NORMAL OPERATORS IN REAL AND QUATERNIONIC HILBERT SPACES

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Abstract
There exists already a concept of quaternionic normal operator, acting on Hilbert bimodules on the quaternionic algebra $\mathbb{H}$, such an operator having a spectral decomposition obtained via a spectral measure defined on subsets of $\mathbb{H}$. Unlike other authors, we start our investigation by a new approach to real normal operators, applying the results to quaternionic normal operators, regarded as special cases of the real ones, having spectrum and spectral measures defined on subsets of the complex plane.

1 Introduction
The concept of normal operator, bounded or not, has a central role in operator theory and its applications, and it has been preponderantly used in the framework of complex Hilbert spaces. Nevertheless, normal operators in real Hilbert spaces have been studied by several authors, including Goodrich [11], Agrawal and Kulkarni [1], and Oreshina [14], to quote some of them, but the oldest contribution in this respect seems to be that of Wong in [23]. We present a new, simplified approach, leading to a more natural formula for the associated spectral measures. Our results recapture the corresponding ones, valid for complex normal operators, as presented in the functional analysis monograph by Rudin in [15]. As a matter of fact, the real case is indispensable for our development in the quaternionic context.

The first investigations in the framework of quaternionic Hilbert spaces seemingly go back to [17] (see also [22], [2], [6] etc.). In the present work, after the discussion concerning the normal operators in real Hilbert spaces, we apply the results to quaternionic normal operators, regarded as a special class of real operators. Although studied by some authors mentioned above, we shall present a
simplified approach, derived directly from the real case, and leading to a more natural formula of the associated spectral measures.

For the case of real normal operators, we think useful to start with the bounded case, the unbounded one being subsequently approached, using some arguments from the former case. Our main results in this respect are Theorem 1 for the bounded case, and Theorem 2 for the unbounded one. These results state and prove the construction of some functional calculi with large classes of Borel functions, employing and recapturing the corresponding results valid for complex normal operators, as developed for instance in [13]. Moreover, using some elements of local spectral theory, specifically the uniqueness of spectral capacities (see [18]), under some general conditions we prove uniqueness results concerning the real spectral measure attached to a real normal operator (see Propositions 1 and 2).

Concerning the quaternionic normal operators, following the author’s ideas from [21] (see also [20]), we treat them in a more classical manner, also acting in Hilbert spaces with quaternionic bimodule structure, but where the inner products are real or complex valued, rather than quaternionic valued, as used by many of the previous authors. Moreover, the spectra and the spectral measures are defined in the complex plane, and not in the algebra $\mathbb{H}$.

The main results in the quaternionic case are Theorem 3 for the bounded case, and Theorem 4 for the unbounded one. It should be stressed that these results are more or less special cases and direct consequences of Theorem 1 and 2 respectively. We should mention that, in the quaternionic case, we also have some uniqueness statements concerning the corresponding spectral measures (see Propositions 3 and 4), derived directly from the results proved in the real case.

We end this work with an example of a quaternionic normal operator, induced by the multiplication with the independent quaternionic variable.

Concerning the case of real normal operators, the starting point of our research was the paper by Oreshina, who made some useful remarks on a first version of this work. Thanks are also due to anonymous reader, who suggested the author to mention other contributions, as for instance those due to Goodrich, Agrawal & Kulkarni, and Wong, whose approaches are different from that of this paper.

2 Normal Operators in Real Hilbert Spaces

2.1 Preliminaries

We start by recalling some elementary facts concerning the linear operators in real Hilbert spaces (see for instance [14] or [21]).

Let $\mathcal{H}$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and corresponding norm $\| \cdot \|$, and let $\mathcal{H}_C$ be the the complexification of $\mathcal{H}$, identified with the direct sum $\mathcal{H} + i\mathcal{H}$. The natural inner product of $\mathcal{H}_C$ is then given by

$$\langle x + iy, u + iv \rangle_C = \langle x, u \rangle + \langle y, v \rangle + i \langle y, u \rangle - i \langle x, v \rangle$$
for all \( x, y, u, v \in \mathcal{H} \).

Let also \( C : \mathcal{H}_C \to \mathcal{H}_C \) be the natural conjugation \( x + iy \mapsto x - iy, x, y \in \mathcal{H} \), which is an \( \mathbb{R} \)-linear isomorphism of \( \mathcal{H}_C \), whose square is the identity.

We denote by \( \mathcal{B}(\mathcal{H}) \) the real algebra of all bounded \( \mathbb{R} \)-linear operators, acting on \( \mathcal{H} \). Similarly, \( \mathcal{B}(\mathcal{H}_C) \) denotes the complex algebra consisting of all bounded \( \mathbb{C} \)-linear operators, acting on \( \mathcal{H}_C \).

Each operator \( T \in \mathcal{B}(\mathcal{H}) \) has a natural extension to an operator \( T_C \in \mathcal{B}(\mathcal{H}_C) \), given by \( T_C(x + iy) = Tx + iTy, x, y \in \mathcal{H} \). Moreover, the map \( \mathcal{B}(\mathcal{H}) \ni T \mapsto T_C \in \mathcal{B}(\mathcal{H}_C) \) is unital, \( \mathbb{R} \)-linear, injective, and multiplicative. In particular, \( T \in \mathcal{B}(\mathcal{H}) \) is invertible if and only if \( T_C \in \mathcal{B}(\mathcal{H}_C) \) is invertible. The operator \( T_C \) will be sometimes called the complex extension of the real operator \( T \).

Fixing an operator \( S \in \mathcal{B}(\mathcal{H}_C) \), we define the operator \( S^0 \in \mathcal{B}(\mathcal{H}_C) \) to be equal to \( C \mathcal{S} C \). It is easily seen that the map \( \mathcal{B}(\mathcal{H}_C) \ni S \mapsto S^0 \in \mathcal{B}(\mathcal{H}_C) \) is a unital conjugate-linear automorphism, whose square is the identity on \( \mathcal{B}(\mathcal{H}_C) \).

Because \( \mathcal{H} = \{ u \in \mathcal{H}_C ; Cu = u \} \), we have \( S^0 = S \) if and only if \( S(\mathcal{H}) \subset \mathcal{H} \). In this case, \( S = (S(\mathcal{H}))_C \), and so \( T_C^0 = T_C \) for all \( T \in \mathcal{B}(\mathcal{H}) \).

In fact, because of the representation

\[
S = \frac{1}{2} (S + S^0) + i\frac{1}{2i} (S - S^0), \quad S \in \mathcal{B}(\mathcal{H}_C),
\]

where \( (S + S^0)(\mathcal{H}) \subset \mathcal{H} \), \( i(S - S^0)(\mathcal{H}) \subset \mathcal{H} \), the algebras \( \mathcal{B}(\mathcal{H}_C) \) and \( \mathcal{B}(\mathcal{H}_C)_C \) are isomorphic and they will be sometimes identified. In other words, the algebra \( \mathcal{B}(\mathcal{H}) \) (in fact \( \{ T_C ; T \in \mathcal{B}(\mathcal{H}) \} \)) may and will be regarded as a (real) subalgebra of \( \mathcal{B}(\mathcal{H}_C) \). In particular, if \( S = U + iV \), with \( U, V \in \mathcal{B}(\mathcal{H}) \), we have \( S^0 = U - iV \), so the map \( S \mapsto S^0 \) is the conjugation of the complex algebra \( \mathcal{B}(\mathcal{H}_C) \) induced by the conjugation \( C \) of \( \mathcal{H}_C \).

For an operator \( T \in \mathcal{B}(\mathcal{H}) \) acting on the real Hilbert space \( \mathcal{H} \), we denote by \( T^* \) its (Hilbert space) adjoint, defined, as usually, via the equality \( (T^*x, y) = (x, Ty) \) for all \( x, y \in \mathcal{H} \). As we have

\[
\langle C(x + iy), u + iv \rangle_C = \langle C(u + iv), x + iy \rangle_C,
\]

for the operator \( T \in \mathcal{B}(\mathcal{H}) \) it is easily seen that \( (T_C)^* = (T^*)_C \), and this operator will be simply denoted by \( T_C^* \). Moreover, \( (T_C^*)^2 = T_C^2 \).

For every operator \( S \in \mathcal{B}(\mathcal{H}_C) \), we denote, as usually, by \( \sigma(S) \) its spectrum. Using a classical idea, for a real operator \( T \in \mathcal{B}(\mathcal{H}) \) its (complex) spectrum is defined by the equality

\[
\sigma_C(T) = \{ u + iv; (u - T)^2 + v^2 \text{ is not invertible, } u, v \in \mathbb{R} \},
\]

where the scalars are identified with the corresponding multiples of the identity. Because the operator \( (u - T)^2 + v^2 \) is invertible if and only if the operator \( (u - T_C)^2 + v^2 \) is invertible, which in turn is invertible if and only if \( u + iv - T_C \) is invertible, it follows that \( \sigma_C(T) = \sigma(T_C) \), as noticed in [14] or in [21]. Note also that the set \( \sigma_C(T) \) is conjugate symmetric, that is \( z \in \sigma_C(T) \) if and only if \( \bar{z} \in \sigma_C(T) \).
Of course, as in the complex case, a bounded linear operator $T$ in a real Hilbert space $\mathcal{H}$ is said to be normal if $TT^* = T^*T$. It is easily seen that $T$ is normal if and only if $T_C$ is normal.

We shall also work with some $\mathbb{R}$-linear operators, not necessarily bounded, called as usually unbounded operators. Such an operator $T$, which is defined on a vector subspace $D(T) \subset \mathcal{H}$, with values in $\mathcal{H}$, has a complex extension $T_C$, defined on $D(T_C) = D(T)_C$, with values in $\mathcal{H}_C$. When $T$ is closed and/or densely defined, then the complex extension $T_C$ is also closed and/or densely defined.

The family of closed and densely defined linear operators on $\mathcal{H}$, or $\mathcal{H}_C$, will be denoted by $\mathcal{C}(\mathcal{H})$, or $\mathcal{C}(\mathcal{H}_C)$, respectively.

The adjoint of an operator $T \in \mathcal{C}(\mathcal{H})$ is defined as for the complex operators. Adapting the corresponding definition valid in complex Hilbert spaces (as done in [14]), an operator $T \in \mathcal{C}(\mathcal{H})$ is said to be normal if $D(T^*) = D(T)$, $D(TT^*) = D(T^*T)$, and $TT^* = T^*T$.

As in the bounded case, the complex extension $T_C$ is normal if and only if $T$ is normal. Moreover, we have $(T_C)^* = (T^*)_C$, which will be denoted by $T^*_C$.

