Stability of the Solitary Manifold of the Perturbed Sine-Gordon Equation

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Abstract
We study the perturbed sine-Gordon equation
\[ \theta_{tt} - \theta_{xx} + \sin \theta = F(\varepsilon, x), \]
where \( F \) is of differentiability class \( C^n \) in \( \varepsilon \) and the first \( k \) derivatives vanish at 0, i.e., \( \partial_l F(0, \cdot) = 0 \) for \( 0 \leq l \leq k \). We construct implicitly a virtual solitary manifold by deformation of the classical solitary manifold in \( n \) iteration steps. Our main result establishes that the initial value problem with an appropriate initial state \( \varepsilon^n \)-close to the virtual solitary manifold has a unique solution which follows up to time \( 1/(C\varepsilon^{k+1}) \) and errors of order \( \varepsilon^n \) a trajectory on the virtual solitary manifold. The trajectory on the virtual solitary manifold is described by two parameters which satisfy a system of ODEs. In contrast to previous works our stability result yields arbitrarily high accuracy as long as the perturbation \( F \) is sufficiently often differentiable.

1 Introduction
The perturbed sine-Gordon equation
\[ \theta_{tt} - \theta_{xx} + \sin \theta = F(\varepsilon, x), \quad t, x \in \mathbb{R}, \quad \varepsilon \ll 1, \tag{1} \]
is a Hamiltonian evolution equation with Hamiltonian given by
\[ H^\varepsilon(\theta, \psi) = \frac{1}{2} \int \psi^2 + \theta^2_x + 2(1 - \cos \theta) - 2F(\varepsilon, x)\theta \, dx \]
and the symplectic form given by
\[ \Omega \left( \left( \begin{array}{c} \theta' \\ \psi' \end{array} \right), \left( \begin{array}{c} \theta \\ \psi \end{array} \right) \right) = \left< \left( \begin{array}{c} \theta' \\ \psi' \end{array} \right), \mathcal{J} \left( \begin{array}{c} \theta \\ \psi \end{array} \right) \right>_{L^2(\mathbb{R}) \otimes L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi'(x)\theta(x) - \theta'(x)\psi(x) \, dx, \tag{2} \]
where
\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

In first order formulation (1) can be written as a system:
\[ \partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} \psi \\ \theta \end{bmatrix}_{xx} - \sin \theta + F(\varepsilon, x) \end{pmatrix}. \tag{3} \]

The unperturbed sine-Gordon equation \( F(\varepsilon, x) = 0 \), admits soliton solutions \( \left( \theta_0(\xi(t), u(t), x), \psi_0(\xi(t), u(t), x) \right) \), where
\[ \dot{\xi} = u, \quad \dot{u} = 0, \quad (\xi(0), u(0)) = (a, v) \in \mathbb{R} \times (-1, 1). \]

Here the functions \( (\theta_0, \psi_0) \) are defined by
\[ \begin{pmatrix} \theta_0(\xi, u, x) \\ \psi_0(\xi, u, x) \end{pmatrix} := \begin{pmatrix} \theta_K(\gamma(u)(x - \xi)) \\ -u\gamma(u)\theta_K'(\gamma(u)(x - \xi)) \end{pmatrix}, \quad u \in (-1, 1), \quad \xi, x \in \mathbb{R}, \tag{4} \]
where
\[ \gamma(u) = \frac{1}{\sqrt{1 - u^2}}, \quad \theta_K(x) = 4 \arctan(e^x), \]
and \( \theta_K \) satisfies \( \theta_K''(x) = \sin \theta_K(x) \) with boundary conditions \( \theta_K(x) \to \begin{pmatrix} 2\pi \\ 0 \end{pmatrix} \) as \( x \to \pm \infty \).

The states \( \begin{pmatrix} \theta_0(a, v, \cdot) \\ \psi_0(a, v, \cdot) \end{pmatrix} \) form the classical two-dimensional solitary manifold
\[ S_0 := \left\{ \begin{pmatrix} \theta_0(a, v, \cdot) \\ \psi_0(a, v, \cdot) \end{pmatrix} : v \in (-1, 1), \quad a \in \mathbb{R} \right\}. \]

In the present paper, we assume that the perturbation term \( F \) in (1) is of differentiability class \( C^n \) in \( \varepsilon \) and that its first \( k \) derivatives vanish at 0, i.e.,
\[ \partial^l F(0, \cdot) = 0 \quad \text{for} \quad 0 \leq l \leq k. \tag{5} \]

We construct a virtual solitary manifold \( S^\varepsilon_n \), which is adjusted to the perturbation \( F \). The construction can be thought of as a successive distortion of the classical solitary manifold \( S_0 \). It is based on an iteration scheme composed of \( n \) steps, where in each iteration step a specific PDE will be solved implicitly. In the last iteration step we obtain an implicit solution \( \begin{pmatrix} \theta_n^\varepsilon(\xi, u, x), \psi_n^\varepsilon(\xi, u, x) \end{pmatrix} \) which defines the virtual solitary manifold
\[ S^\varepsilon_n := \left\{ \begin{pmatrix} \theta_n^\varepsilon(a, v, \cdot) \\ \psi_n^\varepsilon(a, v, \cdot) \end{pmatrix} : v \in (-u_*, u_*), \quad a \in \mathbb{R} \right\}, \quad u_* \in (0, 1]. \tag{6} \]
We consider for $\xi_s \in \mathbb{R}$ and $\varepsilon \ll 1$ the Cauchy problem
\begin{equation}
\partial_t \left( \begin{array}{c} \theta \\ \psi \end{array} \right) = \left( \begin{array}{c} \psi \\ \phi \end{array} \right),
\left( \begin{array}{c} \theta(0, x) \\ \psi(0, x) \end{array} \right) = \left( \begin{array}{c} \theta_n(\xi_s, u_s, x) \\ \psi_n(\xi_s, u_s, x) \end{array} \right) + \left( \begin{array}{c} v(0, x) \\ w(0, x) \end{array} \right),
\end{equation}
with initial data $\varepsilon^n$-close to the virtual solitary manifold $S^\varepsilon_n$, i.e.,
\[ |v(0, \cdot)|^2_{L^2(\mathbb{R})} + |w(0, \cdot)|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n}. \]
Further, we suppose that $(v(0, \cdot), w(0, \cdot))$ is symplectic orthogonal to the tangent space of $S^\varepsilon_n$ at $(\theta_n(\xi_s, u_s, \cdot), \psi_n(\xi_s, u_s, \cdot))$ and that the smallness assumption
\[ |u_s| \leq \tilde{C} \varepsilon^{\frac{1}{2n}} \]
is satisfied.

Our main theorem shows that, under the mentioned assumptions on the initial data, the Cauchy problem (7) has a unique solution $(\theta, \psi)$ for times
\[ 0 \leq t \leq \frac{1}{\tilde{C} \varepsilon^{\frac{1}{2n}}}, \]
which may be written in the form
\[ \left( \begin{array}{c} \theta(t, x) \\ \psi(t, x) \end{array} \right) = \left( \begin{array}{c} \theta_n(\xi(t), \bar{u}(t), x) \\ \psi_n(\xi(t), \bar{u}(t), x) \end{array} \right) + \left( \begin{array}{c} v(t, x) \\ w(t, x) \end{array} \right). \]
Furthermore, the solution remains $\varepsilon^n$-close to the manifold $S^\varepsilon_n$, i.e.,
\[ |v(t, \cdot)|^2_{H^1(\mathbb{R})} + |w(t, \cdot)|^2_{L^2(\mathbb{R})} \leq \tilde{C} \varepsilon^{2n} \]
and the parameters $(\bar{\xi}(t), \bar{u}(t))$ satisfy the ODEs
\[ \dot{\bar{\xi}}(t) = \bar{u}(t), \quad \dot{\bar{u}}(t) = \lambda_n^\varepsilon(\bar{\xi}(t), \bar{u}(t)), \]
with initial data $\bar{\xi}(0) = \xi_s$, $\bar{u}(0) = u_s$, where $\lambda_n^\varepsilon$ is defined implicitly. The time scale (8) is nontrivial and the parameters $\bar{\xi}, \bar{u}$ describe a fixed nontrivial perturbation of the uniform linear motion as $\varepsilon \to 0$ if the perturbation $F$ satisfies condition (19) mentioned below.

This result yields a fairly accurate description of the solution $(\theta, \psi)$ to the Cauchy problem (7), since we are able to control the dynamics on the virtual solitary manifold $S^\varepsilon_n$ by the ODEs (10) and the dynamics of the transversal component $(v(t, \cdot), w(t, \cdot))$ by the upper bound on its norm (9).

The higher the differentiability class $C^n$ of the perturbation $F$ the higher is the accuracy of our stability statement. The time scale of the result is the larger the more first derivatives of $F$ vanish at $0$. A precise statement is found in Section 2.
Let us mention related works and give some background to our paper. Orbital stability of soliton solutions under perturbations of the initial data has been proven for the (unperturbed) sine-Gordon equation (see [HPWS82], [Stu12, Section 4]). In [Stu92] D. M. Stuart investigated the perturbed sine-Gordon equation,

$$\theta_{TT} - \theta_{XX} + \sin \theta + \varepsilon g = 0,$$

where the perturbation $g = g(\theta)$ is a smooth function such that $g_0(Z) = g(\theta_K(Z)) \in L^2(dZ)$ and $\varepsilon \ll 1$. He proved that there exists $T^* = O\left(\frac{1}{\varepsilon}\right)$ such that the corresponding initial value problem with initial data

$$\theta(0, X) = \theta_K(Z(0)) + \varepsilon \tilde{\theta}(0, X), \quad \theta_T(0, X) = \frac{-u(0)}{\sqrt{1 - u(0)^2}} \theta_K'(Z(0)) + \varepsilon \tilde{\theta}_T(0, X),$$

$$(\tilde{\theta}(0, X), \tilde{\theta}_T(0, X)) \in H^1 \oplus L^2,$$

has a unique solution of the form

$$\theta(T, X) = \theta_K(Z) + \varepsilon \tilde{\theta}(T, X), \quad Z = \frac{X - \int^T_0 u - C(T)}{\sqrt{1 - u^2}},$$

where $\tilde{\theta} \in C([0, T_*], H^1)$, $\theta_T \in C([0, T_*], L^2)$ and

$$C(T) = C_0(\varepsilon T) + \varepsilon \tilde{C}, \quad u(T) = u_0(\varepsilon T) + \varepsilon \tilde{u}(T) \quad (\Rightarrow p = \frac{u}{\sqrt{1 - u^2}} = p_0(\varepsilon T) + \varepsilon \tilde{p}(T)).$$

Here $\tilde{p}$, $\tilde{u}$, $\tilde{C}$, $\frac{d\tilde{u}}{dT}$, $\frac{d\tilde{C}}{dT}$, $|\tilde{\theta}|_{H^1(\mathbb{R})}$ are bounded independent of $\varepsilon$ and $u_0, C_0$ are solutions of certain explicitly given modulation equations. The proof is based on an orthogonal decomposition of the solution into an oscillatory part and a one-dimensional "zero-mode" term.

The sine-Gordon equation arises in various physical phenomena such as dynamics of long Josephson junctions [ZH93], [KMS99], dislocations in crystals [FK39], waves in ferromagnetic materials [Mik79], etc. In [Sky61] T. H. R. Skyrme proposed the equation to model elementary particles. Dynamics of solitons under constant electric field were examined numerically in [IC79]. In the present paper, we investigate virtual solitons in the presence of a time independent electric field $F(\varepsilon, x)$, which is a physically relevant problem.

There are also many other long (but finite)-time results for different equations with external potentials such as [FGJS04], [FGS06], [HZ07], [Hol11]. For instance, in [HZ08] J. Holmer and M. Zworski considered the Gross-Pitaevskii equation

$$\begin{cases}
    i\partial_t u + \frac{1}{2} \partial_x^2 u - V(x) u + u|u|^2 = 0, \\
    u(x, 0) = e^{iv_0 x} \text{sech}(x - a_0),
\end{cases}$$

with a slowly varying smooth potential $V(x) = W(\varepsilon x)$, where $W \in C^3(\mathbb{R}, \mathbb{R})$. They proved that up to time $\varepsilon^6(1/\varepsilon)$ and errors of size $\varepsilon^2$ in $H^1$, the solution is a soliton evolving according to the classical dynamics of a natural effective Hamiltonian.
A common method for investigating the interaction of solitons with external potentials is to decompose the dynamics in a neighbourhood of the classical solitary manifold and to apply Lyapunov-type arguments afterwards. Beside other main ingredients this approach has been chosen in \cite{FGJS04, JFGS06, HZ07, HZ08, Hol11}.

In the present paper we extend this method for the perturbed sine-Gordon equation by introducing, the adjusted to the perturbation $F,$ virtual solitary manifold. We decompose the dynamics in a part on the virtual (rather than classical) solitary manifold and a transversal part, before we proceed with a customized Lyapunov method. Utilizing the virtual solitary manifold is crucial for the high accuracy in our stability result.

In the broadest sense, a similar technique has been used in \cite[Section 4, Section 5]{HL12} for the NLS equation. In the present paper, in the iteration scheme for the construction of the virtual solitary manifold, each iteration corresponds to a correction of the classical solitary manifold. These corrections are represented by approximate solutions of a specific PDE \eqref{eq:15} mentioned below, where the accuracy of the approximate solutions increases with each iteration step by order $1$ in $\varepsilon.$ So in \cite[Section 4, Section 5]{HL12}, the solitary manifold has been corrected once, which corresponds to the correction executed in the first iteration in the construction of the virtual solitary manifold in the present paper. In our approach, the construction of the virtual solitary manifold relies on the implicit function theorem. However, the correction in \cite[Section 4, Section 5]{HL12} was done in the form of a direct asymptotic expansion and not by employing the implicit function theorem. In that sense our paper presents a new point of view.

A further remarkable point is that unlike \cite[Section 4, Section 5]{HL12} we are able to carry out arbitrarily many corrections in the case of the sine-Gordon equation as long as the perturbation $F$ is sufficiently often differentiable in $\varepsilon.$ The accuracy of our stability result is the higher the more corrections are possible, whereas the number of possible corrections is determined by the differentiability class of the perturbation $F.$

We abstained from considering perturbations of type $\varepsilon W(\varepsilon x)$ for the following reason. In our approach we do need the assumption that the perturbation $F$ is differentiable with respect to $\varepsilon,$ but there does not exist a function $W \neq 0,$ $W \in L^2(\mathbb{R})$ such that the mapping $\varepsilon \mapsto \varepsilon W(\varepsilon \cdot)$ is differentiable in $L^2(\mathbb{R}).$

Further results on long time soliton asymptotics and orbital stability for different equations can be found in \cite{Wei86, Ben76, Bon75, MP12, SW90, BP92, IKV12, KMMn17, CMnP16}. This paper is based on \cite[Part IV]{Mas16}, where many of the computations are presented in greater detail.

Now let us comment on our techniques. The virtual solitary manifold is constructed by the following iteration scheme. Let firstly $\varepsilon \mapsto \bar{F}(\varepsilon)$ be a general function of differentiability class $C^n$ mapping into a specific Sobolev space such that $\bar{F}(\varepsilon)$ depends on $(\xi, x)$ and $\bar{F}(0) = 0.$ $\bar{F}$ will be specified later. The function $(\theta_0, \psi_0),$ given by \cite, is a solution of

\[
\underbrace{u \partial_x \left( \frac{\partial \theta}{\psi} \right) - \left( \partial^2 \theta - \sin \theta \right)} = 0,
\]

\[
\Rightarrow: G_0(\theta, \psi)
\]
which is the equation characterizing the classical solitons. In the first iteration step we modify \( G_0(\theta, \psi) = 0 \) by introducing an additional unknown variable \( \lambda \) and adding some terms involving \((\theta_0, \psi_0)\) and \( \hat{F} \). The modified equation is of the form

\[
\begin{aligned}
u \partial_{\xi} \left( \begin{array}{c} \theta \\ \psi \end{array} \right) - \left( \begin{array}{c} \psi \\ \theta_{xx} - \sin \theta + \hat{F}(\varepsilon) \end{array} \right) + \lambda \partial_u \left( \begin{array}{c} \theta_0 \\ \psi_0 \end{array} \right) &= 0. \\
\end{aligned}
\]

(11)

Here and in the following iterations the functions \( \theta, \psi \) depend on \((\xi, u, x)\) and \( \lambda \) depends on \((\xi, u)\). We solve \( G_1(\theta, \psi, \lambda) = 0 \) implicitly for \((\theta, \psi, \lambda)\) in terms of \( \varepsilon \) and denote the solution by \((\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_1^\varepsilon)\). In the next iteration step we modify \( G_1(\theta, \psi, \lambda) = 0 \) by adding some terms involving \((\theta_1^\varepsilon, \psi_1^\varepsilon)\) and solve the modified equation of the form

\[
\begin{aligned}
u \partial_{\xi} \left( \begin{array}{c} \theta \\ \psi \end{array} \right) - \left( \begin{array}{c} \psi \\ \theta_{xx} - \sin \theta + \hat{F}(\varepsilon) \end{array} \right) + \lambda \partial_u \left( \begin{array}{c} \theta_1^\varepsilon + \partial_\xi \theta_1^\varepsilon \\ \psi_1^\varepsilon + \partial_\xi \psi_1^\varepsilon \end{array} \right) &= 0. \\
\end{aligned}
\]

(12)

\[
\begin{aligned}
u \partial_{\xi} \left( \begin{array}{c} \theta \\ \psi \end{array} \right) - \left( \begin{array}{c} \psi \\ \theta_{xx} - \sin \theta + \hat{F}(\varepsilon) \end{array} \right) + \lambda \partial_u \left( \begin{array}{c} \sum_{i=0}^{n-1} \frac{\partial_i \theta_{0}^\varepsilon}{i!} \varepsilon^i \\ \sum_{i=0}^{n-1} \frac{\partial_i \psi_{0}^\varepsilon}{i!} \varepsilon^i \end{array} \right) &= 0. \\
\end{aligned}
\]

(13)

We solve \( G_n(\theta, \psi, \lambda) = 0 \) implicitly for \((\theta, \psi, \lambda)\) in terms of \( \varepsilon \) and denote the solution by \((\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)\). The existence of the implicit solutions \( \varepsilon \mapsto (\theta_j^\varepsilon, \psi_j^\varepsilon, \lambda_j^\varepsilon) \) for \( 1 \leq j \leq n \) is ensured by the implicit function theorem. In the actual proof, we consider for functional analytic reasons the translated maps and solve the equations

\[
\hat{G}_j^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) := G_j^\varepsilon(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda) = 0
\]

(14)

for \((\hat{\theta}, \hat{\psi}, \lambda)\) in terms of \( \varepsilon \). This is caused, among others, by the fact that \( \theta_0(\xi, u, x) \not\to 0 \) as \(|x| \to \infty \) for fixed \( \xi \) and \( u \). We denote the solutions to the equations \( \hat{G}_j^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) = 0 \) by \((\hat{\theta}_j^\varepsilon, \hat{\psi}_j^\varepsilon, \lambda_j^\varepsilon)\), where \((\theta_j^\varepsilon, \psi_j^\varepsilon, \lambda_j^\varepsilon) = (\theta_0 + \hat{\theta}_j, \psi_0 + \hat{\psi}_j, \lambda_j^\varepsilon)\). The application of the implicit function theorem relies on the fact that \((0, 0, 0)\) solves all equations in a particular point, i.e., \( \hat{G}_j^\varepsilon(0, 0, 0) = 0 \). As a consequence of the construction, the solution obtained in the \( j \)th iteration \( \varepsilon \mapsto (\theta_j^\varepsilon, \psi_j^\varepsilon, \lambda_j^\varepsilon) \) solves the equation

\[
\begin{aligned}
u \partial_{\xi} \left( \begin{array}{c} \theta \\ \psi \end{array} \right) - \left( \begin{array}{c} \psi \\ \theta_{xx} - \sin \theta + \hat{F}(\varepsilon) \end{array} \right) + \lambda \partial_u \left( \begin{array}{c} \theta \\ \psi \end{array} \right) &= 0. \\
\end{aligned}
\]

(15)
up to errors of order $\varepsilon^{j+1}$ for $1 \leq j \leq n$.

In order to define the virtual solitary manifold we apply this iteration scheme on a specific $\tilde{F}$, which is a truncated version of the perturbation term $F$ in (7), given by

\[
\tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x) \chi(\xi),
\]
where $\chi \in C^{\infty}(\mathbb{R})$, $\chi(\xi) = 1$ for $|\xi| \leq |\xi_s| + 3$ and $\chi(\xi) = 0$ for $|\xi| \geq |\xi_s| + 4$.

