Moduli Spaces of Standard Holomorphic Bundles on a Noncommutative Complex Torus

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Abstract

In this paper we study the moduli space of standard holomorphic structures on a noncommutative complex two torus. It will be shown that the moduli space is naturally identified with the moduli space of stable bundles on an elliptic curve. We also propose that the mirror reflection of the noncommutative complex torus is the mirror reflection of the elliptic curve together with a linear foliation. From this we identify the moduli space of super cycles on the mirror reflection with the moduli space of standard holomorphic bundles on a noncommutative complex torus.
1 Introduction

The noncommutative tori is known to be the most accessible examples of noncommutative geometry [1] and the physics of open strings [2, 3]. A noncommutative torus can be obtained in a number of different ways. It can be thought as a strict deformation quantization [4] and also can be identified with the foliation $C^*$-algebra via Morita equivalence [1]. Here we will choose these two approach to define noncommutative two-tori and we will discuss how they are related with the mirror symmetry [5, 6].

To motivate our result, let us consider D2-brane physics on a two-torus. In Type II string theory compactified on an elliptic curve $X_\tau$, BPS states with fixed charge are associated with the cohomology classes of the D-brane moduli space. Suppose that there is a wrapped 2-brane on $X_\tau$. The characteristic classes of the Chan-Paton bundle and the RR charge vector are related by the Chern character and it gives brane charges associated to a gauge field configuration. The D-brane moduli space is identified with the moduli space of Yang-Mills connections by the Donaldson-Uhlenbeck-Yau Theorem. By taking T-duality, the D-brane moduli space corresponds to the D1-brane moduli space on the dual elliptic curve of $X_\tau$ and as shown in [6], the duality is equivalent to mirror symmetry. On the other hand, open string theory on a D2-brane can be understood from the viewpoint of deformation quantization [2]. Thus it is described by a noncommutative two-torus. The D-brane moduli space on $X_\tau$ corresponds to the moduli space of constant curvature connections [3, 7]. Now D1-brane physics is T-dual to the D2-brane physics. The algebra of open string field on the D1-brane is identified with the irrational rotational $C^*$-algebra [8]. This $C^*$-algebra is Morita equivalent to the $C^*$-algebra of the Kronecker foliation on the dual torus [1]. Thus, the deformation quantization of ordinary two-torus is related to a foliation $C^*$-algebra by T-duality. Since T-duality on two-tori is equivalent to mirror symmetry, it is expected to be relevant to the mirror symmetry on noncommutative two-tori. In other words, a deformation quantization and Kronecker foliation on a two-torus are related by a mirror symmetry as suggested in [9].

In this paper, based on the D2-brane physics, we study the mathematical aspects of the T-duality on a noncommutative complex torus. Holomorphic structures on a noncommutative torus has been introduced in [10, 11]. In the same spirit of [3, 7], we show that the moduli space of stable holomorphic vector bundles on an elliptic curve $X_\tau$ is naturally identified with the moduli spaces of standard holomorphic structures on a basic module on a noncommutative complex torus $T^2_{\theta,\tau}$, whose complex structure is inherited from that of $X_\tau$. Here, a stable bundle on $X_\tau$ is deformed to a basic projective module on $T^2_{\theta,\tau}$ by the curvature condition. This is discussed in section 2. In section 3, motivated by [8], we suggest a linear foliation of slope $\theta^{-1}$ on the mirror dual of $X_\tau$ is the mirror reflection of the noncommutative complex torus $T^2_{\theta,\tau}$. On the dual torus, the Strominger-Yau-Zaslow fibration can be understood as a linear foliation. We find that the foliation is deformed to a Kronecker foliation as $X_\tau$ is deformed to a
noncommutative torus. The deformation of the foliations is seen from the suspension by the diffeomorphisms on a circle, which is a different way of defining Kronecker foliation on a two-torus. Finally, in section 4, we show that the moduli space of super cycles on the mirror reflection with the moduli space of standard holomorphic bundles on a noncommutative complex torus. We conclude in section 5.

2 The moduli spaces of stable bundles on an elliptic curve

In this section, we show that the moduli space of holomorphic stable bundles on an elliptic curve is naturally identified with the moduli space of standard holomorphic bundles on a noncommutative torus, associated to a certain topological type.

