Universal Compressed Sensing of Markov Sources

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Abstract

The main promise of compressed sensing is accurate recovery of high-dimensional structured signals from an underdetermined set of randomized linear projections. Several types of structure such as sparsity and low-rankness have already been explored in the literature. For each type of structure, a recovery algorithm has been developed based on the properties of signals with that type of structure. Such algorithms recover any signal complying with the assumed model from its sufficient number of linear projections. However, in many practical situations the underlying structure of the signal is not known, or is only partially known. Moreover, it is desirable to have recovery algorithms that can be applied to signals with different types of structure.

In this paper, the problem of developing “universal” algorithms for compressed sensing of stochastic processes is studied. First, Rényi’s notion of information dimension is generalized to analog stationary processes. This provides a measure of complexity for such processes and is connected to the number of measurements required for their accurate recovery. Then a minimum entropy pursuit (MEP) algorithm is proposed, and it is proven that it can reliably and robustly recover any Markov process of any order from sufficient number of randomized linear measurements, without having any prior information about the distribution of the process. An implementable version of MEP with the same performance guarantees is also provided.

I. Introduction

Consider the fundamental problem of compressed sensing (CS): a signal $x^n_o \in \mathbb{R}^n$ is observed through a data acquisition system that can be modeled as a linear projection system $y^m_o = Ax^n_o$, where $A \in \mathbb{R}^{m \times n}$ denotes the measurement matrix. The decoder is interested in recovering the signal $x^n_o$ from the measurements $y^m_o$. In CS we are usually interested in the case where $x^n_o$ is high-dimensional and the number of measurements $m$ is much smaller than the ambient dimension of the signal, i.e., $m \ll n$. In such a setup, clearly, without any extra information, it is impossible to recover $x^n_o$ from $y^m_o$, and the system of linear equations described by $y^m_o = Ax^n$ has infinitely many solutions. However, if some extra information about the structure of $x^n_o$ is available, for instance it is known that $x^n_o$ is sparse, i.e., $\|x^n_o\|_0 \ll n$, then it is known that this additional information can be exploited and $x^n_o$ can be recovered from $y^m_o$. For sparse signals, it can be proved that in fact the signal recovery can be achieved efficiently and robustly [1]–[3]. This result can be extended to several other structures as well [4]–[11]. In each of these cases, the signal is known to have some specific structure, and the decoder takes advantage of its knowledge about the structure of the signal to recover it from its linear projections.
It is generally believed that if a signal is “structured” or of “low complexity”, then it is possible to recover it from its under-determined set of linear projections; also, the required number of linear measurements for successful recovery is expected to be proportional to the “structure” of the signal instead of its dimensions. In order to develop a recovery algorithm that can be applied to signals with various structures, one fundamental question to ask is, mathematically, what does it mean for an analog signal to be of low complexity? or how can one measure the structure of a signal? Another relevant theoretical question that is also important from a practical standpoint is: is it possible to design a “universal” compressed sensing decoder that is able to recover structured signals from their randomized linear projections without knowing the underlying structure of the signal?

Universal algorithms, defined as algorithms that achieve the optimal performance without knowing the source distribution, have been well-studied in the information theory literature. Existence of such algorithms has been proved for several different problems such as lossless compression [12], [13], lossy compression [14]–[19], denoising [20], [21] and prediction [22], [23]. For some problems such as universal lossless compression there are well-known efficient algorithms such as the Lempel-Ziv algorithm [12], [13] that are employed in various commercial products, but for some others, such as universal lossy compression, designing implementable algorithms that are both efficient and optimal is still an ongoing effort.

Existence of universal algorithms for compressed sensing has recently been proved in [24]–[26]. In [24], the authors define the Kolmogorov information dimension (KID) of a deterministic analog signal as a measure of its complexity. The KID of a signal \( x^n \) is defined as the growth rate of the Kolmogorov complexity of the quantized version of \( x^n \) normalized by the log of the number of quantization levels, as the number of quantization levels grows to infinity. Employing this measure of complexity, the authors in [24] and [26] propose minimum complexity pursuit (MCP) as a universal signal recovery algorithm. MCP is based on Occam’s razor [27], i.e., among all signals satisfying the linear measurement constraints, MCP seeks the one with the lowest complexity. While MCP proves the existence of universal compressed sensing algorithms, it is not an implementable algorithm, since it is based on minimizing Kolmogorov complexity [28], [29], which is not computable.

In this paper we focus on stochastic signals and develop an “implementable” algorithm for “universal” compressed sensing of stochastic processes. As a measure of complexity for analog stochastic signals, we extend Rényi’s information dimension measure [30] and define the information dimension of a stochastic process. We prove that for an independent and identically distributed (i.i.d.) process \( \{X_i\}_{i=1}^{\infty} \), its information dimension is equal to the Rényi information dimension of \( X_1 \). It has been recently proved that for i.i.d. processes the Rényi information dimension characterizes the fundamental limits of (non-universal) compressed sensing [31].

Consider \( X^n \) generated by an analog stationary process \( \{X_i\}_{i=1}^{\infty} \), and assume that the decoder observes its linear projections \( Y^m = AX^n \), where \( m < n \). To recover \( X^n \) from \( Y^m \), in the same spirit of the MCP algorithm, we propose the minimum entropy pursuit (MEP) algorithm, which among all the signals \( x^n \) satisfying \( Y^n = Ax^n \), outputs the one whose quantized version has minimum conditional empirical entropy. We prove that asymptotically, for proper choice of the number of quantization levels and the order of the conditional empirical entropy, and having slightly more than twice the (upper) information dimension of the process times the ambient dimension of
the process randomized linear measurements, MEP presents a reliable and robust estimate of $X_n^n$. While MEP is not easy to implement, we also present an implementable version with the same asymptotic performance guarantees as MEP. The implementable version of MEP is similar to the heuristic algorithm proposed in [32] and [33] for universal compressed sensing.

The organization of the paper is as follows. Section II introduces the notation used in the paper and reviews some related background. Section III extends Rényi’s information dimension of a random variable [30] to define the information dimension of a stationary process, explores its properties and computes it for some processes. Section IV presents the MEP algorithm for universal compressed sensing of Markov sources of any order, and proves its robustness. Section V concludes the paper.

II. Background

A. Notation

Calligraphic letters such as $\mathcal{X}$ and $\mathcal{Y}$ denote sets. For a discrete set $\mathcal{X}$, let $|\mathcal{X}|$ denote the size of $\mathcal{X}$. Given vectors $u^n, v^n \in \mathbb{R}^n$, let $\langle u^n, v^n \rangle$ denote their inner product, i.e., $\langle u^n, v^n \rangle \triangleq \sum_{i=1}^{n} u_i v_i$. Given $u^n \in \mathbb{R}^n$, $\|u^n\|_0 \triangleq |\{i : u_i \neq 0\}|$ denotes its $\ell_0$-norm defined as the number of non-zero elements in $u^n$. Also, $\|u^n\|_2 \triangleq \left( \sum_{i=1}^{n} u_i^2 \right)^{0.5}$ denotes the $\ell_2$-norm of $u^n$. For $1 \leq i \leq j \leq n$, $u_j^i \triangleq (u_i, u_{i+1}, \ldots, u_j)$. To simplify the notation, $u_j \triangleq u_j^1$. The set of all finite-length binary sequences is denoted by $\{0, 1\}^*$, i.e., $\{0, 1\}^* \triangleq \bigcup_{n \geq 1} \{0, 1\}^n$. Similarly, $\{0, 1\}^\infty$ denotes the set of infinite-length binary sequences. Throughout the paper log refers to logarithm to the basis of 2 and ln refers to the natural logarithm.

Random variables are represented by upper-case letters such as $X$ and $Y$. The alphabet of the random variable $X$ is denoted by $\mathcal{X}$. Given a sample space $\Omega$ and event $\mathcal{A} \subseteq \Omega$, $\mathbb{1}_{\mathcal{A}}$ denotes the indicator function of $\mathcal{A}$. Given $x \in \mathbb{R}$, $\delta_x$ denotes the Dirac measure with an atom at $x$.

Given a real number $x \in \mathbb{R}$, $[x]$ ($\lfloor x \rfloor$) denotes the largest (the smallest) integer number smaller (larger) than $x$. Further, $[x]_b$ denotes the $b$-bit quantized version of $x$ that results from taking the first $b$ bits in the binary expansion of $x$. That is, for $x = [x] + \sum_{i=1}^{\infty} 2^{-i}(x)_i$, where $(x)_i \in \{0, 1\}$,

$$[x]_b \triangleq [x] + \sum_{i=1}^{b} 2^{-i}(x)_i.$$  

Also, for $x^n \in \mathbb{R}^n$, define

$$[x^n]_b \triangleq ([x_1]_b, \ldots, [x_n]_b).$$  

B. Empirical distribution

Consider a stochastic process $X = \{X_i\}_{i=1}^{\infty}$, with discrete alphabet $\mathcal{X}$ and probability measure $\mu(\cdot)$. The entropy rate of a stationary and ergodic process $X$ is defined as

$$\bar{H}(X) \triangleq \lim_{n \to \infty} \frac{H(X_1, \ldots, X_n)}{n}.$$  

Alternatively, for such processes [34], the entropy rate can be defined as the almost sure limit of
\[
\lim_{n \to \infty} \log \frac{1}{\mu(X^n)}.
\]

The k-th order empirical distribution induced by \( x^n \in X^n \) is defined as
\[
p_k(a^k|x^n) = \frac{\{i : x_{i-k}^{i-1} = a^k, 1 \leq i \leq n\}}{n},
\]
where we make a circular assumption such that \( x_j = x_{j+n} \), for \( j \leq 0 \). We define the conditional empirical entropy induced by \( x^n \in X^n \), \( \hat{H}_k(x^n) \), to be equal to \( H(U_{k+1}|U^k) \), where \( U^{k+1} \sim p_{k+1}(|x^n) \).

C. Universal lossless compression

Consider the problem of universal lossless compression of discrete stationary ergodic sources described as follows. A family of source codes \( \{C_n\}_{n \geq 1} \) consists of a sequence of codes corresponding to different blocklengths. Each code \( C_n \) in this family is defined by an encoder function \( f_n \) and a decoder function \( g_n \) such that
\[
f_n : X^n \to \{0,1\}^*,
\]
and
\[
g_n : \{0,1\}^* \to \hat{X}^n.
\]
Here \( \hat{X} \) denotes the reconstruction alphabet which is also assumed to be discrete and in many cases is equal to \( X \). The encoder \( f_n \) maps each source block \( X^n \) to a binary sequence of finite length, and the decoder \( g_n \) maps the coded bits back to the signal space as \( \hat{X}^n = g_n(f_n(X^n)) \). Let \( l_n(f_n(X^n)) = |f_n(X^n)| \) denote the length of the binary sequence assigned to the sequence \( X^n \). We assume that the codes are lossless (non-singular), i.e., \( f_n(x^n) \neq f_n(\tilde{x}^n) \), for all \( x^n \neq \tilde{x}^n \). A family of lossless codes is called universal, if
\[
\frac{1}{n} E[l_n(f_n(X^n))] \to \bar{H}(X),
\]
for any discrete stationary process \( X \). A family of lossless codes is called point-wise universal, if
\[
\frac{1}{n} l_n(f_n(X^n)) \to \bar{H}(X),
\]
almost surely, for any discrete stationary ergodic process \( X \).

