Spectral gaps of simplicial complexes without large missing faces

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Abstract

Let $X$ be a simplicial complex on $n$ vertices without missing faces of dimension larger than $d$. Let $L_j$ denote the $j$-Laplacian acting on real $j$-cochains of $X$ and let $\mu_j(X)$ denote its minimal eigenvalue. We study the connection between the spectral gaps $\mu_k(X)$ for $k \geq d$ and $\mu_{d-1}(X)$. In particular, we establish the following vanishing result: If $\mu_{d-1}(X) > (1 - (k+1)^{-1})n$, then $\tilde{H}^j(X; \mathbb{R}) = 0$ for all $d-1 \leq j \leq k$. As an application we prove a fractional extension of a Hall-type theorem of Holmsen, Martínez-Sandoval and Montejano for general position sets in matroids.

1 Introduction

Let $X$ be a simplicial complex on vertex set $V$. A simplex $\sigma \subset V$ is called a missing face of $X$ if $\sigma \notin X$ but for any $\tau \subset \sigma$, $\tau \in X$. The set of missing faces $\mathcal{M}_X$ of the complex $X$ completely determines $X$:

$$X = \{ \tau \subset V : \sigma \not\subset \tau \text{ for all } \sigma \in \mathcal{M}_X \}.$$ 

Let $h(X) = \max \{ \dim(\sigma) : \sigma \in \mathcal{M}_X \}$.

For $k \geq -1$ let $C^k(X)$ be the space of real valued $k$-cochains of the complex $X$ and let $d_k : C^k(X) \to C^{k+1}(X)$ be the coboundary operator. For $k \geq 0$ the reduced $k$-dimensional Laplacian of $X$ is defined by

$$L_k(X) = d_{k-1}d_{k-1}^* + d_k^*d_k.$$ 

$L_k$ is a positive semidefinite operator from $C^k(X)$ to itself. The $k$-th spectral gap of $X$, denoted by $\mu_k(X)$, is the smallest eigenvalue of $L_k$.

Let $G = (V, E)$ be a graph on $n$ vertices. Its clique complex (or flag complex) $X(G)$ is the simplicial complex on vertex set $V$ whose simplices are the cliques of $G$. Note that clique complexes are exactly the complexes with $h(X) = 1$. Indeed, the missing faces of $X(G)$ are the edges of the complement of $G$. Aharoni, Berger and Meshulam \cite{2} prove the following result:

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Theorem 1.1 (Aharoni, Berger, Meshulam [2]). Let $G = (V, E)$ be a graph, where $|V| = n$, and let $X = X(G)$ be its clique complex. Then for $k \geq 1$

$$k\mu_k(X) \geq (k + 1)\mu_{k-1}(X) - n.$$ 

Our main result is a generalization of Theorem 1.1 to complexes without large missing faces.

Theorem 1.2. Let $X$ be a simplicial complex with $h(X) = d$ on vertex set $V$, where $|V| = n$. Then for $k \geq d$

$$(k - d + 1)\mu_k(X) \geq (k + 1)\mu_{k-1}(X) - dn.$$ 

Our proof combines the approach of [2] with additional new ideas. Both results can be thought of as global variants of Garland’s method, which in its original form relates the spectral gaps of a complex with the spectral gaps of the links of its faces; See [8, 14]. As a consequence of Theorem 1.2 we obtain

Theorem 1.3. Let $X$ be a simplicial complex with $h(X) = d$, on vertex set $V$, where $|V| = n$. If

$$\mu_{d-1}(X) > \left(1 - \left(\frac{k + 1}{d}\right)^{-1}\right)n,$$

then $\tilde{H}^j(X; \mathbb{R}) = 0$ for all $d - 1 \leq j \leq k$.

Remark. In the case $d = 1$ it is shown in [2] that the condition in Theorem 1.3 is the best possible: Let $G$ be the complete $r$-partite graph on $n = \ell r$ vertices, with all sides of size $\ell$. Then $\mu_0(X(G)) = \frac{r-1}{r}n$, but $\tilde{H}^{r-1}(X(G); \mathbb{R}) \neq 0$.

For $d = 2$ we have found such extremal examples only for a few cases:

1. Let $X$ be the simplicial complex whose vertices $V$ are the points of the affine plane over $\mathbb{F}_3$, and whose missing faces are the lines of the affine plane. On the one hand, one can check that $\mu_1(X) = 6 = \frac{2}{3}|V|$. On the other hand, $\tilde{H}^2(X; \mathbb{R}) = \mathbb{R} \neq 0$ (computer checked).

2. Let $X$ be the simplicial complex whose vertices $V$ are the points of the projective space of dimension 3 over $\mathbb{F}_3$, and whose simplices are the sets of points containing at most two points from each line (so the missing faces are the subsets of size 3 of the lines in the projective space). One can show that $\mu_1(X) = 36 = \frac{9}{10}|V|$. On the other hand, $\tilde{H}^4(X; \mathbb{R}) \neq 0$ (computer checked).
We next give some applications of Theorem 1.2 to connectivity bounds and Hall-type theorems for general simplicial complexes.

Let \( \eta(X) = \text{conn}_R(X) + 2 \), where

\[
\text{conn}_R(X) = \min \{ i : \tilde{H}^i(X; \mathbb{R}) \neq 0 \} - 1
\]

is the homological connectivity of \( X \) over \( \mathbb{R} \).

A subset of vertices \( S \subset V \) in a graph \( G = (V, E) \) is called a totally dominating set if for all \( v \in V \) there is some \( u \in S \) such that \( vu \in E \). The total domination number of \( G \), denoted by \( \tilde{\gamma}(G) \), is the minimal size of a totally dominating set. Let \( I(G) \) be the independence complex of the graph, i.e. the simplicial complex whose faces are all the independent sets \( \sigma \subset V \).

The total domination number gives a lower bound on the connectivity of \( I(G) \) (see [13, Theorem 1.2]):

\[
\eta(I(G)) \geq \frac{\tilde{\gamma}(G)}{2}.
\]

(1.1)

(For additional lower bounds on \( \eta(I(G)) \) in terms of other domination parameters, see e.g. [3, 13]).

The inequality (1.1) had been generalized to general simplicial complexes: Let \( X \) be a complex on vertex set \( V \). We say that a subset \( S \subset V \) is totally dominating if for every \( v \in V \) there is some \( \sigma \subset S \) such that \( \sigma \in X \) but \( v\sigma \notin X \). The total domination number of \( X \), denoted \( \tilde{\gamma}(X) \), is the minimal size of a totally dominating set in \( X \). For a graph \( G \) we have \( \tilde{\gamma}(G) = \tilde{\gamma}(I(G)) \) (the totally dominating sets of \( I(G) \) are the same as the totally dominating sets of \( G \)). In [1] it is shown that for any simplicial complex \( X \), \( \eta(X) \geq \tilde{\gamma}(X)/2 \).

Another graphical domination parameter, \( \Gamma(G) \), has been introduced in [2] as follows. A vector representation of the graph \( G \) is an assignment \( P : V \to \mathbb{R}^\ell \) such that \( P(v) \cdot P(w) \geq 1 \) if \( v \) and \( w \) are adjacent in \( G \), and \( P(v) \cdot P(w) \geq 0 \) otherwise. A non-negative vector \( \alpha \in \mathbb{R}^V \) is called dominating for \( P \) if \( \sum_{v \in \sigma} \alpha(v) P(v) \cdot P(w) \geq 1 \) for every \( w \in V \). The value of \( P \) is

\[
|P| = \min \{ \sum_{v \in \sigma} \alpha(v) : \alpha \text{ is dominating for } P \}.
\]

Let \( \Gamma(G) \) be the supremum of \( |P| \) over all vector representations of \( G \). It is easy to check that \( \Gamma(G) \leq \tilde{\gamma}(G) \) (see Proposition 1.5). In [2] the following was proved:

**Theorem 1.4** (Aharoni, Berger, Meshulam [2]).

\[
\eta(I(G)) \geq \Gamma(G).
\]

With a view towards generalizing Theorem 1.4 to an arbitrary simplicial complex \( X \), we define a new domination parameter \( \Gamma(X) \).
For \( k \in \mathbb{N} \) let \( M_X(k) \) be the set of missing faces of \( X \) of dimension \( k \). Let 
\[
J_X = \{ i \in \mathbb{N} : M_X(i) \neq \emptyset \}
\]
be the set of dimensions of simplices in \( M_X \). Define 
\[
S(X) = \bigcup_{i \in J_X} (V_{i-1}^1)
\]

Let \( \sigma \in S(X) \) and fix \( \ell = \ell(\sigma) \in \mathbb{N} \). A *vector representation of \( X \) with respect to \( \sigma \)* is an assignment \( P_{\sigma} : V \rightarrow \mathbb{R}^\ell \) such that the inner product \( P_{\sigma}(v) \cdot P_{\sigma}(w) \geq 1 \) if \( vw\sigma \in M_X(|\sigma|+1) \), and \( P_{\sigma}(v) \cdot P_{\sigma}(w) \geq 0 \) otherwise. We identify the representation \( P_{\sigma} \) with the matrix \( P_{\sigma} \in \mathbb{R}^{|V| \times \ell} \) whose rows are the vectors \( P_{\sigma}(v) \), for \( v \in V \). We call the collection \( \{ P_{\sigma} : \sigma \in S(X) \} \) a *vector representation of \( X \).

For each \( \sigma \in S(X) \), let \( \alpha_{\sigma} \in \mathbb{R}^V \) be a non-negative vector. The set 
\[
\{ \alpha_{\sigma} : \sigma \in S(X) \}
\]
is called *dominating for \( P \)* if 
\[
\sum_{\sigma \in S(X)} \alpha_{\sigma} P_{\sigma} P_{\sigma}^T \geq \mathbf{1}
\]
(where \( \mathbf{1} \in \mathbb{R}^{|V|} \) is the all 1 vector). The *value of \( P \)* is 
\[
|P| = \min \left\{ \sum_{\sigma \in S(X)} \alpha_{\sigma} \cdot \mathbf{1} : \{ \alpha_{\sigma} \}_{\sigma \in S(X)} \text{ is dominating for } P \right\}.
\]

Let \( \Gamma(X) \) be the supremum of \( |P| \) over all vector representations \( P \) of \( X \).

