MAYER-VIETORIS SYSTEMS AND THEIR APPLICATIONS

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Abstract. We introduce notations such as cdp presheaf, cds precosheaf, Mayer-Vietoris system, and give their properties. As applications, we study cohomologies with values in local systems on smooth manifolds and Dolbeault cohomologies with values in locally free sheaves on complex manifolds, where the compactness is unnecessary for both cases. In particular, we prove Poincaré duality theorem, Künneth formula, Leray-Hirsch theorem and write out explicit blow-up formulas for these cohomologies. At last, we compare the blow-up formula given by Rao, S., Yang, S. and Yang, X.-D. with ours, and then deduce that their formula is still an isomorphism in the noncompact case.

Keywords: Mayer-Vietoris system, cdp presheaf, cds precosheaf, Poincaré duality theorem, Künneth formula, Leray-Hirsch theorem, blow-up formula, local system, locally free sheaf, Dolbeault cohomology.

AMS: 53C56, 14F25, 32C35.

1. Introduction

The Mayer-Vietoris sequence is a classical result in algebraic topology. It exists in various homology and cohomology theories satisfying the Eilenberg-Steenrod axioms ([8]). Moreover, it holds in topological K-theory ([13]). In [14, 15, 16], we studied the Morse-Novikov cohomology, Dolbeault cohomology and gave the explicit formulas of blow-ups of these cohomologies. A key approach is that consider the local cases first, and then, extend them to global cases via Mayer-Vietoris sequences. Other applications of this method can be find in [9] and [11]. We develop this method systematically by the language of presheaves and precosheaves as follows.

Theorem 1.1. Let \( X \) be a connected smooth manifold and \( \mathcal{M}^*, \mathcal{N}^* \) M-V systems of cdp presheaves (resp. cds precosheaves) on \( X \). Assume \( F^*: \mathcal{M}^* \to \mathcal{N}^* \) is a M-V morphism satisfying the following hypothesis:

(*) There exists a basis \( \mathcal{U} \) of topology of \( X \), such that, \( F^*(U_1 \cap \ldots \cap U_l) \) is an isomorphism for any finite \( U_1, \ldots, U_l \in \mathcal{U} \).

Then \( F^* \) is an isomorphism.

As applications of this theorem, we generalize the main results in [5, 14, 15, 16, 17, 18]. During our preparation of this article, Rao, S., Yang, S. and Yang, X.-D. ([18]) gave a blow-up formula for bundle-valued Dolbeault cohomology on compact complex manifolds. With the similar way in [18], Chen, Y. and Yang, S. ([5]) gave a blow-up formula for cohomology with values in local systems on compact complex manifolds. By Theorem 1.1, we will give other formulas in a different way and remove the compactness. Moreover, we will prove their formulas are still isomorphic on possibly noncompact bases.
Let $\pi : \tilde{X} \to X$ be the blow-up of a connected complex manifold $X$ along a connected complex submanifold $Y$. We know $\pi|_E : E = \pi^{-1}(Y) \to Y$ is the projectivization $E = \mathbb{P}(N_Y/X)$ of the normal bundle $N_Y/X$ over $Y$. Assume $i_Y : Y \to X$ and $i_E : E \to \tilde{X}$ are inclusions and $r = \text{codim}_C Y$. Set $t = i_E^* \Theta(\mathcal{O}_E(-1)) \in \mathcal{A}^{1,1}(E)$, where $\mathcal{O}_E(-1)$ is the universal line bundle on $E = \mathbb{P}(N_Y/X)$ and $\Theta(\mathcal{O}_E(-1))$ is the Chern curvature of a hermitian metric on $\mathcal{O}_E(-1)$. Clearly, $dt = 0$ and $\partial dt = 0$.

**Theorem 1.2** (Theorem 6.1 and Section 6.3). Assume $X$, $Y$, $\pi$, $\tilde{X}$, $E$, $t$, $i_Y$, $i_E$ are defined as above. Then

$$
\pi^* + \sum_{i=0}^{r-2} (i_E)_* \circ (h^i \cup) \circ (\pi|_E)^* 
$$



gives isomorphisms

$$
H^k(X, \mathcal{V}) \oplus \bigoplus_{i=0}^{r-2} H^{k-2i}(Y, i_Y^* \mathcal{V}) \cong H^k(\tilde{X}, \pi^{-1} \mathcal{V}),
$$

$$
H_c^k(X, \mathcal{V}) \oplus \bigoplus_{i=0}^{r-2} H_c^{k-2i}(Y, i_Y^* \mathcal{V}) \cong H_c^k(\tilde{X}, \pi^{-1} \mathcal{V}),
$$

for any $k$, where $h = [t] \in H^2(E, \mathbb{R})$ or $H^2(E, \mathbb{C})$ and $\mathcal{V}$ is a local system of $\mathbb{R}$ or $\mathbb{C}$-modules of finite rank on $X$, and

$$
H^{p,q}(X, \mathcal{E}) \oplus \bigoplus_{i=0}^{r-2} H^{p-1-i, q-i}(Y, i_Y^* \mathcal{E}) \cong H^{p,q}(\tilde{X}, \pi^* \mathcal{E}),
$$

for any $p$, $q$, where $h = [t] \in H^{1,1}(E)$ and $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-modules of finite rank on $X$.

In the inverse direction,

$$
\phi^v : H^k(\tilde{X}, \pi^{-1} \mathcal{V}) \to H^k(X, \mathcal{V}) \oplus \bigoplus_{i=0}^{r-2} H^{k-2i}(Y, i_Y^* \mathcal{V}),
$$

$$
\phi_c^v : H_c^k(\tilde{X}, \pi^{-1} \mathcal{V}) \to H_c^k(X, \mathcal{V}) \oplus \bigoplus_{i=0}^{r-2} H_c^{k-2i}(Y, i_Y^* \mathcal{V}),
$$

and

$$
\phi : H^{p,q}(X, \mathcal{E}) \oplus \bigoplus_{i=1}^{r-1} H^{p-i, q-i}(Y, i_Y^* \mathcal{E}) \to H^{p,q}(\tilde{X}, \pi^* \mathcal{E})
$$

are isomorphisms, where $\phi^v$, $\phi_c^v$ and $\phi$ are defined by Section 6.2 and Section 6.3.

In Section 2, we introduce cpd presheaf, cps precosheaf, M-V system and prove Theorem 1.1. In Section 3, we define operators on forms and currents with values in local systems and locally free sheaves. In Section 4, several examples are given and generalize R. O. Wells’ main results in [23]. In Section 5, we give some applications of Theorem 1.1 for instance, Poincaré duality theorem, Küneth formula, Leray-Hirsch theorem on cohomology with values in local systems and Dolbeault cohomology with values in locally free sheaves. In Section 6, we prove Theorem 1.2.
2. MAYER-VIETORIS SYSTEMS OF PRESHEAVES AND PRECOSHEAVES

In this section, \( R \) denote a commutative ring with unit. All presheaves, precosheaves and morphisms mentioned refer to them of \( R \)-modules.

2.1. MAYER-VIETORIS systems. Let \( X \) be a topological space and \( \mathcal{M} \) a presheaf (resp. precosheaf) on \( X \). For open sets \( V \subseteq U \), denote by \( \rho_{U,V} \) (resp. \( i_{V,U} \)) the restriction \( \mathcal{M}(U) \to \mathcal{M}(V) \) (resp. the extension \( \mathcal{M}(V) \to \mathcal{M}(U) \)). We say the presheaf (resp. precosheaf) \( \mathcal{M} \) satisfies the countable direct product condition (resp. countable direct sum condition), if for any collection \( \{U_n|n \in \mathbb{N}^+\} \) of disjoint open subsets of \( X \),

\[
(\rho_{U_n,U_m})_{n \in \mathbb{Z}} : \mathcal{M}(\bigcup_{n=1}^{\infty} U_n) \to \prod_{n=1}^{\infty} \mathcal{M}(U_n)
\]

(resp.

\[
\sum_{n=1}^{\infty} i_{U_n} : \bigoplus_{n=1}^{\infty} \mathcal{M}(U_n) \to \mathcal{M}(\bigcup_{n=1}^{\infty} U_n)
\]

is an isomorphism. Briefly, we call \( \mathcal{M} \) a cdp presheaf (resp. cds precosheaf).

A system \( \mathcal{M}^* = \{\mathcal{M}^p, \delta^p|p \in \mathbb{Z}\} \) of presheaves (resp. precosheaves) of \( R \)-modules on \( X \) consists of the following data:

(i) For any \( p \), \( \mathcal{M}^p \) is a presheaf (resp. precosheaf) on \( X \).

(ii) For any open subsets \( U, V \) in \( X \) and all \( p \), \( \delta^p_{U,V} : \mathcal{M}^p(U \cap V) \to \mathcal{M}^{p+1}(U \cup V) \) (resp. \( \delta^p_{U,V} : \mathcal{M}^p(U \cup V) \to \mathcal{M}^{p+1}(U \cap V) \)) are morphisms, such that

\[
\cdots \mathcal{M}^{p-1}(U \cap V) \xrightarrow{\delta^p_{U,V}} \mathcal{M}^p(U \cup V) \xrightarrow{p^p} \mathcal{M}^p(U) \oplus \mathcal{M}^p(V) \xrightarrow{Q^p} \mathcal{M}^p(U \cap V) \xrightarrow{\delta^p_{U,V}} \mathcal{M}^{p+1}(U \cap V) \cdots
\]

(resp.

\[
\cdots \mathcal{M}^{p-1}(U \cup V) \xrightarrow{\delta^p_{U,V}} \mathcal{M}^p(U \cap V) \xrightarrow{p^p} \mathcal{M}^p(U) \oplus \mathcal{M}^p(V) \xrightarrow{Q^p} \mathcal{M}^p(U \cup V) \xrightarrow{\delta^p_{U,V}} \mathcal{M}^{p+1}(U \cap V) \cdots
\]

is a complex, where \( P^p(\alpha) = (\rho^p_{U \cap U,V}(\alpha), \rho^p_{U \cup V,V}(\alpha)), Q^p(\beta, \gamma) = \rho^p_{U \cap V,U}(\beta) - \rho^p_{U \cup V,V}(\gamma) \) (resp. \( P^p(\alpha) = (i_{U \cap U,V}(\alpha), i_{U \cup V,V}(\alpha)), Q^p(\beta, \gamma) = i_{U \cap V,U}(\beta) - i_{U \cup V,V}(\gamma) \)).

Assume \( \mathcal{M}^* \) and \( \mathcal{N}^* \) are systems of presheaves (resp. precosheaves) on \( X \) and \( F^* = \{F^p : \mathcal{M}^p \to \mathcal{N}^p|p \in \mathbb{Z}\} \) is a collection of morphisms of presheaves (resp. precosheaves) satisfying that the diagram

\[
\cdots \mathcal{M}^{p-1}(U \cap V) \xrightarrow{\delta^p_{U,V}} \mathcal{M}^p(U \cap V) \xrightarrow{p^p} \mathcal{M}^p(U) \oplus \mathcal{M}^p(V) \xrightarrow{Q^p} \mathcal{M}^p(U \cap V) \xrightarrow{\delta^p_{U,V}} \mathcal{M}^{p+1}(U \cap V) \cdots
\]

is commutative.
is commutative. Then we say $F^* : \mathcal{M}^* \to \mathcal{N}^*$ is a morphism of systems. Clearly, ker$F^* := \{\ker F^p, \delta^p | p \in \mathbb{Z}\}$ and $\text{Im} F^* := \{\text{Im} F^p, \delta^p | p \in \mathbb{Z}\}$ are systems of presheaves (resp. precosheaves).

For $n \in \mathbb{Z}$ and a system $\mathcal{M}^* = \{\mathcal{M}^p, \delta^p | p \in \mathbb{Z}\}$, we define a system $\mathcal{M}^*[n] = \{\mathcal{M}^p[n], \delta^p[n] | p \in \mathbb{Z}\}$ as $\mathcal{M}^p[n] = \mathcal{M}^{p+n}$ and $\delta^p[n] = \delta^{p+n}$.

Suppose $F^* : \mathcal{M}^* \to \mathcal{N}^*$ is a morphism of systems on $X$.

(1) For an open set $U \subseteq X$, $F^*(U)$ is called a monomorphism (resp. an epimorphism, an isomorphism), if $F^p(U)$ are monomorphisms (resp. epimorphisms, isomorphisms) for all $p$.

(2) $F^*$ is called a monomorphism (resp. an epimorphism, an isomorphism), if $F^*(U)$ is a monomorphism (resp. an epimorphism, an isomorphism) for any open set $U \subseteq X$.

**Definition 2.1.** $\mathcal{M}^*$ is called a Mayer-Vietoris system of presheaves (resp. precosheaves), if for any open subsets $U, V$ in $X$, the sequence $\{1\}$ (resp. $\{2\}$) is exact. Shortly, we say $\mathcal{M}^*$ is a M-V system.

A morphism $F^* : \mathcal{M}^* \to \mathcal{N}^*$ of M-V systems of presheaves (resp. precosheaves) is briefly called a M-V morphism sometimes.

We easily get

**Proposition 2.2.** Suppose $f : X \to Y$ is a continuous map of topological spaces.

1. If $\mathcal{M}$ is a cdp presheaf (resp. cds precosheaf) on $X$, then $f_* \mathcal{M}$ is a cdp presheaf (resp. cds precosheaf) on $Y$.

2. The direct image $f_* \mathcal{M}^* := \{f_* \mathcal{M}^p, f_* \delta^p | p \in \mathbb{Z}\}$ of a system of $\mathcal{M}^* = \{\mathcal{M}^p, \delta^p | p \in \mathbb{Z}\}$ by $f$ is a system of presheaves (resp. precosheaves) on $Y$.

3. If $F^* : \mathcal{M}^* \to \mathcal{N}^*$ is a morphism of systems on $X$, then $f_* F^* : f_* \mathcal{M}^* \to f_* \mathcal{N}^*$ is a morphism of systems on $Y$, where $f_* F^* = \{f_* \mathcal{M}^p \to f_* \mathcal{N}^p | p \in \mathbb{Z}\}$.

4. If $\mathcal{M}^*$ is a M-V system of presheaves (resp. precosheaves) on $X$, then $f_* \mathcal{M}^*$ is a M-V system of presheaves (resp. precosheaves) on $Y$.

5. If $F^* : \mathcal{M}^* \to \mathcal{N}^*$ is a morphism of M-V systems on $X$, then $f_* F^* : f_* \mathcal{M}^* \to f_* \mathcal{N}^*$ is a morphism of M-V systems on $Y$.

2.2. **Elementary Properties.** Following properties can be proved using elementary homological algebra in category of modules. Assume $X$ be a topological space.

For cdp presheaves and cds precosheaves, we have

1. (i) If $\mathcal{M}$ is a cds precosheaf on $X$ and $N$ is a $R$-module, then $\mathcal{M} \otimes_R N$ is a cds precosheaf on $X$.

   (ii) Assume $R$ is Noether ring. If $\mathcal{M}$ is a cdp presheaf on $X$ and $N$ is a finite generalized flat $R$-module. Then $\mathcal{M} \otimes_R N$ is a cdp presheaf on $X$.

   (iii) If $\mathcal{M}$ is a cdp presheaf (resp. cds precosheaf) on $X$ and $N$ is a $R$-module, then $\text{Hom}_{R^X}(N, \mathcal{M})$ (resp. $\text{Hom}_{R^X}(\mathcal{M}, N)$) is a cdp presheaf on $X$, where $N$ is viewed as a constant presheaf (resp. precosheaf).

   (iv) If $\mathcal{M}$ is a cds precosheaf on $X$ and $N$ is a finite generated $R$-module, then $\text{Hom}_{R^X}(N, \mathcal{M})$ is a cds precosheaf on $X$.

2. If $\mathcal{M}_\alpha$ are cdp presheaves (resp. cds precosheaves) for all $\alpha \in \Lambda$ on $X$, then $\prod_{\alpha \in \Lambda} \mathcal{M}_\alpha$ (resp. $\bigoplus_{\alpha \in \Lambda} \mathcal{M}_\alpha$) is a cdp presheaf (resp. cds precosheaf) on $X$. 
(3) If \( F : \mathcal{M} \to \mathcal{N} \) is a morphism of cdp presheaves (resp. cds precosheaves) on \( X \), then \( \ker F, \operatorname{Im} F, \operatorname{coker} F \) are cdp presheaves (resp. cds precosheaves).

For M-V systems, we have

(1') Suppose \( \mathcal{M}^* := \{ \mathcal{M}_p^p, \delta_p^p \mid p \in \mathbb{Z} \} \) is a M-V system of presheaves (resp. precosheaves) on \( X \).

(i) If \( N \) is a flat \( R \)-module, then \( \mathcal{M}^* \otimes_R N = \{ \mathcal{M}_p^p \otimes_R N, \delta_p^p \otimes \text{id}_N \mid p \in \mathbb{Z} \} \) is a M-V system of presheaves (resp. precosheaves).

(ii) If \( N \) is a projective \( R \)-module, \( \mathcal{H}om_R(N, \mathcal{M}^*) \) is a M-V system of presheaves (resp. precosheaves).

(iii) If \( N \) is an injective \( R \)-module, \( \mathcal{H}om_R(\mathcal{M}^*, N) \) is a M-V system of precosheaves (resp. presheaves). In particular, if \( R \) is a divisible ring, \( \mathcal{H}om_R(\mathcal{M}^*, N) \) is a M-V system for any \( R \)-module \( N \).

(2') If \( \mathcal{M}_\alpha^* = \{ \mathcal{M}_\alpha^p, \delta_\alpha^p \mid p \in \mathbb{Z} \} \) are M-V systems of presheaves (resp. precosheaves) on \( X \) for all \( \alpha \in \Lambda \), then

\[
\bigoplus_{\alpha \in \Lambda} \mathcal{M}_{\alpha}^* := \left\{ \bigoplus_{\alpha \in \Lambda} \mathcal{M}_\alpha^p, \bigoplus_{\alpha \in \Lambda} \delta_\alpha^p \mid p \in \mathbb{Z} \right\}
\]

and

\[
\prod_{\alpha \in \Lambda} \mathcal{M}_\alpha^* := \left\{ \prod_{\alpha \in \Lambda} \mathcal{M}_\alpha^p, \prod_{\alpha \in \Lambda} \delta_\alpha^p \mid p \in \mathbb{Z} \right\}
\]

are both M-V systems of presheaves (resp. precosheaves).

