The Influence of the Shear on the Gravitational Waves in the Early Anisotropic Universe

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Abstract We study the singularity of the congruences for both timelike and null geodesic curves using the expansion of the early anisotropic Bianchi type I Universe. In this paper, we concentrate on the influence of the shear of the timelike and null geodesic congruences in the early Universe. Under some natural conditions, we derive the Raychaudhuri type equation for the expansion and the shear-related equations. Recently, scientists working on the LIGO (Laser Interferometer Gravitational-Wave Observatory) have shown many possibilities to observing the anisotropy of the "primordial" gravitational wave background radiation. We deduce the evolution equation for the shear that may be responsible for those observational results.

1 Introduction

The existence of Hawking-Penrose (HP) singularity [Hawking and Penrose 1970] can be assumed at the beginning of the Universe that was born with the Big Bang as an unimaginary hot, dense point. If we presume that the early Universe was filled with a perfect fluid containing massive particles and/or massless particles, then we could find equations of state for each particle, by using the strong energy condition that was used to show the HP singularity theorem.

The motion types of timelike and null geodesic congruences in the Universe can be described in terms of the expansion, the shear, and the rotation. In this article, we will describe the expansion rates of timelike and null geodesic congruences and the shearing motions in the early anisotropic Universe. The aspects of the rotational motions are negligible to produce the Bianchi type I Universe. Additionally, we derive the Raychaudhuri type equation that is an evolution equation for the expansion.

There are many results regarding the dynamics of geodesic surface congruence in the early Universe [Cho and Hong 2007, Cho and Hong 2008, Cho and Hong 2011], [Cho and Hong (An extract from a book) 2011]. In this article, we will go on to discuss the homogeneous and anisotropic Universe. In particular, we consider the well-known models of the homogeneous and anisotropic Universe such as Bianchi type I and Kasner Universe models [Kasner 1921]. The Kasner Universe is the "vacuum" Bianchi type I Universe. We also suggest a straightforward approach to investigate the Universe models. Many scientists have studied the Bianchi type I Universe (e.g., [Byland and Scialom 1998, Cáceres et al. 2010, Chiba et al. 1997, Cho and Speliotopoulos 1995, Jacobs 1969, Tsagas et al. 2008, Pacif and Mishra 2015, Singh and Bishi 2015, Adhav et al. 2011, Adhav et al. 2010, Tiwari 2008, Saha 2007, Pacif and Mishra 2015, Sharif and Zubair 2010, Sharif and Saleem 2015, Singh and Kale 2011, Singh and Bishi 2015, Shamir 2015, Jamil et al. 2012].

This article is organized as follows. In Section 2, we discuss the kinematical quantities in the Bianchi type I Universe and introduce the formalism that describes the geodesic congruence in the early Universe. In Section 3, we describe the timelike geodesic curves in the Bianchi type I Universe. We investigate the expansion rate of timelike geodesic congruence and the aspects of shearing motions in the Bianchi type I Universe model. In Section 4, we describe the null geodesic curves in the Bianchi type I Universe. We investigate the expansion rate of null geodesic congruence and the aspects of the shearing motions, in our Universe model, for null case. Finally, we conclude in Section 5.
2 The Kinematical Quantities in the Bianchi Type I Universe

In this section, we review the research works by Cho and Hong (e.g., [Cho and Hong 2007, Cho and Hong 2008, Cho and Hong (An extract from a book) 2011]) on the expansion, the shear, and the rotation which will be described later. We also discuss the well-known models of homogeneous and anisotropic Universe such as Bianchi type I and Kasner Universe models (e.g., [Ellis 2009, Wainwright and Ellis 1997, Hawking and Ellis 1973, Ellis and van Elst 1999]).

