Dynamical renormalization and universality in classical multifield cosmological models

Calin Iuliu Lazaroiu

Horia Hulubei National Institute of Physics and Nuclear Engineering, Reactorului 30, Bucharest-Magurele, 077125, Romania

E-mail: lcalin@theory.nipne.ro

ABSTRACT: We study the scaling behavior of classical multifield cosmological models with complete scalar manifold \((\mathcal{M}, \mathcal{G})\) and positive smooth scalar potential \(\Phi\), introducing a dynamical renormalization group action which relates their UV and IR limits. We show that the RG flow of such models interpolates between a modification of the geodesic flow of \((\mathcal{M}, \mathcal{G})\) (obtained in the UV limit) and the gradient flow of \((\mathcal{M}, \mathcal{G}, V)\) (obtained in the IR limit), where the classical effective potential \(V\) is proportional to \(\sqrt{2\Phi}\). Using this fact, we show that two-field models whose scalar manifold has constant Gaussian curvature equal to \(-1, 0\) or \(1\) are infrared universal in the sense that they suffice to describe the first order IR approximants of cosmological orbits for all two-field models with positive smooth scalar potential.
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Introduction

The study of cosmological models with more than one real scalar field (known as multfield cosmological models) is crucial for connecting fundamental theories of gravity and matter to the early universe, since the effective description of the generic string or M-theory compactification contains many such fields. In particular, multi-field models are important for cosmological applications of the swampland program [1, 2] (see [3, 4] for reviews), which drew attention to this often overlooked area of cosmology [5–7].

Despite their importance, the current theory of such models is poorly developed, especially as pertains to deeper conceptual and mathematical aspects. In fact, most contributions to the area focus on analyzing the leading orders in the formal cosmological perturbation expansion of such models around a “God given” abstract cosmological curve [8–14] (often considering only two-field models and under highly restrictive conditions such as assuming validity of the SRST approximation of [15, 16]) or on investigations of specific models and cosmological curves aimed at illustrating the feasibility of various inflation scenarios. Since the class of multifield models is continuously infinite, one is left wondering if a more systematic approach is possible which could permit fundamental progress in the field.

At the classical level (and before considering fluctuations) multifield models are described by ODEs to which the powerful methods of the geometric theory of dynamical systems [17–19] could be applied. Even at this level, the current literature remains largely concerned with local descriptions and pays little attention to global aspects, thus hardly amounting to a geometric theory. In particular, there were only a few attempts to develop an approach to the subject that could place it on a firm mathematical foundation and hence could serve as a stepping stone toward the program of connecting string theory to the early universe in a systematic manner. The purpose of this paper is to address some of these limitations by proposing a conceptual viewpoint on the classical dynamics of such models.

As pointed out in [20], classical multifield cosmological models admit a precise global description which leads to their formulation as geometric dynamical systems [17–19] could be applied. Even at this level, the current literature remains largely concerned with local descriptions and pays little attention to global aspects, thus hardly amounting to a geometric theory. In particular, there were only a few attempts to develop an approach to the subject that could place it on a firm mathematical foundation and hence could serve as a stepping stone toward the program of connecting string theory to the early universe in a systematic manner. The purpose of this paper is to address some of these limitations by proposing a conceptual viewpoint on the classical dynamics of such models.

As pointed out in [20], classical multifield cosmological models admit a precise global description which leads to their formulation as geometric dynamical systems. Mathematically, such a model is parameterized by the reduced Planck mass $M$ (equivalently, by the rescaled Planck mass $M_0 \overset{\text{def.}}{=} M \sqrt{\frac{2}{3}}$) and by a scalar triple $(\mathcal{M}, G, \Phi)$, where $\mathcal{M}$ is a (generally non-compact) connected manifold, $G$ is a Riemannian metric on $\mathcal{M}$ and $\Phi$ is a real-valued function defined on $\mathcal{M}$; we assume that these three objects are smooth. Such data enters the classical action of the scalar fields, which is described by a nonlinear sigma model defined on spacetime, where:

- The target space $\mathcal{M}$ is the space in which the scalar fields take values, namely they are described as smooth maps from spacetime into this manifold.

- The scalar field metric $G$ governs the kinetic energy of the scalar fields.
The scalar potential $\Phi$ governs the potential energy of the scalar fields.

To ensure conservation of energy, one requires the metric $G$ to be complete on $\mathcal{M}$; this precludes existence of scalar field configurations which “disappear into nothing”. To preclude global instability of dynamics, one requires that $\Phi$ is bounded from below by zero; for simplicity, we will assume throughout this paper that $\Phi$ is strictly positive. The complete Riemannian manifold $(\mathcal{M}, G)$ is called the scalar manifold.

The classical cosmological model is obtained from this data by considering gravity at rescaled Planck mass $M_0$ coupled to the sigma model parameterized by $(\mathcal{M}, G, \Phi)$. Before considering perturbations, one takes the spacetime to be of FLRW form with cosmological time $t$ and conformal factor $a$, while restricting the scalar fields to depend only on $t$ and hence to be described by a curve $\varphi : I \to \mathcal{M}$, where $I$ is a non-degenerate interval (i.e. an interval on the real axis which is not empty or reduced to a point). With these restrictions, the Einstein equations and scalar field equations of motion reduce to a single second order ODE (known as the cosmological equation) for $\varphi$, whose solutions we call cosmological curves. Since it is geometric in the sense of [21], the cosmological equation is equivalent to a dynamical system defined on the total space of the tangent bundle $T\mathcal{M}$ of the scalar manifold $\mathcal{M}$ by a certain vector field $S$ which we call the cosmological semispray. The latter lives on $T\mathcal{M}$ and is parameterized by $M_0$, by the scalar field metric $G$ and by the scalar potential $\Phi$. The equivalence between the cosmological equation and the cosmological dynamical system follows from the theory of geometric second order ODEs [21–25] (see [26] for a brief account). The flow defined by this dynamical system on $T\mathcal{M}$ is called the cosmological flow of the scalar triple $(\mathcal{M}, G, \Phi)$ at rescaled Planck mass $M_0$.

Since the topology of $\mathcal{M}$ is arbitrary, a systematic approach to such models requires the full force of the geometric theory of dynamical systems as developed for example in [17–19]; one cannot give a satisfactory treatment merely by studying the description of this system in local coordinates on the scalar manifold $\mathcal{M}$. For example, $\mathcal{M}$ need not be simply connected so a maximal cosmological curve need not be contractible. More importantly, $\mathcal{M}$ is generally non-compact in physically interesting applications, so a cosmological curve can “escape to infinity” in the sense that it can approach a Freudenthal end [27–30] of $\mathcal{M}$ for early or late cosmological times depending on the behavior of $\Phi$ and $G$ near that end. Non-compactness of $\mathcal{M}$ complicates the classification of past and future limit points of cosmological curves, which governs the early and late time behavior of the model – an aspect which is of direct physical interest. It also prevents the cosmological flow from being future complete unless appropriate conditions are imposed on $\Phi$ and $G$ in the vicinity of the Freudenthal ends of $\mathcal{M}$. Such global aspects of cosmological dynamics have direct implications for any attempt to construct effective descriptions when one incorporates quantum effects (as envisaged for example in cosmological applications of the swampland program). Indeed, any effective description depends on having an a pri-
ori topological classification of maximal cosmological curves since one cannot expect an effective description to be the same in every topological class. Intuitively, the topological classification of maximal cosmological curves partitions the dynamics of the model into “phases” and every phase will have its own effective description.

The global aspects mentioned above are already important for two-field cosmological models i.e. when $\mathcal{M}$ is a surface, in which case we denote it by $\Sigma$. The topological classification of borderless connected paracompact surfaces [31–33] through their orientability, reduced genus and space of ends (together with its length two chain of distinguished subspaces) shows that the global theory of cosmological dynamical systems is already very rich in the two-field case. This is especially true when the set of ends is infinite, in which case it can be for example a Cantor space. The theory is quite rich even when one restricts to oriented surfaces which are topologically finite in the sense that they have a finitely-generated fundamental group and hence a finite number of Freudenthal ends. The dynamical complexity of two-field models having such surfaces as targets was illustrated in our previous work [20, 34, 35] (see [36] for a brief review) when $\mathcal{G}$ has constant negative curvature; this corresponds to two-field generalized $\alpha$-attractor models, which form a very wide extension of the topologically trivial class of Poincaré disk two-field models considered [37].

Since the connected manifold $\mathcal{M}$, the complete Riemannian metric $\mathcal{G}$ and the positive potential $\Phi$ are arbitrary, the task of classifying multifield cosmological dynamics may seem hopeless at first sight. From a dynamical systems perspective, one could attempt to classify cosmological flows up to the appropriate notions of conjugation or equivalence, a task which is nontrivial when $\mathcal{M}$ is non-compact and was not yet attempted in the generality considered here.

In this paper, we propose a different approach to extract the essential features of such models and to organize them into qualitatively distinct dynamical classes. Our point of view is inspired by ideas akin to those used in the theory of critical phenomena, with the points of $T\mathcal{M}$ playing the role of microscopic states and $\mathcal{G}$ and $\Phi$ playing a role similar to that of non-equilibrium thermodynamic observables. A state of the system is a point $u(t)$ of $T\mathcal{M}$ which evolves in time according to the cosmological flow and has “thermodynamic” parameters $\frac{1}{2}\|u(t)\|_{\mathcal{G}}^2 = \frac{1}{2}\mathcal{G}(\pi(u(t)))(u(t), u(t))$ and $\Phi(\pi(u(t)))$, where $\pi$ is the projection of $T\mathcal{M}$. As in thermodynamics, knowing the values of these parameters does not determine the “microscopic” state $u(t)$. Since the cosmological equation is autonomous, the system is stationary in the sense that its flow is invariant under shifts of the cosmological time by an arbitrary constant. Following the analogy with critical phenomena, we consider the behavior of the model under scale transformations $t \to t/\epsilon$ of the cosmological time (where $\epsilon$ is a positive parameter), showing that such transformations induce a renormalization group action on $\mathcal{M}_0$, $\mathcal{G}$ and $\Phi$. The scaling limits $\epsilon \to \infty$ and $\epsilon \to 0$ capture the high frequency (or ultraviolet) and low frequency (or infrared) behavior of cosmological curves in the sense that they “isolate” the high and low frequency characteristic oscillations of
such curves. We then show that taking $\epsilon$ to be large or small corresponds respectively to replacing the cosmological flow of $(M, G, \Phi)$ with a modification of the geodesic flow of the scalar manifold $(M, G)$ or with the gradient flow of the classical effective potential $V = M_0 \sqrt{2\Phi}$ on this Riemannian manifold. The ultraviolet limit is scale invariant, while the infrared limit (which we consider up to first order in the scale parameter $\epsilon$) is invariant under Weyl transformations of $G$ up to reparameterization of the flow curves; in particular the first order IR approximation of cosmological orbits is Weyl-invariant. This limit is degenerate in that it confines the cosmological flow to the graph of the vector field $-\text{grad}_G V$ inside $TM$; accordingly, the order of the cosmological equation drops by one in the infrared limit. The scaling limits arise as the leading orders of systematic asymptotic approximations (called the UV and IR expansions) of the cosmological flow around the geodesic flow of $(M, G)$ or around the gradient flow of $(M, G, V)$. Such expansions are natural from a physics perspective. They are also mathematically natural since geodesic and gradient flows are well-studied subjects in the theory of dynamical systems – though the generic non-compactness of $M$ complicates the analysis. Use of the scaling limits allows us to classify multifield cosmological models into UV and IR universality classes whose study relates to that of such classical flows.

The Weyl-invariance of infrared approximants to cosmological orbits has striking implications for two-field models. In this case, the uniformization theorem of Poincaré [38] states that the Weyl equivalence class of $G$ contains a unique complete metric $\hat{G}$ (called the uniformizing metric) which has constant Gaussian curvature $K$ equal to $-1$, 0 or $+1$. Thus the first order infrared approximants of cosmological orbits for any cosmological two-field model with positive scalar potential $\Phi$ coincide with those of the model obtained by replacing $G$ with $\hat{G}$ and with the gradient flow orbits of the scalar triple $(M, G, V)$. In particular, models whose scalar manifold metric has constant Gaussian curvature provide distinguished representatives of the infrared universality classes of all two-field models.

The generic case $K = -1$ arises whenever $\Sigma$ is a (compact or non-compact) surface of general type, i.e. not diffeomorphic with a 2-plane, a 2-sphere, a real projective plane, a 2-torus, a Klein bottle, an open 2-cylinder or an open Möbius strip. In this situation, the uniformizing metric $\hat{G}$ is hyperbolic. When $\Sigma$ is diffeomorphic with $S^2$ or $\mathbb{RP}^2$, the metric $\hat{G}$ has Gaussian curvature $+1$, while when $\Sigma$ is diffeomorphic with a torus or Klein bottle the uniformizing metric is flat and complete. When $\Sigma$ is exceptional, i.e. diffeomorphic with a plane, an open annulus or an open Möbius strip, the metric $\hat{G}$ uniformizes to a complete flat metric or to a hyperbolic metric depending on its conformal class. In this situation, a description of universality classes requires considering both\footnote{For the three exceptional surfaces, a hyperbolic metric on $\Sigma$ is conformally flat but it is not conformally equivalent with a complete flat metric.} models with complete flat and hyperbolic
scalar manifold metric. Notice that two-field models with contractible target are of exceptional type and hence the Poincaré disk models of [37] cannot be IR universal among such. The qualitatively different behavior of two-field models with distinct target space topology illustrates the importance of global aspects in cosmological dynamics.

The paper is organized as follows. In Section 1, we recall the global description of multifield cosmological models, outline its dynamical system formulation and some of its basic properties and discuss a universal two-parameter group of similarities of the cosmological equation. We also describe two natural equivalence relations on multifield cosmological models which arise from underlying groupoid structures and make some observations on the early and late time behavior of cosmological curves for models with non-compact target manifold. In Section 2, we discuss the scale transformations and scaling limits of multifield cosmological models, showing that the UV and IR limits recover respectively a modification of the geodesic flow of \((\mathcal{M}, \mathcal{G})\) and the gradient flow of \((\mathcal{M}, \mathcal{G}, V)\), where \(V = M_0\sqrt{2\Phi}\). We also derive consistency conditions for the UV and IR approximations, showing that they differ from other approximations commonly used in cosmology (such as the slow roll and SRST approximations or the gradient flow approximation of [20]). In Section 3, we introduce the dynamical renormalization group of such models. We show invariance of first order IR approximant orbits under Weyl transformations of the scalar manifold metric, define IR universality classes and briefly discuss the late time infrared phase structure. In Section 4 we prove IR universality of two-field models whose scalar manifold metric has constant Gaussian curvature equal to \(-1\), \(0\) or \(1\). Section 5 presents our conclusions and some directions for further research. The Appendices contain some technical notions and results used in the main text.

**Notations and conventions.** All target manifolds \(\mathcal{M}\) considered in this paper are connected, smooth, Hausdorff and paracompact. If \(V\) is a smooth real-valued function defined on \(\mathcal{M}\), we denote by:

\[
\text{Crit}V \overset{\text{def}}{=} \{ c \in \mathcal{M} | (dV)(c) = 0 \}
\]

the set of critical points of \(V\). For any \(c \in \text{Crit}V\), we denote by \(\text{Hess}(V)(c) \in \text{Sym}^2(T^*_c\mathcal{M})\) the Hessian of \(V\) at \(c\), which is a well-defined and coordinate independent symmetric bilinear form defined on the tangent space \(T_c\mathcal{M}\). Recall that a critical point \(c\) of \(V\) is called nondegenerate if \(\text{Hess}(V)(c)\) is a non-degenerate bilinear form. When \(V\) is a Morse function (i.e. all of its critical points are non-degenerate), the set \(\text{Crit}V\) is discrete.

We denote by \(\hat{\mathcal{M}}\) the Freudenthal (a.k.a. end) compactification of \(\mathcal{M}\), which is a Hausdorff topological space containing \(\mathcal{M}\) as a dense subset (see [27–29]). A metric \(\mathcal{G}\) on \(\mathcal{M}\) is called hyperbolic if it is complete and of constant sectional curvature equal
to $-1$. In particular, a metric defined on a surface is hyperbolic if it is complete and of Gaussian curvature $-1$.

1  Multifield cosmological models

Throughout this paper, a multifield cosmological model means a classical cosmological model with a finite number $d > 1$ of scalar fields, which is derived from the following matter-gravity action on a spacetime with topology $\mathbb{R}^4$:

$$S[g, \varphi] = \int \text{vol}_g \mathcal{L}[g, \varphi],$$  \hspace{1cm} (1.1)

where:

$$\mathcal{L}[g, \varphi] = \frac{M^2}{2} R(g) - \frac{1}{2} \text{Tr}_g \varphi^* (\mathcal{G}) - \Phi \circ \varphi.$$  \hspace{1cm} (1.2)

Here $M$ is the reduced Planck mass, $g$ is the spacetime metric on $\mathbb{R}^4$ (taken to be of “mostly plus”) signature, while $\text{vol}_g$ and $R(g)$ are the volume form and Ricci scalar of $g$. The scalar fields are described by a smooth map $\varphi : \mathbb{R}^4 \rightarrow M$, where $M$ is a (generally non-compact) connected, smooth and paracompact manifold of dimension $d$ which is endowed with a smooth Riemannian metric $\mathcal{G}$, while $\Phi : M \rightarrow \mathbb{R}$ is a smooth function which plays the role of potential for the scalar fields. As mentioned in the introduction, we require that $\mathcal{G}$ is complete to ensure conservation of energy.

For simplicity, we will also assume that $\Phi$ is strictly positive on $M$. The quantity $\text{Tr}_g \varphi^*(\mathcal{G})$ is the trace of the $(1, 1)$-tensor obtained by raising one of the indices of the covariant 2-tensor $\varphi^*(\mathcal{G})$ with respect to the spacetime metric $g$, while $\Phi \circ \varphi : \mathbb{R}^4 \rightarrow \mathbb{R}$ is the standard mathematical notation\footnote{This is commonly written as $\Phi(\varphi)$ in the physics literature, though this notation is misleading since $\varphi$ is not the argument of $\Phi$.} for the real-valued function defined on $\mathbb{R}^4$ which is obtained by composing $\Phi$ with $\varphi$:

$$(\Phi \circ \varphi)(x^0, \ldots, x^3) = \Phi(\varphi(x^0, \ldots, x^3)).$$

In local coordinates on $M$, we have $\varphi(x^0, \ldots, x^3) = (\varphi^1(x^0, \ldots, x^3), \ldots, \varphi^d(x^0, \ldots, x^3))$.

The second term in the Lagrangian above takes the familiar sigma model form if one uses local coordinates on $M$:

$$\frac{1}{2} (\text{Tr}_g \varphi^*(\mathcal{G}))(x) = \frac{1}{2} g^{\mu\nu}(x) \mathcal{G}_{\alpha\beta}(\varphi(x)) \partial_\mu \varphi^\alpha \partial_\nu \varphi^\beta.$$  

Notice that the action (1.1) and its Lagrangian density (1.2) are manifestly geometric, i.e. written in coordinate-free form and without making any restrictive assumptions on the differential topology of $M$, which (except for being connected) can be arbitrary since any paracompact manifold admits Riemannian metrics. Also notice that such a model is parameterized by the quadruplet $\mathfrak{M} \overset{\text{def.}}{=} (M_0, M, \mathcal{G}, \Phi)$.\footnote{This is commonly written as $\Phi(\varphi)$ in the physics literature, though this notation is misleading since $\varphi$ is not the argument of $\Phi$.}
1.1 The cosmological equation and cosmological dynamical system

The multifield cosmological model parameterized by \((M_0, M, G, \Phi)\) is obtained by assuming that \(g\) is an FLRW metric with flat spatial section:

\[
ds_g^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} dx_i^2
\]

(1.3)

(where \(a(t) > 0\)) and that \(\varphi\) depends only on \(t \overset{\text{def}}{=} x^0\), which we call cosmological time. In this case, the equations of motion derived from (1.1) (namely the Einstein equations and the equation of motion for \(\varphi\)) amount to the following system of coupled nonlinear ODEs:

\[
\begin{align*}
\nabla_t \dot{\varphi} + 3H \dot{\varphi} + (\text{grad}_g \Phi) \circ \varphi &= 0 \\
\frac{1}{3} \dot{H} + H^2 - \frac{\Phi \circ \varphi}{3M^2} &= 0 \\
\dot{H} + \frac{||\dot{\varphi}||^2_g}{2M^2} &= 0
\end{align*}
\]

(1.4)

where the dot indicates derivation with respect to \(t\) and \(H \overset{\text{def}}{=} \frac{\dot{a}}{a} \in C^\infty(\mathbb{R})\) is the Hubble parameter. The last relation in the system above is called the Friedmann equation. Notice that \(a\) enters this system only through its logarithmic derivative \(H\).

