Weighted Scale-Free Networks with Stochastic Weight Assignments

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Abstract

We propose and study a model of weighted scale-free networks incorporating a stochastic scheme for weight assignments to the links, taking into account both the popularity and fitness of a node. As the network grows the weights of links are driven either by the connectivity with probability $p$ or by the fitness with probability $1-p$. Results of numerical simulations show that the total weight associated with a selected node exhibits a power law distribution with an exponent $\sigma$, the value of which depends on the probability $p$. The exponent $\sigma$ decreases continuously as $p$ increases. For $p = 0$, the total weight distribution displays the same scaling behavior as that of the connectivity distribution with $\sigma = \gamma = 3$, where $\gamma$ is the exponent characterizing the connectivity distribution. An analytical expression for the total weight is derived so as to explain the features observed in the numerical results. Numerical results are also presented for a generalized model with a

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fitness-dependent link formation mechanism.

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I. INTRODUCTION

Many complex systems, including social, biological, physical, economic, and computer systems, can be studied using network models in which the nodes represent the constituents and links or edges represent the interactions between constituents [1,2]. One of important measures of the topological structure of a network is its connectivity distribution \( P(k) \), which is defined as the probability that a randomly selected node has exactly \( k \) edges. In traditional random graphs [3,4] as well as in the small-world networks [5–7] the connectivity distribution shows exponential decay in the tail. However, empirical studies on many real networks showed that the connectivity distribution exhibits a power law behavior \( P(k) \sim k^{-\gamma} \) for large \( k \) [1,2]. Networks with power-law connectivity distributions are called scale-free (SF) networks. Typical examples of SF networks include the Internet [8–10], World-Wide-Web [11–13], scientific citations [14], cells [15,16], the web of actors [17], and the web of human sexual contacts [18]. The first model of SF networks was proposed by Barabási and Albert (BA) [19]. In BA networks two important ingredients are included in order to obtain power law behavior in the connectivity distributions, namely the networks are continuously growing by adding in new nodes as time evolves, and the newly added nodes are preferentially attached to the highly connected nodes. The idea of incorporating preferential attachment in a growing network has led to proposals of a considerable number of models of SF networks [20–26] (see also Refs. [1,2] and references therein).

In most growing network models, all the links are considered equivalent. However, many real systems display different interaction strengths between nodes. It has been shown that in systems such as the social acquaintance network [27], the web of scientists with collaborations [28], and ecosystems [29], links between nodes may be different in their influence and the so-called the weak links play an important role in governing the network’s functions. Therefore, real systems are best described by weighted growing networks with non-uniform strengths of the links. Only recently, a class of models of weighted growing networks was proposed by Yook, Jeong and Barabási (YJB) [30]. In the basic weighted scale-free (WSF) model of YJB,
both the topology and the weight are driven by the connectivity according to the preferential attachment rule as the network grows. It was found that the total weight distribution follows a power law $P(w) \sim w^{-\sigma}$, with an exponent $\sigma$ different from the connectivity exponent $\gamma$. It was also shown analytically that the different scaling behavior in the weight and connectivity distributions are results of strong logarithmic corrections, and asymptotically (i.e., in the long time limit) the weighted and un-weighted models are identical [30].

In real systems one would expect that a link’s weight and/or the growth rate in the number of links of a node depend not only on the “popularity” of the node represented by the connectivity, but also on some intrinsic quality of the node. The intrinsic quality can be collectively represented by a parameter referred to as the “fitness” [31,32]. Besides popularity, the competitiveness of a node in a network may depend, taking for example a node being an individual in a certain community, on the personality, survival skills, character, etc.. A newly added node may take into account of these factors beside popularity in their decision on making connections with existing nodes and on the importance of each of the established links. Clearly, there is always a spectrum of personality among the nodes and therefore a distribution in the fitness. While one may argue that factors determining the popularity may overlap with those in fitness, it is not uncommon that popularity is not the major factor on the importance of a connection. For example, we often hear that a popular person actually has very few good friends, and an influential and powerful figure in a network may often be someone very difficult to work with. In the present work, we generalize the WSF model of YJB to study the effects of fitness. In our model, the weights assigned to the newly added links are determined stochastically either by the connectivity with probability $p$ or by the fitness of nodes with probability $1 - p$. The scaling behavior of the total weight distribution is found to be highly sensitive to the weight assignment mechanism through the parameter $p$.

