Abstract. This paper establishes a bridge between linear logic and mainstream graph theory, building on previous work by Retoré (2003). We show that the problem of correctness for MLL+Mix proof nets is equivalent to the problem of uniqueness of a perfect matching. By applying matching theory, we obtain new results for MLL+Mix proof nets: a linear-time correctness criterion, a quasi-linear sequentialization algorithm, and a characterization of the sub-polynomial complexity of the correctness problem. We also use graph algorithms to compute the dependency relation of Bagnol et al. (2015) and the kingdom ordering of Bellin (1997), and relate them to the notion of blossom which is central to combinatorial maximum matching algorithms.

Alternating cycles in perfect matchings serve as witnesses of non-uniqueness, and in this significantly expanded journal version, we discuss connections with other kinds of constrained cycles known to be equivalent: semicycles in directed graphs, trails avoiding forbidden transitions, and properly colored cycles in edge-colored graphs. While the second one provides an explanation and generalization of Retoré’s “R&B-graphs”, the latter leads us to prove the coNP-hardness of Pagani’s visible acyclicity criterion for MELL proof nets. We also connect Lamarche’s essential nets to R&B-graphs.

1. Introduction

1.1. Algorithmics of proofs in linear logic. One of the major novelties introduced at the birth of linear logic [Gir87] was a representation of proofs as graphs, instead of trees as in natural deduction or sequent calculus. A distinctive property of these proof nets is that checking that a proof is correct cannot be done merely by a local verification of inference steps: among the graphs which locally look like proof nets, called proof structures, some are invalid proofs. Hence the correctness problem: given a proof structure, is it a real proof net?

A lot of work has been devoted to this decision problem, and in the case of the multiplicative fragment of linear logic (MLL), whose proof nets are the most satisfactory, it
can be considered solved from an algorithmic point of view. Indeed, Guerrini [Gue11] and Murawski and Ong [MO06] have found linear-time tests for MLL correctness; the problem has also been shown to be NL-complete by Jacobé de Naurois and Mogbil [JdNM11]. Both the linear-time algorithms we mentioned also solve the corresponding search problem: computing a sequentialization of a MLL proof net, i.e. a translation into sequent calculus.

However, for MLL extended with the Mix rule [FR94] (MLL+Mix), the precise complexity of deciding correctness has remained unknown (though a polynomial-time algorithm was given by Danos [Dan90]). Thus, one of our goals in this paper is to study the following problems:

**Problem 1.1 (MixCorr).** Given a proof structure $\pi$, is it an MLL+Mix proof net?

**Problem 1.2 (MixSeq).** Reconstruct a sequent calculus proof for an MLL+Mix proof net.

### 1.2. Proof nets vs graph theory.

It turns out that a linear-time algorithm for MixCorr follows immediately from already known results. The key is to use a construction by Retoré [Ret99, Ret03] to reduce it to the problem of uniqueness of a given perfect matching, which can be solved in linear time [GKT01]:

**Problem 1.3 (UniquenessPM).** Given a graph $G$, together with a perfect matching $M$ of $G$, is $M$ the only perfect matching of $G$? Equivalently, is there no alternating cycle for $M$?

This brings us to the central idea of this paper: from the point of view of algorithmics, MLL+Mix proof nets and unique perfect matchings are essentially the same thing. This allows us to apply matching theory to the study of proof nets, leading to several new results. Indeed, one would expect graph algorithms to be of use in solving problems on proof structures, since they are graphs! But for this purpose, a bridge between the theory of proof nets and mainstream graph theory is needed, whereas previous work on the former mostly made use of “homemade” objects such as paired graphs (an exception being Murawski and Ong’s use of dominator trees). By building on Retoré’s discovery of a connection with perfect matchings, this paper proposes such a bridge.

Thus, proof structures are revealed to be part of a family of graph-theoretic objects which admit equivalent (as shown by Szeider [Sze04]) “structure from acyclicity” properties. In linear logic, the corresponding acyclicity property has been known for a long time: it is the Danos–Regnier correctness criterion [DR89], a necessary and sufficient condition for a proof structure to be a proof net. These connections have also inspired new results concerning other members of this family, not only perfect matchings but also e.g. edge-colored graphs; that is the subject of another paper by the author [Ngu19].

### 1.3. Contributions.

First, we establish our equivalence by giving a translation from proof structures to graphs equipped with perfect matchings and vice versa (section 3). In the first direction, instead of reusing Retoré’s construction, we propose an alternative having better properties with respect to sequentialization. This yields a new graph-theoretic proof of the sequentialization theorem, i.e. the equivalence between MLL+Mix proof nets and Danos–Regnier acyclic proof structures (end of section 3.1).
1.3.1. Complexity of problems on proof nets. As already mentioned, we give the first linear-time algorithm for \textsc{MixCorr} (section 4.1). As for its sub-polynomial complexity (section 4.2), we show that \textsc{MixCorr} is in randomized NC and in \textsc{quasiNC} (informally, NC is the class of problems with efficient parallel algorithms). On the other hand, we have a sort of hardness result: if \textsc{MixCorr} were in NC – in particular, if it were in NL, as for MLL without Mix – this would imply a solution to a long-standing conjecture concerning the related unique perfect matching problem:

**Problem 1.4** (\textsc{UniquePM} \cite{KV85, GKT01, HMT06}). Given a graph $G$, determine whether it admits exactly one perfect matching and, if so, find this matching.

We then turn to the sequentialization problem, for which we provide a graph-theoretic reformulation, and an algorithm for this reformulation. This gives us a quasi-linear time solution to \textsc{MixSeq} (section 5); to our knowledge, this beats previous algorithms for \textsc{MixSeq}.

As a demonstration of our matching-theoretic toolbox, we also show how to compute some information on the set of all sequentializations, namely Bellin’s kingdom ordering \cite{Bel97} of the links of a MLL+Mix proof net (rediscovered by Bagnol et al. \cite{BDS15} under the name of order of introduction). We give a polynomial time and a \textsc{quasiNC} algorithm (section 6.1), both relying on an effective characterization of this ordering.

1.3.2. Further connections to graph theory. We also show that this notion of kingdom ordering admits a direct counterpart in unique perfect matchings. The above-mentioned characterization, when rephrased in the language of graph theory (section 6.2), turns out to involve objects which play a major role in matching algorithms, namely blossoms \cite{Edm65}. In this way, we obtain a new result of independent interest in combinatorics. The appendix of the conference version of this paper contained a direct proof of this result; instead of reproducing it here, we have moved it to the companion paper \cite{Ngu19}, and limit ourselves here to the equivalence with the already known \cite{Bel97} proof net version.

The rest of the paper consists of new material added for this journal version.

We discuss in section 7 how the Danos–Regnier correctness criterion relates to other notions of constrained cycles in graphs. This includes objects from mainstream graph theory already known \cite{Sze14} to be equivalent to alternating cycles in perfect matchings, such as semicycles in directed graphs \cite{SS79}, as well as Retoré’s “aggregates” \cite{Ret03, Chapter 2}, an early attempt to define a purely graph-theoretic counterpart to the theory of MLL+Mix correctness.

In section 8.2 we analyse Retoré’s “R&B-graphs” reduction \cite{Ret03}, and show that it can be understood in terms of graphs with forbidden transitions \cite{Sze03}, which can be seen as the generalization of paired graphs by dropping a disjointness condition.

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1 More precisely, $O((\log n)^2 \left(\log \log n\right)^2)$ time. Both this and our \textsc{quasiNC} algorithms rely on very recent advances, respectively on dynamic bridge-finding data structures \cite{HRT18} and on the perfect matching existence problem \cite{ST17}. Any further progress on these problems would lead to an improvement of our complexity bounds.

2 To be more accurate, in the reference given, which is a PhD thesis written in French, they are called “agrégats”. However, the word “aggregate” is indeed the official translation, and appeared in the title of the never published note *Graph theory from linear logic: Aggregates* (Preprint 47, Équipe de Logique, Université Paris 7). That title is also a good summary for what we try to achieve in the present paper and in \cite{Ngu19}. 
Finally, we apply graph-theoretic ideas to some other variants of linear logic. We exhibit a connection between R&B-graphs and Lamarche’s essential nets [Lam08] for intuitionistic MLL – whose correctness criterion underlies Murawski and Ong’s linear time algorithm [MO06] for (classical) MLL – via a correspondence between bipartite matchings and directed graphs (section 8.3). And in the last section, we sketch a coNP-hardness proof for Pagani’s visible acyclicity condition [Pag06, Pag12] on Multiplicative-Exponential Linear Logic (MELL) proof structures, by taking inspiration from edge-colored graphs (see e.g. [BJG09, Chapter 16]).

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2. Preliminaries

2.1. Terminology.

2.1.1. Graph theory. By default, “graph” refers to an undirected graph. Our paths and cycles are not allowed to contain repeated vertices. We will sometimes identify them with their sets of edges (which characterize them) and apply set operations on them. A bridge of a graph is an edge whose removal increases the number of connected components.

For directed graphs, the notion of connectedness we consider is weak connectedness, i.e. connectedness of the graph obtained by forgetting the edge directions. A predecessor (resp. successor) of a vertex is the source (resp. target) of some incoming (resp. outgoing) edge.

2.1.2. Complexity classes. We refer to [JdNM11, §1.4] for the logarithmic space classes L (deterministic) and NL (non-deterministic) and to [CSV84] for the class AC⁰ of constant-depth circuits. The class NC⁰ (resp. quasiNC⁰ [Bar92]) consists of the problems which can be solved by a uniform family of circuits of depth O(log⁴ n) and polynomial (resp. quasi-polynomial, i.e. 2O(log⁵ n)) size; NC = ∪k NC⁰ and quasiNC = ∪k quasiNC⁰.

It is well-known that AC⁰ ⊆ NC¹ ⊆ L ⊆ NL ⊆ NC² ⊆ NC ⊆ P.

