Asymptotic Behaviors of Global Solutions to the Two-Dimensional Non-resistive MHD Equations with Large Initial Perturbations

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Abstract

This paper is concerned with the asymptotic behaviors of global strong solutions to the incompressible non-resistive viscous magnetohydrodynamic (MHD) equations with large initial perturbations in two-dimensional periodic domains in Lagrangian coordinates. First, motivated by the oddity conditions imposed in [Arch. Ration. Mech. Anal. 227 (2018), 637–662], we prove the existence and uniqueness of strong solutions under some class of large initial perturbations, where the strength of impressive magnetic fields depends increasingly on the \(H^2\)-norm of the initial perturbation values of both velocity and magnetic field. Then, we establish time-decay rates of strong solutions. Moreover, we find that \(H^2\)-norm of the velocity decays faster than the perturbed magnetic field. Finally, by developing some new analysis techniques, we show that the strong solution convergence in a rate of the field strength to the solution of the corresponding linearized problem as the strength of the impressive magnetic field goes to infinity. In addition, an extension of similar results to the corresponding inviscid case with damping is presented.

Keywords: Incompressible MHD fluids; damping; algebraic decay-in-time; exponential decay-in-time; viscosity vanishing limit.

1. Introduction

We investigate the asymptotic behaviors of the global (-in-time) solutions to the following equations of incompressible magnetohydrodynamic (MHD) fluids with zero resistivity:

\[
\begin{align*}
\rho v_t + \rho v \cdot \nabla v + \nabla p - \mu \Delta v &= \lambda M \cdot \nabla M / 4\pi, \\
M_t + v \cdot \nabla M &= M \cdot \nabla v, \\
\text{div} v &= \text{div} M = 0,
\end{align*}
\]

where the unknowns \(v := v(x,t), M := M(x,t)\) and \(p := p(x,t)\) denote the velocity, magnetic field and the sum of both magnetic and kinetic pressures of MHD fluids respectively, and the three positive (physical) parameters \(\rho, \mu\) and \(\lambda\) stand for the density, shear viscosity coefficient and permeability of vacuum, respectively.

The global well-posedness of the system (1.1), for which the initial data is a small perturbation around a non-zero trivial stationary state (i.e., \(v = 0\), and \(M = \text{a non-zero constant vector} \vec{M}\), often called the impressive magnetic field), has been widely investigated, see [20, 36, 37] and [35] on the 2D and 3D Cauchy problems for (1.1) respectively, and see [25] and [29] on 2D and 3D initial-boundary value problems for (1.1) respectively. The existence of global solutions to the 2D...
Cauchy problem for (1.1) with large initial perturbations was obtained by Zhang under strong impressive magnetic fields [37]. As for the well-posedness of the 3D Cauchy and initial-boundary value problems for (1.1) with large initial perturbations, to our best knowledge, all available results are about the local (-in-time) existence, see [5, 10, 11] for examples. We mention here that the corresponding compressible case has been also widely studied, see [18, 19, 33] and the references cited therein. Since we are interested in the large-time behavior of a global solution and the asymptotic behavior of (a family of) solutions with respect to the strength of the impressive magnetic field, we next briefly introduce the relevant progress on this topic, and our main results in this paper.

1.1. Asymptotic behavior with respect to time

It has been physically conjectured that in MHD fluids, the energy is dissipated at a rate that is independent of the resistivity [4]. Hence, one can easily conclude that a non-resistive MHD fluid may still be dissipative. At present, this conclusion has been mathematically verified for the global small perturbation solutions of equations (1.1) around a non-zero trivial stationary state, in which the impressive magnetic field $M$ is given by

$$\bar{M} = \bar{M}_N e_N.$$ (1.2)

Here and in what follows $\bar{M}_N$ is a non-zero constant, $e_N$ a unit vector with the $N$-th component being 1 and $N$ the spatial dimension. In addition, we define $b := M - \bar{M}_N e_N$ and $\langle t \rangle := (1 + t)^{-1}$.

The first mathematical verification result was probably given by Ren–Wu–Xiang–Zhang for the 2D Cauchy problem of (1.1), they established the following time-decay [26]:

$$\langle t \rangle \langle \nabla^2 \rangle \frac{1}{2} \| \partial_x (v, b) \|_{L^2(\mathbb{R}^2)} \leq c_I,$$ (1.3)

where $\varepsilon \in (0, 1/2)$ is any given, $0 \leq k \leq 2$, and the initial data $(v^0, b^0)$ of $(v, b)$ belongs to $H^8(\mathbb{R}^2)$. Here and in what follows, $c_I$ will denote a generic positive constant, which may depend on the initial data $(v^0, b^0)$. Then, Tan–Wang further obtained the almost exponential decay of solutions to the 2D/3D initial-boundary value problems [29]:

$$\langle t \rangle ^{n-2} \| (v, b) \|_{H^{2n+4}(\Omega)} \leq c_I,$$ (1.4)

where $n \geq 4$, the initial data $(v^0, b^0)$ belongs to $H^{4n}(\Omega)$, and $\Omega$ is a 2D/3D layer domain with finite height. Later, Abidi–Zhang also got a decay rate of solutions to the 3D Cauchy problem [1]:

$$\langle t \rangle ^{1/4} \| (v, b) \|_{H^2(\mathbb{R}^3)} \leq c_I,$$ (1.5)

where $b^0 = 0$ and $v^0 \in H^s$ with $s \in (3/2, 3]$. Very recently, under the assumption that the initial data are sufficiently smooth, Deng–Zhang further established the following faster time-decay than (1.5) [7]:

$$\langle t \rangle \| \nabla v \|_{L^2(\mathbb{R}^3)} + \langle t \rangle ^{1/2} \| (v, b) \|_{H^2(\mathbb{R}^3)} \leq c_I.$$ (1.6)

In addition, Pan–Zhou–Zhu proved the existence of a unique global solution to the initial-boundary value problem of (1.1) defined in a 3D periodic domain $\mathbb{T}^3$, and also obtained the following time-decay as a byproduct [24]:

$$\langle t \rangle ^{(3-\sigma)/2} \| v \|_{H^2(\mathbb{T}^3)} + \sum_{0 \leq i \leq 1} \langle t \rangle ^{(1-\sigma+2i)/2} \| \partial^i_x (v, b) \|_{H^2(\mathbb{T}^3)} \leq c_I.$$ (1.7)
where \(0 < \sigma < 1\) and \((v^0, b^0) \in H^{2s+1}(\mathbb{T}^3)\) with \(s \geq 5\).

Motivated by the existence result on large perturbation solutions to the 2D Cauchy problem of (1.1) in [37] and the time-decay (1.3) for small perturbation solutions, we are interested in the time-decay of large perturbation solutions. To the best of our knowledge, there is no result about time-decay rates of large perturbation solutions. In this paper, we establish the time-decay rates of solutions with some class of large initial perturbations. Our results can be roughly described as follows.

First, we prove the existence of a unique strong solution to the 2D initial-boundary value problem for (1.1) with periodic boundary conditions under some class of large initial perturbations, where the strength of impressive magnetic fields increasingly depend on the \(H^2\)-norm of the initially perturbed values of the velocity and magnetic field, and the initially perturbed data should satisfy some regularity conditions imposed by Pan–Zhou–Zhu in [24]. Then, we show that the global solution enjoys the following decay in time:

\[
\|u(t)\|_{H^1(\mathbb{T}^2)} + \|v(t)\|_{H^2(\mathbb{T}^2)} + \|b(t)\|_{H^2(\mathbb{T}^2)} \leq c_I,
\]

where the initial datum \((b^0, v^0)\) belongs to \(H^2\). We refer the reader to Theorem 2.3 for the details (or see Theorems 2.4 and 2.2 for the version in Lagrangian coordinates). We should point out here that by virtue of (1.8), the velocity in \(H^2\)-norm decays faster than the perturbation magnetic field, while in (1.6), the \(H^2\)-norm of the velocity enjoys the same decay rate as the one of the perturbation magnetic field.

1.2. Asymptotic behavior with respect to the strength of impressive magnetic fields

It is well-known that a non-resistive MHD fluid, the motion of which is described by the system (1.1), exhibits elastic characteristics. In particular, a MHD fluid strains when stretched and will quickly return to its original rest state by the magnetic tension once the stress is removed [15]. This means that the magnetic tension will have stabilizing effects in the motion of MHD fluids. Moreover, the larger the strength of an impressive magnetic field is, the stronger this stabilizing effect will be, see the inhibition phenomenon of flow instabilities by magnetic fields [14, 16, 31]. It is worth to mention here that Bardos–Sulem–Sulem used hyperbolicity of (1.1) with \(\mu = 0\) to establish an interesting global existence result of classical solutions with small initial data in the H"older space \(H^s(\mathbb{R}^3)\) [2]; also see [3, 13] for the case of Sobolev spaces [3, 13]. Remark that such a result is not known for the system (1.1) in 3D when the magnetic field is absent (the 3D incompressible Euler equations).

In [37] Zhang also found an interesting mathematical result that the solution of the nonlinear system (1.1) converges to the solution of some linear equations obtained from (1.1) under the use of the stream function as \(\bar{M}_2 \to \infty\). More precisely, let \(M = (\partial_2, -\partial_1)^T (\psi - \bar{M}_2 x_1)\), then the system (1.1) reduces to the following system:

\[
\begin{aligned}
\rho v_t + \rho v \cdot \nabla v + \nabla p - \mu \Delta v - \bar{M}_2 \lambda \partial_2 (\partial_2 \psi, -\partial_1 \psi)^T/4\pi
&= \lambda (\partial_2 \psi, -\partial_1 \psi) \cdot \nabla (\partial_2 \psi, -\partial_1 \psi)^T/4\pi, \\
\psi_t + v \cdot \nabla \psi + \bar{M}_2 v_1 &= 0, \\
\text{div} v &= 0, \\
(v, \psi)|_{t=0} &= (v^0, \psi^0),
\end{aligned}
\]

(1.9)

where the subscript \(T\) denotes the transposition, \(\partial_i := \partial_{x_i}\) and \(x_i\) is the \(i\)-th component of \(x \in \mathbb{R}^2\). Thus, Zhang proved that the solution \((v, \psi)\) of (1.9) converges to the solution \((v^L, \psi^L)\) of the
following linear pressureless equations as \( \bar{M}_2 \to \infty \):

\[
\begin{aligned}
\rho v^L_t - \mu \Delta v^L - \bar{M}_2 \lambda \partial_2 (\partial_2 \psi^L, -\partial_1 \psi^L)^T / 4\pi &= 0, \\
\psi^L_t + \bar{M}_2 v^L &= 0, \\
(v^L, \psi^L)|_{t=0} &= (v^0, \psi^0).
\end{aligned}
\] (1.10)

However, no convergence rate in the strength \( \bar{M}_2 \) is given in [37]. In this paper we shall prove a similar result in Lagrangian coordinates, and further provide a convergence rate in \( \bar{M}_2 \) of the global solution to (1.1) by developing some new analysis techniques. More precisely, we shall show that the difference between the solution of (1.1) in Lagrangian coordinates and the solution of the corresponding linearized system can be bounded from above by \( c_I \bar{M}_2^{-1/2} \), see Theorem 2.4 for details.

Roughly speaking, the proof of our two results mentioned above is based on a key observation that the deviation function \( \eta \) of MHD fluid particles enjoys the estimate

\[ \| \bar{M}_2 \partial_2 \eta \|_0^2 \leq c_I, \]

where \( c_I \) depends on the initial total mechanical energy. The above estimate can be extended to the case of higher-order derivatives of \( \partial_2 \eta \). Moreover, if \( \eta \) additionally satisfies the odevity conditions, then we formally have an important inverse relation:

\[ \nabla \eta \propto c_I / \bar{M}_2, \]

(1.11)

see Section 2.1 for a detailed discussion. This relation intuitively not only reveals that the (nonlinear) solutions of (1.1) in Lagrangian coordinates can be approximated by the (linear) solutions of the corresponding linearized equations for \( \bar{M}_2 \gg c_I \), but also provides a convergence rate in \( \bar{M}_2 \). Since the nonlinear solutions can be approximated by the linear solutions, we naturally expect the existence of strong solutions under some class of large initial perturbations as in [37]. In fact, in [37] Zhang first obtained the (linear) solution of (1.10), then proved the existence of the small error solution \( (v - v^L, \psi - \psi^L) \) as \( \bar{M}_2 \gg c_I \), and finally got the large solution \( (v, \psi) \) by adding the linear solution and the small error solution together. It is worth to mention that the relation (1.11) allows us to directly establish the existence of solutions under some class of large initial perturbations by one-step procedure, rather than Zhang’s three-step procedure.

We mention that recently, some authors studied the case of inviscid, non-resistive MHD fluids with zero resistivity, i.e., the viscosity term in the system (1.1) is replaced by the velocity damping term \( \kappa \rho v \) with \( \kappa \) being the damping coefficient. Wu–Wu–Xu first proved the existence of a unique global solution with algebraic time-decay to the 2D Cauchy problem, provided that the initial perturbation \((v^0, b^0)\) is small in \( H^n(\mathbb{R}^2) \) with \( n \) sufficiently large [34]. Recently, Du–Yang–Zhou also obtained the existence of a unique global solution with exponential time-decay to the initial-boundary value problem in a strip domain \( \Omega \), provided that the initial perturbation \((v^0, b^0)\) around some non-trivial equilibrium is small in \( H^7(\Omega) \) [8]. Motivated by [8, 34], we can extend our aforementioned results in this paper on the asymptotic behavior of solutions in the viscous case to the inviscid case with the damping term \( \kappa \rho v \), and show that for the inviscid case with damping, the decay in time is exponentially fast, just as in [8], while the convergence rate in \( \bar{M}_2 \) as \( \bar{M}_2 \to \infty \) of a strong solution of the original nonlinear system to the solution of the corresponding linear system is in the form of \( c_I \bar{M}_2^{-1} \), which is faster than that for the viscous case, see Theorem 2.5 for the details.

Finally, we mention that the asymptotic behaviors of solutions with respect to other parameters, such as the Mach and Alfvén numbers, in MHD fluids have been also extensively investigated, see, for example, [6] and the references cited therein.
The rest of this paper is organized as follows: In Section 2 we introduce our main results including the existence of a unique strong solution with some class of large initial data to the 2D equations (1.1) in a periodic domain in Lagrangian coordinates, and the time-decay of the strong solution, and the convergence rate as $\bar{M}_2 \to \infty$ of the strong solution as well as the extension to the inviscid case with damping term, i.e., Theorems 2.1, 2.2, 2.4 and 2.5, the proofs of which are given in Sections 3–6, respectively. Finally, in Section 7, we provide the proof of the local well-posedness for the equations of viscous, non-resistive MHD fluids and inviscid, non-resistive MHD fluids with damping, respectively.

2. Main results

In this section we describe the main results in details. To begin with, we reformulate the equations (1.1) in Lagrangian coordinates. Recalling that (1.1) is considered with in a 2D periodic domain, we see, without loss of generality, that it suffices to consider the periodic domain $T^2$ with $T := \mathbb{R}/\mathbb{Z}$.

Let $(v, M)$ be the solution of the 2D system of equations (1.1), and the flow map $\zeta$ be the solution to

$$
\begin{align*}
\partial_t \zeta(y, t) &= v(\zeta(y, t), t) \quad \text{in } T^2 \times \mathbb{R}^+, \\
\zeta(y, 0) &= \zeta^0(y) \quad \text{in } T^2,
\end{align*}
$$

where $\zeta^0(y)$ satisfies $\det \nabla \zeta^0 = 1$ and “det” denotes the determinant.

Since $v$ is divergence-free, then

$$
\det \nabla \zeta = 1 \quad \text{(2.2)}
$$

as well as $\det \nabla \zeta^0 = 1$. Thus, we define $A^T := (\nabla \zeta)^{-1} := (\partial_j \zeta_i)_{2 \times 2}^{-1}$. In particular, by virtue of (2.2),

$$
A = \begin{pmatrix}
\partial_2 \zeta_2 & -\partial_1 \zeta_2 \\
-\partial_2 \zeta_1 & \partial_1 \zeta_1
\end{pmatrix}.
$$

We temporarily introduce some differential operators involving $A$, which will be used later. The differential operators $\nabla_A$, div$_A$ and $\Delta_A$ are defined by $\nabla_A f := (A_{1k} \partial_k f, A_{2k} \partial_k f)^T$, div$_A(X_1, X_2)^T := A_{ik} \partial_k X_1$, curl$_A f := \partial_1 A_{1k} \partial_k f_2 - A_{2k} \partial_k f_1$ and $\Delta_A f := \text{div}_A \nabla_A f$ for a scalar function $f$ and a vector function $X := (X_1, X_2)^T$, where $A_{ij}$ denotes the $(i, j)$-th entry of the matrix $A$. It should be remarked that we have used the Einstein convention of summation over repeated indices, and $\partial_k = \partial_{y_k}$. In addition, thanks to (2.2), we have

$$
\partial_k A_{ik} = 0. \quad \text{(2.3)}
$$

Let $\mathbb{R}^+ = (0, \infty)$, $\nu = \mu/\rho$ and

$$(u, B, q)(y, t) = (v, M, p/\rho)(\zeta(y, t), t) \quad \text{for } (y, t) \in T^2 \times \mathbb{R}^+.$$ 

By virtue of the equations (1.1) and (2.1), the evolution equations for $(\zeta, u, q)$ in Lagrangian coordinates read as follows.

$$
\begin{align*}
\frac{\partial \zeta}{\partial t} &= u, \\
u \frac{\partial u}{\partial t} + \nabla_A q - \nu \Delta_A u &= \lambda B \cdot \nabla_A B / 4 \pi \rho, \\
B_t - B \cdot \nabla_A u &= 0, \\
\text{div}_A u &= 0, \\
\text{div}_A B &= 0.
\end{align*}
$$

(2.4)
We can derive from (2.4) the differential version of magnetic flux conservation [15]:

\[ A_j B_j = A_{0j} B_{0j}^0, \]

which yields

\[ B = \nabla \zeta A_0^T B^0. \]  \hspace{1cm} (2.5)

Here and in what follows, the notation \( f_0 \) as well as \( f^0 \) denote the value of the function \( f \) at \( t = 0 \). If we assume

\[ A_0^T B^0 = \bar{M} \]  \hspace{1cm} (i.e., \( B^0 = \partial \bar{M} \zeta^0 \)), \hspace{1cm} (2.6)

where \( \bar{M} \) is defined by (1.2) with \( N = 2 \), then (2.5) reduces to

\[ B = \partial \bar{M} \zeta. \]  \hspace{1cm} (2.7)

Here we should point out that \( B \) given by (2.7) automatically satisfies (2.4) and (2.4) 5. Moreover, from (2.7) we see that the magnetic tension in Lagrangian coordinates has the relation

\[ B \cdot \nabla A = \partial^2 \bar{M} \zeta. \]

Let \( I \) denote a \( 2 \times 2 \) identity matrix, \( m^2 = \lambda \bar{M}_2^2 / 4 \pi \rho, \eta = \zeta - y \) and

\[ \tilde{A} = \begin{pmatrix} \partial_2 \eta_2 & -\partial_1 \eta_2 \\ -\partial_2 \eta_1 & \partial_1 \eta_1 \end{pmatrix}. \]

Consequently, under the assumption (2.6), the equations (2.4) are equivalent to the following system:

\[
\begin{cases}
\eta_t = u, \\
u_t + \nabla A q - \nu \Delta A u = m^2 \partial^2 \eta, \\
\text{div}_A u = 0
\end{cases}
\]  \hspace{1cm} (2.8)

where \( B = m(\partial_2 \eta + e_2) \) and \( A = \tilde{A} + I \).

For the well-posedness of (2.8) defined in \( \mathbb{T}^2 \), we impose the initial condition:

\[ (\eta, u)|_{t=0} = (\eta^0, u^0) \quad \text{in} \quad \mathbb{T}^2. \]  \hspace{1cm} (2.9)

Before stating our main results, we introduce some notations which will be frequently used throughout this paper.

(1) Basic notations: \( \mathbb{R}_0^+ := [0, \infty), \mathcal{I}_T := (0, T) \) for \( 0 < T \leq \infty, \mathbb{T}_T := [0, T] \) for \( T \in \mathbb{R}^+ \), \( \Omega_T := \mathbb{T}^2 \times \mathcal{I}_T, \mathcal{I} := \int_{(-1,1)^2}, (w)_{\mathbb{T}^2} := \int \mathcal{I} w dy, \alpha = (\alpha_1, \alpha_2) \) denotes the multi-index with respect to the variable \( y \).

(2) Simplified Banach spaces:

\[
L^r := L^r(\mathbb{T}^2) = W^{0,r}(\mathbb{T}^2), \quad H^i := W^{i,2}(\mathbb{T}^2), \quad H := H^1, \\
H^i_\sigma := \{ u \in H^i \mid \text{div} u = 0 \}, \quad H^{i+1} := \{ \eta \in H^{i+1} \mid \det \nabla (\eta + y) = 1 \}, \\
H^{i+1}_2 := \{ u \in H^i \mid \partial_2 u \in H^i \}, \quad H^{i+1}_{1,2} := H^i \cap H^{i+1}_2, \\
X := \{ w \in X \cap L^2 \mid (w)_{\mathbb{T}^2} = 0 \},
\]

where \( X \) denotes a Banach space, \( 1 < r \leq \infty \) and \( i \geq 0 \) are integers.
2.1. Existence of global solutions

A

In addition, (6) General constants: for integers $i$,

\[ C_{0}^{0}(I_{T}, H^{1}) := \{ u \in L^{2}(\Omega_{T}) \mid u \in C(I_{T} \setminus 3, H^{1}) \text{ for some zero-measurable set } 3 \subset I_{T} \}, \]

\[ U_{i+1,T} := \{ u \in C^{0}(I_{T}, H^{2i+2}) \mid u_{t} \in C^{0}(I_{T}, H^{2i}), (u, u_{t}) \in L^{2}(I_{T}, H^{2i+3} \times H^{2i+1}) \}, \quad (2.10) \]

\[ U_{T} := \{ u \in U_{1,T} \mid (u)_{T} = 0 \}. \]

(4) Simplified function classes: for integer $i \geq 1$,

\[ H_{i}^{*} := \{ \xi \in H^{i} \mid \xi(y) + y : \mathbb{R}^{2} \to \mathbb{R}^{2} \text{ is a } C^{1}\text{-diffeomorphism mapping} \}, \]

\[ H_{i+1,T}^{*} := \{ \eta \in C^{0}(I_{T}, H_{i}^{*}) \mid \eta(t) \in H_{i}^{*} \text{ for each } t \in I_{T} \}, \]

\[ C_{0,B,\text{weak}}^{0}(I_{T}, L^{2}) := C^{0}(I_{T}, L^{2}) \cap L^{\infty}(I_{T}, L^{2}) \cap C_{0,B,\text{weak}}^{0}(I_{T}, L_{\text{weak}}^{2}), \]

\[ C_{0}(I_{T}, H^{1}) := \{ \eta \in C(I_{T}, H^{1}) \mid \partial^{a}_{2} \eta \in C_{0}^{0}(I_{T}, L^{2}) \text{ for any } |\alpha| = i \}, \]

\[ \mathcal{F}_{i+1,T}^{*} := \{ \eta \in C^{0}(I_{T}, H_{i+1}^{*}) \mid \partial^{a}_{2} \eta \in H_{i}^{*} \cap H_{i}^{*} \text{ for each } t \in I_{T} \}, \]

\[ U_{i} := \{ u \in C^{0}(I_{T}, H^{i}) \mid \partial^{a}_{2} u \in C_{0,B,\text{weak}}^{0}(I_{T}, L^{2}) \text{ for any } |\alpha| = i + 1 \}, \]

\[ U_{i}^{+} := \{ u \in U_{T} \mid (u)_{T} = 0 \}. \]

(5) Simplified norms: for integers $i \geq 0$ and $n \geq i$,

\[ || \cdot ||_{i} := || \cdot ||_{H^{i}(T^{2})}, \quad || \nabla^{i} \cdot ||_{0} := \sum_{|\alpha|=i} || \partial^{a}_{2} \cdot ||_{2}^{2}, \quad || \cdot ||_{i+1,2} := \sqrt{|| \cdot ||_{i}^{2} + || \nabla^{i}_{2} \cdot ||_{2}^{2}}, \]

\[ E_{n,i}(t) := || \partial^{i}_{2}(\nabla \eta, u, m \partial^{2} \eta)(t)||_{n-i}^{2}, \quad E_{n,i}^{0} := E_{n,i}(0). \]

(6) General constants: $c_{i}$ ($1 \leq i \leq 3$) and $c_{i}^{*}$ ($1 \leq i \leq 4$) are fixed constants which may depend on the parameters $\nu$ and $\kappa$ respectively, but not on $m$. If not stated explicitly, $c_{0}, c, c^{*}, C$ and $C^{*}$ will denote generic positive constants, which may vary from one place to another. Moreover,

- $c_{0}$ is independent of any parameter;
- $c$ and $c^{*}$ may depend on $\nu$ and $\kappa$ respectively (but not on $m$);
- $C$ depends on $\nu$ and $\sqrt{E_{2,0}}$, and increases with respect to $\sqrt{E_{2,0}}$. In particular, $C$ only depends on $\nu$ and the norm $||u^{0}||_{2}$ for the case $\eta^{0} = 0$;
- $C^{*}$ depends on $\kappa$ and $||(u^{0}, \eta^{0}, m \partial^{2} \eta^{0})||_{4}$, and increases with respect to $||(u^{0}, \eta^{0}, m \partial^{2} \eta^{0})||_{4}$.

In addition, $A \lesssim_{0} B$, $A \lesssim B$ and $A \lesssim_{\kappa} B$ mean that $A \leq c_{0}B$, $A \leq cB$ and $A \leq c^{*}B$, respectively.

2.1. Existence of global solutions

Before stating the global existence result of solutions to the initial-value problem \textbf{(2.8)}–\textbf{(2.9)} in some classes of large data under strong magnetic fields, let us first mention the heuristic idea, which leads us to study this topic.

First, multiplying \textbf{(2.8)} with $u$ in $L^{2}$, we obtain the basic energy identity:

\[ \frac{1}{2} \frac{d}{dt} (||u||_{0}^{2} + ||m \partial^{2} \eta||_{0}^{2}) + \nu ||\nabla A u||_{0}^{2} = 0, \]

\[ (2.11) \]
which implies
\[
\| u \|_0^2 + \| m \partial_2 \eta \|_0^2 + 2\nu \int_0^t \| \nabla A u \|_0^2 d\tau = \| u^0 \|_0^2 + \| m \partial_2 \eta^0 \|_0^2 =: I_0.
\] (2.12)

We call $I_0$ the initial total mechanical energy, which includes the kinetic energy, and the perturbation magnetic energy that could be regarded as the potential energy. We easily see from (2.12) that $\| \partial_2 \eta \|_0 \rightarrow 0$ as $m \rightarrow \infty$ for fixed $I_0$. This basic relation motivates us to expect that the deformation quantity $\nabla \eta$ may be small, when $m$ is sufficiently large. Fortunately, this is indeed the case for $(\eta, u)$ satisfying the additional admissibility conditions imposed by Pan–Zhou–Zhu in [24], see (2.14) for details.

We rewrite (2.8) as a nonhomogeneous system of the Stokes equations:
\[
\begin{align*}
\begin{cases}
&u_t + \nabla q - \nu \Delta u = \vec{\zeta}, \\
&\text{div} u = \text{div} \vec{A}u,
\end{cases}
\tag{2.13}
\end{align*}
\]
where we have defined $\vec{\zeta} := m^2 \partial_2^2 \eta + \mathcal{N}$, $\mathcal{N} := \mathcal{N}^\nu - \nabla A q$, $\mathcal{N}^\nu := \partial_t (\mathcal{N}_{1}^\nu, \mathcal{N}_{2}^\nu)$ and $\mathcal{N}_{1,2}^\nu := \nu (\mathcal{A}_{kl} \mathcal{A}_{km} + \mathcal{A}_{ml}) \partial_m u_j$. This formally reveals that the system (2.13) can be approximated by the corresponding linear system, if $\nabla \eta$ is sufficiently small. Since the linear system admits a global solution, the nonlinear system (2.13) may also admit a global solution in some classes of large data under the strong magnetic fields. This result reads as follows.

**Theorem 2.1.** There are positive constants $c_1 \geq 4$, $c_2 > 0$ and a sufficiently small constant $c_3 \in (0, 1]$, such that for any $(\eta^0, u^0) \in (H^3 \cap H^2) \times H^2$ and $m$ satisfying the incompressible condition $\text{div} \mathcal{A} \eta^0 u^0 = 0$, the admissibility conditions
\[
(\eta_1^0, u_1^0)(y_1, y_2) = (\eta_1^0, u_1^0)(y_1, y_2) \quad \text{and} \quad (\eta_2^0, u_2^0)(y_1, y_2) = -(\eta_2^0, u_2^0)(y_1, y_2),
\] (2.14)
and the condition of strong magnetic fields
\[
m \geq \frac{1}{c_3} \max \left\{ \left( c_1 \mathcal{E}_{2,0}^0 e^{c_2 \mathcal{E}_{2,1}} \right)^{1/4}, c_1 \mathcal{E}_{2,0}^0 e^{c_2 \mathcal{E}_{2,1}} \right\},
\] (2.15)
then the initial value problem (2.8) admits a unique global strong solution $(\eta, u, q) \in H^3 \times U_\infty \times C^0 (\mathbb{R}_+^3, H^2)$. Moreover, the solution $(\eta, u)$ enjoys the stability estimate:
\[
\mathcal{E}_{2,0}(t) + \int_0^t (\| u \|^2 + \| m \partial_2 \eta \|^2) d\tau \lesssim e^{\mathcal{E}_{2,0}^0 e^{c_2 \mathcal{E}_{2,1}}(1 + \mathcal{E}_{2,1})} \quad \text{for any} \ t \geq 0,
\] (2.16)
where $\| \eta \|_3 + \| u \|_2 + \| m \eta \|_{3,2} \lesssim \sqrt{\mathcal{E}_{2,0}}$. In addition,
\[
\| \nabla q \|_1 \lesssim \| \partial_2 u \|_1 + m^2 (\| \partial_2^2 \eta \|_0 \| \partial_2 \eta \|_1 + \| \partial_2 \eta \|_1^2),
\] (2.17)
\[
\| u \|_{i+1,2} \lesssim \begin{cases}
\| \partial_2 u \|_0 + \| \partial_2^2 \eta \|_1 + \| \partial_2 u \|_1 & \text{for } i = 0; \\
\| \partial_2 u \|_i & \text{for } i = 1 \text{ and } 2,
\end{cases}
\] (2.18)
\[
\| \eta \|_{i+1,2} \lesssim \| \partial_2 \eta \|_i, \quad \| \eta \|_{3,2} \lesssim \sqrt{\| \partial_2 \eta \|_0 \| \partial_2 \eta \|_1}.
\] (2.19)

**Remark 2.1.** We can easily construct a family of $(\eta^0, u^0)$ satisfies all the assumptions in Theorem 2.1 where $\eta^0 \neq 0$ and $u^0 \neq 0$. In fact, let $\bar{\eta} = \bar{u} = (\sin x_1 \cos x_2, -\cos x_1 \sin x_2)$. Because
\[ \text{div} \eta = \text{div} \bar{u} = 0, \] for sufficiently small \( \varepsilon \), there exists a function pair \( (\eta^0, u^0) \) enjoying the form \( (\eta^0, u^0) = (\varepsilon \bar{\eta} + \varepsilon^2 \eta^2, \bar{u} + \varepsilon u^2) \), where \( (\eta^2, u^2) \) satisfies \( \|\eta^2\|_3 + \|u^2\|_2 \leq c_0 \),

\[
\begin{align*}
-\Delta \eta^2 + \nabla \beta_1 &= 0, \\
\text{div} \eta^2 &= \text{div} \left( \tilde{\eta}_1 + \varepsilon \tilde{\eta}_2 \right) (-\partial_2 (\eta_2 + \varepsilon \eta_2^2), \partial_1 (\eta_2 + \varepsilon \eta_2^2))^T), \\
(\eta^2)_{T2} &= 0,
\end{align*}
\tag{2.20}
\]

\[
\begin{align*}
-\Delta u^2 + \nabla \beta_2 &= 0, \\
\text{div} u^2 &= \varepsilon^{-1} \text{div} A_0 (u^0 + \varepsilon u^2), \\
(u^2)_{T2} &= 0,
\end{align*}
\tag{2.21}
\]

and for a proof of which we refer to \([17, \text{Proposition 5.1}] \). It is easy to check that for sufficiently small \( \varepsilon \), \( (\eta^0, u^0) \) is non-zero, belongs to \( (H^3 \cap H^2_\varepsilon) \times H^2 \), and satisfies \( \text{div} A_\varepsilon u^0 = 0 \) and \( (2.14) \). We further take \( m = \varepsilon^{-1} \) to immediately see that \( (\eta^0, u^0) \) and \( m \) satisfy the condition of strong magnetic fields \( (2.15) \) for sufficiently small \( \varepsilon \). Furthermore, \( \| (\nabla \eta^0, u^0, m \partial_2 \eta^0) \|_2 \leq c_0 \) for some constant \( c_0 \) independent of \( \varepsilon \) and \( m \).

**Remark 2.2.** Noting that the initial perturbation magnetic field \( B^0 - \bar{M} \) is equal to \( \bar{M}_\varepsilon \partial_2 \bar{\eta} \), we see from \( (2.15) \) that the strength of the impressive magnetic field increasingly depends on the \( H^2 \)-norm of the initial velocity and perturbation magnetic field.

