Instabilities in a running superfluid and stripe supersolid

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The possible instabilities in a running superfluid has been a long-time historical problem since first studied by L. P. Landau. By constructing effective actions in terms of suitable order parameters, we revisit this outstanding open problem. We find that if the instability is driven by the SF Goldstone mode near \( k = 0 \), then there is a quantum Lifshitz transition from the SF to a Boosted SF (BSF) with the dynamic exponent \( z = 3/2 \), \( z_u = 3 \) subject to logarithmic corrections from a marginally irrelevant cubic term. This case may happen to exciton superfluids in bilayer quantum Hall systems or electron-hole bilayer systems, especially in weakly interacting Bose gas in cold atom systems. If the instability is driven by the roton mode near a finite momentum \( k = k_0 \), then there is a SF to a stripe supersolid transition with the dynamic exponent \( z = 1 \) which is in the same universality class as the \( z = 1 \) boosted Mott-SF transition studied previously in a different context. This case may apply to Helium 4 and also cold atom BECs where the rotons in the SF phase plays an important role. Driving a SF sufficiently fast may become an effective way to create a SS which is a long time sought novel state of matter.

1. Introduction: It is well known that many systems become a superfluid at sufficiently low temperatures \( T < T_c \). He4 or He3 is the oldest strongly interacting bosonic or fermionic systems which become a SF below \( T_c \sim 2.17 K \) and \( T_c \sim 1 mK \) respectively. Weakly interacting superfluid systems were created in bosonic \( ^2 \) and fermionic \( ^3 \) charge neutral atoms at even lower temperature \( \sim nK \). There are also some experimental evidences to suggest exciton superfluid of electrons and holes may have been realized in the bilayer quantum Hall systems at the total filling factor \( \nu = 2 \) and \( \nu = 3 \) subject to logarithmic corrections from a marginally irrelevant cubic term. This case may happen to exciton superfluids in bilayer quantum Hall systems or electron-hole bilayer systems, especially in weakly interacting Bose gas in cold atom systems.

The physics of driving an object inside a superfluid (SF) or driving the SF itself has a long history \[10–13\]. It may be necessary to distinguish the two different cases: (1) Controlling the moving object: An impurity moving in a superfluid. It was discussed in \[10, 13\] and more recently in \[14\]. If an object moves in a superfluid at \( T = 0 \) with a velocity below the critical velocity \( v < v_c^O \), there is no viscosity. However, when \( v > v_c^O \), a viscosity arises due to the emission of elementary excitations such as vortex rings \[11, 12\]. (2) Controlling the superfluid: The SF is flowing with a finite velocity \( v \). It was also discussed in \[10, 13\] and more recently in \[15\]. The flow of a SF with \( v > v_c^{SF} \) may not destroy SF, but the order parameter may develop small additional components around a roton minimum, therefore reduce the superfluid density. However, when increasing \( v \) further, the fate of SF is still not known yet. The first class is an non-equilibrium driven system which was experimentally investigated in cold atom BEC, by pulling an optical lattice \[9\], the second is an equilibrium steady one. In this manuscript, we focus on the second class from effective action approach in either phase or dual density representation whichever is suitable. We will also establish some intrinsic connections between this class and a seemingly unrelated problem: the fate of liquid Helium under the increasing pressure.

The global Pressure-Temperature (T-P) phase diagram of the Helium-4 is shown in Fig.1(a) \[13\]. The elementary excitations in the Helium-4 contains the SF phonon part near \( k \sim 0 \) and the roton part near \( k = k_0 \) (Fig.1b). In this well known T-P diagram, there may be a room to host a possible tantalizing supersolid phase which has both the SF order and the solid order \[16\]. In 2005, by using torsional oscillator measurement, Chan’s group \[17\] observed a marked 1 \( \sim 2\% \) non-classical rotational inertial (NCRI) of the solid 4He at \( T \sim 0.2K \), both when embedded in Vycor glass and in bulk Helium 4. The authors suggested that the NCRI may suggest the supersolid state of 4He. These experimental results inspired extensive theoretical \[18, 19\] and experimental interests to examine the very intriguing supersolid phase of 4He. However, a later refined experiment \[20\] excludes the putative SS in Fig.1a. Despite its absence in the He4 system, the SS phase is an interesting phase on its own. It was shown to exist in a lattice system \[21, 27\]. Of course, the SS in a lattice system is quite different than that in a continuum system \[18\]. In this manuscript, we will show that the SS may also exist in Helium 4 under a sufficiently large flow, namely the second class of problem in the last paragraph.

