Further comments on the background field method and gauge invariant effective actions.

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ABSTRACT

The aim of this paper is to give a firm and clear proof of the existence in the background field framework of a gauge invariant effective action for any gauge theory (background gauge equivalence). Here by effective action we mean a functional whose Legendre transform restricted to the physical shell generates the matrix elements of the connected $S$-matrix. We resume and clarify a former argument due to Abbott, Grisaru and Schaefer based on the gauge-artifact nature of the background fields and on the identification of the gauge invariant effective action with the generator of the proper, background field, vertices.

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1. Introduction

The analysis at LEP of the effective couplings of the intermediate bosons of the electro-weak interactions has further increased the need of an efficient parametrization method for the low energy effective actions of gauge theories.

In general these effective actions are identified with the functionals generating the fully renormalized vertices and propagators contributing to the skeleton graphs, technically speaking, the proper, 1-particle irreducible, amplitudes. Therefore they are controlled by the non-linear Slavnov-Taylor identity accounting for the BRS symmetry of the gauge theories.

Since the introduction by De Witt [1] of the background field method it is believed that, computing the $S$ matrix, the above mentioned effective actions are equivalent to those generating the background field amplitudes, that is the amplitudes with only background field external legs. The background field effective actions are gauge invariant and hence allow a much simpler parametrization of couplings than the BRS invariant quantum field effective actions.

The first proofs of this background gauge equivalence, due to De Witt and 't Hooft [2], were limited to the first loop approximation; however more recently Abbott [3] has introduced a complete and consistent renormalization method, based on the background field gauge fixing, and implementing the gauge invariance under background gauge transformations to all perturbative orders. In this framework Abbott, Grisaru and Schaefer (AGS [4]) have suggested how the proof of the background gauge equivalence could be extended to all orders of perturbation theory; however in our opinion a definite and clear proof of this equivalence is still lacking. The purpose of this paper if to fill this gap.

The argument used by AGS in their proof is that the background field is introduced as a gauge-artifact and hence the $S$-matrix should not depend on its choice. They refer to a pure Yang-Mills theory for which the definition of the $S$-matrix is out of reach, since the scattering amplitudes are affected with perturbative and non-perturbative infra-red singularities. However the argument is general enough to be directly extended to any gauge theory.

To make our proof as clear and firm as possible, we shall refer to an $SU(2)$ Higgs model, for which the $S$-matrix is defined in perturbation theory, specifying a complete set of renormalization conditions. We shall also make extended use of the functional formalism that, as it is now well known, allows a precise translation of the diagrammatic framework in which the original AGS argument was formulated.

First of all we give a precise idea of the AGS argument and of the open points in the original proof.

For this we have to recall some general fact about the functional method [5]. Let $j$ label a system of sources of the quantized fields $\phi$, and $\tau$ label the external fields coupled to a system of composite operators. We define by $Z_c[j,\tau]$ the functional generator of the connected amplitudes, those corresponding to connected Feynman graphs. Under the condition: $\frac{\delta}{\delta j} Z_c|_{j=\tau=0} = 0$, the
Legendre transform $\Gamma$ of $Z_c[j, \tau]$ is given by:

$$Z_c[j, \tau] = -\Gamma \left[ \frac{\delta}{\delta j} Z_c, \tau \right] + \int j \frac{\delta}{\delta j} Z_c \ .$$  \hspace{1cm} (1.1)

$\Gamma$ is the functional generator of the one-particle-irreducible amplitudes and identifies a natural choice for an effective action of the theory; in fact (1.1) means that a generic connected amplitude can be written in the form of a tree graph whose lines and vertices are defined from the functional derivatives of $\Gamma$. Assuming the second $j$-derivative of $Z_c$, that is the full propagator, not to be degenerate, one has

$$j = \frac{\delta}{\delta j} \rho \Gamma \left[ \frac{\delta}{\delta j} Z_c, \tau \right] ,$$ \hspace{1cm} (1.2)

$$\frac{\delta}{\delta \tau} Z_c[j, \tau] = -\frac{\delta}{\delta \tau} \Gamma \left[ \frac{\delta}{\delta j} Z_c, \tau \right] .$$ \hspace{1cm} (1.3)

It is worth noticing here that, solving (1.2) with the initial condition $Z_c[0,0] = 0$, one gets $Z_c[j, \tau]$ satisfying (1.1), since both (1.2) and (1.1) identify uniquely the connected functional.

Whenever the theory is infra-red safe, one can introduce the asymptotic field sources $j^{as}$ and use LSZ reduction formulae [6]. This can be achieved by a translation of the field sources $j$. For example, in the case of a scalar field with mass $m$ one has:

$$j(x) \to j(x) + j^{as}(x) \equiv j(x) + \mathcal{Z}^{-1} \int d^4y \delta^4z \ K(x-z;m) \Delta_+(z-y;m) \ f^{as}(y) .$$ \hspace{1cm} (1.4)

where $\mathcal{Z}$ is a normalization constant and $\Delta_+(y-z;m)$ is the positive frequency part of the Wightman function of the free scalar field and $K(z-x;m) \equiv (\Box + m^2) \delta^4(z-x)$. In general the same formula holds replacing $\mathcal{Z}$, $\Delta_+$ and $K$ with matrices in the field components, and (1.4) can be written in the form:

$$j^{as} \equiv \mathcal{Z}^{-1} K * \Delta_+ * f^{as} . \hspace{1cm} (1.5)$$

The introduction of the asymptotic sources allows to define the elements of the connected scattering matrix $S_c$ according to:

$$S_c = Z_c[j^{as}, \tau] . \hspace{1cm} (1.6)$$

*given by

$$\Delta_+(y-z;m) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik(y-z)}}{2\omega_k} = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \theta(k^0) e^{-ik(y-z)}$$
In general only a subset of the asymptotic fields correspond to physical particles and only some of the composite operators have physical meaning. Thus one has to select the physical matrix elements of $S_c$, that is:

$$S_{c}^{ph} \equiv Z_{c} \left[ j^{ph}; \tau^{ph} \right].$$  \hfill (1.7)

On account of (1.1) one has also:

$$S_{c}^{ph} = -\Gamma \left[ \frac{\delta}{\delta j^{ph}} Z_{c} \left[ j^{ph}; \tau^{ph} \right], \tau^{ph} \right] + \int j^{ph} \frac{\delta}{\delta j^{ph}} Z_{c} \left[ j^{ph}; \tau^{ph} \right].$$  \hfill (1.8)

In the framework of gauge theories, introducing the background field $V_{\mu}$ according to Abbot’s all-order scheme, $V_{\mu}$ is identified with the source of a composite operator (such as $\tau$) that appears in the gauge fixing. If $j_{\mu}$ is the source of the gauge field $A_{\mu}$ and we assume the existence of the corresponding asymptotic field and source $j^{as}_{\mu}$ whose restriction to physics is $j^{ph}_{\mu}$, we can define $S_{c}^{ph}$ according to (1.7) and we get:

$$\frac{\delta}{\delta V_{\mu}} Z_{c} \left[ j^{ph}_{\mu}; V_{\mu}, \tau^{ph} \right] = 0 ,$$  \hfill (1.9)

since the background field is a gauge fixing parameter. In this framework (1.1) corresponds to:

$$Z_{c} \left[ j^{as}_{\mu}; V_{\mu}, \tau \right] = -\Gamma \left[ \frac{\delta}{\delta j^{as}_{\mu}} Z_{c} \left[ j^{as}_{\mu}; V_{\mu}, \tau \right], V_{\mu}, \tau \right] + \int j^{as}_{\mu} \frac{\delta}{\delta j^{as}_{\mu}} Z_{c} \left[ j^{as}_{\mu}; V_{\mu}, \tau \right] ,$$  \hfill (1.10)

and hence, using (1.8) we can write:

$$S_{c}^{ph} = -\Gamma \left[ \frac{\delta}{\delta j_{\mu}} Z_{c} \left[ j^{ph}_{\mu}; V_{\mu}, \tau^{ph} \right], V_{\mu}, \tau^{ph} \right] + \int j^{ph}_{\mu} \frac{\delta}{\delta j^{ph}_{\mu}} Z_{c} \left[ j^{ph}_{\mu}; V_{\mu}, \tau^{ph} \right].$$  \hfill (1.11)

