Novel Interaction Phenomena of Localised Waves in the $(2 + 1)$-Dimensional HSI Equation

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Abstract. Localised interaction solutions of the $(2 + 1)$-dimensional generalised Hirota-Satsuma-Ito equation are studied. Using the Hirota bilinear form and Maple symbolic computations, we generate three classes of lump solutions. Specific sets of parameters are chosen to show the dynamic characteristics and evolution of lump and interaction solutions and their energy distribution.

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Key words: $(2+1)$-dimensional generalised Hirota-Satsuma-Ito equation, lump solution, interaction solution, Hirota’s bilinear method.

1. Introduction

The systematic study of non-linear phenomena started in 1960s and experienced rapid growth since then. The solitons, in particular, found various applications in physics, biology, medicine, oceanography, economics and population problems. However, the finding of exact solutions of nonlinear systems requires a lot of effort and the approaches used include the inverse scattering transformation [1], the Bäcklund transformation [15, 16], the Darboux transformation [14], the variable separation [6], the Hirota bilinear methods [4] and some others [2, 7, 8]. One of these techniques — viz. the Hirota bilinear method has been recently utilised to establish lump solutions. Thus Wazwaz [17, 18] employed it to derive multiple soliton solutions of BKP and generalised Ito equations. Zhao [20] applied the Hirota bilinear form to investigate two and three soliton solutions of a multi-component higher-order Ito equation. The lump solution is a rational function solution, which decays in all directions of space variables [9]. Such solutions of partial differential equation (PDE) describe important wave phenomena and are determined for various classes of integrable equations [3, 11, 19]. Thus Ma [10] obtained lumps and interaction solutions of linear
partial differential equations in \((3 + 1)\)-dimensions. A general approach to finding of positive quadratic solutions of bilinear equations is given in [13]. On the other hand, recently Zhou et al. [21] considered the \((2 + 1)\)-dimensional Hirota-Satsuma shallow water wave equation.

In this work, we study the lump and interaction solutions of the \((2 + 1)\)-dimensional generalised Hirota-Satsuma-Ito equation

\[
\nu_t + u_{xxx} + 3(uw)_x + \gamma \cdot u_x = 0, \quad u_y = v_x, \quad u_t = w_x, \tag{1.1}
\]

where \(u, v, w\) are the function of \(x, y, t\) and the subscripts denote partial derivatives with respect to scaled space coordinates \(x, y\) and time \(t\), cf. [5, 12]. This equation arises in the shallow water wave theory and in the Jimbo-Miwa classification.

This paper is structured as follows. In Section 2, we write the bilinear form of the HSI equation through a dependent transformation and lump and interaction solutions of the HSI equation are obtained with assistance of Maple symbolic computations. A number of lump solutions are graphically shown in order to describe the dynamics and properties. A brief conclusion is provided in Section 3.

### 2. Lump and Interaction Solutions of the Eq. (1.1)

Using the mapping,

\[
u = 2(\ln f)_x, \quad v = 2(\ln f)_{xy}, \quad w = 2(\ln f)_{xt}, \tag{2.1}
\]

where \(f(x, y, t)\) is a real function, we transform the Eq. (1.1) into the bilinear form

\[
(D_y D_t + D_x^3 D_t + \gamma D_x^2) f \cdot f = 0. \tag{2.2}
\]

For any positive integers \(m, n\) and \(q\), the operator \(D\) is defined by

\[
D^m D^n D^q f(x, y, t) = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n (\partial_t - \partial_{t'})^q f(x, y, t) g(x', y', t') |_{x'=x, y'=y, t'=t},
\]

and \(g\) is the function of the formal variables \(x', y'\) and \(t'\).

In order to obtain the interaction solutions of the Eq. (1.1), we choose the function \(f\) in the form

\[
f(x, y, t) = g^2 + h^2 + a_9 \cos(a_{10} x + a_{11} y + a_{12} t + a_{13})
+ a_{14} \cosh(a_{15} x + a_{16} y + a_{17} t + a_{18})
+ \exp(-(a_{19} x + a_{20} y + a_{21} t + a_{22}))
+ a_{23} \exp(a_{19} x + a_{20} y + a_{21} t + a_{22}), \tag{2.3}
\]

where \(g = a_1 x + a_2 y + a_3 t + a_4, \quad h = a_5 x + a_6 y + a_7 t + a_8,\)

and the real numbers \(a_i, i = 1, \ldots, 23\) will have to be determined later on.
Substituting (2.3) into (2.2) and equating the coefficients at the corresponding powers of \(x, y\) and \(t\) leads to a system of algebraic equations for \(a_i, i = 1, \ldots, 23\). The solutions of this system can be found after tedious calculation with the aid of Maple symbolic computation. Here we will pay attention to three situations.

