Local topology of a deformation of a function-germ with a one-dimensional critical set *

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Abstract

The Brasselet number of a function $f$ with nonisolated singularities describes numerically the topological information of its generalized Milnor fibre. In this work, we consider two function-germs $f, g : (X, 0) \rightarrow (\mathbb{C}, 0)$ such that $f$ has isolated singularity at the origin and $g$ has a stratified one-dimensional critical set. We use the Brasselet number to study the local topology a deformation $\tilde{g}$ of $g$ defined by $\tilde{g} = g + f^N$, where $N \gg 1$ and $N \in \mathbb{N}$. As an application of this study, we present a new proof of the Lê-Iomdin formula for the Brasselet number.

Introduction

The Milnor number, defined in [16], is a very useful invariant associated to a complex function $f$ with isolated singularity defined over an open neighborhood of the origin in $\mathbb{C}^N$. It gives numerical information about the local topology of the hypersurface $V(f)$ and compute the Euler characteristic of the Milnor fibre of $f$ at the origin.

In the case where the function-germ has nonisolated singularity at the origin, the Milnor number is not well defined, but the Milnor fibre is, what led many authors ([8],[9],[3],[6],[14]) to study an extension for this number in more general settings. For example, if we consider a function with a one-dimensional critical set defined over an open subset of $\mathbb{C}^n$ and a generic linear form $l$ over $\mathbb{C}^n$, Iomdin gave an algebraic proof (Theorem 3.2), in [8], of a relation between the Euler characteristic of the Milnor fibre of $f$ and the Euler characteristic of the Milnor fibre of $f + l^N$, $N \gg 1$ and $N \in \mathbb{N}$, using properties of algebraic sets with one-dimensional critical locus. In [9], Lê proved (Theorem 2.2.2) this same relation in a more geometric approach and with a way to obtain the Milnor fibre of $f$ by attaching a certain number of $n$-cells to the Milnor fibre of $f|_{\{l=0\}}$.

In [14], Massey worked with a function $f$ with critical locus of higher dimension defined over a nonsingular space and defined the Lê numbers and cycles, which provides a way to numerically describe the Milnor fibre of this function with nonisolated singularity. Massey compared (Theorem II.4.5), using appropriate coordinates, the Lê numbers of $f$ and $f + l^N$, where $l$ is a generic linear form over $\mathbb{C}^n$ and $N \in \mathbb{N}$ is sufficiently large, obtaining a Lê-Iomdin type relation between these numbers. He also gave (Theorem II.3.3) a handle decomposition of the Milnor fibre of $f$, where the number of attached cells is a certain Lê

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number. Massey extended the concept of Lé numbers to the case of functions with nonisolated singularities defined over complex analytic spaces, introducing the Lé-Vogel cycles, and proved the Lé-Iomdin-Vogel formulas: the generalization of the Lé-Iomdin formulas in this more general sense.

The Brasselet number, defined by Dutertre and Grulha in [6], also describes the local topological behavior of a function with nonisolated singularities defined over an arbitrarily singular analytic space: if \( f : (X, 0) \to (\mathbb{C}, 0) \) is a function-germ and \( V = \{0\}, V_1, \ldots, V_q \) is a good stratification of \( X \) relative to \( f \) (see Definition 3), the Brasselet number \( B_{f,X}(0) \) is defined by

\[
B_{f,X}(0) = \sum_{i=1}^{q} \chi(V_i \cap f^{-1}(\delta) \cap B_\epsilon) Eu_X(V_i).
\]

In [6], the authors proved several formulas about the local topology of the generalized Milnor fibre of a function germ \( f \) using the Brasselet number, like the Lé-Greuel type formula (Theorem 4.2 in [6]):

\[
B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{\dim_c n},
\]

where \( n \) is the number of stratified Morse critical points of a Morseification of \( g|_{X \cap f^{-1}(\delta) \cap B_\epsilon} \) on \( V_q \cap f^{-1}(\delta) \cap B_\epsilon \). In [5], Dalbelo e Pereira provided formulas to compute the Brasselet number of a function defined over a toric variety and in [1], Ament, Nuño-Ballesteros, Oréficé-Okamoto and Tomazella computed the Brasselet number of a function-germ with isolated singularity at the origin and defined over an isolated determinantal variety (IDS) and the Brasselet number of finite functions defined over a reduced curve. More recently, in [4], Dalbelo and Hartmann calculated the Brasselet number of a function-germ defined over a toric variety using combinatorical properties of the Newton polygons. In the global study of the topology of a function germ, Dutertre and Grulha defined, in [7], the global Brasselet numbers and the Brasselet numbers at infinity of \( V \) and \( g \). We consider analytic function-germs \( f, g : (X, 0) \to (\mathbb{C}, 0) \), a Whitney stratification \( \mathcal{W} \) of \( X \), suppose that \( f \) has isolated singularity at the origin and \( g \) has a one-dimensional stratified critical set. Consider the good stratification of \( X \) induced by \( f \), \( \mathcal{V} = \{W_\lambda \setminus X^f, W_\lambda \cap X^f \setminus \{0\}, \{0\}, \{0\} \in \mathcal{W} \} \) and suppose that \( g \) is tractable at the origin with respect to \( \mathcal{V} \) (see Definition 2.5). Let \( \epsilon \) be sufficiently small such that the local Euler obstruction of \( X^g \) is constant on \( b_j \cap B_\epsilon \). In this case, we denote by \( Eu_{X^g}(b_j) \) the local Euler obstruction of \( X \) at a point of \( b_j \cap B_\epsilon \) and by \( B_{g,X \cap f^{-1}(\delta)}(b_j) \) the Brasselet number of \( g|_{X \cap f^{-1}(\delta)} \) at a point of \( b_j \cap B_\epsilon \). For a deformation of \( g, \tilde{g} = g + f^N, N \gg 1 \), we prove (Proposition 3.5)

\[
B_{g,X^f}(0) = B_{g,X^f}(0) = B_{f,X^g}(0).
\]

and for \( 0 < |\delta| \ll \epsilon \ll 1 \) (Proposition 3.9),

\[
B_{f,X^g}(0) - B_{f,X^g}(0) = \sum_{j=1}^{r} m_{f,b_j}(Eu_{X^g}(b_j) - B_{g,X \cap f^{-1}(\delta)}(b_j)).
\]

As an application of these results, we compare the Brasselet numbers \( B_{g,X}(0) \) and \( B_{\tilde{g},X}(0) \), and we obtain (Theorem 4.4) a topological proof of the Lé-Iomdin formula for the Brasselet
number,
\[ B_{\check{g},X}(0) = B_{g,X}(0) + N \sum_{j=1}^{r} m_{f,b_j} E_{u_{f,X} \cap \check{g}^{-1}(\alpha')} (b_j), \]

where \( 0 \ll |\alpha'| \ll 1 \) is a regular value of \( \check{g} \). This formula generalizes the Lê-Imdonin formula for the Euler characteristic of the Milnor fibre in the case of a function with isolated singularity. We note that an algebraic proof can be obtained using the description (see [3]) of the defect of a function-germ \( f \) in terms of the Euler characteristic of vanishing cycles and the Lê-Vogel numbers associated to \( f \).

In [20], Tibăr provided a bouquet decomposition for the Milnor fibre of \( \check{g} \) and related it with the Milnor fibre of \( g \). As a consequence of this strong result, Tibăr gave a Lê-Imdonin formula to compare the Euler characteristics of these Milnor fibres. In the last section of this work, we apply our results to give an alternative proof for this Lê-Imdonin formula (see Proposition 5.1):

\[ \chi(X \cap \check{g}^{-1}(\alpha') \cap B_\varepsilon) = \chi(X \cap g^{-1}(\alpha) \cap B_\varepsilon) + N \sum_{j=1}^{r} m_{b_j} (1 - \chi(F_j)), \]

where \( F_j = X \cap g^{-1}(\alpha) \cap H_j \cap D_{2\varepsilon} \) is the local Milnor fibre of \( g|_{\{t=\delta\}} \) at a point of the branch \( b_j \) and \( H_j \) denotes the generic hyperplane \( t^{-1}(\delta) \) passing through \( x_t \in b_j \) for \( t \in \{i_1, \ldots, i_{k(j)}\} \).

\section{Local Euler obstruction and Euler obstruction of a function}

In this section, we will see the definition of the local Euler obstruction, a singular invariant defined by MacPherson and used as one of the main tools in his proof of the Deligne-Grothendieck conjecture about the existence and uniqueness of Chern classes for singular varieties.

Let \( (X, 0) \subset (\mathbb{C}^n, 0) \) be an equidimensional reduced complex analytic germ of dimension \( d \) in a open set \( U \subset \mathbb{C}^n \). Consider a complex analytic Whitney stratification \( \mathcal{V} = \{V_\lambda\} \) of \( U \) adapted to \( X \) such that \( \{0\} \) is a stratum. We choose a small representative of \( (X, 0) \), denoted by \( X \), such that \( 0 \) belongs to the closure of all strata. We write \( X = \bigcup_{i=0}^{q} V_i \), where \( V_0 = \{0\} \) and \( V_q = X_{reg} \), where \( X_{reg} \) is the regular part of \( X \). We suppose that \( V_0, V_1, \ldots, V_{q-1} \) are connected and that the analytic sets \( \overline{V}_0, \overline{V}_1, \ldots, \overline{V}_q \) are reduced. We write \( d_i = \text{dim}(V_i), i \in \{1, \ldots, q\} \). Note that \( d_q = d \). Let \( G(d, N) \) be the Grassmannian manifold, \( x \in X_{reg} \) and consider the Gauss map \( \phi : X_{reg} \to U \times G(d, N) \) given by \( x \mapsto (x, T_x(X_{reg})) \).

\textbf{Definition 1.1.} The closure of the image of the Gauss map \( \phi \) in \( U \times G(d, N) \), denoted by \( \check{X} \), is called \textbf{Nash modification} of \( X \). It is a complex analytic space endowed with an analytic projection map \( \nu : \check{X} \to X \).

