RADIATION-DRIVEN WARPING. II. NONISOThERMAL DISKS

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Received 1997 August 5; accepted 1998 April 3

ABSTRACT

Recent work by Pringle and by Maloney, Begelman, & Pringle has shown that geometrically thin, optically thick, accretion disks are unstable to warping driven by radiation torque from the central source. This work was confined to isothermal (i.e., surface density $\Sigma \propto R^{-3/2}$) disks. In this paper we generalize the study of radiation-driven warping to include general power-law surface density distributions, $\Sigma \propto R^{-d}$. We consider the range from $d = 3/2$ (the isothermal case) to $d = -3/2$, which corresponds to a radiation-pressure–supported disk; this spans the range of surface density distributions likely to be found in real astrophysical disks. In all cases there are an infinite number of zero-crossing solutions (i.e., solutions that cross the equator), which are the physically relevant modes if the outer boundary of the disk is required to lie in a specified plane. However, unlike the isothermal disk, which is the degenerate case, the frequency eigenvalues for $d \neq 3/2$ are all distinct. In all cases the location of the zero moves outward from the steady state (pure precession) value with increasing growth rate; thus, there is a critical minimum size for unstable disks. Modes with zeros at smaller radii are damped. The critical radius and the steady state precession rate depend only weakly on $d$. An additional analytic solution has been found for $d = 1$. The case $d = 1$ divides the solutions into two qualitatively different regimes. For $d \geq 1$, the fastest growing modes have maximum warp amplitude, $\beta_{\text{max}}$, close to the disk outer edge, and the ratio of $\beta_{\text{max}}$ to the warp amplitude at the disk inner edge, $\frac{\beta_{\text{max}}}{\beta_0}$, is $\gg 1$. For $d < 1$, $\frac{\beta_{\text{max}}}{\beta_0} \approx 1$, and the warp maximum steadily approaches the origin as $d$ decreases. This implies that nonlinear effects must be important if the warp extends to the disk inner edge for $d \geq 1$, but for $d < 1$ nonlinearity will be important only if the warp amplitude is large at the origin. Because of this qualitative difference in the shapes of the warps, the effects of shadowing of the central source by the warp will also be very different in the two regimes of $d$. This has important implications for radiation-driven warping in X-ray binaries, for which the value of $d$ characterizing the disk is likely to be less than unity. In real accretion disks the outer boundary condition is likely to be different from the zero-crossing condition that we have assumed. In accretion disks around massive black holes in active galactic nuclei, the disk will probably become optically thin before the outer disk boundary is reached, whereas in X-ray binaries there will be an outer disk region (outside the circularization radius) in which the inflow velocity is zero but angular momentum is still transported. We show that in both these cases the solutions are similar to the zero-crossing eigenfunctions.

Subject headings: accretion, accretion disks — binaries: close — galaxies: structure — instabilities — methods: analytical — X-rays: stars

1. INTRODUCTION

Evidence for warped, precessing accretion disks in astrophysical systems ranging from X-ray binaries to active galactic nuclei has steadily accumulated over the last two decades (see Maloney & Begelman 1997a, and references therein). The origin and maintenance of such warped disks has until recently stood as an unsolved theoretical problem. Although it is possible, for example, to generate nonplanar modes with $m = 1$ symmetry in thin, relativistic disks (Kato 1990; Kato & Honma 1991), these modes only exist at small radii ($R \lesssim 10$ Schwarzschild radii), since they rely on trapping of the modes in the non-Newtonian region of the potential. However, an important clue was provided by Peterson (1977), who pointed out that in an optically thick disk with a central source of luminosity, the pressure resulting from reradiation of the intercepted flux will produce a net torque if the disk is warped. Almost 20 years were to pass before it was recognized that radiation-pressure torque actually leads to a warping instability. Pringle (1996, hereafter P96) showed that, for the special case in which the disk surface density $\Sigma \propto R^{-3/2}$ (corresponding to an isothermal disk in the usual $\alpha$-disk formalism, with disk viscosity $\nu \propto R^{3/2}$), even an initially planar disk is unstable to warping by this mechanism. Pringle solved the linearized twisted disk equations in this case using a WKB approximation. Maloney, Begelman, & Pringle (1996, hereafter Paper I) found exact solutions to the linearized twist equations and demonstrated the importance of the outer boundary condition for determining the growth rates of the unstable modes.

These previous works all specialized to the case of an isothermal disk, which simplifies the twist equations. Although this may be a reasonable approximation for some astrophysical disks (e.g., the masing molecular disk in NGC 4258; see Paper I), there are many other systems, such as accretion disks in X-ray binary systems, where this is likely to be a poor assumption. In this paper we extend the work of Paper I by considering disks with power-law surface density profiles, $\Sigma \propto R^{-d}$. We consider the range $-3/2 \leq d \leq 3/2$: the lower limit corresponds to a radiation-pressure–supported disk (e.g., Frank, King, & Raine 1992,
p. 83), and the upper limit is the isothermal value (Paper I). Within the limitations of assuming a constant power law for the surface density, this spans the probable range of surface density laws relevant to real astrophysical accretion disks. For example, the standard Shakura-Sunyaev gas pressure-supported disk is characterized by \( \delta = 0.75 \) (Shakura & Sunyaev 1973).