Now, from (1.3) and (1.9) we have:

$$\frac{\delta}{\delta V_{\mu}} Z_{c} \left[ j^{ph}_{\mu}; V_{\mu}, \tau^{ph} \right] = -\frac{\delta}{\delta V_{\mu}} \Gamma \left[ \frac{\delta}{\delta j_{\mu}} Z_{c} \left[ j^{ph}_{\mu}; V_{\mu}, \tau^{ph} \right], V_{\mu}, \tau^{ph} \right] = 0 ,$$  \hfill (1.12)

meaning that in (1.11) we can, more or less, arbitrarily change the variable $V_{\mu}$ in $\Gamma$ (however not in $Z_{c}$). Therefore replacing: $V_{\mu} \to \frac{\delta Z_{c}}{\delta j_{\mu}}$, we can write:

$$S_{c} = -\Gamma \left[ \frac{\delta}{\delta j_{\mu}} Z_{c} \left[ j^{ph}_{\mu}; V_{\mu}, \tau^{ph} \right], V_{\mu}, \tau^{ph} \right] + \int j^{ph}_{\mu} \frac{\delta}{\delta j_{\mu}} Z_{c} \left[ j^{ph}_{\mu}; V_{\mu}, \tau^{ph} \right].$$  \hfill (1.13)
Formally this equation can be interpreted as equivalent to \((1.11)\) after the substitution:

\[
\Gamma [A_\mu; V_\mu] \rightarrow \Gamma [A_\mu; A_\mu] .
\] (1.14)

On account of the gauge invariance of \(\Gamma [A_\mu; A_\mu]\) this could be interpreted, following AGS, as a general background gauge equivalence proof. More precisely, if \(Z_c\) in \((1.13)\) were solution of the equation:

\[
Z_c [j_\mu] = -\Gamma \left[ \frac{\delta}{\delta j_\mu} Z_c [j_\mu] ; \frac{\delta}{\delta j_\mu} Z_c [j_\mu] \right] 
+ \int j_\mu \frac{\delta}{\delta j_\mu} Z_c [j_\mu] ,
\] (1.15)

\((1.13)\) would give the proof of the existence of a gauge invariant effective action (in fact \(\Gamma [A_\mu; A_\mu]\)) for our gauge theory. This is however not the case; in particular \(\Gamma [A_\mu; A_\mu]\) cannot contain any gauge fixing term; the term existing in \(\Gamma [A_\mu; V_\mu]\) has been cancelled by the substitution \((1.14)\): \(\Gamma_{G.F.} [A_\mu; A_\mu] = \frac{\alpha}{2} \left( \nabla V_\mu (A_\mu - V_\mu) \right)^2 \bigg|_{V_\mu = A_\mu \equiv 0}\). Therefore \((1.13)\) is singular and the former interpretation of \((1.13)\) difficult to verify.

The natural way to overcome this difficulty would be to start from the effective action:

\[
\Gamma [A_\mu; A_\mu, \tau] + \Gamma_{G.F.} [A_\mu; 0] ,
\] (1.16)

and to define the corresponding connected functional generator according to:

\[
\bar{Z}_c [j_\mu; \tau] = -\Gamma \left[ \frac{\delta}{\delta j_\mu} \bar{Z}_c [j_\mu; \tau] ; \frac{\delta}{\delta j_\mu} \bar{Z}_c [j_\mu; \tau] , \tau \right] 
- \Gamma_{G.F.} \left[ \frac{\delta}{\delta j_\mu} \bar{Z}_c [j_\mu; \tau] ; 0 \right] 
+ \int j_\mu \frac{\delta}{\delta j_\mu} \bar{Z}_c [j_\mu; \tau] , \tau \right] ,
\] (1.17)

proving the identity:

\[
\bar{Z}_c [j_\mu; \tau] = Z_c [j_\mu; 0, \tau] \] (1.18)

between the solution of \((1.17)\) and that of \((1.10)\). We shall follow this line in next sections. In particular in section 2 we shall describe the SU(2)-Higgs-model recalling the structure of the background gauge Lagrangian, the functional identities constraining the model and the normalization conditions for the amplitudes. In section 3 we shall briefly discuss the physical functional variables. In section 4 we shall define the gauge invariant effective action and we shall discuss the proof of the background equivalence theorem. The extent of our proof is discussed in section 5.
2. The reference model

In this section we discuss the quantization rules of an \( SU(2) \) Higgs model \[7\]. These rules, beyond the assignment of a classical action, define the symmetry constraints, written in the form of functional differential equations for the connected generator \( Z_c \) and the effective action \( \Gamma \), and the normalization conditions for vertices and propagators. To simplify the symmetry constraints we use the Lautrup-Nakanishi auxiliary fields \[8\] inserting them, as Lagrange multipliers, in the gauge fixing term of the Lagrangian. Since the Lagrangian is quadratic in these auxiliary fields, integrating over them, leads directly to the conventional Feynman-'t Hooft gauge fixing.

Just to fix symbols and functional variables we begin listing the quantum fields and the background ones. For simplicity we avoid spinor fields. Thus the theory is built in terms of the quantum fields:

\[
\varphi = \frac{1}{\sqrt{2}} \left( \frac{\pi_2 + i\pi_1}{\sigma - i\pi_3} \right), \quad A_\mu \equiv (A^1_\mu, A^2_\mu, A^3_\mu) .
\]

The corresponding background fields are:

\[
\phi = \frac{1}{\sqrt{2}} \left( \Pi_2 + i\Pi_1 \right), \quad V_\mu \equiv (V^1_\mu, V^2_\mu, V^3_\mu) .
\]

Following Faddeev and Popov, the gauge fixing procedure requires the introduction of a system of ghosts and anti-ghosts:

\[
c \equiv (c^1, c^2, c^3), \quad \bar{c} \equiv (\bar{c}^1, \bar{c}^2, \bar{c}^3),
\]

and of the above mentioned Lautrup-Nakanishi multipliers:

\[
b \equiv (b^1, b^2, b^3) .
\]