Consider the extension of the tautological bundle \( \mathcal{T} \) over \( U \times G(d, N) \). Since \( \check{X} \subset U \times G(d, N) \), we consider \( \check{T} \) the restriction of \( \mathcal{T} \) to \( \check{X} \), called the \textbf{Nash bundle}, and \( \pi : \check{T} \to \check{X} \) the projection of this bundle.
In this context, denoting by \( \varphi \) the natural projection of \( U \times G(d, N) \) at \( U \), we have the following diagram:

\[
\begin{array}{ccc}
\tilde{T} & \xrightarrow{\pi} & \mathcal{T} \\
\downarrow & & \downarrow \\
\tilde{X} & \xrightarrow{\nu} & U \times G(d, N) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & U \subseteq \mathbb{C}^N
\end{array}
\]

Considering \( ||z|| = \sqrt{z_1\overline{z_1} + \cdots + z_N\overline{z_N}} \), the 1-differential form \( w = d||z||^2 \) over \( \mathbb{C}^N \) defines a section in \( T^*\mathbb{C}^N \) and its pullback \( \varphi^*w \) is a 1-form over \( U \times G(d, N) \). Denote by \( \tilde{w} \) the restriction of \( \varphi^*w \) over \( \tilde{X} \), which is a section of the dual bundle \( \tilde{T}^* \).

Choose \( \epsilon \) small enough for \( \tilde{w} \) be a nonzero section over \( \nu^{-1}(z), 0 < ||z|| \leq \epsilon \), let \( B_\epsilon \) be the closed ball with center at the origin with radius \( \epsilon \) and denote by \( \text{Obs}(\tilde{T}^*, \tilde{w}) \in \mathbb{H}^{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z}) \) the obstruction for extending \( \tilde{w} \) from \( \nu^{-1}(S_\epsilon) \) to \( \nu^{-1}(B_\epsilon) \) and \( O_{\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon)} \) the fundamental class in \( \mathbb{H}^{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z}) \).

**Definition 1.2.** The **local Euler obstruction** of \( X \) at 0, \( Eu_X(0) \), is given by the evaluation

\[
Eu_X(0) = \langle \text{Obs}(\tilde{T}^*, \tilde{w}), O_{\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon)} \rangle.
\]

In [2], Brasselet, Lê and Seade proved a formula to make the calculation of the Euler obstruction easier.

**Theorem 1.3.** (Theorem 3.1 of [2]) Let \( (X, 0) \) and \( \mathcal{V} \) be given as before, then for each generic linear form \( l \), there exists \( \epsilon_0 \) such that for any \( \epsilon \) with \( 0 < \epsilon < \epsilon_0 \) and \( \delta \neq 0 \) sufficiently small, the Euler obstruction of \( (X, 0) \) is equal to

\[
Eu_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\epsilon \cap l^{-1}(\delta)).Eu_X(V_i),
\]

where \( \chi \) is the Euler characteristic, \( Eu_X(V_i) \) is the Euler obstruction of \( X \) at a point of \( V_i, i = 1, \ldots, q \) and \( 0 < |\delta| \ll \epsilon \ll 1 \).

Let us give the definition of another invariant introduced by Brasselet, Massey, Parameswaran and Seade in [3]. Let \( f : X \rightarrow \mathbb{C} \) be a holomorphic function with isolated singularity at the origin given by the restriction of a holomorphic function \( F : U \rightarrow \mathbb{C} \) and denote by \( \nabla F(x) \) the conjugate of the gradient vector field of \( F \) in \( x \in U \),

\[
\nabla F(x) := \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right).
\]

Since \( f \) has an isolated singularity at the origin, for all \( x \in X \setminus \{0\} \), the projection \( \hat{\zeta}_i(x) \) of \( \nabla F(x) \) over \( T_x(V_i(x)) \) is nonzero, where \( V_i(x) \) is a stratum containing \( x \). Using this projection, the authors constructed, in [3], a stratified vector field over \( \tilde{X} \), denoted by \( \nabla f(x) \). Let \( \zeta \) be the lifting of \( \nabla f(x) \) as a section of the Nash bundle \( \tilde{T} \) over \( \tilde{X} \), without singularity over \( \nu^{-1}(X \cap S_\epsilon) \). Let \( \mathcal{O}(\zeta) \in \mathbb{H}^{2n}(\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon)) \) be the obstruction cocycle for extending \( \zeta \) as a nonzero section of \( \tilde{T} \) inside \( \nu^{-1}(X \cap B_\epsilon) \).
**Definition 1.4.** The local Euler obstruction of the function $f$, $Eu_{f,X}(0)$ is the evaluation of $O(\tilde{\zeta})$ on the fundamental class $[\nu^{-1}(X \cap B_{\epsilon}), \nu^{-1}(X \cap S_{\epsilon})]$.

The next theorem compares the Euler obstruction of a space $X$ with the Euler obstruction of function defined over $X$.

**Theorem 1.5.** (Theorem 3.1 of [3]) Let $(X, 0)$ and $V$ be given as before and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a function with an isolated singularity at $0$. For $0 < |\delta| \ll \epsilon \ll 1$, we have

$$Eu_{f,X}(0) = Eu_X(0) - \sum_{i=1}^{q} \chi(V_i \cap B_{\epsilon} \cap f^{-1}(\delta)).Eu_X(V_i).$$

Let us now see a definition we will need to define a generic point of a function-germ. Let $V = \{V_{\lambda}\}$ be a stratification of a reduced complex analytic space $X$.

**Definition 1.6.** Let $p$ be a point in a stratum $V_{\beta}$ of $V$. A degenerate tangent plane of $V$ at $p$ is an element $T$ of some Grassmanian manifold such that $T = \lim_{p_i \rightarrow p} T_{p_i} V_{\alpha}$, where $p_i \in V_{\alpha}, V_{\alpha} \neq V_{\beta}$.

**Definition 1.7.** Let $(X, 0) \subset (U, 0)$ be a germ of complex analytic space in $\mathbb{C}^n$ equipped with a Whitney stratification and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic function, given by the restriction of an analytic function $F : (U, 0) \rightarrow (\mathbb{C}, 0)$. Then $0$ is said to be a generic point of $f$ if the hyperplane $Ker(d_0 F)$ is transverse in $\mathbb{C}^n$ to all degenerate tangent planes of the Whitney stratification at $0$.

Now, let us see the definition of a Morsification of a function.

**Definition 1.8.** Let $W = \{W_0, W_1, \ldots, W_q\}$, with $0 \in W_0$, a Whitney stratification of the complex analytic space $X$. A function $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ is said to be Morse stratified if $\dim W_0 \geq 1, f|_{W_0} : W_0 \rightarrow \mathbb{C}$ has a Morse point at $0$ and $0$ is a generic point of $f$ with respect to $W_i$, for all $i \neq 0$.

A stratified Morsification of a germ of analytic function $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ is a deformation $\tilde{f}$ of $f$ such that $\tilde{f}$ is Morse stratified.

In [18], Seade, Tibăr and Verjovsky proved that the Euler obstruction of a function $f$ is also related to the number of Morse critical points of a stratified Morsification of $f$.

**Proposition 1.9.** (Proposition 2.3 of [18]) Let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of analytic function with isolated singularity at the origin. Then,

$$Eu_{f,X}(0) = (-1)^d n_{\text{reg}},$$

where $n_{\text{reg}}$ is the number of Morse points in $X_{\text{reg}}$ in a stratified Morsification of $f$.

## 2 Brasselet number

In this section, we present definitions and results needed in the development of the results of this work. The main reference for this section is [13].

Let $X$ be a reduced complex analytic space (not necessarily equidimensional) of dimension $d$ in an open set $U \subseteq \mathbb{C}^n$ and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic map. We write $V(f) = f^{-1}(0)$. 

Definition 2.1. A good stratification of $X$ relative to $f$ is a stratification $V$ of $X$ which is adapted to $V(f)$ such that $\{V_\lambda \in V, V_\lambda \not\subseteq V(f)\}$ is a Whitney stratification of $X \setminus V(f)$ and such that for any pair $(V_\lambda, V_\gamma)$ such that $V_\lambda \not\subseteq V(f)$ and $V_\gamma \subseteq V(f)$, the $(a_f)$-Thom condition is satisfied, that is, if $p \in V_\lambda$ and $p_i \in V_\lambda$ are such that $p_i \rightarrow p$ and $T_{p_i}V(f|_{V_\lambda} - f|_{V_\lambda}(p_i))$ converges to some $T_\nu V_\gamma \subseteq T_\nu$.

If $f : X \rightarrow \mathbb{C}$ has a stratified isolated critical point and $V$ is a Whitney stratification of $X$, then

$$\{V_\lambda \setminus X^f, V_\lambda \cap X^f \setminus \{0\}, \{0\}, V_\lambda \in V\}$$

is a good stratification of $X$ relative to $f$, called the good stratification induced by $f$.

Definition 2.2. The critical locus of $f$ relative to $V$, $\Sigma_V f$, is given by the union

$$\Sigma_V f = \bigcup_{V_\lambda \in V} \Sigma(f|_{V_\lambda}).$$

Definition 2.3. If $V = \{V_\lambda\}$ is a stratification of $X$, the symmetric relative polar variety of $f$ and $g$ with respect to $V$, $\Gamma_{f,g}(V)$, is the union $\cup V_\lambda \tilde{\Gamma}_{f,g}(V_\lambda)$, where $\Gamma_{f,g}(V_\lambda)$ denotes the closure in $X$ of the critical locus of $(f,g)|_{V_\lambda \setminus (X^f \cup X^g)}$. $X^f = X \setminus \{f = 0\}$ and $X^g = X \setminus \{g = 0\}$.

Definition 2.4. Let $V$ be a good stratification of $X$ relative to a function $f : (X, 0) \rightarrow (\mathbb{C}, 0)$. A function $g : (X, 0) \rightarrow (\mathbb{C}, 0)$ is prepolar with respect to $V$ at the origin if the origin is a stratified isolated critical point, that is, 0 is an isolated point of $\Sigma_V g$.

Definition 2.5. A function $g : (X, 0) \rightarrow (\mathbb{C}, 0)$ is tractable at the origin with respect to a good stratification $V$ of $X$ relative to $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ if $\dim_0 \tilde{\Gamma}_{f,g}(V) \leq 1$ and, for all strata $V_\alpha \subseteq X^f$, $g|_{V_\alpha}$ has no critical points in a neighbourhood of the origin except perhaps at the origin itself.

We present now the definition of the Brasselet number. Let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a complex analytic function germ and let $V$ be a good stratification of $X$ relative to $f$. We denote by $V_1, \ldots, V_q$ the strata of $V$ that are not contained in $\{f = 0\}$ and we assume that $V_1, \ldots, V_{q-1}$ are connected and that $V_q = X_{\text{reg}} \setminus \{f = 0\}$. Note that $V_q$ could be not connected.

