On Improving Resource Allocations by Sharing

Robert Bredereck\textsuperscript{1}, Andrzej Kaczmarczyk\textsuperscript{2,3}, Junjie Luo\textsuperscript{4}, Rolf Niedermeier\textsuperscript{2}, and Florian Sachse\textsuperscript{2}

\textsuperscript{1}Humboldt-Universität zu Berlin, Berlin, Germany
\textsuperscript{2}Algorithmics and Computational Complexity, TU Berlin, Berlin, Germany
\textsuperscript{3}AGH University, Kraków, Poland
\textsuperscript{4}Nyanyang Technological University, Singapore

robert.bredereck@hu-berlin.de, andrzej.kaczmarczyk@agh.edu.pl, junjie.luo@ntu.edu.sg, rolf.niedermeier@tu-berlin.de, sachse.florian@gmail.com

December 15, 2021

Abstract

Given an initial resource allocation, where some agents may envy others or where a different distribution of resources might lead to higher social welfare, our goal is to improve the allocation without reassigning resources. We consider a sharing concept allowing resources being shared with social network neighbors of the resource owners. To this end, we introduce a formal model that allows a central authority to compute an optimal sharing between neighbors based on an initial allocation. Advocating this point of view, we focus on the most basic scenario where a resource may be shared by two neighbors in a social network and each agent can participate in a bounded number of sharings. We present algorithms for optimizing utilitarian and egalitarian social welfare of allocations and for reducing the number of envious agents. In particular, we examine the computational complexity with respect to several natural parameters. Furthermore, we study cases with restricted social network structures and, among others, devise polynomial-time algorithms in path- and tree-like (hierarchical) social networks.

1 Introduction

The fair allocation of resources undoubtedly is a key challenge for modern societies and economies. Applications can be found in such diverse fields as cloud computing, food banks, or managing carbon loads in the context of global warming. Naturally, this topic received high attention in the scientific literature. This also holds true for the special case of indivisible resources [Bouveret et al., 2016], which we concentrate on here. Moreover, we take into account the role of social networks built by agents, a growing line of research [Abebe et al., 2017, Bei et al., 2017, Bouveret et al., 2017, Bredereck et al., 2018, Chevaleyre et al., 2017, Beynier et al., 2019, Lange and Rothe, 2019, Huang and Xiao, 2019]. We bring one further new aspect into this scenario, reflecting the increasing relevance of “sharing economies” [Belk et al., 2019, Schor and Cansoy, 2019], where agents share resources in a peer-to-peer fashion. Resources to share may be almost everything, for instance, knowledge, machines, time, or natural resources. More specifically, sharing in our scenario, which takes into account the relationships between agents expressed by social networks, means that two adjacent agents in the social network may share the very same resource, thus increasing the utility of the resource allocation for at least one of them (assuming positive utility for each resource). We assume this to be organized and decided by a central authority like, for example, the boss of a company. To get started with this new setting, we focus on a very basic scenario. That is, in our model only two neighbors may share and, reflecting the (very human) principle of protection of acquired possession, no agent shall loose its already allocated resources. This conservative principle naturally makes sharing easier to implement, keeping “restructuring costs” lower, and it may even help to “keep peace” among agents. Moreover, it sometimes comes very naturally as depicted in the subsequent knowledge sharing example. Besides improving egalitarian or utilitarian welfare, we focus on the perhaps most basic fairness criterion,
enjoy-freeness. Since it is not always possible that complete envy-freeness is achieved (consider one indivisible resource and two agents desiring it), we aim at, given an initial resource allocation, improving it by decreasing the number of envious agents through resource sharing. Moreover, we allow for modeling relationship aspects of sharing based on the social network formed by the agents.

Before becoming more specific about our model, let us first introduce the following example related to knowledge sharing. Assume that agents are employees of a company, each having a bundle of qualifications. An agent may “envy” another agent because the other agent has some special qualification. The central authority wants to improve the situation by building teams of two agents where, due to a daily extensive cooperation, one teaches the other the missing qualification (for instance, a realization of this is the concept of pair programming that also has other benefits besides knowledge sharing [Williams et al., 2000]).

Model of sharing allocation. Roughly speaking, our model is as follows (see Section 2 for formal definitions). The input is a set of agents and a set of indivisible resources initially assigned to the agents. Typically, every agent may be assigned several resources. Each agent has an individual utility value for each resource. The general goals are to decrease the overall degree of envy, to increase the sum of “utility scores” of all agents, or to increase the minimal “utility score” among all agents. Importantly, the only way an agent can improve its individual “utility score” is by participating in a sharing with other agents.

We assume that if an owner shares, then this does not decrease its own overall utility value. This approach is justified when the burden of sharing is neutralized by its advantages. Indeed, in our knowledge sharing example a hassle of cooperation is often compensated by a better working experience or higher quality outcomes (as shown by Williams et al., 2000). Note that such complicated mutual dependencies that would be extremely hard to describe formally form a natural field for our approach. Further application examples include irregularly used resources (like printers or compute servers). Here, the coordination with another person is uncritical and splitting the maintenance costs neutralizes the inconvenience of cooperation.

We enrich our model by using two graphs, an undirected sharing graph and a directed attention graph, to model social relations between agents and to govern the following two constraints of our model. The sharing graph models the possibility for two agents to share resources because, e.g., they are close to each other or there is no conflict between the time they use resources. We focus on the case when only neighbors in the sharing graph can share a resource (a missing qualification in our knowledge sharing example). With respect to lowering the degree of envy, we assume that agents may only envy their outneighbors in the directed attention graph. This graph-based envy concept has recently been studied by many works in fair allocation Bredereck et al. (2018), Aziz et al. (2018), Beynier et al. (2019).

Agents may naturally be conservative in the sense of keeping control and not sharing too much. Furthermore, as in our initial example, it might simply be too ineffective to share a qualification among more than two employees simultaneously (due to, e.g., increased communication overhead or additional resources needed). We address this in the most basic way and assume that each resource can be shared to at most one neighbor of its owner and an agent can participate in a bounded number of sharings. This strong restriction already leads to tricky algorithmic challenges and fundamental insights. In particular, the model also naturally extends on well-known matching scenarios in a non-trivial way.

There are numerous options to further extend and generalize our basic model, as discussed in Section 5 and in the concluding Section 6. However, keeping our primary model simple, we aim at spotting its fundamental properties influencing the complexity of related computational problems.

Related work. To the best of our knowledge, so far the model we consider has not been studied. Since obtaining envy-free allocations is not always possible, there has been work on relaxing the concept of envy. In particular, in the literature bounded-maximum envy [Lipton et al., 2004], envy-freeness up to one good [Budish, 2011], envy-freeness up to the least-valued good [Caragiannis et al., 2019b], epistemic envy-freeness [Aziz et al., 2018], and maximin share guarantee [Budish, 2011] have been studied. However, these concepts combat nonexistence of allocations that are envy-free by considering approximate versions of it; they basically do not try to tackle the question of how to achieve less “envy” in an allocation. By way of contrast, our approach tries to find a way to lessen envy not by relaxing the concept of envy, but rather by enabling a small deviation in the model of indivisible, non-shareable resources. To this end, we make resources shareable (in our basic model by two agents). This approach is in line with a series of recent works which try to reduce envy (i) by introducing small amounts of money [Brustle et al., 2020] Halpern and Shah (2019) Caragiannis and Ioannidis (2020). (ii) by donating a small set of resources to
Envy-Reducing Sharing Allocation (ERSA)

| clique | tree- or pathwidth | \( G_s = G_t \) | same utility | few agents | few resources |
|--------|---------------------|----------------|--------------|------------|--------------|
| NP-h   | Thm. 3              | \( G_t = \text{clique} \) | \( n \)     | \( k = 0, \Delta k = 1 \) | 4             |
| P      | Thm. 7              | \( G_s = \text{clique} \) |              |            | p-NP-h       |
| NP-h   | Thm. 8              |                |              |            | p-NP-h       |
| FPT    | Thm. 4              |                |              |            | XP, W[1]-h   |
| p-NP-h | Thm. 5              |                |              |            | p-NP-h       |
| XP, W[1]-h | Obs. 4            |                |              |            | Thm. 8       |
| p-NP-h | Thm. 8              |                |              |            | Thm. 8       |

Table 1: Results overview for ERSA, where \( G_s = G_t \) means that the sharing graph \( (G_s) \) is the same as the underlying graph of the attention graph \( (G_t) \), \( n \) is the number of agents, \( k \) is the number of envious agents after sharing, \( \Delta k \) is a drop in the number of envious agents, and \( m \) is the number of resources.

Our contributions. Introducing a novel model for (indivisible) resource allocation with agents linked by social networks, we provide a view on improving existing allocations for several measures without, conceivably impossible, reallocations.

We analyze the (parameterized) computational complexity of applying our model to improve utilitarian social welfare or egalitarian social welfare (Definition 4), and to decrease the number of envious agents (Definition 5). We show that a central authority can (mostly) find a sharing that improves social welfare (measured in both the egalitarian and utilitarian ways) in polynomial time, while decreasing the number of envious agents is NP-hard even if the sharing graph is a clique and the attention graph is a bidirectional clique. To overcome NP-hardness, we also study the influence of different natural parameters (such as agent utility function values, structural parameters concerning the agent social networks, the number of agents, and the number of resources); Table 1 surveys our results in more detail. We show that the problem is polynomial-time solvable if the underlying undirected graph of the attention graph is the same as the sharing graph and has constant treewidth (close to a tree). We also identify an interesting contrast between the roles of the two graphs: When agents have the same utility function, the problem is solvable in polynomial time if the attention graph is a bidirectional clique, while the problem is NP-hard even if the sharing graph is a clique. Finally, we show that the problem is fixed-parameter tractable (FPT) for the parameter number of agents (giving hope for efficient solutions in case of a small number of agents) and polynomial-time solvable (in XP) for a constant number of resources. However, the problem is NP-hard even if the goal is to reduce the number of envious agents from one to zero.

Altogether, our main technical contributions are with respect to exploring the potential to “overcome” the NP-hardness of decreasing the number of envious agents by exploiting several problem-specific parameters.

2 Preliminaries

For a set \( \mathcal{A} = \{a_1, a_2, \ldots, a_n\} \) of agents and a set \( \mathcal{R} = \{r_1, r_2, \ldots, r_m\} \) of indivisible resources, a (simple) allocation \( \pi: \mathcal{A} \to 2^{\mathcal{R}} \) is a function assigning to each agent a collection of resources—a bundle—such that
the assigned bundles are pairwise disjoint. An allocation is complete if every resource belongs to some bundle.

A directed graph \( G \) consists of a set \( V \) of vertices and a set \( E \subseteq V \times V \) of arcs connecting the vertices; we do not allow self-loops (i.e., there are no arcs of form \((v,v)\) for any vertex \( v \in V \)). A (simple) undirected graph \( G = (V,E) \) consists of a set \( V \) of vertices and a set \( E \) of distinct size-2 subsets of vertices called edges. An underlying undirected graph of a directed graph \( G \) is the graph obtained by replacing all (directed) arcs with (undirected) edges. We say an undirected graph \( G = (V,E) \) is a clique if \( E = \binom{V}{2} \) and a directed graph is a bidirectional clique if \( E = V \times V \). For some vertex \( v \in V \), the set \( I(v) \) of incident arcs (edges) is the set of all arcs (edges) with an endpoint in \( v \).

2.1 Sharing Model

We fix an initial allocation \( \pi \) of resources in \( \mathcal{R} \) to agents in \( \mathcal{A} \). A sharing graph is an undirected graph \( G_s = (\mathcal{A}, \mathcal{E}_s) \) with vertices being the agents; it models possible sharings between the agents. The following definition of sharing says that two agents can only share resources held by one of them.

**Definition 1.** Function \( \delta_s: \mathcal{E}_s \to 2^{\mathcal{R}} \) is a sharing for \( \pi \) if for every two agents \( a_i \) and \( a_j \), with \( \{a_i, a_j\} \in \mathcal{E}_s \), it holds that \( \delta_s(\{a_i, a_j\}) \subseteq \pi(a_i) \cup \pi(a_j) \).

An initial allocation \( \pi \) and a corresponding sharing \( \delta_s \) form a sharing allocation.

**Definition 2.** A sharing allocation induced by allocation \( \pi \) and sharing \( \delta_s \) is a function \( \Pi^\delta_s: \mathcal{A} \to 2^{\mathcal{R}} \) where \( \Pi^\delta_s(a) := \pi(a) \cup \bigcup_{e \in I(a)} \delta_s(e) \).

Since the initial allocation \( \pi \) is fixed, for brevity, we use \( \delta \) and \( \Pi^\delta \), omitting \( \pi \) whenever it is not ambiguous. For simplicity, for every agent \( a \in \mathcal{A} \), we also refer to \( \Pi^\delta(a) \) as a bundle of \( a \).

Naturally, each allocation is also a sharing allocation with a trivial “empty sharing.” Observe a subtle difference in the intuitive meaning of a bundle of an agent between sharing allocations and (simple) allocations. For sharing allocations, a bundle of an agent represents the resources the agent has access to and can utilize, not only those that the agent possesses (as for simple allocations).

