TWO-SITE LOCALISATION IN THE BOUCHAUD TRAP MODEL ON \( \mathbb{Z} \) WITH SUPER-HEAVY-TAILED TRAPS

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Abstract. We consider the Bouchaud trap model on \( \mathbb{Z} \), where the trapping landscape \( \sigma := \{\sigma(z)\}_{z \in \mathbb{Z}} \) consists of independent identically distributed random variables with a super-heavy tail, i.e. whose survival function \( P(\sigma(\cdot) > x) \) has a reciprocal that is slowly varying at infinity.

We prove that the model eventually localises on exactly two sites with overwhelming probability. This is a stronger form of localisation than has previously been established in the literature for the Bouchaud trap model on \( \mathbb{Z} \) with heavy-tailed traps. The fact underlying the result is that the maximum of a sequence of i.i.d. random variables with a super-heavy tail will, with overwhelming probability, asymptotically dominate the sum of the sequence.

We further prove that the proportion of probability mass on each of the two localisation sites is uniformly distributed in the limit, as well as deriving a limit formula for the distance of the localisation sites from the origin.

1. Introduction

1.1. The Bouchaud trap model. We consider the Bouchaud trap model (BTM) on \( \mathbb{Z} \), that is, the continuous-time Markov chain \( X_t \) on \( \mathbb{Z} \) based at the origin with generator

\[
L_\sigma f(z) = \sum_{|y-z|=1} \sigma^{-1}(z)(f(y) - f(z))
\]

where \( \sigma := \{\sigma(z)\}_{z \in \mathbb{Z}} \) is a collection of independent identically distributed (i.i.d.) strictly-positive random variables known as the (random) trapping landscape. The process \( X_t \) describes the position at time \( t \) of a particle undertaking a continuous-time random walk on \( \mathbb{Z} \) based at the origin, where the waiting time at a site \( z \) is distributed exponentially with mean \( \sigma(z) \) and the subsequent site is chosen uniformly at random from among the neighbours of \( z \). The BTM has its origins in the statistical physics literature, where it was proposed as a simple way of modelling the long-term dynamics of certain spin-glass models (see, e.g., [3]). For a general overview of the BTM see [2], for interesting recent work see [6].

Although the BTM may be defined on arbitrary graphs by analogy with the above (see, e.g., [2]), the BTM on \( \mathbb{Z} \) is of particular interest because it demonstrates intermittency; in other words, its dynamics cannot in general be explained with a
simple averaging principle. In the context of the BTM, intermittency manifests in the localisation of the probability mass function

\[ u(t, z) := \mathbb{P}_\sigma(X_t = z) \]

where \( \mathbb{P}_\sigma \) is a random probability measure depending on the particular realisation of the trapping landscape \( \sigma \); we reserve \( \mathbb{P} \) to denote probability taken over \( \sigma \).

Interestingly, intermittency is not in general observed in the BTM on graphs which are highly connected, for instance \( \mathbb{Z}^d \), \( d > 1 \), or the complete graph (the original variant of the BTM studied in [3]). The BTM on a general tree was the subject of a very recent study in [1]. Henceforth, we shall refer exclusively to the BTM on \( \mathbb{Z} \).

1.2. Intermittency and localisation. Broadly speaking, intermittency in the BTM arises because of extremes in the trapping landscape, and so the strength of intermittency naturally depends on the thickness of the upper-tail of \( \sigma(\cdot) \). Indeed, it was proven in [5] that if \( \sigma(\cdot) \) has finite expectation, then the BTM almost surely converges to Brownian motion in the limit. This implies that, for each \( z \) uniformly,

\[ u(t, z) \rightarrow 0 \quad \text{almost surely} \]
as \( t \rightarrow \infty \), and so the model does not exhibit intermittency.

On the other hand, it was proven in [6] that, under certain assumptions, if \( \sigma(\cdot) \) has infinite expectation then \( u(t, z) \) localises in the sense that

\[ \sup_{z \in \mathbb{Z}} u(t, z) \rightarrow 0 \quad \text{almost surely} \]
as \( t \rightarrow \infty \). In other words, for arbitrarily large times \( t \) there is a site where \( X_t \) can be found with non-negligible probability. This was proven, in particular, where the distribution of \( \sigma(\cdot) \) belongs to the domain of attraction of the totally asymmetric \( \alpha \)-stable law with \( \alpha \in (0, 1) \), i.e. where there exists a slowly-varying function \( L \) (i.e. \( \lim_{x \to \infty} L(xs)/L(x) = 1 \) for every \( s > 0 \)) such that

\[ \mathbb{P}(\sigma(\cdot) > x)^{-1} = u^\alpha L(u) . \]

Notably, this class includes the Pareto distributions with parameter \( \alpha \in (0, 1) \).