From now on we denote by $(\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_n)$ the solution obtained in the $n$th iteration by application of the iteration scheme on the specific $\tilde{F}$ given by (16). The first two components of $(\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_n)$ define the virtual solitary manifold (6). A further consequence of the construction and assumption (5) is that the functions $(\theta^\varepsilon_n(\xi(t), u(t), x), \psi^\varepsilon_n(\xi(t), u(t), x))$ solve the perturbed sine-Gordon equation (3) up to an error of order $\varepsilon^{n+k+1}$ as long as $(\xi(t), u(t))$ satisfy the ODE system $\dot{\xi}(t) = u(t)$, $\dot{u}(t) = \lambda^\varepsilon_n(\xi(t), u(t))$. We call these approximate solutions virtual solitons. In the further proof they play a role which is comparable to that of classical solitons, for instance, in the proof of orbital stability ($F = 0$) of classical solitons (see [Stu12, Section 4]).

The idea of deforming the classical solitary manifold and utilizing thereby implicitly defined functions appears in [Stu12] with the purpose of rewriting the Hamiltonian in a neighbourhood of the manifold of virtual solitons (see [Stu12, Section 3]). The virtual solitons in our paper and the corresponding manifold (6) are defined by equations and an iteration scheme that were not considered in [Stu12].

The existence of a local solution of the Cauchy problem (7) follows from the contraction mapping theorem. In the following approach we derive some bounds which imply that the local solution is continuable and that estimate (9) is satisfied on the relevant time scale. We decompose the solution of (7) into a point on the virtual solitary manifold $S^\varepsilon_n$ and a transversal component, i.e.,

\[
\begin{pmatrix}
\theta(t, x) \\
\psi(t, x)
\end{pmatrix} = \begin{pmatrix}
\theta^\varepsilon_n(\xi(t), u(t), x) \\
\psi^\varepsilon_n(\xi(t), u(t), x)
\end{pmatrix} + \begin{pmatrix}
v(t, x) \\
w(t, x)
\end{pmatrix},
\]
where the parameters $(\xi(t), u(t))$ are chosen in such a way that the transversal component $(v(t, \cdot), w(t, \cdot))$ is symplectic orthogonal to the tangent space of $S^\varepsilon_n$ at the corresponding point. This symplectic decomposition is possible in a small uniform distance to the virtual solitary manifold due to the implicit function theorem. The energy

\[H(\theta, \psi) = \frac{1}{2} \int \psi^2 + \theta^2 + 2(1 - \cos \theta) \, dx\]

and the momentum

\[\Pi(\theta, \psi) = \int \psi \theta \, dx\]

are conserved quantities of the unperturbed sine-Gordon equation. We make use of this fact and achieve control over the transversal component of the solution $(v, w)$ by utilizing
an almost conserved Lyapunov function, given by
\[
L^\varepsilon = \int \frac{w^2}{2} + \frac{\partial_x v^2}{2} + \frac{\cos(\theta_n^\varepsilon(\xi, u, \cdot))v^2}{2} + uw\partial_x v \, dx,
\]
where \((v, w)\) and \((\xi, u)\) are such as in \([17]\). \(L^\varepsilon\) is the quadratic part of
\[
H(\theta_n^\varepsilon + v, \psi_n^\varepsilon + w) + u\Pi(\theta_n^\varepsilon + v, \psi_n^\varepsilon + w) - \bigg( H(\theta_n^\varepsilon, \psi_n^\varepsilon) + u\Pi(\theta_n^\varepsilon, \psi_n^\varepsilon) \bigg).
\]
The Lyapunov function is bounded from below in terms of \(\varepsilon\) \([17]\) and of spectral properties of the operator \(-\partial^2_Z + \cos \theta_K(Z)\). The parameters \((\xi, u)\) satisfy ODEs \([10]\) up to errors of order \(\varepsilon^{n+k+1}\), which goes back (among others) to the construction of the virtual solitary manifold, especially to the fact that \((\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)\) solves \([13]\) with \(\tilde{F}\) given by \([16]\). This property of \((\xi, u)\) and once again the mentioned fact about \((\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)\) allow us to control the Lyapunov function from above. Therefore we are able to estimate the norm of the transversal component \((v, w)\) and obtain ultimately bound \([2]\). Using Gronwall’s lemma we pass from the approximate equations for the parameters \((\xi, u)\) to the exact ODEs \([10]\).

Finally let us explain under which conditions the result provides a nontrivial dynamics on the virtual solitary manifold as \(\varepsilon \to 0\). The linearization of \((\hat{\theta}, \hat{\psi}, \lambda) \mapsto \mathcal{G}^\varepsilon_n(\hat{\theta}, \hat{\psi}, \lambda)\) carried out at \((\hat{\theta}, \hat{\psi}, \lambda) = (0, 0, 0)\), \(\varepsilon = 0\) is invertible and we denote the linearization by
\[
\mathcal{M}_1^\varepsilon : (\theta, \psi, \lambda) \mapsto \mathcal{M}_2^\varepsilon(\theta, \psi, \lambda).
\]
Thus there exist functions \((\bar{\theta}, \bar{\psi}, \bar{\lambda})\) such that the \((k+1)\)th derivative of \(\tilde{F}\) with respect to \(\varepsilon\), evaluated at \(\varepsilon = 0\), can be written in the form
\[
\begin{pmatrix}
0 \\
\partial_{\varepsilon}^{k+1} \tilde{F}(0)
\end{pmatrix} = \mathcal{M}_2^\varepsilon(\bar{\theta}, \bar{\psi}, \bar{\lambda}),
\mathcal{M}_2^\varepsilon\text{ given by Proposition } \text{8.2 } n = 2, \alpha = 1. \tag{18}
\]
Here the functions \(\bar{\theta}, \bar{\psi}\) depend on \((\xi, u, x)\) and \(\bar{\lambda}\) depends on \((\xi, u)\). The ODEs \([10]\) can be rescaled in time by introducing \(s = \varepsilon^{\beta(k)}t\) with \(\beta(k) = \frac{k+1}{2}\), \(\hat{\xi}(s) = \xi(s/\varepsilon^{\beta(k)})\), and \(\hat{u}(s) = \frac{1}{\varepsilon^{2\beta(k)}}\hat{u}(s/\varepsilon^{\beta(k)})\) such that the corresponding transformed ODEs have the form
\[
\frac{d}{ds} \hat{\xi}(s) = \hat{u}(s), \quad \frac{d}{ds} \hat{u}(s) = \frac{1}{\varepsilon^{2\beta(k)}} \lambda_n^\varepsilon(\hat{\xi}(s), \varepsilon^{\beta(k)}\hat{u}(s)).
\]
As \(\varepsilon \to 0\), the transformed ODEs converge to ODEs that describe a fixed nontrivial perturbation of the uniform linear motion if the next condition is satisfied:
\[
\left\{
\begin{array}{l}
\text{There exists } \chi \text{ satisfying } \text{[16]} \text{ such that for } \tilde{F} \text{ given by } \text{[16]} \text{ the following holds: The function } \bar{\lambda} \text{ in representation } \text{[18]} \text{ fulfills } \bar{\lambda}(\cdot, 0) \neq 0.
\end{array}
\right. \tag{19}
\]
This is for the following reason. Due to \( \partial_l \tilde{F}(0) = 0 \) for \( 1 \leq l \leq k \) and differentiation of \( G_n^\varepsilon(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon) = 0 \) with respect to \( \varepsilon \) yields (see proof of Theorem 3.4):

\[
\begin{pmatrix}
0 \\
\partial_l \tilde{F}(0)
\end{pmatrix} = M_2^l(\partial^0_\varepsilon \theta_n^0, \partial^0_\varepsilon \psi_n^0, \partial^0_\varepsilon \lambda_n^0), \quad 1 \leq l \leq k + 1.
\] (20)

Using invertibility of \( M_2^l \), condition (19) and the fact that \( \lambda_n^0 = 0 \) it follows that \( 0 \neq \lambda_n^\varepsilon = O(\varepsilon^{k+1}) \), which implies the claim.

The paper is organized as follows. In Section 2, we formulate the main result. In Section 3, we construct the virtual solitary manifold. We prove in Section 4 that in a uniform distance to the virtual solitary manifold a decomposition into symplectically orthonormal components is possible. The existence of a local solution \((\theta, \psi)\) with initial state \( \gamma \) is established in Section 5. In Section 6, we derive modulation equations for the parameters that describe the position on the manifold. We introduce a Lyapunov function and compute its time derivative in Section 7. A lower bound on the Lyapunov function is proved in Section 8. In Section 9, we prove our main result, Theorem 2.2. Some preliminary decompositions are showed in Appendix A. These decompositions are used in Appendix B, where we prove that the linearizations considered in Section 8 are invertible.

**Notation and Conventions**

For a Hilbert space \( H \) its inner product is denoted by \( \langle \cdot, \cdot \rangle_H \), the orthogonal complement of a closed subspace \( M \) in \( H \) by \( M^\perp_H \), the orthogonal projection on \( M \) by \( \langle \cdot \rangle_M \) and the span of \( v_1, \ldots, v_p \in H \) by \( \langle v_1, \ldots, v_p \rangle \). For functions \( \lambda \) depending on \( (\xi, u) \) and functions \( \theta \) depending on \( (\xi, u, x) \) the notation \( \lambda(\xi, u) = \lambda(u)(\xi), \theta(\xi, u, x) = \theta(u)(\xi, x) \) is used. \( \gamma \) without an argument denotes always \( \gamma(u) \). Occasionally we drop the dependence of functions on certain variables. We also denote occasionally by \( \| \cdot \| \) the norm of an operator and drop the spaces in the notation. We write \( L^2_x(\mathbb{R}), H^k_x(\mathbb{R}^2) \) and so on for the Lebesgue and Sobolev spaces when we wish to emphasize the variables of integration.

**2 Main Result**

To formulate our result precisely, we need some definitions.

**Definition 2.1.** Let \( \alpha, n \in \mathbb{N} \) and \( u_\ast > 0 \). Let us denote by \( I(u_\ast) := [-u_\ast, u_\ast] \).

(a) \( H^{k,\alpha}(\mathbb{R}) \) denotes the weighted Sobolev space of functions with finite norm

\[
|\theta|_{H^{k,\alpha}(\mathbb{R})} = |(1 + |x|^2)^{\frac{\alpha}{2}} \theta(x)|_{H^k_x(\mathbb{R})}.
\]

(b) \( H^{k,\alpha}(\mathbb{R}^2) \) denotes the weighted Sobolev space of functions with finite norm

\[
|\theta|_{H^{k,\alpha}(\mathbb{R}^2)} = |(1 + |\xi|^2 + |x|^2)^{\frac{\alpha}{2}} \theta(\xi, x)|_{H^k_{\xi,x}(\mathbb{R}^2)}.
\]
(c) \( Y^{\alpha} \) is the space \( H^{3,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \) with the finite norm
\[
|y|_{Y^{\alpha}} = |\theta|_{H^{3,\alpha}(\mathbb{R}^2)} + |\psi|_{H^{2,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{2,\alpha}(\mathbb{R})}.
\]

(d) \( Y^{\alpha}_n(u_*) \) is the space
\[
\left\{ y = (\theta, \psi, \lambda) \in C^n(I(u_*), Y^{\alpha}) : ||y||_{Y^{\alpha}_n(u_*)} < \infty; \forall u \in I(u_*), \forall \mu \in H^{2,\alpha}(\mathbb{R}) : \right.
\]
\[
\left. \left\langle \left( \frac{\theta(u)(\xi, x)}{\psi(u)(\xi, x)} \right), \mu(\xi) \left( \begin{array}{c} \theta'_{\alpha}(\gamma(x - \xi)) \\ -u\gamma\theta''_{\alpha}(\gamma(x - \xi)) \end{array} \right) \right\rangle_{L^{2,\alpha}_x(\mathbb{R}^2) \oplus L^{2,\alpha}_x(\mathbb{R}^2)} = 0 \right\}
\]
with the finite norm
\[
||y||_{Y^{\alpha}_n(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^{n} |\partial_i y(u)|_{Y^{\alpha}} \right).
\]

(e) For \( l \in \mathbb{N} \) and \( 0 < U < u_* \) we introduce the parameter area
\[
\Sigma(l, U, u_*):= \left\{ (\xi, u) \in \mathbb{R} \times (-1, 1) : u \in (-U - V(l, U, u_*), U + V(l, U, u_*) \right\}.
\]
where \( V(l, U, u_*) := \frac{u_* - U}{U} \).

The weighted Sobolev spaces in Definition 2.1 (a), (b) are defined as in [Kop15]. We are now ready to state our main result.

**Theorem 2.2.** Let \( n, k \in \mathbb{N}, n \geq 1, k + 1 \leq n \). Assume that \( \xi_0 \in \mathbb{R}, F \in C^n((-1, 1), H^{1,1}(\mathbb{R})) \) and \( \partial^l_l F(0, \cdot) = 0 \) for \( 0 \leq l \leq k \). Then there exist \( \varepsilon_0, u_*, \tilde{C} > 0 \) and a map
\[
(-\varepsilon_0, \varepsilon_0) \rightarrow Y^1_{\varepsilon}(u_*), \varepsilon \mapsto (\theta_\varepsilon(x), \psi_\varepsilon(x), \lambda_\varepsilon(x))
\]
of class \( C^n \) such that the following holds. Let \( \varepsilon \in (0, \varepsilon_0) \) and \( 0 < U < u_* \). Consider the Cauchy problem
\[
\partial_t \left( \begin{array}{c} \theta \\ \psi \\ \end{array} \right) = \left( \begin{array}{c} \partial^2_x \theta - \sin \theta + F(\varepsilon, x) \\ \psi(0, x) - F(\varepsilon, x) \end{array} \right) = \left( \begin{array}{c} \theta_\varepsilon(x, u_*, x) \\ \psi_\varepsilon(x, u_*, x) \\ \end{array} \right) + \left( \begin{array}{c} v(0, x) \\ w(0, x) \end{array} \right),
\]
where \( (\theta_\varepsilon, \psi_\varepsilon, \lambda_\varepsilon) = (\theta_0 + \theta_\varepsilon, \psi_0 + \psi_\varepsilon, \lambda_\varepsilon) \) with \( (\theta_0, \psi_0) \) given by (4) such that the following assumptions are satisfied:

(a) \( |u_*| \leq \tilde{C} \varepsilon^{\frac{k + 1}{2}} \);
(b) \( \mathcal{N}^\varepsilon(\theta(0, \cdot), \psi(0, \cdot), \xi_s, u_s) = 0 \), where \( \mathcal{N}^\varepsilon = (\mathcal{N}_1^\varepsilon, \mathcal{N}_2^\varepsilon) : L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times \Sigma(2, U, u_s) \rightarrow \mathbb{R}^2 \) is given by

\[
\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) := \begin{pmatrix}
\Omega 
\begin{pmatrix}
\partial_t \theta^\varepsilon_n(\xi, u, \cdot), & \theta(\cdot) - \theta^\varepsilon_n(\xi, u, \cdot);
\partial_x \psi^\varepsilon_n(\xi, u, \cdot), & \psi(\cdot) - \psi^\varepsilon_n(\xi, u, \cdot)
\end{pmatrix}
\end{pmatrix}
\]

and the symplectic form \( \Omega \) is given by (22);

(c) \( |v(0, \cdot)|^2_{H^1(\mathbb{R})} + |w(0, \cdot)|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n} \), where \((v(0, \cdot), w(0, \cdot))\) is given by (21).

Then the Cauchy problem defined by (21) has a unique solution on the time interval

\[
0 \leq t \leq T, \text{ where } T = T(\varepsilon, k) = \frac{1}{C\varepsilon^{\beta(k)}}, \quad \beta(k) = \frac{k + 1}{2}.
\]

The solution may be written in the form

\[
\begin{pmatrix}
\theta(t, x) \\
\psi(t, x)
\end{pmatrix} = \begin{pmatrix}
\theta^\varepsilon_n(\xi(t), \bar{u}(t), x) \\
\psi^\varepsilon_n(\xi(t), \bar{u}(t), x)
\end{pmatrix} + \begin{pmatrix}
v(t, x) \\
w(t, x)
\end{pmatrix},
\]

where \(v, w\), have regularity \((v(t), w(t)) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\) and \(\tilde{\xi}, \tilde{u}\) solve the ODEs

\[
\dot{\xi}(t) = \tilde{u}(t), \quad \dot{\tilde{u}}(t) = \lambda^\varepsilon_n(\tilde{\xi}(t), \tilde{u}(t)),
\]

with initial data \(\tilde{\xi}(0) = \xi_s, \quad \tilde{u}(0) = u_s\) such that

\[
|v|^2_{L^\infty([0, T], H^1(\mathbb{R}))} + |w|^2_{L^\infty([0, T], L^2(\mathbb{R}))} \leq \tilde{C}\varepsilon^{2n}.
\]

The constant \(\tilde{C}\) depends on \(F\) and \(\xi_s\). The parameters \(\tilde{\xi}, \tilde{u}\) describe a fixed nontrivial perturbation of the uniform linear motion as \(\varepsilon \to 0\) if condition (19) is satisfied.

### 3 Construction of the Virtual Solitary Manifold

#### 3.1 Iteration Scheme

Let \(\alpha, n \in \mathbb{N}\). In this subsection, we establish the iteration scheme presented in the introduction. We implement the scheme for a general function \(\tilde{F} : (-1, 1) \rightarrow H^{1,\alpha}(\mathbb{R}^2), \varepsilon \mapsto \tilde{F}(\varepsilon)\) of class \(C^n\) that satisfies \(\tilde{F}(0) = 0\). For being able to apply the implicit function theorem in the proof of existence of iterative solutions, we need to show that the corresponding linearizations of \((\theta, \psi, \lambda) \mapsto \tilde{G}_j^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda)\) carried out at \((\hat{\theta}, \hat{\psi}, \lambda) = (0, 0, 0), \varepsilon = 0\) are invertible (\(\tilde{G}_j\) given by (11)–(14)). This is done in the following proposition, which is a main ingredient in the construction of the virtual solitary manifold. We start with a definition.
Theorem 3.3. Let the following theorem. The maps \( \tilde{\theta} \) for proof see Appendix A and Appendix B. We formalize the iteration scheme in the regularity of the spaces decreases with increasing \( j \).

Definition 3.1. (a) \( Z^\alpha \) is the space \( H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \) with the finite norm

\[
|z|_{Z^\alpha} = |v|_{H^{2,\alpha}(\mathbb{R}^2)} + |w|_{H^{1,\alpha}(\mathbb{R}^2)}. 
\]

(b) \( Z_n^\alpha = Z_n^\alpha(u_*) \) is the space \( \{ z = (v, w) \in C^n(I(u_*), Z^\alpha) : \|z\|_{Z_n^\alpha(u_*)} < \infty \} \) with the finite norm

\[
\|z\|_{Z_n^\alpha(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^{n} |\partial_{x}^{i} y(u)|_{Z^\alpha} \right).
\]

(c) We denote by \( t_1(\xi, u, x) := \left( \frac{\partial_\theta \theta_0(\xi, u, x)}{\partial_\psi \psi_0(\xi, u, x)} \right) \) and by \( t_2(\xi, u, x) := \left( \frac{\partial u \theta_0(\xi, u, x)}{\partial_\psi \psi_0(\xi, u, x)} \right) \), where \( u \in (-1, 1), \xi, x \in \mathbb{R} \).

Recall that the spaces \( Y^\alpha, Y_n^\alpha(u_*) \) were defined in Section 2. We set \( Y_n^\alpha := Y_n^\alpha(u_*) \).

Proposition 3.2. There exists \( \underline{u}^\alpha > 0 \) such that the operator \( \mathcal{M}_n^\alpha : Y_n^\alpha(u_*) \to Z_n^\alpha(u_*) \), \((\theta, \psi, \lambda) \mapsto \mathcal{M}_n^\alpha(\theta, \psi, \lambda)\), given by

\[
\mathcal{M}_n^\alpha(\theta, \psi, \lambda)(u) = \left( \frac{u \partial_\theta \theta(0) - \psi(u)}{-\partial^2_\theta \theta(u) + \cos(\theta \kappa(\gamma(x - \xi))) \theta(u) + u \partial_\psi \psi(u)} \right) + \lambda(u) t_2(\xi, u, x),
\]

is invertible if \( u_* < \underline{u}^\alpha \).

For proof see Appendix A and Appendix B. We formalize the iteration scheme in the following theorem. The maps \( \bar{G}_j \) are defined on spaces of different regularity in \( u \) such that the regularity of the spaces decreases with increasing \( j \), which ensures well-definedness of \( \bar{G}_j \).

