RVB gauge theory and the topological degeneracy in the honeycomb Kitaev model

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Abstract

We relate the $\mathbb{Z}_2$ gauge theory formalism of the Kitaev model to the $SU(2)$ gauge theory of the resonating valence bond physics. Furthermore, we reformulate a known (Feng et al (2007 Phys. Rev. Lett. 98 087204), Chen and Hu (2007 Phys. Rev. B 76 193101), Chen and Nussinov (2008 J. Phys. A: Math. Theor. 41 075001) and Mandal et al (2006 Int. Conf. on Physics Near the Mott Transition) Jordan–Wigner transformation of the Kitaev model on a torus in a general way that shows that it can be thought of as a $\mathbb{Z}_2$ gauge-fixing procedure. We give an explicit construction of the generators of large gauge transformations on a torus in terms of the spin operators. Using these and the non-trivial loop operators, we construct four mutually anti-commuting operators which commute with the Hamiltonian enabling us to prove that all eigenstates of this model, for the time-reversal symmetric case, are fourfold degenerate in the thermodynamic limit.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Solid state realization of $q$-bits that do not decohere easily is a challenging task in the field of quantum computation. Topological defects in strongly correlated quantum many-body systems are protected from decoherence and have been suggested as $q$-bit candidates [1–5]. In this context, Kitaev constructed a remarkable two-dimensional quantum spin model that exhibits Abelian and non-Abelian anyons and is exactly solvable for its spectrum [6, 7]. It was later shown that the spin correlation functions are also exactly solvable [8]. This model is also extremely interesting from the point of view of frustrated spin models and the physics of the resonating valence bond (RVB) states [9–13]. It realizes, in an exact fashion, the phenomenon of quantum number fractionization and emergent gauge fields that were conjectured and
approximately realized in two-dimensional models for RVB states or quantum spin liquids [14, 15]. It has also been shown that the Jordan–Wigner transformation in this model yields a local fermionic theory [16–19]. This makes the Kitaev model an important one that warrants further investigation, and no wonder that an extensive body of research [20–24] has already been carried out exploring its many fascinating aspects. Kitaev showed that the model has a natural formulation in terms of a Majorana fermion interacting with $Z_2$ gauge fields. The remarkable feature of the Kitaev model is that the gauge fields turn out to be static. This greatly simplifies the dynamics leading to the exact computation of the spectrum and spin–spin correlation functions.

The gauge theory of spin-$1/2$ models has a recent history. It was initially formulated [25] in the context of strongly correlated electronic systems such as a spin-$1/2$ Mott insulator, as a way of implementing the single-electron occupancy constraint,

$$\sum_{\sigma} (c^\dagger_{i\sigma} c_{i\sigma} - 1) |\psi\rangle = 0.$$  \hspace{1cm} (1)

This equation is the Gauss law constraint for a $U(1)$ lattice gauge theory. It resulted in a strongly interacting $U(1)$ gauge theory formalism of spin-$1/2$ models. Soon it was realized that in spin-$1/2$ systems, the $U(1)$ gauge invariance always implied an $SU(2)$ invariance [26–28]. The consequent $SU(2)$ gauge theory formalism was found to be useful in the context of relating apparently different mean field solutions of the model. The extended Hilbert space of the spin-$1/2$ system is much smaller than that of normal lattice gauge theory where the gauge degrees of freedom on the links are $SU(2)$ group elements. Consequently, it was shown that the essential physics of the spin-$1/2$ system is captured by a $Z_2$ gauge theory [29, 30], where the $Z_2$ gauge group is the centre of the original $SU(2)$ gauge group. The $Z_2$ gauge theory formalism has been effectively used to bring out the physics of quantum number fractionization in spin-$1/2$ systems [15].

In this paper we follow the route charted out above in the context of the Kitaev model and show that the $Z_2$ gauge theory can indeed be thought of as the centre of the $SU(2)$ gauge theory of RVB theory. This sheds light on Kitaev’s assertion [6] that the model represents the same universality class of topological order as RVB.

The topological degeneracy of the Kitaev model on a torus has been studied previously by Kells et al [39, 40]. They showed that the ground state is fourfold degenerate in both the phases of the system. They also showed that when time-reversal breaking terms create a gap in the non-Abelian phase, the degeneracy is reduced to a threefold degeneracy. This degeneracy, which characterizes the topological order in the system, arises from the so-called large gauge transformations. Namely gauge field configurations which correspond to the same flux configuration are not related to each other by local gauge transformations. These topologically distinct gauge field configurations can be labelled by the value of the Wilson loops that wind around the torus in the two different directions.

We present an alternate proof of the topological degeneracy that proves the topological degeneracy for all eigenstates. We do this by explicitly constructing the operators that generate the large gauge transformations and showing that they do not change the energy of the system in the thermodynamic limit. This construction is in terms of the original spin operators and the proof is completely algebraic.

The rest of this paper is organized as follows. Section 2 reviews the Kitaev honeycomb model with its main features and some mathematical notions which are used in later sections. The $SU(2)$ gauge symmetry of the model is reviewed in section 3 where we show that Kitaev’s choice of the representation of the spin operators in terms of Majorana fermions amounts to an $SU(2)$ gauge-fixing procedure which fixes the gauge up to the centre $Z_2$ gauge transformations. In section 4, we construct a generalized Jordan–Wigner transformation and show that it is a
Z₂ gauge fixing transformation in the Kitaev formalism. We derive the fermionic Hamiltonian and simplify the conserved quantities in terms of gauge-invariant Jordan–Wigner operators. In section 5, we review the fourfold degeneracy of the ground state Kitaev model on a torus. We derive the fermionic spectrum for a flux-free sector in section 5.1 and show that the finite-sized breaking of the degeneracy is \( \sim 1/L^2 \), where \( L \) is the linear size. Following this, in section 5.2, we present an algebraic proof that every eigenstate of the Kitaev model has fourfold degeneracy. To this end we derive four mutually anti-commuting operators that commute with the Hamiltonian in the thermodynamic limit. The minimum dimension of the representation of the four-dimensional Clifford algebra is 4. Thus we are able to prove that all eigenstates are at least fourfold degenerate. We summarize our results in section 6.

2. The Kitaev model

2.1. The Hamiltonian

The Kitaev model is a spin-1/2 system on a honeycomb lattice. The Hamiltonian is

\[
H = -J_x \sum_{\langle ij \rangle_x} \sigma_i^x \sigma_j^x - J_y \sum_{\langle ij \rangle_y} \sigma_i^y \sigma_j^y - J_z \sum_{\langle ij \rangle_z} \sigma_i^z \sigma_j^z, \tag{2}
\]

where \( i, j \) run over the sites of the honeycomb lattice; \( \langle ij \rangle_\alpha, \alpha = x, y, z \) denotes the nearest-neighbour links oriented in the \( \alpha \)th direction as shown in figure 1. We will be working with periodic boundary conditions which are defined as follows. The honeycomb lattice is a triangular lattice with a basis of two sites. The sites of the triangular lattice are given by

\[
\mathbf{R}_{m,n} = m\mathbf{e}_1 + n\mathbf{e}_2, \tag{3}
\]

where \( m, n \) are integers and \( \mathbf{e}_1 \cdot \mathbf{e}_2 = -\frac{1}{2}, \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 = \mathbf{e}_2 \cdot \mathbf{e}_2 \). The label \( i \) of the sites of the honeycomb lattice therefore stands for \( (m, n, \alpha) \) where \( \alpha = a, b \) is the sub-lattice label. The periodic boundary conditions are then defined by

\[
\sigma_{m,n,\alpha}^i = \sigma_{m+M,n+N,\alpha}^i, \tag{4}
\]
2.2. The conserved quantities