Definition 2.6. Suppose that $X$ is equidimensional. Let $V$ be a good stratification of $X$ relative to $f$. The Brasselet number of $f$ at the origin, $B_{f,X}(0)$, is defined by

$$B_{f,X}(0) = \sum_{i=1}^q \chi(V_i \cap f^{-1}(\delta) \cap B_\epsilon) Eu_X(V_i),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Remark: If $V_q^i$ is a connected component of $V_q$, $Eu_X(V_q^i) = 1$.

Notice that if $f$ has a stratified isolated singularity at the origin, then $B_{f,X}(0) = Eu_X(0) - Eu_{f,X}(0)$ (see Theorem 1.5).
3 Local topology of a deformation of a function-germ with one-dimensional critical set

We begin this section with a discussion about the singular locus of the function \( \tilde{g} = g + f^N \) and a description of the appropriate stratification with which we can compute explicitly the Brasselet numbers we will use.

Let \( f, g : (X, 0) \to (\mathbb{C}, 0) \) be complex analytic function-germs such that \( f \) has isolated singularity at the origin. Let \( \mathcal{W} \) be the Whitney stratification of \( X \) and \( \mathcal{V} \) be the good stratification of \( X \) induced by \( f \). Suppose that \( \Sigma_\mathcal{W}g \) is one-dimensional and that \( \Sigma_\mathcal{W}g \cap \{ f = 0 \} = \{ 0 \} \).

By Lemma 3.1 in [17], if \( \mathcal{V}^f \) denote the set of strata of \( \mathcal{V} \) contained in \( \{ f = 0 \} \), \( \mathcal{V'} = \{ V_i \setminus \Sigma_\mathcal{W}g, V_i \cap \Sigma_\mathcal{W}g, V_i \in \mathcal{V} \} \cup \mathcal{V}^f \) is a good stratification of \( X \) relative to \( f \), such that \( \mathcal{V}_\mathcal{V}(g=0) \) is a good stratification of \( X^g \) relative to \( f \mid_{X^g} \), where

\[
\mathcal{V}_\mathcal{V}(g=0) = \left\{ V_i \cap \{ g = 0 \} \setminus \Sigma_\mathcal{W}g, V_i \cap \Sigma_\mathcal{W}g, V_i \in \mathcal{V} \right\} \cup \mathcal{V}^f \cap \{ g = 0 \},
\]

and \( \mathcal{V}^f \cap \{ g = 0 \} \) denotes the collection of strata of type \( V^f \cap \{ g = 0 \} \), with \( V^f \in \mathcal{V}^f \).

In this whole section, we will use this good stratification of \( X \) relative to \( f \). Suppose that \( g \) is tractable at the origin with respect to \( \mathcal{V} \) and let \( \tilde{g} : (X, 0) \to (\mathbb{C}, 0) \) be the function-germ given by \( \tilde{g}(x) = g(x) + f^N(x), N \gg 1 \).

**Proposition 3.1.** For a sufficiently large \( N \), \( \tilde{g} \) has a stratified isolated singularity at the origin with respect to the Whitney stratification \( \mathcal{W} \) of \( X \).

**Proof.** Let \( x \) be a critical point of \( \tilde{g} \), \( U_x \) be a neighborhood of \( x \) and \( G \) and \( F \) be analytic extensions of \( g \) and \( f \) to \( U_x \), respectively. If \( V(x) \) is a stratum of \( \mathcal{W} \) containing \( x \neq 0 \),

\[
d_x \tilde{g}|_{V(x)} = 0 \iff d_x G|_{V(x)} + N(F(x))^{N-1} d_x F|_{V(x)} = 0.
\]

If \( d_x G|_{V(x)} = 0 \), then \( N(F(x))^{N-1} d_x F|_{V(x)} = 0 \), hence \( x \in \{ F = 0 \} \). Then \( x \in \Sigma_\mathcal{W}g \cap \{ f = 0 \} = \{ 0 \} \). If \( d_x G|_{V(x)} \neq 0 \), we have \( G \neq 0 \). Since \( d_x \tilde{g}|_{V(x)} = 0 \), by Proposition 1.3 of [13], \( \tilde{G} = 0 \), which implies that \( F \neq 0 \). On the other hand, if \( d_x G|_{V(x)} \neq 0 \), \( d_x G|_{V(x)} = -N(F(x))^{N-1} d_x F|_{V(x)} \), and then \( x \in \tilde{G}_{f,g}(V(x)) \). Suppose that \( x \) is arbitrarily close to the origin. Since \( f \) has isolated singularity at the origin, we can define for the stratum \( V(x) \), the function \( \beta : (0, \epsilon) \to \mathbb{R}, 0 < \epsilon \ll 1 \),

\[
\beta(u) = \inf \left\{ \frac{||d_x g|_{V(x)}||}{||d_x f|_{V(x)}||} : z \in \tilde{G}_{f,g}(V(x)) \cap \{|f|_{V(x)}(z) = u, u \neq 0\} \right\},
\]

where \( ||.|| \) denotes the operator norm, (defined, for each linear transformation \( T : V \to W \) between normed vector fields, by \( \sup_{v \in V, ||v||=1} ||T(v)|| \)). Notice that, for each stratum \( W_i \in \mathcal{W}, \tilde{G}_{f,g}(W_i) = \tilde{G}_{f,g}(W_i \setminus \{ f = 0 \}) \). Since \( g \) is tractable at the origin with respect to \( \mathcal{V}, \dim_0 \tilde{G}_{f,g}(V) \leq 1 \). Therefore, \( \dim_0 \tilde{G}_{f,g}(W_i) = \dim_0 \tilde{G}_{f,g}(W_i \setminus \{ f = 0 \}) \leq 1 \). Hence \( \tilde{G}_{f,g}(V(x)) \cap \{|f| = u, u \neq 0\} \) is a finite number of points and \( \beta \) is well defined.

Since the function \( \beta \) is subanalytic, \( \alpha(R) = \beta(1/R), \) for \( R \gg 1 \), is subanalytic. Then, by [10], there exists \( n_0 \in \mathbb{N} \) such that \( \frac{1}{\alpha(R)} < R^{n_0} \), which implies \( \beta(1/R) > (1/R)^{n_0} \), that is, \( \beta(u) > u^{n_0} \). Hence, for \( z \in \tilde{G}_{f,g}(V(x)) \cap \{|f| = u\}, u \ll 1 \), we have
\[ \frac{|d_z g|_{V(x)}|}{|d_z f|_{V(x)}|} \geq \beta(u) > w^{n_0}, \text{ which implies, } |d_z g|_{V(x)}| > |f|_{V(x)}(z)|^{n_0}|d_z f|_{V(x)}|. \]

On the other hand, since \( N \) is sufficiently large, we can suppose \( N > n_0 \). Since \( \tilde{g}(z) = g(z) + f^N(z) \), we obtain using previous inequality that, for the critical point \( x \) of \( \tilde{g} \),

\[ N|f|_{V(x)}(x)|^{N-1}|d_z f|_{V(x)}| = |d_z g|_{V(x)}| > |f|_{V(x)}(x)|^{n_0}|d_z f|_{V(x)}|, \]

which implies that \( N|f|_{V(x)}(x)|^{N-1-n_0} > 1 \). Since \( x \) was taken sufficiently close to the origin, \( f|_{V(x)}(x) \) is close to zero. Hence, \( |f|_{V(x)}(x)| \ll 1 \), which implies that \( N - 1 - n_0 < 0 \). Therefore, \( N \leq n_0 \), what is contradiction. So, there is no \( x \) sufficiently close to the origin such that \( d_x \tilde{g} = 0 \). Therefore, \( \tilde{g} \) has isolated singularity at the origin. \( \blacksquare \)

We will now see how \( \tilde{g} \) behaves with respect to the good stratification \( \mathcal{V} \) of \( X \) induced by \( f \).

**Proposition 3.2.** If \( g \) is tractable at the origin with respect to the good stratification \( \mathcal{V} \) of \( X \) induced by \( f \), then \( \tilde{g} \) is prepolar at the origin with respect to \( \mathcal{V} \).

**Proof.** By Proposition 3.1, \( \tilde{g} \) is prepolar at the origin with respect to \( \mathcal{V} \). So it is enough to verify that \( \tilde{g}|_{V_i \cap \{f = 0\}} \) is nonsingular or has isolated singularity at the origin, where \( V_i \) is a stratum from the Whitney stratification \( \mathcal{V} \) of \( X \). Suppose that \( x \in \Sigma \tilde{g}|_{V_i \cap \{f = 0\}} \). Then \( d_x \tilde{g} = d_x g + N f(x)^{N-1}d_x f = 0 \), which implies that \( d_x g = 0 \). But \( g \) has no critical point on \( V_i \cap \{f = 0\} \), since \( g \) is tractable at the origin with respect to \( \mathcal{V} \). Therefore, \( \tilde{g} \) is prepolar at the origin with respect to \( \mathcal{V} \). \( \blacksquare \)

**Corollary 3.3.** Let \( \tilde{\mathcal{V}} \) be the good stratification of \( X \) induced by \( \tilde{g} \). Then \( f \) is prepolar at the origin with respect to \( \tilde{\mathcal{V}} \).

**Proof.** Use Proposition 3.2 and Lemma 6.1 of [6]. \( \blacksquare \)

Using the previous results, we can relate the relative symmetric polar varieties \( \tilde{\Gamma}_{f, \tilde{g}}(\mathcal{V}) \) and \( \Gamma_{f, g}(\mathcal{V}) \).

