SEQUENCES OF WEAK SOLUTIONS FOR FRACTIONAL EQUATIONS

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Abstract. This work is devoted to study the existence of infinitely many weak solutions to nonlocal equations involving a general integrodifferential operator of fractional type. These equations have a variational structure and we find a sequence of nontrivial weak solutions for them exploiting the \( \mathbb{Z}_2 \)-symmetric version of the Mountain Pass Theorem. To make the nonlinear methods work, some careful analysis of the fractional spaces involved is necessary. As a particular case, we derive an existence theorem for the fractional Laplacian, finding nontrivial solutions of the equation

\[
\begin{aligned}
(-\Delta)^su &= f(x, u) \quad \text{in } \Omega \\
 u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]

As far as we know, all these results are new and represent a fractional version of classical theorems obtained working with Laplacian equations.

1. Introduction

In this paper we obtain an existence result for the following nonlocal problem:

\[
(P_K^f) \quad \begin{aligned}
-\mathcal{L}_K u &= f(x, u) \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]

Here and in the sequel, \( \Omega \) is a bounded domain in \((\mathbb{R}^n, |\cdot|)\) with \( n > 2s \) (where \( s \in (0, 1) \)) and with smooth (Lipschitz) boundary \( \partial\Omega \), \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a suitable continuous function with subcritical growth and \( \mathcal{L}_K \) is the nonlocal operator defined as follows:

\[
\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \left( u(x+y) + u(x-y) - 2u(x) \right) K(y)dy, \quad (x \in \mathbb{R}^n)
\]

where \( K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty) \) is a function with the properties that:

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\((k_1)\) \(\gamma K \in L^1(\mathbb{R}^n), \) where \(\gamma(x) = \min\{|x|^2, 1\};\)
\((k_2)\) There exists \(\lambda > 0\) such that
\[K(x) \geq \lambda |x|^{-(n+2s)},\]
for any \(x \in \mathbb{R}^n \setminus \{0\};\)
\((k_3)\) \(K(x) = K(-x),\) for any \(x \in \mathbb{R}^n \setminus \{0\}.
A typical example for the kernel \(K\) is given by \(K(x) := |x|^{-(n+2s)}.\)

In this case \(L_K\) is the fractional Laplace operator defined as
\[-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n.

Recently, a great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. This type of operators arises in a quite natural way in many different contexts, such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves.

In this paper, problem \((P'_K)\) is studied exploiting classical variational methods. More precisely we apply the \(Z_2\)-symmetric version of the Mountain Pass Theorem (briefly \(Z_2\)-MPT) to this kind of equations motivated by the current literature where the MPT has been intensively applied to find solutions to quasilinear elliptic equations; see \([2, 6, 7, 13]\).

Technically, this approach is realizable checking that the associated energy functional verifies the usual compactness Palais-Smale condition in a suitable variational setting developed in \([10]\). Indeed, the nonlocal analysis that we perform here in order to use Mountain Pass Theorem is quite general and successfully exploited for other goals in several recent contributions; see \([10, 11, 12]\) and \([4]\) for an elementary introduction to this topic and for a list of related references.

This functional analytical context is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation.

Further, we suppose that the right-hand side of equation \((P'_K)\) is a continuous odd function \(f : \overline{\Omega} \times \mathbb{R}^n \rightarrow \mathbb{R}\) verifying the following conditions:
There exist $a_1, a_2 > 0$ and $q \in (2, 2^*)$, $2^* := \frac{2n}{n - 2s}$ such that
\[ |f(x, t)| \leq a_1 + a_2 |t|^{q-1}, \]
for every $x \in \Omega$, $t \in \mathbb{R}$;

(h2) There are two constants $\theta > 2$ and $r > 0$ such that
\[ 0 < \theta F(x, t) \leq tf(x, t), \]
for every $x \in \Omega$, and $|t| \geq r$,

where the function $F$ is the primitive of $f$ with respect to the second variable, that is
\[ F(x, t) := \int_0^t f(x, s)ds, \quad (\forall t \in \mathbb{R}). \]

Under the previous assumptions, we prove the existence of infinitely many weak solutions to problem $(P_f K)$; see Theorem 3.1. Note that the symmetry hypothesis on $f$ allows to remove any condition at zero.

