ON THE FINITENESS OF THE CLASSIFYING SPACE OF
DIFFEOMORPHISMS OF REDUCIBLE THREE MANIFOLDS

SAM NARIMAN

Abstract. Kontsevich ([Kir95, Problem 3.48]) conjectured that BDiff(M, rel ∂) has
the homotopy type of a finite CW complex for all compact 3-manifolds with non-
empty boundary. Hatcher-McCullough ([HM97]) proved this conjecture when M
is irreducible. We prove the homological version of Kontsevich’s conjecture to
show that BDiff(M, rel ∂) has finitely many nonzero homology groups each finitely
generated when M is a connected sum of irreducible 3-manifolds that each has a
nontrivial boundary.

1. Introduction

For a closed surface Σg of genus g > 1, it is well-known that the classifying space
BDiff(Σg) is rationally equivalent to M_g, the moduli space of Riemann surfaces
of genus g. Therefore, in particular, the rational homology groups of BDiff(Σg)
vanish above a certain degree, and in fact, more precisely they vanish above degree
4g − 5, which is the virtual cohomological dimension of the mapping class group
Mod(Σg). And for a surface Σg,k with k > 0 boundary components, the classifying
space BDiff(Σg,k, rel ∂) is in fact homotopy equivalent to the moduli space M_{g,k}.
Therefore, BDiff(Σg,k, rel ∂) has the homotopy type of a finite-dimensional CW-
complex.

Similarly, Kontsevich ([Kir95, Problem 3.48]) conjectured for compact
3-manifold M with non-empty boundary, the classifying space BDiff(M, rel ∂) has
a finite-dimensional model. This conjecture is known to hold for irreducible
3-manifolds with non-empty boundary ([HM97]). In this paper, we shall prove
the homological finiteness of these classifying spaces for reducible 3 manifolds
with a condition on its boundary.

Throughout this paper, for brevity, we write Diff(M, rel ∂) and Homeo(M, rel ∂)
to denote the smooth orientation preserving diffeomorphisms and orientation pre-
serving homeomorphisms respectively whose supports are away from the bound-
ary ∂M and in general when we use rel X in the diffeomorphism group, for some
X ⊂ M, we mean those diffeomorphisms or homeomorphisms whose supports are
away from X.

Theorem 1.1. Let M be an orientable reducible 3-manifold that is a connected sum of
irreducible 3-manifolds that each has a nontrivial boundary. Then the classifying space
BDiff(M, rel ∂) has finitely many nonzero homology groups which are each finitely gen-
erated.

In the irreducible case, the homotopy type of the group Diff(M) is very well
studied. When M admits one of Thurston’s geometries, there has been an en-
compassing program known as the generalized Smale’s conjecture that relates the
homotopy type of Diff(M) with the isometry group of the corresponding geometry
(for more details and history, see the discussions in Problem 3.47 in [Kir95] and
Sections 1.2 and 1.3 in [HKMR12]). For S^3, it was proved by Hatcher ([Hat83]),
and for Haken 3-manifolds, it is a consequence of Hatcher’s work and also un-
derstanding the space of incompressible surfaces ([Wal68, Hat76, Iva76]) inside
such manifolds. Recently Bamler and Kleiner ([BK19, BK21]) used Ricci flow techniques to settle the generalized Smale’s conjecture for all 3-manifolds admitting the spherical geometry and in the Nil geometry. Hence, this recent body of work using Ricci flow techniques addresses all cases of the generalized Smale’s conjecture.

Recall that a 3-manifold $M$ is called prime if it is not diffeomorphic to a connected sum of more than one 3-manifold so that none of which is diffeomorphic to the 3-sphere. The prime decomposition theorem says that every closed 3-manifold is diffeomorphic to the connected sum of prime manifolds. A prime closed 3-manifold is either diffeomorphic to $S^1 \times S^2$ or it is irreducible (i.e. every embedding $S^2$ bounds a ball). On the other hand, geometric manifolds are the building blocks for irreducible manifolds. Given the generalized Smale’s conjecture, we have a good understanding of the homotopy type of the diffeomorphism groups for these atomic pieces. And the JSJ and geometric decomposition theorems (see [Neu96, Chapter 2, section 6] for the statement of these theorems) give a way to cut an irreducible manifold along embedded tori into these building blocks. If the JSJ decomposition is non-trivial for an irreducible manifold, then it will be Haken whose diffeomorphism groups are well studied. Hence, given that we also know the homotopy type of the diffeomorphism group of $S^1 \times S^2$ by Hatcher’s theorem ([Hat81]), we have a complete understanding of the homotopy type of diffeomorphism group of prime manifolds. In the reducible case, the prime decomposition theorem cuts the manifold along separating spheres into its prime factors. The difficulty, however, in understanding the reducible case is to relate the diffeomorphism group of a reducible manifold to the diffeomorphisms of its prime factors.

César de Sá and Rourke ([CdSR79]) proposed to describe the homotopy type of $\text{Diff}(M)$ in terms of the homotopy type of diffeomorphisms of the prime factors and an extra factor of the loop space on “the space of prime decompositions”. Hendriks-Laudenbach ([HL84]) and Hendriks-McCullough ([HM87]) found a model for this extra factor. Later Hatcher, in an interesting unpublished note, proposed a finite dimensional model for this “space of prime decompositions” and more interestingly, he proposed that there should be a “wrong-way map” between $\text{BDiff}(M)$ and the classifying space of diffeomorphisms of prime factors.

Hatcher’s approach, if completed, would also solve Kontsevich’s conjecture in the special case of reducible 3 manifolds such that all the irreducible factors have non-empty boundaries. So our result is the homological version of what Hatcher intended to prove about Kontsevich’s conjecture. However, instead of trying to build this wrong way map, we take the geometric group theory approach by letting the abstract group of diffeomorphisms act on a “huge” simplicial complex inspired by the techniques that Kathryn Mann and the author ([MN20]) used to study the second homology of $\text{BDiff}(M)$.

For technical simplicity, we work with the homeomorphism groups instead of diffeomorphism groups. The reason is that Cerf ([Cer61]) assumed Smale’s conjecture which was later proved by Hatcher ([Hat83]) to show that in these low dimensions, the inclusion $\text{Diff}(M) \hookrightarrow \text{Homeo}(M)$ is, in fact, a weak homotopy equivalence. On the other hand, in all dimensions, by Mather-Thurston’s theorem ([Thu74, Corollary (b) of theorem 5]) for homeomorphisms, we have the natural map

\[(1.2) \quad \text{BHomeo}^\delta(M) \to \text{BHomeo}(M),\]

which is an acyclic map and in particular it induces a homology isomorphism in all degrees. The same statement also holds in the relative case in particular relative to the boundary when it is non-empty (see [McD80]).
Hence to prove the main theorem, we use a homological approach where we consider the action of $\text{Homeo}^0(M, \text{rel } \partial)$ on a simplicial complex $S(M)$ given by the complex of essential spheres, to give a model for $\text{BHomeo}^0(M, \text{rel } \partial)$ suitable for an inductive argument to prove the main theorem.