The model is assumed to be invariant under background field gauge transformations. An infinitesimal background field gauge transformation is defined by:

\[
\delta_W \varphi = ig \frac{1}{2} \lambda \cdot \tau (\varphi + \bar{v}) , \quad \delta_W A_\mu = \nabla A_\mu \lambda ,
\]

\[
\delta_W \phi = ig \frac{1}{2} \lambda \cdot \tau (\phi + \bar{v}) , \quad \delta_W V_\mu = \nabla V_\mu \lambda ,
\]

where the nablas label covariant derivatives whose indices indicate the corresponding connections and:

\[
\bar{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix},
\]

is the vacuum expectation value of the scalar field \( \varphi \). The second main symmetry property of our model is BRS invariance. In particular the classical action is assumed to be invariant under
the system of transformations:

$$\delta_S A_\mu = \nabla A_\mu \cdot c , \quad \delta_S \varphi = i \frac{g}{2} c \cdot \tau (\varphi + \tilde{\varphi}) , \quad \delta_S c = \frac{g}{2} c \wedge c , \quad \delta_S \bar{c} = b , \quad \delta_S b = 0 . \quad (2.7)$$

These transformations commute with the background gauge transformations. As shown by Grassi [10] it is convenient to extend to the background fields the action of BRS transformations introducing a set of anticommuting external fields:

$$\Omega = \frac{1}{\sqrt{2}} \left( \Omega_2 + i \Omega_1 \right) , \quad \Omega_\mu \equiv (\Omega_{\mu 1}, \Omega_{\mu 2}, \Omega_{\mu 3}) , \quad (2.8)$$

and defining:

$$\delta_S V_\mu = \Omega_\mu , \quad \delta_S \varphi = \Omega , \quad \delta_S c = 0 , \quad \delta_S \bar{c} = 0 . \quad (2.9)$$

BRS transformations, being non-linear, transform the elementary fields into composite operators; in the functional framework these operators are coupled to external fields, that in the recent literature are called anti-fields, and appear in the functional form of the Slavnov-Taylor identity. In our case there are anti-fields corresponding to $A_\mu , \varphi$ and $c$, they are:

$$A^*_{\mu} \equiv \left( A^*_{\mu 1}, A^*_{\mu 2}, A^*_{\mu 3} \right) , \quad \varphi^* \equiv \frac{1}{\sqrt{2}} \left( \varphi_2^* - i \varphi_1^* \right) , \quad c^* \equiv \left( c_1^*, c_2^*, c_3^* \right) . \quad (2.10)$$

The action of the model is given by

$$\Gamma_0 = \int (\mathcal{L}_{inv} + \mathcal{L}_{g.f.} + \mathcal{L}_{\Phi \Pi} + \mathcal{L}_{S.T.}) . \quad (2.11)$$

The first term under integral is the well known gauge invariant Lagrangian density of the $SU(2)$ Higgs model [11], the second is the gauge fixing term:

$$\mathcal{L}_{g.f.} = b \nabla V_\mu (A_\mu - V_\mu) + \frac{\rho g}{2} b \left[ i \left( \phi^\dagger + \tilde{\varphi} \right) \tau (\varphi + \tilde{\varphi}) + h.c. \right] + \frac{b^2}{2 \alpha} , \quad (2.12)$$

and the third is the Faddeev-Popov term; the last term defines the BRS transformed fields through their anti-field couplings:

$$\mathcal{L}_{S.T.} = -A^{*\mu} \nabla A_\mu c - i \left[ \varphi^* \frac{g}{2} \tau (\varphi + \tilde{\varphi}) - h.c. \right] c - c^* \frac{g}{2} c \wedge c . \quad (2.13)$$

We shall not discuss here the technical aspects of the renormalization of our model since this is a fairly well known subject that the introduction of the background field does not change
substantially [10]. We assume therefore that the model be quantized respecting all the symmetries of the classical action; this implies, first of all, that the connected functional satisfies the Slavnov-Taylor identity:

\[ S_Z^c = \int \left[ J^\mu \frac{\delta}{\delta A^\mu} + J_\varphi \frac{\delta}{\delta \varphi^i} + \bar{\xi} \frac{\delta}{\delta \bar{c}} - \xi J_b^\mu + \Omega^\mu \frac{\delta}{\delta V^\mu} + \Omega_i \frac{\delta}{\delta \phi^i} \right] Z_c = 0 , \quad (2.14) \]

where the functional variables \( J^\mu, J_\varphi, J_b, \bar{\xi}, \xi \) are the classical sources of the quantized fields \( A^\mu, \varphi, b, c, \) and \( \bar{c} \).

The invariance under the background gauge transformations (2.5) induces the Ward identity

\[ W Z_c = 0 , \quad (2.15) \]

where \( W \) is the differential operator generating the transformations (2.5). Since these gauge transformations commute with BRS ones, one has:

\[ [W, S] = 0 . \quad (2.16) \]

The gauge-fixing term (1.2) is linear, in the sense that the auxiliary field \( b \) multiplies an operator linear in the quantized fields. In these conditions the auxiliary field equation is a linear equation in the quantized fields. Therefore this equation can be translated into a linear functional differential equation for \( Z_c \), that survives renormalization. In fact this equation is a renormalization prescription for our model and is written in the form:

\[ J_b = \nabla_{V^\mu} \left( \frac{\delta Z_c}{\delta J_\mu} - V^\mu \right) + \frac{g \rho}{2} \left[ i \left( \phi^{\dagger} + \bar{v} \right) \tau \left( \frac{\delta Z_c}{\delta J_\varphi} \right) + h.c. \right] \delta \frac{Z_c}{\delta J_\mu} + b , \quad (2.17) \]

Combining this identity with the Slavnov-Taylor identity yields to a further relation which is the BRS transformed of (2.17):

\[ \xi = -\nabla_{V^\mu} \frac{\delta Z_c}{\delta A^\mu} - \nabla_{A^\mu} \Omega^\mu \left[ \frac{g \rho}{2} \left( \phi^{\dagger} + \bar{v} \right) \tau \left( \frac{\delta Z_c}{\delta J_\varphi} \right) + h.c. \right] \left( \frac{\delta Z_c}{\delta J_b} + \bar{v} \right) . \quad (2.18) \]

The equations (2.14), (2.15), (2.17) and (2.18) give a system of functional differential constraints implementing the relevant symmetry properties of the fully quantized version of our model. They can be translated into a corresponding system of functional differential equations for the effective action \( \Gamma \). In particular from (2.17) and (2.18) one has:

\[ \frac{\delta \Gamma}{\delta b} = \nabla_{V^\mu} (A^\mu - V^\mu) + \frac{g \rho}{2} \left[ i \left( \phi^{\dagger} + \bar{v} \right) \tau (\varphi + \bar{v}) + h.c. \right] + b \frac{1}{\alpha} , \quad (2.19) \]
\[
\frac{\delta \Gamma}{\delta \bar{c}} = \nabla_{\nu} \frac{\delta \Gamma}{\delta \bar{A}^*_\mu} - \nabla_{A^*_\mu} \Omega_\mu + \frac{\rho g}{2} \left[ \frac{i}{2} \left( \phi^+ \bar{v} \right) \tau \frac{\delta \Gamma}{\delta \varphi^*} + \frac{i}{2} \Omega_\mu (\varphi + \bar{v}) + \text{h.c.} \right]. \tag{2.20}
\]

