Phantom dark energy from non-local infrared modifications of General Relativity

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We discuss the cosmological consequences of a model based on a non-local infrared modification of Einstein equations. We find that the model generates a dynamical dark energy that can account for the presently observed value of $\Omega_{DE}$, without introducing a cosmological constant. Tuning a free mass parameter $m$ to a value $m \simeq 0.67 H_0$ we reproduce the observed value $\Omega_{DE} \simeq 0.68$. This leaves us with no free parameter and we then get a pure prediction for the EOS parameter of dark energy. Writing $w_{DE}(a) = w_0 + (1 - a) w_a$, we find $w_0 \simeq -1.04$ and $w_a \simeq -0.02$, consistent with the Planck data, and on the phantom side. We also argue that non-local equations of the type that we propose must be understood as purely classical effective equations, such as those derived in semiclassical gravity for the in-in matrix elements of the metric. As such, any apparent ghost instability in such equations only affects the classical dynamics, but there is no propagating degree of freedom associated to the ghost, and no issue of ghost-induced quantum vacuum decay.

I. INTRODUCTION

In recent years there has been an intense search for modifications of General Relativity (GR) that change its behavior in the far infrared, i.e. at cosmological distances, while retaining its successes at solar system scales and below. Beside an intrinsic field-theoretical interest, such studies are motivated by the aim of explaining the behavior in the far infrared, i.e. at cosmological distances. We also argue that non-local equations of the type that we propose must be understood as purely classical effective equations, such as those derived in semiclassical gravity for the in-in matrix elements of the metric. As such, any apparent ghost instability in such equations only affects the classical dynamics, but there is no propagating degree of freedom associated to the ghost, and no issue of ghost-induced quantum vacuum decay.

In a recent paper [17] it has been proposed a different approach, that allows us to introduce a mass parameter in GR while retaining general covariance, and does not require an external reference metric. We found that this can be realized by introducing non-local terms. In particular, in [17] it was studied a model defined by

$$G_{\mu\nu} - m^2 \left( \Box_g^{-1} G_{\mu\nu} \right)^T = 8\pi G T_{\mu\nu}, \quad (1)$$

where the superscript T denotes the extraction of the transverse part, and $\Box_g$ is the d'Alembertian with respect to the metric $g_{\mu\nu}$. Its inverse $\Box_g^{-1}$ is here defined using the retarded Green's function, which ensures causality. In this paper we rather consider the model defined by

$$G_{\mu\nu} - \frac{d - 1}{2d} m^2 \left( g_{\mu\nu} \Box_g^{-1} R \right)^T = 8\pi G T_{\mu\nu}, \quad (2)$$

where $d$ is the number of spatial dimensions and the factor $(d - 1)/2d$ is a convenient normalization of the mass parameter $m^2$. We will see here that the model (2) has particularly interesting cosmological properties (some of which are not shared by the model (1), as we will discuss in [18]).

Independently of their specific form, in [17] the inclusion of non-local terms was considered as part of an attempt to construct a consistent quantum field theory of massive gravity. The purpose of this paper is twofold. First, we will argue that the proper interpretation of equations such as (1) or (2) is actually different, and that they should be understood as classical effective equations, obtained from the smoothing of some underlying more fundamental dynamics. Second, we will explore the cosmological consequences of eq. (2) and we will see that it gives a sensible and predictive model for dark energy.

The paper is organized as follows. In sect. II we discuss conceptual issues that arise in non-local classical equations of motions such as eq. (2). In sect. III we will examine its cosmological consequences, at the level of background evolution. We use the signature $\eta_{\mu\nu} = (\ldots, +, +, +)$ and units $\hbar = c = 1$. 

