Higher order infinitesimal freeness

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Abstract

We define higher order infinitesimal noncommutative probability space and infinitesimal non-crossing cumulant functionals. In this framework, we generalize to higher order the notion of infinitesimal freeness, via a vanishing of mixed cumulants condition. We also introduce and study some non-crossing partitions related to this notion. Finally, as an application, we show how to compute the successive derivatives of the free convolution of two time-indexed families of distributions from their individual derivatives.

1. Introduction

Free probability theory was introduced by Voiculescu in the eighties with motivations from operator algebras [16], but many connections to other fields of mathematics like random matrices (see [18]) or combinatorics appeared. The combinatorial side of free probability, as noticed by Speicher [15], is linked to the convolution on the lattices of non-crossing partitions, which have been first studied by Kreweras [8]. Biane proved in [4] that these lattices of non-crossing partitions can be embedded into the Cayley graphs of the symmetric groups, also known as the type A in the classification of finite reflection groups. Reiner introduced non-crossing partitions related to other types in this classification [14].

In [3], the authors showed that it is possible to build a free probability theory of type B, by replacing the occurrences of the symmetric groups and the non-crossing partitions of type A by their type B analogues, namely the hyperoctaedral groups and the non-crossing partitions of type B. In their work, a central role is played by the boxed convolution which is a combinatorial operation having a natural type B analogue and describing the multiplication of two freely independent noncommutative random variables. The specificity of the boxed convolution of type B led the authors to define a noncommutative probability space of type B as a system \((\mathcal{A}, \varphi, \mathcal{V}, f, \Phi)\), where \((\mathcal{A}, \varphi)\) is a noncommutative probability space, \(\mathcal{V}\) is a complex vector space, \(f : \mathcal{V} \rightarrow \mathbb{C}\) is a...
linear functional, $\Phi : A \times V \times A \rightarrow V$ is a two-sided action of $A$ on $V$. A type B noncommutative random variable is therefore a couple $(a, \xi) \in A \times V$, its distribution is $\mathbb{C}^2$-valued and the non-crossing cumulant functionals of type B introduced in \[3\] are also with values in $\mathbb{C}^2$. An important remark is that the first component of a non-crossing cumulant of type $B$ in $(A, \varphi, V, f, \Phi)$ is simply a non-crossing cumulant of type $A$ in $(A, \varphi)$. It follows that the notion of freeness of type $B$ for $(A_1, V_1), \ldots, (A_m, V_m)$ in $(A, \varphi, V, f, \Phi)$, defined in \[3\] in terms of moments to ensure that the vanishing of mixed cumulants of type $B$ holds, implies the freeness of $A_1, \ldots, A_m$ in $(A, \varphi)$. The free additive convolution of type $B$, denoted by $\boxplus(B)$, which describes the distribution of the sum of two type $B$ noncommutative random variables that are free of type $B$, is an operation on the set of couples $(\mu, \mu')$ of linear functionals on $\mathbb{C}[X]$ satisfying $\mu(1) = 1$ and $\mu'(1) = 0$. Later, Popa stated in \[13\] type $B$ versions of usual limit theorems and defined a $S$-transform for noncommutative random variables of type $B$.

Recently, the analytic aspects of free probability theory of type $B$ were investigated in \[2\] ; in particular, the authors outlined an interesting application of the free probability of type $B$ that they called infinitesimal freeness: defining (when they exist) the zeroth and first derivatives at $0$ of a time-indexed family of distributions $(\mu_t)_{t>0}$ as the couple of distributions $(\mu(0), \mu(1))$ defined by

$$\mu^{(0)} = \lim_{t \to 0} \mu_t$$

and

$$\mu^{(1)} = \frac{d}{dt}_{|t=0} \mu_t = \lim_{t \to 0} \frac{1}{t} (\mu_t - \mu^{(0)}),$$

they prove that, given two such time-indexed families of distributions $(\mu_t)_{t>0}$ and $(\nu_t)_{t>0}$, the zeroth and first derivatives at $0$ of $\mu_t \boxplus \nu_t$, denoted by $((\mu \boxplus \nu)^{(0)}, (\mu \boxplus \nu)^{(1)})$, satisfy :

$$((\mu \boxplus \nu)^{(0)}, (\mu \boxplus \nu)^{(1)}) = (\mu^{(0)}, \mu^{(1)}) \boxplus(B) (\nu^{(0)}, \nu^{(1)}).$$

Following this insight, a new approach of free probability of type $B$ was developed in \[6\], named infinitesimal freeness. The equivalent structures considered there, simplifying and generalizing the noncommutative probability space of type $B$ from \[3\], are the infinitesimal noncommutative probability space $(A, \varphi, \varphi')$ consisting in a noncommutative probability space $(A, \varphi)$ to which has been added another linear functional $\varphi'$ on $A$ satisfying $\varphi'(1_A) = 0$, and the scarce $G$-noncommutative probability space $(A, \hat{\varphi})$, where $\hat{\varphi}$ is a linear map which consolidates the two functionals $\varphi, \varphi'$ in a single one from $A$ into a two-dimensional Grassman algebra $G$ generated by an element $\varepsilon$ which satisfies $\varepsilon^2 = 0$. A scarce $G$-noncommutative probability space appears in the framework of a noncommutative probability space of type $B$ $(A, \varphi, \mathcal{V}, f, \Phi)$ when one considers the link-algebra $A \times \mathcal{V}$ together with the map $(\varphi, f)$. Infinitesimal freeness of unital subalgebras $A_1, \ldots, A_m$ of an infinitesimal noncommutative probability space $(A, \varphi, \varphi')$ is defined as the rewriting of the condition defining freeness of type $B$ in a more general context. More precisely,
\(A_1, \ldots, A_n\) are infinitesimally free if whenever \(i_1, \ldots, i_n \in \{1, \ldots, k\}\) are such that \(i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n\), and \(a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}\) are such that 
\[
\varphi(a_1) = \cdots = \varphi(a_n) = 0, \quad \text{then} \quad \varphi(a_1 \cdots a_n) = 0
\]
and
\[
\varphi'(a_1 \cdots a_n) = \begin{cases} 
\varphi(a_1 a_n) \varphi(a_2 a_{n-1}) \cdots \varphi(a_{(n-1)/2} a_{(n+3)/2}) \cdot \varphi'(a_{(n+1)/2}), & \text{if } n \text{ is odd and } i_1 = i_n, i_2 = i_{n-1}, \ldots, i_{(n-1)/2} = i_{(n+3)/2}, \\
0, & \text{otherwise.}
\end{cases}
\]

It is clear from this definition that infinitesimally free unital subalgebras of an infinitesimal noncommutative probability space \((A, \varphi, \varphi')\) are in particular free in \((A, \varphi)\). A converse is proved in [6]: given free unital subalgebras \(A_1, \ldots, A_m\) of a noncommutative probability space \((A, \varphi)\), \(A_1, \ldots, A_m\) are infinitesimally free in the infinitesimal noncommutative probability space \((A, \varphi, \varphi')\), for instance when we set \(\varphi' := \varphi \circ D\), where \(D : A \rightarrow A\) is a derivation such that \(\forall 1 \leq i \leq m, D(A_i) \subseteq A_i\). Moreover, a method is presented to obtain analogues in the framework of interest of an infinitesimal noncommutative probability space \((A, \varphi, \varphi')\) for results already established in usual free probability. This method is roughly to work in \((A, \tilde{\varphi})\), where the computations are easy in the sense that the combinatorics is exactly the same as in usual noncommutative probability space, and to take advantage of the equivalence between the structures \((A, \tilde{\varphi})\) and \((A, \varphi, \varphi')\). This method is applied in [6] to find the right notion of infinitesimal non-crossing cumulant functional, and to compute the formulas for alternating products of infinitesimally free noncommutative random variables. These formulas make the non-crossing partitions of type B appear, as a reminder of the type B origin of infinitesimal freeness. The present work is in the lineage of [6].

With the motivation to obtain higher order derivatives at 0 of \(\mu_t \boxplus \nu_t\) from those of \(\mu_t\) and \(\nu_t\), we generalize indeed to higher order the notion of infinitesimal noncommutative probability space from [6], by adding to the noncommutative probability space \((A, \varphi^{(0)})\) a certain number \(k\) of other linear functionals \((\varphi^{(i)})_{1 \leq i \leq k}\) on \(A\) satisfying \(\varphi^{(i)}(1_A) = 0\). Following the same idea as [6], some formulas, the infinitesimal analogue of the free moment-cumulant formula for instance, will be simplified in the equivalent scarce \(C_k\) structure \((A, \tilde{\varphi})\), where the \(k + 1\) linear functionals \((\varphi^{(i)})_{0 \leq i \leq k}\) are consolidated in a unique linear map \(\tilde{\varphi}\), but with values in a certain \((k + 1)\)-dimensional algebra \(C_k\). The main benefit coming from this trick is that the formulas in \((A, \tilde{\varphi})\) are the same as in usual free probability, with the only difference that they take place in the \((k + 1)\)-dimensional algebra \(C_k\) instead of the field of complex numbers. In what follows, we will continuously switch from the infinitesimal framework \((A, (\varphi^{(i)})_{0 \leq i \leq k})\) which is the one of interest to the scarce \(C_k\)-structure \((A, \tilde{\varphi})\) which is handy because the computations are easier in it.

As noticed above, in infinitesimal freeness from [6], some formulas involving \(\varphi'\) also involve the lattices of non-crossing partitions of type B, due to the link of infinitesimal freeness with free probability of type B pointed out in [6]. In higher order infinitesimal freeness, new non-crossing partitions appear in the formulas involving \(\varphi^{(k)}\). These so-called non-crossing partitions of type \(k\), generalizing both non-crossing partitions of type A (corresponding to the case \(k = 0\) and
type B (corresponding to the case \( k = 1 \)), are introduced and studied in Section 6.

Our approach is in a sense the opposite of the approach in [3]. Indeed, in [3], the authors substitute the symmetric groups by the hyperoctahedral groups, and by the way non-crossing partitions of type A by their type B analogues and thus they obtain the noncommutative probability space of type B. In the present work, we directly substitute the noncommutative probability space by the \( k \)-th order infinitesimal noncommutative probability space, and we look for the non-crossing partitions of type \( k \) appearing this way.

Following this introduction, the paper is divided in seven other sections. In Section 2, we introduce the two equivalent notions of infinitesimal noncommutative probability space of order \( k \) and of scarce \( C_k \)-noncommutative probability space and discuss their relations with other structures. In Section 3, we introduce infinitesimal non-crossing cumulant functionals of order \( k \), and define infinitesimal freeness of order \( k \) by a condition of vanishing mixed cumulants. Section 4 is devoted to the addition and multiplication of infinitesimally free variables. The formula expressing the infinitesimal cumulants of the product of two infinitesimally free noncommutative random variables may be written as a sum on certain non-crossing partitions generalizing the non-crossing partitions of type B reviewed in Section 5. These special non-crossing partitions, called non-crossing partitions of type \( k \), and the boxed convolution operation associated to them are introduced and studied in Sections 6 and 7. Finally, we give in Section 8 an important application of higher order infinitesimal freeness: a recipe for computing the higher order derivatives of the free convolutions of two distributions.

2. Infinitesimal noncommutative probability space of order \( k \)

Throughout the paper, the integer \( k \in \mathbb{N} \) is fixed. In this section, we introduce the two equivalent structures of infinitesimal noncommutative probability space and of scarce \( C_k \) noncommutative probability space and we discuss their relations to previously defined structures.

2.1 Infinitesimal noncommutative probability space of order \( k \)

The object of this subsection is to introduce the structure which is the framework for our notion of infinitesimal freeness of order \( k \), namely the infinitesimal noncommutative probability space of order \( k \).

**Definition 1.** We call infinitesimal noncommutative probability space of order \( k \) a structure \((\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})\) where \( \mathcal{A} \) is a unital algebra over \( \mathbb{C} \), \( \varphi^{(0)} : \mathcal{A} \to \mathbb{C} \) is a linear map with \( \varphi^{(0)}(1_{\mathcal{A}}) = 1 \), and \( \varphi^{(i)} : \mathcal{A} \to \mathbb{C}, 1 \leq i \leq k \), are linear maps with \( \varphi^{(i)}(1_{\mathcal{A}}) = 0 \).

**Remark 1.** The notion of infinitesimal noncommutative probability space of order 1 coincides with the notion of infinitesimal noncommutative probability space introduced in [6]. The structure defined above is therefore a generalization of this latter object, and the use of the adjective infinitesimal is justified.
An element \( a \in (\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k}) \) of an infinitesimal noncommutative probability space of order \( k \) is called an **infinitesimal noncommutative random variable of order** \( k \). The **infinitesimal distribution of order** \( k \) of a \( n \)-tuple \((a_1, \ldots, a_n) \in \mathcal{A}^n\) of infinitesimal noncommutative random variables of order \( k \) is the \((k + 1)\)-tuple \((\mu^{(i)})_{0 \leq i \leq k}\) of linear functionals on \( \mathbb{C}[X_1, \ldots, X_n] \) defined by:

\[
\mu^{(i)}(P(X_1, \ldots, X_n)) := \varphi^{(i)}(P(a_1, \ldots, a_n)).
\]

The range of infinitesimal distributions is the set of **infinitesimal laws of order** \( k \), introduced below.

**Definition 2.** An infinitesimal law (of order \( k \)) on \( n \) variables is a \((k + 1)\)-tuple of linear functionals \((\mu^{(i)})_{0 \leq i \leq k}\), where \( \mu^{(i)} : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C} \) is defined on the algebra of noncommutative polynomials and satisfies \( \mu^{(i)}(1) = \delta_0^i \).

For some purposes, it is handy to consider, instead of \( k + 1 \) linear functionals as in Definition 1, an equivalent unique linear map with values in a \((k + 1)\)-dimensional algebra. The relevant algebra, denoted by \( \mathbb{C}_k \), is described below.

### 2.2 The algebra \( \mathbb{C}_k \)

In [6], the two linear maps \( \varphi \) and \( \varphi' \) of an infinitesimal noncommutative probability space \((\mathcal{A}, \varphi, \varphi')\) are consolidated in a single linear map \( \tilde{\varphi} \) on \( \mathcal{A} \) with values in the two-dimensional Grassman algebra \( \mathbb{G} \) generated by an element \( \varepsilon \) which satisfies \( \varepsilon^2 = 0 \):

\[
\mathbb{G} = \{ \alpha + \varepsilon \beta \mid \alpha, \beta \in \mathbb{C} \}.
\]

This algebra has a quite natural \((k + 1)\)-dimensional generalization introduced below.

**Definition 3.** Let \( \mathbb{C}_k \) denote the \((k + 1)\)-dimensional complex algebra \( \mathbb{C}^{k+1} \) with usual vector space structure and multiplication given by the following rule: if \( \alpha = (\alpha^{(0)}, \ldots, \alpha^{(k)}) \in \mathbb{C}_k \) and \( \beta = (\beta^{(0)}, \ldots, \beta^{(k)}) \in \mathbb{C}_k \), then

\[
\alpha \cdot \beta = (\gamma^{(0)}, \ldots, \gamma^{(k)})
\]

is defined by

\[
\gamma^{(i)} := \sum_{j=0}^{i} C^i_j \alpha^{(j)} \beta^{(i-j)}.
\]

The algebra \( \mathbb{C}_k \) is a unital complex commutative algebra. Its unit is \( 1_{\mathbb{C}_k} = (1, 0, \ldots, 0) \). An element is invertible in the algebra \( \mathbb{C}_k \) if and only if its first coordinate is non-zero.

