Phragmén–Lindelöf theorems for a weakly elliptic equation with a nonlinear dynamical boundary condition

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Abstract
We establish two Phragmén–Lindelöf theorems for a fully nonlinear elliptic equation. We consider a dynamical boundary condition that includes both spatial variable and time derivative terms. As a spatial term, we consider a non-linear Neumann-type operator with a strict monotonicity in the normal direction of the boundary on the spatial derivative term. Our first result is for an elliptic equation on an epigraph in \( \mathbb{R}^n \). Because we assume a good structural condition, which includes wide classes of elliptic equations as well as uniformly elliptic equations, we can benefit from the strong maximum principle. The second result is for an equation that is strictly elliptic in one direction. Because the strong maximum principle need not necessarily hold for such equations, we adopt the strategy often used to prove the weak maximum principle. Considering such equations on a slab we can approximate the viscosity subsolutions by functions that strictly satisfy the viscosity inequality, and then obtain a contradiction.

Keywords Phragmén–Lindelöf theorem · Maximum principle · Dynamical boundary condition · Viscosity solutions

Mathematics Subject Classification 35B50 · 35D40 · 35J66

1 Introduction

1.1 Background and motivation
The maximum principle is one of the most fundamental properties of the solutions of elliptic partial differential equations. On a bounded domain, the maximum principle is valid for a wide class of elliptic equations. On an unbounded domain, however, it does not always hold, even for uniformly elliptic equations. For example, \( u(x) = x_1x_n \) is harmonic in the half plane,
$\mathbb{R}^+_+ := \{ x \in \mathbb{R}^n \mid x_n > 0 \}$, and $u = 0$ on the boundary. However, since $u(1, 0, \ldots, 1) = 1$, we see the maximum principle does not hold. Under a growth rate assumption on solutions, a similar estimate is valid. We say such assertion as Phragmén–Lindelöf theorem.

In this study, we consider an elliptic equation of the form

$$F(x, t, Du, D^2u) = 0 \quad \text{in } \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$ is a domain, $T > 0$ is a given constant, and $u = u(x, t)$ is an unknown function. $Du$ and $D^2u$ represent the gradient and the Hessian matrix with respect to $x$ of the solution $u$, respectively.

Throughout this paper, we assume that $F : \overline{\Omega} \times (0, T) \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is continuous and degenerate elliptic. That is,

$$F(x, t, p, X) \leq F(x, t, p, Y) \text{ for } X \geq Y \text{ in } \mathbb{S}^n, \quad (x, t, p) \in \overline{\Omega} \times (0, T) \times \mathbb{R}^n.$$

Here, $\mathbb{S}^n$ is the space of $n \times n$ real-valued symmetric matrices. As we do not assume $F$ to be linear, we depend on the viscosity solution theory (cf. [10]). In addition, we always assume that the viscosity subsolution $u$ does not blow up as $t \to T$, that is, we assume that

$$\lim \sup_{t \to T} \max_{x \in K} u(x, t) < +\infty$$

for arbitrary compact subset $K \subset \overline{\Omega}$.

As a boundary condition, we consider the following (4):

$$\partial_t u + B(x, Du) = 0 \quad \text{on } \partial \Omega \times (0, T),$$

where $B$ is a given continuous function on $\partial \Omega \times \mathbb{R}^n$, and $\partial_t u$ is the partial derivative with respect to $t$. The boundary condition that includes a time derivative term is called the “dynamical boundary condition”. This condition describes various diffusion phenomena such as thermal contact with a perfect conductor and solute diffusion from a stirred liquid or vapor. For studies related to the dynamical boundary condition, see [1, 12–17, 20, 22].

We are interested in the case where $B$ is a nonlinear Neumann-type operator. Precisely, we assume that

1. $B(x, rp) = rB(x, p)$ for all $(x, p) \in \partial \Omega \times \mathbb{R}^n$ and all $r \geq 0$.
2. There exists $L_b > 0$ such that

$$|B(x, p) - B(x, q)| \leq L_b|p - q|$$

for all $(x, p) \in \partial \Omega \times \mathbb{R}^n$.
3. There exists $\theta > 0$ such that

$$B(x, p + \tau \nu(x)) - B(x, p) \geq \tau \theta$$

for all $(x, p) \in \partial \Omega \times \mathbb{R}^n$ and $\tau > 0$, where $\nu(x)$ is the unit outer normal vector on $\partial \Omega$.

Typical examples include the Neumann condition $(B(x, p) = \langle \nu(x), p \rangle)$ and the oblique condition $(B(x, p) = \langle l(x), p \rangle)$, where $l$ is a given vector field with $\langle l, \nu \rangle > 0$. Here, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$. We can also deal with the controlled reflection for

$$B(x, p) = \sup_{a \in A} \{ \langle l_a(x), p \rangle - g_a(x) \},$$

with functions $l_a$ and $g_a$. For related results, see [3, 4, 20, 23, 24].
As the initial condition, we assume that
\[
\limsup_{t \to +0} \sup_{|x|=R, x \in \partial \Omega} u(x, t) \leq 0 \quad \text{for all } R > 0. \tag{5}
\]

Elliptic problems with dynamical boundary conditions have been studied extensively. For instance, see [15–17, 20, 26, 36, 37].