In this work, we develop a systematic and unified effective action approach in both the phase and its dual magnitude representation to study all the possible instabilities and quantum phase transitions (QPT) by driving
a SF beyond some critical velocities. If the instability is due to the SF Goldstone mode near $k = 0$, then there is a quantum Lifshitz transition from the SF to a boosted SF (BSF) with the dynamic exponent ($z_x = 3/2, z_y = 3$) and a marginally irrelevant cubic term. We work out the excitation spectrum in both phases, perform the renormalization group analysis and evaluate the scaling functions at a finite $T$ near the QPT. This scenario may apply to the exciton SF in BLQH $[4, 5]$ and ELBL $[6]$ where the magneto-roton exists only at a very high energy, especially in cold atom BEC systems with low critical velocities $[2, 3]$. If the instability is due to the roton mode near $k = k_0$, then there is a SF to a stripe supersolid (SSS) transition with the dynamic exponent $z = 1$ which is in the same universality class of the boosted Mott-SF transition studied in $[8, 29]$ with an emergent C- symmetry. We also analyze the symmetry breaking and the excitation spectrum in the SSS phase. Then as the boost increases further, there is a QPT from the SSS to the stripe solid with $z = 2$. From the general symmetry breaking principle, we argue that the resulting stripe solid may also have vacancies whose BEC may lead to the SSS. We show that the driving is a new and effective mechanism to generate a supersolid which is a long time elusive goal in low temperature physics and can be realized not only in He4, but also in various cold atom systems with long-range dipole-dipole interactions, spin-orbit couplings or dressed by Rydberg atoms.

2. Co-moving frame and the lab frame in a running SF

In the co-moving frame with the SF, the SF is static. In terms of the SF order parameter $\psi = \sqrt{\rho_0 + \delta \rho e^{i\phi}}$, one just takes the effective action inside the SF phase $[28]$: \[ L_{M:SF}[\delta \rho, \phi] = i\delta \rho \partial_x \phi + \rho_0 [v_x^2(\partial_x \phi)^2 + v_y^2(\partial_y \phi)^2] + u(\delta \rho)^2 + w \rho_0 (\partial_y \phi)^3 + \cdots \] (1)

where, in addition to phase-magnitude conjugation $i\delta \rho \partial_x \phi$, the last cubic term also breaks the Charge conjugation (C-) symmetry $\phi \to -\phi$ in the $(\delta \rho, \phi)$ representation.

Now one gets to the lab frame just by performing a Galilean transformation (GT) $[8, 29]$ $\partial_x \to \partial_x - ic \partial_y$ (we drop the $t$): \[ L_{L:SF}[\delta \rho, \phi] = i\delta \rho \partial_x \phi + \rho_0 [v_x^2(\partial_x \phi)^2 + v_y^2(\partial_y \phi)^2] + u(\delta \rho)^2 + w \rho_0 (\partial_y \phi)^3 + c \delta \rho \partial_y \phi \] (2)

where the boost velocity was pinned to be $\vec{c} = \vec{Q}/m$.

The order parameter in the lab frame becomes \[ \psi_{SF} = \sqrt{\rho_0 + \delta \rho e^{i(\vec{Q} \cdot \vec{x} + \phi)}} \] (3)

which carries a SF flow in the lab frame.

In the following, we will study the possible instability when the flow is beyond a critical one from both phase representation and its dual magnitude representation in the lab frame. The C- symmetry case was addressed in $[8]$ in a completely different context: a quantum magnet with SOC in a longitudinal Zeeman filed. The C- symmetry only exists in a lattice system at integer fillings $[8, 29, 30]$. However, there is no C- symmetry in all the continuous SF systems mentioned in the introduction. It is the absence of the C- symmetry which leads to the new QPTs in all the following sections $[31]$.

3. The quantum Lifshitz transition from the SF to the BSF driven by the instability in the Goldstone mode

Taking $\vec{c} = \vec{Q}/m$ an independent tuning parameter, we will study the putative SF to BSF transition tuned by this boost.