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II. CONCEPTUAL ISSUES

A. Absence of vDVZ discontinuity

In order to understand the physical content of the classical theory defined by eq. (2), we begin by studying the matter-matter interaction induced by such a modified Einstein equation. Linearizing over flat space and extracting the transverse part (as in app. B of [17]) we get

\[ E^\mu\nu,\rho\sigma h_{\rho\sigma} - \frac{d-1}{d} m^2 P^\mu\nu P^{\rho\sigma} h_{\rho\sigma} = -16\pi GT^{\mu\nu}, \]  

(3)

where \( E^\mu\nu,\rho\sigma \) is the Lichnerowicz operator (conventions and definitions are as in [17]) and

\[ P^\mu\nu = \eta^\mu\nu - (\partial^\mu \partial^\nu / \Box), \]

(4)

where \( \Box \) is the flat-space d’Alembertian. The corresponding effective matter-matter interaction is given by

\[ S_{\text{eff}} = \int d^{d+1}x \ h_{\mu\nu} T^{\mu\nu}, \]

(5)

where \( h_{\mu\nu} \) is the solution of eq. (3). To solve this equation we use the gauge invariance of the linearized theory to fix the gauge \( \partial^\mu h_{\mu\nu} = 0 \), where \( h_{\mu\nu} = h_{\mu\nu} - (1/2) h \eta_{\mu\nu} \). We also use \( h \equiv \eta^\mu\nu h_{\mu\nu} = -(d-1)h/2 \). Then eq. (3) becomes

\[ \Box h_{\mu\nu} + (m^2/d) P^{\mu\nu} h = -16\pi GT^{\mu\nu}. \]  

(6)

Taking the trace we get

\[ (\Box + m^2) h = -16\pi GT. \]

(7)

We write eq. (6) in momentum space, eliminate \( h \) using

\[ \tilde{h}(k) = \frac{16\pi G}{k^2 - m^2} \tilde{T}(k), \]

(8)

and solve for \( h_{\mu\nu}(k) \), obtaining

\[ \tilde{h}_{\mu\nu}(k) = \frac{16\pi G}{k^2} \left[ \tilde{T}_{\mu\nu}(k) - \frac{\eta^\mu\nu k^2}{(d-1)(k^2 - m^2)} \tilde{T}(k) \right] + \frac{m^2}{d(k^2 - m^2)} \left( \eta_{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \tilde{T}(k). \]

(9)

Plugging the result into \( S_{\text{eff}} \) and using \( k^\mu \tilde{T}_{\mu\nu}(k) = 0 \) to eliminate the term \( k^\mu k^\nu \), we get

\[ S_{\text{eff}} = 16\pi G \int \frac{d^{d+1}k}{{(2\pi)^{d+1}}} \tilde{T}_{\mu\nu}(-k) \Delta^{\mu\nu,\rho\sigma}(k) \tilde{T}_{\rho\sigma}(k), \]

(10)

with

\[ \Delta^{\mu\nu,\rho\sigma}(k) = \frac{1}{2k^2} \left( \eta^\mu\rho \eta^{\nu\sigma} + \eta^\mu\sigma \eta^{\nu\rho} - \frac{2}{d-1} \eta^\mu\nu \eta^{\rho\sigma} \right) \]

\[ + \frac{1}{d(d-1)} \frac{m^2}{k^2(-k^2 - m^2)} \eta^\mu\nu \eta^{\rho\sigma}. \]  

(11)

The term in the first line is the usual GR result, in generic \( d \), due to the exchange of a massless graviton. The term in the second line vanishes for \( m \to 0 \). Therefore this theory has no vDVZ discontinuity and, taking \( m \sim H_0 \), it smoothly reduces to GR inside the horizon. Well inside the horizon \( |k^2| > m^2 \) and the term in the second line of eq. (11) is of order \( m^2/k^4 \), compared to the massless graviton propagator which is of order \( 1/k^2 \). Thus, at least at the level of linearized theory, well inside the horizon the predictions of this non-local theory differ from the predictions of GR by a factor \( 1 + O(m^2/k^2) \). For \( m \sim H_0 \) and \( |k| = (1 \text{ a.u.})^{-1} \) (as appropriate to solar system experiments), \( m^2/k^2 \sim (1 \text{ a.u.} / H_0^{-1})^2 \sim 10^{-30} \), and the predictions of the non-local theory, linearized over flat space, are indistinguishable from that of linearized GR.

