Emergent Anisotropic Non-Fermi Liquid

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Understanding correlation effects in topological phases and their transitions is a cutting-edge area of research in recent condensed matter physics. We study topological quantum phase transitions (TQPTs) between double-Weyl semimetals (DWSMs) and insulators, and argue that a novel class of quantum criticality appears at the TQPT characterized by emergent anisotropic non-Fermi liquid behaviors, in which the interplay between the Coulomb interaction and electronic critical modes induces not only anisotropic renormalization of the Coulomb interaction but also strongly correlated electronic excitation. Using the standard renormalization group methods, large $N_f$ theory and the $\epsilon = 4 - d$ method with fermion flavor number $N_f$ and spatial dimension $d$, we obtain the anomalous dimensions of electrons ($\eta_f = 0.366/N_f$) in large $N_f$ theory and the associated anisotropic scaling relations of various physical observables. Our results may be observed in candidate materials for DWSMs such as HgCr$_2$Se$_4$ or SrSi$_2$ when the system undergoes a TQPT.

**Introduction.** — Quantum criticality and topology play key roles in modern condensed matter physics [1--5], and the two concepts become naturally important near TQPTs. Recently, there has been a surge of interest in TQPTs [6--11]. The simplest class is described by the weakly interacting Dirac fermions, and it is well understood that the sign of the Dirac mass terms determines adjacent topological phases [12--14]. Since quasiparticles are well defined, non-interacting tight-binding models are sufficient to describe TQPTs in this class.

Beyond the simplest class, however, our understanding of TQPTs is far from complete. The long-range Coulomb interaction may drastically change the properties of non-interacting fermions near TQPTs, and the non-interacting tight-binding models cannot describe some classes of TQPTs. The interplay between critical electronic modes and the Coulomb interaction becomes significant, and quantum critical non-Fermi liquid states may appear with emergent particle-hole and rotational symmetries [15--19]. Moreover, the interplay may also give rise to weakly coupled but infinitely anisotropic excitations in a class of TQPTs [20--24]. Thus, it is vital to deepen our understanding of TQPTs beyond the simplest class.

In this work, we uncover a novel class of TQPTs which shows emergent anisotropic non-Fermi liquid behaviors associated with topological nature of electronic wave functions. Our target system is the DWSM adjacent to insulator phases under the long-range Coulomb interaction. Using the standard renormalization group (RG) methods, we find novel quantum critical phenomena at a TQPT between the two phases. Importantly, the TQPT exhibits emergent anisotropy in the sense that the power-law dependences of the energy dispersion and the Coulomb interaction on momentum become anisotropic, even though they are initially set to be the same in all directions. The anisotropic momentum dependence with non-Fermi liquid behaviors is the hallmark of our quantum criticality, and we calculate its experimental signatures in physical observables such as the specific heat, compressibility, diamagnetic susceptibility, and optical conductivity.

**Model.** — We consider a minimal lattice model of DWSMs with a four-fold rotational symmetry $C_4$ with a rotational axis along the $z$ direction [25--29],

\[
\mathcal{H}(k) = 2t_x' [\cos(k_ya_0) - \cos(k_xa_0)] \sigma_x + 2t_y' \sin(k_ya_0) \sin(k_ya_0) \sigma_y + 2M_z(k) \sigma_z,
\]

where $M_z(k) = m_z - t_x' \cos(k_xa_0) + m_0(2 - \cos(k_xa_0) - \cos(k_ya_0))$ and $a_0$ is the lattice constant. In general, $C_4$ symmetry does not imply $t_x' = t_y'$. However, in the presence of the Coulomb interaction, $t_x' = t_y'$ emerges at low energies. A detailed proof on the emergent $t_x' = t_y'$ can be found in the Supplemental Material [29]. For $|m_z| < t_x'$, the Hamiltonian supports two double-Weyl nodes at $k = (0, 0, \pm k^*_z)$, where $k^*_z = a_0^{-1} \cos^{-1}(m_z/t_x')$, which are characterized by the Chern numbers $\pm 2$ around the points [25]. For $|m_z| > t_x'$, the system shows an insulator phase. At $|m_z| = t_x'$, a quantum phase transition

**FIG. 1.** Phase diagram for the TQPT between the DWSM and insulator phases with the tuning parameter $m$. The insets show the energy dispersions for the (a) DWSM, (b) insulator and (c) TQPT.
tion occurs between the DWSM and insulator phases, as shown in Fig. 1. Neglecting \( m_0 \) for simplicity, we obtain the low-energy effective Hamiltonian near the transition point given by

\[
\mathcal{H}_0(k) = t_r \left[ (k_x^2 - k_y^2) \sigma_x + 2k_x k_y \sigma_y \right] + (t_z k_z^2 + m) \sigma_z,
\]

(2)

where \( t_r = t'_r \sigma_z^0 \) and \( t_z = t'_z \sigma_z^0 \). Here, a parameter of the TQPT, \( m \propto |m_z| - t'_z \), is introduced. The energy eigenvalues of the Hamiltonian are given by \( E_{\pm}(k) = \pm \sqrt{t_r^2 (k_x^2 + k_y^2) + (t_z k_z^2 + m)^2} \), and at \( m = 0 \) the energy dispersion becomes quadratic in all three directions.

The corresponding effective action with the long-range Coulomb interaction is

\[
S = \int \! d\tau d^3 x \psi^\dagger \left[ \partial_{\tau} - ig \phi + \mathcal{H}_0(-i\nabla) \right] \psi - \int \! d\tau d^3 x \frac{1}{2} \left\{ a \left( (\partial_x \phi)^2 + (\partial_y \phi)^2 \right) + \frac{1}{\alpha} (\partial_z \phi)^2 \right\},
\]

(3)

where \( \alpha \equiv \frac{\sqrt{2}\pi\epsilon}{\epsilon_0} \) with \( \epsilon_0 \) and \( \epsilon \) being the bare charge and the dielectric constant, respectively, \( \psi \) is a spinor with \( 2N_f \) components, and \( \phi \) is a bosonic field describing the long-range Coulomb interaction. The parameter \( \alpha \) is introduced to characterize the anisotropy ratio of the Coulomb interaction between the \( xy \)-plane and the \( z \)-axis. For later usage, we define the following dimensionless parameters,

\[
\alpha = \frac{A_d - 2g^2}{\sqrt{t_r t_z \Lambda^2}} \quad \beta = \frac{t_z}{t_r} \quad \gamma = \frac{\alpha \sqrt{\beta}}{2},
\]

(4)

with \( A_d = \frac{6\pi(4\pi)^{\frac{3}{2}}}{\Gamma(\frac{3}{2})} \). Here, \( \alpha \) represents the ratio of the Coulomb potential and the electron kinetic energy, \( \beta^{-1} \) is the anisotropy parameter for the fermionic fields, and \( \gamma \) is the combination of the two anisotropy parameters \( \alpha \) and \( \beta \). We assume that all the four-Fermi interactions, \( u_{ijkl} \psi_i^\dagger \psi_j^\dagger \psi_k \psi_l \), are set to be small at the lattice-spacing scale and flow into the trivial fixed point as in the literature [30].

The bare scaling dimensions of the parameters can be determined by setting \( |r| = -z, |x, y| = -z_\perp \) and \( |z| = -1 \). We find \( |\psi| = (2z_\perp + d - 2)/2, |\phi| = (z_\perp + d - 3)/2, \) and \( |g^2| = z - z_\perp - d + 3 \). Here, the dimension of the space along the \( z \)-axis is extended from the physical dimension of \( 1 \) to \( d \). This extension is needed for the \( \epsilon = 4 - d \) expansion method, as will be explained later. If we set \( (t_z, t_r, \alpha) \) marginal, \( z = 2 \) and \( z_\perp = 1 \) thus, for \( d = 3 \) the non-interacting fixed point becomes unstable because of the relevant coupling constant \( g^2 \), similar to that of the Luttinger-Abrikosov-Beneslavskii (LAB) phase [17, 31, 32]. We stress that at our fixed point, the system becomes highly anisotropic, which is fundamentally different from the isotropic LAB phase even though the bare scaling gives the same scaling dimensions.

To deal with the relevant parameter, we employ the large \( N_f \) method and the \( \epsilon = 4 - d \) expansion method, and find that both methods give consistent results for the emergent anisotropic non-Fermi behaviors.

**Large \( N_f \) calculation.** — We first use the large \( N_f \) method since it is naturally extended from the conventional random phase approximation [20, 21, 33–35]. The boson self-energy is

\[
\Pi(i\Omega, \mathbf{q}) = N_f g^2 \int \! d\omega_i \! d\mathbf{k} \text{Tr} \left[ G_0(i\omega_i + i\Omega, \mathbf{k} + \mathbf{q}) G_0(i\omega_i, \mathbf{k}) \right],
\]

(5)

with the fermion propagator \( G_0(i\omega_i, \mathbf{k}) = (-i\omega_i + \mathcal{H}_0(\mathbf{k}))^{-1} \). Here, we use the notation \( \int_{i\omega_i} = \int \! \frac{d\omega_i}{2\pi} \frac{d^d k}{(2\pi)^d} \int \! \frac{d\mathbf{k}}{4\pi} \), where \( \int \! \frac{d\mathbf{k}}{4\pi} \) stands for an integration over \( \mu < |k_z| < \Lambda \) with the infrared (IR) cutoff \( \mu \) and the ultra-violet (UV) cutoff \( \Lambda \). A detailed exposition of the boson self-energy is presented in the Supplemental Material [29]. We propose the following ansatz for the boson self-energy at one-loop level:

\[
\Pi(i\Omega, \mathbf{q}) = -\frac{N_f g^2 |q_r|}{\sqrt{4t_r z}} \left\{ \sqrt{\frac{\pi}{|q_r|}} \sqrt{\frac{\pi}{|q_r|}} \sqrt{\pi} \right\},
\]

(6)

where \( q_r = \sqrt{q_x^2 + q_y^2} \) and

\[
F(x, y) = \sqrt{C_{r_1}^2 + C_{r_2}^2} y^2 \tanh \left( x \sqrt{C_{r_1}^2 + C_{r_2}^2} y^2 \right),
\]

(7)

with \( C_{r_1} = 0.041, C_{r_2} = 1.199, C_{z_1} = 0.016, \) and \( C_{z_2} = 1.267 \). We illustrate the dimensionless function \( F(x, y) \) in Fig. 2(a) and compare it with the exact numerical values in Fig. 2(b).

The one-loop boson self-energy modifies the Coulomb potential in momentum space as

\[
D(i\Omega, \mathbf{q}) = \frac{1}{D_0^{-1}(\mathbf{q}) - \Pi(i\Omega, \mathbf{q})},
\]

(8)

where \( D_0(\mathbf{q}) = (a q_x^2 + \frac{1}{\pi} q_z^2)^{-1} \) is the bare boson propagator. In the static (\( \Omega = 0 \)) and long wave length

![FIG. 2.](image-url)
(q → 0) limit, the self-energy dominates the bare propagator since it linearly depends on the momentum in this limit. Thus, we take the boson self-energy as the main contribution to the renormalized Coulomb interaction, \( D(i\Omega, q) \approx -\frac{1}{\Pi(i\Omega, q)} \). This indicates that the boson is strongly renormalized from the quadratic to a linear momentum dependence, exhibiting the anomalous dimension of order one at the TQPT. This approximation has been well established in large \( N_f \) analysis and is checked afterward.