In order to define the action on the curved space-time manifold, we let \((M, g_{ab})\) be a 4-dimensional manifold associated with the metric \(g_{ab}\). Given \(g_{ab}\), we can have a unique covariant derivative \(\nabla_a\) satisfying [Wald 1984]

\[
\nabla_a g_{bc} = 0, \\
\nabla_a \omega^b = \frac{\partial}{\partial t} \omega^b + \Gamma^b_{ac} \omega^c, \\
(\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_k = R^d_{abc} \omega_d.
\]

(2.1)

We parametrize the surface generated by the world sheet coordinate \(t\), and then we have the corresponding vector field \(\xi^a = (\partial / \partial t)^a\). Then \(\xi^a\) satisfies the timelike condition \(\xi^a \xi_a = -1\). We introduce the tensor field \(B_{ab}\) defined by

\[
B_{ab} = \nabla_b \xi^a,
\]

which satisfies the following identity

\[
B_{ab} \xi^a = 0.
\]

(2.2)

Next, we introduce the metric \(h_{ab}\),

\[
h_{ab} = g_{ab} + \xi_a \xi_b,
\]

(2.3)

which satisfies

\[
h_{ab} \xi^a = 0, \quad h_{ab} \xi^b = 0, \quad h_{ab} g_{cd} \xi^c \xi^d = 0, \quad h_{ab} \xi^b = 0.
\]

(2.4)

Here note that \(h_{ab}\) is the metric on the hypersurfaces orthogonal to \(\xi^a\). Moreover, we can define projection operator \(h_{ab}^\prime\) as

\[
h_{ab}^\prime = g_{ab}^\prime h_{ch}.
\]

(2.5)

This operator fulfills

\[
h_{ab} h_{cb}^\prime = h_{ab} h_{bc} = h_{ac}^\prime, \quad h_{ab} h_{bc}^\prime = h_{ad}, \quad h_{ab} h_{bc}^\prime = 0.
\]

(2.6)

Now, we decompose \(B_{ab}\) into three pieces

\[
B_{ab} = \frac{1}{3} \theta h_{ab} + \sigma_{ab} + \omega_{ab}.
\]

(2.7)

In the above equation, we defined kinematical quantities of the timelike geodesic congruence in the Bianchi type I Universe, such that, the expansion \(\theta\), the shear \(\sigma_{ab}\), and the rotation \(\omega_{ab}\), that are given by

\[
\theta = B_{cb} h_{cb}, \quad \sigma_{ab} = B_{(ab)} - \frac{1}{3} \theta h_{ab}, \quad \omega_{ab} = B_{(ab)}.
\]

(2.8)

We then find

\[
\sigma_{ab} h_{ab} = 0, \quad \omega_{ab} h_{ab} = 0,
\]

(2.9)

and

\[
\xi^c \nabla_c B_{ab} = -B^c_{b} B_{ac} + R_{chab} \xi^c g_{cd}.
\]

(2.10)

Exploiting Equation (2.11), one arrives at

\[
\xi^c \nabla_c \theta = -\frac{1}{3} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - R_{ab} \xi^a \xi^b,
\]

(2.11)

\[
\xi^c \nabla_c \omega_{ab} = -\frac{2}{3} \theta (\omega_{ab} + \xi^c g_{ab} \xi^d) - 2 \sigma^c_{[ab]} \omega^{cd},
\]

(2.12)

\[
\xi^c \nabla_c \sigma_{ab} = -\frac{2}{9} \theta^2 \xi_{[ab]} - \frac{2}{3} \theta h_{(a} \sigma_{bc)} - \sigma_{ac} \sigma^c_b - \omega_{ab} \omega^b + \left( R_{(cd)[ab]} + \frac{2}{3} g_{ab} R_{cd} \right) \xi^c g_{[cd]} + \frac{1}{3} g_{ab} (\sigma_{cd} \sigma^{cd} - \omega_{cd} \omega^{cd})
\]

\[
-\frac{1}{3} \theta^2 (\xi_{(a} \nabla_{c)} \xi^c) - \frac{1}{3} \xi_{(a} \xi_{b} \nabla^c \xi^c) \theta, \quad \xi_{(a} \xi_{b)} = 0,
\]