What has yet to be established, however, is whether the BTM may exhibit a stronger form of localisation in the case \( \alpha = 0 \), i.e. where the distribution \( \sigma(\cdot) \) has a super-heavy tail, such that

\[ \mathbb{P}(\sigma(\cdot) > x)^{-1} = L(u) \]

for \( L \) a slowly-varying function. Indeed, to the best of our knowledge, the literature has not yet considered such super-heavy-tailed traps.

1.3. Our results. We consider the BTM where the distribution \( \sigma(\cdot) \) satisfies the following assumption:

**Assumption 1.1.** Characterise the distribution \( \sigma(\cdot) \) by the positive, non-decreasing and unbounded function

\[ f(x) := -\log \mathbb{P}(\sigma(\cdot) > e^x). \]

Then, as \( x \rightarrow \infty \), the function \( f \) satisfies, for some \( \varepsilon > 0 \):

(a) **(Super-heavy tail)** Eventually

\[ f(x) < x^{1-\varepsilon}; \]
(b) (Tail regularity) Eventually $f$ is continuous, strictly increasing and, for all $y_x \to \infty$, 
\[ |f(x) - f(x \pm y_x)| < |y_x| x^{-\varepsilon}. \]
Note that (b) implies (a); we separate the assumptions for clarity only.

Remark 1.2. An important class of distributions that satisfy Assumption 1.1 are the log-stretched-exponential (or log-Weibull) distributions with parameter $\gamma \in (0, 1)$ (i.e. with $f(x) = x^\gamma$). Heavier-tailed distributions, such as the log-Pareto distribution (i.e. with $f(x) = \log x$), are also included. Remark that the distributions considered in [4] and [5] do not satisfy Assumption 1.1 since these have $f(x) \sim \alpha x$ for some $\alpha \in (0, 1)$.

We prove that under Assumption 1.1 the probability mass function $u(t, z)$ eventually localises on exactly two sites with overwhelming probability (with respect to the trapping landscape $\sigma$). We further prove that the proportion of probability mass on each of the two localisation sites is uniformly distributed in the limit, and derive a limit formula for the distance of the localisation sites from the origin.

To describe these results explicitly, we first define some notation. Fix a value $\varepsilon > 0$ that satisfies Assumption 1.1. For sufficiently large $t$, denote by $r_t$ the unique positive solution to the equation 
\[ \log r_t = f(\log t - \log r_t) \]
remarking that $r_t$ is well-defined for large enough $t$ since $f$ is eventually strictly-increasing and continuous. Further, let $h_t$ be an auxiliary scaling function satisfying 
\[ \log \log t \ll \log h_t \ll (\log t)^c \]
eventually for any constant $c > 0$, where $f(x) \ll g(x)$ denotes $\lim_{x \to \infty} f(x)/g(x) = 0$. Finally, define a level $l_t := th_t^2/r_t$. Denote by $Z_t^{(1)}$ (respectively $Z_t^{(2)}$) the closest site to the origin on the positive (respectively negative) half-line where the trap value exceeds the level $l_t$: 
\[ Z_t^{(1)} := \min\{z \in \mathbb{Z}^+ : \sigma(z) > l_t\} \quad \text{and} \quad Z_t^{(2)} := \max\{z \in \mathbb{Z}^- : \sigma(z) > l_t\}, \]
and let $\Gamma_t := \{Z_t^{(1)}, Z_t^{(2)}\}$. Our main results are the following:

Theorem 1.3 (Two-site localisation in probability). As $t \to \infty$, 
\[ u(t, Z_t^{(1)}) + u(t, Z_t^{(2)}) \to 1 \quad \text{in probability}. \]
Equivalently, as $t \to \infty$, 
\[ \mathbb{P}(X_t \in \Gamma_t) \to 1. \]

Theorem 1.4 (Distribution between localisation sites). For $i = 1, 2$, as $t \to \infty$, 
\[ u(t, Z_t^{(i)}) \Rightarrow U \quad \text{in law} \]
where $U$ denotes the uniform distribution on the unit interval.

Corollary 1.5 (One-site localisation fails). As $t \to \infty$, 
\[ \max_{z \in \mathbb{Z}} u(t, z) \to 1 \quad \text{in probability}. \]

Theorem 1.6 (Distance of localisation sites from origin). As $t \to \infty$, 
\[ r_t^{-1} \left( Z_t^{(1)}, -Z_t^{(2)} \right) \Rightarrow (\text{Exp}_1, \text{Exp}_2) \quad \text{in law} \]
where $\{\text{Exp}_i\}_{i=1,2}$ are independent standard exponential distributions.
Remark 1.7. Theorems 1.3–1.4 collectively state that, for a large fixed time $t$, a particle undertaking the BTM is overwhelmingly likely to be located at either of the two (random) sites in $\Gamma_t$, and that the particle will be uniformly distributed between these two sites. Theorem 1.6 details the location of these sites.