2.2 2-sharing

Definition 1 is very general and only requires that two agents share resources that one of them already has. In particular, Definition 1 allows one agent to share the same resource with many other agents; and does not constrain the number of sharings an agent could participate in. In this paper, we assume that each resource can only be shared by two agents and each agent can participate in at most a bounded number of sharings. We formally express this requirement in Definition 3.

**Definition 3.** A 2-sharing \( \delta \) is a sharing where, for any three agents \( a_i, a_j, \) and \( a_k \), it holds that
\[
\Pi^\delta(a_i) \cap \Pi^\delta(a_j) \cap \Pi^\delta(a_k) = \emptyset.
\]

A \( b \)-bounded 2-sharing \( \delta \) is a 2-sharing where, for each agent \( a \), it holds that \( \left| \bigcup_{e \in I(a)} \delta(e) \right| \leq b \). A simple 2-sharing \( \delta \) is a 1-bounded 2-sharing, i.e., for each agent \( a \), it holds that \( \left| \bigcup_{e \in I(a)} \delta(e) \right| \leq 1 \).

Herein, we count the number of sharings an agent participate in by the number of resources shared with other agents (either shared to other agents or received from other agents). Notably, in simple 2-sharing, each agent can either share or receive a single resource. Thus, every simple 2-sharing can be interpreted as matching in which each edge is labeled with a shared item.

2.3 Welfare and Fairness Measures

We assume agents having additive utility functions. For an agent \( a \) with utility function \( u: \mathcal{R} \to \mathbb{R}_0^+ \) and a bundle \( R \subseteq \mathcal{R} \), let \( u(R) := \sum_{r \in R} u(r) \) be the value of \( R \) as perceived by \( a \). Let us fix a sharing allocation \( \Pi^\delta \) of resources \( \mathcal{R} \) to agents \( \mathcal{A} = \{a_1, a_2, \ldots, a_n\} \) with corresponding utility functions \( u_1, u_2, \ldots, u_n \). The utilitarian social welfare of \( \Pi^\delta \) is
\[
\text{usw}(\Pi^\delta) := \sum_{i \in [n]} u_i(\Pi^\delta(a_i)).
\]
The egalitarian social welfare of $\Pi^k$ is 

$$\text{esw}(\pi) := \min_{i \in [n]} u_i(\Pi^k(a_i)).$$

Notice that we assume each agent $a_i$ gets the full utility for all resources in $\Pi^k(a_i)$. We will discuss a generalization of this assumption in Section 5.

A directed graph $G_t = (A, E_t)$ with vertices being the agents is an attention graph; it models social relations between the agents. We say that an agent $a_i$ looks at another agent $a_j$ if $(a_i, a_j) \in E_t$. An agent is envious on $G_t$ under $\Pi^k$ if it prefers a bundle of another agent it looks at over its own one; formally, $a_i$ envies $a_j$ if $u_i(\Pi^k(a_i)) < u_i(\Pi^k(a_j))$ and $(a_i, a_j) \in E_t$. We denote the set of envious agents in $\Pi^k$ as $\text{Env}(\Pi^k)$. For a given (directed) attention graph $G_t$ over the agents, a sharing allocation is $G_t$-envy-free if no agent envies its out-neighbors.

### 2.4 Useful Problems

We define two variants of Maximum Weighted Matching and show that they are solvable in polynomial time. We will later use them to show polynomial-time solvability of our problems. Given an edge-weighted undirected graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{N}$ and two integers $k_1, k_2 \in \mathbb{N}$, Size-Bounded Maximum Weighted Matching (SBMWM) asks whether there is a matching $M$ such that $|M| \leq k_1$ and $w(M) \geq k_2$, and Weight-Bounded Maximum Matching (WBMM) asks whether there is a matching $M$ such that $w(M) \leq k_1$ and $|M| \geq k_2$, where $w(M) = \sum_{e \in M} w(e)$.

**Lemma 1.** SBMWM and WBMM are solvable in polynomial time.

**Proof.** For SBMWM, we show that it can be reduced to Maximum Weighted Matching, which is known to be solvable in polynomial time. For any instance of SBMWM, since $w(e) \geq 0$ for all $e \in E$, there is a matching $M$ such that $|M| \leq k_1$ and $w(M) \geq k_2$ if and only if there is a matching $M$ such that $|M| = k_1$ and $w(M) \geq k_2$. We add $n - 2k_1$ new vertices into the graph and connect them with all the old vertices by edges of weight $C = \sum_{e \in E} w(e) + 1$. Notice that the weight of every new edge is larger than that of every old edge. Then it is easy to see that every maximum weighted matching in the new graph consists of $k_1$ old edges and $n - 2k_1$ new edges, and the $k_1$ old edges form a matching with the maximum weight among all matchings of size $k_1$ in the old graph. Thus we can apply the algorithm for Maximum Weighted Matching on the new graph and check whether the sum of the weights of these $k_1$ edges in the old graph is at least $k_2$. Therefore, SBMWM is solvable in polynomial time.

For WBMM, we first convert the weight function as follows. Let $W = \max_{e \in E} w(e)$ and build a new weight function $w' : E \rightarrow \mathbb{N}$ where $w'(e) = W - w(e)$ for each $e \in E$. Note that $w(e) \geq 0$ and $w'(e) \geq 0$ for each $e \in E$. Since $w(e) \geq 0$ and $w(M) \geq 0$ for any matching $M$ with $w(M) \leq k_1$ and $|M| \leq k_2$ if and only if there is a matching $M$ with $w(M) \leq k_1$ and $|M| = k_2$. When $|M| = k_2$, we have $w(M) = Wk_2 - w'(M)$, and hence $w(M) \leq k_1$ if and only if $w'(M) \geq Wk_2 - k_1$. Now the problem is to decide whether there is a matching $M$ with $|M| = k_2$ and $w'(M) \geq Wk_2 - k_1$. Since $w'(e) \geq 0$, this is equivalent to decide whether there is a matching $M$ with $|M| \leq k_2$ and $w'(M) \geq Wk_2 - k_1$, which is an instance of SBMWM and, as shown in the last paragraph, is solvable in polynomial time.

### 3 Improving Social Welfare by Sharing

In this section, we study the problem of improving utilitarian (and egalitarian) welfare through sharing, defined as follows.

**Definition 4.** Given an initial complete allocation $\pi$ of resources $R$ to agents $A$, a sharing graph $G_s$, and a non-negative integer $k$, b-Bounded Utilitarian Welfare Sharing Allocation (b-UWSA) asks if there is a b-bounded 2-sharing $\delta$ such that $\text{usw}(\Pi^b) \geq k$; b-Bounded Egalitarian Welfare Sharing Allocation (b-EWSA) asks if there is a b-bounded 2-sharing $\delta$ such that $\text{esw}(\Pi^b) \geq k$.

We first consider b-UWSA. When $b = 1$, since every simple 2-sharing corresponds to a matching, we can easily reduce 1-UWSA to Maximum Weighted Matching. Thus, 1-UWSA is solvable in polynomial time. When $b > 1$, however, the problem is not just finding $c$ matchings such that the total is maximized. Notice that in 2-sharing each resource can only be shared once. Nevertheless, we show that we can still reduce b-UWSA to Maximum Weighted Matching via a more involved reduction.
Theorem 1. $b$-UWSA is solvable in polynomial time for any $b \geq 1$.

Proof. Given an instance of $b$-UWSA, we construct an instance $(G = (V, E), w)$ of MAXIMUM WEIGHTED MATCHING as follows. For simplicity, assume each agent $a_i \in A$ has at least $b$ resources in the initial allocation $\pi$; otherwise we can ensure this by adding enough resources that are valued as 0 by all agents. For each agent $a_i \in A$ and each resource $r_j \in R$, we add a vertex $v_{ij}^r$ into $V$. In addition, for each agent $a_i \in A$, we add $n_i = |\pi(a_i)| - b$ dummy vertices $(v_{ij}^k)_{k=1,2,...,n_i}$ into $V$. For each pair of agents $a_i, a_{j},$ for each vertex $v_{ij}^r$ corresponding to $a_i$ and each vertex $v_{kj}^s$ corresponding to $a_{j}$, we add an edge between $v_{ij}^r$ and $v_{kj}^s$ with weight $\max\{u_{ij}(r_{j2}), u_{kj}(r_{j1})\}$. Finally, for each $v_{ij}^1$ and each $v_{ij}^k$ corresponding to the same agent $a_i$, we add a dummy edge between them with weight $W = \max_{a_i \in A, r_j \in R} u_i(r_j)$. This finishes the construction of the instance $(G = (V, E), w)$. Notice that there always exists a maximum weighted matching that contains $n_i$ dummy edges for each agent $a_i$. Let $P = W \sum_{a_i \in A} n_i$ be the weight of those edges. Next we show that there is a $b$-bounded 2-sharing $\delta$ such that $\mathrm{usw}(\Pi^b) \geq k$ if and only if there is matching $M$ in graph $G$ with weight $\sum_{e \in M} w(e) \geq k - \mathrm{usw}(\pi) + P$, where $\mathrm{usw}(\pi) = \sum_{a_i \in A} u_i(\pi(a_i))$.

$\Rightarrow$: Assume there is a $b$-bounded 2-sharing $\delta$ such that $\mathrm{usw}(\Pi^b) \geq k$. Based on $\delta$, we can find a matching $M$ with the claimed weight by including the corresponding edges for each $e \in E$ with $\delta(e) \neq \emptyset$ and edges between the remaining normal vertices and all dummy vertices. Formally, for each edge $(a_i, a_{j}) \in E$ such that $\delta(a_i, a_{j}) = r_{j2}$ and $r_{j1} \in \pi(a_{i})$, we add the edge $(v_{ij}^r, v_{kj}^s)$ into matching $M$, where $r_{j2} \in \pi(a_{i})$ is an arbitrary resource except for the resource that is shared by $a_{j}$ with some other agent under $\delta$. Notice that $w((v_{ij}^r, v_{kj}^s)) = \max\{u_{ij}(r_{j2}), u_{kj}(r_{j1})\} \geq u_{ij}(r_{j1})$. After this, for each agent $a_{i}$ there are at least $n_i$ normal vertices not matched and exactly $n_i$ dummy vertices not matched, so we can add $n_i$ dummy edges of the same weight $W$ into $M$, each containing one normal vertex and one dummy vertex. Since $\mathrm{usw}(\Pi^b) \geq k$, we have that $\sum_{e \in M} w(e) = \mathrm{usw}(\Pi^b) - \mathrm{usw}(\pi) + P \geq k - \mathrm{usw}(\pi) + P$.

$L\Leftarrow$: Assume there is matching $M$ in graph $G$ with the claimed weight. Without loss of generality, we can assume $M$ contains $n_i$ dummy edges for every agent $a_{i}$ since the weight of dummy edges is no smaller than that of non-dummy edges. Then for each agent $a_{i}$, $M$ contains at most $b$ non-dummy edges. Based on these non-dummy edges in $M$ we can find the corresponding $b$-bounded 2-sharing $\delta$ such that $\mathrm{usw}(\Pi^b) \geq k$ as follows. For each non-dummy edge $(v_{ij}^r, v_{kj}^s) \in M$, we set $\delta(a_{i}, a_{j}) = r_{j2}$ if $u_{ij}(r_{j2}) \geq u_{ij}(r_{j1})$ and $\delta(a_{i}, a_{j}) = r_{j1}$ otherwise. Since $M$ contains at most $b$ non-dummy edges for each $a_{i}$, $a_{i}$ participates in at most $b$ sharings in $\delta$. Moreover, we have that

$$\mathrm{usw}(\Pi^b) = \mathrm{usw}(\pi) + \sum_{e=(v_{ij}^r, v_{kj}^s) : \delta(a_{i}, a_{j}) \neq \emptyset} w(e) = \mathrm{usw}(\pi) + \sum_{e \in M} w(e) - P \geq k.$$ 

Next we consider $b$-EWSA. When $b = 1$, we show in Lemma 2 that we can reduce the problem to MAXIMUM MATCHING. The idea is to partition all agents into two subgroups according to the target $k$ and build a bipartite graph characterizing whether one agent from one group can improve the utility of one agent from the other group to $k$ by sharing, then the problem is just to find a maximum matching on the bipartite graph.

Lemma 2. $1$-EWSA is solvable in polynomial time.

Proof. Depending on the target $k$, we partition the set $A$ of agents into two sets $A^+_k$ and $A^-_k$ containing, respectively, the agents with their bundle value under $\pi$ at least $k$ and smaller than $k$. Now, we construct a bipartite, undirected graph $G_k = (A^+_k, A^-_k, E_k)$. Consider two agents $a_i \in A^+_k$ and $a_j \in A^-_k$ that are neighbors in the sharing graph $G_s$. An edge $e = \{a_i, a_j\}$ belongs to $E_k$ if $a_i$ can share a resource with $a_j$ to raise the utility of the latter to at least $k$; formally, there exists a resource $r \in \pi(a_i)$ such that $u_j(\pi(a_j)) + u_j(r) \geq k$.