**Remarks.**

1. If \( X = I(G) \) for a graph \( G \), then \( \Gamma(X) \) coincides with the parameter \( \Gamma(G) \) defined in [2].

2. In the case when all the missing faces are of the same size, we can bound \( \Gamma(X) \) by the total domination number \( \tilde{\gamma}(X) \):

**Proposition 1.5.** Let \( X \) be a simplicial complex with all its missing faces of dimension equal to \( d \). Then 
\[
\Gamma(X) \leq \left( \frac{\tilde{\gamma}(X)}{d} \right).
\]

Our main application of Theorem 1.2 is the following extension of Theorem 1.4.

**Theorem 1.6.**
\[
\sum_{r \in J_X} r \left( \frac{\eta(X)}{r} \right) \geq \Gamma(X).
\]

Let \( V_1, \ldots, V_m \) be a partition of the vertex set \( V \). We say that a subset \( \sigma \subset V \) is *colorful* if \( |\sigma \cap V_i| = 1 \) for all \( i \in \{ 1, 2, \ldots, m \} \). Theorem 1.6 gives rise to the following Hall-type condition for the existence of colorful simplices:
Theorem 1.7. If for every $\emptyset \neq I \subset \{1, 2, \ldots, m\}$

$$
\Gamma(X|\cup_{i \in I} V_i) > \sum_{r \in J[X|\cup_{i \in I} V_i]} r\binom{|I| - 1}{r},
$$

then $X$ has a colorful simplex.

Next we show an application of Theorem 1.7. Let $M$ be a matroid on vertex set $V$ with rank function $\rho$. Assume $\rho(V) = d + 1$. We identify $M$ with the simplicial complex of its independent sets. For $S \subset V$, define its closure by $\text{cl}(S) = \{ v \in V : \rho(S) = \rho(S \cup \{v\}) \}$. A subset $F \subset V$ is a flat of $M$ if $F = \text{cl}(F)$, i.e. $\rho(F \cup \{v\}) > \rho(F)$ for all $v \notin F$.

We say that a subset $S \subset V$ is in general position with respect to $M$ if for any $1 \leq k \leq d$ every flat of $M$ of rank $k$ contains at most $k$ points of $S$. This is equivalent to requiring that any $S' \subset S$ with $|S'| \leq d + 1$ is an independent set in $M$.

For $S \subset V$ denote by $\phi_M(S)$ the maximal size of a subset of $S$ in general position.

Let $V_1, \ldots, V_m$ be a partition of $V$. The following Hall-type theorem is proved in [10].

Theorem 1.8 (Holmsen, Martínez–Sandoval, Montejano [10]). If for every $\emptyset \neq I \subset \{1, 2, \ldots, m\}$

$$
\phi_M(\cup_{i \in I} V_i) > \begin{cases} 
|I| - 1 & \text{if } |I| \leq d + 1, \\
\frac{1}{d} |I|^{2|I|-2} & \text{if } |I| \geq d + 2,
\end{cases}
$$

then $V$ has a colorful subset in general position.

Let $S \subset V$. A weight function $f : S \to \mathbb{R}_{\geq 0}$ is in fractional general position with respect to $M$ if for any $1 \leq k \leq d$ and for any flat $F$ of $M$ of rank $k$ and $\sigma \subset F \cap S$ of size $k - 1$,

$$
\sum_{v \in S, \text{cl}(v\sigma) = F} f(v) \leq d.
$$

Denote by $\phi_M^*(S)$ the maximum of $\sum_{v \in S} f(v)$ over all functions $f : S \to \mathbb{R}_{\geq 0}$ in fractional general position. Let $f$ be the characteristic function of a set $S' \subset S$ in general position. Let $F$ be a flat of $M$ of rank $k$ for $1 \leq k \leq d$ and $\sigma \subset F \cap S$ of size $k - 1$. Then

$$
\sum_{v \in S, \text{cl}(v\sigma) = F} f(v) = \left| \{ v \in S' : \text{cl}(v\sigma) = F \} \right| \leq |S' \cap F| \leq k \leq d,
$$

so $f$ is in fractional general position. Therefore

$$
\phi_M^*(S) \geq \phi_M(S). \quad (1.2)
$$

Here we prove the following:
Theorem 1.9. If for every $\emptyset \neq I \subset \{1, 2, \ldots, m\}$
\[ \varphi^*_M((\cup_{i \in I} V_i)) > d \sum_{r=1}^{d} r\binom{|I| - 1}{r}, \]
then $V$ contains a colorful subset in general position.

In particular, we obtain a strengthening of Theorem 1.8:

Theorem 1.10. If for every $\emptyset \neq I \subset \{1, 2, \ldots, m\}$
\[ \varphi_M((\cup_{i \in I} V_i)) > \begin{cases} |I| - 1 & \text{if } |I| \leq d + 1, \\ d \sum_{r=1}^{d} r\binom{|I| - 1}{r} & \text{if } |I| \geq d + 2, \end{cases} \]
then $V$ contains a colorful subset in general position.

The paper is organized as follows. In Section 2 we review some basic facts concerning simplicial cohomology and high dimensional Laplacians. We also introduce some notation and results about complexes without large missing faces that we will need later. In Section 3 we prove our main result, Theorem 1.2 and its corollary Theorem 1.3. Section 4 deals with the vector domination parameter $\Gamma(X)$ of the complex $X$. In it we prove Proposition 1.5, Theorem 1.6 and Theorem 1.7. In Section 5 we apply the results of the previous section in order to prove Theorems 1.9 and 1.10, which provide sufficient conditions for the existence of colorful sets in general position in a matroid.

2 Preliminaries

2.1 Simplicial cohomology

Let $X$ be a finite simplicial complex on the vertex set $V$. We denote the set of $k$-dimensional simplices in $X$ by $X(k)$. For each $\sigma \in X(k)$ we choose an order of its vertices $v_0, \ldots, v_k$, which induces an orientation on $\sigma$.

For $\sigma \in X(k)$, let $\text{lk}(X, \sigma) = \{ \tau \in X : \tau \cup \sigma \in X, \tau \cap \sigma = \emptyset \}$ be the link of $\sigma$ in $X$, and $\text{deg}_X(\sigma) = |\{ \eta \in X(k+1) : \sigma \subset \eta \}|$ be the degree of $\sigma$ in $X$. For $U \subset V$, let $X[U] = \{ \sigma \in X : \sigma \subset U \}$ be the subcomplex of $X$ induced by $U$.

For two ordered simplices $\sigma \in X$, $\tau \in \text{lk}(X, \sigma)$, denote by $[\sigma, \tau]$, or simply by $\sigma \tau$, their ordered union. Similarly, for $v \in V$ denote by $v \sigma$ the ordered union of $\{v\}$ and $\sigma$.

For $\sigma \in X$, and $\tau \subset \sigma$, both given an order on their vertices, we define $(\sigma : \tau)$ to be the sign of the permutation on the vertices of $\sigma$ which maps the ordered simplex $\sigma$ to the ordered simplex $[\sigma \setminus \tau, \tau]$ (where the order on the vertices of $\sigma \setminus \tau$ is the one induced by the order on $\sigma$).

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A simplicial $k$-cochain is a real valued skew-symmetric function on all ordered $k$-simplices. That is, $\phi$ is a $k$-cochain if for any two $k$-simplices $\sigma, \tilde{\sigma}$ in $X$ that are equal as sets, it satisfies $\phi(\tilde{\sigma}) = (\sigma: \phi(\sigma))$.

For $k \geq 0$ let $C^k(X)$ denote the space of $k$-cochains on $X$. For $k = -1$ we define $C^{-1}(X) = \mathbb{R}$.

We will use the following lemma implicitly in future calculations.

**Lemma 2.1.** Let $\tau, \eta \in X(k)$ and $\phi \in C^k(X)$. Let $\sigma, \theta \in X$ be ordered simplices such that $\tau, \eta \subset \sigma$ and $\theta \subset \tau \cap \eta$, and let $\tilde{\sigma}, \tilde{\tau}, \tilde{\eta}, \tilde{\theta}$ be equal as sets to $\sigma, \tau, \eta$ and $\theta$ respectively. Then

1. $(\sigma : \tau) = (\sigma : \tilde{\tau}) \cdot (\tilde{\tau} : \tau)$, and if $|\sigma \setminus \tau| = 1$ then $(\sigma : \tau) = (\sigma : \tilde{\sigma}) \cdot (\tilde{\sigma} : \tau)$.
2. $\phi(\tau)^2 = \phi(\tilde{\tau})^2$.
3. $(\sigma : \tau)\phi(\tau) = (\sigma : \tilde{\tau})\phi(\tilde{\tau})$, and if $|\tau \setminus \theta| = 1$ then $(\tau : \theta)\phi(\tau) = (\tilde{\tau} : \theta)\phi(\tilde{\tau})$.
4. If $|\sigma \setminus \tau| = 1$ and $|\sigma \setminus \eta| = 1$ then $(\sigma : \tau)(\sigma : \eta)\phi(\tau)\phi(\eta) = (\tilde{\sigma} : \tilde{\tau})(\tilde{\sigma} : \tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta})$.
5. If $|\tau \setminus \theta| = 1$ and $|\eta \setminus \theta| = 1$ then $(\tau : \theta)(\eta : \theta)\phi(\tau)\phi(\eta) = (\tilde{\tau} : \tilde{\theta})(\tilde{\eta} : \tilde{\theta})\phi(\tilde{\tau})\phi(\tilde{\eta})$.