Note that the form of localisation in Theorem 1.3 is indeed stronger than that in equation (2). For instance, the results in [4] imply that two-site localisation in probability does not hold when $\sigma(\cdot)$ belongs to the domain of attraction of the totally asymmetric $\alpha$-stable law with $\alpha \in (0, 1)$.

Finally, although we do not prove it here, we strongly suspect that $\max_{z \in \mathbb{Z}} u(t, z) \not\to 1$ in probability for arbitrary distributions $\sigma(\cdot)$, meaning that one-site localisation in probability always fails in the BTM, regardless of the assumptions on $\sigma(\cdot)$. The reason for our belief is that Assumption 1.1 is specifically chosen to favour localisation, yet one-site localisation fails.

Remark 1.8. Although $\Gamma_t$ is defined in terms of the scaling function $h_t$, with overwhelming probability it is eventually independent of the specific choice of $h_t$.

1.4. Strategy of the proof. The main idea of the proof is that, for a large fixed $t$, the set $\Gamma_t$ has been constructed to ensure that: (i) a particle undertaking the BTM random walk is very likely to have hit $\Gamma_t$ before time $t$; (ii) if the particle hits $z \in \Gamma_t$ before time $t$, it is very likely to still be at $z$ at time $t$. The underlying fact permitting this construction is that the maximum of a sequence of i.i.d. random variables with a super-heavy tail will, with overwhelming probability, asymptotically dominate the sum of the sequence.

To make this idea precise, fix a trapping landscape $\sigma$ and a large time $t$, and consider running the BTM on this landscape. Define the random time $\tau^1_t := \inf\{s : X_s \in \Gamma_t\}$ and consider $P_\sigma(\tau^1_t > t)$. Denote by $\Sigma_t$ the sum of the traps lying in $\overline{\Gamma}_t := \{z : Z_t^{(2)} < z < Z_t^{(1)}\}$:

$$\Sigma_t := \sum_{z \in \Gamma_t} \sigma(z).$$

By a consideration of the displacement and local time of a simple random walk, it is possible to show that the random time $\tau^1_t$ is very likely to be less than $\Sigma_t d_t h_t$, where $d_t := \max_{z \in \Gamma_t} \{|z|\}$. Hence, on the event (with respect to $\sigma$)

$$\mathcal{E}_t := \{\Sigma_t < t/(r_t h_t^2), \ d_t < r_t h_t\} \subseteq \{\Sigma_t d_t h_t < t\}$$

we conclude that, as $t \to \infty$,

$$P_\sigma(\tau^1_t \leq t) \to 1 \quad \text{in probability.} \ (3)$$

Further, for $i = 1, 2$, define the set $I^i_t := \{z : |z - Z^{(i)}_t| \leq r_t / h_t\}$ and setting

$$Z^i_t := X_{\tau^i_t} \quad \text{and} \quad I^i_t := I^i_t \quad \text{if} \ Z^i_t = Z^{(i)}_t, \ i = 1, 2$$

define a second, strictly later, random time $\tau^2_t := \inf\{s > \tau^1_t : X_s \notin I^1_t\}$. By a similar consideration, we may show that the random time $\tau^2_t$ is very likely to be greater than $r_t h_t^2 = t$, and so we find that, as $t \to \infty$,

$$P_\sigma(\tau^2_t > t) \to 1 \quad \text{in probability.} \ (4)$$
Introduce a new random process \( \hat{X}_t^i \) on the same probability space as \( X_t \) satisfying: (i) \( \hat{X}_t^i = X_t \) for all \( s < \tau_1^i \); and (ii) \( \hat{X}_t^i = [X_s]_{I_t^i} \) for all \( s \geq \tau_1^i \), where \([z]_{I_t^i} \) denotes the site in \( I_t^i \) that shares a conjugacy class with \( z \) in the quotient space \( \mathbb{Z}/I_t^i \). Finally, define, for any \( s > 0 \),

\[
\hat{u}^i(s, z) := \mathbb{P}_\sigma(\hat{X}_s^i = z, \tau_1^i < s).
\]