(2.13)

We consider the comoving coordinates \((t, x, y, z)\) such that the metric takes the form

\[
d s^2 = -dt^2 + X^2(t) dx^2 + Y^2(t) dy^2 + Z^2(t) dz^2.
\]

(2.14)

We redefine the “scale factor” \(l(t)\) by \(l(t) = XYZ\). Here, \(X(t)\), \(Y(t)\), and \(Z(t)\) are the scale factors of \(x\), \(y\), and \(z\) directions, respectively. If \(X = Y = Z\), we get the usual Friedmann-Robertson-Walker (FRW) space-time.

Now, we consider the Kasner vacuum Universe – the “vacuum” case of the Bianchi type I Universe – which satisfies \(r = 0\) and \(P = 0\), where \(r\) and \(P\) are the mass-energy density and pressure of the fluid as measured in its rest frame, respectively [Wald 1984, Misner et al. 1973]. Then, \(X, Y,\) and \(Z\) become distinct functions of \(t\). The metric and the initial conditions of the Kasner vacuum Universe, except \(r\) and \(P\), are exactly the same as those of the Bianchi type I Universe. Now, we set

\[
\xi^a \nabla_a \xi^b = 0, \quad \omega_{ab} = 0, \quad R_{(ab)} = \theta_{ab} = 0, \quad \sigma_{ab} = 0.
\]

(2.15)

The meaningful results of the field equations are [Ellis and van Elst 1999]

\[
\frac{3l^2}{l^3} + 2 \sigma^2 = 0, \quad \left( l^3 \sigma^2 \right)^2 = 0,
\]

(2.16)

where

\[
l = \xi^a \nabla_a l, \quad \sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab} = \frac{\Sigma^2}{l^3}.
\]

(2.17)

From Equations (2.17), we have

\[
\sigma_{ab} = \frac{\Sigma_{ab}}{l^3}, \quad 2 \Sigma^2 = (\Sigma_{11})^2 + (\Sigma_{22})^2 + (\Sigma_{33})^2.
\]

(2.18)

In this case, from Equations (2.9), in orthonormal bases, the symmetric part \(B_{(ab)}\) of \(B_{ab}\) is given by

\[
B_{(ab)} = \sigma_{ab} + \frac{1}{3} \theta h_{ab} + \frac{2}{3} \delta_{ab} + \frac{i}{l} \delta_{ab}.
\]

(2.19)

We then find

\[
X = \frac{\Sigma_{11}}{l^3} + \hat{i}, \quad Y = \frac{\Sigma_{22}}{l^3} + \hat{i}, \quad Z = \frac{\Sigma_{33}}{l^3} + \hat{i}, \quad \frac{\Sigma_{11}}{l^3} + \hat{i},
\]

(2.20)

Then, we can obtain

\[
\frac{X}{X} = \frac{\Sigma_{11}}{l^3} + \hat{i}, \quad \frac{Y}{Y} = \frac{\Sigma_{22}}{l^3} + \hat{i}, \quad \frac{Z}{Z} = \frac{\Sigma_{33}}{l^3} + \hat{i}.
\]

(2.21)
We can also obtain
\[ \Sigma_{11} = \frac{2}{\sqrt{3}} \Sigma \sin \alpha, \]
\[ \Sigma_{22} = \frac{2}{\sqrt{3}} \Sigma \sin \left( \alpha + \frac{2}{3} \pi \right), \]
\[ \Sigma_{33} = \frac{2}{\sqrt{3}} \Sigma \sin \left( \alpha + \frac{4}{3} \pi \right), \] (2.23)

where \( \alpha \left( -\pi/6 < \alpha \leq \pi/2 \right) \) is a constant determining the direction in which the most rapid expansion takes place.