The spectrum $\sigma(T_C)$ of the operator $T_C \in \mathcal{C}(\mathcal{H}_C)$ is defined following [15], as the complement of the set of those points $\zeta \in \mathbb{C}$ such that the operator $\zeta - T_C$ has an everywhere defined bounded inverse. As noticed in [14], the set $\sigma(T_C)$ is actually conjugate symmetric. Because we have $T \in \mathcal{C}(\mathcal{H})$ invertible if and only if $T_C \in \mathcal{C}(\mathcal{H}_C)$ is invertible, and we have $(T_C)^{-1} = (T^{-1})_C$, simply denoted by $T_C^{-1}$, we may define $\sigma_C(T) = \sigma(T_C)$, which will be called the complex spectrum of the real operator $T \in \mathcal{C}(\mathcal{H})$, which is compatible with the bounded case.

2.2 Real and Complex Spectral Measures

In this subsection we recall some definitions, present some simple consequences of them, and fix some notation.

Remark 1 (1) In what follows, we shall often use the concept of spectral measure (see [8], Part III), which is a particular case of the concept of resolution of the identity (see [14]), but we shall not distinguish the former from the latter, designating them as spectral measures.

For our purpose, it is sufficient to consider only spectral measures supported by closed sets in the complex plane. More precisely, a spectral measure in this text is a map defined on a $\sigma$-algebra, say $\Sigma(\mathcal{G})$, consisting of Borel subsets of $\mathcal{G}$, where $\mathcal{G}$ is a closed subset of the complex plane. Specifically, fixing a real or complex Hilbert space $\mathcal{K}$, with the inner product $\langle *, * \rangle_{\mathcal{K}}$, a spectral measure on $\Sigma(\mathcal{G})$ is a set function $F$ on $\Sigma(\mathcal{G})$, whose values are self-adjoint projections on $\mathcal{K}$, with $F(\emptyset) = 0$, $F(\mathcal{G})$ the identity of $\mathcal{K}$, $F(A \cap B) = F(A)F(B)$ for all $A, B \in \Sigma(\mathcal{G})$, and such that the maps $\nu_{x,y}$ on $\Sigma(\mathcal{G})$, given by $\nu_{x,y}(A) = \langle F(A)x,y \rangle_{\mathcal{K}}$ for all $x, y \in \mathcal{K}$, are countably additive measures. The scalar measures $\nu_{x,y}$, called the associated measures to $F$, are complex (resp. real) if the space $\mathcal{K}$ is complex (resp. real; for the real case see also [14]). In particular, if $\{A_k\}_{k=1}^{\infty}$ is a countable family of mutually disjoint sets from $\Sigma(\mathcal{G})$, we have $F(\sum_{k=1}^{\infty} A_k)x = \sum_{k=1}^{\infty} F(A_k)x$ for every $x \in \mathcal{K}$.
(2) In the case of a complex Hilbert space $\mathcal{K}$, we mainly take as $\Sigma(\mathfrak{S})$ the $\sigma$-algebra $\text{Bor}(\mathfrak{S})$, consisting of all Borel subsets of $\mathfrak{S}$.

If $\mathfrak{H}$ is a real Hilbert space, considering its complexification $\mathfrak{H}_{\mathbb{C}}$, and fixing a spectral measure $E : \text{Bor}(\mathfrak{S}) \to \mathcal{B}(\mathfrak{H}_{\mathbb{C}})$, we may consider its restriction $E_{\mathbb{R}}(A) = E(A)\vert_{\mathfrak{H}}$ for all $A \in \text{Bor}(\mathfrak{S})$, where $\text{Bor}(\mathfrak{S}) = \{ A \in \text{Bor}(\mathfrak{S}) : E(A) = E(A)^{\dagger} \}$. The map $E_{\mathbb{R}} : \text{Bor}(\mathfrak{S}) \to \mathcal{B}(\mathfrak{H})$ is a real spectral measure, because $\text{Bor}(\mathfrak{S})$ is a $\sigma$-algebra. Indeed, we have $\mathfrak{S} \setminus \mathfrak{A} \in \text{Bor}(\mathfrak{S})$ whenever $\mathfrak{A} \in \text{Bor}(\mathfrak{S})$, and $E(\bigcup_{k=1}^{\infty} A_k) = E(\bigcup_{k=1}^{\infty} A_k)^{\dagger}$ if $\{ A_k \}_{k=1}^{\infty}$ is a sequence of mutually disjoint sets from $\text{Bor}(\mathfrak{S})$, via the additivity and continuity of the map $\mathcal{B}(\mathfrak{H}_{\mathbb{C}}) \ni \mathcal{S} \to \mathcal{S}' \in \mathcal{B}(\mathfrak{H}_{\mathbb{C}})$. The general case of an arbitrary sequence $\{ B_k \}_{k=1}^{\infty}$ from $\text{Bor}(\mathfrak{S})$ can be reduced to the previous one, replacing it by a sequence of mutually disjoint sets, having the same same.

(3) We denote by $\mathfrak{B}(\mathfrak{S})$ the complex algebra of all complex-valued Borel functions, defined on $\mathfrak{S}$. When $\mathfrak{S}$ is conjugate symmetric, fixing a complex Hilbert space $\mathcal{K}$, and a spectral measure $F : \text{Bor}(\mathfrak{S}) \to \mathcal{B}(\mathcal{K})$, we denote by $\mathfrak{B}_{s,F}(\mathfrak{S})$ the real subalgebra of those functions $f \in \mathfrak{B}(\mathfrak{S})$ with the property $f(\bar{z}) = \bar{f}(z)$ for all $z \in \mathfrak{S}$, except for a set of null $F$-measure, which will be designated as $F$-stem functions.

(4) As in [13], Section 12.20, fixing again a complex Hilbert space $\mathcal{K}$, and a spectral measure $F : \text{Bor}(\mathfrak{S}) \to \mathcal{B}(\mathcal{K})$, we introduce the algebra $L^{\infty}(\mathfrak{S}, F)$ consisting of $F$-essentially bounded measurable functions $f \in \mathfrak{B}(\mathfrak{S})$, so that

$$
\|f\|_{\infty} = \inf \{ r > 0 ; F(\{|f(z)| \geq r\}) = 0 \} < \infty.
$$

Setting $\mathcal{N} = \{ f \in L^{\infty}(\mathfrak{S}, F) ; \|f\|_{\infty} = 0 \}$, which is an ideal in $L^{\infty}(\mathfrak{S}, F)$, we define the quotient $L^{\infty}(\mathfrak{S}, F)/\mathcal{N}$, which is a Banach algebra. In fact, $L^{\infty}(\mathfrak{S}, F)$ is a $C^*$-algebra, with the involution induced by the complex conjugation $f \mapsto \bar{f}$, where $f(z) = \bar{f}(z)$, $z \in \mathfrak{S}$.

Practically, we identify a function $f \in L^{\infty}(\mathfrak{S}, F)$ with its equivalence class $f + \mathcal{N} \in L^{\infty}(\mathfrak{S}, F)$. Moreover, when no confusion is possible, the algebra $L^{\infty}(\mathfrak{S}, F)$ will be simply denoted by $L^{\infty}(\mathfrak{S})$.

In addition, assuming $\mathfrak{S}$ conjugate symmetric, we denote by $L^{\infty}_{s}(\mathfrak{S}, F) = L^{\infty}_{s}(\mathfrak{S})$ the (real) subalgebra of $L^{\infty}(\mathfrak{S})$ consistent of $F$-stem functions.

(5) As in (2), assume that $\mathfrak{S}$ is conjugate symmetric, choose $\mathfrak{H}$ to be a real Hilbert space, take its complexification $\mathfrak{H}_{\mathbb{C}}$, and fix a spectral measure $E : \text{Bor}(\mathfrak{S}) \to \mathcal{B}(\mathfrak{H}_{\mathbb{C}})$. Note that the finite sums of the form $\sum_{j \in J} (r_j \chi_{A_j} + i s_j \theta_{A_j})$, with $r_j, s_j \in \mathbb{R}$ for all $j \in J$, $(A_j)_{j \in J} \subset \text{Bor}(\mathfrak{S})$ a partition of $\mathfrak{S}$, where $\chi_{A}$ is the characteristic function of the set $A$, and $\theta_{A}(z) = 1$, $\theta_{A}(z) = -1$ if $\exists(z) > 0$, $\exists(z) = 0$ or $z \in \mathfrak{S} \setminus A$, $\exists(z) < 0$, respectively, are elements of $L^{\infty}_{s}(\mathfrak{S})$.

The functions of this type will be designated in this text as elementary functions in $L^{\infty}(\mathfrak{S})$.

Let $F^{\infty}_{s}(\mathfrak{S})$ be the real subspace of $L^{\infty}_{s}(\mathfrak{S})$, generated by the elementary functions. We can easily see that the subspace $F^{\infty}_{s}(\mathfrak{S})$ is dense in the algebra $L^{\infty}_{s}(\mathfrak{S})$. Fixing a function $f \in L^{\infty}_{s}(\mathfrak{S})$, we can represent it as $f(z) = f_1(z) + if_2(z)$, with $f_1, f_2$ real valued, and with $f_1(\bar{z}) = f_1(z)$, and $f_2(\bar{z}) = -f_2(z)$.

Taking the restriction $g_1(z) = f_1(z)$ to the set $\{ z \in \mathfrak{S} ; \exists(z) \geq 0 \}$, we can find a simple function $h$ arbitrarily close to the function $g_1$. Extending $h$ to the
whole set \( \mathcal{S} \) by putting \( h(\bar{z}) = h(z) \), we obtain an elementary function, say \( h_1 \), which is arbitrarily close to \( f_1 \). Similarly, we can build an elementary function \( h_2 \), arbitrarily close to \( f_2 \). Consequently, the family of elementary functions is dense in \( L^\infty(\mathcal{S}) \).

### 2.3 Bounded Normal Operators in Real Hilbert Spaces

Following the corresponding part of [15] (see the subsections 12.17-12.26), we recall some general results concerning the bounded normal operators, necessary for our further development.

**Remark 2** Let \( K \) be a complex Hilbert space, and let \( S \in \mathcal{B}(K) \) be a normal operator. If \( S \) is the spectrum of \( S \), there exists a unique spectral measure \( F : \text{Bor}(S) \to \mathcal{B}(K) \) inducing a unital \( \mathbb{C}^* \)-algebra morphism

\[
L^\infty(\mathcal{S}, F) \ni f \mapsto f(S) = \int_{\mathcal{S}} f(z) dF(z) \in \mathcal{B}(K),
\]

meaning that

\[
\langle f(S)x, y \rangle_K = \int_{\mathcal{S}} f(z) d\nu_{x,y}(z), \quad x, y \in K,
\]

where \( \{\nu_{x,y}; x, y \in K\} \) are the scalar measures associated to \( F \).

