The good, the bad and the worse: current, past and future consumption externalities and equilibrium efficiency

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Received: 20 January 2021 / Accepted: 19 April 2022 / Published online: 2 June 2022
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Abstract
We consider an Ak model in which instantaneous utility of the representative agent depends not only on current consumption, but also on a forward-looking external reference level which is specified as an exponentially declining average of future economy-wide average consumption. We show that the decentralized equilibrium is never efficient irrespective of the specification of the utility function. This result differs significantly from the implications of alternative specifications of the external reference level: In case that the latter is given by contemporaneous average consumption, the decentralized equilibrium is always efficient. However, if it is backward-looking, then efficiency obtains only if the arguments of the utility function are perfectly substitutable.

Keywords Consumption externalities · Efficiency · Anticipated consumption · Habits

JEL Classification O41 · H21

1 Introduction
Consumption externalities can be a source of equilibrium inefficiency in dynamic equilibrium models. This may happen when individuals’ utility depends on an external reference level of consumption which is taken as given when they choose their optimal consumption paths, whereas a social planner internalizes the consumption externalities when determining the optimal consumption paths. Thus, both consumption paths might not coincide and, therefore, the market equilibrium could be inefficient.
The literature has widely studied the welfare implications of externalities arising from current consumption (e.g., Rauscher 1997; Fisher and Hof 2000; Dupor and Liu 2003; Abel 2005; Liu and Turnovsky 2005; Aronsson and Johansson-Stenman 2010; Ghosh and Wendner 2017; Pham 2019) and from past consumption associated to the formation of an external reference level (e.g., Ljungqvist and Uhlig 2000; Alonso-Carrera et al. 2004, 2005, 2006; Gómez 2006, 2007; Turnovsky and Monteiro 2007). However, the welfare implications of spillovers associated to (anticipated) future consumption have only recently begun to be explored (Gómez and Monteiro in press).

If spillovers arise from current consumption, Fisher and Hof (2000) and Liu and Turnovsky (2005) show that a necessary and sufficient condition for equilibrium efficiency in the neoclassical model is that the marginal rate of substitution between private and average consumption be constant along the equilibrium path. Alonso-Carrera et al. (2006) show that this condition is immediately satisfied in the $Ak$ endogenous growth model if the requirements for endogenous growth to arise are met, so current consumption externalities do not generate inefficiency. Alonso-Carrera et al. (2006) also analyze the equilibrium efficiency in the $Ak$ model with additive internal habits and current consumption externalities. They prove that constancy of the equilibrium marginal rate of substitution between the arguments of the utility function is also a necessary and sufficient condition for equilibrium efficiency. Unlike the standard $Ak$ model, this condition is not readily satisfied, but if current consumption externalities enter additively into utility, the competitive equilibrium remains efficient.

Spillovers arising from past consumption are typically associated to the formation of external (outward-looking) habits. In these models current utility depends on how current consumption compares to a reference consumption level—the habits stock—which is determined by economy-wide average past consumption levels (e.g., Abel 1990; Carroll et al. 1997). Gómez (2010) proves that external habits do not cause inefficiency in the $Ak$ model if and only if the marginal rate of substitution between the arguments of the utility function is constant, so that they are perfect substitutes, which is shown to be equivalent to habits entering into utility in an additive form.

Following the seminal work of Loewenstein (1987), several authors have examined the consequences of introducing anticipated future consumption in economic growth models; e.g., on portfolio choice (Kuznitz et al. 2008), on the effectiveness of monetary policy (Faria and McAdam 2013), on the dynamics of the neoclassical growth model (Monteiro and Turnovsky 2016) and the $Ak$ model (Gómez and Monteiro in press; Gómez 2021), or on the Green Golden Rule (Faria and McAdam 2018). Among this still scant literature, only Gómez and Monteiro (in press) introduce an external reference level determined by anticipated future consumption, and compares the dynamics of the market and the socially-planned economies. They show that the competitive equilibrium is not efficient in the $Ak$ model with multiplicative anticipated consumption. However, the welfare implications of alternative formulations of external anticipated future consumption have not been systematically studied yet. The purpose of this paper is to fill this gap. Furthermore, Gómez and Monteiro (in press) consider the altruistic case in which an increase in the external anticipated consumption reference level increases the agent’s
own utility. However, the jealous case, in which an increase in the external reference level decreases the individual’s utility, is also plausible. Thus, we extend the previous literature to consider the models with altruism and jealousy.

This paper studies the efficiency of the competitive equilibrium in an endogenous growth model in which individual’s utility depends on current consumption and a forward-looking reference level—the anticipated future consumption reference level—which is formed as an exponentially declining average of economy-wide average future consumption. In order to isolate the effect of introducing anticipated consumption into utility we choose the simplest $Ak$ technology, so that the transitional dynamics is determined just by the individuals’ preferences. Consumption externalities can also affect the intratemporal consumption-leisure margin of choice (see, e.g., Ljungqvist and Uhlig 2000; Dupor and Liu 2003). Thus, to focus on the effects that consumption spillovers have on intertemporal dynamic inefficiency, we assume that labor supply is inelastic. Furthermore, we compare the effects of externalities associated to anticipated future consumption with the effects of spillovers arising from current consumption and from past consumption associated to external habits. Our main result is that the market equilibrium of the $Ak$ model with external anticipated consumption is not efficient irrespective of the specification of the utility function. This result is in stark contrast with the case that spillovers arise from current consumption, in which the market equilibrium is always efficient, or from past consumption associated to external habits, in which the market equilibrium is efficient if and only if habits enters utility in an additive form.

The rest of the paper is organized as follows. Section 2 determines the market equilibrium. Section 3 determines the optimal growth path attainable by a central planner. Section 4 studies the efficiency of the competitive equilibrium. Finally, Section 5 concludes.

2 The market economy

We consider a closed economy inhabited by a constant population of identical infinitely-lived agents.

2.1 Preferences and technology

Lifetime utility of the representative agent is given by

$$U = \int_0^\infty u(C(t), A(t))e^{-\beta t}dt, \quad \beta > 0,$$

where $\beta$ denotes the subjective rate of time preference, $C(t)$ is own consumption at time $t$, while $A(t)$ denotes the forward-looking external reference level of consumption specified as an exponentially declining average of future economy-wide average consumption, $\bar{C}(s), \ s \geq t$:
\[ A(t) = \rho \int_t^{\infty} \tilde{C}(s)e^{-\rho(s-t)}ds, \quad \rho > 0. \] (2)

We assume that the instantaneous utility function \( u \) is twice continuously differentiable and satisfies that \( u_C > 0, \ u_{CC} < 0, \) and \( u_A \neq 0 \). We allow for both \( u_A > 0 \) (altruism) and \( u_A < 0 \) (jealousy). In order to ensure that the integral on the right-hand side of (2) exists, we will henceforth restrict attention to paths of \( \tilde{C} \) that satisfy

\[ \lim_{t \to \infty} \int_0^t \tilde{C}(s)e^{-\rho s} ds < \infty. \] (3)

Obviously, condition (3) rules out that average consumption \( \tilde{C} \) grows too fast. Hereafter, we will make use of the fact that (2) is the solution of the following differential equation with terminal condition (see Appendix A):

\[ \dot{A}(t) = \rho [A(t) - \tilde{C}(t)], \quad \lim_{t \to \infty} A(t)e^{-\rho t} = 0. \] (4)

The flow budget constraint faced by the agent is

\[ \dot{\mathcal{W}}(t) = r(t)\mathcal{W}(t) + w(t) - C(t). \] (5)

Here, \( \mathcal{W}(t) \) denotes assets per capita at time \( t \), \( r(t) \) is the interest rate and \( w(t) \) is the wage rate per capita. The well-known standard version of the no-Ponzi-game condition given by

\[ \lim_{t \to \infty} \mathcal{W}(t)\frac{e^{-\int_0^t r(s)ds}}{C(0)} \geq 0, \] (6)

is unaffected by the introduction of consumption externalities.\(^2\)

The representative agent chooses \( C \) and \( \mathcal{W} \) to maximize the intertemporal utility (1) subject to the budget constraint (5) and the no-Ponzi-game condition (6), taking as given \( \tilde{C} \) and, therefore, the constraint on the accumulation of the anticipated consumption reference level (4), and the initial condition on assets per capita, \( \mathcal{W}(0) = \mathcal{W}_0 > 0 \).

Factor and product markets are competitive. Gross output per capita \( Y \) is determined by the \( Ak \) technology

\[ Y(t) = BK(t), \quad B > 0, \]

where \( K \) is the per capita capital stock. For simplicity, we abstract from capital depreciation. Therefore, the conditions for profit-maximization are that the marginal

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1 The terminal condition \( \lim_{t \to \infty} A(t)e^{-\rho t} = 0 \) in (4) rules out solutions of the form

\[ A(t) = \rho \int_t^{\infty} \tilde{C}(s)e^{-\rho(s-t)}ds + De^{\rho t}, \]

where \( D \) is an arbitrary positive constant.

2 This fact will become obvious in Sect. 2.2 in the context of the decentralized equilibrium version of the transversality condition of the household’s utility maximization problem.
product of capital is equal to the interest rate, \( r(t) = B \), and the wage rate is zero, \( w(t) = 0 \).

### 2.2 Equilibrium

For the sake of simplicity, hereafter the time argument will be deleted when there is no risk of confusion. The current value Hamiltonian of the individual’s maximization problem is

\[
H = u(C, A) + \lambda (rW + w - C).
\]

The first-order conditions for an interior optimum are

\[
\frac{\partial H}{\partial C} = 0 \quad \Rightarrow \quad u_C(C, A) = \lambda, \quad (7)
\]

\[
\frac{\partial H}{\partial W} = \beta \lambda - \dot{\lambda} \quad \Rightarrow \quad r = \beta - \dot{\lambda}/\lambda, \quad (8)
\]

together with the initial condition, \( W(0) = \mathcal{W}_0 \), and the transversality condition

\[
\lim_{t \to -\infty} e^{-\beta t} \lambda W = 0. \quad (9)
\]

Eq. (7) equates the marginal utility of consumption to the shadow price of assets, and Eq. (8) equates the rate of return on assets to the rate of return on consumption. Note that the solution to (8) is \( \lambda(t) = \lambda(0) e^{\beta t} e^{-\int_0^t r(s)ds} \), where (7) entails that \( \lambda(0) = u_C(C(0), A(0)) > 0 \). Hence, the transversality condition (9) is equivalent to

\[
\lim_{t \to -\infty} \mathcal{W}(t) e^{-\beta t} \int_0^t r(s)ds = 0, \quad (10)
\]

which immediately entails that the no-Ponzi-game condition (6) is satisfied with equality.

The only asset in the economy is capital, and so, \( W = K \). Using the fact that \( r = B \) and \( w = 0 \), Eq. (5) gives the flow resource constraint for the overall economy

\[
\dot{K} = BK - C, \quad (11)
\]

with \( K(0) = K_0 \). Henceforth we use that \( \dot{C} = C \) in a symmetric equilibrium because all agents are identical. The evolution of \( C, A, \) and \( K \) is governed by the following dynamic system:

\[
\frac{\dot{C}}{C} = -\frac{u_C(C, A)}{C u_{CC}(C, A)} \left[ B + \frac{A u_{CA}(C, A)}{u_C(C, A)} \frac{\dot{A}}{A} - \beta \right], \quad (12)
\]

\[
\frac{\dot{A}}{A} = \rho \left( 1 - \frac{C}{A} \right), \quad (13)
\]
\[
\frac{\dot{K}}{K} = B - \frac{C}{K},
\]  
(14)

together with the initial condition \( K(0) = K_0 \) and, using that \( r = B \) in Eq. (10), the transversality condition

\[
\lim_{t \to \infty} K(t)e^{-Bt} = 0.
\]  
(15)

Eq. (12), which is obtained from log-differentiating (7) and using (8) with \( r = B \), is the Euler equation for own consumption at the level of the representative agent who takes the time path of the consumption reference level \( A \) as given.

