Observation of Flat Frequency Bands at Open Edges and Antiphase Boundary Seams in Topological Mechanical Metamaterials

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Motivated by the recent theoretical studies on a two-dimensional (2D) chiral Hamiltonian based on the Su-Schrieffer-Heeger chains [L. Zhu, E. Prodan, and K. H. Ahn, Phys. Rev. B 99, 041117(R) (2019)], we experimentally and computationally demonstrate that topological flat frequency bands can occur at open edges of 2D planar metamaterials and at antiphase boundary seams of ring-shaped or tubular metamaterials. Specifically, using mechanical systems made of magnetically coupled spinners, we reveal that the presence of the edge or seam bands that are flat in the entire projected reciprocal space follows the predictions based on topological winding numbers. The edge-to-edge distance sensitively controls the flatness of the edge bands and the localization of excitations. The analogue of the fractional charge state is also observed. Possible realizations of flat bands in a large class of metamaterials, including photonic crystals and electronic metamaterials, are discussed.

Flat energy bands have gained a lot of attention, particularly following the discovery of superconductivity in twisted bilayer graphenes [1–5] and the pursuit of nearly flat bands in the fractional Chern insulators [6–8]. They have also been the focus of intense research interest in photonic crystals, such as the Lieb lattice, due to the possibility of trapping photons, which has technological significance [10–13]. Recently, there has been a theoretical proposal for flat energy bands within antiphase and twin boundaries and at open edges in a system described by a topological two-dimensional (2D) model Hamiltonian [14]. Unlike twisted bilayer graphene or the fractional Chern insulators, the flatness of the bands in the entire projected reciprocal space does not require tuning of parameters. Unlike the Lieb lattice, the flat band states occur only at edges or domain boundaries, giving a unique controllability through patterning. In this Letter, we experimentally demonstrate the realization of the model Hamiltonian and the flat bands in metamaterials, using mechanical systems made of interacting spinners as a specific example [15, 16]. By examining how the width of the edge band narrows in frequency as the edge-to-edge distance increases and directly observing oscillation modes, we show the presence of the topological flat frequency bands localized at the edges. It is revealed that the size of the localized excitations at the edges correlates with the width of the edge frequency band. The analogue to electronic charge fractionalization [17] is also found. We further experimentally verify the presence of a mid-gap mode localized at the antiphase boundary seam of a ring-shaped spinner system, and computationally find a flat antiphase boundary seam band for a tubular system.

Spinners could have very small rotational friction, which makes them ideal components of mechanical metamaterials as well as toys. By attaching magnets at the end of spinner arms, systems of magnetically coupled spinners have been shown to be a versatile experimental platform for various model Hamiltonians [15, 16]. In the mapping between electronic tight-binding Hamiltonians and magnetically coupled spinner systems, the intersite electron hopping corresponds to the interspinner magnetic interaction, which is controlled by the distance between the magnets. The electronic model Hamiltonians for the flat bands at open edges and twin and antiphase boundaries studied in Ref. [14] are based on a particular 2D extension of the Su-Schrieffer-Heeger (SSH) model [18]. Specifically, unlike other 2D SSH models [19, 20], SSH chains with alternating intersite hopping strengths are shifted and stacked in the direction perpendicular to the chains. With only the nearest neighbor hoppings considered, the 2D system preserves the chiral symmetry of the one-dimensional (1D) SSH system. With the constant interchain coupling weaker than the average intrachain coupling, a gap opens between the bulk bands and the topology of the system is characterized by the winding number, which depends on the direction of edges or boundaries. The bulk-boundary correspondence predicts flat zero energy edge or boundary bands of bipartite states for the chiral 2D SSH system, similar to the 1D SSH system.