The analogy between formula (1) defining the product in \( \mathbb{C}_k \) and the well-known Leibniz rule giving the recipe for computing the derivatives of the product of two smooth functions makes it easy to establish the formula for the product
\[ \beta = \alpha_1 \cdots \alpha_n \text{ of } n \text{ elements } \alpha_1, \ldots, \alpha_n \in \mathcal{C}_k. \] Precisely, if \( \alpha_j = (\alpha_j^{(0)}, \ldots, \alpha_j^{(k)}) \) and \( \beta = (\beta^{(0)}, \ldots, \beta^{(k)}) \), one has

\[ \beta^{(i)} = \sum_{\lambda \in \Lambda_{n,i}} C_1^{\lambda_1} \cdots C_n^{\lambda_n} \prod_{j=1}^n \alpha_j^{(\lambda_j)}, \]

where

\[ C_1^{\lambda_1} \cdots C_n^{\lambda_n} = \frac{i!}{\lambda_1! \cdots \lambda_n!}, \]

and

\[ \Lambda_{n,i} := \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum_{j=1}^n \lambda_j = i \}. \quad (2) \]

There is an alternative description of the algebra \( \mathcal{C}_k \): it may be identified with the algebra of \((k + 1)\)-by-\((k + 1)\) upper triangular Toeplitz matrices (with usual matricial operations) as follows:

\[
\begin{pmatrix}
\alpha^{(0)} & \alpha^{(1)} & \cdots & \alpha^{(k-1)} & \alpha^{(k)} \\
0 & \alpha^{(0)} & \cdots & \cdots & \frac{\alpha^{(k-1)}}{(k-1)!} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \alpha^{(0)} & \frac{\alpha^{(1)}}{(1)!} \\
0 & 0 & \cdots & 0 & \alpha^{(0)}
\end{pmatrix}.
\]

Consider

\[
\varepsilon := \begin{pmatrix}
 0 & 1 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 \vdots & \vdots & \cdots & 0 & 1 \\
 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

It is easy to compute the values of \( \varepsilon^i \) for \( 0 \leq i \leq k + 1 \); in particular \( \varepsilon^{k+1} = 0_{\mathcal{C}_k} \) and any element \( \alpha = (\alpha^{(0)}, \ldots, \alpha^{(k)}) \in \mathcal{C}_k \) may be uniquely decomposed

\[
\alpha = \sum_{i=0}^k \alpha^{(i)} \frac{\varepsilon^i}{i!}. \quad (3)
\]

The family \( \{ \varepsilon^i, 0 \leq i \leq k \} \) is thus a linear basis of \( \mathcal{C}_k \), to which we will refer as the canonical basis of \( \mathcal{C}_k \). In particular, \( \mathcal{C}_k \simeq \mathbb{C}[\varepsilon] = \mathbb{C}_k[\varepsilon] \simeq \mathbb{C}[X]/(X^{k+1}) \).

In the definition of a usual noncommutative probability space, if one asks for the state to be \( \mathcal{C}_k \)-valued, one obtains a slightly different structure, introduced in the next section.

2.3 Scarce \( \mathcal{C}_k \)-noncommutative probability space

Definition 4. By scarce \( \mathcal{C}_k \)-noncommutative probability space, we mean a couple \( (\mathcal{A}, \hat{\varphi}) \), where \( \mathcal{A} \) is a unital algebra over \( \mathbb{C} \) and \( \hat{\varphi} : \mathcal{A} \to \mathcal{C}_k \) is a linear map satisfying \( \hat{\varphi}(1_\mathcal{A}) = 1_{\mathcal{C}_k} \).
Remark 2. The notion of scarce noncommutative probability space was introduced in [12], but only the particular case of scarce $G$-noncommutative probability space was considered there. This same structure has been studied later in [6] in connection with infinitesimal noncommutative probability space and free probability of type B.

Remark 3. To any infinitesimal noncommutative probability space of order $k$ $(\mathcal{A},(\varphi^{(i)})_{0 \leq i \leq k})$, we may associate a natural scarce $C_k$-noncommutative probability space $(\mathcal{A},\tilde{\varphi})$, by putting

$$\tilde{\varphi} := \sum_{i=0}^{k} \varphi^{(i)} \frac{x^i}{i!}$$ \hspace{1cm} (4)

Reciprocally, given a scarce $C_k$-noncommutative probability space $(\mathcal{A},\tilde{\varphi})$, the linear decomposition of $\tilde{\varphi}$ in the canonical basis of $C_k$ (see equation (4)) gives rise to $k + 1$ linear functionals $(\varphi^{(i)})_{0 \leq i \leq k}$, and consequently to an infinitesimal noncommutative probability space of order $k$ : $(\mathcal{A},(\varphi^{(i)})_{0 \leq i \leq k})$.

The equivalence between the infinitesimal noncommutative probability space of order $k$ $(\mathcal{A},(\varphi^{(i)})_{0 \leq i \leq k})$ and its associated scarce $C_k$-noncommutative probability space $(\mathcal{A},\tilde{\varphi})$ is fundamental in what follows. Indeed, we will continuously switch from one structure to the other, according to the principle that our interest is in the infinitesimal structure whereas the computations are easier in the scarce $C_k$ structure, in the sense that they mimetize those from usual free probability.

An element $a$ of a scarce $C_k$-noncommutative probability space $(\mathcal{A},\tilde{\varphi})$ is called a $C_k$-noncommutative random variable. We associate to such an $a \in \mathcal{A}$ the sequence of its $C_k$-valued moments $(\tilde{\varphi}(a^n))_{n \in \mathbb{N}^*}$. We call $C_k$-valued distribution of $a$ the whole sequence of its moments, or equivalently, the linear map from $\mathbb{C}[X]$ into $C_k$ which maps any polynomial $P$ to $\tilde{\varphi}(P(a))$. One may find easier to collect all the $C_k$-valued moments in a formal power series, as follows :

**Definition 5.** Let $C$ be a unital commutative algebra over $\mathbb{C}$. We denote by $\Theta_C^{(A)}$ the set of power series of the form

$$f(z) = \sum_{n=1}^{\infty} \alpha_n z^n,$$

where the $\alpha_n$’s are elements of $C$.

**Definition 6.** Let $(\mathcal{A},\tilde{\varphi})$ be a scarce $C_k$-noncommutative probability space. The $C_k$-valued moment series of $a \in \mathcal{A}$ is the power series $\tilde{M}_a \in \Theta_C^{(A)}$ defined as follows:

$$\tilde{M}_a(z) := \sum_{n=1}^{\infty} \tilde{\varphi}(a^n) z^n.$$

The notion of $C_k$-valued distribution is easily generalized to $n$-tuples of variables :
Definition 7. The $\mathcal{C}_k$-valued distribution of a $n$-tuple $(a_1, \ldots, a_n) \in \mathcal{A}^n$ of $\mathcal{C}_k$-noncommutative random variables in a scarce $\mathcal{C}_k$-noncommutative probability space $(\mathcal{A}, \hat{\varphi})$ is the linear map $\hat{\mu}_{(a_1, \ldots, a_n)} : \mathbb{C}(X_1, \ldots, X_n) \to \mathcal{C}_k$ defined by

$$
\hat{\mu}_{(a_1, \ldots, a_n)}(P(X_1, \ldots, X_n)) := \hat{\varphi}(P(a_1, \ldots, a_n)).
$$

As mentioned in [6], scarce $\mathcal{G}$-noncommutative probability space and infinitesimal noncommutative probability space provide a nice framework to do free probability of type B. The equivalent structures defined above are therefore the natural setting for generalizing free probability of type B. There is another structure linked to free probability of type B that one may find interesting to generalize here: the noncommutative probability space of type B, proposed in [3]. Its natural generalization is the noncommutative probability space of type $k$:

Definition 8. By a noncommutative probability space of type $k$ we understand a system $(\mathcal{V}^{(0)}, f^{(0)}, \ldots, \mathcal{V}^{(k)}, f^{(k)}, (\Phi_{i,j})_{0 \leq i,j \leq k})$, where $(\mathcal{V}^{(0)}, f^{(0)})$ is a noncommutative probability space of type $A$, $\mathcal{V}^{(i)}$, $1 \leq i \leq k$, are complex vector spaces, $f^{(i)} : \mathcal{V}^{(i)} \to \mathbb{C}$, $1 \leq i \leq k$, are linear maps, $\Phi_{i,j} : \mathcal{V}^{(i)} \times \mathcal{V}^{(j)} \to \mathcal{V}^{(i+j)}$, $0 \leq i, j \leq k$, are bilinear maps satisfying

$$
\Phi_{h+i,j}(\Phi_{h,i}(x,y), z) = \Phi_{h,i+j}(x, \Phi_{i,j}(y,z)),
$$

$\forall h, i, j \in \mathbb{N}, \forall x \in \mathcal{V}^{(h)}, \forall y \in \mathcal{V}^{(i)}, \forall z \in \mathcal{V}^{(j)}$.

To make the preceding definition work, we put $\mathcal{V}^{(i)} = \{0\}$, when $i \geq k + 1$. The following fact noticed in [3] still holds: noncommutative probability spaces of type $k$ are particular cases of scarce $\mathcal{C}_k$-noncommutative probability spaces. Indeed, given a noncommutative probability space of type $k$ $(\mathcal{V}^{(0)}, f^{(0)}, \ldots, \mathcal{V}^{(k)}, f^{(k)}, (\Phi_{i,j})_{0 \leq i,j \leq k})$, the direct product $\prod_{i=0}^{k} \mathcal{V}^{(i)}$ can be endowed with a complex unital algebra structure, via the maps $(\Phi_{i,j})_{0 \leq i,j \leq k}$. This algebra, together with the linear map $\hat{\varphi}(x_0, \ldots, x_k) := (f^{(0)}(x_0), \ldots, f^{(k)}(x_k))$, forms a scarce $\mathcal{C}_k$-noncommutative probability space.

There are natural equivalent notions of freeness on the structures introduced above, generalizing both infinitesimal freeness from [2] and [3] and freeness of type B from [3]. In [3], infinitesimal freeness in $(\mathcal{A}, \varphi, \varphi')$ is defined by two conditions on the linear functionals $\varphi, \varphi'$; its generalization to an infinitesimal noncommutative probability space of order $k$ denoted by $(\mathcal{A}, (\varphi^{(i)})_{0 \leq i \leq k})$ would require $k + 1$ conditions on the linear functionals $(\varphi^{(i)})_{0 \leq i \leq k}$. Infinitesimal freeness from [3] being also equivalent to the vanishing of the infinitesimal non-crossing cumulants, we adopt this approach and define the infinitesimal freeness of order $k$ by the vanishing of some multilinear functionals, called infinitesimal non-crossing cumulant functionals of order $k$ and introduced in the next section.

3. Infinitesimal non-crossing cumulants of order $k$

We begin this section by reviewing some background on non-crossing partitions.
3.1 Miscellaneous facts on non-crossing partitions of type A

A partition $p$ of a finite set $X$ is a family of disjoint non-empty subsets of $X$, called the blocks of $p$, whose union is $X$. The set of blocks of a partition $p$ of $X$ will be denoted throughout these notes by $\text{bl}(p)$; its cardinal by $|p|$. For $a$ and $b$ in $X$, we write $a \sim_p b$ and say that $a$ and $b$ are linked (in $p$) to denote that $a$ and $b$ are in the same block of the partition $p$ of $X$. The set of partitions of a finite set $X$ together with the reverse refinement order ($p \preceq q$ if every block of $p$ is contained in a block of $q$) is a lattice.

Now suppose $(X, \leq)$ is a totally ordered set. A partition $p$ of $X$ is called non-crossing if, whenever you have $a < b < c < d$ in $X$ such that $a \sim_p c$ and $b \sim_p d$, then $a \sim_p b$.

The set $(\text{NC}^{(A)}(X), \preceq)$ of non-crossing partitions of $X$ together with the reverse refinement order is itself a lattice. Its maximal element $1_X$ has $X$ as its only block; its minimal element $0_X$ has every singleton as a block.

When $X = [m] := \{1 < \ldots < m\}$, we write $\text{NC}^{(A)}(m)$ instead of $\text{NC}^{(A)}([m])$. A nice way to represent a non-crossing partition $p \in \text{NC}^{(A)}(m)$ is to view $1, \ldots, m$ as equidistant clockwisely ordered points on a circle, and to draw for each block of $p$ the convex polygon whose vertices are the elements of this block. It is a necessary and sufficient condition for a partition to be non-crossing that the polygons built this way do not intersect.

Biane found in [4] a bijection between the set of non-crossing partitions of $[m]$ and the set of points lying on a geodesic in the Cayley graph of the symmetric group $S_m$ with generators the set of all transpositions. This bijection $t$ associates to any non-crossing partition $p \in \text{NC}^{(A)}(m)$ the permutation $t(p) \in S_m$ whose restriction to each block $V$ of $p$ is the trace of the cycle $(1, \ldots, m) \in S_m$ on $V$. For $a \in [m]$, $t(p)(a)$ is called the neighbour of $a$ in $p$. Geometrically, it is the first point linked to $a$ that one meets when one goes clockwisely around the circle, starting from $a$.

Let us recall that the Kreweras complementation map, denoted by $\text{Kr}$, is the anti-isomorphism of the lattice $\text{NC}^{(A)}(m)$ of non-crossing partitions of $[m]$ introduced by Kreweras in [8] and defined in the following way: consider a copy $[m] := \{\overline{1} < \ldots < \overline{m}\}$ of $[m]$ and order the elements of $[m] \cup [m]$ as follows:

$$\{1 < \overline{1} < \ldots < m < \overline{m}\}.$$ 

Given $p$ a non-crossing partition of $[m]$, $\text{Kr}(p)$ is the biggest (for the reverse refinement order) partition of $[m]$ such that $p \cup \text{Kr}(p)$ is a non-crossing partition of $[m] \cup [m]$. See [10] for a nice geometric construction of the Kreweras complement.

**Remark 4.** On $[m] \cup [m]$, we could have considered the alternative order

$$\{\overline{1} < 1 < \overline{2} < \ldots < \overline{m} < m\}.$$
This would have led to another anti-isomorphism of $NC^{(A)}(m)$, also called Kreweras complementation map and denoted $Kr'$, which turns out to be the inverse of $Kr$.

There is an important equality (see [8]) verified by the number of blocks of the Kreweras complement of a non-crossing partition:

$$|p| + |Kr(p)| = m + 1, \forall p \in NC^{(A)}(m). \quad (5)$$

Notice that, for $p \in NC^{(A)}(m)$, $Kr^2(p)$ can be easily described in the geometric representation given above: $Kr^2(p)$ is the anti-clockwise rotation of $p$ with angle $\frac{2\pi}{m}$.

We conclude this subsection by the introduction of a total order on the blocks of a fixed non-crossing partition $p$ of $[m]$.

**Definition 9.** Let $p \in NC^{(A)}(m)$, and $V, W \in bl(p)$.

1° $V$ is said to be nested in $W$ if $\min W < \min V \leq \max V < \max W$.

2° $V$ is said to be on the left of $W$ if $\max V < \min W$.

3° $V \sqsubseteq W$ if $V$ is nested in $W$ or $V$ is on the left of $W$.

The proof of the next proposition is trivial and left to the reader.

**Proposition 1.** $\sqsubseteq$ is a total order on $bl(p)$.

If $p \in NC^{(A)}(m)$, we have seen that $p \cup Kr(p)$ is a non-crossing partition of $[m] \cup [m]$ in $m + 1$ blocks. These blocks will be listed in two different ways.

The first way is to list them all together in the increasing order $\sqsubseteq$: we will write Mix($p, i$) for the $i$-th block of $p \cup Kr(p)$ in the increasing order $\sqsubseteq$, for $1 \leq i \leq m + 1$.

