The idea that a quantum system could behave like a liquid crystal holds a longstanding fascination. One widely-discussed example is the quantum spin-nematic, in which magnetic moments order as spin-quadrupoles, breaking spin-rotation symmetry without breaking time-reversal symmetry. However, despite a long history, experimental evidence for the existence of the spin-nematic state remains scarce. Here we show how the NMR $1/T_1$ relaxation rate can be used to distinguish the spin-nematic state from more conventional forms of magnetic order. Building on a symmetry-based theory of spin excitations, we find that, in a spin-nematic phase, $1/T_1$ is dramatically suppressed, does not diverge at continuous phase transitions, and exhibits unconventional power laws at low temperatures. We discuss how this theory could be applied to LiCuVO$_4$, where spin-nematic order has been proposed to exist at high magnetic field.

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In recent years, there has been considerable progress in the understanding of spin-nematic states. A rich vein of candidate systems has been identified among spin-1/2 magnets in which ferromagnetic and antiferromagnetic interactions compete. Bond-centred spin-nematic order is found in the spin-1/2 Heisenberg model relevant to a family of square-lattice vanadates\textsuperscript{19–21}, and the multiple-spin-exchange model describing thin films of \textsuperscript{3}He\textsuperscript{22,26}. Also, the quasi-2D magnet NiGa\textsubscript{2}S\textsubscript{4} has inspired the study of two different types of spin-nematic order in the spin-1 bilinear-biquadratic model on the triangular lattice\textsuperscript{24,25}. Perhaps the most promising candidate is the quasi-1D spin-chain material LiCuVO\textsubscript{4}, which undergoes an unexpected low-temperature phase transition in magnetic fields \( h \approx 43T^{10.14} \). This has been explained in terms of the condensation of bound pairs of magnons to form 2-sublattice, bond-centred, spin-nematic order [cf. Fig. 1(a)].

The high values of magnetic field needed to stabilise spin-nematic order in LiCuVO\textsubscript{4} render most experiments impractical. However measuring the NMR 1\(/ T_1 \) relaxation rates of, e.g. \(^{7}\text{Li} \), is technically feasible\textsuperscript{14}. The relaxation rate can be expressed in terms of the magnet’s dynamical spin susceptibility \( \chi^{\alpha\beta}(q, \omega) \) \textsuperscript{32–34}, with the most general form\textsuperscript{13},

\[
\frac{1}{T_1(h_{\text{ext}})} = \frac{1}{\gamma_N^2} \frac{g_B T}{\omega_N} \sum_{\mathbf{q}, \alpha, \beta} \mathcal{F}^{\alpha\beta}(\mathbf{q}, h_{\text{ext}}) \frac{3m \left \{ \chi^{\alpha\beta}(\mathbf{q}, \omega N) \right \}}{\hbar \omega_N} \tag{1}
\]

where \( \gamma_N \) is the nuclear gyromagnetic ratio, \( \omega_N \) is the NMR frequency, \( h_{\text{ext}} \) is the externally applied magnetic field, \( \alpha \) and \( \beta \) label spin components, and \( \mathcal{F}^{\alpha\beta}(\mathbf{q}, h_{\text{ext}}) \) is a form factor describing the coupling between nuclear and electronic spins. It is therefore possible to look for the ‘finger print’ of spin-nematic order through the coupling of its spin-dipole excitations to nuclear moments.

In what follows we develop a quantitative, symmetry-based theory for the spin excitations of a quantum spin nematic in its critical region (\( T \approx T_N \)), and low-temperature ordered phase (\( T < T_N \)), and use this to make corresponding predictions for 1\(/ T_1 \). For concreteness, we consider a 2-sublattice ‘antiferroquadrupolar’ (AFQ) spin-nematic state in applied magnetic field, of the type proposed to occur in LiCuVO\textsubscript{4}. We contrast this with the behaviour expected of a conventional canted antiferromagnetic (AFM). The results obtained are very general, and can easily be extended to quantum spin nematics in the absence of magnetic field. We note that a detailed theory for 1\(/ T_1 \) relaxation rates in LiCuVO\textsubscript{4}, at higher temperatures where 1D fluctuations predominate, has been developed in Ref. \textsuperscript{36,37}.

We consider first both the overall scale of 1\(/ T_1 \), and its behaviour in the vicinity of the critical point, \( T = T_N \). At any continuous phase transition, the correlation length associated with the order parameter diverges as \( \xi \propto T^{-\nu} \), where \( \nu = (T - T_N)/T_N \). In a conventional antiferromagnet (AFM), the order parameter is a dipole moment of spin, and this divergence is accompanied by a critical slowing in spin fluctuations. This in turn leads to a critical divergence in 1\(/ T_1 \). Calculating this within time-dependent Ginzburg-Landau theory\textsuperscript{21,22}, following Ref. \textsuperscript{1}, we find (see supplementary materials),

\[
\frac{1}{T_1^{\text{AFM}}} \approx \frac{(g_B \mu_B)^2}{4\pi^2} \gamma_N^2 \mathcal{F} \frac{k_B T}{(k_B T N)^2} \frac{1}{\Gamma K^2} f(\Lambda \xi), \tag{2}
\]

where the form factor \( \mathcal{F}^{\alpha\beta}(\mathbf{q}, h_{\text{ext}}) \to \mathcal{F} \) is assumed to be constant, \( \Gamma \) is a phenomenological damping parameter, \( K \) is a generalised elastic constant, \( f(x) = \arctan x - x/(1 + x^2) \) \textsuperscript{21,22}, \( \Lambda \approx \pi \) is a momentum cut-off, lengthscales are measured in units of the lattice spacing, energy in units of \( k_B T N \) and time in units of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{(Color online). Comparison of the NMR 1\(/ T_1 \) relaxation rate in a conventional antiferromagnet (AFM - red shaded area) and an antiferroquadrupolar (AFQ - blue shaded area) spin-nematic state. In the AFM, 1\(/ T_1 \) is seen to diverge approaching a critical point at \( T = T_N \). Within mean field theory, 1\(/ T_1 \) \( \propto \xi \), where \( \xi \) is the correlation length. Predictions are taken from Eq. (A6), with parameters relevant to KNiF\textsubscript{3} [3]: \( T_N = 253K, \mathcal{F} = 4 \left [T/\mu_B^2 \right ]^2, \alpha = 1, u = 1/3, K = 1/9, \chi_\perp = 1/4 \) and \( \Gamma = 0.5 \). The result has been convoluted with a Gaussian of standard deviation 0.03\( T_N = 7.6K \). In the AFQ, predictions are taken from Eq. (11), and the parameters are the same as the AFM, with additionally \( \lambda = 5 \) and \( \gamma = 5 \). The 1\(/ T_1 \) relaxation rate is highly suppressed in the AFQ relative to the AFM and there is no critical divergence at \( T = T_N \). The inset shows that there is instead a small bump at \( T = T_N \). The Feynman diagram describes the most important contribution to 1\(/ T_1 \) relaxation rate [cf. Eq. (10)].}
\end{figure}
\[ \frac{\hbar}{k_b T_1} \propto \xi^{2s-2-d} \propto \tilde{t}^{2-2s+2d}. \]  