By construction it is clear that

\[
\mathbb{P}(X_t^i = z, \tau_1^i \leq s < \tau_2^i) = \mathbb{P}_\sigma(\hat{X}_s^i = z, \tau_1^i \leq s < \tau_2^i)
\]

for each \( z \). Hence equations (3) and (4) imply that, on the event \( \mathcal{E}_t \),

\[
\frac{\hat{u}^i(t, z)}{u(t, z)} \to 1 \quad \text{in probability}
\]

for each \( z \in I_t^i \). Note that \( \hat{u}^i(s, z) \) is a probability mass function supported on \( I_t^i \) which, via Markov chain theory, can be shown to converge monotonically to an equilibrium distribution \( \pi_t^i \) as \( s \to \infty \). Moreover, we can show that \( \pi_t^i \) is proportionate to \( \sigma \). Hence if, as \( t \to \infty \),

\[
\sum_{z \in I_t^i} \sigma(z) \sim \sigma(Z_t^i)
\]

then \( \pi_t^i \) is asymptotically localised at \( Z_t^i \). So defining, for each \( i = 1, 2 \), the sum

\[
\Sigma_t^i := \sum_{z \in I_t^i \setminus \{Z_t^i(\tau)\}} \sigma(z),
\]

on the event (again with respect to \( \sigma \))

\[
\mathcal{F}_i := \{\max\{\Sigma_t^1, \Sigma_t^2\} < l_t/h_t\}
\]

we may conclude that, as \( t \to \infty \),

\[
\hat{u}^i(t, Z_t^i) \to 1 \quad \text{almost surely}.
\]

Since we show that the events \( \mathcal{E}_t \) and \( \mathcal{F}_i \) hold eventually with overwhelming probability, combining equations (3) and (6) yields Theorem 1.3.

In Section 2 we study preliminary asymptotics for \( r_t \) and \( l_t \). In Section 3 we study properties of the trapping landscape \( \sigma \), and show that the events \( \mathcal{E}_t \) and \( \mathcal{F}_i \) hold eventually with overwhelming probability. In Section 4 we study the BTM on the events \( \mathcal{E}_t \) and \( \mathcal{F}_i \), establishing equations (3), (4) and (6), and completing the proof of Theorems 1.3, 1.4 and 1.6.

2. Preliminary asymptotics

**Lemma 2.1** (Asymptotics for \( r_t \)). As \( t \to \infty \), the following hold:

(a) \( \log r_t \sim f(\log t) \) and (b) \( \log r_t = f(\log l_t) + o(1) \).

**Proof.** Note that \( r_t \to \infty \), since \( r_t \) is non-decreasing and if \( \log r_t < C \) for all \( t \) then

\[
C > \log r_t := f(\log t - \log r_t) \geq f(\log t - C)
\]

which contradicts the fact that \( f \) is unbounded. Then by Assumption 1.1, eventually

\[
\log r_t = f(\log t) - f(\log t - \log r_t) = o(\log r_t).
\]
and, since $\log r_t \ll \log t$ and $\log h_t \ll (\log t)\varepsilon$,
\[ |f(\log l_t) - \log r_t| = |f(\log t - \log r_t + 2 \log h_t) - f(\log t - \log r_t)| < 2(\log h_t)(\log t - \log r_t)^{-\varepsilon}(1 + o(1)) = o(1). \]

**Lemma 2.2 (Asymptotics for $l_t$).** As $t \to \infty$, the following hold:
(a) $\log l_t \sim \log t$ and (b) $h_l \sim h_t$.

**Proof.** These follow easily from Assumption 2.1 and the assumptions on $h_t$. □

### 3. The trapping landscape

In this section, we study properties of the trapping landscape $\sigma$, and in particular we prove that the events
\[ \mathcal{E}_t := \{\Sigma_t < t/(r_t h_t^2), d_t < r_t h_t\} = \{\Sigma_t < l_t/h_t^4, d_t < r_t h_t\} \]
and \[ \mathcal{F}_t := \{\max\{\Sigma_1, \Sigma_2\} < l_t/h_t\} \]
hold eventually with overwhelming probability.

We begin by stating three general propositions on sequences of i.i.d random variables with common distribution $\sigma(\cdot)$; let $X := \{X_n\}_{n \in \mathbb{N}}$ be such a sequence. For a level $l$, let
\[ n_l := \min\{n : X_n > l\} \quad \text{and} \quad m_l := \sum_{n < n_l} X_n \]
be, respectively, the index of the first exceedence of the level $l$ and the sum of all previous terms in the sequence.

**Proposition 3.1 (Almost sure asymptotics for the index of first exceedence).** As $l \to \infty$, eventually almost surely
\[ |\log n_l - f(\log l)| < \log \log l. \]

**Proof.** Construct the sequence $Y := \{Y_n\}_{n \in \mathbb{N}}$ with $Y_n := f(\log X_n)$ and remark that $Y$ is a sequence of i.i.d standard exponential distributions. It is well-known (see, e.g. [8, Lemma 4.1]) that, as $n \to \infty$, eventually
\[ |\max_{1 \leq i \leq n} Y_i - \log n| < (\log \log n)^\delta \]
almost surely, for some $\delta \in (0, 1)$. Since $n_l = \min\{n : Y_n > f(\log l)\}$, this yields the result. □

**Proposition 3.2 (Bound on sum prior to first exceedence).** For any $C$, as $l \to \infty$,
\[ \mathbb{P}(\log m_l < \log l - C \log h_t) \to 1. \]

