HOM-ALTERNATIVE, HOM-MALCEV AND HOM-JORDAN SUPERALGEBRAS

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Abstract. Hom-alternative, Hom-Malcev and Hom-Jordan superalgebras are $\mathbb{Z}_2$-graded generalizations of Hom-alternative, Hom-Malcev and Hom-Jordan algebras, which are Hom-type generalizations of alternative, Malcev and Jordan algebras. In this paper we prove that Hom-alternative superalgebras are Hom-Malcev-admissible and are also Hom-Jordan-admissible. Home-type generalizations of some well known identities in alternative superalgebras, including the $\mathbb{Z}_2$-graded Bruck-Kleinfeld function are obtained.

1. Introduction

A Malcev superalgebra is a non-associative superalgebra $A$ with a super skew-symmetric multiplication $[−, −]$ (i.e., $[x, y] = -(−1)^{|x||y|} [y, x]$) such that the Malcev super-identity

\[ 2[t, J_A(x, y, z)] = J_A(t, x, [y, z]) + (−1)^{|x|(|y|+|z|)}J_A(t, y, [z, x]) + (−1)^{|z|(|x|+|y|)}J_A(t, z, [x, y]) \]

(1.1)
is satisfied for all homogeneous elements $x, y, z, t$ in the superspace $A$, where $J_A(x, y, z) = [[x, y], z] − [x, [y, z]] − (−1)^{|y||z|}[[x, z], y]$ is the super-Jacobian. In particular, Lie superalgebras are examples of Malcev superalgebras. Malcev superalgebras play an important role in the geometry of smooth loops.

Closely related to Malcev superalgebras are alternative superalgebras. An alternative superalgebra is a superalgebra whose associator is a super-alternating function. In particular, all associative superalgebras are alternative.

A Jordan superalgebra is a super-commutative superalgebra (i.e. $x ∗ y = (−1)^{|x||y|} y ∗ x$) that satisfies the Jordan super-identity

\[ \sum_{x,y,t} (−1)^{|t|(|x|+|z|)} a_s A(x ∗ y, z, t) = 0, \]

(1.3)
where $\sum_{x,y,t}$ denotes the cyclic sum over $(x, y, t)$ and $a_s A(x, y, z) = (x ∗ y) ∗ z − x ∗ (y ∗ z)$ for all homogeneous elements $x, y, z \in A$.

Starting with an alternative superalgebra $A$, it is known that the super-product

\[ x ∗ y = \frac{1}{2} (x ∗ y + (−1)^{|x||y|} y ∗ x) \]
gives a Jordan superalgebra $A^+ = (A, ∗)$. In other words, alternative superalgebras are Jordan-admissible.

The reader is referred to [3, 24, 25] for discussions about the important role of Jordan superalgebras in physics, especially in quantum mechanics.

The purpose of this paper is to study Hom-type generalizations of alternative superalgebras, Malcev (admissible) superalgebras and Jordan (admissible) superalgebras.

In Section 2, we introduce Hom-alternative superalgebras and prove two construction results Theorems (2.1) and (2.2). Theorem (2.1) says that the category of Hom-alternative superalgebras is closed under self weak morphism.

In Section 3, we introduce Hom-Malcev superalgebra and prove two construction Theorems (3.1) and (3.2). Hom-Malcev superalgebras include Malcev superalgebras and Hom-Lie superalgebras as examples. Theorem (3.1) says that the category of Hom-Malcev superalgebras is closed under self weak morphism.

In Section 4, we show that Hom-alternative superalgebras are Hom-Malcev-admissible (Theorem (4.1)). That is, the super-commutator Hom-superalgebra (Definition (4.1)) of a Hom-alternative superalgebra is a Hom-Malcev superalgebra, generalizing the fact that alternative superalgebras are Malcev-admissible. The proof of the Hom-Malcev-admissibility of Hom-alternative superalgebras involves the Hom-type analogues of certain identities that holds in alternative superalgebra and of the $\mathbb{Z}_2$-graded Bruck-Kleinfeld function.
In Section 5, we introduce Hom-flexible superalgebras and we consider the class of Hom-Malcev-admissible superalgebras. In Proposition (5.1), we give several characterizations of Hom-Malcev-admissible superalgebras that are also Hom-flexible. Hom-alternative superalgebras are Hom-flexible, so by Theorem (4.1) Hom-alternative superalgebras are both Hom-flexible and Hom-Malcev-admissible. In Examples (5.1) and (5.2), we construct Hom-flexible, Hom-Malcev-admissible superalgebras that are not Hom-alternative, not Hom-Lie-admissible, and not Malcev-admissible.

In Section 6, we introduce and study Hom-Jordan (−admissible) superalgebras, which are the Hom-type generalizations of Jordan (−admissible) superalgebras. We show that Hom-alternative superalgebras are Hom-Jordan-admissible (Theorem (6.1)). In other words, the plus Hom-superalgebra (Definition (6.1)) of any Hom-alternative superalgebra is a Hom-Jordan superalgebra, generalizing the Jordan-admissibility of alternative superalgebras. Construction results analogous of Theorems (2.1) and (2.2) are provided for Hom-Jordan (−admissible) superalgebras (Theorems (6.2) and (6.3)).

2. Hom-Alternative Superalgebras

Throughout this paper K is an algebraically closed field of characteristic 0 and A is a linear super-space over K. In this section, we introduce Hom-alternative superalgebras and study their general properties. We provide some construction results for Hom-alternative superalgebras (Theorem (2.1) and Theorem (2.2)).

Now let A be a linear superespace over K that is a Z2-graded linear space with a direct sum A = A0 ⊕ A1. The element of A1, j ∈ Z2, are said to be homogeneous of parity j. The parity of a homogeneous element x is denoted by |x|. In the sequel, we will denote by H(A) the set of all homogeneous elements of A.

Definition 2.1. By a Hom-superalgebra we mean a triple (A, µ, α) in which A is a K−super-module, µ : A × A −→ A is an even bilinear map, and α : A −→ A is an even linear map such that µ ◦ α = µ ◦ α⊗2 (multiplicativity).

Remark 2.1. The multiplicativity of the twisting even map α is built into our definitions of Hom-superalgebra. We chose to impose multiplicativity because many of our results depend on it and all of our concrete examples of Hom-alternative, Hom-Malcev (−admissible) and Hom-Jordan (−admissible) superalgebras have this property.

Definition 2.2. Let (A, µ, α) be a Hom-superalgebra, that is a K-vector superspace A together with a multiplication µ and a linear self-map α.

1. The Hom-associator of A [10] is the trilinear map ˜asA : A × A × A −→ A defined as

(2.4) ˜asA = µ ◦ (µ ⊗ α − α ⊗ µ).

In terms of elements, the map ˜asA is given by

\[ ˜asA(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)) \]

2. The Hom-super-Jacobian of A [9] is the trilinear map ˜JA : A × A × A −→ A defined as

(2.5) ˜JA(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)) - (-1)^{|y||z|}\mu(\mu(x, z), \alpha(y)).

Note that when (A, µ) is a superalgebra (with α = Id), its Hom-associator and Hom-super-Jacobian coincide with its usual associator and super-Jacobian, respectively.

Definition 2.3. A left Hom-alternative superalgebra (resp. right Hom-alternative superalgebra) is a triple (A, µ, α) consisting of Z2−graded vector space A, an even bilinear map µ : A × A −→ A and an even homomorphism α : A −→ A satisfying the left Hom-alternative super-identity, that is for all x, y ∈ H(A),

(2.6) ˜asA(x, y, z) + (-1)^{|z||y|}\tilde{a}sA(y, x, z) = 0,

respectively, right Hom-alternative super-identity, that is

(2.7) ˜asA(x, y, z) + (-1)^{|y||z|}\tilde{a}sA(x, y, z) = 0.

A Hom-alternative superalgebra is one which is both left and right Hom-alternative superalgebra. In particular, if α is a morphism of alternative superalgebras (i.e., α ◦ µ = µ ◦ α⊗2), then we call (A, µ, α) a multiplicative Hom-alternative superalgebra.
Observe that when \( \alpha = Id \), the left Hom-alternative super-identity (2.6) (resp. right Hom-alternative super-identity (2.7)) reduces to the usual left alternative super-identity (resp. right alternative super-identity).

**Definition 2.4.** Let \((\mathcal{A}, \mu, \alpha)\) and \((\mathcal{A}', \mu', \alpha')\) be two Hom-alternative superalgebras. An even linear map \( f : \mathcal{A} \rightarrow \mathcal{A}' \) is called:

1. a weak morphism of Hom-alternative superalgebras if it satisfies \( f \circ \mu = \mu' \circ (f \otimes f) \).
2. a morphism of Hom-alternative superalgebras if \( f \) is a weak morphism and \( f \circ \alpha = \alpha' \circ f \).

**Lemma 2.1.** Let \((\mathcal{A}, \mu, \alpha)\) be a Hom-alternative superalgebra. Then

\[
\tilde{a}_{\mathcal{A}}(x, y, z) = (-1)^{|x||y|} \tilde{a}_{\mathcal{A}}(y, x, z) = (-1)^{|y||z|} \tilde{a}_{\mathcal{A}}(x, z, y).
\]

Proof. Since \((\mathcal{A}, \mu, \alpha)\) is a Hom-alternative superalgebra. Then, for all \( x, y, z \in \mathcal{H}(\mathcal{A}) \), we have

\[
\tilde{a}_{\mathcal{A}}(x, y, z) = (-1)^{|x||y|} \tilde{a}_{\mathcal{A}}(y, x, z) = (-1)^{|y||z|} \tilde{a}_{\mathcal{A}}(x, z, y).
\]

So

\[
\tilde{a}_{\mathcal{A}}(x, y, z) = (-1)^{|y||z|} \tilde{a}_{\mathcal{A}}(z, x, y)
\]

\[
= (-1)^{|y||z|+|x||y|} \tilde{a}_{\mathcal{A}}(z, y, x)
\]

\[
= (-1)^{|y||z|+|x||y|} \tilde{a}_{\mathcal{A}}(z, y, x).
\]

\( \square \)

**Proposition 2.1.** Let \((\mathcal{A}, \mu, \alpha)\) be a Hom-alternative superalgebra. Suppose that \( \mu(x, y) = (-1)^{|x||y|} \mu(y, x) \) for all \( x, y \in \mathcal{H}(\mathcal{A}) \), then

\[
\mu(\alpha(x), \mu(y, z)) = (-1)^{|x||y|} \mu(\alpha(y), \mu(x, z))
\]

and

\[
\mu(\mu(x, z), \alpha(y)) = (-1)^{|x||y|} \mu(\alpha(y), \mu(z, x)).
\]

Proof. Let \( x, y \in \mathcal{H}(\mathcal{A}) \), the left Hom-alternative super-identity gives

\[
\tilde{a}_{\mathcal{A}}(x, y, z) + (-1)^{|x||y|} \tilde{a}_{\mathcal{A}}(y, x, z) = 0.
\]

Then

\[
\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + (-1)^{|x||y|} \mu(\alpha(y), \mu(x, z)) - (-1)^{|x||y|} \mu(\mu(y, x), \alpha(z)) = 0,
\]

or

\[
-\mu(\mu(x, y), \alpha(z)) - (-1)^{|x||y|} \mu(\mu(y, x), \alpha(z)) = (-1)^{|x||y|} \mu(\alpha(y), \mu(x, z)) - (-1)^{|x||y|} \mu(\mu(y, x), \alpha(z))
\]

\[
= 0.
\]

So \( \mu(\alpha(x), \mu(y, z)) + (-1)^{|x||y|} \mu(\alpha(y), \mu(x, z)) = 0 \). Hence \( \mu(\alpha(x), \mu(y, z)) = (-1)^{|x||y|} \mu(\alpha(y), \mu(x, z)) \). \( \square \)

### 2.1. Construction Theorems and Examples.

In this section, we prove that the category of Hom-alternative superalgebras is closed under self weak morphism. This procedure was applied to associative (super-) algebras, G-associative (super-) algebras and Lie (super-) algebras in [31 16]. It was introduced first in [21 Theorem (2.3)] and this procedure is called twisting principle or construction by composition.

**Definition 2.5.** Let \((\mathcal{A}, \mu)\) be a given superalgebra and \( \alpha : \mathcal{A} \rightarrow \mathcal{A} \) be an even superalgebra morphism. Define the Hom-superalgebra induced by \( \alpha \) as

\[
\mathcal{A}_\alpha = (\mathcal{A}, \mu_\alpha = \alpha \circ \mu, \alpha).
\]

**Theorem 2.1.** Let \((\mathcal{A}, \mu, \alpha)\) be a left Hom-alternative superalgebra (resp. right Hom-alternative superalgebra) and \( \beta : \mathcal{A} \rightarrow \mathcal{A} \) be an even left alternative superalgebra endomorphism (resp. right alternative superalgebra).

Then \( \mathcal{A}_\beta = (\mathcal{A}, \mu_\beta = \beta \circ \mu, \beta \alpha) \) is a left Hom-alternative superalgebra (resp. right Hom-alternative superalgebra). Moreover, suppose that \((\mathcal{A}', \mu')\) is an other left alternative superalgebra (resp. right alternative superalgebra) and \( \alpha' : \mathcal{A} \rightarrow \mathcal{A}' \) be a left alternative superalgebra endomorphism (resp. right alternative superalgebra)
Corollary 2.1. Let $A$ be a left alternative superalgebra morphism (resp.
right alternative superalgebra morphism) that satisfies $f \circ \beta = \alpha \circ f$, then

$$f : (A, \mu, \alpha) \longrightarrow (\tilde{A}', \mu', \alpha')$$

is a morphism of left Hom-alternative superalgebras (resp.
right Hom-alternative superalgebras).

Proof. We show that $A \beta = (A, \mu, \beta \alpha)$ satisfies the left Hom-alternative super-identity (2.6) (resp.
right Hom-alternative super-identity (2.7)). Indeed

$$\tilde{\text{as}}_{A \beta}(x, y, z) = \mu(\beta \alpha(x), \mu(y, z)) - \mu(\beta \alpha(x, y), \beta \alpha(z))$$

$$= \beta \circ \mu(\beta \alpha(x), \beta \mu(y, z)) - \beta \circ \mu(\beta \circ \mu(x, y), \beta \alpha(z))$$

$$= \beta \circ \left( \mu(\alpha(x, y), \alpha(z)) - \mu(\mu(x, y), \alpha(z)) \right)$$

The second assertion follows from

$$\tilde{\text{as}}_{A \beta}(y, x, z) = \beta \circ \left( \mu(\alpha(x, y), \alpha(z)) - \mu(\mu(x, y), \alpha(z)) \right)$$

As a particular case we obtain the following Example.