In the nonlocal framework, the simplest example we can deal with is given by the fractional Laplacian, according to the following proposition.

**Theorem 1.1.** Let $s \in (0, 1)$, $n > 2s$ and $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary.

Consider the following equation
\[ \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}}dxdy = \int_{\Omega} f(x, u(x)) \varphi(x)dx \]
for any $\varphi \in H^s(\mathbb{R}^n)$ with $\varphi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an odd continuous function verifying (h1) and (h2), then problem (2) admits a sequence of infinitely many solutions $\{u_j\} \subset H^s(\mathbb{R}^n)$, such that $u_j = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

The above result is the fractional analogous of [7, Theorem 9.38] in which the classical Dirichlet problem
\[ (D_f) \quad \begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]
is studied; see [13, Theorem 6.6]. We also cite [2, 6] where MPT (and some of its variant) has been intensively applied to find solutions to quasilinear elliptic equations.

The plan of the paper is as follows; Section 2 is devoted to our abstract framework and preliminaries. Successively, in Sections 3 we give the main result; see Theorem 3.1. Finally, a concrete example of application is presented in Example 3.6.
2. Abstract Framework

In this subsection we briefly recall the definition of the functional space $X_0$, firstly introduced in [9], and we give some notations. The reader familiar with this topic may skip this section and go directly to the next one.

The functional space $X$ denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $g$ in $X$ belongs to $L^2(\Omega)$ and

$$((x, y) \mapsto (g(x) - g(y))\sqrt{K(x - y)}) \in L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (C\Omega \times C\Omega), dxdy),$$

where $C\Omega := \mathbb{R}^n \setminus \Omega$.

We denote by $X_0$ the following linear subspace of $X$:

$$X_0 := \{ g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.$$

We remark that $X$ and $X_0$ are non-empty, since $C^2_0(\Omega) \subseteq X_0$ by [9, Lemma 11]. Moreover, the space $X$ is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \left( \int_Q |g(x) - g(y)|^2 K(x - y) dxdy \right)^{1/2},$$

where $Q := (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} := (C\Omega) \times (C\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$.

It is easily seen that $\| \cdot \|_X$ is a norm on $X$; see, for instance, [10] for a proof.

By [10, Lemmas 6 and 7] in the sequel we can take the function

$$(3) \quad X_0 \ni v \mapsto \|v\|_{X_0} := \left( \int_Q |v(x) - v(y)|^2 K(x - y) dxdy \right)^{1/2}$$

as norm on $X_0$.

Also $(X_0, \| \cdot \|_{X_0})$ is a Hilbert space with scalar product

$$\langle u, v \rangle_{X_0} := \int_Q \left( u(x) - u(y) \right) \left( v(x) - v(y) \right) K(x - y) dxdy,$$

see [10, Lemmas 7].

Note that in (3) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

In what follows, we denote by $\lambda_k$ be the $k$-th eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, namely the $k$-th eigenvalue of the problem

$$(4) \quad \begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$
The set of the eigenvalues of problem (4) consists of a sequence \( \{ \lambda_k \}_{k \in \mathbb{N}} \) with
\[
0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots
\]
and
\[
\lambda_k \to +\infty \text{ as } k \to \infty.
\]
Further, the following characterization holds:
\[
\lambda_k = \min_{u \in \mathbb{P}_k \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y)dx \, dy}{\int_{\Omega} u(x)^2 \, dx},
\]
where
\[
\mathbb{P}_k := \{ u \in X_0 : \langle u, e_j \rangle_{X_0} = 0 \quad \forall j = 1, \ldots, k \}.
\]
Finally, the symbol \( E(\lambda_k) \) denotes the linear space of eigenfunctions corresponding to \( \lambda_k \). For the existence and the basic properties of this eigenvalue we refer to [11, Proposition 9 and Appendix A], where a spectral theory for these general integrodifferential nonlocal operators was developed. Further properties can be also found in [8].