Let $X_\tau = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$ be an elliptic curve whose complex structure is specified by $\tau \in \mathbb{C}$, $\text{Im } \tau \neq 0$. For $X_\tau$, the algebraic cohomology ring is

$$A(X_\tau) = H^0(X_\tau, \mathbb{Z}) \oplus H^2(X_\tau, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$ 

The Chern character of a holomorphic vector bundle $E$ on $X_\tau$ takes the value in $A(X_\tau)$:

$$\text{Ch}(E) = (\text{rank } E, \deg E) \in H^0(X_\tau, \mathbb{Z}) \oplus H^2(X_\tau, \mathbb{Z}),$$

where $\deg E = c_1(E) = \int_{X_\tau} c_1(E)$. The slope of a vector bundle $E$ is defined by

$$\mu(E) = \frac{\deg E}{\text{rank } E}.$$ 

A bundle $E$ is said to be stable if, for every proper subbundle $E'$ of $E$, $0 < \text{rank } E' < \text{rank } E$, we have

$$\mu(E') < \mu(E).$$

Every stable bundles carries a projectively flat Hermitian connection $\nabla^E$. In other words, there is a complex 2-form $\lambda$ on $X_\tau$ such that the curvature of $\nabla^E$ is

$$R_{\nabla^E} = \lambda \cdot \text{Id}_E$$

where $\text{Id}_E$ is the identity endomorphism of $E$. Since

$$c_1(E) = \frac{i}{2\pi} \text{Tr } R_{\nabla^E} = \frac{i}{2\pi} \lambda \cdot \text{rank } E,$$

we have

$$\lambda = \frac{2\pi}{i} \frac{c_1(E)}{\text{rank } E} = \frac{2\pi}{i} \mu(E).$$

Thus

$$R_{\nabla^E} = -2\pi i \mu(E) \text{Id}_E.$$  (1)
Let us denote by \( \mathcal{M}_{n,m}^s \) the moduli space of holomorphic stable bundles of rank \( n \) and degree \( m \) on \( X_\tau \). In [12], it was shown that \( \mathcal{M}_{n,m}^s \) is isomorphic to \( X_\tau \) when \( n \) and \( m \) are relatively prime. In other words, the points of \( X_\tau \) parameterize isomorphism classes of stable bundles of rank \( n \) and degree \( m \).

The pair of integers \((n, m)\) also determines the topological type of a gauge bundle on a noncommutative torus \( T^2_\theta \). For this, we first recall some notions on vector bundles on a noncommutative torus. A noncommutative torus \( T^2_\theta \) is the deformed algebra of smooth functions on the ordinary torus with the deformation parameter \( \theta \). The algebra is generated by two unitaries \( U_1 \) and \( U_2 \) obeying the relation

\[
U_1 U_2 = e^{2\pi i \theta} U_2 U_1. \tag{2}
\]

The above commutation relation defines the presentation of the involutive algebra

\[
A_\theta = \{ \sum_{n_1, n_2 \in \mathbb{Z}^2} a_{n_1, n_2} U_1^{n_1} U_2^{n_2} \mid a_{n_1, n_2} \in \mathcal{S}(\mathbb{Z}^2) \}
\]

where \( \mathcal{S}(\mathbb{Z}^2) \) is the Schwarz space of sequences with rapid decay. According to [13], the algebra \( A_\theta \) can be understood as the algebra of smooth functions on \( T^2_\theta \) and the vector bundles on \( T^2_\theta \) correspond to finitely generated projective (left) \( A_\theta \)-modules.

The infinitesimal form of the dual action of the torus \( T^2 \) on \( A_\theta \) defines a Lie algebra homomorphism \( \delta : L \to \text{Der} (A_\theta) \), where \( L = \mathbb{R}^2 \) is an abelian Lie algebra and \( \text{Der} (A_\theta) \) is the Lie algebra of derivations of \( A_\theta \). For each \( X \in L \), \( \delta(X) := \delta_X \) is a derivation i.e., for \( u, v \in A_\theta \),

\[
\delta_X(uv) = \delta_X(u)v + u\delta_X(v).
\]

Generators of the Lie algebra \( \delta_1, \delta_2 \) act in the following way:

\[
\delta_i(U_i) = 2\pi i U_i \quad \text{and} \quad \delta_i(U_j) = 0 \quad \text{for} \quad i \neq j.
\]

If \( \mathcal{E} \) is a projective \( A_\theta \)-module, a connection \( \nabla \) on \( \mathcal{E} \) is a linear map from \( \mathcal{E} \) to \( \mathcal{E} \otimes L^* \) such that for all \( X \in L \),

\[
\nabla_X(\xi u) = (\nabla_X \xi) u + \xi \delta_X(u), \quad \xi \in \mathcal{E}, u \in A_\theta.
\]

Equivalently,

\[
[\nabla_i, U_j] = 2\pi i \delta_{ij} \cdot U_j. \tag{3}
\]

The curvature \( F_\nabla \) of the connection \( \nabla \) is a 2-form on \( L \) with values in the algebra of endomorphisms of \( \mathcal{E} \). That is, for \( X, Y \in L \),

\[
F_\nabla(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.
\]
Since $L$ is abelian, we simply have $F_{\nabla}(X, Y) = [\nabla_X, \nabla_Y]$.