Let us consider the time-reverse of the model. For general values of \( \alpha \), i.e., \( -\pi/6 < \alpha < \pi/2 \), the term \( 1 + 2 \sin(\alpha + (4\pi/3)) \) will be negative. Thus, if we consider the forward direction of time, we have a “cigar” singularity: matter collapses in along the \( z \)-axis from infinity, halts, and then starts re-expanding, while in the \( x \)- and \( y \)-directions the matter expands at all times. In the exceptional case (i.e., \( \alpha = \pi/2 \)), the terms \( 1 + 2 \sin(\alpha + (2\pi/3)) \) and \( 1 + 2 \sin(\alpha + (4\pi/3)) \) both vanish. Then we have a “pancake” singularity: matter expands in all directions, starting from an indefinitely high expansion rate in the \( x \)-direction but from zero expansion rates in the \( y \)- and \( z \)-directions (e.g., [Ellis 2009, Wainwright and Ellis 1997, Hawking and Ellis 1973]).

In the “non-vacuum” (i.e., \( \rho \neq 0 \) and \( P \neq 0 \)) anisotropic Bianchi type I Universe, we can assume the perfect fluid. For a perfect fluid, the energy-momentum tensor is given by
\[ T_{ab} = \rho \, \xi_{a} \xi_{b} + P \left( g_{ab} + \xi_{a} \xi_{b} \right). \] (2.24)

### 3 Timelike Geodesic Curves in the Bianchi Type I Universe

The motion types of timelike geodesic congruence in the Universe can be described in terms of the expansion, the shear, and the rotation. In this section, we describe the expansion rate of timelike geodesic congruence and the shearing motions in the early anisotropic Universe.

In the Bianchi type I space-time, Equations (2.12)-(2.14) become
\[ \xi^{a} \nabla_{c} g_{ab} = -\frac{1}{3} \theta^{2} - \sigma_{ab} \sigma^{ab} - R_{ab} \xi^{a} \xi^{b}, \] (3.1)
\[ \xi^{c} \nabla_{c} \sigma_{ab} = 0, \] (3.2)
\[ \xi^{c} \nabla_{c} \sigma_{ab} = -\frac{1}{9} \theta^{2} \xi_{a} \xi_{b} + \frac{2}{3} \theta h_{(c}^{a} \sigma_{b)c} - \sigma_{ac} \sigma_{b}^{c} + \left( R_{(ab)}{}^{cd} + \frac{1}{3} g_{ab} R_{cd} \right) \xi^{a} \xi^{b} + \frac{1}{3} R_{(a}{}^{cd} \sigma^{b)c} \xi^{a} \xi^{b}, \] (3.3)

Since,
\[ \xi^{a} \nabla_{c} \theta = \dot{\theta}, \] (3.4)

and from Equation (3.1), we can obtain a Raychaudhuri type equation
\[ \dot{\theta} = -\frac{1}{3} \theta^{2} - \sigma_{ab} \sigma^{ab} - R_{ab} \xi^{a} \xi^{b} = -\frac{3 l^{2}}{T^{2}} - \frac{2 \Sigma^{2}}{T^{6}} - R_{ab} \xi^{a} \xi^{b}, \] (3.5)

by using the fact that \( \dot{\theta} = 3 l/1 \) and \( \sigma_{ab} = \Sigma_{ab}/l^{3} \). We now assume that the strong energy condition
\[ R_{ab} \xi^{a} \xi^{b} = 8 \pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right) \xi^{a} \xi^{b} \geq 0, \] (3.6)

where \( T_{ab} \) and \( T \) are the energy-momentum tensor and its trace, respectively, if \( R_{ab} \xi^{a} \xi^{b} = 0 \), then this condition corresponds to the strong energy condition for the Kasner “vacuum” (i.e., \( T_{ab} = 0 \) and \( T = 0 \)) Universe. The Raychaudhuri type equation (3.5) then has a solution of the form
\[ \frac{1}{\theta(t)} - \frac{1}{\theta(0)} \geq \frac{1}{2} \left[ \tau + \frac{2}{3} \Sigma^{2} \int_{0}^{\tau} \frac{1}{l^{6} T^{2}} \, dt \right], \] (3.7)

