Linear projections and successive minima

Christophe Soulé

Let $K$ be a number field, $\mathcal{O}_K$ its ring of integers and $E$ a projective $\mathcal{O}_K$-module of finite rank $N$. We endow $E \otimes \mathbb{Z} \subset \mathbb{C}$ with an hermitian metric $h$ and we let $\mu_1, \ldots, \mu_N$ be the logarithm of the successive minima of $(E, h)$. Assume $X_K \subset \mathbb{P}(E_K^*)$ is a smooth geometrically irreducible curve. In this paper we shall find a lower bound for the numbers $\mu_i$, $3 \leq i \leq N$, in terms of the height of $X_K$, $\mu_1$ and the average of the $\mu_i$’s (Theorem 2). This result is a complement to [9], Theorem 4, which gives a lower bound for $\mu_1$. The method of proof is a variant of [9], loc. cit. It relies upon Morrison’s proof of the fact that $X_K$ is Chow semi-stable [7]. We use a filtration $V_1 \supset V_2 \supset \ldots \supset V_N$ of the vector space $E_K$. This filtration is chosen so that, for suitable values of $i$, the projection $\mathbb{P}(V_i^*) \rightarrow \mathbb{P}(V_{i+1}^*)$ does not change the degree of the image of $X_K$ by linear projection. That such a choice is possible follows from a result of C. Voisin, namely an effective version of a theorem of Segre on linear projections of complex projective curves (Theorem 1). I thank her for proving this result and for helpful discussions.

1 Linear projections of projective curves

Let $C \subset \mathbb{P}^n$ be an integral projective curve over $\mathbb{C}$ and $d$ its degree. Assume that $C$ is not contained in some hyperplane, $d \geq 3$ and $n \geq 3$.

**Theorem 1.** (C. Voisin) There exists an integer $A = A(d)$ and a finite set $\Sigma$ of points in $(\mathbb{P}^n - C)(\mathbb{C})$, of order at most $A$, such that, for every point $P \in \mathbb{P}^n(\mathbb{C}) - \Sigma \cup C(\mathbb{C})$, the linear projection $\mathbb{P}^n \rightarrow \mathbb{P}^{n-1}$ of center $P$ maps $C$ birationally onto its image.

**Proof.** The existence of a finite set $\Sigma$ with the property above is a special case of a theorem of C. Segre [3]. The order of $\Sigma$ can be bounded as follows by a function of $d$.

If $n > 3$ a generic linear projection into $\mathbb{P}^3$ will map $C$ isomorphically onto its image [4] and the exceptional set $\Sigma \subset \mathbb{P}^n$ bijectively onto the exceptional set in $\mathbb{P}^3$. Therefore we can assume that $n = 3$.

When the projection with center $P \in \mathbb{P}^3(\mathbb{C})$ is not birational from the curve $C$ to its image $C' \subset \mathbb{P}^2$, we have $d' = \deg(C') \leq \frac{d}{2}$ hence $d' \leq d - 2$, and $P$ is
the vertex of a cone $K$ with base $C'$ containing $C$. So we have to bound the number of such cones.

Let $N$ be the dimension of the kernel of the restriction map

$$\alpha : H^0(\mathbb{P}^3, \mathcal{O}(d')) \to H^0(C, \mathcal{O}(d')).$$

Clearly $N$ is bounded as a function of $d$ and any $f \in \ker(\alpha)$ is an homogeneous polynomial of degree $d'$ which vanishes on $C$.

Let $Z \subset \mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^{N-1}(\mathbb{C})$ be the set of pairs $(P, f)$ such that $f$ is the equation of a cone $K$ of vertex $P$. If $p_1 : \mathbb{P}^3 \times \mathbb{P}^{N-1} \to \mathbb{P}^3$ is the first projection, we have to bound the order of $p_1(Z)$. We note that this order is at most the number $c$ of connected components of $Z$.

Now $Z$ is defined by equations of bidegree $(\delta, 1)$, $\delta \leq d'$. Indeed $f$ is homogeneous of degree $d'$ and $(P, f) \in Z$ when all the derivatives of $f$, except those of order $d'$, vanish at $P$.

Let $L = \mathcal{O}(d', 1)$, $M = \dim H^0(\mathbb{P}^3 \times \mathbb{P}^N, L) - 1$, and

$$j : \mathbb{P}^3 \times \mathbb{P}^N \to \mathbb{P}^M$$

the Segre embedding. Since $j(Z)$ is the intersection of $j(\mathbb{P}^3 \times \mathbb{P}^N)$ with linear hyperplanes, Bézout theorem ([3], § 8.4) tells us that

$$c \leq \deg(j(\mathbb{P}^3 \times \mathbb{P}^N)).$$

Hence $c$ is bounded by a function of $d$.

