Application of the topological sensitivity method to the reconstruction of plasma equilibrium domain in a tokamak

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Abstract
The topological sensitivity method is an optimization technique used in different inverse problem solutions. In this work, we adapt this method to the identification of plasma domain in a Tokamak. An asymptotic expansion of a considered shape function is established and used to solve this inverse problem. Finally, a numerical algorithm is developed and tested in different configurations.

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1 Introduction
The Tokamak is a type of fusion reactor and a method for building these reactors. It is a station that uses a strong magnetic field to confine the hot plasma in the shape of a torus (see Fig. 1). This method works by raising the temperature of the plasma to the point of nuclear fusion. Since the plasma has a very high temperature, it must be separated and removed and made not to touch the components of the reactor, otherwise it will be destroyed and this will cause it to cool and interrupt the fusion reaction. In reactors based on the Tokamak method, the plasma is made to float in the middle of the reactor without touching any of its parts by means of magnets that keep the plasma in a circular path.

This method is still being researched and experimented and has not reached the stage of economic exploitation, yet. In the future, the goal is to produce energy using this technique. The main problem to reach high performance is to construct the plasma magnetic equilibrium domain. This problem has been investigated in [2–4] using parametric optimization or control theory. A new idea using the topological gradient technique is considered in [5–13]. This technique consists in reconstructing the plasma domain by inserting some holes inside a fixed initial one. The position and form of these holes are characterized by an asymptotic expansion of a considered cost function.
The paper is organized as follows. In the next section, we present the model describing the plasma equilibrium and the formulation of the studied problem. Section 3 is devoted to the plasma reconstruction technique using the topological gradient method. In Sect. 4, the plasma reconstruction algorithm is described and some numerical results are presented.

2 Formulation of the problem

Consider the two-dimensional domain $\Omega$ representing the vacuum vessel containing a plasma domain $\Omega_p$ and $\Omega_v = \Omega \setminus \partial \Omega_p$. The plasma equilibrium solves in $\Omega_v$ the equation $G\phi = 0$, where $G = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right) - \frac{\partial}{\partial z} \left( \frac{1}{r^2} \frac{\partial \phi}{\partial z} \right)$ is the Grad–Shafranov operator and $\phi$ is the poloidal flux (see [14]).

We aim to determine the location of $\Gamma_p$, the boundary of $\Omega_p$, from over-specified boundary data on $\Gamma = \partial \Omega$. In this inverse problem, $\phi$ satisfies

$$
\begin{cases}
G\phi = 0 & \text{in } \Omega_v, \\
\frac{1}{r} \frac{\partial \phi}{\partial n} = \Phi & \text{on } \Gamma, \\
\phi = \phi_m & \text{on } \Gamma_r, \\
\phi = 0 & \text{on } \Gamma_p.
\end{cases}
$$

We remark that since $\Gamma_p$ is unknown also $\Omega_v$ is unknown, which makes the problem ill posed in the sense of Hadamard.

In order to determine the location of the unknown plasma boundary $\Gamma_p$, we propose the following formulation for the considered inverse problem:

Knowing the magnetic field $\Phi$ and the poloidal flux $\phi_m$ on the boundary $\Gamma$, the idea consists in identifying the unknown plasma boundary $\Gamma_p = \partial \Omega_p$ where $\Omega_p \subset \Omega$ is the optimal
solution of the topological optimization problem

$$\min_{D \subset \Omega} T(\Omega \setminus \overline{D}).$$  \hspace{1cm} (2)$$

Here $T$ is the boundary tracking function defined by

$$T(\Omega \setminus \overline{D}) = \int_{\Gamma} |\phi_D - \phi_m|^2 \, ds$$  \hspace{1cm} (3)$$

with $\phi_D$ being the solution to

$$\begin{cases}
G \phi_D = 0 & \text{in } \Omega \setminus \overline{\chi}_{z,\epsilon}, \\
\frac{1}{r} \frac{\partial \phi_D}{\partial n} = \Phi & \text{on } \Gamma, \\
\phi_D = 0 & \text{on } \partial D.
\end{cases}$$  \hspace{1cm} (4)$$

