BLOCH AND LANDAU TYPE THEOREMS FOR PLURIHARMONIC MAPPINGS

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Abstract. In this paper, we establish two new versions of Landau-type theorems for pluriharmonic mappings with a bounded distortion. Then using these results, we derive three Bloch-type theorems of pluriharmonic mappings, which improve the corresponding results of Chen and Gauthier.

1. Preliminaries and some basic questions

Let \( \mathbb{C}^n \) denote the \( n \)-dimensional complex Euclidean space so that \( \mathbb{C} := \mathbb{C}^1 \), the complex plane. The conjugate \( \overline{z} \) of \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) is defined by \( \overline{z} = (\overline{z_1}, \ldots, \overline{z_n}) \). For \( z \) and \( w = (w_1, \ldots, w_n) \in \mathbb{C}^n \), we define

\[ \langle z, w \rangle := z_1 w_1 + \cdots + z_n w_n \quad \text{and} \quad |z| := \sqrt{\langle z, z \rangle} = \left( \sum_{k=1}^{n} |z_k|^2 \right)^{1/2}. \]

For \( a \in \mathbb{C}^n \), let \( B^n(a, r) = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z - a| < r \} \) be the ball in \( \mathbb{C}^n \) of radius \( r \) with center \( a \). In the case of \( n = 1 \), we use the standard notation \( D(a, r) := B^1(a, r) \) so that \( D := D(0, 1) \), the open unit disk in \( \mathbb{C} \). We also let \( B^n \) denote the unit ball \( B^n(0, 1) \). Evidently, \( D = B^1 \) (see [17]).

1.1. Planar harmonic mappings. For a continuously differentiable complex-valued mapping \( f(z) = u(z) + iv(z), z = x + iy \), we use the common notation for its formal derivatives:

\[ f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y). \]

We say that \( f \) is a harmonic mapping in a simply connected domain \( D \) if \( f \) is twice continuously differentiable and satisfies the Laplace equation \( \Delta f = 4f_{\overline{z}z} = 0 \) in \( D \).

Let \( \mathcal{H}(\mathbb{D}) \) denote the set of all harmonic mappings in \( \mathbb{D} \). It is well-known that such mappings have the representation \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic functions in \( \mathbb{D} \). It is convenient to introduce the following notations:

\[
\begin{align*}
\mathcal{H}_0(\mathbb{D}) &= \{ f = h + \overline{g} \in \mathcal{H}(\mathbb{D}) : g(0) = 0 \}, \\
\mathcal{A}(\mathbb{D}) &= \{ f = h + \overline{g} \in \mathcal{H}(\mathbb{D}) : g(z) \equiv 0, h'(0) \neq 0 \}, \\
\mathcal{A}_0(\mathbb{D}) &= \{ f \in \mathcal{A}(\mathbb{D}) : f(0) = 0 \}
\end{align*}
\]
Lewy’s theorem [15] from 1936 states that a harmonic mapping \( f = h + \overline{g} \) is locally univalent on \( D \) if and only if the determinant \( |J_f| \) of its Jacobian matrix \( J_f \) does not vanish on \( D \), where \( |J_f| = |f_z|^2 - |f_{\overline{z}}|^2 = |h|^2 - |g'|^2 \). Such a result does not hold in higher dimensions (for details see [18]).

For a continuously differentiable mapping \( f \) on \( D \), the maximum and minimum length distortions of the mapping \( f \) are defined respectively by

\[
\Lambda_f(z) = \max_{0 \leq t \leq 2\pi} |f_z(z) + e^{-2it}f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|, \quad \text{and}
\]

\[
\lambda_f(z) = \min_{0 \leq t \leq 2\pi} |f_z(z) + e^{-2it}f_{\overline{z}}(z)| = |f_z(z)| - |f_{\overline{z}}(z)|.
\]

Methods of Harmonic mappings have been used to study and solve fluid flow problems (see [1, 8]). For example, in 2012, Aleman and Constantin [1] established a connection between harmonic mappings and ideal fluid flows. In fact, they have developed ingenious technique to solve the incompressible two dimensional Euler equations in terms of univalent harmonic mappings. More precisely, the problem of finding all solutions which in Lagrangian variables describing the particle paths of the flow present a labelling by harmonic mappings is reduced to solve an explicit nonlinear differential system in \( \mathbb{C}^n \) (cf. [8]).

Our primary interest in this paper is to establish several new versions of Landau-type theorems and two improved Bloch-type theorems of pluriharmonic mappings.

1.2. Landau-Bloch type theorems of harmonic mappings. The Bloch theorem (1925) asserts the existence of a positive number \( b \) such that for each \( f \in A(D) \) there is a disk of radius \( b|f'(0)| \) which is the univalent image under \( f \) of some subdomain of \( D \). Such a disk is called “schlicht disk” for \( f \). The supremum of all such numbers \( b \) is called the Bloch constant \( B \).