The cosmological equation. When \(H\) is positive (which we assume throughout this paper), it can be eliminated algebraically using last two equations, which give:

\[
H(t) = H_\varphi(t) \overset{\text{def}}{=} \frac{1}{3M_0} \left[ ||\dot{\varphi}(t)||^2_g + 2\Phi(\varphi(t)) \right]^{1/2},
\]

(1.5)

where we defined the rescaled Planck mass \(M_0\) though:

\[
M_0 \overset{\text{def}}{=} M \sqrt{\frac{2}{3}}.
\]

(1.6)

Eliminating \(H\) though (1.5) allows one to reduce the system (1.4) to the following autonomous geometric second order ODE, which we call the cosmological equation:

\[
\nabla_t \dot{\varphi}(t) + \frac{1}{M_0} \left[ ||\dot{\varphi}(t)||^2_{\Phi_0} + 2\Phi_0(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\text{grad}_{\Phi_0} \Phi)(\varphi(t)) = 0.
\]

(1.7)

Here \(\nabla_t \overset{\text{def}}{=} \nabla_{\dot{\varphi}(t)}\). The cosmological equation is equivalent with the system (1.4) when \(H > 0\). Since \(\text{grad}_g \Phi\) is invariant under transformations of the form \((G, \Phi) \rightarrow (\lambda G, \lambda \Phi)\) with \(\lambda\) a positive constant, this equation can be written as:

\[
\nabla_t \dot{\varphi}(t) + \left[ ||\dot{\varphi}(t)||^2_{\Phi_0} + 2\Phi_0(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\text{grad}_{\Phi_0} \Phi)(\varphi(t)) = 0,
\]
while (1.5) reads:
\[ H_\varphi(t) = \frac{1}{3} \left[ ||\dot{\varphi}(t)||_{G_0}^2 + 2\Phi_0(\varphi(t)) \right]^{1/2}, \]
where we defined the rescaled scalar field metric and rescaled scalar potential through:
\[ G_0 \overset{\text{def.}}{=} \frac{1}{M^2} G \quad \text{and} \quad \Phi_0 \overset{\text{def.}}{=} \frac{1}{M^2} \Phi. \]

In particular, the cosmological equation depends only on the rescaled scalar triple \((M, G_0, \Phi_0)\). Given a solution \(\varphi : I \to M\) of this equation, relation (1.5) determines the scale factor \(a(t)\) up to a multiplicative constant \(C > 0\):
\[ a(t) = Ce^{\int_{t_0}^t dt' H_\varphi(t')}, \]
where \(t_0 \in I\) is chosen arbitrarily.

**Cosmological curves and cosmological orbits.** The solutions \(\varphi : I \to M\) of (1.7) (where \(I\) is a non-degenerate interval) are called **cosmological curves**. The images \(\varphi(I)\) of these curves in \(M\) will be called **cosmological orbits**. Notice that a cosmological orbit need not be an immersed submanifold of \(M\). It can be shown that the singular points of a cosmological curve (i.e. those points where its tangent vector vanishes) form an at most countable set whose complement in the corresponding orbit is a union of mutually disjoint embedded submanifolds of \(M\). The cosmological times corresponding to the singular points form a discrete subset of its interval of definition.

**The cosmological semispray.** Since the cosmological equation is manifestly geometric, it is also geometric in the weaker sense that its local description in a coordinate system on \(M\) is invariant under changes of local coordinates. It follows that this equation is equivalent with the flow equation of a special type of vector field (called a **semispray** or **second order tangent vector field**) defined on the total space of the tangent bundle of \(M\). We refer the reader to [24, 25] for an introduction to the well-developed theory of semisprays and second order geometric ODEs; for completeness, let us recall the relevant definitions and properties of such vector fields.

Let \(J \in \text{End}_{T^2M}(TTM)\) be the canonical endomorphism (a.k.a. tangent structure) of the double tangent bundle \(TTM\) and \(C \in \mathcal{X}(TM)\) be its Euler-Liouville vector field. Let \(\kappa : TTM \to TTM\) be the canonical involution (a.k.a. flip) of \(TTM\), \(\pi : TM \to M\) be the bundle projection of \(TM\) and \(\pi_T : TTM \to TM\) be the tangent bundle projection (a.k.a. first projection) of \(TTM\). Then the map \(\pi_* : TTM \to TM\) endows \(TTM\) with a second vector bundle structure such that \((TTM, \pi_T, \pi_*, M)\) is a double vector bundle. By definition, a second order tangent vector to \(M\) is an element \(w \in TTM\) such that \(J(w) = C_w\). This condition is equivalent with \(\kappa(w) = w\) and also with \(\pi_*(w) = \pi_T(w)\). We denote by \(T^2TM\) the vector sub-bundle of \(TTM\) consisting of second order tangent vectors. By definition, a smooth semispray on \(TM\) is a smooth vector field \(S \in \mathcal{X}(TM)\) which is a section
of this sub-bundle, i.e. which has the property that $S_u$ is a double tangent vector to $\mathcal{M}$ for all $u \in T\mathcal{M}$. This amounts to requiring that $S$ satisfies the equivalent conditions:

$$J(S) = C \iff \kappa(S) = S \iff \pi_*(S) = \pi_T \circ S ,$$

where in the right hand side of the last equality $S$ is viewed as a map from $T\mathcal{M}$ to $TT\mathcal{M}$.

The translation between the cosmological equation and the integral curve equation of the cosmological semispray is performed by considering the canonical lift of a cosmological curve $\varphi : I \to \mathcal{M}$, which is defined as its first jet prolongation:

$$c(\varphi) \overset{\text{def.}}{=} j^1(\varphi) = \dot{\varphi} : I \to T\mathcal{M} ,$$

where $\dot{\varphi}(t) \in T_{\varphi(t)}\mathcal{M}$ is the tangent vector to $\mathcal{M}$ at the point $\varphi(t)$. The canonical lift gives an injective map:

$$c : C(\mathcal{M}) \to C(T\mathcal{M})$$

from the set $C(\mathcal{M})$ of smooth curves in $\mathcal{M}$ to the set $C(T\mathcal{M})$ of smooth curves in $T\mathcal{M}$. This is a right inverse of the map $p : C(T\mathcal{M}) \to C(\mathcal{M})$ defined through:

$$p(\gamma) \overset{\text{def.}}{=} \pi \circ \gamma : I \to \mathcal{M}$$

for any smooth curve $\gamma : I \to T\mathcal{M}$. The relation $p \circ c = \text{id}_{C(\mathcal{M})}$ reflects the fact that $\dot{\varphi}(t) \in T\mathcal{M}$ is a tangent vector to $\mathcal{M}$ at the point $\varphi(t)$. Since the canonical lift $\varphi$ satisfies:

$$\pi_* \circ c(c(\varphi)) = \pi_T \circ c(\varphi) ,$$

the image of the map $c$ coincides with the subset $C_s(T\mathcal{M})$ of $C(T\mathcal{M})$ consisting of semispray curves in $T\mathcal{M}$, which are defined as those smooth curves $\gamma : I \to T\mathcal{M}$ whose tangent vector at any point is a double tangent vector to $\mathcal{M}$, i.e. which satisfy $\dot{\gamma}(t) \in T^s_{\gamma(t)} T\mathcal{M}$ for all $t \in I$. The inverse of the bijection $c : C(\mathcal{M}) \to C_s(T\mathcal{M})$ is the restriction of $p$ to $C_s(T\mathcal{M})$.

The canonical lifts of cosmological curves are called cosmological flow curves. The cosmological semispray $S \in \mathcal{X}(T\mathcal{M})$ is defined by the property that its integral curves coincide with the cosmological flow curves. It can be shown that $S$ is given by:

$$S = \mathbb{S} - Q ,$$

where $\mathbb{S}$ is the geodesic spray of the scalar manifold $(\mathcal{M}, \mathcal{G})$ and the cosmological correction $Q \in \mathcal{X}(T\mathcal{M})$ is the vertical vector field:

$$Q = \mathcal{H}C + (\text{grad}_\varphi \Phi)^v .$$

(1.8)

Here the superscript $v$ denotes the vertical lift of vector fields from $\mathcal{M}$ to $T\mathcal{M}$ and $\mathcal{H} : T\mathcal{M} \to \mathbb{R}_{>0}$ is the reduced Hubble function of $(\mathcal{M}, \mathcal{G}, \Phi)$, which is defined through:

$$\mathcal{H}(u) \overset{\text{def.}}{=} \frac{1}{M_0} \sqrt{||u||^2_\mathcal{G} + 2\Phi(\pi(u))} \quad \forall u \in T\mathcal{M} .$$

(1.9)
The cosmological dynamical system and cosmological flow. The cosmological semispray $S$ defines an autonomous geometric dynamical system $(\mathcal{T}M, S)$ on the total space of the tangent bundle to $\mathcal{M}$, whose flow $\Pi : \mathcal{D} \rightarrow \mathcal{T}\mathcal{M}$ is called the cosmological flow of the model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ (equivalently, of the rescaled scalar triple $(\mathcal{M}, \mathcal{G}_0, \Phi_0)$). Here $\mathcal{D} \subset \mathbb{R} \times \mathcal{T}\mathcal{M}$ is the maximal domain of definition of the flow. Little is known about the global behavior of the cosmological flow of a general non-compact scalar triple and especially about its early and late time behavior.

Classical cosmological observables. A basic local cosmological observable is a smooth function $F : \mathcal{T}\mathcal{M} \rightarrow \mathbb{R}$. The on-shell reduction of $F$ on a cosmological flow curve $\gamma : I \rightarrow \mathcal{T}\mathcal{M}$ is the function:

$$\hat{F}_\gamma \overset{\text{def}}{=} F \circ \gamma : I \rightarrow \mathbb{R} ,$$

while its on-shell reduction on a cosmological curve $\varphi : I \rightarrow \mathcal{M}$ is the function:

$$F_\varphi \overset{\text{def}}{=} \hat{F}_\varphi = F \circ \dot{\varphi} = F \circ c(\varphi) : I \rightarrow \mathbb{R} .$$

One can also consider smooth functions $F : J^k(\mathcal{M}) \rightarrow \mathbb{R}$, where $k \geq 2$ and $J^k(\mathcal{M})$ is the $k$-th jet bundle of $\mathcal{M}$. The reduction of such a function on a cosmological curve $\varphi : I \rightarrow \mathcal{M}$ is defined though:

$$F_\varphi \overset{\text{def}}{=} F \circ j^k(\varphi) : I \rightarrow \mathbb{R} ,$$

where $j^k(\varphi) : I \rightarrow J^k(\mathcal{M})$ is the $k$-th jet prolongation of $\varphi$. Repeated use of the cosmological equation allows one to express $F_\varphi$ as:

$$F_\varphi = \tilde{F}_\varphi ,$$

where $\tilde{F} \in C^\infty(\mathcal{T}\mathcal{M})$ is a basic local observable constructed from $F$ and the jets $j^{k-1}(\mathcal{G}) \in J^{k-1}(\text{Sym}^2(\mathcal{T}\mathcal{M}), \mathbb{R})$ and $j^{k-1}(\Phi) \in J^{k-1}(\mathcal{M}, \mathbb{R})$ of $\mathcal{G}$ and $\Phi$.

Dissipativity and stationary points. It is easy to check that the stationary points of the cosmological flow are the images of the critical points of $\Phi$ through the zero section of $\mathcal{T}\mathcal{M}$. Accordingly, the stationary set of the cosmological model coincides with the trivial lift:

$$(\text{Crit}\Phi)_0 \overset{\text{def}}{=} \{0_c | c \in \text{Crit}\Phi\} \subset \mathcal{T}\mathcal{M}$$

of the critical set:

$$\text{Crit}\Phi \overset{\text{def}}{=} \{c \in \mathcal{M} | (d\Phi)(c) = 0\}$$

of $\Phi$, whose complement:

$$\mathcal{M}_0 \overset{\text{def}}{=} \mathcal{M} \setminus \text{Crit}\Phi \subset \mathcal{M}$$
is an open submanifold of $\mathcal{M}$ called the noncritical set of the model.

A cosmological flow curve is constant iff its orbit meets a point of the set $(\text{Crit} \Phi)_0$, in which case its orbit is reduced to that point. Accordingly, a cosmological curve is constant iff its orbit meets a critical point of $\Phi$ at zero speed, in which case the orbit coincides with that critical point. When $\Phi$ is a Morse function, a straightforward computation shows that the stationary points of the cosmological flow are hyperbolic.

It is also easy to see that the cosmological flow is dissipative. For this, consider the total energy observable $E : T\mathcal{M} \to \mathbb{R}$ defined through:

$$E(u) \overset{\text{def}}{=} \frac{1}{2}||u||_G^2 + \Phi(\pi(u)) \quad \forall u \in T\mathcal{M},$$

which is the sum of the kinetic and potential energies:

$$E^k(u) \overset{\text{def}}{=} \frac{1}{2}||u||_G^2 , \quad E^p(u) \overset{\text{def}}{=} \Phi(\pi(u))$$

and is related to the reduced Hubble function by $\mathcal{H} = \frac{1}{M_0} \sqrt{2E}$. The last equation in (1.4) implies that the total energy of a cosmological curve $\varphi$:

$$E_\varphi(t) \overset{\text{def}}{=} E(\dot{\varphi}(t)) = \frac{1}{2}||\dot{\varphi}(t)||_G^2 + \Phi(\varphi(t)) = \frac{9M_0^2}{2}H_\varphi(t)^2$$

satisfies:

$$\frac{dE_\varphi(t)}{dt} = -\frac{1}{M_0} \sqrt{2E_\varphi(t)||\dot{\varphi}(t)||_G^2}$$

and hence decreases strictly with time when $\varphi$ is non-constant. In particular, any non-constant cosmological curve is aperiodic and without self-intersections. Moreover, all non-constant cosmological flow curves are embedded curves in $T\mathcal{M}$. As mentioned before, cosmological curves need not be immersed in $\mathcal{M}$ but their singular times form an at most countable discrete subset of their interval of definition. A cosmological curve $\varphi : I \to \mathcal{M}$ is singular at cosmological time $t_0 \in I$ iff its complete lift $\gamma = c(\varphi)$ meets the zero section of $T\mathcal{M}$ for $t = t_0$. Hence the embedded components of the orbit of $\varphi$ are the $\pi$-projections of the set $\gamma(I) \cap \tilde{T}\mathcal{M}$, where $\tilde{T}\mathcal{M}$ is the slit tangent bundle of $\mathcal{M}$ (which is defined as the complement in $T\mathcal{M}$ of the image of the zero section).

**Future completeness when $\mathcal{M}$ is compact.** Since we assume that $\Phi$ is positive on $\mathcal{M}$, each energy sublevel set:

$$\mathcal{M}_E(C) \overset{\text{def}}{=} \{ u \in T\mathcal{M} \mid E(u) \leq C \} \quad (C > 0)$$

is contained in the tubular neighborhood of the zero section of $T\mathcal{M}$ defined by the inequality $||u||_G \leq C\sqrt{2}$. Relation (1.11) implies that the complete lift $\dot{\varphi}$ of a maximal cosmological curve $\varphi$ is contained for $t \geq t_0$ within the tubular neighborhood with $C = E_\varphi(t_0)$. When $\mathcal{M}$ is compact, this tubular neighborhood is compact and this
observation together with the Escape Lemma implies that $\varphi(t)$ is defined for all $t \geq t_0$, which shows that in this case the cosmological flow is future-complete.

Notice, however, that $\mathcal{M}$ is non-compact in most applications. When $\mathcal{M}$ is not compact, a maximal cosmological curve can “escape to infinity” in the sense that its orbit for large times is not contained in any compact subset of $\mathcal{M}$; this is equivalent with the statement that the curve has a Freudenthal end of $\mathcal{M}$ among its limit points. In this case, the late time behavior of those cosmological curves which escape to infinity depends markedly on the asymptotic form of $\Phi$ and $G$ near the Freudenthal ends of $\mathcal{M}$ and the cosmological flow need not be future-complete.

### 1.2 The universal similarity group

Multifield cosmological models admit a universal two-parameter group of similarities, which relate the cosmological curves of a model with those of another model having the same target manifold $\mathcal{M}$ but different parameters $(\mathcal{M}_0, G, \Phi)$. We first discuss scale transformations of curves in $\mathcal{M}$, which enters the definition of this group action.

**Definition 1.1.** Let $\epsilon > 0$. The $\epsilon$-scale transform of a curve $\varphi : I \to \mathcal{M}$ is the curve $\varphi_\epsilon : I_\epsilon \to \mathcal{M}$ defined through:

$$I_\epsilon \overset{\text{def.}}{=} \epsilon I = \{\epsilon t | t \in I\}$$

and:

$$\varphi_\epsilon(t) \overset{\text{def.}}{=} \varphi(t/\epsilon) \quad \forall t \in I_\epsilon.$$  

The transformations $\varphi \to \varphi_\epsilon$ are called **scale transformations**. They define an action of the multiplicative group $\mathbb{R}_{>0}$ on the set:

$$C(\mathcal{M}) \overset{\text{def.}}{=} \bigsqcup_{I \in \text{Int}} C^\infty(I, \mathcal{M})$$

of all smooth curves in $\mathcal{M}$, where Int is the set of all non-degenerate intervals on the real axis. Scale transformations are not symmetries of the cosmological equation; the scale symmetry is “broken” by the scalar potential $\Phi$, being restored in the limit $\Phi \to 0$ (which, as we will see below, corresponds to the UV limit).

We consider the following similarities of the cosmological equation (1.7):

- **The parameter homothety.** The Lagrangian density $\mathcal{L}$ of (1.2) and the action $S$ of (1.1) are homogeneous of degree one under the parameter homothety:

$$G \to \lambda G, \quad \Phi \to \lambda \Phi, \quad M \to \lambda^{1/2} M \quad (\text{thus} \quad M_0 \to \lambda^{1/2} M_0),$$

where $\lambda$ is a positive constant. As a consequence, the equations of motion (1.4) and the cosmological equation (1.7) are invariant under such transformations\(^3\).

\(^3\)Notice that the Levi-Civita connection $\nabla$ of $\mathcal{G}$ is invariant under constant rescalings of $\mathcal{G}$.
• **The scale similarity.** The equations of motion (1.4) are invariant under the transformations:

\[ t \to \epsilon t , \quad \Phi \to \Phi_\epsilon \overset{\text{def}}{=} \Phi / \epsilon^2 \]

with \( \epsilon > 0 \), which change \( H = \frac{\dot{a}}{a} \) into \( \frac{1}{\epsilon} H \). Accordingly, the cosmological equation (1.7) is invariant under:

\[ \varphi \to \varphi_\epsilon , \quad \Phi \to \Phi_\epsilon \overset{\text{def}}{=} \Phi / \epsilon^2 \ (\epsilon > 0) . \]

(1.13)

Let \( \text{Met}(\mathcal{M}) \) be the set of all Riemannian metrics defined on \( \mathcal{M} \) and \( \text{Pot}_+(\mathcal{M}) \overset{\text{def}}{=} C^\infty(\mathcal{M}, \mathbb{R}_{>0}) \) be the set of all positive smooth functions defined on \( \mathcal{M} \). The **universal cosmological similarity group** is the multiplicative group \( T = \mathbb{R}_{>0} \times \mathbb{R}_{>0} \), where pairs of positive numbers multiply componentwise:

\[ (\lambda_1, \epsilon_1)(\lambda_2, \epsilon_2) \overset{\text{def}}{=} (\lambda_1 \lambda_2, \epsilon_1 \epsilon_2) . \]

The transformations above induce an action \( \rho_{\text{par}} \) of \( T \) on the set \( \text{Par}(\mathcal{M}) \overset{\text{def}}{=} \mathbb{R}_{>0} \times \text{Met}(\mathcal{M}) \times \text{Pot}_+(\mathcal{M}) \) of parameters of models with fixed target space \( \mathcal{M} \):

\[
\rho_{\text{par}}(\lambda, \epsilon)(M_0, G, \Phi) \overset{\text{def}}{=} (\lambda^{1/2} M_0, \lambda G, \frac{\lambda}{\epsilon^2} \Phi) \quad \forall (\lambda, \epsilon) \in T \quad \forall (M_0, G, \Phi) \in \text{Par}(\mathcal{M}) .
\]

(1.14)

This action is free. On the other hand, scale transformations define an action \( \rho_0 \) of \( T \) on the space \( C(\mathcal{M}) \) of all smooth curves in \( \mathcal{M} \):

\[
\rho_0(\lambda, \epsilon)(\varphi) \overset{\text{def}}{=} \varphi_\epsilon \quad \forall (\lambda, \epsilon) \in T \quad \forall \varphi \in C(\mathcal{M}) .
\]

**Definition 1.2.** The **universal similarity action** is the action \( \rho \) of \( T \) on the set \( C(\mathcal{M}) \times \text{Par}(\mathcal{M}) \) given by:

\[
\rho(\lambda, \epsilon)(\varphi, M_0, G, \Phi) = (\rho_0(\lambda, \epsilon)(\varphi), \rho_{\text{par}}(\lambda, \epsilon)(M_0, G, \Phi)) = (\varphi_\epsilon, \lambda^{1/2} M_0, \lambda G, \frac{\lambda}{\epsilon^2} \Phi) .
\]

For any \( (\lambda, \epsilon) \in T \), we have:

\[
\rho_0(\lambda, \epsilon)(\mathcal{S}^{M_0, G, \Phi}(\mathcal{M})) = \mathcal{S}^{\rho_{\text{par}}(\lambda, \epsilon)(M_0, G, \Phi)}(\mathcal{M}) ,
\]

where \( \mathcal{S}^{M_0, G, \Phi}(\mathcal{M}) \) denotes the set of cosmological curves of the model with target \( \mathcal{M} \) and parameters \( (M_0, G, \Phi) \). Thus \( \varphi : I \to \mathcal{M} \) is a cosmological curve for the model with parameters \( (M_0, G, \Phi) \) iff \( \varphi_\epsilon : I_\epsilon \to \mathcal{M} \) is a cosmological curve for the model with parameters \( (\lambda^{1/2} M_0, \lambda G, \frac{\lambda}{\epsilon^2} \Phi) \).