where, within mean field theory, the dynamical exponent \( z = 2 \) and correlation length exponent \( \nu = 1/2 \). Experiment confirms that, at a critical point separating an AFM from the high-temperature paramagnet, 1/T1 is indeed described by Eqs. (3) and (4).

In order to understand the role of critical fluctuations in a spin nematic, it is necessary to generalise this analysis to a quadrupolar order parameter. A continuous phase transition from the paramagnet into the AFQ state is allowed by symmetry, and we find that the energy cost of static fluctuations of AFQ order is given by 15,47

\[ \mathcal{H}_{\text{AFQ}} \approx \int d^3 r \left[ \frac{\alpha t}{2} Q_1^2 \right. \]

\[ \left. + \frac{K}{2} \nabla Q_1 \right] \left( \frac{Q_1^2 - Q_1^2}{Q_1^2} \right), \]

where \( Q_1 \) is a canting field describing spin-dipole fluctuations parallel to applied magnetic field, an irrelevant coupling \( \hbar t^2 \) has been ignored, and the order parameter for the 2-sublattice AFQ state is given by

\[ Q_1(r) = \left( Q_{A1}^2 - Q_{B1}^2 \right), \]

where \( Q_{i1}^2 = (S_i^z)^2 - (S_i^z)^2, Q_{i1}^2 = S_i^z S_i^z + S_i^z S_i^z \) and \( i = A, B \). In order to separate longitudinal and transverse fluctuations we write \( Q_{i1}^2 = \eta \left( (n_i^2) - (n_i^2)^2 \right) \) and \( Q_{i1}^2 = 2m_i^2 n_i^2 \), where \( n_i = (n_i^z, n_i^z, 0) \) is a unit vector aligned with the nematic directors and \( n_A \cdot n_B = 0 \). This results in

\[ \mathcal{H}_{\text{AFQ}} \approx \int d^3 r \left[ \frac{\alpha t}{2} Q_1^2 \right. \]

\[ \left. + \frac{K}{2} \nabla Q_1 \right] \left( \frac{Q_1^2 - Q_1^2}{Q_1^2} \right) + \frac{\rho_0(\eta)}{2} \left( \nabla n_A^2 + \nabla n_B^2 \right) + \frac{1}{2} \chi_2^2 \left( \eta t^2 \right)^2 \]

with the director stiffness, \( \rho_0(\eta) = \eta^2 K \).

The longitudinal fluctuations described by \( \eta(r, t) \) do not couple directly to the nuclear spin lattice because the order parameter \( Q_1 \) is time-reversal symmetric. The 1/T1 relaxation rate is instead controlled by transverse fluctuations of the order parameter, parameterised by \( n_i \) and \( \eta^2 \). The coupled equations of motion, \( \partial_t \eta(r, t) \approx -\Gamma K (\xi^2 - \nabla^2) \delta \eta(r, t) + \zeta_1 \eta(r, t) \)

\[ \partial_t \phi(r, t) \approx \frac{\eta t^2}{\hbar} Q_1^2 \phi(r, t) + \gamma \rho_0 \nabla^2 \phi(r, t) + \zeta_2 \phi(r, t) \]

\[ \partial_t \omega^2 \approx \frac{\rho_0}{\hbar} \nabla^2 \phi(r, t) + \frac{\lambda}{\chi_2^2} \nabla^2 \nabla^2 \phi(r, t) + \zeta_3 \phi(r, t), \]

where \( \Gamma, \gamma \) and \( \lambda \) set the relaxation timescales for the fields \( \eta, \phi \) and \( \omega^2 \), and the mean field approximation \( u \to 0 \) has been assumed. The auxiliary fields \( \zeta_1, \zeta_2 \) and \( \zeta_3 \), which parameterise the damping of long-wavelength fluctuations by interaction with short-wavelength excitations, obey the correlation functions

\[ \langle \zeta_1(r, t) \zeta_1(r', t') \rangle = 2T_k \delta(r - r') \delta(t - t') \]

\[ \langle \zeta_2(r, t) \zeta_2(r', t') \rangle = 2\gamma T k \delta(r - r') \delta(t - t') \]

\[ \langle \zeta_3(r, t) \zeta_3(r', t') \rangle = 2\lambda T k \nabla^2 \delta(r - r') \delta(t - t'). \]

Fluctuations of the total magnetisation commute with \( \mathcal{H}_{\text{AFQ}} \) [Eq. (1)], so the \( q = 0 \) component of \( \eta^2 \) is conserved. Solving the equations of motion, Eq. (15), results in a dispersion,

\[ \omega_q^2 \approx \pm \nu q - \frac{i D q^2}{2}, \quad v = \sqrt{\frac{\rho_0}{\chi_2}}, \quad D = \frac{\lambda}{\chi_2^2} + \rho_q \gamma, \]

where the first term in \( \omega_q^2 \) describes the coherent motion of the quadrupoles and the second term is dissipative.

We are now in a position to calculate the dynamical spin susceptibility of the spin-nematic. This is not given by the dynamical correlations of the order parameter, but rather the associated fluctuations of the spin-dipole moments,

\[ \delta S^2(r, t) \propto \eta^2 \left[ q^2(r, t) + \left[ (\eta) + \delta \eta(r, t) \right] \eta^2(r, t) \approx \delta \eta(r, t) \right] \eta^2(r, t), \]

where the \( \delta S^2(r, t) \) terms are negligible for 1/T1 relaxation. This is because 1/T1 probes the spin susceptibility at \( \omega \to 0 \) [cf. Eq. (1)], and at low energies the \( \eta^2 \) term is suppressed by the fact that \( \eta^2(0, t) = 0 \) is a conserved quantity. The dynamical spin susceptibility is calculated from the diagram shown in Fig. 2 and this leads to

\[ \frac{k_b T}{\hbar} \gamma \rho_0 \eta^2 \left[ (\eta)^2 + \lambda (\omega + \omega')^2 (k + q)^2 \right] \]

\[ \frac{\hbar^2}{\lambda^2} \frac{\pi^2}{\Gamma^2} \left[ (\omega + \omega' - \omega_{k,q}^2) (\omega + \omega' - \omega_{k,q}^2) \right. \]

\[ \frac{\rho_0}{\hbar} \frac{\lambda}{\chi_2^2} \frac{\pi^2}{\Gamma^2} \left. \right] \]

where \( \chi_2 = K^{-1}(\xi^2 + q^2)^{-1} \).

