GENERAL DECAY OF SOLUTIONS OF A BRESSE SYSTEM WITH VISCOELASTIC BOUNDARY CONDITIONS

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Abstract. In this paper we are concerned with a multi-dimensional Bresse system, in a bounded domain, where the memory-type damping is acting on a portion of the boundary. We establish a general decay results, from which the usual exponential and polynomial decay rates are only special cases.

1. Introduction. In their study on networks of flexible beams, Lagnese, Leugering and Schmidt [15] derived a general model for 3-d nonlinear beams. A special case of this model is a linear planar, shearable beam whose motion is governed by the following system of partial differential equations:

\[
\begin{align*}
\rho_1 \frac{\partial^2 u}{\partial t^2} - \kappa \frac{\partial^2 u}{\partial x^2} - \kappa \frac{\partial v}{\partial x} - \kappa l \frac{\partial w}{\partial x} - \kappa_0 l \left( \frac{\partial w}{\partial x} - lu \right) &= 0, \\
\rho_2 \frac{\partial^2 v}{\partial t^2} - EI \frac{\partial^2 v}{\partial x^2} + \kappa \left( \frac{\partial u}{\partial x} + v + lw \right) &= 0, \\
\rho_1 \frac{\partial^2 w}{\partial t^2} - \kappa_0 \frac{\partial^2 w}{\partial x^2} + \kappa_0 l \frac{\partial u}{\partial x} + \kappa l \left( \frac{\partial u}{\partial x} + v + lw \right) &= 0,
\end{align*}
\]

(1)

where \((x,t) \in (0,L) \times (0,\infty)\) and \(u, v\) and \(w\) represent the vertical displacement, rotation angle, and longitudinal displacement, respectively, of the point \(x\) of the beam at the instant \(t\). The coefficients \(\rho_1, \rho_2, E\) and \(I\) denote respectively the mass per unit length, the mass moment of inertia of a cross-section of the beam, Young’s modulus and the moment of inertia of a cross-section of the beam. The coefficient \(\kappa_0, \kappa\) and \(l\) are equals to \(EA, \kappa'GA\) and \(R^{-1}\) respectively and where \(G\) is the modulus of elasticity in shear, \(A\) is the cross sectional area, \(\kappa'\) is the shear factor and \(R\) denotes the radius of the curvature.

We recall that systems (1) are also known as a circular arch problem, consisting of three coupled wave type equations, originally derived by Bresse [4]. A more recent discussion about mathematical modeling of the Bresse system can be found in Lagnese et al. [15], see also Alabau et al. [1].

Let us mention some known results on the decay rates for the Bresse model. In [16], Liu and Rao studied the stabilization for the Bresse system with two different temperature mechanisms affecting the longitudinal displacement and shear angle

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displacement. Under the equal wave speeds, they established an exponential energy decay rate. Otherwise, they showed that the smooth solution decays polynomially to zero with rates $t^{-1/2}$ or $t^{-1/4}$ provided the boundary conditions are Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet type, respectively.

An important problem in the Bresse system is to find a minimum dissipation by which their solutions decay uniformly to zero in time. In this direction we have the paper of Fatori and Rivera [8], which improved the paper by Liu and Rao [16]. They showed that, in general, the Bresse system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave speeds and the regularity of the initial data. Moreover, they introduced a necessary condition to dissipative semigroup decay polynomially. This result allowed them to show some optimality to the polynomial rate of decay.

In [1], the Bresse system with frictional damping was considered by Alabau et al. The authors showed that the Bresse system is exponentially stable if and only if the velocities of waves propagations are equal. Also, they showed that when the wave speeds are not the same, the system is not exponentially stable, and they proved that the solution in this case goes to zero polynomially, with rates that can be improved by taking more regular initial data. The rate of polynomial decay was improved by Fatori and Monteiro [7]. The indefinite damping acting on the shear angle displacement was considered by Palomino et al. in [9].

In [20], Noun and Wehbe extended the results from Alabau-Boussouira et al. [1], by taking into consideration the important case when the dissipation law is locally distributed and improved the polynomial energy decay rate. The authors studied the energy decay rate of the Bresse system with one locally internal distributed dissipation law acting on the equation about the shear angle displacement. Under the equal speed wave propagation condition, they showed that the system is exponentially stable. On the contrary, they established a new polynomial energy decay rate.

Related to this subject, we can mention the work of Khemmoudj and Hamadouche [14]. In that paper, the authors obtained an asymptotic stability of a class of Bresse-type system with three boundary dissipations and with a rigid body attached to its free end. They showed that exponential stabilization can be achieved by applying force and moment feedback boundary controls on the shear, longitudinal and transverse displacement velocities at the point of contact between the mass and the beam.

Concerning the asymptotic behavior of the Bresse system with past memory acting on the three equations we cite the work of Guesmia et al [12]. In that paper the authors showed, under suitable conditions on the initial data and the memories, that the Bresse system converges to zero when time goes to infinity, and they provided a connection between the decay rate of energy and the growth of memories at infinity.

In [2], Santos et al. considered the Bresse system with past history acting only on the shear angle displacement. They show the exponential decay of the solution if and only if the wave speeds are the same. If not, they show that the Bresse system is polynomial stable with optimal decay rate.

In this paper, taking into account the longitudinal displacement $w$, we are concerned with the following generalization of multidimensional Timoshenko problems studied in [10], [17] and [24], that is we consider
\begin{equation}
\begin{aligned}
\rho_1 u_{tt} - \Delta u - \alpha_1 \sum_{i=1}^n \frac{\partial u}{\partial x_i} + (\alpha_1 + \alpha_2) \sum_{i=1}^n \frac{\partial w}{\partial x_i} + \beta_1 u + a(x) f_1(u, v, w) = 0, \\
\rho_2 v_{tt} - \Delta v - \beta_2 v + \beta_2 w + a(x) f_2(u, v, w) = 0, \\
\rho_1 w_{tt} - \Delta w + (\alpha_1 + \alpha_2) \sum_{i=1}^n \frac{\partial w}{\partial x_i} + \beta_2 v + \beta_2 w + a(x) f_3(u, v, w) = 0,
\end{aligned}
\end{equation}

subject to the following boundary conditions
\begin{align}
&u(x, t) = v(x, t) = w(x, t) = 0, &\text{on } \Gamma_0 \times \mathbb{R}^+, \\
&u(x, t) = -\int_0^t h_1(t-s) \left( \frac{\partial u}{\partial \nu} + b_1(x)(v + w) \right) ds, &\text{on } \Gamma_1 \times \mathbb{R}^+, \\
v(x, t) = -\int_0^t h_2(t-s) \frac{\partial v}{\partial \nu} ds, &\text{on } \Gamma_1 \times \mathbb{R}^+, \\
w(x, t) = -\int_0^t h_3(t-s) \left( \frac{\partial w}{\partial \nu} - b_2(x)u \right) ds, &\text{on } \Gamma_1 \times \mathbb{R}^+,
\end{align}

and initial conditions
\begin{align}
(u(0), v(0), w(0)) = (u_0, v_0, w_0), \\
(\sqrt{\rho_1} u_1(0), \sqrt{\rho_2} v_1(0), \sqrt{\rho_1} w_1(0)) = (\sqrt{\rho_1} u^1, \sqrt{\rho_2} v^1, \sqrt{\rho_1} w^1),
\end{align}

where \( \Omega \) is a bounded open set of \( \mathbb{R}^n (n \geq 2) \) with a \( C^2 \)-boundary \( \Gamma = \partial \Omega \). Let \( \Gamma_0 \) and \( \Gamma_1 \) be closed nonempty disjoint subsets of \( \Gamma \) with \( \Gamma = \Gamma_0 \cup \Gamma_1 \), \( \Gamma_0 \cap \Gamma_1 = \phi \) and \( \text{meas}(\Gamma_0) > 0, \text{meas}(\Gamma_1) > 0 \). By \( \nu(x) \) we represent the exterior unit normal vector at \( x \in \Gamma_1 \). We will assume that \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are sufficiently small positive numbers, such that \( \beta_1 > n\alpha_2, \beta_2 > n\alpha_1, \) and
\begin{align}
a \in C^1(\overline{\Omega}), & \quad a(x) \geq a_0 > 0 \quad a.e. \quad in \quad \overline{\Omega},
\end{align}

where \( a_0 \) is a positive constant.