The universal similarity action allows one to eliminate the overall scale of \( \Phi \) and another scale from the problem. For example, one can fix the overall scales of \( G \) and \( \Phi \), in which case one is left with the single scale set by \( M_0 \). Equivalently, one can eliminate the rescaled Planck mass by setting \( M_0 = 1 \) and the overall scale of
Φ, in which case one is left with the single scale set by \( \mathcal{G} \). Notice that one cannot fix the scales of \( M_0 \) and \( \mathcal{G} \) independently. Since \( \Phi \) plays a distinguished role in this regard, it is natural to consider the stabilizer \( T_{\text{ren}} \cong \mathbb{R}_{>0} \) of \( \Phi \) in \( T \) with respect to the universal similarity action:

\[
T_{\text{ren}} = \text{Stab}_T(\Phi) = \{ (\lambda, \epsilon) \in T | \lambda = \epsilon^2 \}
\]  

(1.15)

This is the subgroup of \( T \) which can be used to rescale \( M_0 \) and \( \mathcal{G} \) once the scale of \( \Phi \) has been fixed; it is the renormalization group considered in Section 3.

### 1.3 Cosmological conjugation and equivalence

There exist two natural equivalence relations between cosmological models, which are the isomorphism relations of two underlying groupoids. To describe them, we first note some obvious properties of the canonical lift \( c : C(\mathcal{M}) \to C_s(T\mathcal{M}) \) of smooth curves from a manifold \( \mathcal{M} \) to its tangent bundle. Consider two manifolds \( \mathcal{M}_i \) with tangent bundle projections \( \pi_i : T\mathcal{M}_i \to \mathcal{M}_i \) \((i = 1, 2)\) and canonical curve lifts \( c_i : C(\mathcal{M}_i) \to C_s(T\mathcal{M}_i) \). By definition, a semispray map from \( \mathcal{M}_1 \) to \( \mathcal{M}_2 \) is a smooth map \( f : T\mathcal{M}_1 \to T\mathcal{M}_2 \) which satisfies \( f_*(T^*\mathcal{M}_1) = T^*\mathcal{M}_2 \). Precomposition with a semispray map \( f \) takes semispray curves in \( T\mathcal{M}_1 \) to semispray curves in \( T\mathcal{M}_2 \), i.e. we have:

\[
f \circ (C_s(T\mathcal{M}_1)) = C_s(T\mathcal{M}_2).
\]

Moreover, \( f \) induces a map \( \hat{f} : C(\mathcal{M}_1) \to C(\mathcal{M}_2) \) defined through:

\[
\hat{f}(\varphi_1) \overset{\text{def.}}{=} \pi_2 \circ f \circ c_1(\varphi_1) \quad \forall \varphi_1 \in C(\mathcal{M}_1).
\]  

(1.16)

This determines the precomposition of semispray curves with \( f \) in the sense that two smooth curves \( \varphi_i : I \to \mathcal{M}_i \) \((i = 1, 2)\) satisfy \( \varphi_2 = \hat{f}(\varphi_1) \) iff \( c_2(\varphi_2) = f \circ c_1(\varphi_1) \).

Given a third manifold \( \mathcal{M}_3 \) with tangent bundle projection \( \pi_3 : T\mathcal{M}_3 \to \mathcal{M}_3 \) and a second semispray map \( h : T\mathcal{M}_2 \to T\mathcal{M}_3 \), one easily checks the relation:

\[
\hat{h} \circ f = \hat{h} \circ \hat{f}.
\]

Moreover, for any manifold \( \mathcal{M} \) we have \( \text{id}_{T\mathcal{M}} = \text{id}_{C(\mathcal{M})} \). Hence the maps \( \mathcal{M} \to C(\mathcal{M}) \) and \( f \to \hat{f} \) define a functor from the category of manifolds and semispray maps to the category of sets.

**Cosmological conjugations.** Notice that a cosmological flow curve represents the time evolution of the state of a multifield cosmological model, where the tangent bundle of \( \mathcal{M} \) is the space of classical states. Hence two cosmological models parameterized by \( \mathfrak{M}_i = (M_{0i}, \mathcal{M}_i, G_i, \Phi_i) \) \((i = 1, 2)\) and whose semisprays we denote by \( S_i \) can be identified if there exists a smooth semispray map \( f : T\mathcal{M}_1 \to T\mathcal{M}_2 \) (called smooth cosmological conjugation) which maps the cosmological flow curves...
of the first model into those of the second, i.e. \( \gamma_1 : I \to TM_1 \) is a cosmological flow curve for \( \mathfrak{M}_1 \) iff \( f \circ \gamma_1 : I \to TM_2 \) is a cosmological flow curve for \( \mathfrak{M}_2 \). This amounts to the requirement that \( f \) is a smooth topological conjugation between the cosmological flows of the two models (see Appendix A), which in turn is equivalent with the condition:

\[
f_\sharp(S_1) = S_2,
\]

where \( f_\sharp \) denotes the \( f \)-pushforward of vector fields. Since every cosmological flow curve is the canonical lift of a cosmological curve, a semispray map \( f \) from \( M_1 \) to \( M_2 \) is a cosmological conjugation iff the induced curve map \( \hat{f} \) of (1.16) takes cosmological curves of the model \( \mathfrak{M}_1 \) into those of the model \( \mathfrak{M}_2 \). Since \( f \to \hat{f} \) is a functor, it is clear that cosmological models and smooth cosmological conjugations form a groupoid, whose isomorphism classes we call smooth cosmological conjugacy classes.

When two models \( \mathfrak{M}_1 \) and \( \mathfrak{M}_2 \) are isomorphic in this groupoid, we write \( \mathfrak{M}_1 \equiv \mathfrak{M}_2 \) and say that the models are smoothly conjugate. It is clear that the equivalence relation \( \equiv \) depends only on the rescaled scalar triples of the models, since so do their cosmological equations. In particular, a parameter homothety with parameter \( \lambda \) corresponds to a conjugation with \( f = \text{id}_{TM} \) between the model parameterized by \((M_0, M, G, \Phi)\) and that parameterized by \((\lambda^{1/2} M_0, M, \lambda G, \lambda \Phi)\). If one also requires \( M_{01} = M_{02} \) then the conjugation is called strict. Any smooth cosmological conjugation is the composite of a strict conjugation with a parameter homothety.

A strict smooth cosmological conjugation preserves basic on-shell cosmological observables in the sense that the pullback map \( f^* : C^\infty(TM_2) \to C^\infty(TM_1) \) satisfies:

\[
(f^*(F_2))_{\gamma_1} = (\hat{F}_2)_{f \circ \gamma_1}
\]

for any basic observable \( F_2 \in C^\infty(TM) \) of the second model and any cosmological flow curve \( \gamma_1 \) of the first model.

Example 1.3. A particularly simple class of strict smooth conjugations arises from isomorphisms of scalar triples. We say that two scalar triples \((\mathcal{M}_1, G_1, \Phi_1)\) and \((\mathcal{M}_2, G_2, \Phi_2)\) are isomorphic if there exists an isometry \( f_0 : (\mathcal{M}_1, G_1) \to (\mathcal{M}_2, G_2) \) such that \( \Phi_1 = \Phi_2 \circ f_0 \); in this case, \( f_0 \) is called an isomorphism of scalar triples and we write \((\mathcal{M}_1, G_1, \Phi_1) \simeq (\mathcal{M}_2, G_2, \Phi_2)\). We say that the models parameterized by \( \mathfrak{M}_1 \) and \( \mathfrak{M}_2 \) are isomorphic and write \( \mathfrak{M}_1 \simeq \mathfrak{M}_2 \) if \( M_{01} = M_{02} \) and \((\mathcal{M}_1, G_1, \Phi_1) \simeq (\mathcal{M}_2, G_2, \Phi_2)\). The differential \( df_0 \) of an isomorphism \( f_0 \) of cosmological models is a strict smooth conjugation.

A (necessarily strict) smooth cosmological conjugation between a model parameterized by \((M_0, M, G_0, \Phi_0)\) and itself is called a cosmological symmetry of that model. Automorphisms of a cosmological model are sometimes called visible symmetries while its remaining symmetries are called hidden (see [39–42]).
Remark 1.4. The study of cosmological symmetries of multifield models in the mathematical generality considered here has been limited. If one restricts attention to Lie groups, this becomes the problem of determining the Lie symmetries of the cosmological equation (1.7) and of classifying rescaled scalar triples whose cosmological equation admits given Lie groups of symmetries. Various local results about Lie symmetries of multifield models can be found in [39, 43–46]. The literature contains limited information on the global geometry of the resulting scalar manifolds; see [40, 42] for some results in that direction.

Cosmological equivalences. An equivalence relation weaker than conjugation arises if one identifies cosmological curves up to increasing reparameterization of the cosmological time. We say that a smooth curve \( \varphi : I \to \mathcal{M} \) is a pre-cosmological curve of the model parameterized by \( \mathcal{M} = (M_0, \mathcal{M}, G_0, \Phi) \) if there exists an increasing reparameterization \( \alpha : J \to I \) such that \( \varphi \circ \alpha \) is a cosmological curve of \( \mathcal{M} \). A smooth semispray map \( f : T \mathcal{M}_1 \to T \mathcal{M}_2 \) is called a smooth cosmological equivalence between the models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) if the map \( \hat{f} \) of (1.16) maps the pre-cosmological curves of \( \mathcal{M}_1 \) into those of \( \mathcal{M}_2 \). This amounts to the requirement that \( \hat{f} \) identifies the oriented cosmological orbits of the two models. Notice that cosmological equivalences between \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) differ\(^4\) from smooth dynamical equivalences, which are defined by the requirement \( f \) gives a smooth topological equivalence between the cosmological flows of the models.

It is clear from the properties of \( \hat{f} \) that cosmological models and cosmological equivalence form a groupoid, whose isomorphism classes we call smooth cosmological equivalence classes. When \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are isomorphic in this groupoid, we write \( \mathcal{M}_1 \sim \mathcal{M}_2 \) and say that the two models are smoothly equivalent. This equivalence relation is weaker than cosmological conjugation in the sense that \( \mathcal{M}_1 \equiv \mathcal{M}_2 \) implies \( \mathcal{M}_1 \sim \mathcal{M}_2 \). Like cosmological conjugation, cosmological equivalence depends only on the rescaled scalar triples; we say that the equivalence is strict if \( M_{01} = M_{02} \).

In Section 3, we will introduce other equivalence relations, which arise from the study of UV and IR limits.

1.4 Some remarks on the early and late time behavior of cosmological curves

Consider a maximal cosmological curve \( \varphi : (a_-, a_+) \to \mathcal{M} \), where \( a_{\pm} \in \mathbb{R} \) can be chosen such that \( a < 0 < b \) since the cosmological equation is autonomous. Recall

\(^4\)Indeed, a cosmological equivalence does not identify the cosmological flow curves of the two models up to reparameterization, but up to reparameterization combined with a time-dependent rescaling within the tangent bundle fibers. The notion of topological equivalence of dynamical systems is recalled in Appendix A.
that the ordinary $\alpha$- and $\omega$- limit sets of $\varphi$ are defined through [17]:

\[
\begin{align*}
\text{Lim}_\alpha \varphi & \overset{\text{def.}}{=} \{ m \in \mathcal{M} \mid \exists t_n \to a_+ : \lim_{n \to \infty} \varphi(t_n) = m \} \subset \mathcal{M} \\
\text{Lim}_\omega \varphi & \overset{\text{def.}}{=} \{ m \in \mathcal{M} \mid \exists t_n \to a_- : \lim_{n \to \infty} \varphi(t_n) = m \} \subset \mathcal{M}.
\end{align*}
\]

Let us assume that $\mathcal{M}$ is compact. Then for any $\omega$-limit point $m$ of $\varphi$ and any sequence $t_n \in (a_-, a_+)$ with $t_n \to a_+$ and $\varphi(t_n) \to m$, the tangent vectors $\dot{\varphi}(t_n)$ are contained in a compact subset of $T\mathcal{M}$ since $E_\varphi(t_n)$ stays bounded by (1.11). Hence there exists a subsequence $t'_n$ of $t_n$ such that $\dot{\varphi}(t'_n)$ converges to a point $u \in T\mathcal{M}$ with $\pi(u) = m$. Since $u$ is a limit point of the integral curve $\gamma = c(\varphi)$ of the vector field $S \in \mathcal{X}(T\mathcal{M})$, we must have $S(u) = 0$ i.e. $u$ is a stationary point of the cosmological flow. Hence $u$ belongs to the set $(\text{Crit} \Phi)_0$ and $m = \pi(u)$ belongs to $\text{Crit} \Phi$. This shows that the ordinary $\omega$-limit points of a cosmological curve are critical points of $\Phi$. Moreover, the set $\text{Lim}_\omega \varphi = \pi(\text{Lim}_\omega c(\varphi))$ is nonempty, compact and connected since the map $\pi$ is continuous and the set $\text{Lim}_\omega c(\varphi)$ has these properties by [17, Prop. 1.4].

When $\mathcal{M}$ is not compact, both ordinary limit sets of a maximal cosmological curve $\varphi$ may be empty. Indeed, the orbit of $\varphi$ need not be contained in any compact subset of $\mathcal{M}$, which means that there may exist a sequence of cosmological times $t_n \in (a_-, a_+)$ such that $\varphi(t_n)$ approaches a Freudenthal end of $\mathcal{M}$ when $t_n \to a_-$ or $t_n \to a_+$. To account for this, we consider the end compactification $\hat{\mathcal{M}}$ of $\mathcal{M}$ (which is a compact Hausdorff space containing $\mathcal{M}$ as a dense subset) and view $\varphi$ as the continuous curve $\varphi_e \overset{\text{def.}}{=} \iota \circ \varphi$ in the topological space $\hat{\mathcal{M}}$, where $\iota : \mathcal{M} \hookrightarrow \hat{\mathcal{M}}$ is the inclusion. By definition, the extended limit sets of $\varphi$ are the $\alpha$- and $\omega$-limit sets of $\varphi_e$ in the topological space $\hat{\mathcal{M}}$:

\[
\begin{align*}
\text{Lim}_\alpha \varphi_e & \overset{\text{def.}}{=} \text{Lim}_\alpha \varphi = \{ m \in \hat{\mathcal{M}} \mid \exists t_n \to a : \lim_{n \to \infty} \varphi(t_n) = \hat{m} \} \subset \hat{\mathcal{M}} \\
\text{Lim}_\omega \varphi_e & \overset{\text{def.}}{=} \text{Lim}_\omega \varphi = \{ m \in \hat{\mathcal{M}} \mid \exists t_n \to b : \lim_{n \to \infty} \varphi(t_n) = \hat{m} \} \subset \hat{\mathcal{M}}.
\end{align*}
\]

Let $\text{Ends}(\mathcal{M}) \overset{\text{def.}}{=} \hat{\mathcal{M}} \setminus \mathcal{M}$ be the space of ends of $\mathcal{M}$ (which is a totally disconnected topological space). We have:

\[
\begin{align*}
\text{Lim}_\alpha \varphi & = \text{Lim}_\alpha \varphi \sqcup \Lambda_\alpha \varphi , \quad \text{Lim}_\omega \varphi = \text{Lim}_\omega \varphi \sqcup \Lambda_\omega \varphi ,
\end{align*}
\]

where:

\[
\begin{align*}
\Lambda_\alpha \varphi & \overset{\text{def.}}{=} \text{Lim}_\alpha \varphi_e \cap \text{Ends}(\mathcal{M}) , \quad \Lambda_\omega \varphi \overset{\text{def.}}{=} \text{Lim}_\omega \varphi_e \cap \text{Ends}(\mathcal{M})
\end{align*}
\]

are the sets of $\alpha$- and $\omega$- limit ends of $\varphi$. With these definitions, $\varphi$ is contained in some compact subset of $\mathcal{M}$ iff $\Lambda_\alpha \varphi = \Lambda_\omega \varphi = \emptyset$. When the end compactification $\hat{\mathcal{M}}$ is sequentially compact, an argument similar to that of [17, Prop. 1.4] shows that $\text{Lim}_\alpha \varphi$ and $\text{Lim}_\omega \varphi$ are nonempty, compact and connected subsets of $\hat{\mathcal{M}}$. The fact
that a Freudenthal end of $\mathcal{M}$ can act as a limit point for a cosmological curve is a fundamental feature of models with non-compact target space.

In general, the continuous map $\varphi_\epsilon : (a_-, a_+) \to \mathcal{M}$ need not have limits for $t \to a_-$ or $t \to a_+$ in $\hat{\mathcal{M}}$. When the corresponding limit exists, we call it the $\alpha$- or $\omega$- extended limit of $\varphi$ and denote it by:

$$
\lim_{t \to a_-} \varphi_\epsilon(t) \in \hat{\mathcal{M}} \text{ respectively } \lim_{t \to a_+} \varphi_\epsilon(t) \in \hat{\mathcal{M}}.
$$

2 Scaling limits and approximations

In this section, we study the UV and IR scaling limits of classical multifield cosmological models with rescaled Planck mass $M_0$ and scalar triple $(\mathcal{M}, \mathcal{G}, \Phi)$, showing that cosmological curves are approximated in these limits by a reparameterization of the geodesic flow of $(\mathcal{M}, \mathcal{G})$ and by the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$, where $V = M_0 \sqrt{2\Phi}$ is the classical effective potential of the model. We also study the consistency conditions for these approximations, showing that they differ from commonly used approximations in cosmology, such as the slow roll approximation and its slow roll-slow turn variant as well as from the gradient flow approximation of [20].

2.1 The UV and IR limits

By definition, the behavior of the scale transform $\varphi_\epsilon$ at time $t$ recovers the behavior of $\varphi$ at time $t/\epsilon$. Notice that $\varphi$ can be recovered from its scale transform by setting $\epsilon = 1$:

$$
\varphi = \varphi_1.
$$

A time interval $\Delta t$ is rescaled to $\frac{1}{\epsilon} \Delta t$ under the scale transform by $\epsilon$. Hence cosmological time intervals are compressed for large $\epsilon$ and expanded for small $\epsilon$.

Intuitively, the limits of small and large $\epsilon$ capture the low and high frequency components of $\varphi$ since the large $\epsilon$ limit sharpens the oscillations of $\varphi_\epsilon$ while the small $\epsilon$ limit stretches them out. To make this quantitative, consider for simplicity the case $\mathcal{M} = \mathbb{R}^d$ and a cosmological curve $\varphi$ which is defined on $\mathbb{R}$. Then $\varphi(t) = (\varphi^1(t), \ldots, \varphi^d(t))$, where $\varphi^i : \mathbb{R} \to \mathbb{R}$ (for $i = 1, \ldots, d$) are the projections of $\varphi$ on the Cartesian coordinate axes. Since in this case $\varphi$ is a vector-valued function, it has a Fourier decomposition:

$$
\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\omega e^{i\omega t} \hat{\varphi}(\omega),
$$

where $\hat{\varphi} : \mathbb{R} \to \mathbb{R}^d$ is the Fourier transform of $\varphi$:

$$
\hat{\varphi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dt e^{-i\omega t} \varphi(t).
$$

We have:

$$
\varphi_\epsilon(t) = \varphi(t/\epsilon) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\omega e^{i\omega t/\epsilon} \hat{\varphi}(\omega) = \frac{\epsilon}{\sqrt{2\pi}} \int_{\mathbb{R}} d\omega e^{i\omega \epsilon \varphi(\epsilon \omega)},
$$
where we performed the change of variables \( \omega \to \epsilon \omega \). Hence the Fourier components of \( \varphi_\epsilon \) are:

\[
\hat{\varphi}_\epsilon(\omega) = \epsilon \hat{\varphi}(\epsilon \omega) = \epsilon \hat{\varphi}_{1/\epsilon}(\omega)
\]

and its Fourier transform is given by \( \hat{\varphi}_\epsilon = \epsilon (\hat{\varphi})_{1/\epsilon} \). When \( \epsilon \) is very large or very small, the Fourier component of \( \varphi_\epsilon \) at frequency \( \omega \) reproduces up to a rescaling of the amplitude the Fourier component of \( \varphi \) at the much higher (respectively much lower) frequency \( \epsilon \omega \). Hence the \textit{UV limit} \( \epsilon \to \infty \) corresponds to the high frequency behavior of \( \varphi \) while the \textit{IR limit} \( \epsilon \to 0 \) corresponds to its low frequency behavior; these \textit{scaling limits} correspond to the ultraviolet and infrared limits of the oscillation spectrum of \( \varphi \). Notice that the kinetic energy of a plane wave component \( \hat{\varphi}(\omega) e^{i\omega t} \) of \( \varphi \) equals \( E^k(\omega) \equiv \frac{1}{2} \omega^2 ||\hat{\varphi}(\omega)||_g^2 \) and we have:

\[
E^k(\varphi_\epsilon(t)) = \epsilon^2 \frac{1}{2} \omega^2 ||\hat{\varphi}(\omega)||_{\epsilon g}^2 = \frac{\epsilon^2}{2} \omega^2 ||\hat{\varphi}(\epsilon \omega)||_g^2 = \epsilon^2 E^k(\epsilon \omega) .
\]

**Remark 2.1.** Recall that the generalized curvatures of a curve in a Riemannian manifold do not depend on parameterization and hence are properties of the image of that curve rather than of the curve itself. Hence the generalized curvatures of \( \varphi_\epsilon \) and \( \varphi \) at every point on the orbit \( \text{im}(\varphi_\epsilon) = \text{im}(\varphi) \) coincide. Since scale transforms do not change the generalized curvatures of \( \varphi \), the limits of small and large \( \epsilon \) have no connection with slow or fast turns of the cosmological orbit, which form the focus of other approximation schemes in cosmology.