For some purposes, it is nice to list separately the blocks of $p$ and of $Kr(p)$, and we will write Sep($p, i$) to denote the $i$-th block of $p$ in the increasing order $\sqsubseteq$ if $1 \leq i \leq |p|$ and to denote the $(i - |p|)$-th block of $Kr(p)$ in the increasing order $\sqsubseteq$ if $|p| + 1 \leq i \leq m + 1$.

It is interesting to look at the first blocks in the two resulting lists: Mix($p, 1$) is a singleton in $[m] \cup [m]$, Sep($p, 1$) is an interval in $[m]$. In particular, we can deduce the well-known fact that a non-crossing partition always owns an interval-block.

### 3.2 $C_k$-non-crossing cumulant functionals

In this subsection, we define non-crossing cumulant functionals in the framework of a scarce $C_k$-noncommutative probability space by the free moment-cumulant formula from usual free probability, with the only difference that the computations take place in the algebra $C_k$ instead of the field of complex numbers $\mathbb{C}$. The following notations are commonly used in the combinatorial theory of free probability.

**Notation 1.** Let $(a_1, \ldots, a_n) \in A^n$, and let $V = \{v_1 < \ldots < v_m\} \subseteq [n]$, then we denote

$$(a_1, \ldots, a_n) \mid V := (a_{v_1}, \ldots, a_{v_m}) \in A^m.$$
For a family of multilinear maps \((r_n : \mathcal{A}^n \to \mathcal{C}_k)_{n=1}^\infty\), we define for any \(n \in \mathbb{N}\) and any \(\pi \in NC^{(\Lambda)}(n)\) the \(n\)-linear functional \(r_\pi : \mathcal{A}^n \to \mathcal{C}_k\) by
\[
r_\pi(a_1, \ldots, a_n) := \prod_{V \in \pi} r_{|V|}((a_1, \ldots, a_n) \mid V).
\]

**Definition 10.** Let \((\mathcal{A}, \tilde{\phi})\) be a scarce \(\mathcal{C}_k\)-noncommutative probability space. The \(\mathcal{C}_k\)-non-crossing cumulant functionals are a family of multilinear maps \((\tilde{\kappa}_n : \mathcal{A}^n \to \mathcal{C}_k)_{n=1}^\infty\), uniquely determined by the following equation: for every \(n \geq 1\) and every \(a_1, \ldots, a_n \in \mathcal{A}\),
\[
\sum_{p \in NC^{(\Lambda)}(n)} \tilde{\kappa}_p(a_1, \ldots, a_n) = \tilde{\phi}(a_1 \cdot \ldots \cdot a_n).
\]

(6)

In free probability of type A, the formula above is known as the free moment-cumulant formula [7]. The only difference is that computations here take place in the unital commutative complex algebra \(\mathcal{C}_k\) instead of \(\mathbb{C}\). However, the proofs (see [11]) of the following classical results remain valid in this setting. That is why we record them without proof.

For every \(n \geq 1\) and every \(a_1, \ldots, a_n \in \mathcal{A}\) we have that:
\[
\tilde{\kappa}_n(a_1, \ldots, a_n) = \sum_{p \in NC^{(\Lambda)}(n)} \text{Möb}(p, 1_n) \tilde{\phi}_p(a_1, \ldots, a_n),
\]

(7)

where Möb is the Möbius function of the lattice of non-crossing partitions.

Obviously, the multilinear maps \((\tilde{\phi}_n : \mathcal{A}^n \to \mathcal{C}_k)_{n=1}^\infty\) implicitly used in formula (7) are defined by \(\tilde{\phi}_n(a_1, \ldots, a_n) = \tilde{\phi}(a_1 \cdot \ldots \cdot a_n)\).

**Proposition 2.** One has that \(\tilde{\kappa}_n(a_1, \ldots, a_n) = 0\) whenever \(n \geq 2\), \(a_1, \ldots, a_n \in \mathcal{A}\), and there exists \(1 \leq i \leq n\) such that \(a_i \in \mathcal{C}1_\mathcal{A}\).

**Proposition 3.** Let \(x_1, \ldots, x_s\) be in \(\mathcal{A}\) and consider some products of the form
\[
a_1 = x_1 \cdot \ldots \cdot x_{s_1}, \ a_2 = x_{s_1+1} \cdot \ldots \cdot x_{s_2}, \ldots, \ a_n = x_{s_{n-1}+1} \cdot \ldots \cdot x_s,
\]
where \(1 \leq s_1 < s_2 < \cdots < s_n = s\). Then
\[
\tilde{\kappa}_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(\theta) \text{ such that } \pi \vdash \theta = 1_s} \tilde{\kappa}_n(x_1, \ldots, x_s),
\]

where \(\theta \in NC(s)\) is the partition:
\[
\theta = \{\{1, \ldots, s_1\}, \{s_1 + 1, \ldots, s_2\}, \ldots, \{s_{n-1} + 1, \ldots, s_n\}\}.
\]

Given a \(\mathcal{C}_k\)-noncommutative random variable \(a \in (\mathcal{A}, \tilde{\phi})\), the quantities \(\tilde{\kappa}_n(a, \ldots, a)\) are called its \(\mathcal{C}_k\)-valued cumulants, and they are collected in a power series:
Definition 11. Let \((A, \tilde{\varphi})\) be a scarce \(C_k\)-noncommutative probability space. The \(C_k\)-valued \(R\)-transform of \(a \in A\) is the power series \(\tilde{R}_a \in \Theta^{(A)}_{C_k}\) defined as follows:

\[
\tilde{R}_a(z) := \sum_{n=1}^{\infty} \tilde{\kappa}_n(a, \ldots, a) z^n.
\]

Following the well-known result of [15] stating roughly that, in a usual noncommutative probability space, subsets are free if and only if they satisfy the vanishing of mixed cumulants condition, we generalize this condition to our setting:

Definition 12. Let \((A, \tilde{\varphi})\) be a scarce \(C_k\)-noncommutative probability space and \(M_1, \ldots, M_n\) be subsets of \(A\). We say that \(M_1, \ldots, M_n\) have vanishing mixed \(C_k\)-cumulants if

\[
\tilde{\kappa}_m(a_1, \ldots, a_m) = 0
\]

whenever \(a_1 \in M_{i_1}, \ldots, a_m \in M_{i_m}\) and \(\exists 1 \leq s < t \leq m\), such that, \(i_s \neq i_t\).

As announced, infinitesimal freeness of order \(k\) is defined by the vanishing of mixed \(C_k\)-cumulants condition. More precisely:

Definition 13. We will say that subsets \(M_1, \ldots, M_n \subseteq A\) of a scarce \(C_k\)-noncommutative probability space \((A, \tilde{\varphi})\) are infinitesimally free of order \(k\) if they have vanishing mixed \(C_k\)-cumulants.

Remark 5. Using a classical argument in free probability, one can prove that, if \(A_1, \ldots, A_n\) are unital subalgebras which are infinitesimally free of order \(k\) in a scarce \(C_k\)-noncommutative probability space \((A, \tilde{\varphi})\), then one has:

\[
\tilde{\varphi}(a_1 \cdots a_m) = 0
\]

whenever \(a_1 \in A_{i_1}, \ldots, a_m \in A_{i_m}\) with \(i_1 \neq \ldots \neq i_m\) satisfy \(\tilde{\varphi}(a_1) = \ldots = \tilde{\varphi}(a_m) = 0\).

The converse in our \(C_k\)-valued situation is not true, because one cannot use the nice "centering trick", as noticed in [6] Remark 4.9.

In the next subsection, we switch to the infinitesimal framework, and define infinitesimal non-crossing cumulant functionals, with the intuition that they should appear as the coefficients in the decomposition of the \(C_k\)-non-crossing cumulant functionals in the canonical basis of \(C_k\).

### 3.3 Infinitesimal non-crossing cumulant functionals

In this short subsection, we focus on an infinitesimal noncommutative probability space of order \(k\) structure \((A, (\varphi^{(i)}))_{0 \leq i \leq k}\). The aim is to define cumulants and freeness in this setting, in a consistent way with the last subsection. For convenience, we will use the following notation:
Notation 2. For a family of multilinear maps \((r_n^{(i)} : A^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}\), we define for any \(n \in \mathbb{N}\), any \(\pi = \{V_1 \sqcup \cdots \sqcup V_k\} \in NC^{(A)}(n)\) and any \(\lambda \in \Lambda_{n,h}\) (defined by (2)) the \(n\)-linear functional \(r_{\pi}^{(\lambda)} : A^n \to \mathbb{C}\) by

\[
    r_{\pi}^{(\lambda)}(a_1, \ldots, a_n) := \prod_{i=1}^{k} r_{[V_i]}^{(\lambda_i)}((a_1, \ldots, a_n) | V_i).
\]

The underlying idea is to consider the \(C_k\)-non-crossing cumulant functionals \((\tilde{\kappa}_n : A^n \to \mathbb{C})_{n=1}^{\infty}\) in the associated scarce \(C_k\)-noncommutative probability space \((A, \tilde{\varphi})\) (see Remark 3), and then to define the required \(n\)-th infinitesimal non-crossing cumulant functionals as the \(n\)-linear forms appearing as coefficients in the linear decomposition of \(\tilde{\kappa}_n : A^n \to C_k\) in the canonical basis of \(C_k\). This leads to the following definition:

Definition 14. Let \((A, (\varphi^{(i)})_{0 \leq i \leq k})\) be an infinitesimal noncommutative probability space of order \(k\). The infinitesimal non-crossing cumulant functionals of order \(k\) are a family of multilinear maps \((\kappa^{(i)}_n : A^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}\), uniquely determined by the following equation: for every \(n \geq 1\), every \(0 \leq i \leq k\) and every \(a_1, \ldots, a_n \in A\) we have that:

\[
    \sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_{n,h}} C_{\lambda}^{\lambda_1, \ldots, \lambda_k} \kappa^{(\lambda)}_p(a_1, \ldots, a_n) = \varphi^{(i)}(a_1 \cdots a_n). \quad (8)
\]

Infinitesimal freeness in the framework of an infinitesimal noncommutative probability space of order \(k\) is obviously defined by the vanishing of mixed infinitesimal cumulants.

Definition 15. We will say that subsets \(M_1, \ldots, M_n\) of an infinitesimal noncommutative probability space of order \(k\) are infinitesimally free of order \(k\) if they have vanishing mixed infinitesimal cumulants, which means that, for each \(0 \leq i \leq k\),

\[
    \kappa^{(i)}_m(a_1, \ldots, a_m) = 0
\]

whenever \(a_1 \in M_{i_1}, \ldots, a_m \in M_{i_m}\) and \(\exists 1 \leq s < t \leq m\), such that, \(i_s \neq i_t\).

Remark 6. It is straightforward to check, using formula (8), that the infinitesimal non-crossing cumulant functionals of an infinitesimal noncommutative probability space of order \(k\) are indeed linked to the \(C_k\)-non-crossing cumulant functionals of the associated scarce \(C_k\)-noncommutative probability space by:

\[
    \tilde{\kappa}_n = \sum_{i=0}^{k} \kappa^{(i)}_n \frac{\varepsilon_i}{h}. \quad (9)
\]

A consequence of formulas (7) and (9) is the validity of the following inverse formula:

\[
    \kappa^{(i)}_n(a_1, \ldots, a_n) = \sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_{n,h,i}} \text{Möb}(p, 1_n) C_{\lambda_1}^{\lambda_{i_1}, \ldots, \lambda_k} \rho_p^{(\lambda)}(a_1, \ldots, a_n). \quad (10)
\]
Proposition 4. One has that $\kappa_n^{(i)}(a_1, \ldots, a_n) = 0$ whenever $0 \leq i \leq k$, $a_1, \ldots, a_n \in A$, and there exists $1 \leq j \leq n$ such that $a_j \in C1_A$.

Another consequence of relation (9) is that subsets $M_1, \ldots, M_n$ of an infinitesimal noncommutative probability space of order $k$ are infinitesimally free of order $k$ if and only if they are infinitesimally free of order $k$ in the associated $C_k$-noncommutative probability space.

Remark 7. Let $(A_n(\varphi^{(i)})_{0 \leq i \leq k})$ be an infinitesimal noncommutative probability space of order $k$, and consider its infinitesimal non-crossing cumulant functionals $(\kappa_n^{(i)} : A_n \to \mathbb{C}, 0 \leq i \leq k)^{\infty}_{n=1}$. It is interesting to notice that the multilinear maps $(\kappa_n^{(0)} : A_n \to \mathbb{C})^{\infty}_{n=1}$ and $(\kappa_n^{(1)} : A_n \to \mathbb{C})^{\infty}_{n=1}$ are respectively the usual non-crossing cumulant functionals in the noncommutative probability space $(A, \varphi^{(0)})$ and the infinitesimal non-crossing cumulant functionals of $[6]$ in the infinitesimal noncommutative probability space $(A, \varphi^{(0)}, \varphi^{(1)})$. This implies that subsets that are infinitesimally free of order $k$ are in particular free in $(A, \varphi^{(0)})$ and infinitesimally free in $(A, \varphi^{(0)}, \varphi^{(1)})$ in the sense of [6].

Infinitesimal freeness of unital subalgebras in [6], as well as freeness of type B in [3], is defined in terms of moments. Section 8 will provide such a characterization of the infinitesimal freeness of order $k$ of unital subalgebras of an infinitesimal noncommutative probability space of order $k$ in terms of moments.

As stated in Remark 7, infinitesimal freeness of order $k$ of unital subalgebras $A_1, \ldots, A_n \subseteq (A, (\varphi^{(i)})_{0 \leq i \leq k})$ of an infinitesimal noncommutative probability space of order $k$ implies freeness of $A_1, \ldots, A_n$ in the noncommutative probability space $(A, \varphi^{(0)})$. Conversely, is it possible to "upgrade" freeness of given unital subalgebras of a noncommutative probability space to infinitesimal freeness of order $k$? This question is discussed in the next subsection.