(3') Let \( F^* : \mathcal{L}^* \to \mathcal{M}^* \) and \( G^* : \mathcal{M}^* \to \mathcal{N}^* \) be morphism of systems on \( X \). Suppose

\[
0 \longrightarrow \mathcal{L}^* \xrightarrow{F^*} \mathcal{M}^* \xrightarrow{G^*} \mathcal{N}^* \longrightarrow 0
\]

is an exact sequence of systems of presheaves (resp. precosheaves) on \( X \), i.e. for every \( p \), \( 0 \to L^p \to M^p \to N^p \to 0 \) is an exact sequence. If two among \( \mathcal{L}^*, \mathcal{M}^*, \mathcal{N}^* \) are M-V systems, the other one is also a M-V system.

(4') If \( F_\alpha^* : \mathcal{N}^* \to \mathcal{M}_\alpha^* \) (resp. \( G_\alpha^* : \mathcal{M}_\alpha^* \to \mathcal{N}^* \)) are morphisms of M-V systems of presheaves (resp. precosheaves) on \( X \) for all \( \alpha \in \Lambda \), then

\[
(F_\alpha^*)_{\alpha \in \Lambda} : \mathcal{N}^* \to \prod_{\alpha \in \Lambda} \mathcal{M}_\alpha^*
\]

(resp.

\[
\sum_{\alpha \in \Lambda} G_\alpha^* : \bigoplus_{\alpha \in \Lambda} \mathcal{M}_\alpha^* \to \mathcal{N}^*
\]

is a morphism of M-V systems of presheaves (resp. precosheaves).

Assume \( \Lambda \) is a finite set. Then \((F_\alpha^*)_{\alpha \in \Lambda}\) and \(\sum_{\alpha \in \Lambda} G_\alpha^*\) are both morphisms of M-V systems of presheaves (resp. precosheaves).

(5') If \( F^* : \mathcal{M}^* \to \mathcal{N}^* \) and \( G^* : \mathcal{L}^* \to \mathcal{M}^* \) are both morphisms of M-V morphisms of presheaves (resp. precosheaves) on \( X \), then the composition \( F^* \circ G^* \) is a M-V morphism.

(6') If \( F_1^* : \mathcal{M}^* \to \mathcal{N}^* \) are morphisms of M-V systems of presheaves (resp. precosheaves) on \( X \) for all \( 1 \leq i \leq n \), then \( \bigoplus_{i=1}^n F_i^* \) is a M-V morphism.

(7') If \( F_\alpha^* : \mathcal{M}_\alpha^* \to \mathcal{N}_\alpha^* \) are morphisms of M-V systems of presheaves (resp. precosheaves) on \( X \) for all \( \alpha \in \Lambda \), then

\[
\bigoplus_{\alpha \in \Lambda} F_\alpha^* : \bigoplus_{\alpha \in \Lambda} \mathcal{M}_\alpha^* \to \bigoplus_{\alpha \in \Lambda} \mathcal{N}_\alpha^*
\]
satisfies conditions:

and

\[ \prod_{\alpha \in \Lambda} F_{\alpha}^* : \prod_{\alpha \in \Lambda} \mathcal{M}_{\alpha}^* \to \prod_{\alpha \in \Lambda} \mathcal{N}_{\alpha}^* \]

are M-V morphisms.

2.3. A proof of Theorem 1.1 First, recall Glued Principle, which was proved in previous articles. For readers’ convenience, we give the complete proof here.

Lemma 2.3 (14, 15). Denote \( \mathcal{P}(X) \) a statement on a smooth manifold \( X \). Assume \( \mathcal{P} \) satisfies conditions:

(i) Local condition: There exists a basis \( \mathcal{U} \) of topology of \( X \), such that, \( \mathcal{P}(U_1 \cap \ldots \cap U_1) \) holds for any finite \( U_1, \ldots, U_1 \) in \( \mathcal{U} \).

(ii) Disjoint condition: Let \( \{U_n | n \in \mathbb{N}^+\} \) be a collection of disjoint open subsets of \( X \). If \( \mathcal{P}(U_n) \) hold for all \( n \in \mathbb{N}^+ \), \( \mathcal{P}(\bigcap_{n=1}^{\infty} U_n) \) holds.

(iii) Mayer-Vietoris condition: For open subsets \( U, V \) of \( X \), if \( \mathcal{P}(U), \mathcal{P}(V) \) and \( \mathcal{P}(U \cap V) \) hold, then \( \mathcal{P}(U \cup V) \) holds.

Then \( \mathcal{P}(X) \) holds.

Proof. We first prove the follows:

(c.1) For open subsets \( U_1, \ldots, U_r \) of \( X \), if \( \mathcal{P}(U_{i_1} \cap \ldots \cap U_{i_k}) \) holds for any \( 1 \leq i_1 < \ldots < i_k \leq r \), then \( \mathcal{P}(\bigcup_{i=1}^{r} U_i) \) holds.

For \( r = 1 \), it holds obviously. Suppose (c.1) holds for \( r \). For \( r+1 \), set \( U'_1 = U_1, \ldots, U'_{r-1} = U_{r-1}, U'_r = U_r \cup U_{r+1} \). Then \( \mathcal{P}(U'_{i_1} \cap \ldots \cap U'_{i_k}) \) holds for any \( 1 \leq i_1 < \ldots < i_k \leq r-1 \). Moreover, by Mayer-Vietoris condition, \( \mathcal{P}(U'_{i_1} \cap \ldots \cap U'_{i_k} \cap U'_r) \) also holds for any \( 1 \leq i_1 < \ldots < i_k-1 \leq r-1 \), since \( \mathcal{P}(U_{i_1} \cap \ldots \cap U_{i_k+1} \cap U_r) \), \( \mathcal{P}(U_{i_1} \cap \ldots \cap U_{i_k} \cap U_{r+1}) \) and \( \mathcal{P}(U_{i_1} \cap \ldots \cap U_{i_k-1} \cap U_r \cap U_{r+1}) \) hold. By inductive hypothesis, \( \mathcal{P} \left( \bigcup_{i=1}^{r+1} U_i \right) = \mathcal{P} \left( \bigcup_{i=1}^{r} U'_i \right) \) holds. We proved (c.1).

Let \( \mathcal{U}_l \) be the collection of open sets which is the finite union of open sets in \( \mathcal{U} \). We claim that

(c.2) \( \mathcal{P}(V) \) holds for any finite intersection \( V \) of open sets in \( \mathcal{U}_l \).

Actually, \( V = \bigcap_{i=1}^{r} U_i \), where \( U_i = \bigcup_{j=1}^{\infty} U_{ij} \) and \( U_{ij} \in \mathcal{U} \). Then \( V = \bigcup_{J \in \Lambda} U_J \), where \( \Lambda = \{J = (j_1, \ldots, j_s) | 1 \leq j_1 \leq r_1, \ldots, 1 \leq j_s \leq r_s\} \) and \( U_J = U_{j_1} \cap \ldots \cap U_{j_s} \). For any \( J_1, \ldots, J_t \in \Lambda \), \( \mathcal{P}(U_{J_1} \cap \ldots \cap U_{J_t}) \) holds by Local condition. Hence \( \mathcal{P}(V) = \mathcal{P}(\bigcup_{J \in \Lambda} U_J) \) holds by (c.1).

By [9], p. 16, Prop. II, \( X = V_1 \cup \ldots \cup V_i \), where \( V_i \) is a countable disjoint union of open sets in \( \mathcal{U}_l \). Obviously, for any \( 1 \leq i_1 < \ldots < i_k \leq l \), \( V_{i_1} \cap \ldots \cap V_{i_k} \) is a countable disjoint union of the finite intersection of open sets in \( \mathcal{U}_l \). By Disjoint condition and (c.2), \( \mathcal{P}(V_{i_1} \cap \ldots \cap V_{i_k}) \) holds. So \( \mathcal{P}(X) \) holds by (c.1).

Now, we give a proof of Theorem 1.1.

Proof. We only prove the case of presheaves. The other case can be proved similarly.

Denote by \( \mathcal{P}(U) \) the statement that \( F^*(U) \) is an isomorphism. By the hypothesis (*), \( \mathcal{P} \left( \bigcap_{i=1}^{\infty} U_i \right) \) hold for any finite open sets \( U_1, \ldots, U_l \in \mathcal{U} \). Hence, \( \mathcal{P} \) satisfies Local condition in Lemma 2.3. Clearly, \( \mathcal{P} \) satisfies Disjoint condition. For open sets \( V \subseteq U \), we have a
commutative diagram of long exact sequences

\[
\cdots \to M^{p-1}(U \cap V) \xrightarrow{\delta^p_{U,V}} M^p(U \cup V) \xrightarrow{p^p} M^p(U) \oplus M^p(V) \xrightarrow{Q^p} M^p(U \cap V) \xrightarrow{\delta^p_{U,V}} M^{p+1}(U \cup V) \to \cdots
\]

If \( F^*(U) \), \( F^*(V) \) and \( F^*(U \cap V) \) are isomorphisms, then \( F^*(U \cup V) \) is an isomorphism by Five-Lemma. So \( \mathcal{P} \) satisfies Mayer-Vietoris condition. By Lemma 2.3, \( F^*(X) \) is an isomorphism.

For any open set \( V \) in \( X \), set \( \Omega_V = \{ U \in \Omega | U \subseteq V \} \). By the hypothesis (*), \( \Omega_V \) is a basis of topology of \( V \) and \( F^*(U_1 \cap \ldots \cap U_l) \) is isomorphic for any finite \( U_1, \ldots, U_l \in \Omega_V \). So \( F^*(V) \) is an isomorphism.

\[ \square \]

3. Several Operators

Before giving examples of Mayer-Vietoris systems, we define several operators on forms and currents with values in local systems and locally free sheaves, which are possibly well known for experts. We can’t find references on them, and hence, give all details here by the sheaf-theoretic approach completely. Part of them was defined in a different viewpoint in [19], [23], where they used bundle-valued forms and their topological duals (called bundle-valued currents).

3.1. Complex manifold and locally free sheaf. Let \( X \) be a connected complex manifold and \( \mathcal{E} \) a locally free sheaf of \( \mathcal{O}_X \)-modules of rank \( m \) on \( X \). For an open set \( U \subseteq X \), the elements of \( \Gamma(U, \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}^{p,q}_X) \) and \( \Gamma(U, \mathcal{E} \otimes \mathcal{O}_X \mathcal{D}^{p,q}_X) \) are called \( \mathcal{E} \)-valued \((p,q)\)-forms and currents on \( U \), respectively.

3.1.1. Local representation. Let \( U \) be an open subset of \( X \) such that \( \mathcal{E}|_U \) is a free \( \mathcal{O}_U \)-module and let \( e_1, \ldots, e_m \) be a basis of \( \Gamma(U, \mathcal{E}) \) as \( \mathcal{O}_X(U) \)-module. For \( \omega \in \Gamma(V, \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}^{p,q}_X) \), the restriction \( \omega|_U \) to \( U \) can be written as \( \sum_{i=1}^m e_i \otimes \alpha_i \), where \( \alpha_1, \ldots, \alpha_m \in \mathcal{A}^{p,q}_X(U) \). Similarly, for \( S \in \Gamma(V, \mathcal{E} \otimes \mathcal{O}_X \mathcal{D}^{p,q}_X) \), \( S|_U = \sum_{i=1}^m e_i \otimes T_i \), where \( T_1, \ldots, T_m \in \mathcal{D}^{p,q}_X(U) \). We easily get

**Lemma 3.1.** \( \text{supp} \alpha_i \subseteq \text{supp} \omega \cap U \) and \( \text{supp} \omega \cap U \subseteq \text{supp} S \cap U \).

3.1.2. Extension by zero and restriction. Assume \( j : V \to X \) is the inclusion of an open subset \( V \) of \( X \). Let \( \mathcal{U} \) be an open covering of \( X \) such that, for any \( U \in \mathcal{U} \), \( \mathcal{E}|_U \) is a free \( \mathcal{O}_U \)-module and \( e_1^U, \ldots, e_m^U \) is a basis of \( \Gamma(U, \mathcal{E}) \) as \( \mathcal{O}_X(U) \)-module.

For any \( \omega \in \Gamma(V, \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}^{p,q}_X) \) and \( U \in \mathcal{U} \), the restriction \( \omega|_{V \cap U} \) to \( U \cap V \) is \( \sum_{i=1}^m e_i^U \otimes \alpha_i |_{V \cap U} \), where \( \alpha_1, \ldots, \alpha_m \in \mathcal{A}^{p,q}(V \cap U) \). Clearly, \( \alpha_i = 0 \) on \( (V \cap U) \cap (U - \text{supp} \alpha_i) = V \cap U - \text{supp} \alpha_i \). So \( \alpha_i \) can be extended on \( (V \cap U) \cup (U - \text{supp} \alpha_i) = U \) by zero, denoted by \( \tilde{\alpha}_i \). Set

\[
\tilde{\omega}_U = \sum_{i=1}^m e_i^U \otimes \tilde{\alpha}_i
\]

in \( \Gamma(U, \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}^{p,q}_X) \). Then \( \{ \tilde{\omega}_U | U \in \mathcal{U} \} \) can be glued as a global section of \( \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}^{p,q}_X \) on \( X \), denoted by \( j_* \omega \). It is noteworthy that \( j_* \omega \) does not depend on the choice of the open covering \( \mathcal{U} \). Actually, let \( \mathcal{M} \) be the collection of all open sets in \( X \) satisfying that \( \mathcal{E}|_U \) a free
$\mathcal{O}_U$-module. Any $\mathcal{U}$ is a subcovering of $\mathcal{M}$, then $\{\tilde{w}_U \mid U \in \mathcal{U}\} \subseteq \{\tilde{w}_U \mid U \in \mathcal{M}\}$. So $j_* \omega$ defined by $\mathcal{U}$ and $\mathcal{M}$ coincides. By the definition, $\text{supp} j_* \omega = \text{supp} \omega$ is compact. Therefore, we get a map

$$j_* : \Gamma_c(V, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_{X}^{p,q}) \to \Gamma_c(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_{X}^{p,q}),$$

which is exactly the extension by zero of sections with compact supports.

Let $j^* : \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q}) \to \Gamma(V, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q})$ be the restriction of the sheaf $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q}$. For any $S \in \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q})$ and $U \in \mathcal{U}$, if the restriction $S|_U = \sum_{i=1}^m e_i \otimes T_i$, where $T_1, \ldots, T_m \in \mathcal{D}_X^{p,q}(U)$, then $(j^* S)|_{V \cap U} = \sum_{i=1}^m e_i|_{V \cap U} \otimes T_i|_{V \cap U}$.

3.1.3. **Pullback and pushout.** Let $f : Y \to X$ be a holomorphic map of connected complex manifolds and $r = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X$. Set $f^* \mathcal{E} = f^{-1} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ the inverse image of $\mathcal{E}$ by $f$. The adjunction morphism $\mathcal{E} \to f^*f^{-1} \mathcal{E}$ induces $f^*_\mathcal{U} : \Gamma(U, \mathcal{E}) \to \Gamma(f^{-1}(U), f^* \mathcal{E})$ for any open set $U \subseteq X$, where $f_U : f^{-1}(U) \to U$ is the restriction of $f$ to $f^{-1}(U)$.

The pullback $\mathcal{A}_X^{p,q} \to f_* \mathcal{A}_Y^{p,q}$ induces a morphism of sheaves

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,q} \to \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{A}_Y^{p,q},$$

hence induces a pullback of $\mathcal{E}$-valued $(p,q)$-forms

$$(f^* \omega)|_{f^{-1}(U)} = \sum_{i=1}^m f^*_U e_i \otimes f^*_U \alpha_i,$$

Assume $S$ is a $f^* \mathcal{E}$-valued $(p,q)$-current on $Y$ satisfying $f|_{\text{supp} S} : \text{supp} S \to X$ is proper. Let $\mathcal{U}$ be an open covering of $X$ such that $\mathcal{E}|_U$ is a free $\mathcal{O}_U$-module for any $U \in \mathcal{U}$ and $e_1^U, \ldots, e_m^U$ a basis of $\Gamma(U, \mathcal{E})$ as $\mathcal{O}_X(U)$-module. For $U \in \mathcal{U}$, $S$ can be written as $\sum_{i=1}^m f^*_U e_i^U \otimes T_i$ on $f^{-1}(U)$, where $T_1, \ldots, T_m \in \mathcal{D}_X^{p,q}(f^{-1}(U))$. By Lemma 3.1.1, $\text{supp} T_i \subseteq \text{supp} S \cap f^{-1}(U)$. So $f^*_U|_{\text{supp} T_i} : \text{supp} T_i \to U$ is proper and $f_*T_i$ is well defined. We get a $\mathcal{E}$-valued $(p-r, q-r)$-current

$$(f^* \omega)|_{f^{-1}(U)} = \sum_{i=1}^m f^*_U e_i \otimes f^*_U T_i,$$

on $U$. If $U, U' \in \mathcal{U}$ and $U \cap U' \neq \emptyset$, $\tilde{S}_U = \tilde{S}_{U'}$ on $U \cap U'$. Hence we get a $\mathcal{E}$-valued $(p-r, q-r)$-current on $X$, denoted by $f_* S$, such that $(f_* S)|_U = \tilde{S}_U$. Similarly with $j_*$, the definition of $f_* S$ is independent of the choice of the opening covering $\mathcal{U}$.

**Lemma 3.2.** We have

$$\text{supp} f^* \omega \subseteq f^{-1}(\text{supp} \omega),$$

$$\text{supp} f_* S \subseteq f(\text{supp} S).$$

**Proof.** For any $y \in Y - f^{-1}(\text{supp} \omega)$, $f(y) \in X - \text{supp} \omega$, so there exists an open neighborhood $V$ of $f(y)$ such that $V \subseteq X - \text{supp} \omega$ and $\mathcal{E}|_V$ is a free $\mathcal{O}_V$-module. Then $\omega|_V = 0$. By the local representation of $f^* \omega$, $(f^* \omega)|_{f^{-1}(V)} = 0$, i.e. $\text{supp} f^* \omega \cap f^{-1}(V) = \emptyset$. So $y$ is not in $\text{supp} f^* \omega$. We proved the first part.
For any \( x \in X - f(\text{supp}S) \), there exists an open neighborhood \( U \) of \( x \) such that \( U \subseteq X - f(\text{supp}S) \) and \( E|_U \) is a free \( \mathcal{O}_U \)-module. Clearly, \( f^{-1}(U) \cap \text{supp}S = \emptyset \), so \( S|_{f^{-1}(U)} = 0 \). By the definition of \( f_*S \), \((f_*S)|_U = 0 \), i.e. \( \text{supp}f_*S \cap U = \emptyset \). Therefore, \( x \) is not in \( \text{supp}f_*S \). The second conclusion holds.