The fermion self-energy with the renormalized Coulomb interaction is

\[
\Sigma(i\omega, k) = -(ig)^2 \int_{\Omega, q} G_0(i\omega + i\Omega, k + q) D(i\Omega, q),
\]

and the fermion part of the action is modified by the fermion self-energy as

\[-i\omega + \mathcal{H}_0(k) \rightarrow -i\omega + \mathcal{H}_0(k) - \Sigma(i\omega, k).\]

It is straightforward to show that the corrections from the self-energy are logarithmically divergent in both UV and IR cutoffs, respectively, and we find

\[
\Sigma(i\omega, k) \approx \frac{C_{\omega}}{N_f} (i\omega) \ell - \frac{C_t}{N_f} \epsilon \left( t_r k_z^2 \sigma_z + t_z k_x k_y \sigma_y \right),
\]

where \( C_{\omega} = 0.366, \ C_t = 0.614, \ C_{tz} = 0.341, \) and \( \ell = \log \frac{\Lambda}{\mu} \) is the RG parameter. For the details, see the Supplemental Material [29].

We also evaluate the vertex correction at vanishing external momentum and frequency,

\[
\delta \epsilon = -(ig)^2 \int_{\Omega, q} G_0(i\Omega, q)^2 \frac{1}{-\Pi(i\Omega, q)} = \frac{C_g}{N_f} \epsilon,
\]

where \( C_g = 0.366 \), which is exactly the same as \( C_{\omega} \). This agreement is not a coincidence but instead a consequence of the Ward identity \( \delta \epsilon = \partial \Sigma / \partial (i\omega) \).

Using the logarithmic dependence of the self-energy, one can find various anomalous dimensions. The scale invariance at the critical point forces renormalization of the fermion fields with the anomalous dimension \( \eta_f = \frac{C_{\omega}}{N_f} \). The non-zero anomalous dimension clearly indicates non-Fermi liquid behaviors of the fermionic excitations, which can be understood by the absence of the pole structure in the fermionic Green function.

From Eq. (11), the RG equations for \( t_r \) and \( t_z \) are given by

\[
\frac{1}{t_r} \frac{dt_r}{d\ell} = \frac{C_{tr} - C_\omega}{N_f}, \quad \frac{1}{t_z} \frac{dt_z}{d\ell} = \frac{C_{tz} - C_\omega}{N_f}.
\]

From Eq. (13), we find \( \frac{1}{\beta} \frac{d\beta}{d\ell} = \frac{C_{tr} - C_{tz}}{N_f} > 0 \), indicating that \( \beta^{-1} \) diverges at the TQPT and that the fermionic excitations become highly anisotropic at low energies. Thus, our critical theory is described by an emergent anisotropic non-Fermi liquid.

\[ \epsilon = 4 - d \text{ expansion.} \]

Our large \( N_f \) calculation is further supported by the standard \( \epsilon = 4 - d \) expansion [17, 36–38]. Here, we introduce a new renormalization scheme in which the three spatial dimensions are embedded into a manifold that has more coordinates in the direction of the rotational axis \((z\text{-direction})\). Namely, we extend the coordinates as

\[
\int_\frac{dk_x dk_y}{(2\pi)^2} \int_\frac{dk_z}{2\pi} \int_\frac{d\sigma}{(2\pi)^d-2} \int_\frac{d^d p}{(2\pi)^d-2},
\]

with \( k_z^2 \rightarrow k_z^2 + p_z^2 \), and the momentum \( p \) lives in a \((d-3)\)-dimensional manifold. Recalling \([g^2] = z - z_\perp = 3 - d\) with \( z = 2 \) and \( z_\perp = 1 \), the coupling constant becomes marginal at \( d = 4 \) and the quantum fluctuations give logarithmic divergences. To read off these logarithmic divergences, we introduce the parameter \( \epsilon = 4 - d \) and employ the standard momentum shell RG analysis with \( \epsilon \) expansion. For the momentum shell integration, we impose the UV and IR cutoffs on the \((d-2)\)-dimensional space of \((k_z, p)\) as

\[
\int_{k, \sigma} dk_z dk_y \int_{2\pi} \int_{(2\pi)^d-2} \int_{(2\pi)^d-2},
\]

where \( \delta \Lambda \) represents an infinitesimal momentum shell \( \mu < \sqrt{k_z^2 + p_z^2} < \Lambda \) with \( \mu = \Lambda e^{-\ell} \).

By integrating out the high energy modes, we obtain corrections at one-loop order. The fermion self-energy depicted by the diagram in Fig. 3(a) is given by

\[
\Sigma(i\Omega, q) = -(ig)^2 \int_{i\omega, k} G_0(i\omega, k + q) D_0(i\omega, k)
\]

\[
\approx -\alpha F_\perp(\gamma) \epsilon \left[ t_r (q_x^2 - q_y^2) \sigma_x + 2t_r q_x q_y \sigma_y \right] - \alpha F_z(\gamma) \epsilon \left( t_z q_z^2 \right) \sigma_z,
\]

where \( F_\perp \) and \( F_z \) are dimensionless functions, whose explicit expressions are presented in the Supplemental Material [29]. Note that the frequency part is not renormalized at the one-loop order because of the instantaneous nature of the bare Coulomb interaction. Then, it
is easy to see that the vertex correction [Fig. 3(c)] vanishes due to the Ward identity. For the boson self-energy [Fig. 3(b)], we find

\[
\Pi(q) = N_f g^2 \int \omega \delta \rho \left[ G_0(\omega, \mathbf{k} + \mathbf{q}/2) G_0(\omega, \mathbf{k} - \mathbf{q}/2) \right] 
\approx - N_f \alpha \left[ \frac{a}{\gamma} q_r^2 + \frac{\gamma}{a} q_z^2 \right] \ell.
\]

Renormalizing the wave functions and the coupling constants, we obtain the RG equations for \( \alpha \) and \( \gamma \) as

\[
\frac{1}{\alpha} \frac{d\alpha}{dt} = \epsilon - \frac{N_f \alpha}{2} \left( 1 - \frac{\alpha}{2} F_+(\gamma) \right),
\]

\[
\frac{1}{\gamma} \frac{d\gamma}{dt} = \frac{N_f \alpha}{2} \left( 1 - \frac{\alpha}{2} F_-(\gamma) \right),
\]

where \( F_\pm(x) = F_\pm(x) \pm F_\pm(x) \). We find two fixed points from the RG equations in Eq. (18). The non-interacting fixed point \( \alpha^* = 0 \) with arbitrary \( \gamma^* \) is unstable, whereas there exists a stable interacting fixed point at \( (\alpha^*, \gamma^*) \) with \( \alpha^* > 0 \). For \( N_f = 1 \) and \( \epsilon = 1 \), the stable fixed point is located at \( (\alpha^*, \gamma^*) = (0.671, 0.748) \), and for large \( N_f \), \( (\alpha^*, \gamma^*) \approx (\epsilon/N_f, 1 - 0.358/N_f) \). The RG flows of \( \alpha \) and \( \gamma \) are illustrated in Fig. 4.

At the stable fixed point, the RG equations for the bosonic and fermionic anisotropy parameters are given by, respectively,

\[
\frac{1}{\alpha} \frac{d\alpha}{dt}\bigg|_{\text{f.p.}} = N_f \alpha^2 \left( \frac{1}{\gamma^2} - 1 \right) > 0,
\]

\[
\frac{1}{\beta^-1} \frac{d\beta^-1}{dt}\bigg|_{\text{f.p.}} = - \alpha^* F_-(\gamma^*) > 0,
\]

where f.p. stands for the fixed point. Note that \( \beta^-1 \) diverges at the stable fixed point as in the large \( N_f \) calculation demonstrating an emergent anisotropic non-Fermi liquid, which becomes a sanity check of our analysis. Also note that the anomalous dimension of fermions in the current \( \epsilon \) expansion is zero due to the absence of the frequency dependence in the fermion self-energy. This is an artifact of the \( \epsilon \) expansion up to first order, which can be corrected using the next order calculations, giving a consistent result with the large \( N_f \) calculation (for details, see the Supplemental Material [29]).

Our calculations are controlled by either \( \epsilon \) or \( 1/N_f \). Thus, the scaling dimensions of the four-point interactions at the stable fixed point are the same as the bare one at the leading order, \( [\nu_{ijkl}] = -1 + O(\epsilon \text{ or } 1/N_f) \), which indicates that our fixed point is stable under the four-point interactions.

**Physical observables.** — Recently, several materials [26, 39–42] have been proposed as possible candidates for DWSMs, in which TQPTs may occur by tuning the system parameters. For example, it has been theoretically demonstrated that SrSi\(_2\) can be tuned by changing the lattice constant through doping or strain, leading to a transition from the DWSM to a trivial insulator phase [42]. Since the anisotropic non-Fermi liquid behavior at the TQPT will provide power-law corrections anisotropically to the scaling of physical observables [34, 43, 44], the anisotropic scaling relations will be valuable to experiments.

First, consider the parameter dependence of physical observables in the non-interacting limit [27–29, 45–48]. The details are presented in the Supplemental Material [29]. In the non-interacting limit, the specific heat \( C_V \), compressibility \( \kappa \), diamagnetic susceptibility \( \chi_D \), and optical conductivity \( \sigma \) are given by

\[
C_V \propto \frac{T^{3/2}}{t_r t_z^{1/2}}, \quad \kappa \propto \frac{T^{1/2}}{t_r t_z^{1/2}}, \quad \chi_D \propto \frac{t_r}{t_z^{1/2} T^{1/2}}, \quad \sigma \propto \frac{t_r^{1/2}}{t_z T^{1/2}}.
\]

Here, \( \chi_{D,xx} = \chi_{D,yy} = \chi_{D,\perp} \) and \( \sigma_{xx} = \sigma_{yy} = \sigma_{\perp} \) because of the \( C_4 \) symmetry of the Hamiltonian. We also assume \( t_z = t_y = t_r \) for simplicity.

Now, consider how the anisotropic non-Fermi liquids change the bare scaling behaviors of the physical observables. From the \( \epsilon \) expansion, the RG equations for \( t_r \) and \( t_z \) are given by

\[
\frac{1}{t_r} \frac{dt_r}{d \ln b} = -2 z_\perp + \alpha F_\perp(\gamma),
\]

\[
\frac{1}{t_z} \frac{dt_z}{d \ln b} = -2 + \alpha F_\perp(\gamma),
\]

where \( \ln b \equiv \ell \). Let us choose \( z = 2 \) and \( z_\perp = 1 \) so that \( t_r \) and \( t_z \) are marginal at the tree level. Since \( \frac{d\sigma}{d\ln b} = \sigma O \) for \( O = \omega, T \) with \( z = 2 \), \( O(b) = O b^2 \). Let \( b^* = (A/O)^{1/2} \). Combining this with Eq. (21), we find that \( t_i(b^*) = t_i(0)^{2}\alpha F_i(\gamma^*) \propto O^{-c_i} \) where \( i = \perp, z \), \( c_i = \frac{1}{2} \frac{d t_i}{d \ln b} \bigg|_{\text{f.p.}} = \alpha F_i(\gamma^*) / 2, c_r \approx 0.402 / N_f \), and \( c_z \approx 0.044 / N_f \) in the large \( N_f \) approximation.
Then, near the interacting fixed point, the scaling relations of the physical observables with respect to either temperature or frequency become

\[ C_V \propto T^{1/2+\eta_1}, \quad \kappa \propto T^{1/2+\eta_1}, \quad (22) \]

\[ \chi_D,\perp \propto T^{1/2-\eta_2}, \quad \chi_D,\parallel \propto T^{1/2-\eta_2}, \]

\[ \sigma_{xx} \propto \Omega^{1/2+\eta_2}, \quad \sigma_{zz} \propto \Omega^{1/2+\eta_2}, \]

where \( \eta_1 \equiv c_r + c_e / 2 \equiv 0.423 / N_f, \eta_2 \equiv c_r - c_e / 2 \equiv 0.022 / N_f, \) and \( \eta_3 \equiv c_r - c_e / 2 \equiv 0.380 / N_f. \) (Equivalently, we can obtain the same results by including all the effects of renormalization in the coordinates rather than the system parameters, as presented in the Supplemental Material [29].) Thus, it is easily seen that the diamagnetic susceptibility \( \chi \) and optical conductivity show anisotropic scaling behaviors, \( \chi_D,\perp / \chi_D,\parallel \propto \Omega^{\eta_2-\eta_3} \) and \( \sigma_{zz} / \sigma_{xx} \propto \Omega^{\eta_2-\eta_3}. \) In addition, the permittivity tensor characterizing the electronic modes and the long-range Coulomb interactions also exhibits the anisotropic behavior, \( \varepsilon_\perp / \varepsilon_\parallel = a^2 \propto \Omega^{\eta_2-\eta_3}. \) By measuring these ratios, we can clearly see the anisotropic scaling behaviors at the TQPT.

