $\rho^{AB}$ constitutes a family of isotropic states, $\rho^{p}_P$. In that case, $\forall k, \lambda_k = N^{-1}$, $n = N$, and the generalized quantum correlations are

$$D^{(q,s)}_K(\rho^{p}_P) = \frac{1}{(1-q)s} \left[ \left( \frac{(N^2 - N)(1-p)^q + N \left[ 1 + (N - 1) p \right] q^s }{(N^2 - 1)(1-p)^q + \left[ 1 + (N^2 - 1) p \right] q^s } \right) - 1 \right].$$

Specializing this for Tsallis and Rényi entropies one obtains, respectively,

$$D^{(q,1)}_K(\rho^{p}_P) = \frac{1}{1-q} \frac{1}{N^{2q}} \left[ N(1-p + Np)^q - (1-p + N^2p)^q (N - 1)(1-p)^q \right],$$

$$D^{(q,q)}_K(\rho^{p}_P) = \frac{1}{1-q} \ln \left[ \frac{N(1-p + Np)^q + (N^2 - N)(1-p)^q}{(1-p + N^2p)^q + (N^2 - 1)(1-p)^q} \right].$$

### B. Werner and isotropic states

Although isotropic states are particular cases of Eq. (44), i.e., mixtures of a pure state and the maximally mixed one, we aim to show that both isotropic [31] and Werner states [11], due to their symmetries, are independent of the
local measurements performed. A Werner state is a $N \times N$ dimensional bipartite quantum state that is invariant under local unitary transformations of the form $U \otimes U$, with $U$ an arbitrary unitary acting on $N$ dimensional systems, that is, $\rho^W = U \otimes U \rho U^\dagger \otimes U^\dagger$. On the other hand, an $N \times N$-dimensional isotropic state is invariant under arbitrary local unitaries of the form $U \otimes U^*$, that is, $\rho^I = U \otimes U^* \rho U^\dagger \otimes (U^*)^\dagger$. They can be parametrized, respectively, as

$$
\rho^W_x = \frac{N - x}{N^2 - N} I + \frac{Nx - 1}{N^2 - N} F,
$$

with $F = \sum_{ij} |ij\rangle \langle ji|$, $1 \leq i, j \leq N$, $x \in [-1, 1]$, and

$$
\rho^I_y = \frac{1 - y}{N^2 - 1} I + \frac{N^2 y - 1}{N^2 - 1} |\psi^+\rangle \langle \psi^+|,
$$

with $|\psi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |ii\rangle$ and $y \in [\frac{1}{N^2 - 1}, 1]$. Notice that both definitions of isotropic states –the one derived from Eq. (44) and the one given by Eq. (50)– coincide under the identification $p = \frac{N^2 y - 1}{N^2 - 1}$ and $|\psi^{AB}\rangle = |\psi^+\rangle$.

To see that any local measurement yields the same disturbance over these families of states, let us consider $\Pi^A_1$ as the optimal unilocal measurement over $A$. Any other local measurement is achieved by a unitary transformation over $\Pi^B_1$ as $\Pi^B_1 = V \otimes U^B \Pi^A_1 V^\dagger \otimes U^B$, with $V$ an arbitrary unitary over $A$. Then, using the invariance properties of Werner states, the action of $\Pi^B_1$ is $\Pi^B_1 (\rho^W) = V \otimes V \Pi^B_1 V^\dagger \otimes V^\dagger$. Analogous results holds for isotropic states and measurements over $B$. Invoking the unitary invariance of $(q, s)$-entropies one has that the minimum in $\{19\}$ is attained for any local projective measurement. To prove that nothing changes when considering bilocal measurements, it is sufficient to observe that after any local measurement the state becomes a CC correlated state. Thus, given that the total disturbance can be computed via the partial disturbances (see Eqs. (30)-(31)), the total quantum correlations are equal to the semiquantum ones.

In order to find an explicit formula of the generalized correlations, it is easier to measure on the standard basis (the ones used to define $F$ in Werner states and $|\psi^+\rangle$ in isotropic states), readily obtaining

$$
D_{(q,s)}^K (\rho^W_x) = \frac{1}{(1-q)s} \left[ \left( \frac{2[(N - 1)^q(x + 1)^q + (N - 1)(N - x)^q]}{2(N - 1)^q(x + 1)^q + (N - 1)(N - x + \frac{1}{2}N x - \frac{1}{2})^q + (N - x - \frac{1}{2}N x + \frac{1}{2})^q} \right)^s - 1 \right],
$$

and

$$
D_{(q,s)}^K (\rho^I_y) = \frac{1}{(1-q)s} \left[ \left( \frac{N[(N - 1)(1 - y)^q + (1 - y + N y - \frac{1}{N})^q]}{(N^2 - 1)^q y^q + (N^2 - 1)(1 - y)^q} \right)^s - 1 \right].
$$

Again, it is interesting to observe that these families of states are among the ones that satisfy the triangle-like inequality (Eq. (38)) for any $(q, s)$-entropy.

