A Dichotomy Theorem for First-Fit Chain Partitions

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Abstract

First-Fit is a greedy algorithm for partitioning the elements of a poset into chains. Let $\text{FF}(w, Q)$ be the maximum number of chains that First-Fit uses on a $Q$-free poset of width $w$. A result due to Bosek, Krawczyk, and Matecki states that $\text{FF}(w, Q)$ is finite when $Q$ has width at most 2. We describe a family of posets $Q$ and show that the following dichotomy holds: if $Q \in Q$, then $\text{FF}(w, Q) \leq 2c(\log w)^2$ for some constant $c$ depending only on $Q$, and if $Q \not\in Q$, then $\text{FF}(w, Q) \geq 2^{w-1}$.

1 Introduction

A partially ordered set or poset is a pair $(P, \leq)$ where $P$ is a set and $\leq$ is an antisymmetric, reflexive, and transitive relation on $P$. We use $P$ instead of $(P, \leq)$ when there is no ambiguity in simplifying this notation. We write $x > y$ when $x \geq y$ and $x \neq y$. All posets in this paper are finite.

Two points $x, y \in P$ are comparable if $x \leq y$ or $y \leq x$. Otherwise, $x$ and $y$ are said to be incomparable, denoted $x \parallel y$. We say that $y$ covers $x$ if $y > x$ and there does not exist a point $z \in P$ such that $y > z > x$. A chain $C$ is a set of pairwise comparable elements, and the height of $P$ is the size of a maximum chain. An antichain $A$ is a set of pairwise incomparable elements, and the width of $P$ is the size of a maximum antichain.

A chain partition of a poset $P$ is a partition of the elements of $P$ into nonempty chains. Dilworth’s theorem states that for each poset $P$, the minimum size of a chain partition equals the width of $P$. A Dilworth partition of $P$ is a chain partition of $P$ of minimum size. A poset $Q$ is a subposet of $P$ if $Q$ can be obtained from $P$ by deleting elements. We say that $P$ is $Q$-free if $Q$ is not a subposet of $P$.

First-Fit is a simple algorithm that constructs an ordered chain partition of a poset $P$ by processing the elements of $P$ in a given presentation order. Suppose that First-Fit has already partitioned $\{x_1, \ldots, x_{k-1}\}$ into chains $(C_1, \ldots, C_t)$. First-Fit then assigns $x_k$ to the first chain $C_j$ such that $C_j \cup \{x_k\}$ is a chain; if necessary, we introduce a new chain $C_{t+1}$ containing only $x_k$.

We are concerned with the efficiency of the First-Fit algorithm. A classical example due to Kierstead (see, for example, pages 87 and 88 in [13]) shows that First-Fit may use arbitrarily
many chains even on posets of width 2. However, Bosek, Krawczyk, and Matecki \cite{4} proved that for each fixed poset \(Q\) of width at most 2, the number of chains used by First-Fit on a \(Q\)-free poset \(P\) is bounded in terms of the width of \(P\). Let \(\text{FF}(w, Q)\) be the maximum, over all \(Q\)-free posets \(P\) of width \(w\) and all presentation orders of \(P\), of the number of chains that First-Fit uses. The upper bound on \(\text{FF}(w, Q)\) given by Bosek, Krawczyk, and Matecki’s can be as large as a tower of \(w\)’s with a height that is linear in \(|Q|\).

1.1 Prior work

Aside from the result of Bosek, Krawczyk, and Matecki \cite{4}, prior work has focused on establishing bounds on \(\text{FF}(w, Q)\) when \(Q\) is a particular poset of interest. We outline the history briefly.

Let \(N\) be the 4-element poset with points \(\{a, b, c, d\}\) and relations \(a < c\) and \(b < c, d\). The performance of First-Fit on \(N\)-free posets is closely related to the performance of the greedy coloring algorithm on graphs that contain no induced copies of the 4-vertex path. The clique number of a graph \(G\), denoted \(\omega(G)\), is the maximum size of a set of pairwise adjacent vertices in \(G\). A proper coloring of \(G\) assigns to each vertex a color such that adjacent vertices receive distinct colors. The greedy coloring algorithm gives a proper coloring of \(G\) by processing the vertices of \(G\) in some order, greedily assigning to each vertex \(u\) the first color not already assigned to a neighbor of \(u\). Extending our notation to the analogous problem for graphs, let \(\text{FFG}(w, H)\) be the maximum, over all graphs \(G\) such that \(G\) contains no induced copy of \(H\) and \(\omega(G) \leq w\) and all orderings of the vertices of \(G\), of the number of colors used by the greedy coloring algorithm. Let \(P_n\) be the path on \(n\) vertices. It is well-known that \(\text{FFG}(w, P_4) = w\). If \(P\) is a poset and \(G\) is the incomparibility graph of \(P\), then \(P\) contains \(N\) as a subposet if and only if \(G\) contains an induced copy of \(P_4\). Hence we have \(w \leq \text{FF}(w, N) \leq \text{FFG}(w, P_4) = w\) and so \(\text{FF}(w, N) = w\). Kierstead, Penrice, and Trotter \cite{14} proved that \(\text{FFG}(w, P_5)\) is bounded by a function of \(w\), and a consequence of a theorem of Gyárfás and Lehel \cite{8} is that \(\text{FFG}(w, P_6)\) is unbounded. As noted in \cite{14}, combining results in these two papers gives that, when \(T\) is a tree, \(\text{FFG}(w, T)\) is bounded if and only if \(T\) does not contain \(P_2 + 2P_1\) as an induced subgraph, where \(P_2 + 2P_1\) is the disjoint union of a copy of \(P_2\) and two copies of \(P_1\).

Let \(r\) denote the chain with \(r\) elements. The disjoint union of posets \(P\) and \(Q\) is denoted \(P + Q\), with each element in \(P\) incomparable to every element in \(Q\). An interval order is a poset whose elements are closed intervals with \([x_1, x_2] < [y_1, y_2]\) if and only if \(x_2 < y_1\). Fishburn \cite{7} proved that a poset \(P\) is an interval order if and only if \(P\) is \((2 + 2)\)-free. The problem of determining the performance of First-Fit on interval orders is still open, despite significant efforts by various different research groups over the years. Currently, the best known bounds are \((5 - o(1))w \leq \text{FF}(w, 2 + 2) \leq 8w\). The lower bound is due to Kierstead, D. Smith, and Trotter \cite{11}. The upper bound is due to Brightwell, Kierstead, and Trotter (unpublished), and independently Narayanaswamy and Babu \cite{16}, who improved on the breakthrough column construction method due to Pennmaraju, Raman, and Varadarajan \cite{17}.