In the following, a hat over a variable will denote its steady-state value in the market economy. Now, we will focus on the existence of a balanced growth path (BGP) in which \( C, A, \) and \( K \) grow at a common constant rate \( \hat{g} \), so the ratios \( C/A \) and \( C/K \) are constant. From the Euler equation (12) it is obvious that in order to ensure the existence of such a BGP one has to impose appropriate restrictions on the specification of the utility function such that the two expressions

\[
-\frac{u_C(C,A)}{C u_{CC}(C,A)},
\]

and

\[
\frac{A u_{CA}(C,A)}{u_C(C,A)},
\]

can be represented as functions of \( C/A \equiv c \). This property is ensured by assuming that \( u_C(C,A) \) and \( u_A(C,A) \) are homogeneous of degree \(-v < 0\). This assumption implies that the indifference curves that are implicitly defined by \( u(C,A) = \tilde{u} \), where \( \tilde{u} \) is a constant, have the following property:

\[
\left. \frac{dC}{dA} \right|_{u(C,A) = \tilde{u}} = -\frac{u_A(C,A)}{u_C(C,A)} = \frac{-\tilde{u} u_A(C/A, 1)}{u_C(C/A, 1)}.
\]

Hence, the slope of the indifference curves, if depicted in the \((A, C)\)-plane, is constant along any ray through the origin. Under this homogeneity assumption we have that

\[
-\frac{u_C(C,A)}{C u_{CC}(C,A)} = -\frac{u_C(c, 1)}{c u_{CC}(c, 1)},
\]

and

\[
\frac{A u_{CA}(C,A)}{u_C(C,A)} = -v - \frac{c u_{CC}(c, 1)}{u_C(c, 1)},
\]

so that the Euler equation (12) can be rewritten as
\[
\frac{\dot{C}}{C} = \sigma(c) \left( B - \frac{\dot{A}}{A} - \beta \right) + \frac{\dot{A}}{A},
\]

where

\[
\sigma(c) \equiv -\frac{uc(c, 1)}{u_{cc}(c, 1)} > 0
\]

is the elasticity of intertemporal substitution expressed as a function of \( c \equiv C/A \). We shall assume that the domain of\( u_{c}(c, 1) \) and \( u_{a}(c, 1) \) is the open interval \((c, \infty)\), with \(0 \leq c < 1\), where the assumption \( c < 1 \) has to be imposed to allow for a strictly positive common growth rate \( \dot{g} \) along the BGP.

The dynamics of the market economy in terms of \( c \equiv C/A \) and \( a \equiv A/K \) is then governed by

\[
\begin{align*}
\frac{\dot{c}}{c} &= \sigma(c) [B - \nu \rho (1 - c) - \beta], \\
\frac{\dot{a}}{a} &= \rho (1 - c) - B + c a.
\end{align*}
\]

Solving the system \( \dot{c} = \dot{a} = 0 \), and using the fact that according to (13), the long-run growth rate satisfies the relation \( \dot{g} = \rho (1 - \dot{c}) \), we get the steady-state values

\[
\begin{align*}
\dot{c} &= \frac{\rho - \dot{g}}{\rho}, \\
\dot{a} &= \frac{B - \dot{g}}{\dot{c}},
\end{align*}
\]

where

\[
\dot{g} = \frac{B - \beta}{\nu}.
\]

The transversality condition (15) is equivalent to

\[
\dot{g} - B < 0 \iff \beta > (1 - \nu)B.
\]

Taking into account that \( \dot{C}(t) = C(t) \) grows asymptotically at the rate \( \dot{g} \), condition (3) introduced above to rule out excessive growth of average consumption \( \dot{C} \) requires that

\[
\dot{C}/C - \rho = \dot{g} - \rho < 0 \iff \rho > \dot{g} \iff \beta > B - \nu \rho.
\]

Combining (22) and (23) we have that \( \min\{B, \rho\} > \dot{g} \), which ensures feasibility of the steady state (19)–(20). Finally, positiveness of the long-run growth rate \( \dot{g} > 0 \) requires that
which, combined with (22) and (23), requires that the following condition has to be satisfied\textsuperscript{3}

\[
\max \{ B - \nu \rho, (1 - \nu)B, 0 \} < \beta < B. \tag{25}
\]

The stability analysis performed in Appendix B shows that the market economy does not exhibit transitional dynamics and instantaneously jumps to the balanced growth path which is described by

\[
\begin{align*}
K(t) &= K_0 e^{\delta t}, \\
A(t) &= \hat{\alpha} K(t) = \hat{\alpha} K_0 e^{\delta t}, \\
C(t) &= \hat{\gamma} A(t) = \hat{\gamma} \hat{\alpha} K(t) = \hat{\gamma} \hat{\alpha} K_0 e^{\delta t}.
\end{align*} \tag{26}
\]

These properties of the BGP imply that in the \((A, C)\)–plane the decentralized economy moves along the straight line given by \(C = \hat{\gamma} A\), where \(\hat{\gamma} < 1\), to the northeast. If agents are altruistic, \(u_A > 0\), then instantaneous utility \(u(C(t), A(t))\) increases with time \(t\). However, in case that agents are jealous, \(u_A < 0\), this need not be the case, since the rise in \(A\) exerts a negative effect on \(u\) that might more than offset the positive effect that results from the increase in \(C\). In this paradoxical situation instantaneous utility at any time \(t\) and, hence, also overall utility would depend negatively on the initial capital endowment \(K_0\). Henceforth we restrict attention to the case in which the net effect is strictly positive. Taking into account that

\[
\begin{align*}
\frac{du}{dA}(\hat{\gamma} A, A) &= \hat{\gamma} u_C(\hat{\gamma} A, A) + u_A(\hat{\gamma} A, A) = A^{-\nu} [\hat{\gamma} u_C(\hat{\gamma}, 1) + u_A(\hat{\gamma}, 1)],
\end{align*}
\]

where the second equality is obtained by making use of the assumption that both \(u_C\) and \(u_A\) are homogeneous of degree \(-\nu<0\), it is obvious that this ‘normal’ case obtains if and only if the instantaneous utility function \(u(C, A)\) has the property that

\[
\hat{\gamma} u_C(\hat{\gamma}, 1) + u_A(\hat{\gamma}, 1) > 0.
\]

Thus, in the rest of the paper we will assume that

\[
c u_C(c, 1) + u_A(c, 1) > 0, \quad \text{for all } c \equiv C/A \in (\hat{\gamma}, 1]. \tag{27}
\]

This assumption is equivalent to

\[
- \frac{u_A(C, A)}{u_C(C, A)} = - \frac{u_A(C/A, 1)}{u_C(C/A, 1)} < \frac{C}{A}, \quad \text{for all } C/A \in (\hat{\gamma}, 1]. \tag{28}
\]

It should be noted that \(-u_A(C, A)/u_C(C, A)\) gives the slope of the indifference curves in case that they are depicted in the \((A, C)\)–plane instead of the \((C, A)\)–plane. There is a simple graphical interpretation of (28) for the case in which \(u_A < 0\) so that the

\textsuperscript{3} If \(\hat{\gamma} > 0\), the parameter values must be so the condition \(\hat{\gamma} > \hat{\gamma}\) is satisfied as well.
indifference curves are positively sloped. Condition (28) requires that all points of intersection of indifference curve with an arbitrary straight line \( C = \tilde{c}A \), where \( \tilde{c} < 1 \), have the property that the slope of the indifference curves (that is constant along the any straight line \( C = \tilde{c}A \)) is less than the slope of the straight line given by \( \tilde{c} \). In other words, the positively sloped indifference curve is above (resp., below) the straight line \( C = \tilde{c}A \) on the left (resp., on the right) of the point of intersection. Consequently, as we move on the straight line given by \( C = \tilde{c}A \) to the northeast, we reach indifference curves that represent a higher level of utility.\(^4\)

As \( u_C > 0 \), the condition (27) can be equivalently expressed as

\[
 c + \pi(c) > 0, \quad \text{for all } c \in (\tilde{c}, 1],
\]

where

\[
\pi(c) \equiv \frac{u_A(c, 1)}{u_C(c, 1)}.
\]

Thus, \( -\pi(c) = -u_A(c, 1)/u_C(c, 1) = -u_A(C, A)/u_C(C, A) \) denotes the slope of the indifference curves depicted in the \((A, C)\)-plane irrespective of the sign of \( u_A \). If \( u_A > 0 \), then \( \pi(c) \) is the absolute value of the negative slope and thus gives the standard marginal rate of substitution of \( A \) for \( C \).

3 The centrally-planned economy

The social planner takes into account that \( \tilde{C} = C \), and so, maximizes the lifetime utility (1) subject to the flow resource constraint for the overall economy (11) and the law of motion of the reference level of anticipated consumption,

\[
A = \rho (A - C),
\]

taking as given the initial condition on capital, \( K(0) = K_0 > 0 \).

The current value Hamiltonian of this problem is

\[
\mathcal{H} = u(C, A) + \lambda (BK - C) + \mu \rho (A - C),
\]

where \( \lambda \) and \( \mu \) are the shadow values of capital and the anticipation reference level, respectively. The first-order conditions for an interior optimum are

\[
\frac{\partial \mathcal{H}}{\partial C} = 0 \quad \Rightarrow \quad u_C(C, A) = \lambda + \mu \rho, \tag{32}
\]

\[
\frac{\partial \mathcal{H}}{\partial K} = \beta \lambda - \dot{\lambda} \quad \Rightarrow \quad B = \beta - \dot{\lambda}/\lambda, \tag{33}
\]

\(^4\) I would like to thank an anonymous referee for providing this intuition.
\[ \frac{\partial \mathcal{H}}{\partial A} = \beta \mu - \dot{\mu} \quad \Rightarrow \quad u_A(C, A) + \mu \rho = \beta \mu - \dot{\mu}, \tag{34} \]

together with the initial conditions \( K(0) = K_0 \) and \( \mu(0) = 0 \), and the transversality conditions

\[ \lim_{t \to \infty} e^{-\beta t} \lambda K = \lim_{t \to \infty} e^{-\beta t} \mu A = 0. \tag{35} \]

Eq. (32) equates the marginal utility of own current consumption to its cost, comprised of the (usual) shadow value of the current capital forgone plus the shadow value of future anticipated consumption. Eq. (33) equates the rate of return on capital to the rate of return on own current consumption, whereas Eq. (34) is an arbitrage condition that links the return on own current consumption expressed in terms of units of anticipation, on the right-hand side, to the return on anticipations, given by the left-hand side. Furthermore, as \( A(0) \) is free the shadow value of the reference level of anticipated future consumption must be zero at the initial time, \( \mu(0) = 0 \) (see, e.g., Hestenes 1996, Leitmann 1981, section 13.2, or Léonard and Van Long 1992, theorem 7.8.1).

To ensure that the necessary conditions are also sufficient, besides \( u_{CC} < 0 \) we can impose the condition that \( u_{CC}u_{AA} - u_{CA}^2 \geq 0 \) (which implies that \( u_{AA} \leq 0 \)). This assumption ensures that the utility function and, therefore, the current Hamiltonian are jointly concave with respect to \( C \) and \( A \), so the Mangasarian sufficient conditions would be satisfied (see, e.g., Léonard and Van Long 1992). Appendix D analyzes the fulfillment of the concavity assumption for some particular utility functions, and shows that it is not satisfied by some prominent specifications in the jealousy case. Unfortunately, the question of whether the necessary conditions are also sufficient if \( u \) is not jointly concave in \( C \) and \( A \) remains open, but the following results will be true if this is the case.

Appendix C shows that, along a balanced growth path, consumption \( C \), capital \( K \), and anticipated consumption \( A \), grow at the same rate and, therefore, \( c \equiv C/A \) and \( a \equiv A/K \) are constant. Furthermore, the ratio \( q \equiv \mu/\lambda \) must also be constant if \( \rho \neq B \) while it might not be so if \( \rho = B \). Therefore, we will consider two cases: \( \rho \neq B \) and \( \rho = B \).

