ON THE GEOMETRY OF COSET BRANES

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Abstract

Coset models and their symmetry preserving branes are studied from a representation theoretic perspective, relating e.g. the horizontal branching spaces to a truncation of the space of bulk fields, and accounting for field identification. This allows us to describe the fuzzy geometry of the branes at finite level.
1 Introduction

There are two conceptually rather different approaches to two-dimensional conformal field theory – representation theoretic on the one hand, and geometric on the other. In the former approach, at the chiral level the basic structures are the chiral symmetry algebra – a conformal vertex algebra $\mathfrak{A}$ – and its representations (see e.g. [1–3]), while the full non-chiral theory is obtained by combining the chiral information with algebraic structures in the representation category of the chiral algebra [4,5]. In the geometric description, one deals instead with a sigma model on a suitable target space, which can be studied by Lagrangian field theory methods; see, for instance, [6, 7]. Clearly, on either side a lot can be learned by understanding how the two descriptions are related.

For closed world sheets, the relation between the two approaches is intimately linked to the interpretation of the state space $\mathcal{H}_T$ for the torus – the space of bulk fields, or of closed string states. In the algebraic framework, $\mathcal{H}_T$ is described as a representation space of the direct sum $\mathfrak{A} \oplus \mathfrak{A}$ of two copies of the chiral algebra; the character of this representation is the torus partition function $Z$. In the geometric setting one would like to interpret a subspace $\overline{\mathcal{H}}_T$ of $\mathcal{H}_T$ as a space $\mathcal{F}(\mathcal{M})$ of functions on the target space manifold $\mathcal{M}$, implying in particular that $\overline{\mathcal{H}}_T$ carries the structure of a commutative algebra.

Owing to the truncation to a subspace $\overline{\mathcal{H}}_T$, a relation between the geometric and representation theoretic description is most immediately established for models which come in families depending on some parameter, in a ‘semiclassical’ limit for that parameter – e.g. when the level $k$ of a WZW model gets large. (There are various inequivalent ways of performing such limits; for a discussion see e.g. [8–12] and chapter 16.3 of [13].) In this case a potential description of $\overline{\mathcal{H}}_T$ is as the subspace of those states in $\mathcal{H}_T$ whose conformal weight tends to zero in the limit. On the other hand, at finite values of the parameter an algebra structure on $\overline{\mathcal{H}}_T$ will typically be non-commutative algebra. Accordingly, $\overline{\mathcal{H}}_T$ should rather be understood as a space of functions on a non-commutative manifold, which may be regarded as a quantized version of $\mathcal{M}$ [14–16].

When world sheets with non-empty boundary are admitted, one must in addition account for the state spaces $\mathcal{H}_A^{ab}$ for the annulus with specified conformal boundary conditions $a$ and $b$ – the spaces of boundary fields, or of open string states. In particular, to each boundary condition $a$ there is associated the space $\mathcal{H}_A^{aa}$. Each $\mathcal{H}_A^{ab}$ is a representation space of a single copy of the chiral algebra $\mathfrak{A}$; its character is the annulus amplitude $A^{ab}$. Geometrically, an elementary boundary condition $a$ is, in the simplest case, described as a submanifold $\mathcal{M}_a$ of $\mathcal{M}$ [17, 18]. In the string theory context, $\mathcal{M}_a$ plays the role of a D-brane – the submanifold on which open strings can end. In a semiclassical limit (if it exists), a suitable subspace $\overline{\mathcal{H}}_A^{aa}$ of $\mathcal{H}_A^{aa}$ may be interpreted as the commutative algebra of functions on the brane world volume (and in the same limit, open strings connecting different branes disappear). Again, at finite values of the relevant parameter, and taking into account the background $B$-field, $\overline{\mathcal{H}}_A^{aa}$ carries the structure of a non-commutative algebra and should be interpreted as a space of functions on a non-commutative quantization of $\mathcal{M}_a$ [19–23]. (Accordingly, the low-energy dynamics of the gauge fields on the brane is described by a non-commutative Yang–Mills theory. Here we

1 or, for heterotic theories, of different left and right chiral algebras.
neither address the algebraic structure of $\mathcal{H}^{\text{ua}}_\lambda$ nor brane dynamics, though.)

The geometric interpretation of boundary conditions is quite well understood for (unitary) Wess-Zumino-Witten models, for which the target space is a compact reductive Lie group $G$. In particular, the symmetry preserving branes, labeled by the integrable representations of the relevant affine Lie algebra $\hat{g}$ at level $k$, are conjugacy classes of $G$, while a specific class of symmetry breaking branes is given by ‘twined’ conjugacy classes; see e.g. [24–29, 18]. Since many more rational conformal field theory models can be obtained from WZW models via the coset construction, it is natural to try to extend this analysis to coset models, for which the target is a corresponding quotient $Q$ of compact reductive Lie groups. Indeed, a lot of effort has been devoted to the study of boundary conditions in coset models, a partial list of references being [7, 30–37]. However, it is fair to say that the situation is less satisfactory than in the WZW case, and some interesting questions are still open. It is the purpose of the present paper to collect, with the help of representation theoretic tools, further knowledge about the geometry of branes in coset models.

Among the outcome, what seems to us most relevant is a characterization of the group of identification simple currents in terms of the functions $\mathcal{F}(Q)$ on the target space (see formula (3.22)) and the implementation of field identification in the geometric description of the branes (see the discussion around (5.7)). To a certain extent our discussion is in semiclassical (large level) spirit, but primarily we are interested in what happens at finite values of the level. After all, in any WZW or coset model, the level does have a definite finite value.

The paper is organized as follows. In section 2 we collect information about algebraic aspects of coset models that is needed. In section 3 the geometry of coset models is discussed, leading in particular to a description of $\mathcal{H}_T$ in terms of horizontal branching spaces (formula (3.16)). The study of coset branes starts in section 4 with considerations related to the semiclassical limit of large level, while section 5 contains a detailed description of the branes at finite level. As an application we clarify in section 6 the geometry of parafermion branes at finite level. In an appendix, we comment on general aspects of the large level limit.

## 2 Coset models

Algebraically, a coset model is determined by an embedding $\hat{h} \hookrightarrow \hat{g}$, where $\hat{g}$ and $\hat{h}$ are direct sums of untwisted affine Lie algebras and Heisenberg algebras (with identified centers), together with the choice of a level $k$ of $\hat{g}$-representations, that is, a level for each of the affine ideals of $\hat{g}$ as well as a corresponding integer specifying the rational extension of the vertex algebra associated to each $\hat{u}(1)$ ideal of $\hat{g}$. Here we only consider unitary theories, thus (each component of) $k$ is a positive integer. The chiral algebra $\mathfrak{A}$ of the coset model is the commutant of the vertex algebra associated to $\hat{h}$ in the vertex algebra associated to $\hat{g}$.

Further, an allowed embedding $\hat{h} \hookrightarrow \hat{g}$ must come from a corresponding embedding $h \hookrightarrow g$ of the respective horizontal subalgebras (which are finite-dimensional complex reductive Lie algebras), from which it is obtained via the loop construction. Given the level $k$ of $\hat{g}$, the $\hat{h}$-representations are at level $k' = i_h \hookrightarrow g k$, with $i_h \hookrightarrow g$ the Dynkin index of the embedding $h \hookrightarrow g$.

The coset model obtained this way is often denoted by $G/H$, where $G$ and $H$ are Lie groups
associated to \( \hat{g} \) and \( \hat{h} \) (see the next section). But the relevant target \( Q \) is not the standard space of cosets of \( G \) by the right (or left) action of \( H \), and accordingly we prefer the notation \( \mathfrak{g}/\mathfrak{h} \) or, when we also want to indicate the level of the relevant \( \hat{g} \)-representations, by \( (\mathfrak{g}/\mathfrak{h})_k \).

This notation fits well with the fact that, as a rational conformal field theory, the \( \mathfrak{g}/\mathfrak{h} \) coset model can be analyzed completely through the representation theory of \( \hat{g} \) and of \( \hat{h} \), and thereby through the corresponding WZW models, to which we will refer as the \( \mathfrak{g} \)- (or \( \mathfrak{g}_k \)-) and \( \mathfrak{h} \)-WZW models, respectively. Indeed, the \( \mathfrak{g}/\mathfrak{h} \) model can be realized as a suitable extension of the tensor product \( \mathfrak{g} \times \mathfrak{h} \) of the \( \mathfrak{g} \)-WZW model and a putative\(^2\) theory \( \overline{\mathfrak{h}} \). The latter theory \( \overline{\mathfrak{h}} \) is obtained from the \( \mathfrak{h} \)-WZW model by a specific modification of the representation category of its chiral algebra which, basically, amounts to taking the complex conjugate of all chiral data, hence in particular of the modular group representation (for details and references see [39]).