From these equations one can extract some interesting information on \( \Gamma \). Indeed the general solution to the system (2.19), (2.20) has the following form:

\[
\Gamma = \bar{\Gamma} \left[ A_{\mu}, V_{\mu}, \varphi, c, \bar{\varphi}^*, c^*, \phi, \Omega_{\mu}, \Omega \right] + \int \left[ b \nabla_{\nu} \left( A_{\mu} - V_{\mu} \right) + \frac{\rho g}{2} b \left[ i \left( \phi^+ \bar{v} \right) \tau (\varphi + \bar{v}) + \text{h.c.} \right] + \frac{b^2}{2\alpha} \right]
\]

\[
- \int \bar{c} \left[ \nabla_{A_{\mu}} \Omega_\mu + i \frac{\rho g}{4} \left[ \Omega_\tau (\varphi + \bar{v}) - \text{h.c.} \right] \right], \tag{2.21}
\]

where:

\[
\bar{A}^*_\mu = A^*_\mu - \nabla_{V_{\mu}} \bar{c}, \quad \bar{\varphi}^* = \varphi^* - i \frac{\rho g}{2} \bar{c} \left( \phi^+ \bar{v} \right) \tau. \tag{2.22}
\]

The functional \( \bar{\Gamma} \) is further constrained by the Slavnov-Taylor identity that is:

\[
\int \left[ \frac{\delta \bar{\Gamma}}{\delta A_{\mu}} \frac{\delta \bar{\Gamma}}{\delta \varphi} + \frac{\delta \bar{\Gamma}}{\delta \varphi} \frac{\delta \bar{\Gamma}}{\delta c^*} + \frac{\delta \bar{\Gamma}}{\delta \varphi} \frac{\delta \bar{\Gamma}}{\delta c} + \Omega_\mu \frac{\delta \bar{\Gamma}}{\delta V_{\mu}} + \Omega_\tau \frac{\delta \bar{\Gamma}}{\delta \varphi} \right] = 0, \tag{2.23}
\]

and by the background Ward identity:

\[
W \bar{\Gamma} = \nabla_{A_{\mu}} \frac{\delta \bar{\Gamma}}{\delta A_{\mu}} + \nabla_{V_{\mu}} \frac{\delta \bar{\Gamma}}{\delta V_{\mu}} - ig \left( (\varphi + \bar{v}) \frac{\tau}{2} \frac{\delta \bar{\Gamma}}{\delta \varphi} + \text{h.c.} \right)
\]

\[
- ig \left( (\phi + \bar{v}) \frac{\tau}{2} \frac{\delta \bar{\Gamma}}{\delta \varphi} + \text{h.c.} \right) + \ldots = 0. \tag{2.24}
\]

The dots refer to the contribution of the anti-fields, and of \( c, \Omega_{\mu} \) and \( \Omega \).

We can now discuss the parametrization of our \( \bar{\Gamma} \). Considering the reparametrizations leaving (2.23) and (2.24) unchanged, one has:

\[
A_{\mu} \to \frac{1 - k}{z_g} A_{\mu} + \frac{k}{z_g} V_{\mu}, \quad V_{\mu} \to \frac{1}{z_g} V_{\mu}, \quad \Omega_{\mu} \to \frac{1}{z_g} \Omega_{\mu}, \quad \bar{\varphi}^* \to \frac{z_g}{1 - k} \bar{\varphi}^*, \quad \varphi \to z_{\varphi} \left( \varphi + \frac{l}{1 - l} \phi \right),
\]

\[
\phi \to \frac{z_{\varphi}}{1 - l} \phi, \quad \Omega \to \frac{z_{\varphi}}{1 - l} \Omega, \quad \varphi^* \to \frac{1}{z_{\varphi}} \varphi^*, \quad c \to z_c c, \quad c^* \to \frac{1}{z_c} c^*, \tag{2.25}
\]

accompanied by:

\[
\bar{\Gamma} \to \bar{\Gamma} + \int \left[ k \Omega_{\mu} \bar{A}^*_\mu + \tilde{l} \Omega \varphi \right],
\]

\[
g \to z_g g. \tag{2.26}
\]
Since the theory is renormalizable it is easy to verify that, once the parameters in (2.25) have been fixed, one is left with a single free parameter corresponding to the $\varphi^4$ coupling in the invariant lagrangian. Therefore, taking also into account the parameters appearing in the gauge-fixing, one sees that the theory is identified by the dimensionless parameters

$$\alpha, \rho, g, \lambda, k, l, z_c, z_\varphi,$$

and by $v$. These parameters must be fixed by the normalization conditions.

It is apparent from (2.25) that neither $A_\mu$ nor $\varphi$ are multiplicatively renormalized, while $Q_\mu \equiv A_\mu - V_\mu$ and $\varphi^g \equiv \varphi - \phi$ are. It is well known that $Q_\mu$ and $\varphi^g$ are the natural dynamical variables of the quantized theory and, in fact, all the papers based on the background field method choose the $Q$-framework, that is the natural dynamical variables $Q_\mu$ and $\varphi^g$.\[1\]

However, as discussed in the introduction, the basic argument in favour of the background gauge equivalence, that we consider in the present paper, relies on the fact that, choosing the dynamical variables $A_\mu$ and $\varphi$, the background fields become gauge artifacts.

Then the question arises about the equivalence of the $Q$- and $A$-frameworks. To answer this question the first point to be clarified is that, at least in perturbation theory, the effective action is a formal power series in both quantum and background fields. Indeed the basic idea of the background method is to compute the amplitudes with only background external legs, where the quantum fields ( $Q$ and $\varphi$ ) contribute to the internal propagators. These amplitudes are power series in the background fields whose coefficients correspond to the effective vertices.

In the functional framework Feynman amplitudes are obtained renormalizing the Feynman vacuum functional whose functional integral expression accounts for the diagrammatic structure of the amplitudes. If $Z^Q$ is the vacuum functional in the $Q$-framework and $Z^A$ that in the $A$-framework, at the formal level of the unrenormalized Feynman formula, it is clear that these functionals are related by:

$$Z^A [j^\mu, \ldots] = e^{i \int (j^\mu V_\mu + \ldots)} Z^Q [j^\mu, \ldots].$$

(2.28)

since the “bare” actions of both frameworks coincide. It remains to verify what happens after renormalization.

As discussed before, to renormalize our model corresponds to implement the symmetry constraints that are written in the form: (2.14), (2.15), (2.17) and (2.18) in the $A$-framework and can be easily translated into the form suitable for the $Q$-framework. It is also easy to verify that these constraints are compatible with (2.28). Once the symmetry constraints are implemented, two different renormalization schemes differ in the parametrization; that is, they correspond to different choices of the free parameters listed in (2.27). This means that, given $Z^Q$,\[2\]

(2.28) defines a $Z^A$ corresponding to a particular choice of the parameters in the $A$-framework; in other words (2.28) defines a one-to-one correspondence between the renormalized functional of each framework, proving their equivalence.
To complete the analysis of the SU(2) Higgs-model we identify a system of normalization conditions fixing the free parameters. Assuming the notation: \( \frac{\delta^2 T}{\delta \phi \delta \phi'} |_{\phi = 0} \equiv \Gamma_{\phi \phi'} \), we assign the following wave function normalizations, masses and couplings:

\[
\begin{align*}
\Gamma_{\mu,\nu}^{\mu,\nu}(q^2 = m_Q^2) &= 0, \quad \Gamma_{\sigma \mu \nu}(q^2 = M^2) = 0, \quad \Gamma_{\bar{c}c}(q^2 = m^2_{\Phi \Pi}) = 0, \\
\Gamma_{\mu,\nu}^{\mu,\nu}(q^2 = m_Q^2) &= g^{\mu\nu}, \quad \Gamma_{\sigma \mu \nu}(q^2 = M^2) = 1, \quad \Gamma_{\bar{c}c}'(q^2 = m^2_{\Phi \Pi}) = 1, \\
\Gamma_{\sigma \mu \nu}^{\mu,\nu}(M^2, m_Q^2, m_Q^2) &= g^{\mu\nu} m_Q g^{\mu\nu}, \\
\Gamma_{\Sigma \Sigma'}(0, m^2_{\Phi \Pi}, m^2_{\Phi \Pi}) &= g^{\mu\nu} m^2_{\Phi \Pi}. \quad (2.29)
\end{align*}
\]