The Chern character of a gauge bundle on $T^2_\theta$ is an element in the Grassmann algebra $\wedge(L^*)$, where $L^*$ is the dual vector space of the Lie algebra $L$. Since there is a lattice $\Gamma$ in $L$, we see that there are elements of $\wedge^\Gamma(L^*)$ which are integral. Now the Chern character is the map $\text{Ch} : K_0(A_\theta) \to \wedge^\text{ev}(L^*)$ defined by 

$$\text{Ch}(\mathcal{E}) = e^{i(\theta)}\nu(\mathcal{E}).$$

(4)

Here $i(\theta)$ denotes the contraction with the deformation parameter $\theta$ regarded as an element of $\wedge^2L$ and 

$$\nu(\mathcal{E}) = e^{-i(\theta)}\text{Ch}(\mathcal{E}) \in \wedge^{\text{even}}(\Gamma).$$

The integral element $\nu(\mathcal{E})$ is related with the Chern character on the elliptic curve $X_\tau$.

Now let $(n, m) \in A(X_\tau)$ be the topological type of a stable bundle $E$ on $X_\tau$. The pair of integers defines an integral element $\nu(\mathcal{E})$:

$$\nu(\mathcal{E}) = n + m dx_{12} \in \wedge^2(L^*).$$

Then the Chern character is given by

$$\text{Ch}(\mathcal{E}) = e^{i(\theta)}(n + m dx_{12})$$

$$= (n - m\theta) + m dx_{12}.$$

As in the classical case, we define the slope of $\mathcal{E}$ by the number

$$\mu(\mathcal{E}) = \frac{m}{n - m\theta}.$$

Associated to the curvature condition (1) on the stable bundle $E$ on $X_\tau$, we define a Heisenberg commutation relation by

$$F_{\nabla} = [\nabla_1, \nabla_2] = -2\pi i \frac{m}{n - m\theta} = \frac{2\pi}{i} \mu(\mathcal{E}).$$

(5)

By the Stone-von Neumann theorem, the above relation has a unique representation. As discussed in [7], the representation is just $m$-copies of the Schrödinger representation of the Heisenberg Lie group $\mathbb{R}^3$ on $L^2(\mathbb{R})$, where the product on $\mathbb{R}^3$ is given by

$$(r, s, t) \cdot (r', s', t') = (r + r', s + s', t + t' + sr').$$

Then the operators $\nabla_1$ and $\nabla_2$ are the infinitesimal form of the representation and can be written as

$$(\nabla_1 f)(s) = 2\pi i \frac{ms}{n - m\theta} f(s)$$

(6)

$$(\nabla_2 f)(s) = \frac{df}{ds}(s)$$

(7)
acting on the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/m\mathbb{Z}) \cong \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^m$. To specify the module $\mathcal{E}$, we need to define a module action which is compatible with the relation (2) for $T_\theta^2$. Let us first consider unitary operators $W_1, W_2$ on $\mathcal{S}(\mathbb{Z}/m\mathbb{Z}) = \mathbb{C}^m$ defined by

$$W_1 f(\alpha) = f(\alpha - n)$$
$$W_2 f(\alpha) = e^{-2\pi i \frac{m}{m}} f(\alpha).$$

Then

$$W_1 W_2 = e^{2\pi i \frac{m}{m}} W_2 W_1.$$  

In other words, $W_1$ and $W_2$ provide a representation of the Heisenberg commutation relations for the finite group $\mathbb{Z}/m\mathbb{Z}$. Associated to the representations (6) and (7), we have Heisenberg representations $V_1$ and $V_2$ on the space $\mathcal{S}(\mathbb{R})$ as

$$V_1 f(s) = e^{2\pi i (\frac{m}{m} - \theta)} f(s)$$
$$V_2 f(s) = f(s + 1).$$

The operators obey the relation

$$V_1 V_2 = e^{-2\pi i (\frac{m}{m} - \theta)} V_2 V_1.$$

Finally, the operators

$$U_1 = V_1 \otimes W_1 \quad \text{and} \quad U_2 = V_2 \otimes W_2$$

acting on the space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/m\mathbb{Z})$ satisfy the relation

$$U_1 U_2 = e^{2\pi i \theta} U_2 U_1.$$  

Thus the module $\mathcal{E}$ is the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/m\mathbb{Z})$ with the module action (8). We write the module as $\mathcal{E}_{n,m}(\theta)$. This module admits a constant curvature connection whose curvature is given by (5). There are other connections which have the same constant curvature as (5). It is known that all constant curvature (5) connection on $\mathcal{E}_{n,m}(\theta)$ are given as

$$\nabla_1 = 2\pi i (\frac{m}{n - m\theta}) s + 2\pi i R_1$$
$$\nabla_2 = \frac{d}{ds} + 2\pi i R_2$$

where $R_1, R_2 \in \mathbb{R}/\mathbb{Z}$. Hence the moduli space of constant curvature connections on $\mathcal{E}_{n,m}(\theta)$, specified by (5), is the ordinary torus $T^2$. Denote by $\mathcal{M}_{n,m}^*(\theta)$ the moduli space of constant curvature connections on $\mathcal{E}_{n,m}(\theta)$ with the relation (5). Thus we see that the moduli space $\mathcal{M}_{n,m}^*$ is homeomorphic to the moduli space $\mathcal{M}_{n,m}^*(\theta)$. 