After integrating out the magnitude fluctuations in Eq(2) the quantum phase fluctuations are described by: \[ L_{L:SF}[\phi] = \frac{1}{2u}(\partial_x \phi - ic \partial_y \phi)^2 + \rho_0 [v_x^2(\partial_x \phi)^2 + v_y^2(\partial_y \phi)^2] + \frac{1}{2u} \rho_0 (\partial_y \phi)^3 \] (4)

Now we study how the SF evolves as one increases $\vec{Q}$. The mean-field state can be written as $\phi = \phi_0 + k_0 y$. Substituting it to the effective action Eq(4) leads to: \[ S_0 \propto (2U \rho_0 v_x^2 - c^2)k_0^2 + 2w U \rho_0 k_0^3 \] (5)

At a low boost $c^2 < 2U \rho_0 v_y^2$, $k_0 = 0$ is in the SF phase. Its spectrum is given by: \[ \omega = \sqrt{2uw(\nu_x^2 k_x^2 + \nu_y^2 k_y^2) - ck_y} \] (6)
At the critical boost between the SF and the boosted SF

$$c^2 = 2U\rho_0 v_y^2$$  \hspace{1cm} (7)

which gives the phase boundary in Fig.2a.

At a high boost $c^2 > 2U\rho_0 v_y^2$

$$k_0 = -\frac{c^2 - 2U\rho_0 v_y^2}{3wU\rho_0}$$  \hspace{1cm} (8)

where $w \propto c$, so its sign is completely determined by the driving. So the BSF phase has an additional modulation $k_0$ along the $y-$ axis on top of Eq.8.

$$\psi_{BSF} = \sqrt{\rho_0 + \delta \rho e^{i[(\tilde{q} + k_0) \cdot \vec{x} + \phi]}}$$  \hspace{1cm} (9)

Inside the BSF phase, the quantum phase fluctuations can be written as $\phi \rightarrow \phi_0 + k_0 y + \phi$. Expanding the action up to the second order in the phase fluctuations leads to

$$L_{BSF} = (\partial_\phi \phi - ic\partial_\theta \phi)^2 + 2U\rho_0 v_y^2(\partial_x \phi)^2 + (2c^2 - 2U\rho_0 v_y^2)(\partial_\theta \phi)^2 + 2wU\rho_0 (\partial_\theta \phi)^3 + b(\partial_\phi \phi)^4 + \cdots$$  \hspace{1cm} (10)

which leads to the gapless Goldstone mode inside the BSF phase:

$$\omega_k = \sqrt{2U\rho_0 v_y^2 k_x^2 + (2c^2 - 2U\rho_0 v_y^2)k_y^2 - ck_y}$$  \hspace{1cm} (11)

where one can see $2c^2 - 2U\rho_0 v_y^2 = c^2 + (c^2 - 2U\rho_0 v_y^2) > c^2$ when $c^2 > 2U\rho_0 v_y^2$, thus the $\omega_k$ is stable in BSF phase. It is instructive to expand the first kinetic term in Eq.11 as:

$$2U\mathcal{L} = Z(\partial_\phi \phi)^2 - 2ic\partial_x \phi \partial_y \phi + 2U\rho_0 v_y^2(\partial_x \phi)^2 + \gamma(\partial_\theta \phi)^2 + a(\partial_\phi \phi)^2 + 2wU\rho_0 (\partial_\theta \phi)^3 + b(\partial_\phi \phi)^4$$  \hspace{1cm} (12)

where $Z$ is introduced to keep track of the renormalization of $(\partial_\phi \phi)^2$, $\gamma = 2U\rho_0 v_y^2 - c^2$ is the tuning parameter.

The scaling $\omega \sim k_x^2, k_y \sim k_y^2$ leads to the exotic dynamic exponents ($z_x = 3/2, z_y = 3$). Then one can get the scaling dimension of $[\gamma] = 2$ which is relevant, as expected, to tune the transition, but $[Z] = [\theta] = 2 < 0$, so are two leading irrelevant operators which determine the finite $T$ behaviours and corrections to the leading scalings. However $[w] = 0$ is marginal. The standard field theory one-loop RG finds:

$$\frac{dw}{dt} = \epsilon w - Aw^2$$  \hspace{1cm} (13)

where $\epsilon = 2 - d$ and $A = 1/v_y^2 a > 0$. So it is marginally irrelevant. Setting $Z = w = 0$ in Eq.12 leads to the Gaussian fixed point action at the QCP where $\gamma = 0$, subject to the Logarithmic correction due to the marginally irrelevant $w$ term. Again it is the crossing metric $g_{\tau,T} = g_{\eta,\tau} = -ic$ in Eq.12 which dictates the quantum dynamic scaling near the QCP. It is a direct reflection of the new emergent space-time near the $z = (3/2, 3)$ QPT. Note that here the QPT is a quantum Lifshitz one tuned by $[\gamma] = 2$, so despite the cubic term $[w] = 0$, it could still be a 2nd -order transition in Fig.2a. in contrast to the conventional QPT where a cubic term drives a first order one.

