Federated Singular Vector Decomposition

Di Chai  
dchai@cse.ust.hk  
Hong Kong University of Science and Technology  
Hong Kong, China

Leye Wang  
leyewang@pku.edu.cn  
Peking University  
Beijing, China

Lianzhi Fu  
fullanzhi@clustar.ai  
Clustar  
Shenzhen, China

Junxue Zhang  
jzhangcs@cse.ust.hk  
Hong Kong University of Science and Technology  
Hong Kong, China

Kai Chen  
kaichen@cse.ust.hk  
Hong Kong University of Science and Technology  
Hong Kong, China

Qiang Yang  
qyang@cse.ust.hk  
Hong Kong University of Science and Technology  
Hong Kong, China

ABSTRACT
With the promulgation of data protection laws (e.g., GDPR in 2018), privacy preservation has become a general agreement in applications where cross-domain sensitive data are utilized. Out of many privacy-preserving techniques, federated learning (FL) has received much attention as a bridge for secure data connection and cooperation. Although FL’s research works have surged, some classical data modeling methods are not well accommodated in FL. In this paper, we propose the first masking-based federated singular vector decomposition method, called FedSVD. FedSVD protects the raw data through a singular value invariance mask, which can be further removed from the SVD results. Compared with prior privacy-preserving SVD solutions, FedSVD has lossless results, high confidentiality, and excellent scalability. We provide privacy proof showing that FedSVD has guaranteed data confidentiality. Empirical experiments on real-life datasets and synthetic data have verified the effectiveness of our method. The reconstruction error of FedSVD is around 0.000001% of the raw data, validating the lossless property of FedSVD. The scalability of FedSVD is nearly the same as the standalone SVD algorithm. Hence, FedSVD can bring privacy protection almost without sacrificing any computation time or communication overhead.

1 INTRODUCTION
Two years have passed since the GDPR (General Data Protection Regulation) was released in 2018. Now the private data utilizations in enterprises are closely monitored by the government and customers. Concerns regarding data confidentiality have been growing in applications where cross-domain sensitive data are employed. Privacy preservation has become a general agreement in data-driven applications, and privacy-preserving artificial intelligence (PPAI) has become a hot research topic. Out of many PPAI techniques, federated learning (FL) technology has received much attention. FL is first proposed by Google [23], which targets learning a centralized model based on personal information distributed among mobile devices. While Google’s solution targets individual consumers, WeBank formulated a to-business solution known as vertical federated learning which aims at enterprise-level collaboration under privacy-preserving and security constraints [29].

Although the researches and applications of FL have surged, federated adaptations of classic data modeling techniques are still not gratifying. One representative is singular vector decomposition (SVD), which provides a means of decomposing a matrix into a product of three simpler matrices so that one can discover useful and interesting properties of the original matrix. SVD is widely utilized in machine learning algorithms and applications, e.g., principal component analysis (PCA) [28], latent semantic analysis (LSA) [9], noise filtering [12, 27], recommendation system [26, 30], etc. Prior works have studied privacy-preserving SVD in two main types:

- **Noising based methods**: Polat et al. [26] proposed a private SVD-based collaborative filtering (CF) through randomized perturbation. The data with noise is gathered to the server which will subsequently carry out the SVD and CF. Grammenos et al. [11] proposed a federated and (ε, δ)-differently private principal component analysis method. The differential privacy (DP) based methods typically perform a tradeoff between privacy and data utility. However, the data utility tends to be very low when we have a high privacy requirement, and DP indeed cannot guarantee data confidentiality [16].

- **Distributed training methods**: Han et al. [14] proposed a secure multi-party QR decomposition framework and solved the SVD problem through eigendecompositions twice. However, eigendecomposition is not an efficient way to solve the SVD problem [25]. Hegedűs et al. [15] proposed a gradient descent based fully distributed SVD method in which the participants jointly update the SVD results. The distributed training methods keep the data locally, which reduce the risk of private data leakage. However, the distributed training brings communication overhead because the algorithms usually need to run many rounds to converge [17]. Meanwhile, the distributed training also suffers from system heterogeneities [5], e.g., stragglers and dropouts.

In this paper, different from prior works' strategies, we propose the first masking-based federated singular vector decomposition framework, called FedSVD, which can generate lossless SVD results. FedSVD protects private data using a singular value invariance masking method. To accelerate the computation speed and reduce communication overhead, FedSVD adopts a server-aided secure computation mechanism like [2]. In particular, all the participants will jointly apply a well-designed random mask to the raw data, then expose the masked data to an honest-but-curious third-party which will subsequently run a standard SVD algorithm using the masked...
data. After getting the masked SVD results, all the participants will jointly remove the mask and get the real results. Compared with existing solutions, FedSVD has the following strengths:

- **Lossless SVD results**: The random masks can be removed from the SVD results and thus FedSVD has no performance reduction compared with doing SVD with all data collected in one place. The proof of lossless precision for our masking based method can be found in Section 4.1.
- **Low client engaging**: Our FedSVD solution only bothers the data owners twice. At the beginning and ending of the algorithm, clients jointly add the masks to the data and then remove them from the results.
- **Guaranteed data confidentiality**: We provide a privacy proof in Section 4.3, showing that FedSVD has high data confidentiality. Briefly, we prove that no party could learn extra knowledge apart from the final results.
- **Model-agnostic**: FedSVD does not specify which model is used to solve SVD and thus any existing SVD model could be used (e.g., LAPACK1). In the future, the new state-of-the-art model could also be quickly adopted into FedSVD.