While for a general kernel \( K \) satisfying conditions from (k_1) to (k_3) we have that \( X_0 \subset H^s(\mathbb{R}^n) \), in the model case \( K(x) := |x|^{-(n+2s)} \) the space \( X_0 \) consists of all the functions of the usual fractional Sobolev space \( H^s(\mathbb{R}^n) \) which vanish a.e. outside \( \Omega \); see [12, Lemma 7].

Here \( H^s(\mathbb{R}^n) \) denotes the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)
\[
\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2}.
\]
Before concluding this subsection, we recall the embedding properties of \( X_0 \) into the usual Lebesgue spaces; see [10, Lemma 8].

The embedding \( j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n) \) is continuous for any \( \nu \in [1, 2^*] \), while it is compact whenever \( \nu \in [1, 2^*) \), where \( 2^* := 2n/(n - 2s) \). Hence, for any \( \nu \in [1, 2^*) \) there exists a positive constant \( c_\nu \) such that
\[
\|v\|_{L^\nu(\mathbb{R}^n)} \leq c_\nu \|v\|_{X_0} \quad \text{for any } v \in X_0.
\]
For further details on the fractional Sobolev spaces we refer to [4] and to the references therein, while for other details on \( X \) and \( X_0 \) we

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1 As usual, here we call \( \lambda_1 \) the first eigenvalue of the operator \( -\mathcal{L}_K \). This notation is justified by [5]. Notice also that some of the eigenvalues in the sequence \( \{ \lambda_k \}_{k \in \mathbb{N}} \) may repeat, i.e. the inequalities in (5) may be not always strict.
refer to [9], where these functional spaces were introduced, and also to [8, 10, 11, 12], where various properties of these spaces were proved.

Finally, for the sake of completeness, we recall that a $C^1$-functional $J : E \to \mathbb{R}$, where $E$ is a real Banach space with topological dual $E^*$, satisfies the Palais-Smale condition at level $\mu \in \mathbb{R}$, (briefly $(PS)_\mu$) when:

$$(PS)_\mu \quad \text{Every sequence } \{u_j\}_{j \in \mathbb{N}} \text{ in } E \text{ such that}$$

$$J(u_j) \to \mu, \quad \text{and} \quad \|J'(u_j)\|_{E^*} \to 0,$$

$$\text{as } j \to \infty, \text{ possesses a convergent subsequence.}$$

We say that $J$ satisfies the Palais-Smale condition (in short $(PS)$) if $(PS)_\mu$ holds for every $\mu \in \mathbb{R}$.

With the above notation, our main tool is the following classical result recalled in a convenient form.

**Theorem 2.1.** Let $E$ be an infinite dimensional real Banach space and let $J \in C^1(E; \mathbb{R})$ be even, satisfying the $(PS)$ condition and $J(0_E) = 0$. Suppose $E = E_1 \oplus E_2$, where $E_1$ is finite dimensional and $J$ satisfies:

$(I_1)$ There exist constant $\rho, \alpha > 0$ such that

$$J(u) \geq \alpha,$$

for every $u \in E_2$ and $\|u\|_E = \rho$.

$(I_2)$ For each finite dimensional subspace $W \subset E$, the set

$$\{u \in W : J(u) \geq 0\}$$

is bounded in $E$.

Then $J$ has an unbounded sequence of critical values.

See [7, Theorem 9.12].

We cite the monograph [5] as general reference on variational methods adopted in this paper.

3. **The main Theorem**

Our result is as follows.

**Theorem 3.1.** Let $s \in (0, 1)$, $n > 2s$ and $\Omega$ be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a map satisfying $(k_1)$–$(k_3)$.

In addition, let $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be an odd continuous function verifying $(h_1)$ and $(h_2)$.