**Acknowledgment.** The author was partially supported by NSF grants DMS-2113828, NSF CAREER Grant DMS-2239106, a grant from the Simons Foundation (41000919, SN), and the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No 682922). He thanks Sander Kupers and Andrea Bianchi for their comments on the first draft of this paper.

2. Proof strategy for an inductive argument

In this section we assume that $M$ is a compact reducible 3-manifold with a non-empty boundary. We assume that we do not have spherical boundary components in order to have a prime decomposition in the presence of the boundary ([Hem76, Chapter 3]). To induct on the number of its prime factors to study the homological finiteness of $\text{BHomeo}^0(M, \text{rel } \partial)$, we shall first construct a simplicial complex $\tilde{S}(M)$ on which $\text{Homeo}^0(M, \text{rel } \partial)$ acts simplicially.

**Definition 2.1.** Let $S(M)$ be a simplicial complex whose vertices are given by locally flat embeddings $\phi: S^2 \hookrightarrow M$ whose images are essential spheres, and simplices are given by a collection of locally flat embeddings whose images are disjoint.

**Remark 2.2.** Note that in a simplex there could be vertices that are given by isotopic spheres and since they are disjoint, they bound an embedded $S^2 \times [0,1]$.

Note that the group $\text{Homeo}^0(M, \text{rel } \partial)$ acts on $S(M)$ simplicially. We shall prove in Proposition 2.7 that $S(M)$ is contractible. Therefore, the homotopy quotient $S(M)/\text{Homeo}^0(M, \text{rel } \partial)$ is homotopy equivalent to $\text{BHomeo}^0(M, \text{rel } \partial)$. The stabilizer of each simplex in $S(M)$ is the subgroup of $\text{Homeo}^0(M, \text{rel } \partial)$ that fixes a set of essential spheres pointwise so it is isomorphic to diffeomorphism group of a 3-manifold whose connected components have fewer prime factors. But one issue is that $S(M)$ has simplices of arbitrary large dimensions since we allow parallel spheres. To account for this infinite dimensionality, we use the simplicial complex that Hatcher and McCullough defined in [HM90, Section 1].

**Definition 2.3.** Let $S_{\text{nd}}(M)$ be the simplicial complex whose vertices are the isotopy classes of essential embedded spheres in $M$ and a set of vertices $[S_0], [S_1], \ldots, [S_n]$ constitutes an $n$-simplex if there are pairwise disjoint embedded spheres $S'_i$ in $M$ such that for each $i$, the sphere $S_i$ is isotopic to $S'_i$.

The mapping class group $\text{Mod}(M, \text{rel } \partial) = \pi_0(\text{Homeo}(M, \text{rel } \partial))$ acts on $S_{\text{nd}}(M)$ simplicially. The complex $S_{\text{nd}}(M)$ is finite-dimensional and also by Hatcher-McCullough’s theorem ([HM90, Proposition 2.2]) the set of the orbits of the action of $\text{Mod}(M, \text{rel } \partial)$ on simplices is also finite.

To briefly recall why this is the case, they use a theorem of Scharlemann (see [Bon83, Appendix A, Lemma A.1]) to find a “normal” representative of each orbit. Let the prime decomposition of $M$ be given by $P_1 \# P_2 \# \cdots \# P_g \# S^1 \times S^2$ where $g$ summands are diffeomorphic to $S^1 \times S^2$. Let $B$ be a punctured 3-cell having ordered $r + 2g$ boundary components so that $M$ is obtained by gluing $P_i \setminus \text{int}(D^3)$ to $i$-th sphere boundaries for $1 \leq i \leq r$ and $g$ copies of $S^2 \times [0,1]$ are glues along the remaining $2g$ boundary components (see [Bon83, Appendix A, Lemma A.1] for more detail).
Figure 1. $\sigma$ here is a 2-simplex consisting of 3 separating spheres that are drawn in one dimension lower.

**Lemma 2.4** (Scharlemann). For any simplex $\sigma \subset S(M)$, there is a homeomorphism $f$ such that $f(\sigma) \subset B$.

Now as Hatcher and McCullough observed in [HM90, Proposition 2.2], there are finitely many isotopy classes of essential spheres in $B$ since they are determined by the way they partition the boundary components of $B$. And this observation implies the finiteness of the orbits of the action of $\text{Mod}(M, \text{rel } \partial)$ on simplices of $S_{ad}(M)$.

The skeletal filtration on $S_{ad}(M)$ induces a filtration on the quotient space

$$F_0 \subset F_1 \subset \cdots \subset F_n = S_{ad}(M)/\text{Mod}(M, \text{rel } \partial),$$
and by Hatcher and McCullough’s observation, the filtration quotients are given by the wedge of a finite number of spheres. Let $O_p$ be the set of orbits of the action of $\text{Mod}(M, \text{rel } \partial)$ on $p$-simplices of $S_{ad}(M)$. So $F_p - F_{p-1}$ is homeomorphic to the disjoint union of open $p$-simplices indexed by $O_p$.

Now there is a natural simplicial map $S(M) \to S_{ad}(M)$ which is equivariant with respect to the map $\text{Homeo}\delta(M, \text{rel } \partial) \to \text{Mod}(M, \text{rel } \partial)$. So we have a map $S(M)/\text{Homeo}\delta(M, \text{rel } \partial) \to S_{ad}(M)/\text{Mod}(M, \text{rel } \partial)$ which in turn induces a map

$$\eta: S(M)/\text{Homeo}\delta(M, \text{rel } \partial) \to S_{ad}(M)/\text{Mod}(M, \text{rel } \partial).$$

The preimage filtration $G_p = \eta^{-1}(F_p)$ on $S(M)/\text{Homeo}\delta(M, \text{rel } \partial)$ induces a spectral sequence

$$E^1_{p,q} = H_{p+q}(G_p/G_{p-1}; Z) \Rightarrow H_{p+q}(\text{BHomeo}\delta(M, \text{rel } \partial); Z).$$

To prove the classifying space $\text{BHomeo}\delta(M, \text{rel } \partial)$ has finitely many nonzero homology groups which are each finitely generated, it is enough to prove the same for the filtration quotients $G_p/G_{p-1}$. And this is equivalent to the following theorem.

**Theorem 2.6.** Let us identify $F_p - F_{p-1}$ with the finite disjoint union of open simplices $\bigsqcup_{\sigma \in O_p} \Delta^p_\sigma$. Then for each $\sigma \in O_p$, as $x$ varies in $\Delta^p_\sigma$, the homotopy type of $\eta^{-1}(x)$ does not change and its homology groups are finitely generated and concentrated in finitely many degrees.