To avoid double poles in the propagators (\cite{7}, \cite{11}) we also assume the condition (‘t Hooft):

\[
\begin{align*}
\Gamma_{\mu L b}^{\mu L b} + \Gamma_{b \bar{b} \pi}^{\mu \pi Q L} \bigg|_{q^2 = m^2_{\Phi \Pi}} = 0 . \quad (2.30)
\end{align*}
\]

One can verify that the normalization conditions in (2.29) determine the free parameters (2.27) and \( v \); in particular one has, up to one loop corrections:

\[
\begin{align*}
g &= g^{ph} (1 + O(\hbar)) , \quad \lambda = \frac{g^2 M^2}{2 m_Q^2} (1 + O(\hbar)) , \quad v = \frac{m_Q}{g} (1 + O(\hbar)) , \\
\rho &= \frac{2 m^2_{\Phi \Pi}}{m_Q^2} (1 + O(\hbar)) , \quad k = O(h) , \quad l = O(h) , \quad z_{\varphi} = 1 + O(h) , \quad z_c = 1 + O(h) .
\end{align*}
\]

One has also:

\[
\frac{\rho}{2} - \frac{1}{\alpha} = O(\hbar) ,
\]

from which:

\[
\alpha = \frac{m^2_Q}{m^2_{\Phi \Pi}} (1 + O(\hbar)) .
\]

In Appendix A we list the propagators of our model.

### 3. The physical variables

Having specified the reference gauge model, we must discuss briefly its physical content; that is the physical operators relevant for the construction of the \( S \)-matrix.

First of all, the physical asymptotic fields correspond to the transverse components of the vector field and to the Higgs field \( \sigma \); then we must consider the composite physical operators. We do not need a complete list of these operators; we simply mention an example: the energy

\[11\]
momentum tensor. We associate with every physical operator a corresponding functional variable that is identified, in the case of composite operators, with the $\tau^{ph}$ external fields appearing in the introduction, and, in the case of asymptotic fields, with $j^{ph}$ in Eq. (1.5).

In a general gauge theory one defines a physical variable ($\Xi$) as a functional variable with vanishing Faddeev-Popov charge, that is coupled to a BRS-invariant operator that does not correspond to the BRS-variation of any other operator. In formulae $\Xi$ is a physical variable if and only if:

$$\left[\frac{\delta}{\delta \Xi}, S\right] = 0 \quad (3.1)$$

and

$$\frac{\delta}{\delta \Xi} \neq \left\{\frac{\delta}{\delta X}, S\right\}. \quad (3.2)$$

Notice that the second condition (3.2) is crucial; indeed, for example, the source of the auxiliary field $J_b$ and the background field $V_\mu$ satisfy (3.1), however one has:

$$\frac{\delta}{\delta J_b} = \left\{\frac{\delta}{\delta \xi}, S\right\}, \quad \frac{\delta}{\delta V_\mu} = \left\{\frac{\delta}{\delta \Omega_\mu}, S\right\},$$

and hence these variables are physically trivial.

Notice also that the actually independent asymptotic variables are the components of $f^{ph}$ defined in (1.3); we should therefore use $Z^{-1} K \Delta_+ * f^{ph}$ instead of $j^{ph}$. For simplicity we prefer to use $j^{ph}$ understanding its dependence on $f^{ph}$; however taking functional derivatives we have to refer to $f^{ph}$ using:

$$\frac{\delta}{\delta f^{ph}} = Z^{-1} K \Delta_+ * \frac{\delta}{\delta J}. \quad (3.3)$$

To simplify the notation in the rest of the paper we shall use the following symbols:

$$j \equiv (J_\mu , J_\varphi) , \quad J \equiv J_b , \quad \Omega \equiv (\Omega_\mu , \Omega) ,$$

$$\Phi \equiv (A_\mu , \varphi) , \quad V \equiv (V_\mu , \phi) , \quad b \equiv b ,$$

$$\Phi^* \equiv (A_\mu^* , \varphi^*). \quad (3.4)$$

With these new symbols the Slavnov-Taylor identity (2.14) becomes

$$SZ_c \equiv \int \left(j \frac{\delta}{\delta \Phi^*} + \xi \frac{\delta}{\delta c^*} - \xi \frac{\delta}{\delta J} + \Omega \frac{\delta}{\delta V}\right) Z_c = 0 , \quad (3.5)$$
4. The effective action and the background equivalence theorem

We begin defining the effective action upon which background gauge equivalence is based.

Since we are interested in the physical restriction of the $S$-matrix, the ghost propagator does not appear. Then we restrict our functional variables setting:

\[\Omega = \Phi^* = c^* = \xi = \xi = c = \bar{c} = 0 .\]  \hspace{1cm} (4.1)

After this restriction the effective action, given in (2.21), becomes:

\[\Gamma (\Phi, V, b) = \bar{\Gamma} (\Phi, V) + \Gamma_{gf} (\Phi, b, V) ,\]  \hspace{1cm} (4.2)

where $\Gamma_{gf}$ contains the bosonic part of the gauge fixing term; in the reference model:

\[\Gamma_{gf} = \int \left[ b \nabla_{\mu} (A_{\mu} - V_{\mu}) + \frac{bg}{2} b \left[ i \left( \phi^\dagger + \bar{\nu} \right) \tau (\phi + \bar{\nu}) + h.c. \right] + \frac{b^2}{2\alpha} \right] .\]  \hspace{1cm} (4.3)

It is a crucial and general fact that the dependence of $\Gamma$ on $b$ is restricted to $\Gamma_{gf}$.

To simplify further our notation we shall understand the dependence of the connected functionals and effective actions on the physical variables $\tau_{ph}$, corresponding to physical composite operators, and we concentrate on the asymptotic physical variables $j_{ph}$. Notice that these variables appear in the connected functional $Z_c$ but not in $\Gamma$.