### 3.1 The case \( \rho \neq B \)

Appendix E.1 shows that the dynamics of the socially planned economy in terms of \( c \equiv C/A \), \( a \equiv A/K \) and \( q \equiv \mu/\lambda \), which are constant along the BGP, is governed by the following system:

\[ \frac{\dot{c}}{c} = \sigma(c) \left[ B - \nu p(1 - c) - \frac{\rho q}{1 + \rho q} - \beta \right], \tag{36} \]

\[ \dot{q} = - (\rho - B) q - (1 + \rho q) \pi(c), \quad q(0) = 0, \tag{37} \]
\[ \dot{a} = \rho(1 - c) - B + ca, \]  

(38)

where \( \sigma(c) = -u_C(c, 1)/[c u_{CC}(c, 1)] \) as defined in (16), and \( \pi(c) = u_A(c, 1)/u_C(c, 1) \) as defined in (30). The initial condition \( q(0) = 0 \), follows from the initial condition \( \mu(0) = 0 \) that has been explained above. The steady state of the dynamic system (36)–(38) is given by

\[ \bar{c} = \frac{\rho - \bar{g}}{\rho}, \]  

(39)

\[ \bar{q} = -\frac{\pi(\bar{c})}{\rho[1 + \pi(\bar{c})]} - B, \]  

(40)

\[ \bar{a} = \frac{B - \bar{g}}{\bar{c}}, \]  

(41)

where the long-run growth rate \( \bar{g} \) is

\[ \bar{g} = \rho(1 - \bar{c}) = \frac{B - \beta}{v}. \]  

(42)

Comparison of Eqs. (19), (20) and (21) with Eqs. (39), (41) and (42) shows that the steady-state values of \( c, a \) and \( g \) are the same in the market and the centralized economy, \( \bar{a} = \bar{a}, \bar{c} = \bar{c}, \) and \( \bar{g} = \bar{g}. \) Hence, as in the market economy, the conditions for i) feasibility of the steady state, \( \bar{c} > 0 \) and \( \bar{a} > 0 \) (which entail that \( \rho > \bar{g} \) and \( B > \bar{g} \)), and ii) positiveness of the long-run growth rate, \( \bar{g} > 0 \) (which entails that \( B > \beta \)), imply that condition (25), \( \max\{B - \rho v, (1 - v)B, 0\} < \beta < B \), must be satisfied. Condition (3), with \( \bar{C} = C \), is equivalent to \( -\rho + \bar{g} < 0 \), so that it is satisfied as well. Appendix C shows that, along the BGP,

\[ \bar{K}/K = \bar{A}/A = \bar{g}, \]

and \( q = \mu/\lambda \) is constant, so that

\[ \frac{\bar{\lambda}}{\lambda} = \frac{\bar{\mu}}{\mu} = \beta - B. \]

Hence, the transversality conditions (35), which are equivalent to

\[ -\beta + \frac{\bar{\lambda}}{\lambda} + \bar{K}/K = -\beta + \frac{\bar{\mu}}{\mu} + \bar{A}/A = -\beta + (\beta - B) + \bar{g} = -B + \bar{g} < 0, \]

are satisfied as well. Finally, the condition \( \lim_{t \to \infty} A(t)e^{-\rho t} = 0 \) in (4), which is equivalent to \( -\rho + \bar{g} < 0 \), is also satisfied.

Appendix E.2 shows that that the steady state of the centrally planned economy has the property of local saddle point stability (henceforth, SPS-CP) if and only if

\[ \rho[1 + \pi(\bar{c})] - B > 0. \]  

(43)

Thus, while the socially optimal steady-state values of \( c, a \) and \( g \) coincide with their decentralized counterparts, there is a crucial difference with respect to the stability
properties of the steady states. Recall that there is no transitional dynamics in the market economy. In contrast, if $q \neq B$, then the socially optimal solutions of $c(t), a(t), \text{ and } g(t)$ are not constant functions of time $t$. Obviously, $1 + \pi(\bar{c}) > 0$ is a necessary (but not sufficient) condition for SPS-CP property (43). The validity of $1 + \pi(\bar{c}) > 0$ is ensured by assumption (27) [or, equivalently, (29)], introduced in Sect. 2 to rule out the paradoxical case in which instantaneous utility decreases along the decentralized BGP in spite of positive growth.\(^5\)

Before studying the implications of the SPS-CP stability condition (43), we will show that the existence of the centralized solution when $q \neq B$ requires that $q > B$. Taking into account this additional constraint will simplify the subsequent analysis. The solution of the differential equation (33) for $k$ is

$$k(t) = k(0)e^{-(B-\beta)t} = u_C(C(0), A(0)) e^{-(B-\beta)t} > 0 \quad \text{for } t \geq 0,$$

where we have used the initial condition $\mu(0) = 0$ in Eq. (32) to get that $\lambda(0) = u_C(C(0), A(0)) > 0$. Since $\lambda(t) > 0$ and $u_C > 0$ for $t \geq 0$, Eq. (32) entails that

$$1 + \rho q(t) = \frac{u_C(C(t), A(t))}{\lambda(t)} > 0 \quad \text{for } t \geq 0$$

holds regardless of whether $u_A > 0$ or $u_A < 0$. Hence, it must be that\(^6\)

$$1 + \rho \bar{q} = \lim_{t \to \infty} [1 + \rho q(t)] = \frac{\rho - B}{\rho(1 + \pi(\bar{c})) - B} \geq 0.$$  

In case that $u_A > 0$ so that $\pi(\bar{c}) > 0$, the SPS-CP condition (43) in itself does not rule out that $\rho < B$ holds, but the right-hand side of the last result—in which (43) ensures that the denominator of the fraction is strictly positive—excludes this possibility. Consequently, when $\rho \neq B$, the existence of the centrally planned solution which has the property of local saddle point stability requires that the condition $\rho > B$ is satisfied even in the case of altruism ($u_A > 0$). Combining this condition with (25), the following constraint on the parameters has to be met\(^7\)

$$\max\{(1 - \nu)B, 0\} < \beta < B < \rho. \quad (44)$$

Let us return now to the implications of the SPS-CP condition (43), assuming that (44) holds. A thorough analysis of SPS-CP has to take into account that $\bar{c} = 1 - (\bar{g}/\rho)$ depends on $\rho$, while $\bar{g} = (B - \beta)/\nu$ is independent of $\rho$. Using this fact, we express (43) in the following way:

\(^5\) Taking into account that $\bar{c} < 1$ holds due to $g > 0$, that $u_C > 0$ holds by assumption, and using (29) we obtain

$$0 < u_C(\bar{c}, 1)[\bar{c} + \pi(\bar{c})] < u_C(\bar{c}, 1)[1 + \pi(\bar{c})] \Rightarrow 1 + \pi(\bar{c}) > 0.$$  

\(^6\) I would like to thank an anonymous referee for providing this proof.  

\(^7\) If $\bar{c} > 0$, the parameter values must be so that the condition $\bar{c} > \bar{c}$ is also satisfied.
SPS-CP \iff Z(\rho) > 0,

where

\[ Z(\rho) \equiv \rho[1 + \pi(1 - (\bar{g}/\rho))] - B, \quad \text{for } \rho > \bar{\rho} \equiv \bar{g}/(1 - \bar{c}). \quad (45) \]

The domain of the function \( Z(\rho) \), i.e., the value of \( \bar{c} \), is determined as follows. As discussed in Section 2, we assume that the domain of \( u_C(c, 1) \), \( u_A(c, 1) \) and, therefore, \( \pi(c) \equiv u_A(c, 1)/u_C(c, 1) \) is the open interval \((\hat{c}, \infty)\), with \( 0 \leq \hat{c} < 1 \). In order to ensure that \( \hat{c} = 1 - (\bar{g}/\rho) > \hat{c} \) we have to restrict the domain of \( Z(\rho) \) to \( \rho > \bar{\rho} \equiv \bar{g}/(1 - \hat{c}) \).

First, note that the condition \( \rho > B \) immediately guarantees the fulfillment of the SPS-CP condition \((43)\) for all \( \rho > \max\{\bar{\rho}, B\} \) in the altruistic case, \( u_A > 0 \), because

\[ \rho[1 + \pi(1 - (\bar{g}/\rho))] > \rho > \max\{\bar{\rho}, B\} \geq B. \]

However, in the jealousy case, \( u_A < 0 \), the condition \( Z(\rho) > 0 \) does not necessarily hold for all \( \rho > \max\{\bar{\rho}, B\} \). Appendix E.3 contains a quite general analysis of the implications of the SPS-CP condition in the jealousy model, together with some particular examples. Its main result is that, if there exists at most one solution to \( Z(\rho) = 0 \) for \( \rho > \max\{\bar{\rho}, B\} \), the possibility that SPS-CP obtains for all \( \rho > B \) is ruled out.

### 3.2 The case \( \rho = B \)

If \( \rho = B \), then the differential equation for \( q \) given by \((37)\) simplifies to

\[ \dot{q} = -(1 + Bq)\pi(c). \quad (46) \]

Substitution of this result into the differential equation for \( c \) given by \((36)\) yields

\[ \frac{\dot{c}}{c} = \sigma(c)[B - vB(1 - c) + B\pi(c) - \beta]. \quad (47) \]

Hence, if \( \rho = B \), then \( \dot{c} \) no longer depends on \( q \). The differential equation for \( a \) given by \((38)\) simplifies to

\[ \frac{\dot{a}}{a} = c(a - B). \quad (48) \]

but maintains the property \( \dot{a} = \dot{a}(c, a) \) known from the case \( \rho \neq B \). Obviously, the crucial implication of setting \( \rho = B \) is that the dynamics of \( c \) and \( a \) in the centralized economy is governed by two out of the three differential equations, namely \((47)\) and \((48)\).

From Eq. \((48)\), it is clear that the steady-state value of \( a \) is given by \( \bar{a} = B \). For the analysis of the steady-state value of \( c \) it is very helpful to rewrite \((47)\) in the form \( \dot{c}/c = \sigma(c)h(c) \), where

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8 Some sufficient conditions for this condition to be met are also discussed in Appendix E.3.
\[ h(c) = B[v(c - \hat{c}) + \pi(c)], \quad \text{for } c > \hat{c}. \] (49)

Here,
\[
\hat{c} \equiv \frac{\beta - (1 - \nu)B}{\nu B}
\]
is the long-run value of \( c \) in the market economy that is obtained by using (19) and (21) and setting \( \rho = B \). In principle, the function \( h(c) \) is defined for \( c > \hat{c} \). However, since we restrict attention to the case in which the socially optimal steady-state growth rate is strictly positive, \( \tilde{g} = B(1 - \hat{c}) > 0 \), where \( h(\hat{c}) = 0 \), the relevant segment of the domain is given by \((\hat{c}, 1)\).

In Appendix F.1 it is shown that
\[
h'(c) = B[v + \pi'(c)] = B \frac{c + \pi(c)}{\sigma(c)c} > 0, \quad \text{for } \hat{c} < c \leq 1.
\] (50)

This property implies that if \( h(c) = 0 \) has a solution \( \bar{c} \), with \( \hat{c} < \bar{c} < 1 \), then it is unique. To ensure the actual existence of a solution we introduce the assumption
\[
\lim_{c \to \hat{c}^+} h(c) < 0 < \lim_{c \to 1^-} h(c),
\] (51)

which is equivalent to
\[
\left[1 - \nu(1 - \hat{c}) + \lim_{c \to \hat{c}^+} \pi(c)\right]B < \beta < [1 + \pi(1)]B.
\] (52)

Appendix F.1 gives a proof of this equivalence and, in addition, illustrates condition (52) by means of three specifications of the utility function.

The stability analysis performed in Appendix F.2 shows that in the centralized economy the variables \( c \) and \( a \) have no transitional dynamics, since they instantaneously jump to their steady-state levels \( \bar{c} \) and \( \bar{a} \). The corresponding dynamic evolution of \( K(t), A(t) \) and \( C(t) \) along the BGP is described by equations that have the form given by (26) with the exception that \( \tilde{g}, \hat{c}, \hat{a}, \) and \( \hat{c} \) have to be replaced by \( \bar{g}, \bar{a}, \) and \( \bar{c} \). Appendix F.3 derives the explicit solution of \( q \) for \( \rho = B \) and shows that \( q \) in contrast to \( c \) and \( a \) exhibits transitional dynamics. Using the solution of \( q \), Appendix F.4 proves that the transversality conditions are satisfied.

If there is a feasible steady state \( \bar{c} \), with \( \hat{c} < \bar{c} < 1 \), the condition \( h(\bar{c}) = 0 \) implies that
\[
\hat{c} = \bar{c} + \frac{1}{v} \pi(\bar{c}) \begin{cases} > \bar{c} & \text{if } \pi(\bar{c}) > 0 \iff u_A > 0, \\ < \bar{c} & \text{if } \pi(\bar{c}) < 0 \iff u_A < 0. \end{cases}
\]
Since \( \tilde{g} = B(1 - \hat{c}) \) and \( \bar{g} = B(1 - \bar{c}) \) holds for \( \rho = B \), we also have \( \bar{g} - \tilde{g} = -B(\bar{c} - \hat{c}) \). From this equation and the former result it follows that
\[
\text{sign}(\bar{g} - \tilde{g}) = -\text{sign}(\bar{c} - \hat{c}) = \text{sign}(u_A).
\] (53)

Hence, if \( \rho = B \), the long-run growth rates of the market and the centralized economies do not coincide. Furthermore, long-run growth in the market economy is
suboptimally low in the case of altruism and suboptimally high in the case of jealousy.

