Angular momentum and Horn’s problem

Alain Chenciner & Hugo Jiménez-Pérez
Observatoire de Paris, IMCCE (UMR 8028), ASD
77, avenue Denfert-Rochereau, 75014 Paris, France
chenciner@imcce.fr, jimenez@imcce.fr

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Abstract
We prove a conjecture made in [C1]: given an \(n\)-body central configuration \(X_0\) in the euclidean space \(E\) of dimension \(2p\), let \(\text{Im} F\) be the set of ordered real \(p\)-tuples \(\{\nu_1, \nu_2, \ldots, \nu_p\}\) such that \(\{\pm i\nu_1, \pm i\nu_2, \ldots, \pm i\nu_p\}\) is the spectrum of the angular momentum of some (periodic) relative equilibrium motion of \(X_0\) in \(E\). Then \(\text{Im} F\) is a convex polytope. The proof consists in showing that there exist two, generically \((p-1)\)-dimensional, convex polytopes \(P_1\) and \(P_2\) in \(\mathbb{R}^p\) such that \(P_1 \subset \text{Im} F \subset P_2\) and that these two polytopes coincide.

\(P_1\), introduced in [C1], is the set of spectra corresponding to the hermitian structures \(J\) on \(E\) which are “adapted” to the symmetries of the inertia matrix \(S_0\); it is associated with Horn’s problem for the sum of \(p \times p\) real symmetric matrices with spectra \(\sigma_-\) and \(\sigma_+\) whose union is the spectrum of \(S_0\).

\(P_2\) is the orthogonal projection onto the set of “hermitian spectra” of the polytope \(P\) associated with Horn’s problem for the sum of \(2p \times 2p\) real symmetric matrices having each the same spectrum as \(S_0\).

The equality \(P_1 = P_2\) follows directly from a deep combinatorial lemma, proved in [FFLP], which characterizes those of the sums \(C = A + B\) of two \(2p \times 2p\) real symmetric matrices \(A\) and \(B\) with the same spectrum, which are hermitian for some hermitian structure.

1 Origin of the problem: \(N\)-body relative equilibria and their angular momenta

We recall here the results of [AC] [C1] [C2] which are needed in order to understand the mechanical origin of the purely algebraic conjecture solved in the present paper: given a configuration \(x_0 = (\mathbf{r}_1, \cdots, \mathbf{r}_N) \in E^N\) of \(N\) punctual positive masses in the euclidean space \(E\), a rigid motion of the configuration under Newton’s attraction is a motion in which the mutual distances \(|\mathbf{r}_i - \mathbf{r}_j|\) between the bodies stay constant. It is proved
in \[\text{AC}\] (see also \[\text{C2}\]) that such a motion is necessarily a relative equilibrium. This implies that the motion takes place in a space of even dimension 2p, which can be supposed to coincide with E, and that, in a galilean frame fixing the center of mass at the origin, it is of the form \(x(t) = (e^{i\Omega t}B_1, \cdots, e^{i\Omega t}B_N)\), where \(\Omega\) is a 2p \(\times\) 2p-antisymmetric endomorphism of the euclidean space \(E\) which is non degenerate. Choosing an orthonormal basis where \(\Omega\) is normalized, this amounts to saying that there exists a hermitian structure on the space \(E\) of motion and an orthogonal decomposition \(E \equiv \mathbb{C}^p \equiv \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_r}\) such that

\[x(t) = (x_1(t), \cdots, x_r(t)) = (e^{i\omega t}_1x_1, \cdots, e^{i\omega t}_r x_r),\]

where \(x_m\) is the orthogonal projection on \(\mathbb{C}^{k_m}\) of the \(N\)-body configuration \(x\) and the action of \(e^{i\omega t}\) on \(x_m\) is the diagonal action on each body of the projected configuration. Such quasi-periodic motions exist only for very special configurations, called balanced configurations (see \[\text{AC} \, \text{C2}\] for their characterization). The most degenerate balanced configurations are the central configurations for which all the frequencies \(\omega_i\) are the same; this means that \(\Omega = \omega J\), with \(J\) a hermitian structure on \(E\), and the motion is

\[x(t) = (\vec{r}_1(t), \cdots, \vec{r}_N(t)) = e^{i\omega t}x_0 = (e^{i\omega t}B_1, \cdots, e^{i\omega t}B_N)\]

in the hermitian space \(E \equiv \mathbb{C}^p\); in particular, it is periodic. In a space of dimension at most 3, \(E\) is necessarily of dimension 2 and the configuration of any relative equilibrium is central.

