Horizon spectroscopy in and beyond general relativity

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In this work we generalize the results for the entropy spectra typically derived for black holes in general relativity to a generic horizon within the spherically symmetric (asymptotically flat and non-flat) space-times of more general theories of gravity. We use all the standard — Bekenstein's universal lower bound on the entropy transition, the highly damped quasi-normal modes and reduced phase-space quantization — approaches to derive the spectra. In particular, the three approaches suggest that the semi-classical black hole (BH) entropy spectrum is given by, (let us keep for the rest of the paper Boltzmann constant $k = 1$ and also $G = h = c = 1$):

$$S = 2\pi \gamma \cdot n, \quad \gamma \in O(1).$$

Following Bekenstein, the BH entropy spectrum (often with $\gamma = 1$) was re-derived in many different ways \[3–13\]. These derivations can broadly be classified into three different types of arguments:

1. Bekenstein's original argument about the universality (independence on BH parameters) of the lower bound on the BH entropy/area transition \[2\].
2. Argument via asymptotically highly damped BH quasi-normal modes \[2, 5\].
3. Direct quantization techniques typically applied to a reduced phase space of the black hole parameters \[6–13\].

To these three arguments a fourth independent argument was put forward relatively recently by Kothawala et al. \[1\]. The authors argument relies on the use of effective action with \[\gamma = 1\]. The argument by \[1\], being fairly general, consequently suggests that the result for the BH entropy spectra in general relativity may be generalized in two main directions: First, it may hold for entropy of an arbitrary space-time horizon (observer dependent, or not). Secondly, it may hold for Wald entropy in much more general theories than the general theory of relativity (at least within Lanczos-Lovelock theories).

The purpose of this work is to use the three standard arguments mentioned above (which were originally used to derive the BH entropy spectra within general relativity), to confirm or infirm the conclusions of Kothawala et al \[1\]. (This work also continues previous work done by one of the authors in \[14, 15\].)

The main conclusion of this work is that, indeed, for general horizons in spherically symmetric sector within more general theories than general relativity (GR) one has completely the same evidence for the entropy spectra of the Bekenstein type, as one had in the particular case of BH horizon in GR. This means, the quantization of entropy is a very robust result, such that is related to the general thermodynamics of horizons, rather than only an artefact of the black hole theory in GR. (However it might be not surprising that it was first time discovered in the black hole context.) The basic conclusion of this paper therefore confirms the suggestion of Ref. \[1\]. Our analysis, as in Ref. \[1\], shows that in such theories of gravity where proportionality between the horizon area and the horizon entropy does not hold, one has to expect horizon entropy, not the horizon area, to have an equi-spaced spectrum. Furthermore our conclusion perfectly matches with the longer term development in the field of spacetime thermodynamics, where first the concept of temperature \[16–18\] and subsequently also concept of entropy \[19, 21\] were generalized from black holes to the general horizons and also to more general gravity theories than GR \[22\].

The rest of the paper is organized as follows: In Sec. (II), we discuss the generalization of Bekenstein’s argument to general horizons in spherically symmetric solutions of generic theories of gravity. In Sec. (III), we generalize the asymptotic QNM analysis to generic theories of gravity. In Sec. (IV), we discuss the detailed procedure of generalization of the constrained phase space approach to general horizons in generalized theories of gravity. We would like to point that the detailed analysis in Sec. (IV) generalizes to Lovelock theories of gravity and conclude in Sec. (V). As mentioned earlier, we set Boltzmann constant $k = 1$ and also $G = h = c = 1$) and the metric convention is $(-, +, +, +)$. 

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II. ARGUMENT 1: BEKENSTEIN’S LOWER BOUND ON THE AREA / ENTROPY TRANSITION

Let us consider a $D$ dimensional static spacetime with maximally symmetric $D-2$ dimensional subspace, which can be expressed in suitable fixed coordinates as:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + g^{\perp}_{ij}dx^i dx^j.$$  \hspace{1cm} (2)

Here $i,j = 2, \ldots, D-1$. For example, the $(D-2)$ subspace of a general static spherically symmetric line element \cite{2} can be expressed as

$$g^{\perp}_{ij} dx^i dx^j = h^2(r)d\Omega_{D-2}^2.$$  

Note that the line element \cite{2} includes also Rindler space and therefore the method used in this section applies also to the Rindler horizon. However both the quasi-normal mode analysis and the reduced phase space quantization cannot be applied to Rindler spacetime: The quasi-normal mode analysis does not apply to apparent horizons, such as Rindler horizon, and the reduced phase space approach requires the maximally symmetric subspace to be sphere, which Rindler spacetime does not fulfill.

