A formulation of Rényi entropy on $C^*$-algebras

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Abstract
The entropy of probability distribution defined by Shannon has several extensions. Renyi entropy is one of the general extensions of Shannon entropy and is widely used in engineering, physics, and so on. On the other hand, the quantum analogue of Shannon entropy is von Neumann entropy. Furthermore, the formulation of this entropy was extended to on $C^*$-algebras by Ohya ($S$-mixing entropy). In this paper, we formulate Renyi entropy on $C^*$-algebras based on $S$-mixing entropy and prove several inequalities for the uncertainties of states in various reference systems.

Keywords Quantum information theory · Quantum entropy · $S$-mixing entropy · Rényi entropy · Quantum statistical mechanics · Operator algebras

1 Introduction
Shannon [14] introduced the entropy as the information amount of information systems represented by probability spaces. Rényi defined a general extension of Shannon entropy on probability spaces which is called Rényi entropy [12]. Rényi entropy is more general than Shannon entropy in the sense of a positive number $\alpha$, and it corresponds to Shannon entropy when $\alpha \to 1$. This entropy is useful and widely used in physics, engineering, and so on [3,4].

On the other hand, von Neumann entropy measures the complexity (or the information amount) of a quantum system [17]. In 1984, Ohya formulated the general
extension of von Neumann entropy which is called $S$-mixing entropy on $C^*$-algebras [6–8, 18]. $S$-mixing entropy depends on choosing subset (reference system) of the set of all states on the $C^*$-algebra. Thanks to the property, one can measure the uncertainty of the state depending on reference systems. Mukhamedov and Watanabe formulated an extension of $S$-mixing entropy by taking the set of all quantum channels as the reference system. Moreover, they showed that the entropy can apply to detect entangled states and calculated the complexities of qubit and phase-damping channels [5].

In this paper, we formulate Rényi entropy on $C^*$-algebras based on $S$-mixing entropy and show that the introduced entropy corresponds to $S$-mixing entropy when $\alpha \to 1$. Furthermore, we prove that our Rényi entropy is a general extension of quantum Rényi entropy [9, 15] whenever $\alpha > 1$. Moreover, by using our Rényi entropy, we investigate the uncertainties of states measured from various reference systems. These results obtained by mathematical methods of operator algebras bring new perspectives on quantum information theory.

We organize the paper as follows: In Sect. 2, we recall the notations and some properties of the Rényi entropy on probability spaces. Furthermore, we review the decomposition theory of states on $C^*$-algebras and the definition of $S$-mixing entropy. In Sect. 3, we formulate Rényi entropy on $C^*$-algebras based on the definition of $S$-mixing entropy and show several properties of it. Furthermore, by using the introduced entropy, we prove the equalities or inequalities of the complexities of states measured from different reference systems.

2 Preliminaries

In this section, we review the definitions of Rényi entropy and $S$-mixing entropy, and those several properties.

2.1 Rényi entropy

In this chapter, log denotes the logarithm of base 2.

Definition 1 Let $\{p_1, p_2, \ldots, p_n\}$ be the probability distribution of a random variable $X$. The Rényi entropy is defined by

$$S_\alpha(X) := \frac{1}{1 - \alpha} \log \sum_{k=1}^{n} p_k^\alpha, \quad \alpha \in [0, +\infty) \setminus \{1\}.$$  (1)

This entropy corresponds to the Shannon entropy when $\alpha \to 1$. Namely, the following theorem holds.

Theorem 1 Under the above assumptions,

$$\lim_{\alpha \to 1} S_\alpha(X) = - \sum_{k=1}^{n} p_k \log p_k$$  (2)

is satisfied.
Furthermore, Rényi entropy has the additivity.

**Theorem 2** If \(X\) and \(Y\) are independent random variables, 
\[
S_\alpha(X, Y) = S_\alpha(X) + S_\alpha(Y). 
\]
(3)

Moreover, since 
\[
\frac{\partial}{\partial \alpha} S_\alpha \leq 0, 
\]
(4)
one can see that this entropy is a decreasing function with respect to the parameter \(\alpha\).

Rényi entropy has important roles for the coding theory. For instance, the following theorem exists for the entropy [2,9].

Let \(\mathcal{X}\) be a finite alphabet set and \(X\) be a random variable of \(\mathcal{X}\). Let \(C\) be a source code, that is, a map from \(\mathcal{X}\) to the set of finite-length strings of symbols of a binary alphabet. Then \(C(x)\) denotes the codeword of \(x \in \mathcal{X}\) and \(l(x)\) denotes the length of \(C(x)\). Now, we define the cost of the coding:

\[
L_\beta(C) := \frac{1}{\beta} \log \sum_x p(x) 2^{\beta l(x)}, 
\]
where \(p(x)\) is the probability of \(x\) and \(\beta > -1\).