III. INFORMATION DIMENSION OF STATIONARY PROCESSES

Consider an arbitrarily distributed analog random variable \( X \). For a positive integer \( n \), let
\[
\langle X \rangle_n \triangleq \frac{|nX|}{n}.
\]
By this definition, \( \langle X \rangle_n \) is a finite-alphabet approximation of the random variable \( X \), such that
\[
0 < X - \langle X \rangle_n \leq \frac{1}{n}.
\]

Rényi defined the upper and lower information dimensions of random variable \( X \) in terms of the entropy of \( \langle X \rangle_n \) as
\[
\bar{d}(X) = \limsup_{n \to \infty} \frac{H(\langle X \rangle_n)}{\log n},
\]
and
\[
\underline{d}(X) = \liminf_{n \to \infty} \frac{H(\langle X \rangle_n)}{\log n},
\]
respectively [30]. If \( \bar{d}(X) = \underline{d}(X) \), then the information dimension of the random variable \( X \) is defined as
\[
\underline{d}(X) = \lim_{n \to \infty} \frac{H(\langle X \rangle_n)}{\log n}.
\]

The information involved in analog signals is infinite, which means that they have infinite entropy rate. Define the \( b \)-bit quantized version of stochastic process \( X = \{X_i\}_{i=1}^{\infty} \) as \( [X]_b = \{[X_i]_b\}_{i=1}^{\infty} \). Consider a stationary process \( X = \{X_i\}_{i=1}^{\infty} \); then since \( [X]_b \) is derived from a stationary coding of \( X \), it is also a stationary process. We define the \( k \)-th order upper information dimension of a process \( X \) as
\[
\bar{d}_k(X) = \limsup_{b \to \infty} \frac{H([X_{k+1}]_b|[X^k]_b)}{b}.
\]
Similarly, the \( k \)-th order lower information dimension of \( X \) is defined as
\[
\underline{d}_k(X) = \liminf_{b \to \infty} \frac{H([X_{k+1}]_b|[X^k]_b)}{b}.
\]

**Lemma 1.** Both \( \bar{d}_k(X) \) and \( \underline{d}_k(X) \) are non-increasing in \( k \).

**Proof:** For a stationary process \( [X]_b \), for any value of \( k \),
\[
\frac{H([X_{k+2}]_b|[X^{k+1}]_b)}{b} \leq \frac{H([X_{k+2}]_b|[X^k]_b)}{b} = \frac{H([X_{k+1}]_b|[X^k]_b)}{b}.
\]
Therefore, taking \( \lim \sup \) or \( \lim \inf \) of both sides yields the desired result. \( \blacksquare \)

From Lemma 1, \( \lim_{k \to \infty} \bar{d}_k(X) \) and \( \lim_{k \to \infty} \underline{d}_k(X) \) exist. For stationary process \( X \), we define its upper information dimension and lower information dimension as
\[
\bar{d}_o(X) = \lim_{k \to \infty} \bar{d}_k(X),
\]
and
\[
\underline{d}_o(X) = \lim_{k \to \infty} \underline{d}_k(X),
\]
respectively.

**Lemma 2.** For stationary process $X$,

$$\bar{d}_o(X) = \lim_{k \to \infty} \frac{1}{k} \left( \limsup_{b \to \infty} \frac{H([X^k]_b)}{b} \right).$$

**Proof:** Since the process is stationary,

$$H([X^k]_b) = \sum_{i=1}^{k} H([X_i]_b|[X^{i-1}]_b)$$

$$= \sum_{i=1}^{k} H([X_k]_b|[X_{k-i+1}]_b)$$

$$\geq kH([X_k]_b|[X^{k-1}]_b). \quad (1)$$

Therefore,

$$\frac{1}{k} \left( \limsup_{b \to \infty} \frac{H([X^k]_b)}{b} \right) \geq \limsup_{b \to \infty} \frac{H([X_k]_b|[X^{k-1}]_b)}{b}$$

$$= \bar{d}_k(X).$$

Taking lim inf of both as $k$ grows to infinity proves that

$$\liminf_{k \to \infty} \frac{1}{k} \left( \limsup_{b \to \infty} \frac{H([X^k]_b)}{b} \right) \geq \bar{d}_o(X). \quad (2)$$

On the other hand,

$$\limsup_{b \to \infty} \frac{H([X^k]_b)}{bk} = \limsup_{b \to \infty} \frac{\sum_{i=1}^{k} H([X_k]_b|[X_{k-i+1}]_b)}{bk}$$

$$\leq \frac{1}{k} \sum_{i=1}^{k} \limsup_{b \to \infty} \frac{H([X_k]_b|[X_{k-i+1}]_b)}{b}$$

$$= \frac{1}{k} \sum_{i=0}^{k-1} \bar{d}_i(X). \quad (3)$$

Since

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \bar{d}_i(X) = \lim_{k \to \infty} \bar{d}_k(X) = \bar{d}_o(X),$$

taking lim sup of both sides of (3) yields

$$\limsup_{k \to \infty} \limsup_{b \to \infty} \frac{H([X^k]_b)}{bk} \leq \bar{d}_o(X). \quad (4)$$

The desired result follows from combining (2) and (4).

**Lemma 3.** For stationary process $X$, with $\mathcal{X} = [l, u]$, $\bar{d}_o(X) \leq 1$ and $\underline{d}_o(X) \leq 1$. Also, $\bar{d}_k(X) \leq 1$, $\underline{d}_k(X) \leq 1$. 
for all \( k \).

**Proof:** Note that
\[
H([X_{k+1}]_b|[X^k]_b) \leq \log(u-l)2^b = \log(u-l) + b,
\]
for all \( b \) and all \( k \), and therefore,
\[
\frac{1}{b}H([X_{k+1}]_b|[X^k]_b) \leq 1 + \frac{\log(u-l)}{b}.
\]
Taking \( \lim \sup \) and \( \lim \inf \) of both sides yields that \( \bar{d}_k(X) \leq 1 \) and \( \underline{d}_k(X) \leq 1 \), which in turn proves that \( \bar{d}_o(X) \leq 1 \) and \( d_o(X) \leq 1 \).

**Proposition 1.** For an i.i.d. random process \( X = \{X_i\}_{i=1}^{\infty} \), \( \bar{d}_o(X) \) (\( d_o(X) \)) is equal to \( \bar{d}(X_1) \), the Rényi upper (lower) information dimension of \( X_1 \).

**Proof:** Since the process is memoryless,
\[
\bar{d}_k(X) = \bar{d}_o(X) = \limsup_b \frac{H([X_1]_b)}{b}.
\]
As proved in Proposition 2 of [31],
\[
\bar{d}(X_1) = \limsup_b \frac{H(\langle X_1 \rangle_{2^b})}{b}.
\]
Since \( \langle X_1 \rangle_{2^b} = [X_1]_b \), this yields the desired result.

The following theorem, which follows from Theorem 3 of [30] combined with Proposition 1, characterizes the information dimension of i.i.d processes, whose components are drawn from a mixture of continuous and discrete distribution.

**Theorem 1** (Theorem 3 in [30]). Consider an i.i.d. process \( \{X_i\}_{i=1}^{\infty} \), where each \( X_i \) is distributed according to
\[
(1-p)f_d + pf_c,
\]
where \( f_d \) and \( f_c \) represent a discrete measure and an absolutely continuous measure, respectively. Assume that
\[
H([X_1]) < \infty.
\]
Then,
\[
\bar{d}_o(X) = d_o(X) = \underline{d}_o(X) = p.
\]

From Theorem 1, for an i.i.d. process with components drawn from an absolutely continuous distribution\(^1\) the information dimension is equal to one. As the weight of the continuous part \( p \) decreases, the information dimension decreases linearly with \( p \).

**Theorem 2.** Consider a first-order stationary Markov process \( X = \{X_i\}_{i=1}^{\infty} \), such that conditioned on \( X_{t-1} = x_{t-1} \), \( X_t \) has a mixture of discrete and absolutely continuous distribution equal to \( (1-p)\delta_{x_t} + pf_c \), where \( f_c \) represents the pdf of an absolutely continuous distribution over \( [0,1] \) with bounded differential entropy. Then,
\[
\bar{d}_o(X) = \underline{d}_o(X) = p.
\]

\(^1\)A probability distribution is called absolutely continuous, if it has a probability density function (pdf).
The proof of Theorem 2 is presented in Appendix A.

**Remark 1.** Processes with piecewise constant realizations are one of the standard models in signal processing [35], [36], and are studied in various problems such as denoising [37] and compressed sensing [3], [38].

**Theorem 3.** Consider a stationary Markov process of order \( l \) such that conditioned on \( X_{t-1}^{t-l} = x_{t-1}^{t-l} \), \( X_t \) is distributed as \( \sum_{i=1}^{l} a_i x_{t-i} + Z_t \), where \( a_i \in (0, 1) \), for \( i = 1, \ldots, l \), and \( Z_t \) is an i.i.d. process distributed according to \((1 - p)\delta_0 + pf_c\), where \( f_c \) is the pdf of an absolutely continuous distribution. Let \( Z \) denote the support of \( f_c \) and assume that there exists \( 0 < \alpha < \beta < \infty \), such that \( \alpha < f_c(z) < \beta \), for \( z \in Z \). Then,

\[
\bar{d}_o(X) = \tilde{d}_o(X) = p.
\]

Proof of Theorem 3 is presented in Appendix B.

**IV. Universal Compressed Sensing**

Consider a stationary process \( \{X_i\}_{i=1}^{\infty} \) such that \( \bar{d}_o(X) < 0.5 \). As we argued in Section III, since \( \bar{d}_o(X) < 1 \), we expect this process to be a structured process. In this section we explore universal compressed sensing of such processes. That is, we study acquiring a signal \( X^n \) generated by the source \( X \) through underdetermined linear projections of \( X^n \), without having any prior information about the source distribution.