**Corollary.** Given any projective line $\Lambda \subset \mathbb{P}^n$, there exists a finite set $\Phi$ of order at most $A(d) + d$ in $\Lambda$ such that, if $P \in \Lambda - \Phi$, the linear projection of center $P$ maps $C$ birationally onto its image.

**Proof.** Since $C$ is not equal to $\Lambda$, the cardinality of $C \cap \Lambda$ is at most $d$. So the Corollary follows from Theorem 1.

## 2 Successive minima

### 2.1

Let $K$ be a number field, $[K : \mathbb{Q}]$ its degree over $\mathbb{Q}$, $\mathcal{O}_K$ its ring of integers, $S = \text{Spec}(\mathcal{O}_K)$ the associated scheme and $\Sigma$ the set of complex embeddings of $K$. Consider an hermitian vector bundle $(E, h)$ over $S$, i.e. $E$ is a torsion free $\mathcal{O}_K$-module of finite rank $N$ and, for all $\sigma \in \Sigma$, the associated complex vector space $E_{\sigma} = E \otimes \mathbb{C}$ is equipped with an hermitian scalar product $h_{\sigma}$. If $\bar{\sigma}$ is the conjugate of $\sigma$, we assume that the complex conjugation $E_{\sigma} \simeq E_{\bar{\sigma}}$ is an isometry.

If $i$ is a positive integer, $i \leq N$, we let $\mu_i$ be the infimum of the set of real numbers $r$ such that there exist $v_1, \ldots, v_i \in E$, linearly independent over $K$,
such that \( \log \|v_\alpha\| \leq r \) for all \( \alpha \leq i \). The number \( \mu_i \) is thus the logarithm of the \( i \)-th successive minimum of \((E, h)\). Let
\[
\mu = \frac{\mu_1 + \cdots + \mu_N}{N}.
\] (1)

2.2

If \( E' = \text{Hom}(E, \mathcal{O}_K) \) is the dual of \( E \) we let \( \mathbb{P}(E') \) be the associated projective space, representing lines in \( E' \). Let \( E'_K = E' \otimes K \) and \( X_K \subset \mathbb{P}(E'_K) \) a smooth geometrically irreducible curve of genus \( g \) and degree \( d \). We assume that the embedding of \( X_K \) into \( \mathbb{P}(E'_K) \) is defined by a complete linear series on \( X_K \). We also assume that \( g \geq 2 \) and \( d \geq 2g + 1 \). The rank of \( E \) is thus \( N = d + 1 - g \).

If \( X \) is the Zariski closure of \( X_K \) in \( \mathbb{P}(E') \) and \( \mathcal{O}(1) \) the canonical hermitian line bundle on \( \mathbb{P}(E') \), the Faltings height of \( X_K \) is the real number
\[
h(X_K) = \hat{\deg} (\hat{c}_1(\mathcal{O}(1))^2 \mid X),
\]
see [2] (3.1.1) and (3.1.5).

2.3

For any positive integer \( i \leq N \) we define the integer \( f_i \) by the formulae
\[
f_i = i - 1 \quad \text{if} \quad i - 1 \leq d - 2g
\]
and
\[
f_i = i - 1 + \alpha \quad \text{if} \quad i - 1 = d - 2g + \alpha, \quad 0 \leq \alpha \leq g.
\]

Assume \( k \) and \( i \) are two positive integers, \( k \leq N, i \leq N \). We let
\[
h_{i,k} = \begin{cases} f_i & \text{if} \quad i \leq k, \ i = N - 1 \text{ or } i = N \\ f_k & \text{if} \quad k \leq i \leq N - 2. \end{cases}
\]

Finally, if \( 2 \leq k \leq N \), we let
\[
B_k = \max_{i=2,\ldots,N} \frac{h_{i,k}^2}{(i-1)h_{i,k} - \sum_{j=1}^{i-1} h_{j,k}}.
\]

Theorem 2. There exists a constant \( C = C(d) \) such that, for every \( k \) such that \( 2 \leq k \leq N - 3 \),
\[
B_k(\mu_{N+1-k} - \mu) + \frac{h(X_K)}{[K : \mathbb{Q}]} + 2d \mu \geq (2d - (N + 1)B_k)(\mu - \mu_1) - C.
\]
2.4