**Remark 2.3.** In the above theorem, we have assumed the condition \( (\eta^0)_{T2} = (u^0)_{T2} = 0 \). If \( ((\eta^0)_{T2}, (u^0)_{T2}) \neq 0 \), we can define \( \bar{\eta}^0 := \eta^0 - (\eta^0)_{T2} \) and \( \bar{u}^0 := u^0 - (u^0)_{T2} \). Then, by Theorem \( 2.1 \) there exists a unique global strong solution \( (\bar{\eta}, \bar{u}, \bar{q}) \) to the initial value problem \( (2.8) - (2.9) \) with initial data \( (\bar{\eta}^0, \bar{u}^0) \). It is easy to verify that \( (\eta, u, q) := (\bar{\eta} + t(u)_{T2} + (\eta^0)_{T2}, \bar{u} + (u)_{T2}, \bar{q}) \) is just the unique strong solution of \( (2.8) - (2.9) \) with initial data \( (\eta^0, u^0) \).

**Remark 2.4.** Since \( (\eta^0, u^0) \) satisfies the odevity conditions \( (2.14) \), the strong solution \( (\eta, u) \) of \( (2.8) - (2.9) \) with an associated pressure function \( q \) enjoys the same odevity conditions as \( (\eta^0, u^0) \) does, i.e.,

\[
(\eta_1, u_1)(y_1, y_2, t) = (\eta_1, u_1)(y_1, -y_2, t) \quad \text{and} \quad (\eta_2, u_2)(y_1, y_2, t) = -(\eta_2, u_2)(y_1, -y_2, t). \tag{2.21}
\]

Now, we briefly describe the proof idea of Theorem \( 2.1 \). Motivated by \( (2.12) \), we naturally except that for given value \( \mathcal{E}_{2,0}^0 \), the solution \( (\eta, u) \) enjoys the estimate

\[
\|(u, m \partial_2 \eta)\|_2 \leq C \quad \text{for sufficiently large } m. \tag{2.22}
\]

Thus, we want to derive the \emph{a priori} estimate of \( (\eta, u) \) like

\[
\|(u, m \partial_2 \eta)\|_2 \leq C/2 \tag{2.23}
\]

under the \emph{a priori} assumption \( (2.22) \).

However, if we follow the above idea, we find that the assumption \( (2.22) \) does not suffice to establish the \emph{a priori} estimate \( (2.23) \), expect further requiring the additional assumption:

\[
\|\nabla \eta\|_2^2 \leq C. \tag{2.24}
\]
More precisely, we can conclude that there are constants $K$ (depending on $E^0_{2,0}$, $E^0_{2,1}$ and $m$) and $\delta$, such that
\[ \sup_{0 \leq t \leq T} E_{2,0}(t) \leq K^2/4, \] (2.25)
if
\[ \sup_{0 \leq t \leq T} (\| \nabla \eta, m\partial_2 \eta \|_2^2 + \| \partial_2 u(t) \|_2^2) \leq K^2 \text{ for any given } T > 0 \] (2.26)
and
\[ \max\{K^{1/2}, K^2\}/m \in (0, \delta]. \] (2.27)

The above a priori stability estimate, together with a local well-posedness result on (2.8)–(2.9), immediately yields Theorem 2.1. The detailed proof will be presented in Section 3. In addition, the proof of the existence of a unique local solution will be provided in Section 7.2. It should be remarked that the odevity conditions (2.21) play an important role in the derivation of the a priori stability estimate, see the key Lemma 3.2.

2.2. Asymptotic behaviors of solutions

Now, we turn to stating the asymptotic behaviors of the global solution given in Theorem 2.1. To begin with, we state the result of the asymptotic behavior with respect to the time.

**Theorem 2.2.** Let $(\eta, u, q)$ be the global solution of (2.8)–(2.9) established in Theorem 2.1, then
\[ \sum_{i=1}^{2} \left( \langle t \rangle^{i} \| \partial^{i}_{2} \eta(t) \|_{3-i}^{2} + \langle t \rangle^{i+1} \| \partial^{i}_{2}(u, m\partial_2 \eta)(t) \|_{3-i}^{2} \right) 
+ \int_{0}^{t} \left( \langle \tau \rangle^{i+1} \| \partial^{i}_{2} u \|_{3-i}^{2} + \langle \tau \rangle^{i} \| m\partial^{i+1}_{2} \eta \|_{3-i}^{2} \right) d\tau 
+ \langle t \rangle \| (u, m\partial_2 \eta)(t) \|_{2}^{2} + \int_{0}^{t} \langle \tau \rangle \| u \|_{3}^{2} d\tau \leq C \] (2.28)

and
\[ \sum_{i=1}^{2} \left( \langle t \rangle^{i+1} \| u(t) \|_{3-i}^{2} + \langle t \rangle^{i+1} \int_{0}^{t} e^{\nu(\tau-t)/2} \| u \|_{4-i}^{2} d\tau \right) \leq (1 + m^2)C \text{ for any } t \geq 0. \] (2.29)

**Remark 2.5.** In view of (2.18), (2.19), (2.28) and Poincârè’s inequality, we get from (2.28) that
\[ \langle t \rangle^{5/2} \| m\eta(t) \|_{L^{\infty}(T^2)}^{2} + \sum_{i=1}^{2} \left( \langle t \rangle^{i} \| \eta(t) \|_{4-i,2}^{2} + \int_{0}^{t} \langle \tau \rangle^{i} \| m\partial_2 \eta \|_{3-i,2}^{2} d\tau \right) 
+ \langle t \rangle^{i+1} \| (u, m\partial_2 \eta)(t) \|_{3-i,2}^{2} + \int_{0}^{t} \langle \tau \rangle^{i+1} \| u \|_{4-i,2}^{2} d\tau \right) \leq C, \quad \forall t \geq 0. \] (2.30)

From (2.26) we easily get $K/m \leq \delta$. We should point out that the term $K^{1/2}$ in (2.26) can be replaced by $K$ to establish Theorem 2.1 with “1/2” in place of “1/4” in (2.15). Here we further choose $K^{1/2}$ in order to make sure that $m^{-1}$ does not appear on the right hand of (2.28).
From (2.28) we immediately see that \( u \) enjoys the same time-decay rate as \( m\partial_2\eta \) does. Similar results were obtained for the global solution with small perturbation, see (1.3)-(1.7). Next, we briefly explain the basic idea how to get the faster time-decay of the velocity (2.29).

It is well-known that any solution of the following homogeneous Stokes equations

\[
\begin{aligned}
  u_t + \nabla q - \nu \Delta u &= 0, \\
  \text{div} u &= 0
\end{aligned}
\]  \\
\tag{2.31}

decays exponentially in time. Thus, for the nonhomogeneous case in (2.13), the algebraic time-decay of \( \|u\|_{3-i} \) depends on \( \mathcal{S} \) where \( i = 1 \) and \( 2 \). Since the decay rate of the linear term \( \|\partial_2^2\eta\|_{2-i} \) is \( \langle t \rangle^{-(i+1)} \), we naturally except that \( \|u(t)\|_{3-i} \) also decays in the rate of \( \langle t \rangle^{-(i+1)} \). Fortunately, by employing carefully estimates, one sees that the nonlinear term \( \mathcal{N} \) does not prevent us from obtaining the desired decay rate, see Section 4.2.

Recalling that the solution \( \eta \) in Theorems 2.1 satisfies

\[
\zeta := \eta(y, t) + y : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a } C^1 \text{ diffeomorphism mapping,}
\]  \\
\tag{2.32}

one can easily recover the decay result in Theorem 2.2 from Lagrangian coordinates to the one in Eulerian coordinates:

**Theorem 2.3.** Let \((v, \eta, q)\) be the global solution given in Theorem 2.1. \( \zeta^0 = \eta^0 + y, \zeta^1_0 \) denotes the inverse function and

\[
(v, M, p) = (u, m\partial_2 \zeta, q)|_{y = \zeta^{-1}}.
\]

Then \((v, M, p)\) belongs to \( C^0(\mathbb{R}^0_+, H^2 \times H^2 \times H^2) \), and is a unique global strong solution of the initial value problem (1.1) with initial data \((v, M)|_{t = 0} = (u^0, m\partial_2 \zeta^0)|_{y = \zeta^0_0} \) in \( \mathbb{T}^2 \). Moreover, the solution enjoys the decay estimates

\[
\sum_{i=0}^{2} \left( \langle t \rangle^{i+1} \|v(t)\|_{3-i}^2 + \int_0^t \langle t \rangle^{i+1} \|v(t)\|_{3-i}^2 + \|b(t)\|_{2-i}^2 \right) \, dt \leq C
\]

and

\[
\sum_{i=1}^{2} \left( \langle t \rangle^{i+1} \|v(t)\|_{3-i}^2 + \langle t \rangle^{i+1} \int_0^t e^{\nu(t-\tau)/2} \|v(t)\|_{4-i}^2 \, d\tau \right) \leq (1 + m^2)C,
\]

where \( b := M - \bar{M} \).

**Remark 2.6.** Since the periodic cell of \( \mathbb{T}^2 \) is bounded, we can also establish a result of almost exponential decay, where the decay rate is faster than Tan–Wang’s result (1.4) in [29] under the same regularity on initial data. In fact, if the initial data \((\eta^0, u^0)\) in Theorem 2.1 is in \( H^{n+1} \times H^n \) with \( n \geq 3 \), and \( m \) satisfies (2.13) with \( \mathfrak{E}^0_{2,0} \) in place of \( \mathfrak{E}^0_{0,0} \), then there exists a unique classical solution \((v, M, p) \in C^0(\mathbb{R}^0_+, H^2 \times H^n \times H^n) \) to the initial value problem (1.1) with initial data \((v, M)|_{t = 0} = (u^0, m\partial_2 \zeta^0)|_{y = \zeta^0_0} \). Moreover,

\[
\sum_{i=0}^{n} \left( \langle t \rangle^{i+1} \|v(t)\|_{n-i}^2 + \int_0^t \langle t \rangle^{i+1} \|v(t)\|_{n+1-i}^2 \, d\tau \right) + \sum_{i=1}^{n} \int_0^t \langle t \rangle^i \|b(t)\|_{n-i}^2 \, d\tau \leq c_I,
\]

\[
\sum_{i=1}^{n} \left( \langle t \rangle^{i+1} \|v(t)\|_{n+1-i}^2 + \langle t \rangle^{i+1} \int_0^t e^{\nu(t-\tau)/2} \|v(t)\|_{n+2-i}^2 \, d\tau \right) \leq (1 + m^2)c_I,
\]

where \( b := M - \bar{M} \) and the constant \( c_I \) depends on \( \nu \) and \( \mathfrak{E}^0_{n,0} \).
Next, we further state the asymptotic behavior of solutions with respect to \( m \). Noting that the inhomogeneous term \( \mathfrak{N} \) in \( \mathfrak{F} \) includes \( \partial_i \eta_j \) for \( 1 \leq i, j \leq 2 \), and \( \| \nabla \eta \|_{L^\infty} \lesssim_0 \| \partial_2 \eta \|_2 \leq K/2m \) by (2.25) and (3.9), we formally see that \( \mathfrak{N} \to 0 \) as \( m \to \infty \) for fixed \( \mathfrak{F}_{2,0} \). Thus, the solution \((\eta, u)\) established in Theorem 2.1 converges in the rate \( m^{-1/2} \) to the solution \((\eta^L, u^L)\) of the corresponding linearized system as \( m \to \infty \). More precisely, we have

**Theorem 2.4.** Let \((\eta, u, q)\) be the global solution of (2.8) - (2.9) given in Theorem 2.1.

1. Then, one can use the initial data of \((\eta, u)\) to construct a function pair \((\eta^r, u^r)\) \( \in \mathcal{H}^3 \times \mathcal{H}^2 \), such that the following linear pressureless initial-value problem

\[
\begin{aligned}
\eta^r_t &= u^r, \\
u^r_t - \nu \Delta u^r &= m^2 \partial_2^2 \eta^r, \\
\text{div} u^r &= 0, \\
(\eta^r, u^r)|_{t=0} &= (\eta^0 + \eta^r, u^0 + u^r)
\end{aligned}
\]  

admits a unique strong solution \((\eta^rL, u^rL)\) \( \in C^0(\mathbb{R}^+_0, \mathcal{H}^3) \times \mathcal{U}_\infty \). Moreover,

- \((\eta^rL, u^rL)\) also satisfies the odevity conditions (2.21) as \((\eta, u)\) does;
- the function pair \((\eta^r, u^r)\) satisfies

\[
\text{div}(u^0 + u^r) = \text{div}(\eta^0 + \eta^r) = 0,
\]

\[
\| u^r \|_2 \lesssim_0 \| \partial_2 \eta^0 \|_2 \| u^0 \|_2,
\]

and \( \| \partial_2^j \eta^r \|_{3-j} \lesssim_0 \| \partial_2 \eta^0 \|_2 \| \partial_2^j \eta^0 \|_{3-j}, \quad j = 0, 1. \)

2. Let \((\eta^d, u^d) = (\eta - \eta^L, u - u^L)\), then for any \( t \in \mathbb{R}^+_0 \),

\[
\| \eta^d(t) \|_{3}^2 + \int_0^t \| (u^d, m \partial_2 \eta^d) \|_2^2 d\tau \leq Cm^{-1},
\]  

\[
\sum_{i=1}^2 \left( \langle t \rangle^i \| \partial_2^i \eta^d(t) \|_{3-i}^2 + \langle t \rangle^{-i+1} \| \partial_2^i (u^d, m \partial_2 \eta^d)(t) \|_{2-i}^2 + \right. \]

\[
\left. + \int_0^t \langle \tau \rangle^{-i+1} \| \partial_2^i u^d \|_{3-i}^2 + \langle \tau \rangle^i \| m \partial_2^i \eta^d \|_{2-i}^2 \right) d\tau \leq Cm^{-1},
\]

where the error function \((\eta^d, u^d)\) enjoys the estimates (2.18) and (2.19) with \((\eta^d, u^d)\) in place of \((\eta, u)\).

3. If \( \eta^0 \) further satisfies the additional regularity

\[
\partial_1(\eta^0, u^0) \in \mathcal{H}^3 \times \mathcal{H}^2,
\]

then for any \( t \geq 0 \),

\[
\langle t \rangle \| (u^d, m \partial_2 \eta^d)(t) \|_2^2 + \int_0^t \langle \tau \rangle \| u^d \|_3^2 d\tau \leq Cm^{-1}(1 + \| \partial_1 \eta^0 \|_3^2 + \| \partial_1 u^0 \|_2^2),
\]

We can follow the idea of deriving the estimate (2.28) to establish Theorem 2.4, the proof of which will be presented in Section 3. Here we explain why one has to modify \((\eta^0, u^0)\) to be initial data of \((\eta^L, u^L)\) in (2.33)\(_4\), and why one imposes the additional regularity (2.38) in order to get (2.39).
(1) Since the initial data \( u^L|_{t=0} \) is divergence-free, i.e., \( \text{div}(u^L|_{t=0}) = 0 \), one has to adjust the initial data \( u^0 \) as in \((2.33)_4\).
(2) If the initial data \( \eta^0 \) of \( \eta \) is directly used to be initial data of the corresponding linear problem, then we see that \( \text{div}\eta^L = \text{div}\eta^0 \), and a time-decay of \( \partial_t\eta^d \) in \((2.37)\) can not be expected unless \( \text{div}\eta^0 = 0 \). Hence, we have to modify \( \eta^0 \) as in \((2.33)_4\), so that the obtained new initial data \( \eta^0 + \eta^r \) is also divergence-free.
(3) Subtracting \((2.33)\) from \((2.8) - (2.9)\), one obtains
\[
\begin{aligned}
\eta_t^d &= u^d, \\
u_t^d + \nabla q - \nu \Delta u^d - m^2 \partial_2^2 \eta^d &= \mathcal{N}, \\
\text{div}u^d &= -\text{div} \tilde{A}u, \\
(\eta^d, u^d)|_{t=0} &= -(\eta^r, u^r).
\end{aligned}
\]  
(4.0)

Let \( \alpha = (\alpha_1, \alpha_2) \), an application of \( \partial^\alpha \) with \( |\alpha| = 2 \) to \((4.0)_2\) yields
\[
\partial^\alpha u_t^d + \partial^\alpha \nabla q - \nu \Delta \partial^\alpha u^d - m^2 \partial_2^2 \partial^\alpha \eta^d = \partial^\alpha \mathcal{N} \quad (|\alpha| = 2).  
\]  
(4.1)

We multiply \((4.0)_2\) by \( \partial^\alpha \eta^d \) and \( \partial^\alpha u^d \) in \( L^2 \), respectively, and integrate to obtain the following two energy identities concerning derivatives of \( (\eta^d, u^d) \):
\[
\frac{d}{dt} \int \left( \partial^\alpha \eta^d \cdot \partial^\alpha u^d + \frac{\nu}{2} |\partial^\alpha \nabla \eta^d|^2 \right) dy + \int |m \partial_2 \partial^\alpha \eta^d|^2 dy - \int |\partial^\alpha u^d|^2 dy = \int \partial^\alpha \mathcal{N} \cdot \partial^\alpha \eta^d dy + \int \partial^\alpha \eta^d \text{div} \eta^d dy =: J_1,
\]
(4.2)

and
\[
\frac{1}{2} \frac{d}{dt} \int (|\partial^\alpha u^d|^2 + |m \partial_2 \partial^\alpha \eta^d|^2) dy + \nu \int |\nabla \partial^\alpha u^d|^2 dy = \int \partial^\alpha \mathcal{N} \cdot \partial^\alpha u^d + \int \partial^\alpha \eta^d \text{div} u^d dy =: J_2.
\]
(4.3)

It seems that the integral terms \( J_2 \) for \( \alpha_2 \neq 0 \) and \( J_1 \) could provide the convergence rate \( m^{-1} \) by directly employing the estimate \( \|m \partial_2 \eta\|_2 \leq C \). This idea, can not be directly applied to \( J_2 \) for \( \alpha_2 = 0 \) due to the integral term \( \int \partial^\alpha \eta \partial_2 u \cdot \partial^\alpha u dy \) hidden in \( J_2 \). To circumvent this difficult term, we rewrite it as follows.
\[
\int \partial^\alpha \eta \partial_2 u \cdot \partial^\alpha u dy = \int \partial^\alpha \eta \partial_2 u \cdot \partial^\alpha u dy + \int \partial^\alpha \eta \partial_2 u \cdot \partial^\alpha u dy.
\]
(4.4)

Noting that \( \partial^\alpha \eta^d \) can provide a convergence rate \( m^{-1} \) by using \((4.2)\) and the energy identity \((4.3)\) with \( |\alpha| = 1 \), thus the first term on the right hand of \((4.4)\) also provide a convergence rate \( m^{-1} \). In addition, it is easy to formally derive from the odevity condition of \( \eta^L_2 \) and the linear problem \((2.33)\) that
\[
\|\partial^\alpha \eta^L_2\|_0 \lesssim_0 \|\partial^\alpha \partial_2 \eta^L_2\|_0 \leq C m^{-1} (1 + \|\partial_t \eta^0\|_3^2 + \|\partial_t u^0\|_2^2),
\]
(4.5)

thus we need the regularity condition \((2.38)\) to make the derivation procedure of \((4.5)\) sense. Consequently, \( J_2 \) also implies the convergence rate \( m^{-1} \) for \( \alpha_2 = 0 \) under the additional condition \((2.38)\).
2.3. Extension to the inviscid case with damping

We now describe how to extend the aforementioned results to the inviscid case with damping. The equations of incompressible inviscid MHD fluids with zero resistivity and a low-order damping read as follows:

\[
\begin{align*}
\rho u_t + \rho v \cdot \nabla v + \nabla p - \kappa v &= \lambda M \cdot \nabla M/4\pi, \\
M_t + v \cdot \nabla M &= M \cdot \nabla v, \\
\text{div} v &= \text{div} M = 0,
\end{align*}
\]

where \(\kappa > 0\) is the damping coefficient. We mention that the well-posedness for the idea MHD system with a velocity damping has been widely investigated, see [23, 27, 28, 30] for examples.

Similarly to (2.8), we rewrite (2.46) in the following form of Lagrangian coordinates:

\[
\begin{align*}
\eta_t &= u, \\
u_t + \nabla_A q + \kappa u &= m^2 \partial_2^2 \eta, \\
\text{div}_A u &= 0,
\end{align*}
\]

with initial data

\[
(\eta, u)|_{t=0} = (\eta^0, u^0) \quad \text{in} \ T^2.
\]

Then we have the following results, which can be regarded as an extension of Theorems 2.1, 2.2 and 2.4.

**Theorem 2.5.** Let \(\kappa\) be a positive constant. There are constants \(c_1^k \geq 4\) and \(c_2^k \in (0, 1]\), such that for any \((\eta^0, u^0) \in (H^5_{1,2} \cap H^1_+) \times H^1_+\) and \(\kappa\) satisfying \(\text{div}_A u^0 = 0\), the oddity conditions (2.14) and the condition of strong magnetic fields

\[
m \geq \frac{1}{c_3^k} \max \{R^{1/2}, R^2\},
\]

where \(A^0\) denotes the initial data of \(A\) and

\[
R := \sqrt{c_1^k \|(\eta^0, u^0, m\partial_2 \eta^0)\|^2_4 c_2^k \|(\eta^0, u^0, m\partial_2 \eta^0)\|^2_3},
\]

the initial value problem (2.47) - (2.48) admits a unique global classical solution \((\eta, u, q) \in \mathcal{S}^{5, s}_{1,2, \infty} \times \mathcal{U}^1_+ \times (\mathcal{C}^0(\mathbb{R}_0^+, H^3) \cap \mathcal{C}^0(\mathbb{R}^+, H^1)).\) Moreover, the solution \((\eta, u)\) enjoys

1. **Decay estimate:**

\[
\begin{align*}
\sup_{t \geq 0} \left\{ e^{c_1^k \min(1,m)t} \left( \|(\eta, u)(t)\|^2_4 + \|m\eta(t)\|^2_5 \right) \right\} &+ \int_0^\infty e^{c_1^k \min(1, m)\tau} (\|u\|^2_4 + \|m\eta\|^2_5) d\tau \lesssim \kappa R^2 \\quad \text{(2.50)}
\end{align*}
\]

and

\[
\begin{align*}
\sum_{0 \leq i < 1} \left( \sup_{t \geq 0} \left\{ \langle t \rangle^i \left( \|(\eta, u)(t)\|^2_4 + \|m\eta(t)\|^2_5 \right) \right\} + \int_0^\infty \langle \tau \rangle^i (\|u\|^2_4 + \|m\eta\|^2_5) d\tau \right) \\
+ \int_0^t \langle \tau \rangle \|\eta\|^2_5 d\tau \leq C^{\kappa},
\end{align*}
\]

where we remark that the decay rates in (2.51) do not depend on \(m\) for fixed \(\|(\eta^0, u^0, m\partial_2 \eta^0)\|^2_4\).
(2) Stability around the solution \((\eta^L, u^L)\) of the linear problem:

\[
\sup_{t \geq 0} \left\{ e^{\varepsilon \min(1,m)t} \left( \| (\eta^d, u^d)(t) \|_4^2 + \| m\eta^d(t) \|_{5,2}^2 \right) \right\} \\
+ \int_0^\infty e^{\varepsilon \min(1,m)\tau} \left( \| u^d(t) \|_4^2 + \| m\eta^d(t) \|_{5,2}^2 \right) d\tau \leq C^\kappa m^{-2} \tag{2.52}
\]

and

\[
\sup_{t \geq 0} \left\{ \langle t \rangle^i \left( \| (\eta^d, u^d)(t) \|_4^2 + \| m\eta^d(t) \|_{5,2}^2 \right) \right\} \\
+ \int_0^\infty \langle \tau \rangle^i \left( \| u^d(t) \|_4^2 + (1 + m^2) \| \eta^d(t) \|_{5,2}^2 \right) d\tau \leq C^\kappa m^{-2}, \quad i = 1, 2, \tag{2.53}
\]

where the error function \((\eta^d, u^d) := (\eta - \eta^L, u - u^L)\) and \((\eta^L, u^L) \in C^0(\mathbb{R}^+, H_2^5) \times \mathcal{A}_\infty^1\) is the unique classical solution of the following linear pressureless initial-value problem:

\[
\begin{cases}
\eta_t^L = u^L, \\
u_t^L - \kappa u^L = m^2 \partial_\theta^2 \eta^L, \\
\text{div} u^L = 0, \\
(\eta^L, u^L)|_{t=0} = (\eta^0 + \eta^r, u^0 + u^r),
\end{cases}
\tag{2.54}
\]

for some \((\eta^r, u^r) \in H_2^5 \times H_4^4\) satisfying (2.31) and

\[
\| u^r \|_4 \lesssim_0 \| \partial_\theta^2 u^r \|_4 \| u^0 \|_4 \quad \text{and} \quad \| \partial_\theta^2 u^r \|_4 \lesssim_0 \| \partial_\theta u^0 \|_{3+i} \| \nabla \partial_\theta^2 u^0 \|_3, \quad i = 0, 1. \tag{2.55}
\]

Remark 2.7. We have assumed \((\eta^0)_{T_2} = (u^0)_{T_2} = 0\) in Theorem 2.5. If \(((\eta^0)_{T_2}, (u^0)_{T_2}) \neq 0\), then the unique solution \((\eta, u, q)\) of (2.47)–(2.48) enjoys the following form:

\[
(\eta, u, q) = (\bar{\eta} + \kappa^{-1}(u^0)_{T_2}(1 - e^{-\kappa t}) + (\eta^0)_{T_2}, \bar{u} + (u^0)_{T_2} e^{-t}, \bar{q}),
\]

where \((\bar{\eta}, \bar{u}, \bar{q})\) is the unique solution, established in Theorem 2.5, of the following initial-value problem

\[
\begin{cases}
\bar{\eta}_t = \bar{u}, \\
\bar{u}_t + \nabla A \bar{q} + \kappa \bar{u} = m^2 \partial_\theta^2 \bar{q}, \\
\text{div}_A \bar{u} = 0,
\end{cases}
\]

with initial data \((\bar{\eta}, \bar{u})|_{t=0} = (\eta^0 - (\eta^0)_{T_2}, u^0 - (u^0)_{T_2})\) in \(T^2\).

The key idea in the proof of Theorem 2.5 is similar to that in the proof of Theorems 2.2 2.4. But there are remarkable differences between the decay results for the inviscid and viscous cases, which will be further explained in the proof process of Theorem 2.5, see all the footnotes in Section 6.

2.4. Verification of preserving the odevity of solutions

We end this section by verifying the assertion in Remark 2.4. To this end, let \((\eta, u, q)\) be a strong solution of (2.38)–(2.40), \(\psi = (\eta_1 - \eta_2)(y_1, -y_2, t), w = (u_1, -u_2)(y_1, -y_2, t)\) and \(p = (q_1, q_2)(y_1, -y_2, t)\). Due to the uniqueness of strong solutions, we see that to get the desired
We recall here that the solution automatically satisfies (2.8). It is obvious that $(\psi, w)$ satisfies (2.8)$_1$. Next, we show that $(\psi, w)$ also satisfies (2.8)$_2$ and (2.8)$_3$. Defining
\[
A = \begin{pmatrix}
\partial_2 \eta_2 + 1 & -\partial_1 \eta_2 \\
-\partial_2 \eta_1 & \partial_1 \eta_1 + 1
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
\partial_2 \psi_2 + 1 & -\partial_1 \psi_2 \\
-\partial_2 \psi_1 & \partial_1 \psi_1 + 1
\end{pmatrix},
\]
then
\[
(B_{11}, B_{22}) = (A_{11}, A_{22})|_{y_2 = -y_2} \quad \text{and} \quad (B_{12}, B_{21}) = -(A_{12}, A_{21})|_{y_2 = -y_2}. \tag{2.56}
\]
Let $\chi$ be a function, and $\tilde{\chi} = \chi(y_1, -y_2)$. Then, by (2.56),
\[
\nabla_B \tilde{\chi} = (A_{1i}, -A_{2i}) \partial_i \chi|_{y_2 = -y_2} \tag{2.57}
\]
and
\[
\nabla_B w = \nabla_A u|_{y_2 = -y_2} = 0,
\]
from which we see that $(\eta, u)$ satisfies (2.8)$_3$. By (2.56) and (2.57), we further have
\[
\Delta_B \tilde{\chi} = \Delta_A \chi|_{y_2 = -y_2}. \tag{2.58}
\]
Thus, the identities (2.57) and (2.58) yield
\[
\partial_t w_i + \nabla_B p - \nu \Delta_B w_i = \begin{cases}
(\partial_t u_1 + A_{1i} \partial_i q - \nu \Delta_A u_1)|_{y_2 = -y_2} & \text{for } i = 1, \\
-(\partial_t u_2 + A_{2i} \partial_i q - \nu \Delta_A u_2)|_{y_2 = -y_2} & \text{for } i = 2,
\end{cases}
\]
and
\[
\partial_2^2 (\psi_1, \psi_2) = \partial_2^2 (\eta_1, -\eta_2)|_{y_2 = -y_2}.
\]
Hence, we see that $(\psi, w)$ satisfies (2.8)$_2$ by the above two identity. This proves the property of preserving the odevity of strong solutions.

3. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. First we derive some basic (a priori) estimate for $(\eta, u, q)$ under the a priori assumption (2.26) associated with the smallness condition (2.27) in Subsection 3.1, then further establish the stability estimate (2.16) in Subsection 3.2, and finally, introduce a local well-posedness result for (2.8)–(2.9) and complete the proof of Theorem 2.1 by a standard continuity argument in Subsection 3.3.

3.1. Energy estimates

Let $(\eta, u, q)$ be a solution of the initial-value problem (2.8)–(2.9) defined on $\Omega_T$ for any given $T > 0$, where $(\eta^0, u^0)$ belongs to $H^3 \times H^2$, and satisfies $\nabla_A u^0 = 0$ and the odevity conditions (2.14). We recall here that the solution automatically satisfies $(\eta)_{T^2} = (u)_{T^2} = 0$ and the odevity conditions in (2.21) if $(\eta^0, u^0)$ does. We further assume that $(\eta, u, q)$ and $K$ satisfy $(q)_{T^2} = 0$, (2.26) and (2.27), where $K \geq 1$ will be defined by (3.81) and $\delta \in (0, 1]$ is a sufficiently small constant. It should be noted that the smallness of $\delta$ only depends on the parameter $\nu$. In addition, by Young’s inequality, one easily finds from (2.20) and (2.27) that
\[
\sup_{0 \leq t \leq T} \| \partial_2 \eta(t) \|_2 \leq \frac{K}{m} \leq \frac{1}{3m} (2K^{1/2} + K^2) \leq \delta. \tag{3.1}
\]

Before deriving the energy estimates for $(\eta, u)$, we introduce some basic inequalities and establish some preliminary estimates of $(\eta, u)$ by using the following two lemmas.
Lemma 3.1. We have the following basic inequalities:

(1) Generalized Poincaré’s inequalities:

\[ \|f\|_0 \lesssim_0 \|\partial_2 f\|_0 \quad \text{for any } f \in H^1 \text{ satisfying } \int_{-1}^{1} f(y_1, y_2) dy_2 = 0, \]  
\[ \|f\|_i \leq c_i \|\nabla^i f\|_0 \quad \text{for any } f \in H^i, \]  

where and the positive constant \(c_i\) only depends on \(i\).

(2) Interpolation inequality: for any \(f \in H^2\),

\[ \|f\|_{L^\infty} \lesssim_0 \|f\|_0 (f, \partial_1 f)_0 + \|f\|_1^{1/2} (f, \partial_1 \partial_2 f)_0^{1/2} + \|f\|_0^{1/2} \|\partial_2 f\|_0^{1/2} \|\partial_2 (f, \partial_1 f)\|_0^{1/2}. \]  

In particular, by Young’s inequality,

\[ \|f\|_{L^\infty} \lesssim_0 \|\partial_2 (f, \partial_1 f)\|_0 + \|f, \partial_1 f\|_0. \]  

(3) Product estimate: for any \((f, g, h) \in H^1_2 \times H \times L^2\),

\[ \int |fgh| dy \lesssim \sqrt{\|f\|_0 \|f\|_1^{1/2} (g, \partial_1 g)_0 \|h\|_0 \|f\|_0^{1/2} (f, \partial_2 f)_0 \|g, \partial_1 g\|_0 \|h\|_0. \]  

(4) There is a constant \(\delta \in (0, 1]\), such that for any \((\xi - y) \in H\) satisfying \(\|\nabla (\xi - y)\|_{L^\infty} \leq \delta\),

\[ \|\nabla B f\|_0 \lesssim_0 \|\nabla f\|_0 \|\nabla B f\|_0 \|\nabla f\|_0 \quad \text{for any } \nabla f \in L^2, \]  

where \(B := (\nabla \xi + I)^{-T}\).

Proof. (1) The inequalities (3.2) and (3.3) are obvious to get by virtue of classical Poincaré’s inequality:

\[ \|g\|_{L^r(\Omega)}^{r} \lesssim \|\nabla g\|_{L^r(\Omega)}^{r} + \left( \int_{\Omega} |g| d\gamma \right)^{r} \]  

for any \(g \in W^{1,r}(\Omega)\),

where \(\Omega = \mathbb{T}^n\) with \(n \geq 1\), \(r \geq 1\) is a constant and the constant \(\tilde{c}_{r,\Omega}\) depends only on \(r\) and \(\Omega\).

