A NOTE ON FANO MANIFOLDS WHOSE SECOND CHERN CHARACTER IS POSITIVE

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Abstract. This note outlines some first steps in the classification of Fano manifolds for which \( c_1^2 - 2c_2 \) is positive or nef.

1. Introduction

This note about Fano manifolds \( X \) for which \( (c_1^2 - 2c_2)(T_X) \) is positive, lists what few examples are known, as well as giving many non-examples. Presumably there are many more examples. They do not seem easy to find.

Notation 1.1. Let \( X \) be a projective variety over an algebraically closed field. For every integer \( k \geq 0 \), denote by \( N_k(X) \) the finitely-generated free Abelian group of \( k \)-cycles modulo numerical equivalence, and denote by \( N^k(X) \) the \( k \)th graded piece of the quotient algebra \( A^*(X)/\text{Num}^*(X) \), cf. [Ful84, Example 19.3.9]. For every \( \mathbb{Z} \)-module \( B \), denote \( N_k(X) \otimes B \) and \( N^k(X) \otimes B \). Denote by \( \text{NE}_k(X) \subset N_k(X) \) the semigroup generated by nonzero, effective \( k \)-cycles. For \( \mathbb{R} \) a subring of \( \mathbb{R} \), denote by \( \text{NE}_k(X)_B \) the \( \mathbb{R} \)-semigroup in \( N_k(X) \otimes B \) generated by \( \text{NE}_k(X)_B \).

Definition 1.2. A class in \( N^k(X)_\mathbb{R} \) is nef if it pairs nonnegatively with every element in \( \text{NE}_k(X) \). The corresponding cone is denote \( \text{Nef}^k(X) \). A class is weakly positive if it pairs positively with every element in \( \text{NE}_k(X) \). The corresponding cone is denoted \( \text{WPos}^k(X) \). A class is positive if it is contained in the interior of \( \text{Nef}^k(X) \); the interior of \( \text{Nef}^k(X) \) is denoted \( \text{Pos}^k(X) \). The ample cone is the \( \mathbb{R} \)-semigroup generated by the image of the cup-product map, \( (\text{Pos}^1(X))^k \to N^k(X)_\mathbb{R} \). It is denoted \( \text{Ample}^k(X) \), and its elements are ample classes.

Remark 1.3. There are obvious inclusions,

\[
\text{Ample}^k(X) \subset \text{Pos}^k(X) \subset \text{WPos}^k(X) \subset \text{Nef}^k(X).
\]

For \( k = 1 \), \( \text{Ample}^1(X) = \text{Pos}^1(X) \) by definition. Moreover, by Kleiman’s criterion, this is the \( \mathbb{R} \)-semigroup generated by first Chern classes of ample invertible sheaves. For \( k > 1 \), it can happen that \( \text{Ample}^k(X) \neq \text{Pos}^k(X) \); for instance, because \( (N^1(X))^{\otimes k} \to N^k(X) \) is not surjective. There are also examples where \( \text{Pos}^k(X) \neq \text{WPos}^k(X) \) and \( \text{WPos}^k(X) \neq \text{Nef}^k(X) \).

Problem 1.4. Find smooth, connected, projective varieties \( X \) such that \( \text{ch}_1(T_X) = c_1(T_X) \) is ample and \( \text{ch}_2(T_X) = 1/2(c_1^2 - 2c_2)(T_X) \) is ample, resp. positive, weakly positive, nef. More generally, allow \( X \) to be a smooth, connected, proper Deligne-Mumford stack whose coarse moduli space is projective.
2. Positive Examples

Following are examples of Fano manifolds with $\text{ch}_2(T_X)$ ample or positive.

1. The simplest example is $\mathbb{P}^n$ for $n \geq 2$. Denote by $h \in N^1(\mathbb{P}^n)$ the first Chern class of $\mathcal{O}_{\mathbb{P}^n}(1)$. Using the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0,$$

the Chern character of $T_{\mathbb{P}^n}$ is $(n+1)e^h - 1$. In particular, $\text{ch}_2(T_X) = (n+1)h^k/k!$ for every $k = 1, \ldots, n$. So $\text{ch}_k(T_X)$ is ample for $k = 1, \ldots, n$.

2. Weighted projective spaces are also examples. The weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is the coarse moduli space of a smooth Deligne-Mumford stack $X$, and $\text{ch}_2(T_X) = (n+1)h^k/k!$ where $h$ is the first Chern class of the invertible sheaf $\mathcal{O}_X(1)$ on the stack. Some positive multiple of $h$ is the pullback of an ample class from the coarse moduli space, thus $h$ is an ample class.

3. Let $Y$ be a smooth complete intersection of divisors $D_1, \ldots, D_r$ in $\mathbb{P}$ of respective degrees $d_1, \ldots, d_r$. Using the exact sequences,

$$0 \longrightarrow T_Y \longrightarrow T_{\mathbb{P}}|_Y \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_\mathbb{P}(d_i)|_Y \longrightarrow 0,$$

the Chern character of $T_Y$ is $(n+1)e^h - 1 - \sum_{i=1}^r e^{d_i h}$. Thus $\text{ch}_k(T_Y) = 1/k!(n+1 - (d_1 + \cdots + d_r))h^k$ for $k = 1, \ldots, n - r$. In particular, if $d_1 + \cdots + d_r < n + 1$ then $\text{ch}_1(T_Y)$ and $\text{ch}_2(T_Y)$ are both ample.

