GLOBAL ATTRACTIVITY OF ALMOST PERIODIC SOLUTIONS FOR COMPETITIVE LOTKA–VOLTERRA DIFFUSION SYSTEM

Ahmadjan Muhammadhaji · Zhidong Teng · Mehbuba Rehim

Received: 23 December 2012 / Revised: 16 January 2013 / Accepted: 23 January 2013 / Published online: 13 March 2014
© Institute of Mathematics, Vietnam Academy of Science and Technology (VAST) and Springer Science+Business Media Singapore 2014

Abstract In this paper, two competitive Lotka–Volterra populations in the two-patch system with diffusion are considered. Each of the two species can diffuse independently and discretely between its intrapatch and interpatch. By means of a Lyapunov function, under a moderate condition, the system has a unique almost periodic solution, which is asymptotically stable and globally attractive.

Keywords Lotka–Volterra competitive system · Diffusion · Almost periodic solution · Asymptotic stability · Global attractivity

Mathematics Subject Classification (2010) 34K20 · 34D23 · 34D45 · 34D05 · 34K14

1 Introduction

Diffusion is a ubiquitous phenomenon in the real world. It is population pressure due to the mutual interference between the individuals, describing the migration of species to avoid crowds. It is important for us to understand the dynamics of populations of nature. Basic questions for the dynamics of populations include the persistence, permanence, and extinction of species, global stability of systems and the existence of positive periodic solutions, positive almost periodic solutions, and asymptotically periodic solutions. Recently, many scholars have paid attention to the nonautonomous Lotka–Volterra population models with diffusion. There exists an extensive literature concerning the study of global stability and
the existence of positive periodic solutions, positive almost periodic solutions, and asymptotically periodic solutions of the Lotka–Volterra system with diffusion and periodic parameters; see the papers [1–3, 5–19] and the references cited therein.

In [12], the authors studied the following nonautonomous Lotka–Volterra almost periodic cooperative systems with diffusion:

\[
\begin{align*}
\dot{x}_1 &= x_1 \left[ r_1(t) - a_{11}(t)x_1 + a_{12}(t)y_1 \right] + D_1(t)(x_2 - x_1), \\
\dot{y}_1 &= y_1 \left[ r_2(t) + a_{21}(t)x_1 - a_{22}(t)y_1 \right] + D_2(t)(y_2 - y_1), \\
\dot{x}_2 &= x_2 \left[ s_1(t) - b_{11}(t)x_2 + b_{12}(t)y_2 \right] + D_1(t)(x_1 - x_2), \\
\dot{y}_2 &= y_2 \left[ s_2(t) + b_{21}(t)x_2 - b_{22}(t)y_2 \right] + D_2(t)(y_1 - y_2).
\end{align*}
\]

By means of a Lyapunov function and under an appropriate conditions, sufficient conditions on the existence of a unique almost periodic solution and its global asymptotic stability are established for system (1). Based on system (1), in [13], the authors generalized the almost periodic system (1) to asymptotically periodic systems. Under suitable conditions, the authors proved that asymptotically periodic systems have a unique solution, which is globally asymptotically stable.

In [11], the authors studied the following nonautonomous Lotka–Volterra periodic competitive systems with diffusion:

\[
\begin{align*}
\dot{x}_1 &= x_1 \left[ r_1(t) - a_{11}(t)x_1 - a_{12}(t)y_1 \right] + D_1(t)(x_2 - x_1), \\
\dot{y}_1 &= y_1 \left[ r_2(t) - a_{21}(t)x_1 - a_{22}(t)y_1 \right] + D_2(t)(y_2 - y_1), \\
\dot{x}_2 &= x_2 \left[ s_1(t) - b_{11}(t)x_2 - b_{12}(t)y_2 \right] + D_1(t)(x_1 - x_2), \\
\dot{y}_2 &= y_2 \left[ s_2(t) + b_{21}(t)x_2 - b_{22}(t)y_2 \right] + D_2(t)(y_1 - y_2).
\end{align*}
\]

By using the Brouwer fixed point theorem and constructing a suitable Lyapunov function, under some appropriate conditions, the authors showed that the system has a unique periodic solution, which is globally stable.