Discussion and Conclusion. — So far, for simplicity we ignored \( m_0 \) in Eq. (1) and the corresponding \( s_r(k_x^2 + k_y^2)\sigma_z \) term with \( s_r = m_0 a_0^2 \) in \( H_0, \) which is allowed by symmetry. If we include the effect of this term, we find that there still exists a stable non-Gaussian fixed point at \( (\alpha^*, \gamma^*, \lambda^*) = (0.342 / N_f, 0.799 - 0.079 / N_f, -\text{sgn}(\beta)(0.875 + 0.032 / N_f)) \) in the \( \epsilon \) expansion \( (\lambda = s_r / t_r), \) indicating that the anisotropic non-Fermi liquid behavior is robust against the \( s_r(k_x^2 + k_y^2)\sigma_z \) term. The details are presented in the Supplemental Material [29]. Note that for a TQPT between triple-Weyl semimetals [46], we believe that similar symmetry-allowed parabolic term should be considered.

In summary, we studied TQPTs between DWSMs and insulators using the large \( N_f \) theory and epsilon expansion. We found that a novel class of quantum criticality appears at the TQPT characterized by emergent anisotropic non-Fermi liquid behaviors in which critical electronic modes and the long-range Coulomb interaction are strongly coupled, and the system becomes infinitely anisotropic. The anisotropic behaviors at the TQPT will be demonstrated experimentally by measuring the power-law corrections to the diamagnetic susceptibility \( \chi_D,\perp / \chi_D,\parallel \propto \Omega^{\eta_2-\eta_3} \) and optical conductivity \( \sigma_{zz} / \sigma_{xx} \propto \Omega^{\eta_2-\eta_3}. \)

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See the Supplemental Material for the details of $\epsilon = 4 - d$ expansion method, large $N_f$ calculation, derivations of physical observables in the non-interacting limit, and the sanity check of the power-law corrections for the physical observables, which includes Refs. [43, 48].

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Supplemental Material for “Emergent Anisotropic Non-Fermi Liquid”

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DETAILS OF THE $\epsilon = 4 - d$ METHOD

In this section, we provide detailed calculations of the $\epsilon = 4 - d$ method. First, we prove that $t_x = t_y$ and $a_x = a_y$ at low energies. Next, we derive the renormalization group (RG) equations using the $\epsilon$ expansion. Then we discuss the effect of the symmetry-allowed parabolic term, which is neglected in the main text, demonstrating that the TQPT is still characterized by anisotropic non-Fermi liquids.

Consider the leading-order self-energy corrections for fermions and bosons:

$$\Sigma(0, k) = (-ig)^2 \int_{\Omega, q} G_0(i\Omega, q + k) D_0(i\Omega, q), \quad (S1)$$

$$\Pi(i\Omega, q) = - N_f (-ig)^2 \int_{\omega, k} \text{Tr}[G_0(i\Omega + i\omega, k + q/2) G_0(i\omega, k - q/2)], \quad (S2)$$

where $\int_{\Omega, q, p} = \int_{\Omega} \frac{d\Omega}{(2\pi)^d} \int \frac{dq_d dq_y}{(2\pi)^d} \int_{\partial\Lambda} \frac{dq_d dq_y dq_z}{(2\pi)^d} \pm 3$ with $\partial\Lambda$ being the region $\mu < \sqrt{q_x^2 + q_y^2} < \Lambda$. Here,

$$G_0(i\Omega, k) = \frac{1}{-i\Omega + \varepsilon_\pm(k) \sigma_x + \varepsilon_(k) \sigma_y + \varepsilon_z(k) \sigma_z} = \frac{i\Omega + \varepsilon_x(k) \sigma_x + \varepsilon_y(k) \sigma_y + \varepsilon_z(k) \sigma_z}{\Omega^2 + E(k)^2}, \quad (S3)$$

$$D_0(i\Omega, q) = \frac{1}{a_x q_x^2 + a_y q_y^2 + a_z q_z^2}, \quad (S4)$$

where $\varepsilon_x(k) = t_x k_x^2 - k_y^2$, $\varepsilon_y(k) = 2 t_y k_x k_y$, $\varepsilon_z(k) = t_z k_z^2$, and $E(k) = \sqrt{\varepsilon_x(k)^2 + \varepsilon_y(k)^2 + \varepsilon_z(k)^2}$.

Proof of the emergent rotational symmetry along the $k_z$-axis

Proof of $a_x = a_y$

First, let us prove that $a_x = a_y$ at low energies. From the self-energy of the Coulomb interaction at $\Omega = 0$,

$$\Pi(0, k) = - N_f (-ig)^2 \int_{\omega, q} \text{Tr}[G_0(i\omega, q + k/2) G_0(i\omega, q - k/2)]$$

$$= - N_f g^2 \int_{q, p} \left(1 - \frac{\varepsilon_+ \cdot \varepsilon_\pm}{E_+ E_-}\right) \frac{1}{E_+ + E_-}$$

$$\approx - N_f g^2 \int_{q, p} \left[\frac{1}{a_x} \frac{(q_x^2 + q_y^2)(t_x^2 t_y^2 (q_x^2 + q_y^2)^2 + t_y^2 (t_z^2 + t_y^2) (q_x^2 + p^2)^2)}{2(t_x^2 (q_x^2 - q_y^2)^2 + 4t_y^2 q_x^2 q_y^2 + t_z^2 (q_x^2 + p^2)^2)^{5/2}} \right]$$

$$+ \frac{1}{a_y} \frac{(q_x^2 + q_y^2)(t_x^2 t_y^2 (q_x^2 + q_y^2)^2 + t_y^2 (t_z^2 + t_y^2) (q_x^2 + p^2)^2)}{2(t_x^2 (q_x^2 - q_y^2)^2 + 4t_y^2 q_x^2 q_y^2 + t_z^2 (q_x^2 + p^2)^2)^{5/2}} \right]$$

$$+ \frac{1}{a_z} \frac{t_z^2 (t_y^2 (q_y^2 - q_y^2)^2 + 4t_y^2 q_x^2 q_y^2 + t_z^2 (q_x^2 + p^2)^2)^{5/2}}{a_z k_z^2}, \quad (S5)$$

where $\varepsilon_\pm = \varepsilon_i (q \pm k/2)$ and $E_\pm = \sqrt{\sum \varepsilon_i^2}$. We find that the coefficients of the $k_x^2$ and $k_y^2$ terms are the same, which we denote as $C_a$, are given by

$$C_a = - N_f g^2 \int_{q, p} \frac{q_y^2}{2} \left[\frac{t_x^2 (t_x^2 (q_x^2 + q_y^2)^2 + t_y^2 (t_z^2 + t_y^2) (q_x^2 + p^2)^2)}{t_x^2 (q_x^2 - q_y^2)^2 + 4t_y^2 q_x^2 q_y^2 + t_z^2 (q_x^2 + p^2)^2} \right]$$

""
\[ \alpha = \frac{N_f g^2}{\Lambda - a} \ell, \]  
(6)

where \( \ell = \ln(\Lambda/\mu) \). Let \( C'_{a} = -C_{a}/\ell \), which is positive regardless of \( t_x, t_y \) and \( t_z \). Then, the beta function of \( a_x/a_y \) is

\[ \frac{1}{a_x/a_y} \frac{d(a_x/a_y)}{d\ell} = C_{a} a_y \left( 1 - \frac{a_x}{a_y} \right), \]  
(7)

Since \( C_{a} \) is positive, \( a_x = a_y \) at low energies.

**Proof of \( t_x = t_y \)**

From now on, we employ the following form of the Coulomb interaction propagator with \( a_x = a_y \equiv a \) and \( a_z = 1/a \),

\[ D_0(i\Omega, \mathbf{q}) = \frac{1}{a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a}. \]  
(8)

Then

\[ \Sigma(i\omega, \mathbf{k}) = -(ig)^2 \int \frac{d^4q}{(2\pi)^4} G_0(i\omega + i\Omega, \mathbf{k} + \mathbf{q}) D_0(i\Omega, \mathbf{q}), \]

\[ \approx -\delta_{t_x} \varepsilon_x(k)\sigma_x - \delta_{t_y} \varepsilon_y(k)\sigma_y - \delta_{t_z} \varepsilon_z(k)\sigma_z, \]

where

\[ \delta_{t_x} = \frac{g^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{\varepsilon_x^2 t_x^2 (q_x^4 + 6q_y^2 q_y^2 + q_y^4) - 2\varepsilon_x^2 t_y^2 (q_x^4 + q_y^4) - (2 - t_x^2/t_y^2)\varepsilon_x^2 \varepsilon_x^2 + \varepsilon_x^4}{(\varepsilon_x^2 + \varepsilon_x^2 + \varepsilon_y^2)^{5/2}(a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a)}, \]

(10)

\[ \delta_{t_y} = \frac{g^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{-\varepsilon_x^2 t_y^2 (q_x^4 + 6q_y^2 q_y^2 + q_y^4) + 2\varepsilon_x^2 t_x^2 (q_x^4 + q_y^4) - (2 - t_y^2/t_x^2)\varepsilon_x^2 \varepsilon_x^2 + \varepsilon_x^4}{(\varepsilon_x^2 + \varepsilon_x^2 + \varepsilon_y^2)^{5/2}(a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a)}, \]

(11)

\[ \delta_{t_z} = \frac{g^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{(\varepsilon_x^2 + \varepsilon_y^2 - \varepsilon_z t_z (5q_x^2 - p^2))}{(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2)^{5/2}(a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a)}. \]

(12)

To prove \( t_x = t_y \) at low energies, let us define \( T = t_x/t_y \). Then, the beta function of \( T \) is given by

\[ \frac{1}{T} \frac{dT}{d\ell} = \delta_{t_x} - \delta_{t_y}, \]

(13)

From Eqs. (10) and (11), \( \delta_{t_x} - \delta_{t_y} \) is given by

\[ \delta_{t_x} - \delta_{t_y} = \frac{g^2}{2t_y} \int \frac{(t_x^2 + t_y^2)\varepsilon_x^2 (q_x^4 + 6q_y^2 q_y^2 + q_y^4) - 2(t_x^2 + t_y^2)\varepsilon_x^2 (q_x^4 + q_y^4) - ((2 - t_x^2/t_y^2)\varepsilon_x^2 - (2 - t_x^2/t_y^2)\varepsilon_x^2)\varepsilon_x^2}{(\varepsilon_x^2 + \varepsilon_x^2 + \varepsilon_y^2)^{5/2}(a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a)} + \frac{g^2}{2t_y} \int \frac{\beta^2 (q_x^2 + p^2)^2 q_x^4 + 6q_y^2 q_y^2 q_x^4 + (q_y^2 + q_z^2)^2 T^2 - 8q_z^4 q_x^2}{(T^2 q_x^2 - q_y^2)^2 + 4q_y^2 q_y^2 + \beta^2 (q_z^2 + p^2)^2)^{5/2}(a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a)} \]

(14)

where \( \beta = t_z/t_y \).