(i) \textbf{Assumptions on the relaxation functions.} The relaxation functions \( h_i, i = 1, 2, 3 \) are considered positive, non-increasing and belonging to \( W^{1,2}(0, +\infty) \).

(ii) \textbf{Assumptions on the nonlinear functions.} For the coupling terms \( f_i, i = 1, 2, 3 \), we suppose that
\begin{enumerate}
\item \( f_i \in C^1(\mathbb{R}^3), \quad i = 1, 2, 3. \)
\item Additionally, we assume that there exists a nonnegative function \( F(u, v, w) \in C^2(\mathbb{R}^3) \) such that
\begin{align}
f_1(u, v, w) = \frac{\partial F}{\partial u}, & \quad f_2(u, v, w) = \frac{\partial F}{\partial v}, & \quad f_3(u, v, w) = \frac{\partial F}{\partial w}.
\end{align}
\item Further, we assume that \( F \) is homogeneous of order \( p + 1 \):
\begin{align}
F(\lambda u, \lambda v, \lambda w) = \lambda^{p+1} F(u, v, w), \quad \text{for all } \lambda > 0, \quad (u, v, w) \in \mathbb{R}^3.
\end{align}
\end{enumerate}

Since \( F \) is homogeneous, the Euler Homogeneous Function theorem yields the following useful identity:
\begin{align}
uf_1(u, v, w) + vf_2(u, v, w) + wf_3(u, v, w) = (p + 1) F(u, v, w).
\end{align}
The homogeneity of $F$ implies that there exists a constant $M > 0$ such that

$$|F(u,v,w)| \leq M \left(|u|^{p+1} + |v|^{p+1} + |w|^{p+1}\right). \tag{7}$$

**Remark 1.** There is a large class of functions satisfying the assumptions (5)-(7). For instance functions of the form

$$F(u,v,w) = a |u|^{p+1} + b |v|^{p+1} + c |w|^{p+1},$$

where $a, b, c$ are positive constants, satisfy assumptions (5)-(7) with $p \geq 3$. Indeed, a quick calculation shows that there exists $c_0 > 0$ such that

$$F(u,v,w) = c_0 \left(|u|^{p+1} + |v|^{p+1} + |w|^{p+1}\right).$$

Moreover, it is easy to compute and find that

$$uf_1(u,v,w) + vf_2(u,v,w) + wf_3(u,v,w) = (p + 1)F(u,v,w).$$

In what follows, we are going to assume that there exists $x_0 \in \mathbb{R}^n$ such that

$$\Gamma_0 = \{x \in \Gamma: \nu.m(x) \leq 0\}, \quad \Gamma_1 = \{x \in \Gamma: \nu.m(x) > 0\},$$

where $m(x) = x - x_0$.

Also we assume that $\rho_i \in C^1(\Omega)$, $i = 1, 2, 3$, are positive functions satisfying the following hypothesis

$$\nabla \rho_i.m \geq 0 \quad \text{for} \quad i = 1, 2, 3. \tag{8}$$

We consider the functions $b_j(x) \in W^{1,\infty}(\Gamma_1), \quad j = 1, 2$, given by

$$b_j(x) = \alpha_j \left(\sum_{i=1}^n \nu_i(x)\right), \quad j = 1, 2.$$

The boundary condition of memory type for Timoshenko system, has been studied by Santos [24]. By considering $k_i$ to be the resolvent kernels of $(-h_i'(h_i(0)))$ for $i = 1, 2$, he showed that the energy of the solution decays exponentially (polynomially) when $k_i$ and $-k_i'$, $i = 1, 2$, decay exponentially (polynomially). The same result has been established by Messaoudi and Soufyane [17] without assuming the exponential (polynomial) decay of $k_1$ and $k_2$ but only that their norms are small enough. In [19] the general decay for the same system has been proved.

Models with boundary conditions including a memory term which produces damping were proposed in [5], [6], [18] and [23] for the study of wave propagation, in [22] and [25] for the von Karman plate system and in [13], [11] and [26] in the context of Kirchhoff equations.

The main goal of this paper is to investigate the asymptotic behavior of the system (2-4) for resolvent kernels of general type decay and obtain a more general and explicit energy decay formula, from which the usual exponential and polynomial decay rates are only special cases of our result. The proof is mainly based on the use of a multiplier method and the introduction of a suitable Lyapounov functional.

Our paper is organized as follows. In section 2 we establish the existence and uniqueness for regular and weak solutions of system (2-4). In section 3 we state and prove the general decay of the solutions of our studied system.
2. Existence and regularity. In this section, we study the existence and regularity of solutions for the Bresse system \[2\]–\[4\]. The scalar product and norm of the real Hilbert space \(L^2(\Omega)\) are denoted by \((u, v)\) and \(|u|\), respectively. By \(V\) the Hilbert space is represented as
\[V = \{ u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0 \}.\]

First, we will use the second equation, the third equation and the fourth equation in \[3\] to estimate the terms \(\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu}\) and \(\frac{\partial w}{\partial \nu}\).

Defining the convolution product operator by
\[(h \ast \varphi)(t) = \int_0^t h(t-s)\varphi(s)ds, \quad (9)\]
and differentiating the second, the third and the fourth equations in \[3\], we arrive to the following Volterra equations:
\[\frac{\partial u}{\partial \nu} + b_1(x)(v+w) + \frac{1}{h_1(0)} h_1' \ast \left\{ \frac{\partial u}{\partial \nu} + b_1(x)(v+w) \right\} = -\frac{1}{h_1(0)} u_t, \quad (10)\]
\[\frac{\partial v}{\partial \nu} + \frac{1}{h_2(0)} h_2' \ast \frac{\partial v}{\partial \nu} = -\frac{1}{h_2(0)} v_t, \quad (11)\]
\[\frac{\partial w}{\partial \nu} - b_2(x)u + \frac{1}{h_3(0)} h_3' \ast \left\{ \frac{\partial w}{\partial \nu} - b_2(x)u \right\} = -\frac{1}{h_3(0)} w_t. \quad (12)\]

Applying the Volterra’s inverse operator, we get
\[\frac{\partial u}{\partial \nu} + b_1(x)(v+w) = -\frac{1}{h_1(0)} \{ u_t + k_1 \ast u_t \}, \quad (13)\]
\[\frac{\partial v}{\partial \nu} = -\frac{1}{h_2(0)} \{ v_t + k_2 \ast v_t \}, \quad (14)\]
\[\frac{\partial w}{\partial \nu} - b_2(x)u = -\frac{1}{h_3(0)} \{ w_t + k_3 \ast w_t \}, \quad (15)\]
where the resolvent kernels satisfy
\[k_i + \frac{1}{h_i(0)} h_i' \ast k_i = -\frac{1}{h_i(0)} h_i' \text{ for } i = 1, 2, 3. \quad (16)\]

Denoting by \(\tau_1 = \frac{1}{h_1(0)}, \tau_2 = \frac{1}{h_2(0)}\) and \(\tau_3 = \frac{1}{h_3(0)}\), the normal derivatives of \(u, v\) and \(w\) can be written as
\[\frac{\partial u}{\partial \nu} = -\tau_1 \{ u_t + k_1(0)u - k_1(t)u_0 + k_1' \ast u \} - b_1(x)(v+w), \quad (17)\]
\[\frac{\partial v}{\partial \nu} = -\tau_2 \{ v_t + k_2(0)v - k_2(t)v_0 + k_2' \ast v \}, \quad (18)\]
\[\frac{\partial w}{\partial \nu} = -\tau_3 \{ w_t + k_3(0)w - k_3(t)w_0 + k_3' \ast w \} + b_2(x)u. \quad (19)\]

Reciprocally, taking initial data such that \(u^0 = v^0 = w^0 = 0\) on \(\Gamma_1\), the identities \[17\], \[18\] and \[19\] imply the second, the third and the fourth equations in \[3\] respectively. Since we are interested in relaxation functions of more general decay and the boundary conditions \[17\], \[18\] and \[19\] involving the resolvent kernels...
$k_i, i = 1, 2, 3$, we want to know if $k_i$ has the same decay properties. The following lemma answers this question.