Motivated by the works of Wei and Wang [12] and [13], by the Lyapunov method used in [12–14] we generalize system (2) to almost periodic systems. Under suitable conditions, we proved that the system has a unique almost periodic solution, which is asymptotically stable and globally attractive.

The organization of this paper is as follows. In the next section, we present some basic assumptions, notation, and lemmas. In Sect. 3, conditions for the almost periodic solution and asymptotic stability are considered. In Sect. 4, conditions for the global attractivity are given. In the final section, one example is designed to show that our main results are applicable.

## 2 Preliminaries

In system (2), we have that \(x_1(t), y_1(t)\) are the densities of two competitive species at time \(t\) at the first patch, \(x_2(t), y_2(t)\) are the densities of two competitive species at time \(t\) at the second patch, \(r_i(t)\) and \(s_i(t)\) are the intrinsic growth rates of two competitive species at the first and second patches, respectively, \(a_{ii}(t)\) and \(b_{ii}(t)\) are intrapatch restriction densities of each species in the two-patch-system, \(a_{ij}(t)\) and \(b_{ij}(t)\) \((i \neq j)\) are competitive coefficients between two species, and \(D_i(t)\) are the diffusion coefficients. We always assume that
system (2) satisfies the following assumption:

\[(H_1) \ r_i(t), s_i(t), a_{ij}(t), b_{ij}(t), \text{ and } D_i(t) \text{ are nonnegative continuous bounded almost periodic functions } (i,j = 1,2).\]

From the viewpoint of mathematical biology, for system (2), we only consider the solution with the following initial condition:

\[x_i(t) = \phi_i(t), \quad y_i(t) = \varphi_i(t) \quad \text{for all } t \in [0, +\infty), \quad i = 1,2,\]

where \(\phi_i(t), \varphi_i(t) \ (i = 1,2)\) are nonnegative continuous functions defined on \([0, +\infty)\) satisfying \(\phi_i(0) > 0, \varphi_i(0) > 0 \ (i = 1,2)\).

For a continuous and bounded function \(f(t)\) defined on \([0, +\infty)\), we define \(f^L = \inf_{t \in [0, +\infty)} \{f(t)\}\) and \(f^M = \sup_{t \in [0, +\infty)} \{f(t)\}\).

Now, we present some useful lemmas.

**Lemma 2.1** [11] \(R^4_{+0} = \{(x_1, y_1, x_2, y_2) \in R^4 \mid x_i \geq 0, \ y_i \geq 0, \ (i = 1,2)\}\) is the positive invariance set with respect to system (2).

**Lemma 2.2** [11] Suppose that the following inequalities hold:

\[D^M_1 < r^L_1, \quad D^M_2 < r^L_2, \quad D^M_1 < s^L_1, \quad D^M_2 < s^L_2.\]

Then there exists a compact region that has a positive distance from the coordinate hyperplane, and it attracts all the solutions of system (2) with positive initial values.

**Lemma 2.3** [12] Let \(D\) be an open set of \(R^4_+\), and \(V(t,x,y)\) be a function defined on the region \(R_+ \times D \times D\) or \(R_+ \times R^4_+ \times R^4_+\) satisfying:

(i) \(a(\|x - y\|) \leq V(t,x,y) \leq b(\|x - y\|)\), where \(a(r)\) and \(b(r)\) are continuous increasing positive functions;

(ii) \(\|V(t,x_1,y_1) - V(t,x_2,y_2)\| \leq k(\|x_1 - x_2\| + \|y_1 - y_2\|)\), where \(k > 0\) is a constant;

(iii) \(V'(t,x,y) \leq cV(t,x,y)\), where \(c > 0\) is a constant.

Further, let the solution of system (2) lie in a compact set \(\Omega\) for all \(t \geq t_0 > 0, \ \Omega \in D\). Then system (2) has a unique almost periodic solution \(z(t)\) in \(D\), \(z(t)\) lies in \(\Omega\), and it is uniformly asymptotically stable.

**Lemma 2.4** [4] Let \(f\) be a nonnegative function defined on \([0, \infty)\) such that \(f\) is integrable on \([0, \infty)\) and uniformly continuous on \([0, \infty)\). Then \(\lim_{t \to \infty} f(t) = 0\).