Bloch’s theorem implies the existence of another positive number \( \ell \) such that, for each \( f \in A(D) \), \( f(D) \) contains a disk of radius \( \ell|f'(0)| \). The largest possible value of \( \ell \), denoted by \( L \), is known as the Landau constant. Clearly \( L \geq B \). The exact values of \( B \) and \( L \) are not known although the lower and upper bounds are available in the literature.

The classical Landau theorem asserts that if \( f \in A_0(D) \) such that \( f'(0) = 1 \) and \( |f(z)| < M \) for \( z \in D \), then \( f \) is univalent \( D_{r_0} \), and \( f(D_{r_0}) \) contains a disk \( D_{\sigma_0} \), where

\[
r_0 = \frac{1}{M + \sqrt{M^2 - 1}} \quad \text{and} \quad \sigma_0 = Mr_0^2.
\]

This result is sharp, with the extremal function \( f_0(z) = Mz \left( 1 - \frac{M^2}{M^2 - 1} \right) \).

**Definition 1.** A function \( f \in H(D) \) is said to belong to \( S_H(r; R) \) if it is univalent in \( D_r \) and the range \( f(D_r) \) contains a univalent disk \( D_R \).
In 2000, under a suitable restriction, Chen et al. \[3\] established two versions of Landau-type theorems for bounded harmonic mapping on the unit disk which we now recall them using our notation.

**Theorem A.** ([3, Theorem 3]) Let \( f \in H_0(\mathbb{D}) \) such that \( f\zeta(0) = 0, f_\zeta(0) = 1, \) and \(|f(z)| < M \) for \( z \in \mathbb{D} \). Then \( f \in S_H(r_1; r_1/2) \) with
\[
r_1 = \frac{\pi^2}{16mM} \approx \frac{1}{11.105M},
\]
where \( m \approx 6.85 \) is the minimum of the function \((3 - r^2)/(r(1 - r^2))\) for \( 0 < r < 1 \).

**Theorem B.** ([3, Theorem 4]) Let \( f \in H_0(\mathbb{D}) \) such that \( \lambda_f(0) = 1 \), and \( \Lambda_f(z) \leq \Lambda \) for \( z \in \mathbb{D} \). Then \( f \in S_H(r_2; r_2/2) \), where
\[
r_2 = \frac{\pi}{4(1 + \Lambda)}.
\]

Theorems A and B are not sharp. Better estimates were given in \[2, 5, 6, 9, 10, 11, 12, 13, 14\]. In particular, the sharp version of Theorem A for \( M = 1 \) reads as follows:

**Theorem C.** ([12, Theorems 2.4 and 2.5, Remark 2.6]) Let \( f \in H_0(\mathbb{D}) \) such that either \( J_f(0) = 1 \) or \( \lambda_f(0) = 1 \), and \(|f(z)| < 1 \) for \( z \in \mathbb{D} \). Then \( f \in S_H(1; 1) \). The result is sharp.

Recently, Liu \[14\] proposed the following conjecture which may be regarded as the sharp form of Theorem A for the case \( M > 1 \).

**Conjecture 1.** ([14, Conjecture 3.4]) Let \( f \in H_0(\mathbb{D}) \) such that \( f(0) = 0, \lambda_f(0) = 1 \) and \(|f(z)| < M \) for \( z \in \mathbb{D} \) and for some \( M > 1 \). Then \( f \in S_H(r_0, \sigma_0) \). The two radii \( r_0 \) and \( \sigma_0 \) are sharp, with the extremal mappings \( e^{i\alpha} f_0(z) \) or \( e^{i\alpha} f_0(z) \), where \( \alpha \in \mathbb{R} \) and \( f_0(z) = Mz \left( \frac{1 - Mz}{M - z} \right) \).

**1.3. The Landau-Bloch type theorems of pluriharmonic mappings.** A continuous complex-valued function \( \phi \) defined on a domain \( \Omega \subset \mathbb{C}^n \) is called a pluriharmonic mapping if, for each fixed \( \zeta' \in \Omega \) and \( \theta \in \partial B^n \), the function \( \phi(\zeta' + \theta \zeta) \) is harmonic in the complex variable \( \zeta \), for \(|\zeta| \) smaller than the distance from \( \zeta' \) to \( \partial B^n \). A mapping \( f \) of \( \Omega \) into \( \mathbb{C}^n \) is called a pluriharmonic mapping if every component of \( f \) is pluriharmonic.

A mapping \( f \) of \( B^n \) into \( \mathbb{C}^n \) is pluriharmonic if and only if \( f \) has a representation \( f = g + \overline{h} \), where \( g \) and \( h \) are holomorphic mappings (see [4]).