There is a conserved quantity associated with every plaquette of the lattice. If the plaquette is denoted as \( p \) and its vertices are labelled as shown in figure 1, then, following Kitaev’s notation, the conserved quantity is

\[
B_p = \sigma_x^1 \sigma_z^2 \sigma_y^3 \sigma_x^4 \sigma_z^5 \sigma_y^6.
\]  

We have \( B_p^2 = 1 \) implying that \( B_p \) can take values \( \pm 1 \). It is clear that any product of \( B_p \) will also commute with the Hamiltonian. In fact there is a conserved quantity associated with every closed self-avoiding loop, \( C \), on the lattice defined the following way. At every site, the path will pass through two of the three bonds that emanate from it. We call these two bonds the tangential bonds and the third one the normal bond. We associate two tangential vectors at each site, \( \hat{t}_1 \) and \( \hat{t}_2 \) which are either \( \hat{x}, \hat{y} \) or \( \hat{z} \) according to the direction of the incoming bond and the outgoing bond, respectively. We then define a normal vector \( \hat{n} \) as

\[
\hat{n} \equiv \hat{t}_1 \times \hat{t}_2.
\]

If the sites of \( C \) are \( i_1, i_2, ..., i_N \), then the conserved quantity associated with it is

\[
B_C = \prod_{n=1}^{N} (\hat{n}_n \cdot \sigma_n).
\]

It can be checked that

\[
[B_C, H] = 0, \quad B_C^2 = 1.
\]

We will call \( C \) topologically trivial if it can be written as a product of \( B_p \). On the torus, we have two loops which wind the torus around in the two directions which cannot be expressed as a product of \( B_p \). One cannot be obtained from the other by the multiplication by \( B_p \). We will call these two (Wilson) loops \( W_1 \) and \( W_2 \).

All \( B_p \) are not independent due to the identity

\[
\prod_p B_p = 1.
\]

Thus there are \( N_p - 1 \) independent \( B_p \), where \( N_p = MN \) is the number of plaquettes. Together with \( W_1 \) and \( W_2 \), we have a total of \( N_p + 1 \) conserved quantities on the torus. These two loop operators account for the fourfold degeneracy on a torus.

3. \( SU(2) \) to \( Z_2 \)

3.1. \( U(1) \) and \( SU(2) \) gauge symmetry

Interacting quantum spin systems often lead to spontaneously broken symmetric states such as a ferro, antiferro or spiral magnetic states. Low energy physics of these ordered states is captured by the well-known spin wave approximations, resulting in Goldstone mode type bosonic low energy effective theories [31]. A quantum spin liquid, on the other hand, has no classical long-range order. Experiments in LaCuO\(_4\) and other low spin Mott insulators, according to Anderson, indicated possible presence of neutral fermionic excitations in a quantum spin liquid [32–35]. A theory to describe such a quantum spin liquid or RVB state needed a paradigm shift from spin wave theory. It was also clear that a quantum spin liquid, in view of different possible phase coherence among disordered spin configurations, could offer a variety of quantum spin liquid states to be realized in nature. RVB gauge theory attempted to capture these new possibilities, through an approach involving enlarged Hilbert space and
emergent gauge fields in strongly correlated electron systems. Through the work of Wen [36] and others, it has become clear that there is a plethora of spin liquid phases, characterized by quantum order and projective symmetry groups.

The spin-1/2 Hilbert space can be realized as the subspace of the Hilbert space of two fermions defined by the constraint

$$\left(\sum_{\sigma=\uparrow,\downarrow} c_{\sigma}^\dagger c_{\sigma} - 1\right)|\psi\rangle = 0,$$

(10)

where $c_{\sigma}^\dagger$ and $c_{\sigma}$ are the fermion creation and annihilation operators. As mentioned earlier, equation (10) can be looked upon as the Gauss law constraint for a $U(1)$ gauge theory. The LHS of the equation being the generator of the following $U(1)$ gauge transformations on the fermion operators

$$c_{\sigma} \rightarrow e^{i\Omega} c_{\sigma}, \quad c_{\sigma}^\dagger \rightarrow e^{-i\Omega} c_{\sigma}^\dagger.$$

(11)

The spin operators,

$$S_{a} = \frac{1}{2} c_{\sigma}^\dagger \sigma_{sg}^a c_{\sigma},$$

(12)

are then the gauge-invariant observables of the theory, $\sigma^a$ are the Pauli spin matrices.

The single occupancy constraint in the spin-1/2 theory implies that a spin-↑ hole is the same as a spin-↓ particle in the physical space. This can be mathematically expressed as an $SU(2)$ gauge invariance. It is convenient to express this symmetry in terms of a matrix of the fermion operators

$$\Psi \equiv \begin{pmatrix} c_{\uparrow} & -c_{\downarrow}^\dagger \\ c_{\downarrow} & c_{\uparrow}^\dagger \end{pmatrix}. $$

(13)

In terms of this matrix, the spin operators are given by

$$S_{a} = \frac{1}{4} \text{tr} \Psi^\dagger \sigma_{sg}^a \Psi.$$

(14)

The generators of the $SU(2)$ gauge transformation are given by

$$\tilde{S}_{a} = -\frac{1}{4} \text{tr} \Psi^\dagger \sigma_{sg}^a \Psi.$$

(15)

The $\Psi$ matrix transforms under the $SU(2)$ spin and $SU(2)$ gauge transformations as

$$\Psi \rightarrow U_S \Psi U_G^\dagger,$$

(16)

where $U_S$ and $U_G$ are $SU(2)$ matrices representing the spin and gauge transformations, respectively. It is clear from equations (14)--(16) that the spin operators are gauge invariant and the generators of gauge transformations are spin singlet. The constraint in equation (10) is exactly equivalent to the $SU(2)$ Gauss law

$$\tilde{S}_{a} |\psi\rangle = 0.$$

(17)

Before we close this section, we wish to mention that the above gauge theory formalism offers a possible way to understand quantum spin liquid states as and when they exist. This formalism does not guarantee a simple gauge theory structure at all energy scales in the physics of the problem. It only suggests that in some systems (for some Hamiltonians) at low energy scales, there could be emergent gauge fields and interesting consequences of quantum number fractionization, quantum order, etc. At high energy scales, gauge fields interact and it is no more simple or useful to talk in terms of emergent gauge fields. As we will see soon, the Hamiltonian invented by Kitaev on a honeycomb lattice is very special. It offers static $Z_2$ gauge fields and makes the $Z_2$ gauge theory meaningful at all energy scales.
3.2. Majorana fermions and the $Z_2$ theory

We can make connection to Kitaev’s representation of the spins by writing
\[ c_\uparrow = \frac{c_x - ic_y}{2}, \quad c_\downarrow = \frac{c_x + ic_y}{2}, \]
where $c$, $c_x$, $c_y$ and $c_z$ are Majorana fermions. The single occupancy constraint, equation (10), reduces to exactly Kitaev’s form
\[ cc_x c_y c_z = 1. \]  
Kitaev’s representation of the spins is then written as
\[ \frac{i}{2} cc_x c_y = S^a - \tilde{S}^a. \]  
Note that these three operators are not equal to the gauge-invariant spin operators in the extended Hilbert space but are exactly equivalent to them in the physical Hilbert space. Substituting the expressions in equation (20) for the spin operators in the Hamiltonian is then equivalent to adding gauge fixing terms. Since the $SU(2)$ gauge generators are invariant under the $Z_2$ centre of the gauge group, these terms only fix the gauge up to the central $Z_2$ group represented by $e^{i2\pi \tilde{S}}$. The Hamiltonian will therefore continue to have a $Z_2$ gauge symmetry.