**IV. CONCLUDING REMARKS**

In this work we address the problem of quantifying quantum correlations beyond discord. Specifically, following [16], we obtain entropic measures of bipartite quantum correlations by quantifying the system’s states disturbance under local measurements. Our measures are based on very general entropic forms given by the system’s states disturbance under local unitary transformations of the form $U \otimes U$, that is, $\rho^W = U \otimes U \rho U^\dagger \otimes U^\dagger$. On the other hand, an $N \times N$-dimensional isotropic state is invariant under arbitrary local unitaries of the form $U \otimes U^*$, that is, $\rho^I = U \otimes U^* \rho U^\dagger \otimes (U^*)^\dagger$. They can be parametrized, respectively, as

$$
\rho^W_x = \frac{N - x}{N^2 - N} I + \frac{Nx - 1}{N^2 - N} F,
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with $F = \sum_{ij} |ij\rangle \langle ji|$, $1 \leq i, j \leq N$, $x \in [-1, 1]$, and

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\rho^I_y = \frac{1 - y}{N^2 - 1} I + \frac{N^2 y - 1}{N^2 - 1} |\psi^+\rangle \langle \psi^+|,
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with $|\psi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |ii\rangle$ and $y \in [\frac{1}{N^2 - 1}, 1]$. Notice that both definitions of isotropic states –the one derived from Eq. (44) and the one given by Eq. (50)– coincide under the identification $p = \frac{N^2 y - 1}{N^2 - 1}$ and $|\psi^{AB}\rangle = |\psi^+\rangle$.

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In order to find an explicit formula of the generalized correlations, it is easier to measure on the standard basis (the ones used to define $F$ in Werner states and $|\psi^+\rangle$ in isotropic states), readily obtaining

$$
D_{(q,s)}^K (\rho^W_x) = \frac{1}{(1-q)s} \left[ \left( \frac{2[(N - 1)^q(x + 1)^q + (N - 1)(N - x)^q]}{2(N - 1)^q(x + 1)^q + (N - 1)(N - x + \frac{1}{2}N x - \frac{1}{2})^q + (N - x - \frac{1}{2}N x + \frac{1}{2})^q} \right)^s - 1 \right],
$$

and

$$
D_{(q,s)}^K (\rho^I_y) = \frac{1}{(1-q)s} \left[ \left( \frac{N[(N - 1)(1 - y)^q + (1 - y + N y - \frac{1}{N})^q]}{(N^2 - 1)^q y^q + (N^2 - 1)(1 - y)^q} \right)^s - 1 \right].
$$

Again, it is interesting to observe that these families of states are among the ones that satisfy the triangle-like inequality (Eq. (38)) for any $(q, s)$-entropy.

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In this work we address the problem of quantifying quantum correlations beyond discord. Specifically, following [16], we obtain entropic measures of bipartite quantum correlations by quantifying the system’s states disturbance under local measurements. Our measures are based on very general entropic forms given by the $(q, s)$-entropies. As a consequence, we obtain quantum correlations measures, which include as particular cases or are close to several other measures previously discussed in the literature [16][22][27]. Our main contribution is to propose such quantum correlations measures based on quantum unified $(q, s)$-entropies that are: (i) nonnegative and vanishes only for QC, CQ and CC correlated states, (ii) invariant under local unitary operators, and (iii) invariant under the addition of an uncorrelated ancilla. Regarding with the last property, we show that for $q \not= 1$ or $s \not= 0$, that is when the $(q, s)$-entropies are nonadditive, it is necessary to rescale the disturbances by a generalized purity factor in order to avoid undesirable effects of previous entropic based correlation measures [19][20][24].

Moreover, we distinguish between total and semiquantum correlations, and we naturally obtain that the former are greater than the latter. In addition, we show that a triangle-like inequality is fulfilled for certain families of sates, namely QC, CQ and QQ correlated states, as well as, Werner and Isotropic states, for any entropic measures. In the general case, we only proof this for the von Neumann and Tsallis with entropic index of order 2, which follows from the contractivity property under a projective measurement of quantum relative entropy and Hilbert Schmidt distance, respectively. We provide numerical counterexamples where the local contractivity property of unilocal disturbances fails in a wide range of the entropic index of Rényi and Tsallis entropies, but it remains open if the triangle-like inequality is fulfilled for other entropic measures.
Finally, we provide analytical expressions of the entropic correlations measures for pseudopure, Werner and isotropic states. For these families of states, the optimal measurement of unilocal and bilocal disturbances are independent of the entropic form.

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