Then, problem $(P_{\bar{\Omega}})$ possesses an unbounded sequence of weak solutions.
We recall that a weak solution of problem \( (P_f^K) \) is a function \( u \in X_0 \) such that
\[
\int_Q (u(x) - u(y)) \left( \varphi(x) - \varphi(y) \right) K(x - y) dxdy = \int_\Omega f(x, u(x)) \varphi(x) dx, \quad \forall \varphi \in X_0.
\]

For the proof of Theorem 3.1, we observe that problem \( (P_f^K) \) has a variational structure, indeed it is the Euler-Lagrange equation of the functional \( J^K : X_0 \to \mathbb{R} \) defined as follows
\[
(8) \quad J^K(u) := \frac{1}{2} \int_Q |u(x) - u(y)|^2 K(x - y) dxdy - \int_\Omega F(x, u(x)) dx.
\]
Note that the functional \( J^K \) is Fréchet differentiable in \( u \in X_0 \) and for any \( \varphi \in X_0 \) one has
\[
\langle J'_K(u), \varphi \rangle = \int_Q (u(x) - u(y)) \left( \varphi(x) - \varphi(y) \right) K(x - y) dxdy - \int_\Omega f(x, u(x)) \varphi(x) dx.
\]

Thus, critical points of \( J^K \) are solutions to problem \( (P_f^K) \). In order to find these critical points, we will make use of Theorem 2.1. For this, we have to check that the functional \( J^K \) has a particular geometric structure and satisfies the Palais–Smale compactness condition.

3.1. Proof of Theorem 3.1. In order to prove our result, we apply (as claimed before) Theorem 2.1 to the functional \( J^K \) defined in (8). The conclusion of Theorem 3.1 is equivalent to the assertion that \( J^K \) admits an unbounded sequence of critical points. Hence, let us start observing that, since \( f \) is odd in the second variable i.e. \( f(x, -t) = -f(x, t) \), for every \( t \in \mathbb{R} \), \( J^K \) is even. Moreover, by definition, \( J^K(0_{X_0}) = 0 \).

In the next two lemmas we shall verify the compactness (PS) condition.

**Lemma 3.2.** Every Palais-Smale sequence for the functional \( J^K \) is bounded in \( X_0 \).

**Proof.** Let \( \{u_j\}_{j \in \mathbb{N}} \subset X_0 \) be a Palais-Smale sequence i.e.
\[
(9) \quad J^K(u_j) \to \mu,
\]
for \( \mu \in \mathbb{R} \) and
\[
(10) \quad \|J'_K(u_j)\|_{X_0^*} = \sup \left\{ \left| \langle J'_K(u_j), \varphi \rangle \right| : \varphi \in X_0, \|\varphi\|_{X_0} = 1 \right\} \to 0,
\]
as \( j \to \infty \).

We argue by contradiction. So, suppose that the conclusion is not true. Passing to a subsequence if necessary, we may assume that 

\[
\|u_j\|_{X_0} \to +\infty,
\]

as \( j \to \infty \).

By definition it follows that

\[
J_K(u_j) = \frac{\langle J'_K(u_j), u_j \rangle}{\theta} = \Big( \frac{1}{2} - \frac{1}{\theta} \Big) \int_Q |u_j(x) - u_j(y)|^2 K(x - y) \, dx \, dy
\]

\[
+ \int_{\Omega} \left[ \frac{f(x, u_j(x))}{\theta} u_j(x) - F(x, u_j(x)) \right] \, dx,
\]

for every \( j \in \mathbb{N} \).

Thus

\[
\left( \frac{\theta - 2}{2\theta} \right) \|u_j\|_{X_0}^2 \leq J_K(u_j) - \frac{\langle J'_K(u_j), u_j \rangle}{\theta}
\]

\[
- \int_{|u_j(x)| > r} \left[ \frac{f(x, u_j(x))}{\theta} u_j(x) - F(x, u_j(x)) \right] \, dx,
\]

\[
+ M \text{meas}(\Omega), \quad \forall j \in \mathbb{N},
\]

where “meas(\Omega)” denotes the standard Lebesgue measure of \( \Omega \) and

\[
M := \max \left\{ \left| \frac{f(x,t)t}{\theta} - F(x,t) \right| : x \in \bar{\Omega}, \, |t| \leq r \right\}.
\]