Given a configuration \(x = (\vec{r}_1, \cdots, \vec{r}_N)\) and a configuration of velocities \(y = \dot{x} = (\vec{v}_1, \cdots, \vec{v}_N)\), both with center of mass at the origin: \(\sum_{k=1}^{N} m_k \vec{r}_k = \sum_{k=1}^{N} m_k \vec{v}_k = 0\), the angular momentum of \((x, y)\) is the bivector \(C = \sum_{k=1}^{N} m_k \vec{r}_k \wedge \vec{v}_k\). If we represent \(x\) and \(y\) by the \(2p \times N\) matrices \(X\) and \(Y\) whose \(i\)th column are respectively made of the components \((\vec{r}_1, \cdots, \vec{r}_{2p})\) and \((\vec{v}_1, \cdots, \vec{v}_{2p})\) of \(\vec{r}_i\) and \(\vec{v}_i\) in an orthonormal basis of \(E\) and if \(M = \text{diag}(m_1, \cdots, m_N)\), this bivector is represented by the antisymmetric matrix (we use the french convention \(^tX\) for the transposed of \(X\))

\[C = -XM^tY + YM^tX\]

with coefficients \(c_{ij} = \sum_{k=1}^{N} m_k (\vec{r}_{ik} v_{jk} - \vec{r}_{jk} v_{ik})\).

The dynamics of a solid body is determined by its inertia tensor (with respect to its center of mass), represented in the case of a point masses configuration \(X\) by the symmetric matrix

\[S = XM^tX\]

with coefficients \(s_{ij} = \sum_{k=1}^{N} m_k \vec{r}_{ik} \vec{r}_{jk}\), whose trace is the moment of inertia of the configuration \(x\) with respect to its center of mass. In particular, the angular momentum of a relative equilibrium solution \(X(t) = e^{i\Omega t}X_0\) is represented by the antisymmetric matrix \(C = S_0\Omega + \Omega S_0\), where \(S_0 = X_0M^tX_0\). Restricting to the case of central configurations, that is \(\Omega = \omega J\), and making \(\omega = 1\), we consider
in what follows the spectrum of $J$-skew-hermitian matrices of the form $S_0J + JS_0$ or, what amounts to the same, the spectrum of $J$-hermitian matrices\footnote{Notice that this is the same as the spectrum of the $J_0$-hermitian matrix $\Sigma = J_0^{-1}S_0J_0 + S$, where $S = RSR^{-1}$, which was considered in \cite{C1}.} of the form $J^{-1}S_0J + S_0$.

In the following, we identify $E$ with $\mathbb{R}^{2p}$ by the choice of some orthonormal basis. $\mathbb{R}^{2p}$ is endowed with its canonical basis $e_i = (0, \ldots, 1, \ldots, 0)$ and its canonical euclidean scalar product $x \cdot y = \sum_{i=1}^{2p} x_i y_i$; this allows identifying linear endomorphisms of $E = \mathbb{R}^{2p}$ and $2p \times 2p$ matrices with real coefficients. When we say that $J$ is a hermitian structure, we mean that the standard euclidean structure is given and that $J$ is a complex structure.

2 The frequency map

We recall the definition, given in \cite{C1}, of the frequency map $F$ from the set of hermitian structures on $\mathbb{R}^{2p}$ to the positive Weyl chamber $W^+_p \subset \mathbb{R}^p$: given some $2p \times 2p$ real symmetric matrix $S_0$, we consider the mapping $J \mapsto J^{-1}S_0J + S_0$ from the space of hermitian structures on $E$ to the set of $2p \times 2p$ real symmetric matrices. We are only interested in the spectra of these matrices, hence choosing an orientation for $J$ is harmless and we shall consider only those of the form $J = R^{-1}J_0R$, where $J_0$ is the “standard” structure defined by $J_0 = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ and $R \in SO(2p)$.