(2) Now, we turn to the derivation of (3.4). Here and in what follows, we denote

\[ \|f\|_{L^p_{n}} := \left( \int_{-1}^{1} |f|^p dy \right)^{1/p} \quad \text{for } f \in L^p. \]

Since \(C^2(\mathbb{T}^2)\) is dense in \(H^2\), it suffices to prove (3.4) for \(f \in C^2(\mathbb{T}^2)\). Noting that, for any \(g \in C^0(\mathbb{T}^2)\),

\[ \sup_{y \in (-1,1)} g(y_1, y_2) \]  

is a measurable function defined on \((-1,1)\),

we use the Fubini theorem and the one-dimensional interpolation inequality (see [21, Theorem]) to deduce that

\[ \sup_{s \in (-1,1)} |\phi(s)|^2 \lesssim_0 \|\phi(s)\|_{L^2(-1,1)} \|\phi, \phi'(s)\|_{L^2(-1,1)} \]  

for \(\phi \in H^1(-1,1)\).
Therefore,
\[
\|f\|_{L^\infty}^2 = \sup_{y_1 \in (-1,1)} \left( \sup_{y_2 \in (-1,1)} |f(y_1, y_2)| \right)
\]
\[
\lesssim_0 \sup_{y_1 \in (-1,1)} \left( \|f\|_{L^2_{y_2}} \|\partial_2 f\|_{L^2_{y_2}} + \|f\|_{L^2_{y_2}}^2 \right)
\]
\[
\lesssim_0 \left( \sup_{y_1 \in (-1,1)} \left| \int T \sup_{y_2 \in (-1,1)} |f| \text{d}y_2 \text{d}y_1 \right| \right)^{1/2}
\]
\[
\lesssim_0 \left( \sup_{y_1 \in (-1,1)} \left| \int T \sup_{y_2 \in (-1,1)} |f| \text{d}y_2 \text{d}y_1 \right| \right)^{1/2}
\]
which gives (3.4).

(3) Finally, we prove (3.6). Recalling that \(C^1(\mathbb{T}^2)\) is dense in \(H^2_2\) and \(H\), it suffices to prove (3.6) for \((f, g) \in C^1(\mathbb{T}^2)\). Let \((f, g) \in C^1(\mathbb{T}^2)\), we have
\[
\|f\|_{L^\infty}^2 \lesssim \sqrt{\|f\|_0 \|f\|_{1,2}}, \quad \sup_{y_1 \in (-1,1)} \|g\|_{L^2_{y_2}} \lesssim \sqrt{\|g\|_0 \|(g, \partial_1 g)\|_0},
\]
which, together with the Fubini theorem and Hölder’s inequality, implies
\[
\int |f g h| \text{d}y \lesssim_0 \int_T \sup_{y_2 \in (-1,1)} |f| \int_T |g h| \text{d}y_2 \text{d}y_1
\]
\[
\lesssim_0 \|f\|_{L^2_{y_2}} \sup_{y_1 \in (-1,1)} \|g\|_{L^2_{y_2}} \|h\|_{L^2_{y_2}} \int_T \sup_{y_2 \in (-1,1)} |f| \text{d}y_2 \text{d}y_1
\]
\[
\lesssim_0 \left( \sup_{y_1 \in (-1,1)} |f| \right) \sup_{y_1 \in (-1,1)} \|g\|_{L^2_{y_2}} \|h\|_0
\]
\[
\lesssim_0 \sqrt{\|f\|_0 \|f\|_{1,2} \|g\|_0 \|(g, \partial_1 g)\|_0 \|h\|_0}.
\]
Therefore, an application of Young’s inequality yields (3.6).

(4) The equivalent estimate (3.7) holds obviously.

\[\square\]

**Lemma 3.2.** Under the condition (3.1) with sufficiently small \(\delta \in (0, 1]\), we have

1. Estimates for \(\eta\):
\[
\|\partial_1 \eta\|_{L^\infty} \lesssim_0 \begin{cases} \|\partial_2 \eta\|_2 & \text{for } i = 1 \text{ and } 1 \leq j \leq 2, \\ \|\partial_2 \eta\|_1 & \text{for } i = 2, \end{cases}
\]
\[
\|\eta\|_{i+1,2} \lesssim_0 \|\partial_2 \eta\|_i \text{ for } 0 \leq i \leq 2,
\]
\[
\|\eta\|_{L^\infty} \lesssim_0 \sqrt{\|\partial_2 \eta\|_0 \|\partial_2 \eta\|_1}.
\]

2. Estimates for \(u\):
\[
\|u\|_{i+1,2} \lesssim_0 \begin{cases} \|\partial_2 u\|_0 + \|\partial^2 \eta\|_1 \|\partial_2 u\|_1 & \text{for } i = 0, \\ \|\partial_2 u\|_i & \text{for } i = 1 \text{ and } 2, \end{cases}
\]
\[
\|\partial_1 u\|_{L^\infty} \lesssim_0 \begin{cases} \|\partial_2 u\|_2 & \text{for } i = 1, \\ \|\partial^2 \eta\|_1 \|\partial_2 u\|_1 & \text{for } i = 2. \end{cases}
\]
(3) Product estimates:

\[ \int |\partial^{\alpha} \eta g h| dy \lesssim \begin{cases} 
\| \partial_2 \eta \|_{\alpha_1} \| g \|_1 \| h \|_0 & \text{for } \alpha_2 = 0, \\
\| \partial_2 \partial^{\alpha} \eta \|_{\alpha_1} \| g \|_1 \| h \|_0 & \text{for } \alpha_2 \neq 0,
\end{cases} \tag{3.14} \]

\[ \int |\partial^{\beta} u g h| dy \lesssim \begin{cases} 
\| \partial_2 u \|_{\beta_1} \| g \|_1 \| h \|_0 & \text{for } \beta_2 = 0, \\
\| \partial_2 \partial^{\beta} u \|_{\beta_1} \| g \|_1 \| h \|_0 & \text{for } \beta_2 \neq 0
\end{cases} \tag{3.15} \]

for any \((g, h) \in H \times L^2\), where \(\alpha\) and \(\beta\) satisfy \(1 \leq |\alpha| \leq 2\) and \(|\beta| = 2\).

Remark 3.1. Thanks to (3.1) and (3.9), one has

\[ \| \nabla \eta \|_{L^\infty} \lesssim_0 \| \partial_2 \eta \|_2 \lesssim_0 \delta, \tag{3.16} \]

which, together with (3.7), yields the following equivalent estimate: for sufficiently small \(\delta\), it holds that

\[ \| \nabla A f \|_0 \lesssim_0 \| \nabla f \|_0 \lesssim_0 \| \nabla A f \|_0 \text{ for any } \nabla f \in L^2. \tag{3.17} \]

PROOF. (1) By (3.11) we have that for \(1 \leq i, j \leq 2\),

\[ \| \partial_i \eta_j \|_{L^\infty} \lesssim_0 \| \partial_2 \partial_i (\eta_j, \partial_1 \eta_j) \|_0 + \| \partial_i (\eta_j, \partial_1 \eta_j) \|_0. \tag{3.18} \]

Since \(\eta_2(y_1, y_2)\) is an odd function with respect to \(y_2\) for any given \(y_1\), one sees that

\[ \int_{-1}^{1} \partial_1^k \eta_2 \, dy_2 = 0 \text{ for } 0 \leq k \leq 2. \]

In addition, \(\eta_2(y_1, y_2)\) is a periodic function with respect to \(y_2\) for any given \(y_1\), then

\[ \int_{-1}^{1} \partial_2^l \partial_1^k \eta_2 \, dy_2 = 0 \text{ for } 1 \leq k + l \leq 3 \text{ and } 1 \leq l. \]

Using (3.2) and the above two relations, we find that

\[ \| \partial_1^k \eta_2 \|_0 \lesssim_0 \| \partial_2 \partial_1^k \eta_2 \|_0 \text{ for } 0 \leq k \leq 2 \tag{3.19} \]

and

\[ \| \partial_2^l \partial_1^k \eta_j \|_0 \lesssim_0 \| \partial_2^{l+1} \partial_1^k \eta_j \|_0 \text{ for } 0 \leq l + k \leq 2 \text{ and } 1 \leq l. \tag{3.20} \]

Putting (3.18)–(3.20) together, we get

\[ \| \partial_i \eta_j \|_{L^\infty} \lesssim_0 \begin{cases} 
\| \partial_2 \eta_j \|_2 & \text{for } i = 1 \text{ and } j = 2, \\
\| \partial_2^2 \eta_j \|_1 & \text{for } i = 2 \text{ and } j = 1, 2.
\end{cases} \tag{3.21} \]

To obtain (3.9), we next show that (3.21) also holds for \((i, j) = (1, 1)\).

Keeping in mind that \(\det(I + \nabla \eta) = 1\), one gets from Sarrus’ rule that

\[ \text{div} \eta = \partial_1 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_1 \partial_2 \eta_2. \tag{3.22} \]
In particular,
\[ \partial_1 \eta_1 = \partial_1 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_1 \partial_2 \eta_2 - \partial_2 \eta_2. \]  
(3.23)

Applying \( \nabla \) to (3.23), and then multiplying the resulting identity by \( \nabla \partial_1 \eta_1 \) in \( L^2 \), we obtain
\[ \| \nabla \partial_1 \eta_1 \|_0^2 = \int \nabla (\partial_1 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_1 \partial_2 \eta_2 - \partial_2 \eta_2) \cdot \nabla \partial_1 \eta_1 \, dy. \]  
(3.24)

Making use of Hölder’s inequality, (3.3), (3.6), (3.19) and (3.21), we infer from (3.24) that
\[ \| \partial_1 \eta_1 \|_1^2 \lesssim \| \partial_1 \eta_1 \|_1 (\| \partial_2 \eta_2 \|_1 + \| \partial_2 \eta_2 \|_{L^\infty} \| \partial_1 \eta_1 \|_1 + \| \partial_1 \eta_2 \|_{L^\infty} \| \partial_2 \eta_1 \|_1) \]
\[ + \int |\nabla \partial_1 \eta_2| |\partial_2 \eta_1| |\nabla \partial_1 \eta_1| \, dy + \int |\partial_1 \eta_1| |\nabla \partial_2 \eta_1| |\nabla \partial_1 \eta_1| \, dy \]
\[ \lesssim \| \partial_1 \eta_1 \|_1 (\| \partial_2 \eta_2 \|_1 + \| \partial_1 \eta_1 \|_1 \| \partial_2 \eta_2 \|_2 + (\| \partial_1 \eta_2 \|_{L^\infty} + \| \partial_2 \eta_2 \|_2) \| \partial_2 \eta_1 \|_1). \]
By (3.1), (3.21) and Young’s inequality, we further deduce from the above estimate that
\[ \| \partial_1 \eta_1 \|_1 \lesssim \| \partial_2 \eta_1 \|. \]  
(3.25)

We immediately see from (3.18) and (3.25) that \( \| \partial_1 \eta_1 \|_{L^\infty} \lesssim \| \partial_2 \eta_1 \|_2 \). This completes the proof of (3.9).

Thanks to (3.9), we can get from (3.23) that
\[ \| \partial_1 \eta_1 \|_0 \lesssim \| \partial_2 \eta_0 \|. \]  
(3.26)

Putting (3.19), (3.25) and (3.26) together, one concludes
\[ \| \nabla \eta \|_{i-1} \lesssim \| \partial_2 \eta \|_i \quad \text{for } 1 \leq i \leq 2. \]  
(3.27)

Thus, from (3.3), (3.19), (3.26) and (3.27), the estimate (3.10) follows immediately. Finally, the estimate (3.11) is obvious by virtue of (3.4) and (3.10).

(2) Since \( u \) enjoys the same odevity and the same periodicity as \( \eta \) does, we obtain, similarly to (3.18)–(3.21), that
\[ \| \partial_1^k u_2 \|_0 \lesssim \| \partial_2 \partial_1^k u_2 \|_0 \quad \text{for } 0 \leq k \leq 2, \]  
(3.28)
\[ \| \partial_1^l \partial_1^k u_j \|_0 \lesssim \| \partial_2 \partial_1^{l+1} \partial_1^k u_j \|_0 \quad \text{for } 0 \leq l + k \leq 2 \quad \text{and } 1 \leq l, \]  
(3.29)
\[ \| \partial_1 u_3 \|_{L^\infty} \lesssim \| \partial_2 \partial_1 (u_1, \partial_1 u_1) \|_0 + \| \partial_1 (u_1, \partial_1 u_1) \|_0 \]  
(3.30)

and
\[ \| \partial_1 u_j \|_{L^\infty} \lesssim \begin{cases} \| \partial_2 u_j \|_2 & \text{for } i = 1 \text{ and } j = 2, \\ \| \partial_2^2 u_j \|_1 & \text{for } i = 2 \text{ and } j = 1, 2. \end{cases} \]  
(3.31)

By (2.8)3 and the definition of \( \tilde{A} \), we find that
\[ \text{div} u = -\text{div} \tilde{A} u, \]  
(3.32)

where
\[ \tilde{A} = \begin{pmatrix} \partial_2 \eta_2 & -\partial_1 \eta_2 \\ -\partial_2 \eta_1 & \partial_1 \eta_1 \end{pmatrix}. \]  
(3.33)
If we apply the norm \( \| \cdot \|_0 \) to (3.32), we get
\[
\| \partial_1 u_1 \|_0 \lesssim_0 \| \partial_2 u \|_0 + \| \partial_2 \eta_1 \|_{L^\infty} \| \partial_1 u_2 \|_0 + \delta \| \partial_1 u_1 \|_0,
\]
which, together with (3.9) and (3.28), gives
\[
\| \partial_1 u_1 \|_0 \lesssim_0 \| \partial_2 u \|_0 + \| \partial_2^2 \eta_1 \| \| \partial_2 u \|_1
\]
for sufficiently small \( \delta \).

In addition, following the same process as in the derivation of (3.25), we easily deduce from (3.32) that
\[
\| \partial_1 u_1 \|_1 \lesssim_0 \| \partial_2 u \|_2.
\]
Thus, the estimate (3.12) follows from (3.1), (3.3), (3.28), (3.31) and (3.35).

Plugging (3.35) into (3.30), we get
\[
\| \partial_2 u \|_2 \leq (1 + \| \partial \eta \|_2 + \| \partial u \|_2) + m \| \partial_2 \eta \|_2 \| \partial u \|_2 + \| \partial_2 u \|_2.
\]

(3) The estimate (3.14) follows from (3.6), (3.20) and (3.27). Similarly, in view of (3.6), (3.28), (3.29) and (3.35), we see that (3.15) holds. This completes the proof of Lemma 3.2.

Now, we proceed to derive some basic energy estimates for \((\eta, u, q)\).

**Lemma 3.3.** Under the condition (3.1) with sufficiently small \( \delta \), we have
\[
\| \nabla q \|_i \lesssim_0 \begin{cases} \| (u, \partial_2 u) \|_0 \| \nabla u \|_1 + m^2 \| \partial_2 \eta \|_0 \| \partial_2 \eta \|_2 & \text{for } i = 0; \\ \| \partial_2^2 u \|_0 \| \nabla u \|_1 \text{ or } \| \partial_2 u \|_1^2 & \text{or } i = 1, \\ + m^2 (\| \partial_2 \eta \|_0 \| \partial_2 \eta \|_2 + \| \partial_2^2 \eta \|_2^2) & \text{for } i = 1, 
\end{cases}
\]
\[
\| \nabla q_i \|_1 \lesssim_0 \| \partial_2 u \|_2 \| \nabla u \|_2 (1 + \| \nabla \eta \|_2 + \| \nabla u \|_1) + m^2 \| \partial_2 \eta \|_2 (\| \partial_2 \eta \|_2 \| \nabla u \|_2 + \| \partial_2 u \|_2).
\]

**Remark 3.2.** The estimate (3.37) will be used in the derivation of the error estimate (2.39) in Theorem 2.3 (see the last term in (6.15)).

**Proof.** (1) By (2.3) and (2.8), we have
\[
\text{div}_{\mathcal{A}} u = -\text{div}_{\mathcal{A}} u = -\text{div}(\mathcal{A}_i^T u).
\]
Multiplying (2.8) by \( \nabla_{\mathcal{A}} q \) in \( L^2 \) and using (3.22), (3.38), we integrate by parts and recall the fact
\[
\text{div}_{\mathcal{A}} \Delta_{\mathcal{A}} u = 0
\]
and infer that
\[
\| \nabla_{\mathcal{A}} q \|_0^2 = \int (\mathcal{A}_i^T u) \cdot \nabla q \, dy + m^2 \int \nabla \eta \cdot \nabla_{\mathcal{A}} q \, dy
\]
\[
+ m^2 \int \partial_2 (\partial_1 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_1 \partial_2 \eta_2) \partial_2 q \, dy.
\]
Exploiting (3.10) and (3.14), we deduce from the above identity that
\[
\| \nabla_{\mathcal{A}} q \|_0^2 \lesssim_0 (\| (u, \partial_2 u) \|_0 \| \mathcal{A}_i \|_1 + m^2 (\| \nabla \eta \|_{L^\infty} \| \partial_2^2 \eta \|_0 + \| \partial_2 \eta \|_0 \| \partial_2 \eta \|_2)) \| \nabla q \|_0.
\]
Finally, with the help of the above estimate, and (2.8), (3.9) and (3.17), we obtain (3.36) with \( i = 0 \).

(2) Applying \( \text{div}_A \) to (2.8)2, using then (2.8)1, (3.39) and the first identity in (3.38), we arrive at

\[
\Delta q = f, \tag{3.41}
\]

where

\[
f := (\partial_2 u_2 \partial_1 - \partial_1 u_2 \partial_2)u_1 + (\partial_1 u_1 \partial_2 - \partial_2 u_1 \partial_1)u_2 - (\text{div}_A \nabla_A q + \text{div} \nabla_A \tilde{q}) + m^2 (\partial_2^2 \text{div} \eta + (\partial_2 \eta_2 \partial_1 - \partial_1 \eta_2 \partial_2) \partial_2^2 \eta_1 + (\partial_1 \eta_1 \partial_2 - \partial_2 \eta_1 \partial_1) \partial_2^2 \eta_2).
\]

Multiplying (3.41) by \( \Delta q \) in \( L^2 \), we use the regularity theory of elliptic equations to get

\[
\| \nabla q \|_1^2 \lesssim_0 \| \Delta q \|_0^2 \lesssim_0 \int |f| \Delta q |dy|, \tag{3.42}
\]

where the integral term on the right hand side can be estimated as follows.

\[
\int |f\Delta q|dy \lesssim_0 (\| \partial_2^2 u \|_0 \| \nabla u \|_1 + \| \partial_2 u \|_1^2 + m^2 (\| \partial_2^2 \eta \|_0 (\| \nabla \eta \|_{L^\infty} + \| \partial_2 \eta \|_2) + \| \partial_2 \eta \|_{L^\infty} \| \partial_2^2 \eta \|_1) + (1 + \| \nabla \eta \|_{L^\infty}) (\| \nabla \eta \|_{L^\infty} \| \nabla^2 q \|_0 + \| \partial_2 \eta \|_2 \| \nabla q \|_1) ) \| \Delta q \|_0
\]

\[
\lesssim_0 (\| \partial_2^2 u \|_0 \| \nabla u \|_1 + \| \partial_2 u \|_1^2 + m^2 (\| \partial_2^2 \eta \|_0 \| \partial_2 \eta \|_2 + \| \partial_2^2 \eta \|_1^2) + \| \nabla \eta \|_1 ) \| \Delta q \|_0, \tag{3.43}
\]

where we have used (3.11), (3.15) and (3.22) in the first inequality, and (3.9) and (3.16) in the second inequality. Consequently, combining (3.42) with (3.43), one obtains (3.36) with \( i = 2 \).

(3) Applying \( \nabla \) to (2.8)2 and multiplying the resulting identity by \( \nabla u_i \) in \( L^2 \), we have

\[
\int |\nabla u_i|^2 dy = \int \nabla (m^2 \partial_2^2 \eta + \nu \Delta_A u - \nabla_A \tilde{q}) : \nabla u_i dy.
\]

It is easy to see from the above identity that

\[
\| \nabla u_i \|_0 \lesssim m^2 \| \partial_2^2 \eta \|_1 + \| \nabla u \|_2 (1 + \| \nabla \eta \|_2) + \| \nabla q \|_1,
\]

which, combined with (3.36) with \( i = 1 \), implies that for sufficiently small \( \delta \),

\[
\| \nabla u_i \|_0 \lesssim m^2 \| \partial_2^2 \eta \|_1 + \| \nabla u \|_2 (1 + \| \nabla \eta \|_2 + \| \nabla u \|_1). \tag{3.44}
\]

An application of \( \partial_t \) to (3.41) yields \( \Delta q_t = f_t \). Thus, multiplying this equation with \( \Delta q_t \) in \( L^2 \) and applying the regularity theory of elliptic equations, we get

\[
\| \nabla q_t \|_1^2 \lesssim_0 \| \Delta q_t \|_0^2 \lesssim_0 \int |f_t \Delta q_t|dy. \tag{3.45}
\]

Furthermore, it is easy to verify that

\[
\int |f_t \Delta q_t|dy \lesssim_0 (m^2 \| \partial_2 \eta \|_2 \| \partial_2 u \|_2 + \| \partial_2 u \|_2 \| \nabla u_i \|_0
\]

\[
+ \| \partial_2 u \|_2 \| \nabla q \|_1 + \delta \| \nabla q_t \|_1 ) \| \Delta q_t \|_0. \tag{3.46}
\]

Consequently, if we making use of (3.36) with \( i = 1 \), and (3.44), (3.46) and Young’s inequality, we obtain (3.37) from (3.45). □
Lemma 3.4. Under the conditions of (2.26)–(2.27) with sufficiently small \( \delta \), we have
\[
\frac{d}{dt} \| \nabla u(t) \|^2_0 + \nu \| u(t) \|^2_2 \lesssim (1 + m^2) \| m\partial_2^2 \eta \|_0^2, \quad t \geq 0
\] (3.47)
and
\[
\frac{d}{dt} \| \nabla^2 (u, m\partial_2 \eta)(t) \|^2_0 + \nu \| u(t) \|^2_2 \lesssim F_1 := \| \partial_2^2 \eta \|^2_2 \| m\partial_2 \eta \|^2_2 + \| \nabla \eta \|^2_2 \| \partial_2^2 u \|^2_1 \\
+ \| \nabla \eta \|_2 \| \partial_2 u \|_2 (m^2 \| \partial_2^2 \eta \|_1 \| \partial_2 \eta \|_2 + \| \partial_2 u \|_1), \quad t \geq 0.
\] (3.48)

**Proof.** Let \( \alpha \) satisfy \( 1 \leq |\alpha| \leq 2 \). Applying \( \partial^\alpha \) to (2.13) yields
\[
\partial^\alpha (u_t + \nabla q - \nu \Delta u - m^2 \partial_2^2 \eta) = \partial^\alpha (\mathcal{L} - \nabla \hat{q}^\alpha).
\]
If we multiply the above identity by \( \partial^\alpha u \) in \( L^2 \), integrate by parts and make use of (3.32), we get
\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha u \|^2_0 + \nu \| \nabla \partial^\alpha u \|^2_0 = I_1 + I_2 - \left\{ m^2 \int \partial_2^2 \eta \cdot \partial^{2\alpha} u \, dy \right\} \text{ for } |\alpha| = 1;
\]
\[
\frac{1}{2} \frac{d}{dt} \| m\partial_2 \partial^\alpha \eta \|^2_0 \text{ for } |\alpha| = 2,
\]
where
\[
I_1 := -\int \partial^\alpha \mathcal{L}^\alpha, \partial_t \partial^\alpha u \, dy, \quad I_2 := \int (\partial^\alpha \nabla \hat{q}^\alpha \cdot \partial^{\alpha^+} u - \partial^\alpha q \partial^\alpha \text{div} \hat{q}^\alpha u) \, dy,
\] (3.50)
\[
\alpha^- := \begin{cases} (\alpha_1 - 1, \alpha_2) \quad \text{for } \alpha_2 = 0; \\ (\alpha_1, \alpha_2 - 1) \quad \text{for } \alpha_2 \geq 1 \end{cases} \quad \text{and} \quad \alpha^+ := \begin{cases} (\alpha_1 + 1, \alpha_2) \quad \text{for } \alpha_2 = 0; \\ (\alpha_1, \alpha_2 + 1) \quad \text{for } \alpha_2 \geq 1. \end{cases} 
\] (3.51)

Next, we consider two cases.

(1) **Case** \( |\alpha| = 1 \).

In view of (3.14) and (3.16), we find that
\[
I_1 \lesssim (1 + \| \nabla \eta \|_{L^\infty}) \| \partial_2 \eta \|_2 \| u \|_2 \| \nabla^2 u \|_0 \lesssim \delta \| u \|^2_2.
\] (3.52)

Recalling (2.26) and (2.27), we see that
\[
\| \partial_2 \eta \|_2 (\| \partial_2 u \|_1 + \| (\nabla \eta, m\partial_2 \eta) \|_2) + (\| \nabla \eta \|_2 + \| \nabla \eta \|^2_2 / m) \lesssim_0 \delta.
\] (3.53)

The integral \( I_2 \) can be estimated as follows.
\[
I_2 \lesssim \| \partial_2 \eta \|_2 \| \nabla q \|_0 \| \nabla u \|_1 \\
\lesssim \| \partial_2 \eta \|_2 \| u \|_2 (\| \partial_2 u \|_1 \| u \|_2 + m^2 \| \partial_2^2 \eta \|_0 \| \partial_2 \eta \|_2) \\
\lesssim \delta (\| m\partial_2^2 \eta \|_0^2 + \| u \|^2_3),
\] (3.54)
where we have used (3.9) and (3.14) in the first inequality, (3.12) with \( i = 0 \) and (3.36) with \( i = 0 \) in the second inequality, and (3.53) and Young’s inequality in the last inequality.

Finally, putting (3.52) and (3.54) into (3.49), and utilizing (3.3), (3.20) and Young’s inequality, we get (3.47).

(2) **Case** \( i = 2 \).
We integrate by parts and use (3.15) to deduce

\[ \int \partial_t^2 \tilde{A}_{12} \partial_1 u \cdot \partial_t^2 \partial_2 u dy = - \int \partial_t^2 \tilde{A}_{12} \partial_1 \partial_2 u \cdot \partial_t^2 u dy + \int \partial_1 \partial_2 \tilde{A}_{12} \partial_1 (\partial_1 u \cdot \partial_t^2 u) dy \]

\[ \lesssim \| \nabla^3 \eta \|_0 \| \nabla \partial_t^2 u \|_1 \| \partial_t^2 u \|_1 + \| \nabla^2 \partial_t \eta \|_0 (\| \nabla^2 u \|_1^2 + \| \partial_1 u \|_{L^\infty} \| \nabla^3 u \|_0). \]  

(3.55)

Thus, the integral \( I_1 \) can be estimated as follows.

\[ I_1 \lesssim (1 + \| \nabla \eta \|_{L^\infty}) (\| \nabla \eta \|_{L^\infty} \| \nabla^3 u \|_0 + \| \nabla^2 \partial_t \eta \|_0 (\| \nabla^2 u \|_1 + \| \partial_1 u \|_{L^\infty}) + \| \nabla^3 \eta \|_0 (\| \partial_2 u \|_{L^\infty} + \| \nabla \partial_t^2 u \|_0)) \]

\[ + (\| \partial_2 \eta \|_{L^\infty} + \| \nabla^2 \partial_t \eta \|_0) (\| \nabla \partial_t \eta \|_1 \| \partial_1 u \|_{L^\infty} + \| \nabla^2 \eta \|_1 \| \nabla u \|_{L^\infty})) \| \nabla^2 u \|_1 \]

\[ \lesssim \delta \| u \|_3^2 + \| \nabla \eta \|_2 \| \partial_t^2 u \|_1 \| u \|_3, \]

(3.56)

where we have used (3.14) and (3.55) in the first inequality, and (3.4), (3.13), (3.16) and (3.53) in the second inequality.

Similarly, the second integral \( I_2 \) can be estimated as follows, using (3.36) with \( i = 1 \).

\[ I_2 \lesssim (\| \partial_2 \eta \|_2 \| \partial_2 u \|_2) \| \nabla q \|_1 \]

\[ \lesssim (\| \partial_2 \eta \|_2 \| \partial_2 u \|_2) \| \nabla \eta \|_2 (\| \partial_2 u \|_2^2 + \| \partial_2 \eta \|_2) \]

(3.57)

Consequently, plugging the above two estimates into (3.49), and using Young’s inequality, (3.1), (3.3), and (3.53), we get (3.48).

**Lemma 3.5.** Under the conditions of (2.26) - (2.27) with sufficiently small \( \delta \), we have

\[ \frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^3 \eta \|_0^2 + \sum_{|\alpha| = 2} \left( \int \partial^\alpha \eta \cdot \partial^\alpha u dy + \nu F_2^\alpha \right) \right) + \| m \partial_2 \eta \|_2^2 \leq \| \nabla^2 u \|_0^2 + c\delta \| u \|_3^2, \]

(3.58)

where \( F_2^\alpha := \int \partial^\alpha \tilde{A}_{12} \partial_2 \eta \cdot \partial_1 \partial^\alpha \eta dy \).

**Proof.** Applying \( \partial^\alpha \) \((|\alpha| = 2)\) to (2.13), and multiplying then the resulting identity by \( \partial^\alpha \eta \) in \( L^2 \), one sees that

\[ \frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^3 \eta \|_0^2 + \int \partial^\alpha \eta \cdot \partial^\alpha u dy \right) + \| m \partial_2 \partial^\alpha \eta \|_0^2 = \| \partial^\alpha u \|_0^2 + I_3 + I_4, \]

(3.59)

where

\[ I_3 := - \int \partial^\alpha N_{j,i} \partial_i \partial^\alpha \eta_j dy, \]

\[ I_4 := \int (\partial^{-\alpha} \nabla \partial^\alpha q \cdot \partial^\alpha \eta + \partial^\alpha q \partial^\alpha \text{div} \eta) dy, \]

and \( \alpha^- \) and \( \alpha^+ \) are defined by (3.51).

Noting that

\[ \int \partial^\alpha \tilde{A}_{12} \partial_2 u \cdot \partial_1 \partial^\alpha \eta dy = \frac{d}{dt} F_2^\alpha - \int \partial_i (\partial^\alpha \tilde{A}_{12} \partial_1 \partial^\alpha \eta) \cdot \partial_2 \eta dy, \]

arguing similarly to the derivation of (3.56) and (3.57), we find that

\[ |I_3| \leq c(\| \partial_2 \eta \|_2^2 \| u \|_3 + \| \nabla \eta \|_2(\| \partial_t^2 \eta \|_1 \| u \|_3 + \| \nabla \eta \|_2)(1 + \| \nabla \eta \|_2) + \| \partial_2 \eta \|_2 \| \partial_2 u \|_2)) - \frac{d}{dt} F_2^\alpha, \]

(3.60)

\[ |I_4| \lesssim \| \partial_2 \eta \|_2 \| \nabla \eta \|_2(\| \partial_2 u \|_2^2 + m^2 \| \partial_t^2 \eta \|_1 \| \partial_2 \eta \|_2). \]

(3.61)

Inserting (3.60) and (3.61) into (3.59), and using (3.1), (3.3), (3.53) and Young’s inequality, we obtain (3.58). \( \square \)
Lemma 3.6. Under the assumptions of (2.26)–(2.27) with sufficiently small $\delta$, we have

$$\frac{d}{dt} \|\nabla^{2-i} \partial_t^i (u, m\partial_2 \eta)\|_0^2 + \nu \|\partial_2 u\|_{3-i}^2 \lesssim \begin{cases} m^4 \|\partial_2 \eta\|_2^2 (\|\partial_2^i \eta\|_0^2 \|\partial_2 \eta\|_2^2 + \|\partial_2 \eta\|_1^4) & \text{for } i = 1, \\ F_3 & \text{for } i = 2, \end{cases} \quad (3.62)$$

and

$$\frac{d}{dt} \left( \frac{\nu}{2} \|\nabla^{3-i} \partial_t \eta\|_0^2 + \sum_{|\alpha|=2-i} \left( \int \partial^\alpha \partial_t \eta \cdot \partial^\alpha \partial_t \eta dy + \nu F_\alpha \right) + (i-1)F_5 \right) + m\partial_2^{i+1} \eta\|_{2-i}^2$$

$$\leq \|\partial_2 u\|_{2-i}^2 + \left\{ c(||\partial_2^i \eta||_1 ||\partial_2 \eta||_2 ||\partial_2 u||_2 + \delta ||\partial_2 u||_2^2) \right. \quad \text{for } i = 1,$$

$$\left. F_6 + ||\partial_2^2 \eta||_1 ||\partial_2 \eta||_2 ||\partial_2^2 u||_1 + \delta ||\partial_2 u||_1^2 \right) \quad \text{for } i = 2, \quad (3.63)$$

where

$$F_3 := ||\partial_2^i \eta||_0 ||\partial_2^i u||_0 ||\partial_2 u||_2 + ||\partial_2^i \eta||_2^2 (||\partial_2 u||_2^2 + m||\partial_2 \eta||_2^4 + m^2 ||\partial_2 \eta||_2 ||\partial_2 u||_2)$$

$$+ m||\partial_2^i \eta||_1^2 (||\partial_2^i \eta||_0 ||\partial_2 u||_2 + m||\partial_2 \eta||_2 ||\partial_2^i \eta||_1),$$

$$F_4^\alpha := \left\{ \begin{array}{ll} 0 & \text{for } \alpha = 0, \\ \int \partial^\alpha \partial_2 \tilde{A}_{12} \partial_2 \eta \cdot \partial_1 \partial_2 \partial^\alpha \eta dy & \text{for } \alpha = 1, \end{array} \right.$$

$$F_5 := \int (\partial_2^i (A_{1k} \tilde{A}_{,k1} + \tilde{A}_{11}) \partial_1 \eta \cdot \partial_1 \partial_2 \eta + \partial_2^i (A_{2k} \tilde{A}_{,k1} + \tilde{A}_{12}) \partial_1 \eta \cdot \partial_2 \eta)) dy,$$

$$F_6 := m^2 ||\partial_2^i \eta||_1^2 (||\partial_2^i \eta||_0 ||\partial_2 \eta||_2 + ||\partial_2 \eta||_1^2).$$

**Proof.** Let $\alpha$ satisfy $|\alpha| = 2 - i$. Similar to the derivation of (3.49) and (3.59), one gets

$$\frac{1}{2} \frac{d}{dt} \left( ||\partial^\alpha \partial_t^i u||_0^2 + m||\partial^\alpha \partial_t^{i+1} \eta||_0^2 \right) + \nu ||\partial^\alpha \nabla \partial_t^i u||_0^2 = I_5 + I_6 \quad (3.64)$$

and

$$\frac{d}{dt} \left( \frac{\nu}{2} ||\nabla \partial^\alpha \partial_t \eta||_0^2 + \int \partial^\alpha \partial_t \eta \cdot \partial^\alpha \partial_2 \eta dy \right) + m||\partial^\alpha \partial_t^{i+1} \eta||_0^2 = ||\partial^\alpha \partial_t^i u||_0^2 + I_7 + I_8, \quad (3.65)$$

where

$$I_5 := - \int \partial^\alpha \partial_t^i N_{j,l}^\alpha \cdot \partial_2 \partial^\alpha \partial_2 u_j dy, \quad I_6 := \int (\partial^\alpha \partial_t^{i+1} \nabla A \partial^\alpha \partial_t^{i+1} u - \partial^\alpha \partial_t^i q \partial^\alpha \partial_t^i \div A u) dy,$$

$$I_7 := - \int \partial^\alpha \partial_t^i N_{j,l}^\alpha \cdot \partial_2 \partial^\alpha \partial_2 \eta_j dy, \quad I_8 := \int (\partial^\alpha \partial_t^{i+1} \nabla A \partial^\alpha \partial_t^{i+1} \eta + \partial^\alpha \partial_t^i q \partial^\alpha \partial_t^i \div \eta) dy.$$

Next, we estimate $I_5$–$I_8$ by considering the cases $i = 1$ and $i = 2$ respectively.