Without loss of generality, in the static region of the metric \cite{2}, we can assume $f(r) > 0$ and let us further consider that the line element has at least one horizon defined by $f(r_H) = 0$. Considering the line element \cite{2} let us consider a gravity theory where one can write down a quasi-local version of the first law of thermodynamics (this can be done for a fairly generic theory \cite{19}): \hspace{1cm} (\xi^a \text{ is suitably normalized time-like Killing vector, } T_H \text{ is temperature of the horizon and } S_H \text{ is Wald entropy of the horizon. The integral on the right side is the energy flow through the Killing horizon. Note also that as a matter of principle we want our results and methods to be always quasi-local, as none of the physically relevant results should depend on the global structure of spacetime.})

$$T_H \delta S_H = \int_H T_{ab} \xi^a d\Sigma^b.$$  \hspace{1cm} (3)

Let us consider a point particle moving along the spacetime geodesics passing through the horizon. The point particle is required to contribute to the stress-tensor through its mass and does not change any other additional parameters, (such as the electromagnetic charge). Under this scenario, the change in the entropy will only correspond to the change in the mass parameter.

It is easy to observe that the entropy will have a minimum change when the particle is dropped down the horizon radially. If the particle with a rest mass $m$ has somewhere above the horizon a classical turning point $r_p$, the Killing energy flow through the horizon will correspond to:

$$E = m \sqrt{f(r_p)}.$$  

If the classical turning point of the point particle is arbitrarily close to the horizon, the change in entropy is expected to be arbitrarily small. Analogous to Bekenstein’s \cite{2} argument, this suggests that the Wald entropy is a classical adiabatic invariant. Due to Ehrenfest principle \cite{2, 22, 23}, the Wald entropy should have semi-classically discrete spectrum.

According to Bekenstein \cite{2}, the quantum corrections can be included by attributing the particle an effective size. This places constraints on the entropy transitions. Following Bekenstein, the particle’s center of mass should move along the geodesics. To minimalize the change of entropy, let us place the center of mass of the particle to such a distance from the horizon that corresponds to a proper radius of the particle $b$. Let us approximate the line element \cite{2} near the horizon by a Rindler line element using

$$f(r) = f, r(r_H) \{r - r_H\} + O(r^2) = 2 \kappa \cdot \{r - r_H\} + O(r^2),$$

with $\kappa$ being the surface gravity of the horizon. The line element \cite{2} turns (by redefining $x \equiv r - r_H$) into the Rindler line element:

$$-2 \kappa x \cdot dt^2 + (2 \kappa x)^{-1} dx^2 + dL^2.$$  \hspace{1cm} (4)

The radial position of the particle’s center of mass at the classical turning point is related to the proper radius of the particle as:

$$b = \int_0^{\delta} f^{-1/2}(x) dx = \int_0^{\delta} \frac{dx}{\sqrt{2 \kappa x}} = \sqrt{\frac{2 \delta}{\kappa}}.$$  

Here $\delta$ is the radial position of the particle’s center of mass. The Killing energy that had flown under the horizon is given as

$$E = m \sqrt{2 \kappa \delta} = |\kappa| \cdot mb.$$  

Using the fact that the horizon temperature for the line element \cite{4} is given by

$$T_H = \frac{|\kappa|}{2 \pi},$$

the first law of thermodynamics \cite{3} gives:

$$\frac{|\kappa|}{2 \pi} \delta S_{\text{min}} = |\kappa| \cdot mb \rightarrow \delta S_{\text{min}} = 2 \pi mb,$$

where $\delta S_{\text{min}}$ is the minimal increase of Wald entropy due to the absorption of a particle with parameters $(m, b)$. Then the Bekenstein’s original argument extended to Wald’s entropy goes as follows: One can not choose the parameters $(m, b)$ of the particle arbitrarily, but the particle’s proper radius $b$ has to be bounded either by the reduced Compton wavelength of the particle (Uncertainty principle), or by the Schwarzschild radius, whichever is larger. In Planck units, the reduced Compton wavelength
is larger for \( m < 2^{-1/2} \) and the Schwarzschild radius is larger for \( m > 2^{-1/2} \). If the Compton wavelength is larger, then \( m \geq b^{-1} \), therefore \( mb \geq 1 \) and then \( \delta S_{\text{min}} \) is lower bounded (in Planck units) by

\[
\delta S_{\text{min}} \geq 2\pi.
\]

In the case where the Schwarzschild radius is larger, then \( b \geq 2m \), but then

\[
bm \geq 2m^2 \geq 1,
\]

and one obtains exactly the same lower bound on the entropy transition \( \delta S_{\text{min}} \geq 2\pi \).

Therefore we see that Wald entropy transition has a lower-bound whose value is \( 2\pi \). It is important to note that the result is universal for any horizon, being independent on the spacetime near-horizon parameters (the horizon surface gravity) and the coupling constants of the modified gravity models. The requirement on the gravity theory is that it fulfils the first law of thermodynamics in the form \( \delta S_{\text{min}} \geq 2\pi \).