4. For every integer $k \geq 1$, the Grassmannians $G = \text{Grass}(k, 2k)$ and $G = \text{Grass}(k, 2k + 1)$ have $\text{ch}_1(T_G)$ is ample and $\text{ch}_2(T_G)$ is positive. If $k > 1$, then $\text{ch}_2(T_G)$ is not in the 1-dimensional subspace spanned by $\sigma_1^2$. Therefore $\text{ch}_2(T_G)$ is positive, but not ample.

3. Nef Examples

Given a Fano manifold, there are a few methods of constructing a new Fano manifold $Y$ with $\text{ch}_2(T_Y)$ nef. Typically even if $\text{ch}_2(T_X)$ is positive, $\text{ch}_2(T_Y)$ is not weakly positive.

1. Let $X$ be a Fano manifold with $\text{ch}_2(T_X)$ nef. Let $Y$ be a smooth divisor in $X$. If $c_1(T_X) - [Y]$ is ample, and $\text{ch}_2(T_X) - [Y]^2/2$ is nef, then $Y$ is a Fano manifold and $\text{ch}_2(T_Y)$ is nef. This is essentially the same as Example 3 in Section 2.

2. Let $X$ be a Fano manifold and let $L$ be a nef line bundle such that $c_1(T_X) - c_1(L)$ is ample and $\text{ch}_2(T_X) + c_1(L)^2/2$ is nef. Then the projective bundle $\mathbb{P}(L') \oplus \mathcal{O}_X$ is a Fano manifold and $\text{ch}_2(T_{\mathbb{P}(L')})$ equals $\pi^*(\text{ch}_2(T_X) + c_1(L)^2/2)$. This is nef, but not weakly positive; its restriction to $\pi^{-1}(C)$ is zero for every curve $C \subset X$. Note $\text{ch}_2(T_X)$ need not be nef, e.g., for integers $(n, d, a)$ satisfying $1 \leq d \leq (n^2 + n + 1)/2a$ and $\lceil \sqrt{\max(0, d^2 - n - 1)} \rceil \leq a \leq n - d$, for every smooth degree $d$ hypersurface $X \subset \mathbb{P}^n$, the projective bundle $\mathbb{P}(\mathcal{O}_X(-a) \oplus \mathcal{O}_X)$ is a Fano manifold with $\text{ch}_2(T_{\mathbb{P}(L')})$ nef.

3. Let $X$ and $Y$ be Fano manifolds such that $\text{ch}_2(T_X)$ and $\text{ch}_2(T_Y)$ are nef. The product $X \times Y$ is Fano and $\text{ch}_2(T_{X \times Y}) = \pi_X^*\text{ch}_2(T_X) + \pi_Y^*\text{ch}_2(T_Y)$, which is nef. For rational curves $C_X \subset X$ and $C_Y \subset Y$, the pairing of $\text{ch}_2(T_{X \times Y})$ with $C_X \times C_Y$ is zero, thus $\text{ch}_2(T_{X \times Y})$ is not weakly positive.
4. Projective Bundles

One way to produce new examples of Fano manifolds is to form the projective bundle of a vector bundle of "low degree" over a given Fano manifold.

**Lemma 4.1.** Let $E$ be a vector bundle on $X$ of rank $r$. Denote by $\pi : \mathbb{P}E \to X$ the associated projective bundle. The graded pieces of the Chern character of $T_{\mathbb{P}E}$ are, $c_1(T_{\mathbb{P}E}) = r\zeta + \pi^*(c_1(T_X) + c_1(E))$ and $ch_2(T_{\mathbb{P}E}) = r\zeta^2/2 + \pi^*(c_2(T_X) + \pi^*(\mu_{\mathbb{P}E}E))$, where $\zeta$ equals $c_1(O_{\mathbb{P}E}(1))$.

**Proof.** There is an Euler sequence,

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}E} \longrightarrow \pi^*E \otimes \mathcal{O}_{\mathbb{P}E}(1) \longrightarrow T_{\mathbb{P}E/X} \longrightarrow 0.
$$

Therefore $ch(T_{\mathbb{P}E/X}) = \pi^*ch(E)e^\zeta - 1$, i.e.,

$$(r + \pi^*c_1(E) + \pi^*ch_2(E) + \ldots)(1 + \zeta + \zeta^2/2 + \ldots) - 1 =$$

$$[r - 1] + [r\zeta + \pi^*c_1(E)] + [r\zeta^2/2 + \pi^*c_1(E)\zeta + \pi^*ch_2(E)] + \ldots
$$

Using the exact sequence,

$$
0 \longrightarrow T_{\mathbb{P}E/X} \longrightarrow T_{\mathbb{P}E} \longrightarrow \pi^*T_X \longrightarrow 0,
$$

$ch(T_{\mathbb{P}E})$ equals $ch(T_{\mathbb{P}E/X}) + \pi^*ch(T_X)$. Thus $ch_1(T_{\mathbb{P}E/X}) = r\zeta + \pi^*(c_1(T_X) + c_1(E))$ and,

$$ch_2(T_{\mathbb{P}E}) = r\zeta^2/2 + \pi^*(c_1(E)\zeta + \pi^*(ch_2(T_X) + ch_2(E))).$$

□

**Proposition 4.2.** Let $X$ be a smooth Fano manifold and let $E$ be a vector bundle on $X$ of rank $r$. The projective bundle $\mathbb{P}E$ is Fano iff there exists $\epsilon > 0$ such that for every irreducible curve $B \subset X$,

$$\mu_B(E|B) - \mu_B(E|B) \leq (1 - \epsilon)deg_B(-K_X)/r,$$

where $\mu_B$ and $\mu_B^1$ are the slopes from Definition 6.2, resp. Definition 6.3.

