Abstract

In this paper we propose to use elements of the mathematical formalism of Quantum Mechanics to capture the idea that agents’ preferences, in addition to being typically uncertain, can also be indeterminate. They are determined (i.e., realized, and not merely revealed) only when the action takes place. An agent is described by a state that is a superposition of potential types (or preferences or behaviors). This superposed state is projected (or “collapses”) onto one of the possible behaviors at the time of the interaction. In addition to the main goal of modelling uncertainty of preferences that is not due to lack of information, this formalism seems to be adequate to describe widely observed phenomena of noncommutativity in patterns of behavior. We explore some implications of our approach in a comparison between classical and type indeterminate rational choice behavior. The potential of the approach is illustrated in two examples.

JEL: D80, C65, B41

Keywords: indeterminacy, non-commutativity, quantum state.
1 Introduction

It has recently been proposed that models of quantum games can be used to study how the extension of classical moves to quantum ones (i.e., complex linear combinations of classical moves) can affect the analysis of a game. For example, Eisert et al. (1999) show that allowing the players to use quantum strategies in the Prisoners’ Dilemma is a way of escaping the well-known ‘bad feature’ of this game.\footnote{In the classical version of the dilemma, the dominant strategy for both players is to defect and thereby to do worse than if they had both decided to cooperate. In the quantum version, there are a couple of quantum strategies that are both a Nash equilibrium and Pareto optimal and whose payoff is the one of the joint cooperation.} From a game-theoretical point of view the approach consists in changing the strategy spaces, and thus the interest of the results lies in the appeal of these changes.\footnote{This approach is closely related to quantum computing. It relies on the use of a sophisticated apparatus to exploit q-bits’ property of entanglement in mixed strategies.}

This paper also proposes to use elements of the mathematical formalism of Quantum Mechanics but with a different intention: to model uncertain preferences.\footnote{In this work we borrow the elements of the quantum formalism that concern the measurement process. We will not use the part of the theory theory concerned with the evolution of systems over time.} The basic idea is that the Hilbert space model of Quantum Mechanics can be thought of as a very general contextual predictive tool particularly well-suited to describing experiments in psychology or in ‘revealing’ preferences.

The well-established Bayesian approach suggested by Harsanyi to model incomplete information consists of a chance move that selects the types of the players and informs each player of his own type. For the purposes of this paper, we underline the following essential implication of this approach: all uncertainty about a player’s type exclusively reflects the others player’s incomplete knowledge of it. This follows from the fact that a Harsanyi type is fully determined. It is a complete well-defined description of the characteristics of a player that is known to him. Consequently, from the point of view of the other players, uncertainty as to the type can only be due to lack of information. Each player has a probability distribution over the type of the other players, but her own type is fully determined and is known to her.

This brings us to the first important point at which we depart from the classical approach: we propose that in addition to informational reasons, the uncertainty about preferences is due to indeterminacy: prior to the moment a
player acts, her (behavior) type is indeterminate. The state representing the player is a superposition of potential types. It is only at the moment when the player selects an action that a specific type is actualized. It is not merely revealed but rather determined in the sense that prior to the choice, there is an irreducible multiplicity of potential types. Thus we suggest that in modelling a decision situation, we do not assume that the preference characteristics can always be fully known with certainty (neither to the decision-maker nor even to the analyst). Instead, what can be known is the state of the agent: a vector in a Hilbert space which encapsulates all existing information to predict how the agent is expected to behave in different decision situations.

This idea, daringly imported from Quantum Mechanics to the context of decision and game theory, is very much in line with Tversky and Simonson (Kahneman and Tversky 2000) according to whom “There is a growing body of evidence that supports an alternative conception according to which preferences are often constructed – not merely revealed – in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice”. In Ariely, Prelec and Loewenstein (2003), the authors show in a series of experiments that “valuations are initially malleable but become ”imprinted” after the agent is called upon to make an initial decision” (p. 74). This view is also consistent with that of cognitive psychology, which teaches one to distinguish between objective reality and the proximal stimulus to which the observer is exposed, and to further distinguish between those and the mental representation of the situation that the observer eventually constructs. More generally, this view fits in with the observation that players (even highly rational ones) may act differently in game theoretically equivalent situations that differ only in seemingly irrelevant aspects (framing, prior unrelated events, etc.). Our theory as to why agents act differently in game theoretically equivalent situations is that they are not in the same state; (revealed) preferences are contextual because of (intrinsic) indeterminacy.

The basic analogy with Physics, which makes it appealing to adopt the mathematical formalism of Quantum Mechanics to the social sciences, is the following: we view decisions and choices as something similar to the result of a measurement (of the player’s type). A situation of decision is then similar to an experimental setup to measure the player’s type. It is

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4The associated concept of irreducible uncertainty, which is the essence of indeterminacy, is formally defined in Section 2 of the paper.
modelled by an operator (called observable), and the resulting behavior is an eigenvalue of that operator. The non-commutativity of observables and its consequences (very central features of Quantum Mechanics) is reminiscent of many empirical phenomena like the following one exhibited in a well-known experiment conducted by Leon Festinger. In this experiment people were asked to sort a batch of spools into lots of twelve and give a square pegs a quarter turn to the left. They all agreed that the task was very boring. Then, they were told that one subject was missing for the experiment and asked to convince a potential female subject in the waiting room to participate. They were offered $1 for expressing their enthusiasm for the task. Some refused, but others accepted. Those who accepted maintained afterwards that the task was enjoyable. This experiment aimed at showing that attitudes change in response to cognitive dissonance. The dissonance faced by those who accepted to fake enthusiasm for $1 was due to the contradiction between the self-image of being "a good guy" and that of "being ready to lie for a dollar". Changing one’s attitude and persuading oneself that the task really was interesting was a way to resolve the dissonance. Similar phenomena have been documented in hazardous industries, with employees showing very little caution in the face of a danger. Here too, experimental and empirical studies (e.g., Daniel Ben–Horing 1979) showed how employees change their attitude after they have decided to work in a hazardous industry. More generally, suppose that an agent is presented with the same situation of decision in two different contexts. The contexts may vary with respect to the situation of decision that precedes the investigated one. In Festinger’s experiment the two measurements of the attitude toward the task differ in that the second one was preceded by a question about willingness to lie about the task for a dollar. Two contexts may also vary with respect to the framing of the situation of decision, (cf. Selten 1998)). If we do not observe the same decision in the two contexts, then the classical approach considers that the two situations are not identical and hence that they should be modeled so as to incorporate the context. In many cases, however, such an assumption, i.e., that the situations are different, is difficult to justify.

In contrast, we propose that the difference between the two decisions comes from the fact that the agent is not in the same state. The context,

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5Leon Festinger is the father of the theory of cognitive dissonance (Festinger 1957).
6The experience was richer. People were divided into two groups offered different amounts of money. For our purpose it is sufficient to focus on a single result.
e.g., a past situation of decision to which the agent has been exposed, is represented by an operator that does not commute with the operator associated with the situation of decision currently considered. The consequence is that the initial agent’s state has changed and that the agent is therefore expected to behave differently from what she would have done if confronted directly with the situation. As in Quantum Mechanics, the non-commutativity of certain situations of decision (measurements) leads us to conjecture that the preferences of an agent are represented by a state that is indeterminate and gets determined (with respect to any particular type characteristics) in the course of interaction with the environment. In section 3, we show how this approach can explain the reversal of preferences in a model of rational choice and that it provides a framework for explaining cognitive dissonance and framing effects.

The objective of this paper is to propose a theoretical framework for modelling the KT(Kahneman–Tversky)–man, i.e., for the ”constructive preference perspective”. Our approach amounts to extending the classical representation of uncertainty in Harsanyi’s style to non-classical indeterminacy. This work is a contribution to Behavioral Economics. A distinguishing feature of behavioral theories is that they often focus on rather specific anomalies (e.g., ‘trade-off contrast’ or ‘extremeness aversion’ (Kahneman and Tversky 2000). Important insights have been obtained by systematically investigating the consequences on utility maximization of ‘fairness concerns’ (Rabin 1993), ‘temptation and costly self-control’ (Gul and Pesendorfer 2001) or ‘concerns for self-image’ (Benabou and Tirole 2002). Yet, other explanations appeal to bounded rationality, e.g., ‘superficial reasoning’ or ‘choice of beliefs’ (Selten 1998, Akerlof and Dickens 1982). In contrast, the type indeterminacy model is a framework model that addresses structural properties of preferences, i.e., their intrinsic indeterminacy. A value of our approach is in providing a unified explanation for a wide variety of behavioral phenomena.

In section 2, we present the framework and some basic notions of quantum theory. In Section 3, we develop applications of the theory to social sciences. In Section 4, we discuss some basic assumptions of the model. The appendix provides a brief exposition of some basic concepts of Quantum Mechanics.
2 The basic framework

In this section we present the basic notions of our framework. They are heavily inspired by the mathematical formalism of Quantum Mechanics (see e.g., Cohen-Tannoudji, Diu, Laloë 1973 and Cohen 1989) from which we also borrow the notation.