**Theorem 3** Let \(\alpha = 1/(1+\beta)\). For a uniquely decodable code, the following inequality holds:
\[
L_\beta(C) \geq S_\alpha(X). 
\]
(5)
Furthermore, there exists a uniquely decodable code \(C\) satisfying
\[
L_\beta(C) \leq S_\alpha(X) + 1. 
\]
(6)

### 2.2 Decomposition theory

A quantum state can be decomposed into simpler components. In this section, we recall the mathematical theory on the decompositions of states [1,15] that we need as follows.

First, we briefly recall the definitions of the \(C^*-\)algebra \(\mathcal{A}\), states, and \(*\)-automorphisms on \(\mathcal{A}\).

**Definition 2** \(\mathcal{A}\) is said to be a \(C^*-\)algebra if \(\mathcal{A}\) is a Banach*-algebra satisfying
\[
\|A^*A\| = \|A\|^2. \quad \forall A \in \mathcal{A}. 
\]
(7)

**Definition 3** Denote \(1_\mathcal{A}\) the identity on \(\mathcal{A}\). If a positive linear functional \(\varphi : \mathcal{A} \to \mathbb{C}\) satisfies \(\varphi(1_\mathcal{A}) = 1\), \(\varphi\) is called a state on \(\mathcal{A}\).

**Definition 4** A linear map \(\theta\) on \(\mathcal{A}\) is said to be a \(*\)-automorphism if:
1. \(\theta(AB) = \theta(A)\theta(B)\), \(\forall A, B \in \mathcal{A}\),
2. \( \theta(A^*) = \theta(A)^* \), \( \forall A \in \mathcal{A} \),
3. \( \ker \theta := \{ A \in \mathcal{A} \setminus \theta(A) = 0 \} = \{ 0 \} \).

**Remark 1** The observables of quantum systems are described by positive operators on \( C^*-\text{algebras} \ \mathcal{A} \):

\[ A = A^*, \ \text{Sp}(A) \subset [0, \infty); \ A \in \mathcal{A}. \]

Thus, the dynamics of a quantum system is described by the triplet \( (\mathcal{A}, \mathfrak{S}, \theta(G)) \), where \( \mathfrak{S} \) is the set of all states \( \varphi \) on \( \mathcal{A} \), and \( \theta(G) \) is the set of all \(*\)-automorphisms on \( \mathcal{A} \) associated with a group \( G [1,15] \).

**Example 1** Let \( \mathcal{H} \) be a Hilbert space, \( \mathbf{B}(\mathcal{H}) \) the set of all bounded operators on \( \mathcal{H} \), \( \rho \) a density operator (that is, a quantum state) and \( U_t \) a unitary operator on \( \mathcal{H} \) associated with \( t \in \mathbb{R} \). Put:

\[ \mathcal{A} := \mathbf{B}(\mathcal{H}), \ \varphi(\cdot) := \text{Tr}\rho(\cdot), \ \theta_t(\cdot) := U_t^* \cdot U_t. \]

Then the triplet \( (\mathcal{A}, \mathfrak{S}, \theta(\mathbb{R})) \) represents the dynamics of a quantum system.

Moreover, let \( I(\theta) \) be the set of all \( \theta \)-invariant states (i.e. \( \varphi \circ \theta_g = \varphi, \ \forall g \in G \)), and \( K_\beta(\theta) \) (\( G = \mathbb{R} \)) be the set of all states satisfying KMS condition with respect to \( \theta_t \) \( (t \in \mathbb{R}) \).

**Definition 5** The decomposition from an \( \theta \)-invariant state into extremal \( \theta \)-invariant states is called **ergodic decomposition**.

Since \( I(\theta) \) and \( K_\beta(\theta) \) are weak*-compact and convex subset of \( \mathfrak{S} \), we deal with the case where spaces have such conditions.

Let \( \mathcal{S} \) be a compact and convex subspace of a locally convex Hausdorff space. Moreover, let \( \text{ex} \mathcal{S} \) be the set of all extreme points of \( \mathcal{S} \). According to the Krein–Mil’man theorem [10], \( \text{ex} \mathcal{S} \neq \phi \) and the weak*-closure of convex hull of \( \text{ex} \mathcal{S} \) equals to \( \mathcal{S} \), i.e. \( \overline{\text{co}}^{w^*} \text{ex} \mathcal{S} = \mathcal{S} \).

**Definition 6** The decomposition from \( \mathcal{S} \) into \( \text{ex} \mathcal{S} \) is called **extremal decomposition**.