For the purposes of this paper we restrict our attention to those coset models for which the labelling of fields is related to the one for the \( \mathfrak{g} \) and \( \mathfrak{h} \) theories by means of simple currents [40]. (Otherwise the coset model is called a maverick coset.\(^3\)) Then the primary fields of the \( \mathfrak{g}/\mathfrak{h} \) model are labeled by equivalence classes \([\hat{\Lambda}, \hat{\lambda}; \psi]\), where \( \hat{\Lambda} \) is an integrable highest \( \hat{g} \)-weight of level \( k \) and \( \hat{\lambda} \) an integrable highest \( \hat{h} \)-weight of level \( k' \), while \( \psi \) is a certain degeneracy label [41–45]. These classes are orbits of the action of certain simple currents on (the labels of) primary fields by the fusion product (to be denoted by the symbol \(*\) ), i.e. the equivalence relation reads

\[
(\hat{\Lambda}, \hat{\lambda}; \psi) \sim (J \ast \hat{\Lambda}, J \ast \hat{\lambda}; p(J,J)(\psi)) \quad \text{for} \quad (J, j) \in \mathcal{J}_{\mathfrak{g}/\mathfrak{h}} \tag{2.1}
\]

with

\[
\mathcal{J}_{\mathfrak{g}/\mathfrak{h}} \subseteq \mathcal{J}_{\mathfrak{g}} \times \mathcal{J}_{\mathfrak{h}} \tag{2.2}
\]

a specific subgroup of \( \mathcal{J}_{\mathfrak{g}} \times \mathcal{J}_{\mathfrak{h}} \), where \( \mathcal{J}_{\mathfrak{g}} \) and \( \mathcal{J}_{\mathfrak{h}} \) are the groups of simple currents of the \( \mathfrak{g} \)- and \( \overline{\mathfrak{h}} \)-theories, respectively.\(^4\) This subgroup \( \mathcal{J}_{\mathfrak{g}/\mathfrak{h}} \) is called the identification group of the \( \mathfrak{g}/\mathfrak{h} \) model, and its elements are referred to as identification currents. Also, \( p(J, J) \) in (2.1) is a permutation of the degeneracy labels (compare formula (A.9) in [46]). In the sequel, we will for simplicity assume that the ‘field identification’ (2.1) does not have fixed points, i.e. that all stabilizer subgroups

\[
\mathcal{J}_{\hat{\Lambda}, \hat{\lambda}} := \{(J, j) \in \mathcal{J}_{\mathfrak{g}/\mathfrak{h}} | J \ast \hat{\Lambda} = \hat{\Lambda} \text{ and } J \ast \hat{\lambda} = \hat{\lambda}\} \tag{2.3}
\]

of \( \mathcal{J}_{\mathfrak{g}/\mathfrak{h}} \) are trivial, so that the degeneracy labels \( \psi \) are absent. Let us remark, however, that we do expect that some of our considerations can be extended to coset models with field identification fixed points, though this is definitely beyond the scope of the present paper. On the other hand, it should not be expected that this framework allows one to understand the behavior of maverick cosets, which seem to be a low level phenomenon.

Further, among the classes \([\hat{\Lambda}, \hat{\lambda}]\) only those appear as labels of primary fields of the \( \mathfrak{g}/\mathfrak{h} \) model for which the branching space \( \mathcal{B}_{\hat{\Lambda}, \hat{\lambda}} = \text{Hom}_{\mathfrak{h}}(\mathcal{H}_{\hat{\lambda}}, \mathcal{H}_{\hat{\Lambda}}) \) is non-zero, i.e. for which the irre-

\(^2\) One way to think of \( \overline{\mathfrak{h}} \) is as a combination of a \( \mathfrak{h} \)-WZW model at negative levels and a ghost system [38].

\(^3\) See again [39] for more information and references. The maverick cosets include in particular all conformal embeddings \( \mathfrak{h} \rightarrow \hat{\mathfrak{g}} \), for which the coset theory has zero Virasoro central charge and hence is trivial.

\(^4\) The exceptional simple current of \( E_8 \) at level two is excluded here.
ducible \( \widehat{h} \)-module \( \mathcal{H}_{\widehat{\lambda}} \) with highest weight \( \widehat{\lambda} \) occurs in the decomposition

\[
\mathcal{H}_{\widehat{\lambda}} = \bigoplus_{\lambda} B_{\widehat{\Lambda}, \widehat{\lambda}} \otimes \mathcal{H}_{\widehat{\lambda}}
\]

of the irreducible \( \widehat{g} \)-module \( \mathcal{H}_{\widehat{\Lambda}} \) with highest weight \( \widehat{\Lambda} \):

\[
[\widehat{\Lambda}, \widehat{\lambda}] \text{ allowed label } \iff B_{\widehat{\Lambda}, \widehat{\lambda}} \neq \{0\}.
\] (2.5)

In the absence of field identification fixed points, the branching spaces \( B_{\widehat{\Lambda}, \widehat{\lambda}} \) constitute the irreducible modules of the coset chiral algebra, and the branching functions, i.e. the generating functions for the dimensions of the subspaces with definite conformal weight of branching spaces, are the irreducible characters of the coset theory. Also, branching spaces related by the field identification (2.1) are isomorphic as modules of the coset chiral algebra, \( B_{\widehat{\Lambda}, \widehat{\lambda}} \cong B_{J \ast \hat{\Lambda}, J \ast \hat{\lambda}} \) for \( (J, J) \in \mathcal{J}_{g/h} \).

Just like the field identification, the selection rule (2.5) can be understood in terms of the group \( \mathcal{J}_{g/h} \), too: precisely those pairs \((\widehat{\Lambda}, \widehat{\lambda})\) are allowed whose monodromy charge \([40]\) with respect to all identification simple currents \((J, J) \in \mathcal{J}_{g/h}\) vanishes \([47]\). It is also well known that for non-maverick cosets the selection rules are purely ‘group-theoretical’ in the terminology of \([43]\), i.e. that they are equivalent to imposing compatibility among the weight-conjugacy classes \(^5\) of the respective horizontal projections \( \Lambda \) and \( \lambda \) of the weights \( \widehat{\Lambda} \) and \( \widehat{\lambda} \). The latter description allows one in particular to determine the identification group directly from the relevant embedding \( \widehat{h} \hookrightarrow \widehat{g} \) of horizontal subalgebras (compare e.g. \([48–50]\)), though to the best of our knowledge this hasn’t been investigated in detail for arbitrary (non-maverick) coset models.

3 Geometry of coset models

In the Lagrangian framework, the \( \mathfrak{g}/\mathfrak{h} \) coset model is obtained by coupling the WZW model with target space \( \mathbb{G} \) – the connected and simply connected compact Lie group whose Lie algebra is the compact real form \( \mathfrak{g}_\mathbb{R} \) of \( \mathfrak{g} \) – to a gauge field taking values in \( \mathfrak{h}_\mathbb{R} \) \([51–53]\). By integrating out the gauge field in the path integral, this yields a sigma model with target \( \mathcal{M} = \mathbb{Q} \) given by the space \( \mathbb{G}/\text{Ad}(\mathbb{H}) \) of orbits of the adjoint action of a subgroup \( \mathbb{H} \) on \( \mathbb{G} \) \([54]\),

\[
\mathbb{Q} = \mathbb{G}/\text{Ad}(\mathbb{H}) := \{[g] \mid g \in \mathbb{G}\} \quad \text{with} \quad [g] := \{hgh^{-1} \mid h \in \mathbb{H}\}.
\] (3.1)

Note that here we have to think of \( \mathbb{H} \) not just as a Lie group, but rather as a Lie subgroup embedded in \( \mathbb{G} \) via a concrete embedding \( \mathbb{v} : \mathbb{H} \hookrightarrow \mathbb{G} \), which is determined by the Lie algebra embedding \( \mathfrak{h} \hookrightarrow \mathfrak{g} \). Accordingly, in the sequel group elements \( h \in \mathbb{H} \) will always be regarded as elements of \( \mathbb{G} \), i.e. \( h \equiv \mathbb{v}(h) \in \mathbb{G} \).

[^5]: By the \textit{weight-conjugacy class} of a \( \mathfrak{g} \)-weight \( \Lambda \) we mean the class of \( \Lambda \) in the weight lattice modulo the root lattice of \( \mathfrak{g} \). We avoid the more common term conjugacy class in order that no confusion with the conjugacy classes \( C \) in the Lie group \( \mathbb{G} \) associated (see below) to \( \mathfrak{g} \) can arise.
Also, in (3.1) we take $H$ to be the connected and simply connected compact Lie group whose Lie algebra is $\mathfrak{h}_\mathbb{R}$. However, the subgroup

$$Z(G,H) := Z(G) \cap H$$

of $H$, with $Z(G)$ the center of $G$, acts trivially under the adjoint action, so that in (3.1) we could equally well replace $H$ by the (generically) non-simply connected group $H/Z(G,H)$. Indeed, in the description of the $\mathfrak{g}/\mathfrak{h}$ model as a gauged WZW model, the gauge field is a connection in a principal bundle with structure group $H/Z(G,H)$ rather than $H$. The significance of this observation for the $\mathfrak{g}/\mathfrak{h}$ model becomes evident when one notices that the centers $Z(G)$ and $Z(H)$ are naturally isomorphic to (the duals of) the groups of weight-conjugacy classes of $\mathfrak{g}$- and $\mathfrak{h}$-weights, respectively. The triviality of the action of $Z(G,H)$ therefore corresponds to having a definite relation between the weight-conjugacy classes of the $\mathfrak{g}$- and $\mathfrak{h}$-weights that can appear as labels of primary fields in the $\mathfrak{g}/\mathfrak{h}$ model. Comparison with the description of the selection rules in the previous section shows that indeed $Z(G,H)$ is isomorphic to the identification group,

$$Z(G,H) \cong \mathcal{J}_{\mathfrak{g}/\mathfrak{h}}. \quad (3.3)$$

An alternative description of the target space is [34]

$$Q = \frac{G \times H}{H_l \times H_r} := \{[g,h] \mid g \in G, h \in H\} \quad \text{with} \quad [g,h] := \{(ugv, uhv) \mid u, v \in H\}, \quad (3.4)$$

where $H_l$ and $H_r$ are just equal to $H$ as Lie subgroups of $G$, while their subscript reminds us that they act from the left and right, respectively, on each factor of $G \times H$. A bijection from $G/\text{Ad}(H)$ to $(G \times H)/(H_l \times H_r)$ is given by $\varpi: [g] \mapsto [g,e]$ with $e$ the unit element of $G$ (and $H$), with inverse $\varpi^{-1}: [g,h] \mapsto [gh^{-1}]$ (note that $[g,h] = [gh^{-1}, e]$ in $(G \times H)/(H_l \times H_r)$). Again, a subgroup $Z_{l|r} \subset H_l \times H_r$ acts trivially, and again this subgroup is isomorphic to the identification group,

$$Z_{l|r} = \{(u, u^{-1}) \mid u \in Z(G,H)\} \cong Z(G,H). \quad (3.5)$$

We will denote by

$$\pi_{\text{Ad}} : G \to Q \quad \text{and} \quad \pi_{l|r} : G \times H \to Q \quad (g, h) \mapsto [g, h] \quad (3.6)$$

the projections that correspond to descriptions of $Q$ via (3.1) and via (3.4), respectively.