The simple example of a spin-1/2 in a magnetic field illustrates these issues. If we take the Hamiltonian to be
\[ H = BS^3, \]  
then the theory has $SU(2)$ gauge symmetry, the degenerate ground states in the extended space are
\[ |GS\rangle = \alpha c_\uparrow |0\rangle + \beta |0\rangle + \gamma c_\downarrow c_\uparrow |0\rangle, \]  
for arbitrary $\alpha$, $\beta$ and $\gamma$, $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. The last two states in the RHS of equation (22) transform as a doublet under the $SU(2)$ gauge symmetry. Gauge averaging therefore projects out the ground state in the physical subspace, namely $c_\uparrow |0\rangle$.

If the Hamiltonian is taken to be
\[ H = iBcc_z = B(S^3 - \tilde{S}^3), \]  
the theory has only $Z_2$ gauge invariance, the degenerate ground states in the extended Hilbert space are
\[ |GS\rangle = \alpha c_\uparrow |0\rangle + \gamma c_\downarrow c_\uparrow |0\rangle. \]  
The second state in the above equation transforms non-trivially under the $Z_2$ gauge transformation. Thus again, under gauge averaging, the ground state in the physical sector is projected out. Thus Kitaev’s representation of the spin operators can be interpreted as adding gauge fixing terms to the $SU(2)$ gauge-invariant Hamiltonian which leave an unbroken (unfixed) $Z_2$ gauge symmetry.

4. The $Z_2$ gauge theory of the Kitaev model

4.1. The Hamiltonian
Following Kitaev [6], we write the Hamiltonian in terms of the Majorana fermions
\[ \hat{H} = \sum_{a=1}^{3} \sum_{i,j=1}^{1} i c_j H_{ij} c_i, \]  
for arbitrary $\alpha$, $\beta$ and $\gamma$, $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. The last two states in the RHS of equation (22) transform as a doublet under the $SU(2)$ gauge symmetry. Gauge averaging therefore projects out the ground state in the physical subspace, namely $c_\uparrow |0\rangle$.

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The second state in the above equation transforms non-trivially under the $Z_2$ gauge transformation. Thus again, under gauge averaging, the ground state in the physical sector is projected out. Thus Kitaev’s representation of the spin operators can be interpreted as adding gauge fixing terms to the $SU(2)$ gauge-invariant Hamiltonian which leave an unbroken (unfixed) $Z_2$ gauge symmetry.
where the link variables are defined as $u_{ij} = i c_{ai} c_{aj}$. It is natural to express them in terms of the bond fermions [8] defined as
$$\chi_{ij} = \chi^a_{ij} \equiv c_{ai} + i c_{aj}. \quad (26)$$
The link variables are then given in terms of the occupancy number of the bond fermions
$$u_{ij} = 2 \chi^+_{ij} \chi_{ij} - 1. \quad (27)$$
It is easy to see that
$$u^2_{ij} = 1, \quad [u_{ij}, H] = 0. \quad (28)$$
Thus the link variables can be interpreted as static $Z_2$ gauge fields.

It is remarkable that at one shot, the Kitaev Hamiltonian has been solved exactly for the entire many-body spectrum! In fact, two related phenomena occur: (i) the $Z_2$ gauge theory is exact at all energy scales and (ii) the enlarged Hilbert space gets decomposed into sectors that are identical gauge copies having the same energy eigenvalues (figure 2). The Hilbert space enlargement does not produce any unphysical state, but only gauge copies. In the standard $U(1)$ RVB gauge theory for a Heisenberg antiferromagnet, for example, it is easy to see how unphysical states are brought in by the Hilbert space enlargement. For example, an unphysical state containing $M$ doubly occupied and $M$ empty sites gives the spectrum of a Heisenberg antiferromagnet containing $2M$ missing sites.

4.2. The Jordan–Wigner transformation and $Z_2$ gauge fixing

A remarkable feature of the Kitaev model is that the Jordan–Wigner transformation yields a local fermionic Hamiltonian [16–19]. In this section we show that the Jordan–Wigner transformation in the Kitaev model can be interpreted as a $Z_2$ gauge-fixing procedure resulting in gauge-invariant (gauge-fixed) Majorana and bond fermions. The choice of the Jordan–Wigner path amounts to a choice of the $Z_2$ gauge.

4.2.1. The Jordan–Wigner fermionization. We define the Jordan–Wigner transformation as follows [41]. Take any Hamilton path on the lattice defined by a sequence of sites $i_n, n = 1, \ldots, N_S$, where $N_S$ is the number of sites in the lattice. The path will classify each bond as normal or tangential as defined in section 2.2. The normal bonds will form a dimer covering of the lattice. For a given site ‘$i$’ we attach three vectors, two tangential vectors denoted by $\mathbf{t}_{i1}, \mathbf{t}_{i2}$ and one normal vector $\mathbf{n}_i$, such that they follow equation (6). Then
we can define the Jordan–Wigner transformations in the following compact way discussed below.

4.2.2. Gauge-invariant Jordan–Wigner fermions. To define the Jordan–Wigner transformations we associate two Majorana fermions \( \eta \) and \( \xi \) at a given site \( \nu \). These Jordan–Wigner (Majorana) fermions are defined in terms of the gauge-invariant spin operators in the following way:

\[
\eta_\nu = \hat{t}_1 \cdot \sigma_\nu \mu_\nu,
\]

(29)

\[
\xi_\nu = \hat{t}_2 \cdot \sigma_\nu \mu_\nu,
\]

(30)

\[
\mu_\nu = \prod_{m=1}^{n-1} (\hat{n}_\nu \cdot \sigma_\nu).
\]

(31)

It can be easily checked that the above definitions refer to the usual anti-commutations relations for Majorana fermions

\[
\{ \eta, \xi \} = 2 \delta_{ij},
\]

(32)

\[
\{ \xi, \eta \} = 2 \delta_{ij},
\]

\[
\{ \eta, \eta \} = 0.
\]

(33)

Since the Jordan–Wigner fermions are constructed entirely from the spin operators, they are manifestly gauge invariant. However, it is also interesting to see this by rewriting equations (30) and (29) in terms of the original Majorana fermions and gauge fields

\[
\eta_\nu = i c_\nu (u_\nu - 1 \ldots u_{\nu-1}) \hat{t}_1 \cdot c_\nu,
\]

(33)

\[
\xi_\nu = i c_\nu \cdot \hat{n}_\nu (u_\nu - 1 \ldots u_{\nu-1}) \hat{t}_1 \cdot c_\nu.
\]

(34)

The transformation can be inverted to write the spins in terms of the fermions

\[
\hat{n}_\nu \cdot \sigma_\nu = i \eta_\nu \xi_\nu,
\]

(35)

\[
\hat{t}_1 \cdot \sigma_\nu = \eta_\nu \mu_\nu,
\]

(36)

\[
\hat{t}_2 \cdot \sigma_\nu = \xi_\nu \mu_\nu.
\]

(37)

These complete the definitions of Jordan–Wigner fermitonizations used in this paper.