### 2.2 The rescaled cosmological equation

For any \( n \in \mathbb{N}_{>0} \), we have:

\[
\frac{d^n \varphi_\epsilon}{dt^n} = \frac{1}{\epsilon^n} \left( \frac{d^n \varphi}{dt^n} \right)_\epsilon ,
\]

and:

\[
\nabla^t_n \frac{d \varphi_\epsilon}{dt} = \frac{1}{\epsilon^{n+1}} \left( \nabla^t_n \frac{d \varphi}{dt} \right)_\epsilon .
\]

These relations imply that a curve \( \varphi : I \to \mathcal{M} \) satisfies the cosmological equation (1.7) iff its scale transform \( \varphi_\epsilon \) satisfies the \( \epsilon \)-rescaled cosmological equation:

\[
\epsilon^2 \nabla^t \frac{d \varphi_\epsilon(t)}{dt} + \frac{\epsilon}{M_0} \left[ \epsilon^2 \left( \frac{d \varphi_\epsilon(t)}{dt} \right)_g^2 + 2\Phi(\varphi_\epsilon(t)) \right]^{1/2} \frac{d \varphi_\epsilon(t)}{dt} + (\text{grad}_g \Phi)(\varphi_\epsilon(t)) = 0 .
\]

Upon dividing by \( \epsilon^2 \), the latter is equivalent with:

\[
\nabla^t \frac{d \varphi_\epsilon(t)}{dt} + \frac{1}{M_0} \left[ \left( \frac{d \varphi_\epsilon(t)}{dt} \right)_g^2 + 2\Phi_\epsilon(\varphi_\epsilon(t)) \right]^{1/2} \frac{d \varphi_\epsilon(t)}{dt} + (\text{grad}_g \Phi_\epsilon)(\varphi_\epsilon(t)) = 0 ,
\]

where \( \Phi_\epsilon \equiv \Phi/\epsilon^2 \). Hence \( \varphi \) satisfies the cosmological equation of the scalar triple (\( \mathcal{M}, \mathcal{G}, \Phi \)) iff \( \varphi_\epsilon \) satisfies the cosmological equation of the scalar triple (\( \mathcal{M}, \mathcal{G}, \Phi_\epsilon \)).
This also follows from the fact that (1.13) is a symmetry of the cosmological equation. In particular, the UV limit $\epsilon \to \infty$ amounts to taking the overall scale of $\Phi$ to zero, while the IR limit $\epsilon \to 0$ amounts to taking the overall scale of $\Phi$ to infinity.

The semispray defined by (2.4) is:

$$S_\epsilon = \mathcal{S} - \mathcal{H}_\epsilon C - \frac{1}{\epsilon^2} (\nabla^\mathcal{V}_v \Phi)^v,$$

where the function $\mathcal{H}_\epsilon : T\mathcal{M} \to \mathbb{R}_{\geq 0}$ is defined through:

$$\mathcal{H}_\epsilon(u) = \frac{1}{M_0} \left[ ||u||^2_{G_0} + \frac{2\Phi(\pi(u))}{\epsilon^2} \right]^{1/2}.$$

In the UV limit, $S_\epsilon$ reduces to the $N_0$-modification of the geodesic spray of the rescaled scalar manifold $(\mathcal{M}, \mathcal{G}_0)$:

$$S_{\text{UV}} \overset{\text{def}}{=} \mathcal{S} - N_0 C ,$$

where $N_0 : T\mathcal{M} \to \mathbb{R}_{\geq 0}$ is the norm function defined by $\mathcal{G}_0 \overset{\text{def}}{=} \frac{1}{M_0} \mathcal{G}$ on $T\mathcal{M}$:

$$N_0(u) \overset{\text{def}}{=} ||u||_{\mathcal{G}_0} = \frac{1}{M_0} ||u||_{\mathcal{G}}.$$

Notice that the geodesic spray $\mathcal{S}$ of $(\mathcal{M}, \mathcal{G})$ coincides with that of $(\mathcal{M}, \mathcal{G}_0)$ since it is invariant under constant rescalings of $\mathcal{G}$. Also notice that $N_0$ and the spray (2.5) are continuous on $T\mathcal{M}$ but smooth only on the slit tangent bundle $\dot{T}\mathcal{M}$. In the IR limit $\epsilon \to 0$, $S_\epsilon$ is approximated by the vertical vector field:

$$S_{\text{IR},\epsilon} = -\frac{1}{\epsilon M_0} \sqrt{2\Phi^v C} - \frac{1}{\epsilon^2} (\nabla^\mathcal{V}_v \Phi)^v = -\frac{1}{\epsilon^2 M_0} \sqrt{2\Phi^v S_{\text{IR},\epsilon}},$$

where:

$$S_{\text{IR},\epsilon} \overset{\text{def}}{=} \epsilon C - (\nabla^\mathcal{V}_v V)^v \in \mathcal{X}(T\mathcal{M})$$

and we defined the classical effective scalar potential $V$ through:

$$V \overset{\text{def}}{=} M_0 \sqrt{2\Phi}.$$  

Notice that $J(S_{\text{IR}}) = 0$, hence $S_{\text{IR}}$ is no longer a semispray. This signals that the IR limit is degenerate. Here $\Phi^v$ is the vertical lift of $\Phi$, which is defined through:

$$\Phi^v(u) = \Phi(\pi(u)) \quad \forall u \in T\mathcal{M}.$$

### 2.3 The UV and IR expansions

The rescaled cosmological equation can be used to construct **UV and IR expansions** of cosmological curves and of the cosmological flow. When $\epsilon$ is large, one can seek solutions $\varphi_\epsilon$ to (2.4) which are expanded in positive powers of $\frac{\epsilon}{\sqrt{\epsilon}}$; then $\varphi(t) \overset{\text{def}}{=} \varphi_\epsilon(\epsilon t)$
is a solution of the cosmological equation (1.7) which is expanded in non-negative powers of \( \Phi \). This amounts to treating \( \Phi \) as small, Taylor expanding the reduced Hubble function (1.9) as:

\[
H(u) = \|u\|_g \left[ 1 + \frac{2\Phi(\pi(u))}{\|u\|_g^2} \right]^{1/2} = \|u\|_g \left[ 1 + \frac{\Phi(\pi(u))}{\|u\|_g^2} - \frac{1}{2} \left( \frac{\Phi(\pi(u))}{\|u\|_g^2} \right)^2 + \ldots \right] \tag{2.8}
\]

and seeking solutions \( \varphi \) of the cosmological equation expanded in powers of \( \Phi \). This produces the \textit{UV expansion} of cosmological curves and a corresponding expansion of the cosmological flow. Substituting (2.8) into (1.8) gives an expansion of the cosmological semispray in powers of \( \Phi \).

When \( \epsilon \) is small, one can seek solutions \( \varphi_\epsilon \) of (2.3) which are expanded in powers of \( \epsilon \); then \( \varphi(t) \equiv \varphi_\epsilon(\epsilon t) \) is a solution of (1.7) expanded in powers of \( \frac{1}{\sqrt{2\Phi}} \). This amounts to treating \( \Phi \) as large and expanding the reduced Hubble function as:

\[
H(u) = \sqrt{2\Phi(\pi(u))} \left[ 1 + \left( \frac{\|u\|_g}{\sqrt{2\Phi(\pi(u))}} \right)^2 \right]^{1/2} = \sqrt{2\Phi(\pi(u))} \left[ 1 + \frac{1}{2} \left( \frac{\|u\|_g}{\sqrt{2\Phi(\pi(u))}} \right)^2 - \frac{1}{8} \left( \frac{\|u\|_g}{\sqrt{2\Phi(\pi(u))}} \right)^4 + \ldots \right] \tag{2.9}
\]

Substituting (2.9) into (1.8) produces an expansion of the cosmological semispray in powers of \( \frac{1}{\sqrt{2\Phi}} \). Substituting (2.9) into the cosmological equation and dividing both sides by \( \sqrt{2\Phi(\varphi(t))} \) brings (1.7) to the form:

\[
\frac{1}{\sqrt{2\Phi(\varphi(t))}} \nabla \varphi(t) + \frac{1}{M_0} \left[ 1 + \left( \frac{\|\varphi(t)\|_g}{\sqrt{2\Phi(\varphi(t))}} \right)^2 - \frac{1}{8} \left( \frac{\|\varphi(t)\|_g}{\sqrt{2\Phi(\varphi(t))}} \right)^4 + \ldots \right]^{1/2} \varphi(t) + (\text{grad}_G \sqrt{2\Phi})(\varphi(t)) = 0 \quad (2.10)
\]

and one can seek solutions expanded in powers of \( \frac{1}{\sqrt{2\Phi}} \). This produces an asymptotic expansion of cosmological curves called the \textit{IR expansion} and a corresponding expansion of the cosmological flow. Notice that the small expansion parameter multiplies the highest order term in (2.10). As a consequence, the first order approximant \( \varphi_{IR} \) is obtained by solving a first order ODE, which means that the corresponding approximant \( \Pi_{IR} \) to the cosmological flow \( \Pi \) is not defined on the whole tangent bundle \( T\mathcal{M} \) but on a closed submanifold of the latter. It is only higher order approximants of the cosmological flow which are defined on the entire tangent bundle of \( \mathcal{M} \). In the next subsections, we discuss the first order UV and IR approximants of cosmological curves and of the cosmological flow.

\textit{Remark 2.2}. The UV and IR expansions admit a geometric description obtained by writing \( \varphi_\epsilon \) as:

\[
\varphi_\epsilon(t) = \exp_{\varphi \epsilon(t)} \left( \sum_{n \geq 1} \frac{1}{\epsilon^n} v_n(t) \right) \quad \text{with} \quad v_n(t) \in T_{\varphi \epsilon(t)} \mathcal{M} \tag{2.11}
\]

respectively:

\[
\varphi_\epsilon(t) = \exp_{\varphi \epsilon(t)} \left( \sum_{n \geq 1} \epsilon^n w_n(t) \right) \quad \text{with} \quad w_n(t) \in T_{\varphi(t)} \mathcal{M} \quad , \tag{2.12}
\]
where $\varphi_{\text{UV}}$ and $\varphi_{\text{IR}}$ are the first order UV and IR approximants of $\varphi$ discussed below and $\exp_m : T_m \mathcal{M} \to \mathcal{M}$ denotes the exponential map of the Riemannian manifold $(\mathcal{M}, G_0)$ at the point $m \in \mathcal{M}$. The vector-valued functions $v_n(t)$ and $w_n(t)$ can be determined by substituting (2.11) and (2.12) into the rescaled cosmological equation and expanding respectively in powers of $1/\epsilon$ or $\epsilon$; we will explain the technical details of this procedure in a different publication. When $\epsilon$ is large (respectively small), the arguments of the exponential maps in the expressions above tend to zero and hence the right hand sides reduce to the UV and IR approximants.

2.4 The first order UV and IR approximants

Let $\varphi : I = (a_-, a_+ \to \mathcal{M}$ be a non-constant maximal cosmological curve with $0 \in (a_-, a_+)$, where $a_- \in \mathbb{R} \cup \{-\infty\}$ and $a_+ \in \mathbb{R} \cup \{+\infty\}$.

The strict UV limit and first order UV approximant. Let us fix $t \in I_\epsilon = (\epsilon a_-, \epsilon a_+)$ and set $\lambda \text{ def.} = \text{sign}(t) \in \{-1, 0, 1\}$. In the strict UV limit $\epsilon \to \infty$, we have

$$\lim_{\epsilon \to \infty} I_\epsilon = \mathbb{R}$$

and:

$$\lim_{\epsilon \to \infty} \varphi(\epsilon t) = \lim_{\epsilon \to \infty} \varphi(t/\epsilon) = \varphi(0), \quad \lim_{\epsilon \to \infty} \frac{d\varphi(\epsilon t)}{dt} = \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \dot{\varphi}(t/\epsilon) = 0 .$$

Hence the limiting curve $\varphi_\infty$ is the constant curve defined on $I_\infty = \mathbb{R}$ whose image is the point $\varphi(0)$.

When $\epsilon$ is large but finite, the rescaled cosmological equation (2.4) becomes:

$$\nabla_t \frac{d\varphi(\epsilon t)}{dt} + \frac{1}{M_0} \left\| \frac{d\varphi(\epsilon t)}{dt} \right\| G \frac{d\varphi(\epsilon t)}{dt} = O(1/\epsilon^2)$$

and we have:

$$\varphi(\epsilon t) = \varphi(t/\epsilon) = \varphi(0) + O(1/\epsilon), \quad \frac{d\varphi(\epsilon t)}{dt} = \frac{1}{\epsilon} \dot{\varphi}(t/\epsilon) = \frac{1}{\epsilon} \dot{\varphi}(0) + O(1/\epsilon^2)$$

for $|t| \ll \epsilon$. Thus $\varphi_\epsilon$ is approximated up to first order in $1/\epsilon$ by a curve $t \to \varphi_{\text{UV},\epsilon}(t)$ which satisfies:

$$\nabla_t \frac{d\varphi_{\text{UV},\epsilon}(t)}{dt} + \frac{1}{M_0} \left\| \frac{d\varphi_{\text{UV},\epsilon}(t)}{dt} \right\| G \frac{d\varphi_{\text{UV},\epsilon}(t)}{dt} = 0 \quad (2.13)$$

and:

$$\varphi_{\text{UV},\epsilon}(0) = \varphi(0), \quad \left. \frac{d\varphi_{\text{UV},\epsilon}(t)}{dt} \right|_{t=0} = \frac{1}{\epsilon} \dot{\varphi}(0) .$$

Since (2.13) is scale-invariant, the curve $\varphi_{\text{UV}} : \mathbb{R} \to \mathcal{M}$ defined through $\varphi_{\text{UV}}(t) \text{ def.} = \varphi_{\text{UV},\epsilon}(\epsilon t)$ satisfies the same equation:

$$\nabla_t \frac{d\varphi_{\text{UV}}(t)}{dt} + \frac{1}{M_0} \left\| \frac{d\varphi_{\text{UV}}(t)}{dt} \right\| G \frac{d\varphi_{\text{UV}}(t)}{dt} = 0 \quad (2.14)$$
and the conditions:

$$\varphi_{UV}(0) = \varphi(0), \quad \left. \frac{d\varphi_{UV}}{dt} \right|_{t=0} = \dot{\varphi}(0).$$  \hspace{1cm} (2.15)$$

It follows that \( \varphi(t) = \varphi(\epsilon t) \) is approximated by \( \varphi_{UV}(t) \) for \( |t| \ll 1 \). Notice that (2.5) is the spray defined by equation (2.14). The complete lift of maximal solutions to (2.14) defines the flow \( \Pi_{UV} \) of the \( N_0 \)-modification \( S_{UV} \) of the geodesic spray \( S \).

An increasing reparameterization \( \sigma \rightarrow \sigma(t) \) shows that (2.14) is equivalent with the geodesic equation:

$$\nabla_\sigma \frac{d\varphi_{UV}}{d\sigma} = 0,$$  \hspace{1cm} (2.16)

of \((\mathcal{M}, \mathcal{G})\), where the affine parameter \( \sigma \) is obtained by solving the second order ODE:

$$\dot{\sigma}(t) + \frac{1}{M_0} ||\varphi'_{UV}(\sigma)|| \dot{\sigma}(t)^2 = 0.$$  \hspace{1cm} (2.17)

Here the prime indicates derivation with respect to \( \sigma \). Since the Riemannian manifold \((\mathcal{M}, \mathcal{G})\) is complete, its geodesic flow is complete by the Hopf-Rinow theorem. As a consequence, the maximal geodesic \( \sigma \rightarrow \varphi_{UV}(\sigma) \) is defined on the entire real axis. Equation (2.17) can be written as:

$$\frac{d}{dt} \left( \frac{1}{\dot{\sigma}} \right) = \frac{1}{M_0} ||\varphi'_{UV}(\sigma)|| \dot{\sigma}.$$

Writing the left hand side as \( \dot{\sigma} \frac{d}{d\sigma} \left( \frac{dt}{d\sigma} \right) = \dot{\sigma} \frac{d^2 t}{d\sigma^2} \), this is equivalent with:

$$\frac{d^2 t}{d\sigma^2} = \frac{1}{M_0} ||\varphi'_{UV}(\sigma)|| \frac{d\dot{\sigma}}{d\sigma} = \frac{1}{M_0} ||\varphi'_{UV}(\sigma)|| \dot{\sigma} u,$$

where we set \( u = \frac{dt}{d\sigma} \). Thus \( u(\sigma) = B e^{\int_0^\sigma \frac{1}{M_0} \int_0^\sigma ||\varphi'_{UV}(\sigma')|| \dot{\sigma} \sigma'} \) and:

$$t(\sigma) = A + B \int_0^\sigma d\sigma' e^{\int_0^{\sigma'} \frac{1}{M_0} \int_0^{\sigma'} ||\varphi'_{UV}(\sigma'')|| \dot{\sigma} \sigma''},$$  \hspace{1cm} (2.18)

where \( A \) and \( B \) are integration constants. Since \( \sigma \) is an increasing parameter for \( \varphi_{UV} \) (i.e. we require \( \dot{\sigma} > 0 \) and hence \( \frac{dt}{d\sigma} > 0 \)), we must take \( B > 0 \). Notice that (2.17) is invariant under affine transformations of \( \sigma \), which correspond to affine transformations of \( t \), i.e. to changing the constants \( A \) and \( B \) in (2.18). Since we require \( \dot{\sigma} > 0 \), only affine transformations of the form \( \sigma \rightarrow \alpha + \beta \sigma \) with \( \beta > 0 \) are allowed. Using such transformations, we can set \( \sigma|_{t=0} = 0 \), which amounts to taking \( A = 0 \) in (2.18). Suppose that \( \dot{\varphi}(0) \neq 0 \), so that the maximal geodesic \( \varphi_{UV} \) is not constant. Then the remaining freedom of changing \( B \) allows us to take \( \sigma \) to be the proper length parameter:

$$s(t) = \int_0^t dt' ||\dot{\varphi}_{UV}(t')|| \dot{\sigma}.$$
in which case equation (2.18) (with \( A = 0 \)) gives:

\[
t(s) = B \int_0^s ds' e^{\frac{s'}{M_0}} = B_0 \left[ e^{\frac{s}{M_0}} - 1 \right],
\]

(2.19)

where \( B_0 \overset{\text{def.}}{=} M_0 B \) and we used the relation \( \left\| \frac{d\varphi_{UV}(s)}{ds} \right\|_G = 1 \). Moreover, (2.16) becomes:

\[
\nabla_s \frac{d\varphi_{UV}}{ds} = 0
\]

(2.20)

while conditions (2.15) require \( B_0 = \frac{M_0}{\|\varphi(0)\|_G} \) and:

\[
\varphi_{UV}|_{s=0} = \varphi(0), \quad \left. \frac{d\varphi_{UV}}{ds} \right|_{s=0} = \frac{\dot{\varphi}(0)}{\|\dot{\varphi}(0)\|_G}.
\]

To arrive at the last relations, we noticed that \( \frac{d\varphi_{UV}}{ds} \bigg|_{s=0} = \dot{\varphi}_{UV}(0)(s) \bigg|_{s=0} = \frac{B_0}{M_0} \dot{\varphi}(0) \), where we used the second of conditions (2.15). Since \( \left\| \frac{d\varphi_{UV}}{ds} \right\|_G > 1 \), this requires \( B_0 = \frac{M_0}{\|\varphi(0)\|_G} \). In particular, (2.19) reads:

\[
t(s) = \frac{M_0}{\|\varphi(0)\|_G} \left[ e^{\frac{s}{M_0}} - 1 \right].
\]

(2.21)

Notice that (2.21) requires \( t(s) > -\frac{M_0}{\|\varphi(0)\|_G} \), so the maximal curve \( t \to \varphi_{UV}(t) \) is defined on the interval \( \left( -\frac{M_0}{\|\varphi(0)\|_G}, +\infty \right) \). In particular, the first order UV approximation must break down if \( t \in I \) is smaller than \( -\frac{M_0}{\|\varphi(0)\|_G} \). In practice, the approximation becomes inaccurate for negative times which are smaller in absolute value since it is only appropriate for small \( \frac{|s|}{M_0} \). We have \( t(s) \approx \frac{1}{\|\varphi(0)\|_G} s \) when \( \frac{|s|}{M_0} \ll 1 \).