**A. Noiseless measurements**

Consider the standard compressed sensing setup: instead of observing \( x_o^n \), the decoder observes \( y_o^n = Ax_o^n \), where \( A \in \mathbb{R}^{m \times n} \) denotes the linear measurement matrix, and typically \( m \ll n \). To recover \( x_o^n \) from \( y_o^n \), we employ the following optimization algorithm:

\[
\hat{x}_o^n = \arg \min_{Ax^n = y_o^n} \hat{H}_k([x^n]_b). \tag{5}
\]

We name this algorithm minimum entropy pursuit (MEP). The following theorem proves that MEP is a universal recovery algorithm for Markov processes of any order.

**Theorem 4.** Consider an aperiodic stationary Markov process \( \{X_i\}_{i=1}^{\infty} \) of order \( l \), with \( X = [0, 1] \) and upper information dimension \( \bar{d}_o(X) \). Let \( b = b_n = \lfloor \log \log n \rfloor \), \( k = k_n = o(\frac{\log n}{\log \log n}) \) and \( m = m_n \geq 2(1 + \delta)\bar{d}_o(X)n \), where \( \delta > 0 \). For each \( n \), let the entries of the measurement matrix \( A = A_n \in \mathbb{R}^{m \times n} \) be drawn i.i.d. according to \( \mathcal{N}(0, 1) \). For \( X_o^n \) generated by the source \( X \) and \( Y_o^n = AX_o^n \), let \( \hat{X}_o^n = \hat{X}_o^n(Y_o^n, A) \) denote the solution of (5), i.e., \( \hat{X}_o^n = \arg \min_{Ax^n = y_o^n} \hat{H}_k([x^n]_b) \). Then,

\[
\frac{1}{\sqrt{n}} \|X_o^n - \hat{X}_o^n\|_2 \overset{p}{\to} 0.
\]

Proof: We show that for any \( \epsilon > 0 \),

\[
P\left( \frac{1}{\sqrt{n}} \|X_o^n - \hat{X}_o^n\|_2 > \epsilon \right) \to 0,
\]
as \( n \to \infty \). Let \( X^n_o = [X^n_o]_b + q^n_o \) and \( \hat{X}^n_o = [\hat{X}^n_o]_b + \hat{q}^n_o \). By assumption, \( AX^n_o = A\hat{X}^n_o \), and therefore, 
\[
A([X^n_o]_b - [\hat{X}^n_o]_b) = A(q^n_o - \hat{q}^n_o). \]
Note that
\[
\|A([X^n_o]_b - [\hat{X}^n_o]_b)\|_2 = \|A(q^n_o - \hat{q}^n_o)\|_2 \leq \sigma_{\text{max}}(A)\|q^n_o - \hat{q}^n_o\|_2.
\]
Define the event
\[
\mathcal{E}_1 \triangleq \{ \sigma_{\text{max}}(A) \leq \sqrt{n} + 2\sqrt{m} \}.
\]
From [39], \( P(\mathcal{E}_1^c) \leq e^{-m/2} \). But,
\[
\|q^n_o - \hat{q}^n_o\|_2 \leq \|q^n_o\|_2 + \|\hat{q}^n_o\|_2 \leq \sqrt{n}2^{-b+1}.
\]
Hence, conditioned on \( \mathcal{E}_1 \),
\[
\|A(q^n_o - \hat{q}^n_o)\|_2 \leq \sigma_{\text{max}}(A)\|q^n_o - \hat{q}^n_o\|_2 \leq n\left(1 + 2\sqrt{\frac{m}{n}}\right)2^{-b+1}.
\]
As the next step, we want to lower bound \( \|A([X^n_o]_b - [\hat{X}^n_o]_b)\|_2 \). For a fixed vector \( u^n \) and for any \( \tau \in (0, 1) \), by Lemma 4 in Appendix C,
\[
P(\|Au^n\|_2 \leq \sqrt{m(1-\tau)}\|u^n\|_2) \leq e^{\frac{\pi}{2}(\tau + \ln(1-\tau))}.
\]
However, \( [X^n_o]_b - [\hat{X}^n_o]_b \) is not a fixed vector. But as we will show next, with high probability, we can upper bound the number of such vectors possible.

Since \( \hat{X}^n_o \) is the solution of (5), we have
\[
\hat{H}_k([\hat{X}^n_o]_b) \leq \hat{H}_k([X^n_o]_b).
\]
On the other hand as proved in Appendix D,
\[
\frac{1}{n} \ell_{LZ}([\hat{X}^n_o]_b) \leq \hat{H}_k([\hat{X}^n_o]_b) + \frac{b(kb + b + 3)}{(1 - \epsilon_n)\log n - b} + \gamma_n,
\]
where \( \gamma_n = o(1) \), and does not depend on \( [\hat{X}^n_o]_b \) or \( b \), and
\[
\epsilon_n = \frac{\log((2^b - 1)(\log n)/b + 2^b - 2) + 2b}{\log n}.
\]
Combining (7) and (8) and dividing both sides by \( b = b_n \) yields
\[
\frac{1}{nb_n} \ell_{LZ}([\hat{X}^n_o]_{b_n}) \leq \frac{\hat{H}_k([X^n_o]_{b_n})}{b_n} + \frac{kb_n + b_n + 3}{(1 - \epsilon_n)\log n - b_n} + \frac{\gamma_n}{b_n}.
\]
As the next step, we find an upper bound on \( \hat{H}_k([X^n_o]_{b_n})/b_n \) that holds with high probability. The upper information dimension of process \( X_o \), \( \hat{d}_o(X) \), is defined as \( \lim_{k \to \infty} \hat{d}_k(X) \). By Lemma 1, \( \hat{d}_k(X) \) is a non-increasing
function of \( k \). Therefore, given \( \delta_1 > 0 \), there exists \( k_{\delta_1} > 0 \), such that for any \( k > k_{\delta_1} \),

\[
\bar{d}_o(X) \leq \bar{d}_k(X) \leq \bar{d}_o(X) + \delta_1.
\]

By the definition of \( d_{k_{\delta_1}}(X) \), given \( \delta_2 > 0 \), there exists \( b_{\delta_2} \), such that for \( b \geq b_{\delta_2} \),

\[
\frac{H([X_{k_{\delta_1}+1}]_b || X_{k_{\delta_1}})_b}{b} \leq d_{k_{\delta_1}}(X) + \delta_2.
\]

But, \( \hat{H}_k([X^n]_{b_n})/b_n \) is a decreasing function of \( k \). Therefore, since \( k = k_n \) by construction is a diverging sequence, for \( n \) large enough, \( k_n > k_{\delta_1} \), and

\[
\frac{\hat{H}_{k_n}([X^n]_{b_n})}{b_n} \leq \frac{\hat{H}_{k_{\delta_1}}([X^n]_{b_n})}{b_n}.
\]

We now prove that, given our choice of parameters, for large values of \( n \), \( \hat{H}_{k_{\delta_1}}([X^n]_{b_n})/b_n \) is close to

\[
\frac{H([X_{k_{\delta_1}+1}]_{b_n} || X_{k_{\delta_1}})_b}{b_n},
\]

with high probability.

Since the original process is an stationary aperiodic Markov chain of order \( l \), \( \{[X_i]_b \}_{i=1}^\infty \) is also an aperiodic stationary Markov chain of order \( l \), which has discrete alphabet. Theorem 7 states that for such a process, given \( \epsilon_1 > 0 \), there exists \( g \in \mathbb{N} \), only depending on \( \epsilon_1 \) and transition probabilities of process \( \{[X_i]_b \}_{i=1}^\infty \), such that for any \( k > l \) and \( n > 6(k + g)/\epsilon_1 + k \),

\[
P(||p_k(\cdot|[X^n])_b - \mu_k||_1 \geq \epsilon_1) \leq 2^{e_2^2/8(k + g)n^{2k}2^\frac{n \epsilon_1^2}{2(k + g)^2}},
\]

where \( c = 1/(2 \ln 2) \). Let

\[
\mathcal{E}_2 \triangleq \{ ||p_{k_{\delta_1}+1}(\cdot|[X^n])_{b_n} - \mu_{k_{\delta_1}+1}||_1 \leq \epsilon_1/(k_{\delta_1} + 1) \}.
\]

Letting \( k = k_{\delta_1} + 1 \), where \( k_{\delta_1} > l \), and \( b = b_n \), for \( n \) large enough,

\[
P(\mathcal{E}_2^c) \leq 2^{\frac{\epsilon_1^2}{2 \ln 2}} (k_{\delta_1} + g + 1)n^{2n(k_{\delta_1} + 1)}2^\frac{n \epsilon_1^2}{2(k_{\delta_1} + g + 1)^2}.
\]

Let \( U^{k_{\delta_1}+1} \sim p_{k_{\delta_1}+1}(\cdot|[x^n])_{b_n} \). Then, conditioned on \( \mathcal{E}_2 \),

\[
|\hat{H}_{k_{\delta_1}}([x^n]_{b_n}) - H([H_{k_{\delta_1}+1}]_{b_n} || X_{k_{\delta_1}})_b) | = |H(U_{k_{\delta_1}} U^{k_{\delta_1}+1}) - H([H_{k_{\delta_1}+1}]_{b_n} || X_{k_{\delta_1}})_b) |
\]

\[
= |H(U^{k_{\delta_1}+1}) - H(U^{k_{\delta_1}}) - H([X_{k_{\delta_1}+1}]_{b_n}) + H([X_{k_{\delta_1}})_b) |
\]

\[
= |H(U^{k_{\delta_1}+1}) - H([X_{k_{\delta_1}+1}]_{b_n})| + |H(U^{k_{\delta_1}}) - H([X_{k_{\delta_1}})_b) |
\]

\[
\leq \frac{2 \epsilon_1}{k_{\delta_1} + 1} \log \left( \frac{\epsilon_1}{k_{\delta_1} + 1} \right) + \epsilon_1 b_n,
\]

(13)

\[
\leq \frac{2 \epsilon_1}{k_{\delta_1} + 1} \log \left( \frac{\epsilon_1}{k_{\delta_1} + 1} \right) + \epsilon_1 b_n,
\]

(13)
Define the event bounded as number of sequences in each binary string corresponding to an LZ-coded sequence corresponds to a unique uncoded sequence. Hence, the enough. Furthermore, given our choice of parameters

By the union bound,

or

where (a) follows from Lemma 6 in Appendix C. Dividing both sides of (13) by \( b_n \) yields

On the other hand, \( b_n \) is a diverging sequence of \( n \). Therefore, for \( n \) large enough, \( b_n \geq b_δ \), and as a result, combining (10), (11) and (14) yields that, for \( n \) large enough, conditioned on \( \mathcal{E}_2 \),

where \( δ_3 ≡ δ_1 + δ_2 - \frac{2ε_1}{(k_{δ_1} + 1)b_n} \log \frac{ε_1}{κ_{δ_1} + 1} + ε_1 \) can be made arbitrarily small by choosing \( ε_1, δ_1 \) and \( δ_2 \) small enough. Furthermore, given our choice of parameters \( b_n \) and \( k_n \), from (9) and (15), conditioned on \( \mathcal{E}_2 \),

and

where \( δ_4 ≡ (k_n b_n + b_n + 3)/(1 - ε_1) \log n - b_n) + γ/n + δ_3 \) can be made arbitrarily small.