2.2. Perfect matchings, alternating cycles and sequentialization.

Definition 2.1. Let G = (V, E) be a graph. A matching (resp. perfect matching) M in G is a subset of E such that every vertex in V is incident to at most one (resp. exactly one) edge in M. An alternating path (resp. cycle) for M is a path (resp. cycle) where, for every pair of consecutive edges, one of them is in the matching and the other one is not.

Testing the existence of a perfect matching in a graph – or, more generally, finding a maximum cardinality matching – is one of the central computational problems in graph

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This choice of terminology is common, see e.g. [BjG09, §1.4].

For NC⁰ and quasiNC⁰, we may take this to mean that there is a deterministic logarithmic space Turing machine which, given n in unary, computes the circuit for inputs of size n. We will not enter into the details of AC⁰ uniformity.
theory. Combinatorial maximum matching algorithms, starting with Edmonds’s blossom algorithm [Edm65], use alternating paths to iteratively increase the size of the matching; similarly, alternating cycles are important for the problems UNIQUENESSPM and UNIQUEPM because they witness the non-uniqueness of perfect matchings.

**Lemma 2.2** (Berge). Let $G$ be a graph and $M$ be a perfect matching of $G$. Then if $M' \neq M$ is a perfect matching, the symmetric difference $M \Delta M'$ is a vertex-disjoint union of cycles, which are alternating for both $M$ and $M'$. Conversely, if $C$ is an alternating cycle for $M$, then $M \Delta C$ is another perfect matching.

As an example, consider fig. 1a. The matching on the left admits an alternating cycle, the outer square; by taking the symmetric difference between this matching and the set of edges of the cycle, one gets the matching on the right. Conversely, the symmetric difference between both matchings (which, in this case, is their union) is the square. Note also that in fig. 1b, there is no alternating cycle because vertex repetitions are disallowed.

Another approach to finding perfect matchings, using linear algebra, was initiated by Lovász [Lov79] and leads to a randomized NC algorithm by Mulmuley et al. [MVV87]. Recently, Svensson and Tarnawski have shown that this algorithm can be derandomized to run in deterministic quasiNC [ST17].

There is also a considerable body of purely mathematical work on matchings, starting from the 19th century. Let us mention for our purposes a result dating from 1959.

**Theorem 2.3** (Kotzig [Kot59]). Let $G$ be a graph. Suppose that $G$ admits a unique perfect matching $M$. Then $M$ contains a bridge of $G$.

As remarked by Retoré [Ret03], Kotzig’s theorem leads to an inductive characterization of the set of graphs equipped with a unique perfect matching.

**Theorem 2.4** (Sequentialization for unique perfect matchings [Ret03]). The class $\mathcal{UPM}$ of graphs equipped with an unique perfect matching is inductively generated as follows:

- The empty graph (with the empty matching) is in $\mathcal{UPM}$.
- The disjoint union of two non-empty members of $\mathcal{UPM}$ is in $\mathcal{UPM}$.
- Let $(G = (V, E), M \subseteq E) \in \mathcal{UPM}$ and $(G' = (V', E'), M' \subseteq E') \in \mathcal{UPM}$, with $V$ and $V'$ disjoint. Let $U \subseteq V$, $U' \subseteq V'$ such that $U \neq \emptyset$ (resp. $U' \neq \emptyset$) unless $G$ (resp. $G'$) is the empty graph, and let $x, x'$ be two fresh vertices not in $V$ nor $V'$. Then $(G'' = (V'', E''), M'' \subseteq E'') \in \mathcal{UPM}$, where
  - $V'' = V \cup V' \cup \{x, x'\}$
  - $E'' = E \cup E' \cup \{(x, x')\} \cup (U \times \{x\}) \cup (U' \times \{x'\})$
  - $M'' = M \cup M' \cup \{(x, x')\}$

**Remark 2.5.** By relaxing the non-emptiness condition on $U$ and $U'$, the disjoint union operation becomes unnecessary; this is actually the original statement [Ret03, Theorem 1].

The inspiration for the above theorem comes from linear logic: it is a graph-theoretic version of the sequentialization theorems for proof nets, with Kotzig’s theorem being analogous to the “splitting lemmas” which appear in various proofs of sequentialization.

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5Note that the problem was solved long before in the special case of bipartite graphs. In fact, a solution for this case was found in Jacobi’s posthumous papers [Jac65, JO09].

6This paper is one of the first to propose defining efficient algorithms as polynomial-time algorithms; it also contributed to the birth of the field of polyhedral combinatorics.
2.3. Proof structures, proof nets and the correctness criterion. A proof structure is some kind of graph-like object made of “nodes” (or “formulae”) and “links”, with the precise definition varying in the literature (we will come back to this point in Remark 8.5). Since our aim is to apply results from graph theory, it will be helpful to commit to a representation of proof structures as graphs.

We write \( \deg^- \) for the indegree and \( \deg^+ \) for the outdegree of a vertex.

**Definition 2.6.** A proof structure is a non-empty directed acyclic multigraph \((V, A)\) with a labeling of the vertices \( l : V \to \{\text{ax}, \otimes, \varnothing\} \) such that, for \( v \in V \):

- if \( l(v) = \text{ax} \), then \( \deg^-(v) = 0 \) and \( \deg^+(v) \leq 2 \),
- if \( l(v) \in \{\otimes, \varnothing\} \), then \( \deg^-(v) = 2 \) and \( \deg^+(v) \leq 1 \).

Vertices of a proof structure will also be called links. A terminal link is a link with outdegree 0. A sub-proof structure is a vertex-induced subgraph which is a proof structure.

**Definition 2.7.** The set of MLL proof nets is the subset of proof structures inductively generated by the following rules:

- **ax-rule:** a proof structure with a single \( \text{ax} \)-link is a proof net.
- **\( \otimes \)-rule:** if \( N \) and \( N' \) are proof nets, \( u \) is a link of \( N \) and \( v \) is a link of \( N' \), then taking the disjoint union of \( N \) and \( N' \), adding a new \( \otimes \)-link \( w \), an edge from \( u \) to \( w \) and an edge from \( v \) to \( w \) gives a proof net, as long as the resulting graph is a proof structure (i.e. the degree constraints are satisfied).
- **\( \varnothing \)-rule:** if \( N \) is a proof net and \( u, v \) are links of \( N \), then adding a new \( \varnothing \)-link \( w \), an edge from \( u \) to \( w \) and an edge from \( v \) to \( w \) gives a proof net, with the same proviso as above.

The set of MLL+Mix proof nets is inductively generated by the above rules together with the Mix rule: if \( N \) and \( N' \) are proof nets, their disjoint union is a proof net.

![Proof net and its sequentialization](image)
Figure 4. 2 switchings out of 4 possibilities for the proof structure of fig. 2

A proof structure is said to be correct if it is a MLL+Mix proof net.

Remark 2.8. As with any inductively defined set, membership proofs for the set of MLL (resp. MLL+Mix) proof nets may be presented as inductive derivation trees, which are isomorphic to the usual sequent calculus proofs of MLL (resp. MLL+Mix): see fig. 2 for an example, and fig. 3 for the inference rules of the sequent calculus.

Remark 2.9. The proof structures and proof nets defined here are cut-free. This restriction is without loss of generality, since cut link has exactly the same behavior as a terminal $\otimes$-link with respect to correctness and sequentialization.

To tackle the problem of correctness, it is useful to have non-inductive characterizations of proof nets, called correctness criteria, at our disposal. Many of them are formulated using the notion of paired graphs. We will state a criterion first discovered by Danos and Regnier for MLL [DR89] and extended to MLL+Mix by Fleury and Retoré [FR94].

Definition 2.10. A paired graph consists of an undirected graph $G = (V, E)$ and a set $\mathcal{P}$ of unordered pairs of edges such that:
- if $\{e, f\} \in \mathcal{P}$, then $e$ and $f$ have a vertex in common;
- the pairs are disjoint: if $p, p' \in \mathcal{P}$ and $p \neq p'$, then $p \cap p' = \emptyset$.

When $\{e, f\} \in \mathcal{P}$, the edges $e$ and $f$ are said to be paired.

A switching of this paired graph is a spanning subgraph of $G$ which intersects each pair of $\mathcal{P}$ exactly once. A feasible cycle is a cycle which intersects each pair of $\mathcal{P}$ at most once.

Remark 2.11. Equivalently, feasible cycles are cycles which exist in some switching.

Definition 2.12. Let $\pi$ be a proof structure. Its correctness graph $C(\pi)$ is the paired graph obtained by forgetting the directions of the edges and the labels of the vertices in $\pi$, and pairing together two edges when their targets are the same $\otimes$-link.

A feasible cycle in $\pi$ is a sequence of edges of $\pi$ whose image in $C(\pi)$ is a feasible cycle.

Examples of switchings of a correctness graph are given in fig. 4.

Theorem 2.13 (Danos–Regnier correctness criterion). $\pi$ is a MLL (resp. MLL+Mix) proof net if and only if all the switchings of $C(\pi)$ are trees (resp. forests).

Remark 2.14. Equivalently, $\pi$ is a MLL+Mix proof net iff it contains no feasible cycle.

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$^7$That is, the targets of the directed edges in $\pi$ they come from.
Before graphification. The left \( \otimes \)-link and the right \( \otimes \)-link of (b) correspond to the \( \text{ax} \)-links here.

After proofification.

Figure 5. Composing graphification and proofification: as we will see, the graphification of the proof net of (a) is the graph of fig. 1b and, in turn, the proofification of fig. 1b is (b).

The above is usually called a sequentialization theorem: it means that a proof structure which satisfies the correctness criterion admits a sequent calculus derivation.

The analogy with Theorem 2.4 is that proof nets are to proof structures what unique perfect matchings are to perfect matchings. The next section is dedicated to formalizing this analogy into an equivalence.

3. AN EQUIVALENCE THROUGH MUTUAL REDUCTIONS

We will now see how to turn a proof structure into a graph equipped with a perfect matching, in such a way that feasible cycles become alternating cycles, and vice versa.