**Proof.** The invertible sheaf $\omega_{\mathbb{P}E}$ is $\pi$-relatively ample. By hypothesis, $\omega_X^\vee$ is ample. By Lemma 4.3, $\omega_{\mathbb{P}E}$ is ample iff there exists a real number $\epsilon > 0$ such that

$$deg_B(g^*\omega_{\mathbb{P}E}^\vee) \geq rdeg_B(g^*\pi^*\omega_X^\vee),$$

for every finite morphism $g : B \to \mathbb{P}E$ of a smooth, connected curve to $X$ for which $\pi \circ g$ is also finite. Using the universal property of $\mathbb{P}E$, this holds iff for every finite morphism $f : B \to X$ and every invertible quotient $f^*E^\vee \to L^\vee$,

$$deg_B(g^*\omega_{\mathbb{P}E}^\vee) \geq rdeg_B(g^*\pi^*\omega_X^\vee),$$

where $g : B \to \mathbb{P}E$ is the associated morphism. By Lemma 4.1, $deg_B(\omega_{\mathbb{P}E}^\vee)$ equals $rc_1(L^\vee) + c_1(f^*E) + c_1(f^*T_X)$, i.e.,

$$r[c_1(f^*T_X)/r - (\mu_B(L) - \mu_B(f^*E))].$$

So, finally, $\omega_{\mathbb{P}E}$ is ample iff there exists $\epsilon > 0$ such that for every finite morphism $f : B \to X$ and every invertible quotient $f^*E^\vee \to L^\vee$,

$$\mu_B(L) - \mu_B(f^*E) \leq (1 - \epsilon)deg_B(f^*c_1(T_X))/r.$$

Taking the supremum over covers of $B$ and invertible quotients of the pullback of $E$, this is,

$$\mu_B^1(f^*E) - \mu_B(f^*E) \leq (1 - \epsilon)deg_B(-f^*K_X)/r.$$
Since every finite morphism \( f : B \to X \) factors through its image, it suffices to consider only irreducible curves \( B \) in \( X \).

For \( r = 2 \), there is a necessary and sufficient condition for \( \text{ch}_2(T_{FY}) \) to be nef.

**Proposition 4.3.** Let \( E \) be a vector bundle on \( X \) of rank 2. Denoting by \( \pi : \mathbb{P}E \to X \) the projection, \( \text{ch}_2(T_{FY}) = \pi^*(\text{ch}_2(E) + 1/2(c_1^2 - 4c_2)(E)) \). Therefore \( \text{ch}_2(T_{FY}) \) is nef iff \( \text{ch}_2(E) + 1/2(c_1^2 - 4c_2)(E) \) is nef. If \( \dim(X) > 0 \), \( \text{ch}_2(T_{FY}) \) is not weakly positive.

**Proof.** By Lemma 4.1, \( \text{ch}_2(T_{FY}) \) equals \( \zeta^2 + \pi^*c_1(E)\zeta + \pi^*(\text{ch}_2(E)) \). By definition of the Chern classes of \( E \), \( \zeta^2 + \pi^*c_1(E)\zeta + \pi^*c_2(E) \) equals 0. So the class above is \( -\pi^*c_2(E) + \pi^*(\text{ch}_2(E)) \). Finally, \( \text{ch}_2(E) - c_2(E) \) equals \( 1/2(c_1^2 - 2c_2)(E) - c_2(E) = 1/2(c_1^2 - 4c_2)(E) \).

Applying Proposition 4.2 and Proposition 4.3 to the vector bundle \( E \) on \( \mathbb{P}E \) gives Example 2 in Section 3.

Finally, for \( r > 2 \), there is a necessary condition for \( \text{ch}_2(T_{FY}) \) to be nef.

**Proposition 4.4.** Let \( E \) be a vector bundle of rank \( r > 2 \) on \( X \). If \( \text{ch}_2(T_{FY}) \) is nef, then the pullback of \( E \) to every smooth, projective, connected curve is semistable. Also, \( \text{ch}_2(T_{FY}) \) is not weakly positive if \( \dim(X) > 0 \) and if the pullback of \( E \) to some curve is strictly semistable, e.g., if \( X \) contains a rational curve.

**Proof.** If the pullback of \( E \) to some smooth, projective, connected curve is not semistable, then by Corollary 6.11 there exists a smooth, projective, connected curve \( B \), a morphism \( f : B \to X \), and a rank 2 locally free subsheaf \( F \) of \( f^*E \) such that \( f^*E/F \) is locally free and \( \mu_B(F) > \mu_B(E) \). There is an induced morphism \( g : \mathbb{P}F \to \mathbb{P}E \) such that \( \pi \circ g = f \circ \pi \). By Lemma 4.1, \( g^*\text{ch}_2(T_{FY}) \) equals \( \pi^*f^*(\zeta + \pi^*(\text{ch}_2(E)) \). Since \( B \) is a curve, \( f^*(\text{ch}_2(E)) \) equals 0. Also, by definition of the Chern classes of \( F \), \( \zeta^2 + \pi^*c_1(F)\zeta = 0 \). Substituting in,

\[
g^*\text{ch}_2(T_{FY}) = 1/2\pi^*(2c_1(f^*E) - r c_1(F))\zeta.
\]

In particular, \( \deg_B(g^*\text{ch}_2(T_{FY})) \) equals \( 1/2(2\deg_B(c_1(f^*E)) - r \deg_B(F)) \). This equals \( r(\mu_B(f^*E) - \mu_B(F)) \), which is negative by construction. Therefore \( \text{ch}_2(T_{FY}) \) is not nef.