**Proof.**

1. Let $\pi_1$ be the permutation on the vertices of $\sigma$ that maps $\sigma$ to $[\sigma \setminus \tilde{\tau}, \tilde{\tau}] = [\sigma \setminus \tau, \tilde{\tau}]$, and let $\pi_2$ be the permutation on the vertices of $\tau$ that maps $\tilde{\tau}$ to $\tau$. Extend $\pi_2$ to a permutation $\tilde{\pi}_2$ on the vertices of $\sigma$, which maps $[\sigma \setminus \tau, \tilde{\tau}]$ to $[\sigma \setminus \tau, \tau]$. It satisfies $\text{sign}(\pi_2) = \text{sign}(\tilde{\pi}_2)$. Define $\pi = \tilde{\pi}_2 \circ \pi_1$. $\pi$ maps $\sigma$ to $[\sigma \setminus \tau, \tau]$, therefore

\[
(\sigma : \tau) = \text{sign}(\pi) = \text{sign}(\tilde{\pi}_2) \cdot \text{sign}(\pi_1) \]

\[
= \text{sign}(\pi_2) \cdot \text{sign}(\pi_1) = (\tilde{\tau} : \tau) \cdot (\sigma : \tilde{\tau}).
\]

Assume now that $|\sigma \setminus \tau| = 1$ and let $\{v\} = \sigma \setminus \tau$. Let $\pi_3$ be the permutation on the vertices of $\sigma$ that maps $\sigma$ to $\tilde{\sigma}$, and $\pi_4$ be the permutation which maps $\tilde{\sigma}$ to $[\tilde{\sigma} \setminus \tau, \tau] = v\tau = [\sigma \setminus \tau, \tau]$. Then the permutation $\pi' = \pi_4 \circ \pi_3$ maps $\sigma$ to $[\sigma \setminus \tau, \tau]$, therefore

\[
(\sigma : \tau) = \text{sign}(\pi') = \text{sign}(\pi_4) \cdot \text{sign}(\pi_3) = (\sigma : \tilde{\sigma}) \cdot (\tilde{\sigma} : \tau).
\]

2. Since $\phi$ is a cochain, we have $\phi(\tau)^2 = (\tau : \tilde{\tau})^2 \phi(\tilde{\tau})^2 = \phi(\tilde{\tau})^2$. 

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3. By the first part of this lemma

\[(\sigma : \tau)\phi(\tau) = (\sigma : \tilde{\tau})(\tilde{\tau} : \tau)\phi(\tau),\]

and since \(\phi\) is a cochain

\[(\sigma : \tilde{\tau})(\tilde{\tau} : \tau)\phi(\tau) = (\sigma : \tilde{\tau})\phi(\tilde{\tau}).\]

The second equality is similar: By the first part of the lemma

\[(\tau : \theta)\phi(\tau) = (\tau : \tilde{\tau})(\tilde{\tau} : \theta)\phi(\tau),\]

and since \(\phi\) is a cochain

\[(\tau : \tilde{\tau})(\tilde{\tau} : \theta)\phi(\tau) = (\tau : \tilde{\tau})\phi(\tilde{\tau}).\]

4. By part 3 of this lemma we have

\[(\sigma : \tau)(\sigma : \eta)\phi(\tau)\phi(\eta) = (\sigma : \tilde{\tau})(\sigma : \tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta}).\]

Then by part 1

\[(\sigma : \tilde{\tau})(\sigma : \tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta}) = (\tilde{\sigma} : \tilde{\tau})(\tilde{\sigma} : \tilde{\eta})\phi(\tilde{\tau})\phi(\tilde{\eta}) = (\tilde{\sigma} : \tilde{\tau})\phi(\tilde{\tau})\phi(\tilde{\eta}).\]

5. The proof is similar to the proof of part 4.

\[\square\]

For \(k \geq 0\) let the coboundary operator \(d_k : C^k(X) \to C^{k+1}(X)\) be the linear operator defined by

\[d_k \phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i),\]

where for an ordered \((k+1)\)-simplex \(\sigma = [v_0, \ldots, v_{k+1}]\), \(\sigma_i\) is the ordered simplex obtained by removing the vertex \(v_i\), that is \(\sigma_i = [v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}]\). Equivalently, we can write

\[d_k \phi(\sigma) = \sum_{\tau \in \sigma(k)} (\sigma : \tau)\phi(\tau),\]

where \(\sigma(k) \subset X(k)\) is the set of all \(k\)-dimensional faces of \(\sigma\), each given some fixed order on its vertices.

For \(k = -1\) we define \(d_{-1} : C^{-1}(X) = \mathbb{R} \to C^0(X)\) by \(d_{-1} a(v) = a\), for every \(a \in \mathbb{R}, v \in V\).

Let \(\tilde{H}^k(X; \mathbb{R}) = \text{Ker}(d_k)/\text{Im}(d_{k-1})\) be the \(k\)-th reduced cohomology group of \(X\) with real coefficients.
2.2 Higher Laplacians

For each $k \geq -1$ we define an inner product on $C^k(X)$ by

$$\langle \phi, \psi \rangle = \sum_{\sigma \in X(k)} \phi(\sigma) \psi(\sigma).$$

This induces a norm on $C^k(X)$:

$$\|\phi\| = \left(\sum_{\sigma \in X(k)} \phi(\sigma)^2\right)^{1/2}.$$

Let $d^*_k : C^{k+1}(X) \to C^k(X)$ be the adjoint of $d_k$ with respect to this inner product. We can write $d^*_k \phi$ explicitly:

$$d^*_k \phi(\tau) = \sum_{v \in \text{lk}(X, \tau)} \phi(v \tau).$$

For $k \geq 0$ define the lower $k$-Laplacian of $X$ by $L^-_k(X) = d_{k-1} d^*_k - d^*_k d_k$. The reduced $k$-Laplacian of $X$ is the positive semidefinite operator on $C^k(X)$ given by $L_k(X) = L^-_k(X) + L^+_k(X)$.

Let $k \geq 0$ and $\sigma \in X(k)$. We define the $k$-cochain $1^e_\sigma$ by

$$1^e_\sigma(\tau) = \begin{cases} (\sigma : \tau) & \text{if } \sigma = \tau \text{ (as sets)}, \\ 0 & \text{otherwise}. \end{cases}$$

The set $\{1^e_\sigma\}_{\sigma \in X(k)}$ forms a basis of the space $C^k(X)$, which we will call the standard basis.

For a linear operator $T : C^k(X) \to C^k(X)$, let $[T]$ be the matrix representation of $T$ with respect to the standard basis. We denote by $[T]_{\sigma, \tau}$ the matrix element of $[T]$ at index $(1^e_\sigma, 1^e_\tau)$.

One can write explicitly the matrix representation of the Laplacian operators in the standard basis (see e.g. [6, 9]):

**Claim 2.2.** For $k \geq 0$

$$[L^-_k]_{\sigma, \tau} = \begin{cases} k + 1 & \text{if } \sigma = \tau, \\ (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, \\ 0 & \text{otherwise}, \end{cases}$$

$$[L^+_k]_{\sigma, \tau} = \begin{cases} \text{deg}_X(\sigma) & \text{if } \sigma = \tau, \\ -(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, \sigma \cup \tau \in X(k+1), \\ 0 & \text{otherwise}, \end{cases}$$

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and

\[ [L_k]_{\sigma,\tau} = \begin{cases} 
  k + 1 + \deg_X(\sigma) & \text{if } \sigma = \tau, \\
  (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, \sigma \cup \tau \notin X(k + 1), \\
  0 & \text{otherwise.}
\end{cases} \]

The following upper bound on the eigenvalues of the Laplacian is implicit in [6]:

**Lemma 2.3.** Let \( X \) be a simplicial complex on vertex set \( V \), with \(|V| = n\). Let \( k \geq 0 \) and let \( \lambda \) be an eigenvalue of \( L_k(X) \). Then

\[ \lambda \leq n. \]

The following discrete version of Hodge’s theorem had been observed by Eckmann in [7].

**Theorem 2.4 (Simplicial Hodge theorem).**

\[ \tilde{H}^k(X; \mathbb{R}) \cong \text{Ker} L_k. \]

As a consequence of Hodge theorem we obtain

**Corollary 2.5.** \( \tilde{H}^k(X; \mathbb{R}) = 0 \) if and only if \( \mu_k(X) > 0 \).

### 2.3 Missing faces and sums of degrees

Let \( X \) be a complex on vertex set \( V \) with \( h(X) = d \). Let \( k \geq d \) and \( \theta \in \binom{V}{k+1} \).

Define

\[ T(\theta) = \left\{ \tau \in \binom{\theta}{d+1} : \tau \notin X(d) \right\}. \]

So \( T(\theta) \) is the set of all \( d \)-dimensional simplices in \( \theta \) that do not belong to \( X \), and \( \theta \in X \) if and only if \( T(\theta) = \emptyset \). Let

\[ \text{Mis}(\theta) = \bigcap_{\tau \in T(\theta)} \tau \]

and

\[ m(\theta) = \left| \bigcap_{\tau \in T(\theta)} \tau \right|. \]

Since every \( \tau \in T(\theta) \) has \( d+1 \) vertices it follows that \( m(\theta) \leq d+1 \). Another simple observation is the following:

**Lemma 2.6.** Let \( \sigma, \tau \in X(k) \) such that \(|\tau \cap \sigma| = k\). Then if \( \sigma \cup \tau \in X(k+1) \), \( m(\sigma \cup \tau) = 0 \), otherwise \( 2 \leq m(\sigma \cup \tau) \leq d+1 \).
Proof. Denote $\sigma \setminus \tau = \{v\}$ and $\tau \setminus \sigma = \{w\}$. If $\sigma \cup \tau \in X(k+1)$ then $T(\sigma \cup \tau) = \emptyset$, therefore $m(\sigma \cup \tau) = 0$. If $\sigma \cup \tau \notin X(k+1)$, then every $\eta \in T(\sigma \cup \tau)$ must contain both $v$ and $w$ (otherwise $\eta$ will be contained in $\sigma$ or in $\tau$, a contradiction to $\eta \notin X(d)$). Therefore $m(\sigma \cup \tau) \geq 2$. \hfill \Box

The following is a known result about clique complexes (see [2, Claim 3.4], [4]):

**Lemma 2.7.** Let $X$ be a clique complex with $n$ vertices and let $\sigma \in X(k)$. Then
\[
\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) - k \deg_X(\sigma) \leq n.
\]

We will need a version of this lemma for complexes without large missing faces:

**Lemma 2.8.** Let $X$ be a simplicial complex on vertex set $V$ with $h(X) = d$. Let $k \geq d$ and $\sigma \in X(k)$. Then
\[
\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = k + 1 + (k+1) \deg_X(\sigma) + \sum_{r=2}^{d+1} (r-1) \cdot \left| \{ v \in V : m(v\sigma) = r \} \right|.
\]

**Proof.**
\[
\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = \sum_{\tau \in \sigma(k-1)} \sum_{v \in \text{lk}(X,\tau)} 1 = \sum_{v \in V} \sum_{\tau \in \sigma(k-1), \tau \in \text{lk}(X,v)} 1
\]
\[
= \sum_{v \in \sigma} \sum_{\tau \in \sigma(k-1), \tau \in \text{lk}(X,v)} 1 + \sum_{v \in \text{lk}(X,\sigma)} \sum_{\tau \in \sigma(k-1), \tau \in \text{lk}(X,v)} 1 + \sum_{v \in V \setminus \sigma} \sum_{\tau \in \sigma(k-1), \tau \in \text{lk}(X,v)} 1. \tag{2.1}
\]

We consider separately the three summands on the right hand side of (2.1):

1. For $v \in \sigma$, there is only one $\tau \in \sigma(k-1)$ such that $\tau \in \text{lk}(X,v)$, namely $\tau = \sigma \setminus \{v\}$. Thus the first summand is $k+1$.