If \( \dot{\varphi}(0) = 0 \), then \( \varphi_{UV} \) is the constant geodesic at \( \varphi(0) \) since it satisfies \( \varphi_{UV}|_{s=0} = \varphi(0) \) and \( \frac{d\varphi_{UV}}{ds} \bigg|_{s=0} = 0 \). In this case, \( \varphi_{UV} \) satisfies (2.14) with respect to any parameter \( t \). Notice that relation (2.18) for \( A = 0 \) formally gives \( t = B\sigma \) with \( B > 0 \).

For any non-zero vector \( u \) of \( T,M \) such that \( m \overset{\text{def.}}{=} \pi(u) \in M_0 = M \setminus \text{Crit}V \), there exists a unique maximal cosmological curve \( \varphi_u : I_u \to M \) (defined on an open interval \( I_u \) which contains zero) which satisfies \( \varphi_u(0) = m \) and \( \dot{\varphi}_u(0) = u \). Setting \( n \overset{\text{def.}}{=} \frac{u}{\|u\|_G} \), this cosmological curve is approximated in the IR limit by the reparameterization (2.21) of the unique maximal normalized geodesic \( \psi_n : \mathbb{R} \to M \) of \( (M, G) \) which satisfies \( \psi_n|_{s=0} = m \) and \( \frac{d\psi_n}{ds} \bigg|_{s=0} = n \). Hence the cosmological time along \( \varphi_u \) is recovered in this approximation as:

\[
t(s) = \frac{M_0}{\|u\|_G} \left[ e^{\frac{s}{M_0}} - 1 \right].
\]

Under the parameter homothety (1.12), the proper length parameter \( s \) of \( \psi_u \) changes as \( s \to \lambda^{1/2} s \) while the norm of \( u \) changes as \( \|u\|_G \to \lambda^{1/2} \|v\|_G \). Thus we can use that similarity of the cosmological equation to set \( M_0 = 1 \); this absorbs \( M_0 \) into \( G \) and
changes the normalization of the geodesic flow without changing the cosmological time $t$. Hence the first order of the UV approximation of the cosmological flow of the model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ is determined by the normalized geodesic flow of the rescaled scalar field metric $\mathcal{G}_0 \overset{\text{def.}}{=} \frac{1}{M^2_0} \mathcal{G}$, which has proper length parameter $s_0 = \frac{1}{M_0} s$ and norm $\| \|_{\mathcal{G}_0} = \frac{1}{M_0} \| \|_\mathcal{G}$. Summarizing the discussion above gives the following:

**Proposition 2.3.** Consider the cosmological model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ and set $\mathcal{G}_0 \overset{\text{def.}}{=} \frac{1}{M^2_0} \mathcal{G}$. Let:

$$o(\mathcal{M}) = \{0_m | m \in \mathcal{M}\} \subset T\mathcal{M}$$

be the image of the zero section of $T\mathcal{M}$ and:

$$\dot{T}\mathcal{M} \overset{\text{def.}}{=} T\mathcal{M} \setminus o(\mathcal{M})$$

be the slit tangent bundle of $\mathcal{M}$. For each $u \in T\mathcal{M}$, let $\varphi_u : I \to \mathcal{M}$ be the maximal cosmological curve of this model which satisfies:

$$\varphi_u(0) = \pi(u) \quad \text{and} \quad \dot{\varphi}_u(0) = u.$$  

When $u \in \dot{T}\mathcal{M}$, this curve is approximated in the UV limit (for $t \in I$ with $t > -\frac{1}{\|u\|_{\mathcal{G}_0}}$) by a reparameterization of the maximal normalized geodesic $\psi_n : \mathbb{R} \to \mathcal{M}$ of the rescaled scalar manifold $(\mathcal{M}, \mathcal{G}_0)$ which satisfies:

$$\psi_n|_{s_0=0} = \pi(u) \quad , \quad \frac{d\psi_n}{ds_0}|_{s_0=0} = n \overset{\text{def.}}{=} \frac{u}{\|u\|_{\mathcal{G}_0}},$$

where the cosmological time $t$ is related to the proper length parameter $s_0$ of $\psi_n$ through:

$$t(s_0) = e^{s_0} - 1 \|u\|_{\mathcal{G}_0}.$$  

When $u \in o(\mathcal{M})$, the cosmological curve $\varphi_u$ is approximated in the UV limit by the constant geodesic of $(\mathcal{M}, \mathcal{G}_0)$ at $\pi(u)$. Accordingly, the cosmological flow $\Pi : D \to T\mathcal{M}$ of the model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ is approximated in the UV limit by the flow $\Pi_{\mathcal{G}_0} : D_{\mathcal{G}_0} \to T\mathcal{M}$ of the $N_0$-modification (2.5) of the geodesic spray of $(\mathcal{M}, \mathcal{G}_0)$, which has maximal domain of definition:

$$D_{\mathcal{G}_0} = \{ (t, u) \in \mathbb{R} \times T\mathcal{M} | t > -\frac{1}{\|u\|_{\mathcal{G}_0}} \}.$$  

Notice that the approximations in the proposition are only asymptotic.
The strict IR limit and first order IR approximant. When $\epsilon$ is small but not zero, the rescaled cosmological equation (2.3) gives:

$$
\frac{1}{M_0} \epsilon \sqrt{2\Phi(\varphi_{\epsilon}(t))} \frac{d\varphi_{\epsilon}(t)}{dt} + (\text{grad}_\varphi \Phi)(\varphi_{\epsilon}(t)) = O(\epsilon^2)
$$

which is equivalent with:

$$
\epsilon \frac{d\varphi_{\epsilon}(t)}{dt} + (\text{grad}_\varphi V)(\varphi_{\epsilon}(t)) = O(\epsilon^2),
$$

where the classical effective scalar potential $V$ was defined in (2.7). Hence $\varphi_{\epsilon}(t)$ is approximated to first order in $\epsilon$ by the solution $\varphi_{\text{IR},\epsilon}(t)$ of the equation:

$$
\epsilon \frac{d\varphi_{\text{IR},\epsilon}(t)}{dt} + (\text{grad}_\varphi V)(\varphi_{\text{IR},\epsilon}(t)) = 0
$$

which satisfies:

$$
\varphi_{\text{IR},\epsilon}(0) = \varphi(0).
$$

It follows that $\varphi(t) = \varphi_{\epsilon}(\epsilon t)$ is approximated by the solution $\varphi_{\text{IR}}(t) \overset{\text{def.}}{=} \varphi_{\text{IR},\epsilon}(\epsilon t)$ of the equation:

$$
\frac{d\varphi_{\text{IR}}(t)}{dt} + (\text{grad}_\varphi V)(\varphi_{\text{IR}}(t)) = 0
$$

which satisfies:

$$
\varphi_{\text{IR}}(0) = \varphi(0).
$$

Notice that the first order approximant $\varphi_{\text{IR}}$ of $\varphi$ is entirely determined by $\varphi(0)$ and does not depend on $\dot{\varphi}(0)$; one must consider higher orders of the IR expansion in order to obtain an approximant of $\varphi$ which also depends on $\dot{\varphi}(0)$. In the first order IR approximation, the initial speed $\dot{\varphi}(0) \in T_{\varphi(0)}M$ is approximated by the vector $-(\text{grad}_\varphi V)(\varphi(0)) \in T_{\varphi(0)}M$; this approximation can only be accurate when $||\dot{\varphi}(0) + (\text{grad}_\varphi V)(\varphi(0))||_\varphi$ is small.

Equation (2.25) is equivalent with the condition:

$$
S_{\text{IR}}(\dot{\varphi}_{\text{IR}}(t)) = 0,
$$

where:

$$
S_{\text{IR}} \overset{\text{def.}}{=} -C - (\text{grad}_\varphi V)^e \in \mathcal{X}(TM)
$$

is obtained from (2.6) for $\epsilon = 1$. Condition (2.27) confines the canonical lift of $\varphi_{\text{IR}}$ to the gradient flow shell $\text{Grad}_\varphi V \subset TM$ of the effective scalar triple $(M, G, V)$, which is defined as the graph of the vector field $-\text{grad}_\varphi V$:

$$
\text{Grad}_\varphi V \overset{\text{def.}}{=} \text{graph}(-\text{grad}_\varphi V) = \{u \in TM|u = -(\text{grad}_\varphi V)(\pi(u))\} = \{u \in TM|S_{\text{IR}}(u) = 0\}.
$$

To first order of the IR expansion, the cosmological flow curves of the model degenerate to the canonical lifts of the gradient flow curves of $(M, G, V)$, which lie within
the gradient flow shell $\text{Grad}_G V$. More precisely, consider a point $u \in T_M$ and set $\pi(u) = m \in \mathcal{M}$. Then the maximal cosmological curve $\varphi_u$ of this model which satisfies:

$$\varphi_u(0) = m \quad \text{and} \quad \dot{\varphi}_u(0) = u$$

is approximated to first order of the IR expansion by the gradient flow curve $\eta_m$ of $(\mathcal{M}, G, V)$ which satisfies:

$$\eta_m(0) = m$$

This approximation is good for small $|t|$ only when $||u + (\text{grad}_G V)(\pi(u))||_G$ is small, i.e. when $u$ is close the gradient flow shell $\text{Grad}_G V$ and it is most precise when $u \in \text{Grad}_G V$.

In this approximation, a basic cosmological observable $F : T\mathcal{M} \to \mathbb{R}$ reduces to the function:

$$F_{\text{IR}} \overset{\text{def.}}{=} F \circ (\text{grad}_G V) \in \mathcal{C}^\infty(\mathcal{M})$$

defined on $\mathcal{M}$. For any cosmological curve $\varphi$, we have:

$$F_\varphi \approx_{\text{IR}} F_{\text{IR}} \circ \varphi_{\text{IR}}$$

Summarizing the discussion above gives the following:

**Proposition 2.4.** Consider the cosmological model parameterized by $\mathfrak{M} = (M_0, \mathcal{M}, G, \Phi)$ and define its classical effective potential by $V \overset{\text{def.}}{=} M_0 \sqrt{2\Phi}$. For each $u \in T\mathcal{M}$ with $\pi(u) = m \in \mathcal{M}$, the maximal cosmological curve $\varphi_u$ of the model which satisfies:

$$\varphi_u(0) = m \quad \text{and} \quad \dot{\varphi}_u(0) = u$$

is approximated to first order of the IR expansion by the gradient flow curve $\eta_m$ of the effective scalar triple $(\mathcal{M}, G, V)$ which satisfies:

$$\eta_m(0) = m$$

This approximation is optimal for small $|t|$ when $u \in \text{Grad}_G V$. Accordingly, the cosmological flow $\Pi : \mathcal{D} \to T\mathcal{M}$ of the model is approximated to first order of the IR expansion by the map $\Pi_{\text{IR}} : \mathcal{D}_{\text{IR}} \to \text{Grad}_G V$ defined through:

$$\Pi_{\text{IR}}(t, u) \overset{\text{def.}}{=} -(\text{grad}_G V)(\Pi_V(t, \pi(u))) \quad \forall (t, u) \in \mathcal{D}_{\text{IR}}$$

where $\Pi_{G,V} : \mathcal{D}_V \to \mathcal{M}$ is the gradient flow of the effective scalar triple $(\mathcal{M}, G, V)$, whose maximal domain of definition we denote by $\mathcal{D}_{G,V} \subset \mathbb{R} \times \mathcal{M}$. Here:

$$\mathcal{D}_{\text{IR}} \overset{\text{def.}}{=} \{(t, u) \in T\mathcal{M} \mid (t, \pi(u)) \in \mathcal{D}_{G,V}\}$$

As mentioned above, the first order IR approximation of a cosmological curve $\varphi$ is optimal when $|t| \ll 1$ for those cosmological curves which satisfy $\varphi(0) \in \text{Grad}_G V$. This motivates the following:
Definition 2.5. A cosmological curve $\varphi$ of the model parameterized by $M = (M_0, M, G, \Phi)$ is called infrared optimal if its orbit meets the gradient flow shell $\text{Grad}_G V$ of the effective scalar triple $(M, G, V)$, where $V \overset{\text{def.}}{=} M_0\sqrt{2\Phi}$.

Suppose that $\varphi : I \to M$ is an infrared optimal cosmological curve and let $t_0 \in I$ be such that $\dot{\varphi}(t_0) = - (\text{grad}_G V)(\varphi(t_0))$. Shifting $t$ by a constant we can assume that $t_0 = 0$ without loss of generality. Then the first order IR approximant of $\varphi$ satisfies $\varphi_{IR}(0) = \varphi(0)$ and $\dot{\varphi}_{IR}(0) = \dot{\varphi}(0)$. Thus $\varphi_{IR}$ osculates in first order to $\varphi$ at $t = 0$ and hence approximates $\varphi$ to first order in $t$ for $|t| \ll 1$. Notice that the covariant accelerations of $\varphi$ and $\varphi_{IR}$ need not agree at $t = 0$ and hence the approximation need not hold to second order in $t$.

Remark 2.6. Equation (2.25) can also be written as:
\[
\frac{d\varphi_{IR}}{d\tau} + (\text{grad}\Phi)(\varphi_{IR}(\tau)) = 0 ,
\]
where $\tau$ is the increasing parameter defined though:
\[
\tau(t) = \tau_0 + M_0 \int_{t_0}^{t} \frac{dt'}{\sqrt{2\Phi(\varphi_{IR}(t'))}} ,
\]
with $\tau_0$ an arbitrary constant. Hence the cosmological curves of the model parameterized by $M$ are approximated by the gradient flow curves of the original scalar triple $(M, G, \Phi)$ up to such a curve-dependent reparameterizations of the gradient flow curves. It is more natural to work with the effective potential $V$ since its gradient flow parameter coincides with the cosmological time $t$.

2.5 Consistency conditions for the UV and IR approximations

The UV and IR approximations discussed in the previous subsection are accurate when the conditions given below are satisfied.

For the UV approximation. Comparison of (2.14) with the cosmological equation (1.7) shows that the UV approximation amounts to neglecting the contributions $2\Phi(\varphi(t))$ and $(\text{grad}_G \Phi)(\varphi(t))$ in (1.7). Hence this approximation is accurate when the following conditions are satisfied:
\[
\nu_1(\varphi(t)) \ll 1 \quad \text{and} \quad \nu_2(\varphi(t)) \ll 1 ,
\]
where we defined the first and second UV parameters of $\varphi$ through:
\[
\nu_1(\varphi(t)) \overset{\text{def.}}{=} \frac{2\Phi(\varphi(t))}{||\dot{\varphi}(t)||_G^2} \quad \text{and} \quad \nu_2(\varphi(t)) \overset{\text{def.}}{=} M_0 \frac{||\text{grad}(\varphi(t))||_G}{||\dot{\varphi}(t)||_G^2} .
\]

This agrees with the fact that the UV expansion is equivalent with an expansion in the overall scale of $\Phi$. 

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Indeed, the cosmological equation (1.7) can be written as:

$$\nabla_t \dot{\phi}(t) + \frac{1}{M_0} \left[ 1 + \nu_1(t) \right]^{1/2} ||\dot{\phi}(t)||_g \dot{\phi}(t) + (\text{grad}_g \Phi)(\phi(t)) = 0 \, .$$

The first condition in (2.29) allows us to approximate this with:

$$\nabla_t \dot{\phi}(t) + \frac{1}{M_0} ||\dot{\phi}(t)||_g \dot{\phi}(t) + (\text{grad}_g \Phi)(\phi(t)) = 0 \, ,$$

(2.31)

while the second condition allows us to approximate (2.31) with (2.14).

When conditions (2.29) are satisfied, the cosmological curve $\phi$ is well-approximated by the reparameterized geodesic $\phi_{UV}$ of $(\mathcal{M}, \mathcal{G})$ which satisfies (2.15). Using the proper length parameter $s$ on the latter, equation (2.21) gives:

$$t'(s) = \frac{1}{||\dot{\phi}(0)||_g} e^{\frac{2s}{M_0}} = \frac{1}{||\dot{\phi}(0)||_g} + \frac{1}{M_0} t(s) \quad \text{(2.32)}$$

and we have $\left\| \frac{d\phi_{UV}(s)}{ds} \right\|_g = 1$. Thus:

$$||\dot{\phi}(t)||_g \approx ||\dot{\phi}_{UV}(t)||_g = \frac{1}{t'(s)} = \frac{1}{||\dot{\phi}(0)||_g} e^{-\frac{2s}{M_0}} = \frac{M_0 ||\dot{\phi}(0)||_g}{M_0 + t ||\dot{\phi}(0)||_g}$$

and:

$$\nu_1(\phi) \approx \frac{2\Phi(\phi(0))}{||\dot{\phi}(0)||_g^2} e^{\frac{2s}{M_0}} \quad \text{and} \quad \nu_2(\phi) \approx \frac{M_0 ||(d\Phi)(\phi(s))||_g}{||\dot{\phi}(0)||_g^2} e^{\frac{2s}{M_0}} \, ,$$

(2.33)

i.e.:\n
$$\nu_1(\phi) \approx \frac{2\Phi(\phi(s))}{M_0^2 ||\dot{\phi}(0)||_g^2} (M_0 + t ||\dot{\phi}(0)||_g)^2$$

$$\nu_2(\phi) \approx \frac{||(d\Phi)(\phi(0))||_g}{||\dot{\phi}(0)||_g^2} (M_0 + t ||\dot{\phi}(0)||_g^2)^2$$

For $t = 0$, conditions (2.29) require:

$$\frac{2\Phi(\phi_{UV}(0))}{||\dot{\phi}(0)||_g^2} \ll 1 \quad \text{and} \quad M_0 \frac{||(d\Phi)(\phi_{UV}(0))||_g}{||\dot{\phi}(0)||_g^2} \ll 1 \, ,$$

while for $t \neq 0$ they constrain the cosmological time interval on which the approximation is accurate. When these conditions are not satisfied, the leading UV approximation has to be corrected by higher order terms in the UV expansion.
For the IR approximation. Comparison of (2.23) with the cosmological equation (1.7) shows that the IR approximation amounts to neglecting the terms $\nabla_t \dot{\varphi}(t)$ and $||\dot{\varphi}(t)||^2_G$ in (1.7). Hence this approximation is good when the cosmological curve $\varphi$ satisfies:

$$\kappa_{1\varphi}(t) \ll 1 \quad \text{and} \quad \kappa_{2\varphi}(t) \ll 1,$$

(2.34)

where we defined the first and second IR parameters of $\varphi$ through:

$$\kappa_{1\varphi}(t) \overset{\text{def}}{=} \frac{||\dot{\varphi}(t)||_G^2}{2\Phi(\varphi(t))} \quad \text{and} \quad \kappa_{2\varphi}(t) \overset{\text{def}}{=} \frac{||\nabla_t \dot{\varphi}(t)||_G}{||(d\Phi)(\varphi(t))||_G}.$$  \hfill (2.35)

The first condition in (2.34) requires the kinetic energy of $\varphi$ to be much smaller than its potential energy. Notice that:

$$\kappa_{1\varphi}(t) = \kappa_{1}(\dot{\varphi}(t)),$$

where the function $\kappa_1 : TM \to \mathbb{R}_{>0}$ is defined through:

$$\kappa_1(u) \overset{\text{def}}{=} \frac{||u||_G^2}{2\Phi(\pi(u))}.$$  \hfill (2.36)

The cosmological equation takes the form:

$$\nabla_t \dot{\varphi}(t) = -\frac{1}{M_0} \sqrt{2\Phi(\varphi(t))} \left[ \sqrt{1 + \kappa_{1\varphi}(t)\dot{\varphi}(t) + (\text{grad}_G V)(\varphi(t))} \right].$$  \hfill (2.37)

Hence the second IR parameter of $\varphi$ can be written as:

$$\kappa_{2\varphi}(t) = \frac{1}{M_0} \sqrt{2\Phi(\varphi(t))} \frac{||\sqrt{1 + \kappa_{1\varphi}(t)\dot{\varphi}(t) + (\text{grad}_G V)(\varphi(t))}||_G}{||(d\Phi)(\varphi(t))||_G} = \kappa_{2}(\dot{\varphi}(t)),$$