**Proof.** As in the proof of Proposition 3.1, construct the sequence $Y$ with $Y_n := f(\log X_n)$. Fix a $0 < \delta < \varepsilon/2$ and define
\[ K := K_{\varepsilon, \delta} = \max\left\{K : K < \frac{1}{\varepsilon - \delta}\right\} \geq 1. \]
Define a decreasing sequence of levels $\{l_i\}_{0 \leq i \leq K}$ by
\[ \log l_i := \log l - (\log l)^{(\varepsilon - \delta)} > 0 \]
and, for each $1 \leq i \leq K$, let
\[ N_i := |\{n < n_l : X_n > l_i\}| = |\{n < n_l : Y_n > f(\log l_i)\}| \]
be the number of values in the sequence \( X \) that exceed the level \( l_i \). Remark that the sum \( m_i \) is bounded above by
\[
\sum_{1 \leq i \leq K} N_i^i l_{i-1}
\]
and so
\[
\log m_i \leq \log l + \log K + \max_{1 \leq i \leq K} \left\{ \log N_i^i - (\log l)^{(i-1)(\varepsilon-\delta)} \right\}.
\]
Hence it is sufficient to prove that, for any \( C \),
\[
\mathbb{P} \left( \max_{1 \leq i \leq K} \left\{ \log N_i^i - (\log l)^{(i-1)(\varepsilon-\delta)} \right\} < -C \log h_l \right) \to 1.
\]
Consider first each \( N_i^i \) for \( 2 \leq i \leq K \). Since, for \( n < n_i \), the random variable \( Y_n \) is distributed as a standard exponential distribution conditioned on not exceeding \( l \), we have
\[
N_i^i \preceq M_i^i := \left\{ \{n < n_i : Z_n > f(\log l_i)\} \right\}
\]
where \( Z := \{Z_n\}_{n \in \mathbb{N}} \) is a sequence of i.i.d. unconditioned standard exponential distributions, and \( \preceq \) denotes the usual stochastic ordering. Clearly,
\[
\log \mathbb{E}(M_i^i) = \log \mathbb{E}(n_i) - f(\log l_i)
\]
and so, applying Proposition 3.1 almost surely
\[
\log \mathbb{E}(M_i^i) \sim f(\log l) - f(\log l_i) = f(\log l) - f \left( \log l - (\log l)^{i(\varepsilon-\delta)} \right).
\]
Moreover, by Assumption 1.1 eventually
\[
f \left( \log l - (\log l)^{i(\varepsilon-\delta)} \right) > f(\log l) - (\log l)^{(i-1)(\varepsilon-\delta)-\delta/2}
\]
and so, almost surely
\[
\log \mathbb{E}(M_i^i) = O \left( (\log l)^{(i-1)(\varepsilon-\delta)-\delta/2} \right).
\]
Since \( \text{Var}(M_i^i) \sim \mathbb{E}(M_i^i) \), by Chebyshev’s inequality there exists a \( C_1 > 0 \) such that
\[
\mathbb{P} \left( \log M_i^i < C_1 (\log l)^{(i-1)(\varepsilon-\delta)-\delta/2} \right) \to 1
\]
as \( l \to \infty \). Finally, since
\[
\log h_l \ll (\log l)^{(i-1)(\varepsilon-\delta)-\delta/2} \ll (\log l)^{(i-1)(\varepsilon-\delta)}
\]
we conclude that
\[
\mathbb{P} \left( \log N_i^i - (\log l)^{(i-1)(\varepsilon-\delta)} < -C \log h_l \right) \to 1.
\]
Consider now \( N_1^i := |\{n < n_i : Y_n > l_i\}| \). Denote by \( E := \{E_n\}_{n \in \mathbb{N}} \) the sequence of progressive maxima of \( Y \). By the memoryless property of the exponential distribution, for each \( i \) we have that
\[
E_i - E_{i-1} \overset{d}{=} Y_1
\]
and so the sequence \( E \) can be considered as the arrival times of a Poisson process on \( \mathbb{R}^+ \) with unit intensity. By the time-reversibility of a Poisson process,
\[
f(\log l) - \max\{Y_n : n < n_i\} \overset{d}{=} Y_1
\]
and so
\[
\mathbb{P}(\max\{Y_n : n < n_i\} < f(\log l) - (\log l)^{-\delta/2}) \to 1
\]
since \((\log l)^{-\delta/2} \ll 1\). On the other hand, by Assumption 1.1 eventually 
\[ f(\log l_1) = f(\log l - (\log l)^{\varepsilon - \delta}) > f(\log l) - (\log l)^{-\delta/2} \]
and so 
\[ \mathbb{P}(\max\{X_n : n < n_1\} < l_1) \rightarrow 1. \]
This implies that 
\[ \mathbb{P}(N_1^1 = 0) \rightarrow 1. \]
Combining equations (9) and (10) establishes equation (8) as required. \(\square\)