Further work (in progress) is needed to study the behavior of perturbations both around non-trivial static solutions, as well as around the cosmological solutions that will be presented below.

B. Apparent ghosts and effective classical equations

The above computation of the matter-matter interaction stresses the purely classical nature of the derivation. One might try to be more ambitious, and interpret directly eq. (2) in terms of a quantum field theory. Formally the quadratic Lagrangian corresponding to the linearized equation of motion (3) is

\[ \mathcal{L}_2 = \frac{1}{2} h_{\mu\nu} E^{\mu\nu,\rho\sigma} h_{\rho\sigma} - \frac{d-1}{2d} m^2 h_{\mu\nu} P^{\mu\nu} P^{\rho\sigma} h_{\rho\sigma}. \]

(12)

Adding the usual gauge fixing term of linearized massless gravity, \( \mathcal{L}_g = -(\partial^\mu h_{\mu\nu})(\partial^\rho h^{\rho\nu}) \), and inverting the quadratic form we get the propagator \( \tilde{D}^{\mu\nu,\rho\sigma}(k) \). The explicit computation shows that, as expected, \( \tilde{D}^{\mu\nu,\rho\sigma}(k) = -i\Delta^{\mu\nu,\rho\sigma}(k) \), with \( \Delta^{\mu\nu,\rho\sigma}(k) \) given in eq. (11) (plus, as usual, terms proportional to \( k^\mu k^\nu \), \( k^\mu k^\rho \) and \( k^\mu k^\nu k^\rho k^\sigma \), that give zero when contracted with a conserved energy-momentum tensor). Thus, the term on the first line of eq. (11) gives the usual propagator of a massless graviton, which describes only two massless states with helicities \( \pm 2 \). The second line gives an extra term in the saturated propagator \( \tilde{T}_{\mu\nu}(-k) \Delta^{\mu\nu,\rho\sigma}(k) \tilde{T}_{\rho\sigma}(k) \), equal to

\[ \frac{1}{d(d-1)} \tilde{T}(-k) \left[ \frac{i}{k^2} + \frac{i}{k^2 - m^2} \right] \tilde{T}(k). \]

(13)

This term apparently describes the exchange of a healthy massless scalar plus a ghostlike massive scalar. However, such an interpretation would not be correct. The difficulty of promoting this classical theory to a full quantum field theory becomes apparent once one examines the \( i\epsilon \) factor in these propagators. As emphasized in [19], depending on the choice of the \( i\epsilon \) prescription in the propagator, a ghost either carries negative norm (and therefore ruins the probabilistic interpretation) and
positive energy, or positive norm and negative energy. The choice of the $i\epsilon$ prescription is not specified by the Lagrangian, and is made during the quantization procedure. For a normal particle the usual scalar propagator is $-i/(k^2 + m^2 - i\epsilon)$ (with our $(-, +, +, +)$ signature). This $i\epsilon$ prescription propagates positive energies forward in time. For a ghost the sign in front of the kinetic term changes, and we have in principle two options for the $i\epsilon$ prescription, $i/(k^2 - m^2 \pm i\epsilon)$ (furthermore, now a positive $m^2$ gives rise to a tachyonic instability in the classical equations). The $+i\epsilon$ choice propagates negative energies forward in time but preserves the unitarity of the theory and the optical theorem. With the $-i\epsilon$ choice, in contrast, ghosts carry positive energy but negative norm, and the probabilistic interpretation of QFT is lost. This latter choice is therefore unacceptable. If in eq. (13) we use the prescription that preserves positive norms, the term in brackets becomes

$$-rac{i}{k^2 - i\epsilon} + \frac{i}{k^2 - m^2 + i\epsilon}$$

(14)

