Constructing spectra using cone injectivity

J. Jurka, T. Perutka, L. Vokřínek

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Abstract

We provide a generalization of the construction of a spectrum of a commutative ring as a locally ringed space, applicable to cone injectivity classes in general contexts. In this generality, it is not always the case that the spectrum functor is fully faithful and so we study necessary and sufficient conditions for this to happen. If it does, we introduce a generalization of another concept from algebraic geometry – the functor of points – and prove equivalence of the two resulting notions of schemes.

1 Introduction

This paper is motivated by modern algebraic geometry – more concretely, by its basic objects of study: schemes. To define them, one must first define a functor \( \text{Spec}: \text{CRing}^{\text{op}} \to \text{LRSp} \), the codomain being the category of locally ringed spaces. This is quite a non-trivial and non-obvious construction: for a commutative ring \( R \), the topological space of \( \text{Spec} R \) is usually not even Hausdorff and its points are given by prime ideals of \( R \); moreover, forming the structure sheaf is quite a painful procedure. But it pays off, since powerful methods of scheme-based algebraic geometry have proved useful in many parts of mathematics, e.g. number theory. To some extent, we can thus say that in order to apply these methods in new contexts, we first need to construct some analogue of the functor \( \text{Spec} \), landing in some analogue of the category \( \text{LRSp} \). Many of these \( \text{Spec} \)-like functors have been already used (see Section 16).

\[ \text{LRSp} \]

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But how to come up with such a construction? Here, category theory gives a hint: we can characterize the functor Spec formally as the right adjoint to a functor $\Gamma$: $\text{LRSp} \to \text{CRing}^{\text{op}}$ sending a locally ringed space $(X, \mathcal{O}_X)$ to a ring of global sections $\mathcal{O}_X(X)$. This is useful for if we try to construct some analogue of Spec, we can make sure that our construction makes sense by verifying the appropriate analogue of this adjunction.

This observation is useful in proving that some construction is Spec-like, but not quite helpful in finding the construction: in literature, one first constructs Spec (or its analogue) and then proves that it is the right adjoint to $\Gamma$. However, in this paper, we are going to reverse the process: first we prove formally that the right adjoint to $\Gamma$ exists and use that to deduce the particular construction. Since we are doing this in great generality, this might be quite helpful for constructing new Spec-like functors. Our general framework will be a locally finitely presentable category $\mathcal{A}$ (in place of $\text{CRing}$) and instead of using local rings in defining $\text{LRSp}$, we will use some cone injectivity class in $\mathcal{A}$.

By concentrating on the existence, by means of the adjoint functor theorem, one is led naturally to considering colimits of $\mathcal{A}$-spaces and $(\mathcal{A}, \mathcal{P})$-spaces that are of great importance regardless of the proof; of course, the verification of the solution set condition is slightly annoying, but once this is done, we know that the spectrum functor (however abstract) readily satisfies all the axioms asked of an $(\mathcal{A}, \mathcal{P})$-space. Unlike the proof techniques, the results in the first part are not new – they are very much just categorical versions of the logic-oriented results of [4], see also [2]. In addition, there is a more general approach of [6] that we comment on later.

The price for the generalization is that, unlike in the classical case of commutative rings, the spectrum functor Spec needs not be fully faithful, so that the “affine scheme” Spec $R$ does not give a faithful picture of the algebraic object $R$. While this question was treated in the above mentioned papers, we did not find the criteria very practical. In our take, we identify a number of obstructions to fully faithful Spec. For most of the obstructions, if these do not vanish, it is possible to replace $\mathcal{A}$ by a better-behaved subcategory and finally arrive at one for which Spec is fully faithful.

The full faithfulness of the spectrum functor is important for the final part of our paper that generalizes the so-called functor of points approach which studies a locally ringed space $X$ through maps from affine schemes: $N \mathcal{X} = \text{LRSp}(\text{Spec }-, X) \in [\mathcal{A}, \text{Set}]$. This presheaf $N \mathcal{X}$ satisfies a natural sheaf condition; when $N$ is considered as a functor from locally ringed spaces to these sheaves, it restricts to an equivalence of categories of schemes.
2 Main results

Let $\mathcal{A}$ be a complete and cocomplete category such that:

- For any topological space $X$, the inclusion $\text{Sh}_X \to \text{PSh}_X$ of $\mathcal{A}$-valued sheaves into $\mathcal{A}$-valued presheaves admits a left adjoint – the sheafification functor $\text{sh}$. We will use the easily verified fact that $\text{sh}$ then automatically preserves stalks.

- The isomorphisms of sheaves are reflected by the (total) stalk functor: $F \to G$ is an isomorphism if and only if the induced map $F_p \to G_p$ is an isomorphism, for each point $p \in X$.

Both these properties hold if $\mathcal{A}$ is locally finitely presentable; we will elaborate on this in Section 11.3.

Definition 1. A cone in $\mathcal{A}$ is a family of maps $(a_j: A \to B_j)_{j \in J}$ with a specified common domain which we sometimes call a summit of the cone; thus, empty cones (with $J = \emptyset$) with different summits are considered different. Let $C = \{(a_{ij}: A_i \to B_{ij}) \mid i \in I\}$ be a set of cones in $\mathcal{A}$ with finitely presentable domains and codomains (the index $i$ ranges over a set $I$, while, for a fixed $i \in I$, the index $j$ ranges over a set $J_i$). We say that an object $R \in \mathcal{A}$ is local (or that $R$ is cone injective w.r.t. $C$) if, for each $i$, and for each map $A_i \to R$, there exists $j \in J_i$ and an extension

$\begin{array}{ccc}
A_i & \longrightarrow & R \\
\downarrow^{a_{ij}} & & \downarrow \\
B_{ij} & \rightarrow & \\
\end{array}$

We will always assume that $A_i \to B_{ij}$ are epimorphisms so that the extension is unique (the index $j$ need not be). For a set of cones $C$ as above, denote by $C^\dagger$ the collection of all the maps $a_{ij}$ appearing in the cones from $C$. A map $f: R \to S$ is admissible if it has the right lifting property w.r.t. $C^\dagger$, i.e. for each commutative square as below, there exists a lift making both triangles commute:

$\begin{array}{ccc}
A_i & \longrightarrow & R \\
\downarrow^{a_{ij}} & & \downarrow^{f} \\
B_{ij} & \longrightarrow & S \\
\end{array}$

We note that in this case, we do not make use of the common domain $A_i$ for the maps $a_{ij}$, with varying $j$. Notationally, we will emphasize that $f$ is admissible by decorating the map as $f: R \leadsto S$. 

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We define $\mathcal{P} \subseteq \mathcal{A}$ to be the non-full subcategory of local objects and admissible maps.

Finally, if $f: R \to S$ is obtained as a transfinite composition of pushouts of coproducts of the $a_{ij}$'s, we say that $f$ is a localization (or that $f$ is cellular w.r.t. $\mathcal{C}^i$). Notationally, we will emphasize this by decorating the map as $f: R \to S$. Further, we say that $f$ is a finite localization if the total number of the $a_{ij}$'s used is finite (each counted as many times as it is used). Importantly, any finite localization is a finitely presentable object of $R/\mathcal{A}$.

Localizations have the left lifting property w.r.t. admissible maps, implying in particular that a map that is both a localization and admissible must be an isomorphism.

Further, we define a local form of $R$ to be a localization $p: R \to P$ with local codomain $P \in \mathcal{P}$. It follows essentially from the small object argument that for any object $R$ there exists a set of local forms $p_\alpha: R \to P_\alpha$ such that any map $R \to Q$ to a local object factors through some $P_\alpha$; although the factorization itself is unique, the local form needs not be. We will prove a refined version: There exists a set of local forms for which both the local form and the factorization is unique if we further require that the factorization $P_\alpha \xrightarrow{\sim} Q$ is admissible:

$$
\begin{array}{ccc}
R & \xrightarrow{\pi} & Q \\
\downarrow & & \downarrow \sim \\
P_\alpha & \xrightarrow{\sim} & \\
\end{array}
$$

This means that the local forms comprise a multi-reflection onto $\mathcal{P}$.

Taking all the local forms together, we thus obtain a map $R \to \prod_\alpha P_\alpha$ and the factorization system generated by $\mathcal{C}^i$ then produces a factorization

$$
R \xrightarrow{rR} \xrightarrow{\sim} \prod_\alpha P_\alpha
$$

in which the object $rR$ is called the reduction of $R$. We will show that $r$ is a reflection functor onto the full subcategory $r\mathcal{A}$ of “reduced” objects (the objects $R$ for which $R \to \prod_\alpha P_\alpha$ is admissible; put differently, the collection of all local forms $R \to P_\alpha$ is jointly admissible). We will also show that

$$
\mathcal{P} \subseteq r\mathcal{A} \subseteq \mathcal{A}
$$

and that the spectrum functor does not see difference between $\mathcal{A}$ and $r\mathcal{A}$, so that one could safely restrict from $\mathcal{A}$ to $r\mathcal{A}$ and thus assume that all objects are reduced.

**Example 2.** Our main illustrating example is in the category of commutative rings. Here, there is a single index $i$ with two corresponding indices $j$: the
The first map is
\[ Z[x, y]/(x + y - 1) \to Z[x, y, x^{-1}]/(x + y - 1) \]
and the second is the symmetric version of the first
\[ Z[x, y]/(x + y - 1) \to Z[x, y, y^{-1}]/(x + y - 1) \]
(that can be seen also as \( Z[x] \to Z[x, x^{-1}] \) and \( Z[x] \to Z[x, (1 - x)^{-1}] \)). Here localness of a commutative ring \( R \) means that it is a local ring or the trivial ring. In order to get rid of the trivial ring we add another cone \((A_i \to B_{ij})\) with a single index \( i \) and \( J_i = \emptyset \) (giving thus an empty cone) and with \( A_i = \{1 = 0\} \), the trivial ring. Localness with respect to this empty cone simply means that there is no map \( \{1 = 0\} \to A \), otherwise we would get a contradiction with \( J_i \) being empty. Admissibility of a ring homomorphism \( f \) means simply that it reflects invertibility: \( \text{if } r \text{ invertible then } f(r) \text{ invertible}. \) (Finite) localizations are obtained by adding inverses to (a finite number of) elements. All objects are reduced.

There are two modifications whose spectra have the same underlying set as the spectrum resulting from the above example. The first modification consists of
\[ Z[x] \to Z[x, x^{-1}], \quad Z[x] \to Z[x]/(x) \]
(plus the exact same empty cone) for which the local objects are exactly the fields, and which are obtained as localizations (fraction fields) of the quotients by various prime ideals. The second modification consists of
\[ Z[x, y]/(xy) \to Z[x, y]/(x), \quad Z[x, y]/(xy) \to Z[x, y]/(y) \]
(plus the exact same empty cone) for which the local objects are exactly the integral domains, and which are obtained as quotients by various prime ideals. In both these modifications, reduced objects are the reduced rings in the usual sense, i.e. those with trivial nilradical.

Next we define \( \text{Top}_A \), the category of \( A \)-spaces, to be the category of topological spaces \( X \) equipped with an \( A \)-valued sheaf \( \mathcal{O}_X : \text{Op}(X)^{\text{op}} \to A \). Maps are continuous maps \( \varphi : X \to Y \) together with \( \varphi^\sharp : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X \) or equivalently \( \varphi_* : \varphi^* \mathcal{O}_Y \to \mathcal{O}_X \). The non-full subcategory \( \text{Top}_{A,P} \subseteq \text{Top}_A \), consisting of \( (A, \mathcal{P}) \)-spaces, is the category consisting of the \( A \)-spaces whose stalks \( \mathcal{O}_{X,p} \), for all \( p \in X \), lie in \( \mathcal{P} \) and maps for which the induced maps on stalks \( \varphi_* : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p} \) also lie in \( \mathcal{P} \). We will sometimes abuse the notation and write just \( X \) instead of full \( (X, \mathcal{O}_X) \).

In our main example, \( \text{Top}_A \) are the ringed spaces and \( \text{Top}_{A,P} \) are the locally ringed spaces.
We have the following categories, functors and adjunctions:

\[
\text{Top}_{\mathcal{A}, \mathcal{P}} \xleftarrow{\text{in}} \text{Spec} \xrightarrow{\Gamma} \mathcal{A}^{\text{op}}
\]

where, \(\text{in}\) denotes the inclusion, \(\Gamma\) is the functor of global sections \(\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)\), the right adjoint of which is \(R \mapsto R^*\), the one-point space with global sections \(R\). The other, dashed, right adjoint is the content of our first main theorem.

**Theorem 3.** The inclusion \(\text{in}: \text{Top}_{\mathcal{A}, \mathcal{P}} \subseteq \text{Top}_{\mathcal{A}}\) admits a right adjoint \(\text{Spec}\). Its image \(\text{Spec} R = \text{spec} R^*\) on the one-point space \(R^*\) is the \((\mathcal{A}, \mathcal{P})\)-space with:

- underlying set consisting of all isomorphism classes of local forms of \(R\),
- topology with basis given by the sets \(\text{Pts}(K)\) of (isomorphism classes of) local forms of \(R\) which factor through a given localization \(R \twoheadrightarrow K\), with \(K\) varying over all finite localizations,
- the structure sheaf obtained as the sheafification of the right Kan extension of the canonical presheaf \(\mathcal{O}_R^{\text{can}}(\text{Pts}(K)) = rK\).

We denote the composite adjunction as \(\Gamma \dashv \text{Spec}\). We say that \(R \in \mathcal{A}\) is a fixed point of this adjunction if the counit \(\varepsilon: R \to \overline{R} := \Gamma \text{Spec} R\) (written here in \(\mathcal{A}\) rather than \(\mathcal{A}^{\text{op}}\)) is an isomorphism. We will write \(\mathcal{A}_f\) for the full subcategory of \(\mathcal{A}\) on these fixed points. Dually, an \((\mathcal{A}, \mathcal{P})\)-space \(X\) is a fixed point for this adjunction, and we write \(X \in \text{f Top}_{\mathcal{A}, \mathcal{P}}\), if the unit \(\eta: X \to \text{Spec} \Gamma X\) is an isomorphism. It is classical that any adjunction induces an equivalence

\[
\mathcal{A}_f^{\text{op}} \simeq \text{f Top}_{\mathcal{A}, \mathcal{P}}
\]

and we may thus study the fixed points equivalently via their spectra. We will show that

\[
\mathcal{P} \subseteq \mathcal{A}_f \subseteq r\mathcal{A} \subseteq \mathcal{A}.
\]

Thus, in order for \(R \in \mathcal{A}\) to be a fixed point, it is necessary that \(R\) is reduced and, consequently, it is not only convenient to restrict to \(r\mathcal{A}\), but when fully faithful \(\text{Spec}\) is required, it is also necessary. At the same time, we may restrict to the subcategory \(m\mathcal{A}\) of objects, for which the collection of all local forms is jointly monic; this again includes all fixed points. For our next theorem, we say that a map \(f: R \to S\) is flat if the cobase change \(f_*: R/\mathcal{A} \to S/\mathcal{A}\) preserves finite limits.
Theorem 4. Assume that $\mathcal{A} = r\mathcal{A}$ and $\mathcal{A} = m\mathcal{A}$. If all local forms of $R$ are flat then $\varepsilon: R \to \mathcal{R}$ induces an isomorphism on spectra and $\mathcal{R}$ is then a fixed point. If this happens for all $R \in \mathcal{A}$, then the adjunction $\Gamma \dashv \text{Spec}$ is idempotent and thus:

- $f\mathcal{A} \subseteq \mathcal{A}$ is reflective (with reflection $\varepsilon$) and equals the full image of $\Gamma$,
- $f\text{Top}_{\mathcal{A},\mathcal{P}} \subseteq \text{Top}_{\mathcal{A},\mathcal{P}}$ is also reflective and equals the full image of $\text{Spec}$, i.e. the full subcategory $\text{AffSch}_{\mathcal{A},\mathcal{P}}$ of affine schemes,
- There results an equivalence $f\mathcal{A}^{\text{op}} \simeq \text{AffSch}_{\mathcal{A},\mathcal{P}}$.

In general, this does not imply that $\varepsilon$ itself is an isomorphism, i.e. that $R$ is a fixed point. However, we have the following recognition result.

Proposition 5. Assume that $\mathcal{A} = r\mathcal{A}$. A map $f: R \to S$ induces an isomorphism on spectra $f^*: \text{Spec} S \xrightarrow{\cong} \text{Spec} R$ if and only its pushout along any local form of $R$

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
\text{P} & \cong & f_*\text{P}
\end{array}
\]

is an isomorphism.

We may then say that local forms detect isomorphisms if the condition implies that $f$ itself is an isomorphism. This is known to be the case for local rings and we give a counterexample for (reduced) integral domains. We may conclude:

Theorem 6. Assume that $\mathcal{A} = r\mathcal{A}$ and $\mathcal{A} = m\mathcal{A}$. If all local forms of $R$ are flat and detect isomorphisms then $R$ is a fixed point.

Finally, we study the relationship between $(\mathcal{A},\mathcal{P})$-spaces and certain sheaves on $\mathcal{A}^{\text{op}}$, assuming that $\mathcal{A} = f\mathcal{A}$, i.e. Spec being fully faithful, e.g. by replacing $\mathcal{A}$ by $f\mathcal{A}$. Namely, consider the functor $N X = \text{Top}_{\mathcal{A},\mathcal{P}}(\text{Spec} -, X)$, something that algebraic geometers usually call the functor of points of $X$ (whereas category theorists prefer to call this a nerve functor). It has a partial left adjoint given by the left Kan extension $| | = \text{lan}_y \text{Spec}$:

$$
\begin{array}{ccc}
[\mathcal{A}, \text{Set}] & \xleftarrow{\varepsilon} & \longrightarrow & \text{Set} \\
& & \mathcal{A}^{\text{op}}\downarrow & \downarrow |
\end{array}
\xleftarrow{\text{lan}_y \text{Spec}}
\begin{array}{ccc}
& & \longrightarrow \text{Top}_{\mathcal{A},\mathcal{P}}
\end{array}
$$

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Our assumption on Spec being fully faithful implies that both triangles commute. For our last theorem, we say that an \((\mathcal{A}, \mathcal{P})\)-space is an affine scheme if it lies in the image of Spec and a scheme is then an \((\mathcal{A}, \mathcal{P})\)-space that admits an open cover by affine schemes. To give a corresponding definition in \([\mathcal{A}, \mathcal{Set}]\), we have to restrict to a certain subcategory of sheaves, by introducing a sheaf condition that forces an affine scheme to be a union of affine subschemes if this happens on the level of \((\mathcal{A}, \mathcal{P})\)-spaces. Working in this category of sheaves, a functor is an affine scheme if it lies in the image of \(y\) and a scheme if it admits an “open cover” by affine schemes; we postpone the precise definition of an open cover. This is our last main theorem concerning \((\mathcal{A}, \mathcal{P})\)-spaces.

**Theorem 7.** Assume that Spec is fully faithful and that a regular epi–mono factorization exists in \(\mathcal{A}\). When restricted to full subcategories of schemes on both sides the partial adjunction \(| | \dashv N\) becomes an equivalence of categories.

### 3 Proof of the existence of Spec

In the first part of the paper, we deal with the existence and description of the spectrum functor – Theorem 3. We start proving the existence of the adjoint spec to the inclusion of \(\text{Top}_{\mathcal{A}, \mathcal{P}}\) into \(\text{Top}_{\mathcal{A}}\). This will be an application of the adjoint functor theorem, so we will prove that all colimits exist in \(\text{Top}_{\mathcal{A}}\) and that \(\text{Top}_{\mathcal{A}, \mathcal{P}}\) is closed under them and finally we will verify the solution set condition.

#### 3.1 Characterization of local spaces and maps

First we prove a useful characterization of \((\mathcal{A}, \mathcal{P})\)-spaces among \(\mathcal{A}\)-spaces. Let \(X \in \text{Top}_{\mathcal{A}}\) and \(f: A_i \to \mathcal{O}_X(U)\) be a map. For each \(j \in J_i\), we define a subset \(U_j^{\max} \subseteq U\) to consist of all the points \(p \in U\) for which an extension through \(B_{ij}\) exists in the diagram

\[
\begin{array}{ccc}
A_i & \xrightarrow{f} & \mathcal{O}_X(U) \\
\downarrow & & \downarrow \\
B_{ij} & \xrightarrow{\mathcal{O}_X} & \mathcal{O}_{X,p}
\end{array}
\]

By the finite presentability of \(A_i\) and \(B_{ij}\), one may then replace the stalk \(\mathcal{O}_{X,p}\) by \(\mathcal{O}_X(V_p)\) for some open \(V_p \ni p\) and, by uniqueness of the extensions, these are compatible with restrictions and the sheaf condition for \(\mathcal{O}_X\) thus
provides an extension

\[ \begin{array}{c}
A_i \xrightarrow{f} O_X(U) \\
\downarrow \\
B_{ij} \xrightarrow{g_j} O_X(U_{j}^{\max})
\end{array} \]

Moreover, we see that \( U_j^{\max} = \bigcup_{p \in U_j^{\max}} V_p \) is open. When \( X \in \text{Top}_{A,P} \) the sets \( U_j \) give a particular open cover of \( U \) which we refer to as the canonical open cover. This proves the necessity part of the following criterion in which any open cover suffices.

**Lemma 8.** An \( A \)-space \( X \) is an \((A,P)\)-space if and only if for each map \( f : A_i \rightarrow O_X(U) \) there exists an open cover \( U = \bigcup U_j \) together with "partial" extensions \( g_j : B_{ij} \rightarrow O_X(U_j) \) as above.

**Proof.** It remains to prove sufficiency. For any extension problem

\[ \begin{array}{c}
A_i \xrightarrow{f} O_{X,p} \\
\downarrow \\
B_{ij}
\end{array} \]

we may first replace the stalk by \( O_X(U) \), then use the property from the statement to get an open cover \( U = \bigcup U_j \), choose \( j \) so that \( p \in U_j \) and then get an extension

\[ \begin{array}{c}
A_i \xrightarrow{f} O_X(U_j) \\
\downarrow \\
B_{ij} \xrightarrow{g_j} O_{X,p}
\end{array} \]

to \( O_X(U_j) \) and thus also to \( O_{X,p} \), as required. \( \square \)

Next we give a similar criterion for maps.