**Case 1.** The first solution set of the corresponding algebraic system is

\[
\begin{align*}
a_1 &= 0, & a_2 &= 0, & a_3 &= 0, & a_5 &= 0, \\
a_6 &= 0, & a_7 &= 0, & a_9 &= 0, & a_{14} &= 0, \\
a_{20} &= -\frac{a_{19}^2(a_{19}a_{21} + \gamma)}{a_{21}}, & a_{23} &= 0, & a_{21} &\neq 0.
\end{align*}
\]

Substituting the Eqs. (2.4) into (2.3) produces the function

\[
f(x, y, t) = a_4^2 + a_8^2 + \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right).
\]

Using this function in the Eqs. (2.1), we obtain different kind solutions of the Eq. (1.1). In particular, the soliton solution \(u\) is

\[
u(x, y, t) = \frac{2a_{19}^2 \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right)}{a_4^2 + a_8^2 + \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right)}
\]

\[\times \frac{- 2a_{19} (\exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right))^2}{(a_4^2 + a_8^2 + \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right))^2},
\]

where \(a_4, a_8, a_{19}, a_{21}, a_{22}\) and \(\gamma\) are arbitrary constants and \(a_{21} \neq 0\).

The dynamics of the solution \(u\) shows that a one lump, which is anti-bell-shaped one-soliton, moves. The surfaces of the function \(u\) for \(t = 2, -5\) are shown in Fig. 1.

Analogously, we derive the function \(v\) in the Eq. (2.1). Thus

\[
v(x, y, t) = -\frac{2a_{19}^3(a_{19}a_{21} + \gamma) \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right)}{a_{21} (a_4^2 + a_8^2 + \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right))}
\]

\[\times \left[2a_{19} (a_{19}a_{21} + \gamma) \left(\exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right)\right)^2\right]
\]

\[\times \frac{a_{21} (a_4^2 + a_8^2 + \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right))^2}{a_{21} (a_4^2 + a_8^2 + \exp\left(-a_{19}x + (a_{19}^2(a_{19}a_{21} + \gamma)y) / a_{21} - a_{21}t - a_{22}\right))^2},
\]

where \(a_4, a_8, a_{19}, a_{21}, a_{22}\) and \(\gamma\) are free constants.

Fig. 2 shows the motion of the lump solution at different time. We observe that the lump solution moves with regular speed without changing its state and shape. Fig. 2 demonstrates the evolution of solution \(v\).
Figure 1: $a_4 = 2, a_8 = 2, a_{19} = 1, a_{21} = 1, a_{22} = 2, \gamma = 2$. Solution $u$: a) $t = 2$; b) $t = -5$. Contour plot: c) $t = 2$; d) $t = -5$.

Figure 2: $a_4 = 2, a_8 = 2, a_{19} = 1, a_{21} = 1, a_{22} = 2, \gamma = 2$. Solution $v$: a) $t = 1$; b) $t = -3$. Contour plot: c) $t = 1$; d) $t = -3$. 
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Figure 3: $a_4 = 2, a_8 = 2, a_{19} = 1, a_{21} = 1, a_{22} = 2, \gamma = 2$. Solution: a) $t = 1$; b) $t = -1$. Contour plot: c) $t = 1$; d) $t = -1$.

Cumbersome calculations are needed to express the remaining function $w$ in the Eq. (2.1). It is a soliton solution of the form

$$w(x, y, t) = \frac{2a_{19}a_{21} \exp \left(-a_{19}x + \left(a_{19}^2(a_{19}a_{21} + \gamma)y\right)/a_{21} - a_{21}t - a_{22}\right)}{a_4^2 + a_8^2 + \exp \left(-a_{19}x + \left(a_{19}^2(a_{19}a_{21} + \gamma)y\right)/a_{21} - a_{21}t - a_{22}\right)}$$

where $a_4, a_8, a_{19}, a_{21}, a_{22}$ and $\gamma$ are free constants. Modifying these parameters changes the structure of soliton solution, as is shown in Fig. 3.

**Case 2.** In this case

$$a_1 = \frac{2}{3} \frac{a_7 \gamma}{a_{21}^2}, \quad a_2 = \frac{4}{9} \frac{a_3 \gamma^3}{a_{21}^4}, \quad a_9 = 0,$$

$$a_5 = \frac{2}{3} \frac{a_3 \gamma}{a_{21}^2}, \quad a_6 = \frac{4}{9} \frac{a_7 \gamma^3}{a_{21}^4}, \quad a_{14} = 0,$$

$$a_{19} = \frac{2}{3} \frac{\gamma}{a_{21}}, \quad a_{20} = \frac{4}{27} \frac{\gamma^3}{a_{21}^2}, \quad a_{23} = 0, \quad a_{21} \neq 0.$$
Analogously, we obtain the soliton solution and they create a new one-line soliton with a higher amplitude. Fig. 4 shows that through evolution of time, the line soliton moves to the lump solution and they create a new one-line soliton with a higher amplitude.