We are especially interested by the particular case of a real Hilbert space \( H \) and a normal operator \( T \in \mathcal{B}(H) \), applying the previous discussion to the normal operator \( T_C \in \mathcal{B}(H_C) \), with the spectral measure \( E_C : \text{Bor}(\mathcal{S}) \to \mathcal{B}(H_C) \). As the real Hilbert space \( H \) is not necessarily invariant under the spectral measure \( E_C(\cdot) \), we shall try to characterize those Borel subsets \( A \subset \mathcal{S} \) such that \( E_C(A^\circ) = E_C(A) \).

For an arbitrary subset \( A \subset \mathbb{C} \), we put \( A^\circ := \{\bar{z}; z \in A\} \). For a later use, we designate by \([A]\) the closure of the arbitrary subset \( A \subset \mathbb{C} \).

**Lemma 1** With \( T, T_C \) and \( E_C \) as above, we have the equality \( E_C(A)^\circ = E_C(A) \) for some \( A \in \text{Bor}(\mathcal{S}) \) if and only if \( E_C(A^\circ \setminus A) = 0 \), or if and only if \( E_C(A \setminus A^\circ) = 0 \), or if and only if \( E_C(A) = E_C(A^\circ) \).

**Proof** We first note that the map \( E_C^\circ : \text{Bor}(\mathcal{S}) \to \mathcal{B}(\mathcal{S}) \), with \( E_C^\circ(A) = E_C(A)^\circ \), is also a spectral measure, which is easily checked. Let us find the operator \( S = \int_{\mathcal{S}} z dE_C^\circ(z) \). Note that for all \( \xi, \eta \in \mathcal{S} \), and \( A \subset \mathcal{S} \) a Borel subset, we have

\[
\langle E_C^\circ(A)\xi, \eta \rangle_C = \langle E_C(A)C\xi, C\eta \rangle_C,
\]

and so \( \mu_{\xi,\eta}^C = \mu_{C\xi,C\eta} \), where \( \mu_{\xi,\eta}^C \) is the scalar measure \( \langle E_C^\circ(\xi)\eta \rangle_C \). Therefore,

\[
\langle S\xi, \eta \rangle_C = \int_{\mathcal{S}} zd\mu_{\xi,\eta}^C = \int_{\mathcal{S}} zd\mu_{C\xi,C\eta} = \int_{\mathcal{S}} zd\mu_{C\xi,C\eta} = \langle T_C^\circ C\xi, C\eta \rangle_C = \langle CT_C^\circ C\xi, \eta \rangle_C = \langle T_C^\circ \xi, \eta \rangle_C,
\]

where \( (\cdot)^c \) denotes the complex conjugate.
showing that $E^C_z$ is precisely the spectral measure of the normal operator $T^*_C$.

The spectral measure of the adjoint $T^*_C$ of the normal operator $T_C$ can be also obtain by regarding $T^*_C$ as a function of $T_C$, via the map $C \ni z \mapsto \overline{z} \in \mathbb{C}$. Applying a particular case of the change of measure principle (see [12], Theorem 13.28), we obtain the equality $E^C_z(A) = E^C_{\overline{z}}(A^c)$ for all $A \in \text{Bor}(\mathcal{S})$. Consequently, $E^C(A^c) = CE^C(A)C$. This implies that $E^C(A^c) = 0$ if and only if $E^C(A) = 0$, or if and only if $E^C_z(A) = 0$. In addition, $E^C_z(A) = E^C(A)$ if $A = A^c$.

Assume now that $E^C(A^c \setminus A) = 0$. In this case we also have $E^C(A \setminus A^c) = 0$ because $(A \setminus A^c)^c = A \setminus A^c$. Writing $A^c = (A^c \setminus A) \cup (A \cap A^c)$, and $A = (A \setminus A^c) \cup (A \cap A^c)$, we infer that $E^C(A^c) = E^C(A \cap A^c) = E^C(A)$ for any $A \in \text{Bor}(\mathcal{S})$ with $E^C(A^c \setminus A) = 0$ because of the the equality $E^C(A \cap A^c) = E^C_z(A \cap A^c)$, via the equality $(A \cap A^c)^c = A^c \setminus A$ and the previous discussion.

Conversely, assume that $E^C(A)^c = E^C(A) = E^C(A^c)$ for some $A \in \text{Bor}(\mathcal{S})$. We use again the equalities $A = (A \setminus A^c) \cup (A \cap A^c)$, and $A^c = (A^c \setminus A) \cup (A \cap A^c)$. Having the equality $E^C(A \cap A^c) = E^C_z(A \cap A^c)$, via $(A \cap A)^c = A^c \cap A$, we obtain $E^C(A \setminus A^c) = E^C(A^c \setminus A)$. On the other hand, $E^C(A \setminus A^c) = E^C(A \cap A^c) = E^C(\emptyset) = 0$. Therefore, $E^C(A \setminus A^c) = E^C(A^c \setminus A) = 0$, which completes the proof.

**Remark 3** Lemma [1] establishes a partial invariance of the real Hilbert space $\mathcal{H}$, with respect to the spectral measure $E^C$ of the normal operator $T_C \in \mathcal{B}(\mathcal{H}_C)$, obtained via a normal operator $T$, acting on $\mathcal{H}$. More precisely, setting $E^R(A) = E^C(A)|\mathcal{H}$ for all $A \in \text{Bor}(\mathcal{S})$ with $E^C(A^c \setminus A) = 0$, where $\mathcal{S} = \sigma(T_C)$, we get a family of self-adjoint projections on $\mathcal{H}$. As a matter of fact, Lemma [1] shows the equality

$$\{ A \in \text{Bor}(\mathcal{S}); E^C(A^c) = E^C(A) \} = \{ A \in \text{Bor}(\mathcal{S}); E^C(A^c \setminus A) = 0 \},$$

which is a $\sigma$-algebra denoted by $\text{Bor}_{E^C}(\mathcal{S})$ (see Remark [1]2). Therefore, the map $\text{Bor}_{E^C}(\mathcal{S}) \ni A \mapsto E^R(A) \in \mathcal{B}(\mathcal{H})$ is a real spectral measure (as in [14], page 90). Clearly, the $\sigma$-algebra $\text{Bor}_{E^C}(\mathcal{S})$ depends eventually on the normal operator $T$.

Some important properties of a real spectral measure $E^R$ can and will be deduced from the properties of the spectral measure $E^C$, proved in [15], as already mentioned.

The next result is a functional calculus for real normal operators with a large class of Borel functions, in the spirit of Theorem 12.21 from [15]. Our approach is different from the corresponding ones in [11], [1], and [13].

**Theorem 1** Let $T \in \mathcal{B}(\mathcal{H})$ be a real normal operator, let $T_C \in \mathcal{B}(\mathcal{H}_C)$ be its complex extension, and let $E_C : \text{Bor}(\mathcal{S}) \mapsto \mathcal{B}(\mathcal{H}_C)$ be the spectral measure of $T_C$, where $\mathcal{S} = \sigma(T_C)$. Then the restriction $E^R$ of the complex spectral measure $E^C$ to the $\sigma$-algebra $\text{Bor}_{E^C}(\mathcal{S})$ induces a real spectral measure with values in $\mathcal{B}(\mathcal{H})$, such that the map $\Phi$, given by the formula

$$\langle \Phi(f)x, y \rangle = \int_{\mathcal{S}} f d\mu_{x,y}, \ f \in L^\infty(\mathcal{S}), \ x, y \in \mathcal{H},$$

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with \( \mu_{x,y}(A) = \langle E_\mathbb{R}(A)x, y \rangle \) for all \( A \in \text{Bor}_E(\mathcal{G}) \), is a unital real algebra isometric morphism from \( L^\infty_s(\mathcal{G}, E_C) \) into \( B(\mathcal{H}) \), where \( \text{Bor}_E(\mathcal{G}) \) is the \( \sigma \)-algebra \( \{ A \in \text{Bor}(\mathcal{G}); E_C(A) = E_C(A^c) \} \). Moreover, we have the following.

(i) For every polynomial with real coefficients \( p(z) = \sum_{j=1}^n a_j z^j, z \in \mathbb{C} \), one has \( \Phi(p) = \sum_{j=1}^n a_j T^j \).

(ii) \( \Phi(f) \) is a real normal operator and \( \Phi(f^*) = \Phi(f)^* \) for all \( f \in L^\infty_s(\mathcal{G}, E_C) \).

(iii) \( \| \Phi(f)x \|^2 = \int_\mathcal{G} |f|^2 d\mu_{x,x} \) for all \( f \in L^\infty_s(\mathcal{G}, E_C) \) and \( x \in \mathcal{H} \).

(iv) \( \sigma_C(T| E_\mathbb{R}(A)\mathcal{H}) \subseteq [A] \) for all \( A \in \text{Bor}_E(\mathcal{G}) \).

(v) If \( S \in \mathcal{B}(\mathcal{H}) \) and \( ST = TS \), then \( SE_\mathbb{R}(A) = E_\mathbb{R}(A)S \) for all \( A \in \text{Bor}_E(\mathcal{G}) \).

Proof: Put \( \Phi(f) = f(T_C)|\mathcal{H} \) for all \( f \in L^\infty_s(\mathcal{G}) \). We first show that that the map \( \Phi \) has values in \( \mathcal{B}(\mathcal{H}) \). For \( A \in \text{Bor}_E(\mathcal{G}) \) we set \( A^+ = \{ z \in A; \Im z > 0 \}, A^- = \{ z \in A; \Im z < 0 \}, A^0 = \{ z \in A; \Im z = 0 \} \). Clearly, \( A^{+c} = A^- \).

Moreover, with \( \theta_A \) as in Remark 1(5),

\[
\int_\mathcal{G} \theta_A dE_C = E_C(A^+) - E_C(A^-),
\]

and \( C[i(E_C(A^+) - E_C(A^-))]C = i(E_C(A^+) - E_C(A^-)) \), showing that the space \( \mathcal{H} \) is invariant under the operator \( i\theta_A(T_C) = i(E_C(A^+) - E_C(A^-)) \).

With \( f = \sum_{j \in J}(r_j \chi_{A_j} + is_j \theta_{A_j}) \), that is, an elementary function, and for \( x, y \in \mathcal{H} \), we have

\[
\langle \Sigma_{j \in J}(r_j E(x) + is_j \theta_A(T_C))x, y \rangle
\]

and so \( \Phi(f) = \sum_{j \in J}(r_j E(x) + is_j \theta_A(T_C))|\mathcal{H} = f(T_C)|\mathcal{H} \).

Let \( L^\infty_0(\mathcal{G}) \) be the subalgebra of \( L^\infty_s(\mathcal{G}) = L^\infty_0(\mathcal{G}, E_C) \), generated by the elementary functions. Because the map \( L^\infty_s(\mathcal{G}) \ni f \mapsto f(T_C) \in \mathcal{B}(\mathcal{H}) \) is an algebra morphism, it follows, in particular, that for every \( f \in L^\infty_0(\mathcal{G}) \), the space \( \mathcal{H} \) is invariant under \( f(T_C) \), and so \( \Phi(f) = f(T_C)|\mathcal{H} \) is \( \mathcal{H} \)-valued.