Example 2.1. Let $(A, \mu)$ be an alternative superalgebra and $\alpha$ be an even alternative superalgebra morphism,
then $A \alpha = (A, \mu, \alpha)$ is a multiplicative Hom-alternative superalgebra.

Remark 2.2. Let $(A, \mu, \alpha)$ be a Hom-alternative superalgebra, one may ask whether this Hom-alternative
superalgebra is induced by an ordinary alternative superalgebra $(A, \tilde{\mu})$, that is $\alpha$ is an even superalgebra
endomorphism with respect to $\tilde{\mu}$ and $\mu = \alpha \circ \tilde{\mu}$. This question was addressed and discussed for Hom-associative
algebras in [9].

First observation, if $\alpha$ is an even superalgebra endomorphism with respect to $\tilde{\mu}$. Then $\alpha$ is also an even
superalgebra endomorphism with respect to $\mu$. Indeed

$$\mu(\alpha(x), \alpha(y)) = \alpha \circ \tilde{\mu}(\alpha(x), \alpha(y))$$

$$= \alpha \circ \mu(\alpha(x, y))$$

$$= \alpha \circ \mu(x, y).$$

Second observation, if $\alpha$ is bijective then $\alpha^{-1}$ is also an even superalgebra automorphism. Therefore one may
use an untwist operation on the Hom-alternative superalgebra in order to recover the alternative superalgebra
($\tilde{\mu} = \alpha^{-1} \circ \mu$).

Definition 2.6. Let $(A, \mu)$ be a Hom-superalgebra and $n \geq 0$. Define the $n$th derived Hom-superalgebra
of $A$ by

$$A^n = (A, \mu(\alpha), = \alpha^{2n-1} \circ \mu, \alpha^{2n}).$$

Note that $A^0 = A$, $A^1 = (A, \mu(\alpha) = \alpha \circ \mu, \mu^2)$, and $A^{n+1} = (A^n)^1$.

Corollary 2.1. Let $(A, \mu, \alpha)$ be a multiplicative Hom-alternative superalgebra. Then the $n$th derived Hom-
superalgebra $A^n = (A, \mu(\alpha) = \alpha^{2n-1} \circ \mu, \alpha^{2n})$ is also a multiplicative Hom-alternative
superalgebra for each $n \geq 0$. 

□
3. Hom-Malcev Superalgebras

We introduce and study in this section Hom-Malcev superalgebras. Other characterizations of the Hom-Malcev super-identity are given by Proposition 6.11. We provide some construction results for Hom-Malcev superalgebras Theorem 6.2 and Theorem 6.1. Then using Theorem 6.1, we construct (non-Hom-Lie) Hom-Malcev superalgebra (Example 6.2).

**Definition 3.1.**

1. A Hom-Lie superalgebra [3] is a Hom-superalgebra \((\mathcal{A}, [-, -], \alpha)\) such that \([-,-]\) is super skewsymmetric (i.e. \([x, y] = -(−1)^{|x||y|}[y, x]\)) and that the Hom-Jacobian super-identity

\[
\mathcal{J}_A(x, y, z) = [[x, y], \alpha(z)] - [\alpha(x), [y, z]] - (−1)^{|y||z|}[[x, z], \alpha(y)] = 0
\]  

is satisfied for all \(x, y, z\) in \(\mathcal{H}(\mathcal{A})\).

2. A Hom-Malcev superalgebra is a Hom-superalgebra \((\mathcal{A}, [-, -], \alpha)\) such that \([-,-]\) is super skewsymmetric (i.e. \([x, y] = -(−1)^{|x||y|}[y, x]\)) and that the Hom-Malcev super-identity

\[
2[\alpha^2(t), \mathcal{J}_A(x, y, z)] = \mathcal{J}_A(\alpha(t), \alpha(x), [y, z]) + (−1)^{|x||y|+(|z|)}\mathcal{J}_A(\alpha(t), \alpha(y), [z, x])
\]

(3.9)

is satisfied for all \(x, y, z, t\) in \(\mathcal{H}(\mathcal{A})\).

Observe that when \(\alpha = Id\), the Hom-Jacobi super-identity reduces to the usual Jacobi super-identity

\[
\mathcal{J}_A(x, y, z) = [[x, y], z] - [x, [y, z]] - (−1)^{|y||z|}[[x, z], y] = 0
\]

for all \(x, y, z \in \mathcal{H}(\mathcal{A})\). Likewise, when \(\alpha = Id\), by the super anti-symmetry of \([-,-]\), the Hom-Malcev super-identity reduces to the Malcev super-identity 3.10 or equivalently

\[
2[t, \mathcal{J}_A(x, y, z)] = \mathcal{J}_A(t, x, [y, z]) + (−1)^{|x||y|+(|z|)}\mathcal{J}_A(t, y, [z, x])
\]

(3.10)

for all \(x, y, z, t\) in \(\mathcal{H}(\mathcal{A})\).

**Example 3.1.** A Lie (resp. Malcev [6] [7]) superalgebra \((\mathcal{A}, [-, -], \alpha)\) is a Hom-Lie (resp. Hom-Malcev) superalgebra with \(\alpha = Id\), since the Hom-Jacobi super-identity 6.32 (resp. Hom-Malcev super-identity 6.9) reduces to the usual Jacobi (resp. Malcev) super-identity. Moreover, every Hom-Lie superalgebra is a Hom-Malcev superalgebra because the Hom-Jacobi super-identity 6.32 clearly implies the Hom-Malcev super-identity 6.9.

**Lemma 3.1.** If \((\mathcal{A}, [-, -], \alpha)\) is a Hom-Malcev superalgebra. Then for all \(x, y, z \in \mathcal{H}(\mathcal{A})\)

1. \(\mathcal{J}_A(x, y, z) = -\mathcal{J}_A(y, x, z)\),
2. \(\mathcal{J}_A(x, y, z) = -\mathcal{J}_A(y, z, x)\),
3. \(\mathcal{J}_A(x, y, z) = -\mathcal{J}_A(z, y, x)\).

**Proof.** Straightforward calculations.

Now, we aim to provide examples of Hom-Malcev superalgebras, using twisting principle. Let us give some characterizations of the Hom-Malcev super-identity.

**Proposition 3.1.** Let \((\mathcal{A}, [-, -], \alpha)\) be a multiplicative Hom-superalgebra where \([-,-]\) is super skewsymmetric. The following statements are equivalent

1. \((\mathcal{A}, [-, -], \alpha)\) is a Hom-Malcev superalgebra, i.e. the Hom-Malcev super-identity 3.9 holds.
2. The equality

\[
\mathcal{J}_A(\alpha(x), \alpha(y), [t, z]) + (−1)^{|x||y|+(|t||x|+|y|)}\mathcal{J}_A(\alpha(t), \alpha(y), [x, z])
\]

(3.11)

\[
= (−1)^{|x||y|}\mathcal{J}_A(x, y, z) + (−1)^{|x||y|+(|t||x|+|y|)}\mathcal{J}_A(t, y, z, \alpha^2(x))
\]

holds for all \(x, y, z, t \in \mathcal{H}(\mathcal{A})\).
The equality
\[
(-1)^{|x||y|+|t|(|x|+|y|)}\alpha([t, y], [x, z]) + \alpha([x, y], [t, z])
\]
holds for all \(x, y, z\) and \(t\) in \(\mathcal{H}(\mathcal{A})\).

**Proof.** To prove the equivalence between (3.11) and (3.12), observe that the left hand side of the identity (3.11) is
\[
\tilde{J}_\mathcal{A}(\alpha(x), \alpha(y), [t, z]) + (-1)^{|x||y|+|t|(|x|+|y|)}\tilde{J}_\mathcal{A}(\alpha(t), \alpha(y), [x, z])
\]
\[
= \alpha([x, y], [t, z]) - \alpha^2(x), \alpha(y), [t, z]) - (-1)^{|y||t|+|z|}[\alpha(x), [t, z]], \alpha^2(y)
\]
\[
+ (-1)^{|x||y|+|t|(|x|+|y|)}\alpha([t, y], [x, z]) - (-1)^{|x||y|+|t|(|x|+|y|)}[\alpha^2(t), [\alpha(y), [x, z]]
\]
\[
- (-1)^{|x||y|+|t|(|x|+|y|)}\alpha([x, y], [t, z]), \alpha^2(y)
\]
In the last equality above, we use the multiplicativity of \(\alpha\) and the super anti-symmetry of \([-,-]\). Likewise, the right-hand side of the identity (3.11) is
\[
(-1)^{|t||z|}\tilde{J}_\mathcal{A}(x, y, z), \alpha^2(t)
\]
\[
= (-1)^{|t||x|}[\alpha(x), [y, z]) + (-1)^{|x||y|+|t|(|x|+|y|)}[\alpha(x), [t, z]], \alpha^2(t)
\]
\[
- (-1)^{|x||y|+|t|(|x|+|y|)}[\alpha^2(t), [\alpha(y), [x, z]]
\]
\[
- (-1)^{|x||y|+|t|(|x|+|y|)}\alpha([x, y], [t, z]), \alpha^2(t)
\]
Since the two summands \((-\alpha^2(x), [\alpha(y), [t, z]])\) and \((-(-1)^{|x||y|+|t|(|x|+|y|)}[\alpha^2(t), [\alpha(y), [x, z]]\)) appears on both sides of (3.11), the above calculation and a rearrangement of terms imply the equivalence between (3.11) and (3.12).

To state our next result, we need the following Lemma.

**Lemma 3.2.** Let \((\mathcal{A}, [-,-], \alpha)\) be a multiplicative Hom-superalgebra. Then we have
\[
\tilde{J}_\mathcal{A} \circ \alpha^{\otimes 3} = \alpha \circ \tilde{J}_\mathcal{A}
\]
and
\[
\tilde{J}_\mathcal{A}^n = \alpha^{2(2^n-1)} \circ \tilde{J}_\mathcal{A}
\]
for all \(n \geq 0\).

**Proof.** Let \(x, y, z \in \mathcal{H}(\mathcal{A})\). We have
\[
\tilde{J}_\mathcal{A}(\alpha(x), \alpha(y), \alpha(z)) = ([\alpha(x), \alpha(y)], \alpha^2(z) - [\alpha^2(x), [\alpha(y), \alpha(z)] - (-1)^{|y||z|}[\alpha(x), [\alpha(y), \alpha(z)]], \alpha^2(y)]
\]
\[
= \alpha([x, y], [x, z]) - \alpha([\alpha(x), [y, z]]) - (-1)^{|y||z|}\alpha([x, [z], \alpha(y)]) \text{ (by multiplicativity of } \alpha)
\]
\[
= \alpha \circ \tilde{J}_\mathcal{A}(x, y, z).
\]
So condition (3.13) holds. For (3.14), we have
\[
\tilde{J}_\mathcal{A}^n(x, y, z) = [[x, y]^{(n)}, \alpha^{2^n}(z)]^{(n)} - \alpha^{2^n}(x), [y, z]^{(n)}(n) - (-1)^{|y||z|}[\alpha^{2^n}(x), [x, z]^{(n)}, \alpha^{2^n}(y)]^{(n)}
\]
\[
= \alpha^{2^n-1} \circ [\alpha^{2^n-1}([x, y]), \alpha^{2^n}(z)] - \alpha^{2^n-1} \circ [\alpha^{2^n}(x), \alpha^{2^n-1}([y, z])]
\]
\[
- (-1)^{|y||z|}\alpha^{2^n-1} \circ [\alpha^{2^n-1}([x, z]), \alpha^{2^n}(y)]
\]
\[
= \alpha^{2(2^n-1)} \circ \tilde{J}_\mathcal{A}(x, y, z),
\]
where the last equality follows from the multiplicativity of \(\alpha\) with respect to the bracket \([-,-]\). \(\square\)

The following result shows that the category of Hom-Malcev superalgebras is closed under taking derived Hom-superalgebras.

**Proposition 3.2.** Let \((\mathcal{A}, [-,-], \alpha)\) be a multiplicative Hom-Malcev superalgebra. Then the derived Hom-superalgebra \(\mathcal{A}^n = (\mathcal{A}, [-,-]^{(n)}, \alpha^{2^n-1} \circ [-,-], \alpha^{2^n})\) is also a Hom-Malcev superalgebra for any \(n \geq 0\).
Proof. Since $\mathcal{A}^0 = \mathcal{A}$, $\mathcal{A}^1 = (\mathcal{A}, [-,-]^{(1)} = \alpha \circ [-,-], \alpha^2)$, and $\mathcal{A}^{n+1} = (\mathcal{A}^n)^1$, by an induction argument it is enough to prove the case $n = 1$.

To show that $\mathcal{A}^1$ is a Hom-Malcev superalgebra, first note that $[-,-]^{(1)}$ is super skewsymmetric because $[-,-]$ is super skewsymmetric and $\alpha$ is linear. Since $\alpha^2$ is multiplicative with respect to $[-,-]^{(1)}$, it remains to show the Hom-Malcev super-identity for $\mathcal{A}^1$. For all $x, y, z$ and $t$ in $\mathcal{H}(\mathcal{A})$, we have

$$J_{\mathcal{A}^1}(\alpha^2(x), \alpha^2(y), [t, z]^{(1)}) + (-1)^{|x||y|+|t||[x]+|y]}J_{\mathcal{A}^1}(\alpha^2(t), \alpha^2(y), [x, z]^{(1)})$$

$$= \alpha^2 \circ \left( J_{\mathcal{A}^1}(\alpha^2(x), \alpha^2(y), \alpha([t, z])) + (-1)^{|x||y|+|t||[x]+|y|} J_{\mathcal{A}^1}(\alpha^2(t), \alpha^2(y), \alpha([x, z])) \right)$$

$$= \alpha^3 \circ \left( J_{\mathcal{A}^1}(\alpha(x), \alpha(y), [t, z]) + (-1)^{|x||y|+|t||[x]+|y|} J_{\mathcal{A}^1}(\alpha(t), \alpha(y), [x, z]) \right)$$

$$= \alpha^3 \circ \left( (-1)^{|t||z|} J_{\mathcal{A}^1}(x, y, z), \alpha^2(t) \right) + (-1)^{|x||y|+|t||[x]+|y|} J_{\mathcal{A}^1}(\alpha^2(t), \alpha^2(x))$$

$$= (-1)^{|t||z|} J_{\mathcal{A}^1}(x, y, z), (\alpha^2(t))^{(1)} + (-1)^{|x||y|+|t||[x]+|y|} j_{\mathcal{A}^1}(\alpha^2(t), \alpha^2(x))^{(1)}$$

This establishes the Hom-Malcev super-identity in $\mathcal{A}^1$ and finishes the proof. □

The next result shows that given a Hom-Malcev superalgebra and an even alternative superalgebra morphism, the induced Hom-superalgebra is a Hom-Malcev superalgebra.