Now we come to the main subject of this paper: the proof of background equivalence following the lines presented in the introduction. As already discussed, we must compare the connected $S$-matrix corresponding to the effective action (4.2) that is by no means invariant under gauge transformations of $\Phi$ and $b$ at $V = 0$, with that corresponding to the alternative effective action:

\[\Gamma'_{eff} (\Phi) = \bar{\Gamma} (\Phi, \Phi) + \Gamma_{gf} (\Phi, b, 0) ,\]  \hspace{1cm} (4.4)

which identifies our prescription for the gauge-fixed, gauge-invariant, effective action (1.16). We call this effective action gauge invariant since its gauge invariance is only broken by the gauge fixing term ((4.3) at $V = 0$) that is by the choice of the propagators. Indeed $\bar{\Gamma} (\Phi, \Phi)$ is gauge invariant² owing to (2.24). The connected functional of our model is identified with the solution to:

\[Z_c [j, J, V] = -\bar{\Gamma} \left[ \frac{\delta}{\delta j} Z_c [j, J, V], V \right] -\Gamma_{gf} \left[ \frac{\delta}{\delta j} Z_c [j, J, V], \frac{\delta}{\delta J} Z_c [j, J, V], V \right] + \int \left( j \frac{\delta}{\delta j} + J \frac{\delta}{\delta J} \right) Z_c [j, J, V] ,\]  \hspace{1cm} (4.5)

*But not gauge independent! ²
vanishing in the origin of the functional variable space; while that corresponding to the gauge invariant effective action (4.4) is identified with:

\[ \bar{Z}_c[j, J, V] = -\Gamma \left[ \frac{\delta}{\delta j} \bar{Z}_c[j, J, V], V + \frac{\delta}{\delta J} \bar{Z}_c[j, J, V] \right] - \Gamma_{gf} \left[ \frac{\delta}{\delta j} \bar{Z}_c[j, J, V], \frac{\delta}{\delta J} \bar{Z}_c[j, J, V], 0 \right] \]

+ \int \left( J \frac{\delta}{\delta j} + J \frac{\delta}{\delta J} \right) \bar{Z}_c[j, J, V] \]

The corresponding connected \( S \)-matrices are given by \( Z_c[j^{ph}, 0, 0] \) and \( \bar{Z}_c[j^{ph}, 0, 0] \).

For background equivalence to hold true they should coincide.

To prove this, we introduce a further connected functional \( \hat{Z}_c \) depending on two background fields \( V \) and \( W \); \( \hat{Z}_c \) is defined by:

\[ \hat{Z}_c[j, J, V, W] = -\Gamma \left[ \frac{\delta}{\delta j} \hat{Z}_c[j, J, V, W], V \right] - \Gamma_{gf} \left[ \frac{\delta}{\delta j} \hat{Z}_c[j, J, V, W], \frac{\delta}{\delta J} \hat{Z}_c[j, J, V, W], W \right] \]

+ \int \left( J \frac{\delta}{\delta j} + J \frac{\delta}{\delta J} \right) \hat{Z}_c[j, J, V, W] \]

(4.7)

We shall use \( \hat{Z}_c \) to verify the dependence of \( Z_c \) on the background field appearing in \( \Gamma_{gf} \). It is obvious that:

\[ \hat{Z}_c[j, J, V, V] = Z_c[j, J, V] \]

(4.8)

Taking the \( \Omega \)-derivative of (3.5) in the point specified by (4.1) and \( j = j^{ph} \), we have:

\[ \frac{\delta}{\delta V} Z_c[j^{ph}, J, V] = \left( \frac{\delta}{\delta V} + \frac{\delta}{\delta W} \right) \hat{Z}_c[j^{ph}, J, V] = 0 \]

(4.9)

since \( f^{ph} \) satisfies (3.1). Taking the \( \xi \) derivative in the same conditions, we have:

\[ \frac{\delta}{\delta J} Z_c[j^{ph}, J, V] = \frac{\delta}{\delta J} \hat{Z}_c[j^{ph}, J, V] = 0 \]

(4.10)

Using (1.3) we have:

\[ \frac{\delta}{\delta W} \hat{Z}_c[j, J, V, W] = -\frac{\delta}{\delta W} \Gamma_{gf} \left[ \frac{\delta}{\delta j} \hat{Z}_c[j, J, V, W], \frac{\delta}{\delta J} \hat{Z}_c[j, J, V, W], W \right] \]

(4.11)

The right-hand side of (4.11) can be easily computed taking into account the explicit form of \( \Gamma_{gf} \) given in (1.3). We exploit in particular the fact that the background functional derivative of \( \Gamma_{gf} \) is linear in the field \( b \):

\[ \frac{\delta}{\delta V} \Gamma_{gf}[\Phi, b, V] = L[\Phi, V] b \]

(4.12)
Combining (4.11) and (4.12), written as a functional differential equation for \( \hat{Z}_c \), we get:

\[
\frac{\delta}{\delta W} \hat{Z}_c [j, J, V, W] = -L \left[ \frac{\delta}{\delta j} \hat{Z}_c [j, J, V, W], W \right] \frac{\delta}{\delta J} \hat{Z}_c [j, J, V, W]. \tag{4.13}
\]

Then starting from (4.10) and taking multiple \( J \) and \( W \) derivatives of (4.13) one finds recursively that:

\[
\left( \frac{\delta}{\delta W} \right)^n \hat{Z}_c [j^{ph}, J, V] = 0, \tag{4.14}
\]

for any \( n \). A more detailed analysis of this point is given in Appendix B.

From (4.9) and (4.10) (see Appendix B), one finds that:

\[
\hat{Z}_c [j^{ph}, J, V, W] \equiv Z_c [j^{ph}, J, V] \equiv Z_c [j^{ph}, 0, 0]. \tag{4.15}
\]

That is: the \( S \)-matrices corresponding to \( Z_c \) and \( \hat{Z}_c \) coincide.

Now we compare the functional \( \hat{Z}_c \) with \( \bar{Z}_c \). Setting \( j = j^{ph} \) and \( J = 0 \) and applying (1.2) one finds:

\[
\left( \frac{\delta}{\delta \Phi} + \frac{\delta}{\delta V} \right) \Gamma \left[ \frac{\delta}{\delta j} Z_c [j^{ph}, 0, V], V + \frac{\delta}{\delta j} Z_c [j^{ph}, 0, V] \right] + \frac{\delta}{\delta \Phi} \Gamma_{gfj} \left[ \frac{\delta}{\delta j} \bar{Z}_c [j^{ph}, 0, V], \frac{\delta}{\delta J} \bar{Z}_c [j^{ph}, 0, V], 0 \right] = j^{ph},
\]

\[
\frac{\delta}{\delta b} \Gamma_{gfj} \left[ \frac{\delta}{\delta j} \bar{Z}_c [j^{ph}, 0, V], \frac{\delta}{\delta J} \bar{Z}_c [j^{ph}, 0, V], 0 \right] = 0. \tag{4.16}
\]

As mentioned in the introduction, this system determines \( \frac{\delta}{\delta j} Z_c [j^{ph}, 0, V] \) and \( \frac{\delta}{\delta j} \bar{Z}_c [j^{ph}, 0, V] \) uniquely.

One has furthermore from (1.3):

\[
\frac{\delta}{\delta V} Z_c [j^{ph}, 0, V] = -\frac{\delta}{\delta V} \Gamma \left[ \frac{\delta}{\delta j} Z_c [j^{ph}, 0, V], V + \frac{\delta}{\delta j} Z_c [j^{ph}, 0, V] \right]. \tag{4.17}
\]

In much the same way, considering \( \hat{Z}_c \) one has:

\[
\frac{\delta}{\delta \Phi} \Gamma \left[ \frac{\delta}{\delta j} \hat{Z}_c [j^{ph}, 0, V], V \right] + \frac{\delta}{\delta \Phi} \Gamma_{gfj} \left[ \frac{\delta}{\delta j} \hat{Z}_c [j^{ph}, 0, V], \frac{\delta}{\delta J} \hat{Z}_c [j^{ph}, 0, V], 0 \right] = j^{ph},
\]

\[
\frac{\delta}{\delta b} \Gamma_{gfj} \left[ \frac{\delta}{\delta j} \hat{Z}_c [j^{ph}, 0, V], \frac{\delta}{\delta J} \hat{Z}_c [j^{ph}, 0, V], 0 \right] = 0. \tag{4.18}
\]
that determine $\frac{\delta}{\delta \gamma} \dot{Z}_c [j^{ph}, 0, V, 0]$ and $\frac{\delta}{\delta \gamma} \dot{Z}_c [j^{ph}, 0, V, 0]$ uniquely.