According to Bekenstein’s original argument, the existence of the universal lower bound of the BH area/entropy transition arises due to the fact that the BH horizon entropy/area is equispaced as given by Eq. \( 1 \). Following the same reasoning in this section applied to any quasi-local horizon of spherically symmetric static spacetime in any generic theory of gravity implies that that the entropy is equispaced and is given by formula \( 2 \). (\( \gamma \) parameter reflects some ambiguities in the derivation of the lower bound.)

(We would like to mention that the Bekenstein area / entropy lower bound was derived within GR for the Kerr-deSitter spacetime and the deSitter horizon in Ref. \( 24 \)).

**III. ARGUMENT 2: THE HIGHLY DAMPED QUASI-NORMAL MODES**

The second argument used to derive the black hole entropy / area spectra comes originally from the work of Hod and Bekenstein \( 3, 6 \). It was suggested that due to the Bohr’s correspondence principle “the transition frequencies at high quantum numbers equate the classical oscillation frequencies” one could possibly identify the transition frequency between different BH states (in the semi-classical limit) with the BH quasi-normal mode frequencies. It was observed in Ref. \( 3 \) that since the transitions at high quantum numbers are supposed to have relaxation times close to zero, the relevant quasi-normal frequencies are the ones in the limit of high damping. (We would refer the readers to Refs. \( 23, 26 \) for comprehensive reviews on quasi-normal modes.)

The original conjecture showed some difficulties (for example, see Ref. \( 27–29 \)). However, difficulties in Hod’s conjecture was overcome by Maggiore’s conjecture\( 4 \) to link the asymptotic highly damped frequencies to the transition mass (in Planck units) as:

\[
\Delta m = \lim_{n \to \infty} \Delta_{(n,n-1)} \sqrt{\omega^2_{nR} + \omega^2_{nI}} = \lim_{n \to \infty} \Delta_{(n,n-1)} \omega_n I,
\]

where \( \Delta_{(n,n-1)} \) refers to the difference between two adjacent levels. (Here we write the quasi-normal frequencies as \( \omega = \omega_R + i\omega_I \). Also, the reason for the upper equality between the limits is that for all the relevant cases the real part of the frequency is bounded, while the imaginary part is unbounded.) (For the detailed reasoning why the upper connection is made see again the original paper \( 4 \). Alternatively some more detailed analysis of the conjectures linking the asymptotic quasi-normal modes and the transition masses was offered recently by one of the authors \( 15 \). In the rest of this section, we will show that Maggiore’s modification can be used to support the main conclusions of this paper.

In Ref. \( 28, 29 \), one of the authors derived general transcendental formulas for the asymptotic quasi-normal frequencies of tensor perturbations for a generic single horizon black hole in asymptotically flat, de Sitter and Anti-de Sitter static spherically symmetric spacetime. In particular the analysis of \( 28, 29 \) depends only on the properties of the metric near the horizon and singularity, and is independent on the form of gravity theory and matter contribution. Let us put aside the asymptotically anti-deSitter spacetime, where it is for specific reasons (absence of asymptotically highly damped frequencies) unsuitable to use Maggiore’s conjecture.

Let us first proceed for the generic asymptotically flat spherically symmetric static spacetime with a single horizon. For such a spacetime it was shown in Ref. \( 28 \) that:

\[
\omega_n = \text{(offset)} + in \cdot \kappa + O(n^{-1/2}),
\]

where “offset” is some complex number and \( \kappa \) is the surface gravity of the horizon. This universal spacing of quasi-normal modes in spherically symmetric static spacetimes can be confirmed via the intuition obtained through the Born approximation \( 30, 31 \), or approximations by the analytically solvable models \( 32 \). Using \( 4 \), the following relation holds:

\[
\lim_{n \to \infty} \Delta_{(n,n-1)} \sqrt{\omega^2_{nR} + \omega^2_{nI}} = \lim_{n \to \infty} \Delta_{(n,n-1)} \omega_n I = \kappa = 2\pi T_H,
\]

where \( T_H \) is the horizon’s temperature. One can further plug the result \( 4 \) to Maggiore’s conjecture and derive for the mass quantum the following:

\[
\delta M = 2\pi T_H.
\]

Assuming again that the spherically symmetric static spacetime is a solution of a theory in which the quasi-
local first law of thermodynamics \[^{[3]}\] hold\[^{[4]}\], one automatically obtains for the entropy:
\[
T_H \delta S_H = \delta M = 2\pi T_H, \quad \rightarrow \quad \delta S_H = 2\pi.
\]

One can extend the above result to the generic spherically symmetric static two horizon asymptotically de-Sitter spacetimes. In Ref. \[^{[29]}\] it was shown that the formula for the asymptotic frequencies can be for the generic case put in the form which was later on reasonably general grounds analysed in \[^{[33]}\]. The formula for the frequencies can be written as:
\[
\sum_{i=1}^{K} \left\{ A_i \exp \left( B_{i1} \omega \frac{1}{T_1} + B_{i2} \omega \frac{1}{T_2} \right) \right\} = 0. \tag{7}
\]

