Electromagnetic wave focusing with 2-D gridded parabolic reflector made of parallel circular PEC wires at oblique incidence

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Abstract
We consider the two-dimensional (2-D) scattering of the TE- and TM-polarized plane electromagnetic waves from a discrete grid-like parabolic reflector consisting of parallel circular perfectly electrically conducting (PEC) wires at oblique incidence. Our analysis is performed in the exact formulation, fully taking into account the wave interactions between the wires. We use the method of separation of variables in the local cylindrical coordinate systems in combination with Graf’s addition formulas for cylindrical functions. The near field in the vicinity of the reflector is calculated at different angles of incidence. The dependences of the maximum field amplitude in the focal area and the position of this maximum on the angle of incidence are plotted. The far field scattered patterns, total scattering cross-sections and backward scattering cross-sections are calculated and discussed.

Keywords Circular wire · Gridded parabolic reflector · Addition theorem for cylindrical functions · Focusing

1 Introduction
This work is a continuation of our previous studies (Velichko 2021a, b) on the scattering of a plane electromagnetic wave by a discrete parabolic reflector of circular PEC wires for the TE- and TM-polarizations. In these works, the scattering problem was considered in the full-wave formulation taking into account the electromagnetic interactions between the wires. In (Velichko 2021a), the accent was made on the near-field characteristics of the discrete reflector, namely, the focusing ability, while in Velichko (2021b) the far-field characteristics, such as the angular scattering pattern and the total scattering cross-section were calculated and discussed. Continuing this study, here we consider the oblique incidence of a plane electromagnetic wave on a parabolic reflector made of PEC wires. Following (Velichko 2021a, b), full-wave self-consistent calculation taking into account all interactions...
between the wires is performed by the method of separation of variables in the local polar coordinate systems in combination Graf’s addition theorem for the cylindrical functions.

It should be reminded that grid-like reflectors, which have reduced weight and wind load, serve, since long ago, as reasonable alternative to solid metal reflectors in harsh atmospheric and precipitation conditions (Kraus 1940). Their design needs sub-wavelength spacing of wires, otherwise the reflector performance is lost. Crossed-wire gridded reflectors are more frequent because of the obvious polarization insensitivity, however, reflectors made of parallel metal wires are also used. They provide a polarization selectivity because, due to the “Hertz effect,” only the TM-polarization (i.e. the parallel or E-polarization) regime provides sufficiently high reflectivity. This effect was found by H. Hertz in 1888 and soon became so common that association with its discoverer was lost. Still, it is worth citing a historical review (Ramsay 1958):

“His [Hertz’s] principal invention, however, was the screen or grating, a grid of parallel wires, which he described as follows: "I next had made an octagonal frame, 2 meters high and 2 meters broad; across this were stretched copper wires 1 mm thick, the wires being parallel to each other and 3 cm apart." At a wavelength of 66 cm as used by Hertz, this screen would reflect 96 per cent of the incident power in a plane wave normally incident and polarized linearly parallel to the wires. If, however, the incident wave were polarized at right angles to the wires, most of its power would be transmitted through the screen.”

2 Scattering problem formulation

Consider a set of $N_c$ identical infinite parallel circular PEC wires placed in vacuum, the axes of which are located on the parabola, $x = F - (0.25/F) \cdot y^2$, where $F$ is the focal distance (Fig. 1). Such a parabola is optimal in the sense that its segment with endpoints at $(0, \pm D/2)$ corresponds to the value of $F/D = 0.25$, which provides the best focusing by a solid PEC reflector. This conclusion was first obtained in the geometric-optics approximation (Varga and Torok n.d.,2000) and later was supported by the full-wave modeling (Bulygin et al. 2012).

The direction of the $X$ axis is chosen to coincide with the symmetry line of such a discrete gridded parabolic reflector cross-section. Suppose that a plane time-harmonic electromagnetic wave of the unit amplitude is incident at this reflector at the angle $\alpha$ to the $X$ axis, so that its wave vector is $\vec{k} = (k_x, k_y, 0)$, where $k_x = k \cos \alpha$, $k_y = k \sin \alpha$.