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We now consider holomorphic structures on the moduli space $\mathcal{M}_{n,m}^*(\theta)$. Let us fix a complex number $\tau \in \mathbb{C}$ such that $\text{Im} \, \tau \neq 0$. The parameter $\tau$ defines a complex structure on $T_\theta^2$ via derivation $\delta_\tau$ spanning $\text{Der}(A_\theta)$:

$$\delta_\tau \left( \sum_{(n_1,n_2) \in \mathbb{Z}^2} a_{n_1,n_2} U^{n_1} U^{n_2} \right) = 2\pi i \sum_{(n_1,n_2) \in \mathbb{Z}^2} (n_1 \tau + n_2) a_{n_1,n_2} U^{n_1} U^{n_2}. \quad (9)$$

The noncommutative torus equipped with such a complex structure is denoted by $T_{\theta,\tau}^2$. A holomorphic structure on the module $E_{n,m}(\theta)$ compatible with the complex structure is an operator $\nabla : E_{n,m}(\theta) \rightarrow E_{n,m}(\theta)$ such that

$$\nabla(\xi \cdot u) = \nabla(\xi) \cdot u + e \cdot \delta_\tau(\xi), \quad \xi \in E_{n,m}(\theta), u \in A_\theta.$$

A holomorphic structure on $E_{n,m}(\theta)$ is specified by $\nabla = \tau \nabla_1 + \nabla_2$. Along with the connections defined in (6), (7), and for $a \in \mathbb{C}$, let

$$(\nabla_z)(f)(s, \beta) = \frac{\partial f}{\partial s}(s, \beta) + 2\pi i \left( \frac{m}{n - m \theta} \right) s + z \right) f(s, \beta).$$

Then $\nabla_z$ defines a standard holomorphic structure on $E_{n,m}(\theta)$. Other holomorphic structures are determined by translations of $R_1$ and $R_2$ in (3) and (7). In other words, all the holomorphic structures are determined by the complex number $z = \tau R_1 + R_2$, where $R_1, R_2 \in \mathbb{R}/\mathbb{Z}$. And $z \sim z'$ if and only if $z \equiv z' \mod \tau \mathbb{Z} + \mathbb{Z}$. Furthermore the holomorphic structures are compatible with the holomorphic structure defined by the holomorphic structure on the stable bundle $E$ on the elliptic curve $X_\tau$. This is easily seen when $\theta = 0$. Now the moduli space of holomorphic structures on $E_{n,m}(\theta)$ is $\mathbb{C}/\tau \mathbb{Z} + \mathbb{Z} = X_\tau$. Thus we see that the moduli space $\mathcal{M}_{n,m}^*$ is isomorphic to the moduli space $\mathcal{M}_{n,m}^*(\theta)$.

3 The mirror reflection of a noncommutative complex torus

In this section, we briefly review the topological mirror symmetry of [5] for one-dimensional Calabi-Yau manifolds. We will mainly follow the lines of [6]. Then we describe a Kronecker foliation on the mirror reflection of the elliptic curve $X_\tau$ as the mirror partner of the noncommutative complex torus $T_{\theta,\tau}^2$.

A complex orientation of an elliptic curve $X_\tau$ is given by a holomorphic 1-form $\Omega$, which determines a Calabi-Yau manifold structure on $X_\tau$. A special Lagrangian cycle of $X_\tau$ is a 1-dimensional Lagrangian submanifold $L$ such that

$$\text{Im} \, \Omega|_L = 0 \quad \text{and} \quad \text{Re} \, \Omega|_L = \text{Vol}(L).$$
where the volume form is determined by the Euclidean metric on $X_\tau$. A special Lagrangian cycle is just a closed geodesic and hence it is represented by a line with rational slope on the universal covering space of $X_\tau$.

Let us fix a smooth decomposition

$$X_\tau = S^1_+ \times S^1_-$$

which induces a decomposition of cohomology group

$$H^1(X_\tau, \mathbb{Z}) \cong \mathbb{Z}_+ \oplus \mathbb{Z}_-.$$ 

Let $[B] \in \mathbb{Z}_-$ and $[F] \in \mathbb{Z}_+$ be generators of the cohomology group. Then the cohomology class $[F] \in H^1(X_\tau, \mathbb{Z})$ is represented by a smooth cycle in $X_\tau$ and is a special Lagrangian cycle. The family of special Lagrangian cycles representing the class $[F]$ gives a smooth fibration

$$\pi : X_\tau \longrightarrow S^1_- := B$$

and the base space $B = S^1_-$ is just the moduli space of special Lagrangian cycles associated to $[F] \in H^1(X_\tau, \mathbb{Z})$. The unitary flat connections on the trivial line bundle $S^1_+ \times \mathbb{C} \rightarrow S^1_+$ are parameterized by $\hat{S}^1 := \text{Hom}(\pi_1(S^1_+), U(1))$, up to gauge equivalences. Thus we have the dual fibration

