Simplified tetrahedron equations: Fermionic realization

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Abstract

The natural generalization of the (two-dimensional) Yang-Baxter equations to three dimensions is known as the Zamolodchikov’s tetrahedron equations. We consider a simplified version of these equations which still ensures the commutativity of the transfer matrices with different spectral parameters and we present a family of free fermionic solutions.
1 Introduction

The Algebraic Bethe Ansatz (ABA) is a constructive method for solving 2d integrable models and it is natural to try to generalize the method to 3d integrable systems. The key relations in ABA are the Yang-Baxter equations (YBE), which ensure the commutativity of two transfer matrices with different spectral parameters. Zamolodchikov presented a three-dimensional generalization of the YBE in [1]. His so-called tetrahedron equations (ZTE) imply the factorization of the scattering matrices of more than three one-dimensional objects - strings, thereby playing the same role for strings as the YBE do for the scattering of more than two particles in two-dimensional integrable field theories [2,3]. ZTE also generalize another function of YBE - integrability condition for the lattice models. They ensure the commutativity of two transfer matrices with different spectral parameters [1,4]. In the three-dimensional case the scattering matrices (R-matrices) in the ZTE play the role of local evolution operators for the states on a 2d lattice (quantum spin model), or of the Boltzmann weights in the classical statistical models on 3d lattice. There exists a number of reformulations of ZTE in the context of integrable 3d models. In [4,14] the 3d conditions for integrability were formulated with the spin variables assigned to the lattice sites (interaction around the cube), while the spin variables in the original Zamolodchikov model were assigned to the faces of the 3d lattice. ZTE-equations for spin models where the spin variables live on the edges of a 3d cubic lattice (the so-called vertex formulation of ZTE) can be found in [5,7,13,17]. In the above listed articles a number of special solutions have been found and various parameterizations of ZTE were investigated (see also [15,16,18]). Finally 3d integrable models have been formulated in terms of free fermions in [6].

In the first section of this article we once more address the commutativity of two transfer matrices defined on a 3d lattice in the vertex formulation and we present a simplification of ZTE (STE), sufficient for the commutation of transfer matrices with different spectral parameters and defined on a 2d plane. The proof that STE implies the
commutation of the transfer matrices is similar the proof in the case of the full ZTE. We show that in a particular case the ZTE is formally equivalent to STE. In the last section we present a description of some solutions of STE which can be represented in terms of free fermions and we find the spectrum of the corresponding Hamiltonian.

2 Commutativity of the transfer matrices.

We consider a 3d classical statistical lattice vertex model on a cubic lattice with periodic boundary conditions. If $R_{i,j,k}^{i',j',k'}$ is the Boltzmann weight of a vertex which joins the links with variables labeled by $i, j, k$, and $i', j', k'$, then the partition function $Z$ of the model is defined as the product of the $R$’s over all lattice sites:

$$Z = \sum_{\{i,j,k\}} \prod_{\text{all vertices}} R_{i,j,k}^{i',j',k'}.$$  \hfill (1)

We can rewrite it by defining a monodromy matrix $T$ and the transfer matrix $\tau$, so that:

$$T = \prod' R_{i,j,k}^{i',j',k'}, \quad \tau = \text{tr} T, \quad Z = \text{tr} \tau^{N_z}. \hfill (2)$$

The monodromy matrix $T$ acts on the $N \times N \times N^2$ dimensional spaces (we assume that the lattice is cubic with the equal numbers of sites in each direction: $N_x = N_y = N_z = N$) and $\prod'$ denotes the product only over the horizontal (x y) plane of the 3d lattice. In the transfer matrix $\tau$ the trace is taken over the $N \times N$ dimensional spaces of variables living on the edge sites of the horizontal (xy) lattice. The variables on the vertical links (labeled by $z_{ij}$) can be regarded as an elements of the $N^2$ dimensional quantum space, while the horizontal ones form a $N \times N$ dimensional so-called auxiliary space, labeled by $x_i, z_j$ (see Fig. 1 for an illustration of the notation $R_{x_i y_j z_{ij}}$).

In order to find a condition which ensures that two transfer matrices $\tau(u)$ and $\tau(v)$ with different spectral parameters, $u$ and $v$, commute, we are going to define an extension of the YBE known from 2d models to the 3d models. In analogy with the method used for the
YBE we will define two intertwiner $R(u, v)$ matrices which will ensure the commutativity of the transfer matrices. At this point it should be mentioned that when we talk about spectral parameters in higher than two dimensions they should not be thought of as single real variable like in 2d, but as a set of continuous variables or angles on which the matrices $R_{ijk}$ depend.

The transfer matrix constructed from the $R_{ijk}$-matrices can be represented as

$$\tau(u) = \sum_{\{xy\}} \prod_{i=1}^{N} \prod_{j=1}^{N} R_{x_i, y_j, z_{ij}}(u)$$

Let us denote the variables of the auxiliary spaces by $x_i$ and $y_j$ in the first transfer matrix $\tau(u)$ and by $a_i, b_j$ in the second transfer matrix $\tau(v)$. As above the $N \times N$ states on the vertical links are denoted by $z_{ij}$ (see Fig. 1). Then the product of two transfer matrices is

$$\tau(u)\tau(v) =$$

$$\sum_{\{xyz\}} (R_{x_1, y_1, z_{11}}(u) \cdots R_{x_N, y_N, z_{1N}}(u) R_{x_1, y_1, z_{21}}(u) R_{x_2, y_1, z_{22}}(u) \cdots R_{x_2, y_N, z_{2N}}(u) \cdots R_{x_N, y_1, z_{NN}}(u) \cdots R_{x_1, y_N, z_{NN}}(u))$$

$$\times (R_{a_1, b_1, z_{11}}(v) \cdots R_{a_1, b_N, z_{1N}}(v) R_{a_2, b_1, z_{21}}(v) \cdots R_{a_2, b_N, z_{2N}}(v) \cdots R_{a_N, b_1, z_{NN}}(v) \cdots R_{a_N, b_N, z_{NN}}(v)) .$$

We expect, that there is an intertwiner operator $R(u, v)$ ensuring the commutativity of the transfer matrices with different spectral parameters:

$$R(u, v)T(u)T(v) = T(v)T(u)R(u, v),$$
\[
\tau(u)\tau(v) = \text{tr} (T(u)T(v)) = \text{tr} (R(u, v)T(v)T(u)R^{-1}(u, v)).
\] (6)

\( R \) matrix acts on the space labeled by the variables \( a_i, b_i, x_i, y_i \). Let us define the intertwiner \( R \) matrix as the product of the local intertwiners:

\[
R_{a,b;x,y}(u, v) = R_{x_N,a_N}(u, v) \cdots R_{x_1,a_1}(u, v) R_{y_N,b_N}(u, v) \cdots R_{y_1,b_1}(u, v).
\] (7)