We denote the initial-boundary value problem defined by equations (1), (4), and (5) as DBP. Our aim is to establish Phragmén–Lindelöf theorems for viscosity subsolutions of DBP.

The Phragmén–Lindelöf theorem has been extensively studied. Classically, for subharmonic functions in \( \Omega \subset \mathbb{R}^n_+ \), the theorem states that if \( u \leq 0 \) on \( \partial \Omega \) and \( u = o(|x|) \) as \( |x| \to \infty \), then \( u \leq 0 \) in \( \Omega \). Gilbarg proved it for \( n = 2 \) in [19] and Hopf did so for \( n \geq 3 \) in [21]. Later, Oddson (\( n = 2 \)) and Miller (\( n \geq 3 \)) dealt with the general linear equations ([27, 30]). For classical arguments, we refer the reader to [19, 21, 27–30, 32, 34].

For nonlinear equations with Dirichlet boundary conditions, Capuzzo Dolcetta and Vitolo proved the Phragmén–Lindelöf theorem via the Alexandrov–Bakelman–Pucci estimate and the weak boundary Harnack inequality for viscosity solutions (see [6]). Later, the maximum principle and other estimates including the Phragmén–Lindelöf property were studied. For related issues, we refer the reader to [6–9, 38]. Furthermore, the Phragmén–Lindelöf theorems have been shown in the framework of \( L^p \)-viscosity solutions. Koike and Nakagawa proved this in [25] for elliptic equations, and Tateyama did so in [35] for parabolic equations.

### 1.2 Uniformly elliptic equations

Ishige and Nakagawa proved the Phragmén–Lindelöf theorem for fully nonlinear elliptic problems with a linear Neumann-type dynamical boundary condition in [22], which assumed (6) as a structure condition:
\[
\mathcal{P}_{\lambda, \Lambda}^-(X) - L(x)|p'| \leq F(x, t, p, X), \tag{6}
\]
for \((x, t, p, X) \in \overline{\Omega} \times (0, T) \times \mathbb{R}^n \times S^n\). Here, \( L \) is a positive continuous function on \( \overline{\Omega} \), \( p = (p', p_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \), and \( \mathcal{P}_{\lambda, \Lambda}^-(X) \) is Pucci’s minimal operator:
\[
\mathcal{P}_{\lambda, \Lambda}^-(X) := \min\{-\text{Tr} (AX) \mid \lambda I \leq A \leq \Lambda I, \; A \in S^n\},
\]
where \( 0 < \lambda \leq \Lambda \).

A simple calculation confirms that (6) is satisfied if \( F \) is uniformly elliptic and Lipschitz continuous with respect to the \( p' \in \mathbb{R}^{n-1} \) variable. Here, \( F \) is uniformly elliptic if there exists \( 0 < \lambda \leq \Lambda \) such that
\[
\mathcal{P}_{\lambda, \Lambda}^-(X - Y) \leq F(x, t, p, X) - F(x, t, p, Y) \leq \mathcal{P}_{\lambda, \Lambda}^+(X - Y)
\]
for all \((x, t, p) \in \overline{\Omega} \times (0, T) \times \mathbb{R}^n \) and \( X, Y \in S^n \). Here, \( \mathcal{P}_{\lambda, \Lambda}^+(X) := -\mathcal{P}_{\lambda, \Lambda}^-(-X) \) is Pucci’s maximal operator. We note that (6) does not necessarily imply the uniform ellipticity (see [6]).

Here, we present a result from [22], which we prove in Sect. 3 as one of the main results of this paper.

**Proposition 1** [22, Theorem 5] Let \( u \in C(\mathbb{R}^n_+ \times (0, T)) \) be a viscosity subsolution of
\[
\begin{align*}
F(x, t, Du, D^2u) &= 0 \quad \text{in } \mathbb{R}^n_+ \times (0, T), \\
\partial_t u + \partial_
u u &= 0 \quad \text{on } \partial \mathbb{R}^n_+ \times (0, T),
\end{align*}
\]
where \( F \in C(\mathbb{R}_+^n \times (0, T) \times \mathbb{R}^n \times S^n) \) satisfies (2) and (6). If \( u \) satisfies (5) and

\[
\liminf_{R \to \infty} \sup_{|x|=R, t \in (0, T)} \frac{u(x, t)}{1 + x_n} \leq 0,
\]

then \( u \leq 0 \) in \( \mathbb{R}_+^n \times (0, T) \).

Next, we state one form of the main results in this paper, which is proven in Sect. 3. We extend the result of Proposition 1 to nonlinear boundary problems.

**Theorem 2** Assume that \( \Omega \) is an epigraph in \( \mathbb{R}_+^n \) such that

\[
\Omega = \{ x \in \mathbb{R}^n \mid x_n > \rho(x') \}
\]

for a non-negative function \( \rho \in C^2(\mathbb{R}^n-1) \), \( F \) satisfies (2) and (6), and \( B \) satisfies (B1), (B2), and (B3). Let \( u \in C(\overline{\Omega} \times (0, T)) \) be a viscosity subsolution of DBP and satisfy (3) and (7), then, \( u \leq 0 \) in \( \overline{\Omega} \times (0, T) \).