The plan of the paper is as follows. In Sec. II, we present our model and simulation results. In Sec. III, we derive an analytical expression between the total weight and the total connectivity of a node and provide a theoretical explanation on the features observed in the
II. THE MODEL AND NUMERICAL RESULTS

The topological structure of our model follows that of the BA model of SF networks [19]. A small number \((m_0)\) of nodes are created initially. At each time step, a new node \(j\) with \(m\) \((m \leq m_0)\) links is added to the network. These \(m\) links will connect to \(m\) pre-existing nodes in the system according to the preferential attachment rule that the probability \(\Pi_i\) of an existing node \(i\) being selected for connection is proportional to the total number of links \(k_i\) that node \(i\) carries, i.e.,

\[
\Pi_i = \frac{k_i}{\sum_{i} k_i}. \tag{1}
\]

The procedure creates a network with \(N = t + m_0\) nodes and \(mt\) links after \(t\) time steps. Geometrically, the network displays a connectivity distribution with a power law decay in the tail with an exponent \(\gamma = 3\), regardless of the value of \(m\) [19,33].

A weighted growing network is constructed by assigning weights to the links as the network grows. To incorporate a fitness-dependent weight assignment mechanism, a fitness parameter \(\eta_i\) is assigned to each node [31,32]. The fitness \(\eta_i\) is chosen randomly from a distribution \(\rho(\eta)\), which is assumed to be a uniform distribution in the interval \([0,1]\) for simplicity. With probability \(p\), each newly established link \(j \leftrightarrow i\) is assigned a weight \(w_{ji}\) (= \(w_{ij}\) ) given by

\[
w_{ji} = \frac{k_i}{\sum_{i'} k_{i'}}, \tag{2}
\]

where \(\sum_{i'}\) is a sum over the \(m\) nodes to which the new node \(j\) is connected. With probability \(1 - p\), \(w_{ji}\) is determined by the fitness through

\[
w_{ji} = \frac{\eta_i}{\sum_{i'} \eta_{i'}}. \tag{3}
\]
In Eqs. (2) and (3), $w_{ji}$ is normalized so that the sum of the weights for the $m$ new links is unity, i.e., $\sum_{i'} w_{ji'} = 1$ [30]. For $p = 1$, our model reduces to the basic WSF model of YJB in which the weights are driven by the connectivity alone [30]. For $p = 0$, the weights are driven entirely by the fitness of the nodes. For $0 < p < 1$, the present model provides a possible stochastic weight assignment scheme in which a newly added node, e.g. representing some newcomer into a web, considers either the popularity, or the fitness of its connected neighbor in determining the influence of such a connection.

We performed extensive numerical simulations on the model. In our simulations, we studied networks up to $N = 5 \times 10^5$ nodes with $m = m_0 = 5$. For each value of $p$, results are obtained by averaging over 10 independent runs. First we study the total weight distribution $P(w)$, which is defined as the probability that a randomly selected node has a total weight $w$. The total weight of a node $i$ is given by the sum of the weights of all links connected to it, i.e., $w_i = \sum_j w_{ij}$. Figure 1 shows that $P(w)$ behaves as a power law $P(w) \sim w^{-\sigma}$, with an exponent $\sigma$ that decreases from the value of 3 at $p = 0$ continuously as $p$ increases. For $p = 1$, $\sigma = 2.4$, a result in agreement with that of the WSF model of YJB [30]. For $p = 0$, $\sigma = 3$ ($= \gamma$) showing that $P(w)$ follows the same scaling behavior as $P(k)$. YJB found that the scaling behavior of $P(w)$ depends strongly on $m$ [30] in their model. In the present model, we found that the $m$-dependence persists for all $p > 0$. Only when $p = 0$, $\sigma$ becomes independent of $m$.

It is also interesting to study the dynamical behavior of the total weight $w_i(\eta_i, t)$ of some node $i$ with fitness $\eta_i$. Fig. 2 shows that $w_i(\eta_i, t)$ grows with a power law behavior with time with an exponent $\delta$ that depends on $p$. For $p > 0$, $\delta > \beta$, where $\beta = 1/2$ is the exponent characterizing the dynamical behavior of the connectivity $k_i(t)$ [19]. For $p = 0$, $w_i(\eta_i, t)$ shows the same scaling behavior as $k_i(t)$ with $\delta = \beta = 1/2$. For $0 < p < 1$, $\delta$ depends on the node’s fitness $\eta_i$. Thus, the total weight actually shows a multi-scaling dynamical behavior in the range $0 < p < 1$ [31].