Let $h$ be a relaxation function and $k$ its resolvent kernel, that is,  
$$k(t) - (k * h)(t) = h(t). \tag{20}$$

**Lemma 2.1.** (See Lemma 2.1, [10]) If $h$ is a positive continuous function, then $k$ is also positive and continuous. Suppose that $h(t) \leq c_0 e^{-\int_0^t \gamma(\zeta) d\zeta}$, where $\gamma : [0, +\infty) \rightarrow \mathbb{R}^+$, is a nonincreasing function satisfying, for some positive constant $\varepsilon < 1$,

$$c_1 = \int_0^{+\infty} e^{-\int_0^t (1-\varepsilon) \gamma(\zeta) d\zeta} < \frac{1}{c_0}.$$  

Then $k$ satisfies

$$k(t) \leq \frac{c_0}{1 - c_0 c_1} e^{-\varepsilon \int_0^t \gamma(\zeta) d\zeta}.$$  

**Corollary 1.** (See [18]) Suppose that

$$h(t) \leq c_0 e^{-\gamma t}, \tag{21}$$

for $\gamma > c_0$. Then, there exists a positive constant $\varepsilon < 1$ such that

$$k(t) \leq \beta e^{-\varepsilon \gamma t}, \tag{22}$$

where $\beta > 0$ is a constant.

**Corollary 2.** (See [18]) Suppose that

$$h(t) \leq \frac{c_0}{(1+t)^p}, \tag{23}$$

for $c_0 < p - 1$. Then, there exists a positive constant $\varepsilon < 1$ such that

$$k(t) \leq \frac{\beta}{(1+t)^{\varepsilon p}}, \tag{24}$$

where $\beta > 0$ is a constant.

Based on Lemma 2.1, we will use the boundary relations (17), (18) and (19) instead of the second, the third and the fourth equations in (3). Let us define

$$(h \circ \varphi)(t) = \int_0^t h(t-s)[(\varphi(t) - \varphi(s))|^2 ds, \tag{25}$$

and

$$(h \circ \varphi)(t) = \int_0^t h(t-s)(\varphi(t) - \varphi(s))ds. \tag{26}$$

By using Hölder’s inequality, we have

$$|(h \circ \varphi)(t)|^2 \leq \left( \int_0^t |g(s)|ds \right) (|g| \circ \varphi)(t). \tag{27}$$

The next lemma gives an identity for the convolution product.
Lemma 2.2. (See Lemma 2.2, [21]) For real functions \( h, \varphi \in C^1(\mathbb{R}^+) \), we have
\[
(h \ast \varphi)\varphi_t = -\frac{1}{2} |\varphi(t)|^2 + \frac{1}{2} h' \varphi - \frac{1}{2} \frac{d}{dt} \left( h \varphi - \left( \int_0^t h(s)ds \right) |\varphi(t)|^2 \right).
\]

The well-posedness of system (2) - (4) is given by the following theorem.

Theorem 2.3. Let \( k_i \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+) \), \( \rho_i \in C^1(\mathbb{R}) \), \( i = 1, 2 \). Assume that \((u^0, v^0, w^0) \in (H^2 \cap V)^3 \) and \((u^1, v^1, w^1) \in V^3 \) with
\[
\begin{align*}
\frac{\partial u^0}{\partial t} + \tau_1 u^0 + b_1 (v^0 + w) &= 0 \quad \text{on } \Gamma_1, \\
\frac{\partial v^0}{\partial t} + \tau_2 v^1 &= 0 \quad \text{on } \Gamma_1, \\
\frac{\partial w^0}{\partial t} + \tau_3 w^0 - b_2 w^0 &= 0 \quad \text{on } \Gamma_1.
\end{align*}
\]
Then there exists only one strong solution \((u, v, w)\) of the Bresse system (2) - (4) satisfying
\[
\begin{align*}
&u, v, w \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap V), \quad \sqrt{\rho_1} u_t, \sqrt{\rho_2} v_t, \sqrt{\rho_1} w_t \in L^\infty([0, \infty), L^2(\Omega)), \\
&u_t, v_t, w_t \in L^\infty(\mathbb{R}^+; V), \quad \sqrt{\rho_1} u_{tt}, \sqrt{\rho_2} v_{tt}, \sqrt{\rho_1} w_{tt} \in L^\infty([0, \infty), L^2(\Omega)).
\end{align*}
\]

Proof. The theorem can be proved, by making use of standard semi group arguments (see for instance, P. Pei et al. [21]).

3. Decay of solutions. In this section we study the asymptotic behavior of the solutions of system (2) - (4) when the resolvent kernels \( k_i, i = 1, 2, 3 \), satisfy
\[
k_i(0) > 0, \quad k_i(t) \geq 0, \quad k_i'(t) \leq 0, \quad k_i''(t) \geq \gamma_i(t)(-k_i'(t)), \quad (30)
\]
where \( \gamma_i : \mathbb{R}^+ \to \mathbb{R}^+ \) is a function satisfying the following conditions
\[
\gamma_i(t) > 0, \quad \gamma_i'(t) \leq 0, \quad \text{and } \int_0^{+\infty} \gamma_i(t)dt = +\infty \quad i = 1, 2, 3. \quad (31)
\]
By multiplying the first equation in (2) by \( u_t \), the second equation by \( v_t \) and the third equation by \( w_t \), integrating over \( \Omega \) using integration by parts, the boundary conditions, and \( [17] - [19] \), one can easily find that the first order energy of system (2) is given by
\[
E(t) = \frac{1}{2} \int_{\Omega} \left( |\rho_1 u_t|^2 + |\rho_2 v_t|^2 + |\rho_1 w_t|^2 + (\beta_1 - n\alpha_2)|u|^2 \right) dx \\
+ \frac{1}{2} \int_{\Omega} \left( (\beta_2 - n\alpha_1)(|v| + |w|)^2 \right) dx \\
+ \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + \frac{(1 - \alpha_1)}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{(1 - \alpha_2)}{2} \int_{\Omega} |\nabla w|^2 dx \\
+ \frac{1}{2} \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} + v + w \right|^2 dx + \frac{\alpha_2}{2} \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial w}{\partial x_i} - u \right|^2 dx \\
+ \int_{\Gamma_1} a(x) F(u, v, w) dx + \frac{\tau_1}{2} \int_{\Gamma_1} (k_1(t)|u|^2 - k_1'(t) u) d\Gamma_1
\]
Theorem 3.1. Given \((u^0, v^1), (w^0, w^1)\) \(\in (V \times L^2(\Omega))^3\). Assume that \((5)-(8), (30)\) and \((31)\) hold, with

\[
\lim_{t \to \infty} k_i(t) = 0, \quad i = 1, 2, 3,
\]

\[
((-n + \varepsilon_0)\delta - n + \varepsilon_0)a + 2m.\nabla a < 0, \quad \forall x \in \Omega.
\]