By Lemma 3.2 we get a pushout
\[
f_* : \Gamma_c(Y, f^*E \otimes_{\mathcal{O}_Y} D_Y^{p,q}) \to \Gamma_c(X, E \otimes_{\mathcal{O}_X} D_X^{p-r,q-r}).
\]
In particular, for the inclusion \( j : V \to X \) of an open subset \( V \) of \( X \), \( j_* \) is the extension by zero, whose restriction to \( \Gamma_c(V, E \otimes_{\mathcal{O}_X} A_X^{p,q}) \) coincides with (9).

In addition, assume \( f \) is proper. By Lemma 3.2 the pullback \( f^\prime \) gives
\[
f^\prime_* : \Gamma_c(X, E \otimes_{\mathcal{O}_X} A_X^{p,q}) \to \Gamma_c(Y, f^*E \otimes_{\mathcal{O}_Y} D_Y^{p,q}),
\]
and by Lemma 3.1 (5) define a pushout
\[
f_* : \Gamma(Y, f^*E \otimes_{\mathcal{O}_Y} D_Y^{p,q}) \to \Gamma(X, E \otimes_{\mathcal{O}_X} D_X^{p-r,q-r}),
\]
which is actually induced by the pushout \( f_*D_Y^{p,q} \to D_X^{p-r,q-r} \).

**Proposition 3.3.** Let \( f : Y \to X \) be a proper holomorphic map of connected complex manifolds and \( E \) a locally free sheaf of \( \mathcal{O}_X \)-modules of finite rank on \( X \). For an open set \( V \) of \( X \), let \( f_V : f^{-1}(V) \to V \) be the restriction of \( f \) to \( f^{-1}(V) \) and let \( j : V \to X \) and \( j' : f^{-1}(V) \to Y \) be inclusions. Then, \( j'_*f^*_V = f^*j_* \) on \( \Gamma_c(V, E \otimes_{\mathcal{O}_X} A_X^{*,*}) \) and \( f_Vj^*j' = j^*f_* \)
on \( \Gamma(Y, f^*E \otimes_{\mathcal{O}_Y} D_Y^{*,*}) \).

**Proof.** Let \( U \) be arbitrary open set of \( X \) such that \( E|_U \) is a free \( \mathcal{O}_U \)-module and \( e_1, \ldots, e_m \) a basis of \( \Gamma(U, E) \) as \( \mathcal{O}_X(U) \)-module.

For \( \omega \in \Gamma_c(V, E \otimes_{\mathcal{O}_X} A_X^{*,*}) \), \( \omega|_{V \cap U} = \sum_{i=1}^m e_i|_{V \cap U} \otimes \alpha_i \), where \( \alpha_1, \ldots, \alpha_m \in A^{*,*}(V \cap U) \).

By definition,
\[
(j_*\omega)|_U = \sum_{i=1}^m e_i \otimes \tilde{\alpha}_i,
\]
\[
(f^*j_*\omega)|_{f^{-1}(U)} = \sum_{i=1}^m f^*_V e_i \otimes f^*_V \tilde{\alpha}_i,
\]
\[
(f^*\omega)|_{f^{-1}(V \cap U)} = \sum_{i=1}^m f^*_V e_i |_{V \cap U} \otimes f^*_V \tilde{\alpha}_i
\]
\[
= \sum_{i=1}^m (f^*_V e_i)|_{f^{-1}(V \cap U)} \otimes f^*_V \tilde{\alpha}_i,
\]
\[
(j'_*f^*_V \omega)|_{f^{-1}(U)} = \sum_{i=1}^m f^*_V e_i \otimes f^*_V \tilde{\alpha}_i,
\]
where \( \tilde{\alpha}_i \in A^{*,*}(U) \) and \( f^*_V \tilde{\alpha}_i \in A^{*,*}(f^{-1}(U)) \) denote the extension of \( \alpha_i \) and \( f^*_V \alpha_i \) by zero, respectively. Since \( f^*_V \tilde{\alpha}_i = f^*_V \alpha_i \), \( (f^*j_*\omega)|_{f^{-1}(U)} = (j'_*f^*_V \omega)|_{f^{-1}(U)} \). We proved the first part.
For $S \in \Gamma(Y,f^*E \otimes_{\mathcal{O}_Y} \mathcal{D}^{p,q})$, $S|_{f^{-1}(U)} = \sum_{i=1}^m f_i^* e_i \otimes T_i$, where $T_1, \ldots, T_m \in \mathcal{D}^{p,q}(f^{-1}(U))$. Then

$$(j_i^* S)|_{f^{-1}(V \cap U)} = \sum_{i=1}^m (f_i^* e_i)|_{f^{-1}(V \cap U)} \otimes T_i|_{f^{-1}(V \cap U)}$$

$$= \sum_{i=1}^m f_i^* (e_i|_{V \cap U}) \otimes T_i|_{f^{-1}(V \cap U)},$$

$$(f_* j_* S)|_{V \cap U} = \sum_{i=1}^m e_i|_{V \cap U} \otimes f_i|_{V \cap U} (T_i|_{f^{-1}(V \cap U)}),$$

$$(f_* S)|_U = \sum_{i=1}^m e_i \otimes f_i|_U T_i,$$

$$(j_* f_* S)|_{V \cap U} = \sum_{i=1}^m e_i|_{V \cap U} \otimes (f_i|_U T_i)|_{V \cap U}.$$  

It is easily to check that $f_i|_{V \cap U} (T_i|_{f^{-1}(V \cap U)}) = (f_i|_U T_i)|_{V \cap U}$, hence $(f_* j_* S)|_{V \cap U} = (j_* f_* S)|_{V \cap U}$. We complete the proof. 

3.1.4. Operators on cohomology. We still denote by $\overline{\partial}$ the differentials $1 \otimes \overline{\partial} \colon \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,0} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,0+1}$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q+1}$. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^p$ has two soft resolutions

$$0 \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^p \underset{i}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,0} \underset{\partial}{\longrightarrow} \cdots \underset{\partial}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,n} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^p \underset{i}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,0} \underset{\partial}{\longrightarrow} \cdots \underset{\partial}{\longrightarrow} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,n} \longrightarrow 0,$$

where $\dim_{\mathbb{C}} X = n$. So

$$H^q(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_X^p) = H^q(\Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,*})) = H^q(\Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,*})).$$

We briefly denote them by $H^{p,q}(X, \mathcal{E})$. It coincides with the bundle-valued Dolbeault cohomology $H^{p,q}(X, E)$ (see [2], p. 268, Prop. 11.5), where $E$ is the holomorphic vector bundle associated to $\mathcal{E}$. Similarly, denote by $H^{p,q}_c(X, \mathcal{E})$ the cohomology with compact support.

Clearly, all operators defined in Secion 2.1.2 and 2.1.3 commutate with $\partial$, hence induce the morphisms at the level of cohomology.

3.1.5. Wedge, cup and cartesian products. Assume $\mathcal{E}$ and $\mathcal{F}$ are locally free sheaves of $\mathcal{O}_X$-modules of rank $m$ and $n$ on $X$ respectively. Let $\mathcal{U}$ be an open covering of $X$ such that, for any $U \in \mathcal{U}$, $\mathcal{E}|_U$ and $\mathcal{F}|_U$ are free $\mathcal{O}_U$-modules, and let $e_1^U, \ldots, e_m^U$ and $f_1^U, \ldots, f_n^U$ be bases of $\Gamma(U, \mathcal{E})$ and $\Gamma(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules, respectively.

For $S \in \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p,q})$, $\omega \in \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{s,n})$ and $U \in \mathcal{U}$, $S$ and $\omega$ are represented by $\sum_{i=0}^m e_i^U \otimes T_i$ and $\sum_{i=0}^m f_i^U \otimes \alpha_i$ on $U$, where $T_i \in \mathcal{D}^{p,q}(U)$ and $\alpha_i \in \mathcal{A}^{s,n}(U)$ for any $i$. Then

$$\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} e_i^U \otimes f_j^U \otimes (T_i \wedge \alpha_j)$$
gives a $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$-valued $(p + r, q + s)$-current on $U$. By them, we construct a global section
of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X^{p+r,q+s}$ on $X$, which is independent of the choice of the open covering $\mathcal{U}$. We call it *wedged product*, denoted by $S \wedge \omega$. Similarly, we can define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$-valued current $\omega \wedge S$ and $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$-valued form $\psi \wedge \omega$ on $X$, where $\psi \in \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_X^{p,q})$. Clearly,

$$\overline{\partial}(S \wedge \omega) = \overline{\partial}S \wedge \omega + (-1)^{p+q}S \wedge \overline{\partial}\omega,$$

where $f : Y \to X$ is a holomorphic map of complex manifolds. In addition, if $T$ is a $f^* \mathcal{E}$-valued current on $Y$ satisfying that $f|_{\text{supp}(T)} : \text{supp}(T) \to X$ is proper, then

$$f_*(T \wedge f^*\omega) = f_*T \wedge \omega,$$

which is called the *projection formula*. Locally, it is the classical projection formula for currents.

Define the *cup product* on cohomology groups

$$\cup : H^{p,q}(X, \mathcal{E}) \times H^{r,s}(X, \mathcal{F}) \to H^{p+r,q+s}(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

as $[\psi] \cup [\omega] = [\psi \wedge \omega]$ or $[S] \cup [\omega] = [S \wedge \omega]$. Similarly, we can define the cup products between $H^{p,q}(X, \mathcal{E})$ or $H^{r,s}(X, \mathcal{F})$ and $H^{r,s}(X, \mathcal{F})$ or $H^{r,s}(X, \mathcal{F})$. By the projection formula (6)

$$f_*(\varphi \cup f^*\eta) = f_*\varphi \cup \eta,$$

for $\varphi \in H^{r,s}(Y, f^*\mathcal{E})$, $\eta \in H^{r,s}(X, \mathcal{F})$. Moreover, if $f$ is proper, they also hold for $\varphi \in H^{r,s}(Y, f^*\mathcal{E})$, $\eta \in H^{r,s}(X, \mathcal{F})$, or $\varphi \in H^{r,s}(Y, f^*\mathcal{E})$, $\eta \in H^{r,s}(X, \mathcal{F})$, or $\varphi \in H^{r,s}(Y, f^*\mathcal{E})$, $\eta \in H^{r,s}(X, \mathcal{F})$.

Let $\mathcal{G}$ be a locally free sheaf of $\mathcal{O}_Z$-modules of finite rank on a complex manifold $Z$ and $pr_1, pr_2$ the projections from $X \times Z$ to $X$, $Z$ respectively. Then $pr_1^*(\bullet) \cup pr_2^*(\bullet)

$$H^p(X, \mathcal{E}) \times H^r(Z, \mathcal{G}) \to H^{p+r,q+s}(X \times Z, \mathcal{E} \boxtimes \mathcal{G})$$

is called *cartesian product*, where $\mathcal{E} \boxtimes \mathcal{G} = pr_1^*\mathcal{E} \otimes_{\mathcal{O}_{X \times Z}} pr_2^*\mathcal{G}$ is the *external tensor product* of $\mathcal{E}$ and $\mathcal{G}$.

3.2. Smooth manifold and local system. Recall that, for a topology space $X$, a local system of $\mathbb{R}$-modules on $X$ is a locally constant sheaf of $\mathbb{R}$-modules on $X$, or equivalently, a locally free sheaf of $\underline{\mathbb{R}}_X$-modules on $X$, where $\underline{\mathbb{R}}_X$ is the constant sheaf with stalks $\mathbb{R}$ on $X$. For a smooth map $f : Y \to X$, local systems $\mathcal{V}, \mathcal{W}$ of $\mathbb{R}$-modules of finite ranks on $X$, all notations in Section 3.1 can be similarly defined and all results there hold. Actually, we need only to replace $\mathcal{O}_X, \mathcal{E}, \mathcal{A}_X, \mathcal{D}_X, f^*\mathcal{E}, H^*(X, \mathcal{E})$, $H^*_c(X, \mathcal{E}), \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $\mathcal{E} \boxtimes \mathcal{G} = pr_1^*\mathcal{E} \otimes_{\mathcal{O}_{X \times Z}} pr_2^*\mathcal{G}$ with $\underline{\mathbb{R}}_X, \mathcal{V}, \mathcal{A}_X, \mathcal{D}_X, f^*\mathcal{V}, H^*(X, \mathcal{V}), H^*_c(X, \mathcal{V}), \mathcal{V} \otimes_{\underline{\mathbb{R}}_X} \mathcal{W}$ and $\mathcal{V} \boxtimes \mathcal{U} = pr_1^{-1}\mathcal{V} \otimes_{\underline{\mathbb{R}}_{X \times Z}} pr_2^{-1}\mathcal{U}$. It is noteworthy that, if the notations is related to currents, the corresponding manifolds are necessarily orientable.

We list out the main results as follows.

Lemma 3.4. Let $f : Y \to X$ be a smooth map of connected smooth manifolds and $\mathcal{V}$ a local system of $\mathbb{R}$-modules of finite rank on $X$. Set $r = \dim Y - \dim X$.

(1) For a $\mathcal{V}$-valued form $\omega$ on $X$,

$$\text{supp} f^*\omega \subseteq f^{-1}(\text{supp}\omega).$$
(2) Suppose $X$ and $Y$ are oriented. For a $f^{-1}\mathcal{V}$-valued current $S$ on $Y$ satisfying that $f|_{\text{supp}S} : \text{supp}S \to X$ is proper,

\[ \text{supp}f_*S \subseteq f(\text{supp}S). \]

**Proposition 3.5.** Let $f : Y \to X$ be a proper smooth map of connected smooth manifolds and $\mathcal{V}$ a local system of $\mathbb{R}$-modules of finite rank on $X$. For an open set $V$ of $X$, let $f_V : f^{-1}(V) \to V$ be the restriction of $f$ to $f^{-1}(V)$ and let $j : V \to X$ and $j' : f^{-1}(V) \to Y$ be inclusions. Then $j'_*f^*_V = f^*j_*$ on $\Gamma_c(V, \mathcal{V} \otimes \mathbb{R}_X, A_X^\bullet)$. Moreover, if $X$ and $Y$ are oriented, $f_{V}j'^* = j^*f_*$ on $\Gamma(Y, f^{*}\mathcal{V} \otimes \mathbb{R}_Y, D_Y^\bullet)$.

Let $f : Y \to X$ be a smooth map between orientable smooth manifolds, and let $\mathcal{V}$ and $\mathcal{W}$ be local systems of $\mathbb{R}$-modules of finite rank on $X$. If $\omega$ is a $\mathcal{W}$-valued form on $X$ and $T$ is a $f^{-1}\mathcal{V}$-valued current on $Y$ satisfying that $f|_{\text{supp}T}$ is proper, we have the *projection formula*

\[ f_*(\omega \wedge T) = f_*\omega \wedge f*T. \]

So

\[ f_*(\varphi \cup f^*\eta) = f_*\varphi \cup f^*\eta \]

for $\varphi \in H_c^*(Y, f^{-1}\mathcal{V})$, $\eta \in H^*(X, \mathcal{W})$. Moreover, if $f$ is proper, they also hold for $\varphi \in H^*(Y, f^{-1}\mathcal{V})$, $\eta \in H^*(X, \mathcal{W})$, or $\varphi \in H^*_c(Y, f^{-1}\mathcal{V})$, $\eta \in H^*_c(X, \mathcal{W})$, or $\varphi \in H^*(Y, f^{-1}\mathcal{V})$, $\eta \in H^*_c(X, \mathcal{W})$.

Suppose $X$ is a connected oriented smooth manifold with dimension $n$ and $\mathcal{V}$ a local system of $\mathbb{R}$-modules of rank $m$. Denote by $\mathcal{V}^\vee$ the dual of $\mathcal{V}$. Let $\mathcal{U}$ be an open covering of $X$ such that $\mathcal{V}|_U$ and $\mathcal{V}^\vee|_U$ are constant for any $U \in \mathcal{U}$. Assume $e_i^U, \ldots, e_m^U$ and $f_i^U, \ldots, f_m^U$ are bases of $\Gamma(U, \mathcal{V})$ and $\Gamma(U, \mathcal{V}^\vee)$, respectively. For $\Omega \in \Gamma_c(X, \mathcal{V} \otimes \mathbb{R}_X, \mathcal{V}^\vee \otimes \mathbb{R}_X, A_X^\bullet)$, the restriction of $\Omega$ on $U$

\[ \Omega|_U = \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} e_i^U \otimes f_j^U \otimes \Omega_{ij}, \]

where $\Omega_{ij}$ is a smooth $n$-form on $U$ for $1 \leq i, j \leq m$. Then

\[ \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} (e_i^U, f_j^U) \Omega_{ij} \]

is a smooth $n$-form on $U$, where $\langle \cdot, \cdot \rangle$ is the contraction between $\mathcal{V}$ and $\mathcal{V}^\vee$. By them, we construct a smooth $n$-form on $X$, denoted by $\text{tr} \Omega$, which does not depend on the choice of the open covering $\mathcal{U}$. Obviously, $\text{supp}(\text{tr} \Omega) \subseteq \text{supp} \Omega$. $\text{tr}$ gives a *trace map*

\[ \Gamma_c(X, \mathcal{V} \otimes \mathbb{R}_X, \mathcal{V}^\vee \otimes \mathbb{R}_X, A_X^\bullet) \to \Gamma_c(X, A_X^\bullet). \]

### 3.3. Cohomology with compact vertical supports

Assume $\pi : E \to X$ is a smooth fiber bundle on a smooth manifold $X$. Set

\[ cv = \{ Z \subseteq E | \text{Z is closed in E and } \pi|_Z : Z \to X \text{ is proper} \}. \]

Any element in $cv$ is called a *compact vertical support*. Clearly, $Z \in cv$, if and only if, $\pi^{-1}(K) \cap Z$ is compact for any compact subset $K \subseteq X$. $cv$ has following properties:

(i) $cv$ is a paracompactifying family of supports on $E$ (\cite{3}, IV, 5.3 (b), 5.5).

(ii) $A_E^\bullet$ is $cv$-soft (\cite{3}, II, 9.4, 9.16).

