Calculation of the $B \to \pi$ Transition Matrix Element in QCD

A. Khodjamirian$^{a,\dagger}$ and R. Rückl$^{a,b}$

$^a$ Institut für Theoretische Physik, Universität Würzburg, D-97074 Würzburg, Germany
$^b$ Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, D-80805 München, Germany

Abstract

We describe the calculation of the $B \to \pi$ hadronic matrix element from operator product expansion near the light-cone in QCD. In the resulting sum rules, the form factors $f^+$ and $F_0$ determining this matrix element are expressed in terms of pion light-cone wave functions. We present predictions for $f^+$ and $F_0$ in the region of momentum transfer $0 \leq p^2 < m_b^2 - \mathcal{O}(1 \text{ GeV}^2)$ taking into account the wave functions up to twist 4. We also discuss the extrapolation of the form factors to higher $p^2$ and their asymptotic dependence on the heavy quark mass.

$\dagger$ on leave from Yerevan Physics Institute, 375036 Yerevan, Armenia

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1 Introduction

The matrix element of the $B \to \pi$ transition can be written in terms of two independent form factors $f^+$ and $f^-$:

$$\langle \pi(q) | \bar{u} \gamma_{\mu} b | B(p + q) \rangle = 2f^+(p^2)q_{\mu} + (f^+(p^2) + f^-(p^2))p_{\mu},$$

where $p + q$ and $q$ denote the initial and final state four-momenta, respectively, and $\bar{u} \gamma_{\mu} b$ is the relevant quark vector current. The form factor $f^-$ is usually combined with $f^+$ into the scalar form factor

$$F_0(p^2) = f^+(p^2) + \frac{p^2}{m_B^2 - m_{\pi}^2} f^-(p^2).$$

A reliable estimate of the amplitude (1), or equivalently, of the form factors $f^+$ and $f^-(F_0)$ is very desirable, for example, in order to extract the CKM parameter $V_{ub}$ from measurements of the $B$ meson exclusive semileptonic widths $B \to \pi l \nu_l$. Obviously, such an estimate demands nonperturbative methods.

The current state-of-art in calculating $B \to \pi$ and other heavy-to-light form factors makes use of lattice simulations, heavy quark effective theory (HQET) combined with chiral perturbation theory, and QCD sum rules. Each of these methods has its limitations. For example, lattice calculations suffer from uncertainties connected with the necessary extrapolations to physical quark masses. Applications of HQET are restricted to the kinematical region of large momentum transfer, $p^2 \geq m_b^2 - O(1 \text{GeV}^2)$, i.e. to low momentum of the pion in the $B$ rest frame. Also, the corrections to the heavy mass expansion are often substantial and cannot be calculated directly within HQET.

Using QCD sum rule (1) techniques one can approach the problem of calculating the matrix element (1) in two different ways. One way is based on the operator product expansion (OPE) at short distances in terms of local operators. The second approach employs the more economical technique of OPE near the light-cone. In both methods one has to introduce new elements which cannot (yet) be calculated directly in QCD. In the more familiar short-distance approach, these are vacuum condensates. In the light-cone approach, on the other hand, one is led to the so-called light-cone wave functions. These nonperturbative input quantities are of universal nature so that sum rules have high predictive power. Furthermore, QCD sum rules provide a unique possibility to perform calculations in the experimentally favourable region of small and intermediate values of $p^2$:

$$0 \leq p^2 < m_b^2 - O(1 \text{GeV}^2).$$

In what follows we will report on the results of a calculation of the form factors $f^+$ and $F_0$ using the light-cone technique. We begin in section 2 with a short description of the method and present some of the main results. In section 3, we consider the extrapolation of these form factors to larger $p^2$, beyond the limit (3). Finally, in section 4 we discuss the behaviour of the form factors in the heavy mass limit.