**Lemma 9.** A map of \( A \)-spaces \( \varphi : X \rightarrow Y \) is local if and only if the map of structure sheaves \( \varphi^\#$ : \varphi^* O_Y(U) \rightarrow O_X(U) \) is local for every \( U \).

**Proof.** It is quite easy to verify that a map of sheaves is objectwise local iff it is stalkwise local: The forward implication follows from the fact that admissible maps are closed under filtered colimits and it works also for presheaves. For
the backward implication one finds lifts locally in a neighbourhood of every point, by uniqueness and the sheaf condition for \( \mathcal{F} \) they provide a global lift:

\[
\begin{array}{cccc}
A_1 & \to & \mathcal{F}(U) & \to \mathcal{F}(U_p) & \to \mathcal{F}_p \\
\downarrow & & \downarrow & & \downarrow \\
B_{ij} & \to & \mathcal{G}(U) & \to \mathcal{G}_p
\end{array}
\]

Now apply this to the map \( \varphi^* : \mathcal{O}_Y \to \mathcal{O}_X \) whose effect on stalks is

\[
\varphi_p : \mathcal{O}_{Y, \varphi(p)} = (\varphi^* \mathcal{O}_Y)_p \to \mathcal{O}_{X,p}
\]

(the stalk at \( p \) is given by the inverse image along the inclusion of \( p \) in \( X \)).

### 3.2 Colimits in \( \text{Top}_A \) and \( \text{Top}_{A,P} \)

We proceed with the proof of the main theorem. First of all, we can see that the category \( \text{Top}_A \) of \( A \)-spaces is cocomplete using the Grothendieck fibration \( \text{Top}_A \to \text{Top} \) (see Example 66). Secondly, we show that \( \text{Top}_{A,P} \) is closed under colimits, i.e. when all the stalks in a diagram \( X_k, k \in \mathcal{K} \), are local and maps between them are admissible, the same is true for the colimit \( X \) and the components \( \lambda_k \) of the colimit cocone.

Consider an open subset \( U \subseteq X = \text{colim} X_k \) and a map \( A_i \to \mathcal{O}_X(U) \). Then, for each \( k \), the preimage \( U_k = (\lambda_k)^{-1}(U) \subseteq X_k \) admits, by Lemma 8, the canonical open cover \( U_k = \bigcup U_{k,j} \). The \( U_{k,j} \) are compatible under taking inverse images in the diagram, i.e. \( p \in U_{k,j} \) iff \( \alpha_* p \in U_{l,j} \). For the diagram

\[
\begin{array}{cccc}
A_i & \to & \mathcal{O}_{X_i, \alpha_* p} \\
\downarrow & & \downarrow \\
B_{ij} & \to & \mathcal{O}_{X_i, p}
\end{array}
\]

one extension exists if and only if the other exists, by virtue of the admissibility of the map between stalks. Therefore, by Example 66, the collection of open sets \( U_{k,j}^{\text{max}} \) describes an open set \( U_j \subseteq X \). Since also \( \mathcal{O}_X(U_j) = \lim \mathcal{O}_X(U_{k,j}^{\text{max}}) \) by Example 66, the (unique) partial extensions \( B_{ij} \to \mathcal{O}_{X_i, \lambda_{k,j}^{\text{max}}} \) yield a partial extension \( B_{ij} \to \mathcal{O}_X(U_j) \) and Lemma 8 proves \( X \in \text{Top}_{A,P} \).
The admissibility of the components of the colimit cocone $\lambda_k$ is similar, using Lemma 9. We need to find a lift in

$$
\begin{array}{ccc}
A_i & \xrightarrow{\lambda_k^* O_X(U)} & \colim_{V \supseteq \lambda_k(U)} O_X(V) \\
\downarrow{a_{ij}} & & \downarrow \\
B_{ij} & \rightarrow & O_{X_k}(U)
\end{array}
$$

By finite presentability of $A_i$ we get a factorization through some $O_X(V)$ in the colimit. Expressing this as $\lim O_{X_l}(\lambda_l^{-1}(V))$ we consider, for each $l$, the maximal open subset $W_l \subseteq \lambda_l^{-1}(V)$ where the extension

$$
\begin{array}{ccc}
A_i & \rightarrow & O_{X_l}(\lambda_l^{-1}(V)) \\
\downarrow{a_{ij}} & & \downarrow \\
B_{ij} & \rightarrow & O_{X_l}(W_l)
\end{array}
$$

exists; by assumption, $U \subseteq W_k$. As in the previous part, the $W_l$ describe an open subset $W \subseteq V$ with $\lambda_k(U) \subseteq W$ and so the extensions $B_{ij} \rightarrow O_{X_l}(W_l)$ describe a single extension $B_{ij} \rightarrow O_X(W)$ and finally a lift in the original diagram.

### 3.3 Solution set condition

This is the technical part of the proof: While colimits of $(\mathcal{A}, \mathcal{P})$-spaces are useful outside of the existence proof, we do not find other uses for the arguments of this part. The solution set condition for the inclusion in: $\text{Top}_{\mathcal{A}, \mathcal{P}} \rightarrow \text{Top}_{\mathcal{A}}$ is the condition that the comma category in/$Y$ contains a weakly terminal set of objects. We will be using more comma categories associated with inclusions of subcategories and we do not want to introduce names for each of the inclusions, so we will use an alternative notation:

**Notation.** Let $i: \mathcal{D} \rightarrow \mathcal{C}$ be the inclusion of the subcategory $\mathcal{D}$ into $\mathcal{C}$. Then for any object $A$ of $\mathcal{C}$ we will denote the comma category $i/A$ by $\mathcal{D}/_iA$.

This should be reminiscent of the usual notation for pullback (a lax version of which the comma category is an example).

Thus, given $Y \in \text{Top}_{\mathcal{A}}$ we need to find a weakly terminal subset of $\text{Top}_{\mathcal{A}, \mathcal{P}} /_{/\mathcal{A}} Y$. Let $(\varphi, \varphi^T): (X, O_X) \rightarrow (Y, O_Y)$ be an arbitrary map with $X \in \text{Top}_{\mathcal{A}, \mathcal{P}}$. Using the fibration structure on a forgetful functor $\text{Top}_{\mathcal{A}} \rightarrow \text{Top}$ as in Example 66 we obtain the vertical-cartesian factorization of $(\varphi, \varphi^T)$ as follows:

$$
\varphi: (X, O_X) \xrightarrow{(1, \varphi_T)} (X, \varphi^* O_Y) \xrightarrow{(\varphi, 1)} (Y, O_Y)
$$
This essentially reduces the possible structure sheaves $\mathcal{O}_X$ to those of the form $\varphi^* \mathcal{O}_Y$ except the latter does not have stalks in $\mathcal{P}$. Thus, for each open $U \subseteq X$, apply the usual small object argument to $\varphi_\sharp : \varphi^* \mathcal{O}_Y(U) \to \mathcal{O}_X(U)$, thus producing by functoriality a factorization

$$
\varphi^* \mathcal{O}_Y \xrightarrow{\psi} \mathcal{F} \xrightarrow{\chi} \mathcal{O}_X
$$

in $\mathbb{P}Sh_X$ and upon replacing $\mathcal{F}$ by its sheafification also in $Sh_X$. Lemma 9 shows that the presheaf version of $\mathcal{F} \to \mathcal{O}_X$ is also stalkwise admissible. Since the stalks of $\mathcal{O}_X$ are local, so are those of $\mathcal{F}$.

This means that $\varphi$ factors through the $(\mathcal{A}, \mathcal{P})$-space $(X, \mathcal{F})$, 

$$
\varphi : (X, \mathcal{O}_X) \xrightarrow{(1, \chi_\sharp)} (X, \mathcal{F}) \xrightarrow{(\varphi, \psi_\sharp)} (Y, \mathcal{O}_Y)
$$

and in effect we reduced all the possible $\mathcal{O}_X$ to those obtained from $\varphi^* \mathcal{O}_Y$ by localization followed by sheafification; for a fixed $\varphi : X \to Y$ there is a set of such, by the proof of Theorem 13. However, the underlying spaces $X$ and maps $\varphi$ still form a proper class. Thus, form an equivalence relation on $X$: $p \sim p'$ if their images in $Y$ agree, i.e. $\varphi(p) = q = \varphi(p')$ and if the two induced maps

$$
\begin{array}{ccc}
\mathcal{F}_p & \xrightarrow{\sim} & \mathcal{O}_{Y,q} \\
\downarrow \cong & & \downarrow \\
\mathcal{F}_{p'} & \leftarrow & \\
\end{array}
$$

are isomorphic – note that, as localizations (we obtained $\mathcal{F}$ via small object argument), they are epimorphic so the isomorphism will be unique. Now we need to show that $X/\sim$ inherits a structure sheaf from $X$ with local stalks and that $\varphi$ factors through it:

$$
\varphi : (X, \mathcal{O}_X) \xrightarrow{(1, \varphi)} (X, \mathcal{F}) \xrightarrow{pr} (X/\sim, \mathcal{F}/\sim) \xrightarrow{\varphi} (Y, \mathcal{O}_Y).
$$

This will finish the proof since the $(\mathcal{A}, \mathcal{P})$-spaces of the form $(X/\sim, \mathcal{F}/\sim)$ form, up to isomorphism, a set: localizations of each $\mathcal{O}_{Y,q}$ form a set by Theorem 13, thus we get a set of all possible underlying sets $X/\sim$, thus a set of possible topologies and, finally, a set of possible structure sheaves as above.

To finish the proof, we remark that $X/\sim$ is a colimit; namely, it is a coequalizer

$$
\begin{array}{ccc}
\sim & \xrightarrow{pr_1} & X \\
\downarrow pr_2 & & \downarrow \\
X & & \\
\end{array}
$$

with the equivalence relation $\sim \subseteq X \times X$ topologized discretely and given the structure sheaf with stalks at $(p, p')$ the local $\mathcal{F}_p \cong \mathcal{F}_{p'}$. Since this diagram lies in $\text{Top}_{\mathcal{A}, \mathcal{P}}$, so does its colimit.
4 Factorization system

To conclude the existence part of Theorem 3, we have to prove Theorem 13 about the number of local forms of a given object $R \in A$. At the same time, we will prepare ground for the description part of Theorem 3.

There is a factorization system associated with the collection $C^j$ of maps $a_{ij}$. A factorization is obtained by the small object argument and uniqueness is guaranteed by the generating maps $a_{ij}$ being epic; this implies that the factorization is functorial. Also note that by uniqueness every cofibration is in fact cellular (apply the small object argument to the cofibration and apply uniqueness).

We will call the maps in the left class localizations and denote them as $\rightarrow$, and the maps from the right class admissible maps and denote them as $\sim \rightarrow$. It follows easily from the lifting properties that a map that is both a localization and admissible is an isomorphism.

We will make a frequent use of the following simple lemma:

Lemma 10. If $f$ is epic then the following square is cartesian:

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow{h} & & \downarrow{g} \\
T & = & T
\end{array}
\]

Proof. Since $h = gf$, the square can be decomposed as

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow{f} & & \downarrow{g} \\
S & = & S \\
\downarrow{g} & & \downarrow{g} \\
T & = & T
\end{array}
\]

with the upper square cartesian since $f$ is assumed epic. $\square$

Lemma 11. In a commutative triangle

\[
\begin{array}{ccc}
R & \xrightarrow{h} & T \\
\downarrow{f} & & \downarrow{g} \\
S & & \\
\end{array}
\]

- $h$ admissible $\Rightarrow$ $f$ admissible;
- $h$ localization and $f$ epi $\Rightarrow$ $g$ localization;
- $h$ finite localization and $f$ epi $\Rightarrow$ $g$ finite localization.
Proof. The first point is easy to verify (using that the $a_{ij}$ are epic), as is the second, we proceed with the third point. The square

\[
\begin{array}{c}
R \xrightarrow{f} S \\
\downarrow{h} \quad \downarrow{g} \\
T \xrightarrow{} T
\end{array}
\]

is a pushout by Lemma 10, i.e. $g$ is a pushout of $h$, so a finite localization. \qed

**Lemma 12.** The components of the comma category $R/A\mathcal{P}$ are in bijection with isomorphism classes of local forms of $R$.

Every local form $R \to P$ is an initial object of the respective component.

**Proof.** By factoring a map $f: R \to Q$ with $Q \in \mathcal{P}$ into a localization $p$ followed by an admissible map (thus $p$ is a local form), we obtain a map $p \to f$ in $R/A\mathcal{P}$; for a fixed $f$, the object $p$ is unique up to isomorphism and, upon fixing it, the map $p \to f$ is unique. For a map $f \to f'$ in $R/A\mathcal{P}$ the factorization gives a diagram on the left, interpreted in $R/A\mathcal{P}$ on the right:

\[
\begin{array}{cc}
P & \xrightarrow{p} & Q \\
\downarrow{p} & & \downarrow{f} \\
P' & \xrightarrow{} & Q'
\end{array}
\]

By the previous lemma, the map $P \to P'$ is both a localization and admissible, thus an isomorphism and hence $p \cong p'$. This implies that the object $p \in R/A\mathcal{P}$ is shared by the whole component and as such is initial. \qed

The previous lemma essentially says that the inclusion $\mathcal{P} \to A$ admits a multi-adjoint (see [11]), i.e. that $\mathcal{P}$ is a multi-reflective subcategory, except for the size issues that we address now.

**Theorem 13.** The class of components $\pi_0(R/A\mathcal{P})$ is small, i.e. a set. In other words, the collection of local forms of $R$ up to isomorphism is a set.

**Proof.** We will present a more refined argument later. For the time being, start with a local form $p: R \to P$. By finite presentability, the small object argument gives a factorization of $p$ into a countable relative cell complex followed by an admissible map. The uniqueness of the factorization gives that $p$ is, in fact, itself a countable relative cell complex. In each step, there is no need to glue a cell of one shape $a_{ij}: A_i \to B_{ij}$ along one attaching map $f: A_i \to R$ multiple times, since the $a_{ij}$’s are epic. Thus, each step allows for a set of alternatives (given by sets of triples $(i, j, f)$ used in that step) and there is a countable number of steps. \qed
In fact, we proved that the collection of localizations of \( R \), up to isomorphism, is small.

5 Proof of the concrete description

We continue with the proof of Theorem 3, namely its descriptive part.

5.1 Points and stalks

Consider \( \mathcal{P}^* \subseteq \text{Top}_{A,P} \) the full subcategory consisting of one-point spaces (they are exactly the spaces \( P^* \) for \( P \in \mathcal{P} \)). We will now describe the points and the stalks of \( (X, \mathcal{O}_X) \in \text{Top}_{A,P} \) categorically, in a way similar to Lemma 12. Since we will need to distinguish between an \((A, \mathcal{P})\)-space and its underlying topological space, we will use \((X, \mathcal{O}_X)\) for the first and \(X\) for the second.

**Lemma 14.** The components of the comma category \( \mathcal{P}^*/\text{Top}_{A,P}(X, \mathcal{O}_X) \) are in bijection with points of \( X \).

Every point \( p: * \to X \) induces a map \( \overline{p}: (\mathcal{O}_{X,p})^* \to (X, \mathcal{O}_X) \) (a cartesian lift of \( p \)) that is a terminal object of the respective component.

**Proof.** Similarly to Example 66, \( \text{Top}_{A,P} \to \text{Top} \) is a fibrational Grothendieck construction for the diagram \( \text{Top}^{op} \to \text{CAT} \), sending a space \( X \) to the opposite of the category of \( A \)-valued sheaves on \( X \) with stalks in \( \mathcal{P} \) (the inverse image preserves stalks). Thus, the cartesian lift of \( p: * \to X \) is the map \( \overline{p}: (p^*\mathcal{O}_X)^* \to (X, \mathcal{O}_X) \) whose sheaf component is the identity; clearly \( p^*\mathcal{O}_X = \mathcal{O}_{X,p} \). Thus, the vertical-cartesian factorization of a map \( \varphi: P^* \to (X, \mathcal{O}_X) \),

\[
\varphi: P^* \to (\mathcal{O}_{X,p})^* \to (X, \mathcal{O}_X),
\]

gives a unique map from \( \varphi \) to \( \overline{p} \) in the comma category \( \mathcal{P}^*/\text{Top}_{A,P}(X, \mathcal{O}_X) \). \( \square \)

The adjunction \( \Gamma \dashv \text{Spec} \), when restricted to \( \mathcal{P}^* \subseteq \text{Top}_{A,P} \) yields

\[
\text{Top}_{A,P}(P^*, \text{Spec } R) \cong \mathcal{A}(R, \Gamma P^*) = \mathcal{A}(R, P),
\]
naturally in \( P \) and \( R \), thus giving

\[
\mathcal{P}^*/\text{Top}_{A,P} \text{Spec } R \cong (R/\mathcal{A} P)^{op},
\]

still natural in \( R \). Lemmata 12 and 14 give the following conclusions:

- (components) The points \( p \in \text{Spec } R \) are identified with isomorphism classes of local forms \( p: R \to P \).
• (terminal objects) The stalk at \( p \in \text{Spec } R \) is identified with \( P \).

The naturality in \( R \) gives an interpretation of the map induced by \( f : R \to S \) on the spectra: Let \( q \in \pi_0(S/\mathcal{A}\mathcal{P})^{\text{op}} \) be a point of \( \text{Spec } S \), represented by a local form \( q : S \to Q \). Its image under \( f^* : \pi_0(S/\mathcal{A}\mathcal{P})^{\text{op}} \to \pi_0(R/\mathcal{A}\mathcal{P})^{\text{op}} \) is represented by a local form \( p : R \to P \) is obtained by factoring \( qf \) into a localization \( p \) followed by an admissible map \( g \), as in

\[
\begin{array}{ccc}
R & \xrightarrow{p} & P \\
\downarrow{f} & \sim & \downarrow{g} \\
S & \xrightarrow{q} & Q
\end{array}
\]

Thus \( f^*(q) = p \) and the square on the right commutes since it corresponds to \( qf = gp \) in \( \mathcal{A}(R,Q) \) under the above correspondence. Since \( p \) and \( q \) are cartesian and \( g^* \) is vertical, the map on stalks \((f^*)_q : P \to Q\) is just \( g \).

### 5.2 Distinguished open sets

We will now exhibit certain open subsets of \( \text{Spec } R \). It will be convenient to denote the structure sheaf of \( \text{Spec } R \) simply as \( \mathcal{O}_R := \mathcal{O}_{\text{Spec } R} \). The counit of the adjunction gives a canonical map \( \varepsilon_R : R \to \Gamma \text{Spec } R \) (for commutative rings, this happens to be an isomorphism, but generally, this is not the case). Thus, \( R \) admits a canonical map into all objects in the structure sheaf \( \mathcal{O}_R(U) \).

Let \( k : R \to K \) be a finite localization. In our identification of points with local forms, \( k^* : \text{Spec } K \to \text{Spec } R \) is given by composing a local form \( K \to P \) with the epic \( k \) (resulting in a local form of \( R \)) and as such is injective on points. We will now show that its image is an open subset: to match the classical terminology, we will call these open sets distinguished and will denote them \( \text{Pts } K \).

Starting from the other side, a local form \( p : R \to P \) lies in \( \text{Pts } K \) iff \( p \) factors through \( k \); if this is the case, it factors through \( \mathcal{O}_R(U) \), for some \( U \ni p \):

\[
p : R \xrightarrow{k} K \quad \xrightarrow{y} \quad P = \colim_{U \ni p} \mathcal{O}_R(U)
\]

(by finite presentability of \( K \in R/\mathcal{A} \)). Consequently, for any \( p' \in U \), the corresponding localization \( p' : R \to P' \) factors through \( K \) as well and thus \( U \subseteq \text{Pts } K \).
We will now show that, in fact, Spec $K \to$ Spec $R$ is an open embedding: that is, a map of $\mathcal{A}$-spaces $(i, i^\flat): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ which is a homeomorphism $i$ onto an open subset on the topological level and on the level of sheaves, $i^\flat: i^* \mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism. Since the image Pts $K$ of $k^*$ is open, we may consider the restriction of Spec $R$ to this image and thus obtain a map Spec $K \to$ Spec $R|_{\text{Pts} K}$. We will now construct a map in $\text{Top}_{\mathcal{A}, \mathcal{P}}$ in the opposite direction by showing that a factorisation

$$\begin{align*}
\text{Spec } K & \quad \text{Spec } R|_{\text{Pts} K} \\
& \quad \text{Spec } R
\end{align*}$$

exists by translating it to an equivalent extension problem (where the bottom middle term is identified with the global sections of Spec $R|_{\text{Pts} K}$):

$$\begin{align*}
\mathcal{O}_R(U) & \quad \mathcal{O}_R(\text{Pts } K) \\
& \quad R
\end{align*}$$

By the above arguments, such an extension exists locally in some neighbourhood $U$ of any given $p \in \text{Spec } K$ – the dotted arrow in the diagram. Since these local extensions are unique and $\mathcal{O}_R$ is a sheaf, they glue to give an extension to $\mathcal{O}_R(\text{Pts } K)$ as required. It follows that $k^*: \text{Spec } K \to \text{Spec } R$ is an open embedding and since it is also a stalkwise iso, it gives an isomorphism Spec $K \cong \text{Spec } R|_{\text{Pts} K}$. Using this identification, the counit $K \to \Gamma \text{Spec } K$ then becomes $K \to \mathcal{O}_R(\text{Pts } K)$.

### 5.3 Topology and structure sheaf

Finally, we will show that the distinguished opens Pts $K$ form a basis of the topology of Spec $R$ and give a description of the structure sheaf $\mathcal{O}_R$, slightly different from that claimed in the main statement; the relationship of the two approaches is addressed in Section 9.

