Continuous solutions to two iterative functional equations

**Karol Baron**

*Dedicated to Professor Ludwig Reich on his 80th birthday.*

**Abstract.** Based on iteration of random-valued functions we study the problem of solvability in the class of continuous and Hölder continuous functions $\varphi$ of the equations

$$\varphi(x) = F(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),$$

$$\varphi(x) = F(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),$$

where $P$ is a probability measure on a $\sigma$-algebra of subsets of $\Omega$.

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**1. Introduction**

Fix a probability space $(\Omega, \mathcal{A}, P)$, a complete and separable metric space $(X, \rho)$ with the $\sigma$-algebra $\mathcal{B}$ of all its Borel subsets, and a $\mathcal{B} \otimes \mathcal{A}$-measurable function $f : X \times \Omega \to X$.

We continue the research of continuous solutions $\varphi : X \to \mathbb{R}$ of the equations

$$\varphi(x) = F(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega), \tag{1}$$

$$\varphi(x) = F(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega). \tag{2}$$
We refer mainly to [2,5]. Like in these papers we focus on the iteration of random-valued functions:

\[ f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n) \]

for \( n \in \mathbb{N} \), \( x \in X \) and \( (\omega_1, \omega_2, \ldots) \) from \( \Omega^\infty \) defined as \( \Omega^\infty \). Note that for \( n \in \mathbb{N} \) the \( n \)th iterate \( f^n \) mapping \( X \times \Omega^\infty \) into \( X \) is \( B \otimes A_n \)-measurable, where \( A_n \) denotes the \( \sigma \)-algebra of all sets of the form

\[ \{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A \} \]

with \( A \) from the product \( \sigma \)-algebra \( A \). (See [13, section 1.4], [10].)

Let \( \pi^f_n(x, \cdot) \) denote the distribution of \( f^n(x, \cdot) \), i.e.,

\[ \pi^f_n(x, B) = P_{\infty}(f^n(x, \cdot) \in B) \quad \text{for} \quad n \in \mathbb{N} \cup \{0\}, \quad x \in X \text{ and } B \in B. \]

If

\[ \int_{\Omega} \rho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \rho(x, z) \quad \text{for } x, z \in X \]

with a \( \lambda \in (0,1) \), and

\[ \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X, \]

then (see [2, Theorem 3.1]) there exists a probability Borel measure \( \pi^f \) on \( X \) such that for every \( x \in X \) the sequence \( (\pi^f_n(x, \cdot))_{n \in \mathbb{N}} \) converges weakly to \( \pi^f \), and (see [11, Corollary 5.6 and Lemma 3.1], also [3, Lemma 2.2]) for every non-expansive \( u : X \to \mathbb{R} \) the inequality

\[ \left| \int_X u(z) \pi^f_n(x, dz) - \int_X u(z) \pi^f(dz) \right| \leq \frac{\lambda^n}{1-\lambda} \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \]

holds for \( x \in X \) and \( n \in \mathbb{N} \).

This limit distribution \( \pi^f \) plays an important role in solving (1) and (2), see [5, Theorem 3.1], [2, Corollary 4.1], [3, Theorem 2.1]. In particular:

(I) If \( F : X \to \mathbb{R} \) is continuous and bounded, then any continuous and bounded solution \( \varphi : X \to \mathbb{R} \) of (1) has the form

\[ \varphi(x) = F(x) - \frac{1}{2} \int_X F(z) \pi^f(dz) \]

\[ + \sum_{n=1}^{\infty} (-1)^n \left( \int_X F(z) \pi^f_n(x, dz) - \int_X F(z) \pi^f(dz) \right) \quad \text{for } x \in X; \]

if additionally \( F \) is Lipschitz, then (6) defines a Lipschitz solution \( \varphi : X \to \mathbb{R} \) of (1).