4.2.3. The gauge-fixed Hamiltonian. The Hamiltonian can be written in terms of the gauge-invariant fermions as

\[
\hat{H} = J_x \sum_{\langle ij \rangle} \tilde{u}_{ij} \eta_i \eta_j + J_y \sum_{\langle ij \rangle} \tilde{u}_{ij} \xi_i \xi_j + J_z \sum_{\langle ij \rangle} \tilde{u}_{ij} \eta_i \xi_j,
\]

(38)

where the gauge-fixed \( Z_2 \) fields, \( \tilde{u}_{ij} \) are

\[
\tilde{u}_{ij} = i \tilde{\xi}_i \tilde{\xi}_j \quad \text{normal bonds},
\]

\[
= 1 \quad \text{tangential bonds except } \langle ij \rangle = (i_1 i_2),
\]

\[
\tilde{u}_{i_1 i_2} = S
\]

\[
= \prod_{n=1}^{N} (\hat{n}_n \cdot \sigma_\nu)
\]

\[
= (u_{i_1 i_2} u_{i_2 i_3} \ldots u_{i_{N-1} i_N}) \prod_{n=1}^{N} \eta_n.
\]

(39)
Figure 3. The Jordan–Wigner path on a torus for $4 \times 4$ lattice. The numerics at the lattice sites describe how the Jordan–Wigner path traverses the lattice.

$S$ is a well-known conserved quantity in the one-dimensional applications where it corresponds to the total number of fermions modulo 2 and determines the boundary conditions on the fermions. Thus the Jordan–Wigner transformation is equivalent to a gauge-fixing procedure where all the gauge fields on the tangential bonds (except one) are set equal to 1. The choice of the Hamilton path amounts to a gauge choice since it defines which of the bonds are tangential. It also defines the sign in the definition of $u_{ij}$ for the normal bonds. In equation (39), the sign corresponds to a Hamilton path which winds regularly in the $\hat{e}_1$ direction as shown in figure 3. Hereinafter all explicit computations will be with respect to this path. A general algorithm to go to this gauge, which we will refer to as the Jordan–Wigner gauge, is given in the appendix.

4.2.4. The fermionic conserved quantities. All the gauge-fixed $Z_2$ fields are conserved quantities. It is convenient to express them in terms of the gauge-fixed bond fermions on the normal bonds

$$\chi_{ij} = \frac{\xi_i + i\xi_j}{2}, \quad \chi_{ij}^\dagger = \frac{\xi_i - i\xi_j}{2}. \tag{40}$$

The conserved quantities are then the occupation numbers of the bond fermions,

$$\tilde{u}_{ij} = 2\chi_{ij}^\dagger\chi_{ij} - 1. \tag{41}$$

There are $N_p$ normal bonds and hence the bond fermion occupation numbers form a set of $N_p$ conserved quantities. Thus along with $S$, we have $N_p + 1$ conserved quantities consistent with the analysis in terms of spin variables in section 2.2. To see the meaning of $S$, it is convenient to define complex fermions in the matter sector from the two $\xi$ fermions on every normal bond

$$\psi_{ij} = \frac{\eta_i + i\eta_j}{2}, \quad \psi_{ij}^\dagger = \frac{\eta_i - i\eta_j}{2}. \tag{42}$$

It can then be shown that

$$S = (-1)^{(N_p + N_\psi + 1)},$$

$$N_\psi = \sum_{\text{bonds}} \psi_{ij}^\dagger\psi_{ij},$$

$$N_\chi = \sum_{\text{normal bonds}} \chi_{ij}^\dagger\chi_{ij}. \tag{43}$$
4.2.5. From Kitaev gauge to Jordan–Wigner gauge. We have explained that the Jordan–Wigner gauge is a special realization of the Kitaev gauge where all the gauge fields residing on the tangential bonds are fixed to unity. One may wonder whether there exists a gauge transformation on the lattice which renders equation (25) to that of equation (38). Indeed there exists such a gauge transformation. Referring to the Jordan–Wigner path given in figure 3 we do the following gauge transformations at a site ‘n’

$$c_n \rightarrow \prod_{i=1,n-1} u_{i,i+1} c_n, \quad 1 < n < N. \tag{44}$$

In the above equation the various indices correspond to the Jordan–Wigner path in figure 3. $u_{i,i+1}$ are the $Z_2$ gauge fields that would normally exist on the link joining site ‘i’ and ‘$i + 1$’ if we apply the Majorana fermionization as adopted by Kitaev. After implementing the above gauge transformations, $Z_2$ gauge fields appear only on the normal bonds. The above gauge transformations yield a new conserved quantity $S'$ which appear on the bond where the Jordan–Wigner end path meets. The expression for $S'$ is

$$S' = \prod_{i=1,N-1} u_{i,i+1}. \tag{45}$$

The significance of $S'$ is very similar to $S$ introduced in equation (39).

5. Fourfold degeneracy on a torus

As mentioned earlier, the explicit computation of the degenerate ground states on a torus has been done by Kells et al [39]. We review the calculation here using our form of the Jordan–Wigner transformation for the sake of making this paper self-contained.

Since the gauge fields are static, the problem is reduced to one of the non-interacting fermions on a lattice. It is known [6, 37] that the lowest fermionic ground state energy is obtained for the flux-free configuration, i.e. $B_p = 1, \forall p$. On the torus, there are four gauge inequivalent configurations for every configurations of $B_p$ as argued in section 2.2. These correspond to the four values of the gauge-invariant conserved quantities $W_1$ and $W_2$.

First we examine these four different gauge field configurations for the flux-free sector and compute the corresponding fermionic ground state. Following this we show that this leads to fourfold degeneracy of ground states in the thermodynamic limit. Next we demonstrate explicitly how to obtain the fourfold degeneracy for every configurations of fluxes. To this end we derive the required operators which enable us to obtain any one of the inequivalent gauge field configurations from the other for arbitrary flux configurations.