We denote the radius of the wires as $a$ and introduce a global Cartesian coordinate system $(x, y, z)$ with the $z$-axis collinear to the wires and $N_c$ local Cartesian $(x_q, y_q)$ and

![Fig. 1 Cross-sectional geometry of finite number of circular wires placed on parabola](image-url)
polar \((r, \varphi)\) coordinate systems, \(q = 1, \ldots, N_c\), with the origins in the centers of all wire cross-sections. The time dependence \(\exp(-i\omega t)\) is used and will be omitted.

As usual in the 2-D problems, we consider two alternative field polarizations, \(TE\) and \(TM\). Then, in the global Cartesian coordinate system, the equation of the incident plane \(z\)-component is \(E_i = H_x = 0\) for the \(TE\)-polarization and \(E_i = E_y = H_x = 0\) for the \(TM\)-polarization. In the polar coordinate system, these equations are of the form \(E_i = H_x = 0\) and \(E_i = E_y = 0\), respectively.

The full-wave numerical solution can be obtained by expanding the field function in the terms of the Fourier series in azimuth exponents in the local polar coordinates associated with each cylinder. Subsequent transformations are performed similarly to those given in (Velichko 2021a, b). Our self-consistent treatment takes into account the wave interactions by using the method of separation of variables in the local polar coordinates in combination with Graf’s addition theorem for the cylindrical functions (Abramowitz and Stegun 1964). This approach has been widely used in electromagnetics since the 1900s (Honl et al. 1961; Lax 1951, 1952; Twersky 1952; Row 1955; King and Wu 1959; Richmond 1965; Olaofe 1970; Ragheb and Hamid 1985; Frezza et al. 1985). However, as stressed in (Velichko 2021a, b), to provide the convergence of the numerical solution, the resulting infinite-matrix equation has to be regularized, i.e. cast to the Fredholm second-kind type. This enables reducing the error of computations to the machine precision, if desired, by increasing the matrix truncation order. Note that similar regularized matrix equations were also used in Ivanov (1970); Antoine et al. 2010; Natarov et al. 2011; Natarov et al. 2013; Natarov et al. 2014a; Natarov et al. 2014b; Yevtushenko and Dukhopelynykov 2020).

### 3 Basic equations

All the details of mathematical calculations are presented in detail in Velichko (2021a). Therefore, in this work, we present only the final equations. For the TM-polarization of the incident wave we obtain the following set of linear equations for the field Fourier coefficients, \(a_n^{(q)}\), where \(n = 0, \pm 1, \pm 2, \ldots\) and \(q = 1, \ldots, N_c\).

\[
d_n^{(q)} + \left[H_n^{(1)}(ka)\right]^{-1} \sum_{p=1, p\neq q}^{N_c} \sum_{m=-\infty}^{+\infty} A_m^{(p)} J_m(ka) H_{m-n}^{(1)}(kL_{pq}) e^{i(m-n)\varphi_{pq}} = -E_{c_{pq}}^{(q)} H_n^{(1)}(ka)^{-1},
\]

(1)

In the case of the TE polarization:

\[
d_n^{(q)} + \left[H_n^{(1)}(ka)\right]^{-1} \sum_{p=1, p\neq q}^{N_c} \sum_{m=-\infty}^{+\infty} A_m^{(p)} J_m(ka) H_{m-n}^{(1)}(kL_{pq}) e^{i(m-n)\varphi_{pq}} = -H_{c_{pq}}^{(q)} H_n^{(1)}(ka)^{-1}
\]

(2)

The quantities included in the matrix elements can be found in Velichko (2021a). Note that these systems of equations are the Fredholm equations of the second kind. This guarantees the convergence of the solutions obtained with truncation number \(N\) to the exact solution if \(N\) is taken greater (Velichko 2021a).
4 Results of numerical modeling

In computations, we take the number of cylinders to be \( N_c = 51 \), the focal distance of the parabola is \( F = 24 \text{ mm} \), the radii of the wires are \( a = 0.5 \text{ mm} \), the projection of the distance between their centers to the \( y \)-axis is \( \Delta y = 1.5 \text{ mm} \), the aperture of the reflector, so that reflector’s aperture, i.e. the distance between the most out-standing wire centers, is \( D = 76 \text{ mm} \).

In Figs. 2 and 3, we show the near electric and magnetic fields portraits around explained above 51-wire discrete parabolic reflector at different incident angles for the \( TM \) and \( TE \)-polarizations, respectively. Calculations were carried out at the frequency of 77.25 GHz for both polarizations.