The instabilities of driving a SF. In the absence of a roton, the instability happens near the origin (the phonon mode), it leads to a BSF phase in (a). If there exists a roton such as in He4, the instability at $k = 0$ will always be pre-empted by that near the roton. Then the instability near the roton minimum leads to a stripe supersolid (SSS) phase in (b). The QPT from the SF to the SSS is in the same universality class as that from the boosted Mott to the SF with $z = 1$. There is also a QPT from the SSS to the stripe solid with $z = 2$. The finite temperature melting transition from the stripe solid to the normal liquid is also in the 3d XY universality class. The vacancies BEC in the stripe solid leads to the SSS intervening between the SF and the stripe solid. The order parameters in all the 4 phases are: Normal liquid $\langle \psi_0 \rangle = 0, \langle \psi_G \rangle = 0$, SF $\langle \psi_0 \rangle \neq 0, \langle \psi_G \rangle = 0$, Stripe solid $\langle \psi_0 \rangle = 0, \langle \psi_G \rangle \neq 0$, SSS $\langle \psi_0 \rangle \neq 0, \langle \psi_G \rangle \neq 0$.

Now we evaluate the conserved currents in both SF and BSF phase. The $U(1)$ symmetry in the normal phase transpire as $\phi \rightarrow \phi + a$ for any shift $a$ inside the $U(1)$ symmetry broken SF phase, so the Noether current can be derived from Eq.11 as:

$$J_\tau = 2(\partial_\tau \phi - ic\partial_\phi \phi)$$

$$J_x = 4u\rho_0 v_y^2 \partial_x \phi$$

$$J_y = J_y - icJ_\tau = 2u\rho_0 [2v_y^2(\partial_y \phi) + 3w(\partial_\phi \phi)^2] - icJ_\tau$$  \hspace{1cm} (14)

In the SF phase, $\phi = \phi_0$, then $J_\tau, J_x, J_y = (0, 0, 0)$ and $J_\tau, J_x, J_y = (0, 0, 0)$ also. In the BSF phase, $\phi = \phi_0 + k_0 y$ where $k_0$ is given by Eq.8 then $(J_\tau, J_x, J_y) = (-ic2k_0, 0, 0)$, but $(J_\tau, J_x, J_y) = (-ic2k_0, 0, 2c^2k_0)$. So the conserved currents $(J_\tau, J_x, J_y)$ can still be used to distinguish the BSF from the SF phase.

However, if there exists roton shown in Fig.1c, then this SF to BSF transition will be preempted by the SF to a solid transition triggered by the roton touchdown as shown in the following sections.

4. The SF to the SF density wave transition driven by the instability in the roton mode

In Helium 4, due to the long-range Van der Waals interaction, the density-density interaction $V_d(q)$ develops
a roton minimum which drives the transition from the SF to a solid \cite{19}. Now the density-density interaction $U$ becomes long-ranged in Helium 4, so we adopt the notation in \cite{28} as $U = V_d(k) = a - bk^2 + \alpha k^4$ which can be written as $V_d(k) = r + \alpha (k^2 - k_0^2)^2$ near the roton minimum (Fig.1c). We also consider the isotropic case $v_x^2 = v_y^2$, then $\rho_x = \rho_0 v_x^2 = \rho_0 v_y^2$ is the superfluid density, $\kappa^- = \lim_{k \to 0} V_d(k) = a$ is the compressibility and $v^2(k) = \rho_s V_d(k)$. 

From Eq.(3) one can see that the density stays the same in both the lab frame and the co-moving frame. The dynamic structure factor is:

$$S_n^x(k,\omega) = S_n(\tilde{k})\delta(\omega - \epsilon_+ (\tilde{k})), \quad S_n(\tilde{k}) = \frac{\pi \rho_s k}{2v(k)}$$

(15)

where $\epsilon_+ (\tilde{k}) = v(k)k - ck_x$ is the quasi-particle excitation energy.