2 Light-cone sum rules for $B \to \pi$ form factors

In order to obtain the sum rules for the form factors $f^\pm$ defined in (1), we start from the vacuum-to-pion correlation function

$$F_\mu(p, q) = i \int d^4x \ e^{ipx} \langle \pi(q) | T\{\bar{u}(x)\gamma_{\mu} b(x), \bar{b}(0)i\gamma_5 d(0)\} | 0\rangle$$

$$= i \int d^4x \ e^{ipx} \langle \pi(q) | T\{\bar{u}(x)\gamma_{\mu} b(x) \} | 0\rangle.$$
\[ F(p^2, (p+q)^2) q_\mu + \tilde{F}(p^2, (p+q)^2) p_\mu. \]  

(4)

The form factors \( f^+ \) and \( (f^+ + f^-) \) enter the dispersion relations for the invariant amplitudes \( F \) and \( \tilde{F} \) through the ground state \( B \)-meson contribution:

\[
F(p^2, (p+q)^2) = \frac{2f_Bm_B^2f^+(p^2)}{m_b(m_B^2 - (p+q)^2)} + \int_{s_0}^\infty ds \frac{\rho^+(p^2, s)}{s - (p+q)^2},
\]

(5)

\[
\tilde{F}(p^2, (p+q)^2) = \frac{f_Bm_B^2(f^+(p^2) + f^-(p^2))}{m_b(m_B^2 - (p+q)^2)} + \int_{s_0}^\infty ds \frac{\tilde{\rho}^+(p^2, s)}{s - (p+q)^2},
\]

(6)

where the defining relation of the \( B \) meson decay constant \( f_B \),

\[
\langle B | \bar{b}i\gamma_5d | 0 \rangle = \frac{m_b^2f_B}{m_b},
\]

(7)

has been used. The integrals in (5) and (6) over the spectral densities \( \rho^h \) and \( \tilde{\rho}^h \) take into account the contributions from the excited states and the continuum with \( B \) meson quantum numbers, \( s_0 \) denoting the effective mass threshold.

In certain momentum regions the correlation function \( F_\mu \) can be calculated by expanding the \( T \)-product of the currents near the light-cone \( x^2 = 0 \). Here, one uses the fact that in the region of large space-like momenta \( (p+q)^2 < 0 \) the virtual \( b \) quark is far off-shell. Simultaneously, one may keep the momentum \( q \) at the physical point \( q^2 = m^2_x \), and the momentum \( p \) in the timelike region \( p^2 < m^2_B - O(1 \text{ GeV}^2) \). In the following calculation we set \( m_x = 0 \). The leading contribution to the OPE of (4) arises from the contraction of the \( b \)-quark operators to the free \( b \)-quark propagator. Gluon emission from the virtual \( b \) quark contributes at next-to-leading order. After substituting the relevant expressions for the above, the correlation function (4) is expressed in terms of vacuum-to-pion matrix elements of the type

\[
\langle \pi(q)|\bar{u}(x)\Gamma_i d(0)|0\rangle \\
\langle \pi(q)|\bar{u}(x)\Gamma_j\lambda^a G^a_{\mu\nu}(vx) d(0)|0\rangle
\]

(8)

where \( \Gamma_i \) denote certain products of \( \gamma \)-matrices, \( G^a_{\mu\nu} \) is the gluon field tensor and \( 0 \leq v \leq 1 \). The path-ordered exponential gauge factors of the gluon field are implied in (8), but omitted in correspondence to the Fock-Schwinger gauge which is used here.

Each of the nonlocal operators in (8) is equivalent to an infinite series of local operators. The latter can be distinguished by their twist. A detailed study of this expansion can be found in [2]. The concept of twist turns out to be very useful, because the contributions of higher twist components of the matrix elements (8) to the correlation function \( F_\mu \) are suppressed by powers of the heavy quark propagator. If the pion momentum does not vanish, which is just the case under consideration, one has to keep all local operators with given twist in this series. This would demand introducing an infinite number of unknown local matrix elements. It is then better to work directly with the nonlocal matrix elements (8). Usually, one parametrizes the components of different twist in these matrix elements by certain distribution functions. These distributions, known as the light-cone wave functions of the pion, were already introduced [3, 4] before the invention of QCD sum rules in the context of exclusive hard processes. They play a similar role as the vacuum condensates entering the more conventional short-distance sum rules. Like the condensates, they may be determined [3] by confronting suitable sum rules with experimental data.
The final expressions for the light-cone sum rules for $\tilde{f}$ and the matrix elements from the continuum exponentially, one applies a Borel transformation in the variable $\tilde{g}$ the leading twist 2 wave function, and $g_1$ and $g_2$ denoting twist 4 components. Furthermore, the matrix elements