For a continuously differentiable mapping \( f : B^n \rightarrow \mathbb{C}^m, w = f(z) = (f_1(z), \ldots, f_m(z)), z = (z_1, \ldots, z_n), \) we denote by \( f_z \) and \( f_\overline{z} \) for the matrices \((\partial f_j/\partial z_k)_{m \times n}\) and \((\partial f_j/\partial \overline{z}_k)_{m \times n}\), respectively.

Denote the maximum length distortion \( \Lambda_f \) and minimum length distortion \( \lambda_f \) by
\[
\Lambda_f(z) = \max_{\theta \in \partial B^n} |f_z(z)\theta + f_\overline{z}(z)\overline{\theta}| \quad \text{and} \quad \lambda_f(z) = \min_{\theta \in \partial B^n} |f_z(z)\theta + f_\overline{z}(z)\overline{\theta}|,
\]
respectively, where \( \theta \) is regarded as a column vector.
For a continuously differentiable mapping \( w = f(z) = (f_1(z), \ldots, f_n(z)) \), \( z = (z_1, \ldots, z_n) \), of a domain \( \Omega \subset \mathbb{C}^n \) into \( \mathbb{C}^n \), let \( z_k = x_k + iy_k \) and \( f_j = u_j + iv_j \). Note also that \( f \) can be regarded as a mapping of a domain in \( \mathbb{R}^{2n} \) into \( \mathbb{R}^{2n} \). We denote the real Jacobian matrix of this mapping by \( J_f \):

\[
J_f = \begin{pmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1} & \cdots & \frac{\partial u_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \cdots & \frac{\partial u_n}{\partial y_n}
\end{pmatrix}.
\]

Let \( B^n \) (resp. \( \mathbb{B}^{2n} \)) denote the unit ball in \( \mathbb{C}^n \) (resp. \( \mathbb{R}^{2n} \)). With this notation, the maximum length distortion \( \Lambda_f \) and minimum length distortion \( \lambda_f \) have another equivalent representation:

\[
\Lambda_f = \max_{\theta \in \partial B^n} |f_2 \theta + f_3 \overline{\theta}| = \max_{\theta \in \partial B^n} |J_f \theta| \quad \text{and} \quad \lambda_f = \min_{\theta \in \partial B^n} |f_2 \theta + f_3 \overline{\theta}| = \min_{\theta \in \partial B^{2n}} |J_f \theta|.
\]

For a complex or real \( n \times n \) matrix \( A \), the operator norm of \( A \) is defined by

\[
|A| = \sup_{z \neq 0} \frac{|Az|}{|z|} = \max_{\theta \in \partial B^{2n}} |A \theta|.
\]

Thus the maximum distortion \( \Lambda_f \) is the \( L^2 \) operator norm of the Jacobi matrix \( J_f \). From now onwards, we identify a point in \( \mathbb{C}^n \) or \( \mathbb{R}^{2n} \) (real space of dimension \( 2n \)) with a complex or real column vector.

A pluriharmonic mapping \( f \) of \( B^n \) into \( \mathbb{C}^n \) is said to be a \( K \)-mapping if

\[
|J_f(z)| \leq K|\det J_f(z)|^{1/(2n)} \quad \text{for} \quad z \in B^n.
\]

In 2011, Chen and Gauthier [2] obtained Landau theorems and Bloch theorems for pluriharmonic mappings \( f : B^n \rightarrow \mathbb{C}^n \). Recently, Xu and Liu obtained a new version of Landau theorem and a Bloch theorem for pluriharmonic mappings in [19]. Before we recall the work of Chen and Gauthier, it is convenient to use the following notations.

**Definition 2.** Define

\[
\mathcal{PH}(B^n) = \{ f : B^n \rightarrow \mathbb{C}^n : f \text{ is pluriharmonic mapping} \},
\]

\[
\mathcal{PH}^\alpha(B^n) = \{ f \in \mathcal{PH}(B^n) : \det J_f(0) = \alpha \},
\]

\[
\mathcal{PH}_K(B^n) = \{ f : B^n \rightarrow \mathbb{C}^n : f \text{ is pluriharmonic } K \text{-mapping} \},
\]

\[
\mathcal{PH}^\alpha_K(B^n) = \{ f \in \mathcal{PH}_K(B^n) : \det J_f(0) = \alpha \}, \quad \text{and}
\]

\[
\mathcal{PH}_{loc, K}(B^n) = \{ f \in \mathcal{PH}_K(B^n) : f \text{ is locally univalent} \}.
\]

A mapping \( f \in \mathcal{PH}(B^n) \) is said to belong to \( \mathcal{S}_{PH}(r; R) \) if \( f \) is univalent on the ball \( B^n(0, r) \) and the range \( f(B^n(0, r)) \) covers the ball \( B^n(0, R) \).