Therefore, using the density of \( L^\infty_0(\mathcal{G}) \ni F^\infty(\mathcal{G}) \) in \( L^\infty_s(\mathcal{G}) \) (see Remark 1(5)), a direct continuity argument shows that the space \( \mathcal{H} \) is invariant under \( f(T_C) \) for all \( f \in L^\infty_s(\mathcal{G}) \).

In fact, the map \( \Phi(f) : L^\infty_s(\mathcal{G}) \ni f \mapsto f(T_C) \in \mathcal{B}(\mathcal{H}) \), which is the functional calculus of \( T_C \) with bounded Borel functions. Therefore, it inherits most of the properties of the latter. In particular, it is an isometric morphism of real algebras, whose image consists of normal operators. As the function \( \bar{f}(z) = \overline{f(z)}, z \in \mathcal{G} \), is an element of \( L^\infty(\mathcal{G}) \) when \( f \in L^\infty_s(\mathcal{G}) \), we must have (ii). The properties (i), (iii) are also direct consequences from the properties of the map \( f \mapsto f(T_C) \).

The inclusion (iv) is obtained in the following way:

\[
\sigma_C(T| E_\mathbb{R}(A)\mathcal{H}) = \sigma(T_C| ((E_\mathbb{R}(A))|\mathcal{H}) = \sigma(T_C| E_C(A)\mathcal{H}) \subseteq [A]
\]

for all \( A \in \text{Bor}_E(\mathcal{G}) \), via the corresponding spectral inclusion property valid for the normal operator \( T_C \) (see [S], Part II, Corollary X.2.6).
If \( S \in \mathcal{B} (\mathfrak{H}) \) and \( ST = TS \), we must also have \( S \sigma T = T \sigma S \). This implies \( S \sigma E_C (A) = E_C (A) \sigma S \) for all \( A \in \text{Bor}(\mathfrak{H}) \), by Theorem 12.23 from [15]. Passing to restrictions to \( \mathfrak{H} \), we obtain \( SE_R (A) = E_R (A) S \) for all \( A \in \text{Bor}_{E_R} (\mathfrak{H}) \), showing that (v) also holds.

The next assertion is a uniqueness result of the real spectral measure attached to a real normal operator.

**Proposition 1** Let \( T \in \mathcal{B} (\mathfrak{H}) \) be a real normal operator, and let \( F_R : \text{Bor}_{E_C} (\mathfrak{H}) \to \mathcal{B} (\mathfrak{H}) \) be a real spectral measure with the following properties.

(a) \( TF_R (A) \delta = F_R (A) \delta \) for all \( A \in \text{Bor}_{E_C} (\mathfrak{H}) \).

(b) We have the inclusion \( \sigma (T) F_R (A) \mathfrak{H} \subseteq [A] \) for all \( A \in \text{Bor}_{E_C} (\mathfrak{H}) \).

Then the real spectral measure \( F_R \) coincides with the real spectral measure obtained by the restriction to the \( \sigma \)-algebra \( \text{Bor}_{E_C} (\mathfrak{H}) \) of the spectral measure \( E_C \) of the complex extension \( T_C \) of \( T \).

**Proof** For every \( A \in \text{Bor}_{E_C} (\mathfrak{H}) \) we set \( F_C (A) = F_R (A) \mathfrak{H} \). Denoting by \( \mathcal{F}^c (\mathfrak{H}) \) the family of all closed subset in \( \text{Bor}_{E_C} (\mathfrak{H}) \), we get what is called a pseudoring of closed subsets of \( \mathfrak{H} \) (see [18], Definition IV.1.1). Moreover, the restrictions

\[
\mathcal{F}^c (\mathfrak{H}) \ni A \mapsto F_C (A) \delta_C \subseteq \delta_C, \quad \mathcal{F}^c (\mathfrak{H}) \ni A \mapsto E_C (A) \delta_C \subseteq \delta_C,
\]

are spectral capacities attached to the normal operator \( T_C \) (see [18], Definitions IV.1.5 and IV.1.6), because a normal operator is decomposable (see [5] for the original definition, which is a special case of Definition IV.1.6 from [18]). Then a particular case of Theorem IV.1.9 from [18] implies the equality \( F_C (A) = E_C (A) \) for all \( A \in \mathcal{F}^c (\mathfrak{H}) \). The regularity of the measures leads to the equality \( F_C (A) = E_C (A) \) for all \( A \in \text{Bor}_{E_C} (\mathfrak{H}) \), whence \( F_R (A) = E_R (A) \) for all \( A \in \text{Bor}_{E_C} (\mathfrak{H}) \).

**Remark 4** A real normal operator may be said to be \( \mathcal{F}^c (\mathfrak{H}) \)-decomposable, in the spirit of Definition IV.1.6 from [18]. We also recall that the original concept of spectral capacity has been introduced in [3].

### 2.4 Unbounded Normal Operators in Real Hilbert Spaces

As in the bounded case, some important properties of a real normal operator may and will be obtain from the well-known properties of its complex normal extension. For some properties of unbounded normal operators used in what follows, we refer to [15] (especially the subsections 13.22-13.25).

If the operator \( T \) is a normal operator in the real Hilbert space \( \mathfrak{H} \), and so its complex extension \( T_C \) is a normal operator in \( \mathfrak{H}_C \), the latter has a unique spectral measure \( E_C : \text{Bor}(\mathfrak{H}_C) \to \mathcal{B}(\mathfrak{H}_C) \), whose values are orthogonal projections commuting with \( T_C \), that is \( E_C (A) T_C \xi = T_C E_C (A) \xi \) for all \( \xi \in D(T_C) \) and \( A \in \text{Bor}(\mathfrak{H}_C) \), where \( \mathfrak{H}_C = \sigma (T_C) \). We also denote by \( \{ \mu_{\xi, \eta} ; \xi, \eta \in \mathfrak{H}_C \} \) the family of complex valued associated measures with the spectral measure \( E_C \).

With this notation, given an arbitrary function \( f \in \mathfrak{B} (\mathfrak{H}) \), we put

\[
\mathcal{D}_f = \{ \xi \in \mathfrak{H}_C ; \int_{\mathfrak{H}_C} |f|^2 d\mu_{\xi, \xi} < \infty \}.
\]
We recall that the functional calculus with unbounded Borel functions of the unbounded normal operator $T_C$ is given by

$$
\langle \Psi(f) \xi, \eta \rangle = \int_\mathcal{S} f d\mu_{\xi,\eta}, \quad f \in \mathcal{B}(\mathcal{S}), \ \xi, \eta \in \mathcal{H}_C
$$

(see Theorem 13.24 from [15]).

The next result is a version of Lemma 1, valid for unbounded operators.

**Lemma 2** With $T$, $T_C$ and $E_C$ as above, we have the equality $E_C(A)^\# = E_C(A)$ for some $A \in \text{Bor}(\mathcal{S})$ if and only if $E_C(A^c \setminus A) = 0$, or if and only if $E_C(A \setminus A^c) = 0$, or if and only if $E_C(A) = E_C(A^c)$.

**Proof** As in the proof of Lemma 1, the map $E_C^\# : \text{Bor}(\mathcal{S}) \mapsto \mathcal{B}(\mathcal{H}_C)$, with $E_C^\#(A) = E_C(A)^\#$, is a spectral measure. A similar argument used in the proof of Lemma 1 leads to the equality

$$
\int_\mathcal{S} zd\mu_{\xi,\eta}^\# = \langle T_C^* \xi, \eta \rangle_C \quad \forall \xi, \eta \in D(T_C) = D(T_C^*),
$$

where, as before, $\mu_{\xi,\eta}^\#$ is the scalar measure $\langle E_C^\#(\xi), \eta \rangle_C$, showing that $E_C^\#$ is precisely the spectral measure of the normal operator $T_C^*$.

As in the proof of Lemma 1, the spectral measure of the adjoint $T_C^*$ of the normal operator $T_C$ can be also obtain by regarding $T_C^*$ as a function of $T_C$, via the map $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$, and leading to the equality $E_C^\#(A) = E_C(A^c)$ for all $A \in \text{Bor}(\mathcal{S})$. Consequently, $E_C(A^c) = CE_C(A)C$.

The rest of the proof regards only the spectral measures $E_C$ and $E_C^\#$, and can be performed as in the proof of Lemma 1 leading to the desired conclusion.

As in the bounded case, we set

$$
\text{Bor}_{E_C}(\mathcal{S}) = \{ A \in \text{Bor}(\mathcal{S}); E_C(A^c) = E_C(A) \} = \{ A \in \text{Bor}(\mathcal{S}); E_C(A^c \setminus A) = 0 \},
$$

which is a $\sigma$-algebra. Using the notation from the next statement (see also Remark 2), a functional calculus with functions from the space $\mathcal{B}(s_{E_C}(\mathcal{S}))$ (that is, $E_C$-stem functions), associated to a real unbounded normal operator is given by the following result.

**Theorem 2** Let $T : D(T) \subset \mathcal{H} \mapsto \mathcal{H}$ be a normal operator, let $T_C : D(T_C) \subset \mathcal{H}_C \mapsto \mathcal{H}_C$ be its complex extension, and let $E_C : \text{Bor}(\mathcal{S}) \mapsto \mathcal{B}(\mathcal{H}_C)$ be the spectral measure of $T_C$, where $\mathcal{S} = \sigma(T_C)$. Then the restriction $E_R$ of the complex spectral measure $E_C$ to $\text{Bor}_{E_C}(\mathcal{S})$ induces a real spectral measure with values in $\mathcal{B}(\mathcal{H})$, such that

$$
\langle Tx, y \rangle = \int_\mathcal{S} zd\mu_{x,y}, \quad x \in D(T), \ y \in \mathcal{H},
$$

where $\mu_{x,y}(A) = \langle E_R(A)x, y \rangle$ for all $A \in \text{Bor}_{E_C}(\mathcal{S})$.

Setting $D_{f,\mathcal{H}} = D_f \cap \mathcal{H}$, for every $E_C$-stem function $f \in \mathcal{B}(s_{E_C}(\mathcal{S}))$ we define the map $\Phi(f)$ via the equality

$$
\langle \Phi(f)x, y \rangle = \int_\mathcal{S} fd\mu_{x,y}, \quad x \in D_{f,\mathcal{H}}, \ y \in \mathcal{H},
$$

(3)
which is a normal operator with domain \( D(\Phi(f)) = D_{f,\mathcal{B}} \), satisfying

\[ ||\Phi(f)x||^2 = \int_\mathcal{S}|f|^2d\mu_{x,x}, \ x \in D_{f,\mathcal{B}}. \]

Moreover, \( \Phi(f)\Phi(g) \subset \Phi(fg) \), with \( D(\Phi(f)\Phi(g)) = D_{g,\mathcal{B}} \cap D_{fg,\mathcal{B}} \). When \( D(\Phi(f)\Phi(g)) = D_{g,\mathcal{B}} \), we actually have \( \Phi(f)\Phi(g) = \Phi(fg) \).