Theorem 3.3. Let \( J = (-1, 1), u_* < \underline{u}^\alpha \) and let \( \bar{F} : J \to H^{1,\alpha}(\mathbb{R}^2), \varepsilon \mapsto \bar{F}(\varepsilon) \) be a \( C^n \) function such that \( \bar{F}(0) = 0 \). Let \( \bar{G}_1 \) be given by

\[
\bar{G}_1 : J \times Y_{n+1}^\alpha(u_*) \to Z_{n+1}^\alpha(u_*), (\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) \mapsto \bar{G}_1^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) := G_1^\varepsilon(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda),
\]

where \( G_1 \) is defined by (11). Then there exists \( \varepsilon^* > 0 \) and a map

\[
(-\varepsilon^*, +\varepsilon^*) \to Y_{n+1}^\alpha(u_*), \varepsilon \mapsto (\tilde{\theta}_1^\varepsilon, \tilde{\psi}_1^\varepsilon, \lambda_1^\varepsilon),
\]

of class \( C^n \) such that \( \bar{G}_1^\varepsilon(\tilde{\theta}_1^\varepsilon, \tilde{\psi}_1^\varepsilon, \lambda_1^\varepsilon) = 0 \). Let \( \bar{G}_2 \) be given by

\[
\bar{G}_2 : J \times Y_n^\alpha(u_*), (\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) \mapsto \bar{G}_2^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) := G_2^\varepsilon(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda),
\]

where \( G_2 \) is defined by (12) with \((\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_1^\varepsilon) = (\theta_0 + \hat{\theta}_1^\varepsilon, \psi_0 + \hat{\psi}_1^\varepsilon, \lambda_1^\varepsilon)\). Then there exists a map

\[
(-\varepsilon^*, +\varepsilon^*) \to Y_n^\alpha(u_*), \varepsilon \mapsto (\tilde{\theta}_2^\varepsilon, \tilde{\psi}_2^\varepsilon, \lambda_2^\varepsilon),
\]

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of class $C^n$ such that $\tilde{G}^\varepsilon_2(\hat{\theta}_2, \hat{\psi}_2, \lambda_2^\varepsilon) = 0$. This process can be continued successively to arrive at $\tilde{G}_n$ be given by

$$\tilde{G}_n : J \times Y_2^n(u_*) \to Z_2^n(u_*) \to \tilde{G}^\varepsilon_n(\hat{\theta}, \hat{\psi}, \lambda) := G^\varepsilon_n(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda),$$

where $G_n$ is defined by (13) with $(\theta_{n-1}^\varepsilon, \psi_{n-1}^\varepsilon, \lambda_{n-1}^\varepsilon) = (\theta_0 + \hat{\theta}_{n-1}, \psi_0 + \hat{\psi}_{n-1}, \lambda_{n-1})$. Ultimately there exists a map

$$(-\varepsilon^*, +\varepsilon^*) \to Y_2^n(u_*) \to \tilde{G}^\varepsilon_n(\hat{\theta}_n, \hat{\psi}_n, \lambda_n^\varepsilon),$$

of class $C^n$ such that $\tilde{G}^\varepsilon_n(\hat{\theta}_n, \hat{\psi}_n, \lambda_n^\varepsilon) = 0$ and we set $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon) = (\theta_0 + \hat{\theta}_n, \psi_0 + \hat{\psi}_n, \lambda_n^\varepsilon)$.

**Proof.** We skip $u_*$ in the notation. Notice that $\tilde{G}^0_1(0, 0, 0) = G^0_1(\theta_0, \psi_0, 0) = 0$. The derivative of $\tilde{G}_1 : J \times Y_{n+1}^0 \to Z_{n+1}^0$ with respect to $(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda)$ evaluated at $(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) = (0, 0, 0, 0)$ is $M^0_{n+1}$, which is invertible due to Proposition 3.2. By the implicit function theorem there exists a $\varepsilon^*_n > 0$ and a map

$$(-\varepsilon^*_1, +\varepsilon^*_1) \to Y_{n+1}^0 \to (\hat{\theta}_1^\varepsilon, \hat{\psi}_1^\varepsilon, \lambda_1^\varepsilon)$$

of class $C^n$ such that $\tilde{G}^\varepsilon_1(\hat{\theta}_1, \hat{\psi}_1^\varepsilon, \lambda_1^\varepsilon) = 0$. We continue successively this process until we obtain that the derivative of $\tilde{G}_n : J \times Y_{n+1}^0 \to Z_{n+1}^n$ is $M^0_n$. This yields by using the same argument combined with $\tilde{G}_0^n(0, 0, 0) = 0$ that there exists $\varepsilon^*_n > 0$ and a map

$$(-\varepsilon^*_n, +\varepsilon^*_n) \to Y_2^n \to (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon)$$

of class $C^n$ such that $\tilde{G}^\varepsilon_n(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon) = 0$. We set $\varepsilon^* = \min\{\varepsilon^*_1, \varepsilon^*_2, \ldots, \varepsilon^*_n\}$. $\square$

In the following theorem we state the properties of the $n$th iterative solution from Theorem 3.3.

**Theorem 3.4.** Let the assumptions of Theorem 3.3 hold. Then the following relations are satisfied.

(a) $(\partial^j_\theta \theta_{n-1}^\varepsilon, \partial^j_\psi \psi_{n-1}^\varepsilon, \partial^j_\lambda \lambda_{n-1}^\varepsilon) = (\partial^j_\theta \theta_0^\varepsilon, \partial^j_\psi \psi_0^\varepsilon, \partial^j_\lambda \lambda_0^\varepsilon)$ for $j = 0, \ldots, n - 1$, where $n \geq 2$.

(b) $\forall u \in I : \quad u \delta_\varepsilon \left( \frac{\theta_n^\varepsilon}{\psi_n^\varepsilon} - \left[ \theta_{n, xx}^\varepsilon - \sin \theta_n^\varepsilon \hat{F}(\varepsilon) \right] \right) + \lambda_n^\varepsilon \partial_u \left( \frac{\theta_n^\varepsilon}{\psi_n^\varepsilon} \right) + R_n^\varepsilon = 0$, where

$$R_n^\varepsilon := \frac{\lambda_n^\varepsilon}{\theta_n^\varepsilon} \left( \sum_{i=0}^{n-1} \frac{\partial^i_\theta \theta_0^\varepsilon}{\partial^i_\psi \psi_0^\varepsilon} \varepsilon_i - \theta_n^\varepsilon \right) - \frac{\lambda_n^\varepsilon}{\psi_n^\varepsilon} \left( \sum_{i=0}^{n-1} \frac{\partial^i_\lambda \lambda_0^\varepsilon}{\partial^i_\psi \psi_0^\varepsilon} \varepsilon_i - \psi_n^\varepsilon \right).$$

The following rates of convergence hold.

$$\left\| \begin{pmatrix} \sum_{i=0}^{n-1} \frac{\partial^i_\theta \theta_0^\varepsilon}{\partial^i_\psi \psi_0^\varepsilon} \varepsilon_i - \theta_n^\varepsilon \\ \sum_{i=0}^{n-1} \frac{\partial^i_\lambda \lambda_0^\varepsilon}{\partial^i_\psi \psi_0^\varepsilon} \varepsilon_i - \psi_n^\varepsilon \\ 0 \end{pmatrix} \right\|_{Y_2^n(u_*)} = O(\varepsilon^n), \quad \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\|_{Y_2^n(u_*)} = O(\varepsilon).$$
\textbf{Remark 3.5.} The derivatives of the iterative solutions coincide at 0 in the following way:

\[ \frac{\partial_j \theta^0_n, \partial_j \psi^0_n, \partial_j \lambda^0_n}{\partial \varepsilon} = \left( \frac{\partial_j \theta^0_n, \partial_j \psi^0_n, \partial_j \lambda^0_n}{\partial \varepsilon} \right) \text{ for } j = 0, 1; \left( \frac{\partial_j \theta^0_n, \partial_j \psi^0_n, \partial_j \lambda^0_n}{\partial \varepsilon} \right) = \left( \frac{\partial_j \theta^0_n, \partial_j \psi^0_n, \partial_j \lambda^0_n}{\partial \varepsilon} \right) \text{ for } j = 0, 1, 2 \text{ and so on up to the identities for the } n\text{th iterative solution stated in Theorem 3.3 (a).}

\textbf{Proof} (of Theorem 3.4). (a) The claim can be proved by induction on \( n \). We show the induction step. Assume that the claim is true for all integers less than or equal to \( n - 2 \). Let \( 0 \leq j \leq n - 1 \). The fact that the solutions from Theorem \ref{thm:iterative_solution} satisfy

\[ \forall u \in I(u_*): \quad (\theta^0_j(u), \psi^0_j(u), \lambda^0_j(u)) \in Y^\alpha = H^{3, \alpha}(\mathbb{R}^2) \oplus H^{2, \alpha}(\mathbb{R}^2) \oplus H^{2, \alpha}(\mathbb{R}) \]

and that the injections \( H^1(\mathbb{R}) \subset L^\infty(\mathbb{R}), H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \) are continuous \cite[Corollary 9.13]{Brezis} yields the justification for using the Leibniz's formula and Faà di Bruno’s formula. Thus we obtain for the \( j \)-th derivatives with respect to \( \varepsilon \), evaluated at \( \varepsilon = 0 \):

\[ 0 = \frac{\partial_j \mathcal{G}^0_n(\theta^0_n, \psi^0_n, \lambda^0_n)}{\partial \varepsilon} + \left( \sum_{l_j, n_1 \leq \ldots \leq n_j} \varepsilon^{l_j} \right) \sin(\theta^0_n) \left( \frac{\partial^1 \theta^0_n}{\partial \theta^0_n} \right)^{l_1} \left( \frac{\partial^2 \theta^0_n}{\partial ^2 \theta^0_n} \right)^{l_2} \ldots \left( \frac{\partial^j \theta^0_n}{\partial ^j \theta^0_n} \right)^{l_j} \]

\[ 0 = \frac{\partial_j \mathcal{G}^0_{n-1}(\theta^0_{n-1}, \psi^0_{n-1}, \lambda^0_{n-1})}{\partial \varepsilon} + \left( \sum_{l_j, n_1 \leq \ldots \leq n_j} \varepsilon^{l_j} \right) \sin(\theta^0_{n-1}) \left( \frac{\partial^1 \theta^0_{n-1}}{\partial \theta^0_{n-1}} \right)^{l_1} \left( \frac{\partial^2 \theta^0_{n-1}}{\partial ^2 \theta^0_{n-1}} \right)^{l_2} \ldots \left( \frac{\partial^j \theta^0_{n-1}}{\partial ^j \theta^0_{n-1}} \right)^{l_j} \]

where \( l = l_1 + l_2 + \ldots + l_j \) and the sum is taken over all \( l_1, l_2, \ldots, l_j \) for which \( l_1 + 2 l_2 + \ldots + j l_j = j \). Subtracting \eqref{eq:virtual_solitary_manifold_0} from \eqref{eq:virtual_solitary_manifold_1} yields the claim due to Proposition 3.2 (b) follows from (a), Theorem 3.3 and Taylor’s formula. \hfill \Box

\subsection{Virtual Solitary Manifold}

From now on we set \( \alpha := 1 \). In this subsection, we apply Theorem 3.3 on a specific \( \tilde{F} \) and define the virtual solitary manifold by the solution obtained in the \( n \)th iteration.

\textbf{Definition 3.6.} Let \( F, \xi \) be from Theorem 2.2 and \( \Xi := \Xi(\xi) := |\xi| + 3 \). We set \( \tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x) \chi(\xi) \), where \( \chi \) is a smooth cutoff function with \( \chi(\xi) = 1 \) for \( |\xi| \leq \Xi \) and \( \chi(\xi) = 0 \) for \( |\xi| \geq \Xi + 1 \).
The next lemma follows immediately from Theorem 3.4 and from the assumptions on $F$ in Theorem 2.2.

**Lemma 3.7.** Let $F$ be from Theorem 2.2, $\tilde{F}$ from Definition 3.6. Then it holds that

(a) $\forall (\varepsilon, \xi, x) \in (-1, 1) \times [-\Xi, \Xi] \times \mathbb{R} : \tilde{F}(\varepsilon, \xi, x) = F(\varepsilon, x)$.

(b) $\tilde{F} \in C^n((-1, 1), H^{1, 1}(\mathbb{R}^2))$ and $\partial^l_\varepsilon \tilde{F}(0, \cdot, \cdot) = 0$ for $0 \leq l \leq k$.

(c) $\left\| \begin{pmatrix} 0 \\ 0 \\ \lambda^\varepsilon_n \end{pmatrix} \right\|_{Y^1_2(u_\ast)} = \mathcal{O}(\varepsilon^{k+1})$.

**Lemma 3.8.** Let $v \in H^1(\mathbb{R}^2)$. Then there exists $b > 0$ such that

$\forall \xi \in \mathbb{R} : |v(\xi, x)|_{L^2(\mathbb{R})} \leq b |v(\xi, x)|_{H^1(\mathbb{R}^2)}$.

**Proof.** This follows from applying Morrey’s embedding Theorem to the variable $\xi$. □

We solve iteratively the equations in Theorem 3.3 with the specific $\tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x) \chi(k, \xi)$ from Definition 3.6 and define by the $n$th solution $(\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_n)$ the virtual solitary manifold $S^\varepsilon_n$. We utilize the truncated version of $F$ rather than $F$ itself in order to make sure that the maps $\tilde{G}_j$ in Theorem 3.3 are well defined. Theorem 3.3 is applicable to $\tilde{F}$ due to Lemma 3.7.

**Definition 3.9.** Let $u^1$ be from Proposition 3.2 (case $\alpha = 1$). We fix a specific $u_\ast$ such that $0 < u_\ast < u^1$. Let $F$ be from Definition 3.6. Let $\varepsilon^* > 0$ be the constant and let $(\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_n)$ be the $n$th solution obtained from application of Theorem 3.3 to $\tilde{F}$. We set

$$S^\varepsilon_n := \left\{ \left( \begin{array}{c} \theta^\varepsilon_n(\xi, u, \cdot) \\ \psi^\varepsilon_n(\xi, u, \cdot) \end{array} \right) : u \in (-u_\ast, u_\ast), \xi \in \mathbb{R} \right\}$$

for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$, and call $S^\varepsilon_n$ the virtual solitary manifold.

**Remark 3.10.**

(a) From now on we denote by $(\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_n) = (\theta_0 + \theta^\varepsilon_n, \psi_0 + \psi^\varepsilon_n, \lambda^\varepsilon_n)$ always the $n$th solution utilized in Definition 3.9.

(b) The vectors

$$t^\varepsilon_{1,n}(\xi, u, x) := \left( \begin{array}{c} \partial_\xi \theta^\varepsilon_n(\xi, u, x) \\ \partial_\xi \psi^\varepsilon_n(\xi, u, x) \end{array} \right) \quad \text{and} \quad t^\varepsilon_{2,n}(\xi, u, x) := \left( \begin{array}{c} \partial_u \theta^\varepsilon_n(\xi, u, x) \\ \partial_u \psi^\varepsilon_n(\xi, u, x) \end{array} \right)$$

are tangent vectors of the manifold $S^\varepsilon_n$ at the point $(\theta^\varepsilon_n(\xi, u, \cdot), \psi^\varepsilon_n(\xi, u, \cdot))$ and form a basis of the tangent space at this point.
4 Symplectic Orthogonal Decomposition

Let from now on $u_*, \varepsilon^*$ be always from Definition 3.9 and let $U$ be fixed such that $0 < \varepsilon < u_*$. We consider $V(l, U, u_*)$, $\Sigma(l, U, u_*)$ introduced in Definition 2.1 (e) and the function $\mathcal{N}^\varepsilon : L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times \Sigma(2, U, u_*) \rightarrow \mathbb{R}^2$ defined as in (22) for these specific $u_*, U$. For simplicity of further notation we set $\Sigma(l, U, u_*) = \Sigma(l)$ and $V(l, U, u_*) = V(l)$.

In this chapter we will choose $\varepsilon_0$ sufficiently small and consider $\varepsilon \in (0, \varepsilon_0]$. We show that if $(\theta, \psi) \in L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ is close enough (in the $L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})$ norm) to the region

$$\mathcal{S}_n^\varepsilon(U) := \left\{ \left( \begin{array}{c} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{array} \right) : (\xi, u) \in \Sigma(4) \right\},$$

of the virtual solitary manifold $\mathcal{S}_n^\varepsilon$, then there exists a unique $(\xi, u) \in \Sigma(2)$ such that we are able to decompose the solution

$$\begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} + \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix},$$

in a point on the virtual solitary manifold $(\theta_n^\varepsilon(\xi, u, \cdot), \psi_n^\varepsilon(\xi, u, \cdot))$ and a transversal component $(v(\cdot), w(\cdot))$, which is symplectic orthogonal to the tangent vectors $t_{1,n}^\varepsilon(\xi, u, \cdot)$ and $t_{2,n}^\varepsilon(\xi, u, \cdot)$ at the corresponding point of the manifold $\mathcal{S}_n^\varepsilon$, i.e., the orthogonality condition

$$\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) = 0$$

is satisfied. We prove that the symplectic decomposition is possible in a small uniform distance to the manifold $\mathcal{S}_n^\varepsilon$, where the distance might depend on $\varepsilon_0$ but does not depend on $\varepsilon$.

**Remark 4.1.** In Theorem 3.3 we have solved the equations defining $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon)$ in weighted spaces. One of the reasons for working in weighted Sobolev spaces was to make sure that $\mathcal{N}^\varepsilon : L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times \Sigma(2) \rightarrow \mathbb{R}^2$ is well defined.

We start with a definition and some elementary lemmas which will be used later.

**Definition 4.2.** Let $\varepsilon \in (0, \varepsilon^*)$. We set

(a) $m := \int |\theta_K(Z)|^2 dZ,$

(b) $m_n^\varepsilon(\xi, u) := \int -\partial_{\xi} \psi_n^\varepsilon(\xi, u, x) \partial_u \theta_n^\varepsilon(\xi, u, x) + \partial_{\xi} \theta_n^\varepsilon(\xi, u, x) \partial_u \psi_n^\varepsilon(\xi, u, x) d\xi d\mu(\xi, u, x),$

(c) $k_n^\varepsilon(\xi, u) := \int -\partial_{\xi} \psi_0(\xi, u, x) \partial_u \tilde{\theta}_n^\varepsilon(\xi, u, x) - \partial_u \theta_0(\xi, u, x) \partial_{\xi} \tilde{\psi}_n^\varepsilon(\xi, u, x)$

$$+ \partial_{\xi} \theta_0(\xi, u, x) \partial_u \tilde{\psi}_n^\varepsilon(\xi, u, x) + \partial_u \psi_0(\xi, u, x) \partial_{\xi} \tilde{\theta}_n^\varepsilon(\xi, u, x)$$

$$- \partial_{\xi} \tilde{\psi}_n^\varepsilon(\xi, u, x) \partial_u \theta_n^\varepsilon(\xi, u, x) + \partial_{\xi} \tilde{\theta}_n^\varepsilon(\xi, u, x) \partial_u \psi_n^\varepsilon(\xi, u, x) d\xi d\mu(\xi, u, x).$$

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A straightforward computation yields the following lemma.

**Lemma 4.3.** Let \( \varepsilon \in (0, \varepsilon^*) \). It holds that
\[
\forall (\xi, u) \in \mathbb{R} \times [-U - V(2), U + V(2)] : \quad m_n^\varepsilon(\xi, u) = \gamma^3(u)m + k_n^\varepsilon(\xi, u).
\]

**Lemma 4.4.** Let \( \varepsilon_0 > 0 \) be sufficiently small. There exist constants \( c = c(U) > 0 \), \( C = C(U) > 0 \), such that \( \forall \varepsilon \in (0, \varepsilon_0] \), \( (\xi, u) \in \mathbb{R} \times [-U - V(2), U + V(2)] \):
\[
c \leq \frac{\gamma^3(u)m}{2} \leq m_n^\varepsilon(\xi, u) \leq 2\gamma^3(u)m \leq C.
\]

**Proof.** Using Lemma [3.8] and continuity of \( \varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon) \) (see Theorem [3.3]) we obtain for sufficiently small \( \varepsilon_0 \): \( \forall \varepsilon \in (0, \varepsilon_0] \), \( (\xi, u) \in \mathbb{R} \times [-U - V(2), U + V(2)] \): \( |k_n^\varepsilon(\xi, u)| < m/2 \), which implies the claim. \( \square \)

The next lemma provides that the symplectic decomposition described above is possible. In the proof we will take derivatives of \( (\theta_n^\varepsilon, \psi_n^\varepsilon) \) up to second order with respect to \( \xi \) and \( u \). This was the reason for solving, in Section 3, the equations defining \( (\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon) \) in spaces of higher regularity in \( \xi \) and \( u \).

**Lemma 4.5.** Let \( \varepsilon_0 > 0 \) be sufficiently small. Let
\[
\mathcal{O} = \mathcal{O}_{U, p} = \left\{ (\theta, \psi) \in L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) : \inf_{(\xi, u) \in \Sigma(4)} \left| \begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \\ \theta_n^\varepsilon(\xi, u, \cdot) \\ \psi_n^\varepsilon(\xi, u, \cdot) \end{pmatrix} \right|_{L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})} < p \right\}.
\]

There exists \( r > 0 \) such that if \( \varepsilon \in (0, \varepsilon_0] \) and \( p \leq r \) then for any \( (\theta, \psi) \in \mathcal{O}_{U, p} \) there exists a unique \( (\xi, u) \in \Sigma(2) \) such that
\[
\mathcal{N}^\varepsilon(\theta, \psi, \xi, u) = 0
\]
and the map \( (\theta, \psi) \mapsto (\xi(\theta, \psi), u(\theta, \psi)) \) is in \( C^1(\mathcal{O}_{U, p}, \Sigma(2)) \).