5.1. Degenerate ground states on a torus

To start with, we briefly recapitulate the notions of the Jordan–Wigner transformation and refer to figure 3. The normal bonds are the ones that form the basis of the triangular lattice except for the (0, n, a) line. On this line the normal bonds are the ones between (0, n, a) and (0, n + 1, b). We choose the first site of the path to be $i_1 = (0, 0, b)$. The four flux-free configurations are then explained as follows. To this end we write the exact Hamiltonian and $B_p$ for this particular realization of the Jordan–Wigner transformation. We divide the Hamiltonian in three parts: $H_{int}$, $H_{bound}$ and $H_{end}$. $H_{int}$ includes all the internal bonds and $H_{bound}$ includes all the boundary bonds except one where the Jordan–Wigner end points meet. $H_{end}$ includes the interaction for the bond where the Jordan–Wigner end points meet each other. Similarly all $B_p$ are categorized in the above three different ways. Below we write the various parts of the Hamiltonian and $B_p$. 
Conserved quantities appeared in the final form of the Hamiltonian. The conserved quantity $B_p$ gauge fields. Thus, $H_{\text{int}} = \sum_{m,n} i J_i \eta_{m,n}^a \eta_{m+1,n+1}^b + \sum_{m,n} i J_j \eta_{m,n}^a \eta_{m,n+1}^b + \sum_{m,n} i J_k \eta_{m,n}^a \eta_{m,n+1}^b$.

where $\tilde{u}_{m,n}$ is defined on each internal $z$-bonds.

$H_{\text{bound}} = \sum_{m,n} i J_l \tilde{u}_{m,n+1}^a \tilde{\eta}_{m,n+1}^a \eta_{m,n}^b + \sum_{m,n} J_s \tilde{\eta}_{m,n}^a \eta_{m,n}^b$.

where $\tilde{u}_{m,n}$ is defined on each boundary $y$-bond. The Hamiltonian for the end bond is given by

$H_{\text{end}} = -S \tilde{\eta}_{M,N}^a \eta_{M,N}^b$.

Now with the definition of $\psi$ fermion and $\chi$ fermion, we obtain equations (46) and (47) to be rewritten as

$H_{\text{int}} = \sum_{m,n} J_s (\psi_{m,n}^+ + \psi_{m,n})(\psi_{m+1,n+1}^+ - \psi_{m+1,n+1})$

$+ J_l (\psi_{m,n}^+ + \psi_{m,n})(\psi_{m,n+1}^+ - \psi_{m,n+1}) + J_s \tilde{u}_{m,n}(2 \psi_{m,n}^+ \psi_{m,n} - 1)$. (49)

The Hamiltonian for the slanting bonds

$H_{\text{bound}} = \sum_{m,n} J_l \tilde{u}_{m,n+1} (\psi_{m,n}^+ + \psi_{m,n})(\psi_{m,n+1}^+ - \psi_{m,n+1}) + J_s (2 \psi_{m,n}^+ \psi_{m,n} - 1)$. (50)

Lastly the Hamiltonian term for the end bond where the end points of the Jordan–Wigner path meet each other is given by

$H_{\text{end}} = -S J_l (2 \psi_{M,N}^+ \psi_{M,N} - 1)$. (51)

Following equations (39), (30) and (42), we rewrite the complete expressions for various conserved quantities appeared in the final form of the Hamiltonian. The conserved quantity $\tilde{u}_{m,n}$ defined on each internal $z$-link is given by

$\tilde{u}_{m,n} = (2 \chi_{m,n}^+ \chi_{m,n} - 1)$. (52)

Similarly the conserved quantity defined on each boundary $y$-bond (which is labelled by the $z$-bonds it is connected with (i.e $\tilde{u}_{m,n+1}$) ) is given by

$\tilde{u}_{m,n+1} = \left(2 \chi_{m,n+1}^+ \chi_{m,n+1} - 1 \right)$. (53)

$S$, for the Jordan–Wigner gauge, is given by

$S = -(-1)^{MN+N_s-N_s}$.

If $P^G$ and $P^M$ denote the parity operators for the gauge fermions and the matter fermions, respectively, then we can write $S = -(-1)^{MN} P^M P^G$. From figure 3, we see that a single hexagon always contains two normal bonds where each normal bond is associated with a conserved (static) $Z_2$ gauge field. $B_p$ for any plaquette is the product of these two conserved $Z_2$ gauge fields. Thus, $B_p = \tilde{u}_{ij} b_{jk}$, where ‘$ij$’ and ‘$kl$’ are the normal bonds for the plaquette ‘$p$’.

For the end plaquette where the Jordan–Wigner path terminates $B_p$ is given by $B_p = -S \tilde{u}_{ij} \tilde{u}_{kl}$.

Now we are in a position to show the ground state degeneracy in the thermodynamic limit. To this end we explicitly write the four inequivalent gauge field configurations corresponding to the flux-free configurations and write down the corresponding fermionic Hamiltonian. Finally we find the spectra for each of these four fermionic Hamiltonians.
5.1.1. Choice 1. Here the flux-free configuration is obtained by making all the \( \hat{u} \) to be 1 and \( S = -1 \). The loop conserved quantities are having the following eigenvalues: \( W_1 = 1 \) and \( W_2 = 1 \). This particular choice makes the resulting fermionic Hamiltonian transitionally invariant and the usual periodic boundary condition in both the directions can be used to diagonalize the Hamiltonian. We will explicitly write the Hamiltonian in terms of complex fermions as well as \( \psi \) fermions. The complete translational invariant Hamiltonian is given by

\[
H = \sum_{m,n} J_x (\psi_{m,n}^\dagger + \psi_{m,n}) (\psi_{m+1,n+1}^\dagger - \psi_{m+1,n+1}) \\
+ J_y ((\psi_{m,n}^\dagger + \psi_{m,n})(\psi_{m,n+1}^\dagger - \psi_{m,n+1}) + J_z (2\psi_{m,n}^\dagger \psi_{m,n} - 1). \tag{55}
\]

This is a manifestly p-wave superconducting Hamiltonian and can be easily diagonalized by going to the momentum space. The constraint on the number of \( \psi \) fermions becomes

\[
\prod (2\psi_{m,n}^\dagger \psi_{m,n} - 1) = 1, \tag{56}
\]

which implies that we are to fill up only the even number of \( \psi \) fermions. We define the Fourier transform of the \( \psi_{m,n} \) as follows:

\[
\psi_{m,n} = \frac{1}{\sqrt{MN}} \sum_{p,q} e^{i(k_{m+1/2} + k_{n+1/2})}, \tag{57}
\]

where \( k_1 = 2\pi p/M \) and \( k_2 = 2\pi q/N \). This is obtained by noting the fact that we can write \( \vec{k} = \frac{G_1}{2} + \frac{G_2}{2} \), where \( G_{1/2} \) are the reciprocal lattice vectors which are given by

\[
G_1 = \frac{4\pi}{\sqrt{3}} \left( \frac{\sqrt{3}}{2} \vec{e}_x + \frac{1}{2} \vec{e}_y \right), \quad G_2 = \frac{4\pi}{\sqrt{3}} \vec{e}_y. \tag{58}
\]

Substituting this we obtain the resulting Hamiltonian in the momentum space as

\[
H = \sum_k (\epsilon_k \psi_k^\dagger \psi_k - \epsilon_k \psi_{-k}^\dagger \psi_{-k} + i\delta_k \psi_k^\dagger \psi_{-k}^\dagger - i\delta_k \psi_{-k} \psi_k) \\
+ \epsilon_{0,0} \psi_{0,0}^\dagger \psi_{0,0} + \epsilon_{\pi,0} \psi_{\pi,0}^\dagger \psi_{\pi,0} + \epsilon_{0,\pi} \psi_{0,\pi}^\dagger \psi_{0,\pi} + \sum_k \epsilon_k - MNJ_z, \tag{59}
\]

where \( \epsilon_k = 2(J_x \cos k_x + J_y \cos k_y + J_z) \) and \( \delta_k = 2(J_x \sin k_x + J_y \sin k_y) \). \( k_x = k \cdot n_x, k_y = k \cdot n_y \), and \( n_x, n_y = \frac{1}{2} \vec{e}_x, \frac{1}{2} \vec{e}_y \) are unit vectors along the \( x \)- and \( y \)-type bonds.