It is easy to see that a generalized Feynman relation still holds under the driving:

$$\epsilon_+ (\tilde{k}) = \int_0^\infty d\omega S_n^x(k,\omega)$$

(16)

A similar relation for the quasi-hole excitation energy $\epsilon_- (\tilde{k}) = v(k)k + ck_x$ can be derived by replacing $S_n^x(k,\omega)$ by $S_n^y(k,\omega)$.

After integrating out the phase fluctuations in Eq.(4) the quantum magnitude fluctuations can be used to describe such a transition under a driving:

$$\mathcal{L}[\delta \rho] = \frac{1}{2} \delta \rho(-k, -\omega)|\frac{\omega^2 + i2c\omega k_x}{\rho_s k^2} + (r - \frac{c^2 k^2}{\rho_s k^2}) + \alpha(k^2 - k_0^2)^2|\delta \rho(k, \omega) - w(\delta \rho)^3 + u(\delta \rho)^4 + \cdots$$

(17)

where $r$ is the roton gap near $k = k_0$. Again the cubic term in the density-density channel need to be included at the very beginning.

The $U$ term nails down the momentum $k$ to be in the roton ring $k = k_0$, the boost term pins it to be in the $k_x$ axis $k_x = \pm k_0$. So the boost term just introduce an easy-axis to the isotropic roton mode. So the resulting solid has only two shortest reciprocal lattice vectors $\vec{G} = \pm k_0\hat{x}$:

$$n = n_0 + (\psi_G e^{ik_0 x} + \psi^*_G e^{-ik_0 x})$$

$$= n_0 + 2|\psi_G| \cos(k_0 x + \alpha)$$

(18)

where $\psi_G$ is the complex order parameter. Its phase $\alpha$ is the gapless phonon mode due to the translational symmetry breaking. It is the stripe solid phase. In fact, as shown in \cite{34}, even without such an easy axis term which explicitly breaks the rotational symmetry, a strip solid phase is most likely to be the ground state lattice structure due to the spontaneously lattice symmetry breaking. In the presence of such an easy axis term, the stripe solid is the ground state.

Then writing $k_x = \pm k_0 + q_x, k_y = q_y$ and expanding up to the quartic term, we obtain:

$$\mathcal{L}[\delta \rho] = \frac{1}{2} \delta \rho(-k, -\omega)|\frac{\omega^2 + i2c\omega q_x}{\rho_s k_0^2} + 4\alpha k_0^2 q_x^2 + \frac{c^2}{\rho_s k_0^2} q_y^2 + \hat{r}|\delta \rho(k, \omega) - w(\delta \rho)^3 + u(\delta \rho)^4 + \cdots$$

(19)

where $\hat{r} = r - c^2/\rho_s$ is the boosted roton gap and $k_x = \pm k_0 + q_x, k_y = q_y$ need to be summed around both regimes near $\pm k_0$. Using the decomposition Eq.(18) one obtain the effective action describing the SF to the stripe solid transition \cite{36}:

$$\mathcal{L}[\psi_G] = \frac{1}{2} \psi^*_G(k,\omega)|\frac{\omega^2 + i2c\omega q_x}{\rho_s k_0^2} + 4\alpha k_0^2 q_x^2 + \frac{c^2}{\rho_s k_0^2} q_y^2 + \hat{r}|\psi_G(k, \omega) + u(\psi_G(k, \omega))^4 + \cdots$$

(20)

where due to the stripe structure, the cubic term plays no role. In fact as shown in \cite{34}, the cubic term play an important role only in a triangular lattice where the three shortest reciprocal lattice vectors form a closed triangle.

Substituting Eq.(15) into Eq.(4) we get the corresponding order parameter:

$$\psi_{SSDW} = \sqrt{\rho_s} e^{i(Q_x x + \phi)} [1 + |\psi_G| \cos(k_0 x + \alpha)/\rho_0]$$

(21)

which establishes the relation between the physical quantity $\psi_{SSDW}$ and the order parameter $\psi_G$ in the effective action Eq.(20). The translational symmetry $x \to x + a$ in Eq.(18) translates into the $U(1)_T$ symmetry of $\psi_G \to e^{i\alpha} \psi_G$ where its phase $\theta = k_0 a$ is any continuous real number \cite{35}. The $U(1)_T$ symmetry breaking leads to the Goldstone mode which is the phonon mode $\alpha$ in Eq.(18) due to the translational symmetry breaking to the SSDW phase. It breaks the $U(1)_T \times U(1)_T$ symmetry leading to the two Goldstone modes $\phi, \alpha$ which are is the superfluid and lattice Goldstone mode respectively.