In addition, if \( S \in \mathcal{B}(\mathcal{F}) \) and \( STx = TSx \) for all \( x \in D(T) \), then \( SE_{\mathbb{R}}(A) = E_{\mathbb{R}}(A)S \) for all \( A \in \text{Bor}_{\mathbb{R}}(\mathcal{S}) \).

Proof. Note that if \( \{\mu_{\xi,\eta}, \xi, \eta \in \mathcal{H}_\mathbb{C}\} \) is the family of scalar measures associated to the spectral measure \( E_\mathbb{C} \), it "contains" the family \( \{\mu_{x,y}, x, y \in \mathcal{F}\} \), defined in the statement, via a natural identification.

Let \( f \in \mathcal{B}_{s,E_\mathbb{C}}(\mathcal{S}) \) be bounded. Then the operator \( \Psi(f) \), given by the equation \( \ref{eq:2} \), is a bounded normal operator, by Theorem 13.24 from [15]. Because \( f \) is bounded, it is possible to approximate it with elementary functions, as in Remark \ref{rem:4}, leading to the conclusion that the space \( \mathcal{F} \) is invariant under the operator \( \Psi(f) \), via the corresponding argument from the proof of Theorem \ref{thm:1}.

Moreover, the formula

\[ \langle \Psi(f)x, y \rangle = \int_\mathcal{F} f d\mu_{x,y}, \ x, y \in \mathcal{F} \]

clearly holds.

If \( f \in \mathcal{B}_{s,E_\mathbb{C}}(\mathcal{S}) \) is not bounded, we define the sets \( \mathcal{S}_n = \{z \in \mathcal{S}; |f(z)| \leq n\} \), for every integer \( n \geq 1 \), which are conjugate symmetric. Setting \( f_n = \chi_{\mathcal{S}_n}f \), we have

\[ \lim_{n \to \infty} \|\Psi(f_n)x - \Psi(f)x\|^2 \leq \lim_{n \to \infty} \int_\mathcal{S}|f_n - f|^2d\mu_{x,x} = 0, \ x \in D_f \cap \mathcal{F}, \]

by Legesque’s theorem of dominated convergence (as in formula (7) from the proof of Theorem 13.24 in [15]). Consequently, \( \Psi(f)x \in \mathcal{F} \).

We therefore set \( \Phi(f) = \Psi(f)|D_{f,\mathcal{B}} \) for all \( f \in \mathcal{B}_{s,E_\mathbb{C}}(\mathcal{S}) \), which is precisely the map from \( \ref{eq:3} \). Note that we have

\[ ||\Phi(f)x||^2 = ||\Psi(f)x||^2 = \int_\mathcal{S}|f|^2d\mu_{x,x}, \ x \in D_{f,\mathcal{B}}, \]

via the corresponding property of the spectral measure \( E_\mathbb{C} \), proved in Theorem 13.24 from [15].

Next, if \( x \in D_{g,\mathcal{B}} \cap D_{fg,\mathcal{B}} \), then \( \Phi(f)\Phi(g)x = \Psi(f)\Phi(g)x = \Psi(fg)x = \Phi(fg)x \), because \( D_{g,\mathcal{B}} \cap D_{fg,\mathcal{B}} \subset D_g \cap D_{fg} \). Then, if \( D_{g,\mathcal{B}} \cap D_{fg,\mathcal{B}} = D_{g,\mathcal{B}} \), we must have \( \Phi(f)\Phi(g) = \Phi(fg) \), again as in in Theorem 13.24 from [15].

Finally, if \( S \in \mathcal{B}(\mathcal{F}) \) and \( STx = TSx \), we must have \( S\mathcal{C}T\xi = T\xi S\mathcal{C} \) for all \( \xi \in D(T\mathcal{C}) \). Therefore, \( S\mathcal{C}E_\mathbb{C}(A) = E_\mathbb{C}(A)S\mathcal{C} \) for all \( A \in \text{Bor}(\mathcal{S}) \), via Theorem 13.33 from [15]. Passing to restrictions to \( \mathcal{F} \), we obtain \( SE_{\mathbb{R}}(A) = E_{\mathbb{R}}(A)S \) for all \( A \in \text{Bor}_{\mathbb{R}}(\mathcal{S}) \).

A uniqueness result of the real spectral measure attached to a real normal operator is also valid for unbounded operators.
Proposition 2 Let \( T \in C(\mathcal{S}) \) be a real normal operator, and let \( F_R : \text{Bor}_{E_C}(\mathcal{S}) \rightarrow \mathcal{B}(\mathcal{S}) \) be a real spectral measure with the following properties.

(a) The inclusion \( T(F_R(A)\mathcal{S}) \cap D(T) \subseteq F_R(A)\mathcal{S} \) holds for all \( A \in \text{Bor}_{E_C}(\mathcal{S}) \).

(b) The inclusion \( \sigma_C(T)F_R(A)\mathcal{S} \subseteq [A] \) holds for every \( A \in \text{Bor}_{E_C}(\mathcal{S}) \).

Then the real spectral measure \( F_R \) coincides with the real spectral measure obtained by the restriction to the \( \sigma \)-algebra \( \text{Bor}_{E_C}(\mathcal{S}) \) of the spectral measure \( E_C \) of the complex extension \( T_C \) of \( T \).

Proof. If \( A \in \text{Bor}_{E_C}(\mathcal{S}) \) is a compact set, the inclusion \( \sigma_C(T)F_R(A)\mathcal{S} \cap D(T) \subseteq A \) shows that \( F_R(A)\mathcal{S} \subseteq D(T) \) and that \( T_A = T|F_R(A)\mathcal{S} \) is bounded. Applying Proposition 1 to the bounded normal operator \( T_A \), we deduce that its spectral measure \( F_R,A(B) = F_R(A \cap B), B \in \text{Bor}_{E_C}(\mathcal{S}) \) is uniquely determined. Then the regularity of the spectral measure \( F_R \) leads to its uniqueness.

3 Normal Operators in Quaternionic Hilbert Spaces

3.1 Quaternionic Algebra as a Hilbert Space

Let us recall some known definitions and elementary facts (see, for instance, [7, Section 4.6, and/or [19]).

Let \( \mathbb{H} \) be the abstract algebra of quaternions, which is the four-dimensional \( \mathbb{R} \)-algebra with unit 1, generated by the "imaginary units" \( \{j, k, l\} \), which satisfy

\[
\begin{align*}
jk &= -kj = l, \\
kl &= -lk = j, \\
lj &= -jl = k, \\
jj &= kk = ll = -1.
\end{align*}
\]

We may assume that \( \mathbb{H} \supset \mathbb{R} \) identifying every number \( x \in \mathbb{R} \) with the element \( x1 \in \mathbb{H} \).

The algebra \( \mathbb{H} \) has a natural involution

\[
\mathbb{H} \ni x = x_0 + x_1j + x_2k + x_3l \mapsto x^* = x_0 - x_1j - x_2k - x_3l \in \mathcal{S}.
\]

For an arbitrary quaternion \( x = x_0 + x_1j + x_2k + x_3l, x_0, x_1, x_2, x_3 \in \mathbb{R} \), we set \( \Re x = x_0 = (x + x^*)/2 \), and \( \Im x = x_1j + x_2k + x_3l = (x - x^*)/2 \), that is, the real and imaginary part of \( x \), respectively.

The algebra \( \mathbb{H} \) may be regarded as a real Hilbert space, with the inner product

\[
(x, y) = \sum_{m=0}^{3} x_m y_m,
\]

denoted simply by \( (x, y) \) when no confusion is possible.

A direct computation shows that

\[
qx, y) = (x, q^*y), \quad (xq, y) = (x, yq^*) \quad x, y, q \in \mathbb{H}.
\]

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The natural norm associated to the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{H}}$ is given by
\[
\|\mathbf{x}\|_{\mathbb{H}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad \mathbf{x} = x_0 + x_1 \mathbf{j} + x_2 \mathbf{k} + x_3 \mathbf{l}, \quad x_0, x_1, x_2, x_3 \in \mathbb{R},
\]
which is a multiplicative norm, simply denoted by $\|\mathbf{x}\|$.

Note that $\mathbf{x}\mathbf{x}^* = \mathbf{x}^*\mathbf{x} = \|\mathbf{x}\|^2$, implying, in particular, that every element $\mathbf{x} \in \mathbb{H} \setminus \{0\}$ is invertible, and $\mathbf{x}^{-1} = \|\mathbf{x}\|^{-2}\mathbf{x}^*$.

As an element of a real algebra, each $\mathbf{x} \in \mathbb{H}$ has a complex spectrum given by $\sigma_C(\mathbf{x}) = \{\Re\mathbf{x} \pm i|\Im\mathbf{x}|\}$ (see [19] for details). In particular, if $\lambda \in \sigma_C(\mathbf{x})$, then $|\lambda| = \|\mathbf{x}\|$.

We now consider the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ of the real-algebra $\mathbb{H}$ (see also [10]), which will be identified with the direct sum $\mathbb{M} = \mathbb{H} + i\mathbb{H}$. Of course, the algebra $\mathbb{M}$ contains the complex field $\mathbb{C}$. Moreover, in the algebra $\mathbb{M}$, the elements of $\mathbb{H}$ commute with all complex numbers. In particular, the "imaginary units" $\mathbf{j}, \mathbf{k}, \mathbf{l}$ of the algebra $\mathbb{H}$ are independent of and commute with the imaginary unit $i$ of the complex plane $\mathbb{C}$.

The algebra $\mathbb{M}$ is an involutive one, whose involution is given by $(\mathbf{x} + i\mathbf{y})^* = \mathbf{x}^* - i\mathbf{y}^*$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{H}$.

In the algebra $\mathbb{M}$, there also exists a natural conjugation defined by $\bar{\mathbf{a}} = \mathbf{b} - i\mathbf{c}$, where $\mathbf{a} = \mathbf{b} + i\mathbf{c}$ is arbitrary in $\mathbb{M}$, with $\mathbf{b}, \mathbf{c} \in \mathbb{H}$ (see also [10]). Note that $\bar{\mathbf{a}} + \mathbf{b} = \bar{\mathbf{a}} + \mathbf{b}$, and $\bar{\mathbf{ab}} = \bar{\mathbf{a}}\mathbf{b}$, in particular $\bar{\mathbf{a}x} = r\bar{\mathbf{a}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{M}$, and $r \in \mathbb{R}$. Moreover, $\bar{\mathbf{a}} = \mathbf{a}$ if and only if $\mathbf{a} \in \mathbb{H}$, which is a useful characterization of the elements of $\mathbb{H}$ among those of $\mathbb{M}$.