We see that now for $m = 0$ these two terms no longer cancel. Thus, if one uses this prescription for the propagators, one finds a rather bizarre situation in which at the classical level the limit $m \to 0$ is smooth, while at the quantum level is not. The ghost is now apparently a radiative field that destabilizes the vacuum, despite the fact that, classically, in the limit $m \to 0$ it reduces to a non-radiative degree of freedom of GR. If we instead impose continuity as $m \to 0$, we are forced to chose the $i\epsilon$ prescription for the ghost propagator that violates the probabilistic interpretation.\footnote{For this reason the argument in \cite{17} that the vacuum decay amplitude induced by the ghost is suppressed by powers of $m^2$ is unfortunately incorrect. This argument was based on the continuity of the $m \to 0$ limit, which however only holds with the $-i\epsilon$ prescription that ruins the probabilistic interpretation.}

None of these two options is meaningful. Indeed, the $m^2/\Box$ operator in eqs. (2) or (3) automatically comes with a retarded prescription. Since it is this term that gives rise to the two propagators in eq. (13), these terms inherit the retarded prescription and cannot be promoted to Feynman propagators. They describe classical radiation effects from already existing degrees of freedom, and not new propagating degrees of freedom.

A related observation is that a Lagrangian involving a $\Box^{-1}$ operator never gives a retarded $\Box^{-1}$ in the equations of motion, independently of the Green’s function used in the definition of the $\Box^{-1}$ operator that appears in the Lagrangian. This is most easily illustrated in the case of a scalar field. Consider for illustration a non-local term in an action of the form

$$\int d^4x \phi \Box^{-1} \phi,$$

(15)

where $\phi$ is some scalar field, and $\Box^{-1}$ is defined with respect to some Green’s function $G(x; x')$. Taking the variation with respect to $\phi(x)$ we get

$$\frac{\delta}{\delta \phi(x)} \int d^4x' \phi(x')(\Box^{-1} \phi)(x') = \int d^4x' d^4x'' \phi(x') G(x'; x'') \phi(x'') = \int d^4x' [G(x; x') + G(x'; x)] \phi(x').$$

(16)

We see that the variational of the action automatically symmetrizes the Green’s function \cite{17, 20, 21}. Similarly, taking the variation of eq. (12) does not really give eq. (3) as equation of motion: even if we use $h_{\mu\nu} P_{\text{ret}}^{\mu\nu} P_{\text{ret}}^{\rho\sigma} h_{\rho\sigma}$ in the action, in the equation of motions we rather get $(P_{\text{ret}}^{\mu\nu} P_{\text{ret}}^{\rho\sigma} + P_{\text{adv}}^{\mu\nu} P_{\text{adv}}^{\rho\sigma}) h_{\rho\sigma}$. We can still use this Lagrangian as part of a formal trick for obtaining the classical equations of motion from an action principle, in which after taking the variation we replace by hand $\Box^{-1}_{\text{sym}} \to \Box^{-1}_{\text{ret}}$. However, any direct connection to a fundamental quantum field theory is then lost.

This suggests that we should not attempt to promote the model defined by eq. (2) directly to a full quantum field theory, but we should rather consider it just as a classical effective equation of motion. The connection to a fundamental QFT will be less direct, and we expect that it will typically involve some form of classical or quantum averaging. For instance, such effective non-local (but causal) equations govern the dynamics of the in-in matrix elements of quantum fields, such as $\langle 0_{\text{in}} | \hat{\phi} | 0_{\text{in}} \rangle$ or $\langle 0_{\text{in}} | \hat{\phi}_{\mu\nu} | 0_{\text{in}} \rangle$, and encode quantum corrections to the classical dynamics \cite{22}. Similar non-local equations also emerge in a purely classical context when one separates the dynamics of a system into a long-wavelength and a short-wavelength part. One can then obtain an effective non-local equation for the long-wavelength modes by integrating out the short-wavelength modes, see e.g. \cite{23} for a recent example in the context of cosmological perturbation theory. One more example comes from the standard post-Newtonian/post-Minkowskian formalisms for GW production in GR \cite{24, 25}. In linearized theory the gravitational wave (GW) amplitude $h_{\mu\nu}$ is determined by $\Box h_{\mu\nu} = -16\pi GT_{\mu\nu}$, which in such a radiation problem is solved with the retarded Green’s function, $h_{\mu\nu} = -16\pi GT_{\mu\nu}$. When the non-linearities of GR are included, the GWs generated at some perturbative order become themselves sources for the GW generation at the next order. In the far-wave zone, this iteration gives rise to effective equations involving $\Box^{-1}_{\text{ret}}$.