**Theorem 3.1.** Let $(\mathcal{A}, [-,-], \alpha)$ be a Hom-Malcev superalgebra and $\beta : \mathcal{A} \rightarrow \mathcal{A}$ be an even Malcev superalgebra endomorphism. Then $\mathcal{A}_\beta = (\mathcal{A}, [-,-], \beta \circ [-,-], \beta \circ \alpha)$ is a Hom-Malcev superalgebra.

**Proof.** Since $[-,-]_\beta$ is super skewsymmetric, it remains to prove the Hom-Malcev super-identity (3.9) for $\mathcal{A}_\beta$. Here we regard $\mathcal{A}$ as the Hom-Malcev superalgebra $(\mathcal{A}, [-,-], 1\mathcal{D})$ with identity twisting map. For any superalgebra $(\mathcal{A}, [-,-])$, by the multiplicativity of $\beta$ with respect to $[-,-]$, we have

$$[-,-]_\beta \circ ([-,-]_\beta \otimes \beta \alpha) = \beta^2 \circ [-,-] \circ ([-,-] \otimes \beta)$$

and

$$[-,-] \circ ([-,-] \otimes 1\mathcal{D}) \circ \beta \otimes 3 \beta = \beta \circ [-,-] \circ ([-,-] \otimes 1\mathcal{D}).$$

Pre-composing these identities with the cyclic sum $(1\mathcal{D} + \sigma + \sigma^2)$ (3.5) (which commutes with $\beta \otimes 3 \beta$), we obtain

$(3.15)$  
$$J_{\mathcal{A}_\beta} = \beta^2 \circ J_\mathcal{A}$$

and

$(3.16)$  
$$J_\mathcal{A} \circ \beta \otimes 3 \beta = \beta \circ J_\mathcal{A}.$$

To prove the Hom-Malcev super-identity for $\mathcal{A}_\beta$, we compute

$$J_{\mathcal{A}_\beta}(\beta\alpha(x), \beta\alpha(y), [t, z]_\beta) + (-1)^{|x||y|+|t||[x]+|y|}J_{\mathcal{A}_\beta}(\beta\alpha(t), \beta\alpha(y), [x, z]_\beta)$$

$$= \beta^2 \circ \left( J_{\mathcal{A}_\beta}(\beta\alpha(x), \beta\alpha(y), \beta([t, z])) + (-1)^{|x||y|+|t||[x]+|y|} J_{\mathcal{A}_\beta}(\beta\alpha(t), \beta\alpha(y), \beta([x, z])) \right)$$

$$= \beta^3 \circ \left( J_{\mathcal{A}_\beta}(\alpha(x), \alpha(y), [t, z]) + (-1)^{|x||y|+|t||[x]+|y|} J_{\mathcal{A}_\beta}(\alpha(t), \alpha(y), [x, z]) \right)$$

$$= \beta^3 \circ \left( (-1)^{|t||z|} J_{\mathcal{A}_\beta}(x, y, z), \alpha^2(t) \right) + (-1)^{|x||y|+|t||[x]+|y|} J_{\mathcal{A}_\beta}(\alpha^2(t), \alpha^2(x))$$

This shows that the Hom-Malcev super-identity holds in $\mathcal{A}_\beta$. □

**Example 3.2.** (Hom-Malcev superalgebra of dimension 4). Let $M^3(3,1)$ be a non-Lie Malcev superalgebra defined with respect to a basis $\{e_1, e_2, e_3, e_4\}$, where $(M^3(3,1))_0 = span\{e_1, e_2, e_3\}$ and $(M^3(3,1))_1 = span\{e_4\}$, by the following multiplication table (see [2]):

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|------|------|------|------|
| $e_1$ | 0     | 0     | $-e_1$ | 0     |
| $e_2$ | 0     | 0     | 2$e_2$ | 0     |
| $e_3$ | $e_1$ | $-2e_2$ | 0     | $-e_4$ |
| $e_4$ | 0     | 0     | $e_4$  | $e_1 + e_2$ |
The even superalgebra endomorphism $\alpha_1$ with respect to the same basis is defined by the matrix

$$\alpha_1 = \begin{pmatrix} a^2 & 0 & b & 0 \\ 0 & a^2 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

where $a, b, c \in \mathbb{K}$. For each such even superalgebra morphism $\alpha_1 : M^3(3,1) \to M^3(3,1)$, by Theorem 3.1, there is a Hom-Malcev superalgebra $M^3(3,1)_{\alpha_1} = (M^3(3,1), [-, -]_{\alpha_1}, \alpha_1)$ whose multiplication table is:

| $[,]_{\alpha_1}$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|------------------|-------|-------|-------|-------|
| $e_1$            | 0     | 0     | $-a^2 e_1$ | 0     |
| $e_2$            | 0     | 0     | $2a^2 e_2$ | 0     |
| $e_3$            | $a^4 e_1$ | $-2a^4 e_2$ | 0     | $-ae_4$ |
| $e_4$            | 0     | 0     | $ae_4$    | $a^4(e_1 + e_2)$ |

Note that $(M^3(3,1), [-, -]_{\alpha_1})$ is in general not a Hom-Lie superalgebra. Indeed, we have

$$\bar{J}(e_3, e_4, e_4) \neq 0$$

Then $\bar{J}(e_3, e_4, e_4) \neq 0$ whenever $\alpha_1 \neq 0$. So $(M^3(3,1), [-, -]_{\alpha_1})$ is not a Hom-Lie superalgebra. Also, in general $(M^3(3,1), [-, -]_{\alpha_1})$ is not a Malcev superalgebra. Indeed, on the one hand we have

$$\bar{J}(e_3, e_4, [e_4, e_3]_{\alpha_1}) - \bar{J}(e_4, e_4, [e_3, e_3]_{\alpha_1}) = a^5(2e_2 - e_1)$$

which is not equal to 0 whenever $\alpha_1 \neq 0$. On the other hand, we have

$$[\bar{J}(e_3, e_4, e_3), e_4]_{\alpha_1} - [\bar{J}(e_4, e_4, e_3), e_4]_{\alpha_1} = -(a^6 + a^5)e_1 + (2a^5 - 4d^6)e_2 - a^3e_4,$$

which is not equal to 0 whenever $\alpha_1 \neq 0$. Then

$$\bar{J}(e_3, e_4, [e_4, e_3]_{\alpha_1}) - \bar{J}(e_4, e_4, [e_3, e_3]_{\alpha_1}) \neq [\bar{J}(e_3, e_4, e_3), e_4]_{\alpha_1} - [\bar{J}(e_4, e_4, e_3), e_4]_{\alpha_1}.$$

So $(M^3(3,1), [-, -]_{\alpha_1})$ is not a Malcev superalgebra. Now we provide a twisting of $M^3(3,1)$ into a Hom-Lie superalgebra. For example, consider the class of even superalgebra morphisms $\alpha_2 : M^3(3,1) \to M^3(3,1)$ given by the matrix

$$\alpha_2 = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & c & d \\ 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $b, c, d \in \mathbb{K}$. By Theorem 3.1, $M^3(3,1)_{\alpha_2} = (M^3(3,1), [-, -]_{\alpha_2}, \alpha_2)$ is a Hom-Malcev superalgebra with multiplication table:

| $[,]_{\alpha_2}$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|------------------|-------|-------|-------|-------|
| $e_1$            | 0     | 0     | 0     | 0     |
| $e_2$            | 0     | 0     | 0     | 0     |
| $e_3$            | 0     | 0     | $-d e_4$ | 0     |
| $e_4$            | 0     | $d e_4$ | 0     | 0     |

It turns out that $\bar{J}_{M^3(3,1)_{\alpha_2}} = 0$, which implies that $M^3(3,1)_{\alpha_2}$ is a Hom-Lie superalgebra and clearly $(M^3(3,1), [-, -]_{\alpha_2})$ is a Lie superalgebra.

4. Hom-Alternative Superalgebras Are Hom-Malcev-admissible

The main purpose of this section is to show that every Hom-alternative superalgebra gives rise to a Hom-Malcev superalgebra via the super-commutator bracket (Theorem 4.1). This means that Hom-alternative superalgebras are all Hom-Malcev-admissible superalgebras, generalizing the well-known fact that alternative superalgebras are Malcev-admissible. At the end of this section, we consider a 6-dimensional, (non-Hom-Lie) Hom-Malcev superalgebra arising from the alternative superalgebra (Example 4.1).

**Definition 4.1.** Let $(A, \mu, \alpha)$ be a Hom-superalgebra. Define its super-commutator Hom-superalgebra as the Hom-superalgebra

$$A^- = (A, [-, -], \alpha)$$
where \([x, y] = \mu(x, y) - (-1)^{|x||y|} \mu(y, x)\) for all \(x, y \in \mathcal{H}(A)\).

The multiplication \([- , -]\) is called the super-commutator bracket of \(\mu\). We call a Hom-superalgebra \(A\) Hom-Malcev-admissible (resp. Hom-Lie-admissible \([3]\)) if \(A^\perp\) is a Hom-Malcev (resp. Hom-Lie) superalgebra (Definition 4.1).

It is proved in [3] that, given a Hom-associative superalgebra \(A\), its super-commutator Hom-superalgebra \(A^\perp\) is a Malcev-admissible superalgebra. Also, it is known that the super-commutator superalgebra of any alternative superalgebra is a Malcev-admissible superalgebra. The following main result of this section generalizes both of these facts. It gives us a large class of Hom-Malcev-admissible superalgebras that are in general not Hom-Lie admissible.

**Theorem 4.1.** Every Hom-alternative superalgebra is Hom-Malcev-admissible.

The proof of Theorem 4.1 depends on the Hom-type analogues of some identities on alternative superalgebras, see for classical case [1, 29]. We will first establish some identities about the Hom-associator and the Hom-superalgebras, see for classical case [1, 29]. We will first establish some identities about the Hom-associator and the Hom-superalgebras, see for classical case [1, 29].

**Lemma 4.1.** Let \((A, \mu, \alpha)\) be a Hom-alternative superalgebra. Then the identity

\[
\alpha^2(t) \hat{s}_A(x, y, z) + \hat{s}_A(t, x, y) \alpha^2(z)
\]

holds for all \(x, y, z\) and \(t\) in \(\mathcal{H}(A)\).

The identity (4.17) is said to be the Hom-Teichmüller super-identity (see [4]).

**Proof.** Let \(x, y, z\) and \(t\) in \(\mathcal{H}(A)\), we have

\[
\alpha^2(t) \hat{s}_A(x, y, z) + \hat{s}_A(t, x, y) \alpha^2(z) = \alpha^2(t)(\hat{s}_A(x, y, z) - \hat{s}_A(x, y, z) - (\hat{s}_A(x, y, z) - \hat{s}_A(x, y, z))
\]

In the third equality above, we used the multiplicity of \(\alpha\) twice. \(\square\)

In a Hom-alternative superalgebra, the Hom-associator is an super-alternating map on three variables. We now build a map on four variables using the Hom-associator that is super-alternating in a Hom-alternative superalgebra.

**Definition 4.2.** Let \((A, \mu, \alpha)\) be a Hom-superalgebra. Define the \(\mathbb{Z}_2\)-graded Hom-Bruck-Kleinfeld function \(f : A \times A \times A \times A \to A\) as the even multi-linear map

\[
f(t, x, y, z) = \alpha^2(t)(\hat{s}_A(x, y, z) - \hat{s}_A(x, y, z) - (\hat{s}_A(x, y, z) - \hat{s}_A(x, y, z))
\]

for all \(x, y, z\) and \(t\) in \(\mathcal{H}(A)\). Define another even multi-linear map \(F : A \times A \times A \times A \to A\)

\[
F = \alpha^2 \otimes \hat{s}_A \circ (\text{Id} - \zeta + \zeta^2 - \zeta^3)
\]

where \([- , -]\) is the super-commutator bracket of \(\mu\) and \(\zeta\) is the cyclic permutation

\[
\zeta(t, x, y, z) = (z, t, x, y).
\]

In terms of elements, the map \(F\) is given by

\[
F(t, x, y, z) = \alpha^2(t)(\hat{s}_A(x, y, z) - \hat{s}_A(x, y, z) - (\hat{s}_A(x, y, z) - \hat{s}_A(x, y, z))
\]

for all \(x, y, z\) and \(t\) in \(\mathcal{H}(A)\).
Lemma 4.2. Given a Hom-alternative superalgebra \((A, \mu, \alpha)\), we have:

\[
F = f \circ (\text{Id} - \rho + \rho^2)
\]

where \(\rho = \zeta^3\) is the cyclic permutation \(\rho(t, x, y, z) = (x, y, z, t)\).

In terms of elements, the even map \(F\) is given by

\[
F(t, x, y, z) = f(t, x, y, z) - (-1)^{|t|(|x|+|y|+|z|)}f(x, y, z, t)
\]

\[
+(-1)^{|t|(|x|+|y|+|z|)}f(y, z, t, x)
\]

for all \(x, y, z\) and \(t\) in \(\mathcal{H}(A)\).