1. **Case $i = 1$**

   From (3.11), (3.13) and (3.15) we get

   $$I_5 \lesssim ||\partial_2 \eta||_2 ||\partial_2 u||_2^2 \lesssim \delta ||\partial_2 u||_2^2. \quad (3.66)$$

   The integral $I_6$ can be estimated in the following way.

   $$I_6 \lesssim ||\partial_2 \eta||_2 ||\partial_2 u||_2 ||\nabla q||_1 \lesssim m^2 ||\partial_2 \eta||_2 (||\partial_2^2 \eta||_0 ||\partial_2 \eta||_2 + ||\partial_2 \eta||_1^2) ||\partial_2 u||_2 + \delta ||\partial_2 u||_2^2, \quad (3.67)$$
where we have used (3.9) and (3.13)–(3.15) in the first inequality, and (3.36) with \( i = 1 \), (3.53) and Young’s inequality in the second inequality.

Noting that for \( \alpha_1 = 1 \),
\[
\int \partial^\alpha \partial_2 A_{12} \partial_2 u \cdot \partial_1 \partial_2 \partial^\alpha \eta dy = \frac{d}{dt} F^\alpha_i - \int \partial_i (\partial^\alpha \partial_2 A_{12} \partial_1 \partial_2 \partial^\alpha \eta) \cdot \partial_2 \eta dy,
\]
we argue, similarly to (3.66) and (3.67), to infer that
\[
I_7 \leq c \| \partial^2_2 \eta \|_1 \| \partial_2 \eta \|_2 \| \partial_2 u \|_2 - \frac{d}{dt} F^\alpha_i, \quad (3.68)
\]
\[
I_8 \lesssim \| \partial^2_2 \eta \|_1 \| \partial_2 \eta \|_2 \| \nabla q \|_1 \lesssim \delta (\| m \partial^2_2 \eta \|_1^2 + \| \partial_2 u \|_2^2). \quad (3.69)
\]

Thanks to the four estimates (3.66)–(3.69) and Young’s inequality, we derive (3.62) and (3.63) from (3.64) and (3.65) with \( i = 1 \), respectively.

\( \text{(2) Case } i = 2. \)

By employing a partial integration, (3.13) and (3.15), one can easily see that
\[
\int \partial^2_2 \tilde{A}_{12} \partial_1 u \cdot \partial^2_2 u dy = \int \partial^2_2 \partial_1 \partial_2 \partial_2 u \cdot \partial^2_2 u dy - \int \partial^2_2 \partial_1 \partial_2 (\partial_1 u \cdot \partial^2_2 u) dy \\
\lesssim \| \partial^2_2 \eta \|_0 \| \partial^2_2 u \|_1 \| \partial_2 u \|_2 + \| \partial_2 \eta \|_2 \| \partial^2_2 u \|_1^2.
\]

Similarly, we can also obtain
\[
\int \partial^2_2 \tilde{A}_{21} \partial_1 u \cdot \partial^2_2 u dy \lesssim \| \partial^2_2 \eta \|_0 \| \partial^2_2 u \|_1 \| \partial_2 u \|_2 + \| \partial_2 \eta \|_2 \| \partial^2_2 u \|_1^2.
\]

Thanks to the above two estimates, we easily deduce that
\[
I_5 \lesssim \| \partial^2_2 \eta \|_0 \| \partial^2_2 u \|_1 \| \partial_2 u \|_2 + \| \partial_2 \eta \|_2 \| \partial^2_2 u \|_1^2 + \int \partial^2_2 \tilde{A}_{12} \partial_1 u \cdot \partial^2_2 u dy + \int \partial^2_2 \tilde{A}_{21} \partial_1 u \cdot \partial^2_2 u dy \\
\lesssim \| \partial^2_2 \eta \|_0 \| \partial^2_2 u \|_1 \| \partial_2 u \|_2 + \| \partial_2 \eta \|_2 \| \partial^2_2 u \|_1^2.
\]

Noting that
\[
\int (\partial^2_2 (A_{k1} \tilde{A}_{k1} + \tilde{A}_{11}) \partial_1 u \cdot \partial_1 \partial^2_2 \eta + \partial^2_2 (A_{k2} \tilde{A}_{k1} + \tilde{A}_{12}) \partial_1 u \cdot \partial^2_2 \eta )) dy \\
\leq \frac{d}{dt} F_5 + c \| \partial^2_2 \eta \|_1 \| \partial_2 \eta \|_2 \| \partial^2_2 u \|_1, \quad (3.70)
\]

making use of (3.9) and (3.13)–(3.15), we can control \( I_6 - I_8 \) as follows.

\[
I_6 \lesssim (\| \partial^2_2 \eta \|_0 \| \partial_2 u \|_2 + \| \partial_2 \eta \|_2 \| \partial^2_2 u \|_1) (\| \partial^2_2 u \|_0 \| \partial_2 \eta \|_2 + \| \partial^2_2 \eta \|_1^2),
\]
\[
I_7 \leq - \frac{d}{dt} F_5 + c \| \partial^2_2 \eta \|_1 \| \partial_2 \eta \|_2 \| \partial^2_2 u \|_1,
\]
\[
I_8 \lesssim (\| \partial^2_2 \eta \|_0 \| \partial_2 \eta \|_2 + \| \partial^2_2 \eta \|_1^2) (\| \partial_2 u \|_2^2 + \| \partial^2_2 \eta \|_1^2 + \| \partial^2_2 \eta \|_1^2)).
\]

Utilizing the estimates for \( I_6 - I_8 \), (3.1), and (3.2), (3.53) and Young’s inequality, we get (3.62) and (3.63) from (3.64) and (3.65) with \( i = 2 \), respectively.
3.2. Stability estimates

With the energy estimates in Lemmas 3.4–3.6 in hand, we are in a position to establish the stability estimate (2.16).

By virtue of (3.16), it is easy to see that

\[ |F_\alpha^2| \leq \delta \| \nabla \eta \|_2^2, \]
\[ |F_\alpha^4| \leq \delta \| \partial_2 \eta \|_2^2, \]
\[ |F_\alpha^5| \leq \delta \| \partial_2^\alpha \eta \|_1^2. \]

(3.71) (3.72) (3.73)

Thus, we can use (3.3), (3.53), (3.71) and Young’s inequality to derive from (3.48) and (3.58) the two-order energy inequality:

\[
\frac{d}{dt} E + c(\| u \|_3^2 + \| m \partial_2 \eta \|_2^2) \lesssim \| (\nabla \eta, u, m \partial_2 \eta) \|_2^2 (\| \partial_2 u \|_2^2 + \| m \partial_2^2 \eta \|_1^2) \]

(3.74)

for sufficiently small \( \delta \), where

\[ E := c \| \nabla^2 (u, m \partial_2 \eta) \|_0^2 + \frac{\nu}{2} \| \nabla^3 \eta \|_0^2 + \sum_{|\alpha|=2} \left( \int \partial^\alpha \eta \cdot \partial^\alpha u \, dy + \nu F_\alpha^2 \right), \]

satisfying

\[ \mathcal{E}_{2,0} \lesssim E \lesssim \mathcal{E}_{2,0}. \]

(3.75)

Furthermore, if one utilizes (3.3), (3.53), (3.72) and Young’s inequality, one gets from (3.62) and (3.63) that

\[
\frac{d}{dt} \mathcal{E}_i + c(\| \partial_i^2 u \|_3^2 + \| m \partial_i \partial_2 \eta \|_2^2) \lesssim \| (\nabla \eta, u, m \partial_2 \eta) \|_2^2 (\| \partial_2 u \|_2^2 + \| m \partial_2^2 \eta \|_1^2) \]

(3.76)

for sufficiently small \( \delta \), where \( F_7 := c F_3 + F_6 + \| \partial_2^2 \eta \|_2^2 \| \partial_2 \eta \|_2^2 \) and

\[
\mathcal{E}_i := c \| \nabla^{2-i} \partial_i^2 (u, m \partial_2 \eta) \|_0^2 + \frac{\nu}{2} \| \nabla^{3-i} \partial_i \partial_2 \eta \|_0^2 \]

\[ + \sum_{|\alpha|=2-i} \left( \int \partial^\alpha \partial_i^2 \eta \cdot \partial^\alpha \partial_i \partial_2 \eta \, dy + \nu F_i^2 \right) + (i-1) F_5, \]

satisfying

\[ \mathcal{E}_{2,i} \lesssim \mathcal{E}_i \lesssim \mathcal{E}_{2,i}. \]

(3.77)

From (3.76) with \( i = 1 \) we find that

\[
\mathcal{E}_{2,1}(t) + c \int_0^t (\| \partial_2 u \|_2^2 + \| m \partial_2^2 \eta \|_1^2) \, dt \lesssim \mathcal{E}_{2,1}^0. \]

(3.78)

Thanks to (3.75), an application of Gronwall’s lemma to (3.74) yields

\[ \mathcal{E}_{2,0} \lesssim \mathcal{E}_{2,0}^0 e^{c \int_0^T (\| \partial_2 u \|_2^2 + \| m \partial_2^2 \eta \|_1^2) \, dt}. \]
Combining (3.75) and (3.78) with the above estimate, we see that there is $\delta_1 \in (0, 1]$, such that for any $\delta \leq \delta_1$,

$$E_{2,0}(t) + \int_0^t (\|\partial_2 u\|^2_2 + \|m\partial_2^2 \eta\|_1^2) d\tau \leq c_1 E_{2,0}^{\eta,u,q} e^{c_2 e_{2,1}} / 4 \quad \text{for any } t \in I_T,$$

(3.79)

where $c_1 \geq 4$. In addition, due to (3.78) and (3.79), we can obtain from (3.74) that

$$\int_0^t (\|u\|^2_2 + \|m\partial_2 \eta\|_2^2) dt \leq E_{2,0}^{\eta,u,q} (1 + E_{2,1}^{\eta,u,q} e^{c_2 e_{2,1}}).$$

(3.80)

Finally, if we take

$$K := \sqrt{c_1 E_{2,0}^{\eta,u,q} e^{c_2 e_{2,1}}} > 0.$$  

(3.81)

we complete the derivation of (2.25) under the conditions (2.26) and (2.27) with $\delta \leq \delta_2$.

### 3.3. Proof of Theorem 2.1

We start with introducing a local (-in-time) well-posedness result for the initial value problem (2.8)-(2.9) and a result concerning diffeomorphism mappings.

**Proposition 3.1.** Let $(\eta^0, u^0) \in H^3 \times H^2$ satisfy $\|\nabla \eta^0, u^0\|_2 \leq B$ and $\text{div} \mathcal{A}^0 u^0 = 0$, where $B$ is a positive constant, $\zeta^0 := \eta^0 + y$ and $\mathcal{A}^0$ is defined by $\zeta^0$. Then there is a constant $\delta_2 \in (0, 1]$, such that for any $(\eta^0, u^0)$ satisfying

$$\|\nabla \eta^0\|_{2,2} \leq \delta_2,$$

(3.82)

there exist a local existence time $T > 0$ (depending possibly on $B, \nu, m$ and $\delta_2$) and a unique local strong solution $(\eta, u, q) \in C^0(\overline{I_T}, H^3) \times \mathcal{U}_{1,T} \times C^0(\overline{I_T}, H^2)$ to the initial value problem (2.8)-(2.9), satisfying $0 < \inf_{(y,t) \in \mathbb{R}^2 \times \overline{I_T}} \text{det}(\nabla \eta + I)$ and $\sup_{t \in I_T} \|\nabla \eta\|_{2,2} \leq 2\delta_2$.

**Proof.** The proof of Proposition 3.1 will be given in Section 7.

**Remark 3.3.** If $(\eta^0, u^0)$ in Proposition 3.1 further satisfies $(\eta^0, u^0) \in H^3_1 \times H^2$ and the odevity conditions (2.14), then $(\eta, u)$ belongs to $C^0(\overline{I_T}, H^3_1) \times \mathcal{U}_{1,T}$ and satisfies the odevity conditions (2.21) as $(\eta^0, u^0)$ does.

**Proposition 3.2.** There is a positive constant $\delta_3$, such that for any $\varphi \in H^3$ satisfying $\|\nabla \varphi\|_{2,2} \leq \delta_3$, we have (after possibly being redefined on a set of measure zero) $\text{det}(\nabla \varphi + I) > 1/2$ and

$$\psi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a } C^1 \text{ homeomorphism mapping,}$$

(3.83)

where $\psi := \varphi + y$.

**Proof.** In view of (3.4), we see that $\text{det}(\nabla \varphi + I) > 1/2$ and $\|\nabla \varphi\|_{L\infty} \lesssim_0 \|\nabla \varphi\|_{2,2} \leq \delta_3$. Thus, we easily verify that for sufficiently small $\delta_3$, $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^1$ homeomorphism mapping, please refer to [15, Lemma 4.2] for a detailed proof.

---

2Here the uniqueness means that if there is another solution $(\tilde{\eta}, \tilde{u}, \tilde{q}) \in C^0(\overline{I_T}, H^3) \times \mathcal{U}_{1,T} \times C^0(\overline{I_T}, H^2)$ satisfying

$$0 < \inf_{(y,t) \in \mathbb{R}^2 \times \overline{I_T}} \text{det}(\nabla \tilde{\eta} + I),$$

then $(\tilde{\eta}, \tilde{u}, \tilde{q}) = (\eta, u, q)$ by virtue of the smallness condition “$\sup_{t \in I_T} \|\nabla \eta\|_{2,2} \leq 2\delta_2$.”
With the a priori estimate (3.79) (under the assumptions (2.26) and (2.27) with \( \delta \leq \delta_2 \)) and Propositions 3.1 and 3.2 in hand, we can easily establish Theorem 2.1. We briefly give the proof below.

Let \( m \) and \((\eta^0, u^0) \in (H^1_t \cap H^3_x) \times H^2\) satisfy the odevity conditions (2.14) and

\[
\max\{K^{1/2}, K^2\}/m \leq \min\{\delta_1, \delta_2/c_0, \delta_3/c_0\} =: c_3 \leq 1, 
\]

(3.84)

where \( K \) is defined by (3.81), and the constant \( c_0 \) is the same as in (3.10). Thus we see that \( \eta^0 \) satisfies (3.82) by (3.10) and (3.84). Thus, by virtue of Proposition 3.1 and Remark 3.3, there exists a unique local solution \((\eta, u, q)\) of (2.8)–(2.9) with a maximal existence time \( T^{\max} \), satisfying

- for any \( T \in I_{T^{\max}} \), the solution \((\eta, u, q)\) belongs to \( C^0(\bar{T}_T; H^3_x) \times U_T \times C^0(\bar{T}_T; H^2) \) and

\[
\sup_{t \in \bar{T}_T} \|\nabla \eta\|_{2,2} \leq 2\delta_2;
\]

- \( \limsup_{t \to T^{\max}} \|\nabla \eta(t)\|_{2,2} > \delta_2 \) or \( \limsup_{t \to T^{\max}} \|u(t)\|_2 = \infty \), if \( T^{\max} < \infty \).

In addition, the solution enjoys the odevity conditions (2.21).

Let

\[
T^* = \sup\{T \in I_{T^{\max}} \mid \varepsilon_{2,0}(t) \leq K^2 \text{ for any } t \leq T\}.
\]

Recalling the definition of \( K \) and the condition \( c_1 \geq 4 \), we easily see that the definition of \( T^* \) makes sense and \( T^* > 0 \).

By (3.10) we have \( \|\nabla \eta\|_{2,2} \leq \delta_3 \) for all \( t \in I_{T^*} \), then \( \eta(t) \in H^3_x \) for all \( t \in I_{T^*} \) by Proposition 3.2. Thus, to obtain the existence of a global solution, it suffices to verify \( T^* = \infty \). Now, we show this by contradiction.

Assume \( T^* < \infty \). Keeping in mind that \( T^{\max} \) denotes the maximal existence time and \( K/m \leq \delta_2/c_0 \) by virtue of (3.84), we apply Proposition 3.1 to find that \( T^{\max} > T^* \) and

\[
\varepsilon_{2,0}(T^*) = K^2. 
\]

(3.85)

Since \( \max\{K, K^2\}/m \leq \delta_1 \) and \( \sup_{0 \leq t \leq T^*} \varepsilon_{2,0}(t) \leq K^2 \), we can still show that the solution \((\eta, u)\) enjoys the stability estimate (3.79) with \( T^* \) in place of \( T \) by the regularity of \((\eta, u, q)\). More precisely, one has \( \sup_{0 \leq t \leq T^*} \varepsilon_{2,0}(t) \leq K^2/4 \), which contradicts with (3.85). Hence, \( T^* = \infty \), and thus \( T^{\max} = \infty \).

Obviously, the global solution \((\eta, u)\) enjoys the stability estimate (2.16) by using (3.79) and (3.80), and the estimates (2.17)–(2.19) are easily obtained from (3.10)–(3.12) and (3.36) with \( i = 1 \). The uniqueness of the global solutions is obvious due to the uniqueness of the local solutions in Proposition 3.1 and the fact \( \sup_{t \geq 0} \|\nabla \eta\|_{2,2} \leq 2\delta_2 \). This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

We now proceed to the derivation of the time-decay estimates stated in Theorem 2.2. It should be noted that, under the assumptions of Theorem 2.2, the solution \((\eta, u, q)\) of (2.8)–(2.9), established in Theorem 2.1 satisfies the differential inequalities in Lemma 3.4 (3.62) and (3.76) for a.e. \( t > 0 \), and the estimates (3.77) for any \( t > 0 \).
4.1. Decay estimates for $\partial_2 \eta$

We begin with the derivation of the decay estimates for $\eta$. Multiplying (3.48), (3.62) and (3.76) by $(t)$, $(t)^{i+1}$ and $(t)^i$ respectively, we obtain for a.e. $t > 0$,

\[
\frac{d}{dt} \left( \langle t \rangle \| \nabla^2(u, m \partial_2 \eta) \|_0^2 \right) + \nu \langle t \rangle \| u \|_3^2 \lesssim \| \nabla^2(u, m \partial_2 \eta) \|_0^2 + \langle t \rangle F_1
\]

(4.1)

\[
\frac{d}{dt} \left( \langle t \rangle^{i+1} \| \nabla^2_i \partial_2^i(u, m \partial_2 \eta) \|_0^2 \right) + \nu \langle t \rangle^{i+1} \| \partial_2^i u \|_{3-i}^2
\]

\[
\lesssim \langle t \rangle^i \| \nabla^2_i \partial_2^i(u, m \partial_2 \eta) \|_0^2 + \left\{ \begin{array}{ll}
t^2 m^4 \| \partial_2^i \eta \|_2^2 (\| \partial_2^i \eta \|_0^2 \| \partial_2^i \eta \|_2^2 + \| \partial_2^i \eta \|_1^4) & \text{for } i = 1, \\
t^3 \| \partial_2^i \eta \|_1^2 & \text{for } i = 2,
\end{array} \right.
\]

(4.2)

and

\[
\frac{d}{dt} \langle t \rangle^i \mathcal{E}_i + c \langle t \rangle^i (\| \partial_2^i u \|_{3-i}^2 + \| m \partial_2^i \eta \|_{2-i}^2) \lesssim \langle t \rangle^{i-1} \mathcal{E}_i + \left\{ \begin{array}{ll}0 & \text{for } i = 1, \\
\langle t \rangle^2 F_7 & \text{for } i = 2. \end{array} \right.
\]

(4.3)

Integrating (4.3) with $i = 1$ over $(0, t)$, and then using (2.15), (2.16) and (3.77), we have

\[
\langle t \rangle \mathcal{E}_{2,1} + \int_0^t \langle \tau \rangle (\| \partial_2^i u \|_1^2 + \| m \partial_2^i \eta \|_1^2) d\tau
\]

\[
\leq \left( \mathcal{E}^0_{2,0} e^{\nu t} e^{\nu^2 t} + \mathcal{E}^0_{2,0} e^{\nu t} e^{\nu^2 t} \right) (1 + \mathcal{E}^0_{2,1}) \text{ for any } t > 0.
\]

(4.4)

Denoting

\[
\mathcal{F}_7 = \| u \|_3^2 + \| \partial_2 \eta \|_2^2 (1 + m^2 + m^4 \| \partial_2 \eta \|_2^2),
\]

one gets by (2.15) and (2.16) that

\[
\int_0^t \mathcal{F}_7 d\tau \leq C.
\]

Applying Gronwall’s lemma to (4.3) with $i = 2$, and utilizing (3.77), (4.4) and the above estimate, and the fact $F_7 \lesssim (\| \partial_2^i \eta \|_2^2 + \| \partial_2^i u \|_1^2 + \| m \partial_2^i \eta \|_1^2) \mathcal{F}_7$, one can infer from (4.3) with $i = 2$ that

\[
\langle t \rangle^{2} \mathcal{E}_{2,2} + \int_0^t \langle \tau \rangle^2 (\| \partial_2^i u \|_1^2 + \| m \partial_2^i \eta \|_0^2) d\tau \lesssim \mathcal{E}^0_{2,2} + \int_0^t \langle \tau \rangle \mathcal{E}_{2,2} d\tau e^{\nu t} \mathcal{F}_7 d\tau
\]

\[
\leq C \text{ for all } t > 0.
\]

(4.5)

Integrating (4.1) and (4.2) over $(0, t)$, using (2.15), (2.16), (4.4) and (4.5), we conclude

\[
\langle t \rangle^{i+1} \| \partial_2^i(u, m \partial_2 \eta) \|_{2-i}^2 + \int_0^t \langle \tau \rangle^{i+1} \| \partial_2^i u \|_{3-i}^2 d\tau \leq C \text{ for } 0 \leq i \leq 2.
\]

(4.6)

4.2. Decay estimates for $u$

We proceed to derive decay estimates of higher derivatives of the velocity. Multiplying (3.48) by $e^{\nu t/2}$ and using Young’s inequality, we obtain

\[
\frac{d}{dt} (e^{\nu t/2} \| \nabla^2 u \|_0^2) + c \int_0^t e^{\nu \tau/2} \| u \|_3^2 d\tau \lesssim e^{\nu \tau/2} (m^4 \| \partial_2 \eta \|_1^2 + F_1),
\]

which implies

\[
\| u \|_2^2 + \int_0^t e^{\nu (\tau-t)/2} \| u \|_3^2 d\tau \lesssim \| \nabla^2 u \|_0^2 e^{-\nu t/2} + \int_0^t e^{\nu (\tau-t)/2} (m^4 \| \partial_2 \eta \|_1^2 + F_1) d\tau.
\]

(4.7)

In addition,
for any given $a$, $\theta \geq 0$, there is a positive constant $c$, depending only on $a$ and $\theta$, such that
\[ e^{-at} \leq c(t)^{-\theta} \] for any $t \geq 0$. (4.8)

for any given $r_1 > 0$ and $r_2 \in [0, r_1]$, there is a positive constant $c$, depending only on $r_1$ and $r_2$, such that (see Lemma 2.5 in [3])
\[ \int_0^t \langle t - \tau \rangle^{-r_1} \langle \tau \rangle^{-r_2} d\tau \leq c(t)^{-r_2}. \] (4.9)

for any $\tau \in (0, t)$,
\[ \langle t \rangle \leq 4 \langle t - \tau \rangle \langle \tau \rangle. \] (4.10)

Therefore, we make use of (2.16), (4.6) with $i = 1$, (4.8)–(4.10) and Hölder’s inequality to deduce from (4.7) that
\[ \|u\|_2^2 + \int_0^t e^{\nu(t-t)/2} \|u\|_2^2 d\tau \leq (1 + m^2) C \langle t \rangle^{-2}. \] (4.11)

Finally, from (4.4)–(4.6) the estimate (2.28) follows, while from the above two estimates we obtain (2.29).

5. Proof of Theorem 2.4

This section is devoted to the proof of Theorem 2.4. Let $(\eta^0, u^0)$ satisfy all the assumptions in Theorem 2.1 and $(\eta, u, q)$ be the solution constructed by Theorem 2.1. Then, by the regularity theory of the Stokes problem, there exists a unique solution $(\eta^r, u^r, Q_1, Q_2)$ satisfying
\[
\begin{aligned}
-\Delta \eta^r + \nabla Q_1 &= 0, \\
\text{div} \eta^r &= -\text{div} \eta^0, \\
(\eta^r)_{T^2} &= 0
\end{aligned}
\] (5.1)

and
\[
\begin{aligned}
-\Delta u^r + \nabla Q_2 &= 0, \\
\text{div} u^r &= \text{div} \tilde{A}^0 u^0, \\
(u^r)_{T^2} &= 0,
\end{aligned}
\] (5.2)

where $\tilde{A}^0 := A^0 - I$. Moreover, $(\eta^r, u^r)$ satisfies (2.35), also cf. the derivation of (3.36).

Let $\tilde{\eta}^0 = \eta^0 + \eta^r$ and $\tilde{u}^0 = u^0 + u^r$. Thus, it is easy to see that $(\tilde{\eta}^0, \tilde{u}^0)$ belongs to $H^3_\sigma \times H^2_\sigma$ and enjoys the odevity conditions as $(\eta^0, u^0)$ does. Therefore, there exists a unique global solution $(\eta^L, u^L, q^L) \in C^0(\mathbb{R}_0^+, H^3_\sigma) \times \mathcal{U}_\infty \times C^0(\mathbb{R}_0^+, H^2_\sigma)$ to the initial-value problem (2.33). Moreover, the solution enjoys the odevity conditions as $(\eta, u)$ does.
Similarly to (2.28), we employ (2.15) and (2.35) to see that the solution \((\eta^L, u^L)\) of the linearized problem enjoys the following estimate:

\[
\|\nabla \eta^L\|_2^2 + \sum_{i=1}^{2} \left( \langle t \rangle^i \| \partial_i^2 \eta^L \|_{3-i}^2 + \int_0^t \langle \tau \rangle^i \| m \partial_2^{i+1} \eta^L \|_{2-i}^2 d\tau \right) + \sum_{i=0}^{2} \left( \langle t \rangle^{i+1} \| \partial_i^2 (u^L, m \partial_2 \eta^L) \|_{2-i}^2 + \int_0^t \langle \tau \rangle^{i+1} \| \partial_2^2 u^L \|_{3-i}^2 d\tau \right) \lesssim C_{2,0}^L.
\] (5.3)

Let \((\eta^d, u^d) = (\eta - \eta^L, u - u^L)\), then the error function \((\eta^d, u^d)\) satisfies (2.40). It is easy to see from (2.40) that \((u^d)_{T^2} = (\eta^d)_{T^2} = 0\) for any \(t > 0\), since \((\eta^d)_{T^2} = (u^d)_{T^2} = 0\). Moreover, \((\eta^d, u^d)\) also enjoys the odevity conditions as \((\eta, u)\) does.

Recalling that \((\eta, u, q)\) is constructed by Theorem 2.1 the solution \((\eta, u)\) satisfies all the estimates in Lemma 3.2 Hence, we can follow the arguments in the proof of Lemmas 3.4–3.6 with slight modifications to derive from (2.40) that

\[
\frac{d}{dt} \| \nabla (u^d, m \partial_2 \eta^d) \|_0^2 + \nu \| u^d \|_2^2 \lesssim F_1^d,
\] (5.4)

\[
\frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^3 \eta^d \|_0^2 + \sum_{|\alpha|=2} \int \partial^\alpha \eta^d \cdot \partial^\alpha u^d dy + \int \partial_2^2 \Phi_2 \partial_2 \eta^d \cdot \partial_1^2 \eta^d dy \right) + \| m \partial_2 \eta^d \|_2^2 \lesssim \| u^d \|_2^2 + F_2^d,
\] (5.5)

\[
\frac{d}{dt} \| \nabla^2 \partial_2^2 (u^d, m \partial_2 \eta^d) \|_0^2 + \nu \| \partial_2^2 u^d \|_{3-i}^2 \lesssim F_{3^d}^d,
\] (5.6)

\[
\frac{d}{dt} \left( \frac{\nu}{2} \| \nabla^3 \partial_2 \eta^d \|_0^2 + \sum_{|\alpha|=2-i} \int \partial^\alpha \partial_2 \eta^d \cdot \partial^\alpha \partial_2 u^d dy \right) + \| m \partial_2 \eta^d \|_{2-i}^2 \lesssim \| \partial_2^2 u^d \|_{2-i}^2 + F_{4^d}^d,
\] (5.7)

for \(i = 1, 2\), where

\[
F_1^d := \| \partial_2^2 \eta^d \|_2 \| u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \nabla q \|_2^2 + \| \nabla q \|_0 (\| \partial_2 \eta^d \|_1 \| u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2 u^d \|_1),
\]

\[
F_2^d := \| \partial_2 \eta^d \|_2 (\| \nabla \eta^d \|_2 \| \nabla u^d \|_2 + \| \eta^d \|_2 \| \nabla \eta^d \|_2) + \| \partial_2 \eta^d \|_2 \| \nabla (\eta, \eta^d) \|_2 \| \nabla q \|_1,
\]

\[
F_{3^d}^d := \begin{cases}
\| \partial_2^2 \eta^d \|_2 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2 u^d \|_2 & \text{for } i = 1; \\
\| \partial_2 \eta^d \|_2 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2^2 u^d, \nabla q \|_1^2 & \text{for } i = 2;
\end{cases}
\]

and

\[
F_{4^d}^d := \begin{cases}
\| \partial_2 \eta^d \|_2 (\| \partial_2 \eta^d \|_1 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2^2 u^d \|_1) + \| \nabla q \|_1 (\| \partial_2 \eta^d \|_1 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2^2 u^d \|_1) & \text{for } i = 1; \\
\| \partial_2 \eta^d \|_2 \| \nabla q \|_1 (\| \partial_2 \eta^d \|_1 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2^2 u^d \|_1) + \| \nabla q \|_1 (\| \partial_2 \eta^d \|_1 \| \partial_2 u^d \|_2 + \| \partial_2 \eta^d \|_2 \| \partial_2^2 u^d \|_1) & \text{for } i = 2.
\end{cases}
\]

If one integrates by parts, one gets from (2.35), (5.4) and (5.5) that

\[
\| \eta^d \|_3^2 + \| (u^d, m \partial_2 \eta^d) \|_1^2 + \int_0^t \| (u^d, m \partial_2 \eta^d) \|_2^2 d\tau \lesssim C m^{-1} + \| \partial_2 \eta \|_2 \| \nabla \eta \|_2 \| \nabla \eta^d \|_2 + \int_0^t (F_1^d + F_2^d) d\tau.
\] (5.8)
Recalling that \((\eta, u, q)\) satisfies \((3.36)\), we make use of \((2.16)\), \((3.36)\) and \((5.3)\) to get \((2.36)\) from \((5.8)\).