$$
\langle \pi(q) | \tilde{u}(x) \gamma_\mu \gamma_5 d(0) | 0 \rangle = -i q_\mu f_\pi \int_0^1 du \, e^{i u q x} \left( \varphi_\pi(u) + x^2 g_1(u) + O(x^4) \right) + f_\pi \left( x_\mu - \frac{x^2 q_\mu}{q \cdot x} \right) \int_0^1 du \, e^{i u q x} g_2(u) + ... ,
$$

where the r.h.s. represents the first few terms of the light-cone expansion in $x^2$, with $\varphi_\pi$ being the leading twist 2 wave function, and $g_1$ and $g_2$ denoting twist 4 components. Contributions of twist larger than 4 are neglected. Referring to \cite{6,7,8} for more details, we only mention here that the invariant amplitude $F$ receives the main contribution from the leading wave function $\varphi_\pi$, while $\varphi_\pi$ does not contribute to $\bar{F}$. The correction to the correlation function \cite{4} from gluon emission by the heavy quark involves quark-antiquark-gluon matrix elements of the type outlined in \cite{8}. The corresponding 3-particle wave functions are described in \cite{6,7}.

Direct calculation shows that the twist 3 and 4 three-particle terms contribute to the invariant amplitude $F$, but not to the invariant amplitude $\bar{F}$. Hence, to twist 4 accuracy, the result for $\bar{F}$ turns out to be remarkably simple:

$$
\bar{F}_{QCD}(p^2, (p + q)^2) = f_\pi \int_0^1 du \, \frac{1}{m_b^2 - (p + u q)^2} \left\{ \frac{\mu_\pi \varphi_\pi(u) + \mu_\pi \varphi_\sigma(u)}{6 u} \right\}
$$

$$
\times \left[ 1 - \frac{m_b^2 - p^2}{m_b^2 - (p + u q)^2} \right] + \frac{2 m_b g_2(u)}{m_b^2 - (p + u q)^2}.
$$

This expression as well as the analogous result for $F_{QCD}$ given in \cite{6,7} determine the l.h.s. of the expressions \cite{4} and \cite{5} respectively. In accordance with the standard procedure we use quark-hadron duality and replace $\rho^h$ and $\bar{\rho}^h$ at $s > s_0$ by the imaginary parts of $F_{QCD}$ and $\bar{F}_{QCD}$, respectively. Then, in order to suppress the contributions from the excited states and from the continuum exponentially, one applies a Borel transformation in the variable $(p + q)^2$.

The final expressions for the light-cone sum rules for $f^+$ and $f^+ + f^-$ read

$$
f^+(p^2) = \frac{m_b^2}{2 f_B M_B^2} \exp \left( \frac{m_B^2}{M^2} \right) \left\{ \int_0^1 du \, \exp \left[ -\frac{m_b^2 - p^2 (1 - u)}{u M^2} \right] \right\} \times \left( \varphi_\pi(u) + \frac{\mu_\pi}{m_b} \left[ u \varphi_p(u) + \frac{\varphi_\sigma(u)}{3} \left( 1 + \frac{m_b^2 + p^2}{2 u M^2} \right) \right] - \frac{4 m_b^2 g_1(u)}{u^2 M^2} \right) + \frac{2}{u M^2} \int_0^u g_2(v) dv \left( 1 + \frac{m_b^2 + p^2}{u M^2} \right) + f^+_G(p^2, M^2) + ... ,
$$
and

\[ f^+(p^2) + f^-(p^2) = \frac{f_B \mu_p m_b}{f_{B \pi} m_B^2} \exp \left( \frac{m_B^2}{M^2} \right) \left\{ \int_0^1 \frac{du}{u} \exp \left[ -\frac{m_b^2 - p^2(1-u)}{u M^2} \right] \right\} \times \left[ \varphi_p(u) + \frac{\varphi_{\pi}(u)}{6u} \left( 1 - \frac{2m_b^2 - p^2}{u M^2} \right) \right] + \ldots, \]

where \( M \) is the Borel parameter and \( \Delta = (m_b^2 - p^2)/(s_0 - p^2) \). In (13), \( f^+_c \) denotes the gluon corrections of twist 3 and 4 not shown here for brevity. The ellipses in (13) and (14) indicate threshold terms which emerge in addition to the integrals over \( u \) after subtraction of the spectral density of the higher states. These terms are investigated in detail in [8] and found to be quantitatively unimportant.

