Gauge Independence of the Effective Potential Revisited

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Abstract

We apply the formalism of extended BRS symmetry to the investigation of the gauge dependence of the effective potential in a spontaneously symmetry broken gauge theory. This formalism, which includes a set of Grassmann parameters defined as the BRS variations of the gauge-fixing parameters, allows us to derive in a quick and unambiguous way the related Nielsen identities, which express the physical gauge independence, in a class of generalized 't Hooft gauges, of the effective potential. We show in particular that the validity of the Nielsen identities does not require any constraint on the gauge-fixing parameters, to the contrary of some claims found in the literature. We use the method of algebraic renormalization, which leads to results independent of the particular renormalization scheme used.
1 Introduction

The notion of effective potential was first introduced by Euler, Heisenberg and Schwinger \[1\] and later applied to studies of spontaneous symmetry breakdown by Goldstone, Salam, Weinberg and Jona-Lasinio \[2\]. Unfortunately, an exact computation of the effective potential is very hard, often the best answer being given for the first few terms in a loop expansion \[3\]-\[6\]. This is a difficult task, in particular when several interactions are present, as it is the case in spontaneously broken gauge theories. In such theories, the calculation of the radiative corrections to the effective potential has long been of interest, specially in view of its gauge dependence \[6\]-\[13\].

However, the problem of gauge (in)dependence may have been obscured by some confusion between the \textit{classical gauge invariant potential} \( V_{\text{class}} \), used in order to determine, at the classical level, the field configuration \( \phi = v \) corresponding to the minimum of the energy density, \textit{i.e.}, the classical ground state, and the \textit{effective potential} \( V_{\text{eff}} \), defined after the gauge has been “fixed”. To the contrary of the classical potential, the effective potential is gauge dependent, even in the tree approximation, \textit{i.e.}, it depends on the gauge-fixing parameters \( \xi \). However its gauge dependence is restricted by the following \textit{Nielsen identities} \[7\]:

\[
\frac{\partial V_{\text{eff}}(\phi, \xi)}{\partial \xi_{\alpha}} + C_{\alpha i}(\phi, \xi) \frac{\partial V_{\text{eff}}(\phi, \xi)}{\partial \phi_i} = 0 \quad . \tag{1.1}
\]

Here, the argument \( \phi \) of \( V_{\text{eff}} \) denotes the set of “classical fields” \( \phi_i \), \( i = 1, 2, \cdots \), corresponding to the scalar fields of the theory, and \( \xi \) the set of gauge-fixing parameters \( \xi_{\alpha} \), \( \alpha = 1, 2, \cdots \). The function \( C_{\alpha i}(\phi, \xi) \) is calculable.

The Nielsen identities imply that the potential

\[
V(\phi) = V_{\text{eff}}(\hat{\phi}(\phi, \xi), \xi) \quad ,
\]

where \( \hat{\phi}(\phi, \xi) \) is solution of the set of differential equations

\[
\frac{\partial \hat{\phi}_i}{\partial \xi_{\alpha}} = C_{\alpha i}(\hat{\phi}, \xi) \quad ,
\]

with some boundary condition \( \hat{\phi}(\phi, \xi_0) = \phi \), is gauge independent:

\[
\frac{\partial V(\phi)}{\partial \xi_{\alpha}} = 0 \quad .
\]

For suitable boundary conditions, this potential \( V(\phi) \) coincides, in the tree approximation, with the classical gauge invariant potential \( V_{\text{class}} \).

The problem is particularly of relevance in spontaneously broken gauge theories quantized with a \textquoteleft t \textquoteright Hooft-like gauge condition implying some scalar fields. In this case, at the value \( \hat{\phi} = v \) of the scalar fields which minimizes the effective potential – in fact at any stationary point of the effective potential:

\[
\frac{\partial V_{\text{eff}}(\phi, \xi)}{\partial \phi_i} \bigg|_{\phi=v} = 0 \quad , \tag{1.2}
\]
we get, from the Nielsen identities (1.1), the “physical” gauge independence of the effective potential, \( i.e. \), the gauge independence of its minimum:
\[
\frac{\partial V_{\text{eff}}(\phi, \xi)}{\partial \xi^\alpha} \bigg|_{\phi=v} = 0 .
\] (1.3)

The meaning of the Nielsen identities thus is that the vacuum that realizes the minimum of the effective potential – \( e.g. \) the spontaneous symmetry breaking – is a physical minimum.