Furthermore from (4.13) and (1.3) one has:

$$\frac{\delta}{\delta V} \dot{Z}_c [j^{ph}, 0, V, 0] = -\frac{\delta}{\delta V} \dot{\Gamma} \left[ \frac{\delta}{\delta \gamma} \dot{Z}_c [j^{ph}, 0, V, 0], V \right] = 0 .$$ (4.19)

To compare $\dot{Z}_c$ and $\bar{Z}_c$ we consider the following system of functional equations:

$$\zeta [j^{ph}, V] = \frac{\delta}{\delta \gamma} \dot{Z}_c [j^{ph}, 0, V + \zeta [j^{ph}, V], 0] ,$$
$$\eta [j^{ph}, V] = \frac{\delta}{\delta \gamma} \dot{Z}_c [j^{ph}, 0, V + \zeta [j^{ph}, V], 0] .$$ (4.20)

It is rather apparent that the iterative solution to (4.20) leads to a unique solution $(\zeta, \eta)$. A detailed analysis supporting this conclusion is given in Appendix B.

Therefore if we replace $V \rightarrow V + \zeta$ everywhere into the system (4.18), on account of (4.20), we get:

$$\frac{\delta}{\delta \Phi} \dot{\Gamma} [\zeta, V + \zeta] + \frac{\delta}{\delta \Phi} \Gamma_{gf} [\zeta, \eta, 0] = j^{ph} ,$$
$$\frac{\delta}{\delta b} \Gamma_{gf} [\zeta, \eta, 0] = 0 .$$ (4.21)

Furthermore the same substitution into (4.19) gives:

$$\frac{\delta}{\delta V} \dot{\Gamma} [\zeta, V + \zeta] = 0 .$$ (4.22)

Owing to (4.22) we see that (4.21) is equivalent to:

$$\left( \frac{\delta}{\delta \Phi} + \frac{\delta}{\delta V} \right) \dot{\Gamma} [\zeta, V + \zeta] + \frac{\delta}{\delta \Phi} \Gamma_{gf} [\zeta, \eta, 0] = j^{ph} ,$$
$$\frac{\delta}{\delta b} \Gamma_{gf} [\zeta, \eta, 0] = 0 .$$ (4.23)

Since this system identifies uniquely its solution $(\zeta, \eta)$, comparing (4.23) with (4.16), we have:

$$\zeta [j^{ph}, V] = \frac{\delta}{\delta \gamma} \bar{Z}_c [j^{ph}, 0, V] = \frac{\delta}{\delta \gamma} \bar{Z}_c [j^{ph}, 0, V + \zeta [j^{ph}, V], 0] ,$$ (4.24)

$$\eta [j^{ph}, V] = \frac{\delta}{\delta \gamma} \bar{Z}_c [j^{ph}, 0, V] = \frac{\delta}{\delta \gamma} \bar{Z}_c [j^{ph}, 0, V + \zeta [j^{ph}, V], 0] .$$ (4.25)
If, using (3.3), we restrict the $j$-functional derivatives in (4.24) to the physical shell and we take into account the $V$-independence of $\hat{Z}_c \left[ j^{ph}, 0, V, 0 \right]$ shown in (4.17), we get:

$$
\frac{\delta}{\delta f^{ph}} \hat{Z}_c \left[ j^{ph}, 0, V \right] = Z^{-1} K * \Delta_+ * \frac{\delta}{\delta j} \hat{Z}_c \left[ j^{ph}, 0, V \right] = \frac{\delta}{\delta f^{ph}} \hat{Z}_c \left[ j^{ph}, 0, V + \zeta \left[ j^{ph}, V \right], 0 \right]
$$

$$
= \frac{\delta}{\delta f^{ph}} \hat{Z}_c \left[ j^{ph}, 0, 0, 0 \right].
$$

(4.26)

Excluding the physical composite operators ($\tau^{ph} = 0$), the last identity can be integrated over $f^{ph}$ with the initial condition: $\hat{Z}_c \left[ 0, 0, 0, 0 \right] = \bar{Z}_c \left[ 0, 0, 0 \right] = 0$ ensuring, on account of (4.15), the identity of the connected $S$-matrices:

$$
Z_c \left[ j^{ph}, 0, 0 \right] = \hat{Z}_c \left[ j^{ph}, 0, 0, 0 \right] = \bar{Z}_c \left[ j^{ph}, 0, 0 \right],
$$

(4.27)

and hence proving the background equivalence of the $S$-matrix elements. It is however possible to extend this results to the matrix elements between physical asymptotic states of $T$-ordered products of physical operators, proving that $\hat{Z}_c \left[ 0, 0, 0, 0 \right] = \bar{Z}_c \left[ 0, 0, 0 \right]$ for any choice of $\tau^{ph}$.

This is easily done using (1.3), (4.24) and (4.25). Indeed, applying (1.3) to $\bar{Z}_c$ and $\hat{Z}_c$, we get for $j^{ph} = J = 0$:

$$
\frac{\delta}{\delta \tau} \bar{Z}_c \left[ 0, 0, V \right] = -\frac{\delta}{\delta \tau} \bar{Z}_c \left[ 0, 0, V \right] + \frac{\delta}{\delta j} \bar{Z}_c \left[ 0, 0, V \right] V
$$

$$
- \frac{\delta}{\delta \tau} \bar{Z}_c \left[ 0, 0, V \right] + \frac{\delta}{\delta j} \bar{Z}_c \left[ 0, 0, V \right] V + \frac{\delta}{\delta j} \bar{Z}_c \left[ 0, 0, V, 0 \right],
$$

(4.28)

and:

$$
\frac{\delta}{\delta \tau} \hat{Z}_c \left[ 0, 0, V, 0 \right] = -\frac{\delta}{\delta \tau} \hat{Z}_c \left[ 0, 0, V, 0 \right] + \frac{\delta}{\delta j} \hat{Z}_c \left[ 0, 0, V, 0 \right] V
$$

$$
- \frac{\delta}{\delta \tau} \hat{Z}_c \left[ 0, 0, V, 0 \right] + \frac{\delta}{\delta j} \hat{Z}_c \left[ 0, 0, V, 0 \right] V + \frac{\delta}{\delta j} \hat{Z}_c \left[ 0, 0, V, 0, 0 \right],
$$

(4.29)

If we replace in (4.28) $V \rightarrow V + \frac{\delta}{\delta j} \bar{Z}_c \left[ 0, 0, V \right]$, the left-hand side does not change, owing to (4.19), and, on account of (4.24) and (4.25), the right-hand side becomes equal to that of (4.28). We thus conclude that:

$$
\frac{\delta}{\delta \tau} \bar{Z}_c \left[ 0, 0, V \right] = \frac{\delta}{\delta \tau} \bar{Z}_c \left[ 0, 0, V, 0 \right].
$$

(4.30)

Integrating with respect to $\tau$ with the initial condition: $\hat{Z}_c \left[ 0, 0, 0, 0 \right] = \bar{Z}_c \left[ 0, 0, 0 \right] = 0$, we prove the identity: $\hat{Z}_c \left[ j^{ph}, 0, 0, 0 \right] = \bar{Z}_c \left[ j^{ph}, 0, 0 \right]$, and hence (4.27) for any $\tau^{ph}$. 
5. Comments

We stress, first of all, that our proof is based on the existence of a fully renormalized theory satisfying a set of renormalization prescriptions \((2.14), (2.13), (2.17)\) and \((2.18)\); the only explicit references to the perturbative construction concern the reference model and the discussion of the existence and uniqueness of the solution of the system \((1.20)\).