For the numerical analysis of the sum rules (13) and (14) we use the parameters and light-cone wave functions specified and discussed in detail in [6, 7]. In particular, we take the parameters \( f_B \) and \( s_0 \) from a QCD sum rule for the correlator of two \( B \) and \( \pi \) states. For consistency, this two-point sum rule is used without \( O(\alpha_s^0) \) corrections, since these corrections are also absent in the present sum rules (13) and (14). The formal expressions for the pion wave functions are collected in [7] and will not be presented again for the sake of brevity.

In order to extract reliable and selfconsistent values for \( f^+ \) and \( (f^+ + f^-) \), we restrict the Borel parameter \( M \) to the fiducial interval in which the twist 4 contributions to (13) and (14) do not exceed 10\%, and simultaneously the higher states do not contribute more than 30\%. For \( 0 \leq p^2 < 20 \text{ GeV}^2 \) these criteria are satisfied at \( 8\text{ GeV}^2 < M^2 < 12 \text{ GeV}^2 \). Within this interval both sum rules are found to be very stable.

The prediction for \( f^+ \) obtained from (13) at the central value, \( M^2 = 10 \text{ GeV}^2 \), of the fiducial interval is plotted in Fig. 1. Combining (13) and (14) and using the definition (2) one easily obtains the prediction for the scalar form factor \( F_0 \). This is also shown in Fig. 1.

3 Extrapolation to higher momentum transfer

At \( p^2 \rightarrow (m_B - m_\pi)^2 \), i.e. close to the kinematical threshold at zero pion recoil, the form factor \( f^+(p^2) \) is expected to have a simple pole behaviour,

\[ f^+_{\text{pole}}(p^2) = \frac{f_B \cdot g_{B^* B_\pi}}{2m_B \cdot (1 - p^2/m_B^2)} , \]

determined by the vector ground state \( B^* \). Here, the parameters \( f_{B^*} \) and \( g_{B^* B_\pi} \) are defined by the matrix elements

\[ \langle 0 \mid \bar{u} \gamma_\mu b \mid B^* \rangle = m_B \cdot f_{B^*} \cdot \epsilon_\mu , \]

and

\[ \langle B^{*+}(p) \pi^+(q) \mid \bar{B}^0(p + q) \rangle = -g_{B^* B_\pi} q_\mu \epsilon^\mu , \]

\( \epsilon_\mu \) being the polarization vector of the \( B^* \). The physical argument for the validity of the single-pole approximation (13) relies on the fact that the mass of the \( B^* \) is very close to the threshold

\[ \text{threshold} \approx m_B - m_\pi \approx 5.279 \text{ GeV} \]
of the $B \to \pi$ transition. However, there is a priori no reason to expect the approximation (15) to be valid at small $p^2$, i.e. far away from this threshold.

In [7] we investigated this question by comparing the prediction of the light-cone sum rule for $f^+$ with the extrapolation of the single-pole approximation. The $B^*B\pi$ coupling was estimated from the correlation function (4) using a double dispersion relation. We found the two descriptions of $f^+$, (13) and (15), to be very similar in shape and magnitude. Hence, they smoothly match at $p^2 \approx 15 \div 17 \text{ GeV}^2$ (see Fig. 1). From this, one may conclude that the contributions of excited $B^*$ states to $f^+$ are numerically not very important at intermediate momentum transfer. At smaller $p^2$, the numerical disagreement between the pole model extrapolation and the light-cone sum rule prediction for $f^+$ increases amounting to about 50 \% at $p^2 = 0$. This difference is insensitive to the choice of parameters, simply because the input used in the sum rule for $g_{B^*B\pi}$ is exactly the same as in the sum rule (13).

For several reasons the form factor $F_0$ cannot be approximated by a single scalar pole, not even at large $p^2$. Although the scalar $B$ states are still poorly known, it seems clear that the relevant distance between the lowest lying scalar pole and the physical region of the $B \to \pi$ transition is larger than in the case of $f^+$ and the vector $B^*$ pole. Therefore, excited scalar states may have substantial influence on $F_0$ even at $p^2$ close to the kinematical threshold $(m_B - m_\pi)^2$. Moreover, not only the scalar but also the vector states in principle contribute to the dispersion relation for $F_0$.