Let \( C_n ≡ \{ x^n \} : l_{LZ}(x^n) \leq nb_n(d_o(X) + δ_4) \}. Since the Lempel-Ziv code is a uniquely decodable code, each binary string corresponding to an LZ-coded sequence corresponds to a unique uncoded sequence. Hence, the number of sequences in \( C_n \), i.e., the number of quantized sequences satisfying the upper bound of (17), can be bounded as

Define the event \( \mathcal{E}_3 ≡ \{ \forall x_1^n, x_2^n \in [0, 1]^n : [x_1^n]_{b_n} = [x_2^n]_{b_n} \in C_n, \| A([x_1^n]_{b_n} - [x_2^n]_{b_n}) \|_2 \geq \sqrt{m(1 - τ)} \| [x_1^n]_{b_n} - [x_2^n]_{b_n} \|_2 \}. Combining the union bound, (6) and (18), it follows that

Finally, conditioned on \( \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \),

or

By the union bound, \( P(\mathcal{E}_1) + P(\mathcal{E}_2) + P(\mathcal{E}_3) \}. Clearly, \( P(\mathcal{E}_1) \rightarrow 0 \), as \( n \rightarrow ∞ \). Also, for our
choice of parameter $b = b_n$, from (12), as $n \to \infty$, $P(E^c_2) \to 0$ as well. Let

$$\tau_n = 1 - \frac{1}{(\log n)^{2/(1+\nu)}},$$

where $\nu \in (0, 1)$. Then from (19), it follows that

$$P(E^c_3) \leq 2^{2nb_n(\bar{d}_o(X) + \delta_4)} + 2e^{\frac{2}{n}(1 - \frac{2}{\delta_5} \log \log n)} \leq 2^{2n(\log \log n + 1)(\bar{d}_o(X) + \delta_4) + 2\delta_4(1 + \delta_4) \bar{d}_o(X) n(\log e - \frac{2}{\delta_4} \log \log n)} = 2^{2n(\log \log n + 1)(\bar{d}_o(X) + \delta_4) + \frac{1}{\delta_4} - \frac{1}{\delta_4} + \rho_n},$$

(21)

where $\delta_5 \triangleq \frac{d_4}{\bar{d}_o(X)}$ and $\rho_n = o(1)$. Given $\delta > 0$, choosing $\nu < (\delta - \delta_5)/(1 + \delta_5)$ ensures that (21) converges to zero, as $n \to \infty$. Note that $\delta_4$ as a result $\delta_5$ can be made arbitrarily small for $n$ large enough. On the other hand, from (20), since always $m/n \leq 1$, we have

$$\frac{1}{\sqrt{n}} \|X^n_o - \hat{X}^n_o\|_2 \leq \frac{3(\log n)^{1/(1+\nu)}}{\sqrt{2(1 + \delta)\bar{d}_o(X)}} 2^{-\log \log n + 1} \leq \frac{3(\log n)^{1/(1+\nu)}}{\sqrt{2(1 + \delta)\bar{d}_o(X)}} 2^{-\log \log n + 1} \leq \frac{6}{\sqrt{2(1 + \delta)\bar{d}_o(X)}} 2^{-\log \log n(1 - 1/(1+\nu))},$$

(22)

which goes to zero as $n$ grows to infinity. This concludes the proof.

\begin{remark}
For i.i.d. processes, the number of measurements required by MEP for reliable recovery is two times the fundamental limits of the non-universal setup characterized in [31]. Since we do not yet have a lower bound on the number of measurements required by a universal compressed sensing algorithm, it is not clear to us at this stage whether this factor of two is the price of universality, sub-optimality of MEP or an artifact of our proof technique.
\end{remark}

The optimization presented in (5) is not easy to handle. While the search domain is a hyperplane, i.e., the set of points satisfying $Ax^n = y_o^m$, the cost function is defined on a discretized space. To move towards designing an implementable universal compressed sensing algorithm, consider the following algorithm:

$$\hat{x}_o^n = \arg\min \left( \tilde{H}_k([x^n]_b) + \frac{\lambda}{n^2} \|A[x^n]_b - y_o^m\|_2^2 \right).$$

(23)

We refer to this algorithm as Lagrangian-MEP. The main difference between (5) and (23) is that in (23) the search is done over the discretized space. This algorithm is similar to the heuristic algorithm proposed in [32] and [33] based on universal priors for universal compressed sensing. The advantage of Lagrangian-MEP compared to MEP is that, as discussed in [32] and [33], Markov chain Monte Carlo (MCMC) and simulated annealing [40]–[42] can be employed to approximate the optimizer of Lagrangian-MEP. The following theorem shows that (23) approximates
the solution of MEP with no loss in the asymptotic performance.

**Theorem 5.** Consider an aperiodic stationary Markov process \( \{X_t\}_{t=1}^\infty \) of order \( l \), with \( \mathcal{X} = [0,1] \) and upper information dimension \( \bar{d}_o(X) \). Let \( b = b_n = [r \log \log n] \), where \( r > 1 \), \( k = k_n = o(\frac{\log n}{\log \log n}) \), \( \lambda = \lambda_n = (\log n)^{2r} \) and \( m = m_n \geq 2(1+\delta)\bar{d}_o(X)n \), where \( \delta > 0 \). For each \( n \), let the entries of the measurement matrix \( A = A_n \in \mathbb{R}^{m \times n} \) be drawn i.i.d. according to \( \mathcal{N}(0,1) \). Given \( X_n^o \) generated by source \( X \) and \( Y_n^m = AX_n^o \), let \( \hat{X}_n^o \) denote the solution of (23), i.e., \( \hat{X}_n^o = \arg \min \{ \hat{H}_k([x]^n)_b + \frac{\lambda}{n^2}\|A[x]^n)_b - Y_n^m\|^2 \} \). Then,

\[
\frac{1}{\sqrt{n}} \| X_n^o - \hat{X}_n^o \|_2 \overset{p}{\rightarrow} 0.
\]

**Proof:** We need to prove that for any \( \epsilon > 0 \),

\[
P\left( \frac{1}{\sqrt{n}} \| X_n^o - \hat{X}_n^o \|_2 > \epsilon \right) \rightarrow 0,
\]
as \( n \to \infty \). Throughout the proof \( \bar{d}_o \) refers to \( \bar{d}_o(X) \). As before, let \( X_n^o = [X_n^o]_b + q_n^o \) and \( \hat{X}_n^o = [\hat{X}_n^o]_b + \hat{q}_n^o \), where as we proved earlier, \( \|q_n^o\|_2, \|\hat{q}_n^o\|_2 \leq \sqrt{n}2^{-b} \). Since

\[
\hat{X}_n^o = \arg \min \{ \hat{H}_k([x]^n)_b + \frac{\lambda}{n^2}\|A[x]^n)_b - Y_n^m\|^2 \},
\]

we have

\[
\hat{H}_k([\hat{X}_n^o]_b) + \frac{\lambda}{n^2}\|A[\hat{X}_n^o]_b - Y_n^m\|^2 \leq \hat{H}_k([X_n^o]_b) + \frac{\lambda}{n^2}\|Aq_n^o\|^2,
\]

\[
\leq \hat{H}_k([X_n^o]_b) + \frac{\lambda\sigma_{\max}(A)2^{-2b}}{n}. \tag{24}
\]

Let \( \mathcal{E}_1 \triangleq \{ \sigma_{\max}(A) \leq \sqrt{n} + 2\sqrt{m} \} \), where from [39], \( P(\mathcal{E}_1^c) \leq e^{-m/2} \). Also given \( \epsilon > 0 \), define

\[
\mathcal{E}_2 \triangleq \{ \frac{1}{b}\hat{H}_k([X_n^o]_b) \leq \bar{d}_o + \epsilon \}.
\]

In the proof of Theorem 4, we showed that, for any \( \epsilon > 0 \), given our choice of parameters, \( P(\mathcal{E}_2^c) \) converges to zero as \( n \) grows to infinity. Conditioned on \( \mathcal{E}_1 \cap \mathcal{E}_2 \), from (24), we derive

\[
\frac{1}{b}\hat{H}_k([\hat{X}_n^o]_b) + \frac{\lambda}{bn^2}\|A[\hat{X}_n^o]_b - Y_n^m\|^2 \leq \bar{d}_o + \epsilon + \frac{\lambda 2^{-2b}}{b}(1 + 2\sqrt{m/n})^2
\]

\[
\leq \bar{d}_o + \epsilon + \frac{9\lambda 2^{-2b}}{b}, \tag{25}
\]

where the last line holds since for \( m \leq n \), \( 1 + \sqrt{m/n} \leq 2 \). On the other hand, for \( b = b_n = [r \log \log n] \) and \( \lambda = \lambda_n = (\log n)^{2r} \), we have

\[
\frac{9\lambda 2^{-2b}}{b} \leq \frac{9(\log n)^{2r}}{r(\log n)^{2r}\log \log n} = \frac{9}{r \log \log n},
\]

which goes to zero as \( n \) grows to infinity. For \( n \) large enough, \( \frac{9}{r \log \log n} \leq \epsilon \), and hence from (25),

\[
\frac{1}{b}\hat{H}_k([\hat{X}_n^o]_b) + \frac{\lambda}{bn^2}\|A[\hat{X}_n^o]_b - Y_n^m\|^2 \leq \bar{d}_o + 2\epsilon,
\]
which implies that
\[
\frac{1}{b} \hat{H}_k([\hat{X}_o^n]_b) \leq \bar{d}_o + 2\epsilon, \tag{26}
\]
and since \(\bar{d}_o \leq 1\),
\[
\sqrt{\frac{\lambda}{bn^2}} \|A[\hat{X}_o^n]_b - Y_o^n\|_2 \leq \sqrt{1 + 2\epsilon}. \tag{27}
\]