**Remark 4.5.** A vector bundle on a product of projective spaces whose restriction to every curve is semistable is of the form \( L^\oplus r \), where \( L \) is an invertible sheaf, [OSS80] Thm. 3.2.1]. In this case, \( \mathbb{P}E \) is also a product of projective spaces.

**Corollary 4.6.** Let \( X \) be a Fano manifold. For every vector bundle \( E \) on \( X \) of rank \( r > 1 \), \( \text{ch}_2(T_{FY}) \) is not weakly positive.

5. **BLOWINGS UP**

Let \( X \) be a smooth, connected, projective variety, let \( i : Y \to X \) be the closed immersion of a smooth, connected subvariety of \( X \) of codimension \( c \). Denote by \( \nu : \tilde{X} \to X \) the blowing up of \( X \) along \( Y \). Denote by \( \pi : E \to Y \) the exceptional divisor. Denote by \( j : E \to \tilde{X} \) the obvious inclusion. Then \( E = \mathbb{P}N_{Y/X} \) and \( i^*\mathcal{O}_{\tilde{X}}(E) \) is canonically isomorphic to \( \mathcal{O}_{\mathbb{P}N}(-1) \).
Lemma 5.1. The graded pieces of the Chern character of $\tilde{X}$ are, $c_1(T_{\tilde{X}}) = \nu^*c_1(T_X) - (c - 1)|E|$ and $ch_2(T_{\tilde{X}}) = \nu^*ch_2(T_X) + (c + 1)|E|^2/2 - i^*\pi^1_1(N_{Y/X})$

Proof. Using the short exact sequence,

\[
\begin{array}{cccc}
0 & \longrightarrow & \nu^*\Omega_X & \longrightarrow & \Omega_{\tilde{X}} & \longrightarrow & j_*\Omega_\pi & \longrightarrow & 0,
\end{array}
\]

$\text{ch}(\Omega_{\tilde{X}})$ equals $\nu^*\text{ch}(\Omega_X) + \text{ch}(j_*\Omega_\pi)$. Grothendieck-Riemann-Roch for the morphism $j$ gives,

\[
\text{ch}(Rj_*a) = j_*\left(\text{ch}(a)(1 - e^{-|E|})/|E|\right).
\]

Using the Euler sequence for $\Omega_\pi$,

\[
\begin{array}{cccc}
0 & \longrightarrow & \Omega_\pi & \longrightarrow & \pi^*N^\vee_{Y/X} \otimes \mathcal{O}_{P^N}(-1) & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0,
\end{array}
\]

$\text{ch}(\Omega_\pi)$ equals $\pi^*\text{ch}(N^\vee_{Y/X})i^*(1 + e^{[E]}) - 1$. Putting the pieces together gives the lemma. \hfill $\square$

When is $\tilde{X}$ Fano? Denote by $\mathcal{C}_1$ the collection of finite morphisms $g : B \to X$ from a smooth, connected curve to $X$ whose image is not contained in $Y$. Denote by $\mathcal{C}_2$ the collection of finite morphisms $g : B \to Y$ from a smooth, connected curve to $Y$. The following result is well-known.

Proposition 5.2. Let $h$ be the first Chern class of an ample invertible sheaf on $X$, e.g., $h = c_1(T_X)$ if $X$ is Fano. The blowing up $\tilde{X}$ is Fano iff there exists $\epsilon > 0$ such that,

(i) for every $g : B \to X$ in $\mathcal{C}_1$,

\[
\text{deg}_B(g^{-1}Y) \leq \frac{1}{c-1}\left(\text{deg}_B(g^*c_1(T_X)) - \epsilon \text{deg}_B(g^*h)\right),
\]

and

(ii) for every $g : B \to Y$ in $\mathcal{C}_2$,

\[
\mu_B(g^*N_{Y/X}) \leq \frac{1}{c - 1}\left(\text{deg}_B(g^*c_1(T_X)) - \epsilon \text{deg}_B(g^*h)\right).
\]

The proof is similar to the proof of Proposition 122. Using an analogue of Proposition 123 no blowing-up of $\mathbb{P}^n$ is a Fano manifold with $\text{ch}_2$ nef.

6. Theorems about vector bundles on curves

There are two theorems in this section. The first goes back to Shou-Wu Zhang, though possibly it is older. The second is a variation of the first.

Definition 6.1. Let $B$ be a smooth, projective curve. A cover of $B$ is a finite, flat morphism $f : C \to B$ of constant, positive degree. A vector bundle on $B$ is a locally free $\mathcal{O}_B$-module of constant rank.

Definition 6.2. Let $B$ be a smooth, projective curve. For every non-zero vector bundle $\mathcal{E}$ on $B$, the slope is,

\[
\mu_B(\mathcal{E}) = \text{deg}(\mathcal{E})/\text{rank}(\mathcal{E}) = \chi(\mathcal{B}, \mathcal{E})/\text{rank}(\mathcal{E}) - \chi(\mathcal{B}, \mathcal{O}_B).
\]

For every cover $f : C \to B$ and every non-zero vector bundle $\mathcal{E}$ on $C$, the $B$-slope is,

\[
\mu_B(f, \mathcal{E}) := \text{deg}(\mathcal{E})/\text{deg}(f)\text{rank}(\mathcal{E})) = \mu_B(f_*\mathcal{E}) - \mu_B(f_*\mathcal{O}_C).
\]

When there is no chance of confusion, this is denoted simply $\mu_B(\mathcal{E})$. 