In summary, we have the following contributions in this paper:

- We present the first **federated, lossless, low client engagement, data confidentiality guaranteed, and highly scalable SVD framework**.
- We provide formal proof showing that FedSVD has lossless precision (Section 4.1) and the data confidentiality is guaranteed in FedSVD (Section 4.3). To the best of our knowledge, this is the first paper in FL that formally defines and proves the framework’s privacy protection.
- We implement FedSVD and evaluate it on real-world and synthetic datasets. The results show that (1) FedSVD has lossless data utility compared with state-of-the-art differential private based method [11], (2) FedSVD has the same scalability as the standalone SVD algorithm, which is remarkable because we usually need to pay excessive time or communication to guarantee the data confidentiality [4, 23, 25].

## 2 PRELIMINARIES

In this section, we briefly introduce several technologies that are closely related to our method.

### 2.1 Singular Vector Decomposition (SVD)

Given an arbitrary matrix \( X \in \mathbb{R}^{m \times n} \), SVD decomposes \( X \) into a product of three matrices: \( X = U \Sigma V^T \), where \( U \in \mathbb{R}^{m \times m} \), \( \Sigma \in \mathbb{R}^{m \times n} \), and \( V \in \mathbb{R}^{n \times n} \). According to the property of SVD, \( U \) and \( V \) are orthogonal matrices.

Through many years of exploration, there are many methods to solve the SVD problem [25], e.g., through eigendecomposition, householder bidiagonalization, and divide-and-conquer based methods. In this paper, we do not specify the SVD solver, and FedSVD works perfectly with all existing methods.

1http://www.netlib.org/lapack/explore-html/

### 2.2 Secure Aggregation

Secure aggregation is a set of secure multi-party computation algorithms wherein a group of \( K \) parties each hold a private value \( x_i \ (1 \leq i \leq K) \) and collaborate to compute an aggregate value \( x = \sum_i x_i \), such that \( x_i \) remains oblivious to each participant.

Here we introduce a practical secure aggregation protocol proposed by [3] using One-Time Pads. The algorithm builds on an assumption that all parties complete the protocol and process pairwise secure communication channels with ample bandwidth. Each pair of parties first agree on a matched pair of input perturbations. That is, party \( i \) samples a vector \( r_{i,j} \) uniformly from \([0, R)\) for each other party \( j \). Party \( u \) and \( v \) exchange \( r_{i,j} \) and \( r_{j,i} \) over their secure channel and compute perturbations \( p_{i,j} = r_{i,j} - r_{j,i} \ (\text{mod} \ R) \), noting that \( p_{i,j} = -p_{j,i} \) and \( p_{i,j} = 0 \) when \( i = j \). Each user sends distorted data \( y_i = x_i + \sum_{j=1}^{K} p_{i,j} \ (\text{mod} \ R) \). The server simply sums the perturbed values: \( \bar{x} = \sum_{i=1}^{K} y_i \). Correctness is guaranteed because the paired perturbations in \( y_i \) cancel out:

\[
\bar{x} = \sum_{i=1}^{K} x_i + \sum_{i=1}^{K} \sum_{j=1}^{K} p_{i,j} = \sum_{i=1}^{K} x_i + \sum_{j=1}^{K} \sum_{i=1}^{K} r_{i,j} = \sum_{i=1}^{K} r_{i,j} = 0
\]

In this paper, we use the above practical secure aggregation protocol, and denote it as a function: \( x = \text{SecureAggregation}(x_1, ..., x_K) \).

### 2.3 Security Definitions

In the cryptographic study, we usually categorize the adversaries into two types based on how willing they are to deviate from the protocol. And they are *semi-honest* and *malicious* adversaries [21]. The definitions are listed below:

- **Semi-honest** means that the adversaries will strictly follow the prespecified protocol, however, will collect all the received data and try to derive knowledge from it.
- **Malicious** means that the adversaries are corrupted and can deviate from the protocol, e.g., changing the inputs and outputs.

Following the typical security definition in FL [29], we assume that all the participants are semi-honest in this paper.

## 3 PROBLEM FORMULATION

In this section, we formally define the federated SVD problem. Assume we have \( K \) parties, and each party has data matrix \( x_i \in \mathbb{R}^{m \times n} \). Those \( K \) parties would like to do an SVD jointly on data \( X = [x_1, x_2, ..., x_K] \), where \( X \in \mathbb{R}^{m \times n} \) and \( n = \sum_i n_i \).

Thus we want to design a federated SVD method, such that the \( i \)-th party (\( 1 \leq i \leq K \)) gets \( x_i = U \Sigma V_i^T \), where \( U, \Sigma \) are shared among all participants and \( V_i^T \) is the secret result possessed by party-\( i \). Figure 1 illustrates the problem formulation. The \( V_i^T \) are vertically partitioned and secretly distributed to different parties. We want to guarantee that party-\( i \)’s data (i.e., \( x_i \) ) is not leaked to any other parties during the computation.

In this paper, we assume that \( m < n \) holds for all parties. Thus the full SVD results of \( X \) are: \( U \in \mathbb{R}^{m \times m} \), \( \Sigma \in \mathbb{R}^{m \times n} \) and \( V_i^T \in \mathbb{R}^{n_i \times n} \), because \( m < n \). After the federated SVD algorithm, the data holder \( i \) gets \( U \in \mathbb{R}^{m \times m} \), \( \Sigma \in \mathbb{R}^{m \times n} \) and \( V_i^T \in \mathbb{R}^{n_i \times n_i} \), which have exactly the same size with local SVD on \( x_i \). Thus all data holders
get no extra knowledge apart from a group of new SVD results which has the same size with local SVD.

4 METHOD

In this section, we introduce the FedSVD method. According to the functionality, we formulate three types of roles in our method: factorization server, masking server, and the data holders. We show that the system is confidential when all the participants are semi-honest.