For \(x^n \in [0,1]^n\), from (D.27) in Appendix D,
\[
\frac{1}{n} \ell_{LZ}([x^n]_b) \leq \hat{H}_k([x^n]_b) + \frac{b(kb + b + 3)}{(1 - \epsilon_n) \log n - b} + \gamma_n,
\]
where \(\epsilon_n = o(1)\) and \(\gamma_n = o(1)\) are both independent of \(x^n\). Therefore there exists \(n_\epsilon\), such that for \(n > n_\epsilon\),
\[
\frac{1}{nb} \ell_{LZ}([x^n]_b) \leq \frac{1}{b} \hat{H}_k([x^n]_b) + \epsilon.
\]
On the other hand, conditioned on \(E_1 \cap E_2\), \(\frac{1}{b} \hat{H}_k([X_o^n]_b) \leq \bar{d}_o + \epsilon\), and \(\frac{1}{b} \hat{H}_k([\hat{X}_o^n]_b) \leq \bar{d}_o + 2\epsilon\). Therefore, conditioned on \(E_1 \cap E_2\), \([X_o^n]_b, \hat{X}_o^n)_{b} \in C_n\), where
\[
C_n \triangleq \{[x^n]_{b_n} : \frac{1}{nb} \ell_{LZ}([x^n]_b) \leq \bar{d}_o + 3\epsilon\}.
\]
Define the event \(E_3\) as
\[
E_3 \triangleq \{\|A([\hat{X}_o^n]_b - [X_o^n]_b)\|_2 \geq \|\hat{X}_o^n - X_o^n\|_2 \sqrt{(1 - \tau)m}\},
\]
where \(\tau > 0\). Note that
\[
P((E_1 \cap E_2 \cap E_3)) \leq P(E_1^c) + P(E_2^c) + P(E_1 \cap E_2 \cap E_3^c)
\leq P(E_1^c) + P(E_2^c) + P(E_3^c|E_1 \cap E_2). \tag{28}
\]
Since conditioned on \(E_1 \cap E_2\), \([X_o^n]_b, \hat{X}_o^n)_{b} \in C_n\), as we argued in the proof of Theorem 4, by the union bound, it follows that
\[
P(E_3^c|E_1 \cap E_2) \leq 2^{2(\bar{d}_o + 3\epsilon)bn} e^{-\frac{\gamma}{\lambda} (\tau + \ln(1-\tau))},
\]
Let \(\tau = 1 - (\log n)^{-\frac{2\epsilon}{\lambda}}\), where \(f > 0\). Then, since \(b \leq r \log \log n + 1\), and \(m = m_n > 2(1 + \delta)\bar{d}_o\),
\[
P(E_3^c|E_1 \cap E_2) \leq 2^{2(\bar{d}_o + 3\epsilon)(r \log \log n + 1)n} e^{-\frac{\gamma}{\lambda} (\tau + \ln(1-\tau))}
\leq 2^{2r (\log \log n) n (\bar{d}_o + 3\epsilon - (\frac{\gamma}{\lambda} + \alpha_n))}
\leq 2^{2r (\log \log n) n (\bar{d}_o + 3\epsilon - (\frac{\gamma}{\lambda} + \alpha_n))}, \tag{29}
\]
where \(\alpha_n = o(1)\). Since \(\epsilon > 0\) and \(f > 0\) can be selected arbitrarily small, they can be chosen such that
\(3\epsilon < (\delta - f)\bar{d}_o/(1 + f)\), which yields \(P(E_3^c|E_1 \cap E_2) \to 0\), as \(n \to \infty\).
On the other hand, from the triangle inequality, \( \| A[\hat{X}^n_o]_b - Y^m_o \|_2 \geq \| A([\hat{X}^n_o]_b - [X^n_o]_b) \|_2 - \| Aq^n_o \|_2 \). Therefore, conditioned on \( E_1 \cap E_2 \cap E_3 \), from (27),
\[
\sqrt{\frac{\lambda(1 - \tau)m}{bn^2}} \| [\hat{X}^n_o]_b - [X^n_o]_b \|_2 \leq \sqrt{\frac{\lambda}{bn^2}} \| Aq^n_o \|_2 + \sqrt{1 + 2\epsilon} \\
\leq \sqrt{\frac{\lambda}{b}(1 + 2\sqrt{m/n})2^{-b} + \sqrt{1 + 2\epsilon}} \\
\leq \sqrt{\frac{9\lambda}{b}2^{-b} + \sqrt{1 + 2\epsilon}},
\]
(30)
or
\[
\frac{1}{\sqrt{n}} \| [\hat{X}^n_o]_b - [X^n_o]_b \|_2 \leq 2^{-b} \sqrt{\frac{9n}{(1 - \tau)m} + \frac{(1 + 2\epsilon)bn}{(1 - \tau)\lambda m}}.
\]
Hence, for \( \tau = 1 - (\log n)^{1/\lambda} \), \( \lambda = (\log n)^{2\tau} \), and \( b = \lceil r \log \log n \rceil \),
\[
\frac{1}{\sqrt{n}} \| [\hat{X}^n_o]_b - [X^n_o]_b \|_2 \leq \frac{3 + \sqrt{(1 + 2\epsilon)(r\log \log n + 1)}}{\sqrt{2d_o(1 + \delta)(\log n)^{1/r}}},
\]
(31)
which can be made arbitrarily small.

\[\blacksquare\]

B. Robustness to noise

In the previous section we assumed that the measurements are perfect and noise-free. In this section we study the case where the measurements are corrupted by noise. In this case, instead of \( Ax^n_o \), the decoder observes \( y^m_o = Ax^n_o + z^m \), where \( z^m \) denotes the noise in the measurement system, and employs the following optimization algorithm to recover \( x^n_o \):
\[
\hat{x}^n = \arg \min_{H_k(\{x^n\}_b) \leq C} \| Ax^n - y^m_o \|_2,
\]
(32)
where \( C \) is a constant depending on the source complexity.

**Theorem 6.** Consider an aperiodic stationary Markov process \( \{X_i\}_{i=1}^\infty \) of order \( l \), with \( \mathcal{X} = [0, 1] \) and upper information dimension \( d_o(X) \). Consider a measurement matrix \( A = A_n \in \mathbb{R}^{m \times n} \) with i.i.d. entries distributed according to \( \mathcal{N}(0, 1) \). For \( X^n_o \) generated by the source \( X \), we observe \( Y^m_o = AX^n_o + Z^m \), where \( Z^m \) denotes the measurement noise. Assume that there exists a deterministic sequence \( c_m \) such that \( \lim_{m \to \infty} P(\| Z^m \|_2 > c_m) = 0 \), and \( c_m = o(\frac{m}{\log \log m}) \). Let \( \hat{X}^n_o = \hat{X}^n_o(Y^m_o, A) \) denote the solution of (32) with \( C = b(1 + \delta)d_o(X) \), where \( \delta > 0 \), i.e.,
\[
\hat{X}^n_o = \arg \min_{H_k(\{x^n\}_b) \leq b(1 + \delta)d_o(X)} \| Ax^n - Y^m_o \|_2.
\]
(33)
Let \( b = b_n = \lceil \log \log n \rceil \), \( k = k_n = o(\frac{\log n}{\log \log n}) \) and \( m = m_n \geq 2(1 + 3\delta)d_o(X)n \). Then,
\[
\frac{1}{\sqrt{n}} \| X^n_o - \hat{X}^n_o \|_2 \overset{P}{\to} 0.
\]
Proof: Throughout the proof for simplicity $\delta_o$ denotes $\delta_o(X)$. Define event $E_0$ as

$$E_0 \triangleq \{ \hat{H}_k([X^n_o]_b) \leq b(1 + \delta)\delta_o \}.$$ 

In the proof of Theorem 4, we proved that, given our choice of parameters, for any fixed $\delta > 0$, $P(E_0^c) \to 0$, as $n$ grows to infinity. Conditioned on $E_0$, $X^n_o$ satisfies the constraint of (33), and since $\hat{X}^n_o$ is the minimizer of (33), we have

$$\|AX^n_o - Y^n_o\|_2 \leq \|AX^n_o - Y^n_o\|_2 = \|Z^n\|_2.$$

Let $X^n_o = [X^n_o]_b + q^n_o$ and $\hat{X}^n_o = [\hat{X}^n_o]_b + \hat{q}^n_o$. Then, from (34),

$$\|A([X^n_o]_b - [\hat{X}^n_o]_b) + A(q^n_o - \hat{q}^n_o) - Z^n\|_2 \leq \|Z^n\|_2,$$

and by the triangle inequality,

$$\|A([X^n_o]_b - [\hat{X}^n_o]_b)\|_2 \leq \|A(q^n_o - \hat{q}^n_o)\|_2 + 2\|Z^n\|_2.$$

Let

$$U^n \triangleq \frac{A([X^n_o]_b - [\hat{X}^n_o]_b)}{\|A([X^n_o]_b - [\hat{X}^n_o]_b)\|_2}.$$

By this definition, (35) can be rewritten as

$$\|X^n_o - \hat{X}^n_o\|_2 \|U^n\|_2 \leq \|A(q^n_o - \hat{q}^n_o)\|_2 + 2\|Z^n\|_2.$$

Define the following events:

$$E_1 \triangleq \{ \sigma_{\text{max}}(A) \leq \sqrt{n} + 2\sqrt{m} \},$$

$$E_2 \triangleq \{ \|U^n\|_2 \geq \sqrt{(1 - \tau)m} \},$$

$$E_3 \triangleq \{ \|Z^n\|_2 \leq c_m \},$$

where $\tau > 0$. Conditioned on $E_0 \cap E_1 \cap E_2$, since $\|q^n_o - \hat{q}^n_o\|_2 \leq 2^{-b+1}\sqrt{n}$, it follows from (36) that for $n$ large enough

$$\frac{1}{\sqrt{n}}\|X^n_o - \hat{X}^n_o\|_2 \leq \frac{2^{-b+1}(1 + 2\sqrt{m/n})\sqrt{n}}{\sqrt{(1 - \tau)m}} + \frac{2c_m}{\sqrt{(1 - \tau)mn}},$$

and since $m \leq n$,

$$\frac{1}{\sqrt{n}}\|X^n_o - \hat{X}^n_o\|_2 \leq \frac{6}{2\sqrt{m}}\sqrt{\frac{n}{(1 - \tau)m}} + \frac{2c_m}{\sqrt{(1 - \tau)mn}}.$$
To set the parameter $\tau$, we first study the probability of $\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$. Note that

$$P\left(\left(\bigcap_{i=0}^{3} \mathcal{E}_i\right)^c\right) \leq P(\mathcal{E}_0^c) + P(\mathcal{E}_1^c) + P(\mathcal{E}_2^c|\mathcal{E}_0) + P(\mathcal{E}_3^c).$$