**Functions on $Q$.** For studying the geometric interpretation of boundary conditions, we will have to work with the space $\mathcal{F}(Q)$ of (square integrable $\mathbb{C}$-valued) functions on $Q$, or on certain subsets of $Q$. The functions on $Q$ can be identified with the $H_l \times H_r$-invariant functions on $G \times H$. Accordingly, the projection $\pi_{l|r}^* f \in \mathcal{F}(Q)$ of a function $f \in \mathcal{F}(G \times H)$ is given by

$$\pi_{l|r}^* f([g,h]) := \frac{1}{|H|} \int_{H \times H} du \, dv \, f(ugv, uhv), \quad (3.7)$$

To deal with fixed point resolution, also gauge fields in non-trivial $H/Z(G,H)$ bundles must be included [41, 55, 56].
where $d\mu$ is the (unnormalized\(^7\)) Haar measure on $H$ and $|H| = \int_H d\mu$ the volume of $H$. A function $f \in \mathcal{F}(G \times H)$ is $H$-invariant iff $\pi^*_\mu f = f$.

The space $\mathcal{F}(G \times H)$ is spanned over $\mathbb{C}$ by the functions

$$D_{mn,ab}^{\Lambda,\lambda} := D_{mn}^{\Lambda} D_{ab}^{\lambda}$$

with $\Lambda$ and $\lambda$ ranging over the dominant integral weights of $G$ and $H$, respectively, and $D_{mn}^{\Lambda}(g)$ and $D_{ab}^{\lambda}(h)$, with $m,n \in \{1,2,\ldots,d_\Lambda\}$ and $a,b \in \{1,2,\ldots,d_\lambda\}$, the entries of the corresponding representation matrices (the complex conjugation on $D_{ab}^{\lambda}$ is chosen for later convenience). The functions (3.8) actually form a basis of $\mathcal{F}(G \times H)$ and (when choosing orthonormal bases of the representation spaces) satisfy orthogonality relations which follow from the relation

$$\int_G d\mu D_{mn}^{\Lambda} D_{m'n'}^{\Lambda'} = d_{\Lambda}^{-1} |G| \delta_{mm'} \delta_{nn'} \delta_{\Lambda\Lambda'}$$

for $\mathcal{F}(G)$ together with the analogous relation for $\mathcal{F}(H)$.

Let us investigate the behavior of the basis functions (3.8) under the projection (3.7). It turns out that among the functions (3.8), only those $D_{mn,ab}^{\Lambda,\mu}$ give rise to non-zero functions on $Q$ for which the irreducible $H$-representation $R^\mu$ with highest weight $\mu$ occurs in the decomposition of the irreducible $G$-representation $R^\Lambda$ with highest weight $\Lambda$ as a $H$-representation, a property that we will indicate by writing $\mu \prec \Lambda$. All other functions $D_{mn,ab}^{\Lambda,\lambda}$ have vanishing projection to $\mathcal{F}(Q)$. In short, the space $\mathcal{F}(Q)$ is spanned by the functions $\pi^*_\mu D_{mn,ab}^{\Lambda,\lambda}$ with $\lambda \prec \Lambda$. Note that this constitutes a horizontal counterpart of the selection rules (2.5) of the coset model\(^3\) – when studying functions on $Q$, precisely those pairs $(\Lambda,\lambda)$ are allowed labels for which $\lambda \prec \Lambda$, i.e. those for which the horizontal branching space $B_{\Lambda,\lambda} = \mathbb{C}^{b_{\Lambda,\lambda}}$ that appears in the decomposition

$$\mathcal{H}_\Lambda \cong \bigoplus_{\lambda \prec \Lambda} B_{\Lambda,\lambda} \otimes \mathcal{H}_\lambda$$

is non-zero.

To analyze this issue in more detail, we formulate the fact that the irreducible $G$-representation $R^\Lambda$ decomposes as a direct sum of irreducible $H$-representations $R^\mu$ in terms of the representation matrices. Namely, upon suitable basis choices, for $h \in H$ the representation matrix $D^\Lambda$ decomposes into blocks along the diagonal,

$$D^\Lambda(h) = \bigoplus_{\lambda \prec \Lambda} \bigoplus_{\ell=1} b_{\Lambda,\lambda} D^{\lambda(\ell)}(h)$$

(3.10)

where the summation is over all irreducible $H$-representations that appear in the branching of $R^\Lambda$, counting multiplicities, and where the symbol $D^{\lambda(\ell)}$ denotes the matrix block within the big matrix $D^\Lambda$ that corresponds to the $\ell$th occurrence of $R^\Lambda$ in $R^\Lambda$. Thus the labels $\lambda(\ell)$ enumerate the matrix blocks, while $\lambda$ labels (equivalence classes of) irreducible representations. For a general choice of orthonormal bases in the representation spaces of $G$ and $H$, the equality (3.10) holds up to a similarity transformation. The matrix elements are then related by

$$D_{mn}^{\Lambda}(h) = \sum_{\lambda} \sum_{\ell=1} b_{\Lambda,\lambda} \sum_{a,b=1}^{d_\lambda} c_{m,a}^{\lambda(\ell);\ell} c_{n,b}^{\lambda(\ell);\ell} D_{ab}^{\lambda}(h).$$

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\(^7\) Since the action of a WZW sigma model, and hence the metric on the group manifold $G$ to be used here, involves a factor of $k$, the volume of $G$ scales as $k^{\dim(G)/2}$ with the level.
The numbers $c^\mu<\Lambda;\ell$ appearing here are the coefficients in the expansion
\begin{equation}
\epsilon^\Lambda_m = \sum_{\mu,\ell,a} c^\mu<\Lambda;\ell \beta^{(\ell)} \otimes \epsilon^\mu_a
\end{equation}
of the vectors $\epsilon^\Lambda_m$ in the chosen basis of $\mathcal{H}_\Lambda$ as a linear combination of vectors $\epsilon^\mu_a$ in the chosen bases of the irreducible H-modules $\mathcal{H}_\mu$ and $\beta^{(\ell)}$ in bases of the multiplicity spaces $\mathcal{B}_{\Lambda,\mu}$ (in particular, $c^\mu<\Lambda;\ell$ vanishes unless $\mu<\Lambda$). Being coefficients of a basis transformation between orthonormal bases, they form unitary matrices.

Using (3.11) together with the representation property and the orthogonality relations for the representation matrices, we get
\begin{equation}
\pi^*_{l|r} \mathcal{D}_{mn,ab}^{\Lambda,\lambda}(g,h) = \frac{1}{|H|^2} \int_{H \times H} du \, dv \, \sum_{p,q,d,e} D^\Lambda_{mp}(u) D^\lambda_{pq}(g) D^\Lambda_{qn}(v) D^\lambda_{ad}(u)^* D^\lambda_{de}(h)^* D^\lambda_{eb}(v)^* \\
= \frac{1}{d^2} \sum_{\ell,\ell'=1}^{b_{\Lambda,\lambda}} c^\prec<\Lambda;\ell \prec \lambda<\Lambda;\ell' c^\prec\mu<\Lambda;\prec \delta_{\mu<\Lambda,\lambda} \delta_{\mu<\Lambda,\lambda}
\end{equation}
Provided that the right hand side of (3.13) is non-zero, up to a scalar factor it does not depend on the labels $m$, $n$ and $a$, $b$; hence this way each allowed pair $(\Lambda, \lambda)$ of dominant integral G- and H-weights gives rise to $b_{\Lambda,\lambda}^2$ functions on $Q$.

Let us also remark that the calculations simplify when one makes the following adapted Gelfand-Zetlin type basis choice: First select arbitrary orthonormal bases in all $\mathcal{H}_\Lambda$ (constructions). With this adapted basis choice the branching coefficients (i.e. as an equality of matrices rather than just as an equality between linear transformations) corresponding to formula (3.10) holding exactly, not only up to a similarity transformation (i.e. as an equality of matrices rather than just as an equality between linear transformations). With this adapted basis choice the branching coefficients $c^\mu<\Lambda;\ell$ are $c^\mu<\Lambda;\ell = \delta_{a\midd,\delta_{\mu<\Lambda,\lambda}}$, and thus we have
\begin{equation}
\mathcal{D}_{mn}^\Lambda(h) = \mathcal{D}_{\tilde{m}\tilde{n}}^\Lambda(h),
\end{equation}
where the labels $\tilde{m}_\ell$ and $\tilde{n}_\ell$ are the row and column labels of the matrix block $\mathcal{D}^{\Lambda(\ell)}$ in $\mathcal{D}^\Lambda$ that according to (3.10) correspond to the row and column labels $m$ and $n$ of the big matrix. The result (3.13) then reduces to
\begin{equation}
\pi^*_{l|r} \mathcal{D}_{mn,ab}^{\Lambda,\lambda}(g,h) = \begin{cases}
d^{-2} \sum_{\ell,\ell'} \delta_{a\midd} \delta_{b\midd} \sum_{p,q=1}^{d_{\Lambda}} D_{pq,\ell\ell'}^\Lambda(g,h) & \text{for } \lambda<\Lambda, \\
0 & \text{else}
\end{cases}
\end{equation}
In particular, when the branching rule for $\Lambda$ is multiplicity-free, then formula (3.13) reduces to $\pi^*_{l|r} \mathcal{D}_{mn,ab}^{\Lambda,\lambda}(g,h) = \delta_{a\midd} \delta_{b\midd} d^2 \sum_{p,q} D_{pq,\ell\ell'}^\Lambda(g,h)$ for $\lambda<\Lambda$. (This applies e.g. to all branching rules in the description of the unitary Virasoro minimal models as $\mathfrak{su}(2) \oplus \mathfrak{su}(2)/\mathfrak{su}(2)_{\text{diag}}$ coset models, for which the coefficients $c^\mu<\Lambda$ are just ordinary Clebsch–Gordan coefficients.)