4.1 Singular Value Invariance Masking

The privacy-preserving SVD methods emphasize the confidentiality of the data matrix. Here we propose a method to mask the raw data, and the masks are removable from the final results. Our idea is inspired by the similar matrices: matrix $B$ is similar to $A$ when $B = P^{-1}AP$. It is well known that similar matrices (e.g., $A$ and $B$) have the same eigenvalues. In SVD, denote the data matrix as $X$, we use two random orthogonal matrices $P$, $Q$ to mask the data $X$ as $X' = PXQ$. $X'$ has the same singular value with $X$, which is proved in Theorem 4.1. The singular vectors of $X$ and $X'$ can be orthogonal transformed to each other, and the transformation matrices are $P$ and $Q$. Thus we can get the singular vectors of $X$ through an orthogonal transformation on the singular vectors of $X'$ (i.e., the masks can be removed from the results).

**Theorem 4.1.** For an arbitrary matrix $X \in \mathbb{R}^{m \times n}$, we can use two random orthogonal matrices $P \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{n \times n}$ to mask $X$ into $X' = PXQ$. $X'$ has the same singular values with $X$, and their singular vectors can be orthogonal transformed to each other.

**Proof.** According to the definition of SVD, we can decompose $X$ into $U \Sigma V^T$, thus $X' = PXQ = P(U \Sigma V^T)Q$. Denote $U' = PU$, $V' = V^T Q$, then $X'$ can be decomposed into $X' = U' \Sigma V'^T$ if $U'$ and $V'^T$ are orthogonal matrices.

$$
U'^{-1} = (PU)^{-1} = U^{-1} P^{-1} = U^T P^T = (PU)^T = U'^T
t
(V'^T)^{-1} = (V^T Q)^{-1} = Q^{-1} (V^T)^{-1}

= Q^T V = (V^T Q)^T = (V'^T)^T
$$

Thus $U'$ and $V'^T$ are orthogonal, $X'$ and $X$ have the same singular values $\Sigma$, and the singular vectors can be orthogonal transformed to each other through: $U = P^T U'$, $V = Q V'$.

---

**Algorithm 1:** Generate Random Orthogonal Matrix

**Input:** Dimension of the matrix $d$

**Output:** Orthogonal matrix $Q \in \mathbb{R}^{d \times d}$

1. Function Orthogonal ($d$);  
2. Random sample matrix $R \in \mathcal{N}(0, 1)^{d \times d}$
3. $[Q, \sim] = QR(R) \ // \ QR$ Decomposition
4. End Function

We present a random orthogonal matrix generation method in Algorithm 1 using the QR decomposition. It is proved in prior work [13] that QR decomposition of Gaussian matrices produces uniformly distributed orthogonal matrices. However, the complexity of QR decomposition on a $n$ dimensional square matrix is $O(n^3)$ using Gram-Schmidt process [6]. Thus we propose an efficient random orthogonal matrix generation algorithm through building blocks, which is presented in Algorithm 2. Briefly, we decompose the problem of generating a $d$ dimensional orthogonal matrix into generating $b$ small orthogonal matrices, which are subsequently placed at the diagonal position forming a $d$ dimensional matrix. It is worth noting that we cannot use an overlarge $b$ because the orthogonal matrix’s degree of freedom decreases with the increase of $b$, which may reduce the effectiveness of privacy protection. We performed an empirical experiment in Section 6.4, and the results show that the data is still confidential when we set $b = 100$ for 100,000 MNIST images. If we fix the size of block matrices (i.e., $Q_i$ in Algorithm 2) to $c$ and $b = \frac{d}{c}$, then the complexity of Algorithm 2 is $O(c^3n) = O(c^2n) = O(n)$.

4.2 FedSVD

Using the singular value invariance masking method, we propose the FedSVD method, which is presented in Figure 2 and Algorithm 3. According to the functionality, we specify three types of roles in our system:

- Factorization Server: The factorization server receives the masked data $X'$, and runs a standard SVD algorithm factorizing $X'$ into $U' \Sigma V'^T$.
- Masking Server: The masking server generates the random orthogonal matrices $P$ and $Q$.
- Data Holders: The participants that jointly hold the data matrix $X$.

Briefly, FedSVD has the following four steps:
Algorithm 2: Efficient Orthogonal Matrix Generation Through Building Blocks

Input: Dimension of the matrix d
Input: Number of building blocks b
Output: Orthogonal matrix Q ∈ ℝ^{d×d}

1 Function EfficientOrthogonal(d, b):
  2 Uniformly divide d into [d₁, ..., d_b], where \( \sum_{i=1}^{b} d_i = d \)
  3 for i = 1 → b do
  4     \( Q_i = \text{Orthogonal}(d_i) \) / \( Q_i \in \mathbb{R}^{d_i \times d_i} \)
  5 end
  6 \( Q = \begin{bmatrix} Q_1 & \cdots & 0 \\
                     \vdots & \ddots & \vdots \\
                     0 & \cdots & Q_b \end{bmatrix} \)

End Function

Algorithm 3: FedSVD

Input: \( X = [X_1 | ... | X_K] \)
Output: \( U, \Sigma \) and \( V^T = [V_1^T | ... | V_K^T] \)
Constraint: Data holder \( i \)'s data is not leaked to the others, and it gets \( U, \Sigma, \) and \( V_i^T \) (i.e., \( X_i = U \Sigma V_i^T \)) as the results.