2. For $v \in \text{lk}(X,\sigma)$, any $\tau \in \sigma(k-1)$ is in $\text{lk}(X,v)$, therefore the second summand is $(k+1) \deg_X(\sigma)$.

3. Let $v \in V \setminus \sigma$ such that $v \notin \text{lk}(X,\sigma)$. Let $\tau \in \sigma(k-1)$ and let $u$ be the unique vertex in $\sigma \setminus \tau$. If $\tau \in \text{lk}(X,v)$ then every missing face of $X$ contained in $v\sigma$ must contain $u$, so $u \in \text{Mis}(v\sigma)$. If $\tau \notin \text{lk}(X,v)$, then there is a missing face of $X$ contained in $v\tau$, and therefore it doesn’t contain the vertex $u$. Hence, $u \notin \text{Mis}(v\sigma)$. Since $v \in \text{Mis}(v\sigma)$, the
number of \( \tau \in \sigma(k - 1) \) such that \( \tau \in \text{lk}(X, v) \) is exactly \( m(v\sigma) - 1 \). Hence the third summand is

\[
\sum_{v \in V \setminus \sigma, v \notin \text{lk}(X, \sigma)} (m(v\sigma) - 1) = \sum_{r=2}^{d+1} (r - 1) \left| \{ v \in V : m(v\sigma) = r \} \right|.
\]

We obtain

\[
\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = k + 1 + (k + 1) \deg_X(\sigma) + \sum_{r=2}^{d+1} (r - 1) \left| \{ v \in V : m(v\sigma) = r \} \right|.
\]

\[\square\]

### 3 Spectral gaps

In this section we prove Theorems 1.2 and 1.3.

Let \( X \) be a simplicial complex with \( h(X) = d \) on vertex set \( V \), where \( |V| = n \), and let \( k \geq d \). For \( \phi \in C^k(X) \) and \( u \in V \) we define \( \phi_u \in C^{k-1}(X) \) by

\[
\phi_u(\tau) = \begin{cases} 
\phi(u\tau) & \text{if } u \in \text{lk}(X, \tau), \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( B_k : C^k(X) \to C^k(X) \) be the linear transformation whose matrix representation in the standard basis is

\[
[B_k]_{\tau, \sigma} = \begin{cases} 
k \deg_X(\sigma) - \sum_{\eta \in \sigma(k-1)} \deg_X(\eta) & \text{if } \sigma = \tau, \\
(m(\sigma \cup \tau) - 2)(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, \sigma \cup \tau \notin X(k+1), \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( R_k = (d-1)L_k - B_k \), and let \( \lambda_k \) be the largest eigenvalue of \( R_k \).

The proof of Theorem 1.2 depends on the following two ingredients:

**Proposition 3.1.** Let \( \phi \in C^k(X) \). Then

\[
(k - d + 1) \langle L_k \phi, \phi \rangle = \sum_{u \in V} \langle L_{k-1} \phi_u, \phi_u \rangle - \langle R_k \phi, \phi \rangle.
\]

**Proposition 3.2.** \( \lambda_k \leq dn \).

We postpone the proof of these propositions to the end of this section, and first show how they imply Theorem 1.2.
Proof of Theorem 1.2. Let \( 0 \neq \phi \in C^k(X) \) be an eigenvector of \( L_k \) with eigenvalue \( \mu_k(X) \). By Proposition 3.1 we obtain

\[
(k - d + 1)\mu_k(X) \|\phi\|^2 = (k - d + 1) \langle L_k \phi, \phi \rangle = \sum_{u \in V} \langle L_{k-1} \phi_u, \phi_u \rangle - \langle R_k \phi, \phi \rangle \geq \mu_{k-1}(X) \sum_{u \in V} \|\phi_u\|^2 - \lambda_k \|\phi\|^2.
\]

But

\[
\sum_{u \in V} \|\phi_u\|^2 = \sum_{u \in V} \sum_{\tau \in X(k-1)} \phi_u(\tau)^2 = \sum_{\tau \in X(k-1)} \sum_{u \in \text{lk}(X,\tau)} \phi(u\tau)^2
= (k + 1) \sum_{\sigma \in X(k)} \phi(\sigma)^2 = (k + 1) \|\phi\|^2.
\]

Therefore

\[
(k - d + 1)\mu_k(X) \geq (k + 1)\mu_{k-1}(X) - \lambda_k,
\]

and by Proposition 3.2

\[
(k - d + 1)\mu_k(X) \geq (k + 1)\mu_{k-1}(X) - dn.
\]

For the proof of Theorem 1.3 we will need the following result, which will also be used in Section 4.

Claim 3.3. For \( k \geq d - 1 \),

\[
\mu_k(X) \geq \binom{k+1}{d} \mu_{d-1}(X) - \binom{k+1}{d} n. \tag{3.1}
\]

If in addition \( X \) has complete \( (d - 1) \)-dimensional skeleton, then there is equality in (3.1) for \( 0 \leq k \leq d - 1 \).

Proof. We argue by induction on \( k \). The case \( k = d - 1 \) is clear. Let \( k \geq d \). By Theorem 1.2 and the induction hypothesis we obtain

\[
\mu_k(X) \geq \frac{k+1}{k-d+1} \mu_{k-1}(X) - \frac{d}{k-d+1} n
\]

\[
\geq \frac{k+1}{k-d+1} \left[ \binom{k}{d} \mu_{d-1}(X) - \binom{k}{d} n \right] - \frac{d}{k-d+1} n
= \binom{k+1}{d} \mu_{d-1}(X) - \binom{k+1}{d} n.
\]

Now assume that \( X \) has complete \( (d - 1) \)-dimensional skeleton, and let \( k < d - 1 \). Then we have \( \binom{k+1}{d} = 0 \), therefore the inequality in the claim is just \( \mu_k(X) \geq n \). But one can see by Claim 2.2 that in this case \( L_k \) is the scalar matrix with diagonal elements \( n \), thus \( \mu_k(X) = n \). \( \square \)
Proof of Theorem 1.3. Let \( d - 1 \leq j \leq k \). We have by Claim 3.3

\[
\mu_j(X) \geq \binom{j + 1}{d} \mu_{d-1}(X) - \left( \binom{j + 1}{d} - 1 \right) n
\]

\[
> \binom{j + 1}{d} \cdot \left( 1 - \binom{k + 1}{d} \right) n - \left( \binom{j + 1}{d} - 1 \right) n
\]

\[
\geq \binom{j + 1}{d} \cdot \left( 1 - \left( \frac{j + 1}{d} \right)^{k+1} \right) n - \left( \frac{j + 1}{d} \right) n = 0.
\]

Thus, by Corollary 2.5, \( \tilde{H}^j(X; \mathbb{R}) = 0 \). \( \square \)

In order to prove Proposition 3.1 we will need the following claims.

**Claim 3.4** (see [2, Claim 3.1]). For \( \phi \in C^k(X) \)

\[
\| d_k \phi \|^2 = \sum_{\sigma \in X(k)} \deg X(\sigma) \phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in \text{lk}(X, \eta)} \phi(v \eta) \phi(w \eta).
\]

**Proof.**

\[
\| d_k \phi \|^2 = \sum_{\tau \in X(k+1)} d_k \phi(\tau)^2
\]

\[
= \sum_{\tau \in X(k+1)} \left( \sum_{\theta_1 \in \tau(k)} (\tau : \theta_1) \phi(\theta_1) \right) \left( \sum_{\theta_2 \in \tau(k)} (\tau : \theta_2) \phi(\theta_2) \right)
\]

\[
= \sum_{\tau \in X(k+1)} \sum_{\sigma \in \tau(k)} \phi(\sigma)^2
\]

\[
+ \sum_{\tau \in X(k+1)} \sum_{\theta_1 \in \tau(k)} \sum_{\theta_2 \in \tau(k), \theta_2 \neq \theta_1} (\tau : \theta_1)(\tau : \theta_2) \phi(\theta_1) \phi(\theta_2)
\]

\[
= \sum_{\sigma \in X(k)} \deg(\sigma) \phi(\sigma)^2
\]

\[
+ \sum_{\tau \in X(k+1)} \sum_{\theta_1 \in \tau(k)} \sum_{\theta_2 \in \tau(k), \theta_2 \neq \theta_1} (\tau : \theta_1)(\tau : \theta_2) \phi(\theta_1) \phi(\theta_2).
\]

Now look at the map

\[
\left\{ (\eta, v, w) : \eta \in X(k-1), v, w \in V, v \neq w, \ vw \in \text{lk}(X, \eta) \right\} \rightarrow \left\{ (\tau, \theta_1, \theta_2) : \tau \in X(k+1), \ \theta_1, \theta_2 \in \tau(k), \theta_1 \neq \theta_2 \right\}
\]

defined by \((\eta, v, w) \mapsto (v \eta, v \eta, w \eta)\). For each \((\tau, \theta_1, \theta_2)\) in the codomain, let \( \eta = \theta_1 \cap \theta_2 \), \( \{v\} = \theta_1 \setminus \theta_2 \) and \( \{w\} = \theta_2 \setminus \theta_1 \). \((\eta, v, w)\) is the unique
Let $\eta$ split into two different cases: $u$ or $u \neq k$. So the map is a bijection, therefore we obtain

$$
\|d_k\phi\|^2 = \sum_{\sigma \in X(k)} \deg_X(\sigma)\phi(\sigma)^2
+ \sum_{\eta \in X(k-1)} \sum_{v \in V \setminus \{v\}} \sum_{w \in \text{lk}(X,\eta)} (vw : \eta)(vw : \eta)\phi(vw)\phi(\eta)
= \sum_{\sigma \in X(k)} \deg_X(\sigma)\phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in \text{lk}(X,\eta)} \phi(vw)\phi(\eta).
$$

\[ \square \]