In this paper, we revisit the problem of the gauge dependence of the effective potential and extend the discussion to all orders of perturbation theory. As we have mentioned in the beginning, indeed, in spite of the large number of papers which have looked carefully at it, a lot of confusion has arisen in the literature, in the context of a class of gauge models quantized with generalized 't Hooft gauges. For instance, it is claimed in [9] that for a gauge-fixing term of the form
\[
\Sigma_{gf} = -\int d^4x \frac{1}{2\alpha} \left( \partial_\mu A_\mu + \rho^{ai} \phi_i \right)^2 ,
\] (1.4)
where \( \rho^{ai} \) and \( \alpha \) are the gauge parameters, the Nielsen identities can be derived only if the \( \rho^{ai} \)'s are \( \alpha \)-independent and if \( \rho^{ai} v_i = 0 \), \( v_i \) being the vacuum expectation value of \( \phi_i \). On the other hand, the authors of [10, 11] have derived the Nielsen identities also for the case of \( \rho^{ai} \) depending on \( \alpha \), \( \rho^{ai} = f(\alpha) \lambda^{ai} \). But they still demand that \( \lambda^{ai} \) be perpendicular to the direction of symmetry breakdown, \( i.e. \), \( \lambda^{ai} v_i = 0 \). In both cases, this condition of transversality arose from the procedure followed there, which consists in including first the gauge-fixing term in the action and only then minimizing the potential – which of course is gauge dependent, already at the tree level. Moreover, if one tries to minimize this gauge dependent potential, one finds, as presented in [6], spurious gauge dependent solutions corresponding to other stationary points, in addition to the usual gauge independent symmetry breaking minima. Therefore, for the Nielsen identities to hold – thus removing these unphysical minima – such constraints as \( \rho^{ai} v_i = 0 \) should be imposed, according to the argumentation of [9]-[13]. At this point we would like to make another comment concerning the orthogonality condition, \( \rho^{ai} v_i = 0 \), imposed, \( a \text{ posteriori} \), as a necessary condition to have the Nielsen identities satisfied by the effective potential. It sounds at least strange such a condition by the fact that, if we think in the other way around by choosing a particular direction in the space of the gauge parameters \( (\rho^{ai}) \), \( \hat{\rho}^{ai} \), the condition \( \hat{\rho}^{ai} \hat{v}_i = 0 \) would establish an orthogonal subspace of the “correct” directions of symmetry breaking, \( \hat{v}_i \) ! But the \( \hat{\rho}^{ai} \)'s are simply gauge parameters, roughly speaking, they could never dictate the rules.

Our aim is to show that, to the contrary of the claims above, it is possible to derive the Nielsen identities generally, without any restriction on the gauge parameters. In order to achieve this, we define the vacuum – characterized by the vacuum expectation value \( v \) of the scalar field \( \phi \), around which perturbation theory is developed – at the classical level already, \( i.e. \), without any gauge dependent ambiguity. The gauge-fixing is next introduced in terms of the shifted field \( \hat{\phi} = \phi - v \), which has a vanishing vacuum expectation value.

We shall derive in this way the Nielsen identities for a general non-Abelian Yang-Mills gauge theory with scalar and spinor matter fields in a 't Hooft-like gauge. We use the
techniques of “extended BRS invariance” \cite{14, 15, 16}, which has been introduced precisely in view of investigating and controlling the gauge dependence in any gauge theory, in particular the gauge independence of the physical quantities. As we shall see, the Nielsen identities follow straightforwardly from the Slavnov-Taylor identity associated to extended BRS invariance\footnote{The formalism of extended BRS symmetry has also been used by the author of \cite{10}.}. Although we specialize the analysis to the case of a semi-simple Lie group for the sake of simplicity, our results obviously hold for the case of a general compact gauge group too, provided all the fields remain massive. The renormalizability of such a theory has indeed been proven in \cite{17}, the generalization to extended BRS invariance being straightforward.

**Note.** A derivation of the Nielsen identities, based on BRS invariance, has already been given in \cite{18}, for the Abelian Higgs model in the ’t Hooft gauge, and in \cite{19} for a more general gauge model. Our presentation, however, differs from the latters in the amount that we emphasize the use of extended BRS invariance, which allows to derive the result in a quick and elegant way and staying on the firm basis of rigorous renormalization theory.