(II) If \( F : X \to \mathbb{R} \) is continuous and bounded and (2) has a continuous and bounded solution \( \varphi : X \to \mathbb{R} \), then

\[ \int_X F(x) \pi^f(dz) = 0, \]
and any such solution has the form
\[
\varphi(x) = c + F(x) + \sum_{n=1}^{\infty} \int_X F(z) \pi_n^f(x, dz) \quad \text{for } x \in X
\]
with a real constant \(c\).

(III) If \(F : X \to \mathbb{R}\) is Lipschitz, then it is integrable for \(\pi^f\) and (2) has a Lipschitz solution \(\varphi : X \to \mathbb{R}\) if and only if (7) holds.

The limit distribution \(\pi^f\) and facts cited above will be used in the main part of the paper. A characterization of this limit for some special random-valued functions in Hilbert spaces have been given by [3, Theorem 3.1] and, in Banach spaces, by [4, Theorem 2.1].

Actually we do not have a sufficiently satisfactory theorem to guarantee the existence of continuous solutions to the equations considered. An explanation of this situation is given in the paper [9] by Witold Jarczyk (see also [13, Note 3.8.4]). Namely, in the case where \(\Omega\) is a singleton and \(X\) is a compact real interval, for the appropriate \(f\) the set of continuous \(F : X \to \mathbb{R}\) such that the equation has a continuous solution is small in the sense of Baire category. It is also small from the measure point of view (see [1]). We will go also in this direction but, above all, we are looking for conditions under which Eqs. (1) and (2) have continuous and Hölder continuous solutions \(\varphi : X \to \mathbb{R}\). In the case where \(\Omega\) is a singleton, see [12, Chapter II, §7].

2. Results

We will consider Eqs. (1) and (2) assuming the following hypothesis (H).

(H) \((\Omega, \mathcal{A}, P)\) is a probability space, \((X, \rho)\) is a complete and separable metric space, \(f : X \times \Omega \to X\) is \(\mathcal{B} \otimes \mathcal{A}\)-measurable, (3) holds with a \(\lambda \in (0, 1)\) and (4) is satisfied.

We regard \(\lambda\) as fixed in \((0, 1)\), and for any metric space \(X\) we define \(\mathcal{F}(X)\) as the set of all continuous functions \(F : X \to \mathbb{R}\) such that there are a sequence \(\{F_n\}_{n \in \mathbb{N}}\) of real functions on \(X\) and constants \(\vartheta \in (0, 1), L \in (0, \frac{1}{\lambda})\) and \(\alpha, \beta \in (0, \infty)\) such that
\[
|F(x) - F_n(x)| \leq \alpha \vartheta^n \quad \text{for } x \in X, \ n \in \mathbb{N},
\]
and
\[
|F_n(x) - F_n(z)| \leq \beta L^n \rho(x, z) \quad \text{for } x, z \in X, \ n \in \mathbb{N}.
\]
Clearly any real Lipschitz function defined on \(X\) belongs to \(\mathcal{F}(X)\).

**Theorem 2.1.** Assume (H). If \(F \in \mathcal{F}(X)\), then formula (6) defines a continuous solution \(\varphi : X \to \mathbb{R}\) of (1), and if additionally (7) holds, then the formula
\[
\varphi_0(x) = F(x) + \sum_{n=1}^{\infty} \int_X F(z) \pi_n^f(x, dz) \quad \text{for } x \in X
\](8)
defines a continuous solution \( \varphi_0 : X \to \mathbb{R} \) of (2).

The proof will be based on three lemmas. In each of them we assume \((H)\).

**Lemma 2.2.** If \( F \in \mathcal{F}(X) \), then the integrals

\[
\int_{\Omega} |F(f(x, \omega))| P(d\omega) \quad \text{for} \ x \in X, \quad \int_{X} |F(z)| \pi^f(dz)
\]

are finite, and the function

\[
x \mapsto \int_{\Omega} F(f(x, \omega)) P(d\omega), \quad x \in X,
\]

is continuous.