**Claim 3.5.** For $\phi \in C^k(X)$

$$
\sum_{u \in V} \|d_{k-1}\phi_u\|^2 = \sum_{\sigma \in X(k)} \sum_{\tau \in \sigma(k-1)} \deg_X(\tau)\phi(\sigma)^2
- 2k \sum_{\tau \in X(k-1)} \sum_{vw \in \text{lk}(X,\tau)} \phi(v\tau)\phi(w\tau)
- 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X,\eta)} \sum_{u \in \text{lk}(X,vw)} \phi(vu)\phi(\eta) \cdot \phi(vw).
$$

**Proof.** First we apply Claim 3.4 to $\phi_u \in C^{k-1}(X)$:

$$
\|d_{k-1}\phi_u\|^2 = \sum_{\tau \in X(k-1)} \deg_X(\tau)\phi_u(\tau)^2 - 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X,\eta)} \phi_u(vw)\phi_u(\eta).
$$

Summing over all vertices we obtain

$$
\sum_{u \in V} \|d_{k-1}\phi_u\|^2 = \sum_{u \in V} \sum_{\tau \in X(k-1)} \deg_X(\tau)\phi_u(\tau)^2
- 2 \sum_{u \in V} \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X,\eta)} \phi_u(vw)\phi_u(\eta)
= \sum_{u \in V} \sum_{\tau \in X(k-1)} \deg_X(\tau)\phi(u\tau)^2
- 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X,\eta)} \sum_{u \in \text{lk}(X,vw)} \phi(vu)\phi(\eta) \cdot \phi(vw)
= \sum_{\sigma \in X(k)} \sum_{\tau \in \sigma(k-1)} \deg_X(\tau)\phi(\sigma)^2
- 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X,\eta)} \sum_{u \in \text{lk}(X,vw)} \phi(vu)\phi(\eta) \cdot \phi(vw).
$$

Let $\eta \in X(k-2)$, $vw \in \text{lk}(X,\eta)$, and $u \in \text{lk}(X,vw) \cap \text{lk}(X,\eta)$. We split into two different cases: $u \in \text{lk}(X,vw)$ or $u \notin \text{lk}(X,vw)$. Assume
\[ u \in \text{lk}(X, vw\eta), \text{ and let } \tau = u\eta. \text{ Then we have } vw \in \text{lk}(X, \tau). \text{ This defines a map } \]
\[ \{ (\eta, vw, u) : \eta \in X(k-2), u \in \text{lk}(X, \eta), \} \rightarrow \{ (\tau, vw) : \tau \in X(k-1), \} \].

Each pair \((\tau, vw)\) has a preimage of size \(k\) (these are the tuples \((\tau \setminus u, vw, u)\) for each \(u \in \tau\)). Therefore we obtain
\[
\sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X, \eta)} \sum_{u \in \text{lk}(X, \eta)} \deg_X(\tau) \phi(\sigma)^2
- 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X, \eta)} \sum_{u \in \text{lk}(X, \eta)} \phi(vu\eta) \phi(wu\eta)
- 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X, \eta)} \sum_{u \in \text{lk}(X, \eta) \cap \text{lk}(X, w\eta)} \phi(vu\eta) \phi(wu\eta)
= \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X, \eta)} \sum_{u \in \text{lk}(X, \eta) \cap \text{lk}(X, w\eta)} \phi(vu\eta) \phi(wu\eta).
\]

**Remark.** If \(X\) is a clique complex and \(u \in \text{lk}(X, \eta) \cap \text{lk}(X, w\eta)\) for \(\eta \in X(k-2)\) and \(vw \in \text{lk}(X, \eta)\), then all the 1-dimensional faces of the simplex \(uvw\eta\) belong to \(X\), therefore \(uvw\eta \in X\) (i.e. \(u \in \text{lk}(X, vw\eta)\)). Therefore in this case the last term of the previous equation vanishes (see \[2, Claim 3.2\]).

**Claim 3.6 (see \[2, Claim 3.3\]).** For \(\phi \in \mathcal{C}^k(X)\)
\[
\sum_{u \in V} \|d^*_{k-1}\phi_u\|^2 = k \|d^*_{k-1}\phi\|^2.
\]

**Proof.**
\[
\|d^*_{k-1}\phi\|^2 = \sum_{\tau \in X(k-1)} d^*_{k-1}\phi(\tau)^2 = \sum_{\tau \in X(k-1)} \left( \sum_{v\in\text{lk}(X,\tau)} \phi(v\tau) \right)^2.
\]
Similarly,

\[
\sum_{u \in V} \left\| d_{k-2}^* \phi_u \right\|^2 = \sum_{u \in V} \sum_{\eta \in X(k-2)} \left( \sum_{v \in \text{lk}(X, \eta)} \phi_u(v\eta) \right)^2 
\]

\[
= \sum_{\eta \in X(k-2)} \sum_{u \in \text{lk}(X, \eta)} \left( \sum_{v \in \text{lk}(X, u\eta)} \phi(vu\eta) \right)^2 
\]

\[
= k \sum_{\tau \in X(k-1)} \left( \sum_{v \in \text{lk}(X, \tau)} \phi(v\tau) \right)^2 = k \left\| d_{k-1}^* \phi \right\|^2.
\]

\[\square\]

Let \( A_k : C^k(X) \to C^k(X) \) be the linear transformation whose matrix representation in the standard basis is

\[
[A_k]_{\sigma, \tau} = \begin{cases} 
(m(\sigma \cup \tau) - 2) \cdot (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) \cdot \phi(\tau) \phi(\sigma) & \text{if } |\sigma \cap \tau| = k, \sigma \cup \tau \notin X(k+1), \\
0 & \text{otherwise}.
\end{cases}
\]

Claim 3.7. For \( \phi \in C^k(X) \)

\[
\langle A_k \phi, \phi \rangle = 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(X, \eta)} \sum_{u \in \text{lk}(X, v\eta) \cap \text{lk}(X, w\eta), \ u \notin \text{lk}(X, vw\eta)} \phi(vu\eta) \phi(wu\eta).
\]

Proof.

\[
\langle A_k \phi, \phi \rangle 
= \sum_{\tau \in X(k)} \sum_{\sigma \in X(k), |\sigma \cap \tau| = k, \sigma \cup \tau \notin X(k+1)} (m(\sigma \cup \tau) - 2)(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) \cdot \phi(\tau) \phi(\sigma)
\]

\[
= \sum_{\theta \in X(k-1)} \sum_{v \in \text{lk}(X, \theta), \ w \in \text{lk}(X, \theta), \ v < w, \ vw \notin \text{lk}(X(k+1))} (m(vw\theta) - 2) \phi(v\theta) \phi(w\theta).
\]

Let \( < \) be an order on the vertices of \( X \). Look at the map

\[
\{ (\eta, u, vw) : \eta \in X(k-2), u \in \text{lk}(X, \eta), \ vw \in \text{lk}(X, u\eta), \ u \notin \text{lk}(X, vw\eta) \} \to \{ (\theta, v, w) : \theta \in X(k-1), \ v, w \in \text{lk}(X, \theta), \ v < w, \ vw \notin \text{lk}(X(k+1)) \}
\]

defined by \((\eta, u, vw) \mapsto (u\eta, v, w)\). Note that for any \((\eta, u, vw) \) in the domain, we must have \(u, v, w \in \text{Mis}(uvw\eta)\). Let \((\theta, v, w) \) in the codomain,
and let \( u \in \text{Mis}(vw\theta) \setminus \{v, w\} \) and \( \eta = \theta \setminus \{u\} \). Then \( vw \in \text{lk}(X, \eta) \) (since \( vwn \) doesn’t contain \( u \), therefore can’t contain any missing face). Similarly, \( u \in \text{lk}(X, v\eta) \cap \text{lk}(X, w\eta) \), but \( u \notin \text{lk}(X, vw\eta) \) (otherwise \( uvw\eta = vw\theta \in X(k+1) \)). Therefore \( (\eta, u, vw) \) is in the preimage of \( (\theta, v, w) \). Hence \( (\theta, v, w) \) has a preimage of size \( m(vw\theta) - 2 \). So we have

\[
\langle A_k \phi, \phi \rangle = \sum_{\theta \in X(k-1)} \sum_{\nu \in \text{lk}(X, \theta)} \sum_{\nu \notin \text{lk}(X, \theta)(k+1)} (m(vw\theta) - 2) \phi(v\theta) \phi(w\theta)
\]

\[
= 2 \sum_{\theta \in X(k-1)} \sum_{\nu \in \text{lk}(X, \theta)}, \nu < w, \nu \notin \text{lk}(X(k+1)) \sum_{m(vw\theta) - 2) \phi(v\theta) \phi(w\theta)
\]

\[
= 2 \sum_{\eta \in X(k-2)} \sum_{\nu \in \text{lk}(X, \eta)} \sum_{\nu \notin \text{lk}(X, \nu\eta)} \phi(v\nu\eta) \phi(w\nu\eta).
\]

**Proof of Proposition 3.7** Let \( \phi \in C^k(X) \). By Claim 3.7 we have

\[
\langle B_k \phi, \phi \rangle = \sum_{\sigma \in X(k)} \left( k \deg_X(\sigma) - \sum_{\tau \in \sigma(k-1)} \deg_X(\tau) \right) \phi(\sigma)^2 + 2 \sum_{\eta \in X(k-2)} \sum_{\nu \in \text{lk}(X, \eta)} \sum_{\nu \notin \text{lk}(X, \nu\eta)} \phi(v\nu\eta) \phi(w\nu\eta).
\]