**Proof.** Let \( \varepsilon_0 \in (0, \varepsilon^*) \). We will specify \( \varepsilon_0 \) later in this proof. Let \( \varepsilon \in (0, \varepsilon_0] \). Notice that the map \( \varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon) \) from Theorem [3.3] is continuous and it holds \( \Sigma(4) \subset \Sigma(3) \subset \Sigma(2) \). Consider \( (\xi_0, u_0) \in \Sigma(3) \). Lemma [4.3] yields that
\[
D_{\xi, u}\mathcal{N}^\varepsilon(\theta_n^\varepsilon(\xi_0, u_0, \cdot), \psi_n^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) = \left( \gamma^3(u_0)m + k_n^\varepsilon(\xi_0, u_0) \right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Using Lemma [3.8] we obtain for sufficiently small \( \varepsilon_0 \) for all \( \varepsilon \in (0, \varepsilon_0] \): \( |k_n^\varepsilon(\xi_0, u_0)| \leq \frac{m}{2} \) and thus
\[
det D_{\xi, u}\mathcal{N}^\varepsilon(\theta^\varepsilon(\xi_0, u_0, \cdot), \psi^\varepsilon(\xi_0, u_0, \cdot), \xi_0, u_0) \neq 0.
\]
We prove that there exist \( r > 0, \tilde{\delta} > 0, \varepsilon_0 > 0 \) such that \( \forall \varepsilon \in (0, \varepsilon_0], (\xi_0, u_0) \in \Sigma(3) \) there exist balls \( B_r(\theta^e_n(\xi_0, u_0, \cdot), \psi^e_n(\xi_0, u_0, \cdot)) \subseteq L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}) \), \( B_{\tilde{\delta}}(\xi_0, u_0) \subseteq \Sigma(2) \), and a map

\[
T^e_{\xi_0, u_0} : B_r(\theta^e_n(\xi_0, u_0, \cdot), \psi^e_n(\xi_0, u_0, \cdot)) \to B_{\tilde{\delta}}(\xi_0, u_0)
\]

such that \( \mathcal{N}^e(\theta, \psi, T^e_{\xi_0, u_0}(\theta, \psi)) = 0 \) on \( B_r(\theta^e_n(\xi_0, u_0, \cdot), \psi^e_n(\xi_0, u_0, \cdot)) \). Therefore we refer to [Dei85 Theorem 15.1] and check their proof of the implicit function theorem, whereas we show that \( r \) and \( \tilde{\delta} \) do not depend on \( \varepsilon \) and on \((\xi_0, u_0)\). We introduce

\[
\tilde{\mathcal{N}}^e_{\xi_0, u_0}(\theta, \psi, \xi, u) := \mathcal{N}^e(\theta(\cdot) + \theta^e_n(\xi_0, u_0, \cdot), \psi(\cdot) + \psi^e_n(\xi_0, u_0, \cdot), \xi + \xi_0, u + u_0).
\]

Notice that \( \tilde{\mathcal{N}}^e_{\xi_0, u_0}(0, 0, 0, 0) = (0, 0) \). We set \( K^e_{\xi_0, u_0} := D_{(\xi, u)} \tilde{\mathcal{N}}^e_{\xi_0, u_0}(0, 0, 0, 0) \) and

\[
S^e_{\xi_0, u_0}(\theta, \psi, \xi, u) := \left[ K^e_{\xi_0, u_0} \right]^{-1} \tilde{\mathcal{N}}^e_{\xi_0, u_0}(\theta, \psi, \xi, u) - I(\xi, u),
\]

which is well defined due to (27). Due to Lemma 3.8 it holds for a sufficiently small \( \varepsilon_0 \) that

\[
\forall \varepsilon \in (0, \varepsilon_0], (\xi, u) \in \mathbb{R} \times [-U - V(2), U + V(2)], \beta_1 + \beta_2 \leq 2, \ p = 1, 2 :
\]

\[
\left| \partial_\xi^{\beta_1} \partial_u^{\beta_2} \mathcal{N}^e(\theta, \psi, \xi, u, x) \right|_{L^p(\mathbb{R})} \leq B, \quad \left| \partial_\xi^{\beta_1} \partial_u^{\beta_2} \mathcal{N}^e(\theta, \psi, \xi, u, x) \right|_{L^p(\mathbb{R})} \leq B.
\]

In this proof we denote by \( \| \cdot \| \) the maximum row sum norm of a \( 2 \times 2 \) matrix induced by the maximum norm \( \| \cdot \|_\infty \) in \( \mathbb{R}^2 \). We claim that \( \exists k \in (0, 1), \tilde{\delta} > 0, \varepsilon_0 > 0 \) \( \forall \varepsilon \in (0, \varepsilon_0], (\xi_0, u_0) \in \Sigma(3) \) \( \forall ((\theta, \psi), (\xi, u)) \in B_{\tilde{\delta}}(0) \times B_{\tilde{\delta}}(0) : \| D_{(\xi, u)} S^e_{\xi_0, u_0}(\theta, \psi, \xi, u) \| \leq k < 1 \). Due to (20) it holds that

\[
D_{(\xi, u)} S^e_{\xi_0, u_0}(\theta, \psi, \xi, u)
\]

\[
= \frac{1}{\gamma(\xi_0)^2 m + k^e_{\xi_0}(\xi_0, u_0)} \begin{pmatrix}
-\partial_\xi \tilde{\mathcal{N}}^e_{\xi_0, u_0, 1}(\theta, \psi, \xi, u) & -\partial_u \tilde{\mathcal{N}}^e_{\xi_0, u_0, 2}(\theta, \psi, \xi, u) \\
\partial_\xi \tilde{\mathcal{N}}^e_{\xi_0, u_0, 1}(\theta, \psi, \xi, u) & \partial_u \tilde{\mathcal{N}}^e_{\xi_0, u_0, 2}(\theta, \psi, \xi, u)
\end{pmatrix} - \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

The claim follows by using Lemma 3.8 and estimating each entry of \( D_{(\xi, u)} S^e_{\xi_0, u_0}(\theta, \psi, \xi, u) \), for instance:

\[
| - \frac{1}{m^e_{\xi_0}(\xi_0, u_0)} \partial_\xi \tilde{\mathcal{N}}^e_{\xi_0, u_0, 2}(\theta, \psi, \xi, u) - 1 |
\]

\[
\leq \frac{1}{m^e_{\xi_0}(\xi_0, u_0)} \left( | \partial_\xi \partial_u \psi^e(n)(\xi, \bar{u}, x) |_{L^2(\mathbb{R})} | \hat{\theta}^e_n(\xi, u_0, x) - \hat{\theta}^e_n(\xi, \bar{u}, x) |_{L^2(\mathbb{R})} \\
+ | \partial_\xi \partial_u \psi^e(n)(\xi, \bar{u}, x) |_{L^2(\mathbb{R})} | \theta(x) + \theta(\xi_0, u_0, x) - \theta(\xi, \bar{u}, x) |_{L^\infty(\mathbb{R})} \\
+ | \partial_\xi \partial_u \psi^e(n)(\xi, \bar{u}, x) |_{L^2(\mathbb{R})} | \psi(x) + \psi(\xi_0, u_0, x) - \psi(\xi, \bar{u}, x) |_{L^2(\mathbb{R})} \\
+ | \partial_\xi \partial_u \psi^e(n)(\xi, \bar{u}, x) |_{L^2(\mathbb{R})} | \psi(\xi_0, u_0, x) - \psi(\xi, \bar{u}, x) |_{L^\infty(\mathbb{R})} + | m^e_{\xi_0}(\xi, \bar{u}) - m^e_{\xi_0}(\xi_0, u_0) | \right).
\]

Similarly as above one shows that \( \exists r \leq \tilde{\delta}, \varepsilon_0 > 0 \) \( \forall \varepsilon \in (0, \varepsilon_0], (\xi_0, u_0) \in \Sigma(3) \) \( \forall (\theta, \psi) \in B_r(0) : | S^e_{\xi_0, u_0}(\theta, \psi, 0, 0) |_{\infty} < \tilde{\delta}(1 - k) \), which completes the proof. \( \square \)
5 Existence of Dynamics and the Orthogonal Component

We argue similar to [Stu98, Proof of theorem 2.1]. Let \( \varepsilon_0 \) be from Lemma 4.5 and \( \varepsilon \in (0, \varepsilon_0] \). In order to make use of existence theory we consider the problem

\[
\begin{pmatrix}
\bar{v}(0, x) \\
\bar{w}(0, x)
\end{pmatrix} = \begin{pmatrix}
\theta(0, x) - \theta_n^\varepsilon(\xi, u, x) \\
\psi(0, x) - \psi_n^\varepsilon(\xi, u, x)
\end{pmatrix},
\]

(28)

\[
\partial_t \begin{pmatrix}
\bar{v}(t, x) \\
\bar{w}(t, x)
\end{pmatrix} = \begin{pmatrix}
\bar{w}(t, x) - \psi_n^\varepsilon(\xi, u, x) \\
[\bar{v}(t, x) + \theta_n^\varepsilon(\xi, u, x)]_{xx} - \sin(\bar{v}(t, x) + \theta_n^\varepsilon(\xi, u, x)) + F(\varepsilon, x)
\end{pmatrix}.
\]

(29)

By [Mar76, Theorem VIII 2.1, Theorem VIII 3.2] there exists a local solution (see also [Stu98, Proof of theorem 2.1], [Stu92, p.434]), where

\[
(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})).
\]

(\( \theta, \psi \)) given by \( \theta(t, x) = \bar{v}(t, x) + \theta_n^\varepsilon(\xi, u, x) \) and \( \psi(t, x) = \bar{w}(t, x) + \psi_n^\varepsilon(\xi, u, x) \) solves obviously locally the Cauchy problem (21) and \( (\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})) \) due to Morrey’s embedding theorem. We are going to obtain a bound in Section 9 which will imply that the local solutions are indeed continuably. So from now we assume that \((\bar{v}, \bar{w}) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\) is a solution of (28)-(29) and \((\theta, \psi)\) is a solution of (21) such that \((\theta, \psi) \in C^1([0, T], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\), where \( T > 0 \).

In the following we define, similar to \( \Sigma(l, U, u_\ast) \), a new parameter area, where the parameter \( \xi \) is bounded.

Definition 5.1. We set for \( u_\ast, U \) as in Section 4 and \( \Xi \) from Definition 3.6

\[
\Sigma(l, \Xi) := \Sigma(l, U, u_\ast, \Xi) := \left\{ (\xi, u) \in (-\Xi + 1 - V(l), \Xi - 1 + V(l)) \times (-U - V(l), U + V(l)) \right\}.
\]

Given \((\theta, \psi)\) we choose the parameters \((\xi(t), u(t))\) according to Lemma 4.5 and define \((v, w)\) as follows:

\[
v(t, x) = \theta(t, x) - \theta_n^\varepsilon(\xi(t), u(t), x),
\]

(30)

\[
w(t, x) = \psi(t, x) - \psi_n^\varepsilon(\xi(t), u(t), x).
\]

(31)

\((v(t, x), w(t, x))\) is well defined for \( t \geq 0 \) so small that \(|v(t)|_{L^\infty(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})} \leq r \) and \((\xi(t), u(t)) \in \Sigma(\varepsilon, \Xi)\), where \( r \) is from Lemma 4.5. We formalize this in the following definition.

Definition 5.2. Let \( r \) be from Lemma 4.5, \( t^* \) is the "exit time"

\[
t^* := \sup \left\{ T > 0 : |v|_{L^\infty(\mathbb{R})L^\infty([0,t])} + |w|_{L^\infty([0,t],L^2(\mathbb{R}))} \leq r, (\xi(t), u(t)) \in \Sigma(\varepsilon, \Xi), 0 \leq t \leq T \right\}.
\]

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Notice that \((\xi, u, v, w) = (\xi(0), u(0)) \in \Sigma(4, \Xi)\). The transversal component \((v(t, x), w(t, x))\) is well defined for \(0 \leq t \leq t^*\) since, among others, we choose \(\varepsilon\) such that \(\varepsilon \in (0, \varepsilon_0]\) with \(\varepsilon_0\) from Lemma 4.5 and the initial data such that \(|v(0)|_{L^\infty(\mathbb{R})} + |w(0)|_{L^2(\mathbb{R})} \leq \frac{\varepsilon}{2}\) with \((v(0), w(0))\) given by (21).

**Lemma 5.3.** Let \(T = \min\{t^*, \bar{T}\}\) and let \((v, w)\) be defined by (30)–(31). Then \((v, w) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\).

**Proof.** This follows by using (30)–(31) and the fact that \((\bar{v}, \bar{w}) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\), since the difference \((\theta_K(\gamma(u_0)(\cdot - \xi_0)) - \theta_K(\gamma(\bar{u})(\cdot - \bar{\xi})))\) is in \(L^2(\mathbb{R})\) for all \((\xi_0, u_0), (\bar{\xi}, \bar{u}) \in \mathbb{R} \times (-1, 1)\). \(\square\)

In the following lemma we point out the relation between \(F\) and \((\theta^e_n, \psi^e_n, \lambda^e_n)\). Notice that there appears \(F\) instead of \(\tilde{F}\) in the equation above. Moreover, we state the rates of convergence of \(R^e_n(\xi, u, \cdot)\) and \(\lambda^e_n(\xi, u)\) which will be needed in the proof of the modulation equations for the parameters \((\xi(t), u(t))\) in the next section and in the proof of the main result in Section 6.

**Lemma 5.4.** It holds that for a.e. \((\xi, u, x) \in \Sigma(4, \Xi) \times \mathbb{R}\)

\[
\begin{align*}
&u \partial_x \left( \theta^e_n(\xi, u, x) \right) - \left( \psi^e_n(\xi, u, x) \right) - \left( \partial^2_x \theta^e_n(\xi, u, x) - \sin \theta^e_n(\xi, u, x) + F(\varepsilon, x) \right) \\
&+ \lambda^e_n(\xi, u) \partial_u \left( \theta^e_n(\xi, u, x) \right) - \left( \psi^e_n(\xi, u, x) \right) + R^e_n(\xi, u, x) = 0
\end{align*}
\]

and \(|[R^e_n(\xi, u, \cdot)]_{L^2(\mathbb{R})}| = O(\varepsilon^{n+1})\), \(|[R^e_n(\xi, u, \cdot)]_{L^2(\mathbb{R})}| = O(\varepsilon^{n+1})\), \(|\lambda^e_n(\xi, u)| = O(\varepsilon^{k+1})\), \(|\partial_1 \lambda^e_n(\xi, u)| = O(\varepsilon^{k+1})|\) uniformly in \((\xi, u) \in \Sigma(4, \Xi)\).

**Proof.** The first identity follows due to Theorem 3.4 and Lemma 3.7. Using Theorem 3.4, Lemma 3.5, and Morrey’s embedding theorem we obtain for all \((\xi, u) \in \Sigma(4, \Xi)\):

\[
|[R^e_n(\xi, u, \cdot)]_{L^2(\mathbb{R})}| \leq c \left\| \begin{pmatrix} 0 \\ \lambda^e_n \end{pmatrix} \right\|_{Y^0_2(u_+)^*} \left\| \begin{pmatrix} \sum_{i=0}^{n-1} \partial^i \theta^e_n \cdot \varepsilon^i - \theta^e_n \\ \sum_{i=0}^{n-1} \partial^i \psi^e_n \cdot \varepsilon^i - \psi^e_n \end{pmatrix} \right\|_{Y^0_2(u_+)} = O(\varepsilon^{n+1})
\]

and

\[
|\partial_1 \lambda^e_n(\xi, u)| \leq c \left\| \begin{pmatrix} 0 \\ \lambda^e_n \end{pmatrix} \right\|_{Y^0_2(u_+)} = O(\varepsilon^{k+1}).
\]

The other cases can be treated analogously. \(\square\)

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We compute the time derivatives of \( v \) and \( w \), which will be needed in the following sections.

**Lemma 5.5.** Let \((v, w)\) be given by \(30\) - \(31\). Then it holds
\[
\dot{v}(x) = w(x) - \xi \partial_n \theta^e_n(\xi, u, x) - \dot{u} \partial_n \theta^e_n(\xi, u, x)
\]
\[
+ u \partial_n \theta^e_n(\xi, u, x) + \lambda^e_n(\xi, u) \partial_u \theta^e_n(\xi, u, x) + [\mathcal{R}^e_n(\xi, u, x)]_1,
\]
\[
\dot{w}(x) = \partial_n^2 v(x) - \cos \theta^e_n(\xi, u, x)v(x) + \frac{\sin \theta^e_n(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x) + u \partial_n \psi^e_n(\xi, u, x)
\]
\[
+ \lambda^e_n(\xi, u) \partial_u \psi^e_n(\xi, u, x) + [\mathcal{R}^e_n(\xi, u, x)]_2 - \dot{\xi} \partial_n \psi^e_n(\xi, u, x) - \dot{u} \partial_u \psi^e_n(\xi, u, x),
\]
for times \( t \in [0, t^*] \), where \( \tilde{R}(v) = \mathcal{O}(\|v\|_{L^2(\mathbb{R})}^3) \) and \( \mathcal{R}^e_n(\xi, u, x) \) is from Theorem \(3.4\) (b).

**Proof.** By taking the time derivatives of \((v, w)\) and using Lemma \(5.4\) \(21\) we obtain
\[
\dot{v}(x) = w(x) + \psi^e_n(\xi, u, x) - \dot{\xi} \partial_n \theta^e_n(\xi, u, x) - \dot{u} \partial_n \theta^e_n(\xi, u, x)
\]
\[
= w(x) - \dot{\xi} \partial_n \theta^e_n(\xi, u, x) - \dot{u} \partial_n \theta^e_n(\xi, u, x)
\]
\[
+ u \partial_n \theta^e_n(\xi, u, x) + \lambda^e_n(\xi, u) \partial_u \theta^e_n(\xi, u, x) + [\mathcal{R}^e_n(\xi, u, x)]_1
\]
and
\[
\dot{w}(x) = \partial_n^2 \theta(x) - \sin \theta(x) + F(\varepsilon, x) - \dot{\theta}_\xi \partial_n \psi^e_n(\xi, u, x) - \dot{\theta}_u \partial_u \psi^e_n(\xi, u, x)
\]
\[
= \partial_n^2 \theta^e_n(\xi, u, x) + \partial_n^2 v(x) - \sin \theta^e_n(\xi, u, x) - \cos \theta^e_n(\xi, u, x)v(x)
\]
\[
+ \frac{\sin \theta^e_n(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x) + F(\varepsilon, x) - \dot{\theta}_\xi \partial_n \psi^e_n(\xi, u, x) - \dot{\theta}_u \partial_u \psi^e_n(\xi, u, x)
\]
\[
= \partial_n^2 v(x) - \cos \theta^e_n(\xi, u, x)v(x) + \frac{\sin \theta^e_n(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x) + u \partial_n \psi^e_n(\xi, u, x)
\]
\[
+ \lambda^e_n(\xi, u) \partial_u \psi^e_n(\xi, u, x) + [\mathcal{R}^e_n(\xi, u, x)]_2 - \dot{\theta}_\xi \partial_n \psi^e_n(\xi, u, x) - \dot{\theta}_u \partial_u \psi^e_n(\xi, u, x),
\]
where we have expanded the term \( \sin(\theta^e_n(\xi, u, x) + v(x)) \).

## 6 Modulation Equations

In the following lemma we derive modulation equations for the parameters \((\xi(t), u(t))\).

**Lemma 6.1.** There exists an \( \varepsilon_0 > 0 \) such that the following statement holds. Let \( \varepsilon \in (0, \varepsilon_0] \) and let \((v, w)\) be given by \(30\) - \(31\) with \((\xi, u)\) obtained from Lemma \(4.5\). Let
\[
|v|_{L^\infty([0,t^*],H^1(\mathbb{R}))}, |w|_{L^\infty([0,t^*],L^2(\mathbb{R}))} \leq \varepsilon_0.
\]
Then it holds for $t \in [0, t^*]$ that

$$|\dot{v}(t) - u(t)| \leq C|v(t)|_{H^1(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})}|\varepsilon|^{k+1} + C|v(t)|_{H^1(\mathbb{R})}^2 + C\varepsilon^{n+k+1},$$

$$|\ddot{v}(t) - \lambda^v_n(\xi(t), u(t))| \leq C|v(t)|_{H^1(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})}|\varepsilon|^{k+1} + C|v(t)|_{H^1(\mathbb{R})}^2 + C\varepsilon^{n+k+1},$$

where $C$ depends on $F$ and $\xi_s$.