By use of a so-called railway argument one can prove the commutativity condition (6):

\[
\text{tr} \{T(u)T(v)\} = \text{tr} \{R_{a,b;x,y}(u, v)T(u)T(v)R^{-1}_{a,b;x,y}\} =
\]

\[
= \sum_{\{a,b;x,y\}} \{R_{x_N,a_N}(u, v) \cdots R_{y_N,b_N}(u, v) \prod_{i=1}^{N} \prod_{j=1}^{N} (R_{x_i,y_j,z_{ij}}(u)R_{a_i,y_j,z_{ij}}(v))
\]

\[
\times R^{-1}_{y_N,b_N}(u, v) \cdots R^{-1}_{x_N,a_N}(u, v)\} = \sum_{\{a,b;x,y\}} \prod_{i=1}^{2} R_{x_i,a_i}(u, v) R_{y_i,b_i}(u, v) R_{x_1,a_1}(u, v) R_{y_1,b_1}(u, v)
\]

\[
\times R_{x_1,y_1,z_{11}}(u)R_{a_1,b_1,z_{11}}(v) \prod_{i=2}^{N} R_{x_i,y_i,z_{ij}}(u) R_{a_i,y_i,z_{ij}}(v)
\]

\[
\times \prod_{i=2}^{N} \prod_{j=1}^{N} R_{x_i,y_j,z_{ij}}(u) R_{a_i,y_j,z_{ij}}(v) \prod_{i=1}^{1} R^{-1}_{y_i,b_i}(u, v) \prod_{i=1}^{N} R^{-1}_{x_i,a_i}(u, v).
\]

While the ordering of the \( R_{x_1,a_j} \) and \( R_{b_i,y_j} \) operators is not important since they commute, we shall nevertheless treat them as non-commuting as long as possible (see later).

If the following equations for local \( R \)-matrices are fulfilled for every \( i, j = 1, ..., N \):

\[
R_{x_i,a_i}(u, v) R_{y_j,b_j}(u, v) R_{x_i,y_j,z_{ij}}(u) R_{a_i,b_j,z_{ij}}(v) = R_{a_i,b_j,z_{ij}}(v) R_{x_i,y_j,z_{ij}}(u) R_{y_j,b_j}(u, v) R_{x_i,a_i}(u, v).
\] (9)

then by taking them into account in the figure brackets of the equation (8) and successively continuing that procedure along of all plane we will get

\[
\tau(u)\tau(v) = \sum_{a,b;x,y} \prod_{i=1}^{2} R_{x_i,a_i}(u, v) \{R_{a_1,b_1,z_{11}}(v)R_{x_1,y_1,z_{11}}(u)R_{y_1,b_1}(u, v)R_{x_1,a_1}(u, v)\}
\]

\[
\times R_{y_2,b_2}(u, v) R_{x_1,y_2,z_{12}}(u) R_{a_1,b_2,z_{12}}(v) \prod_{i=3}^{N} R_{y_i,b_i}(u, v) R_{x_i,y_i,z_{11}}(u) R_{a_1,b_i,z_{11}}(v)
\]
Equations (9) can be represented graphically as in (2). We call them Semi-tetrahedron equations (STE) (referring to the known equations of Zamolodchikov).

If we force the intertwiner matrices to have a structure similar to the $R_{ijk}$ (the Boltzmann weights), we must consider the $R_{ab}^{cd}$ matrix elements as operators acting on an additional (auxiliary) space (the same space for all local intertwiners): $R_{c,x_i,a_i}$, $R_{c,y_i,b_i}$, and then take the trace in the transfer matrices over that space. In this case the intertwiner in the equation (7) will be

$$R_{a,b,x,y(c)}(u,v) = R_{c,x_N,y_N}(u,v) \cdots R_{c,x_1,y_1}(u,v)R_{c,a_1,b_1}(u,v) \cdots R_{c,a_N,b_N}(u,v).$$

(12)
and the commutativity conditions of transfer matrices \cite{9} are defined by Zamolodchikov’s tetrahedron equations:

\[
R_{c,x_i,a_i}(u,v)R_{c,y_i,b_i}(u,v)R_{x_i,y_i,z_{ii}}(u)R_{a_i,b_i,z_{ii}}(v) = R_{a_i,b_i,z_{ii}}(v)R_{x_i,y_i,z_{ii}}(u)R_{c,y_i,b_i}(u,v)R_{c,x_i,a_i}(u,v).
\] (13)

Up to the last step, all equations in (10) are valid in this general case as well, but the commutativity of intertwiners $R_{x_i,a_i}$ and $R_{y_i,b_i}$ is broken for general $R_{c,x_i,a_i}$ and $R_{c,y_i,b_i}$ and we have to make a more detailed analysis. First we should take trace in the space of new auxiliary variable $c$

\[
\text{tr}_c\{R_{y,b,c} \otimes R_{x,a,c}\}\{T_{x,y} \otimes T_{a,b}\} = \{T_{a,b} \otimes T_{x,y}\} \text{tr}_c\{R_{x,a,c} \otimes R_{y,b,c}\},
\] (14)

\[
R_{x,a,c} = \prod_i R_{c,x_i,a_i}, \quad R_{y,b,c} = \prod_i R_{c,x_i,y_i}.
\] (15)

As the variables $x, a$ and $y, b$ are independent, it is possible to change the positions of the R-matrices under the trace

\[
\text{tr}_c\{R_{x,a,c} \otimes R_{y,b,c}\} = \text{tr}_c\{R_{y,b,c} \otimes R_{x,a,c}\}
\] (16)
and following the by now familiar steps we arrive at the desired result

\[
\text{tr}_c \{ R_{y,b,c} \otimes R_{x,a,c} \} \{ T_{x,y} \otimes T_{a,b} \} \{ \text{tr}_c \{ R_{x,a,c} \otimes R_{y,b,c} \} \}^{-1} = \\
= \text{tr}_c \{ R_{y,b,c} \otimes R_{x,a,c} \} \{ T_{x,y} \otimes T_{a,b} \} \{ \text{tr}_c \{ R_{y,b,c} \otimes R_{x,a,c} \} \}^{-1},
\]

(17)

\[
\text{tr}_{a,b,x,y} \{ T_{x,y} (u) \otimes T_{a,b} (v) \} = \\
= \text{tr}_{a,b,x,y} \{ \text{tr}_c \{ R_{y,b,c} \otimes R_{x,a,c} \} \{ T_{x,y} (u) \otimes T_{a,b} (v) \} \{ \text{tr}_c \{ R_{y,b,c} \otimes R_{x,a,c} \} \}^{-1} \} = \\
= \text{tr}_{a,b,x,y} \{ T_{a,b} (u) \otimes T_{x,y} (v) \}
\]

(18)

In the Figure 2 we present graphically the semi-tetrahedron and the tetrahedron equations, respectively. It is seen that the semi-tetrahedron equations can be decomposed in two connected triangle equations.