Following the arguments of deriving \((4.1)\)–\((4.3)\), using \((5.6)\) and \((5.7)\), we arrive at

\[
\frac{d}{dt} \langle t \rangle^i \mathcal{E}_i^d + c(t)^i (\| \partial_2^i u^1 \|^2 + m \| \partial_2^i \eta^d \|^2) \lesssim \langle t \rangle^{i-1} \mathcal{E}_i^d + \langle t \rangle^i (F_{3}^{d,i} + F_{4}^{d,i})
\]  
(5.9)

and

\[
\frac{d}{dt} \langle t \rangle^{i+1} \| \nabla^{-2-i} \partial_2^i (u^d, m \partial_2 \eta^d) \|^2 \lesssim \langle t \rangle \| \nabla^{-2-i} \partial_2^i (u^d, m \partial_2 \eta^d) \|^2 + \langle t \rangle^{i+1} F_{3}^{d,i}
\]  
for \(i = 1, 2\),

(5.10)

where

\[
\mathcal{E}_i^d := c \| \nabla^{-2-i} \partial_2^i (u^d, m \partial_2 \eta^d) \|^2 + \frac{\nu}{2} \| \nabla^{2-i} \partial_2^i \eta^d \|^2 + \sum_{|\alpha|=2-i} \int \partial_0^\alpha \partial_2^i \eta^d \cdot \partial_0 \partial_2^i u^d dy,
\]

and \(\mathcal{E}_i^d\) satisfies

\[
\| \partial_2^i (\nabla \eta^d, u^d, m \partial_2 \eta^d) \|^2 \lesssim \mathcal{E}_i^d \lesssim \| \partial_2^i (\nabla \eta^d, u^d, m \partial_2 \eta^d) \|^2.
\]

Utilizing \((2.28)\), \((2.35)\), \((3.36)\) with \(i = 1\) and \((5.3)\), we easily obtain \((2.37)\) from \((5.9)\) and \((5.10)\).

Next, we further assume that \(\eta^0\) satisfies the additional regularity condition \(\partial_1 (\eta^0, u^0) \in H^3 \times H^2\). Thus \(\partial_1 (\eta^i, u^i) \in H^3 \times H^2\) satisfies the estimates:

\[
\| \partial_1 u^i \|_2 \lesssim \| \partial_1 \partial_2 \eta^0 \|_2 + \| \partial_2 \partial_1 \eta^0 \|_2 + \| \partial_1 \eta^0 \|_2 + \| \partial_1 u^0 \|_2,
\]

\[
\| \partial_1 \partial_2 \eta^i \|_2 \lesssim \| \partial_2 \eta^0 \|_2 + \| \partial_1 \partial_2 \eta^0 \|_2.
\]

(5.11) \hspace{2cm} (5.12)

Keeping in mind that \(\partial_1 (\eta^0, u^0) \in H^3 \times H^2\), we can further get, using \((5.11)\) and \((5.12)\), that

\[
\| \partial_2^3 (u^L_2, m \partial_2 \eta^L_2)(t) \|_0^2 + \nu \int_0^t \| \partial_2^3 \nabla u^L_2 \|^2_0 dt = \| \partial_2^3 (u^L_2, m \partial_2 \eta^L_2)(t) \|_0^2 \lesssim (1 + \| \partial_1 \eta^0 \|_2^2 + \| \partial_1 u^0 \|_2^2). \]

\[
\lesssim (1 + \| \partial_1 \eta^0 \|_2^2 + \| \partial_1 u^0 \|_2^2). \]

(5.13)

If we make use of the estimate

\[
\int \partial_1^2 A_{k2} \partial_2 u_k \partial_1^2 q dy = \frac{d}{dt} \int \partial_1^2 A_{k2} \partial_2 \eta_k \partial_1^2 q dy - \int \partial_2 \eta_k (\partial_t \partial_1^2 A_{k2} \partial_1^2 q + \partial_1^2 A_{k2} \partial_1^2 \eta_k) dy,
\]

\[
- \int \partial_2^2 A_{12} \partial_2 u \cdot \partial_1^2 u^d dy = \int \partial_1^2 (u^L_2 + \eta^L_2) \partial_2 u \cdot \partial_1^2 u^d dy \]

and \(\| \partial_1^3 \eta^L_0 \|_0 \lesssim \| \partial_1^3 \partial_2 \eta^L \|_0\),

we can deduce from \((2.40)\) that

\[
\frac{d}{dt} \left( \| \nabla^2 (u^d, m \partial_2 \eta^d) \|^2_0 + \int \partial_2^2 A_{k2} \partial_2 \eta_k \partial_1^2 q dy \right) + c \| u^d \|^2_0 \lesssim F_0^d,
\]

(5.14)

where

\[
F_0^d := \| \partial_2^2 \eta \|^2_0 \| u \|_0^2 + \| \partial_2 \eta \|^2_0 \| \partial_2 u \|_0^2 + \| \partial_1^2 \eta^d \|^2_0 \| \partial_2 u \|_0^2 + \| \partial_1^2 \partial_2 \eta^d \|^2_0 \| \partial_2 u \|_0^2 + \| \partial_2 \eta \|^2_0 \| \nabla q \|_1^2 + \| \nabla q \|_1 (\| \partial_2^2 \eta \|_1 \| u \|_3 + \| \partial_2 \eta \|_2 \| \partial_2 u \|_2 + \| \partial_2 \eta \|_2 \| \nabla \eta \|_2 \| \nabla q \|_1). \]

(5.15)

Consequently, with the help of \((2.16)\), \((2.28)\), and \((2.36)\), \((3.37)\) and \((5.13)\), we easily obtain \((2.39)\) from \((5.14)\). This completes the proof of Theorem 2.4.
6. Proof of Theorem 2.5

This section is devoted to the proof of Theorem 2.5. The key ideas to establish Theorem 2.5 are similar to those in the proof of Theorems 2.1, 2.2 and 2.3. We break up the proof of Theorem 2.5 into two subsections.

6.1. Existence and uniqueness of a global time-decay classical solution

We start with the derivation of the \(a \text{ priori}\) stability estimates (2.50) and (2.51) in Theorem 2.5. To this end, let \((\eta, u, q)\) be a solution of the initial-value problem (2.47)–(2.48) defined on \(\Omega_T\) for any given \(T > 0\), where \((\eta^0, u^0)\) belongs to \(H^4 \times H^2\), and satisfies \(\text{div} A_0 u^0 = 0\) and the odevity conditions (2.14). It should be remarked that the solution automatically satisfies \((\eta)_{T^2} = (u)_{T^2} = 0\) and the odevity conditions (2.21). We further assume that \((\eta, u, q)\) and \(K_\kappa\) satisfy

\[
\sup_{0 \leq t \leq T} \|\eta(t)\|_4 \leq K_\kappa \quad \text{for any given } T > 0,
\]

and

\[
\max \left\{ K^{1/2}_\kappa, K^2_\kappa \right\} / m \leq \delta,
\]

where \(K_\kappa\) will be given in (6.29) and \(\delta\) is a sufficiently small constant. We should point out here that the smallness of \(\delta\) depends only on \(\nu\) and \(\kappa\).

By virtue of (6.1) and (6.2), it is easy to see that

\[
\sup_{0 \leq t \leq T} \|\partial_2 \eta(t)\|_4 \leq \delta
\]

and

\[
\sup_{0 \leq t \leq T} (m^{-1} \|\eta(t)\|_4 + \|u(t)\|_4 \|\partial_2 \eta(t)\|_4) \lesssim_0 \delta.
\]

Now, we establish some basic energy estimates for \((\eta, u, q)\) which will be used later.

**Lemma 6.1.** Under the condition (6.3) with sufficiently small \(\delta\), one has

\[
\begin{align*}
\|\eta\|_{i+2, 2} &\lesssim_0 \|\partial_2 \eta\|_{i+1}, \\
\|\nabla q\|_3 &\lesssim \|u\|_3^2 + m^2 \|\partial_2 \eta\|_3 \|\partial_2 \eta\|_4, \\
\|\chi\|_{i+1} &\lesssim_0 \|\nabla^2 \chi\|_0 \lesssim_0 \|\partial_2 \eta\|_4, \quad 0 \leq i \leq 3,
\end{align*}
\]

where \(\chi\) denotes \(\eta\), or \(\partial_2 \eta\), or \(u\).

**Proof.** We can argue in the same manner as in the derivation of (3.10) and (3.36) with \(i = 1\) to deduce (6.5) and (6.6). To show (6.7), keeping in mind that \(\Delta = \nabla^2 \chi\) where \(\chi = \partial_1 \chi_2 - \partial_2 \chi_1\) and \(\nabla^\perp := (-\partial_2, \partial_1)\), we easily obtain

\[
\|\nabla \chi\|_i^2 = \|\nabla^2 \chi\|_i^2 + \|\text{div} \chi\|_i^2.
\]

Recalling the product estimate

\[
\begin{cases} 
\|fg\|_j \lesssim \|f\|_1 \|g\|_1 & \text{for } j = 0, \\
\|fg\|_j \lesssim \|f\|_j \|g\|_2 & \text{for } 0 \leq j \leq 2, \\
\|fg\|_j \lesssim \|f\|_2 \|g\|_j + \|f\|_j \|g\|_2 & \text{for } 3 \leq j \leq 4,
\end{cases}
\]

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and noting that \((\eta, u)\) satisfies (3.22) and (3.32), we find that
\[
\|\text{div}\chi\|_{i} \lesssim_{0} \|\eta\|_{i} \|\nabla\chi\|_{i}, \quad 0 \leq i \leq 3.
\] (6.10)

Thus, with the help of (3.3), (6.3) and (6.5), we get from (6.8) and (6.10) that
\[
\|\chi\|_{i+1} \lesssim_{0} \|\nabla^{i}\text{curl}\chi\|_{0}.
\] (6.11)

In addition, by (6.3), (6.5) and (6.9),
\[
\|\nabla^{i}\text{curl}\chi\|_{0} \lesssim \|\nabla^{i}\text{curl}\chi_{0}\|_{0} \lesssim \|\chi\|_{i+1},
\] (6.12)
which, together with (6.11), yields (6.7).

\textbf{Lemma 6.2.} Under the condition (6.3) with sufficiently small \(\delta\), one has
\[
\frac{d}{dt}\|\nabla^{3}\text{curl}_{A}(u, m\partial_{2}\eta)\|^{2}_{0} + \kappa \|u\|^{2}_{4} \lesssim_{0} \|(u, m\partial_{2}\eta)\|_{3}(u, m\partial_{2}\eta)\|^{2}_{4}
\] (6.13)

and
\[
\frac{d}{dt}\left(\sum_{|\alpha|=3} \int \partial^{\alpha}\text{curl}_{A}\eta \partial^{\alpha}\text{curl}_{A}ud\eta + \frac{\kappa}{2} \|\nabla^{3}\text{curl}_{A}\eta\|^{2}_{0}\right) + \|m\eta\|^{2}_{5,2} \lesssim_{\kappa} \|u\|^{2}_{4}.
\] (6.14)

\textbf{Proof.} One can not directly apply \(\partial^{\beta}\) to (2.47)2 to derive (6.13) and (6.14) for the case \(|\beta|=4\), since \(q\) does not have the fifth-order derivatives. Instead, we apply \(\partial^{\alpha}\text{curl}_{A}\) to (2.47)2 to get
\[
\partial^{\alpha}\text{curl}_{A}u_{t} + \kappa \partial^{\alpha}\text{curl}_{A}u = m^{2}\partial^{\alpha}\text{curl}_{A}\partial_{2}^{2}\eta,
\] (6.15)
where \(|\alpha|=3\). Multiplying (6.15) by \(\partial^{\alpha}\text{curl}_{A}u\), resp. \(\partial^{\alpha}\text{curl}_{A}\eta\), in \(L^{2}\), we have
\[
\frac{1}{2} \frac{d}{dt}\|\partial^{\alpha}\text{curl}_{A}(u, m\partial_{2}\eta)\|^{2}_{0} + \kappa \|\partial^{\alpha}\text{curl}_{A}u\|^{2}_{0}
= \int \partial^{\alpha}\text{curl}_{A}u \partial^{\alpha}\text{curl}_{A}ud\eta + m^{2} \int \left(\partial^{\alpha}\text{curl}_{A}\partial_{2}\eta \partial^{\alpha}\text{curl}_{A}\partial_{2}\eta + \partial^{\alpha}\text{curl}_{A}\partial_{2}\eta \partial^{\alpha}\text{curl}_{A}\partial_{2}\eta \partial^{\alpha}\text{curl}_{A}u\right)dy =: I_{9},
\] (6.16)
resp.
\[
\frac{d}{dt}\left(\int \partial^{\alpha}\text{curl}_{A}u \partial^{\alpha}\text{curl}_{A}\eta dy + \frac{\kappa}{2} \|\partial^{\alpha}\text{curl}_{A}\eta\|^{2}_{0}\right) + \|m\partial^{\alpha}\text{curl}_{A}\partial_{2}\eta\|^{2}_{0}
= \int \left(\partial^{\alpha}\text{curl}_{A}u \partial^{\alpha}\text{curl}_{A}\eta + (\partial^{\alpha}\text{curl}_{A}u)^{2} + \partial^{\alpha}\text{curl}_{A}u \partial^{\alpha}\text{curl}_{A}\eta\right)dy
- m^{2} \int \partial^{\alpha}\text{curl}_{A}\partial_{2}\eta \partial^{\alpha}\text{curl}_{A}\partial_{2}\eta dy + \kappa \int \partial^{\alpha}\text{curl}_{A}\eta \partial^{\alpha}\text{curl}_{A}\eta dy =: I_{10},
\] (6.17)

\footnote{We explain why to take \(\alpha\) satisfying \(|\alpha|=3\). Let us consider the first term \(\int \partial^{\alpha}\text{curl}_{A}u \partial^{\alpha}\text{curl}_{A}ud\eta\) on the right hand of (6.16). Obviously, \(\int \partial^{\alpha}\text{curl}_{A}u \partial^{\alpha}\text{curl}_{A}ud\eta \lesssim_{0} \|\nabla u\|_{L^{\infty}} \|\partial^{\alpha}u\|_{0}^{2}. \) Since \(H^{2} \hookrightarrow L^{\infty}\), obviously, we need at least \(|\alpha| \geq 3\). However, for \(|\alpha|=3\), it seems difficult to close the energy estimates by (6.16) and (6.17). To overcome this difficulty, we adopt the two-layers energy method, i.e., we close the lower-order energy estimate \(\|\nabla (u, m\partial_{2}\eta)\|_{2}\) by a lower-order energy inequality, and then further close the highest-order energy estimate by a highest-order energy inequality. Thus apparently, we need at least \(|\alpha| \geq 4\). Here we remark that the initial-value problem (2.47) with small initial data \((\eta^{0}, u^{0}) \in (H^{4,1,2}_{0} \cap H^{4,2}_{0}) \times H^{3}\) admits a unique global strong solution \((\eta, u, q) \in H^{4,1,2}_{0} \times C^{0}(R^{+}, H^{3})\).}
where $I_9$ and $I_{10}$ can be bounded as follows.

\[
I_9 \lesssim_0 \| \nabla (u, m \partial_2 \eta) \|_2 \| \nabla (u, m \partial_2 \eta) \|_3^2;
I_{10} \lesssim_0 (1 + \| \nabla \eta \|_3) \| \nabla u \|_3^2 + m^2 \| \nabla \eta \|_3 \| \nabla \partial_2 \eta \|_3^2 + \| \nabla u \|_3 \| \nabla \eta \|_3^2.
\]

Consequently, inserting the above two estimates into (6.16) and (6.17) respectively, using (6.3)–(6.5) and Young’s inequality, we obtain (6.13) and (6.14).

\[\Box\]

**Lemma 6.3.** Under the conditions (6.3)–(6.4) with sufficiently small $\delta$, we have

\[
\frac{d}{dt} \| \nabla^3 (u, m \partial_2 \eta) \|_0^2 + \kappa \| u \|_3^2 \lesssim_0 \delta \| m \eta \|_{1,2}^2
\]

(6.18)

and

\[
\frac{d}{dt} \left( \sum_{|\alpha|=3} \int \partial^\alpha \eta \cdot \partial^\alpha u \, dy + \frac{\kappa}{2} \| \nabla^3 \eta \|_0^2 \right) + \| m \eta \|_{1,2}^2 \lesssim_0 \| u \|_3^2.
\]

(6.19)

**Proof.** It is easy to see from the proof of Lemma 6.2 that we cannot directly establish the estimates (6.18) and (6.19) by the curl-$A$-estimate method, since we shall use some smallness properties which are hidden in some integral terms involving $\nabla A q$.

To exploit the smallness properties hidden in the pressure term $\nabla A q$, we apply $\partial^\alpha \ (|\alpha| = 3)$ to (2.47) to get

\[
\partial^\alpha u_t + \partial^\alpha \nabla A q + \kappa \partial^\alpha u = m^2 \partial^\alpha \partial_2^2 \eta. \quad (|\alpha| = 3)
\]

Thus, multiplying the above identity by $\partial^\alpha u$ and $\partial^\alpha \eta$ in $L^2$ respectively, we have

\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha (u, m \partial_2 \eta) \|_0^2 + \kappa \| \partial^\alpha u \|_0^2 = I_{11} := - \int \partial^\alpha \nabla A q \cdot \partial^\alpha u \, dy,
\]

(6.20)

and

\[
\frac{d}{dt} \left( \int \partial^\alpha \eta \cdot \partial^\alpha u \, dy + \frac{\kappa}{2} \| \partial^\alpha \eta \|_0^2 \right) + \| m \partial^\alpha \partial_2 \eta \|_0^2 - \| \partial^\alpha u \|_0^2
\]

\[
= I_{12} := - \int \partial^\alpha \nabla A q \cdot \partial^\alpha \eta \, dy,
\]

(6.21)

respectively, where the right-hand sides can be bounded as follows, using (3.22), (3.32) and (6.6).

\[
I_{11} \lesssim_0 \| \nabla \eta \|_3 \| \nabla u \|_2 (\| u \|_3^2 + m^2 \| \partial_2 \eta \|_3 \| \partial_2 \eta \|_4),
I_{12} \lesssim_0 \| \nabla \eta \|_3 \| \nabla \eta \|_2 (\| u \|_3^2 + m^2 \| \partial_2 \eta \|_3 \| \partial_2 \eta \|_4),
\]

which, together with (6.20), (6.21) and (6.3)–(6.5), implies (6.18) and (6.19).

\[\Box\]

Now, we are in a position to show the *a priori* stability estimate (2.50). Firstly, using (6.7), we derive from Lemmas 6.2–6.3 that

\[
\frac{d}{dt} \varepsilon_t^\kappa + c^\kappa (\| u \|_4^2 + \| m \eta \|_{1,2}^2) \lesssim_\kappa \| (u, m \partial_2 \eta) \|_3^2 \| (u, m \partial_2 \eta) \|_1^2,
\]

(6.22)

\[
\frac{d}{dt} \varepsilon_t^\kappa + c^\kappa (\| u \|_3^2 + \| m \eta \|_2^2) \leq 0,
\]

(6.23)
where
\[
\mathcal{E}^\kappa := c\|
abla^3\text{curl}_A(u, m\partial_2\eta)\|_0^2 + \sum_{|\alpha|=3} \int \partial^\alpha\text{curl}_A\eta \partial^\alpha\text{curl}_Audy + \frac{K}{2}\|
abla^3\text{curl}_A\eta\|_0^2, \tag{6.24}
\]
and
\[
\mathcal{E}^{\kappa}_t := c\|
abla^3(u, m\partial_2\eta)\|_0^2 + \sum_{|\alpha|=3} \int \partial^\alpha\eta \cdot \partial^\alphaudy + \frac{K}{2}\|
abla^3\eta\|_0^2,
\]
\[
\||\eta, u\|_4^2 + \|m\eta\|_{5, 2}^2 \leq \kappa \mathcal{E}^\kappa \leq \kappa \||\eta, u, m\partial_2\eta\|_4^2;
\]
\[
\||\eta, u\|_3^2 + \|m\eta\|_{5, 2}^2 \leq \kappa \mathcal{E}^{\kappa}_t \leq \kappa \||\eta, u, m\partial_2\eta\|_3^2.
\]

Thus, we further obtain
\[
\frac{d}{dt}e^{c_4^\kappa \min\{1, m\}t}\mathcal{E}^\kappa + c_4^\kappa e^{c_4^\kappa \min\{1, m\}t}\left(\|u\|_4^2 + \|m\eta\|_{5, 2}^2\right)
\leq c_4^\kappa \|\eta, u, m\partial_2\eta\|_4^2
\|
\leq \kappa \mathcal{E}^{\kappa}_t + \|u\|_4^2 + \|m\eta\|_{5, 2}^2
\leq \kappa \||\eta, u\|_4^2 + \|m\eta\|_{5, 2}^2\| \leq \kappa \||\eta, u, m\partial_2\eta\|_4^2.
\tag{6.25}
\]

Let \(c_4^\kappa\) be the positive constant in (6.41). In view of (6.25) and (6.27), we find that
\[
e^{c_4^\kappa \min\{1, m\}t}\left(\|\eta, u\|_4^2 + \|m\eta\|_{5, 2}^2\right) \leq c_4^\kappa \|\eta, u, m\partial_2\eta\|_4^2 e^{c_4^\kappa \min\{1, m\}t}/c_4^\kappa,
\tag{6.28}
\]
where \(c_4^\kappa \geq 4\). From now on, we take
\[
K_\kappa = \sqrt{c_4^\kappa \|\eta, u, m\partial_2\eta\|_4^2 e^{c_4^\kappa \min\{1, m\}t}/c_4^\kappa}.
\tag{6.29}
\]
Thus, there is a \(\delta_1^\kappa \in (0, 1]\), such that for any \(\delta \leq \delta_1^\kappa\),
\[
\||\eta, u\|_4^2 + \|m\eta\|_{5, 2}^2 \leq e^{c_4^\kappa \min\{1, m\}t}\left(\|\eta, u\|_4^2 + \|m\eta\|_{5, 2}^2\right) \leq K_\kappa^2/c_4^\kappa.
\tag{6.30}
\]

Besides, we get from (6.25) and (6.30) that
\[
\int_0^t e^{c_4^\kappa \min\{1, m\}\tau}\left(\|u\|_4^2 + \|m\eta\|_{5, 2}^2\right)d\tau \leq \kappa K_\kappa^2/c_4^\kappa,
\tag{6.31}
\]
which, together with (6.30), yields the desired stability estimate:
\[
e^{c_4^\kappa \min\{1, m\}t}\left(\|\eta, u\|_4^2 + \|m\eta\|_{5, 2}^2\right) + \int_0^t e^{c_4^\kappa \min\{1, m\}\tau}\left(\|u\|_4^2 + \|m\eta\|_{5, 2}^2\right)d\tau \leq \kappa K_\kappa^2.
\tag{6.32}
\]

\(^4\)We can get the exponential decay in the fourth-order energy inequality (6.32), since the energy functional \(\mathcal{E}^\kappa\) is equivalent to the dissipative functional \(\mathcal{D}^\kappa\) in (6.22), where \(\mathcal{D}^\kappa := \|u\|_4^2 + \|m\partial_2\eta\|_{5, 2}^2\). Obviously, for the viscous fluid case, a similar equivalent relation does not hold in the second-order energy inequality (3.71), in which one sees that the second-order energy functional \(\mathcal{E}\) only controls the second-order dissipative functional \(\mathcal{D} := \|u\|_3^2 + \|m\partial_2\eta\|_{5, 2}^2\). Similarly, if we further derive the first-order energy inequality, then the first-order energy functional, denoted by \(\mathcal{E}^1\), also controls the the first-order dissipative functional. However, the first-order energy functional \(\mathcal{E}^1\) can be controlled by the second-order dissipative functional \(\mathcal{D}\); and this relation admits us to expect at most algebraic decay rates in the viscous fluid case.
Next, we proceed to derive algebraic time-decay stated in (2.51). To begin with, using (6.2) and (6.31), we have
\[ \int_0^t \| \eta \|_{5,2}^2 \ d\tau \lesssim K_\kappa. \] (6.33)

Applying Gronwall’s lemma to (6.26) and using (6.27), we conclude
\[ (t)^i(\| \eta, u \|_4^2 + \| m\eta \|_{5,2}^2) \lesssim \kappa \left( \| (\eta^0, u^0, m\partial_2 \eta^0) \|_4^2 + \int_0^t \langle \tau \rangle^{i-1} \| (\eta, u, m\partial_2 \eta) \|_{4,1}^2 \ d\tau \right) e^{c_2\kappa\| (\eta^0, u^0, m\partial_2 \eta^0) \|_3^2}. \] (6.34)

Thanks to (6.27) and (6.34), we further deduce from (6.26) that
\[ \int_0^t \langle \tau \rangle^{i}(\| u \|_4^2 + \| m\eta \|_{5,2}^2) \ d\tau \lesssim \kappa \left( 1 + \| (\eta^0, u^0, m\partial_2 \eta^0) \|_3^2 \right) \times \left( \| (\eta^0, u^0, m\partial_2 \eta^0) \|_4^2 + \int_0^t \langle \tau \rangle^{i-1} \| (\eta, u, m\partial_2 \eta) \|_{4,1}^2 \ d\tau \right) e^{c_2\kappa\| (\eta^0, u^0, m\partial_2 \eta^0) \|_3^2}. \] (6.35)

Consequently, we conclude from (6.2), (6.31), (6.33) and (6.35) that for \( i = 0 \) and 1,
\[ \int_0^t \langle \tau \rangle^{i}(\| u \|_4^2 + (1 + m^2)\| \eta \|_{5,2}^2) \ d\tau \leq C^\kappa. \] (6.36)

Finally, we get from (6.34) and (6.36) that for any \( i = 1 \) and 2,
\[ (t)^i(\| (u, \eta) \|_4^2 + \| m\eta \|_{5,2}^2) \leq C^\kappa, \]
which, together with (6.36), yields
\[ \sum_{0 \leq i \leq 1} \left( (t)^i(\| (\eta, u) \|_4^2 + \| m\eta \|_{5,2}^2) + \int_0^t \langle \tau \rangle^{i}(\| u \|_4^2 + \| m\eta \|_{5,2}^2) \ d\tau \right) + \int_0^t \langle \tau \rangle \| \eta \|_{5,2}^2 \ d\tau \leq C^\kappa. \] (6.37)

Next, we introduce a global well-posedness result for the linear initial-value problem (2.54) and a local well-posedness result for the nonlinear initial-value problem (2.47)–(2.48).

**Proposition 6.1.** Let \( i \geq 1 \) be an integer, \( \kappa \) and \( m \) be positive constants. If \( (\tilde{\eta}^0, \tilde{u}^0) \in (H^{i+1}_2 \cap H^i_\sigma) \times H^i_\sigma \), then there exists a unique strong solution \((\eta^L, u^L) \in C^0(\mathbb{R}^+, H^{i+1}_2) \times \Omega_{\infty}^i\) to the following linear initial-value problem:
\[ \begin{align*}
\eta^L_t &= u^L, \\
u^L_t - \kappa u^L &= m^2 \partial_2^2 \eta^L, \\
\text{div} u^L &= 0, \\
(\eta^L, u^L)|_{t=0} &= (\tilde{\eta}^0, \tilde{u}^0).
\end{align*} \] (6.38)

**Proof.** The proof of Proposition 6.1 is trivial, and hence we omit it here.

**Proposition 6.2.** Let \( B^\kappa > 0 \) and \((\eta^L, u^L) \in C^0(\mathbb{R}^+, H^2_2) \times \Omega_{\infty}^4\) be the unique global solution of (6.38) with \((\tilde{\eta}^0, \tilde{u}^0) \in (H^2_2 \cap H^1_\sigma) \times H^1_\sigma\). Assume that \((\eta^0, u^0) \in H^2_2 \times H^4\), \(\|(u^0, \partial_2 \eta^0)\|_4 \leq B^\kappa\) and
\( \text{div}_A u^0 = 0 \), where \( A^0 \) is defined by \( \zeta^0 \) and \( \zeta^0 = \eta^0 + y \). Then, there is a constant \( \delta_2^\kappa \in (0,1] \), such that if, in addition,

\[
\| \nabla \eta^0 \|_3 \leq \delta_2^\kappa; \tag{6.39}
\]

the initial value problem \( (2.47)-(2.48) \) possesses a unique local classical solution \( (\eta, u, q) \in C^0(I_T, H^3_T) \times U_T \times (C^0(I_T, H^3) \cap C^0(I_T, H^4)) \) for some \( T > 0 \) dependent of \( B^\kappa, \kappa, m \) and \( \delta_2^\kappa \). Moreover, \( (\eta, u) \) satisfies

\[
0 < \inf_{(y,t) \in \mathbb{R}^2 \times I_T} \det(\nabla \eta + I), \quad \sup_{t \in I_T} \| \nabla \eta \|_3 \leq 2\delta_2^\kappa; \tag{6.40}
\]

and

\[
\sup_{t \in I_T} \| (\text{curl} \, A \eta^0, \text{curl} \, A u^0, \text{curl} \, A m \partial_2 \eta^0)(\tau) \|_3^2 \\
\leq e^{t(\| \eta^0, u^0, m \partial_2 \eta^0 \|_4)} \left( \| \eta^0 - \tilde{\eta}^0, u^0 - \tilde{u}^0, m \partial_2 (\eta^0 - \tilde{\eta}^0) \|_4^2 + t \left( \| (u^0, m \partial_2 \eta^0, \tilde{u}^0, m \partial_2 \tilde{\eta}^0) \|_4^3 + \| (u^0, m \partial_2 \eta^0, \tilde{u}^0, m \partial_2 \tilde{\eta}^0) \|_4^2 \right) \right), \quad \forall t \in I_T \tag{6.42}
\]

where the constant \( c_4^\kappa \geq 1 \) depends on \( \kappa \) at least, and \( (\eta^0, u^0) := (\eta - \eta^1, u - u^1) \).

**Proof.** We postpone the proof of Proposition 6.2 to Section 7.3. \( \square \)

**Remark 6.1.** If \( (\eta^0, u^0) \) in Proposition 6.2 further satisfies \( (\eta^0, u^0) \in H_{1,2}^{5} \times H^{4} \) and the odevity conditions \( (2.14) \), then for each fixed \( t \in (0, T] \), \( (\eta, u)(t) \) also belongs to \( H_{1,2}^{5} \times H^{4} \) and satisfies the odevity conditions \( (2.21) \).

**Remark 6.2.** Since \( (u, \partial_2 \eta) \) may do not belong to \( C^0(\overline{T}_T, H^{4}) \), one needs the additional estimates \( (6.41) \) and \( (6.42) \) to further establish the existence result of a global solution and its asymptotic behavior with respect to \( m \), respsectively.

With the priori estimate \( (6.30) \) and Propositions 3.2 and 6.2 in hand, we can easily establish the existence and uniqueness of a global time-decay smooth solution stated in Theorem 2.5. Next, we sketch the proof.

Let \( m \) and \( (\eta^0, u^0) \in (H_{1,2}^{5} \cap H_{2}^{4}) \times H^{4} \) satisfy the odevity conditions \( (2.14) \) and

\[
\max \{ K_{\kappa}^{1/2}, K_{\kappa}^{2} \}/m \leq \min \{ \delta_1^\kappa, \delta_2^\kappa/c_0, \delta_3/c_0 \} =: c_3^\kappa \leq 1; \tag{6.43}
\]

where \( K_{\kappa} \) is defined by \( (6.29) \), and the constant \( c_0 \) is the same as in \( (6.5) \). Thus, \( \eta^0 \) satisfies \( (6.39) \) by virtue of \( (6.5) \) and \( (6.43) \). Moreover, by Proposition 6.2 and Remark 6.1, one sees that there is a unique local solution \( (\eta, u, q) \) of \( (2.47)-(2.48) \) defined on a maximal existence time interval \([0, T_{\text{max}}]\), such that

- for any \( T \in I_{T_{\text{max}}} \), the solution \( (\eta, u, q) \) belongs to \( \mathfrak{D}_{1,2}^{5,*} \times U_T \times (C^0(\overline{T}_T, H^{3}) \cap C^0(I_T, H^{4})) \) and satisfies \( \sup_{t \in \overline{T}_T} \| \nabla \eta \|_3 \leq 2\delta_2^\kappa \);

- \( \limsup_{t \to T_{\text{max}}} \| \nabla \eta(t) \|_3 > \delta_2^\kappa \), or \( \limsup_{t \to T_{\text{max}}} \| (u, \partial_2 \eta)(t) \|_4 = \infty. \)

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In addition, the solution enjoys the odevity conditions (2.21).

Let

$$T^* = \sup \{ T \in T_{max} \mid \| (\eta, u, m\partial_2 \eta) \|_4 \leq K_\kappa \text{ for any } t \leq T \}.$$ 

Recalling the definition of $K_\kappa$ and the condition $c_4^* \geq 4$, we see that the definition of $T^*$ makes sense and $T^* > 0$. Moreover, $\eta(t) \in H^4_\kappa$ for any $t \in T_{max}$ by Proposition 3.2 and (6.5). To obtain the existence of a global solution, we prove $T^* = \infty$ by contradiction.