1 Function FedSVD(\( [X_1 | ... | X_K] \)):
  2 Masking Server do:
    3 Generates orthogonal matrices \( P, Q \).
    4 Then split \( Q^T \) into \( [Q_1^T | ... | Q_K^T] \).
  5 end
  6 Data Holders do:
    7 for \( i = 1 \rightarrow K, \) Data holder \( i \) do
      8 Download \( P, Q_i^T \) from masking server, compute \( X_i' = PX_iQ_i \).
    9 end
  10 Factorization Server do:
    11 Compute \( X' = \text{SecureAggregation}(X_1', ..., X_K') \).
    12 Factorize \( X' \) into \( U' \Sigma V'^T \).
  13 end
  14 Data Holders do:
    15 for \( i = 1 \rightarrow K, \) Data holder \( i \) do
      16 Download \( U', \Sigma \) from the factorization server.
      17 Recover \( U \) through \( P^T U' \).
      18 Mask \( Q_i^T \) through: \( [Q_i^T]^R = Q_i^T R_i \), where \( R_i \in \mathbb{R}^{n\times n} \) is a random matrix
      19 Send \( [Q_i^T]^R \) to the factorization server.
    20 end
  21 Factorization Server wait to receive data and do:
    22 if Receive \( [Q_i^T]^R \) then
      23 Compute \( [V_i^T]^R = V'^T [Q_i^T]^R \).
      24 Send \( [V_i^T]^R \) back to data holder \( i \).
    25 end
  26 Data Holders do:
    27 for \( i = 1 \rightarrow K, \) Data holder \( i \) do
      28 Receive \( [V_i^T]^R \) from factorization server.
      29 Recover \( V_i^T \) by \( V_i^T = [V_i^T]^R R_i^{-1} \).
    30 end
  31 end
End Function

Figure 2: Framework of FedSVD

- **Step 1**: The masking server generates two random orthogonal matrices \( P \in \mathbb{R}^{n \times m} \) and \( Q \in \mathbb{R}^{m \times n} \). The matrix \( Q \) is horizontally split into \( K \) parts.
- **Step 2**: All the data holders download \( P, Q_i \) from the masking server, then compute \( X_i' = PX_iQ_i \). The factorization server gets \( X' \) through secure aggregation on \( X_i' \). The detailed computation is introduced in Section 4.2.1.
- **Step 3**: The factorization server runs a standard SVD algorithm, factorizing \( X' \) into \( U' \Sigma V'^T \).
- **Step 4**: The data holders recover the real federated SVD results:
  - \( U \) and \( \Sigma \): Each data holder downloads \( U', \Sigma \), and recover \( U \) by \( P^T U' \).
  - \( V_i^T \) is recovered under the protection of random masks, which is introduced in Section 4.2.2.

4.2.1 Compute \( X' \) through secure aggregation. As introduced in Section 4.1, we mask the raw data \( X \) through two orthogonal matrices \( P \) and \( Q \), i.e., \( X' = PXQ \). While applying the random masks, the major challenge is the data dispersity. The masking server holds \( P, Q \), the data holders jointly possess \( X \), and \( X' \) could only be exposed to the factorization server.

Equation 3 shows our idea of jointly computing \( X' \). The raw data \( X \) is vertically partitioned by \( [X_1 | ... | X_K] \) and \( X_i \in \mathbb{R}^{m \times n_i} \). We horizontally split the right orthogonal matrix \( Q \) (i.e., vertically split \( Q^T \rightarrow [Q_1^T | ... | Q_K^T] \)) such that \( Q_i \in \mathbb{R}^{n_i \times n} \). According to the rule of block matrix multiplication, we can represent \( PXQ \) into \( \sum_{i=1}^{K} PX_iQ_i \).
Thus the computation of \(X'\) can be divided into two steps. Firstly, the data holders download \(P_i Q_i\) and compute \(PX_i Q_i\). Secondly, the factorization server runs a secure aggregation to get the sum of \(\{PX_i Q_i\}_{1 \leq i \leq K}\). The secure aggregation conceals the intermediate results (i.e., \(PX_i Q_i\)), and guarantees that the factorization server only learns \(X'\).

\[
X' = P X Q\]

\[
\begin{bmatrix}
Q_1 \\
\vdots \\
Q_i \\
\vdots \\
Q_K
\end{bmatrix}
\]

\[
= PX_1 Q_1 + \ldots + PX_i Q_i + \ldots + PX_K Q_K = \sum_{i=1}^{K} PX_i Q_i
\]

4.2.2 Recover \(V^T_i\) using random masks. Another challenge in FedSVD is recovering the singular vectors \(V^T_i\). The \(V^T\) is possessed by the factorization server, and \(Q^T\) is jointly held by data holders. Data holder \(i\) can recover \(V^T_i\) through \(V^T_i = V^T Q^T_i\). During the computation, we want to guarantee the confidentiality of \(Q^T_i\) and \(V^T\), i.e., the data holders cannot directly get \(V^T\) and the factorization server cannot learn \(Q^T_i\).

\[
\begin{align}
|Q^T_i|^R &= Q^T_i R_i \\
|V^T_i|^R &= V^T Q^T_i R_i \\
V^T_i &= [V^T_i]^R R_i^{-1}
\end{align}
\]

Given \(Q^T_i \in \mathbb{R}^{n \times n_i}\), our idea is first mask \(Q^T_i\) using a random matrix \(R_i \in \mathbb{R}^{n \times n_i}\), according to Equation 4. Then data holder \(i\) sends the \([Q^T_i]^R\) (i.e., the masked matrix) to the factorization server, which will subsequentially compute \([V^T_i]^R\) according to Equation 5 and send \([V_i]^R\) back to data holder \(i\). The data holder \(i\) can remove the random mask according to Equation 6 and get the final result (i.e., \(V^T_i\)).