4 Equilibrium (in)efficiency

In this section we analyze the efficiency of the competitive equilibrium in the $Ak$ model with external anticipated future consumption. We can state our main result.

**Proposition 1** The market equilibrium of the $Ak$ model with external anticipated future consumption is inefficient irrespective of the specification of the utility function.

**Proof** As shown in Sect. 2.2, the market economy does not exhibit transitional dynamics and, for all $t \geq 0$, stays at its steady state $(\hat{c}, \hat{a})$, which is given by (19) and (20). Hence, coincidence of the market equilibrium and the optimal growth path requires that the efficient paths of $c$ and $a$ in the centralized economy are also constant with $c(t) = \hat{c} = \bar{c}$ and $a(t) = \hat{a} = \bar{a}$ for all $t$. If $\rho > B$, substituting $c = \hat{c} = \bar{c}$ and $a = \hat{a} = \bar{a}$, with $\hat{c}$ and $\hat{a}$ given by (39) and (41), into the dynamic equation (36) and equating to zero, we get that $\dot{q}$ must be zero, which implies that $q(t) = \bar{q}$ for all $t$. However, this condition cannot hold as $q$ is subject to the initial condition $q(0) = 0 \neq \bar{q}$. Hence, the market equilibrium is not efficient in this case. If $\rho = B$, the centralized economy does not have transitional dynamics, as it happens in the market economy. However, substituting $\hat{c}$ into (49) we have that

$$h(\hat{c}) = B \pi(\hat{c}) \neq 0,$$

and, therefore, $\hat{c} \neq \bar{c}$, so the market equilibrium is not efficient in this case either. □

Hence, irrespective of the specification of the utility function $u$, the market equilibrium is not efficient in the presence of externalities associated to anticipated future consumption. This result is in stark contrast to what happens in the $Ak$ model with externalities associated to current or past consumption.

Let us first recall the case of current consumption externalities. If spillovers arise from current consumption we have that $A = \tilde{C}$ and $u(C, A) = u(C, \tilde{C})$. Fisher and Hof (2000, Proposition 4) show that the equilibrium is efficient if the effective intertemporal elasticities of substitution in the market and the socially planned economies coincide. They also show that this condition is equivalent to constancy of the equilibrium marginal rate of substitution between private and average consumption, $u_C(C, C) / u_A(C, C)$, which entails that the slope of the iso-utility curves $u(C, \tilde{C})$ in the $(\tilde{C}, C)$ space is constant along the $45^\circ$ line. Alonso-Carrera et al. (2006, Proposition 1) show that this condition is satisfied in the $Ak$ endogenous growth model if the requirements for endogenous growth to arise are met. Thus, current consumption externalities do not generate inefficiency in the $Ak$ model. It should be noted that, given the assumption made that $u_C$ and $u_A$ are homogeneous of the same degree, in our model this condition would be trivially satisfied so the market equilibrium would be efficient. Alonso-Carrera et al. (2006) also analyze the equilibrium efficiency in the $Ak$ model with additive internal habits and current consumption externalities and show that the equilibrium is efficient under general conditions.
consumption externalities, $u(C_t - \gamma H_t, \tilde{C}_t) = u(C_t - \gamma C_{t-1}, \tilde{C}_t) = u(Z_t, \tilde{C}_t)$. They prove that a necessary and sufficient condition for equilibrium efficiency is that $u_C(C_t - \gamma C_{t-1}, \tilde{C}_t)/u_Z(C_t - \gamma C_{t-1}, \tilde{C}_t)$ be constant along the equilibrium path. Thus, constancy of the marginal rate of substitution between the arguments of the utility function along the equilibrium path is also a necessary and sufficient condition for equilibrium efficiency in this case. Alonso-Carrera et al. (2006) show that, unlike the standard Ak model, this condition does not always hold, though it does if current consumption externalities enter additively into utility.

If spillovers arise from externalities associated to past consumption through the formation of external habits, we have that $A = H$ and $u(C, A) = u(C, H)$. The external habits stock, $H$, is the weighted sum of average past consumption:

$$H(t) = \rho \int_{-\infty}^{t} \tilde{C}(s)e^{\rho(t-s)} ds,$$

which after differentiation is equivalent to

$$\dot{H} = \rho (\tilde{C} - H), \quad H(0) = H_0.$$

Gómez (2010, Proposition 1) shows that in the Ak model with external habits the competitive equilibrium is efficient if and only if the marginal rate of substitution between the arguments of the utility function, $u_{H}(C, H)/u_C(C, H)$, is constant along the equilibrium path. Then, Gómez (2010, Proposition 2) proves that this condition holds if and only if $u_{H}(C, H)/u_C(C, H)$ is constant along any indifference curve so that consumption and habits are perfect substitutes. Finally, Gómez (2010, Proposition 3) proves that perfect substitutability between $C$ and $H$ is equivalent to habits entering utility in a subtractive form. Hence, the market equilibrium of the Ak model with external habits is socially optimal if and only if habits enter into utility in a subtractive way, i.e., $u(C, H) = u(C - \gamma H)$.

In summary, constancy of the equilibrium marginal rate of substitution between the arguments of the utility function is a necessary and sufficient condition for equilibrium efficiency in the Ak model with externalities associated to current consumption—with and without (internal) habits—and past consumption, but not when spillovers arise from anticipated future consumption. Alonso-Carrera et al. (2006) show that this condition is met in the Ak model when spillovers arise from current consumption—and there are no habits. Gómez (2010) shows that, in the Ak model with externalities associated to past consumption through the formation of external habits, constancy of the equilibrium marginal rate of substitution holds if and only if habits enter utility in a subtractive way. In contrast, in Proposition 1 we have shown that the market equilibrium is never efficient in the model with future consumption externalities associated to anticipation.

To further clarify the different implications that past, present and future consumption externalities have for equilibrium efficiency, let us consider the utility function\footnote{For a detailed analysis, see the exposition in Example #3 of Appendix D for the case $0 < \phi < 1$, and in Example #2 for the case $\phi = 1$.}
The parameter $\gamma$ reflects the strength of consumption externalities, $\epsilon$ is the inverse of the elasticity of intertemporal substitution of consumption when $\gamma = 0$, and $1/(1 - \phi)$ is the elasticity of substitution between own current consumption and the external reference consumption level. This elasticity is guaranteed to be greater than unity, and less than infinity if $\phi < 1$. The assumption $1 + \gamma > 0$ is required to ensure that $u_C > 0$. If $\gamma > 0$ then $u_A > 0$ so the agent is altruistic, whereas if $\gamma < 0$ then $u_A < 0$ so the agent is jealous. In this case, the domain of the utility function has to be conveniently restricted to guarantee that $C + \gamma A > 0$ to be well-defined. The condition $\epsilon > 1 - \phi$ is introduced to ensure that $u_C < 0$ in the jealousy case $\gamma < 0$. If $\phi = 1$, the expression (54) yields the additive specification—similar to (D.3) in Appendix D,

$$u(C, A) = \frac{1}{1 - \epsilon} \left( \frac{C^\phi + \gamma A^\phi}{1 + \gamma} \right)^{(1-\epsilon)/\phi}, \quad 0 < \phi \leq 1, \quad \epsilon > 1 - \phi, \quad \epsilon \neq 1, \quad \gamma > -1.$$  

(54)

and as $\phi \to 0$ the utility function (54) converges to the multiplicative specification—similar to (D.1) in Appendix D,

$$u(C, A) = \frac{1}{1 - \epsilon} \left( \frac{C + \gamma A}{1 + \gamma} \right)^{1-\epsilon},$$

and as $\phi \to 0$ the utility function (54) converges to the multiplicative specification—similar to (D.1) in Appendix D,

$$u(C, A) = \frac{1}{1 - \epsilon} \left( \frac{C + \gamma A}{1 + \gamma} \right)^{1-\epsilon} = \frac{1}{(1 + \gamma)(1 - \sigma)} (CA^\gamma)^{1-\sigma},$$

where $\sigma = (\epsilon + \gamma)/(1 + \gamma)$.

If spillovers arise from current consumption, so that $A = \bar{C}$ is the economy-wide average current consumption, the competitive equilibrium would be efficient for all $\phi$. If spillovers arise from past consumption via habit formation, so that $A = H$ is the external habits stock, the market equilibrium is efficient if and only if habits enter utility in an additive form; i.e., if and only if $\phi = 1$. In contrast, if spillovers arise from anticipated future consumption, so that $A$ is the reference level of external anticipated future consumption, the equilibrium would be inefficient for all $\phi$. Thus, spillovers associated to current, past or future consumption have quite different implications for equilibrium efficiency.

5 Conclusions

This paper studies the effect of introducing consumption externalities associated to anticipated future consumption on the efficiency of the competitive equilibrium. To isolate the effect of preferences, we consider that technology is $Ak$. We show that, in this model, current consumption externalities are the good—they do not cause inefficiency—, past consumption externalities are the bad—they cause inefficiency unless external habits enter utility in an additive manner—, but future consumption externalities are the worse—they always cause inefficiency. As it is well-known, changing the underlying simplistic assumptions affects the equilibrium efficiency in models with current or past consumption externalities. Thus, when labor supply is
elastic and/or there are diminishing returns to capital, current consumption externalities cause inefficiency (Dupor and Liu 2003; Liu and Turnovsky 2005), and past consumption externalities associated to external additive habits do so as well (Ljungqvist and Uhlig 2000; Alonso-Carrera et al. 2004). However, (anticipated) future consumption externalities are qualitatively different from past and present consumption externalities from the welfare viewpoint, since its presence always causes equilibrium inefficiency without the need of interacting with other model features.

Appendix A: Dynamics of anticipated consumption

The solution of the differential equation given in (4) can be expressed in the form

\[ A(t) = e^{pt} \left[ A(0) - \rho \int_0^t \tilde{C}(s)e^{-\rho s} ds \right]. \]  

Consequently, since (3) holds by assumption, the terminal condition becomes

\[ \lim_{t \to \infty} e^{-pt} A(t) = A(0) - \rho \int_0^\infty \tilde{C}(s)e^{-\rho s} ds = 0. \]

Hence, we have

\[ A(0) = \rho \int_0^\infty \tilde{C}(s)e^{-\rho s} ds < \infty, \]

which, after substitution in (A.1), implies that

\[ A(t) = \rho \int_t^\infty \tilde{C}(s)e^{-\rho(s-t)} ds < \infty. \]

Appendix B: Local stability analysis in the market economy

Linearizing (17)–(18) around its steady state (19)–(20) we have that

\[
\begin{pmatrix}
\dot{c} \\
\dot{\hat{a}}
\end{pmatrix}
\approx
\begin{pmatrix}
\hat{c} \sigma(\hat{c}) \nu \rho & 0 \\
\hat{a} \left( \hat{\alpha} - \rho \right) & \hat{a} \hat{\alpha}
\end{pmatrix}
\begin{pmatrix}
c - \hat{c} \\
\hat{a} - \hat{\alpha}
\end{pmatrix}.
\]

The jacobian matrix is triangular so its eigenvalues are its diagonal elements, which are both positive. Hence, the steady state is unstable and, as both \( c \) and \( \hat{a} \) are jump variables, they jump at \( t = 0 \) to their respective stationary values and stay there henceforth.

Appendix C: Balanced growth path of the centralized economy

In the centralized economy, the dynamic evolution of \( C, A, K, \lambda, \) and \( \mu \) is governed by the following differential and static equations and initial conditions:
\[
\frac{\dot{K}}{K} = B - \frac{C}{K}, \quad K(0) = K_0, \quad (C.1)
\]
\[
\frac{\dot{A}}{A} = \rho \left(1 - \frac{C}{A}\right), \quad (C.2)
\]
\[
u_C(C, A) = \lambda + \rho\mu, \quad (C.3)
\]
\[
\dot{\lambda} = -(B - \beta)\lambda, \quad (C.4)
\]
\[
\dot{\mu} = -(\rho - \beta)\mu - u_A(C, A), \quad \mu(0) = 0. \quad (C.5)
\]

Along a balanced growth path, the growth rates of \(K, A\) and \(C\) are constant. From (C.1) and (C.2), the long-run growth rates of \(A, C\) and \(K\) are identical,
\[
\frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \bar{\gamma},
\]
and so, the long-run ratios \(c \equiv C/A\) and \(a \equiv A/K\) are constant.