Expanding \( \delta_{t_x} - \delta_{t_y} \) in terms of \( \delta T = T - 1 \), we then have

\[ \delta_{t_x} - \delta_{t_y} \approx \frac{g^2}{2t_y} \int \frac{((q_x^2 - 6q_y^2 q_y^2 + q_y^4) (2q_x^4 + q_y^4)^2 - (q_z^2 + p^2)^2 \beta^2}{(q_x^2 + q_y^2)^2 + \beta^2 (q_z^2 + p^2)^2)^{5/2}(a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a)} - \frac{g^2}{2t_y} \int \beta \frac{4(q_x^2 + q_y^2)^2 (q_y^4 - 2q_x^2 q_y^2 + 5q_x^4 q_y^2 - 2q_x^2 q_y^4 + q_y^4)}{(q_x^2 + q_y^2)^2 + \beta^2 (q_z^2 + p^2)^2)^{5/2}(a(q_x^2 + q_y^2) + (q_z^2 + p^2)/a)} \]

(14)
where \( \alpha = \frac{A_d}{\sqrt{\pi} \Lambda A} \), \( \gamma = \frac{a\sqrt{\pi}}{2} \), and \( A_d = \frac{1}{6\pi x^2 (d/2)} \). Here, we introduce the function \( G_T(x) \) defined by

\[
G_T(x) = 72x^5 \int_0^\infty \frac{dr}{r^5 \left( x^4 + 16x^4 \right)^{7/2}} \left( \frac{4r^2 (4x^2 + 1 + 16x^4)}{1 + 48x^4 - 16x^4} \right).
\]

Note that \( G_T(\gamma) \) is positive for all \( \gamma \). Then, we see

\[
\frac{1}{dT} \frac{d\delta T}{dt} \approx -\alpha G_T(\gamma) \delta T \tag{S17}
\]

is negative (positive) for positive (negative) \( \delta T \). Therefore, \( \delta T \) flows to 0, which means \( T = 1 \) is a stable fixed point and we arrive at the conclusion that \( t_x = t_y \equiv t_r \) at the low energies. Combining the results of Secs. and , we can use the following form of action at low energies,

\[
S = \int d\tau d^dx \left[ \bar{\psi} \left( \partial_\tau - ig \phi + \bar{H}_0(-i\nabla) \right) \psi + \frac{1}{2} \left( \alpha \left( (\partial_x \phi)^2 + (\partial_y \phi)^2 \right) + \frac{1}{a} (\partial_z \phi)^2 \right) \right],
\]

where

\[
\bar{H}_0(k) = t_x (k_x^2 - k_y^2) \sigma_x + 2t_x k_x k_y \sigma_y + t_z k_z^2 \sigma_z.
\]

Renormalization group equations in the \( \epsilon = 4 - d \) expansion

In this section, we will show the details of the RG analysis using the \( \epsilon = 4 - d \) expansion. From Eqs. (S5) and (S10)–(S12) with \( t_x = t_y = t_r \) and \( a_x = a_y = a_z = a \), we obtain the fermion and boson self-energies, respectively, given by

\[
\Sigma(i\Omega, q) = (-ig)^2 \int_{\omega, k, p} G_0(\omega + i\Omega, k + q) D_0(\omega, k)
\]

\[
\approx -\alpha F_\perp(\gamma) \ell \left[ t_x (q_x^2 - q_y^2) \sigma_x + 2t_x q_x q_y \sigma_y \right] - \alpha F_z(\gamma) \ell \left( t_x q_x^2 \right) \sigma_z,
\]

\[
\Pi(q) = g^2 \int_{\omega, k, p} \text{Tr} \left[ G_0(\omega, k + q/2) G_0(\omega, k - q/2) \right]
\]

\[
\approx -N_f \alpha \left[ \frac{a}{\gamma} q_x^2 + \frac{\gamma}{a} q_y^2 \right] \ell,
\]

where \( F_\perp(\gamma) \) and \( F_z(\gamma) \) are given by

\[
F_\perp(x) \equiv \frac{\delta_x}{\alpha \ell} \left| t_x = t_y = t_r, \ a_x = a_y = a_z = a \right.
\]

\[
= 48x^5 \int_0^\infty dr \frac{r (32x^4 - r^4)}{(r^4 + 16x^4)^{5/2}(r^2 + 1)}
\]
\[ F(x) \equiv \frac{\delta \psi}{\alpha \ell} \bigg|_{t_x=t_y=t_r, \ \alpha_x=\alpha_y=\alpha_z=1=\alpha} = 6x \int_0^\infty dr \frac{\sqrt{r^5(r^4 - 32x^4)}}{(r^4 + 16x^4)^{5/2}(r^2 + 1)} \]

\[ = \frac{3x}{(1 + 16x^4)^{5/2}} \left[ \sqrt{1 + 16x^4(-2 + 12x^2 + 16x^4)} + (1 - 32x^4) \ln \left( \frac{4x^2 (4x^2 + \sqrt{1 + 16x^4})}{-1 + \sqrt{1 + 16x^4}} \right) \right]. \] (S23)

Figure S1 shows the plots of \( F_1(x) \) and \( F_2(x) \). Then, after rescaling \( z \rightarrow z e^\ell \), \( (x, y) \rightarrow (x, y)e^{z\ell} \), and \( \tau \rightarrow e^{\gamma \ell} \tau \), and introducing the renormalization constant, \( \psi \rightarrow \psi/Z_\psi^\ell \), \( \phi \rightarrow \phi/Z_\phi^\ell \), \( t_r \rightarrow t_r/Z_{t_r} \), \( t_z \rightarrow t_z/Z_{t_z} \), \( a \rightarrow a/Z_a \), and \( g \rightarrow g/Z_g \), we arrive at the following renormalized action,

\[ S_{\text{renorm}} = \int d\tau d^d x \left[ \psi^\dagger (\partial_\tau - i \phi + H_0(-i\nabla) - \Sigma(-i\nabla)) \psi + \frac{1}{2} \left( \alpha \left\{ (\partial_\tau \phi)^2 + (\partial_y \phi)^2 \right\} + \frac{1}{a} (\partial_\gamma \phi)^2 \right) - \frac{1}{2} (\partial_\gamma \phi) \right] + \frac{e^{-(z+1)d-2}}{2Z_\phi} \left[ e^{-(z+1)d-2} \frac{(1 + z N_f \alpha \ell)}{Z_\phi} \left( \partial_\tau \phi \right)^2 + (\partial_y \phi)^2 + e^{-2\gamma} Z_a (1 + (N_f \alpha \ell) \left( \frac{1}{a} (\partial_\gamma \phi)^2 \right) \right]. \] (S24)

Requiring the scaling invariance of the action, we obtain the renormalization constants as follows:

\[ Z_\psi = 1 + [z + (d - 2)] \ell, \] (S25)
\[ Z_{t_r} = 1 + [z - z + (d - 2)] \ell, \] (S26)
\[ Z_{t_z} = 1 + [z - 2 + (d - 2)] \ell, \] (S27)
\[ Z_\phi = 1 + \left[ z + z + (d - 3) + \frac{N_f \alpha}{2} \left( \frac{1}{\gamma} + \gamma \right) \right] \ell, \] (S28)
\[ Z_a = 1 + \left[ 1 - z + \frac{N_f \alpha}{2} \left( \frac{1}{\gamma} - \gamma \right) \right] \ell, \] (S29)
\[ Z_g = 1 + \left[ \frac{z - z - (d - 3)}{2} - \frac{N_f \alpha}{4} \left( \frac{1}{\gamma} + \gamma \right) \right] \ell. \] (S30)
From these renormalization constants, we can obtain the following RG equations for \( d = 4 - \epsilon \),

\[
\frac{1}{t_r} \frac{dt_r}{dt} = z - 2z_\perp + \alpha F_\perp(\gamma), \tag{S31}
\]

\[
\frac{1}{t_z} \frac{dt_z}{dt} = z - 2 + \alpha F_z(\gamma), \tag{S32}
\]

\[
\frac{1}{a} \frac{da}{dt} = 1 - z_\perp + \frac{N_f \alpha}{2} \left( \frac{1}{\gamma} - \gamma \right), \tag{S33}
\]

\[
\frac{1}{g^2} \frac{dg^2}{dt} = z - z_\perp - 1 + \epsilon - \frac{N_f \alpha}{2} \left( \frac{1}{\gamma} + \gamma \right). \tag{S34}
\]

Thus, we find the RG equations for the dimensionless parameters \( \alpha \) and \( \gamma \) as follows:

\[
\frac{1}{\alpha} \frac{d\alpha}{dt} = \epsilon - \frac{N_f \alpha}{2} \left( \frac{1}{\gamma} + \gamma \right) - \frac{\alpha}{2} \left( F_z(\gamma) + F_\perp(\gamma) \right), \tag{S35}
\]

\[
\frac{1}{\gamma} \frac{d\gamma}{dt} = \frac{N_f \alpha}{2} \left( \frac{1}{\gamma} - \gamma \right) + \frac{\alpha}{2} \left( F_z(\gamma) - F_\perp(\gamma) \right). \tag{S36}
\]

**Effects of the symmetry-allowed parabolic term**

If we include the symmetry-allowed parabolic term, \( s_r(k_x^2 + k_y^2)\sigma_z \), the non-interacting Hamiltonian \( \mathcal{H}_0 \) is modified as

\[
\mathcal{H}_0 = t_r(k_x^2 - k_y^2)\sigma_x + 2t_r k_x k_y \sigma_y + \left[ B t_z k_z^2 + s_r(k_x^2 + k_y^2) \right] \sigma_z, \tag{S37}
\]

where \( B = \pm 1 \) for the topologically trivial and nontrivial insulator phases, respectively.

**Boson self-energy**

Similarly as in Eq. (S21), we can obtain the boson self-energy in the presence of the symmetry-allowed parabolic term as

\[
\Pi(i\Omega, q) = -N_f |(i\Omega)|^2 \int_{\omega, k, p} \text{Tr}[G_0(i\Omega + i\omega, k + q)G_0(i\omega, k)]
\approx -N_f \alpha \left[ \frac{1}{\gamma} \left( 2 + \frac{\lambda^2}{2} - B \frac{\lambda(5 + 2\lambda^2)}{4\sqrt{1 + \lambda^2}} \right) aq^2 + \gamma \frac{1 + 2\lambda^2}{\sqrt{1 + \lambda^2} - 2B\lambda} \frac{1}{a'q_z^2} \right], \tag{S38}
\]

where \( \lambda = \frac{\pi}{t_r} \).