Here \(K\) and \(A_i\) are numbers of no particular importance from the point of what we want to show, on the other hand it is important to mention that \(B_{i1}, B_{i2}\) are integers. Further by \(T_1, T_2\) we mean temperatures of the two horizons (the BH horizon and the deSitter cosmological horizon). It can be shown \[^{[32]}\] that if the ratio of the two temperatures is a rational number, the solutions of \(^{[7]}\) split in a finite number of families labeled by \(a\) of the form:
\[
\text{(offset)}_a + in \cdot 2\pi \cdot \text{lcm}(T_1, T_2).
\]

Here again the “offset” is some complex number depending this time on the family and by “lcm” we mean the least common multiple of the two temperatures in question, therefore:
\[
\text{lcm}(T_1, T_2) = p_1T_1 = p_2T_2,
\]

where \(p_1, p_2\) are relatively prime integers.

Let us further employ the reasoning from \[^{[14]}\] and assume what would happen from the point of Maggiore’s conjecture if we assumed both semi-classical entropy spectra of the both horizons to be in Planck units \(2\pi\gamma n\). Let us, for example, imagine that a quantum of mass appears from the white hole horizon and disappears eventually behind the cosmological horizon. Due to the first law of thermodynamics \[^{[3]}\], if the Killing energy between the horizons remains eventually unchanged the quantum of mass has to fulfil:
\[
\delta M = -T_1 \delta S_1 = T_2 \delta S_2.
\]

If the spectra of \(S_1, S_2\) are both of the form \(2\pi\gamma n\) then the transition in entropy of the horizons can be of only the form:
\[
\delta S_{1,2} = 2\pi\gamma m_{1,2},
\]

where \(m_{1,2}\) are some integers. Therefore the mass quantum has to fulfil:
\[
\delta M = -T_1 \cdot 2\pi\gamma m_1 = T_2 \cdot 2\pi\gamma m_2,
\]

but if we want the quantum to be as small as possible, such that it is consistent with the previous condition then necessarily:
\[
\delta M = 2\pi\gamma \cdot \text{lcm}(T_1, T_2).
\]

Maggiore’s conjecture then implies that:
\[
\lim_{n \to \infty} \Delta(n,n-1)\omega_n = 2\pi\gamma \cdot \text{lcm}(T_1, T_2),
\]

which is precisely the case we observe for each of the families if \(\gamma = 1\). (One has to therefore attribute the genuine physical meaning only to the families of the frequencies. Also let us mention that the physics of the situation considered, together with the universal \(2\pi n\) entropy spectra has as a consequence the rational ratio of temperatures of the two horizons. However, one does not want to speculate if this has any real physical meaning, but notice in this sense a complementary result obtained in Ref. \[^{[34]}\].)

IV. ARGUMENT 3: QUANTIZATION OF GENERIC HORIZONS IN GEOMETRODYNAMICS OF SPHERICALLY SYMMETRIC REGIONS OF SPACETIMES

The third argument for the spectra \[^{[1]}\] is obtained through a direct quantization of a constrained phase space corresponding to the spacetime variables. There is more than one way on how to derive the form of entropy spectra within this approach. We consider the approach of \[^{[7-9]}\] to be the most straightforward and based on least assumptions. We will further use the 2D effective dilaton geometrodynamical approach \[^{[35–38]}\] to apply it to general horizons in generalized gravity theories. The 2D dilaton gravity is chosen because it describes dimensionally reduced spherically symmetric sector of \(D\) dimensional general relativity with the cosmological constant, and near horizon-limit of the general \(D\) dimensional Lovelock gravity. After introducing the geometrodynamics of the 2D dilaton gravity with the EM field, we review the suggestion by \[^{[7-9]}\] of how to quantize the Euclidean sector of the theory.

Let us mention that the authors of \[^{[6,8]}\] claim that their quantization results are fairly general within black holes in generalized theories, however we suggest that there is no reason to be constrained by black holes: the result relates to general spherically symmetric horizons. Therefore, as we will see, the quantization techniques could be used in the same way to derive the entropy spectra (for example) of the deSitter cosmological horizon, or of the inner Cauchy horizon of the Reissner-Nordström black hole. In this section, we further show that the technique can be used to derive the entropy spectrum for the

\^2 Similarly to the previous section, perturbations considered here do not carry any additional properties beyond energy (like electromagnetic charge).
spherically symmetric sector in GR in arbitrary dimension with cosmological constant, and, with some approximation, it further generalizes to Wald entropy within general $D$ dimensional spherically symmetric Lovelock gravity.

We will describe in detail the Hamiltonian formulation of geometrodynamics of a 2D dilaton theory coupled to the Maxwell field, despite of the fact that most of the different pieces of the formalism used here can be found in relatively broad literature on the subject [35, 36]. In Ref. [35, 36], the authors have looked at a general 2D dilaton gravity with EM fields, however, this section offers a more comprehensive analysis necessary to reduce the theory to effective 1-dimensional action. At the end of the detailed analysis we offer a relatively broad discussion that reflects the shift in the viewpoint that we suggest.