In the rest of this subsection we show the implications of the balanced growth assumption on \(q \equiv \lambda/\bar{\gamma}\). Using that \(u_C(C, A) = \lambda + \rho\mu\), we can rewrite (C.3) as
\[
\frac{\lambda + \rho\mu}{A^{-\nu}} = u_C(c, 1).
\]

Along the BGP the right-hand side of the last equation is constant, since \(c = \bar{c}\). Hence, the left-hand side has to be constant, too. The latter property requires that the growth rate of the numerator equals the constant growth rate of the denominator:
\[
\frac{\dot{\lambda} + \rho\dot{\mu}}{\lambda + \rho\mu} = -\nu\frac{\dot{A}}{A} = -\nu\bar{\gamma}. \quad (C.6)
\]

Using (C.4)–(C.5) and taking into account that \(u_A(C, A) = A^{-\nu}u_A(c, 1)\), we obtain
\[
\dot{\lambda} + \rho\dot{\mu} = -(B - \beta)\lambda + \rho\left[-(\rho - \beta)\mu - u_A(C, A)\right]
= -(B - \beta)(\lambda + \rho\mu) - \rho(\rho - B)\mu - \rho A^{-\nu}u_A(c, 1),
\]
and, therefore,
\[
\frac{\dot{\lambda} + \rho\dot{\mu}}{\lambda + \rho\mu} = -(B - \beta) - \rho(\rho - B) - \rho\pi(c). \quad (C.7)
\]

Now, we have two cases. The first case occurs when \(\rho \neq B\). From (C.6) it follows that along the BGP the left-hand side of (C.7) is constant. Hence, the right-hand side must be constant, too. This, in turn, requires that \(\mu/(\lambda + \rho\mu)\) and, therefore, \(q = \mu/\lambda\) are constant as well. This is the case analyzed in Sect. 3.1. The second case occurs when \(\rho = B\). In this case Eq. (C.7) becomes
\[
\frac{\dot{\lambda} + \rho \dot{\mu}}{\lambda + \rho \mu} = \frac{\dot{\lambda} + B \dot{\mu}}{\lambda + B \mu} = -(B - \beta) - B \pi(c),
\] (C.8)

so taking into account (C.6) and (C.2), along the BGP it must be that

\[-(B - \beta) - B \pi(c) = -v \bar{\pi} = -v \beta (1 - \bar{c}) = -v B (1 - \bar{c}).\] (C.9)

It should be noted that in this case it is not required the constancy of \(q = \mu / \lambda\) along a BGP in which \(C, K\) and \(A\) grow at constant rates. The case \(\rho = B\) is analyzed in Sect. 3.2.

**Appendix D: Particular specifications of the utility function—parameter restrictions that ensure their well-behavedness**

In the following we consider three specifications of the utility function and check whether they satisfy the following six properties (henceforth, “P” stands for “Property”):

P#1: \(\bar{c} < 1\), where \(\bar{c}\) denotes the left endpoint of the left-open interval that represents the domain of \(u_C(c, 1)\) and \(u_A(c, 1)\) as functions of \(c\),

P#2: \(u_C > 0\),

P#3: \(u_{CC} < 0\),

P#4: \(u_C(C, A)\) and \(u_A(C, A)\) are homogeneous of degree \(-v < 0\),

P#5: Condition (27) which which rules out the paradoxical case in which instantaneous utility decreases along the decentralized BGP in spite of positive growth is satisfied:

\[cu_C(c, 1) + u_A(c, 1) > 0, \quad \forall c \in (\bar{c}, 1],\]

P#6: either \(\det Hu(C, A) > 0\) or \(\det Hu(C, A) = 0\), where \(\det Hu(C, A) = u_{CC}u_{AA} - u_{CA}^2\) is the determinant of the Hessian matrix of \(u(C, A)\).

While P#3 implies that \(u(C, A)\) is strictly concave in \(C\), P#6 together with P#3 implies that \(u(C, A)\) is either strictly concave or concave in \(C\) and \(A\).

**Example #1**: Let us consider the multiplicative specification

\[u(C, A) = \frac{(C A^\gamma)^{1-\varepsilon}}{1 - \varepsilon}, \quad \gamma > -1, \quad \varepsilon > \max\{0, \gamma/(1 + \gamma)\}, \quad \varepsilon \neq 1.\] (D.1)

It is easily verified that
\[ u_C(C, A) = C^{-\epsilon}A^{(1-\epsilon)} > 0, \]
\[ u_A(C, A) = \gamma C^{1-\epsilon}A^{\gamma(1-\epsilon)-1}, \]
\[ Hu(C, A) = \begin{pmatrix} -\epsilon C^{-\epsilon-1}A^{\gamma(1-\epsilon)} & \gamma(1 - \epsilon) C^{-\epsilon}A^{\gamma(1-\epsilon)-1} \\ \gamma(1 - \epsilon) C^{-\epsilon}A^{\gamma(1-\epsilon)-1} & -\gamma[1 + \gamma(\epsilon - 1)]C^{1-\epsilon}A^{\gamma(1-\epsilon)-2} \end{pmatrix}, \] (D.2)

where \( Hu(C, A) \) denotes the Hessian matrix of \( u(C, A) \). Obviously, the utility function and its derivatives are defined for \( A > 0 \) and \( c \equiv C/A > \check{c} \), where \( \check{c} = 0 \) regardless of whether \( \gamma > 0 \) or \( \gamma < 0 \). With respect to the properties P#1–P#6 we thus obtain:

P#1: We have \( \check{c} = 0 < 1 \) for both \( \gamma > 0 \) or \( \gamma < 0 \).

P#2: \( u_C > 0 \) holds over the complete domain of \( u \) irrespective of the values of \( \epsilon \) and \( \gamma \).

P#3: From \( u_{CC} < 0 \iff \epsilon > 0 \) and \( \epsilon > \max \{0, \gamma/(1 + \gamma)\} \) it follows that \( u_{CC} < 0 \).

P#4: \( u_C(C, A) \) and \( u_A(C, A) \) are homogeneous of degree \(-\nu = -(\epsilon + \gamma(\epsilon - 1)) < 0\), where the assumptions \( \gamma > -1 \) and \( \epsilon > \max \{0, \gamma/(1 + \gamma)\} \) ensure that \(-\nu < 0\) for both \( \gamma > 0 \) and \( \gamma < 0 \).

P#5: Condition (27) becomes

\[ cu_C(c, 1) + u_A(c, 1) = (1 + \gamma)c^{1-\epsilon} > 0, \quad \forall c \in (0, 1]. \]

It is clearly satisfied because i) (27) obviously obtains if and only if \( 1 + \gamma > 0 \), and ii) \( \gamma > -1 \) holds by assumption.

P#6: The determinant of the Hessian matrix is given by

\[ \det Hu(C, A) = \gamma [(1 + \gamma)\epsilon - \gamma]C^{-2\epsilon}A^{2\gamma(1-\epsilon)-2}. \]

The assumptions \( \gamma > -1 \) and \( \epsilon > \max \{0, \gamma/(1 + \gamma)\} \) ensure that \((1 + \gamma)\epsilon - \gamma > 0\) holds for both \( \gamma > 0 \) and \( \gamma < 0 \). Hence, we have that \( \text{sign} [\det Hu(C, A)] = \text{sign} (\gamma) \). Taking into account that \( u_{CC} < 0 \) due to \( \epsilon > 0 \), it is obvious that the utility function \( u(C, A) \) is strictly concave in the case of altruism \( (\gamma > 0) \), while it is not concave in the presence of jealousy \( (\gamma < 0) \).

**Example #2:** Let us consider the additive specification

\[ u(C, A) = \frac{1}{1-\epsilon} (C + \gamma A)^{1-\epsilon}, \quad \epsilon > 0, \quad \epsilon \neq 1, \quad \gamma > -1. \] (D.3)

It has the following properties
\[ u_C(C, A) = (C + \gamma A)^{-\epsilon}, \]
\[ u_A(C, A) = \gamma (C + \gamma A)^{-\epsilon}, \]
\[ Hu(C, A) = \begin{pmatrix} -\epsilon(C + \gamma A)^{-\epsilon-1} - \gamma \epsilon(C + \gamma A)^{-\epsilon-1} \\ -\gamma \epsilon(C + \gamma A)^{-\epsilon-1} - \gamma^2 \epsilon(C + \gamma A)^{-\epsilon-1} \end{pmatrix}. \]

Obviously, the utility function and its derivatives are defined for \( A > 0 \) and \( c \equiv C/A > \tilde{c} \), where
\[ \tilde{c} = \begin{cases} 0, & \text{if } \gamma > 0 \text{ (altruism)}, \\ -\gamma, & \text{if } \gamma < 0 \text{ (jealousy)}. \end{cases} \]

Using these results we obtain:

P#1: i) If \( \gamma > 0 \), then \( \tilde{c} = 0 < 1 \). ii) If \( \gamma < 0 \), then it follows from \( \gamma > -1 \) that \( \tilde{c} = -\gamma < 1 \).

P#2: \( u_C > 0 \) holds over the complete domain of \( u \).

P#3: \( \epsilon > 0 \Rightarrow u_{CC} < 0 \).

P#4: \( u_C(C, A) \) and \( u_A(C, A) \) are homogeneous of degree \( -\nu = -\epsilon \), where the assumption \( \epsilon > 0 \) ensures that \( -\nu < 0 \).

P#5: Condition (27) is satisfied because
\[ cu_C(c, 1) + u_A(c, 1) = (c + \gamma)^{1-\epsilon} > 0, \quad \forall c \in (\tilde{c}, 1]. \]

P#6: \( \det Hu(C, A) = 0 \) holds irrespective of the assumptions made with respect to \( \epsilon \) and \( \gamma \).

From these considerations it follows that the two parameter restrictions \( \epsilon > 0 \) and \( \gamma > -1 \) ensure that the specification (D.3) is well-behaved in the sense that the properties P#1–P#5 are satisfied. Moreover, it is jointly concave (but not strictly concave) in \( C \) and \( A \) regardless of whether \( \gamma > 0 \) or \( \gamma < 0 \).

**Example #3**: Let us consider the generalized CES utility function
\[ u(C, A) = \frac{1}{1 - \epsilon} \left( \frac{C^\phi + \gamma A^\phi}{1 + \gamma} \right)^{(1-\epsilon)/\phi}, \quad 0 < \phi < 1, \quad \epsilon > 1 - \phi, \quad \epsilon \neq 1, \quad \gamma > -1. \]

(D.5)

The generalized CES specification has the following properties
The function $u_C(C, A) = \frac{C^{\phi-1}}{1 + \gamma} \left( \frac{C^{\phi} + \gamma A^{\phi}}{1 + \gamma} \right)^{(1-\epsilon-\phi)/\phi}$, and

$u_A(C, A) = \frac{\gamma A^{\phi-1}}{1 + \gamma} \left( \frac{C^{\phi} + \gamma A^{\phi}}{1 + \gamma} \right)^{(1-\epsilon-\phi)/\phi}$,

$Hu(C, A) = - (1 + \gamma)^{-(1-\epsilon)/\phi} (C^\phi + \gamma A^\phi)^{(1-\epsilon)/\phi - 2} \times \left( C^{\phi - 2} [\epsilon C^{\phi} + \gamma(1 - \phi)A^{\phi}] - \gamma(\epsilon + \phi - 1)A^{\phi-1}C^{\phi-1} - \gamma A^{\phi - 2} [(1 - \phi)C^{\phi} + \gamma \epsilon A^{\phi}] \right)$.

(D.6)

Since $\gamma > -1$, and, hence, $1 + \gamma > 0$ holds by assumption, the utility function and its derivatives are defined for $A > 0$ and $c \equiv C/A > \tilde{c}$, where

$$\tilde{c} = \begin{cases} 0, & \text{if } \gamma > 0 \text{ (altruism)}, \\ (-\gamma)^{1/\phi}, & \text{if } \gamma < 0 \text{ (jealousy)}. \end{cases}$$

The assumption $0 < \phi < 1$ guarantees that the elasticity of substitution between $C$ and $A$ given by $1/(1 - \phi)$ is greater than unity and less than infinity. With respect to the properties P#1–P#6 we obtain the following results:

P#1: i) If $\gamma > 0$, then $\tilde{c} = 0 < 1$. ii) If $\gamma < 0$, then it follows from $\phi > 0$ and $\gamma > -1$ that $\tilde{c} = (-\gamma)^{1/\phi} < 1$.