Wei et al. [11] obtained that system (2) has a bounded closed and convex set

\[\Omega = \{z : z \in R^4_+, S(z) \leq \beta, x^L_i \leq x_i \leq x^M_i, y^L_i \leq y_i \leq y^M_i \ (i = 1,2), \ R_+ = [0, +\infty)\},\]

where \(S(z), \beta, x^L_i, x^M_i, y^L_i, y^M_i\) are defined in [11, Theorems 3.1 and 4.1].
We discuss system (2) in \( \Omega \). In order to obtain an almost periodic solution and asymptotic stability of system (2), we introduce the following adjoint system of (2):

\[
\begin{align*}
\dot{x}_1 &= x_1 \left[ r_1(t) - a_{11}(t)x_1 - a_{12}(t)y_1 \right] + D_1(t)(x_2 - x_1), \\
\dot{y}_1 &= y_1 \left[ r_2(t) - a_{21}(t)x_1 - a_{22}(t)y_1 \right] + D_2(t)(y_2 - y_1), \\
\dot{x}_2 &= x_2 \left[ s_1(t) - b_{11}(t)x_2 - b_{12}(t)y_2 \right] + D_1(t)(x_1 - x_2), \\
\dot{y}_2 &= y_2 \left[ s_2(t) - b_{21}(t)x_2 - b_{22}(t)y_2 \right] + D_2(t)(y_1 - y_2), \\
\end{align*}
\]

(3)

Such an adjoint system can be found in [7, 12, 13].

3 Almost periodic solution and asymptotic stability

In this section, we derive some sufficient conditions for the existence of an almost periodic solution of system (2) and its asymptotic stability.

**Theorem 3.1** Let the conditions of (H1), Lemma 2.2, and Lemma 2.3 be satisfied. Further, assume that system (2) satisfies

\[
a^L_{11} + a^L_{21} > \frac{D^M_1}{x^L_2}, \quad b^L_{11} + b^L_{21} > \frac{D^M_1}{x^L_2}, \quad a^L_{12} + a^L_{22} > \frac{D^M_2}{y^L_2}, \quad b^L_{12} + b^L_{22} > \frac{D^M_2}{y^L_2}.
\]

Then (2) has a unique almost periodic solution, which is uniformly asymptotically stable.

**Proof** Let \( z(t) = (x_1(t), x_2(t), y_1(t), y_2(t)) \) be any solution of (2) from the attraction of \( \Omega \). We discuss our problem in \( \Omega \) and denote

\[
(t) = (X_1(t), X_2(t), Y_1(t), Y_2(t)).
\]

We consider the adjoint system (3) of (2). Let

\[
\begin{align*}
x_i(t) &= \ln x_i(t), \quad y_i(t) = \ln y_i(t), \quad \bar{x}_i(t) = \ln \bar{x}_i(t), \\
\bar{y}_i(t) &= \ln \bar{y}_i(t) \quad (i = 1, 2),
\end{align*}
\]

where \( x_i(t), y_i(t), \bar{x}_i(t), \bar{y}_i(t) \) \((i = 1, 2)\) are the solutions of the adjoint system (3) on \( \Omega \times \Omega \).

Define the Lyapunov function

\[
V(t) = V(t, Z(t), \bar{Z}(t)) = \sum_{i=1}^{2} \left| x_i(t) - \bar{x}_i(t) \right| + \sum_{i=1}^{2} \left| y_i(t) - \bar{y}_i(t) \right|.
\]

\( \textcopyright \) Springer
Taking

\[ a(r) = b(r) = \sum_{i=1}^{2} |X_i(t) - \tilde{X}_i(t)| + \sum_{i=1}^{2} |Y_i(t) - \tilde{Y}_i(t)|, \]

\( a(r) \) and \( b(r) \) are continuous increasing positive functions; then \( V(t) \) satisfies condition (i) of Lemma 2.3. Again from

\[
\sum_{i=1}^{2} |X_{i1}(t) - \tilde{X}_{i1}(t)| + \sum_{i=1}^{2} |Y_{i1}(t) - \tilde{Y}_{i1}(t)| \\
- \left( \sum_{i=1}^{2} |X_{i2}(t) - \tilde{X}_{i2}(t)| + \sum_{i=1}^{2} |Y_{i2}(t) - \tilde{Y}_{i2}(t)| \right) \\
\leq \sum_{i=1}^{2} |X_{i1}(t) - X_{i2}(t)| + \sum_{i=1}^{2} |Y_{i1}(t) - Y_{i2}(t)| \\
+ \sum_{i=1}^{2} |\tilde{X}_{i1}(t) - \tilde{X}_{i2}(t)| + \sum_{i=1}^{2} |\tilde{Y}_{i1}(t) - \tilde{Y}_{i2}(t)|
\]