(iii) $A_E^\bullet$ is $cv$-acyclic (\cite{3}, II, 9.11).
Let $W$ be a local system of $\mathbb{R}$-modules of finite rank on $E$. Then $0 \to W \to W \otimes_{\mathbb{R}} A^*_E$ is a $cv$-soft resolution of $W$ on $X$. By [3], II, 4.1, the compact vertical cohomology can be computed by

$$H^*_{cv}(E, W) \cong H^*(\Gamma_{cv}(E, W \otimes_{\mathbb{R}} A^*_E)).$$

Let $i_x : E_x \to E$ be the inclusion of the fiber $E_x$ of $E$ over $x \in X$ into $E$. For $\omega \in \Gamma_{cv}(E, W \otimes_{\mathbb{R}} A^*_E)$, supp$(i_x^*\omega) \subseteq E_x \cap$ supp$\omega$ is compact by Lemma 3.4. The restriction gives a morphism

$$\Gamma_{cv}(E, W \otimes_{\mathbb{R}} A^*_E) \to \Gamma(E_x, i_x^{-1}W \otimes_{\mathbb{R}} A^*_E),$$

which induces

$$H^*_{cv}(E, W) \to H^*_c(E_x, i_x^{-1}W).$$

If $E$ is an oriented manifold, $D^p_E$ is also a $cv$-soft sheaf for any $p$. $A^*_E \to D^p_E$ induces an isomorphism $H^*(\Gamma_{cv}(E, W \otimes_{\mathbb{R}} A^*_E)) \cong H^*(\Gamma_{cv}(E, W \otimes_{\mathbb{R}} D^p_E))$.

Moreover, assume $X$ and $E$ are both oriented manifolds. Let $i : X \to E$ be the inclusion and $r = \text{rank } E$. For $T \in \Gamma(X, i^{-1}W \otimes_{\mathbb{R}} D^p_E)$, $i_*T \in \Gamma_{cv}(E, W \otimes_{\mathbb{R}} D^{p+r})$ by Lemma 3.4. So $i_*$ induce a morphism $i_* : H^*(X, i^{-1}W) \to H_{cv}^{r+p}(E, W)$.

Let $\pi : E \to X$ be an oriented smooth vector bundle of rank $r$ on (possibly unorientable) smooth manifold $X$ and $\omega \in A^p_{cv}(E)$. Denote by $\pi_*\omega$ the integral along fibers, seeing [13], Sec. 2.4. Let $V$ be a local system of $\mathbb{R}$-modules of rank $m$ on $X$ and $\Omega \in \Gamma_{cv}(E, \pi^{-1}V \otimes_{\mathbb{R}} A^p_E)$. Let $U$ be an open subset of $X$ such that $V\big|_U$ is constant and $v_1, \ldots, v_m$ a basis of $\Gamma(U, V)$. By Lemma 3.1

$$\Omega\big|_{U_x} = \sum_{i=1}^m \pi_U^*v_i \otimes \alpha_i,$$

where $\alpha_1, ..., \alpha_m \in A^p_{cv}(E_U)$. We get a $V$-valued $(p-r)$-form

$$\sum_{i=1}^m v_i \otimes \pi_U^*\alpha_i$$

on $U$. For all such $U$, these $V$-valued forms can be glued as a global section on $X$, denoted by $\pi_*\Omega$. $\pi_*$ gives a morphism

$$A^*_E(E, \pi^{-1}V) \to A^*(X, \pi^{-1}V).$$

Clearly, supp$(\pi_*\Omega) \subseteq \pi_*(\text{supp } \Omega)$. So the restriction of $\pi_*$ to $A^*_E(E, \pi^{-1}V) \subseteq A^*_E(E, \pi^{-1}V)$ gives a morphism

$$A^*_E(E, \pi^{-1}V) \to A^*_E(X, \pi^{-1}V).$$

On the level of cohomology, $\pi_*$ induces $H^*_E(E, \pi^{-1}V) \to H^*(X, \pi^{-1}V)$ and $H^*_c(E, \pi^{-1}V) \to H^*_c(X, \pi^{-1}V)$, since $\pi_! d = d\pi_*$.

If $W$ is a local system of $\mathbb{R}$-modules of finite rank on $X$ and $\omega \in \Gamma(X, W \otimes_{\mathbb{R}} A^*_X)$, we have the projection formula

$$\pi_! (\Omega \wedge \pi^* \omega) = \pi_* \Omega \wedge \omega,$$

which can be checked locally by [2], Prop. 6.15.

Let $pr : X \times Y \to X$ be a trivial smooth fiber bundle over $X$ and $W$ a local system of $\mathbb{R}$-modules of finite rank on $X \times Y$. For open subsets $V \subseteq U$ of $Y$, set $j_{UV} : X \times V \to X \times U$ the inclusion.
Lemma 3.6. Suppose $\omega$ is in $\mathcal{A}_{\alpha}^*(X \times V, \mathcal{W})$, where $X \times V$ is viewed as a smooth fiber bundle over $X$. Then $(X \times U) \cap \text{supp}\omega$ is closed in $X \times U$.

Proof. Suppose $\{B_\alpha\}$ is an open covering of $X$ such that $\overline{B_\alpha}$ are compact for all $\alpha$. Since $\omega$ has compact vertical support, $\text{supp}\omega \cap (\overline{B_\alpha} \times V)$ is compact. Then

$$B_\alpha \times U - \text{supp}\omega = B_\alpha \times U - (B_\alpha \times V) \cap \text{supp}\omega = [\overline{B_\alpha} \times U - (\overline{B_\alpha} \times V) \cap \text{supp}\omega] \cap (B_\alpha \times U)$$

is open in $B_\alpha \times U$. So

$$X \times U - \text{supp}\omega = \bigcup_\alpha (B_\alpha \times U - \text{supp}\omega)$$

is open in $X \times U$. \hfill $\square$

By this lemma, any $\omega \in \mathcal{A}_{\alpha}^*(X \times V, \mathcal{W})$ can be extension by zero on $(X \times V) \cup (X \times U - \text{supp}\omega)$, where $j_{V,U}$ is the inclusion. Clearly, $\text{supp}(j_{V,U}*\omega) = \text{supp}\omega$, so $j_{V,U}$ gives an operator

$$\mathcal{A}_{\alpha}^*(X \times V, \mathcal{W}) \rightarrow \mathcal{A}_{\alpha}^*(X \times U, \mathcal{W}),$$

where $X \times U$ is also viewed as a smooth fiber bundle over $X$.

4. Examples

4.1. $\mathcal{H}^{p,q}_X(\mathcal{E})$ and $\mathcal{H}^{p,q}_{X,c}(\mathcal{E})$. Let $X$ be a connected complex manifold and $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_X$-modules of finite rank on $X$. Define $\mathcal{H}^{p,q}_X(\mathcal{E})$ as

$$\mathcal{H}^{p,q}_X(\mathcal{E})(U) = H^{p,q}(U, \mathcal{E})$$

for any open set $U$ in $X$, and the restriction

$$\rho_{U,V} = j_{V,U}^* : \mathcal{H}^{p,q}_X(\mathcal{E})(U) \rightarrow \mathcal{H}^{p,q}_V(\mathcal{E})(V)$$

for open sets $V \subseteq U$, where $j_{V,U} : V \rightarrow U$ is the inclusion. Define $\mathcal{H}^{p,q}_{X,c}(\mathcal{E})$ as

$$\mathcal{H}^{p,q}_{X,c}(\mathcal{E})(U) = H^{p,q}_{c}(U, \mathcal{E})$$

for any open set $U$ in $X$, and the extension

$$i_{V,U} = j_{V,U,*} : \mathcal{H}^{p,q}_{X,c}(\mathcal{E})(V) \rightarrow \mathcal{H}^{p,q}_{X,c}(\mathcal{E})(U)$$

for open sets $V \subseteq U$, where $j_{V,U} : V \rightarrow U$ is the inclusion. It is easily to check that $\mathcal{H}^{p,q}_X(\mathcal{E})$ is a cdp presheaf and $\mathcal{H}^{p,q}_{X,c}(\mathcal{E})$ is a cdps presheaf of $\mathbb{R}$-modules on $X$.

Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules of finite rank on $X$ and $\Omega$ an element in $H^{p,q}(X, \mathcal{F})$. Assume the $\mathcal{E}$-valued $(s, t)$-form $\omega$ is a representative of $\Omega$. For convenience, denote $\mathcal{F}^{p,q} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}^{p,q}_c$ and $\mathcal{G}^{p,q} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}^{p,q}_c$, for any $p, q$. Given $p$, for open subsets $U$ and $V$ of $X$, there exist two exact sequences of complexes

$$0 \rightarrow \Gamma(U \cup V, \mathcal{F}^{p,*}) \xrightarrow{(j_1, j_2)^*} \Gamma(U, \mathcal{F}^{p,*}) \oplus \Gamma(V, \mathcal{F}^{p,*}) \xrightarrow{j_2^* - j_1^*} \Gamma(U \cap V, \mathcal{F}^{p,*}) \rightarrow 0$$

and

$$0 \rightarrow \Gamma(U \cup V, \mathcal{G}^{p+*,*+t}) \xrightarrow{(j_1, j_2)^*} \Gamma(U, \mathcal{G}^{p+*,*+t}) \oplus \Gamma(V, \mathcal{G}^{p+*,*+t}) \xrightarrow{j_2^* - j_1^*} \Gamma(U \cap V, \mathcal{G}^{p+*,*+t}) \rightarrow 0$$

where $\mathcal{F}^{p,*}$ and $\mathcal{G}^{p+*,*+t}$ are the pullbacks of $\mathcal{F}$ and $\mathcal{G}$, respectively.
and

\[ 0 \longrightarrow \Gamma_c(U \cap V, F^{p,*}) \xrightarrow{(j_2, j_4)} \Gamma_c(U, F^{p,*}) \oplus \Gamma_c(V, F^{p,*}) \xrightarrow{j_{1*}} \Gamma_c(U \cup V, F^{p,*}) \longrightarrow 0 \]

\[ 0 \longrightarrow \Gamma_c(U \cap V, G^{p+q,*+1}) \xrightarrow{(j_2, j_4)} \Gamma_c(U, G^{p+q,*+1}) \oplus \Gamma_c(V, G^{p+q,*+1}) \xrightarrow{j_{1*}} \Gamma_c(U \cup V, G^{p+q,*+1}) \longrightarrow 0, \]

where \( j_i \) are self-explanatory inclusions and \( j_i^*, j_{1*} \) are restrictions, extensions by zero, respectively, for \( i = 1, 2, 3, 4 \). For the exactness, we refer to [2], Prop. 2.3, 2.7. They induce two commutative diagrams of long exact sequences on the level of cohomology

\[ \cdots \xrightarrow{\partial(U, U \cup V)} H^{p,q}(U) \oplus H^{p,q}(V) \xrightarrow{j_{1*}} H^{p,q}(U \cup V) \longrightarrow \cdots \]

\[ \cdots \xrightarrow{\partial(U, U \cup V)} H^{p,q+1}(U) \oplus H^{p,q+1}(V) \xrightarrow{j_{1*}} H^{p,q+1}(U \cup V) \longrightarrow \cdots \]

Hence, by cup products, \( \Omega \) gives M-V morphisms \( \Omega \cup : \mathcal{H}_{Xr,*}^r(\mathcal{E}) \to \mathcal{H}_{Xr,*}^{p+r,*+s}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \) and \( \Omega \cup \bullet : \mathcal{H}_{Xr,*}^r(\mathcal{E}) \to \mathcal{H}_{Xr,*}^{p+r,*+s}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) \).

Let \( f : Y \to X \) be a holomorphic map of connected complex manifolds and \( r = \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X \). Denote \( \tilde{U} = f^{-1}(U) \) for any open subset \( U \) in \( X \), and denote \( F^{p,q} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{A}_{Xr}^{p,q} \), \( G^{p,q} = f^* \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{A}_{Y}^{p,q} \), \( H^{p,q} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}^{p,q}_{Xr} \) for any \( p, q \). For open subsets \( U, V \) of \( X \), there exists two commutative diagrams of exact sequences of complexes

\[ 0 \longrightarrow \Gamma(U \cup V, F^{p,*}) \xrightarrow{(j_2, j_4)} \Gamma(U, F^{p,*}) \oplus \Gamma(V, F^{p,*}) \xrightarrow{j_{1*}} \Gamma(U \cap V, F^{p,*}) \longrightarrow 0 \]

\[ 0 \longrightarrow \Gamma(U \cup V, G^{p,*}) \xrightarrow{(j_2, j_4)} \Gamma(U, G^{p,*}) \oplus \Gamma(V, G^{p,*}) \xrightarrow{j_{1*}} \Gamma(U \cap V, G^{p,*}) \longrightarrow 0 \]

and

\[ 0 \longrightarrow \Gamma(U \cap V, G^{p,*}) \xrightarrow{(j_2, j_4)} \Gamma(U, G^{p,*}) \oplus \Gamma(V, G^{p,*}) \xrightarrow{j_{1*}} \Gamma(U \cup V, G^{p,*}) \longrightarrow 0 \]

\[ 0 \longrightarrow \Gamma(U \cap V, H^{p-r,*-r}) \xrightarrow{(j_2, j_4)} \Gamma(U, H^{p-r,*-r}) \oplus \Gamma(V, H^{p-r,*-r}) \xrightarrow{j_{1*}} \Gamma(U \cup V, H^{p-r,*-r}) \longrightarrow 0, \]

where \( j_i, j_i^*, \tilde{j_i}, j_{1*}, \tilde{j_{1*}} \) are self-explanatory as above case, for \( i = 1, 2, 3, 4 \). So we get two commutative diagrams of long exact sequences

\[ \cdots \xrightarrow{\partial(U, U \cup V)} H^{p,q}(U) \oplus H^{p,q}(V) \xrightarrow{j_{1*}} H^{p,q}(U \cup V) \longrightarrow \cdots \]

\[ \cdots \xrightarrow{\partial(U, U \cup V)} H^{p,q}(U) \oplus H^{p,q}(V) \xrightarrow{j_{1*}} H^{p,q}(U \cup V) \longrightarrow \cdots \]
Proposition 4.1. Let \( \check{\text{check the commutative diagrams on the level of forms and currents}} \)
\( R \) of \( O \) and \( 1 \) \( L \) \( \text{LINGXU MENG} \) \( \text{Hence, pullbacks and pushouts define M-V morphisms.} \)
\( \text{Proposition 4.2.} \)
\( \text{are M-V morphisms.} \)
\( \text{Moreover, assume f is proper. It is similar to prove that} \)
\( \text{and} \)
\( \text{Moreover, if} f \text{ is proper, then} \)
\( \text{are M-V morphisms.} \)
\( \text{We summarize above conclusions as follows.} \)

**Proposition 4.1.** Let \( X \) be a connected complex manifold and \( E \) a locally free sheaf of \( O_X \)-modules of finite rank on \( X \). Fixed a \( p \in \mathbb{Z} \).

1. \( H^{p,*}_X(E) \) (resp. \( H^{p,*}_{X,c}(E) \)) is a M-V system of cdp presheaves (resp. cds precosheaves) of \( \mathbb{R} \)-modules on \( X \).

2. Assume \( F \) is a locally free sheaf of \( O_X \)-modules of finite rank on \( X \) and \( \Omega \) is an element in \( H^*(X,F) \). Then

\[ \Omega \cup \bullet : H^{p,*}_X(E) \rightarrow H^{p+s,*}_X(E \otimes O_X \mathbb{F}) \]

and

\[ \Omega \cup \bullet : H^{p,*}_{X,c}(E) \rightarrow H^{p+s,*}_{X,c}(E \otimes O_X \mathbb{F}) \]

are M-V morphisms.

3. Assume \( f : Y \rightarrow X \) is a holomorphic map of connected complex manifolds and \( r = \dim_{\mathbb{C}}Y - \dim_{\mathbb{C}}X \). Then

\[ f^* : H^{p,*}_Y(E) \rightarrow f_* H^{p,*}_X(f^*E) \]

and

\[ f_* : f_* H^{p,*}_Y(f^*E) \rightarrow H^{p-r,*}_{X,c}(E)[-r] \]

are M-V morphisms.

Moreover, if \( f \) is proper, then

\[ f^* : H^{p,*}_{X,c}(E) \rightarrow f_* H^{p,*}_{X,c}(f^*E) \]

and

\[ f_* : f_* H^{p,*}_{X,c}(f^*E) \rightarrow H^{p-r,*}_{X,c}(E)[-r] \]

are M-V morphisms.

4. \( H^*_X(V) \) and \( H^*_{X,c}(V) \). Analogue to Section 3.2.1, we can define \( H^*_X(V) \) and \( H^*_{X,c}(V) \) for any local system \( V \) of \( \mathbb{R} \)-modules of finite rank on \( X \). As Proposition 4.1.1 we get the following proposition.

**Proposition 4.2.** Let \( X \) be a connected smooth manifold and \( V \) a local system of \( \mathbb{R} \)-modules of finite rank on \( X \).

1. \( H^*_X(V) \) (resp. \( H^*_{X,c}(V) \)) is a M-V system of cdp presheaves (resp. cds precosheaves).
(2) Assume \( W \) is a local system of \( \mathbb{R} \)-modules of finite rank on \( X \) and \( \Omega \) is an element in \( H^s(X,W) \). Then
\[
\Omega \cup : \mathcal{H}_X^s(V) \to \mathcal{H}_X^s(V \otimes \mathbb{R}_X W)[s]
\]
and
\[
\Omega \cup : \mathcal{H}_{X,c}^s(V) \to \mathcal{H}_{X,c}^s(V \otimes \mathbb{R}_X W)[s]
\]
are \( M-V \) morphisms.

(3) Assume \( f : Y \to X \) is a smooth map of connected smooth manifolds and \( r = \dim Y - \dim X \).

(i) \( f^* : \mathcal{H}_X^s(V) \to f_*\mathcal{H}_Y^s(f^{-1}V) \) is a \( M-V \) morphism. Moreover, if \( f \) is proper, then \( f^* : \mathcal{H}_{X,c}^s(V) \to f_*\mathcal{H}_{Y,c}^s(f^{-1}V) \) is a \( M-V \) morphism.

(ii) Suppose \( X \) and \( Y \) are oriented. \( f_* : f_*\mathcal{H}_Y^s(f^{-1}V) \to \mathcal{H}_X^s(V)[-r] \) is a \( c-ds \) \( M-V \) morphism. Moreover, if \( f \) is proper, then \( f_* : f_*\mathcal{H}_Y^s(f^{-1}V) \to \mathcal{H}_X^s(V)[-r] \) is a \( c-dp \) \( M-V \) morphism.

4.3. \( \mathcal{H}_{E,cv}^p(W) \) and \( \mathcal{H}_{X \times Y,cv}^p(W) \). Let \( \pi : E \to X \) be a smooth fiber bundle on a smooth manifold \( X \) and \( W \) a local system of \( \mathbb{R} \)-modules of finite rank on \( E \). Define \( \mathcal{H}_{E,cv}^p(W) \) as follows:
\[
\mathcal{H}_{E,cv}^p(W)(U) = H_{cv}^p(E_U,W)
\]
for any open set \( U \) in \( X \), and the restriction
\[
\rho_{U,V} = j_{U,V}^* : \mathcal{H}_{E,cv}^p(W)(U) \to \mathcal{H}_{E,cv}^p(W)(V)
\]
for any open sets \( V \subseteq U \) in \( X \), where \( j_{U,V} : E_V \to E_U \) is the inclusion.