One of the 2D spinner systems and its schematic diagram are shown in Figs. 1(a) and 1(b), respectively. The spinners have six arms, separated by 60°. With magnets attached to the 0°, 60°, 180° and 240° direction arms and the spinners arranged in quasitriangular lattices, the spinner systems are equivalent to the electronic systems of quasisquare lattices with electron hoppings in the 0°, 90°, 180° and 270° directions, studied in Ref. [14]. The spinner systems are assembled with the edges in the 0° and 120° directions, equivalent to the 0° and 135° directions for the quasisquare electron systems. The spinners are indexed as (n₁, n₂) with n₁ = 1, ..., N₁ and n₂ = 1, ..., N₂ (See Fig. 1(b)). The SSH chains run along the horizontal direction with alternating intrachain couplings, represented by the red and the blue lines in Fig. 1(b). The chains are coupled along the 60° direction with a constant interchain coupling, represented by the green lines. A unit cell is marked by a purple el-
The azimuthal direction and an antiphase boundary seam, obtained by joining the through the interaction between a magnet on either is driven to oscillate at a chosen frequency by an actuator in measurements, one of the spinners and 1(b), providing the necessary potential energies to spinners placed around the edges, as shown in Figs. 1(a) and 1(b), (c) Tubular model system with thirteen spinners in the azimuthal direction and an antiphase boundary seam, obtained by joining the (1, n2) and (13, n2 + 1) spinners of a planar system like (b). The grey balls represent fixed spinners interacting with the spinners at the seam. The magnification shows the antiphase boundary seam with the weak azimuthal couplings on both sides, represented by the blue lines, and the couplings with the fixed spinners, represented by the purple lines.

![Diagram](image)

FIG. 1. (a) Photograph of a 6 × 6 spinner system, where rotatable spinners and magnetically coupled arms are highlighted and fixed spinners and decoupled arms without magnets are shaded. (b) Illustration of the 6 × 6 spinner system pictured in (a), where the orange and grey balls represent rotatable and fixed spinners, respectively. The purple ellipse represents a unit cell. The blue [red] lines indicate the couplings within [between] the unit cells within the 1D SSH chains along the 0° direction. The green lines indicate the constant interchain coupling, which is chosen smaller than the average intrachain coupling. Coordinates in the form of (n1, n2) are used to describe the position of the spinners. Coordinates in the form of (n1, n2) are used to describe the position of the spinners. (c) Tubular model system with thirteen spinners in the azimuthal direction and an antiphase boundary seam, obtained by joining the (1, n2) and (13, n2 + 1) spinners of a planar system like (b). The grey balls represent fixed spinners interacting with the spinners at the seam. The magnification shows the antiphase boundary seam with the weak azimuthal couplings on both sides, represented by the blue lines, and the couplings with the fixed spinners, represented by the purple lines.

With the parameter values found in Ref. 15 for various intermagnet distances, the spectra are calculated to decide which spinners to actuate and measure at, so that the bulk and the edge band widths are well represented in the spectra. It is found that actuating and measuring at the (N1, 2) spinner [(N1 − 1, 1) spinner] gives the spectrum that represents the edge [bulk] band width reasonably well for the topological systems. By choosing the intermagnet distances of 5.0, 8.0, and 9.0 mm for the interactions represented by the red, blue, and green lines respectively in Fig. 1(b), we realize topological systems with the winding numbers \( \nu(120°) = \nu(0°) = 0 \), and by choosing 8.0, 5.0, and 9.0 mm, nontopological systems with the winding numbers \( \nu(120°) = \nu(0°) = 0 \), according to Ref. 14.

Results for the topological systems are displayed in Fig. 2. Spectra obtained by actuating and measuring at the (N1 − 1, 1) spinners for the 4 × 6, 6 × 6 and 8 × 6 topological systems are shown in Figs. 2(a)-2(c), respectively, each of which reveals upper and lower bulk bands, marked by blue areas, and a gap in between. Inside the gap, some of the edge modes are detected for the 4 × 6 system, as one of them marked by a red arrow in Fig. 2(a).