**Proposition 3.3** \(\text{(Bound on partial sum)}\).
For any \(C\), as \(t \to \infty\),
\[ \mathbb{P}\left( \log \sum_{i < r_t/h_t} \sigma(X_i) < \log l_t - C \log h_t \right) \rightarrow 1. \]

**Proof.** First remark that if \(r_t/h_t = O(1)\) then the statement is immediate. So assume \(r_t/h_t \to \infty\). As in the proof of Proposition 3.1, construct the sequence \(Y\) with \(Y_n := f(\log X_n)\). Applying equation (7), eventually
\[ \left| \max_{i < r_t/h_t} \{Y_i\} - \log(r_t/h_t) \right| < \log \log r_t \]
almost surely. Recalling that \(\log \log r_t \ll \log h_t\), this implies that
\[ \max_{i < r_t/h_t} \{f(\log X_i)\} = \max_{i < r_t/h_t} \{Y_n\} < \log r_t = f(\log l_t - 2 \log h_t) < f(\log l_t) \]
eventually almost surely. Since eventually \(f\) is invertable, eventually almost surely
\[ \sum_{i < r_t/h_t} \sigma(X_i) < m_t, \]
and so, applying Proposition 3.2
\[ \mathbb{P}\left( \log \sum_{i < r_t/h_t} \sigma(X_i) < \log l_t - C \log h_t \right) \rightarrow 1 \]
where we have replaced \(h_t\) with \(h_t\) by Lemma 2.2. \(\square\)

We are now in a position to prove that the events \(E_t\) and \(F_t\) hold eventually with overwhelming probability.

**Proposition 3.4.** As \(t \to \infty\), 
\[ \mathbb{P}(E_t) \rightarrow 1. \]

**Proof.** Applying Proposition 3.2 to the sequences \(\{\sigma(z)\}_{z \in \mathbb{N}^+}\) and \(\{\sigma(z)\}_{z \in \mathbb{N}^- \cup \{0\}}\) and setting \(l = l_t\), we have that
\[ \mathbb{P}(\log \Sigma_l - \log l_t < -C \log h_t) \rightarrow 1 \]
where we have again replaced \(h_t\) with \(h_t\) by Lemma 2.2. Similarly, applying Proposition 3.1 to the same sequences,
\[ \mathbb{P}(\log d_t < f(\log l_t) + \log \log l_t) = 1 \]
eventually. Recalling that \(\log \log l_t \ll \log h_t\) and \(f(\log l_t) = \log r_t + o(1)\), we have that
\[ \mathbb{P}(\log d_t < \log r_t + \log h_t) = 1 \]
eventually. Combining equations (11) and (12) yields Proposition 3.3.

Proposition 3.5. As $t \to \infty$, 
\[ \mathbb{P}(\mathcal{F}_t) \to 1. \]

Proof. By symmetry, it is sufficient to prove that 
\[ \mathbb{P}(\log \sum_{z \in I_{1}^{+}} \sigma(z) < \log l_t - C \log h_t) \to 1. \]

Consider the set $I_{1}^{+} := \{ z : Z_{t}^{(1)} < z \leq Z_{t}^{(1)} + r_t / h_t \}$, which may be empty. Note that the sequence $\{ \sigma(z) \}_{z \in I_{1}^{+}}$ is distributed like $X$, and so by Proposition 3.3 
\[ \mathbb{P}(\log \sum_{z \in I_{1}^{+}} \sigma(z) < \log l_t - C \log h_t) \to 1. \]

(13) 
Remark that, on the event $\{ I_{1}^{+} \cap I_{2}^{+} = \emptyset \}$, 
\[ \Sigma_{t}^{1} \leq \Sigma_{t} + \sum_{z \in I_{1}^{+}} \sigma(z) \]
and so, combining equations (11) and (13), 
\[ \mathbb{P}(\Sigma_{t}^{1} < l_t / h_t, I_{1}^{+} \cap I_{2}^{+} = \emptyset) \to 1. \]
Finally, notice that $\mathbb{P}(I_{1}^{+} \cap I_{2}^{+} = \emptyset) = 1$ eventually by equation (12).

4. COMPLETING THE PROOF

In this section we establish rigorously equations (3), (4) and (6), and complete the proof of Theorems 1.3, 1.4 and 1.6. First we state some general results on random walks and Markov chains. Let $D_n$ be the simple discrete-time random walk on $\mathbb{Z}$ based at the origin. For a level $l$, define the stopping time 
\[ a_{l} := \min\{ n : |D_n| = l \} \]
and, for each $z$, the local time 
\[ L_{l}^{z} := |\{ i < a_{l} : D_i = z \}|. \]
Further, for a site $z \in \mathbb{Z}$, define the stopping time 
\[ b_{z} := \min\{ n > 0 : D_n = z \}. \]

Proposition 4.1 (Bounds on local time for the simple discrete-time random walk).
There exists a $C > 0$ such that, as $l \to \infty$, eventually 
\[ \max_{z} L_{l}^{z} < Cl \log \log l \quad \text{and} \quad L_{0}^{l} > \frac{l}{(\log l)^C} \]
both hold almost surely.