By Claims 3.3 and 3.5 we obtain

\[
k \|d_k \phi\|^2
\]

\[
= \sum_{u \in V} \|d_{k-1}\phi_u\|^2 + \sum_{\sigma \in X(k)} \left( k \deg_X(\sigma) - \sum_{\tau \in \sigma(k-1)} \deg_X(\tau) \right) \phi(\sigma)^2 + 2 \sum_{\eta \in X(k-2)} \sum_{\nu \in \text{lk}(X, \eta)} \sum_{\nu \notin \text{lk}(X, \nu\eta)} \phi(v\nu\eta) \phi(w\nu\eta)
\]

\[
= \sum_{u \in V} \|d_{k-1}\phi_u\|^2 + \langle B_k \phi, \phi \rangle.
\]

Then by the previous equation and Claim 3.6 we obtain

\[
k \langle L_k \phi, \phi \rangle = k \langle d_k \phi + d_{k-1}d_k \phi, \phi \rangle = k \|d_k \phi\|^2 + k \|d_{k-1} \phi\|^2
\]

\[
= \sum_{u \in V} \|d_{k-1}\phi_u\|^2 + \langle B_k \phi, \phi \rangle + \sum_{u \in V} \|d_{k-2}\phi_u\|^2
\]

\[
= \sum_{u \in V} \langle L_{k-1} \phi_u, \phi_u \rangle + \langle B_k \phi, \phi \rangle.
\]
Substracting \((d - 1) \langle L_k \phi, \phi \rangle\) from both sides of the equation we get
\[
(k - d + 1) \langle L_k \phi, \phi \rangle = \sum_{u \in V} \langle L_{k-1} \phi_u, \phi_u \rangle - \langle((d - 1)L_k - B_k)\phi, \phi \rangle \\
= \sum_{u \in V} \langle L_{k-1} \phi_u, \phi_u \rangle - \langle R_k \phi, \phi \rangle.
\]

For the proof of Proposition 3.2 we will need the next result, which follows from the definition of \(B_k\) and Claim 2.2.

**Claim 3.8.** The matrix representation of \(R_k\) in the standard basis is
\[
[R_k]_{\sigma,\tau} = \begin{cases} 
\sum_{\eta \in \sigma(k-1)} \deg_X(\eta) - (k-d+1) \deg_X(\sigma) + (d-1)(k+1) & \text{if } \sigma = \tau, \\
(d + 1 - m(\sigma \cup \tau))(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, m(\sigma \cup \tau) = r, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof of Proposition 3.2.** Let \(K_r\) be the \((k+1)\)-dimensional simplicial complex on vertex set \(V\), with full \(k\)-skeleton, whose \((k+1)\)-dimensional faces are the simplices \(\eta \in \binom{V}{k+2}\) such that \(m(\eta) = r\). By Claim 2.2, we have
\[
[L_k^+ (K_r)]_{\sigma,\tau} = \begin{cases} 
\deg_X(\sigma) & \text{if } \sigma = \tau, \\
-(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, m(\sigma \cup \tau) = r, \\
0 & \text{otherwise}.
\end{cases}
\]

Denote by \(M_{k,r}\) the principal submatrix of \([L_k^+ (K_r)]\) obtained by keeping only the rows and columns corresponding to simplices in \(X(k)\). \(M_{k,r}\) is a positive semidefinite matrix (as a principal submatrix of a positive semidefinite matrix).

Define a new matrix
\[
M_k = [R_k] + \sum_{r=2}^{d} (d + 1 - r)M_{k,r}.
\]

For a matrix \(A\), denote by \(\lambda_{\text{max}}(A)\) the largest eigenvalue of \(A\). Since \(M_{k,r}\) is positive semidefinite it follows that \(\lambda_{\text{max}}(-M_{k,r}) \leq 0\) for all \(2 \leq r \leq d\) and therefore
\[
\lambda_k = \lambda_{\text{max}}(R_k) \leq \lambda_{\text{max}}(M_k) + \sum_{r=2}^{d} (d + 1 - r)\lambda_{\text{max}}(-M_{k,r}) \leq \lambda_{\text{max}}(M_k). \quad (3.3)
\]
By equation (3.2), Lemma 2.6 and Claim 3.8 we see that the matrix $M_k$ is diagonal, and

$$(M_k)_{\sigma,\sigma} = \sum_{\eta \in \sigma(k-1)} \deg_X(\eta) - (k - d + 1) \deg_X(\sigma) + (d - 1)(k + 1)$$

$$+ \sum_{r=2}^{d} (d + 1 - r) \cdot |\{ v \in V : m(v\sigma) = r \}|.$$

Let $\sigma \in X(k)$. We can write

$$\deg_X(\sigma) = |\{ v \in V : v \in \text{lk}(X,\sigma) \}|$$

and

$$k + 1 = |\{ v \in V : v \in \sigma \}|,$$

and by Lemma 2.8

$$\sum_{\eta \in \sigma(k-1)} \deg_X(\eta) = |\{ v \in V : v \in \sigma \}| + (k + 1) \cdot |\{ v \in V : v \in \text{lk}(X,\sigma) \}|$$

$$+ \sum_{r=2}^{d+1} (r - 1) \cdot |\{ v \in V : m(v\sigma) = r \}|.$$

Hence,

$$(M_k)_{\sigma,\sigma} = d \cdot |\{ v \in V : v \in \sigma \}| + d \cdot |\{ v \in V : v \in \text{lk}(X,\sigma) \}|$$

$$+ \sum_{r=2}^{d+1} d \cdot |\{ v \in V : m(v\sigma) = r \}| \leq d \cdot |V| = dn.$$

Therefore $\lambda_{\text{max}}(M_k) \leq dn$, so by inequality (3.3): $\lambda_k \leq dn$. □

4 Vector domination

In this section we study the vector domination number $\Gamma(X)$ of a simplicial complex $X$, leading up to the proof of Theorem 1.6 that provides an upper bound on $\Gamma(X)$ in terms of the homological connectivity of $X$. First we prove Proposition 1.5 relating $\Gamma(X)$ to the total domination number $\tilde{\gamma}(X)$.

Proof of Proposition 1.5. Let $S$ be a totally dominating set in $X$. Let $\sigma \in S(X) = \binom{V}{d-1}$. Let $f_\sigma$ be the characteristic vector of $S \setminus \sigma$. Define $\alpha_\sigma = \frac{1}{d} f_\sigma$ if $\sigma \subset S$, and $\alpha_\sigma = 0$ otherwise. Then for every vector representation $P$ of $X$ and every $w \in V$ we have

$$\sum_{\sigma \in S(X)} \sum_{v \in V} \alpha_\sigma(v) P_\sigma(v) \cdot P_\sigma(w) = \sum_{\sigma \in \binom{V}{d-1}} \sum_{\sigma \setminus \sigma} \frac{1}{d} P_\sigma(v) \cdot P_\sigma(w).$$
Proof. We argue by induction on $m$. For $m = 1$ the statement is trivial. Assume $m = 2$. For any complex $C$ on vertex set $V$ containing $A_1 \cap A_2$, denote by $\tilde{L}_k(C)$ the principal submatrix of $[L_k(C)]$ obtained by keeping only the rows and columns corresponding to simplices of $A_1 \cap A_2$.

Let $\lambda_{\min}(\tilde{L}_k(C))$ and $\lambda_{\max}(\tilde{L}_k(C))$ be respectively the minimal and maximal eigenvalues of $\tilde{L}_k(C)$.

We have $\lambda_{\min}(\tilde{L}_k(C)) \geq \mu_k(C)$ and $\lambda_{\max}(\tilde{L}_k(C)) \leq \lambda_{\max}(L_k(C)) \leq n$ (by Lemma 2.3).

It is easy to check by Claim 2.2 that

$$[L_k(A_1 \cap A_2)] = \tilde{L}_k(A_1) + \tilde{L}_k(A_2) - \tilde{L}_k(A_1 \cup A_2).$$

Therefore

$$\mu_k(A_1 \cap A_2) \geq \lambda_{\min}(\tilde{L}_k(A_1)) + \lambda_{\min}(\tilde{L}_k(A_2)) - \lambda_{\max}(\tilde{L}_k(A_1 \cup A_2)) \geq \mu_k(A_1) + \mu_k(A_2) - n.$$
For $m > 2$ we get by the case $m = 2$ and the induction hypothesis
\[
\mu_k(\cap_{i=1}^m A_i) \geq \mu_k(A_1) + \mu_k(\cap_{i=2}^m A_i) - n
\]
\[
\geq \mu_k(A_1) + \left( \sum_{i=2}^m \mu_k(A_i) - (m-2)n \right) - n = \sum_{i=1}^m \mu_k(A_i) - (m-1)n.
\]
\)

For $i \in J_X$ let $Y_i$ be the $i$-dimensional complex on vertex set $V$ with full $(i-1)$-dimensional skeleton whose $i$-dimensional faces are the sets in $\mathcal{M}_X(i)$. Denote the maximal eigenvalue of $L_{i-1}^+(Y_i)$ by $\lambda^i_{\max}(X)$.