**Fermion self-energy**

Similarly as in Eq. (S20), we can obtain the fermion self-energy as

\[
\Sigma(i\omega, k) = (i\Omega)|(i\Omega + i\omega, k + q)D_0(i\Omega, q)
\approx -\delta_{t_r} \left[ t_r(k_x^2 - k_y^2)\sigma_x + 2t_r k_x k_y \sigma_y \right] - \delta_{s_r} \left[ B t_z k_z^2 + s_r(k_x^2 + k_y^2) \right] \sigma_z, \tag{S39}
\]

where \( \delta_{t_r}, \delta_{t_z}, \) and \( \delta_{s_r} \) are, respectively, given by

\[
\delta_{t_r} = \frac{g^2}{2} \int_{k, p} \frac{t_r^2 k_x k_y^2 (2(B t_z k_z^2 + s_r k_x^2)^2 - t_r^2 k_y^2)}{2(t_r^2 k_x^2 + (B t_z k_z^2 + s_r k_y^2)^2)^{3/2}(a k_x^2 + 1/a k_y^2)}
\]
\[
\begin{align*}
&= A_{d-2g^2} \ell \int dr \frac{3(2\gamma)^5r(-r^4 + 2(4B\gamma^2 + \lambda r^2)^2)}{2(1 + r^2)(r^4 + (4B\gamma^2 + \lambda r^2)^2)^{5/2}} \\
&= \alpha F_\perp(\gamma, \lambda) \ell, \\
\delta_\perp &= \frac{g^2}{2} \int_{k_p} \left( t_z^4 k_z^4 - 3(t_z^4 k_z^4) - (Bt_z k_z^2)^2 (Bt_z k_z^2 + s r^2)ight) \right) \right) \right) \\
&= \frac{B A_{d-2g^2} \ell}{\lambda \sqrt{t_z t_A}} \int dr \frac{6\gamma r^5 (r^4 - (8B\gamma^2 - \lambda r^2)(2B\gamma^2 + \lambda r^2))}{(1 + r^2)(r^4 + (4B\gamma^2 + \lambda r^2)^2)^{5/2}} \\
&= \frac{B}{\lambda} \alpha F_\perp(\gamma, \lambda) \ell. \\
F_\perp(\gamma, \lambda) &= \int_0^\infty \frac{3\gamma}{1 + (4B\gamma^2 - \lambda)^2} \left[ 1 + \lambda^2 \right] \left[ 1 + \lambda^2 \right] \\
&= \frac{16\gamma^4 r^4}{\left(1 + (4B\gamma^2 + \lambda r^2)^2\right)^{5/2}} \right) \right) \\
&= \frac{1}{\lambda} \cdot \left( \frac{4\gamma^2 - B\lambda + \sqrt{1 + (4B\gamma^2 - \lambda)^2}}{1 + 4B\gamma^2 - \lambda^2 + \sqrt{1 + \lambda^2} \sqrt{1 + (4B\gamma^2 - \lambda)^2}} \right) \\
F_\parallel(\gamma, \lambda) &= \int_0^\infty \frac{24\gamma^3 r^3}{(1 + r^2)(r^4 + (4B\gamma^2 + \lambda r^2)^2)^{5/2}} \\
&= \frac{1}{\lambda^2} \cdot \left( \frac{4\gamma^2 - B\lambda + \sqrt{1 + (4B\gamma^2 - \lambda)^2}}{1 + 4B\gamma^2 - \lambda^2 + \sqrt{1 + \lambda^2} \sqrt{1 + (4B\gamma^2 - \lambda)^2}} \right) \\
&= -4\gamma^2 F_\perp(\gamma, \lambda). \\
\end{align*}
\]

Note that in the limit \( \lambda = 0 \), \( F_\perp(\gamma, \lambda) = F_\perp(\gamma) \), and \( F_\parallel(\gamma, \lambda) = F_\parallel(\gamma) \).

\[
\text{RG flow equation}
\]

From Sec. and Sec. , we can obtain the following RG flow equations,

\[
\begin{align*}
\frac{1}{t_\perp} \frac{dt_\perp}{dt} &= z - 2z_\perp + \alpha F_\perp(\gamma, \lambda), \\
\frac{1}{t_\parallel} \frac{dt_\parallel}{dt} &= z - 2 + \alpha F_\parallel(\gamma, \lambda).
\end{align*}
\]
\[
\frac{1}{s_r} \frac{ds_r}{d\ell} = z - 2z_\perp - 4b_0^2 \gamma^2 F_\perp(\gamma, \lambda),
\]
\[
\frac{1}{g^2} \frac{dg^2}{d\ell} = -z - z_\perp + 1 - \epsilon - N_f \alpha \left[ \frac{1}{\gamma} \left( \frac{2 + \lambda^2}{2} - B \frac{\lambda(5 + 2\lambda^2)}{4\sqrt{1 + \lambda^2}} \right) - \gamma \left( \frac{1 + 2\lambda^2}{\sqrt{1 + \lambda^2}} - 2B\lambda \right) \right].
\]

Then, the RG equations for the dimensionless parameters, \( \alpha, \gamma \) and \( \lambda \) are given by
\[
\frac{1}{\alpha} \frac{d\alpha}{d\ell} = \epsilon - \frac{\alpha}{2} \left[ N_f \left\{ \frac{1}{\gamma} \left( \frac{2 + \lambda^2}{2} - B \frac{\lambda(5 + 2\lambda^2)}{4\sqrt{1 + \lambda^2}} \right) + \gamma \left( \frac{1 + 2\lambda^2}{\sqrt{1 + \lambda^2}} - 2B\lambda \right) \right\} + F_\gamma(\gamma, \lambda) + F_\perp(\gamma, \lambda) \right],
\]
\[
\frac{1}{\gamma} \frac{d\gamma}{d\ell} = \frac{\alpha}{2} \left[ N_f \left\{ \frac{1}{\gamma} \left( \frac{2 + \lambda^2}{2} - B \frac{\lambda(5 + 2\lambda^2)}{4\sqrt{1 + \lambda^2}} \right) - \gamma \left( \frac{1 + 2\lambda^2}{\sqrt{1 + \lambda^2}} - 2B\lambda \right) \right\} + F_\gamma(\gamma, \lambda) - F_\perp(\gamma, \lambda) \right],
\]
\[
\frac{1}{\lambda} \frac{d\lambda}{d\ell} = -\frac{\alpha}{\lambda} \left[ 4B\gamma^2 F_\gamma(\gamma, \lambda) + \lambda F_\perp(\gamma, \lambda) \right].
\]

For given \( N_f \), the RG equations have unstable fixed point, \( \alpha^* = 0 \) with arbitrary \( \gamma^* \) and \( \lambda^* \), and stable interacting fixed point, \((\alpha^*, \gamma^*, \lambda^*) = (0.342\epsilon/N_f, 0.799 - 0.079/N_f, -B(0.875 + 0.032/N_f)) \) for large \( N_f \). Then, near the interacting fixed point,
\[
\frac{1}{\alpha} \frac{d\alpha}{d\ell} \bigg|_{f.p.} = \frac{N_f}{2} \left\{ \frac{1}{\gamma^*} \left( \frac{2 + \lambda^*}{2} - B \frac{\lambda^*(5 + 2\lambda^*)}{4\sqrt{1 + \lambda^*}} \right) - \gamma^* \left( \frac{1 + 2\lambda^*}{\sqrt{1 + \lambda^*}} - 2B\lambda^* \right) \right\} > 0,
\]
\[
\frac{1}{\beta^{-1}} \frac{d\beta^{-1}}{d\ell} \bigg|_{f.p.} = -\alpha^* \left( F_\gamma(\gamma^*, \lambda^*) - F_\perp(\gamma^*, \lambda^*) \right) > 0.
\]

Thus, the bosonic and fermionic anisotropy parameters \( a \) and \( \beta^{-1} \) diverge at the stable interacting fixed point. Therefore, even if we keep \( s_r(k_x^2 + k_y^2)\sigma_z \), the interacting fixed point still exhibits anisotropic non-Fermi liquid behaviors.

**Details of the Large \( N_f \) Calculation**

In this section, we will show the detailed calculations of the large \( N_f \) method.

**Boson self-energy**

Consider the self-energy of the Coulomb interaction given by
\[
\Pi(i\Omega, \mathbf{q}) = -N_f (-ig)^2 \int d\omega \langle \mathbf{k} \rangle \text{Tr} \left[ G_0(i\Omega + i\omega, \mathbf{k} + \mathbf{q}) G_0(i\omega, \mathbf{k}) \right]
\]
\[
= -N_f g^2 \int \epsilon_+ + \epsilon_- \frac{E_+ + E_-}{(E_+ + E_-)^2 + \Omega^2} \left( 1 - \frac{\epsilon_+ \cdot \epsilon_-}{E_+ E_-} \right),
\]
where \( \epsilon_{\pm} = \epsilon_i(k \pm q/2) \) and \( E_{\pm} = \sqrt{\sum \epsilon_{i\pm}^2} \).

**\( q_r \) dependence**

Let us find the \( q_r \) dependence in \( \Pi(i\Omega, q_r) \) with non-zero \( i\Omega \). Because of the emergent rotational symmetry along the \( k_z \)-axis, we put \( q_r = q_r \hat{x} \) for simplicity. After changing the integration variables, \( k_x \rightarrow q_r x, k_y \rightarrow q_r y, k_z \rightarrow (t_r/t_z)^{1/2} q_r z \), we get
\[
\Pi(i\Omega, q_r) = -\frac{N_f g^2 |q_r|}{8\pi^3 s_t t_z} \int d^3x \sqrt{\left( \frac{(x+1)^2 + y^2}{z^4} + \frac{(x^2 + y^2)^2 + z^4}{\Omega^2} \right)}
\]
where $\xi_r = \sqrt{|q_r|}/t_r$, $C_{r1} = 0.042$, and $C_{r2} = 1.199$. The final result is a fitting function using an ansatz obtained from $\Pi(i\Omega, q_r) \propto \xi_r^2$ for $\xi_r \ll 1$, and $\Pi(i\Omega, q_r) \propto \xi_r$ for $\xi_r \gg 1$.

**$q_z$ dependence**

Similarly, after changing the integration variables, $k_r \rightarrow (t_z/t_r)^{1/2} q_r r$, $k_z \rightarrow q_z z$, we get

$$
\Pi(i\Omega, q_z) = -\frac{N_f g^2 |q_z| t_z}{4\pi^2 t_r} \int_0^\infty dr r \int_{-\infty}^\infty dz \sqrt{r^4 + (z + 1)^4 + r^4 + z^4} \left[1 - \frac{r^4 + (z + 1)^2 z^2}{\sqrt{r^4 + (z + 1)^4 + r^4 + z^4}}\right] \times \left[1 - \frac{(x + 1)^2 - y^2}{\sqrt{(x + 1)^2 + y^2}^2 + \frac{x^4}{\xi_r^2} (z + 1)^4 + \sqrt{(x^2 + y^2)^2 + \frac{x^4}{\xi_r^2} z^4}^2} + \left(\frac{\Omega}{t_r q_r}\right)^2\right]
$$

$$
= -\frac{C_{z1} N_f g^2}{\sqrt{t_r^2 t_z}} \sqrt{t_z q_z^2} \tanh(C_{z2} \xi_z),
$$

where $\xi_z = \sqrt{|q_z|}/t_z$, $C_{z1} = 0.016$, and $C_{z2} = 1.267$. The final result is a fitting function using an ansatz obtained from $\Pi(i\Omega, q_z) \propto \xi_z^2$ for $\xi_z \ll 1$, and $\Pi(i\Omega, q_z) \propto \xi_z$ for $\xi_z \gg 1$.