2.1 The notions of state and superposition

The object of our investigation is the individual choice behavior, which we interpret as the revelation of an agent’s preferences in a situation of decision that we shall call a DS (Decision Situation). In this paper, we focus on decision situations that do not involve any strategic thinking. Examples of such DS include the choice between buying a Toshiba or a Compaq laptop, the choice between investing in a project or not, the choice between a sure gain of $100 or a bet with probability 0.5 to win $250 and 0.5 to win $0, etc. When considering games, we view them as decision situations from the perspective of a single player who plays once.

An agent is represented by a state which encapsulates all the information on the agent’s expected choice behavior. The formalism that we present below allows for a variety of underlying models. In particular, some states may correspond to a choice. For instance if the choice set is \{a, b, c\}, we may identify three possible states corresponding to the respective choices a, b and c. A state may also correspond to an ordering of the 3 items in which case we have six possible states (e.g. a state could be (a,b,c) in this order). We may also consider other models. In section 3.1 we examine consequences of the second case. For the ease of exposition, the basic framework is presented in terms of the first case where states are identified with choices (but the reader is invited to keep in mind that other interpretations of the model are possible). However what we just wrote must be considered in the light of the principle of superposition as explained below.

Mathematically, a state |ψ⟩ is a vector in a Hilbert space \(\mathcal{H}\) of finite or countably infinite dimensions, over the field of the real numbers \(\mathbb{R}\). The

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7All information (beliefs) and strategic considerations are embedded in the definition of the choices. Thus an agent’s play of cooperation in a Prisoner’s Dilemma, is a play of cooperation given his information (knowledge) about the opponent.

8In Quantum Mechanics the field that is used is the complex numbers field. However, for our purposes the field of real numbers provides the structure and the properties needed.
relationship between $\mathcal{H}$ and a decision situation will be specified later. For technical reasons related to the probabilistic content of the state, each state vector has to be of length one, that is, $\langle \psi | \psi \rangle^2 = 1$ (where $\langle \cdot | \cdot \rangle$ denotes the inner product in $\mathcal{H}$). So all vectors of the form $\lambda |\psi\rangle$ where $\lambda \in \mathbb{R}$, correspond to the same state, which we represent by a vector of length one.

A key ingredient in the formalism of indeterminacy is the principle of superposition. This principle states that the linear combination of any two states is itself a possible state.\(^9\) Consider two states $|\varphi_1\rangle, |\varphi_2\rangle \in \mathcal{H}$. If $|\psi\rangle = \lambda_1 |\varphi_1\rangle + \lambda_2 |\varphi_2\rangle$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ then $|\psi\rangle \in \mathcal{H}$. The principle of superposition implies that, unlike the Harsanyi type space, the state space is non-Boolean.\(^{10}\)

### 2.2 The notions of measurement and of observable

Measurement is a central notion in our framework. A measurement is an operation (or an experiment) performed on a system. It yields a result, the outcome of the measurement. A defining feature of a measurement is the so-called first-kindness property.\(^{11}\) It refers to the fact that if one performs a measurement on a system and obtains a result, then one will get the same result if one performs again that measurement on the same system immediately after. Thus, the outcome of a first-kind measurement is reproducible but only in a next subsequent measurement. First-kindness does not entail that the first outcome is obtained when repeating a measurement if other measurements are performed on the system in between. Whether an operation is a measurement, i.e., is endowed with the property of first-kindness is an empirical issue. When it comes to decision theory, it means that we do not, a priori, assume that any choice set can be used to measure preferences.

\(^{9}\)We use the term state to refer to ‘pure state’. Some people use the term state to refer to mixture of pure states. A mixture of pure states combines indeterminacy with elements of incomplete information. They are represented by the so called density operators.

\(^{10}\)The distributivity condition defining a Boolean space is dropped for a weaker condition called ortho-modularity. The basic structure of the state space is that of a logic, i.e., an orthomodular lattice. For a good presentation of Quantum Logic, a concept introduced by Birkhoff and Von Neuman (1936), and further developed by Mackey (2004, 1963), see Cohen (1989).

\(^{11}\)The term first-kind measurement was introduced by Pauli.
In particular, the product set of two sets each of which is associated with a first-kind measurement, is not in general associated with a first-kind measurement. The set of decision problems that we consider consists exclusively of decision problems that can be associated with first-kind measurements. We call those decision problems Decision Situations (DS)\textsuperscript{12}

A Decision Situation $A$ can be thought of as an experimental setup where the agent is invited to choose a particular action among all the possible actions allowed by this Decision Situation. In this paper, we will consider only the case of finitely many possible outcomes. They will be labelled from 1 to $n$ by convention. When an agent selects an action, we say that she ‘plays’ the DS. To every Decision Situation $A$, we will associate an observable, namely, a specific symmetric operator on $\mathcal{H}$ which, for notational simplicity, we also denote by $A$\textsuperscript{13}. If we consider only one Decision Situation $A$ with $n$ possible outcomes, we can assume that the associated Hilbert space is $n$-dimensional and that the eigenvectors of the corresponding observable, which we denote by $|1_A\rangle, |2_A\rangle, \ldots, |n_A\rangle$, all correspond to different eigenvalues, denoted by $1_A, 2_A, \ldots, n_A$ respectively. By convention, the eigenvalue $i_A$ will be associated with choice $i$.

$$A |k_A\rangle = k_A |k_A\rangle, \quad k = 1, \ldots, n.$$  

As $A$ is symmetric, there is a unique orthonormal basis of the relevant Hilbert space $\mathcal{H}$ formed with its eigenvectors. The basis $\{|1_A\rangle, |2_A\rangle, \ldots, |n_A\rangle\}$ is the unique orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $A$. It is thus possible to represent the agent’s state as a superposition of vectors of this basis:

$$|\psi\rangle = \sum_{k=1}^{n} \lambda_k |k_A\rangle,$$  \hspace{1cm} (1)

where $\lambda_k \in \mathbb{R}, \forall k \in \{1, \ldots, n\}$ and $\sum_{k=1}^{n} \lambda_k^2 = 1$. 

\textsuperscript{12}Even standard decision theory implicitly restricts its application to decision problems satisfying the first-kindness property (or that can be derived from such decision problems). In contrast, random utility models do not require choice behavior to satisfy the first-kindness property in the formulation used in this paper.

\textsuperscript{13}Observables in Physics are represented by Hermitian operators because QM is defined over the field of complex numbers. Here, we confine ourselves to the field of real numbers which is why observables are represented by symmetric operators.
The Hilbert space can be decomposed as follows
\[
\mathcal{H} = \mathcal{H}_1^A \oplus \ldots \oplus \mathcal{H}_n^A, \quad \mathcal{H}_i^A \perp \mathcal{H}_j^A, \quad i \neq j,
\tag{2}
\]
where \( \oplus \) denotes the direct sum of the subspaces \( \mathcal{H}_1^A, \ldots, \mathcal{H}_n^A \) spanned by \( |1_A\rangle, \ldots, |n_A\rangle \) respectively. Or, equivalently, we can write \( I_\mathcal{H} = P_1^A + \ldots + P_n^A \) where \( P_i^A \) is the projection operator on \( \mathcal{H}_i^A \) and \( I_\mathcal{H} \) is the identity operator on \( \mathcal{H} \).

A Decision Situation \( A \) is an experimental setup and the actual implementation of the experiment is represented by a measurement of the associated observable \( A \). According to the so-called Reduction Principle (see Appendix), the result of such a measurement can only be one of the \( n \) eigenvalues of \( A \). If the result is \( m_A \), i.e., the player selects action \( m \), the superposition \( \sum \lambda_i |i_A\rangle \) “collapses” onto the eigenvector associated with the eigenvalue \( m_A \). The initial state \( |\psi\rangle \) is projected into the subspace \( \mathcal{H}_{m_A}^A \) (of eigenvectors of \( A \) with eigenvalue \( m_A \)). The probability that the measurement yields the result \( m_A \) is equal to \( \langle m_A | \psi \rangle^2 = \lambda_{m_A}^2 \), i.e., the square of the corresponding coefficient in the superposition. The coefficients themselves are called ‘amplitudes of probability’. They play a key role when studying sequences of measurements (see Section 2.3). As usual, we interpret the probability of \( m_A \) either as the probability that one agent in state \( |\psi\rangle \) selects action \( m_A \) or as the proportion of the agents who will make the choice \( m_A \) in a population of many agents, all in the state \( |\psi\rangle \).

In our theory an agent is represented by a state. We shall also use the term type (and eigentype) to denote a state corresponding to one eigenvector, say \( |m_A\rangle \). An agent in this state is said to be of type \( m_A \). An agent in a general state \( |\psi\rangle \) can be expressed as a superposition of all eigentypes of the DS under consideration. Our notion of type is closely related to the notion introduced by Harsanyi. Consider a simple choice situation e.g., when an employee faces a menu of contracts. The type captures all the agent’s characteristics (taste, subjective beliefs, private information) of relevance for uniquely predicting the agent’s behavior. In contrast to Harsanyi, we shall not assume that there exists an exhaustive description of the agent that enables us to determine the agent’s choice uniquely and simultaneously in all possible Decision Situations.