The filtration $G_i$’s are sub-CW-complexes of realization of the semisimplicial set given by the bar construction that realizes to the homotopy quotient $S(M)/\text{Homeo}\delta(M, \text{rel } \partial)$. Hence, the inclusions $G_{p-1} \hookrightarrow G_p$ are cofibrations. On the other hand, given Theorem 2.6, the homotopy type of $\eta^{-1}(x)$ does not change.
as $x$ varies in each open simplex $\Delta^p_q$. So $E^{1}_{p,q}$ in the spectral sequence is isomorphic to

$$\bigoplus_{\sigma \in \hat{O}_p} H^{p+q}_{p,q}(\partial_\Delta \Delta^p \wedge \eta^{-1}(b_\sigma); \mathbb{Z}),$$

where $b_\sigma$ is the barycenter of the open simplex $\Delta^p_q$. Hence, given Theorem 2.6, the homology groups of the filtration quotients $G_p/G_{p-1}$ are finitely generated and concentrated in finitely many degrees which implies Theorem 1.1.

In the following sections, we find a model for $\eta^{-1}(x)$ to which we can apply the induction hypothesis (i.e. the homological finiteness of $\text{BHomeo}(M, \text{rel } \partial)$ for $M$ with fewer prime factors) when $M$ is connected sum of irreducible factors that each has a non-trivial boundary. And we end this section with the proof of the contractibility of $S(M)$.

**Proposition 2.7.** The simplicial complex $S(M)$ is contractible.

In fact, it is a weakly Cohen-Macaulay complex of dimension infinity. Recall that a simplicial complex $X$ is called weakly Cohen-Macaulay of dimension $n$ if it is $(n - 1)$-connected and the link of any $p$-simplex is $(n - p - 2)$-connected. In this case, we denote this property by $wCM(X) \geq n$ (see [HW10, Definition 3.4]).

**Proof.** It is enough to show that for each $k$, any continuous map $f : S^k \to S(M)$ is nullhomotopic. Note that by the simplicial approximation theorem, we can assume that $f$ is a PL map concerning a triangulation $K$ of $S^k$ and we shall change $f$ up to a simplicial homotopy such that there exists a vertex $v$ in $S(M)$, that cones off $f(S^k)$ in $S(M)$.

It is easy to modify the map $f$ via a simplicial homotopy given by moving vertices to their parallel copies to make sure that the images of vertices of $f(K)$ are pairwise transverse $^1$ (see [Nar20, Lemma 3.31 and Lemma 4.3] for treating transversality in the locally flat settings). Now we choose a vertex $v$ in $S(M)$ so that as an embedded sphere, it is transverse to all vertices in $f(K)$. The intersection of spheres in $f(K)$ with $v$ gives a collection of circles on the sphere given by $v$. From this collection, choose a maximal family of disjoint circles, and let $C$ be the innermost circle in this family. Then $C$ is given by the intersection of the sphere $v$ and a sphere $w = f(x)$ in $f(K)$. Note that by the innermost circle, we mean that $C$ bounds a disk $D$ on $v$ whose interior is disjoint from all spheres in the maximal collection. We cut $w$ along $c$ and glue two nearby parallel copies of the disk $D$ to obtain two disjoint embedded spheres $w'$ and $w''$ (see Figure 2).

![Figure 2. Surgery on spheres in one dimension lower](image)

We can arrange this so that $w, w'$, and $w''$ are disjoint spheres. Since $w$ is an essential sphere, $w'$ and $w''$ both cannot bound a ball so let us assume that $w'$ is

---

$^1$One can alternatively change $f$ up to simplicial homotopy to replace each vertex $f(x)$ by a smooth nearby parallel copy and use transversality in the smooth category.
essential and we give it an arbitrary parametrization to consider it as a vertex in \( S(M) \). Let \( z \) be a vertex in the Star(w) where star here means as a subcomplex of \( f(K) \). The corresponding sphere for \( z \) cannot intersect \( w' \), since if it does intersect \( w' \), it has to intersect \( D \), which contradicts that \( D \) is innermost on \( v \). Hence, vertices in Star(w) represent spheres that are disjoint from the representative sphere of \( w' \). So we can define a simplicial homotopy \( G: K \times [0,1] \rightarrow S(M) \) such that \( G(\cdot,1) \) is the same as \( f(\cdot) \) on all vertices but \( x \) and \( G(x,1) = w' \). Note that the vertices in the image of \( G(\cdot,1): K \rightarrow S(M) \) have fewer numbers of circles in their intersection with \( v \). By repeating this process to reduce the number of circles in the intersection of spheres in \( f(K) \) with \( v \), we could homotope the map \( f \) to a map whose image lies in the star of \( v \). Therefore, \( f \) is nullhomotopic.

\[ \square \]

**Remark 2.8.** To prove the weak Cohen-Macaulay condition, we need to prove that the links of simplices in \( S(M) \) are contractible. To do so, it is easier to consider auxiliary complexes. Let \( R \) be a collection of boundary components that is diffeomorphic to the union of spheres. Let \( S(M,R) \) be a complex whose vertices are given by embedded essential separating spheres that could be parallel to components of \( R \) and whose simplices are given by a set of such spheres that are disjoint. If \( R \) is empty, this complex is the same as \( S(M) \), and if \( R \) is non-empty, then the complex \( S(M,R) \) is still contractible. In fact, the proof, in this case, is easier since for any PL map \( f: K \rightarrow S(M,R) \) where \( K \) is a finite complex, we can find an embedded sphere \( v \) that is sufficiently close to a component of \( R \) which is disjoint from all vertices in \( f(K) \). Hence, the vertex \( v \) cones off \( f(K) \) which implies that \( f \) is nullhomotopic.

Now let \( \sigma \) be a simplex in \( S(M) \) consisting of spheres \( \{S_0,S_1,\ldots,S_t\} \). Then the link of \( \sigma \) is the join of complexes \( S(M_i,R_i) \) where \( M_i \)'s are the components obtained by cutting \( M \) along spheres in \( \sigma \) and \( R_i \)'s are non-empty. Therefore, the link of \( \sigma \) is contractible.

### 3. Parallel spheres and bar constructions

In this section, we shall use the hypothesis that the prime decomposition of \( M \) consists of irreducible factors that each has non-empty boundary. The goal is for each \( p \) and each \( x \in F_p/F_{p-1} \) to find a semi-simplicial space \( X_x \) whose realization is homology equivalent to \( \eta^{-1}(x) \) and sits in a fibration sequence so that by induction on the number of prime factors we could argue that the fiber and the base have finite CW complex model.