Physically, these equations describe a longitudinal fluctuation of the quadrupole order parameter, which then cant, resulting in a spin-dipole fluctuation. In order for this process to occur, it is necessary that \( \tau_\rho \gg \tau_1 \), where the correlation time for the \( \eta^2 \) field is given by \( \tau_\rho^2 \approx 2\chi_2^2/\lambda q^2 \) in the paramagnet and \( \tau_\rho^2 \approx K^{-1} \chi_2^2 \). Close to the critical point this is equivalent to requiring \( \chi_2^2 K \lambda \ll 1 \).

Substituting Eq. (10) into Eq. (11) results in a relaxation rate,

\[ \frac{1}{T_1^{\text{AFQ}}} \approx \frac{\rho_0}{\hbar} \frac{(g\mu_B)^2}{8\pi^3} \frac{F h^2}{k_b T^2} \frac{1}{k_b T} \int d\omega \frac{d\omega'}{2\pi} \frac{G^2}{\lambda^2} \frac{\omega^2}{\chi_2^2} \]

\[ \frac{\hbar^2}{\lambda^2} \frac{\pi^2}{\Gamma^2} \left[ (\omega + \omega' - \omega_{k,q}^2) (\omega + \omega' - \omega_{k,q}^2) \right. \]

\[ \left. \right] \]

\[ \left( \omega - \omega_{k,q}^2 \right)^2 + \frac{\omega_{k,q}^4}{4} \left( \omega + \omega_{k,q}^2 \right)^2 + \frac{\omega_{k,q}^4}{4} \right). \]
where the form factor, $F$, is again assumed constant.

A numerical integration of this expression for an isotropic, 3D AFQ state results in the temperature dependence shown in Fig. 2. In direct contrast to the AFM, there is no divergence at the critical point, but instead a small bump. The relaxation rate in the AFQ is suppressed relative to the AFM even away from the critical point. For $t \approx O(1)$, assuming comparable parameters, the ratio of the two is small as,

$$\frac{1}{T_{1}^{\text{AFQ}}}/\frac{1}{T_{1}^{\text{AFM}}} \propto \left(\frac{\chi_{\perp}^{2} \Gamma_{K}}{\chi_{\perp}}\right)^{\frac{3}{2}} \ll 1.$$  \hspace{1cm} (12)

The enhancement of the relaxation rate in the paramagnetic state to the ordered phase (cf. Fig. 2) is due to the greater phase space for fluctuations.

We now turn to the low-temperature ordered state. In a conventional AFM, at low temperatures, $1/T_1$ is driven by the Raman scattering of thermally excited spin waves.\textsuperscript{32,33} In the absence of magnetic anisotropy, a canted AFM in field has a single linearly-dispersing Goldstone mode, associated with rotations of the order parameter about the direction of magnetic field. This controls $1/T_1$ for $T \ll T_N$ and leads, in 3D, to\textsuperscript{13,32,33},

$$\frac{1}{T_{1}^{\text{AFM}}} \approx \frac{\hbar V_{\text{cell}}^2}{48\pi m_{s}^{2} \gamma N} \left(\frac{k_{B}T}{\chi_{\perp}^{2} v^{6}_{s}}\right)^{3},$$  \hspace{1cm} (13)

where $m_s$ is the staggered ordered moment, $V_{\text{cell}}$ is the volume of a unit cell, $\bar{v}_{s} = (v_{x} v_{y} v_{z})$ is the mean spin-wave velocity and $\chi_{\perp}$ the transverse susceptibility. A constant form factor $F$ has been assumed.

The NMR $1/T_1$ relaxation rate at low temperatures in a quantum spin nematic is controlled by the long-wavelength excitations of the nematic order parameter. We have constructed a sigma-model model description of the low-energy excitations of AFQ order, based on the SU(3) algebra needed to describe general rotations of the quadrupolar order parameter. Details of this work will be presented elsewhere\textsuperscript{23} – here we concentrate on the implications for NMR $1/T_1$ relaxation rates in a 2-sublattice AFQ state of the type proposed for LiCuVO$_4$\textsuperscript{10,11}.

In the 2-sublattice AFQ there are four long-wavelength modes, one of which is a linearly dispersing Goldstone mode associated with rotations of the quadrupole moments within the plane perpendicular to magnetic field. There are also three gapped modes, two associated with out-of-plane rotations of quadrupole moments, and one that has the character of a fluctuating spin-density wave. This last mode has a strong dipole moment, but is gapped even in the absence of magnetic field.

At temperatures small compared to all energy gaps, only the Goldstone mode contributes to relaxation. Spin-dipole fluctuations are associated with time derivatives of the field, and, in 3D and for $T \ll T_N$ we find,

$$\frac{1}{T_{1}^{\text{AFQ}}} \approx \frac{9 \hbar V_{\text{cell}}^2}{256 \pi^{5}} \left(\frac{m_{s}^{2} \gamma_{N}^{2}}{\chi_{\perp}^{2} v^{6}_{s}}\right)^{3} \left(\frac{k_{B}T}{\chi_{\perp}^{2} v^{6}_{s}}\right),$$  \hspace{1cm} (14)

where $\hat{h}_{K T} \approx 146.5$, $(m_{s}^{2} \gamma_{N}^{2})^{2} = \sqrt{3}(\gamma_{B}^{2})^{2} |\langle Q_{\perp}\rangle|/2$, $\chi_{\perp}$ is the susceptibility associated with the Goldstone mode, and $v_{s}$ the algebraic mean of its velocity.\textsuperscript{12} Comparing Eq. (13) and Eq. (14), we see that the low-temperature prediction $1/T_1^{\text{AFQ}} \propto T^7$ for a spin-nematic state in magnetic field is dramatically suppressed relative to the $1/T_1^{\text{AFM}} \propto T^3$ predicted for a canted antiferromagnet.