\footnote{That is, for \( i \neq j \) any vector in \( \mathcal{H}_i^A \) is orthogonal to any vector in \( \mathcal{H}_j^A \) and any vector in \( \mathcal{H} \) is a sum of \( n \) vectors, one in each component space.}
Instead, our types are characterized by an irreducible uncertainty that is revealed when the agent is confronted with a sequence of DS (see Section 2.3.2 below for a formal characterization of irreducible uncertainty).

Remark: Clearly, when only one DS is considered, the above description is equivalent to the traditional probabilistic representation of an agent by a probability vector \((\alpha_1, \ldots, \alpha_n)\) in which \(\alpha_k\) is the probability that the agent will choose action \(k\) and \(\alpha_k = \lambda_k^2\) for \(k = 1, \ldots, n\). The advantage of the proposed formalism consists in enabling us to study several decision situations and the interaction between them.

2.3 More than one Decision Situation

When studying more than one DS, say \(A\) and \(B\), the key question is whether the corresponding observables are commuting operators in \(H\), i.e., whether \(AB = BA\). Whether two DS can be represented by two commuting operators or not is an empirical issue. We next study its mathematical implications.

2.3.1 Commuting Decision Situations

Let \(A\) and \(B\) be two DS. If the corresponding observables commute then there is an orthonormal basis of the relevant Hilbert space \(H\) formed by eigenvectors common to both \(A\) and \(B\). Denote by \(|i\rangle\) (for \(i = 1, \ldots, n\)) these basis vectors. We have

\[ A|i\rangle = i_A|i\rangle \quad \text{and} \quad B|i\rangle = i_B|i\rangle. \]

In general, the eigenvalues can be degenerated (i.e., for some \(i\) and \(j\), \(i_A = j_A\) or \(i_B = j_B\)). Any normalized vector \(|\psi\rangle\) of \(H\) can be written in this basis:

\[ |\psi\rangle = \sum_i \lambda_i |i\rangle, \]

where \(\lambda_i \in \mathbb{R}\), and \(\sum_i \lambda_i^2 = 1\). If we measure \(A\) first, we observe eigenvalue \(i_A\) with probability

\[ p_A(i_A) = \sum_{j: j_A = i_A} \lambda_j^2. \]  

\[ \text{In the argument that we develop in section 3.1, the pure states are linear orders therefore choice experiments are observables with degenerated eigenvalues} \]
If we measure \( B \) first, we observe eigenvalue \( j_B \) with probability
\[
p_B(j_B) = \sum_{k:k_B=j_B} \lambda_k^2.
\]
After \( B \) is measured and the result \( j_B \) is obtained, the state \( \psi \rangle \) is projected into the eigensubspace \( \mathcal{E}_{j_B} \) spanned by the eigenvectors of \( B \) associated with \( j_B \). More specifically, it collapses onto the state:
\[
|\psi_{j_B}\rangle = \frac{1}{\sqrt{\sum_{k:k_B=j_B} \lambda_k^2}} \sum_{k:k_B=j_B} \lambda_k |k\rangle
\]
(the factor \( \frac{1}{\sqrt{\sum_{k:k_B=j_B} \lambda_k^2}} \) is necessary to make \( |\psi_{j_B}\rangle \) a unit vector).

When we measure \( A \) on the agent in the state \( |\psi_{j_B}\rangle \), we obtain \( i_A \) with probability
\[
p_A(i_A|j_B) = \frac{1}{\sum_{k:k_B=j_B} \lambda_k^2} \sum_{k:k_B=j_B} \lambda_k^2.
\]
So when we measure first \( B \) and then \( A \), the probability of observing the eigenvalue \( i_A \) is
\[
p_{AB}(i_A) = \sum_j p_B(j_B) p_A(i_A|j_B)
\]
\[
= \sum_j \frac{1}{\sum_{k:k_B=j_B} \lambda_k^2} \sum_{k:k_B=j_B} \lambda_k^2 \sum_{l_A=i_A} \lambda_l^2
\]
\[
= \sum_j \sum_{l_A=i_A} \lambda_l^2 = \sum_{i_A} \lambda_l^2.
\]
Hence, \( p_{AB}(i_A) = p_A(i_A) \), \( \forall i \), and similarly \( p_{BA}(j_B) = p_B(j_B) \), \( \forall j \).

When dealing with commuting observables it is meaningful to speak of measuring them simultaneously. Whether we measure first \( A \) and then \( B \) or first \( B \) and then \( A \), the probability distribution on the joint outcome is
\[
p(i_A \land j_B) = \sum_{k:k_B=j_B} \lambda_k^2,
\]
so \( (i_A, j_B) \) is a well-defined event. Formally, this implies that the two \( DS \) can be merged into a single \( DS \). When we measure it, we obtain a vector as the outcome, i.e., a value in \( A \) and a value in \( B \). To each eigenvalue of the merged observable we associate a type that captures all the characteristics of the agent relevant to her choices (one in each \( DS \)).

Remark: Note that, as in the case of a single \( DS \), for two such commuting \( DS \) our model is equivalent to a standard (discrete) probability space in which the elementary events are \{\((i_A, j_B)\)\} and \( p(i_A \land j_B) = \sum_{k:k_B=j_B} \lambda_k^2 \).

In particular, in accordance with the calculus of probability we see that the conditional probability formula holds:
\[
p_{AB}(i_A \land j_B) = p_A(i_A) p_B(j_B|i_A).
\]
This also means that the type space associated with type characteristics represented by commuting observables is equivalent to the Harsanyi type space. In particular, if the Decision Situations $A$ and $B$ together provide a full characterization of the agent, then all types $i_{Aj_{B}}$ are mutually exclusive: knowing that the agent is of type $1_{A}2_{B}$ it is sure that she is not of type $i_{A}j_{B}$ for $i \neq 1$ and/or $j \neq 2$.

As an example, consider the following two Decision Situations. Let $A$ be the Decision Situation presenting a choice between a week vacation in Tunisia and a week vacation in Italy. And let $B$ be the choice between buying 1000 euros of shares of Bouygues Telecom or of Deutsche Telecom. It is quite plausible that $A$ and $B$ commute, but whether or not this is the case is of course an empirical question. If $A$ and $B$ commute, we expect that a decision on portfolio ($B$) will not affect the decision-making concerning the vacation ($A$). And thus the order in which the decisions are made does not matter as in the classical model.

Note finally that the commutativity of the observables does not exclude statistical correlations between observations. To see this, consider the following example in which $A$ and $B$ each have two degenerated eigenvalues in a four dimensional Hilbert space. Denote by $|i_{A}j_{B}\rangle$ ($i = 1,2, j = 1,2$) the eigenvector associated with eigenvalues $i_{A}$ of $A$ and $j_{B}$ of $B$, and let the state $|\psi\rangle$ be given by

$$|\psi\rangle = \sqrt{\frac{3}{8}}|1_{A}1_{B}\rangle + \sqrt{\frac{1}{8}}|1_{A}2_{B}\rangle + \sqrt{\frac{1}{8}}|2_{A}1_{B}\rangle + \sqrt{\frac{3}{8}}|2_{A}2_{B}\rangle$$

Then, $p_{A}(1_{A}|1_{B}) = \frac{3}{8+\frac{3}{8}} = \frac{3}{7}$, $p_{A}(2_{A}|1_{B}) = \frac{3}{8+\frac{3}{8}} = \frac{1}{7}$.

So if we first measure $B$ and find, say, $1_{B}$, it is much more likely (with probability $\frac{3}{7}$) that when measuring $A$ we will find $1_{A}$ rather than $2_{A}$ (with probability $\frac{1}{7}$). But the two interactions (measurements) do not affect each other, i.e., the distribution of the outcomes of the measurement of $A$ is the same whether or not we measure $B$ first.

### 2.3.2 Non-commuting Decision Situations

It is when we consider Decision Situations associated with observables that do not commute that the predictions of our model differ from those of the probabilistic one. In such a context, the quantum probability calculus ($p(i_{A} | \psi) = \langle i_{A} | \psi \rangle^{2}$) generates cross-terms also called interference terms.
These cross-terms are the signature of indeterminacy. In the next section, we demonstrate how this feature can capture the phenomenon of cognitive dissonance as well as that of framing.

Consider two Decision Situations $A$ and $B$ with the same number $n$ of possible choices. We shall assume for simplicity that the corresponding observables $A$ and $B$ have non-degenerated eigenvalues $1_A, 2_A, ..., n_A$ and $1_B, 2_B, ..., n_B$ respectively. Each set of eigenvectors $\{|1_A\rangle, |2_A\rangle, ..., |n_A\rangle\}$ and $\{|1_B\rangle, |2_B\rangle, ..., |n_B\rangle\}$ is an orthonormal basis of the relevant Hilbert space. Let $|\psi\rangle$ be the initial state of the agent

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |i_A\rangle = \sum_{j=1}^{n} \nu_j |j_B\rangle.$$  \hspace{1cm} (4)

We note that each set of eigenvectors of the respective observables forms a basis of the state space. The multiplicity of alternative basis is a distinguishing feature of this formalism. It implies that there is no single or privileged way to describe (express) the type of the agent. Instead, there exists a multiplicity of equally informative alternative ways to characterize the agent.