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For every cover \( g : C' \to C \), \( f \circ g : C' \to B \) is a cover and \( \mu_B(f \circ g, g^*E) \) equals \( \mu_B(f, E) \).

**Definition 6.3.** Let \( B \) be a smooth, projective curve and let \( E \) be a vector bundle on \( B \) of rank \( r > 0 \). For every integer \( 1 \leq k \leq r \), define \( \mu^k_B(E) \) to be,

\[
\sup \{-\mu_B(f, F^\vee) | f : C \to B \text{ a cover}, f^*E^\vee \to F^\vee \text{ a rank } k \text{ quotient}\} = \sup \{\mu_B(f, F) | f : C \to B \text{ a cover}, F \subset f^*E \text{ a rank } k \text{ subbundle whose cokernel is locally free}\}.
\]

Let \( f : X \to Y \) be a morphism of projective varieties. Denote by \( C_1 \) the collection of all irreducible curves in \( X \) not contained in a fiber of \( f \). Denote by \( C_2 \) the collection of finite morphisms \( g : C \to X \) occurring as the normalization of an irreducible curve in \( X \) not contained in a fiber of \( f \). Finally, denote by \( C_3 \) the collection of all finite morphisms from smooth, connected curves to \( X \) whose image is not contained in a fiber of \( f \).

**Lemma 6.4.** Let \( f : X \to Y \) be a morphism of projective varieties and let \( L \) be an ample invertible \( \mathcal{O}_Y \)-module. An \( f \)-ample invertible \( \mathcal{O}_X \)-module \( M \) is ample iff there exists a real number \( \epsilon > 0 \) such that for every morphism \( g : C \to X \) in \( C_1 \), \( C_2, C_3 \), \( \deg_C(g^*M) \geq \epsilon \deg_C(g^*f^*L) \).

**Proof.** Because \( M \) is \( f \)-ample and \( L \) is ample, there exists an integer \( n > 0 \) such that \( M \otimes f^*L^\otimes n \) is ample. By Kleiman’s criterion, \( M \) is ample iff there exists a real number \( 0 < \delta < 1 \) such that for every irreducible curve \( C \) in \( X \),

\[
\deg_C(M) \geq \delta \deg_C(M \otimes f^*L^\otimes n).
\]

Simplifying, this is equivalent to,

\[
\deg_C(M) \geq \frac{n\delta}{1-\delta} \deg_C(f^*L).
\]

As \( M \) is \( f \)-ample, this holds if \( C \) is contained in a fiber of \( f \). So \( M \) is ample iff the inequality holds for every curve in \( C_1 \). Setting \( \epsilon = n\delta/(1-\delta) \), \( \delta = \epsilon/(n + \epsilon) \), gives the lemma.

Since \( C_2 \subset C_3 \), the condition for \( C_3 \) implies the condition for \( C_2 \). Since degrees on a curve can be computed after pulling back to the normalization, the condition for \( C_2 \) implies the condition for \( C_1 \). Finally, for every morphism \( g : C \to X \) in \( C_3 \), \( g(C) \) is in \( C_1 \). The inequality for \( g(C) \) implies the inequality for \( C \). Thus the condition for \( C_1 \) implies the condition for \( C_3 \).

**Lemma 6.5.** Let \( B \) be a smooth, connected, projective curve. A nonzero vector bundle \( E \) on \( B \) is ample iff there exists a positive real number \( \delta \) such that for every cover \( f : C \to B \) and every invertible quotient \( f^*E \to L \), \( \mu_B(L) \geq \delta \). In other words, \( E \) is ample iff \( \mu^1_B(L^\vee) < 0 \).

**Proof.** Denote by \( \pi : \mathbb{P}E^\vee \to B \) the projective bundle associated to \( E^\vee \), and denote by \( \pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1) \) the tautological invertible quotient. By definition, \( E \) is ample iff \( \mathcal{O}_{\mathbb{P}E^\vee}(1) \) is an ample invertible sheaf. Of course \( \mathcal{O}_{\mathbb{P}E^\vee}(1) \) is \( \pi \)-relatively ample. Let \( M \) be an invertible \( \mathcal{O}_B \)-module of degree 1. Then \( M \) is ample. By Lemma 6.3, \( \mathcal{O}_{\mathbb{P}E^\vee}(1) \) is ample iff there exists \( \epsilon > 0 \) such that for every smooth, connected curve \( C \) and every finite morphism \( g : C \to \mathbb{P}E^\vee \) such that \( \pi \circ g \) is finite,
\[ \text{deg}_C(g^*O_{E'})(1) \geq c \text{deg}_C(g^*\pi^*M). \] Of course \( \text{deg}_C(g^*\pi^*M) = \text{deg}(\pi \circ g) \). Using the universal property of \( P^E \), this holds if for every cover \( f : C \to B \) and every invertible quotient \( f^*E \to L \),

\[ \text{deg}_C(L) \geq c \text{deg}(f) \Leftrightarrow \mu_B(L) \geq \epsilon. \]

\[ \square \]

**Lemma 6.6.** For every ample vector bundle \( E \) on \( B \), there exists a cover \( f : C \to B \), invertible \( O_C \)-modules \( L_1, \ldots, L_r \), and a morphism of \( O_C \)-modules, \( \phi : f^*E \to (L_1 \oplus \cdots \oplus L_r) \) such that,

(i) the support of \( \text{coker}(\phi) \) is a finite set,

(ii) for every \( i = 1, \ldots, r \), the projection \( \text{deg}(\phi_i) \to \oplus_{j \neq i} L_j \) is surjective, and

(iii) for every \( i = 1, \ldots, r \), \( \mu_B(L_i) = \text{deg}_B(E) \).