We note that relaxation due to excitation of acoustic phonons also has a $1/T_1^{\text{AFQ}} \propto T^7$ behaviour,\textsuperscript{45} but could in principle be distinguished through its (lack of) magnetic field dependence.

The temperature dependence of $1/T_1$ in the spin nematic crosses over to $1/T_1 \propto T^5$ when the temperature becomes comparable with the energy gap, $\Delta$, to dynamical spin-density wave fluctuations, which couple more strongly to nuclear spins\textsuperscript{15}. We also also find $1/T_1 \propto T^5$ for a 3D AFQ state in the absence of magnetic field, in agreement with the $1/T_1 \propto T^{2d-1}$ quoted in Ref. \textsuperscript{23}\textsuperscript{23,30}. However the $1/T_1 \propto T^3$ found in magnetic field arises from a different physical process (see supplementary material).

In conclusion, we have shown that the NMR $1/T_1$ relaxation rate can be used to distinguish a spin-nematic state from conventionally ordered magnets. We find that: (1) $1/T_1$ is strongly suppressed relative to conventional magnets; (2) unlike conventional magnets, $1/T_1$ does not diverge at a continuous phase transition; and (3) $1/T_1$ exhibits unusual power-law behaviour at low temperature. The intrinsic smallness of $1/T_1$ in quantum spin nematics may render low-temperature power laws difficult to observe. However, the absence of a critical divergence in $1/T_1$ is a robust, qualitative feature, which should be experimentally testable even at the high magnetic fields and low temperatures needed to stabilise spin-nematic order in systems like LiCuVO$_4$.

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49. Details of the derivation can be found in the supplementary materials, together with a justification of the use of a constant form factor $F$ for $^7$Li NMR, and a discussion of the role of magnetic anisotropy.
50. We note that power laws $1/T_1 \propto T^{-7}$ and $1/T_1 \propto T^{-12}$ for a spin-nematic in the absence of magnetic field are quoted, without explanation, in Ref. [15].
Appendix A: Theory of NMR relaxation rates in a conventional canted antiferromagnet

For completeness and ease of comparison, we develop a theory of the relaxation rate in a conventional canted antiferromagnet (AFM), where $1/T_1$ is known to diverge for $T \to T_N$. Spin excitations in the AFM can be described phenomenologically using time-dependent Ginzburg-Landau theory, written in terms of the order parameter $m_\perp(r)$, and its associated canting field $\phi(r)$.

$$\mathcal{H}_{\text{AFM}} \approx \int d^3r \left[ \frac{\alpha^2}{2} \mathbf{m}_\perp^2 + \frac{K}{2} (\nabla \mathbf{m}_\perp)^2 + \frac{u}{4} \mathbf{m}_\perp^4 + \frac{1}{2\chi_\perp} (\phi^2)^2 \right]$$

where $\alpha$ is a term $h\phi^2$ ignored as irrelevant. Here $t = (T - T_N)/T_N$ is a generalised elastic constant, $\chi_\perp$ the transverse susceptibility, $\alpha > 0$ and $u > 0$ are positive constants and magnetic field is applied parallel to the $z$-axis. Length scales are measured in units of the lattice spacing and energy scales in units of $k_B T_N$. Expanding,

$$m_\perp(r) = \eta(r) \mathbf{n}(r) = [\langle \eta \rangle + \delta \eta(r)] \mathbf{n}(r),$$

where $\mathbf{n}(r)$ is a 2D unit vector, results in,

$$\mathcal{H}_{\text{AFM}} \approx \int d^3r \left[ \frac{\alpha^2}{2} \mathbf{n}^2 + \frac{K}{2} (\nabla \eta)^2 + \frac{u}{4} \eta^4 \right.$$

$$\left. + \frac{\rho_\eta(\eta)}{2} (\nabla \mathbf{n})^2 + \frac{(\phi^2)^2}{2\chi_\perp} \right] \tag{A1}$$

where the spin stiffness is defined by $\rho_\eta(\eta) = K\eta^2$. The Ginzburg-Landau theory predicts a diverging correlation length $\xi \propto t^{-\nu}$ approaching the critical point, with mean field exponent $\nu = 1/2$.

The critical divergence of $\xi$ is accompanied by a critical slowing down in fluctuations of AFM order. This can be described by adding dynamics to the static Ginzburg-Landau model, to capture the way in which the slow, long-wavelength fluctuations of AFM order are damped by interaction with short-wavelength fluctuations. Following Ref.’s [2–4], we write,

$$\partial_t \delta \eta(r, t) = -\Gamma K (\xi^2 - \nabla^2 \delta \eta(r, t)) + \zeta(r, t), \tag{A2}$$

where time is measured in units of $\hbar/k_B T_N$, $\Gamma^{-1}$ sets the rate of damping, $\nu \to 0$, and $\zeta(r, t)$ is a (white) noise term with correlation function,

$$\langle \zeta(r, t) \zeta(r', t') \rangle = 2k_B T \delta(r - r') \delta(t - t'). \tag{A3}$$

It follows that the correlation time is given by $\tau_\eta(q) \approx \Gamma^{-1} \chi_q$, with $\chi_q = K^{-1} (\xi^2 + q^2)^{-1}$. Approaching $T_N$, $\tau_\eta(0)$ diverges as $\tau_\eta(0) \propto \xi^z$ with dynamical exponent $z = 2$.

In the case of the antiferromagnet, fluctuations of the field $\eta$ are directly related to fluctuations of the spin-dipole moments, since, for moments canting in the $x$-$z$ plane, $S^z(r, t) \propto \pm \eta(r, t)$. Therefore, the dynamic susceptibility associated with $\eta$ controls the critical behaviour of the $1/T_1$ relaxation rate. The fluctuation-dissipation theorem allows the imaginary part of the dynamic susceptibility to be calculated from Eq. (A3) and Eq. (A4) as,

$$\frac{k_B T}{\hbar \omega} \Im \{\chi_\eta^zz(\mathbf{q}, \omega)\} \approx (g \mu_B)^2 \frac{k_B T}{(k_B T_N)^2} \frac{1}{\Gamma^2 \chi_\eta^2 + \omega^2}, \tag{A5}$$

where $g$ is the Landé g-factor. Thus the dynamic susceptibility diverges approaching the critical point for both $q \to 0$ and $\omega \to 0$.