The algebra $\mathbb{M}$ may also be regarded as a (complex) Hilbert space, with the inner product given by
\[
\langle \mathbf{x} + i\mathbf{y}, \mathbf{u} + i\mathbf{v} \rangle_{\mathbb{M}} = \langle \mathbf{x}, \mathbf{u} \rangle - \langle \mathbf{y}, \mathbf{v} \rangle + i\langle \mathbf{y}, \mathbf{u} \rangle - i\langle \mathbf{x}, \mathbf{v} \rangle
\]
for all $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathbb{H}$, which is the natural extension of the inner product of $\mathbb{H}$.

It will be simply denoted by $(\mathbf{x} + i\mathbf{y}, \mathbf{u} + i\mathbf{v})$, when no confusion is possible.

It is also clear that
\[
\langle \mathbf{q}(\mathbf{x} + i\mathbf{y}), \mathbf{u} + i\mathbf{v} \rangle = \langle \mathbf{x} + i\mathbf{y}, \mathbf{q}^*(\mathbf{u} + i\mathbf{v}) \rangle,
\]
\[
\langle (\mathbf{x} + i\mathbf{y})\mathbf{q}, \mathbf{u} + i\mathbf{v} \rangle = \langle \mathbf{x} + i\mathbf{y}, (\mathbf{u} + i\mathbf{v})\mathbf{q}^* \rangle,
\]
for all $\mathbf{x}, \mathbf{y}, \mathbf{q}, \mathbf{u}, \mathbf{v} \in \mathbb{H}$

### 3.2 Quaternionic Hilbert spaces

**Remark 5** Following [7], a right $\mathbb{H}$-vector space $\mathcal{V}$ is a real vector space having a right multiplication with the elements of $\mathbb{H}$, such that $x1 = x$, $(x + y)q = xq + yq$, $x(q + s) = xq + xs$, $x(qs) = (xq)s$ for all $x, y \in \mathcal{V}$ and $q, s \in \mathbb{H}$.

If $\mathcal{V}$ is a Hilbert space, which is a right $\mathbb{H}$-vector space, the operator $T \in B(\mathcal{V})$ is right $\mathbb{H}$-linear if $T(xq) = T(x)q$ for all $x \in \mathcal{V}$ and $q \in \mathbb{H}$. The set of right $\mathbb{H}$-linear operators will be denoted by $B^r(\mathcal{V})$, which is, in particular, a unital real algebra.
In a similar way, one defines the concept of a left \(\mathbb{H}\)-vector space. A real vector space \(V\) will be said to be an \(\mathbb{H}\)-vector space if it is simultaneously a right \(\mathbb{H}\)- and a left \(\mathbb{H}\)-vector space. As noticed in [7], it is the framework of \(\mathbb{H}\)-vector spaces an appropriate one for the study of right \(\mathbb{H}\)-linear operators.

If \(\mathcal{H}\) is \(\mathbb{H}\)-vector space which is also a Hilbert space with the inner product \(\langle *, * \rangle\) and norm \(\| * \|\), assuming that

\[
\langle qx, y \rangle = \langle x, q^* y \rangle, \quad \langle xq, y \rangle = \langle x, yq^* \rangle, \quad x, y \in \mathcal{H}, \quad q \in \mathbb{H},
\]

then \(\mathcal{H}\) is said to be a Hilbert \(\mathbb{H}\)-space. In this case, we have that \(R_q \in \mathcal{B}(\mathcal{H})\), and the map \(\mathbb{H} \ni q \mapsto R_q \in \mathcal{B}(\mathcal{H})\) is norm continuous, where \(R_q\) is the right multiplication of the elements of \(\mathcal{H}\) by a given quaternion \(q \in \mathbb{H}\). In fact, this map is actually an isometry because we have \(\|R_q x\| = \|q\| \|x\|\) for all \(q \in \mathbb{H}\) and \(x \in \mathcal{H}\).

Similarly, if \(L_q\) is the left multiplication of the elements of \(\mathcal{H}\) by the quaternion \(q \in \mathbb{H}\), we have \(L_q \in \mathcal{B}(\mathcal{H})\) for all \(q \in \mathbb{H}\), because \(\|L_q x\| = \|q\| \|x\|\) for all \(q \in \mathbb{H}\) and \(x \in \mathcal{H}\). Note also that

\[
\mathcal{B}'(H) = \{ T \in \mathcal{B}(\mathcal{H}) ; TR_q = R_q T, \quad q \in \mathbb{H} \}.
\]

**Remark 6** To adapt the discussion regarding the real algebras to this case, we first consider the complexification \(\mathcal{H}_{\mathbb{C}}\) of \(\mathcal{H}\). Because \(\mathcal{H}\) is an \(\mathbb{H}\)-bimodule, the space \(\mathcal{H}_{\mathbb{C}}\) is actually an \(\mathbb{M}\)-bimodule, via the multiplications

\[
(q + is)(x + iy) = qx - sy + i(qy + sx), \quad (x + iy)(q + is) = xq - ys + i(yq + xs),
\]

for all \(q + is \in \mathbb{M}\), \(q, s \in \mathbb{H}\), \(x + iy \in \mathcal{H}_{\mathbb{C}}\), \(x, y \in \mathcal{H}\). Moreover, if the operator \(T \in \mathcal{B}(\mathcal{H})\) is right \(\mathbb{H}\)-linear, the operator \(T_{\mathbb{C}}\) is right \(\mathbb{M}\)-linear, that is \(T_{\mathbb{C}}(x + iy)(q + is) = T_{\mathbb{C}}(x + iy)(q + is)\) for all \(q + is \in \mathbb{M}\), \(x + iy \in \mathcal{H}_{\mathbb{C}}\), via a direct computation.

Let \(C\) be the natural conjugation of \(\mathcal{H}_{\mathbb{C}}\). As in the real case, for every \(S \in \mathcal{B}(\mathcal{H}_{\mathbb{C}})\), we put \(S^0 = CSC\). The left and right multiplication with the quaternion \(q\) on \(\mathcal{H}_{\mathbb{C}}\) will be also denoted by \(L_q, R_q\), respectively, as elements of \(\mathcal{B}(\mathcal{H}_{\mathbb{C}})\). We set

\[
\mathcal{B}'(\mathcal{H}_{\mathbb{C}}) = \{ S \in \mathcal{B}(\mathcal{H}_{\mathbb{C}}) ; SR_q = R_q S, \quad q \in \mathbb{H} \},
\]

which is a unital complex algebra containing all operators \(L_q, q \in \mathbb{H}\). Note that if \(S \in \mathcal{B}'(\mathcal{H}_{\mathbb{C}})\), then \(S^0 \in \mathcal{B}'(\mathcal{H}_{\mathbb{C}})\). Indeed, because \(CR_q = R_q C\), we also have \(S^0 R_q = R_q S^0\). In fact, as we have \((S + S^0)(\mathcal{H}_{\mathbb{C}}) \subset \mathcal{H}_{\mathbb{C}}\) and \(i(S - S^0)(\mathcal{H}_{\mathbb{C}}) \subset \mathcal{H}_{\mathbb{C}}\), it follows that the algebras \(\mathcal{B}'(\mathcal{H}_{\mathbb{C}})\), \(\mathcal{B}'(\mathcal{H})_{\mathbb{C}}\) are isomorphic, and they will be often identified, where \(\mathcal{B}'(\mathcal{H})_{\mathbb{C}} = \mathcal{B}'(\mathcal{H}) + i\mathcal{B}'(\mathcal{H})\) is the complexification of \(\mathcal{B}'(\mathcal{H})\), which is also a unital complex Banach algebra.

### 3.3 Bounded Normal Operators in Quaternionic Hilbert Spaces

Let \(\mathcal{H}\) be a Hilbert \(\mathbb{H}\)-space, and let \(\mathcal{H}_{\mathbb{C}}\) be its complexification. Being, in particular, a real Hilbert space, the adjoint \(T^* \in \mathcal{B}(\mathcal{H})\) of an operator \(T \in \mathcal{B}(\mathcal{H})\)
and its normality are defined as in the framework of real operators. We are mainly interested by normal operators \( T \in \mathcal{B}^r(\mathfrak{H}) \). In this case we also have 
\( T^* \in \mathcal{B}^r(\mathfrak{H}) \) and, of course, \( T \) is normal if \( TT^* = T^*T \).

If \( T \in \mathcal{B}^r(\mathfrak{H}) \) is normal, then \( T_C \in \mathcal{B}^r(\mathfrak{H}_C) \) is normal, and it has a spectral measure \( E_C \) defined on \( \text{Bor}(\mathfrak{H}) \), having a priori values in \( \mathcal{B}(\mathfrak{H}_C) \), where \( \mathfrak{H} = \sigma(T_C) \). Because the operators \( R_q(\mathfrak{q} \in \mathbb{H}) \) commute with \( T_C \), they should also commute with its spectral measure, implying that the spectral measure \( E_C \) takes values actually in \( \mathcal{B}^r(\mathfrak{H}_C) \).

Then we have the following spectral theorem for bounded quaternionic normal operators.

**Theorem 3** Let \( T \in \mathcal{B}^r(\mathfrak{H}) \) be a normal operator, let \( T_C \in \mathcal{B}^r(\mathfrak{H}_C) \) be its complex extension, and let \( E_C : \text{Bor}(\mathfrak{H}) \rightarrow \mathcal{B}^r(\mathfrak{H}_C) \) be the spectral measure of \( T_C \), where \( \mathfrak{H} = \sigma(T_C) \). Then the restriction \( E_R \) of the complex spectral measure \( E_C \) to the \( \sigma \)-algebra \( \text{Bor}_{\mathfrak{H}_C}(\mathfrak{H}) \) induces a real spectral measure with values in \( \mathfrak{H} \), such that the map \( \Phi \), given by the formula

\[
\langle \Phi(f)x, y \rangle = \int_{\mathfrak{H}} f \mu_{x,y}, \ f \in L^\infty(\mathfrak{H}), \ x, y \in \mathfrak{H},
\]

with \( \mu_{x,y}(A) = \langle E_R(A)x, y \rangle \) for all \( A \in \text{Bor}_{\mathfrak{H}_C}(\mathfrak{H}) \), is a unital real algebra isomorphic morphism from \( L^\infty(\mathfrak{H}, E_C) \) into \( \mathfrak{H} \), where \( \text{Bor}_{\mathfrak{H}_C}(\mathfrak{H}) \) is the \( \sigma \)-algebra \( \{ A \in \text{Bor}(\mathfrak{H}); E_C(A) = E_C(A^c) \} \). Moreover,

(i) For every polynomial with real coefficients \( p(z) = \sum_{j=1}^{n} a_j z^j \), \( z \in \mathbb{C} \), one has \( \Phi(p) = \sum_{j=1}^{n} a_j \Phi(T^j) \).