Two elements $R$ and $R'$ of $SO(2p)$ defining the same $J$ if and only if there exists an element $U \in U(p)$ such that $R' = U R'$. It follows that the space of (oriented) hermitian structures is identified to the homogeneous space $U(p) \backslash SO(2p)$. The symmetric matrix $J^{-1}S_0J + S_0$ is $J$-hermitian, that is, it commutes with $J$. This implies that its spectrum is real, of the form $\{\nu_1, \nu_2, \ldots, \nu_p\}$ if considered as a $p \times p$ complex matrix (for the identification of $\mathbb{R}^{2p}$ to $\mathbb{C}^p$ defined by $J$) and of the form $\{(\nu_1, \nu_2, \ldots, \nu_p), (\nu_1, \nu_2, \ldots, \nu_p)\}$ if considered as a $2p \times 2p$ real matrix (see the next section for the trivial proof).

Definition 1 The frequency mapping

$$F : U(p) \backslash SO(2p) \to W^+_p = \{(\nu_1, \ldots, \nu_p) \in \mathbb{R}^p, \nu_1 \geq \cdots \geq \nu_p\}$$

associates to each hermitian structure $J$ the ordered spectrum $\nu_1, \ldots, \nu_p$ of the $J$-hermitian matrix $J^{-1}S_0J + S_0$.

3 Hermitian spectra

Lemma 1 Let $C : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ be a symmetric endomorphism. The following assertions are equivalent:

1) There exists a hermitian structure $J = R^{-1}J_0R$ such that $C$ be $J$-hermitian (i.e. $JC = CJ$);

2) The spectrum $\sigma(C)$ of $C$ is of the form

$$\sigma(C) = \{(\nu_1, \nu_2, \cdots, \nu_p), (\nu_1, \nu_2, \cdots, \nu_p)\}.$$
Proof. Let $J = R^{-1}J_0R$; the mapping $C$ is $J$-hermitian if and only if $RCR^{-1}$ is $J_0$-hermitian. This is equivalent to the existence of an isomorphism $U \in U(p) \subset SO(2p)$ such that
\[ U^{-1}RCR^{-1}U = \text{diag}(\nu_1, \cdots, \nu_p, (\nu_1, \cdots, \nu_p)). \]
Conversely, the identity
\[ R^{-1}CR = \text{diag}(\nu_1, \cdots, \nu_p, (\nu_1, \cdots, \nu_p)) \]
implies the commutation of $R^{-1}CR$ with $J_0$ and hence the commutation of $C$ with $J = R^{-1}J_0R$.

Notations. We shall call hermitian the spectra of this form and the diagonal the linear subspace $\Delta$ of $W^+_p$ defined by
\[ \Delta = \{(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{2p} \}, \mu_1 = \mu_2, \mu_3 = \mu_4, \cdots, \mu_{2p-1} = \mu_{2p}\}. \]
Hence the ordered hermitian spectra are the ones belonging to $\Delta$.

4 Two convex polytopes
Let $S_0 : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ be a symmetric endomorphism with spectrum
\[ \sigma(S_0) = \{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2p}\}. \]
To $S_0$ we associate the subsets $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\mathbb{R}^p$ (in fact of the positive Weyl chamber $W^+_p$ of ordered real $p$-tuples), defined as follows:
1) $\mathcal{P}_1$ is the set of ordered spectra
\[ \sigma(c) = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p\} \]
of symmetric endomorphisms $c$ of $\mathbb{R}^p$ of the form $c = a + b$, where $a : \mathbb{R}^p \to \mathbb{R}^p$ and $b : \mathbb{R}^p \to \mathbb{R}^p$ are arbitrary symmetric endomorphisms with respective spectra
\[ \sigma(a) = \sigma_- := \{\sigma_1, \sigma_3, \cdots, \sigma_{2p-1}\}, \quad \sigma(b) = \sigma_+ := \{\sigma_2, \sigma_4, \cdots, \sigma_{2p}\}; \]
2) $\mathcal{P}_2$ is the set of $p$-tuples $\{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p\}$ such that
\[ \{(\nu_1, \nu_2, \cdots, \nu_p), (\nu_1, \nu_2, \cdots, \nu_p)\} \]
is the spectrum of some symmetric endomorphism $C$ of $\mathbb{R}^{2p}$ of the form $C = A + B$, where $A : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ and $B : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ are arbitrary symmetric endomorphisms with the same spectrum
\[ \sigma(A) = \sigma(B) = \sigma(S_0). \]
In other words, identifying canonically the diagonal $\Delta$ with $\mathbb{R}^p$, one can write
\[ \mathcal{P}_2 = \mathcal{P} \cap \Delta, \]
where $\mathcal{P}$ is the $(2p - 1)$-dimensional Horn polytope which describes the ordered spectra of sums $C = A + B$ of $2p \times 2p$ real symmetric matrices $A, B$ with the same spectrum as $S_0$.  