P#2: $\gamma > -1 \Rightarrow 1 + \gamma > 0 \Rightarrow u_C > 0$.

P#3: Since $1 + \gamma > 0$ by assumption it follows from the expression for $u_{CC}$ given in the Hessian matrix that $u_{CC} < 0 \iff \epsilon C^\phi + \gamma(1 - \phi)A^\phi > 0$. It is easily verified that the latter inequality holds regardless of whether $\gamma > 0$ or $\gamma < 0$: i) In the case of altruism ($\gamma > 0$), $\epsilon C^\phi + \gamma(1 - \phi)A^\phi$ obtains because of $\epsilon > 1 - \phi > 0$. ii) In the presence of jealousy ($\gamma < 0$), the fact that $C^\phi + \gamma A^\phi > 0$ holds in the domain of the utility function together with $\epsilon > 1 - \phi > 0$ implies that $\epsilon C^\phi + \gamma(1 - \phi)A^\phi > (1 - \phi)(C^\phi + \gamma A^\phi) > 0$.

P#4: $u_C(C, A)$ and $u_A(C, A)$ are homogeneous of degree $-\nu = -\epsilon$, where $-\nu < 0$ follows from $\epsilon > 1 - \phi > 0$.

P#5: Condition (27) is satisfied because

$$cu_C(c, 1) + u_A(c, 1) = \left( \frac{c^{\phi} + \gamma}{1 + \gamma} \right)^{(1-\epsilon)/\phi} > 0, \quad \forall c \in [\tilde{c}, 1].$$

From these considerations it follows that the parameter restrictions $0 < \phi < 1$, $\epsilon > 1 - \phi$, and $\gamma > -1$ ensure that the specification (D.5) is well-behaved in the sense that the properties P#1–P#5 are satisfied. Unfortunately, we obtain a mixed result with respect to concavity.
Since $\epsilon > 1 - \phi > 0$ holds by assumption it follows from
\[
\det Hu(C, A) = \gamma \epsilon (1 - \phi) A^{\phi - 2} C^{\phi - 2} (1 + \gamma)^{2(1 - \phi)} (C^\phi + \gamma A^\phi)^{\frac{2(1 - \phi - 1)}{\phi}},
\]
that $\text{sign} [\det Hu(C, A)] = \text{sign} (\gamma)$. Taking into account that $u_{CC} < 0$ it is obvious that the utility function $u(C, A)$ is strictly concave in the case of altruism ($\gamma > 0$), while it is not concave in the presence of jealousy ($\gamma < 0$).

**Appendix E: The centralized economy when $\rho \neq B$**

**Dynamics**

From (32), as $q \equiv \mu/\check{\lambda}$, we have that
\[
\check{\lambda} = u_C(C, A)/(1 + \rho q), \quad (E.1)
\]
\[
\check{\mu} = q u_C(C, A)/(1 + \rho q). \quad (E.2)
\]

Using that $\check{q}/q = \check{\mu}/\mu - \check{\lambda}/\lambda$, taking into account (33) and (34), together with (E.2) and the homogeneity of degree $-v$ of $u_C$ and $u_A$, we get Eq. (37).

Log-differentiating (E.1), after simplification we get
\[
\frac{\check{C}}{C} = \frac{u_C(C, A)}{C u_{CC}(C, A)} \left[ \frac{\check{\lambda}}{\check{\lambda}} - A \frac{u_A(C, A)}{u_C(C, A)} \frac{A}{A} + \frac{\rho \check{q}}{1 + \rho q} \right].
\]

Now, using (33) and the homogeneity of degree $-v$ of $u_C$, we can obtain that
\[
\frac{\check{C}}{C} = \sigma(c) \left[ B - v \frac{\check{A}}{A} - \frac{\rho \check{q}}{1 + \rho q} - \beta \right] + \frac{\check{A}}{A}, \quad (E.3)
\]
where
\[
\sigma(c) \equiv - \frac{u_C(c, 1)}{c u_{CC}(c, 1)} > 0.
\]

From $\check{c}/c = \check{C}/C - \check{A}/A$, using (E.3) and (31), we immediately get Eq. (36). Finally, Eq. (38) is obtained using that $\check{\alpha}/\alpha = \check{A}/A - \check{K}/K$, taking into account (31) and (11).

**Stability analysis**

The following presentation (and further subsections of the main text and the appendix, respectively) make use of the fact that the first derivative of $\pi(c) = u_A(c, 1)/u_C(c, 1)$ given by
\[
\pi'(c) = \frac{u_C'(c,1)u_C(c,1) - u_A'(c,1)u_{CC}(c,1)}{[u_C(c,1)]^2}
\]  
(E.4)

can be rewritten in the following simple form:
\[
\pi'(c) = -\nu + \frac{c + \pi(c)}{\sigma(c)c}.
\]  
(E.5)

The two essential elements of the simple proof are that i) \(u_{CA}(c,1) = -\nu u_C(c,1) - c u_{CC}(c,1)\) holds due to the assumption that both \(u_{C}(C,A)\) and \(u_{A}(C,A)\) are homogeneous of degree \(-\nu < 0\), and ii) \(\sigma(c) = -u_{C}(c,1) / [c u_{CC}(c,1)]\) holds by definition.

Linearizing the dynamic system (36)–(38) around its steady state (39)–(41),
\[
\begin{align*}
\bar{c} &= \frac{\rho - \bar{g}}{\rho}, \\
\bar{q} &= -\frac{\pi(\bar{c})}{\rho[1 + \pi(\bar{c})] - B}, \\
\bar{a} &= \frac{B - \bar{g}}{\bar{c}}, \\
\bar{g} &= \rho(1 - \bar{c}) = (B - \beta) / \nu,
\end{align*}
\]
we obtain
\[
\begin{pmatrix}
\dot{c} \\
\dot{q} \\
\dot{a}
\end{pmatrix} \approx \begin{pmatrix}
m_{11} & m_{12} & 0 \\
m_{21} & m_{22} & 0 \\
a(\bar{a} - \rho) & 0 & \bar{a}\bar{c}
\end{pmatrix} \begin{pmatrix}
c - \bar{c} \\
q - \bar{q} \\
a - \bar{a}
\end{pmatrix} = M \begin{pmatrix}
c - \bar{c} \\
q - \bar{q} \\
a - \bar{a}
\end{pmatrix}
\]
where
\[
\begin{align*}
m_{11} &= \frac{\partial \dot{c}}{\partial c}(\bar{c}, \bar{q}, \bar{a}) = \rho \bar{c} \sigma(\bar{c})[\nu + \pi'(\bar{c})], \\
m_{12} &= \frac{\partial \dot{c}}{\partial q}(\bar{c}, \bar{q}, \bar{a}) = -\bar{c} \sigma(\bar{c}) \frac{\rho}{1 + \rho\bar{q}} m_{22}, \\
m_{21} &= \frac{\partial \dot{q}}{\partial c}(\bar{c}, \bar{q}, \bar{a}) = -(1 + \rho\bar{q})\pi'(\bar{c}) \\
m_{22} &= \frac{\partial \dot{q}}{\partial q}(\bar{c}, \bar{q}, \bar{a}) = \frac{1}{\bar{q}}\pi(\bar{c}) = -\{\rho[1 + \pi(\bar{c})] - B\}.
\end{align*}
\]  
(E.9)

Let \(T = m_{11} + m_{22}\) and \(D = m_{11}m_{22} - m_{12}m_{21}\) denote the trace and the determinant, respectively, of the submatrix that is obtained by deleting the last row and the last column of the Jacobian matrix. The characteristic equation of the complete Jacobian is obviously given by
\[ P(\xi) = -(\xi - \bar{a}c)(\xi^2 - T\xi + D) = 0. \]

It is easily verified that
\[
T = \rho[\bar{c} + \pi(\bar{c})] - \{\rho[1 + \pi(\bar{c})] - B\} = -\rho(1 - \bar{c}) + B = B - \bar{g} = \bar{c}\bar{a} > 0.
\]

In the first equation we used (E.5) to rewrite the element \(m_{11} = \rho\bar{c}\sigma(\bar{c})[v + \pi'(\bar{c})]\) in the form \(m_{11} = \rho[\bar{c} + \pi(\bar{c})]\). In the second and third equations we took into account that \(\bar{g} = \rho(1 - \bar{c})\) and \(\bar{a} = (B - \bar{g})/\bar{c}\). The crucial result is, however, given by
\[
D = \bar{c}\sigma(\bar{c})\rho v m_{22} = -\bar{c}\sigma(\bar{c})\rho v\{\rho[1 + \pi(\bar{c})] - B\},
\]
which, in turn, implies that
\[
D < 0 \iff \rho[1 + \pi(\bar{c})] - B > 0. \tag{E.10}
\]

The dynamic system (36)–(38) features two control-like variables, \(c\) and \(a\), and one state-like variable, \(q\), as its initial value is predetermined, \(q(0) = 0\). The last diagonal element of the matrix \(M\), \(\bar{a}\bar{c}\), is a positive eigenvalue of \(M\). If \(D > 0\) the real parts of the other two eigenvalues of \(M\) have the same sign and, as \(T > 0\), both are positive, so the steady state is unstable. If \(D < 0\), the other two eigenvalues of \(M\) are real numbers of opposite sign, and, therefore, the steady state is locally saddle-path stable. Thus, saddle-path stability requires that \(D < 0\), so it must hold the condition (E.10), which is equivalent to (43).

**Restrictions for \(\rho\) that result from the SPS-CP condition in the jealousy model, \(u_A < 0\)**

The fulfillment of the condition for the existence of a centralized solution, \(\rho > B\), immediately guarantees that the SPS-CP condition is satisfied in the altruistic case \(u_A > 0\). However, this Appendix illustrates that this condition might not be sufficient for the SPS-CP condition to be met in the jealous case, \(u_A < 0\), in which \(\pi(c) < 0\) holds.

Let us start with the function
\[
Z(\rho) \equiv \rho[1 + \pi(1 - (\bar{g}/\rho))] - B, \quad \text{for } \rho > \bar{\rho} \equiv \bar{g}/(1 - \bar{c}),
\]
defined in (45). Using that \(\lim_{\rho \to \infty} [1 + \pi(1 - (\bar{g}/\rho))] = 1 + \pi(1) > 0\), where the positive sign follows from (29), we have that
\[
\lim_{\rho \to \infty} Z(\rho) = \infty. \tag{E.11}
\]

A general analysis of the solution to \(Z(\rho) > 0\) is probably intractable. So, let us consider that there exists at most one solution to \(Z(\rho) = 0\) for \(\rho > \max\{\bar{\rho}, B\}\). Two cases may arise:

i) If \(\bar{\rho} \leq B\), we have that \(Z(B) = B\pi(1 - (\bar{g}/B)) < 0 < \lim_{\rho \to \infty} Z(\rho)\), and so, there is a unique solution \(\rho^* > B\) to \(Z(\rho) = 0\) such that \(Z(\rho) > 0\) if and only if \(\rho > \rho^* > B\).
ii) If $\hat{\rho} > B$, then there are two cases. If $\lim_{\rho \to \hat{\rho}^+} Z(\rho) \geq 0$, then we have that $Z(\rho) > 0$ for all $\rho > \hat{\rho} > B$ and, hence, the stability SPS-CP condition (43) is satisfied for all $\rho > \hat{\rho}$. If $\lim_{\rho \to \hat{\rho}^+} Z(\rho) < 0$, then there is a unique solution $\rho^S > \hat{\rho}$ to $Z(\rho) = 0$ such that $Z(\rho) > 0$ if and only if $\rho > \rho^S > \hat{\rho} > B$.

In any case, it should be noted that the possibility that SPS-CP obtains for all $\rho > B$ is ruled out.