The advantage of working with 3 manifolds that are connected sum of irreducible pieces such that each has a non-empty boundary is:

- When we cut along essential separating spheres, the remaining pieces each has a non-spherical boundary component that is fixed and we shall use this for the inductive argument.
- Since each irreducible factor has a non-trivial boundary that is fixed, homeomorphic irreducible factors cannot be permuted under the action of \( \text{Homeo}(M,\text{rel } \partial) \).

To start, let \( x \) be in \( F_0 \) in the filtration 2.5 which is the image of a separations sphere \( S \subset M \). Let \( S(M,[S]) \) be the full subcomplex of \( S(M) \) whose vertices are the orbits of \( S \) under the action of \( \text{Homeo}^3(M,\text{rel } \partial) \). Note that the preimage of \( \eta^{-1}(x) \) is homotopy equivalent to

\[ S(M,[S])/\text{Homeo}^3(M,\text{rel } \partial). \]

Suppose that \( S \) cuts the manifold \( M \) into two pieces \( M_1 \) and \( M_2 \) where \( M_1 \) contains \( \partial M \), the boundary component of \( M \) with the base point. . And let
Homeo$(M_1,S,\partial M)$ be the subgroup of Homeo$(M_1)$ that fixes the boundary component $S$ set-wise and the rest of the boundary components pointwise. Towards our goal in this section, we prove that $\eta^{-1}(x)$ is homology equivalent to the realization of a semi-simplicial space (in fact a two-sided bar construction) $X_{\bullet}$ that fits in a fibration

\begin{equation}
B\text{Homeo}(M_2,\partial) \rightarrow ||X_{\bullet}|| \rightarrow B\text{Homeo}(M_1,S,\partial M).
\end{equation}

By induction the fiber $B\text{Homeo}(M_2,\partial)$ has a finite CW complex model and in Theorem 3.13, we show that the base also has a finite CW complex model. First, we note that there is a natural order on vertices of each simplex in $S(M,[S])$.

**Lemma 3.2.** When $M$ is a compact 3-manifold with a non-empty boundary and its prime decomposition of $M$ consists of irreducible factors that each have a non-empty boundary, then vertices in each simplex in $S(M,[S])$ have a natural order.

**Definition 3.3.** Let $S_{\bullet}(M,[S])$ denote the semisimplicial set given by this ordering of vertices.

**Proof of Lemma 3.2.** Since the prime decomposition of $M$ has irreducible factors with non-empty boundaries, an edge in $S(M,[S])$ consists of two disjoint isotopic spheres in the orbit of $S$. This is because if we had two disjoint non-isotopic spheres in the orbit of $S$, given that $S$ is separating, these two spheres cut out homeomorphic pieces that are permuted by an element in Homeo$(M,\partial)$ which contradicts the hypothesis. Hence each simplex in $S(M,[S])$ consists of disjoint isotopic spheres. We call them parallel spheres.

Now we show that there is an induced order on parallel separating spheres when the manifold $M$ has a non-empty boundary. To describe this a priori order on parallel spheres, we choose once and for all a base point on one of the boundary components. We denote this boundary component by $\partial M$. Each separating sphere $S$ separates $M$ into connected components and one of them $P_S$ contains the base point. If we have isotopic disjoint separating spheres $S_i$’s, we order them by the inclusion of the components $P_S$’s. In other words, we can put a metric on $M$ and order $S_i$’s by their distance to the base point. We call this order from inside to outside direction.

Now we define a semi-simplicial space whose underlying semi-simplicial set is $S_{\bullet}(M,[S])$.

**Definition 3.4.** $S_{\bullet}(M,[S])$ is a semi-simplicial space whose 0-simplices as a set is the same as $S_0(M,[S])$ but it is topologized as the subspace of locally flat embeddings $\text{Emb}^0(S^2,M)$. And for each $p > 0$ the space $S_p^\tau(M,[S])$ as a set is the same as $S_p(M,[S])$ but it is topologized as a subspace of $S_p^\tau(M,[S])^{p+1}$.

Since the action of Homeo$(M,\partial)$ on the set of $p$-simplices $S_p(M,[S])$ is transitive, it is easy to use Thurston’s homology isomorphism 1.2 to obtain that for each $p$, the natural map

$$S_p^\tau(M,[S])//\text{Homeo}^\tau(M,\partial) \rightarrow S_p^\tau(M,[S])//\text{Homeo}(M,\partial),$$

induces a homology isomorphism. Therefore, by the spectral sequence that calculates the homology of the realization (see [ERW19, Section 1.4]), we have a homology isomorphism between fiber realizations

$$||S_p^\tau(M,[S])//\text{Homeo}^\tau(M,\partial)|| \rightarrow ||S_p^\tau(M,[S])//\text{Homeo}(M,\partial)||$$

Hence, to prove homological finiteness for $\eta^{-1}(x)$, we find a model for

$$||S_{\bullet}^\tau(M,[S])//\text{Homeo}(M,\partial)||.$$
that sits in a fibration 3.1.

We can define the smooth version of $S^*_i(M,[S])$ and work with diffeomorphism groups. But in this dimension and for codimension 1 embeddings, the corresponding objects in the $C^0$ and $C^\infty$-category are weakly homotopy equivalent. So we stick to $C^0$-category.

3.A. Moduli spaces of manifold models. Let $\io: \partial M \hookrightarrow [0] \times \mathbb{R}^\infty$ be a fixed locally flat embedding of the boundary component that contains the base point and let $\text{Emb}_\partial^f(M,[0,\infty) \times \mathbb{R}^\infty)$ be the space of locally flat embeddings of $M$ whose intersection with $[0] \times \mathbb{R}^\infty$ is $\io(\partial M)$. By [Las76, Appendix, theorem 1] and [Kup15, Lemma 2.2], the space $\text{Emb}_\partial^f(M,[0,\infty) \times \mathbb{R}^\infty)$ is weakly contractible. Hence, the semi-simplicial space

$$\mathcal{M}_\bullet(M,[S]) := \frac{S^*_i(M,[S]) \times \text{Emb}_\partial^f(M,[0,\infty) \times \mathbb{R}^\infty)}{\text{Homeo}(M,\text{rel } \partial)},$$

is level-wise weakly equivalent to $S^*_i(M,[S])/\text{Homeo}(M,\text{rel } \partial)$ and we think of $\mathcal{M}_\bullet(M,[S])$ as a configuration space of the manifolds in $[0,\infty) \times \mathbb{R}^\infty$ that are homeomorphic to $M$ satisfying the boundary condition and with a choice of parallel spheres in the orbit of $S$.

We shall define a two-sided bar construction model for $\mathcal{M}_\bullet(M,[S])$. Let $t_0: S^2 \hookrightarrow [0] \times \mathbb{R}^\infty$ be a fixed embedding and we denote the embedding $t_0 + t \cdot e_1$ in $[t] \times \mathbb{R}^\infty$ by $t_1$.