4
Lemma 2 $\mathcal{P}_1$ and $\mathcal{P}_2$ are both contained in the hyperplane of $\mathbb{R}^p$ with equation
\[ \sum_{i=1}^{p} \nu_i = \sum_{j=1}^{2p} \sigma_j. \]
Moreover, $\mathcal{P}_1$ and $\mathcal{P}_2$ are both $(p - 1)$-dimensional convex polytopes and $\mathcal{P}_1 \subset \text{Im} F \subset \mathcal{P}_2$.

Proof. The first identity comes from the additivity of the trace function. The fact that both $\mathcal{P}_1$, $\mathcal{P}$ and hence $\mathcal{P}_2 = \mathcal{P} \cap \Delta$, are convex polytopes is a general fact coming from the interpretation of the Horn problem as a moment map problem. Finally, the second inclusion comes from the very definition of $F$ and the first comes from Lemma 1 and the following identity, where $\sigma_-$ and $\sigma_+$ are considered as $p \times p$ diagonal matrices and $\rho \in SO(p)$:
\[
\left\{ \begin{array}{l}
\begin{pmatrix}
\sigma_- & 0 \\
0 & \sigma_+
\end{pmatrix} + \begin{pmatrix}
0 & -\rho^{-1} \\
\rho & 0
\end{pmatrix}^{-1} \begin{pmatrix}
\sigma_- & 0 \\
0 & \sigma_+
\end{pmatrix} \begin{pmatrix}
0 & -\rho^{-1} \\
\rho & 0
\end{pmatrix} \\
= \begin{pmatrix}
\sigma_- + \rho^{-1} \sigma_+ \rho & 0 \\
0 & \rho \sigma_- \rho^{-1} + \sigma_+
\end{pmatrix}.
\end{array}
\right.
\]

Remark. The choice of the partition $\sigma = \sigma_- \cup \sigma_+$ of $\sigma$ is dictated by the following theorem, which proves that any other partition of $\sigma$ into two subsets with $p$ elements will lead to a smaller polytope $\mathcal{P}_1$:

Theorem 3 ([FFLP Proposition 2.2]) Let $A$ and $B$ be $p \times p$ Hermitian matrices. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2p}$ be the eigenvalues of $A$ and $B$ arranged in descending order. Then there exist Hermitian matrices $\tilde{A}$ and $\tilde{B}$ with eigenvalues $\sigma_1 \geq \sigma_3 \geq \cdots \geq \sigma_{2p-1}$ and $\sigma_2 \geq \sigma_4 \geq \cdots \geq \sigma_{2p}$ respectively, such that $\tilde{A} + \tilde{B} = A + B$.

This was used in [1] to prove that $\mathcal{P}_1 = \text{Im} A$ is the image under the frequency map $F$ of the adapted hermitian structures, i.e. those $J$ which send some $p$-dimensional subspace of $\mathbb{R}^{2p}$ generated by eigenvectors of $S_0$ onto its orthogonal.

5 The projection property

In this section, we prove the

Theorem 4 The two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide.

Corollary 5 $\text{Im} F = \mathcal{P}_1 = \text{Im} A$. In other words, the image by the frequency map $F$ of the adapted structures is already the full image $\text{Im} F$.

We need recall the inductive definition of the Horn inequalities which define the Horn polytopes (see [1]). For a sum $a + b = c$ of symmetric $p \times p$ matrices with respective (ordered in decreasing order) spectra $\alpha = (\alpha_1, \cdots, \alpha_p)$, $\beta = (\beta_1, \cdots, \beta_p)$, $\gamma = (\gamma_1, \cdots, \gamma_p)$,
they read

\[(^* IJK) \quad \forall r < p, \ \forall (I, J, K) \in T^p_r, \quad \sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,\]

where \(T^p_r\) (notation of [F], noted \(LR^p_r\) by reference to Littlewood-Richardson coefficients in [FLLP]) is defined as follows: let \(U^p_r\) be the set of triples \((I, J, K)\) of subsets of cardinal \(r\) of \(\{1, 2, \cdots, p\}\) such that

\[
\sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \frac{r(r + 1)}{2}.
\]