The mean-field behaviour of the $1/T_1$ relaxation rate can be calculated from Eq. (1) in the main text and Eq. (A4) as,

$$\frac{1}{T_1} \approx \frac{(g \mu_B)^2}{4 \pi^2} \frac{k_B T}{(k_B T_N)^2} \frac{1}{\Gamma^2 \chi_\eta^2} f(\Lambda \xi) \xi, \tag{A6}$$

where a constant form factor, $F^{\alpha \beta}(\mathbf{q}, \mathbf{h}_{\text{ext}}) \to F$, is assumed, $\Lambda$ is a momentum cut-off and $f(x) = \arctan x - x/(1 + x^2)/2$, where $f(\infty) = \pi/4$. Approaching the critical point Eq. (A6) predicts $1/T_1 \propto \xi \propto t^{-\nu}$, i.e. the NMR $1/T_1$ relaxation rate diverges with mean field exponent $\nu = 1/2$. More generally this can be written as,

$$\frac{1}{T_1} \propto \xi^{z+2-d} \propto t^{(d-2-z)\nu}, \tag{A7}$$

where the exponents $z$ and $\nu$ exponents take on values appropriate for an $O(2)$ phase transition. However experiment confirms that mean field theory is qualitatively correct in predicting a divergence in $1/T_1$ [2–4].

Appendix B: NMR $1/T_1$ in LiCuVO$_4$ at high magnetic field

It has been suggested that a 2-sublattice antiferroquadrupolar (AFQ) spin nematic state may be realised in LiCuVO$_4$ at high magnetic field [10,11] [see Fig. 1 in the main text]. We here extend the analysis of $1/T_1$ in the AFQ state presented in the main text to the specific example of LiCuVO$_4$. The difference in the analysis is that LiCuVO$_4$ is quasi-1-dimensional, and, in consequence, the stiffness, correlation length and all related properties are anisotropic.

The anisotropic Ginzburg-Landau Hamiltonian [cf. Eq. (1) in the main text] is given by,

$$\mathcal{H} = \int d^3r \left[ \frac{\alpha^2}{2} \mathbf{Q}_\perp^2 + \frac{K}{2} (\partial_\theta \mathbf{Q}_\perp)^2 + \frac{u}{4} \mathbf{Q}_\perp^4 + \frac{1}{2\chi_\perp^2} (\phi^2)^2 \right], \tag{B1}$$

where $\mu = (a, b, c)$ labels the crystallographic axes, the Einstein summation convention is assumed, $t = (T - T_N)/T_N$ is the reduced temperature, $u > 0$, $\chi_\perp^2$
is the perpendicular susceptibility and \( z \) defines the field direction.

The order parameter can be rewritten using \( Q^z_{1} = \eta \nu \) and \( Q^\nu = 2\eta n^x n^y \), where \( \mathbf{n} = (n^x, n^y, 0) \) is a unit vector aligned with the nematic directors and \( \mathbf{n}_A \cdot \mathbf{n}_0 = 0 \). This results in,

\[
\mathcal{H} = \int d^3r \left[ \frac{\alpha t}{2} \eta^2 + \frac{K_{\mu} (\partial_{\eta} \eta)^2}{2} + \frac{\mu}{4} \eta^4 \right. \\
+ \frac{\rho_{\mu} (\eta)}{2} \left( (\partial_{\mu} \mathbf{n}_A)^2 + (\partial_{\mu} \mathbf{n}_B)^2 \right) + \frac{1}{2 \chi_{\perp}} (l^2)^2 \right],
\]

where the director stiffness, \( \rho_{\mu} (\eta) = 2\eta K_{\mu} \), is spatially anisotropic.

The phenomenological equation of motion associated with the longitudinal order parameter field is given by,

\[
\partial_t \delta \mathbf{r} (r,t) = -\Gamma K_{\mu} (\xi_{\mu}^z - \partial_{\eta}^z) \delta \mathbf{r} (r,t) + \zeta (r,t),
\]

where \( \xi_{\mu} \approx \sqrt{K_{\mu}/\alpha t} \) in the paramagnet, \( \xi_{\mu} \approx \sqrt{-K_{\mu}/2\alpha t} \) in the ordered state and,

\[
(\zeta (r,t) \zeta (r',t')) = 2k_B T \delta (r-r') \delta (t-t').
\]

The equations of motion for the transverse fields are,

\[
\partial_t \phi (r,t) \approx \frac{L^2 (r,t)}{\lambda^2} + \gamma \rho_{\mu} \partial^2_{\eta} \phi (r,t) + \zeta_{\phi} (r,t)
\]

\[
\partial_t L^2 (r,t) \approx \rho_{\mu} \partial^2_{\eta} L^2 (r,t) + \frac{\lambda_{\mu}}{\lambda^2} \partial^2_{\eta} \phi (r,t) + \zeta_{L^2} (r,t),
\]

where \( \lambda_{\mu} \) is anisotropic and,

\[
(\zeta_{\phi} (r,t) \zeta_{\phi} (r',t')) = 2\gamma k_B T \delta (r-r') \delta (t-t')
\]

\[
(\zeta_{L^2} (r,t) \zeta_{L^2} (r',t')) = 2\lambda_{\mu} k_B T \partial_{\eta}^2 \delta (r-r') \delta (t-t').
\]

Solving these equations results in the dispersion relation,

\[
\omega^\pm_q \approx \pm \frac{\sqrt{v_{\mu}^2 q^2}}{2} - \frac{i D_{\mu} q^2}{2},
\]

with,

\[
v_{\mu} = \sqrt{\frac{\rho_{\mu}}{\lambda^2}},
\]

and,

\[
D_{\mu} = \frac{\lambda_{\mu}}{\lambda^2} + \rho_{\mu} \gamma = \frac{K_{\mu}}{K} \left( \frac{\bar{\rho}}{\lambda^2} + \bar{\rho} \gamma \right) = \frac{K_{\mu}}{K} \bar{D},
\]

where \( \bar{K} = (K_\| K_\perp K_\gamma)^{1/3} \) and \( \bar{\rho} = \bar{\rho} \gamma \) are geometric means and we have made the assumption that \( \lambda_{\mu} = \lambda K_{\mu}/\bar{K} \).