We will prove the claims by constructing from Spec $R$ a minimalistic version of it that is still an $(\mathcal{A}, \mathcal{P})$-space with the same stalks; the universal property will then ensure that it is in fact isomorphic to Spec $R$. Concretely, let Spec $' R$ be Spec $R$ equipped with a topology generated by the basis Pts $K$, for all finite localizations $R \to K$ – these are clearly closed under finite intersections (these correspond to pushouts of localizations). The identity of the underlying sets is a continuous map $\varphi$: Spec $R \to \text{Spec }' R$ that we will
now promote to a map of \((A, \mathcal{P})\)-spaces and then show to be an isomorphism. In order to do so, we need to define a structure sheaf \(\mathcal{O}'_R\) on \(\text{Spec}' R\) and provide it with a map \(\mathcal{O}'_R \rightarrow \varphi_*\mathcal{O}_R\) where the codomain is easily seen to be just the restriction of \(\mathcal{O}_R\) to the opens of \(\text{Spec}' R\). We know that the counit is a map \(K \rightarrow \mathcal{O}_K(\text{Pts } K) = \mathcal{O}_R(\text{Pts } K)\) and it is thus tempting to set \(\mathcal{O}'_R(\text{Pts } K) = K\). This prescription is only defined on the distinguished opens and does not satisfy the sheaf condition; most importantly, it is not well defined since \(\text{Pts } K\) does not determine \(K\).

We can correct the above deficiency by considering, for any distinguished open \(U\), the diagram \(\mathcal{F}\mathcal{L}_U\) of all finite localizations \(K\) with \(\text{Pts } K = U\) and defining \(R_U = \text{colim}_{K \in \mathcal{F}\mathcal{L}_U} K\), i.e.
the union of all finite localizations with the same associated point set \(U\). This is easily seen to be a localization, possibly infinite. At the same time, for any \(K \in \mathcal{F}\mathcal{L}_U\), the corresponding component of the colimit cocone is a localization \(K \rightarrow R_U\) that induces an isomorphism

\[
\text{Spec } K \cong \lim_{K \in \mathcal{F}\mathcal{L}_U} \text{Spec } K \cong \text{Spec } R_U,
\]

since the diagram of spectra is cofiltered and constant. By our identification of these with subsets of \(\text{Spec } R\), this means in particular that \(\text{Pts } R_U = U\).

**Lemma 15.** If \(V \subseteq U\) then a factorization \(R_U \rightarrow R_V\) exists.

**Proof.** Let \(L \in \mathcal{F}\mathcal{L}_V\) and define a functor \(\mathcal{F}\mathcal{L}_U \rightarrow \mathcal{F}\mathcal{L}_V\) by a pushout along \(R \rightarrow L\). This is well-defined since \(\text{Spec}\) takes pushouts in \(A\) to pullbacks in \(\text{Top}_{A, \mathcal{P}}\). The bottom map in the following diagram

\[
\begin{array}{ccc}
R & \rightarrow & L \\
\downarrow & & \downarrow \\
K & \rightarrow & L +_R K
\end{array}
\]

then gives a natural transformation

\[
\mathcal{F}\mathcal{L}_U \rightarrow \mathcal{F}\mathcal{L}_V
\]

with the diagonal functors the canonical inclusions and thus induces the required map \(R_U \rightarrow R_V\) by taking colimits. \(\square\)

We thus have a partial presheaf defined on the distinguished opens of \(\text{Spec}' R\), given by \(U \mapsto R_U\) and we extend it to a presheaf on \(\text{Spec}' R\) using
a right Kan extension along the inclusion of distinguished opens into all opens; since the inclusion is full, the result is indeed an extension which we call the **canonical presheaf** and denote \( O^\text{can}_R \). Finally, we define \( O'_R = \text{sh} \, O^\text{can}_R \) as the sheafification.

The counits \( K \to O_R(\text{Pts} \, K) \) induce a natural map \( R_U \to O_R(U) \) for \( U \) distinguished open and thus, for any open \( V \subseteq \text{Spec} \, R \), also

\[
\lim_{\text{d.o. } U \subseteq V} R_U \to \lim_{\text{d.o. } U \subseteq V} O_R(U)
\]

where the left hand side is the usual formula for the right Kan extension \( O^\text{can}_R(V) \) and the right hand side is \( O_R(V) \) by the sheaf property of \( O_R \), since the collection of all distinguished open subsets of \( V \) forms a particular open cover of \( V \). Finally, the resulting map \( O^\text{can}_R \to \varphi_* O_R \) induces a map from the sheafification \( O'_R \to \varphi_* O_R \) and this finishes the construction of \( \varphi : \text{Spec} \, R \to \text{Spec} \, R' \). We need to verify that this is a map of \((\mathcal{A}, \mathcal{P})\)-spaces. Both the right Kan extension and the sheafification preserve stalks (the first does since a stalk can be computed as a colimit over any neighbourhood basis, e.g. that formed by all distinguished open neighbourhoods, the second by the general property of the sheafification) and, thus, the stalk of \( O'_R \) at \( p : R \to P \) is

\[
O'_{R,p} = \text{colim}_{U \ni p} R_U \cong \text{colim}_{\text{Pts} \, K \ni p} K = \text{colim}_{p : R \to K \to P} K \cong P
\]

where the last isomorphism comes from the general procedure of rewriting a relative cell complex as a directed colimit of finite cell complexes, see [9]. This implies easily that \( \varphi : \text{Spec} \, R \to \text{Spec'} R \) is a stalkwise isomorphism and as such is, in particular, a map of \((\mathcal{A}, \mathcal{P})\)-spaces. Since \( \text{Spec'} R \) admits a canonical map into \( R^* \), the universal property of \( \text{Spec} \, R \) gives a filler \( \psi \) in the diagram:

\[
\begin{array}{ccc}
\text{Spec} \, R & \xrightarrow{\varphi} & \text{Spec'} R \\
\downarrow \varphi & & \nearrow \psi \\
\text{Spec} \, R & & R^*
\end{array}
\]

\(^1\)With respect to the arguments in the next paragraph, a more reasonable choice seems to be that of the left Kan extension. However, the result would almost never be equal to the structure sheaf \( O_R \), while Proposition [34] shows that the right Kan extension works better in this respect. At the same time, the sheafification of both Kan extensions is the same (they agree on distinguished opens).
As \( \varphi \) and \( \psi \) are the identity maps on the underlying point sets, this shows that the topologies coincide; since \( \varphi \) was also shown to be a stalkwise isomorphism, it is then an isomorphism of \((\mathcal{A}, \mathcal{P})\)-spaces.

6 Connection to the spectrum of Diers

We relate our spectral construction to a Diers spectrum as described in [6] and [11]. A Diers context is a right multiadjoint \( U: \mathcal{P} \to \mathcal{A} \) with \( \mathcal{A} \) locally finitely presentable satisfying certain technical condition: any local form \( p: R \to \mathcal{P} \) is a filtered colimit of all maps \( k: R \to K \) factorizing \( p \) with \( K \) finitely presentable in \( R/\mathcal{A} \) such that \( k \) is left orthogonal to the maps in the image of \( U \). We then define the category \( \text{Top}_U \) of \( U \)-spaces as the following pseudo-pullback in \( \text{CAT} \) (where the map \( \text{Top}_A \to \prod \mathcal{A} \) is a joint stalk functor):

\[
\begin{array}{ccc}
\text{Top}_U & \longrightarrow & \text{Top}_A \\
\downarrow & & \downarrow_{\text{stalk}} \\
\prod \mathcal{P} & \longrightarrow & \prod \mathcal{A}
\end{array}
\]

In particular, if \( \mathcal{P} \) is the subcategory of local objects and admissible maps with respect to a set of cones satisfying the prerequisites of Theorem 3, the inclusion \( U: \mathcal{P} \to \mathcal{A} \) is a Diers context (it is a multireflection by Lemma 12 and the technical condition is satisfied since any local form is a filtered colimit of finite localizations factoring through it). The category \( \text{Top}_U \) is then precisely \( \text{Top}_{\mathcal{A}, \mathcal{P}} \). We have the following theorem of Osmond, building heavily on the work of Diers in [6]:

Theorem 16. ([11], Thm. 2.14) The functor \( \text{Top}_U \to \text{Top}_A \) admits a right adjoint \( \text{spec} \).

This shows that if \( \mathcal{A} \) is locally finitely presentable, our Theorem 3 is a special case of the theorem above. What differs are the proof methods: in both [6] and [11] the spectrum is first described explicitly and only then, this description is used to prove that \( \text{spec} \) is an adjoint.

7 Obstructions to fully faithful Spec

In the second part of the paper, we want to study conditions, under which \( \text{Spec} \) is fully faithful. This is the case in classical algebraic geometry and it turns out to be very important also in the generalized context. It is well
known that this is equivalent to the counit $\varepsilon$ of $\Gamma \dashv \text{Spec}$ being an isomorphism. We define $f_\mathcal{A} \subseteq \mathcal{A}$ to be the full subcategory of the fixed points of the adjunction, i.e. those objects $R \in \mathcal{A}$ at which the counit $\varepsilon_R: R \to \overline{R}$ (interpreted in $\mathcal{A}$) is an isomorphism, where we abbreviate $\overline{R} = \Gamma \text{Spec} R$.

There is a slightly weaker condition of $\varepsilon_R$ being a geometrical isomorphism, i.e. inducing an isomorphism on spectra. In such a situation, $f_\mathcal{A}$ is exactly the full image of $\Gamma$, it is a reflective subcategory and $\text{Spec}|_{f_\mathcal{A}}$ is fully faithful. This is the content of a general well known theorem that we summarize first.

### 7.1 Idempotent adjunctions

A monad $T$ on $\mathcal{A}$ is said to be idempotent if its multiplication $\mu$ is an isomorphism or, equivalently, if its unit $\eta$ satisfies $T\eta = \eta T$. In this case, the Eilenberg-Moore category $\mathcal{A}^T$ of algebras happens to be a reflective subcategory of all objects at which $\eta$ is an isomorphism, i.e. the fixed points of $T$, and the reflection is given by $T$. There is an obvious dual notion of an idempotent comonad. An adjunction $L \dashv R$ is said to be idempotent if the associated monad $RL$ is idempotent or, equivalently, if the associated comonad $LR$ is idempotent. In this case the fixed points of $RL$ are the full image of $R$ and the fixed points of $LR$ are the full image of $L$ and the adjunction restricts to an equivalence between these, very much like for a Galois connection. Since the multiplication for $RL$ is given by $\mu = R\varepsilon L$ we thus obtain the following theorem.

**Theorem 17.** Given an adjunction $L \dashv R$, assume that the transformation $R\varepsilon L$ is an isomorphism. Then the adjunction restricts to an equivalence between the full image of $R$ and the full image of $L$. \hfill \Box

### 7.2 The consecutive obstructions

In our situation, the condition from the previous theorem reads specifically that $\text{Spec} \varepsilon \Gamma$ should be an isomorphism. In such a situation, we will ultimately replace $\mathcal{A}$ by $f_\mathcal{A}$, so it is not a big deal to restrict from $\mathcal{A}$ to an arbitrary reflective subcategory $\mathcal{B}$ satisfying $f_\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{A}$. This will enable us to prove useful properties of $\text{Spec}|_{\mathcal{B}}$ and finally give conditions under which the counit becomes a geometric isomorphism on $\mathcal{B}$. We will introduce two such reflective subcategories $r_\mathcal{A}$ and $m_\mathcal{A}$: The subcategory $r_\mathcal{A}$ of reduced objects enables for a tighter (bijective, in fact) correspondence between distinguished open sets and finite localizations. The subcategory $m_\mathcal{A}$ of mono-reduced objects allows for more efficient computation with the sheafification (e.g. all
the canonical presheaves are monopresheaves). Both are introduced in Section 9, together with a precise description of geometric isomorphisms that allows to determine when these are actually isomorphisms. Assuming that $\mathcal{A} = r\mathcal{A}$ and $\mathcal{A} = m\mathcal{A}$, Section 13 gives conditions for $\varepsilon$ to be a geometric isomorphism.

We now summarize the obstructions to full faithfulness of Spec:

- $\mathcal{A} = r\mathcal{A}$ and $\mathcal{A} = m\mathcal{A}$; if not, replace $\mathcal{A}$ by $r\mathcal{A} \cap m\mathcal{A}$.
- $\varepsilon$ should be a geometric isomorphism (at least on the image of $\Gamma$).
- Every geometric isomorphism should be an isomorphism; if not, replace $\mathcal{A}$ by $f\mathcal{A}$.

8 Cone small object argument

Here we present a way of seeing local objects and admissible maps at the same time in a uniform way. This will allow us to present a variation of the small object argument that constructs all the local forms at once and endows this collection with a universal property – it is a (multi)reflection. Our formulation takes place in the product completion of $\mathcal{A}$.

8.1 Product completion

First a quick reminder on product completions. We define $\prod \mathcal{A}$ as the opfibrational Grothendieck construction of the functor $\text{Set}^{\text{op}} \to \text{CAT}$, $I \mapsto \mathcal{A}^I$, so that it forms an opfibration over $\text{Set}^{\text{op}}$. We will now describe this category in more elementary terms. An object of $\prod \mathcal{A}$ is a pair $(I, (A_i)_{i \in I})$ where $I$ is a set and $(A_i)_{i \in I}$ is an $I$-indexed collection in $\mathcal{A}$; since the indexing set $I$ is implicit in the collection, we will use just $(A_i)_{i \in I}$ or even $(A_i)$ to denote this object. A singleton family consisting of $\mathcal{A}$ will be denoted simply by $\mathcal{A}$, i.e. we consider $\mathcal{A}$ as a subcategory of $\prod \mathcal{A}$. Finally the empty collection will be denoted $(\ )$. A map $f: (A_i)_{i \in I} \to (B_j)_{j \in J}$ consists of a map of sets $J \to I$.
and a collection of maps $A_{f(j)} \to B_j$ for each $j \in J$. Such a map is then a product of the maps $f_i: A_i \to (B_j)_{j \in J_i}$, where $J_i = f^{-1}(i)$.

$$
\begin{array}{ccc}
A_i & \xrightarrow{f_i} & B_j \\
& /\swarrow & \searrow / \\
& B_j & \cdots & B'_j
\end{array}
$$

We think of each $f_i$ as a cone, since it is just a collection of maps $f_j: A_i \to B_j$ with a common domain, the components of the cone. In the opposite direction, we will say that the $f_i$ are factors of $f$ and, more generally, a factor is the product of the $f_i$ over any subset of $I$.

8.2 Cone weak factorization systems

We will now study lifting properties which we denote by $f \boxslash g$. A lifting problem in $\prod A$ looks like

$$
\begin{array}{ccc}
(A_i) & \xrightarrow{(X_k)} & (Y_l) \\
& \searrow \swarrow f & \searrow \swarrow g \\
(B_j) & \xrightarrow{(Y_l)} & (Y_l)
\end{array}
$$

and is similarly a product of lifting problems (the products in the diagram are taken in $\prod A$)

$$
\begin{array}{ccc}
A_i & \xrightarrow{(X_k)_{k \in K_i}} & \prod_{k \in K_i} X_k \\
& \searrow \swarrow f & \searrow \swarrow g \\
(B_j)_{j \in J_i} & \xrightarrow{(Y_l)_{l \in L_i}} & \prod_{l \in L_i} Y_l
\end{array}
$$

and solving the original problem is equivalent to solving each of these problems. We see that $f \boxslash g$ iff for every decomposition of $g$ into factors $g_i$ we have $\forall i: f_i \boxslash g_i$. Since one can use identities on the empty collection for some of these factors $g_i$, it is easy to see that $f \boxslash g$ implies $f_i \boxslash g$ for all $i$. In other words, the left class $\mathcal{L}$ in any weak factorization system $(\mathcal{L}, \mathcal{R})$ is closed under factors. As usual, $\mathcal{R}$ is closed under products. The above analysis also shows the implications: $\mathcal{L}$ is closed under products with the identity on the initial object $\Rightarrow$ $\mathcal{R}$ is closed under factors $\Rightarrow$ $\mathcal{L}$ is closed under products.

Lemma 18. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system on $\prod A$. Then $\mathcal{L}$ is closed under products iff $\mathcal{R}$ is closed under factors.

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Definition 19. A cone weak factorization systems is a system in the product completion that satisfies the equivalent conditions of the previous lemma.

Intuitively, both classes \( \mathcal{L} \) and \( \mathcal{R} \) are determined by the cones that they contain. We will thus say that a cone weak factorization system is generated by a collection \( \mathcal{C} \) of cones \( a_i : A_i \to (B_{ij})_{j \in J_i} \) if \( \mathcal{R} \) is the class of all factors of maps in \( I^\oplus \) and \( \mathcal{L} \) is the corresponding class \( \mathcal{C} \mathcal{R} \) (we do not need to add products since \( \mathcal{R} \) is closed under factors).

Again, we call the elements of \( \mathcal{L} \) localizations and the elements of \( \mathcal{R} \) admissible maps. It will be of some importance to describe explicitly what it means for \( R \to (S_t) \) to be admissible, depending on the cardinality of the indexing set. For the empty indexing set, the lifting problem

\[
\begin{array}{ccc}
A_i & \to & R \\
\downarrow \quad a_i \downarrow & & \downarrow \quad R \\
(B_{ij}) & \to & (S_t)
\end{array}
\]

is exactly the localness of \( R \) with respect to the cone \( a_i \) from the introduction. For a singleton, the lifting problem

\[
\begin{array}{ccc}
A_i & \to & R \\
\downarrow \quad a_i \downarrow & & \downarrow \quad f \\
(B_{ij}) & \to & S
\end{array}
\]

is exactly the injectivity of \( f \) with respect to each component \( a_{ij} \), i.e. the admissibility from the introduction. The case of more than one factor is of limited use for us, except when related to the collection \( \mathcal{C}^\downarrow \) of the cone components rather than the collection \( \mathcal{C} \) of cones:

\[
\begin{array}{ccc}
A_i & \to & R \\
\downarrow \quad a_{ij} \downarrow & & \downarrow \quad f \\
B_{ij} & \to & (S_t)
\end{array}
\]

is exactly the joint admissibility of \( f \) with respect to the components \( a_{ij} \) or, equivalently, the admissibility of the map \( R \to \prod S_t \) in \( \mathcal{A} \) (with the product interpreted in \( \mathcal{A} \)).

8.3 Cone small object argument

We will now present a variation of the small object argument that takes into account the compatibility with products. We need to assume smallness of
the $A_i$; to make the notation simpler, we will assume that these are finitely presentable. Because of our application, we will formulate the small object argument directly for a map $R \to ( )$, i.e. it produces an $\mathcal{L}$-injective (=local) replacement of $R$.

Making $(R_k) \to ( )$ local is equivalent to making local each of the factors $R_k \to ( )$. This is achieved as usual by attaching cells in all possible ways:

$$\coprod_{i \in I, f} A_i \to \coprod_{i \in I, f} R_k \to (B_{ij}) \to R_k'$$

where the bottom left is $(\coprod_{i \in I, f} A_i \to R_k B_{ij}(i,f))$ with the product indexed over all functions $j$ taking a pair $(i, f)$ to some $j(i, f) \in J_i$; in plain words, the function chooses for each attaching map a shape of the attached cell and the product then ranges over all possible choices and then so does the pushout $R_k'$. Applying this procedure for each factor $R_k$ and taking product of the results then produces $(R_k)' = \prod_k R_k'$; this is then indexed by the factor $k$ and a choice function $j$ for the cell shapes. Finally, one applies this procedure countably many times $R^{(0)} = R$, $R^{(n+1)} = (R^n)'$ and takes the colimit $R^{(\infty)} = \operatorname{colim}_n R^{(n)}$. The factors of $R^{(\infty)}$ are now indexed by a sequence of choice functions, one for each step; in effect, in each step one glues cells along all possible attaching maps, choosing always cell shapes in all possible ways. One proves in the usual way that this produces a local object, using finite presentability of the $A_i$.

We will now prove a crucial property of this version of small object argument when applied to cones $a_i : A_i \to (B_{ij})$ with all components epic. To make things easier, we add to this cone also all the possible wide pushouts of the components (this clearly does not affect the cone injectivity). We may then determine, for any map $f : A_i \to P$ to a local object, the maximal $B_{ij}$ to which $f$ admits an extension. We will now show that $R^{(\infty)}$ contains all local forms of $R$. Thus, let $p : R \to P$ be a local form. For each $f : A_i \to R$ consider the maximal $B_{ij}$ for which an extension in

$$A_i \xrightarrow{f} R \xrightarrow{p} (B_{ij}) \to P$$

exists. Using this particular choice of $j = j(i, f)$ we obtain a factor $R_1$ of $R^{(1)}$ through which $p$ factors, so that every lifting problem as below has a
solution in $R_1$:

$$
\begin{array}{c}
A_i \\
\downarrow \\
(B_{ij})
\end{array} \xrightarrow{f} \begin{array}{c}
R \\
\downarrow \\
\sim \\
P
\end{array}
$$

(this is so for the maximal $B_{ij}$ and thus also for any smaller). Now proceed inductively with $R$ replaced by $R_1$ etc. and finally obtain a particular factor $R_\infty$ of $R^{(\infty)}$ such that

$$
\begin{array}{c}
A_i \\
\downarrow \\
(B_{ij})
\end{array} \xrightarrow{f} \begin{array}{c}
R_\infty \\
\downarrow \\
P
\end{array}
$$

i.e. the map $R_\infty \to P$ belongs to $\mathcal{R}$. Since it also belongs to $\mathcal{L}$ by Lemma 11 (cancellation lemma), it must be an isomorphism and thus $P$ is one of the factors of $R^{(\infty)}$.