**Proof.** Corresponding to \( F \) choose a sequence \((F_n)_{n \in \mathbb{N}}\) of real functions on \( X \) and constants \( \vartheta \in (0, 1) \), \( L \in (0, \frac{1}{X}) \) and \( \alpha, \beta \in (0, \infty) \) as in the definition of \( \mathcal{F}(X) \). Then

\[
\int_{\Omega} |F(f(x, \omega))| P(d\omega) \leq \alpha \vartheta + \beta L \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) + |F_1(x)|
\]

for \( x \in X \), and

\[
\int_{X} |F(z)| \pi^f(dz) \leq \alpha \vartheta + \int_{X} |F_1(z)| \pi^f(dz),
\]

see also (III). Moreover, for every \( n \in \mathbb{N} \) the function

\[
x \mapsto \int_{\Omega} F_n(f(x, \omega)) P(d\omega), \quad x \in X,
\]

is Lipschitz:

\[
\left| \int_{\Omega} F_n(f(x, \omega)) P(d\omega) - \int_{\Omega} F_n(f(z, \omega)) P(d\omega) \right| \leq \beta L^n \lambda \rho(x, z)
\]

for \( x, z \in X \), and therefore function (9), as their uniform limit, is continuous. \( \square \)

**Lemma 2.3.** If \( F \in \mathcal{F}(X) \), then

\[
\int_{X} |F(z)| \pi^f_n(x, dz) < \infty \quad \text{for} \ x \in X \text{ and } n \in \mathbb{N},
\]

and for every \( n \in \mathbb{N} \) the function

\[
x \mapsto \int_{X} F(z) \pi^f_n(x, dz), \quad x \in X,
\]

is continuous.
Proof. By induction, (3) and (4),
\[
\int_{\Omega} \rho(f^n(x,\omega), f^n(z,\omega)) P^{\infty}(d\omega) \leq \lambda^n \rho(x,z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N}
\]
and
\[
\int_{\Omega} \rho(f^n(x,\omega), x) P^{\infty}(d\omega) < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.
\]

Since
\[
\int_X F(z)\pi^n f(x, dz) = \int_{\Omega} F(f^n(x,\omega)) P^{\infty}(d\omega) \quad \text{for } x \in X \text{ and } n \in \mathbb{N},
\]
an application of Lemma 2.2 with \( f \) replaced by \( f^n, n \in \mathbb{N} \), finishes the proof. \( \square \)

Lemma 2.4. If \( F \in \mathcal{F}(X) \), then there are constants \( \theta \in (0, 1) \) and \( M \in (0, \infty) \) such that
\[
\left| \int_X F(z)\pi^n f(x, dz) - \int_X F(z)\pi^n f(dz) \right| \leq M\theta^n (1 + \rho(x,x_0) + \int_{\Omega} \rho(f(x_0,\omega),x_0) P(d\omega))
\]
for \( x, x_0 \in X \) and \( n \in \mathbb{N} \).

Proof. Corresponding to \( F \) choose a sequence \( (F_n)_{n \in \mathbb{N}} \) of real functions on \( X \) and constants \( \vartheta \in (0, 1), L \in (0, \frac{1}{\lambda}) \) and \( \alpha, \beta \in (0, \infty) \) as in the definition of \( \mathcal{F}(X) \), and put
\[
\theta = \max\{\vartheta, \lambda L\}, \quad M = 2\max\left\{ \alpha, \frac{\beta}{1 - \lambda} \right\}.
\]