To prove Theorem 2 fix a positive integer \( k \leq N - 3 \) and choose elements \( x_1, \ldots, x_N \) in \( E \), linearly independent over \( K \) and such that
\[
\log \| x_i \| = \mu_{N - i + 1}, \quad 1 \leq i \leq N.
\]

Fix integers \( n_\alpha, \alpha = k + 1, \ldots, N - 2 \), to be specified later (in § 2.6). If \( 1 \leq i \leq N \) we define
\[
v_i = \begin{cases} x_i + n_i x_{i-1} & \text{if } k + 1 \leq i \leq N - 2 \\ x_i & \text{else.} \end{cases} \tag{2}
\]

We get a complete flag \( E = V_1 \supset V_2 \supset \ldots \supset V_N \) by defining \( V_i \) to be the linear span of \( v_i, v_{i+1}, \ldots, v_N \).

When \( m \) is large enough the cup-product map
\[
\varphi : E_K^{\otimes m} \to H^0(X_K, \mathcal{O}(m))
\]
is surjective, hence \( H^0(X_K, \mathcal{O}(m)) \) is generated by the monomials
\[
v_1^{\alpha_1} \cdots v_N^{\alpha_N} = \varphi(v_1^{\otimes \alpha_1} \cdots v_N^{\otimes \alpha_N}),
\]
\( \alpha_1 + \cdots + \alpha_N = m \). A special basis of \( H^0(X_K, \mathcal{O}(m)) \) is a basis made of such monomials.

Let \( r_1 \geq r_2 \geq \cdots \geq r_N \) be \( N \) real numbers and \( \mathbf{r} = (r_1, \ldots, r_N) \). We define the weight of \( v_i \) to be \( r_i \), the weight of a monomial in \( E_K^{\otimes m} \) to be the sum of the weights of the \( v_i \)'s occuring in it, and the weight of a monomial \( u \in H^0(X_K, \mathcal{O}(m)) \) to be the minimum \( \text{wt}_\mathbf{r}(u) \) of the weights of the monomials in the \( v_i \)'s mapping to \( u \) by \( \varphi \). The weight \( \text{wt}_\mathbf{r}(\mathcal{B}) \) of a special basis \( \mathcal{B} \) is the sum of the weights of its elements, and \( w_\mathbf{r}(m) \) is the minimum of the weight of a special basis of \( H^0(X_K, \mathcal{O}(m)) \).

When \( r_1 \geq r_2 \geq \cdots \geq r_N \) are natural integers there exists \( e_\mathbf{r} \in \mathbb{N} \) such that, as \( m \) goes to infinity,
\[
w_\mathbf{r}(m) = e_\mathbf{r} \frac{m^2}{2} + O(m)
\]
([8], [7] Corollary 3.3).

Our next goal is to find an upper bound for \( e_\mathbf{r} \).

2.5

For every positive integer \( i \leq N \) we let \( e_i \) be the drop in degree of \( X_K \) when projected from \( \mathbb{P}(E_K) \) to \( \mathbb{P}(V_i) \). A criterion of Gieseker ([5], [7] Corollary 3.8) tells us that \( e_\mathbf{r} \leq S \) with
\[
S = \min_{1 \leq i_0 < \cdots < i_\ell = N} \left( \sum_{j=0}^{\ell-1} (r_{i_j} - r_{i_{j+1}}) (e_{i_j} + e_{i_{j+1}}) \right).
\]
Note that $S$ is an increasing function in each variable $e_i$. Furthermore, it follows from Clifford’s theorem and Riemann-Roch that

\[ e_i \leq f_i \]  

for every positive $i \leq N$ – see [7] proof of Theorem 4.4 (N.B.: in [7] Theorem 4.4 the filtration of $V_0$ has length $n + 1$, while $n = \dim V_0$. In our case, we start the filtration with $V_1$, hence the discrepancy between our definition of $f_i$ and [7] loc. cit.).

### 2.6

Let $w_1, \ldots, w_N \in E_K^\vee$ be the dual basis of $v_1, \ldots, v_N$. The linear projection from $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ has center the image $\hat{w}_i$ of $w_i$.