The plan of the paper is as follows. First, in order to provide the necessary basis for the unfamiliarized reader, we review in Section 2 the construction of the BRS and extended BRS invariant theory in the tree-graph approximation, described by a classical action obeying functional identities characterizing the gauge-fixing and the (extended) BRS invariance. We show, at the end of this section, how extended BRS invariance does control the gauge (in)dependence. The renormalized theory is described at the beginning of Section 3, which continues with the derivation of the Nielsen identities for the effective potential.

\textbf{N.B.} In order to avoid infrared problems in the definition of the effective potential, we shall work with massive fields only. The mechanism of spontaneous symmetry breaking will thus be assumed to provide nonzero masses to all physical fields – scalars, fermions and gauge bosons. The choice of a generalized ’t Hooft gauge then ensures nonvanishing masses for all unphysical (ghost) fields which, in any case, decouple from the physical sector of the theory due to BRS invariance, as it is well-known \cite{20, 21}.

## 2 Extended BRS Symmetry in the Classical Approximation

### 2.1 The Classical Theory: Spontaneous Breakdown of Gauge Invariance

Matter is described by a set of scalar and spinor fields, \( \phi_i(x) \) (\( i = 1, \ldots, n \)) and \( \Psi_I(x) \) (\( I = 1, \ldots, N \)), respectively, belonging to some unitary representation of a semi-simple Lie group \( G \). The matter fields carrying an anti-Hermitian fully reducible representation\footnote{It should be stressed that, as pointed out previously, it could be considered here a nonsemi-simple gauge group, but for the sake of simplicity we restrict our case to a semi-simple one.}
of $G$ transforms as
\[ \delta \phi_i(x) = \omega^a(x) T_a^{(\phi)} i^j \phi_j(x) , \quad (2.1) \]
\[ \delta \Psi_I(x) = \omega^a(x) T_a^{(\Psi)} I^J \Psi_J(x) , \quad \delta \bar{\Psi}_I(x) = -\omega^a(x) T_a^{(\Psi)} I^J \bar{\Psi}_J(x) , \quad (2.2) \]
where the matrices $T_a$ are anti-Hermitian and obey the commutation relations of the Lie algebra of the group $G$:
\[ [T_a, T_b] = f_{abc} T_c . \quad (2.3) \]

Because of the local character of the transformations one has to introduce covariant derivatives
\[ D_\mu \phi_i(x) = \partial_\mu \phi_i(x) - A_\mu^a(x) T_a^{(\phi)} i^j \phi_j(x) , \quad (2.4) \]
\[ \not{D} \Psi_I(x) = \not{\partial} \Psi_I(x) - A_\mu^a(x) T_a^{(\Psi)} I^J \Psi_J(x) , \quad (2.5) \]
the connection being given in terms of vector fields $A_\mu^a(x)$ transforming as
\[ \delta A_\mu^a(x) = \partial_\mu \omega^a(x) - f_{abc} \omega^b(x) A_\mu^c(x) . \quad (2.6) \]

A gauge invariant action is built up with the matter fields, $\phi_i$ and $\Psi_I$, and the gauge fields, $A_\mu^a$, as
\[ \Sigma_{\text{inv}} = \int d^4 x \left\{ -\frac{1}{4 g^2} F_{\mu \nu}^a F^{\mu \nu}_a + i \bar{\Psi}_I \not{D} \Psi_I + m_{IJ} \bar{\Psi}_I \Psi_J + D_\mu \phi_i D_\mu \phi_i - \mu_{ij}^2 \phi_i \phi_j + \right. \]
\[ \left. + \lambda_{IJK} \bar{\Psi}_I \Psi_J \phi_k - h_{ijkl} \phi_i \phi_j \phi_k \phi_l \right\} , \quad (2.7) \]
where $F_{\mu \nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a - f_{abc} A_{\nu}^b A_{\mu}^c$ and the Yukawa and quartic coupling constants, $\lambda_{IJK}$ and $h_{ijkl}$, respectively, are invariant tensors of the gauge group $G$.