Substituting the Eqs. (2.6) into (2.3) produces the quadratic function $f$, 

$$f(x, y, t) = k_1^2 + k_2^2 + \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right),$$  \hspace{1cm} (2.7)

where

$$k_1 = \frac{2}{3} \frac{a_2 \gamma}{a_{21}^2} x + \frac{4}{9} \frac{a_2 \gamma^3}{a_{21}^3} y + a_3 t + a_4,$$

$$k_2 = -\frac{2}{3} \frac{a_2 \gamma}{a_{21}^2} x + \frac{4}{9} \frac{a_2 \gamma^3}{a_{21}^3} y + a_7 t + a_8.$$

Substituting function (2.7) into (2.1) generates another class of lump and interaction solutions of the Eq. (1.1). In particular, $u$ is the soliton solution,

$$u(x, y, t) = 2 \left( \frac{8 a_2^2 \gamma^2}{9 a_{21}^4} + \frac{8 a_2^2 \gamma^2}{9 a_{21}^4} + \frac{4}{9} \frac{\gamma^2 \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right)}{a_{21}^2} \right)$$

$$\times \left( k_1^2 + k_2^2 + \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right) \right)^{-1}$$

$$- 2 \left( \frac{4 k_1 a_2 \gamma}{3 a_{21}^2} - \frac{4 k_2 a_3 \gamma}{3 a_{21}^2} + \frac{2}{3} \frac{\gamma \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right)}{a_{21}} \right)^2$$

$$\times \left( k_1^2 + k_2^2 + \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right) \right)^{-2}.$$  

Fig. 4 shows that through evolution of time, the line soliton moves to the lump solution and they create a new one-line soliton with a higher amplitude.

Analogously, we obtain the soliton solution $v$,

$$v(x, y, t) = \frac{16}{81} \left( \gamma^4 \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right) \right)$$

$$\times \left( a_{21}^4 \left( k_1^2 + k_2^2 + \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right) \right) \right)^{-1}$$

$$- 2 \left( \frac{4 k_1 a_2 \gamma}{3 a_{21}^2} - \frac{4 k_2 a_3 \gamma}{3 a_{21}^2} + \frac{2}{3} \frac{\gamma \exp \left( \frac{2}{3} \gamma x + \frac{4}{27} \gamma^3 y - a_{21} t - a_{22} \right)}{a_{21}} \right)^2.$$  

Case 3. Another particular soliton solution is presented in Fig. 5, where a lump and one-line solitons get closer and merge. The function \( v \) is shown at time \( t = 2 \) and \( t = -2 \).

Another application of the above procedure generates the functions \( w \),

\[
w(x, y, t) = -\frac{4}{3} \left( \gamma \exp \left( \frac{2}{3} \frac{y}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right) \times \left( k_1^2 + k_2^2 + \exp \left( \frac{2}{3} \frac{y}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right)^2 \times \left( \frac{8 k_1 a_3 \gamma^3}{9 a_{21}^3} + \frac{8 k_2 \gamma^3 a_7}{9 a_{21}} + \frac{\gamma^3 \exp \left( \frac{2}{3} \frac{y}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right)}{a_{21}^3} \right)
\]

\[
\times \left( k_1^2 + k_2^2 + \exp \left( \frac{2}{3} \frac{y}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right)^{-2}.
\]

The three-dimensional and contour plots of such lump solution are displayed in Fig. 6.

Case 3. Another solution set of the corresponding system of algebraic equations is

\[
\begin{align*}
    a_1 &= -\frac{2}{3} \frac{a_7 \gamma}{a_{21}^2}, & a_2 &= \frac{4}{9} \frac{a_3 \gamma^3}{a_{21}^3}, & a_9 &= 0, \\
    a_5 &= \frac{2}{3} \frac{a_3 \gamma^3}{a_{21}^3}, & a_6 &= \frac{4}{9} \frac{a_7 \gamma^3}{a_{21}^3}, & a_{14} &= 0, \\
    a_{19} &= -\frac{2}{3} \frac{\gamma}{a_{21}}, & a_{20} &= -\frac{4}{27} \frac{\gamma^3}{a_{21}^3}, & a_{23} &= 0, & a_{21} \neq 0.
\end{align*}
\]
Figure 4: \( a_3 = 1, a_4 = 2, a_7 = 1, a_8 = 2, a_{21} = 2, a_{22} = 2, \gamma = 2 \). Solution \( u \): a) \( t = 1 \); b) \( t = -1 \). Contour plot: c) \( t = 1 \); d) \( t = -1 \).