**Definition 3.5.** Let $BD$ be the topological monoid given by space of pairs $(t,f) \in [0,\infty) \times \text{Emb}_\partial^f(S^2 \times [0,1],([0,\infty) \times \mathbb{R}^\infty)$ where $f(S^2 \times [0,1]) \subset [0,t] \times \mathbb{R}^\infty$ and the restriction of $f$ to $S^2 \times [0]$ and $S^2 \times [1]$ are given by embeddings $t_0$ and $t_1$ respectively. The monoid structure is given by adding the $t$-coordinates and stacking the embeddings next to each other.

It is standard to see that the topological monoid $BD$ is homotopy equivalent to $B\text{Homeo}(S^2 \times [0,1],\text{rel } \partial)$. But since the homotopy type of $\text{Homeo}(S^2 \times [0,1],\text{rel } \partial)$ is known ([Hat83, Appendix]) to be the loop space $\Omega(\text{SO}(3))$, we can also determine the homotopy type of the delooping $BD$.

**Lemma 3.6.** The space $BBD$ which is the classifying space of the topological monoid $BD$ is homotopy equivalent to $B\text{SO}(3)$.

**Proof.** It is easier to see this by describing $BBD$ as the realization of a bi-semi-simplicial space and then realize it in two different simplicial directions. Consider the standard embedding of $[0,\infty) \times S^2 \hookrightarrow [0,\infty) \times \mathbb{R}^\infty$. Let $D$ be the subspace of pairs $(t,f) \in [0,\infty) \times \text{Homeo}_c([0,\infty) \times S^2)$, such that supp$(f) \subset [0,t] \times S^2$. The multiplication of $(t,f) \cdot (t',f') = (t+t', f \cup f')$ where $f \cup f'$ is in Homeo$([0,t+t'] \times S^2,\text{rel } \partial)$ given by concatenating $f$ and $f'$. Let $D_k$ be a simplicial monoid where the space of $k$-simplices is given by the tuples $(t,f_1,\ldots,f_k)$ where supp$(f_i) \subset [0,t] \times S^2$ for all $i$. The face maps are given by the composition of homeomorphisms and the degeneracies are given by inserting the identity. It is easy to see that the realization $||D_\bullet||$ is a topological monoid homotopy equivalent to $BD$.

Let $\Omega(\text{SO}(3))$ be the Moore monoid model for the space of loops on $\text{SO}(3)$. This is a submonoid of $D$ by sending a loop $f: [0,t] \rightarrow \text{SO}(3)$ to the homeomorphism of $[0,t] \times S^2$ that sends $(s,x)$ to $(s,f(s)(x))$. This inclusion is a weak equivalence and also respects the composition of homeomorphism groups. Hence, if we define the simplicial monoid $\Omega(\text{SO}(3))$, similar to $D_\bullet$, then its nerve is a bi-simplicial space that can be realized in both directions. The homotopy type of the realization of this
bi-simplicial space is the same as BBD. So if we realize $\Omega_2SO(3)_*$ in the monoid direction first and then in the simplicial direction, we obtain the homotopy type of $BSO(3)$. □

Recall that when we cut $M$ along $S$, we obtain two pieces $M_1$ and $M_2$ where $M_1$ contains $\partial M$, the boundary component of $M$ with the base point. Now we define moduli space models for $B\text{Homeo}(M_1, \text{rel } \partial)$ and $B\text{Homeo}(M_2, \text{rel } \partial)$ that are modules over the topological monoid BD.

**Definition 3.7.** Let $BL$ be the space of pairs $(t, f) \in [0, \infty) \times \text{Emb}_{\partial}(M_1, [0, \infty) \times \mathbb{R}^\infty)$ such that

- The image $f(M_1)$ lies in the strip $[0, t] \times \mathbb{R}^\infty$.
- The intersection $f(M_1) \cap [0] \times \mathbb{R}^\infty$ is given by the embedding $e$ and $f(M_1) \cap \{t\} \times \mathbb{R}^\infty$ is given by $\iota_t$.

And similarly, let $BR$ to be the space of pairs $(t, f) \in [0, \infty) \times \text{Emb}_{\partial}(M_2, [0, \infty) \times \mathbb{R}^\infty)$ such that

- The image $f(M_2)$ lies in $[t, \infty) \times \mathbb{R}^\infty$.
- The intersection $f(M_2) \cap \{t\} \times \mathbb{R}^\infty$ is given by $\iota_t$.

It is easy to see that $BL$ and $BR$ are weakly equivalent to $B\text{Homeo}(M_1, \text{rel } \partial)$ and $B\text{Homeo}(M_2, \text{rel } \partial)$ respectively.

Note that there is a right BD-module structure on $BL$ such that the action of $(t, f) \in BD$ on $(t', f') \in BL$ is the pair $(t + t', f' \sqcup (f + t' \cdot e_1))$ where $f + t' \cdot e_1$ is the embedding $f$ shifted in the first coordinate to the right by $t'$. Similarly, there is a left BD-module structure on $BR$.

![Figure 3. Schematic picture in one dimension lower on how BD acts on BR and BL](image)

We consider the two-sided bar resolution given by the semi-simplicial space

$$B_p(BL, BD, BR) = BL \times BD^p \times BR$$

where the face map $d_0$ and $d_p$ are given by the actions of BD on BL and BR respectively and other face maps are induced by the monoid structure of BD.

Note that there is a natural semi-simplicial map

$$b_p : B_p(BL, BD, BR) \to M_p(M, [S])$$

by gluing the embeddings in order and choosing the spheres along which the embeddings are glued as a choice of parallel spheres in the orbit of $S$. On the other hand, recall that the action of $\text{Homeo}(M, \text{rel } \partial)$ on $S^2_p(M, [S])$ is transitive.
for each \( p \). So for a \( p \)-simplex \( \sigma \in S_p^2(M,[S]) \), the homotopy quotient \( S_p^2(M,[S])/\text{Homeo}(M,\text{rel} \partial) \) is weakly equivalent to \( \text{BStab}(\sigma) \). Hence, the semi-simplicial map \( h_* \) is level-wise a weak equivalence, and we have weak equivalences between the (fat) realizations

\[
\|B_*(\text{BL},\text{BD},\text{BR})\| \xrightarrow{\sim} \|M_*(M,[S])\| \simeq \|S_p^2(M,[S])/\text{Homeo}(M,\text{rel} \partial)\|.
\]

Note that we have a fibration

\[
(3.8) \quad \text{BR} \to \|B_*(\text{BL},\text{BD},\text{BR})\| \to \|B_*(\text{BL},\text{BD},\ast)\|.
\]

We shall use the technique of the Weiss fibration as was explained in [Kup19] to show that this is the desired fibration 3.1.

3.B. **Kupers’ bar resolution for self-embeddings.** We shall use Kupers’ theorem ([Kup19, Section 4]) to determine the homotopy type of \( \|B_*(\text{BL},\text{BD},\ast)\| \) to be able to say that has finite CW complex model.