The interval orders are the \((2 + 2)\)-free posets; we obtain a larger class of posets by
forbidding the disjoint union of longer chains. Bosek, Krawczyk, and Szczypta [5] showed that when \( r \geq s \), \( \text{FF}(w, r + s) \leq (3r - 2)(w - 1)w + w \). Joret and Milans [10] improved the bound to \( \text{FF}(w, (r + s)) \leq 8(r - 1)(s - 1)w \). Dujmović, Joret, and Wood [6] further improved the bound to \( \text{FF}(w, (r + r)) \leq 8(2r - 3)w \), which is best possible up to the constants.

The ladder of height \( n \), denoted \( L_n \), consists of two disjoint chains \( x_1 < \cdots < x_n \) and \( y_1 < \cdots < y_n \) with \( x_i \leq y_j \) if and only if \( i \leq j \) and no relations of the form \( y_i \leq x_j \). Kierstead and M. Smith [12] showed that \( \text{FF}(w, L_2) = w^2 \) and \( \text{FF}(2, L_n) \leq 2n \). They also proved the general bound \( \text{FF}(w, L_n) \leq w^{\gamma(lg(w)+lg(n))} \), where \( lg(x) \) denotes the base-2 logarithm; this result plays an important role in our main theorem.

### 1.2 Our Results

Our aim is to say something about the behavior of \( \text{FF}(w, Q) \) in terms of the structure of \( Q \). We obtain subexponential bounds on \( \text{FF}(w, Q) \) when \( Q \) belongs to a particular family of posets \( Q \), and we also give an exponential lower bound on \( \text{FF}(w, Q) \) when \( Q \not\in Q \). From the point of view of the First-Fit algorithm, efficiency is vastly improved if a single poset in \( Q \) is forbidden. From the point of view of an adversary, forcing First-Fit to use exponentially many chains requires all posets in \( Q \) to appear.

For each \( x \in P \), we define the above set of \( x \), denoted \( A(x) \), to be \( \{ y \in P : y > x \} \); also, when \( S \) is a set of points, we define \( A(S) \) to be \( \bigcup_{x \in S} A(x) \). Similarly, the below set of \( x \), denoted \( B(x) \), is \( \{ y \in P : y < x \} \) and we extend this to sets via \( B(S) = \bigcup_{x \in S} B(x) \). We define \( A[x] = A(x) \cup \{ x \} \) and similarly for \( B[x] \). The series composition of posets \( S_1, \ldots, S_n \), denoted \( S_1 \oplus \cdots \oplus S_n \), produces a poset \( S \) which has disjoint copies of \( S_1, \ldots, S_n \) arranged so that \( x < y \) whenever \( x \in S_i \), \( y \in S_j \) and \( i < j \). The blocks of \( S \) are the subposets \( S_1, \ldots, S_n \).

### 2 Dichotomy Theorem

A poset is ladder-like if its elements can be partitioned into two chains \( C_1 \) and \( C_2 \) such that if \( (x, y) \in C_1 \times C_2 \) and \( x \) is comparable to \( y \), then \( x < y \). Our first lemma shows that every ladder-like poset is contained in a sufficiently large ladder.

**Lemma 1.** If \( P \) is a ladder-like poset of size \( n \), then \( P \) is a subposet of \( L_n \).

**Proof.** Let \( P \) be a ladder-like poset of size \( n \). Clearly the 1-element poset is a subposet of \( L_1 \), and so we may assume \( n \geq 2 \). Let \( C_1 \) and \( C_2 \) be a chain partition of \( P \) such that whenever \( (x, y) \in C_1 \times C_2 \) and \( x \) and \( y \) are comparable, we have \( x < y \). Suppose that \( P \) has a maximum element \( u \). Recall that \( L_n \) consists of chains \( x_1 < \cdots < x_n \) and \( y_1 < \cdots < y_n \) with \( x_i \leq y_j \) if and only if \( i \leq j \). By induction, \( P - u \) can be embedded into the copy of \( L_{n-1} \) in \( L_n \) induced by \( \{ x_1, \ldots, x_{n-1} \} \cup \{ y_1, \ldots, y_{n-1} \} \). Allowing \( y_n \) to play the role of \( u \) completes a copy of \( P \) in \( L_n \). Next, suppose that \( P \) has no maximum element. Let \( u = \max C_2 \), let \( S = \{ v \in C_1 : v \parallel u \} \), and let \( s = |S| \). Since \( P \) has no maximum element, it follows that \( s \geq 1 \). By induction, \( P - S \) can be embedded in the copy of \( L_{n-s} \) in \( L_n \) induced by
\{x_1, \ldots, x_{n-s}\} \cup \{y_1, \ldots, y_{n-s}\}$. Allowing \(x_{n-s+1}, \ldots, x_n\) to play the role of \(S\) completes a copy of \(P\) in \(L_n\).

The performance of First-Fit on a poset \(P\) can be analyzed using a static structure. A wall of a poset \(P\) is an ordered chain partition \((C_1, \ldots, C_i)\) such that for each element \(x \in C_j\) and each \(i < j\), there exists \(y \in C_i\) such that \(y \parallel x\). It is clear that every ordered chain partition produced by First-Fit is a wall, and conversely, each wall \(W\) of \(P\) is output by First-Fit when the elements of \(P\) are presented in order according to \(W\). Hence, the worst-case performance of First-Fit on \(P\) is equal to the maximum size of a wall in \(P\). A subwall of a wall \(W\) is obtained from \(W\) by deleting zero or more of the chains in \(W\). Note that if \(W\) is a wall of \(P\), then each subwall of \(W\) is a wall of the corresponding subposet of \(P\).

For each positive integer \(k\), we construct a poset called the reservoir of width \(k\), denoted \(R_k\), and a corresponding wall \(W_k\) of size \(2^k - 1\). The reservoirs provide an example of a family of posets which are good at avoiding subposets and yet still have exponential First-Fit performance.