The semi-tetrahedron equations are cases of the tetrahedron equations. It is seen that if

\[
\text{tr}_c (R_{a,b,c} R_{b,y,c}) = R_{x,a} R_{y,b}
\]

(19)

the tetrahedron equations reduce to the semi-tetrahedron equations. Thus the structure of the STE is somewhat simpler than the structure of the ZTE and they consist of fewer equations than ZTE.

3 Solutions of the STE using free fermionic representations.

We can present rather simple solutions of the STE by using a free fermionic realization of R-operator when the representation spaces where it acts are two-dimensional, i.e. the spaces of spin 1/2 states on the links of the 3d lattice.

In the two-dimensional integrable models considered in [10, 11, 12] the three-particle \( R_{ijk} \) matrices were represented in terms of fermionic operators:

\[
R_{123} : |i_1 \rangle \otimes |i_2 \rangle \otimes |i_3 \rangle \rightarrow |j_1 \rangle \otimes |j_2 \rangle \otimes |j_3 \rangle,
\]
Figure 3: Graphical representation of the semi-tetrahedron equations (the upper pair) and the tetrahedron equation (the lower pair).

\[ R_{123} = (-1)^{(p(i_1)+p(i_2))}{p(i_3)+p(j_3)}{p(i_1)+p(i_2)}{p(j_2)} \]
\[ \left( R_{123} \right)_{i_1j_2j_3} \left( |i_1\rangle \langle i_2| \right) \left( |j_2\rangle \langle j_3| \right) \left( |j_3\rangle \langle i_3| \right) \]
\[ (i,j)_{(1,2,3)} = 0, 1, \] (20)

with the fermionic creation and annihilation operators for each representation space defined as

\[ (|0\rangle\langle 0|)_i = 1 - n_i, \quad (|1\rangle\langle 1|)_i = n_i, \quad (|0\rangle\langle 1|)_i = c_i, \quad (|1\rangle\langle 0|)_i = c_i^+, \quad n_i = c_i^+c_i. \] (21)

Here \( p(\alpha) = \alpha \) are the parities of the states, due to the graded character of fermionic states [12]. The R-matrices we will use here differ from the above matrices by a permutation operator

\[ P_{13}|1\rangle \otimes |2\rangle \otimes |3\rangle = |3\rangle \otimes |2\rangle \otimes |1\rangle, \] (22)
\[ R_{123} = R_{13} R_{123} = (-1)^{p(i_1)p(j_2) + p(j_3)} (R_{123})^{j_1 j_2 j_3}_{i_1 i_2 i_3} (|j_1\rangle \langle i_1|) (|j_2\rangle \langle i_2|) (|j_3\rangle \langle i_3|). \]  

Similarly a representation for the two-particle R-matrix can be defined as

\[ R_{12} : |1\rangle|2\rangle \rightarrow |2\rangle|1\rangle, \quad R_{12} = (-1)^{p(i_1)p(j_2)} (R_{12})^{j_1 j_2}_{i_1 i_2} (|j_1\rangle \langle i_1|) (|j_2\rangle \langle i_2|). \]  

A representation of the three-particle \( R \) operators with \( R_{ijk} \)-matrices which cannot be reduced to products of two-particle R-matrices was found in [12]:

\[ R_{123} = e^{(a_{ij} - \delta_{ij}) c_i^+ c_j}; \quad i, j = 1, 2, 3, \quad a_{ii} = 0, \]  

where : () : means normal ordering.

In general, when \( a_{ii} \neq 0 \), the representation above has 9 independent parameters \( a_{ij} \) (not counting the normalization parameter). On the other hand the \( R_{123} \) operator in the so-called "free-fermionic" representation, \( e^{\sum_i (a_{ij} - \delta_{ij}) c_i^+ c_j} \), (called so since the fermions enter quadratically) conserves the number of the fermions and has 20 matrix elements. So in this case there are 10 constrains between the matrix elements in (23).

Similarly, the \( R_{12} \) operator : \( e^{\sum_i (a_{ij} - \delta_{ij}) c_i^+ c_j} \) : has 4 independent parameters \( a_{ij} \) and one normalization parameter, so one constraint, known from the XX model is imposed on the six matrix elements [24]:

\[ R_{00}^{10} R_{11}^{10} + R_{01}^{10} R_{10}^{10} = R_{01}^{01} R_{10}^{10}. \]  

Let us write down all ten relations between the \( R_{123} \) matrix elements, having the above "free-fermionic" representation.

\[
\begin{align*}
R_{010}^{10} R_{100}^{10} + R_{110}^{10} R_{000}^{10} &= R_{010}^{10} R_{100}^{10}, & R_{001}^{10} R_{100}^{10} + R_{101}^{10} R_{000}^{10} &= R_{001}^{10} R_{100}^{10}, \\
R_{110}^{10} R_{000}^{10} + R_{100}^{10} R_{010}^{10} &= R_{010}^{10} R_{100}^{10}, & R_{011}^{10} R_{000}^{10} + R_{001}^{10} R_{100}^{10} &= R_{001}^{10} R_{100}^{10}, \\
R_{011}^{10} R_{000}^{10} + R_{100}^{10} R_{010}^{10} &= R_{010}^{10} R_{100}^{10}, & R_{010}^{10} R_{000}^{10} + R_{001}^{10} R_{100}^{10} &= R_{001}^{10} R_{100}^{10}, \\
R_{110}^{10} R_{000}^{10} + R_{100}^{10} R_{010}^{10} &= R_{010}^{10} R_{100}^{10}, & R_{011}^{10} R_{000}^{10} + R_{001}^{10} R_{100}^{10} &= R_{001}^{10} R_{100}^{10}, \\
R_{101}^{10} R_{000}^{10} + R_{001}^{10} R_{100}^{10} &= R_{001}^{10} R_{100}^{10}, & R_{101}^{10} R_{000}^{10} + R_{001}^{10} R_{100}^{10} &= R_{001}^{10} R_{100}^{10}.
\end{align*}
\]
\[ R^{000}_{100} R^{111}_{010} = \det \begin{pmatrix} R^{100}_{100} \\ R^{100}_{100} \\ R^{010}_{010} \\ R^{010}_{010} \\ R^{001}_{001} \\ R^{001}_{001} \end{pmatrix}. \] (27)

In terms of \( a_{ij} \) parameters equations (25) contain 25 constraints - the independent ones from the \( 2^5 \times 2^5 \) equations for matrix elements. Substituting (23) and (24) in the STE equations (9) we find

\[
\sum_{j_0} R_{i_{a_1} a_{12} z} (u, v) R_{i_{b_1} b_{12} z} (u, v) R_{i_1 k z} (u, v) R_{i_{a_2} a_{21} z} (u, v) (1 - i_{j_0})^{p(j_0) + p(k_j) + p(k_a)} = \\
\sum_{j_0} R_{i_{a_1} a_{12} z} (u, v) R_{i_{b_1} b_{12} z} (u, v) R_{i_1 k z} (u, v) R_{i_{a_2} a_{21} z} (u, v) (1 - i_{j_0})^{p(j_0) + p(i_y) + p(i_a)}, \quad i_x, i_y, \ldots = 0, 1. \tag{28}
\]

Note that when dealing with fermionic spaces as here, in the matrix representation of the vertex ZTE (13), some additional signs will appear in the equations (see the Appendix for details), compared to the usual form of these equations (5). These signs are reflecting fermionic (graded) nature of our redefinitions of ZTE.