4.3 Privacy Proof

In this section, we analyze the confidentiality of FedSVD (i.e., Algorithm 3). According to the secure definition in Section 2.3, we assume that all parties are semi-honest. We are going to prove that FedSVD is confidential under the semi-honest setting.

Firstly, we formally define the confidentiality of FedSVD, which is presented in Definition 1.

**Definition 1.** FedSVD is private and confidential as long as:

1. Apart from the final result, each data holder cannot learn any knowledge about the raw data (i.e., \(X\)) held by the others.
2. For an arbitrary data holder \(i\), the third parties (i.e., factorization server and masking server) cannot learn more knowledge about \(i\)'s data than arbitrary data holder \(j\), when \(j \neq i\).

Table 1 shows the sent and received messages of all parties, where \([\cdot]^R\) denotes messages protected by the random mask. Since the sent and received messages are symmetric, we only analyze the received data.

**Lemma 1.** Given three messages \(A, B\) and \(C\). If \(C\) can be derived from \(A, B\) (e.g., \(C = A + B\)), then the knowledge that \(C\) includes is no more than the total knowledge of \(A\) and \(B\), i.e., \(\text{Knowledge}(A, B, C) = \text{Knowledge}(A, B)\).

**Lemma 2.** Given two messages \(A\) and \(B\). If \(B\) is randomly generated and independent of \(A\). Then Knowledge \(A, B\) = Knowledge \(A\).

**Theorem 4.2.** For \(X' = PXQ\), where \(X \in \mathbb{R}^{m \times n}\) is the data matrix, \(P \in \mathbb{R}^{m \times m}\) and \(Q \in \mathbb{R}^{n \times n}\) are two orthogonal matrices. Assume \(X\) can be decomposed into \(U \Sigma V^T\). Then the knowledge that the adversary learns about \(X\) from \(X'\) is equal to \(\Sigma\), i.e., the singular values.

**Proof.** We denote the set of all orthogonal matrices with dimension \(z\) as \(S(z)\). Then the set of all possible \(P\) and \(Q\) should be \(S(m)\) and \(S(n)\), respectively. The set of \(X'\) is \(S_X' = \{PXQ|P \in S(m), Q \in S(n)\}\). Since we pick \(P\) and \(Q\) randomly from \(S(m)\) and \(S(n)\), \(X'\) follows a random uniform distribution in set \(S_X'\). Thus, the adversary only learns that \(X'\) is a random uniform sample from set \(S_X'\). After running the SVD on \(X'\), the adversary get the singular value \(\Sigma\). A common similarity of matrices in \(S_X'\) is that they have the same singular value, i.e., \(\Sigma\). The adversary can also build a set \(S_X = \{P'XQ'|P' \in S(m), Q' \in S(n)\}\). Then the adversary learns the same amount of knowledge from \(X'\) and \(\Sigma\) if \(S_X = S_X'\). For an arbitrary element \(P_X Q_X\) in \(S_X\), there exists element \(P_X Q_X\) in \(S_X\) equals to \(P_X Q_X\) by setting \(P_X = P_X U, Q_X = V^T\). For an arbitrary element \(P_X Q_X\) in \(S_X\), there exists element \(P_X Q_X\) in \(S_X\) equals to \(P_X Q_X\) by setting \(P_X = P_X U^T, Q_X = V Q_X\). Thus there exists a one-to-one mapping between \(S_X\) and \(S_X'\) (i.e., \(S_X' = S_X\)). The adversary learns the same amount of knowledge from \(X'\) and \(\Sigma\).

**Theorem 4.3.** In FedSVD, data holder \(i\) (\(1 \leq i \leq K\)) learns no more knowledge than its own final results, i.e., \(U, \Sigma\), and \(V^T_i\).

**Proof.** After finishing the computation, each data holder has the receiving messages which is presented in Table 1, the local generated random matrix \(R_i\), and the final results. We denote one party's total knowledge as the set of all the messages that it holds. Then the total knowledge possessed by data holder \(i\) can be denoted as: \(K_i = \text{Knowledge}(P, Q_i, U', \Sigma, U, V_i^T, R_i)\). Next we use Lemma 1 and 2 to gradually simplify the terms in \(K_i\).

According to Equation 6, \([V^T_i]^R = V_i^T R_i\), thus we have:

\[
K_i = \text{Knowledge}(P, Q_i, U', \Sigma, U, V_i^T)
\]

\(R_i\) is randomly generated and independent of the other messages, so we have:

\[
K_i = \text{Knowledge}(P, Q_i, U', \Sigma, U, V_i^T) = \text{Knowledge}(P, Q_i, \Sigma, U, V_i^T)
\]

According to Theorem 4.1, \(U' = PU\), so we have:

\[
K_i = \text{Knowledge}(P, Q_i, \Sigma, U, V_i^T)
\]
Since \( P, Q_i \) are randomly generated by the masking server and they are both independent of \( \Sigma, U, V_i^T \), we have:

\[
K_i = \text{Knowledge}(\Sigma, U, V_i^T)
\]

Thus each data holder learns no more knowledge than their own final results, which are \( U, \Sigma, \) and \( V_i^T \).

**Theorem 4.4.** In FedSVD, the factorization server learns no more knowledge than \( X' \), which has the same amount of knowledge as \( X \) according to Theorem 4.2.

**Proof.** The total messages held by the factorization server are \( X' = \sum_i [X_i^T]^R \) and \( [Q_i^T]^R \). According to the prior work [3], no information is leaked during the secure aggregation, and the factorization server only learns the final results \( X' \). According to the prior work [31], the masked data \( [Q_i^T]^R \) cannot be computationally distinguished from a random matrix, thus contains no information. Then the only valid information that the factorization server gets is \( X' \).