Then \( \theta \in (0, 1) \), and by Lemmas 2.3 and 2.2, (5) with \( u = \frac{F_n}{\beta L^n} \) and (3) for every \( x, x_0 \in X \) and \( n \in \mathbb{N} \) we have
\[
\left| \int_X F(z) \pi_n^f(x, dz) - \int_X F(z) \pi^f(dz) \right| \\
\leq \left| \int_X F(z) \pi_n^f(x, dz) - \int_X F_n(z) \pi_n^f(x, dz) \right| \\
+ \left| \int_X F_n(z) \pi_n^f(x, dz) - \int_X F_n(z) \pi^f(dz) \right| \\
+ \left| \int_X F_n(z) \pi^f(dz) - \int_X F(z) \pi^f(dz) \right| \\
\leq \int_X |F(z) - F_n(z)| \pi_n^f(x, dz) + \beta L^n \frac{\lambda^n}{1 - \lambda} \int_\Omega \rho(f(x, \omega), x) P(d\omega) \\
+ \int_X \rho(F_n(z) - F(z)| \pi^f(dz) \leq 2\alpha \theta^n + \beta L^n \frac{\lambda^n}{1 - \lambda} \int_\Omega \rho(f(x, \omega), x) P(d\omega) \\
\leq 2\alpha \theta^n + \frac{\beta}{1 - \lambda} \theta^n (\lambda \rho(x, x_0) + \int_\Omega \rho(f(x_0, \omega), x_0) P(d\omega) + \rho(x, x_0)) \\
\leq M \theta^n (1 + \rho(x, x_0) + \int_\Omega \rho(f(x_0, \omega), x_0) P(d\omega)).
\]

\[\square\]

**Proof of Theorem 2.1.** It follows from Lemmas 2.2–2.4 that formula (6) defines a continuous function \( \varphi : X \to \mathbb{R} \) and arguing like in the proof of Theorem 3.1(ii) of [5] (see also the calculations below) we show that it solves (1).

Assume now that also (7) holds. Then it follows from Lemmas 2.3 and 2.4 that formula (8) defines a continuous function \( \varphi_0 : X \to \mathbb{R} \). Applying (11), Lemma 2.4, the Lebesgue dominated convergence theorem and the Fubini theorem we observe that for every \( x \in X \) the function \( \varphi_0 \circ f(x, \cdot) \) is integrable for \( P \) and

\[
\int_\Omega \varphi_0(f(x, \omega)) P(d\omega) = \int_\Omega F(f(x, \omega)) P(d\omega) \\
+ \int_\Omega \sum_{n=1}^{\infty} \left( \int_X F(z) \pi_n^f(f(x, \omega), dz) \right) P(d\omega) = \int_X F(z) \pi_1^f(x, dz) \\
+ \sum_{n=1}^{\infty} \int_\Omega \left( \int_{\Omega^n} F(f^n(f(x, \omega_1), \omega_2, \omega_3, \ldots)) P^{\infty}(d(\omega_2, \omega_3, \ldots)) \right) P(d\omega_1) \\
= \int_X F(z) \pi_1^f(x, dz) + \sum_{n=1}^{\infty} \int_{\Omega^{\infty}} F(f^{n+1}(x, \omega_1, \omega_2, \ldots)) P^{\infty}(d(\omega_1, \omega_2, \ldots)) \\
= \int_X F(z) \pi_1^f(x, dz) + \sum_{n=1}^{\infty} \int_X F(z) \pi_{n+1}^f(x, dz) = \varphi_0(x) - F(x).
\]

\[\square\]
Proposition 2.5. If \( F \) is a real function on a metric space \( X \) and
\[
|F(x) - F(z)| \leq \beta \rho(x, z)^{\alpha} \quad \text{for} \quad x, z \in X
\]
with some constants \( \alpha \in (0, 1) \), \( \beta \in [0, \infty) \), then \( F \in \mathcal{F}(X) \).