**Theorem 2.** For each \(k \geq 1\), the reservoir \(R_k\) has width \(k\) and a wall \(W_k\) of size \(2^k - 1\).

**Proof.** Let \(R_1\) be the 1-element poset, and let \(W_1\) be the chain partition of \(R_1\). For \(k \geq 2\), we first construct \(R_k\) using \(R_{k-1}\) and \(W_{k-1}\). Then, we give a presentation order for \(R_k\) which forces First-Fit to use at least \(2^k - 1\) chains. Let \(W_{k-1} = (C_1, \ldots, C_m)\) where \(m = 2^{k-1} - 1\), and for \(0 \leq i \leq m\), let \(\hat{S}_i\) be the subwall \((C_1, \ldots, C_i)\) with corresponding subposet \(S_i\). (Although \(S_0\) and \(\hat{S}_0\) are empty, they are convenient for describing \(R_k\).) Let \(S\) be the series composition of disjoint copies of \(S_m, S_{m-1}, \ldots, S_0, R_{k-1}\) in this order, so that \(S = S_m \circ S_{m-1} \circ \cdots \circ S_0 \circ R_{k-1}\). The poset \(R_k\) consists of a copy of \(S\) and a chain \(X\) where \(X = \{x_{m+1} < \cdots < x_1\}\) and each \(x_i\) satisfies \(A(x_i) \cap S = \emptyset\) and \(B(x_i) \cap S = S_i \cup \cdots \cup S_m\). See Figure 1.

Note that since \(S\) is a series composition of posets of width at most \(k - 1\), it follows that \(S\) has width at most \(k - 1\). Adding \(X\) increases the width by at most 1, and so \(R_k\) has width at most \(k\). An antichain in the top copy of \(R_{k-1}\) of size \(k - 1\) and \(x_1\) form an antichain in \(R_k\) of size \(k\).

It remains to show that First-Fit might use as many as \(2^k - 1\) chains to partition \(R_k\). Consider the partial presentation order given by \(\hat{S}_m, x_{m+1}, \hat{S}_{m-1}, x_m, \ldots, \hat{S}_1, x_2, \hat{S}_0, x_1\). We claim that First-Fit assigns color \(j\) to \(x_j\) for \(1 \leq j \leq m + 1\). Indeed, when \(\hat{S}_{j-1}\) is presented, the points in \(S_{j-1}\) are above all previously presented points except \(\{x_{j+1}, \ldots, x_{m+1}\}\), which have already been assigned colors larger than \(j\). It follows that First-Fit uses colors \(\{1, \ldots, j - 1\}\) on \(S_{j-1}\). Next, \(x_j\) is presented; since \(x_j\) is above all previously presented points except those in \(S_{j-1}\), it follows that First-Fit assigns color \(j\) to \(x_j\).

In the final stage, we present the top copy of \(R_{k-1}\) in order given by \(W_{k-1}\). This copy of \(R_{k-1}\) is incomparable to each point in \(X\) and it follows that First-Fit uses \(m\) new colors on these points. In total, First-Fit uses \((m + 1) + m\) colors, and \(2m + 1 = 2^k - 1\).

If \(Q\) is a poset such that \(\text{FF}(w, Q)\) is subexponential in \(w\), then Theorem 2 implies that \(Q\) is a subposet of a sufficiently large reservoir \(R_k\). These posets have a nice description.
Definition 3. Let $Q$ be the minimal poset family which contains the ladder-like posets and is closed under series composition.

Our next lemma shows that $Q$ characterizes the posets of width 2 that appear in reservoirs.

Lemma 4. Let $Q$ be a poset of width 2. Some reservoir $R_k$ contains $Q$ as a subposet if and only if $Q \in Q$.

Proof. If $Q$ is ladder-like and has $t$ elements, then $Q$ is a subposet of $L_t$ by Lemma 1, and $L_t$ is a subposet of a sufficiently large reservoir. Suppose that $Q = Q_1 \oplus Q_2$ for some $Q_1, Q_2 \in Q$ with $|Q_1|, |Q_2| < |Q|$. By induction, $Q_1$ and $Q_2$ are subposets of $R_k$ for some $k$. Since $R_{k+1}$ contains the series composition of two copies of $R_k$, it follows that $Q$ is a subposet of $R_{k+1}$.

Let $Q$ be a poset of width 2 that is contained in some reservoir. We show that $Q \in Q$ by induction on $|Q|$. Let $k$ be the least positive integer such that $Q \subseteq R_k$, and let $S_0, \ldots, S_m$, $S$, and $X$ be as in the definition of $R_k$. If $Q \cap S$ is a chain, then $(Q \cap S, Q \cap X)$ is a chain partition witnessing that $Q$ is ladder-like, and so $Q \in Q$. Let $y, z$ be a maximal incomparable pair in $Q \cap S$, meaning that if $y', z' \in Q \cap S$, $y' \geq y, z' \geq z$ and $(y', z') \neq (y, z)$, then $y'$ and $z'$ are comparable. We claim that if $u \in Q$ and $u$ is above one of $\{y, z\}$, then $u$ is above both $y$ and $z$. This holds for $u \in Q \cap S$ by maximality of the pair $y, z$. This holds for $u \in Q \cap X$ since $y \parallel z$ implies that $y$ and $z$ belong to the same block in $S$, and all comparison relations between $u \in X$ and elements in $S$ depend only on their block in $S$.

Since $Q$ has width 2, it follows that $Q = Q_1 \oplus Q_2$ where $Q_1 = B[y] \cup B[z]$ and $Q_2 = A(y) \cup A(z)$. Unless $Q_2$ is empty and $Q_1 = Q$, it follows by induction that $Q_1, Q_2 \in Q$ and...
therefore $Q \in \mathcal{Q}$ also. Suppose that no point in $Q$ is above $y$ or $z$. Since no point in $X$ is below a point in $S$, it follows that $Q \cap X = \emptyset$, or else a point in $Q \cap X$ would complete an antichain of size 3 with $\{y, z\}$.

Therefore $Q \subseteq S$. Note that $Q$ is not contained in one of the blocks in $S$ by minimality of $k$ since each such block is a subposet of $R_{k-1}$. It follows that $Q = Q_1 \oplus Q_2$ for posets $Q_1$ and $Q_2$ with $|Q_1|, |Q_2| < |Q|$. By induction, $Q_1, Q_2 \in \mathcal{Q}$ and so $Q \in \mathcal{Q}$ also.