To summarize, each allowed pair $(\Lambda, \lambda)$ provides us with $b_{\Lambda,\lambda}^2$ functions on $Q$. Also, from the construction it is apparent that all these functions are linearly independent. It is then natural to identify the $b_{\Lambda,\lambda}^2$-dimensional space spanned by these functions with the tensor product
\[ B_{\Lambda,\lambda} \otimes B^*_{\Lambda,\lambda} \] of the branching space \[ B_{\Lambda,\lambda} = \mathbb{C}^{b_{\Lambda,\lambda}} \] with its dual space. The latter, in turn, is naturally isomorphic to the branching space \[ B^*_{\Lambda',\lambda'} \] for the dual modules. Thus we arrive at a description of \( \mathcal{F}(Q) \) as
\[
\mathcal{F}(Q) \cong \bigoplus_{\Lambda' \prec \Lambda} B_{\Lambda,\lambda} \otimes B^*_{\Lambda',\lambda'} .
\] (3.16)

This is nothing but the analogue of the Peter-Weyl isomorphism
\[
\mathcal{F}(G) \cong \bigoplus_{\Lambda} \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda'}
\] (3.17)
for the functions on the target space \( G \) of the \( g \)-WZW model.

It is tempting to think of the horizontal branching spaces \( B_{\Lambda,\lambda} \) as subspaces of the branching spaces \( B_{\hat{\Lambda},\hat{\lambda}} \) that appear in the branching rules (2.4) of \( \hat{g} \)-representations, and thereby to identify \( \mathcal{F}(Q) \) with the horizontal subspace \( \overline{\mathcal{H}}_T \) of the space of bulk fields,
\[
\mathcal{F}(Q) \approx \overline{\mathcal{H}}_T \subset \mathcal{H}_T .
\] (3.18)

Actually, as a consequence of field identification, a few specific states present in the \( B_{\Lambda,\lambda} \) can be absent in the \( B_{\hat{\Lambda},\hat{\lambda}} \). Indeed, it can happen that several different horizontally allowed pairs correspond to one and the same state in the coset model.\(^8\) However, this appears to be a low-level effect: the number of such missing states is small, and it does not depend on the level, and hence with increasing level this mismatch becomes less and less relevant.

**Simple current action on \( \mathcal{F}(Q) \).** The ‘horizontal selection rules’ just described are the closest analogue of the \( g/h \) selection rules that can reasonably be expected to hold for \( \mathcal{F}(Q) \). It is thus tempting to seek also for a manifestation of the field identification (2.1) in the space \( \mathcal{F}(Q) \). Since the fusion rules are (strongly) level-dependent, there cannot possibly be any nice relation between the functions \( D_{\Lambda,\lambda} \) and \( D_{\Lambda',\lambda'} \) (by a slight abuse of notation, by \( J_{\Lambda} \) we mean the horizontal part of the affine weight \( J_{\Lambda} \)). However, there turns out to be another distinctive role of the identification group.

For \( g \) and \( h \) semisimple, an action of the simple currents \((J,\lambda) \in \mathcal{J}_g \times \mathcal{J}_h \) on functions on \( Q \) can be defined as follows. Let us denote by \( t_\Lambda \) the element of the Cartan subalgebra \( g_c \) of \( g \) that is dual to the weight \( \Lambda \in g_c^* \), i.e. \( t_\Lambda = (t,\Lambda) = \sum_{i,j} \gamma_{i,j} t^i \Lambda^j \) with \( \gamma \) the inverse of the symmetrized Cartan matrix. Further, for \( J \) a simple current of the \( g \)-WZW model, denote by \( \Lambda_{(J)} \) the corresponding [58] cominimal fundamental \( g \)-weight and set
\[
g_{(J)} := \exp \left( 2\pi i \, t_{\Lambda_{(J)}} \right) \in G .
\] (3.19)

The group elements \( h_{(J)} := \exp \left( 2\pi i \, (t',\lambda_{(J)}) \right) \in G \) are defined analogously for the simple currents of the \( h \)-WZW model. Now via the fusion product, the set \( \mathcal{J}_g = \{J\} \) of simple currents of the

---

\(^8\) For instance, as noted in [57], for the parafermions (see section 6 below, where also the notation used here is explained in detail), there is precisely one such state: at level \( k \) there exist \( k \) allowed fields labeled as \((k,n)\), while the number of horizontally allowed pairs of the form \((k,n)\) is \( k+1 \).
The g-WZW model is a finite abelian group, and (see e.g. section 14.2.3 of [59]) is isomorphic as a group to the center $Z(G)$ of $G$, with an isomorphism provided by the mapping $J \mapsto g(J)$. Likewise, the mapping $j \mapsto h(j)$ furnishes an isomorphism $\{j\} = \mathfrak{z}_h \xrightarrow{\cong} Z(H)$. For an abelian ideal of $\mathfrak{g}$ (and analogously for $\mathfrak{h}$), the situation is a bit simpler. Indeed, then every primary field is a simple current, and the set of primary fields endowed with the fusion product is an abelian group that is naturally isomorphic to a subgroup of $Z(G) = G$, and we denote the elements of that subgroup by $g(J)$. We now define, for $(J, j) \in \mathfrak{g} \times \mathfrak{h}$, an action $F_{(J,j)}$ on $\mathcal{F}(Q)$ by

$$
(F_{(J,j)} f)([g, h]) := f([gg(J), hh(j)])
$$

for $f \in \mathcal{F}((G \times H)/(H_L \times H_R))$ and $g \in G, h \in H$. This can be rewritten as

$$
(F_{(J,j)} f)([g, h]) = f([gg(J)h^{-1}(j), h])
$$

thus implying in particular that the subgroup $\{(J, j) \in \mathfrak{g} \times \mathfrak{h} | g(J) = h(j)\}$ of $\mathfrak{g} \times \mathfrak{h}$ acts trivially. Now it is precisely in this way that the weight-conjugacy classes of the weights $\Lambda(J)$ and $\lambda(j)$ must be related in order for $(J, j)$ to form an allowed pair. Moreover, via $(J, j) \mapsto (g(J), h(j))$ this subgroup is isomorphic to $Z(G, H)$, and hence by comparison with (3.3) we see that

$$
\{(J, j) \in \mathfrak{g} \times \mathfrak{h} | g(J) = h(j)\} \cong \mathfrak{g}/\mathfrak{h}.
$$

Thus the identification group $\mathfrak{g}/\mathfrak{h}$ of the $\mathfrak{g}/\mathfrak{h}$ model can be characterized as the subgroup of $\mathfrak{g} \times \mathfrak{h}$ acting trivially on $\mathcal{F}(Q)$. This is the counterpart of field identification in a geometric analysis of the $\mathfrak{g}/\mathfrak{h}$ model in terms of the functions on the target space $Q$. Since this invariance property is shared (by definition) by the space $\mathcal{H}_T$ of bulk fields, this observation provides further evidence for the correctness of the identification (3.18).

4 Branes at large level

The description of coset models given above together with the knowledge about D-branes in WZW models can be combined to draw conclusions about the geometry of D-branes of coset models at large level. In this paper we consider only such boundary conditions which preserve the full chiral symmetry of the coset model. For these boundary conditions, at large level the branes on $G \times H$ are concentrated on the submanifolds

$$
C_{\Lambda, \lambda} := \{(g, h) \in G \times H | g \in C_{\Lambda}^G, h \in C_{\lambda}^H\} = C_{\Lambda}^G \times C_{\lambda}^H \subset G \times H.
$$

Here

$$
C_{\Lambda}^G := \{g'g_{\Lambda}g'^{-1} | g' \in G\}
$$

is the conjugacy class in $G$ of the group element

$$
g_{\Lambda} := \exp\left(\frac{2\pi i}{k+g^{\vee}}t_{\lambda+\rho}\right),
$$

10
with $\rho$ and $g^\vee$ the Weyl vector and dual Coxeter number of $\mathfrak{g}$, respectively, and $t_{\Lambda+\rho} = (t, \Lambda+\rho)$. Analogously, $C^H_\Lambda = \{ h'h_\Lambda h'^{-1} \mid h' \in H \}$ is the conjugacy class in $H$ of $h_\Lambda := \exp \left( \frac{2\pi i}{k+h^\vee} t_{\Lambda+\rho}^i \right)$, with $\rho'$ and $h^\vee$ the Weyl vector and dual Coxeter number of $\mathfrak{h}$. Further, the labels $\Lambda$ and $\lambda$ are (the horizontal parts of) those of the primary fields of the $\mathfrak{g}$- and $\mathfrak{h}$-WZW models, i.e. dominant integral weights satisfying $(\Lambda, \theta) \leq k$ and $(\lambda, \vartheta) \leq k'$, with $\theta$ and $\vartheta$ the highest roots of $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

Let us remark that instead of (4.1) we may equally well use the sets

$$C^{(-)}_{\Lambda,\lambda} := \{ (g, h^{-1}) \in G \times H \mid g \in C^G_\Lambda, h \in C^H_\Lambda \} \subset G \times H. \quad (4.4)$$

to describe the subsets on which the $G \times H$-branes are concentrated. Indeed, this just amounts to choosing a different labelling, owing to

$$C^{(-)}_{\Lambda,\lambda} = C_{\Lambda,\lambda^+} \quad \text{(4.5)}$$

with $\lambda^+$ the $\mathfrak{h}$-weight charge conjugate to $\lambda$. The equality (4.5) holds because $h \in C^H_\Lambda$ iff $h^{-1} \in C^H_{\lambda^+}$, which in turn follows from the fact that the map $\lambda \mapsto -\lambda^+$ is a Weyl transformation (namely the one corresponding to the longest element of the Weyl group of $\mathfrak{h}$) and that the Weyl vector is self-conjugate.