Proof. Fix \( L \in (1, \frac{1}{\lambda}) \), put
\[
\vartheta = L^{- \frac{\alpha}{1 - \alpha}} \quad \text{and} \quad \theta = \vartheta^\frac{1}{\alpha},
\]
and for every \( n \in \mathbb{N} \) let \( A_n \) be a maximal for inclusion subset of \( X \) such that
\[
\rho(x, z) \geq \theta^n \quad \text{for every pair of distinct points} \quad x, z \ \text{of} \ A_n.
\]
By the maximality,
\[
X = \bigcup_{z \in A_n} \{ x \in X : \rho(x, z) < \theta^n \} \quad \text{for} \quad n \in \mathbb{N}.
\]
If \( n \in \mathbb{N} \) and \( x, z \) are distinct points of \( A_n \), then by (12),
\[
|F(x) - F(z)| \leq \beta \rho(x, z)^{\alpha - 1} \rho(x, z) \leq \beta \theta^{(\alpha - 1) n} \rho(x, z) = \beta L^n \rho(x, z).
\]
It follows from this, using Kirszbraun–McShane extension theorem [7, Theorem 6.1.1], that for every \( n \in \mathbb{N} \) there exists an \( F_n : X \to \mathbb{R} \) such that
\[
F_n |_{A_n} = F |_{A_n} \quad \text{and} \quad |F_n(x) - F_n(z)| \leq \beta L^n \rho(x, z) \quad \text{for} \quad x, z \in X.
\]
If \( n \in \mathbb{N} \) and \( x, z \in X \), then there is a \( z \in A_n \) such that \( \rho(x, z) < \theta^n \), and
\[
|F(x) - F_n(x)| \leq |F(x) - F(z)| + |F_n(z) - F_n(x)|
\leq \beta \rho(x, z)^{\alpha} + \beta L^n \rho(x, z)
\leq \beta \theta^{\alpha n} + \beta L^n \theta^n = 2 \beta \vartheta^n.
\]
\[ \square \]

Corollary 2.6. Assume (H). If \( F : X \to \mathbb{R} \) satisfies (12) with some constants \( \alpha \in (0, 1) \), \( \beta \in [0, \infty) \), then formula (6) defines a solution \( \varphi : X \to \mathbb{R} \) of (1) such that
\[
|\varphi(x) - \varphi(z)| \leq \frac{\beta}{1 - \lambda^\alpha} \rho(x, z)^{\alpha} \quad \text{for} \quad x, z \in X,
\]
and if additionally (7) holds, then formula (8) defines a solution \( \varphi_0 : X \to \mathbb{R} \) of (2) such that
\[
|\varphi_0(x) - \varphi_0(z)| \leq \frac{\beta}{1 - \lambda^\alpha} \rho(x, z)^{\alpha} \quad \text{for} \quad x, z \in X.
\]
Proof. By Proposition 2.5 and Theorem 2.1 formula (6) defines a solution \( \varphi : X \to \mathbb{R} \) of (1). Using (6), (11), (12), Jensen’s inequality and (10) for every \( x, z \in X \) we have

\[
|\varphi(x) - \varphi(z)| \leq |F(x) - F(z)| \\
+ \sum_{n=1}^{\infty} \int_{\Omega} |F(f^n(x, \omega)) - F(f^n(z, \omega))| P^\infty(d\omega)
\]

\[
\leq \beta \rho(x, z)^\alpha + \sum_{n=1}^{\infty} \int_{\Omega} \beta \rho(f^n(x, \omega), f^n(z, \omega))^{\alpha} P^\infty(d\omega)
\]

\[
\leq \beta \rho(x, z)^\alpha + \sum_{n=1}^{\infty} \left( \int_{\Omega} \rho(f^n(x, \omega), f^n(z, \omega)) P^\infty(d\omega) \right)^\alpha
\]

\[
\leq \beta \rho(x, z)^\alpha + \sum_{n=1}^{\infty} \left( \lambda^n \rho(x, z) \right)^\alpha = \frac{\beta}{1 - \lambda^\alpha} \rho(x, z)^\alpha.
\]

For the second part we argue similarly. \( \Box \)

Regarding the uniqueness of solutions, we have the following theorem.