We shall now compute the probability for type $i_A$ under two different scenarios. In the first scenario we measure $A$ on the agent in state $|\psi\rangle$. In the second scenario we first measure $B$ on the agent in state $|\psi\rangle$ and thereafter measure $A$ on the agent in the state resulting from the first measurement. We can write observable $B'$'s eigenvector $|j_B\rangle$ in the basis made of $A$'s eigenvectors:

$$|j_B\rangle = \sum_{i=1}^{n} \mu_{ij} |i_A\rangle.$$  \hspace{1cm} (5)

Using the expression above we write the last term in equation (4)

$$|\psi\rangle = \sum_{j=1}^{n} \nu_j \sum_{i=1}^{n} \nu_j \mu_{ij} |i_A\rangle.$$  \hspace{1cm} (6)

From expression (6), we derive the probability $p_A(i_A)$ that the agent chooses $i_A$ in the first scenario: $p_A(i_A) = \left( \sum_{j=1}^{n} \nu_j \mu_{ij} \right)^2$. In the second scenario, she first plays $B$. By (4) we see that she selects action $j_B$ with probability
The state $| \psi \rangle$ is then projected onto $| j_B \rangle$. When the state is $| j_B \rangle$, the probability for $i_A$ is given by (5), it is $\nu^2_{ij}$. Summing up the conditionals, we obtain the (ex-ante) probability for $i_A$ when the agent first plays $B$ and then $A : p_{AB}(i_A) = \sum_{j=1}^{n} \nu^2_{ij}$, which is, in general, different from $p_A(i_A) = \left( \sum_{j=1}^{n} \nu_{ij} \right)^2$. Playing first $B$ changes the way $A$ is played. The difference stems from the so-called interference terms

$$p_A(i_A) = \left( \sum_{j=1}^{n} \nu_{ij} \right)^2 = \sum_{j=1}^{n} \nu^2_{ij} + 2 \sum_{j \neq j'} \left[ (\nu_{ij'} \mu_{ij}) (\nu_{ij} \mu_{ij'}) \right]$$

$$= p_{AB}(i_A) + \text{interference term}$$

The interference term is the sum of cross-terms involving the amplitudes of probability (the Appendix provides a description of interference effects in Physics).

Some intuition about interference effects may be provided using the concept of “propensity” due to Popper (1992). Imagine an agent’s mind as a system of propensities to act (corresponding to different possible actions). As long as the agent is not required to choose an action in a given $DS$, the corresponding propensities coexist in her mind; the agent has not “made up her mind”. A decision situation operates on this state of ”hesitation” to trigger the emergence of a single type (of behavior). But as long as alternative propensities are present in the agent’s mind, they affect choice behavior by increasing or decreasing the probability of the choices in the $DS$ under investigation.

An illustration of this kind of situation may be supplied by the experiment reported in Knetz and Camerer (2000). The two studied $DS$ are the Weak Link (WL) game and the Prisoners’ Dilemma (PD) game. They compare the distribution of choices in the Prisoners’ Dilemma (PD) game when it

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16 The Weak Link game is a type of coordination game where each player picks an action from a set of integers. The payoffs are defined in such a manner that each player wants to select the minimum of the other players but everyone wants that minimum to be as high as possible.
is preceded by a Weak Link (WL) game and when only the PD game is being played. Their results show that playing the WL game affects the play of individuals in the PD game. The authors appeal to an informational argument, which they call the “precedent effect”. However, they cannot explain the high rate of cooperation (37.5 %) in the last round of the PD game (Table 5, p. 206). Instead, we propose that the WL and the PD are two DS that do not commute. In such a case we expect a difference in the distributions of choices in the (last round of the) PD depending on whether or not it was preceded by a play of the WL or another PD game. This is because the type of the agent is being modified by the play of the WL game.

Remark: In the case where A and B do not commute, they cannot have simultaneously defined values: the state of the agent is characterized by an irreducible uncertainty. Therefore, and in contrast with the commuting case, two non-commuting observables cannot be merged into one single observable. There is no probability distribution on the events of the type “to have the value \( i_A \) for \( A \) and the value \( j_B \) for \( B \)”. The conditional probability formula does not hold:

\[
p_A(i_A) \neq \sum_{j=1}^{n} p_B(j_B) \cdot p(i_A | j_B).
\]

Indeed,

\[
\left( \sum_{j=1}^{n} \nu_{ij} \mu_{ij} \right)^2 = p_A(i_A) \neq \sum_{j=1}^{n} p_B(j_B) \cdot p(i_A | j_B) = \sum_{j=1}^{n} \nu_{ij}^2 \mu_{ij}^2.
\]

Consequently, the choice experiment consisting of asking the agent to select a pair \((i_A, j_B)\) out the set of alternatives \( A \times B \) is NOT a DS.

We must here acknowledge a fundamental distinction between the type space of our TI-model (Type Indeterminacy model) and that of Harsanyi. In the Harsanyi type space the (pure) types are all mutually exclusive: the agent is either of type \( \theta_i \) or of type \( \theta_j \) but she cannot be both! In the TI-model this is not always the case. When dealing with (complete) non-commuting DS, the types associated with respective DS are not mutually exclusive: knowing

\[\text{Remark:} \quad \text{The precedent effect hypothesis is as follows: “The shared experience of playing the efficient equilibrium in the WL game creates a precedent of efficient play strong enough to (...) lead to cooperation in a finitely repeated PD game”, Knetz and Camerer (2000 see p.206).} \]

\[\text{Remark:} \quad \text{We say that a DS is complete when its outcome provides a complete characterization of the agent.}\]
that the agent is of type $1_A$, which is a full description of her type, it cannot be said that she is not of type $1_B$. The eigentypes of non-commuting $DS$ are “connected” in the sense that the agent can transit from one type to another under the impact of a measurement. When making a measurement of $B$ on the agent of type $1_A$, she is projected onto one of the eigenvectors of $DS_B$. Her type changes from being $1_A$ to being some $j_B$.

3 The Type Indeterminacy model and Social Sciences

The theory of choice exposed in this paper does not allow for a straightforward comparison with standard choice theory. Significant further elaboration is required. Yet, some implications of the type indeterminacy approach can be explored. First, we shall be interested in comparing the behavior of an agent of indeterminate type with the one of a classical agent in the case where both satisfy the Weak Axiom of Revealed Preference (WARP) which is a basic axiom of rational choice. Then, we will show that this framework can be used to explain two instances of behavioral anomalies that have been extensively studied in the literature.

3.1 The TI-model and the classical rational man

In standard decision theory, it is assumed that an individual has preferences (i.e. a complete ranking or a complete linear ordering) on the universal set of alternatives $X$. The individual knows her preferences while the outsiders may not know them. But it is also possible that the individual only knows what she would choose from some limited sets of alternatives and not from the whole set $X$. Thus, a less demanding point of view consists in representing the choice behavior by a choice structure (i.e. a family $\mathcal{B}$ of subsets of the universal set of alternatives $X$ and a choice rule $C$ that assigns a non empty set of chosen elements $C(A)$ for all $A \in \mathcal{B}$). The link between the two points of view is well-known. From preferences, it is always possible to build a choice structure but the reverse is not always true. For it to be true, the choice structure must display a certain amount of consistency (satisfying the Weak Axiom of Revealed Preference) and the family $\mathcal{B}$ must include all

\footnote{Samuelson (1947)}
subsets of \( X \) of up to three elements\(^{20}\). So, preferences can be revealed by asking the individual to make several choices from subsets of \( X \). How does this simple scheme changes when we are dealing with an individual whose type is indeterminate?

Consider a situation where an individual is invited to make a choice of one item out of a set \( A, A \subseteq X \). If this experiment satisfies the first-kindness property, we can consider it to be a measurement represented by an observable. The set of possible outcomes of this experiment is the set \( A \). We also denote the observable by \( A \).\(^{21}\)

We make two key assumptions on the individual choice behavior:

A1. Choices out of a “small” subset satisfy the first-kindness property (the meaning of ”small” will be made precise).

A2. Choices out of a “small” subset respect our Weak Axiom of Revealed Preference (WARP’, see below).

Assumption A1 means that ”small” subsets are associated with \( DS \), i.e., the experiment consisting in letting an individual choose an item out of a ”small” subset of item is a measurement. Assumption A2 means that choices from “small” subsets are rational. The idea behind these assumptions is that an individual can, in her mind, structure any ”small” set of alternatives, i.e., simultaneously compare those alternatives. She may not be able to do that within a “big” set though. But this does not mean that our individual cannot make a choice from a “big” set. For example, she might use an appropriate sequence of binary comparisons and select the last winning alternative. However, such a compound operator would not in general satisfy the first-kindness property i.e., there may not exist any \( DS \) representing such an operation.

A standard formulation of the WARP can be found in Mas-Colell et al. (1995)\(^{22}\). We shall use a stronger version by assuming that \( C(B) \), for any \( B \in B \), is a singleton.\(^{23}\) For our purpose, it is also useful to formulate the axiom in two parts:

\(^{20}\)See for example Mas-Colell, Whinston and Green, Microeconomic Theory, p.13.