**Theorem D.** ([2 Theorem 5]) Let \( f \in \mathcal{PH}^\alpha(B^n) \) such that \( \alpha > 0 \), \( \alpha \geq 0 \), \( (4M/\pi)^{2n} \leq \alpha > 0 \). Then \( f \in \mathcal{S}_{PH}(\rho_0; R_0) \), where

\[
\rho_0 = \frac{\alpha 2^{2n+1}}{4m(4M)^{2n}} \quad \text{and} \quad R_0 = \frac{\alpha^2 \pi^{4n}}{8m(4M)^{4n-1}}.
\]
where \( m \approx 4.2 \) is the minimum of the function \( \frac{2-r^2}{r(1-r^2)} \) on the interval \((0,1)\).

**Theorem E.** ([2, Theorem 6]) Let \( f \in \mathcal{PH}_K^1(B^n) \) with \( n > 1 \). Then \( f(B^n) \) contains a disk of radius \( b_f \), with
\[
b_f \geq R'_n = \frac{k_n\pi}{8m} \left( \frac{k_n\pi}{8K\log(1/(1-k_n))} \right)^{4n-1} = \frac{\pi}{8me(2n-1)} \left( \frac{\pi}{4} \right)^{4n-1} \frac{1}{K^{4n-1}} (1 + o(1)),
\]
where \( 0 < k_n < 1 \) is the unique number such
\[
4n\log \frac{1}{1-k_n} = (4n - 1) \cdot \frac{k_n}{1-k_n}.
\]

**Theorem F.** ([2, Theorem 7]) Let \( f \in \mathcal{PH}_{loc, K}^\alpha(B^n) \) such that \( n > 1 \), \((4M/\pi)^{2n} \geq \alpha > 0\), \( f(0) = 0 \) and \( |f(z)| < M \) for \( z \in B^n \). Then there exists a domain \( \Omega \subset B^n(0, \rho'_0) \) such that \( 0 \in \Omega \) and \( f \) maps \( \Omega \) onto a ball \( B^n(0, R'_0) \) injectively, where
\[
\rho'_0 = \frac{\pi^2(\alpha)^{1/(2n)}}{16mMK^{2n-1}}, \quad R'_0 = \frac{\pi^2\alpha^{1/n}}{32mMK^{4n-2}},
\]
and \( m \) is the same number as in Theorem D

**Theorem G.** ([2, Theorem 8]) Let \( f \in \mathcal{PH}_{loc, K}^1(B^n) \) and \( n > 1 \). Then \( f(B^n) \) contains a disk of radius \( b_f \), with
\[
b_f \geq \frac{1}{134K^{4n-1}}.
\]

Our main aim of this article is to consider the following natural questions.

**Problem 1.** Can we establish some new versions of Landau-type theorems for pluriharmonic mapping, which are different with Theorems D and F?

**Problem 2.** Can we improve Theorems E and G?

The paper is organized as follows. In Section 2, we present the main results of this paper. In Theorems 1-2, we present an affirmative answer to Problem 1. In Section 3, we state and prove four related theorems which improve two results of Chen and Gauthier [2]. In particular, Theorems 3 and 4 provide an affirmative answer to Problem 2.

2. **Main Results**

We first state two Landau-type theorems of pluriharmonic mapping with bounded distortion, which are analogues versions of Theorem D.
Theorem 1. Let $f \in \mathcal{PH}^1(B^n)$ such that $f(0) = 0$ and $\Lambda_f(z) \leq \Lambda$ for $z \in B^n$. Then $\Lambda \geq 1$ and $f \in \mathcal{S}_{PH}(\rho_1; R_1)$, where
\[ \rho_1 = \frac{\pi}{4\Lambda f^2(0)(\Lambda f(0) + \Lambda)} \quad \text{and} \quad R_1 = \frac{\pi}{8\Lambda f^4(0)(\Lambda f(0) + \Lambda)}. \]
If, in addition, $\Lambda f(0) = 1$, then $f \in \mathcal{S}_{PH}(\rho'_1; R'_1)$, where
\[ \rho'_1 = \frac{\pi}{4(1 + \Lambda)} \quad \text{and} \quad R'_1 = \frac{\pi}{8(1 + \Lambda)}. \]

Next, we state a Landau-type theorem of locally univalent pluriharmonic $K$-mapping with bounded distortion, which is the analogues version of Theorem F.

Theorem 2. Let $f \in \mathcal{PH}_{loc,K}^1(B^n)$, $n > 1$, such that $f(0) = 0$ and $\Lambda_f(z) \leq \Lambda$ for $z \in B^n$. Then $\Lambda \geq 1/K^{2n-1}$ and that there exists a domain $\Omega \subset B^n(0, \rho_2)$ such that $0 \in \Omega$ and $f$ maps $\Omega$ onto a ball $B^n(0, R_2)$ injectively, where
\[ \rho_2 = \frac{\pi}{4K^{2n-1}(K + \Lambda)} \quad \text{and} \quad R_2 = \frac{\pi}{8K^{4n-2}(K + \Lambda)}. \]

Now, we state a Bloch-type theorem of pluriharmonic $K$-quasiregular mapping.