Looking for solution for STE let us to choose the parameterization for \( R \) matrices as

\[
R_{x a} (u, v) =: e^{(a_{ij}^i (u,v) - \delta_{ij}) c_i^+ c_j}, \quad R_{y s} (u) =: e^{(a_{ij}^j (u,v) - \delta_{ij}) c_i^+ c_j}, \\
R_{y b} (u, v) =: e^{(a_{ij}^i (u,v) - \delta_{ij}) c_i^+ c_j}, \quad R_{a z} (u) =: e^{(a_{ij}^j (u,v) - \delta_{ij}) c_i^+ c_j}. \tag{29}
\]

Then the constraints can be rewritten as (we are omitting three equations, which are identities, and we are suppressing the arguments of the \( a_{jk}^i \) functions)

\[
\begin{align*}
a_{21} a_{21}^3 + a_{11} a_{23}^3 a_{31} = a_{21}^2 a_{21}^4 & \quad a_{21}^2 a_{12}^2 + a_{11} a_{13}^2 a_{32} = a_{12}^2 a_{21}^4, \\
a_{21} a_{22}^3 + a_{21} a_{23}^3 a_{32} = a_{22}^2 a_{21}^4 & \quad a_{21} a_{11}^3 + a_{11} a_{13}^2 a_{31} = a_{21} a_{11}^4, \\
a_{31} a_{33}^3 = a_{13} a_{32}^4 + a_{13} a_{33}^4 a_{21}^2 & \quad a_{22} a_{33}^3 = a_{22}^3 a_{22}^2 + a_{22} a_{33}^2 a_{21}^2, \\
a_{21} a_{12}^2 = a_{22} a_{12}^2 + a_{23} a_{32}^2 a_{11} & \quad a_{12} a_{21}^4 = a_{21} a_{12}^2 + a_{23} a_{31} a_{11}^2, \\
a_{22} a_{31}^3 + a_{12} a_{33}^3 a_{31} = a_{33}^2 a_{31}^4 & \quad a_{22} a_{32}^3 + a_{12} a_{32}^2 a_{33} = a_{33} a_{32}^4, \\
a_{22} a_{12}^4 = a_{12} a_{12}^2 + a_{13} a_{32} a_{11} & \quad a_{12} a_{22} = a_{22} a_{12}^2 + a_{23} a_{32} a_{11}. 
\end{align*}
\]
We will pay special attention to the homogeneous case where the above equations can be converted into the following equations:

\[
\begin{align*}
  a_{22}^3 a_{22}^3 + a_{12}^2 a_{23} a_{32}^4 &= a_{22}^3 a_{22}^2 + a_{23} a_{32}^2 a_{21}^2, \\
  a_{22}^3 a_{11}^3 + a_{12}^3 a_{13} a_{31}^4 &= a_{11}^3 a_{22}^1 + a_{13} a_{31}^1 a_{21}^1, \\
  a_{22}^2 a_{21}^2 + a_{12}^2 a_{23} a_{41}^4 &= a_{21}^2 a_{22}^2 + a_{23} a_{41}^2 a_{21}^2, \\
  a_{22}^2 a_{12}^2 + a_{12}^2 a_{13} a_{32}^4 &= a_{12}^1 a_{22}^2 + a_{13} a_{32}^1 a_{21}^1, \\
 a_{21}^3 a_{32}^3 + a_{11}^3 a_{33} a_{32}^4 &= a_{32}^4, \\
 a_{23}^4 &= a_{13} a_{12}^2 + a_{33}^3 a_{11}^2, \\
 a_{11}^4 a_{21}^{12} &= a_{21}^2 a_{11}^2, \\
 a_{11}^2 a_{12}^2 &= a_{12}^1 a_{11}^2.
\end{align*}
\]

In the general case one has the following constraints on the \( a_{ij} \) parameters:

\[
\begin{align*}
  a_{11}^{(1)} &= a_{11}^{(2)}, \\
  a_{12}^{(1)} &= a_{12}^{(2)}, \\
  \frac{a_{23}^{(3)} a_{32}^{(3)}}{a_{13}^{(3)} a_{31}^{(3)}} &= \frac{a_{23}^{(4)} a_{32}^{(4)}}{a_{13}^{(4)} a_{31}^{(4)}}, \\
  \frac{\chi_{13}^{(1)} a_{23}^{(3)}}{a_{13}^{(3)}} &= \frac{\chi_{13}^{(2)} a_{23}^{(4)}}{a_{13}^{(4)}}, \\
  a_{11}^{(1)} &= a_{11}^{(2)} - a_{11}^{(4)}, \\
  a_{12}^{(2)} &= a_{12}^{(2)} - a_{12}^{(4)}, \\
  \frac{a_{33}^{(3)} a_{11}^{(3)} - \chi_{13}^{(4)}}{a_{33}^{(3)} a_{11}^{(3)} - \chi_{13}^{(4)}} &= \frac{a_{33}^{(4)} a_{11}^{(4)} - \chi_{13}^{(4)}}{a_{33}^{(4)} a_{11}^{(4)} - \chi_{13}^{(4)}}, \\
  \chi_{12}^{(2)} &= \frac{a_{11}^{(4)} a_{12}^{(3)}}{a_{11}^{(3)} a_{12}^{(4)}}, \\
  \chi_{12}^{(1)} &= \frac{a_{11}^{(4)} a_{12}^{(3)}}{a_{11}^{(3)} a_{12}^{(4)}}, \\
  a_{11}^{(1)} &= a_{11}^{(2)} - a_{11}^{(4)}, \\
  a_{12}^{(2)} &= a_{12}^{(2)} - a_{12}^{(4)}, \\
  \frac{a_{33}^{(3)} a_{11}^{(3)} - \chi_{13}^{(4)}}{a_{33}^{(3)} a_{11}^{(3)} - \chi_{13}^{(4)}} &= \frac{a_{33}^{(4)} a_{11}^{(4)} - \chi_{13}^{(4)}}{a_{33}^{(4)} a_{11}^{(4)} - \chi_{13}^{(4)}}, \\
  \chi_{ij}^{(\alpha)} &= a_{ki}^{(\alpha)} a_{rj}^{(\alpha)} - a_{kj}^{(\alpha)} a_{ri}^{(\alpha)}, \quad \alpha = 3, 4.
\end{align*}
\]