First, from a comparison of Fig. 2a and Fig. 3a, it can be seen that the maximum of the absolute value of the normalized total field is near the geometrical focus (GF) of the parabola, however, not exactly at GF. Its magnitude is higher for the \( TM \)-polarized plane wave than the \( TE \)-polarized one, and much deeper shadow is formed behind discrete reflector in the case of the \( TM \) polarization (see (Velichko 2021b)). This is a manifestation of the Hertz effect of 1888 (Ramsay 1958), explained in Introduction. Secondly, as can be seen, with an increase in the angle of incidence, the focusing of the field

![Image](image_url)

**Fig. 2** TM polarization. Electric field pattern around PEC circular wires arranged in a parabola at the frequency of 77.25 GHz for the incident angles \( \alpha = 0^\circ \) (a), \( \alpha = 2^\circ \) (b), \( \alpha = 10^\circ \) (c), \( \alpha = 80^\circ \) (d).
deteriorates, the value of the maximum field amplitude decreases, and the focal domain itself gets blurred for both polarizations.

Detailed analysis of the field in the focal area is very important in applications – see (Hung and Mittra 1983; Hongo and Ji 1988; Ivashina and van’t Klooster 2003) A more detailed study of the “true focus”, i.e. the field maximum, is presented in Fig. 4, where

Fig. 3 The same as in Fig. 2, however, for the TE polarization.

Fig. 4 The field maximum position in the focal area \((X_{\text{max}}, Y_{\text{max}})\) versus the incidence angle, in the TM (a) and the TE (b) polarization case at 77.25 GHz
we show the field maximum position and amplitude versus the incidence angle, for the TM and TE polarizations.

As one can see, with an increase in the angle of incidence, the position of the field maximum along the $X$ coordinate changes slightly (i.e. remains near $X = 0$ mm), but its position along the $Y$ coordinate changes larger and in almost linear manner. It is also seen that with an increase in the angle of incidence, the field amplitude at the maximum decreases. This is true for both polarizations.

The magnitude of the normalized total field at GF, where a small horn antenna is usually placed, can be considered as the simplest figure of merit of reflector in the reception regime. This value can be called the focusing ability (FA). The plots in Fig. 5 demonstrate the FA dependences for the grid-like reflector on the angle of plane-wave arrival. As visible, FA displays oscillations and can drop to just 3% of the maximum value if the angle of incidence lies within $10^\circ$. In contrast, the varying-position maximum of the absolute value of the normalized total field deteriorates only by 10%. This comparison explains why the shifting of a horn receiver can provide the scanning of the targets by the receiving reflector antenna. This observation lies in the core of the design of so-called “focal plane arrays” of today’s reflector antennas in radio astronomy and other applications (Ivashina and van’t Klooster 2003).

In Fig. 6, we present the value of FA (i.e. field magnitude in GF) versus the incidence angle in wide region, up to $180^\circ$ for low (77.25 GHz) and high frequencies (500 GHz) for both polarizations. Besides, here we indicate a critical angle called the spillover angle, $\alpha_{sp} = 180^\circ - \alpha_1$ (see Fig. 1). The angle $\alpha_1 \approx 76^\circ$ can be found from Fig. 1, given the coordinates of the “edge” wire (9.35; 37.5) mm. Thus, for the chosen parameters of the parabolic reflector, $\alpha_{sp} \approx 104^\circ$. These plots confirm earlier conclusions that a discrete parabolic reflector focuses the TM-waves better than the TE-waves and that, with an increase in the angle of incidence, the field magnitude at GF decreases and oscillates. The oscillations become enhanced with increasing the frequency of the incident wave.

In Fig. 7, we show the far-field scattering patterns for the explained above 51-wire discrete parabolic reflector in both polarizations for different incident angles at the low frequency of 77.25 GHz (i.e. $\lambda = 3.88$ mm and $ka \approx 0.809$) This means that the electric size of reflector’s aperture is $D = 19.33\lambda$.
Electromagnetic wave focusing with 2-D gridded parabolic...

Fig. 6 The focusing ability (the value of field amplitude in the geometrical focus) versus the incidence angle in the TM and TE polarization cases, for low (a) and high (b) frequencies. Note the effect of the spillover angle at 104°, better visible in the low-frequency TM-polarization case.