We prove this assertion by a similar argument in [22]. After transforming the solution appropriately, it is attributed to an argument on a bounded domain, and the maximum principle is used to derive the contradiction.

Although we do not require \( F \) to be uniformly elliptic in Theorem 2, we still obtain the benefit of ellipticity by the structure condition (6) because Pucci’s minimal operator is uniformly elliptic. In Sect. 2, we review the strong maximum principle and Hopf’s boundary point lemma, as they play a significant role in our proof.

### 1.3 Directionally elliptic equations

Next, we consider the case in which the strong maximum principle may not hold. Thus, we cannot expect significant ellipticity in the structure. In this case, we deal with a slab such that \( \Omega = \{ x \in \mathbb{R}^n \mid 0 < x_n < 1 \} \).

Instead of the structure condition (6), we assume the following properties.

1. For \( (x, t, 0, O) = 0 \) for \( (x, t) \in \overline{\Omega} \times (0, T) \).
2. There exists a positive continuous and bounded function \( L(x, t) \) such that
   \[
   |F(x, t, p, X) - F(x, t, q, X)| \leq L(x, t)|p - q|
   \]
   for all \( (x, t, X) \in \overline{\Omega} \times (0, T) \times S^n \) and \( p, q \in \mathbb{R}^n \).
3. There exists a positive function \( \gamma(x, t) \) such that \( \liminf_{|x| \to \infty} \inf_{t \in (0, T)} \gamma(x, t) > 0 \) and
   \[
   F(x, t, p, X + \tau e_n \otimes e_n) - F(x, t, p, X) \leq -\gamma(x, t)\tau
   \]
   for all \( (x, t, p, X) \in \overline{\Omega} \times (0, T) \times \mathbb{R}^n \times S^n \) and \( \tau > 0 \). Here, we mean \( \xi \otimes \eta := (\xi_i \eta_j)_{ij} \) for \( \xi, \eta \in \mathbb{R}^n \) and \( e_n := (0, \cdots, 0, 1) \in \mathbb{R}^n \).
4. For any sequence \( \{(x_\varepsilon, t_\varepsilon)\} \subset \overline{\Omega} \times (0, T) \) such that \( |x_\varepsilon| \to \infty \) as \( \varepsilon \to +0 \),
   \[
   \liminf_{\varepsilon \to +0} F(x_\varepsilon, t_\varepsilon, 0, 0, \frac{e_1}{|x_\varepsilon|}) = 0.
   \]

Here, \( I' = I - e_n \otimes e_n \).
(F3) refers to the directional ellipticity, which is a strictly monotone property in the $e_n$ direction. With a small calculation, we can see that all uniformly elliptic functions are directional elliptic in any direction. However, directional ellipticity does not imply uniform ellipticity. Furthermore, it does not ensure the strong maximum principle. (F4) applies a growth condition in the unbounded direction. For Dirichlet problems, the Phragmén–Lindelöf property is already known (see [7–9]). These assumptions are equivalent to those made in [8].

Example 1 [8, Example 1.1, 1.3] Consider the linear case

$$F(p, X) = -\text{Tr}(AX) + \langle b, p \rangle,$$

where $A = (a_{ij})_{i, j} \in \mathbb{R}^n$ is positive-semidefinite, $b \in \mathbb{R}^n$. In this case, $F$ satisfies (F3) if $a_{nn} > 0$ and the other properties are clearly satisfied. When $F$ depends on $(x, t)$, (F3) is fulfilled if $a_{nn} = a_{nn}(x, t) > 0$ for all $(x, t) \in \Omega \times (0, T)$. (F4) is equivalent to

$$\limsup_{\varepsilon \to +0} \varepsilon |x_\varepsilon| \text{Tr}(I'A(x_\varepsilon, t_\varepsilon)) = 0.$$

Furthermore, if $F$ is Isaacs type such as

$$F(x, t, p, X) = \sup_{\alpha, \beta} \inf \{ -\text{Tr}(A^{\alpha\beta}(x, t)X) + \langle b^{\alpha\beta}(x, t), p \rangle \},$$

$F$ satisfies (F1)–(F4) if, for all $\alpha$ and $\beta$,

$$A^{\alpha\beta} \geq 0, \quad a^{\alpha\beta}_{nn} > 0, \quad a^{\alpha\beta}_{ij}(x, t) = O(|x|) (|x| \to \infty), \quad \text{and} \quad b^{\alpha\beta} \text{ is bounded}.$$

Under these hypotheses, we establish the second main result of this paper. We provide proof in Sect. 4.

Theorem 3 Assume $F$ and $B$ satisfy hypotheses (2), (F1)–(F4), and (B1)–(B3). Let $u \in C(\Omega \times (0, T))$ be a viscosity subsolution of DBP. If $u$ satisfies (3) and (8):

$$\liminf_{R \to \infty} \sup_{|x|=R, t \in (0, T)} \frac{u(x, t)}{1 + |x|} \leq 0,$$

then $u \leq 0$ in $\Omega \times (0, T)$.