Such features become less prominent as the edge-to-edge distance \( N_1 \) increases to 6 and 8, as shown in Figs. 2(b) and 2(c). Figures 2(d)-2(f) show spectra obtained by actuating and measuring at the (N1, 2) spinners for the same topological systems as in Figs. 2(a)-2(c), respectively. Edge bands, marked by red areas, appear within the gaps of the bulk spectra. Similarly to Fig. 2(a), the bulk modes are also seen in Fig. 2(d) for the 4 × 6 system, as two of them marked by blue arrows. Figures 2(a)-2(f) show systematic changes in the bulk and the edge band widths, which are plotted as solid symbols in Fig. 2(g), along with the results from theoretical calculations for up to \( N_1 = 12 \), shown as open symbols. The experimental results show both upper and lower bulk band widths increase as the edge-to-edge distance \( N_1 \) increases from 4 to 6, and to 8, due to the finite size effect. The theoretical results predict that the bulk band widths saturate as \( N_1 \) increases further up to 12. In contrast to the bulk bandwidth, the edge band width from the experiments narrows rapidly as the edge-to-edge distance increases, consistent with the theoretical results. The results in Fig. 2(g) clearly indicate that all edge
FIG. 2. Results for the topological systems. (a)-(c) Bulk mode spectra experimentally obtained by actuating and measuring at the \((N_1 - 1, 1)\) spinner for the \(4 \times 6, 6 \times 6\) and \(8 \times 6\) topological systems, respectively. The blue areas indicate the lower and upper bulk bands. (d)-(f) Edge mode spectra experimentally obtained by actuating and measuring at the \((N_1, 2)\) spinner for the same topological systems as in (a)-(c), respectively. Red areas indicate the edge bands. A red arrow in (a) [blue arrows in (d)] indicates an example of the edge [bulk] modes appearing in the bulk [edge] spectra due to the short edge-to-edge distance for the \(4 \times 6\) system. (g) Experimental and theoretical bulk and edge band widths versus the system size \(N_1 \times N_2\).

modes would have an identical frequency and the edge band would be completely flat in the reciprocal space, as the edge-to-edge distance \(N_1\) increases further, confirming the main predictions in Ref. \[14\].

By exchanging the strong and the weak intrachain couplings, the topological systems turn into nontopological systems. The experimental results for \(6 \times 6\) topological and nontopological systems are shown in Figs. 3(a) and 3(b), respectively. For the topological system, the edge band is prominent in the spectra obtained from the \((6, 2)\) and \((6, 1)\) spinners, located within the bulk band gap in the spectrum obtained from the \((5, 1)\) spinner. In contrast, for the nontopological system, the edge band disappears from the gap, leaving only the lower and upper bulk bands in the spectra. The results clearly show the difference between the topological and nontopological systems and the topological origin of the edge states \[14\].

For the topological systems, the presence of the edge modes depends on the direction of the open edges, determined by the winding numbers \(\nu(120^\circ) = 1\) and \(\nu(0^\circ) = 0\). With the winding number \(\nu = 0\) outside the open edges, the edge modes are expected to occur only along the \(120^\circ\) direction edges, not along the \(0^\circ\) direction edges. To directly test these predictions, we build a \(4 \times 4\) topological spinner system with the same number of spinners along the \(0^\circ\) and \(120^\circ\) directions, and actuate the \((4, 1)\) spinner, which belongs to both \(0^\circ\) and \(120^\circ\) direction edges, at a frequency of 23.6 Hz within the edge band to see along which direction the edge mode appears. The oscillation patterns are displayed in Fig. 4(a), in which the colors approximately represent the oscillation amplitudes of individual spinners, estimated from the slow motion movies by eyes (See the Supplemental Material for video recordings. \[21\]). The actuated spinners are marked by stars in Fig. 4. It is clear that the edge mode appears along the edges in the \(120^\circ\) direction, not in the \(0^\circ\) direction, consistent with the topological bulk-boundary correspondence. It also confirms that the bands in the bulk band gap in Figs. 2(d)-2(f) and 3(a) are indeed the edge bands. For the \(4 \times 4\) topological
systems with \( N_1 = 6 \) and \( N_1 = 8 \) are studied by actuating \((N_1, 1)\) spinner at edge mode frequencies. The results shown in Figs. 2(a)-2(c) and 3(a) are indeed the bulk bands. As the edge-to-edge distance \( N_1 \) increases and the edge band becomes flatter in the reciprocal space, the excitation at the edges is expected to be more localized along the edges, which could be useful, for example, to trap photons in photonic crystals. To directly verify this, the \( N_1 \times 4 \) systems with \( N_1 = 6 \) and 8 are studied by actuating \((N_1, 1)\) spinner at edge mode frequencies. The results shown in Figs. 2(c) and 3(d) (See the Supplemental Material for the video recordings.) reveal that edge modes decay much faster along the edges compared to the \( 4 \times 4 \) system in Fig. 4(a), consistent with the enhanced localization as the edge band gets narrower (See Fig 2(e)).

For the \( 8 \times 4 \) system shown in Fig. 4(d), it is striking that only \((1, 4)\) spinner, other than the actuated \((8, 1)\) spinner, shows an appreciable oscillation, while the oscillations of all other spinners between them are almost negligible, which is the analogue of the fractional charge state [17] and may have technological applications, such as secure communications deterring eavesdropping.