The form of the potential,
\[ V(\phi) = \mu_{ij}^2 \phi_i \phi_j + h_{ijkl} \phi_i \phi_j \phi_k \phi_l , \quad (2.8) \]
is chosen such as to ensure the broken regime, i.e. $\mu_{ij}^2 < 0$, and $h_{ijkl}$ must be positive-definite for the sake of stability of the system, guaranteeing, therefore, the validity of perturbation theory. In the classical theory, the potential is the energy density for constant scalar fields – all other fields vanishing. The equilibrium state (the "fundamental state", or "vacuum state") is given by the field configuration which minimizes the energy density. This minimum is obtained at some value $v_i = \phi_i$ function of the parameters $\mu_{ij}^2$, $h_{ijkl}$. This value is interpreted in the corresponding quantum theory as the vacuum expectation value of the field $\phi$. The equilibrium – or vacuum – state not being invariant under the gauge transformations, gauge invariance is said to be spontaneously broken.\footnote{In fact, the condition of minimum does not determine $v_i$ uniquely, but only up to a group transformation: it fixes an orbit in the space of scalar fields. One has to choose one particular – arbitrary – value in the orbit.}
In order to study the small oscillations around the equilibrium/vacuum state – which gives rise to the physical particle interpretation in the quantum case – one proceeds to the change of variables
\[ \phi_i = v_i + \tilde{\phi}_i , \] (2.9)
where \( \tilde{\phi}_i \) are the Higgs scalars, all with vanishing vacuum expectation values, such that
\[ \frac{\partial \tilde{V}_{\text{eff}}(\tilde{\phi})}{\partial \tilde{\phi}_i} \bigg|_{\tilde{\phi}=0} = 0 , \]
where \( \tilde{V}_{\text{eff}}(\tilde{\phi}) = V(v + \tilde{\phi}) . \) (2.10)

Let us recall that the mass matrix of the scalar fields satisfies the eigenvalue equation
\[ M^2_{ij} (T^a)^j_k v^k = \frac{\partial^2 \tilde{V}_{\text{eff}}(\tilde{\phi})}{\partial \tilde{\phi}_i \partial \tilde{\phi}_j} \bigg|_{\tilde{\phi}=0} = 0 , \]
whereas the mass matrix \( m^2_{ab} \) of the gauge vector fields is given by
\[ m^2_{ab} = (T^a)^j_k v^k . \] (2.11)

In the following we shall assume that all the vector fields acquire a mass. For the scalar fields the same will be achieved by a suitable choice of the gauge-fixing condition.

2.2 Gauge-fixing

Since from now on we shall work in terms of the shifted fields \( \tilde{\phi}_i \) (the Higgs scalars) with vanishing vacuum expectation value, we will omit, therefore, the tilde symbol.

In order to quantize the theory one has to fix the gauge. We first require invariance under the following BRS transformations:
\[ sA^a_{\mu} = D_{\mu} c^a_{\mu} \equiv \left( \partial_{\mu} c^a_{\mu} - f^a_{\beta c} A^b_{\mu} c^c \right) , \]
\[ sc^a = \frac{1}{2} f^a_{\beta c} c^b c^c , \]
\[ s\Psi_I = c^a T^a_{(\Psi)} I J \Psi_J , \]
\[ s\Psi_I = \Psi_J T^a_{(\Psi)} I J c^a , \]
\[ sc^a = b^a , \]
\[ sb^a = 0 . \] (2.13)

The BRS transformations of the matter and gauge fields are their gauge transformations (2.1), (2.2) and (2.6), with the infinitesimal parameters \( \omega^a(x) \) being replaced by the anti-commuting Faddeev-Popov ghost fields \( c^a(x) \). We have also introduced the antighost fields, \( \bar{c}^a(x) \), and the Lagrange multiplier fields, \( b^a(x) \), which will be used in order to define the gauge-fixing condition. The transformation of the ghosts, \( c^a \), was chosen such as to make the BRS operator nilpotent:
\[ s^2 = 0 . \]

The gauge-fixing is then defined through the introduction in the action of the gauge breaking term – BRS invariant due to the nilpotency of \( s \):
\[ \Sigma_{\text{gf}} = \int d^4 x \left\{ b_a \left( \partial^\mu A^a_{\mu} + \rho^a \tilde{\phi}_i \right) + \frac{1}{2} \alpha b_a b^a - \bar{c}_a \left[ \delta^a_b \partial^\mu D_{\mu} + \rho^a T^a_{(\phi)} (v_j + \phi_j) \right] c^b \right\} \]
\[ = s \int d^4 x \left\{ \bar{c}_a \left( \partial^\mu A^a_{\mu} + \rho^a \tilde{\phi}_i \right) + \frac{1}{2} \alpha \bar{c}_a b^a \right\} , \] (2.14)
where \( \rho_{ai} \) and \( \alpha \) are the “gauge parameters”. For a generic value of the ’t Hooft parameters \( \rho_{ai} \), all scalar fields become massive.
2.3 Gauge Independence and Extended BRS Invariance

One observes that the gauge dependence of the classical theory is given by

\[ \frac{\partial \Sigma}{\partial \alpha} = s \int d^4x \, \frac{1}{2} \bar{c}_a b^a , \quad \frac{\partial \Sigma}{\partial \rho_{ai}} = s \int d^4x \, \bar{c}_a \phi_i , \]  

(2.15)

where \( \Sigma \) is the total action, sum of (2.7) and (2.14). The right-hand sides of (2.15) appear as a BRS-variation, which expresses the unphysical character of the gauge parameters. This means that the physical quantities such as the \( S \)-matrix elements and the Green functions of gauge invariant operators are independent of these parameters \[16\].