Our proof is a modified version of the argument in [8, 9], which discuss the Dirichlet problem.

The rest of this paper is organized as follows. In Sect. 2, we introduce the definition of viscosity solutions of DBP and review some standard facts. In Sects. 3 and 4, we prove the Phragmén–Lindelöf theorems for DBP.

2 Preliminaries and basic results

First, we define the viscosity subsolution for the initial-boundary value problem DBP.

Definition 1 Let $u \in C(\Omega \times (0, T))$, and $\varphi \in C^2(\Omega \times (0, T))$.

We say that $u$ is a viscosity subsolution of DBP if $u$ satisfies (5) and

1. $F(\hat{x}, \hat{t}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})) \leq 0$

holds whenever $u - \varphi$ attains its local maximum value at $(\hat{x}, \hat{t}) \in \Omega \times (0, T)$.
2.\[\begin{align*}
\min\{F(\hat{x}, \hat{t}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})), \partial_t\varphi(\hat{x}, \hat{t}) + B(\hat{x}, D\varphi(\hat{x}, \hat{t}))\} \leq 0
\end{align*}\]
holds whenever \(u - \varphi\) attains its local maximum value at \((\hat{x}, \hat{t}) \in \partial\Omega \times (0, T)\).

We require the strong maximum principle for our proof of Theorem 2. For related studies, refer to [2, 5, 11, 18, 31].

**Proposition 4** [11, Theorem 2.1] Let \(\Omega \subset \mathbb{R}^n\) be a domain and \(u \in USC(\overline{\Omega} \times [0, T])\) be a viscosity subsolution of

\[G(x, t, u, \partial_tu, Du, D^2_{xx}u) = 0 \text{ in } \Omega \times (0, T),\]

where \(G : \overline{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n+1} \rightarrow \mathbb{R}\) is a locally bounded and lower semicontinuous function. In addition, assume that \(G\) is proper in the sense of [10] and satisfies the following properties:

(G1) There exists \(\rho_0 > 0\) such that for some \(\gamma_0 \geq 0\),

\[G(x, t, 0, s, p, I - \gamma(s, p) \otimes (s, p)) > 0\]

for all \(0 < |(s, p)| < \rho_0\) with \(p \in \mathbb{R}^n \setminus \{0\}, \gamma > \gamma_0,\) and \((x, t) \in \Omega \times (0, T)\).

(G2) For all \(\eta > 0\), there exists a function \(\varphi : (0, 1) \rightarrow (0, +\infty), \epsilon_\eta > 0\) and \(\gamma_0 \geq 0\) such that for all \(\epsilon \in (0, \epsilon_\eta)\) and \(\gamma > \gamma_0\) the following condition holds uniformly:

\[G(x, t, \epsilon r, \epsilon s, \epsilon p, \epsilon(I - \gamma(p, s) \otimes (p, s))) \geq \varphi(\epsilon)G(x, t, r, s, p, I - \gamma(p, s) \otimes (p, s))\]

for all \((x, t) \in B(\xi)\) \((\eta)\) with the given \((\xi, \eta) \in \Omega \times (0, T), r \in [-1, 0], 0 < |p| \leq \eta, |s| \leq \eta\). Here, \(B(\xi)\) is a ball in \(\mathbb{R}^{n+1}\) with its center at \((\xi, \eta)\) and with a radius of \(\eta\).

Then, the strong maximum principle is valid. Namely, if \(u\) attains a non-negative maximum at \((\xi, \eta) \in \Omega \times (0, T)\), \(u\) is constant in \(\overline{\Omega} \times [0, T]\).

Although Da Lio [11] proved this assertion with a similar argument in the classical parabolic case, \(G\) is not necessarily parabolic. Particularly, elliptic equations on \(\Omega \times (0, T)\) are also included.

In the classical argument for linear elliptic problems with the mixed boundary condition (see [32] for example), Hopf’s boundary point lemma plays an important role in the proof. Thus, we require the interior sphere condition. Namely, for all \(x \in \partial\Omega,\) there exists a ball \(B \subset \Omega\) satisfying \(x \in \partial B\). For \(x \in \partial\Omega,\) we set the radius \(R(x)\) and the center \(c(x)\).

**Proposition 5** (Hopf’s boundary point lemma) Consider the same hypotheses as in Proposition 4. In addition, assume that \(\partial\Omega\) satisfies the interior sphere condition. Assume that \(u(x_*, t_*) = M \geq 0\) for \((x_*, t_*) \in \partial\Omega \times (0, T)\), and \(u(x, t) < M\) for all \((x, t) \in B_{c(x)}(R(x_*)) \times (t_* - R(x_*), t_*)\). Then,

\[\liminf_{s \to +0} \frac{u(x_* + sw, t_* + s\tau) - u(x_*, t_*)}{s} < 0,\]

where \((w, \tau) \in \mathbb{R}^{n+1}\) satisfies \((x_* + sw, t_* + s\tau) \in B_{c(x)}(R(x_*)) \times (t_* - R(x_*), t_*)\).