$$\hat{\pi} : \hat{X}_\tau \longrightarrow B = S^1_-$$

with fibers

$$\hat{\pi}^{-1}(b) = \text{Hom}(\pi_1(\pi^{-1}(b), U(1)) = \hat{S}^1.$$ 

The dual fibration admits the section $s_0 \in \hat{X}_\tau$ with

$$s_0 \cap \hat{\pi}^{-1}(b) = 1 \in \text{Hom}(\pi_1(\pi^{-1}(b), U(1)),$$

so that we have a decomposition

$$\hat{X}_\tau = \hat{S}^1 \times S^1_-.$$ 

Hence, associated to the class $[F] \in H^1(X_\tau, \mathbb{Z})$, the space $\hat{X}_\tau$ is the moduli space of special Lagrangian cycles endowed with unitary flat line bundles. Furthermore, $\hat{X}_\tau$ admits a Calabi-Yau manifold structure. In other words, $\hat{X}_\tau$ is the mirror reflection of $X_\tau$ in the sense of [5]. Under the Kähler-Hodge mirror map (see [3]), a complexified Kähler parameter $\rho = b + ik$ defines a complex structure on $\hat{X}_\tau$, where $k$ is a Kähler form on $X_\tau$ and $b$ defines a class in $H^2(X_\tau, \mathbb{R})/H^2(X_\tau, \mathbb{Z})$. Then $\hat{X}_\tau$ is the elliptic curve $\mathbb{C}^*/e^{2\pi i \rho \mathbb{Z}}$. Similarly, the modular parameter $\tau$ of $X_\tau = \mathbb{C}^*/q \mathbb{Z}$, $q = \exp(2\pi i \tau)$, $\text{Im}(\tau) > 0$, corresponds to a complexified Kähler parameter $\hat{\rho}$ on $\hat{X}_\tau$. 

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Noncommutative tori can be obtained in many different ways. In Section 2, we understood $T^2_\theta$ as a strict deformation quantization, \cite{4}, and bundles on $T^2_\theta$ were constructed as a deformation of bundles on an ordinary torus. On the other hand, the algebra $A_\theta$ of functions on $T^2_\theta$ can be defined as the irrational rotation $C^*$-algebra, \cite{14}. This definition will allow us to study $T^2_\theta$ from a geometrical point of view. In fact, such a $C^*$-algebra is obtained from a Kronecker foliation on a torus, (cf. \cite{1}). In below, we propose that such a foliation structure on $\hat{X}_\tau$ define the mirror reflection of the noncommutative complex torus $T^2_{\theta,\tau}$.

The Kronecker or linear foliation of $\hat{X}_\tau$ associated to the irrational number $\theta^{-1}$ is defined by the differential equation $dy = (\theta^{-1} - 1) dx$, with natural coordinates $(x, y)$ on the flat torus determined by the symplectic form on $\hat{X}_\tau$. On the covering space, the leaves are represented by straight lines with fixed slope $\theta^{-1}$ and every closed geodesic of $\hat{X}_\tau$ yields a compact transversal which meets every leaf of the foliation. Such a transversal is represented by a line with rational slope, which means that a compact transversal is a special Lagrangian cycle in $\hat{X}_\tau$.

The linear foliation can be described by the suspension of diffeomorphisms, \cite{15}. Let $\alpha_{\theta^{-1}} : \hat{S}^1 \rightarrow \hat{S}^1$ be the diffeomorphism defined by

$$\alpha_{\theta^{-1}}(z) = \exp(2\pi i \theta^{-1}) \cdot z, \quad z \in \hat{S}^1.$$ 

It is the rotation through an angle $2\pi \theta^{-1}$. Consider the product manifold $S^1 \times \mathbb{R}$ with projections $p_1$ and $p_2$:

$$\begin{array}{ccc}
\hat{S}^1 \times \mathbb{R} & \xrightarrow{p_1} & \hat{S}^1 \\
p_2 & \downarrow & \\
\mathbb{R} & & \\
\end{array}$$

The product manifold $\hat{S}^1 \times \mathbb{R}$ is foliated by the leaves $p_1^{-1}(z) = \{z\} \times \mathbb{R}$ and the foliation transverses to the fibers of $p_2 : \hat{S}^1 \times \mathbb{R} \rightarrow \mathbb{R}$. Consider the $\mathbb{Z}$-action on $\hat{S}^1 \times \mathbb{R}$:

$$(z, b)^n := (\alpha_{\theta^{-1}}^n(z), b + n) = (\exp(2\pi i n \theta^{-1}) \cdot z, b + n), \quad n \in \mathbb{Z}. \quad (12)$$