Then for some \(t_0\) large enough, if \((u_0, v_0, w_0) = (0, 0, 0)\) on \(\Gamma_1\), we have

\[
E(t) \leq cE(0)e^{-\omega \int_0^t \gamma(s)ds}, \forall t \geq t_0.
\]

Otherwise, If \((u_0, v_0, w_0) \neq (0, 0, 0)\) on \(\Gamma_1\) then

\[
E(t) \leq c \left\{ E(0) + \int_0^t k_0(s)[1 + e^{\int_0^s \gamma(\zeta)d\zeta}]ds \right\} e^{-\omega \int_0^t \gamma(s)ds},
\]

where

\[
\gamma(t) = \min\{\gamma_1(t), \gamma_2(t), \gamma_3(t)\},
\]

\[
k_0(t) = k_1^2(t) \int_{\Gamma_1} |u|^2 d\Gamma_1 + k_2^2(t) \int_{\Gamma_1} |v|^2 d\Gamma_1 + k_3^2(t) \int_{\Gamma_1} |w|^2 d\Gamma_1,
\]

and \(\omega\) is a fixed positive constant and \(c\) is a generic positive constant.

Lemma 3.2. Under the assumption of Theorem 3.1, the energy of the solution of \((2) - (4)\), satisfies

\[
\frac{dE}{dt} \leq -\frac{\tau_1}{2} \int_{\Gamma_1} |u|^2 d\Gamma_1 + \frac{\tau_1}{2} \int_{\Gamma_1} k_1^2(t)|u|^2 d\Gamma_1 - \frac{\tau_1}{2} \int_{\Gamma_1} k_1' \circ ud\Gamma_1
\]

\[
- \frac{\tau_2}{2} \int_{\Gamma_1} |v|^2 d\Gamma_1 + \frac{\tau_2}{2} \int_{\Gamma_1} k_2^2(t)|v|^2 d\Gamma_1 - \frac{\tau_2}{2} \int_{\Gamma_1} k_2' \circ vd\Gamma_1
\]

\[
- \frac{\tau_3}{2} \int_{\Gamma_1} |w|^2 d\Gamma_1 + \frac{\tau_3}{2} \int_{\Gamma_1} k_3^2(t)|w|^2 d\Gamma_1 - \frac{\tau_3}{2} \int_{\Gamma_1} k_3' \circ wd\Gamma_1.
\]

Proof. Multiplying the first equation in \((2)\) by \(u_i\) and integrating by parts over \(\Omega\) we obtain

\[
\frac{1}{2} \int_{\Omega} \left( |\rho_1|u_i|^2 + |\nabla u|^2 + \beta_1|u|^2 \right) dx
\]

\[
- \alpha_1 \int_{\Omega} \sum_{i=1}^n \frac{\partial v}{\partial x_i} u_i dx - (\alpha_1 + \alpha_2) \int_{\Omega} \sum_{i=1}^n \frac{\partial w}{\partial x_i} u_i dx
\]

\[
+ \int_{\Omega} \alpha(x) f_1(u,v,w) u_i dx
\]

\[
= \int_{\Gamma_1} \frac{\partial u}{\partial n} u_i d\Gamma_1.
\]
Using Gauss’s Theorem, we get
\[
\alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial v}{\partial x_i} u_t \, dx = \alpha_1 \int_{\Gamma_1} v u_t \left( \sum_{i=1}^n \nu_i \right) \, d\Gamma_1 - \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} v \, dx \quad (38)
\]
and
\[
\alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial w}{\partial x_i} u_t \, dx = \alpha_1 \int_{\Gamma_1} w u_t \left( \sum_{i=1}^n \nu_i \right) \, d\Gamma_1 - \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} w \, dx. \quad (39)
\]
Plugging the estimates (38) and (39) into (37), we find that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( \rho_1 |u_t|^2 + |\nabla u|^2 + \beta_1 |u|^2 \right) \, dx
\]
\[
+ \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} v \, dx + \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u_t}{\partial x_i} w \, dx \quad (40)
\]
\[
- \alpha_2 \int_\Omega \sum_{i=1}^n \frac{\partial w}{\partial x_i} u_t \, dx + \int_\Omega a(x) f_1(u, v, w) u_t \, dx
\]
\[
= \int_{\Gamma_1} \frac{\partial v}{\partial \nu} u_t d\Gamma_1 + \alpha_1 \int_{\Gamma_1} v u_t \left( \sum_{i=1}^n \nu_i \right) d\Gamma_1 + \alpha_1 \int_{\Gamma_1} w u_t \left( \sum_{i=1}^n \nu_i \right) d\Gamma_1.
\]
The second equation in (2) multiplied by \( v_t \) in \( L^2(\Omega) \), and integration by parts give
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( \rho_2 |v_t|^2 + |\nabla v|^2 + \beta_2 |v|^2 \right) \, dx
\]
\[
+ \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} v_t \, dx + \alpha_2 \int_\Omega w v_t \, dx
\]
\[
+ \int_\Omega a(x) f_2(u, v, w) v_t \, dx
\]
\[
= \int_{\Gamma_1} \frac{\partial v}{\partial \nu} u_t \, d\Gamma_1. \quad (41)
\]
Finally, the third equation in (2) multiplied by \( w_t \) in \( L^2(\Omega) \), and integrating by parts over yield
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( \rho_3 |w_t|^2 + |\nabla w|^2 + \beta_2 |w|^2 \right) \, dx
\]
\[
+ (\alpha_1 + \alpha_2) \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} w_t \, dx + \alpha_2 \int_\Omega w v_t \, dx
\]
\[
+ \int_\Omega a(x) f_3(u, v, w) w_t \, dx
\]
\[
= \int_{\Gamma_1} \frac{\partial w}{\partial \nu} u_t \, d\Gamma_1. \quad (42)
\]
Again, using Gauss’s Theorem, we have
\[
\alpha_2 \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} w_i \, dx = \alpha_2 \int_{\Gamma_1} u w_t (\sum_{i=1}^n v_i) \, d\Gamma_1 - \alpha_2 \int_\Omega \sum_{i=1}^n \frac{\partial w_t}{\partial x_i} u \, dx. \tag{43}
\]
Substituting the equation (43) in (42), we deduce
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (\rho_1 |u|^2 + |\nabla w|^2 + \beta_2 |w|^2) \, dx + \alpha_1 \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} w_i \, dx - \alpha_2 \int_\Omega \sum_{i=1}^n \frac{\partial w_t}{\partial x_i} u \, dx + \beta_2 \int_\Omega v w_t \, dx + \int_\Omega a(x) f_3(u,v,w) w_t \, dx
\]
\[
= \int_{\Gamma_1} \frac{\partial w}{\partial \nu} w_t d\Gamma_1 - \alpha_2 \int_{\Gamma_1} u w_t (\sum_{i=1}^n v_i) d\Gamma_1. \tag{44}
\]
Adding up (40), (41) and (44), using (17), (18), (19) and Lemma 2.2 we obtain the desired result.

Let \( \epsilon_0 > 0 \) be a small constant and define the following functional:
\[
\Phi(t) = \int_\Omega \left[ (2m \cdot \nabla u + (n - \epsilon_0)u) \rho_1 u_t + (2m \cdot \nabla v + (n - \epsilon_0)v) \rho_2 v_t \right] \, dx
\]
\[
+ \int_\Omega (2m \cdot \nabla w + (n - \epsilon_0)w) \rho_1 w_t \, dx.
\]
The following lemma plays an important role for the construction of the Lyapunov functional.