Proof. By Lemma 4.1 and definition 4.18, we have

\[
\alpha^2(t)\overline{\text{as}}_A(x, y, z) + \overline{\text{as}}_A(t, x, y)\alpha^2(z) = \overline{\text{as}}_A(tx, \alpha(y), \alpha(z))
\]

\[
-(-1)^{|t|(|x|+|y|+|z|)}\overline{\text{as}}_A(xy, \alpha(z), \alpha(t))
\]

\[
+(-1)^{|t|(|x|+|y|+|z|)}\overline{\text{as}}_A(yz, \alpha(t), \alpha(x))
\]

\[
= f(t, x, y, z) + (-1)^{|t|(|x|+|y|+|z|)}\overline{\text{as}}_A(x, y, z)\alpha^2(t)
\]

\[
+(-1)^{|x|t}\alpha^2(x)\overline{\text{as}}_A(t, y, z) - (-1)^{|t|(|x|+|y|+|z|)}f(x, y, z, t)
\]

\[
-(-1)^{|t|(|y|+|z|)+|x|}f(y, z, t, x)
\]

\[
+(-1)^{|t|(|x|+|y|+|z|)+|z|}\alpha^2(y)\overline{\text{as}}_A(x, z, t, x)
\]

Since the Hom-associator \(\overline{\text{as}}_A\) is super-alternating, we have:

\[
\overline{\text{as}}_A(y, t, x) = (-1)^{|y|(|t|+|x|)}\overline{\text{as}}_A(y, t, x),
\]

\[
\overline{\text{as}}_A(x, z, t) = (-1)^{|z|(|t|+|x|)}\overline{\text{as}}_A(z, t, x),
\]

and

\[
\overline{\text{as}}_A(t, y, z) = (-1)^{|y|(|t|+|x|)}\overline{\text{as}}_A(y, t, z).
\]

Therefore, rearranging terms in the above equality, we have:

\[
[\alpha^2(t), \overline{\text{as}}_A(x, y, z)] = \alpha^2(t)\overline{\text{as}}_A(x, y, z)
\]

\[
-(-1)^{|z|(|t|+|x|+|y|)}\alpha^2(z)\overline{\text{as}}_A(t, x, y)
\]

\[
= \overline{\text{as}}_A(x, y, z)\alpha^2(t)
\]

\[
-(-1)^{|t|(|x|+|y|+|z|)}\overline{\text{as}}_A(x, y, z)\alpha^2(t),
\]

\[
+(-1)^{|t|(|x|+|y|+|z|)}\overline{\text{as}}_A(y, t, x)
\]

\[
+(-1)^{|t|(|x|+|y|+|z|)}\overline{\text{as}}_A(z, t, x)\alpha^2(y),
\]

and

\[
-(-1)^{|t|(|x|+|y|+|z|)}\alpha^2(x)\overline{\text{as}}_A(y, z, t)
\]

\[
= (-1)^{|x|t}\alpha^2(x)\overline{\text{as}}_A(y, z, t)
\]

\[
+(-1)^{|t|(|x|+|y|+|z|)}\overline{\text{as}}_A(y, z, t)\alpha^2(x).
\]

Then, we obtain \(F = f \circ (\text{Id} - \rho + \rho^2)\) in the explicit form (4.20). \(\square\)

Proposition 4.1. Let \((A, \mu, \alpha)\) be a Hom-alternative superalgebra. Then the \(\mathbb{Z}_2\)-graded Hom-Bruck-Kleinfeld function \(f\) is super-alternating.
Proof. Let $x, y, z$ and $t$ in $\mathcal{H}(A)$, by definition (4.20) we have

\[ F(t, x, y, z) + (-1)^{|t|(|x|+|y|+|z|)} F(x, y, z, t) \]

\[ = [\alpha^2(t), \tilde{\sigma}_A(t, x, y)] - (-1)^{|\alpha^2(t)|(|x|+|y|+|z|)} [\alpha^2(z), \tilde{\sigma}_A(t, x, y)] \]

\[ + (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(y), \tilde{\sigma}_A(t, y, z)] - (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(x), \tilde{\sigma}_A(t, x, y)] \]

\[ + (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(y), \tilde{\sigma}_A(t, y, z)] - (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(y), \tilde{\sigma}_A(t, z, x)] \]

\[ = 0. \]

In other hand by Lemma (4.2), we have:

\[ F(t, x, y, z) = f(t, x, y, z) - (-1)^{|t|(|x|+|y|+|z|)} f(x, y, z, t) \]

\[ = (-1)^{|t|(|x|+|y|+|z|)} f(y, z, t, x). \]

Then

\[ F(t, x, y, z) + (-1)^{|t|(|x|+|y|+|z|)} F(x, y, z, t) = f(t, x, y, z) + (-1)^{|t|(|x|+|y|+|z|)} f(z, t, x, y) \]

\[ = 0, \]

which implies that $f$ is super-alternating.

Next, we provide further properties of the $\mathbb{Z}_2$–graded Hom-Bruck-Kleinfeld function $f$ (Definition (4.18)). The following result gives two characterizations of the $\mathbb{Z}_2$–graded Hom-Bruck-Kleinfeld function in a Hom-alternative superalgebra $(A, \mu, \alpha)$.

**Corollary 4.1.** Let $(A, \mu, \alpha)$ be a Hom-alternative superalgebra. Then the $\mathbb{Z}_2$–graded Hom-Bruck-Kleinfeld function $f$ satisfies

\[ f(t, x, y, z) = \frac{1}{3} F(t, x, y, z) \]

(4.21)

\[ = \tilde{\sigma}_A([t, x], \alpha(y), \alpha(z)) + (-1)^{|\alpha^2(t)|(|x|+|y|+|z|)} \tilde{\sigma}_A([y, z], \alpha(t), \alpha(x)) \]

for all $x, y, z$ and $t$ in $\mathcal{H}(A)$, where $[x, y] = xy - (-1)^{|x||y|}yx$.

**Proof.** by Lemma (4.2), we have $F = f \circ (Id - \rho + \rho^2)$ but $f$ is super-alternating, then $F = 3f$. This proves the first identity in (4.21). It remains to prove that $f$ is equal to the last entry in (4.21).

Since $f$ is super-alternating, from its definition (4.18), we have

\[ 2f(t, x, y, z) = f(t, x, y, z) - (-1)^{|x||t|} f(x, t, y, z) \]

\[ = \tilde{\sigma}_A([t, x], \alpha(y), \alpha(z)) - (-1)^{|\alpha^2(t)|(|x|+|y|+|z|)} \tilde{\sigma}_A(x, y, z) \alpha^2(t) \]

\[ - (-1)^{|x||t|} \alpha^2(t) \tilde{\sigma}_A(t, y, z) - (-1)^{|x||t|} \tilde{\sigma}_A(t, x, y) \alpha^2(z) \]

\[ + (-1)^{|x||t|} \tilde{\sigma}_A(t, y, z) \alpha^2(x) + \alpha^2(t) \tilde{\sigma}_A(x, y, z). \]

Rearranging terms we obtain

\[ \tilde{\sigma}_A([t, x], \alpha(y), \alpha(z)) = (-1)^{|x||t|} [\alpha^2(x), \tilde{\sigma}_A(t, y, z)] - [\alpha^2(t), \tilde{\sigma}_A(x, y, z)] + 2f(t, x, y, z) \]

(4.22)

\[ = (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(x), \tilde{\sigma}_A(y, z, t)] \]

\[ - \alpha^2(t), \tilde{\sigma}_A(x, y, z)] + 2f(t, x, y, z), \]

in which $\tilde{\sigma}_A(t, y, z) = (-1)^{|t|(|y|+|z|)} \tilde{\sigma}_A(y, z, t)$ because $\tilde{\sigma}_A$ is super-alternating. Interchanging $(t, x)$ with $(y, z)$ in (4.22) and using the super-derivativity of $f$, we obtain

\[ (-1)^{|t|(|x|+|y|+|z|)} \tilde{\sigma}_A([y, z], \alpha(t), \alpha(x)) = (-1)^{|y||t|(|x|+|y|+|z|)} [\alpha^2(z), \tilde{\sigma}_A(t, x, y)] \]

\[ - (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(y), \tilde{\sigma}_A(z, t, x)] \]

\[ + 2(-1)^{|t|(|x|+|y|+|z|)} f(y, z, t, x) \]

\[ = (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(z), \tilde{\sigma}_A(t, x, y)] \]

\[ - (-1)^{|t|(|x|+|y|+|z|)} [\alpha^2(y), \tilde{\sigma}_A(z, t, x)] \]

\[ + 2f(t, x, y, z). \]

(4.23)
Adding (4.22) and (4.23), we have:
\[
\tilde{a}s_A([t, x], \alpha(y), \alpha(z)) + (-1)^{|t|+|x|+|y|+|z|}a\tilde{s}_A([y, z], \alpha(t), \alpha(x)) = (4f - F)(t, x, y, z).
\]
Since \( F = 3f \). This proves that \( f \) is equal to to the last entry in (4.24).

The following result is the Hom-type analogue of part of [14] and says that every Hom-alternative superalgebra satisfies a variation of the Hom-Malcev super-identity (3.9) in which the Hom-super-Jacobian is replaced by the Hom-associator.

**Proposition 4.2.** Let \((A, \mu, \alpha)\) be a Hom-alternative superalgebra. Then
\[
2[\alpha^2(t), \tilde{a}s_A(x, y, z)] = \tilde{a}s_A(\alpha(t), \alpha(x), [y, z]) + (-1)^{|x||y||z|}\tilde{a}s_A(\alpha(t), \alpha(y), [z, x]) + (-1)^{|z||x||y|}\tilde{a}s_A(\alpha(t), \alpha(z), [x, y])
\]
(4.24)
for all \( x, y, z \) and \( t \) in \( \mathcal{H}(A) \), where \([-,-]\) is the super-commutator bracket.

Next we consider the relationship between the Hom-associator (6.1) in a Hom-superalgebra \((A, \mu, \alpha)\) and the Hom-super-Jacobian (6.32) in its super-commutator Hom-superalgebra \( A^- = (A, [-,-], \alpha) \) (Definition 4.1).

**Lemma 4.3.** Let \((A, \mu, \alpha)\) be a Hom-superalgebra. Then
\[
\tilde{J}_{A^-}(x, y, z) = \tilde{a}s_A(x, y, z) + (-1)^{|x||y||z|}\tilde{a}s_A(y, z, x) + (-1)^{|z||x||y|}\tilde{a}s_A(z, x, y)
\]
\[
+ (-1)^{|z||y||x|}\tilde{a}s_A(z, y, x) + (-1)^{|x||y||z|}\tilde{a}s_A(y, x, z) + (-1)^{|z||x||y|}\tilde{a}s_A(z, y, x)
\]
\[
+ (-1)^{|y||z||x|}\tilde{a}s_A(x, z, y) + (-1)^{|x||y||z|}\tilde{a}s_A(x, y, z) + (-1)^{|z||x||y|}\tilde{a}s_A(z, x, y)
\]
for all \( x, y, z \) in \( \mathcal{H}(A) \).

**Proof.** Let \((A, \mu, \alpha)\) be a Hom-superalgebra. Then for all \( x, y, z \) in \( \mathcal{H}(A) \), we have
\[
\tilde{J}_{A^-}(x, y, z) = [[x, y], \alpha(z)] - [\alpha(x), [y, z]] - (-1)^{|y||z|}[x, z], \alpha(y)]
\]
\[
= [xy - (-1)^{|x||y|}yx, \alpha(z)] - [\alpha(x), yz - (-1)^{|y||z|}zy] - (-1)^{|y||z|}[xz - (-1)^{|z||y||z|}zx, \alpha(y)]
\]
\[
= \left(\begin{array}{c}
(xy)\alpha(z) - (-1)^{|x||y|+|y||z|}\alpha(z)(xy) - (-1)^{|x||y|}\alpha(z)(xy) + (-1)^{|z||y||z|}(xy)\alpha(z)
\end{array}\right)
\]
\[
- \left(\begin{array}{c}
(-1)^{|y||z||z|}(xy)\alpha(z) - (-1)^{|y||z||z|}(xy)\alpha(z) - (-1)^{|x||y||z|}(xy)\alpha(z) + (-1)^{|z||y||z|}(xy)\alpha(z)
\end{array}\right)
\]
\[
= \tilde{a}s_A(x, y, z) + (-1)^{|x||y||z|}\tilde{a}s_A(y, z, x) + (-1)^{|z||x||y|}\tilde{a}s_A(z, x, y)
\]
\[
+ (-1)^{|x||y||z|}\tilde{a}s_A(x, z, y) + (-1)^{|y||z||x|}\tilde{a}s_A(y, x, z) + (-1)^{|x||y||z|}\tilde{a}s_A(x, y, z) + (-1)^{|z||x||y|}\tilde{a}s_A(z, x, y)
\]
which completes the proof.

**Proposition 4.3.** In a Hom-alternative superalgebra \((A, \mu, \alpha)\), the identity
\[
\tilde{J}_{A^-} = 6\tilde{a}s_A
\]
holds.

**Proof.** Since the Hom-associator \( \tilde{a}s_A \) is super-alternating, with the notations in Lemma 4.3, we have:
\[
(-1)^{|x||y||z|}\tilde{a}s_A(y, z, x) = \tilde{a}s_A(x, y, z),
\]
\[
(-1)^{|z||x||y|}\tilde{a}s_A(z, x, y) = \tilde{a}s_A(x, y, z),
\]
\[
-(-1)^{|x||y|}\tilde{a}s_A(y, x, z) = \tilde{a}s_A(x, y, z),
\]
\[
-(-1)^{|y||z|}\tilde{a}s_A(x, z, y) = \tilde{a}s_A(x, y, z),
\]
\[
-(-1)^{|z||x||y|}\tilde{a}s_A(z, y, x) = \tilde{a}s_A(x, y, z).
\]
Then the result follows from Lemma 4.3.
Now, we are ready to prove Theorem 4.1.