Eq.(20) is nothing but in the same universality class of QPT from the boosted Mott to SF along the Path-I in Fig.3a in \cite{29}. So it has the dynamic exponent $z = 2$, the boost $c$ is exactly marginal. Setting $\hat{r} = \Delta - c^2/\rho_s = 0$ leads to the critical velocity:

$$c^2 = \rho_s \Delta$$

(22)

which gives the SF to the SSDW phase boundary in Fig.2b. When $\hat{r} > 0, \langle \psi_G \rangle = 0$, it is in the SF phase. When $\hat{r} < 0, \langle \psi_G \rangle \neq 0$, it is in the Stripe SF density wave (SDFW) phase. Despite the original action Eq.(2) has no C- symmetry, it has an emergent C- symmetry in the effective action Eq.(20).

The main differences between the BSF in Eq.(3) and the SSDW in Eq.(21) is that in the former, it is completely a phase driven QPT, so there is only one wavevector component $\tilde{Q} + \tilde{k}_0$, it has only one gapless Goldstone mode $\phi$, while in the latter, it is magnitude driven QPT, so there are three wavevector components $\tilde{Q}$ and $\tilde{Q} \pm \tilde{k}_0$ which
leads to the magnitude modulation in Eq\[21\] it has two gapless Goldstone modes $\phi$ and $\alpha$.

5. Crossover from the SSDW to the stripe supersolid. Eq\[21\] holds near the SF to the SSDW transition. As one increases the boost further, the superfluid density $\rho_s$ starts to decrease as the normal solid component $|\psi_G|$ develops in Eq\[15\]. Then the density order Eq\[15\] emerges as an independent order parameter. The SSDW crossovers to the stripe supersolid (SSS). One can write down a GL theory \[23\] in terms of the two independent order parameters $\psi$ and $\delta n$ and their mutual couplings. For example, the periodic potential $\delta n = 2|\psi_G|\cos(k_0 x + \alpha)$ in Eq\[15\] acts as a periodic potential $\delta n|\psi(x)|^2$ on the superfluid component $\psi$, so one can write down the generic expression for $\psi$:

$$\psi_{SSS} = \psi_0[1 + A|\psi_G|\cos(k_0 x + \alpha)] \quad (23)$$

where $\psi_0 = \sqrt{\alpha}e^{i(Q\cdot x + \phi)}$ and $A$ is a numerical factor of the order 1. Well inside the stripe supersolid, the coupling between the two gapless modes $\phi$ and $\alpha$ leads to two branches of supersolidons \[19\].

As the boost increase further, the normal solid component $|\psi_G|$ increases, the superfluid component $\psi_{SSS}$ decreases and eventually disappears. There is a QPT from the SSS where $\langle \psi_0 \rangle \neq 0$, $\langle \psi_G \rangle \neq 0$ in the normal solid $\langle \psi_0 \rangle = 0$, $\langle \psi_G \rangle = 0$ in Fig\[2\]. Just in terms of symmetry breaking, there is really no difference between the SSDW and the SSS, so Eq\[21\] where $a \sim \rho_0$ and Eq\[23\] $a \ll \rho_0$ have the same symmetry breaking structure. But the former works best near the SF to the SSDW transition with $z = 1$ where the SF component is non-critical, while the latter near the SSS to the stripe solid transition with $z = 2$ where the SF component becomes critical.

So it resembles the extended boson Hubbard model \[26\], \[27\] where a stripe supersolid was shown to always exist slightly away from 1/2 filling both by microscopic calculations \[21\], \[23\] and effective actions in the original basis \[21\], \[24\] and the dual vortex basis \[26\], \[27\]. Due to the lack of C-symmetry, the vacancies usually have lower energies than that of interstitials, the stripe solid always host some vacancies. If their excitation energies $E_v$ are positive, so can only be thermally excited. But when they become negative, they may undergo BEC. So approaching from the stripe solid side, the $\psi_{SSS}$ in Eq\[23\] can be interpreted as the BEC of vacancies in the spontaneously formed stripe solid in Eq\[15\]. The vacancies behave similarly as the holes on the top of the CDW or Valence Bond (VB) state at 1/2 filling examined in \[21\], \[24\] with $z = 2$. We expect it to be in the same class as the CDW to CDW-SS or VB to VB-SS transition \[21\], \[24\] with the dynamic exponent $z = 2$.