We can prove this assertion using the same argument (cf. [2, 11]).
Clearly, we find that $\alpha > 0$. If $\epsilon > 0$ such that for sufficiently large $\alpha$, by the lower semicontinuity of $G$, we can find a small $r > 0$ such that

$$G(x, t, h(x, t), D_h(x, t), D^2_{xx}h(x, t)) \geq C > 0 \quad ((x, t) \in D_s)$$

where

$$D_s := \{(x, t) \in \Omega \times (0, T) \mid |(x, t) - (c(x_s), t_s)| < R_s, |(x, t) - (x_s, t_s)| < r, r < t < \tau\}.$$  

Clearly, we find that $\epsilon h$ is also strict supersolution of (14) in a neighborhood of $(x_s, t_s)$. Now, for small $\epsilon > 0$, we prove that

$$u(x, t) \leq \epsilon h(x, t) + M \quad (\forall (x, t) \in \overline{D_s})$$

by contradiction.

Suppose that $\max_{\overline{D_s}} \{u - \epsilon h - M\} = (u - \epsilon h - M)(\hat{z}) > 0$ where $\hat{z} = (\hat{x}, \hat{t}) \in \overline{D_s}$. Consider the case $|\hat{z} - (x_s, t_s)| = r$ and $|\hat{z} - (c(x_s), t_s)| < R_s$. Since $\inf_{|z - (x_s, t_s)| = r} |z - z| > 0$, we have $u - M > \exists C > 0$. Also considering (10), we obtain $u - \epsilon h - M \leq 0$ for sufficiently small $\epsilon > 0$, thus this case is unsuitable.

Therefore $\hat{z} \in D_s$ and thus we can regard $\epsilon h + M$ as a test function. So the following inequality has to hold:

$$G(\hat{z}, \epsilon h(\hat{z}), \epsilon \partial_t h(\hat{z}), \epsilon D_h(\hat{z}), \epsilon D^2_{xx}h(\hat{z})) \leq G(\hat{z}, u(\hat{z}), \epsilon \partial_t h(\hat{z}), \epsilon D_h(\hat{z}), \epsilon D^2_{xx}h(\hat{z})) \leq 0.$$  

However, this contradicts the fact that $\epsilon h$ is a strict supersolution.

Thus, we find $u(x_s + sw, t_s + st) \leq \epsilon h(x_s + sw, t_s + st) + M$. Then,

$$u(x_s + sw, t_s + st) - u(x_s, t_s) \leq \epsilon h(x_s + sw, t_s + st) - h(x_s, t_s)$$

$$\leq \epsilon \frac{h(x_s + sw, t_s + st) - h(x_s, t_s)}{s} \to \epsilon \{D_h(x_s, t_s), w\} + \epsilon t \partial_t h(x_s, t_s) (s \to +0)$$

< 0.

Then, we reach the conclusion. □

**Remark 1** If $\rho \in C^2(\mathbb{R}^{n-1})$, the epigraph satisfies the interior sphere condition.

Next, we review a lemma regarding the time at which the maximum value of the subsolution is achieved.
Lemma 6 Let \( u \in C(\overline{\Omega} \times (0, T)) \) be a viscosity subsolution of DBP and satisfy (3). Then, for \( \varepsilon > 0 \), there exists a viscosity subsolution \( u_\varepsilon \in C(\overline{\Omega} \times (0, T)) \) which satisfies \( u_\varepsilon \to u \) as \( \varepsilon \to 0 \) and \( u_\varepsilon(x, t) \to -\infty \) as \( t \to T \).

Considering \( u_\varepsilon(x, t) = u(x, t) - \frac{\varepsilon}{T-t} \), we obtain \( u_\varepsilon(x, t) \to -\infty \) as \( t \to T \) by (3). We omit detailed proofs here.

If we obtain \( u_\varepsilon \leq 0 \), we also find \( u \leq 0 \) by taking \( \varepsilon \to +0 \). So, it is sufficient to prove our results for subsolutions that do not attain their non-negative maximum value at \( t = T \).

Finally, we prepare a lemma for proof of Theorem 3. In this case, recall that \( \Omega = \{ x \in \mathbb{R}^n \mid 0 < x_n < 1 \} \). We set \( \partial_0 \Omega := \{ x \in \mathbb{R}^n \mid x_n = 0 \} \) and \( \partial_1 \Omega := \{ x \in \mathbb{R}^n \mid x_n = 1 \} \). Lemma 7 is the standard technique for proof of the weak maximum principle and comparison results. See [10] for instance.

Lemma 7 Assume that \( F \) satisfies (F2) and (F3). Let \( u \in C(\overline{\Omega} \times (0, T)) \) be a viscosity subsolution of DBP. Then, there exists \( u_\varepsilon \in C(\overline{\Omega} \times (0, T)) \) such that \( u_\varepsilon(x, t) \to u(x, t) \) as \( \varepsilon \to +0 \), \( u_\varepsilon(x, t) \leq u(x, t) \) for all \( (x, t) \in \overline{\Omega} \times (0, T) \), and it is a viscosity subsolution of (11):

\[
\begin{align*}
F(x, t, Du_\varepsilon, D^2u_\varepsilon) + \varepsilon C &= 0 \quad \text{in } \Omega \times (0, T), \\
\partial_t u_\varepsilon + B(x, Du_\varepsilon) + \varepsilon C &= 0 \quad \text{on } \partial\Omega \times (0, T), \\
\limsup_{t \to +0} \sup_{|x| = R, x \in \partial\Omega} u_\varepsilon(x, t) &\leq 0 \quad \text{for all } R > 0,
\end{align*}
\]

where \( C = C(x, t) > 0 \) is a positive continuous function such that \( \liminf_{|x| \to \infty} \inf_{t \in (0, T)} C(x, t) \).