**Lemma 3.3.** Under the assumption of Theorem 3.1, the solution of (2)-(4), satisfies
\[
\frac{d}{dt} \Phi(t) \leq \int_{\Gamma_1} m.\nu (\rho_1 |u|^2 + \rho_2 |v|^2 + \rho_1 |w|^2) \, d\Gamma_1 - \int_{\Gamma_1} m.\nu |\nabla u|^2 \, d\Gamma_1
\]
\[
- c_1 \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} + v + w \right|^2 \, dx + \int_{\Gamma_1} \frac{\partial u}{\partial \nu} (2m.\nabla u + (n - \epsilon_0)u) \, d\Gamma_1
\]
\[
- (1 - \epsilon_0) \int_\Omega \left( |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \right) \, dx - c_2 \sum_{i=1}^n \int_\Omega \left| \frac{\partial w}{\partial x_i} - u \right|^2 \, dx
\]
\[
+ \int_{\Gamma_1} \frac{\partial v}{\partial \nu} (2m.\nabla v + (n - \epsilon_0)v) \, d\Gamma_1 + \int_{\Gamma_1} \frac{\partial w}{\partial \nu} (2m.\nabla w + (n - \epsilon_0)w) \, d\Gamma_1
\]
\[
- \int_{\Gamma_1} m.\nu |\nabla v|^2 \, d\Gamma_1 - \int_{\Gamma_1} m.\nu |\nabla w|^2 \, d\Gamma_1 - \epsilon_0 \int_\Omega (\rho_1 |u|^2 + \rho_2 |v|^2 + \rho_1 |w|^2) \, dx
\]
\[
+ \int_\Omega ((-n + \epsilon_0) \delta - n + \epsilon_0) a + 2m.\nabla a) F(u,v,w) \, dx.
\]
Proof. We multiply the first equation in (2) by \(2 m \cdot \nabla u + (n - \varepsilon_0)u\) to obtain
\[
\frac{d}{dt} \int_{\Omega} \left(2 m \cdot \nabla u + (n - \varepsilon_0)u\right) \rho_1 u_t dx = \int_{\Omega} 2 \rho_1 m \cdot \nabla u_t u_t dx + (n - \varepsilon_0) \int_{\Omega} \rho_1 |u_t|^2 dx \\
+ \int_{\Omega} 2 m \cdot \nabla u \Delta u dx + (n - \varepsilon_0) \int_{\Omega} u \Delta u dx \\
+ \alpha_1 \sum_{i=1}^{n} \int_{\Omega} (2 m \cdot \nabla u + (n - \varepsilon_0)u) \frac{\partial v}{\partial x_i} dx \\
+ (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \int_{\Omega} (2 m \cdot \nabla u + (n - \varepsilon_0)u) \frac{\partial w}{\partial x_i} dx \\
- \int_{\Omega} (a(x) f_1(u, v, w) + \beta_1 u) [2 m \cdot \nabla u + (n - \varepsilon_0)u] dx.
\]
Integrating by parts and using the relation \(\text{div} m = n\), we get
\[
\frac{d}{dt} \int_{\Omega} \left(2 m \cdot \nabla u + (n - \varepsilon_0)u\right) \rho_1 u_t dx \\
\leq \int_{\Gamma_1} m \nu \rho_1 |u_t|^2 d\Gamma_1 - \varepsilon_0 \int_{\Omega} \rho_1 |u_t|^2 dx - \int_{\Gamma_1} \nabla \rho_1 m |u_t|^2 d\Gamma_1 - \int_{\Omega} m \nu |\nabla u_t|^2 d\Omega \\
+ \int_{\Gamma_1} (2 m \cdot \nabla u + (n - \varepsilon_0)u) \frac{\partial u}{\partial n} d\Gamma_1 + (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \int_{\Omega} (2 m \cdot \nabla u + (n - \varepsilon_0)u) \frac{\partial w}{\partial x_i} dx \\
+ \alpha_1 \sum_{i=1}^{n} \int_{\Omega} (2 m \cdot \nabla u + (n - \varepsilon_0)u) \frac{\partial v}{\partial x_i} dx - (1 - \varepsilon_0) \int_{\Omega} |\nabla u|^2 dx \\
- \beta_1 \int_{\Gamma_1} m \nu |u|^2 d\Gamma_1 + \beta_1 \varepsilon_0 \int_{\Omega} |u|^2 dx - \int_{\Omega} a(x) f_1(u, v, w) [2 m \cdot \nabla u + (n - \varepsilon_0)u] dx.
\]
Similarly, multiplying the second equation in (2) by \((2 m \cdot \nabla v + (n - \varepsilon_0)v)\) and integrating over \(\Omega\), using integration by parts, we arrive at
\[
\frac{d}{dt} \int_{\Omega} \left(2 m \cdot \nabla v + (n - \varepsilon_0)v\right) \rho_2 v_t dx \\
\leq \int_{\Gamma_1} m \nu \rho_2 |v_t|^2 d\Gamma_1 - \varepsilon_0 \int_{\Omega} \rho_2 |v_t|^2 dx - \int_{\Gamma_1} \nabla \rho_2 m |v_t|^2 d\Gamma_1 \\
+ \int_{\Gamma_1} (2 m \cdot \nabla v + (n - \varepsilon_0)v) \frac{\partial v}{\partial n} d\Gamma_1 - \int_{\Omega} m \nu |\nabla v_t|^2 d\Omega - (1 - \varepsilon_0) \int_{\Omega} |\nabla v|^2 dx \\
+ \beta_2 \varepsilon_0 \int_{\Omega} |v|^2 dx + \beta_2 \varepsilon_0 \int_{\Omega} vwdx - \alpha_1 \sum_{i=1}^{n} \int_{\Omega} (2 m \cdot \nabla v + (n - \varepsilon_0)v) \frac{\partial u}{\partial x_i} dx \\
- \beta_2 \int_{\Gamma_1} m \nu |v|^2 d\Gamma_1 - \beta_2 \int_{\Gamma_1} m \nu vwdx - \int_{\Omega} a(x) f_2(u, v, w) [2 m \cdot \nabla v + (n - \varepsilon_0)v] dx.
\]
Finally, multiplying the third equation in (2) by \((2m_1m + (n - \varepsilon_0)w)\) and integrating over \(\Omega\), using integration by parts, to arrive at

\[
\frac{d}{dt} \int_{\Omega} (2m_1m + (n - \varepsilon_0)w)\rho_1 w_i dx
\]

\[
\leq \int_{\Gamma_1} m_1 \rho_1 |w_i|^2 \, d\Gamma_1 - \varepsilon_0 \int_{\Omega} \rho_1 |w_i|^2 \, dx - \int_{\Omega} \nabla \rho_1 \cdot m_1 |w_i|^2 \, dx - \int_{\Gamma_1} m_1 |\nabla w|^2 \, d\Gamma_1
\]

\[
+ \int_{\Gamma_1} (2m_1m + (n - \varepsilon_0)w) \frac{\partial w}{\partial \nu} \, d\Gamma_1 + \beta_2 \varepsilon_0 \int_{\Omega} vwdx - (1 - \varepsilon_0) \int_{\Omega} |\nabla w|^2 \, dx
\]

\[
+ \beta_2 \varepsilon_0 \int_{\Omega} |w|^2 \, dx - (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \int_{\Omega} (2m_1m + (n - \varepsilon_0)w) \frac{\partial u}{\partial x_i} \, dx
\]

\[- \int_{\Omega} a(x)f_3(u, v, w)[2m_1m + (n - \varepsilon_0)w] dx - \beta_2 \int_{\Gamma_1} m_1 |w|^2 \, d\Gamma_1 - \beta_2 \int_{\Gamma_1} m_1 |v|^2 \, d\Gamma_1.
\]