Fig. 7 Normalized angular far-field scattering patterns (in dB) for 51 cylinders arranged on parabola and illuminated by the normally incident plane wave for incident angles $\alpha = 0^\circ$ (a), $\alpha = 2^\circ$ (b), $\alpha = 10^\circ$ (c), $\alpha = 80^\circ$ (d).

Fig. 8 Normalized TSCS as a function of parameter $ka$ for different incident angles, for 51 PEC wires arranged on the parabola, for the TM (a) and TE (b) polarizations.
Figure 8 shows the plots of the normalized total scattering cross-section (TSCS) (see eq. 12 in Velichko (2021b)),

\[ \frac{\sigma^{E,H}_s}{4a} = \frac{1}{2\pi ka} \int_0^{2\pi} |F_{E,H}(\varphi)|^2 d\varphi. \]  

(3)

as a function of parameter \( ka \), for the gridded parabolic reflector in the \( TM \) - and \( TE \) polarizations, normalized by the high-frequency limit for single wire, \( 4a \). Here, \( \varphi \) is the polar angle in the global polar coordinates with the origin in the geometrical focus and \( F_{E,H}(\varphi) \) is the far-field angular scattering pattern, defined in (Velichko 2021a, b). As one can see, the plots show some oscillations, however, follow the plot of TSCS of single wire times the number of wires, \( N_c = 51 \). The largest deviations appear at lower frequencies. This proves that at high frequencies a gridded reflector becomes more and more transparent in either polarization regime, similarly to the reflectors made of graphene (Oguzer et al. 2017; Oguzer and Altintas 2021).

Besides, the \( TE \) polarization case shows more ripples than the \( TM \) case. Note also that for the larger angles \( \alpha \) the spectra of TSCS show lower values than for \( \alpha = 0 \)—this can be explained by the overshadowing of distant wires by the forward ones. Such overshadowing is clearly visible in Fig. 2d and Fig. 3d and leads to the reduced effective aperture seen by the incident field.

Finally, in Fig. 9 we present the plots of the backward scattering cross-section (BSCS) (for \( \alpha = 0^\circ, 10^\circ \)). Its definition is given, for instance, in Honl et al. (1961) as the ratio of the total power scattered by a fictitious isotropic scatterer, creating a field in all directions equal to the field from true obstacle towards the source, to the absolute value of the Poynting vector of a plane wave incident at the scatterer. After some algebra and normalization by its high-frequency limit, \( \pi a \), it reduces to

\[ \frac{\sigma^{E,H}_b}{\pi a} = \frac{4}{\pi ka} |F_{E,H}(0)|^2. \]  

(4)

In these calculations, the step in parameter \( ka \) is taken as 0.1. For both polarizations, BSCS plots show numerous small and sharp oscillations, especially at higher frequencies,

Fig. 9 Normalized BSCS as a function of parameter \( ka \) for 51 PEC wires arranged on the parabola, for the TM (a) and TE (b) polarizations

in the either polarization regime. Note that the peak values of BSCS are larger in the case of the symmetrical illumination ($\alpha = 0^\circ$) than at the oblique incidence.

5 Conclusions

We have studied in full-wave manner the near and far field characteristics of the discrete parabolic reflector made of parallel PEC circular wires at oblique incidence. This analysis has confirmed the expectation that to be efficient, i.e. provide high focusing ability, a gridded reflector should be used in the TM-polarization regime. With an increase in the angle of incidence, the focusing ability of the gridded parabolic reflector deteriorates in both polarizations. However, it is possible to minimize this degradation by adapting the position of receiver to the true focus, which is shifted from the geometrical focus of parabola. This shift almost linearly depends on the incidence angle, counted from the line of symmetry.

A natural question arises on the implications appearing in the case of 3-D gridded reflector performance. The analysis of such reflectors, which are closer to the real-life configurations, is much more complicated. This is because even the scattering of waves from a single straight finite-length PEC circular wire is a complicated mathematical problem, which has no explicit solution (unlike the scattering from infinite single circular PEC wire). From the physical point of view, 3-D gridded reflector will combine the focusing features of the solid 3-D paraboloidal reflector and the polarization-discrimination features of the gridded 2-D reflector. In the plane, which cuts normally through the centers of finite curved wires, the 3-D focusing will have the features, similar to the features in the studied here 2-D case.

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