Trying to quantize eq. (2) is like trying to quantize such effective non-local equations, and makes no sense. Simply, eq. (2) must be regarded as an effective classical equation and any issue of quantization, ghost, etc. can only be addressed in the underlying fundamental theory (see also the discussion in \cite{26}, where it is nicely shown that the very existence of a ghost depends on the UV completion of the theory). A more extended discussion of issues related to this “fake” ghost will be given in \cite{27}.
III. COSMOLOGICAL EVOLUTION

A. Evolution equations

Having better understood the conceptual status of this model, we can now move to extracting its cosmological consequences. To obtain the cosmological equations governing the background we proceed as in Sect. 8 of [17]. We introduce a scalar field $U$ from $U = \Box^{-1} R$, so

$$G_{\mu\nu} + \frac{d-1}{2d} m^2 (U g_{\mu\nu})^T = 8\pi G T_{\mu\nu}, \quad (17)$$

where

$$\Box U = R. \quad (18)$$

The introduction of the auxiliary field $U$ is technically convenient because it allows us to transform the original integro-differential equation into a set of differential equations. However, it is important to observe that at the same time this procedure introduces spurious solutions. This is due to the fact that the most general solution of eq. (18) is given by a particular solution of the inhomogeneous equation, plus the most general solution of the homogeneous equation $\Box U = 0$. However, once we define the operator $\Box^{-1}$ that enters in the original non-local equation (i.e. we specify the corresponding Green’s function), the initial conditions on $U$, and hence the homogeneous solution, are uniquely fixed.

For instance, in a FRW metric in $d$ spatial dimensions, $ds^2 = -dt^2 + a^2(t)dx^2$, the d’Alembertian operator on a scalar is given by $\Box f = -a^{-d} \partial_0 (a^d \partial_0 f)$. A possible inversion is then given by [20]

$$\Box^{-1} R(t) = -\int_{t_s}^t dt' \frac{1}{a^d(t')} \int_{t_s}^{t'} dt'' a^d(t'') R(t''), \quad (19)$$

where $t_s$ is some initial value of time (that can be taken for instance as a value of time for which an effective description in terms of the non-local equation (2) becomes appropriate). With this definition, $U \equiv -\Box^{-1} R$ is such that $U(t_s) = 0$ and $U'(t_s) = 0$, so the initial conditions on $U$ are fixed once we specify what we mean by $\Box^{-1} R$. More generally, we could define $\Box^{-1}$ such that

$$U(t) \equiv -\Box^{-1} R \equiv U_{\text{hom}}(t) + \int_{t_s}^t dt' \frac{1}{a^d(t')} \int_{t_s}^{t'} dt'' a^d(t'') R(t''), \quad (20)$$

where $U_{\text{hom}}(t)$ is a given solution of $\Box U = 0$. The point is that each definition of the $\Box^{-1}$ operator, i.e. each definition of the original non-local theory, corresponds to one and only one choice of the homogeneous solution and therefore of the initial conditions for $U$. The “free field” that satisfies the homogeneous equation $\Box U = 0$ seems a propagating degree of freedom from the point of view of the local formulation (17), (18), but in fact in the original non-local theory it is not a propagating degree of freedom. Rather, each possible choice of the homogeneous solution corresponds to one definition of $\Box^{-1}$, and therefore to one specific non-local theory. In [27] we will discuss in great detail this issue, as well as its relation with the apparent ghost degree of freedom.