**Theorem 4.5.** FedSVD satisfies the Definition 1, thus is private and confidential.

**Proof.** According to Theorem 4.3, FedSVD satisfies the Definition 1(1) because each data holder only learns knowledge about the final results. According to Theorem 4.2 and 4.4, FedSVD satisfies the Definition 1(2) because the knowledge learned by the factorization server is no more than \( \Sigma \), which is the final result of all the data holders. Meanwhile, the masking server cannot learn any knowledge because it receives nothing during the computation.

### 5 APPLICATIONS OF FEDSVD

In this section, we present two applications using FedSVD: federated principle component analysis and federated latent semantic analysis.

#### 5.1 Principle Component Analysis using FedSVD

Principle component analysis (PCA) [28] is used in exploratory data analysis for dimensionality reduction, and it is closely related to SVD. Given a data matrix \( X \in \mathbb{R}^{m \times n} \) jointly held by many parties, we assume each column represents a data sample, then \( X \) has \( m \) features and \( n \) samples. All the parties run the FedSVD algorithm together, getting \( X = U \Sigma V^T \). We pick the \( k \) left singular vectors with the largest singular values as the projection matrix for dimension reduction.

It is worth noting that we actually do not need to run an entire FedSVD for PCA, because only the left singular vectors with the largest \( k \) singular values are required.

#### 5.2 Latent Semantic Analysis through FedSVD

Latent semantic analysis (LSA) [9] is a natural language processing technique, analyzing the relationship between a set of documents and the words they contain. LSA assumes that words close in meaning are likely to appear in similar pieces of documents.

A typical solution of LSA is using SVD. Initially, we have a data matrix \( X \in \mathbb{R}^{m \times n} \), where \( x_{i,j} \) represents the TF-IDF value of word-\( i \) over document-\( j \). Then we factorize \( X \) using SVD and get the word-vectors and document-vectors. Afterward, we can get the similarity between words or documents by computing the distance (e.g., cosine similarity) between the vectors.

In the federated learning setting, we assume different parties possess different documents. Thus they can perform an LSA by running a full FedSVD algorithm. After the FedSVD, each party gets all the word-vectors, and the document vectors are distributed according to the documents’ ownership.

### 6 EXPERIMENTS

In this section, we provide a comprehensive evaluation of FedSVD regarding the accuracy, scalability, and masking based privacy protection. All the experiments run on a Linux server with Intel(R) Xeon(R) E5-2620 32-core 2.10GHz CPU and 378GB RAM. We have implemented FedSVD and the programming language is Python3.5.

Apart from the scalability test in Figure 6(c), we fix the number of data holders to ten. We uniformly distribute \( n \) data samples to all data holders, e.g., if \( n = 10K \) and we have 10 data holders, then each holder has \( 1K \) data samples.

#### 6.1 Datasets

We evaluate FedSVD on the following four datasets:

- **MNIST [20]:** A standard handwritten digits image testset, and each image contains 784 (i.e., \( 28 \times 28 \)) features. We take 10K labeled images in the experiment, thus \( X_{mnist} \in \mathbb{R}^{784 \times 10K} \).
- **CIFAR10 [19]:** A standard 10-classes image testset, and each image includes 3072 features (i.e., \( 32 \times 32 \times 3 \)). We also take 10K samples in the trial, thus \( X_{cifar} \in \mathbb{R}^{3072 \times 10K} \).
- **Wine [7]:** The physicochemical data for 6498 variants of red and white wine, and each sample has 12 features. Thus \( X_{wine} \in \mathbb{R}^{12 \times 6499} \).
- **Synthetic data [11]:** Apart from the real-world datasets, we also use synthetic data in the evaluation. The synthetic data is generated from a power-law spectrum \( Y_{\alpha} \sim \text{Synth}(\alpha)_{m \times n} \) using \( \alpha \in \{0.01, 0.1, 0.5, 1\} \). More specifically, \( Y = U \Sigma V^T \), where \( \alpha \sim [U, \sim] = QR(N_{m \times n}) \), \( V = QR(N_{m \times n}) \), \( \Sigma = I^\alpha \), and \( N_{m \times n} \) is an matrix with i.i.d entries drawn from \( N(0, 1) \).

In the following experiments, we assess FedSVD regarding lossless precision, scalability and the protection of masking. And the dataset usages in each evaluation are summarized in Table 2.

| Evaluation Target   | Datasets                          |
|---------------------|-----------------------------------|
| Lossless Precision  | MNIST, Wine, and Synthetic data   |
| Scalability         | Synthetic data                    |
| Protection by masking | MNIST and CIFAR10                |

#### 6.2 Lossless Precision

We have proved in Theorem 4.1 that the masking based protection in FedSVD is lossless. Here we would like to use more experimental results to show that the FedSVD framework is overall lossless in

---

\(^2\)The code and data are available at: https://github.com/Di-Chai/FedSVD
Federated Singular Vector Decomposition

precision. We first evaluate the reconstruction error of FedSVD, i.e., distance to the ground truth: $||X - UV^T||$. We use mean absolute percentage error (MAPE) as the metric and the results are presented in Figure 3. The figure shows that FedSVD has a very low reconstruction error, which is around 0.000001% of the raw data. It is worth noting that FedSVD’s tiny reconstruction error in the experiment is brought by the approximation while generating the orthogonal matrices and decomposing the matrix. Theoretically, as proved in Theorem 4.1, FedSVD is lossless in precision.