**Theorem 2.7.** Assume (H) and let \( F : X \to \mathbb{R} \).

(i) If \( \varphi_1, \varphi_2 \in F(X) \) are solutions of (1), then \( \varphi_1 = \varphi_2 \).

(ii) If \( \varphi_1, \varphi_2 \in F(X) \) are solutions of (2), then \( \varphi_1 - \varphi_2 \) is a constant function.

**Proof.** Let \( \varphi_1, \varphi_2 \in F(X) \) and put \( \varphi = \varphi_1 - \varphi_2 \). Then \( \varphi \in F(X) \). Corresponding to \( \varphi \) choose a sequence \( (F_n)_{n \in \mathbb{N}} \) of real functions on \( X \) and constants \( \vartheta \in (0, 1), L \in (0, \frac{1}{\lambda}) \) and \( \alpha, \beta \in (0, \infty) \) as in the definition of \( F(X) \).

If \( \varphi_1, \varphi_2 \) are solutions of (1), then \( \varphi \) solves (1) with \( F = 0 \), and, by induction,

\[
\varphi(x) = (-1)^n \int_{\Omega} \varphi(f^n(x, \omega)) P^\infty(d\omega) \quad \text{for } x \in X, \ n \in \mathbb{N}.
\]

If \( \varphi_1, \varphi_2 \) are solutions of (2), then \( \varphi \) solves (2) with \( F = 0 \), and

\[
\varphi(x) = \int_{\Omega} \varphi(f^n(x, \omega)) P^\infty(d\omega) \quad \text{for } x \in X, \ n \in \mathbb{N}.
\]

In both cases

\[
|\varphi(x) - \varphi(z)| \leq \int_{\Omega} |\varphi(f^n(x, \omega)) - \varphi(f^n(z, \omega))| P^\infty(d\omega)
\]

for \( x, z \in X, \ n \in \mathbb{N} \). Moreover,

\[
|\varphi(x) - \varphi(z)| \leq |\varphi(x) - F_n(x)| + |F_n(x) - F_n(z)| + |F_n(z) - \varphi(z)|
\]

\[
\leq 2\alpha \vartheta^n + |F_n(x) - F_n(z)| \quad \text{for } x, z \in X, \ n \in \mathbb{N}.
\]
Consequently, applying among others (10),
\[
|\varphi(x) - \varphi(z)| \leq 2\alpha \vartheta^n + \int_{\Omega} \left| F_n(f^n(x, \omega)) - F_n(f^n(z, \omega)) \right| P^\infty(d\omega)
\leq \beta L^n \lambda^n \rho(x, z) \quad \text{for } x, z \in X, \ n \in \mathbb{N},
\]
whence $\varphi(x) = \varphi(z)$ for $x, z \in X$, i.e., $\varphi$ is a constant function. Noting that if a constant $\varphi$ solves (1) with $F = 0$, then $\varphi = 0$, we end the proof. \hfill \square

We finish with a qualitative result.

Following [6] by Jens Peter Reus Christensen we say that a Borel subset $B$ of an abelian Polish group $G$ is a Haar zero set if there is a probability Borel measure $\mu$ on $G$ such that $\mu(B + x) = 0$ for every $x \in G$. See also [8] where measurability in abelian Polish groups related to Christensen’s Haar zero set is studied.

Assume

$(H_0)$ $(\Omega, \mathcal{A}, P)$ is a probability space, $(X, \rho)$ is a compact metric space, $f : X \times \Omega \rightarrow X$ is $\mathcal{B} \otimes \mathcal{A}$-measurable and (3) holds with a $\lambda \in (0, 1)$.