To write down the equations for the cosmological evolution we now define $S_{\mu\nu} = -U g_{\mu\nu}$ and we split it into its transverse and longitudinal parts,

$$S_{\mu\nu} = S_{\mu\nu}^T + (1/2)(\nabla_\mu S_\nu + \nabla_\nu S_\mu). \quad (21)$$

To determine $S_\mu$ we apply $\nabla^\mu$ to both sides of this equation. We henceforth specialize to a flat Friedmann-Robertson-Walker (FRW) spacetime. In FRW, the three-vector $S^\mu$ vanishes because there is no preferred spatial direction, while for $S_0$ we get

$$\ddot{S}_0 + 2H \dot{S}_0 - dH^2 S_0 = \dot{U}, \quad (22)$$

In FRW, eq. (18) becomes

$$\dot{U} + dH \dot{U} = 2dH + (d + 1)H^2. \quad (23)$$

Finally, since the left-hand side of eq. (2) is transverse by construction, the energy-momentum tensor $T_{\mu\nu}$ is covariantly conserved. Thus, to study the cosmological evolution we only need the $(0, 0)$ component of eq. (2), i.e. the Friedmann equation,

$$H^2 - \frac{m^2}{d^2} (U - \dot{S}_0) = \frac{16\pi G}{d(d-1)} \rho. \quad (24)$$

Equations (23), (24) and (25) give three differential equations for the three variables $\{H(t), U(t), S_0(t)\}$. We now take $\rho$ to be the sum of the energy densities of matter and radiation, $\rho = \rho_m + \rho_r$, and we set $d = 3$. We do not include a cosmological constant term $\Lambda$. Indeed, our aim is to see whether a phenomenologically viable dark energy model can be obtained from the term proportional to $\Lambda$. Again, here we have reduced the non-local operation of taking the transverse part to a local differential equation, involving now the operator $\nabla^2 = \partial_0^2 + 2dH \partial_0 - dH^2$. Just as for the inversion of the $\Box^{-1}$ operator in $CU = -R$, the initial conditions on $S_0$ are not free parameters. Rather, the definition of the non-local theory is completed once we define $\Box^{-1}$, which in turn fixes the initial conditions on $S_0$, see the more extended discussion in [18].
to \( m^2 \) in eq. (25). We parametrize the temporal evolution using \( x = \ln \alpha(t) \) instead of \( t \), we denote \( df/dx = f' \), and we define
\[
Y = U - \dot{S}_0. \tag{26}
\]
We also use the standard notation \( h = H/H_0 \), \( \Omega_i(t) = \rho_i(t)/\rho_0(t) \) (where \( i \) labels radiation, matter and dark energy), and \( \Omega_i = \Omega_i(t_0) \), where \( t_0 \) is the present value of cosmic time. After simple manipulations, the final form of the evolution equations is as follows. The Friedmann equation reads
\[
h^2(x) = \Omega_M e^{-3x} + \Omega_R e^{-4x} + \gamma Y(x), \tag{27}
\]
where
\[
\gamma \equiv m^2/(9H_0^2). \tag{28}
\]
This shows that there is an effective DE density
\[
\rho_{DE}(t) = \rho_0 \gamma Y(x), \tag{29}
\]
where \( \rho_0 = 3H_0^2/(8\pi G) \). The evolution of \( Y(x) \) is obtained from the coupled system
\[
Y'' + (3 - \zeta)Y' - 3(1 + \zeta)Y = 3U'' - 3(1 + \zeta)U, \tag{30}
\]
\[
U'' + (3 + \zeta)U' = 6(2 + \zeta), \tag{31}
\]
\[
\zeta(x) \equiv \frac{h'}{h} = \frac{3\Omega_M e^{-3x} + 4\Omega_R e^{-4x} - \gamma Y'}{2(\Omega_M e^{-3x} + \Omega_R e^{-4x} + \gamma Y)}. \tag{32}
\]
We define \( w_{DE} \) from
\[
\dot{\rho}_{DE} + 3(1 + w_{DE})H \rho_{DE} = 0. \tag{33}
\]
Using \( \dot{\rho} = H \rho' \) we see that the equation of state (EOS) parameter of DE is
\[
w_{DE}(x) = -1 - \frac{Y'(x)}{3Y(x)}. \tag{34}
\]
The same expression for \( w_{DE}(x) \) can be obtained taking the trace of the \((i,j)\) component of eq. (2). In a \( d = 3 \) FRW space-time this gives
\[
2\dot{H} + 3H^2 - \frac{m^2}{3}(U - HS_0) = -8\pi G p, \tag{35}
\]
which can be rewritten as
\[
2\dot{H} + 3H^2 = -8\pi G(p + p_{DE}), \tag{36}
\]
with
\[
p_{DE} = -\rho_0 \gamma (U - HS_0). \tag{37}
\]
From this we get
\[
w_{DE} = \frac{\rho_{DE}}{\rho_{DE}} = -\frac{U - HS_0}{U - S_0}. \tag{38}
\]
This can be rewritten as
\[
w_{DE} = -1 - \frac{\dot{S}_0 - HS_0}{U - S_0}, \tag{39}
\]
Using eq. (26), together with eq. (23) in \( d = 3 \), we see that eq. (39) is equivalent to eq. (34). This is of course a consequence of the fact that eq. (2) can be rewritten as
\[
G_{\mu\nu} = 8\pi G (T_{\mu\nu} + T_{\mu\nu}^{DE}), \tag{40}
\]
where (in \( d = 3 \))
\[
T_{\mu\nu}^{DE} = \gamma \rho_0 (g_{\mu\nu} - \frac{1}{2} g) R, \tag{41}
\]
and by construction \( \nabla^\mu T_{\mu\nu}^{DE} = 0 \).