We claim that there is a simple 2-sharing $\delta$ with $\mathrm{esw}(\Pi^b) \geq k$ if and only if there is matching $M$ in graph $G_k$ with $|M| \geq |A^-_k|$. The backward direction is clear according to the construction of $G_k$. For the forward direction, if there is a simple 2-sharing $\delta$ with $\mathrm{esw}(\Pi^b) \geq k$, we can build a matching $M$ in graph $G_k$ by adding an edge $e = \{a_i, a_j\}$ to $M$ if $\delta(\{a_i, a_j\}) \neq \emptyset$. Notice that since $\mathrm{esw}(\Pi^b) \geq k$, the edge $e = \{a_i, a_j\}$ must be in $E_k$. Since $\mathrm{esw}(\Pi^b) \geq k$, according to construction of $G_k$ and $M$, we have that
for each \(a_j \in A_k\), there exists an agent \(a_i \in A_k\) such that \(\{a_i, a_j\} \in M\), and hence \(|M| \geq |A_k|\). Thus, we just need to check whether the maximum matching in graph \(G_k\) has size at least \(|A_k|\), which is solvable in polynomial time.

On the other hand, we show that \(b\)-EWSA is NP-hard when \(b \geq 2\). Notice that there is a trivial reduction from 3-Partition to \(b\)-EWSA if \(b \geq 3\). To show the result for any \(b \geq 2\), we reduce from the strongly NP-hard Numerical Three-dimensional Matching (N3DM) \cite{GareyJohnson1979}.

**Theorem 2.** \(b\)-EWSA is NP-hard for any constant \(b \geq 2\).

**Proof.** We present a polynomial-time many-one reduction from the NP-hard N3DM. Therein, given 3 multisets of positive integers \(X, Y, Z\), each containing \(m\) elements, and a bound \(T\), the task is to decide whether there is a partition \(S_1, S_2, \ldots, S_m\) of \(X \cup Y \cup Z\) such that each \(S_i\) contains exactly one element from each of \(X, Y, Z\) and the sum of numbers in each \(S_i\) is equal to \(T\). Given an instance \((X, Y, Z)\) of N3DM, we construct an instance of \(b\)-EWSA as follows. Without loss of generality, assume all elements from \(X \cup Y \cup Z\) are smaller than \(T\) and the sum of them is equal to \(B := mT\). We set the goal \(k = (B^2 + B + 1)T\). We create 3 groups of agents corresponding to the 3 multisets \(X, Y, Z\). For each \(x_i \in X\), we create an agent \(a_i^1\) in group 1 who holds a large resource that is valued as \(k\) by itself and \(B^2T + x_i\) by all other agents. For each \(y_i \in Y\), we create an agent \(a_i^2\) in group 2 who holds a middle resource that is valued as \(k\) by itself and \(BT + y_i\) by all other agents. For each \(z_i \in Z\), we create an agent \(a_i^3\) in group 3 who holds a small resource that is valued as \(z_i\) by itself and 0 by all other agents. In the initial allocation, all \(2m\) agents in group 1 and group 2 have utility exactly \(k\) and all \(m\) agents from group 3 have utility less than \(k\).

\[\Rightarrow:\] If \((X, Y, Z)\) is a “yes”-instance of N3DM, then based on the partition \(S_1, S_2, \ldots, S_m\) with the claimed properties, we can find a sharing \(\delta\) such that each agent participates in at most two sharings and has utility at least \(k\) under \(\Pi^\delta\) as follows. For each \(S_i = \{x_{i_1}, y_{i_2}, z_{i_3}\}\) with \(x_{i_1} + y_{i_2} + z_{i_3} = T\), we let agent \(a^1_{i_1}\) share its big resource with \(a^1_{i_2}\) and let agent \(a^2_{i_2}\) share its middle resource with \(a^3_{i_3}\) such that agent \(a^3_{i_3}\) has utility \(B^2T + x_{i_1} + BT + y_{i_2} + z_{i_3} = (B^2 + B + 1)T = k\). Since each \(S_i\) contains exactly one element from each of \(X, Y, Z\), we have that each agent in group 3 has value exactly \(k\) under \(\Pi^\delta\). Thus all agents have value at least \(k\) under \(\Pi^\delta\). Moreover, every agent in group 1 and group 2 participates in exactly one sharing and every agent in group 3 participates in exactly two sharings.

\[\Leftarrow:\] If the instance of \(b\)-EWSA is a “yes”-instance, then there is a sharing \(\delta\) such that each agent has utility at least \(k\) under \(\Pi^\delta\). For each agent \(a^3\) in group 3, let \(S_i^3 = \Pi^\delta(a^3)\) be the set of resources \(a^3\) has access to under \(\Pi^\delta\). Since each agent \(a^3\) in group 3 values all small items held by other agents as 0, without loss of generality, we can assume \(S_i^3\) contains no small items except for the one initially held by \(a^3\) itself. Notice that from the viewpoints of agents in group 3, the sum of value of all middle resources is smaller than one big resource. Since each \(a^3\) has utility at least \(k\) under \(\Pi^\delta\), \(S_i^3\) contains at least one big resource. It follows that each \(S_i^3\) contains exactly one big resource as each big resource can be shared by at most one agent from group 3 and there are \(m\) agents in group 3 and \(m\) big resources. Next, to guarantee that each agent \(a^3\) has utility at least \(k\), each \(S_i^3\) should contain at least one middle resource. Again, since each middle resource can be shared by at most one agent and there are \(m\) agents in group 3 and \(m\) middle resources, each \(S_i^3\) contains exactly one middle resource. Thus, each \(S_i^3\) contains exactly one big resource, one middle resource, and one small resource, and each resource is contained in exactly one \(S_i^3\). According to the construction, based on \(S_1^3, S_2^3, \ldots, S_m^3\), we can find a partition \(S_1, S_2, \ldots, S_m\) of \(X \cup Y \cup Z\) such that each \(S_i\) contains exactly one element from each of \(X, Y, Z\) and the sum of numbers in each \(S_i\) is at least \(k - B^2T - BT = T\). Since the sum of all elements from \(X \cup Y \cup Z\) is \(mT\), the sum of numbers in each \(S_i\) is exactly \(T\). Therefore, \((X, Y, Z)\) is a “yes”-instance of N3DM.

\[\square\]

## 4 Reducing Envy by Sharing

In this section, we study the problem of reducing envy through sharing, defined as follows.

**Definition 5.** Given an initial complete allocation \(\pi\) of resources \(\mathcal{R}\) to agents \(\mathcal{A}\), a sharing graph \(G_s\), an attention graph \(G_a\), and a non-negative integer \(k\), Envy Reducing Sharing Allocation (ERSA) asks if there is a simple 2-sharing \(\delta\) such that the number of envious agents \(|\text{Env}(\Pi^\delta)|\) \(\leq k\).

Notice that in the above definition we restrict that the sharing is a simple 2-sharing, i.e., 1-bounded 2-sharing, because the problem for reducing envy in this setting is already NP-hard, even in a special case when the attention graph and the sharing graph are (bidirectional) cliques and the goal is to decrease the number of envious agents by one, as shown in Theorem 3.
Table 2: Utility functions in the proof of Theorem 3. We use the Iverson bracket notation: for some logical expression $X$, $[X]$ is one if $X$ is true and zero otherwise.

| $u$ | $r^a_j$ | $r^a_{j'}$ | $r^p_j$ | $r^s$ |
|-----|--------|----------|-----|----|
| $a_i$ | $[\{v_i, v_j\} \in E]$ | 0 | 3 | 3 |
| $p_i$ | 1 | 1 | 3 | 3 |
| $s$ | 0 | 0 | 0 | 3 |

**Theorem 3.** ERSA is NP-hard even if the attention graph and the sharing graph are (bidirectional) cliques, and the goal is to reduce the number of envious agents by at least one.

**Proof.** We present a polynomial-time many-one reduction from the NP-hard INDEPENDENT SET problem. Therein, for an undirected graph $G = (V, E)$ and an integer $\ell$, we ask whether there is a subset of $V$ of at least $\ell$ vertices that are pairwise non-adjacent.

For an instance $(G = (V, E), \ell)$ of INDEPENDENT SET with $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance of ERSA as follows. For each vertex $v_i \in V$, we create an agent $a_i$ who initially has two resources $r_i^a$ and $r_i^{a'}$. Moreover, we add $\ell$ more agents: providers $p_1, p_2, \ldots, p_{\ell-1}$ and a special provider $s$. Initially, we allocate resource $r_i^p$ to each provider $p_i$ and resource $r^s$ to the special provider. Next we specify utility functions (see also Table 2). Each agent $a_i$ has value 1 for each resource $r_i^a$ for which $\{v_i, v_j\} \in E$, $j \in [n]$, and value 0 for other resources $r_i^a$ (note that, in particular, agent $a_i$ gives value 0 to its own resource). Each agent $a_i$ has value 0 for all $r_i^a$, $j \in [n]$, and value 3 for other resources. Each provider $p_i$ gives value 1 to resources in $\{r_i^a, r_i^a, \ldots, r_i^a\} \cup \{r_i^{a'}, r_i^{a'}, \ldots, r_i^{a'}\}$ and value 3 to other resources. The special provider $s$ has value 3 for its own resource $r^s$ and value 0 for all other ones. The attention graph is a (bidirectional) clique and the sharing graph is also a clique, so every two agents can share their resources. By the construction, initially there are $n$ envious agents: $a_1, a_2, \ldots, a_n$. Thus, we end the construction by setting the target number $k = n - 1$, so we aim at decreasing the number of envious agents by at least 1.

In what follows we show that there is an independent set of size at least $\ell$ in $G$ if and only if there is simple 2-sharing $\delta$ such that $|\text{Env}(\Pi^b)| \leq k$. Notice that in every simple 2-sharing, the special provider $s$ is not envious.

$\Rightarrow$: Suppose that there is an independent set $S$ of size $\ell$ in $G$. Let $S'$ be a set of $\ell$ agents from $\{a_1, a_2, \ldots, a_n\}$ corresponding to $S$. Then, each agent in $S'$ can share with a different provider (including the special provider) and increase its own value to 3. After this sharing, denoted by $\delta$, agents in $S'$ do not envy any provider since from their point of view each provider has a bundle of value 3. In addition, since vertices in $S$ are pairwise non-adjacent, agents in $S'$ do not envy each other as they see each other having a bundle of value $3 + 0 + 0 = 3$. Together with the special provider, there are at least $\ell + 1$ non-envious agents. Consequently, $|\text{Env}(\Pi^b)| \leq k$.

$\Leftarrow$: Suppose that there is a simple 2-sharing $\delta$ such that $|\text{Env}(\Pi^b)| \leq k$. Let $N$ be the set of non-envious agents excluding the special provider $s$ after the sharing. Since overall there are $n+\ell$ agents and at most $k$ of them are envious, we have $|N| \geq (n+\ell)-k-1 = \ell$ ($s$ is not included in $N$). Denote $N_a = N \cap \{a_1, a_2, \ldots, a_n\}$ and $N_p = N \cap \{p_1, p_2, \ldots, p_{\ell-1}\}$. Then $|N_a| + |N_p| = |N| \geq \ell$. Since $|N_p| \leq \ell - 1$, we have $N_a \neq \emptyset$. We will show that $N_p = \emptyset$. Suppose, towards a contradiction, that $N_p \neq \emptyset$ and let $a_i \in N_a$ and $p_j \in N_p$. Initially, $u_{a_i}(\pi(a_i)) = 0$ and $u_{a_i}(\pi(p_j)) = 3$. Since $a_i$ is not envious after the sharing $\delta$, $a_i$ must share with a provider. Consequently, $u_{p_j}(\Pi^b(a_i)) = 1 + 1 + 3 = 5$. Since initially $u_{p_j}(\pi(p_j)) = 3$ and $p_j$ is not envious after the sharing $\delta$, $p_j$ must share with another provider. However, this will make $a_i$ envious again. Thus, $N_p = \emptyset$, and hence $N = N_a \subseteq \{a_1, a_2, \ldots, a_n\}$. Now to make all agents in $N$ non-envious, all of them have to share with one provider. For any two agents $a_i, a_j \in N$, let $r_i^p$ and $r_j^p$ be the resources shared to $a_i$ and $a_j$, respectively. Since $a_i$ is not envious towards $a_j$, we have that $u_{a_i}(\Pi^b(a_j)) = 0$ and $u_{a_i}(\Pi^b(a_i)) = 3$. Thus the corresponding vertices for agents in $N$ form an independent set of size at least $\ell$ in $G$. □

Theorem 3 in fact constitutes a strong intractability result and it calls for further studies on other specific features of the input. We counteract the intractability result of ERSA (Theorem 3) by considering cases with few agents, tree-like graphs, identical utility functions, or few resources.
4.1 Reducing Envy for Few Agents

The simple fact that for \( n \) agents and \( m \) resources there are at most \( m^n \) possible 2-sharings leads to a straightforward brute-force algorithm that runs in polynomial time if the number of agents is constant.