In order to translate later on these equations into a functional form, we introduce new Grassmann variables, \( \chi \) and \( \eta_{ai} \), and define the BRS transformations of the gauge parameters as follows:

\[ s\alpha = \chi , \quad s\chi = 0 , \]
\[ s\rho_{ai} = \eta_{ai} , \quad s\eta_{ai} = 0 . \]  

(2.16)

We shall now require invariance under the “extended BRS transformations” \[14, 15, 16\], i.e., under the transformations (2.16) taken together with the field BRS transformations (2.13). This implies the modification of the gauge breaking term (2.14) into:

\[ \Sigma_{\text{gf}} = \int d^4x \, \left\{ \bar{c}_a \left( \partial_{\mu} A_{\mu}^a + \rho^{ai} \phi_i \right) + \frac{1}{2} \alpha \bar{c}_a b^a \right\} \]
\[ = \int d^4x \, \left\{ b_a \left( \partial_{\mu} A_{\mu}^a + \rho^{ai} \phi_i \right) + \frac{1}{2} \alpha b_a b^a - \bar{c}_a \left[ \delta^a_b \partial_{\mu} D_{\mu} + \rho^{ai} T_b^j (\phi) (v_j + \phi_j) \right] \Phi^a + \right. \]
\[ + \left. \frac{1}{2} \chi \bar{c}_a b^a + \eta^{ai} \bar{c}_a \phi_j \right\} . \]  

(2.17)

The extended BRS invariance will allow us to control the gauge parameter dependence of the theory, in particular it will automatically ensure the conditions (2.13), as we shall show in Subsection 2.5.

2.4 The Functional Identities

The BRS symmetry\[7\] of the model, as well as the gauge-fixing we have chosen may be expressed as functional identities obeyed by the classical action (2.19) defined below.

Let us first write down the Slavnov-Taylor identity expressing the BRS invariance of the theory. Because of the nonlinearity of some of the BRS transformations (2.13), we have to add to the action a term giving their couplings with external fields, the “antifields”, \( A_{a}^{\mu} \), \( c_{a} \), \( \Psi_{I} \), \( \phi_{i}^{*} \):

\[ \Sigma_{\text{ext}} = \int d^4x \sum_{\Phi = A_{a}^{\mu}, c_{a}, \Psi_{I}, \phi_{i}} \Phi^{*} S \Phi . \]  

(2.18)

\[7\]From now on “BRS” will mean “extended BRS”.

7
The antifields are BRS invariant. Thus, from now on, the total classical action is given by
\[ \Gamma^{(0)} = \Sigma_{\text{inv}} + \Sigma_{\text{gf}} + \Sigma_{\text{ext}} \],
such that its BRS invariance is expressed through the Slavnov-Taylor (ST) identity \[ S(\Gamma^{(0)}) = \int d^4x \left( \sum_{\Phi=A_{\mu}^a, c^a, \Psi_I, \phi_i} \frac{\delta \Gamma^{(0)}}{\delta \Phi^*} \frac{\delta \Gamma^{(0)}}{\delta \Phi} + b^a \frac{\delta \Gamma^{(0)}}{\delta c^a} \right) + \chi \frac{\partial \Gamma^{(0)}}{\partial \alpha} + \eta^{ai} \frac{\partial \Gamma^{(0)}}{\partial \rho^{ai}} = 0 \] (2.20).

For later use we introduce the linearized ST operator defined as the derivation of the nonlinear operator \( S \),
\[ B_{\Gamma^{(0)}} = \frac{\partial S(\Gamma^{(0)})}{\partial \Gamma^{(0)}} \],
\text{i.e.:}
\[ B_{\Gamma^{(0)}} = \int d^4x \left\{ \sum_{\Phi=A_{\mu}^a, c^a, \Psi_I, \phi_i} \left( \frac{\delta \Gamma^{(0)}}{\delta \Phi^*} \frac{\delta}{\delta \Phi} + \frac{\delta \Gamma^{(0)}}{\delta \Phi^*} \frac{\delta}{\delta \Phi} \right) + b^a \frac{\delta}{\delta c^a} \right\} + \chi \frac{\partial}{\partial \alpha} + \eta^{ai} \frac{\partial}{\partial \rho^{ai}} \].