**Remark 3.4.** Let us describe \( \tilde{\Gamma}_{f, \tilde{g}}(\mathcal{V}) \). Let \( \Sigma(\tilde{g}, f) = \{ x \in X; rk(d_x \tilde{g}, d_x f) \leq 1 \} \). Since \( f \) is prepolar at the origin with respect to the good stratification induced by \( \tilde{g}, f|_{W_i \cap \{\tilde{g} = 0\}} \) is nonsingular, for all \( W_i \in \mathcal{W}, i \neq 0 \). Also \( \tilde{g} \) is prepolar at the origin with respect to the good stratification induced by \( f \), which implies that \( \tilde{g}|_{W_i \cap \{f = 0\}} \) is nonsingular, for all \( W_i \in \mathcal{W}, i \neq 0 \). Nevertheless, since \( f \) and \( \tilde{g} \) have stratified isolated singularity at the origin, \( \Sigma_{\mathcal{W}} \tilde{g} \cup \Sigma \mathcal{W} f = \{0\} \). Therefore, the map \( (f, g) \) has no singularities in \( \{g = 0\} \) or in \( \{f = 0\} \). Hence, \( \Sigma(\tilde{g}, f) = \tilde{\Gamma}_{f, \tilde{g}}(\mathcal{V}) \). So, it is sufficient to describe \( \Sigma(\tilde{g}, f) \). Let \( x \in \Sigma(\tilde{g}, f) \), then

\[ rk(d_x \tilde{g}, d_x f) \leq 1 \iff (d_x \tilde{g} = 0) \text{ or } (d_x f = 0) \text{ or } (d_x \tilde{g} = \lambda d_x f) \]

\[ \iff (d_x \tilde{g} = 0) \text{ or } (d_x f = 0) \text{ or } (d_x g = (-N f(x)^{N-1} + \lambda)d_x f) \]

Since \( x \notin \{f = 0\}, d_x f \neq 0 \). And since \( \tilde{g} \) has isolated singularity at the origin, \( d_x \tilde{g} \neq 0 \). If \( -N f(x)^{N-1} + \lambda = 0 \), then \( d_x g = 0 \), that is, \( x \in \Sigma_{\mathcal{W}} g \). If \( -N f(x)^{N-1} + \lambda \neq 0 \), then \( d_x g \) is a nonzero multiple of \( d_x f \), that is, \( x \in \tilde{\Gamma}_{f, \tilde{g}}(\mathcal{V}) \). Therefore,

\[ \Sigma(\tilde{g}, f) \subseteq \Sigma_{\mathcal{W}} g \cup \tilde{\Gamma}_{f, \tilde{g}}(\mathcal{V}). \]

On the other hand, if \( x \in \Sigma_{\mathcal{W}} g \), then \( d_x g = 0 \), and

\[ d_x \tilde{g} = d_x g + N f(x)^{N-1}d_x f = N f(x)^{N-1}d_x f. \]
So, \( x \in \Sigma(\tilde{g}, f) \). If \( x \in \tilde{\Gamma}_{f,g}(\mathcal{V}) \), \( d_x g = \lambda d_x f \), and
\[
d_x \tilde{g} = d_x g + N f(x)^{N-1} d_x f = (\lambda + N) f(x)^{N-1} d_x f,
\]
which implies \( x \in \Sigma(\tilde{g}, f) \). Therefore, \( \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \Sigma(\tilde{g}, f) = \Sigma_W g \cup \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) \).

**Proposition 3.5.** Suppose that \( g \) is tractable at the origin with respect to the good stratification \( \mathcal{V} \) of \( X \) induced by \( f \). Then, for \( N \gg 1 \),
\[
B_{g,X}(0) = B_{\tilde{g},X}(0) = B_{f,X}(0).
\]

**Proof.** Since \( \tilde{g} = g + f^N \), over \( \{ f = 0 \} \), \( \tilde{g} = g \). Therefore, \( B_{g,X}(0) = B_{\tilde{g},X}(0) \). On the other hand, by Corollary 3.6, \( f \) is prepolar at the origin with respect to the good stratification \( \mathcal{V} \) of \( X \) induced by \( \tilde{g} \) and so is \( \tilde{g} \) with respect to \( \mathcal{V} \), by Proposition 3.2. Hence, by Corollary 6.3 of [6], \( B_{f,X}(0) = B_{\tilde{g},X}(0) \).

**Corollary 3.6.** Let \( l \) be a generic linear form in \( \mathbb{C}^n \) and denote \( l^{-1}(0) \) by \( H \). Then
\[
B_{g,X \cap H}(0) = B_{\tilde{g},X \cap H}(0) = Eu_{X'}(0).
\]

**Proof.** By [17], \( g \) is tractable at the origin with respect to the good stratification \( \mathcal{V} \) of \( X \) induced by \( l \). Hence, the formula follows directly by Proposition 3.2 using the equality \( B_{g,X'}(0) = B_{\tilde{g},X'}(0) \), and Corollary 6.6 of [6].

**Corollary 3.7.** Let \( N \in \mathbb{N} \) be a sufficiently large number.

1. If \( d \) is even, \( Eu_{X'}(0) \geq Eu_{X}(0) \);
2. If \( d \) is odd, \( Eu_{X'}(0) \leq Eu_{X}(0) \).

**Proof.** Use Corollary 4.11 of [17] and Corollary 3.6.

In order to compare the Brasselet numbers \( B_{f,X'}(0) \) and \( B_{f,X}(0) \) we need to understand the stratified critical set of \( g \). We use the description presented in [17]. Consider a decomposition of \( \Sigma_W g \) into branches \( b_j \),
\[
\Sigma_W g = \bigcup_{\alpha=1}^{q} \Sigma g|_{W_\alpha} \cup \{ 0 \} = b_1 \cup \ldots \cup b_r,
\]
where \( b_j \subseteq W_\alpha \), for some \( \alpha \in \{ 1, \ldots, q \} \). Notice that a stratum \( W_\alpha \) can contain no branch and that a stratum \( V_j \) can contain more than one branch, but a branch can not be contained in two different strata. Let \( \delta \) be a regular value of \( f \), \( 0 < |\delta| \ll 1 \), and let us write, for each \( j \in \{ 1, \ldots, r \} \), \( f^{-1}(\delta) \cap b_j = \{ x_{i_1}, \ldots, x_{i_{k(j)}} \} \). So, in this case, the local degree \( m_{f,b_j} \) of \( f|_{b_j} \) is \( k(j) \). Let \( \epsilon \) be sufficiently small such that the local Euler obstruction of \( X \) and of \( X^g \) are constant on \( b_j \cap B_\epsilon \). Denote by \( Eu_{X}(b_j) \) (respectively, \( Eu_{X'}(b_j) \)) the local Euler obstruction of \( X \) (respectively, \( X^g \)) at a point of \( b_j \cap B_\epsilon \).

**Remark 3.8.** If \( \epsilon \) is sufficiently small and \( x_l \in b_j \), \( l \in \{ i_1, \ldots, i_{k(j)} \} \), \( B_{g,X \cap f^{-1}(\delta)}(x_l) \) is constant on \( b_j \cap B_\epsilon \) (see Remark 4.5 of [17]). Then we denote \( B_{g,X \cap f^{-1}(\delta)}(x_l) \) by \( B_{g,X \cap f^{-1}(\delta)}(b_j) \).
Since \( B_{g,X \cap f^{-1}(\delta)}(x_l) = Eu_{X \cap f^{-1}(\delta)}(x_l) - Eu_{g,X \cap f^{-1}(\delta)}(x_l) \), we also denote \( Eu_{g,X \cap f^{-1}(\delta)}(x_l) \) by \( Eu_{g,X \cap f^{-1}(\delta)}(b_j) \).
**Proposition 3.9.** Suppose that $g$ is tractable at the origin with respect to the good stratification $\mathcal{V}$ of $X$ relative to $f$. Then, for $0 < |\delta| \ll \epsilon \ll 1$,

$$B_{f,X\delta}(0) - B_{f,X\delta}(0) = \sum_{j=1}^{r} m_{f,b_j}(Eu_{X\delta}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

**Proof.** Use Corollary 4.8 of [17] and Proposition 3.5. \hfill \square

**Corollary 3.10.** For $0 < |\delta| \ll \epsilon \ll 1$,

$$Eu_{X\delta}(0) - Eu_{X\delta}(0) = \sum_{j=1}^{r} m_{b_j}(Eu_{X\delta}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$  (2)

**Proof.** By [17], $g$ is tractable at the origin with respect to the good stratification $\mathcal{V}$ of $X$ induced by a generic linear form $l$. Hence, the formula follows directly from Proposition 3.9 using that $B_{l,X\delta}(0) = Eu_{X\delta}(0)$ and that $B_{l,X\delta}(0) = Eu_{X\delta}(0)$. \hfill \square

**Remark 3.11.** Since $l$ is a generic linear form over $\mathbb{C}^n$, $l^{-1}(\delta)$ intersects $X \cap \{g = 0\}$ transversely and using Corollary 6.6 of [6], we have $Eu_{X\delta}(b_j) = Eu_{X\cap l^{-1}(\delta)}(b_j \cap l^{-1}(\delta)) = B_{g,X\cap l^{-1}(\delta)\cap l}(b_j \cap l^{-1}(\delta))$, where $L$ is a generic hyperplane in $\mathbb{C}^n$ passing through $x_l \in b_j \cap l^{-1}(\delta), j \in \{1, \ldots, r\}$ and $l \in \{i_1, \ldots, i_{k(j)}\}$. Denoting $B_{g,X\cap l^{-1}(\delta)\cap l}(b_j \cap l^{-1}(\delta))$ by $B_{g,X\cap l^{-1}(\delta)}(b_j)$, the formula obtained in Corollary 3.10 can be written as

$$Eu_{X\delta}(0) - Eu_{X\delta}(0) = \sum_{j=1}^{r} m_{b_j}(B_{g,X\cap l^{-1}(\delta)}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)).$$

Let $m$ be the number of stratified Morse points of a partial Morseification of $g|_{X\cap f^{-1}(\delta)\cap B\epsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B\epsilon$ and $m$ be the number of stratified Morse points of a Morseification of $g|_{X\cap f^{-1}(\delta)\cap B\epsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B\epsilon$. The next lemma shows how to compare $m$ and $\tilde{m}$. In the following we keep the same description of $\Sigma_{\mathcal{W}g}$.