The manifold \( M_1 \) has a sphere boundary component \( \partial_0 M_1 = S \) which we call the free boundary component and we denote the union of the rest of the boundary components by \( \partial_1 M_1 \) which we call the fixed boundary components. Let \( \text{Homeo}(M_1,S,\text{rel} \partial_1) \) be the group homeomorphisms of \( M_1 \) that fix \( S \) set-wise and fix \( \partial_1 M_1 \) point-wise.

**Theorem 3.9.** There is a zig-zag of weak equivalences between the bar resolution \( \|B_*(\text{BL},\text{BD},\ast)\| \) and the classifying space \( \text{BHomeo}(M_1,S,\text{rel} \partial_1) \).

Kupers in [Kup19] gives a model for Weiss fiber sequence where the set-up is we have an \( n \)-dimensional manifold \( M \) with a non-empty boundary and we fix an embedded \( \mathbb{D}^{n-1} \to \partial M \). Let \( \text{Emb}_{0/2}(M) \) be the space of self-embeddings of \( M \) that are identity on \( \partial M \setminus \text{int}(\mathbb{D}^{n-1}) \) and are isotopic to a diffeomorphism that fixes the boundary through isotopies fixing \( \partial M \setminus \text{int}(\mathbb{D}^{n-1}) \). There exists a fiber sequence named after Michael Weiss

\[
\text{BDiff}(M,\text{rel} \partial) \to \text{BEEmb}_{0/2}(M) \to \text{BDiff}(\mathbb{D}^n,\text{rel} \partial),
\]

where the delooping \( \text{BDiff}(\mathbb{D}^n,\text{rel} \partial) \) is defined by considering \( \text{BDiff}(\mathbb{D}^n,\text{rel} \partial) \) as a topological monoid similar to Definition 3.5 and the \( E_1 \)-structure on this topological monoid is given by stacking along the first coordinate when we consider the interior of the cube as a model for the interior of the disk. We want to use a similar fiber sequence for a compact 3-manifold \( M \) with a non-empty boundary whose one of its boundary components is homeomorphic to \( S^2 \).

**Proof of Theorem 3.9.** Let \( \text{Emb}_{0/2}^{S^2}(M) \) the space of self locally flat embeddings of \( M_1 \) that are the identity on the fixed boundary components \( \partial_1 M_1 \) and are isotopic to a homeomorphism that fixes the boundary through isotopies fixing \( \partial_1 M_1 \).

Given that in dimension 3, the corresponding objects in \( C^0 \) and \( C^\infty \) category are weakly equivalent, we may apply the proof of [Kup19, Theorem 4.17] mutatis mutandis to conclude that there is a fiber sequence

\[
(3.10) \quad \text{BHomeo}(M_1,\text{rel} \partial) \to \text{BEEmb}_{0/2}^{S^2}(M_1) \to \text{BBD},
\]

and a weak equivalence \( \|B_*(\text{BL},\text{BD},\ast)\| \simeq \text{BEEmb}_{0/2}^{S^2}(M_1) \). On the other hand, we have a fiber sequence

\[
(3.11) \quad \text{BHomeo}(M_1,\text{rel} \partial) \to \text{BHomeo}(M_1,S,\text{rel} \partial_1) \to \text{BHomeo}_0(S^2),
\]

where the last map is the restriction to \( S \). Since homeomorphisms in \( \text{Homeo}(M_1,S,\text{rel} \partial_1) \) fix at least one boundary component, they are orientation preserving so they restrict to \( \text{Homeo}_0(S^2) \).
Recall from the proof of Lemma 3.6 that there is a map from BSO(3) to BBD which is a weak equivalence. And also the inclusion SO(3) \hookrightarrow \text{Homeo}(S^2) is a weak equivalence ([Ham74]). Hence, there is a map from the fiber sequence 3.11 to the fiber sequence 3.10 that induces weak equivalences between bases and the fibers. Therefore, their total spaces are also weakly equivalent. □

In the next section, we prove in Theorem 3.13 that BH\text{omeo}(M_1, S, \text{rel } \partial) has a finite CW complex model by induction on the number of prime factors. The space BR in the fibration 3.8 also by induction has a finite CW complex model. So in the fibration 3.8 the base and the fiber have a finite CW complex model which implies the same for the total space. Hence, this implies that \( \eta^{-1}(x) \) when \( x \in F_0 \) is homology isomorphic to a finite CW complex.

Remark 3.12. For \( M \) being connected sum of two irreducible 3-manifolds with non-empty boundaries, Hatcher’s theorem ([Hat81]) about 3-manifolds also implies Kontsevich’s finiteness. Let \( S \) be a separating sphere in \( M \), then his theorem implies that \( r: \text{BDiff}(M, S) \to \text{BDiff}(M) \) is a homotopy equivalence where \( \text{Diff}(M, S) \) is the subgroup of \( \text{Diff}(M) \) that fixes \( S \) set wise. And we have a fibration

\[
\text{BDiff}(M_2, \text{rel } \partial) \to \text{BDiff}(M, S) \to \text{BDiff}(M_1, S, \text{rel } \partial M),
\]

where \( M_1 \) and \( M_2 \) are obtained by cutting \( M \) along \( S \). This fibration is similar to our fibration 3.1.

3.C. Higher filtrations and finishing the proof of Theorem 2.6. For \( p > 0 \) suppose \( x \in F_p - F_{p-1} \). We want to generalize the above bar resolution model by iterating the same construction \((p + 1)\) times for each separating sphere in different orbits. And then write this iterated bar construction in a fiber sequence whose base and fiber, by induction, have finite CW complex models.

Let \( S = \{S_0, S_1, \ldots, S_p\} \) be a set of \( p + 1 \) separating spheres where \( S_i \)'s are pairwise in different orbit classes under the action of \( \text{Homeo}^0(M, \text{rel } \partial) \). We pick an order on these spheres and note that they cannot be permuted via the action of \( \text{Homeo}^0(M, \text{rel } \partial) \). We similarly define \( S(M,[S]) \) to be the full subcomplex of \( S(M) \) whose vertices are in the orbits of spheres in \( S \) under the action of \( \text{Homeo}^0(M, \text{rel } \partial) \). Note that the preimage of \( \eta^{-1}(x) \) is homotopy equivalent to

\[
S(M,[S]) / \text{Homeo}^0(M, \text{rel } \partial).
\]

In each simplex in \( S(M,[S]) \), the spheres parallel to \( S_i \) for each \( i \) has a natural inside to outside order (see Lemma 3.2). So there is a natural multi-semi-simplicial set structure on \( S(M,[S]) \) that we denote by \( S_{\star, \star}(M,[S]) \) where the number of simplicial directions is \( p + 1 \). Similar to Definition 3.4, we have a multi-semi-simplicial space \( S_{\star, \star}(M,[S]) \) and a homology isomorphism

\[
\|S_{\star, \star}(M,[S]) / \text{Homeo}^0(M, \text{rel } \partial)\| \to \|S_{\star, \star}(M,[S]) / \text{Homeo}(M, \text{rel } \partial)\|.
\]

To finish the proof of Theorem 2.6, it is enough to show that \( \|S_{\star, \star}(M,[S]) / \text{Homeo}(M, \text{rel } \partial)\| \) has a finite CW complex model by putting it in a fibration whose base and the fiber have finite CW complex model by induction.