In the description (3.4) of $Q$, we can obtain the branes on the target space $Q$ of the coset model as the projections

$$\pi_{\text{lr}}(C_{\Lambda,\lambda}) = \{ [g, h] \mid (g, h) \in C_{\Lambda,\lambda} \} \quad \text{(4.6)}$$

of the sets (4.1). In the alternative description (3.1), one deals instead with projections of the sets [32, 7, 34, 60]

$$\tilde{C}_{\Lambda,\lambda} := \{ gh^{-1} \in G \mid g \in C^G_\Lambda, h \in C^H_\Lambda \subset G \} \subset G. \quad (4.7)$$

These subsets are $\text{Ad}(H)$-invariant, i.e. satisfy $u \tilde{C}_{\Lambda,\lambda} u^{-1} = \tilde{C}_{\Lambda,\lambda}$ for all $u \in H$, and hence they trivially project on $Q$, i.e.

$$\pi_{\text{Ad}}(\tilde{C}_{\Lambda,\lambda}) = \{ [gh^{-1}] \in Q \mid g \in C^G_\Lambda, h \in C^H_\Lambda \} \subset Q. \quad (4.8)$$

It follows directly from the existence of the bijection $\varpi$ defined after (3.4) that these descriptions of the coset branes at large level are equivalent:

$$\pi_{\text{Ad}}(\tilde{C}_{\Lambda,\lambda}) = \pi_{\text{lr}}(C^{(-)}_{\Lambda,\lambda}). \quad (4.9)$$

Expressed in terms of functions on $Q$, the discussion above amounts to the statement that the shape of a brane is a delta function on the subset (4.9). This description is adequate in the limit of large level, whereas at any finite value of the level the shape of the brane is smeared about this subset. The extent of localization increases with the level; this will be analyzed quantitatively in the next section.

To see explicitly that in the limit one indeed deals with a delta function requires the information that in the WZW case the limit yields (a multiple of) a delta function $\mathcal{D}^G_\Lambda = \delta^G_{C^G_{\Lambda}} / |C^G_{\Lambda}|$ on a conjugacy class $C^G_{\Lambda}$ of the group, see formula (A.4) in the appendix. The corresponding
function on the coset is (including a compensating factor of $|H|$, see the discussion after (5.6) below)

\[ D_{\Lambda,\lambda}([g,h]) = \frac{1}{|H|} \int_{H \times H} du \, dv \, D_{\Lambda}^G(ugv) \, D_{\lambda}^H(uhv). \]  \hspace{1cm} (4.10)

One of the two integrations over $H$ is trivial, and the other can be performed with the help of the identity $\delta_{g\Lambda}^G(\nu v v^{-1} h^{-1} g) = \delta_{g\Lambda}^G(u h^{-1} g)$, which for $u, v, h \in H$ and $g \in G$ is valid as an equality of functions on $Q$. The result is

\[ D_{\Lambda,\lambda}([g,h]) = \left| C_{g\Lambda}^{G,\lambda} \right| \delta_{g\Lambda}^G(\lambda h h^{-1} g) = \left| C_{g\Lambda}^{G,\lambda} \right| \delta_{C_{g\Lambda}^{G,\lambda}}(h^{-1} g). \]  \hspace{1cm} (4.11)

**Simple current action.** Let us consider the action of simple currents $J$ of the $g$-WZW model on the conjugacy classes $C_{\Lambda}^G \subset G$ that is obtained by mapping $C_{\Lambda}^G$ to

\[ C_{\Lambda}^G \cdot g_{(J)} := \{ gg_{(J)} \mid g \in C_{\Lambda}^G \} \subset G. \]  \hspace{1cm} (4.12)

Now the horizontal part of $J \cdot \tilde{\Lambda}$ can be written as (see e.g. section 14.2.2 of [59])

\[ J \cdot \tilde{\Lambda} = w_{(J)} w_{\omega}(\Lambda) + k \Lambda_{(J)}, \]  \hspace{1cm} (4.13)

where $w_\omega$ is the longest element of the Weyl group $W$ of $g$ and $w_{(J)}$ the longest element of the subgroup of $W$ that is generated by all simple Weyl reflections except the one corresponding to the simple root that is dual to $\Lambda_{(J)}$. It follows that

\[ J \cdot \tilde{\Lambda} + \rho = w_{(J)} w_{\omega}(\Lambda + \rho) + (k + g^\vee) \Lambda_{(J)}, \]  \hspace{1cm} (4.14)

and thus the group elements (4.3) satisfy

\[ g_{J \cdot \tilde{\Lambda}} = \left( w_{(J)} w_{\omega}(g_{\Lambda}) \right) g_{(J)}. \]  \hspace{1cm} (4.15)

Since Weyl transformations do not change the conjugacy class of a group element, this implies that

\[ C_{\Lambda}^G \cdot g_{(J)} = C_{\Lambda}^{G,\pi}. \]  \hspace{1cm} (4.16)

Thus the simple current group $\mathcal{J}_g$ acts in a natural way via the fusion product on conjugacy classes of $G$. Analogously, the simple current group $\mathcal{J}_h$ of the $h$-WZW model acts via the fusion product on conjugacy classes of $H$, as $C_{\Lambda}^H \mapsto C_{\Lambda}^H \cdot h_{(J)} = C_{\Lambda}^{H,\pi}$, and hence $\mathcal{J}_g \times \mathcal{J}_h$ acts on the submanifolds (4.1) of $G \times H$ as

\[ C_{\Lambda,\lambda} \mapsto C_{\Lambda,\pi \Lambda,\pi \lambda} = C_{\Lambda,\Lambda} \cdot (g_{(J)} h_{(J)}). \]  \hspace{1cm} (4.17)

In particular, owing to $[g,h] = [gh^{-1},e]$ in $(G \times H)/(H_t \times H_r)$, it follows that for $g_{(J)} = h_{(J)}$ we have

\[ \pi_{\|r}(C_{\Lambda,\pi \Lambda,\pi \lambda}) = \pi_{\|r}(C_{\Lambda,\lambda}) \]  \hspace{1cm} (4.18)

for the coset branes (4.6). That is, if the simple current $(J,J) \in \mathcal{J}_g \times \mathcal{J}_h$ is an identification current,\(^9\) then it acts trivially on the coset branes (4.6).

---

\(^9\) This is the only generic reason for a simple current $(J,J)$ to act trivially on branes. On some specific branes, also simple currents $(J,J) \in \mathcal{J}_g \times \mathcal{J}_h \setminus \mathcal{J}_g/h$ can act trivially, though.
5 Branes at finite level

Like in any rational conformal field theory, the symmetry preserving boundary states $B^Q$ of the coset theory are naturally labelled by the labels of primary fields, and when expressing them in terms of the Ishibashi functionals (or boundary blocks) $I^Q$, which form bases of the spaces of conformal blocks for the one-point correlators on the disk, the coefficients are given by the modular $S$-matrix. Thus, in the absence of field identification fixed points (which we assume), the $B^Q$ are labelled by the $\mathcal{J}_{gh}$-orbits $[\tilde{\Lambda}, \tilde{\lambda}]$, and are related to the Ishibashi states as

$$B^Q_{[\tilde{\Lambda}, \tilde{\lambda}]} = \sum_{[\tilde{\Lambda}', \tilde{\lambda}]} \frac{S_{[\Lambda, \lambda], [\Lambda', \lambda']}}{\sqrt{S_{[\Lambda', \lambda'], [0, 0]}}} I^Q_{[\Lambda', \lambda']}.$$  \hfill (5.1)

(Here and below we slightly abuse notation by writing $\Lambda$ in place of $\tilde{\Lambda}$ in the subscripts of $S$-matrices.) Also (again, in the absence of fixed points), the coset $S$-matrix is expressible through the modular $S$-matrices of the $\mathfrak{g}$- and $\mathfrak{h}$-WZW models as

$$S_{[\Lambda', \lambda'], [\Lambda, \lambda]} = |\mathcal{J}_{gh}| S^\mathfrak{g}_{\Lambda, \Lambda'} S^\mathfrak{h}_{\lambda, \lambda'}^*,$$ \hfill (5.2)

where on the right hand side arbitrary representatives $(\tilde{\Lambda}, \tilde{\lambda})$ and $(\tilde{\Lambda}', \tilde{\lambda}')$ of the orbits $[\tilde{\Lambda}, \tilde{\lambda}]$ and $[\tilde{\Lambda}', \tilde{\lambda}']$ are chosen [42, 43].

We would like to associate functions on the target space to the Ishibashi functionals $I^Q$, and thereby to the boundary functionals (5.1). This allows one, via the approach of [61, 26], to probe the geometry of a brane by bulk fields.\[^{10}\] For general background geometries it is a difficult task to find suitable functions. But in the case of our interest a helpful strategy is to start with a discussion of the branes on the group manifold $G \times H$, following the lines of [26] for WZW branes. For group manifolds there is a preferred way of relating the functions on the target to the horizontal subspace $\overline{\mathcal{H}}_T \subset \mathcal{H}_T$ of the space of bulk fields, and thereby to the boundary states which live in the dual space $\overline{\mathcal{H}}_T$: the Peter–Weyl isomorphism (3.17), which associates to a vector $v \otimes w \in (\mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^+})^*$ in the dual of $\overline{\mathcal{H}}_T$ the function

$$f^G_{PW}(v \otimes w) := \sqrt{d_\Lambda/|G|} \langle v | D^\Lambda | w \rangle \in \mathcal{F}(G).$$ \hfill (5.3)

For any $\Lambda$, the normalization of $f^G_{PW}$ is determined, up to a phase, by the requirement that the mapping associates to vectors in an orthonormal basis of $(\mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^+})^*$ functions that are orthonormal with respect to the Haar measure.

Using $I^G_{\Lambda'}(e^\Lambda_m \otimes e^{\Lambda^+}_n) = \delta_{\Lambda\Lambda'} \delta_{mn}$ (with $e^\Lambda_m$ elements of an orthonormal basis of $\mathcal{H}_\Lambda$, as introduced after formula (3.11) above), it follows that the function $I^G_\Lambda$ associated this way to the Ishibashi state $I^G_\Lambda$ is given by

$$I^G_\Lambda(g) = \sqrt{d_\Lambda/|G|} \sum_{m} \langle e^{\Lambda^+}_m | D^\Lambda(g) | e^\Lambda_m \rangle = \sqrt{d_\Lambda/|G|} \langle X_\Lambda(g),$$ \hfill (5.4)

\[^{10}\] In the WZW case the analysis was done for bulk fields corresponding to, in string terminology, the graviton, dilaton and Kalb–Ramond field [26]. To avoid having to deal explicitly with the coset chiral algebra, here we consider the bulk fields corresponding to the tachyon instead. As observed in [26], different choices of bulk fields give the same results qualitatively.
where $\mathcal{X}_\lambda$ is the character of the $G$-representation $R_\Lambda$; we will refer to this function as the shape of the Ishibashi state. Analogously, the Ishibashi states of the $g\oplus h$-WZW model are given by

$$
\mathcal{I}^{G\times H}_{\Lambda,\hat{\lambda}}(g,h) = \sqrt{\frac{d_\Lambda d_\lambda}{|G||H|}} \mathcal{X}_\lambda(g) \mathcal{X}_\Lambda(h)^* = \sqrt{\frac{d_\Lambda d_\lambda}{|G||H|}} \sum_{m,a} D^{\Lambda,\lambda}_{mm,aa}(g,h).
$$

(5.5)

Here we have chosen a different convention on phases for the Peter–Weyl mapping $f^h_{PW}$ of the $h$-WZW model, in order to conform with the fact that in the algebraic description instead of the $h$-WZW model it is the ‘complex conjugate theory’ $\bar{h}$ that matters.