**Proof.** Denote \( r = \text{rank}(E) \). The claim is that for every \( k = 1, \ldots, r \), there exists a cover \( f_k : C_k \to B \), effective \( O_{C_k} \)-modules \( L_{k,1}, \ldots, L_{k,k} \), and a morphism of \( O_{C_k} \)-modules, \( \phi_k : f_k^*E \to (L_{k,1} \oplus \cdots \oplus L_{k,k}) \) satisfying (ii) and (iii) above and the following variant of (i): for \( k < r \), \( \phi_k \) is surjective and for \( k = r \), the support of \( \text{coker}(\phi_k) \) is a finite set. The lemma is the case \( k = r \). The claim is proved by induction on \( k \).

The base case is \( k = 1 \). Denote by \( \pi : P^E \to B \) the projective bundle associated to \( E' \), and denote by \( \pi^*E \to O_{P^E}(1) \) the tautological invertible quotient. By hypothesis, \( O_{P^E}(1) \) is ample. Denote \( \text{deg}(E) \) by \( d_1 \). By Bertini’s theorem, for \( d_1, \ldots, d_{r-1} > 0 \), there exist effective Cartier divisors \( D_1, \ldots, D_{r-1} \) with \( D_i \in |O_{P^E}(d_1)| \) such that the intersection \( C_1 = D_1 \cap \cdots \cap D_r \) is a smooth, connected curve, cf. [JouX3]. Denote \( f_1 : C_1 \to B \) the restriction of \( \pi \). Denote by \( \phi_1 : f_1^*E \to L_1 \) the restriction of \( \pi^*E \to O_{P^E}(1) \). This satisfies (i) because \( \pi^*E \to O_{P^E}(1) \) is surjective. It satisfies (ii) trivially. Finally, \( \text{deg}(f) = d_1 \times \cdots \times d_{r-1} \), and \( \mu_B(L_1) = \text{deg}_B(E) \). Therefore \( \mu_B(L_1) = \text{deg}_B(E) \). This satisfies (iii).

By way of induction, assume the result is known for \( k < r \), and consider the case \( k + 1 \). Since \( \phi_k \) is surjective, there is an induced closed immersion \( P(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee \to P(f_k^*E)^\vee \). The image is irreducible and has codimension \( k \geq 1 \). For every \( i = 1, \ldots, k \), the image of \( P(\oplus_{j \neq i} L_{k,j})^\vee \) is irreducible and has codimension \( r - k + 1 \geq 2 \). Associated to the finite morphism \( f_k \), there is a finite morphism \( P(f_k^*E)^\vee \to P^E \). The pullback of an ample invertible sheaf by a finite morphism is ample; hence \( O_{P(f_k^*E)^\vee}(1) \) is ample. By Bertini’s theorem, for \( d_1, \ldots, d_{r-1} > 0 \), there exist effective Cartier divisors \( D_1, \ldots, D_{r-1} \) with \( D_i \in |O_{P(f_k^*E)^\vee}(d_1)| \) such that the intersection \( C_{k+1} = D_1 \cap \cdots \cap D_{r-1} \) is a smooth, connected curve, disjoint from \( P(\oplus_{j \neq i} L_{k,j})^\vee \) for every \( i = 1, \ldots, k \), and either disjoint from \( P(\oplus_{i=1}^k L_i)^\vee \) if \( k < r - 1 \), or else intersecting \( P(\oplus_{i=1}^k L_i)^\vee \) in finitely many points if \( k = r - 1 \). Define \( g_{k+1} : C_{k+1} \to C_k \) to be the restriction of the projection. Define \( f_{k+1} = f_k \circ g_{k+1} \), define \( L_{k+1,i} = g_{k+1}^*L_{k,i} \) for \( i = 1, \ldots, k \), and define \( L_{k+1,k+1} \) to be the restriction of \( O_{P(f_k^*E)^\vee}(1) \). Define \( \phi_{k+1} \) to be the obvious morphism.

The cokernel of \( \phi_{k+1} \) is supported on the intersection of \( C_{k+1} \) with \( P(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee \). By construction, this is empty if \( k < r - 1 \), and is a finite set if \( k = r - 1 \). Thus \( \phi_{k+1} \) satisfies (i). By the induction hypothesis, \( f_{k+1}^*E \to (L_{k+1,1} \oplus \cdots \oplus L_{k+1,k}) \), which is the pullback under \( g_{k+1} \) of \( \phi_k \), is surjective. For \( i = 1, \ldots, k \), the
By way of contradiction, assume Hypothesis 6.8. Let $f$ smooth cover of degree $d$.

Certainly an effective version of the following argument can be given, but a simpler argument as in the base case. The claim is proved by induction on $k$. 

Theorem 6.7. For every non-zero vector bundle $E$ on $B$, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and an invertible quotient $f^*E \to L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. In other words, $\mu_B(E^\vee) \geq \mu_B(E^\vee)$.

Proof. Denote $r = \text{rank}(E)$. If $r = 1$, set $f = \text{Id}_B$ and $L = E$. Then $L$ is an invertible quotient of $f^*E$, and $\mu_B(L)$ equals $\mu_B(E)$ which is less than $\mu_B(E) + \epsilon$. Therefore assume $r > 1$.

Certainly an effective version of the following argument can be given, but a simpler argument is by contradiction.

Hypothesis 6.8. For every cover $f : C \to B$ and every invertible quotient $f^*E \to L$, $\mu_B(L) \geq \mu_B(E) + \epsilon$, i.e., $\mu_B(E^\vee) < \mu_B(E^\vee) - \epsilon$.