it follows that \( V(t) \) satisfies condition (ii) of Lemma 2.3. To check condition (iii) of Lemma 2.3, we need to calculate the upper-right derivative of system (3). For convenience of statements, we denote

\[
A_i = \text{sign}(|X_i(t) - \tilde{X}_i(t)|), \quad B_i = \text{sign}(|Y_i(t) - \tilde{Y}_i(t)|) \quad (i = 1, 2),
\]

\[
D^+ V(t) = D^+ \left( \sum_{i=1}^{2} |X_i(t) - \tilde{X}_i(t)| + \sum_{i=1}^{2} |Y_i(t) - \tilde{Y}_i(t)| \right) \\
= \sum_{i=1}^{2} D^+ \left( |X_i(t) - \tilde{X}_i(t)| \right) + \sum_{i=1}^{2} D^+ \left( |Y_i(t) - \tilde{Y}_i(t)| \right) \\
= \sum_{i=1}^{2} A_i (\dot{X}_i(t) - \dot{\tilde{X}}_i(t)) + \sum_{i=1}^{2} B_i (\dot{Y}_i(t) - \dot{\tilde{Y}}_i(t)) \\
= \sum_{i=1}^{2} A_i \left( \frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{\tilde{x}}_i(t)}{\tilde{x}_i(t)} \right) + \sum_{i=1}^{2} B_i \left( \frac{\dot{y}_i(t)}{y_i(t)} - \frac{\dot{\tilde{y}}_i(t)}{\tilde{y}_i(t)} \right) \\
= -A_1 a_{11}(t) (x_1 - \tilde{x}_1) - A_1 a_{12}(t) (y_1 - \tilde{y}_1) + A_1 D_1(t) \left( \frac{x_2}{x_1} - \frac{\tilde{x}_2}{\tilde{x}_1} \right) \\
- A_2 b_1(t) (x_2 - \tilde{x}_2) - A_2 b_2(t) (y_2 - \tilde{y}_2) + A_2 D_2(t) \left( \frac{y_1}{y_2} - \frac{\tilde{y}_1}{\tilde{y}_2} \right) \\
- B_1 a_{21}(t) (x_1 - \tilde{x}_1) - B_1 a_{22}(t) (y_1 - \tilde{y}_1) + B_1 D_1(t) \left( \frac{y_2}{y_1} - \frac{\tilde{y}_2}{\tilde{y}_1} \right) \\
- B_2 b_{21}(t) (x_2 - \tilde{x}_2) - B_2 b_{22}(t) (y_2 - \tilde{y}_2) + B_2 D_2(t) \left( \frac{y_1}{y_2} - \frac{\tilde{y}_1}{\tilde{y}_2} \right)
\]
\[
\begin{align*}
&\leq -(a_{11}^L + a_{21}^L)|x_1 - \tilde{x}_1| - (b_{11}^L + b_{21}^L)|x_2 - \tilde{x}_2| \\
&\quad - (a_{12}^L + a_{22}^L)|y_1 - \tilde{y}_1| - (b_{12}^L + b_{22}^L)|y_2 - \tilde{y}_2| + \sum_{i=1}^{4} \tilde{D}_i(t),
\end{align*}
\]

where

\[
\tilde{D}_1(t) = \begin{cases} 
D_1(t)(\frac{x_1}{x_2} - \frac{\tilde{x}_1}{\tilde{x}_2}), & X_2(t) - \tilde{X}_2(t) \geq 0, \\
D_1(t)(\frac{\tilde{x}_1}{\tilde{x}_2} - \frac{x_1}{x_2}), & X_2(t) - \tilde{X}_2(t) < 0,
\end{cases}
\]