The probability distribution $P(w_{ij})$ of the weights $w_{ij}$ of individual links is also worth investigating. To suppress statistical fluctuations, Fig. 3 shows the cumulative distribution,
\( P(x > w_{ij}) \), instead of \( P(w_{ij}) \), on a log-linear scale. For \( p = 0 \), \( P(x > w_{ij}) \) decays exponentially in the tail. Recall that \( P(w) \) and \( w_i(\eta_i, t) \) show identical behavior as \( P(k) \) and \( k_i(t) \) for \( p = 0 \) respectively, and the latter two quantities are not sensitive to the weight assignment scheme. Here \( P(x > w_{ij}) \) shows an exponentially decaying behavior, implying that the weighted and un-weighted models are not totally identical even for \( p = 0 \). For \( p > 0 \), the tail deviates from an exponential decaying form and decays faster as \( p \) increases. For \( p = 1 \), we recover the results in the YJB model [30].

### III. ANALYTICAL SOLUTION

To understand the different behavior between \( w_i(\eta_i, t) \) and \( k_i(t) \) (as well as between \( P(w) \) and \( P(k) \)) found in numerical simulations, we derive an analytical expression for the total weight \( w_i(\eta_i, t) \) of a node \( i \) with fitness \( \eta_i \) at time \( t \). Following YJB [30], \( w_i(\eta_i, t) \) can be expressed as

\[
    w_i(\eta_i, t) = 1 + \int_{\delta_i}^t \int_m^\infty \int_0^1 \tilde{P}_i(m, t') w_{ji}((\eta_l, k_l)) \varrho(k_l) \rho(\eta_l) d\eta_l dk_l dt',
\]

where \( \tilde{P}_i(m, t) \) is the probability that node \( i \) is selected for connection to a new node \( j \) at time \( t \) for given \( m \) and it is related to \( \Pi_i \) in Eq.(1) by a factor of \( m \). Here, \( \delta_i \) is the time at which the node \( i \) has been added to the system. \( w_{ji}(\eta_l, k_l) \) is the weight assigned to the link between node \( j \) and node \( i \). \( \varrho(k) \) and \( \rho(\eta) \) are the probability distributions of \( k \) and \( \eta \), respectively. According to the stochastic weight assignment scheme modelled by Eqs. (2) and (3), the weight \( w_{ji}(\eta_l, k_l) \), on the average, can be written as

\[
    w_{ji}(\eta_l, k_l) = p \frac{k_i}{k_i + k_l} + (1 - p) \frac{\eta_i}{\eta_i + \eta_l},
\]

for the simple case of \( m = 2 \). Generalization to arbitrary value of \( m \) is straightforward.

From the connectedness of the SF model, \( \tilde{P}_i(m, t) \), \( \varrho(k) \) and \( k_i(t) \) are given by [33,30]

\[
    \tilde{P}_i(m, t) = m \Pi_i = \frac{k_i(t)}{2t},
\]

\[
    \varrho(k) = mk^{-2},
\]
\[ k_i(t) = \frac{m}{\sqrt{t_i^0}} \sqrt{t}. \quad (8) \]

Substituting Eqs. (5) - (8) into Eq. (4) and noticing that \( \rho(\eta) \) is assumed to be a uniform distribution in the interval \([0, 1]\), the integration in Eq. (4) can be carried out to give

\[ w_i(\eta_i, t) \simeq [p + 2(1 - p)\eta_i \ln \frac{1 + \eta_i}{\eta_i} |k_i(t)| - \frac{1}{4} p [(\ln \frac{4t}{t_i^0})^2 - 4 \ln 2 \ln \frac{t}{t_i^0}] + C, \quad (9) \]

where \( C \) is an integration constant. Eq. (9) implies that the different scaling behavior in \( w_i(\eta_i, t) \) and \( k_i(t) \) as shown in the simulations are results of the logarithmic correction term, which can be tuned by the parameter \( p \). For \( p \rightarrow 0 \), Eq. (9) gives

\[ w_i(\eta_i, t) \sim 2\eta_i \ln \frac{1 + \eta_i}{\eta_i} k_i(t), \quad (10) \]

leading to the same scaling behavior of \( w_i(\eta_i, t) \) and \( k_i(t) \), as observed in the simulation results. For \( p = 1 \) corresponding to the WSF model of YJB [30], the dynamical behavior of \( w_i(\eta_i, t) \) deviates most from that of \( k_i(t) \). For arbitrary \( m \), \( w_i(\eta_i, t) \) follows a similar form with \( m \) dependence coming into the second term on the right hand side of Eq. (9).