**Arbitrary $q$ dependence**

For arbitrary $q$,

$$
\Pi(i\Omega, q) = -\frac{N_f g^2 |q| \xi_r}{8\pi^2 \sqrt{t_r t_z} \xi_r} \int d^3 x \sqrt{\left((x + 1)^2 + y^2\right)^2 + \frac{x^4}{\xi_r^2} (z + 1)^4 + \sqrt{(x^2 + y^2)^2 + \frac{x^4}{\xi_r^2} z^4}^2} 
$$

$$
\times \left[1 - \frac{(x + 1)^2 - y^2}{\sqrt{(x + 1)^2 + y^2}^2 + \frac{x^4}{\xi_r^2} (z + 1)^4 + \sqrt{(x^2 + y^2)^2 + \frac{x^4}{\xi_r^2} z^4}^2} + \left(\frac{\Omega}{t_r q_r}\right)^2\right]
$$

$$
= -\frac{N_f g^2}{\sqrt{t_r^2 t_z}} \sqrt{C_{r1} t_r q_r^2 + C_{z1} t_z q_z^2} \tanh\left(\sqrt{C_{r2} \xi_r^2 + C_{z2} \xi_z^2}\right).
$$

The comparison between the exact numerical values and ansatz for the Coulomb interaction self-energy is presented in Fig. 1 in the main text.

**Fermion self-energy**

Using the boson self-energy obtained in Sec. , we can obtain the fermion self-energy as follows:

$$
\Sigma(i\omega, \mathbf{k}) = (ig)^2 \int_{\Omega, q} G_0(i\Omega + i\omega, \mathbf{q} + \mathbf{k}) D(i\Omega, \mathbf{q})
$$
After changing the integration variables, the corrections

\[ \Delta \approx g^2 \int_{\Omega, q} \frac{i(\Omega + \omega) + \varepsilon_x(k + q)\sigma_x + \varepsilon_y(k + q)\sigma_y + \varepsilon_z(k + q)\sigma_z}{(\Omega + \omega)^2 + \varepsilon_x^2(k + q) + \varepsilon_y^2(k + q) + \varepsilon_z^2(k + q)} \frac{1}{a(q_x^2 + q_y^2) + q_z^2/a - \Pi(i\Omega, q)} \]

\[ \approx g^2 \int_{\Omega, q} \frac{i(\Omega + \omega) + \varepsilon_x(k + q)\sigma_x + \varepsilon_y(k + q)\sigma_y + \varepsilon_z(k + q)\sigma_z}{(\Omega + \omega)^2 + \varepsilon_x^2(k + q) + \varepsilon_y^2(k + q) + \varepsilon_z^2(k + q)} \frac{\delta}{-\Pi(i\Omega, q)} \]

\[ \approx i\omega \Delta - \delta_t^x (\varepsilon_x(k)\sigma_x + \varepsilon_y(k)\sigma_y - \delta_t^x \varepsilon_z(k))\sigma_z. \]  

(S60)

The corrections \( \Delta, \delta_t^x, \) and \( \delta_t^z \) are evaluated in the following subsections.

\[ \omega \text{ correction } \delta \]

The correction \( \delta \) is given by

\[ \delta = -g^2 \int_{\Omega, q} \frac{t_q^2(\beta q^2 + \Omega^2)^2 + t_{\Omega}^2 q_z^2 - \Omega^2 \coth \left( \frac{\sqrt{C_{t_q}^2(q_z^2 + \Omega^2)} + C_{t_{\Omega}^2}(\beta q^2)}{(\Omega t_q)^2} \right)}{\left[ \Omega^2 + t_q^2(q_z^2 + \Omega^2)^2 + t_{\Omega}^2 q_z^2 + \Omega^2 \right]^2} \frac{\delta}{N(t_q)^2 \sqrt{C_{t_q}^2(q_z^2 + \Omega^2) + C_{t_{\Omega}^2}(\beta q^2)}} \]

(S61)

After changing the integration variables, \( q_r \to \sqrt{\beta} q_z a \) and \( \Omega \to \beta t_r q^2 b \), we have

\[ \delta = \frac{(t_r t_z)^{1/2}}{N_f} \frac{t_r b \beta^{3/2}}{t_r^2} \ln(\Lambda/\mu) \int_0^\infty \frac{d\alpha}{2\pi^3} \frac{a - a^2 - 1 + b^2}{(a^4 + 1 + b^2)^2} \coth \left( \frac{\sqrt{C_{t_q}^2(a^2 + C_{t_{\Omega}^2}(\beta q^2))}}{C_{t_q}^2 a^2 + C_{t_{\Omega}^2}(\beta q^2)} \right) \]

(S62)

where \( C_\omega = 0.366072 \). Note that \( \delta \) has a logarithmic divergence both in the UV and IR cutoffs.

\[ t_r \text{ correction } \delta_t^x \]

The correction \( \delta_t^x \) is given by

\[ \delta_t^x = g^2 \int_{\Omega, q} \frac{(\Omega^2 + t_q^2 q_z^2)(\Omega^2 - 3t_q^2(q_z^2 + \Omega^2)^2 + t_{\Omega}^2 q_z^2)}{\left[ \Omega^2 + t_q^2(q_z^2 + \Omega^2)^2 + t_{\Omega}^2 q_z^2 + \Omega^2 \right]^3} \frac{\coth \left( \frac{\sqrt{C_{t_q}^2(q_z^2 + \Omega^2)} + C_{t_{\Omega}^2}(\beta q^2)}{(\Omega t_q)^2} \right)}{N(t_q)^2 \sqrt{C_{t_q}^2(q_z^2 + \Omega^2) + C_{t_{\Omega}^2}(\beta q^2)}} \]

(S63)

After changing the integration variables, \( q_r \to \sqrt{\beta} q_z a \) and \( \Omega \to \beta t_r q^2 b \), we have

\[ \delta_t^x = \frac{(t_r t_z)^{1/2}}{t_r^2 N_f} \frac{t_r^2 b \beta^{3/2}}{t_r^2} \ln(\Lambda/\mu) \int_0^\infty \frac{d\alpha}{2\pi^3} \frac{a - a^2 - 1 + b^2}{(a^4 + 1 + b^2)^3} \coth \left( \frac{\sqrt{C_{t_q}^2(a^2 + C_{t_{\Omega}^2}(\beta q^2))}}{C_{t_q}^2 a^2 + C_{t_{\Omega}^2}(\beta q^2)} \right) \]

(S64)

where \( C_t = 0.614362 \). Note that \( \delta_t^x \) has a logarithmic divergence both in the UV and IR cutoffs.
The correction \( \delta_{t_z} \) is given by

\[
\delta_{t_z} = g^2 \int_{q, r} \frac{16t_r^4 q_z^8 + \left( \Omega^2 + t_r^2 (q_z^2 + q_y^2)^2 + t_z^2 k_z^4 \right) (\Omega^2 + t_r^2 (q_z^2 + q_y^2)^2 - 13t_r^2 k_z^4)}{[\Omega^2 + t_r^2 (q_z^2 + q_y^2)^2 + t_z^2 q_z^4]^3} \coth \left( \frac{\sqrt{k_z^2 (q_z^2 + q_y^2) + C_{1z} \beta q_z^2}}{(t_r t_z)^{1/2}} \right) \times \frac{N_f}{\mu^2} \frac{1}{\sqrt{C_r t_r^2 (q_z^2 + q_y^2) + C_{2z} \beta q_z^2}}.
\]

(S65)

After changing the integration variables, \( q_r \to \sqrt{\beta} q_z a \) and \( \Omega \to \beta t_r q_z^2 b \), we have

\[
\delta_{t_z} = \frac{(t_r t_z)^{1/2} t_r^2 b}{t_z^2 N_f} \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} \int_{|q_z| < \Lambda} dq_z \int_{|q_r| < \Lambda} dq_r \frac{16(\alpha^2 + 1 + b^2)(\alpha^2 - 13 + b^2) \coth \left( \frac{C_{2z} a^2 + C_{2z}}{\sqrt{C_{1z} a^2 + C_{2z}}} \right)}{C_{1z}}
\]

(S66)

\[= \frac{C_{1z}}{N_f} \ln(\Lambda/\mu),\]

where \( C_{1z} = 0.341231 \). Note that \( \delta_{t_z} \) has a logarithmic divergence both in the UV and IR cutoffs.

**Vertex correction**

The correction \( \delta_{v} \) is given by

\[
\delta_{v} = (ig)^2 \int_{q, r} \frac{1}{2} \text{Tr} [G_0(i\omega, q)G_0(i\omega, q)] D(i\omega, q)
\]

\[= -g^2 \int_{q, r} \frac{-\Omega^2 + t_r^2 (q_z^2 + q_y^2)^2 + t_z^2 q_z^4 \coth \left( \frac{\sqrt{k_z^2 (q_z^2 + q_y^2) + C_{1z} \beta q_z^2}}{(t_r t_z)^{1/2}} \right)}{[\Omega^2 + t_r^2 (q_z^2 + q_y^2)^2 + t_z^2 q_z^4]^2} \times \frac{N_f}{\mu^2} \frac{1}{\sqrt{C_r t_r^2 (q_z^2 + q_y^2) + C_{2z} \beta q_z^2}}.
\]

(S68)

After changing the integration variables, \( q_r \to \sqrt{\beta} q_z a \) and \( \Omega \to \beta t_r q_z^2 b \), we have

\[
\delta_{v} = \frac{(t_r t_z)^{1/2} t_r^2 b}{t_z^2 N_f} \int_0^\infty \frac{d\Omega}{\sqrt{\Omega}} \int_{|q_z| < \Lambda} dq_z \int_{|q_r| < \Lambda} dq_r \frac{-\Omega^2 / t_r^2 + q_z^4 + \beta^2 q_z^4 \coth \left( \frac{C_{2z} a^2 + C_{2z}}{\sqrt{C_{1z} a^2 + C_{2z}}} \right)}{[\Omega^2 / t_r^2 + q_z^4 + \beta^2 q_z^4]^2} \times \frac{N_f}{\mu^2} \frac{1}{\sqrt{C_r t_r^2 (q_z^2 + q_y^2) + C_{2z} \beta q_z^2}}.
\]

(S69)

where \( C_{v} = C_{\omega} \), which is consistent with the Ward identity.

**CONSISTENCY BETWEEN THE LARGE N_F CALCULATION AND \( \epsilon \) EXPANSION**

In this section, we will show the correspondence between the large \( N_f \) calculation and the \( \epsilon \) expansion. In the static (\( \Omega = 0 \)) and long wavelength limit (\( q \rightarrow 0 \)), the boson propagator in the large \( N_f \) approximation has the following form for the momentum dependence:

\[
D(i\omega = 0, q \rightarrow 0)^{-1} \sim q, |q_z|.
\]

(S70)
Let us consider the $\epsilon$ expansion case. In the $\epsilon$ expansion, near the interacting fixed point,

$$\alpha^* \gamma^* = \frac{\epsilon}{N_f} \left( 1 - \frac{c_{Nf}}{N_f} \right) \approx \frac{\epsilon}{N_f}, \quad (S71)$$

$$\frac{\alpha^*}{\gamma^*} = \frac{\epsilon}{N_f} \frac{1}{1 - c_{Nf}/N_f} \approx \frac{\epsilon}{N_f} \left( 1 + \frac{c_{Nf}}{N_f} \right) \approx \frac{\epsilon}{N_f}, \quad (S72)$$

where we only keep up to $N_f^{-1}$ order because we consider the large $N_f$ limit. Using these results,

$$D(i\omega = 0, q \rightarrow 0)^{-1} = a q_r^2 + \frac{1}{a} q_z^2 - \Pi(i\omega, q) \approx a (1 - \epsilon \ell) q_r^2 + \frac{1}{a} (1 + \epsilon \ell) q_z^2 \approx \epsilon q_r^2 + e^{\epsilon \ell} q_z^2 \approx q_r^{2-\epsilon/z_{\perp}} + |q_z|^{2-\epsilon}. \quad (S73)$$

Here, in the fourth line, we absorbed the momentum dependence of $a$ into $q_r$ and $q_z$. For a sufficiently large $N_f$, $z_{\perp} \approx 1$, thus for $\epsilon = 1$ with $d = 3$, $D(0, q)^{-1} \sim q_r + |q_z|$. Therefore, the result of the $\epsilon$ expansion is consistent with the large $N_f$ calculation.