\[
\tilde{D}_2(t) = \begin{cases} 
D_1(t)(\frac{x_2}{x_1} - \frac{\tilde{x}_2}{\tilde{x}_1}), & X_1(t) - \tilde{X}_1(t) \geq 0, \\
D_1(t)(\frac{\tilde{x}_2}{\tilde{x}_1} - \frac{x_2}{x_1}), & X_1(t) - \tilde{X}_1(t) < 0,
\end{cases}
\]

\[
\tilde{D}_3(t) = \begin{cases} 
D_2(t)(\frac{y_1}{y_2} - \frac{\tilde{y}_1}{\tilde{y}_2}), & Y_2(t) - \tilde{Y}_2(t) \geq 0, \\
D_2(t)(\frac{\tilde{y}_1}{\tilde{y}_2} - \frac{y_1}{y_2}), & Y_2(t) - \tilde{Y}_2(t) < 0,
\end{cases}
\]

\[
\tilde{D}_4(t) = \begin{cases} 
D_2(t)(\frac{y_2}{y_1} - \frac{\tilde{y}_2}{\tilde{y}_1}), & Y_1(t) - \tilde{Y}_1(t) \geq 0, \\
D_2(t)(\frac{\tilde{y}_2}{\tilde{y}_1} - \frac{y_2}{y_1}), & Y_1(t) - \tilde{Y}_1(t) < 0.
\end{cases}
\]

There are the following three cases to consider for \(\tilde{D}_1(t)\):

(i) If \(X_2(t) > \tilde{X}_2(t)\) and \(t \geq t^*\), then

\[
\tilde{D}_1(t) \leq \frac{D_1(t)}{x_2(t)} (x_1(t) - \tilde{x}_1(t)) \leq \frac{D_1^M}{x_2(t)} |x_1(t) - \tilde{x}_1(t)|.
\]

(ii) If \(X_2(t) < \tilde{X}_2(t)\) and \(t \geq t^*\), then

\[
\tilde{D}_1(t) \leq \frac{D_1(t)}{x_2(t)} (\tilde{x}_1(t) - x_1(t)) \leq \frac{D_1^M}{x_2(t)} |x_1(t) - \tilde{x}_1(t)|.
\]

(iii) If \(X_2(t) = \tilde{X}_2(t)\), arguing similarly as in the above analysis, we can get the same result as (i) and (ii).

From (i)–(iii) we have

\[
\tilde{D}_1(t) \leq \frac{D_1^M}{x_2} |x_1(t) - \tilde{x}_1(t)| \quad \text{for } t \geq t^*.
\]

Considering \(\tilde{D}_2(t), \tilde{D}_3(t), \tilde{D}_4(t)\) in the same way, we can obtain

\[
\tilde{D}_2(t) \leq \frac{D_2^M}{x_1} |x_2(t) - \tilde{x}_2(t)|, \quad \tilde{D}_3(t) \leq \frac{D_2^M}{y_1} |y_1(t) - \tilde{y}_1(t)|, \quad \tilde{D}_4(t) \leq \frac{D_2^M}{y_1} |y_2(t) - \tilde{y}_2(t)|.
\]
for $t \geq t^*$. It then yields that

$$D^+ V(t) \leq -\left( a^{L}_{11} + a^{L}_{21} - \frac{D^M_1}{x_1^L} \right) |x_1 - \tilde{x}_1| - \left( b^{L}_{11} + b^{L}_{21} - \frac{D^M_1}{x_1^L} \right) |x_2 - \tilde{x}_2|$$

$$- \left( a^{L}_{12} + a^{L}_{22} - \frac{D^M_2}{y_2^L} \right) |y_1 - \tilde{y}_1| - \left( b^{L}_{12} + b^{L}_{22} - \frac{D^M_2}{y_1^L} \right) |y_2 - \tilde{y}_2|.$$ 

According to the condition of Theorem 3.1, now we let

$$P_1 := a^{L}_{11} + a^{L}_{21} - \frac{D^M_1}{x_1^L}, \quad P_2 := b^{L}_{11} + b^{L}_{21} - \frac{D^M_1}{x_1^L},$$

$$P_3 := a^{L}_{12} + a^{L}_{22} - \frac{D^M_2}{y_2^L}, \quad P_4 := b^{L}_{12} + b^{L}_{22} - \frac{D^M_2}{y_1^L},$$

and $\eta = \min\{P_1, P_2, P_3, P_4\} > 0$. Then we get that

$$D^+ V(t) \leq -\eta \left( \sum_{i=1}^{2} |x_i - \tilde{x}_i| + \sum_{i=1}^{2} |y_i - \tilde{y}_i| \right). \quad (4)$$