Let \( M(\mathcal{S}) \) be the set of all normal Borel measures on \( \mathcal{S} \). Furthermore, define

\[ M_1(\mathcal{S}) := \{ \mu \in M(\mathcal{S}), \mu(\mathcal{S}) = 1 \}. \]  \hspace{1cm} (8)

**Definition 7** For any \( \mu \in M(\mathcal{S}) \),

\[ b(\mu) := \int_{\mathcal{S}} \omega d\mu(\omega) \hspace{1cm} (9) \]

is called the **barycenter** of \( \mu \).
Moreover, let $C_{\mathbb{R}}(S)$ be the set of all real continuous functions on $S$ and

$$K(S) := \{ f \in C_{\mathbb{R}}(S); \ f \text{ are convex functions} \}.$$ 

For two measures $\mu, \nu \in M(S)$, define “$<$” as follows:

$$\mu < \nu \iff \mu(f) \leq \nu(f), \ \forall f \in K(S).$$

Then $<$ gives an ordering on $M(S)$. Let us denote $M^m(S)$ as the set of all maximal elements with respect to the ordering.

Furthermore, we recall the following theorems.

**Theorem 4** If $S$ is a metricable compact convex set,

1. $\text{ex} S$ is a $G_\delta$ set.
2. $\mu \in M_1^m(S)$ iff $\mu(\text{ex} S) = 1$.
3. For any $\varphi \in S$, there exist $\mu \in M_1^m(S)$ such that $\varphi = b(\mu)$.

**Theorem 5** If $S$ is a compact convex set,

1. Any $\mu \in M_1^m(S)$ has $\text{ex} S$ as their pseudo-support (i.e. for any Bair sets $Q$ such that $\text{ex} S \subset Q \subset S$, $\mu(Q) = 1$).
2. For any $\varphi \in S$, there exist $\mu$ which satisfy (1) such that $\varphi = b(\mu)$.

Moreover, we have the following theorem for uniqueness of maximal measure $\mu$.

Let $\mathcal{X}$ be a locally convex Hausdorff space, $S$ be a compact convex subset of $\mathcal{X}$, and $\mathcal{K}$ be a convex cone whose vortex is 0. Furthermore, let $S$ be the base of $\mathcal{K}$, i.e.

$$\mathcal{K} = \{ \lambda \omega; \ \lambda \geq 0, \ \omega \in S \}.$$ 

Then $\mathcal{K}$ is the convex cone generated by $\{1\} \times S$. Defining

$$\omega_1 \geq \omega_2 \iff \omega_1 - \omega_2 \in \mathcal{K},$$

then $\geq$ gives an ordering on $\mathcal{K}$.

**Definition 8** If $\mathcal{K}$ is the lattice with respect to the above $\geq$, $S$ is called *Choquet simplex*.

**Theorem 6** If $S$ is compact convex, the following are equivalent:

1. $S$ is a Choquet simplex.
2. For any $\varphi \in S$, there exists a unique maximal probability measure $\mu$.

Let $M_\varphi(S)$ be the set of all $\mu$ which is its barycenter equal to the state $\varphi$ on the $C^*$-algebra, i.e.

$$M_\varphi(S) := \{ \mu \in M_1(S), \ b(\mu) = \varphi \}. \quad (10)$$

For $\varphi$ satisfying (10), one obtains the integral representation of $\varphi$:

$$\varphi = \int_S \omega d\mu(\omega). \quad (11)$$
It is called the *barycentric decomposition* of $\varphi$. According to Theorem 6, this decomposition is not unique unless $S$ is a Choquet simplex.

Furthermore, we review the orthogonality of states. Let $\{H_\varphi, \pi_\varphi, x_\varphi\}$ be the GNS representation defined by $\varphi$. For $\varphi_1, \varphi_2 \in \mathcal{S}$, set $\varphi := \varphi_1 + \varphi_2 \in \mathcal{A}_+^*$. Then the following are equivalent:

1. Let $\psi \in \mathcal{A}_+^*$. If $\psi \leq \varphi_1$ and $\psi \leq \varphi_2$, $\psi = 0$.
2. There exists a projection $E \in \pi_\varphi(A)'$ such that $\varphi_1(A) = \langle x_\varphi, E\pi_\varphi(A)x_\varphi \rangle$, $\varphi_2(A) = \langle x_\varphi, (I - E)\pi_\varphi(A)x_\varphi \rangle$.
3. $H_\varphi = H_{\varphi_1} \oplus H_{\varphi_2}$, $\pi_\varphi = \pi_{\varphi_1} \oplus \pi_{\varphi_2}$, $x_\varphi = x_{\varphi_1} \oplus x_{\varphi_2}$.

**Definition 9** The states $\varphi_1, \varphi_2$ satisfying the above conditions are called *mutually orthogonal* and denoted by $\varphi_1 \perp \varphi_2$.