Assuming $(H_0)$ we have in particular (4):
\[
\int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \leq \text{diam}(X) \quad \text{for } x \in X.
\]

Moreover one can consider the Banach space $C(X)$ of all continuous real functions on $X$ with the uniform norm and its subspace $C_f$,
\[
C_f = \left\{ F \in C(X) : \int_X F(x) \pi^f(dx) = 0 \right\}.
\]
Clearly $C_f$ is a closed linear subspace of $C(X)$ and (see, e.g., [7, Corollary 11.2.5]) $C(X)$ is a separable Banach space. We have also the following lemma.

**Lemma 2.8.** Assume $(H_0)$. If $F : X \rightarrow \mathbb{R}$ is continuous, then so is the function
\[
x \mapsto \int_{\Omega} F(f(x, \omega)) P(d\omega), \quad x \in X.
\]

**Proof.** Fix $\varepsilon \in (0, \infty)$ and choose $\delta \in (0, \infty)$ such that
\[
|F(x) - F(z)| \leq \varepsilon \quad \text{for } x, z \in X \text{ with } \rho(x, z) \leq \delta.
\]
Then, by (3), for all $x, z \in X$,
\[
\left| \int_{\Omega} F(f(x, \omega)) P(d\omega) - \int_{\Omega} F(f(z, \omega)) P(d\omega) \right|
\leq \int_{\Omega} \left| F(f(x, \omega)) - F(f(z, \omega)) \right| P(d\omega)
\leq \varepsilon + \int_{\{\omega \in \Omega : \rho(f(x, \omega), f(z, \omega)) > \delta\}} \left| F(f(x, \omega)) - F(f(z, \omega)) \right| P(d\omega)
\leq \varepsilon + 2\|F\| P\{\{\omega \in \Omega : \rho(f(x, \omega), f(z, \omega)) > \delta\}\}
\leq \varepsilon + 2\|F\|\frac{1}{\delta} \int_{\Omega} \rho(f(x, \omega), f(z, \omega)) P(d\omega)
\leq \varepsilon + \frac{2\lambda\|F\|}{\delta} \rho(x, z),
\]
and therefore the discussed function is continuous. \hfill \square

Let
\[
\mathcal{F}_1 = \{F \in C(X) : \text{equation (1) has a continuous solution } \varphi : X \to \mathbb{R}\},
\]
\[
\mathcal{F}_2 = \{F \in C_f : \text{equation (2) has a continuous solution } \varphi : X \to \mathbb{R}\}.
\]

**Theorem 2.9.** Under the assumptions $$(H_0)$$:

(i) $\mathcal{F}_1$ is a Borel and dense subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then $\mathcal{F}_1$ is of first category in $C(X)$ and a Haar zero subset of $C(X)$.

(ii) $\mathcal{F}_2$ is a Borel and dense subset of $C_f$, and if $\mathcal{F}_2 \neq C_f$, then $\mathcal{F}_2$ is of first category in $C_f$ and a Haar zero subset of $C_f$.

**Proof.** By Lemma 2.8 the formulas
\[
T_1(\varphi)(x) = \varphi(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),
\]
\[
T_2(\varphi)(x) = \varphi(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) \quad \text{for } \varphi \in C(X) \text{ and } x \in X,
\]
define self-mappings $T_1, T_2$ of $C(X)$. Clearly, these operators are linear and continuous. Moreover,
\[
T_1(C(X)) = \mathcal{F}_1.
\]
Furthermore, for every $F \in T_2(C(X))$ Eq. (2) has a continuous solution $\varphi : X \to \mathbb{R}$. Hence (II) gives $T_2(C(X)) \subset C_f$, and
\[
T_2(C(X)) = \mathcal{F}_2.
\]
Applying now [1, Lemma] we see that $\mathcal{F}_1$ is a Borel subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then $\mathcal{F}_1$ is of first category in $C(X)$ and a Haar zero subset of $C(X)$, and $\mathcal{F}_2$ is a Borel subset of $C_f$, and if $\mathcal{F}_2 \neq C_f$, then $\mathcal{F}_2$ is of first category in $C_f$ and a Haar zero subset of $C_f$.