A simple sufficient condition for the existence of at most one solution to $Z(\rho) = 0$ for $\rho > \max\{\hat{\rho}, B\}$ is that $Z'(\rho) > 0$. The derivative of $Z(\rho)$ is

$$Z'(\rho) = 1 + \pi(1 - (\bar{g}/\rho)) + (\bar{g}/\rho)\pi'(1 - (\bar{g}/\rho))$$

$$= 1 - (\bar{g}/\rho) + \pi(1 - (\bar{g}/\rho)) + (\bar{g}/\rho)[1 + \pi'(1 - (\bar{g}/\rho))],$$

where the positiveness of the first term follows from (29). Thus, a sufficient (but not necessary) condition for $Z'(\rho) > 0$ is that

$$1 + \pi'(c) \geq 0, \quad \text{for } c > \check{c}. \quad (E.12)$$

We now present some sufficient (but not necessary) conditions that ensure the fulfillment of the condition (E.12):

i) Starting from the representation of $\pi'(c)$ given by (E.4), and using that

$$c u_{CC}(c, 1) + u_{CA}(c, 1) = -v u_{C}(c, 1),$$

$$u_{AC}(c, 1) + u_{AA}(c, 1) = -v u_{A}(c, 1),$$

holds, since by assumption $u_C$ and $u_A$ are homogeneous of degree $-v$, $\pi'(c)$ can we rewritten as

$$\pi'(c) = \frac{u_{CC}(c, 1)u_{AA}(c, 1) - [u_{CA}(c, 1)]^2}{v[u_{C}(c, 1)]^2}.$$ 

Hence, if $u$ is concave then we have that $\pi'(c) \geq 0$ and, therefore, $1 + \pi'(c) > 0$.

ii) Using the condition (E.5), a sufficient condition for $1 + \pi'(c) \geq 0$ is that $v \leq 1$. This condition also applies if $u$ is not concave, so that $\pi'(c)$ could be negative for some $c > \check{c}$.

In the following we will analyze the implications of the specifications #1–#3 that were analyzed in Appendix D for the properties of i) the function $Z(\rho)$, ii) the solutions of $Z(\rho) = 0$, and iii) the resulting conditions for the occurrence of SPS-CP. Recall that all specifications are well-behaved in the sense that they satisfy the properties P#1–P#5. In contrast, specifications #1 and #3 are not concave in the presence of jealousy, while specification #2 is concave (but not strictly concave) regardless of whether agents are altruistic or jealous. Since P#5 requires that conditions (27) and (29) are satisfied, all specifications have the property that (E.11) is satisfied, i.e., $\lim_{\rho \to -\infty} Z(\rho) = \infty$. 

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To illustrate the former results we now analyze some specific utility functions.

**Example #1:** Let us consider the specification (D.1) in the jealousy case ($\gamma < 0$),

$$u(C, A) = \frac{(CA^\gamma)^{1-\epsilon}}{1 - \epsilon}, \quad -1 < \gamma < 0, \quad \epsilon > \max\{0, \gamma/(1 + \gamma)\}, \quad \epsilon \neq 1.$$

From (D.2), we get that

$$u_C(c, 1) = c^{-\epsilon}, \quad u_A(c, 1) = \gamma c^{1-\epsilon},$$

which are defined for $c > \bar{c} = 0$. We have that $\pi(c) = \gamma c < 0$ for $c > \bar{c} = 0$, and so, the function $Z(\rho)$ defined in (45) takes the following linear form:

$$Z(\rho) = \rho - B + \gamma(\rho - \bar{g}), \quad \text{for } \rho > \bar{\rho} = \bar{g}.$$

Its derivative is positive, $Z'(\rho) = 1 + \gamma > 0$ for $\rho > \bar{\rho} = \bar{g}$, and $Z(B) = \gamma(B - \bar{g}) < 0$. Please note that $Z'(\rho) > 0$ in spite of the fact that $u$ is not concave. This feature illustrates that concavity—as stressed above in the context of i)—is a sufficient, but not a necessary condition for $Z'(\rho) > 0$. Taking into account (E.11), this implies that there exists a unique solution $\rho^S > B$ of $Z(\rho) = 0$ that can be computed as

$$\rho^S = \frac{B + \gamma \bar{g}}{1 + \gamma} = B - \frac{\gamma(B - \bar{g})}{1 + \gamma} > B.$$

Hence, the fulfillment of the stability SPS-CP condition (43) requires that $\rho > \rho^S > B$.

**Example #2:** Let us consider the specification (D.3) in the jealousy case ($\gamma < 0$),

$$u(C, A) = \frac{1}{1 - \epsilon} (C + \gamma A)^{1-\epsilon}, \quad \epsilon > 0, \quad \epsilon \neq 1, \quad -1 < \gamma < 0.$$

Using (D.4), the partial derivatives $u_C(c, 1)$ and $u_A(c, 1)$ are given by

$$u_C(c, 1) = (c + \gamma)^{-\epsilon}, \quad u_A(c, 1) = \gamma(c + \gamma)^{-\epsilon},$$

which are defined for $c > \bar{c} = -\gamma > 0$. Hence, $\pi$ is a constant function of $c$, $\pi(c) = \gamma$, for $c > \bar{c} = -\gamma$ so that $\pi'(c) = 0$, for $c > \bar{c}$. The latter property illustrates the considerations made above: Since the specification of $u$ is concave, but not strictly concave, if follows from the representation of $\pi'(c)$ given in i) that $\pi'(c) = 0$. The function $Z(\rho)$ defined in (45) takes the form:

$$Z(\rho) = \rho(1 + \gamma) - B, \quad \text{for } \rho > \bar{\rho} = \bar{g}/(1 + \gamma) > \bar{g}.$$

Its derivative is positive,

$$Z'(\rho) = 1 + \gamma > 0, \quad \text{for } \rho > \bar{\rho} = \bar{g}/(1 + \gamma).$$

The unique solution of $Z(\rho) = 0$ is given by
\[
\rho^S = \frac{B}{1 + \gamma} = \tilde{\rho} + \frac{B - \tilde{g}}{1 + \gamma}.
\]

Taking into account that \(\gamma < 0\) and \(1 + \gamma > 0\) it is obvious from these equivalent representations of \(\rho^S\) that i) \(\rho^S > \tilde{\rho}\) and ii) \(\rho^S > B\). From \(Z'(\rho) = 1 + \gamma > 0\), for \(\rho > \tilde{\rho}\), it then follows that \(Z(\rho) > 0\) (i.e., SPS-CP obtains) if and only if \(\rho > \rho^S\), where \(\rho^S > \text{max}\{\tilde{\rho}, B\}\).

**Example #3:** Let us consider the specification (D.5) in the jealousy case \((\gamma < 0)\),

\[u(C, A) = \frac{1}{1 - \epsilon} \left( \frac{C^\phi + \gamma A^\phi}{1 + \gamma} \right)^{(1-\epsilon)/\phi}, \quad 0 < \phi < 1, \quad \epsilon > 1 - \phi, \quad \epsilon \neq 1, \quad -1 < \gamma < 0.\]

The partial derivatives \(u_C(c, 1)\) and \(u_A(c, 1)\) are given by

\[u_C(c, 1) = \frac{c^\phi - 1}{1 + \gamma} \left( \frac{c^\phi + \gamma}{1 + \gamma} \right)^{(1-\epsilon-\phi)/\phi}, \quad u_A(c, 1) = \frac{\gamma}{1 + \gamma} \left( \frac{c^\phi + \gamma}{1 + \gamma} \right)^{(1-\epsilon-\phi)/\phi},\]

for \(c > \tilde{c} = (\gamma)^{1/\phi} > 0\). Hence, the function \(\pi\) has the form \(\pi(c) = \gamma c^{1-\phi}\), for \(c > \tilde{c} = (\gamma)^{1/\phi}\). Consequently, the function \(Z(\rho)\) defined in (45) takes the form:

\[Z(\rho) = \rho \left\{ 1 + \gamma \left[ 1 - \left( \frac{\tilde{g}}{\rho} \right) \right]^{1-\phi} \right\} - B, \quad \text{for } \rho > \tilde{\rho} = \frac{\tilde{g}}{1 - (\gamma)^{1/\phi}} > \tilde{g}.\]

The condition \(\rho > \tilde{g}/[1 - (\gamma)^{1/\phi}]\) entails that \([1 - (\tilde{g}/\rho)]^{\phi} > -\gamma\) and, therefore, \(1 + \gamma \left[ 1 - \left( \frac{\tilde{g}}{\rho} \right) \right]^{1-\phi}\). Hence, we have that

\[Z'(\rho) = 1 + \gamma \left[ 1 - \phi \left( \frac{\tilde{g}}{\rho} \right) \right] [1 - \left( \frac{\tilde{g}}{\rho} \right)]^{-\phi} = \phi \left( \frac{\tilde{g}}{\rho} \right) [1 - \phi \left( \frac{\tilde{g}}{\rho} \right)] \left\{ 1 + \gamma \left[ 1 - \left( \frac{\tilde{g}}{\rho} \right) \right]^{-\phi} \right\} > \phi \left( \frac{\tilde{g}}{\rho} \right) > 0.\]

1) If \(B \leq \tilde{\rho}\), then it follows from

\[\lim_{\rho \to \tilde{\rho}^+} Z(\rho) = -(B - \tilde{g}) < 0,\]

\(Z'(\rho) > 0\), and \(\lim_{\rho \to -\infty} Z(\rho) = \infty\) that there exists a unique solution \(\rho^S\) of \(Z(\rho) = 0\), where \(B \leq \tilde{\rho} < \rho^S\). 2) In the opposite case \(\tilde{\rho} < B\), it follows from

\[Z(B) = \gamma B \left[ 1 - \left( \frac{\tilde{g}}{B} \right) \right]^{1-\phi} < 0,\]

\(Z'(\rho) > 0\), and \(\lim_{\rho \to -\infty} Z(\rho) = \infty\) that there exists a unique solution \(\rho^S\) of \(Z(\rho) = 0\), where \(\tilde{\rho} < B < \rho^S\). Since \(Z'(\rho) > 0\), these properties imply that regardless of whether \(B < \tilde{\rho}\) or \(\tilde{\rho} < B\), there exists a unique value \(\rho^S > \text{max}\{\tilde{\rho}, B\}\) such that SPS-CP holds if and only if \(\rho > \rho^S\).
Appendix F: The centralized economy when $\rho = B$

Existence of steady state

We shall now derive conditions for the existence of a feasible steady state in the centralized economy when $q = B$. Assumption (27) introduced in Sect. 2 to rule out the paradoxical case in which instantaneous utility decreases along the decentralized BGP in spite of positive growth is equivalent to (29) and, hence, ensures that

$$h_0(c) = B \left[ v + \pi(c) \right] > 0, \quad \text{for } \bar{c} < c \leq 1,$$

where the second equality follows from using (E.5). $h'(c) > 0$, for $\bar{c} < c \leq 1$, implies that if there exists a steady state $\bar{c}$ with $\bar{c} < \bar{c} < 1$, then it is unique.

In the following we prove the equivalence of (51) and (52). Using the definition $h(c)$ given in (49),

$$h(c) = B \left[ v(c) + \pi(c) \right],$$

it is obvious that (51),

$$\lim_{c \to \bar{c}^+} \{B[v(c) + \pi(c)]\} < 0 < \lim_{c \to 1^-} \{B[v(c) + \pi(c)]\},$$

and, hence, equivalent to

$$B \left[ v(\bar{c}) + \lim_{c \to \bar{c}^+} \pi(c) \right] < 0 < B[v(1 - \bar{c}) + \pi(1)].$$

Using the definition $\bar{c} \equiv \frac{\beta - (1 - v)B}{vB}$ it can be shown that the last representation is equivalent to

$$-\beta + \left[ 1 - v(1 - \bar{c}) + \lim_{c \to \bar{c}^+} \pi(c) \right] B < 0 < -\beta + [1 + \pi(1)]B,$$

and, hence, also equivalent to

$$\left[ 1 - v(1 - \bar{c}) + \lim_{c \to \bar{c}^+} \pi(c) \right] B < \beta < [1 + \pi(1)]B,$$

where the last representation is identical to (52). Next, we will illustrate condition (52) by means of the three specifications:

**Example #1**: In the case of the multiplicative specification (D.1),

$$u(C, A) = \left( \frac{CA^\gamma}{1 - \epsilon} \right)^{1 - \epsilon}, \quad \gamma > -1, \quad \epsilon > \max\{0, \gamma/(1 + \gamma)\}, \quad \epsilon \neq 1,$$

we have $v = (1 + \gamma)\epsilon - \gamma > 0$. From (D.2) it follows that $\pi(c) = \gamma c$ for $c > \bar{c}$, where $\bar{c} = 0$ holds irrespective of the sign of $\gamma$. Substitution of the expressions for $v$, $\pi(c)$, and $\bar{c}$ into (52) yields the following condition:

$$(1 + \gamma)(1 - \epsilon)B < \beta < (1 + \gamma)B.$$

**Example #2**: In the case of the additive specification (D.3),

\[ \text{(continued...)} \]
\[ u(C, A) = \frac{1}{1 - \epsilon} \left( C + \gamma A \right)^{1-\epsilon}, \quad \epsilon > 0, \quad \epsilon \neq 1, \quad \gamma > -1, \]

we have \( v = \epsilon > 0 \). From (D.4) it follows that

\[
\pi(c) = \gamma, \quad \text{for } c > \bar{c} = \begin{cases} 
0, & \text{if } \gamma > 0, \\
-\gamma, & \text{if } -1 < \gamma < 0,
\end{cases}
\]