**Observation 1.** ERSA with \( n \) agents and \( m \) resources is solvable in \( O\left(\left(\frac{m}{2}\right)^n m n\right) \) time.

*Proof.* We can simply enumerate all possible sharings and compute for each sharing the number of envious agents. A sharing contains at most \( \binom{m}{2} \) shared resources and each shared resource has at most \( n \) destinations, thus there are at most \( \left(\frac{m}{2}\right)^n \) different sharings. Computing the number of envious agents under a sharing can be done in \( O(mn) \) time.

However, due to the factor \( m \) in the base, the running time of such an algorithm skyrockets with large number of resources (even for small, fixed values of \( n \)). We improve this by showing that ERSA is fixed-parameter tractable with respect to the number of agents.

**Theorem 4.** ERSA with \( n \) agents and \( m \) resources is solvable in \( O((2n)^mn^2) \) time.

The high-level idea behind Theorem 4 is as follows. In order to find a desired sharing, our algorithm guesses target agents—a set of at least \( n - k \) agents that do not envy in the desired sharing—and a sharing configuration—a set of ordered pairs of agents that share some resource in the desired sharing. Then, for such a guessed pair, the algorithm tests whether the desired sharing indeed exists. If it is true for at least one guessed pair, then the algorithm returns “yes”; otherwise, it returns “no.” The main difficulty in checking the existence of the desired sharing is that we need to maintain the envy-freeness within target agents while increasing their utilities.

Before stating the algorithm more formally, we give some notation and definitions. Let \( C \) be a fixed set of target agents.

**Definition 6.** A sharing configuration \( M \) for a set \( C \) of target agents is a set of arcs such that

1. \( M \) is a set of vertex-disjoint arcs and
2. if \( (i,j) \in M \), then \( \{i,j\} \in E_s \) and \( j \in C \).

A simple 2-sharing \( \delta \) is called a realization of \( M \) if \( \delta \) only specifies the shared resource for each arc in \( M \); formally, for each \( (i,j) \in M \), we have that \( \delta(\{i,j\}) \neq \emptyset \) \( \land \delta(\{i,j\}) \in \pi(i) \), and for each \( \{i,j\} \) with \( \delta(\{i,j\}) \neq \emptyset \), we have that \((i,j) \in M \land \delta(\{i,j\}) \in \pi(i) \) \( \lor \{(j,i) \in M \land \delta(\{i,j\}) \in \pi(j)\}) \). A realization \( \delta \) is feasible if \( C \cap \text{Env}(\Pi^\delta) = \emptyset \), i.e., no agent in \( C \) will be envious under \( \delta \).

Note that a sharing configuration does not only describe shares of a proper simple 2-sharing but also ensures that the shared resources are indeed shared “to” the target agents; we justify this restriction later in Lemma 4.

Let us fix a sharing configuration \( M \) for \( C \). For each target agent \( a_i \), we define a set \( P_i^0 \) of initially possible resources that \( a_i \) might get in some realization of \( M \). For convenience, we augment each \( P_i^0 \) with a dummy resource \( d_i \) that has utility zero for every agent. Formally, we have

\[
P_i^0 := \begin{cases} \pi(j) \cup \{d_i\} & \text{if } \exists j \text{ such that } (j,i) \in M, \\ \{d_i\} & \text{otherwise.} \end{cases}
\]

For each target agent \( a_i \in C \), we define a utility threshold \( t_i \)—the smallest utility agent \( a_i \) must have such that \( a_i \) will not envy agents outside \( C \). Formally, if there is at least one agent \( a_j \notin C \) such that \((a_i,a_j) \in G_i\), then

\[
t_i := \max_{a_j \notin C, (a_i,a_j) \in G_i} u_i(\pi(j)),
\]

otherwise, \( t_i := 0 \). If some target agent cannot achieve its utility threshold by obtaining at most one of its initially possible resources, then there is no realization of \( M \) such that the agent does not envy. We express it more formally in Observation 2.

**Observation 2.** There is no feasible realization of \( M \) in which some agent \( a_i \in C \) gets a resource \( r \in P_i^0 \) such that \( u_i(\pi(i) \cup \{r\}) < t_i \).
For each target agent $a_i \in C$, we define a set of forbidden resources.

**Definition 7.** Let $C = \{a_1, a_2, \ldots, a_q\}$ and let $\mathcal{P} = \{P_1, P_2, \ldots, P_q\}$ be a family of sets of possible resources for the target agents. Then resource $r \in P_i$ is a forbidden resource for some target agent $a_i$ if there is some target agent $a_j$ with $(a_j, a_i) \in \mathcal{G}_i$ such that

$$\max\{u_j(\pi(j) \cup \{r'\}) | r' \in P_j\} < u_j(\pi(i) \cup \{r\}),$$

that is, if agent $a_i$ gets resource $r$, then agent $a_j$ will envy $a_i$ even if $a_j$ gets its most valuable resource from $P_j$. We denote the set of all forbidden resources for $a_i$ as $F_i(\mathcal{P})$.

Observe that in every feasible realization no target agent gets one of its forbidden resources since otherwise there is another target agent that envies.

**Observation 3.** Let $\mathcal{P}$ be a family of possible resources for the target agents. In every feasible realization no target agent $a_i$ gets a resource from $F_i(\mathcal{P})$.

Based on the above observations, Algorithm 1 tests whether for a pair of a set $C$ of target agents and a sharing configuration $M$ there is a feasible realization. The algorithm keeps track of the possible resources $P_i$ for each target agent $a_i$. Starting with each $P_i$ equal to the corresponding set of initially possible resources, it utilizes Observation 2 and removes the “low-utility” resources. Then, utilizing Observation 3, the algorithm finds all forbidden resources for a particular collection of the possible resources for the target agents and eliminates the forbidden resources. This procedure is repeated exhaustively. Finally, if at least one of the possible resource sets is empty, the algorithm outputs “no.” Otherwise, the algorithm returns “yes” since at least one resource remained in the set of possible resources for every target agent.

After applying Observation 2 and 3 to Algorithm 1 and proving its correctness (Lemma 3 and Lemma 4), we finish the proof of Theorem 1

**Lemma 3.** In Algorithm 1, if some $P_i$ is empty after the repeat-loop, then there is no feasible realization for $M$ for $C$.

**Proof.** Suppose towards a contradiction that $P_i$ is empty after the repeat-loop and there is a feasible realization $\delta$ for $M$ such that $C \cap \text{Env}(\Pi^{\delta}) = \emptyset$. Let $t^* \in \Pi^{\delta}(i) \setminus \pi(i)$ be the resource shared to agent $i$. For any agent $j \in C$ with $(j, i) \in \mathcal{E}_i$, let $t_j \in \Pi^{\delta}(j) \setminus \pi(j)$ be the resource shared to agent $j$. We remark that here resources $t^*$ and $t_j$ could be dummy resources. Since $j \in C$ and $C \cap \text{Env}(\Pi^{\delta}) = \emptyset$, we have $j \notin \text{Env}(\Pi^{\delta})$, and hence $j$ does not envy $i$ after $\delta$. Thus $u_j(\pi(j)) + u_j(t_j) \geq u_j(\pi(i)) + u_j(t^*)$. Since this holds for any agent $j \in C$ with $(j, i) \in \mathcal{E}_i$, resource $t^* \in S_i$ is not a forbidden resource, which contradicts that $S_i$ becomes empty after deleting blocking resources.

**Lemma 4.** If there is a simple 2-sharing $\sigma$ such that $C \cap \text{Env}(\Pi^\sigma) = \emptyset$, then there is a sharing configuration $M$ for $C$ that has a feasible realization and Algorithm 1 outputs “yes”; otherwise, the algorithm outputs “no”.

**Proof.** We first show that if there is a simple 2-sharing $\sigma$ such that $C \cap \text{Env}(\Pi^\sigma) = \emptyset$, then there is a sharing configuration $M$ for $C$ that has a feasible realization $\delta$. Recall that all sharing configurations have the restriction that the shared resources are indeed shared “to” the target agents. We now justify this...
restriction. Let \( E = \{ (i, j) \in E \mid \sigma((i, j)) \neq \emptyset \} \) be the set of edges where there is a sharing in \( \sigma \). Let \( E_1 \subseteq E \) be the set of edges that could be used in a sharing configuration for \( C \), i.e.,
\[
E_1 = \{ (i, j) \in E \mid (\sigma((i, j)) \in \pi(i) \land j \in C) \\
\lor (\sigma((i, j)) \in \pi(j) \land i \in C) \}.
\]

We construct a new simple 2-sharing \( \delta \) by restricting \( \delta \) on \( E_1 \):
\[
\delta((i, j)) = \begin{cases} 
\sigma((i, j)) & \text{if } (i, j) \in E_1, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Then we define the sharing configuration as
\[
M = \{ (i, j) \mid \delta((i, j)) \neq \emptyset \land \delta((i, j)) \in \pi(i) \}.
\]

Now it is clear that \( M \) is a sharing configuration for \( C \) and \( \delta \) is a realization of \( M \). Moreover, since all sharings through edges in \( E \setminus E_1 \) only increase the bundles of agents in \( \mathcal{A} \setminus C \), we have that
\[
\begin{align*}
\forall i \in C, & \quad u_i(\Pi^\delta) = u_i(\Pi^\sigma) \\
\text{otherwise.} & \quad u_i(\Pi^\delta) \leq u_i(\Pi^\sigma)
\end{align*}
\]

Since \( C \cap \text{Env}(\Pi^\sigma) = \emptyset \), we have that \( C \cap \text{Env}(\Pi^\delta) = \emptyset \). Therefore, \( \delta \) is a feasible realization.

Next, we show that there is a sharing configuration \( M \) for \( C \) that has a feasible realization if and only if Algorithm 1 outputs “yes”. According to Lemma 3 if the algorithm output “no”, then there is no feasible realization for \( M \) for \( C \). On the other hand, if the algorithm output “yes”, then at the end of the algorithm we get \( P_i \neq \emptyset \) for each agent \( a_i \in C \). We can build a simple 2-sharing \( \delta \) by choosing an arbitrary resource from \( P_i \) for each agent \( a_i \in C \). It is clear that \( \delta \) is a realization for \( M \) for \( C \). Moreover, for any agent \( a_i \in C \), \( a_i \) does not envy agents outside \( C \) (as resources that do not have high enough utility for \( a_i \) such that \( a_i \) does not envy agents outside \( C \) are removed from \( P_i \) and \( P_i \neq \emptyset \) at the end) and \( a_i \) does not envy agents in \( C \) (as \( B = \emptyset \)). Therefore, \( \delta \) is a feasible realization for \( M \) for \( C \).

Eventually, we are set to present the proof of Theorem 4.

**Proof of Theorem 4.** According to Lemma 4 to solve an instance of ERSA, it is enough to test whether there is a pair of a target subset and a sharing configuration that has a feasible realization. Since, checking a feasible realization, due to Lemma 4, can be done by Algorithm 1, we check all such possible pairs and return “yes” if there is (at least) one that has a feasible realization.

There are \( O(2^n) \) possible target sets and at most \( n^3 \) possible sharing configurations per target set, which gives \( O((2n)^n) \) cases. For each case, we apply Algorithm 1. Therein, the for-loop takes \( O(nm) \) time. Concerning the repeat-loop that runs at most \( m \) times, computing the set \( B \) takes \( O(nm) \) time; thus, the repeat-loop takes \( O(nm^2) \) time. Finally, Algorithm 1 runs in \( O(nm^2) \) time and ERSA can be solved in \( O((2n)^n \cdot m^2) \) time.

Next, we show that restricting the parameter \( k \) “number of envious agents” does not help to make ERSA solvable in polynomial time.

**Theorem 5.** ERSA is NP-hard even if the goal is to reduce the number of envious agents from one to zero.

**Proof.** We provide a polynomial-time many-one reduction from the NP-hard 3-CNF-SAT problem that asks whether a given Boolean expression in conjunctive normal form with each clause of size at most three is satisfiable. From now on we stick to the following notation. Let \( \phi = \bigwedge_{i \in [n]} C_i \) be a 3-CNF formula over a set \( X = \{ x_1, x_2, \ldots, x_n \} \) of variables where a clause \( C_i, i \in [n] \), is of the form \( (\ell_1^i \lor \ell_2^i \lor \ell_3^i) \). We use a standard naming scheme and call each \( \ell_j^i \) a literal.

We first describe the **leader gadget**, the **clause gadget**, and the **variable gadget**. See Figure 1 for an overview on the construction. Then, we show how to connect our gadgets to achieve a desired instance of ERSA. Eventually, we show the reduction’s correctness preceded by a discussion on the structure of the built instance. Notably, in our construction we use a unanimous utility function, that is, each agent we introduce has the same utility function over all introduced resources.
is the same. The envious agents are framed. Numbers in brackets indicate an initial allocation (e.g., (0) gets for formula (a) The leader gadget for formula \( \phi \). (b) The variable gadget for variable \( x \). (c) The clause gadget for clause \( \ell_i \). Figure 1: The gadgets in the construction in the proof of Theorem 4. The utility function for all agents is the same. The envious agents are framed. Numbers in brackets indicate an initial allocation (e.g., (0) denotes an empty bundle; (1, 2) means a bundle of a value-one and a value-two resource).