Then set \(T^1_r = U^1_r\) and define recursively \(T^p_r\) by

\[
T^p_r = \left\{ (I, J, K) \in U^p_s, \forall s < r, \ \forall (F, G, H) \in T^r_s, \left( \sum_{i \in F} i + \sum_{j \in G} j \leq \sum_{k \in H} k + \frac{s(s + 1)}{2} \right) \right\}.
\]

An immediate computation gives the following

**Lemma 6** Let

\[
I_2 = (2i_1 - 1, 2i_2 - 1, \cdots, 2i_r - 1, 2j_1, 2j_2, \cdots, 2j_r),
\]

\[
J_2 = (2i_1 - 1, 2i_2 - 1, \cdots, 2i_r - 1, 2j_1, 2j_2, \cdots, 2j_r),
\]

\[
K_2 = (2k_1 - 1, 2k_2 - 1, \cdots, 2k_r - 1, 2k_1, 2k_2, \cdots, 2k_r).
\]

Then \((I_2, J_2, K_2) \in U^p_{2r}\).

**Proof.** It suffices to check that

\[
2 \left[ \sum_{i \in I} (2i - 1) + \sum_{j \in J} (2j) \right] = \sum_{k \in K} (2k - 1) + \sum_{k \in K} 2k + \frac{2r(2r + 1)}{2}.
\]

Now, comes the key fact:

**Theorem 7** ([FLLP], lemma 1.18) For any triple \((I, J, K)\) in \(T^p_r\), the triple \((I_2, J_2, K_2)\) is in \(T^p_{2r}\).

The proof of this theorem, which concerns the so-called “domino-decomposable Young diagrams”, is based on a version of the Littlewood-Richardson rule due to Carré and Leclerc [CL].

It implies that, for any a sum \(A + B = C\) of real symmetric \(2p \times 2p\) matrices with respective (ordered in decreasing order) spectra

\[
\hat{\alpha} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_{2p}), \quad \hat{\beta} = (\hat{\beta}_1, \cdots, \hat{\beta}_{2p}), \quad \hat{\gamma} = (\hat{\gamma}_1, \cdots, \hat{\gamma}_{2p}),
\]

\[(^* I_2, J_2, K_2)\) holds, that is

\[
\sum_{k \in K} (\hat{\gamma}_{2k - 1} + \hat{\gamma}_{2k}) \leq \sum_{i \in I} (\hat{\alpha}_{2i - 1} + \hat{\beta}_{2i - 1}) + \sum_{j \in J} (\hat{\alpha}_{2j} + \hat{\beta}_{2j}).
\]

In particular, if

\[
\hat{\alpha} = \hat{\beta} = \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_{2p}),
\]

we get that

\[
\sum_{k \in K} \frac{\hat{\gamma}_{2k - 1} + \hat{\gamma}_{2k}}{2} \leq \sum_{i \in I} \sigma_{2i - 1} + \sum_{j \in J} \sigma_{2j}.
\]
Note that the mapping
\[(\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_{2p-1}, \hat{\gamma}_{2p}) \mapsto \left(\frac{\hat{\gamma}_1 + \hat{\gamma}_2}{2}, \frac{\hat{\gamma}_1 + \hat{\gamma}_2}{2}, \cdots, \frac{\hat{\gamma}_{2p-1} + \hat{\gamma}_{2p}}{2}, \frac{\hat{\gamma}_{2p-1} + \hat{\gamma}_{2p}}{2}\right)\]
is the orthogonal projection of the ordered set \((\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_{2p-1}, \hat{\gamma}_{2p})\) on the diagonal \(\Delta\) of \(\mathbb{R}^{2p}\) defined by the equations \(\hat{\gamma}_1 = \hat{\gamma}_2 = \cdots = \hat{\gamma}_{2p} = \hat{\gamma}_{2p-1} = \hat{\gamma}_{2p}\), that is on the subset of “hermitian” spectra. Hence a paraphrase of the above theorem is

**Theorem 8** Let \(C = A + B\) be the sum of two \(2p \times 2p\) real symmetric matrices with the same spectrum \(\{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2p}\}\).