As a consequence of Lemma 4 and Theorem 2 it follows that $\text{FF}(w, Q) \geq 2^w - 1$ when $Q \not\in \mathcal{Q}$. It turns out that the performance of First-Fit is subexponential when $Q \in \mathcal{Q}$. Our next theorem shows how upper bounds on $\text{FF}(w, Q_1)$ and $\text{FF}(w, Q_2)$ can be used to obtain an upper bound on $\text{FF}(w, Q_1 \oplus Q_2)$. A Dilworth coloring of a poset $P$ of width $w$ is a function $\varphi: P \to [w]$, where $[w] = \{1, \ldots, w\}$ such that the preimages of $\varphi$ form a Dilworth partition.

**Theorem 5.** Let $Q_1$ and $Q_2$ be posets, let $w$, $s$, and $t$ be integers such that $\text{FF}(w, Q_1) < s$ and $\text{FF}(w, Q_2) < t$, and let $Q = Q_1 \oplus Q_2$. We have $\text{FF}(w, Q) \leq stw^2 + (s + t)w$.

**Proof.** For an ordered chain partition $\mathcal{C}$ of a poset $P$, an ascending $\mathcal{C}$-chain is a chain $x_1 < \cdots < x_k$ such that the chain in $\mathcal{C}$ containing $x_i$ precedes the chain containing $x_j$ for $i < j$. Similarly, a descending $\mathcal{C}$-chain is a chain $x_1 > \cdots > x_k$ such that the chain in $\mathcal{C}$ containing $x_i$ precedes the chain containing $x_j$ for $i < j$. The $\mathcal{C}$-depth of a point $x$, denoted $d_C(x)$, is the size of a maximum ascending $\mathcal{C}$-chain with bottom element $x$ and the $\mathcal{C}$-height of a point $x$, denoted $h_C(x)$, is the size of a maximum descending $\mathcal{C}$-chain with top element $x$.

Let $P$ be a $Q$-free poset of width at most $w$, and let $\mathcal{C}$ be a wall of $P$. We show that $|\mathcal{C}| \leq stw^2 + (s + t)w$. We claim that for each $x \in P$, at least one of the inequalities $h_C(x) \leq s$, $d_C(x) \leq t$ holds. Otherwise, if $h_C(x) \geq s + 1$ and $d_C(x) \geq t + 1$, then we obtain a copy of $Q$ in $P$ as follows. Let $x > y_1 > y_2 > \cdots > y_s$ be a descending $\mathcal{C}$-chain and let $x < z_1 < z_2 < \cdots < z_t$ be an ascending $\mathcal{C}$-chain. Let $P_1$ be the subposet of $P$ consisting of all $u \in P$ such that for some $y_i$, the points $u$ and $y_i$ share a chain in $\mathcal{C}$ and $u \leq y_i$. Let $\mathcal{C}_1$ be the restriction of $\mathcal{C}$ to $P_1$ and observe that $\mathcal{C}_1$ is a wall of $P_1$. Indeed, suppose that $\mathcal{C}, \mathcal{C}' \in \mathcal{C}_1$ where $\mathcal{C}$ precedes $\mathcal{C}'$ and let $(y_i, y_j) = (\max C, \max C')$. Let $v \in \mathcal{C}'$ and note that $v$ and $y_j$ share a chain in $\mathcal{C}$. Let $u$ be a point in $P$ such that $u$ belongs to the same chain in $\mathcal{C}$ as $y_i$ and $u \parallel v$. Note that $u \leq y_i$, since otherwise $u > y_i > y_j \geq v$, contradicting $u \parallel v$. Therefore $u \in P_1$ and $u \in C$. Since $\mathcal{C}_1$ is a wall of $P_1$ of size $s$ and $s > \text{FF}(w, Q_1)$, it follows that $P_1$ contains a copy of $Q_1$. Similarly, we let $P_2$ be the subposet of $P$ consisting of all $u \in P$ such that for some $z_i$, the points $u$ and $z_i$ share a chain in $\mathcal{C}$ and $u \geq z_i$. Restricting $\mathcal{C}$ to $P_2$ gives a wall $\mathcal{C}_2$ of size $t$ analogously, and since $t > \text{FF}(w, Q_2)$, it follows that $P_2$ contains a copy of $Q_2$. Since every element in $P_1$ is less than $x$ and $x$ is less than every element in $P_2$, it follows that $P$ contains a copy of $Q$.

The lower part of $P$, denoted by $L$, is $\{x \in P : h_C(x) \leq s\}$ and the upper part of $P$, denoted by $U$, is $P - L$. Note that $\{L, U\}$ is a partition of $P$, that $h_C(x) \leq s$ for $x \in L$, and that $d_C(x) \leq t$ for $x \in U$. Let $\mathcal{C}_U$ be the subwall of $\mathcal{C}$ consisting of all chains that are contained in $U$, and let $\mathcal{C}_{U,j}$ be the subwall of $\mathcal{C}_U$ consisting of the chains $C \in \mathcal{C}_U$ such that
Suppose \( C, C' \in C_{U,j} \) and that \( C \) precedes \( C' \). Since \( C \) precedes \( C' \), it is not possible for \( \min C > \min C' \). Therefore if \( \min C \) and \( \min C' \) are comparable, then it must be that \( \min C < \min C' \), and it would follow that \( d_C(\min C) > d_C(\min C') \). Hence \( |C_{U,j}| < w \) for \( 1 \leq j \leq t \) and \( |C_U| \leq tw \). A symmetric argument shows that the sublist \( C_L \) consisting of all chains that are contained in \( L \) satisfies \( |C_L| \leq sw \).

It remains to bound the number of chains in \( C \) that contain points in both \( U \) and \( L \). Let \( C_{LU} \) be the sublist of \( C \) consisting of these chains. Note that for each \( C \in C \), we have that \( y, z \in C \) and \( y < z \) implies that \( h_C(y) \leq h_C(z) \) and \( d_C(y) \geq d_C(z) \). It follows that each point in \( C \cap L \) is less than each point in \( C \cap U \). Let \( \varphi: P \rightarrow [w] \) be a Dilworth coloring. For each \( C \in C_{LU} \) with \( y = \max(C \cap L) \) and \( z = \min(C \cap U) \), we assign to \( C \) the signature \( (\varphi(y), h_C(y), \varphi(z), d_C(z)) \). We claim that the signatures are distinct. Suppose that \( C, C' \in C_{LU} \) have the same signature and that \( C \) precedes \( C' \). Let \( y = \max(C \cap L), z = \min(C \cap U), y' = \max(C' \cap L), \) and \( z' = \min(C' \cap U) \). Note that \( y < z \) is a cover relation in \( C \) and \( y' < z' \) is a cover relation in \( C' \). Since \( \varphi(y) = \varphi(y') \), it follows that \( y \) and \( y' \) are comparable. Since \( h_C(y) = h_C(y') \), it must be that \( y < y' \). Since \( \varphi(z) = \varphi(z') \), it follows that \( z' \) and \( z \) are comparable. Since \( d_C(z') = d_C(z) \), it must be that \( z' < z \). We now have that \( y < z \) is a cover relation in \( C \) but \( y < y' < z' < z \) for points \( z', y' \) that appear in a chain \( C' \) that follows \( C \), contradicting that \( C \) is a wall.