Now, we observe that, the Ambrosetti Rabinowitz condition yields

\[
\int_{|u_j(x)| > r} \left[ \frac{f(x, u_j(x))}{\theta} u_j(x) - F(x, u_j(x)) \right] \, dx \geq 0.
\]

So, we deduce that

\[
\left( \frac{\theta - 2}{2\theta} \right) \|u_j\|_{X_0}^2 \leq J_K(u_j) - \frac{\langle J'_K(u_j), u_j \rangle}{\theta} + M \text{meas}(\Omega),
\]

for every \( j \in \mathbb{N} \).

Then, for every \( j \in \mathbb{N} \) one has

\[
C \|u_j\|_{X_0}^2 \leq J_K(u_j) + \theta \|J'_K(u_j)\|_{X_0^*} \|u_j\|_{X_0} + M \text{meas}(\Omega),
\]

where \( C := m_0 \left( \frac{\theta - 2}{2\theta} \right) > 0 \).

In conclusion, dividing by \( \|u_j\|_{X_0} \) and letting \( j \to \infty \), we obtain a contradiction. \( \square \)
The above Lemma implies that the $C^1$-functional $J_K$ verifies the Palais-Smale condition as proved in the next result.

**Lemma 3.3.** The functional $J_K$ satisfies the compactness (PS) condition.

**Proof.** Let $\{u_j\}_{j \in \mathbb{N}} \subset X_0$ be a Palais-Smale sequence and, in order to simplify the notations, let us put

$$\Phi(u) := \frac{1}{2} \int_Q |u(x) - u(y)|^2 K(x - y) dxdy,$$

for every $u \in X_0$.

Taking into account Lemma 3.2, the sequence $\{u_j\}_{j \in \mathbb{N}}$ is necessarily bounded in $X_0$. Since $X_0$ is reflexive, we may extract a subsequence that for simplicity we call again $\{u_j\}_{j \in \mathbb{N}}$, such that $u_j \rightharpoonup u_\infty$ in $X_0$. This means that

$$\int_Q (u_j(x) - u_j(y))(\varphi(x) - \varphi(y)) K(x - y) dxdy \to$$

$$\int_Q (u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y)) K(x - y) dxdy,$$

for any $\varphi \in X_0$, as $j \to \infty$.

We will prove that $u_j$ strongly converges to $u_\infty \in X_0$. Exploiting the derivative $J'_K(u_j)(u_j - u_\infty)$, we obtain

$$\langle \Phi'(u_j), u_j - u_\infty \rangle = \langle J'_K(u_j), u_j - u_\infty \rangle$$

$$+ \int_\Omega f(x, u_j(x))(u_j - u_\infty)(x) dx,$$

where

$$\langle \Phi'(u_j), u_j - u_\infty \rangle = \int_Q |u_j'(x) - u_j(y)|^2 K(x - y) dxdy$$

$$- \int_Q (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y)) K(x - y) dxdy$$

Since $\|J'_K(u_j)\|_{X_0^*} \to 0$ and the sequence $\{u_j - u_\infty\}$ is bounded in $X_0$, taking into account that $\|J'_K(u_j), u_j - u_\infty\| \leq \|J'_K(u_j)\|_{X_0^*} \|u_j - u_\infty\|_{X_0}$, one has

$$\langle J'_K(u_j), u_j - u_\infty \rangle \to 0,$$

as $j \to \infty$.

At this point, we observe that, since the embedding $X_0 \hookrightarrow L^q(\Omega)$ is compact, clearly $u_j \to u_\infty$ strongly in $L^q(\Omega)$. So by condition (h$_1$), we
easily obtain that
\begin{equation}
\int_{\Omega} |f(x, u_j(x))| |u_j(x) - u_\infty(x)| dx \to 0,
\end{equation}
as \( j \to \infty \).

By (12) relations (13) and (14) yield
\begin{equation}
\langle \Phi'(u_j), u_j - u_\infty \rangle \to 0,
\end{equation}
as \( j \to \infty \).