\(^{21}\)The use of the same symbol for sets of items and observables should not confuse the reader. Either the context unambiguously points to the right interpretation or we make it precise.

\(^{22}\)"If for some \( B \in B \) with \( x, y \in B \) we have \( x \in C(B) \), then for any \( B' \in B \) with \( x, y \in B' \) and \( y \in C(B') \), we must also have \( x \in C(B') \)." (Mas-Colell et al. (1995) p.10).

\(^{23}\)At this stage of the research we do not want to deal with indifference relations.
Consider two subsets $B, B' \in \mathcal{B}$ such that $B \subset B'$.

(a) Let $x, y \in B$, $x \neq y$ with $x \in C(B)$ then $y \notin C(B')$.

(b) Let $C(B') \cap B \neq \emptyset$ then $C(B) = C(B')$.

The intuition for (a) is that as we enlarge the set of items from $B$ to $B'$ a rational decision-maker never chooses from $B'$ an item that is available in $B$ but is not chosen. The intuition for (b) is that as we reduce the set of alternatives an item chosen in the large set, is also chosen in the smaller set containing that item and no item previously not chosen becomes chosen.

It can be shown that in the classical context (b) implies (a) (see Arrow (1959)). In our context where choice experiments can be non-commuting, we do not have such an implication. Moreover, the notion of choice function is not appropriate because it implicitly assumes the commutativity of choice (see below). Since our purpose is to investigate this issue explicitly, we express the axiom more immediately in terms of (observable) choice behavior in the following way and we call it WARP':

Consider two subsets $B, B' \in \mathcal{B}$ with $B \subset B'$.

1) Suppose the agent chooses from $B$ some element $x$. In a next subsequent measurement of $B'$ the outcome of the choice is not in $B \setminus \{x\}$.

2) Suppose the agent chooses from $B'$ some element $x$ that belongs to $B$. In a next subsequent measurement of $B$ the agent’s choice is $x$.

Points (1) and (2) capture the classical intuitions about rational choice behavior associated with (a) and (b) above. A distinction with the standard formulations of WARP is that we do not refer to a choice function but to choice behavior and that our axiom only applies to two subsequent choices. We shall see below that in the classical context where choices commute WARP' is equivalent to WARP. That is not longer true if we allow for some choices not to commute.

We now investigate the choice behavior of a type indeterminate agent under assumptions A1 and A2 above i.e., under the assumption that choices from small subsets satisfy the property of first-kindness and that they respect WARP'. We consider in turn two cases. In the first one, we assume that all the DS considered pairwise commute. In the second one, we allow for non-commuting DS. In the following we define “small” subsets as subsets of size 3 or less.

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\[24\text{Conditions (a) and (b) are equivalent to C2 and C4 in Arrow (1959).}\]
Case 1

The assumption here is that all experiments of choice from small subsets are compatible with each other, i.e., the corresponding observables commute. A first implication of the commutativity of choice experiments is that for any \( B \subseteq X \), the outcome of the choice experiment \( B \) only depends on \( B \) (whatever was done before the choice in \( B \) will always be the same and in particular the outcome does not depend on choices experiments that were performed before). Therefore, we can express the outcome in terms of a choice function \( C(X) : B \rightarrow x \) that associates to each \( B \in \mathcal{B} \) a chosen item \( x \in B \). A second consequence of commutativity is that the restriction of WARP' to two subsequent choices is inconsequential. This is because the outcome of any given choice experiment must be the same in any series of two consecutive experiments. In particular, WARP' implies the transitivity of choices. This can easily be seen taking an example on a subset of 3 items and performing choice experiments on pairs in different orders. So for the case choice experiments commute, WARP' is equivalent to WARP and we recover the standard results. We know that if \((\mathcal{B}, C(\cdot))\) is a choice structure satisfying WARP' and defined for \( \mathcal{B} \) including all subsets of \( X \) of up to three elements, then there exists a rational preference relation that rationalizes choice behavior. It is therefore natural to identify the states (types) of the individual with the preference orders that rank all the elements of \( X \). The type space \( \mathcal{H} \) has dimension \(|X|! \). We obtain the well-known classical model.

In this model all DS are coarse measurements of the type (the preference order) i.e., their outcomes are degenerated eigenvalues (see Section 2.3.1). When all DS commute and satisfy WARP', the type indeterminate rational agent behaves in all respects as a classical rational agent. This should not surprise us since we know that when observables commute they

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25 Consider the choice set \( \{a, b, c\} \). We do the following two series of experiments. In the first series we let the agent choose from \( \{a, b\} \), then from \( \{b, c\} \) and last from \( \{a, b, c\} \). In the second series the agent first choose from \( \{b, c\} \) then from \( \{a, b\} \) and last from \( \{a, b, c\} \). Assume the outcomes of the two first experiments are \( a \) and \( b \) then by WARP' the third choice maybe either \( a \), or \( b \). In the second series, the outcomes must be (because of commutativity) \( b \) and \( a \) respectively. So the third choice may be either \( a \) or \( c \). By commutativity the outcome of \( \{a, b, c\} \) must be the same in the two series which uniquely selects \( a \) so a choice behavior respecting WARP' is transitive when the choices commute.

26 See e.g., Mas-Colell et al. (1995) p.13.
can be merged into a single observable and the TI-model is equivalent to the classical model.

Case 2

We now consider a case where some DS do not commute. We need to emphasize that, in contrast with the commuting case, a large amount of non-commuting models are possible. The models differ from each other according to which DS commute and which do not. We investigate here a simple example that illustrates interesting issues.

Assume the set $X$ consists of four items: $a, b, c$ and $d$. As before, any subset $A \subset X$ consisting of 3 elements or less is associated with a DS and we assume that any two consecutive choices made from small subsets respect WARP'. Our non-commuting model is defined by the following assumption:

(nc) Choices out of $A$ and $B$ commute if and only if $A \cup B \neq X$.

In particular DS associated with different triples e.g., a choice out of \{a, b, c\} and a choice out of \{b, c, d\} are represented by observables that do not commute with each other. So for instance the agent may choose $a$ out of \{a, b, c\}, $b$ out of \{b, c, d\} and thereafter $c$ out of \{a, c\}. This may happened because the order (partially) revealed by the choice in \{a, b, c\} has been modified by the performance of the \{b, c, d\} choice experiment. This choice behavior satisfies WARP' but violates transitivity. Nevertheless, the satisfaction of WARP' induces a certain amount of consistency in behavior. In particular, any observable $A$ with $A \subset B$ where $B$ is a triple, commutes with $B$ (since $A \cup B \subset X$). We may perform choice experiments in pairs and in triples so as to elicit a preference relation on each triple taken separately. It is therefore natural to identify the types with preferences orders on triples. Since observables representing choice experiments on different triples do not commute with each other, they are alternative ways to measure the individual's preferences. The type space representing the individual is a six dimensional Hilbert space because there are six ways to rank three items. There are four alternative bases spanned by the eigenvectors corresponding to the six possible rankings in each one of the four triples.

In this model the agent only has preferences over triples of items. One may wonder what happens when she must make a choice out of the set \{a, b, c, d\}. Many scenarios are possible. Suppose that the individual proceeds by making a series of two measurements. For instance, she first selects from a pair
followed by a choice from a triple consisting of the first selected item and the two remaining ones. We can call such a behavior "procedural rational" because she acts as if she had preferences over the four items.

As an example consider the following line of events and for the ease of presentation we let the DS associated with the set \{a, b, c\} be denoted \(abc\) and similarly for the other DS. Suppose the individual just made a choice in \{a, b, c\} and picked up c. So her initial state is some superposition of type \([c > b > a]\) and type \([c > a > b]\). We now ask her to choose from \{a, b, c, d\}. Assume the individual first plays \(ab\) then \(bcd\) (which by assumption of procedural rationality means that the outcome of \(ab\) is \(b\)). The choice of \(b\) from \{a, b\} changed the type of the individual. In particular, suppose the \(ab\) observable is considered as a coarse measurement of \(abd\). The new type, expressed in the basis of preference orders on \{a, b, d\}, is a superposition of types \([b > a > d]\), \([b > d > a]\) and \([d > b > a]\). In order to make her choice out of \{a, b, c, d\}, she now plays \(bcd\). The result of that last measurement (performed on the individual of the type resulting from the first measurement) is therefore \(b\) with positive probability. But this violates the principle of independence of irrelevant alternative (IIA). She effectively selects \(b\) from \{a, b, c, d\} while she initially picked up \(c\) in \{a, b, c\} where \(b\) was available. Yet, it is easy to check that in this example any two consecutive choices satisfy the WARP' so the choice behavior of our type indeterminate agent is "rational".

We can now draw some first conclusions from this short exploration. If we demand that the agent’s choice behavior respects the Weak Axiom of Revealed Preference (in our WARP’ formulation) and conforms to procedural rationality, then

i. when all the DS associated with subsets of the universal set commute, a type indeterminate agent does not distinguish herself from a classical agent;

ii. when some DS associated with subsets of the universal set do not commute with each other, a type indeterminate agent does not have a preference order over the universal set \(X\). But she may have well-behaved preferences over subsets of \(X\). Generally, she does not behave as a classical rational agent. In particular, her behavior may exhibit standard phenomena of preference reversal.