Interestingly, there exists a model independent constraint on the behaviour of the form factor $F_0$ at $p^2 \sim m_B^2$. From current algebra and PCAC one can obtain [9] a Callan-Treiman type relation:

$$\lim_{p^2 \to m_B^2} F_0(p^2) = f_B/f_\pi .$$

Unfortunately, current theoretical estimates of $f_B$ are still quite uncertain. A recent compilation of lattice data [10] gives

$$f_B/f_\pi = 1.0 \div 1.7.$$  \hspace{1cm} (19)

This is consistent with various QCD sum rule estimates (see e.g. [11] and [12]), but not yet very useful for constraining $F_0$.

4 \hspace{1cm} The heavy quark limits of $f^+$ and $(f^+ + f^-)$

The light-cone sum rules presented in section 2 provide a unique possibility to investigate the heavy mass dependence of the $B \to \pi$ transition matrix element. To this end, one introduces the scale-independent effective parameters $\bar{\Lambda}$, $\omega_0$ and $\tau$, through the relations

$$m_B = m_b + \bar{\Lambda},$$

$$s_0 = m_b^2 + 2m_b \omega_0,$$

$$M^2 = 2m_b \tau ,$$

and uses the familiar scaling laws for the coupling constants in the heavy quark limit:

$$f_B = \hat{f}_B/\sqrt{m_b}, \quad f_{B^*} = \hat{f}_{B^*}/\sqrt{m_b}$$ \hspace{1cm} (21)

These substitutions allow to readily extract the leading power of the heavy mass expansion of (13) and (14). We find that both sum rules have a consistent heavy mass expansion, that is the higher-twist contributions either have the same heavy mass behaviour as the leading
twist term, or they are suppressed by extra powers of the heavy quark mass. Furthermore, the heavy-mass behaviour of the form factors sharply differs at small and large momentum transfers. At \( p^2 = 0 \) and \( m_b \to \infty \) one has

\[
f^+(0) = F_0(0) \sim m_b^{-3/2},
\]

\[
f^+(0) + f^-(0) \sim m_b^{-3/2}.
\]

The fact that the light-cone sum rule predicts \( f^+(0) \sim m_b^{-3/2} \) was first noticed in [13]. In contrast, at large momentum transfers characterized by

\[
p^2 \sim m_b^2 - 2m_b\chi,
\]

where \( \chi \) is finite and does not scale with \( m_b \), so that the sum rules are still valid, we get

\[
f^+(p^2) \sim m_b^{1/2},
\]

\[
f^+(p^2) + f^-(p^2) \sim F_0(p^2) \sim m_b^{-1/2}.
\]

The sum rules thus nicely reproduce the asymptotic dependence of the form factors \( f^+ \) and \( f^- \) on the heavy quark mass \( m_b \) derived in [9, 14] in the kinematical region where the pion has small momentum in the rest frame of the \( B \) meson.

It should be stressed that there are no constraints from HQET when \( p^2 \to 0 \). Also the pole-dominance model which is sometimes used for extrapolating the form factors from the region (24) to zero momentum transfer cannot be trusted outside of (24). As our analysis for the \( f^+ \) form factor shows, at small momentum transfer higher states are expected to be important. Therefore, the change in the heavy mass behaviour between small and large \( p^2 \) predicted by the light-cone sum rules is very reasonable.

5 Conclusions

Sum rules on the light-cone are an economical and reliable method for calculating exclusive hadronic amplitudes involving a single pion (or kaon). Here, we have described the first complete evaluation of the \( B \to \pi \) transition matrix element in this framework.

The accuracy of the method is conservatively estimated to be around 20-30%. Currently, the main sources of uncertainty are our limited knowledge of the nonasymptotic terms in the wave functions and the lack of the calculation of the perturbative \( \alpha_s \)-corrections to the correlation function (4). There is room for improvement.

In Fig. 1, we compare our predictions for the form factors \( f^+ \) and \( F_0 \) with recent results of lattice calculations [15] taken from [16]. In the region of overlap, one observes encouraging agreement.

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Figure 1: $B \to \pi$ form factors $f^+(p^2)$ (upper) and $F_0(p^2)$ (lower). The solid curves show the sum rule predictions, the dashed curve depicts the pole approximation. The dots with error bars are lattice results (from $^{16}$).