**Corollary 3.12.** Suppose that $g$ is tractable at the origin with respect to the good stratification $\mathcal{V}$ of $X$ relative to $f$. Then

$$\tilde{m} = (-1)^{d-1} \sum_{j=1}^{r} m_{f,b_j} Eu_{g,X\cap f^{-1}(\delta)}(b_j) + m.$$

**Proof.** By Theorem 3.2 of [17],

$$B_{f,X}(0) - B_{f,X\delta}(0) - \sum_{j=1}^{r} m_{f,b_j}(Eu_{X}(b_j) - Eu_{X\delta}(b_j)) = (-1)^{d-1} m,$$

and by Proposition 3.2 $\tilde{g}$ is prepolar at the origin with respect to $\mathcal{V}$, by Theorem 4.4 of [6],

$$B_{f,X}(0) - B_{f,X\delta}(0) = (-1)^{d-1} \tilde{m}.$$  

Using Proposition 3.9 we obtain that

$$\tilde{m} = m + (-1)^{d-1} \sum_{j=1}^{r} m_{f,b_j} (Eu_{X}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Since $f$ has isolated singularity at the origin, $f^{-1}(\delta)$ intersects each stratum out of $\{f = 0\}$ transversely. So, $Eu_{X}(V_i) = Eu_{X\cap f^{-1}(\delta)}(S)$, for each connected component of $V_i \cap f^{-1}(\delta)$. In particular, $Eu_{X}(b_j) = Eu_{X\cap f^{-1}(\delta)}(b_j)$. The formula holds by Theorem 1.5

$$Eu_{X}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j) = Eu_{g,X\cap f^{-1}(\delta)}(b_j).$$ \hfill \square
\textbf{Proposition 3.13.} Let $\tilde{\alpha}$ be a regular value of $\tilde{g}$ and $\alpha_t$ a regular value of $f$, $0 \ll |\tilde{\alpha}| \ll |\alpha_t| \ll 1$. If $g$ is tractable at the origin with respect to $\mathcal{V}$ relative to $f$, then $B_{g,X \cap f^{-1}(\alpha_t)}(b_j) = B_{f,X \cap \tilde{g}^{-1}(\tilde{\alpha})}(b_j)$.

\textbf{Proof.} Let $x_t \in \{f = \alpha_t\} \cap b_j$, $D_{x_t}$ be the closed ball with center at $x_t$ and radius $r_t$, $0 < |\alpha - \delta| \ll |\alpha_t| \ll r_t \ll 1$. We have

\[
B_{g,X \cap f^{-1}(\alpha_t)}(x_t) = \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap g^{-1}(\alpha - \delta) \cap D_{x_t})E u_{X \cap f^{-1}(\alpha_t)}(W_i \cap f^{-1}(\alpha_t))
\]

\[
= \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap \tilde{g}^{-1}(\tilde{\alpha}) \cap D_{x_t})E u_{X \cap \tilde{g}^{-1}(\tilde{\alpha})}(W_i \cap \tilde{g}^{-1}(\tilde{\alpha}))
\]

Let $g(x_t) = \alpha$, $\tilde{g}(x_t) = \alpha'$ and $f(x_t) = \alpha_t$. Then

\[
p \in f^{-1}(\alpha_t) \cap g^{-1}(\alpha - \delta) \iff g(p) = \alpha - \delta \text{ and } f(p) = \alpha_t
\]

\[
\iff g(p) = g(x_t) - \delta \text{ and } f(p) = \alpha_t
\]

\[
\iff g(p) + \alpha_t^N = \alpha + \alpha_t^N - \delta \text{ and } f(p) = \alpha_t
\]

\[
\iff g(p) + f^N(p) = g(x_t) + f^N(x_t) - \delta \text{ and } f(p) = \alpha_t
\]

\[
\iff \tilde{g}(p) = \tilde{g}(x_t) - \delta \text{ and } f(p) = \alpha_t
\]

\[
\iff \tilde{g}(p) = \alpha' - \delta \text{ and } f(p) = \alpha_t.
\]

Therefore, denoting $\tilde{\alpha} = \alpha' - \delta$,

\[
B_{g,X \cap f^{-1}(\alpha_t)}(x_t) = \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap g^{-1}(\alpha - \delta) \cap D_{x_t})E u_X(W_i)
\]

\[
= \sum \chi(W_i \cap f^{-1}(\alpha_t) \cap \tilde{g}^{-1}(\tilde{\alpha}) \cap D_{x_t})E u_{X \cap \tilde{g}^{-1}(\tilde{\alpha})}(W_i \cap \tilde{g}^{-1}(\tilde{\alpha}))
\]

\[
= B_{f,X \cap \tilde{g}^{-1}(\tilde{\alpha})}(x_t).
\]

An immediate consequence of the last proposition is the following.

\textbf{Corollary 3.14.} Let $\tilde{\alpha}$ be a regular value of $\tilde{g}$ and $\alpha_t$, a regular value of $f$, $0 \ll |\tilde{\alpha}| \ll |\alpha_t| \ll 1$. If $g$ is tractable at the origin with respect to $\mathcal{V}$, then $E u_{g,X \cap f^{-1}(\alpha_t)}(b_j) = E u_{f,X \cap \tilde{g}^{-1}(\tilde{\alpha})}(b_j)$.

\textbf{Proof.} Let $x_t \in \{f = \alpha_t\} \cap b_j$, $D_{x_t}$ be the closed ball with center at $x_t$ and radius $r_t$, $0 < |\alpha' - \delta| \ll |\alpha_t| \ll r_t \ll 1$. The equality holds by Proposition 3.13.

\section{Lê-Iomdin formula for the Brasselet number}

Let $f, g : (X, 0) \to (\mathbb{C}, 0)$ be complex analytic function-germs such that $f$ has isolated singularity at the origin. Let $\mathcal{W}$ be the Whitney stratification of $X$ and $\mathcal{V}$ be the good stratification of $X$ induced by $f$. Suppose that $\Sigma_{\mathcal{W}g}$ is one-dimensional and that $\Sigma_{\mathcal{W}g} \cap \{f = 0\} = \{0\}$.

Suppose that $g$ is tractable at the origin with respect to $\mathcal{V}$. By [17],

\[
\mathcal{V'} = \left\{V_i \setminus \Sigma_{\mathcal{W}g}, V_i \cap \Sigma_{\mathcal{W}g}, V_i \in \mathcal{V} \right\} \cup \mathcal{V}^f (3)
\]

is a good stratification of $X$ relative to $f$, where $\mathcal{V}^f$ denotes the collection of strata of $\mathcal{V}$ contained in $\{f = 0\}$ and

\[
\mathcal{V''} = \left\{V_i \setminus \{g = 0\}, V_i \cap \{g = 0\} \setminus \Sigma_{\mathcal{W}g}, V_i \cap \Sigma_{\mathcal{W}g}, V_i \in \mathcal{V} \right\} \cup \{0\},
\]
is a good stratification of $X$ relative to $g$. Let us denote by $\tilde{\mathcal{V}}$ the good stratification of $X$ induced by $\tilde{g} = g + f^N, N \gg 1$.

Let $\alpha$ be a regular value of $g$, $\alpha'$ a regular value of $\tilde{g}$, $0 < |\alpha|, |\alpha'| \ll \epsilon \ll 1$, $n$ be the number of stratified Morse points of a Morseification of $f|_{X \cap g^{-1}(\alpha) \cap B_\epsilon}$ appearing on $X_{\text{reg}} \cap g^{-1}(\alpha) \cap \{ f \neq 0 \} \cap B_\epsilon$, and $\tilde{n}$ be the number of stratified Morse points of a Morseification of $f|_{X \cap g^{-1}(\alpha') \cap B_\epsilon}$ appearing on $X_{\text{reg}} \cap g^{-1}(\alpha') \cap \{ f \neq 0 \} \cap B_\epsilon$.

**Proposition 4.1.** Suppose that $g$ is tractable at the origin with respect to $\mathcal{V}$. Then,

$$B_{g,X}(0) - B_{\tilde{g},X}(0) = (-1)^{d-1}(n - \tilde{n}).$$

**Proof.** By [17],

$$B_{g,X}(0) - B_{f,X}(0) = (-1)^{d-1}(n - m) - \sum_{j=1}^r m_{f,b_j}(E u_{X}(b_j) - B_{g,X\cap\{f=0\}}(b_j)),$$

where $m$ is the number of stratified Morse points of a Morseification of $g|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing on $X_{\text{reg}} \cap f^{-1}(\delta) \cap \{ g \neq 0 \} \cap B_\epsilon$, for $0 < |\delta| \ll \epsilon \ll 1$.

By Lemma 3.2, $\tilde{g}$ is prepolar at the origin with respect to $\mathcal{V}$. So, by Corollary 6.5 of [6],

$$B_{\tilde{g},X}(0) - B_{f,X}(0) = (-1)^{d-1}(\tilde{n} - \tilde{m}),$$

where $\tilde{m}$ is the number of stratified Morse points of a Morseification of $\tilde{g}|_{X \cap f^{-1}(\delta) \cap B_\epsilon}$ appearing on $X_{\text{reg}} \cap f^{-1}(\delta) \cap \{ \tilde{g} \neq 0 \} \cap B_\epsilon$.

Using Corollary 3.12 and Theorem 1.5 we have the formula. $\blacksquare$

**Lemma 4.2.** Suppose that $g$ is tractable at the origin with respect to $\mathcal{V}$ relative to $f$. If $N \gg 1$ is bigger than the maximum gap ratio of all components of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathcal{V})$ such that Proposition 3.7 is satisfied, then

$$\left( [\tilde{\Gamma}_{f,g}(\mathcal{V})], [V(g)] \right)_0 = \left( [\tilde{\Gamma}_{f,g}(\mathcal{V})], [V(\tilde{g})] \right)_0.$$

**Proof.** Since $g$ is tractable at the origin with respect to $\mathcal{V}$, $\tilde{\Gamma}_{f,g}(\mathcal{V})$ is a curve. Let us write $[\tilde{\Gamma}_{f,g}(\mathcal{V})] = \sum_v m_v[v]$, where each component $v$ of $\tilde{\Gamma}_{f,g}(\mathcal{V})$ is a reduced irreducible curve at the origin. Let $\alpha_v(t)$ be a parametrization of $\mathcal{V}$ such that $\alpha_v(0) = 0$. By page 974 [13], each component $v$ intersects $V(g - g(p))$ at a point $p \in v$, $p \neq 0$, sufficiently close to the origin and such that $g(p) \neq 0$. So,

$$\text{codim}_X \{ 0 \} = \text{codim}_X V(g) + \text{codim}_X v.$$

Also, each component (reduced irreducible curve at the origin) $\tilde{v}$ of $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V})$ intersects $V(\tilde{g} - g(p))$ at such point $p \in \tilde{v}$, $p \neq 0$ and $\tilde{g}(p) \neq 0$. But since $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \tilde{\Gamma}_{f,g}(\mathcal{V}) \cup \sum Wg$, we also have that $v$ intersects $V(\tilde{g} - g(p))$ at the point $p$, so

$$\text{codim}_X \{ 0 \} = \text{codim}_X V(\tilde{g}) + \text{codim}_X v.$$

Therefore, by A.9 of [14],

$$\begin{align*}
([v],[V(g)])_0 &= \text{mult}_tg(\alpha_v(t)) \\
([v],[V(\tilde{g})])_0 &= \min\{\text{mult}_tg(\alpha_v(t)), \text{mult}_tf^N(\alpha_v(t))\}
\end{align*}$$