Let us assume $T^* < \infty$. Then, for any given $T** \in T^*$, it holds that

$$\sup_{0 \leq t \leq T^{**}} \| (\eta, u, m\partial_2 \eta) \|_4 \leq K_\kappa.$$  (6.44)

Thanks to (6.41) and (6.44), the condition “$\max\{K_\kappa^{1/2}, K_\kappa^2\}/m \leq \delta^*_\kappa$” and the fact

$$\sup_{t \in T} \| f \|_0 = \| f \|_{L^\infty(t_T, L^2)} \text{ for any } f \in C(T_T, L^2_\text{weak}),$$  (6.45)

we can verify that the solution $(\eta, u)$ indeed satisfies the stability estimate (6.30) by a standard regularity method. More precisely, we have

$$\sup_{0 \leq t \leq T^*} \| (\eta, u, m\partial_2 \eta)(t) \|_4 \leq K_\kappa/\sqrt{c_4^*}.$$  

Now, we take $(\eta(T^{**}), u(T^{**}))$ as an initial data. Noting that

$$\| (u, \partial_2 \eta)(T^{**}) \|_4 \leq B^*_\kappa := 2 \max\{K_\kappa/\sqrt{c_4^*}, K_\kappa/m\sqrt{c_4^*}\} \text{ and } \| \nabla \eta(T^{**}) \|_3 \leq \delta^*_\kappa,$$

we apply Proposition 3.1 to see that there exists a unique local classical solution, denoted by $(\eta^*, u^*, q^*)$, to the initial-value problem (2.47)–(2.48) with $(\eta(T^{**}), u(T^{**}))$ in place of $(\eta^0, u^0)$. Moreover,

$$\sup_{t \in [T^{**}, T]} \| (\eta^*, u^*, m\partial_2 \eta^*) \|_4 \leq K_\kappa \text{ and } \sup_{t \in [T^{**}, T]} \| \nabla \eta^* \|_3 \leq 2\delta^*_\kappa,$$  (6.46)

where the local existence time $T > T^{**}$ depends possibly on $B^*_\kappa$, $\kappa$, $m$ and $\delta^*_\kappa$.

In view of the uniqueness in Proposition 6.2 and the fact that $T_{max}$ is the maximal existence time, we immediately see that $T_{max} > T^* + T/2$ and $\sup_{t \in [0, T^* + T/2]} \| (\eta, u, m\partial_2 \eta) \|_4 \leq K_\kappa$. This contradicts with the definition of $T^*$. Hence, $T^* = \infty$ and thus $T_{max} = \infty$.

In addition, we also verify that the global solution $(\eta, u)$ indeed enjoys the time-decay estimates (2.50) and (2.51) by the a priori estimates (6.32) and (6.37). Finally, the uniqueness of the global solutions is obvious due to the uniqueness of the local solutions in Proposition 6.2 and the fact $\sup_{t \geq 0} \| \nabla \eta \|_3 \leq 2\delta^*_\kappa$. This completes the proof of the existence and uniqueness of global time-decay solutions stated in Theorem 2.5.

6.2. Stability around the solutions of the linearized problem

We now turn to the proof of stability around the solutions of (2.51). Thanks to the regularity of $(\eta^0, u^0)$ in Theorem 2.5, there exists a unique solution $(\eta^t, u^t, Q_1, Q_2)$ satisfying (6.1) and (6.2). Furthermore, $(\eta^t, u^t)$ satisfies (2.34) and (2.55).

Let $\tilde{\eta}^0 = \eta^0 + \eta^t$ and $\tilde{u}^0 = u^0 + u^t$. Then it is easy to see that $(\tilde{\eta}^0, \tilde{u}^0)$ is in $(H^5_2 \cap H^4_\sigma) \times H^4_\sigma$ and enjoys the odevity conditions as $(\eta^0, u^0)$ does. Hence, there is a unique global solution $(\eta^L, u^L) \in C^0(T_T, H^5_2 \times H^4_\sigma)$ to the problem (2.54). Furthermore, the solution enjoys the odevity conditions as $(\eta, u)$ does.
Let \((\eta^d, u^d) = (\eta - \eta^L, u - u^L)\), then the error function \((\eta^d, u^d)\) satisfies

\[
\begin{align*}
\eta_t^d &= u^d, \\
\eta_i^d + \nabla_x q + \kappa u^d &= m^2 \partial_x^2 \eta^d, \\
\text{div}u^d &= -\text{div}_x u, \\
(\eta^d, u^d)|_{t=0} &= -(\eta^r, u^r).
\end{align*}
\] (6.47)

Moreover, \((u^d)_{T^2} = (\eta^d)_{T^2} = 0\), \text{div} \eta^d = \text{div} \eta, and \((\eta^d, u^d)\) also satisfies the ovdeity conditions as \((\eta, u)\) does.

Employing the arguments used for (6.25) and (6.26), together with a standard regularity method, we deduce from (6.47) that for a.e. \(t > 0\),

\[
\frac{d}{dt} e^{C^\epsilon t} \min(1,m)t \mathcal{E}_d^\epsilon + C^\epsilon e^{C^\epsilon t} \min(1,m)t \left( \|u^d\|^2_4 + \|m \eta^d\|^2_{5,2} \right) \\
\leq \kappa e^{C^\epsilon t} \min(1,m)t \left( \|\eta, u, m \partial_2 \eta\|_4^4 \mathcal{E}_d^\epsilon + \|\nabla \eta\|_3^2 \|\eta, m \partial_2 \eta\|_4^2 \right),
\] (6.48)

\[
\frac{d}{dt} \langle t \rangle^i \mathcal{E}_d^\epsilon + C^\epsilon \langle t \rangle^i \left( \|u^d\|^2_4 + \|m r^d\|^2_{5,2} \right) \\
\leq \kappa \langle t \rangle^{i-1} \mathcal{E}_d^\epsilon + \langle t \rangle^i \left( \|\eta, u, m \partial_2 \eta\|_4^4 \mathcal{E}_d^\epsilon + \|\nabla \eta\|_3^2 \|\eta, m \partial_2 \eta^2\|_4^2 \right),
\] (6.49)

where \(\mathcal{E}_d^\epsilon\) is defined by (6.25) with \((u^d, \eta^d)\) in place of \((u, \eta)\), in \(W^{1,\infty}(\mathbb{R}^+), \) and satisfies that for any \(t \geq 0\),

\[
\|\eta^d, u^d\|_4^4 + \|m r^d\|_{5,2}^2 \leq \kappa \mathcal{E}_d^\epsilon + \|\nabla \eta\|_3^2 \|\eta, m \partial_2 \eta\|_4^2, \quad \mathcal{E}_d^\epsilon \leq \kappa \|\eta^d, u^d, m \partial_2 \eta^d\|_4^2.
\] (6.50)

In view of (6.42), one sees that there is a \(T^0 > 0\), such that for any \(t < T^0\),

\[
\sup_{\tau \in [t]} \mathcal{E}_d^\epsilon \leq \kappa (1 + t) e^{\epsilon t \|\eta^0, u^0, m \partial_2 \eta^0\|_4^4 \|\eta^r, u^r, m \partial_2 \eta^r\|_4^4} \\
+ t \left( \|u^0, m \partial_2 \eta^0, \tilde{\eta}^0, m \partial_2 \tilde{\eta}^0\|_{4,1}^4 + t \|u^0, m \partial_2 \eta^0, \tilde{\eta}^0, m \partial_2 \tilde{\eta}^0\|_{4,1}^4 \right),
\] which, together with (2.55), yields that

\[
\limsup_{t \to 0} \sup_{\tau \in [t]} \mathcal{E}_d^\epsilon \leq \kappa \|\eta^r, u^r, m \partial_2 \eta^r\|_4^4 \|\eta^0, u^0, m \partial_2 \eta^0\|_4^4.
\] (6.51)

Thus, if we apply Gronwall’s lemma to (6.48), and make use of (2.49), (2.50), (6.4), (6.50) and (6.51), we obtain

\[
\sup_{t > 0} \left( e^{C^\epsilon t} \min(1,m)t \left( \|\eta^d, u^d\|_4^4 + \|m \eta^d\|_{5,2}^2 \right) + \int_0^t e^{C^\epsilon \tau} \min(1,m)\tau \left( \|u^d\|^2_4 + \|m \eta^d\|^2_{5,2} \right) d\tau \right) \\
\leq \kappa \left( 1 + \mathcal{R}^4 \right) \|\eta^0, u^0, m \partial_2 \eta^0\|_4^4 e^{c_\epsilon \mathcal{R}^2} m^{-2},
\]

which gives (2.52). Finally, following the same process as in the derivation of (6.37) with necessary modifications in arguments, we obtain (2.53). This completes the proof of Theorem 2.5.

\(^5\)It is easy to see from the proof of (2.52) and (2.53) that due to the special structure of energy inequalities of \((\eta^d, u^d)\), we can use (6.51) and \(m \nabla \eta\|_{5,2}^2 \leq C^\epsilon\) to obtain the convergence rate \(m^{-2}\). For the viscous fluid case, however, we mainly exploit \(m \partial_2 \eta\|_{2} \leq C\) to get the convergence rate \(m^{-1}\). This consequently yields some nonlinear terms of third-order derivatives, such as \(\|\partial_2 \eta\|_2 \nabla \eta\|_2 \nabla q\|_1\), on the right-hand side of the energy inequalities for \((\eta^d, u^d)\) which can only provide the convergence rate \(m^{-1}\).
7. Local well-posedness

This section is devoted to the proof of the local well-posedness results in Propositions 3.1 and 6.2, and is organized as follows.

First we establish the existence of both strong and classical solutions to the following linear initial-value problem in Section 7.1:

\[
\begin{cases}
  u_t + \nabla_A q - \nu \Delta_A u + \kappa u = f, \\
  \text{div}_A u = 0, \\
  u|_{t=0} = u^0 \text{ in } \mathbb{T}^2,
\end{cases}
\]

where \( \nu > 0, \kappa \geq 0, (\eta^0, u^0, w) \) are given,

\[ A = (\nabla \zeta)^{-T} \quad \text{and} \quad \zeta = \int_0^t w dy + \eta^0 + y, \]

see Propositions 7.1 and 7.2 for the details. Then we give the proof of Proposition 3.1 based on Proposition 7.1 by a standard iterative method in Section 7.2. Similarly, we also give the local existence of a unique classical solution to the following initial-value problem:

\[
\begin{cases}
  \eta_t = u, \\
  u_t + \nabla_A q - \nu \Delta_A u + \kappa u = f, \\
  \text{div}_A u = 0, \\
  (\eta, u)|_{t=0} = (\eta^0, u^0) \text{ in } \mathbb{T}^2,
\end{cases}
\]

see Proposition 7.2 in Section 7.2 for the details. Finally, thanks to Proposition 7.2, we complete the proof of Proposition 6.2 by a standard method of vanishing viscosity limit in Section 7.3.

Finally, we introduce new notations appearing in this section.

- \( H^{-1} = \) the dual space of \( H \), \( H^{-1}_\sigma = \) the dual space of \( H^1_\sigma \),
- \( \langle \cdot, \cdot \rangle_{X^{-1},X} \) denotes dual product,
- \( \mathcal{U}_T^i := \{ u \in \mathcal{U}_i, T \mid u_{tt} \in L^2(I_T, H) \} \),
- \( \mathcal{G}_T := \{ f \in C^0(I_T, L^2) \mid (f, f_t) \in L^2(I_T, H \times H^{-1}) \} \),
- \( \| v \|_{\mathcal{U}_i, T} := \sqrt{\sum_{0 \leq j \leq 1} \left( \| \partial_t^j v \|_{C^0(I_T, H^2(1-j))}^2 + \| \partial_t^j v \|_{L^2(I_T, H^2(1-j)+1)}^2 \right)} \),
- \( \| v \|_{\mathcal{U}_i^2, T} := \sqrt{\sum_{0 \leq j \leq 2} \| \partial_t^j v \|_{C^0(I_T, H^2(2-j))}^2 + \sum_{0 \leq j \leq 2} \| \partial_t^j v \|_{L^2(I_T, H^2(2-j)+1)}^2} \),

\[ A \lesssim_L B \text{ means } A \leq c_L B, \]

where \( \mathcal{U}_i, T \) for \( i = 1, 2 \) is defined by (2.10), and \( c_L \) denotes a generic positive constant depending on \( \nu, \kappa \) and \( m \), and may vary from one place to another (if not stated explicitly).

7.1. Unique solvability of the linear initial-value problem (7.1)

This section is devoted to establishing the existence and uniqueness of both strong and classical solutions to (7.1). We start with the existence of a unique strong solution.
Proposition 7.1. Let $B_1 > 0$, $\delta > 0$, $(\eta^0, u^0) \in H^3 \cap H^2$, $A^0 = (\nabla \eta^0 + I)^{-1}$, $w \in \mathcal{U}_{1,T}$, $f \in \mathcal{G}_T$, $A$ and $\zeta$ be defined by (7.2), $\eta = \zeta - y$ and

$$T := \min\{1, (\delta / B_1)^4\}. \tag{7.4}$$

Assume that

$$\|\nabla \eta^0\|_{2,2} \leq \delta, \quad \text{div}_{A^0} u^0 = 0, \quad w|_{t=0} = u^0,$$

$$\sqrt{\|\nabla w\|^2_{C^0(T_0, H)} + \|\nabla w\|^2_{L^2(I_T, H^2)} + \|\nabla w_t\|^2_{L^2(I_T, L^2)}} \leq B_1, \tag{7.5}$$

then there is a sufficiently small constant $\delta_1^e \in (0, 1]$ independent of $m$ and $\nu$, such that for any $\delta \leq \delta_1^e$, there exists a unique local strong solution $(u, q) \in \mathcal{U}_{1,T} \times (C^0(T, H) \cap L^2(I_T, H^2))$ to the initial-value problem (7.1). Moreover, the solution enjoys the following estimate:

$$1 \leq 2 \det \zeta \leq 3, \quad \|\nabla \eta\|_{2,2} \leq 2\delta,$$

$$\|u\|_{\mathcal{U}_{1,T}} + \|q\|_{C^0(T_0, H)} + \|q\|_{L^2(I_T, H^2)} \lesssim \sqrt{\mathcal{B}_1(u^0, f)}, \tag{7.6}$$

where

$$\mathcal{B}_1(u^0, f) := (1 + \|\nabla \eta^0\|^2_2) \mathcal{B}_1(u^0, f),$$

$$\mathcal{B}_1(u^0, f) := \|u^0\|^2_2 + \|u^0\|^4_2 + \|f\|^2_{C^0(T_0, L^2)} + \|f\|^2_{L^2(I_T, H)} + \|f_t\|^2_{L^2(I_T, H^{-1})} + \|\nabla w\|_{C^0(T_0, H)}(1 + \|u^0\|_1)(\|u^0\|^2_2 + \|f\|^2_{L^2(Q_T)}).$$

Moreover, if $f = \partial_2^2 \eta$, then

$$\sqrt{\mathcal{B}_1(u^0, \partial_2^2 \eta)} \lesssim 1 + \|u^0\|^2_2 + \sqrt{\|\nabla w\|_{C^0(T_0, H)}(1 + \|u^0\|^3_2)}, \tag{7.7}$$

$$\Delta_{A^0} q = m^2 \text{div}_{A^0} \partial_2^2 \eta + \text{div}_{A^0} u \text{ holds for any } t \in T, \tag{7.8}$$

$$\|q\|_{C^0(T_0, H^2)} \lesssim 1 + \|\nabla w\|_{C^0(T_0, H)}(1 + \|u^0\|^2_2 + \sqrt{\|\nabla w\|_{C^0(T_0, H)}(1 + \|u^0\|^3_2)}), \tag{7.9}$$

$$\|q_t\|_{L^2(I_T, H^2)} \lesssim (1 + \|u^0\|^3_2 + \|\nabla w\|_{C^0(T_0, H)})(1 + \|\nabla u^0\|_0 + \|\nabla (w, w_t)\|_{L^2(I_T, H^2 \times L^2)}). \tag{7.10}$$

Remark 7.1. For the case $f = \partial_2^2 \eta$, from (7.6) and (7.7) we get

$$\|u\|_{\mathcal{U}_{1,T}} + \|q\|_{C^0(T_0, H)} + \|q\|_{L^2(I_T, H^2)} \leq c^1(1 + \|u^0\|^3_2) + \|\nabla w\|_{C^0(T_0, H)} / 2. \tag{7.11}$$

Proof. We shall break up the proof into three steps.

1. **Existence of local strong solutions**

   Recalling that $\|\nabla \eta^0\|_{2,2} \leq \delta$, the definition (7.4) and

   $$\eta = \int_0^t w \, \text{d}t + \eta^0, \tag{7.12}$$

   we make use of the regularity of $w$, (7.4) and (7.5) to find that $\eta \in C^0(T, H^3)$, and

   $$\|\nabla \eta(t)\|_2 \leq \|\nabla \eta^0\|_2 + \sqrt{\delta} \|\nabla w\|_{L^2(I_T, H^2)} \leq 1 + \|\nabla \eta^0\|_2, \tag{7.13}$$

   and

   $$\|\nabla \eta(t)\|_{2,2} \leq \delta + \sqrt{\delta} \|\nabla w\|_{L^2(I_T, H^2)} \leq 2\delta \quad \text{for all } t \in T.$$
By \( (3.5) \) one has
\[
\|\nabla \eta(t)\|_{L^\infty} \lesssim_0 \|\nabla \eta(t)\|_{2,2} \lesssim_0 \delta \quad \text{for any } t \in \overline{I_T}.
\]
(7.14)

Thanks to the estimate \( (7.14) \), we have for sufficiently small \( \delta \) that \( 1 \leq 2J \leq 3 \), where and in what follows, \( J := \det \nabla \zeta \). Therefore, \( A \) makes sense and is given by the following formula:
\[
A = J^{-1} \begin{pmatrix}
\partial_2 \eta_2 + 1 & -\partial_1 \eta_2 \\
-\partial_2 \eta_1 & \partial_1 \eta_1 + 1
\end{pmatrix}.
\]

We remark that the smallness of \( \delta \) (independent of \( \nu \) will be often used in the derivation of some estimates and conclusions later, and we shall omit to mention it for the sake of simplicity.

Inspired by the proof in [12, Theorem 4.3], we next solve the linear problem (7.1) by applying the Galerkin method. Let \( \{ \varphi^i \}_{i=1}^\infty \) be a countable orthogonal basis in \( H^2 \). For each \( i \geq 1 \) we define \( \psi^i = \psi^i(t) := \nabla \zeta \varphi^i \). Let \( \mathcal{H}(t) = \{ v \in H^2 \mid \text{div} v = 0 \} \). Then \( \psi^i(t) \in \mathcal{H}(t) \) and \( \{ \psi^i(t) \}_{i=1}^\infty \) is a basis of \( \mathcal{H}(t) \) for each \( t \in \overline{I_T} \). Moreover,
\[
\psi^i_t = R \psi^i,
\]
(7.15)
where \( R := \nabla w A^T \).

For any integer \( n \geq 1 \), we define the finite-dimensional space \( \mathcal{H}^n(t) := \text{span} \{ \psi^1, \ldots, \psi^n \} \subset \mathcal{H}(t) \), and write \( \mathcal{P}^n(t) : \mathcal{H}(t) \to \mathcal{H}^n(t) \) for the \( \mathcal{H} \) orthogonal projection onto \( \mathcal{H}^n(t) \). Clearly, for each \( v \in \mathcal{H}(t) \), \( \mathcal{P}^n(t)v \to v \) as \( n \to \infty \) and \( \| \mathcal{P}^n(t)v \|_2 \leq \|v\|_2 \).

Now, we define an approximate solution
\[
u^n(t) = a^n_j(t) \psi^j \quad \text{with } a^n_j : \overline{I_T} \to \mathbb{R} \quad \text{for } j = 1, \ldots, n,
\]
where \( n \geq 1 \) is given. We want to choose the coefficients \( a^n_j \), so that for any \( 1 \leq i \leq n \),
\[
\int u^n_i \cdot \psi^j dy + \nu \int \nabla u^n : \nabla \psi dy + \kappa \int u^n : \psi^i dy = \int f \cdot \psi^i dy
\]
(7.16)
with initial data \( u^n(0) = \mathcal{P}^n u_0 \in \mathcal{H}^n \).

Let
\[
X = (a^n_j)_{n \times 1}, \quad \mathfrak{M} = \left( \int f \cdot \psi^j dy \right)_{n \times 1}, \quad \mathfrak{C}^1 = \left( \int \psi^i \cdot \psi^j dy \right)_{n \times n}, \quad \\
\mathfrak{C}^2 = \left( \int R \psi^i \cdot \psi^j dy + \nu \int \nabla \psi^i : \nabla \psi^j dy + \kappa \int \psi^i \cdot \psi^j dy \right)_{n \times n}.
\]
Recalling the regularity of \( w \), we easily verify that
\[
\mathfrak{C}^1 \in C^{1,1/2}(\overline{I_T}), \quad \mathfrak{C}^2 \in C^{0,1/2}(\overline{I_T}), \quad \mathfrak{M} \in C^0(\overline{I_T}) \quad \text{and} \quad \mathfrak{M}_t \in L^2(I_T).
\]
(7.17)

Noting that \( \mathfrak{C}^1 \) is invertible, we can rewrite (7.16) as follows.
\[
X_t + (\mathfrak{C}^1)^{-1}(\mathfrak{C}^2 X - \mathfrak{M}) = 0
\]
(7.18)
with initial data
\[
X|_{t=0} = \left( \int \mathcal{P}^n u_0 \cdot \psi^j dy \right)_{n \times 1},
\]
where \((\mathfrak{C}^1)^{-1}\) denotes the inverse matrix of \( \mathfrak{C}^1 \). By virtue of the well-posedness theory of ODEs (see [32, Section 6 in Chapter II]), the equation (7.18) has exactly one solution \( X \in C^1(\overline{I_T}) \).
Thus, one has established the existence of the approximate solution $u^n(t) = a^n_j(t)\psi^j$. Next, we derive uniform-in-$n$ estimates for $u^n$.

Due to (7.14), we easily get from (7.16) with $u^n$ in place of $\psi$ that for sufficiently small $\delta$,

$$
\frac{d}{dt}\|u^n\|_0^2 + cL\|u^n\|_1^2 \lesssim_L \|f\|_0^2.
$$

(7.19)

By (7.15),

$$
u u^n_t - Ru^n = \dot{a}^n_i \psi^i.
$$

(7.20)

Obviously, we can replace $\psi$ by $\dot{a}^n_i \psi^j$ in (7.16) and use (7.20) to deduce that

$$
\|u^n_t\|_0^2 + \nu \int {\nabla_A u^n} : {\nabla_A u^n} dy + \kappa \int u^n : u^n dy
$$

$$
= \int (u^n_t + \kappa u^n) \cdot (Ru^n) dy + \nu \int {\nabla_A u^n} : {\nabla_A (Ru^n)} dy + \int f \cdot (u^n_t - Ru^n) dy.
$$

(7.21)

Thanks to (3.4), (3.6) and (7.14), one can further obtain from (7.21) that

$$
\frac{d}{dt} (\|u^n\|_0^2 + \nu \|\nabla_A u^n\|_0^2) + \|u^n_t\|_0^2
$$


\[\lesssim_L \|\nabla u^n\|_0 (\|R\|_{L^\infty} \|\nabla u^n\|_0 + \|\nabla u^n\|_0) + \int |\nabla u^n| |\nabla u^n| dy
$$

$$
+ \|f, Ru^n\|_0^2 + \|R\|_{L^\infty} \|u^n\|_0^2
$$

$$\lesssim_L \|\nabla w\|_{L^2} (\|u^n\|_0^2 + \|\nabla_A u^n\|_0^2) + \|\nabla w\|_{L^1} \|\nabla w\|_{L^2} \|u^n\|_0^2 + \|f\|_0^2.
$$

(7.22)

With the help of Gronwall’s lemma, (7.4) and (7.14), we infer from (7.19) and (7.22) that for any $t \in I_T$,

$$
\|u^n\|_1^2 + \int_0^t \|u^n\|_0^2 dt \lesssim_L \left( \|\mathcal{P}^n u^0\|_1^2 + \int_0^t \|f\|_0^2 dt \right) e^{\int_0^t c(\|\nabla w\|_{L^2} + \|\nabla w\|_{L^1} + \|\nabla w\|_{L^2}) dt}
$$

$$\lesssim_L \|u^0\|_{L^2}^2 + \|f\|_{L^2(I_T)}^2.
$$

(7.23)

Recall (see [22, Theorem 1.67])

$$
\int f(\tau) \psi^j(\tau) d\tau \bigg|_{\tau=0}^{\tau=t} = \int_s^t \left( < f_\tau, \psi^j >_{H^{-1}, H} + \int f \cdot \psi^j dy \right) d\tau,
$$

which gives

$$
\frac{d}{dt} \int f(t) \psi^j(t) dt = < f_\tau, \psi^j >_{H^{-1}, H} + \int f \cdot \psi^j dy \quad \text{for a.e. } t \in I_T.
$$

(7.24)

Hence, $\mathcal{C}^2_t \in L^2(I_T)$. In view of (7.17) and (7.18), we have $\ddot{a}^n_j(t) \in L^2(I_T)$. This means that $u^n_t$ makes sense. So, with the help of (7.15) and (7.24), we get from (7.16) that

$$
\int u^n_t \cdot \psi^j dy + \nu \int {\nabla_A u^n} : {\nabla_A \psi^j} dy + \kappa \int u^n_t : \psi^j dy
$$

$$= < f_t, \psi^j >_{H^{-1}, H} + \int (f - u^n_t - \kappa u^n) \cdot (R\psi^j) dy
$$

$$\quad - \nu \int (\nabla_A, u^n) : (\nabla_A \psi^j + \nabla_A u^n : (\nabla_A \psi^j + \nabla_A (R\psi^j)) dy \quad \text{a.e. in } I_T.
$$

(7.25)
Noting that (also see [22, Theorem 1.67])

\[
\frac{1}{2}\|u_t^n\|_0^2 - \int u_t^n \cdot (R u^n) dy - \left( \frac{1}{2}\|u_t^n\|_0^2 - \int u_t^n \cdot (R u^n) dy \right) \bigg|_{t=0}^t \\
= \int_0^t \left( \int u_{\tau \tau}^n \cdot (u_\tau^n - R u^n) dy - \int u_\tau^n \cdot (R u^n) dy \right) d\tau
\]

and

\[
\int f(\tau) \cdot (R u^n)(\tau) dy \bigg|_{\tau=0}^{\tau=t} = \int_0^t \left( <f_\tau, R u^n >_{H^{-1}, H} + \int f \cdot (R u^n)_\tau dy \right) d\tau,
\]

we utilize (7.20) and the above two identities to infer from (7.25) with \(\psi^i\) replaced by \((u_t^n - R u^n)\) that

\[
\frac{1}{2}\|u_t^n\|_0^2 - \int u_t^n \cdot (R u^n) dy + \int f \cdot (R u^n) dy + \int (\kappa\|u_\tau^n\|_0^2 + \nu\|\nabla A u^n\|_0^2) d\tau
\]

\[
= \left( \frac{1}{2}\|u_t^n\|_0^2 - \int u_t^n \cdot (R u^n) dy + \int f \cdot (R u^n) dy \right) \bigg|_{t=0} + I_7,
\]

(7.26)

where

\[
I_7 := \int_0^t \left( <f_\tau, u_\tau^n >_{H^{-1}, H} + \int \left( f \cdot (2R u_\tau^n + R u_\tau^n - R^2 u^n) + (u_\tau^n + \kappa u^n) \cdot (R (u_\tau^n - R u^n)) \right) dy + \int (\kappa u_\tau^n (R u^n) - u_\tau^n \cdot (R u^n)) dy - \nu \int (\nabla A u^n : (\nabla A (u_\tau^n - R u^n) + \nabla A (R(u_\tau^n - R u^n))) + \nabla A_{u^n} : \nabla A_{(u_\tau^n - R u^n) - \nabla A_{u_\tau^n} : \nabla A (R u^n)) dy \right) d\tau.
\]

Keeping in mind that

\[
\|\nabla w\|_0 \leq \int_0^t \|\nabla w_\tau\|_0 d\tau + \|\nabla u^n\|_0,
\]

(7.27)

we get from (7.26) that

\[
\|u_t^n\|_0^2 + \int_0^t (\kappa\|u_\tau^n\|_0^2 + \nu\|\nabla u^n\|_0^2) d\tau
\]

\[
\lesssim_L \|\nabla w\|_0 \|\nabla w_1\|_1 \|u_t^n\|_0^2 + \|u_0^n\|_2^4 + \|f\|_{C^0(\overline{\Omega}, L^2)}^2 + \|u_t^n\|_{t=0}^2 + I_9
\]

\[
\lesssim_L \|\nabla w\|_1 (1 + \|u_0^n\|_1) \left( \|u_t^n\|_2^2 + \|f\|_{L^2(\Omega)}^2 \right) + \|u_0^n\|_2^2 + \|f\|_{C^0(\overline{\Omega}, L^2)}^2 + \|u_t^n\|_{t=0}^2 + I_9,
\]

(7.28)

where we have used (3.6) in the first inequality, (7.4), (7.23) and (7.27) in the second inequality. Below, we shall bound the last two terms in (7.28).

Replacing \(\psi^i\) by \((u_t^n - R u^n)\) in (7.16), one sees that

\[
\|u_t^n\|_0^2 = \int f \cdot (u_t^n - R u^n) dy + \nu \int \Delta A u^n : (u_t^n - R u^n) dy
\]

\[
+ \int u_t^n \cdot (R u^n) dy - \kappa \int u^n \cdot (u_t^n - R u^n) dy,
\]

(7.29)
which implies

$$
\|u^n_0\|_0^2 \lesssim_L \|f\|_2^2 + \|u^n\|_2^2 + \|\nabla w\|_1^2 \|u^n\|_2^2, \quad \forall t \in [0, T).
$$

In particular,

$$
\|u^n_t|_{t=0}\|_0^2 \lesssim_L \|u^0\|_2^2 + \|u^n\|_4^4 + \|f^0\|_2^2.
$$

(7.30)

Thus, the last term on the right-hand side of (7.26) can be estimated as follows, using the first inequality in (3.6) and (7.31):

$$
I_7 \lesssim L \int_0^t \left( \|f_t\|_1 \|u^n_t\|_1 + \|u^n\|_1 \left( \|f_0\|_1 \|\nabla w\|_L^\infty \|\nabla w\|_1 + \sqrt{\|f_0\|_1 \|f_t\|_1 \|\nabla w\|_0} \right) 
+ \|f\|_1 \|\nabla w\|_1 \|u^n_t\|_0 + \|u^n\|_1^2 \|\nabla w\|_L^\infty \left( \|\nabla w\|_L^\infty + \sqrt{\|\nabla w\|_1 \|\nabla w\|_{2,2}} \right) 
+ \|u^n\|_0 (\|\nabla w\|_2 \|u^n\|_0 + (1 + \|\nabla w\|_L^\infty) \|\nabla w\|_1 \|u^n\|_1) + \|\nabla w\|_0 \|u^n\|_1 \sqrt{\|u^n\|_0 \|u^n\|_1} 
+ \|u^n\|_1 \|u^n_t\|_1 \left( \|\nabla w\|_L^\infty + \sqrt{\|\nabla w\|_1 \|\nabla w\|_{2,2}} \right) \right) d\tau 
\lesssim L \|u^0\|_2^2 + \|f\|_{C^0(I_T, L^2)}^2 + \|f\|_{L^2(I_T, H)}^2 + \int_0^t \|f_t\|_{H^{-1}} \|u^n_t\|_1 d\tau 
+ \left( \|u^0\|_2^2 + \|f\|_{L^2(I_T, H)} \right) \left( \|u^n_t\|_{C^0(I_T, L^2)} + \|\nabla u^n_t\|_{L^2(I_T, L^2)} + \|\nabla w\|_0 \|u^n\|_1 \right) + \|u^n\|_1^2 \lesssim L \mathfrak{B}_1(u^0, f).
$$

(7.31)

Substituting (7.30) and (7.31) into (7.28), and applying Young's inequality, we arrive at

$$
\|u^n_t\|_{C^0(I_T, L^2)}^2 + \|u^n_t\|_{L^2(I_T, H)}^2 \lesssim L \mathfrak{B}_1(u^0, f).
$$

(7.32)

Summing up (7.25) and (7.32), we conclude

$$
\|(u^n, u^n_t)\|_{C^0(I_T, H \times L^2)}^2 + \|(u^n, u^n_t)\|_{L^2(I_T, H)}^2 \lesssim L \mathfrak{B}_1(u^0, f).
$$

(7.33)

In view of (7.33), the Banach–Alaoglu and Arzelà–Ascoli theorems, up to the extraction of a subsequence (still labelled by $u^n$), we have, as $n \to \infty$, that

$$
(u^n, u^n_t) \to (u, u_t) \quad \text{weakly-* in } L^\infty(I_T, H \times L^2),
$$

$$
(u^n, u^n_t) \to (u, u_t) \quad \text{weakly in } L^2(I_T, H \times H),
$$

$$
u_n \to u \quad \text{strongly in } C^0(I_T, L^2),
$$

$$
\text{div}_A u = 0 \quad \text{a.e. in } \Omega_T \text{ and } u(0) = u_0,
$$

where $u$ and $u_t$ are measurable functions defined on $\Omega_T$. Moreover,

$$
\|(u, u_t)\|_{L^\infty(I_T, L^2)} + \|(u, u_t)\|_{L^2(I_T, H)} \lesssim L \sqrt{\mathfrak{B}_1(u^0, f)}.
$$

(7.34)

Therefore, we can take to the limit in (7.16) as $n \to \infty$, and obtain

$$
\int u_t \cdot \zeta dy + \nu \int \nabla A u : \nabla A \zeta dy + \kappa \int u \cdot \zeta dy = \int f \cdot \zeta dy \quad \text{a.e. in } I_T, \quad \forall \zeta \in \mathfrak{F}.
$$

(7.35)

Now, we begin to show spatial regularity of $u$. Let us further assume that $\delta$ is so small that $\eta$ satisfies (2.32) by virtue of Proposition 3.2. Denoting $F := f - \kappa u - u_t$, $\tilde{F} := F(\zeta^{-1}, t)$ and $\tilde{J} := J(\zeta^{-1}, t)$, we see that $\tilde{F}$ has the same regularity as that of $F$, i.e.,

$$
\|\tilde{F}\|_{L^\infty(I_T, L^2)} + \|\tilde{F}\|_{L^2(I_T, H)} < \infty.
$$

(7.36)
Moreover,
\[ \int F(y,t)dy = \int \tilde{F} \tilde{J}^{-1}dx. \]

Applying the regularity theory of the Stokes problem, we see that there is a unique strong solution \( \alpha \in L^\infty(I_T,H^2) \cap L^2(I_T,H^3) \) with a unique associated function \( p \in L^\infty(I_T,H) \cap L^2(I_T,H^2) \), such that
\[
\begin{cases}
\nabla p - \nu \Delta \alpha = \tilde{F}, \\
\text{div} \, \nu = 0.
\end{cases}
\]

(7.37)

Let \( \varpi = \alpha(\zeta, t) \) and \( q = p(\zeta, t) - (p(\zeta, t))_{\tau^2} \), then \( (\varpi, q) \in (L^\infty(I_T,H^2) \cap L^2(I_T,H^3)) \times (L^\infty(I_T,H) \cap L^2(I_T,H^2)) \) satisfies the following system:
\[
\begin{cases}
\nabla_A q - \nu \Delta_A \varpi = F, \\
\text{div}_A \, \varpi = 0 \quad \text{for a.e. } t \in I_T.
\end{cases}
\]

(7.38)

By a density argument, the identity (7.35) also holds for \( \zeta \in H \) with \( \text{div}_A \zeta = 0 \). This fact, together with (7.38), implies \( \text{div} \, u = \nabla \varpi \).