**Definition 10** For any Borel sets $Q \subset \mathcal{S}$ (i.e. $Q \in \mathcal{B}(\mathcal{S})$), $\mu \in M(\mathcal{S})$ satisfying $\left( \int_Q \omega d\mu \right) \perp \left( \int_{\mathcal{S} \setminus Q} \omega d\mu \right)$ is called *orthogonal measure* on $\mathcal{S}$.

We define $O_\varphi(\mathcal{S})$ as the set of all orthogonal probability measures whose barycenters are $\varphi$.

### 2.3 $S$-mixing entropy

If $\mu \in M_\varphi(S)$ has countable supports, that is, (11) can be written as

$$\varphi = \sum \lambda_k \varphi_k$$

where $\lambda_k > 0; \sum \lambda_k = 1$ and $\{\varphi_k\} \subset \text{ex}S$, we denote the set of all such measures as $D_\varphi(S)$.

**Definition 11** Under the above assumptions, the entropy of $\varphi \in S$ is given by

$$S^S(\varphi) := \begin{cases} \inf \left\{ - \sum \lambda_k \log \lambda_k : \mu = \{\lambda_k\} \in D_\varphi(S) \right\} \\
+\infty & (\mu \notin D_\varphi(S)) \end{cases}$$

The above entropy is called *$S$-mixing entropy*. Since one can regard that the complexity of the system is $+\infty$ if $\varphi$ has uncountable states, Ohya defined $S^S(\varphi) := +\infty (\mu \notin D_\varphi(S))$.

$S^S(\varphi)$ depends on the set $S$ chosen, thus it represents the amount of complexity of the state measured from the reference system $S$. That is, this entropy takes measuring the uncertainty of states from various reference systems into account.
Furthermore, if \( \varphi \) is faithful normal and \( S = \mathcal{S} \), this entropy corresponds to von Neumann entropy [6,15].

By the way, since one can regard that the complexities of real physical systems are finite, we denote the subset of \( S \) as

\[
S_r := \{ \varphi \in S; S^S(\varphi) < \infty \}.
\]

Since \( S = \overline{\text{co}^w} \text{ex}^S \), the following proposition holds.

Proposition 1

\[
\overline{S}^w_r = S. \quad (14)
\]

3 Rényi entropy on \( C^* \)-algebras

In this section, we define Rényi entropy on \( C^* \)-algebras based on \( S \)-mixing entropy and show that the introduced entropy includes \( S \)-mixing entropy and quantum Rényi entropy as the special cases. Furthermore, by using our Rényi entropy, we investigate the uncertainty of states in different reference systems.

Definition 12 Under the same assumptions and notations with Definition 11, we define:

\[
S^S_\alpha(\varphi) := \inf \left\{ (1 - \alpha)^{-1} \log \sum_k \lambda^\alpha_k \right\}; \quad \alpha \in [0, +\infty) \setminus \{1\}, \quad (15)
\]

where the infimum is taken over all \( \mu = \{\lambda_k\} \in D_\psi(S) \). Moreover, if \( \mu \notin D_\psi(S) \),

\[
S^S_\alpha(\varphi) := \infty.
\]

We call (15) \( S \)-mixing Rényi entropy.

From the analogue of classical case, one can see the following theorems:

Theorem 7 \( S^S_\alpha(\varphi) \) is monotone decreasing with respect to the parameter \( \alpha \).

Proof For any \( \varphi \in \mathcal{S} \), there holds

\[
\frac{\partial}{\partial \alpha} \left\{ (1 - \alpha)^{-1} \log \sum_k \lambda^\alpha_k \right\} \leq 0.
\]

Taking the infimum over all \( \{\lambda_k\} \), we have the theorem. \( \square \)

Proposition 2 For any \( \alpha \in [0, +\infty) \setminus \{1\} \), the positivity \( S^S_\alpha(\varphi) \geq 0 \) is satisfied.

Proof If \( \alpha \in [0, 1) \), due to the monotonicity of log,

\[
1 - \alpha > 0, \quad \sum_k \lambda^\alpha_k \geq 1 \Rightarrow \log \sum_k \lambda^\alpha_k > 0.
\]
Hence,
\[(1 - \alpha)^{-1} \log \sum_k \lambda_k^\alpha \geq 0, \quad \forall \alpha \in [0, 1).\]  
(16)

Similarly, if \( \alpha \in (1, +\infty) \),
\[1 - \alpha < 0, \quad \sum_k \lambda_k^\alpha < 1 \Rightarrow \log \sum_k \lambda_k^\alpha < 0.\]

Thus, we have
\[(1 - \alpha)^{-1} \log \sum_k \lambda_k^\alpha \geq 0, \quad \forall \alpha \in (1, +\infty).\]  
(17)

(16) and (17) give the positivity of \( S_\alpha(\varphi) \) for any \( \alpha \in [0, +\infty) \backslash \{1\} \).