(2.37)

where the function $\kappa_2 : TM \to \mathbb{R}_{>0}$ is defined through:

$$\kappa_2(u) \overset{\text{def}}{=} \frac{1}{M_0} \sqrt{2\Phi(\pi(u))} \frac{||\sqrt{1 + \kappa_1(u)u + (\text{grad}_G V)(\pi(u))}||_G}{||(d\Phi)(\pi(u))||_G}.$$  \hfill (2.38)

Let $\eta = \varphi_{\text{IR}}$ be the first order IR approximant of $\varphi$ at $t = 0$. Since $\eta$ satisfies the gradient flow equation of $(M, G, V)$, we have:

$$\kappa_{1\eta}(t) \overset{\text{def}}{=} \kappa_1(\dot{\eta}(t)) = M_0^2 \frac{||(d\Phi)(\eta(t))||_G^2}{4\Phi(\eta(t))^2} = \hat{\kappa}_1(\eta(t)),$$

where the function $\hat{\kappa}_1 : M \to \mathbb{R}_{>0}$ is defined through:

$$\hat{\kappa}_1 \overset{\text{def}}{=} M_0^2 \frac{||d\Phi||_G^2}{4\Phi^2}.$$
Since $\nabla V = -\frac{M_0}{\sqrt{2}\Phi} \nabla \Phi$, we have\footnote{We temporarily denote $\|\|_G$ by $\|\|$ and $\nabla_g$ by $\nabla$ to simplify notation.}:

\[
\nabla \dot{\eta}(t) = \nabla \eta(t) \dot{\eta}(t) = (\nabla \nabla V \cdot \nabla V)(\eta(t)) = \left[ \frac{M_0^2}{2\Phi} \nabla \nabla \Phi \nabla \Phi - \kappa \nabla \Phi \right]_{\eta(t)}
\]

because $\nabla \nabla \Phi = (d\Phi)(\nabla \Phi) = ||d\Phi||^2$. Thus:

\[
\kappa_{2\eta}(t) \overset{\text{def.}}{=} \frac{||\nabla \eta(t)||_G}{||(d\Phi)(\eta(t))||_G} = \frac{M_0^2}{2\Phi} \left[ \frac{||\nabla \nabla \Phi \nabla \Phi||}{||d\Phi||^2} + \frac{1}{4\Phi^2} ||d\Phi||^4 - \frac{1}{2\Phi} \nabla \nabla \Phi ||d\Phi||^2 \right]^{1/2}_{\eta(t)} = \hat{\kappa}_2(\eta(t)) ,
\]

where we used the relation:

\[
G(X, \nabla X) = \frac{1}{2} \nabla X ||X||^2 ,
\]

which holds for any vector field $X \in \mathcal{X}(\mathcal{M})$ since $G$ is covariantly constant. Here the function $\hat{\kappa}_2 : \mathcal{M} \to \mathbb{R}_{>0}$ is defined through:

\[
\hat{\kappa}_2 \overset{\text{def.}}{=} \frac{\frac{M_0^2}{2\Phi} \nabla \nabla \Phi \nabla \Phi - \kappa \nabla \Phi}{||d\Phi||} = \frac{M_0^2}{2\Phi} \left[ \frac{||\nabla \nabla \Phi \nabla \Phi||}{||d\Phi||^2} + \frac{1}{4\Phi^2} ||d\Phi||^4 - \frac{1}{2\Phi} \nabla \nabla \Phi ||d\Phi||^2 \right]^{1/2}
\]

On the other hand, we have:

\[
\kappa_{2\eta}(\eta(t)) = \sqrt{1 + \kappa_{1\eta}(t) - 1} ,
\]

where we used (2.38) and the fact that $\eta$ satisfies the gradient flow equation of $(\mathcal{M}, G, V)$.

Using the first order IR approximation for $\varphi$ and its first time derivative amounts to replace $\varphi(t)$ by $\eta(t)$ and $\dot{\varphi}(t)$ by $\dot{\eta}(t)$. Then $\kappa_{1\varphi}(t)$ is replaced by $\kappa_{1\eta}(t)$ and $\kappa_{2\varphi}(t) = \kappa_{2\eta}(\dot{\varphi}(t))$ is replaced by $\kappa_{2\eta}(\dot{\eta}(t))$. Accuracy of this approximation requires $\kappa_{1\eta}(t) \ll 1$, which implies $\kappa_{2\eta}(\dot{\eta}(t)) \ll 1$ by (2.40). For the approximation to be accurate up to the second time derivative of $\varphi$ and $\eta$, we must also have $\kappa_{2\varphi}(t) \approx \kappa_{2\eta}(t)$, which requires $\kappa_{2\eta}(t) \ll 1$.

\begin{remark}
Suppose that $\varphi$ is an infrared optimal cosmological curve. In this case, we have $\varphi(0) = \eta(0) := m \in \mathcal{M}$ and $\dot{\varphi}(0) = \dot{\eta}(0)$, which gives:

\[
\kappa_{1\varphi}(0) = \kappa_{1\eta}(0) = \kappa_1(m) \quad \text{and} \quad \kappa_{2\varphi}(0) = \kappa_{2\eta}(0) = \sqrt{1 + \kappa_1(m)} - 1 .
\]

The curves $\varphi$ and $\eta$ osculate in order two at the point $m$ iff $\nabla \dot{\varphi}(0) = \nabla \dot{\eta}(0)$, i.e. (see (2.36) and (2.39)):

\[
(\sqrt{1 + \kappa_1(m)} + \kappa_1(m) - 1)(\nabla_g \Phi) = \frac{M_0^2}{2\Phi(m)} (\nabla \nabla \Phi \nabla \Phi)(m) .
\]

This requires $\kappa_{2\varphi}(0) = \kappa_{2\eta}(0)$, i.e.:

\[
\kappa_2(m) = \sqrt{1 + \kappa_1(m)} - 1 .
\]

When (2.41) is satisfied, $\varphi(t)$ is approximated by $\eta(t)$ to order two in $t$ for $|t| \ll 1$.
\end{remark}
2.6 Relation to some other approximations used in cosmology

Recall that the slow roll parameter of a cosmological curve $\varphi$ is defined through:

$$\varepsilon_\varphi(t) \overset{\text{def}}{=} - \frac{\dot{H}_\varphi(t)}{H_\varphi(t)^2} = - \frac{1}{H_\varphi(t)} \frac{d}{dt} \log H_\varphi(t) \ .$$

We have:

$$\frac{d}{dt} \log H_\varphi(t) = \frac{\mathcal{G}(\dot{\varphi}(t), \nabla_t \dot{\varphi}(t)) + (d\Phi)(\varphi(t))(\dot{\varphi}(t))}{||\dot{\varphi}(t)||^2_G + 2\Phi(\varphi(t))} = - \frac{3H_\varphi(t)||\dot{\varphi}(t)||^2_G}{||\dot{\varphi}(t)||^2_G + 2\Phi(\varphi(t))} \ ,$$

where in the last equality we used the cosmological equation and the relation:

$$\mathcal{G}(\dot{\varphi}(t), (\text{grad}_G \Phi)(\varphi(t))) = (d\Phi)(\varphi(t))(\dot{\varphi}(t))$$

to simplify the numerator. Thus:

$$\varepsilon_\varphi(t) = \frac{3||\dot{\varphi}(t)||^2_G}{||\dot{\varphi}(t)||^2_G + 2\Phi(\varphi(t))} = \frac{3}{1 + \nu_1 \varphi(t)} \ ,$$

where $\nu_1 \varphi(t) = \frac{2\Phi(\varphi(t))}{||\dot{\varphi}(t)||^2_G} = \frac{1}{\kappa_1(t)}$ is the first UV parameter and $\kappa_1 \varphi(t)$ is the first IR parameter of $\varphi$. Hence the slow roll condition $\varepsilon_\varphi(t) \ll 1$ holds iff $\nu_1 \varphi(t) \gg 1$ i.e. $\kappa_1 \varphi(t) \ll 1$. In particular, the slow roll approximation does not hold when the UV approximation is accurate but it automatically applies when the first order IR approximation is accurate up to first time derivatives. Notice, however, that conditions (2.34) for the IR approximation to be accurate up to second time derivatives are stronger (i.e. more restrictive) than the slow roll condition, since they also require the parameter $\kappa_2 \varphi(t)$ to be small. In particular, the slow roll approximation by itself is not equivalent with the first order IR approximation.

Define the scalar gradient flow parameter of $\varphi$ through:

$$\eta_\varphi(t) \overset{\text{def}}{=} \frac{1}{3} ||\eta_\varphi(t)||_G = \frac{1}{3H_\varphi(t)} \frac{||\nabla_t \dot{\varphi}(t)||_G}{||\dot{\varphi}(t)||_G} \ ,$$

where $\eta_\varphi(t)$ is the vector gradient flow parameter of [20, Sec. 1.5]. Since $||\nabla_t \dot{\varphi}(t)||_G = \kappa_2 \varphi(t)||(d\Phi)(\varphi(t))||_G$ and $||\dot{\varphi}(t)||_G = \sqrt{2\Phi(\varphi(t))\kappa_1 \varphi(t)}$, we have:

$$H_\varphi(t) = \frac{1}{3M_0} \sqrt{2\Phi(\varphi(t))[1 + \kappa_1 \varphi(t)]}$$

and:

$$\eta_\varphi(t) = \frac{M_0}{2} \frac{||(d\Phi)(\varphi(t))||_G \kappa_2 \varphi(t)}{\Phi(\varphi(t)) \sqrt{\kappa_1(t)[1 + \kappa_1(t)]}} \ .$$

When the IR approximation is accurate to second order in time derivatives, we have $\kappa_1 \varphi(t), \kappa_2 \varphi(t) \ll 1$ and the previous relation gives:

$$\eta_\varphi(t) \approx \frac{M_0}{2} \frac{||(d\Phi)(\varphi(t))||_G \kappa_2 \varphi(t)}{\Phi(\varphi(t)) \sqrt{\kappa_1(t)}} \ . \quad (2.42)$$
Hence the gradient flow approximation of [20] applies within the IR approximation only when this quantity is small. For two-field models, it was shown in [20, Sec. 1.9] that the SRST approximation of [15, 16] is a further specialization of the gradient flow approximation which applies only when certain conditions on the Hessian of $\Phi$ are satisfied; these conditions are much stronger than those for accuracy of the IR approximation. Hence the SRST approximation can be applied within the IR approximation only under very restrictive conditions. Thus the IR approximation is conceptually and quantitatively different from the gradient flow approximation of [20]. It is markedly different from (and much less restrictive than) the SRST approximation and its $n$-field variant.

3 Renormalization group and dynamical universality classes

In this section, we discuss the action of the subgroup $T_{\text{ren}} \overset{\text{def}}{=} \text{Stab}_T(\Phi)$ of the universal similarity group $T$ of Subsection 1.2 on the parameters $(M_0, G)$ of a multi-field cosmological model with target manifold $\mathcal{M}$, showing that it plays a role similar to that of the renormalization group action in the theory of critical phenomena. This action defines the $RG$ flow of the model, which proceeds by homothety transformations of the rescaled Planck mass $M_0$ and of the scalar manifold metric $G$. We also show that the first order infrared approximation is invariant up to reparameterization under Weyl transformations of $G$ and define UV and IR universality classes of classical cosmological models. After briefly discussing the infrared phase structure which arises from the classification of limit points of cosmological curves, we use these results and the uniformization theorem of Poincaré to show that two-field models whose scalar manifold metric has constant curvature are IR universal in the sense that they provide distinguished representatives of the universality classes of all two-field models.

3.1 $RG$ similarities and $RG$ transformations

For fixed target space $\mathcal{M}$, the cosmological equation (1.7) is invariant under the following action of the multiplicative group $\mathbb{R}_{>0}$ on the curve $\varphi$ and on the parameters $(M_0, G, \Phi)$:

$$\varphi \rightarrow \varphi_\epsilon \ , \ M_0 \rightarrow \epsilon M_0 \ , \ G \rightarrow \epsilon^2 G \ , \ \Phi \rightarrow \Phi \ (\epsilon > 0) \ . \quad (3.1)$$

The $RG$ similarity (3.1) is the composition of the parameter homothety (1.12) at parameter $\lambda = \epsilon^2$ with the scale similarity (1.13) at parameter $\epsilon$. These transformations give the restriction $\rho_{\text{ren}}$ of the similarity action $\rho$ of the group $T = (\mathbb{R}_{>0})^2$ to the renormalization subgroup $T_{\text{ren}} = \text{Stab}_T(\Phi) \simeq \mathbb{R}_{>0}$ discussed in Subsection 1.2:

$$\rho_{\text{ren}}(\epsilon)(\varphi, M_0, G, \Phi) \overset{\text{def}}{=} \rho(\epsilon^2, \epsilon)(\varphi, M_0, G, \Phi) = (\varphi_\epsilon, \epsilon M_0, \epsilon^2 G, \Phi) \ (\epsilon > 0) \ .$$
The RG similarities (3.1) can be used to absorb $M_0$ into the overall scale of $G$ or vice versa without changing the scale of the scalar potential $\Phi$. Notice that RG similarities are adiabatic in the sense that they preserve the total energy $E_{\varphi}^{G,\Phi}(t) \overset{\text{def}}{=} \frac{1}{2}||\dot{\varphi}(t)||_G^2 + \Phi(\varphi(t))$ of a cosmological curve $\varphi$. Namely, we have:

$$E_{\varphi_{\epsilon}}^{\epsilon^2G,\Phi}(t) = E_{\varphi}^{G,\Phi}(t/\epsilon) \ .$$

The time $t$ cosmological curves of the model with parameters $(\epsilon M_0, \epsilon^2 G, \Phi)$ coincide with the time $t/\epsilon$ cosmological curves of the model with parameters $(M_0, G, \Phi)$. Hence the infrared and ultraviolet limits of the latter can be described equivalently by taking $\epsilon$ to zero and infinity in the former. Due to these properties, the RG transformations:

$$M_0 \rightarrow \epsilon M_0 \ , \ G \rightarrow \epsilon^2 G \ , \ \Phi \rightarrow \Phi \quad (3.2)$$

play a role akin to that familiar from the theory of critical phenomena. These transformations give the restriction $\rho_{\text{rg}}$ of the parameter action $\rho_{\text{par}}$ of $T$ (see (1.14)) to the renormalization group $T_{\text{ren}}$:

$$\rho_{\text{rg}}(\epsilon)(M_0, G, \Phi) = \rho_{\text{par}}(\epsilon^2, \epsilon)(M_0, G, \Phi) = (\epsilon M_0, \epsilon^2 G, \Phi) \quad (\epsilon > 0) \ .$$

Under RG transformations, the rescaled scalar field metric $G_0 = \frac{1}{M_0^2} G$ is invariant, while the classical effective potential $V = M_0 \sqrt{2\Phi}$ and its gradient with respect to $G$ change as:

$$V \rightarrow \epsilon V \ , \ \text{grad}_G V \rightarrow \frac{1}{\epsilon} \text{grad}_G V \ .$$

Hence $\text{grad}_G V$ tends to zero in the UV limit $\epsilon \rightarrow \infty$, while it tends to a current supported on the non-critical locus $M_0 = M \setminus \text{Crit}\Phi$ in the strict IR limit $\epsilon \rightarrow 0$. Notice that the geodesic flow of $(M, G_0)$ is invariant under RG transformations, since the covariant derivative of $G_0$ satisfies:

$$\nabla^{\epsilon^2 G_0} = \nabla^{G_0} \ .$$

This also follows from the fact that the cosmological equation is invariant under RG similarities while the geodesic equation is invariant under affine reparameterizations. On the other hand, the gradient flow of $(M, G, V)$ is invariant under RG transformations up to a reparameterization $\varphi_{\text{IR}} \rightarrow \varphi_{\text{IR},\epsilon}$ with positive constant factor $1/\epsilon$; this also follows from invariance of the cosmological equation under RG similarities.

### 3.2 The dynamical renormalization group flow

Let $M$ be a smooth manifold and $T_2(M) \overset{\text{def}}{=} \Gamma(\text{Sym}^2(T^*M))$ be the infinite-dimensional space of smooth symmetric covariant 2-tensor fields defined on $M$. A Riemannian homothety line on $M$ is a one-dimensional linear subspace $L \subset T_2(M)$ which contains a Riemannian metric defined on $M$. In this case, all elements of $L$ which are
positively-homothetic with $G$ are Riemannian metrics on $\mathcal{M}$; such elements form an open half-line $L_+$ contained in $L$ which satisfies $L = L_+ \cup (-L_+) \cup \{0\}$. The \textit{cosmological homothety plane} defined by $L$ is the linear space $\Pi(\mathcal{M}, L) \equiv \mathbb{R} \oplus L \subset \mathbb{R} \times T_2(\mathcal{M})$, which contains the \textit{cosmological homothety cone} $C(\mathcal{M}, L) \equiv \mathbb{R}_{>0} \oplus L_+$. The \textit{cosmological RG action} on $C(\mathcal{M}, L)$ is the action of the group $T_{\text{ren}} = \mathbb{R}_{>0}$ defined through:

$$\rho_{\text{RG}}(\epsilon)(M_0, G) \overset{\text{def}}{=} (\epsilon M_0, \epsilon^2 G) \quad \forall (M_0, G) \in C(\mathcal{M}, L) \quad \forall \epsilon > 0 .$$

Setting $\epsilon = e^\lambda$ with $\lambda \in \mathbb{R}$, this action describes the flow on the homothety cone of the Euler vector field $E_L$ defined through:

$$E_L(M_0, G) \overset{\text{def}}{=} M_0 \oplus 2G \in \Pi(\mathcal{M}, G) \equiv T_{M_0,G}C(\mathcal{M}, L) \quad \forall (M_0, G) \in C(\mathcal{M}, L) .$$

This flow is called the \textit{cosmological RG flow} of $(\mathcal{M}, L)$.

Any choice of a reference metric $G_{\text{ref}} \in L_+$ induces coordinates $w_1, w_2$ on $C(\mathcal{M}, L)$ given by:

$$M_0 = w_1 , \quad G = w_2 G_{\text{ref}} ,$$

which extend to coordinates on the homothety plane $\Pi(\mathcal{M}, L)$ and hence identify the latter with $\mathbb{R}^2$ and $C(\mathcal{M}, L)$ with the first quadrant. Then $\rho_{\text{RG}}$ identifies with the action:

$$\rho_{\text{RG}}(\epsilon)(w_1, w_2) = (\epsilon w_1, \epsilon^2 w_2) \quad \forall \epsilon > 0$$

and $E_L$ identifies with the vector field $E$ on $\mathbb{R}_{>0}^2$ given by:

$$E(w_1, w_2) = (w_1, 2w_2) \quad \forall w_1, w_2 > 0.$$ 

Moreover, the integral curves of the RG flow identify with the solutions of the system:

$$\frac{d w_1(\lambda)}{d\lambda} = E_1(w_1(\lambda), w_2(\lambda)) = w_1(\lambda)$$

$$\frac{d w_2(\lambda)}{d\lambda} = E_2(w_1(\lambda), w_2(\lambda)) = 2w_2(\lambda) ,$$

namely:

$$w_1(\lambda) = e^\lambda w_1(0) , \quad w_2(\lambda) = e^{2\lambda} w_2(0) .$$

The limit $\lambda \to +\infty$ recovers the UV limit $\epsilon \to +\infty$ while $\lambda \to -\infty$ corresponds to the IR limit $\epsilon \to 0$. These limits correspond to the fixed points of the RG flow on the one-point compactification of the closure of the homothety cone, which are the apex of the cone and the point at infinity (see Figure 1).
Figure 1: Integral curves of the RG flow on the homothety cone and on the one-point compactification of its closure. In the second figure we identified the Alexandroff compactification of the homothety plane with the unit sphere through stereographic projection. Here $\phi \in [0, \pi]$ is the spherical altitude angle on the unit sphere and $\theta \in [0, 2\pi]$ is the azimuth/longitude angle (which coincides with the polar angle in the homothety plane), while $r = \sqrt{w_1^2 + w_2^2} = \cot(\frac{\phi}{2})$ is the distance from the origin in the homothety plane. The homothety cone identifies with the region $(\phi, \theta) \in [0, \pi] \times [0, \frac{\pi}{2}]$ of the unit sphere. The RG flow on the one-point compactification has fixed points at $(\phi, \theta) = (\pi, 0)$ (the red dot) and $(\phi, \theta) = (0, \frac{\pi}{2})$ (the blue dot), which correspond respectively to the apex of the homothety cone and its point at infinity, i.e. to the south and north poles of the sphere. These fixed points give the IR limit (the red dot) and the UV limit (the blue dot).