The NMR relaxation rate can be calculated from the Feynman diagram shown in Fig. 3, and is given by,

\[
\frac{1}{T_1^{AFQ}} \approx \frac{\gamma^2 (g\mu_B)^2}{8\pi^5} \frac{F_{\Gamma}}{(k_B T)^2} \int d\omega d\omega_{\text{ext}} \frac{\Gamma k_i^2}{k_F^2} \left( \frac{k_F^2}{4} + \bar{\lambda}^2 \right) \left( \omega + \bar{\omega} k_i^2 \right),
\]

where \( \delta k_i = \bar{K}^{-1} (\bar{\xi}^{-1} + k_F^2)^{-1} \) and the approximation of a constant form factor, \( \mathcal{F} \rightarrow \mathcal{F} \), is justified for \( ^7\text{Li} \) NMR in \( \text{LiCuVO}_4 \) in Section C. Length-scales have been rescaled using \( \sqrt{K_{\mu}/K} \), and therefore the momentum integral can be taken over a sphere. The expression for \( 1/T_1 \) given in Eq. (B10) is equivalent to Eq. (11) in the main text, except for the fact that the correlation length, \( \xi \) stiffness \( \rho_{\mu} \) damping \( \lambda \) and all related quantities have been replaced with their geometric mean.

The result of numerically integrating this equation for parameters appropriate to \( \text{LiCuVO}_4 \) is shown in Fig. 4, where the response has been convoluted with a Gaussian of width 0.03\( T_N \). This clearly shows that there is no critical divergence at \( T = T_N \), merely a small bump. For a reasonable set of parameters (quoted in the figure caption) the \( 1/T_1 \) is also extremely small compared with a conventional AFM.

Appendix C: The form factor for \(^7\text{Li} \) NMR in the spin-nematic state

In Section 2 above, we assumed the NMR form factor to be a constant. We now justify this assumption for the case of \(^7\text{Li} \) NMR in \( \text{LiCuVO}_4 \).

We assume that the Li nuclear spin interacts via a transferred hyperfine interaction with the electronic spins on the 4 nearest-neighbour Cu atoms [see Fig. 4]. The
FIG. 4: (Color online). Predictions for the NMR $1/T_1$ relaxation rate in the spin nematic phase proposed to exist in LiCuVO$_4$ at high magnetic field$^{10,11}$. There is no critical divergence at $T = T_N$, but instead a small bump. Predictions are taken from Eq. $^{9}$ \( \chi^\perp \) with $T_N = 2K$, \( \mathcal{F} = 2.5 \times 10^{-3} [T/\mu_B]^2 \), $\alpha = 1$, $u = 1/3$, $K = 0.22$, $\chi = 1/20$, $\Gamma = 5$ and $\gamma = 5$, where the static parameters have been adapted from Ref. $^{12}$. The result has been convoluted with a Gaussian of standard deviation $0.03T_N$.

Internal magnetic field at the Li site is given by,

$$ h_{\text{int}}(t) = \sum_i \mathcal{A}_i \cdot m_i(t), $$

(C1)

where $\mathcal{A}_i$ is the nuclear-electron coupling tensor and $m_i(t)$ is the magnetic moment associated with the $i$th Cu atom in the 4-site plaquette. The form factor for $1/T_1$ relaxation is given by$^{13}$,

$$ \mathcal{F}^{\alpha\beta}(q, h_{\text{ext}}) = \sum_{\gamma, \delta} R_{h_{\text{int}}}^{\gamma} R_{h_{\text{int}}}^{\delta} + R_{h_{\text{int}}}^{\gamma} R_{h_{\text{int}}}^{\delta} \mathcal{A}_i^{\gamma, \alpha} \mathcal{A}_i^{\delta, \beta}, $$

(C2)

where $R_{h_{\text{int}}}$ is a rotation matrix relating the direction of the external magnetic field to the crystallographic coordinate axes and,

$$ \mathcal{A}_i^{\alpha, \beta} = \sum_i e^{i q \cdot r_i} \mathcal{A}_i^{\alpha, \beta}. $$

(C3)

In the spin nematic phase the Cu spin-1/2's form time-reversal-symmetric triplets, and the quadrupolar order parameter is associated with the bonds between Cu atoms. Excitations of this state mix a small spin-dipole component into the wavefunction, and these spin-dipole fluctuations are also associated with the bonds. The internal magnetic field at the Li site can be expressed in terms of the bond spin fluctuations by,

$$ h_{\text{int}}(t) = \mathcal{A}_{13} m_{13}(t) + \mathcal{A}_{24} m_{24}(t), $$

(C4)

where $\mathcal{A}_{13} = (\mathcal{A}_1 + \mathcal{A}_3)/2$, $\mathcal{A}_{24} = (\mathcal{A}_2 + \mathcal{A}_4)/2$ and $m_{ij}(t)$ describes the fluctuating spin-dipole moment associated with the bond joining the site $i$ with the site $j$ [see Fig. 4 for the numbering of Cu atoms]. Using the symmetry of the Li environment, these tensors can be expressed as,

$$ \mathcal{A}_{13} = \mathcal{A}_{24} = \begin{pmatrix} A_{aa} & 0 & A_{ac} \\ 0 & A_{bb} & 0 \\ A_{ac} & 0 & A_{cc} \end{pmatrix}, $$

(C5)

and in consequence,

$$ \mathcal{A}_q = 2 \begin{pmatrix} A_{aa} c_{ac} & 0 & i A_{ac} c_{ac} \\ 0 & A_{bb} c_{ac} & 0 \\ i A_{ac} c_{ac} & 0 & A_{cc} c_{ac} \end{pmatrix}, $$

(C6)

where,

$$ c_{ac} = \cos \left[ \frac{q_x a_0 - q_z c_0}{2} \right]. $$

(C7)

The dominant spin-dipole fluctuations are in-phase on neighbouring bonds (ie. $q \approx 0$) and parallel to the direction of applied magnetic field. In consequence, the relevant form factors are given by,

$$ \mathcal{F}^{aa}(q \approx 0, h^a) \approx 4 (A_{ac})^2 $$

$$ \mathcal{F}^{bb}(q \approx 0, h^b) \approx 0 $$

$$ \mathcal{F}^{cc}(q \approx 0, h^c) \approx 4 (A_{ac})^2. $$

(C8)

Thus, for magnetic field applied in the $a$ and $c$ directions the form factor is well approximated by a constant. The nuclear-electron tensor can be probed in the low-field, magnetically-ordered phase to find $A_{ac} \approx 0.025T/\mu_B$. In consequence $\mathcal{F}^{aa}(q \approx 0, h^a) \approx \mathcal{F}^{cc}(q \approx 0, h^c) \approx 2.5 \times 10^{-3}[T/\mu_B]^2$.