where \( \tau \) is the proper time. We assume that \( \theta(0) \) is negative so that the congruence is initially converging as in the point particle case. The inequality (3.7) implies that \( \theta(t) \) must pass through the singularity within a proper time (see Figure 2),
\[ \tau \leq \frac{3}{\left| \theta(0) \right|} \int_{0}^{\tau} \frac{2}{l^{2} T^{2}} \, dt, \] (3.8)

since,
\[ \frac{3}{\left| \theta(0) \right|} > 0. \] (3.9)

If we choose \( \tau > 0 \) such that,
\[ \frac{3}{\left| \theta(0) \right|} \geq \frac{2}{3} \Sigma^{2} \int_{0}^{\tau} \frac{1}{l^{4} T^{2}} \, dt, \] (3.10)

then the right-hand side of Equation (3.8) is greater than or equal to zero.

For a perfect fluid, the strong energy condition (3.6) yields one inequality equation
\[ 4 \pi (\rho + 3P) \geq 0. \] (3.11)

Then we have the following two inequalities
\[ \rho + 3P \geq 0, \quad \rho + P \geq 0. \] (3.12)

If we neglect the shearing motions (i.e., \( \sigma_{ab} = 0 \)), then we have the differential inequality equation
\[ \frac{d \theta}{d\tau} + \frac{1}{3} \theta^{2} \leq 0, \] (3.13)

which has a solution in the following form
\[ \frac{1}{\theta(\tau)} \geq \frac{1}{\theta(0)} + \frac{1}{3} \tau. \] (3.14)

If we assume that \( \theta(0) \) is negative, then the expansion \( \theta(\tau) \) must go to the negative infinity along that geodesic within a proper time
\[ \tau \leq \frac{3}{\left| \theta(0) \right|}. \] (3.15)
whose consequence coincides with that of Hawking and Penrose [Hawking and Penrose 1970].

From now on, we will consider the shear of timelike geodesic congruence in the early anisotropic Universe. Using Equation (3.3), we obtain an evolution equation for the shear,

\[
\frac{d\sigma_{ab}}{dt} = -\frac{1}{9} \theta^a \xi^b + \frac{2}{3} \left[ \theta \left( \frac{1}{3} h^c_{(a} \xi_{b)c} \right) \right] - \frac{1}{3} \Sigma^a \Sigma_b + R_{c(ab)} \xi^c \xi^b + \frac{2}{3} \theta^a \xi^b \xi (\sqrt{\sigma} \nabla_b \xi^c) \Sigma^c - \frac{1}{3} \Sigma_a \xi_b \xi^c \nabla_c \theta = 0.
\]

(3.16)

Substituting Equation (3.5) into (3.16), we obtain

\[
\frac{d\sigma_{ab}}{dt} = -\frac{2}{3} \theta^a \xi^b + \frac{2}{3} \left[ \theta \left( \frac{1}{3} h^c_{(a} \xi_{b)c} \right) \right] - \frac{1}{3} \Sigma^a \Sigma_b + R_{c(ab)} \xi^c \xi^b + \frac{2}{3} \theta^a \xi^b \xi (\sqrt{\sigma} \nabla_b \xi^c) \Sigma^c - \frac{1}{3} \Sigma_a \xi_b \xi^c \nabla_c \theta = 0.
\]

(4.1)

In the standard point-particle inflationary cosmology, the influence of the shear on the ensuing Universe evolution are negligible to produce the homogeneous and isotropic Universe features. It is worthy to note that in the homogeneous and “anisotropic” Universe, one can have the condition \(\sigma_{ab} \neq 0\). Here the non-vanishing \(\sigma_{ab}\) evolves and dominates in the early anisotropic Universe.