As mentioned earlier, $P(\mathcal{E}_0^c) \rightarrow 0$, as $n$ grows to infinity, and also from [39], $P(\mathcal{E}_1^c) \leq e^{-m/2}$. Furthermore, since $m$ grows linearly with $n$, by the theorem’s assumption, $P(\mathcal{E}_3^c) \rightarrow 0$, as $n \rightarrow \infty$. In (D.27) of Appendix D, we showed that for $x^n \in [0,1]^n$,

$$\frac{1}{n} l_{\text{LZ}}([x^n]_b) \leq \hat{H}_k([x^n]_b) + \frac{b(kb + b + 3)}{(1 - \epsilon_n) \log n - b} + \gamma_n,$$

where $\epsilon_n = o(1)$ and $\gamma_n = o(1)$ are both independent of $x^n$. Therefore, given $\delta' > 0$, $b = b_n = \lfloor \log \log n \rfloor$ and $k = k_n = o\left(\frac{\log n}{\log \log n}\right)$, there exits $n_{\delta'}$, such that for $n > n_{\delta'}$,

$$\frac{1}{n} n b l_{\text{LZ}}([x^n]_b) \leq \frac{1}{b} \hat{H}_k([x^n]_b) + \delta'.$$

On the other hand, by construction, $\frac{1}{b} \hat{H}_k([X^n_o]_b) \leq \bar{d}_o(1 + \delta)$, and conditioned on $\mathcal{E}_0$, $\frac{1}{b} \hat{H}_k([X^n_o]_b) \leq \bar{d}_o(1 + \delta)$. Choosing $\delta' = \delta \bar{d}_o$, for $n$ large enough, $\frac{1}{n} l_{\text{LZ}}([X^n_o]_b) \leq \bar{d}_o(1 + 2\delta)$ and $\frac{1}{n} l_{\text{LZ}}([\hat{X}^n_o]_b) \leq \bar{d}_o(1 + 2\delta)$.

As argued in the proof of Theorem 4, given the fact that the Lempel-Ziv code is uniquely decodable, $\{|[x^n]_b : \frac{1}{n} l_{\text{LZ}}([x^n]_b) \leq \bar{d}_o(1 + 2\delta)\} \leq 2n b (1 + 2\delta) d_o + 1$. Therefore, by the union bound and Lemma 4,

$$P(\mathcal{E}_0^c|\mathcal{E}_0) \leq 2^{2nb(1+2\delta)d_o + 1} e^{2(\tau - \log(1 - \tau))}.$$  (38)

Let $\tau = \tau_n = 1 - (\log n)^{-\frac{1}{1 + 3\delta}_f}$, where $f > 0$. Then, for $b = \lfloor \log \log n \rfloor$, and $m > 2n(1 + 3\delta)d_o$, from (38),

$$P(\mathcal{E}_0^c|\mathcal{E}_0) \leq 2^{2n(\log \log n + 1)(1 + 2\delta)d_o + 1} e^{2(\tau \log e^{-\frac{1}{1 + 3\delta}_f} + \log \log n)}$$

$$\leq 2^{2n(\log \log n + 1)(1 + 2\delta)d_o + 1} e^{2(\frac{2n(1 + 3\delta)}{1 + \tau} + \log \log n)}$$

$$= 2^{2nd_o \log n(1 + 2\delta - \frac{4n + 9\delta}{1 + \tau} + \eta_n)},$$  (39)

where $\eta_n \rightarrow 0$, as $n \rightarrow \infty$. Choosing $f$ small enough such that $1 + 2\delta < \frac{4n + 9\delta}{1 + \tau}$ ensures that $P(\mathcal{E}_0^c|\mathcal{E}_0) \rightarrow 0$, as $n \rightarrow \infty$. For the selected value of $\tau$, it follows from (37) that

$$\frac{1}{\sqrt{n}} \|X^n_o - [\hat{X}^n_o]_b\|_2 \leq \frac{6(\log n)^{-f(1 + f)/2}}{\sqrt{2(1 + 3\delta)d_o}} + \frac{2c_m(\log n)^{1 + \tau}}{m}$$

and by the triangle inequality,

$$\frac{1}{\sqrt{n}} \|X^n_o - \hat{X}^n_o\|_2 \leq \frac{1}{\sqrt{n}} \|X^n_o - [\hat{X}^n_o]_b\|_2 + \frac{1}{\sqrt{n}} \|\tilde{q}_o^n + \hat{q}_o^n\|_2$$

$$\leq \frac{6(\log n)^{-f(1 + f)/2}}{\sqrt{2(1 + 3\delta)d_o}} + \frac{2c_m \log n}{m} + 2^{-b + 1}.$$  (40)
Since by assumption \( c_m = o\left(\frac{m}{\log m}\right) \), and \( m \) grows linearly by \( n \), for \( n \) large enough, for any \( \epsilon > 0 \),

\[
\frac{6(\log n)^{-f/(1+f)}}{\sqrt{2(1+3\delta)d_o}} + \frac{2c_m \log n}{m} + 2^{-b+1} < \epsilon,
\]

which proves that \( \lim_{n \to \infty} P\left(\frac{1}{\sqrt{n}} \| X^o_n - \hat{X}^o_n \|_2 > \epsilon\right) = 0 \).

V. CONCLUSIONS

In this paper we have studied the problem of universal compressed sensing, i.e., the problem of recovering “structured” signals from their under-determined set of random linear projections without having prior information about the structure of the signal. We have considered structured signals that are modeled by stationary processes. We have generalized Rényi’s information dimension and defined information dimension of a stationary process, as a measure of complexity for such processes. We have also calculated the information dimension of some stationary processes, such as Markov processes used to model piecewise constant signals.

We then have introduced the MEP algorithm for universal signal recovery. The algorithm is based on Occam’s Razor and among the signals satisfying the measurements constraints seeks the “simplest” signal, where the complexity of a signal is measured in terms of the conditional empirical entropy of its quantized version. We have proved that for an aperiodic stationary Markov chain with upper information dimension of \( \bar{d}_o(X) \), the algorithm requires slightly more than \( 2\bar{d}_o(X)n \) random linear measurements to recover the signal. We have also provided an implementable version of the MEP algorithm with the same asymptotic performance as the original MEP.

APPENDIX A
PROOF OF THEOREM 2

Since the process is a stationary first-order Markov process,

\[
H([X_{k+1}]_b|[X_k]_b) = H([X_{k+1}]_b|X_k)_b) = H([X_2]_b|[X_1]_b),
\]

and therefore,

\[
\bar{d}_k(X) = \bar{d}_1(X),
\]

and

\[
d_k(X) = d_1(X),
\]

for all \( k \geq 1 \). Let \( \mathcal{X}_b = \{ \sum_{i=1}^b b_i 2^{-i} : b_i \in \{0,1\}, i = 1, \ldots, b \} \) denote the alphabet at resolution \( b \). Clearly, \( |\mathcal{X}_b| = 2^b \). Then,

\[
H([X_2]_b|[X_1]_b) = \sum_{c \in \mathcal{X}_b} P([X_1]_b = c) H([X_2]_b|[X_1]_b = c).
\]

We next prove that \( \frac{1}{b} H([X_2]_b|[X_1]_b = c_1) \) uniformly converges to \( p \), as \( b \) grows to infinity, for all values of \( c_1 \).

Define the indicator random variable \( I = 1_{X_2=X_1} \). Given the transition probability of the Markov chain, \( I \) is independent of \( X_1 \), and \( P(I = 1) = 1 - p \). Also define a random variable \( U \), independent of \( (X_1, X_2) \), and
distributed according to $f_c$. Then, it follows that

$$
P([X_2]_b = c_2|[X_1]_b = c_1) = P([X_2]_b = c_2, I = 0|[X_1]_b = c_1)$$

$$+ P([X_2]_b = c_2, I = 1|[X_1]_b = c_1)$$

$$= p P([X_2]_b = c_2|[X_1]_b = c_1, I = 0)$$

$$+ (1 - p) P([X_2]_b = c_2|[X_1]_b = c_1, I = 1)$$

$$= p P([U]_b = c_2) + (1 - p) \mathbb{I}_{c_2 = c_1},$$

where the last line follows from the fact that conditioned on $X_2 \neq X_1$, $X_2$, independent of the value of $X_1$, is distributed according to $f_c$. For $a \in \mathcal{X}_b$,

$$P([U]_b = a) = \int_a^{a+2^{-b}} f_c(u) du. \quad (A.1)$$

On the other hand, by the mean value theorem, there exists $x_a \in [a, a + 2^{-b}]$, such that

$$2^{-b} \int_a^{a+2^{-b}} f_c(u) du = f_c(x_a).$$

Therefore, $P([U]_b = a) = 2^{-b} f_c(x_a)$, and

$$H([X_2]_b|[X_1]_b = c_1) = - \sum_{a \in \mathcal{X}_b, a \neq c_1} -(p2^{-b} f_c(x_a)) \log(p2^{-b} f_c(x_a))$$

$$- (p2^{-b} f_c(x_{c_1}) + 1 - p) \log(p2^{-b} f_c(x_{c_1}) + 1 - p)$$

$$= \sum_{a \in \mathcal{X}_b, a \neq c_1} -(p2^{-b} f_c(x_a))(\log p - b + \log(f_c(x_a)))$$

$$- (p2^{-b} f_c(x_{c_1}) + 1 - p) \log(p2^{-b} f_c(x_{c_1}) + 1 - p). \quad (A.2)$$

Dividing both sides of (A.2) by $b$ yields

$$\frac{H([X_2]_b|[X_1]_b = c_1)}{b} = \frac{(b - \log p)p}{b} \sum_{a \in \mathcal{X}_b, a \neq c_1} 2^{-b} f_c(x_a)$$

$$- \left(\frac{p}{b}\right) \sum_{a \in \mathcal{X}_b, a \neq c_1} 2^{-b} f_c(x_a) \log(f_c(x_a))$$

$$- \left(\frac{p2^{-b} f_c(x_{c_1}) + 1 - p}{b}\right) \log(p2^{-b} f_c(x_{c_1}) + 1 - p). \quad (A.3)$$
On the other hand, from (A.1),
\[
\sum_{a \in X_0} 2^{-b} f_c(x_a) = \sum_{a \in X_0} \int_a^{a+2^{-b}} f_c(u) du \\
= \int_0^1 f_c(u) du \\
= 1.
\]
(A.4)

Also, since \( \int_0^1 f_c(u) du = 1 \),
\[
\lim_{b \to \infty} \sum_{a \in X_0} 2^{-b} f_c(x_a) \log(f_c(x_a)) = h(f_c),
\]
where \( h(f_c) = -\int f_c(u) \log f_c(u) du \) denotes the differential entropy of \( U \). Therefore, for any \( \epsilon > 0 \), there exists \( b_\epsilon \in \mathbb{N} \), such that for \( b > b_\epsilon \),
\[
\left| \sum_{a \in X_0} 2^{-b} f_c(x_a) \log(f_c(x_a)) - h(f_c) \right| \leq \epsilon.
\]

Since \( f_c \) is bounded by assumption, \( M = \sup_{x \in [0,1]} f_c(x) < \infty \), and \( h(f_c) \leq \log M < \infty \). Finally, \( -q \log q \leq e^{-1} \log e \), for \( q \in [0,1] \). Therefore, combining (A.3), (A.4) and (A.5), it follows that, for \( b > b_\epsilon \),
\[
\left| H([X_2]_b | [X_1]_b = c_1) - p \right| \leq \frac{M}{2^b} - p \log p + \frac{p(h(f_c) + \epsilon)}{b} + \frac{\log e}{eb},
\]
(A.5)
for all \( c_1 \in X_0 \). Since the right hand side of the above equation does not depend on \( c_1 \), and goes to zero as \( b \to \infty \), for any \( \epsilon' > 0 \), there exists \( b_{\epsilon'} \), such that for \( b > \max\{b_\epsilon, b_{\epsilon'}\} \),
\[
\left| H([X_2]_b | [X_1]_b = c_1) - p \right| \leq \epsilon',
\]
(A.6)
and
\[
\left| H([X_2]_b | [X_1]_b) - p \right| \leq \sum_{c_1 \in X_0} P([X_1]_b = c_1) \left| H([X_2]_b | [X_1]_b = c_1) - p \right| \\
\leq \epsilon' \sum_{c_1 \in X_0} P([X_1]_b = c_1) \\
= \epsilon',
\]
which concludes the proof.