By the mean value theorem we have

$$|x_i - \tilde{x}_i| = |e^{X_i} - e^{\tilde{X}_i}| = \zeta_i(t) |X_i - \tilde{X}_i| \geq x_i^L |X_i - \tilde{X}_i|,$$

$$|y_i - \tilde{y}_i| = |e^{Y_i} - e^{\tilde{Y}_i}| = \xi_i(t) |Y_i - \tilde{Y}_i| \geq y_i^L |Y_i - \tilde{Y}_i| \quad (i = 1, 2),$$

where $\zeta_i(t) \in (x_i(t), \tilde{x}_i(t))$ $(i = 1, 2)$ and $\xi_i(t) \in (y_i(t), \tilde{y}_i(t))$ $(i = 1, 2)$, respectively; then $\zeta_i(t) \in \Omega$ and $\xi_i(t) \in \Omega$.

By the above formulas, taking $c = \min\{x_1^L \eta, x_2^L \eta, y_1^L \eta, y_2^L \eta\} > 0$, we get that

$$D^+ V(t) \leq -c \left( \sum_{i=1}^{2} |X_i(t) - \tilde{X}_i(t)| + \sum_{i=1}^{2} |Y_i(t) - \tilde{Y}_i(t)| \right) = -c V(t),$$

which means that $V(t)$ satisfies condition (iii) of Lemma 2.3. By Lemma 2.3, system (2) has a unique almost periodic solution $z(t)$ on the region $\Omega$, which is uniformly asymptotically stable on the compact set $\Omega$. Since $\Omega$ is the ultimately bounded region and compact set of system (2), we get that the solution $z(t)$ is ultimately bounded on $\Omega$, and therefore, when the conditions of Lemma 2.3 hold, the almost periodic solution $z(t)$ is uniformly asymptotically stable. It shows that system (2) has a unique almost periodic solution, which is uniformly asymptotically stable. This completes the proof. □

4 Global attractivity

In this section, we derive some sufficient conditions for the global attractivity of system (2).

**Theorem 4.1** If system (2) satisfies all the conditions of Theorem 3.1, then the unique almost periodic solution of system (2) is globally attractive.
Proof Let \( z(t) = (x_1(t), x_2(t), y_1(t), y_2(t)) \) be a definitive almost periodic solution of (2), and \( \tilde{z}(t) = (\tilde{x}_1(t), \tilde{x}_2(t), \tilde{y}_1(t), \tilde{y}_2(t)) \) be any solution of system (2).

Consider the same Lyapunov function as defined in the proof of Theorem 3.1,

\[
V(t) = V(t, Z(t), \tilde{Z}(t)) = \sum_{i=1}^{2} |X_i(t) - \tilde{X}_i(t)| + \sum_{i=1}^{2} |Y_i(t) - \tilde{Y}_i(t)|. \tag{5}
\]

Integrating both sides of (4) from 0 to \( t \), we derive

\[
V(t) + \eta \int_{0}^{t} \left( \sum_{i=1}^{2} |x_i(s) - \tilde{x}_i(s)| + \sum_{i=1}^{2} |y_i(s) - \tilde{y}_i(s)| \right) ds \leq V(0). \tag{6}
\]

Expression (6) shows that

\[
0 \leq V(t) \leq V(0) = \sum_{i=1}^{2} |X_i(0) - \tilde{X}_i(0)| + \sum_{i=1}^{2} |Y_i(0) - \tilde{Y}_i(0)| < +\infty, \quad t \geq 0, \tag{7}
\]

and

\[
\int_{0}^{t} \left( \sum_{i=1}^{2} |x_i(s) - \tilde{x}_i(s)| + \sum_{i=1}^{2} |y_i(s) - \tilde{y}_i(s)| \right) ds \leq \frac{V(0)}{\eta} < +\infty, \quad t \geq 0. \tag{8}
\]

Expression (8) implies that

\[
\sum_{i=1}^{2} |x_i(s) - \tilde{x}_i(s)| + \sum_{i=1}^{2} |y_i(s) - \tilde{y}_i(s)| \in L^{1}[0, +\infty). \tag{9}
\]