**Proof.** (Theorem 4.1) Let \((A, \mu, \alpha)\) be a Hom-alternative superalgebra and \(A^- = (A, [\cdot, \cdot], \alpha)\) be its super-commutator Hom-superalgebra. The super-commutator bracket \(\cdot, \cdot\) is a super-skew-symmetric. Thus, it remains to show that the Hom-Malcev super-identity \((3.9)\) holds in \(A^-\), that is

\[
2[\alpha^2(t), J_{A^-}(x, y, z)] = J_{A^-}(\alpha(t), \alpha(x), [y, z]) + (-1)^{|x||y||z|} J_{A^-}(\alpha(t), \alpha(y), [z, x]) + (-1)^{|x||y|} J_{A^-}(\alpha(t), \alpha(z), [x, y]).
\]

To prove this, we compute

\[
2[\alpha^2(t), J_{A^-}(x, y, z)] = 2[\alpha^2(t), 6\tilde{\alpha} A^-(x, y, z)] \quad \text{(by Proposition 4.3)}
\]

\[
= 6\left(\tilde{A} \tilde{\alpha}(\alpha(t), \alpha(x), [y, z]) + (-1)^{|x||y||z|}\tilde{A} \tilde{\alpha}(\alpha(t), \alpha(y), [z, x]) + (-1)^{|x||y|}\tilde{A} \tilde{\alpha}(\alpha(t), \alpha(z), [x, y])\right) \quad \text{(by Proposition 4.2)}
\]

\[
= J_{A^-}(\alpha(t), \alpha(x), [y, z]) + (-1)^{|x||y||z|} J_{A^-}(\alpha(t), \alpha(y), [z, x]) + (-1)^{|x||y|} J_{A^-}(\alpha(t), \alpha(z), [x, y]) \quad \text{(by Proposition 4.3)}.
\]

Example 4.1. In this example, we describe a Hom-alternative superalgebra (hence a Hom-Malcev-admissible superalgebra by Theorem (4.1)) that is not Hom-Lie-admissible and not alternative. Let \(F\) be a field of characteristic 3 and let \(B(4, 2) = B_0 \oplus B_1\) be a 6-dimensional simple alternative superalgebra (see [14, 23]), where \(B_0 = M_2(F)\) and \(B_1 = F \cdot m + F \cdot m_2\) is the 2-dimensional irreducible Cayley bi-module over \(B_0\). Now \(B(4, 2)\) has basis \(\{e_{11}, e_{12}, e_{21}, e_{22}, m_1, m_2\}\), where the product, with respect to the basis, is given by the following table:

| \mu | \(e_{11}\) | \(e_{12}\) | \(e_{21}\) | \(e_{22}\) | \(m_1\) | \(m_2\) |
|-----|-----|-----|-----|-----|-----|-----|
| \(e_{11}\) | 0 | 0 | 0 | 0 | \(e_{11}\) | \(m_1\) |
| \(e_{12}\) | 0 | 0 | 0 | 0 | \(e_{12}\) | 0 |
| \(e_{21}\) | \(e_{21}\) | 0 | 0 | 0 | \(e_{21}\) | 0 |
| \(e_{22}\) | 0 | 0 | 0 | 0 | \(e_{22}\) | 0 |
| \(m_1\) | 0 | 0 | 0 | 0 | \(m_1\) | \(-e_{21}\) |
| \(m_2\) | 0 | 0 | 0 | 0 | \(-m_1\) | \(-e_{22}\) |

The even superalgebra endomorphisms \(\alpha\) with respect to the basis \(\{e_{11}, e_{12}, e_{21}, e_{22}, m_1, m_2\}\) are defined by the matrix

\[
\alpha = \begin{pmatrix}
1 & 0 & \frac{-a}{b} & 0 & 0 & 0 \\
0 & b & \frac{-a}{b} & -a & 0 & 0 \\
0 & 0 & \frac{b}{a} & 0 & 0 & 0 \\
0 & 0 & \frac{a}{b} & 1 & 0 & 0 \\
0 & 0 & 0 & \pm \sqrt{\frac{1}{F}} & 0 & 0 \\
0 & 0 & 0 & \pm \frac{a}{b} & \pm b \sqrt{\frac{1}{F}} & \frac{a}{b}
\end{pmatrix}
\]
where \( a, b \neq 0 \). According to Theorem [2.1], the even linear map \( \alpha \) on the following multiplication

\[
\begin{align*}
\mu_\alpha(e_{11}, e_{11}) &= e_{11} + ae_{12}, \quad \mu_\alpha(e_{11}, e_{12}) = \mu_\alpha(e_{11}, e_{22}) = \mu_\alpha(e_{11}, m_1) = 0, \\
\mu_\alpha(e_{11}, e_{21}) &= -\frac{a}{b} e_{11} - \frac{1}{b} e_{12} + \frac{1}{b} e_{21} + \frac{a}{b} e_{22}, \quad \mu_\alpha(e_{11}, m_2) = \pm b \sqrt{\frac{1}{b} m_2}, \\
\mu_\alpha(e_{12}, e_{11}) &= be_{12}, \quad \mu_\alpha(e_{12}, e_{12}) = \mu_\alpha(e_{12}, e_{22}) = \mu_\alpha(e_{12}, m_2) = 0, \\
\mu_\alpha(e_{12}, e_{21}) &= -ae_{12} + \frac{1}{b} e_{21} + e_{22}, \quad \mu_\alpha(e_{12}, m_1) = \pm b \sqrt{\frac{1}{b} m_2}, \\
\mu_\alpha(e_{21}, e_{12}) &= e_{11} + ae_{12}, \quad \mu_\alpha(e_{21}, e_{11}) = \mu_\alpha(e_{21}, e_{21}) = \mu_\alpha(e_{21}, m_1) = 0, \\
\mu_\alpha(e_{21}, e_{22}) &= -\frac{a}{b} e_{11} - \frac{1}{b} e_{12} + \frac{1}{b} e_{21} + \frac{a}{b} e_{22}, \quad \mu_\alpha(e_{21}, m_2) = \pm \sqrt{\frac{1}{b} m_1} \pm \mu_\sqrt{\frac{1}{b} m_2}, \\
\mu_\alpha(e_{22}, e_{11}) &= \mu_\alpha(e_{22}, e_{21}) = \mu_\alpha(e_{22}, m_2) = 0, \\
\mu_\alpha(e_{22}, e_{12}) &= be_{12}, \quad \mu_\alpha(e_{22}, e_{22}) = -ae_{12} + e_{22}, \quad \mu_\alpha(e_{22}, m_1) = \pm b \sqrt{\frac{1}{b} m_1} \pm b \sqrt{\frac{1}{b} m_2}, \\
\mu_\alpha(m_1, e_{11}) &= \pm \sqrt{\frac{1}{b} m_1} \pm a \sqrt{\frac{1}{b} m_2}, \quad \mu_\alpha(m_1, e_{11}) = \mu_\alpha(m_1, e_{21}) = 0, \\
\mu_\alpha(m_1, e_{12}) &= \pm b \sqrt{\frac{1}{b} m_2}, \quad \mu_\alpha(m_1, m_1) = \frac{a}{b} e_{11} + \frac{1}{b} e_{12} - \frac{1}{b} e_{21} - \frac{a}{b} e_{22}, \quad \mu_\alpha(m_1, m_2) = ae_{12} - e_{22}, \\
\mu_\alpha(m_2, e_{21}) &= \pm \sqrt{\frac{1}{b} m_1} \pm \sqrt{\frac{1}{b} m_2}, \quad \mu_\alpha(m_2, e_{11}) = \mu_\alpha(m_2, e_{21}) = 0, \\
\mu_\alpha(m_2, e_{22}) &= \pm b \sqrt{\frac{1}{b} m_2}, \quad \mu_\alpha(m_2, m_1) = e_{11} + ae_{12}, \quad \mu_\alpha(m_2, m_2) = be_{12},
\end{align*}
\]

where \( a, b \neq 0 \). determine a 6-dimensional Hom-alternative (hence Hom-Malcev-admissible and Hom-flexible) superalgebra.

Note that \( B(4, 2)_{\alpha} \) is not is not alternative because

\[
a s_{B(4, 2)_{\alpha}}(e_{11}, e_{21}, e_{22}) + a s_{B(4, 2)_{\alpha}}(e_{21}, e_{11}, e_{22}) = \frac{a}{b} (e_{11} + e_{22}) \neq 0.
\]

Also, \( B(4, 2)_{\alpha} \) is not Hom-Lie-admissible, that is \( B(4, 2)_{\alpha}^{-} \) is not Hom-Lie superalgebra. Indeed, we have

\[
\begin{align*}
\overline{J}_{B(4, 2)_{\alpha}}(e_{11}, e_{21}, e_{22}) &= 6 a s_{B(4, 2)_{\alpha}}(e_{11}, e_{21}, e_{22}) \\
&= 6 \frac{a}{b} (e_{11} + e_{22}) \\
&\neq 0.
\end{align*}
\]

Therefore, \( B(4, 2)_{\alpha} \) is a Hom-alternative (and hence Hom-Malcev-admissible and Hom-flexible) superalgebra that is neither alternative nor Hom-Lie-admissible. Also, \( (B(4, 2), [-, -]_{\alpha}) \) is not a Malcev superalgebra. Indeed, on the one hand we have

\[
[\overline{J}(e_{11}, e_{12}, e_{21}), e_{22}]_{\alpha} + [\overline{J}(e_{22}, e_{12}, e_{21}), e_{11}]_{\alpha} = \left( \frac{2a}{b^2} + 1 + \frac{1}{b^2} \right)e_{11} + (4ab^2 - 2a + \frac{3}{b} + \frac{1}{b^2})e_{12}
\]

\[
+ \left( \frac{1}{b^2} + 1 - \frac{2}{b^2} \right)e_{21} + (1 - \frac{2a}{b} + \frac{1}{b^2})e_{22},
\]

on the other hand, we have

\[
\overline{J}(e_{11}, e_{12}, [e_{22}, e_{21}]_{\alpha}) + \overline{J}(e_{22}, e_{12}, [e_{11}, e_{21}]_{\alpha}) = \left( \frac{2a}{b^2} - 2 + \frac{2}{b^2} \right)e_{11} + (2 + \frac{2a}{b^2} - \frac{2}{b^2})e_{22}
\]

\[
+ \frac{2}{b^2}e_{21} + (2b^2 + a + \frac{4a}{b^2} - \frac{2}{b^3} + 2ab + \frac{5a}{b})e_{12},
\]

where \( a, b \neq 0 \). Then

\[
[\overline{J}(e_{11}, e_{12}, e_{21}), e_{22}]_{\alpha} + [\overline{J}(e_{22}, e_{12}, e_{21}), e_{11}]_{\alpha} \neq \overline{J}(e_{11}, e_{12}, [e_{22}, e_{21}]_{\alpha}) + \overline{J}(e_{22}, e_{12}, [e_{11}, e_{21}]_{\alpha}).
\]

So, \( (B(4, 2), [-, -]_{\alpha}) \) does not satisfy the Malcev super-identity.
5. Hom-Malcev-Admissible Superalgebras

In Theorem (5.1) we showed that every Hom-alternative superalgebra is Hom-Malcev-admissible. The purpose of this section is to introduce and study the class of Hom-flexible, Hom-Malcev-admissible superalgebras. We give several characterizations of Hom-flexible superalgebra that are Hom-Malcev-admissible in terms of the cyclic Hom-associator (Proposition (5.1)). Then we prove the analogue of the construction results, Theorem (2.1) and Theorem (2.1). We consider examples of Hom-flexible, Hom-Malcev-admissible superalgebras that are neither Hom-alternative nor Hom-Lie-admissible.

To state our characterizations of Hom-flexible superalgebras that are Hom-Malcev-admissible, we need the following definitions:

**Definition 5.1.** A **Hom-flexible superalgebra** is a triple $(\mathcal{A}, \mu, \alpha)$ consisting of $\mathbb{Z}_2$-graded vector space $\mathcal{A}$, an even bilinear map $\mu : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ and an even homomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ satisfying the **Hom-flexible super-identity**, that is for any $x, y, z \in \mathcal{H}(\mathcal{A})$,

$$\widetilde{a_{S\mathcal{A}}}(x, y, z) + (-1)^{|x||y|+|z|} \widetilde{a_{S\mathcal{A}}}(y, z, x) + (-1)^{|z||x|+|y|} \widetilde{a_{S\mathcal{A}}}(z, x, y) = 0.$$  

(5.25)

It follows from the definition and by Lemma (2.1) that Hom-alternative superalgebra is Hom-flexible superalgebra. Also, when $\alpha = Id$ in Definition (5.1), we recover the usual notion of flexible superalgebra.

**Definition 5.2.** Let $(\mathcal{A}, \mu, \alpha)$ be any Hom-superalgebra. Define the cyclic **Hom-associator** $S_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ as the even multi-linear map

$$S_{\mathcal{A}}(x, y, z) = \widetilde{a_{S\mathcal{A}}}(x, y, z) + (-1)^{|x||y|+|z|} \widetilde{a_{S\mathcal{A}}}(y, z, x) + (-1)^{|z||x|+|y|} \widetilde{a_{S\mathcal{A}}}(z, x, y),$$

for all $x, y, z \in \mathcal{H}(\mathcal{A})$, where $\widetilde{a_{S\mathcal{A}}}$ is the Hom-associator (2.4).

We will use the following preliminary observations about the relationship between the cyclic Hom-associator and the Hom-super-Jacobian (2.5) of the super-commutator Hom-superalgebra (Definition (1.1)):

**Lemma 5.1.** Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-flexible superalgebra. Then we have

$$J_{\mathcal{A}^-} = 2S_{\mathcal{A}},$$

where $\mathcal{A}^- = (\mathcal{A}, [-, -], \alpha)$ is the super-commutator Hom-superalgebra.

**Proof.** Assume that $x, y, z \in \mathcal{H}(\mathcal{A})$, by Lemma (4.1) we have

$$J_{\mathcal{A}^-}(x, y, z) = \widetilde{a_{S\mathcal{A}}}(x, y, z) + (-1)^{|x||y|+|z|} \widetilde{a_{S\mathcal{A}}}(y, z, x) + (-1)^{|z||x|+|y|} \widetilde{a_{S\mathcal{A}}}(z, x, y)$$

$$= (-1)^{|x||y|+|z|} \widetilde{a_{S\mathcal{A}}}(x, y, z) + (1)^{|z||x|+|y|} \widetilde{a_{S\mathcal{A}}}(y, z, x)$$

$$= (1)^{|z||x|+|y|} \widetilde{a_{S\mathcal{A}}}(x, y, z) + (-1)^{|x||y|+|z|} \widetilde{a_{S\mathcal{A}}}(y, z, x)$$

$$= (1)^{|x||y|+|z|} \widetilde{a_{S\mathcal{A}}}(x, y, z) + (-1)^{|x||y|+|z|} \widetilde{a_{S\mathcal{A}}}(y, z, x)$$

$$= 2S_{\mathcal{A}}$$  

(by Hom-super flexibility).

The following result gives characterizations of Hom-Malcev-admissible superalgebras in terms of cyclic Hom-associator, assuming Hom-flexibility:

**Proposition 5.1.** Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-flexible superalgebra and $\mathcal{A}^- = (\mathcal{A}, [-, -], \alpha)$ be the super-commutator Hom-superalgebra. Then the following statements are equivalent:

1. $\mathcal{A}$ is Hom-Malcev-admissible superalgebra.
2. The equality

$$J_{\mathcal{A}^-}((x, \alpha(x), [t, z]) + (-1)^{|t||x|+|z|} J_{\mathcal{A}^-}(\alpha(t), \alpha(y), [x, z])$$

$$= (1)^{|t||x|} J_{\mathcal{A}^-}(x, y, z), \alpha^2(t) + (-1)^{|t||x|+|z|} J_{\mathcal{A}^-}(t, y, z), \alpha^2(x)]$$

holds for all $x, y, z, t \in \mathcal{H}(\mathcal{A})$.
3. The equality

$$S_{\mathcal{A}}((x, \alpha(x), [t, z]) + (-1)^{|t||x|+|z|} S_{\mathcal{A}}(\alpha(t), \alpha(y), [x, z])$$

$$= (1)^{|t||x|} S_{\mathcal{A}}(x, y, z), \alpha^2(t) + (-1)^{|t||x|+|z|} S_{\mathcal{A}}(t, y, z), \alpha^2(x)]$$

holds for all $x, y, z, t \in \mathcal{H}(\mathcal{A})$.