Hence by (15) we can write
\begin{equation}
\int_Q |u_j(x) - u_j(y)|^2 K(x - y) dxdy - \int_Q (u_j(x) - u_j(y)) \left( u_\infty(x) - u_\infty(y) \right) K(x - y) dxdy \to 0,
\end{equation}
as \( j \to \infty \).

Thus, by (16) and (11) it follows that
\begin{equation*}
\lim_{j \to \infty} \int_Q |u_j(x) - u_j(y)|^2 K(x - y) dxdy = \int_Q |u_\infty(x) - u_\infty(y)|^2 K(x - y) dxdy.
\end{equation*}

In conclusion, thanks to [3, Proposition III.30], \( u_j \to u_\infty \) in \( X_0 \). The proof is complete. \( \square \)

The proof of the main result is concluded if we prove that hypotheses \((I_1)\) and \((I_2)\) in Theorem 2.1 are verified. We show these facts in the next two propositions.

**Proposition 3.4.** *The functional \( J_K \) satisfies condition \((I_1)\).*

*Proof.* We claim that there exists \( k_0 \in \mathbb{N} \) sufficiently large and two positive constants \( \rho \) and \( \alpha \) such that for every
\[
\|u\|_{X_0} = \rho,
\]
with \( \|u\|_{X_0} = \rho \), there holds \( J_K(u) \geq \alpha \).

Indeed, since \( q \in (2, 2^*) \) one has
\begin{equation}
\|u\|_{L^q(\Omega)} \leq \|u\|_{L^p(\Omega)}^{\beta} \|u\|_{L^{q-\beta}(\Omega)}^{\frac{q-\beta}{p-\beta}}, \quad (\forall u \in X_0)
\end{equation}
where we set \( \beta := 2 \left( \frac{2^* - q}{2^* - 2} \right) \); see, for instance, [11, p. 105].
Now, let us denote by \( \{e_k\}_{k \in \mathbb{N}} \) the sequence of eigenfunctions. By Proposition 9 and Appendix A, we have that the sequence \( \{e_k\}_{k \in \mathbb{N}} \) is an orthonormal basis of \( L^2(\Omega) \) and an orthogonal basis of \( X_0 \).

More precisely, for every \( k \in \mathbb{N} \) one has

\begin{equation}
\langle e_k, e_k \rangle_{X_0} = \lambda_k \int_{\Omega} e_k(x)^2 \, dx = \lambda_k,
\end{equation}

and

\begin{equation}
\langle e_i, e_j \rangle_{X_0} = \int_{Q} (e_i(x) - e_i(y)) (e_j(x) - e_j(y)) K(x - y) \, dx \, dy = \delta_{ij},
\end{equation}

for all \( i, j \in \mathbb{N} \).

If \( u \in E_2, u = \sum_{j=k_0}^{+\infty} \beta_j e_j \), for suitable \( \beta_j \in \mathbb{R} \), where \( j \in \mathbb{N} \) and \( j \geq k_0 \). Hence, by using (18) and (19) one has

\[ \|u\|_{L^2(\Omega)}^2 = \sum_{j=k_0}^{+\infty} \beta_j^2 \int_{\Omega} e_j(x)^2 \, dx = \sum_{j=k_0}^{+\infty} \frac{\beta_j^2}{\lambda_j} \langle e_j, e_j \rangle_{X_0} \leq \frac{1}{\lambda_{k_0}} \|u\|_{X_0}^2, \]

i.e.,

\begin{equation}
\|u\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\lambda_{k_0}}} \|u\|_{X_0},
\end{equation}

for each \( u \in E_2 \).

By the growth condition \( (h_1) \) there is a positive constant \( C_1 \) such that

\[ |F(x, \xi)| \leq C_1 (1 + |t|^q), \]

for every \( x \in \bar{\Omega}, \xi \in \mathbb{R} \).