We will pay special attention to the homogeneous case

\[
\begin{align*}
  a_{ij}^{(r)} &= a_{ji}^{(r)}, \quad r = 1, 2, 3, 4, \\
  a_{11}^{(r)} &= a_{22}^{(r)}, \quad r = 1, 2, \\
  a_{11}^{(\alpha)} &= a_{22}^{(\alpha)} = a_{33}^{(\alpha)}, \quad \alpha = 3, 4, \\
  \chi_{13}^{(r)} &= \chi_{13}^{(\alpha)} + 1, \quad \chi_{23}^{(\alpha)} + 1 = \chi_{23}^{(r)} + 1 = \text{const} = \Delta, \quad \alpha = 1, 2, r = 3, 4, \\
  a_{ij}^{(1)} &= a_{ij}^{(2)}, \\
  a_{13}^{(r)} &= a_{23}^{(r)}, \\
  a_{12}^{(r)} &= a_{11}^{(r)} + \tilde{\Delta}, \quad r = 3, 4, \tilde{\Delta} = \text{const}.
\end{align*}
\]
The solution of these equations for $R_2$ can be parameterized via trigonometric functions

$$a_{11} = a_{22} = \sin(u)/\cos(u), \quad a_{12} = a_{21} = 1/\cos(u), \quad \Delta = 0.$$ \hfill (35)

This is the parameterization of the XX-model. For for the $R_3$ we then obtain

$$a_{12} = a_{21} = a_{11} = a_{22} = a_{33} = \sin(u)/\cos(u),$$

$$a_{13} = a_{31} = a_{23} = a_{32} = 1/\cos(u), \quad \bar{\Delta} = 0,$$ \hfill (36)

which is an extension of the XX R-matrix to three dimensional case. Then, by using the technique of coherent states developed in [9, 8] one can find a fermionic expressions of the transfer matrix $\tau(u)$ and the partition function $Z = \prod^N \tau(u)$:

$$\tau(u) = \int D\bar{\psi} D\psi : \exp\left\{ \sum_{i,j} \left( a_{12}\bar{\psi}_{x,i,j}\psi_{y,i-1,j-1} + a_{21}\bar{\psi}_{y,i,j}\psi_{x,i-1,j-1} + a_{11}\bar{\psi}_{x,i,j}\psi_{x,i-2,j} + a_{22}\bar{\psi}_{y,i,j}\psi_{y,i-2,j} - a_{13}\bar{\psi}_{x,i,j}\bar{c}_{z,i,j} + a_{31}\bar{c}_{z,i,j}\psi_{x,i,j} + a_{23}\bar{\psi}_{y,i,j}\bar{c}_{z,i,j} + a_{32}\bar{c}_{z,i,j}\psi_{x,i,j} + (a_{33} - 1)\bar{c}_{z,i,j}\bar{c}_{z,i,j} - \bar{\psi}_{x,i,j}\psi_{x,i,j} - \bar{\psi}_{y,i,j}\psi_{y,i,j} \right) \right\} :. \hfill (37)$$

After Fourier transformation we will have

$$\tau(u) = \int D\bar{\psi} D\psi \prod_{p,q}^{N\times N} : \exp\left\{ (a_{12}e^{i(p+q)}\bar{\psi}_{x,p,q}\psi_{y,p,q} + a_{21}e^{i(p+q)}\bar{\psi}_{y,p,q}\psi_{x,p,q} + a_{11}e^{2ip}\bar{\psi}_{x,p,q}\psi_{x,p,q} + a_{22}e^{2iq}\bar{\psi}_{y,p,q}\psi_{y,p,q} + a_{13}e^{ip}\bar{\psi}_{x,p,q}\bar{c}_{z,p,q} + a_{31}e^{ip}\bar{c}_{z,p,q}\psi_{x,p,q} + a_{23}e^{iq}\bar{\psi}_{y,p,q}\bar{c}_{z,p,q} + a_{32}e^{iq}\bar{c}_{z,p,q}\psi_{y,p,q} + (a_{33} - 1)\bar{c}_{z,p,q}\bar{c}_{z,p,q} - \bar{\psi}_{x,p,q}\psi_{x,p,q} - \bar{\psi}_{y,p,q}\psi_{y,p,q} \right) \right\} = \prod_{p,q}^{N\times N} a(p,q) : \exp\{ A_{p,q}\bar{c}_{z,p,q}\psi_{x,p,q} \} :. \hfill (38)$$

Here the normal ordering notion $::$ refers to the $\bar{c}_z$ and $c_z$ operators and

$$a(p,q) = 1 + (a_{11}a_{22} - a_{12}a_{21})e^{2i(p+q)} - (a_{11}e^{2ip} + a_{22}e^{2iq}),$$

$$A_{p,q} = a_{33} - 1 - \frac{1}{a(p,q)} \left( (a_{22}e^{2iq} - 1)a_{13}a_{31}e^{2ip} + (a_{11}e^{2ip} - 1)a_{32}a_{23}e^{2iq} - a_{12}a_{31}a_{23}e^{2i(p+q)} - a_{21}a_{13}a_{32}e^{2i(p+q)} \right). \hfill (39)$$
The solution corresponding to \((36)\) reads

\[
a(p, q) = 1 - \sin(u) / \cos(u) (e^{2ip} + e^{2iq}),
\]

\[
A_{p,q} = \sin(u) / \cos(u) - 1 + \frac{e^{2ip} + e^{2iq}}{\cos(u)^2 (1 - \sin(u) / \cos(u) (e^{2ip} + e^{2iq}))}.
\]

At the point \(u = 0\) we have

\[
a(p, q) = 1 - u (e^{2ip} + e^{2iq}),
\]

\[
A_{p,q} = u - 1 + (e^{2ip} + e^{2iq}) (1 + u (e^{2ip} + e^{2iq})),
\]

and the transfer matrix \(\tau(u)\) takes the following form for small values of the spectral parameter \(u\)

\[
\tau = \prod_{p,q}^{N,N} (a(p, q) : \exp\{A_{p,q} \bar{c}_{z,p,q} c_{z,p,q}\} : ) = \\
= \prod_{p,q} (a(p, q) : e^{(u-1+(e^{2ip}+e^{2iq})(1+u(e^{2ip}+e^{2iq})))\bar{c}_{z,p,q}c_{z,p,q}} : ) = \\
= \prod_{p,q} e^{P(p,q)\bar{c}_{z,p,q}c_{z,p,q} + u \varepsilon(p,q)\bar{c}_{z,p,q}c_{z,p,q}},
\]

where

\[
e^{P(p,q)} = e^{2ip} + e^{2iq}, \quad \varepsilon(p, q) = 2 \cosh P(p, q).
\]

When \(ip \to -\infty\) (or \(iq \to -\infty\)) this transfer matrix decouples into the product of \(N\) independent XX model’s transfer matrices defined on a 1D chain.