Since by (I) the set
\[
\{F \in C(X) : F \text{ is Lipschitz}\}
\]
is contained in $\mathcal{F}_1$ and (see [7, Theorem 11.2.4]) dense in $C(X)$, the set $\mathcal{F}_1$ is dense in $C(X)$.

To show that $\mathcal{F}_2$ is dense in $C_f$, fix $F \in C_f$ and $\varepsilon \in (0, \infty)$. Choose a Lipschitz $F_1 : X \to \mathbb{R}$ so that $\|F - F_1\| < \frac{\varepsilon}{2}$. According to (III), $F_1 - \int_X F_1 d\pi^f \in \mathcal{F}_2$. Moreover,

$$\left\| F - (F_1 - \int_X F_1 d\pi^f) \right\| \leq \|F - F_1\| + \left| \int_X F_1 d\pi^f - \int_X F d\pi^f \right| < \varepsilon.$$

□

**Remark 2.10.** It is possible that $(H_0)$ holds and $\mathcal{F}_1 = C(X)$, $\mathcal{F}_2 = C_f$.

To see it consider an $\mathcal{A}$-measurable $\xi : \Omega \to X$ and let $f(x, \omega) = \xi(\omega)$ for $(x, \omega) \in X \times \Omega$. Then $f^n(x, \omega) = \xi(\omega_n)$ for $(x, \omega) \in X \times \Omega^\infty$, so $\pi_n^f(x, B) = P(\xi \in B) = \pi^f(B)$ for $n \in \mathbb{N}$, $x \in X$, $B \in \mathcal{B}$, and $\int_X F d\pi^f = \int_\Omega F \circ \xi dP$ for $F \in C(X)$. Consequently for every $F \in C(X)$ the function $F - \frac{1}{2} \int_X F d\pi^f$ solves (1), and every $F \in C_f$ solves (2).

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**References**

[1] Baron, K.: A remark on linear functional equations in the indeterminate case. Glasnik Mat. 20(40), 373–376 (1985)
[2] Baron, K.: On the convergence in law of iterates of random-valued functions. Aust. J. Math. Anal. Appl. 6, no. 1, Art. 3 (2009)
[3] Baron, K.: Weak limit of iterates of some random–valued functions and its application. Aequ. Math. 94, 415–425; 427 (Correction) (2020)
[4] Baron, K.: Around the weak limit of iterates of some random-valued functions. Ann. Univ. Budapest. Sect. Comput. 51, 31–37 (2020)
Baron, K., Kapica, R., Morawiec, J.: On Lipschitzian solutions to an inhomogeneous linear iterative equation. Aequ. Math. 90, 77–85 (2016)

Christensen, J.P.R.: On sets of Haar measure zero in abelian Polish groups. Isr. J. Math. 13, 255–260 (1972)

Dudley, R.M.: Real Analysis and Probability. Cambridge Studies in Advanced Mathematics, vol. 74. Cambridge University Press, Cambridge (2002)

Fischer, P., Słodkowski, Z.: Christensen zero sets and measurable convex functions. Proc. Amer. Math. Soc. 79, 449–453 (1980)

Jarczyk, W.: On a set of functional equations having continuous solutions. Glasnik Mat. 17(37), 59–64 (1982)

Kapica, R.: Sequences of iterates of random-valued vector functions and solutions of related equations. Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 213, 113–118 (2005)

Kapica, R.: The geometric rate of convergence of random iteration in the Hutchinson distance. Aequ. Math. 93, 149–160 (2019)

Kuczma, M.: Functional equations in a single variable. Monografie Matematyczne, vol. 48. PWN–Polish Scientific Publishers, Warszawa (1968)

Kuczma, M., Choczewski, B., Ger, R.: Iterative Functional Equations. Encyclopedia of Mathematics and Its Applications, vol. 32. Cambridge University Press, Cambridge (1990)

Karol Baron
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: baron@us.edu.pl

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