### 3.4 Upgrading freeness to infinitesimal freeness of order $k$

Given a noncommutative probability space $(A, \varphi)$ and free unital subalgebras $A_1, \ldots, A_n$ of $A$, the question of how to build a linear form $\varphi'$ on $A$ such that $A_1, \ldots, A_n$ are infinitesimally free in the infinitesimal noncommutative probability space $(A, \varphi, \varphi')$ is addressed in [6]. Among the answers given there, there is the idea to define $\varphi' := \varphi \circ D$, where $D$ is a derivation of $A$ (a linear map $D : A \to A$ satisfying $\forall a, b \in A, D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$) such that $D(A_j) \subseteq A_j$ for each $1 \leq j \leq n$. We examine the question of how to build linear forms $\varphi^{(1)}, \ldots, \varphi^{(k)}$ on $A$ such that $A_1, \ldots, A_n$ are infinitesimally free of order $k$ in the infinitesimal noncommutative probability space $(A, \varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$. The natural idea consisting in defining $\varphi^{(i)} := \varphi \circ D^i$ where $D$ is a derivation of $A$ such that $D(A_j) \subseteq A_j$ for each $1 \leq j \leq n$ is a possible answer, as proved below:

Proposition 5. Let $(A, \varphi)$ be a noncommutative probability space and let $D : A \to A$ be a derivation. Define $\varphi^{(i)} := \varphi \circ D^i$. Let the infinitesimal non-crossing cumulant functionals associated to $(A, \varphi, \varphi^{(1)}, \ldots, \varphi^{(k)})$ be denoted by
$(\kappa_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}$. Then, for every $n \geq 1$, $0 \leq i \leq k$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has

$$\kappa_n^{(i)}(a_1, \ldots, a_n) = \sum_{\lambda \in \Lambda_n,i} C_i^{\lambda_1,\ldots,\lambda_n} \kappa_n(D^{\lambda_1}(a_1), \ldots, D^{\lambda_n}(a_n)).$$

Proof. Define the family of multilinear functionals $(\eta_n^{(i)} : \mathcal{A}^n \to \mathbb{C}, 0 \leq i \leq k)_{n=1}^{\infty}$ by the following formulas : for every $n \geq 1$, $0 \leq i \leq k$ and $b_1, \ldots, b_n \in \mathcal{A}$

$$\eta_n^{(i)}(b_1, \ldots, b_n) = \sum_{\lambda \in \Lambda_n,i} C_i^{\lambda_1,\ldots,\lambda_n} \kappa_n(D^{\lambda_1}(b_1), \ldots, D^{\lambda_n}(b_n)).$$

Our aim is then to prove that, for every $n \geq 1$, $0 \leq i \leq k$, $\eta_n^{(i)} = \kappa_n^{(i)}$. We verify that the functionals $(\eta_n^{(i)}, 0 \leq i \leq k)_{n=1}^{\infty}$ satisfy the equations (8) defining the infinitesimal non-crossing cumulant functionals. The left-hand side of this formula writes :

$$\sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_n,i} C_i^{\lambda_1,\ldots,\lambda_n} \eta_p^{(\lambda)}(a_1, \ldots, a_n). \quad (11)$$

Each $\eta_{\{V_j|\}}^{(\lambda)}((a_1, \ldots, a_n) | V_j)$ in the latter is a sum indexed by $\Lambda_{|V_j|,\lambda_j}$, involving variables $a_i, i \in V_j$. Given $p := \{V_1, \ldots, V_k\} \in NC^{(A)}(n)$, there is a very natural bijection between $\{((\lambda_1, \lambda_2, \ldots, \lambda_h)) \in \Lambda_{n,i} \times \Lambda_{n,i} \times \ldots \times \Lambda_{n,i} | \lambda_j \in \Lambda_{|V_j|,\lambda_j}\}$ and the set $\Lambda_{n,i}$. Thus, the quantity (11) rewrites :

$$\sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_n,i} C_i^{\lambda_1,\ldots,\lambda_n} \kappa_p(D^{\lambda_1}(a_1), \ldots, D^{\lambda_n}(a_n)).$$

By exchanging the summation signs, the usual free moment-cumulant formula appears, and one obtains :

$$\sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_n,i} C_i^{\lambda_1,\ldots,\lambda_n} \eta_p^{(\lambda)}(a_1, \ldots, a_n) = \sum_{\lambda \in \Lambda_n,i} C_i^{\lambda_1,\ldots,\lambda_n} \varphi(D^{\lambda_1}(a_1) \cdots D^{\lambda_n}(a_n)). \quad (12)$$

Using Leibniz rule in the right-hand side of (12), one may conclude :

$$\sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_n,i} C_i^{\lambda_1,\ldots,\lambda_n} \eta_p^{(\lambda)}(a_1, \ldots, a_n) = \varphi(D^{\lambda_1}(a_1) \cdots D^{\lambda_n}(a_n)) = \varphi(D^i(a_1 \cdots a_n)) = \varphi^{(i)}(a_1 \cdots a_n).$$
Corollary 1. In the notations of Proposition \( \Box \) let \( A_1, \ldots, A_n \) be unital subalgebras of \( A \) which are free in \( (A, \varphi) \), and such that \( D(A_j) \subseteq A_j \) for \( 1 \leq j \leq n \). Then \( A_1, \ldots, A_n \) are infinitesimally free of order \( k \) in \( (A, \varphi, \varphi^{(1)}, \ldots, \varphi^{(k)}) \).

4. Addition and multiplication of infinitesimally free random variables

In this section, we consider \( n \)-tuples of infinitesimal noncommutative random variables \( (a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A^n \) (where \( (A, (\varphi^{(i)}))_{0 \leq i \leq k} \) is an infinitesimal noncommutative probability space of order \( k \)), with respective infinitesimal distributions \( (\mu^{(i)})_{0 \leq i \leq k} \) and \( (\nu^{(i)})_{0 \leq i \leq k} \). We assume that the sets \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \) are infinitesimally free of order \( k \) and we are interested in the distributions of the sum \( (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \) and of the product \( a_1b_1, \ldots, a_nb_n \).

4.1. Addition of infinitesimally free random variables

We do not provide a proof of the following result, which is a straightforward calculation using multilinearity of the infinitesimal cumulant functionals and definition of infinitesimal freeness.

Proposition 6. Let \( (A, (\varphi^{(i)})_{0 \leq i \leq k}) \) be an infinitesimal noncommutative probability space of order \( k \). Consider subsets \( M_1, M_2 \) of \( A \) that are infinitesimally free of order \( k \). Then, one has, for each \( n \geq 1 \), each \( n \)-tuples \( (a_1, \ldots, a_n) \in M_1^n \), \( (b_1, \ldots, b_n) \in M_2^n \) and each \( 0 \leq i \leq k \):

\[
\kappa^{(i)}_n(a_1 + b_1, \ldots, a_n + b_n) = \kappa^{(i)}_n(a_1, \ldots, a_n) + \kappa^{(i)}_n(b_1, \ldots, b_n). \tag{13}
\]

Using formulas \( \Box \) and \( \Box \), the quantities \( \kappa^{(i)}_m(a_1, \ldots, a_m), \kappa^{(i)}_m(b_1, \ldots, b_m) \) for each \( 0 \leq i \leq k \), each \( m \geq 1 \) and each subsets \( \{i_1, \ldots, i_m\}, \{j_1, \ldots, j_m\} \subseteq [n] \) called respectively infinitesimal cumulants of \( (a_1, \ldots, a_n) \) and \( (b_1, \ldots, b_n) \) completely determine and are completely determined by the infinitesimal distributions of \( (a_1, \ldots, a_n) \) and \( (b_1, \ldots, b_n) \). Proposition \( \Box \) thus has the following corollary.

Corollary 2. Let \( (A, (\varphi^{(i)})_{0 \leq i \leq k}) \) be an infinitesimal noncommutative probability space of order \( k \), and \( (a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A^n \) with respective infinitesimal distributions \( (\mu^{(i)})_{0 \leq i \leq k} \) and \( (\nu^{(i)})_{0 \leq i \leq k} \). If the sets \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \) are infinitesimally free of order \( k \), then the infinitesimal distribution of \( (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \) only depends on \( (\mu^{(i)})_{0 \leq i \leq k} \) and \( (\nu^{(i)})_{0 \leq i \leq k} \). It is called the infinitesimal free additive convolution of order \( k \) of \( (\mu^{(i)})_{0 \leq i \leq k} \) and \( (\nu^{(i)})_{0 \leq i \leq k} \) and denoted by \( (\mu^{(i)})_{0 \leq i \leq k} \boxplus^{(k)} (\nu^{(i)})_{0 \leq i \leq k} \).

The corollary above means that the infinitesimal free additive convolution of order \( k \) defines an operation on infinitesimal laws. The practical way to compute the infinitesimal free additive convolution of order \( k \) of two infinitesimal laws is to use consecutively the inverse of the infinitesimal version of the free
moment-cumulant formula (formula (10)), the additivity of infinitesimal cumulants (formula (13)), and finally the infinitesimal version of the free moment-cumulant formula (formula (8)). One may find easier to make the computations in a scarce $C_k$-noncommutative probability space.

Taking into account the link (9) between infinitesimal cumulant functionals and $C_k$-non-crossing cumulant functionals, Proposition 6 admits the following corollaries:

**Corollary 3.** Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $C_k$-noncommutative probability space. Consider subsets $M_1, M_2$ of $\mathcal{A}$ that are infinitesimally free of order $k$. Then, one has, for each $n \geq 1$ and each $n$-tuples $(a_1, \ldots, a_n) \in M_1^n$, $(b_1, \ldots, b_n) \in M_2^n$

$$\tilde{\kappa}_n(a_1 + b_1, \ldots, a_n + b_n) = \tilde{\kappa}_n(a_1, \ldots, a_n) + \tilde{\kappa}_n(b_1, \ldots, b_n).$$

**Corollary 4.** Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $C_k$-noncommutative probability space. Consider $a, b \in \mathcal{A}$ that are infinitesimally free of order $k$ then

$$\tilde{\check{R}}_{a + b} = \tilde{\check{R}}_a + \tilde{\check{R}}_b.$$

**Remark 8.** Using Corollary 3, it is possible to state and prove $C_k$-valued versions of some famous limit theorems of free probability. We discuss this without going into the details; for a more complete discussion of limit theorems in free probability of type B, we refer to [13] and [2]. In a scarce $C_k$-noncommutative probability space $(\mathcal{A}, \tilde{\varphi})$, consider a sequence $(a_n)_{n \in \mathbb{N}} \in \mathcal{A}^\mathbb{N}$ of centered infinitesimally free identically distributed $C_k$-valued noncommutative random variables. Then the moments of the (rescaled by a $\frac{1}{\sqrt{N}}$ factor) sum $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n$ converge to a $C_k$-valued distribution characterized by the vanishing of all of its cumulants except the second one: this is the $C_k$-valued version of the free central limit theorem. The distributions that appear as limits in the preceding result deserve to be named $C_k$-valued semicircular elements. Their moments may be computed using the $C_k$-valued free moment-cumulant formula. Parallely, a $C_k$-valued version of the free Poisson theorem may also be stated and proved, and thus a $C_k$-valued Poisson distribution may be defined.

### 4.2 Multiplication of infinitesimally free random variables

We now investigate the distribution of the product of $n$-tuples of noncommutative random variables that are infinitesimally free of order $k$. We first focus on a $C_k$-noncommutative probability space because, the combinatorics being the same in this setting as in usual free probability, the proofs and results will be straightforward adaptations of the usual ones, which can be found in [11] for instance.

**Proposition 7.** Let $(\mathcal{A}, \tilde{\varphi})$ be a scarce $C_k$-noncommutative probability space. Consider subsets $M_1, M_2$ of $\mathcal{A}$ that are infinitesimally free of order $k$. Then, one has, for each $n \geq 1$ and each $n$-tuples $(a_1, \ldots, a_n) \in M_1^n$, $(b_1, \ldots, b_n) \in M_2^n$

$$\tilde{\kappa}_n(a_1b_1, \ldots, a_nb_n) = \sum_{p \in NC^{(\mathcal{A})(n)}} \tilde{\kappa}_p(a_1, \ldots, a_n)\tilde{\kappa}_K(p)(b_1, \ldots, b_n).$$

(14)
Proof. Using Proposition 3, the left-hand side of (14) is equal to
\[
\sum_{\pi \in NC(2n) \text{ such that } \pi \vee \theta = 1} \kappa_{\pi}(a_1, b_1, a_2, \ldots, b_{n-1}, a_n, b_n),
\]
where \(\theta\) is the partition \(\{1, 2, \ldots, 2n-1, 2n\}\).

By the vanishing of mixed cumulants condition, the only contributing terms are those indexed by non-crossing partitions \(\pi\) which are reunion of a non-crossing partition \(p\) of \(\{1, 3, \ldots, 2n-1\}\) and a non-crossing partition \(q\) of \(\{2, 4, \ldots, 2n\}\).

The condition \(\pi \vee \theta = 1\), for such a partition \(\pi\) may be reinterpreted as \(q = Kr(p)\) (up to the identifications \(\{1, 3, \ldots, 2n-1\} \leftrightarrow [n]\) and \(\{2, 4, \ldots, 2n\} \leftrightarrow [n]\)).

Switching to the infinitesimal framework, one can state the following result.

**Corollary 5.** Let \((A, \varphi_{(i)}^{(j)})_{0 \leq i, j \leq k}\) be an infinitesimal noncommutative probability space of order \(k\), and \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A^n\) be \(n\)-tuples with respective infinitesimal distributions \((\mu_{(i)})_{0 \leq i \leq k}\) and \((\nu_{(i)})_{0 \leq i \leq k}\).

If the sets \(\{a_1, \ldots, a_n\}\) and \(\{b_1, \ldots, b_n\}\) are infinitesimally free of order \(k\), then the infinitesimal distribution of \((a_1b_1, \ldots, a_nb_n)\) only depends on \((\mu_{(i)})_{0 \leq i \leq k}\) and \((\nu_{(i)})_{0 \leq i \leq k}\). It is denoted by \((\mu_{(i)})_{0 \leq i \leq k} \boxtimes_k (\nu_{(i)})_{0 \leq i \leq k}\) and called the infinitesimal free multiplicative convolution of order \(k\) of \((\mu_{(i)})_{0 \leq i \leq k}\) and \((\nu_{(i)})_{0 \leq i \leq k}\).

If \(a, b \in A\) are \(C_k\)-noncommutative random variables that are infinitesimally free of order \(k\) in a scarce \(C_k\)-noncommutative probability space, the \(C_k\)-valued R-transform of \(a \cdot b\) is \(\hat{R}_{\alpha : b} = \hat{R}_a \boxtimes \hat{R}_b\), where \(\boxtimes\) is the version of the boxed convolution operation introduced in [9] with scalars in \(C_k\). We recall in the next subsection the definition and main properties of this operation.

### 4.3 Boxed convolution of type A

An operation on formal power series in several noncommuting indeterminates and with complex coefficients is introduced in [9], and called boxed convolution. It is defined as a convolution on the lattices of non-crossing partitions (of type A). We recall here this definition, but for series in only one variable (for simplicity) and with coefficients in any unital complex algebra. This is already the point of view adopted in [3].

**Definition 16.** Let \(C\) be a unital commutative algebra over \(\mathbb{C}\). On \(\Theta_C^{(A)}\) we define a binary operation \(\boxtimes\), as follows. If
\[
f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \in \Theta_C^{(A)},
\]
and
\[
g(z) = \sum_{n=1}^{\infty} \beta_n z^n \in \Theta_C^{(A)},
\]
then
\[
f(z) \boxtimes g(z) = \sum_{n=1}^{\infty} \left( \sum_{\{0 < i < n\}} \alpha_i \beta_{n-i} \right) z^n \in \Theta_C^{(A)}.
\]
then $f[A]g$ is the series $\sum_{n=1}^{\infty} \gamma_n z^n$, where

$$
\gamma_m = \sum_{p \in NC^{(A)}(m)} \left( \prod_{i=1}^{h} \alpha_{\text{card}(E_i)} \right) \cdot \left( \prod_{j=1}^{l} \beta_{\text{card}(F_j)} \right).
$$

**Remark 9.** It is obviously possible to define a boxed convolution operation for power series in several noncommuting indeterminates and with coefficients in $\mathbb{C}$. The formulas are the same as in the case of complex coefficients, which first appeared in [9] and can also be found in [11].

The operation $[A]C$ is associative, commutative and the series

$$
\Delta^{(A)}_C(z) := 1_C z
$$

is its unit element. There is another important series in $\Theta^{(A)}_C$, namely

$$
\zeta^{(A)}_C(z) := \sum_{n=1}^{\infty} 1_C z^n.
$$

Notice that a series $f \in \Theta^{(A)}_C$ is invertible with respect to $[A]C$ if and only if its coefficient of degree one is itself invertible in the algebra $C$. In particular, $\zeta^{(A)}_C$ is invertible with respect to $[A]C$, and its inverse is called the Möbius series, and denoted by $\text{Möb}^{(A)}_C$. The proofs of these claims may be obtained by a straightforward adaptation of the proofs given in [9]. The free moment-cumulant relation of free probability (and its inverse) may be read at the level of power series: more precisely, in a noncommutative probability space $(A, \varphi)$, the moment series and the R-transform of $a \in A$ satisfy the following relations:

$$
M_a = R_a [A]C \zeta^{(A)}_C, \quad R_a = M_a [A]C \text{Möb}^{(A)}_C.
$$

These formulas have infinitesimal analogues, as stated in the next proposition. It is indeed straightforward to check that, in the particular case of single variables, the formulas (6) and (7) may be read at the level of power series as follows:

**Proposition 8.** Let $(A, \varphi)$ be a scarce $C_k$-noncommutative probability space and consider a $C_k$-noncommutative random variable $a \in A$. Then the $C_k$-valued moment series $M_a$ and the $C_k$-valued R-transform $R_a$ of $a$ are related by the equivalent formulas : $M_a = R_a [A]C \zeta^{(A)}_C, \quad R_a = M_a [A]C \text{Möb}^{(A)}_C$.