We assume the following condition to investigate the evolution of shear in the “extremely early” Universe (e.g., in the inflationary epoch),

\[
\frac{d\sigma_{ab}}{dt} \approx -\frac{1}{3} \Sigma^c \Sigma_b + \frac{2}{3} \Sigma^c \Sigma_b = 0
\]

(3.18)

where we left only two \(t^{-6}\) terms. If we substitute Equations (2.23) into (3.18), then we finally get

\[
\frac{d\sigma_{ab}}{dt} \approx -\frac{2}{3} \Sigma^c \Sigma_b \left( -1 + \frac{1}{X^2} + \frac{1}{Y^2} + \frac{1}{Z^2} \right).
\]

(3.19)

By integrating Equation (3.19), we have the following approximation on the shear,

\[
\sigma_{ab}(\tau) \approx \sigma_{ab}(0) - 2 \Sigma^2 \int_0^\tau \left( \frac{1}{t^6} \right) dt.
\]

(3.20)

Suppose one considers the time-reverse of the model, then the evolution equation for the shear depends on the scale factor \(l(t)\) in the “extremely early” anisotropic Universe.

### 4 Null Geodesic Curves in the Bianchi Type I Universe

In this section, we consider the evolution of the vectors in a 2−dimensional subspace of spatial vectors normal to the null tangent vector field \(k^a = (\partial / \partial \lambda)^a\), where \(\lambda\) is the affine parameter, and to an auxiliary null vector \(l^a\) which points in the opposite spatial direction to \(k^a\), normalized by [Carroll 2004]

\[
l^a k_a = -1,
\]

(4.1)

and is parallel transported, namely,

\[
k^a \nabla_b l^b = 0.
\]

(4.2)

The spatial vectors in the 2−dimensional subspace are then orthogonal to both \(k^a\) and \(l^a\). We now introduce the metric \(n_{ab}\),

\[
n_{ab} = g_{ab} + k_a k_b + l_a k_b.
\]

(4.3)

Similar to the timelike case, we introduce tensor fields

\[
B_{ab} = \nabla_b k_a,
\]

(4.4)

satisfying the identity

\[
B_{ab} k^b = 0.
\]

(4.5)

We decompose \(B_{ab}\) into three pieces

\[
B_{ab} = \frac{1}{2} \partial n_{ab} + \sigma_{ab} + \omega_{ab},
\]

(4.6)

where the expansion, shear, and rotation of the null geodesic congruence along the affine direction are defined as

\[
\theta = B_{ab} n^{ab}, \quad \sigma_{ab} = B_{(ab)} - \frac{1}{2} \partial n_{ab}, \quad \omega_{ab} = B_{[ab]}.
\]

(4.7)

The metric \(n_{ab}\) also satisfies the identities

\[
\sigma_{ab} n^{ab} = \omega_{ab} n^{ab} = 0,
\]

(4.8)

and

\[
n_{ab} k^a n_{bc} = n_{ab} k^a n_{bc} = n_{ab} k^a n_{bc} = 0,
\]

(4.9)

We define \(n_b^a\) as

\[
n_b^a = g^{ac} n_{cb} = \delta^a_b + k^a l_b + l^a k_b.
\]

(4.10)

which fulfills the following identities

\[
k^a \nabla_c n_b^a = 0,
\]

(4.11)

and

\[
n_b^a k_b = n_b^a k_b = n_b^a k_b = n_b^a k_b = 0,
\]

(4.12)

Then, we have the identities

\[
B_{ab} k^a = 0, \quad \sigma_{ab} k^b = 0, \quad \omega_{ab} k^b = 0,
\]

(4.13)

and

\[
k^a \nabla_c B_{ab} = -B_{b}^c B_{ac} + R_{cad} k^d.
\]