Such a translation from proof structures to perfect matchings was first proposed by Retoré [Ret03], under the name of R&B-graphs (defined in section 8.2). However, we would like to deduce Theorem 2.13 as an immediate corollary of sequentialization for unique perfect matchings (Theorem 2.4), which is not possible with R&B-graphs – instead, one must resort to a proof of induction using Kotzig’s theorem (Theorem 2.3), see [Ret99, §2.4]. Thus, we propose here our own graphification construction. We also define the proofification construction, going from perfect matchings to proof structures.

Remark 3.1. The nature of the object corresponding to a matching edge in a proof structure will vary depending on the translation considered: for graphifications, they correspond to links, whereas in the case of proofifications, they are translated into \( \otimes \)-links (and for R&B-graphs, they correspond to edges or terminal links).

Thus, by taking the proofification of a graphification of a proof structure, one gets a different proof structure, with the \( \text{ax} \)-links and \( \otimes \)-links of the former being sent to \( \otimes \)-links of the latter (see fig. 5 for an example). It is unclear whether this transformation has any meaning in terms of linear logic; in particular it does not preserve correctness for MLL without Mix.

3.1. FROM PROOF STRUCTURES TO PERFECT MATCHINGS

Definition 3.2. Let \( \pi \) be a proof structure and \( L \) be its set of links. The graphification of \( \pi \) is the graph \( G = (V, E) \) equipped with a perfect matching \( M \subseteq E \) with
Translation rules for sets of incoming edges.

Graphification of the proof structure of fig. 2.

(a) Translation rules for sets of incoming edges.
(b) Graphification of the proof structure of fig. 2

Figure 6. The graphification construction.

- the matching edges corresponding to the links: \( V = \bigcup_{l \in L} \{a_l, b_l\}, M = \{(a_l, b_l) \mid l \in L\} \),
- and the remaining edges in \( E \setminus M \) reflect the incoming edges of the \( \otimes \)-links and \( \& \)-links, as specified by fig. 6a.

Figure 6b shows an example of this construction. As another example, fig. 1b is the graphification of fig. 5a.

**Proposition 3.3** (Graphification-based correctness criterion). A proof structure satisfies the Danos–Regnier criterion for MLL+Mix if and only if the perfect matching of its graphification is unique.

**Proof.** By negating the two sides of the equivalence, the goal becomes proving that a proof structure \( \pi \) contains a feasible cycle if and only if its graphification \((G, M)\) contains an alternating cycle.

Consider any alternating cycle for \( M \) in \( G \) of length \( 2n \), and take the \( \mathbb{Z}/(n) \)-indexed sequence of vertices corresponding to the matching edges in the cycle. By construction of the graphification, if two edges in \( M \) are incident to a common non-matching edge, then the corresponding links in \( \pi \) are adjacent: thus, in our sequence, each vertex is adjacent to the previous and the next one, and thus we have a cycle. If it were not feasible, it would contain three consecutive links \( p, q, r \) with \( q \) a \( \& \)-link and \( p, r \) its predecessors\(^8\), but then the alternating cycle would have to cross two incident non-matching edges (from \( p \) to \( q \) and from \( q \) to \( r \)), which is impossible. Thus, \( \pi \) contains a feasible cycle.

To show the converse we will exhibit a right inverse to the map from alternating cycles to feasible cycles defined above. Consider a feasible cycle: it can be partitioned into directed paths from \( \text{ax} \)-links to \( \otimes \)-links. Let \( l \) be an intermediate link in such a path, and \( e, p, s \) be matching edges corresponding respectively to \( l \), its predecessor, and its successor in the directed path. \( s \) has a unique endpoint \( u \) which is incident to both endpoints of \( e \); \( e \) has a unique endpoint \( v \) which is *not* incident to both endpoints of \( p \). To join \( e \) with \( s \), we use the edge \((u, v)\). By taking all these non-matching edges for all maximal directed paths in the cycle, as well as a choice of two edges incident to each matching edge corresponding to

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\(^8\)To expand on this point: this is because we have prohibited vertex repetitions in our definition of cycles. This is legitimate since a graph is a forest if and only if it does not contain a non-vertex-repeating cycle.
Proof. Suppose for contradiction that \( k \) involves \( " \text{criterion. Therefore, this criterion characterizes MLL+Mix proof nets: as we wanted, we } \)

In the case of R&B-graphs, there is an actual bijection between the feasible cycles of a proof structure and the alternating cycles of its R&B-graph. That said, the main technical advantages of graphifications over R&B-graphs are summarized by the following properties.

**Lemma 3.4.** Let \( \pi \) be a proof structure with graphification \((G, M)\) and \( l \) be a link of \( \pi \) such that \((a_l, b_l) \in M\) is a bridge of \( G \). Then \( l \) is a terminal link in \( \pi \), and if \( l \) is a \( \otimes\)-link, then removing \( l \) from \( \pi \) disconnects its predecessors.

**Proof.** Suppose for contradiction that \( l \) is not a terminal link, and let \( l' \) be a successor of \( l \). Then for some endpoint \( v \) of \((a_v, b_v)\), \((a_l, v)\) and \((b_l, v)\) are both edges in \( G \), and they make up a path between \( a_l \) and \( b_l \) not going through \((a_l, b_l)\). Thus, \((a_l, b_l)\) cannot be a bridge.

The fact that \((a_l, b_l)\) is a bridge means that by removing this edge, \( a_l \) and \( b_l \) are in different connected components; if \( l \) is a \( \otimes\)-link, each of these connected components contain the matching edge corresponding to one predecessor of \( l \).

**Theorem 3.5.** Let \( \pi \) be a proof structure and \((G, M)\) be its graphification. There is a bijection between the sequent calculus proofs corresponding to \( \pi \) (if any) and the sequentializations (i.e. the derivation trees for the inductive definition of Theorem 2.14) of \((G, M)\) (if any), through which occurrences of Mix rules correspond to disjoint unions and conversely.

**Proof.** We convert a sequentialization \( S \) of \((G, M)\) into a sequentialization \( \Sigma \) of \( \pi \) inductively as follows. Since \( G \neq \emptyset \), the last rule of \( S \) is either a disjoint union or the introduction of a bridge \( e = (a_l, b_l) \in M \) by joining together \((G_a, M_a)\) and \((G_b, M_b)\) with respective sequentializations \( S_a \) and \( S_b \). In the latter case, \( l \) is a terminal link of \( \pi \).

- If \( G_a = G_b = \emptyset \), then \( l \) is an \( \text{ax}\)-link, and \( \Sigma \) consists of a single \( \text{ax}\)-rule.
- If \( G_a \neq \emptyset \) and \( G_b = \emptyset \), then \( l \) is a \( \otimes\)-link, and the removal of \( l \) from \( \pi \) yields a proof structure \( \pi' \) whose graphification is \((G_a, M_a)\). \( \Sigma \) then consists of a \( \otimes\)-rule introducing \( l \) applied to the sequentialization of \( \pi' \) corresponding to \( S_a \).
- If \( G_a \neq \emptyset \) and \( G_b \neq \emptyset \), then \( l \) is a \( \otimes\)-link. Since \( e \) is a bridge, the removal of \( l \) from \( \pi \) yields two proof structures \( \pi_a \) and \( \pi_b \) whose respective graphifications are \((G_a, M_a)\) and \((G_b, M_b)\). \( \Sigma \) then consists of an \( \otimes\)-rule applied to the translations of \( S_a \) and \( S_b \).

If the last rule of \( S \) is a disjoint union rule, it is translated into a Mix rule in \( \Sigma \).

The bijectivity can be proven by defining the inverse transformation and by checking that it is indeed its inverse.

In particular, \( \pi \) is a MLL+Mix proof net if and only \((G, M)\) admits a sequentialization, that is, according to Theorem 2.14 if and only if \( M \) is the only perfect matching of \( G \). Proposition 3.3 tells us that this is equivalent to \( \pi \) satisfying the Danos–Regnier acyclicity criterion. Therefore, this criterion characterizes MLL+Mix proof nets: as we wanted, we just proved the sequentialization theorem for MLL+Mix (Theorem 2.13).

### 3.2. From perfect matchings to proof structures.

The translation we present below involves “\( k \)-ary \( \otimes\)-links”. When \( k > 1 \), these are just binary trees of \( k - 1 \) \( \otimes\)-links (correctness is independent of the choice of binary tree: semantically, this is associativity of \( \otimes \)) with \( k \) leaves (incoming edges) and a single root (outgoing edge); the \( k = 1 \) case corresponds to a single edge and no link.
**Definition 3.6.** Let $G = (V, E)$ be a graph and $M$ be a perfect matching of $G$. We define the *proofification* of $(G, M)$ as the proof structure $\pi$ built as follows:

- For each non-matching edge $e = (u, v) \in E \setminus M$, we create an $\text{ax}$-link $\text{ax}_e$ whose two outgoing edges we will call $A_{u,v}$ and $A_{v,u}$.
- For each vertex $u \in V$, if $\text{deg}(u) > 1$, we add a $k$-ary $\otimes$-link with $k = \text{deg}(u) - 1$, whose incoming edges are the $A_{u,v}$ for all neighbors $v$ of $u$ such that $(u, v) \notin M$, and we call its outgoing edge $B_u$. If $\text{deg}(u) = 1$, we add an $\text{ax}$-link calling one of its outgoing edges $B_u$.
- For each matching edge $(u, v) \in M$, we add an $\otimes$-link whose incoming edges are $B_u$ and $B_v$. These $\otimes$-links are the terminal links of $\pi$.

See Fig. 7 for an annotated example of proofification. The reader may also check that the proof net in Fig. 5b is the proofification of the graph in Fig. 1b.

**Proposition 3.7.** Let $G$ be a graph and $M$ be a perfect matching of $G$. The alternating cycles for $M$ in $G$ are in bijection with the feasible cycles in the proofification of $(G, M)$.

**Proof.** Let $\pi$ be the proofification of $(G, M)$. Any feasible cycle in $\pi$ changes direction only at $\text{ax}$-links and $\otimes$-links, and therefore can be partitioned into an alternation of $\otimes$-links, corresponding to matching edges, and of paths starting with some $B_u$, ending with some $B_v$ and crossing some $\text{ax}_e$, corresponding to non-matching edges $e = (u, v)$. Therefore, it corresponds to an alternating cycle for $M$, and the mapping defined this way is bijective.