To further evaluate the precision of FedSVD, we compare FedSVD with the start-of-the-art ($\epsilon$, $\delta$)-differential private PCA method [11], denoted as DP-FedPCA. We run PCA tasks using both FedSVD and DP-FedPCA, then compare their results with the standalone PCA

\[ \mu = 0.9 \]  

\[ \nu = 0.1 \]  

\[ \alpha = 0.1 \]  

\[ \beta = 0.01 \]  

\[ \gamma = 0.01 \]  

\[ \delta = 0.1 \]  

\[ \epsilon = 0.01 \]  

For DP-FedPCA, we fix $\delta = 0.1$ and set $\epsilon \in \{0.01, 0.05, 0.1, 0.5, 1\}$. FedSVD does not utilize the DP method, thus we plot a horizontal line in the figure for comparison. In all the figures, FedSVD shows lossless precision (i.e., cosine similarity to standalone PCA equals to 1). On MNIST and Wine dataset (i.e., Figure 5(a) to 5(d)), the DP-FedPCA tends to need a larger $\epsilon$ when more explained variance is required. For example, we need to set $\epsilon > 1$ to get an accurate result that covers 90% of variance in MNIST data. In the synthetic data, we fix $m = 1K$, $n = 10K$, and set $\alpha = \{0.01, 0.1, 0.5, 1\}$. DP-FedSVD failed to preserve the data utility and all the results have cosine similarity smaller than 0.2, while the FedSVD still has lossless data utility.

To summarize, the results show that differential privacy based methods bring significant data utility loss, especially when we use a tight privacy policy (i.e., small $\epsilon$). In contrast, FedSVD is lossless and preserves the full data utility as well as making no privacy compromises.

6.3 Remarkable Scalability

Figure 6 shows the scalability test for FedSVD regarding time consumption and communication size. The scalability evaluation uses synthetic data with $m = 1K$ and $\alpha = 1.0$. Figure 6(a) shows the time consumption of FedSVD when varying the data size (i.e., $n$) from 10K to 50K. We test FedSVD using different number of building blocks when generating the orthogonal matrices (i.e., $b$ in Algorithm 2), and the overall time consumption significantly decreases with the increase of $b$.

We also test the time consumption of using fixed-size building block when generating the orthogonal matrices. We set $b = \frac{m}{1000}$ such that each block has size 1000, and Figure 6(b) shows the time consumption when varying the data size (i.e., $n$) from 10K to 100K. The time consumption of FedSVD is very close to the standalone SVD. We repeat the experiments ten times and plot the average ratio $\frac{\text{FedSVD}}{\text{Standalone SVD}}$ of the time consumption. The ratio decreases with the increase of data size, showing that FedSVD has the same scalability with standalone SVD.

Figure 6(c) shows the amount of communication data per data holder when we change the total data size and the number of data holders. When the number of data samples is fixed, the amount of transmitted data first dramatically decreases then slowly rises with the increasing number of data holders. The curve first dramatically decreases because each party’s data reduces when there are more data holders. Then the curve slowly rises because the amount of transmitted random numbers grows during the secure aggregation.

6.4 Protection by Masking

Figure 7 shows the raw and masked images (i.e., mask using orthogonal matrices) when setting $b$ to different values in FedSVD. In the MNIST dataset, the raw images consist of 0s and 1s, which is one of the most simple datasets and thus vulnerable for attacks. However, we cannot specify any information from the masked data, which shows the high confidentiality of our method. When the number of building blocks $b = \frac{m}{c}$, $c = \{10, 100, 1000\}$, the masked data shows no difference and leaks no information about the raw data.

7 RELATED WORKS

In Section 1, we have introduced the existing privacy-preserving SVD methods and categorized them into noising-based and distributed training based methods. Apart from these studies, there are also some other research topics that are closely related to our work:

- Privacy-preserving Funk-SVD: The Funk-SVD [18] is proposed in the recommendation system to explore the latent correlation

\[ \text{Running standard PCA algorithm with data collected at one place.} \]

\[ \text{Due to space limitation, we put the results when } b = 1 - 64 \text{ in Appendix B.} \]
between users and items. The major difference between Funk-SVD and SVD is that Funk-SVD runs on the sparse rating matrix and adopts the gradient descent method to learn the user and item vectors. The privacy-preserving matrix factorization in recommender system (i.e., Funk-SVD) is well explored in recent works. Chai et al. [4] proposed a secure federated matrix factorization method, in which the participants jointly train a set of global item vectors using gradient descent. Kim et al. [16] and Nikolaenko et al. [24] solved the privacy-preserving Funk-SVD problem using cryptographic techniques: homomorphic encryption and garbled circuits, respectively. Berlioz et al. [1] proposed a differential private matrix factorization method in the recommender system.

- Outsourcing matrix factorization techniques: The secure outsourcing computation is a traditional research topic, and the outsourcing matrix factorization techniques had also been explored in the past few years. Zhang et al. [31] proposed a secure outsourcing computation framework for PCA-based face recognition. Duan et al. [8] and Fu et al. [10] proposed outsourcing computation frameworks for non-negative matrix factorization through random permutation and homomorphic encryption, respectively. Luo et al. [22] proposed a masking based outsourcing computation method for QR and LU factorization.

FedSVD could also be applied in the recommender system if we manually fill in the rating matrix’s empty values and exploit the similarity between user vectors for collaborative filtering [26,
Federated Singular Vector Decomposition

8 CONCLUSION AND FUTURE WORKS

In this paper, we proposed a masking-based federated SVD method. Compared with the existing privacy-preserving SVD frameworks, FedSVD has lossless precision, low client engagement, guaranteed data confidentiality, and high scalability. We implement and test FedSVD regarding precision, scalability, and privacy protection by masking. The experiment results show that: (1) FedSVD has lossless data utility compared with differential privacy based method, (2) After using the building block strategy (i.e., Algorithm 2), FedSVD has the same scalability as standalone SVD algorithm, (3) FedSVD provides very high data confidentiality, not only the raw data cannot be exactly recovered, but we also cannot specify any image information from the masked data (e.g., the digits in MNIST cannot be distinguished).