Obviously \( x_i(t) \) and \( y_i(t) \) \((i = 1, 2)\) are uniformly bounded, so \( X_i(t) \) and \( Y_i(t) \) \((i = 1, 2)\) are also uniformly bounded. In addition, by (5)–(7) we can know that \( \tilde{X}_i(t) \) and \( \tilde{Y}_i(t) \) \((i = 1, 2)\) are uniformly bounded, so \( \tilde{x}_i(t) \) and \( \tilde{y}_i(t) \) \((i = 1, 2)\) are also uniformly bounded. Combining this fact with system (2), we see that \( \dot{x}_i, \dot{y}_i, \dot{\tilde{x}}_i, \dot{\tilde{y}}_i \) \((i = 1, 2)\) are uniformly bounded. Therefore, we can easily check that \([x_i(t) - \tilde{x}_i(t)]\) and \([y_i(t) - \tilde{y}_i(t)]\) \((i = 1, 2)\) and their derivatives remain bounded on \([0, +\infty)\). As a consequence, \( \sum_{i=1}^{2} |x_i(t) - \tilde{x}_i(t)| + \sum_{i=1}^{2} |y_i(t) - \tilde{y}_i(t)| \) is uniformly continuous on \([0, +\infty)\). From expression (9) it follows that \( \sum_{i=1}^{2} |x_i(t) - \tilde{x}_i(t)| + \sum_{i=1}^{2} |y_i(t) - \tilde{y}_i(t)| \) is integrable on \([0, +\infty)\). By Lemma 2.4 it follows that

\[
\lim_{t \to \infty} \left( \sum_{i=1}^{2} |x_i(t) - \tilde{x}_i(t)| + \sum_{i=1}^{2} |y_i(t) - \tilde{y}_i(t)| \right) = 0.
\]

Hence,

\[
\lim_{t \to \infty} |x_i(t) - \tilde{x}_i(t)| = 0, \quad \lim_{t \to \infty} |y_i(t) - \tilde{y}_i(t)| = 0 \quad (i = 1, 2).
\]

This result implies that the unique almost periodic solution of system (2) is stable and attracts all positive solutions of system (2). This completes the proof. \( \square \)
5 One example

Consider the system

\[
\begin{aligned}
\dot{x}_1(t) &= x_1(t)(5 + 0.5(\sin \sqrt{2}t + \sin t) - (2.5 + 0.5(\cos \sqrt{2}t + \cos t))x_1(t) \\
&\quad - (2.2 + 0.3(\sin \sqrt{2}t + \sin t))y_1(t)) \\
&\quad + (1 + 0.1(\cos \sqrt{2}t + \cos t))(x_2(t) - x_1(t)), \\
\dot{y}_1(t) &= y_1(t)(5 + 0.4(\sin \sqrt{2}t + \sin t) - (2.25 + 0.6(\cos \sqrt{2}t + \cos t))x_1(t) \\
&\quad - (2.4 + 0.4(\sin \sqrt{2}t + \sin t))y_1(t)) \\
&\quad + (1 + 0.2(\sin \sqrt{2}t + \sin t))(y_2(t) - y_1(t)), \\
\dot{x}_2(t) &= x_2(t)(4 + 0.5(\cos \sqrt{2}t + \cos t) - (2.4 + 0.7(\sin \sqrt{2}t + \sin t))x_2(t) \\
&\quad - (2.3 + 0.5(\cos \sqrt{2}t + \cos t))y_2(t)) \\
&\quad + (1 + 0.1(\cos \sqrt{2}t + \cos t))(x_2(t) - x_1(t)), \\
\dot{y}_2(t) &= y_2(t)(4 + 0.3(\cos \sqrt{2}t + \cos t) - (2.3 + 0.5(\sin \sqrt{2}t + \sin t))x_2(t) \\
&\quad - (2.5 + 0.3(\cos \sqrt{2}t + \cos t))y_2(t)) \\
&\quad + (1 + 0.2(\sin \sqrt{2}t + \sin t))(y_1(t) - y_2(t)).
\end{aligned}
\]

It is easy to verify that all the conditions required in Theorems 3.1 and 4.1 are satisfied. Then (10) has a unique almost periodic solution and which is asymptotically stable and globally attractive.