In the case \(a_{ii} = 0\) there are another solutions for the fermionic STE equations \((31)\). They are defined by three multiplicative relations for the parameters of three-particle R-matrices

\[
a_{21}^k a_{12}^k = \beta, \quad a_{13}^k = \alpha a_{12}^k a_{23}^k, \quad a_{31}^k = \gamma a_{32}^k a_{21}^k, \quad k = 3, 4,
\]
where the constants $\alpha, \beta, \gamma$ can be considered as model parameters. The relations can be reformulated as

$$a_{21}(u)a_{12}(u) = \beta, \quad a_{13}(u) = \alpha a_{12}(u)a_{23}(u), \quad a_{31}(u) = \gamma a_{32}(u)a_{21}(u). \quad (45)$$

The non zero matrix elements of corresponding $R_{ijk}$ operators in this case are the following (with above relations)

$$R_{000}^{00} = 1, \quad R_{100}^{00} = a_{12}, \quad R_{001}^{100} = a_{21},$$
$$R_{011}^{011} = a_{32}a_{23}, \quad R_{110}^{011} = a_{23}a_{31}, \quad R_{100}^{010} = a_{31},$$
$$R_{110}^{110} = a_{13}a_{31}, \quad R_{010}^{110} = a_{31}a_{12}, \quad R_{100}^{100} = a_{32},$$
$$R_{101}^{101} = -a_{12}a_{21}, \quad R_{011}^{101} = a_{23}a_{12}, \quad R_{101}^{010} = -a_{23}a_{31},$$
$$R_{111}^{111} = -a_{13}a_{21}, \quad R_{111}^{110} = a_{13}a_{32}a_{23}a_{12}a_{31}a_{12}. \quad (46)$$

While the non vanishing matrix elements of two $(R^{1,2})_{ij}$ intertwiners as solutions of the semi-tetrahedron equations have the following expressions in terms of the $a_{ij}$ functions of $R_{123}$’s

$$\left(R^1(u,v)\right)^{00}_{00} = a_{32}(u)/a_{32}(v), \quad \left(R^1(u,v)\right)^{11}_{11} = a_{21}(v)/a_{21}(u),$$
$$\left(R^1(u,v)\right)^{01}_{01} = a_{32}(u)/a_{32}(v)/a_{21}(v), \quad \left(R^1(u,v)\right)^{01}_{10} = 1,$$
$$\left(R^2(u,v)\right)^{00}_{00} = a_{12}(u)/a_{12}(v), \quad \left(R^2(u,v)\right)^{11}_{11} = a_{23}(v)/a_{23}(u),$$
$$\left(R^2(u,v)\right)^{10}_{01} = a_{12}(u)a_{23}(v)/a_{12}(v)a_{23}(u), \quad \left(R^2(u,v)\right)^{01}_{10} = 1. \quad (47)$$

As in two dimensional case we can extract from the commuting transfer matrices exact solvable quantum models (in our case on a two-dimensional lattice rather than on one-dimensional lattice), identifying first the logarithmic derivative of the transfer matrix with the Hamiltonian:

$$\ln \tau(u) = \ln \left( tr_{x_i,y_i} \prod_{i,j} R_{x_i,y_i,z_i,i}(u) \right) =$$
$$= \ln \tau(u_0) + u(\tau(u_0))^{-1} d\tau(u)/du|_{u=u_0} + ....$$
By inserting the representation \( R_{x_i,y_i,z_{i,j}}(u) \) for this equation and taking the trace over the auxiliary spaces, due to the gaussian form of the exponents in R-operators, \( \tau \) will have the similar form (in the following we shall omit the \( z_{ij} \) notation, keeping only \( ij \))

\[
\tau = (1 - \beta^N)^N : e^{\frac{1}{\alpha \beta \gamma} H(c^+, c)} :,
\]

\[
H(c^+, c) = \sum_{i,j} \sum_r A_r c_{i,j}^+ c_{i+r,j-r} + \sum_{i,j} \sum_r B_r c_{i,j}^+ c_{i+r+1,j-r} + \sum_{i,j} \sum_r C_r c_{i,j}^+ c_{i+r,j-r-1},
\]

\[
A_r = (\beta)^r \left( \frac{1}{\alpha} + \frac{1}{\gamma} \right) a_{13} a_{31}, \quad B_r = (\beta)^r a_{13} a_{31}, \quad C_r = (\beta)^r a_{32} a_{23}.
\]

From (22) it follows that

\[
a_{23} a_{32} = a_{13} a_{31} / (\alpha \beta \gamma).
\]

If \( a_{13} a_{31} \) is small it is possible to expand \( \tau \) in terms of the \( a_{13} a_{31} \) variables and \( H \) in the expression above manifests itself as a non-local Hamiltonian. From the structure of \( H \) it follows that although Hamiltonian is not local, there is localization around the path \((i, j \rightarrow i + 1, j + 1 \rightarrow ... \rightarrow i + r, j - r...)\). In momentum space this becomes more apparent

\[
H(c_{p_x,p_y}^+, c_{p_x,p_y}) = \sum_{p_x,p_y = px+p_0} \frac{1}{\beta} \left( \frac{1}{\alpha} + \frac{1}{\gamma} + \beta e^{ip_x} + \frac{1}{\alpha \gamma} e^{-ip_x + ip_0} \right) c_{p_x,p_y}^+ c_{p_x,p_y+p_0}^+, c_{p_x,p_y+p_0},
\]

\[
p_0 = i \ln \beta.
\]

In the case where

\[
\alpha \gamma = (\beta)^2, \quad \frac{\alpha}{\gamma} = e^{p_1}
\]

the Hamiltonian is real and can be written as

\[
H(c_{p_x,p_y}^+, c_{p_x,p_y}) = 2 \sum_p (\cosh p_1 + \cos p) c_{p,p+p_0}^+, c_{p,p+p_0},
\]

\[
p_0 = i \ln \alpha \gamma / 2, \quad p_1 = \ln \frac{\alpha}{\gamma}.
\]

In the limits \( \beta \rightarrow 0, \infty \) Hamiltonian \( (48) \) becomes completely local.
The expression (51) allows us to represent the transfer matrix as $\tau = \exp H$, and thus to find the exact discrete time Hamiltonian not only for small values of $a_{12}a_{21}$. The Hamiltonian follows directly from:

$$\tau = A : e^H := A \prod_k : e^{\epsilon(k)c_k^+c_k} = Ae^{\epsilon(k)c_k^+c_k}, \quad \epsilon(k) = \ln (1 + \varepsilon(k)).$$