Furthermore, in analogy with the classical case, we have the following theorem.

**Theorem 8** For any \( \varphi \in S \),
\[\lim_{\alpha \to 1} S_\alpha(\varphi) = S(\varphi)\]  
(18)

**Proof** According to the classical case, for \( \mu \in D(\varphi(S)) \),
\[\lim_{\alpha \to 1} (1 - \alpha)^{-1} \log \sum_k \lambda_k^\alpha = -\sum_k \lambda_k \log \lambda_k \]  
(19)

holds. We shall denote \( \tilde{S}_\alpha^S(\varphi) \) as \( (1 - \alpha)^{-1} \log \sum_k \lambda_k^\alpha \), \( \tilde{S}(\varphi) \) as \( -\sum_k \lambda_k \log \lambda_k \).

Then we have
\[0 \leq \inf_{[\lambda_k]} \tilde{S}_\alpha^S(\varphi) - \inf_{[\lambda_k']} \tilde{S}_\alpha(\varphi) = \sup \left( -\tilde{S}_\alpha^S(\varphi) \right) - \sup \left( -\tilde{S}_\alpha(\varphi) \right) \]
\[\leq \sup \left( \tilde{S}_\alpha^S(\varphi) - \tilde{S}_\alpha^S(\varphi) \right), \quad \forall \alpha > 1.\]  
(20)

\[0 \leq \inf_{[\lambda_k]} \tilde{S}_\alpha^S(\varphi) - \inf_{[\lambda_k']} \tilde{S}(\varphi) \leq \sup \left( \tilde{S}_\alpha^S(\varphi) - \tilde{S}(\varphi) \right), \quad 0 \leq \forall \alpha < 1.\]  
(21)

Due to (19), the right-hand sides of (20) and (21) go to 0 when \( \alpha \to 1 \). Therefore, we obtain the theorem.

Now, we prove that our \( S \)-mixing Rényi entropy includes the density case [9,15] whenever \( \alpha > 1 \). Let \( T(H) \) be the set of all trace class operators on a Hilbert space \( H \), and \( T(H)_{+,1} := \{ A \in T(H) : \text{Tr} A = 1 \} \).

**Definition 13** For any \( \rho \in T(H)_{+,1} \) and any \( \alpha \in [0, +\infty) \backslash \{1\} \), the quantum Rényi entropy is defined by
\[S_\alpha(\rho) := (1 - \alpha)^{-1} \log \text{Tr} \rho^\alpha.\]  
(22)
Lemma 1 Let \( \rho = \sum_n \lambda_n \rho_n \) be the decomposition into pure states (i.e. \( \dim(\text{ran} \rho_n) = 1 \)). For any \( \alpha > 1 \),
\[
S_\alpha(\rho) \leq (1 - \alpha)^{-1} \log \sum_n \lambda_n^\alpha
\]
holds. If \( \rho_n \perp \rho_m (n \neq m) \), one obtains the equality.

**Proof** Let \( \rho = \sum_k p_k E_k \) be the Schatten decomposition \([13]\) of \( \rho \). Then for any \( n \in \mathbb{N} \),
\[
\sum_{k=1}^n p_k \geq \sum_{k=1}^n \lambda_n
\]
is satisfied \([15]\). Therefore, we have \( \sum_{k=1}^n p_k^\alpha \geq \sum_{k=1}^n \lambda_n^\alpha (\forall \alpha \in [0, +\infty) \setminus \{1\}) \). Moreover, according to the monotonicity of log,
\[
(1 - \alpha)^{-1} \log \sum_{k=1}^n p_k^\alpha \leq (1 - \alpha)^{-1} \log \sum_{k=1}^n \lambda_n^\alpha, \quad \forall \alpha > 1.
\]
Since \( 0 \leq \sum_{k=1}^n p_k^\alpha < 1 \) (resp. \( 0 \leq \sum_{k=1}^n \lambda_k^\alpha < 1 \)), there exists the limit:
\[
\lim_{n \to \infty} \log \sum_{k=1}^n p_k^\alpha \text{ (resp. } \lim_{n \to \infty} \log \sum_{k=1}^n \lambda_k^\alpha \text{).}
\]
Thus, we have
\[
(1 - \alpha)^{-1} \log \sum_{k=1}^\infty p_k^\alpha \leq (1 - \alpha)^{-1} \log \sum_{k=1}^\infty \lambda_k^\alpha, \quad \forall \alpha > 1.
\]
This gives the inequality (23).

Moreover, if \( \rho_n \perp \rho_m (n \neq m) \), \( \rho = \sum_n \lambda_n \rho_n \) becomes the Schatten decomposition of \( \rho \). Thus, \( \lambda_n = p_n \). Therefore,
\[
S_\alpha(\rho) = (1 - \alpha)^{-1} \log \sum_n \lambda_n^\alpha.
\]
\( \square \)

Using this lemma, we prove the following theorem.