By comparison with the discussion of $\mathcal{F}(Q)$ in section 3, we expect that in order to obtain the shape of Ishibashi states of the coset model, we should consider the projection

$$
\sqrt{\frac{d_\Lambda d_\lambda}{|G|}} \sum_{m,a} \pi_{\Lambda,\lambda} D^{\Lambda,\lambda}_{mm,aa} = \sqrt{\frac{d_\Lambda d_\lambda}{|G|}} \sum_{\ell,\ell'=1}^{b_\Lambda,\lambda} \sum_{m,a} \sum_{\ell'=1}^{d_\Lambda} \sum_{\ell'=1}^{d_\lambda} c_{m,a}^{\Lambda<\Lambda;\ell;\ell'} c_{a,c}^{\lambda<\Lambda;\ell;\ell'} D^{\Lambda,\lambda}_{pq,bc}.
$$

(5.6)

of the functions (5.5) to the coset manifold $Q$ (in the second equality, the unitarity of the matrices $c$ is used). However, a remark on the choice of normalization made in (5.6) is in order. In the WZW case, the prefactor in (5.4) is chosen in such a way that the integral $\int_G dg \mathcal{B}^G(g)$ is of order $k^0$ at large level. Now whereas the normalization of the Ishibashi states $\mathcal{I}^G$ is unique up to a phase, the normalization of the functions $\mathcal{I}^G$ (5.4) involves the one of the mappings $f^h_{PW}$. The latter normalization is not intrinsic. In the WZW case we have fixed it up to a phase by imposing orthonormality with respect to the Haar measure. Similarly, for the coset model we may require orthonormality of basis functions in $\mathcal{F}(Q)$. Since our formulation starts with functions on $G \times H$ rather than $G/H$, this requires a compensating factor of $|H|$.

Observe, however, that while the projections (5.6) are well-defined functions on $Q$, they are still labelled by (horizontally) allowed pairs $(\Lambda, \lambda)$, whereas the coset Ishibashi states must be labelled by the allowed $\mathcal{J}_{g/h}$-orbits $[\Lambda, \hat{\lambda}]$. Further, there is, in general, no bijection between allowed highest $g\oplus h$-weights and $\mathcal{J}_{g/h}$-orbits of highest $\hat{g}\oplus h$-weights. However, the selection rules are compatible in the sense that for every allowed pair $(\Lambda, \lambda)$ the associated pair $(\hat{\Lambda}, \hat{\lambda})$ belongs to an allowed $\mathcal{J}_{g/h}$-orbit. Moreover, inspection of various examples indicates that every allowed $\mathcal{J}_{g/h}$-orbit contains at least one representative whose horizontal projection is a horizontally allowed pair, and that orbits with more than one such representative are rare. (We are, however, not aware of any proof that this is actually true for all coset models.) Thus the projection still yields functions on $Q$ that closely match the properties of Ishibashi states, and accordingly we define the Ishibashi states as orbit sums of the functions (5.5), i.e. set

$$
\mathcal{J}^Q_{[\Lambda,\hat{\lambda}]}([g,h]) := \sqrt{\frac{|H|}{|G|}} \sum_{(J,j) \in \mathcal{J}_{g/h}} \sum_{m,a} \pi_{\Lambda,\lambda}^* D^{\Lambda,\lambda}_{mm,aa}(g,h).
$$

(5.7)
The branes, i.e. the shapes of the boundary states, of the coset model, are then given by

\[ B_{\hat{\Lambda}',\hat{\lambda}'}([g,h]) = \sqrt{|H|/|G|} \sum_{\hat{\Lambda},\hat{\lambda}} \frac{S_{\hat{\Lambda}',\hat{\lambda}'}[\Lambda,\lambda]}{S_{\hat{\Lambda},\hat{\lambda},[0,0]}} \sum_{(J,c) \in J_g/h} \sqrt{d_{\pi\Lambda}/d_{\pi\lambda}} \]

\[ \sum_{p,q} \sum_{b,c} c_{q,c}^{\pi\lambda<\pi\Lambda;\ell} f_{p,b}^{\pi\Lambda<\pi\lambda;\ell} D_{pq,bc}^{\pi\Lambda,\pi\lambda}(g,h) \]

\[ = |J_{g/h}| \sqrt{|H|/|G|} \sum_{\hat{\Lambda}} \frac{\sqrt{d_{\lambda}} S_{\lambda,0}^{\hat{\lambda}}}{S_{\lambda,0}^{\hat{\lambda}}} \sum_{\hat{\lambda}} \frac{S_{\lambda,0}^{\hat{\lambda}}}{\sqrt{d_{\lambda} S_{\lambda,0}^{\hat{\lambda}}}} \delta_{\lambda<\Lambda} \sum_{\ell} D_{\ell q}^{\hat{\lambda},\hat{\lambda}}(h). \]  

(5.8)

Here in the second equality we switched to use adapted bases, and combined the summation over allowed orbits and over the identification group to a summation over pairs \((\hat{\Lambda},\hat{\lambda})\). This yields of course only allowed pairs. But indeed we can sum over all pairs, including those which are forbidden by the selection rules, because forbidden pairs are also horizontally forbidden and hence do not contribute owing to the presence of \(\delta_{\lambda<\Lambda}\).

The formula (5.8) is our result for the shape of symmetry preserving branes in a coset model without field identification fixed points. It expresses the coset branes entirely in terms of quantities for \(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}\) and \(G, H\). Notice that field identification is built in through the summation over the identification group in (5.7). Still, in the formula (5.8) the only explicit remnant of field identification is the overall factor of \(|J_{g/h}|\); such a simplification will certainly no longer arise in models with field identification fixed points.

In the limit of large level the results presented in this section reproduce those of the previous section. Indeed one can also study the precise way in which the large level description emerges, but the analysis is somewhat involved and we refrain from going into it in this paper. But anyhow it should be kept in mind that the limit of letting the level approach infinity is not unique; some pertinent aspects of the large level limit will be discussed, in the context of WZW rather than coset models, in appendix A.

There is an interesting class of coset models in which the formula (5.8) simplifies considerably – the case that \(H = T\) is a maximal torus of \(G\), and hence \(\mathfrak{h} = \mathfrak{g}_0\) the Cartan subalgebra of \(\mathfrak{g}\). These coset models are known as generalized parafermions [62]. It turns out that in this case the representation matrices appearing in the formula can be combined to characters of \(G\).

All irreducible \(T\)-representations are one-dimensional, with representation matrices the numbers \(D^\lambda(e^{\iota\mu}) = e^{\iota\lambda\mu}\). (Here we use the notation for Cartan subalgebra elements that was introduced before formula (3.19).) The primary fields of the \(\mathfrak{h}\)-theory are labelled by the weight lattice of \(\mathfrak{g}\) modulo \(k\) times the root lattice, so that their number \(N_T\) is \(k^{\text{rank} \mathfrak{g}}\) times the number of weight-conjugacy classes of \(\mathfrak{g}\), i.e. \(N_T = k^{\text{rank} \mathfrak{g}} |Z(G)|\). The modular S-matrix of the \(\mathfrak{g}_0\)-theory is

\[ S_{\lambda,\lambda'}^{\mathfrak{g}_0} = N_T^{-1/2} e^{-2\pi i \lambda^\vee (\lambda')/k}, \]

and the identification group \(J_{\mathfrak{g}/\mathfrak{g}_0}\) is isomorphic as an abelian group to the group \(J_{\mathfrak{g}}\) of simple
currents of \( \mathfrak{g} \). It follows that

\[
B^Q_{[\mathfrak{X},\mathfrak{Y}]}([g,h]) = |d_{\mathfrak{g}|} N^{-1/4}_T \sqrt{|T|/|G|} \sum_{\Lambda} \sum_{\lambda < \Lambda} \sum_{\ell} \frac{\sqrt{d_{\mathfrak{g}} S^g_{\lambda,0}} e^{2\pi i (\lambda,\lambda')/k} D_{pp}^\Lambda(g) D_{pp}^\Lambda(h)}{\sqrt{S^g_{\lambda,0}}}.
\]

Here in the first line \( \ell \) labels the occurrences of the weight \( \lambda \) in \( \Lambda \), and we use the short-hand \( p \equiv (\lambda; \ell) \). Thus \( \mathcal{X}_\Lambda = \sum_p D^\Lambda \). The second equality holds because the diagonal entries of the representation matrices \( D^\Lambda \) satisfy \( D^\Lambda_{pp}(gh) = D^\Lambda_{pp}(g) D^\Lambda(h) \) for any \( g \in G \) and \( h \in T \).

\section{Parafermions}

Parafermions can be realized by the \( \mathfrak{su}(2)_k/\mathfrak{u}(1)_k \) coset construction \cite{59}. The (horizontal part of) labels of the primary fields are pairs \( (\Lambda, \lambda) \) of integers in the range \( 0 \leq \Lambda \leq k \) and \( -k < \lambda < k \), subject to the field identification

\[
(\Lambda, \lambda) \sim (k-\Lambda, \lambda+k)
\]

and the selection rule \( 2|\Lambda+\lambda \), as well as \( \lambda \sim \lambda+2k \) (stemming from the presence of an extended chiral algebra in the \( \mathfrak{u}(1)_k \) theory). The identification group is a \( \mathbb{Z}_2 \) generated by the pair \( (k,k) \). The nontrivial \( \mathfrak{su}(2)_k \) simple current \( \Lambda = k \) corresponds to the element \( g_{(k)} = -\mathbb{1} \in G \cap H = \mathbb{U}(1) \) with \( \mathbb{U}(1) = \{ e^{it \sigma_3} | t \in [0, 2\pi) \} \subset \mathbb{U}(2) \), while for the primary fields of \( \mathfrak{u}(1)_k \) (all of which are simple currents) we have \( h_{(\Lambda)} = e^{i \pi \lambda \sigma_3/2} \).