Now,
\[ \text{mult}_f N(\alpha_v(t)) = N([v],[V(f)])_0 \text{ and } \text{mult}_g(\alpha_v(t)) = ([v],[V(g)])_0. \]

The gap ratio of \( v \) at the origin for \( g \) with respect to \( f \) is the ratio of intersection numbers \( \frac{([v],[V(g)])_0}{([v],[V(f)])_0} \). So, if \( N > \frac{([v],[V(g)])_0}{([v],[V(f)])_0} \), then \( \text{mult}_f N(\alpha_v(t)) > \text{mult}_g(\alpha_v(t)). \)

Making the same procedure over each component \( v \) of \( \tilde{\Gamma}_{f,g}(\mathcal{V}) \) and using that \( N \) is bigger then the maximum gap ratio of all components \( v \) of \( \tilde{\Gamma}_{f,g}(\mathcal{V}) \) and such that Proposition 3.1 is satisfied, we conclude that

\[ \left( [\tilde{\Gamma}_{f,g}(\mathcal{V})],[V(g)] \right)_0 = \left( [\tilde{\Gamma}_{f,g}(\mathcal{V})],[V(\tilde{g})] \right)_0. \]

\[ \text{Lemma 4.3. If } N \text{ is bigger than the maximum gap ratio of all components of the symmetric relative polar curve } \tilde{\Gamma}_{f,g}(\mathcal{V}) \text{ and such that Proposition 3.1 is satisfied, if } 0 < |\alpha|, |\alpha'| \ll \epsilon \ll 1, \text{ then } \]

\[ \tilde{n} = n + (-1)^{d-1}N \sum_{j=1}^r m_{f,b_j} E_{u_{f,X}} g^{-1}(\alpha') (b_j). \]

\[ \text{Proof. We start describing the critical points of } f|_{g^{-1}(\alpha) \cap B_{\epsilon}}. \text{ We have } \]

\[ x \in \Sigma f|_{g^{-1}(\alpha) \cap B_{\epsilon}} \iff x \in g^{-1}(\alpha) \cap B_{\epsilon} \text{ and } rk(d_x g, d_x f) \leq 1 \]

\[ \iff x \in g^{-1}(\alpha) \cap B_{\epsilon} \text{ and } (d_x g = 0) \text{ or } (d_x f = 0) \text{ or } (d_x g = \lambda d_x f, \lambda \neq 0). \]

Since \( f \) has isolated singularity at the origin and, by Proposition 1.3 of [13], \( \Sigma_{\mathcal{W}g} \subset \{g = 0\} \), we have that \( \Sigma f|_{g^{-1}(\alpha) \cap B_{\epsilon}} = g^{-1}(\alpha) \cap B_{\epsilon} \cap \tilde{\Gamma}_{f,g}(\mathcal{V}) \). Therefore, \( n \) counts the number of Morse points of a Morsification of \( f|_{g^{-1}(\alpha) \cap B_{\epsilon}} \) coming from \( g^{-1}(\alpha) \cap B_{\epsilon} \cap \tilde{\Gamma}_{f,g}(\mathcal{V}) \).

Now, let us describe \( \Sigma f|_{\tilde{g}^{-1}(\alpha') \cap B_{\epsilon}} \).

\[ x \in \Sigma f|_{\tilde{g}^{-1}(\alpha') \cap B_{\epsilon}} \iff x \in \tilde{g}^{-1}(\alpha') \cap B_{\epsilon} \text{ and } rk(d_x \tilde{g}, d_x f) \leq 1 \]

\[ \iff x \in \tilde{g}^{-1}(\alpha') \cap B_{\epsilon} \text{ and } (d_x \tilde{g} = 0) \text{ or } (d_x f = 0) \text{ or } (d_x \tilde{g} = \lambda' d_x f, \lambda' \neq 0). \]

Since \( f \) and \( \tilde{g} \) has isolated singularity at the origin, we have that

\[ \Sigma f|_{\tilde{g}^{-1}(\alpha') \cap B_{\epsilon}} = \tilde{g}^{-1}(\alpha') \cap B_{\epsilon} \cap \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}). \]

Since \( \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \tilde{\Gamma}_{f,g}(\mathcal{V}) \cup \Sigma_{\mathcal{W}g}, \)

\[ \Sigma f|_{\tilde{g}^{-1}(\alpha') \cap B_{\epsilon}} = (\Sigma_{\mathcal{W}g} \cap \tilde{g}^{-1}(\alpha') \cap B_{\epsilon}) \cup (\tilde{\Gamma}_{f,g}(\mathcal{V}) \cap \tilde{g}^{-1}(\alpha') \cap B_{\epsilon}). \]

Notice that, since \( \Sigma_{\mathcal{W}g} \cap \{f = 0\} = \{0\} \), \( \Sigma_{\mathcal{W}g} \cap \tilde{g}^{-1}(\alpha') \cap B_{\epsilon} \subset \{f \neq 0\} \). Also, by definition, \( \tilde{\Gamma}_{f,g}(\mathcal{V}) \setminus \{0\} \subset \{f \neq 0\} \). Therefore, \( \tilde{n} \) counts the number of Morse points of a Morsification of \( f|_{\tilde{g}^{-1}(\alpha') \cap B_{\epsilon}} \) coming from \( \tilde{g}^{-1}(\alpha') \cap B_{\epsilon} \cap \Sigma_{\mathcal{W}g} \cap \{f \neq 0\} \cap \{g = 0\} \) and from \( \tilde{g}^{-1}(\alpha') \cap B_{\epsilon} \cap \tilde{\Gamma}_{f,g}(\mathcal{V}) \cap \{f \neq 0\} \cap \{g \neq 0\} \).

By Lemma 4.2, the number of Morse points of a Morsification of \( f|_{\tilde{g}^{-1}(\alpha') \cap B_{\epsilon}} \) appearing on \( \tilde{g}^{-1}(\alpha') \cap B_{\epsilon} \cap \tilde{\Gamma}_{f,g}(\mathcal{V}) \cap \{f \neq 0\} \cap \{g \neq 0\} \) is precisely \( n \). Let us describe the number of
Morse points of a Morsification of \( f|_{\tilde{g}^{-1}(\alpha') \cap B_c} \) appearing on \( \tilde{g}^{-1}(\alpha') \cap B_c \cap \Sigma_Wg \cap \{ f \neq 0 \} \cap \{ g = 0 \} \). Using that \( \Sigma_Wg \subset \{ g = 0 \} \),
\[
x \in \tilde{g}^{-1}(\alpha') \cap B_c \cap \Sigma_Wg \iff \tilde{g}(x) = \alpha' \text{ and } d_xg = 0
\]
\[
\iff g(x) + f(x)^N = \alpha' \text{ and } d_xg = 0
\]
\[
\iff f(x)^N = \alpha' \text{ and } d_xg = 0
\]
\[
\iff f(x) \in \{ \alpha_0, \ldots, \alpha_{N-1} \} \text{ and } d_xg = 0,
\]
where \( \{ \alpha_0, \ldots, \alpha_{N-1} \} \) are the \( N \)-th roots of \( \alpha' \). Therefore,
\[
\tilde{g}^{-1}(\alpha') \cap B_c \cap \Sigma_Wg = \bigcup_{i=0}^{N-1} f^{-1}(\alpha_i) \cap B_c \cap \Sigma_Wg.
\]

Since \( \Sigma_Wg \) is one-dimensional, \( f^{-1}(\alpha_i) \cap \Sigma_Wg \) is a finite set of critical points of \( f|_{\tilde{g}^{-1}(\alpha') \cap B_c} \). Since \( \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \Sigma_Wg \cup \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) \), each branch \( b_j \) of \( \Sigma_Wg \) is a component of \( \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) \). If \( V_{i(j)} \) is the stratum of \( \mathcal{V}' \) containing \( b_j \), then \( f|_{\tilde{g}^{-1}(\alpha')} \) has isolated singularity at each point \( x_l \in b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha') \), \( j \in \{ 1, \ldots, r \} \) and \( l \in \{ i_1, \ldots, i_{k(j)} \} \) (page 974, [13]). Using Proposition 1.9 we can count the number \( n_l \) of Morse points of a Morsification of \( f|_{\tilde{g}^{-1}(\alpha') \cap B_c} \) in a neighborhood of each \( x_l \),
\[
Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(x_l) = (-1)^{d-1}n_l.
\]

Since the Euler obstruction of a function is constant on each branch \( b_j \), so is the Euler obstruction of a function and we can denote \( Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(x_l) \) by \( Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j) \), for all \( x_l \in b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha') \). Therefore, if \( b_j \cap \tilde{g}^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha') = \{ x_{j_1}, \ldots, x_{j_{m_f(b_j)}} \} \), the number of Morse points of a Morsification of \( f|_{\tilde{g}^{-1}(\alpha') \cap B_c} \) appearing on \( (X_{reg} \setminus \{ \tilde{g} = 0 \}) \cap b_j \cap \{ \tilde{g} = \alpha' \} \cap B_c \cap \{ f = \alpha_i \} \) is
\[
n_{j_1} + \cdots + n_{j_{m_f(b_j)}} = (-1)^{d-1}m_{f,b_j}Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(x_l).
\]

Making the same analysis over each \( \alpha_i \in \sqrt[N]{\alpha'} \), the number of Morse points of a Morsification of \( f|_{\tilde{g}^{-1}(\alpha') \cap B_c} \) appearing in \( X_{reg} \setminus \{ \tilde{g} = 0 \} \cap \{ g = 0 \} \cap \{ \tilde{g} = \alpha' \} \cap B_c \) is
\[
(\alpha')^d \sum_{j=1}^{r} m_{f,b_j}Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j).
\]

\[\blacklozenge\]

**Theorem 4.4.** Suppose that \( g \) is tractable at the origin with respect to \( \mathcal{V} \). If \( 0 < |\alpha|, |\alpha'| \ll \epsilon \) and \( N \) is bigger than the maximum gap ratio of each component of the symmetric relative polar curve \( \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) \) and such that Proposition 3.7 is satisfied, then
\[
B_{\tilde{g},X}(0) = B_{g,X}(0) + N \sum_{j=1}^{r} m_{f,b_j}Eu_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j).
\]

**Proof.** It follows by Proposition 4.1 and Lemma 4.3. \[\blacksquare\]

This formula gives a way to compare the numerical data associated to the generalized Milnor fibre of a function \( g \) with a one-dimensional singular locus and and to the generalized
Milnor fibre of the deformation $\tilde{g} = g + f^N$, for $N \gg 1$ sufficiently large. This is what Lê [9] and Iomdin [8] have done in the case where $g$ is defined over a complete intersection in $\mathbb{C}^n$, $g$ has a one-dimensional critical locus and $f$ is a generic linear form over $\mathbb{C}^n$. Therefore, Theorem 4.4 generalizes this Lê-Iomdin formula.