Adding up the above three inequalities and taking into account (5), we easily deduce

\[
\frac{d}{dt} \Phi(t)
\]

\[
\leq \int_{\Gamma_1} m_1 \rho_1 |w_i|^2 \, d\Gamma_1 - \varepsilon_0 \int_{\Omega} \rho_1 |w_i|^2 \, dx - \int_{\Omega} \nabla \rho_1 \cdot m_1 |w_i|^2 \, dx - \int_{\Gamma_1} m_1 |\nabla w|^2 \, d\Gamma_1
\]

\[- \int_{\Gamma_1} m_1 |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \, d\Gamma_1 + \int_{\Gamma_1} \frac{\partial u}{\partial \nu} (2m_1m + (n - \varepsilon_0)u) \, d\Gamma_1
\]

\[- \varepsilon_0 \int_{\Omega} (\rho_1 |w_i|^2 + \rho_2 |v_i|^2 + \rho_1 |w_i|^2) dx - (1 - \varepsilon_0) \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2) \, dx
\]

\[+ \int_{\Gamma_1} \frac{\partial v}{\partial \nu} (2m_1m + (n - \varepsilon_0)v) \, d\Gamma_1 + \alpha_1 \sum_{i=1}^{n} \int_{\Omega} (2m_1m + (n - \varepsilon_0)v) \frac{\partial v}{\partial x_i} \, dx
\]

\[+ \int_{\Gamma_1} \frac{\partial w}{\partial \nu} (2m_1m + (n - \varepsilon_0)v) \, d\Gamma_1 - \alpha_1 \sum_{i=1}^{n} \int_{\Omega} (2m_1m + (n - \varepsilon_0)v) \frac{\partial u}{\partial x_i} \, dx
\]

\[+ (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \int_{\Omega} (2m_1m + (n - \varepsilon_0)v) \frac{\partial w}{\partial x_i} \, dx - 2\beta_2 \int_{\Gamma_1} m_1 |v|^2 \, d\Gamma_1
\]

\[- (\alpha_1 + \alpha_2) \sum_{i=1}^{n} \int_{\Omega} (2m_1m + (n - \varepsilon_0)v) \frac{\partial u}{\partial x_i} \, dx + 2\beta_2 \varepsilon_0 \int_{\Omega} vwdx
\]

\[- \int_{\Omega} a(x)f_1(u, v, w)[2m_1m + (n - \varepsilon_0)v] dx - \beta_1 \int_{\Gamma_1} m_1 |u|^2 \, d\Gamma_1
\]

\[- \int_{\Omega} a(x)f_3(u, v, w)[2m_1m + (n - \varepsilon_0)v] dx + \beta_1 \varepsilon_0 \int_{\Omega} |u|^2 \, dx
\]

\[- \int_{\Omega} a(x)f_2(u, v, w)[2m_1m + (n - \varepsilon_0)v] dx + \beta_2 \varepsilon_0 \int_{\Omega} (|v|^2 + |w|^2) \, dx.
\]
Exploiting conditions (1) and (5), we get
\[- \int_{\Omega} a(x)f_1(u, v, w)[2m.\nabla u + (n - \varepsilon_0)u]dx \]
\[- \int_{\Omega} a(x)f_2(u, v, w)[2m.\nabla v + (n - \varepsilon_0)v]dx \]
\[- \int_{\Omega} a(x)f_3(u, v, w)[2m.\nabla w + (n - \varepsilon_0)w]dx \]
\[- \int_{\Omega} a(x)\frac{\partial F}{\partial u}[2m.\nabla u + (n - \varepsilon_0)u]dx - \int_{\Omega} a(x)\frac{\partial F}{\partial v}[2m.\nabla v + (n - \varepsilon_0)v]dx \]
\[- \int_{\Omega} a(x)\frac{\partial F}{\partial w}[2m.\nabla w + (n - \varepsilon_0)w]dx \]
\[\leq (n + \varepsilon_0)(1 + \delta) \int_{\Omega} a(x)F(u, v, w)dx - 2 \int_{\Omega} a(x)m.\nabla F(u, v, w)dx \]
\[\leq ((n + \varepsilon_0)\delta - n + \varepsilon_0) \int_{\Omega} a(x)F(u, v, w)dx \]
\[+ \int_{\Gamma_1} (2m.\nabla a)F(u, v, w)dx - \int_{\Gamma_1} a(x)(2m.\nabla a)F(u, v, w)d\Gamma_1 \]
\[\leq ((n + \varepsilon_0)\delta - n + \varepsilon_0) + 2m.\nabla a)F(u, v, w)dx. \]

Next, we use the fact that there exists a positive constant \(c\) such that
\[\alpha_1 \sum_{i=1}^{n} \int_{\Omega} \left( \frac{\partial v}{\partial x_i} m.\nabla u + \frac{\partial u}{\partial x_i} m.\nabla v + \frac{\partial w}{\partial x_i} m.\nabla u + \frac{\partial u}{\partial x_i} m.\nabla w \right) dx \]
\[+ \alpha_2 \sum_{i=1}^{n} \int_{\Omega} \left( \frac{\partial w}{\partial x_i} m.\nabla u + \frac{\partial u}{\partial x_i} m.\nabla w \right) dx \]
\[\leq c \max \{ \alpha_1, \alpha_2 \} \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2) dx, \] (46)

to obtain
\[
\frac{d}{dt}\Phi(t) \leq \int_{\Gamma_1} m.\nu(\rho_1|u_1|^2 + \rho_2|v_i|^2 + \rho_1|w_i|^2) d\Gamma_1 - \int_{\Gamma_1} m.\nu|\nabla v|^2 d\Gamma_1 \]
\[- \varepsilon_0 \int_{\Omega} (\rho_1|u_1|^2 + \rho_2|v_i|^2 + \rho_1|w_i|^2) dx - \int_{\Gamma_1} m.\nu|\nabla w|^2 d\Gamma_1 \]
\[+ \int_{\Gamma_1} \frac{\partial u}{\partial \nu} (2m.\nabla u + (n - \varepsilon_0)u) d\Gamma_1 - \int_{\Gamma_1} m.\nu|\nabla u|^2 d\Gamma_1 \]
\[- (1 - \varepsilon_0) \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2) dx - c_1 \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} + v + w \right|^2 dx \]
\[ + \int_{\Gamma_1} \frac{\partial v}{\partial \nu}(2m \nabla v + (n - \varepsilon_0)v) d\Gamma_1 + \int_{\Gamma_1} \frac{\partial w}{\partial \nu}(2m \nabla w + (n - \varepsilon_0)w) d\Gamma_1 \]

\[ - \beta_1 \int_{\Gamma_1} m.\nu |u|^2 d\Gamma_1 - \beta_2 \int_{\Gamma_1} m.\nu (|v| + |w|)^2 d\Gamma_1 - c_2 \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial w}{\partial x_i} - u \right|^2 dx \]

\[ + \int_{\Omega} \left( (n + \varepsilon_0) - n + \varepsilon_0 \right) \delta n + \varepsilon_0 a + 2m \nabla a) F(u, v, w) dx. \]