The representation matrices for \( H \) are the numbers

\[
D^\Lambda(e^{it \sigma_3}) = e^{i \Lambda t}.
\]

The horizontal branching spaces \( \mathcal{B}_{\Lambda,\lambda} \), where now \( \Lambda, \lambda \in \mathbb{Z}_{\geq 0} \) and \( \lambda \in \mathbb{Z} \), are one-dimensional if \( 2|\Lambda+\lambda \) and \( |\lambda| \leq \Lambda \), and are zero else. Every orbit of the identification group contains precisely one horizontally allowed pair, except for the orbit of the identity field, for which both representatives \( (0,0) \) and \( (k,k) \) are horizontally allowed pairs.

\textbf{Target space geometry.} The target space of the parafermion theory can be described as follows. We parametrize, as in \cite{31}, the points on the manifold \( \mathbb{U}(2) \cong S^3 \) as

\[
g = g(\psi, \theta, \phi) = \exp(i \psi \sigma_3) = \cos \psi \mathbb{1} + i \sin \psi \sigma_\tilde{n},
\]

where \( \tilde{n} \) is a point on the unit two-sphere with the standard coordinates \( \theta \) and \( \phi \), and \( \sigma_\tilde{n} := \tilde{n} \cdot \tilde{\sigma} \) with \( \tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) the Pauli matrices. Then the coordinate ranges are \( \psi \in [0, \pi], \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \), and taking the radius of the three-sphere to be \( \sqrt{k} \), the metric is \( ds^2 = k \left( d\psi^2 + \sin^2 \psi \, d\theta^2 + \sin^2 \psi \, \sin^2 \theta \, d\phi^2 \right) \), so that \( |G| = 2\pi^2 k^{3/2} \) and \( |H| = 2\pi k^{1/2} \). The non-trivial simple current of the \( \mathfrak{su}(2) \) theory acts on \( \mathbb{U}(2) \) as

\[
g = g(\psi, \theta, \phi) \mapsto g g_{(J)} = -g = g(\pi-\psi, \pi-\theta, \pi+\phi),
\]
and the simple currents $\lambda$ of the $u(1)$ theory act on the maximal torus as multiplication by $e^{i\pi\lambda\sigma_3/k}$, i.e. as a rigid rotation by the angle $\lambda\pi/k$.

The subgroup $H = U(1)$ whose adjoint action is gauged is the maximal torus, which in the parametrization (6.3) is given by

$$H = \{ e^{it\sigma_3} \mid t \in [0, 2\pi) \} = \{ g(\psi, \theta, \phi) \mid \phi = 0, \theta = 0, \pi \}. \quad (6.5)$$

Conjugation by $e^{it\sigma_3}$ amounts to a shift in the $\phi$-coordinate, and hence the parafermion target space $Q_{PF}$ is parametrized by $\psi \in [0, \pi]$ and $\theta \in [0, \pi]$. In terms of the coordinates

$$x := \frac{\pi}{2} \cos \psi \quad \text{and} \quad y := -\frac{\pi}{2} \cos \theta \sin \psi, \quad (6.6)$$

$Q_{PF}$ is the set of points $(x, y) \in \mathbb{R}^2$ in the range $x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$, $y \in \left[ -\sqrt{\left(\frac{\pi}{2}\right)^2 - x^2}, \sqrt{\left(\frac{\pi}{2}\right)^2 - x^2} \right]$, i.e. a (round) disk of diameter $\pi/2$. Embedding this disk $D_{\pi/2}$ in its covering space $SU(2)$ at $\phi = 0$ describes it as the set of group elements

$$g(x, y) \equiv g(\arccos \frac{2x}{\pi}, \arccos \frac{-2y}{\sqrt{\pi^2 - 4x^2}}, 0) = \frac{2x}{\pi} \mathbb{I} - \frac{2y}{\pi} i\sigma_3 + \frac{1}{\pi} \sqrt{\pi^2 - 4x^2 - 4y^2} i\sigma_1. \quad (6.7)$$

**Brane geometry.** The parafermions constitute the simplest case $G = SU(2)$ of the $H = T$ situation, in which the formula (5.8) reduces to (5.10). Plugging in the data for the parafermions into (5.10), we obtain

$$\mathcal{B}_{[\Lambda', \lambda]} \left( \left[ g, e^{i\sigma_3} \right] \right) = \frac{2}{\sqrt{\pi k (k+2)^{1/4}}} \mathcal{X}_{\Lambda}(e^{i\sigma_3(-t+\pi\lambda'/k)} g). \quad (6.8)$$
Figure 2: The straight lines on $Q_{\text{PF}} = D_{x/2}$ at which the parafermion branes at level $k = 6$ are centered: (a) The branes $(\Lambda, 0)$. (b) The branes $(\Lambda, 1)$. (c) All 21 branes. Note that some of the branes meet on the boundary $\partial D_{x/2}$, but that this is not the generic situation.

Figure 3: The shape $\tilde{B}_{PF}$ of the brane $(k - 4/6, 0)$ as a function of the coordinate $x$, at four different values of the level. At large level, the brane is concentrated at $x = \frac{5\pi}{12}$.
While the right hand side of (6.8) is written as a function of \( g' = e^{i\sigma_3(-t+\pi\lambda'/k)}g \), i.e. like a function on the group SU(2), it is indeed a function on \( Q_{PF} \), since \( g'(x,y) \) as given by (6.7) just projects to \((x,y)\) ∈ \( Q \). Also, as the group characters \( \chi_\Lambda \) depend only on a single variable, each brane actually depends on a definite combination of \( x \) and \( y \). Concretely, with our choice of coordinates they are constant along some direction on the disk. This is most directly seen for branes with \( \lambda = 0 \); they depend only on \( x \), but not on \( y \). Further, the factor \( e^{i\pi\lambda\sigma_3/k} \in U(1) \) amounts to a rigid rotation of \( \partial D_{\pi/2} \) by an angle \( \pi\lambda/k \) about its center, so that the straight line for the brane labelled by \((\Lambda, \lambda+\lambda')\) is obtained from the one for the brane \((\Lambda, \lambda')\) by such a rotation (see also [60]). We illustrate this behavior in figure 1 for a brane at level 10. The same observation also shows that the branes are mapped to themselves by the action of the non-trivial identification current.

In the large level limit, according to (4.11) the shape of each brane converges to a multiple of the delta function on the projection of the product of the relevant conjugacy classes, i.e. of the class of \( e^{it\sigma_3}g \) in SU(2) and of the point \( e^{i\pi\lambda\sigma_3/k} \in U(1) \). By the previous remarks, this yields just a delta function on a straight line in \( D_{\pi/2} \); these lines are shown, for level \( k \) = 6, in figure 2.

At finite level the branes are peaked along these straight lines, but they are smeared significantly about these subsets. As an illustration, in figure 3 we display the shape of the brane \((\Lambda, 0)\) as it evolves with the level, plotted as a as a function of the coordinate \( x \), for \( \Lambda = \Lambda(k) \) chosen such that \( \Lambda(k)+1 = (k+2)/6 \). Since we draw the shapes as a function of the variable along the straight line, rather than on the disk, we must account for the different extension in perpendicular direction by an appropriate measure factor; in the case at hand, this amounts to replacing \( B_{PF}(x) \) by \( \tilde{B}_{PF}(x) = 2(\pi^2 - x^2)^{1/2} B_{PF}(x) \).

A The large level limit

In the limit of large level the world volumes of the (symmetry preserving) coset branes – the subsets of \( Q \) on which the branes are concentrated – are lower-dimensional submanifolds of \( Q \). When performing the limit for our result (5.8), one is led to conclude that these submanifolds have the same dimension for all branes. For a more explicit description, the particular way of taking the large level limit must be specified, however. Rather than performing this analysis for coset models, in this appendix we present the corresponding results for WZW models in some detail. The coset case can then be treated analogously, but requires more notational complexity. As already mentioned, there is no unique way of taking the limit \( k \to \infty \). That one better does not draw conclusions about the large level behavior too quickly is already apparent from the observation that naively at \( k \to \infty \) the target space is a group manifold with infinite radius, which superficially looks as flat space even though it should still be compact.

One possibility is to consider the situation that one deals with a definite conjugacy class \( C \). This requires to let the weight \( \Lambda \) labelling a brane depend on the level in such a manner that the associated group elements \( g_\Lambda \) (4.3) belong to the desired conjugacy class \( C \). This limit has been already described in detail in [26]. (The analogous procedure for parafermions amounts to a level dependence of the type occuring in figure 3.) In the WZW case one finds that, analogously as at any finite level, in the limit all brane world volumes are concentrated on regular conjugacy.
Figure 4: The mapping (A.1) of the fundamental Weyl alcove $\mathcal{W}$ for $\mathfrak{su}(3)$ at level 3. The left picture shows the alcove $\mathcal{W}$, with the dots indicating the location of the integral weights, while the right picture shows its image $\tilde{\mathcal{W}}$ under (A.1) inside the original region. The points on the boundary $\partial \tilde{\mathcal{W}}$ are those weights $\tilde{\Lambda}$ whose associated Cartan subalgebra elements are mapped by the exponential mapping to points on exceptional conjugacy classes of SU(3).

classes, and already at small level the overlap with the exceptional lower-dimensional conjugacy classes is negligible.

An alternative limit consists in keeping instead the weight $\Lambda$ fixed. It is easily seen that in this case at large levels for any $\Lambda$ the position of the center of the brane is driven to the unit element of $G$, i.e. to an exceptional conjugacy class. In other words, each brane tends to a D0-brane. On the other hand, upon closer inspection it turns out that even in this limit all branes are, in a specific sense, still well separated from the unit element. The rest of this appendix is devoted to show how one arrives at this conclusion.