**Proposition 3.8.** Let $G$ be a graph with a unique perfect matching $M$ and let $\pi$ be the proofification of $(G, M)$. A matching edge $e \in M$ is a bridge of $G$ if and only if its corresponding $\otimes$-link is introduced by the last rule of some sequentialization of $\pi$.

**Proof.** This follows from the fact that a $\otimes$-link may be introduced by the last rule of a sequentialization if and only if it is splitting, i.e. its removal disconnects its two predecessors.

However, unlike the case of graphifications, this does not give us a bijection between the sequentializations of a unique perfect matching and those of its proofification.

### 4. On the Complexity of MLL+Mix Correctness

Through the translations of the previous section, MLL+Mix proof nets become *unique* perfect matchings and conversely: these translations provide *reductions* between the problems MIXCORR and UNIQUENESSPM, allowing us to draw complexity-theoretic conclusions on
proof nets from known results in graph theory. We first look at the time complexity of MixCorr, then turn to its complexity under constant-depth (AC⁰) reductions.

4.1. A linear-time algorithm. Since graphifications (section 3.1) can be computed in linear time, and UniquenessPM can also be decided in linear time [GKT01, §3], we immediately get:

**Theorem 4.1.** MixCorr can be decided in linear time.

**Remark 4.2.** By using the “Euler–Poincaré lemma” [BDS15] to count the uses of the Mix rule in a proof net, this also allows us to decide the correctness of a proof structure for MLL without Mix in linear time. Our decision procedure has the advantage of being simpler to describe than the previously known linear-time algorithms for MLL correctness [Gue11, MO06].

That said, this apparent simplicity is due to our use of the algorithm of Gabow et al. [GKT01] as a black box. Looking inside the black box reveals, for instance, that it uses the incremental tree set union data structure of Gabow and Tarjan [GT85], which is also a crucial ingredient of the above-mentioned previous algorithms.

**Remark 4.3.** This algorithm for UniquenessPM relies on the technique of blossom shrinking pioneered by Edmonds [Edm65], a kind of graph contraction which may remind us of the contractibility correctness criterion [Dan90] for MLL without Mix. Indeed, there exists a formal connection: a rewrite step of big-step contractibility [BDS15] corresponds, when translated to graphifications, to contracting a blossom. However, not all blossoms are re-dexes for big-step contractibility. See section 6.2 for further discussion of blossoms.

4.2. Characterizing the sub-polynomial complexity. For MLL proof nets without Mix, correctness is known to be NL-complete under AC⁰ reductions thanks to the Mogbil–Naurois criterion [JdNM11]. What about MLL+Mix? Since the reductions of section 3 can be computed in constant depth, we have:

**Theorem 4.4.** MixCorr and UniquenessPM are equivalent under AC⁰ reductions.

Thus, it will suffice to study the complexity of UniquenessPM. Let us start with a positive result, using the parallel algorithms for finding a perfect matching mentioned in section 2.2.

**Proposition 4.5.** UniquenessPM is in randomized NC and in deterministic quasiNC.

**Proof.** Let G = (V, E) be a graph and M be a perfect matching of G. M is not unique if and only if, for some e ∈ M, the graph G_e = (V, E \ {e}) has a perfect matching. To test the uniqueness of M, run the |M| parallel instances, one for each G_e, of a randomized NC [MVV87] or deterministic quasiNC [ST17] algorithm for deciding the existence of a perfect matching, and compute the disjunction of their answers in AC⁰. □
Being in quasiNC is a much weaker result than being in NL. But as we shall now see, even showing that \textsc{UniquenessPM} is in NC (recall that NL ⊂ NC) would be a major result. It would answer in the affirmative the following conjecture dating back from the 1980’s:

**Conjecture 4.6** (Lovász\textsuperscript{10}). \textsc{UniquePM} is in NC.

Indeed, the following shows that \textsc{UniquenessPM} ∈ NC ⇒ \textsc{UniquePM} ∈ NC (and the converse follows from the definitions).

**Proposition 4.7.** There is a NC\textsuperscript{2} reduction from \textsc{UniquePM} to \textsc{UniquenessPM}.

**Proof.** This is a consequence of a NC\textsuperscript{2} algorithm by Rabin and Vazirani [RV89, §4] which, given a graph \(G\), computes a set of edges \(M\) such that if \(G\) admits a unique perfect matching, then \(M\) is this matching. Starting from any graph \(G\), run this algorithm and test whether its output is a perfect matching. If not, then \(G\) does not admit a unique perfect matching; if it is, then \(G\) is a positive instance of \textsc{UniquePM} if and only if \((G, M)\) is a positive instance of \textsc{UniquenessPM}.

To sum up these results about \textsc{UniquenessPM}, which apply to \textsc{MixCorr}:

**Theorem 4.8.** \textsc{MixCorr} is in randomized NC and in deterministic quasiNC; it is in deterministic NC if and only if Conjecture 4.6 is true.

### 5. Sequentializing MLL+Mix proof nets

In section 4.1, we managed to solve MLL+Mix correctness in linear time, matching the known time complexity for MLL correctness. But the algorithms for MLL correctness still have an advantage: they can compute a sequentialization in linear time, whereas we only have a decision procedure for \textsc{MixCorr} which returns a yes/no answer\textsuperscript{11}. We do not know how to compute MLL+Mix sequentializations in linear time. Nevertheless, by applying our bridge between proof nets and graph theory, we get the first quasi-linear time algorithm for \textsc{MixSeq}. The beginning of the next section will discuss why the problem seems harder with Mix.

Our algorithm proceeds in a “top-down” way: it starts by determining the root of the derivation tree and the link it introduces. To obtain the children of the root, it suffices to recurse on the connected components created by removing this link.

Furthermore, through the correspondence of Theorem 3.5 finding a link which is introduced by the last rule of some sequentialization amounts to finding a bridge in the matching of the graphification of the proof net (cf. section 3.1). This is in fact a bit more convenient with graphifications than with general unique perfect matchings, thanks to the following property:

**Lemma 5.1.** All bridges in the graphification of some proof structure are matching edges.

\textsuperscript{9}In fact, one can show that NL ⊊ NSPACE\((O(\log^{3/2} n))\) ⊊ quasiNC\textsuperscript{3}, and the latter is where Svensson and Tarnawski’s analysis puts finding a perfect matching.

\textsuperscript{10}The conjecture is attributed to Lovász by a paper by Kozen et al. [KVV85] which claims to solve it. But Hoang et al. [HMT06] note that “this was later retracted in a personal communication by the authors”. Still, the proposed solution works for bipartite graphs.

\textsuperscript{11}It can find a feasible cycle, witnessing incorrectness, but cannot produce a certificate of correctness.
Proof. Let $e$ be a non-matching edge. Then there are matching edges $(u, v)$ and $(s, t)$ such that the link corresponding to $(u, v)$ is the predecessor of the one for $(s, t)$, and $e = (u, s)$. The non-matching edge $(v, s)$ is then also present in the graph, and so $e$ cannot be a bridge.

The algorithm will alternate between finding and deleting bridges; a deletion may cut cycles and thus create new bridges, which we want to detect without traversing the entire graph each time. To do so, we use a dynamic bridge-finding data structure designed for this kind of use case by Holm et al. \cite{HRTS}. It keeps an internal state corresponding to a graph, whose set of vertices is immutable but whose set of edges may vary, and supports the following operations in $O((\log n)^2(\log \log n)^2)$ amortized time:

- updating the graph by inserting or deleting an edge;
- computing the number of vertices of the connected component of a given vertex;
- finding a bridge in the connected component of a given vertex;
- determining whether two vertices are in the same connected component.

Theorem 5.2. MIXSEQ can be solved in $O(n(\log n)^2(\log \log n)^2)$ time.

Proof. Let $\pi$ be a MLL+Mix proof net with $n$ links, and $(G = (V, E), M)$ be its graphification. Both $V$ and $E$ are have cardinality $O(n)$ (in fact, $|V| = 2n$ and $|M| = n$).

The algorithm starts by initializing the bridge-finding data structure $D$ with the graph $G$, computing the weakly connected components of $\pi$ in linear time, and selecting a link in each component. On each selected link $l$, we call the following recursive procedure; its role is to sequentialize the sub-proof net of $\pi$ containing $l$ whose graphification is a current connected component of $G$ ($G$ and $D$ being mutable global variables):

- Let $u$ be one endpoint of the matching edge corresponding to $l$. Using the bridge-finding structure, find a bridge $e = (v, w)$ in the component of $u$; necessarily, $e \in M$. Remove the edge $e$ from $G$ (and reflect this change on $D$ with a deletion operation).
- If both $v$ and $w$ are isolated vertices, $e$ corresponds to an $\texttt{ax}$-link and the entire sub-proof net consisted of this link. In this case, return a sequentialization with a single $\texttt{ax}$-rule.
- If one of $v$ and $w$ is isolated, and the other is not — by symmetry, let us assume the latter is $v$ — then $e$ corresponds to a $\otimes$-link $l'$. Let $p$ and $p'$ be its predecessors.
  - Remove all edges incident to $v$.
  - If the matching edges corresponding to $p$ and $p'$ are in the same connected component of $G$, recurse on $p$, add a final $\otimes$-link and return the resulting sequentialization.
  - If $p$ and $p'$ are in different connected components of $G$, recurse on $p$ and $p'$, use the results as the two premises of a Mix rule, add a final $\otimes$-link and return the resulting sequentialization.
- If neither $v$ nor $w$ is isolated, $e$ corresponds to a $\otimes$-link. This is handled similarly to the $\otimes$+Mix case above.