1. Geometrodynamics of the 2D dilaton gravity with the E-M field

As mentioned earlier, we will consider a general version of a dilaton-EM field action (some of the notation and factors match [35]):

$$S_{2D} = \frac{1}{2} \int \sqrt{-g} \left[ \phi R^{(2)} + V(\phi) - \frac{W(\phi)}{2} F_{\mu \nu} F^{\mu \nu} \right].$$

and parametrize the 2D metric as:

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = -N^2 dt^2 + B^2(dx + N^x dt)^2.$$  \hspace{1cm} (8)

One can express the action (8) as:

$$S_{2D} = \int dtdx \cdot \left[ P_B \dot{B} + P_\phi \dot{\phi} + P_A \dot{A}_x - NH \right.$$

$$\left. -N^x H_x - \dot{A}_t H_A \right],$$  \hspace{1cm} (9)

where $A_\mu$ is the electromagnetic potential,

$$P_B = \frac{-\dot{\phi} + N^x \phi'}{N},$$

$$P_\phi = \frac{(N^x B)'}{N},$$

$$P_{A_x} = \frac{W(\phi)(\dot{A}_x - A'_t)}{BN},$$

$$\dot{A}_t = A_t - N^x A_x,$$

and the constraint variables read as:

$$H_t = -P_B P_\phi - \frac{1}{2} B V(\phi) + \frac{\phi''}{B} - \frac{B' \phi'}{B^2} + \frac{B P_{A_x}^2}{2 W(\phi)},$$

$$H_x = P_\phi \phi' - \Delta P_A^t - A_x P_{A_x}',$$

$$H_A = -P_{A_x}'.$$

Dot denotes $t$-derivative and prime denotes $x$-derivative.

In order to define the variational problem, one has to supplement it with the correct fall-off conditions of the functions at the boundaries. One obtains suitable boundary conditions by simply updating the fall-off conditions given by [37]. We will further follow the ideas of [7] and to remain quasi-local we will be constrained in a box. The basic physical requirement of quasi-locality that is reflected by the box quantization can be also expressed as: any quantum spectra that would heavily depend on the physics outside a constrained region are necessarily unphysical, as the configurations of variables outside that region are in some sense always arbitrary. Let us further mention that the boundary conditions which are adapted at the walls of the box already assume that one of the walls of the box is a spacetime horizon. The constrain by the box means we have a limited domain in the $x$ coordinate, with the walls of the box corresponding to $x = x_1$ and $x = x_2$, where $x_1$ lies on some bifurcate horizon $x_1 = x_H$, and $x_2$ is just a fixed point in the static region of spacetime.

The 2D metric in a suitable coordinates has only one degree of freedom. The Birkhoff theorem holds for our case [35], which means the solutions of our theory are static and have a simple structure. Let us therefore express the line element corresponding to the solution in the static coordinates as:

$$ds^2 = -F \cdot dT^2 + F^{-1} d\phi^2.$$  \hspace{1cm} (10)

$T$ is the Killing time (in the static region we are dealing with). The Birkhoff theorem for the solutions of the 2D dilaton-E-M theory further tells us that the function $F$ can be obtained [35] as

$$F = I(\phi) - 2M - Q^2 K(\phi),$$

where

$$I(\phi) = \int^{\phi} V(\phi')d\phi',$$

$$K(\phi) = \int^{\phi} \frac{d\phi'}{W(\phi')}.$$  \hspace{1cm} (11)

$F$ can be expressed through the ADM data as:

$$F = \frac{\phi'^2}{B^2} - P_B^2.$$  \hspace{1cm} (12)

Let us further define:

$$M \doteq 1/2 (F + I(\phi) - P_{A_x}^2 K(\phi))$$.
\( P_A \doteq Q \).

By connecting the physical parameters of the solution with the ADM data one obtains the following relations

\[
B = \sqrt{-FT'^2 + F^{-1}\phi'^2} \quad (10)
\]

\[
N^x = \frac{-F'T'^2 + F^{-1}\phi'^2}{-FT'^2 + F^{-1}\phi'^2}, \quad (11)
\]

\[
N = \frac{\phi'T' - T'\phi}{\sqrt{-FT'^2 + F^{-1}\phi'^2}}. \quad (12)
\]

Then plugging this in the expression for \( P_B \) one can prove the following:

\[
T' = -\frac{B P_B}{F}. \quad \text{(13)}
\]

One can further show that by taking the Poisson brackets \( T' \) commutes with \( Q \) and behaves as a canonical conjugate to \( M \). Therefore let us further call it \( P_M \)

and suggest that after some appropriate transformation of variables it will play role of a canonical conjugate to \( M \). One can express the \( P_M \) variable also as:

\[
P_M = T' = -\frac{B^3 P_B}{B^2 P_B^2 - \phi'^2}. \quad \text{(14)}
\]