First notice that the result that at finite level all branes are concentrated at regular conjugacy classes has its origin in the fact that it is not the integrable highest weights $\Lambda$ that specify the locations of the conjugacy classes on the maximal torus of $G$, but rather there is a shift combined with a compression,

$$\Lambda \mapsto \tilde{\Lambda} := \frac{k}{k+g^\vee} (\Lambda + \rho), \quad (A.1)$$

which maps the fundamental Weyl alcove to its interior. Since with increasing $k$ the shift $k \mapsto k+g^\vee$ is more and more difficult to detect, certain semiclassical formulas ignore the modification (A.1), and hence this aspect of the branes is missed when applying such formulas directly to the finite-level situation. The absence of exceptional conjugacy classes is particularly significant at low levels, at which the fraction of dominant integral weights lying on the boundary of the fundamental alcove is large; this is illustrated with an SU(3) example in figure 4.

Let us now present for fixed $\Lambda$ the large-level behavior of the boundary coefficients $B^G_{\Lambda}$ that appear in the expansion of boundary states in terms of Ishibashi states, i.e. in the WZW analogue of the formula (5.1). We use the the fact that $S_{\Lambda,\Lambda}/S_{\Lambda,0} = \mathcal{K}_\Lambda (g_\Lambda^{-1})$ (see e.g. formula (13.8.9) of [63]), where $g_\Lambda$ is the group element associated to the weight $\Lambda$, see (4.3), and that according to

$$\frac{S_{0,\Lambda}}{S_{0,0}} = \prod_{\alpha \in \Phi} \sin \left( \frac{(\Lambda + \rho, \alpha_\alpha)}{k+g^\vee} \pi \right) \quad \xrightarrow{k \to \infty} \prod_{\alpha \in \Phi} \left( \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)} \right) = d_\Lambda$$

(A.2)
with $\Phi$ the set of positive roots of $\mathfrak{g}$, at large level the quantum dimensions approach the ordinary dimensions. Following the arguments in [26] we then obtain
\begin{equation}
\frac{S_{N',\Lambda}}{\sqrt{S_{0,N'}}} \rightarrow \sqrt{\frac{d_A}{\sqrt{d_{N'}}}} \mathcal{X}_{N'}(g_A^{-1}).
\end{equation}

It is here that the shift (A.1) enters the story.

Implementing the orthogonality and completeness of the characters, it follows that in the large level limit the shape of the branes behaves as
\begin{equation}
\mathcal{B}^G_{\Lambda}(g) \rightarrow d_A \sqrt{\frac{S_{0,0}}{|G|}} \sum_{\Lambda'} \mathcal{X}_{\Lambda'}(g_A^{-1}) \mathcal{X}_{\Lambda}(g) = d_A \sqrt{\frac{S_{0,0}|G|}{|C_{g_A}|}} \delta_{C_{g_A}}^G(g),
\end{equation}

where $C_{g_A} \subset G$ is the conjugacy class of $g_A$ and $\delta_C$ is the class delta distribution, which acts on class functions $f$ as $\int_G du \delta^G_C(u) f(u) = |C_g| f(g)$ for $g \in G$. Note that $\delta^G_C/|C_{g_A}| = \delta^T_{g_A}$ is nothing but a delta distribution on the maximal torus of $G$, and that in the limit $k \to \infty$ the product $S_{0,0}|G|$ approaches a $k$-independent constant. Thus in short, for fixed $\Lambda$ in the large level limit the shape of the brane labeled by $\Lambda$ is a constant multiple of a delta function concentrated at $g_A$ on the maximal torus.

For continuing the discussion, let us restrict our attention to $G = SU(2)$.

In the $su(2)_k$ WZW model, the horizontal weights labelling the branes are the integers in the range $0 \leq \Lambda \leq k$. Parametrizing $g \in SU(2)$ as in (6.3), the branes are given by
\begin{equation}
\mathcal{B}^{SU(2)}_{\Lambda}(g) = \frac{1}{\sqrt{|SU(2)|}} \frac{2^{1/4} \sin^{-1} \psi}{k+2} \sum_{\Lambda'=0}^k \sqrt{\frac{\Lambda'+1}{k+2}} \frac{\sin \left((\Lambda'+1)\psi_{\Lambda}\right)}{\sin \left((\Lambda'+1)\frac{\pi}{k+2}\right)}
\end{equation}
with
\begin{equation}
\psi_{\Lambda} := \frac{(\Lambda + 1) \pi}{k + 2},
\end{equation}
compare formula (D.4) of [31]. According to (A.4) this function behaves for large $k$ as
\begin{equation}
\mathcal{B}^{SU(2)}_{\Lambda} \rightarrow 2^{3/4} \pi^{3/2} (\Lambda + 1) \frac{\delta_{C_{g_A}}}{|C_{g_A}|} = 2^{3/4} \pi^{3/2} (\Lambda + 1) \delta^T_{g_A}.
\end{equation}

The volume of the conjugacy class is $|C_{g_A}| = 4\pi k \sin^2 \psi_{\Lambda}$, and hence in particular decreases as $k^{-1}$ at large level.

In agreement with the general remarks above, (A.7) means that for any $\Lambda$ the brane $\mathcal{B}^{SU(2)}_{\Lambda}$ approaches the D0-brane located at $g = 1$. To see in more detail how this happens, we need to have a closer look at formula (A.5). Since $\mathcal{B}^{SU(2)}_{\Lambda}$ only depends on the coordinate $\psi$, it is natural to regard it as a function on the maximal torus, parametrized (modulo the Weyl group) by the angle $\psi \in [0, \pi]$, rather than on $SU(2)$. When expressing $\mathcal{B}^{SU(2)}_{\Lambda}$ as a function of $\psi$, we

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11 Apart from the different normalization convention for Ishibashi states, a factor of $S_{0,\Lambda}/S_{0,N'}$ is lacking on the right hand side of formula (2.4) of [26]. This does not affect the qualitative behavior of the branes.
must take into account the (Weyl) measure on the maximal torus, which is nothing but the volume of $C_g$; thus when we want to visualize the way the brane tends to a delta function, we should study

$$\tilde{B}^{SU(2)}_\Lambda(\psi) := |C_g| \tilde{B}^{SU(2)}_\Lambda(\psi) = 4\pi k \sin^2\psi \tilde{B}^{SU(2)}_\Lambda(\psi)$$

(A.8)

rather than $\tilde{B}^{SU(2)}_\Lambda$ itself. Also, when we want to account for the growing radius $r = \sqrt{k}$ of the group manifold $S^3$ (and hence of the torus), we must measure distances as seen by a ‘comoving’ observer on the group, i.e. use the scaled variable $\sqrt{k} \psi$. However, as we are particularly interested in the vicinity of $\psi = 0$, it is indeed convenient to introduce in addition the ‘blow-up’ coordinate $a := (k+2)\psi_0 / \sqrt{k}$.

In terms of this parameter we have

$$\tilde{B}^{SU(2)}_\Lambda(\psi) \bigg|_{\psi = a\psi_0} = \frac{4\pi k}{\sqrt{2k}} \left( \frac{2}{k+2} \right)^{\frac{k+2}{2}} \sin^2\psi_0 \left( \sum_{\Lambda' = 0}^k \frac{\psi_{\Lambda'}}{\sin\psi_{\Lambda'}} \sin((\Lambda+1)\psi_{\Lambda'}) \sin(a\psi_{\Lambda'}) \right).$$

(A.9)

In the limit $k \to \infty$ the $\Lambda'$-summation turns into $(k/\pi)$ times an integration over $t = \psi_{\Lambda'}$:

$$\lim_{k \to \infty} \tilde{B}^{SU(2)}_\Lambda(a) = 2^{7/4} \sqrt{\frac{k}{\pi}} a \int_0^{\pi} f_\Lambda(a, t) \, dt$$

(A.10)

with

$$f_\Lambda(a, t) := \sqrt{\frac{t}{\sin t}} \sin((\Lambda+1)t) \sin(at).$$

(A.11)

For any $\Lambda$ the integral $F_\Lambda(a) := \int_0^{\pi} f_\Lambda(a, t) \, dt$ is a continuous function of $a$ independent of $k$; it has its center of mass at $a = \Lambda + 1$, i.e. at $\psi = \psi_\Lambda$, the maximum being located at a slightly ($<0.01$) smaller value of $a$. According to (A.10), in terms of the coordinate $a$, asymptotically at large level the shape grows with the level uniformly as $\sqrt{k}$. Indeed, the shape stabilizes already at small level; this is illustrated in figure 5 for $\Lambda = 0$, i.e. for the brane closest to the exceptional conjugacy class $\{1\}$, and for $\Lambda = 5$.

In terms of the parameter $\sqrt{k} \psi = a/\sqrt{k}$ which accounts for the growing radius, the width of the peak of the function $F_\Lambda$ shrinks with the level as $\sqrt{k}$, and so does the distance from its peak to the origin, as well as the distance between the peaks for any two different branes; the area under the peak stabilizes for large $k$. Since the distance between the peaks for different $\Lambda$, and between peak and origin, shrinks exactly at the same rate as the width of the peaks, even at arbitrarily large level we can distinguish the individual branes and distinguish their location from the origin. It is in this sense that in the large limit the branes keep being well separated from each other, and also well separated from the exceptional conjugacy class. To put it more sloppily, even at arbitrarily large level the branes insist on being well located at $\psi_\Lambda \simeq \Lambda/k$ rather than at zero.
Obviously these arguments generalize from SU(2) to other groups. In particular in the limit
the summation over weights again reduces to a Riemann integral, which up to an over-all factor
is a level-independent smooth function of (appropriately scaled) coordinates on the maximal
torus. Technically the analysis is, however, quite a bit more involved.

Figure 5: The brane shape (A.9) as a function of the blow-up variable $a$ for $\Lambda = 0$ and $\Lambda = 5$
at levels $k = 10, 100$ and $10,000$. For $k = 10$ and $k = 100$ the whole range $[0, k+2]$ of $a$ is shown;
for $k = 10,000$ the range $[0, 12]$ is displayed instead, in order to facilitate comparison with the
diagrams in the first row.

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