Figure 2: Connections of gadgets for literal \( \ell_i \) (over variable \( x \)) of some clause \( C_i \). Envious agents are framed. Other possible literals of \( C_i \) are omitted for clarity.

Construction. The leader gadget consists of two agents: the leader \( a_L(\phi) \) and the follower \( a_F(\phi) \). The follower is not assigned any resource, while the leader has a resource of value two. There is a directed arc from the follower to the leader. Thus, initially, the follower envies the leader.

The variable gadget for a variable \( x \in X \) consists of two value agents \( a(x) \) and \( \bar{a}(x) \), and two dummy agents \( d(x, 1) \) and \( d(x, 2) \). The two value agents represent, respectively, assigning “true” and “false” to \( x \). Both dummy agents are paying attention to both value agents (but not vice versa). Initially, three resources are allocated. The resource of value three is allocated to \( d(x, 1) \) and two resources, one of value one and one of value two, are allocated to \( d(x, 2) \). As a result, initially, no agent within the gadget envies.

To describe the clause gadget, let us fix a clause \( C_i = (\ell_1^i \lor \ell_2^i \lor \ell_3^i) \). For each literal \( \ell \) in the clause we add two agents: the donor \( \bar{d}(\ell) \) and the recipient \( \bar{d}(\ell) \). The donor initially has a single one-valued resource, while the recipient initially gets two one-valued resources. As for the attention relation, for each literal \( \ell \), there is an arc \( (\bar{d}(\ell), \bar{d}(\ell)) \), that is, the arc points from the recipient to the donor. Eventually, the gadget contains the root agent \( a(C_i) \) that gets a single resource of value two. The root agent pays attention to recipients \( \bar{d}(\ell_1^i), \bar{d}(\ell_2^i), \) and \( \bar{d}(\ell_3^i) \). Observe that, initially, no agent within this gadget envies.

We obtain the full construction by interconnecting the gadgets (see Figure 2). First, we connect every value agent with an arc directed from the value agent to the follower. Note that we do not introduce any envy because the follower has no resources initially. Finally, for each literal \( \ell_j^i, j \in \{1, 2, 3\}, i \in [\bar{m}] \), we add an arc from a respective recipient \( \bar{d}(\ell_j^i) \) to the value agent of the corresponding variable agent; for example, for \( \ell_j^i = \neg x_y, y \in [\bar{m}] \), we would add an arc \( (\bar{d}(\ell_j^i), \bar{a}(x_y)) \). The sharing graph is just the same as the underlying graph of the attention graph.

The constructed attention graph, the sharing graph, the introduced agents, the resources allocated by the initial allocation, and their utilities together with the desired number \( k := 0 \) of envious agents form an instance of ERS, clearly computable in polynomial time.

Correctness. We show that \( \phi \) is satisfiable if and only if there exists a simple 2-sharing for the above instance such that no agent will be envious.

\( \iff \): Suppose there exists a simple 2-sharing for the above instance such that no agent will be envious. Since none of value agents has any resource, the only way to make a follower non-envious is to share the leader’s resource. This makes all value agents envious because they look at the follower that has obtained a resource of value two. Hence, to decrease the number of envious agents to zero, one has to actually provide
each value agent with a resource of value at least two. Since respective recipients that are connected to the value agents have only one-valued resources, value agents have to share the resources of the dummy agents. Among these only two suitable resources exist, two- and three-valued ones. Thus, in each variable gadget, one of the value agents needs to get a resource of value two and the other a resource of value three.

Next, we consider the clause gadget. As was already discussed, each value agent gets either a two-valued or a three-valued resource (as a result of a sharing that eliminates envy). Notably, every value agent with a three-valued resource makes every recipient agent paying attention on the value agent envious. Let us fix some literal \( \ell \in C \) whose value agent \( a(x) \) got a three-valued resource. One can easily verify that the only way to fix envy of \( d(\ell) \) is to share a resource from the donor \( \overrightarrow{d}(\ell) \). However, then the root agent \( a(C) \) becomes envious. A major observation is that this envy can only be fixed if there exists a recipient \( d(\ell') \) of another literal of clause \( C \) that does not need to share with its donor \( \overrightarrow{d}(\ell') \), which implies that the corresponding value agent for recipient \( d(\ell') \) got a two-valued resource. As a result, for each clause, at least one of the corresponding value agents has to get a two-valued resource. Then, the truth assignment that sets \( x \) true if and only if agent \( a(x) \) is shared with a two-value resource can satisfies every clause of \( \phi \).

\[ \Rightarrow: \] Suppose there exists a satisfiable truth assignment \( x \) of \( \phi \). We first make the follower share with the leader, making the follower non-envious. Next, in each variable gadget, for every variable \( x \) setting to “True”, we let agent \( a(x) \) get a two-value resource and \( a(x) \) get a three-value resource through sharing with the corresponding dummy agents; otherwise, we let agent \( a(x) \) get a three-value resource and \( a(x) \) get a two-value resource. Finally, for each clause \( C \), since \( x \) is a satisfiable truth assignment, there must be a literal \( \ell \in C \) that is assigned “True”, which means the corresponding value agent is shared with a two-value resource. Then in the corresponding clause gadget, recipient \( d(\ell) \) can share with the root agent \( a(C) \) such that \( a(C) \) has value three and the other two recipients can share with their donors such that they have value three. It is easy to verify that no agent will be envious under this simple 2-sharing.

4.2 Reducing Envy for Tree-like Graphs

We study how the tree-like structure of the sharing graph and the attention graph influences the computational complexity of ERSA. Studying tree-likeness, we hope for tractability for quasi-hierarchical social networks, where agents at the same level of the hierarchy influence each other but they rather do not do so in a cross-hierarchical manner.

Theorem \[3\] shows that when both graphs are (bidirectional) cliques ERSA is NP-hard. We continue to focus on the case when the underlying graph of the attention graph is the same as the sharing graph. Note that this restriction appears naturally when assuming that one may envy everybody one knows and one may share only with known people. Theorem \[6\] shows that in this case, if the sharing graph is a path, a tree or being very close to a tree (corresponding to a “hierarchical network”), then we can solve ERSA in polynomial time, while, intuitively, for sharing graphs being “far from a path,” presumably there is no algorithm whose exponential growth in the running time depends only on the “distance from path”.

**Theorem 6.** When the underlying graph of the attention graph is the same as the sharing graph, ERSA can be solved in polynomial time if the sharing graph has a constant treewidth (assuming the tree decomposition is given), and ERSA is W[1]-hard with respect to the pathwidth of the sharing graph.

**Proof of the first part.** Denote by \( tw \) the treewidth of the underlying graph. Let \( T = (T, \{V_t\}_{t \in V(T)}) \) be a nice tree decomposition (see [Kloks 1994] for the definition) of the underlying graph that has width \( tw \). Before defining the target values for each bag to be computed, we first define some concepts and notations.

A bundle configuration \( b_t \) for bag \( X_t \) is a function which maps every agent in \( X_t \) to a bundle of resources. We say a sharing \( \delta \) realizes a bundle configuration \( b_t \) if \( \Pi^t(a) = b_t(a) \) for each \( a \in X_t \). A bundle configuration \( b_t \) is feasible if there exists a simple 2-sharing \( \delta \) which can realize \( b_t \). For a node \( t \) of \( T \), let \( B_t \) be the set of all feasible bundle configurations for bag \( X_t \), and let \( V_t \) be the union of all the bags in the subtree of \( T \) rooted at \( t \) (including \( X_t \) itself). Denote by \( \text{Env}_{V_t}\{\delta\} \) the set of envious agents in the sub-instance induced by agents in \( V_t \) under the sharing \( \delta \). Note that \( \text{Env}_{V_t}\{\delta\} \) does not contain an agent \( a \) if \( a \) only envies agents outside \( V_t \) under \( \delta \). For a bundle configuration \( b_t \) and a bag \( X_{t'} \subseteq X_t \), denote by \( b_t[X_{t'}] \) the bundle configuration of \( b_t \) restricted on \( X_{t'} \).

Now we define our target values. For every node \( t \), every bundle configuration \( b_t^i \in B_t \) and every subset \( S_t \subseteq X_t \), define the following value:

\[
f[t, b_t^i, S_t] := \min \{|\text{Env}_{V_t}\{\delta\} | \delta \text{ realizes } b_t^i, \text{Env}_{V_t}\{\delta\} \cap X_t = S_t\}.
\]
That is, \( f[t, b'_i, S_t] \) is the minimum number of envious agents in the sub-instance induced by \( V_t \) under a sharing \( \delta \) which realizes \( b'_i \) and \( \text{Env}_{X_t}(\delta) \cap X_t = S_t \). If no such sharing exists, then \( f[t, b'_i, S_t] = +\infty \). It is easy to see that our goal is to compute \( \min \{ f[r, b'_i, S_r] \mid b'_i \in B_r, S_r \subseteq X_r \} \), where \( r \) is the root node. Next we show how to compute \( f[t, b'_i, S_t] \) for all \( t, b'_i \) and \( S_t \).

**Introduce node.** Suppose \( t \) is an introduce node with child \( t' \) such that \( X_t = X_{t'} \cup \{ v \} \) for some \( v \not\in X_{t'} \). For any \( b'_i \in B_t \) and \( S_t \subseteq X_t \), we first check whether \( b'_i \) and \( S_t \) are compatible. To this end, we compute a set \( S^0_t \subseteq X_t \) of agents who envies agents in \( X_t \) under \( b'_i \). If \( S^0_t \not\subseteq \emptyset \), then \( b'_i \) and \( S_t \) are not compatible, and we set \( f[t, b'_i, S_t] = +\infty \). Since all neighbors of \( v \) in \( V_{t'} \) are in \( X_t \), if \( v \not\in S^0_t \), then \( v \) will not be envious in the sub-instance induced by \( V_t \) under any sharing which can realize \( b'_i \). Thus, if \( v \in S_t \setminus S^0_t \), then \( b'_i \) and \( S_t \) are not compatible, and we set \( f[t, b'_i, S_t] = +\infty \). Now for any \( b'_i \in B_t \) and \( S_t \subseteq X_t \) with \( S^0_t \subseteq S_t \) and \( v \in S^0_t \Leftrightarrow v \in S_t \), we have that:

\[
f[t, b'_i, S_t] = \min\{ f[t', b'_i[X_{t'}], S_{t'}] + |S_t \setminus S_{t'}| \mid S_t \setminus S^0_t \subseteq S_t \}.\]

**Forget node.** Suppose \( t \) is a forget node with child \( t' \) such that \( X_t = X_{t'} \setminus \{ w \} \) for some \( w \in X_{t'} \). For any \( b'_i \in B_t \) and \( S_t \subseteq X_t \), we have that:

\[
f[t, b'_i, S_t] = \min\{ f[t', b'_i[X_{t'}], S_{t'}] \mid b'_i[X_{t'}] = b'_i, S_{t'} \cap X_t = S_t \}.\]

**Join node.** Suppose \( t \) is a join node with children \( t_1 \) and \( t_2 \) such that \( X_t = X_{t_1} \cup X_{t_2} \). For any \( b'_i \in B_t \) and \( S_t \subseteq X_t \), we have that:

\[
f[t, b'_i, S_t] = f[t_1, b'_i, S_t] + f[t_2, b'_i, S_t] - |S_t|.
\]

Note that \( V_{t_1} \cap V_{t_2} = X_t \) and \( S_t \) is the set of envious agents that have been counted in both \( f[t_1, b'_i, S_t] \) and \( f[t_2, b'_i, S_t] \).

Overall, for each node \( t \), we have \( |B_t| \leq m^{tw+1} \) since in any bundle configuration for \( X_t \), each agent in \( X_t \) could get at most one additional resource compared to its initial bundle. So we have at most \( n(2m)^{tw+1} \) values to be computed. Each value of an introduce node can be computed in \( O(2^{tw+1}) \) time. Each value of an forget node can be computed in constant time. So the running time is \( O(n(4m)^{tw+2}) \).

**Proof of the second part.** We present a parameterized reduction from the \( W[1] \)-hard problem \textsc{Multicolored Clique}; here, for an integer \( \ell \), and an undirected graph \( G = (V, E) \) in which each vertex is colored with one of \( \ell \) colors, the goal is to find a set of \( \ell \) pairwise adjacent, differently colored vertices. Without loss of generality, we assume that set \( V \) can be partitioned into \( \ell \) size-\( n \) sets \( V_i = \{ v^1_i, v^2_i, \ldots, v^n_i \} \), \( i \in [\ell] \), where each \( V_i \) consists of vertices of color \( i \). Similarly, we assume that there are no edges between vertices of the same color. Given an instance \( (G, \ell) \) of \textsc{Multicolored Clique}, we construct an instance of \textsc{ERSA} as follows (see also Figure \ref{fig:ersa}).