If \(\{\nu_1 \geq \cdots \geq \nu_p\}\) is the orthogonal projection on the diagonal \(\Delta \equiv \mathbb{R}^p\) of the spectrum \(\{\hat{\gamma}_1 \geq \hat{\gamma}_2 \geq \cdots \geq \hat{\gamma}_p\}\) of \(C\), that is if \(\nu_k = \frac{\hat{\gamma}_{2k-1} + \hat{\gamma}_{2k}}{2}\), the triple of ordered spectra
\[
\alpha = (\sigma_1, \sigma_2, \cdots, \sigma_{2p-1}), \quad \beta = (\sigma_2, \sigma_4, \cdots, \sigma_{2p}), \quad \gamma = (\nu_1, \nu_2, \cdots, \nu_p)
\]
satisfies the Horn inequality \((I, J, K)\).

This implies the following extremal property of the subset of “hermitian” spectra:

**Corollary 9** The orthogonal projection on the diagonal \(\Delta\) of the \((2p-1)\)-dimensional Horn polytope \(\mathcal{P} \subset \mathbb{R}^{2p}\) associated with the spectra
\[\sigma(A) = \sigma(B) = \{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2p}\}\]
coincides with the \((p-1)\)-dimensional Horn polytope \(\mathcal{P}_1 \subset \mathbb{R}^p\) associated with the spectra
\[\sigma(a) = (\sigma_1, \sigma_3, \cdots, \sigma_{2p-1}), \quad \sigma(b) = (\sigma_2, \sigma_4, \cdots, \sigma_{2p}).\]

In particular, the intersection \(\mathcal{P}_2 = \mathcal{P} \cap \Delta\) corresponding to the hermitian spectra, that is those such that \(\hat{\gamma}_1 = \hat{\gamma}_2 = \cdots = \hat{\gamma}_{2p-1} = \hat{\gamma}_{2p}\), coincides with \(\mathcal{P}_1\). Indeed, \(\mathcal{P}_2\) coincides with the projection of \(\mathcal{P}\), which itself coincides with \(\mathcal{P}_1\).

**Remark.** The equality \(\text{Im} \mathcal{F} = \mathcal{P}_2\) implies the following

**Corollary 10** Let \(C : \mathbb{R}^{2p} \to \mathbb{R}^{2p}\) be the sum \(C = A + B\) of two symmetric endomorphisms \(A\) and \(B\) with the same spectrum \(\sigma(A) = \sigma(B) = \sigma(S_0)\).

Then \(C\) is \(J\)-hermitian for some hermitian structure \(J\) on \(\mathbb{R}^{2p}\) if and only if it is conjugate by an element of \(SO(2p)\) to a matrix of the form \(S_0 + J^{-1}S_0J\), where \(J\) is a hermitian structure on \(\mathbb{R}^{2p}\).

**Note.** The proof of the above results has been written by the first author after he was convinced by the numerical experiments made by the second author that the equality \(\mathcal{P}_1 = \mathcal{P}_2\) was plausible when \(p = 3\) and more precisely that not only the intersection \(\mathcal{P}_2 = \mathcal{P} \cap \Delta\) but also the orthogonal projection of the Horn polytope \(\mathcal{P}\) on \(\Delta\) was contained in \(\mathcal{P}_1\) after the canonical identification of \(\Delta\) with \(\mathbb{R}^p\). This led first to a proof when \(p = 2\) or 3, obtained by coping directly with Horn’s inequalities and then to the discovery that the general case followed from a lemma which turned out to be exactly the lemma 1.18 of [FFLP]. The numerical experiments are described in the next section.
6 Numerical experiments

The numerical checking of the conjecture that \( \text{Im} \mathcal{F} = \mathcal{P}_1 \), was made on the matrix \( S_0 = \frac{1}{2} \text{diag} \{13, 8, 5, 3, 2, 1\} \), whose spectrum satisfies the inequalities in [C1] (section 8). We wrote a program in TRIP [GL11] producing different rotation matrices \( R \in SO(2p) \) in a random way. Starting from the canonical basis \( \xi = \{\xi_1, \ldots, \xi_m\} \), \( m = p(2p - 1) \), of \( \mathfrak{so}(2p) \), we created a list containing the \( m \) one-parameter subgroups \( G_i(t) = e^{t\xi_i} \subset SO(2p) \). We created a second list of \( m \) random values \([t_1, t_2, \ldots, t_m] \) and a random permutation \([1, 2, \ldots, m] \rightarrow [i_1, i_2, \ldots, i_m] \). The random rotation matrix was defined as

\[
R = \prod_{j=1}^{m} G_{i_j}(t_j), \quad 0 \leq t_i \leq 2\pi. \tag{1}
\]