Consider a model parameterized by $(M_0, M, G, \Phi)$, where $G \in L_+$. The discussion of the previous subsection implies that the cosmological RG flow curve of $(M, L)$ which passes through the point $(M_0, G) \in C(M, L)$ induces a curve in the space of all flows defined on $TM$ which interpolates when $\lambda$ runs from $-\infty$ to $+\infty$ between the gradient flow of $(M, G, V)$ (where $V = M_0 \sqrt{2\Phi}$) and a modification of the geodesic flow of $(M, G_0)$ (where $G_0 = \frac{1}{M_0^2} G$).

3.3 Conformal invariance in the infrared

The gradient flow of a scalar potential $V$ defined on a Riemannian manifold $(M, G)$ is invariant under Weyl rescalings of the metric $G$ up to reparameterization of the gradient flow curves. More precisely, consider a solution $\eta : I \to M$ of the gradient flow equation:

$$\frac{d\eta(t)}{dt} = -(\text{grad}_G V)(\eta(t))$$  \hspace{1cm} (3.4)
of $V$ with respect to the metric $\mathcal{G}$ and let $\Omega$ be a smooth and everywhere-positive real-valued function defined on $\mathcal{M}$. Then the $\eta$-dependent increasing reparameterization $t \to \tau$ defined through:

$$\tau(t) \overset{\text{def}}{=} \tau_\eta(t) = \int_{t_0}^t \Omega(t') + C,$$

(where $t_0 \in I$ and $C$ is an arbitrary constant) takes $\eta(t)$ into a solution $\eta(\tau)$ of the gradient flow equation:

$$\frac{d\eta(\tau)}{d\tau} = -(\text{grad}_{\mathcal{G}_\eta}V)(\eta(\tau)),$$

of $V$ with respect to the metric $\mathcal{G}_\eta \overset{\text{def}}{=} \Omega \mathcal{G}$. This implies that the gradient flow orbits of $V$ with respect to the metrics $\mathcal{G}$ and $\Omega \mathcal{G}$ coincide as oriented submanifolds of $\mathcal{M}$.

In particular, these orbits depend only on the Weyl-equivalence class of $\mathcal{G}$.

This observation can be generalized as follows. Recall that a smooth map $f : (\mathcal{M}_1, \mathcal{G}_1) \to (\mathcal{M}_2, \mathcal{G}_2)$ between Riemannian manifolds $(\mathcal{M}_1, \mathcal{G}_1)$ and $(\mathcal{M}_2, \mathcal{G}_2)$ is called a conformal diffeomorphism if $f^*(\mathcal{G}_2) = \Omega \mathcal{G}_1$ for some positive smooth function $\Omega$ on $\mathcal{M}_1$.

**Definition 3.1.** Two scalar triples $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ are called smoothly conformally equivalent if there exists a conformal diffeomorphism $f : (\mathcal{M}_1, \mathcal{G}_1) \to (\mathcal{M}_2, \mathcal{G}_2)$ such that $V_1 = V_2 \circ f$. In this case, $f$ is called a (smooth) conformal equivalence between the two triples. A conformal automorphism of a scalar triple $(\mathcal{M}, \mathcal{G}, V)$ is a conformal equivalence from $(\mathcal{M}, \mathcal{G}, V)$ to itself.

With this definition, we show in Appendix B that the gradient flows of conformally-equivalent scalar triples are smoothly topologically equivalent. In particular, the smooth topological equivalence class of the gradient flow of a scalar triple is invariant under conformal automorphisms of that triple. Since in the gradient flow of the function $V = M_0 \sqrt{2 \Phi}$ with respect to $\mathcal{G}$ gives the first order IR approximant of cosmological curves for the model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$, we conclude that:

*Up to topological equivalence, the first order IR approximant of the cosmological flow of a multifield cosmological model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ depends only on the conformal equivalence class of the effective scalar triple $(\mathcal{M}, \mathcal{G}, V)$, where $V \overset{\text{def}}{=} M_0 \sqrt{2 \Phi}$. Up to curve-dependent increasing reparameterization of the flow curves, this approximant depends only on $\mathcal{M}, V$ and on the Weyl-equivalence class of $\mathcal{G}$.***

### 3.4 Dynamical universality classes

Consider two cosmological models parameterized by $\mathcal{M}_i = (M_0i, \mathcal{M}_i, \mathcal{G}_i, \Phi_i) \ (i = 1, 2)$ whose tangent bundle projections and Levi-Civita connections we denote by $\pi_i$ and $\nabla_i$. Let $\mathcal{G}_{0i} \overset{\text{def}}{=} \frac{1}{M_{0i}} \mathcal{G}_i$ be the rescaled scalar manifold metrics and $V_i \overset{\text{def}}{=} M_{0i} \sqrt{2 \Phi_i}$ be the effective scalar potentials of the two models. Notice that $\nabla_i$ coincides with the Levi-Civita connection of $\mathcal{G}_{0i}$. 
**UV conjugations and UV equivalences.** Recall the map (1.16) induced between the sets of smooth curves in two manifolds by a smooth semispray map.

**Definition 3.2.** A smooth geodesic conjugation between two connected Riemannian manifolds \((M_1, \mathcal{G}_{01})\) and \((M_2, \mathcal{G}_{02})\) is a smooth semispray map \(f : TM_1 \to TM_2\) which restricts to a topological conjugation between their normalized geodesic flows, i.e. \(f\) restricts to a diffeomorphism \(f_S : STM_1 \to STM_2\) between the unit sphere bundles \(STM_1\) and \(STM_2\) and has the property that \(\hat{f}(\psi_1)\) is a normalized geodesic of \((M_2, \mathcal{G}_{02})\) whenever \(\psi_1\) is a normalized geodesic of \((M_1, \mathcal{G}_{01})\).

**Example 3.3.** Consider a diffeomorphism \(f_0 : M_1 \to M_2\). Then it is easy to see that \(f_0\) maps normalized geodesics of \((M_1, \mathcal{G}_{01})\) into normalized geodesics of \((M_2, \mathcal{G}_{02})\) iff it is an isometry, which in turn is equivalent with the condition that its differential \(df_0\) maps \(STM_1\) into \(STM_2\). In this case, the restriction \(df_0|_{STM_1} : STM_1 \to STM_2\) is a geodesic conjugation.

**Remark 3.4.** Recall that an affine mapping from \((M_1, \mathcal{G}_{01})\) to \((M_2, \mathcal{G}_{02})\) is a diffeomorphism \(f_0 : M_1 \to M_2\) such that \(f_0^* (\nabla_2) = \nabla_1\). Such a map takes affinely parameterized geodesics of \((M_1, \mathcal{G}_{01})\) into affinely parameterized geodesics of \((M_2, \mathcal{G}_{02})\) but it need not take normalized geodesics into normalized geodesics. An affine mapping from a Riemannian manifold \((M, \mathcal{G}_0)\) to itself is called an affine transformation of \((M, \mathcal{G}_0)\); such transformations form a Lie group. An infinitesimal affine mapping is sometimes called an affine collineation. When \((M, \mathcal{G}_0)\) is complete, irreducible and of dimension greater than one, any affine transformation of \((M, \mathcal{G}_0)\) is an isometry (see [48, 49]).

**Definition 3.5.** A smooth geodesic equivalence between two Riemannian manifolds \((M_1, \mathcal{G}_{01})\) and \((M_2, \mathcal{G}_{02})\) is a smooth semispray \(f : TM_1 \to TM_2\) which identifies geodesic orbits, i.e. \(\psi_1 : I_1 \to M_1\) is an arbitrarily parameterized geodesic of \((M_1, \mathcal{G}_{01})\) iff \(\hat{f}(\psi_1)\) is an arbitrary parameterized geodesic of \((M_2, \mathcal{G}_{02})\).

**Example 3.6.** Consider a smooth geodesic (a.k.a. projective) mapping \(f_0 : (M_1, \mathcal{G}_{01}) \to (M_2, \mathcal{G}_{02})\), i.e. a smooth diffeomorphism from \(M_1\) to \(M_2\) which maps the geodesic orbits of \((M_1, \mathcal{G}_{01})\) into those of \((M_2, \mathcal{G}_{02})\), a condition which amounts to the requirement that the connections \(\nabla_1\) and \(f_0^* (\nabla_2)\) be projectively-equivalent. Then \(df_0 : TM_1 \to TM_2\) is a smooth geodesic equivalence. The study of geodesic mappings is a classical subject in Riemannian geometry (see, for example [50]) which is intimately connected (see [51]) to Cartan’s projective differential geometry [52].

The results of Section 2 motivate the following:

**Definition 3.7.** Consider two classical cosmological models parameterized by \(\mathfrak{M}_1 = (M_{01}, \mathcal{M}_1, \mathcal{G}_1, \Phi_1)\) and \(\mathfrak{M}_2 = (M_{02}, \mathcal{M}_2, \mathcal{G}_2, \Phi_2)\) and let \(V_i = M_{0i} \sqrt{2\Phi_i}\) and \(\mathcal{G}_{0i} = \frac{1}{M_{0i}} \mathcal{G}_i\) \((i = 1, 2)\). We say that the models are:
• **smoothly UV conjugate** and write $\mathcal{M}_1 \equiv_{UV} \mathcal{M}_2$ if there exists a smooth geodesic conjugation:

$$f : (\mathcal{M}, \mathcal{G}_{01}) \to (\mathcal{M}, \mathcal{G}_{02}) .$$

• **smoothly UV equivalent** and write $\mathcal{M}_1 \sim_{UV} \mathcal{M}_2$ if there exists a smooth geodesic equivalence:

$$f : (\mathcal{M}, \mathcal{G}_{01}) \to (\mathcal{M}, \mathcal{G}_{02}) .$$

In the situations above, $f$ is called respectively a smooth *UV conjugation* or *UV equivalence* between $\mathcal{M}_1$ and $\mathcal{M}_2$.

**IR conjugations and IR equivalences.** The results of Section 2 show that the IR limits of the two models can be identified when the gradient flows of the scalar triples $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ are smoothly topologically conjugate, i.e. related by a diffeomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ (see Appendix A). This amounts to the condition that the vector fields $\text{grad}_1 V_1$ and $\text{grad}_2 V_2$ be $f$-related, i.e. $\text{grad}_2 V_2 = f_\sharp(\text{grad}_1 V_1)$, where $f_\sharp : \mathcal{X}(\mathcal{M}_1) \to \mathcal{X}(\mathcal{M}_2)$ denotes the push-forward of vector fields by $f$:

$$f_\sharp(X) = (df) \circ X \circ f^{-1} \forall X \in \mathcal{X}(\mathcal{M}_1) .$$

Moreover, the two gradient flows can be identified through $f$ up to increasing re-parameterization (and hence are smoothly topologically equivalent, see Appendix A) iff $\frac{1}{\Omega} \text{grad}_1 V_1$ is $f$-related to $\text{grad}_2 V_2$ for some positive smooth function $\Omega$ defined on $\mathcal{M}_1$ (which induces a Weyl rescaling $\mathcal{G}_1 \to \Omega\mathcal{G}_1$ of $\mathcal{G}_1$) i.e. iff we have $f_\sharp(\frac{1}{\Omega} \text{grad}_1 V_1) = \text{grad}_2 V_2$. These observations motivate the following:

**Definition 3.8.** Consider two scalar triples $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$. A smooth diffeomorphism $f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \to (\mathcal{M}_2, \mathcal{G}_2, V_2)$ is called:

• **smooth gradient conjugation**, if it satisfies the condition:

$$f_\sharp(\text{grad}_1 V_1) = \text{grad}_2 V_2 . \quad (3.5)$$

• **smooth gradient equivalence**, if it satisfies the condition:

$$f_\sharp(\Omega^{-1} \text{grad}_1 V_1) = \text{grad}_2 V_2 \quad (3.6)$$

for some positive smooth function $\Omega : \mathcal{M}_1 \to \mathbb{R}_{>0}$.

The two scalar triples are called smoothly *gradient conjugate/equivalent* if there exists a smooth gradient conjugation/equivalence $f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \to (\mathcal{M}_2, \mathcal{G}_2, V_2)$.

Since the differential pull-back $f^* : \mathcal{X}(\mathcal{M}_2) \to \mathcal{X}(\mathcal{M}_1)$ is given by $f^* = (f_\sharp)^{-1}$, condition (3.6) is equivalent with:

$$\text{grad}_1 V_1 = \Omega \text{grad}_{f^\ast(\mathcal{G}_2)} f^\ast V_2 . \quad (3.7)$$

where $f^\ast(V_2) = V_2 \circ f$. Gradient conjugation and equivalence are equivalence relations on scalar triples.

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Example 3.9. Any conformal equivalence of scalar triples \( f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \rightarrow (\mathcal{M}_2, \mathcal{G}_2, V_2) \) is a gradient equivalence, as can be seen from its defining relations \( f^*(\mathcal{G}_2) = \Omega \mathcal{G}_1 \) and \( f^*(V_2) = V_1 \). However, condition (3.6) admits solutions \( f \) which need not be conformal equivalences of triples. In local coordinates \((x^1, \ldots, x^d)\) on \( \mathcal{M}_1 \) and \((y^1, \ldots, y^d)\) on \( \mathcal{M}_2 \), this condition takes the form:

\[
\frac{1}{\Omega(x)} G_1^{jk}(x) \partial_j f^i(x) \partial_k V_1(x) = G_2^{ij}(x)(\partial_i V_2)(f(x)) ,
\]

which is weaker than the defining conditions of a conformal transformation from \((\mathcal{M}_1, \mathcal{G}_1, V_1)\) to \((\mathcal{M}_2, \mathcal{G}_2, V_2)\):

\[
(\mathcal{G}_2)_{ij}(f(x)) \partial_k f^i(x) \partial_j f^j(x) = \Omega(x)(\mathcal{G}_1)_{kl}(x) \quad \text{and} \quad V_2(f(x)) = V_1(x) .
\]

Here \( f^i(x) = y^i(f(x)) \). If one chooses coordinates such that \( x^i = y^i(f(x)) \) then \( f \) is locally given by \( f^i(x) = x^i \) and (3.8) becomes:

\[
\frac{1}{\Omega(x)} G_1^{ij}(x) \partial_j V_1(x) = G_2^{ij}(x) \partial_j V_2(x)
\]

while (3.9) reduces to:

\[
\frac{1}{\Omega(x)} G_1^{ij}(x) = G_2^{ij}(x) \quad \text{and} \quad V_1(x) = V_2(x) .
\]

A gradient conjugation of a scalar triple to itself is called a *gradient symmetry*, while a gradient equivalence to itself is called a *weak gradient symmetry* of that triple. The condition that a diffeomorphism \( f : \mathcal{M} \rightarrow \mathcal{M} \) be a weak gradient symmetry of the scalar triple \((\mathcal{M}, \mathcal{G}, V)\) reduces to the requirement that the gradient vector field \( v \overset{\text{def}}{=} \text{grad}_G V \) (which has components \( v^i(x) = G^{ij}(x) \partial_j V(x) \)) satisfies \( f_*(\frac{1}{\Omega} v) = v \), i.e.:

\[
\partial_j f^i(x) v^j(x) = \Omega(x) v^i(f(x)) .
\]

With these preparations, the remarks made above motivate the following:

**Definition 3.10.** Consider two classical cosmological models parameterized by \( \mathfrak{M}_1 = (\mathcal{M}_{01}, \mathcal{G}_1, \Phi_1) \) and \( \mathfrak{M}_2 = (\mathcal{M}_{02}, \mathcal{G}_2, \Phi_2) \) and let \( V_i \overset{\text{def.}}{=} M_{0i} \sqrt{2 \Phi_i} \) \((i = 1, 2)\). We say that the models are

- **smoothly IR conjugate** and write \( \mathfrak{M}_1 \equiv_{\text{IR}} \mathfrak{M}_2 \) if there exists a smooth gradient conjugation:

  \[
  f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \rightarrow (\mathcal{M}_2, \mathcal{G}_2, V_2) .
  \]

- **smoothly IR equivalent** and write \( \mathfrak{M}_1 \sim_{\text{IR}} \mathfrak{M}_2 \) if there exists a smooth gradient equivalence

  \[
  f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \rightarrow (\mathcal{M}_2, \mathcal{G}_2, V_2) .
  \]
In the situations above, $f$ is called respectively a smooth IR conjugation or IR equivalence between $\mathcal{M}_1$ and $\mathcal{M}_2$.

A (necessarily strict) IR or UV smooth conjugation between a model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ and itself is called an IR or UV symmetry of that model.

The binary relations introduced above are equivalence relations on the class of multifield cosmological models; in fact, they are the isomorphism relations of associated groupoids. The equivalence classes of $\sim_{\text{UV}}$ and $\sim_{\text{IR}}$ are called smooth UV and IR cosmological universality classes.

### 3.5 The late time infrared phase structure

As explained in Section 2, the cosmological curves of a model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ are approximated by gradient flow curves of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$ in the IR limit. With sufficiently strong assumptions on the behavior of $\mathcal{M}$ and $V$ near the Freudenthal ends of $\mathcal{M}$, the $\omega$-extended limit points of such curves coincide with those of certain gradient flow curves of $(\mathcal{M}, \mathcal{G}, V)$. In good cases, a cosmological curve $\varphi$ approaches a gradient flow curve $\varphi_{\text{IR}}^{(+)}$ in the distant future and hence $\varphi$ can be replaced in the infrared by $\varphi_{\text{IR}}^{(+)}$ for late times.

This statement is familiar when $\mathcal{M}$ is compact and $V$ is a Morse function. In that case, the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$ is complete and hence its maximal flow curves are defined on the entire real axis. Moreover, any non-constant maximal gradient flow curve $\eta$ satisfies:

$$\exists \lim_{t \to -\infty} \eta(q) = c_i \quad \text{and} \quad \exists \lim_{t \to \infty} \eta(q) = c_f,$$

where $c_i$ and $c_f$ are critical points of $V$ whose Morse indices obey $\text{ind}(c_i) > \text{ind}(c_f)$. In this situation, the moduli space of gradient flow orbits has components indexed by the pair $(c_i, c_f)$, while the early and late time behavior of $\eta$ depend respectively on $c_i$ and $c_f$. In this case, the late time infrared dynamics of the cosmological model parameterized by $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ has a phase structure in which different phases are indexed by the critical points of $V$; every cosmological curve “settles” at late times in a phase indexed by a critical point $c$ of Morse index greater than $d \overset{\text{def.}}{=} \dim \mathcal{M}$. The asymptotic behavior of the cosmological flow for late times in the phase indexed by $c$ depends on the Morse index of $c$ and on the behavior of $\mathcal{G}$ near $c$.

This simple characterization of limit points of the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$ no longer holds when $V$ is Morse but $\mathcal{M}$ is non-compact. With strong enough conditions on the topology of $\mathcal{M}$ and on the asymptotic behavior of $V$, the gradient flow curves will still have $\alpha$- and $\omega$-extended limits but each of these can be either a critical point of $V$ or a Freudenthal end of $\mathcal{M}$. In this case, the IR phases of the model are indexed by critical points of $V$ and by those ends of $\mathcal{M}$ which can arise as $\omega$-limit points of a cosmological curve.
For general effective potentials $V$ with non-compact $\mathcal{M}$, a maximal cosmological curve can have more than one extended $\omega$-limit point. In this situation, IR phases can be classified partially by the allowed $\omega$-limit sets, but specifying these need not provide a full classification and more detailed analysis is required. In particular, the classification of such phases can be quite involved and a simple relation to the late time behavior of the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$ may not exist.

4 IR universality of two-field models with hyperbolic scalar manifold

The IR conformal invariance property discussed in Subsection 3.3 has a striking implication for two-field cosmological models when combined with the uniformization theorem of Poincaré. According to the latter, the Weyl equivalence class of any Riemannian metric $\mathcal{G}$ defined on a borderless connected surface $\Sigma$ contains a unique complete metric $G$ (called the uniformizing metric of $\mathcal{G}$) of constant Gaussian curvature $K$ equal to one of the values $-1$, $0$ or $+1$. The case $K = -1$ is generic; in this case, the metric $\mathcal{G}$ (and its conformal class) is called hyperbolizable and $G$ is called the hyperbolization of $\mathcal{G}$. The cases $K = +1$ and $K = 0$ occur only for seven topologies, as follows:

- When $K = +1$, the surface $\Sigma$ must be diffeomorphic with the 2-sphere $S^2$ or with the real projective plane $\mathbb{RP}^2 \simeq S^2/\mathbb{Z}_2$. Both of these surfaces admit a unique metric of unit Gaussian curvature.

- When $K = 0$, the surface $\Sigma$ must be diffeomorphic with the 2-torus $T^2$, the Klein bottle $K^2 = \mathbb{RP}^2 \times \mathbb{RP}^2 \simeq T^2/\mathbb{Z}_2$, the open annulus $A^2$ (which is diffeomorphic with the open cylinder and with the twice punctured sphere), the open Möbius strip $\mathbb{M}^2 \simeq A^2/\mathbb{Z}_2$ (which is diffeomorphic with the once-punctured real projective plane) or with the plane $\mathbb{R}^2$. The 2-torus admits a three-parameter family of flat metrics while the Klein bottle admits a two-parameter family of such. The open annulus and open Möbius strip admit a one-parameter family of complete flat metrics while $\mathbb{R}^2$ admits a unique complete flat metric.