Then, by (17), (20) and using the Sobolev’s embedding \( X_0 \hookrightarrow L^{2^*}(\Omega) \) we have

\[
J_K(u) \geq \frac{\|u\|_{X_0}^2}{2} - C_1 \|u\|_{L^q(\Omega)}^q - C_1 \text{meas}(\Omega)
\geq \frac{\|u\|_{X_0}^2}{2} - C_1 \|u\|_{L^2(\Omega)}^{q-\beta} \|u\|_{L^{2^*}(\Omega)}^{2^*-\beta} - C_1 \text{meas}(\Omega)
\geq \frac{\|u\|_{X_0}^2}{2} - C_1 \frac{c_2^{q-\beta}}{\lambda_{k_0}^{\beta/2}} \|u\|_{X_0}^{q} - C_1 \text{meas}(\Omega)
\geq \left( \frac{1}{2} - C_1 \frac{c_2^{q-\beta}}{\lambda_{k_0}^{\beta/2}} \|u\|_{X_0}^{q-2} \right) \|u\|_{X_0}^2 - C_1 \text{meas}(\Omega),
\]

for every \( u \in E_2 \).
We may let $\rho := 2^{1/2}C_1 \text{meas}(\Omega) + 1$ and choose $k_0 \in \mathbb{N}$ such that

$$\lambda_{k_0} \geq \left(2^{q-2}C_1 c_q^\beta (C_1 \text{meas}(\Omega) + 1)^{(q-2)/2}\right)^{2/\beta},$$

to achieve that

$$J_K(u) \geq 1, \quad (\alpha := 1)$$

for every $u \in E_2$ and $\|u\|_E = \rho$.

Hence, condition $(I_1)$ is verified. The proof is complete. \hfill $\Box$

**Proposition 3.5.** The functional $J_K$ satisfies condition $(I_2)$.

**Proof.** Let $W \subset X_0$ be a finite dimensional space. We prove the set

$$\{u \in W : J_K(u) \geq 0\}$$

is bounded in $X_0$.

Indeed, let $u \in X_0$ arbitrary and denote

$$\Omega_\prec := \{x \in \Omega : |u(x)| < r\},$$

as well as

$$\Omega_\succ := \{x \in \Omega : |u(x)| \geq r\}.$$

We shall prove that $J_K$ satisfies the following estimate

$$J_K(u) \leq \frac{\|u\|_{X_0}^2}{2} - \int_\Omega \gamma(x)|u(x)|^\theta dx + \kappa,$$

where $\kappa$ is a suitable positive constant and $\gamma \in L^\infty(\Omega)$, with $\gamma > 0$ in $\Omega$.

To show the above inequality, let us start observing that by $(h_1)$, as pointed out before, the function $F$ satisfies

$$|F(x, \xi)| \leq C_1 (1 + |\xi|^\theta), \quad \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}.$$

We claim that there exists $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ in $\Omega$, such that

$$F(x, \xi) \geq \gamma(x)|\xi|^\theta, \quad \forall x \in \bar{\Omega}, \forall |\xi| \geq r.$$  

Indeed, since $F$ is $\theta$-superhomogeneous, we have that

$$F(x, \xi) \geq \gamma_1(x)|\xi|^\theta, \quad \forall x \in \bar{\Omega}, \forall |\xi| \geq r,$$

where $\gamma_1(x) := F(x, r)/r^\theta$. It is easy to see that $\gamma_1 \in L^\infty(\Omega)$ and $\gamma_1 > 0$ in $\Omega$.

In a similar way, it follows that

$$F(x, \xi) \geq \gamma_2(x)|\xi|^\theta, \quad \forall x \in \bar{\Omega}, \forall |\xi| \leq -r,$$

with $\gamma_2(x) := F(x, -r)/r^\theta$. Also in this case $\gamma_2 \in L^\infty(\Omega)$ and $\gamma_2 > 0$ in $\Omega$. 

Then (23) holds with
\[ \gamma(x) := \min\{\gamma_1(x), \gamma_2(x)\}, \]
for every \( x \in \bar{\Omega} \).