The following construction results for Hom-flexible and Hom-Malcev-admissible superalgebras are the analogues of Theorems (2.1) and (2.1).
Theorem 5.1. Let \((A, \mu, \alpha)\) be a Hom-Malcev-admissible superalgebra and \(\beta : A \rightarrow A\) be an even Malcev-admissible superalgebra endomorphism. Then \(A_{\beta} = (A, \mu_{\beta} = \beta \circ \mu, \beta \alpha)\) is a Hom-Malcev-admissible superalgebra.

Proof. The super-commutator Hom-superalgebra of \((A, \mu, \alpha)\) is \(A^- = (A, [-, -], \alpha)\), where \([x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(y, x)\) which is a Hom-Malcev superalgebra by assumption. In particular, the Malcev super-identity

\[
\mathcal{J}_A(x, y, [t, z]) + (-1)^{|x||y|+|t||z|+|y||z|} \mathcal{J}_A(t, y, [x, z]) = (-1)^{|t||z|} [\mathcal{J}_A(x, y, z), t] + (-1)^{|x||y|+|t||z|+|x||y|} [\mathcal{J}_A(t, y, z), x]
\]

(5.27) holds. The super-commutator Hom-superalgebra of the induced Hom-superalgebra \(A_{\beta} = (A, [-, -], \beta \alpha)\), where

\[
[x, y]_{\beta} = \mu_{\beta}(x, y) - (-1)^{|x||y|}\mu_{\beta}(y, x)
\]

(5.28) is super skewsymmetric. We must show that \(A_{\beta}^+\) satisfies the Hom-Malcev super-identity. We have

\[
\mathcal{J}^-_{A_{\beta}}(x, y, z) = [[x, y]_{\beta}, \beta \alpha(z)]_{\beta} - [\beta \alpha(x), [y, z]_{\beta}]_{\beta} - (-1)^{|y||z|}[x, z]_{\beta} - [\beta \alpha(y)]_{\beta} = \beta^2 \circ \mathcal{J}^-_{A_{\beta}}(x, y, z).
\]

Therefore, we have:

\[
\mathcal{J}^-_{A_{\beta}}(\beta \alpha(x), \beta \alpha(y), [t, z]) + (-1)^{|x||y|+|t||z|+|y||z|} \mathcal{J}^-_{A_{\beta}}(\beta \alpha(t), \beta \alpha(y), [x, z])
\]

\[
= \beta^2 \circ \left( \mathcal{J}^-_{A_{\beta}}(\beta \alpha(x), \beta \alpha(y), \beta([t, z])) + (-1)^{|x||y|+|t||z|+|y||z|} \mathcal{J}^-_{A_{\beta}}(\beta \alpha(t), \beta \alpha(y), \beta([x, z])) \right)
\]

\[
= \beta^3 \circ \left( \mathcal{J}^-_{A_{\beta}}(\alpha(x), \alpha(y), [t, z]) + (-1)^{|x||y|+|t||z|+|y||z|} \mathcal{J}^-_{A_{\beta}}(\alpha(t), \alpha(y), [x, z]) \right)
\]

\[
= \beta^3 \circ \left( (-1)^{|t||z|} \mathcal{J}^-_{A_{\beta}}(x, y, z), \alpha^2(t) \right) + (-1)^{|x||y|+|t||z|+|t||y|} \mathcal{J}^-_{A_{\beta}}(t, y, z), \alpha^2(x)).
\]

This shows that the Hom-Malcev super-identity holds in \(A_{\beta}^+\).

Theorem 5.2. Let \((A, \mu, \alpha)\) be a Hom-Malcev-admissible superalgebra. Then the derived Hom-superalgebra \(A^n = (A, \mu^{(n)} = \alpha^{2^n-1} \circ \mu, \alpha^{2^n})\) is also a Hom-Malcev-admissible superalgebra for each \(n \geq 0\).

Proof. Assume that \((A, \mu, \alpha)\) is a Hom-Malcev-admissible superalgebra. Note that the super-commutator Hom-superalgebra of the nth derived Hom-superalgebra \(A^n\) is \((A^n)^- = (A, [-, -], \alpha^{2^n})\), where

\[
[x, y]^{(n)} = \mu^{(n)}(x, y) - (-1)^{|x||y|}\mu^{(n)}(y, x)
\]

\[
= \alpha^{2^n-1} \circ (\mu(x, y) - (-1)^{|x||y|}\mu(y, x)) = \alpha^{2^n-1} \circ [x, y].
\]

Thus, we have

\[
(A^n)^- = (A^-)^n
\]

where \(A^- = (A, [-, -], \alpha)\) is the super-commutator Hom-superalgebra of \(A\) and \((A^-)^n\) is its nth derived Hom-superalgebra. Since \([-, -]^{(n)}\) is super skewsymmetric. We must show that \((A^n)^-\) satisfies the Hom-Malcev super-identity. To do that, observe that

\[
\mathcal{J}^-_{(A^n)} = \mathcal{J}^-_{(A^-)}^{(n)} = \alpha^{2^n-1} \circ \mathcal{J}^-_{A^-}.
\]
Proof. \( (\mathcal{A}^n)^{-1} (\mathcal{A}^n) - (\mathcal{A}^n)^{-1} (\mathcal{A}^n) \) we compute as follows
\[
\begin{align*}
\mathcal{J}_{\mathcal{A}^n} (\mathcal{A}^n) (\mathcal{A}^n) (\mathcal{A}^n) - (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) - (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) & = (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) - (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) + (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) \\
& = (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) - (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) + (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n)
\end{align*}
\]
Thus, \( \mathcal{J}_{\mathcal{A}^n} (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) - (\mathcal{A}^n)^{-1} (\mathcal{A}^n) (\mathcal{A}^n) \) is also a Hom-flexible superalgebra for each \( n \geq 0 \).

We have shown that \( (\mathcal{A}^n)^{-1} \) satisfies the Hom-Malcev super-identity. So \( \mathcal{A}^n \) is Hom-Malcev-admissible superalgebra.

\[ \square \]

**Theorem 5.3.**

(1) Let \( (\mathcal{A}, \mu, \alpha) \) be a Hom-flexible superalgebra and \( \beta : \mathcal{A} \longrightarrow \mathcal{A} \) be an even flexible superalgebra endomorphism. Then \( \mathcal{A}_\beta = (\mathcal{A}, \mu, \alpha) \) is a Hom-flexible superalgebra.

(2) Let \( (\mathcal{A}, \mu, \alpha) \) be a Hom-flexible superalgebra. Then the derived Hom-superalgebra \( \mathcal{A}^n = (\mathcal{A}, \mu^{(n)} = \alpha^{-n} \circ \mu, \alpha^n) \) is also a Hom-flexible superalgebra for each \( n \geq 0 \).

**Proof.** For the first assertion, for any superalgebra \( (\mathcal{A}, \mu) \), we regard it as the Hom-superalgebra \( (\mathcal{A}, \mu, \text{Id}) \) with identity twisting map. Then we have:
\[
\begin{align*}
\tilde{\alpha}_{\mathcal{A}_\beta} (x, y, z) &= \mu_\beta (\mu_\beta (x, y), \beta (z)) - \mu_\beta (\beta (x), \mu (y, z)) \\
&= \beta^2 \circ (\mu (\mu (x, y), \alpha (z)) - \mu (\alpha (x), \mu (y, z))) \\
&= \beta^2 \circ \tilde{\alpha}_{\mathcal{A}} (x, y, z).
\end{align*}
\]
Then
\[
\tilde{\alpha}_{\mathcal{A}_\beta} = \beta^2 \circ \tilde{\alpha}_{\mathcal{A}}.
\]

Now for a flexible superalgebra \( (\mathcal{A}, \mu) \), this implies that
\[
\begin{align*}
\tilde{\alpha}_{\mathcal{A}_\beta} (x, y, z) &= \beta^2 \circ \tilde{\alpha}_{\mathcal{A}} (x, y, z) \\
&= - (1) \beta (\beta (x), \alpha (y, z)) - \beta (x, \mu (y, z)) \\
&= - (1) \beta (\beta (x), \alpha (y, z)) - \beta (x, \mu (y, z)).
\end{align*}
\]
Then
\[
\tilde{\alpha}_{\mathcal{A}_\beta} (x, y, z) + (1) \beta (\beta (x), \alpha (y, z)) - \beta (x, \mu (y, z)) = 0.
\]
So \( \mathcal{A}_\beta \) is Hom-flexible superalgebra.

\[ \square \]

**5.1. Examples of Hom-flexible superalgebras.** We construct examples of Hom-flexible superalgebras using Theorem 5.3.

**Example 5.1. (Hom-flexible superalgebra of dimension 3).** We consider the simple flexible superalgebra \( K_3 (\beta, \gamma, \eta) = (K_3)_0 \oplus (K_3)_1 \) (see [20]), where \( (K_3)_0 = \text{span} \{ e_1 \} \) and \( (K_3)_1 = \text{span} \{ e_2, e_3 \} \), endowed with a product given by the following multiplication table:

| \( \mu \) | \( e_1 \) | \( e_2 \) | \( e_3 \) |
|---|---|---|---|
| \( e_1 \) | \( e_1 \) | \( \beta e_2 + \gamma e_3 \) | \( (1 - \beta)e_3 + \eta e_2 \) |
| \( e_2 \) | \( 1 - \beta e_2 - \gamma e_3 \) | \( -2 \gamma e_1 \) | \( 2 \beta e_1 \) |
| \( e_3 \) | \( -\eta e_2 + \beta e_3 \) | \( -2 (1 - \beta) e_1 \) | \( 2 \eta e_1 \) |

Even superalgebra endomorphisms \( \alpha \) with respect to the basis \( \{ e_1, e_2, e_3 \} \) are defined by the matrix
\[
\alpha = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{a}{\gamma} & \frac{b}{\gamma} \\
0 & \frac{-a + \sqrt{1 - \eta} a^2 + 4 \gamma a^2}{2b} & \frac{\gamma a - b}{\gamma}
\end{pmatrix}
\]
with $\beta = 1$ and $\gamma, \eta \neq 0$. According to Theorem [5.3], the even linear map $\alpha$ on the following multiplication

$$
\mu_\alpha(e_1, e_1) = e_1, \quad \mu_\alpha(e_1, e_2) = (a(1 - \beta) - \eta b)e_2 + \left(1 - \beta\right)\frac{-a + \sqrt{-4\gamma \eta + a^2 + 4\gamma \eta a^2}}{2\eta} + (b - \gamma a)e_3,
$$

$$
\mu_\alpha(e_1, e_3) = (\beta \eta - \eta \gamma)e_2 + \left(1 - \beta\right)\frac{a - \sqrt{-4\gamma \eta + a^2 + 4\gamma \eta a^2}}{2} + \beta(\gamma a - b)e_3,
$$

$$
\mu_\alpha(e_2, e_1) = (\beta a + \gamma b)e_2 + \left(1 - \beta\right)\frac{(\gamma a - b) - a + \sqrt{-4\gamma \eta + a^2 + 4\gamma \eta a^2}}{2\eta}e_3,
$$

$$
\mu_\alpha(e_2, e_2) = -2\gamma e_1, \quad \mu_\alpha(e_2, e_3) = -2(1 - \beta)e_1,
$$

$$
\mu_\alpha(e_3, e_1) = (\eta a + \frac{(1 - \beta)\eta b}{\gamma})e_2 + \left(1 - \beta\right)\frac{(\gamma a - b) + a + \sqrt{-4\gamma \eta + a^2 + 4\gamma \eta a^2}}{2}e_3,
$$

$$
\mu_\alpha(e_3, e_3) = 2\beta e_1, \quad \mu_\alpha(e_3, e_3) = 2\eta e_1,
$$

where $a, b \in \mathbb{K}$ and $\gamma, \eta \neq 0$, determine 3-dimensional Hom-flexible, Hom-Malcev-admissible superalgebra.

**Example 5.2.** (Hom-flexible superalgebras of dimension 4). Let $\beta \in \mathbb{K}$ and $t \neq 0$. Define a superalgebra $U = D_t(\beta)$ (see [20]), where $U_0 = \text{span}\{e_1, e_2\}$ and $U_1 = \text{span}\{x, y\}$ by setting

| $e_1$ | $e_2$ | $x$ | $y$ |
|-------|-------|-----|-----|
| $e_1$ | $e_1$ | $e_2$ | $(1 - \beta)y$ |
| $e_2$ | $0$ | $e_2$ | $\beta x$ |
| $x$ | $\beta x$ | $0$ | $1 - \beta y$ |
| $y$ | $(1 - \beta)y$ | $\beta y$ | $2(\beta e_1 + (1 - \beta)te_2)$ |

Then $(U, \mu)$ is a simple flexible, Malcev-admissible superalgebra. Even superalgebra endomorphisms $\alpha$ with respect to the basis $\{e_1, e_2, x, y\}$ are defined by the matrix

$$
\alpha = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & a & 0 \\
0 & 0 & 0 & \frac{1}{a}
\end{pmatrix}
$$

where $a \neq 0$. According to Theorem [5.3], the even linear map $\alpha$ on the following multiplication table:

| $e_1$ | $e_2$ | $x$ | $y$ |
|-------|-------|-----|-----|
| $e_1$ | $e_1$ | $e_2$ | $(1 - \beta)y$ |
| $e_2$ | $0$ | $e_2$ | $\beta x$ |
| $x$ | $\beta x$ | $0$ | $1 - \beta y$ |
| $y$ | $(1 - \beta)y$ | $\beta y$ | $2(\beta e_1 + (1 - \beta)te_2)$ |

where $a \neq 0$, determine 4-dimensional Hom-flexible, Hom-Malcev-admissible superalgebra. Note that $U_\alpha$ is not Hom-alternative superalgebra because

$$
asu_\alpha(e_2, x, y) + asu_\alpha(x, e_2, y) = 2\beta^2 e_1 + (\beta - 2)(1 - \beta)te_2
$$

$asu_\alpha(e_2, x, y) + asu_\alpha(x, e_2, y) \neq 0$ whenever $\beta \neq 0, 1, 2$. So $(U, \mu, \alpha)$ does not satisfy the Hom-alternative super-identity, therefore, is not Hom-alternative. Also, $U_\alpha$ is not Hom-Lie-admissible superalgebra because

$$
\overline{J}_{(U, \alpha)}(e_1, e_2, x) = \frac{(1 - \beta)(2\beta - 1)}{a^2}\frac{y}{y}
$$

$\overline{J}_{(U, \alpha)}(e_1, e_2, x) \neq 0$ whenever $\beta \neq 1, \frac{1}{2}$ and $a \neq 0$.