**PHYSICAL OBSERVABLES IN THE NON-INTERACTING LIMIT**

In this section, we will calculate the physical observables such as the specific heat, compressibility, diamagnetic susceptibility, and optical conductivity at the TQPT between DWSM and insulating phases in the non-interacting limit. For simplicity, we assume $t_x = t_y = t_z$, the rotational symmetry along the $k_z$-axis.

**Density of states**

Through the analytic continuation $i\omega \rightarrow \omega + i\delta$ in $G_0(i\omega, k)$, the retarded Green’s function $G_0^{\text{ret}}$ is obtained as

$$G_0^{\text{ret}}(\omega + i\delta, k) = \frac{1}{-(\omega + i\delta) + H_0(k)}, \quad (S74)$$

and the imaginary part of $G_0^{\text{ret}}$ and the spectral function are

$$\operatorname{Im}[G_0^{\text{ret}}(\omega, k)] = \frac{\pi \operatorname{sgn}(\omega)}{2E_k} (\omega + H_0(k)) (\delta(\omega - E_k) + \delta(\omega + E_k)), \quad (S75)$$

$$S_F(\omega) = - \frac{1}{\pi} \text{Tr}[G_0^{\text{ret}}(\omega, k)] = \delta(\omega + E_k) + \delta(\omega - E_k). \quad (S76)$$

The density of states is given by

$$\rho(\omega) = \int \frac{d^3k}{(2\pi)^3} S_F(\omega, k) = \frac{\omega}{\pi^2} \int_0^\infty dk_r \int_0^\infty dk_z k_r \delta(\omega^2 - (t_r^2 k_r^2 + t_z^2 k_z^2)) = \frac{\Gamma(5/4)}{4\pi^{5/2}\Gamma(3/4)} \frac{\omega^{1/2}}{t_r t_z^{1/2}} \quad (S77)$$

where $\Gamma(x)$ is the gamma function and we use the identity,

$$\int_0^1 dR \frac{R}{(1-R^2)^{3/4}} = \sqrt{\pi}\Gamma(5/4)/\Gamma(3/4). \quad (S78)$$
Free energy

In this section, we will calculate the free energy at the TQPT in the non-interacting limit from which the specific heat and the compressibility are derived. The finite-temperature propagator of fermion is
\[ G_0(i\omega_n, k)^{-1} = (-i\omega_n - \mu) + \mathcal{H}_0(k), \]  
where we introduce the chemical potential \( \mu \) for deriving the compressibility. The partition function and its logarithmic form are given by
\[ Z = \text{Det}[\beta G_0^{-1}] = \prod_{i\omega_n} [\beta^2((\omega_n - i\mu)^2 + E(k)^2)], \]  
\[ \ln Z = V \int \frac{d^3k}{(2\pi)^3} T \sum_{i\omega_n} \ln [\beta^2((\omega_n - i\mu)^2 + E(k)^2)] \]
\[ = \frac{V}{2} \int \frac{d^3k}{(2\pi)^3} T \sum_{i\omega_n} [\ln \{\beta^2(\omega_n^2 + (E(k) - \mu)^2)\} + \ln \{\beta^2(\omega_n^2 + (E(k) + \mu)^2)\}], \]  
where \( \beta = T^{-1} \) and we use the relation
\[ [\omega_n - i\mu)^2 + E(k)^2] [\omega_n + i\mu)^2 + E(k)^2] = [\omega_n^2 + (E(k) - \mu)^2] [\omega_n^2 + (E(k) + \mu)^2]. \]  
By using
\[ \sum_{i\omega_n} [\beta^2(\omega_n^2 + E(k)^2)] = E(k)/T + 2\ln(1 + e^{-E(k)/T}) + \text{const.}, \]  
we obtain the free energy density as
\[ F = -\frac{T}{V} \ln Z \]
\[ = -T \int \frac{d^3k}{(2\pi)^3} \left[ E(k)/T + \ln(1 + e^{-(E(k)-\mu)/T}) + \ln(1 + e^{-(E(k)+\mu)/T}) + \text{const.} \right]. \]  
Subtracting \( T = 0 \) contribution, \( \delta F(T) := F(T) - F(0) \) is given by
\[ \delta F(T, \mu) = -T \int \frac{d^3k}{(2\pi)^3} \left[ \ln(1 + e^{-(E(k)-\mu)/T}) + \ln(1 + e^{-(E(k)+\mu)/T}) \right] \]
\[ = \frac{\Gamma(5/4)}{8\pi \Gamma(3/4)} \frac{T^{5/2}}{t_r t_{z}^{1/2}} \left[ \text{Li}_{5}(\mu/T) + \text{Li}_{5}(-\mu/T) \right], \]  
where \( \text{Li}_{n}(x) \) is the polylogarithm function.

Specific heat

For \( \mu = 0 \), using \( \text{Li}_{5}(-1) = -\frac{(4 - \sqrt{2})\zeta(5/2)}{4} \) with the zeta function \( \zeta(x) \), we get the free energy \( \delta F(T, 0) \) as
\[ \delta F(T, 0) = -\frac{(4 - \sqrt{2})\Gamma(5/4)\zeta(5/2)}{16\pi \Gamma(3/4)} \frac{T^{5/2}}{t_r t_{z}^{1/2}}. \]  
The specific heat at \( \mu = 0 \) is then given by
\[ C_V = -T \frac{\partial^2 \delta F(T, 0)}{\partial T^2} \]
\[ = \frac{15(4 - \sqrt{2})\Gamma(5/4)\zeta(5/2)}{64\pi \Gamma(3/4)} \frac{T^{3/2}}{t_r t_{z}^{1/2}}. \]
The compressibility is given by
\[ \kappa = - \frac{\partial^2 \delta F(T, \mu)}{\partial \mu^2} \]
\[ = - \frac{\Gamma(5/4)}{8\pi \Gamma(3/4)} \frac{T^{1/2}}{t_r t_z^{1/2}} \left[ \text{Li}_{1/2}(-e^{\mu/T}) + \text{Li}_{1/2}(-e^{-\mu/T}) \right]. \] (S88)

At \( \mu = 0 \), we have
\[ \kappa = - \frac{(\sqrt{2} - 1)\Gamma(5/4)\zeta(1/2)}{4\pi \Gamma(3/4)} \frac{T^{1/2}}{t_r t_z^{1/2}}, \] (S89)

where \( \text{Li}_{1/2}(-1) = (\sqrt{2} - 1)\zeta(1/2) \) is used. Note that \( \zeta(1/2) < 0 \), hence, \( \kappa > 0 \). (S90)

**Diamagnetic susceptibility**

Using the Fukuyama formula [1], the diamagnetic susceptibility is given by
\[ \chi_{D,x} = e_0^2 T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \text{Tr}[J_y G(i\omega_n, k)J_z G(i\omega_n, k)J_y G(i\omega_n, k)J_z G(i\omega_n, k)], \] (S91)

where \( J_i \equiv \frac{\partial H_0}{\partial c_i} \) is the current operator,
\[ J_x = 2t_r k_x \sigma_x + 2t_r k_y \sigma_y, \] (S92)
\[ J_y = -2t_r k_y \sigma_x + 2t_r k_x \sigma_y, \] (S93)
\[ J_z = 2t_z k_x \sigma_z. \] (S94)

Note that because of the \( C_4 \) symmetry of the Hamiltonian, \( \chi_{D,x} = \chi_{D,y} = \chi_{D,z} \). Subtracting the zero temperature contribution to obtain a finite result, we have
\[ \chi_{D,z} = e_0^2 T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \text{Tr}[J_y G(i\omega_n, k)J_z G(i\omega_n, k)J_y G(i\omega_n, k)J_z G(i\omega_n, k)] \]
\[ - e_0^2 \int d\omega d^3k \frac{1}{(2\pi)^3} \text{Tr}[J_y G(i\omega, k)J_z G(i\omega, k)J_y G(i\omega, k)J_z G(i\omega, k)], \]
\[ = e_0^2 t_r t_z^2 \int \frac{d^3k}{(2\pi)^3} \left[ -32(k_x^2 + k_y^2)k_z^2 M_2 + 128t_r^2 t_z^2 (k_x^2 + k_y^2)^3 k_z^6 M_4 \right] \]
\[ = e_0^2 t_z^{1/2} t_r^{1/2} \kappa_{\chi,z}, \] (S95)

where \( c_{\chi,z} = 0.054 \). Here, we use
\[ \int_0^{\pi/2} \cos \theta_R \sin^{1/2} \theta_R \, d\theta_R = \frac{2}{3}, \] (S96)
\[ \int_0^{\pi/2} \cos^3 \theta_R \sin^{5/2} \theta_R \, d\theta_R = \frac{8}{11}, \] (S97)

and the following Matsubara frequency summations (where the zero-temperature contribution has been subtracted)
\[ M_1(\xi/T) = T \sum_{\omega_n} \frac{1}{(\omega_n^2 + \xi^2)} - \int_{-\infty}^\infty \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \xi^2)} \]
In summary,

\[ M_2(\xi/T) = T \sum_{\omega_n} \frac{1}{(\omega_n^2 + \xi^2)^2} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \xi^2)^2} \]

\[ = \frac{1}{4\xi^3} \left[ \tanh\left( \frac{\xi}{2T} \right) - 1 \right] - \frac{1}{8\xi^2T^2} \frac{1}{\cosh^2\left( \frac{\xi}{2T} \right)} \]  (S98)

Similarly, \( \chi_{D,z} \) is given by

\[ \chi_{D,z} = e_0^2 T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \text{Tr}[J_z G(\omega_n, k) J_z G(\omega_n, k) J_z G(\omega_n, k)] \]

\[ - e_0^2 \int \frac{d\omega d^3k}{(2\pi)^4} \text{Tr}[J_z G(\omega, k) J_z G(\omega, k) J_z G(\omega, k) J_z G(\omega, k)] \]

\[ = e_0^2 \int \frac{d^3k}{(2\pi)^3} \left[-32(k_x^2 + k_y^2)^2 M_2 + 256t_r^4(k_x^2 + k_y^2)^4 k_z^2 M_4 \right], \]

\[ = \frac{e_0^2 t_r}{t_z^{1/2}} T^{1/2} \chi_{D,z} \]  (S102)

where \( c_{\chi,z} = 0.107 \). Here, we used

\[ \int_0^{\pi/2} d\theta \frac{\cos^2\theta}{\sqrt{\sin\theta}} = \frac{4\pi^{1/2} \Gamma(5/4)}{3\Gamma(3/4)}, \]  (S103)

\[ \int_0^{\pi/2} d\theta \frac{\cos^6\theta}{\sqrt{\sin\theta}} = \frac{80\pi^{1/2} \Gamma(5/4)}{77\Gamma(3/4)}. \]  (S104)

In summary,

\[ \chi_{D,\perp} = c_{\chi,\perp} e_0^2 t_r^{1/2} T^{1/2}, \quad \chi_{D,z} = c_{\chi,z} e_0^2 t_r T^{1/2}. \]  (S105)