Theorem 3. Let $f \in \mathcal{PH}_K(B^n)$ and $n > 1$. Then $f(B^n)$ contains a disk of radius $b_f$, with
\[ b_f \geq R_3 = \frac{3 - 2\sqrt{2}}{8} \left( \frac{\pi}{K^{4n-1}} \right). \]

Remark 1. Note that $R_3$ in Theorem 3 has a simple expression, and
\[ \frac{3 - 2\sqrt{2}}{8} > 0.0214466 \]
which is independent of $n$. Moreover, from Theorem E, we have
\[ R_n' = \frac{\pi}{8me(2n - 1)} \left( \frac{\pi}{4} \right)^{4n-1} \frac{1}{K^{4n-1}}(1 + o(1)) \approx \frac{1}{33.6(2n - 1)e} \left( \frac{\pi}{4} \right)^{4n-1} \frac{\pi}{K^{4n-1}}(1 + o(1)), \]
the number $\frac{1}{33.6(2n - 1)e} \left( \frac{\pi}{4} \right)^{4n-1}$ is a decreasing function of $n > 1$. In particular, when $n = 2$, we have
\[ \frac{1}{33.6(2n - 1)e} \left( \frac{\pi}{4} \right)^{4n-1} \approx 0.0006727816777, \]
which implies that $R_3 > 31.8 \cdot R_n'$ for all $n > 1$ which shows that the constant in Theorem 3 has a more substantial improvement when compared to the estimate of Chen and Gauthier [2], namely, Theorem E.

Corollary 1. Let $f \in \mathcal{PH}_{K}^1(B^n)$, $n > 1$ and such that $\Lambda_f(0) = 1$. Then $f(B^n)$ contains a disk of radius $b_f$, with
\[ b_f \geq R_3' = \frac{(2 - \sqrt{2})\pi}{8(1 + (\sqrt{2} + 1)K)} \geq \frac{3 - 2\sqrt{2}}{8 \cdot \frac{\pi}{K}} \cdot \frac{\pi}{K} > 0.0214466 \cdot \frac{\pi}{K}. \]
Finally, we state a Bloch-type theorem of locally univalent pluriharmonic mappings, which also has bigger improvement compared to the estimate of Theorem G due to Chen and Gauthier [2].

**Theorem 4.** Let \( f \) be a locally univalent pluriharmonic \( K \)-mapping of the unit ball \( B^n \) into \( \mathbb{C}^n, \ n > 1 \), such that \( \det J_f(0) = 1 \). Then \( f(B^n) \) contains a disk of radius \( b_f \), with

\[
b_f \geq \frac{3 - 2\sqrt{2}}{8} - \frac{\pi}{K^{4n-1}} > \frac{1}{14.8K^{4n-1}} > 9 \cdot \frac{1}{134K^{4n-1}}.
\]

**3. Proofs of the main results**

First we recall the following well-known lemma.

**Lemma H.** (cf. Chen et al. [2, Theorem 4]) Let \( f \) be a pluriharmonic mapping of \( B^n \) into \( B^m \). Then

\[
\Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{1 - |z|^2} \text{ for } z \in B^n.
\]

If \( f(0) = 0 \), then

\[
|f(z)| \leq \frac{4}{\pi} \arctan |z| \leq \frac{4}{\pi} |z| \text{ for } z \in B^n.
\]

**3.1. Proof of Theorem 1**. By assumption \( f(0) = 0 \), \( \Lambda_f(z) \leq \Lambda \) for \( z \in B^n \) and \( \det J_f(0) = 1 \). Thus, we have

\[
\Lambda \geq \Lambda_f(0) \geq \lambda_f(0) \geq \frac{|\det J_f(0)|}{\Lambda_f^{2n-1}(0)} = \frac{1}{\Lambda_f^{2n-1}(0)},
\]

which implies that \( \Lambda \geq \Lambda_f(0) \geq 1 \).

Fix two distinct points \( z', z'' \in B^n(0, \rho_1) \), let \( z'' - z' = |z'' - z'|\theta \) and define the pluriharmonic mapping \( \phi_\theta \) by

\[
\phi_\theta(z) = (f_z(z) - f_z(0))\theta + (f_{\overline{z}}(z) - f_{\overline{z}}(0))\overline{\theta}.
\]

Then, it follows from the definition of \( \Lambda_f(z) \) that

\[
|\phi_\theta(z)| \leq \Lambda_f(0) + \Lambda_f(z) \leq \Lambda_f(0) + \Lambda \text{ for } z \in B^n.
\]