In equation (59), the sum over ‘\( k \)’ runs over first half of the Brillouin zone and does not include the ‘\( k \)’-points (\( \pi, 0 \)), \( (0, \pi) \), and \( (0, 0) \). The first line of the Hamiltonian is diagonalized by the following transformations:

\[
\begin{pmatrix}
\alpha_k \\
\beta_k
\end{pmatrix} = \begin{pmatrix}
\cos \theta_k & -i \sin \theta_k \\
-i \sin \theta_k & \cos \theta_k
\end{pmatrix} \begin{pmatrix}
\psi_k \\
\psi_{-k}^\dagger
\end{pmatrix}, \tag{60}
\]

where \( \cos 2\theta_k = \epsilon_k/E_k \), with \( E_k = \sqrt{\epsilon_k^2 + \delta_k^2} \). Then re-writting the Hamiltonian we obtain

\[
H = \sum_k E_k (\alpha_k^\dagger \alpha_k - \beta_k^\dagger \beta_k) + \epsilon_{0,0} \psi_{0,0}^\dagger \psi_{0,0} + \epsilon_{\pi,0} \psi_{\pi,0}^\dagger \psi_{\pi,0} + \epsilon_{0,\pi} \psi_{0,\pi}^\dagger \psi_{0,\pi} \\
- \frac{1}{2}(\epsilon_{0,0} + \epsilon_{0,\pi} + \epsilon_{\pi,0}) + \left( \sum_k \frac{1}{2} \epsilon_k - MNJ_z \right). \tag{61}
\]

Here the sum over \( k \) runs over the full Brillouin zone. The last term in the parenthesis is always zero for torus.
5.1.2. Choice 2. Here the flux-free configuration is obtained by making all $\bar{u} = 1$ and $\bar{S} = -1$. The corresponding values of loop conserved quantity are given by $W_1 = -1$ and $W_2 = -1$. To implement the Fourier transformation we do the following steps. We make the following gauge transformation: $\eta^{1}_{m,n} = -\eta^{b}_{m,n}$ for all ‘$n$’. In terms of $\psi$ fermion, the necessary gauge transformation is $\psi^{1}_{M,n} = -\psi^{3}_{m,n}$ for all ‘$n$’. Then the Hamiltonian requires $\eta^{1}_{M+1,n} = -\eta^{1}_{1,n}$ and $\eta^{1}_{m,N+1} = \eta^{1}_{m,1}$ (or alternatively $\psi^{3}_{M+1,n} = -\psi^{1}_{1,n}$ and $\psi^{3}_{m,N+1} = \psi^{1}_{m,1}$). This is equivalent to an anti-periodic boundary condition in the $e_1$ direction and the periodic boundary condition in the $e_2$ direction. The necessary Fourier transform is defined with
\[\psi_{m,n} = \frac{1}{\sqrt{MN}} \sum_{p,q} e^{i(k_{m}+k_{n})},\] where $k_{1} = \frac{2\pi}{M}(m + \frac{1}{2})$; $k_{2} = \frac{2\pi}{N}n$. Substituting this in the Hamiltonian and diagonalizing straightforwardly we obtain
\[H = \sum_{k} E_{k}^{2}(\alpha_{k}^{\dagger}\alpha_{k} - \beta_{k}^{\dagger}\beta_{k} + \sum_{\nu} \frac{1}{2} \epsilon_{\nu} - N_{x}J_{x} \).\]

Here also ‘$k$’ runs over first half of the Brillouin zone and ‘$k'$’ runs over the full Brillouin zone. Note the absence of $(0, \pi), (\pi, 0), (0, 0)$ mode. They do not appear here for this anti-periodic boundary condition. This will be true for choices 3 and 4 also. Various parameters appearing in equation (63) are given as follows:
\[E_{k}^{2} = \sqrt{(\epsilon_{k}^{2} + \delta_{k}^{2})}, \]
\[\epsilon_{k} = 2(J_{x} \cos k_{x} + J_{y} \cos k_{y} - J_{z}), \]
\[\delta_{k} = 2(J_{x} \sin k_{x} + J_{y} \sin k_{y}).\]

5.1.3. Choice 3. For this case the flux-free configuration is obtained by making $\bar{u}_{M,n} = -1$ where ‘$n$’ runs from 1 to $N - 1$. All other $\bar{u}$ are 1 and $\bar{S} = 1$. The loop conserved quantity $W_1$ takes value 1 and $W_2 = -1$. Similar to the previous case we need to do following gauge transformations in order to apply the Fourier transform. We make $\eta^{b}_{m,N} = -\eta^{b}_{m,n}$ for all ‘$n$’. In terms of $\psi$ fermion the necessary gauge transformation is $\psi^{1}_{m,N} = -\psi^{1}_{m,N}$ for all ‘$m$’. The resulting Hamiltonian requires $\eta^{1}_{m,N+1} = -\eta^{1}_{m,1}$ and $\eta^{1}_{m+1,n} = \eta^{1}_{1,n}$ (or alternatively $\psi^{1}_{m,N+1} = -\psi^{1}_{m,1}$ and $\psi^{1}_{m+1,n} = \psi^{1}_{m,1}$). This indicates the anti-periodic boundary condition in the $e_1$ direction and the periodic boundary condition in the $e_2$ direction. The resulting Fourier transform is defined with
\[k_{1} = \frac{2\pi}{M}(m + \frac{1}{2}); \quad k_{2} = \frac{2\pi}{N}(n + \frac{1}{2}).\]

The resulting Hamiltonian in ‘$k$’ space is similar to equation (63) with $\epsilon_{k} = 2(J_{x} \cos k_{x} + J_{y} \cos k_{y} + J_{z})$.

5.1.4. Choice 4. For these choices we need, $\bar{u}_{M,N} = 1$ where ‘$n$’ runs from 1 to $N - 1$. All other $\bar{u}$ are 1 and $\bar{S} = 1$. $W_1 = -1$ and $W_2 = 1$. In this case one requires combined gauge transformation mentioned for choices 2 and 3. This makes the Hamiltonian anti-periodic in both the directions. The required Fourier transform is defined with
\[k_{1} = \frac{2\pi}{M}\left(m + \frac{1}{2}\right); \quad k_{2} = \frac{2\pi}{N}\left(n + \frac{1}{2}\right).\]

Proceeding as before we exactly obtain equation (63) with an identical expression for $\epsilon_{k}$. 

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5.1.5. Ground state energy in the thermodynamic limit. To obtain the ground state energy for the gauge choices 2–4, one fills up the negative energy states consistent with the boundary condition (i.e. to satisfy the constraint $\mathcal{S}$ which restricts the total number of particles (odd or even number) to be taken). In the limit of $M, N \to \infty$ all the above four choices, the ground state energy is obtained as

$$E_G = \frac{\sqrt{3}}{16\pi^2} \int_{\mathbb{BZ}} E(k_x, k_y) \, dk_1 \, dk_2,$$

with $E(k_x, k_y)$ defined before. The appearance of a ‘$-$’ sign in the expression of $\epsilon_k$ for choices 2 and 4 can be accounted for by shifting the $k_1$ integral to $\pi - k_1$. Thus it is clear that in the thermodynamic limit, ground state has fourfold degeneracy.