To conveniently present the construction of the instance of \textsc{ERSA}, we introduce some handy notation. Let us fix a pair of distinct colors \( i \) and \( j \). We refer to the set of edges connecting the vertices of these colors as \( E_{i,j} = \{ (v, v') \in E \mid v \in V_i, v' \in V_j \} \). Complementarily, let \( \overline{E}_{i,j} = \{ (v, v') \mid v \in V_i, v' \in V_j \} \setminus E_{i,j} \) and \( \overline{E} = \cup_{i,j \in [\ell]} \overline{E}_{i,j} \). We say that an edge \( e \) is **forbidden** if \( e \in \overline{E} \). Indeed, since forbidden edges are not part of \( G \), they cannot appear between vertices of any clique in \( G \).

**Construction.** Our construction consists of the **vertex selection** gadget and the **certification gadget**. We describe in the given order specifying how they relate to each other in the description of the certification gadget. For better understanding, we refer to Figure \ref{fig:ersa} showing the construction and Table \ref{tab:ersa} showing the utility functions.

For each color \( i \), we build the **vertex selection** gadget consisting of three agents: a **selector** \( s_i \), a **provider** \( p_i \), and a **dummy** \( d_i \). As for the attention relation, the provider \( p_i \) attends the selector \( s_i \), that, in turn, attends the dummy \( d_i \). For each vertex \( v \in V_i \) we create a **vertex** resource \( r(v) \) and give all of them to the provider. The selector initially gets two resources \( r(s_i, 1) \) and \( r(s_i, 2) \) as well as the dummy that gets \( r(d_i, 1) \) and \( r(d_i, 2) \). The desired goal of this gadget is that the provider shares with the selector exactly one vertex resource. We get this by making the selector initially envious of the dummy and ensuring that the dummy cannot share with the selector to remove the envy. To implement such a behavior, we set the utility function of the dummy and the provider to be 0 for all resources. The selector, however, gives utility 1 to both \( r(d_i, 1) \) and \( r(d_i, 2) \) and utility 2 to all vertex resources; the selector gives utility 0
to every other resource. Thus, under the initial allocation, the selector has utility 0 and envies the dummy whose bundle has utility 2 for the selector.

For each forbidden edge, we build the certification gadget, the purpose of which is to forbid selectors from being shared vertex resources representing vertices that are not adjacent in $G$. To demonstrate the gadget, let us fix a pair $\{i,j\}$ of distinct colors such that $i<j$ and a forbidden edge $e = \{v,v'\} \in E_{i,j}$. The certification gadget consists of two certification agents $c_e^i$ and $c_e^j$, each having a resource $r(c_{d,e}^i)$ and $r(c_{d,e}^j)$, respectively. Agent $c_e^i$ attends $c_e^j$. We connect the certification gadget with the respective vertex selection gadgets by letting both certification agents attend their respective selectors, that is $c_e^i$ attends $s_i$ and $c_e^j$ attends $s_j$. We make two certification agents being envious of the respective selectors according to the initial allocation, and ensure that unless the selector are shared nonadjacent (with respect to $G$) vertex resources, then we can make one of the certification agents envious. Additionally, we make sure that sharing a resource from a selector to a certification agent cannot make the certification agent envious. To this end, $c_e^i$ gives utility 1 to both resources initially given to $s_i$. Further, $c_e^j$ gives utility 2 for vertex resource $r(v)$ and utility 1 for all other vertex resources of color $i$. We set the utility function of $c_e^j$ for the initial resources of $s_j$ and vertex resources of color $j$ analogously. Finally, both certification agents give each other’s resource utility 3 and utility 0 for every other resource.

Finally we set $k = \binom{\ell}{2} n^2 - |E|$, which is the number of all forbidden edges. This finishes our construction of the instance of ERSA. Concerning the pathwidth, denote the set of all selectors, all providers and all dummies as $B$. There is a path decomposition $P = (X_1, X_2, \ldots, X_m)$, where $m = |E|$ is the number of forbidden edges, and each bag $X_i$ contains all vertices in $B$ and two certification agents for some forbidden edge. Therefore, the pathwidth of the underlying graph of the attention graph $G_\ell$ is at most $|B| + 2 = 3\ell + 2$.

**Correctness.** We show that the instance of Multicolored Clique is a “yes”-instance if and only if the constructed instance of ERSA is a “yes”-instance. Note that under the initial allocation, every selector $s_i$ envies the corresponding dummy $d_i$ since $u_{s_i}(\pi(s_i)) = 0 < u_{s_i}(\pi(d_i)) = 2$. For each forbidden edge $e$, both certification agents $c_e^i$ and $c_e^j$ are envious because $u_{c_e^i}(\pi(c_e^i)) = u_{c_e^j}(\pi(c_e^j)) = 0 < u_{c_e^i}(\pi(s_i)) = u_{c_e^j}(\pi(s_j)) = 2$.

$\Rightarrow$: Suppose there is a clique $C^*$ of size $\ell$ in $G$ containing one vertex from each color. Based on $C^*$, we construct a sharing function $\delta$ such that after this sharing, all selectors are not envious, and for each forbidden edge $e$, exactly one agent from $\{c_e^i, c_e^j\}$ is envious. For each color $i$, let $v_{e,i} \in C^*$. Then in sharing $\delta$ let provider $p_i$ share resource $r(v_{e,i}^i)$ to selector $s_i$. Hence, each selector $s_i$ receives a resource of utility 2

![Figure 3: The vertex selection gadget for colors $i$ and $j$ and the certification gadget for a forbidden edge $e$ with endpoints of colors $i$ and $j$ in the constructed instance.](image_url)

| $u$ | $r(v)$ | $r(d_i,t)$ | $r(s_i,t)$ | $r(c_{d,e}^i)$ |
|-----|--------|------------|------------|---------------|
| $d_i$ | 0 | 0 | 0 | 0 |
| $p_i$ | 0 | 0 | 0 | 0 |
| $s_i$ | 2 | 1 | 0 | 0 |
| $c_e^i$ | \{2 | $v \in e \cap V_i$ | 0 | 1 | \{3 | $e = e' \land i \neq h$ |
| | 1 | o.w. | | 0 | o.w. |
and they become not envious. Then for each forbidden edge \(e = \{v^i_x, v^j_y\}\), since \(C^*\) is a clique, we have that \(\{v^i_x, v^j_y\} \notin C^*\). According to our current sharing function, the following two events could not happen at the same time:

1. resource \(r(v^i_x)\) is shared to selector \(s_i\);
2. resource \(r(v^j_y)\) is shared to selector \(s_j\).

Without loss of generality, assume that resource \(r(v^i_x)\) is shared to selector \(s_i\) and resource \(r(v^j_y)\) instead of \(r(v^i_x)\) is shared to selector \(s_j\). Then let agent \(c^i_x\) share resource \(r(c^i_x)\) to agent \(c^j_y\) and this makes agent \(c^j_y\) not envious, since \(u_{c^j_y}(\pi(c^j_y) \cup \{r(c^i_x)\}) = 3 = u_{c^j_y}(\{\pi(s_j) \cup \{r(v^j_y)\})\). This finishes our definition of sharing \(\delta\).

After this sharing, exactly one certification agent for each forbidden edge is envious and no other agent is envious, thus there are \(k = n^2 - |E|\) envious agents after sharing \(\delta\).

\[\Leftarrow\]: Suppose there is a sharing \(\delta\) such that only \(k\) agents will be envious after this sharing. First, for each forbidden edge \(e\), the only way to make agent \(c^i_x\) not envious is to share resource \(r(c^i_x)\) from \(c^i_x\) to \(c^j_y\) and the only way to make agent \(c^j_y\) not envious is to share resource \(r(c^j_y)\) from \(c^j_y\) to \(c^i_x\). Since there could be at most one sharing between \(c^i_x\) and \(c^j_y\), at least one agent of \(c^i_x\) and \(c^j_y\) will be envious. Since \(k = \left(\frac{n}{2}\right)n^2 - |E|\) is exactly the number of all forbidden edges, it must be that after sharing \(\delta\), no selector \(a_i\) is not envious, and for each forbidden edge \(e\), exactly one agent from \(\{c^i_x, c^j_y\}\) is envious. Based on this fact, we show in the following that there is a clique \(C^*\) of size \(\ell\) containing one vertex from each color.

To make selector \(s_i\) not envious, \(s_i\) has to receive a resource from the provider \(p_i\). For each \(i \in [\ell]\), let \(r(v^i_x)\) be the resource shared to \(s_i\), we claim that \(C^* = \{v^i_{x_1}, v^i_{x_2}, \ldots, v^i_{x_l}\}\) is the desired clique. Suppose \(C^*\) is not a clique, then there are two vertices \(v^i_{x_1}\) and \(v^i_{x_j}\) such that \(\{v^i_{x_1}, v^i_{x_j}\} \notin E\), which means that there is a forbidden edge \(e = \{v^i_{x_1}, v^i_{x_j}\}\). Since every selector \(s_i\) shares with provider \(p_i\), agents \(c^i_{x_1}\) and \(c^i_{x_j}\) can only share with each other and the maximum utility for them after such a sharing is at most 3. However, after the sharing, \(u_{c^i_x}(\Pi^a(s_i)) = u_{c^i_x}(\{\pi(s_i), r^i_{x_1}\}) = 4\) and \(u_{c^i_x}(\Pi^a(s_j)) = u_{c^i_x}(\{\pi(s_j), r^i_{x_j}\}) = 4\), then both \(c^i_{x_1}\) and \(c^i_{x_j}\) will be envious, which is a contradiction. Therefore, \(C^*\) is a clique of size \(\ell\) containing one vertex from each color.

4.3 Reducing Envy for Identical Utility Functions

We proceed by studying the natural special case where all agents have the same utility function. Already in this constrained setting of homogeneous agents allocation problems frequently become hard [Bouveret and Lang 2008]. This is why this scenario also attracts quite some attention in the fair allocation literature [Nguyen et al. 2013, Biswas and Barman 2018, Barman et al. 2018, Bouveret and Lang 2008]. Here, we focus on cliques that naturally model small, dense communities and allow for convenient comparison with the classical setting of indivisible, non-shareable resources (where the attention graph is implicitly assumed to be a bidirectional clique).

Theorem 4 already shows that restricting the attention graph and the sharing graph to be cliques is not enough to make ERSA solvable in polynomial time. However, if all agents have the same utility function, Theorem 7 shows that restricting the attention graph to be a clique alone can achieve polynomial-time solvability. The idea behind the proof of Theorem 7 is that unenvious agents are exactly those with the highest utility, so we can guess the highest utility and then reduce the problem to a bounded number of Maximum Matching.

Theorem 7. ERSA is solvable in polynomial time if the attention graph is a bidirectional clique and all agents have the same utility function.

Proof. Let \(u\) be the utility function for all agents and let \(N\) be the set of non-envious agents after a sharing \(\delta\). For every two agents \(a_i, a_j \in N\), since the attention graph is a clique, we have that \(u(\Pi^a(a_i)) \geq u(\Pi^a(a_j))\) and \(u(\Pi^a(a_j)) \geq u(\Pi^a(a_i))\); hence \(u(\Pi^a(a_i)) = u(\Pi^a(a_j))\). Denote the utility of all agents in \(N\) by \(u^*\). Clearly, \(u(\Pi^a(a_k)) < u^*\) for every agent \(a_k \notin N\). Thus, under any sharing allocation, the set of unenvious agents is exactly the set of agents who have the highest utility. Based on this observation, it suffices to compute, for each possible target utility \(u^*\), the largest number of agents that can have exactly utility \(u^*\) after some sharing.

We first show that the number of different possibilities for the target utility \(u^*\) is \(O(nm)\). Each agent can get at most one resource through a simple 2-sharing and there are overall \(m\) different resources, so each agent could end with \(O(m)\) different utilities. Together with the number \(n\) of agents, we get the
desired upper bound \(O(nm)\). Let \(S\) be the set of all these utilities, that is, \(S = \{t \mid t = u(\pi(a_i)) + u(r)\ \text{for some agent} \ a_i \in A \ \text{and some resource} \ r \in R \ \setminus \pi(a_i)\}\). Let \(u_0 = \max_{i \in A} u(\pi(a_i))\) be the largest utility of some agent before a sharing. We only need to consider utility values in \(S' = \{t \in S \mid t \geq u_0\}\) as any target utility \(u^*\) should be at least \(u_0\).

Next we show that for every target utility \(u^* \in S'\) computing the largest number of agents that can have utility \(u^*\) after a sharing \(\delta\) can be reduced to \textsc{Maximum Matching}. We first find the set \(N_0\) of agents who already have utility \(u^*\) before sharing, that is, \(N_0 = \{a_i \in A \mid u(\pi(a_i)) = u^*\}\). Notice that \(N_0 = \emptyset\) if \(u^* > u_0\). Then we construct a graph \(G = (A, E)\) where vertices being the agents and an edge \(e = \{a_i, a_j\}\) belongs to \(E\) if one of agents \(a_i\) and \(a_j\) can strictly increase its utility to \(u^*\) through a sharing between \(a_i\) and \(a_j\). Formally, \(e = \{a_i, a_j\} \in E\) if \(\{a_i, a_j\} \in E_s\), \(a_i \notin N_0\), and there exists a resource \(r \in \pi(a_j)\) such that \(u(\pi(a_i)) + u(r) = u^*\) or \(a_j \notin N_0\) and there exists a resource \(r \in \pi(a_i)\) such that \(u(\pi(a_j)) + u(r) = u^*\).