The use of the simplified symbols introduced in \((3.4)\) should put into evidence the general nature of our proof. Indeed the essential ingredients of the analysis can be divided into two sets: the basic, physical, ingredient is Slavnov-Taylor identity \((3.5)\) ensuring that the background field and the auxiliary field are gauge artifacts and have no influence on the physical amplitudes. The second, technical, ingredient is the linear gauge choice, that has allowed us to separate from the effective action the gauge fixing part (see \((2.11), (2.12)\) and \((2.21)\)) guaranteeing in particular the property \((4.12)\). Of course everything is based on the systematic use of the functional framework and in particular on \((1.2)\) and \((1.3)\) whose validity is completely general.

Therefore it should be clear that our proof extends directly to any gauge model, provided one can define the asymptotic fields and hence the S-matrix. In fact, in the models in which the gauge group contains abelian invariant factors there are further constraints that are conveniently introduced to guarantee the radiative stability of abelian charges \((12)\) and \((13)\). These constraints, that correspond to the prescription of the field equations of the abelian anti-ghosts, further specify the gauge fixing prescription without any interference with the ingredients of our proof.

A further point that requires a short discussion concerns the dependence of the gauge invariant effective theory on the gauge fixing prescription. First of all, we should notice that our construction is based on two, in principle independent, gauge fixing procedures. The first quantum gauge fixing is needed to compute from the lagrangian the effective action, the second one allows the construction of the S-matrix from the effective action\(^\star\). It has been convenient for us to identify these gauge fixings, since we had to compare the S-matrix of the effective theory to that obtained directly from the theory in a trivial background. However it is fairly well known that, once a gauge invariant effective action is given, the S-matrix is independent of the gauge fixing necessary to define the effective theory propagators. It is also independent of the first, quantum, gauge fixing, since this is true for the theory in a trivial background \(^\dagger\), however, it is known that the gauge invariant effective action is not \(^\ddagger\). This could appear a little paradoxical, since one could think that all the parameters appearing into a gauge invariant effective action should carry an independent physical information, but it is easy to show that this is not true, and there is wide room to change the gauge invariant effective couplings that are proportional to the field equations, thus keeping the S-matrix fixed. Concerning the parametrization of gauge effective actions see also \(^\triangleright\).

\(^\star\)the existence of two distinct gauge fixings justifies the introduction of the \textit{interpolating} functional \(\hat{Z}_c\) depending on two background fields \(V\) and \(W\).
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Appendix A

Taking into account the normalization conditions (2.29), the propagators of the reference model are:

\[
G_{\mu\nu}^{QQ}(q) = \frac{I}{q^2 - m_Q^2} \left( g_{\mu\nu} - \left( \frac{m_Q^2 + m_\Phi^2}{m_Q^2} \right) \frac{q_\mu q_\nu}{q^2 - m_\Phi^2} \right),
\]

\[
G_{\nu}^{\pi Q L}(q) = 0, \quad G_{\nu}^{b Q L}(q) = i \frac{q_\nu}{q^2 - m_\Phi^2},
\]

\[
G^{\pi\pi}(q) = \frac{m_Q}{q^2 - m_\Phi^2}, \quad G^{bb}(q) = 0,
\]

\[
G^{\pi\pi}(q) = \frac{1}{q^2 - m_\Phi^2}, \quad G^{\sigma\sigma} = \frac{1}{q^2 - M^2},
\]

\[
G^{cc}(q) = \frac{I}{q^2 - m_\Phi^2}.
\] (A.1)

Appendix B

We begin this appendix considering (4.9), (4.10) and (4.13) and proving recursively (4.14) for any n.

First of all, from (4.10) and (4.13) we have:

\[
\delta \frac{\delta W}{\delta J} \hat{Z}_c \left[ j^{ph}, J, V, V \right] = 0.
\] (B.1)

Let us assume (4.14) to hold true for any \( n \leq m - 1 \), up to order \( m - 1 \) we have:

\[
\delta \frac{\delta J}{\delta W} \left( \delta \frac{\delta W}{\delta J} \right)^n \hat{Z}_c \left[ j^{ph}, J, V, V \right] = 0.
\] (B.2)
We can compute the $m^{th}$ $W$-derivative of $\hat{Z}_c \left[j^{ph}, J, V, W \right]$ for $V = W$ taking the $(m-1)^{th}$ $W$-derivative of both sides of (4.13) and putting $W = V$. The right-hand side of the resulting equation is the sum of many terms, each proportional to a derivative (3.2) with $n < m$, therefore it vanishes and hence (4.14) holds true for any $n$.

We notice furthermore that $\hat{Z}_c \left[j^{ph}, J, V, W \right]$ is independent of $V$ and $W$. Indeed, Taylor expanding this functional around $V = W$, and using (4.14), we see that it is independent of $W$. Then, on account of (4.8), it coincides with $Z_c \left[j^{ph}, J, V \right]$ which, according to (4.9), is $V$-independent. Thus we have proved (4.15).

Now we consider the system (4.20). For our purposes it is sufficient to study the iterative solutions of this system that are formal power series in $V$ and $j^{ph}$, since the physical amplitudes, that we are considering, are identified with the coefficients of an analogous series. The iterative solution of (4.20) is built developing the right-hand side of this equation in series of $\zeta$ getting:

$$\zeta \left[j^{ph}, V \right] (x) = \frac{\delta}{\delta J(x)} \hat{Z}_c \left[j^{ph}, 0, V, 0 \right] + \int dy \frac{\delta^2}{\delta J(x) \delta V(y)} \hat{Z}_c \left[j^{ph}, 0, V, 0 \right] \zeta [j^{ph}, V](y) + O (\zeta^2) ,$$

that can be written in the form:

$$\int dy \left[ \delta(x - y) - \frac{\delta^2}{\delta J(x) \delta V(y)} \hat{Z}_c \left[j^{ph}, 0, V, 0 \right] \right] \zeta [j^{ph}, V](y) = \frac{\delta}{\delta J(x)} \hat{Z}_c \left[j^{ph}, 0, V, 0 \right] + O (\zeta^2) .$$

(B.4)

Now it is clear that (4.20) is solvable provided the "matrix"

$$\delta(x - y) - \frac{\delta^2}{\delta J(x) \delta V(y)} \hat{Z}_c \left[j^{ph}, 0, V, 0 \right] ,$$

(B.5)

be non-degenerate at $V = 0$. This is certainly true in perturbation theory, since the second term in the left-hand side of (B.4) vanishes in the tree approximation. Indeed, owing to (1.3) and (4.7), $\frac{\delta}{\delta V} \hat{Z}_c$ can be computed from $\frac{\delta}{\delta \eta} \tilde{\Gamma}$ and $\tilde{\Gamma}$ in the tree approximation reduces to $\int L_{inv}$, the classical action deprived of the gauge fixing part, that is independent of $V$.

It is also clear that, the $\zeta$-component of the solution of (4.20) identifies uniquely the $\eta$-component. Thus, at least perturbatively, the system (4.20) has a unique solution.
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