Figure 5: \( a_3 = 1, a_4 = 2, a_7 = 1, a_8 = 2, a_{21} = 2, a_{22} = 2, \gamma = 2 \). Solution \( v \): a) \( t = 2 \); b) \( t = -2 \). Contour plot: c) \( t = 2 \); d) \( t = -2 \).
Substituting (2.8) into (2.3) leads to the function $f$ of the form

$$f(x, y, t) = l_1^2 + l_2^2 + \exp \left( \frac{2}{3} \gamma x + \frac{1}{9} \gamma^2 y - a_{21} t - a_{22} \right),$$

(2.9)

where

$$l_1 = \frac{2 a_{2} \gamma}{3 a_{21}} x + \frac{4 a_{3} \gamma^3}{9 a_{21}^2} y + a_3 t + a_4,$$

$$l_2 = \frac{2 a_{3} \gamma}{3 a_{21}} x + \frac{4 a_{7} \gamma^3}{9 a_{21}^2} y + a_7 t + a_8.$$ 

Substituting function (2.9) into the Eq.(2.1), we obtain the soliton solutions of the Eq. (1.1). In particular,

$$u(x, y, t) = 2 \left( \frac{8 a_2^2 \gamma^2}{9 a_{21}^4} + \frac{8 a_3^2 \gamma^2}{9 a_{21}^4} + \frac{4 \gamma^2 \exp \left( \frac{2}{3} \gamma x + \frac{4 \gamma^3}{27 a_{21}^2} y - a_{21} t - a_{22} \right)}{a_{21}^2} \right)$$

$$\times \left( l_1^2 + l_2^2 + \exp \left( \frac{2}{3} \gamma x + \frac{4 \gamma^3}{27 a_{21}^2} y - a_{21} t - a_{22} \right) \right)^{-1}.$$
\[-2 \left( -\frac{4 l_1 a_7 \gamma}{3 a_{21}^2} + \frac{4 l_2 a_3 \gamma}{3 a_{21}^2} + \frac{2}{3} \gamma \exp \left( \frac{2}{3} \frac{\gamma}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right) \right]^{2}
\times \left( l_1^2 + l_2^2 + \exp \left( \frac{2}{3} \frac{\gamma}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right)^{-2}.

Fig. 7 shows the propagation of wave and fusion. As $t$ grows, the lump solution is admitted by the line soliton and holds the shape and speed without changing.

![Figure 7](image)

Figure 7: $a_3 = 1, a_4 = 2, a_7 = 1, a_8 = 2, a_{21} = 2, a_{22} = 2, \gamma = 2$. Solution $u$: a) $t = 1$; b) $t = -1$. Contour plot: c) $t = 1$; d) $t = -1$.

Proceeding as before, we obtain solutions $v$ and $w$, viz.

\[ v(x, y, t) = \frac{16}{81} \left( \gamma^4 \exp \left( \frac{2}{3} \frac{\gamma}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right) \]
\[ \times a_{21}^4 \left( l_1^2 + l_2^2 + \exp \left( \frac{2}{3} \frac{\gamma}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right)^{-1} \]
\[-2 \left( -\frac{4 l_1 a_7 \gamma}{3 a_{21}^2} + \frac{4 l_2 a_3 \gamma}{3 a_{21}^2} + \frac{2}{3} \gamma \exp \left( \frac{2}{3} \frac{\gamma}{a_{21}} x + \frac{4}{27} \frac{\gamma^3}{a_{21}^3} y - a_{21} t - a_{22} \right) \right) \]
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We establish lump and interaction solutions of the $(2 + 1)$-dimensional generalised Hirota-Satsuma-Ito equation. Using a logarithmic transformation and the Hirota bilinear form, we obtain a quadratic function $f$, which allows us to determine various solutions of the equation under consideration. The parameters defining function $f$ satisfy a system of algebraic equations and we use Maple symbolic computations to find them. Finally, specific sets of parameters are employed to show the dynamic characteristics and evolution of lump and interaction solutions and their energy distribution.

3. Conclusions
Figure 8: $a_3 = 1, a_4 = 2, a_7 = 1, a_8 = 2, a_{21} = 2, a_{22} = 2, \gamma = 2$. Solution $v$: a) $t = 1$; b) $t = -1$. Contour plot: c) $t = 1$; d) $t = -1$.

Figure 9: $a_3 = 1, a_4 = 2, a_7 = 1, a_8 = 2, a_{21} = 2, a_{22} = 2, \gamma = 2$. Solution $w$: a) $t = 1$; b) $t = -1$. Contour plot: c) $t = 1$; d) $t = -1$. 

a) 

b) 

c) 

d)
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