The projection \( p_{1,Y} : X \times Y \to X \) is viewed as a smooth fiber bundle over \( X \) and suppose \( W \) is a local system of \( \mathbb{R} \)-modules of finite rank on \( X \times Y \). Define \( \mathcal{H}_{X \times Y,cv}^p(W) \) as follows:
\[
\mathcal{H}_{X \times Y,cv}^p(W)(U) = H_{cv}^p(X \times U,W)
\]
for any open set \( U \) in \( Y \), and the extension
\[
i_{U,V} = j_{U,V}^* : \mathcal{H}_{X \times Y,cv}^p(W)(V) \to \mathcal{H}_{X \times Y,cv}^p(W)(U)
\]
for any open sets \( V \subseteq U \) in \( Y \), where \( j_{U,V} : X \times V \to X \times U \) is the inclusion.

For any open set \( U, V \) in \( X \), there exists an short exact sequence of complexes
\[
0 \to \mathcal{A}_{cv}^p(E_{U \cup V}) \to \mathcal{A}_{cv}^p(E_U) \oplus \mathcal{A}_{cv}^p(E_V) \to \mathcal{A}_{cv}^p(E_{U \cap V}) \to 0,
\]
and for any open set \( U, V \) in \( Y \), there exists an short exact sequence of complexes
\[
0 \to \mathcal{A}_{cv}^p(X \times (U \cap V)) \to \mathcal{A}_{cv}^p(X \times U) \oplus \mathcal{A}_{cv}^p(X \times V) \to \mathcal{A}_{cv}^p(X \times (U \cup V)) \to 0.
\]
By the two exact sequences, we easily get the following propositions.

**Proposition 4.3.** Let \( \pi : E \to X \) be a smooth fiber bundle on a connected smooth manifold \( X \) and \( V, W \) local systems of \( \mathbb{R} \)-modules of finite ranks on \( X, E \) respectively.

1. \( \mathcal{H}_{E,cv}^p(W) \) is a \( M-V \) system of cdp presheaves on \( X \).

2. Assume \( \mathcal{U} \) is a local system of \( \mathbb{R} \)-modules of finite rank on \( E \) and \( \Omega \) is an element in \( H^s(E,W) \) (resp. \( H^s_{cv}(E,W) \)). Then
\[
\Omega \cup : \mathcal{H}_{E,cv}^p(\mathcal{U}) \to \mathcal{H}_{E,cv}^p(\mathcal{U} \otimes \mathbb{R}_E W)[s]
\]
and
\[ \Omega \cup \bullet : \mathcal{H}_{E}^{\ast}(\mathcal{U}) \to \mathcal{H}_{E,\text{cv}}^{\ast}(\mathcal{U} \otimes_{\mathbb{R}} \mathcal{W})[s] \]
are M-V morphisms.

(3) Assume \( \Omega \) is an element in \( \mathcal{H}_{E}^{\ast}(E,\mathcal{W}) \). Then
\[ \Omega \cup \pi^{\ast}(\bullet) : \mathcal{H}_{X,\text{c}}^{\ast}(\mathcal{V}) \to \mathcal{H}_{E,\text{c}}^{\ast}(\pi^{-1}\mathcal{V} \otimes_{\mathbb{R}} \mathcal{W})[s] \]
are M-V morphisms.

(4) If \( E \) is an orientable fiber bundle, then \( \pi_{\ast} : \mathcal{H}_{E,\text{cv}}^{\ast}(\pi^{-1}\mathcal{V}) \to \mathcal{H}_{X}^{\ast}(\mathcal{V})[-r] \) is a M-V morphism. In addition, if \( X \) is also orientable, then \( i_{\ast} : \mathcal{H}_{X}^{\ast}(i^{-1}\mathcal{W}) \to \mathcal{H}_{E,\text{cv},c}(\mathcal{W})[r] \) is a M-V morphism.

Proposition 4.4. Suppose \( X \) and \( Y \) are connected smooth manifolds. Let \( \mathcal{V} \) and \( \mathcal{W} \) be local systems of \( \mathbb{R} \)-modules of finite ranks on \( Y \) and \( X \times Y \), respectively. If we view \( pr_{1} : X \times Y \to X \) as a smooth fiber bundle, then

1. \( \mathcal{H}_{X \times Y,\text{cv}}^{\ast}(\mathcal{W}) \) is a M-V system of cds presheaves on \( Y \);
2. \( pr_{2}^{\ast} : \mathcal{H}_{Y,\text{c}}^{\ast}(\mathcal{V}) \to \mathcal{H}_{X \times Y,\text{cv}}^{\ast}(pr_{1}^{-1}\mathcal{V}) \) is a M-V morphism, where \( pr_{2} : X \times Y \to Y \) is the second projection.

4.4. Generalizations of R. O. Wells’ results. In [23], R. O. Wells compared de Rham and Dolbeault cohomology for proper surjective maps. We extend his results to more general cases.

[23], Thm. 3.3 was generalized as follows.

Proposition 4.5. Let \( f : Y \to X \) be a proper surjective smooth map of connected oriented smooth manifolds with the same dimensions and \( \deg f \neq 0 \). If \( \mathcal{V} \) is a local system of \( \mathbb{R} \) or \( \mathbb{C} \)-modules of finite rank on \( X \), then, for any \( p \),

1. \( f^{\ast} \) gives M-V monomorphisms \( \mathcal{H}_{X}^{\ast}(\mathcal{V}) \to f_{\ast}\mathcal{H}_{Y}^{\ast}(f^{-1}\mathcal{V}) \) and \( \mathcal{H}_{X,\text{c}}^{\ast}(\mathcal{V}) \to f_{\ast}\mathcal{H}_{Y,\text{c}}^{\ast}(f^{-1}\mathcal{V}) \);
2. \( f_{\ast} \) gives M-V epimorphisms \( f_{\ast}\mathcal{H}_{Y}^{\ast}(f^{-1}\mathcal{V}) \to \mathcal{H}_{X}^{\ast}(\mathcal{V}) \) and \( f_{\ast}\mathcal{H}_{Y,\text{c}}^{\ast}(f^{-1}\mathcal{V}) \to \mathcal{H}_{X,\text{c}}^{\ast}(\mathcal{V}) \).

Proof. By Proposition 4.4, \( f^{\ast} \) and \( f_{\ast} \) are M-V morphisms. For any open set \( U \) in \( X \), set \( \tilde{U} = f^{-1}(U) \). Notice that \( f_{\ast}1_{U} = \deg f \cdot 1_{\tilde{U}} \neq 0 \) in \( H^{0}(U,\mathbb{R}) \), where \( 1_{U} \) and \( 1_{\tilde{U}} \) are classes of the constant 1 in \( H^{0}(U,\mathbb{R}) \) and \( H^{0}(\tilde{U},\mathbb{R}) \), respectively. By the projection formula [1], \( f_{\ast}f^{\ast}\eta = \deg f \cdot \eta \), where \( \eta \in H^{p}(U,\mathcal{V}) \) or \( H_{p}^{c}(U,\mathcal{V}) \). We get the proposition immediately.

Recall that a complex manifold \( X \) is called \( p \)-Kählerian, if it admits a closed strictly positive \((p,p)\)-form \( \Omega \) ([1], Def. 1.1, 1.2). In such case, \( \Omega|_{Z} \) is a volume form on \( Z \), for any complex submanifold \( Z \) of pure dimension \( p \) of \( X \). Any complex manifold is 0-Kählerian and any Kähler manifold \( X \) is \( p \)-Kählerian for every \( p \leq \dim_{\mathbb{C}}X \). We generalize [23], Thm. 3.1 and 4.1 as follows.

Proposition 4.6. Suppose \( f : Y \to X \) is a proper surjective holomorphic map between connected complex manifolds and \( Y \) is \( r \)-Kählerian, where \( r = \dim_{\mathbb{C}}Y - \dim_{\mathbb{C}}X \). Let \( \mathcal{V} \) be a local system of \( \mathbb{R} \) or \( \mathbb{C} \)-modules of finite rank and \( \mathcal{E} \) a locally free sheaf of \( \mathcal{O}_{X} \)-modules of finite rank on \( X \), respectively. Assume Then, for any \( p, q \),

1. For any \( p, f^{\ast} \) gives M-V monomorphisms
   \( \mathcal{H}_{X}^{\ast}(\mathcal{V}) \to f_{\ast}\mathcal{H}_{Y}^{\ast}(f^{-1}\mathcal{V}) \),
   \( \mathcal{H}_{X,\text{c}}^{\ast}(\mathcal{V}) \to f_{\ast}\mathcal{H}_{Y,\text{c}}^{\ast}(f^{-1}\mathcal{V}) \),
Proof. By Proposition 4.4 and 4.5, \( f^* \) and \( f_* \) are M-V morphisms. Let \( \Omega \) be a strictly positive closed \((r,r)\)-form on \( Y \). Then \( c = f_* \Omega \) is a closed current of degree \( 0 \), hence a constant. By Sard’s theorem, the set \( X_0 \) of regular values of \( f \) is nonempty. For any \( x \in X_0 \), \( Y_x = f^{-1}(x) \) is a compact complex submanifold of pure dimension \( r \), so \( c = \int_{Y_x} \Omega \mid_{Y_x} > 0 \).

For any open set \( U \) in \( X \), set \( \bar{U} = f^{-1}(U) \). Then \( f_*(\Omega \mid \bar{U}) = c \). By the projection formula \( \mathfrak{1} \) (resp. \( \mathfrak{I} \)), \( f_*(\Omega \mid \bar{U}) \cup f^* \eta \) is a current where \( \Omega \mid \bar{U} \in H^{2r}(\bar{U}, \mathbb{R}) \) or \( H^{2r}(\bar{U}, \mathbb{C}) \) (resp. \( H^{r+1}(\bar{U}) \)) and \( \eta \in H^p(U, \mathcal{V}) \) (resp. \( H^{p,q}(U, \mathcal{V}) \) or \( H^p(U, \mathcal{V}) \) (resp. \( H^{p,q}(U, \mathcal{V}) \)). It is easily to deduce the proposition. \( \square \)

4.5. A result of Vietoris-Belge type.

Proposition 4.7. Let \( f : Y \to X \) be a closed surjective smooth map between smooth manifolds and \( \mathcal{V} \) a local system of \( \mathbb{R} \)-modules of finite rank on \( X \).

1. Suppose that every fibre \( f^{-1}(x) \) is connected and \( H^i(f^{-1}(x), \mathbb{R}) = 0 \) for \( 1 \leq i \leq r \). Then \( f^* : H^p(X, \mathcal{V}) \to H^p(Y, f^{-1}\mathcal{V}) \) is an isomorphism for \( 0 \leq p \leq r \), and a monomorphism for \( p = r + 1 \).

In addition, if \( f \) is proper, \( f_* \) also holds for the cases of compact supports.

2. Suppose that every fibre \( f^{-1}(x) \) is connected and \( H^i(f^{-1}(x), \mathbb{R}) = 0 \) for \( i > 0 \). Then \( f^* : H^r(X, \mathcal{V}) \to f_! H^r(f^{-1}\mathcal{V}) \) is an isomorphism.

In addition, if \( f \) is proper, \( f^* : H^r(X, \mathcal{V}) \to f_! H^r(f^{-1}\mathcal{V}) \) is an isomorphism.

Proof. (1) The first part holds by Vietoris-Belge Mapping theorem \((20)\).

For a proper map \( f \),

\[
(R^p f_* \mathcal{F}_Y)_x = H^q(f^{-1}(x), \mathcal{F}_X)\]

So \( R^p f_* \mathcal{F}_Y = 0 \) for \( 1 \leq p \leq r \), and the pullback \( \mathcal{F}_X \to f_* \mathcal{F}_Y \) is an isomorphism, since every fibre \( f^{-1}(x) \) is connected. By the projection formula,

\[
R^q f_* f^{-1}\mathcal{V} = \mathcal{V} \otimes \mathcal{F}_X, R^q f_* \mathcal{F}_Y = \begin{cases} \mathcal{V}, & q = 0 \\ 0, & 1 \leq q \leq r. \end{cases}
\]

Consider the Leray spectral sequence

\[
E_2^{p,q} = H^p(X, R^q f_* f^{-1}\mathcal{V}) \Rightarrow H^{p+q} = H^{p+q}(X, \mathcal{V}).
\]

Then \( E_2^{p,q} = 0 \) for \( 1 \leq q \leq r \). So \( E_2^{0,0} \cong H^p \) for \( 0 \leq p \leq r \) and \( E_2^{r+1,0} \to H^{r+1} \) is injective, where the morphisms are all induced by the pullback of sheaves. We prove the second part.

(2) For any open set \( U \) in \( X \) and any \( p, f^* : H^p(U, \mathcal{V}) \to H^p(f^{-1}(U), f^{-1}\mathcal{V}) \) and \( f^* : H^p_c(U, \mathcal{V}) \to H^p_c(f^{-1}(U), f^{-1}\mathcal{V}) \) are isomorphisms by (1). We proved (2). \( \square \)
5. Some applications

5.1. Poincaré duality theorem. Denote by $M^\vee = \text{Hom}_\mathbb{R}(M, \mathbb{R})$ the dual space of a $\mathbb{R}$-vector space $M$ and denote by $\rho^\vee : N^\vee \to M^\vee$ the dual of a linear map $\rho : M \to N$.

Let $X$ be a connected oriented smooth manifold with dimension $n$ and $\mathcal{V}$ a local system of $\mathbb{R}$- or $\mathbb{C}$-modules of finite rank on $X$. For any $p$, denote $\mathcal{F}^p = \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{A}_X^p$ and $\mathcal{G}^p = \mathcal{V}^\vee \otimes_{\mathcal{O}_X} \mathcal{A}_X^{n-p}$. For any open set $U$, define

$$PD_U(\alpha)(\beta) = \int_U \text{tr}(\alpha \wedge \beta)$$

for $\alpha \in \Gamma(U, \mathcal{F}^p)$ and $\beta \in \Gamma_c(U, \mathcal{G}^p)$, which gives a linear map $PD_U : \Gamma(U, \mathcal{F}^p) \to \Gamma_c(U, \mathcal{G}^p)^\vee$.

For open subsets $U$, $V$ of $X$ and $\varphi \in \Gamma_c(U, \mathcal{G}^p)^\vee$, $\psi \in \Gamma_c(V, \mathcal{G}^p)^\vee$,

$$I^p(\varphi, \psi) = \varphi \circ \text{pr}_1 - \psi \circ \text{pr}_2$$

gives an isomorphism

$$I^p : \Gamma_c(U, \mathcal{G}^p)^\vee \oplus \Gamma_c(V, \mathcal{G}^p)^\vee \to (\Gamma_c(U, \mathcal{G}^p) \oplus \Gamma_c(V, \mathcal{G}^p))^\vee,$$

where $\text{pr}_1$ and $\text{pr}_2$ are projections from $\Gamma_c(U, \mathcal{G}^p) \oplus \Gamma_c(V, \mathcal{G}^p)$ onto $\Gamma_c(U, \mathcal{G}^p)$ and $\Gamma_c(V, \mathcal{G}^p)$ respectively. There is a commutative diagram

$$\begin{array}{ccc}
0 & \to & \Gamma(U \cup V, \mathcal{F}^\bullet) \\
& \downarrow{PD_{U\cup V}} & \downarrow{\{PD_U, PD_V\}} \\
0 & \to & \Gamma_c(U \cup V, \mathcal{G}^\bullet)^\vee \\
& \downarrow{id} & \downarrow{id} \\
0 & \to & \Gamma_c(U \cup V, \mathcal{G}^\bullet)^\vee \\
\end{array}$$

where $j_i$, $j_i^*$ and $j_i^!$ are self-explanatory inclusions, for $i = 1, 2, 3, 4$. Clearly the first and second rows are exact sequences of complexes, so does the third row, since $I^\bullet$ is isomorphic.

Therefore, we get a commutative diagram of long exact sequences

$$\begin{array}{ccc}
\cdots & H^p(U \cup V, \mathcal{V}) & \to \cdots \\
& \downarrow{PD_{U\cup V}} & \downarrow{\{PD_U, PD_V\}} \\
\cdots & H^p(U, \mathcal{V}) \oplus H^p(V, \mathcal{V}) & \to \\
& \downarrow{PD_{U,V}} & \downarrow{PD_{U\cap V}} \\
\cdots & H^p(U \cap V, \mathcal{V}) & \to \\
& \downarrow{PD_{U\cap V}} & \downarrow{PD_{U\cap V}} \\
\cdots & H^p(U \cup V, \mathcal{V})^\vee & \to \\
& \downarrow{PD_{U\cup V}} & \downarrow{PD_{U\cup V}} \\
\cdots & H^p(U, \mathcal{V})^\vee \oplus H^p(V, \mathcal{V})^\vee & \to \\
& \downarrow{PD_{U\cap V}} & \downarrow{PD_{U\cap V}} \\
\cdots & H^p(U \cap V, \mathcal{V})^\vee & \to \\
& \downarrow{PD_{U\cap V}} & \downarrow{PD_{U\cap V}} \\
\cdots & H^p(U \cup V, \mathcal{V})^\vee & \to \\
\end{array}$$

which implies that $PD_X : \mathcal{H}_X^\bullet(\mathcal{V}) \to (\mathcal{H}_{X,c}^{n-n}(\mathcal{V}^\vee))^\vee = \text{Hom}_{\mathcal{O}_X}(\mathcal{H}_{X,c}^{n-n}(\mathcal{V}^\vee), \mathbb{R})$ is a morphsim of $\mathcal{M}$-$\mathcal{V}$ systems of cpl presheaves by Section 2.2 (1) (iii).

Let $\mathfrak{U}$ be a basis for topology of $Y$ such that $\mathcal{V}|_U$ is constant for every $U \in \mathfrak{U}$. For any $U_1, \ldots, U_t \in \mathfrak{U}$, $\mathcal{V}|_{\bigcap_{i=1}^t U_i}$ is constant. By classical Poincaré duality theorem, $PD_X(\bigcap_{i=1}^t U_i) = PD_{\bigcap_{i=1}^t U_i}$ is isomorphic. By Theorem 5.1, $PD_X = PD_X(X)$ is an isomorphism. We get Poincaré duality theorem for local systems.