By way of contradiction, assume Hypothesis 6.8. Let $f : C \to B$ be a connected, smooth cover of degree $d$. For every $a/d \in \mathbb{Q}$, there exists an invertible sheaf $M$ on $C$ of degree $a$, and thus $\mu_B(M) = a/d$. In particular, for $d$ sufficiently large, there exists an invertible quotient $M$ such that $0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1)$. Denote $\delta = \mu_B(E) - \mu_B(M)$. Denote $F = f^*E \otimes M^\vee$. Then $\mu_B(F)$ equals $\delta$, and $0 < \delta < \epsilon/(r-1)$.

Let $g : C' \to C$ be any cover and let $g^*F \to N$ be any invertible quotient. Then $f \circ g : C' \to B$ is a cover and $(f \circ g)^*E = g^*F \otimes g^*M \to N \otimes g^*M$ is an invertible quotient. By Hypothesis 6.8,

$$\mu_C(N) = \deg(f)\mu_B(N) = \deg(f)(\mu_B(N \otimes g^*M) - \mu_B(M))$$

$$\geq \deg(f)((\mu_B(E) + \epsilon) - \mu_B(M)) > \deg(f)\epsilon.$$ 

By Lemma 6.6, $F$ is an ample vector bundle on $C$. By Lemma 6.6, there exists a cover $g : C' \to C$ and an invertible quotient $g^*F \to P$ such that $\mu_B(P) = r\mu_B(F) = r\delta$. Therefore $L := g^*M \otimes P$ is an invertible quotient of $g^*f^*E$ and,

$$\mu_B(L) = \mu_B(g^*M \otimes P) = \mu_B(M) + r\delta = \mu_B(E) + (r-1)\delta.$$ 

By hypothesis, $(r-1)\delta < \epsilon$. So $\mu_B(L) < \mu_B(E) + \epsilon$, contradicting Hypothesis 6.8. The proposition is proved by contradiction.

Corollary 6.9. For every non-zero vector bundle $E$ on $B$, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a sequence of vector bundle quotients,

$$f^*E = E^r \to E^{r-1} \to \cdots \to E^1,$$

such that each $E^k$ is a vector bundle of rank $k$ and $\mu_B(E^k) < \mu_B(E) + \epsilon$.

Proof. The proof is by induction on the rank $r$ of $E$. If rank$(E) = 1$, defining $f = \text{Id}_B$ and $E^1 = E$, the result follows. Thus, assume $r > 1$ and the result is known for smaller values of $r$. By Theorem 6.7, there exists a cover $g : B' \to B$ and a rank 1 quotient $g^*E \to L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. Denote by $K$ the kernel of $g^*E \to L$. Then rank$(K) = r - 1$ and $\mu_B(K) = (r\mu_B(E) - \mu_B(L))/(r-1)$. By
the induction hypothesis, there exists a cover \( h : C \to B' \) and a sequence of vector bundle quotients,
\[
h^* K = K'^{-1} \to \cdots \to K^1,
\]
such that each \( K^k \) is a vector bundle of rank \( k \), and \( \mu_{B'}(K^k) \leq \mu_B(K) + \deg(g)\epsilon \).
Of course \( \mu_B(F) = \mu_{B'}(F)/\deg(g) \) for every \( F \). Thus \( \mu_B(K^k) \leq \mu_B(K) + \epsilon \).

Define \( f = h \circ g \), define \( E^1 = h^* L \), and for every \( k = 2, \ldots, r \), define \( f^* E \to E^k \) to be the unique quotient whose kernel is contained in \( h^* K \) and such that \( h^* K \to E^k \) has image \( K^{k-1} \). Then \( \mu_B(E^1) = \mu_B(L) \leq \mu_B(E) + \epsilon \), and for \( k = 2, \ldots, r \),
\[
\mu_B(E^k) = 1/k(\mu_B(L) + (k-1)\mu_B(K^{k-1})) < 1/k(\mu_B(L) + (k-1)\mu_B(K) + (k-1)\epsilon) = \\
\frac{r(k-1)}{(r-1)k} \mu_B(E) + \frac{r-k}{(r-1)k} \mu_B(L) + \frac{(r-1)(k-1)}{(r-1)k} \epsilon < \mu_B(E) + \epsilon.
\]

For semistable bundles in characteristic zero, there is a more precise result.

**Theorem 6.10 (Zhang).** Let \( B \) be a smooth, projective curve over an algebraically closed field of characteristic 0. Let \( E \) be a semistable vector bundle on \( B \). Let \( \epsilon \) be a positive real number. There exists a cover \( f : C \to B \), invertible sheaves \( L_1, \ldots, L_r \) on \( C \), and a morphism of \( \mathcal{O}_C \)-modules, \( \phi : f^* E \to (L_1 \oplus \cdots \oplus L_r) \) such that

(i) the support of \( \text{coker}(\phi) \) is a finite set,
(ii) for every \( i = 1, \ldots, r \), the projection \( f^* E \to \oplus_{j \neq i} L_j \) is surjective,
(iii) for every \( i = 1, \ldots, r \), \( \mu_B(L_i) \leq \mu_B(E) + \epsilon \).

**Proof.** Denote \( r = \text{rank}(E) \). If \( r \) equals 1, the theorem is trivial. Thus assume \( r > 1 \). As in the proof of Theorem 6.7 there exists a cover \( g : C' \to B \) and an invertible sheaf \( M \) on \( C' \) such that \( 0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1) \). Denote \( \delta = \mu_B(E) - \mu_B(M) \) and denote \( F = g^* E \otimes M^\epsilon \). Then \( \mu_B(F) \) equals \( \delta \), and \( 0 < \delta < \epsilon/(r-1) \).