To solve problem (2), we use the topological gradient method. It corresponds to developing an asymptotic expansion of the function $T$ as

$$T(\Omega \setminus \overline{\chi}_{z,\epsilon}) - T(\Omega) = f(\epsilon)\delta T(z) + o(f(\epsilon)),$$

where $f(\epsilon) > 0$ with $\lim_{\epsilon \to 0} f(\epsilon) = 0$ and $\chi_{z,\epsilon}$ is a geometric perturbation created near the point $z = (z_1, z_2) \in \Omega$ having a small size $\epsilon > 0$ (chosen in such a way that $\chi_{z,\epsilon} \subset \Omega$) and the shape $\chi_{z,\epsilon} = z + \epsilon \chi$ with $\chi \subset \mathbb{R}^2$ is a given, regular, and bounded domain containing the origin.

Further, $\delta T$ is called the topological gradient and the minimum of $T$ is obtained when $\delta T$ is the most negative.

Using this technique, the optimal design of the unknown plasma $\Omega_p$ is constructed using a level set curve of the scalar function $\delta T$, namely

$$\Omega_p = \{ x \in \Omega \text{ such that } \delta T \leq c \},$$

where $c$ is a negative constant, chosen in such a way that the shape function $T$ decreases as much as possible.

3 Plasma reconstruction technique

Using the axisymmetric configuration and a horizontal cut, the first formulation can be rewritten as follows: find the unknown domain $\Omega_p$ occupied by the plasma as the optimal solution to the optimization problem (2), where $\phi_D$ is solution to the anisotropic system

$$\begin{cases}
-\text{div}(\gamma(x) \nabla \phi_D) = 0 & \text{in } \Omega \setminus \overline{D}, \\
\gamma(x) \nabla \phi_D \cdot n = \Phi & \text{on } \Gamma, \\
\phi_D = 0 & \text{on } \partial D,
\end{cases}$$

with $\gamma$ being a scalar positive function defined by

$$\gamma(x) = \frac{1}{|x_1|}, \quad \forall x = (x_1, x_2) \in \Omega \setminus \overline{D}.$$
In the presence of the perturbation $\chi_{z,\varepsilon}$, we have

$$T(\Omega \setminus \chi_{z,\varepsilon}) = \int_\Gamma |\phi_{\varepsilon} - \phi_m|^2 \, ds$$

where $\phi_{\varepsilon}$ is the solution to

$$\begin{cases}
-\text{div}(\gamma(x) \nabla \phi_{\varepsilon}) = 0 & \text{in } \Omega_{z,\varepsilon}, \\
\gamma(x) \nabla \phi_{\varepsilon} \cdot \mathbf{n} = \Phi & \text{on } \Gamma, \\
\phi_{\varepsilon} = 0 & \text{on } \partial \chi_{z,\varepsilon},
\end{cases} \quad (5)$$

with $\Omega_{z,\varepsilon}$ being the perturbed domain defined by $\Omega_{z,\varepsilon} = \Omega \setminus \chi_{z,\varepsilon}$.

One can establish the following asymptotic expansion for the shape function $T$ (see [15]).

**Theorem 1** The shape function $T$ admits the following asymptotic expansion:

$$T(\Omega \setminus \chi_{z,\varepsilon}) - T(\Omega) = \frac{-1}{\log(\varepsilon)} \delta T(z) + o\left(\frac{-1}{\log(\varepsilon)}\right),$$

where $\delta T$ is the topological gradient given by

$$\delta T(z) = 2\pi \gamma(z) \phi_0(z) \psi_0(z), \quad z \in \Omega,$$

with $\phi_0(z)$ being the solution to problem (5) for $\varepsilon = 0$ and $\psi_0$ the solution of the associated adjoint problem.