(4.14)
Using Equation (4.14), we get
\[ k^a \nabla_a \theta = -\frac{1}{2} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - R_{ab} k^a k^b, \] (4.15)
\[ k^c \nabla_c \omega_{ab} = -\theta (\omega_{ab} - k^c k_{abc} + 2 \sigma^c_{\beta} \omega_{\beta c}), \] (4.16)
\[ k^c \nabla_c \sigma_{ab} = -\frac{1}{4} \theta^2 k_{abc} - \theta h_{(a}(\sigma)_{bc) - \sigma_{ac} \sigma_b - \omega_{ac} \omega_b + \left( R_{(ab) c d} + \frac{1}{2} g_{ab} R_{cd} \right) k^d k^c + \frac{1}{2} g_{abc} \sigma_{cd} \sigma^{ac} - \omega_{ac} \omega^{cd} - \frac{1}{2} \theta k^c k^d (\nabla_c k^b), \] (4.17)
In the Bianchi type I space-time, Equations (4.15)-(4.17) become
\[ k^a \nabla_a \theta = -\frac{1}{2} \theta^2 - \sigma_{ab} \sigma^{ab} - R_{ab} k^a k^b, \] (4.18)
\[ k^c \nabla_c \omega_{ab} = 0, \] (4.19)
\[ k^c \nabla_c \sigma_{ab} = -\frac{1}{4} \theta^2 k_{abc} - \theta h_{(a}(\sigma)_{bc) - \sigma_{ac} \sigma_b + \sigma_{bc} \sigma^c - \frac{1}{2} \theta k^c k^d (\nabla_c k^b), \] (4.20)
where
\[ k^a \nabla_a \theta = \frac{d \theta}{d \lambda}. \] (4.21)
From Equation (4.18), we can obtain the Raychaudhuri type equation for null case,
\[ \frac{d \theta}{d \lambda} = -\frac{1}{2} \theta^2 - \sigma_{ab} \sigma^{ab} - R_{ab} k^a k^b = -\frac{9}{4} \mathbf{I} \theta^2 - 2 \Sigma^2 - R_{ab} k^a k^b, \] (4.22)
which resembles Equation (3.5).

By Einstein’s equation, we obtain
\[ R_{ab} k^a k^b = 8 \pi T_{ab} k^a k^b, \] (4.23)
where \( T_{ab} \) is the energy-momentum tensor. Using the laws of physics, we may assume that the energy density \( T_{ab} \geq 0 \) for timelike case. By continuity, we may also assume \( T_{ab} k^a k^b \geq 0 \) for null case.

The Raychaudhuri type equation (4.22) for null case then has a solution of the form
\[ \frac{1}{\theta(\lambda)} \geq \frac{1}{\theta(0)} + \frac{1}{9 \Sigma^2} \int_0^\lambda \frac{1}{14 \lambda^2} d \lambda, \] (4.24)
where \( \theta(0) \) is the initial value of \( \theta \) at \( \lambda = 0 \). We assume again that \( \theta(0) \) is negative. The inequality (4.24) then implies that \( \theta \) must pass through the singularity within an affine length
\[ \lambda \leq 2 \frac{2}{\theta(0)} - \frac{4 \Sigma^2}{9} \int_0^\lambda \frac{1}{14 \lambda^2} d \lambda \leq \frac{2}{\theta(0)}. \] (4.25)