The program which plots \( \mathcal{P}_1 \) is very simple (the fact that we replaced the conjugation of \( J_0 \) by the conjugation of \( S_0 \) is immaterial and comes from the formulation of the conjecture in [C1]):

```plaintext
create \( S_0 \) and \( J_0 \)
for \( i = 1 \) to \( N_{\text{max}} \) do
    create \( R \) and \( R^{-1} \);
    \( S = RS_0R^{-1} \);
    \( C = S - J_0SJ_0 \);
    lst = eigenvalues(C);
    plot ( lst[5], lst[3], lst[1] );
end for.
```

We have assigned the value \( N_{\text{max}} = 25000 \) obtaining the results shown in Figure 1. The figure shows also the simplex \( \gamma_1 + \gamma_2 + \gamma_3 = 1 \) and its intersection with \( W_3^+ \).

![Figure 1: \( \mathcal{P}_1 = \text{Im} \mathcal{A} \): 25000 random adapted hermitian structures](image)
The modified algorithm to estimate the shape of $P_2 = P \cap \Delta$ in $W_3^+$ is similar. For a random $R$ in $SO(6)$, the ordered spectrum $\text{spec}(C) = (\gamma_1, \ldots, \gamma_6)$ of $C = S_0 + R^{-1}S_0R$ is projected orthogonally onto the diagonal $\Delta$ by the map $\pi_\Delta : \mathbb{R}^6 \to \Delta$:

$$(\gamma_1, \gamma_2, \ldots, \gamma_6) \mapsto \left(\frac{\gamma_1 + \gamma_2}{2}, \frac{\gamma_3 + \gamma_4}{2}, \frac{\gamma_5 + \gamma_6}{2}\right).$$

At first, when $\text{spec}(C)$ was $\varepsilon$-close to $\Delta$, i.e., if $\sum_{k=1}^{4} |\gamma_{2k-1} - \gamma_{2k}|^2 < 2\varepsilon^2$ for $\varepsilon$ small, the projected point was plotted in green; otherwise it was plotted in red. No particular pattern was found for the green points meanwhile the red ones were all contained in $P_1$. The algorithm to plot $\pi_\Delta(P)$ is

```python
create S0
for i = 1 to N_max do
    create R and R^(-1);
    C = S0 + R^(-1)S0R;
    lst = eigenvalues(C);
    sort(lst);
    plot(lst[6]+lst[5], lst[4]+lst[3], lst[2]+lst[1]);
end for.
```

The results of the projection $\pi_\Delta(\text{spec}(C))$ for 50000 random rotation matrices are shown in Figure 2.

![Figure 2: Projection of $P$: 50000 random rotations](image)

The matrix $S_0$ and hence the polytope $P_1$ are the same as in Figure 1 (the interior lines correspond to the polytopes associated to different partitions of the spectrum of $S_0$, as depicted in the corresponding figure in [C1]). Recall that the polytope $P$ has dimension 5; this explains that in order to get a better filling one should have taken many more points. This was not done because the evidence was sufficiently convincing to ask for a proof.
7 Acknowledgements

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References

[AC] A. Albouy, A. Chenciner Le Problème des N corps et les distances mutuelles, Inventiones mathematicae 131 (1998), 151-184.

[C1] A. Chenciner The angular momentum of a relative equilibrium, arXiv:1102.0025v1, final version to appear in D.C.D.S.

[C2] A. Chenciner The Lagrange reduction of the N-body problem: a survey, to appear in Acta Mathematica Vietnamica

[CL] C. Carré, B. Leclerc Splitting the Square of a Schur Function into its Symmetric and Antisymmetric Parts, Journal of Algebraic Combinatorics 4 (1995), 201-231

[F] W. Fulton Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc. (N.S.) 37 no. 3, 209-249 (2000)

[FFLP] S. Fomin, W. Fulton, C.K. Li, Y.T. Poon, Eigenvalues, singular values, and Littlewood-Richardson coefficients, Amer. J. Math. 127, no. 1, 101–127 (2005)

[GL11] M. Gastineau and J. Laskar, 2011. TRIP 1.1.18, TRIP Reference manual. IMCCE, Paris Observatory. http://www.imcce.fr/trip/