The importance of boxed convolution (with complex coefficients) in free probability also comes from the fact, proved in [9], that $[A]C$ provides the combinatorial description for the multiplication of two free noncommutative random variables, in terms of their R-transforms. More precisely, we have, for free $a, b$ in a noncommutative probability space $(A, \varphi) : R_{a \cdot b} = R_a [A]C R_b$.

Interestingly, $[A]C$ also provides the combinatorial description for the multiplication of two infinitesimally free infinitesimal noncommutative random variables, in terms of their R-transforms.
Proposition 9. Let \((A, \tilde{\varphi})\) be a scarce \(C_k\)-noncommutative probability space. Consider \(a, b \in A\) that are infinitesimally free of order \(k\), then
\[
\tilde{R}_{a \cdot b} = \tilde{R}_a \mathbf{E}^{(A)}_{k} \tilde{R}_b.
\]

In \([10]\), a "Fourier transform" is introduced for multiplicative functions on non-crossing partitions. It is barely a map \(\mathcal{F}\) which associates to \(f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \in \Theta(A)\), with \(\alpha_1 \neq 0\) (to ensure that \(f\) is invertible with respect to the composition of formal power series), the series \(\mathcal{F}(f)(z) := \frac{1}{z} f^{(-1)}(z)\). The map \(\mathcal{F}\) has the important property to convert the boxed convolution into the multiplication of formal power series : \(\mathcal{F}(\mathbf{E}^{(A)}_{k} g) = \mathcal{F}(f) \cdot \mathcal{F}(g)\). If \(a\) is a noncommutative random variable with non-zero mean and \(R\)-transform \(R_a\) in a noncommutative probability space, the series \(\mathcal{F}(R_a)\) is of great importance : this is a combinatorial approach to Voiculescu’s S-tranform \([17]\). As noticed in \([13]\), the combinatorial proofs remain valid for series with \(C_k\)-valued coefficients such that the coefficient of degree one is invertible.

Definition 17. Let \((A, \tilde{\varphi})\) be a scarce \(C_k\)-noncommutative probability space. The \(C_k\)-valued S-transform of an infinitesimal noncommutative random variable \(a \in A\) such that \(\tilde{\varphi}(a)\) is invertible in \(C_k\) is the power series \(\tilde{S}_a \in \Theta^{(k)}\) defined as follows:
\[
\tilde{S}_a(z) := \frac{1}{z} \tilde{R}_a^{(-1)}(z).
\]

Proposition 10. Let \((A, \tilde{\varphi})\) be a scarce \(C_k\)-noncommutative probability space. Consider \(a, b \in A\) that are infinitesimally free of order \(k\), and such that \(\tilde{\varphi}(a)\) and \(\tilde{\varphi}(b)\) are invertible in \(C_k\), then the \(C_k\)-valued S-transform \(\tilde{S}_{a \cdot b}\) of \(a \cdot b\) satisfies:
\[
\tilde{S}_{a \cdot b}(z) = \tilde{S}_a(z) \tilde{S}_b(z).
\]

Practically speaking, the computation of the distribution of the product of two infinitesimally free infinitesimal noncommutative random variables requires a good understanding of the \(C_k\)-valued version of the boxed convolution. More precisely, in the notations of Definition \([10]\) it would be of interest to have a formula for \(\gamma_m^{(i)}\) as a function of the \(\alpha_n^{(j)}\)'s and the \(\beta_n^{(j)}\)'s.

As mentioned before, the version of the boxed convolution with scalars in \(C_0 = \mathbb{C}\) is a classical operation in free probability. The version of the boxed convolution with scalars in \(C_1 = \mathbb{G}\) has already been considered in \([3]\), where it is shown to coincide with the boxed convolution based on non-crossing partitions of type B, in connection with free probability of type B. This leads to the natural question : does the operation \(\mathbf{E}^{(A)}_{k}\) coincide with a boxed convolution based on a certain set of special non-crossing partitions. In Section 7, we will give a positive answer to this problem, by introducing the non-crossing partitions of type \(k\). Before that, we review some background on non-crossing partitions and boxed convolution of type B.

5. Non-crossing partitions and boxed convolution of type B
This section is devoted to some background on non-crossing partitions of type B and on the boxed convolution of type B.

5.1 Non-crossing partitions of type B

As recalled in Section 2, there is a close link between the lattice of non-crossing partitions and the Cayley graph of the symmetric group. Actually, one may interpret the lattices of non-crossing partitions in terms of the root systems of type A, justifying the notation $NC^{(A)}(n)$. This led Reiner to introduce in [14] the type B analogue $NC^{(B)}(n)$ of the lattice of non-crossing partitions. To this aim, consider the totally ordered set

$$[\pm n] = \{1 < 2 < \ldots < n < -1 < -2 < \ldots < -n\}.$$

One defines $NC^{(B)}(n)$ to be the subset of $NC^{(A)}([\pm n])$ consisting of non-crossing partitions that are invariant under the inversion map $x \mapsto -x$. In such a partition $\pi \in NC^{(B)}(n)$, there is at most one block that is inversion-invariant, called, whenever it exists, the zero-block of $\pi$. The other blocks of $\pi$ come two by two: if $F$ is a block which is not inversion-invariant, then $-F$ is another block (obviously not inversion-invariant).

It is immediate that $NC^{(B)}(n)$ is a sublattice of $NC^{(A)}([\pm n])$, with the same minimal and maximal elements.

Moreover, $NC^{(B)}(n)$ is closed under the Kreweras complementation maps $Kr$ and $Kr'$ (considered on $NC^{(A)}([\pm n])$). When restricted from $NC^{(A)}([\pm n])$ to $NC^{(B)}(n)$, these maps will then give two anti-isomorphisms of $NC^{(B)}(n)$, inverse to each other, and which will also be called (without ambiguity) Kreweras complementation maps (on $NC^{(B)}(n)$). In this case, the important relation [5] becomes

$$|\pi| + |Kr(\pi)| = 2n + 1, \forall \pi \in NC^{(B)}(n).$$

As a consequence, for $\pi \in NC^{(B)}(n)$, exactly one of the two partitions $\pi$ and $Kr(\pi)$ has a zero-block. In the description of a non-crossing partition of type B, a role is played by the absolute value map $\text{Abs} : [\pm n] \rightarrow [n]$ sending $\pm i$ to $i$.

**Notation 3.** Any map $f$ defined from $[m]$ into $[n]$ is naturally extended to a map from $[m] \cup [m]$ into $[n] \cup [n]$ by simply requiring that $f(\overline{i}) = f(i)$. Moreover, if $Y$ is a subset of $[m] \cup [m]$, we will use the notation $f(Y)$ for the set $\{f(y), y \in Y\} \subset [n] \cup [n]$.

Finally, given a collection $\Upsilon$ of subsets of $[m] \cup [m]$, we will denote by $f(\Upsilon)$ the collection $\{f(Y), Y \in \Upsilon\}$ of subsets of $[n] \cup [n]$.

Let us now state a key result of [3].

**Theorem 1.** $\pi \mapsto \text{Abs}(\pi)$ is a $(n+1)$-to-1 map from $NC^{(B)}(n)$ onto $NC^{(A)}(n)$.

We refer to the paper [3] for the proof. In the next subsection, we recall the definition of the type B analogue of the boxed convolution operation and give the announced result stating that this operation is a boxed convolution of type A on the algebra $C_1$. 21
5.2 Boxed convolution of type B

Definition 18. 1. We denote by $\Theta^{(B)}$ the set of power series of the form

$$f(z) = \sum_{n=1}^{\infty} (\alpha_n', \alpha_n'') z^n,$$

where the $\alpha_n'$s and $\alpha_n''$s are complex numbers.

2. Let $f(z) := \sum_{n=1}^{\infty} (\alpha_n', \alpha_n'') z^n$ and $g(z) = \sum_{n=1}^{\infty} (\beta_n', \beta_n'') z^n$ be in $\Theta^{(B)}$. For every $m \geq 1$, consider the numbers $\gamma_m'$ and $\gamma_m''$ defined by

$$\gamma_m' = \sum_{p \in NC^{(A)}(m) \text{ with zero-block}} \prod_{i=1}^{h} \alpha_{\text{card}(E_i)} \cdot \prod_{j=1}^{l} \beta_{\text{card}(F_j)},$$

$$\gamma_m'' = \sum_{\substack{p \in NC^{(B)}(m) \text{ without zero-block} \\ p=\{Z,X_1,-X_1,...,X_h,-X_h\} \\ Kr(p)=\{Y_1,-Y_1,...,Y_l,-Y_l\}}} \prod_{i=1}^{h} \alpha_{\text{card}(X_i)} \cdot \beta_{\text{card}(Z)/2} \cdot \prod_{j=1}^{l} \beta_{\text{card}(Y_j)} + \sum_{\substack{p \in NC^{(B)}(m) \text{ without zero-block} \\ p=\{X_1,-X_1,...,X_h,-X_h\} \\ Kr(p)=\{Z,Y_1,-Y_1,...,Y_l,-Y_l\}}} \prod_{i=1}^{h} \alpha_{\text{card}(X_i)} \cdot \beta_{\text{card}(Z)/2} \cdot \prod_{j=1}^{l} \beta_{\text{card}(Y_j)}.$$

Then the series $\sum_{n=1}^{\infty} (\gamma_n', \gamma_n'') z^n$ is called the boxed convolution of type B of $f$ and $g$, and is denoted by $f \boxtimes^{(B)} g$.

Theorem 2. $[3]$ Theorem 5.3 $\boxtimes^{(B)} = \boxtimes^{(A)}$

We now introduce the non-crossing partitions of type $k$, generalizing non-crossing partitions of type A and B.

6. Non-crossing partitions of type $k$

This section is devoted to the introduction and study of a set of non-crossing partitions, namely the set of non-crossing partitions of type $k$, which has to be a cover of $NC(A)(n)$ related to the version of the boxed convolution with scalars in $C_k$.

6.1 Definition and first properties

Definition 19. Let $n$ be a positive integer. We call reduction mod $n$ map the map

$$\text{Red}_{n}^{(k)} : [(k+1)n] \rightarrow [n]$$

sending each $i \in [(k+1)n]$ to its congruence class mod $n$. 
Remark 10. For $k = 0$, the map $\text{Red}_n^{(0)}$ is simply the identity map on $[n]$.
For $k = 1$, up to identifying $[2n]$ with $[\pm n]$, the map $\text{Red}_n^{(1)}$ is identified with Abs.

Definition 20. A non-crossing partition $\pi$ of $[(k+1)n]$ is said to satisfy the mod $n$ reduction property if $\text{Red}_n^{(k)}(\pi)$ is a non-crossing partition of $[n]$ and if $\text{Red}_n^{(k)}(\text{Kr}(\pi))$ is a non-crossing partition of $[n]$.

Non-crossing partitions of type $k$ are the non-crossing partitions of $[(k+1)n]$ satisfying the mod $n$ reduction property.

Definition 21. We write $NC^{(k)}(n)$ for the set of non-crossing partitions of type $k$, that is non-crossing partitions of $[(k+1)n]$ satisfying the mod $n$ reduction property.

Remark 11. All non-crossing partitions of $[n]$ trivially satisfy the mod $n$ reduction property (since $\text{Red}_n^{(0)}$ is simply the identity map). Hence $NC^{(0)}(n) = NC^{(A)}(n)$.

The next proposition states that the non-crossing partitions of type $k$ are a generalization of the non-crossing partitions of type $B$.

Proposition 11. If we identify $[\pm n]$ with $[2n]$ and Abs with $\text{Red}_n^{(1)}$, then $NC^{(B)}(n) = NC^{(1)}(n)$.

Proof. That $\pi \in NC^{(B)}(n)$ satisfies the mod $n$ reduction property is a corollary of Proposition 1.3 and Lemma 1 in [3]. For the converse, let $\pi \in NC^{(1)}(n)$ satisfy the mod $n$ reduction property, and assume that there exist two elements $x, y \in [\pm n]$ such that $x \sim_\pi x \sim_\pi y$. By reduction mod $n$ property, we necessarily have $-y \sim_\pi x \sim_\pi y \sim_\pi -x$, which is a contradiction. 

Remark 12. The proof above and Lemma 1 in [3] show that, for a non-crossing partition $\pi$ of $[2n]$, the mod $n$ reduction property is equivalent to the only requirement that $\text{Red}_n^{(1)}(\pi)$ is a non-crossing partition of $[n]$.

In Definition 20 the reduction mod $n$ property for a non-crossing partition $\pi$ of $[(k+1)n]$ consists of two requirements: $\text{Red}_n^{(k)}(\pi)$ has to be a non-crossing partition of $[n]$ and $\text{Red}_n^{(k)}(\text{Kr}(\pi))$ has to be a non-crossing partition of $[n]$. Actually, there is a slightly stronger characterization stated in the next proposition.

Proposition 12. A non-crossing partition $\pi$ of $[(k+1)n]$ satisfies the reduction mod $n$ property if and only if $\text{Red}_n^{(k)}(\pi \cup \text{Kr}(\pi))$ is a non-crossing partition of $[n] \cup [n]$.

Proof. If $\text{Red}_n^{(k)}(\pi \cup \text{Kr}(\pi))$ is a non-crossing partition of $[n] \cup [n]$, since $\text{Red}_n^{(k)}(\pi)$ is a family of subsets of $[n]$ and $\text{Red}_n^{(k)}(\text{Kr}(\pi))$ is a family of subsets of $[n]$, they have to be non-crossing partitions of $[n]$ and $[n]$ respectively; in other words $\pi$
has to satisfy the reduction mod $n$ property.

We assume now that $\pi$ is a non-crossing partition of $[(k + 1)n]$ satisfying the reduction mod $n$ property, and aim at proving that $\text{Red}^{(k)}_n(\pi \cup \text{Kr}(\pi))$ is a non-crossing partition of $[n] \cup [\overline{n}]$.

By the reduction property, $\text{Red}^{(k)}_n(\pi \cup \text{Kr}(\pi)) = \text{Red}^{(k)}_n(\pi) \cup \text{Red}^{(k)}_n(\text{Kr}(\pi))$ is the union of a partition of $[n]$ and of a partition of $[\overline{n}]$, and hence a partition of $[n] \cup [\overline{n}]$. To prove that this partition is non-crossing, consider four elements $a < b < c < d$ of $[n] \cup [\overline{n}]$, such that $a \sim \text{Red}^{(k)}_n(\pi \cup \text{Kr}(\pi)) c$ and $b \sim \text{Red}^{(k)}_n(\pi \cup \text{Kr}(\pi)) d$. We have to show that $a \sim \text{Red}^{(k)}_n(\pi \cup \text{Kr}(\pi)) b$.