The local object $R^{(\infty)}$ contains factors multiple times. By disposing off the extra occurrences (which is achieved by a retract so that this is still an $(\mathcal{L}, \mathcal{R})$-factorization), we obtain a map $R \to (P_\alpha)$ that has a strong universal property, namely it is a multi-reflection of $R$ into $\mathcal{P}$, i.e. it satisfies the following universal lifting property:

$$
\begin{array}{c}
R \\
\downarrow \\
(P_\alpha)
\end{array} \xrightarrow{f} \begin{array}{c}
Q \\
\downarrow \\
\sim \\
\exists
\end{array}
$$

The existence of a lift follows from factoring $f: R \to P \sim Q$ using the ordinary weak factorization system associated with $C$; since $P$ is then a local form of $R$, it is one of the $P_\alpha$ and, by construction, a unique such; the factorization is also unique.

**Theorem 20.** The inclusion $\mathcal{P} \to \mathcal{A}$ admits a multi-reflection. For each $R \in \mathcal{A}$, this is the collection $R \to (P_\alpha)$ of all local forms of $R$ (one representative of each isomorphism class).

\[\qed\]

### 9 Reduction

Using the formalism of the previous section, we will now reinterpret the construction of the localization $R_U$ associated with a distinguished open set
U. Namely, we will construct a reduction functor $r: \mathcal{A} \to \mathcal{A}$ – a reflection on the full subcategory $r\mathcal{A}$ of reduced objects. We will then show that the localization $R_{\text{Pts}K}$ associated with the distinguished open set $\text{Pts}K$ is exactly the reduction $rK$, in particular independent of the localization $K$ used to describe $U = \text{Pts}K$.

The unit of the multi-reflection from Theorem 20 is a map $\ell_R: R \to (P_\alpha)$ in the product completion $\prod \mathcal{A}$. Its components are the local forms $p_\alpha: R \to P_\alpha$ and these are in bijection with points of Spec $R$. For this reason, we will index the local forms of $R$ by points of Spec $R$, i.e. we will write $\alpha \in \text{Pts}R$.

### 9.1 General reduction

For the later use, we will construct a reduction functor associated with any factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{A}$ with $\mathcal{E}$ composed of epimorphisms. We will then apply this to the localization–admissible factorization and later also to the regular epi–mono factorization. The previous section describes an extension of $(\mathcal{E}, \mathcal{M})$ to a cone weak factorization system on $\prod \mathcal{A}$; alternatively, the factorization of $\ell_R$ in the display below can be executed in $\mathcal{A}$ by replacing the formal product in $\prod \mathcal{A}$ by the actual product in $\mathcal{A}$.

We say that $R$ is $\mathcal{M}$-reduced if the map $\ell_R: R \to (P_\alpha)$ lies in $\mathcal{M}$. We denote the full subcategory of $\mathcal{M}$-reduced objects by $r\mathcal{M}\mathcal{A}$. For a general $R$, there is a unique $(\mathcal{E}, \mathcal{M})$-factorization

$$\ell_R: R \xrightarrow{\eta_R} r\mathcal{M}R \to (P_\alpha).$$

By the uniqueness, $R$ is $\mathcal{M}$-reduced iff $\eta_R$ is an isomorphism.

We say that $f: R \to S$ is a geometric isomorphism, if it induces an isomorphism of spectra $f^*: \text{Spec} S \to \text{Spec} R$.

**Theorem 21.** $r\mathcal{M}\mathcal{A}$ is an epi-reflective subcategory of $\mathcal{A}$, the unit of the corresponding adjunction is $\eta_R: 1 \to r\mathcal{M}$ and it is a geometric isomorphism.

**Proof.** Let $R \in \mathcal{A}$ and $S \in r\mathcal{M}\mathcal{A}$. Since $\eta_R: R \to r\mathcal{M}R$ is epic (it belongs to $\mathcal{E}$), the precomposition map

$$\eta^*_R: r\mathcal{A}(rR, S) \to \mathcal{A}(R, S)$$

is injective. It is also surjective – a preimage of $f$ is obtained as $\eta_S^{-1} \circ r\mathcal{M}f$:

$$\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow \eta_R & & \downarrow \eta_S \\
r\mathcal{M}R & \xrightarrow{r\mathcal{M}f} & r\mathcal{M}S
\end{array}$$

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The second point is a simple application of Proposition 21. Since \( \eta_R \) is epic, and since every local form \( p_\alpha \) factors through \( r_M \) by construction, the square

\[
\begin{array}{ccc}
R & \xrightarrow{\eta_R} & r_M R \\
p_\alpha & & \\
\downarrow & & \\
P_\alpha & = & P_\alpha
\end{array}
\]

is a pushout by the proof of Lemma 11.

Thus, from the point of view of the associated spectrum, we may replace \( R \) by its reduction \( r_M R \). In fact, it is useful to work instead in the category \( r_M A \), but in order to do so, we need to equip it with a collection of cones and from the point of view of this section also with a factorization system.

The cones in \( r_M A \) are taken to be the \( M \)-reductions \( r_M a_i : r_M A_i \to (r_M B_{ij}) \) of the cones in \( A \). The factorization system \( (E, M) \) in \( A \) restricts to one in \( r_M A \), since the following lemma shows that any \( (E, M) \)-factorization of a map in \( r_M A \) stays in \( r_M A \).

**Lemma 22.** If \( f : R \to S \) lies in \( M \) then \( S \in r_M A \Rightarrow R \in r_M A \).

**Proof.** Using that \( E \) consists of epis, the proof of Lemma 11 gives that \( \ell_R \in M \) in the following square:

\[
\begin{array}{ccc}
R & \xrightarrow{f \in M} & S \\
\ell_R & & \ell_s \in M \\
(P_\alpha) & = & (Q_\beta)
\end{array}
\]

Now we compare the above introduced notions in \( r_M A \) with those in \( A \).

**Theorem 23.** The following claims hold:

- \( \Gamma \) takes values in \( r_M A \), so that \( P \subseteq \text{im} \Gamma \subseteq r_M A \).

- The notions of a local object and of an admissible map in \( r_M A \) coincide with those in \( A \). Consequently, \( r_M P = P \).

- The adjunction \( \Gamma \dashv \text{Spec} \) restricts to the adjunction \( r_M \Gamma \dashv r_M \text{Spec}. \) In particular, the notion of a local form in \( r_M A \) coincides with that in \( A \).

- All objects of \( r_M A \) are \( M \)-reduced.
Proof. The adjunction $\Gamma \dashv \text{Spec}$ gives equivalence

$$
\begin{array}{ccc}
R & \overset{f}{\longrightarrow} & \Gamma X \\
\eta_R & \downarrow & \approx \\
r_M R & \quad & \text{Spec } r_M R \\
\end{array}
$$

and Theorem [21] gives the iso on the right. Thus, $\Gamma X$ is injective w.r.t. the unit $\eta_R$; since $\eta_R$ is epic, this easily gives that $\Gamma X \in r_M A$ (take $f = 1_{\Gamma X}$).

The second point follows easily from the reflectivity: For $R \in r_M A$,

$$
\begin{array}{ccc}
A_i & \overset{f}{\longrightarrow} & R \\
a_i & \downarrow & \approx \\
(B_{ij}) & \quad & \text{Spec } r_M A_i \\
\end{array}
$$

Together with $\mathcal{P} \subseteq r_M A$ of the previous point, it follows that $r_M \mathcal{P} = \mathcal{P}$.

The previous two points yield $\text{Top}_{r_M A, r_M \mathcal{P}} = \text{Top}_{A, \mathcal{P}}$ so that $\text{Spec}$ takes values in $\text{Top}_{r_M A, r_M \mathcal{P}}$ and $\Gamma$ takes values in $r_M A$; this easily gives the third point – local forms $R \to P$ are the adjuncts of those maps $P^* \to \text{Spec } R$ that are terminal in the components of $\mathcal{P}^*/\text{Top}_{A, \mathcal{P}} \text{Spec } R$.

For the last point, the unit of the multi-reflection for $r_M A$ is the same as that for $A$ and as such lies in $M$. \hfill \Box

**Proposition 24.** Let $f : R \to S$ be a map whose pushout along any local form

$$
\begin{array}{ccc}
R & \overset{f}{\longrightarrow} & S \\
\downarrow & \approx & \downarrow f_* P \\
P & \quad & f_* P \\
\end{array}
$$

is an isomorphism. Then $f$ is a geometric isomorphism.

Later, we will also prove a converse to this statement, see Theorem [30].

**Proof.** The action of $f^*$ on a point $q \in \text{Spec } S$, i.e. on a local form $q : S \to Q$, is obtained by factoring the composite $q f$ into a localization followed by an admissible map

$$
\begin{array}{ccc}
R & \overset{f}{\longrightarrow} & S \\
\downarrow f^* q & \approx & \downarrow q \\
f^* Q & \quad & Q \\
\end{array}
$$
and we show now that $f_*$ and $f^*$ are mutually inverse: Clearly, the pushout square in the statement can be seen as such a factorization for $Q = f_* P$ and thus $P \cong f^* f_* P$. The factorization of the above square through the pushout is a map $f_* f^* Q \to Q$ and Lemma 11 shows that it is both a localization and admissible, hence an iso. The action of $f^*$ on stalks is identified with the bottom map $P \to f^* P$ in the square from the statement and is thus an iso. It remains to show that $f^*$ is open, so let $S \to L$ be a finite localization and $P \in (f^*)^{-1} \text{Pts} L$, i.e. $f_* P \in \text{Pts} L$. We thus have the following diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\sim} & S \\
\downarrow & & \downarrow \\
P & \xrightarrow{\sim} & f_* P
\end{array}
\]

and we need to prove existence of a finite localization $R \to K$ containing $P$ for which

\[
(f^*)^{-1} \text{Pts} K \subseteq \text{Pts} L,
\]

i.e. such that there is a factorization $S \to L \to f_* K$. Since $P$ is a filtered colimit of finite localizations $K$, its pushout $f_* P$ is a filtered colimit of the pushouts $f_* K$ and the map $L \to f_* P$ factors through some of the $f_* K$, by finiteness of the localization $L$.

\[ \square \]

### 9.2 Admissible case

The most important instance of the previous construction is the case of the factorization system generated by the components of the cones $C^\dagger$. The resulting reduction will be denoted simply by $r$ instead of $r_M$ for the class of admissible maps $\mathcal{M}$. In concrete terms, we say that $R$ is reduced if $\ell_R: R \to (P_a)$ is admissible; for general $R$, we get a factorization

\[
R \twoheadrightarrow r R \xrightarrow{\sim} (P_a).
\]

**Proposition 25.** $r R$ is the largest localization of $R$ among those localizations $R \twoheadrightarrow K$, for which $\text{Pts} K = \text{Pts} R$.

**Proof.** In the diagram below, the bottom map exists by the assumption,

\[
\begin{array}{ccc}
R & \xrightarrow{\sim} & r R \\
\downarrow & & \downarrow \\
K & \xrightarrow{\sim} & (P_a)
\end{array}
\]
hence also the diagonal. 

This easily implies the claim from the section introduction.

**Corollary 26.** $R_U = rK$ for any distinguished open set $U = \text{Pts } K$. 

We have already proved that the distinguished open sets $U$ are faithfully described by the localizations $R_U$, here we present a short alternative proof in terms of reductions.

**Lemma 27.** Let $K$ and $L$ be two localizations of $R$. Then $\text{Pts } K \subseteq \text{Pts } L$ iff there exists a factorization $rL \rightarrow rK$.

**Proof.** Similarly to the previous proof:

$$
\begin{array}{ccc}
R & \rightarrow & K \\
\downarrow & & \downarrow rK \\
rL & \rightarrow & (P_a)_{a \in \text{Pts } K}
\end{array}
$$

In $rA$, we obtain a “reduced” cone weak factorization system, generated by cones $ra_i: rA_i \rightarrow (rB_{ij})$. According to Theorem 23, the local objects, admissible maps and local forms are exactly those of $A$. Finite localizations are more complicated and it is thus not immediately clear that the distinguished opens in $\text{Spec } R$ and in $\text{Spec } rR$ agree; this is addressed in Section 12. Theorem 23 gives that all objects are reduced w.r.t. $A$-admissible maps in $rA$, but these are equal to $rA$-admissible maps, so that we get:

**Proposition 28.** All objects of $rA$ are reduced.

Finally, we conclude with the summary of the most important properties of reduced objects.

**Proposition 29.** Assuming $A = rA$, finite localizations $R \rightarrow K$ are in bijection with distinguished opens $\text{Pts } K \subseteq \text{Spec } R$ and the canonical presheaf is obtained as the right Kan extension of the partial presheaf $\text{Pts } K \mapsto K$. 

### 9.3 Geometric isomorphisms

In general, Spec is not fully faithful and, in particular, it does not reflect isomorphisms. We recall that a map is called a geometric isomorphism if its image under Spec is an isomorphism.
**Theorem 30.** A map $f: R \to S$ is a geometric isomorphism if and only if the pushout of $f$ along any local form becomes an isomorphism in $r\mathcal{A}$,

$$
\begin{array}{c}
R \\ f \\
\downarrow \\
S \\
\downarrow \\
P \\
\downarrow \downarrow \\
f_*P = r(f_*P)
\end{array}
$$

i.e. if and only if the composite across the bottom is an isomorphism.

**Proof.** According to Theorem 21, the units $\eta_R$, $\eta_S$ are geometric isomorphisms, so $f$ is a geometric isomorphism iff $rf$ is a geometric isomorphism:

$$
\begin{array}{c}
R \\ f \\
\downarrow \eta_R \\
S \\
\downarrow \eta_S \\
rR \\
\downarrow \downarrow rf \\
rS
\end{array}
$$

In addition, Theorem 23 says that the two potential notions of a geometric isomorphism, defined through Spec and $r\text{Spec}$, are the same. We may thus work in the category $r\mathcal{A}$ all the time and ignore all the reductions. The backward implication is then exactly Proposition 24.

For the forward implication, i.e. assuming that $f$ is a geometric iso, denote the unique preimage of $p$ under $f^*$ by $q$; we get a square

$$
\begin{array}{c}
R \\ f \\
\downarrow \\
S \\
\downarrow q \\
P \\
\downarrow \sim \\
Q
\end{array}
$$

with the bottom map iso since it is the action of $f^*$ on stalks. It remains to show that this square is cocartesian. To this end, express $P$ as a colimit of all its finite sublocalizations $K$; then the distinguished open sets $\text{Pts } K$ form a neighbourhood basis of $p \in \text{Spec } R$. Since $f^*$ is a homeomorphism, their preimages $\text{Pts}(f_*K)$ form a neighbourhood basis of $q \in \text{Spec } S$ and thus the $f_*K$ form a cofinal subdiagram in the diagram of all finite sublocalizations of $Q$, by Proposition 29. In particular, $Q$ is the colimit of the $f_*K$ and is thus itself a pushout of $P$.

We can view the theorem as an obstruction to Spec being fully faithful, expressed solely in terms of $\mathcal{A}$ (or $r\mathcal{A}$); coupled with results of Section 13, this
will provide a sufficient condition for Spec being fully faithful. Say that local forms reflect isomorphisms if pushouts of $f: R \to S$ along all local forms of $R$ are iso $\Rightarrow f$ is iso.

**Corollary 31.** Assume that $A = rA$. Then $\text{Spec}: A^{\text{op}} \to \text{Top}_{A,P}$ reflects isomorphisms iff local forms reflect isomorphisms.

### 9.4 Monomorphism case

Here we assume that $A$ is locally presentable, even though some parts only require that a strong epi–mono factorization exists in $A$.

We can apply the general reduction construction to the strong epi–mono factorization system, denoting the reduction by $m$ instead of $r_M$ for the class of monos $M$. In concrete terms, we say that $R$ is mono-reduced if $\ell_R: R \to (P_\alpha)$ is monic. This will be utilized in Section 11. Here we want to elaborate on the last point of Theorem 23, saying that all objects of $mA$ are reduced w.r.t. $A$-monos in $mA$. Since $mA$ is a reflective subcategory, the $A$-monos in $mA$ are exactly the $mA$-monos, so we can conclude:

**Proposition 32.** All objects of $mA$ are mono-reduced.

**Remark.** Assuming $A = rA$, the condition $A = mA$ is equivalent to the canonical presheaves $\mathcal{O}_{\text{mon}}^R$ on $\text{Spec} R$ being monopresheaves (this is almost clear for distinguished opens and extends easily to all opens by taking appropriate limits). Thus, in order to produce a sheafification $\mathcal{O}_R$, it is enough to perform the plus-construction once.

**Remark.** Interestingly, we can cofibrantly generate a factorization system $(E_\lambda, M_\lambda)$ by all epis between $\lambda$-presentable objects and by the general theory, the canonical map $\ell_R: R \to (P_\alpha)$ lies in $M_\lambda$ for any fixed point $R \in fA$. Since this holds for every $\lambda$, this canonical map $\ell_R$ is a strong mono.

**Remark.** There is also a cofibrantly generated factorization system $(E, M)$ whose right class consists of maps that are monic and admissible at the same time. We then get a reflection onto the intersection $mA \cap rA$.

### 10 First consequences of fully faithful Spec

For the further use, we record the following implication:

**Lemma 33.** If $\text{Spec}$ is fully faithful then $A = rA$.

**Proof.** If $fA \subseteq rA \subseteq A$ is equality, both inclusions have to be. □
First we study a simple criterion for $\text{Spec}$ being fully faithful. By the above lemma, we may restrict to the case $\mathcal{A} = r\mathcal{A}$.

**Proposition 34.** Assume that $\mathcal{A} = r\mathcal{A}$. Then $\text{Spec}: \mathcal{A}^{\text{op}} \to \text{Top}_{\mathcal{A}, P}$ is fully faithful iff the canonical presheaf $\mathcal{O}^{\text{can}}_R$ is a sheaf for each $R$ (then it equals $\mathcal{O}_R$).

**Proof.** $\text{Spec}$ is fully faithful if and only $\varepsilon$ is an isomorphism:

$$\varepsilon_R: R = \mathcal{O}^{\text{can}}_R(R) \to \mathcal{O}_R(R) = \Gamma \text{Spec} R.$$ 

Thus, the counit is an isomorphism for all finite localizations $K$ of $R$ iff $\mathcal{O}^{\text{can}}_R \to \mathcal{O}_R$ is an isomorphism on all distinguished opens $\text{Pts} K \subseteq \text{Spec} R$. It remains to show that in this case, it is an isomorphism for every open $U \subseteq \text{Spec} R$. As a right Kan extension, $\mathcal{O}^{\text{can}}_R(U)$ is given by the limit

$$\mathcal{O}^{\text{can}}_R(U) = \lim_{\text{Pts} K \subseteq U} \mathcal{O}^{\text{can}}_R(\text{Pts} K)$$

over the canonical cover of $U$ by all distinguished opens. Since the same is true for any sheaf such as $\mathcal{O}_R$, the canonical map $\mathcal{O}^{\text{can}}_R(U) \to \mathcal{O}_R(U)$ is a limit of isomorphisms and thus an isomorphism.

For the rest of this section, we assume that $\text{Spec}$ is fully faithful. We say that $X \in \text{Top}_{\mathcal{A}, P}$ is affine if it is isomorphic to $\text{Spec} R$ for some $R \in \mathcal{A}$. Since any isomorphism $\text{Spec} R \cong \text{Spec} S$ is now induced by a unique isomorphism $R \cong S$ and this induces a bijection of sets of finite localizations, we may then say that a subset $U \subseteq X$ of an affine is distinguished open if $U \cong \text{Spec} K$ is also affine and the inclusion is induced by a finite localization $R \rightarrow K$.

**Lemma 35.** Assume that $\text{Spec}$ is fully faithful. Let $Z \subseteq Y \subseteq X$ be a chain of open embeddings of affines.

- (composition) If $Z$ is distinguished open in $Y$ and $Y$ is distinguished open in $X$ then $Z$ is distinguished open in $X$.

- (cancellation) If $Z$ is distinguished open in $X$ then it is distinguished open in $Y$.

**Proof.** The first point is easier, so we concentrate on the second which translates easily to the claim: If the composition $R \to S \to K$ is a finite localization then so is the second map. This holds by Lemma 11 provided that the first map is an epi. Since Spec is assumed fully faithful, we only need $Y \subseteq X$ monic, which is guaranteed by the next lemma. 

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Lemma 36. Any open embedding of $A$-spaces is a monomorphism.

Proof. Let $(\iota, \iota')$ be an open embedding of $X$ into $Y$, i.e. $\iota: X \to Y$ is an open embedding of topological spaces and $\iota': \iota^*O_Y \to O_X$ is an isomorphism. Consider a pair of maps

$$(\varphi, \varphi') : T \xrightarrow{(\psi_0, \psi_0')} X \xrightarrow{(\iota, \iota')} Y$$

coequalized by $(\iota, \iota')$. Since $\iota$ is injective, we get $\psi_0 = \psi = \psi_1$. On the level of structure sheaves,

$$\psi^*\iota^*O_Y \xrightarrow{\psi^*\iota_1} \psi^*O_X \xrightarrow{\psi_0} O_T$$

the two maps $\psi_0, \psi_1$ get equalized by an isomorphism $\psi^*\iota_2$ and are thus equal.

Now, we are ready to prove the statement we chose to call “Affine Communication Lemma” as it is a generalization of a technical lemma (5.3.1 in [13]) which subsumes the principle usually called the Affine Communication Lemma by algebraic geometers (5.3.2 in [13]).

Theorem 37. Assume that Spec is fully faithful. Let $U, V \subseteq X$ be two affine opens and $x \in U \cap V$. Then there exists an open affine $W \ni x$ that is distinguished open both in $U$ and $V$.

Proof. Since distinguished open subsets form a basis of topology on affines, we can find $W' \ni x$ contained in $U \cap V$ that is distinguished open in $U$. Further, we can find $W \ni x$ contained in $W'$ that is distinguished open in $V$.

By Lemma 35 we know that $W$ is also distinguished open in $W'$ and thus also in $U$.