**Theorem 5.1.** Let $X$ be a connected oriented smooth manifold with dimension $n$ and $\mathcal{V}$ a local system of $\mathbb{R}$ or $\mathbb{C}$-modules of finite rank on $X$. Then, for every $p$,

$$PD_X : H^p(X, \mathcal{V}) \to (H^{n-p}_c(X, \mathcal{V}^\vee))^\vee$$
is an isomorphism, where PD_{X}(\alpha \wedge \beta) = \int_X tr(\alpha \wedge \beta), for \[ \alpha \in H^{p}(X,\mathcal{V}) \text{ and } [\beta] \in H^{n-p}_{c}(X,\mathcal{V}^{\vee}). \]

**Remark 5.2.** Poincaré duality theorem is a special case of Verdier Duality Theorem, refer to [7], Cor. 3.3.12.

If \( f : Y \to X \) is a smooth map of connected oriented smooth manifolds, the diagram

\[
\begin{array}{ccc}
H^{p}(X,\mathcal{V}) & \xrightarrow{=} & (H^{n-p}_{c}(X,\mathcal{V}^{\vee}))^{\vee} \\
\downarrow (-1)^{p} f^{*} & & \downarrow (f^{*})^{\vee} \\
H^{p}(Y, f^{-1}\mathcal{V}) & \xrightarrow{=} & (H^{n-p}_{c}(Y, f^{-1}\mathcal{V}^{\vee}))^{\vee}
\end{array}
\]

is commutative, where \( m = \dim Y \) and \( r = m - n \).

Moreover, if \( f \) is proper, we have a commutative diagram

\[
\begin{array}{ccc}
H^{p+r}(Y, f^{-1}\mathcal{V}) & \xrightarrow{=} & (H^{n-p}(Y, f^{-1}\mathcal{V}^{\vee}))^{\vee} \\
\downarrow f^{*} & & \downarrow (f^{*})^{\vee} \\
H^{p}(X,\mathcal{V}) & \xrightarrow{=} & (H^{n-p}_{c}(X,\mathcal{V}^{\vee}))^{\vee}
\end{array}
\]

5.2. **Künneth formulas.**

**Lemma 5.3.** Let \( \mathcal{V} \) be a local system of \( \mathbb{R} \)-modules of finite rank on a smooth manifold \( Y \). Assume \( pr_{1} \) and \( pr_{2} \) are projections from \( \mathbb{R}^{n} \times Y \) onto \( \mathbb{R}^{n} \) and \( Y \) respectively. If \( pr_{1} : \mathbb{R}^{n} \times Y \to \mathbb{R}^{n} \) is viewed as a smooth fiber bundle over \( \mathbb{R}^{n} \), then

\[
pr_{2}^{*} : H^{*}_{c}(Y,\mathcal{V}) \to H^{*}_{c}(\mathbb{R}^{n} \times Y, pr_{2}^{-1}\mathcal{V})
\]

is an isomorphism.

**Proof.** By Proposition 4.4 (2), \( pr_{2}^{*} : H^{*}_{c}(\mathcal{V}) \to H^{*}_{c}(\mathbb{R}^{n} \times Y, pr_{2}^{-1}\mathcal{V}) \) is a M-V morphism. Let \( U \) be a basis for topology of \( Y \) such that \( \mathcal{V}|_{U} \) is constant for every \( U \in U \). For any \( U_{1},...,U_{i} \in U \), \( \mathcal{V}|_{\bigcap_{i=1}^{U} U_{i}} \) is constant. By [14], Thm. 3.7, \( pr_{2}^{*} \) is isomorphic on \( \bigcap_{i=1}^{U} U_{i} \). By Theorem 4.1, \( pr_{2}^{*} : H^{*}_{c}(Y,\mathcal{V}) \to H^{*}_{c}(\mathbb{R}^{n} \times Y, pr_{2}^{-1}\mathcal{V}) \) is an isomorphism. \( \square \)

**Proposition 5.4.** Let \( \mathcal{V} \) and \( \mathcal{W} \) be local systems of \( \mathbb{R} \)-modules of finite ranks on smooth manifolds \( X \) and \( Y \) respectively. Assume \( pr_{1} \) and \( pr_{2} \) are projections from \( X \times Y \) onto \( X \) and \( Y \) respectively.

1. The cartesian product gives an isomorphism

\[
H^{*}_{c}(X,\mathcal{V}) \otimes_{\mathbb{R}} H^{*}_{c}(Y,\mathcal{W}) \to H^{*}_{c}(X \times Y,\mathcal{V} \boxtimes \mathcal{W}).
\]

2. If \( H^{*}_{c}(Y,\mathcal{W}) = \bigoplus_{p \geq 0} H^{p}_{c}(Y,\mathcal{W}) \) has finite dimension, the cartesian product gives an isomorphism

\[
H^{*}(X,\mathcal{V}) \otimes_{\mathbb{R}} H^{*}_{c}(Y,\mathcal{W}) \to H^{*}_{c}(X \times Y,\mathcal{V} \boxtimes \mathcal{W}),
\]

where \( pr_{1} : X \times Y \to X \) is viewed as a smooth fiber bundle over \( X \).

3. If \( H^{*}(X,\mathcal{V}) \) or \( H^{*}(Y,\mathcal{W}) \) has finite dimension, the cartesian product gives an isomorphism

\[
H^{*}(X,\mathcal{V}) \otimes_{\mathbb{R}} H^{*}(Y,\mathcal{W}) \to H^{*}(X \times Y,\mathcal{V} \boxtimes \mathcal{W}).
\]
Proof. (1) is a special case of [3], II, 15.2.

(2) The proof is similar with [14], Thm. 3.2. For readers’ convenience, we give a complete proof. Assume \( \dim X = n \) and \( U = \{ U_\alpha \} \) is a good covering of \( X \) satisfying that every \( \mathcal{V}|_{U_\alpha} \) is constant. Define double complexes

\[
K^{p,q} = \bigoplus_{r+s=q} C^p(U, \mathcal{V} \otimes_{\mathbb{R}_X} \mathcal{A}^r_X) \otimes_{\mathbb{R}} H^s_c(Y, \mathcal{W}),
\]

where \( \mathcal{A}^r_X \) is isomorphic by Lemma 5.3. Therefore, the morphism \( f: \mathcal{V}|_{U_\alpha} \otimes_{\mathbb{R}} H^s_c(Y, \mathcal{W}) \) induces an isomorphism

\[
d' = \sum_{r+s=q} \delta \otimes \id_{H^s_c(Y, \mathcal{W})}, \quad d'' = \sum_{r+s=q} d \otimes \id_{H^s_c(Y, \mathcal{W})},
\]

and

\[
L^{p,q} = C^p_{cv}(pr^{-1}_1 \mathcal{U}, \mathcal{V} \boxtimes \mathcal{W} \otimes_{\mathbb{R}_X} \mathcal{A}^r_X |_{\mathcal{Y}}),
\]

where \( (C^*_{cv}(\mathcal{U}, \mathcal{F}), \delta) \) denote the Čech complex of the sheaf \( \mathcal{F} \) with supports in \( \Phi \) associated to the open covering \( \mathcal{U} \).

Let \( \{ \gamma_i \} \) be in \( \Gamma(\mathcal{W} \otimes_{\mathbb{R}_X} \mathcal{A}^p_X) \) with pure degrees, such that \( \{ [\gamma_i] \} \) is a basis of \( H^p_c(Y, \mathcal{W}) \). Define a morphism \( f: K^{*,*} \to L^{*,*} \) of double complexes as

\[
\{ \eta_{\alpha_0, \ldots, \alpha_p} \} \otimes [\gamma_i] \mapsto \{ pr_1^* \eta_{\alpha_0, \ldots, \alpha_p} \wedge pr_2^* \gamma_i \}.
\]

Suppose \( \text{rank} \mathcal{V} = k \). \( H^r(U_{\alpha_0, \ldots, \alpha_p}, \mathcal{V}) \cong H^r(U_{\alpha_0, \ldots, \alpha_p})^{\oplus k} \) is \( \mathbb{R}^k \) for \( r = 0 \) and zero otherwise, since \( \mathcal{V}|_{U_\alpha} \) is constant on \( U_{\alpha_0, \ldots, \alpha_p} \cong \mathbb{R}^n \). So

\[
H^0_{d^r}(K^{p,*}) \cong \prod_{\alpha_0, \ldots, \alpha_p} H^0(U_{\alpha_0, \ldots, \alpha_p}, \mathcal{V}) \otimes_{\mathbb{R}} H^0_c(Y, \mathcal{W})
\]

\[
\cong \prod_{\alpha_0, \ldots, \alpha_p} H^0_{d^r}(Y, \mathcal{W})^{\oplus k},
\]

where the finiteness of dimension of \( H^2_c(Y, \mathcal{W}) \) implies the second isomorphism, and

\[
H^0_{d^r}(L^{p,*}) = \prod_{\alpha_0, \ldots, \alpha_p} H^0_{d^r}(U_{\alpha_0, \ldots, \alpha_p} \times Y, \mathcal{V} \boxtimes \mathcal{W})
\]

\[
\cong \prod_{\alpha_0, \ldots, \alpha_p} H^0_{d^r}(U_{\alpha_0, \ldots, \alpha_p} \times Y, pr_2^{-1}\mathcal{W})^{\oplus k}.
\]

The morphism \( H^0_{d^r}(K^{p,*}) \to H^0_{d^r}(L^{p,*}) \) induced by \( f \) is just

\[
\prod_{\alpha_0, \ldots, \alpha_p} (pr_2^*)^{\oplus k} : \prod_{\alpha_0, \ldots, \alpha_p} H^0_c(Y, \mathcal{W})^{\oplus k} \to \prod_{\alpha_0, \ldots, \alpha_p} H^0_{d^r}(U_{\alpha_0, \ldots, \alpha_p} \times Y, pr_2^{-1}\mathcal{W})^{\oplus k},
\]

which is isomorphic by Lemma 5.3. Therefore, \( f \) induces an isomorphism \( H^p(f): H^p(K^*) \to H^p(L^*) \), where \( K^* \) and \( L^* \) are the simple complexes associated to \( K^{*,*} \) and \( L^{*,*} \), respectively.

Consider the spectral sequence \( E^{p,q}_r = H^p_{d^r}(H^{r-1}) \to H^{p+q}(K^*) \). By Leray Theorem,

\[
E^{p,q}_r = \begin{cases} \bigoplus_{r+s=q} H^r(X, \mathcal{V}) \otimes_{\mathbb{R}} H^s_c(Y, \mathcal{W}), & p = 0 \\ 0, & \text{otherwise}, \end{cases}
\]

which implies \( H^i(K^*) = \bigoplus_{p+q=i} H^p(X, \mathcal{V}) \otimes_{\mathbb{R}} H^q_c(Y, \mathcal{W}) \). Similarly, \( H^i(L^*) = H^i_{d^r}(X \times Y, \mathcal{V} \boxtimes \mathcal{W}) \). We get (2).
(3) Assume $H^*(Y,W)$ has finite dimension and $\mathcal{U} = \{U_\alpha\}$ a good covering of $X$ satisfying that every $\mathcal{V}|_{U_\alpha}$ is constant. Consider two double complexes

$$R^{p,q} = \bigoplus_{r+s=q} C^p(\mathcal{U}, \mathcal{V} \otimes_{\mathbb{R}^X} \mathcal{A}_{X}) \otimes_{\mathbb{R}} H^s(Y,W),$$

$$d' = \sum_{r+s=q} \delta \otimes \text{id}_{H^r(Y,W)}, \quad d'' = \sum_{r+s=q} d \otimes \text{id}_{H^r(Y,W)},$$

and

$$L^{p,q} = C^p(\text{pr}_1^{-1}\mathcal{U}, \mathcal{V} \boxtimes \mathcal{W} \otimes_{\mathbb{R}^X \times Y} \mathcal{A}_{X \times Y}),$$

$$d' = \delta, \quad d'' = d.$$

Following the proof of (2), we can prove (3). \hfill \Box

5.3. Leray-Hirsch theorem.

**Theorem 5.5.** (1) Let $\pi : E \to X$ be a smooth fiber bundle over a connected smooth manifold $X$ and $\mathcal{V}$ a local system of $\mathbb{R}$-modules of finite rank on $X$.

(i) Assume there exist classes $e_1, \ldots, e_r$ of pure degrees in $H^*(E, \mathbb{R})$, such that, for every $x \in X$, their restrictions $e_1|_{E_x}, \ldots, e_r|_{E_x}$ freely linearly generate $H^*(E_x, \mathbb{R})$. Then, $\pi^*(\bullet) \cup \bullet$ gives an isomorphism of graded vector spaces

$$H^*(X,\mathcal{V}) \otimes_{\mathbb{R}} \text{span}_R \{e_1, \ldots, e_r\} \to H^*(E, \pi^{-1}\mathcal{V}).$$

(ii) Assume there exist classes $e_1, \ldots, e_r$ of pure degrees in $H^*_c(E, \mathbb{R})$, such that, for every $x \in X$, their restrictions $e_1|_{E_x}, \ldots, e_r|_{E_x}$ freely linearly generate $H^*_c(E_x, \mathbb{R})$. Then, $\pi^*(\bullet) \cup \bullet$ gives isomorphisms of graded vector spaces

$$H^*_c(X,\mathcal{V}) \otimes_{\mathbb{R}} \text{span}_R \{e_1, \ldots, e_r\} \to H^*_c(E, \pi^{-1}\mathcal{V}).$$

(iii) Assume there exist classes $e_1, \ldots, e_r$ of pure degrees in $H^*_c(E, \mathbb{R})$, such that, for every $x \in X$, their restrictions $e_1|_{E_x}, \ldots, e_r|_{E_x}$ freely linearly generate $H^*_c(E_x, \mathbb{R})$. Then, $\pi^*(\bullet) \cup \bullet$ gives isomorphisms of graded vector spaces

$$H^*_c(X,\mathcal{V}) \otimes_{\mathbb{R}} \text{span}_R \{e_1, \ldots, e_r\} \to H^*_c(E, \pi^{-1}\mathcal{V}).$$

(2) Let $\pi : E \to X$ be a holomorphic fiber bundle over a connected complex manifold $X$ whose fiber $F$ is compact, and let $\mathcal{E}$ be a local free sheaf of $\mathcal{O}_X$-modules of finite rank on $X$.

Assume there exist classes $e_1, \ldots, e_r$ of pure degrees in $H^{**}(E)$, such that, for every $x \in X$, their restrictions $e_1|_{E_x}, \ldots, e_r|_{E_x}$ freely linearly generate $H^{**}(E_x)$. Then, $\pi^*(\bullet) \cup \bullet$ gives isomorphisms of bigraded vector spaces

$$H^{**}(X,\mathcal{E}) \otimes_{\mathbb{C}} \text{span}_{\mathbb{C}} \{e_1, \ldots, e_r\} \to H^{**}(E, \pi^*\mathcal{E}).$$

**Proof.** (1) Set $k_i = \text{deg}e_i$, for $0 \leq i \leq r$.

(i) By Proposition 12, (2), (3) (i),

$$\pi^* : \mathcal{H}^*_X(\mathcal{V})[-k_i] \to \pi_*\mathcal{H}^*_E(\pi^{-1}\mathcal{V})[-k_i]$$

and

$$e_i \cup : \mathcal{H}^*_E(\pi^{-1}\mathcal{V})[-k_i] \to \mathcal{H}^*_E(\pi^{-1}\mathcal{V})$$

\hfill \Box
are morphisms of M-V systems of cdp presheaves. By Proposition 2.2 (5),
\[ e_i \cup : \pi_* \mathcal{H}_E^*(\pi^{-1}V)[-k_i] \to \pi_* \mathcal{H}_E^*(\pi^{-1}V) \]
is also a morphism of M-V systems of cdp presheaves, so is the composition
\[ \pi^*(\bullet) \cup e_i : \mathcal{H}_X^*(V)[-k_i] \to \pi_* \mathcal{H}_E^*(\pi^{-1}V) \]
by Section 2.2 (5'). Hence the sum
\[ F^* = \sum_{i=1}^{r} \pi^*(\bullet) \cup e_i : \bigoplus_{i=1}^{r} \mathcal{H}_X^*(V)[-k_i] \to \pi_* \mathcal{H}_E^*(\pi^{-1}V) \]
is a morphism of M-V systems by Section 2.2 (4'). Let \( U \) be a basis of topology of \( X \) such that \( V|_U \) is constant for any \( U \in U \). For \( U_1, \ldots, U_l \in U \), \( V|_{U_1 \cap \ldots \cap U_l} \) is constant. By [14], Thm. 3.10 (1), \( F^* \) is an isomorphism on \( U_1 \cap \ldots \cap U_l \). By Theorem 1.1, we get (ii) immediately.

(ii) Similarly with (i), by Proposition 4.2 (3), Proposition 2.2 (5) and Section 2.2 (4), (5'),
\[ \sum_{i=1}^{r} \pi^*(\bullet) \cup e_i : \bigoplus_{i=1}^{r} \mathcal{H}_X^* c^c(V)[-k_i] \to \pi_* \mathcal{H}_E^*(\pi^{-1}V) \]
is a morphism of M-V systems of cdp presheaves.

By Proposition 4.2 (3) (i), Proposition 4.2 (2), Proposition 2.2 (5) and Section 2.2 (4), (5'),
\[ \sum_{i=1}^{r} \pi^*(\bullet) \cup e_i : \bigoplus_{i=1}^{r} \mathcal{H}_X^* c^c(V)[-k_i] \to \pi_* \mathcal{H}_E^*(\pi^{-1}V) \]
is a cdp M-V morphism of M-V systems of cdp presheaves. The rest of the proof is similarly with that of (i), except that we use [14], Thm. 3.10 (2) instead of [14], Thm. 3.10 (1).

(iii) Let \( \hat{e}_1, \ldots, \hat{e}_r \) be the image of \( e_1, \ldots, e_r \) under the natural map \( H^*_c(E) \to H^*_c(E) \). By the hypothesis, it gives an isomorphism \( \text{span}_R\{e_1, \ldots, e_r\} = \text{span}_R\{\hat{e}_1, \ldots, \hat{e}_r\} \), which immediately implies (iii) by (ii).