Note that \( \phi_\theta(0) = 0 \). Using (3.2), we obtain

\[
|\phi_\theta(z)| \leq \frac{4(\Lambda_f(0) + \Lambda)|z|}{\pi} \text{ for } z \in B^n.
\]
Consequently,

\[ |f(z'') - f(z')| = \left| \int_{[z',z'']} f_z(z) \, dz + f_{\overline{z}}(z) \, d\overline{z} \right| \]

\[ \geq \left| \int_{[z',z'']} f_{z}(0) \, dz + f_{\overline{z}}(0) \, d\overline{z} \right| \]

\[ - \left| \int_{[z',z'']} (f_z(z) - f_z(0)) \, dz + (f_{\overline{z}}(z) - f_{\overline{z}}(0)) \, d\overline{z} \right| , \]

\[ \geq |z' - z''| \left| \phi_f(0) - \int_{[z',z'']} |\phi_z(z)| \, ds \right| \]

\[ = \frac{4(\Lambda_f(0) + \Lambda)}{\pi} |z' - z''| \left[ \frac{\pi}{4\Lambda_f^{2n-1}(0)\Lambda_f(0) + \Lambda} - \rho_1 \right] = 0, \]

which implies that \( f(z') \neq f(z'') \) and thus, \( f \) is univalent in \( B^n(0, \rho_1) \).

Now let \( z' \in \partial B^n(0, \rho_1) \). As \( f(0) = 0 \), as above, we have

\[ |f(z')| \geq \left| \int_{[0,z']} f_z(0) \, dz + f_{\overline{z}}(0) \, d\overline{z} \right| - \left| \int_{[0,z']} (f_z(z) - f_z(0)) \, dz + (f_{\overline{z}}(z) - f_{\overline{z}}(0)) \, d\overline{z} \right| \]

\[ \geq \lambda_f(0)\rho_1 - \int_0^{\rho_1} \frac{4(\Lambda_f(0) + \Lambda)\rho}{\pi} \, dr \]

\[ \geq \frac{\rho_1}{\Lambda_f^{2n-1}(0)} - \frac{2(\Lambda_f(0) + \Lambda)\rho_1^2}{\pi} = \frac{\pi}{8\Lambda_f^{4n-2}(0)(\Lambda_f(0) + \Lambda)} = R_1, \]

which shows that \( f(B^n(0, \rho_1)) \) covers the ball \( B^n(0, R_1) \), and the proof is complete. \( \square \)

3.2. Proof of Theorem 2. Let \( f \in \mathcal{PH}_{loc, K}(B^n) \) with \( n > 1 \), i.e. \( f \) is a locally univalent pluriharmonic \( K \)-mapping of the unit ball \( B^n \) into \( \mathbb{C}^n \) such that \( \det J_f(0) = 1 \). As \( \Lambda_f(z) \leq \Lambda \) for \( z \in B^n \) and

\[ \lambda_f(0) \geq \frac{|\det J_f(0)|}{K^{2n-1}} = \frac{1}{K^{2n-1}}, \]

it follows that

\[ \Lambda_f(0) = |J_f(0)| \leq K|\det J_f(0)|^{1/(2n)} = K \quad \text{and} \quad \Lambda \geq \Lambda(0) \geq \lambda_f(0) \geq \frac{1}{K^{2n-1}}. \]

As with the previous theorem, let \( z', z'' \in B^n(0, \rho_2) \) be two distinct points, \( z'' - z' = |z'' - z'|\theta \) and

\[ \phi_\theta(z) = (f_z(z) - f_z(0))\theta + (f_{\overline{z}}(z) - f_{\overline{z}}(0))\overline{\theta}. \]

Then, the definition of \( \Lambda_f(z) \) gives that

\[ |\phi_\theta(z)| \leq \Lambda_f(0) + \Lambda_f(z) \leq K + \Lambda \quad \text{for} \quad z \in B^n. \]

Again, as \( \phi_\theta(0) = 0 \), (3.2) implies that

\[ |\phi_\theta(z)| \leq \frac{4(K + \Lambda)|z|}{\pi} \quad \text{for} \quad z \in B^n. \]
Then, using the analogous proof of Theorem 11 we have

\[ |f(z'') - f(z')| \geq |z' - z''| \lambda_f(0) - \int_{[z', z'']} |\phi_\theta(z)| \, ds \]

\[ > \frac{|z' - z''|}{K^{2n-1}} - \frac{4(K + \Lambda)\rho_2}{\pi}|z' - z''| = 0, \]

which shows that \( f \) is univalent in \( B^n(0, \rho_2) \).

Now let \( z' \in \partial B^n(0, \rho_2) \) and observe that \( f(0) = 0 \). As in the proof of Theorem 11 we have

\[ |f(z')| \geq \lambda_f(0)\rho_2 - \int_0^{\rho_2} \frac{4(K + \Lambda)r}{\pi} \, dr \]

\[ = \frac{\rho_2}{K^{2n-1}} - \frac{2(K + \Lambda)\rho_2^2}{\pi} = \frac{\pi}{8K^{2n-2}(K + \Lambda)} = R_2. \]

This shows that \( f(B^n(0, \rho_2)) \) covers the ball \( B^n(0, R_2) \), and the proof is complete. \( \square \)

3.3. Proof of Theorem 3. By means of Theorem 11, we may use arguments similar to those in the proof of [2, Theorem 6]. For the sake of readability, we provide the details.