Let $1 \leq i_0 \leq (k+1)n + 1$ be minimal with the property that $\text{Mix}(\pi, i_0)$ contains an element $x$ such that $\text{Red}^{(k)}_n(x) \in \{a, b, c, d\}$. Choose also the smallest such $x$. We may assume that $\text{Red}^{(k)}_n(x) = a$ (the other cases are similar). By assumption, $c \in \text{Red}^{(k)}_n(\text{Mix}(\pi, i_0))$ : there is an element $z \in \text{Mix}(\pi, i_0)$ such that $\text{Red}^{(k)}_n(z) = c$. Our choice of $x$ ensures that $x < z$ and there is necessarily an element $x < y < z$ such that $\text{Red}^{(k)}_n(y) = b$. By minimality of $i_0$, $y \in \text{Mix}(\pi, i_0)$, hence $b \in \text{Red}^{(k)}_n(\text{Mix}(\pi, i_0))$ is linked to $a$ in $\text{Red}^{(k)}_n(\pi \cup \text{Kr}(\pi))$ and we are done.

\begin{proof}[Remark 13] When $k = 0$, the reduction mod $n$ property is satisfied by any non-crossing partition of $[n]$ and is in particular equivalent to the only empty requirement : $\pi \in NC^{(A)}(n)$ satisfies the reduction mod $n$ property if and only if $\text{Red}^{(0)}_n(\pi)$ is a non-crossing partition of $[n]$.

As explained in Remark 12, this is also the case when $k = 1 : \pi \in NC^{(A)}(2n)$ satisfies the reduction mod $n$ property if and only if $\text{Red}^{(1)}_n(\pi)$ is a non-crossing partition of $[n]$.

Assume now that $k \geq 2$ ; the situation then is different. As an example, for $k = 2$ and $n = 2$, consider the partition

$$\pi := \{\{1, 2, 3\}, \{4, 5, 6\}\} \in NC^{(A)}(6).$$

It is straightforward to check that $\text{Red}^{(2)}_2(\pi) = \{1, 2\}$ is a non-crossing partition of $[2]$. However, from the easy computation $\text{Kr}(\pi) = \{\{1\}, \{2\}, \{4\}, \{5\}, \{3, 6\}\}$, we deduce that $\text{Red}^{(2)}_2(\text{Kr}(\pi))$ is not a partition of $[6]$ and consequently that $\pi$ does not satisfy the reduction mod $2$ property.

The following proposition states that the Kreweras complementation maps may be considered as two order-reversing bijections of $NC^{(k)}(n)$.

\begin{proposition}
The restrictions from $NC^{(A)}([(k + 1)n])$ to $NC^{(k)}(n)$ of $\text{Kr}$ and $\text{Kr}'$ are order-reversing bijections of $NC^{(k)}(n)$.
\end{proposition}

The name of Kreweras complementation map and the notations $\text{Kr}$, $\text{Kr}'$ will be conserved as there is no ambiguity about the meaning of $\text{Kr}(\pi)$ or $\text{Kr}'(\pi)$ whether $\pi$ is viewed as an element of $NC^{(k)}(n)$ or of $NC^{(A)}([(k + 1)n])$.

\begin{proof}
It is clearly sufficient to prove that the non-crossing partition $\text{Kr}(\pi)$ of $[(k + 1)n]$ satisfies the reduction mod $n$ property whenever $\pi$ does. Assume that
the non-crossing partition \( \pi \) of \([(k+1)n] \) satisfies the reduction mod \( n \) property. By assumption, \( \text{Red}_n^{(k)}(\text{Kr}(\pi)) \) is a non-crossing partition of \([n]\). It remains to prove that \( \text{Red}_n^{(k)}(\text{Kr}_2(\pi)) \) is a non-crossing partition of \([n]\).

From the geometric description of \( \text{Kr}_2(\pi) \) given in Section 3, we deduce that \( \text{Red}_n^{(k)}(\text{Kr}_2(\pi)) \) is obtained from \( \text{Red}_n^{(k)}(\pi) \) by a rotation. By reduction mod \( n \) property, \( \text{Red}_n^{(k)}(\pi) \) is a non-crossing partition of \([n]\), so \( \text{Red}_n^{(k)}(\text{Kr}_2(\pi)) \) is itself a non-crossing partition of \([n]\). Thus the proof is complete. □

Given \( \pi \in NC^{(k)}(n), \text{Kr}(\text{Red}_n^{(k)}(\pi)) \) and \( \text{Red}_n^{(k)}(\text{Kr}(\pi)) \) are thus two non-crossing partitions of \([n]\). The following lemma, generalizing Lemma 1 of [9], states that these two partitions coincide.

**Proposition 14.** \( \forall \pi \in NC^{(k)}(n), \text{Kr}(\text{Red}_n^{(k)}(\pi)) = \text{Red}_n^{(k)}(\text{Kr}(\pi)) \).

**Proof.** Let \( \pi \) be a non-crossing partition of type \( k \). By Proposition [12], \( \text{Red}_n^{(k)}(\pi) \cup \text{Red}_n^{(k)}(\text{Kr}(\pi)) = \text{Red}_n^{(k)}(\pi \cup \text{Kr}(\pi)) \) is a non-crossing partition of \([n] \cup [n] \). Since \( \text{Kr}(\text{Red}_n^{(k)}(\pi)) \) is maximal with the property that \( \text{Red}_n^{(k)}(\pi) \cup \text{Kr}(\text{Red}_n^{(k)}(\pi)) \) is non-crossing, it follows that

\[
\text{Red}_n^{(k)}(\text{Kr}(\pi)) \preceq \text{Kr}(\text{Red}_n^{(k)}(\pi)).
\]

There is equality if, for any \( \pi \) having a neighbour \( \gamma > \pi \) in \( \text{Kr}(\text{Red}_n^{(k)}(\pi)) \), \( \gamma \) is linked to \( \pi \) in \( \text{Red}_n^{(k)}(\text{Kr}(\pi)) \). For such elements \( \pi, \gamma \in [n] \), we call \( V \) the block of \( \pi \) containing \( x + 1 \). The reduction property implies that \( \text{Red}_n^{(k)}(V) \) is a block of the partition \( \text{Red}_n^{(k)}(\pi) \). By construction of the Kreweras complement, \( x + 1 \) is the smallest element of both \( V \) and \( \text{Red}_n^{(k)}(V) \), and \( y \) is the greatest element of \( \text{Red}_n^{(k)}(V) \). Consider now the greatest element \( z \) of \( V \). Notice that \( x + 1 \leq \text{Red}_n^{(k)}(z) \leq y \). By construction of the Kreweras complement again, \( \pi \) is linked to \( \pi \) in \( \text{Kr}(\pi) \), then \( \overline{\pi} \) is linked to \( \text{Red}_n^{(k)}(\overline{\pi}) = \text{Red}_n^{(k)}(z) \in \text{Red}_n^{(k)}(\text{Kr}(\pi)) \) and therefore in \( \text{Kr}(\text{Red}_n^{(k)}(\pi)) \). This means that, if \( \text{Red}_n^{(k)}(z) < y, \gamma \) would not be the neighbour of \( \pi \) in \( \text{Kr}(\text{Red}_n^{(k)}(\pi)) \), which is a contradiction. So \( \text{Red}_n^{(k)}(z) = y \) or, in other words, \( \pi \) is linked to \( \gamma \) in \( \text{Red}_n^{(k)}(\text{Kr}(\pi)) \). □

A deeper description of non-crossing partitions of type \( k \) is given in the next subsection.

**6.2 Structure of non-crossing partitions of type \( k \)**

The goal of this subsection is to describe the structure of a non-crossing partition of type \( k \). In the next proposition, \( t \) denotes the bijection between non-crossing partitions and permutations lying on a geodesic in the Cayley graph of the symmetric group, introduced by Biane in [4], and described in Section 3. We warn the reader that we choose to use the same notation \( t \) for this bijection, defined either on \( NC(n) \) or \( NC((k + 1)n) \). We hope that this choice, made in the sake of simplicity, will not be a source of confusion in the reader’s mind. The content of this proposition is, roughly speaking, that a type
$k$ non-crossing partition $\pi$ is characterized by the two requirements: $\text{Red}_n^{(k)}(\pi)$ is a non-crossing partition of $NC^{(A)}(n)$ and the elements of each of the blocks of $\pi$ come in the same order as their congruence classes in its reduction $\text{Red}_n^{(k)}(\pi)$.

**Proposition 15.** For $\pi \in NC^{(A)}((k+1)n)$ such that $\text{Red}_n^{(k)}(\pi) \in NC^{(A)}(n)$, $\pi \in NC^{(k)}(n)$ if and only if

$$\forall x \in [(k+1)n], \text{Red}_n^{(k)}(t(\pi)(x)) = t(\text{Red}_n^{(k)}(\pi))(\text{Red}_n^{(k)}(x)).$$ (15)

**Proof.** Assume first that $\pi \in NC^{(k)}(n)$ and fix $x \in [(k+1)n]$. Set $y := t(\pi)(x)$. By construction of $t$, $y$ is the neighbour of $x$ in $\pi$ and $t(\text{Red}_n^{(k)}(\pi))(\text{Red}_n^{(k)}(x))$ is the neighbour of $\text{Red}_n^{(k)}(x)$ in $\text{Red}_n^{(k)}(\pi)$. By construction of the Kreweras complement, $\overline{\pi}$ is the neighbour of $y-1$ in $\text{Kr}(\pi)$, and $\text{Red}_n^{(k)}(x)$ is the neighbour of $t(\overline{\text{Red}_n^{(k)}(\pi)})(\text{Red}_n^{(k)}(x)) - 1$ in $\text{Kr}(\text{Red}_n^{(k)}(\pi)) = \text{Red}_n^{(k)}(\text{Kr}(\pi))$ (the latter equality holds because of Proposition 13). By reduction property, $\text{Red}_n^{(k)}(y-1)$ is linked to $\text{Red}_n^{(k)}(x)$. It follows that the neighbour of $\text{Red}_n^{(k)}(x)$ in $\text{Red}_n^{(k)}(\pi)$, $t(\text{Red}_n^{(k)}(\pi))(\text{Red}_n^{(k)}(x))$, is the first point coming after $\text{Red}_n^{(k)}(y-1)$ linked to $\text{Red}_n^{(k)}(x)$: it is $\text{Red}_n^{(k)}(y)$ and we are done. For the converse, let $\pi \in NC^{(A)}((k+1)n)$ be such that $\text{Red}_n^{(k)}(\pi) \in NC^{(A)}(n)$ and assume that condition 2 holds. We have to prove that $\text{Red}_n^{(k)}(\text{Kr}(\pi))$ is a non-crossing partition of $[n]$. Let $\overline{\pi} \in [(k+1)n]$, its neighbour in $\text{Kr}(\pi)$ is $t(\pi)^{-1}(x+1)$, by construction of the Kreweras complement. It follows of condition 2 that $\text{Red}_n^{(k)}(t(\pi)^{-1}(x+1)) = t(\text{Red}_n^{(k)}(\pi))^{-1}(\text{Red}_n^{(k)}(x+1))$. Hence the congruence class of the neighbour of $\overline{\pi}$ in $\text{Kr}(\pi)$ only depends on the congruence class of $x$, and moreover $\text{Red}_n^{(k)}(\text{Kr}(\pi)) = \text{Kr}(\text{Red}_n^{(k)}(\pi))$ and we are done. \[\square\]

The preceding proposition has some important consequences.

**Corollary 6.** Let $\pi \in NC^{(A)}((k+1)n)$ and $V$ be a block of $\pi \cup \text{Kr}(\pi)$. The cardinal of $\text{Red}_n^{(k)}(V)$ divides the cardinal of $V$. We call multiplicity of $V$ the quotient

$$\text{mult}_{\pi \cup \text{Kr}(\pi)}(V) := \frac{\text{card}(V)}{\text{card}((\text{Red}_n^{(k)}(V)))}.$$ 

This is a positive integer lower or equal than $k+1$. The blocks of multiplicity 1 will be called simple.

**Proof.** For $x \in V$, the cardinal of $V$ is the smallest positive $i$ verifying

$$(t(\pi))^i(x) = x.$$ 

A repeated use of Proposition 15 gives that, for such an $i$,

$$(t(\text{Red}_n^{(k)}(\pi)))^i(\text{Red}_n^{(k)}(x)) = \text{Red}_n^{(k)}(x).$$ (16)

Thus $i$ is a multiple of the cardinal of $\text{Red}_n^{(k)}(V)$, which is also characterized by the fact that it is the smallest positive $i$ verifying condition 13. \[\square\]
It is not so difficult to see that, if there is a block of multiplicity \( k + 1 \) in \( \pi \cup \text{Kr}(\pi) \), for \( \pi \in NC^{(k)}((k + 1)n) \), the other blocks are necessarily simple, because one cannot link two elements of the same congruence class without crossing the block of multiplicity \( k + 1 \). This is in fact a particular case of the following result:

**Corollary 7.** For \( \pi \in NC^{(k)}(n) \),

\[
\sum_{V \in bl(\pi \cup \text{Kr}(\pi))} (\text{mult}_{\pi \cup \text{Kr}(\pi)}(V) - 1) = k.
\]

**Proof.** This is a simple computation. First notice that

\[
\sum_{V \in bl(\pi \cup \text{Kr}(\pi))} (\text{mult}_{\pi \cup \text{Kr}(\pi)}(V) - 1) = \sum_{V \in bl(\pi \cup \text{Kr}(\pi))} \text{mult}_{\pi \cup \text{Kr}(\pi)}(V) - |\pi \cup \text{Kr}(\pi)|.
\]

The first term in (17) is

\[
\sum_{W \in bl(\text{Red}^{(k)}_{n}(\pi \cup \text{Kr}(\pi)))} \sum_{V \in bl(\pi \cup \text{Kr}(\pi)) : \text{Red}^{(k)}_{n}(V) = W} \text{mult}_{\pi \cup \text{Kr}(\pi)}(V).
\]

But for any block \( W \) of \( \text{Red}^{(k)}_{n}(\pi \cup \text{Kr}(\pi)) \), one has

\[
\sum_{V \in bl(\pi \cup \text{Kr}(\pi)) : \text{Red}^{(k)}_{n}(V) = W} \text{mult}_{\pi \cup \text{Kr}(\pi)}(V) = k + 1.
\]

Applying twice formula (5), we get

\[
\sum_{V \in bl(\pi \cup \text{Kr}(\pi))} (\text{mult}_{\pi \cup \text{Kr}(\pi)}(V) - 1) = (k + 1)|\text{Red}^{(k)}_{n}(\pi \cup \text{Kr}(\pi))| - |\pi \cup \text{Kr}(\pi)|
\]

\[
= (k + 1)(n + 1) - ((k + 1)n + 1)
\]

\[
= k.
\]

\[\square\]

For a partition \( \pi \in NC^{(k)}(n) \), one may define a vector \( \lambda_{\pi} \) with integer coordinates as follows:

\[
(\lambda_{\pi})_{i} = \sum_{V \in bl(\pi \cup \text{Kr}(\pi)) : \text{Red}^{(k)}_{n}(V) = \text{Mix}(\text{Red}^{(k)}_{n}(\pi \cup \text{Kr}(\pi)), i)} (\text{mult}_{\pi \cup \text{Kr}(\pi)}(V) - 1).
\]

The vector \( \lambda_{\pi} \in \Lambda_{n+1,k} \) is called the *shape* of \( \pi \).
Remark 14. A type B non-crossing partition $\pi$ is determined by its absolute value $p := \text{Abs}(\pi)$ and the choice of the block $Z \in \text{bl}(p \cup \text{Kr}(p))$, which has to be lifted to the zero-block of $\pi$. This latter choice is encoded in the shape $\lambda_\pi$ of $\pi$. Indeed, type B corresponds to the case $k = 1$ of non-crossing partitions of type $k$ and therefore the shape $\lambda_\pi$ belongs to the set $\Lambda_{n+1,k}$ consisting of the $n + 1$ vectors $e_i = (\delta^j_i)_{1 \leq j \leq n+1}, 1 \leq i \leq n + 1$. That $\lambda_\pi = e_i$ means exactly that we have to choose the block $\text{Mix}(p, i)$ as the absolute value of the zero-block. The conclusion is that a type B non-crossing partition, considered as a non-crossing partition of type 1, is determined by its reduction (or absolute value in the type B language) and its shape. Unfortunately, this is not the case when $k \geq 2$. It is interesting to ask how to determine a general non-crossing partition of type $k$. This question is investigated in the proof of the next proposition.