Since the assigned signatures are distinct, we have that \( |C_{LU}| \leq stw^2 \). It follows that

\[
|C| \leq |C_{LU}| + |C_L| + |C_U| \leq stw^2 + sw + tw.
\]

\[ \square \]

\textbf{Corollary 6}. Let \( Q = Q_1 \otimes \cdots \otimes Q_k \). If \( \text{FF}(w, Q_i) \leq 2^{c_i(\lg w)^2} \) for \( 1 \leq i \leq k \), then \( \text{FF}(w, Q) \leq 2^{c(\lg w)^2} \), where \( c = \sum_{i=1}^k c_i \).

\textbf{Proof}. By induction on \( k \). For \( k = 1 \), the claim is clear. Suppose \( k \geq 2 \). Since \( \text{FF}(1, Q) \leq 1 \), we may assume \( w \geq 2 \). Let \( R = Q_1 \otimes \cdots \otimes Q_{k-1} \). By induction, \( \text{FF}(w, R) \leq 2^{c'(\lg w)^2} \), where \( c' = \sum_{i=1}^{k-1} c_i \). By Theorem 5 with \( s \leq 1 + 2^{(c'+6(k-1))(\lg w)^2} \) and \( t \leq 1 + 2^{c_k(\lg w)^2} \), we have \( \text{FF}(w, Q) \leq stw^2 + (s + t)w \leq 3stw^2 < 2^2 \cdot 2^{(c'+6(k-1))(\lg w)^2+1} + 2^{c_k(\lg w)^2+1} \cdot 2^{2\lg w} \). It follows that \( \lg[\text{FF}(w, Q)] < (c' + c_k + 6(k - 1))(\lg w)^2 + 4 + 2\lg w \leq (c + 6k)(\lg w)^2 \).

\[ \square \]

The following key result due to Kierstead and M. Smith [12] shows that First-Fit uses a subexponential number of chains on ladder-free posets. We follow with the characterization of posets \( Q \) for which \( \text{FF}(w, Q) \) is subexponential.

\textbf{Theorem 7} (Kierstead–M. Smith [12]). \textit{For some constant }\gamma, \textit{we have }\text{FF}(w, L_n) \leq w^{\gamma(\lg w + \lg(n))}.

\textbf{Theorem 8} (Dichotomy Theorem). \textit{Let }\mathcal{Q} \textit{be an }n\text{-element poset of width 2. If }\mathcal{Q} \in \mathcal{Q}, \textit{then there exists a constant }C \textit{(depending only on }\mathcal{Q}) \textit{such that }\text{FF}(w, Q) \leq 2^{C(\lg w)^2}; \textit{in fact, }C = O(n) \textit{suffices. If }\mathcal{Q} \notin \mathcal{Q}, \textit{then }\text{FF}(w, Q) \geq 2^w - 1.

\textbf{Proof}. Suppose \( \mathcal{Q} \notin \mathcal{Q} \). By Theorem 2 and Lemma 4 we have \( \text{FF}(w, Q) \geq 2^w - 1 \). Suppose that \( \mathcal{Q} \in \mathcal{Q} \). Since \( \text{FF}(1, Q) \leq 1 \), we may assume \( w \geq 2 \). Since \( Q \in \mathcal{Q} \), it follows that \( Q = Q_1 \otimes \cdots \otimes Q_k \) for some ladder-like posets \( Q_1, \ldots, Q_k \). For \( 1 \leq i \leq k \), let \( n_i = |Q_i| \). Since \( Q_i \) is ladder-like, Theorem 7 implies that \( \text{FF}(w, Q_i) \leq 2^{c_i(\lg w)^2} \) where \( c_i = (1 + \frac{\lg(n_i)}{\lg(w)}) \leq (1 + \frac{\lg(n)}{\lg(w)}) \leq (1 + \frac{\lg(n)}{\lg(n)}) \leq \gamma(\lg w + \lg(n)) \).
it follows by convexity that \( \sum_{i=1}^{k} \) natural logarithm. Using suffices to take \( C = 6k + c = 6k + \sum_{i=1}^{k} c_i \leq (6 + \gamma)k + \gamma \sum_{i=1}^{k} \lg n_i \). Since \( \sum_{i=1}^{k} n_i = n \), it follows by convexity that \( \sum_{i=1}^{k} \lg n_i \leq k\lg(n/k) \leq (n/e)\lg e \), where \( e \) is the base of the natural logarithm. Using \( k \leq n \), we conclude \( C \leq (6 + \gamma)n + \gamma(n/e)\lg e = O(n) \).

Theorem 8 provides a large separation in the behavior of First-Fit on \( Q \)-free posets according to whether or not \( Q \in Q \). It may be that even stronger results are possible. Theorem 5 shows that if \( \text{FF}(w,Q_1) \) and \( \text{FF}(w,Q_2) \) are polynomial in \( w \), then so is \( \text{FF}(w,Q_1 \oplus Q_2) \). For large \( n \), the best known lower bound on \( \text{FF}(w,L_n) \) is \( w^{k(n-1)/(n-1)} \), due to Bosek, Kierstead, Krawczyk, Matecki, and M. Smith [3]. This leaves open the possibility that \( \text{FF}(w,L_n) \) is polynomial in \( w \) for each fixed \( n \). If so, then the separation provided by the Dichotomy Theorem would improve, yielding that \( \text{FF}(w,Q) \) is polynomial when \( Q \in Q \) and exponential when \( Q \notin Q \).