The foliation $\{\{z\} \times \mathbb{R}\}_{z \in \hat{S}^1}$ is preserved by the action (12), and thus descends to a foliation on $\hat{X}_\tau$. Now the quotient $\hat{S}^1 / \mathbb{Z}$ carries a 1-dimensional foliation whose leaves are the inverse images of the projection $\tilde{p}_1 : \hat{S}^1 \times \mathbb{R} \rightarrow \hat{S}^1 / \mathbb{Z}$ under the action (12). Note that the quotient $\hat{S}^1 / \mathbb{Z}$ is identified with the space of the leaves of the foliation on $\hat{S}^1 \times \mathbb{R}$. On the universal covering space of $\hat{S}^1$, the $\mathbb{Z}$-action on $\hat{S}^1$ gives an identification $\theta^{-1} \sim n(\theta^{-1} - 1)$, $n \in \mathbb{Z}$ and hence $\hat{S}^1 / \mathbb{Z} \cong \mathbb{R} / \mathbb{Z} + \theta^{-1} \mathbb{Z}$. The leaves are represented by straight lines of the fixed slope $\theta^{-1}$. On the other hand, the second projection $p_2 : \hat{S}^1 \times \mathbb{R} \rightarrow \mathbb{R}$ becomes a fibration

$$\tilde{p}_2 : \hat{S}^1 \times \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R} / \mathbb{Z} \cong S^1 \quad (13)$$
whose fibers are compact transversals of the linear foliation on $\hat{S}^1 \times_\mathbb{Z} \mathbb{R}$. Now at the cost of defining the foliation on $\hat{X}_\tau$, the dual fibration (11) is modified to the fibration (13) and we have the following projections:

$$\begin{align*}
\hat{S}^1 \times_\mathbb{Z} \mathbb{R} & \overset{\hat{p}_1}{\longrightarrow} \hat{S}^1 \cong \mathbb{R}/\mathbb{Z} + \theta^{-1}\mathbb{Z} \\
\hat{p}_2 & \downarrow \\
\mathbb{R}/\mathbb{Z} & = S^1,
\end{align*}$$

which determines not only a foliation but also the compact transversals for the foliation. On the other hand, other structures on $\hat{X}_\tau$ such as the complex structure and the Kähler structure are irrelevant to define a linear foliation on $\hat{X}_\tau$. So, the Calabi-Yau 1-manifold $\hat{X}_\tau$ equipped with the $\theta^{-1}$-linear foliation will be denoted by $\hat{X}_{\tau,\theta^{-1}}$. When $\theta = 0$, the foliated manifold, denoted by $\hat{X}_{\tau,\infty}$, is just the dual fibration (11) and $\hat{X}_\tau$ is foliated by the fibers of the fibration. Now by identifying the base space of the fibration (13) with that of the fibration $\pi : X_\tau \to S^1$, we see that the foliated manifold $\hat{X}_{\tau,\theta^{-1}}$ is the moduli space of special Lagrangian cycles endowed with a unitary flat line bundle and the leaves of the $\theta^{-1}$-linear foliation. Thus, on the mirror side, the deformation quantization $T^2_{\theta,\tau}$ of an elliptic curve $X_\tau$ can be seen as to define a new foliation, $\theta^{-1}$-foliation, on the foliated manifold $\hat{X}_{\tau,\infty}$ and the leaves of $\hat{X}_{\tau,\infty}$ are served as compact transversals of the new foliation. In other words, a linear foliation on $\hat{X}_\tau$ is uniquely determined by the deformation parameter. In this sense, we may conclude that the foliated manifold $\hat{X}_{\tau,\theta^{-1}}$ is the mirror reflection of the noncommutative complex torus $T^2_{\theta,\tau}$. Finally, we remark from [1] that the foliation $C^*$-algebra for the $\theta^{-1}$-linear foliation on $\hat{X}_\tau$ is Morita equivalent to a $C^*$-algebra defined from a compact transversal for the foliation. In particular, if we take the line $y = 0$ in $\hat{X}_\tau \cong \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$, then the action of leaves of the foliation on the line defines the $\theta$ rotation $C^*$-algebra, which is known to be the noncommutative torus $A_\theta$, [14]. In Section 2, we understood $T^2_{\theta}$ as a deformation quantization of an ordinary torus. Thus we see that the deformation quantization and the linear foliation $C^*$-algebra on a tours is related by the mirror symmetry.

4 The moduli spaces of supercycles

As discussed in Section 2, a stable bundle of topological type $(n, m)$ defines a standard holomorphic bundle $\mathcal{E}_{n,m}(\theta)$ on $T^2_{\theta,\tau}$ by deforming the slope $\mu(E) = \frac{m}{n}$ to $\frac{m - \theta n}{n - \theta m}$. In this section, we show that the moduli space of supercycles of the slope $\frac{m}{n}$ on $\hat{X}_\tau$ is naturally identified with the moduli space of standard holomorphic structures on $\mathcal{E}_{n,m}(\theta)$.