If $y_1, \ldots, y_N \in E_K^\vee$ is the dual basis of $x_1, \ldots, x_N$, we get

\[ w_i = \begin{cases} 
  y_i + n_i z_i & \text{if } k \leq i \leq N - 3 \\
  y_i & \text{else,}
\end{cases} \]

where $z_i + y_{i+1}$ is a linear combination of $y_{i+2}, y_{i+3}, \cdots$ with coefficients depending only on $n_{i+1}, n_{i+2}, \cdots$. When $n \neq m$ are two integers, the vectors $y_i + n z_i$ and $y_i + m z_i$ are linearly independent over $K$, therefore their images in $\mathbb{P}(V_i^\vee)$ are distinct. Since $e_{N-3} \leq f_{N-3}$ and $g \geq 2$ we get $e_{N-3} \leq d - 3$, therefore the image of $X_K$ in $\mathbb{P}(V_i^\vee)$, $i \leq N - 3$, has degree at least 3. Furthermore $\dim \mathbb{P}(V_i^\vee) \geq 3$. By Theorem 1 and its Corollary, it follows that we can choose $n_i$ such that $0 \leq n_i < A(d) + d$ and the projection of $\mathbb{P}(V_i^\vee)$ to $\mathbb{P}(V_{i+1}^\vee)$ does not change the degree of the image of $X_K$. We fix the integers $n_i$, $k \leq i \leq N - 3$, with this property. Hence we have

\[ e_i = e_k \quad \text{whenever} \quad k \leq i \leq N - 2. \]  

\[ (4) \]

### 2.7

From (3) and (4) we conclude that

\[ e_i \leq h_{i,k} \quad \text{if} \quad 1 \leq i \leq N \]

(see 2.3). Hence, by Morrison’s main combinatorial theorem, [7] Corollary 4.3, for any decreasing sequence of real numbers $r_1 \geq r_2 \geq \cdots \geq r_N$ we have, if $k \geq 2$,

\[ S \leq \psi(r) \]

with

\[ \psi(r) = B_k \cdot \sum_{j=1}^{N} (r_j - r_{N}). \]

So, when $r_1 \geq r_2 \geq \cdots \geq r_N = 0$ is a decreasing sequence of real numbers,

\[ e_r \leq \psi(r). \]
From the proof of Theorem 1 in [9] we deduce that, letting

\[ s_i = \log \| v_i \| - \log \| v_N \|, \quad 1 \leq i \leq N, \]

\[ h(X_K) \left[ \frac{1}{K : \mathbb{Q}} \right] + 2d \log \| v_N \| + \psi(s_1, s_2, \ldots, s_{N-1}, 0) \geq 0. \]  
(5)

From (2) above we get

\[ \log \| v_i \| \leq \log \| x_i \| + \log(1 + n_i) \quad \text{if} \quad k+1 \leq i \leq N-2 \]

and \( \log \| v_i \| = \log \| x_i \| \) otherwise.

Since \( \log \| x_i \| = \mu_{N+1-i} \) and \( n_i < A + d \) we deduce that

\[ \psi(s_1, s_2, \ldots, s_{N-1}, 0) \leq B_k \left( \sum_{i=1}^{N} (\mu_i - \mu_1) + \mu_{N+1-k} - \mu_3 + (N-2-k) \log(A+d) \right). \]  
(6)

From (1), (5) and (6) it follows that

\[ \frac{h(X_K)}{[K : \mathbb{Q}]} + 2d \mu_1 + B_k (N(\mu - \mu_1) + \mu_{N+1-k} - \mu_3) + C \geq 0 \]  
(7)

for some constant \( C = C(d) \). Since \( \mu_3 \geq \mu_1 \) the inequality in Theorem 2 follows from (7).
References

[1] Arbarello, E.; Cornalba, M.; Griffiths, P.A.; Harris, J. Geometry of algebraic curves. Volume I. Grundlehren der mathematischen Wissenschaften, 267. New York etc.: Springer-Verlag (1985).

[2] Bost, J.-B.; Gillet, H.; Soulé, C. Heights of projective varieties and positive Green forms. J. Am. Math. Soc. 7, No.4, 903-1027 (1994).

[3] Calabri, A.; Ciliberto, C.: On special projections of varieties: epitome to a theorem of Beniamino Segre. Adv. Geom. 1, no. 1, 97–106 (2001).

[4] Fulton, W. Intersection theory. 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 2. Berlin: Springer-Verlag (1998).

[5] Gieseker, D. Global moduli for surfaces of general type. Invent. Math. 43, 233-282 (1977).

[6] Hartshorne, R. Algebraic geometry. Corr. 3rd printing. Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag (1983).

[7] Morrison, I. Projective stability of ruled surfaces. Invent. Math. 56, 269-304 (1980).

[8] Mumford, D. Stability of projective varieties. Enseign. Math. 23, 39-110 (1977).

[9] Soulé, C. Successive minima on arithmetic varieties. Compos. Math. 96, No.1, 85-98 (1995).

soule@ihes.fr