(ii) \( \Phi(f) \) is a real normal operator and \( \Phi(f)^* = \Phi(f)^\ast \) for all \( f \in L^\infty(\mathfrak{H}, E_C) \).

(iii) \( \|\Phi(f)x\|^2 = \int_{\mathfrak{H}} |f|^2 \mu_{x,x} \) for all \( f \in L^\infty(\mathfrak{H}, E_C) \) and \( x \in \mathfrak{H} \).

(iv) \( \sigma_C(T)E_R(A)\mathfrak{H} \subseteq [A] \) for all \( A \in \text{Bor}_{\mathfrak{H}_C}(\mathfrak{H}) \).

(v) If \( S \in \mathfrak{H} \) and \( ST = TS \), then \( SE_R(A) = E_R(A)S \) for all \( A \in \text{Bor}_{\mathfrak{H}_C}(\mathfrak{H}) \).

**Proof** This is a version of Theorem [1] whose proof is similar to the quoted result, via some minor modifications. For instance, when \( f = \sum_{j \in J} (r_j x A_j + i s_j \theta_{A_j}) \) is an elementary function, we must have

\[
f(T_C)R_q = \sum_{j \in J} (r_j E_C(A_j) + i s_j \theta_{A_j}(T_C))R_q = R_qf(T_C), \quad q \in \mathbb{H},
\]

because the spectral measure \( E_C \) is \( \mathcal{B}^r(\mathfrak{H}_C) \)-valued. This commutation property also holds by passing to limits, showing that \( f(T_C) \in \mathcal{B}^r(\mathfrak{H}_C) \) for all \( f \in L^\infty(\mathfrak{H}, E_C) \). Consequently, \( \Phi(f) = f(T_C)\mathfrak{H} \in \mathfrak{H} \) for all \( f \in L^\infty(\mathfrak{H}, E_C) \).

Other details are left to the reader.

We also have a uniqueness result corresponding to Proposition [1] having a similar proof, which will be omitted.

**Proposition 3** Let \( T \in \mathcal{B}^r(\mathfrak{H}) \) be a real normal operator, and let \( F_R : \text{Bor}_{\mathfrak{H}_C}(\mathfrak{H}) \rightarrow \mathcal{B}^r(\mathfrak{H}) \) be a real spectral measure with the following properties.

(a) \( TF_R(A)\mathfrak{H} \subseteq F_R(A)\mathfrak{H} \) for all \( A \in \text{Bor}_{\mathfrak{H}_C}(\mathfrak{H}) \).
obtained by the restriction to the $\sigma_T$ whose values are orthogonal projections commuting with the latter space consists of those operators from $C^* H$ of the complex extension $E$ measures with the spectral measure $T$.

Let $H$ be a Hilbert space, which is, in particular, a real Hilbert space. Let also $T$ be an operator from the family $C(H)$. Therefore, the adjoint $T^* \in C(H)$ of $T$ and its normality are defined as in the framework of real operators.

The operator $T \in C(H)$ is said to be right $\mathbb{H}$-linear if $R_qD(T) \subset D(T)$, and $T(xq) = T(x)q$ for all $x \in D(T)$ and $q \in \mathbb{H}$. The family of right $\mathbb{H}$-linear operators from $C(H)$ will be denoted by $C^r(H)$. A direct calculation shows that, if $T \in C^r(H)$, we have $\langle (T^*R_q)x, y \rangle = \langle (R_qT^*)x, y \rangle$ for all $x \in D(T^*)$, $y \in D(T)$, showing that $T^* \in C^r(H)$.

As before, we are particularly interested by normal operators $T \in C^r(H)$. Then $T^* \in C^r(H)$, and $T$ is normal if $D(T^*) = D(T)$, $D(TT^*) = D(T^*T)$, and $TT^* = T^*T$.

We now consider the complexification $H_C$ of $H$. If $T \in C(H)$, then $T_C \in C(H_C)$, and $D(T_C) = D(T)_C$. Moreover, if $T \in C^r(H)$, then $T_C \in C^r(H_C)$, where the latter space consists of those operators from $C(H_C)$ which are right $\mathbb{H}$-linear.

When the operator $T \in C(H)$ is normal, and so its complex extension $T_C \in C(H_C)$ is also normal, having a unique spectral measure $E_C : \text{Bor}(H_C) \mapsto \mathcal{B}(H_C)$, whose values are orthogonal projections commuting with $T_C$, as in Subsection 2.4. Let also $\{\mu_{\xi, \eta} ; \xi, \eta \in H_C\}$ be the family of complex valued associated measures with the spectral measure $E_C$. If $T \in C^r(H)$, and so $T_C \in C^r(H_C)$ because $R_qT_C\xi = T_CR_q\xi$ for all $\xi \in D(T_C)$ and $q \in \mathbb{H}$, we must actually have $E_C : \text{Bor}(H_C) \mapsto \mathcal{B}^r(H_C)$, as follows from Theorem 13.33 from [15] or Proposition 5.26 from [16].

The next result is a functional calculus with a class of Borel functions, valid for unbounded normal operators in Hilbert $\mathbb{H}$-spaces. The notation $D_f$ is given by the equality [1].

**Theorem 4** Let $H$ be an $\mathbb{H}$-space, and let $T \in C^r(H)$ be a normal operator. Let also $T_C \in C^r(H_C)$ be its complex extension, and let $E_C : \text{Bor}(H_C) \mapsto \mathcal{B}^r(H_C)$ be the spectral measure of $T_C$, where $\mathcal{G} = \sigma(T_C)$. Then the restriction $E_R$ of the complex spectral measure $E_C$ to $\text{Bor}_C(\mathcal{G})$ induces a real spectral measure $E_R$ with values in $\mathcal{B}^r(H)$, such that

$$(Tx, y) = \int_\mathcal{G} zd\mu_{x,y}, \ x \in D(T), y \in \mathcal{G},$$

where $\mu_{x,y}(A) = \langle E_R(A)x, y \rangle$ for all $A \in \text{Bor}_C(\mathcal{G})$.

Setting $D_{f,H} = D_f \cap H$, for every function $f \in \mathcal{B}_{s,H}(\mathcal{G})$ we put

$$\langle \Phi(f)x, y \rangle = \int_\mathcal{G} fd\mu_{x,y}, \ x \in D_{f,H}, y \in \mathcal{G},$$

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Theorem 3. For an arbitrary unbounded function \( f \) belongs to \( \Psi(\mathfrak{h}) \), with domain \( D(\Phi(f)) = D_{f,\mathfrak{h}} \), satisfying
\[
\| \Phi(f)x \|^2 = \int_{\mathfrak{h}} |f|^2 d\mu_{x,x}, \ x \in D_{f,\mathfrak{h}}.
\]
Moreover, \( \Phi(f)\Phi(g) \subseteq \Phi(fg) \), with \( D(\Phi(f)\Phi(g)) = D_{g,\mathfrak{h}} \cap D_{fg,\mathfrak{h}} \). When \( D(\Phi(f)\Phi(g)) = \mathcal{D}_{g,\mathfrak{h}} \), we actually have \( \Phi(f)\Phi(g) = \Phi(fg) \).

In addition, if \( S \in \mathcal{B}(\mathfrak{h}) \) and \( STx = TSx \) for all \( x \in D(T) \), then \( SE_{\mathbb{R}}(A) = E_{\mathbb{R}}(A)S \) for all \( A \in \text{Bor}_{\mathbb{C}}(\mathfrak{S}) \).

Proof The proof of this assertion is similar to that of Theorem 2 except for some details. For instance, it is clear that for every \( E_\mathbb{C} \)-stem function \( f \in \mathfrak{B}_{s,\mathbb{C}}(\mathfrak{S}) \) the operator \( \Phi(f) \) is normal as an element of \( \mathcal{C}(\mathfrak{S}) \). In fact, it actually belongs to \( \mathcal{C}(\mathfrak{S}) \). Of course, this is true when \( f \) is bounded, as in the proof of Theorem 3. For an arbitrary unbounded function \( f \in \mathfrak{B}_{s,\mathbb{C}}(\mathfrak{S}) \), we refine an argument from the proof of Theorem 2.

We set \( \mathfrak{S}_n = \{ z \in \mathfrak{S} ; |f(z)| \leq n \} \), for every integer \( n \geq 1 \), which are conjugate symmetric. Setting also \( f_n = \chi_{\mathfrak{S}_n} f \), as we have \( \lim_{n \to \infty} \Psi(f_n)x = \Psi(f)x \), and \( R_q \Psi(f_n)x = R_q \Psi(f)x \), it follows
\[
\lim_{n \to \infty} R_q \Psi(f_n)x = \lim_{n \to \infty} \Psi(f_n)R_q x = R_q \Psi(f)x,
\]
showing that \( R_q x \in D(\Psi(f)) \), and \( R_q \Psi(f)x = \Psi(f)R_q x \) for all \( x \in D_f \cap \mathfrak{S} \). Therefore, \( \Psi(f) \in \mathcal{C}(\mathfrak{S}) \).

Other details are left to the reader.

As in some previous cases, we still have a uniqueness result.

Proposition 4 Let \( \mathfrak{S} \) be \( \mathbb{H}\)-space, let \( T \in \mathcal{C}(\mathfrak{S}) \) be a real normal operator, and let \( F_{\mathbb{R}} : \text{Bor}_{\mathbb{C}}(\mathfrak{S}) \to \mathcal{B}(\mathfrak{S}) \) be a real spectral measure with the following properties.

(a) The inclusion \( T(F_{\mathbb{R}}(A)\mathfrak{S}) \cap D(T) \subseteq F_{\mathbb{R}}(A)\mathfrak{S} \) holds for all \( A \in \text{Bor}_{\mathbb{C}}(\mathfrak{S}) \).

(b) The inclusion \( \sigma_C(T)F_{\mathbb{R}}(A)\mathfrak{S} \subseteq |A| \) holds for every \( A \in \text{Bor}_{\mathbb{C}}(\mathfrak{S}) \).

Then the real spectral measure \( F_{\mathbb{R}} \) coincides with the real spectral measure obtained by the restriction to the \( \sigma \)-algebra \( \text{Bor}_{\mathbb{C}}(\mathfrak{S}) \) of the spectral measure \( E_\mathbb{C} \) of the complex extension \( T_\mathbb{C} \) of \( T \).

4 An Example

We shall give a basic example of a quaternionic normal operator, defined by the multiplication with the independent quaternionic variable.