Now, by condition (22) we conclude that
\[ \int_{\Omega_<} F(x, u(x)) \, dx \geq -C_1 (r^q + 1) \text{meas} (\Omega). \]

Further, inequality (23) yields
\[ \int_{\Omega_>} F(x, u(x)) \, dx \geq \int_{\Omega_>} \gamma(x) |u(x)|^\theta \, dx. \]

Then
\[
J_K(u) \leq \frac{\|u\|_{X_0}^2}{2} - \left( \int_{\Omega_<} F(x, u(x)) \, dx + \int_{\Omega_>} F(x, u(x)) \, dx \right)
\leq \frac{\|u\|_{X_0}^2}{2} - \int_{\Omega_>} \gamma(x) |u(x)|^\theta \, dx + C_1 (r^q + 1) \text{meas} (\Omega)
\leq \frac{\|u\|_{X_0}^2}{2} - \int_{\Omega} \gamma(x) |u(x)|^\theta \, dx + \kappa,
\]
where
\[ \kappa := (\|\gamma\|_\infty r^\theta + C_1 (r^q + 1)) \text{meas} (\Omega). \]

Hence, inequality (21) is proved.

At this point, observe that the functional \( \| \cdot \|_\gamma : X_0 \to \mathbb{R} \) defined by
\[ \|u\|_\gamma := \left( \int_{\Omega} \gamma(x) |u(x)|^\theta \, dx \right)^{1/\theta}, \]
is a norm in \( X_0 \).

Since in \( W \) the norms \( \| \cdot \|_{X_0} \) and \( \| \cdot \|_\gamma \) are equivalent (\( W \) is finite dimensional), there exists a positive constant \( \kappa_W \) such that
\[ \|u\|_{X_0} \leq \kappa_W \|u\|_\gamma, \]
for every \( u \in X_0 \).

Consequently, we have that
\[ J_K(u) \leq \frac{\kappa_W^2}{2} \|u\|_\gamma^2 - \|u\|_\gamma^\theta + \kappa, \]
for every \( u \in W \).

Hence
\[ \frac{\kappa_W^2}{2} \|u\|_\gamma^2 - \|u\|_\gamma^\theta + \kappa \geq 0, \]
for every
\[ u \in \{ u \in W : J_K(u) \geq 0 \}. \]

Since \( \theta > 2 \) we conclude that the above set is bounded in \( X_0 \).

**Proof of Theorem 3.1 concluded.** With the above notations, we can write
\[ X_0 = E_1 \oplus E_2, \]
where \( E_1 \), given by \( \text{Span}\{e_j : j < k_0\} \), is the orthogonal complement of \( E_2 \). Thanks to Lemma 3.3 and Propositions 3.4-3.5, Theorem 2.1 implies that the functional \( J_K \) possesses an unbounded sequence of critical value \( \{ J_K(u_k) \}_{k \in \mathbb{N}} \), where \( u_k \) is a weak solution of \( (P_K) \). Since \( J'_K(u_k)(u_k) = 0 \),
\begin{equation}
\int_Q |u_k(x) - u_k(y)|^2 K(x-y) dx dy = \int_\Omega f(x, u_k(x)) dx,
\end{equation}
and it follows that
\begin{equation}
J_K(u_k) = \int_\Omega \left[ \frac{1}{2} f(x, u_k(x))u_k(x) - F(x, u_k(x)) \right] dx \to +\infty,
\end{equation}
as \( k \to \infty \). Hence by (24)-(25) and (h2), the sequence \( \{u_k\}_{k \in \mathbb{N}} \) must be unbounded in \( X_0 \) and in \( L^\infty(\Omega) \). The proof is complete.

In conclusion, we present a simple and direct application of Theorem 3.1.

**Example 3.6.** Let \( s \in (0,1) \), \( n > 2s \) and \( \Omega \) be an open bounded set of \( \mathbb{R}^n \) with Lipschitz boundary and consider the following nonlocal problem:
\begin{equation}
(P_K) \quad \begin{cases}
-L_K u = u^3 + u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\end{equation}
where \( K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty) \) is a map satisfying (k1)-(k3). Owing to Theorem 3.1 problem \( (P_K) \) admits infinitely many weak solutions.

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