\(27\) We remind that "playing a DS" means performing the corresponding measurement (see section 2.2).
In this example the distinction between the classical rational and the type indeterminate rational agent is only due to the non-commutativity of $DS$ associated with different subsets of items. This distinctive feature of the TI-model of choice delivers a formulation of bounded rationality in terms of the impossibility to compare and order all items simultaneously. Non-commutativity also implies that as the agent makes a choice her type (preferences) changes. A type indeterminate agent does not have a fixed type (preferences). Instead her preferences are shaped in the process of elicitation as proposed by Kahneman and Tversky (2000).

### 3.2 Examples

In this section we demonstrate how type indeterminacy can explain two well-documented examples of so-called behavioral anomalies. With these examples we want to suggest that one contribution of our approach is to provide a unified framework which can accommodate a variety of behavioral anomalies. These anomalies are currently explained by widely different theories.

#### 3.2.1 Cognitive dissonance

The kind of phenomena we have in mind can be illustrated as follows. Numerous studies show that employees in risky industry (like nuclear plants) often neglect safety regulations. The puzzle is that before moving into the risky industry those employees were typically able to estimate the risks correctly. They were reasonably averse to risk and otherwise behaved in an ordinary rational manner.

Psychologists developed a theory called cognitive dissonance (CD) according to which people modify their beliefs or preferences in response to the discomfort arising from conflicting beliefs or preferences. In the example above, they identify a dissonance as follows. On the one hand, the employee holds an image of himself as ”a smart person” and on the other hand he understands that he deliberately chose to endanger his health (by moving to the risky job), which is presumably ”not smart”. So in order to cancel the dissonance, the employee decides that there is no danger and therefore no need to follow the strict safety regulation.

We propose a formal model that is very much in line with psychologists’ theory of cognitive dissonance. We shall compare two scenarios involving a
sequence of two non-commuting Decision Situations each with two options. Let \( A \) be a DS about jobs with options \( a_1 \) and \( a_2 \) corresponding to taking a job with a dangerous task (adventurous type) and respectively staying with the safe routine (habit-prone type). Let \( B \) be a DS about the willingness to use safety equipment with choices \( b_1 \) (risk-averse type) and \( b_2 \) (risk-loving type) corresponding to the choice of using and respectively not using the safety equipment.

**First scenario:** The dangerous task is introduced in an existing context. It is *imposed* on the workers. They are only given the choice to use or not to use the safety equipment (\( B \)). We write the initial state of the worker in terms of the eigenvectors of observable \( A \):

\[
|\psi\rangle = \lambda_1 |a_1\rangle + \lambda_2 |a_2\rangle, \quad \lambda_1^2 + \lambda_2^2 = 1.
\]

We develop the eigenvectors of \( A \) on the eigenvectors of \( B \):

\[
|a_1\rangle = \mu_{11} |b_1\rangle + \mu_{21} |b_2\rangle, \\
|a_2\rangle = \mu_{21} |b_1\rangle + \mu_{22} |b_2\rangle.
\]

We now write the state in the basis of the \( B \) operator:

\[
|\psi\rangle = (\lambda_1 \mu_{11} + \lambda_2 \mu_{21}) |b_1\rangle + (\lambda_1 \mu_{12} + \lambda_2 \mu_{22}) |b_2\rangle.
\]

The probability that a worker chooses to use the safety equipment is

\[
p_B(b_1) = \langle b_1 | \psi \rangle^2 = (\lambda_1 \mu_{11} + \lambda_2 \mu_{21})^2 = \lambda_1^2 \mu_{11}^2 + \lambda_2^2 \mu_{21}^2 + 2 \lambda_1 \lambda_2 \mu_{11} \mu_{21}.
\]

**Second scenario:** First \( A \) then \( B \). The workers choose between taking a new job with a dangerous task or staying with the current safe routine. Those who chose the new job then face the choice between using safety equipment or not. Those who turn down the new job offer are asked to answer a questionnaire about their willingness to use the safety equipment for the case they would be working in the risky industry. The ex-ante probability for observing \( b_1 \) is

\[
p_{BA}(b_1) = p_A(a_1) p_B(b_1|a_1) + p_A(a_2) p_B(b_1|a_2) = \lambda_1^2 \mu_{11}^2 + \lambda_2^2 \mu_{21}^2.
\]
The phenomenon of cognitive dissonance can now be formulated as the following inequality
\[ p_{BA}(b_1) < p_B(b_1), \]
which occurs in our model when \( 2\lambda_1\lambda_2\mu_{11}\mu_{21} > 0 \). We next show that interference effects may be quantitatively significant.

**Numerical example**

Assume for simplicity that \( |\psi\rangle = |b_1\rangle \) which means that everybody in the first scenario is willing to use the proposed safety equipment. Let \( \text{prob}(a_1 | \psi) = 0.25 \), and \( \text{prob}(a_2 | \psi) = 0.75 \) so \( |\lambda_1| = \sqrt{0.25} \), and \( |\lambda_2| = \sqrt{0.75} \). It is possible to show that in this case we have \( |\mu_{11}| = \sqrt{0.25} \), and \( |\mu_{21}| = \sqrt{0.75} \). We now compute \( p_B(b_1) \) using the formula in (7) and recalling that \( |\psi\rangle = |b_1\rangle \) (so \( p_B(b_1) = 1 \)):

\[
1 = \langle b_1 | \psi \rangle^2 = \lambda_1^2\mu_{11}^2 + \lambda_2^2\mu_{21}^2 + 2\lambda_1\lambda_2\mu_{11}\mu_{21}
\]

\[ = 0.0625 + 0.5625 + 2\lambda_1\lambda_2\mu_{11}\mu_{21} = 0.625 + 2\lambda_1\lambda_2\mu_{11}\mu_{21}, \]  

which implies that the interference effect is positive and equal to \( 1 - 0.625 = 0.375 \). We note that it amounts to a third of the probability.

Under the second scenario the probability for using safety equipment is given by the formula in (8) i.e., it is the same sum as in (9) without the interference term:

\[ p_{BA}(b_1) = 0.625. \]

So we see that our TI-model "delivers" cognitive dissonance: \( p_{BA}(b_1) < p_B(b_1) \).

The key assumption that drives our result is that the choice between jobs and the choice between using or not using the safety equipment are measurements of two incompatible type characteristics (represented by two non-commuting observables). A possible behavioral interpretation is as follows. The job decision appeals to an abstract perception of risk, while the decision to use the safety equipment appeals to an emotional perception of concrete risks. In this interpretation, our assumption is that the two modes

\[28\] We note that \( p_{BA}(b_1) \) includes the probability of a choice of safety measures both in the group that chose the risky job and in the group that chose the safe job. This guarantees that we properly distinguish between the CD effect (change in attitude) and the selection bias.

\[29\] This uses the fact that \( \begin{pmatrix} \mu_{11} & \mu_{22} \\ \mu_{21} & \mu_{22} \end{pmatrix} \) is a rotation matrix.
of perceptions are incompatible. This is consistent with evidence that shows a gap between perceptions of one and the same issue when the agent is in a “cold” (abstract) state of mind compared to when she is in a “hot” emotional state of mind (see for instance Loewenstein 2005).

Comments

In their article from 1981 Akerlof and Dickens explain the behavior attributed to cognitive dissonance in terms of a rational choice of beliefs. Highly sophisticated agents choose their beliefs to fit their preferences

They are fully aware of the way their subjective perception of the world is biased and yet they keep to the wrong views. This approach does explain observed behavior but raises serious questions as to what rationality means. The type indeterminacy approach is consistent with psychologists’ thesis that cognitive dissonance prompts a change in preferences (or attitude). We view its contribution as follows. First, the TI-model provides a model that explains the appearance of cognitive dissonance. Indeed, if coherence is such a basic need, as proposed by L. Festinger and his followers, why does dissonance arise in the first place? In the TI-model dissonance arises when resolving indeterminacy in the first DS because of the ‘limitations’ on possible psychological types (see Section 2.3.2). Second, the TI-model features a dynamic process such that the propensity to use safety measures is actually altered (reduced) as a consequence of the act of choice. This dynamic effect is reminiscent of psychologists’ “drive-like property of dissonance” which leads to a change in attitude.

3.2.2 Framing Effects

When alternative descriptions of one and the same decision problem induce different decisions from agents we talk about ”framing effects”. Below we attend a well-known experiment which showed that two alternative formulations of the Prisoner Dilemma (the standard presentation in a 2 by 2 matrix and a presentation in a decomposed form, see below) induced dramatically different rates of cooperation.

Kahneman and Tversky (2002, p. xiv) address framing effects using a two-steps (nonformal) model of the decision-making process. The first step

\footnote{Akerlof and Dickens allow workers to freely choose beliefs (about risk) so as to optimize utility which includes psychological comfort. The workers are highly rational in the sense that when selecting beliefs, they internalize their effect their own subsequent bounded rational behavior.}
corresponds to the construction of a representation of the decision situation. The second step corresponds to the evaluation of the choice alternatives. The crucial point is that “the true objects of evaluation are neither objects in the real world nor verbal description of those objects; they are mental representations.” To capture this feature we suggest modelling the ”process of constructing a representation” in a way similar to the process of constructing preferences, i.e., as a measurement performed on the state of the agent. This is consistent with psychology that treats attitudes, values (preferences), beliefs, and representations as mental objects of the same kind.