First of all, we recall some definitions and properties from [19] (see especially Remark 4), also adding a few simple observations. A subset \( S \) in \( \mathbb{H} \) is said to be spectrally saturated if whenever for an element \( q \in \mathbb{H} \) we have \( \sigma_C(q) = \sigma_C(s) \) for some \( s \in S \), then \( q \in S \). Setting \( \mathfrak{S}(\Omega) = \bigcup_{q \in \Omega} \sigma_C(q) \), for an arbitrary subset \( \Omega \in \mathbb{H} \), we have that \( \Omega \) is spectrally saturated if an only if \( \mathfrak{S}(\Omega) \) is conjugate symmetric. If \( A \subset \mathbb{C} \) is conjugate symmetric, we put \( A_\mathbb{H} = \{ q \in \mathbb{H} ; \sigma_C(q) \subset A \} \).
which is spectrally saturated. The assignment $A \mapsto A_{\mathbb{H}}$ is a bijection between the conjugate symmetric subsets of $\mathbb{C}$ and the spectrally saturated subsets of $\mathbb{H}$.

Moreover, we have the properties $(A \cap B)_{\mathbb{H}} = A_{\mathbb{H}} \cap B_{\mathbb{H}}$ and $(A \cup B)_{\mathbb{H}} = A_{\mathbb{H}} \cup B_{\mathbb{H}}$ for all conjugate symmetric subsets $A, B$ of $\mathbb{C}$, as one can easily see. In fact, if $(A_j)_{j \geq 1}$ is a sequence of conjugate symmetric subsets of $\mathbb{C}$, setting $A = \cup_{j \geq 1} A_j$, we still have $A_{\mathbb{H}} = \cup_{j \geq 1} A_{j\mathbb{H}}$.

Let $\mathbb{K} \subset \mathfrak{S}$ be a compact subset, which is spectrally saturated. Then the conjugate symmetric set $\mathfrak{S}(\mathbb{K})$ is also compact because it is bounded and closed.

Let $\lambda$ be the Lebesgue measure on $\mathbb{K}$ (identified with a subset of $\mathbb{R}^4$). As usually, we denote by $L^2(\mathbb{K}, \mathbb{H})$ the space (of equivalence classes) of square integrable functions $f : \mathbb{K} \mapsto \mathbb{H}$, regarded as a real vector space. Using the inner product defined on $\mathbb{H}$ (see Subsection 3.1), we consider the inner product

$$\langle f, g \rangle = \int_{\mathbb{K}} \langle f(q), g(q) \rangle d\lambda(q) \quad f, g \in L^2(\mathbb{K}, \mathbb{H}),$$

on the vector space $L^2(\mathbb{K}, \mathbb{H})$, which becomes a Hilbert $\mathbb{H}$-space, simply denoted by $\mathfrak{S}$. Note that $\mathfrak{S}_{\mathbb{C}}$ is isomorphic to the complex Hilbert space $L^2(\mathbb{K}, \mathbb{M})$, whose inner product is induced by that of $\mathbb{M}$.

Let $T$ be the map given by $Tf(q) = qf(q)$ for all $f \in \mathfrak{S}$ and $q \in \mathbb{K}$, that is, the left multiplication with the quaternionic independent variable $q$ in the space $\mathfrak{S}$. Then $T \in \mathcal{B}(\mathfrak{S})$ and it is a normal operator, because $T^*f(q) = q^*f(q)$ for all $f \in \mathfrak{S}$ and $q \in \mathbb{K}$, and obviously $TT^* = T^*T$.

Note that for every $h \in \mathfrak{S}_{\mathbb{C}}$ we have $(\zeta - T_{\mathbb{C}})h = (\zeta - T_{\mathbb{C}})(1)h$. Therefore, writing $h = 1h = [(\zeta - T_{\mathbb{C}})(\zeta - T_{\mathbb{C}})^{-1}(1)]h = (\zeta - T_{\mathbb{C}})[(\zeta - T_{\mathbb{C}})^{-1}(1)]h$, we obtain

$$[(\zeta - T_{\mathbb{C}})^{-1}(1)]h = (\zeta - T_{\mathbb{C}})^{-1}(\zeta - T_{\mathbb{C}})^{-1}[(\zeta - T_{\mathbb{C}})(\zeta - T_{\mathbb{C}})^{-1}(1)h] = (\zeta - T_{\mathbb{C}})^{-1}(\zeta - T_{\mathbb{C}})^{-1}(1)]h = [(\zeta - T_{\mathbb{C}})^{-1}(1)]h.$$

Let us compute the complex spectrum $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$ of $T$. The complex extension $T_{\mathbb{C}}$ of $T$ is clearly the left multiplication with the independent quaternionic variable $q$ in the space $\mathfrak{S}_{\mathbb{C}}$. We shall prove that the spectrum $\sigma(T_{\mathbb{C}})$ is given by the conjugate symmetric set $\mathfrak{S}(\mathbb{K})$. If $\zeta \notin \mathfrak{S}(\mathbb{K})$, the function $q \mapsto (\zeta - q)^{-1}$ is well defined on $\mathbb{K}$ and the left multiplication with this function on $\mathfrak{S}_{\mathbb{C}}$ is equal to the operator $(\zeta - T_{\mathbb{C}})^{-1}$, and so $\zeta \notin \sigma_{\mathbb{C}}(T)$.

To prove the converse, we use an argument from above, and fix a point $\zeta \in \mathfrak{S}(\mathbb{K})$. So there exists a point $q_0 \in \mathbb{K}$ such that $\zeta \in \sigma_{\mathbb{C}}(q_0)$. Let us show that $\zeta \in \sigma_{\mathbb{C}}(T)$.

Assuming that $\zeta \notin \sigma(T_{\mathbb{C}})$, we can write $[(\zeta - q_0)(\zeta - T_{\mathbb{C}})^{-1}(1)](q) = 1$ for all $q \in \mathbb{K}$. We also have $[(\zeta - T_{\mathbb{C}})^{-1}(\zeta - T_{\mathbb{C}})^{-1}1](q) = [(\zeta - T_{\mathbb{C}})^{-1}1](q)(\zeta - q_0) = 1$, implying, in particular, that $\zeta - q_0$ is invertible in $\mathbb{M}$. Hence $\zeta \notin \mathfrak{S}(\mathbb{K})$, which is impossible. Consequently, $\zeta \in \sigma_{\mathbb{C}}(T)$, proving the equality $\mathfrak{S}(\mathbb{K}) = \sigma_{\mathbb{C}}(T)$.

Let $\text{Bor}_{\mathcal{C}}(\mathfrak{S}(\mathbb{K})) = \{ A = A^*; A \in \text{Bor}(\mathfrak{S}(\mathbb{K})) \}$, which is a $\sigma$-algebra. For every subset $A$ in $\mathfrak{S}(\mathbb{K})$, we put $A_{\mathbb{K}} = \{ q \in \mathbb{K} ; \sigma_{\mathbb{C}}(q) \subset A \}$. Because $\mathbb{K}$ is spectrally saturated, we have in fact $A_{\mathbb{K}} = A_{\mathbb{H}}$. Put also $\mathfrak{S}_T(A) = \{ f \in \mathfrak{S} ; \supp(f) \subset A_{\mathbb{K}} \}$. If $F(A)$ is the orthogonal projection on the subspace $\mathfrak{S}_T(A)$, the map $\text{Bor}_{\mathcal{C}}(\mathfrak{S}(\mathbb{K})) \ni A \mapsto F(A)$ is a spectral measure. This can be seen because the
projection $F(A)$ is the multiplication operator with the characteristic function $\chi_{A_k}$. Specifically, if $A, B \in \mathcal{S}(\mathbb{K})$, we have

$$F(A \cap B)f = \chi_{(A \cap B)k}f = \chi_{A_k} \chi_{B_k}f = F(A)F(B)f$$

for each $f \in \mathcal{H}_C$. Moreover, if $(A_j)_{j \geq 1}$ is a sequence of mutually disjoint subsets in $\text{Bor}_c(\mathcal{S}(\mathbb{K}))$, and $A = \bigcup_{j \geq 1} A_j$, we have

$$\langle F(A)f, g \rangle = \int_{\mathbb{K}} \langle \chi_{A_k}(q)f(q), g(q) \rangle d\lambda(q) = \sum_{j \geq 1} \int_{\mathbb{K}} \langle \chi_{A_k}(q)f(q), g(q) \rangle d\lambda(q) = \sum_{j \geq 1} \langle F(A_j)f, g \rangle,$$

showing that the set map $A \mapsto \langle F(A)f, g \rangle$ is countably additive, so it is a measure.

Let us finally remark that $TF(A)\mathcal{S} \subset F(A)\mathcal{S}$, and that $\sigma_C(T|F(A)\mathcal{S}) \subset [A]$, via a slight modification of the proof that $\sigma_C(T) = \mathcal{S}(\mathbb{K})$ from above. Consequently, the map $A \mapsto F(A)$ is the spectral measure of the normal operator $T$, by Proposition 3.

Supposing that the set $\mathbb{K}$ is unbounded and closed, yet spectrally saturated, the similar multiplication operator with the independent quaternionic variable will be an unbounded normal operator. Specifically, if $Tf(q) = qf(q)$ for all $f \in \mathcal{S}$, with $\mathcal{S}$ as above, and $q \in \mathbb{K}$, then $T$ is an unbounded operator with $D(T) = \{ f \in \mathcal{S} : \exists f \in \mathcal{S} \}$. Moreover, $T^*f(q) = q^*f(q)$ for all $f \in D(T)$ and $q \in \mathbb{K}$. Clearly, $D(T^*) = D(T)$, $D(TT^*) = D(T^*T)$, and $TT^* = T^*T$. Therefore, the operator $T$ is normal.

The set $\mathcal{S}(\mathbb{K})$ is closed and, as in the bounded case, we have the equality $\sigma(T_C) = \mathcal{S}(\mathbb{K})$. Indeed, the inclusion $\sigma(T_C) \subset \mathcal{S}(\mathbb{K})$ can be obtained as in the bounded case. For the converse, we adopt an appropriate argument.

Fixing $\zeta \in \mathcal{S}(\mathbb{K})$, and choosing a point $q_0 \in \mathbb{K}$ such that $\zeta \in \sigma_C(q_0)$, we want to use the equality $\zeta \in \sigma_C(T_C)$. Assuming that $\zeta \notin \sigma(T_C)$, we take a spectrally saturated compact subset $\mathbb{K}_0 \subset \mathbb{K}$ containing $q_0$, and consider the characteristic function $\chi_0$ of $\mathbb{K}_0$. We can write $(\zeta - q_0)^{-1} = (\zeta - T_C)^{-1} \chi_0(q) = \chi_0(q)$ for all $q \in \mathbb{K}_0$. We also have $(\zeta - T_C)^{-1} \chi_0(q) = (\zeta - T_C)^{-1} \chi_0(q)$, implying, that $\zeta - q_0$ is invertible in $\mathbb{M}$. As $\zeta \notin \mathcal{S}(\mathbb{K})$ is not possible, this completes the proof of the equality $\mathcal{S}(\mathbb{K}) = \sigma_C(T_C)$.

The spectral measure of $T$ is also given by multiplications with characteristic functions of sets from $\text{Bor}_c(\mathcal{S}(\mathbb{K}))$. We omit the details, left to the reader.

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