**Appendix D: NMR $1/T_1$ at low temperature in the spin nematic state**

Low temperature measurement of the NMR $1/T_1$ relaxation rate could also be used to distinguish the spin-
nematic state from conventional magnets. At low temperatures \( 1/T_1 \) is activated above an energy gap and then follows a power law behaviour as a function of \( T \), with different exponents in the spin nematic compared to conventional magnets. We consider here the spin nematic state that may be realised in LiCuVO4.

In a conventional collinear antiferromagnet at low temperature \( T \approx \frac{\text{F}}{\pi T_1} \),

\[
\frac{1}{T_1} \approx \frac{\mathcal{F} N_\text{cell} m_s^2 \gamma N}{8\pi \hbar k_B T} \Delta^2 \phi^2 (k_B T/\Delta),
\]

where a constant form factor \( \mathcal{F} \) is assumed, \( m_s \) is the ordered moment, \( V_\text{cell} = ab_a b_c c_0 \) is the volume of a unit cell, \( \Delta \) is an energy gap, \( \bar{\nu}_s \) is the geometric mean of the spin wave velocities, \( \chi_\perp \) is the perpendicular susceptibility and,

\[
\Phi(x) = x^2 \bar{\nu}_1 (e^{-1/x}) + x^3 \bar{\nu}_2 (e^{-1/x}),
\]

with \( \bar{\nu}_m(z) = \sum_{l=0}^\infty z^l I_m(z) \), the \( m \)th polylogarithm of \( z \). For temperatures \( T > \Delta \),

\[
\frac{1}{T_1} \approx \frac{\mathcal{F} N_\text{cell} m_s^2 \gamma N}{48\pi} \frac{(k_B T)^3}{\chi_\perp \bar{\nu}_s^3}.
\]

while for temperatures \( T \ll \Delta \),

\[
\frac{1}{T_1} \approx \frac{\mathcal{F} N_\text{cell} m_s^2 \gamma N}{8\pi} \frac{(k_B T)^2 \Delta}{\chi_\perp \bar{\nu}_s^3} e^{-\frac{\Delta}{\hbar k_B T}}.
\]

For a discussion of cases where the form factor is not constant see Ref. [13].

In a 2-sublattice AFQ state in large applied magnetic field, \( 1/T_1 \) can be calculated by an analogous method. The excitations of the quadrupolar order parameter can be described by a non-linear sigma model field theory, and this can be linearised to give the effective action,

\[
S_{2SL} [\phi_1, \phi_0, \phi_x, \phi_y, h] \approx \int_0^\beta d\tau \int d^3r \left\{ \chi_0^2 (h) (\partial_\tau \phi_1)^2 + \sum_\mu \rho_x^2 (h) (\partial_\mu \phi_1)^2 + \chi_0^2 (h) \Delta^2 \phi_1^2 + \chi_x^2 (h) (\theta_x \phi_4)^2 + \chi^2 (h) (\Delta_x \phi_x^2 + \Delta_y \phi_y^2 + \Delta_{xy} \phi_{xy}^2) \right. \\
+ \sum_\mu \rho_{xy} (h) (\partial_\mu \phi_0)^2 + \chi_{xy}^2 (h) \Delta_{xy} \phi_0^2 + \sum_\mu \rho_x \phi_0 \phi_0^2 \left. \right\},
\]

where \( \rho \)'s denote order parameter stiffness’ \( \chi \)'s susceptibilities, \( \Delta \)'s energy gaps and \( \mu = \{a, b, c\} \). This action describes four gapped modes with dispersia of the form \( \omega_k = \sqrt{\Delta^2 + v_k^2 k^2} \), where \( v_k = \sqrt{\rho_\mu/\chi} \). At low temperatures physical properties are controlled by rotations of the order parameter in the plane perpendicular to the magnetic field, and these are described by the field \( \phi_1 \).

The gap \( \Delta_{xy} \) allows for a small exchange anisotropy in this plane, as has been measured in LiCuVO4 [16]. At higher temperatures three additional modes with much larger energy gaps become important. The fields \( \phi_0 \) and \( \phi_{\pi} \) describe in-phase and anti-phase rotations of the order parameter towards the direction of the magnetic field.

The field \( \phi_4 \) describes fluctuations of a spin-dipole moment parallel to the field direction. It is not clear without solving a specific microscopic model, which of the gaps, \( \Delta_x (h), \Delta_{xy} (h) \) or \( \Delta_{xy, \pi} (h) \), is smallest. However, as far as the \( 1/T_1 \) relaxation rate is concerned, the three modes behave in an equivalent manner for \( ^7 \text{Li} \) NMR in LiCuVO4.

![Diagram](image.png)

FIG. 6: Diagrams used to calculate the low-temperature, dynamic spin susceptibility in the spin-nematic phase proposed to exist in LiCuVO4. At low temperatures the dynamic spin susceptibility is controlled by the field \( \phi_1 \), and is calculated using diagrams (a), (b) and (c). Slashes denote differentiation of the fields with respect to time. It follows that \( 1/T_1 \propto T^2 \). At temperatures comparable to the gap \( \Delta_x \), excitations described by the field \( \phi_4 \) become important, and diagram (d) is used to calculate the dynamic spin susceptibility. It follows from this diagram that \( 1/T_1 \propto T^3 \).

The action can be used to calculate the dynamic spin susceptibility, and \( 1/T_1 \) follows from Eq. (1) in the main text. At low temperature we just consider the field \( \phi_1 \). The lowest order contribution to the relaxation rate is the ‘three magnon’ term, which involves averages over six \( \phi_1 \) fields with two time derivatives. Evaluating this gives,

\[
\frac{1}{T_1} \approx \mathcal{F} \frac{9hV_{\text{cell}}^3}{32} (m_s^{\text{eff}})^2 \frac{\gamma N}{\chi_x^2} I(T),
\]
where \((m_{\text{eff}}^*)^2 = \sqrt{3}(g\mu_B)^2 |\langle Q_1 \rangle| / 2\) and,

\[
I(T) = \frac{\Delta_{\text{ex}}}{8\pi^5(v_1^*)^9} \left(\frac{k_B T}{\Delta_{\text{ex}}}\right)^7 \frac{1}{\Delta_{\text{ex}}/k_B T} \int_0^\infty \frac{dx_1 dx_2 \sqrt{x_1^2 + x_2^2}}{e^{x_1 + x_2}} \left(x_1^2 + x_1 x_2 + x_2^2\right)
\]

\[
\times \left(\frac{\Delta_{\text{ex}}}{k_B T}\right)^2 \sqrt{x_1^2 - \left(\frac{\Delta_{\text{ex}}}{k_B T}\right)^2} \left(x_1 + x_2\right)^2 - \left(\frac{\Delta_{\text{ex}}}{k_B T}\right)^2 e^{x_1 + x_2}\right)^2 \left(x_1 + x_2 - 1\right)\left(e^{x_1 + x_2} - 1\right) - 1\right),
\]