**Optical conductivity**

The optical conductivity is given by

\[ \sigma_{ij}(\Omega, T) = e_0^2 \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{n_F(\omega) - n_F(\omega + \Omega)}{\Omega} \int \frac{d^3k}{(2\pi)^3} \text{Tr} [J_i \text{Im}[G^\omega_0(\omega, k)] J_j \text{Im}[G_0(\omega + \Omega, k)]] , \]  (S106)
where \( n_F(x) = \frac{1}{1 + e^{x}} \). Because of the \( C_4 \) symmetry of the Hamiltonian, \( \sigma_{xx} = \sigma_{yy} \). Hence, we only need to consider \( \sigma_{xx} \) and \( \sigma_{zz} \).

\[
\sigma_{xx}(\Omega, T) = e_0^2 \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{n_F(\omega) - n_F(\omega + \Omega)}{\Omega} \int \frac{d^3 k}{(2\pi)^3} \text{Tr} \left[ J_x \text{Im} [G_0^{\text{ret}}(\omega, k)] J_x \text{Im} [G_0^{\text{ret}}(\omega + \Omega, k)] \right]
\]
\[
= \frac{e_0^2 T^{3/2}}{5t_z^{1/2}} \delta(\Omega) \int_0^\infty dR \frac{R^{3/2}}{\cosh^2 \left( \frac{R}{2} \right)} + \frac{3}{20\sqrt{2}\pi t_z^{1/2}} [\Omega]^{1/2} \tanh \left( \frac{\Omega}{4T} \right),
\]
\[
\sigma_{zz}(\Omega, T) = e_0^2 \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{n_F(\omega) - n_F(\omega + \Omega)}{\Omega} \int \frac{d^3 k}{(2\pi)^3} \text{Tr} \left[ J_z \text{Im} [G_0^{\text{ret}}(\omega, k)] J_z \text{Im} [G_0^{\text{ret}}(\omega + \Omega, k)] \right]
\]
\[
= \frac{e_0^2 T^{3/2}}{t_r t_z^{1/2}} \frac{3\Gamma(-1/4)^2}{160\sqrt{2}\pi^{5/2}} \delta(\Omega) \int_0^\infty dR \frac{R^{3/2}}{\cosh^2 \left( \frac{R}{2} \right)} + \frac{\sqrt{\pi} \Gamma(3/4)}{40\sqrt{2}\Gamma(5/4) t_r t_z^{1/2}} |\Omega|^{1/2} \tanh \left( \frac{\Omega}{4T} \right).
\]

(S107)

(S108)

Here, we used the following identities,

\[
\int_0^{\infty} dR \frac{R^{3/2}}{\cosh^2 \left( \frac{R}{2} \right)} = 4.06856,
\]
\[
\int_0^{\pi/2} \frac{\cos^{7/2} \theta}{\sqrt{\cos \theta \sin \theta}} d\theta = \frac{8}{5},
\]
\[
\int_0^{\pi/2} \sin^{5/2} \theta d\theta = \frac{3\Gamma(-1/4)^2}{40\sqrt{2}},
\]
\[
\int_0^{1} dR \frac{R^4 - 2}{(1 - R^4)^{3/4}} = -\frac{6}{5},
\]
\[
\int_0^{1} dR \frac{R^5}{\sqrt{1 - R^4}} = \frac{\sqrt{\pi} \Gamma(3/4)}{10\Gamma(5/4)},
\]
\[
\lim_{\Omega \to 0} \frac{n_F(A) - n_F(A \pm \Omega)}{\Omega} = \pm \frac{1}{4T} \frac{1}{\cosh^2 (A/2T)}.
\]

(S109)

(S110)

(S111)

(S112)

(S113)

(S114)

For \( T = 0 \),

\[
\sigma_{xx}(\Omega) = \frac{3}{20\sqrt{2}\pi} \frac{e_0^2}{t_z^{1/2}} |\Omega|^{1/2}, \quad \sigma_{zz}(\Omega) = \frac{\sqrt{\pi} \Gamma(3/4)}{40\sqrt{2}\Gamma(5/4) t_r t_z^{-1/2}} |\Omega|^{1/2}.
\]

(S115)

**SANITY CHECK OF THE POWER-LAW CORRECTION**

In the main text, we included all the renormalization effects in the system parameters. Here, for a sanity check, equivalently we will include all the renormalization effects in the coordinates and obtain the associated anomalous dimensions.

Recall that the RG equations for \( t_r \) and \( t_z \) are given by

\[
\frac{1}{t_r} \frac{dt_r}{dt} = z - 2z_\perp + \alpha F_\perp(\gamma),
\]
\[
\frac{1}{t_z} \frac{dt_z}{dt} = z - 2 + \alpha F_z(\gamma).
\]

(S116)

(S117)

Imposing \( t_r \) and \( t_z \) as constants, then we have

\[
z = 2 - \alpha F_z(\gamma),
\]
\[
z_\perp = 1 + \frac{\alpha}{2} [F_\perp(\gamma) - F_z(\gamma)].
\]

(S118)

(S119)

At the fixed point \( (\alpha, \gamma) = (\alpha^*, \gamma^*) \),

\[
z^* = 2 - \alpha^* F_z(\gamma^*),
\]

(S120)
\[ z^*_\perp = 1 + \frac{\alpha^*}{2} [F_{\perp}(\gamma^*) - F_{\perp}(\gamma^*)]. \]  

(S121)

Now, let us find the power-law corrections of the physical observables by using scaling hypothesis with the renormalized quantity \( O_R \) and the scaling dimension \( \delta \) for an observable \( O \). For the density of states, we have

\[ \rho = b^{-(2z^*_\perp + 1)} \rho_R, \]  
(S122)

whereas for the free energy,

\[ \mathcal{F} = b^{-(z+2z^*_\perp + 1)} \mathcal{F}_R. \]  
(S123)

From Eq. (S123), we obtain the specific heat and the compressibility, respectively, as

\[ C_V = -T \frac{\partial^2 \mathcal{F}}{\partial T^2} = b^{-(2z^*_\perp + 1)} C_{V,R}, \]  
(S124)

\[ \kappa = -\frac{\partial^2 \mathcal{F}}{\partial \mu^2} = b^{-(2z^*_\perp + 1)} \kappa_R. \]  
(S125)

To determine the scaling relation of the optical conductivities and the diamagnetic susceptibilities, we use the minimal coupling \(-i\partial_t \rightarrow -i\partial_t + e_0 A_i(\tau, \mathbf{x})\), where \( A_i(\tau, \mathbf{x}) \) is a gauge-field. Since \( e_0 \) receives no renormalization at all, the scaling dimension of \( A_i \) is the same as that of \( \partial_t \). The optical conductivities and the diamagnetic susceptibilities can be obtained from the current-current response function \( K_{ij}(i\omega_n, \mathbf{q}) = \frac{1}{(2\pi)^d} \int d\mathbf{\Omega} \langle \mathcal{J}_i(i\omega_n, \mathbf{q}), \mathcal{J}_j(i\mathbf{\Omega}, \mathbf{p}) \rangle \) with \( \mathcal{J}_i(i\omega_n, \mathbf{q}) = e_0 \int d\mathbf{k} \psi^\dagger(i\omega_n, \mathbf{k} + \mathbf{q}) \frac{\partial H_{\text{int}}(\mathbf{k})}{\partial \mathbf{k}_i} \psi(i\omega_n, \mathbf{k}) \) by the following relations [1, 2]:

\[ \sigma_{ij}(\omega) = \frac{1}{2\omega} \text{Im} K^{\text{ret}}_{ij}(\omega, \mathbf{q} = 0), \]  
(S126)

\[ \chi_{D,ij}(\omega) = -\lim_{q \rightarrow 0} \frac{\epsilon_{ijk}}{2\mathbf{q}_j} K_{jk}(0, \mathbf{q}). \]  
(S127)

Here, the repeated indices are not summed. Becaus (\( \mathcal{J}_i(i\omega_n, \mathbf{q}), \mathcal{J}_j(i\mathbf{\Omega}, \mathbf{p}) \)) is obtained by differentiating the logarithm of the partition function \( Z[A] \) with respect to \( A_i(i\omega_n, \mathbf{q}) \) and \( A_j(i\omega_n, \mathbf{p}) \), the scaling dimension of \( K_{ij}(i\omega_n, \mathbf{q}) \), namely \( [K_{ij}] \), is given by

\[ [K_{ij}] = [\frac{\delta}{\delta A_i(i\omega_n, \mathbf{q})}] + [\frac{\delta}{\delta A_j(i\omega_n, \mathbf{q})}] - [d\mathbf{r}] - [d^4\mathbf{x}] \]  
(S128)

\[ = -[\partial_t] - [\partial_j] + (z + 2z^*_\perp + d - 2). \]

Equipped with this scaling relation of \( K_{ij} \), we can derive the following relations:

\[ \sigma_{\perp\perp} = b^{d-2} \sigma_{\perp\perp,R}, \]  
(S129)

\[ \sigma_{zz} = b^{2z^*_\perp + d + 2} \sigma_{zz,R}, \]  
(S130)

\[ \chi_{D,\perp} = b^{-z^*_\perp + d + 2} \chi_{D,\perp,R}, \]  
(S131)

\[ \chi_{D,z} = b^{-z^*_\perp + 2z^*_\perp + d + 2} \chi_{D,z,R}. \]  
(S132)

The RG equation of the temperature and frequency is

\[ \frac{d\mathcal{O}}{d\ln b} = z\mathcal{O}, \]  
(S133)

where \( \mathcal{O} = T, \Omega \). Let \( z = z^* \) and \( z^*_\perp = z^*_\perp \). Solving this, we obtain \( \mathcal{O}(b) = b^{z^*} \mathcal{O} \). Let \( b^* \) be the cutoff value, so that \( \mathcal{O}(b^*) = (b^*)^{z^*} \mathcal{O} = \Lambda \), then \( b^* = (\Lambda/\mathcal{O})^{1/z^*} \mathcal{O}^{-1/z^*} \). Using this, we can obtain the power-law corrections of the observables in terms of the temperature and frequency.

For the density of states, we have

\[ \rho \propto |\Omega|^{(2z^*_\perp + 1 - z^*)/z^*} \propto |\Omega|^{1/2 + c_\rho + \frac{1}{2} c_\rho}. \]  
(S134)

For the specific heat and compressibility,

\[ C_V \propto T^{(2z^*_\perp + 1)/z^*} \approx T^{3/2 + c_v + \frac{1}{2} c_v}. \]  
(S135)
\[ \kappa \propto T^{(2z^*_{\perp} + 1 - z^*)/z^*} \approx T^{1/2 + c_r + \frac{1}{2} c_z}. \]  

(S136)

For the diamagnetic susceptibility,

\[ \chi_{D,\perp} \propto T^{(z^* - 1)/z^*} \approx T^{1/2 - \frac{1}{2} c_z}, \]  

(S137)

\[ \chi_{D,z} \propto T^{(z^* - 2 z^*_{\perp} + 1)/z^*} \approx T^{1/2 - c_r + \frac{1}{2} c_z}. \]  

(S138)

For the optical conductivity,

\[ \sigma_{xx} \propto \Omega^{1/z^*} \approx \Omega^{1/2 + c_z}, \]  

(S139)

\[ \sigma_{zz} \propto \Omega^{(2z^*_{\perp} - 1)/z^*} \approx \Omega^{1/2 + c_r - \frac{1}{2} c_z}. \]  

(S140)

Here, \( c_r \approx 0.402/N_f \) and \( c_z \approx 0.044/N_f \). Thus, we obtain the same results as in the main text. If the symmetry-allowed parabolic term is included, we have \( c_r \approx 0.145/N_f \) and \( c_z \approx 0.050/N_f \).

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