As one can see from the expression (51) rotational symmetry is broken in the above model. In order to recover it we can suggest another integrable model, where the transfer matrix is constructed with the help of the same solutions $R_3$ and $R_2$ (satisfying (46) and (47)) of the local STE equations (9), as in the previous section, but with some chess like disposition of the $R_3$ and rotated $R_3$ matrices

$$\bar{\tau}(u) = tr \bar{\mathcal{T}}(u) = \sum_{\{x,y \}} \prod_j \left( \prod_i R_{x2i,y2j,z_{2i};2j}(u)R_{x2i+1,y2j,z_{2i+1};2j}(u) \prod_i R_{x2i,y2j+1,z_{2i};2j+1}(u)R_{x2i+1,y2j+1,z_{2i+1};2j+1}(u) \right), (52)$$

where $\tau_i, i = 1, 2, 3,$ refer to the rotated matrices

$$R_{\tau_i}^{i'j'k'} = R_{ijk}', \quad R_{\tau_2}^{i'j'k'} = R_{i'jk}', \quad R_{\tau_3}^{i'j'k'} = R_{ij'k'}. (53)$$

In this case the multiplicative intertwiner matrix, necessary for the commutativity of two transfer matrices with different spectral parameters:

$$\bar{R}_{a,b;x,y}(u,v)\bar{\mathcal{T}}(u)\bar{\mathcal{T}}(v) = \bar{\mathcal{T}}(v)\bar{\mathcal{T}}(u)\bar{R}_{a,b;x,y}(u,v), (54)$$

is modified a little compared to (7)

$$\bar{R}_{a,b;x,y}(u,v) = \prod_j R_{y_{2j},b_{2j}}(u,v)R_{y_{2j+1},b_{2j+1}}^{-1}(u,v) \prod_i R_{x_{2i},a_{2i}}(u,v)R_{x_{2i+1},a_{2i+1}}^{-1}(u,v). (55)$$

Here $\tau$ denotes

$$R_{cd}^{cd\tau} = R_{cd}^{ab} (56)$$
The commutativity can be verified by putting the right hand sides of (52) and (56) into the equation (54) and observing that it is fulfilled if four local equations are satisfied (as it was done in (8) and (10)). Here we are omitting all arguments of the operators:

\[
R_{yb}R_{xa}R_{xyz}R_{abz} = R_{abz}R_{xyz}R_{xa}R_{yb},
\]

(57)

\[
(R_{xa} \tau_{x} \tau_{a})^{-1} R_{yb} R_{xy} R_{abz} R_{abz} \tau_{a} = R_{abz} \tau_{a} R_{xy} \tau_{a} R_{yb} (R_{xa} \tau_{x} \tau_{a})^{-1},
\]

(58)

\[
(R_{yb} \tau_{y} \tau_{b})^{-1} R_{xa} R_{xyz} \tau_{y} R_{abz} R_{abz} \tau_{b} = R_{abz} \tau_{a} R_{xyz} \tau_{y} R_{xa} (R_{yb} \tau_{y} \tau_{b})^{-1},
\]

(59)

\[
(R_{yb} \tau_{y} \tau_{b})^{-1} (R_{xa} \tau_{x} \tau_{a})^{-1} R_{xy} \tau_{x} \tau_{y} R_{abz} \tau_{a} \tau_{b} = R_{abz} \tau_{a} \tau_{b} R_{xyz} \tau_{y} \tau_{x} (R_{xa} \tau_{x} \tau_{a})^{-1} (R_{yb} \tau_{y} \tau_{b})^{-1}.
\]

(60)

By \(\tau_{x}, \tau_{y}, \tau_{a}, \tau_{b}\) we mean the rotations or transpositions defined in (53) and (56):

\[
R_{xa} \tau_{x} \tau_{a} = R_{xa}, \quad R_{yb} \tau_{y} \tau_{b} = R_{yb},
\]

\[
R_{xy} \tau_{x} = R_{xy}, \quad R_{abz} \tau_{a} = R_{abz},
\]

\[
R_{xyz} \tau_{y} = R_{xyz}, \quad R_{abz} \tau_{b} = R_{abz},
\]

\[
R_{xyz} \tau_{x} \tau_{y} = R_{xyz}, \quad R_{abz} \tau_{a} \tau_{b} = R_{abz}.
\]

(61)

The solutions of the first equation (57) (which are the STE defined in (9)) are also solutions for other three equations. It can be shown for (58) by applying corresponding \(\tau\) operation on the standard STE equations as done below:

\[
(R_{yb} R_{xa} R_{xy} R_{abz}) \tau_{x} \tau_{a} = (R_{abz} R_{xy} R_{xa} R_{yb}) \tau_{x} \tau_{a},
\]

(62)

\[
R_{yb} R_{xy} R_{abz} \tau_{a} R_{xa} \tau_{x} \tau_{a} = R_{xa} \tau_{x} \tau_{a} R_{abz} \tau_{a} R_{xy} \tau_{y} R_{yb},
\]

(63)

Two other equations, (59) and (60), can be derived analogously.

The corresponding 2d quantum Hamiltonian in the fermionic representation for this model, unlike the previous one, has only nearest-neighbor and next to nearest-neighbor interactions, and exhibits a Manhattan like structure on the 2d square lattice shown in Fig. 3.

\[
\bar{\tau} = (1 - \beta^{2}) \sum_{N} C_{N} \frac{1}{2} \tilde{H}(c^{\dagger}, c) : ,
\]

(64)
\( H(c^+, c) = a_{31} a_{13} \sum_{i,j} (c_{2i+1,2j}^+ c_{2i,2j} + 1/\alpha c_{2i+1,2j-1}^+ c_{2i,2j} + \beta c_{2i,2j-1}^+ c_{2i,2j} + \beta/\alpha c_{2i,2j}^+ c_{2i,2j}) + \ldots \)

The \( H(c^+, c) \) consists of mass terms and hopping terms only along the elementary circuits, described by the arrows drawn on the links of the square lattice in the Fig 3 (if continued for all the links, they form a square Manhattan lattice). In (65) we find terms with \( c_{2i,2j} \) operators. They correspond to the circulations around two elementary circuits with clockwise and counterclockwise orientations, connected by the \( 2i, 2j \) vertex. The other three terms with the annihilation operators on the even-odd, odd-odd, even-even sites can be obtained from that expression, following to the directions of the arrows, outgoing from that site and writing the same horizontal, diagonal and vertical hopping parameters (and also the mass terms) as in the first term.