**Proof.** The technique we use is similar to that in the proof of [IKV12] Lemma 6.2. Let $\varepsilon_0 \in (0, \varepsilon^*)$ with $\varepsilon^*$ from Definition 3.9 and let $\varepsilon \in (0, \varepsilon_0)$. Further in the proof we will make some more assumptions on $\varepsilon$. We start with some definitions and set

$$\Omega_n^v(\xi, u) := \begin{pmatrix} \Omega(t^v_{1,n}(\xi, u, \cdot), t^v_{1,n}(\xi, u, \cdot)) & \Omega(t^v_{2,n}(\xi, u, \cdot), t^v_{2,n}(\xi, u, \cdot)) \end{pmatrix} = (\gamma(u)^3m + m^v_n(\xi, u)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now we consider for any $(\xi, \bar{u}) \in \mathbb{R} \times [-U - V(2), U + V(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ the matrix:

$$M_n^v(\xi, \bar{u}, \bar{v}, \bar{w}) = \begin{pmatrix} \left\langle \partial^2_{\xi} \psi_n^v(\xi, \bar{u}, \cdot), (\bar{v}(\cdot), \bar{w}(\cdot)) \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \left\langle \partial_v \partial^2_{\xi} \psi_n^v(\xi, \bar{u}, \cdot), (\bar{v}(\cdot), \bar{w}(\cdot)) \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \\ \left\langle \partial_{\xi} \partial_v \psi_n^v(\xi, \bar{u}, \cdot), (\bar{v}(\cdot), \bar{w}(\cdot)) \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \left\langle \partial_v \partial^2_{\xi} \psi_n^v(\xi, \bar{u}, \cdot), (\bar{v}(\cdot), \bar{w}(\cdot)) \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \end{pmatrix}.$$

We use Lemma 4.4 Lemma 3.8 and Hölder’s inequality similar to the proof of Lemma 4.5 and obtain for all $(\xi, \bar{u}) \in \mathbb{R} \times [-U - V(2), U + V(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$:

$$\| \Omega_n^v(\xi, \bar{u})^{-1} M_n^v(\xi, \bar{u}, \bar{v}, \bar{w}) \| \leq C(|\bar{v}|_{H^1(\mathbb{R})} + |\bar{w}|_{L^2(\mathbb{R})}),$$

where we denote by $\| \cdot \|$ a matrix norm. Let $I = I_2$ be the identity matrix of dimension 2. Due to (32) we are able to choose $\varepsilon_0 > 0$ such that if $|\bar{v}|_{H^1(\mathbb{R})}, |\bar{w}|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then the matrix

$$I + [\Omega_n^v(\xi, \bar{u})^{-1} M_n^v(\xi, \bar{u}, \bar{v}, \bar{w})]$$

is invertible by von Neumann’s theorem. Using (30)-(31) we express the orthogonality condition $\mathcal{N}^v(\theta, \psi, \xi, u) = 0$ from Lemma 4.5 in terms of $(v, w, \xi, u)$ and take its derivative with respect to $t$. For simplicity of notation, we drop $(\theta, \psi, \xi, u)$ and obtain in matrix form:

$$0 = \frac{d}{dt} \begin{pmatrix} \mathcal{N}^v_1 \\ \mathcal{N}^v_2 \end{pmatrix} = \Omega \begin{pmatrix} \dot{\xi} - u \\ \dot{u} - \lambda_n^v(\xi, u) \end{pmatrix} + M \begin{pmatrix} \dot{\xi} - u \\ \dot{u} - \lambda_n^v(\xi, u) \end{pmatrix} + P_1, \quad P_2,$$
where $M = M^\varepsilon_n(\xi, u, v, w)$, $\Omega = \Omega^\varepsilon_n(\xi, u)$, $P_1 = P_{1,n}^\varepsilon(\xi, u, v, w)$, $P_2 = P_{2,n}^\varepsilon(\xi, u, v, w)$,

$\begin{align*}
P_{1,n}^\varepsilon(\xi, u, v, w) &= \int \partial_\xi \psi^\varepsilon_n(\xi, u, x) w(x) - \partial_\xi \theta^\varepsilon_n(\xi, u, x) \left( \partial_x^2 v(x) - \cos \theta^\varepsilon_n(\xi, u, x) v(x) \right) \, dx \\
&\quad + \int u \partial_\xi^2 \psi^\varepsilon_n(\xi, u, x) v(x) - u \partial_\xi^2 \theta^\varepsilon_n(\xi, u, x) w(x) \, dx \\
&\quad + \int \partial_\xi \partial_\xi \psi^\varepsilon_n(\xi, u, x) v(x) - \partial_\xi \partial_\xi \theta^\varepsilon_n(\xi, u, x) w(x) \, dx - \lambda^\varepsilon_n(\xi, u) + \int \partial_\xi \psi^\varepsilon_n(\xi, u, x) [R^\varepsilon_n(\xi, u, x)] \, dx \\
&\quad - \int \partial_\xi \theta^\varepsilon_n(\xi, u, x) \left( \sin \theta^\varepsilon_n(\xi, u, x) v^2(x) \right) + \tilde{R}(v)(x) + [R^\varepsilon_n(\xi, u, x)]_2 \, dx
\end{align*}$

and

$\begin{align*}
P_{2,n}^\varepsilon(\xi, u, v, w) &= \int \partial_\xi \psi^\varepsilon_n(\xi, u, x) w(x) - \partial_\xi \theta^\varepsilon_n(\xi, u, x) \left( \partial_x^2 v(x) - \cos \theta^\varepsilon_n(\xi, u, x) v(x) \right) \, dx \\
&\quad + \int u \partial_\xi \partial_\xi \psi^\varepsilon_n(\xi, u, x) v(x) - u \partial_\xi \partial_\xi \theta^\varepsilon_n(\xi, u, x) w(x) \, dx \\
&\quad + \int \partial_\xi \partial_\xi \psi^\varepsilon_n(\xi, u, x) v(x) - \partial_\xi \partial_\xi \theta^\varepsilon_n(\xi, u, x) w(x) \, dx - \lambda^\varepsilon_n(\xi, u) + \int \partial_\xi \psi^\varepsilon_n(\xi, u, x) [R^\varepsilon_n(\xi, u, x)] \, dx \\
&\quad - \int \partial_\xi \theta^\varepsilon_n(\xi, u, x) \left( \sin \theta^\varepsilon_n(\xi, u, x) v^2(x) \right) + \tilde{R}(v)(x) + [R^\varepsilon_n(\xi, u, x)]_2 \, dx.
\end{align*}$

If $|v|_{H^1(\mathbb{R})}, |w|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then we obtain as mentioned above by von Neumann’s theorem that

$$\begin{bmatrix}
\dot{\xi} - u \\
\dot{u} - \lambda^\varepsilon_n(\xi, u)
\end{bmatrix} = - \left( I + \Omega^{-1} M \right)^{-1} \left[ \Omega^{-1} P \right].$$

We make a further assumption on $\varepsilon_0$, namely that $\varepsilon_0$ should be so small that the convergence rates in Lemma 5.4 are satisfied. Now we consider $P_1$ and $P_2$. The zeroth-order Taylor’s approximations (in $\varepsilon$) of expressions 33-34 and 35-36 respectively are

$$\begin{align*}
\left\langle \mathcal{L}_{\xi,u} \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_\xi \psi_0(\xi, u, \cdot) \\ \partial_\xi \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})}
&\quad \text{and} \quad \left\langle \mathcal{L}_{\xi,u} \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_\xi \psi_0(\xi, u, \cdot) \\ \partial_\xi \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})},
\end{align*}$$

where $\mathcal{L}_{\xi,u}$ is given in Definition A.4. Integration by parts and symplectic orthogonality yield that these approximations vanish, which can also be deduced from Lemma A.5. Thus
we obtain from Lemma 5.4, and similar arguments as above
\[ |P_1| \leq C [v]_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}^\varepsilon \right] + C [v]_{H^1(\mathbb{R})} + C \varepsilon^{n+k+1}, \]
\[ |P_2| \leq C [v]_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}^\varepsilon \right] + C [v]_{H^1(\mathbb{R})} + C \varepsilon^{n+k+1} . \]

\[ \square \]

7 Lyapunov Function

In this section we introduce the Lyapunov function and calculate its time derivative.

Definition 7.1. Let \((v, w)\) be given by (30)–(31), with \((\xi, u)\) obtained from Lemma 4.5.

We define the Lyapunov function \(L^\varepsilon\) by
\[ L^\varepsilon = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_n^\varepsilon(\xi, u, x))v^2(x)}{2} + uw(x)\partial_x v(x) \, dx \]  

(37)

and the auxiliary function \(L\) by
\[ L = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(x - \xi)))v^2(x)}{2} + uw(x)\partial_x v(x) \, dx . \]

Lemma 7.2. It holds that
\[ \frac{d}{dt} L^\varepsilon = (u - \dot{\xi}) \left[ \int -u\partial_x v(x) \left\{ -\partial_\xi \psi_n^\varepsilon(\xi, u, x) \right\} - w(x) \{ -\partial_\xi \psi_n^\varepsilon(\xi, u, x) \} \right] \]
\[ + [\cos(\theta_n^\varepsilon(\xi, u, x))v(x) - \partial_x^2 v(x)]\partial_x \theta_n^\varepsilon(\xi, u, x) + uw(x)\partial_x \theta_n^\varepsilon(\xi, u, x) \, dx \]
\[ - (\dot{u} - \lambda_2^\varepsilon(\xi, u)) \left[ \int -u\partial_x v(x) \left\{ -\partial_\xi \psi_n^\varepsilon(\xi, u, x) \right\} - w(x) \{ -\partial_\xi \psi_n^\varepsilon(\xi, u, x) \} \right] \]
\[ + [\cos(\theta_n^\varepsilon(\xi, u, x))v(x) - \partial_x^2 v(x)]\partial_x \theta_n^\varepsilon(\xi, u, x) + uw(x)\partial_x \theta_n^\varepsilon(\xi, u, x) \, dx \]
\[ - \dot{u} \int \frac{\sin(\theta_n^\varepsilon(\xi, u, x))}{2} \partial_x \theta_n^\varepsilon(\xi, u, x) v^2(x) \, dx + (\dot{\xi} - u) \int \cos(\theta_n^\varepsilon(\xi, u, x))v(x)\partial_x v(x) \, dx \]
\[ + \dot{u} \int w(x)\partial_x v(x) \, dx + \int w(x) \left[ \frac{\sin(\theta_n^\varepsilon(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}_n^\varepsilon(\xi, u, x)]_2 \right] \]
\[ + u\partial_x v(x) \left[ \frac{\sin(\theta_n^\varepsilon(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x) + [\mathcal{R}_n^\varepsilon(\xi, u, x)]_2 \right] + \partial_x v(x)\partial_x [\mathcal{R}_n^\varepsilon(\xi, u, x)]_1 \]
\[ + \cos(\theta_n^\varepsilon(\xi, u, x))v(x)[\mathcal{R}_n^\varepsilon(\xi, u, x)]_1 + uw(x)\partial_x [\mathcal{R}_n^\varepsilon(\xi, u, x)]_1 \, dx . \]
Proof. We use a similar technique as in the proof of [KSK97 Lemma 2.1]. We can assume that the initial data of our problem have compact support. This allows us to do the following computations (integration by parts etc.). The claim for non-compactly supported initial data follows by density arguments. We obtain the stated formula by taking the time derivative of (37), where we use

\[ \int \frac{\partial_t \left[ \cos \theta_n^\varepsilon(\xi, u, x) \right]}{2} v^2(x) \, dx \]

\[ = - \int \frac{\sin(\theta_n^\varepsilon(\xi, u, x))}{2} \left[ \partial_\xi \theta_n^\varepsilon(\xi, u, x) + \partial_\varepsilon \theta_n^\varepsilon(\xi, u, x) \right] v^2(x) \, dx \]

\[ + \int \xi \cos(\theta_n^\varepsilon(\xi, u, x)) v(x) \partial_\xi v(x) - \frac{\sin(\theta_n^\varepsilon(\xi, u, x))}{2} \partial_\varepsilon \theta_n^\varepsilon(\xi, u, x) v^2(x) \, dx \]

and \( \int \partial_\varepsilon v(x) \partial_\varepsilon^2 v(x) + w(x) \partial_\varepsilon w(x) \, dx = 0. \)

8 Lower Bound

Here we introduce a functional \( E \) and prove a lower bound on \( E \) by using symplectic orthogonality combined with functional analytic arguments. This will imply a lower bound on the Lyapunov function \( L^\varepsilon \), which will play a key role in the proof of the main result.

**Definition 8.1.** For \((v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), (\xi, u) \in \mathbb{R} \times (-1, 1)\) we set

\[ E(v, w, \xi, u) := \frac{1}{2} \int (w(x) + u \partial_\varepsilon v(x))^2 + v^2_Z(x) + \cos(\theta_K(Z)) v^2(x) \, dx, \]

where \( Z = \gamma(x - \xi) \) and \( v_Z(x) = \partial_\varepsilon v(\frac{Z}{\gamma} + \xi) = \frac{1}{\gamma} \partial_\varepsilon v(x) \).

A straightforward computation yields the following lemma.

**Lemma 8.2.** For \((v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), (\xi, u) \in \mathbb{R} \times (-1, 1)\) it holds that

\[ E(v, w, \xi, u) = \int \frac{w^2(x)}{2} + \left( \frac{\partial_\varepsilon v(x)}{2} \right)^2 + \frac{\cos(\theta_K(\gamma(x - \xi)) v^2(x)}{2} + uw(x) \partial_\varepsilon v(x) \, dx. \]

Recalling the relations (30)-(31) we introduce a notation in order to be able to express the orthogonality conditions in terms of the variables \((v, w, \xi, u)\) instead of the variables \((\theta, \psi, \xi, u)\).

**Definition 8.3.** For \((v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), (\xi, u) \in \mathbb{R} \times (-u_*, u_*)\) we set

\[ \hat{N}_1^\varepsilon(v, w, \xi, u) = \int \partial_\xi \psi_n^\varepsilon(\xi, u, x) v(x) - \partial_\xi \theta_n^\varepsilon(\xi, u, x) w(x) \, dx, \]

\[ \hat{N}_2^\varepsilon(v, w, \xi, u) = \int \partial_u \psi_n^\varepsilon(\xi, u, x) v(x) - \partial_u \theta_n^\varepsilon(\xi, u, x) w(x) \, dx. \]
Now we prove a lower bound on the functional $\mathcal{E}$.

**Lemma 8.4.** Let $\varepsilon_0 > 0$ be sufficiently small. There exists $c > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, $(\xi, u) \in [-\Xi, \Xi] \times [-U - V(2), U + V(2)] \subset \mathbb{R} \times (-1, 1)$ and $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfy

$$
\mathcal{N}_2^\varepsilon(v, w, \xi, u) = 0
$$

then

$$
\mathcal{E}(v, w, \xi, u) \geq c(\|v\|^2_{H^1(\mathbb{R})} + \|w\|^2_{L^2(\mathbb{R})}).
$$

**Proof.** We follow closely [Stu12] and [Stu98]. This proof is a slight modification of the proof of [Stu12] Lemma 4.3]. First of all we choose $\varepsilon_0$ such that $\varepsilon_0 \in (0, \varepsilon^*)$ with $\varepsilon^*$ from Definition 3.9. We will specify $\varepsilon_0$ later. Notice that the operator $-\partial^2_x + \cos \theta_K(Z)$ is nonnegative. It has (see [Stu92]) an one dimensional null space spanned by $\theta_K(\cdot)$ and the essential spectrum $[1, \infty)$. We argue by contradiction and assume that the result claimed is false: $\forall j \in \mathbb{N} \exists \varepsilon_j \in (0, \varepsilon_0)$, $(\xi_j, u_j) \in [-\Xi, \Xi] \times [-U - V(2), U + V(2)]$, $(\bar{v}_j, \bar{w}_j) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$:

$$
\mathcal{N}_2^\varepsilon_j(\bar{v}_j, \bar{w}_j, \xi_j, u_j) = 0, \quad \mathcal{E}(\bar{v}_j, \bar{w}_j, \xi_j, u_j) < \frac{1}{f}(\|\bar{v}_j\|^2_{H^1(\mathbb{R})} + \|\bar{w}_j\|^2_{L^2(\mathbb{R})}). \quad (38)
$$

This statement is also true for the sequences $v_j := \bar{v}_j((\bar{v}_j)^2_{H^1(\mathbb{R})} + \|\bar{w}_j\|^2_{L^2(\mathbb{R})})^{-\frac{1}{2}}$ and $w_j := \bar{w}_j((\bar{v}_j)^2_{H^1(\mathbb{R})} + \|\bar{w}_j\|^2_{L^2(\mathbb{R})})^{-\frac{1}{2}}$. Assuming that $\|v_j\|_{L^2(\mathbb{R})} \xrightarrow{\varepsilon \to 0} 0$ we obtain $\|v_j\|_{L^2(\mathbb{R})} \xrightarrow{\varepsilon \to 0} 0$ and $\|w_j\|_{L^2(\mathbb{R})} \xrightarrow{\varepsilon \to 0} 0$. This is a contradiction to the fact that $\|v_j\|^2_{H^1(\mathbb{R})} + \|w_j\|^2_{L^2(\mathbb{R})} = 1 \forall j \in \mathbb{N}$. By passing to a subsequence we may assume (without loss of generality) that there exists $\delta > 0$ such that

$$
\|v_j\|^2_{L^2(\mathbb{R})} \geq \delta \forall j \in \mathbb{N}. \quad (39)
$$

Since $(v_j, w_j)$ is bounded in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ we may assume that $v_j \xrightarrow{H^1(\mathbb{R})} v$ and $w_j \xrightarrow{L^2(\mathbb{R})} w$ by taking subsequences. Due to Rellich’s theorem we may assume again by passing to subsequences that $v_j \xrightarrow{L^2(\Omega)} v$, where $\Omega \subset \mathbb{R}$ is bounded and open. Passing to a further subsequence we assume almost everywhere convergence. Due to the fact that

$$
\exists r > 0 \text{ s.t. } |\cos(\theta_K(Z))| > \frac{1}{2} \text{ for } |Z| > r \quad (40)
$$

and that $-\partial^2_x + \cos \theta_K(Z)$ is a nonnegative operator we obtain the estimate

$$
\mathcal{E}(v_j, w_j, \xi_j, u_j) \geq \frac{1}{4} \int_{-\infty}^{\gamma(v_j) + \xi_j} v_j^2(x) \, dx + \frac{1}{4} \int_{\gamma(v_j) + \xi_j}^\infty v_j^2(x) \, dx,
$$

where we used integration by parts and substitution. We may extract a subsequence such that $u_j \xrightarrow{r} u, \xi_j \xrightarrow{\mathbb{R}} \xi$ and $\varepsilon_j \xrightarrow{r} \varepsilon$. It follows from (38) and from the previous estimate that

$$
\int_{\{x \in \mathbb{R} : |x| > r\}} v_j^2(x) \, dx \xrightarrow{j \to \infty} 0
$$

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for a sufficiently large \( \tilde{r} \). As a consequence, (39) and the strong convergence on bounded domains yield \( \int_{x \in \mathbb{R} : |x| \leq \tilde{r}} v^2(x) \, dx \geq \delta \), from which it follows that \( v \neq 0 \). Weak convergence and the continuity of \( \varepsilon \mapsto (\tilde{\theta}_n^\varepsilon, \tilde{\psi}_n^\varepsilon, \lambda_n^\varepsilon) \) (see Theorem 3.3) imply using the triangle inequality that

\[
\nabla_2^\varepsilon(v, w, \xi, u) = 0
\]

(41)

and

\[
\frac{1}{2} \int \left( w(x) + uv'(x) \right)^2 \, dx \leq \liminf_{j \to \infty} \frac{1}{2} \int \left( w_j(x) + u_j v_j'(x) \right)^2 \, dx,
\]

(42)

\[
\frac{1}{2} \int \left( \frac{1}{\gamma(u)} v'(x) \right)^2 \, dx \leq \liminf_{j \to \infty} \frac{1}{2} \int \left( \frac{1}{\gamma(u_j)} (v_j)'(x) \right)^2 \, dx.
\]

(43)

Due to (40) we are able to apply Fatou’s lemma for a sufficiently large \( \tilde{r} \) and obtain

\[
\frac{1}{2} \int_{\{x \in \mathbb{R} : |x| > \tilde{r} \}} \cos(\theta_K(\gamma(u)(x - \xi))) v^2(x) \, dx
\]

\[
\leq \liminf_{j \to \infty} \frac{1}{2} \int_{\{x \in \mathbb{R} : |x| > \tilde{r} \}} \cos(\theta_K(\gamma(u_j)(x - \xi_j))) v_j^2(x) \, dx,
\]

(44)

where we have used that \( (v_j) \) converges almost everywhere. The dominated convergence theorem yields:

\[
\frac{1}{2} \int_{\{x \in \mathbb{R} : |x| \leq \tilde{r} \}} \cos(\theta_K(\gamma(u)(x - \xi))) v^2(x) \, dx
\]

\[
= \lim_{j \to \infty} \frac{1}{2} \int_{\{x \in \mathbb{R} : |x| \leq \tilde{r} \}} \cos(\theta_K(\gamma(u_j)(x - \xi_j))) v_j^2(x) \, dx.
\]

(45)

(38) together with (42)-(45) imply that \( \mathcal{E}(v, w, \xi, u) = 0 \). This yields \( (v(x), w(x)) = \eta(\theta_K(\gamma(u)(x - \xi)), -u \gamma(u) \theta_K(\gamma(u)(x - \xi))) \) for some \( \eta \neq 0 \), since \( v \neq 0 \). Using Lemma 3.8 and the notation from Definition 4.2 we choose \( \varepsilon_0 \) sufficiently small so that for all \( (\xi, u) \)

\[
\frac{1}{\gamma(u)} \left( \left| \partial_u \tilde{\psi}_n^\varepsilon(\xi, u, x) \right|_{L^2(\mathbb{R})} \right) \left( \left| \partial_x \psi_0(\xi, u, x) \right|_{L^2(\mathbb{R})} \right) \left( \left| \partial_x \tilde{\psi}_n^\varepsilon(\xi, u, x) \right|_{L^2(\mathbb{R})} \right) \leq \frac{m}{2},
\]

which implies that \( \nabla_2^\varepsilon(v, w, \xi, u) \neq 0 \). This yields a contradiction to (41).

\[ \Box \]

Remark 8.5. Let \( (v, w) \) be given by (30)-(31), with \( (\xi, u) \) obtained from Lemma 4.5. It holds that \( L(t) = \mathcal{E}(v(t), w(t), \xi(t), u(t)) \).