Proposition 16. Let $\lambda \in \Lambda_{n+1,k}$. The number of $\pi \in NC^{(k)}(n)$ having shape $\lambda$ and reduction a fixed non-crossing partition $p \in NC^{(A)}(n)$ is the same for any choice of $p \in NC^{(A)}(n)$. We will denote this quantity by $r(\lambda)$.

Proof. As announced, we investigate how to determine a type $k$ non-crossing partition $\pi \in NC^{(k)}(n)$, once its reduction $p \in NC^{(A)}(n)$ and its shape $\lambda \in \Lambda_{n+1,k}$ are given. We know that $\text{Mix}(p, 1)$ is a singleton of $[n] \cup [n]$. For simplicity, we assume that it is a singleton $\{i\}$ of $[n]$. We need to know how to form the blocks of $\pi$ reducing to $\{i\}$. The number of admissible ways to form these blocks depends on the value of $\lambda_i$ but of course not on $p$, because the actual value of $i$ does not come into the game. Assume that these blocks are formed; this gives a decomposition of $[(k + 1)n] \setminus \{x \mid \text{Red}^{(k)}_n(x) = i\} \cup [(k + 1)n]$ into $\lambda_i + 1$ sets, according to the following process: let us denote by $\{i + l_1 n, \ldots, i + l_m n\}$ the smallest (with respect to $\subseteq$) of the blocks we have just formed that is not simple (if there is no such block, i.e. when $\lambda_1 = 0$, our decomposition is trivial); each of the $\{i + l_j n, \ldots, i + l_j n\}$ becomes a set in our decomposition after erasing the $i + ln, l_j \leq l \leq l_{j+1}$, for each $1 \leq j \leq m - 1$. Then remove all elements $x$ such that $i + l_j n \leq x \leq i + l_{m} n$ and repeat the process by considering the new smallest block with respect to $\subseteq$ among the remaining blocks that are not simple. Notice that the sets obtained this way may be identified with sets of the form $[i(n - 1)] \cup [i(n - 1)]$, for some $l \leq k + 1$, up to identifying the first and last elements of the sets. This can be done, because these elements are necessarily linked by construction of the Kreweras complement. On each of these sets, $\pi$ induces a non-crossing partition that belongs to $NC^{(l)}(n - 1)$. All such induced non-crossing partitions have the same reduction $\tilde{p}$ obtained by erasing in $p \cup \text{Kr}(p)$ the element $i$ and by identifying $i - 1$ with $i$ (which are also necessarily linked in Kr(p)). The shapes of the induced partitions sum to the shape $\lambda$ of $\pi$. Hence a non-crossing partition of type $k$ is determined by its reduction $p$, its shape $\lambda$, an admissible way to form the blocks reducing to $\text{Mix}(p, 1)$, an admissible decomposition of $\lambda$ and the choice of the induced non-crossing partitions in sets $NC^{(l)}(n - 1)$, having reduction $\tilde{p}$ and shape the summands in the decomposition of $\lambda$.

Our argument goes by induction on $n$. For $n = 1$ and any $k$, there is only one possible reduction, because $\#NC^{(A)}(1) = 1$ and consequently there is noth-
ing to prove in that case. Assume that, for any \( l \), the number of partitions in \( NC^{(l)}(n-1) \) with given shape and reduction does not depend on the choice of the reduction. According to our analysis of the first part of the proof, the number of partitions \( \pi \in NC^{(k)}(n) \) with given shape \( \lambda \) and reduction \( p \) does not depend on the choice of the reduction, because we noticed that the number of admissible ways to form the blocks reducing to \( \text{Mix}(p,1) \) does not depend on \( p \), the shape decomposition depend only on \( \lambda \) and the way the latter blocks are formed, and by induction, the numbers of choices for the induced partitions only depend on their shapes.

\[ \square \]

**Remark 15.** For small values of \( k \), one may easily compute the values of \( r(\lambda) \) for each \( \lambda \in \Lambda_{n+1,k} \).

In the simplest case \( k = 0 \),

\[ \Lambda_{n+1,0} = \{(0, \ldots ,0)\}, \]

\[ r((0, \ldots ,0)) = 1. \]

For \( k = 1 \),

\[ \Lambda_{n+1,1} = \{e_i\}_{i=1,\ldots,n+1}, \]

and one has

\[ \forall 1 \leq i \leq n+1, r(e_i) = 1. \]

For \( k = 2 \),

\[ \Lambda_{n+1,2} = \{e_i + e_j\}_{i,j=1,\ldots,n+1}. \]

The value of \( r(e_i + e_j) \) depends on whether \( i = j \) or not:

\[ \forall 1 \leq i \leq n+1, r(2e_i) = 1. \]

\[ \forall 1 \leq i < j \leq n+1, r(e_i + e_j) = 3. \]

We investigate in the next subsection some properties of the set \( NC^{(k)}(n) \).

### 6.3 Study of the poset \( NC^{(k)}(n) \)

The set \( NC^{(k)}(n) \), being a subset of \((NC^{(A)}((k+1)n), \preceq)\), inherits its partially ordered set (abbreviated poset) structure. Contrary to \( NC^{(B)}(n) \), which is a sublattice of \((NC^{(A)}(2n), \preceq)\) (up to the identification \([\pm n] = [2n]\)), \((NC^{(k)}(n), \preceq)\) is unfortunately not a sublattice of \((NC^{(A)}((k+1)n), \preceq)\), when \( k \geq 2 \).

**Remark 16.** When \( k = 2 \) and \( n = 2 \), consider the partitions

\[ \pi := \{2,3,4,5\}, \{1,6\} \in NC^{(2)}(2) \]

and

\[ \rho := \{1,2\}, \{3,4,5,6\} \in NC^{(2)}(2). \]
It is an easy exercise to determine the meet of these two partitions in the lattice $(NC^{(A)}(6), \leq)$:

\[ \pi \wedge_{NC^{(A)}(6)} \rho = \{ \{1\}, \{2\}, \{3, 4, 5\}, \{6\} \}. \]

It is immediate that $\pi \wedge_{NC^{(A)}(6)} \rho$ is not an element of $NC^{(2)}(2)$ which is consequently not a sublattice of $(NC^{(A)}(6), \leq)$; the same kind of argument would prove that $NC^{(k)}(n)$ is never a sublattice of $(NC^{(A)}((k + 1)n), \leq)$, as soon as $k \geq 2$.

It is natural to ask whether $NC^{(k)}(n)$ is or not a lattice in its own right for the reverse refinement order $\preceq$. We do not know the answer to this question.

We now state and prove the main result of this section.

**Theorem 3.** $\pi \mapsto \text{Red}_{n}^{(k)}(\pi)$ is a $\displaystyle \frac{1}{(k + 1)n + 1}C^{k+1}_{(n+1)(k+1)}$-to-1 map from $NC^{(k)}(n)$ onto $NC^{(A)}(n)$.

**Proof.** We fix $p \in NC^{(A)}(n)$. The shape $\lambda_{\pi}$ of a $\pi \in NC^{(k)}(n)$ satisfying $\text{Red}_{n}^{(k)}(\pi) = p$ is an element of the set $\Lambda_{n+1,k}$, and for each $\lambda \in \Lambda_{n+1,k}$, there are exactly $r(\lambda)$ non-crossing partitions of type $k$ with reduction $p$ and shape $\lambda$. Hence there are $\sum_{\lambda \in \Lambda_{n+1,k}} r(\lambda)$ non-crossing partitions of type $k$ with reduction $p$, and we know by Proposition 16 that this number does not depend on $p$. It remains to prove that $\sum_{\lambda \in \Lambda_{n+1,k}} r(\lambda) = \frac{1}{(k + 1)n + 1}C^{k+1}_{(n+1)(k+1)}$, by counting the non-crossing partitions of type $k$ with reduction $1_{[n]}$. The set formed by these partitions is precisely the set $NC_{n}(k)$ of non-crossing partitions of $[(k + 1)n]$ having blocks of size divisible by $n$. The latter set appears in [1], where it is proved that its cardinal is $\frac{1}{(k + 1)n + 1}C^{k+1}_{(n+1)(k+1)}$. \(\square\)

We end this section by defining a subset of $NC^{(k)}(n)$ that will be used in Section 7.

**Definition 22.** We write $NC^{(k)}_{\star}(n)$ for the set of non-crossing partitions of type $k$ without non-simple blocks in their Kreweras complement.

**Remark 17.** In the shape of a non-crossing partition $\pi \in NC^{(k)}_{\star}(n)$, the coordinates corresponding to blocks of Kr($\pi$) are zero; there is therefore a straightforward bijection between the set of shapes of non-crossing partitions $\pi \in NC^{(k)}_{\star}(n)$ satisfying $\text{Red}_{n}^{(k)}(\pi) = p$ and the set $\Lambda_{|p|,k}$. Notice also that, given $p \in NC^{(A)}(n)$ and $\lambda \in \Lambda_{|p|,k}$, there are exactly $r(\lambda)$ non-crossing partitions $\pi \in NC^{(k)}_{\star}(n)$ with reduction $p$ and, with a small abuse of language, shape $\lambda$.

Non-crossing partitions of type $k$ give a combinatorial description of the version of the boxed convolution with scalars in $C_{k}$, as explained in the next section.

7. Boxed convolution of type $k$
As for type A and B, there is a boxed convolution operation associated to the non-crossing partitions of type k. It is defined on formal power series with coefficients in $\mathbb{C}^{k+1}$, as follows.

**Definition 23.** 1. We denote by $\Theta^{(k)}$ the set of power series of the form

$$f(z) = \sum_{n=1}^{\infty} (\alpha_n^{(0)}, \ldots, \alpha_n^{(k)}) z^n,$$

where, for each $n \geq 1$ and $0 \leq i \leq k$, $\alpha_n^{(i)}$ is a complex number.

2. Let $f(z) = \sum_{n=1}^{\infty} (\alpha_n^{(0)}, \ldots, \alpha_n^{(k)}) z^n$ and $g(z) = \sum_{n=1}^{\infty} (\beta_n^{(0)}, \ldots, \beta_n^{(k)}) z^n$ be in $\Theta^{(k)}$. For every $m \geq 1$ and every $0 \leq i \leq k$, consider the numbers $\gamma_m^{(i)}$ defined by

$$\gamma_m^{(i)} = \sum_{\pi \in NC^{(i)}(m)} \frac{C_{\nu(\pi)}^{i}}{r(\nu)} \prod_{j=1}^{m+1} \alpha_{\text{card}(\text{Sep}(\text{Red}_m^{(i)}(\pi),j))}^{(\nu(j))},$$

$$\prod_{j=|\text{Red}_m^{(i)}(\pi)|+1}^{m+1} \beta_{\text{card}(\text{Sep}(\text{Red}_m^{(i)}(\pi),j))}^{(\nu(j))}. $$

Then the series $\sum_{n=1}^{\infty} (\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) z^n$ is called the boxed convolution of type k of f and g, and is denoted by $f \Box^{(k)} g$.

It turns out that, up to identifying the two sets $\Theta^{(k)}$ and $\Theta^{(A)}_{\mathbb{C}^k}$, the two operations $\Box^{(k)}$ and $\Box^{(A)}_{\mathbb{C}^k}$ are actually the same, as stated in the next theorem.

**Theorem 4.** $\Box^{(k)} = \Box^{(A)}_{\mathbb{C}^k}$

**Proof.** Let $f(z) = \sum_{n=1}^{\infty} (\alpha_n^{(0)}, \ldots, \alpha_n^{(k)}) z^n$ and $g(z) = \sum_{n=1}^{\infty} (\beta_n^{(0)}, \ldots, \beta_n^{(k)}) z^n$ be in $\Theta^{(k)}$.

Write $f \Box^{(k)} g = \sum_{n=1}^{\infty} (\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) z^n$ and $f \Box^{(A)}_{\mathbb{C}^k} g = \sum_{n=1}^{\infty} (\delta_n^{(0)}, \ldots, \delta_n^{(k)}) z^n$.

We fix a positive integer $n$, for which we will show that

$$(\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) = (\delta_n^{(0)}, \ldots, \delta_n^{(k)}).$$

Let us look at $\gamma_n^{(i)}$. First, we have

$$\gamma_n^{(i)} = \sum_{\pi \in NC^{(i)}(n)} \frac{C_{\nu(\pi)}^{i}}{r(\nu)} \prod_{j=1}^{m+1} \alpha_{\text{card}(\text{Sep}(\text{Red}_m^{(i)}(\pi),j))}^{(\nu(j))},$$

$$\prod_{j=|\text{Red}_m^{(i)}(\pi)|+1}^{m+1} \beta_{\text{card}(\text{Sep}(\text{Red}_m^{(i)}(\pi),j))}^{(\nu(j))}. $$

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On the other hand, by recalling the definition of the operation \( \delta_n \), we obtain the proof of Theorem 3, one gets
\[
\phi(\lambda)(p, j) := \begin{cases} 
\alpha(\lambda) \text{card}(\text{Sep}(p, j)) & \text{if } j \leq |p|, \\
\beta(\lambda) \text{card}(\text{Sep}(p, j)) & \text{if } j > |p|,
\end{cases}
\]
The summation over \( NC(i)(n) \) can be reduced to one over \( NC^{(A)}(n) \), by using the cover \( \text{Red}^{(i)}_n : NC^{(i)}(n) \to NC^{(A)}(n) \). When doing so, and taking into account the explicit description of \( (\text{Red}^{(i)}_n)^{-1}(p), p \in NC^{(A)}(n) \) provided by the proof of Theorem \( 3 \), one gets
\[
\gamma_n^{(i)} = \sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_{n+1, i}} C_{\lambda_1, \ldots, \lambda_{n+1}}^n \prod_{j=1}^{n+1} \phi^{(\lambda_j)}(p, j).
\]
On the other hand, by recalling the definition of the operation \( \mathfrak{c}_k^{(A)} \), we see that \( \delta_n^{(i)} \) equals
\[
\sum_{p \in NC^{(A)}(n)} \sum_{\lambda \in \Lambda_{n+1, i}} C_{\lambda_1, \ldots, \lambda_{n+1}}^n \prod_{j=1}^{n+1} \phi^{(\lambda_j)}(p, j).
\]
By comparing, we obtain \( (\gamma_n^{(0)}, \ldots, \gamma_n^{(k)}) = (\delta_n^{(0)}, \ldots, \delta_n^{(k)}) \), as desired. \( \square \)

Corollary 8. The operation \( \mathfrak{c}_k^{(c)} \) is associative, commutative and the series \( \Delta^{(k)}(z) = \Delta^{(A)}_{\mathfrak{c}_k}(z) \) is its unit element. A series \( f \in \Theta^{(k)}(z) \) is invertible with respect to \( \mathfrak{c}_k^{(c)} \) if and only if its coefficient of degree one has a non-zero first component.

Remark 18. Theorem 4 tells us that the operation \( \mathfrak{c}_k^{(A)} \) is a boxed convolution of type A, for which it is noticed in Remark 4 that one may define a generalization to power series in several noncommuting indeterminates. This means that there exists an operation \( \mathfrak{c}_k^{(c)} \) on power series in several noncommuting indeterminates. We do not find interesting to record here the formulas involved in this operation.

Non-crossing partitions of type \( k \) are thus the combinatorial objects describing the version of the boxed convolution of type A with scalars in the algebra \( \mathfrak{c}_k \).