**Theorem 9** Let \( \mathcal{A} \) be a \( C^* \)-algebra. If a state \( \varphi \) can be written as \( \varphi(A) = \text{Tr} \rho A (\forall A \in \mathcal{A}) \),
\[
S_\alpha^\mathcal{S}(\varphi) = S_\alpha(\rho), \quad \forall \alpha > 1,
\]
where \( \mathcal{S} \) is the set of all states on \( \mathcal{A} \).

**Proof** Let \( \rho = \sum_k \lambda_k \rho_k \) be the decomposition into pure states \( \rho_k \) (i.e. \( \rho_k^2 = \rho_k \), \( \forall k \)). Denoting
\[
\varphi_k(A) = \text{Tr} \rho_k A (\forall A \in \mathcal{A}),
\]

\( \square \) Springer
then $\varphi = \sum \lambda_k \varphi_k$ is the extremal decomposition. Furthermore, if $\varphi \in \text{ex} \mathcal{S}$, $\rho$ is a pure state (i.e. $\rho = \rho^2$). Therefore according to Lemma 1,

$$S_\alpha(\varphi) = \inf \left\{ (1 - \alpha)^{-1} \log \sum \lambda_k^\alpha \right\} = S_\alpha(\rho)$$

holds. \hfill \Box

Therefore, if $\alpha > 1$, $\mathcal{S}$-mixing Rényi entropy includes the quantum Rényi entropy as the special case. On the other hand, if $0 \leq \alpha < 1$, the following inequality holds.

**Theorem 10** Under the above settings, for any $0 \leq \alpha < 1$,

$$S_\alpha^\mathcal{S}(\varphi) \leq S_\alpha(\rho). \tag{25}$$

**Proof** If $0 \leq \alpha < 1$, there holds

$$(1 - \alpha)^{-1} \log \sum \lambda_n^\alpha \leq (1 - \alpha)^{-1} \log \sum p_n^\alpha.$$ 

This result induces the inequality (25). \hfill \Box

### 3.1 Density case

Since $\mathcal{S}$-mixing Rényi entropy depends on $\mathcal{S}$, we can consider the complexity of the state measured from the reference system $\mathcal{S}$. In this chapter, we study the complexities of density operators by taking different reference systems.

Let $\mathbf{C}(\mathcal{H})$ be the set of all compact operators on $\mathcal{H}$. Then $\mathcal{A} := \mathbf{C}(\mathcal{H}) + \mathbb{C} I$ is a $C^*$-algebra. Now, let $\theta(\mathbb{R})$ be the set of all 1-parameter strongly continuous automorphisms on $\mathcal{A}$ and let

$$\theta_t(\cdot) := U_t \cdot U_{-t}, \quad \theta_t \in \theta(\mathbb{R}),$$

where $U_t$ is a unitary operator on $\mathcal{A}$.

Furthermore, when $\mathcal{S} = \mathcal{S}$, we simply denote $S_\alpha^\mathcal{S}(\varphi)$ by $S_\alpha(\varphi)$.

**Theorem 11** If $\varphi$ is faithful and $\theta$-invariant, and if eigenvalues of $\rho$ are non-degenerate,

$$S_\alpha^{I(\theta)}(\varphi) = S_\alpha(\varphi) \tag{26}$$

holds.

**Proof** Since $\varphi \in I(\theta)$, for any $t \in \mathbb{R}$ and unitaries $U_t$, $[U_t, \rho] = 0$ holds. Moreover, if $\varphi$ is faithful, $\rho > 0$ is satisfied. Furthermore, since the eigenvalues of $\rho$ are non-degenerate, we can put $\rho = |x_k\rangle\langle x_k|$, where $x_k$ are any eigenvectors of $\rho$.

Therefore, for any $t \in \mathbb{R}$ and any $k$,

$$[U_t, \rho_k] = 0$$
holds. Hence, $\rho_k \in I(\theta)$. Thus, we obtain the following inequality:

$$S_\alpha(\varphi) \geq S_\alpha^I(\theta)(\varphi).$$

Next, we show the opposite inequality. Let $\varphi = \sum \lambda_k \varphi_k$ be the ergodic decomposition (i.e. $\varphi_k \in \text{ex} I(\theta)$), and $\rho_k$ be a density adjusted $\varphi_k$. Then $\rho_k$ is a pure state. Therefore, $\varphi_k \in \text{ex} \mathcal{G}$. Hence,

$$S_\alpha(\varphi) \leq S_\alpha^I(\theta)(\varphi).$$

\[ \square \]