The operators \( S \) and \( B_{\Sigma} \) obey the algebraic identities
\[ (B_{\Sigma} S(\mathcal{F})) = 0 \], \( \forall \mathcal{F} \),
\[ (B_{\Sigma} S(\mathcal{F}))^2 = 0 \text{ if } S(\mathcal{F}) = 0 \].

In particular, since the action \( \Gamma^{(0)} \) obeys the ST identity (2.20), we have the nilpotency property (2.23):
\[ (B_{\Gamma^{(0)}})^2 = 0 \].

In addition to the ST identity (2.20), the action (2.19) satisfies the following constraints:
– the gauge condition:
\[ \frac{\delta \Gamma^{(0)}}{\delta b_a} = \partial^\mu A^a_{\mu} + \rho^{ai} \phi_i + \alpha b^a + \frac{1}{2} \chi c^a \],
(2.25)

– the ghost equation, which follows from the former by commuting the functional derivation \( \delta/\delta b_a \) with the ST identity (2.20):
\[ G_a \Gamma^{(0)} = \left( \frac{\delta}{\delta c_a} + \partial^\mu \frac{\delta}{\delta A^a_{\mu}} + \rho^{ai} \frac{\delta}{\delta \phi_i} \right) \Gamma^{(0)} = -\frac{1}{2} \chi b^a - \eta^{ai} \phi_i \].
(2.26)

It is worth noting that, the right-hand sides of eqs. (2.25)–(2.26) being linear in the quantum fields, will not get renormalized.

Notice that the classical action obeys the decomposition
\[ \Gamma^{(0)} = \hat{\Gamma}^{(0)} + \int d^4x \left\{ b_a \left( \partial^\mu A^a_{\mu} + \rho^{ai} \phi_i \right) + \frac{1}{2} \alpha b_a b^a + \frac{1}{2} \chi c_a b^a + \eta^{ai} \bar{c}_a \phi_i \right\} \],
(2.27)
where $\hat{\Gamma}^{(0)}$ satisfies the homogeneous gauge condition and the homogeneous ghost equation

$$\frac{\delta \hat{\Gamma}^{(0)}}{\delta b_a} = 0, \quad G_a \hat{\Gamma}^{(0)} = 0,$$

which means that $\hat{\Gamma}^{(0)}$ is independent from $b_a$ and depends on $\bar{c}_a$ and on the antifields, $A^\mu_a$ and $\phi_i^*$, only through the combinations

$$A^\mu_a = A^\mu_a + \partial^\mu \bar{c}_a, \quad \phi_i^* = \phi_i^* - \rho_a \bar{c}_a.$$

### 2.5 Gauge Independence

Gauge independence follows from differentiating the ST identity (2.20) with respect to $\chi$ and $\eta^i_{ai}$ and later setting $\chi = \eta^i_{ai} = 0$:

$$\frac{\partial \Gamma^{(0)}}{\partial \alpha} \bigg|_{\chi = \eta = 0} = B\Gamma^{(0)} \left( \frac{\partial \Gamma^{(0)}}{\partial \chi} \bigg|_{\chi = \eta = 0} \right), \quad \frac{\partial \Gamma^{(0)}}{\partial \rho^a_{ai}} \bigg|_{\chi = \eta = 0} = B\Gamma^{(0)} \left( \frac{\partial \Gamma^{(0)}}{\partial \eta^i_{ai}} \bigg|_{\chi = \eta = 0} \right).$$

Thus, the dependence of the theory on $\chi$ and $\eta^i_{ai}$ is automatically restricted to a BRS-variation, as already announced, which means that the physical quantities do not depend on the gauge parameters. In other words, extended BRS invariance takes care of the non-physical character of the gauge parameters, and the problem of the gauge independence is reduced to the problem of implementing the ST identity to all orders.