**4 Numerical tests**

For the numerical tests, we consider two cases of computational domain $\Omega$. The unknown plasma boundary is defined by the level set curve of the topological gradient

$$\delta T(x_1, x_2) = \frac{2\pi}{x_1} \phi_0(x_1, x_2) \psi_0(x_1, x_2), \quad \forall (x_1, x_2) \in \Omega. \quad (6)$$

**4.1 Plasma reconstruction algorithm**

The main steps of our numerical algorithm are the following.

**The reconstruction algorithm**

- Solve the direct and adjoint problems in $\Omega$,
- Compute the topological gradient $\delta T$ defined in (6),
- Determine the plasma location

$$\Omega_P = \{(x_1, x_2) \in \Omega; \delta T(x_1, x_2) \leq (1 - \varepsilon)\delta T_{\text{min}} < 0\},$$

where $\delta T_{\text{min}} = \min_{(x_1, x_2) \in \Omega} \delta T(x_1, x_2)$ and $\varepsilon \in [0, 1]$ is chosen in such a way that the function $T$ decreases as much as possible.

Next, we apply this reconstruction procedure for two numerical tests.
4.2 First test

In this case we have used the following data:
- The vacuum vessel region is defined by the disc $\Omega = B(C_1, 1)$, with $C_1 = (2, 0)$.
- The exact plasma domain is defined by the disc $\Omega_{p}^{ex} = B(C_2, 0.2)$, with $C_2 = (3/2, 0)$.
- The Dirichlet and Neumann boundary data are given by

\[
\phi_m(x, y) = \frac{4-x-y}{x} \sin(0.5 + x + y), \quad (x, y) \in \Gamma,
\]
\[
\Phi(x, y) = \frac{1}{x} \frac{\partial \phi_m}{\partial n}(x, y), \quad (x, y) \in \Gamma.
\]

In Fig. 2, we present the obtained isovalues of the solution $\phi_0$.

The isovalues of the topological gradient $\delta T$ are described in Fig. 3.

To evaluate the accuracy of our approach, we introduce the following error function which defines the Hausdorff distance between the exact $\Omega_{p}^{ex}$ and obtained $\Omega_{p}^{e}$ plasma domain

\[
er(\varepsilon) = \frac{\text{mes}(\Omega_{p}^{ex} \cup \Omega_{p}^{e}) - \text{mes}(\Omega_{p}^{ex} \cap \Omega_{p}^{e})}{\text{mes}(\Omega_{p}^{ex})}
\]

with

\[
\Omega_{p}^{e} = \{ X \in \Omega; \delta T(X) < (1 - \varepsilon)\delta T_{\text{min}} \}.
\]

The variation of the error function $er$ is illustrated in Fig. 4. As one can see in Fig. 4, the optimal choice of the parameter $\varepsilon$ is $\varepsilon = 0.88$. In this test, we identify the plasma domain $\Omega_{p}^{ex}$. 
4.3 Second test
In this case we have used the following data:
- $\Omega = \left[ \frac{\pi}{3}, \frac{2\pi}{3} \right] \times \left[ 0, \frac{\pi}{3} \right]$.
- $\Omega^{ex}_2 = B(C_2, 0.2)$, with $C_2 = \left( \frac{\pi}{7}, \frac{\pi}{8} \right)$. 
- The Dirichlet and Neumann boundary data are given by

\[
\phi_m(x, y) = 2e^x \cos(y) - e^y \sin(x) - 1.75, \ (x, y) \in \Gamma,
\]

\[
\Phi(x, y) = \frac{1}{x} \frac{\partial \phi_m}{\partial n}(x, y), (x, y) \in \Gamma.
\]
The result of this test are described in Figs. 5, 6, and 7. In Fig. 5, we plot the isovalues of the solution $\phi_0$.

Figure 6 describes the isovalues of the topological gradient $\delta T$. The variation of the error function $er$ describing the Hausdorff distance between the exact $\Omega_{P}^{ex}$ and obtained $\Omega_{P}$ plasma domain is illustrated in Fig. 7. The optimal choice of the parameter $\varepsilon$ corresponds to $\varepsilon = 0.82$.

We remark that the plasma domain $\Omega_{P}^{ex}$ has been well identified in one iteration.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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