APPENDIX B

PROOF OF THEOREM 3

Since the process is stationary and Markov of order \( l \), \( d_l(X) = \bar{d}_l(X) \) and \( \tilde{d}_l(X) = \tilde{d}_l(X) \). To compute \( d_l(X) \), \( \bar{d}_l(X) \), \( \tilde{d}_l(X) \), we need to study \( H([X_{l+1}]_b | [X_l]_b) \). Let \( X_0 \) denote the quantized version of \( X \) at resolution \( b \).

Define the indicator random variable \( I = \mathbb{1}_{Z_l=0} \). By the definition of the Markov chain, \( P(I = 1) = 1 - p \).
Then, given \( c_1, \ldots, c_{l+1} \in \mathcal{X}_b \), since \( I \) is independent of \( X^l \),

\[
H([X_{l+1}]_b|X^l) \leq H([X_{l+1}]_b, I|X^l)_b
\]

\[
\leq 1 + H([X_{l+1}]_b|X^l)_b, \quad \text{(B.8)}
\]

where \( U \) is independent of \( X^l \) and is distributed according to \( f_c \). Similarly,

\[
H([X_{l+1}]_b|X^l) \geq H([X_{l+1}]_b, I|X^l)_b
\]

\[
= pH((\sum_{i=1}^{l} a_i X_{l-i} + U)_b|X^l)_b, \quad \text{(B.9)}
\]

Conditioned on \( [X^l]_b = c^l \), we have

\[
c_i \leq X_i < c_i + 2^{-b},
\]

for \( i = 1, \ldots, l \), and

\[
\sum_{i=1}^{l} a_i X_{l-i} - \sum_{i=1}^{l} a_i c_{l-i} \leq 2^{-b} \sum_{i=1}^{l} |a_i|.
\]

Let \( M = \sum_{i=1}^{l} |a_i| \) and \( c = \sum_{i=1}^{l} a_i c_{l-i} \). Then,

\[
\sum_{i=1}^{l} a_i X_{l-i} \in [c - M2^{-b}, c + M2^{-b}].
\]

Therefore, \( [\sum_{i=1}^{l} a_i X_{l-i}]_b \) can take only \( 2M + 1 \) different values, and as a result \( H(\sum_{i=1}^{l} a_i X_{l-i} | [X^l]_b) \leq \log(2M + 1) \). Since \( M \) does not depend on \( b \), it follows that

\[
\lim_{b \to \infty} \frac{H(\sum_{i=1}^{l} a_i X_{l-i} | [X^l]_b)}{b} = 0.
\]

We next prove that

\[
\lim_{b \to \infty} \frac{H(\sum_{i=1}^{l} a_i X_{l-i} + U | [X^l]_b)}{b} = 1,
\]

for any absolutely continuous distribution with pdf \( f_c \). This combined with the lower and upper bounds in (B.8) and (B.9), respectively, yields the desired result.

To bound \( H(\sum_{i=1}^{l} a_i X_{l-i} + U | [X^l]_b) \), we need to study \( P(\sum_{i=1}^{l} a_i X_{l-i} + U = c | [X^l]_b = c^l) \), where \( c^l \in \mathcal{X}_b^l \), and \( c \in \mathcal{X}_b \). Let \( \mathcal{N}(c^l, b) = \{ x^l : c_i \leq x_i \leq c_i + 2^{-b}, i = 1, \ldots, l \} \), and define the function \( g : \mathbb{R}^l \to \mathbb{R} \),
by \( g(x^l) = \sum_{i=1}^{l} a_i x_{l-i} \). Note that

\[
P \left( \left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b = c \left| X^l \right|_b = c^l \right) = \frac{P(\left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b, \left| X^l \right|_b = c^l)}{P(\left| X^l \right|_b = c^l)}
\]

\[
= \frac{\int_{N(c^l, b)} \int_{c-g(x^l)}^{c-\min(g(x^l) + 2, b)} f(x^l) f_c(u) du \, dx^l}{P(\left| X^l \right|_b = c^l)} \tag{B.10}
\]

where \( f(x^l) \) denotes the pdf of \( X^l \). By the mean value theorem, there exists \( \delta(x^l) \in (0, 2^{-b}) \), such that

\[
\int_{c-g(x^l)}^{c-\min(g(x^l) + 2, b)} f_c(u) du = 2^{-b} f_c(c - g(x^l) + \delta(x^l)). \tag{B.11}
\]

Combining (B.10) and (B.11) yields that

\[
P \left( \left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b = c \left| X^l \right|_b = c^l \right) = \frac{2^{-b} \int_{N(c^l, b)} \int_{c-g(x^l)}^{c-\min(g(x^l) + 2, b)} f(x^l) f_c(c - g(x^l) + \delta(x^l)) \, dx^l}{\int_{N(c^l, b)} \int_{c-g(x^l)}^{c-\min(g(x^l) + 2, b)} f(x^l) \, dx^l}. \tag{B.12}
\]

Define the pdf \( p_{c^l, b}(y^l) \) over \( N(c^l, b) \) as

\[
p_{c^l, b}(y^l) = \frac{f(y^l)}{\int_{N(c^l, b)} f(x^l) \, dx^l}.
\]

Then, \( P(\left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b = c \left| X^l \right|_b = c^l) = 2^{-b} E[f_c(c - g(Y^l) - \delta(Y^l))], \) where \( Y^l \sim p_{c^l, b} \). Hence,

\[
H \left( \left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b \big| X^l \right) = \sum_{c} \sum_{c^l} \left( b - \log E[f_c(c - g(Y^l) - \delta(Y^l)])
\times P \left( \left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b = c \big| X^l \right) = c^l \right) \n\]

\[
= b - \sum_{c} \sum_{c^l} \log E[f_c(c - g(Y^l) - \delta(Y^l)])
\times P \left( \left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b = c \big| X^l \right) = c^l \right).
\]

Since by assumption \( f_c \) is bounded on its support between \( \alpha \) and \( \beta \), then for \( b \) large enough, if \( c - \sum_{i=1}^{l} a_i c_{l-i} \in \mathbb{Z} \), then \( E[f_c(c - g(Y^l) - \delta(Y^l))] \) is also bounded between \( \alpha \) and \( \beta \), and hence the desired result follows. That is,

\[
\lim_{b \to \infty} b^{-1} H \left( \left[ \sum_{i=1}^{l} a_i X_{l-i} + U \right]_b \big| X^l \right) = 1.
\]

**APPENDIX C**

**USEFUL LEMMAS**

The first two lemmas in the following are from [26], and are useful in our proofs.

**Lemma 4** (\( \chi^2 \) concentration). Fix \( \tau > 0 \), and let \( U_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1), i = 1, 2, \ldots, m \). Then,

\[
P \left( \sum_{i=1}^{m} U_i^2 < m(1 - \tau) \right) \leq e^{\frac{m}{\tau}(\tau + \ln(1-\tau))}
\]
\[ P \left( \sum_{i=1}^{m} U_i^2 > m(1 + \tau) \right) \leq e^{-\frac{m}{2}(\tau - \ln(1+\tau))}. \] (C.14)

**Lemma 5.** Let \( U^n \) and \( V^n \) denote two independent Gaussian vectors of length \( n \) with i.i.d. elements distributed as \( \mathcal{N}(0,1) \). Then the distribution of \( \langle U^n, V^n \rangle = \sum_{i=1}^{n} U_i V_i \) is the same as the distribution of \( \| U^n \|_2 G \), where \( G \sim \mathcal{N}(0,1) \) and is independent of \( \| U^n \|_2 \).

**Lemma 6.** Consider distributions \( p \) and \( q \) on finite alphabet \( \mathcal{X} \) such that \( \| p - q \|_1 \leq \epsilon \). Then,

\[ |H(p) - H(q)| \leq -\epsilon \log \epsilon + \epsilon \log |\mathcal{X}|. \]

**Proof:** Define \( f(y) = -y \ln y \), for \( y \in [0,1] \), and \( g(y) = f(y + \epsilon) - f(y) \). Since \( g'(y) = \ln(y/(y + \epsilon)) < 0 \), \( g \) is a decreasing function of \( y \). Therefore,

\[ g(y) \leq -\epsilon \ln \epsilon. \]

For \( x \in \mathcal{X} \), let \( |p(x) - q(x)| = \epsilon_x \). By our assumption,

\[ \sigma \triangleq \sum_{x \in \mathcal{X}} \epsilon_x \leq \epsilon. \] (C.15)

On the other hand, we just proved that

\[ |- p(x) \ln p(x) + q(x) \ln q(x)| \leq -\epsilon_x \ln \epsilon_x. \]

Therefore,

\[ |H(p) - H(q)| = \sum_{x \in \mathcal{X}} (- p(x) \ln p(x) + q(x) \ln q(x)) \]

\[ \leq \sum_{x \in \mathcal{X}} |- p(x) \ln p(x) + q(x) \ln q(x)| \]

\[ \leq \sum_{x \in \mathcal{X}} -\epsilon_x \ln \epsilon_x. \] (C.16)

Also

\[ \sum_{x \in \mathcal{X}} -\epsilon_x \log \epsilon_x = \sum_{x \in \mathcal{X}} -\epsilon_x \log \frac{\epsilon_x \sigma}{\sigma} \]

\[ = -\sigma \log \sigma + \sigma H \left( \frac{\epsilon_x}{\sigma} : x \in \mathcal{X} \right) \]

\[ \leq -\epsilon \log \epsilon + \epsilon \log |\mathcal{X}|, \] (C.17)

where the last line follows because \( f(y) \) is an increasing function for \( y \leq e^{-1} \).
APPENDIX D

CONNECTION BETWEEN $\ell_{LZ}$ AND $\hat{H}_k$

In this appendix, we adapt the results of [43] to the case where the source is non-binary. Since in this work in most cases we deal with real-valued sources that are quantized at different number of quantization levels and the number of of quantization levels usually grows to infinity with blocklength, we need to derive all the constants carefully to make sure that the bounds are still valid, when the size of the alphabet depends on the blocklength.