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9 Proof of Theorem 2.2

9.1 Description of the Dynamics with Approximate Equations for the Parameters \((\xi, u)\)

We prove first a version of Theorem 2.2 with approximate equations for the parameters \((\xi, u)\).

**Theorem 9.1.** Assume that the assumptions of Theorem 2.2 on \(C\) of class \(C^n\) are satisfied. There exist positive constants \(c, C\) such that the following assumptions are satisfied:

- \(\|u\| \leq \tilde{C} \varepsilon^{\beta(k)}\).
- \(\mathcal{N}^c(\theta(0, \cdot), \psi(0, \cdot), \xi, u) = 0\).
- \(|v(0, \cdot)|^2_{H^1(\mathbb{R})} + |w(0, \cdot)|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n}\), where \((v(0, \cdot), w(0, \cdot))\) is given by (47).

Then the Cauchy problem defined by (47) has a unique solution on the time interval

\[0 \leq t \leq T, \text{ where } T = T(\varepsilon, k) = \frac{1}{C\varepsilon^{\beta(k)}}, \beta(k) = \frac{k + 1}{2}.\]

The solution may be written in the form

\[
\begin{pmatrix}
\theta(t, x) \\
\psi(t, x)
\end{pmatrix} = \begin{pmatrix}
\theta^c_n(\xi(t), u(t), x) \\
\psi^c_n(\xi(t), u(t), x)
\end{pmatrix} + \begin{pmatrix}
v(t, x) \\
w(t, x)
\end{pmatrix},
\]

where \(v, w, \xi, u\) have regularity \((v(t), w(t)) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\) and \((\xi(t), u(t)) \in C^1([0, T], \mathbb{R} \times (-1, 1))\) such that the symplectic orthogonality condition

\[\mathcal{N}^c(\theta(t, \cdot), \psi(t, \cdot), \xi(t), u(t)) = 0\]

is satisfied. There exist positive constants \(c, C\) such that

\[|\dot{\xi}(t) - u(t)| \leq C\varepsilon^{n+k+1}, \quad |\dot{u}(t) - \lambda^c_n(\xi(t), u(t))| \leq C\varepsilon^{n+k+1},\]

and

\[|v|^2_{L^\infty([0, T], H^1(\mathbb{R}))} + |w|^2_{L^\infty([0, T], L^2(\mathbb{R}))} \leq c\varepsilon^{2n}.\]

The constants \(c, C\) depend on \(F\) and \(\xi_s\).
Notice that the previous theorem describes the dynamics less precisely than Theorem 2.2. However, in the previous theorem the orthogonality condition is satisfied which does not have to hold in Theorem 2.2.

The proof of Theorem 9.1 needs some preparation. The existence of $\varepsilon_0 > 0$, $u_\ast > 0$ and the map (46) is ensured by Theorem 3.3. Now we suppose that (47) has a solution and we make some assumptions on $(v, w)$ given by (30)-(31) and on $(\xi, u)$ obtained from Lemma 4.5. Then the following lemma yields us more accurate information on $(v, w)$ and $(\xi, u)$.

**Lemma 9.2.** Assume that the assumptions of Theorem 2.2 on $n, k, \xi, F$ are satisfied and let $0 < \delta < 1/32$. There exist $\varepsilon_0, \tilde{C} > 0$ such that the following statement holds. Let $\varepsilon \in (0, \varepsilon_0)$. Assume that (47) has a solution $(\theta, \psi)$ on $[0, T]$ such that

$$\theta, \psi \in C^1([0, T], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})).$$

Suppose that $0 \leq T \leq t^* \leq T$. Suppose that $(v, w)$ is given by (30)-(31), with $(\xi, u)$ obtained from Lemma 4.5, such that

$$|u_s| \leq \tilde{C}\varepsilon^{k+1}, \quad |v|^2_{L^\infty([0, T], H^1(\mathbb{R}))} + |w|^2_{L^\infty([0, T], L^2(\mathbb{R}))} \leq \varepsilon^{2n-\delta}.$$

Then, provided

$$0 \leq T \leq \frac{1}{C\varepsilon^{2(k)}}, \quad \beta(k) = \frac{k+1}{2},$$

there exist $c, C > 0$ such that

1. $\forall t \in [0, T]$ $(\xi(t), u(t)) \in \Sigma(5, \Xi),$
2. $|v|^2_{L^\infty([0, T], H^1(\mathbb{R}))} + |w|^2_{L^\infty([0, T], L^2(\mathbb{R}))} \leq \frac{1}{k}(L(0) + C\varepsilon^{2n})$, where $c$ is from Lemma 8.4 and $C$ depends on $F, \xi_s$.

**Remark 9.3.** Notice that the assumption $T \leq t^*$ yields us the information:

$\forall t \in [0, T]$ $(\xi(t), u(t)) \in \Sigma(4, \Xi).$

**Proof.** Choose $\varepsilon_0$ sufficiently small, in particular such that the lemmas used below can be applied and such that for any $\varepsilon \in (0, \varepsilon_0)$ the following statement holds: if $(v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfies $|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n-\delta}$ then it holds that $|v|_{L^\infty(\mathbb{R})} + |w|_{L^2(\mathbb{R})} \leq \frac{\varepsilon}{2}$, where $r$ is from Lemma 4.5. This can be ensured by Morrey’s embedding theorem.

Lemma 6.1 yields $\forall t \in [0, T]$:  

\[
|\dot{\xi}(t) - u(t)| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C|v|_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1} \leq C\varepsilon^{n+k+1-\delta},
\]

\[
|\dot{\xi}(t) - \lambda_{\xi}(\xi(t), u(t))| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}]\varepsilon^{k+1} + C|v|_{H^1(\mathbb{R})} + C\varepsilon^{n+k+1} \leq C\varepsilon^{n+k+1-\delta}.
\]

Then, using Lemma 5.4, it follows that there exists $\tilde{C} > 1$ such that $\forall t \in [0, T]$:  

\[
|u(t) - u(0)| \leq \int_0^t |\dot{u}(s)| \, ds \leq \tilde{C}\varepsilon^{k+1}t,
\]

\[
|\xi(t) - \xi(0)| \leq \int_0^t |\dot{\xi}(s)| \, ds \leq \tilde{C}\varepsilon^{n+k+1-\delta}t + \tilde{C}\varepsilon^{k+1}t^2 + |u(0)|t.
\]
This implies (1) by choosing \( \varepsilon_0 \) small enough and utilizing \( |u_s| \leq \tilde{C}\varepsilon^{k+1} \). By using Lemma 8.4, Lemma 8.2, Lemma 7.2 and Lemma 5.4 we obtain for times \( 0 \leq t \leq T \leq \frac{1}{C_\varepsilon^2(\varepsilon)} \), estimate (2):

\[
c(|v(t)|^2_{H^1(\mathbb{R})} + |w(t)|^2_{L^2(\mathbb{R})}) \leq L(t) = L^\varepsilon(t) + C\varepsilon^{k+1} |v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} = L^\varepsilon(0) + \int_0^t \frac{d}{dt}L^\varepsilon(t)\, dt + C\varepsilon |v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} \leq L^\varepsilon(0) + C\varepsilon^{2n}.
\]

\[ \square \]

Theorem 9.4. Assume that the assumptions of Theorem 2.2 on \( n, k, \xi_s \) and \( F \) are satisfied. There exists \( \varepsilon_0, C > 0 \) such that the following statement holds. Let \( \varepsilon \in (0, \varepsilon_0) \). Assume that (37) has a solution \((\theta, \psi)\) on \([0, T]\) such that

\[ (\theta, \psi) \in C^1([0, T], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})). \]

Suppose that \( 0 \leq T \leq T \) and that the assumptions (a),(b),(c) of Theorem 9.1 are satisfied. Then, provided

\[ 0 \leq T \leq \frac{1}{C_\varepsilon^{\beta(k)}}, \quad \beta(k) = \frac{k + 1}{2}, \]

it holds that \((v, w)\) given by (30)–(31) is well defined for times \([0, T]\) and there exists \( \hat{c} > 0 \) such that

1. \( |v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \hat{c}\varepsilon^{2n}, \)
2. \( \forall t \in [0, T] \ (\xi(t), u(t)) \in \Sigma(5, \Xi). \)

Proof. Let \( \delta \) and \( \tilde{C} \) be as in Lemma 9.2. Choose \( \varepsilon_0 \) sufficiently small, in particular such that

\[ \forall \varepsilon \in (0, \varepsilon_0) : \frac{2}{c}(L(0) + C\varepsilon^{2n}) < \varepsilon^{2n-\delta}, \]

where \( L(0) = E(v(0), w(0), \xi_s, u_s) \) and the constants \( c, C \) are from Lemma 9.2 (2). Let \( \varepsilon \in (0, \varepsilon_0) \). Notice that \( \Sigma(5, \Xi) \subset \Sigma(4, \Xi) \). We define an exit time

\[ t_* := \sup \left\{ T > 0 : |v|^2_{L^\infty([0,t],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,t],L^2(\mathbb{R}))} \leq \frac{2}{c}(L(0) + C\varepsilon^{2n}), \right. \]

\[ \left. (\xi(t), u(t)) \in \Sigma(5, \Xi), \ 0 \leq t \leq T \right\}. \]

Suppose \( t_* < \frac{1}{C_\varepsilon^{\beta(k)}} \). Then there exists a time \( \hat{t} \) such that \( \frac{1}{C_\varepsilon^{\beta(k)}} > \hat{t} > t_* \), with

\[ \forall t \in [0, \hat{t}] : (\xi(t), u(t)) \in \Sigma(4, \Xi), \quad (\xi(\hat{t}), u(\hat{t})) \notin \Sigma(5, \Xi) \]
or
\[
\frac{1}{c}(L(0) + C\varepsilon^{2n}) < \frac{2}{c}(L(0) + C\varepsilon^{2n}) < |v|^2_{L^\infty([0,\hat{t}], H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,\hat{t}], L^2(\mathbb{R}))} < \varepsilon^{2n-\delta}.
\]
This leads a contradiction to Lemma 9.2. 

The previous theorem implies that the local solution of (47) discussed in Section 5 is indeed continuabe up to times \(1/(\tilde{C}\varepsilon^{\beta(k)})\) for \(\varepsilon \in (0, \varepsilon_0)\). Theorem 9.4 and Lemma 6.1 yield the approximate equations for the parameters \((\xi, u)\), which conclude the proof of Theorem 9.1.

### 9.2 ODE Analysis

In this subsection we lay the groundwork for passing from the approximate equations for the parameters \((\xi, u)\) in Theorem 9.1 to the ODEs given by (23). We start with a preparing lemma.

**Lemma 9.5.** There exists \(\varepsilon_0 > 0\) such that the following statement holds. Let \(\varepsilon \in (0, \varepsilon_0)\). Let \(\beta(k) = \frac{k+1}{2}\). Let \(\xi = \tilde{\xi}(s), \, \hat{u} = \tilde{u}(s), \, \epsilon_1 = \epsilon_1(s), \, \epsilon_2 = \epsilon_2(s)\) be \(C^1\) real-valued functions. Suppose that
\[
|\epsilon_j(s)| \leq \bar{c}\varepsilon^n
\]
on \([0, T]\) for \(j = 1, 2\). Assume that on \([0, T]\),
\[
\frac{d}{ds}\tilde{\xi}(s) = \tilde{\xi}(s) = \epsilon_1(s), \quad \tilde{\xi}(0) = \tilde{\xi}_0,
\]
\[
\frac{d}{ds}\tilde{u}(s) = \frac{1}{\varepsilon^{2\beta(k)}}(s, \epsilon_1(s)) + \epsilon_2(s), \quad \tilde{u}(0) = \tilde{u}_0.
\]
Let \(\hat{\xi} = \hat{\xi}(s)\) and \(\hat{u} = \hat{u}(s)\) be \(C^1\) real-valued functions which satisfy the exact equations
\[
\frac{d}{ds}\hat{\xi}(s) = \hat{\xi}(s), \quad \hat{\xi}(0) = \hat{\xi}_0,
\]
\[
\frac{d}{ds}\hat{u}(s) = \frac{1}{\varepsilon^{2\beta(k)}(s, \epsilon_1(s), \epsilon_2(s))), \quad \hat{u}(0) = \hat{u}_0.
\]
Then there exists \(c > 0\) such that the estimates
\[
|\hat{\xi}(s) - \tilde{\xi}(s)| \leq c\varepsilon^n, \quad |\hat{u}(s) - \tilde{u}(s)| \leq c\varepsilon^n,
\]
hold on \([0, T]\).

**Proof.** We follow very closely [HZ08, Lemma 6.1]. We choose \(\varepsilon_0\) so small that the convergence rates in Lemma 5.4 are satisfied for all \(\varepsilon \in (0, \varepsilon_0)\). Let \(\varepsilon \in (0, \varepsilon_0)\). Let \(x = x(s)\) and \(y = y(s)\) be \(C^1\) real-valued functions, \(C \geq 1\), and let \((x, y)\) satisfy the differential inequalities:
\[
\begin{align*}
|\dot{x}| &\leq |y|, \quad x(0) = x_0, \\
|\dot{y}| &\leq C|x| + C|y|, \quad y(0) = y_0.
\end{align*}
\]
For \( z(s) = x^2 + y^2 \) the following estimate holds
\[
|\dot{z}| = |2x\dot{x} + 2y\dot{y}| \leq 2|x||\dot{y}| + 2C|x||y| + 2C|y||\dot{y}| \leq 4C(x^2 + y^2) = 4Cz.
\]
It follows from Gronwall’s lemma that \( z(s) \leq z(0)e^{4Cs} \). Thus
\[
|x(s)| \leq \sqrt{2}\max(|x_0|, |y_0|) \exp(2Cs), \quad |y(s)| \leq \sqrt{2}\max(|x_0|, |y_0|) \exp(2Cs).
\]
(48)

Now we recall Duhamel’s formula. Let \( X(s) : \mathbb{R} \to \mathbb{R}^2 \) be a two-vector function, \( X_0 \in \mathbb{R}^2 \) a two-vector, and \( A(s) : \mathbb{R} \to (2 \times 2 \text{ matrices}) \) a \( 2 \times 2 \) matrix function. We consider the ODE system
\[
\dot{X}(s) = A(s)X(s), \quad X(0) = X_0
\]
and denote its solution by \( X(s) = S(s, s')X_0 \) such that
\[
\frac{d}{ds}S(s, s')X_0 = A(s)S(s, s')X_0, \quad S(s, s')X_0 = X_0.
\]
Let \( F(s) : \mathbb{R} \to \mathbb{R}^2 \) be a 2-vector function. We can express the solution of the inhomogeneous ODE system
\[
\dot{X}(s) = A(s)X(s) + F(s)
\]
with initial condition \( X(0) = 0 \) by Duhamel’s formula
\[
X(s) = \int_0^s S(s, s')F(s')ds'.
\]
Let \( U = \hat{u} - \check{u} \) and \( \Xi = \check{\xi} - \hat{\xi} \). These functions satisfy
\[
\frac{d}{ds}\Xi(s) = U(s) + \epsilon_1(s), \quad \frac{d}{ds}U(s) = \frac{1}{\epsilon^{2\beta(k)}} \left[ \lambda_0^\epsilon(\check{\xi}(s), \epsilon^{\beta(k)}\check{u}(s)) - \lambda_0^\epsilon(\hat{\xi}(s), \epsilon^{\beta(k)}\hat{u}(s)) \right] + \epsilon_2(s).
\]
Let
\[
g(s) = \begin{cases} 
\frac{1}{\epsilon^{2\beta(k)}} \lambda_0^\epsilon(\check{\xi}(s), \epsilon^{\beta(k)}\check{u}(s)) - \lambda_0^\epsilon(\hat{\xi}(s), \epsilon^{\beta(k)}\hat{u}(s)), & \text{if } \hat{\xi}(s) \neq \check{\xi}(s), \\
\frac{1}{\epsilon^{2\beta(k)}} \partial_1 \lambda_0^\epsilon(\check{\xi}(s), \epsilon^{\beta(k)}\check{u}(s)), & \text{if } \hat{\xi}(s) = \check{\xi}(s),
\end{cases}
\]
and
\[
h(s) = \begin{cases} 
\frac{1}{\epsilon^{2\beta(k)}} \lambda_0^\epsilon(\check{\xi}(s), \epsilon^{\beta(k)}\check{u}(s)) - \lambda_0^\epsilon(\hat{\xi}(s), \epsilon^{\beta(k)}\hat{u}(s)), & \text{if } \check{u}(s) \neq \hat{u}(s), \\
\frac{1}{\epsilon^{\beta(k)}} \partial_2 \lambda_0^\epsilon(\check{\xi}(s), \epsilon^{\beta(k)}\check{u}(s)), & \text{if } \check{u}(s) = \hat{u}(s).
\end{cases}
\]
We set
\[
A(s) = \begin{bmatrix} 0 & g(s) \\ h(s) & 0 \end{bmatrix}, \quad F(s) = \begin{bmatrix} \epsilon_1(s) \\ \epsilon_2(s) \end{bmatrix}, \quad X(s) = \begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix}
\]
and obtain by Duhamel's formula:

$$\begin{bmatrix} \Xi(s) \\ U(s) \end{bmatrix} = \int_0^s \begin{bmatrix} \epsilon_1(s') \\ \epsilon_2(s') \end{bmatrix} ds'.$$

(49)

We use Lemma 5.4 and apply (48) with function theorem according to Lemma 4.5 and the solutions (\(\hat{\xi}\)).

It follows that

$$|\Xi| \leq \sqrt{2} \exp(2CT) \max_{0 \leq s \leq T} (|\epsilon_1(s)|, |\epsilon_2(s)|),$$

$$|U| \leq \sqrt{2} T \exp(2CT) \sup_{0 \leq s \leq T} \max(|\epsilon_1(s)|, |\epsilon_2(s)|),$$

which yields the claim.

In the following we show the relation between the parameters \((\xi, u)\) selected by the implicit function theorem according to Lemma 4.5 and the solutions \((\hat{\xi}, \hat{u})\) of the exact ODEs from the previous lemma.