**Claim 4.2.** For all $i \in J_X$
\[
\mu_{i-1}(X_i) = n - \lambda^i_{\max}(X).
\]

**Proof.** By Claim 2.2 we have
\[
[L_{i-1}^+(Y_i)]_{\sigma,\tau} = \begin{cases} \deg_{Y_i}(\sigma) & \text{if } \sigma = \tau, \\
-(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = i-1, \\
0 & \text{otherwise.} \end{cases}
\]

Therefore $L_{i-1}^+(Y_i) = nI - L_{i-1}(X_i)$. So every eigenvector of $L_{i-1}(X_i)$ with eigenvalue $\lambda$ is an eigenvector of $L_{i-1}^+(Y_i)$ with eigenvalue $n - \lambda$. In particular, $n - \mu_{i-1}(X_i)$ is the largest eigenvalue of $L_{i-1}^+(Y_i)$. \qed

**Claim 4.3.** For $k \geq 0$,
\[
\mu_k(X) \geq n - \sum_{i \in J_X} \binom{k+1}{i} \lambda^i_{\max}(X).
\]

**Proof.** By Lemma 4.1 we obtain
\[
\mu_k(X) = \mu_k(\cap_{i \in J_X} X_i) \geq \sum_{i \in J_X} \mu_k(X_i) - (|J_X| - 1)n.
\]

Applying Claim 3.3 to each of the complexes $X_i$ (note that $h(X_i) = i$ and $X_i$ has full $(i-1)$-dimensional skeleton) we get
\[
\mu_k(X) \geq \sum_{i \in J_X} \left[ \binom{k+1}{i} \mu_{i-1}(X_i) - \binom{k+1}{i} (n - (|J_X| - 1)n) \right].
\]
Then by Claim 4.2

\[ \mu_k(X) \geq \sum_{i \in J_X} \left[ \binom{k+1}{i} (n - \lambda_{\max}^i(X)) - \binom{k+1}{i} - 1 \right] n - \left( |J_X| - 1 \right)n \]

\[ = n - \sum_{i \in J_X} \binom{k+1}{i} \lambda_{\max}^i(X). \]

Claim 4.4.

\[ \sum_{i \in J_X} \binom{\eta(X)}{i} \lambda_{\max}^i(X) \geq n. \]

Proof. Let \( k \) be the integer such that

\[ \sum_{i \in J_X} \binom{k-1}{i} \lambda_{\max}^i(X) < n \leq \sum_{i \in J_X} \binom{k}{i} \lambda_{\max}^i(X). \]

Let \( j \leq k - 2 \). By Claim 4.3,

\[ \mu_j(X) \geq n - \sum_{i \in J_X} \binom{j+1}{i} \lambda_{\max}^i(X) > 0, \]

therefore by Corollary 2.5 we have \( \tilde{H}_j(X; \mathbb{R}) = 0 \). So \( \eta(X) \geq k \), thus

\[ \sum_{i \in J_X} \binom{\eta(X)}{i} \lambda_{\max}^i(X) \geq \sum_{i \in J_X} \binom{k}{i} \lambda_{\max}^i(X) \geq n. \]

Claim 4.5. Let \( i \in J_X \). Then for \( \phi \in C^{i-1}(Y_i) \),

\[ \langle L_{i-1}^\dagger(Y_i) \phi, \phi \rangle \leq \sum_{\sigma \in \{V\} \setminus V} \sum_{v \in \text{lk}(Y_i, \sigma)} (\phi(v \sigma) - \phi(w \sigma))^2. \]

Proof.

\[ \sum_{\sigma \in Y_i(i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} (\phi(v \sigma) - \phi(w \sigma))^2 \]

\[ = \sum_{\sigma \in Y_i(i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} \text{deg}_{Y_i}(v \sigma) \phi(v \sigma)^2 - 2 \sum_{\sigma \in Y_i(i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} \phi(v \sigma) \phi(w \sigma) \]

\[ = i \cdot \sum_{\eta \in Y_i(i-1)} \text{deg}_{Y_i}(\eta) \phi(\eta)^2 - 2 \sum_{\sigma \in Y_i(i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} \phi(v \sigma) \phi(w \sigma). \]
By Claim 3.4 we have

\[ \langle L_{i-1}^+(Y_i) \phi, \phi \rangle = \|d_{i-1}\phi\|^2 = \sum_{\eta \in Y_{i}(i-1)} \deg Y_i(\eta)\phi(\eta)^2 - 2 \sum_{\sigma \in Y_{i}(i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} \phi(v\sigma)\phi(w\sigma). \]

Hence

\[ \langle L_{i-1}^+(Y_i) \phi, \phi \rangle = \sum_{\sigma \in Y_{i}(i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} (\phi(v\sigma) - \phi(w\sigma))^2 - (i - 1) \cdot \sum_{\eta \in Y_{i}(i-1)} \deg Y_i(\eta)\phi(\eta)^2 \leq \sum_{\sigma \in Y_{i}(i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} (\phi(v\sigma) - \phi(w\sigma))^2. \]

Thus $Y_i$ has full $(i-1)$-dimensional skeleton, therefore $Y_i(i-2) = \binom{V}{i-1}$. Thus

\[ \langle L_{i-1}^+(Y_i) \phi, \phi \rangle \leq \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in \text{lk}(Y_i, \sigma)} (\phi(v\sigma) - \phi(w\sigma))^2. \]

\[ \square \]

Claim 4.6. Let $P$ be a vector representation of $X$. Then for all $i \in J_X$

\[ \lambda_{\max}(X) \leq i \cdot \max_{\sigma \in \binom{V}{i-1}} \left( P_\sigma(v) \cdot \sum_{w \in V} P_\sigma(w) \right). \]

Proof. Let $\phi \in C^{i-1}(Y_i)$. For $\sigma \in Y_{i}(i-2) = \binom{V}{i-1}$ and $v, w \in V \setminus \sigma, v \neq w$, we have, by the definition of $P$, $P_\sigma(v) \cdot P_\sigma(w) \geq 1$ if $vw \in \text{lk}(Y_i, \sigma)$, and $P_\sigma(v) \cdot P_\sigma(w) \geq 0$ otherwise. Therefore we obtain

\[ \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in \text{lk}(Y_i, \sigma)} (\phi(v\sigma) - \phi(w\sigma))^2 \leq \frac{1}{2} \sum_{\sigma \in \binom{V}{i-1}} \sum_{v, w \in V \setminus \sigma} (\phi(v\sigma) - \phi(w\sigma))^2 P_\sigma(v) \cdot P_\sigma(w) \]

\[ = \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)^2 P_\sigma(v) \cdot \sum_{w \in V \setminus \sigma} P_\sigma(w) - \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)P_\sigma(v) \]

\[ \leq \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)^2 P_\sigma(v) \cdot \sum_{w \in V \setminus \sigma} P_\sigma(w) \]

\[ \leq \left( \sum_{\sigma \in \binom{V}{i-1}} \sum_{v \in V \setminus \sigma} \phi(v\sigma)^2 \right) \cdot \max_{\sigma \in \binom{V}{i-1}} P_\sigma(v) \cdot \sum_{w \in V \setminus \sigma} P_\sigma(w). \quad (4.1) \]
Since $Y_i$ has full $(i - 1)$-dimensional skeleton, we have

$$
\sum_{\sigma \in (\mathcal{V})^{i-1}} \sum_{v \in V \setminus \sigma} \phi(v) \cdot \phi(v) = \sum_{\sigma \in Y_i (i-2)} \sum_{v \in \text{lk}(Y_i, \sigma)} \phi(v) \cdot \phi(v) = i \sum_{\eta \in Y_i (i-1)} \phi(\eta) = i \|\phi\|^2. \quad (4.2)
$$

Combining (4.1), (4.2) and Claim 4.5 we obtain

$$\langle L^+_{i-1} (Y_i) \phi, \phi \rangle \leq \sum_{\sigma \in (\mathcal{V})^{i-1}} \sum_{uv \in \text{lk}(Y_i, \sigma)} (\phi(v) - \phi(w)) \cdot \phi(v) \cdot \sum_{\sigma \in (\mathcal{V})^{i-1}} \phi(w) \cdot \sum_{v \in V \setminus \sigma} \phi(v) \cdot \sum_{w \in V \setminus \sigma} \phi(w).
$$

Thus

$$\lambda^i_{\text{max}}(X) = \max_{0 \neq \phi \in C^{V_i}} \frac{\langle L^+_{i-1} \phi, \phi \rangle}{\|\phi\|^2} \leq i \cdot \max_{\sigma \in (\mathcal{V})^{i-1}, v \in V} \left( \phi(v) \cdot \sum_{w \in V} \phi(w) \right).$$

Lemma 4.7. Let $P$ be a vector representation of $X$. Then

$$|P| = \max \{ \alpha \cdot 1 : \alpha \geq 0, \alpha P_{\sigma} P_{\sigma}^T \leq 1 \forall \sigma \in S(X) \}.$$

Proof. Let $\sigma_1, \ldots, \sigma_m$ be all the sets in $S(X)$. For each $i \in \{1, 2, \ldots, m\}$ let $A_i = P_{\sigma_i} P_{\sigma_i}^T \in \mathbb{R}^{|V| \times |V|}$. Note that $A_i = A_i^T$. Define the matrix

$$A = (A_1 | A_2 | \cdots | A_m)^T \in \mathbb{R}^{(m|V|) \times |V|}.$$

Let $x \in \mathbb{R}^{m|V|}$. Write $x = (\alpha_{\sigma_1} | \alpha_{\sigma_2} | \cdots | \alpha_{\sigma_m})$, where $\alpha_{\sigma_i} \in \mathbb{R}^{|V|}$ for each $i \in \{1, 2, \ldots, m\}$. We have

$$xA = \sum_{i=1}^{m} \alpha_{\sigma_i} A_i = \sum_{\sigma \in S(X)} \alpha_{\sigma} P_{\sigma} P_{\sigma}^T,$$

therefore

$$|P| = \min \{ \sum_{\sigma \in S(X)} \alpha_{\sigma} \cdot 1 : \alpha_{\sigma} \geq 0 \forall \sigma \in S(X), \sum_{\sigma \in S(X)} \alpha_{\sigma} P_{\sigma} P_{\sigma}^T \geq 1 \} = \min \{ x \cdot 1 : x \geq 0, x A \geq 1 \}.$$
By linear programming duality

\[ |P| = \max \{ y \cdot 1 : y \geq 0, yA^T \leq 1 \} . \]

But \( yA^T = (yA_1|yA_2|\ldots|yA_m) \), so \( yA^T \leq 1 \) if and only if \( yA_i \leq 1 \) for all \( i \in \{1,2,\ldots,m\} \). Therefore

\[ |P| = \max \{ y \cdot 1 : y \geq 0, yP_\sigma P_\sigma^T \leq 1 \quad \forall \sigma \in S(X) \} . \]

\[ \square \]

Let \( \mathbb{Z}_+ \) denote the positive integers, and \( \mathbb{Q}_+ \) the positive rationals. Let \( a \in \mathbb{Z}_+^V \) and

\[ V_a = \{ (v,i) : v \in V, 1 \leq i \leq a(v) \} . \]

Define the projection \( \pi : V_a \rightarrow V \) by \( \pi((v,i)) = v \), and let

\[ X_a = \pi^{-1}(X) = \{ \sigma \subset V_a : \pi(\sigma) \in X \} . \]

The missing faces of \( X_a \) are the sets \( \sigma \subset V_a \) such that \( |\pi(\sigma)| = |\sigma| \) and \( \pi(\sigma) \) is a missing face of \( X \).