Finally $(U, \mu_\alpha)$ is not Malcev-admissible, i.e., $(U, [\cdot, \cdot])_\alpha$ is not Malcev superalgebra. Indeed, let $J$ the usual super-Jacobian of $(U, [\cdot, \cdot])$ as in (1.2). Then, on the one hand, we have

$$
J(e_1, e_2, [x, y]) = \frac{2}{a^2}(2\beta - 1)(1 - \beta)\frac{e_1 + 2\sqrt{2(\beta - 1)(1 - \beta)}e_2}{a^2}.
$$
On the other hand, we have
\[
\mathcal{J}(e_1, e_2, y, x) + [\mathcal{J}(x, e_2, y), e_1] = [[[e_1, e_2, y, x], x]_\alpha - [[[e_1, y, e_2, x], x]_\alpha - [[[e_1, e_2, y, x], x]_\alpha + [[[x, e_2, y], e_1], x]_\alpha - [[[x, e_2, y], e_1], x]_\alpha - [[[x, e_2, y], e_1], x]_\alpha - [[[x, y, e_2], e_1], x]_\alpha = \frac{4\beta(\beta - 1)(2\beta - 1)}{a^2} e_1 + \frac{4\beta(\beta - 1)(1 - 2\beta)}{a^2} e_2.
\]
\[
\mathcal{J}(e_1, e_2, [x, y]_\alpha) + \mathcal{J}(x, e_2, [e_1, y]_\alpha) = \mathcal{J}(e_1, e_2, y, x) + [\mathcal{J}(x, e_2, y), e_1]_\alpha whenever $\beta \neq 0, \frac{1}{2}, 1, 2$ and $\alpha \neq 0$. So $(U, [-, -])$ does not satisfy the Malcev super-identity and, therefore, is not a Malcev superalgebra. Then $(U, \mu, \alpha)$ does not in general a Malcev-admissible superalgebra.

6. Hom-Alternative Superalgebras are Hom-Jordan-Admissible

We introduce in this section Hom-Jordan (−admissible) superalgebras and we show in Theorem [6.1] that every Hom-alternative superalgebras are Hom-Jordan-admissible. Then we prove Theorems [6.2] and [6.3], which are constructions results for Hom-Jordan and Hom-Jordan admissible superalgebras. In Examples [6.2] and [6.3] we construct Hom-Jordan superalgebras from the 3-dimensional Kaplansky Jordan superalgebra and from the 4-dimensional simple Jordan superalgebra $D_4$ where $t \neq 0$.

Let us begin with some relevant definitions.

**Definition 6.1.** Let $(A, \mu, \alpha)$ be a Hom-superalgebra. Define its plus Hom-superalgebra as the Hom-superalgebra $A^+ = (A, *, \alpha)$, where the product $*$ is given by
\[
x * y = \frac{1}{2}(\mu(x, y) + (-1)^{|x||y|}\mu(y, x)) = \frac{1}{2}(xy + (-1)^{|x||y|}yx),
\]
with $\mu(x, y) = xy$. The product $*$ is super-commutative.

**Definition 6.2.**

1. A Hom-Jordan superalgebra is a Hom-superalgebra $(A, \mu, \alpha)$ such that $\mu$ is super-commutative (i.e. $\mu(x, y) = (-1)^{|x||y|}\mu(y, x)$) and the Hom-Jordan super-identity
\[
(6.31) \sum_{x, y, t} (-1)^{|t||x||z|} a_{A}(xy, \alpha(z), \alpha(t)) = 0
\]
is satisfied for all $x, y, z$ and $t$ in $\mathcal{H}(A)$, where $\sum$ denotes the cyclic sum over $(x, y, t)$ and $\tilde{a}_{A}$ is the Hom-associator (2.3).

2. A Hom-Jordan-admissible superalgebra is a Hom-superalgebra $(A, \mu, \alpha)$ whose plus Hom-superalgebra $A^+ = (A, *, \alpha)$ is a Hom-Jordan superalgebra.

The Hom-Jordan super-identity (6.31) can be rewritten as
\[
(6.32) \sum_{x, y, t} (-1)^{|t||x||z|} \left(\mu(\mu(x, y), \alpha(z)), \alpha^2(t) - \mu(\alpha(\mu(x, y)), \mu(\alpha(z), \alpha(t)))\right) = 0.
\]

Since the product $*$ is super-commutative, a Hom-superalgebra $(A, \mu, \alpha)$ is Hom-Jordan-admissible if and only if $A^+ = (A, *, \alpha)$ satisfies the Hom-Jordan super-identity (6.32) for all $x, y, z$ and $t$ in $\mathcal{H}(A)$, where $\sum$ denotes summations over the cyclic permutation on $x, y, t$.

Or equivalently
\[
\sum_{x, y, t} (-1)^{|t||x||z|} \left((x * y) * \alpha(z) * \alpha^2(t) - \alpha(x * y) * (\alpha(z) * \alpha(t))\right) = 0
\]
for all $x, y, z$ and $t$ in $\mathcal{H}(A)$, where $\sum$ denotes the cyclic sum over $(x, y, t)$.

**Example 6.1.** A Jordan (−admissible) superalgebra is a Hom-Jordan (−admissible) superalgebra with $\alpha = Id$, since the Hom-Jordan super-identity (6.32) with $\alpha = Id$ is the Jordan super-identity (1.3). We refer to [1] [2] [12] [18] [21] [24] [28] for discussions about structure of Jordan superalgebras.
Remark 6.1. In [13], a Hom-Jordan algebra is defined as a commutative Hom-algebra satisfying
\[ \sum_{x,y,t} a_s A(xy, z, \alpha(t)) = 0, \] which becomes our Hom-Jordan super-identity \( (6.32) \) if \( z \) is replaced by \( \alpha(z) \). Using this definition of a Hom-Jordan algebra, Hom-alternative superalgebras are not Hom-Jordan-admissible.

Theorem 6.1. Let \( (A, \mu, \alpha) \) be a multiplicative Hom-alternative superalgebra. Then \( A \) is Hom-Jordan-admissible, in the sense that the Hom-superalgebra
\[ A^+ = (A, *, \alpha) \]
is a multiplicative Hom-Jordan superalgebra, where \( x * y = \frac{1}{2}(xy + (-1)^{|x||y|}yx) \).

To prove Theorem 6.1, we will follow the following preliminary observation.

Lemma 6.1. Let \( (A, \mu, \alpha) \) be any Hom-superalgebra and \( A^+ = (A, *, \alpha) \) be its plus Hom-superalgebra. Then we have
\[ \sum_{x,y,t} 4(-1)^{|t|(|x|+|z|)} a_s A^+(x * y, \alpha(z), \alpha(t)) \]
\[ = \sum_{x,y,t} \left( (-1)^{|t|(|x|+|z|)+|x|} a_s A(xy, \alpha(z), \alpha(t)) + (-1)^{|t|(|x|+|z|)+|y|} a_s A(y, \alpha(z), \alpha(t)) \right) \]
\[ - (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} a_s A(\alpha(t), \alpha(z), \alpha(y)x) - (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} a_s A(\alpha(t), \alpha(z), yx) \]
\[ - (-1)^{|x|(|y|+|t|)+|y|(|x|+|t|)} a_s A(\alpha(t), \alpha(y), \alpha(z)) - (-1)^{|y|(|y|+|t|)+|y|(|x|+|t|)} a_s A(\alpha(t), \alpha(y), \alpha(z)) \]
\[ + (-1)^{|x|(|y|+|t|)+|y|(|x|+|t|)} a_s A(\alpha(t), \alpha(y), \alpha(z)) + (-1)^{|y|(|y|+|t|)+|y|(|x|+|t|)} a_s A(\alpha(t), \alpha(y), \alpha(z)) \]
\[ - (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} a_s A(\alpha(z), \alpha(t), xy) + (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} a_s A(\alpha(z), \alpha(t), xy) \]
\[ (6.33) \]
for all \( x, y, z, t \in H(A) \), where \([- , -]\) is the super-commutator bracket
\( (i.e. [x, y] = xy - (-1)^{|x||y|}yx) \) and where \( \sum_{x,y,t} \) denotes the cyclic sum over \( (x,y,t) \).

Proof. As usual we write \( \mu(x, y) \) as the juxtaposition \( xy \). Starting from the left-hand side of (6.33), we have:
\[ \sum_{x,y,t} 4(-1)^{|t|(|x|+|z|)} a_s A^+(x * y, \alpha(z), \alpha(t)) \]
\[ = \sum_{x,y,t} 4(-1)^{|t|(|x|+|z|)} \left( (x * y) * \alpha(z) * \alpha^2(t) - \alpha(x * y) * (\alpha(z) * \alpha(t)) \right) \]
\[ = \sum_{x,y,t} \left( (-1)^{|t|(|x|+|z|)+|x|} \mu(xy)(\alpha(z)) \alpha^2(t) + (-1)^{|t|(|x|+|z|)+|y|} \mu(\alpha(z))(\alpha(t)) \alpha^2(t) \right) \]
\[ + (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} \alpha(\alpha(xy))(\alpha(z)) \alpha^2(t) + (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) \]
\[ + (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) + (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) \]
\[ + (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} (\alpha(xy))(\alpha(z)) \alpha^2(t) + (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} (\alpha(xy))(\alpha(z)) \alpha^2(t) \]
\[ + (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) + (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) \]
\[ + (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) + (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) \]
\[ + (-1)^{|x|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) + (-1)^{|y|(|x|+|y|)+|y|(|x|+|t|)} \alpha(xy)(\alpha(z)) \alpha^2(t) \]
\[ = \sum_{x,y,t} 4(-1)^{|t|(|x|+|z|)} a_s A^+(x * y, \alpha(z), \alpha(t)) \]
\begin{align*}
+&(-1)^{t(|x|+|z|)+|z||y|} \overline{a}_{\mathcal{A}}(y, x, \alpha(t), y) \alpha(z) - (-1)^{y(|x|+|t|)+|z||x|+|y|} \alpha^{2}(t) \\
&+(-1)^{|x||y|} \alpha^{2}(t)((xy)\alpha(t)) + (-1)^{|\alpha|(|x|+|z|)+|z||x|+|y|} \alpha^{2}(t) \\
&+(-1)^{|y|(|x|+|t|)} \alpha^{2}(t)((y)\alpha(t)) + (-1)^{|z||t|+|y|+|t|} \alpha^{2}(t) \\
&-(-1)^{|\alpha|(|x|+|z|)+|z||x|+|y|} \alpha^{2}(t) \alpha(y) - (-1)^{|t||x|+|y|} \alpha^{2}(t) \alpha(y) \\
(6.34) &= -(-1)^{|y||t|+|x|+|z||y|+|t|} \alpha^{2}(t) \alpha(y)(\alpha(t)\alpha(z)) \Big)
\end{align*}

Using the definition of the Hom-associator \textcolor{red}{(2.4)}, the last eight terms in \textcolor{red}{(6.34)} are:

\begin{itemize}
  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]

  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]

  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]

  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]

  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]

  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]

  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]

  \item \[
  \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \alpha^{2}(t)\big((\alpha(z)(y)\alpha(t))\big)
  \]
  \[
  = \sum_{x,y,t} (-1)^{|x||t|+|y|+|z||x|+|y|+|t|} \big(\alpha(z)(y)\alpha(t) + \alpha^{2}(t)\big)\big((y)\alpha(t)\big)
  \]
\end{itemize}
• \[-\sum_{x,y,t} (-1)^{|x|(|y|+|t|)} \alpha(yx)(\alpha(t)\alpha(z))\]

(6.35)

\[= \sum_{x,y,t} (-1)^{|x|(|y|+|t|)} \left(\widetilde{\alpha A}(yx, \alpha(t), \alpha(z)) - ((yx)\alpha(t))\alpha^2(z)\right),\]

or

\[\sum_{x,y,t} \left((-1)^{|x|(|y|+|t|)+|z|(|x|+|y|+|t|)} \alpha^2(z)((yx)\alpha(t)) + (-1)^{|t||y|} \alpha(t)(xy))\alpha^2(z)\right.
\]

\[\left.\hspace{1cm} + (-1)^{|t||x|}((xy)\alpha(t))\alpha^2(z) - (-1)^{|y||x|+|x|+|y|+|t|} \alpha^2(z)\right)\alpha(t)(yx)\right)
\]

\[-(-1)^{|y||x|+|x|+|y|+|t|}((xy)\alpha(t))\alpha^2(z)\right)
\]

\[= (-1)^{|x||y|+|z|(|x|+|y|+|t|)}[\alpha^2(z), \widetilde{\alpha A}(y, t, x)] + (-1)^{|y|||x|+|t|+|z||x|+|y|+|t|} \alpha^2(z), \widetilde{\alpha A}(t, y, x)] + (-1)^{|t||x|+|y|+|z||x|+|y|+|t|} \alpha^2(z), \widetilde{\alpha A}(x, y, t)]
\]

\[+(-1)^{|x||y|+|z|(|x|+|y|+|t|)} \alpha^2(z), \widetilde{\alpha A}(y, x, t)] + (-1)^{|x||y|+|z|(|x|+|y|+|t|)} \alpha^2(z), \widetilde{\alpha A}(y, x, t)].\]

Note that

\[(-1)^{|x||y|+|z|(|x|+|y|+|t|)} \alpha^2(z), \widetilde{\alpha A}(y, t, x)] = \]

\[\left((-1)^{|x||y|+|z|(|x|+|y|+|t|)} \alpha^2(z), (yt)\alpha(x) - \alpha(y)(tx)\right)
\]

\[\left((-1)^{|x||y|+|z|(|x|+|y|+|t|)} \alpha^2(z)(((yt)\alpha(x))\alpha^2(z)\right)
\]

\[-(-1)^{|y||x|+|z||x|+|y|+|t|}((yt)\alpha(x))\alpha^2(z)\right)
\]

\[+(-1)^{|y||x|+|z||x|+|y|+|t|}((yt)\alpha(x))\alpha^2(z)\right).
\]

(6.36)

and

\[(-1)^{|y|||x|+|t|+|z||x|+|y|+|t|} \alpha^2(z), \widetilde{\alpha A}(t, y, x)] = \]

\[\left((-1)^{|y|||x|+|t|+|z||x|+|y|+|t|} \alpha^2(z), (ty)\alpha(x) - \alpha(t)(yx)\right)
\]

\[\left((-1)^{|y|||x|+|t|+|z||x|+|y|+|t|} \alpha^2(z)(((ty)\alpha(x))\alpha^2(z)\right)
\]

\[-(-1)^{|y||x|+|z||x|+|y|+|t|}((ty)\alpha(x))\alpha^2(z)\right)
\]

\[+(-1)^{|y||x|+|z||x|+|y|+|t|}((ty)\alpha(x))\alpha^2(z)\right).
\]

(6.37)

The desired condition (6.33) now follows from (6.34), (6.35) and (6.36).