**Theorem 12** If $\varphi \in K_\beta(\theta)$, $S^K_\alpha(\theta) = 0$.

**Proof** Let $H$ be a Hamiltonian of a physical system, and $\beta := 1/kT$ ($k$; the Boltzmann constant, $T$; the temperature). Denote

$$\rho = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}}, \quad e^{-\beta H} \in T(H)$$

and

$$\varphi(A) := \text{Tr} \rho A, \quad A \in A.$$ 

Then $\varphi$ is a unique KMS state for $\beta$ and $\theta_t(A) := u_t Au_{-t}$ ($u_t := \exp(itH)$). Therefore, if $\varphi \in K_\beta(\theta)$, from uniqueness of a state,

$$S^K_\alpha(\theta)(\varphi) = 0.$$

\[ \square \]

**3.2 General case**

In this section, we study the complexities of general states by taking different $S$.

**Theorem 13** For any KMS states $\varphi \in K_\beta(\theta)$ and any $\alpha \in [0, +\infty) \setminus \{1\}$, the following inequalities hold:

1. $S_\alpha^I(\theta)(\varphi) \geq S^K_\alpha(\theta)(\varphi)$.
2. $S_\alpha(\varphi) \geq S^K_\alpha(\theta)(\varphi)$.

**Proof** 1. The decomposition from $\varphi \in K_\beta(\theta)$ into $\text{ex} K_\beta(\theta)$ is unique [1]. We put the decomposition $\varphi = \sum \lambda_n \varphi_n$. Then $\varphi_n \perp \varphi_m$ ($n \neq m$) holds. On the other hand, since $\text{ex} K_\beta(\theta) \subset I(\theta)$, $\varphi_n$ can be decomposed into the elements of $\text{ex} I(\theta)$, that is, ergodic states. Let

$$\varphi_n = \sum_k \mu_k^{(n)} \psi_k, \quad \psi_k \in \text{ex} I(\theta)$$
be the ergodic decomposition. Because of the uniqueness of the decomposition into $\varphi_n$, we can regard $(1 - \alpha)^{-1} \log \sum_n (\lambda_n)^\alpha$ as the constant. Therefore, for any $\alpha \in [0, +\infty) \setminus \{1\}$,

$$
(1 - \alpha)^{-1} \log \sum_{k,n} (\lambda_n \mu_k^{(n)})^\alpha \\
= (1 - \alpha)^{-1} \log \sum_n (\lambda_n)^\alpha \sum_k (\mu_k^{(n)})^\alpha \\
= \frac{1}{\alpha - 1} \left\{ -\log \sum_n \lambda_n^\alpha + \left( -\log \sum_k (\mu_k^{(n)})^\alpha \right) \right\} \\
\geq (1 - \alpha)^{-1} \log \sum_n \lambda_n^\alpha = S^K_\alpha (\varphi).
$$

By taking the infimum over all $\{\mu_k^{(n)}\}$, we obtain $S^I_\alpha (\varphi) \geq S^K_\alpha (\varphi)$.

2. Due to $\text{ex} K_\beta (\theta) \subset \mathcal{S}$, $\varphi_n \in \text{ex} K_\beta (\theta)$ is decomposed as:

$$
\varphi_n = \sum_l \tilde{\mu}_l^{(n)} \tilde{\psi}_l, \quad \tilde{\psi}_l \in \text{ex} \mathcal{S}.
$$

Therefore, the $\mathcal{S}$-mixing Rényi entropy of $\varphi \in K_\beta (\theta)$ measured from $\mathcal{S} = \mathcal{S}$ is given by

$$
S_\alpha (\varphi) = \inf_{\{\tilde{\mu}_l^{(n)}\}} \left\{ (1 - \alpha)^{-1} \log \sum_{l,n} (\lambda_n \tilde{\mu}_l^{(n)})^\alpha \right\} \\
= \inf_{\{\tilde{\mu}_l^{(n)}\}} \left\{ (1 - \alpha)^{-1} \log \sum_l (\tilde{\mu}_l^{(n)})^\alpha \right\} + S^K_\beta (\theta) (\varphi),
$$

$\forall \alpha \in [0, +\infty) \setminus \{1\}$.

This completes the proof. \hfill $\Box$

Moreover, in order to investigate the inequality between $S^I_\alpha (\varphi)$ and $S_\alpha (\varphi)$, we need $G$-commutativity of $(\mathcal{A}, \theta(G))$. Thus, we recall the definition.

Let $(\mathcal{H}_\varphi, \pi_\varphi, \chi_\varphi)$ be the GNS-representation defined by $\varphi$ and $\{u_\varphi^g; \ g \in G\}$ be the strongly continuous unitary group on $\mathcal{H}_\varphi$.