Let \( h : C \to C' \) be any cover and let \( h^* F \to N \) be an invertible quotient. The composition \( g \circ h : C \to B \) is a cover. By Kempf’s theorem, which ultimately relies on the theorem that every stable vector bundle admits a Hermite-Einstein metric, \((g \circ h)^* E \) is semistable. (Note, there are counterexamples in positive characteristic.) Therefore \( h^* F \) is semistable. So \( \mu_{C'}(L) \geq \mu_C(h^* F) \), i.e., \( \mu_{C'}(L) \geq \mu_C(F) = \delta \). Thus by Lemma 6.6 \( F \) is an ample vector bundle on \( C' \). Thus by Lemma 6.6 there exists a cover \( h : C \to C' \), invertible \( \mathcal{O}_C \)-modules \( N_1, \ldots, N_r \), and a morphism of \( \mathcal{O}_C \)-modules \( \psi : h^* F \to (N_1 \oplus \cdots \oplus N_r) \) satisfying (i), (ii) and (iii) of Lemma 6.6.

Define \( f = g \circ h \), \( L_i = N_i \otimes h^* M \) and \( \phi \) is the twist of \( \psi \) by \( \text{Id}_{h^* M} \). Then \( \phi \) satisfies (i) and (ii). And for every \( i = 1, \ldots, r \),
\[
\mu_B(L_i) = \mu_B(N_i) + \mu_B(M) = \mu_{C'}(N_i)/\deg(g) + \mu_B(E) - \delta = \\
\mu_B(E) + r\delta/\deg(g) - \delta \leq \mu_B(E) + (r-1)\delta/\deg(g) < \mu_B(E) + \epsilon.
\]

Of course, \( \mu'_{B'}(E) \) equals \( \mu_B(E) \). The other values are more interesting.

**Corollary 6.11.** The slopes \( \mu_k^B(E) \) satisfy \( \mu_1^B(E) \geq \mu_2^B(E) \geq \cdots \geq \mu_k^B(E) = \mu_B(E) \). For each \( 1 \leq k < r \), \( \mu_k^B(E) = \mu_B(E) \) iff \( f^* E \) is semistable for every cover \( f : C \to B \).
Proof. By Corollary 6.14 for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a rank $k$ quotient $f^*E \to E^k$ such that $\mu_B(E^k) < \mu_B(E) + \epsilon$. Thus $\mu^k_B(E) \geq \mu_B(E)$. Applying the same reasoning to rank $k - 1$ quotients of rank $k$ quotients of $f^*E$, $\mu^{k-1}_B(E) \geq \mu^k_B(E)$.

If $f^*E$ is semistable for every cover $f : C \to B$, then every vector bundle quotient of $f^*E$ has slope $\geq \mu_G(f^*E)$, and thus has $B$-slope $\geq \mu_B(f^*E)$. Therefore $\mu^k_B(E) \leq \mu_B(E)$, i.e., $\mu^k_B(E) = \mu_B(E)$.

Conversely, suppose there is a cover $f : C \to B$ such that $f^*E$ is not semistable. Then there exists a vector bundle quotient $f^*E \to F$ such that $\mu_B(F) < \mu_B(E)$. Denote the rank by $l$. Suppose first that $l \geq k$, and define $\epsilon = \deg(f)(\mu_B(E) - \mu_B(F))$. Then by Corollary 6.14 there exists a cover $g : C' \to C$ and a rank $k$ quotient $g^*F \to G$ such that $\mu_C(G) < \mu_C(F) + \epsilon$. Therefore $g^*f^*E \to g^*F \to G$ is a rank $k$ quotient of $g^*f^*E$ and $\mu_B(G) < \mu_C(F) + (\mu(B) - \mu_B(F)) = \mu_B(E)$. Therefore $\mu^k_B(E) > \mu_B(E)$.

Next suppose that $l < k$. Denote by $K$ the kernel of $f^*E \to F$. Then $r\mu_B(E) = l\mu_B(F) + (r - l)\mu_B(K)$. Define,

$$\epsilon = \frac{(r - k)\deg(f)(\mu_B(E) - \mu_B(F))}{(r - l)(k - l)}.$$  

By Corollary 6.14 there exists a cover $g : C' \to C$ and a rank $k - l$ quotient $g^*K \to G'$ such that $\mu_C(G') < \mu_C(K) + \epsilon$. Therefore $\mu_B(G') < \mu_B(K) + \epsilon/\deg(f)$. Define $g^*f^*E \to G$ to be the unique vector bundle whose kernel is contained in $g^*K$ and such that the image of $g^*K \to G$ equals $G'$. Then,

$$k\mu_B(G) = l\mu_B(F) + (k - l)\mu_B(G') < l\mu_B(F) + (k - l)\mu_B(K) + (k - l)\epsilon/\deg(f) =$$

$$l\mu_B(F) + \frac{k - l}{\deg(f)}(r\mu_B(E) - l\mu_B(F)) + \frac{k - l}{\deg(f)}\epsilon =$$

$$k\mu_B(E) - \frac{(r - k)}{r - l}(\mu_B(E) - \mu_B(F)) + \frac{(r - k)}{r - l}(\mu_B(E) - \mu_B(F)) = k\mu_B(E).$$

Thus $\mu_B(G) < \mu_B(E)$, and therefore $\mu^k_B(E) > \mu_B(E)$. \qed

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