11 Covers, sheafification

In order to study the counit $\varepsilon : R \to \overline{R} = \Gamma \text{Spec } R$, we need to get hold of the global sections of the structure sheaf $O_R$, i.e. of the sheafification
of the canonical presheaf. We will make use of the classical formula for the sheafification, so we will start by setting up notation for (hyper)covers. For us, a (hyper)cover will be a certain simplicial object in the coproduct completion, defined via a condition on its matching maps.

11.1 Semi-simplicial objects, matching maps

For an augmented semi-simplicial object

\[ \cdots \rightarrow U_1 \rightarrow U_0 \rightarrow U_{-1} \]

we define the \( n \)-th matching object \( M_n U \) by truncating the object \(<n\), then right Kan extending and finally taking the \( n \)-th object of the result. We will suffice with \( M_0 U = U_{-1} \) and \( M_1 U = U_0 \times_{U_{-1}} U_0 \) (the kernel pair of the augmentation map). The \( n \)-th matching map is the canonical map \( U_n \rightarrow M_n U \).

Dually, for an augmented semi-cosimplicial object

\[ K^{-1} \rightarrow K^0 \rightarrow K^1 \rightarrow \cdots \]

the latching objects \( L^n K \) and latching maps \( L^n K \rightarrow K^n \) are defined using left Kan extensions, with \( L^0 K = K^{-1} \) and \( L^1 K = K^0 +_{K^{-1}} K^0 \) the cokernel pair of the augmentation map.

11.2 Covers

We consider a category \( \mathcal{C} \) equipped with a Grothendieck topology \( J \). By a cover of \( C \in \mathcal{C} \), we will understand a map \( U_0 \rightarrow C \) in the coproduct completion \( \coprod \mathcal{C} \) with a singleton codomain whose components form a \( J \)-cover of \( C \) in \( \mathcal{C} \). More generally, \( U_0 \rightarrow (C^n)_{i \in I} \) is a cover if each of its summands is a cover of \( C^n \) in the previous sense. For us, a hypercover will be an augmented semi-simplicial object \( U : \Delta^{op} \rightarrow \coprod \mathcal{C} \) such that the matching maps \( U_n \rightarrow M_n U \) are covers (for our purposes, only \( n \leq 1 \) will be relevant):

\[ \cdots \rightarrow U_1 \rightarrow U_0 \rightarrow C \]

This means explicitly that \( U_0 = (U^i_0)_{i \in I_0} \rightarrow C \) is a cover and that for any \( i_0, i_1 \in I_0 \) the collection \((U^j_0)_{j \in I_1 / (i_0, i_1)} \rightarrow U_{i_0}^{i_0} \times_{U_0} U^{i_1}_0 \) is a cover (of this pullback), where we denote by \( I_1 / (i_0, i_1) \) the subset of the index set \( I_1 \) of elements mapping to \( i_0 \) and \( i_1 \) via the face maps \( d_1 \) and \( d_0 \).

The semi-simplicial object \( U \) is a convenient presentation of a larger diagram \( \text{El} U \) that contains the same data but not grouped into formal coproducts, see Appendix A. Concretely, \( \text{El} U \) takes \((n, i) \) to \( U^i_n \) and \( d_k : i \mapsto j \) to \( d_k^i : U^i_n \rightarrow U^j_{n-1} \). We abbreviate \( H_0 U = \text{colim} \text{El} U \).
Example 38. In $\text{Top}$ or $\text{Top}_{A,P}$, where the $J$-covers are understood to be jointly surjective collections of open embeddings, $H_0U$ is obtained by gluing the $U_i^0$ together along all the relevant $U_i^j$ and is thus isomorphic to $C$.

By Lemma 68, $H_0U$ is equally obtained by taking colim $U$ in $\prod C$ and subsequently taking the coproduct of all the components (there may be multiple when $E I$ is disconnected). Denoting the latter operation by $\Sigma$: $\prod C \rightarrow C$, we thus have alternative presentations

$$H_0U \cong \Sigma \text{colim } U \cong \text{colim } \Sigma U$$

(the last since $\Sigma$ clearly preserves colimits).

Assume that $A = rA$. A distinguished hypercover of $\text{Spec } R$ is given by an augmented semi-simplicial object in $\prod \text{Top}_{A,P}$ with matching maps distinguished covers, and is induced by an augmented semi-cosimplicial object

$$R \rightarrow K^0 \rightarrow K^1 \rightarrow \cdots$$

in the product completion $\prod A$ whose latching maps are distinguished opcovers in the sense of the following definition. We will then say that the above is a distinguished hyperopcover.

Definition 39. A cone $R \rightarrow (K_i)$ whose components are finite localizations is opcovering or a distinguished opcover if every local form $R \rightarrow P$ factors through one of these components; diagrammatically in $\prod A$,

$$R \rightarrow P \rightarrow (K_i)$$

We will also have a chance to encounter opcovers by localizations that are not necessarily finite; their definition is obvious.

Lemma 40. Distinguished opcovers are closed under pushouts and compositions.

11.3 Heller–Rowe formula

Later, we will need a concrete description of the sheafification functor, given by the Heller–Rowe formula

$$(\text{sh } F)V = \text{colim}_{U \rightarrow V} H^0(FU)$$
where the colimit runs over hypercovers $U$ of $V$. This formula works under various sets of assumptions, we will now briefly comment on the case of a locally finitely presentable $A$. A hypercover version of Proposition III.2.2 of [3] shows that $sh F$ is a presheaf through which any map from $F$ to a sheaf factors uniquely (this is a bit technical), so that it remains to prove that $sh F$ is a sheaf. Since the functors $A(R, -)$, for finitely presentable $R$, preserve limits and jointly reflect isomorphisms, it is enough to show that $A(R, sh F)$ satisfies the sheaf condition; but since $A(R, -)$ also preserves filtered colimits, we easily get that $A(R, sh F) \cong sh A(R, F)$ and as such is a sheaf by the classical result in $Set$.

A related result, with analogous proof, says that for a monopresheaf $F$ (such as the canonical presheaves in the case $A = m A$), one may restrict the colimit to covers (instead of hypercovers). In both versions, when $V$ is compact, one may further restrict to finite (hyper)covers as they form a cofinal subcategory.

Proposition 29 now easily implies that $R$ is given by the formula

$$R = \lim_{R \to K} H^0 K$$

(the colimit ranges over all distinguished hyperopcovers of $R$). This is why it is important to have a condition for $R \to H^0 K$ being an isomorphism. The lemma below represents a rather degenerate case of a hyperopcover (where one of the constituents is everything).

**Lemma 41.** Assume that $A = rA$. If one of the components $R \to K^0_{i_0}$ of an augmented semi-cosimplicial object is an isomorphism and $d_i^0: K^0_i \to (K^1_j)_{j \in \Gamma/(i_0, i)}$ is monic for all $i$, then the map $R \to H^0 K$ is an isomorphism.

**Proof.** We will show that the inverse is given by $H^0 K \to K^0_{i_0} \to R$ where the first map is the limit projection and the the second is the inverse to the isomorphism from the statement. Clearly one of the composites is $1_R$ and the other composite $H^0 K \to H^0 K$ is induced by the cone whose component $H^0 K \to K^0_i$ is the composition across the top
The composition with $d^0_j$ easily gives the same map as the limit projection $H^0 K \to K^0_i$. This holding for each $j$, the joint monicity shows that $H^0 K \to H^0 K$ is also the identity.

Assuming in addition that $A = mA$, the monicity condition is automatically satisfied for distinguished hyperopcovers: The square in the previous proof is equivalently a map from the pushout $K^0_i \amalg_R K^0_i \cong K^0_i$ and the hyperopcover condition thus says that $K^0_i \to (K^1_j)_{j \in I^1/(i_0, i)}$ is a distinguished opcover, i.e. there exists a factorization

$$K^0_i \to (K^1_j)_{j \in I^1/(i_0, i)} \to (P_{\alpha})_{\alpha \in K^0_i}.$$ 

Since the composite is monic, so is the first map.

**Proposition 42.** Assume that $A = rA$ and $A = mA$. If one of the components $R \to K^0_{i_0}$ of a distinguished hyperopcover is an isomorphism, then the map $R \to H^0 K$ is an isomorphism. 

12 Compactness of Spec $R$

In this section, we assume that the cones in $C$ are finite, i.e. that each defining cone $a_i: A_i \to (B_{ij})_{j \in J_i}$ has a finite indexing set $J_i$. We will prove that Spec $R$ is then compact. This was proved in [2] by identifying Spec $R$ with a spectrum of a distributive lattice that is a spectral space hence compact. By combining this proof with that of compactness of a spectrum of a distributive lattice, one obtains the following proof which we present here for completeness, but we strongly recommend reading the proof in [2].

**Theorem 43.** Assume that all cones in $C$ are finite. Then Spec $R$ is compact for every $R \in A$.

**Proof.** By Alexander lemma, it suffices to find a finite subcover for any cover by subbasic opens, in our case this means by distinguished opens. Thus, let $\mathcal{L} = \{L_k\}$ be an opcovering family of finite localizations. Assuming that no finite sub-opcover exists, we will produce a local form that does not factor through any of the $L_k$, yielding a contradiction.

An adapted filter is a subset $\mathcal{F} \subseteq \text{FinLoc}(R)$ satisfying

- $\mathcal{F}$ is upward closed,
- $\mathcal{F}$ is closed under finite distinguished opcovers, i.e. if $L$ admits a finite opcover by elements of $\mathcal{F}$ then $L$ belongs to $\mathcal{F}$. 

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An adapted filter \( \langle L \rangle \) generated by \( L \) is obtained by first closing the set \( L \) upwards and then by closing it under finite opcovers. Thus, the bottom element \( R \in \langle L \rangle \) iff \( R \) admits a finite opcover from \( L \).

Assuming that no finite subcover exists, \( \langle L \rangle \) does not contain \( R \), i.e. it is a proper adapted filter. Let now \( F \) be a maximal proper adapted filter above \( \langle L \rangle \), which exists by Zorn lemma. We will now show that it is prime in the following sense: If the pushout of \( L_0 \) and \( L_1 \) belongs to \( F \) then one of the \( L_i \) belongs to \( F \). For otherwise, there exist two finite opcovers of \( R \) – each by \( L_i \) and some finite subset of \( F \). Taking pushouts of these finite opcovers, \( R \) admits an opcover by \( L_0 + R L_1 \in F \) and a finite subset of \( F \), and thus \( R \in F \), a contradiction. It is now easy to see that the complement \( I \) of \( F \) is a collection of finite localizations that is non-empty, downward closed and closed under pushouts, so its colimit \( P = \text{colim} I \) is a localization. By filteredness of \( I \) and by the second property of \( F \), this \( P \) is then a local form. Since \( L_k \notin I \) and \( I \) is downward closed, \( P \) does not factor through any of the \( L_k \), a contradiction. Thus, a finite sub-opcover exists.

13 Flat local forms

Throughout this section, we assume that \( A \) is locally finitely presentable with \( A = rA \) and \( A = mA \).

We say that a map \( f : R \to S \) is flat if the cobase change \( f_* : R/A \to S/A \) preserves finite limits. We will study the counit of the adjunction \( \Gamma \dashv \text{Spec} \); for simplicity we denote \( \overline{R} = \Gamma \text{Spec} R \) so that we write the counit as \( \varepsilon_R : R \to \overline{R} \).

**Theorem 44.** If \( \text{Spec} R \) is compact and all local forms of \( R \) are flat then the counit \( \varepsilon_R : R \to \overline{R} \) is a geometric isomorphism.

The same conclusion holds if all finite localizations of \( R \) are flat. This may be obtained as a special case of Corollary 17 or by a simple modification of the proof below.

**Proof.** Consider a finite distinguished hyperopcover

\[
R \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots
\]

By taking a pushout along any local form \( p : R \to P \), we obtain

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{ R \ar[r]^-{f} \ar[d]_-{p} & H^0 K \\
K^0 \ar[r] & K^1 \\
\end{array} \\
\begin{array}{c}
\xymatrix{ P \ar[r]^-{f} \ar[d]_-{p} & H^0 K \\
K^0 \ar[r] & K^1 \\
\end{array}
\end{array}
\]

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where the leftmost square is cocartesian, since \( p \) is assumed flat. By picking some \( K_0 \) through which \( p \) factors, we get \( P \cong \widetilde{K}_0 \) by Lemma 10. Proposition 42 then shows that \( P \to H^0 \widetilde{K} \) is an iso, so that Theorem 30 concludes that \( f \) is a geometric iso. By passing to the limit

\[
\text{Spec } \overline{R} = \text{Spec } \text{colim}_K H^0 K = \lim_K \text{Spec } H^0 K \longrightarrow \text{Spec } R,
\]

we obtain the iso from the statement.

\[\Box\]

**Example 45.** In the category \( \mathcal{A} \) of commutative rings with \( \mathcal{P} \) the local rings and their local maps, every local form is flat. We conclude that every \( R \) is a fixed point, since Spec reflects isomorphisms by virtue of Theorem 30 and the well known fact that a map of \( R \)-modules is an isomorphism iff it is so after localization with respect to every prime of \( R \).

**Lemma 46.** Let \( f: R \to S \) be a map for which there exists an opcovering collection of localizations \( S \to (L_i) \) whose composition with \( f \) is an opcovering collection of finite localizations \( R \to (L_i) \) and such that the squares

\[
\begin{array}{ccc}
S & \longrightarrow & L_i \\
\downarrow & & \downarrow \\
L_j & \longrightarrow & L_i +_R L_j
\end{array}
\]

commute. Then \( f \) is a geometric isomorphism.

**Proof.** By assumptions, we have a factorization

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow g_i & & \downarrow h_i \\
L_i & \xrightarrow{g^*_i} & L_i
\end{array}
\]

which means that the restriction of \( f^* \): \( \text{Spec } S \to \text{Spec } R \) along the injective \( h^*_i : \text{Spec } L_i \to \text{Spec } S \) is the open embedding \( g^*_i : \text{Spec } L_i \to \text{Spec } R \). The usual equalizer–cokernel pair Galois connection between monic maps into \( (L_i) \) and jointly epic parallel pairs of maps out of \( (L_i) \) gives that \( S \) lies between \( R \) and the regular image of \( R \). Since these give the same cokernel pair, it must further equal to that of \( S \), i.e. \( L_i +_R L_j = L_i +_S L_j \). Thus, the intersection of the inclusions \( h^*_i, h^*_j \) is the same as the intersection of the inclusions \( g^*_i, g^*_j \) and this easily yields that \( f^* \) is a homeomorphism with the \( h^*_i \) open embeddings, too, and thus \( f^* \) is an isomorphism.

\[\Box\]
We will now apply this to the situation $S = H^0K$ of the limit of any distinguished opcover $R \to K$. Then the condition on the commutativity of the square is satisfied and we thus obtain the following instance.

**Corollary 47.** Assume that for any distinguished opcover $K$ of $R$ the induced map $H^0K \to K^0$ from the limit is an opcovering collection of localizations. Then the canonical map $\varepsilon_R: R \to \overline{R}$ is a geometric isomorphism.

If $\text{Spec} R$ is compact, this condition is required only for finite distinguished opcovers. \[ \square \]

Finally, we specialize the above general result to an important case easily applicable to the integral domains.

**Theorem 48.** Suppose that admissible maps are monic and that jointly monic families of localizations are opcovering. Then the counit $\varepsilon_R: R \to \overline{R}$ is a geometric isomorphism; finite families are sufficient for compact spectra.

**Proof.** We will deduce this from the last corollary, so we need to verify its hypothesis. For any distinguished opcover $K$ and the factorization

$$R \to H^0K \to K^0,$$

the dual of Lemma 11.1 shows that $H^0K \to K^0$ consists of localizations. Since limit projections are always jointly monic, they are opcovering, by assumption. \[ \square \]

**Example 49.** In the category $\mathcal{A}$ of reduced commutative rings with $\mathcal{P}$ integral domains and their monomorphisms, the above condition holds. This is because a finite family of localizations, i.e. of quotients $R \to R/I_k$, is jointly monic iff $I_1 \cap \cdots \cap I_n = 0$. In that case, any prime $P$ contains some $I_k$ and so the corresponding local form $R \to R/P$ factors through the localization $R \to R/I_k$, as required.

We have seen that for every commutative ring $R$ the counit $\varepsilon_R$ is a geometric iso. We will now give an example of a reduced ring where $\varepsilon_R$ is not an iso, i.e. Spec is not fully faithful even on $r\mathcal{A}$. Every noetherian ring $R$ possesses the smallest open cover $(R/P_i)$ with $P_i$ ranging over all minimal primes of $R$. Thus, $\overline{R}$ is the $H^0$ of

$$(R/P_i) \to (R/\text{rad}(P_i + P_j)) \to \cdots$$

In particular, let $R$ be the coordinate ring of a union $X$ of two smooth curves $X_1$, $X_2$ meeting at a single point $X_0$ non-transversely. Then the minimal prime $P_i$ consists of functions vanishing on $X_i$ and clearly $P_1 \cap P_2 = 0$, while

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$P_1 + P_2$ is a non-radical ideal whose radical is the prime $P$ consisting of functions vanishing at $X_0$. Thus, $R$ is the pullback of

$$R/P_1 \to R/(P_1 + P_2) \leftarrow R/P_2$$

while $\overline{R}$ is the pullback of

$$R/P_1 \to R/\text{rad}(P_1 + P_2) \leftarrow R/P_2$$

which describes the same two curves intersecting transversely at the point $X_0$. These commutative rings are not isomorphic, for the minimal primes $\overline{P}_i$ of $\overline{R}$ are obtained as kernels of the projection onto $R/P_i$ and it is easy to see that $\overline{P}_1 + \overline{P}_2$, being the kernel of the projection onto $R/P$, is also prime, hence radical, unlike the situation in $R$.

**Remark.** There is the following converse to Theorem 44. If Spec $R$ is compact and $\ell_R$ is an isomorphism for all $R$, then the local forms $p: R \to P$ preserve limits of finite hyperopcovers. This is so since $\mathcal{O}_R^{\text{can}}$ is then a sheaf and $R \to H^0K$ is thus an isomorphism; since $P \to H^0\mathcal{K}$ is still an iso, the leftmost square in the proof is cocartesian trivially.

Similarly, there is the following converse to Corollary 47. If $\ell_R$ is an isomorphism for all $R$, then the hypothesis of Corollary 47 on the factorization of distinguished opcovers holds. This is so because $R \to H^0K$ is an iso, as above, and thus the condition becomes trivial.

In both cases, in order for $\ell_R$ to be a geometric isomorphism, the fixed points must form a reflective subcategory where the reverse implication holds, so it is a good idea to look for such objects, i.e. objects whose local forms are flat w.r.t. finite hypercovers (this may be a bit complicated) or objects for which limit cones are opcovering.

**Remark.** We will now give a third instance of idempotency of the adjunction $\Gamma \dashv \text{Spec}$ which, however, does not apply to any of our standard examples. If monos are pushout stable and non-empty products are couniversal we get an epi–regular mono factorization in the product completion. Now consider a distinguished hypercover of Spec $R$, given by a semi-cosimplicial object

$$R \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots$$

in the product completion, where the latching maps are opcovering collections of finite localizations. Thanks to $\mathcal{A} = \mathfrak{m}\mathcal{A}$, such collections are jointly monic and the dual of Theorem 73 gives that the induced map $R \to H^0K$ is epic. Since a filtered colimit of epis is epic we get that the map from the
canonical presheaf to its sheafification is objectwise epic; this is the counit of the adjunction \( \varepsilon_R : R \to \Gamma \text{Spec} R \). Applied to \( R = \Gamma X \), it follows from the triangle identity

\[
1 : \Gamma X \to \Gamma \text{Spec} \Gamma X \to \Gamma X
\]

that the counit \( \varepsilon_{\Gamma X} \) is an isomorphism; Theorem 17 then gives idempotency.

As usual, for compact \( \text{Spec} R \), finite hypercovers are sufficient, so that finite product completion is sufficient and we only need binary products universal.

## 14 Functor of points approach

The last goal of this paper is to explore what algebraic geometers call the functor of points approach. In the classical case where \( \mathcal{A} \) is the category of commutative rings and \( \mathcal{P} \) is the subcategory of local rings and local homomorphisms, instead of a scheme \( X \) in \( \text{Top}_{\mathcal{A},\mathcal{P}} \), we can consider the functor

\[
\mathcal{N}X := \text{Top}_{\mathcal{A},\mathcal{P}}(\text{Spec } -, X) : \mathcal{A}^{\text{op}} \to \text{Set}.
\]

This is a sheaf in the Zariski topology \( J \) on \( \mathcal{A}^{\text{op}} \) and it turns out that the functor \( \mathcal{N} : \text{Top}_{\mathcal{A},\mathcal{P}} \to \text{SH}_J^{\mathcal{A}^{\text{op}}} \) valued in \( J \)-sheaves is fully faithful when restricted to schemes. This fact is crucial since it enables us to view schemes as a full subcategory of the category of Zariski sheaves which is much better behaved from the categorical standpoint than the category \( \text{Top}_{\mathcal{A},\mathcal{P}} \).

In the rest of the paper, we will generalize this to our context. Having a set of cones \( \mathcal{C} \) in \( \mathcal{A} \) as before, we can consider the Grothendieck topology \( J \) on \( \mathcal{A}^{\text{op}} \) defined as follows: \( J \)-covering families correspond to distinguished opcovers \( (R \to K_i) \) in \( \mathcal{A} \). Now, we want to consider the category of \( J \)-sheaves on \( \mathcal{A}^{\text{op}} \) – but as we will see later, we need to handle some issues coming from the fact that \( \mathcal{A} \) need not be small. In particular, working with sheaves on a large category, it is in general not clear whether the left adjoint to the inclusion of sheaves into presheaves exists. Furthermore we would wish to consider the left Kan extension of \( \text{Spec} \) along the Yoneda embedding (composed with sheafification) and we want this to be cocontinuous, but this requires some care if \( \mathcal{A} \) is large.