Let \( f \in \mathcal{PH}_K(B^n) \) and \( n > 1 \). Without loss of generality, we assume that \( f \) is pluriharmonic on \( \overline{B^n} \). Then \( (1 - |z|)^2n|\det J_f(z)| \) is continuous and bounded on \( \overline{B^n} \). Moreover,

\[ (1 - |z|)^2n|\det J_f(z)| \bigg|_{z=0} = |\det J_f(0)| = 1 \quad \text{and} \quad \lim_{|z| \to 1 - } (1 - |z|)^2n|\det J_f(z)| = 0. \]

This implies that there exists a point \( z' \in B^n \) such that

\[ (1 - |z'|)^2n|\det J_f(z')| = 1 \quad \text{and} \quad (1 - |z|)^2n|\det J_f(z)| \leq 1 \text{ for } z' = r \leq |z| \leq 1. \]

In particular, we have

\[ |\det J_f(z)| \leq |\det J_f(z')| \text{ for } |z| = r. \]

In the following, we need to consider two cases,

**Case 1.** \( r > 0 \): Fix a point \( z_0 \) with \( 0 < |z_0| \leq r \) and consider \( \Lambda_f(z_0) = |f_z(z_0)\theta + f_{\overline{z}}(z_0)\overline{\theta}| \) with \( \theta \in \partial B^n \). Define the function \( \phi \) by

\[ \phi(\zeta) = f_z(\zeta z_0/|z_0|)\theta + f_{\overline{z}}(\zeta z_0/|z_0|)\overline{\theta} \quad \text{for} \quad \zeta \in \mathbb{D}. \]

Since \( \phi \) is harmonic, by the maximum principle, there exists a point \( \zeta' \) with \( |\zeta'| = r \), such that

\[ \Lambda_f(z_0) = |\phi(|z_0|)| \leq |f_z(\zeta' z_0/|z_0|)\theta + f_{\overline{z}}(\zeta' z_0/|z_0|)\overline{\theta}|. \]

Let \( z'' = \zeta' z_0/|z_0| \). Note that \( |z''| = r \), by the definition of \( K \)-mappings and (3.4), we have

\[ \Lambda_f(z_0) \leq |f_z(z'')\theta + f_{\overline{z}}(z'')\overline{\theta}| \leq \Lambda_f(z'') = |J_f(z'')| \leq K|\det J_f(z'')|^{1/(2n)} \leq K|\det J_f(z')|^{1/(2n)}. \]
On the other hand, by the definition of \( K \)-mappings, (3.3) and \(|\det J_f(0)| = 1\), we have

\[
\Lambda_f(0) = |J_f(0)| \leq K|\det J_f(0)|^{1/(2n)} = K \\
geq K(1 - r)|\det J_f(z')|^{1/(2n)} \\
\leq K|\det J_f(z')|^{1/(2n)}.
\]

Thus we conclude that

\[
\Lambda_f(z) \leq |\det J_f(z')|^{1/(2n)} \quad \text{for} \quad |z| \leq r.
\]

Now we define the functions \( g \) and \( F \) by

\[
g(\zeta) = z' + (2 - \sqrt{2})(1 - r)\zeta \quad \text{and} \quad F(\zeta) = \frac{1}{2 - \sqrt{2}}(f(g(\zeta)) - f(z'))
\]

for \( \zeta \in B^n \). Then

\[
F(0) = 0 \quad \text{and} \quad |\det J_F(0)| = (1 - r)^{2n}|\det J_f(z')| = 1.
\]

If \( |g(\zeta)| \leq r \), by (3.5) and (3.3), we have

\[
\Lambda_F(\zeta) = (1 - r)\Lambda_f(g(\zeta)) \leq K(1 - r)|\det J_f(z')|^{1/(2n)} = K;
\]

and if \( |g(\zeta)| \geq r \),

\[
\Lambda_F(\zeta) = (1 - r)\Lambda_f(g(\zeta)) \leq K(1 - r)|\det J_f(g(\zeta))|^{1/(2n)} \\
= K \left( \frac{1 - r}{1 - |g(\zeta)|} \right) (1 - |g(\zeta)|)|\det J_f(g(\zeta))|^{1/(2n)} \\
\leq K \left( \frac{1 - r}{1 - |g(\zeta)|} \right) \frac{K(1 - r)}{1 - r - (2 - \sqrt{2})(1 - r)|\zeta|} \\
= \frac{K}{1 - (2 - \sqrt{2})|\zeta|}.
\]

\[\text{(3.6)}\]

**Case 2.** \( r = 0 \): Consider the functions \( g \) and \( F \) defined as above. Then \( |g(\zeta)| \geq r = 0 \) and it follows from (3.6) that

\[
\Lambda_F(\zeta) \leq \frac{K}{1 - (2 - \sqrt{2})|\zeta|} \quad \text{for} \quad \zeta \in B^n.
\]

Thus we conclude that

\[
\Lambda_F(\zeta) \leq \frac{K}{1 - (2 - \sqrt{2})|\zeta|} < (\sqrt{2} + 1)K \quad \text{for} \quad \zeta \in B^n.
\]

In particular, \( \Lambda_F(0) \leq K \).