Now let us try to define a new canonical chart \( \{ M, P_M, Q, P_Q, \phi, \tilde{P}_\phi \} \). This chart is much more relevant to capture the physics of the problem we are interested in. In the new chart we already know the variables \( M, P_M, Q, \phi \) and want to define the remaining \( P_Q \) and \( \tilde{P}_\phi \). We proceed as in [40, 41], where the variables were derived particularly for the Reissner-Nordström BH. Since \( M, Q, \phi \) are all scalars, the constraint variable \( H_x \) has to be given as:

\[
H_x = P_\phi \phi' - \Lambda P_A' - A_x P_A' = P_M M' + P_Q Q' + \tilde{P}_\phi \phi', \quad \text{(15)}
\]

and finding the most natural solution of (15) one obtains:

\[
P_Q = -\left( A_x + \frac{P_A K(\phi) B^3 P_B}{B^2 P_B^2 - \phi'^2} \right) = -A_x + P_A K(\phi) P_M,
\]

and

\[
\tilde{P}_\phi = \frac{\phi'(BP_B)' - \phi''BP_B + \frac{1}{2} P_\phi P_B^2 W'}{B^2 P_B^2 - \phi'^2} + \frac{1}{2} P_M \left( V(\phi) - \frac{P_A^2 W'}{W(\phi)} \right).
\]

It is important to note that these variables are generalizations of the variables obtained in Ref. [40].

It is well known that the Hamiltonian constrains \( H_e = H_x = H_A = 0 \) can be solved [32] by:

\[
M' = Q' = 0,
\]

with the last condition that we get from the

superhamiltonian constraint

\[
P_M M' + P_Q Q' + \tilde{P}_\phi \phi' = 0,
\]

and this is:

\[
\tilde{P}_\phi = 0.
\]

Further, one needs to show that the transformation to the new variables \( \{ M, Q, \phi \} \) is canonical. One can generalize the result of Ref. [37] as follows:

\[
\int_{x_1}^{x_2} dx \left[ MP_M + \phi \tilde{P}_\phi + Q P_Q - B P_B - \phi P_\phi - A_x P_A \right] = \left( \int dx \cdot \dot{G} \right) + S(x_2) - S(x_1), \quad \text{(16)}
\]

where the function \( G \) is:

\[
G = \left[ \phi' \cdot \text{Arctanh} \left( \frac{\phi'}{F P_M} \right) - FP_M + Q \{ P_Q - Q K(\phi) P_M \} \right],
\]
and the surface term function is
\[ S(x) = \dot{\phi} \cdot \text{Arctanh} \left( \frac{\phi'}{P_M F} \right). \]

This surface term can be shown to vanish by requiring that \( \dot{\phi} \) vanishes sufficiently quickly at the boundaries. This shows that the transformation is canonical.

Our constraints show that the action \( S \) rewritten in the new variables can be reduced by extremalizing the variables at each of the hypersurfaces (or in other words by solving the constraints) to a 1-dimensional reduced action as:
\[ S_{\text{red}} = \int dt \cdot (P_M \dot{M} + P_Q \dot{Q} - H), \]  
where \( H \) is some Hamiltonian to which also the surface terms of the action contribute and
\[ P_M = \int_{x_1}^{x_2} P_M dx = T(x_1) - T(x_2), \]
\[ P_Q = \int_{x_1}^{x_2} P_Q dx. \]

2. Quantization in the Euclidean sector

For a general horizon of the spherically symmetric static line element \( ds^2 = \) there is a well defined concept of horizon temperature linked to the time periodicity of the regular solution in the Euclidean sector of the theory. Let us further argue as in Refs. \cite{7,8}: Peri periodicity in \( T \) in the Euclidean sector implies that \( P_M \) is also periodic with the same period \( T_H^{-1} \). Imposing the periodicity condition on the action \( S \) one can provide a transformation of variables:
\[ X = \sqrt{\frac{E(M)}{\pi}} \cos \left( 2\pi P_M T_H \right), \]
and
\[ P_X = \sqrt{\frac{E(M)}{\pi}} \sin \left( 2\pi P_M T_H \right). \]

For this transformation to be canonical, as shown in \cite{9,10}, direct calculation implies that the following must hold:
\[ \frac{\partial E(M)}{\partial M} = T_H^{-1}. \]  
(17)

Now consider theory in which \( M \) corresponds to some reasonably defined mass, then \( S \) means \( E \) is from the definition related to the entropy of a horizon and some theory of gravity, in particular
\[ E(M) + S_{\text{ext}}(Q) = S_H, \]
so that one can easily observe:
\[ S_H - S_{\text{ext}}(Q) = 2\pi \left( \frac{1}{2} P_M^2 + \frac{1}{2} X^2 \right), \]
with the spectrum of the harmonic oscillator:
\[ S_H = 2\pi(n + 1/2) + S_{\text{ext}}(Q). \]  
(18)

This spectrum describes deviation of entropy from entropy of an extremal state.