Let us evaluate the time complexity. At each recursive call, one bridge is eliminated from $G$, so the number of recursive calls is $n$. The cost of each recursive call is $O(1)$ except for the updates and queries of the bridge-finding data structure. In total, there are $|E| = O(n)$ deletions, $|M| = n$ bridge queries, and at most $n$ connectedness tests, and each of those takes $O((\log n)^2(\log \log n)^2)$ amortized time. Hence the $O(n(\log n)^2(\log \log n)^2)$ bound.
Remark 5.3. If we want to compute a sequentialization for a unique perfect matching, in general, a complication is the existence of bridges which are not in the matching.

Interestingly, one can determine whether a bridge $e$ is in $M$ without looking at $M$: it is the case if and only if both of the connected components created by removing $e$ have an odd number of vertices. This leads to an algorithm for UniquePM; it is virtually the same as the one proposed by Gabow et al. [GKT01, §2] from which we took our inspiration.

Remark 5.4. One needs to use a sparse representation for derivation trees: the size of a fully written-out sequent calculus proof is, in general, not linear in the size of its proof net.

6. On the kingdom ordering of links

One may wonder if we could not have just tweaked an algorithm for MLL sequentialization into an algorithm for MixSeq. In order to argue to the contrary, let us briefly mention a difference between Bellin and van de Wiele’s study of the sub-proof nets of MLL proof nets [BvdW95] and its extension to the MLL+Mix case by Bellin [Bel97]. Any MLL sub-proof net of a MLL proof net may appear in the sequentialization of the latter; however, for MLL+Mix, fig. 8 serves as a counterexample: the sub-proof structure containing all links but the $\otimes$-link is correct for MLL+Mix, but it cannot be an intermediate step in a sequentialization of the entire proof net. A normality condition is needed to distinguish those sub-proof nets which may appear in a sequentialization, and this is why sequentialization algorithms which are morally based on a greedy parsing strategy, such as Guerrini’s linear-time algorithm [Gue11], do not adapt well to the presence of the Mix rule.

Any link $l$ in a MLL+Mix proof net $\pi$ admits a minimum normal sub-proof net of $\pi$ containing $l$, its kingdom [Bel97]. Bellin’s kingdom ordering is the partial order on links corresponding to the inclusion between kingdoms. We give an algorithm to compute this order for any MLL+Mix proof net: this is yet another application of matching theory. It uses a characterization of the kingdom ordering in terms of a relation called dependency by Bagnol et al. [BDS15] (who, in turn, take this name from the closely related dependency graph of Mogbil and Naurois [JdNM11]). We will also see how this dependency relation can be reformulated, through our correspondence between proof structures and perfect matchings, in terms of the blossoms mentioned in section 2.2 and section 4.1.

One may in fact define the kingdom ordering, written $\ll_{\pi}$, without reference to the notion of normal sub-proof net (we will not introduce the latter formally here):

Definition 6.1. Let $\pi$ be a MLL+Mix proof net. For any two links $p, q$ of $\pi$, $p \ll_{\pi} q$ if and only if, in any sequentialization of $\pi$, the rule introducing $q$ has, among its premises, a proof net containing $p$.

From this point of view, the kingdom ordering gives us information about the set of all sequentializations. Let us give some examples. The proof net of fig. 5b admits a unique sequentialization, so this directly gives us the kingdom ordering: for instance the middle $\otimes$-link is the greatest element. On the other hand, in the proof net of fig. 8 both $\otimes$-links may be introduced by a last rule, so there is no greatest element. In fact, the kingdom

\[12\text{Not to be confused with their algorithm for UniquenessPM [GKT01, §3] that we used in section 4.1. They only claim a bound of } O(m \log^4 n) \text{ because the best dynamic 2-edge-connectivity data structure known at the time has operations in } O(\log^5 n) \text{ amortized time.}\]
ordering coincides with the predecessor relation. So it does not distinguish between the 3 terminal links even though, unlike the 2 others, the $\otimes$-link cannot be introduced last.

Before proceeding further, here is another property of MLL proof nets which is contradicted by fig. 8 for MLL+Mix proof nets, providing more evidence that MixSeq is trickier than MLL sequentialization.

**Proposition 6.2.** Let $\pi$ be a MLL proof net and $l$ be a maximal link for $\ll \pi$. Then there exists a sequentialization of $\pi$ whose last rule introduces $l$.

**Proof.** If $l$ is a terminal $\&$-link, no other assumption is needed for the existence of such a sequentialization. Else, $l$ is a terminal $\otimes$-link and it suffices to show that $l$ is **splitting**, i.e. that the removal of $l$ splits $\pi$ into two connected components.

Suppose that it is not the case, and consider some sequentialization of $\pi$: it must contain a $\&$-rule, applied to a sub-proof net $\pi'$ for which $l$ is splitting, which turns it into a sub-proof net for which $l$ is not splitting anymore. Let $p$ be the $\&$-link introduced by that rule; its predecessors lie in different connected components of $\pi' \setminus \{l\}$. Since $\pi'$ is a MLL proof net, the predecessors of $p$ are connected by a feasible path in $\pi'$, which must cross $l$. This shows that $l$ is a dependency of $p$ in the sense of Definition 6.3, contradicting the maximality of $l$. (This only uses the fact that $D(\pi) \subseteq \ll \pi$, which is the “easy” part of Bellin’s theorem.)

6.1. **Computing the kingdom ordering.**

**Definition 6.3.** Let $\pi$ be a proof structure. We write $D(\pi)$ for the dependency relation defined as follows: for any two links $p \neq q$ of $\pi$, $p$ is a dependency of $q$ when $q$ is a $\&$-link and there exists a feasible path between the predecessors of $q$ going through $p$.

For instance, in the proof net of fig. 5b, the left $\&$-link depends on the left $\otimes$-link, but not on the other $\otimes$-links or $\&$-links; the middle $\otimes$-link has no dependency. In the case of fig. 8, the dependency relation is empty.

**Theorem 6.4** (Bellin [Bel97, Lemma 2]). Let $\pi$ be a MLL+Mix proof net. The transitive closure of $D(\pi) \cup S(\pi)$ is $\ll \pi$, where $(p, q) \in S(\pi)$ means that $p$ is a predecessor of $q$.

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This theorem was rediscovered in the special case of MLL proof nets by Bagnol et al. [BDS15 Theorem 11], who refer to the kingdom ordering as the “order of introduction”. We borrow the notations $D(\pi)$ and $S(\pi)$ from them.
The dependency relation can be computed by reduction to a matching problem in the case of MLL+Mix proof nets: even though it is well-defined in arbitrary proof structures, we need MLL+Mix correctness to compute it, because our matching algorithm relies on the absence of alternating cycles.

**Lemma 6.5** ([Ngu19]). Let $G = (V, E)$ be a graph and $M$ be a matching of $G$. Suppose that:

- there are no alternating cycles for $M$ – equivalently, $M$ is the unique perfect matching of the subgraph induced by the vertices matched by $M$;
- there are exactly two unmatched vertices $u, v$.

Then the existence of an alternating path for $M$ with endpoints $u, v$ and crossing a prescribed matching edge $e \in M$ can be reduced to the existence of a perfect matching; and such a path can be found in linear time.

**Remark 6.6.** An alternating path between unmatched vertices is often called an augmenting path; combinatorial maximum matching algorithms generally work by iteratively searching for augmenting paths, see e.g. [Tar83, Chapter 9].

**Theorem 6.7.** Let $\pi$ be a MLL+Mix proof net with a link $p$ and a $\otimes$-link $q$. Deciding whether $(p, q) \in D(\pi)$ can be done in linear time, in randomized NC and in quasiNC.

**Proof.** A degenerate case is when $p$ is a predecessor of $q$: in this case, $p$ depending on $q$ is equivalent to $\pi$ becoming incorrect if $q$ is turned into a $\otimes$-link, and thus the complexity is the same as that of (the complement of) the correctness problem.

When $p$ is not a predecessor of $q$, the definition of dependency translates into the problem defined in the above lemma by taking the graphification of $\pi$, and removing the matching edge corresponding to $q$. The endpoints of this edge then become unmatched, and we choose as prescribe intermediate edge the matching edge corresponding to $p$. The fact that $\pi$ is a proof net ensures that the acyclicity assumption of Lemma 6.5 is satisfied.

We directly obtain the linear time complexity, and since the existence of a perfect matching can be decided in randomized NC or quasiNC (cf. section 2.2), so can our problem. 

A transitive closure can be computed in polynomial time, and reachability in a directed graph can be decided in NL \(\subseteq\) quasiNC, so we get in the end:

**Corollary 6.8.** There are a polynomial-time algorithm and a quasiNC algorithm to compute the kingdom ordering \(\ll_\pi\) of any MLL+Mix proof net $\pi$.

6.2. **Dependencies and blossoms in unique perfect matchings.** We will now see how, through the correspondence of section 3, Bellin’s theorem can be rephrased as a statement on unique perfect matchings.

**Definition 6.9.** Let $G$ be a graph and $M$ be a perfect matching of $G$. A blossom for $M$ is a cycle whose vertices are all matched within the cycle, except for one, its root. The matching edge incident to the root, is called the stem of the blossom.

That is, a blossom consists of an alternating path between two vertices, starting and ending with a matching edge, together with a non-matching edge from the root to each of these two vertices. See fig. 6 for an illustration; as another example, in fig. 11, the two
triangles are blossoms with a common stem. The stem of a blossom is not part of the cycle. Blossom are central to combinatorial matching algorithms, e.g., [Edm65, GKT01], as we have previously mentioned.

**Definition 6.10.** When $e \in M$ is in some blossom with stem $f \in M$, we write $e \rightarrow f$.

This is the graph-theoretical counterpart of the dependency relation, as is shown by the following two propositions.

**Proposition 6.11.** Let $\pi$ be a MLL+Mix proof net and $(G, M)$ be its graphification. Let $p, q$ be links in $\pi$ with corresponding matching edges $e_p, e_q \in M$. Then $e_p \rightarrow e_q$ if and only if $p$ is a dependency of $q$ or a predecessor of $q$, i.e., $(p, q) \in D(\pi) \cup S(\pi)$.

Both the cases $(p, q) \in S(\pi)$ and $(p, q) \in D(\pi)$ occur in the proof net of fig. 2, see respectively fig. 10 and fig. 11.