A supercycle or a brane on $\hat{X}_\tau$ is given by a pair $(\mathcal{L}, A)$, where $\mathcal{L}$ is a special Lagrangian submanifold of $\hat{X}_\tau$ and $A$ a flat connection on the trivial line bundle $\mathcal{L} \times \mathbb{C} \to \mathcal{L}$. A special Lagrangian cycle in $\hat{X}_\tau \cong \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ is represented by a line of rational
slopes, so can be given by a pair of relatively prime integers. The lines of a fixed rational slope are parameterized by the points of intersection with the line \( y = 0 \). Let \( \mathcal{L}_{n,m} \) be a special Lagrangian submanifold of \( \hat{X}_\tau \) given by

\[
\mathcal{L}_{n,m} = \{(ns + R_1, ms) \mid s \in \mathbb{R}/\mathbb{Z}\},
\]

so that the line has slope \( \frac{m}{n} \) and \( x \)-intercept \( R_1 \). The shift of \( \mathcal{L}_{n,m} \) is represented by the translation of \( R_1 \). Note that a unitary flat line bundle on \( \mathcal{L}_{n,m} \) is specified by the monodromy around the circle. On the trivial line bundle \( \mathcal{L} \times \mathbb{C} \rightarrow \mathcal{L} \), we have a connection one-form given by

\[
A = 2\pi i R_2 dx, \quad x \in \mathbb{R}^2, \quad R_2 \in \mathbb{R}/\mathbb{Z}
\]

so that the monodromy between points \((x_1, y_1)\) and \((x_2, y_2)\) is \( \exp[2\pi i R_2(x_2 - x_1)] \). Thus, the shift of connections is represented by monodromies. Let us denote by \( \mathcal{S}\mathcal{M}_{n,m} \) the moduli space of supercycles on \( \hat{X}_\tau \), whose slope is \( \frac{m}{n} \). It was shown in [1] that the moduli space \( \mathcal{S}\mathcal{M}_{n,m} \) is isomorphic to \( \mathcal{M}^s_{n,m} \), the moduli space of topological type \((n, m)\) stable bundles on \( X_\tau \). In Section 2, we have shown that the moduli space \( \mathcal{M}^s_{n,m} \) is identified with the moduli space \( \mathcal{M}^s_{n,m}(\theta) \) of standard holomorphic bundles of slope \( \frac{m}{n-m\theta} \). Thus we see that \( \mathcal{S}\mathcal{M}_{n,m} \cong \mathcal{M}^s_{n,m}(\theta) \). This isomorphism can also be obtained from a geometric point of view. In other words, we construct standard holomorphic bundles on \( T^2_{\theta,\tau} \) from a supercycle on \( \hat{X}_\tau \). This will give us a more clear picture of mirror symmetry between \( T^2_{\theta,\tau} \) and \( \hat{X}_{\tau,\theta^{-1}} \).

Under the identification \( \mathcal{M}^s_{n,m} \cong \mathcal{S}\mathcal{M}_{n,m} \), the special Lagrangian cycle \( \mathcal{L}_{n,m} \) on \( \hat{X}_\tau \), given by (14), corresponds to a stable bundle \( E \) with slope \( \mu(E) = \frac{m}{n} \). By the deformation of the slope \( \mu(E) \) to \( \frac{m}{n-m\theta} \), we get a standard holomorphic bundle \( \mathcal{E}_{n,m}(\theta) \) on \( T^2_{\theta,\tau} \). The bundle \( \mathcal{E}_{n,m}(\theta) \) can be defined directly from the special Lagrangian cycle \( \mathcal{L}_{n,m} \). Our construction is basically based on [1]. Let us first consider a simple case when \( n = 1, m = 0 \). In this case, the special Lagrangian cycle is represented by the line \( y = 0 \) in \( \hat{X}_\tau \cong \mathbb{R}^2/\mathbb{Z}^2 \) and it corresponds to the trivial line bundle on \( X_\tau \). On the foliated torus \( \hat{X}_{\tau,\theta^{-1}} \), the line \( y = 0 \) is a compact transversal for the \( \theta^{-1} \)-linear foliation and each leaf meets the line countably many times. Associated to the intersection points with the line \( y = 0 \), each leaf defines the rotation through the angle \( \theta \) on \( S^1 \), which gives a \( \mathbb{Z} \)-action on \( S^1 \). The \( C^* \)-algebra of the group action, defined from the algebra of compactly supported smooth functions on \( S^1 \times \mathbb{Z} \), is known to be the irrational rotation \( C^* \)-algebra or the noncommutative torus \( A_\theta \) (see [1] for details). Regarding \( A_\theta \) as a free module \( \mathcal{E}_{1,0}(\theta) \) on \( T^2_{\theta} \), connections on \( \mathcal{E}_{1,0}(\theta) \) are simply the derivations on \( A_\theta \). Together with the complex parameter \( \tau \), determined by the complexified Kähler form on \( \hat{X}_\tau \), the derivations on \( A_\theta \) give a holomorphic structure on the free module \( \mathcal{E}_{1,0}(\theta) \) as given in [1]. Thus we see that the special Lagrangian cycle represented by the line \( y = 0 \) naturally defines the trivial line bundle \( \mathcal{E}_{1,0}(\theta) \cong A_\theta \), just as it defines
the trivial line bundle on $X$. This is one reason why we take $\hat{X}_{\tau,\theta^-1}$ instead of $\hat{X}_{\tau,\theta}$ as the mirror reflection of $T^2_{\theta,\tau}$.