Proof. These bounds can be derived by combining the law of the iterated logarithm with well-known almost sure bounds on the local time of the simple discrete-time random walk in term of the number of steps (see, e.g., [7, Theorems 11.1, 11.3]).

Proposition 4.2 (Hitting probability for the simple discrete-time random walk).
For any $x \in \mathbb{Z}^{+}$ and $y \in \mathbb{Z}^{-}$, 
\[ \mathbb{P}(b_{x} < b_{y}) = \frac{y}{x + y}. \]
Proof. This is a well-known property of the simple discrete-time random walk, following easily from the optional stopping theorem.

**Proposition 4.3** (Monotonic convergence of Markov chain distribution). Let $M_t$ be a time-homogeneous continuous-time Markov chain, initialised at a state $0$, whose transition rates $\tau$ satisfy the detailed balance condition, i.e. there exists a function $f$ such that

$$f(x)\tau(x \rightarrow y) = f(y)\tau(y \rightarrow x)$$

for all states $x$ and $y$. Then, there exists a constant $c \in [0, 1]$ such that, as $t \to \infty$,

$$\mathbb{P}(M_t = 0) \downarrow c$$

monotonically.

Moreover, if an equilibrium distribution $\pi$ exists, then $\pi(0) = c$.

Proof. This is a well-known result from Markov chain theory. It is proved by considering the spectral representation of $\mathbb{P}(M_t = 0)$ in terms of the eigenvalues $\lambda_i$ and eigenfunctions $\varphi_i$ of the generator of $M_t$:

$$\mathbb{P}(M_t = 0) = \sum_i e^{\lambda_i t} \varphi_i^2(0)$$

recalling that the detailed balance condition ensures that $\lambda_i$ and $\varphi_i$ are real. Since $\mathbb{P}(M_t = 0)$ stays bounded in time, each $\lambda_i \leq 0$, resulting in monotonic convergence to a certain constant, which must be the equilibrium distribution at 0 if it exists. □

We are now in a position to establish equations (3), (4) and (6):

**Proposition 4.4.** As $t \to \infty$,

$$\mathbb{1}_{E_t} \mathbb{P}_\sigma(\tau^I_1 > t) \to 0 \quad \text{in probability.}$$

Proof. Let $Q_z$ denote the local time at $z$ of the geometric path induced by $\{X_s : s \leq \tau^I_1\}$, which follows the simple discrete-time random walk. By Proposition 4.1 there exists a constant $C > 0$ such that eventually

$$\max_z Q_z \prec \max_z L^d_t z \prec Cd \log \log d_t$$

almost surely. Consider $\tau^I_1$ as the sum of the jump times along the geometric path. Since on the event $E_t$,

$$C \log \log d_t < C \log(\log r_t + \log h_t) \ll h_t/2$$

we have that

$$\tau^I_1 < \sum_{z \in \Gamma_t} \text{Gam}(d_t h_t/2, \sigma(z))$$

where each $\text{Gam}(n, \mu)$ is an independent gamma distribution with mean $n\mu$ and variance $n\mu^2$. By Chebyshev’s inequality,

$$\mathbb{P}(\Gamma(d_t h_t/2, \sigma(z)) > d_t h_t \sigma(z)) \ll \frac{1}{d_t h_t}$$

uniformly in $z$, and so

$$\mathbb{P} \left( \sum_{z \in \Gamma_t} \text{Gam}(d_t h_t/2, \sigma(z)) > \Sigma d_t h_t \right) \to 0.$$  

Since $\Sigma d_t h_t < t$ on $E_t$, combining equations (14) and (15) yields the result. □
Proposition 4.5. As \( t \to \infty \),
\[
\mathbb{P}_\sigma(\tau_2^t \geq t) \to 0 \quad \text{in probability}.
\]