**Proof** Define

\[
u(x, t) := u(x, t) - \varepsilon(e^\alpha - e^{\alpha x_n}) - \frac{\delta}{T-t} \tag{12}\]

for \( \varepsilon, \delta, \alpha > 0 \). Then, in \( \Omega \times (0, T) \), we obtain the following estimation in the viscosity sense:

\[
0 \geq F(x, t, Du_\varepsilon - \varepsilon \alpha e^{\alpha x_n} e_n, D^2u_\varepsilon - \varepsilon \alpha^2 e^{\alpha x_n} e_n \otimes e_n) \\
\geq F(x, t, Du_\varepsilon, D^2u_\varepsilon) + \varepsilon e^{\alpha x_n} (\alpha^2 \gamma - \alpha L) .
\]

Thus \( F(x, t, Du_\varepsilon, D^2u_\varepsilon) + C \leq 0 \) for sufficiently large \( \alpha > 0 \).

On \( \partial_0 \Omega \times (0, T) \),

\[
0 \geq \partial_t u_\varepsilon + \frac{\delta}{(T-t)^2} + B(x, Du_\varepsilon - \varepsilon \alpha v(x)) \\
\geq \partial_t u_\varepsilon + \frac{\delta}{(T-t)^2} + B(x, Du_\varepsilon) - \varepsilon \alpha L_b \\
\geq \alpha e L_b \varepsilon + \partial_t u_\varepsilon + B(x, Du_\varepsilon)
\]

for \( \delta = 2T^2 \alpha L_b \varepsilon \). We also calculate on \( \partial_1 \Omega \times (0, T) \) by a similar line. \(\square\)

### 3 Proof of Theorem 2

**Proof of Theorem 2** Let

\[
u(x, t) := \frac{u(x, t)}{e^{L_b t}(1 + x_n)} .
\]
Then, (7) implies \( \lim \inf_{R \to \infty} \sup_{|x| = R, t \in (0, T)} v(x, t) \leq 0 \). Particularly, for all \( \varepsilon > 0 \), there exists \( \{R_k\} \) such that \( R_k \to \infty \) as \( k \to \infty \) and

\[
\sup_{|x| = R_k, t \in (0, T)} v(x, t) \leq \varepsilon
\]

for large \( k \).

Set

\[
\Omega_k := \{ x \in \Omega \mid |x| < R_k \}, \quad \Gamma_k := \partial \Omega \cap \partial \Omega_k, \quad \Gamma'_k := \Omega \cap \partial \Omega_k.
\]

We argue by contradiction. Suppose that there exists \( k \) such that \( \sup_{|x| \leq R_k, t \in (0, T)} v(x, t) > \varepsilon \). Let \( M_k := \sup_{|x| \leq R_k, t \in (0, T)} v(x, t) \) and choose \( (x_*, t_*) \in \overline{\Omega_k} \times (0, T) \) as \( v(x_*, t_*) = M_k \).

We note that in view of Lemma 6, we can argue as \( t_0 < T \).

By using (6), (B1), and (B2), we find that

\[
\begin{align*}
\mathcal{P}_{-1, \Lambda}^-(D^2 v) - \tilde{L}|Dv| &= 0 \quad \text{in } \Omega_k \times (0, T), \\
\partial_t v + B(x, Dv) + L_b v - L_b |v| &= 0 \quad \text{on } \Gamma_k \times (0, T), \\
v &= \varepsilon \quad \text{on } \Gamma'_k \times (0, T).
\end{align*}
\]

(13)

Here, \( \tilde{L} := \frac{2n \Lambda}{1 + x_0} + L \).

Therefore, we can apply the strong maximum principle on \( v \). Considering the initial condition, we obtain \( (x_*, t_*) \in \Gamma_k \times (0, T) \) and \( v < M_k \) in \( \Omega_k \times (0, T) \).

Define

\[
\varphi(x, t) := \delta h(x, t) + M_k,
\]

where \( h \) is the auxiliary function defined by (9), and \( \delta > 0 \) is sufficiently small. Then, by the same argument used in the proof of Proposition 5, we observe that \( \varphi \geq v \in \{(x, t) \mid |(x, t) - (c(x_0), t_*)| \leq R(x_0)\} \). Additionally, because \( \varphi > M_k \) holds in the outside of the ball, there exists a neighborhood of \( (x_*, t_*) \) such that \( \varphi \geq v \) holds. Therefore, we can apply \( \varphi \) as a test function.