**Lemma 9.6.** Assume that the assumptions of Theorem 9.1 are satisfied. There exists \(\varepsilon_0 > 0\) such that the following statement holds. Let \(\varepsilon \in (0, \varepsilon_0), \beta(k) = \frac{k+1}{2}\) and \(s = \varepsilon(\beta(k)) t\), where

$$0 \leq s \leq \frac{1}{C}, \quad 0 \leq t \leq \frac{1}{C\varepsilon(\beta(k))}.$$

Let \((\xi, u)\) be the parameters selected according to Lemma 4.5 and \((\hat{\xi}, \hat{u})\) from Lemma 9.3. Then there exists \(c > 0\) such that

$$|\xi(t) - \hat{\xi}(\varepsilon(\beta(k)) t)| \leq c\varepsilon^n, \quad |u(t) - \varepsilon(\beta(k)) \hat{u}(\varepsilon(\beta(k)) t)| \leq c\varepsilon^{n+\beta(k)}.$$

**Proof.** We choose \(\varepsilon_0\) as in Theorem 9.1. Let \(\varepsilon \in (0, \varepsilon_0)\) and

$$\hat{\xi}(s) = \xi(s/\varepsilon(\beta(k))), \quad \hat{u}(s) = \frac{1}{\varepsilon(\beta(k))} u(s/\varepsilon(\beta(k))).$$

For times \(0 \leq t \leq (\varepsilon(\beta(k)))^{-1}\) Theorem 9.1 yields that

$$|\xi(t) - u(t)| \leq C\varepsilon^{n+k+1}, \quad \varepsilon(\beta(k)) \hat{u}(\varepsilon(\beta(k)) t)| \leq C\varepsilon^{n+k+1}.$$

Thus \((\hat{\xi}, \hat{u})\) satisfy the assumptions of Lemma 9.3, which implies that

$$|\hat{\xi}(s) - \hat{\xi}(s)| = |\xi(t) - \hat{\xi}(\varepsilon(\beta(k)) t)| \leq c\varepsilon^n, \quad |\hat{u}(s) - \hat{u}(s)| = \frac{|u(t)}{\varepsilon(\beta(k))} - \hat{u}(\varepsilon(\beta(k)) t)| \leq c\varepsilon^n.$$
9.3 Completion of the Proof of Theorem 2.2

Theorem 9.1 yields the dynamics with the parameters \((\xi, u)\) selected by the implicit function theorem according to Lemma 4.5 on the time interval \(0 \leq t \leq (C\varepsilon^{\beta(k)})^{-1}\). Using Lemma A.6 and the triangle inequality we can replace \((\xi(t), u(t))\) with \((\hat{\xi}(\varepsilon^{\beta(k)}t), \varepsilon^{\beta(k)}\hat{u}(\varepsilon^{\beta(k)}t))\). The claim follows after possibly increasing the constant \(C\) in the proof of Theorem 9.1.

\[\square\]

A Preliminary Decompositions

Let \(\alpha, n \in \mathbb{N}\). Here we prove some decompositions for certain Sobolev spaces on \(\mathbb{R}\) and on \(\mathbb{R}^2\). We start with the spaces on \(\mathbb{R}\) and prove an orthogonal decomposition of \(H^1(\mathbb{R}) \oplus L^2(\mathbb{R})\).

**Definition A.1.** We define the following spaces.

(a) \(L^{2,\alpha}(\mathbb{R}) := H^{0,\alpha}(\mathbb{R})\).

(b) \(H^{2,\alpha}_{\xi,u,\bot}(\mathbb{R}) := \{\theta \in H^{2,\alpha}(\mathbb{R}) : \langle \theta(\cdot), \theta_K'(\gamma(\cdot - \xi)) \rangle_{L^2(\mathbb{R})} = 0\}\).

(c) \(L^{2,\alpha}_{\xi,u,\bot}(\mathbb{R}) := \{\theta \in L^2(\mathbb{R}) : \langle \theta(\cdot), \theta_K'(\gamma(\cdot - \xi)) \rangle_{L^2(\mathbb{R})} = 0\}\).

We define the following operators.

(d) \(L^{\alpha}_{\xi,u} : H^{2,\alpha}(\mathbb{R}) \subset L^{2,\alpha}(\mathbb{R}) \rightarrow L^{2,\alpha}(\mathbb{R})\) given by

\[
(L^{\alpha}_{\xi,u}\theta)(x) = -(1-u^2)\partial^2\theta(x) + \cos(\theta_K(\gamma(x-\xi)))\theta(x).
\]

(e) \(M^{\alpha}_{\xi,u} : H^{2,\alpha}(\mathbb{R}) \oplus \mathbb{R} \rightarrow L^{2,\alpha}(\mathbb{R})\) given by

\[
(M^{\alpha}_{\xi,u}(\theta, \lambda))(x) = (L^{\alpha}_{\xi,u}\theta)(x) + \lambda\theta_K'(\gamma(x-\xi)).
\]

(f) \(\hat{L}^{\alpha}_{\xi,u} = L^{\alpha}_{\xi,u}\big|_{H^{2,\alpha}_{\xi,u,\bot}(\mathbb{R})}, \hat{M}^{\alpha}_{\xi,u} = M^{\alpha}_{\xi,u}\big|_{H^{2,\alpha}_{\xi,u,\bot}(\mathbb{R}) \oplus \mathbb{R}}\).

**Lemma A.2.** \(\text{ran} \ L^{\alpha}_{\xi,u}\) is closed with respect to \(L^{2,\alpha}(\mathbb{R})\).

**Proof.** We prove the case \(\alpha = 0\) that implies the claim. We consider the case \((\xi, u) = (0, 0)\). The proof for a general \((\xi, u) \in \mathbb{R} \times (-1, 1)\) works in the same way. \(L^0_{0,0}\) is self-adjoint and 0 is an isolated point of \(\sigma(L^0_{0,0})\). \(l := L^0_{0,0}|_{H^2(\mathbb{R})(\gamma(\theta_K))}\) is self-adjoint and has a bounded inverse (see [HS96 Proposition 6.6]). Notice that \(\text{ran} L^0_{0,0} = \text{ran} l\). Let \(y_n = Mx_n \xrightarrow{L^2} y\).

Boundness yields \(x_n = l^{-1}y_n \xrightarrow{L^2} l^{-1}y\), where \(l^{-1}\) denotes the bounded extension of \(l^{-1}\) on the closure \(\text{ran} l\). Since \(l^* = l\) is a closed operator (see [HS96 Proposition 4.9]), we obtain \(l(l^{-1}y) = y\). \[\square\]
Lemma A.3. (a) $\ker L_{\xi,u}^2 = \langle \theta'(\gamma(\cdot - \xi)) \rangle$, $L^{2, \alpha}(\mathbb{R}) = \text{ran} L_{\xi,u}^2 \oplus \langle \theta'(\gamma(\cdot - \xi)) \rangle$.

(b) $L^{2, \alpha}(\mathbb{R}) = \text{ran} \hat{L}_{\xi,u}^2 \oplus \langle \theta'(\gamma(\cdot - \xi)) \rangle$.

(c) $\hat{L}_{\xi,u} \in L(H_{\rho_{\alpha, \perp}}^2(\mathbb{R}), L_{\rho_{\alpha, \perp}}^2(\mathbb{R}))$.

(d) $[\hat{L}_{\xi,u}^2]^{-1} \in L(L_{\rho_{\alpha, \perp}}^2(\mathbb{R}), H_{\rho_{\alpha, \perp}}^2(\mathbb{R}))$.

(e) $\hat{M}_{\xi,u}^2 \in L(H_{\rho_{\alpha, \perp}}^2(\mathbb{R}) \oplus \mathbb{R}, L_{\rho_{\alpha, \perp}}^2(\mathbb{R}))$ and $\hat{M}_{\xi,u}$ is one-to-one and onto.

Definition A.4. We define the following operators.

(a) $L_{\xi,u} : H^2(\mathbb{R}) + H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$ given by

$$
L_{\xi,u} \left( \begin{array}{c} \theta \\ \psi \end{array} \right) (x) = \left( \begin{array}{c} -u \partial_x \theta(x) - \psi(x) \\ -\partial_x^2 \theta(x) + \cos(\theta(\gamma(\cdot - \xi))) \theta(x) - u \partial_x \psi(x) \end{array} \right).
$$

(b) $\hat{L}_{\xi,u} : \left[ H^2(\mathbb{R}) \oplus H^1(\mathbb{R}) \right] \cap (\ker L_{\xi,u})^{\perp \otimes L^2} \rightarrow H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$ given by

$$
\hat{L}_{\xi,u} \left( \begin{array}{c} \theta \\ \psi \end{array} \right) (x) = \left( \begin{array}{c} -u \partial_x \theta(x) - \psi(x) \\ -\partial_x^2 \theta(x) + \cos(\theta(\gamma(\cdot - \xi))) \theta(x) - u \partial_x \psi(x) \end{array} \right).
$$

Lemma A.5 (orthogonal sum).

$$
H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) = \hat{L}_{\xi,u} \left( \left[ H^2(\mathbb{R}) \oplus H^1(\mathbb{R}) \right] \cap \{ t(\xi, u, \cdot) \}^{\perp \otimes L^2} \right) \oplus \langle \{ t(\xi, u, \cdot) \} \rangle.
$$

Proof. "$\supseteq"$ clear. "$\subseteq"$ Let $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$. Orthogonal decomposition of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ yields that there exists $\mu(\xi, u) \in \mathbb{R}$ and $(\theta_n, \psi_n) \in H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$ such that

$$
(\bar{v}, \bar{w}) = \lim_{n \to \infty} L_{\xi,u} \left( \begin{array}{c} \theta_n \\ \psi_n \end{array} \right) + \mu(\xi, u) t(\xi, u, \cdot),
$$

since $\ker L_{\xi,u}^* = \langle \{ t(\xi, u, \cdot) \} \rangle$ due to Lemma A.3. Hence $\left( \begin{array}{c} \bar{v} \\ \bar{w} \end{array} \right) := \lim_{n \to \infty} L_{\xi,u} \left( \begin{array}{c} \theta_n \\ \psi_n \end{array} \right) \in \langle \{ t(\xi, u, \cdot) \} \rangle^{\perp \otimes L^2} \cap \left[ H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \right]$ and thus $(-w \gamma \nu(\cdot) + \gamma \omega(\cdot)) \in \langle \theta_K'(\gamma(\cdot - \xi)) \rangle^{\perp \otimes L^2} = \text{ran} \hat{L}_{\xi,u}$ due to Lemma A.3 ($\alpha = 0$). By setting $\hat{\theta}(x) := [\hat{L}_{\xi,u}]^{-1} \left( -w \gamma \partial_x v(x) + \gamma w(x) \right)$, $\hat{\psi}(x) := -u \partial_x \hat{\theta}(x) - \nu(x)$ we obtain that $(\hat{\theta}, \hat{\psi}) \in H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$ due to Lemma A.3 and that $\hat{L}_{\xi,u} \left( \begin{array}{c} \hat{\theta} \\ \hat{\psi} \end{array} \right) \in \langle \{ t(\xi, u, \cdot) \} \rangle^{\perp \otimes L^2} = \left( \begin{array}{c} \bar{v} \\ \bar{w} \end{array} \right)$. □
We define the following spaces.

**Definition A.6.** We define the following spaces.

(a) $H^{2,\alpha}_1(\mathbb{R}^2) := \{ \theta \in H^{2,\alpha}(\mathbb{R}^2) \mid \forall \lambda \in H^2(\mathbb{R}) : \langle \theta(\xi, Z), \lambda(\xi)\partial_\lambda(Z) \rangle_{L^2,\alpha, 2(\mathbb{R}^2)} = 0 \}$.

(b) $H^{2,\alpha}_{u,1}(\mathbb{R}^2) := \{ \theta \in H^{2,\alpha}(\mathbb{R}^2) \mid \forall \lambda \in H^{2,\alpha}(\mathbb{R}) : \langle \theta(\xi, x), \lambda(\xi)\partial_\lambda(\gamma(x-\xi)) \rangle_{L^2,\alpha, 2(\mathbb{R}^2)} = 0 \}$.

(c) $L^{2,\alpha}_{u,1}(\mathbb{R}^2) := \{ \theta \in L^{2,\alpha}(\mathbb{R}^2) \mid \forall \lambda \in H^{2,\alpha}(\mathbb{R}) : \langle \theta(\xi, x), \lambda(\xi)\partial_\lambda(\gamma(x-\xi)) \rangle_{L^{2,\alpha}, 2(\mathbb{R}^2)} = 0 \}$.

(d) $\tilde{Y}^\alpha$ is the space $H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)$ with the finite norm

\[ |y|_{\tilde{Y}^\alpha} = |\theta|_{H^{2,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{1,\alpha}(\mathbb{R}^2)}. \]

(e) $\tilde{Z}^\alpha$ is the space $L^{2,\alpha}(\mathbb{R}^2)$ with the finite norm

\[ |z|_{\tilde{Z}^\alpha} = |z|_{L^{2,\alpha}(\mathbb{R}^2)}. \]

(f) $Y^\alpha = Y^\alpha(u_\ast)$ is the space

\[ \left\{ y = (\theta, \lambda) \in C(I(u_\ast), \tilde{Y}^\alpha) : \|y\|_{Y^\alpha(u_\ast)} < \infty, \forall u \in I(u_\ast) : \theta(u) \in H^{2,\alpha}_{u,1}(\mathbb{R}^2) \right\} \]

with the finite norm

\[ \|y\|_{Y^\alpha(u_\ast)} = \sup_{u \in I(u_\ast)} |y|_{\tilde{Y}^\alpha}. \]

(g) $Z^\alpha = Z^\alpha(u_\ast)$ is the space

\[ \left\{ z \in C(I(u_\ast), \tilde{Z}^\alpha) : \|z\|_{Z^\alpha(u_\ast)} < \infty \right\} \]

with the finite norm

\[ \|z\|_{Z^\alpha(u_\ast)} = \sup_{u \in I(u_\ast)} |z|_{\tilde{Z}^\alpha}. \]

We define the following operators.

(h) $L^\alpha : H^{2,\alpha}(\mathbb{R}^2) \subset L^{2,\alpha}(\mathbb{R}^2) \to L^{2,\alpha}(\mathbb{R}^2)$ given by

\[ (L^\alpha \theta)(\xi, Z) = -\partial_Z^2 \theta(\xi, Z) + \cos(\theta_K(Z))\theta(\xi, Z). \]

(i) $L^\alpha_u : H^{2,\alpha}(\mathbb{R}^2) \subset L^{2,\alpha}(\mathbb{R}^2) \to L^{2,\alpha}(\mathbb{R}^2)$ given by

\[ (L^\alpha_u \theta)(\xi, x) = -(1-u^2)\partial_x^2 \theta(\xi, x) + \cos(\theta_K(\gamma(x-\xi)))\theta(\xi, x). \]

(j) $M^\alpha_u : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to L^{2,\alpha}(\mathbb{R})$ given by

\[ \left( M^\alpha_u \left( \begin{array} {c} \theta \\ \lambda \end{array} \right) \right)(\xi, x) = (L^\alpha_u \theta)(\xi, x) + \lambda(\xi)\partial_\lambda(\gamma(x-\xi)). \]
Lemma A.7.

Proof. The claim follows from Lemma A.3 combined with the fact that the operator \( \theta \) null space spanned by

\[
\begin{bmatrix}
  \alpha \\
  \gamma
\end{bmatrix}
\]

Using that Lemma A.2 the following lemma.

Lemma A.8.

Proof. Analogously to Lemma A.3 we obtain the next lemma.

Lemma A.10.

Let \( m^\alpha : \mathcal{Y}^\alpha \to \mathcal{Z}^\alpha, (\theta, \lambda) \mapsto m^\alpha(\theta, \lambda) \) be the linear operator, given by

\[
m^\alpha(\theta, \lambda)(u) = \hat{M}^\alpha(\theta(u), \lambda(u)).
\]

Then \( m^\alpha \) is one-to-one, onto and bounded, i.e., \( [m^\alpha]^{-1} \) is bounded.

Proof. \( m^\alpha \) is well defined and \( m^\alpha \) one-to-one due to Lemma A.9. In order to see that \( m^\alpha \) is onto let \( v \in \mathcal{Z}^\alpha \). Due to Lemma A.9 for each \( u \in I \) there exists \( (\theta(u), \lambda(u)) \in H_{u,\bot}^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \) such that

\[
v(u)(\xi, \gamma Z + \xi) = \hat{M}^\alpha \begin{bmatrix} \bar{\theta}(u) \\ \lambda(u) \end{bmatrix} \begin{bmatrix} \xi \\ Z \end{bmatrix} = \begin{bmatrix} \hat{\theta}(u) \\ \lambda(u) \end{bmatrix} \begin{bmatrix} \xi \\ Z \end{bmatrix} + \lambda(u)(\xi)\theta_K(Z), \] (50)
where \( \bar{\theta}(u)(\xi, Z) = \theta(u)(\xi, \frac{Z}{\gamma} + \xi) \). It holds for \( h \in H^2(\mathbb{R}^2) \) the inequality

\[
|h(\xi, x)|_{H^2_{\xi,x}(\mathbb{R}^2)} \leq \sqrt{\gamma} \left| h(\xi, \frac{Z}{\gamma(u)} + \xi) \right|_{H^2_{\xi,x}(\mathbb{R}^2)}. \tag{51}
\]

Using (51) for \( h(\xi, x) = (1 + |\xi|^2 + |x|^2)^{\frac{\gamma}{2}} [\theta(u) - \theta(\bar{u})](\xi, x) \), Lemma A.9 and Lemma A.11 we obtain:

\[
\left\| \left( \begin{array}{c}
\theta(u) - \theta(\bar{u}) \\
\lambda(u) - \lambda(\bar{u})
\end{array} \right) \right\|_{\bar{Y}^\alpha} \leq \gamma(u)^{\frac{3}{2}}C(\alpha) \left\| M_\alpha^{-1} \right\| \left( 1 + |\xi|^2 + |Z|^2 \right)^{\frac{\gamma}{2}} \left| v(\xi, \frac{Z}{\gamma(u)} + \xi) - v(\xi, \frac{Z}{\gamma(\bar{u})} + \xi) \right|_{L^2_{\xi,x}(\mathbb{R}^2)} \\
+ \gamma(u)^{\frac{3}{2}}C(\alpha) \left\| M_\alpha^{-1} \right\| \left( 1 + |\xi|^2 + |Z|^2 \right)^{\frac{\gamma}{2}} \left| v(\xi, \frac{Z}{\gamma(u)} + \xi) - v(\xi, \frac{Z}{\gamma(\bar{u})} + \xi) \right|_{L^2_{\xi,x}(\mathbb{R}^2)} \\
+ \gamma(u)^{\frac{3}{2}}C(\alpha) \left( 1 + |\xi|^2 + |Z|^2 \right)^{\frac{\gamma}{2}} \left| \theta(\bar{u})(\xi, \frac{Z}{\gamma(\bar{u})} + \xi) - \theta(u)(\xi, \frac{Z}{\gamma(u)} + \xi) \right|_{H^2_{\xi,x}(\mathbb{R}^2)}.
\]

This implies that \((\theta, \lambda) \in \bar{Y}^\alpha\), since \( v \in Z^\alpha \). The inverse mapping theorem yields that \([m^\alpha]^{-1}\) is bounded, since \( m^\alpha \) is bounded. \( \square \)

In the following we introduce the operator \( \hat{L}_u^\alpha \) that will be used for the main decomposition of this appendix in Corollary A.13.

**Definition A.11.** We define the following operators.

(a) \( L_u^\alpha : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2) \) given by

\[
\left( \begin{array}{c}
\theta \\
\psi
\end{array} \right) = \left( \begin{array}{c}
-u \partial_\xi \theta(\xi, x) - \psi(\xi, x) \\
-\partial^2_\xi \theta(\xi, x) + \cos(\theta_K(\gamma(x - \xi))) \theta(\xi, x) - u \partial_\xi \psi(\xi, x)
\end{array} \right).
\]

(b) \( \hat{L}_u^\alpha = \hat{L}_u^\alpha \left|_{H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)} \cap \ker L_u^\alpha \right. \cap L^{2,\alpha}(\mathbb{R}^2) \).

**Lemma A.12 (orthogonal sum).**

\[
H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)
\]

\[
= \hat{L}_u^\alpha \left( H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \right) \cap \ker L_u^\alpha \cap L^{2,\alpha}(\mathbb{R}^2) \oplus \left\{ \lambda I_1(u), \lambda \in H^{2,\alpha}(\mathbb{R}) \right\}.
\]

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Proof. It follows from Lemma A.7 that \( \ker [\mathcal{L}^\alpha_u]^* = \{ \lambda \mathbb{J}_1(t_1, \lambda) \in H^{2,\alpha}(\mathbb{R}) \} \). One proves first the case \( \alpha = 0 \) analogously to the proof of Lemma A.3 which can be used to deduce the case \( \alpha \neq 0 \).