Figure 1 shows that system (10) converges to an almost periodic solution. Figure 2 shows that the almost periodic solution of system (10) is globally attractive.
Fig. 2 Global attractivity of almost periodic solutions for system (10). Here, we take different initial values

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant Nos. 11261061, 61362039, 10661010, 11271312, 11261058, 2013211B12), the doctoral program of Xinjiang Normal University (Grant No. XJNUBS1402), the Natural Science Foundation of Xinjiang (Grant No. 200721104).

References

1. Chen, F.: Persistence and global stability for nonautonomous cooperative system with diffusion and time delay. Acta Sci. Nat. Univ. Pekin. 39, 22–28 (2003)
2. Ding, X., Wang, F.: Positive periodic solution for a semi-ratio-dependent predator–prey system with diffusion and time delays. Nonlinear Anal., Real World Appl. 9, 239–249 (2008)
3. Dong, L., Chen, L., Shi, P.: Periodic solutions for a two-species nonautonomous competition system with diffusion and impulses. Chaos Solitons Fractals 32, 1916–1926 (2007)
4. Gopalasamy, K.: Stability and Oscillation in Delay Equation of Population Dynamics. Kluwer Academic, Dordrecht (1992)
5. Kishimoto, K.: Coexistence of any number of species in the Lotka–Volterra competition system over two-patches. Theor. Popul. Biol. 38, 149–158 (1990)
6. Liang, Y., Li, L., Chen, L.: Almost periodic solutions for Lotka–Volterra systems with delays. Commun. Nonlinear Sci. Numer. Simul. 14, 3660–3669 (2009)
7. Liu, C., Chen, L.: Periodic solution and global stability for nonautonomous cooperative Lotka–Volterra diffusion system. J. LanZhou Univ. Nat. Sci. 33, 33–37 (1997)
8. Liu, Z., Zhong, S.: Permanence and extinction analysis for a delayed periodic predator–prey system with Holling type II response function and diffusion. Appl. Math. Comput. 216, 3002–3015 (2010)
9. Meng, X., Chen, L.: Periodic solution and almost periodic solution for a nonautonomous Lotka–Volterra dispersal system with infinite delay. J. Math. Anal. Appl. 339, 125–145 (2008)
10. Song, X., Chen, L.: Uniform persistence and global attractivity for nonautonomous competitive systems with dispersion. J. Syst. Sci. Complex. 15, 307–314 (2002)
11. Wei, F., Lin, Y., Que, L., Chen, Y., Wu, Y., Xue, Y.: Periodic solution and global stability for a nonautonomous competitive Lotka–Volterra diffusion system. Appl. Math. Comput. 216, 3097–3104 (2010)
12. Wei, F., Wang, K.: Almost periodic solution and stability for nonautonomous cooperative Lotka–Volterra diffusion system. SongLiao J. Nat. Sci. Ed. 3, 1–4 (2002)
13. Wei, F., Wang, K.: Global stability and asymptotically periodic solution for nonautonomous cooperative Lotka–Volterra diffusion system. Appl. Math. Comput. 182, 161–165 (2006)
14. Wei, F., Wang, S.: Almost periodic solution and global stability for cooperative L–V diffusion system. J. Math. Res. Expo. 30, 1108–1116 (2010)
15. Zeng, G., Chen, L.: Persistence and periodic orbits for two-species nonautonomous diffusion Lotka–Volterra models. Math. Comput. Model. 20, 69–80 (1994)
16. Zhang, J., Chen, L.: Permanence and global stability for two-species cooperative system with delays in two-patch environment. Math. Comput. Model. 23, 17–27 (1996)
17. Zhang, J., Chen, L., Chen, X.: Persistence and global stability for two-species nonautonomous competition Lotka–Volterra patch-system with time delay. Nonlinear Anal. 37, 1019–1028 (1999)
18. Zhang, Z., Wang, Z.: Periodic solution for a two-species nonautonomous competition Lotka–Volterra patch system with time delay. J. Math. Anal. Appl. 265, 38–48 (2002)
19. Zhou, X., Shi, X., Song, X.: Analysis of nonautonomous predator–prey model with nonlinear diffusion and time delay. Appl. Math. Comput. 196, 129–136 (2008)