3. The theories reduced to the dilaton action

Let us consider now a reduced theory to a 2D dilaton gravity. The 2D reduced metric \( g^{\text{red}}_{ab} \) of the theory generally relates to the dilaton metric \( g_{ab} \) via some conformal transformation:
\[ g^{\text{red}}_{ab} = \Omega^{-2} g_{ab}. \]
The metric of the higher dimensional solution is therefore \( (a, b = 0, 1) \)
\[ ds^2 = \Omega^{-2} g_{ab} dx^a dx^b + r^2 d\Omega_{D-2}. \]  
(19)

(The dilaton field \( \phi \) from the action \( S \) is a particular function of \( r \) that depends on the reduced theory.) The horizon of the solution is defined as:
\[ F(\phi_H) = 0 \rightarrow M = \frac{1}{2} \left[ I(\phi_H) - Q^2 K(\phi_H) \right]. \]

Now let us recover the thermodynamical concepts of the original metric \( S \) that was reduced. The concepts of entropy and temperature must depend beyond the dilaton theory also on the reduction, which includes conformal transformation and redefinition of variables. Taking care of this, one can easily observe that the temperature of the horizon becomes:
\[ T_H = \frac{\Omega^{-2}(\phi_H) F_{\phi_H} \phi_H, r | \phi_H}{4\pi}, \]
and if we suppose that \( M \) is the mass of the theory, the entropy of the horizon turns out to be \( (dM = \frac{1}{2} F_{\phi | \phi_H} d\phi_H) \):
\[ S_H = \int T_H^{-1} dM = \int \frac{2\pi \Omega^2}{\phi, r | \phi_H} d\phi_H = \int 2\pi \Omega^2(r_H) dr_H. \]  
(20)

The above expression raises the following two important questions: In the spherically symmetric sector, which gravity theories reduce to the 2D dilaton theory? Also, what kind of quantization spectra will this dilaton gravity give for Wald entropy? As expected \( D > 2 \) dimensional GR with cosmological constant is included in our picture, and \( M \) represents the "correct" mass. In
particular for GR in $D$ dimensions with cosmological constant we can write the functions $\Omega^2$, $V(\phi)$, $W(\phi)$ and $\phi(r)$ from the action \[8\] as follows:

\[
V(\phi) = 2(D - 3) \left( \frac{\alpha \phi}{2} \right) \frac{D-2}{D-2} - 4\Lambda \left( \frac{\phi}{2} \right) \frac{D-2}{D-2},
\]

\[
W(\phi) = (D - 2) \alpha \frac{D-2}{D-2} \left( \frac{\phi}{2} \right) \frac{D-2}{D-2},
\]

\[
\phi(r) = \alpha^{-1} r^{D-2},
\]

\[
\Omega^2(r_H) = 2(D - 2) \alpha^{-1} r^{D-3}.
\]

Here we define the symbol $\alpha$ to be $\alpha = 16\pi/V_{D-2}$ with $V_{D-2}$ being the volume of a unit $D-2$ sphere. Then the upper expression for entropy \[20\] gives in the case of GR in $D$ dimensions:

\[
S_H = \frac{V_{D-2} r_H^{D-2}}{4} = \frac{A_H}{4}.
\]

The Lovelock action can be reduced in the spherically symmetric sector to \[42\]:

\[
S = V_{D-2} \int dt dx \sum_{m=0}^{[D/2]} \left\{ \lambda_m \frac{(D - 2)!}{(D - 2m)!} mR^{(2)} r^2 [1 - (\nabla r)^2] + \tilde{L}_m(r, \nabla r, \nabla^2 r) \right\},
\]

where $\tilde{L}_m$ is some complicated highly nonlinear function of the covariant derivatives of $r$. However, near the horizon one can apply the condition of $(\nabla r)^2$ being small, and neglect any higher powers of $(\nabla r)^2$, simplifying $L$ in a way that one obtains \[42\], (after suitable redefinitions of variables and a conformal transformation), the action \[8\]. The near horizon approximation means that we take the approximation of the box in which we quantize not extending “too far” from the horizon.