Corollary A.13 (direct sum).

\[
H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2)
= \mathcal{L}^\alpha_u \left[ H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \right] \cap (\ker \mathcal{L}^\alpha_u)^\perp \mathcal{L}^{2,\alpha} \mathcal{L}^{2,\alpha} \{ \lambda t_2(u), \lambda \in H^{2,\alpha}(\mathbb{R}) \}.
\]

Proof. "\( \supset \)" clear. "\( \subset \)". Let \( (v, w) \in H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2) \) then there exists due to Lemma A.12 \( (\theta, \psi) = (\theta(u), \psi(u)) \in H^{2,\alpha}(\mathbb{R}) \oplus H^{1,\alpha}(\mathbb{R}) \cap (\ker \mathcal{L}^\alpha_u)^\perp \mathcal{L}^{2,\alpha} \) and \( \lambda = \lambda(u) \in H^{2,\alpha}(\mathbb{R}) \) such that

\[
\begin{pmatrix} v \\ w \end{pmatrix} = \mathcal{L}^\alpha_u \begin{pmatrix} \theta \\ \psi \end{pmatrix} + \lambda(u) \mathbb{J}_1(t_1).
\]

Assume without loss of generality \( |\lambda|_{H^{2,\alpha}(\mathbb{R})} \neq 0 \), then \( \langle \lambda t_1(u), \lambda \mathbb{J}_2(t_2(u)) \rangle_{L^{2,\alpha}(\mathbb{R})} \neq 0 \). Thus due to Lemma A.12 there exist \( (\tilde{\theta}, \tilde{\psi}) = (\tilde{\theta}(u), \tilde{\psi}(u)) \in H^{2,\alpha}(\mathbb{R}) \oplus H^{1,\alpha}(\mathbb{R}) \cap (\ker \mathcal{L}^\alpha_u)^\perp \mathcal{L}^{2,\alpha} \) and \( \lambda(u) \in H^{2,\alpha}(\mathbb{R}) \) such that

\[
\lambda(u) t_2(u) = \mathcal{L}^\alpha_u \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \end{pmatrix} + \tilde{\lambda}(u) \mathbb{J}_1(t_1).
\]

This is an identity in \( H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2) \). Fixing \( \xi \) and pairing this identity with \( \mathbb{J}_1(\xi, u, \cdot) \) in \( L^2_\mathbb{R} \oplus L^2_\mathbb{R} \) yields due to Lemma A.5 for a.e. \( \xi \in \mathbb{R} \) the identity \( \lambda(\xi, u) = \eta(u) \tilde{\lambda}(\xi, u) \), where \( \eta(u) := \gamma(u)^{-3} m^{-1} \left( u^2 \gamma^3 |\theta|^2_{L^2_\mathbb{R}} + \gamma |\theta|^2_{L^2_\mathbb{R}} \right) \in \mathbb{R} \). Thus using (52) we obtain

\[
\begin{pmatrix} v \\ w \end{pmatrix} = \mathcal{L}^\alpha_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} + \lambda(u) \mathbb{J}_1(t_1(u)) = \mathcal{L}^\alpha_u \begin{pmatrix} \theta(u) \\ \psi(u) \end{pmatrix} - \eta(u) \begin{pmatrix} \tilde{\theta}(u) \\ \tilde{\psi}(u) \end{pmatrix} + \eta(u) \lambda(u) t_2(u).
\]

The sum is direct due to (52).

\[\square\]

B Proof of Proposition 3.2

Let \( \alpha, n \in \mathbb{N} \). We want to show that the linear operator \( \mathfrak{M}^\alpha_n : Y^\alpha_{n*} \to Z^\alpha_{n*} \) is invertible if \( u_n \) is small. The operator \( \mathfrak{M}^\alpha_n \) contains derivatives with respect to \( \xi \) and \( x \) which makes it difficult to analyze it. Therefore we consider first an operator \( \mathfrak{M}^\alpha : Y^\alpha \to Z^\alpha \), which contains only derivatives with respect to \( x \). Using Corollary A.13 we prove that \( \mathfrak{M}^\alpha_n \) is invertible.

Definition B.1. We define the following operators.

\[
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\]
(a) \( \mathcal{M}_u^\alpha : H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \to H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}) \) given by

\[
(\mathcal{M}_u^\alpha(\theta, \psi, \lambda))(\xi, x) = \left( \mathcal{L}_u^\alpha \left( \begin{array}{c} \theta \\ \psi \end{array} \right) \right)(\xi, x) + \lambda(\xi)\theta(\xi, u, x).
\]

(b) \( \tilde{\mathcal{M}}_u^\alpha = \mathcal{M}_u^\alpha \bigg|_{H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \cap (\ker \mathcal{L}_u^\alpha) \cap L^{2,\alpha}(\mathbb{R})} \).

We define the following spaces.

(c) \( Y^\alpha \) is the space \( H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \) with the finite norm

\[
|y|_{Y^\alpha} = |\theta|_{H^{2,\alpha}(\mathbb{R}^2)} + |\psi|_{H^{1,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{2,\alpha}(\mathbb{R})}.
\]

(d) \( \tilde{Z}^\alpha \) is the space \( H^{1,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R}^2) \) with the finite norm

\[
|z|_{\tilde{Z}^\alpha} = |v|_{H^{1,\alpha}(\mathbb{R}^2)} + |w|_{L^{2,\alpha}(\mathbb{R}^2)}.
\]

(e) \( Y^\alpha = Y^\alpha(u_*) \) is the space

\[
\left\{ y = (\theta, \psi, \lambda) \in C(I(u_*), \tilde{Y}^\alpha) : |y|_{Y^\alpha(u_*)} < \infty ; \forall u \in I(u_*), \forall \mu \in H^{2,\alpha}(\mathbb{R}) : \right. \\
\left. \left\langle \begin{array}{c} \theta(u)(\xi, x) \\ \psi(u)(\xi, x) \end{array} , \mu(\xi) \right\rangle_{L^{2,\alpha}(\mathbb{R}^2) \oplus L^{2,\alpha}(\mathbb{R})} = 0 \right\}
\]

with the finite norm

\[
|y|_{Y^\alpha(u_*)} = \sup_{u \in I(u_*)} |y|_{Y^\alpha}.
\]

(f) \( Z^\alpha = Z^\alpha(u_*) \) is the space \( \left\{ z = (v, w) \in C(I(u_*), \tilde{Z}^\alpha) : |z|_{Z^\alpha(u_*)} < \infty \right\} \) with the finite norm

\[
|z|_{Z^\alpha(u_*)} = \sup_{u \in I(u_*)} |z|_{Z^\alpha}.
\]

**Lemma B.2.** The linear operator \( \tilde{\mathfrak{M}}^\alpha : Y^\alpha(u_*) \to Z^\alpha(u_*) \), \( (\theta, \psi, \lambda) \mapsto \tilde{\mathfrak{M}}^\alpha(\theta, \psi, \lambda) \), given by

\[
\tilde{\mathfrak{M}}^\alpha(\theta, \psi, \lambda)(u) = \tilde{\mathcal{M}}_u^\alpha(\theta(u), \psi(u), \lambda(u)),
\]

is invertible.
**Proof.** $\mathcal{M}_u^\alpha$ is onto. Let $(v, w) \in Z^\alpha$. Due to Corollary A.13 for all $u \in I$ there exist $(\theta(u), \psi(u)) \in [H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2)] \cap (\ker \mathcal{L}_u^\alpha)^{1, L^2,\alpha} \oplus L^2,\alpha$ and $\lambda(u) \in H^{2,\alpha}(\mathbb{R})$ such that

$$
\begin{pmatrix}
 v(u) \\
 w(u)
\end{pmatrix} = \mathcal{L}_u^\alpha \begin{pmatrix}
 \theta(u) \\
 \psi(u)
\end{pmatrix} + \lambda(u) t_2(u). \quad (53)
$$

This is an identity in $H^{1,\alpha}(\mathbb{R}^2) \oplus L^2,\alpha$. By fixing $\xi$, using Lemma A.5 and pairing (53) with $\mathcal{M}_u(\xi, u, x)$ in $L^2,\alpha(\mathbb{R}) \oplus L^2,\alpha(\mathbb{R})$ we obtain $\lambda \in C(I, L^{2,\alpha}(\mathbb{R}))$. Further (53) yields

$$
\begin{pmatrix}
 ((1 + u^2)^{-1} \gamma^{-3} \theta)_{\ker \mathcal{L}_u^\alpha} \\
 \lambda \\
 \psi
\end{pmatrix} = \begin{pmatrix}
 \hat{M}_u^\alpha \\
 (1 + u^2)^{-1} \gamma^{-3} (w - u \partial_x v - 2\lambda [u^2 \gamma^4 (x - \xi) \theta''_K(Z)]) \\
 -u \partial_y - v + \lambda [u^2 \gamma^3 (x - \xi) \theta''_K(Z)]
\end{pmatrix},
$$

where $Z = \gamma(x - \xi)$. Hence $((1 + u^2)^{-1} \gamma^{-3} \theta)_{\ker \mathcal{L}_u^\alpha}, \lambda \in Y^\alpha$ due to Lemma A.10, since $(1 + u^2)^{-1} \gamma^{-3} (w - u \partial_x v) \in Z^\alpha$ and $\lambda \in C(I, L^{2,\alpha}(\mathbb{R}))$. Thus $(\theta, \psi, \lambda) \in Y^\alpha$ and $\mathcal{M}_u^\alpha(\theta, \psi, \lambda) = (v, w)$. $\mathcal{M}_u^\alpha$ is one-to-one due to Corollary A.13. The inverse mapping theorem yields the claim.

Next, we want to show that the operator norm of $\hat{M}_u^{-1}$ is bounded by a function, which is continuous in $u$. We start with a preparing lemma.

**Lemma B.3 (Norm of $\left[\hat{M}_u^\alpha\right]^{-1}$).** There exists a constant $c^\alpha > 0$ such that

$$
\left\| \left[\hat{M}_u^\alpha\right]^{-1} \right\|_{L(L^2,\alpha(\mathbb{R}^2), H^2,\alpha(\mathbb{R}))} \leq \gamma(u) c^\alpha \left\| \left[\hat{M}_u^\alpha\right]^{-1} \right\|_{L(L^2(\mathbb{R}^2), H^2,\alpha(\mathbb{R})))}.
$$

**Proof.** Let $|v|_{L^2,\alpha(\mathbb{R}^2)} \leq 1$. Due to Lemma A.9 and Lemma A.7 there exists $(\theta, \lambda) \in H^{2,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R})$, such that

$$
v(u)(\xi, \gamma \frac{Z}{\gamma} + \xi) = \hat{M}_u^\alpha \begin{pmatrix}
 \bar{\theta}(u) \\
 \lambda(u)
\end{pmatrix}(\xi, Z) = \begin{pmatrix}
 \hat{L}_u^\alpha(\bar{\theta}(u)) \\
 \lambda(u)(\xi) + \lambda(u)(\xi) \theta''_K(Z)
\end{pmatrix},
$$

where $\bar{\theta}(u)(\xi, Z) = \theta(u)(\xi, \gamma \frac{Z}{\gamma} + \xi)$. Using (51) for $h(\xi, x) = (1 + |\xi|^2 + x^2) \frac{\gamma}{\pi} \theta(\xi, x)$ we obtain

$$
\left\| \left[\hat{M}_u^\alpha\right]^{-1} v(\xi, x) \right\|_{H^2,\alpha(\mathbb{R}^2)} \leq \sqrt{\gamma(u)} \left\| \left[\hat{M}_u^\alpha\right]^{-1} \right\|_{L(L^2(\mathbb{R}^2), H^2,\alpha(\mathbb{R})))}.
$$

**Lemma B.4 (Norm of $\left[\hat{M}_u^\alpha\right]^{-1}$).** There exists a continuous function $C^\alpha : (-1, 1) \to \mathbb{R}$ such that

$$
\left\| \left[\hat{M}_u^\alpha\right]^{-1} \right\| \leq C^\alpha(u).
$$
Proof. Let \( |(v,w)|_{H^{1,0}(\mathbb{R}^2) \oplus L^{2,0}(\mathbb{R}^2)} \leq 1 \). Due to Corollary B.13 there exists \((\theta, \psi, \lambda) \in \left[ H^{2,0}(\mathbb{R}^2) \oplus H^{1,0}(\mathbb{R}^2) \right] \cap \ker \mathcal{L}_u^{1,0,1,0,1,0} \) such that \( \begin{bmatrix} v \\ w \end{bmatrix} = \hat{\mathcal{N}}_u^\alpha(\theta, \psi, \lambda) \). Lemma B.3 and the expression for \((\theta, \psi, \lambda)\) from the proof of Lemma B.2 imply the claim. \(\square\)

**Definition B.5.** We define the following operators.

(a) \( K_u^\alpha : H^{2,0}(\mathbb{R}^2) \oplus H^{1,0}(\mathbb{R}^2) \to H^{1,0}(\mathbb{R}^2) \oplus L^{2,0}(\mathbb{R}^2) \) given by

\[
\left( K_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix} \right)(\xi, x) = \begin{pmatrix} u \partial_x \theta(\xi, x) - \psi(\xi, x) \\ -\partial_x^2 \theta(\xi, x) + \cos(\theta \xi, x) + u \partial_x \psi(\xi, x) \end{pmatrix}.
\]

(b) \( N_u^\alpha : H^{2,0}(\mathbb{R}^2) \oplus H^{1,0}(\mathbb{R}^2) \oplus H^{2,0}(\mathbb{R}) \to H^{1,0}(\mathbb{R}^2) \oplus L^{2,0}(\mathbb{R}^2) \) given by

\[
( N_u^\alpha(\theta, \psi, \lambda)) (\xi, x) = \left( K_u^\alpha \begin{pmatrix} \theta \\ \psi \end{pmatrix} \right)(\xi, x) + \lambda(\xi) t_2(\xi, u, x).
\]

(c) \( \hat{N}_u^\alpha = N_u^\alpha \bigg|_{H^{2,0}(\mathbb{R}^2) \oplus H^{1,0}(\mathbb{R}^2) \cap \ker \mathcal{L}_u^{1,0,1,0,1,0} \oplus H^{2,0}(\mathbb{R})} \).

Using von Neumann’s theorem we are able to prove now that for small \( u_* \) an extension of the operator \( \mathcal{M}_u^\alpha : Y_u^\alpha(u_*) \to Z_u^\alpha(u_*) \) is invertible. Before proceeding to the proof we introduce the following definition in order to specify \( u_* \).

**Definition B.6.** Let \( C^\alpha \) be a specific fixed function from Lemma B.4. Set

\[
\tilde{\alpha}^\alpha = \tilde{\alpha} \left( \frac{1}{\left| [\hat{M}^\alpha]^{-1} \right|} \right) = \sup \{ u \in (-1, 1) \mid \forall s, t \in \mathbb{R} : |s|, |t| \leq |u| : |s| C^\alpha(t) < 1 \}. \quad (54)
\]

**Corollary B.7.** The linear operator \( \mathcal{M} : Y^\alpha(u_*) \to Z^\alpha(u_*) \), \( (\theta, \psi, \lambda) \mapsto \mathcal{M}(\theta, \psi, \lambda) \),

is invertible if \( u_* < \tilde{\alpha}^\alpha \).

**Proof.** The operator \( \mathcal{M}^\alpha \) is invertible by Lemma B.2 and it holds for its operator norm that \( \left\| [\hat{M}^\alpha]^{-1} \right\| \leq \sup_{|u| \leq u_*} C^\alpha(u) \). Let \( \mathcal{P}^\alpha : Y^\alpha(u_*) \to Z^\alpha(u_*) \) be given by

\[
\mathcal{P}^\alpha(\theta, \psi, \lambda)(u) = u \begin{pmatrix} \partial_x \theta(u) \\ \partial_x \psi(u) \end{pmatrix} + u \begin{pmatrix} \partial_x^2 \theta(u) \\ \partial_x^2 \psi(u) \end{pmatrix}.
\]

It holds that \( \| \mathcal{P}^\alpha \| \leq \sup_{|u| \leq u_*} |u| \) and thus

\[
\| \mathcal{P}^\alpha \| \left\| [\hat{M}^\alpha]^{-1} \right\| \leq \sup_{|u| \leq u_*} |u| \sup_{|u| \leq u_*} C^\alpha(u) < 1,
\]

due to (54), since \( u_* < \tilde{\alpha}^\alpha \). Hence \( \mathcal{P}^\alpha + \mathcal{M}^\alpha = \mathcal{M}^\alpha \) is invertible by von Neumann’s theorem. \(\square\)

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Completion of the Proof of Proposition 3.2

Analogously to Corollary 3.7 one shows first that the corresponding operator on spaces of higher regularity in $(\xi, x)$ is invertible. The claim of Proposition 3.2 for the operator on spaces of higher regularity in $u$ and in $(\xi, x)$ follows by using difference quotients, orthogonal projection and the inverse mapping theorem. □

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References

[Ben76] T. Brooke Benjamin. Applications of Leray-Schauder degree theory to problems of hydrodynamic stability. *Math. Proc. Cambridge Philos. Soc.*, 79(2):373–392, 1976.

[Bon75] J. Bona. On the stability theory of solitary waves. *Proc. Roy. Soc. London Ser. A*, 344(1638):363–374, 1975.

[BP92] V. S. Buslaev and G. S. Perel’man. On nonlinear scattering of states which are close to a soliton. *Astérisque*, (210):6, 49–63, 1992. Méthodes semi-classiques, Vol. 2 (Nantes, 1991).

[Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[CMnPS16] Raphaël Côte, Claudio Muñoz, Didier Pilod, and Gideon Simpson. Asymptotic stability of high-dimensional Zakharov-Kuznetsov solitons. *Arch. Ration. Mech. Anal.*, 220(2):639–710, 2016.

[Dei85] Klaus Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[FK39] Y. I. Frenkel, T. Kontorova. *J. Phys. Acad. Sci. USSR* 1, 137, 1939.

[FGJS04] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, and I. M. Sigal. Solitary wave dynamics in an external potential. *Comm. Math. Phys.*, 250(3):613–642, 2004.

[HL12] Justin Holmer and Quanhui Lin. Phase-driven interaction of widely separated nonlinear Schrödinger solitons. *J. Hyperbolic Differ. Equ.*, 9(3):511–543, 2012.

[Hol11] Justin Holmer. Dynamics of KdV solitons in the presence of a slowly varying potential. *Int. Math. Res. Not. IMRN*, (23):5367–5397, 2011.
[HPW82] Daniel B. Henry, J. Fernando Perez, and Walter F. Wreszinski. Stability theory for solitary-wave solutions of scalar field equations. *Comm. Math. Phys.*, 85(3):351–361, 1982.

[HS96] P. D. Hislop and I. M. Sigal. *Introduction to spectral theory*, volume 113 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. With applications to Schrödinger operators.

[HZ07] Justin Holmer and Maciej Zworski. Slow soliton interaction with delta impurities. *J. Mod. Dyn.*, 1(4):689–718, 2007.

[HZ08] Justin Holmer and Maciej Zworski. Soliton interaction with slowly varying potentials. *Int. Math. Res. Not. IMRN*, (10):Art. ID rnn026, 36, 2008.

[IC79] Masahiro Inoue and S. G. Chung. Bion dissociation in sine-gordon system. *Journal of the Physical Society of Japan*, 46(5):1594–1601, 1979.

[IKV12] Valery Imaykin, Alexander Komech, and Boris Vainberg. Scattering of solitons for coupled wave-particle equations. *J. Math. Anal. Appl.*, 389(2):713–740, 2012.

[JFGS06] B. Lars G. Jonsson, Jürg Fröhlich, Stephen Gustafson, and Israel Michael Sigal. Long time motion of NLS solitary waves in a confining potential. *Ann. Henri Poincaré*, 7(4):621–660, 2006.

[KM89] Yuri S. Kivshar and Boris A. Malomed. Dynamics of solitons in nearly integrable systems. *Rev. Mod. Phys.*, 61:763–915, Oct 1989.

[KMMn17] Michal Kowalczyk, Yvan Martel, and Claudio Muñoz. Kink dynamics in the $\phi^4$ model: Asymptotic stability for odd perturbations in the energy space. *J. Amer. Math. Soc.*, 30(3):769–798, 2017.

[Kop15] Elena Kopylova. *Habilitationsschrift, Asymptotic stability of solitons for nonlinear hyperbolic equations*. Universität Wien, 2015.

[KSK97] Alexander Komech, Herbert Spohn, and Markus Kunze. Long-time asymptotics for a classical particle interacting with a scalar wave field. *Comm. Partial Differential Equations*, 22(1-2):307–335, 1997.

[Mar76] Robert H. Martin, Jr. *Nonlinear operators and differential equations in Banach spaces*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976. Pure and Applied Mathematics.

[Mas16] Timur Mashkin. *Stability of the Solitary Manifold of the Sine-Gordon Equation*. Universität zu Köln, 2016.
[Mik78] H. J. Mikeska. Solitons in a one-dimensional magnet with an easy plane. *Journal of Physics C: Solid State Physics*, 11(1):L29, 1978.

[MP12] Tetsu Mizumachi and Dmitry Pelinovsky. Bäcklund transformation and $L^2$-stability of NLS solitons. *Int. Math. Res. Not. IMRN*, (9):2034–2067, 2012.

[Sky61] T. H. R. Skyrme. Particle states of a quantized meson field. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 262(1309):237–245, 1961.

[Stu92] David M. A. Stuart. Perturbation theory for kinks. *Comm. Math. Phys.*, 149(3):433–462, 1992.

[Stu98] David M. Stuart. Solitons on pseudo-Riemannian manifolds. I. The sine-Gordon equation. *Comm. Partial Differential Equations*, 23(9-10):1815–1837, 1998.

[Stu12] David M. A. Stuart. *Sine Gordon notes*. Unpublished notes, 2012.

[SW90] A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering for nonintegrable equations. *Comm. Math. Phys.*, 133(1):119–146, 1990.

[Wei86] Michael I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39(1):51–67, 1986.

[ZHQ95] L. Zhang, L. Huang, and X. M. Qiu. Josephson junction dynamics in the presence of microresistors and an ac drive. *Journal of Physics: Condensed Matter*, 7(2):353, 1995.