In line with Kahneman and Tversky we describe the process of decision-making as a sequence of two steps. The first step consists of a measurement of the agent’s mental representation of the decision situation. The second step corresponds to a measurement of the agent’s preferences. Its outcome is a choice. Note that here we depart from standard decision theory. We propose that agents do not always have a unique representation of a decision situation, which can be revealed when thinking about it. Instead, uncertainty about what the decision situation actually is about can be resolved in a variety of ways some of which may be reciprocally incompatible. The decision situation itself is, as in standard theory, defined by the monetary payoffs associated with the choices i.e., in a unique way. The utility associated with the options generally depends both on the representation and the monetary payoffs.

To illustrate this approach we revisit the experiment reported in Pruitt (1970) and discussed in Selten (1998). Two groups of agents are invited to play a Prisoners’ Dilemma. The game is presented to the first group in the usual matrix form with the options labelled $1_G$ and $2_G$ (instead of $C$ and $D$, presumably to avoid associations with the suggestive terms ‘cooperate’ and ‘defect’):

|     | $C$ | $D$ |
|-----|-----|-----|
| $1_G$ | 3   | 4   |
|     | 3   | 0   |
| $2_G$ | 0   | 1   |
|     | 4   | 1   |

The game is presented to the second group in a decomposed form as follows:
The payoffs are computed as the sum of what you take for yourself and what you get from the other player. So for instance if player 1 plays 1₁ and player 2 plays 2₁, player 1 receives 0 from his own play and 0 from the other’s play) which sums to 0. Player 2 receives 1 from his own play and 3 from player 1’s play which sums to 4. So we recover the payoffs (0, 4) associated with the play of Cooperate for player 1 and Defect for player 2. Game theoretically it should make no difference whether the game is presented in a matrix form or in a decomposed form. Pruitt’s main experimental result is that one observes dramatically more cooperation in the game presented in decomposed form.

We now provide a possible TI-model for this situation. Let us consider a two-dimensional state space and a sequence of two measurements. The first measurement determines the mental representation of the DS. We call A the (representation) observable corresponding to the matrix presentation. It has two non-degenerated eigenvalues denoted |a₁⟩ and |a₂⟩. Similarly, B is the observable corresponding to the decomposed presentation with two eigenvalues |b₁⟩ and |b₂⟩. If |ψ⟩ is the initial state of the agent, we can write |ψ⟩ = α₁ |a₁⟩ + α₂ |a₂⟩ or |ψ⟩ = β₁ |b₁⟩ + β₂ |b₂⟩. The second measurement i.e., the decision observable is unique, we called it G. G has also two eigenvalues denoted 1₁ and 2₁.

For the sake of concreteness we may think of the four alternative mental representations as follows:\(^{31}\)

- |a₁⟩: G is perceived as an (artificial) small-stake game;
- |a₂⟩: G is perceived by analogy as a real life PD-like situation (often occurring in a repeated setting);
- |b₁⟩: G is perceived as a test of generosity;
- |b₂⟩: G is perceived as a test of smartness.

When confronted with a presentation of the DS the agent forms her mental representation of the DS which prompts a change in her state from |ψ⟩ to some |aᵢ⟩ (if the frame is A) or |bᵢ⟩ \( i, j = 1, 2 \) (if the frame is B). The new state can be expressed in terms of the eigenvectors of the decision situation: |aᵢ⟩ = γ₁₁ |1₁⟩ + γ₂₁ |2₁⟩ or |bᵢ⟩ = δ₁₁ |1₁⟩ + δ₂₁ |2₁⟩.

\(^{31}\)This is only meant as a suggestive illustration.
We can now formulate the framing effect as the following difference

\[ p_{GA}(i_G) \neq p_{GB}(i_G), \quad i = 1, 2. \]  \hspace{1cm} (10)

Using our result in Section 2.3.2 we get

\[ p_{GA}(1_G) = p_G(1_G) - 2\alpha_1\gamma_{11}\alpha_2\gamma_{12} \quad \text{and} \quad p_{GB}(1_G) = p_G(1_G) - 2\beta_1\delta_{11}\beta_2\delta_{12}, \]

where \( p_G(1_G) \) is the probability of choosing 1 in an (hypothetical) unframed situation. So we have a framing effect whenever \( 2\alpha_1\gamma_{11}\alpha_2\gamma_{12} \neq 2\beta_1\delta_{11}\beta_2\delta_{12} \).

The experimental results discussed in Selten (1998) namely that the decomposed presentation induces more cooperation require that \( 2\alpha_1\gamma_{11}\alpha_2\gamma_{12} - 2\beta_1\delta_{11}\beta_2\delta_{12} > 0 \). The inequality says that the interference term for \( 1_G \) is larger in the standard matrix presentation \( A \) than in the decomposed form corresponding to the \( B \) presentation. In order to better understand the meaning of this difference we shall consider a simple numerical example.

Set \( \alpha_1 = \sqrt{.8}, \alpha_2 = \sqrt{.2}, \beta_1 = \sqrt{0.4}, \beta_2 = \sqrt{0.6} \). The key variables are the correlations between the "representation types" i.e., \( |a_i\rangle \) or \( |b_j\rangle \) and the "game type" \( 1_G \), i.e., the numbers \( \gamma_{11}, \gamma_{12} \) and \( \delta_{11}, \delta_{12} \). We propose that \( \gamma_{11} = \sqrt{.3} \) which is interpreted as when the agent views the game as a small stake game she plays \( 1_G \) with probability .3. Similarly \( \gamma_{12} = \sqrt{.7} \) which means that when the agent perceives the game by analogy with real life, she "cooperates" with probability .7. In the alternative presentation \( B \), we propose that \( \delta_{11} = 1 \), i.e., when the game is perceived as a test of generosity, the agent cooperates with probability 1. When the game is perceived as a test of smartness \( \delta_{12} = 0 \) (because the agent views the play of \( 1_G \) as plain stupid). Computing these figures, we get

\[ 2\alpha_1\gamma_{11}\alpha_2\gamma_{11} - 2\beta_1\delta_{11}\beta_2\delta_{12} = 0.366 - 0 > 0 \]

In the \( A \) presentation the contribution of both the \( |a_1\rangle \) and the \( |a_2\rangle \) type is positive and significant. When the agent is indeterminate both types positively contribute reinforcing each other. In contrast in the \( B \) presentation the contribution from \( |b_2\rangle \) is null so there is no interference between the types. When the agent determines herself i.e., selects a representation, the contribution from indeterminacy is lost and that loss is positive with \( A \) while it is null with \( B \). Therefore, the probability for \( 1_G \) when the game is presented in the matrix form is larger (here by .36) than in the game presented in the decomposed form.
Comments

Selten proposes a ‘bounded rationality’ explanation: players make a superficial analysis and do not perceive the identity of the games presented under the two forms. Our approach is closer to Kahneman and Tversky who suggest that prior to the choice, a representation of the decision situation must be constructed. The TI-model provides a framework for ‘constructing’ a representation such that it delivers framing effects in choice behavior. Of course framing effects can easily be obtained when assuming that the mental images are incomplete or biased. In the TI-model we do not need to appeal to such self-explanatory arguments. In the TI-model framing effects arise as a consequence of (initial) indeterminacy of the agent’s representation of the decision situation. Since alternative (non-commuting) presentations are equally valid and their corresponding representations (eigenvalues) equally informative, highly rational agent can exhibit framing effects.

4 Discussion

In this section we briefly discuss some formal features of our model.\(^\text{32}\)

Our approach to decision-making yields that the type of the agents rather than being exogenously given, emerges as the outcome of the interaction between the agent and the decision situations. This is modelled by letting a decision situation be represented by an operator (observable). Decision-making is modelled as a measurement process. It projects the initial state of the agent into the subspace of the state (type) space associated with the eigenvalue corresponding to the choice made. Observables may either pairwise commute or not. When the observables commute, the corresponding type space has the properties of the Harsanyi type space. From a formal point of view this reflects the fact that all (pure) types are then mutually exclusive.\(^\text{33}\) When the observables do not commute, the associated pure types are not all mutually exclusive. Instead, an agent who is in a pure state after the measurement of an observable will be in a different pure state after the measurement of another observable that is incompatible with the first

\(^{32}\)For a systematic investigation of the mathematical foundations of the HSM in view of their relevance for social sciences see Danilov and Lambert-Mogiliansky (2005).

\(^{33}\)We say that a type is pure when it is obtained as the result of a complete measurement i.e., the measurement of a complete set of commuting observables (CSCO).
one. As a consequence, the type space cannot be associated with a classical probability space and we obtain an irreducible uncertainty in behavior. The Type Indeterminacy model provides a framework where we can deal both with commuting and non-commuting observables. In the TI-model any type (state) corresponds to a probability measure on the type space which allows to make predictions about the agent’s behavior. It is in this sense that the TI-model generalizes Harsanyi’s approach to uncertainty.