The conformal transformation is such that the function $\Omega^2$ is given as $\Omega^2 = d(\Phi^2)/dr$ with $\Phi$ being a function of $r$ as (but the exact form of the function actually turns not to be important for the consequence we are trying to make):

\[
\Phi^2 = 2V_{D-2} \sum_{m=0}^{[D/2]} \left( \frac{m \lambda_m}{(D - 2m)!} r^{D-2m} \right).
\]

The one can easily observe from \[20\]:

\[
S_H = 2\pi \Phi_H^2.
\]

This is precisely the result for the entropy in the Lovelock gravity \[42\]. This means that in the near horizon approximation the previous result contains also arbitrary Lovelock gravity in $D$ dimensions.

\[\text{3 However, it is nice to see that the entropy spectra \[18\] can be obtained for the 5D Gauss-Bonnet BH without this approximation by using the geometrodynamical approach from \[19\].}\]

### 4. Discussion of the results

In the previous section we obtained the results for 2D dilaton gravity with arbitrary potential and presence of E-M field. First of all in papers \[7\] the ideas were applied to the BH horizons. However since there is nothing that in the previous analysis distinguishes the BH horizon from other horizons, we suggest here that if a static spherically symmetric spacetime is a solution of GR, the quantization as described above can be applied in the same way to the entropy (or more specifically to the deviation of entropy from the extremality) of any arbitrary horizon of the line element \[2\]. Therefore, as mentioned before, (since cosmological constant is included) we can apply the quantization from \[7\] to quantize entropy of the de Sitter cosmological horizon, or even (considering the interior region inside the Cauchy horizon), to the inner Cauchy horizon of the R-N black hole. Furthermore it was shown how the result concerning quantization of entropy can be (as some kind of asymptotic result) generalized to the entropy of the horizons within Lovelock gravity. Therefore we claim that the results of \[7\], when pushed to their consequences, support the basic conclusion of this paper.

However, let us point the fact that we trust the spectra \[18\] only in the semi-classical limit. This is due to the fact that the calculation from \[7\] is essentially a semi-classical calculation, combining classical and quantum ideas. (In order to be able to exactly quantize the theory one has to freeze most of the degrees of freedom by requiring spherical symmetry and more importantly, by reducing the action to one dimension by plugging into the action the classical solutions of the theory.) The fact that the above spectra are valid only in the semi-classical limit implies that the spectra may not be of the form of
Bekenstein and Mukhanov [43]:

\[ \delta S_H = \ln(k), \quad k \in \mathbb{N}_+. \]

At the end of this section let us discuss one more problem: Like in Refs. [7, 8], let us identify the \( S_{\text{ext}}(Q) \) function in the spectrum [18] with the entropy of the extremal case. The \( S_{\text{ext}}(Q) \) function can be then identified with some classical expression for the entropy of the extremal spacetime as a function of charge. (For the Reissner-Nordström spacetime it is for example \( S_{\text{ext}} = \pi Q^2 \).) If the charge has to be quantized in integer multiples of an elementary charge, one might wonder if this condition does not contradict the general entropy quantization rule for each of the spacetime horizons. The somewhat naive calculation shows, that even in the Reissner-Nordström spacetime one can derive from the spectrum [18] (plugging in the correct quantization of charge) an area/entropy spectrum of the inner Cauchy horizon and it will not be equispaced.

However we want to point one significant insufficiency of the reasoning that leads to this apparent contradiction. Certainly every classical spacetime has to be quasi-locally approximated by a quantum spacetime in the semi-classical limit (Bohr’s principle of correspondence). However one cannot apply this reasoning to the spacetime as a whole. Take a simple example of the Schwarzschild spacetime: despite of the fact that quasi-locally (in limited regions) it certainly has to be approached by some quantum configuration in the semi-classical limit, globally this does not hold. Unlike the classical Schwarzschild spacetime which is static, any quantum configuration would Hawking radiate and on sufficiently large time scale it would rapidly deviate from the classical static solution. In Reissner-Nordström spacetime the deviation on the semi-classical level would be even faster: The inner Cauchy horizon would be relatively rapidly destroyed by a mass-inflation instability triggered by a backscattered Hawking radiation coming from the horizon itself.

Therefore one cannot claim that a classical \( n \)-parameter family of solutions has to be globally approached by a corresponding quantum family with semiclassically quantized parameters. Any of the results we presented is for fundamental reasons quasi-local and so is the classical limit. Accepting the quasi-locality of the result removes the above apparent contradiction and all the horizon spectra could easily co-exist.

V. CONCLUSIONS

In this work we have generalized different results typically derived for the black hole horizons in GR, to general horizons in generalized gravity theories. We showed that all three standard arguments of the entropy spectra — Bekenstein’s universal lower bound on the entropy transition, the highly damped quasi-normal modes and the reduced phase-space quantization — can be generalized, and imply the following: The entropy quantization of the Bekenstein form [1] is a robust result and holds for a generic horizon (at least in the static spherically symmetric sector) of a wide class of gravity theories. Furthermore the results lead to the choice of \( \gamma = 1 \) in the Bekenstein type spectra [1].

Similarly to Ref. [1] we have shown that in the theories where proportionality between horizon area and horizon entropy does not hold, entropy is the quantity that remains equispaced. The entropy spectra therefore provide another example of a result, such that was originally derived within the black hole thermodynamics, but generalizes to the horizon thermodynamics in reasonably general gravity theories.

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