\section*{IV. DISCUSSION}

Our model can be easily generalized to allow for a fitness-dependent link formation mechanism [31,32]. In the basic model with fitness [31], the probability \( \Pi_i \) that a new link is established with an existing node \( i \) is determined jointly by the node’s connectivity \( k_i \) and fitness \( \eta_i \) with

\[ \Pi_i = \frac{\eta_i k_i}{\sum_i \eta_i k_i}. \quad (11) \]

To study the effects of fitness, we study a generalization of our model by replacing Eq. (1) by Eq. (11) for link formation, while keeping Eqs. (2) and (3) for weights assignments. The connectivity distribution follows a generalized power law [31] with an inverse logarithmic correction of the form

\[ P(k) \sim \frac{1}{\log k} k^{-\gamma'}, \quad (12) \]
with $\gamma' = 2.255$. Fig. 4 shows the numerical results for the total weight distribution $P(w)$ for three different values of $p = 0, 0.5$ and 1. It is found that $P(w)$ follows the same generalized power law form as $P(k)$, but with a different exponent $\sigma'$ that depends on $p$. For $p > 0$, $\sigma' < \gamma'$. Only for $p = 0$, $P(w)$ and $P(k)$ have the same exponent of $\sigma' = 2.25 \sim \gamma'$. For the cumulative distribution $P(x > w_{ij})$ of weights of individual links, the numerical results are similar to those shown in Fig. 3.

In summary, we proposed and studied a model of weighted scale-free networks in which the weights assigned to links as the network grows are stochastically determined by the connectivity of nodes with probability $p$ and by the fitness of nodes with probability $1 - p$. The model leads to a power law probability distribution for the total weight characterized by an exponent $\sigma$ that is highly sensitive to the probability $p$. If the weight is driven solely by the fitness, i.e., $p = 0$, $P(w)$ follows the same scaling behavior of $P(k)$ with the same exponent $\sigma = \gamma$. Similar results were also found in a generalized model with a fitness dependent link formation mechanism. An expression relating the total weight and the total connectivity of a node was derived analytically. The analytical result was used to explain the features observed in the results of numerical simulations. In closing, we note that although the total weight distribution $P(w)$ and the connectivity distribution $P(k)$ carry different exponents $\sigma$ and $\gamma$ for $p > 0$ in our model, $P(w)$ still follows a power law, i.e., has the same functional form as $P(k)$. The same feature was also found in the generalized model. However, one would expect that in some complex real systems even the functional forms of $P(w)$ and $P(k)$ may be different. It remains a challenge to introduce simple and yet non-trivial models that give one behavior for the geometrical connection among the constituents and another behavior for the extend of connectivity between constituents.

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FIGURE CAPTIONS

Figure 1: The weight distribution $P(w)$ as a function of the total weight $w$ on a log-log scale for different values of $p = 0, 0.1, 0.5, 1.0$. The two solid lines are guide to the eye corresponding to the exponents $\sigma = 2.4$ and $3.0$, respectively.

Figure 2: The total weight $w_i(\eta_i, t)$ of a randomly selected node $i$ with fitness $\eta_i (= 0.75)$ as a function of time $t$ on a log-log scale for different values of $p = 0, 0.5, 1.0$. The solid line is a guide to the eye corresponding to an exponent $\sigma = 0.5$.

Figure 3: The cumulative distribution $P(x > w_{ij})$ of the weights of individual links as a function of $w_{ij}$ on a log-linear plot for different values of $p = 0, 0.1, 0.5, 1.0$.

Figure 4: The weight distribution $P(w)$ as a function of the total weight $w$ on a log-log scale for different values of $p = 0, 0.5, 1.0$ in a model with fitness-dependent link formation mechanism. The two solid lines are plotted according to the form of Eq.(12), but with an exponent $\sigma'$ characterizing $P(w)$ that takes on the values $1.82$ and $2.25$ respectively.
Figure 1
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Figure 3
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