Now, applying Theorem I to the mapping \( F(z) \), we see that \( F(B^n) \) contains a schlicht ball with center 0 and radius

\[
R = \frac{\pi}{8K^{4n-1}(K + (\sqrt{2} + 1)K)} = \frac{1}{8(2 + \sqrt{2})} \cdot \frac{\pi}{K^{4n-1}}.
\]

Consequently, \( f(B^n) \) contains a schlicht ball with center \( f(z') \) and radius \( R \), that is,

\[
b_f \geq R_3 = (2 - \sqrt{2})R = \frac{3 - 2\sqrt{2}}{8} \cdot \frac{\pi}{K^{4n-1}} > 0.0214466 \cdot \frac{\pi}{K^{4n-1}}.
\]
This proves the theorem.

3.4. **Proof of Theorem 4.** By means of Theorem 2 (replacing Theorem 1), and the analogous proof of Theorem 3, we may finish the proof of Theorem 4.

**Acknowledgments.** This research of the first author was partly supported by Guangdong Natural Science Foundations (Grant No. 2021A1515010058). The work of the second author was supported by Mathematical Research Impact Centric Support (MATRICS) of the Department of Science and Technology (DST), India (MTR/2017/000367).

**References**

1. Aleman A. and Constantin A., Harmonic maps and ideal fluid flows, *Arch. Ration. Mech. Anal.*, 204, 479–513 (2012)
2. Chen H. H. and Gauthier P. M., The Landau theorem and Bloch theorem for planar harmonic and pluriharmonic mappings, *Proc. Amer. Math. Soc.* 139(2), 583–595 (2011)
3. Chen H. H., Gauthier P. M. and Hengartner W., Bloch constants for planar harmonic mappings, *Proc. Amer. Math. Soc.* 128(11), 3231–3240 (2000)
4. Chen S., Ponnusamy S. and Rasila A., Coefficient estimates, Landau’s theorem and Lipschitz-type spaces on planar harmonic mappings, *J. Aust. Math. Soc.*, 96(2), 198–215 (2014)
5. Chen S., Ponnusamy S. and Wang X., Properties of some classes of planar harmonic and planar biharmonic mappings, *Complex Anal. Oper. Theory*, 5, 901–916 (2011)
6. Chen S., Ponnusamy S. and Wang X., Coefficient estimates and Landau-Bloch’s theorem for planar harmonic mappings, *Bull. Malaysian Math. Sciences Soc.*, 34(2), 255–265 (2011)
7. Clunie J. G. and Sheil-Small T., Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A. I.* 9, 3–25 (1984).
8. Constantin O. and Martin M. J., A harmonic maps approach to fluid flows, *Math. Ann.* 369, 1–16 (2017)
9. Grigoryan A., Landau and Bloch theorems for harmonic mappings, *Complex Variable Theory Appl.* 51(1), 81–87 (2006)
10. Huang X. Z., Sharp estimate on univalent radius for planar harmonic mappings with bounded Fréchet derivative (in Chinese), *Sci. Sin. Math.*, 44(6), 685–692 (2014)
11. Liu M. S., Estimates on Bloch constants for planar harmonic mappings, *Sci. China Ser. A-Math.*, 52(1), 87–93 (2009)
12. Liu M. S., Landau’s theorem for planar harmonic mappings, *Comput. Math. Appl.*, 57(7), 1142–1146 (2009)
13. Liu M. S. and Chen H. H., The Landau-Bloch type theorems for planar harmonic mappings with bounded dilation, *J. Math. Anal. Appl.*, 468(2), 1066–1081 (2018).
14. Liu M. S., Luo L. F. and Luo X., Landau-Bloch type theorems for strongly bounded harmonic mappings, *Monatsh. Math.*, 191(1), 175–185 (2020).
15. Lewy H., On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Am. Math. Soc.*, 42, 689–692 (1936).
16. Marden A. and Rickman S., Holomorphic mappings of bounded distortion. *Proc. Amer. Math. Soc.*, 46, 226–228 (1974)
17. Rudin W., Function theory in the unit ball of $C^n$, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
18. Wood J. C., Lewy’s theorem fails in higher dimensions, *Math. Scand.*, 69(2), 166–166 (1991)
19. Xu Z. F. and Liu M. S., On pluriharmonic $ν$-Bloch-type mappings and hyperbolic-harmonic mappings, *Monatsh. Math.*, 192(4), 965–978 (2020).
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