**Question 9.** Is it true for each fixed \( n \) that \( \text{FF}(w,L_n) \) is bounded by a polynomial in \( w \)?

It is clear that \( \text{FF}(w,L_1) = w \) and Kierstead and M. Smith [12] proved that \( \text{FF}(w,L_2) = w^2 \). Note that \( L_3 = Q_1 \oplus Q_2 \oplus Q_3 \) where \( Q_1 \) and \( Q_3 \) are 1-element posets and \( Q_2 \) is the \( N \) poset. Since \( \text{FF}(w,Q_1) = \text{FF}(w,Q_3) = 0 \) and \( \text{FF}(w,Q_2) = w \), it follows from Theorem 5 that \( \text{FF}(w,L_3) \) is polynomial in \( w \). A more careful analysis, along the lines of Kierstead and M. Smith’s proof of \( \text{FF}(w,L_2) = w^2 \), shows that \( \text{FF}(w,L_3) \leq w^2(w+1) \). Question 9 is open for \( n \geq 4 \).

It would also be interesting to better understand the behavior of First-Fit on \( Q \)-free posets when \( Q \notin Q \). The smallest poset of width 2 that is not in \( Q \) is the skewed butterfly, denoted \( B \), which consists of the chains \( x_1 < x_2 < x_3 \) and \( y_1 < y_2 \) with relations \( x_1 < y_2 \) and \( y_1 < x_3 \). What is \( \text{FF}(w,B) \)?

### 3 First-Fit on Butterfly-Free Posets

The butterfly poset, denoted \( B \), is \( Q \oplus Q \), where \( Q \) is the 2-element antichain. In this section, we obtain the asymptotics of \( \text{FF}(w,B) \). The performance of First-Fit on butterfly-free posets is strongly related to the bipartite Turán number for \( C_4 \). Kövari, Sós, Turán [15] showed that the maximum number of edges in a subgraph of \( K_{n,n} \) that excludes \( C_4 \) is \( (1+o(1))n^{3/2} \).

**Lemma 10** (Kövari–Sós–Turán [15]). Let \( q \) be a prime power, and let \( n = q^2 + q + 1 \). There exists a \( (q+1) \)-regular spanning subgraph of \( K_{n,n} \) that has no 4-cycle.

We also need a standard result about the density of primes.

**Theorem 11** (Hoheisel [9]). There exists a real number \( \theta \) with \( \theta < 1 \) such that for all sufficiently large real numbers \( x \), there is a prime in the interval \( [x - x^\theta, x] \).

Since the result of Hoheisel [9], many research groups have improved the bound on \( \theta \); see Baker and Harman [1] for the history. The current best bound is \( \theta = 0.525 \), due to Baker, Harman, and Pintz [2].
Theorem 12. \( \text{FF}(w, B) \geq (1 - o(1))w^{3/2} \).

Proof. By Theorem 11 and standard asymptotic arguments, we may assume that \( w \) has the form \( q^2 + q + 1 \), where \( q \) is prime. By Lemma 10 there exists a \((q+1)\)-regular \((X,Y)\)-bigraph \( G \) with parts of size \( w \) that has no 4-cycle. Since \( G \) is a regular bipartite graph, it follows from Hall’s Theorem that \( G \) has a perfect matching \( M \). Let \( G' = G - M \), and let \( L \) be an ordering of \( E(G') \).

Using \( G' \), we construct a \( B \)-free poset \( P \) of width \( w \) and a wall of \( P \) size \( |E(G)| \). It will then follow that \( \text{FF}(w, B) \geq |E(G)| = (q + 1)w = (1 - o(1))w^{3/2} \). Let \( I_X \) be the set of all pairs \((x, e)\) such that \( x \in X \), \( e \in E(G') \), and \( e \) is incident to \( x \). Similarly, let \( I_Y \) be the set of all pairs \((y, e)\) such that \( y \in Y \), \( e \in E(G') \) and \( e \) is incident to \( y \). We construct \( P \) so that \( M \) is a maximum antichain, \( B(M) = I_X \), and \( A(M) = I_Y \). The subposet induced by \( I_X \cup M \) consists of \( w \) incomparable chains, indexed by \( M \). For \( x_i y_i \in M \) with \( x_i \in X \) and \( y_i \in Y \), the chain associated with \( x_i y_i \) consists of all pairs \((x_i, e)\) \in \( I_X \) in order according to \( L \) followed by top element \( x_i y_i \). The subposet induced by \( M \cup I_Y \) also consists of \( w \) incomparable chains, indexed by \( M \). For \( x_i y_i \in M \) with \( x_i \in X \) and \( y_i \in Y \), the chain associated with \( x_i y_i \) in the subposet induced by \( M \cup I_Y \) consists of bottom element \( x_i y_i \) followed by all pairs \((y_i, e)\) \in \( I_Y \) in reverse order according to \( L \). Note that if \( e \) is the first edge in \( L \) and \( e = xy \), then \((x,e)\) is minimal in \( P \) and \((y,e)\) is maximal. The chains in \( I_X \cup M \) and the chains in \( M \cup I_Y \) combine to form a Dilworth partition of \( P \) of size \( w \); let \( D_i \) be the Dilworth chain containing \( x_i y_i \). It remains to describe the relations between points in \( I_X \) and points in \( I_Y \). For \((x, e_1) \in I_X \) and \((y, e_2) \in I_Y \), we have that \((x, e_1)\) is covered by \((y, e_2)\) if and only if \( e_1 = e_2 = xy \in E(G') \).

We claim that \( P \) is \( B \)-free. For each element \( z \in I_X \cup M \), we have that \( B(z) \) is a chain. Hence, a maximal element in a copy of \( B \) must belong to \( I_Y \). Similarly, since \( A(z) \) is a chain when \( z \in M \cup I_Y \), a minimal element in a copy of \( B \) must belong to \( I_X \). In a chain of cover relations from \((x, e_1) \in I_X \) up to \((y, e_2) \in I_Y \), either all points stay in the same Dilworth chain \( D_i \), implying that \( xy = x_i y_i \in M \), or there is a cover relation from a point in \( D_i \) to a point in \( D_j \), that implying \( xy = x_j y_j \) with \( x_j y_j \in E(G') \). In both cases, \((x, e_1) \leq (y, e_2)\) implies that \( xy \in E(G) \), and it follows that a copy of \( B \) in \( P \) corresponds to a 4-cycle in \( G \), a contradiction.