### 14.1 Presheaves and sheaves over large categories

The material of this section will be applied to presheaves and sheaves on \( \mathcal{A}^{\text{op}} \), but in order to decrease the number of \( \text{op} \)'s, and also with a view towards applications outside of this paper, we will be considering presheaves on a category \( \mathcal{C} \). It will be important that \( \mathcal{C} \) is not assumed small. In this case, we assume the axiom of universes and consider a universe \( \text{SET} \) with respect
to which $C$ is small. In addition, we require that the normal universe $\text{Set}$ consists of $\kappa$-presentable objects in $\text{SET}$ for some inaccessible cardinal $\kappa$. Then $\text{Set}$ is closed in $\text{SET}$ under all small (i.e. $\kappa$-small) limits and colimits and, in particular, these are computed the same way in the two categories. Since $C$ is small w.r.t. $\text{SET}$, the usual theory shows that $\text{PSH}_C = [\text{C}^{\text{op}}, \text{SET}]$ is a free completion of $C$ under “large” colimits. Also, for any Grothendieck topology $J$ on $C$, there exists a sheafification functor $\text{sh}$ on $\text{PSH}_C$, a reflection onto the full subcategory $\text{SH}_J^C$ of $J$-sheaves. The sheafification is given locally by a formula

$$(\text{sh} F) C = \text{colim}_{U \to C} \text{lim} FU$$

where the colimit runs over all $J$-hypercovers $U$ of $C$. Finally, we need to consider the full subcategory $\text{PSh}_C \subseteq \text{PSH}_C$ of small presheaves, i.e. the closure of the image $y_C$ of the Yoneda embedding under small colimits in $\text{PSH}_C$; most importantly, $\text{PSh}_C$ is the free completion of $C$ under small colimits, as is easy to be seen (or apply Theorem 50 to the trivial topology). Similarly, the full subcategory $\text{SH}_J^C \subseteq \text{SH}_J^C$ of small $J$-sheaves is the closure of the image $y'C$ of the “sheaffied Yoneda embedding” $y' = \text{sh} y$ under small colimits in $\text{SH}_J^C$.

We will now show that it has a crucial universal property.

**Theorem 50.** For any cocomplete category $\mathcal{D}$, there exists an adjoint equivalence, given by the left Kan extension and restriction

$$[C, \mathcal{D}]_J \simeq [\text{Sh}_J^C, \mathcal{D}]_{\text{ccts}},$$

between functors $K: C \to \mathcal{D}$ that turn $J$-hypercovers to colimit cocones and cocontinuous functors $L: \text{Sh}_J^C \to \mathcal{D}$.

By definition, a $J$-hypercover of $C$ is a semi-simplicial object

$$\cdots \xrightarrow{=} U_1 \xrightarrow{=} U_0 \longrightarrow C$$

in the coproduct completion $\coprod C$, augmented by $C$, such that all the matching maps are $J$-covers. A functor $K$ turns this into a colimit cocone if the canonical map $H_0 KU \to KC$ is an isomorphism.

We will now compare the condition from the theorem for $J$-hypercovers with that for $J$-covers which might be easier to check; here a $J$-cover is interpreted as a particular $J$-hypercover – one where $U_1 = U_0 \times_C U_0$, etc.

**Lemma 51.** If a functor $K: C \to \mathcal{D}$ sends $J$-covers to colimit cocones, then it also sends $J$-hypercovers to colimit cocones.

\footnote{While $\text{PSh}_C$ is easily seen to consist precisely of small colimits of representables, the corresponding claim for sheaves, of being precisely small colimits of sheafified representables, is much harder but still true, see [12].}
Proof. We will make use of $H_0 KU \cong \text{colim} \Sigma KU$ where $\Sigma : \bigsqcup \mathcal{D} \to \mathcal{D}$ is the functor from Section 11 (taking coproducts of the components). The hypothesis then guarantees that, for any $J$-cover $U_0$ of $C$, the induced map $\Sigma KU_0 \to KC$ is a coequalizer, hence an epimorphism.

Now consider an arbitrary $J$-hypercover $U$ of $C$, compare it with the underlying $J$-cover $U_0$, apply $K$ and take coproducts:

$\cdots \longrightarrow \Sigma KU_1 \longrightarrow \Sigma KU_0 \longrightarrow KC$

$\cdots \longrightarrow \Sigma K(U_0 \times_C U_0) \longrightarrow \Sigma KU_0 \longrightarrow KC$

The map $\Sigma KU_1 \to \Sigma K(U_0 \times_C U_0)$ is a coproduct of epimorphisms by the first paragraph, and thus itself an epimorphism. By assumption, the lower row is a coequalizer, hence so is the upper one.

14.2 Weighted colimits

It will be convenient to consider colimits weighted by possibly large weights – these may well exist, e.g. a coproduct of an arbitrary number of copies of the initial object. Thus, a weight is just $F \in \text{PSH}_C$. For a functor $K \in [C, \mathcal{D}]$ the weighted colimit $F \ast_C K$ is a representing object of the covariant functor on the left

$\text{PSH}_C(F, \mathcal{D}(K, -)) \cong \mathcal{D}(F \ast_C K, -)$,

should it exist. By Yoneda lemma, $yC \ast_C K \cong KC$. We have the following important cocontinuity property of the weighted colimit:

Lemma 52. Let $F : \mathcal{I} \to \text{PSH}_C$ be a diagram of weights such that for each $i \in \mathcal{I}$ the weighted colimit $F_i \ast_C K$ exists. Then

$\text{colim}_{i \in \mathcal{I}} (F_i \ast_C K) \cong (\text{colim}_{i \in \mathcal{I}} F_i) \ast_C K$

in the sense that one side exists iff the side exists, in which case the canonical comparison map is an isomorphism.

Proof. They are, respectively, representing objects for the functors

$\lim_{i \in \mathcal{I}} \text{PSH}_C(F_i \ast_C K, -) \cong \lim_{i \in \mathcal{I}} \text{PSH}_C(F_i, \mathcal{D}(K, -)) \cong \text{PSH}_C(\text{colim}_{i \in \mathcal{I}} F_i, \mathcal{D}(K, -))$

that are always isomorphic. \qed
One of the applications of weighted colimits is a pointwise formula for the left Kan extension: Given functors $K : C \to D$ and $I : C \to C'$, we have

$$(\text{lan}_I K)C' \cong C'(I -, C') \ast_C K;$$

more precisely, if the weighted colimit exists then so does the left Kan extension and it is then given by this formula.

### 14.3 Proof of the main theorem

The proof revolves around left Kan extensions of a functor $K : C \to D$ along the Yoneda embedding to various full subcategories of presheaves. There is a pointwise formula for the left Kan extension at $F \in \text{PSH}_C$ in terms of a weighted colimit

$$(\text{lan}_y K)F = \text{PSH}_C(y -, F) \ast_C K \cong F \ast_C K,$$

where the simplifying isomorphism is the Yoneda lemma. The weights for which this colimit exists form a full subcategory $\hat{C}_K$ and a pointwise left Kan extension of $K$ along the restricted Yoneda embedding $y : C \to \hat{C}_K$ then exists and is given by the weighted colimit as above. For simplicity, we will denote it as $\hat{K} = \text{lan}_y K$:

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{K} & \text{D} \\
\downarrow{y} & & \downarrow{\hat{K}} \\
\hat{C}_K
\end{array}
\]

Although the theorem concerns the pointwise left Kan extension along the composite $y' = \text{sh} y$, the isomorphism

$$(\text{lan}_{y'} K)F = \text{Sh}_C^J(y' -, F) \ast_C K \cong \text{PSH}_C(y -, F) \ast_C K \cong F \ast_C K \cong \hat{K}F$$

shows that this is equally a pointwise left Kan extension along the restricted Yoneda embedding $y : C \to \text{Sh}_C^J$, so that an important step will be to show $\text{Sh}_C^J \subseteq \hat{C}_K$:

**Lemma 53.** $\hat{C}_K$ is closed under the following constructions:

- $\hat{C}_K$ contains all representables.
- If $\mathcal{D}$ is (small) cocomplete then $\hat{C}_K$ is closed under small colimits and $\hat{K}$ is cocontinuous.
If $K$ turns $J$-hypercovers to colimit cocones then $F \in \hat{C}_K \Rightarrow \text{sh} F \in \hat{C}_K$; moreover, the image $\tilde{K}F \to \tilde{K}\text{sh} F$ of the sheafification map is an isomorphism.

Proof. The first point is obvious and the second is just Lemma 52 so we proceed with the third. In order to apply Lemma 52 again, we need to express the sheafification as a colimit in the category of presheaves. $J$-sheaves are exactly the objects orthogonal to the collection of maps $\alpha^0_U : H_0 yU \to yC$ coming from hypercovers $U \to C$ or equivalently objects injective w.r.t. the $\alpha^0_U$ and the corresponding codiagonal maps from the cokernel pair $\alpha^1_U : yC + H_0 yU yC \to yC$. To simplify the notation, we write $\alpha^n_U : G^n_U \to H^n_U$.

The main idea now is that $\text{sh} F$ is built from $F$ by gluing cells of the shape $\alpha^n_U$, which we make more precise in the next paragraph, and these induce isomorphisms on weighted colimits: for $\alpha^0_U$ this is just the condition

$$H_0 yU \ast C K \cong H_0 KU \to KC = yC \ast C K,$$

and it easily implies the corresponding claim for $\alpha^1_U$. Consequently, the sheafification map $F \to \text{sh} F$ also induces an isomorphism on weighted colimits.

The sheafification $\text{sh} F$, being the reflection of $F$ onto this injectivity class, can be produced by the small object argument, since the domains $G^n_U$ of $\alpha^n_U$ are $\kappa$-presentable in $\text{PSH}_C$. There results a smooth chain $F_\lambda$, for $\lambda \leq \kappa$, with $F_0 = F$ and $F_\kappa = \text{sh} F$. We prove by transfinite induction that $F_\lambda \in \hat{C}_K$ and that $F_{\leq \lambda} \ast C K$ consists of isomorphisms. For a limit $\lambda$, this is straightforward by Lemma 52. For a successor $\lambda$, we have a pushout square

$$\begin{array}{ccc}
\coprod G^n_{u_i} & \coprod H^n_{u_i} \\
\downarrow & \downarrow \\
F_{\lambda - 1} & \to & F_{\lambda}
\end{array}$$

It is easy to rewrite this as $F_\lambda$ being a colimit of a large diagram with objects $F_{\lambda - 1}$, $G^n_{u_i}$ and $H^n_{u_i}$, so that Lemma 52 can be applied again, giving that $F_\lambda \ast C K$ is a colimit of a diagram consisting of the $F_{\lambda - 1} \ast C K$ and isomorphisms; therefore this colimit exists and is isomorphic to $F_{\lambda - 1} \ast C K$, as required.

Corollary 54. If $\mathcal{D}$ is (small) cocomplete and $K$ turns $J$-hypercovers to colimit cocones then $\text{Sh}^J \subseteq \hat{C}_K$.

Proof. A colimit in $\text{SH}^J_C$ is the sheafification of the colimit in $\text{PSH}_C$; thus, the previous lemma shows that $\hat{C}_K \cap \text{Sh}^J_C$ is closed under colimits in $\text{SH}^J_C$. Since it also contains sheafified representables $y'C$, it must contain $\text{Sh}^J_C$. 

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We are now in the position to prove the main theorem of this section:

**Proof of Theorem 50.** We have seen that a left Kan extension along \( y' \) is a restriction of the left Kan extension along \( y : \mathcal{C} \to \mathcal{C}_K \) and the previous corollary shows its existence under the assumption that \( K \) turns \( J \)-hypercovers to colimit cocones. Clearly the left Kan extension is left adjoint to the precomposition functor; now for \( K \in [\mathcal{C}, \mathcal{D}] \) and \( L \in [\text{Sh}^J_{\mathcal{C}}, \mathcal{D}] \) we need to verify:

- If \( K \) turns \( J \)-hypercovers to colimit cocones, then \( \mathcal{K} \) is cocontinuous.
- If \( K \) turns \( J \)-hypercovers to colimit cocones, then \( K = \mathcal{K} y' \).
- If \( L \) is cocontinuous, then \( L y' \) turns \( J \)-hypercovers to colimit cocones.
- If \( L \) is cocontinuous, then \( L = \mathcal{K} y' \).

The first point follows from the previous lemma, since a colimit in \( \text{Sh}^J_{\mathcal{C}} \) is computed as a sheafification of a colimit in \( \text{PSH}_{\mathcal{C}} \) and \( \mathcal{K} \) preserves colimits of presheaves and turns the sheafification map into an isomorphism:

\[
\mathcal{K}(\text{sh colim}_i F_i) \cong \text{colim}_i \mathcal{K} F_i.
\]

For the second point, we have \( \mathcal{K} y' \cong \mathcal{K} y \cong K \), since the Yoneda embedding is fully faithful.

For the third point, the map

\[
H_0 L y' \to L y' C
\]

is isomorphic to the \( L \)-image of \( \text{sh } \alpha_{y U}^0 : \text{sh } H_0 y U \to \text{sh } y C \), which is an isomorphism (as follows e.g. from the analogous well known fact in \( \text{PSH}_{\mathcal{C}} \)).

The fourth point follows since the collection of \( J \)-sheaves, at which the canonical map \( \mathcal{L} y' \to L \) is an isomorphism, is clearly closed under colimits and contains all sheafified representables, by the iso in the proof of the second point (applied to \( K = L y' \)).

Next we promote the extension \( \mathcal{K} : \text{Sh}^J_{\mathcal{C}} \to \mathcal{D} \) to an adjunction. Namely, consider the “nerve functor” \( N_K : \mathcal{D} \to \text{PSH}_{\mathcal{C}} \), given as \( N_K D = \mathcal{D}(K -, D) \).

**Proposition 55.** If \( \mathcal{D} \) is (small) cocomplete and \( K \) turns \( J \)-hypercovers to colimit cocones then the nerve functor takes values in \( \text{SH}^J_{\mathcal{C}} \) and there results a partial adjunction \( \mathcal{K} \dashv N_K \).

If, in addition, \( K \) is fully faithful then \( y' = y \), i.e. the topology \( J \) is subcanonical. Moreover, \( y C \in \text{Sh}^J_{\mathcal{C}} \) and \( K c \in \mathcal{D} \) constitute a fixed point of this adjunction, for any \( C \in \mathcal{C} \); in particular, \( N_K \) maps the image of \( K \) to \( \text{Sh}^J_{\mathcal{C}} \).
Proof. For the first part, the nerve functor is the composite

\[ N_K D = D(K -, D) \]

where \( K \) takes any \( J \)-hypercover to a colimit cocone and \( D( -, D) \) turns this into a limit cone, as required for a presheaf to be a \( J \)-sheaf. Then

\[ D(\hat{K} F, D) \cong D(F *_C K, D) \cong PSH_C(F -, D(K -, D)) \cong SH^J_C(F, N_K D). \]

The second part is easy:

\[ N_K(KC) = D(K -, KC) \cong C( -, C) = yC \]

which must then be a sheaf by the first part, and thus equals \( y/C \). The other direction \( \hat{K} yC = yC *_C K \cong KC \) is just Yoneda lemma.

We finish with proving a sufficient condition for local smallness of \( \text{Sh}_C \).

**Proposition 56.** Assume that, for any fixed \( C \), the \( J \)-hypercovers of \( C \) up to isomorphism form a (small) set. Then \( \text{Sh}_C \) is locally small.

**Proof.** Consider the collection of all \( J \)-sheaves \( F \in \text{Sh}_C \) for which the corresponding hom-functor \( \text{Sh}_C(F, -) \) takes values in small sets. Our aim is thus to show that this collection equals \( \text{Sh}_C \). It is clear that it is closed under small colimits and it thus remains to show that it contains sheafified representables. Since we have already seen that

\[ \text{Sh}_C(y/C, G) \cong GC, \]

this in turn amounts to showing that \( \text{Sh}_C \subseteq [C^{\text{op}}, \text{Set}] \).

The formula for the sheafification shows that \( \text{sh} \) restricts to an endofunctor on \( [C^{\text{op}}, \text{Set}] \) and thus provides a reflection onto a full subcategory \( S \). Since a colimit in \( S \) is computed in the same way as in \( \text{SH}_C^J \), i.e. as the sheafification of the colimit in \( PSH_C \), the subcategory \( S \) is closed in \( \text{SH}_C^J \) under small colimits and thus \( \text{Sh}_C \subseteq S \subseteq [C^{\text{op}}, \text{Set}] \).

**15 Schemes and functors**

Throughout this section, we assume that \( A = fA \), i.e. that Spec is fully faithful. By Lemma \( 33 \) this implies \( A = rA \).

Here we add an assumption on \( A \): We demand that there exists a regular epi–mono factorization system both in \( A \) and in the coproduct completion \( \coprod A \). This is the case e.g. when colimits are universal in \( A \), see Theorem \( 74 \).
We equip \( \mathcal{A}^{\text{op}} \) with a Grothendieck topology \( J \) given by the distinguished opcovers in \( \mathcal{A} \). Since \( \text{Spec} \) takes distinguished \( J \)-hyperopcovers to colimit cocones, Proposition 55 gives a partial adjunction which we denote for simplicity by \( \| \) and \( N \):

\[
\begin{array}{ccc}
\mathcal{A}^{\text{op}} & \xleftarrow{\text{Spec}} & \mathcal{J}_{\mathcal{A}^{\text{op}}} \\
\downarrow & \| & \downarrow \\
\mathcal{J}_{\mathcal{A}^{\text{op}}} & \xleftarrow{N} & \text{Top}_{\mathcal{A},\mathcal{P}}
\end{array}
\]

In addition, since we assume that \( \text{Spec} \) is fully faithful, \( y' = y \) and \( yR \in \mathcal{J}_{\mathcal{A}^{\text{op}}} \), and Spec \( R \in \text{Top}_{\mathcal{A},\mathcal{P}} \) provide a fixed point, for each \( R \in \mathcal{A} \). It is easy to see that in any adjunction, the right adjoint is fully faithful on maps from a fixed point, giving the following lemma:

**Lemma 57.** \( N \) is fully faithful on maps from affine schemes, i.e.

\[
N: \text{Top}_{\mathcal{A},\mathcal{P}}(\text{Spec } R, X) \xrightarrow{\cong} \mathcal{J}_{\mathcal{A}^{\text{op}}}(N \text{Spec } R, N.X).
\]

Clearly the fixed points of this adjunction are closed in \( \text{Top}_{\mathcal{A},\mathcal{P}} \) under all colimits that the right adjoint \( N \) preserves. Since our goal is to show that all schemes are fixed points, this will be our main theorem for this section:

**Theorem 58.** The nerve functor \( N \) preserves colimits of hypercovers. Consequently, \( N \) is fully faithful on schemes.

**Proof.** Let \( U \to X \) be a hypercover, i.e. an augmented semisimplicial object

\[
\cdots \xrightarrow{\alpha} U_1 \xrightarrow{\beta} U_0 \xrightarrow{\gamma} X
\]

in the coproduct completion of \( \text{Top}_{\mathcal{A},\mathcal{P}} \). Our aim is to show that the map \( H_0NU \to NX \) is an isomorphism in \( \mathcal{J}_{\mathcal{A}^{\text{op}}} \),

\[
\cdots \xrightarrow{\alpha} NU_1 \xrightarrow{\beta} NU_0 \xrightarrow{\gamma} H_0NU \xrightarrow{\delta} NX
\]

i.e. we need to show that \( \mathcal{J}_{\mathcal{A}^{\text{op}}}(yR, H_0NU) \to \mathcal{J}_{\mathcal{A}^{\text{op}}}(yR, NX) \) is an isomorphism for every \( R \in \mathcal{A} \) (since \( y\mathcal{A}^{\text{op}} \) is a generator).
First we prove surjectivity by finding a lift of an arbitrary $yR \to NX$. Such a map lies in the image of $N$ by the previous lemma. Pull back the hypercover $U \to X$ to a hypercover $V \to \text{Spec } R$ and apply $N$ to obtain the top part of the following diagram and choose a distinguished refinement $\text{Spec } L$ of $V$ to obtain the bottom part.

\[
\begin{array}{cccccccc}
\cdots & NU_1 & \to & NU_0 & \to & H_0NU & \to & NX \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & NV_1 & \to & NV_0 & \to & H_0NV & \to & N \text{Spec } R \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & yL_1 & \to & yL_0 & \to & H_0yL & \cong & yR
\end{array}
\]

The map $H_0yL \to yR$ is an isomorphism in $\text{Sh}^J_{\text{Avp}}$ and we thus obtain a lift as the composition $yR \xleftarrow{\sim} H_0yL \to H_0NU$.

The first part shows, in particular, that any hypercover yields a jointly epic family on nerves. Thus, the hypothesis of Theorem 73 is satisfied for $NU \to NX$ and we may conclude that the map $H_0NU \to NX$ is monic. Together with the first part, it is thus an isomorphism.

We finish by giving a more explicit description of the fixed points in $\text{Sh}^J_{\text{Avp}}$, closely connected to the definition of schemes in $\text{Top}_{\mathcal{A}, \mathcal{P}}$. We will thus define open covers in $\text{Sh}^J_{\text{Avp}}$ and schemes will then be objects admitting an open cover by representables. We define open subfunctors of $yR$ to be exactly the nerves of open subspaces of $\text{Spec } R$. Thus, let $U \subseteq \text{Spec } R$ be an open subset and define a subfunctor of $yR = \mathcal{A}(R, -)$ by

\[
yR_U(S) = \left\{ f: R \to S \mid \text{for each local form } q: S \to Q, \text{ the composite } qf: R \to S \to Q \text{ factors through some } P \in U \right\}
\]

Remark. In the case that $U = \text{Pts } K$ is a distinguished open then this is equivalently the representable associated with the localization $K = R_U$ so that the notation is consistent.