Using Equation (4.20), we obtain an evolution equation for the shear of null geodesic congruence,
\[ \frac{d \sigma_{ab}}{d \lambda} = -\frac{9}{4 \mathbf{I}} \theta (\omega_{ab} - \omega_{ac} \omega^c) + \left( R_{(ab) c d} + \frac{1}{2} g_{ab} R_{cd} \right) k^d k^c - \frac{1}{16} \Sigma^c g_{ab} R_{cd} k^d k^c - \frac{3}{2} k^c k^d (\nabla_c k^b), \] (4.26)
where
\[ k^c \nabla_c \theta = \theta / d \lambda. \] (4.27)
Substituting Equation (4.22) into (4.26), we obtain
\[ \frac{d \sigma_{ab}}{d \lambda} = -\frac{3}{16} \Sigma^c g_{ab} R_{cd} + \left( R_{(ab) c d} + \frac{1}{2} g_{ab} R_{cd} \right) k^d k^c - \frac{1}{16} \Sigma^c g_{ab} R_{cd} k^d k^c - \frac{3}{2} k^c k^d (\nabla_c k^b), \] (4.28)
where we left only two \( I = 0 \) terms. If we substitute Equations (2.23) into (4.28), then we finally get
\[ \frac{d \sigma_{ab}}{d \lambda} \approx -\frac{1}{16} \Sigma^2 + \frac{1}{16} \Sigma^2 (g_{ab} + k_a k_b), \] (4.29)
By integrating Equation (4.29), we have the following approximation on the shear,
\[ \sigma_{ab}(\lambda) \approx \sigma_{ab}(0) + \frac{1}{16} \Sigma^2 \int_0^\lambda \frac{1}{16} \Sigma^2 (g_{ab} - 2 g_{ab}) d \lambda. \] (4.30)

5 Conclusions

In the standard point-particle inflationary cosmology, the influence of the shear on the ensuing Universe evolution are negligible to produce the homogeneous and isotropic Universe features. It is worthy to note that in the homogeneous and “anisotropic” Universe, one can have the condition \( \sigma_{ab} \neq 0 \). Here the non-vanishing \( \sigma_{ab} \) evolves and dominates in the early anisotropic Universe. In the “extremely early” Universe, if the scale factor \( l(t) \) increases, then the evolution equation for the shear, \( d \sigma_{ab} / dt \), decreases. This means that the influence of the shear decreases as the scale of the Universe increases.

Recently, the two detectors of the LIGO (Laser Interferometer Gravitational-Wave Observatory) simultaneously observed a transitory gravitational-wave signal which directly
matches the waveform predicted by Einstein’s general relativity for the inspiral and merger of a pair of stellar-mass black holes and the ringdown of the resulting single black hole [LIGO PRL 2016]. In the source frame, the initial black hole masses are $36M_\odot$ and $29M_\odot$, and the final black hole mass is $62M_\odot$, with $3.0M_\odot c^2$ radiated in gravitational waves [LIGO APJL 2016]. This detection is the first step to discovery of the gravitational wave background (GWB) radiation. Unfortunately, the LIGO’s detection sensitivity at low frequencies is limited by the largest practical arm lengths, by terrestrial gravity gradient noise, and by interference from nearby moving objects. Future gravitational wave observatories like the Evolved Laser Interferometer Space Antenna (eLISA), might show “primordial” gravitational waves generated during cosmological inflation, relics of the early Universe, up to less than a second of the Big Bang [eLISA 2012].

If we can observe any anisotropy of the “primordial” gravitational wave background radiation, then the evolution equation for the shear in this article will be responsible to the future observational anticipations for the gravitational waves.

There are many possibilities to develop our model of the early anisotropic Universe. For instance, we may further consider the rotational motions of the early Universe because of the initial conditions of the Bianchi type I Universe. Instead of using the Bianchi type I, we may consider other anisotropic Universe models that might describe the rotational motions. Applying string theory [Polchinski 1998, Green et al. 1987] to anisotropic Universe models woul further lead us to consider the rotational motions and geodesic surface congruence in the early Universe. There are previous results about the anisotropic Universe with cosmic strings and bulk viscosity (e.g., [Tripathy et al. 2008, Tripathy et al. 2009a, Tripathy et al. 2009b, Tripathy et al. 2010, Pradhan and Chouhan 2011]) and Bianchi type I string cosmological model in general relativity.

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