It remains to construct a wall \( W \) of \( P \) of size \( |E(G)| \). The wall contains \( |E(G')| \) chains of size 2 arranged in order according to \( L \), followed by \( w \) singleton chains. For \( e \in L \) with \( e = xy \), the corresponding chain in the wall is \((x, e) \prec (y, e)\). These chains are followed by \( w \) singleton chains, each consisting of a point in \( M \). Let \( C_i \) and \( C_j \) be chains in \( W \) with \( i < j \), and let \( z \in C_j \). We show that \( z \) is incomparable to some point in \( C_i \). Since \( M \) is an antichain, we may assume that \( C_i \) is a chain of the form \((x, e) \prec (y, e)\). If \( C_j \) is a singleton chain containing only \( z \), then \( z \) is incomparable to every element in \( P \) outside its Dilworth chain. Since \((x, e)\) and \((y, e)\) are in distinct Dilworth chains, it follows that \( C_i \) contains a point incomparable to \( z \). Otherwise, \( C_j \) has the form \((x', e') \prec (y', e')\), and since \( i < j \), it follows that \( e \) precedes \( e' \) in \( L \). Suppose that \( z = (x', e') \). If \((x', e') \parallel (x, e)\), then \((x, e)\) is the desired point in \( C_i \). Otherwise, \((x', e')\) is comparable to \((x, e)\), implying that \((x, e)\) and \((x', e')\) are in the same Dilworth chain and \( x = x' \). Since \( e \) precedes \( e' \) in \( L \), we have \((x, e) \prec (x', e')\). If \((x', e')\) is also comparable to \((y, e)\), it must be that \((x', e') \prec (y, e)\).
But now \((x,e) < (x',e') < (y,e)\) contradicts that \((y,e)\) covers \((x,e)\) in \(P\). The case that 
\(z = (y',e')\) is analogous.

In a poset \(P\) with a set of elements \(S\), an **extremal point** of \(S\) is a minimal or maximal element in \(S\).

**Lemma 13.** Let \(C\) and \(D\) be chains in \(P\). If \(\min C \parallel \max D\) and \(\max C \parallel \min D\), then \(C\) and \(D\) are pairwise incomparable. Consequently if \(C'\) and \(D'\) are chains and \((x_1,y_1),(x_2,y_2)\) \(\in C'\times D'\) are incomparable pairs, then \(\min \{x_1,x_2\} \parallel \min \{y_1,y_2\}\) and \(\max \{x_1,x_2\} \parallel \max \{y_1,y_2\}\).

**Proof.** If \(u \leq v, u \in C\), and \(v \in D\), then \(\min C \leq u \leq v \leq \max D\). If \(u \leq v, u \in D\), and \(v \in C\), then \(\min D \leq u \leq v \leq \max C\). For the second part, either the statement is trivial or we apply the first part to the subchains of \(C'\) and \(D'\) with extremal points \(\{x_1,x_2\}\) and \(\{y_1,y_2\}\) respectively.

Starting with an arbitrary chain partition \(C\), iteratively moving elements to earlier chains produces a wall \(W\) with \(|W| \leq |C|\). Beginning with a Dilworth partition, it follows that each poset \(P\) of width \(w\) has a **Dilworth wall** consisting of \(w\) chains. If \(R\) and \(S\) are sets of points in \(P\), we write \(R < S\) if \(u < v\) when \((u,v) \in R \times S\).

**Theorem 14.** \(\text{FF}(w,B) \leq (1 + o(1))w^{3/2}\).

**Proof.** Let \(P\) be a \(B\)-free poset and let \(D\) be a Dilworth wall of \(P\) with \(D = (D_1,\ldots,D_w)\). Let \(R\) be the set of points \(x \in P\) such that \(A(x)\) is a chain. Let \(R' = P - R\), and note that \(B(x)\) is a chain for each \(x \in R'\) since \(P\) is \(B\)-free.

Let \(C\) be a wall of \(P\) with \(C = (C_1,\ldots,C_i)\); we bound \(|C|\). Since \(|D| = w\), at most \(2w\) chains in \(C\) contain an extremal point from a chain in \(D\). Also, no two chains in \(C\) are contained in the same chain in \(D\), and so at most \(w\) chains in \(C\) are contained in a chain in \(D\). Let \(C'\) be the subwall of \(C\) consisting of all chains \(C \in C\) that do not contain an extremal point of a chain in \(D\) but contain points from at least two chains in \(D\). We have that \(|C| \leq |C'| + 3w\). We claim that for each chain \(C_i \in C'\), we have that \(C_i \cap R\) is contained in a chain in \(D\). Suppose that \(C_i \cap R\) contains elements from at least two chains in \(D\). Let \(D_\alpha\) be the Dilworth chain containing \(C_i\), let \(x = \max(C_i - D_\alpha)\), and let \(D_\beta\) be the Dilworth chain containing \(x\). Let \(m = \max D_\beta\), and note that \(C_i \in C'\) implies \(m \notin C_i\). It follows that \(m \in C_j\) for some \(C_j \in C\) with \(j \neq i\); since \(A(x)\) is a chain and \(m > x\), it follows that \(m\) is comparable to every element in \(C_i\) and therefore \(j < i\). Let \(y\) be the element covering \(x\) in \(C_i\). Note that \(y \in D_\alpha\) and \(y\) is comparable to everything in \(D_\beta\) since \(A(x)\) is a chain, and this implies \(\alpha < \beta\). Since \(m,y \in A(x)\) and \(A(x)\) is a chain, either \(m < y\) or \(m > y\). If \(m > y\), then \(m\) is comparable to everything in \(D_\alpha\), contradicting \(m \in D_\beta\) and \(\alpha < \beta\). Similarly, if \(m < y\), then \(y\) is comparable to every element in \(C_j\), contradicting \(y \in C_i\) and \(j < i\). Therefore \(C_i \cap R\) is contained in a single chain in \(D\). By a symmetric argument, \(C_i \cap R'\) is contained in a single chain in \(D\).

It remains to bound \(|C'|\). Note that for each \(C \in C'\), we have that \(C \cap R\) is contained in some Dilworth chain \(D_\alpha\) \(\in D\) and \(C \cap R'\) is contained in some Dilworth chain \(D_\gamma\) \(\in D\), with \(\alpha \neq \gamma\); we say that \((\alpha,\gamma)\) is the signature of \(C \in C'\) if \(C \cap R \subseteq D_\alpha\) and \(C \cap R' \subseteq D_\gamma\). Note
that if $C_i, C_j \in \mathcal{C}'$ with $i < j$, then it is not possible for both $C_i$ and $C_j$ to have the same signature $(\alpha, \gamma)$, or else $C_i \cap R' < C_j \cap R$. Since the signatures are distinct, it follows that $|\mathcal{C}'| \leq w^2$ and so $\text{FF}(w, B) \leq (1 + o(1))w^2$.