An open embedding is a monic $\alpha: G \to F$ such that for any map from a representable $yR$, the resulting pullback of $\alpha$ is an inclusion of some open subfunctor $yR_U \subseteq yR$:

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & F \\
\downarrow & & \downarrow \\
yR_U & \subseteq & yR
\end{array}
\]

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An open cover is a collection of open embeddings $G_i \to F$ such that for each local object $P \in \mathcal{P}$ the components $G_i(P) \to F(P)$ are jointly surjective (the argument of Lemma 62 implies that $\coprod G_i \to F$ is then epic in $\text{Sh}_{\mathcal{A}_{op}}^J$). Finally, $F$ is said to be a scheme, if it admits an open cover by representables.

**Lemma 59.** The nerve of every scheme in $\text{Top}_{\mathcal{A}, \mathcal{P}}$ is a scheme in $\text{Sh}_{\mathcal{A}_{op}}^J$.

**Proof.** Since $N$ is fully faithful on maps from affine schemes by Lemma 57 and preserves pullbacks as a right adjoint, it is easy to see that any open cover $U \to X$ yields a collection of open subfunctors $NU_i \to NX$.

$$NU^i \to NX$$

$$yR_{\varphi^{-1}U^i} = N(\varphi^{-1}U^i) \to N\text{Spec} \, R = yR$$

The joint surjectivity means that any map $\varphi : \text{Spec} \, P \to X$ factors through one of the $U^i$. But since the preimages $\varphi^{-1}U^i$ form an open cover of $\text{Spec} \, P$, one of them must equal $\text{Spec} \, P$, since this is the only open containing the point $1_P$. \qed

**Theorem 60.** Every scheme $F \in \text{Sh}_{\mathcal{A}_{op}}^J$ is a fixed point, i.e. $F \cong N|F|$.

**Corollary 61.** The adjunction $| \dashv N$ restricts to an adjoint equivalence between the respective categories of schemes. \qed

**Proof of Theorem 60.** Let $yR_0^i \to F$ be an open cover. Since the pullbacks $yR_0^i \times_F yR_0^j$ are open subfunctors of both $yR_0^i$ and $yR_0^j$, Theorem 37 together with the previous lemma produce a distinguished open cover $yR_i$ and we thus obtain an augmented semisimplicial object

$$\cdots \to yR_i \to yR_0 \to F$$

whose higher matching maps are distinguished open covers and in particular are jointly epic in $\text{Sh}_{\mathcal{A}_{op}}^J$. Theorem 73 then shows that $H_0yR \to F$ is monic and it is easily seen to be an open embedding:

$$\cdots \to yR_i \to yR_0 \to H_0yR \to F$$

namely, $yS_U$ consists of nerves of open subspaces of $\text{Spec} \, S$ with higher matching maps covers, and since $N$ preserves such colimits, its colimit is indeed an open subfunctor of $yS$. The proof is finished by an application of the following lemma, applied to $H_0yR \to F$. \qed
Lemma 62. An open embedding $\alpha: G \to F$ with surjective components on local objects is an isomorphism.

Proof. Since $\alpha$ is monic, we need only show that $G(R) \to F(R)$ is surjective for every $R \in A$. Thus, take any $yR \to F$ and form a pullback

$$
\begin{array}{ccc}
G & \longrightarrow & F \\
\downarrow & & \downarrow \\
yR_U & \longrightarrow & yR
\end{array}
$$

Clearly, the components $yR_U(P) \to yR(P)$ are also surjective and it follows that the open subspace $U \subseteq \text{Spec } R$ contains all points and this inclusion is in fact an isomorphism. A lift is thus obtained as $yR \xrightarrow{\sim} yR_U \to G$. \qed

16 Concrete examples

We are now going to identify spectra and schemes for some particular choices of $A$ and $P$.

16.1 Ordinary (relative) schemes

If we start with $A$ the category of commutative $R$-algebras for any commutative ring $R$ and $P$ the subcategory of local $R$-algebras (i.e. the underlying ring is local) and local homomorphisms, we can recognize $P$ as a subcategory of local objects and admissible maps with respect to a cone

$$
\begin{array}{ccc}
\mathbb{Z}[x,y]/(x+y-1) & \longrightarrow & \mathbb{Z}[y, y^{-1}]/(x+y-1) \\
\mathbb{Z}[x, x^{-1}, y]/(x+y-1) & \xrightarrow{\text{ }} & \mathbb{Z}[x, y, y^{-1}]/(x+y-1)
\end{array}
$$

together with an empty cone with summit being a trivial ring. $A$ and $P$ satisfy conditions of the main theorem, hence we can construct the spectrum functor, which is just the relative Spec used in algebraic geometry. Schemes in this context are thus just $R$-schemes in the usual sense. For an $R$-algebra $A$, points of Spec $A$ correspond to prime $R$-ideals of $A$. 

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16.2 Deitmar’s $\mathbb{F}_1$-schemes

$\mathbb{F}_1$-geometry is a part of algebraic geometry which tries to define a geometry behaving like “schemes over field with one element” in order to prove the Riemann hypothesis in the similar way the Weil conjectures were proved. So far, most of the results are just conjectural as the optimal setting have not been found yet. For a nice exposition of different proposals for $\mathbb{F}_1$-schemes, see [8].

Here, we are going to take a look at $\mathbb{F}_1$-schemes in the sense of Deitmar and Kato ([5], [8]). We start with the category $\mathcal{A} = \text{CMon}$ of commutative monoids and their homomorphisms and we consider the cone injectivity with respect to the following cone with additive monoids of natural numbers and integers (where the map on the left is the identity and the map on the right a is an inclusion):

\[
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{i} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{N} & \xleftarrow{1} & \mathbb{Z}
\end{array}
\]

Therefore, any monoid is local (obviously, any map $\mathbb{N} \to M$ factors through the identity on $\mathbb{N}$). Admissible maps are homomorphisms reflecting invertible elements; this is precisely what the injectivity w.r.t. $i$ says. So, the subcategory $\mathcal{P}$ of local objects and admissible maps is a wide subcategory given by all monoids and maps that reflect invertibility.

Schemes defined in $(\mathcal{A}, \mathcal{P})$-spaces in this context are $\mathbb{F}_1$-schemes in sense of Deitmar [5]. For a commutative monoid $M$, points of $\text{Spec} M$ looks fairly similar to a spectrum of a ring in the previous example. Indeed, if we have a map $\mathbb{N} \to M$ sending 1 to $a \in M$, pushout of $i$ corresponds adding inverse of $a$ to $M$:

\[
\begin{array}{ccc}
\mathbb{N} & \xrightarrow{1 \to a} & M \\
\downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{} & M[a^{-1}]
\end{array}
\]

Recall that an ideal of a monoid $M$ is a subset $I \subseteq M$ such that $I \cdot M \subseteq I$. An ideal $P$ of $M$ is said to be prime if $xy \in P$ implies $x \in P$ or $y \in P$ for any $x, y \in M$. Then the topological spec underlying $\text{Spec} M$ is precisely the set of prime ideals of $M$. 

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16.3 $C^\infty$-schemes

In this section, we are going to define a framework in which smooth manifolds arise as affine schemes for certain algebraic objects. This was developed in [10] and further studied for example in [7].

In this context, $\mathcal{A} = C^\infty\text{Ring}$ is the category of smooth rings, i.e. the category of product preserving functors $\text{Euc} \to \text{Set}$ where $\text{Euc}$ is the category with $\mathbb{R}^n$ as objects (for $n \geq 0$) and smooth functions $\mathbb{R}^m \to \mathbb{R}^n$ as morphisms. In other words, $\mathcal{A}$ is a category of $\text{Set}$-valued models for the Lawvere theory $\text{Euc}$ and as such it is locally finitely presentable.

For any smooth ring $\mathcal{C} : \text{Euc} \to \text{Set}$, the “underlying set” $\mathcal{C}(\mathbb{R})$ has a structure of a commutative $\mathbb{R}$-algebra. The assignment $\mathcal{C} \mapsto \mathcal{C}(\mathbb{R})$ then gives a functor from smooth rings to commutative $\mathbb{R}$-algebras.

Denote $\mathcal{P}$ the subcategory of those smooth rings $\mathcal{C}$ (and morphisms) such that the underlying $\mathbb{R}$-algebra $\mathcal{C}(\mathbb{R})$ (and $\mathbb{R}$-algebra homomorphism) is local. We claim that the inclusion $\mathcal{P} \to \mathcal{A}$ is a right multiadjoint. Since $C^\infty\text{Ring}$ is a category of models for a Lawvere theory, we can build a free smooth ring $C^\infty\{a, b\}$ on two generators $a, b$ as well as a localization $\mathcal{C}[a^{-1}]$ inverting an element $a \in \mathcal{C}$. It is known that we can also form a quotient $\mathcal{C}/I$ for any $R$-ideal $I$ of underlying $\mathbb{R}$-algebra $\mathcal{C}(\mathbb{R})$ (cf. [10]). Hence we are able to recognize $\mathcal{P}$ as the subcategory of local objects and admissible maps with respect to the following cone

$$\begin{align*}
C^\infty\{a, b\}/(a + b - 1) & \longleftarrow C^\infty\{a, b\}[a^{-1}]/(a + b - 1) \\
C^\infty\{a, b\}[b^{-1}]/(a + b - 1) & \longleftarrow
\end{align*}$$

together with an empty cone with summit being the trivial $C^\infty$ ring. It is not hard to see that both maps appearing in the cone above are epics and all the objects are finitely presentable. Statement then follows from Theorem 3.

The corresponding spectrum functor was explicitly constructed in [10]; points of Spec $\mathcal{C}$ are so-called $C^\infty$-radical prime ideals of $\mathcal{C}$.

A Grothendieck constructions and their completeness

For a pseudofunctor $R : B^{\text{op}} \to \text{CAT}$ valued in (possibly large) categories, we consider the associated fibration

$$P : \int R \to B,$$
also called the Grothendieck construction of $R$. The following is well-known:

**Theorem 63.** If $B$ is complete and $R$ is valued in complete categories and continuous functors then $fR$ is also complete and the canonical projection $P$ is also continuous.

More precisely, for a diagram $F: K \to fR$ consider the universal cone $\lambda_k: \lim PF \to PFk$ in $B$ and denote the cartesian lifts of its components as $\lambda_k^*Fk \to Fk$. Then the $\lambda_k^*Fk$'s form a diagram in the fibre over $\lim PF$ whose limit equals $\lim F$.

Dually, any pseudofunctor $L: B \to \text{CAT}$ induces an opfibration $P: fL \to B$. Since this is the opposite of the first Grothendieck construction, the previous theorem now holds with limits replaced by colimits.

Finally, we will discuss bifibrations and their bicompleteness. A fibration $fR \to B$ is easily seen to be a bifibration iff the diagram $R: B^{op} \to \text{CAT}$ is valued in categories and right adjoint functors. If $L: B \to \text{CAT}$ is the diagram consisting of the same categories but with all functors replaced by their left adjoints, one observes that $fL \simeq fR$ over $B$ and we thus obtain the following theorem. In this situation we will say that the bifibration is associated to the diagram $A: B \to \text{CAT}_{ad}$ of adjunctions $L \dashv R$.

**Theorem 64.** If $B$ is bicomplete and $A$ is a diagram of bicomplete categories and adjunctions between them then $fA$ is also bicomplete and the canonical projection $P$ is bicontinuous.

We will now describe two examples that are relevant for this paper.

**Example 65.** Consider the diagram $A: \text{Set} \to \text{CAT}_{ad}$ associating to each set $X$ the poset $AX = \text{Top}^{op}_X$ of all topologies on $X$ ordered by the reverse inclusion; the adjunction associated to a mapping $f: X \to Y$ consists of the direct and inverse image $f_* \dashv f^*$, where the direct image $f_* X$ is the topology generated by $\{ V \subseteq Y \mid f^{-1}V \in X \}$. The associated bifibration is then easily seen to be the category $\text{Top}$ of topological spaces and continuous maps. The theorem says that limits and colimits in $\text{Top}$ are created from those in $\text{Set}$.

Although colimits of topological spaces are well known and the general theorem above gives the usual description of the topology as the largest topology making the components of the universal cocone continuous, we will make use of an alternative description of open subsets of the colimit. This description starts with the observation that an open subset of $X$ is given equivalently by a continuous map into the Sierpiński space $\Sigma$. Thus, open subsets of $\text{colim} X_k$ are given equivalently by cocones $X_k \to \Sigma$, i.e. by collections of open subsets $U_k \subseteq X_k$ that are compatible in the sense: if $\alpha: k \to l$ lies in $\mathcal{K}$ then $(\alpha_*)^{-1}(U_l) = U_k$; in terms of points this reads $p \in U_k$ iff $\alpha_*p \in U_l$.  

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Example 66. Consider the diagram $A: \text{Top} \to \text{CAT}_{\text{ad}}$ associating to each space $X$ the category $AX = \text{Sh}_X^{\text{op}}$ of $A$-valued sheaves on $X$, or rather its opposite; the adjunction associated to a map $f: X \to Y$ consists of the direct and inverse image functors $f_* \dashv f^*$. The associated bifibration is then easily seen to be the category $\text{Top}_A$ of $A$-spaces. The theorems says that limits and colimits in $\text{Top}_A$ are created from those in $\text{Top}$.

More explicitly, the colimit of a diagram $(X_k, O_{X_k})$ is the colimit $X = \text{colim} X_k$ of the underlying spaces with universal cocone $\lambda_k: X_k \to X$ and with the structure sheaf $O_X = \lim(\lambda_k^* O_{X_k})$.

Example 67. For a category $C$, consider the diagram $R: \text{Set}^{\text{op}} \to \text{CAT}$ associating to each set $I$ the category $C^I$, i.e. $R = \text{CAT}(\cdot, C)$. When $C$ has coproducts, each functor in the diagram $R$ has a left adjoint given by the left Kan extension, i.e. summation along fibres. The associated (bi)fibration is then easily seen to be the free completion $\coprod C$ of $C$ under coproducts. The limits and colimits as given in the previous theorems can then be spelled out explicitly as in the following lemma.

Let $D_k$ be a diagram in $\coprod C$, i.e. it consists of a diagram of sets $I_k$ and a collection $(D^i_k)_{i \in I_k}$ of objects together with maps $D^i_k \to D^j_{\alpha(l)}$ for each $\alpha: k \to l$ that are closed under composition. This describes a diagram in $C$, indexed by the category of elements associated with $I: K \to \text{Set}$ (Grothendieck construction). We denote this diagram by $\text{El} D: \text{El} I \to C$, $(k, i) \mapsto D^i_k$. In the other direction, $D = \text{lan}_P \text{El} D$ is the left Kan extension in

$$
\begin{array}{ccc}
\text{El} I & \xrightarrow{\text{El} D} & C \\
\downarrow P & & \downarrow \text{lan}_P \text{El} D \\
K & \to & \coprod C
\end{array}
$$

along the canonical projection $P: \text{El} I \to K$.

Lemma 68. The limit of the diagram $D$ in the coproduct completion has indexing set $\text{lim} I$ and, for each $\sigma \in \text{lim}_{k \in K} I_k$, i.e. a section of the projection $\text{El} I \to K$, the corresponding component of the limit is

$$
H^0 D|_\sigma = \lim_{(k, i) \in \sigma} D^i_k = \lim_{k \in K} D^{\sigma(k)}_k.
$$

The colimit of the diagram $D$ in the coproduct completion has indexing set $\text{colim} I$ and, for each $\gamma \in \text{colim}_{k \in K} I_k$, i.e. a path component of $\text{El} I$, the corresponding component of the colimit is

$$
H_0 D|_\gamma = \text{colim}_{(k, i) \in \gamma} D^i_k = \text{colim}_{k \in K} \coprod_{i \in \gamma(k)} D^i_k.
$$
Dually, colimits in the product completion are computed over sections and limits over components. Thus, an object \((A_i)_{i \in I} \in \prod A\) is finitely presentable if it is a finite collection of finitely presentable objects, i.e. if \(I\) is finite and each \(A_i\) is finitely presentable.

Any map \(f: (A^i)_{i \in I} \to (B^j)_{j \in J}\) can be decomposed into a coproduct of maps into singletons: define \(I^j \subseteq I\) to be the inverse image of \(j \in J\) and then \(f\) is a coproduct of \(f^j: (A^i)_{i \in I^j} \to (B^j)\) with singleton target.

**Corollary 69.** A map \(f: (A^i)_{i \in I} \to (B^j)_{j \in J}\) in the coproduct completion is mono iff its underlying map of index sets is mono and each summand \(f^j\) is mono in \(C\). The map \(f\) is epi iff its underlying map of index sets is epi and each summand consists of a jointly epi family in \(C\).

\[\square\]

## B  Cone small object argument

Here we present a more detailed account of the cone small object argument; our aim here is the greatest generality, so we will not assume the existence of all small colimits – this may prove useful in homotopically oriented applications.

Let us formulate and prove this variation in a general categorical context. In order to substantially simplify the notation in the argument it is useful to consider the free product completion, see the following well-known definition.

**Definition 70.** Let \(\mathcal{K}\) be a category. Then the **free product completion** \(\text{Prod}(\mathcal{K})\) of \(\mathcal{K}\) is defined as follows: It has as objects pairs \((X, A)\), where \(X\) is a set and \(A: X \to \mathcal{K}\) is a functor whose domain is viewed as a discrete category. A morphism is a pair \((p, f): (X, A) \to (Y, B)\), where \(p: Y \to X\) is a function and \(f: A \cdot p \Rightarrow B\) is a natural transformation, i.e. a collection \((f_i: A(p^i) \to B^i)_{i \in Y}\).

**Remark.** We will sometimes write \((Ax)_{x \in X}\) instead of \((X, A)\).

Note that one can use the free product completion to derive the definitions of cone injectivity of an object and of cone injectivity of a morphism.

The reason why the free product completion is going to simplify the notation is that at each stage in the cone small object argument we are dealing with possibly a collection of objects rather than with a single object as is the case in the usual small object argument, and the free product completion allows us to consider this collection as a single object instead.

When trying to perform the cone small object argument in a category \(\mathcal{K}\) one might be tempted to simply pass to \(\text{Prod}(\mathcal{K})\) and perform the usual small object argument there. However, this gives a different result than
we are looking for: The reason is that an injective object \((Ax)_{x \in X}\) of the free product completion can happen to have a non-injective component. For example, consider the category with only four objects \(A, B, C, D\) and only two non-identity morphisms \(A \to B, A \to C\). If we consider injectivity of objects with respect to \(A \to B\), then the object \(C\) is not injective because there does not exist an extension of \(A \to C\) to \(B\), whereas the object \((C, D)\) of the free product completion is injective because there does not exist a morphism \(A \to (C, D)\).

**Remark.** If a category \(K\) has transfinite composites, then its free product completion also has transfinite composites and they’re explicitly given as follows: Suppose that we have a diagram 

\[(X_0, A_0) \xrightarrow{(p_0, f_0)} (X_1, A_1) \xrightarrow{(p_1, f_1)} (X_2, A_2) \xrightarrow{(p_2, f_2)} \cdots\]

whose objects are indexed by ordinals less than some limit ordinal \(\lambda\). Let \((q_\mu : X \to X_\mu)_{\mu < \lambda}\) be the limit of the diagram 

\[\cdots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1 \xrightarrow{p_0} X_0\]

in \(\mathbf{Set}\). Hence, \(X\) consists precisely of the elements \((x_\mu)_{\mu < \lambda} \in \prod_{\mu < \lambda} X_\mu\) such that \(p_{\mu, \kappa}(x_\mu) = x_\kappa\) for each \(p_{\mu, \kappa} : X_\mu \to X_\kappa\) from the diagram. Each element \(x = (x_\mu)_{\mu < \lambda} \in X\) induces a diagram 

\[A_0(x_0) \xrightarrow{f_{0,x_0}} A_1(x_1) \xrightarrow{f_{1,x_1}} A_2(x_2) \xrightarrow{f_{2,x_2}} \cdots\]

Let \((f_{\mu, x} : A_\mu(x_\mu) \to A(x))_{\mu < \lambda}\) be the colimit of that diagram in \(K\). This defines a functor \(A : X \to \mathbf{Set}\). Finally, it is easy to verify that the cocone 

\[((q_\mu, f_\mu) : (X_\mu, A_\mu) \to (X, A))_{\mu < \lambda}\]

is a colimit in \(\text{Prod}(K)\).

The following theorem is already known (under mildly stronger assumptions than in the theorem statement below), see Theorem 2.53 and Proposition 4.16 in [1]. However, the proof there uses a different approach than the explicit construction that we give below.

**Theorem 71.** Let \(K\) be a category with pushouts and transfinite composites and let \(I\) be a set of cones in \(K\) whose domains are \(\lambda\)-presentable for some regular cardinal \(\lambda\). Then the full subcategory \(I\)-\text{Inj} of \(K\) consisting of cone injective objects with respect to \(I\) is cone-reflective in \(K\). Explicitly, this means that for each object \(A\) in \(K\) there exists a cone \((f_i : A \to A_i)_{i \in I'}\) such that each \(A_i\) is in \(I\)-\text{Inj} and for each morphism \(g : A \to B\) whose codomain \(B\) belongs to \(I\)-\text{Inj} there exists \(i \in I'\) and a morphism \(h : A_i \to B\) such that \(h \cdot f_i = g\).
Proof. We will transfinitely define pairwise disjoint sets $I_\alpha$, where $\alpha \in \lambda$, and a diagram $D: \bigcup_{\alpha \in \lambda} I_\alpha \to K$ such that on $\bigcup_{\alpha \in \lambda} I_\alpha$ there is the obvious partial order such that for $x \in I_\alpha$, $y \in I_\beta$: $x < y$ iff $\alpha < \beta$, $I_0 = \{0\}$, $D0 = A$, each morphism $D(x \to y)$, where $x \in I_\alpha$, $y \in I_{\alpha+1}$, is a transfinite composition of pushouts of morphisms contained in cones in $I$ for each ordinal $\alpha < \lambda$, and $DI_\alpha$ will be a “fat colimit” of all $Dx$, where $x \in I_\beta$, $\beta < \alpha$ for each limit ordinal $\alpha < \lambda$. The precise meaning of this imprecisely stated last fact will be clear from the construction.