In the momentum space

\[
\begin{align*}
C_{2j,2k} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{-\frac{2\pi ip+2\pi k}{N}} C_{1p,q}, \\
C_{2j+1,2k} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{\frac{-2\pi (2j+1)p+2\pi k}{N}} C_{2p,q}, \\
C_{2j+1,2k+1} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{\frac{-2\pi (2j+1)p+2\pi (k+1)}{N}} C_{3p,q}, \\
C_{2j,2k+1} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{\frac{-2\pi (2j)p+2\pi (k+1)}{N}} C_{4p,q}, \\
C_{2j,2k} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{\frac{2\pi ip+2\pi k}{N}} C_{1p,q}, \\
C_{2j+1,2k} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{\frac{2\pi (2j+1)p+2\pi k}{N}} C_{2p,q}, \\
C_{2j+1,2k+1} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{\frac{2\pi (2j+1)p+2\pi (k+1)}{N}} C_{3p,q}, \\
C_{2j,2k+1} &= \frac{1}{N} \sum_{p,q=1}^{N/2} e^{\frac{2\pi (2j)p+2\pi (k+1)}{N}} C_{4p,q}
\end{align*}
\]

the Hamiltonian (65) will take a more compact form

\[
\begin{align*}
\hat{H}(p,q) &= c_{1p,q}^+ c_{1p,q} (a_{31} a_{13} \beta e^{-ip} + a_{23} a_{32} e^{ip}) + c_{2p,q}^+ c_{1p,q} (a_{31} a_{13} \beta e^{-ip} + a_{23} a_{32} \beta e^{-ip}) \\
&\quad + c_{1p,q} c_{3p,q} (a_{31} a_{13} \frac{1}{\alpha} e^{-ip+iq} + a_{23} a_{32} \beta \alpha e^{ip-iq}) + c_{3p,q} c_{1p,q} (a_{31} a_{13} \frac{1}{\alpha} e^{ip-iq} + a_{23} a_{32} \beta \alpha e^{-ip+iq}) \\
&\quad + c_{1p,q} c_{4p,q} (a_{31} a_{13} \beta e^{iq} + a_{23} a_{32} e^{-iq}) + c_{4p,q} c_{1p,q} (a_{31} a_{13} \beta e^{-iq} + a_{23} a_{32} e^{iq}) \\
&\quad + c_{2p,q} c_{3p,q} (a_{31} a_{13} \beta e^{ip} + a_{23} a_{32} e^{-ip}) + c_{3p,q} c_{2p,q} (a_{31} a_{13} \beta e^{-ip} + a_{23} a_{32} \frac{1}{\alpha} e^{ip}) \\
&\quad + c_{2p,q} c_{4p,q} (a_{31} a_{13} \beta e^{ip} + a_{23} a_{32} e^{-ip}) + c_{4p,q} c_{2p,q} (a_{31} a_{13} \beta e^{-ip} + a_{23} a_{32} \frac{1}{\alpha} e^{ip}) \\
&\quad + (c_{1p,q} c_{1p,q} + c_{3p,q} c_{3p,q}) (a_{31} a_{13} \frac{1}{\alpha} + a_{23} a_{32} \beta^2) \alpha
\end{align*}
\]
For the simple choice of the parameters $a_{31}a_{13} = a_{23}a_{32} = a^2$, $\alpha = 1$, $\beta = 1$, the Hamiltonian $\bar{H}(p, q)$ acquires simple matrix form

$$2a^2 \begin{pmatrix} 1 & \cos p & \cos (p - q) & \cos q \\ \cos p & 1 & \cos q & \cos (p + q) \\ \cos (p - q) & \cos q & 1 & \cos p \\ \cos q & \cos (p + q) & \cos p & 1 \end{pmatrix},$$

(67)

two nonzero eigenvalues of which gives us the spectrum

$$e_{\pm} = 8a^2(1 \pm \cos p \cos q).$$

(68)

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Appendix: ZTE equations for free fermionic representation \((29)\) of R-operators

In this appendix we present the independent set of ZTE equations \((13)\) in case when the three state R-operators admit the: \(e^{(\alpha_i - \delta_{ij})c_i^\dagger c_j} \): free-fermionic realization. For the operators \((23)\) the equations \((13)\), written in matrix form, are:

\[
\sum_{j,a} R_{ij,i} R_{j,j} R_{k,l} R_{l,k} (u) (-1)^{p(j) + p(k)} (p(\alpha_i) + p(\beta_j) + p(\gamma_i) + p(\delta_j)) = 0
\]

In the equations we have omitted spectral parameters and we have distinguished the elements of different matrices by labels 1, 2, 3, 4. In terms of \(a_{ij}^k\) elements the non-identity equations are

\[
\begin{align*}
    a_{12}^1 a_{21}^2 a_{13}^3 & = a_{21}^3 a_{12}^2 a_{21}^1, \\
    a_{22}^1 a_{31}^2 a_{12}^3 & = a_{31}^3 a_{22}^2 a_{12}^1, \\
    a_{12}^2 a_{21}^3 a_{13}^4 & + a_{13}^1 a_{12}^4 a_{13}^1 & = a_{13}^1 a_{12}^4 a_{13}^1 + a_{13}^1 a_{12}^4 a_{13}^1, \\
    a_{12}^3 a_{21}^4 a_{13}^2 & = a_{13}^1 a_{12}^4 a_{13}^1, \\
    a_{12}^4 a_{21}^5 a_{13}^3 & + a_{13}^1 a_{12}^6 a_{13}^1 & = a_{13}^1 a_{12}^6 a_{13}^1 + a_{13}^1 a_{12}^6 a_{13}^1, \\
    a_{12}^5 a_{21}^6 a_{13}^4 & + a_{13}^1 a_{12}^7 a_{13}^1 & = a_{13}^1 a_{12}^7 a_{13}^1 + a_{13}^1 a_{12}^7 a_{13}^1, \\
    a_{12}^6 a_{21}^8 a_{13}^5 & + a_{13}^1 a_{12}^9 a_{13}^1 & = a_{13}^1 a_{12}^9 a_{13}^1 + a_{13}^1 a_{12}^9 a_{13}^1, \\
    a_{12}^7 a_{21}^10 a_{13}^6 & + a_{13}^1 a_{12}^11 a_{13}^1 & = a_{13}^1 a_{12}^11 a_{13}^1 + a_{13}^1 a_{12}^11 a_{13}^1, \quad (70)
\end{align*}
\]
\[ a_{13}^1 a_{11}^2 = a_{21}^4 a_{13}^2 a_{11}^1 + a_{31}^4 a_{23}^2 a_{12}^1 + a_{31}^4 a_{13}^1 a_{12} + a_{11}^4 a_{13}, \]
\[ a_{22}^2 a_{12}^2 = a_{12}^2 a_{22}^1 + a_{22}^2 a_{12} a_{12}, \]
\[ a_{22}^3 a_{13}^1 + a_{32}^2 a_{14}^1 = a_{22}^3 a_{13}^1 a_{12}^1 + a_{32}^1 a_{14}^1, \]
\[ a_{22}^3 a_{13}^2 a_{12}^3 + a_{32}^4 a_{14}^2 = a_{22}^3 a_{13}^2 a_{14} + a_{32}^4 a_{14}^2, \]
\[ a_{22}^3 a_{13}^4 + a_{32}^4 a_{14}^3 = a_{22}^3 a_{13}^4 + a_{32}^4 a_{14}, \]
\[ a_{22}^3 a_{13}^5 + a_{32}^4 a_{14}^4 = a_{22}^3 a_{13}^5 + a_{32}^4 a_{14}. \]

(71)

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