In order to get a physical interpretation, let us write the ST identity in terms of the generating functional of the Green functions $^8$, $Z(J_\Phi, \Phi^*, q_I)$, where $J_\Phi$ denotes the sources of the fields $\Phi, \Phi^*$ the associated antifields and $q_I$ a set of BRS invariant sources coupled to the gauge invariant operators $^9$ of the theory. The ST identity (2.20) then writes

$$S\frac{\delta Z}{\delta J_\Phi} + J_\Phi \delta Z + J_{\bar{c}} \delta \bar{c} + J_{\psi_i} \delta \psi_i - \rho_{ai} \delta Z = 0.$$ 

(2.31)

In case of vanishing gauge invariant sources $q_I$, one may deduce from (2.31) the gauge independence of the $S$-matrix:

$$\frac{\partial}{\partial \alpha} S = \frac{\partial}{\partial \rho^a_{ai}} S = 0,$$

(2.32)

whereas for vanishing sources, $J_\Phi$ and $J_{\bar{c}}$, one gets the gauge independence of the generating functional of the Green functions of the gauge operators:

$$\frac{\partial}{\partial \alpha} Z(0,0,q_I) = \frac{\partial}{\partial \rho^a_{ai}} Z(0,0,q_I) = 0.$$

(2.33)

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$^8$In the classical approximation considered in this section, the Green functions are made of tree-graph contributions only.

$^9$The generating functional of the connected Green functions is obtained from the generating functional of the vertex functions – coinciding with the classical action in the tree-graph approximation – through a Legendre transformation with respect to the dynamical fields $\Phi$. Exponentiation then yields the generating functional of the general Green functions. See, e.g., $^5$, $^6$.

$^{10}$A gauge invariant operator is defined as an equivalence class of BRS-invariant operators modulo BRS-variations – what is called a cohomology class of the nilpotent BRS operator.
3 The Nielsen Identities

3.1 Renormalization

For the classical – or tree-graph – approximation of the theory described previously, the renormalization program consists in preserving all the symmetry properties of the classical theory in the perturbative construction of a quantum theory. It is well known\(^\text{11}\) that this is feasible for the class of models considered in the present paper, up to a possible obstruction by the Adler-Bardeen gauge anomaly – which we shall suppose to be absent\(^\text{12}\).

Concretely, the resulting renormalized theory is given by the vertex functional or generating functional of amputated 1-particle irreducible Green functions

\[
\Gamma(A, \Psi, \phi, c, \bar{c}, b, A^*, \Psi^*, \phi^*, c^*) = \Gamma^{(0)}(A, \Psi, \phi, c, \bar{c}, b, A^*, \Psi^*, \phi^*, c^*) + \mathcal{O}(\hbar),
\]

which, in the limit \(\hbar = 0\), coincides with the classical action \((2.19)\) and corresponds to the tree-graph approximation\(^\text{13}\). The vertex functional \(\Gamma\) obeys all the functional identities depicted in Subsection 2.4 and which define the theory, namely, the ST identity \((2.20)\), the gauge condition \((2.25)\) and the ghost equation \((2.26)\).

3.2 The Effective Potential and the Nielsen Identities

The control of the gauge dependence of the Green functions is given by the identities \((2.30)\) for \(\Gamma\) which, as we have already mentioned, follow from differentiating the ST identity \((2.20)\) with respect to the gauge parameters:

\[
\frac{\partial \Gamma}{\partial \xi} = B_{\Gamma}\left(\frac{\partial \Gamma}{\partial \sigma}\right), \quad \xi = \alpha, \rho^{ai}, \quad \sigma = \chi, \eta^{ai}.
\]

By setting from now on

\[
\chi = \eta^{ai} = 0,
\]

we can write eq.\((3.2)\) explicitly as (see eq.\((2.21)\) for the definition of the linearized ST operator \(B_{\Gamma}\))

\[
\frac{\partial \Gamma}{\partial \xi} = \int d^4x \left( \sum_{\Phi = A^a, \psi, \phi_i} \frac{\delta \Gamma}{\delta \Phi^*} \frac{\delta}{\delta \Phi} + \frac{\delta \Gamma}{\delta \Phi} \frac{\delta}{\delta \Phi^*} + b^a \frac{\delta}{\delta \bar{c}^a} \right) \Delta_\xi \cdot \Gamma, \quad \xi = \alpha, \rho^{ai}, \quad (3.3)
\]

where the operator insertion in the right-hand side, \(\Delta_\xi \cdot \Gamma\), is defined by

\[
\frac{\partial \Gamma}{\partial \sigma} \bigg|_{\sigma = 0} = \Delta_\xi \cdot \Gamma, \quad \sigma = \chi, \eta^{ai}.
\]

---

\(^{11}\)See \([16]\) and the references to the original literature therein.

\(^{12}\)This amounts to choose a convenient representation for the spinor fields \([16]\).