Following the derivation of (3.36) with slight modification, we can get from (7.38) that for a.e. \( t \in I_T \),
\[
\|q\|_{i+1} \lesssim_0 \|\nabla q\|_i \lesssim_0 \|f(u_t)\|_i \quad \text{for } i = 0, 1.
\]

(7.39)

Similarly to the derivation of (7.39) with \( i = 1 \), we can derive from (7.38) that
\[
\begin{align*}
\|\varpi\|_2 & \lesssim_0 \|(u, u_t, \nabla q, f)\|_0, \\
\|\varpi\|_{3,2} & \lesssim_0 \|(u, u_t, \nabla q, f)\|_1, \\
\|\varpi\|_3 & \lesssim_0 \|(u, u_t, \nabla q, f)\|_1 + \|\nabla \eta\|_2 \|\varpi\|_{3,2}.
\end{align*}
\]

(7.40) – (7.42)

So, it follows from (7.13), (7.34) and (7.39) – (7.42) that
\[
\|(u, u_t, q)\|_{L^\infty(I_T,H^2 \times L^2 \times H)} \lesssim_0 \|(u, u_t, f)\|_{L^\infty(I_T,L^2)} \lesssim_0 \sqrt{\mathcal{B}_1(u_0, f)}
\]

(7.43)

and
\[
\|(u, u_t, q)\|_{L^2(I_T,H^2 \times H \times H)} \lesssim_0 (1 + \|\nabla \eta^0\|_2) \|(u, u_t, f)\|_{L^2(I_T,H)} \lesssim_0 \sqrt{\mathcal{B}_1(u_0, f)}.
\]

(7.44)

Combining (7.43) with (7.44), one obtains
\[
\|(u, u_t, q)\|_{L^\infty(I_T,H^2 \times L^2 \times H)} + \|(u, u_t, q)\|_{L^2(I_T,H^3 \times H \times H)} \lesssim_0 \sqrt{\mathcal{B}_1(u_0, f)}.
\]

(7.45)

This completes the existence of local strong solutions. Moreover, a strong solution, which enjoys the regularity of \((\eta, u)\) constructed above, is obviously unique.

(2) Strong continuity of \((u_t, u, q)\) on \(T_T\) with values in \(L^2 \times H^2 \times H\).

For any given \( \varphi \in H \), let \( \psi = \varphi(\zeta(y,t)) \). Noting \( J_t = J \text{div}_A w \), we can derive from (7.38) with \( v \) in place of \( w \) that for any \( \phi \in C_0^\infty(I_T) \),
\[
- \int_0^t \phi_t \int \mathbf{j} \cdot \mathbf{j} y \, dy \, d\tau = \int_0^t \phi \int (f + \nu \Delta_A u - \kappa u - \nabla_A q) \psi J \, dy \, d\tau - \int_0^t \phi \int w \cdot \nabla_A u \cdot \psi J \, dy \, d\tau.
\]

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Let \( v = u(\zeta^{-1}(x, t), t), \omega = w(\zeta^{-1}(x, t), t) \) and \( g = f(\zeta^{-1}(x, t), t) \), then from the above identity we can get that

\[
- \int_0^t \phi_t \int v \cdot \varphi dx \, d\tau = \int_0^t \phi \int (f + \nu \Delta_A u - \bar{\kappa} u - \nabla_A q + w \cdot \nabla_A u) \cdot \psi \, J dy \, d\tau
= \int_0^t \phi \int (g + \nu \Delta v - \bar{\kappa} v - \nabla p - \bar{w} \cdot \nabla v) \cdot \varphi dx \, d\tau,
\]

which immediately results in

\[
v_t = g + \nu \Delta v - \bar{\kappa} v - \nabla p - \bar{w} \cdot \nabla v \in L^\infty(I_T, L^2)
\]

and

\[
\int v_t \cdot \varphi dx = \int (f + \nu \Delta_A u - \bar{\kappa} u - \nabla_A q - w \cdot \nabla_A u) \cdot \psi J dy.
\]

We find by (7.38) and (7.37) that

\[
u_t = (v_t + \bar{w} \cdot \nabla v)|_{x=\zeta} = v_t|_{x=\zeta} + w \cdot \nabla_A u.
\]

Now let us further assume \( \varphi \in H^1_\sigma \), then \( \text{div}_A \psi = 0 \). Recalling \( \partial_j (J A_{ij}) = 0 \), the identity (7.48) implies

\[
\frac{d}{dt} \int v_t \cdot \varphi dy = \langle f_t, \psi J \rangle_{H^{-1}, H} + \int (f(\psi J)_t - \partial_t ((\kappa u + w \cdot \nabla_A u) \cdot \psi + \nu \nabla_A u : \nabla_A \psi)) dy =: \langle \chi, \varphi \rangle_{H^{-1}, H}.
\]

Noting that \( ||\psi||_1 \lesssim \langle ||\varphi||_1 \lesssim \langle ||\psi||_1 \) for any \( t \in I_T \), and recalling the definition of \( \langle \chi, \varphi \rangle_{H^{-1}, H} \), we have \( \chi \in L^2(I_T, H^{-1}_\sigma) \). Therefore, \( v_{tt} = \chi \) and \( v_t \in C^0(I_T, L^2) \).

Consequently, the identity (7.49) implies \( u_t \in C^0(I_T, L^2) \). Since \( u \in L^2(I_T, H^3) \) and \( u_t \in L^2(I_T, H) \), \( u \in C^0(I_T, H^2) \). Hence, \( u \in U_{1,T} \). In addition, we can derive from (7.38) that \( q \in C^0(I_T, H^1) \) for sufficiently small \( \delta \). Thanks to the strong continuity of \( (u_t, u, q) \) on \( I_T \) with values in \( L^2 \times H^2 \times H \), we immediately get (7.6) from (7.45).

(3) More regularities of \( q \) under the case “\( f = \partial^2_2 \eta \)”.

Obviously, (7.41) holds for \( f = \partial^2_2 \eta \). Keeping in mind that \( p \) is in \( C^0(T_T, H) \), we obtain from (7.37) that

\[
\int \nabla p \cdot \nabla \varphi dx = \int (g - \bar{w} \cdot \nabla v) \cdot \nabla \varphi dx \quad \text{for any } t \in T_T,
\]

which implies that \( p(t) \in H^2 \) (\( t \in T_T \)) satisfies

\[
\Delta p = \text{div} g - \nabla \bar{w} : \nabla v^T \quad \text{for any } t \in T_T.
\]

Then, \( q(t) \in H^2 \) satisfies (7.8). Thus, the estimate (7.9) follows from (7.6)–(7.8).

Recalling the derivation of (3.36) with \( i = 1 \) and the fact \( ||u(\tau)||_{T_s} \leq \int_0^s ||u||_1 \, d\tau \), we see that \( q \in AC^0(T_T, H) \), i.e., \( q \) is absolutely contiguous in \( T_T \) with respect to the norm \( H \). Next, we proceed to show that \( q_t \) exists and enjoys the estimate (7.10).

By virtue of the Riesz representation theorem, it is easy to show that there is a unique function \( \chi \in L^2(I_T, H^1) \), such that

\[
- \int \nabla_A \chi \cdot \nabla_A \varsigma dy = \int (\partial_t (m^2 A^T \partial^2_2 \eta + A^T u) + A^T \nabla_A q + A^T \nabla_A q) \cdot \nabla \varsigma dy, \quad \varsigma \in H^1.
\]
Moreover, \(w\) enjoys the following estimate
\[
\|\chi\|_{L^2(I_T, H^2)} \lesssim_{\omega} \|(u, u_0, q)\|_{C^0(\overline{T}, H^2 \times L^2)} \left( 1 + \|\nabla u_0\|_0 \right) \\
+ \|\nabla(w, w_0)\|_{L^2(I_T, H^2 \times L^2)} + \|\nabla w\|_{C^0(\overline{T}, H^2)}.
\] (7.54)

Let \(t \in I_T\) and \(D_h\vartheta = (\vartheta(y, t + h) - \vartheta(y, t))/h\) where \(t + h \in I_T\). Multiplying (7.8) by \(\varsigma\) in \(L^2\) and applying then \(D_h\) to the resulting equation, we get
\[
- \int \nabla A D_h q \cdot \nabla \varsigma dq = \int \left( D_h (m^2 A^T \partial_y^2 \eta + A_i^T u) + A^T \nabla D_h A q(t + h) \right) \\
+ D_h (A^T \nabla A(y, t + h) q(y, t + h)) \cdot \nabla \varsigma dq.
\] (7.55)

Subtracting (7.53) from (7.55) and denoting \(\varsigma = D_h q - \chi\), we have
\[
\|D_h q - \chi\|_1 \lesssim_{\omega} \|(D_h - \partial_1)(m^2 A^T \partial_y^2 \eta + A_i^T u)\|_0 + \|A^T \nabla D_h A q(y, t + h)\|_0 \\
+ \|A^T \nabla A \vartheta(q(t + h) - q(t))\|_0 + \|(D_h - \partial_1) A^T \nabla A(t + h) q(y, t + h)\|_0 \\
+ \|A^T \nabla A(y, t + h) q(y, t + h) - \nabla A q\|_0 =: \Theta(t) \quad \text{for a.e. } t \in I_T.
\]

Noting that the generalized derivative with respect to \(t\) is automatically strong derivative, we easily see that \(\Theta(t) \to 0\) for a.e. \(t \in I_T\). So, \(\|D_h q - \chi\|_1^2 \to 0\) as \(h \to 0\) for a.e. \(t \in I_T\). This means that the strong derivative of \(q\) with respect to \(t\) is equal to that of \(\chi\). Because of \(q \in AC^0(\overline{T}, H^2)\), \(q_t = \chi\), where \(q_t\) denotes the generalized derivative of \(q\). Hence, \(q_t \in L^2(I_T, H^2)\) satisfies the estimate (7.10) by (7.7), (7.43) and (7.54). This completes the proof of Proposition 7.1. \(\square\)

Now, we turn to establishing the existence and uniqueness of classical solutions to the initial-value problem (7.3).

**Proposition 7.2.** Let \(\delta > 0\). Under the assumptions of Proposition 7.1 with \(f = \partial_y^2 \eta\), assume further that \((\eta^0, u^0) \in H^5 \times H^4\), \(\|\nabla \eta^0\|_4 \leq \delta\) and
\[
\sqrt{\|\nabla(w, w_0)\|_{C^0(\overline{T}, H^5 \times H^4)}^2 + \|w_H\|_{C^0(\overline{T}, L^2)}^2 + \|\nabla(w_t, w_{tt})\|_{L^2(I_T, H^2 \times L^2)}^2} \leq B_1.
\] (7.56)

Then, there is a sufficiently small constant \(\delta_2^{1, 4} \leq \delta_2^{1, 4}\), such that for any \(\delta \leq \delta_2^{1, 4}\), the solution \((u, q)\) constructed by Proposition 7.1 belongs to \(U^2 \times C^0(\overline{T}, H^4)\), where \(\delta_2^{1, 4}\) is independent of \(m\) and \(\nu\). Moreover,
\[
\|u\|_{U^2} + \|q\|_{C^0(\overline{T}, H^2)} \lesssim_{\omega} \left( 1 + \|\nabla \eta^0\|_4 \right) \left( 1 + \|u^0\|_4^6 + \|u\|_2^6 \right) \|\nabla w_0\|_0, (7.57)
\]
\[
\|q\|_{C^0(\overline{T}, H^4)} \lesssim_{\omega} \|\nabla \eta^0\|_4 + \|w\|_{C^0(\overline{T}, H^4)} \|u\|_{C^0(\overline{T}, H^4)}, (7.58)
\]
\[
\|\nabla \eta\|_4 \leq 2\delta, \quad \text{and } \|u_t\|_{L^2} \lesssim 1 + \|u^0\|_4. (7.59)
\]

**Proof.** Let \((u, q)\) be constructed in Proposition 7.1 and \(\|\nabla \eta^0\|_{4, 2} \leq \delta \leq \delta_1^{1, 4}\). Recalling
\[
T = \min\{1, (\delta/B_1)^4\},
\] (7.60)
we see that
\[
\|\nabla \eta\|_4 \leq 2\delta \quad \text{for any } t \in \overline{T}.
\] (7.61)
We remark here that the smallness of $\delta$ will be used in the derivation of some estimates later. From now on, we denote $D_t \sigma := \sigma_t - R \sigma$.

By (7.8) we see that $q^0 \in H^2$ satisfies
\[
\Delta_A q^0 = \text{div}_A \partial_2^2 \eta^0 + \text{div}_A |_{t=0} u^0.
\]
Thanks to (7.61), we can apply a standard difference quotient method to deduce that $q^0 \in H^3$. Moreover,
\[
\| \nabla q^0 \|_2 \lesssim_0 \| \partial_2^2 \eta^0 \|_2 + \| u^0 \|_3 \lesssim_0 1 + \| u^0 \|_3. \tag{7.62}
\]
Recalling $u^0_t + \nabla A v q^0 - \nu \Delta_A v u^0 + \kappa u^0 = \partial_2^2 \eta^0$, we have
\[
\| u^0_t \|_2 \lesssim_0 1 + \| \nabla q^0 \|_2 + \| u^0 \|_4 \lesssim_0 1 + \| u^0 \|_4, \tag{7.63}
\]
which gives
\[
\| D_t u \|_{t=0} \|_2 \lesssim_0 1 + \| u^0 \|_4^2. \tag{7.64}
\]
Noting that
\[
\| \nabla w \|_2 \leq \int_0^t \| \nabla w_t \|_2 d \tau + \| u^0 \|_3, \tag{7.65}
\]
we make use of (7.7), (7.56), (7.60) and (7.65) to deduce from the definition of $\mathcal{B}_1(u^0, \partial_2^2 \eta)$ that
\[
\mathcal{B}_1(u^0, \partial_2^2 \eta) \lesssim_L 1 + \| u^0 \|_2. \tag{7.66}
\]
Similarly, we can obtain by using (7.6), (7.9) and (7.10) that
\[
\| u \|_{C_T^0(T, H^2)} + \| q_t \|_{L^2(I_T, H^2)} \lesssim_L 1 + \| u^0 \|_4. \tag{7.67}
\]
Let $F = \nu (\text{div}_A \nabla_A u^T + \text{div}_A \nabla_A u^T + \Delta_A (Ru) - R \Delta_A u) + R \nabla_A q - \nabla_A q - R u$. One may use (7.56), (7.60), and (7.61), (7.64) and (7.65) to verify that
\[
\mathcal{B}_1(D_t u \|_{t=0}, D_t \partial_2^2 \eta + F) \lesssim_0 1 + \| u^0 \|_4 + \| F \|_{C_T^0(T, L^2)} + \| F \|_{L^2(I_T, H^2)} + \| F \|_{L^2(I_T, H^{-1})}^2 + \| F \|_{L^2(Q_T)}^2 \tag{7.68}
\]
Thanks to (7.10), (7.56), (7.60), (7.65) and (7.67), we have the following upper bounds for $F$ and $F_i$:
\[
\| F \|_{L^2(I_T, H^2)} \lesssim_L T \sup_{t \in I_T} (\| u_2 \|_2 (\| \nabla w \|_3 + \| \nabla w \|_2^2 + \| \nabla w \|_1) + \| \nabla w \|_2 (\| u \|_{L^2(I_T, H^3)} + \| \nabla q \|_{L^2(I_T, H^2)})) \lesssim_L 1 + \| u^0 \|_4^4
\]
and
\[
\| F_i \|_{L^2(I_T, H^{-1})} \lesssim_L T \sup_{t \in I_T} \left( (\| \nabla w \|_3 + \| \nabla w \|_2^2 + \| \nabla w_i \|_1 + \| \nabla w \|_2 (\| \nabla w \|_1) (\| q \|_2 + \| u_t \|_0 + \| \nabla q \|_1) \right)
\]
\[
+ \sup_{t \in I_T} (\| \nabla w \|_2 (\| u_i \|_{L^2(I_T, H)} + \| \nabla q_i \|_{L^2(I_T, L^2)}) + \| \nabla w \|_2 (\| u_v \|_0, \| u_2 \|_2 + \| \nabla w_i \|_1 + \| \nabla w \|_2 (\| u_2 + \| u_t \|_0 + \| \nabla q \|_1) \right) dt
\]
\[
\lesssim_L 1 + \| u^0 \|_4^5.
\]
Moreover,
\[ \| F \|_{C^0(T,L^2)} \lesssim_0 \| F \|_{L^2(I,T,H)} + \| F \|_{L^2(I,T,H^{-1})} \lesssim_0 1 + \| u^0 \|_4^5. \]

Substitution of the above three estimates into (7.68) yields
\[ \sqrt{\mathcal{B}_1(D_tu|_{t=0}, D_t\partial^2_2 \eta + F)} \lesssim L 1 + \| u^0 \|_4^5. \] (7.69)

Now, let us consider the problem
\[
\begin{aligned}
U_t + \nabla A Q - \nu \Delta A U + \kappa U &= D_t \partial^2_2 \eta + F, \\
\text{div}_A U &= 0, \\
w|_{t=0} &= D_t u|_{t=0}.
\end{aligned}
\] (7.70)

Recalling \( \text{div} A^0(D_tu)|_{t=0} = 0 \), and using (7.64), (7.65) and (7.69), we can apply Proposition 7.1 to (7.70) to see that the initial-value problem (7.70) admits a unique strong solution \((U, Q) \in U_{1,T} \times (C^0(T,H) \cap L^2(I,T,H^2)) \) , which satisfies
\[ \| U \|_{\sigma_t} + \| Q \|_{C^0(\sigma_{t,H^2})} \lesssim L 1 + \| u^0 \|_4^5. \] (7.71)

Let \( \tilde{U} = U(\zeta^{-1}(x,t),t) \) and \( \tilde{U}(x,0) = (D_t u|_{y=\zeta^{-1}(x,t)})|_{t=0} \). Similarly to (7.46), we can obtain from (7.70) that
\[ -\int_0^T \phi \int \tilde{U} \cdot \xi dx dt + \int \phi \int (\nu \nabla \tilde{U} : \nabla \xi + \kappa \tilde{U} \cdot \xi) dx dt \]
\[ = \int_0^T \phi \int ((D_t \partial^2_2 \eta + F)|_{y=\zeta^{-1}(x,t)} - \tilde{w} \cdot \nabla \tilde{U}) \cdot \xi dx dt, \quad \phi \in C^\infty_0(I_T), \ \xi \in H^4_\sigma, \]
which implies
\[ \frac{d}{dt} \int \tilde{U} \cdot \xi dx + \int (\nu \nabla \tilde{U} : \nabla \xi + \kappa \tilde{U} \cdot \xi) dx \]
\[ = \int ((D_t \partial^2_2 \eta + F)|_{y=\zeta^{-1}(x,t)} - \tilde{w} \cdot \nabla v) \cdot \xi dx \quad \text{for a.e. } t \in I_T. \] (7.72)

Since \( u \) solves (7.3) with \( f = \partial^2_2 \eta \), we utilize the regularity of \( u \) to find that
\[ \frac{d}{dt} \int \tilde{Du}_t \cdot \xi dx + \int (\nu \nabla \tilde{Du}_t : \nabla \xi + \kappa \tilde{Du}_t \cdot \xi) dx \]
\[ = \int ((D_t \partial^2_2 \eta + F)|_{y=\zeta^{-1}(x,t)} - \tilde{w} \cdot \nabla v) \cdot \xi dx \quad \text{for a.e. } t \in I_T, \] (7.73)

where \( \tilde{Du}_t := Du_t|_{x=\zeta^{-1}(y,t)} \) and \( \tilde{Du}_t|_{t=0} = \tilde{U}(x,0) \). Hence, from (7.72) and (7.73) it follows that \( \tilde{U} = \tilde{Du}_t \), which gives \( U = D_t u \). Thus, in view of (7.6), (7.66) and (7.71), we have
\[ \| u_t \|_{\sigma_t} \lesssim L 1 + \| u^0 \|_4^4 + \| u^0 \|_2^4 \| \nabla w_t|_{t=0} \|_0. \] (7.74)

Taking into account the regularity of \( u \), we get from (7.3) that
\[ (D_t u)_t + \nabla A q_t - \nu \Delta A D_t u + \kappa D_t u = D_t \partial^2_2 \eta + F. \] (7.75)
Thus, \( Q = q_t \) by virtue of (7.70) and (7.75).

Similarly to the derivation of (7.60), we can obtain

\[
\|u\|_{C^0(T, H^1)} + \|u\|_{L^2(I_T, H^2)} \\
\lesssim L \left( 1 + \|\nabla \eta^0\|_4 \right) (1 + \|u^0\|_4^6 + \|u^0\|_4^6 \|\nabla \omega_t|_{t=0}\|_0),
\]

which, together with (7.71) with \( q_t \) in place of \( Q \) and (7.74), yields (7.57). Employing the same arguments as in the proof of (7.59), one gets (7.58). Finally, the two estimates in (7.59) are obvious to get by using (7.61) and (7.63). This completes the proof of Proposition 7.2. \( \square \)

7.2. Proof of Proposition 3.1

Now we are in a position to show Proposition 3.1. To start with, let \((\eta^0, u^0)\) satisfy all the assumptions in Proposition 3.1 and \(\|\nabla \eta^0\|_{2,2} \leq \delta \leq \delta_1^L \). We should remark here that the smallness of \( \delta \) (independent of \( m \) and \( \nu \)) will be frequently used in the calculations that follow.

Denote

\[
B_1 := 2c^L(1 + B^3),
\]

where \( B \) comes from Proposition 3.1 and the constant \( c^L \) is the same as in (7.11). By Proposition 7.1 with \( B_1 \) defined by (7.77) and Remark 7.11, one can easily construct a function sequence \( \{u^k, q^k\}_{k=1}^\infty \) defined on \( \Omega_T \) with \( T \) satisfying (7.4). Moreover,

- for \( k \geq 1 \), \((u^{k+1}, q^{k+1}) \in \mathcal{U}_{1,T} \times C^0(T, H^2) \) and

\[
\begin{cases}
\eta^k = \int_0^t u^k d\tau + \eta^0, \\
u u^{k+1} + \nabla_{\mathcal{A}^k} q^{k+1} - \nu \Delta_{\mathcal{A}^k} u^{k+1} = m^2 \partial_2^2 \eta^k, \\
\text{div}_{\mathcal{A}^k} u^{k+1} = 0
\end{cases}
\]

with initial condition \( u^{k+1}|_{t=0} = u^0 \), where \( \mathcal{A}^k \) is defined by \( \zeta^k := \eta^k + y \);

- \((u^1, q^1)\) is constructed by Proposition 7.1 with \( w = 0 \) and \( \partial_2^2 \eta^0 \) in place of \( f \);

- the solution sequence \( \{u^k, q^k\}_{k=1}^\infty \) satisfies the following uniform estimates: for all \( k \geq 1 \),

\[
1 \leq 2 \det(\nabla \eta^k + I) \leq 3, \quad \|\nabla \eta^k\|_{2,2} \leq 2\delta \quad \text{for all } t \in T_T, \\
\|u^k\|_{\mathcal{U}_{1,T}} \leq B_1 \quad \text{and } \|\nabla \eta^k\|_2 + \|q^k\|_{C^0(T, H^2)} \lesssim 1 + B_1^4.
\]

In order to take limits in (7.78) as \( k \to \infty \), we have to show that \( \{u^k, q^k\}_{k=1}^\infty \) is a Cauchy sequence. To this end, we define for \( k \geq 2 \),

\[
(\tilde{\eta}^k, \bar{u}^{k+1}, \tilde{\mathcal{A}}^k, \bar{q}^{k+1}) := (\eta^k - \eta^{k-1}, u^{k+1} - u^k, \tilde{\mathcal{A}}^k - \tilde{\mathcal{A}}^{k-1}, q^{k+1} - q^k),
\]

which satisfies

\[
\begin{cases}
\tilde{\eta}^k = \int_0^t \tilde{u}^k d\tau, \\
\Delta \bar{q}^{k+1} = \mathcal{M}_k, \\
\bar{u}^{k+1} + \nabla \bar{q}^{k+1} - \nu \Delta \bar{u}^{k+1} - m^2 \partial_2^2 \bar{\eta}^k = \mathcal{N}_k, \\
\text{div}\bar{u}^{k+1} = -(\text{div}_{\mathcal{A}^k} u^{k+1} + \text{div}_{\tilde{\mathcal{A}}^{k-1}} \bar{u}^{k+1}), \\
\bar{u}^{k+1}|_{t=0} = 0,
\end{cases}
\]

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where\[M_k := m^2 \partial_3^2 (\text{div} \bar{\eta}_k + \text{div} \bar{\eta}_k \eta_k + \text{div} \bar{\eta}_k^{-1} \eta_k^2) + \text{div} \bar{\eta}_k^{-1} u^{k+1} + \text{div} \bar{\eta}_k^{-1} u^{-k} - (\text{div} \bar{\eta}_k \nabla \bar{\eta}_k q^{k+1} + \text{div} \bar{\eta}_k^{-1} \nabla \bar{\eta}_k^{-1} q^{k+1}) + \text{div} \bar{\eta}_k^{-1} \nabla \bar{\eta}_k^{-1} q^{k+1} + \text{div} (\nabla \bar{\eta}_k q^{k+1} + \nabla \bar{\eta}_k^{-1} q^{k+1})],\]

\[N_k := \nu (\text{div} \bar{\eta}_k \nabla \bar{\eta}_k u^{k+1} + \text{div} \bar{\eta}_k^{-1} \nabla \bar{\eta}_k u^{k+1} + \text{div} \bar{\eta}_k^{-1} \nabla \bar{\eta}_k^{-1} u^{k+1} + \text{div} \nabla \bar{\eta}_k u^{k+1} + \text{div} \nabla \bar{\eta}_k^{-1} u^{k+1} - (\nabla \bar{\eta}_k q^{k+1} + \nabla \bar{\eta}_k^{-1} q^{k+1}).\]

Keeping in mind that\
\[
\begin{align*}
\|\bar{A}^k\|_2 &\lesssim_0 (1 + B^2_1) \|\nabla \bar{\eta}_k\|_2 \lesssim_0 T^{1/2} (1 + B^2_1) \|\nabla \bar{u}_k\|_{L^2(I_T,H^2)}, \\
\|\bar{A}^k\|_i &\lesssim_0 \|\nabla \bar{u}_k\|_i + B_1 \|\nabla \bar{\eta}_k\|_{2,2}, \quad i = 0, 1, \\
\int \|\bar{A}^k\| \|\nabla \bar{u}_k\| \|\Delta \bar{q}\| dy &\lesssim_0 \|A_0^k\|_1 \|\nabla \bar{u}_k\|_1 \|\Delta \bar{q}\|_0, \\
\int \|\bar{A}^k\| \|\nabla \bar{u}_k\| \|\Delta \bar{q}\| dy &\lesssim_0 \|A_1^k\|_1 \|\nabla \bar{u}_k\|_1 \|\Delta \bar{q}\|_0, \\
\|\nabla \bar{u}_k\|_0 &\lesssim T^{1/2} \|\nabla \bar{u}_k\|_{L^2(I_T,H^2)},
\end{align*}
\]

we make use of (3.4), (3.6), (7.79), (7.80) and the above five estimates to deduce from (7.81)\_2–(7.81)\_4 that
\[
\begin{align*}
\|\nabla \bar{q}^{k+1}\|_{C^0(I_T,H^2)} &\lesssim_0 T^{1/4} (1 + B_1^2) \|\nabla \bar{u}_k\|_{L^2(I_T,H^2)} + \|\nabla \bar{u}_k\|_{L^2(I_T,L^2)} + \|\nabla \bar{u}_k\|_{C^0(I_T,H^2)}, \\
\|\nabla^2 \bar{u}^{k+1}\|_{C^0(I_T,L^2)} + \|\nabla^3 \bar{u}^{k+1}\|_{L^2(I_T,L^2)} &\lesssim_0 T (1 + B_1^2) \|\nabla \bar{u}_k\|_{L^2(I_T,H^2)} + \|\nabla^3 \bar{u}^{k+1}\|_{L^2(I_T,L^2)}, \\
\|\bar{u}_k^{k+1}\|_{C^0(I_T,L^2)} + \|\bar{u}_k^{k+1}\|_{L^2(I_T,H^2)} &\lesssim_0 (1 + T^{1/4} (1 + B^2_1)) \|\nabla^2 \bar{u}^{k+1}\|_{C^0(I_T,L^2)} + \|\nabla \bar{u}_k^{k+1}\|_{C^0(I_T,L^2)} + \|\nabla^2 \bar{u}^{k+1}\|_{C^0(I_T,H^2)} + T^{1/2} (1 + B^5_1) \|\nabla \bar{u}_k\|_{L^2(I_T,H^2)}.
\end{align*}
\]

Recalling \((\bar{u}^{k+1})_{T_2} = 0\), we put (7.82)–(7.84) together to conclude that for sufficiently small \(T\) (depending possibly on \(B_1, \nu\) and \(m\)),
\[
\|\bar{u}_k^{k+1}\|_{\mathcal{U}_T} + \|\bar{q}_k^{k+1}\|_{C^0(I_T,H^2)} \leq \|\bar{u}_k\|_{\mathcal{U}_T}/2 \quad \text{for any } k \geq 1,
\]

which implies
\[
\sum_{k=2}^{\infty} (\|\bar{u}_k\|_{\mathcal{U}_T} + \|\bar{q}_k\|_{C^0(I_T,H^2)}) < \infty.
\]

Hence, \(\{u_k, q_k\}_{k=1}^{\infty}\) is a Cauchy sequence in \(\mathcal{U}_{1,T} \times C^0(I_T,H^2)\) and
\[
(\eta_k, u_k, q_k) \to (\eta, u, q) \quad \text{strongly in } C^0(I_T,H^3) \times \mathcal{U}_{1,T} \times C^0(I_T,H^2),
\]

where
\[
\eta := \int_0^t u dt + \eta_0.
\]
Remembering that (7.86) implies $\eta_t = u$, we infer from (7.78) and (7.85) that the limit $(\eta, u, q)$ is a solution to the initial-value problem (2.8)–(2.9). The uniqueness of solutions to (2.8)–(2.9) in the function class $C^0(\overline{I_T}, H^3) \times \mathcal{U}_{I_T} \times C^0(\overline{I_T}, H^2)$ is easily verified by a standard energy method, and its proof will be omitted here. The proof of Proposition 3.1 is complete.

Similar to Proposition 3.1, we can use Proposition 7.2 to establish the following existence and uniqueness of a classical solution to the problem (7.3) with an additional damping term.