Proof. (Theorem 6.1) Let \((A, \mu, \alpha)\) be a Hom-alternative superalgebra. To show that it is Hom-Jordan-admissible, it suffices to prove the Hom-Jordan super-identity (6.32) for its plus Hom-superalgebra \(A^+ = (A, *, \alpha)\).

Using again the super-antialternativity of \(\widetilde{\alpha A}\), this implies that

\[(\widetilde{\alpha A} \circ \theta)(xy, \alpha(z), \alpha(t)) = 0\]

and

\[(\widetilde{\alpha A} \circ \theta)(yx, \alpha(z), \alpha(t)) = 0\]

for any permutation \(\theta\) on three letters. Since

\[\alpha^2(z), (-1)^{|x||y|+|z||x|+|y|+|t|}(\widetilde{\alpha A}(y, t, x) + (-1)^{|y||t|} \widetilde{\alpha A}(t, y, x)) = 0,\]

\[\alpha^2(z), (-1)^{|t||x|+|z||x|+|y|+|t|}(\widetilde{\alpha A}(t, x, y) + (-1)^{|x||t|} \widetilde{\alpha A}(x, t, y)) = 0,\]

and

\[\alpha^2(z), (-1)^{|x||t|+|z||x|+|y|+|t|}(\widetilde{\alpha A}(x, y, t) + (-1)^{|x||y|} \widetilde{\alpha A}(x, y, t)) = 0.\]

As well, it follows from Lemma 6.1 that

\[\sum_{x,y,t} 4(-1)^{|t||x|+|z|}(\widetilde{\alpha A}^+(x * y, \alpha(z), \alpha(t)) = 0,\]

from which the desired Hom-Jordan super-identity for \(A^+\) (6.32) follows.

The following constructions result are the analogues of Theorems (2.1) and (2.1) for Hom-Jordan and Hom-Jordan-admissible superalgebras.
Theorem 6.2.

(1) Let $\mathcal{A}(\mu, \alpha)$ be a Hom-Jordan superalgebra and $\beta : \mathcal{A} \rightarrow \mathcal{A}$ be an even Jordan superalgebra endomorphism. Then $\mathcal{A}_\beta = (\mathcal{A}, \mu_\beta = \beta \circ \mu, \alpha_\beta)$ is a Hom-Jordan superalgebra.

(2) Let $\mathcal{A}(\mu, \alpha)$ be a Hom-Jordan superalgebra. Then the derived Hom-superalgebra $\mathcal{A}^n = (\mathcal{A}, \mu^{(n)} = \alpha^{2^n - 1} \circ \mu, \alpha^{2^n})$ is also a Hom-Jordan superalgebra for each $n \geq 0$.

Proof. For the first assertion, first note that $\mu_\beta = \beta \circ \mu$ is super-commutative. To prove the Hom-Jordan super-identity (6.31) in $\mathcal{A}_\beta$, regard $(\mathcal{A}, \mu)$ as the Hom-superalgebra $(\mathcal{A}, \mu, Id)$. Then for all $x, y, z, t \in H(\mathcal{A})$ we have:

$$
\sum_{x,y,t} (-1)^{|t|(|x|+|z|)} \overline{a_{\mathcal{A}_\beta}}(\mu_\beta(x, y), \beta\alpha(z), \beta\alpha(t)) = \sum_{x,y,t} (-1)^{|t|(|x|+|z|)} \overline{a_{\mathcal{A}}} (\beta(\mu(x, y)), \beta\alpha(z), \beta\alpha(t)) \\
= \beta^2 \left( \sum_{x,y,t} (-1)^{|t|(|x|+|z|)} \overline{a_{\mathcal{A}}} (\beta(\mu(x, y)), \beta\alpha(z), \beta\alpha(t)) \right) \\
= \beta^3 \left( \sum_{x,y,t} (-1)^{|t|(|x|+|z|)} \overline{a_{\mathcal{A}}} (\mu(x, y), \alpha(z), \alpha(t)) \right) \\
= 0.
$$

This shows that $\mathcal{A}_\beta$ is a Hom-Jordan superalgebra.

For the second assertion, first note that $\mu^{(n)} = \alpha^{2^n - 1} \circ \mu$ is super-commutative. To prove the Hom-Jordan super-identity (6.31) in $\mathcal{A}^n$, we compute

$$
\sum_{x,y,t} (-1)^{|t|(|x|+|z|)} \overline{a_{\mathcal{A}^n}} (\mu^{(n)}(x, y), \alpha^{2^n}(z), \alpha^{2^n}(t)) \\
= \alpha^{2(2^n-1)} \left( \sum_{x,y,t} (-1)^{|t|(|x|+|z|)} \overline{a_{\mathcal{A}}} (\alpha^{2^n-1}(\mu(x, y)), \alpha^{2^n}(z), \alpha^{2^n}(t)) \right) \\
= \alpha^{3(2^n-1)} \left( \sum_{x,y,t} (-1)^{|t|(|x|+|z|)} \overline{a_{\mathcal{A}}} (\mu(x, y), \alpha(z), \alpha(t)) \right) \\
= 0.
$$

This shows that $\mathcal{A}^n$ is a Hom-Jordan superalgebra. \qed

6.1. Examples of Hom-Jordan superalgebras. We construct examples of Hom-Jordan superalgebras according to Theorem (6.2).

Example 6.2. (Hom-Jordan superalgebra of dimension 3). We consider the 3-dimensional Kaplansky superalgebra $K_3 = \mathbb{K}e \oplus (\mathbb{K}x + \mathbb{K}y)$ (see [11]), with characteristic of $\mathbb{K}$ is different to 2. The product is defined as:

$$
\mu(e, e) = e, \ \mu(e, x) = \frac{1}{2}x, \ \mu(e, y) = \frac{1}{2}y, \ \mu(x, y) = e.
$$

$K_3$ is a simple Jordan superalgebra.

Even superalgebra endomorphisms $\alpha$ with respect to the basis $\{e, x, y\}$ are defined by

$$
\alpha(e) = e, \ \alpha(x) = \frac{1}{c}x \ \alpha(y) = \frac{1}{c}y,
$$

with $c \neq 0$. According to Theorem (6.2), the even linear map $\alpha$ and the following multiplication

$$
\mu_\alpha(e, e) = e, \ \mu_\alpha(e, x) = \frac{1}{2c}x, \ \mu_\alpha(e, y) = \frac{1}{2c}y, \ [x, y]_\alpha = e.
$$

determine a 3-dimensional Hom-Jordan superalgebra.

In general, $(K_3, \mu_\alpha)$ is not a Jordan superalgebra. Indeed, we have

$$
\sum_{e,e,y} (-1)^{|y|(|e|+|x|)} a_{K_3}(\mu_\alpha(e, e), x, y) = \left( \frac{1}{2c} - 1 \right)e
$$

$$
\neq 0,
$$

for $c \neq 0, \frac{1}{2}$. Then $(K_3, \mu_\alpha)$ does not satisfy the Jordan super-identity.
Example 6.3. (Hom-Jordan superalgebra of dimension 4). Let \( D_t = (D_t)_0 \oplus (D_t)_1 \) (see [18]), where \( (D_t)_0 = \text{span}\{e_1, e_2\} \) and \( (D_t)_1 = \text{span}\{x, y\} \) be the 4-dimensional superalgebra, \( t \neq 0 \), with the product given by
\[
\mu(e_i, e_i) = e_i, \quad \mu(e_1, e_2) = 0, \quad \mu(e_i, x) = \frac{1}{2} x, \quad \mu(e_i, y) = \frac{1}{2} y, \quad [x, y] = e_1 + te_2, \quad i = 1, 2.
\]
This family of Jordan superalgebras (that depend on the parameter \( t \)) corresponds to the family of 17-dimensional Lie superalgebras \( D(2, 1, \beta) \).

Superalgebra endomorphisms \( \alpha \) with respect to the basis \( \{e_1, e_2, x, y\} \) are defined by
\[
\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \alpha(x) = ax + by, \quad \alpha(y) = cx + \frac{1 + bc}{a} y.
\]
According to Theorem 6.2, the even linear map \( \alpha \) and the following multiplication
\[
\mu_{\alpha}(e_i, e_i) = e_i, \quad \mu_{\alpha}(e_1, e_2) = 0, \quad \mu_{\alpha}(e_i, x) = \frac{1}{2} ax + \frac{1}{2} by,
\]
\[
\mu_{\alpha}(e_i, y) = \frac{1}{2} cx + \frac{1}{2} (\frac{1 + bc}{a}) y, \quad \mu_{\alpha}(x, y) = e_1 + te_2, \quad i = 1, 2
\]
where \( a, b, t \) are parameter in \( \mathbb{K}^* \) and \( c \in \mathbb{K} \) satisfying \( 1 + bc \neq 0 \), determine a 4-dimensional Hom-Jordan superalgebra.

Note that \( (D_t, \mu_{\alpha}) \) in general is not a Jordan superalgebra. Indeed, we have
\[
\sum_{x, y, t} (-1)^{|x||y|} a_{x, y}^{D_t}(\mu_{\alpha}(e_1, e_2), x, y) = \left( \frac{1}{2} - \frac{1}{2} \left( \frac{1 + bc}{a} \right) \right)(e_1 + te_2) \\
\neq 0,
\]
for \( 1 + bc \neq 0, a \). So, \( (D_t, \mu_{\alpha}) \) does not satisfy the Jordan super-identity.

Theorem 6.3.

(1) Let \( (A, \mu, \alpha) \) be a Hom-Jordan-admissible superalgebra and \( \beta : A \rightarrow A \) be an even Jordan-admissible superalgebra endomorphism. Then \( A_{\beta} = (A, \mu, \beta \alpha) \) is a Hom-Jordan-admissible superalgebra.

(2) Let \( (A, \mu, \alpha) \) be a Hom-Jordan-admissible superalgebra. Then the derived Hom-superalgebra \( A^n = (A, \mu^{(n)} = \alpha^{2^n - 1} \circ \mu, \alpha^{2^n}) \) is also a Hom-Jordan-admissible superalgebra for each \( n \geq 0 \).

Proof. For the first assertion, first note that the plus Hom-superalgebra \( (A_{\beta})^+ = (A, \ast_{\beta}, \beta \alpha) \) satisfies, for all \( x, y \in H(A) \),
\[
x \ast_{\beta} y = \frac{1}{2} (\mu_{\beta}(x, y) + (-1)^{|x||y|} \mu_{\beta}(y, x)) \\
= \beta \circ \frac{1}{2} (\mu(x, y) + (-1)^{|x||y|} \mu(y, x)) \\
= \beta \circ (x \ast y).
\]
Then \( \ast_{\beta} = \beta \circ \ast \).

Therefore, we have \( (A_{\beta})^+ = (A^+)^{\beta} \), where \( A^+ \) is the Hom-Jordan-superalgebra \( (A, \ast, \alpha) \). Since \( \ast_{\beta} \) is super-commutative, it remains to prove the Hom-Jordan super-identity in \( (A_{\beta})^+ = (A^+)^{\beta} \). We have
\[
\sum_{x, y, t} (-1)^{|x||y| + |z|} a_{\beta}(x, y, z, \alpha, \beta \alpha(t)) \\
= \beta^2 \left( \sum_{x, y, t} (-1)^{|x||y| + |z|} a_{\beta}(\beta(x, y), z, \alpha(t)) \right) \\
= \beta^3 \left( \sum_{x, y, t} (-1)^{|x||y| + |z|} a_{\beta}(\mu(x, y), z, \alpha(t)) \right) \\
= 0.
\]
This shows that \( (A_{\beta})^+ \) satisfies the Hom-Jordan super-identity, so \( A_{\beta} \) is Hom-Jordan-admissible superalgebra.
For the second assertion, first note that the plus Hom-superalgebra \((\mathcal{A}^n)^+ = (\mathcal{A}, \star^{(n)}, \alpha^n)\) satisfies, for all \(x, y \in \mathcal{H}(\mathcal{A})\),

\[
x \star^{(n)} y = \frac{1}{2} \mu^{(n)}(x, y) + (-1)^{|x||y|} \mu^{(n)}(y, x)
\]

\[
= \alpha^{2^n-1} \circ \frac{1}{2} (\mu(x, y) + (-1)^{|x||y|} \mu(y, x))
\]

\[
= \alpha^{2^n-1} \circ (x \star y).
\]

Therefore, we have \((\mathcal{A}^n)^+ = (\mathcal{A}^+)^n\), where \(\mathcal{A}^+\) is the Hom-Jordan superalgebra \((\mathcal{A}, \star, \alpha)\) and \((\mathcal{A}^n)^+\) is its \(n\)th derived Hom-superalgebra. Since \(\star^{(n)}\) is super-commutative, it remains to prove the Hom-Jordan super-identity in \((\mathcal{A}^n)^+ = (\mathcal{A}^+)^n\). We have

\[
\sum_{x, y, t} (-1)^{|x||y|+|z|} \alpha_{\mathcal{A}^n}^{(2^n-1)} (\mu(x, y), \alpha^{2^n}(z), \alpha^{2^n}(t))
\]

\[
= \alpha^{2^n-1} \left( \sum_{x, y, t} (-1)^{|x||y|+|z|} \alpha_{\mathcal{A}^n}^{(2^n-1)} (\mu(x, y), \alpha^{2^n}(z), \alpha^{2^n}(t)) \right)
\]

\[
= \alpha^{3(2^n-1)} \left( \sum_{x, y, t} (-1)^{|x||y|+|z|} \alpha_{\mathcal{A}^n} (\mu(x, y), \alpha(z), \alpha(t)) \right)
\]

\[
= 0.
\]

This shows that \((\mathcal{A}^n)^+\) is Hom-Jordan superalgebra, so \(\mathcal{A}^n\) is a Hom-Jordan-admissible superalgebra. \(\square\)

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