We know that \( Dh(x_*, t_*) = 2\alpha e^{-\alpha R(x_*)} R(x_*) v(x_*) \) and \( \partial_t h(x_*, t_*) = 0 \). Thus, we find

\[
\mathcal{P}_{-1, \Lambda}^-(D^2 \varphi(x_*, t_*)) - \tilde{L}(x_*)|D\varphi(x_*, t_*)| > 0.
\]

Furthermore, we obtain

\[
\begin{align*}
\partial_t \varphi(x_*, t_*) + B(x_*, D\varphi(x_*, t_*)) + L_b v(x_*, t_*) - L_b |v(x_*, t_*)| \\
= \delta \partial_t h(x_*, t_*) + \delta B(x_*, Dh(x_*, t_*)) \\
= 2\delta \alpha e^{-\alpha R(x_*)} R(x_*) B(x_*, v(x_*)) \\
\geq 2\delta \alpha e^{-\alpha R(x_*)} R(x_*) \theta \\
> 0,
\end{align*}
\]

and this is a contradiction. Therefore, we obtain \( M_k \leq \varepsilon \) for all \( k \). Considering \( k \to \infty \) and \( \varepsilon \to +0 \), we reach the conclusion. \( \square \)

**Proof of Proposition 1** Apply Theorem 2 with \( \rho \equiv 0 \) and \( B(x, p) = -p_n \). Then the conclusion immediately follows. \( \square \)

**Remark 2** We can apply the same argument as above to the parabolic equations \( \partial_t u + F(x, t, Du, D^2 u) \leq 0 \) because the strong maximum principle similarly holds.
Remark 3 The growth condition (7) is essential. Consider the case in which $|D\rho| < C$ for some constant $C > 0$, $F(x, t, p, X) = -\text{Tr}(X)$, and $B(x, p) = \langle \nu(x), p \rangle$. Then, $u(x, t) = \frac{t}{\sqrt{1+|x_n|^2}} + x_n$ is a subsolution of DBP and $u > 0$ in $\Omega \times (0, T)$.

In consideration of this observation, some alternative ideas would be as follows. Instead of (7), we consider the following growth condition,

$$\liminf_{R \to \infty} \sup_{|x| = R, t \in (0, T)} u(x, t) \leq 0.$$ 

If $g \in C^2(\overline{\Omega} \times [0, T))$ is non-negative and

$$\begin{cases}
P_{\lambda, \Lambda}(D^2 g) - L|D' g| \geq 0 & \text{in } \Omega \times (0, T), \\
g_t + B(x, Dg) \geq 0 & \text{on } \partial \Omega \times (0, T), \\
g(x, 0) \geq 0 & \text{in } \overline{\Omega},
\end{cases} \tag{14}$$

we can apply the same argument used above. However, if $g$ is just continuous, and not expected to be differentiable, our argument does not work. If $g$ satisfies (14) in the viscosity sense, a method using the comparison theorem can be employed. This approach is treated in [33].

Example 2 Now we consider the minimal surface equation, in which

$$F(p, X) = -\text{Tr} \left\{ \left( I - \frac{p \otimes p}{1+|p|^2} \right) X \right\}.$$ \tag{15}

As argued in [18, Remark 2.7], $F$ is uniformly elliptic if there exists $M > 0$ such that $|p| \leq M$ with the ellipticity constants $\lambda = \frac{1}{1+M^2}$ and $\Lambda = 1$. In addition, $p \mapsto F(p, X)$ is Lipschitz continuous if $|p|$ and $\|X\|$ are bounded. Namely, the benefits of a good structure can only be localized. In addition, without some modification of the argument or additional assumptions, it seems unlikely that equation corresponding to (13) can be expected to have a good structure. In order to apply Theorem 2, we need to make some modifications.

4 Proof of Theorem 3

Proof of Theorem 3 It suffices that we prove Theorem 3 for subsolutions of (11).

(8) implies that for all $\varepsilon > 0$, there exists a subsequence $\{R_\varepsilon\}$ such that $R_\varepsilon \to \infty$ as $\varepsilon \to +0$, and

$$\sup_{|x| = R_\varepsilon, t \in (0, T)} \frac{u(x, t)}{\psi(x)} \leq \varepsilon, \tag{16}$$

where $\psi(x', x_n) := \sqrt{1 + |x'|^2}$.

Let $u^\varepsilon := u - \varepsilon \psi$ and suppose $M_\varepsilon := \sup_{\overline{\Omega} \times (0, T)} u^\varepsilon > 0$. Because we know $u^\varepsilon(x, t) \leq 0$ for all $|x| \geq R_\varepsilon$ and $t \in (0, T)$ form (16), we obtain

$$M_\varepsilon = \sup_{|x| \leq R_\varepsilon, t \in (0, T)} u^\varepsilon(x, t).$$

Considering the initial condition and (12), we can find $(x_\varepsilon, t_\varepsilon) \in \overline{\Omega_\varepsilon} \times (0, T)$ such that $M_\varepsilon = u^\varepsilon(x_\varepsilon, t_\varepsilon)$. Here, $\Omega_\varepsilon := \{x \in \Omega \mid |x| < R_\varepsilon\}$. 