**Proof.** If $(p, q) \in S(\pi)$, then by construction there exists a blossom of length 3 containing $p$ with stem $q$. If $(p, q) \in D(\pi)$, then for the same reason as Proposition 5.3, we can get, from the feasible path between the predecessors of $q$ visiting $p$, an alternating path for $M$ starting and ending with the edges corresponding to those predecessors and crossing the edge corresponding to $p$. By adding two non-matching edges to the same endpoint of the matching edge for $q$, we get a blossom with stem $q$. 
Conversely, let $q$ be a link, $e$ the corresponding matching edge, and $B$ be a blossom with stem $q$. Let us first note that if $B$ contains an non-matching edge joining $e$ with the matching edge corresponding to a successor of $q$, then by replacing this non-matching edge with its twin incident to the other endpoint of $q$, we get an alternating cycle; this is impossible because we have assumed $\pi$ to be a MLL+Mix proof net. Therefore, the first and last matching edges in $B$ are both predecessors of $q$. If they are the same – that is, if $B$ has length 3 and contains a single matching edge – then this edge corresponds to a predecessor $p$ of $q$. Otherwise, $B$ gives an alternating path between two distinct predecessors of $q$; necessarily $q$ is a $\otimes$-link (otherwise, there would be an alternating cycle), and all links corresponding to matching edges in $B$ are dependencies of $q$.

**Proposition 6.12.** Let $G$ be a graph, $M$ be a perfect matching of $G$ and $\pi$ be the proofification of $(G, M)$. Let $e, f \in M$ with corresponding $\otimes$-links $l_e, l_f \in M$. Then $e \rightarrow f$ if and only if $l_e$ is a dependency of some $\otimes$-link from which $l_f$ is reachable (by a directed path).

**Proof.** Let $B$ be a blossom with stem $f$, whose two non-matching edges incident to $f$ are $a$ and $b$. $B$ translates into a feasible path between $ax_a$ and $ax_b$ in $\pi$. Now, $ax_a$ and $ax_b$ are also leaves of a binary tree of $\otimes$-links whose root has the single successor $l_f$; by taking $q$ to be the lowest common ancestor of $ax_a$ and $ax_b$ in this tree, $l_f$ is reachable from $q$, and every link in the path between $ax_a$ and $ax_b$ depends on $q$. Conversely, any feasible path between the two predecessors of a $\otimes$-link corresponds to a blossom for $M$ in $G$. 

**Remark 6.13.** In Proposition 6.11, the "if" direction holds even for incorrect proof structures; in Proposition 6.12 note that no uniqueness property is required of the perfect matching.

Thus, we see that Bellin’s theorem is equivalent to the following theorem where $\rightarrow^+$ is the transitive closure of $\rightarrow$.

**Theorem 6.14.** Let $G$ be a graph with a unique perfect matching $M$, and $e, f \in M$. The edge $e$ occurs before $f$ in all sequentializations for $M$ if and only if $e \rightarrow^+ f$.

For instance, in fig. 1b, the middle edge $e$ is the only bridge, and it is the stem of the two triangular blossoms which contain the other matching edges.

This graph-theoretic version is somewhat simpler to state than the original theorem: one takes the transitive closure of a single relation, instead of a union of two unrelated relations.
relations. And as far as we know, this is a new result in graph theory. We have included it in the companion paper [Ngu19], aimed at a broader audience of graph theorists, where we present a direct combinatorial proof with no mention of proof nets.

7. Notions of constrained cycles in proof nets

This section connects the cycles involved in the correctness criterion, as well as the notion of paired graphs introduced in section 2.3 to objects studied in mainstream graph theory.

To do so, we will need to distinguish between different notions of paths and cycles in undirected graphs. Let us remind the reader of a terminological point made at the start of the paper (§2.1): paths and cycles are not allowed to repeat vertices. As in e.g. [BJG09, §1.4], we will use the word trail (resp. closed trail) for a relaxed notion of path (resp. cycle) allowed to have repeating vertices, but no repeating edges. (Thus, paths are trails, but the converse does not always hold.)

Definition 7.1. An edge-colored graph is an undirected graph $G = (V, E)$ equipped with a mapping from the edges to a finite set of colors.

A path, cycle, trail or closed trail in $G$ is properly colored (PC) if it contains no two consecutive edges with the same color.

A rainbow subgraph of $G$ is a subgraph of $G$ whose edges all have different colors. Rainbow paths, cycles, trails and closed trails are defined analogously.

We refer to [BJG09, Chapter 16] for general results on edge-colored graphs and properly colored paths and cycles. For rainbow subgraphs, see the survey [KL08]. One can now observe that edge-colored graphs generalize paired graphs (Definition 2.10):

Proposition 7.2. Let $(G, \mathcal{P})$ be a paired graph. Equip $G$ with an edge coloring such that two edges have the same color iff they are paired in $\mathcal{P}$ – this is possible since the pairs of $\mathcal{P}$ are disjoint by definition.

Then switchings correspond to maximal rainbow subgraphs of $G$ for this coloring, and feasible cycles to rainbow cycles.

Remark 7.3. Paired graphs may actually be defined as the subclass of edge-colored graphs such that, for each color $c$:

• all edges with color $c$ share a common endpoint;
• there are at most 2 of them.

While the former condition is important as we will see next, the latter is useless and may be relaxed. It is actually usual in proof nets to consider paired graphs with “pairs” of any cardinality, when dealing with correctness criteria involving jumps, such as Girard’s treatment of quantifiers [Gir91]. (See also [DGF08] for another use of jumps.) In such cases, maximal rainbow subgraphs still correspond to the right definition of switching.

Unfortunately, finding rainbow paths is NP-hard [CFMY11], so this generalization does not explain why MixCorr can be solved efficiently. To do so one should consider properly colored cycles instead.

Proposition 7.4. In a paired graph, properly colored cycles are the same as rainbow cycles without vertex repetitions. (Note that any rainbow cycle contains a subset of edges forming a cycle without vertex repetitions.)

14 Also called a heterochromatic or multicolored subgraph in the literature.
Proof. Suppose that a properly colored cycle crosses two edges \( e, f \) with the same color. Since we consider a paired graph, there is a vertex \( u \) which is both an endpoint of \( e \) and \( f \). Since \( e \) and \( f \) cannot occur consecutively in the PC cycle, \( u \) must appear twice, contradicting the condition which distinguishes cycles from closed trails.

In the companion paper [Ngu19], we showed that the existence of a PC cycle in an edge-colored graph reduces to (and is actually equivalent to) \textsc{UniquenessPM}, by adapting the reduction given for properly colored cycles in [BJG09 §16.4]. This leads to another linear time algorithm for \textsc{MixCorr}, with an additional level of indirection compared to section 4.1.

More than that: we mentioned in section 2.3 that the “splitting lemmas” that can be used to prove the MLL+Mix sequentialization theorem are equivalent to Kotzig’s theorem on bridges in unique perfect matchings (Theorem 2.3). Edge-colored graphs also have an analogous property, first proved by Yeo [Yeo97], see [BJG09 §16.3]. Szeider showed that Kotzig’s and Yeo’s theorems are in fact equivalent, through mutual reductions sending properly colored cycles to alternating cycles and vice versa.

**Remark 7.5.** Although associating an edge-colored graph to a proof structure suffices to study the complexity of \textsc{MixCorr} and seems more natural at first sight than our graphification construction (section 3.1), the latter is more informative and plays a crucial role in the results of sections 5 and 6. See Remark 7.9 for further discussion of the advantages of graphifications.

**Remark 7.6.** Proposition 7.4 shows that paired graphs are a tractable case of the rainbow cycle problem. Retoré considers a slightly more general case in his PhD thesis [Ret93 Chapter 2], which he called “aggregates”, and proved a splitting lemma for aggregates. In [Ngu19], we show that both the tractability of rainbow path-finding and Retoré’s theorem generalize to a larger class of edge-colored graphs: the condition is that all color classes must be complete multipartite (whereas in aggregates, color classes are complete bipartite).

Among Szeider’s list of equivalent theorems, there is also a “splitting lemma” for directed graphs, due to Shoesmith and Smiley [SS79]. We will not state it precisely here, but we recall the notion of cycle involved, because it relates directly to feasible cycles in proof structures.

**Definition 7.7.** Let \( G \) be a directed graph. A semicycle in \( G \) is a cycle in the underlying undirected graph. A vertex \( v \) is a turning vertex of a semicycle \( C \) if \( C \) contains two incoming edges for \( v \), or two outgoing edges for \( v \). For a set of vertices \( S \), a \( S \)-semicycle is a semicycle whose turning vertices are not in \( S \).

**Proposition 7.8.** Let \( \pi \) be a proof structure, and let its correctness graph (Definition 2.12) \( C(\pi) \) be equipped with the edge coloring of Proposition 7.2. Then the following are the same:

1. feasible cycles in \( \pi \);
2. \( \forall \)-semicycles in \( \pi \), i.e. \( S \)-semicycles for \( S = \{ \forall \text{-links} \} \);
3. rainbow cycles in \( C(\pi) \);
4. properly colored cycles in \( C(\pi) \);
5. properly colored closed trails in \( C(\pi) \).

**Proof.** The last two are equivalent because in a paired graph coming from a proof structure, for each vertex \( v \), there is at most one pair whose common vertex is \( v \) (note that this argument generalizes to “pairs” of cardinality > 2 obtained from \( \forall \text{-links} \) à la [Gir91]).

\[15\] The adaptation is mostly straightforward, but some definitions have to be tweaked.

\[16\] A color class is a subgraph edge-induced by all \( c \)-colored edges for some color \( c \).
So paired graphs already collapsed the global notion of rainbow path and the local notion of properly colored path, and proof structures go even further. Note that PC cycles and PC closed trails need not be the same for paired graphs which do not come from proof structures, as demonstrated by fig. 12. In the next section, this distinction between cycles and closed trails will become relevant.