(2) For \( 0 \leq i \leq r \), assume the bedegree of \( e_i \) is \( (k_i, l_i) \). Fixed \( p \in \mathbb{Z} \). By Proposition 4.1 (2), (3), Proposition 2.2 (5) and Section 2.2 (4'), (5'),
\[ F^* = \sum_{i=1}^{r} \pi^*(\bullet) \cup e_i : \bigoplus_{i=1}^{r} \mathcal{H}_X^{p-k_i, \bullet}(E)[-l_i] \to \pi_* \mathcal{H}_E^{p, \bullet}(\pi^* E) \]
is a morphism of M-V systems of cdp presheaves. As the proof in (1) (i), by [16], Thm. 1.2, \( F^* \) satisfies the hypothesis (*) in Theorem 1.1. We proved (2).

We immediately get

**Corollary 5.6.** Let \( \pi : P(E) \to X \) be the projectivization of a complex vector bundle \( E \) of rank \( r \) on a complex manifold \( X \). Set \( t = \frac{1}{\pi^*} \Theta(\mathcal{O}_{P(E)}(-1)) \in \mathcal{A}^d(P(E)) \), where \( \mathcal{O}_{P(E)}(-1) \) is the universal line bundle on \( P(E) \) and \( \Theta(\mathcal{O}_{P(E)}(-1)) \) is a curvature of a hermitian metric on \( \mathcal{O}_{P(E)}(-1) \).

(1) For any local system \( V \) of \( \mathbb{R} \)-modules of finite rank, \( \pi^* (\bullet) \cup \bullet \) gives isomorphisms of graded vector spaces
\[ H^*(X, V) \otimes_{\mathbb{R}} \text{span}_{\mathbb{R}}\{1, \ldots, h^{r-1}\} \to H^*(P(E), \pi^{-1}V) \]
and
\[ H^*_c(X, V) \otimes_{\mathbb{R}} \text{span}_{\mathbb{R}} \{1, ..., h^{r-1}\} \rightarrow H^*_c(\mathbb{P}(E), \pi^{-1}V), \]
where \( h = [t] \in H^2(\mathbb{P}(E), \mathbb{R}). \)

(2) Assume \( E \) is holomorphic and \( \Theta(O_{\mathbb{P}(E)}(-1)) \) is the Chern curvature of a hermitian metric on \( O_{\mathbb{P}(E)}(-1). \) For any locally free sheaf \( E \) of \( O_X \)-modules of finite rank, \( \pi^*(\bullet) \cup \bullet \) gives an isomorphism of graded vector spaces
\[ H**(X, E) \otimes_{\mathbb{C}} \text{span}_{\mathbb{C}} \{1, ..., h^{r-1}\} \rightarrow H**(\mathbb{P}(E), \pi^*E) \]
where \( h = [t] \in H^{1,1}(\mathbb{P}(E)). \)

**Remark 5.7.** (1) Theorem 5.5 and Corollary 5.6 (1) also hold if we use \( \mathbb{C} \) instead of \( \mathbb{R}. \)

(2) Following the step of [4], Rao, S., Yang, S. and Yang, X.-D. gave another version of Hirsch theorem ([18], Lemma 3.2), where they assumed that \( H**(F) \) is a free bigraded algebra. Using this, they proved Corollary 5.6 (2) ([18], Lemma 3.3).

By [2], Thm. 6.17 and Remark 6.17.1, \( \pi_* : H^*_c(E, \mathbb{R}) \rightarrow H^{*-r}(X, \mathbb{R}) \) is an isomorphism. Let \( \Phi \in A^*_c(E) \) be a closed form satisfying \( \pi_*(\Phi)_{cv} = 1 \) in \( H^0(X, \mathbb{R}). \) \( [\Phi]_{cv} \in H^*_c(E, \mathbb{R}) \) is the Thom class of \( E. \) Clearly, \( \pi_* \Phi = 1 \) in \( A^0(X). \) We get the Thom isomorphism theorem for local systems.

**Corollary 5.8.** Let \( \pi : E \rightarrow X \) be an oriented smooth vector bundle of rank \( r \) on a (possibly unorientable) smooth manifold \( X. \) Assume \( V \) is a local system of \( \mathbb{R} \)-modules of finite rank on \( X. \) Then, \( \Phi \wedge \pi^*(\bullet) \) gives isomorphisms
\[ H^{*-r}_c(X, V) \rightarrow H^*_c(E, \pi^{-1}V) \]
and
\[ H^{*-r}(X, V) \rightarrow H^*_c(E, \pi^{-1}V), \]
which have the inverse isomorphism \( \pi_. \) Moreover, if \( X \) is oriented, they coincide with the pushout \( i_. \)

**Proof.** By [2], Prop. 6.18, the restriction \( [\Phi]_{E_x} \) is a generator of \( H^*_c(E_x, \mathbb{R}). \) By Theorem 5.5 (1) (ii), \( \Phi \wedge \pi^*(\bullet) \) gives the two isomorphisms. For \( \omega \in \Gamma(X, V \otimes_{\mathbb{R}} A^*_X), \) \( \pi_*(\Phi \wedge \pi^*\omega) = \omega \) by the projection formula [10]. So \( \pi_* \) is their inverse isomorphisms.

If \( X \) is oriented, the pushout \( i_. \) is defined well in both cases and \( \pi_* i_. = id. \) So
\[ i_. = \pi^{-1} = \Phi \wedge \pi^*. \]

**5.4. Formulas of proper modifications.** A proper holomorphic map \( \pi : X \rightarrow Y \) between connected complex manifolds is called a proper modification, if there is a nowhere dense analytic subset \( F \subset Y, \) such that \( \pi^{-1}(F) \subset X \) is nowhere dense and \( \pi : X - f^{-1}(F) \rightarrow Y - F \) is biholomorphic. If \( F \) is such a minimal analytic subset, then \( E = \pi^{-1}(F) \) is called the exceptional set of the proper modification \( \pi. \)

**Proposition 5.9.** Let \( \pi : X \rightarrow Y \) be a proper modification of connected complex manifolds and \( E \) the exceptional set of \( \pi. \)

(1) Suppose \( V \) is a local system of \( \mathbb{R} \) or \( \mathbb{C} \)-modules of finite rank on \( Y. \)
are inclusions and local systems and Dolbeault cohomology with values in locally free sheaves. In follows, we study blow-up formulas for cohomology with values in various cohomologies of blow-ups. There are several results of this aspect, seeing [10, 12, 14, 15, 17, 22, 24, 25].

We can prove (1) in the similar way with [14], Lemma 4.6 and Thm. 4.7.

Proof. We can prove (1) in the similar way with [14], Lemma 4.6 and Thm. 4.7.

(2) Let $\mathfrak{U}$ be a basis for topology of $Y$ such that $\mathcal{E}|\mathfrak{U}$ is a free $\mathcal{O}_Y$-module for every $U \in \mathfrak{U}$. By Proposition [1, 1](3), $\pi^*: \mathcal{H}^0_{Y,E}(\mathcal{E}) \to \pi_*(\mathcal{H}^0_{X,E}(\pi^*\mathcal{E}))$ is a morphism of $\mathcal{M}$-$\mathcal{V}$ systems of c.s. presheaves. For $U_1, ..., U_l \in \mathfrak{U}$, $\mathcal{E}|\bigcap_{i=1}^l U_i$ is free. By [10, Prop. 4.3, $\pi^*$ is an isomorphic on $\bigcap_{i=1}^l U_i$, i.e. $\pi^*$ satisfies the hypothesis (*) in Theorem [11]. So $\pi^*: H^0_c(Y,\mathcal{E}) \to H^0_c(X,\pi^*\mathcal{E})$ is an isomorphism.

Other cases can be proved similarly, where we still use [16, Prop. 4.3 when we check the hypothesis (*) in Theorem [11].

6. Blow-up formulas

Blow-up is a fundamental transform on complex manifolds and algebraic varieties, which play an important role in complex and algebraic geometries. So, it is significant to calculate various cohomologies of blow-ups. There are several results of this aspect, seeing [10, 12, 14, 15, 17, 22, 24, 25]. In follows, we study blow-up formulas for cohomology with values in local systems and Dolbeault cohomology with values in locally free sheaves.

Let $\pi: \tilde{X} \to X$ be the blow-up of a connected complex manifold $X$ along a connected complex submanifold $Y$. We know $\pi|_E: E = \pi^{-1}(Y) \to Y$ is the projectivization $E = \mathbb{P}(N_Y/X)$ of the normal bundle $N_Y/X$ over $Y$. Assume $i_Y: Y \to X$ and $i_E: E \to \tilde{X}$ are inclusions and $r = \text{codim}_c Y$. Set $t = \frac{\text{deg}}{n-r} \Theta(\mathcal{O}_E(-1)) \in \mathcal{A}^{1,1}(E)$, where $\mathcal{O}_E(-1)$ is the universal line bundle on $E = \mathbb{P}(N_Y/X)$ and $\Theta(\mathcal{O}_E(-1))$ is the Chern curvature of a hermitian metric on $\mathcal{O}_E(-1)$. Clearly, $\partial t = 0$ and $\bar{\partial} t = 0$.

6.1. Blow-up formulas.

Theorem 6.1. Assume $X, Y$, $\pi$, $\tilde{X}$, $E$, $t$, $i_Y$, $i_E$ are defined above. Then,

$$\pi^* + \sum_{i=0}^{r-2} (i_E)_* \circ (h^i) \circ (\pi|_E)^*$$
gives isomorphisms

\[(11) \quad H^k(X, \mathcal{V}) \oplus \bigoplus_{i=0}^{r-2} H^{k-2-2i}(Y, i_Y^{-1}\mathcal{V}) \cong H^k(\bar{X}, \pi^{-1}\mathcal{V}),\]

\[(12) \quad H^k_c(X, \mathcal{V}) \oplus \bigoplus_{i=0}^{r-2} H^{k-2-2i}(Y, i_Y^{-1}\mathcal{V}) \cong H^k(\bar{X}, \pi^{-1}\mathcal{V}),\]

for any \(k\), where \(h = [t] \in H^2(E, \mathbb{R})\) or \(H^2(E, \mathbb{C})\) and \(\mathcal{V}\) is a local system of \(\mathbb{R}\) or \(\mathbb{C}\)-modules of finite rank on \(X\), and,

\[(13) \quad H^{p,q}(X, \mathcal{E}) \oplus \bigoplus_{i=0}^{r-2} H^{p-1-i,q-1-i}(Y, i_Y^*\mathcal{E}) \cong H^{p,q}(\bar{X}, \pi^*\mathcal{E}),\]

for any \(p, q\), where \(h = [t] \in H^{1,1}(E)\) and \(\mathcal{E}\) is a locally free sheaf of \(O_X\)-modules of finite rank on \(X\).

**Proof.** By Proposition 2.1 (3),

\[\pi^* : \mathcal{H}^p_X(\mathcal{E}) \to \pi_* \mathcal{H}^p_X(\pi^*\mathcal{E}),\]

\[(\pi|_E)^* : \mathcal{H}^{p-1-i,*}_Y(i_Y^*\mathcal{E})[-1 - i] \to (\pi|_E)_* \mathcal{H}^{p-1-i,*}_E((\pi|_E)^*i_Y^*\mathcal{E})[-1 - i],\]

\[(i|_E)_* : i_*\mathcal{H}^{p-1,i,*}_E(i_Y^*\pi^*\mathcal{E})[-1] \to \mathcal{H}^p_X(\pi^*\mathcal{E})\]

and, by Proposition 2.1 (2),

\[h^i \cup : \mathcal{H}^{p-1-i,*}_E((\pi|_E)^*i_Y^*\mathcal{E})[-1 - i] \to \mathcal{H}^{p-1-i,*}_E((\pi|_E)^*i_Y^*\mathcal{E})[-1]\]

are morphisms of M-V systems of cdp presheaves, so are

\[(14) \quad (\pi|_E)^* : i_*\mathcal{H}^{p-1-i,*}_Y(i_Y^*\mathcal{E})[-1 - i] \to i_*\pi_*(\pi|_E)_* \mathcal{H}^{p-1-i,*}_E((\pi|_E)^*i_Y^*\mathcal{E})[-1 - i],\]

\[h^i \cup : i_*\pi_*(\pi|_E)_* \mathcal{H}^{p-1-i,*}_E((\pi|_E)^*i_Y^*\mathcal{E})[-1 - i] \to i_*\pi_*(\pi|_E)_* \mathcal{H}^{p-1,i,*}_E((\pi|_E)^*i_Y^*\mathcal{E})[-1]\]

\[= i_*\pi_*(\pi|_E)_* \mathcal{H}^{p-1,i,*}_E(i_Y^*\pi^*\mathcal{E})[-1],\]

and

\[(15) \quad (i|_E)_* : \pi_* i_* \mathcal{H}^{p-1,i,*}_E(i_Y^*\pi^*\mathcal{E})[-1] \to \pi_* \mathcal{H}^p_X(\pi^*\mathcal{E})\]

by Proposition 2.2 (5). Combined with (14) and (16), we obtain a M-V morphism

\[(i|_E)_* \circ (h^i \cup) \circ (\pi|_E)^* : i_*\mathcal{H}^{p-1-i,*}_Y(i_Y^*\mathcal{E})[-1 - i] \to \pi_* \mathcal{H}^p_X(\pi^*\mathcal{E})\]

by Section 2.2 (5'). By Section 2.2 (4), (4'), the sum

\[F^* = \pi^* + \sum_{i=1}^{r-2} (i|_E)_* \circ (h^i \cup) \circ (\pi|_E)^*\]

gives a morphism

\[\mathcal{H}^p_X(\mathcal{E}) \oplus \bigoplus_{i=0}^{r-2} i_*\pi_*(\pi|_E)_* \mathcal{H}^{p-1-i,*}_E(i_Y^*\mathcal{E})[-1 - i] \to \pi_* \mathcal{H}^p_X(\pi^*\mathcal{E})\]

of M-V systems of cdp presheaves.
Let $\mathcal{U}$ be a basis for topology of $X$ such that, $\mathcal{E}|_U$ is a free $\mathcal{O}_U$-module for every $U \in \mathcal{U}$. For any $U_1, \ldots, U_l \in \mathcal{U}$, $\bigcap_{i=1}^l U_i$ is free. By [15], Thm 1.2, $F^*$ is isomorphic on $\bigcap_{i=1}^l U_i$. Hence, $F^*$ satisfies (*) in Theorem 1.1 and then $F^*$ is an isomorphism. We proved (13).

The proofs of (11) and (12) are similar with that of (13), except that we use Proposition 4.2 and [14], Thm. 1.3 instead of Proposition 4.1 and [15], Thm. 1.2. □

6.2. Comparison with the formula given by Rao, S., Yang, S. and Yang, X.-D..

6.2.1. The formula given by Rao, S. et al. In [18], Rao, S., Yang, S. and Yang, X.-D. gave an explicit formula of blow-ups for bundle-valued Dolbeault cohomology on compact complex manifolds. We recall their construction as follows:

Suppose $X$ is compact. For any $\alpha$ in $H^{p,q}(\tilde{X}, \pi^* \mathcal{E})$, by [18], Lemma 3.3, there exist unique $\alpha^{p-i,q-i}$ in $H^{p-i,q-i}(Y, i^*_Y \mathcal{E})$, for $i = 0, \ldots, r-1$ such that

$$i^*_Y \alpha = \sum_{i=0}^{r-1} h^i \cup (\pi|_E)^* \alpha^{p-i,q-i}.$$  \hspace{1cm} (17)

Define

$$\phi : H^{p,q}(\tilde{X}, \pi^* \mathcal{E}) \to H^{p,q}(X, \mathcal{E}) \oplus \bigoplus_{i=1}^{r-1} H^{p-i,q-i}(Y, i^*_Y \mathcal{E})$$

$$\alpha \mapsto (\pi_* \alpha, \alpha^{p-1,q-1}, \ldots, \alpha^{p-r+1,q-r+1}).$$  \hspace{1cm} (18)

Clearly, $\phi$ a linear map of vector spaces. Rao, S. et al. proved that $\phi$ is injective and

$$H^{p,q}(\tilde{X}, \pi^* \mathcal{E}) \cong H^{p,q}(X, \mathcal{E}) \oplus \bigoplus_{i=1}^{r-1} H^{p-i,q-i}(Y, i^*_Y \mathcal{E}),$$  \hspace{1cm} (19)

which imply $\phi$ is an isomorphism.

From now on, assume $X$ is a (unnecessarily compact) connected complex manifold. We notice that, for the definition (18) of $\phi$, the compactness of $X$ is unnecessary, since we can use Corollary 5.6 instead of [18], Lemma 3.3 for the general case. With the same proof in [18], the injectivity of $\phi$ and the isomorphism (19) also hold for the noncompact base $X$. However, we don’t know whether $\phi$ is isomorphic in this case, since the dimensions of cohomology are possibly infinite in this case. In follows, through comparison of $\phi$ and the formula given in Theorem 6.1, we will prove that $\phi$ is still isomorphic on general complex manifolds.

6.2.2. Relative Dolbeault sheaves. We recall two sheaves defined in [17] and their properties. For more details, we refer to [18], Sec. 4.2 for compact cases, or [15], Sec. 4 for general cases.

Let $X$ be a connected complex manifold and $i : Y \to X$ a closed complex submanifold. For any $p, q$, denote $\mathcal{F}^{p,q}_{X,Y} = \ker(A^{p,q}_X \to i_* A^{p,q}_Y)$. For any $p$,

$$0 \to \mathcal{F}^{p,q}_{X,Y} \to A^{p,q}_X \to i_* A^{p,q}_Y \to 0.$$  \hspace{1cm} (20)

is an exact sequence of sheaves. Define $\mathcal{F}^{0,0}_{X,Y} = \ker(\bar{\partial} : \mathcal{F}^{0,0}_{X,Y} \to \mathcal{F}^{1,0}_{X,Y})$. There is a resolution of soft sheaves of $\mathcal{F}^p_{X,Y}$

$$0 \longrightarrow \mathcal{F}^p_{X,Y} \longrightarrow i^* \mathcal{F}^{p,0}_{X,Y} \longrightarrow \mathcal{F}^{p,1}_{X,Y} \longrightarrow \cdots \longrightarrow \mathcal{F}^{p,n}_{X,Y} \longrightarrow 0.$$
\( \mathcal{F}_{X,Y}^p \) and \( \mathcal{F}_{X,Y}^{p,q} \) are called the relative Dolbeault sheaves of \( X \) with respect to \( Y \). If \( \mathcal{E} \) is a locally free sheaf of \( \mathcal{O}_X \)-modules of finite rank on \( X \), we have an exact sequence of sheaves

\[
0 \to \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^{p,q} \to \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}_X^{p,q} \xrightarrow{i_Y^*} (i_Y^* \mathcal{E} \otimes \mathcal{O}_Y \mathcal{A}_X^{p,q}) \to 0
\]

by (20) and the projection formula of sheaves. Then

\[
0 \to \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^{p,q}) \to \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}_X^{p,q}) \xrightarrow{i_Y^*} \Gamma(Y, i_Y^* \mathcal{E} \otimes \mathcal{O}_Y \mathcal{A}_Y^{p,q}) \to 0.
\]

is exact, since \( \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^{p,q} \) is \( \pi \)-acyclic.