The plane $\mathbb{R}^2$, the open annulus $A^2$ and the open Möbius strip $M^2$ are the only three surfaces which admit two types of uniformizing metrics, namely a Riemannian metric defined on one of these surfaces is uniformized either by a complete flat metric or by a hyperbolic metric depending on its conformal class. Namely:

- The plane $\mathbb{R}^2$ admits both a complete flat metric and a hyperbolic metric (the Poincaré metric).

- The open annulus $A^2$ admits a one-parameter family of complete flat metrics, a hyperbolic metric which makes it into the hyperbolic punctured disk and
a one-parameter family of hyperbolic metrics which produce the hyperbolic annuli.

- The open Möbius strip \( M^2 \) admits a one-parameter family of complete flat metrics and a one-parameter family of hyperbolic metrics which is obtained by quotienting the hyperbolic annuli through a \( \mathbb{Z}_2 \) group of isometries.

Surfaces on which any metric is hyperbolizable are called of \textit{general type}; these include all borderless connected surfaces except \( S^2, \mathbb{R}P^2, T^2, K^2, A^2, M^2 \) and \( \mathbb{R}^2 \). When \( \Sigma \) is diffeomorphic with \( A^2, M^2 \) or \( K^2 \), only certain conformal classes are hyperbolizable, while the remaining conformal classes uniformize to a complete flat metric. When \( \Sigma \) is diffeomorphic with \( T^2 \) or \( K^2 \), every conformal class uniformizes to a complete flat metric, while when it is diffeomorphic with \( S^2 \) or \( \mathbb{R}P^2 \) any conformal class uniformizes to a complete metric with Gaussian curvature equal to \( +1 \).

Consider a two-field cosmological model parameterized by \((M_0, \Sigma, G, \Phi)\). The observations of Subsection 3.3 imply that the gradient flow of \( V = M_0 \sqrt{2\Phi} \) computed with respect to \( G \) has the same oriented orbits as the gradient flow of \( V \) computed with respect to the uniformizing metric \( G \) and hence differs from the latter only by an increasing reparameterization of the gradient flow curves. Thus models whose scalar field metric has constant Gaussian curvature equal to \(-1, 0\) or \(+1\) provide distinguished representatives of the IR universality classes of all two-field cosmological models. More precisely:

\textit{Up to (curve-dependent) increasing reparameterization, the first order IR approximant of the cosmological flow of a two-field model with scalar triple \((\Sigma, G, \Phi)\) and rescaled Planck mass \( M_0 \) is described by the gradient flow of the scalar triple \((\Sigma, G, V)\), where \( G \) is the uniformizing metric of \( G \) and \( V = M_0 \sqrt{2\Phi} \) is the classical effective scalar potential of the model.}

When \( \Sigma \) is of general type, this statement implies that IR universality classes of two-field models with target \( \Sigma \) are classified by gradient equivalence classes of hyperbolic scalar triples \((\Sigma, G, V)\); more generally, this holds for two-field models with hyperbolizable scalar manifold \((\Sigma, G)\). Two-field models whose complete scalar manifold metric \( G \) has constant negative curvature are called \textit{generalized two-field \( \alpha \)-attractor models}. In this case, one has \( G = 3\alpha G \) for some positive constant \( \alpha \) and some hyperbolic metric \( G \). Such models were introduced and studied in [20, 34, 35] and form a very wide extension of the two-field cosmological \( \alpha \)-attractor models previously introduced in [37], which correspond to the topologically trivial case when \((\Sigma, G)\) is the Poincaré disk. It follows that generalized two-field \( \alpha \)-attractor models are \textit{IR universal} among two-field cosmological models with hyperbolizable target manifold. This gives a conceptual reason to single out generalized two-field \( \alpha \)-attractor models for special study.
Remark 4.1. The notion of IR universality employed in this paper differs from the colloquial use of the same word in the \(a\)-attractor literature (see for example [47]), where certain predictions extracted from models whose scalar manifold is a Poincaré disk were claimed to be ‘universal’ in the sense that they are insensitive to small enough perturbations of the scalar potential (a notion which corresponds to structural stability of the cosmological dynamical system and differs conceptually from our notion of IR universality). Poincaré disk models cannot be IR universal even in the class of models with contractible target, since the open disk \(D^2 \simeq \mathbb{R}^2\) admits both metrics which uniformize to the Poincaré metric and metrics which uniformize to the complete flat metric\(^7\). As we show in a separate publication, IR universality classes of two-field models with hyperbolizable scalar manifold are much more complicated than one might expect from such works. When \(\Sigma\) is not topologically trivial, such models can have distinct late time IR phases associated to the Freudenthal ends of \(\Sigma\), all of which differ from the single IR phase associated to the plane end of the Poincaré disk; in fact, the latter is the only hyperbolic surface which admits a plane end and hence Poincaré disk models are uniquely special among two-field models with hyperbolizable target manifold. Notice that Freudenthal end phases arise in addition to those indexed by the critical points of the scalar potential.

5 Conclusions and further directions

We studied the scaling behavior and scaling limits of classical multifield cosmological models with scalar triple \((\mathcal{M}, G, \Phi)\) and rescaled Planck mass \(M_0\) and their dynamical renormalization group, showing that the latter deforms the cosmological flow of such models into a family of flows defined on \(T\mathcal{M}\) which interpolates between a modification of the geodesic flow of the scalar manifold \((\mathcal{M}, G)\) and a certain lift of the gradient flow of the classical effective potential \(V \overset{\text{def}}{=} M_0 \sqrt{2\Phi}\) on this Riemannian manifold. Using the invariance of oriented gradient flow orbits under Weyl transformations of \(G\), we found that the first order IR approximants of cosmological orbits are insensitive to such transformations. This allowed us to give a mathematical description of dynamical UV and IR universality classes of classical cosmological models. In the infrared limit, the late time dynamics of such models is partitioned into “phases” which in good cases are indexed by critical points of \(V\) and by Freudenthal ends of \(\mathcal{M}\). These results provide a realization of ideas familiar from the theory of critical phenomena within the context of classical cosmology and open up new directions for the study and classification of such models.

Since the UV and IR approximations are controlled by different conditions from those governing the slow roll approximation and its slow roll - slow turn variant, the

\(^7\text{While the Poincaré metric is conformally flat, it is not conformally equivalent to the complete flat metric on } D^2 \simeq \mathbb{R}^2.\)
study of the IR and UV behavior of classical cosmological models requires new ideas and tools. The UV approximation recovers a modification of the geodesic flow of \((\mathcal{M}, \mathcal{G})\), thus making contact with a classical subject in Riemannian geometry and dynamical systems theory. The IR approximation recovers the gradient flow of the classical effective potential \(V\), thus making contact with another classical subject. Since the scalar manifold \((\mathcal{M}, \mathcal{G})\) is generally non-compact, one has to analyze such flows without the compactness assumptions made in most of the mathematics literature – and hence the study of cosmological scaling limits does not reduce to a simple application of known results. For example, one cannot directly apply classical results of Morse and gradient flow theory to the study of the IR limit even when a natural smooth compactification of \(\mathcal{M}\) exists and \(\Phi\) admits a smooth and Morse extension to that compactification, because the scalar manifold metric \(\mathcal{G}\) does not generally extend.

For two-field models, the target manifold \(\mathcal{M}\) is a connected surface \(\Sigma\). In this case, our results and the uniformization theorem of Poincaré imply that the IR behavior of the model parameterized by \((M_0, \Sigma, \mathcal{G}, \Phi)\) coincides with that of the model parameterized by \((M_0, \Sigma, G, \Phi)\) to first order in the scale expansion, where \(G\) is the uniformizing metric of \(\mathcal{G}\). This implies that two-field models whose scalar manifold metric has Gaussian curvature equal to \(-1, 0\) or \(1\) are IR universal in the Wilsonian sense that they provide distinguished representatives of the infrared universality classes of all two-field models. This gives a powerful conceptual reason to single out such models for detailed study. In [53], we study the IR behavior of a very large class of two-field cosmological models using the methods and ideas of the present paper.

One direction for further research concerns the systematic study of higher orders of the UV and IR expansions, on which we hope to report shortly. In this regard, it would be interesting to extract precise asymptotic bounds which control the error terms at each order. More ambitiously, one can look for improvements of these expansions which produce uniformly convergent series; such improvements can be extracted in principle using the method of multiple scales (see [54–57]). In this context, the renormalization group action constructed in this paper relates to the work of [58–61].

It would also be interesting to study the implications of scale expansions for cosmological perturbation theory with a view toward developing new approximation schemes and constructing effective descriptions which take into account the fact that the underlying classical dynamics of cosmological models has a dynamical RG flow of its own. In our opinion, this could address the conceptual criticism of current approaches to effective descriptions in cosmology made in papers such as [62, 63].

Since the dynamics of multifield cosmological models admits a constrained Hamiltonian description in the minisuperspace formalism, the RG flow and scale approximations considered in this paper can be reformulated in that framework, which allows
one to use methods from the theory of Hamiltonian systems. It would be interesting
to see what insight can be gained by pursuing Wilsonian ideas in that context.

Finally, one can expect that the models with hidden symmetries considered in
[39, 40, 42] (see [64, 65] for some phenomenological implications) play a special
role in the infrared limit, by analogy to similar situations in the theory of critical
phenomena; it would be interesting to address this conjecture.

When the target manifold $M$ has dimension greater than two, the existence of
a metric $G$ with special properties in the Weyl-equivalence class of $\mathcal{G}$ is a classical
problem. When $M$ is non-compact and one requires $G$ to be complete and of con-
stant scalar curvature, this is the Yamabe problem for non-compact manifolds, which
generally has a negative answer unless one imposes further restrictions on $(M, \mathcal{G})$.
When $M$ is three-dimensional, one can instead use Thurston uniformization, which
is different in character. It would be interesting to see if analogues of our arguments
exist that would allow one to find good representatives of universality classes of mo-
dels with more than two scalar fields when appropriate conditions are imposed on
the scalar manifold and potential.

A Topological and smooth equivalence of dynamical systems

Recall that a (smooth) autonomous dynamical system is a pair $(M, X)$, where $M$ is
a manifold and $X$ is a smooth vector field defined on $M$ (see [17]).

Definition A.1. A flow curve of the autonomous dynamical system $(M, X)$ is a
smooth integral curve $\gamma : I \to M$ of the vector field $X$, i.e. a solution of the equation:

$$\frac{d\gamma(t)}{dt} = X(\gamma(t)) \quad \forall t \in I$$

where $I$ is a non-degenerate interval, i.e. a non-empty (open, closed or semi-closed)
interval which is not reduced to a point. The image $\text{im} \gamma \equiv \gamma(I) \subset M$ of a flow
curve is called a flow orbit.

Definition A.2. The flow of the autonomous dynamical system $(M, X)$ is the flow
$\Pi : \mathcal{D} \to M$ of the vector field $X$ considered on its maximal domain of definition
$\mathcal{D} \subset \mathbb{R} \times M$.

Recall that $\mathcal{D}$ is an open subset of $\mathbb{R} \times M$ which contains the set $\widehat{M} \equiv \{0\} \times M$.
It is fibered over $M$ with fiber at $m \in M$ given by the interval of definition $I_m$ of
the maximal integral curve $\gamma_m : I_m \to M$ of $X$ which satisfies $0 \in I_m$ and $\gamma(0) = m$.
Notice that the interval $I_m$ is open for all $m \in M$. By definition, we have:

$$\Pi(q, m) = \gamma_m(q) \quad \forall (q, m) \in \mathcal{D}$$
We denote by $\pi : \mathcal{D} \to M$ the projection on the second factor. For each $m \in M$, we have $\Pi(0, m) = m$. Moreover, for all $t_0 \in I_m$ we have $I_{\gamma_m(t_0)} = I_m - t_0$ and for all $t \in I_{\gamma_m(t_0)}$ we have:

$$\Pi(t, \Pi(t_0, m)) = \Pi(t + t_0, m) .$$

**Definition A.3.** Let $(M_1, X_1)$ and $(M_2, X_2)$ be autonomous dynamical systems with flows $\Pi_k : \mathcal{D}_k \to M_k$ and domain projections $\pi_k : \mathcal{D}_k \to M_k$ ($k = 1, 2$). Denote by $I_m^{(k)} \overset{\text{def.}}{=} \pi_k^{-1}(m_k)$ the fiber of $\mathcal{D}_k$ at $m_k \in M_k$. Let $h : M_1 \to M_2$ be a homeomorphism and $f : \mathcal{D}_1 \to \mathcal{D}_2$ be an unbased isomorphism of topological fiber bundles above $h$, thus:

$$f(t, m) = (f_m(t), h(m)) \ \forall m \in M_1 \ \forall t \in I_m^{(1)} ,$$

where $f_m : I_m^{(1)} \to I_{h(m)}^{(2)}$ is a homeomorphism for all $m \in M_1$. The pair $(f, h)$ is called:

1. **topological equivalence**, if the following conditions are satisfied:

   - We have:

     $$\Pi_2 \circ f = h \circ \Pi_1 ,$$

     i.e.:

     $$h(\gamma_{I_m}^{(1)}(t)) = \gamma_{I_{h(m)}}^{(2)}(f_m(t)) \ \forall m \in M \ \forall t \in I_m^{(1)} ,$$

     where $\gamma_{I_m}^{(k)} : I_m^{(k)} \to M_k$ is the flow curve of $\Pi_k$ which satisfies $\gamma_{I_m}^{(k)}(0) = m_k$.

   - $f_m : I_m^{(1)} \to I_{h(m)}^{(2)}$ is a strictly increasing function for all $m \in M$.

2. **smooth topological equivalence**, if it is a topological equivalence and the maps $f$ and $h$ are smooth diffeomorphisms.

3. **topological conjugation** if it is a topological equivalence and for all $m \in M$ we have $I_m^{(1)} = I_{h(m)}^{(2)} := I_m$ and $f_m = \text{id}_{I_m}$.

4. **smooth topological conjugation** if it is a topological conjugation and the maps $f$ and $h$ are smooth diffeomorphisms.

The two dynamical systems and their flows are called **(smoothly) topologically conjugate/equivalent** if there exists a (smooth) topological conjugation/equivalence $(f, h)$ from $\Pi_1$ to $\Pi_2$.

( Smooth) topological equivalence and conjugation are equivalence relations. If $\Pi_1$ and $\Pi_2$ are topologically equivalent through a pair $(f, h)$, then $h$ induces a bijection between the sets of orbits of $\Pi_1$ and $\Pi_2$ which preserves their orientation.
B Invariance of gradient flow orbits under conformal transformations

Proposition B.1. Let $\Omega : \mathcal{M} \to \mathbb{R}_{>0}$ be an everywhere-positive smooth function defined on $\mathcal{M}$, $\eta : I \to \mathcal{M}$ be a gradient flow curve of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$ and $t_0 \in I$ be arbitrary. Consider the increasing reparameterization $\tau : I \to J$ defined through:

$$
\tau(t) := \tau_\eta(t) \overset{\text{def.}}{=} \int_{t_0}^t dt' \Omega(\eta(t')) + C ,
$$

where $C$ is an arbitrary constant. Then the reparameterized curve $\eta_r := \eta \circ \tau^{-1} : J \to \mathcal{M}$ satisfies the gradient flow equation of the scalar triple $(\mathcal{M}, \Omega \mathcal{G}, V)$. In particular, the orbits of the gradient flows of $(\mathcal{M}, \mathcal{G}, V)$ and $(\mathcal{M}, \Omega \mathcal{G}, V)$ coincide.

Proof. Consider the Weyl transformation:

$$
\mathcal{G} \to \mathcal{G}' \overset{\text{def.}}{=} \Omega \mathcal{G} ,
$$

and notice that $\text{grad}_{\mathcal{G}'} V = \frac{1}{\Omega} \text{grad} V$. The reparameterized curve $\eta_r$ satisfies:

$$
\frac{d\tau}{dt} d\eta_r + (\text{grad} \mathcal{G} V) \circ \eta_r = 0 .
$$

(B.2)

Since $\text{grad}_{\Omega \mathcal{G}} V = \frac{1}{\Omega} \text{grad} V$, this is equivalent to the gradient flow equation of the scalar triple $(\mathcal{M}, \Omega \mathcal{G}, V)$ iff $\frac{d\tau(t)}{dt} = \Omega(\eta(t))$, which gives (B.1). $\square$

Corollary B.2. With the notations of the previous proposition, the gradient flows of the scalar triples $(\mathcal{M}, \mathcal{G}, V)$ and $(\mathcal{M}, \Omega \mathcal{G}, V)$ are smoothly topologically equivalent.

Proof. Let $\mathcal{D}$ and $\mathcal{D}_r$ be the maximal domains of definition of the gradient flows of $(\mathcal{M}, \mathcal{G}, V)$ and $(\mathcal{M}, \Omega \mathcal{G}, V)$. Consider the map $f : \mathcal{D} \to \mathcal{D}_r$ defined through:

$$
f(t, m) \overset{\text{def.}}{=} (\tau_{\eta_m}(t), m) , \quad \forall (t, m) \in \mathcal{D} ,
$$

where $\eta_m$ is the gradient flow curve of $(\mathcal{M}, \mathcal{G}, V)$ which satisfies $\eta_m(0) = m$. Then $f$ is a diffeomorphism and $(f, \text{id}_{\mathcal{M}})$ is a smooth topological equivalence from the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$ to that of $(\mathcal{M}, \Omega \mathcal{G}, V)$. $\square$

Definition B.3. A conformal equivalence from the scalar triple $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ to the scalar triple $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ is a smooth conformal diffeomorphism $\Psi : (\mathcal{M}_1, \mathcal{G}_1) \to (\mathcal{M}_2, \mathcal{G}_2)$ which satisfies the condition $V_2 \circ \Psi = V_1$. The scalar triples $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ are called conformally equivalent if there exists a conformal equivalence from $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ to $(\mathcal{M}_2, \mathcal{G}_2, V_2)$.

It is clear that conformal equivalence is an equivalence relation on the collection of all scalar triples.
Proposition B.4. Suppose that the scalar triples $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ are conformally equivalent. Then the gradient flows of $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ are smoothly topologically equivalent.

Proof. Let $\Psi : (\mathcal{M}_1, \mathcal{G}_1, V_1) \rightarrow (\mathcal{M}_2, \mathcal{G}_2, V_2)$ be a conformal diffeomorphism and let $\Pi_i : \mathcal{D}_i \rightarrow \mathcal{M}_i$ be the gradient flows of $(\mathcal{M}_i, \mathcal{G}_i, V_i)$ for $i = 1, 2$. By the definition of conformal diffeomorphisms, there exists $\Omega \in C^\infty(\mathcal{M}_1, \mathbb{R}^+)$ such that $\mathcal{G}' \overset{\text{def}}{=} \Psi^*(\mathcal{G}_2) = \Omega \mathcal{G}_1$. Thus $\Psi$ is an isometry from $(\mathcal{M}_1, \mathcal{G}')$ to $(\mathcal{M}_2, \mathcal{G}_2)$ which satisfies $\Psi^*(V_2) = V_1$, i.e. an isomorphism of scalar triples from $(\mathcal{M}_1, \mathcal{G}', V_1)$ to $(\mathcal{M}_2, \mathcal{G}_2, V_2)$. Let $\Pi' : \mathcal{D}' \rightarrow \mathcal{M}$ be the gradient flow of $(\mathcal{M}_1, \mathcal{G}', V_1)$. It is easy to see that a curve $\eta : I \rightarrow \mathcal{M}_1$ is a maximal gradient flow curve of $(\mathcal{M}_1, \mathcal{G}', V_1)$ which satisfies $\eta(0) = m$ (with $m \in \mathcal{M}_1$) iff $\eta' \overset{\text{def}}{=} \Psi \circ \eta$ is a maximal gradient flow curve of $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ which satisfies $\eta'(0) = \Psi(m)$. Thus we have $\mathcal{D}' = \mathcal{D}_2$ and $(\text{id}_{\mathcal{D}_2}, \Psi)$ is a smooth topological equivalence from $\Pi'$ to $\Pi_2$. Since $\mathcal{G}' = \Omega \mathcal{G}_1$, Corollary B.2 shows that $\Pi_1$ is smoothly topologically equivalent with $\Pi'$. Since smooth topological equivalence of flows is an equivalence relation, we conclude. \qed

Definition B.5. A conformal automorphism of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$ is a conformal equivalence from this triple to itself.

Conformal automorphisms of $(\mathcal{M}, \mathcal{G}, V)$ form the stabilizer of $V$ under the group of conformal transformations of $(\mathcal{M}, \mathcal{G})$.

Corollary B.6. The smooth topological equivalence class of the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$ is invariant under conformal automorphisms of $(\mathcal{M}, \mathcal{G}, V)$.

Proof. Follows immediately from Proposition B.4 \qed

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