Let $X$ and $Y$ be disjoint copies of $\mathcal{D}$, and let $G$ be the $(X, Y)$-bigraph in which $D_\alpha \in X$ and $D_\gamma \in Y$ are adjacent if and only if some chain in $\mathcal{C}'$ has signature $(\alpha, \gamma)$. We claim that $G$ has no 4-cycle, implying $|\mathcal{C}'| = |E(G)| \leq (1 + o(1))w^{3/2}$.

Suppose for a contradiction that $G$ has a 4-cycle on $D_\alpha, D_\beta \in X$ and $D_\gamma, D_\delta \in Y$. Let $C_i, C_j, C_k, C_\ell$ be chains in $\mathcal{C}'$ with signatures $(\alpha, \gamma)$, $(\alpha, \delta)$, $(\beta, \gamma)$, and $(\beta, \delta)$, respectively. Assume, without loss of generality, that $C_i$ precedes $C_j$ in $\mathcal{C}'$, and let $y_1 \in C_j \cap R' \subseteq D_\delta$. Since $y_1$ is in a later chain, it must be that $x_1 \parallel y_1$ for some $x_1 \in C_i$. Since $C_j \cap R$ and $C_i \cap R$ are both contained in $D_\alpha$ and $y_1 \in C_j \cap R' < C_j \cap R < C_i \cap R$, it follows that $x_1 \in C_i \cap R' \subseteq D_\gamma$. Therefore there is an incomparable pair $(x_1, y_1) \in (C_i \cap R') \times (C_j \cap R')$. A similar argument applied to $C_k$ and $C_\ell$ with top parts in $D_\beta$ shows that there is an incomparable pair $(x_2, y_2) \in (C_k \cap R') \times (C_\ell \cap R')$. Since $C_i \cap R', C_k \cap R' \subseteq D_\gamma$ and $C_j \cap R', C_\ell \cap R' \subseteq D_\delta$, it follows from Lemma \[13\] that there is an incomparable pair $(x, y) \in D_\gamma \times D_\delta$ with $x \leq \min\{\max C_i \cap R', \max C_k \cap R'\}$ and $y \leq \min\{\max C_j \cap R', \max C_\ell \cap R'\}$. Similarly, there is an incomparable pair $(x', y') \in D_\alpha \times D_\beta$ with $x' \geq \max\{\min C_i \cap R, \min C_j \cap R\}$ and $y' \geq \max\{\min C_k \cap R, \min C_\ell \cap R\}$. Since $x, y < x', y'$, it follows that $\{x, y, x', y'\}$ induces a copy of $B$ in $P$.

Since $|\mathcal{C}| \leq |\mathcal{C}'| + 3w \leq (1 + o(1))w^{3/2}$, the bound on $\text{FF}(w, B)$ follows. \hfill \qed

**Corollary 15.** $\text{FF}(w, B) = (1 + o(1))w^{3/2}$.

The **stacked butterfly** of height $t$, denoted $B_t$, is $Q_1 \oplus \cdots \oplus Q_t$, where each $Q_i$ is a 2-element antichain. Note that $B_{2k}$ is the series composition of $k$ copies of $B$. A consequence of our results is that $\text{FF}(w, B_t)$ is bounded by a polynomial in $w$ for each fixed $t$.

**Corollary 16.** $\text{FF}(w, B_{2k}) \leq (1 + o(1))w^{3.5k-2}$

**Proof.** From Theorem \[5\] and Corollary \[15\] we have that

$$\text{FF}(w, B_{2k}) \leq (1 + o(1))w^2 \text{FF}(w, B_{2(k-1)}) \text{FF}(w, B_2) = (1 + o(1))w^{3.5k-2}.$$

\hfill \qed

It would be interesting to find lower bounds on $\text{FF}(w, B_{2k})$. In particular, is $\text{FF}(w, B_{2k})$ bounded below by a polynomial in $w$ whose degree grows linearly in $k$?

## 4 Conclusions and Open Problems

A consequence of Theorem \[8\] is that $Q$ is the family of posets $Q$ such that $\text{FF}(w, Q)$ is subexponential in $w$. It may be that $Q$ is also the family of posets $Q$ such that $\text{FF}(w, Q)$ is polynomial in $w$. This is the case if and only if Question \[9\] has a positive answer. Alternatively, if Question \[9\] has a negative answer, then it would be interesting to understand what structural properties of $Q$ lead to polynomial behavior of $\text{FF}(w, Q)$. 

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Problem 17. Characterize the posets $Q$ for which $\text{FF}(w, Q)$ is bounded above by a polynomial in $w$.

We have focused on upper bounds for posets in $Q$ and lower bounds for posets outside $Q$. It would be nice to obtain better bounds for posets outside $Q$. The smallest poset of width 2 that is outside $Q$ is the skewed butterfly $\hat{B}$ consisting of disjoint chains $x_1 < x_2 < x_3$ and $y_1 < y_2$ with the cover relations $x_1 < y_2$ and $y_1 < x_3$. According to Theorem 2, we have $\text{FF}(w, \hat{B}) \geq 2^w - 1$. What is $\text{FF}(w, \hat{B})$? Although Bosek, Krawczyk, and Matecki [4] provide tower-type upper bounds on $\text{FF}(w, Q)$, there may be room for significant improvement.

Question 18. Is there any poset $Q$ of width 2 for which $\text{FF}(w, Q)$ is superexponential?

We have studied the behavior of First-Fit on families that forbid a single poset $Q$, but it is also natural to ask about families that forbid a set of posets. If $S$ is a set of posets, we say that a poset $P$ is $S$-free if no poset in $S$ is a subposet of $P$. Let $\text{FF}(w, S)$ be the maximum number of chains that First-Fit uses on an $S$-free poset of width $w$.

Problem 19. Characterize the sets $S$ for which $\text{FF}(w, S)$ is bounded by a polynomial in $w$.

If $P$ is a poset family that is closed under taking subposets, then $P$ is exactly the set of posets that is $S$-free, where $S$ is the set of minimal posets not in $P$. A solution to Problem 19 is therefore equivalent to a characterization of all subposet-closed families $P$ such that First-Fit has polynomial behavior when restricted to $P$. We suspect that this is a challenging problem, but the restriction of Problem 19 to $|S| \leq 2$ is likely more accessible and even partial progress would still be interesting.

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