Now, we go back to the case of blow-ups. By [15], Lemma 4.2,

\[
R^q \pi_* (\pi^* \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,E}^p) = \mathcal{E} \otimes \mathcal{O}_X R^q \pi_* \mathcal{F}_{X,E}^p = \begin{cases} \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^p, & q = 0 \\ 0, & q \geq 1. \end{cases}
\]

By Leray spectral sequence, \( \pi^* \) induces an isomorphism

\[
H^q(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^p) = H^q(X, \mathcal{E} \otimes \mathcal{O}_X \pi_* \mathcal{F}_{X,E}^p) \cong H^q(\tilde{X}, \pi^* \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,E}^p).
\]

For a fixed \( p \), we have a commutative diagram of complexes

\[
0 \to \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^{p,q}) \to \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{A}_X^{p,q}) \xrightarrow{i_Y^*} \Gamma(Y, i_Y^* \mathcal{E} \otimes \mathcal{O}_Y \mathcal{A}_Y^{p,q}) \to 0
\]

where all the differentials are naturally induced by \( \tilde{\partial} \) and the two rows are exact by (21). It induces a commutative diagram of long exact sequences

\[
\ldots \to H^q(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^p) \xrightarrow{i_Y^*} H^q(Y, i_Y^* \mathcal{E}) \xrightarrow{(\pi|_E)^*} H^{q+1}(X, \mathcal{E} \otimes \mathcal{O}_X \mathcal{F}_{X,Y}^p) \to \ldots
\]

where \( \pi^* \) and \( (\pi|_E)^* \) are injective by Corollary 4.6 and 5.6 respectively. By Snake Lemma, \( i_Y^* \) induces an isomorphism \( \text{coker}(\pi|_E)^* \to \text{coker}(\pi|_E)^* \). We get a commutative diagram of exact sequences

\[
(22) \quad 0 \to H^q(X, \mathcal{E}) \xrightarrow{\pi^*} H^q(\tilde{X}, \pi^* \mathcal{E}) \xrightarrow{\text{coker}(\pi|_E)^*} 0,
\]

\[
0 \to H^q(Y, i_Y^* \mathcal{E}) \xrightarrow{(\pi|_E)^*} H^q(E, i_Y^* \pi^* \mathcal{E}) \xrightarrow{\text{coker}(\pi|_E)^*} 0
\]

for any \( p, q \).
2.3. Comparison of two formulas. Denote
\[ \psi = \pi^* + \sum_{i=0}^{r-2} (i_E)_* \circ (h^i \cup) \circ (\pi|_E)^* : H^{p,q}(X, \mathcal{E}) \oplus \bigoplus_{i=0}^{r-2} H^{p-1-i,q-1-i}(Y, i_Y^* \mathcal{E}) \to H^{p,q}(\tilde{X}, \pi^* \mathcal{E}). \]

A natural problem is:

**Question 6.2.** Are \( \phi \) and \( \psi \) inverse to each other?

It is true in following cases.

**Proposition 6.3.** Suppose
\[ (23) \quad i_E^* \pi_E^* \sigma = h \cup \sigma \]
for any \( \sigma \in H^{*,*}(E, i_E^* \pi^* \mathcal{E}). \) Then \( \phi \) and \( \psi \) are inverse isomorphisms to each other.

**Proof.** For any \( \gamma \in H^{p,q}(\tilde{X}, \pi^* \mathcal{E}) \), suppose that
\[ \phi(\gamma) = (\beta^{p,q}, \alpha^{p-1,q-1}, ..., \alpha^{p-r+1,q-r+1}), \]
where \( \beta^{p,q} \in H^{p,q}(X, \mathcal{E}) \) and \( \alpha^{p-i,q-1} \in H^{p-i,q-1}(Y, i_Y^* \mathcal{E}) \) for \( i = 1, ..., r-1 \). Then \( \pi_* \gamma = \beta^{p,q} \) and there exists \( \alpha^{p,q} \in H^{p,q}(Y, i_Y^* \mathcal{E}) \) such that
\[ i_E^* \gamma = \sum_{i=0}^{r-1} h^i \cup (\pi|_E)^* \alpha^{p-i,q-i}. \]

By (23),
\[ i_E^* \left[ \gamma - \sum_{i=0}^{r-2} (i_E)_* \left( h^i \cup (\pi|_E)^* \alpha^{p-1-i,q-1-i} \right) \right] = (\pi|_E)^* \alpha^{p,q}, \]
which is zero in \( \text{coker}(\pi|_E)^* \). By the commutative diagram (22),
\[ \gamma - \sum_{i=0}^{r-2} (i_E)_* \left( h^i \cup (\pi|_E)^* \alpha^{p-1-i,q-1-i} \right) = \pi^* \gamma^{p,q}, \]
for some \( \gamma^{p,q} \in H^{p,q}(X, \mathcal{E}). \) Through \( \pi_* \), by the projection formula, \( \gamma^{p,q} = \pi_* \gamma = \beta^{p,q} \), where we used the fact that \( (\pi|_E)_* h^i = 0 \) for \( 0 \leq i \leq r-2 \). Hence
\[ \gamma = \psi(\beta^{p,q}, \alpha^{p-1,q-1}, ..., \alpha^{p-r+1,q-r+1}). \]

So \( \psi \phi = \text{id} \). By Theorem 6.1, \( \psi \) is isomorphic, and then, \( \phi \) is inverse to \( \psi \).

**Proposition 6.4.** If \( \mathcal{E} \) is a free sheaf of \( \mathcal{O}_X \)-modules of finite rank on \( X \) and one of following conditions is satisfying:

1. \( X \) is in the Fujiki class \( \mathcal{C} \),
2. \( Y \) is a Stein manifold,
we have \( i_E^* \pi_E^* \sigma = h \cup \sigma \), for \( \sigma \in H^{*,*}(E, i_E^* \pi^* \mathcal{E}) \). For the two cases, \( \phi \) and \( \psi \) are inverse isomorphisms to each other.

**Proof.** We need only to prove the case \( \mathcal{E} = \mathcal{O}_X \).

1. Clearly, \( \tilde{X} \) is in the Fujiki class \( \mathcal{C} \), so \( \tilde{X} \) has the canonical Hodge decomposition
\[ H^k(\tilde{X}, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(\tilde{X}), \]
for any $k$. By [14], Lemma 1.2, $i_E^*\sigma = h \cup \sigma$, for any $\sigma \in H^*(\tilde{X}, \mathbb{C})$. Since $i_E^*$ and $i_E^*$ preserve the bidegrees, we proved the case (1).

(2) It is just [15], Lemma 4.4.

For any $i \in \{1, \cdots, r-1\}$, define polynomials $P^i_j(T_1, \ldots, T_{r-2}) \in \mathbb{Z}[T_1, \ldots, T_{r-2}]$, $0 \leq j \leq r-1-i$, by recursion relations

$$P^i_j(T_1, \ldots, T_{r-2}) = \begin{cases} (-1)^r \sum_{k=i+1}^{r-1-j} T_{k-i} P^k_j(T_1, \ldots, T_{r-2}), & j < r-1-i \\ (-1)^{r-1}, & j = r-1-i. \end{cases}$$

Suppose

$$P^i_j(T_1, \ldots, T_{r-2}) = \sum_{d_1, \ldots, d_{r-2}} a^i_{j, d_1, \ldots, d_{r-2}} T_1^{d_1} \cdots T_{r-2}^{d_{r-2}}.$$ 

It is easily checked that, for every nonzero term $a^i_{j, d_1, \ldots, d_{r-2}} T_1^{d_1} \cdots T_{r-2}^{d_{r-2}},$

$$\sum_{k=1}^{r-2} kd_k = r-1-i-j.$$ 

We can explicitly represent $\phi(\alpha)$ by $\alpha$ as follows.

**Lemma 6.5.** For any $\alpha \in H^{p,q}(\tilde{X}, \pi_* \mathcal{E})$, if

$$\phi(\alpha) = (\pi_* \alpha, \alpha^{p-1,q-1}, \cdots, \alpha^{p-r+1,q-r+1}),$$

then

$$\alpha^{p-i,q-i} = \sum_{j=0}^{r-1-i} \left[ P^j_i((\pi|_E)_* h^r, \ldots, (\pi|_E)_* h^{2r-3}) \right] \cup (\pi|_E)_*(h^l \cup i_E^* \alpha),$$

for $1 \leq i \leq r-1$.

**Proof.** Obviously, $(\pi|_E)_* h^i = 0$ for $0 \leq i \leq r-2$. Moreover, $(\pi|_E)_* h^{r-1} = (-1)^{r-1}$ in $H^{0,0}(Y) = \mathcal{O}(Y)$. Actually, $(\pi|_E)_* t^{r-1}$ is a $d$-closed smooth 0-form on $Y$, hence a constant. For any $y$ in $Y$,

$$(\pi|_E)_* t^{r-1} = \int_{E_y} t^{r-1}|_{E_y}$$

$$\quad = \int_{E_y} \left[ \frac{i}{2\pi} \Theta(\mathcal{O}_{E_y}(-1)) \right]^{r-1}$$

$$\quad = \int_{p^{r-1}} c_1(\mathcal{O}_{p^{r-1}}(-1))^{r-1}$$

$$\quad = (-1)^{r-1}.$$ 

Push out (17) by $\pi|_E$, we get

$$\alpha^{p-r+1,q-r+1} = (-1)^{r-1}(\pi|_E)_* i_E^* \alpha$$

using the projection formula (7). The conclusion holds for $i = r-1$. Suppose that it holds for any $i \geq l+1$. Cup product with (17), and then push out by $\pi|_E$, we get

$$\alpha^{p-l,q-l} = (-1)^{r-1}(\pi|_E)_* (h^{r-1-l} \cup i_E^* \alpha) + (-1)^r \sum_{k=l+1}^{r-1} (\pi|_E)_* (h^{k+r-1-l}) \cup \alpha^{p-k,q-k}.$$
By inductive hypothesis, the conclusion holds for \( i = l \). We complete the proof. \( \square \)

**Theorem 6.6.** \( \phi \) is an isomorphism for any connected complex manifold \( X \).

**Proof.** Fix an integer \( p \). By Proposition 4.1 (3),

\[
i_E^*: \mathcal{H}_X^{p,*}(\pi^*E) \to i_E^*\mathcal{H}_E^{p,*}(i_E^*\pi^*E)
\]

\[
(\pi|E)_*: (\pi|E)_*\mathcal{H}_E^{p+j,*}(i_Y^*\pi^*E)[j] \to \mathcal{H}_Y^{p+j-r+1,*}(i_Y^*E)[j-r+1],
\]

and, by Proposition 4.1 (2),

\[
h^j \cup: \mathcal{H}_E^{p,*}(i_E^*\pi^*E) \to \mathcal{H}_E^{p+j,*}(i_E^*\pi^*E)[j]
\]

\[
\prod_{k=1}^{r-2}[(\pi|E)_*\mathcal{H}_E^{p+j-r+1,*}(i_Y^*E)[j-r+1] \to \mathcal{H}_Y^{p-r,*}(i_Y^*E)[-i]
\]

are morphisms of M-V systems of cdp presheaves, so are

\[
i_E^*: \pi_*\mathcal{H}_X^{p,*}(\pi^*E) \to \pi_*i_E^*\mathcal{H}_E^{p,*}(i_E^*\pi^*E),
\]

\[
h^j \cup: \pi_*i_E^*\mathcal{H}_E^{p,*}(i_E^*\pi^*E) \to \pi_*i_E^*\mathcal{H}_E^{p+j,*}(i_E^*\pi^*E)[j]
\]

\[
= i_Y^*(\pi|E)_*\mathcal{H}_E^{p+j,*}(i_Y^*\pi^*E)[j],
\]

\[
(\pi|E)_*: i_Y^*(\pi|E)_*\mathcal{H}_E^{p+j,*}(i_Y^*\pi^*E)[j] \to i_Y^*\mathcal{H}_Y^{p+j-r+1,*}(i_Y^*E)[j-r+1],
\]

\[
\prod_{k=1}^{r-2}[(\pi|E)_*\mathcal{H}_Y^{p-r,*}(i_Y^*E)[-i]
\]

by Proposition 2.2 (5), where \( \sum_{k=1}^{r-2}kd_k = r - i - j \). Combining with (25)-(28), by Section 2.2 (5'), (6'),

\[
\Phi^{p-r,*-i} := \sum_{j=0}^{r-1-i} \left[ P_j^i ((\pi|E)_*h^r,...,(\pi|E)_*h^{2r-3}) \cup (\pi|E)_* \circ (h^{j} \cup) \circ i_E^* \right]
\]

is a morphism \( \pi_*\mathcal{H}_X^{p,*}(\pi^*E) \to i_Y^*\mathcal{H}_Y^{p-r,*}(i_Y^*E)[-i] \) of M-V systems of cdp presheaves on \( X \) for any \( 1 \leq i \leq r - 1 \), so is

\[
\Phi^* := \pi_*\mathcal{H}_X^{p,*}(\pi^*E) \to \mathcal{H}_X^{p,*}(E) \oplus \bigoplus_{i=1}^{r-1} i_Y^*\mathcal{H}_Y^{p-r,*}(i_Y^*E)[-i],
\]

which is given by

\[
\Phi^* = (\pi_*\mathcal{H}_X^{p-*},...,\Phi^{p-1,*-1},\Phi^{p-r+1,*-1+}).
\]

Let \( \mathfrak{U} \) be a basis of topology of \( X \) such that \( U \) is Stein and \( \mathcal{E}|_U \) is a free \( \mathcal{O}_U \)-module for every \( U \in \mathfrak{U} \). Since \( U_1 \cap ... \cap U_l \cap Y \) is Stein for \( U_1,...,U_l \in \mathfrak{U} \), \( \Phi^* \) is an isomorphism on \( U_1 \cap ... \cap U_l \) by Proposition 6.3, i.e. \( \Phi^* \) satisfies the hypothesis (*). Then \( \Phi^*(X) \) is an isomorphism by Theorem 1.1 so is \( \phi = \Phi^q(X) \). \( \square \)
6.3. Several questions. Let $\pi, X, Y, \tilde{X}, E, t, V, i_Y, i_E$ be defined as above. Denote isomorphisms (11) and (12) by $\psi^V$ and $\psi^c_V$ respectively. With a similar proof of Theorem 6.6, we get isomorphisms as (13)

$$\phi^V : H^k(\tilde{X}, \pi^{-1}V) \rightarrow H^k(X, V) \oplus \bigoplus_{i=0}^{r-2} H^{k-2-2i}(Y, i_Y^{-1}V)$$

and

$$\phi^c_V : H^k(\tilde{X}, \pi^{-1}V) \rightarrow H^k_c(X, V) \oplus \bigoplus_{i=0}^{r-2} H^{k-2-2i}_c(Y, i_Y^{-1}V),$$

where we use [14], Lemma 4.3 instead of Proposition 6.4 in the proof. As Question 6.2, we can ask

**Question 6.7.** (1) Are $\phi^V$ and $\psi^V$ inverse to each other?

(2) Are $\phi^c_V$ and $\psi^c_V$ inverse to each other?

Analogous to Proposition 6.3, it is easily to prove that

**Proposition 6.8.** (1) If $i^*_E i_E^* \sigma = h \cup \sigma$ for any $\sigma \in H^*(E, i^{-1}_E \pi^{-1}V)$, then $\phi^V$ and $\psi^V$ are inverse to each other.

(2) If $i^*_E i_E^* \sigma = h \cup \sigma$ for any $\sigma \in H^*_c(E, i^{-1}_E \pi^{-1}V)$, then $\phi^c_V$ and $\psi^c_V$ are inverse to each other.

For Question 6.7, we need to check the hypothesis in Proposition 6.8. Actually, we may consider a more general case:

**Question 6.9.** Let $Y$ be an oriented connected submanifold of an oriented connected smooth manifold $X$ and $i : Y \rightarrow X$ the inclusion. Assume $V$ is a local system of $\mathbb{R}$-modules of finite rank on $X$ and $[Y]$ is the fundamental class of $Y$ in $X$.

(1) For $\sigma \in H^*(Y, i^{-1}V)$,

$$i^* i_* \sigma = [Y]_V \cup \sigma?$$

(2) For $\sigma \in H^*_c(Y, i^{-1}V)$,

$$i^* i_* \sigma = [Y]_V \cup \sigma?$$

It is true for weight $\theta$-sheaf $V = \mathbb{R}_\theta$, seeing [14], Lemma 4.3.

An analogue of Theorem 6.1 on Bott-Chern cohomology is as follows.

**Question 6.10.** Assume $X, Y, \pi, \tilde{X}, E, t, i_Y, i_E, r$ are defined as above. Set $h = [t] \in H^{1,1}_{BC}(E)$. For any $p, q$, is

$$\pi^* + \sum_{i=0}^{r-2} (i_E)_* \circ (h^i \cup) \circ (\pi|_E)^* : H^{p,q}_{BC}(X) \oplus \bigoplus_{i=0}^{r-2} H^{p-1-i,q-1-i}_{BC}(Y) \rightarrow H^{p,q}_{BC}(\tilde{X})$$

an isomorphism?

For a compact complex manifold satisfying $\partial \bar{\partial}$-Lemma, there is a canonical isomorphism between Bott-Chern cohomology and Dolbeault cohomology, so the question is true by
Proposition 6.1. On general compact manifolds, Yang, S., Yang, X.-D. ([24]) and Stelzig, J. ([21]) proved that there exists an isomorphism

\[ H^p_{BC}(X) \oplus \bigoplus_{i=0}^{r-2} H^p_{BC}(Y) \cong H^p_{BC}(\tilde{X}). \]

However, it seems difficult to write out the isomorphism explicitly. For noncompact cases, a possible approach is similar with the proof of Theorem 6.1, where \( i^*_E \sigma = h \cup \sigma \), for \( \sigma \in H^*_BC(E) \) and the Mayer-Vietoris sequence for Bott-Chern cohomology are necessary, but we don’t know if they exist.

Acknowledgements. I would like to express my gratitude to School of Mathematics and Statistics, Wuhan University for the warm hospitality during my visit. I sincerely thank Prof. Jiangwei Xue for pointing out to me the meaningless notations in previous version of present article. I would like to thank Prof. Sheng Rao and Xiang-Dong Yang for sending me their articles [18], [24] and their helpful discussions.

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