By computing a maximum matching in \(G\), which can be done in polynomial time \cite{hopcroft1973}, we can find the largest number of agents in \(A \setminus N_0\) who can increase their utilities to \(u^*\) through a sharing.

We complement Theorem 7 by showing in Theorem 8 that identical utility functions together with the sharing graph being a clique are not sufficient to make \textsc{ERSA} solvable in polynomial time. Note that Theorem 7 and Theorem 8 show an interesting contrast between the influence of the completeness of the attention graph and the sharing graph on the problem’s computational complexity.

**Theorem 8.** \textsc{ERSA} is \(\text{NP-hard}\) even if the sharing graph is a clique, all agents have the same utility function, and the maximum initial bundle size is one. For the same constraints, \textsc{ERSA} is \(\text{W[1]-hard}\) with respect to the parameter “number of resources.”

**Proof.** We give a polynomial-time many-one reduction from \textsc{Clique} (that is \(\text{NP-hard}\); and \(\text{W[1]-hard}\) with respect to the number of vertices in a sought clique), where we are given an undirected graph and an integer \(\ell\), and the question is whether there is a set of \(\ell\) mutually connected vertices. To this end, we fix an instance \(I = (G, \ell)\) of \textsc{Clique}, where \(V = \{v_1, v_2, \ldots, v_n\}\) and \(E = \{e_1, e_2, \ldots, e_m\}\). For brevity, let \(\ell := \binom{n}{2}\) be the number of edges in a clique of size \(\ell\). Without loss of generality we assume that \(4 \leq \ell < n\) and \(\ell \leq m\). We build an instance \(I\) of \textsc{ERSA} corresponding to \(I\) as follows.

For each vertex \(v_i \in V\) we create a vertex agent \(a(v_i)\) and for each edge \(e_i \in E\) we create an edge agent \(a(e_i)\). We also add \(2m\) dummy agents \(\{d_1, d_2, \ldots, d_{2m}\}\) and \(\ell\) happy agents \(\{h_1, h_2, \ldots, h_\ell\}\). We introduce \(\ell\) resources \(r_1, r_2, \ldots, r_\ell\). In the initial allocation \(\pi\), the resources are given to the happy agents such that each happy agent gets one resource. The resources are indistinguishable for the agents, that is, each agent gives all of them the same utility value; for convenience, we fix this utility to be one. The sharing graph is a clique. In the attention graph \(\mathcal{G}_t\), all edge agents have outgoing arcs to all happy agents, each vertex agent \(a(v), v \in V\), has an outgoing arc to every edge agent corresponding to an edge incident to \(v\), and all dummy agents have outgoing arcs to all vertex agents. The described ingredients, together with the desired number of envious agents at most \(k := m - \ell + 1\), build an instance \(I\) of \textsc{ERSA} corresponding to \textsc{Clique} instance \(I\). The whole construction can be done in polynomial time.

Observe that all edge agents are envious under \(\pi\). Since vertex agents and dummy agents do not pay attention to happy agents, no more agents are envious. Thus, there are exactly \(m > k\) envious agents under \(\pi\). We now show that instance \(I\) of \textsc{Clique} is a “yes”-instance if and only if instance \(I\) of \textsc{ERSA} is a “yes”-instance.

\(\Rightarrow:\) Assume that there exists a clique of size \(\ell\) in \(G\). Let us call each agent corresponding to an edge in the clique a clique agent; by definition, there are exactly \(\ell\) of them. We construct a desired sharing for \(I\) by giving every clique agent a resource. Indeed, it is achievable since we have exactly \(\ell\) resources, which can be shared by the happy agents with exactly \(\ell\) distinct clique agents. After such a sharing, all clique agents do not envy any more. Observe that the sharing made \(\ell\) agents corresponding to the vertices of the clique envious. Thus, the total number of envious agents is decreased by \(\ell - \ell\). This gives exactly the desired number \(k\), resulting in \(I\) being a “yes”-instance.

\(\Leftarrow:\) Let \(\delta\) be a sharing such that there are at most \(k\) envious agents under the sharing allocation \(\Pi^\delta\). Since the sharing graph is a clique, happy agents could share resources with any one of the remaining agents. However, if a happy agent shares with a vertex agent, then all \(2m\) dummy agents will envy this vertex agent. Since there are only \(\ell\) resources, which means at least \(2m - \ell \geq m > k\) dummy agents will be envious at the end, which is a contradiction. Hence, we can assume that no vertex agent is involved in \(\delta\). Consequently, no dummy agent will envy as they only look at vertex agents. Hence, without loss of
generality, we can additionally assume that no dummy agent is involved in $\delta$. Then all sharings in $\delta$ are restricted to edge agents and happy agents.

Now, we show that $\delta$ actually encodes a clique of size $\ell$ in $G$. When an edge agent gets a resource through sharing, it stops being envious, however, all vertex agents that looks at this edge agent start becoming envious. So, an ultimate goal for an optimal sharing is to share resources to the highest possible number of edge agents that are connected with an outgoing arc with as few vertex agents as possible. Formally, let $S$ be the set of edge agents who get shared resources in $\delta$ and $T$ the set of the corresponding vertex agents who will become envious. Denote $s = |S|$ and $t = |T|$. Naturally, $s \leq \binom{\ell}{2}$ and $s \leq \ell$. Then, the number of envious agents after the sharing in the solution is at least $m - s + t$ and this number should be at most $k = m - \ell + \ell$, that is,

$$m - s + t \leq m - \ell + \ell \Rightarrow s - t \geq \ell - \ell.$$

Since $\ell = \binom{\ell}{2}$ and $s \leq \binom{\ell}{2}$, if $t < \ell$, then $s - t \leq \binom{\ell}{2} - t < \binom{\ell}{2} - \ell$, which contradicts with the above $s - t \geq \ell - \ell$. Hence $t \geq \ell$. Since $s \leq \ell$, the only way to satisfy $s - t \geq \ell - \ell$ is that $s = \ell$ and $t = \ell$. Thus, the corresponding vertices of $T$ form a clique of size $\ell$ in $G$.

Observe that we use exactly $\ell$ resources, which proves the claimed W[1]-hardness.

Finally, we observe that for the scenario with a constant number of shared resources, there is a na"ive brute-force algorithm running in polynomial time.

Observation 4. ERSA is solvable in polynomial time if the number of shared resources (or the number of resources) is a constant.

Proof. Let $b \leq m$ be the upper bound on the number of shared resources. There are at most $m^b$ different choices of resources to be shared, and each resource has at most $n - 1$ possible receivers, so overall we need to check at most $m^bn^b$ cases.

5 Extensions

We introduce two natural extensions of our model, both of which describe costs which can be incurred by sharing—either for the central authority or agents. For each extension, we discuss which results can easily be adapted to cover them. Finally, we formally present how to modify the proofs of the relevant results.

5.1 Extension 1: Loss by Sharing

So far, we have assumed that agents get the full utility of the shared resources. This does not hold for situations in which sharing causes more inconvenience than owning a resource alone. Nonetheless, many of our algorithms can be easily adapted to deal with (computational) issues that arise in such situations. Consider the case when agents get only a fraction of the full utility from shared resources. Then our algorithms for few agents (Theorem 4) or identical utility functions (Theorem 7) still work with minor changes. For reducing envy it might be that after sharing agents lose some utility and thus become envious. Again, our algorithms for few agents (Theorem 4) or identical utility functions (Theorem 7) still work with minor changes.

Formally, we introduce two parameters $\alpha, \beta \in [0, 1]$ to quantify the effect that agents do not get the full utility of the shared resources, that is, for any resource $r$ initially assigned to agent $a_i$ in $\pi$ and shared to $a_j$ under $\delta$, the utility of resource $r$ for $a_i$ is $\alpha \cdot u_i(r)$ and that for $a_j$ is $\beta \cdot u_j(r)$. Recall that we refer to $\Pi^i(a)$ as a bundle of $a$. Now we refer to $\Pi^i_\delta(a)$ as the set of resources shared to $a$ by other agents, $\Pi^\delta_\delta(a)$ the set of resources shared to other agents by $a$, and $\Pi^\delta_0(a)$ the set of remaining unshared resources. Then the utility of agent $a$ under $\Pi^\delta$ is

$$u_i(\Pi^\delta(a)) = \sum_{r \in \Pi^\delta_\delta(a)} u_i(r) + \sum_{r \in \Pi^\delta_\delta(a)} \alpha u_i(r) + \sum_{r \in \Pi^\delta_0(a)} \beta u_i(r).$$

Later in this section, we consider this extension in the proofs of Theorem 1, Lemma 2, Theorem 4, and Theorem 7. Notice that the original setting corresponds to the case with $\alpha = \beta = 1$. 

18
5.2 Extension 2: Cost of Sharing

It is also natural to assume that the central authority would need to pay some cost for each sharing to incentivize agents to share resources. In this case, there would be a limited budget that the central authority can spend and the goal would be to improve the allocation through sharings whose costs do not exceed the budget. So far, our model was not capable of modeling the described scenario. However, for this generalized setting, our algorithms for improving egalitarian welfare (Lemma 2), reducing envy for few agents (Theorem 4) or identical utility functions (Theorem 7) still work with minor changes.

More precisely, for a simple 2-sharing $\delta$, we introduce a sharing cost $c_g: E_s \to \mathbb{N}$ and a budget $C \in \mathbb{N}$ for the central authority to incentivise sharing. Notice that the sharing cost is defined for each pair of agents. The cost of a sharing $\delta$ for the central authority is

$$c(\delta) = \sum_{\delta(a_i, a_j) \neq \emptyset} c_g(\{a_i, a_j\}).$$

Now the goal of the central authority is to find a sharing $\delta$ with $c(\delta) \leq B$ to improve the allocation.

Later in this section, we consider this extension in the proofs of Lemma 2, Theorem 4, and Theorem 7. Notice that the original setting corresponds to the case with $B = \sum_{(a_i, a_j) \in E} c_g(\{a_i, a_j\})$, i.e., $c(\delta) \leq B$ holds for any sharing $\delta$.

5.3 Respective Proofs’ Modifications

In the ensuing paragraphs, we list the necessary modifications of our results to make them work correctly for the aforementioned extensions.

Proof of Theorem 4 for Extension 1. We adapt the proof of Theorem 4 given earlier to work for Extension 1. To this end, we just need to change the weight of edges in the constructed graph as follows. The weight of the edge between $v^A_{i_1}$ and $v^A_{i_2}$ is now defined as

$$\max\{\delta u_i(r_{j_2}) - (1 - \alpha)u_i(r_{j_2}), \beta u_i(r_{j_1}) - (1 - \alpha)u_i(r_{j_1})\}$$

instead of $\max\{u_i(r_{j_2}), u_i(r_{j_1})\}$, i.e., the weight is the increased utilitarian social welfare in the new setting. Accordingly, in the “⇐” part, for each non-dummy edge $(v^{A}_{i_1}, v^{A}_{i_2}) \in M$, we set $\delta(a_{i_1}, a_{i_2}) = r_{j_1}$ if the increased utilitarian social welfare by sharing $r_{j_1}$ to agent $a_{i_1}$ is smaller than that of sharing $r_{j_2}$ to agent $a_{i_2}$, and $\delta(a_{i_1}, a_{i_2}) = r_{j_2}$ otherwise. Then, using the same arguments, we can show that there is a 6-bounded 2-sharing $\delta$ such that usw($\Pi^\delta$) $\geq k$ if and only if there is matching $M$ in graph $G$ with weight $\sum_{e \in M} w(e) \geq k - \text{usw}(\pi) + P$, and thus, the problem can be reduced to MAXIMUM WEIGHTED MATCHING.

Proof of Lemma 2 for Extensions 1 and 2. We adapt the original proof of Lemma 2 to work for the case with both Extension 1 and Extension 2. Similarly as before, we partition the set $A$ of agents into two sets $A_k^+$ and $A_k^-$ containing, respectively, the agents with their bundle value under $\pi$ at least $k$ and smaller than $k$. When constructing the graph $G_k = (A_k^+, A_k^-, E_k)$, for two agents $a_i \in A_k^+$ and $a_j \in A_k^-$ that are neighbors in the sharing graph $G_\pi$, we add an edge $e = \{a_i, a_j\}$ belongs to $E_k$ if $a_i$ can share a resource with $a_j$ to raise the utility of the latter to at least $k$ and after this sharing the utility of $a_i$ is at least $k$; formally, there exists a resource $r \in \pi(a_i)$ such that $u_j(\pi(a_j)) + \beta u_j(r) \geq k$ and $u_i(\pi(a_j)) - (1 - \alpha)u_i(r) \geq k$. In addition, for each edge $e = \{a_i, a_j\} \in E_d$ we assign it a weight $c_g(\{a_i, a_j\})$. With similar arguments, we have that there is a simple 2-sharing $\delta$ with $c(\delta) \leq B$ and esw($\Pi^\delta$) $\geq k$ if and only if there is matching $M$ in graph $G_k$ with $\sum_{e \in M} c_g(e) \leq B$ and $|M| \geq |A_k^-|$. Thus, we just need to check whether there is matching $M$ in graph $G_k$ with $\sum_{e \in M} c_g(e) \leq B$ and $|M| \geq |A_k^-|$, which is an instance of WBMM and is solvable in polynomial time according to Lemma 1.