**Definition 14** Let $E_\varphi$ be a projection from $\mathcal{H}_\varphi$ to the set of $u_\varphi^g$-invariant vectors. If $E_\varphi \pi_\varphi (\mathcal{A})'' E_\varphi$ is a commutative von Neumann algebra, $(\mathcal{A}, \theta(G))$ is called $G$-commutative for $\varphi$.

Furthermore, we mention the following theorem.

**Theorem 14** For $\varphi \in I(\theta)$, the following are satisfied:
1. There exists $\mu \in O_{\psi}(I(\theta))$ whose pseudo-support is $\text{ex}I(\theta)$.
2. If $(A, \theta(G))$ is $G$-commutative, $I(\theta)$ is a Choquet simplex. Therefore, the above $\mu$ is a unique maximal measure.

**Proof** See the proof of Corollary 4.3.32 of [1].

Now, we prove the following inequalities.

**Theorem 15** If $(A, \theta(\mathbb{R}))$ is $G$-commutative for $\varphi$,

$$S_\alpha(\varphi) \geq S^{I(\theta)}_\alpha(\varphi) \geq S^{K(\theta)}_\alpha(\varphi).$$

(27)

**Proof** According to Theorem 14, the ergodic decomposition of $\varphi$ is unique. Hence, the first inequality is satisfied. The second one holds from Theorem 13.

4 Conclusion

In this paper, we have formulated a Rényi entropy on $C^*$-algebras, investigated its properties, and shown that it is a generalization of the $S$-mixing entropy. Furthermore, we have derived that the $S$-mixing Rényi entropy includes the traditional quantum Rényi entropy whenever $\alpha > 1$. Besides, we have seen that the new Rényi entropy is smaller than the quantum Rényi entropy if $0 \leq \alpha < 1$. Moreover, using the $S$-mixing Rényi entropy, we have given inequalities about uncertainty when a state is measured from different reference systems. These results tell us that the $S$-mixing Rényi entropy is applicable to both studies of quantum information theory and operator algebras. Therefore, using the new entropy, one can expect to discover new connections between quantum information theory and theory of operator algebras.

Incidentally, the classical Rényi entropy has extensions to mutual information and unified entropy [11,16]. We emphasize that the $S$-mixing Rényi entropy is also capable of such extensions.

**References**

1. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics I. Springer, New York (1987)
2. Campbell, L.L.: A coding theorem and Rényi entropy. Inf. Control 8, 429–523 (1965)
3. Hughes, M.S., Marsh, J.N., Arment, J.M., Neumann, R.G., Fuhrho, R.W., Wallace, K.D., Thomas, L., Smith, J., Agyem, K., Lanza, G.M., Wickline, S.A., McCarthy, J.E.: Application of Rényi entropy for ultrasonic molecular imaging. J. Acoust. Soc. Am. 125(5), 3141–3145 (2007)
4. Kusaki, Y., Takayanagi, T.: Renyi entropy for local quenches in 2D CFT from numerical conformal blocks. J. High Energy Phys. 01, 115 (2018)
5. Mukhamedov, F., Watanabe, N.: On $S$-mixing entropy of quantum channels. Quantum Inf. Process 17, 148–168 (2018)
6. Ohy, M.: Entropy transmission in $C^*$-dynamical systems. J. Math. Anal. Appl. 100, 222–235 (1984)
7. Ohy, M., Petz, D.: Quantum Entropy and Its Use. Springer, Berlin (1993)
8. Ohy, M., Watanabe, N.: Foundation of Quantum Communication Theory. Makino Pub. Co., Tokyo (1998)
9. Petz, D.: Quantum Information Theory and Quantum Statistics. Springer, Berlin (2008)
10. Phelps, R.R.: Lecture on Choquet’s Theorem. Van Nostrand, New York (1966)
11. Rathie, P.N.: Unified \((r, s)\)-entropy and its bivariate measures. Inf. Sci. 54, 23 (1991)
12. Rényi, A.: On the foundations of information theory. Rev. Int. Stat. Inst. 33, 1–14 (1965)
13. Schatten, R.: Norm Ideals of Completely Continuous Operators. Springer, Berlin (1970)
14. Shannon, C.E.: Mathematical theory of communication. Bell Syst. Tech. J. 27, 379–423, 623–656 (1948)
15. Umegaki, H., Ohya, M.: Quantum Entropies. Kyoritsu, Tokyo (1984)
16. Verdú, S.: \(\alpha\)-mutual information. In: Proceedings of ITA, San Diego, CA, USA, pp. 1–6 (2015)
17. von Neumann, J.: Die Mathematischen Grundlagen der Quantenmechanik. Springer, Berlin (1932)
18. Watanabe, N.: Note on entropies of quantum dynamical systems. Found. Phys. 41, 549–563 (2011)

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