For $X = \mathbb{C}^n$, let us consider $\mathcal{V} = \{\mathbb{C}^n \setminus \{0\}, \{0\}\}$ the Whitney stratification of $\mathbb{C}^n$. If $f$ has isolated singularity at the origin, the good stratification $\mathcal{V}$ of $\mathbb{C}^n$ induced by $f$ is given by $\mathcal{V} = \{\mathbb{C}^n \setminus \{f = 0\}, \{f = 0\} \setminus \{0\}, \{0\}\}$.

**Corollary 4.5.** Suppose that $g$ is tractable at the origin with respect to $\mathcal{V}$ relative to $f$. If $\alpha$ and $\alpha'$ are regular values of $g$ and $\tilde{g}$, respectively, with $0 < |\alpha|, |\alpha'| \ll \epsilon$, then

$$\chi(g^{-1}(\alpha') \cap B_\epsilon) = \chi(g^{-1}(\alpha) \cap B_\epsilon) + (-1)^{n-1} N \sum_{j=1}^r m_{f,b_j} \mu(g|_{f^{-1}(\delta_{j,i})}, b_j),$$

where $\mu(g|_{f^{-1}(\delta_{j,i})}, b_j)$ denotes the Milnor number of $g|_{X \cap f^{-1}(\delta_{j,i}) \cap B_\epsilon}$ at a point $x_{j,i}$ of the branch $b_j$, with $f(x_{j,i}) = \delta_{j,i}$.

**Proof.** By [17], $\mathcal{V}' = \{\mathbb{C}^n \setminus \{f = 0\} \cup \Sigma_{\mathcal{W}g}, \{f = 0\} \setminus \{0\}, \Sigma_{\mathcal{W}g}, \{0\}\}$ is a good stratification of $\mathbb{C}^n$ relative to $f$. Also by [17], $\mathcal{V}''$, given by

$$\{\mathbb{C}^n \setminus \{f = 0\} \cup \{g = 0\}, \{f = 0\} \setminus \{g = 0\}, \{g = 0\} \setminus \{f = 0\} \cup \Sigma_{\mathcal{W}g}, \{f = 0\} \cap \{g = 0\} \setminus \Sigma_{\mathcal{W}g}, \Sigma_{\mathcal{W}g}, \{0\}\},$$

is a good stratification of $\mathbb{C}^n$ relative to $g$.

By definition of the Brasselet number, if $0 < |\alpha| \ll \epsilon \ll 1$,

$$B_{g,x}(0) = \sum_{V_i \in \mathcal{V}'} \chi(V_i \cap g^{-1}(\alpha) \cap B_\epsilon) Eu_{\mathbb{C}^n}(V_i)$$

$$= \chi((\mathbb{C}^n \setminus \{f = 0\} \cup \{g = 0\}) \cap g^{-1}(\alpha) \cap B_\epsilon) Eu_{\mathbb{C}^n}(\mathbb{C}^n \setminus \{f = 0\} \cup \{g = 0\})$$

$$+ \chi((\{f = 0\} \setminus \{g = 0\}) \cap g^{-1}(\alpha) \cap B_\epsilon) Eu_{\mathbb{C}^n}(\{f = 0\} \setminus \{g = 0\})$$

$$= \chi((\mathbb{C}^n \setminus \{g = 0\}) \cap g^{-1}(\alpha) \cap B_\epsilon)$$

$$= \chi(g^{-1}(\alpha) \cap B_\epsilon).$$

The good stratification of $\mathbb{C}^n$ induced by $\tilde{g}$ is $\mathcal{V} = \{\{\tilde{g} = 0\}, \mathbb{C}^n \setminus \{\tilde{g} = 0\}, \{0\}\}$ and then, if $0 < |\alpha'| \ll \epsilon \ll 1$,

$$B_{\tilde{g},x}(0) = \chi(\mathbb{C}^n \setminus \{\tilde{g} = 0\} \cap g^{-1}(\alpha) \cap B_\epsilon) Eu_{\mathbb{C}^n}(\mathbb{C}^n \setminus \{0\}) = \chi(g^{-1}(\alpha') \cap B_\epsilon).$$

Since $f|_{\tilde{g}^{-1}(\alpha') \cap B_\epsilon}$ is defined over $\mathbb{C}^n$ and has isolated singularity at each $x_{j,i} \in b_j$, considering a small ball $B_\epsilon(x_{j,i})$ with radius $\epsilon$ and center at $x_{j,i}$, for $0 < |\delta| \ll \epsilon \ll 1$,

$$Eu_{f,\tilde{g}^{-1}(\alpha')}(x_{j,i}) = (-1)^{n-1} \mu(f|_{\tilde{g}^{-1}(\alpha')}, x_{j,i})$$

$$= (-1)^{n-1} \chi(f|_{\tilde{g}^{-1}(\alpha')})^{-1}(\delta) \cap B_\epsilon(x_{j,i}) - 1$$

$$= \chi(f^{-1}(\delta_{j,i}) \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon(x_{j,i})) - 1, f(x_{j,i}) = \delta_{j,i}$$

$$= \chi(f^{-1}(\delta_{j,i}) \cap g^{-1}(\alpha' - \delta) \cap B_\epsilon(x_{j,i})) - 1$$

$$= \chi((g|_{f^{-1}(\delta_{j,i})})^{-1}(\delta) \cap B_\epsilon(x_{j,i})) - 1$$

$$= (-1)^{n-1} \mu(g|_{f^{-1}(\delta_{j,i})}, x_{j,i}),$$

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where the equality \((\ast)\) is justified by Proposition 6.2 of \([6]\). Therefore, applying Theorem \([4,4]\) we obtain
\[
\chi(\tilde{g}^{-1}(\alpha') \cap B) = \chi(g^{-1}(\alpha) \cap B) + (-1)^{n-1} N \sum_{j=1}^{r} m_{f,b_j} \mu(g|_{f^{-1}(\delta_j)}, b_j).
\]

Another consequence of Theorem \([4,4]\) is a different proof for the Lé-Iomdin formula proved by Massey in \([14]\) in the case of a function with a one-dimensional singular locus. For that we will need the definition of the Lé-numbers. We present here the case for functions defined over a nonsingular subspace of \(\mathbb{C}^n\), and we recommend Part I of \([14]\) for the general case. Let \(h : (U, 0) \subseteq (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) be an analytic function such that its critical locus \(\Sigma h\) is a \(s\)-dimensional set. For \(0 \leq k \leq n\), the \(k\text{-th relative polar variety} \Gamma_{h,z}^k\) of \(h\) with respect to \(z\) is the scheme \(V \left( \partial h / \partial z^k, \ldots, \partial h / \partial z^n \right) / \Sigma h\), where \(z = (z_1, \ldots, z_n)\) are fixed local coordinates and the \(k\text{-th polar cycle} \) of \(h\) with respect to \(z\) is the analytic cycle \([\Gamma_{h,z}^k]\). The \(k\text{-th Lé cycle} \) \([\Lambda_{h,z}^k]\) of \(h\) with respect to \(z\) is the difference of cycles \([\Gamma_{h,z}^{k+1} \cap V (\partial h / \partial z^k)] - [\Gamma_{h,z}^k]\).

**Definition 4.6.** The \(k\text{-th Lé number} \) of \(h\) in \(p\) with respect to \(z\), \(\lambda_{h,z}^k\), is the intersection number
\[
(\Lambda_{h,z}^k \cap V(z_0 - p_0, \ldots, z_{k-1} - p_{k-1}))_p,
\]
provided this intersection is purely zero-dimensional at \(p\).

If this intersection is not purely zero-dimensional, the \(k\text{-th Lé number} \) of \(h\) at \(p\) with respect to \(z\) is said to be undefined.

**Corollary 4.7.** Let \(\mathcal{V}\) be the good stratification of an open set \((U, 0) \subseteq (\mathbb{C}^n+1, 0)\) induced by a generic linear form \(l\) defined over \(\mathbb{C}^{n+1}\). Let \(N \geq 2\), \(z = (z_0, \ldots, z_n)\) be a linear choice of coordinates such that \(\lambda_{l,z}^i(0)\) is defined for \(i = 0, 1\), and \(\tilde{z} = (z_1, \ldots, z_n, z_0)\) be the coordinates for \(\tilde{g} = g + l^n\) such that \(\lambda_{\tilde{g},\tilde{z}}^0\) is defined. If \(N\) is greater then the maximum gap ratio of each component of the symmetric relative polar curve \(\tilde{\Gamma}_{f,g}(\mathcal{V})\) and such that Proposition \([3,1]\) is satisfied, then

\[
\lambda_{\tilde{g},\tilde{z}}^0(0) = \lambda_{l,z}^0(0) + (N - 1)\lambda_{g,z}^1(0).
\]

**Proof.** By \([17]\), \(g\) is tractable at the origin with respect to the good stratification \(\mathcal{V}\) induced by \(l\). Without loss of generality, we can suppose that \(l = z_0\). Let \(F_{g,0}\) be the Milnor fibre of \(g\) at the origin and \(F_{\tilde{g},0}\) be the Milnor fibre of \(\tilde{g}\) at the origin. Since \(g\) has a one-dimensional critical set, the possibly nonzero Lé numbers are \(\lambda_{g,z}^0(0)\) and \(\lambda_{g,z}^1(0)\) and, since \(\tilde{g}\) has isolated singularity at the origin, the only possibly nonzero Lé number is \(\lambda_{g,z}^0(0)\). By Theorem 4.3 of \([15]\),
\[
\chi(F_{g,0}) = 1 + (-1)^n \lambda_{g,z}^0(0) + (-1)^{n-1} \lambda_{g,z}^1(0)
\]
and
\[
\chi(F_{\tilde{g},0}) = 1 + (-1)^n \lambda_{\tilde{g},\tilde{z}}^0(0).
\]

In \([14]\), on page 49, Massey remarked that for \(0 < |\delta| \ll \epsilon \ll 1\),
\[
\lambda_{g,z}^1(0) = \sum_{j=1}^{r} m_{b_j} \mu(g|_{l^{-1}(\delta)}, b_j).
\]

Therefore, the formula holds by Corollary \(4.5\) ■
5 Applications to generic linear forms

Let \( g : (X, 0) \to (\mathbb{C}, 0) \) be a complex analytic function-germ and \( l \) be a generic linear form in \( \mathbb{C}^n \). Let \( \mathcal{W} = \{ \{ 0 \}, W_1, \ldots, W_q \} \) be a Whitney stratification of \( X \) and \( \mathcal{V} \) be the good stratification of \( X \) induced by \( l \). Suppose that \( \Sigma_{\mathcal{V} \mathcal{W}}g \) is one-dimensional.