By doing the bar construction model in each simplicial direction, we have fibrations similar to the fibration 3.8. And by applying Theorem 3.9 we obtain a fibration similar to the fibration 3.1 whose fiber and the base are realizations of multi-semi-simplicial spaces with fewer simplicial directions to which we can apply induction.

Let \( M_1 \) and \( M_2 \) be the submanifolds obtained by cutting \( M \) along \( S_0 \) and \( M_1 \) containing the boundary component \( \partial M \). Suppose that \( k \) of spheres in \( \{S_1, \ldots, S_p\} \)
lies in $M_1$ and the rest are in $M_2$. By doing the bar construction model and applying Theorem 3.9, we obtain a fibration
\[ \|R_{x=k}(M_2)\| \to \|S^*\|_{\text{free}}(M,[S])/{\text{Homeo}(M,\partial \sigma)} \to \|L_{x=k}(M_1)\|, \]
where the number of simplicial directions in $R_{x=k}(M_2)$ and $L_{x=k}(M_1)$ are respectively $p-k$ and $k$. Hence, it is easy to see that we can exhaust simplicial directions by considering fibrations and using Theorem 3.9. So to finish the proof of Theorem 2.6, it is enough to prove the following base case.

Let $P$ be an irreducible 3-manifold with a non-empty boundary. Let $e_i: \mathbb{D}^3 \hookrightarrow P$ for $1 \leq i \leq k+l$ be disjoint embeddings and let $N$ be the 3 manifold obtained from $P$ by removing $e_i(\text{int}(\mathbb{D}^3))$ for all $1 \leq i \leq k+l$. So the boundary of $N$ is the union of $\partial P$ with sphere boundary components $S_i$'s. We denote the union of the sphere boundary components $\{S_i\}_{i=1}^k$ by $S_{\text{free}}$ and union of the rest of sphere boundary components by $S_{\text{fixed}}$. Let $\text{Homeo}(N,S_{\text{free}},\partial P \cup S_{\text{fixed}})$ be the subgroup of $\text{Homeo}(N)$ whose elements fix each sphere in $S_{\text{free}}$ set-wise and fix $\partial P \cup S_{\text{fixed}}$ pointwise.

**Theorem 3.13.** Then $\text{BDiff}(N,S_{\text{free}},\partial P \cup S_{\text{fixed}})$ has a finite CW complex model.

Since the corresponding diffeomorphism groups and homeomorphism groups are weakly equivalent, we shall instead prove that $\text{BDiff}(N,S_{\text{free}},\partial P \cup S_{\text{fixed}})$ has a finite CW complex model. We already know the homotopical finiteness for an irreducible 3-manifold with a non-empty boundary ([HM97]).

**Theorem 3.14 (Hatcher-McCullough).** If $M$ is an irreducible 3-manifold with a non-empty boundary, then $\text{BDiff}(M,\partial \sigma)$ has the homotopy type of a finite CW-complex.

So we know that $\text{BDiff}(P,\partial \sigma)$ has a finite CW complex model. And we want to inductively fix $e_i(\mathbb{D}^3)$ either set-wise or pointwise and still get a finite CW complex model.

**Lemma 3.15.** Suppose $M$ is a 3-manifold with possibly nonempty boundary. Let $\partial_1$ be a subset of boundary components containing the non-spherical components and let $S_{\text{free}}$ be the union of remaining spherical components. Let $e: \mathbb{D}^3 \hookrightarrow M$ be an embedding of a ball inside $M$. If $\text{BDiff}(M,S_{\text{free}},\partial_1)$ has the homotopy type of a finite CW-complex, so does $\text{BDiff}(M,S_{\text{free}},\partial_1 \cup e(\mathbb{D}^3))$.

**Proof.** We give the argument when $M$ has a non-trivial boundary. The case of the closed manifold is similar. The fibration
\[ \text{Diff}(M,S_{\text{free}},\partial_1 \cup e(\mathbb{D}^3)) \to \text{Diff}(M,S_{\text{free}},\partial_1) \to \text{Emb}^+(\mathbb{D}^3,M) \cong \text{Fr}^+(M), \]
where $\text{Emb}^+(\mathbb{D}^3,M)$ is the space of orientation preserving embeddings and $\text{Fr}^+(M)$ is the oriented frame bundle of $M$, induces the fibration
\[ \text{Fr}^+(M) \to \text{BDiff}(M,S_{\text{free}},\partial_1 \cup e(\mathbb{D}^3)) \to \text{BDiff}(M,S_{\text{free}},\partial_1). \]
The base and the fiber of this fibration have the homotopy type of a finite CW-complex. Therefore, the total space also has a finite-dimensional model. □

**Proof of Theorem 3.13.** We shall prove that $\text{BDiff}(N,S_{\text{free}},\partial P \cup S_{\text{fixed}})$ has a finite CW complex model. Let $M$ be the manifold obtained from $P$ by removing $e_i(\text{int}(\mathbb{D}^3))$ for all $1 \leq i \leq k$. Given Lemma 3.15, it is enough to prove that $\text{BDiff}(M,S_{\text{free}},\partial P)$ has a finite CW complex model.

Let $x_i$ be a point in $P$ given by the image of the center of the ball $e_i(\text{int}(\mathbb{D}^3))$. And let $\text{Diff}(P,[x_1,\ldots,x_k],\partial P)$ be the subgroup of $\text{Diff}(P,\partial P)$ consisting of those elements that fix each $x_i$.

**Claim.** The classifying space $\text{BDiff}(M,S_{\text{free}},\partial P)$ is homotopy equivalent to $\text{BDiff}(P,[x_1,\ldots,x_k],\partial P)$.
Proof of the claim: Consider the fibration
\[
\text{BDiff}(M, \text{rel } \partial P) \to B(\text{Diff}_g(S^2)^k),
\]
given by the restriction of diffeomorphisms of $M$ to sphere boundary components. The fiber of this map is homotopy equivalent to $\text{BDiff}(M, \text{rel } \partial M)$ i.e. the classifying space of diffeomorphisms of $M$ that fix all boundary components pointwise.