### B. Perturbative solutions

Before performing the numerical integration, we can get some analytic insight into the equations. In particular we can work perturbatively in \( \gamma \), assuming that the contribution of the function \( Y \) to \( \zeta(x) \) is negligible at \( x \) large and negative, so that we recover standard cosmology at early times, and we then check a posteriori the self-consistency of the procedure. In this case, in each given era \( \zeta(x) \) can be approximated by a constant \( \zeta_0 \), with \( \zeta_0 = -2 \) in RD and \( \zeta_0 = -3/2 \) in MD. Then eq. (31) can be integrated analytically,
\[
U(x) = \frac{6(2 + \zeta_0)}{3 + \zeta_0} x + u_0 + u_1 e^{-(3+\zeta_0)x}, \tag{42}
\]
where the coefficients \( u_0, u_1 \) parametrize the general solution of the homogeneous equation \( U'' + (3 + \zeta_0)U = 0 \). For the moment we consider the most general homogeneous solution. However, as discussed below eq. (18), each definition of the non-local theory corresponds to one and only one choice of the homogeneous solution. Plugging eq. (42) into eq. (30) and solving for \( Y(x) \) we get
\[
Y(x) = -\frac{2(2 + \zeta_0)\zeta_0}{(3 + \zeta_0)(1 + \zeta_0)} x + u_0 + u_1 e^{-(3+\zeta_0)x} + \frac{-6(2 + \zeta_0)u_1}{2\zeta_0^2 + 3\zeta_0 - 3} e^{-(3+\zeta_0)x} + a_1 e^{\alpha_+ x} + a_2 e^{\alpha_- x}, \tag{43}
\]
where
\[
\alpha_\pm = \frac{1}{2} \left[ -3 + \zeta_0 \pm \sqrt{21 + 6\zeta_0 + \zeta_0^2} \right]. \tag{44}
\]
Observe that in RD \( \zeta_0 = -2 \) and the inhomogeneous solutions for \( U \) and \( Y \) vanish. This is a consequence of the fact that, in RD, the Ricci scalar vanishes, so \( \Box U = 0 \) and the only contributions to \( U \) and to \( (U \Box g_{\mu\nu})^T \) come from the solutions of the homogeneous equations. The inhomogeneous solution is self-consistent with our perturbative approach. Indeed, in a pure RD phase it just vanishes, and in a generic epoch, as \( x \to -\infty \), \( Y(x) \propto x \).
so its contribution to $\zeta(x)$ is anyhow negligible compared to the term $\Omega_M e^{-3x}$ and $\Omega_R e^{-4x}$ in eq. (32). Furthermore, in RD, $\alpha_\pm = (-5 \pm \sqrt{13})/2$ are both negative. The same happens in MD, where $\alpha_\pm = (-9 \pm \sqrt{57})/4$. Therefore in RD and MD all exponentials in eqs. (42) and (43) are decaying with $x$ and, apart for the constant mode $w_0$, in the perturbative regime the inhomogeneous solution is an attractor. If we start the evolution deep into RD, with initial conditions that are not too far from the inhomogeneous term of the perturbative solution, we will be quickly driven toward it, so within this attraction basin we can set $a_1 = a_2 = u_1 = 0$. The situation is different for $u_0$. We see from eq. (21) that a constant shift $U \to U + u_0$ is equivalent to introducing a cosmological constant, with $\Omega_A = \gamma u_0$. It is clear that, whenever one finds a model that gives a dynamical dark energy, one can always put on top of it a constant cosmological constant term. This defines a non-minimal model with one more parameter. In this paper we focus on a minimal model in which the $\Box^{-1}$ operator is defined so that $u_0 = 0$ at an initial time $t_*$ chosen deep in the RD phase, and we also set $a_1 = a_2 = u_1 = 0$. A more general analysis of the dependence on the definition of the $\Box^{-1}$ and $D^{-1}$ operators, which in the local formulation corresponds to a different choice of initial conditions, will be presented in [18].