In the future, we will keep working on the FedSVD framework and try to further improve it from the following directions:

- Removing the third parties: Based on the functionality, we have proposed three types of roles in FedSVD: the factorization server, masking server, and the data holders. The factorization server and masking server are third parties in the application. However, the trusted parties may not be easy to find in the real world. Thus it is always necessary for us to consider whether the third parties are removable in the FL systems. In the future, we will try to remove the third parties to improve the practicality. The basic idea is that we want to prove the system is confidential when the server colludes with some of the data holders. Then we can assign the role of factorization server and masking server to the data holders.

- Handling the scenario when $m > n$ (e.g., more feature than samples): In the problem formulation of FedSVD, we assume $m < n$ such that the results of federated SVD have the same size with running SVD locally. If $m > n$, then each party gets extra singular values from the final results, and we need more theoretical proof or system improvements to guarantee the data confidentiality.

REFERENCES

[1] Arnaud Berlli, Arik Friedman, Mohamed Ali Kafaar, Roksana Boreli, and Shlomo Berkovsky. 2015. Applying differential privacy to matrix factorization. In Proceedings of the 9th ACM Conference on Recommender Systems. 107–114.
[2] P Bogetoft, DL Christensen, I Damgaard, M Geisler, T Jakobsen, M Kroigaard, JD Nielsen, JB Nielsen, K Nielsen, J Pagter, et al. 2009. Secure Multiparty Computation Goes Live. Financial Cryptography and Data Security, R. Dingledine and P. Golle (eds), LNCS Vol. 5628. Springer-Verlag 10 (2009), 978–3.
[3] Keith Bonawitz, Vladimir Ivanov, Ben Kreuter, Antonio Marcedone, H Brendan McMahan, Sarvar Patel, Daniel Ramage, Aaron Segal, and Karn Seth. 2016. Practical secure aggregation for federated learning on user-held data. arXiv preprint arXiv:1611.04482 (2016).
[4] Di Chai, Leye Wang, Kai Chen, and Qiang Yang. 2020. Secure federated matrix factorization. IEEE Intelligent Systems (2020).
[5] Fei Chen, Mi Luo, Zhenhua Dong, Zhenguo Li, and Xiuqiang Li. 2020. Federated meta-learning with fast convergence and efficient communication. arXiv preprint arXiv:1802.07876 (2018).
[6] James W Daniel, Walter Bill Gragg, Linda Kaufman, and Gilbert W Stewart. 1976. Reorthogonalization and stable algorithms for updating the Gram-Schmidt QR factorization. Math. Comp. 30, 136 (1976), 772–795.
[7] Dheeru Dua and Casey Graff. 2017. UCI Machine Learning Repository. http://archive.ics.uci.edu/ml
[8] Jia Duan, Jiantao Zhou, and Yuanman Li. 2019. Secure and verifiable outsourcing of large-scale nonnegative matrix factorization (NMF). IEEE Transactions on Services Computing (2019).
[9] Susan T Dumas. 2004. Latent semantic analysis. Annual review of information science and technology 38, 1 (2004), 188–230.
[10] Ammin Fu, Zhenzhu Chen, Yi Mu, Willy Susilo, Yinxia Sun, and Jie Wu. 2019. Cloud-based outsourcing for enabling privacy-preserving large-scale nonnegative matrix factorization. IEEE Transactions on Services Computing (2019).
[11] Andreas Grammenos, Rodrigo Mendoza Smith, Jon Crowcroft, and Cecilia Mascolo. 2020. Federated Principal Component Analysis. Advances in Neural Information Processing Systems 33 (2020).
[12] Qiang Guo, Caiming Zhang, Yunfeng Zhang, and Hui Liu. 2015. An efficient SVD-based method for image denoising. IEEE transactions on Circuits and Systems for Video Technology 26, 5 (2015), 868–880.
[13] Arjun K Gupta and Daya K Nagar. 2018. Matrix variate distributions. Vol. 104. CRC Press.
[14] Shuguang Han, Wei Keong Ng, and S Yu Philip. 2009. Privacy-preserving singular value decomposition. In 2009 IEEE 25th International Conference on Data
Appendix A  LOSSLESS PRECISION

Figure 8 shows the data utility evaluation when the PCs capture 50% and 90% of variance on synthetic data. And it is supplementary to Figure 5. Based on Figure 8, we can have the same conclusion that the DP-FedPCA failed to preserve the data utility on synthetic data and FedSVD consistently has lossless precision.

Appendix B  PROTECTION BY MASKING

Figure 9 shows the full evaluation of protection by masking. When the number of building blocks $b$ changes from 1 to 64, or we fix the size of the block by setting $b = \frac{n}{c}$, $c = \{10, 100, 1000\}$, the masked data shows no difference and leaks no information about the raw data.

Appendix C  DATA AND CODE

We have released the code and data for reproducing our experiment results at https://github.com/Di-Chai/FedSVD.
Federated Singular Vector Decomposition

(a) Synthetic a=0.01, 50% variance explained
(b) Synthetic a=0.1, 50% variance explained
(c) Synthetic a=0.5, 50% variance explained
(d) Synthetic a=1.0, 50% variance explained

(e) Synthetic a=0.01, 90% variance explained
(f) Synthetic a=0.1, 90% variance explained
(g) Synthetic a=0.5, 90% variance explained
(h) Synthetic a=1.0, 90% variance explained

Figure 8: Data utility evaluation through the cosine similarity to the standalone PCA results. Supplementary to Figure 5.

Figure 9: The raw and masked image data.