It is now easy to rewrite the main formulas involving infinitesimal non-crossing cumulants with sums indexed by the set of non-crossing partitions of type \( k \). This is the content of the next proposition:

Proposition 17. Let \( (\mathcal{A}, (\varphi^{(i)}_{\leq i \leq k})) \) be an infinitesimal noncommutative probability space of order \( k \). The infinitesimal non-crossing cumulant functionals satisfy, for every \( n \geq 1 \), every \( 0 \leq i \leq k \) and every \( a_1, \ldots, a_n \in \mathcal{A} \):
\[
\varphi^{(i)}(a_1 \cdots a_n) = \sum_{\pi \in NC^{(i)}_n(n)} C_{\gamma_1, \ldots, \gamma_{n-1}}^{(\lambda_1, \ldots, \lambda_{n-1})} r(\lambda_n) \kappa(\lambda_n) \text{Red}^{(i)}_{\pi}(a_1, \ldots, a_n).
\]
Proposition 18. Let \( (A, (\varphi^{(i)})_{0 \leq i \leq k}) \) be an infinitesimal noncommutative probability space of order \( k \). Consider subsets \( M_1, M_2 \) of \( A \) that are infinitesimally free of order \( k \). Then, one has, for each \( n \geq 1 \), each \( n \)-tuples \( (a_1, \ldots, a_n) \in M_1^n \), \( (b_1, \ldots, b_n) \in M_2^n \) and each \( 0 \leq i \leq k \) :

\[
\kappa_n^{(i)}(a_1, b_1, \ldots, a_n, b_n) = \sum_{\pi \in \mathcal{N}(n)} \frac{C_1^{(\lambda_1), \ldots, (\lambda_n)_{n+1}} \pi^{(\lambda_1)} \cdot \text{Red}^{(i)}_{\pi}(\kappa, Kn(\pi))}{r(\lambda_1)}\langle a_1, b_1, \ldots, a_n, b_n \rangle.
\]

We move to the main application of infinitesimal freeness.

8. Application to derivatives of the free convolution

In this final section, we give an application of infinitesimal freeness of order \( k \). We consider the situation already examined in [2] : let \( \{a_u^v(t) \mid 1 \leq v \leq m_u\}_{t \in K} \) be \( s \) families of noncommutative random variables in a (usual) noncommutative probability space \( (A, \varphi) \). These families are indexed by a subset \( K \) of \( \mathbb{R} \) having zero as an accumulation point, and we are interested in the joint distribution \( \mu_t \) of \( \{a_u^v(t) \mid 1 \leq v \leq m_u, 1 \leq u \leq s\} \) when \( t \) is going to 0, in other words for infinitesimal values of \( t \). Recall that \( \mu_t \) is the linear functional on \( C(X_u^v, 1 \leq v \leq m_u, 1 \leq u \leq s) \) defined by :

\[
\mu_t(P((X_u^v)_{1 \leq v \leq m_u, 1 \leq u \leq s})) = \varphi(P((a_u^v(t))_{1 \leq v \leq m_u, 1 \leq u \leq s})).
\]

In what follows, we will consider a family \( \{\mu_t\}_{t \in K} \) of linear functionals on \( C(X_u^v, 1 \leq v \leq m_u, 1 \leq u \leq s) \) without any further reference to the variables \( \{a_u^v(t) \mid 1 \leq v \leq m_u, 1 \leq u \leq s\}_{t \in K} \). For each value of \( t \) in \( K \), one may obviously define the non-crossing cumulant functionals \( (\kappa_i)_n : (C(X_u^v, 1 \leq v \leq m_u, 1 \leq u \leq s))^{\min} \rightarrow C(X_u^v, 1 \leq v \leq m_u, 1 \leq u \leq s, \mu_t) \). A way to capture the behavior of \( \mu_t \) for infinitesimal values of \( t \) is to introduce recursively its derivatives at 0 by :

\[
\mu_{(0)} := \lim_{t \rightarrow 0} \mu_t,
\]

\[
\mu_{(i)} := \lim_{t \rightarrow 0} \frac{1}{t^i}(\mu_t - \sum_{j=0}^{i-1} \frac{t^j}{j!}\mu_{(j)}), 1 \leq i \leq k.
\]

We will assume that the limits in formulas (18) and (19) exist and use the notation \( \mu^{(i)} = \frac{d^{i}}{dt^{i}}|_{t=0}\mu_t \). Notice that, in [2], only \( \mu_{(0)} \) and \( \frac{d}{dt}|_{t=0}\mu_t \) were studied. It follows from formulas (18) and (19) that

\[
\mu_t = \sum_{i=0}^{k} \frac{\mu_{(i)}}{i!} t^i + o(t^k).
\]

Notice that \( (\mu^{(i)})_{0 \leq i \leq k} \) is an infinitesimal law (of order \( k \)) on \( \sum_{u=1}^{s} m_u \) variables and therefore \( (C(X_u^v, 1 \leq v \leq m_u, 1 \leq u \leq s), (\mu^{(i)})_{0 \leq i \leq k}) \) is an infinitesimal noncommutative probability space of order \( k \). Associated to this infinitesimal
noncommutative probability space of order $k$, we have infinitesimal non-crossing cumulant functionals $(\kappa_n^{(i)} : \mathbb{A}_n \rightarrow \mathbb{C}, 0 \leq i \leq k)^{\infty}_{n=1}$, as defined by formula (8). These infinitesimal cumulant functionals are linked to $((\kappa_t)_n)_{n=1}^{\infty}$ as follows:

**Proposition 19.** For every $n \geq 1$ and $0 \leq i \leq k$,

$$\kappa_n^{(i)} = \frac{d^i}{dt^i}|_{t=0} (\kappa_t)_n.$$ 

**Proof.** By the inverse of the free moment-cumulant formula, one has

$$\forall t \in K, (\kappa_t)_n = \sum_{p \in NC^{(\Lambda)}(n)} \text{Möb}(p, 1_n)(\mu_t)_p.$$ 

By the assumption made above, the right-hand side of formula (20) has $k$ derivatives at 0, hence $\frac{d^k}{dt^k}|_{t=0} (\kappa_t)_n$ is well-defined and, using linearity of derivation and Leibniz rule, one obtains:

$$\frac{d^k}{dt^k}|_{t=0} (\kappa_t)_n = \sum_{p \in NC^{(\Lambda)}(n)} \sum_{\lambda \in \Lambda_n} \sum_{\nu \in \Lambda_{n1}} \text{Möb}(p, 1_n)\lambda_1^{\Lambda_1} \cdots \lambda_k^{\Lambda_k} \mu_p^{(\Lambda)}.$$ 

One recognizes in the right-hand side above the right-hand side of formula (10), and we are done. \(\square\)

This proposition will be the main tool to characterize infinitesimal freeness of order $k$ in terms of moments in Theorem 5. We first give a recipe to deduce the infinitesimal behaviour of the free convolution of two families of distributions from their individual infinitesimal behaviours.

**Proposition 20.** Let $\{\mu_i \in K \text{ (resp. } \nu_i \in K) \}$ be a family of linear functionals on $\mathbb{C}(X_u, 1 \leq u \leq m)$ (resp $\mathbb{C}(Y_u, 1 \leq u \leq m)$) such that $\mu^{(i)} = \frac{d^i}{dt^i}|_{t=0} \mu_t$ (resp. $\nu^{(i)} = \frac{d^i}{dt^i}|_{t=0} \nu_t$) exist for $0 \leq i \leq k$. Set:

$$(\eta^{(i)})_{0 \leq i \leq k} := (\mu^{(i)})_{0 \leq i \leq k} \boxplus (\nu^{(i)})_{0 \leq i \leq k};$$

$$(\theta^{(i)})_{0 \leq i \leq k} := (\mu^{(i)})_{0 \leq i \leq k} \boxtimes (\nu^{(i)})_{0 \leq i \leq k}.$$ 

Then $\eta^{(i)} = \frac{d^i}{dt^i}|_{t=0} \mu_t \boxplus \nu_t$ and $\theta^{(i)} = \frac{d^i}{dt^i}|_{t=0} \mu_t \boxtimes \nu_t$.

**Proof.** For each $t \in K$, we consider the free product

$$\mathbb{C}(X_u, Y_u, 1 \leq u \leq m), \mu_t \ast \nu_t).$$

Since $\frac{d^i}{dt^i}|_{t=0} \mu_t$ and $\frac{d^i}{dt^i}|_{t=0} \nu_t$ exist by assumption for each $0 \leq i \leq k$, we obtain the existence of $\frac{d^i}{dt^i}|_{t=0} (\mu_t \ast \nu_t)$ for each $0 \leq i \leq k$ and these functionals are completely determined by the $\mu^{(i)}$’s and the $\nu^{(i)}$’s. In the infinitesimal noncommutative probability space $(\mathbb{C}(X_u, Y_u, 1 \leq u \leq m), (\frac{d}{dt}|_{t=0}(\mu_t \ast \nu_t))_{0 \leq i \leq k})$, the unital
subalgebras $A_1 = \mathbb{C}(X, 1 \leq u \leq m)$ and $A_2 = \mathbb{C}(Y, 1 \leq u \leq m)$ are infinitesimally free of order $k$; indeed, if $n \geq 1, 0 \leq i \leq k$ and $P_1 \in A_{i_1}, \ldots, P_n \in A_{i_n}$ are such that $i_1, \ldots, i_n$ are not all equal, then

$$
\kappa_n^{(i)}(P_1, \ldots, P_l) = \frac{d^i}{dt^i|_{t=0}} (\kappa_n(P_1, \ldots, P_l)),
$$

where $(\kappa_n)^{(i)}$ is the $n$-th non-crossing cumulant functional in the noncommutative probability space $(\mathbb{C}(X, Y, 1 \leq u \leq m), \mu \ast \nu)$, by Proposition 19. But it follows from the construction of the free product that $(\kappa_n)^{(i)}(P_1, \ldots, P_l) = 0$ for each $t \in K$. In particular $\kappa_n^{(i)}(P_1, \ldots, P_l) = 0$. The infinitesimal distribution of the $m$-tuple $(X_1 + Y_1, \ldots, X_m + Y_m)$ (resp. $(X_1 \cdot Y_1, \ldots, X_m \cdot Y_m)$) is, on the one hand, $(\frac{d^i}{dt^i|_{t=0}}(\mu_1 \ast \nu_1))_{0 \leq i \leq k}$ (resp. $(\frac{d^i}{dt^i|_{t=0}}(\mu_1 \circ \nu_1))_{0 \leq i \leq k}$) by construction of the free product and, on the other hand, $(\theta^{(i)})_{0 \leq i \leq k}$ (resp. $(\theta^{(i)})_{0 \leq i \leq k}$) by the argument above.

We conclude by a characterization of infinitesimal freeness of order $k$ in terms of moments. Its formulation and proof rely on the Proposition 19.

**Theorem 5.** Let $(A_1, (\varphi^{(i)}_{\infty})_{0 \leq i \leq k})$ be an infinitesimal noncommutative probability space of order $k$, and $A_1, \ldots, A_n$ be unital subalgebras of $A$. Then $A_1, \ldots, A_n$ are infinitesimally free of order $k$ if and only if for any positive integer $l \in \mathbb{N}$, and any $a_1 \in A_{i_1}, \ldots, a_l \in A_{i_l}$, one has

$$
\varphi_l((a_1 - \varphi_l(a_1)) \cdots (a_l - \varphi_l(a_l))) = o(t^k),
$$

whenever $i_1 \neq \ldots \neq i_l$, where $\varphi_l := \sum_{i=0}^k \varphi^{(i)}_{\infty} t^i$. The condition (21) translates into $k + 1$ requirements:

$$
\forall i \in \{0, \ldots, k\}, \sum_{j=0}^i \sum_{\lambda \in A_{i_1}, \ldots, A_{i_j}} (-1)^{\#\{m, \lambda_m > 0\}} \hat{\mu}^{(j)}(\hat{\mu}^{(\lambda_1)}(P_1) \cdots \hat{\mu}^{(\lambda_i)}(P_l)) = 0,
$$

where $\hat{\mu}^{(\lambda)}(P) := P - \mu^{(0)}(P)$ if $\lambda = 0$, and $\hat{\mu}^{(\lambda)}(P) := \mu^{(\lambda)}(P)$ else.

**Proof.** We assume that condition (21) holds and have to prove that $A_1, \ldots, A_n$ satisfy the vanishing of mixed infinitesimal cumulants condition. Using Proposition 19 it is equivalent to prove that for $l \geq 2$, and $a_1 \in A_{i_1}, \ldots, a_l \in A_{i_l}$

$$
(\kappa_l)_{l}(a_1, \ldots, a_l) = o(t^k)
$$

whenever $\exists r \neq s, i_r \neq i_s$, where $(\kappa_l)_{l}$ is the $l$-th non-crossing cumulant functional in $(A, \varphi_l)$. We proceed by induction on $l \geq 2$. It is easy to see that

$$
(\kappa_l)_{2}(a_1, a_2) = \varphi_l((a_1 - \varphi_l(a_1))(a_2 - \varphi_l(a_2))).
$$

35
If $a_1 \in A_{i_1}, a_2 \in A_{i_2}$ with $i_1 \neq i_2$, the right-hand side of (24) is $o(t^k)$ by assumption. We assume then that the vanishing of mixed infinitesimal cumulants is proved for $2, 3, \ldots, l - 1$ variables, and consider $(\kappa_t)_l(a_1, \ldots, a_l)$ with $a_1 \in A_{i_1}, \ldots, a_l \in A_{i_l}$ such that $\exists r \neq s, i_r \neq i_s$. By Propositions 2, 3 and the induction hypothesis, we may assume that $\varphi_t(a_1) = \ldots = \varphi_t(a_l) = 0$ and $i_1 \neq \ldots \neq i_l$. Write then the free moment-cumulant formula:

$$\forall t \in K, (\varphi_t)(a_1 \cdots a_l) - \sum_{p \in NC^{(A)}(l)} \kappa_t)_p(a_1, \ldots, a_l) = (\kappa_t)_i(a_1, \ldots, a_l).$$

By assumption, $(\varphi_t)(a_1 \cdots a_l) = o(t^k)$. Any non-crossing partition $p \neq 1_l$ owns an interval-block $V_0$, as noticed in Section 3. If $V_0$ is a singleton, 

$$(\kappa_t)_p(a_1, \ldots, a_l) = (\varphi_t)|_{V_0}((a_1, \ldots, a_l) \mid V_0) \prod_{V \neq V_0} (\kappa_t)|_V((a_1, \ldots, a_l) \mid V) = 0.$$

Otherwise, $V_0$ contains two following, hence distinct, indices, and, by induction hypothesis,

$$(\kappa_t)|_{V_0}((a_1, \ldots, a_l) \mid V_0) = o(t^k).$$

Since, for each $V \in bl(p)$, $(\kappa_t)|_V((a_1, \ldots, a_l) \mid V)$ is bounded in a neighborhood of 0, one may affirm that

$$(\kappa_t)_p(a_1, \ldots, a_l) = o(t^k).$$

We conclude that

$$(\kappa_t)_i(a_1, \ldots, a_l) = o(t^k),$$

as required.

For the converse, we assume that the vanishing of mixed infinitesimal cumulants is satisfied, or equivalently that equation (23) holds. We write then the free moment-cumulant formula:

$$\forall t \in K, (\varphi_t)(a_1 - \varphi_t(a_1) \cdots a_l - \varphi_t(a_l)) = (25)$$

$$\sum_{p \in NC^{(A)}(l)} (\kappa_t)_p(a_1 - \varphi_t(a_1), \ldots, a_l - \varphi_t(a_l)).$$

(26)

If $a_1 \in A_{i_1}, \ldots, a_l \in A_{i_l}$ with $i_1 \neq \ldots \neq i_n$, the same argument as above gives that (26) is $o(t^k)$. This concludes the proof.

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