Remark 7.9. Both Yeo’s theorem and Shoesmith and Smiley’s theorem could be used to prove MLL+Mix sequentialization. But the edge coloring only encodes the pairing of the incoming edges of \( \land \)-links, forgetting information about the directed graph structure of proof nets, and therefore Yeo’s theorem would give us a splitting vertex which is not necessarily a terminal link. In the same way, when applied to \( \land \)-semicycles, the Shoesmith–Smiley theorem does not care whether the edges incident to a non-\( \land \) link are incoming or outgoing.

Therefore, to pursue this proof strategy, one needs to reason inductively on a class of “proof structures with premises” which are closed under splitting at non-terminal links. This has been done for instance by Danos [Dan90] with his own “splitting pair” lemma.

The same happens in Retoré’s proof of sequentialization based on R&B-graphs – cf. the discussion at the beginning of section 3 – and it is why the sequentializations of a proof net are not in bijection with those of its R&B-graph. By introducing graphifications, we fixed this issue: the predecessor relation of a proof structure is encoded in its graphification (by blossoms of length 3, see Proposition 6.11).

8. R&B-graphs, forbidden transitions and essential nets

In this section, we explain the combinatorial content of Retoré’s R&B-graph criterion [Ret03] – which will finally be defined in section 8.2 – by factorizing it as a composition of:

- the Danos–Regnier correctness graph;
- a reduction to perfect matchings for a general notion of locally constrained closed trails, which, in the case of edge-colored graphs, corresponds to properly colored closed trails.

We introduced the latter in [Ngu19], but here the logical order of exposition is the reverse of the order of discovery: it was by attempting to understand Retoré’s R&B-graphs that we found this reduction.

After that, we draw a connection between R&B-graphs and Lamarche’s essential nets. However at the current stage it remains rather superficial, in particular we have no satisfactory purely combinatorial explanation of how their correctness criteria relate.
8.1. Graphs with forbidden transitions. A natural generalization of paired graphs is to drop the disjointness requirement on pairs: switchings do not make sense anymore, but feasible cycles will become those which avoid forbidden transitions, i.e. two paired edges occurring consecutively. Equivalently, one may specify the allowed transitions, as in the following definition taken from [Sze03].

Definition 8.1. Let $G = (V, E)$ be a graph. A transition graph for a vertex $v \in V$ is a graph whose vertices are the edges incident to $v : T(v) = (\partial(v), E_v)$. A transition system on $G$ is a family $T = (T(v))_{v \in V}$ of transition graphs.

A path (resp. trail) $v_1, e_1, v_2, \ldots, e_{k-1}, v_k$ is said to be compatible if for $i = 1, \ldots, k - 1$, $e_i$ and $e_{i+1}$ are adjacent in $T(v_{i+1})$. For a cycle (resp. closed trail), we also require $e_{k-1}$ and $e_1$ to be adjacent in $T(v_1) = T(v_k)$.

This subsumes properly closed cycles and closed trails (and therefore feasible cycles in paired graphs) by taking the transition system where two edges are adjacent iff they have different colors.

Finding a compatible path is proved to be NP-complete in [Sze03]. However, for compatible trails, we showed the problem to be tractable [Ngu19] by using a “edge-colored line graph” construction $L_{EC}$. This construction has other uses but, in the case of compatible (closed) trails, it can be replaced by a version $L_{PM}$ using perfect matchings – which in fact is $L_{EC}$ composed with a previously known reduction, see [Ngu19] for details. Anyway, all this arguably goes to show that we are manipulating rather natural objects which are not contrived to fit with R&B-graphs (actually, it is by proposing a small tweak to the definition of R&B-graphs that the fit is ensured, while the graph-theoretic side stays unaffected.)

Definition 8.2. Let $G$ be a graph and $T$ be a transition system on $G$. The PM-line graph $L_{PM}(G, T)$ is defined as the graph:

- with vertex set $\{u_e \mid e \in E, u \text{ is an endpoint of } e\}$;
- with edge set $M \sqcup E'$, where
  - $M = \{(u_e, v_e) \mid e = (u, v) \in E\}$;
  - $E' = \{(u_e, u_f) \mid u \in V, e, f \in \partial(u) \text{ are adjacent in } T(u)\}$;
- equipped with the perfect matching $M$.

Proposition 8.3 (Ngu19). Closed trails of length $k$ in $G$ compatible with $T$ correspond bijectively to alternating cycles of length $2k$ in $L_{PM}(G, T)$.

8.2. Recovering Retoré’s R&B-graphs. We are now going to apply this PM-line graph construction to proof structures, but before that, we need to give a slightly altered definition of proof structures a bit.

Definition 8.4. A proof structure with conclusions is a non-empty directed acyclic multi-graph $(V, A)$ with a partial labeling of the vertices $l : V \rightarrow \{\text{ax}, \otimes, \otimes\}$ such that, for $v \in V$:

- if $l(v) = \text{ax}$, then $\deg^-(v) = 0$ and $\deg^+(v) = 2$;
- if $l(v) \in \{\otimes, \otimes\}$, then $\deg^-(v) = 2$ and $\deg^+(v) = 1$;
- else, $v$ is unlabeled, and then $\deg^-(v) = 1$ and $\deg^+(v) = 0$.

In the latter case, $v$ is called a conclusion vertex and its unique incoming edge is called a conclusion edge.
Compared with Definition 2.6, the bounds on the outdegree have become equalities, while a new kind of vertex has been added. The idea is that, when the inequality on the outdegree is strict, there are “missing” outgoing edges, which are materialized here as conclusion edges. There is a unique way to add conclusions to a proof structure, and conversely, given a proof structure with conclusions, the subgraph induced by the labeled vertices is a proof structure according to Definition 2.6; this correspondence is bijective. See fig. 13 for an example.

Remark 8.5. Here we are confronted with the fact that there is no single canonical definition of MLL proof structures (although two given definitions are always canonically isomorphic). Depending on the task at hand, different combinatorial formalizations of the same object may be more or less convenient.

To define proof nets inductively, and describe the sequentialization algorithm of section 5, it was easier to use proof structures without conclusions and rely on the notion of terminal link. But the conclusion edges are logically significant, they correspond to the formulas in the sequent being proven.

Now, let \( \pi \) be a proof structure with conclusions and \( C(\pi) \) its correctness graph (adapting Definition 2.12 to handle conclusion vertices/edges), and \( T \) be the transition system corresponding to the paired edges of \( C(\pi) \). The uniqueness of the perfect matching of \( L_{PM}(C(\pi), T) \) is equivalent to \( \pi \) being a MLL+Mix proof net, according to propositions 8.3 and 7.8 so this gives us a correctness criterion. It turns out that this PM-line graph can be described directly from \( \pi \) by local translation rules:

**Proposition 8.6.** With the above notations, \( L_{PM}(C(\pi), T) \) the graph equipped with a perfect matching

- which has, for every edge in \( \pi \), two vertices and a matching edge between them,
- and whose non-matching edges are given by the translations of links given in fig. 14.

(The conclusion vertices disappear in the translation, but the conclusion edges do not.)

At this point we have succeeded in our reconstruction of R&B-graphs: the rules defining them in the table of [Ret03, §1.2] are almost exactly those given in fig. 14. The only difference is that Retoré translates an ax-link with its two outgoing edges as a single matching edge,

\footnote{An annoying point, however, is that the conclusion vertices have no significance, so sometimes proof structures are defined with “dangling edges” with no target. However, dangling edges drag us out of the world of graphs, and into hypergraphs – indeed, they are hyperedges of arity 1. Proof structures are also often defined as the dual hypergraph: links are hyperedges, and formulas are vertices. For our purposes, we have chosen to keep proof structures (with or without conclusions) as actual graphs, to make the connections with graph theory clearer.}
whereas the translation we obtained consists of two matching edges (one for each conclusion) joined by a non-matching edge. Our modified notion of R&B-graph has the advantage that the edges of a proof structure are in bijection with the matching edges in its R&B-graph.

8.3. **Essential nets as (almost) subgraphs of R&B-graphs.** Essential nets were introduced by Lamarche [Lam08] as a notion of proof net for *Intuitionistic* Multiplicative Linear Logic (IMLL). Instead of recalling the formulas and rules of IMLL here, we will directly describe a translation from proof structures endowed with a *polarization* to essential nets; the reader familiar with the one-sided presentation of IMLL will recognize the system of intuitionistic polarities at work.

We reuse the definition of proof structures with conclusion vertices and edges introduced in the previous subsection. Our presentation of essential nets is derived from [MO06].

**Definition 8.7.** Let $\pi$ be a proof structure with conclusions, and let $E$ be the set of directed edges of $\pi$. An *IMLL polarization* of $\pi$ is a labeling of the edges $p : E \to \{+, -\}$ such that there is at most 1 positive (labeled $+$) conclusion edge, and for each non-conclusion vertex $v$:

- if $l(v) = \text{ax}$, then $v$ is incident to one positive edge and one negative edge;
- if $l(v) = \otimes$, then depending on the outgoing edge of $v$,
  - if it is positive, then both incoming edges are positive;
  - if it is negative, then the incoming edges have different polarities;
- if $l(v) = \otimes$, then depending on the outgoing edge of $v$,
  - if it is negative, then both incoming edges are negative;
  - if it is positive, then the incoming edges have different polarities.

Given a proof structure $\pi$ and an acyclic and connected switching of $\pi$, the *trip translation* algorithm[18] described in [MO06] computes an IMLL polarization of $\pi$, such that the essential net associated to this polarization – that we define next – is correct (i.e. corresponds to an IMLL proof) if and only if $\pi$ is a MLL proof net. Murawski and Ong’s linear time correctness criterion [MO06] for MLL actually tests the correctness of essential nets, using the trip translation as a linear-time reduction.

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[18]Due to Bellin and van de Wiele, and communicated on the Types mailing list in 1992.
Figure 15. Top row: proof structure links with an IMLL-polarized neighborhood. Middle row: translation into (modified) essential nets. Bottom row: corresponding subgraph of R&B-graph (compare fig. 14).