C. Full background evolution and prediction of the DE equation of state

Eventually the perturbative treatment breaks down as we approach the present epoch, and we need to integrate the equations numerically. We use the Planck best-fit values $\Omega_M = 0.3175$, $\Omega_R = 4.15 \times 10^{-5} h_0^{-2}$, $h_0 = 0.6711$ [33] (and $\Omega_A = 0$). We set the initial conditions at values of $x$ deep in the RD phase (matter-radiation equilibrium is at $x \simeq -8.1$) such that we are on the perturbative solution found above. The numerical solution shown in the figures is obtained setting $\gamma = 0.0504$, which corresponds to $m \simeq 0.67 H_0$. This value of $\gamma$ has been tuned so that, at $x = 0$, $\Omega_{DE} = 1 - \Omega_M - \Omega_R \simeq 0.6825$. The behavior of $\rho_{DE}(x)$ (normalized to $\rho_0$) is shown in Fig. 1. Having fixed $\gamma$ so that today $\Omega_{DE} = 1 - \Omega_M - \Omega_R$, we get a prediction (with no more free parameter) for $\rho_{DE}(x)$ at any other time. Then, from eq. (34), we get a pure prediction for the dark energy EOS parameter $w_{DE}(x)$. The result is shown in Fig. 2. In MD $w_{DE}(x)$ is on the phantom side and grows slowly, and at the present epoch is close but still slightly smaller than $-1$. The fact that the EOS parameter is on the phantom side is generically a consequence of the fact that in our model the DE density starts from zero in RD and then grows during MD. Thus, in this regime $\rho_{DE} > 0$ and $\dot{\rho}_{DE} > 0$, and then $w_{DE}$ is always put on top of it a constant cosmological constant term. This defines a non-minimal model with one more parameter. In this paper we focus on a minimal model in which the $\Box^{-1}$ operator is defined so that $u_0 = 0$ at an initial time $t_*$ chosen deep in the RD phase, and we also set $a_1 = a_2 = u_1 = 0$. A more general analysis of the dependence on the definition of the $\Box^{-1}$ and $D^{-1}$ operators, which in the local formulation corresponds to a different choice of initial conditions, will be presented in [18].
In the next few years the DES survey should measure \( \omega_0 \) to an accuracy of about \( \Delta \omega_0 \approx 0.04 \) and later EU-CLID should measure it to an accuracy \( \Delta \omega_0 \approx 0.01 \) (and \( \omega_a \) to an accuracy \( \Delta \omega_a \approx 0.1 \)) \cite{36}. Such measurements will provide a stringent test of the prediction given in eq. (46).

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