\( \pi \) induces an homotopy equivalence between \( X_a \) and \( X \) (see \[11, Lemma 2.6\]), therefore \( \eta(X_a) = \eta(X) \).

**Proof of Theorem 1.6.** Let \( P = \{ P_\sigma \}_{\sigma \in S(X)} \) be a vector representation of \( X \). Let \( \alpha \in \mathbb{Q}_+^{V_\pi} \) such that \( \alpha P_\sigma P_\sigma^T \leq 1 \) for all \( \sigma \in S(X) \). Write \( \alpha = a/k \) where \( k \in \mathbb{Z}_+ \) and \( a \in \mathbb{Z}_+^V \). Denote \( N = |V_a| = \sum_{v \in V} a(v) \). For \( \sigma \in S(X_a) \) and \( (v,j) \in V_a \) define

\[ Q_\sigma((v,j)) = \begin{cases} P_{\pi(\sigma)}(v) & \text{if } |\pi(\sigma)| = |\sigma|, \\ 0 & \text{otherwise}. \end{cases} \]

\( Q = \{ Q_\sigma : \sigma \in S(X_a) \} \) is a vector representation of \( X_a \): Let \( \sigma \in S(X_a) \) of size \( r - 1 \), and let \( \tilde{v} = (v,i), \tilde{u} = (u,j) \in V_a \) such that \( \tilde{u} \tilde{v} \sigma \in M_{X_a}(r) \). Then \( \pi(\tilde{u} \tilde{v} \sigma) = u \pi(\sigma) \in M_X(r) \). In particular \( |\pi(\sigma)| = |\sigma| \), therefore, since \( P \) is a representation of \( X \),

\[ Q_\sigma(\tilde{v}) \cdot Q_\sigma(\tilde{u}) = P_{\pi(\sigma)}(v) \cdot P_{\pi(\sigma)}(u) \geq 1. \]

Let \( r \in J_X \). By Claim 4.6

\[ \chi^r_{\max}(X_a) \leq r \cdot \max_{\sigma \in (\tau_{\sigma})_{(v,j)} \in V_a} \left( Q_\sigma((v,j)) \cdot \sum_{(w,k) \in V_a} Q_\sigma((w,k)) \right) \]

\[ = r \cdot \max_{\tau \in (\tau_{\tau})_{(v,j)} \in V} \left( P_\tau(v) \cdot \sum_{w \in V} a(w) P_\tau(w) \right) \leq r \cdot k. \]

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By Claim 4.4 we obtain
\[ \sum_{r \in J_{X_a}} \left( \eta(X_a) \right)^r \cdot k \geq \sum_{r \in J_{X_a}} \left( \eta(X_a) \right)^{r \cdot \lambda_{\max}(X_a)} \geq N. \]

Therefore
\[ \alpha \cdot 1 = \frac{1}{k} \sum_{v \in V} \eta(v) = \frac{N}{k} \leq \sum_{r \in J_{X_a}} r \left( \eta(X_a) \right)^r = \sum_{r \in J_X} r \left( \eta(X) \right)^r. \]

Thus by Lemma 4.7
\[ |P| = \max \{ \alpha \cdot 1 : \alpha \geq 0, \alpha P_{\sigma} P_{\sigma}^T \leq 1 \forall \sigma \in S(X) \} \]
\[ \leq \sup \{ \alpha \cdot 1 : \alpha \in \mathbb{Q}_+, \alpha P_{\sigma} P_{\sigma}^T \leq 1 \forall \sigma \in S(X) \} \leq \sum_{r \in J_X} r \left( \eta(X) \right)^r, \]

therefore \( \Gamma(X) \leq \sum_{r \in J_X} r \left( \eta(X) \right)^r. \)

For the proof of Theorem 1.7 we need the following Hall-type condition for the existence of colorful simplices, which appears in [3, 12], and more explicitly in [13]:

**Proposition 4.8.** Let \( Z \) be a simplicial complex on vertex set \( W = \bigcup_{i=1}^m W_i \).
If for all \( \emptyset \neq I \subset \{1, 2, \ldots, m\} \)
\[ \eta(Z[\cup_{i \in I} W_i]) \geq |I| \]
then \( Z \) contains a colorful simplex.

**Proof of Theorem 1.7.** Let \( \emptyset \neq I \subset \{1, 2, \ldots, m\} \). By Theorem 1.6 we have
\[ \sum_{r \in J_X[\cup_{i \in I} V_i]} r \left( \eta(X[\cup_{i \in I} V_i]) \right)^r \geq \Gamma(X[\cup_{i \in I} V_i]) > \sum_{r \in J_X[\cup_{i \in I} V_i]} r \left( |I| - 1 \right)^r, \]
therefore
\[ \eta(X[\cup_{i \in I} V_i]) > |I| - 1. \]
Thus by Proposition 4.8 \( X \) has a colorful simplex.

5 Colorful sets in general position

Let \( M \) be a matroid of rank \( d + 1 \) on vertex set \( V \). Let \( \tilde{M} \) be the simplicial complex on vertex set \( V \) whose simplices are the subsets \( S \subset V \) in general position with respect to \( M \). The missing faces of \( M \) are the dependent sets \( S \subset V \) with \( |S| \leq d + 1 \) such that any \( |S| - 1 \) points in \( S \) are independent in \( M \).
Claim 5.1. For $U \subset V$,

$$\varphi^*_M(U) \leq d \cdot \Gamma(\tilde{M}[U]).$$

Proof. We construct a vector representation of the complex $\tilde{M}[U]$. Let $1 \leq r \leq d$ and let $\mathcal{F}_r$ be the set of flats of $M$ of rank $r$.

Let $\sigma \in S(\tilde{M}[U])$ with $|\sigma| = r - 1$, and let $v \in U$. Define $P_\sigma(v) \in \mathbb{R}^{\mathcal{F}_r}$ by

$$P_\sigma(v)(F) = \begin{cases} 1 & \text{if } \text{cl}(v\sigma) = F, \\ 0 & \text{otherwise.} \end{cases}$$

For $v, w \in U$, if $vw\sigma$ is a missing face of $\tilde{M}[U]$ of dimension $r$ then $vw\sigma$ lies in a flat of rank $r$, which is spanned by any $r$ points in $vw\sigma$. In particular $\text{cl}(v\sigma) = \text{cl}(w\sigma) \in \mathcal{F}_r$, therefore

$$P_\sigma(v) \cdot P_\sigma(w) = 1.$$

Hence $P$ is a vector representation of $\tilde{M}[U]$.

Let $f : U \to \mathbb{R}_{\geq 0}$ be a function in fractional general position with

$$\sum_{v \in U} f(v) = \varphi^*_M(U).$$

Define $\alpha \in \mathbb{R}^U$ by $\alpha(v) = f(v)/d$.

Let $w \in U$, and let $F = \text{cl}(w\sigma)$. If $F \notin \mathcal{F}_r$ then $P_\sigma(w) = 0$, therefore

$$\sum_{v \in U} \alpha(v)P_\sigma(v) \cdot P_\sigma(w) = 0 \leq 1.$$

If $F \in \mathcal{F}_r$ then

$$\sum_{v \in U, \text{cl}(v\sigma) = F} \alpha(v) \leq \frac{1}{d} \sum_{v \in U, \text{cl}(v\sigma) = F} f(v) \leq 1.$$

So $\alpha P_\sigma P_\sigma^T \leq 1$ for each $\sigma \in S(\tilde{M}[U])$, therefore by Lemma 4.7

$$\Gamma(\tilde{M}[U]) \geq |P| \geq \alpha \cdot 1 = \frac{\varphi^*_M(U)}{d}.$$ 

Proof of Theorem 1.9. Let $\emptyset \neq I \subset \{1, 2, \ldots, m\}$. By Claim 5.1

$$\Gamma(\tilde{M}[U]) \geq |P| \geq \alpha \cdot 1 = \frac{\varphi^*_M(U)}{d}.$$ 

Thus by Theorem 1.7 there is a colorful simplex of $\tilde{M}$, i.e. a colorful subset of $V$ in general position.

Proof of Theorem 1.10. Let $\emptyset \neq I \subset \{1, 2, \ldots, m\}$. Assume $|I| \leq d + 1$. The $d$-dimensional skeleton of $\tilde{M}[\cup_{i \in I} V_i]$ is $M[\cup_{i \in I} V_i]$, therefore for all $0 \leq k \leq d - 1$

$$\tilde{H}^k \left( \tilde{M}[\cup_{i \in I} V_i]; \mathbb{R} \right) = \tilde{H}^k \left( M[\cup_{i \in I} V_i]; \mathbb{R} \right).$$
$M$ is a matroid, therefore $\tilde{H}^k(M[\cup_{i\in I}V_i];\mathbb{R}) = 0$ for $0 \leq k \leq \rho(\cup_{i\in I}V_i) - 2$ (see [5]). So $\eta(M[\cup_{i\in I}V_i]) \geq \rho(\cup_{i\in I}V_i)$. But

$$\rho(\cup_{i\in I}V_i) = \min\{d + 1, \varphi_M(\cup_{i\in I}V_i)\},$$

so if $\varphi_M(\cup_{i\in I}V_i) > |I| - 1$, then $\eta(M[\cup_{i\in I}V_i]) > |I| - 1$.

Assume now that $|I| \geq d + 2$. If $\varphi_M(\cup_{i\in I}V_i) > d \sum_{r=1}^{d} r \binom{|I| - 1}{r}$, then, by inequality (1.2), $\varphi_M^*(\cup_{i\in I}V_i) > d \sum_{r=1}^{d} r \binom{|I| - 1}{r}$, and therefore by Theorem 1.6 and Claim 5.1

$$\sum_{r=1}^{d} \frac{r}{d} \eta(M[\cup_{i\in I}V_i]) \geq \Gamma(M[\cup_{i\in I}V_i]) \geq \frac{\varphi_M^*(\cup_{i\in I}V_i)}{d} > \sum_{r=1}^{d} \frac{r}{d} \binom{|I| - 1}{r},$$

so $\eta(M[\cup_{i\in I}V_i]) > |I| - 1$. Therefore by Proposition 4.8 there is a colorful subset of $V$ in general position.

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