Although the homogeneous systems do not host flat bands inside the bulk, inhomogeneous systems are predicted to host flat bands at the antiphase or twin boundaries separating domains of different winding numbers [14, 22]. It is well known that the antiphase boundary of the 1D SSH chain hosts the zero energy state in the bulk band gap, because it separates domains with the winding number \( \nu = 0 \) and \( \nu = 1 \) [17, 23]. To force the system to have an antiphase boundary but no edges, a ring-shaped 1D SSH spinner system with an odd number (fifteen) of the spinners is built, as schematically shown in the inset of Fig. 5(a), where the red and the blue lines represent the interactions between magnets separated by 5.0 and 8.0 mm, respectively. A fixed spinner, shown as a grey ball in the inset, is placed just outside the seam with the intermagnet distance of 6.5 mm with the spinner #1, so that the chiral symmetry is preserved. By actuating and measuring at the spinner #1 [spinner #2] at the seam [next to the seam], the antiphase boundary seam mode [bulk mode] is revealed in the spectrum, as shown in the red [blue] line in the main panel of Fig. 5(a). The seam mode peak is present in the bulk gap, consistent with the theory [14, 17, 24]. The slow motion movie of the seam mode (See the Supplemental Material.) reveals an oscillation of every other spinners in both directions from the spinner #1 with fast decaying amplitudes, consistent with bipartite antiphase boundary modes found in the theory [17, 23] and experiments for other 1D SSH metamaterials [23, 25].

We also theoretically consider tubular SSH spinner systems with odd numbers of the spinners in the azimuthal direction, so that the antiphase boundary seams are forced to be present, while edges are absent. Specifically, by rolling a ribbon with an odd number \( N_1 \) and an infinite \( N_2 \) in Fig. 4(b), and joining the \((1, n_2)\) and \((N_1, n_2 + 1)\) spinners at the 120° direction edges, we theoretically create a tubular model system with an antiphase boundary seam, as shown in Fig. 5(c) and the inset of Fig. 5(b) for \( N_1 = 13 \). This seam is locally equivalent to the 120° antiphase boundary separating topological and nontopological domains with \( \nu(120°) = 1 \) and \( \nu(120°) = 0 \) respectively [14, 17], and, therefore, is expected to host topological flat bands. With a periodic boundary condition along the axis of the tube, the frequency versus the wave vector \( k_2 \) along the \( n_2 \)-direction is calculated and shown in Fig. 5(b), which indeed reveals an almost flat antiphase boundary seam band (red line) inside the gap between the upper and lower bulk bands (blue lines). The width of the seam band approaches zero as \( N_1 \) increases, similar to the edge band in the planar systems. In spite of the different geometries between the tubular and planar systems, the flat band persists, which demonstrates the robustness of the flat bands localized at open edges and antiphase and twin boundaries in these 2D chiral systems based on the SSH chains.

The experimental results for the spinner systems presented here have broad implications for other metamaterials, particularly for a large class of metamaterials experimentally shown to realize the 1D SSH model and its edge or antiphase boundary states. It includes photonic [19, 25, 29], electronic [30], plasmonic [31], acoustic [29], and microwave [32] metamaterials, and lattices of the Bose-Einstein condensates [33], circuit elements...
In particular, flat bands and localized modes have significant technological applications in photonics. Mapping between models for photonic crystals and the electronic tight-binding SSH Hamiltonian, described in Ref. [39], indicates that the chiral 2D SSH Hamiltonian in Ref. [14] could be realized in photonic crystals, in which photons could be trapped at the edges or twin/antiphase boundaries and seams, or could be slowly guided along designed paths [14]. Further, in electronic metamaterials, the effects of electronic correlations would be enhanced for the flat bands, which could result in, for example, edge magnetism [40]. Experimental realizations of such flat bands, localized states, and slowly guided states in various metamaterials could lead to new device applications.

In summary, using the systems of the magnetically coupled spinners, we have experimentally demonstrated the presence of the flat bands of the topological origin localized at open edges and antiphase boundary seams in metamaterials. It has been shown that the flatness of the edge bands and the size of the localized states at the open edges are controlled by the edge-to-edge distance. Analyses for the ring-shaped and tubular systems with antiphase boundary seams reveal the midgap state and the flat band localized at the seams. The results found here would apply to other metamaterials also, potentially leading to novel phenomena and applications.

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