Definition 8.8. Let $\pi$ be a proof structure with conclusions equipped with an IMLL polarization $p$. The essential net associated to $(\pi, p)$ is a directed graph whose vertices correspond to the edges of $\pi$, and whose edges are given by locally translating configurations in $\pi$ (taking $p$ into account) as described by the first two rows of fig. 15.

Remark 8.9. Here, again, we need to amend the definition to ensure the fit! (Hence the “almost”.) Compare with [MO06, Figure 1]: the case of the negative tensor differs. However this change has virtually no influence on the directed paths which exist in the graph, and thus does not affect the correctness criterion of essential nets.

The connection with R&B-graphs is obtained through a classical bijection between directed graphs and bipartite perfect matchings. It works as follows: given a directed graph $G = (V, A)$, one builds the undirected graph having

- for each vertex $v \in V$, two vertices $v_{\text{in}}, v_{\text{out}}$;
- for each directed edge $(u, v) \in A$, one undirected edge $(u_{\text{out}}, v_{\text{in}})$.

The graph thus obtained is bipartite: indeed, there is a proper vertex 2-coloring $\{\text{in, out}\}$. $M = \{v_{\text{in}}, v_{\text{out}} \mid v \in V\}$ is a perfect matching, and alternating paths (resp. cycles) for $M$ correspond to directed paths (resp. cycles) in $G$.

We may now state the main result of this subsection:

Proposition 8.10. Let $\pi$ be a proof structure with conclusions, and $\nu$ be an essential net associated with some IMLL polarization of $\pi$. Then the bipartite graph with perfect matching
corresponding to \( \nu \) by the above bijection embeds into the R&B-graph of \( \pi \) as a maximum spanning bipartite subgraph, the matching \( M \) defined above being sent to the matching of the R&B-graph.

This is illustrated by the bottom row of fig. 15.

**Remark 8.11.** The orientations of the edges in \( \pi \) can be transferred to orientations of its R&B-graph. We can also consider the (non-proper) vertex coloring \{in, out\} of the R&B-graph obtained via the embedding of the essential net for some IMLL polarization \( p \).

Then a matching edge \( e \) is directed from out to in if and only if \( p(e) = + \).

The proposition leads us to conjecture that the graph-theoretic “domination condition” characterizing the correctness of essential nets should admit a low-complexity reduction to UniquenessPM. Perhaps such a reduction can be derived from analyzing the role of the edges outside the spanning bipartite subgraph in the R&B-graph, but we leave this to future work. Relatedly, see Remark 4.2; one of the “previous algorithms” mentioned is indeed Murawski and Ong’s linear time decision procedure for this domination condition.

### 9. Edge-colored graphs and visible acyclicity (sketch)

The visible acyclicity condition was first introduced by Pagani for Multiplicative-Exponential Linear Logic (MELL) proof structures [Pag06] and later extended to differential interaction nets [Pag12]. It is motivated by semantics, and also behaves well under cut-elimination: it is stable under reduction and guarantees strong normalization. Thus, it can be seen as a sort of correctness criterion relaxing the usual MELL+Mix correctness.

Analogously to MLL+Mix correctness, the visible acyclicity of a MELL proof structure can be defined as:

- the acyclicity of its visible graphs which generalize the Danos–Regnier switchings;
- equivalently, the absence of visible cycles – this is closer to our point of view when we reduce MIXCORR to UniquenessPM, i.e. the absence of alternating cycle.

A novelty of visible graphs with respect to switchings is that the former are directed graphs. Similarly, visible cycles are directed cycles with some constraints. This suggests connections with directed versions of alternating cycles, or of the other graph-theoretic problems mentioned in section 7. For instance, properly colored directed paths have been studied:

**Theorem 9.1 (GLMM13).** Deciding whether a 2-edge-colored\(^{23}\) directed graph with no properly colored cycle contains a properly colored path between two given vertices is NP-complete.

**Corollary 9.2.** Deciding whether a 2-edge-colored directed graph contains a properly colored cycle is NP-complete.

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\(^{21}\)Visible acyclic proof structures (resp. differential interaction nets) are characterized by having a sound denotation in non-uniform coherence spaces (resp. finiteness spaces).

\(^{22}\)To our knowledge, this strong normalization result is claimed in both Pag06, Pag12 but has no published proof.

\(^{23}\)That is, using only 2 different edge colors, e.g. blue and red. For instance, to express alternating cycles as properly colored cycles, it suffices to color the matching edges blue and the non-matching edges red.
In fact, this problem can be reduced to the existence of a visible cycle, leading to an NP-hardness result for the latter. We now provide a high-level overview of how to obtain this reduction – giving the details would require recalling long definitions.

It is instructive to start with the undirected case:

**Proposition 9.3.** There is a reduction from properly colored cycles in 2-edge-colored graphs to Danos–Regnier cycles in proof structures, sending edges to \( \text{ax-links} \).

**Proof.** The proofification construction (section 3.2) reduces alternating cycles to Danos–Regnier cycles. One can compose that with a simple reduction from 2-edge-colored graphs to perfect matchings, attributed to Edmonds in [Man95, Lemma 1.1]: basically, make two copies the graph, each having all the original vertices but keeping only the edges of a single color, and join the corresponding vertices with matching edges; see fig. 16 for an example (and [Ngu19] for a slight generalization).

Explicitly, given a 2-edge-colored graph \( G = (V, E) \), with \( E = R \sqcup B \) being the partition into colors, this composition results in the proof structure \( \pi \) built as follows:

- For each edge \( e = (u, v) \in E \), we create an \( \text{ax-link} \ \text{ax}_e \) with outgoing edges \( A_{u,v} \) and \( A_{v,u} \).
- For each vertex \( u \in V \):
  - If \( u \) has \( k \) incident edges in \( R \), we add a \( k \)-ary \( \otimes \)-link whose incoming edges are the \( A_{u,v} \) for all neighbors \( v \) of \( u \) such that \( (u, v) \in R \), and we call its outgoing edge \( R_u \). (The 0-ary case is an \( \text{ax-link} \).)
  - Similarly for \( B \): we introduce a \((\deg(u) - k)\)-ary \( \otimes \)-link whose outgoing edge is \( B_u \).
  - Then we add an \( \otimes \)-link whose incoming edges are \( R_u \) and \( B_u \). These \( \otimes \)-links are the terminal links of \( \pi \).

Now, it suffices to replace the \( \text{ax} \)-links above by a gadget realizing a “directed \( \text{ax} \)-link” in order to handle directed edges. The construction of such a gadget requires examining the definition of visible acyclicity, for which we refer the reader to Pagani’s papers so that she may convince herself of the following: an exponential box with a single auxiliary door will do the trick, as long as the box contains a MLL+Mix proof net with no path between the premise of the principal door and that of the auxiliary door.

In the end, we have:

**Theorem 9.4.** Visible acyclicity is \( \text{coNP-hard} \).
However, it is not clear whether we can express the visible acyclicity of a general MELL proof structure as the absence of properly colored directed cycles in a graph without a super-polynomial size increase. More generally, we do not know whether visible acyclicity is in \text{coNP}.

10. Conclusion

We have presented a correspondence between proof nets and perfect matchings, and demonstrated its usefulness through several applications of graph theory to linear logic: our results give the best known complexity for MLL+Mix correctness and sequentialization, by taking advantage of sophisticated graph algorithms. Beyond that, we have also contextualized this correctness problem as a member of a family of equivalent constrained cycle-finding problems in graphs, and used this to shed some light on earlier work on proof nets. These connections also have some benefits for graph theory, as the rephrasing of Bellin’s theorem illustrates; this is what we attempt to demonstrate in the companion paper [Ngu19]. In general, we hope to see fruitful interactions arise between those two domains.

10.1. Open questions. Now that we have shed a new light on MLL+Mix proof nets, it would be interesting to revisit the well-studied theory of MLL proof nets. Therefore, we would like to find the right graph-theoretical counterpart to the connectedness condition in the Danos–Regnier criterion for MLL. The goal would be to extract the combinatorial essence of the statics of MLL proof structures, forgetting about logic; without having to handle the dynamics (cut-elimination), one could hope to distill some simpler combinatorial object, in the same way that perfect matchings are simpler than MLL+Mix proof structures.

But unique perfect matchings do not seem to be the right setting to do so; and one year after the conference version of this paper, despite the connections described here with \textit{e.g.} edge-colored graphs, we still have not found a natural graph-theoretic decision problem equivalent to correctness for MLL without Mix. (As far as naturality is concerned, perfect matchings set a high bar, given their importance in discrete mathematics!)

Here by “equivalent” we mean, in particular, through low-complexity reductions (hopefully computable both in linear time and in \text{AC}^0). Though the \text{NL}-completeness of MLL correctness means that it is equivalent to directed reachability, Mogbil and Naurois’s correctness criterion \textbf{[JdNM11]} uses a subroutine for connectivity in undirected forests, a \text{L}-complete problem, in its reduction. The same objection holds for the reduction to essential nets (section \textbf{8.3}). A related question is to understand why all known linear-time correctness criteria for MLL – including the one presented here – rely on the same sophisticated data structure, as mentioned in section \textbf{4.1}.

In the same vein, the present paper does not treat at all the \textit{contractibility} criterion introduced by Danos \textbf{[Dan90]}, despite its importance in recent developments in proof nets (\textit{e.g.} \textbf{[BH18, HH16]}). It is also part of the divide between MLL and MLL+Mix proof nets: contractibility, reformulated as graph parsing, underlies a linear-time sequentialization algorithm for MLL \textbf{[Gue11]}, while no such algorithm is known for MLL+Mix. Aside from the obvious question of sequentializing MLL+Mix nets in linear time, looking for a mainstream graph-theoretic account of contractibility is also of interest.

\textsuperscript{24}Except for the short Remark \textbf{4.3}.
The last two sections also raised a few open problems: understand how correctness of essential nets relates to uniqueness of perfect matchings, and find the exact complexity of Pagani’s visible acyclicity. Unlike further progress on the complexity of MLL+Mix correctness or sequentialization, these do not seem to depend on difficult topics of active research in algorithmics.

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