5.2. Fourfold Topological degeneracy for any eigenstate

In the preceding section, we have shown that four degenerate ground states are characterized by four different topologically distinct gauge field configurations. From section 4, we infer that every flux configuration is characterized by such four topologically distinct gauge field configurations. This leads to a fourfold degeneracy for every eigenstate for any arbitrary vortex configuration, including the ground state, in the thermodynamic limit.

We now demonstrate this explicitly. Our proof is similar to that of Wen and Niu [38] which shows the topological degeneracy of fractional quantum Hall states on a torus. We construct two operators which we call $V_1$ and $V_2$ that act on states with a given gauge field configuration and produce states with a different gauge field configuration without changing the values of the flux operators, $B_p$. They however change the values of the Wilson loop operators, $W_{1,2}$. These two operators are therefore the generators of the so-called large gauge transformations. There are four topologically different sectors of gauge field configurations corresponding to $W_i = \pm 1$. We further show that the only effect of the large gauge transformations on the matter sector is to change the boundary conditions of the Majorana fermions from periodic to anti-periodic or vice versa. Thus the energy eigenvalues only change by $\sim 1/L$, where $L$ is the length of the torus. The eigenstates in the four sectors (related by the action of $V_i$) are therefore degenerate in the thermodynamic limit.

The four operators $V_i, W_i$ characterize the topological degeneracy. We show that $V_i$ and $W_i$ satisfy the following algebra:

$$\{V_1, V_2\} = 0, \quad [V_i, W_j] = 0, \quad [V_i, W_j]_{i \neq j} = 0.$$  \hspace{1cm} (68)

We can then construct four operators, $T_a, a = 1, \ldots, 4$,

$$T_1 = V_1 W_1, \quad T_2 = V_2 W_2, \quad T_3 = V_1, \quad T_4 = V_2.$$  \hspace{1cm} (69)

These satisfy the Clifford algebra

$$\{T_a, T_b\} = 2\delta_{ab}.$$  \hspace{1cm} (70)

Thus we show that the fourfold topological degeneracy on the torus is characterized by the four-dimensional Clifford algebra.

We will first write down and discuss the expressions for $V_1$ and $V_2$ for the 32-site system illustrated in figure 3. The construction easily generalizes for any even–even lattice.

$$V_1 = \sigma_x^1 \sigma_y^1 \sigma_x^2 \sigma_y^2$$  \hspace{1cm} (71)

$$V_2 = \sigma_1^x \sigma_2^y \sigma_3^x \sigma_4^y \prod_{i=2}^{7} \sigma_i^x \prod_{j=18}^{23} \sigma_j^y.$$  \hspace{1cm} (72)
The two Wilson loops for this lattice are

\[ W_1 = \sigma_1^y \sigma_2^y \sigma_3^y \sigma_4^y \sigma_5^y \sigma_6^y \sigma_7^y \sigma_8^y \]  

(73)

\[ W_2 = \prod_{i=1}^{8} \sigma_i^z. \]  

(74)

It can be verified that the above constructions satisfy the algebra in equation (68) and hence the topological operators defined in equation (69) satisfy the Clifford algebra (70). It can also be verified that \( V_i \) commute with all the \( B_p \).

Now consider simultaneous eigenstates of the Wilson loop operators

\[ W_i |w_1, w_2) = w_i |w_1, w_2), \quad w_j = \pm 1. \]  

(75)

The algebra in equation (68) implies

\[ V_i |w_1, w_2) = |w_1, w_2) \]  

\[ V_i |w_1, w_2) = |w_1, w_2), \]  

(76)

Thus we have shown that \( V_i \) are the generators of large gauge transformations.

We next consider their action on the Hamiltonian

\[ V_i H V_i^{-1} = H^i, \quad V_2 H V_2^{-1} = H^2, \]  

(77)

where \( H^1 \) is the same as \( H \), except that the bonds on one non-trivial loop in the \( e_1 \) direction have a changed sign, namely the loop \((1, 2, 3, 4, 5, 6, 7, 8, 1)\). In \( H^2 \), a line of parallel bonds in the \( e_1 + e_2 \) direction have the changed sign, namely the bonds \((7–8), (31–32), (23–24)\) and \((16–17)\). We will now write down the operators for a general even–even lattice and then show that the transformed Hamiltonians are degenerate in the thermodynamic limit. We will show that in the fermionized theory, these changes of sign can be absorbed into the single particle eigenfunctions of the Majorana fermions and change the single particle energy eigenvalues by \( \sim 1/L \). Thus the energies do not change in the thermodynamic limit, making every many-body eigenstate fourfold degenerate.

The general expressions for \( V_i(2) \) are given by

\[ V_1 = \prod_{m=1}^{M} \sigma_{m,0}^{b,z}, \]  

(78)

\[ V_2 = \prod_{n=1}^{N/2} \sigma_{0,2n}^{a,z} \sigma_{0,2n}^{b,y} \prod_{n=1,N/2}^{N} \sigma_{m,2n-1}^{a,z} \sigma_{m,2n-2}^{b,z}. \]  

It can be verified that these constructions satisfy the algebra in equation (68) and also commute with all \( B_p \). In \( H^1 \), the bonds \((m, 0, b) \rightarrow (m + 1, 1, a) \) and \((m, 1, a) \rightarrow (m, 0, b) \) change sign and in \( H^2 \), the bonds \((M – 1, n, b) \rightarrow (0, n + 1, a) \) change sign.

In the fermionized theory, the single particle eigenfunctions satisfy the equation

\[ \sum_j A_{ij} \phi_i^a = \epsilon^a \phi_i^a, \]  

(79)

where \( A_{ij} \) is an antisymmetric matrix coupling the nearest neighbours of the honeycomb lattice. The eigenvalues come in pairs and we denote \((\phi^a)^* = \phi^{-a}, \) \(\epsilon^{-n} = -\epsilon^n \). \( n \) will then go from 1, \(\ldots\), \( NM \). The Hamiltonian is diagonal in terms of the complex fermions defined by

\[ \alpha_n = \sum_i \phi_i^{a} \eta_i^{a} + \phi_i^{b} \eta_i^{b} \]  

(80)

\[ H = \sum_n \epsilon^n (2\alpha_n \alpha_n - 1). \]  

(81)
We now make the transformation
\[ \phi_{nm,a}^{\prime} = e^{i\pi N} \phi_{nm,a}. \]  
(82)

Equation (79) then gets written as
\[ \sum_j A_{ij} \phi_n^j = \sum_j (i A_{1ij} + i \delta A_{ij}) \phi_n^{j'}, = \epsilon_n \phi_n^{j'}, \]  
(83)

where \( A^1 \) is the antisymmetric matrix corresponding to \( H^1 \) and \( \delta A_{ij} \propto 1/N \) when \( N \) is very large. Thus the single particle energy eigenvalues of \( A \) and \( A^1 \) are identical in the thermodynamic limit when \( N \to \infty \). The spectrum of \( H \) and \( H^1 \) is also therefore identical.