Now recall that tangent bundles for orientable 3-manifolds are trivial, so the derivative map at the marked points $x_i$'s gives
\[
\text{BDiff}(P, \{x_1, \ldots, x_k\}, \text{rel } \partial P) \to B(\text{GL}_3^+(\mathbb{R})^k).
\]
Recall that Smale’s theorem ([Sma59]) implies that $\text{Diff}_g(S^2) \cong \text{GL}_3^+(\mathbb{R}) \cong \text{SO}(3)$, and the fiber of this fibration is also homotopy equivalent to $\text{BDiff}(M, \text{rel } \partial M)$.

Hence, the natural map
\[
\text{BDiff}(M, \text{rel } \partial P) \to \text{BDiff}(P, \{x_1, \ldots, x_k\}, \text{rel } \partial P),
\]
that is induced by capping off the sphere boundary components and extending the diffeomorphisms by the identity to the bounding balls, is a homotopy equivalence.

Let $\text{PConf}_k(P)$ be the space of ordered configuration space of $k$ points in the interior of $P$. We have a fiber sequence
\[
\text{PConf}_k(P) \to \text{BDiff}(P, \{x_1, \ldots, x_k\}, \text{rel } \partial P) \to \text{BDiff}(P, \text{rel } \partial P).
\]
Since both $\text{BDiff}(P, \text{rel } \partial P)$ and $\text{PConf}_k(P)$ have a finite CW complex model so does $\text{BDiff}(P, \{x_1, \ldots, x_k\}, \text{rel } \partial P)$.

\[\square\]

Remark 3.16. To consider the general case of a 3-manifold with a non-empty boundary, this method runs into two difficulties. One is that in the prime decomposition, not all the prime factors necessarily contain a boundary component. So diffeomorphic prime factors may be permuted so the homological model for $\eta^{-1}(x)$ for $x \in \mathcal{F}_p - \mathcal{F}_{p-1}$ should be modified in this case. But the author thinks the more difficult issue is to deal with the case where we only have sphere boundary components. More concretely, let $P$ be a closed prime 3 manifold and let $\mathbb{D}^3 \hookrightarrow P$ be an embedded ball inside $P$. Given the resolution of the generalized Smale’s conjecture ([BK19, BK21]), the author thinks one could show that $\text{BDiff}(P, \text{rel } \mathbb{D}^3)$ has a finite CW complex model. But at the moment he does not know how to show that $\text{BDiff}(M, \text{rel } \mathbb{D}^3)$ has a finite CW complex model when $M$ is a closed reducible 3 manifold.

References

[BK19] Richard H Bamler and Bruce Kleiner. Ricci flow and contractibility of spaces of metrics. arXiv preprint arXiv:1909.08710, 2019.

[BK21] Richard H Bamler and Bruce Kleiner. Diffeomorphism groups of prime 3-manifolds. arXiv preprint arXiv:2108.03302, 2021.

[Bon83] Francis Bonahon. Cobordism of automorphisms of surfaces. Ann. Sci. École Norm. Sup. (4), 16(2):237–270, 1983.

[CdSR79] Eugénia César de Sá and Colin Rourke. The homotopy type of homeomorphisms of 3-manifolds. Bull. Amer. Math. Soc. (N.S.), 1(1):251–254, 1979.

[Cer61] Jean Cerf. Topologie de certains espaces de plongements. Bulletin de la Société Mathématique de France, 89:227–380, 1961.

[ERW19] Johannes Ebert and Oscar Randal-Williams. Semisimplicial spaces. Algebr. Geom. Topol., 19(4):2099–2150, 2019.

[Ham74] Mary-Elizabeth Hamstrom. Homotopy in homeomorphism spaces, $T0$ and PL. Bulletin of the American Mathematical Society, 80(2):207–230, 1974.

[Hat76] Allen Hatcher. Homeomorphisms of sufficiently large $P^2$-irreducible 3-manifolds. Topology, 15(4):343–347, 1976.
A. Hatcher. On the diffeomorphism group of $S^1 \times S^2$. Proc. Amer. Math. Soc., 83(2):427–430, 1981.

Allen E. Hatcher. A proof of the Smale conjecture, $\text{Diff}(S^3) = O(4)$. Annals of Mathematics, pages 553–607, 1983.

John Hempel. 3-Manifolds. Annals of Mathematics Studies, No. 86. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1976.

Sungbok Hong, John Kalliongis, Darryl McCullough, and J. Hyam Rubinstein. Diffeomorphisms of elliptic 3-manifolds, volume 2055 of Lecture Notes in Mathematics. Springer, Heidelberg, 2012.

Harrie Hendriks and François Laudenbach. Difféomorphismes des sommes connexes en dimension trois. Topology, 23(4):423–443, 1984.

Harrie Hendriks and Darryl McCullough. On the diffeomorphism group of a reducible 3-manifold. Topology Appl., 26(1):25–31, 1987.

Allen Hatcher and Darryl McCullough. Finite presentation of 3-manifold mapping class groups. In Groups of Self-Equivalences and Related Topics, pages 48–57. Springer, 1990.

Allen Hatcher and Darryl McCullough. Finiteness of classifying spaces of relative diffeomorphism groups of 3-manifolds. Geom. Topol., 1:91–109, 1997.

Allen Hatcher and Nathalie Wahl. Stabilization for mapping class groups of 3-manifolds. Duke Mathematical Journal, 155(2):205–269, 2010.

N. V. Ivanov. Groups of diffeomorphisms of Waldhausen manifolds. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 66:172–176, 209, 1976. Studies in topology, II.

Rob Kirby. Problems in low-dimensional topology. In Proceedings of Georgia Topology Conference, Part 2. Citeseer, 1995.

Alexander Kupers. Proving homological stability for homeomorphisms of manifolds. arXiv preprint arXiv:1510.02456, 2015.

Alexander Kupers. Some finiteness results for groups of automorphisms of manifolds. Geometry & Topology, 23(5):2277–2333, 2019.

R. Lashof. Embedding spaces. Illinois J. Math., 20(1):144–154, 03 1976.

Dusa McDuff. The homology of some groups of diffeomorphisms. Commentarii Mathematici Helvetici, 55(1):97–129, 1980.

Kathryn Mann and Sam Nariman. Dynamical and cohomological obstructions to extending group actions. Mathematische Annalen, 377(3):1313–1338, 2020.

Sam Nariman. A local to global argument on low dimensional manifolds. Transactions of the American Mathematical Society, 373(2):1307–1342, 2020.

Walter D Neumann. Notes on geometry and 3-manifolds. Topology Atlas, 1996.

Stephen Smale. Diffeomorphisms of the 2-sphere. Proc. Amer. Math. Soc., 10:621–626, 1959.

William Thurston. Foliations and groups of diffeomorphisms. Bulletin of the American Mathematical Society, 80(2):304–307, 1974.

Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. Ann. of Math. (2), 87:56–88, 1968.