The more controversial feature of the TI-model as a framework for describing human behavior is related to the modelling of the impact of measurement on the state i.e., how the type of the agent changes with decision-making. The rules of change are captured in the geometry of the type space and in the projection postulate. It is more than justified to question whether this seemingly very specific process should have any relevance to Social Sciences.

It has been shown that the crucial property that gives all its structure to the process of change can be stated as a "minimal perturbation principle". The substantial content of that principle is that we require that when a coarse DS resolves some uncertainty about the type of an agent, the remaining uncertainty is left unaffected. Recall our example in section 3.1 case 2. When the agent chooses an item out of a subset $A$ of 3 items, this prompts a resolution of some uncertainty. The type is projected into the eigenspace spanned by the two orderings consistent with the choice made. The minimal perturbation principle says that uncertainty relative to the ordering of the two remaining items is the same as before. In behavioral terms this can be expressed as follows. When confronted with the necessity to make a choice, the agent only ”makes the effort” to select her preferred item while leaving the order relationship between the other items uncertain as initially.

It may be argued that the minimal perturbation principle is quite demanding. Returning again to our example, if the mental processes involved in the search for the preferred items fully upset the initial state, the principle is violated. It could also be argued that the mental processes involved in decision-making determine the whole ranking. That would also violate the minimal perturbation principle. This short discussion suggests that selecting a good TI-model requires careful thinking and possibly some trial and error.

We do not expect the Type Indeterminacy model to be a fully realistic description of the human behavior rather we propose it as an idealized model of agents characterized by the fact that their type change with decision-making. In particular some features of the TI-model, like the symmetry of the correlation matrix in simple examples may seem very constraining from
a behavioral point of view. Consider two DS with two options e.g., the Prisoner Dilemma and the Ultimatum game (with option ”share fairly” and ”share egoistically”). Assuming those DS have nondegenerated eigenvalues, the symmetry of correlation matrix means that the probability to play e.g. defect after having played say egoist is exactly the same as the probability to play egoist after having played defect. We do not expect this kind of equality to hold in general. Nevertheless, keeping in mind some reservations as to its realism, our view is that the Type Indeterminacy model can provide a fruitful framework for analyzing, explaining and predicting human behavior. Clearly much additional work is needed to extend of the TI-model to strategic and repeated decision-making. We are currently exploring this second stage of our research program.

As a final remark it should be emphasized that not all instances of non-commutativity in choice behavior call for Hilbert space modelling. Theories of addiction feature effects of past choices on future preferences. And in standard consumer theory, choices do have implications for future behavior, i.e., when goods are substitutes or complements. But in those cases we do expect future preferences to be affected by the choices. The Type Indeterminacy model of decision-making can be useful when we expect choice behavior to be consistent with the standard probabilistic model, because nothing justifies a modification of preferences. Yet, actual behavior contradicts those expectations.

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5 Appendix: Elements of Quantum Mechanics

5.1 States and Observables

In Quantum Mechanics the state of a system is represented by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. According to the superposition principle, every complex linear combination of state vectors is a state vector. A Hermitian operator called an observable is associated to each physical property of the system.

**Theorem 1**
A Hermitian operator $A$ has the following properties:
- Its eigenvalues are real.
- Two eigenvectors corresponding to different eigenvalues are orthogonal.
- There is an orthonormal basis of the relevant Hilbert space formed with the eigenvectors of $A$.

Let us call $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ the normalized eigenvectors of $A$ forming a basis of $\mathcal{H}$. They are associated with eigenvalues $\alpha_1, \alpha_2, ..., \alpha_n$, so: $A |v_i\rangle = \alpha_i |v_i\rangle$. The eigenvalues can possibly be degenerated, i.e., for some $i$ and $j$, $\alpha_i = \alpha_j$. This means that there is more than one linearly independent eigenvector associated with the same eigenvalue. The number of these eigenvectors defines the degree of degeneracy of the eigenvalue which in turn defines the dimension of the eigensubspace spanned by these eigenvectors. In this case, the orthonormal basis of $\mathcal{H}$ is not unique because it is possible to replace the eigenvectors associated to the same eigenvalue by any complex linear combination of them to get another orthonormal basis. When an observable $A$ has no degenerated eigenvalue there is a unique orthonormal basis of $\mathcal{H}$ formed with its eigenvectors. In this case (see below), it is by itself a Complete Set of Commuting Observables.

**Theorem 2**
If two observables $A$ and $B$ commute there is an orthonormal basis of $\mathcal{H}$ formed by eigenvectors common to $A$ and $B$.

Let $A$ be an observable with at least one degenerated eigenvalue and $B$ another observable commuting with $A$. There is no unique orthonormal basis formed by $A$ eigenvectors. But there is an orthonormal basis of the relevant
Hilbert space formed by eigenvectors common to $A$ and $B$. By definition, 
$\{A, B\}$ is a Complete Set of Commuting Observables (CSCO) if this basis is unique. Generally, a set of observables $\{A, B, ...\}$ is said to be a CSCO if there is a unique orthonormal basis formed by eigenvectors common to all the observables of the set.

### 5.2 Measurements

An observable $A$ is associated to each physical property of a system $S$. Let $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ be the normalized eigenvectors of $A$ associated respectively with eigenvalues $\alpha_1, \alpha_2, ..., \alpha_n$ and forming a basis of the relevant state space. Assume the system is in the normalized state $|\psi\rangle$. A measurement of $A$ on $S$ obeys the following rules, collectively called ‘Wave Packet Reduction Principle’ (the Reduction Principle).

**Reduction Principle**

1. When a measurement of the physical property associated with an observable $A$ is made on a system $S$ in a state $|\psi\rangle$, the result only can be one of the eigenvalues of $A$.

2. The probability to get the non-degenerated value $\alpha_i$ is $P(\alpha_i) = |\langle v_i | \psi \rangle|^2$.

3. If the eigenvalue is degenerated then the probability is the sum over the eigenvectors associated with this eigenvalue: $P(\alpha_i) = \sum |\langle \nu_j | \psi \rangle|^2$.

4. If the measurement of $A$ on a system $S$ in the state $|\psi\rangle$ has given the result $\alpha_i$ then the state of the system immediately after the measurement is the normalized projection of $|\psi\rangle$ onto the eigensubspace of the relevant Hilbert space associated with $\alpha_i$. If the eigenvalue is not degenerated then the state of the system after the measurement is the normalized eigenvector associated with the eigenvalue.

If two observables $A$ and $B$ commute then it is possible to measure both simultaneously: the measurement of $A$ is not altered by the measurement of $B$. This means that measuring $B$ after measuring $A$ does not change the value obtained for $A$. If we again measure $A$ after a measurement of $B$, we again get the same value for $A$. Both observables can have a definite value.
5.2.1 Interferences

The archetypal example of interferences in quantum mechanics is given by the famous two-slits experiment. A parallel beam of photons falls on a diaphragm with two parallel slits and strikes a photographic plate. A typical interference pattern showing alternate bright and dark rays can be seen. If one slit is shut then the previous figure becomes a bright line in front of the open slit. This is perfectly understandable if we consider photons as waves, as it the assumption is in classical electromagnetism. The explanation is based on the fact that when both slits are open, one part of the beam goes through one slit and the other part through the other slit. Then, when the two beams join on the plate, they interfere constructively (giving bright rays) or destructively (giving dark ones), depending on the difference in length of the paths they have followed. But a difficulty arises if photons are considered as particles, as can be the case in quantum mechanics. Indeed, it is possible to decrease the intensity of the beam so as to have only one photon travelling at a time. In this case, if we observe the slits in order to detect when a photon passes through (for example, by installing a photodetector in front of the slits), it is possible to see that each photon goes through only one slit. It is never the case that a photon splits to go through both slits. The photons behave like particles. Actually, the same experiment was done with electrons instead of photons, with the same result. If we do the experiment this way with electrons (observing which slit the electrons go through, i.e., sending light through each slit to “see” the electrons), we see that each electron goes through just one slit and, in this case, we get no interference. If we repeat the same experiment without observing which slit the electrons pass through then we recover the interference pattern. Thus, the simple fact that we observe which slit the electron goes through destroys the interference pattern (two single slit patterns are observed). The quantum explanation is based on the assumption that when we don’t observe through which slit the electron has gone then its state is a superposition of both states “gone through slit 1” and “gone through slit 2”, while when we observe it, its state collapses onto one of these states. In the first case, the position measurement is made on electrons in the superposed state and gives an interference pattern since both states are manifested in the measurement. In the second case, the position is

\[35\]
See, e.g., Feynman (1965) for a very clear presentation.

\[36\]
This doesn’t mean that the photon actually went through both slits. This state simply can’t be interpreted from a classical point of view.
measured on electrons in a definite state and no interference arises. In other words, when only slit 1 is open we get a spectrum, say \( S_1 \) (and \( S_2 \) when only slit 2 is open). We expect to get a spectrum \( S_{12} \) that sums of the two previous spectra when both slits are open, but this is not the case: \( S_{12} \neq S_1 + S_2 \).