\(^{13}\)Perturbation theory as usual is ordered according to the number of loops in the Feynman graphs or, equivalently, to the powers of \(\hbar\).
The definition of the effective potential \[ \text{[23, 22]} \] involves the vertex functional \( \Gamma \), with the dependence of \( \Gamma \) restricted only to the scalar fields \( \phi_i \) as follows\[^{14} \]

\[
\Gamma_{\text{scalar}}(\phi) = \Gamma|_{A_\mu = \Psi = \bar{c} = \epsilon = b = 0, \text{ and all } \Phi^* = 0} = \sum_{n=0}^\infty \sum_{i_1, \ldots, i_n} \frac{1}{i_1! \cdots i_n!} \int d^4x_1 \cdots d^4x_n \Gamma^{i_1 \cdots i_n}(x_1, \ldots, x_n) \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n) . \tag{3.5}
\]

One defines the effective potential as the zeroth order term \( V_{\text{eff}}(\phi) \) in the expansion of \( \Gamma_{\text{scalar}}(\phi) \) involving higher and higher derivatives in the fields \( \phi_i \):

\[
\Gamma_{\text{scalar}}(\phi) = \int d^4x \left\{ -V_{\text{eff}}(\phi) + Z^{ij}(\phi) \partial_\mu \phi_i \partial^\mu \phi_j + \cdots \right\} , \tag{3.6}
\]

where the first term involves the sum of all proper functions at zero external momenta, the second sums all second derivatives at the same point, and so on. In principle the functions \( \phi_i(x) \) remains arbitrary. However, since we wish to compute \( V_{\text{eff}} \), this can be achieved by assuming \( \phi_i(x) \) constant, \( \phi_i(x) = \phi_i \). Now, bearing this in mind, the effective potential can be written as

\[
V_{\text{eff}}(\phi) = -\sum_{n=0}^\infty \sum_{i_1, \ldots, i_n} \frac{1}{i_1! \cdots i_n!} \tilde{\Gamma}^{i_1 \cdots i_n}(0, \ldots, 0) \phi_{i_1} \cdots \phi_{i_n} , \tag{3.7}
\]

where the \( \tilde{\Gamma}^{i_1 \cdots i_n}(0, \ldots, 0) \)'s are the momentum-space vertex functions taken at zero external momenta.

It then follows from the definition of the effective potential and from the gauge dependence equations (3.3) for the vertex functional, that the gauge dependence of the effective potential \( V_{\text{eff}}(\phi) \) is given by

\[
\frac{\partial V_{\text{eff}}}{\partial \alpha} + C_i(\phi, \alpha) \frac{\partial V_{\text{eff}}}{\partial \phi_i} = 0 , \quad \frac{\partial V_{\text{eff}}}{\partial \rho^{ai}} + C_{aij}(\phi, \rho) \frac{\partial V_{\text{eff}}}{\partial \phi_j} = 0 , \tag{3.8}
\]

where

\[
C_i(\phi, \alpha) = -\int d^4x \frac{\delta(\Delta_\alpha \cdot \Gamma)|_{\phi_i(x) = \phi_i}}{\delta \phi_i} , \quad C_{aij}(\phi, \rho) = -\int d^4x \frac{\delta(\Delta_\rho \cdot \Gamma)_{ai}}{\delta \phi_j} \bigg|_{\phi_i(x) = \phi_i} .
\]

The equations (3.8) are the Nielsen identities \[^7\] announced in the Introduction. As we have previously mentioned, there are no constraints on the space of gauge parameters to be imposed in order to have the Nielsen identities satisfied by the effective potential. This could be suspected by the main purpose of Nielsen identities, the control of the gauge dependence of the effective potential.

4 Conclusions

We have thus be able to show how simply and unambiguously the application of the idea of extended BRS invariance \[^{14, 13}\] to the study of the gauge dependence of the

\[^{14}\]Recall that the arguments of the vertex functional \( \Gamma \), the “classical fields”, are Schwartz fast decreasing test functions.
effective potential leads to the Nielsen identities [7] which control this dependence. In particular, for theories quantized in a generalized 't Hooft gauge, if one properly defines the theory, no restriction on the gauge parameters is required at all, contrary to some claims already published in the literature [9]-[13]. This algebraic proof of the validity of the Nielsen identities at all orders in perturbation theory is independent of any particular renormalization scheme. In spite of this proof was done in the context of simple gauge groups, the generalization to general compact gauge groups is straightforward, since the renormalizability of such theories has been rigorously shown in [17].

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