Let $J$ be the set of all spans $(1, E) (f \in I) (Z, C)$ in $\text{Prod}(K)$. When interpreted in $K$, $g$ is a morphism and $f$ is a cone, and we can reinterpret the set $J$ as $\{(f_j, z_j) \mid j \in J, z \in Z_j\}$ for some indexing set $J$. Well-order the set $J$, i.e. let $J = \delta$, where $\delta$ is an ordinal. Using transfinite induction we will obtain an object $(X_\beta, A_\beta)$ for each ordinal $\beta \leq \delta$. More precisely, we will define a chain $(X_0, A_0) (X_1, A_1) (X_2, A_2) \cdots (X_\delta, A_\delta)$ in $\text{Prod}(K)$. Such a chain induces for each pair $\beta \leq \beta' \leq \delta$ of ordinals a map $(p_{\beta, \beta'}, I_{\beta, \beta'})$: $(X_\beta, A_\beta) \to (X_{\beta'}, A_{\beta'})$.

Base Case: Define $D0 := A$. This gives us an object $(X_0, A_0)$ of $\text{Prod}(K)$, where $X_0 := 1$ and $A_0(0) := A$.

Successor step: If $\beta < \delta$ is an ordinal such that we’ve already performed the construction for all the ordinals less or equal to $\beta$, then for each $z \in Z_\beta$ and each $x \in X_\beta$ consider the following pushout

$$E_\beta(0) \xrightarrow{\ell_{\beta, x} \cdot g}\ A_\beta(x)$$

$$\downarrow f_{\beta, x}$$

$$C_\beta(z) \xrightarrow{h_{\beta+1, z, x}} A_{\beta + 1}(z, x)$$

where $\ell_{\beta, x}$ is the transfinite composite

$$A_0(p_{\beta, 0}(x)) \xrightarrow{T_{1, p_{\beta, 1}(x)}} A_1(p_{\beta, 1}(x)) \xrightarrow{T_{2, p_{\beta, 2}(x)}} A_2(p_{\beta, 2}(x)) \xrightarrow{T_{3, p_{\beta, 3}(x)}} \cdots \to A_\beta(x).$$

In this way we obtain an object $(X_{\beta + 1}, A_{\beta + 1})$ of $\text{Prod}(K)$ that’s defined as follows: $X_{\beta + 1} := Z_\beta \times X_\beta$ and $A_{\beta + 1}(z, x)$ is the pushout from the diagram

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above. Moreover, the morphism \((p_{\beta+1}, \mathcal{J}_{\beta+1})\): \((X_\beta, A_\beta) \rightarrow (X_{\beta+1}, A_{\beta+1})\) is defined as follows: \(p_{\beta+1}: \mathcal{J}_\beta \times X_\beta \rightarrow X_\beta\) is the product projection and the component \(\mathcal{J}_{\beta+1,z,x}\) of \(\mathcal{J}_{\beta+1}\) at \((z, x)\) is the vertical morphism from the right side of the pushout square above.

Limit Step: If \(\beta \leq \delta\) is a limit ordinal such that we’ve already performed the construction for all the ordinals less than \(\beta\), then we define the object \((X_\beta, A_\beta) := \text{colim}_{\gamma<\beta}(X_\gamma, A_\gamma)\), where the transfinite composite is taken in \(\text{Prod}(\mathcal{K})\).

The first transfinite construction is finished. In this way we obtain a cone \((\tilde{f}_{0,x}: D_0 \rightarrow A_\delta(x))_{x \in X_\delta}\), and we define \(I_1 := X_\delta\) and \(D\) on \(I_1\) to be \(A_\delta\). Furthermore, let \((Y_0, B_0) := (X_0, A_0)\) and \((Y_1, B_1) := (X_\delta, A_\delta)\). There is an obvious morphism \((Y_0, B_0) \rightarrow (Y_1, B_1)\). We repeat this transfinite procedure for each \(x \in I_1\) by using \(Dx\) instead of \(D0\) and in this way we obtain a cone \((g_{x,y}: Dx \rightarrow Dy)_{y \in I_{2,x}}\). Putting together all the codomains from these cones, where \(x \in I_1\), we obtain the definition of \(D\) on \(I_2 := \bigcup_{x \in I_1} I_{2,x}\). This gives us an object \((Y_2, B_2)\) of \(\text{Prod}(\mathcal{K})\) in an obvious way and also an obvious morphism \((Y_1, B_1) \rightarrow (Y_2, B_2)\). In this way we continue in each successor step. In limit steps \(\beta \leq \lambda\) we take \((Y_\beta, B_\beta) := \text{colim}_{\gamma<\beta}(Y_\gamma, B_\gamma)\). This transfinite construction finally gives us the functor \(D\). Now define the cone \((f_i: A \rightarrow A_i)_{i \in I'}\) to be the colimit injection \((Y_0, B_0) \rightarrow (Y_\lambda, B_\lambda)\).

Now that we’ve finished the construction let us show that each \(A_i\), where \(i \in I'\), belongs to \(I\)-Inj. Suppose that we have a cone \((h_j: A \rightarrow L_j)_{j \in J'}\) from \(I\) and a morphism \(g: A \rightarrow A_i\). By construction, there exists a transfinite composition from \(A\) to \(A_i\), in other words for each \(\alpha < \lambda\) there exists \(x_\alpha \in I_\alpha\) such that \(A_i = \text{colim}_{\alpha<\lambda} Dx_\alpha\). Using the fact that \(A\) is \(\lambda\)-presentable we get that there exists a factorization of \(g\) through some \(Dx_\alpha\) via some morphism \(f: A \rightarrow Dx_\alpha\). The diagram

\[
\begin{array}{ccc}
(1, A) & \xrightarrow{f} & (1, Dx_\alpha) \\
\downarrow{h} & & \downarrow{\text{inj}} \\
(J', L) & & (J', L)
\end{array}
\]

is one of the diagrams from the construction. Therefore there exists an index \(j \in J'\) and a morphism \(u: L_j \rightarrow Dx_{\alpha+1}\). Composing this morphism with the colimit injection \(\iota_{\alpha+1}: Dx_{\alpha+1} \rightarrow A_i\) we obtain a morphism \(\iota_{\alpha+1} \cdot u: L_j \rightarrow A_i\) and this morphism satisfies

\[
\iota_{\alpha+1} \cdot u \cdot h_j = \iota_{\alpha+1} \cdot (Dx_\alpha \rightarrow Dx_{\alpha+1}) \cdot f = \iota_\alpha \cdot f = g.
\]

Now suppose that \(g: A \rightarrow B\) is a morphism whose codomain \(B\) belongs to \(I\)-Inj. We want to show that there exists \(i \in I'\) and a morphism \(h: A_i \rightarrow B\)
such that $h \cdot f_i = g$. Using $B \in I$-Inj we get that there exists $z \in Z_0$ and
$g': C_0(z) \to B$ such that $g \cdot g_0 = g' \cdot f_0,z$. Using the universal property of the
pushout $A_1(z,0)$ we get a morphism $h_1: A_1(z,0) \to B$ such that $h_1 \cdot f_{1,z,0} = g$
and $h_1 \cdot h'_{1,z,0} = g'$. The previous two sentences are summed up in the following
diagram.

\[
\begin{array}{ccc}
E_0(0) & \xrightarrow{g_0} & A \\
\downarrow f_{0,x} & & \downarrow g \\
C_0(z) & \xrightarrow{h_{1,z,0}} & A_1(z,0) \\
& \downarrow h_1 & \downarrow g' \\
& B & \\
\end{array}
\]

Repeating the previous procedure we get $z' \in Z_1$ and $h_2: A_2(z',(z,0)) \to B$
such that the obvious diagrams commute. We continue in this way in each
successor step. In the limit steps we use the universal property of colimits.
After this transfinite construction is finished we obtain an element $x \in I_1$
and a morphism $h_x: Dx \to B$ such that the obvious diagrams commute.
Repeating this procedure we obtain $y \in I_2$ and a morphism $h_y: Dy \to B$
such that the obvious diagrams commute. In each successor step we repeat
this. In limit steps we use the universal property of the colimit. In the end
we obtain $i \in I'$ and a morphism $h: A_i \to B$ such that $h \cdot f_i = g$. \hfill \Box

\section{Regular epi–mono factorization in coproduct completion}

We give a brief account on the regular epi–mono factorization with applica-
tions towards semi-simplicial objects; most importantly, this will be applied
to the coproduct completion of a category in order to study colimits of covers.
Therefore, it will also be important that the regular epi–mono factorization
exists in the coproduct completion.

\subsection{Factorization and colimits of semi-simplicial ob-
jects}

Assume that $A$ is a complete and cocomplete category that admits a fac-
torization system with the left class the regular epis and the right class the
monos. Since

- every regular epi is the coequalizer of its kernel pair and
• in the regular epi–mono factorization \( f = me \), the kernel pair of \( f \) is the same as that of \( e \),

the factorization of \( f \) is the so called coimage factorization, i.e. the middle object is the coequalizer of the kernel pair of \( f \):

\[
\begin{array}{ccc}
A \times_B A \\
\downarrow d_0 \\
\downarrow d_1 \\
A \overset{f}{\longrightarrow} \text{coeq}(d_0, d_1) \longrightarrow B
\end{array}
\]

Quite easily, the coimage is also the coequalizer of any pair \( K \longrightarrow A \) for which the induced map \( K \longrightarrow A \times_B A \) is epic, since this condition implies that the coequalizer is the same as that of the kernel pair, thus proving the following theorem:

**Theorem 72.** Assume that a regular epi–mono factorization exists in \( \mathcal{A} \). Let

\[
\cdots \longrightarrow A_1 \longrightarrow A_0 \overset{f}{\longrightarrow} B
\]

be an augmented semi-simplicial object with matching maps epic. Then the induced factorization

\[
f: A_0 \longrightarrow \text{colim } A \longrightarrow B
\]

consists of a regular epi followed by a mono.

We will now apply this to the coproduct completion \( \coprod \mathcal{A} \), assuming that it admits a regular epi–mono factorization – its existence will be addressed later in the section. Considering for simplicity the case of a singleton \( B \), a direct translation of the above theorem is simple enough: We require an augmented semi-simplicial object with matching maps epic, which in \( \coprod \mathcal{A} \) means that all summands are non-empty jointly epic families; it is the non-emptiness assumption that we want to remove. On a related note, \( \text{colim } A \) may have more than one component and the object of interest is their coproduct in \( \mathcal{A} \) – by Lemma [68] it is \( H_0 A = \text{colim } \text{El } A \), where \( \text{El } A \) is the diagram composed of all the components of all the \( A_n \) and the maps between them the components of all the maps in the semi-simplicial object. In some sense, we are really thinking \( \text{El } A \) but pack the information into \( A \) for simplicity.

Formally, introduce the summation functor \( \Sigma: \coprod \mathcal{A} \rightarrow \mathcal{A} \), given on \( A = (A^i)_{i \in I} \) as \( \Sigma A = \sum_{i \in I} A^i \), clearly left adjoint to the inclusion functor, hence cocontinuous. Applying \( \Sigma \), we replace the augmented semi-simplicial object in \( \coprod \mathcal{A} \), together with its colimit, by a similar picture in \( \mathcal{A} \):

\[
\cdots \longrightarrow \Sigma A_1 \longrightarrow \Sigma A_0 \longrightarrow \Sigma \text{colim } A \longrightarrow B
\]

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We still desire that this gives the regular epi–mono factorization of $\Sigma A_0 \to B$, and the first map, being a coequalizer, is indeed a regular epi. We will now repair the deficiency of the matching maps being possibly empty, by adding to each $A_n$ a number of dummy components consisting of the initial object $\emptyset$, in a consistent way. This does not alter the above diagram, while making the general theorem directly applicable.

For $B \in \coprod A$, denote by $\hat{B}$ the family with the same indexing set as $B$ but with all components consisting of the initial object, and let $\hat{B} \to B$ be the unique map that is the identity on the underlying index sets. For $f : A \to B$, define $A' = A \amalg \hat{B}$ and $f' : A' \to B$ to be $f$ on $A$ and the above canonical map on $\hat{B}$. Now starting with an augmented semi-simplicial object $A$ in $\coprod A$, inductively w.r.t. $n$, consider the $n$-th matching map $m_n : A_n \to M_n A$ and apply the above construction $m'_n : A'_n \to M_n A$. Replacing $A_n$ by $A'_n$, but keeping the rest, the map $m'_n$ says how to make this into an augmented semi-simplicial object. The final result is an augmented semi-simplicial object with all matching maps surjective on the indexing sets (i.e. all summands non-empty families) and with the same $\Sigma$-image. We thus obtain:

**Theorem 73.** Assume that a regular epi–mono factorization exists in $\coprod A$. Let

\[
\cdots \xrightarrow{=} A_1 \xrightarrow{=} A_0 \xrightarrow{f} B
\]

be an augmented semi-simplicial object in $\coprod A$ with a singleton $B$ and with matching maps jointly epic. Then the induced factorization

\[
f : \Sigma A_0 \xrightarrow{\Sigma \colim A} \cong_{H_0 A} B
\]

consists of a regular epi followed by a mono. 

\[\square\]

### C.2 Factorization in coproduct completion

**Theorem 74.** If colimits are universal in $\mathcal{A}$ then the regular epi–mono factorization exists in $\coprod \mathcal{A}$.

The proof will consist of a series of implications.

**Lemma 75.** If pushouts are universal then epis are closed under pullbacks.

**Proof.** This comes from the characterization of epis as maps $f$ for which

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow f & & \downarrow 1 \\
B & \xrightarrow{1} & B
\end{array}
\]
is a pushout square. Pulling back such a square along $B' \to B$ yields another such square and thus the pullback $f': A' \to B'$ is epi too.

Here is the main technical statement of the theory of Reedy categories applied to pullbacks:

**Lemma 76.** Fix a class $\mathcal{E}$ of maps closed under pullbacks and composition. In the diagram

\[
\begin{array}{ccc}
A' & \rightarrow & C' \\
| & \downarrow c & | \\
A & \rightarrow & C
\end{array}
\quad \begin{array}{ccc}
B' & \leftarrow & B' \\
| & \downarrow b & | \\
B & \leftarrow & B
\end{array}
\quad \begin{array}{ccc}
A' \times_C B' & \rightarrow & A' \\
| & \downarrow a \times_c b & | \\
A \times_C B & \rightarrow & A
\end{array}
\]

if $c: C' \to C$ and both pullback corner maps $A' \to A \times_C C'$ and $B' \to B \times_C C'$ belong to $\mathcal{E}$ then so does the induced map $a \times_c b$.

More generally, the same conclusion holds if $a$ and the right pullback corner map belong to $\mathcal{E}$.

By applying this to the kernel pair, obtainable as a pullback with its two sides identical, we easily obtain:

**Proposition 77.** Assume that epis are closed under pullbacks and that in the square

\[
\begin{array}{ccc}
A' & \rightarrow & B' \\
\downarrow a & & \downarrow b \\
A & \rightarrow & B
\end{array}
\]

both $b$ and the pullback corner map are epis. Then so is the induced map $\ker f' \to \ker f$ on the kernel pairs.

Now we apply this to factorization through the coimage:

**Theorem 78.** Assume that epis are closed under pullbacks and $f: A \to B$. Then the factorization through the coimage

\[
f: A \rightarrow C \\
\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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Proof. The first map is a regular epi by definition. Form the square

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow f & & \downarrow 1 \\
C & \rightarrow & B
\end{array}
\]
Since 1 is epi and so is the pullback corner map (being the map $A \to C$), so is the induced map on kernel pairs:

\[
\begin{array}{c}
\begin{array}{c}
A 
\times_B \ B \\
\downarrow \\
C \times_B \ C
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A \\
f
\downarrow \\
C
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow \\
B
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow \\
B
\end{array}
\end{array}
\end{array}
\]

Now the two composites $A \times_B A \to C$ are equal by the definition of $C$ as the coequalizer; since the left vertical map is epic, the two structure maps $C \times_B C \to C$ must be equal and consequently $C \to B$ is monic.

Lastly, in order to apply this theorem to the coproduct completion $\coprod A$, we need to show that epis are pullback stable in $\coprod A$. Since a map $f$ is epic iff each factor of $f$ is non-empty with components jointly epic, this translates to the requirement that (non-empty) jointly epic families should be pullback stable in $A$. A jointly epic family $(A') \to B$ is equivalently a single epi $\coprod A' \to B$. This implies easily the following claim:

**Proposition 79.** Assume that (non-empty) coproducts are universal in $A$. If epis are closed under pullbacks in $A$ then the same holds in $\coprod A$. □

## D Reduction and distinguished open sets

While an $rA$-localization happens to be the same as an $A$-localization, as can be shown quite easily, the analogue for finite localizations is not true. Since these define distinguished open sets, it is not immediately clear whether the notions of distinguished opens w.r.t. $rA$ and $A$ coincide in $\text{Spec } rR = \text{Spec } R$. It is the purpose of this section to show that they do.

Clearly, the reduction of a finite $A$-localization $R \rightarrowtail K$ is a finite $rA$-localization $rR \rightarrowtail rK$ and they give the same points, inducing a diagram

\[
\begin{array}{c}
\text{FinLoc}_A(R) \\
\downarrow \\
\text{FinLoc}_{rA}(rR)
\end{array}
\begin{array}{c}
\text{Pts} \\
\downarrow \\
\text{Pts}
\end{array}
\begin{array}{c}
\text{DistOp}(\text{Spec } R)_{\text{op}} \\
\downarrow \\
\text{DistOp}(\text{Spec } rR)_{\text{op}}
\end{array}
\]

with the bottom map bijective by Proposition 29.

**Theorem 80.** The map $\text{DistOp}(\text{Spec } R) \to \text{DistOp}(\text{Spec } rR)$ is bijective.

**Proof.** The map on the left is surjective by Proposition 82 hence so is the map on the right. □
Lemma 81. Let \( f: R \to S \) be a localization. Then the pushout along \( f \) gives a surjective map

\[
f_*: \text{FinLoc}(R) \to \text{FinLoc}(S).
\]

In other words, the induced map on spectra, \( f^*: \text{Spec } S \to \text{Spec } R \) when (co)restricted to its image takes distinguished opens to distinguished opens.

Proof. Clearly finite complexes are closed under pushouts, so the map is well defined. The main idea of the proof is quite simple: Thinking of \( S \) as a localization of \( R \), express it as a filtered colimit of its finite sublocalizations \( S_\alpha \). Any finite localization of \( S \) is obtained by attaching a cell along a map \( A_i \to S \) and this lands in some \( S_\alpha \). When this cell is attached to \( S_\alpha \) instead of \( S \), one obtains the required finite localization of \( R \). Performing this inductively for all cells of \( R \) produces the required finite localization of \( R \).

In more detail, let \( S \to L \) be a finite localization and by induction we may assume that \( S \to L' \) lies in the image and \( L' \to L \) is a pushout of a single generator \( a_{ij} \) as in the rectangle on the right of

\[
\begin{array}{ccc}
R & \xrightarrow{a_{ij}} & B_{ij} \\
S & \xrightarrow{K'} & L' \\
S & \xrightarrow{K} & L
\end{array}
\]

Now since \( K' \to L' \) is a filtered colimit of its finite sublocalizations, the map \( A_i \to L' \) factors through some finite subcomplex \( K'' \); denoting by \( K \) the pushout as in

\[
\begin{array}{ccc}
R & \xrightarrow{a_{ij}} & B_{ij} \\
R & \xrightarrow{K''} & K \\
S & \xrightarrow{K'} & L' \\
S & \xrightarrow{K} & L
\end{array}
\]

we observe that the bottom left square is also pushout (since \( K' \to K'' \) is epi) and conclude that \( L \cong f_* K \). □

Proposition 82. Let \( R \in \mathcal{A} \). Then \( r: \text{FinLoc}_\mathcal{A}(R) \to \text{FinLoc}_\mathcal{A}(rR) \) is surjective.

Proof. Since the factorization system \( r\mathcal{A} \) is generated by \( ra_{ij} \), every step of the localization in \( r\mathcal{A} \) as on the left can be viewed as the reduction of the
localization in $\mathcal{A}$ as on the right

\[
\begin{array}{ccc}
A_i \xrightarrow{r_{ij}} & A_i \xrightarrow{a_{ij}} & B_{ij} \\
\downarrow & \downarrow & \downarrow \\
X \xrightarrow{r} & Y = rY' & X \xrightarrow{r} Y' \xrightarrow{r} rY'
\end{array}
\]
i.e. as a step of a localization in $\mathcal{A}$ followed by a reduction. Lemma 81 then shows that we may reorganize such a composition in the top row

\[
\begin{array}{ccc}
R \xrightarrow{r} & R \xrightarrow{r} & L_1 \xrightarrow{r} L_1 \xrightarrow{r} L_2 \xrightarrow{r} L_2 \xrightarrow{r} \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
K_1 \xrightarrow{r} & K_1 \xrightarrow{r} K_1 \xrightarrow{r} K_1 \xrightarrow{r} \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
K_2 \xrightarrow{r} & K_2 \xrightarrow{r} K_2 \xrightarrow{r} \cdots \\
\end{array}
\]

by replacing the first three maps (the composition across the bottom left in the diagram below)

\[
\begin{array}{ccc}
R \xrightarrow{r} & K \xrightarrow{r} & rK \\
\downarrow & \downarrow & \downarrow \approx \\
rR \xrightarrow{r} & L \xrightarrow{r} & rL
\end{array}
\]
by the composition across the top (the map on the right is a pushout in $r\mathcal{A}$ of the reduction of $R \xrightarrow{r} rR$ on the left and as such is an iso); repeating this process, any finite localization in $r\mathcal{A}$ is obtained as a finite localization in $\mathcal{A}$ followed by a reduction.

\[\blacksquare\]

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Jan Jurka, Tomáš Perutka, Lukáš Vokřínek
Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
jurka@math.muni.cz
xperutkat@math.muni.cz
vokrinek@math.muni.cz

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