On the other hand, the leaves of the $\theta^{-1}$-linear foliation rotate the special Lagrangian cycle $L_{n,m}$, $m \neq 0$, different angle $\theta'$ from $\theta$. Thus the leaf action on $L_{n,m}$ defines another noncommutative torus $A_{\theta'}$, which is strongly Morita equivalent to $A_{\theta}$, so that $A_{\theta} \cong \text{End}_{A_{\theta}}(E)$. The finitely generated projective $A_{\theta}$-module $E = E_{n,m}(\theta)$ is obtained from the space of leafwise paths starting from the line $y = 0$ and ending at $L_{n,m}$. Associated to each intersection points of the leaves and the line $L_{n,m}$, the $x$-intercepts of the leaves are parameterized by

$$x = \frac{n - m\theta}{m} t + R_1 \pmod{1}, \quad t \in \mathbb{R}.$$  \hfill (16)

which determines the manifold

$$E_{n,m} = \{((x,0), t) \in \hat{X}_{\tau} \times \mathbb{R} | m(x - R_1) = (n - m\theta)t \pmod{1}\}.$$

From the equation (16), one finds that $E_{n,m}$ is the disjoint union of $m$-copies of $\mathbb{R}$. Thus the space of compactly supported smooth functions on $E_{n,m}$ is the Schwarz space $S(\mathbb{R}) \otimes \mathbb{C}^m$. Let $W_1$ and $W_2$ be unitary operators in $\mathbb{C}^m$ such that $W_1^m = W_2^m = 1$ and $W_1W_2 = \exp(2\pi i \frac{m}{n})W_2W_1$. Also, we define operators $V_1$ and $V_2$ on $S(\mathbb{R})$, from the equation (16), by

$$V_1f(t) = \exp(2\pi i R_1) \exp[2\pi i(n - m\theta)t]f(t)$$

$$V_2f(t) = f(t + 1),$$

so that $V_1V_2 = \exp[-2\pi i(\frac{n}{m})]V_2V_1$. Then the action of $T^2_{\theta}$ on $E_{n,m}(\theta)$ is determined by $U_1 = V_1 \otimes W_1$ and $U_2 = V_2 \otimes W_2$. In other words, the space $C_c^\infty(E_{n,m}) \cong S(\mathbb{R}) \otimes \mathbb{C}^m$ is the basic module $E_{n,m}(\theta)$ on $T^2_{\theta}$, discussed in Section 2. Now the manifold together with the module action is uniquely determined up to translation of $R_1$. Thus the moduli space is identified with $S^1$. To get the full information of $E_{n,m}(\theta)$, one has to take into account of the constant curvature connections on $E_{n,m}(\theta)$. Form the definition 3 of connection, constant curvature connections are uniquely determined up to translations and we can define the connections as in Section 2,

$$\nabla_1 = 2\pi i(n - m\theta)t + 2\pi i R_1$$

$$\nabla_2 = \frac{d}{dt} + 2\pi i R_2$$

so that $\nabla_1$ determines the position of $E_{n,m}$ or the special Lagrangian cycles and $\nabla_2$ determines the holonomy or the monodromies of the connection $A$. Finally, the complexified Kähler form on $\hat{X}_{\tau}$ defines the complex structure $\nabla_1 + \tau \nabla_2$ so that we see that $S \mathcal{M}_{n,m} \cong \mathcal{M}_{n,m}(\theta)$.\"
5 Discussion

In this paper, we proposed that a linear foliation on an ordinary symplectic two-torus can be regarded as a mirror reflection of a noncommutative complex two-torus. Analogously we have discussed that the deformation quantization and the linear foliation $C^*$-algebra are related by mirror symmetry. Also, we showed that the relevant moduli spaces are naturally identified. We may extend our study to the homological mirror symmetry of Kontsevich [16] based on the works [17, 8, 19] and a recent work [18]. In doing so, one may need to consider the Floer homology for foliations which was introduced in [9]. In the case of 2-dimensional complex tori, it is not exactly the noncommutative version in the sense of [9]. However, it will be interesting to compute explicitly the Floer homology for the foliations discussed in this paper and compare with the tensor product of holomorphic vectors on a noncommutative complex torus, computed in [11].

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