Let \( \mathcal{V}' \) be the good stratification of \( X \) relative to \( l \), \( \mathcal{V}'' \) be the good stratification of \( X \) relative to \( g \) and \( \mathcal{V} \) be the good stratification of \( X \) induced by \( \tilde{g} = g + l^N, N \gg 1 \), taken as in Section 4.

Let \( \alpha \) be a regular value of \( g \), \( \alpha' \) a regular value of \( \tilde{g} \), \( 0 < |\alpha|, |\alpha'| \ll \epsilon \ll 1 \), \( n \) be the number of stratified Morse points of a Morsification of \( l|_{X\cap g^{-1}(\alpha)\cap B_\epsilon} \) appearing on \( X_{\text{reg}} \cap g^{-1}(\alpha) \cap \{ l \neq 0 \} \cap B_\epsilon \), \( n_i \) be the number of stratified Morse points of a Morsification of \( l|_{W_i \setminus \{ (g=0) \cup \{ l=0 \} \} \cap g^{-1}(\alpha) \cap B_\epsilon} \) appearing on \( W_i \cap g^{-1}(\alpha) \cap \{ l \neq 0 \} \cap B_\epsilon \), \( \tilde{n} \) be the number of stratified Morse points of a Morsification of \( l|_{X\cap \tilde{g}^{-1}(\alpha')\cap B_\epsilon} \) appearing on \( X_{\text{reg}} \cap \tilde{g}^{-1}(\alpha') \cap \{ l \neq 0 \} \cap B_\epsilon \), and \( \tilde{n}_i \) be the number of stratified Morse points of a Morsification of \( l|_{W_i \setminus \{ \tilde{g}=0 \} \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon} \) appearing on \( W_i \cap \tilde{g}^{-1}(\alpha') \cap \{ l \neq 0 \} \cap B_\epsilon \), for each \( W_i \in \mathcal{W} \).

As before, we write \( \Sigma_{\mathcal{V} \mathcal{W}}g \) as a union of branches \( b_1 \cup \ldots \cup b_r \) and we suppose that \( \{ l = \delta \} \cap b_j = \{ x_{i_1}, \ldots, x_{i_{k(j)}} \} \). For each \( t \in \{ i_1, \ldots, i_{k(j)} \} \), let \( D_{x_t} \) be the closed ball with center at \( x_t \) and radius \( r_t, 0 < |\alpha|, |\alpha'| < |\delta| < r_t < \epsilon \ll 1 \), sufficiently small for the balls \( D_{x_t} \) be pairwise disjoint and the union of balls \( D_j = D_{x_{i_1}} \cup \ldots \cup D_{x_{i_{k(j)}}} \) be contained in \( B_\epsilon \) and \( \epsilon \) is sufficiently small such that the local Euler obstruction of \( X \) at a point of \( b_j \cap B_\epsilon \) is constant.

In [20], Tibar gave a bouquet decomposition to for the Milnor fibre of \( \tilde{g} \) in terms of the Milnor fibre of \( g \). Let us denote by \( F_g \) the local Milnor fibre of \( g \) at the origin, \( F_{\tilde{g}} \) the local Milnor fibre of \( \tilde{g} \) at the origin and \( F_j \) is the local Milnor fibre of \( g|_{\{ l=\delta \}} \) at a point of the branch \( b_j \). Then there is a homotopy equivalence

\[
F_{\tilde{g}} \overset{ht}{\simeq} (F_g \cup E) \bigvee_{j=1}^r M_j S(F_j),
\]

where \( \bigvee \) denotes the wedge sum of topological spaces, \( M_j = Nm_{b_j} - 1 \), \( S(F_j) \) denotes the topological suspension over \( F_j, E := \bigcup_{j=1}^r \text{cone}(F_j) \) and \( F_g \cup E \) is the attaching to \( F_g \) of one cone over \( F_j \subset F_g \) for each \( j \in \{ 1, \ldots, r \} \). As a consequence of this theorem, Tibar proved a Lê-Iomdin formula for the Euler characteristic of these Milnor fibres.

In the following, we present a new proof for this formula using our previous results.

**Proposition 5.1.** Suppose that \( g \) is tractable at the origin with respect to \( \mathcal{V} \). If \( 0 < |\alpha|, |\alpha'| \ll |\delta| \ll \epsilon \ll 1 \), then

\[
\chi(X \cap g^{-1}(\alpha) \cap B_\epsilon) - \chi(X \cap g^{-1}(\alpha) \cap B_\epsilon) = N \sum_{j=1}^r m_{b_j} (1 - \chi(F_j)),
\]

where \( F_j = X \cap g^{-1}(\alpha) \cap H_j \cap D_{x_t} \) is the local Milnor fibre of \( g|_{\{ l=\delta \}} \) at a point of the branch \( b_j \) and \( H_j \) denotes the generic hyperplane \( l^{-1}(\delta) \) passing through \( x_t \in b_j \), for \( t \in \{ i_1, \ldots, i_{k(j)} \} \).

**Proof.** For a stratum \( V_i = W_i \setminus (\{ g = 0 \} \cup \{ l = 0 \}) \) in \( \mathcal{V}'' \), \( W_i \in \mathcal{W} \), let \( N_i \) be a normal slice to \( V_i \) at \( x_t \in b_j \), for \( t \in \{ i_1, \ldots, i_{k(j)} \} \) and \( D_{x_t} \) a closed ball of radius \( r_t \) centered at \( x_t \).
Considering the constructible function $1_X$, the normal Morse index along $V_i$ is given by

$$
\eta(V_i, 1_X) = \chi(W_i \setminus \{g = 0\} \cup \{l = 0\} \cap N_i \cap D_{x_i}) - \chi((W_i \setminus \{g = 0\} \cap N_i \cap \{g = \alpha\} \cap D_{x_i})
$$

For a stratum $V_i = W_i \setminus \{g = 0\} \in \mathcal{V}$, $W_i \in \mathcal{W}$, let $\tilde{N}_i$ be a normal slice to $V_i$ at $x_t \in b_j$, for $t \in \{i_1, \ldots, i_{k(j)}\}$. Considering the constructible function $1_X$, the normal Morse index along $V_i$ is given by

$$
\eta(\tilde{V}_i, 1_X) = \chi((W_i \setminus \{g = 0\}) \cap \tilde{N}_i \cap D_{x_i}) - \chi((W_i \setminus \{g = 0\}) \cap \tilde{N}_i \cap \{\tilde{g} = \alpha'\} \cap D_{x_i})
$$

Then applying Theorem 4.2 of [6] for $1_X$, we obtain that

$$
\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_e) - \chi(X \cap \tilde{g}^{-1}(\alpha') \cap l^{-1}(0) \cap B_e) = \sum_{i=1}^{q} (-1)^{d_i-1} n_i (1 - \chi(l_{W_i}))
$$

and that

$$
\chi(X \cap g^{-1}(\alpha) \cap B_e) - \chi(X \cap g^{-1}(\alpha) \cap l^{-1}(0) \cap B_e) = \sum_{i=1}^{q} (-1)^{d_i-1} n_i (1 - \chi(l_{W_i}))
$$

where $d_i = \dim W_i$.

Therefore, since $\chi(X \cap \tilde{g}^{-1}(\alpha' \cap l^{-1}(0) \cap B_e) = \chi(X \cap g^{-1}(\alpha) \cap l^{-1}(0) \cap B_e),$

$$
\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_e) - \chi(X \cap g^{-1}(\alpha) \cap B_e) = \sum_{i=1}^{q} (-1)^{d_i-1} (\tilde{n}_i - n_i) (1 - \chi(l_{W_i})).
$$

Applying Lemma 4.3 and Corollary 3.14, we obtain, for each $i$,

$$
\tilde{n}_i = n_i + (-1)^{d_i-1} N \sum_{j=1}^{r} m_{b_j} \chi(\tilde{l}_{W_i \cap \tilde{g}^{-1}(\alpha')} \cap b_j)
$$

$$
= n_i + (-1)^{d_i-1} N \sum_{j=1}^{r} m_{b_j} \chi(g_{W_i \cap H_j} \cap b_j),
$$

where $H_j$ denotes the generic hyperplane $l^{-1}(\delta)$ passing through $x_t \in b_j$, for $t \in \{i_1, \ldots, i_{k(j)}\}$.

Hence

$$
\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_e) - \chi(X \cap g^{-1}(\alpha) \cap B_e) = \sum_{i=1}^{q} \left( \sum_{j=1}^{r} m_{b_j} \chi(g_{W_i \cap H_j} \cap b_j) \right) (1 - \chi(l_{W_i}))
$$

$$
= \sum_{j=1}^{r} m_{b_j} \left( 1 - \chi(X \cap g^{-1}(\alpha) \cap H_j \cap D_{x_i}) \right)
$$

$$
= \sum_{j=1}^{r} m_{b_j} (1 - \chi(F_j)),
$$

for $t \in \{i_1, \ldots, i_{k(j)}\}$.  

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