Using Poincaré inequality and taking \( \varepsilon_0 \) small enough, we get

\[ \frac{d}{dt} \Phi(t) \leq \int_{\Gamma_1} m.\nu (|u|^2 + |v|^2 + |w|^2) d\Gamma_1 \]

\[ - \varepsilon_0 \int_{\Omega} \left( \rho_1 |u|^2 + \rho_2 |v|^2 + \rho_1 |w|^2 \right) dx \]

\[ + \int_{\Gamma_1} \frac{\partial u}{\partial \nu} (2m \nabla u + (n - \varepsilon_0)u) d\Gamma_1 - \int_{\Gamma_1} m.\nu |u|^2 d\Gamma_1 \]

\[ - (1 - \varepsilon_0) \int_{\Omega} \left( |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \right) dx \]

\[ - c_1 \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} + v + w \right|^2 dx - c_2 \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial w}{\partial x_i} - u \right|^2 dx \]

\[ + \int_{\Gamma_1} \frac{\partial v}{\partial \nu} (2m \nabla v + (n - \varepsilon_0)v) d\Gamma_1 - \int_{\Gamma_1} m.\nu |v|^2 d\Gamma_1 \]

\[ + \int_{\Gamma_1} \frac{\partial w}{\partial \nu} (2m \nabla w + (n - \varepsilon_0)w) d\Gamma_1 - \int_{\Gamma_1} m.\nu |w|^2 d\Gamma_1 \]

\[ + \int_{\Omega} \left( (n + \varepsilon_0) - n + \varepsilon_0 \right) \delta n + \varepsilon_0 a + 2m \nabla a) F(u, v, w) dx. \]

The proof of Lemma 3.3 is completed. \( \square \)

Let us introduce the Lyapunov functional

\[ \mathcal{L}(t) = NE(t) + \Phi(t) \]

(47)

where \( N > 0 \) is large enough.

Using Young and Poincaré inequalities to the boundary integrals, we have, for \( \varepsilon > 0 \)

\[ \int_{\Gamma_1} \frac{\partial u}{\partial \nu} (2m \nabla u + (n - \varepsilon_0)u) d\Gamma_1 \]

\[ \leq \varepsilon \left( \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma_1} m.\nu |\nabla u|^2 d\Gamma_1 \right) + C_\varepsilon \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma_1, \]

\[ \int_{\Gamma_1} \frac{\partial v}{\partial \nu} (2m \nabla v + (n - \varepsilon_0)v) d\Gamma_1 \]
Rewriting the boundary conditions (17), (18) and (19) as

\[
\begin{align*}
\frac{\partial u}{\partial \nu} &= -\tau_1 \{ u_t + k_1(t)u - k_1(t)u_0 - k'_1 \circ u \}, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\
\frac{\partial v}{\partial \nu} &= -\tau_2 \{ v_t + k_2(t)v - k_2(t)v_0 - k'_2 \circ v \}, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\
\frac{\partial w}{\partial \nu} &= -\tau_3 \{ w_t + k_3(t)w - k_3(t)w_0 - k'_3 \circ w \}, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+,
\end{align*}
\]

and combining all above relations, we obtain

\[
\begin{align*}
\frac{dL}{dt}(t) &\leq \left( \frac{N\tau_1}{2} - C_\varepsilon - m\nu\rho_1 \right) \int_{\Gamma_1} |u_t|^2 \, d\Gamma_1 + \frac{N\tau_1}{2} \int_{\Gamma_1} k'_1(t)|w|^2 \, d\Gamma_1 \\
&\quad - (1 - \varepsilon) \int_{\Gamma_1} m\nu |\nabla u|^2 \, d\Gamma_1 - \frac{N\tau_1}{2} \int_{\Gamma_1} k''_1 \circ u \, d\Gamma_1 \\
&\quad - (1 - \varepsilon_0 - \varepsilon - C_\varepsilon k'_1(t)) \int_{\Omega} |\nabla u|^2 \, dx \\
&\quad - \left( \frac{N\tau_2}{2} - C_\varepsilon - m\nu\rho_2 \right) \int_{\Gamma_1} |v_t|^2 \, d\Gamma_1 + \frac{N\tau_2}{2} \int_{\Gamma_1} k''_2(t)|v|^2 \, d\Gamma_1 \\
&\quad - (1 - \varepsilon) \int_{\Gamma_1} m\nu |\nabla v|^2 \, d\Gamma_1 - \frac{N\tau_2}{2} \int_{\Gamma_1} k''_2 \circ v \, d\Gamma_1 \\
&\quad - (1 - \varepsilon_0 - \varepsilon - C_\varepsilon k'_2(t)) \int_{\Omega} |\nabla v|^2 \, dx \\
&\quad - \left( \frac{N\tau_3}{2} - C_\varepsilon - m\nu\rho_3 \right) \int_{\Gamma_1} |w_t|^2 \, d\Gamma_1 + \frac{N\tau_3}{2} \int_{\Gamma_1} k''_3(t)|w|^2 \, d\Gamma_1 \\
&\quad - (1 - \varepsilon) \int_{\Gamma_1} m\nu |\nabla w|^2 \, d\Gamma_1 - \frac{N\tau_3}{2} \int_{\Gamma_1} k''_3 \circ w \, d\Gamma_1 \\
&\quad - (1 - \varepsilon_0 - \varepsilon - C_\varepsilon k'_3(t)) \int_{\Omega} |\nabla w|^2 \, dx \\
&\quad - \varepsilon_0 \int_{\Omega} (\rho_1 |u_t|^2 + \rho_2 |v_t|^2 + \rho_1 |w_t|^2) \, dx.
\end{align*}
\]
\[-c_1 \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} + v + w \right|^2 dx - c_2 \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial w}{\partial x_i} - u \right|^2 dx \]

\[+ \int_{\Omega} ((-n + \varepsilon_0)\delta + n + \varepsilon_0) a + 2m.\nabla a)F(u, v, w)dx \]

\[+ C_\varepsilon \int |k_1' \circ u|^2 d\Gamma_1 + C_\varepsilon \int |k_2' \circ v|^2 d\Gamma_1 + C_\varepsilon \int |k_3' \circ w|^2 d\Gamma_1 \]

\[+ C_\varepsilon k_1^2(t) \int |u|^2 d\Gamma_1 + C_\varepsilon k_2^2(t) \int |v|^2 d\Gamma_1 + C_\varepsilon k_3^2(t) \int |w|^2 d\Gamma_1. \]

At this point, we take

\[\varepsilon = \varepsilon_0 < \frac{1}{2}. \quad (48)\]

Once \(\varepsilon\) is fixed, we pick \(N\) large enough so that

\[N > \sup \left\{ \frac{2(C_\varepsilon + \max |m.\nu_1|)}{\tau_1}, \frac{2(C_\varepsilon + \max |m.\nu_2|)}{\tau_2}, \frac{2(C_\varepsilon + \max |m.\nu_3|)}{\tau_3} \right\}. \quad (49)\]

Using the fact that \(\lim_{t \to \infty} k_i(t) = 0\) for \(i = 1, 2, 3\), and \([27]\), we arrive at

\[\frac{d\mathcal{L}}{dt}(t) \leq -C_1 E(t) \]

\[+ C_2 \int_{\Gamma_1} \left[ k_1^2(t)|u|^2 + k_2^2(t)|v|^2 + k_3^2(t)|w|^2 \right] d\Gamma_1 \]

\[\quad - C_3 \int_{\Gamma_1} \left[ k_1' \circ u + k_2' \circ v + k_3' \circ w \right] d\Gamma_1, \quad \forall t \geq t_0, \quad (50)\]

for \(t_0\) large enough and some positive constants \(C_1, C_2, C_3\).