(D7)

where \(\bar{v}_z^2 = (v_{z,x}^2 v_{z,y}^2 v_{z,z}^2)^{1/3}\) is the geometric mean of the velocities and \(v_{z,\mu}^2 = \sqrt{\rho_{z,\mu}^2/\chi_{z,\mu}^2}\). At temperatures \(T \ll \Delta_{\text{ex}}\) this can be approximated by,

\[
\frac{1}{T_1} \approx I_{T_1} \approx \frac{F_7 \sqrt{3} I_{T_1} h V_{\text{cell}}^3}{256\pi^5} \left(m_{\text{eff}}^*\right)^2 \gamma_N^2 \frac{\Delta_{\text{ex}}^4 (k_B T)^3}{\chi_{z,\mu}^2} e^{-\frac{2\bar{v}_z}{\gamma_{\text{eff}}}}.
\]

where,

\[
\tilde{I}_{T_1} = \int_0^\infty dy_1 dy_2 \sqrt{y_1 y_2} e^{-(y_1 + y_2)} \approx 0.79.
\]

For temperatures \(T \gg \Delta_{\text{ex}}\),

\[
\frac{1}{T_1} \approx \frac{F_7 I_{T_1} h V_{\text{cell}}^3}{16\pi^5} \left(m_{\text{eff}}^*\right)^2 \gamma_N^2 \frac{(k_B T)^7}{(\chi_{z,\mu}^2)^2} e^{-\frac{2\bar{v}_z}{\gamma_{\text{eff}}}},
\]

where,

\[
I_{hT} = \int_0^\infty dx_1 dx_2 x_1 x_2 (x_1 + x_2) \left(x_1^2 + x_1 x_2 + x_2^2\right)
\]

\[
\times \left(\frac{\Delta_{\text{ex}}}{k_B T}\right)^2 \sqrt{x_1^2 - \left(\frac{\Delta_{\text{ex}}}{k_B T}\right)^2} \left(x_1 + x_2\right)^2 - \left(\frac{\Delta_{\text{ex}}}{k_B T}\right)^2 e^{x_1 + x_2}\right)^2 \left(x_1 + x_2 - 1\right)\left(e^{x_1 + x_2} - 1\right) - 1\right)\right) \approx 146.5.
\]

(D11)

At higher temperatures the gapped modes associated with the fields \(\phi_4, \phi_0\) and \(\phi_+\) become important, and dominate the \(1/T_1\) relaxation. It is not \textit{a priori} clear what the relative size of the three gaps is, or how they compare to \(T_N\). However, the contribution to the \(1/T_1\) relaxation rate associated with each mode is the same, and so we just consider in detail the \(\phi_4\) mode. The contribution to the relaxation rate is given by,

\[
\frac{1}{T_1} \approx \frac{3h V_{\text{cell}}^3}{16\pi^5} \left(m_{\text{eff}}^*\right)^2 \gamma_N^2 \frac{I(T)}{(\chi_{z,\mu}^2)^3},
\]

where,

\[
I(T) = \left(\frac{k_B T}{\Delta_z}\right)^5 \frac{1}{\Delta_z/\sqrt{k_B T}} \int_0^\infty \frac{dx_1 dx_2 \sqrt{x_1^2 + x_2^2}}{e^{x_1 + x_2}} \left(x_1^2 + x_1 x_2 + x_2^2\right)
\]

\[
\times \left(\frac{\Delta_z}{k_B T}\right)^2 \sqrt{x_1^2 - \left(\frac{\Delta_z}{k_B T}\right)^2} \left(x_1 + x_2\right)^2 - \left(\frac{\Delta_z}{k_B T}\right)^2 e^{x_1 + x_2}\right)^2 \left(x_1 + x_2 - 1\right)\left(e^{x_1 + x_2} - 1\right)\right).
\]

(D13)

At temperatures \(T \ll \Delta_z\) this can be approximated by,

\[
\frac{1}{T_1} \approx \frac{3\sqrt{3} I_{T_1} h V_{\text{cell}}^3}{8\pi^5} \left(m_{\text{eff}}^*\right)^2 \gamma_N^2 \frac{\Delta_z^2 (k_B T)^3}{(\chi_{z,\mu}^2)^3} e^{-\frac{2\bar{v}_z}{\gamma_{\text{eff}}}},
\]

where,

\[
\tilde{I}_{T_1} = \int_0^\infty dy_1 dy_2 \sqrt{y_1 y_2} e^{-(y_1 + y_2)} \approx 0.79.
\]

For temperatures \(T \gg \Delta_z\),

\[
\frac{1}{T_1} \approx \frac{3 I_{T_1} h V_{\text{cell}}^3}{16\pi^5} \left(m_{\text{eff}}^*\right)^2 \gamma_N^2 \frac{(k_B T)^5}{(\chi_{z,\mu}^2)^3} e^{-\frac{2\bar{v}_z}{\gamma_{\text{eff}}}},
\]

where,

\[
I_{hT} = \int_0^\infty dx_1 dx_2 x_1 x_2 (x_1 + x_2)
\]

\[
\times \left(\frac{\Delta_z}{k_B T}\right)^2 \sqrt{x_1^2 - \left(\frac{\Delta_z}{k_B T}\right)^2} \left(x_1 + x_2\right)^2 - \left(\frac{\Delta_z}{k_B T}\right)^2 e^{x_1 + x_2}\right)^2 \left(x_1 + x_2 - 1\right)\left(e^{x_1 + x_2} - 1\right) - 1\right)\right) \approx 8.66.
\]

(D17)

We have shown that the NMR \(1/T_1\) relaxation rate has a distinctive temperature dependence at low \(T\) in the AFQ spin nematic state. In principle this could be used to recognise the spin nematic state. However, this may prove experimentally difficult due to the long relaxation times.

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