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First, consider the case \( x_\varepsilon \in \partial \Omega \). Because \( \psi \) is smooth, we can regard \( \psi \) as a test function. We thus know that
\[
0 \geq C + B(x_\varepsilon, \varepsilon D\psi(x_\varepsilon)) \geq C - \varepsilon L_b.
\]
Thus, this is a contradiction for small \( \varepsilon > 0 \).

Next, consider the case \( x_\varepsilon \in \Omega \). We know that
\[
F(x_\varepsilon, t_\varepsilon, \varepsilon D\psi(x_\varepsilon), \varepsilon D^2\psi(x_\varepsilon)) + C \leq 0.
\]
By directional calculation, we have
\[
D\psi(x) = \frac{1}{\psi(x)}x', \quad D^2\psi(x) = \frac{1}{\psi(x)}I' - \frac{1}{\psi(x)^3}x' \otimes x'.
\]
Here, we identify \( x' \in \mathbb{R}^{n-1} \) with \( (x', 0) \in \mathbb{R}^n \).

Take \( \varepsilon \to +0 \). Because \( \{t_\varepsilon\} \) is bounded, there exists a subsequence and \( \hat{\varepsilon} \) such that \( t_\varepsilon \to \hat{\varepsilon} \).

The initial condition implies \( \hat{\varepsilon} > 0 \). By examining (12), we observe \( u(x, t) \to -\infty \) as \( t \to T \). Thus, \( \hat{\varepsilon} < T \) because \( M_\varepsilon > 0 \).

If \( \{x_\varepsilon\} \) is bounded, there exists a subsequence such that \( (x_\varepsilon, t_\varepsilon) \to (\hat{x}, \hat{\varepsilon}) \) for \( (\hat{x}, \hat{\varepsilon}) \in \overline{\Omega} \times (0, T) \).

Then,
\[
0 \geq C + \liminf_{\varepsilon \to +0} F(x_\varepsilon, t_\varepsilon, \varepsilon D\psi(x_\varepsilon), \varepsilon D^2\psi(x_\varepsilon)) > F(\hat{x}, \hat{\varepsilon}, 0, O) = 0.
\]
If \( \{x_\varepsilon\} \) is unbounded, there exists \( x_\varepsilon \in \Omega \) with \( |x'_\varepsilon| \to \infty \). Thus,
\[
0 \geq C + F \left( x_\varepsilon, t_\varepsilon, 0, \frac{\varepsilon}{\psi(x_\varepsilon)}I' \right) - \varepsilon L(x_\varepsilon, t_\varepsilon),
\]
and taking \( \varepsilon \to +0 \), we find a contradiction by (F4).

**Remark 4** The aforementioned argument is still valid for domains with a curved boundary. Precisely, \( \Omega = \{x \in \mathbb{R}^n \mid \rho_0(x') < x_n < \rho_1(x')\} \), where \( \rho_0, \rho_1 \in C^1(\mathbb{R}^{n-1}) \) are bounded functions, which satisfy \( \rho_0 < \rho_1 \) and \( |D'\rho_0|, |D'\rho_1| \) are bounded in \( \mathbb{R}^{n-1} \).

**Remark 5** Instead of (B2), we can prove this assertion when we assume the following condition (B4):

(B4) For all \( x_\varepsilon \in \partial_0 \Omega \) with \( |x'_\varepsilon| \to \infty \) as \( \varepsilon \to +0 \),
\[
\liminf_{\varepsilon \to +0} \varepsilon B \left( x_\varepsilon, \frac{x'_\varepsilon}{\psi(x_\varepsilon)} \right) > 0.
\]

First, return to Lemma 7. By the same definition of \( u_\varepsilon \), the argument in \( \Omega \times (0, T) \) and \( \partial_1 \Omega \times (0, T) \) are still valid. Therefore, in the proof of Theorem 3, the same argument still works if \( x_\varepsilon \in \Omega \) or \( x_\varepsilon \in \partial_1 \Omega \).

On \( \partial_0 \Omega \times (0, T) \), we obtain
\[
0 \geq \partial_1 u_\varepsilon + \frac{\delta}{T^2} + B(x, Du_\varepsilon - \varepsilon \alpha v).
\]
Consider the case in which \( x_\varepsilon \in \partial_0 \Omega \times (0, T) \). If the maximum value \( M_\varepsilon \) is achieved at some interior point, we already know that it is a contradiction. Thus, the maximum value is achieved only on the boundary. Applying the same argument for Hopf’s boundary point.
lemma, we can find that there exists a smooth function $\varphi(x, t)$ such that $u^\varepsilon - \varphi$ attains its local maximum value at $(x^\varepsilon, t^\varepsilon)$, and $D\varphi(x^\varepsilon, t^\varepsilon) = -An = Av(x^\varepsilon)$ for some $A > 0$.

Therefore, we have

$$0 \geq \frac{\delta}{T^2} + A'\theta + \varepsilon B\left(x^\varepsilon, \frac{x^\varepsilon}{\varphi(x^\varepsilon)}\right),$$

where $A' < A$ is a positive constant. Taking $\varepsilon \to +0$, we have a contradiction.

**Remark 6** The minimal surface equation (15) is not directional elliptic with fixed direction because the direction depends on $p$. Thus we can not apply Theorem 3.

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