If \(\gamma(t) = \min\{\gamma_1(t), \gamma_2(t), \gamma_3(t)\}\), for \(t \geq t_0\), we multiply both sides of \((50)\) by \(\gamma_1(t)\) to get

\[\gamma(t) \frac{d\mathcal{L}}{dt}(t) \leq -C_1 \gamma(t) E(t) \]

\[+ C_2 \gamma(t) \int_{\Gamma_1} \left[ k_1^2(t)|u|^2 + k_2^2(t)|v|^2 + k_3^2(t)|w|^2 \right] d\Gamma_1 \]

\[\quad - \gamma(t) C_3 \int_{\Gamma_1} \left[ k_1' \circ u + k_2' \circ v + k_3' \circ w \right] d\Gamma_1 \]

\[\leq -C_1 \gamma(t) E(t) \]

\[+ C_2 \int_{\Gamma_1} \left[ \gamma_1(t)k_1^2(t)|u|^2 \right] \]

\[+ \gamma_2(t)k_2^2(t)|v|^2 + \gamma_3(t)k_3^2(t)|w|^2 \right] d\Gamma_1 \]

\[+ C_3 \int_{\Gamma_1} \left[ \gamma_1(t)(-k_1') \circ u \right] \]

\[+ \gamma_2(t)(-k_2') \circ v + \gamma_3(t)(-k_3') \circ w \right] d\Gamma_1, \quad \forall t \geq t_0.
Using (30) and the fact that $\gamma_1(t)$ is nonincreasing, we obtain
\[
\gamma(t)\frac{dL}{dt}(t) \leq - C_1 \gamma(t) E(t) \\
+ C_2 \int_{\Gamma_1} [\gamma_1(t)k_{12}(t)|u^0|^2 \\
+ \gamma_2(t)k_{22}(t)|v^0|^2 + \gamma_3(t)k_{32}(t)|w^0|^2] \, d\Gamma_1 \\
+ C_3 \int_{\Gamma_1} [\gamma_1(t)k_{11}' \circ u \, d\Gamma_1 \\
+ \gamma_2(t)k_{22}' \circ v + \gamma_3(t)k_{32}' \circ w] \, d\Gamma_1, \quad \forall t \geq t_0.
\]

Next, using (36), we easily see that
\[
\gamma(t)\frac{dL}{dt}(t) \leq - C_1 \gamma(t) E(t) + C_4 \int_{\Gamma_1} [k_{11}'(t)|u^0|^2 \\
+ k_{22}'(t)|v^0|^2 + k_{32}'(t)|w^0|^2] \, d\Gamma_1 \\
- C \frac{dE}{dt}, \quad \forall t \geq t_0,
\]
which yields
\[
\gamma(t)\frac{dL}{dt}(t) + C \frac{dE}{dt} \leq - C_1 \gamma(t) E(t) + C_4 \int_{\Gamma_1} [k_{11}'(t)|u^0|^2 \\
+ k_{22}'(t)|v^0|^2 + k_{32}'(t)|w^0|^2] \, d\Gamma_1,
\]
or
\[
\frac{d}{dt} \{ \gamma(t)L(t) + CE(t) \} - \gamma'(t)L(t) \leq - C_1 \gamma(t) E(t) + C_4 \int_{\Gamma_1} [k_{11}'(t)|u^0|^2 \\
+ k_{22}'(t)|v^0|^2 + k_{32}'(t)|w^0|^2] \, d\Gamma_1. \tag{51}
\]

Again, using the fact that $\gamma_1(t)$ is nonincreasing and setting
\[
L_1(t) = \gamma(t)L(t) + CE(t) \sim E(t), \tag{52}
\]
the estimate (51) gives
\[
\frac{dL_1}{dt}(t) \leq - \omega \gamma(t)L_1(t)(t) + c \int_{\Gamma_1} k_{11}'|u^0|^2 d\Gamma_1 \\
+ c \int_{\Gamma_1} k_{22}'(t)|v^0|^2 d\Gamma_1 + c \int_{\Gamma_1} k_{32}'(t)|w^0|^2 d\Gamma_1, \quad \forall t \geq t_0. \tag{53}
\]

(i) If $u^0 = v^0 = w^0 = 0$ on $\Gamma_1$, then (53) reduces to
\[
\frac{dL_1}{dt}(t) \leq - \omega \gamma(t)L_1(t), \quad \forall t \geq t_0. \tag{54}
\]
A integration over $(t_0, t)$ yields
\[
L_1(t) \leq L_1(t_0)e^{-\omega \int_{t_0}^{t} \gamma(s)ds}, \quad \forall t \geq t_0. \tag{55}
\]
Using (52), then we obtain

$$E(t) \leq cE(t_0)e^{-\omega \int_0^t \gamma(s)ds}, \quad \forall t \geq t_0. \quad (56)$$

where \(c\) is a positive constant.

Then, we get

$$E(t) \leq cE(0)e^{-\omega \int_0^t \gamma(s)ds}, \quad \forall t \geq t_0. \quad (57)$$

(ii) If \((u^0, v^0, w^0) \neq (0, 0, 0)\) on \(\Gamma_1\), then (53) gives

$$\frac{dL_1}{dt}(t) \leq -\omega \gamma(t)L_1(t) + C_1 k_1^2(t) + C_2 k_2^2(t) + C_3 k_3^2(t) \quad \forall t \geq t_0, \quad (58)$$

where \(C_1 = c \int_{\Gamma_1} |u^0|^2 \, ds, \ C_2 = c \int_{\Gamma_1} |v^0|^2 \, ds\) and \(C_3 = c \int_{\Gamma_1} |w^0|^2 \, ds\).

In this case, we introduce

$$L_2(t) = L_1(t) - C_1 e^{-\omega \int_0^t \gamma(s)ds} \int_{t_0}^t k_1^2(s)e^{\omega \int_0^s \gamma(\zeta)d\zeta} \, ds$$

$$- C_2 e^{-\omega \int_0^t \gamma(s)ds} \int_{t_0}^t k_2^2(s)e^{\omega \int_0^s \gamma(\zeta)d\zeta} \, ds$$

$$- C_3 e^{-\omega \int_0^t \gamma(s)ds} \int_{t_0}^t k_3^2(s)e^{\omega \int_0^s \gamma(\zeta)d\zeta} \, ds.$$

Using a differentiation of \(L_2\) and (53), we find that

$$\frac{dL_2}{dt}(t) \leq -\omega \gamma(t)L_2(t), \quad \forall t \geq t_0. \quad (59)$$

Again, integrating over \((t_0, t)\), we obtain

$$L_2(t) \leq L_2(t_0)e^{-\omega \int_0^t \gamma(s)ds}, \quad \forall t \geq t_0, \quad (60)$$

which implies

$$L_1(t) \leq L_1(t_0)e^{-\omega \int_0^t \gamma(s)ds}$$

$$+ \left( C_1 \int_{t_0}^t k_1^2(s)e^{\omega \int_0^s \gamma(\zeta)d\zeta} \, ds + C_2 \int_{t_0}^t k_2^2(s)e^{\omega \int_0^s \gamma(\zeta)d\zeta} \, ds + C_3 \right) \int_{t_0}^t k_3^2(s)e^{\omega \int_0^s \gamma(\zeta)d\zeta} \, ds \right) e^{-\omega \int_0^t \gamma(s)ds}$$

$$+ C_3 \left( \int_{t_0}^t k_3^2(s)e^{\omega \int_0^s \gamma(\zeta)d\zeta} \, ds \right) e^{-\omega \int_0^t \gamma(s)ds}, \quad \forall t \geq t_0.$$

Using (52) and (36), then we obtain

$$E(t) \leq \left\{ E(0) + C_1 \int_0^t k_1^2(s) \left[ \frac{\tau_1}{2c} + e^{\omega \int_0^s \gamma(\zeta)d\zeta} \right] ds \right\} e^{-\omega \int_0^t \gamma(s)ds} e^{-\omega \int_0^t \gamma(s)ds}$$
and polynomial decay for particular cases of (34) and (35). In fact, we obtain exponential decay for Remark 2. Note that the exponential and polynomial decay estimates are just their valuable suggested comments. The authors would like to thank the anonymous referees for

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