6. Normal Liquid to stripe solid transitions. In fact, by only keeping the $\omega = 0$ component in Eq\[20\] it may also be used to describe the normal liquid to stripe solid transition at a finite $T$ in Fig\[2\]. One need only replace the superfluid density $\rho_s$ by the normal liquid density $\rho_n$. The gap $\Delta$ is the one in the density-density correlation function (the static structure factor) $S(k) \sim \frac{1}{\Delta + \alpha(k^2 - k_0^2)}$ in the normal liquid. Then $\dot{r} > 0$, $\langle \psi_G \rangle = 0$, it is in the normal liquid phase. $\dot{r} < 0$, $\langle \psi_G \rangle \neq 0$, it is in the Stripe solid phase. Eq\[22\] still holds. The boost $Q$ still plays an important role. It is in the 3d XY universality class which may also be understood as a stripe lattice melting transition driven by the phonons $\alpha$ from the low temperature stripe solid side.

7. Experimental realizations. We study two kinds of instabilities: the one induced by the SF Goldstone mode near $k = 0$ in the absence of rotons and the one induced by the roton mode near $k = k_0$. Here we discuss their experimental realizations respectively. The sound velocity in He4 is about $v \sim 238 m/s$. In a conventional lab on the earth, taking a high way (magnetic levitated) train moving with a velocity $300 km/h \sim 83 m/s$ is still below this characteristic velocity. A civil air-craft flight can reach even higher $800 km/h \sim 240 m/s$ which just reaches the sound velocity in the Helium4. So the chance to see the instability near $k = 0$ is unlikely on the earth. As mentioned in the introduction, in addition to the well known SF in the He4, there are also excitonic SF in the electronic systems in semi-conductors. The Bilayer Quantum Hall systems (BLQH) \[4\], \[5\] hosts the exciton SF in the charge neutral sector with the Goldstone mode velocity $v_{BL} \sim 1.4 \times 10^4 m/s$. The electron-hole bilayer system (EHBL) holds the exciton SF \[3\] with $v_{EH} \sim 5 \times 10^3 m/s$. It is essentially impossible except going to a satellite orbiting around the earth. In this regard, the weakly interacting cold atom BEC systems \[21\] become better candidates with the SF Goldstone mode velocity $v \sim 1 cm/s$.

The critical velocity due to the rotons in He4 is reduces to $v \sim 60 m/s$, so Fig\[2\] can be mapped out by driving the SF beyond this critical velocity \[17\], \[18\]. Cold atom BEC systems with a long-range interaction such as a dipole-dipole interaction \[38\], \[41\] or with spin-orbital couplings \[44\], \[45\] or dressed by Rydberg atoms \[42\], \[43\] also support rotons with much smaller critical velocities. So Fig\[2\] may find wide applications in these cold atom systems with tunable roton gaps.

8. Conclusions. Obviously, the SF and the stripe solid phase break two completely different symmetries: the former the internal $U(1)_J$ symmetry whose breaking leading to the gapless Goldstone mode, the latter the translational symmetry whose breaking leading to the gapless lattice phonon modes. Just from general symmetry principle, there are two possibilities from the SF to the solid transition (1) a direct 1st order transition. This is the case driven by the pressure $P$ in He4 in Fig\[1\]. The QPT near $k = k_0$ is first order one driven by the pressure resulting a lcp solid structure, then the SF just disappears suddenly across the QPT. (2) It splits into two
second order ones with an intervening SS phase. This is the case in He4 driven by the boost or cold atom BEC with tunable rotons in Fig.2. The QPT near $k = k_0$ is second order one driven by the boost $Q$ resulting in a stripe solid structure, then the SF undergoes an accompanying second order QPT to a superfluid density wave (SDW). Then the Stripe SDW evolves into the stripe supersolid phase where the solid component is given by $\delta \rho$, the superfluid density wave component is given by its phase $\psi_{SDW}$. Finally the SSS phase gets into a stripe solid phase by kicking out of its SDW component $\psi_{SS}$ which stands for vacancies near the SSS to the stripe solid QPT. A microscopic calculation such as a QMC simulation is needed to test the existence of these vacancies and if they are stable against phase separations. If so, we may have discovered a new mechanism to realize a stable supersolid phase. Then the boost becomes an effective way to generate a supersolid which does not happen when increasing the pressure.

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