Proof of Theorem 4 for Extension 1 and 2. We adapt the original proof of Theorem 4 to work for the case with both Extension 1 and Extension 2. The extended algorithm is given in Algorithm 2.

For Extension 2 where there is cost for sharing, notice that the cost of any realization $\delta$ of $M$ is fixed: $c(\delta) = c(M)$. Therefore, for any guessed $M$, the extended algorithm checks in the first line whether the cost of the configuration $M$ is at most $B$. 

19
DoesFeasibleRealizationExist+(Gt, π, {ut}t∈C, B, C, M)
if c(M) > B then return "no";
for each agent ai ∈ C do
    P_i ← P^0_i \ {r ∈ P_i | u_i(π(i)) + βu_i(r) < t_i};
repeat
    B ← \bigcup_{ai ∈ C} F_i(\{P_1, P_2, \ldots, P_{|C|}\});
    P_i ← P_i \ B;
until B = ∅;
if ∃i with P_i = ∅ then return "no" else return "yes";

Algorithm 2: Generalization of Algorithm 1 for two extensions.

For Extension 1 where agents only get a fraction of utility from shared resources, in the for-loop of Algorithm 2, the set of deleted resources from P^0_i are changed accordingly. In addition, since agent may lose utility after sharing, it is important to guarantee that target agents in C who will share resources to other agents according to the configuration M does not becomes envious after the sharing. To guarantee this, we partition all agents in C into three subsets C_+ ∪ C_- ∪ C_0, where

\[ C_+ = \{a_i ∈ C | ∃j such that (j, i) ∈ M\}; \]
\[ C_- = \{a_i ∈ C | ∃j such that (i, j) ∈ M\}; \]
\[ C_0 = \{a_i ∈ C | ∃j such that (i, j) ∈ M or (j, i) ∈ M\}. \]

Then we change the definition of P^0_i (defined in Eq. 1) as follows:

\[ P^0_i := \begin{cases} 
\pi(j) \cup \{d_i\} & \text{if } a_i ∈ C_+; \\
\pi(i) \cup \{d_i\} & \text{if } a_i ∈ C_-; \\
\{d_i\} & \text{if } a_i ∈ C_0. 
\end{cases} \quad (3) \]

Finally, the forbidden resource defined in Definition 7 should be changed accordingly, i.e., resource r ∈ P_i is a forbidden resource for some target agent a_i if there is some target agent a_j ∈ C_+∪C_0 with (a_j, a_i) ∈ G_i such that

\[ \max\{u_j(\pi(j)) + βu_i(r') | r' ∈ P_j\} < u_j(\pi(i)) + βu_j(r), \]
or there is some target agent a_j ∈ C_- with (a_j, a_i) ∈ G_i such that

\[ \max\{u_j(\pi(j)) - (1 - α)u_i(r') | r' ∈ P_j\} < u_j(\pi(i)) + βu_j(r). \]

Proof of Theorem 7 for Extensions 1 and 2. We adapt the earlier version of the proof to work for the case with both Extension 1 and Extension 2. For extension 1 where agents only get a fraction of utility for shared resources, if α = 1, then as no agent will lose utility after sharing, we just need to change the edge set E as follows. We add e = {a_i, a_j} to E if {a_i, a_j} ∈ E, α_i \notin N_0, and there exists a resource r ∈ π(a_j) such that u(π(a_i)) + βu(r) = u^* or a_j ∈ N_0 and there exists a resource r ∈ π(a_i) such that u(π(a_j)) + βu(r) = u^*. If α < 1, then in additional to the above change, when constructing the graph G = (A, E), we do not consider those agents who already have utility u^*, as they cannot help other agents increase utility without making their utility less than u^*. For Extension 2 where there is a cost for sharing, we need to convert the graph G = (A, E) into a weighted graph by assigning each edge e ∈ E a weight c_g(e). Then by computing a maximum-cardinality matching with weight at most C in G, which can be done in polynomial time according to Lemma 1, we can find the largest number of agents in A ∖ N_0 who can increase their utilities to u^* through a sharing.

6 Conclusion

We brought together two important topics—fair allocation of resources and resource sharing. Already our very basic and simple model where each resource can be shared by neighbors in a social network and
each agent can participate in a bounded number of sharings led to challenging computational problems. We shed light at their fundamental computational complexity limitations (in the form of computational hardness) and provided generalizable algorithmic techniques (as mentioned in Section 5). Our results are of broader interest in at least two respects. First, we gained insight into a recent line of research aiming at achieving fairness without relaxing its requirements too much. Second, there is rich potential for future research exploring our general model of sharing allocations (in its full power described by Definition 1).

**Beyond 2-sharing.** We focused on sharing resources between neighbors in a social network. Yet, there are many scenarios where sharing resources among a large group of agents can be very natural and wanted. If every resource can be shared with everyone, then there is a trivial envy-free allocation. Hence, it is interesting to further study the limits of existence of envy-free allocations under various sharing relaxations (concerning parameters such as number of shared resources, number of agents sharing a resource, etc.).

**No initial allocation.** In our model, the sharing builds up on an initial allocation. It is interesting to study the case without initial allocation, i.e., allocating indivisible but shareable resources to achieve welfare and/or fairness goal.

**Combinations of graph classes.** We showed that ERSA is NP-hard even if both input graphs are (bidirectional) cliques, but it is polynomial-time solvable if two graphs are the same and have constant treewidth. Based on this, analyzing various combinations of graph classes of the two social networks might be valuable.

**Strategic concerns and robustness.** We have assumed that all utility values are truthfully reported as well as correct and that the agents need not to be incentivized to share resources. Neither of these assumptions might be justified in some cases—the agents might misreport their utility, the utility values might be slightly incorrect, or a sharing can come at a cost for agents (splitting utility from shared resources, as described in Section 5 is an example of the latter). Tackling this kind of issues opens a variety of directions, which includes studying strategic misreporting of utilities, robustness of computed solutions against small utility values perturbations, and finding allocations that incentivize sharing.

**Acknowledgments**

We are grateful to the AAAI’22 reviewers for their insightful comments. This work was started when all authors were with TU Berlin. Andrzej Kaczmarczyk was supported by the DFG project “AFFA” (BR 5207/1 and NI 369/15) and by the European Research Council (ERC). Junjie Luo was supported by the DFG project “AFFA” (BR 5207/1 and NI 369/15) and the Singapore Ministry of Education Tier 2 grant (MOE2019-T2-1-045).

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 101002854).

**References**

Rediet Abebe, Jon Kleinberg, and David C. Parkes. Fair division via social comparison. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems (AAMAS’17)*, pages 281–289, 2017.

Haris Aziz, Sylvain Bouveret, Ioannis Caragiannis, Ira Giagkousi, and Jérôme Lang. Knowledge, fairness, and social constraints. In *Proceedings of the 32nd AAAI Conference on Artificial Intelligence (AAAI’18)*, pages 4638–4645, 2018.
Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. Finding fair and efficient allocations. In *Proceedings of the 19th ACM Conference on Economics and Computation (EC’19)*, pages 557–574, 2018.

Xiaohui Bei, Youming Qiao, and Shengyu Zhang. Networked fairness in cake cutting. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI’17)*, pages 3632–3638, 2017.

Russell Belk, Giana Eckhardt, and Fleura Bardhi. *Handbook of the Sharing Economy*. Edward Elgar Pub, 2019.

Aurélie Beynier, Yann Chevaleyre, Laurent Gourvès, Ararat Harutyunyan, Julien Lesca, Nicolas Maudet, and Anaëlle Wilczynski. Local envy-freeness in house allocation problems. *Autonomous Agents and Multi-Agent Systems*, 33:591–627, 2019.

Arpita Biswas and Siddharth Barman. Fair division under cardinality constraints. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI’18)*, pages 91–97, 2018.

Sylvain Bouveret and Jérôme Lang. Efficiency and envy-freeness in fair division of indivisible goods: Logical representation and complexity. *Journal of Artificial Intelligence Research*, 32(1):525–564, 2008.

Sylvain Bouveret, Yann Chevaleyre, and Nicolas Maudet. Fair allocation of indivisible goods. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 13, pages 311–329. Cambridge University Press, 2016.

Sylvain Bouveret, Katarína Cechlárová, Edith Elkind, Ayumi Igarashi, and Dominik Peters. Fair division of a graph. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI’17)*, pages 135–141, 2017.

Robert Bredereck, Andrzej Kaczmareczyk, and Rolf Niedermeier. Envy-free allocations respecting social networks. In *Proceedings of the 17th Conference on Autonomous Agents and MultiAgent Systems (AAMAS’18)*, pages 283–291, 2018.

Johannes Brustle, Jack Dippel, Vishnu V. Narayan, Mashbat Suzuki, and Adrian Vetta. One dollar each eliminates envy. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC’20)*, pages 23–39, 2020.

Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

Ioannis Caragiannis and Stavros Ioannidis. Computing envy-freeable allocations with limited subsidies. *CoRR*, abs/2002.02789, 2020.

Ioannis Caragiannis, Nick Gravin, and Xin Huang. Envy-freeness up to any item with high Nash welfare: The virtue of donating items. In *Proceedings of the 20th ACM Conference on Economics and Computation (EC’19)*, pages 527–545, 2019b.

Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation*, 7(3):12:1–12:32, 2019b.

Bhaskar Ray Chaudhury, Telikepalli Kavitha, Kurt Mehlhorn, and Alkmini Sgouritsa. A little charity guarantees almost envy-freeness. *SIAM Journal on Computing*, 50(4):1336–1358, 2021.

Yann Chevaleyre, Ulrich Endriss, and Nicolas Maudet. Allocating goods on a graph to eliminate envy. In *Proceedings of the 22nd AAAI Conference on Artificial Intelligence (AAAI’07)*, pages 700–705, 2007.

Yann Chevaleyre, Ulle Endriss, and Nicolas Maudet. Distributed fair allocation of indivisible goods. *Artificial Intelligence*, 242:1–22, 2017.

Eduard Eiben, Robert Ganian, Thekla Hamm, and Sebastian Ordyniak. Parameterized complexity of envy-free resource allocation in social networks. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI’20)*, pages 7135–7142, 2020.
Eric Friedman, Christos-Alexandros Psomas, and Shai Vardi. Dynamic fair division with minimal disruptions. In *Proceedings of the 16th ACM Conference on Economics and Computation (EC’15)*, pages 697–713, 2015.

Eric Friedman, Christos-Alexandros Psomas, and Shai Vardi. Controlled dynamic fair division. In *Proceedings of the 18th ACM Conference on Economics and Computation (EC’17)*, pages 461–478, 2017.

M. R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

Laurent Gourvès, Julien Lesca, and Anaëlle Wilczynski. Object allocation via swaps along a social network. In Carles Sierra, editor, *Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI'17)*, pages 213–219, 2017.

Daniel Halpern and Nisarg Shah. Fair division with subsidy. In *Proceedings of the 12th International Symposium on Algorithmic Game Theory (SAGT’19)*, pages 374–389, 2019.

Jiafan He, Ariel D. Procaccia, Alexandros Psomas, and David Zeng. Achieving a fairer future by changing the past. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI’19)*, pages 343–349, 2019.

John E. Hopcroft and Richard M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM Journal on Computing*, 2(4):225–231, 1973.

Sen Huang and Mingyu Xiao. Object reachability via swaps along a line. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI’19)*, pages 2037–2044, 2019.

Ton Kloks. *Treewidth, Computations and Approximations*, volume 842 of *Lecture Notes in Computer Science*. Springer, 1994.

Pascal Lange and Jörg Rothe. Optimizing social welfare in social networks. In *Proceedings of the 6th International Conference on Algorithmic Decision Theory (ADT’19)*, pages 81–96, 2019.

R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM Conference on Electronic Commerce (EC’04)*, pages 125–131, 2004.

Trung Thanh Nguyen, Magnus Roos, and Jörg Rothe. A survey of approximability and inapproximability results for social welfare optimization in multiagent resource allocation. *Annals of Mathematics and Artificial Intelligence*, 68(1-3):65–90, 2013.

Fedor Sandomirskiy and Erel Segal-Halevi. Fair division with minimal sharing. *CoRR*, abs/1908.01669, 2019.

Juliet B. Schor and Mehmet Cansoy. The sharing economy. In Frederick F. Wherry and Ian Woodward, editors, *The Oxford Handbook of Consumption*, pages 51–74. Oxford University Press, 2019.

Erel Segal-Halevi. Fair division with bounded sharing. *CoRR*, abs/1912.00459, 2019.

L. Williams, R. R. Kessler, W. Cunningham, and R. Jeffries. Strengthening the case for pair programming. *IEEE Software*, 17(4):19–25, 2000.