The mapping of the eigenvalues of \( H \) and \( H^2 \) can be similarly shown using the transformation
\[ \phi_{nm,a}^{\prime} = e^{i\pi M} \phi_{nm,a}. \]  
(84)

We can also write equation (78) in terms of the gauge-invariant fermions. \( V_i \) or \( V_2 \) can be written as
\[ V_i = V_i^M V_i^G \]  
(85)

From the above expressions we can easily find the commutation relations of parity operator for gauge fermions and the matter fermions. We find that
\[ \{ P_G, V_1 \} = 0, \quad [ P_G, V_2 ] = 0, \quad [ P_M, V_1 ] = 0, \quad [ P_M, V_2 ] = 0. \]  
(86)

The fact that the parity of the matter fermions is conserved is consistent with the fourfold degeneracy discussed here and in section 5.1.

6. Discussion

In this paper we have discussed several important aspects of the Kitaev model. We have shown how the \( SU(2) \) gauge symmetry is contained in the Kitaev model. We show that the \( Z_2 \) gauge symmetry in Kitaev’s fermionization is a gauge-fixed version of the \( SU(2) \) symmetry, where the gauge has been fixed up to the centre symmetry. We have solved the Kitaev model using the Jordan–Wigner method in a general way. We have shown explicitly that the fermionized Hamiltonian obtained in the Jordan–Wigner transformation is exactly the Hamiltonian obtained by the Kitaev fermionization, in a particular gauge which we call the Jordan–Wigner gauge. We have explicitly constructed the generators of large gauge transformations on a torus in terms of the spin operators. This enabled us to give an algebraic proof of the fourfold degeneracy of all eigenstates on the torus in the thermodynamic limit. Our proof holds for both the phases of the time-reversal symmetric model.

Our analysis indicates that the Jordan–Wigner analysis can be used in quantum spin liquid problems, as a general method, to bring out non-trivial gauge field content, thereby providing an alternative method in resonating valence bond theories.
We note that for \( m \neq 0 \),
\[
\hat{t}^a_{1,m,n} = \hat{\gamma}; \quad \hat{t}^a_{2,m,n} = \hat{\zeta}; \quad \hat{n}^a_{1,m,n} = -\hat{\zeta},
\]
(A.1)
\[
\hat{t}^b_{1,m,n} = \hat{\xi}; \quad \hat{t}^b_{2,m,n} = \hat{\eta}; \quad \hat{n}^b_{1,m,n} = \hat{\eta}.
\]
(A.2)

Now for \( m = 0 \),
\[
\hat{t}^a_{1,m,n} = \hat{\gamma}; \quad \hat{t}^a_{2,m,n} = \hat{\zeta}; \quad \hat{n}^a_{1,m,n} = -\hat{\gamma},
\]
\[
\hat{t}^b_{1,m,n} = \hat{\xi}; \quad \hat{t}^b_{2,m,n} = \hat{\eta}; \quad \hat{n}^b_{1,m,n} = -\hat{\eta}.
\]

Then starting with the ferromagnetic Hamiltonian we obtain, for \( m = 0, n = 0 \),
\[
-\sigma_{0,0}^{ac} \sigma_{0,0}^{bc} = -i \eta_{0,0}^a \eta_{0,0}^b \mathcal{S}.
\]
(A.3)

For \( m \neq 0 \)
\[
-\sigma_{0,n}^{ac} \sigma_{n,0}^{bc} = -i \eta_{0,n}^a \eta_{n,0}^b (i \xi_{0,n}^a \xi_{n,0}^b).
\]
(A.4)

Now for \( m = 0, n \neq 0 \)
\[
-\sigma_{0,n}^{ac} \sigma_{0,n}^{bc} = i \eta_{0,n}^a \eta_{n,0}^b.
\]
(A.5)

The above three equations give a complete description of all the \( z-z \) interaction. Now for the \( y \)-bond we obtain for \( m = 0 \)
\[
-\sigma_{0,n}^{ay} \sigma_{0,n+1}^{by} = -i \eta_{0,n}^a \eta_{n,n+1}^b (i \xi_{0,n}^a \xi_{n,n+1}^b).
\]
(A.6)

For \( m \neq 0 \) we obtain
\[
-\sigma_{m,n}^{ay} \sigma_{m,n+1}^{by} = i \eta_{m,n}^a \eta_{m,n+1}^b.
\]
(A.7)

At last we write the Hamiltonian for the \( x \)-interaction. For \( m = 0 \)
\[
-\sigma_{0,n}^{ax} \sigma_{1,n+1}^{bx} = -i \eta_{0,n}^a \eta_{1,n+1}^b.
\]
(A.8)

For \( m \neq 0 \) we obtain
\[
-\sigma_{m,n}^{ax} \sigma_{m+1,n+1}^{bx} = -i \eta_{m,n}^a \eta_{m+1,n+1}^b.
\]
(A.9)

Now we make the following gauge transformation which we call the Jordan–Wigner gauge.
\[
\eta_{m,n}^a \rightarrow (-1)^m \eta_{m,n}^a.
\]
(A.10)

Then we choose for each normal bond \( \tilde{\eta}_{ij} = -i \xi_{m}^{a} \xi_{m}^{b} \). This gives the Jordan–Wigner Hamiltonian given by equations (46)–(48). Then we define complex fermion \( \chi_{m,n} \) on each normal internal \( z \)-bond in the following way:
\[
\xi_{m,n}^a = (\chi_{m,n} + \chi_{m,n}^\dagger); \quad \xi_{m,n}^b = \frac{1}{i} (\chi_{m,n} - \chi_{m,n}^\dagger).
\]
(A.11)
Similarly on each normal slanted $y$-bond which is joined with a $z$-link $(m, n)$ and $(m, n + 1)$ we define
\[
\xi^u_{m,n+1} = \left(\frac{X_{n,n}^+ + X_{n,n+1}^+}{A_{n,n}^+} \right) \quad \text{and} \quad \xi^b_{m,n} = -\frac{1}{i} \left(\frac{X_{n,n}^+ - X_{n,n+1}^+}{A_{n,n}^+} \right).
\] (A.12)

With this we have always $\tilde{u}_{i,j} = (2\chi_{i,j}^+X_{i,j}^+ - 1)$. At last on each $z$-link we define $\psi$ in the following way:
\[
\eta^u_{m,n} = (\psi_{m,n} + \psi_{m,n}^\dagger); \quad \eta^b_{m,n} = \frac{1}{i} (\psi_{m,n} - \psi_{m,n}^\dagger).
\] (A.13)

Now the quantity $S$ for lattice of dimension $(M, N)$ is given by $S = -(1)^{MN+N_g+N_s}$. Here $N_g$ and $N_s$ are the number of $\psi$ and $\chi$ fermions, respectively. Now noting the number of gauge transformations needed for various flux-free configurations, we find that we need to fill an even number of $\psi$ fermions for each of the different gauge choices. All the results derived here are based on a representative lattice of dimensions $M$ and $N$, where $M$ and $N$ are both even. It is straightforward to carry the analogous calculation for lattice where $M$ and $N$ can be anything; odd or even. However the results obtained here should not change in the thermodynamic limit.

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