Proof. First note that if \( \frac{r_t}{h_t} = O(1) \) then the result follows easily, since then \( t \ll l_t \)
and
\[
\mathbb{P}_\sigma(\tau_2^t < t) \leq \mathbb{P}_\sigma(\tau_2^t - \tau_1^t < t) \leq \mathbb{P}(\text{Exp}(l_t) < t) \to 0
\]
where \( \text{Exp}(l_t) \) denotes the exponential distribution with mean \( l_t \). So assume \( \frac{r_t}{h_t} \to \infty \). Let \( Q \) denote the local time at \( Z_{\tau_1^t} \) of the geometric path induced by \( \{ X_s : \tau_1^t \leq s \leq \tau_2^t \} \). By Proposition 4.1, there exists a constant \( C > 0 \) such that eventually \( Q \) almost surely, and so eventually
\[
(\tau_2^t - \tau_1^t) \gtrsim \text{Gam} \left( \frac{r_t}{h_t (\log r_t)^C}, l_t \right).
\]
On the other hand, by Lemma 2.1 and the conditions on \( h_t \),
\[
\frac{r_t l_t}{h_t (\log r_t)^C} = \frac{th_t}{(\log r_t)^C} \gg t
\]
and so Chebyshev’s inequality gives
\[
\mathbb{P} \left( \text{Gam} \left( \frac{r_t}{h_t (\log r_t)^C}, l_t \right) < t \right) \to 0.
\]
The result follows from combining equations (16) and (17). □

Proposition 4.6. As \( t \to \infty \),
\[
\mathbb{1}_{F_t} \hat{u}_t(t, Z_{\tau_1^t}) \to 1 \quad \text{almost surely}.
\]

Proof. Remark that the BTM satisfies the detailed balance condition. Applying Proposition 4.3 to the Markov chain \( \{ \hat{X}_s^{i_1} \}_{s > \tau_1^t} \), we conclude that
\[
\hat{u}_t(s, Z_{\tau_1^t}^{i_1}) \gtrsim \pi_t^{i_1}(Z_{\tau_1^t}^{i_1})
\]
for each \( s \), where \( \pi_t^{i_1} \) is the equilibrium distribution of the BTM on \( I_t^{i_1} \) with periodic boundary condition, which must exist since \( I_t^{i_1} \) is almost surely finite. By the definition of the BTM, this equilibrium distribution satisfies
\[
(\Delta_t \sigma^{-1}) \pi_t^{i_1} = \Delta_t (\sigma^{-1} \pi_t^{i_1}) = 0
\]
where \( \Delta_t \) is the discrete Laplacian on \( I_t^{i_1} \) with periodic boundary condition. Since the equilibrium distribution of \( \Delta_t \) is uniform, the distribution \( \pi_t^{i_1} \) is proportional to \( \sigma \) and so
\[
\pi_t^{i_1}(z) = \frac{\sigma(z)}{\sigma(Z_t^{i_1})} \pi_t^{i_1}(Z_t^{i_1}) \leq \frac{\sigma(z)}{\sigma(Z_t^{i_1})}.
\]
for all \( z \in I_t^{i_1} \). On the event \( F_t \),
\[
\sum_{z \in I_t^{i_1} \setminus \{Z_t^{i_1}\}} \sigma(z) \leq \max \{ \Sigma_1, \Sigma_2 \} \lesssim \frac{h_t}{l_t} \ll l_t \lesssim \sigma(Z_t^{(i)})
\]
and so, as \( t \to \infty \),
\[
\sum_{z \in I_t^{i_1} \setminus \{Z_t^{(i)}\}} \pi_t^{i_1}(z) \to 0 \quad \text{almost surely}.
\]
Combining equations (18) and (19) completes the proof. □
Finally, we complete the proof of Theorems 1.3, 1.4 and 1.6.

Proof of Theorem 1.3. This follows from combining Propositions 4.4 – 4.6 with equation (5), since \( E_t \) and \( F_t \) hold with overwhelming probability eventually. □

Proof of Theorem 1.6. For fixed \( x, y > 0 \),

\[
P(Z_t^{(1)} > x r_t, Z_t^{(2)} > y r_t) = P(\sigma(\cdot) \leq l_t) \cdot [x r_t] + [y r_t]
\]

\[
= (1 - \exp(-\log r + o(1))) \cdot [x r_t] + [y r_t]
\]

where we applied Lemma 2.1 in the last equality. Then

\[-\log P(|Z_t^{(1)}| > x r_t, |Z_t^{(2)}| > y r_t) = \frac{[x r_t] + [y r_t]}{e^{\log r + o(1)}} + o(1) \to x + y\]

as required. □

Proof of Theorem 1.4. Recall that by Propositions 4.4 – 4.6 we have that

\[ u(t, Z_t^{(1)}) \to 1 \text{ in probability.} \]

Considering the BTM as a time-changed simple discrete-time random walk, it follows from Proposition 4.2 that

\[ P(\sigma(\cdot) = Z_t^{(1)}) = \frac{|Z_t^{(2)}|}{\sum_{z \in \Gamma_t} |z|}. \]

By Theorem 1.6 this implies that

\[ u(t, Z_t^{(1)}) \Rightarrow \frac{\operatorname{Exp}_2}{\sum_{i=1,2} \operatorname{Exp}_i} \overset{d}{\to} U \text{ in law} \]

with the last identity easy to verify. An identical argument works for \( Z_t^{(2)} \). □

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