Using these results, the condition (52) can be expressed as

\[
\begin{cases} 
(1 + \gamma - \epsilon)B < \beta < (1 + \gamma)B, & \text{if } \gamma > 0, \\
(1 - \epsilon)(1 + \gamma)B < \beta < (1 + \gamma)B, & \text{if } -1 < \gamma < 0.
\end{cases}
\]

**Example #3:** The generalized CES utility function (D.5),

\[ u(C, A) = \frac{1}{1 - \epsilon} \left( \frac{C^\phi + \gamma A^\phi}{1 + \gamma} \right)^{(1-\epsilon)/\phi}, \quad 0 < \phi < 1, \quad \epsilon > 1 - \phi, \quad \epsilon \neq 1, \quad \gamma > -1, \]

has the property that \( v = \epsilon \). From (D.6) it follows that

\[
\pi(c) = \gamma c^{1-\phi}, \quad \text{for } c > \bar{c} = \begin{cases} 
0, & \text{if } \gamma > 0, \\
(-\gamma)^{1/\phi}, & \text{if } -1 < \gamma < 0.
\end{cases}
\]

Substitution of these results into (52) yields

\[
\begin{cases} 
(1 - \epsilon)B < \beta < (1 + \gamma)B, & \text{if } \gamma > 0, \\
\left[1 - (-\gamma)^{1/\phi}\right](1 - \epsilon)B < \beta < (1 + \gamma)B, & \text{if } -1 < \gamma < 0.
\end{cases}
\]

**Stability analysis**

Linearizing (47)–(48) around this steady state we have that

\[
\begin{pmatrix} \dot{c} \\ \dot{a} \end{pmatrix} \approx \begin{pmatrix} B[\bar{c} + \pi(\bar{c})] & 0 \\ 0 & \bar{a} \end{pmatrix} \begin{pmatrix} c - \bar{c} \\ a - \bar{a} \end{pmatrix},
\]

where, using (E.5),

\[
\frac{\partial \dot{c}}{\partial c} (\bar{c}, \bar{a}) = \bar{c} \sigma'(\bar{c}) h'(\bar{c}) = B[\bar{c} + \pi(\bar{c})] > 0.
\]

Here, the sign follows from (29) and that \( \bar{c} < 1 \). Given that the coefficient matrix is diagonal, its eigenvalues are its diagonal elements, which are both positive. Hence, the steady state of the centralized economy is unstable in the sense that the variables \( c \) and \( a \) have no transitional dynamics, i.e., they instantaneously jump to their steady-state levels \( \bar{c} \) and \( \bar{a} \). In the next subsection it will become obvious that in contrast to \( c \) and \( a \) the variable \( q = \mu/\lambda \) exhibits transitional dynamics.
Explicit solution of $q(t)$

We shall now illustrate the assertion in Appendix C that the ratio $q = \mu/\lambda$ does not have to be constant along a BGP when $\rho = B$. To this end, we shall determine the explicit solution of $q(t)$. In the following, we make use of the fact that if $\rho = B$, then the solution of the centrally planned economy has the property that $a(t) = \bar{a}$ and $c(t) = \bar{c}$ for $t \geq 0$, where $\bar{c}$ is the unique solution of the equation

$$B - \nu B(1 - c) + B\pi(c) - \beta = 0,$$

and the growth rate is given by $\bar{g} = B(1 - \bar{c})$.

Substituting $c = \bar{c}$ into Eq. (46), the equilibrium dynamics of $q$ is governed by the following differential equation with initial condition:

$$\dot{q} = -(1 + Bq)\pi(\bar{c}), \quad q(0) = 0.$$

It is easily verified that its solution is given by

$$q(t) = \frac{1}{B} \left[ e^{-\pi(\bar{c})Bt} - 1 \right].$$

Thus,

$$\lim_{t \to \infty} q(t) = \begin{cases} -1/B, & \text{if } \pi(\bar{c}) > 0 \iff u_A > 0, \\ \infty, & \text{if } \pi(\bar{c}) < 0 \iff u_A < 0, \end{cases}$$

and

$$\lim_{t \to \infty} [1 + Bq(t)] = \begin{cases} 0, & \text{if } \pi(\bar{c}) > 0 \iff u_A > 0, \\ \infty, & \text{if } \pi(\bar{c}) < 0 \iff u_A < 0. \end{cases}$$

Furthermore, the long-run growth rate of $q(t)$ is

$$\lim_{t \to \infty} \frac{\dot{q}}{q} = \begin{cases} 0, & \text{if } \pi(\bar{c}) > 0 \iff u_A > 0, \\ -B\pi(\bar{c}), & \text{if } \pi(\bar{c}) < 0 \iff u_A < 0. \end{cases}$$

Transversality conditions

Finally, we show that the transversality conditions $\lim_{t \to \infty} e^{-\beta t} \lambda K = 0$ and $\lim_{t \to \infty} e^{-\beta t} \mu A = 0$ are satisfied. For this purpose we employ the results for $\lim_{t \to \infty} \dot{q}/q$ given in Appendix F.3 as well as the following facts: (i) $\dot{\lambda}/\lambda = \beta - B$ holds for $t \geq 0$ [see (C.4)], (ii) $\dot{K}/K = \dot{A}/A = \bar{g} = B(1 - \bar{c})$ holds for $t \geq 0$, and (iii) $\dot{\mu}/\mu = \dot{\lambda}/\lambda + \dot{q}/q$ holds due to the definition $q \equiv \mu/\lambda$.

The fulfillment of the transversality condition $\lim_{t \to \infty} e^{-\beta t} \lambda K = 0$ follows from the fact that the growth rate of $e^{-\beta t} \lambda K$ is constant over time is strictly negative.
\[-\beta + \dot{\lambda}/\lambda + \dot{K}/K = -\beta + (\beta - B) + B(1 - \bar{c}) = -B\bar{c} < 0, \quad t \geq 0.\]

With respect to the transversality condition \(\lim_{t \to \infty} e^{-\beta t} \mu A = 0\), in which the growth rate \(\dot{\mu}/\mu\) in contrast to \(\dot{A}/A\) is not constant over time, we have to distinguish the following two cases:

1. If \(u_A > 0\), we have \(\bar{q}/\bar{q} = \lim_{t \to \infty} \bar{q}/q = 0\), so that \(\bar{\mu}/\bar{\mu} = \lim_{t \to \infty} \bar{\mu}/\mu = \beta - B\) and

\[-\beta + \bar{\mu}/\mu + \bar{A}/A = -\beta + (\beta - B) + B(1 - \bar{c}) = -B\bar{c} < 0.\]

2. If \(u_A < 0\), we have \(\bar{q}/\bar{q} = -B\pi(\bar{c})\) which, in turn, implies that \(\bar{\mu}/\mu = (\beta - B) - B\pi(\bar{c})\) and

\[-\beta + \bar{\mu}/\mu + \bar{A}/A = -\beta + [\beta - B - B\pi(\bar{c})] + B(1 - \bar{c}) = -B[\bar{c} + \pi(\bar{c})] < 0,\]

where the negative sign is obtained by taking into account that i) \(\dot{c} < \bar{c} < 1\) and ii) according to (29) \(c + \pi(c) > 0\) holds for \(\dot{c} < c \leq 1\).

From these results it is obvious that the transversality condition \(\lim_{t \to \infty} e^{-\beta t} \mu A = 0\) is satisfied regardless of whether \(u_A > 0\) or \(u_A < 0\).

Acknowledgements I gratefully acknowledge the helpful comments of two anonymous referees. Without implicating, I am specially indebted to one referee for detailed and insightful suggestions that contributed to improve the paper in a substantial manner.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. This work has been supported by the Spanish Ministerio de Economía, Industria y Competitividad and the Fondo Europeo de Desarrollo Regional (FEDER) under Grant No. ECO2017-85701-P, and the Spanish Ministerio de Ciencia e Innovación and the Fondo Europeo de Desarrollo Regional (FEDER) under Grant No. PID2021-127599NB-I00.

Availability of data and material Not applicable.

Code availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

Abel AB (1990) Asset prices under habit formation and catching up with the Joneses. Am Econ Rev 80 (2):38–42
Abel AB (2005) Optimal taxation when consumers have endogenous benchmark levels of consumption. Rev Econ Stud 72(1):21–42
Alonso-Carrera J, Caballé J, Raurich X (2004) Consumption externalities, habit formation and equilibrium efficiency. Scand J Econ 106(2):231–251
Alonso-Carrera J, Caballé J, Raurich X (2005) Growth, habit formation, and catching-up with the Joneses. Eur Econ Rev 49(6):1665–1691
Alonso-Carrera J, Caballé J, Raurich X (2006) Welfare implications of the interaction between habits and consumption externalities. Int Econ Rev 47(2):557–571
Aronsson T, Johansson-Stenman O (2010) Positional concerns in an OLG model: optimal labor and capital income taxation. Int Econ Rev 51(4):1071–1095
Carroll CD, Overland J, Weil DN (1997) Comparison utility in a growth model. J Econ Growth 2(4):339–367
Dupor B, Liu WF (2003) Jealousy and equilibrium overconsumption. Am Econ Rev 93(1):423–428
Faria JR, McAdam P (2013) Anticipation of future consumption: a monetary perspective. J Money Credit Bank 45(2–3):423–447
Faria JR, McAdam P (2018) The green golden rule: habit and anticipation of future consumption. Econ Lett 172:131–133
Fisher WH, Hof FX (2000) Relative consumption, economic growth, and taxation. J Econ 72(3):241–262
Ghosh S, Wendner R (2017) Positional preferences and efficient capital accumulation when households exhibit a preference for wealth. Oxford Econ Pap 70(1):114–140
Gómez MA (2006) Optimal consumption taxation in a model of endogenous growth with external habit formation. Econ Lett 93(3):427–435
Gómez MA (2010) A note on external habits and efficiency in the AK model. J Econ 99(1):53–64
Gómez MA (2021) On the closed-form solution of an endogenous growth model with anticipated consumption. J Math Econ 95:102471
Hestenes MR (1996) Calculus of variations and optimal control theory. Wiley, New York
Kuznitz A, Kandel S, Fos V (2008) A portfolio choice model with utility from anticipation of future consumption and stock market mean reversion. Eur Econ Rev 52(8):1338–1352
Leitmann G (1981) The calculus of variations and optimal control, mathematical concepts and methods in science and engineering, vol 24. Springer, New York
Léonard D, Van Long N (1992) Optimal control theory and static optimization in economics. Cambridge University Press, Cambridge
Liu WF, Turnovsky SJ (2005) Consumption externalities, production externalities, and long-run macroeconomic efficiency. J Public Econ 89(5–6):1097–1129
Ljungqvist L, Uhlig H (2000) Tax policy and aggregate demand management under catching up with the Joneses. Am Econ Rev 90(3):356–366
Loewenstein G (1987) Anticipation and the valuation of delayed consumption. Econ J 97(387):666–684
Monteiro G, Turnovsky SJ (2016) Anticipated consumption and its impact on capital accumulation and growth: ‘forward-looking’ vs. ‘backward-looking’ consumption reference. Int J Econ Theory 12 (3):203–232
Pham TKC (2019) Keeping up with or running away from the Joneses: the Barro model revisited. J Econ 126(2):179–192
Rauscher M (1997) Conspicuous consumption, economic growth, and taxation. J Econ 66(1):35–42
Turnovsky SJ, Monteiro G (2007) Consumption externalities, production externalities, and efficient capital accumulation under time non-separable preferences. Eur Econ Rev 51(2):479–504
Gómez MA (2007) Equilibrium efficiency in the Ramsey model with habit formation, Stud Nonlinear Dyn Econom 11(2), Article 2
Gómez MA, Monteiro G (in press) Anticipated future consumption in an endogenous growth model, Macroecon Dyn

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