Consider a finite-alphabet sequence $z^n \in \mathbb{Z}^n$, where $|\mathbb{Z}| = r$ and $r = 2^b$, $b \in \mathbb{N}^2$. Let $N_{LZ}(z^n)$ denote the number of phrases in $z^n$ after parsing it according the Lempel-Ziv algorithm [13].

For $k \in \mathbb{N}$, let

$$n_k = \sum_{j=1}^{k} j r^j = \frac{r^{k+1}(kr - (k + 1)) + r}{(r - 1)^2}.$$ \hfill (D.18)

Let $N_{LZ}(z^n)$ denote the maximum possible number of phrases in a parsed sequence of length $n$. For $n = n_k$,

$$N_{LZ}(n_k) \leq \sum_{j=1}^{k} j r^j = \frac{r(r^k - 1)}{r - 1} = \frac{r^{k+1}(kr - (k + 1))}{(r - 1)(kr - (k + 1))} - \frac{r}{r - 1} \leq \frac{(r - 1)n_k}{kr - (k + 1)} \leq \frac{n_k}{k - 1}.$$ \hfill (D.19)

Now given $n$, assume that $n_k \leq n < n_{k+1}$, for some $k$. It is straightforward to check that for $k \geq 2$, $n \geq n_k \geq r^k$. Therefore, if $n > r(1 + r)$,

$$k \leq \frac{\log n}{\log r}.$$  

On the other hand, $n < n_{k+1}$, where from (D.18)

$$n_{k+1} = \frac{r^{k+2}((k + 1)r - (k + 2)) + r}{(r - 1)^2} = \frac{r^{k+2}(r - 1)\log_r n + r - 2)}{(r - 1)^2}.$$ 

Therefore,

$$k + 2 \geq \log_r \frac{n(r - 1)^2 - r}{(r - 1)\log_r n + r - 2}.$$

Restricting the alphabet size to satisfy this condition is to simplify the arguments, but the results can be generalized to any finite-alphabet source.
or

\[ k \geq (1 - \epsilon_n) \frac{\log n}{\log r}, \tag{D.20} \]

where

\[ \epsilon_n = \frac{\log((r-1)\log_2 n + r - 2)r^2)}{\log n}. \tag{D.21} \]

To pack the maximum possible number of phrases in a sequence of length \( n \), we need to first pack all possible phrases of length smaller than or equal to \( k \), then use phrases of length \( k + 1 \) to cover the rest. Therefore,

\[ N_{LZ}(n) \leq N_{LZ}(n_k) + \frac{n - nk}{k + 1} \leq \frac{nk}{k - 1} + \frac{n - nk}{k + 1} \leq \frac{n}{k - 1}. \tag{D.22} \]

Combining (D.22) with (D.20), and noting that \( \log r = b \), yields

\[ \frac{N_{LZ}(n)}{n} \leq \frac{b}{(1 - \epsilon_n) \log n - b}. \tag{D.23} \]

Taking into account the number of bits required for describing the blocklength \( n \), the number of phrases \( N_{LZ} \), the pointers and the extra symbols of phrases, we derive

\[ \frac{1}{n} \ell_{LZ}(z^n) = \frac{1}{n} N_{LZ} \log N_{LZ} + \frac{b}{n} N_{LZ} + \eta_n, \tag{D.24} \]

where

\[ \eta_n = \frac{1}{n}(\log n + 2 \log \log n + \log N_{LZ} + 2 \log \log N_{LZ} + 2), \tag{D.25} \]

On the other hand, straightforward extension of the analysis presented in [43] to the case of general non-binary alphabets yields

\[ \frac{1}{n} N_{LZ} \log N_{LZ} \leq \hat{H}_k(z^n) + \frac{N_{LZ}}{n}((\mu + 1) \log(\mu + 1) - \mu \log \mu + k \log r), \tag{D.26} \]

where \( \mu = N_{LZ}/n \). But, \((\mu + 1) \log(\mu + 1) - \mu \log \mu = \log(\mu + 1) + \mu \log(1 + 1/\mu) \leq \log(\mu + 1) + 1/\ln 2 < \log \mu + 2 \).

Also, it is easy to show that for any value of \( r \) and \( z^n \), \( n \leq \sum_{\ell=1}^{N_{LZ}} l \), or \( N_{LZ}(z^n) \geq \sqrt{2n} - 1 \), or \( n/N_{LZ}(z^n) \leq \sqrt{n} \), for \( n \) large enough. Therefore, since \( \mu^{-1} \log \mu \) is an increasing function of \( \mu \),

\[ \frac{\log \mu}{\mu} \leq \frac{\log n}{2\sqrt{n}}. \]

Hence, combining (D.24), (D.23) and (D.26), we conclude that, for \( n \) large enough,

\[ \frac{1}{n} \ell_{LZ}(z^n) \leq \hat{H}_k(z^n) + \frac{b(kb + b + 3)}{(1 - \epsilon_n) \log n - b} + \gamma_n, \tag{D.27} \]
where $\gamma_n = \eta_n + \frac{\log n}{2\sqrt{n}}$, and $\epsilon_n$ and $\eta_n$ are defined in (D.21) and (D.25), respectively. Note that $\gamma_n = o(1)$ and does not depend on $b$ or $z^n$.

APPENDIX E
EXponential convergence rates

Consider an aperiodic Markov process $\{Z_i\}_{i=1}^\infty$ of order $l$ with probability measure $\mu$, i.e.,

$$\mu(z^n) = \mu(z^l) \prod_{i=1}^n \mu(z_i^{|z_{i-1}^{|z_{i-2}^{|...^{|z_1|}}}}).$$

Let $r \triangleq |Z|$.

**Definition 1.** The process $Z = \{Z_i\}_{i=1}^\infty$ is called $\Psi$-mixing, if there exists a non-decreasing function $\Psi(g)$ such that $\Psi(g) \rightarrow 1$, as $g \rightarrow \infty$, and

$$\mu(u^{l_1}v^gw^{l_3}) \leq \mu(u^{l_1})\mu(w^{l_3})\Psi(g),$$

for all $l_1, l_2 \in \mathbb{N}$, $u^{l_1} \in Z^{l_1}$, $v^g \in Z^g$ and $w^{l_3} \in Z^{l_3}$.

First order Markov chains are known to be $\Psi$-mixing [34]. We prove that Markov chains of order $l$ have a weaker property similar to being $\Psi$-mixing. As we will show later, this is enough to prove that an aperiodic stationary Markov process of order $l$ has exponential rates for frequencies of all orders.

**Lemma 7.** Consider an aperiodic Markov process $\{Z_i\}_{i=1}^\infty$ of order $l$. There exists a non-decreasing function $\Phi : \mathbb{N} \rightarrow \mathbb{R}^+$, such that for all $l_1 \geq l, l_3 \geq l$, $u^{l_1} \in Z^{l_1}$, $v^g \in Z^g$ and $w^{l_3} \in Z^{l_3}$, we have

$$\mu(u^{l_1}v^gw^{l_3}) \leq \mu(u^{l_1})\mu(w^{l_3})\Psi(g),$$

and $\lim_{g \rightarrow \infty} \Psi(g) = 1$.

**Proof:** The proof follows directly from the case of $l = 1$. For general $l$, $\Psi$ is defined as

$$\Psi(g) = \max_{a^{l_1}b^l \in Z^{l_1} \times Z^l} \frac{M_a^{l_1}b^l}{\pi(b^l)},$$

where $M = [M_{a^{l_1}b^l}]_{a^{l_1} \times b^l}$ denotes the transition probability matrix of the process $\{Z_i\}_{i=1}^\infty$ and $\pi(\cdot)$ denotes its stationary distribution.

The following theorem uses Lemma 7 and extends Theorem III.1.7 of [34] to Markov processes of order $l$.

**Theorem 7.** An aperiodic Markov of order $l$ has exponential rates for frequencies of all orders. More precisely, given process $\{Z_i\}$, an aperiodic Markov chain of order $l$, for any $\epsilon > 0$, there exists $g \in \mathbb{N}$, depending only on $\epsilon$ and the transition probabilities of process $\{Z_i\}$, such that for any $k > l$ and $n > 6(k + g)/\epsilon + k$,

$$P(\{z^n : \|p_k(\cdot | z^n) - \mu_k\|_1 \geq \epsilon\}) \leq 2^{\epsilon^2/8}(k + g)n^{l|Z|^k}2^{-n^{\epsilon^2/2}},$$
where $c = 1/(2 \ln 2)$.

**Proof:** Since for any $z^n \in \mathbb{Z}^n$, $\|\mu_{k_1} - p_{k_1}(\cdot|z^n)\|_1 \leq \|\mu_{k_2} - p_{k_2}(\cdot|z^n)\|_1$, for all $k_1 \leq k_2$, it is enough to prove the statement for $k$ large.

Employing Lemma 7 and the technique used in Chapter III of [34], for $n > 6(k + g)/\epsilon + k$ and $k \geq l$, we derive

$$P(\{z^n : \|p_k(\cdot|z^n) - \mu_k\|_1 \geq \epsilon\}) \leq (k + g)|\Psi(g)|^t (t + 1)^{|Z| 2^{-tc\epsilon^2/4}},$$

where $c = 1/(2 \ln 2)$, and $t \in \mathbb{N}$ is the integer that satisfies

$$\frac{n}{k + g} - 1 \leq t < \frac{n}{k + g}.$$

On the other hand, by Lemma 7, $\Psi(g)$ is a non-increasing function that converges to 1. Hence, for $g$ large enough, $\Psi(g) < 2^{-c\epsilon^2/8}$, and

$$P(\{z^n : \|p_k(\cdot|z^n) - \mu_k\|_1 \geq \epsilon\}) \leq (k + g)(t + 1)^{|Z| 2^{-tc\epsilon^2/8}}.$$

Note that choice of $g$ depends only on the Markov chain transition matrix and $\epsilon$. □

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