**Proposition 7.3.** Let $\delta > 0$, $(\eta^0, u^0) \in H^5 \times H^4$ and $\|\nabla \eta^0\|_3 \leq \delta$. Then, there is a sufficiently small constant $\delta^0 \in (0, \frac{\delta}{2})$, independent of $m$ and $\nu$, such that for any $\delta \leq \delta^0$, the initial-value problem (7.1) admits a unique local strong solution $(\eta^r, u^r, q^r) \in C^0(\overline{I_T}, H^5) \times \mathcal{U}_{I_T} \times C^0(\overline{I_T}, H^4)$, where $T := \min\{2e^c\|(u^0, m\partial_2\eta^0, \sqrt{\nu}\nabla \eta^0)\|_4\}^{-1}, \delta(\sqrt{2e^c\|(u^0, m\partial_2\eta^0, \sqrt{\nu}\nabla \eta^0)\|_4)^{-1}\}$, and the constant $c^e$ is the same as in the definition of $T$. Moreover,

$$1 \leq 2 \det(\nabla \eta + I) \leq 3, \quad \|\nabla \eta^r\|_3 \leq 2\delta,$$  

$$\|\eta^r\|_4 \leq \|\eta^0\|_4 + t\|(u^0, m\partial_2\eta^0, \sqrt{\nu}\nabla \eta^0)\|_4 \quad \text{for any } t \in I_T,$$  

$$\|(u^r, m\partial_2 q^r)\|_{C^0(\overline{I_T}, H^5)} \leq \delta \|(u^0, m\partial_2\eta^0, \sqrt{\nu}\nabla \eta^0)\|_4,$$  

$$\|(u^r, q^r, q^c)\|_{C^0(\overline{I_T}, H^5 \times H^4)} \leq \delta I_0,$$  

where

$$I_0 := (1 + \|(u^0, m\partial_2\eta^0, \sqrt{\nu}\nabla \eta^0)\|_4)(\|(u^0, m\partial_2\eta^0, \sqrt{\nu}\nabla \eta^0)\|_4^2 + (1 + \nu + m^2)\|(u^0, m\partial_2\eta^0, \sqrt{\nu}\nabla \eta^0)\|_4^2).$$

**Proof.** We divided the proof into three steps.

1. Let $\delta \in (0, \delta^0]$, $(\eta^0, u^0) \in H^5 \times H^4$ and $\|\nabla \eta^0\|_3 \leq \delta$. Let $B^c > 0$ be an undetermined constant that satisfies $B^c \geq \|(\nabla \eta^0, u^0)\|_4$ and will be defined in (7.103). Thanks to Proposition 7.2, we can follow the same arguments as in Section 7.2 to deduce that there is a constant $\delta_{1}^0$ independent of $\nu$ and $B^c$, such that for any $\delta \leq \delta_{1}^0$,

- there are a function sequence $\{\eta^k, u^k, q^k\}_{k=1}^{\infty} \in C^0(\overline{I_{T^k}}, H^5) \times \mathcal{U}_{I_{T^k}} \times C^0(\overline{I_{T^k}}, H^4)$ and a limit function $(\eta^r, u^r, q^r)$, such that as $k \to \infty$,

$$\eta^k, u^k, q^k \to (\eta^0, u^0, q^0),$$

in $C^0(\overline{I_{T^k}}, H^5) \times \mathcal{U}_{I_{T^k}} \times C^0(\overline{I_{T^k}}, H^4) \times C^0(I_{T^k}, H^2)$,

$$\begin{cases}
\eta^k_t = u^k, \\
u u^k + \nabla_A q^k - \nu \Delta_A u^k + \kappa u^k = m^2 \partial_2^2 \eta^k, \\
\text{div } A^r u^r = 0,
\end{cases}
$$

$$\big(\eta^k, u^k\big)|_{t=0} = (\eta^0, u^0),$$

$$1 \leq 2 \det(\nabla \eta^r + I) \leq 3, \quad \sup_{t \in I_{T^r}} \|\nabla \eta^r\|_3 \leq \|\nabla \eta^0\|_3 + \delta < \delta' := 4\delta,$$

where the local existence time $T^r_{\delta} \in (0, 1]$ may depend on $B^c$, $\nu$, $m$ and $\delta$.

- the function $(\eta^r, u^r, q^r)$ is just the unique solution of (7.92), i.e., if there is another solution $(\tilde{\eta}^r, \tilde{u}^r, \tilde{q}^r)$ in $C^0(\overline{I_{T^r}}, H^5) \times \mathcal{U}_{I_{T^r}} \times C^0(\overline{I_{T^r}}, H^4)$ satisfying $0 < \inf_{(y, t) \in \mathbb{R}^2 \times I_{T^r}} \det(\nabla \tilde{\eta}^r + I)$, then $(\eta^r, u^r, q^r) = (\tilde{\eta}^r, \tilde{u}^r, \tilde{q}^r)$ by using the smallness condition $\sup_{t \in I_{T^r}} \|\nabla \eta^r\|_3 \leq 4\delta'^{0}$. 


From now on, we further take \( \delta \leq \delta_1^a / 2 \), then the definition \( T_{\min}^\nu := \min\{T_0^\nu, T_{23}^\nu\} \) makes sense.

(2) Noting that (6.8) holds for any \( \chi \in L^2 \), and

\[
\text{div} \eta'' = \partial_1 \eta''_2 \partial_2 \eta''_1 - \partial_1 \eta''_1 \partial_2 \eta''_2 + \det(\nabla \eta + I)|_{t=0} - 1,
\]

we have the inequality:

\[
\|(u'', m \partial_2 \eta'')\|_4^2 \leq_0 \|(u'', \nabla^3 \text{curl}_A v (u'', m \partial_2 \eta''))\|_0^2 + \|\eta''\|_4^2 \|(u'', m \partial_2 \eta'')\|_4^2 + \|\eta''\|_4^2 \|u''\|_5^2 + \|m \partial_2 \eta''\|_2^2.
\]

(7.94)

On the one hand, remembering that \( \sup_{t \in T_{\min}^\nu} \|\nabla \eta\|_3 \leq \delta' \) in (7.93), we can use (7.92)1, (7.92)2 and (7.94), and follow the same process (under slight modifications) as in the derivation of (6.13), to deduce that there is a constant \( \delta_2^a \) independent of \( \nu \) and \( B^\kappa \), such that for any \( \delta' \leq \delta_2^a \), the solution \( (\eta'', u'') \) satisfies

\[
\frac{d}{dt} \|(u'', m \partial_2 \eta''), \nabla^3 \text{curl}_A v (u'', m \partial_2 \eta''))\|_0^2 + \nu \|u''\|_5^2 + \kappa \|u''\|^4 / 2 \\
\leq_\kappa \|(u'', m \partial_2 \eta'')\|_4^2 + \nu \|u''\|_3^2 \|\nabla \eta''\|_4^2,
\]

where

\[
\|(u'', m \partial_2 \eta'')\|_4^2 \leq_0 \|(u'', \nabla^3 \text{curl}_A v (u'', m \partial_2 \eta''))\|_0^2 + \|m \partial_2 \eta''\|_1^2,
\]

\[
\|(u'', \nabla^3 \text{curl}_A v (u'', m \partial_2 \eta''))\|_0^2 \leq_0 \|(u'', m \partial_2 \eta'')\|_4^2.
\]

On the other hand, by (7.92)1, we find that

\[
\|\nabla \eta''(t)\|_4 \leq \|u''\|_{L^2(I, H^2)} + \|\nabla \eta^0\|_4,
\]

(7.96)

Thus, one concludes from (7.95)–(7.96) that for any \( t \in (0, T_{\min}^\nu) \),

\[
\|(u'', m \partial_2 \eta'')\|_4^2 + \nu \int_0^t \|u''\|_5^2 \, dt \leq \frac{c^\kappa}{1 - c^\kappa t} \|(u^0, m \partial_2 \eta^0, \sqrt{\nu} \nabla \eta^0)\|_4^2.
\]

(7.97)

In particular, taking \( \bar{T} = 1 / 2c^\kappa \|(u^0, m \partial_2 \eta^0, \sqrt{\nu} \nabla \eta^0)\|_4^2 \), we derive from (7.97) that for any \( t \leq \min\{\bar{T}, T_{\min}^\nu\} \),

\[
\|(u'', m \partial_2 \eta'')\|_4 + \sqrt{\nu} \|u''\|_{L^2(I,H^2)} \leq \sqrt{2c^\kappa} \|(u^0, m \partial_2 \eta^0, \sqrt{\nu} \nabla \eta^0)\|_4 =: B_1^\kappa.
\]

(7.98)

Let \( T_{\min} = \min\{\bar{T}, T_{\min}^\nu, \delta / B_1^\kappa\} \). With the help of (7.97)–(7.98), we then infer from (7.92)1 that for any \( t \leq T_{\min}^\nu \),

\[
\|\eta''\|_4 \leq \|\eta^0\|_4 + t B_1^\kappa,
\]

(7.99)

\[
\|\nabla \eta''\|_3 \leq \|\nabla \eta^0\|_3 + t B_1^\kappa \leq 2 \delta,
\]

(7.100)

\[
\|\nabla \eta''\|_4 \leq_\kappa \|\nabla \eta^0\|_4 + \|(u^0, m \partial_2 \eta^0, \sqrt{\nu} \nabla \eta^0)\|_4 \sqrt{2c^\kappa / \nu} =: B_2^\kappa.
\]

(7.101)

Furthermore, by (7.100), (7.92)2 and (7.92)3, we find that there is a constant \( \delta_3^a \) independent of any parameters, such that for any \( \delta \leq \delta_3^a \),

\[
\|(u''_i, q''_i, q_{i,0}'')\|_{C^0(T_{\min}^\nu, H^2 \times H^4 \times H^2)} \leq_\kappa I_0.
\]

(7.102)
(3) Now, if we take
\[
B^\kappa := \max\{B_1^\kappa /2, B_2^\kappa /2, \| \nabla \eta^0, u^0 \|_4 \},
\]
then we have by (7.98) and (7.101) that
\[
\| \nabla \eta, u^\prime \|_4 \leq B^\kappa \quad \text{for any } t \leq T^{\min}.
\]  

Denote \(T^{\min} = \{ \bar{T}, \delta / B_1^\kappa \} \). If \(T^{\min} \leq T^\nu\), we easily see that the conclusion in Proposition 7.3 holds for any given \(\delta \in (0, \delta^{\min}]\), where \(\delta^{\min} := \min\{\delta_1^0, \delta_2^0 / 4, \delta_3^0\}\). Next, we consider the case \(T^{\min} > T^\nu\).

For the case \(T^{\min} > T^\nu\), we can take \((\eta(T^{\min}_\nu), u(T^{\min}_\nu))\) as initial data. By (7.100), (7.104) and Step (1), we see that for any \(\delta \leq \delta^{\min}/2\), there exists a unique local solution \((\eta^\nu, u^\nu, q^\nu)\) defined on \(\mathbb{T}^2 \times (T^{\min}_\nu, 2T^{\min}_\nu]\) of the initial-value problem:
\[
\begin{cases}
\eta^\nu_t = u^\nu_t, \\
\eta^\nu_t + \nabla_A q^\nu - \nu \Delta_A u^\nu + \kappa u^\nu = m^2 \partial^2_2 \eta^\nu, \\
\text{div}_A u^\nu = 0, \\
(\eta^\nu, u^\nu)|_{t=T^\nu} = (\eta^\nu(T^\nu), u^\nu(T^\nu)).
\end{cases}
\]

Moreover, \(1 \leq 2 \det(\nabla \eta^\nu + I) \leq 3\) and \(\| \nabla \eta^\nu(t) \|_3 \leq \delta^\prime\) for any \(t \in (T^{\min}_\nu, 2T^{\min}_\nu]\). Due to the uniqueness, we can get a new local solution defined on \(\mathbb{T}^2 \times (0, 2T^{\min}_\nu]\), still denoted this new solution by \((\eta^\nu, u^\nu, q^\nu)\). By Step (2) we find that for any \(\delta \leq \delta^{\min}/2\), the new solution satisfies (7.98)–(7.100), (7.102) and (7.104) for any \(t \in (0, \min\{T^{\min}_\nu, 2T^{\min}_\nu\})\). Let \(n = [(T^{\min}_\nu - T^{\min}_\nu)/T^{\min}_\nu] + 1\), where \([\cdot]\) means the integer part. Therefore, by performing \(n\)-times extension with respect to time, we obtain the desired conclusion in Proposition 7.3.

7.3. Proof of Proposition 6.2

With the help of Proposition 7.3 we are able to prove Proposition 6.2 by the method of vanishing viscosity limit.

Let \(B^\kappa > 0\), \(\delta \in (0, \delta^a]\), \((\eta^0, u^0)\) satisfy the assumptions in Proposition 6.2 and \(\| \nabla \eta^0 \|_3 \leq \delta\), where \(\delta^a\) is the same constant as in Proposition 7.3.

Let \(\epsilon \in (0, 1)\) and \(S_\epsilon\) be a standard mollifier (or a regularizing operator, see [22, Section 1.3.4.4] for definition). It is well-known that \(S_\epsilon(\vartheta) \in C^\infty(\mathbb{T}^2)\), and \(\| S_\epsilon(\vartheta) \|_i \leq \tilde{c}_i \| \vartheta \|_i\) for \(\vartheta \in H^i\) and \(i \geq 0\), where the positive constant \(\tilde{c}_i\) depends on \(i\) only. Let \((\eta^0_\epsilon, u^0_\epsilon, \tilde{\eta}^0_\epsilon, \tilde{u}^0_\epsilon) = S_\epsilon(\eta^0, u^0, \tilde{\eta}^0, \tilde{u}^0)\).

We now fix \(\epsilon > 0\). By virtue of Proposition 7.3, there is a sufficiently small \(\nu^0\) (depending possibly on \(\| S_\epsilon(\eta^0) \|_5\), \(\| (u^0, m\partial_2^0 \eta^0) \|_4\) and \(\| \nabla \eta^0 \|_3\)), such that for any \(\nu < \nu^0\), there exists a unique solution \((\eta^\nu, u^\nu, q^\nu) \in C^0(\bar{T}_T, H^5) \times U^T_k \times C^0(\bar{T}_T, H^4)\) to the initial-value problem:
\[
\begin{cases}
\eta^\nu = \int_0^t u^\nu d\tau + \eta^0, \\
\eta^\nu_t + \nabla_A^v q^\nu + \kappa u^\nu - \nu \Delta_A u^\nu = m^2 \partial^2_2 \eta^\nu, \\
\text{div}_A u^\nu = 0, \\
(\eta^\nu, u^\nu)|_{t=0} = \eta^0_\epsilon \quad \text{in } \mathbb{T}^2,
\end{cases}
\]

where \(A^\nu = (\nabla \eta^\nu + I)^{-T}, T := \min\{(1 + 2c^\kappa \| (u^0, m\partial_2^0 \eta^0) \|_4^{-1}, \delta(1 + \sqrt{2c^\kappa}) \| (u^0, m\partial_2^0 \eta^0) \|_4^{-1})\}^{-1}\).
Moreover, the solution satisfies the uniform estimates:

\[
1 \leq 2 \det(\nabla \eta^\nu + I) \leq 3, \quad \|\nabla \eta^\nu\|_3 \lesssim_0 \delta, \tag{7.106}
\]
\[
\|\eta^\nu\|_4 \lesssim_0 \|(\eta^0, u^0, m \partial_2 \eta^0, \sqrt{\nu} \nabla \eta^0)\|_4 \quad \text{for each } t \in I_T, \tag{7.107}
\]
\[
\|(u^\nu, m \partial_2 \eta^\nu)\|_{C^1(I_T, H^2)} + \sqrt{\nu} \|u^\nu\|_{L^2(I_T, H^5)} \lesssim_0 \|(u^0, m \partial_2 \eta^0)\|_4, \tag{7.108}
\]
\[
\|(u^\nu_t, q^\nu, \eta^\nu_t)\|_{C^0(I_T, H^2 \times H^4)} \lesssim_0 (1 + m^2)(1 + \|(u^0, m \partial_2 \eta^0)\|_4)^3, \tag{7.109}
\]
\[
\nu \|\eta^\nu\|_3^2 \lesssim_0 \nu \|\eta_{\xi}^0\|_3^2 + t \nu \int_0^t \|u^\nu\|_5^2 \, dt \lesssim_0 \nu \|\eta_{\xi}^0\|_3^2 + t \|(u^0, m \partial_2 \eta^0)\|_4^2. \tag{7.110}
\]

From now on, we take \(\nu = \nu_\infty = 1/n\) with \(n \geq 1/\nu^0\), and renew to define \((\eta^\nu, u^\nu, q^\nu)\) and \(A^\nu\) by \((\eta^n, u^n, q^n)\) and \(A^n\), respectively.

Let \((\eta^L, u^L) \in C^0(\mathbb{R}^+, H^2_0) \times U^L_\infty\) be the unique global solution of the linear initial-value problem \((6.38)\). In view of Proposition \((6.11)\) we see that there is a unique strong solution \((\eta^{\varepsilon,L}, u^{\varepsilon,L})\) to \((6.38)\) with \((\tilde{\eta}^0, \tilde{u}^0)\) in place of \((\eta^0, u^0)\). Moreover,

- for \(i \geq 0\), \((\eta^{\varepsilon,L}, u^{\varepsilon,L}) \in C^0(\mathbb{R}^+, H^{i+1}_2) \times U^L_\infty\), and for any \(t \geq 0\),

\[
\|(u^{\varepsilon,L}, m \partial_2 \eta^{\varepsilon,L})\|_1^2 + \int_0^t \|u^{\varepsilon,L}\|_1^2 \, dt \lesssim_0 \|(\tilde{u}^0, m \partial_2 \eta^0)\|_1^2, \tag{7.111}
\]

- for any \(t \in I_T\),

\[
(\eta^{\varepsilon,L}, u^{\varepsilon,L}, \partial_2 \eta^{\varepsilon,L}) \rightarrow (\eta^L, u^L, \partial_2 \eta^L) \quad \text{weakly-* in } L^\infty(I_T, H^4) \quad \text{as } \varepsilon \rightarrow 0. \tag{7.112}
\]

Defining \((\eta^{n,\varepsilon,d}, u^{n,\varepsilon,d}) := (\eta^n, u^n) - (\eta^{\varepsilon,L}, u^{\varepsilon,L})\), then

\[
\begin{cases}
\eta^{n,\varepsilon,d} = u^{n,\varepsilon,d}, \\
u_t^{n,\varepsilon,d} + \nabla A^n q^n + \kappa u^{n,\varepsilon,d} = m^2 \partial_2 \eta^{n,\varepsilon,d} + \nu \Delta A^n u^\nu, \\
\text{div} u^{n,\varepsilon,d} = -\text{div}_{A^n} u^n, \\
(\eta^{n,\varepsilon,d}, u^{n,\varepsilon,d})|_{t=0} = (\eta_{\xi}^0 - \tilde{\eta}^0, u^0 - \tilde{u}^0).
\end{cases} \tag{7.113}
\]

It is easy to see from \((7.105)_3, (7.106)\) and \((7.113)\) that for any \(t \in I_T\),

\[
\|\text{curl}_{A^n}(u^{n,\varepsilon,d}, m \partial_2 \eta^{n,\varepsilon,d})\|_3 \lesssim_0 e^{t\|u^n, m \partial_2 \eta^n\|_{C^0(I_T, H^4 \times H^4)}} \left(\|(u^{n,\varepsilon,d}, m \partial_2 \eta^{n,\varepsilon,d})|_{t=0}\|_4^2 + t \sup_{\tau \in I_T} \|(u^n, m \partial_2 \eta^n, m \partial_2 \eta^{\varepsilon,L})\|_4^3\right) + \nu t \sup_{\tau \in I_T} \left(\|\eta^n\|_3^2 \|(u^{\varepsilon,L}, u^n)\|_1^2 + \|u^n\|_5^2 \|u^n\|_4^2\right) + \nu \int_0^t \|u^{\varepsilon,L}\|_5^2 \, dt,
\]

and

\[
\|\text{curl}_{A^n} \eta^{n,\varepsilon,d}\|_3 \lesssim_0 \|\text{curl}_{A^n} \eta^{n,\varepsilon,d}|_{t=0}\|_3 + t \sup_{\tau \in I_T} \|\text{curl}_{A^n} u^{n,\varepsilon,d}\|_3.
\]
If we making use of (7.108), (7.110), (7.111) and the above two estimates, we further obtain

\[
\sup_{\tau \in I_T} \| \text{curl}_{A^\alpha}(\eta^{n,\varepsilon,d}, u^{n,\varepsilon,d}, m\partial_2 \eta^{n,\varepsilon,d})(\tau) \|_{L^3}^2
\]

\[
\lesssim_{\varepsilon} e^{2 \| (\eta^0, u^0, m\partial_2 \eta^0) \|_{L^4}} (\| (\eta^0 - \tilde{\eta}^0, u^0 - \tilde{u}^0, m\partial_2 (\eta^0 - \tilde{\eta}^0)) \|_{L^4}^2
\]

\[
+ t(\| (u^0, m\partial_2 \eta^0, \tilde{u}^0, m\partial_2 \tilde{\eta}^0) \|_{L^4}^3 + t(\| (u^0, m\partial_2 \eta^0) \|_{L^4}^2
\]

\[
+ \nu \| \eta^0 \|_4 ||(\tilde{u}^0, m\partial_2 \tilde{\eta}^0) \|_{L^4}^3 + \| (u^0, m\partial_2 \eta^0) \|_{L^4}^2 + \nu || (u^0, m\partial_2 \eta^0) \|_{L^4}^2). \tag{7.114}
\]

Thanks to the uniform estimates (7.106)–(7.109) and (7.114), we can choose a sequence of \( \{\eta^n, u^n, q^n\} \) (still labelled by \( \{\eta^n, u^n, q^n\} \) for the sake of simplicity), such that as \( n \to \infty \),

\[
(\eta^n, \partial_2 \eta^n, u^n, q^n) \to (\eta^\varepsilon, \partial_2 \eta^\varepsilon, u^\varepsilon, q^\varepsilon) \text{ strongly in } C^0(I_T, H^3 \times H^3 \times H^3 \times H^3),
\]

\[
(\eta^n, u^n, t^n, q^n, q^n) \to (\eta^\varepsilon, u^\varepsilon, t^\varepsilon, q^\varepsilon, q^\varepsilon) \text{ weakly-* in } L^\infty(I_T, H^2 \times H^4 \times H^2 \times H^2),
\]

\[
(\text{curl}_{A^\alpha} \eta^{n,\varepsilon,d}, \text{curl}_{A^\alpha} u^{n,\varepsilon,d}, \text{curl}_{A^\alpha} \partial_2 \eta^{n,\varepsilon,d}) \to (\text{curl}_{A^\alpha} \eta^\varepsilon,d, \text{curl}_{A^\alpha} u^\varepsilon,d, \text{curl}_{A^\alpha} \partial_2 \eta^\varepsilon,d)
\]

weakly-* in \( L^\infty(I_T, H^3) \) for any \( t \in (0, T] \),

where \( (\eta^\varepsilon,d, u^\varepsilon,d) := (\eta^\varepsilon - \eta^\varepsilon,L, u^\varepsilon - u^\varepsilon,L) \). Moreover, the limit functions \( \eta^\varepsilon, u^\varepsilon \) and \( q^\varepsilon \) solve the initial-value problem

\[
\begin{aligned}
\eta^\varepsilon &= u^\varepsilon, \\
u^\varepsilon + \nabla A q^\varepsilon + \kappa u^\varepsilon &= m^2 \partial_2^2 \eta^\varepsilon, \\
\text{div}_A u^\varepsilon &= 0,
\end{aligned}
\]

(7.115)

and satisfies

\[
1 \leq 2 \det(\nabla \eta^\varepsilon + I) \leq 3, \quad \| \nabla \eta^\varepsilon \|_{L^\infty(I_T, H^3)} \leq 2\delta, \tag{7.116}
\]

\[
\text{ess sup}_{t \in I_T} \| (\eta^\varepsilon, u^\varepsilon, m\partial_2 \eta^\varepsilon) \|_{L^4} \lesssim_{\varepsilon} (\| (\eta^0, u^0, m\partial_2 \eta^0) \|_{L^4})^2, \tag{7.117}
\]

\[
\| (u^\varepsilon, q^\varepsilon, q^\varepsilon) \|_{L^\infty(I_T, H^2 \times H^4 \times H^2)} \lesssim_{\varepsilon} (1 + m^2) (1 + \| (\eta^0, u^0, m\partial_2 \eta^0) \|_{L^4})^3, \tag{7.118}
\]

\[
\text{ess sup}_{\tau \in I_T} \| (\text{curl}_{A^\alpha} \eta^\varepsilon,d, \text{curl}_{A^\alpha} u^\varepsilon,d, \text{curl}_{A^\alpha} m \partial_2 \eta^\varepsilon,d)(\tau) \|_{L^3}^2 \lesssim_{\varepsilon} e^{2\| (\eta^0,u^0,m\partial_2 \eta^0) \|_{L^4}} \| (\eta^0 - \tilde{\eta}^0, u^0 - \tilde{u}^0, m\partial_2 (\eta^0 - \tilde{\eta}^0)) \|_{L^4}^2
\]

\[
+ t(\| (u^0, m\partial_2 \eta^0, \tilde{u}^0, m\partial_2 \tilde{\eta}^0) \|_{L^4}^3 + \| (u^0, m\partial_2 \eta^0) \|_{L^4}^2). \tag{7.119}
\]

In addition, by \( u^\varepsilon \in L^\infty(I_T, H^1) \), one has

\[
\eta^\varepsilon \in C^0(I_T, H^3). \tag{7.120}
\]

Let \( \alpha \) and \( \beta \) satisfy \( |\alpha| = 4 \) and \( |\beta| = 3 \). With the help of the uniform estimates (7.106)–(7.109), we can use (7.105)1 and (7.105)2 to deduce that the sequences \( \{\partial^\alpha u^n\}, \{\partial^\beta (\text{curl}_{A^\alpha} u^n)\}\), \( \{\partial^\alpha \partial_2 \eta^n\} \) and \( \{\partial^\alpha \text{curl}_{A^\alpha} \partial_2 \eta^n\} \) are uniformly continuous in \( H^{-1} \) on \( \bar{T} \). Besides, they are also uniformly bounded in \( L^2 \). Therefore, there is a sequence of \( \{\eta^n, u^n, q^n\} \) (still denoted by \( \{\eta^n, u^n, q^n\} \) ), such that (see [22] Lemma 6.2 for example),

\[
\partial^\alpha (u^n, \partial_2 \eta^n) \to \partial^\alpha (u^\varepsilon, \partial_2 \eta^\varepsilon) \text{ in } C^0(I_T, L^2_{\text{weak}}).
\]
where we have relabelled $\partial^\alpha (u^\varepsilon, \partial_2 \eta^\varepsilon)$ on a set of zero-measure in $T$. Thus, by (6.45), (7.120) and the above result of weak continuity, the notation “ess” can be removed in (7.117) and (7.119).

Thanks to the uniform estimates (7.116)–(7.118) and the limit behavior (7.112), we can again take to the limit as $\varepsilon \to 0$ by employ the same arguments as used in obtaining $(\eta^\varepsilon, u^\varepsilon, q^\varepsilon)$, and thus obtain a limit function $(\eta, u, q)$ which is a solution of the initial-value problem (2.47)–(2.48) and satisfies the estimates (7.118) with $(\eta, u, q)$ in place of $(\eta^\varepsilon, u^\varepsilon, q^\varepsilon)$, (6.40)–(6.42), and the same regularity as that of $(\eta^\varepsilon, u^\varepsilon, q^\varepsilon)$. Moreover, the obtained solution $(\eta, u, q)$ is unique, provided that $\delta$ is sufficiently small. To complete the proof of Proposition 7.3 obviously, it suffices to show the

 Thanks to the above two identities, we further get that for a.e. $t$ in $I_T$,

$$\frac{1}{2} \int_0^T \partial_t S_\varepsilon(\partial^\beta \text{curl}_A u) S_\varepsilon(\partial^\beta \text{curl}_A u) dy - \int \partial_2 S_\varepsilon(\partial^\beta \text{curl}_A \partial_2 \eta) S_\varepsilon(\partial^\beta \text{curl}_A u) dy = \frac{1}{2} \int \frac{d}{dt} \int |S_\varepsilon(\partial^\beta \text{curl}_A u)|^2 dy
$$

and

$$\partial_t S_\varepsilon(\partial^\beta \text{curl}_A \partial_2 \eta) = \partial_2 S_\varepsilon(\partial^\beta \text{curl}_A u) + S_\varepsilon(\partial^\beta \text{curl}_A \partial_2 \eta) - S_\varepsilon(\partial^\beta \text{curl}_A \partial_2 \eta).$$

Thanks to the above two identities, we further get that for a.e. $t \in I_T$,

$$\int_0^T \partial_t S_\varepsilon(\partial^\beta \text{curl}_A u) S_\varepsilon(\partial^\beta \text{curl}_A u) dy = \frac{1}{2} \int \partial_t S_\varepsilon(\partial^\beta \text{curl}_A u) S_\varepsilon(\partial^\beta \text{curl}_A u) dy$$

where $\|S_\varepsilon(\partial^\beta \text{curl}_A u)\|_0^2$ and $\|S_\varepsilon(\partial^\beta \text{curl}_A \partial_2 \eta)\|_0^2 \in AC^0(I_T \setminus \tilde{Z})$ for some zero-measurable set $\tilde{Z}$.

For $\phi \in C_0^\infty (I_T)$, we multiply (7.121) by $S_\varepsilon(\partial^\beta \text{curl}_A u) \phi$ in $L^2(\Omega_T)$, and take then to the limits as $\varepsilon \to 0$ to obtain

$$-\frac{1}{2} \int_0^T \int (|\partial^\beta \text{curl}_A u|^2 + m^2 \partial^\beta \text{curl}_A \partial_2 \eta^2) dy \phi \tau d\tau + \kappa \int_0^T \int |\partial^\beta \text{curl}_A u|^2 dy \phi d\tau$$

$$= \int_0^T \int (\partial^\beta \text{curl}_A u \partial^\beta \text{curl}_A u dy + m^2 (\partial^\beta \text{curl}_A \partial_2 \eta (\partial^\beta \text{curl}_A \partial_2 \eta - \partial_2 \eta \partial^\beta \text{curl}_A \partial_2 \eta)

- \partial^\beta \text{curl}_A \partial_2 \eta \partial^\beta \text{curl}_A u) dy \phi d\tau,$$

which implies that for any $\beta$ satisfying $|\beta| = 3$,

$$\frac{1}{2} \int_0^T \int (|\partial^\beta \text{curl}_A u|^2 + m^2 \partial^\beta \text{curl}_A \partial_2 \eta^2) dy \phi \tau d\tau + \kappa \int_0^T \int |\partial^\beta \text{curl}_A u|^2 dy \phi d\tau$$

$$= \int (\partial^\beta \text{curl}_A u \partial^\beta \text{curl}_A u dy + m^2 (\partial^\beta \text{curl}_A \partial_2 \eta \partial^\beta \text{curl}_A \partial_2 \eta

- \partial^\beta \text{curl}_A \partial_2 \eta \partial^\beta \text{curl}_A u - \partial^\beta \text{curl}_A \partial_2 \eta \partial^\beta \text{curl}_A u) dy \phi d\tau,$$

whence,

$$\|\partial^\beta (\text{curl}_A u, m\text{curl}_A \partial_2 \eta)|_0^2 \in AC^0(I_T \setminus Z) \text{ for some zero-measurable set } Z. \quad (7.122)$$
By (7.122) and the fact “$\partial^3 (\text{curl}_A u, \text{curl}_A \partial_2 \eta) \in C^0(\overline{T_T}, L^2_{\text{weak}})$”, we immediately get

$$\partial^\alpha (\text{curl}_A u, \text{curl}_A \partial_2 \eta) \in C(I_T \setminus Z, L^2).$$  \hfill (7.123)

Since $\partial^\alpha (u, \partial_2 \eta) \in C^0(\overline{T_T}, L^2_{\text{weak}})$, one has by (6.45) that

$$\sup_{t \in \overline{T_T}} \|\partial^\alpha (u, \partial_2 \eta)\|_0 = \text{ess sup} \|\partial^\alpha (u, \partial_\alpha \partial_2 \eta)\|_0.$$

Keeping in mind that for any $t$ and $t_0 \in \overline{T_T}$,

$$\|\partial^\beta \nabla (u(t) - u(t_0))\|_0 = \sqrt{\|\partial^\beta \text{curl}(u(t) - u(t_0))\|_0^2 + \|\partial^\beta \text{div}(u(t) - u(t_0))\|_0^2}$$

$$\lesssim_{t_0} \|\partial^\beta (\text{curl}_A u(t) - \text{curl}_A u(t_0))\|_0 + \|\partial^\beta (\text{curl}_A \partial_2 \eta(t) - \text{curl}_A \partial_2 \eta(t_0))\|_0$$

$$+ \|\partial^\beta (\text{div}_A u(t) - \text{div}_A u(t_0))\|_0 + \|\partial^\beta (\text{div}_A \partial_2 \eta(t) - \text{div}_A \partial_2 \eta(t_0))\|_0$$

and

$$\|\partial^\beta \nabla \partial_2 (\eta(t) - \eta(t_0))\|_0^2$$

$$= \|\partial^\beta \text{curl}\partial_2 (\eta(t) - \eta(t_0))\|_0^2 + \|\partial^\beta \text{div}\partial_2 (\eta(t) - \eta(t_0))\|_0^2$$

$$\lesssim_{t_0} \|\partial^\beta (\text{curl}_A \partial_2 \eta(t) - \text{curl}_A \partial_2 \eta(t_0))\|_0 + \|\partial^\beta (\text{curl}_A \partial_2 \eta(t) - \text{curl}_A \partial_2 \eta(t_0))\|_0$$

$$+ \|\partial^\beta \partial_2 ((\partial_1 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_2 \partial_2 \eta_2) (t) - (\partial_1 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_2 \partial_2 \eta_2) (t_0))\|_0,$$

we employ the regularity $(\eta, u) \in C^0(\overline{T_T}, H^1 \times H^2)$ to arrive at $\partial^\alpha (u, \partial_2 \eta) \in C(I_T \setminus Z, L^2)$. Consequently, $\partial^\alpha q \in C(I_T \setminus Z, L^2)$. The proof of Proposition 7.3 is complete.

Acknowledgements. The research of Fei Jiang was supported by NSFC (Grant Nos. 11671086 and 12022102) and the Natural Science Foundation of Fujian Province of China (2020J02013), and the research of Song Jiang by National Key R&D Program (2020YFA0712200), National Key Project (GJXM92579), and NSFC (Grant No. 11631008), the Sino-German Science Center (Grant No. GZ 1465) and the ISF-NSFC joint research program (Grant No. 11761141008).

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