In §2 we discuss the twist equation, including the effect of radiation torque, and cast it into a more convenient form. We solve the equation numerically in §3 and discuss both the time-dependent and steady state solutions. As in the isothermal case, the outer boundary condition is crucial for determining the stability of the disk and the growth rates of the unstable modes. In §4 we discuss the important issue of the appropriate outer boundary condition for accretion disks around stellar-mass objects and active galactic nuclei (AGNs). Finally, in §5 we discuss the implications of the results and their application to real accretion disks.

2. THE DISK EVOLUTION EQUATION

As in Paper I and in Paper P96, we adopt a Cartesian coordinate system with the \( Z \)-axis aligned with the normal to the unwarped disk and define \( \beta \) to be the local angle of tilt of the disk axis with respect to the \( Z \)-axis, and \( \gamma - \pi/2 \) is the angle between the descending line of nodes and the \( X \)-axis. The equation governing the evolution of the local tilt vector, \( \dot{R}, (R, t) = (\sin \beta \cos \gamma, \sin \beta \sin \gamma, \cos \beta) \), including the radiation torque term, is then (P96)

\[
\frac{\partial \ell}{\partial t} + \left[ V_R - \frac{v_\Omega}{\Omega} - \frac{1}{2} v_2 \frac{(\Sigma R^3 \Omega)}{\Sigma R^2 \Omega} \right] \frac{\partial \ell}{\partial R} = \frac{\partial}{\partial R} \left( \frac{1}{2} v_2 \frac{\partial \ell}{\partial R} \right) + \frac{1}{2} \frac{\partial v_2}{\partial R} \frac{\partial \ell}{\partial R} \frac{I}{\Sigma R^2 \Omega},
\]

(1)

where \( v_1 \) and \( v_2 \) are the disk viscosity in the azimuthal and vertical directions (with a ratio \( \eta \equiv v_2/v_1 \) that is assumed constant but not necessarily unity), primes denote derivatives with respect to \( R \), \( \Sigma \) is the disk surface density, \( \Omega \) is the Keplerian angular velocity, \( V_R \) is the radial inflow speed, and the radiation torque term is

\[
\mathcal{G} = \frac{1}{2 \pi R} \frac{dG}{dR},
\]

(2)

where \( dG \) is the torque exerted on a ring of width \( dR \) and radius \( R \) and is given by equation (2.18) of P96.

As in Paper I, we assume the disk viscosity can be written \( v_1(R) = v_0(R/R_0)^{\delta} \), where \( R_0 \) is an arbitrary fiducial radius, but we now allow the radial power law to be arbitrary, rather than specializing to the case \( \delta = 3/2 \). In a steady state disk far from the boundaries \( \Sigma = M/3\pi v_1 \), so this radial dependence of the viscosity translates directly into a power-law surface density \( \Sigma \propto R^{-\delta} \). Furthermore, in a steady disk the radial inflow velocity \( V_R = v_1 \Omega/\Omega = v_1 \) and so the first two terms inside the brackets on the left-hand side of equation (1) cancel, and the third term becomes

\[
\frac{(\Sigma R^3 \Omega)}{\Sigma R^2 \Omega} = \left( \frac{3}{2} - \delta \right) \frac{1}{R}.
\]

(3)

We then linearize to obtain the more general version of equation (1) of Paper I for \( W = \beta e^{i\gamma} \),

\[
\frac{\partial W}{\partial t} = \left( \frac{3}{2} \frac{v_2}{R} - i\ell \right) \frac{\partial W}{\partial R} + \left( \frac{1}{2} v_2 \frac{\partial^2 W}{\partial R^2} \right),
\]

(4)

where the radiation torque term is

\[
\mathcal{G} = \frac{L}{12\pi \Sigma R^3 \Omega c},
\]

(5)

where \( L \) is the luminosity of the central source. Assuming an accretion-fueled source with radiative efficiency \( \epsilon \equiv L/Mc^2 \), this term can be written as

\[
\mathcal{G} = \frac{\epsilon}{2\sqrt{2k} \Sigma_0 R_0^{3/2} \Omega} R^{3/2 - 1/2} = \Gamma_0 R^{3/2 - 1/2},
\]

(6)

where \( R_0 \) is the Schwarzschild radius (cf. eq. [3] of Paper I).

Transforming to the new radius variable \( x = aR^{1/2} \) using equation (6) for \( \Gamma \) and Fourier transforming with respect to time, we find that the twist evolution equation becomes

\[
\frac{\partial W}{\partial t} = i\sigma W = \frac{\eta v_0}{8R_0^3} a^{-2\delta} x^{2\delta - 3} \times \left[ x \frac{\partial^2 W}{\partial x^2} + \left( 2 - i\frac{\sqrt{2}e}{\eta a^{1/2} R_0} \right) \frac{\partial W}{\partial x} \right],
\]

(7)

where in general \( \sigma \) is complex. We set \( a = (2)^{1/2} \eta R_0^{1/2} \eta \) (note that this differs from the definition of radius variable \( z \) in Paper I only in making \( x \) real, rather than pure imaginary). Since \( R_0 \) is arbitrary, we now define \( R_0 \) so that \( x(R_0) = 1; v_0 \) is thus the value of \( v_1 \) at \( x = 1 \). Therefore

\[
R_0 = a^{-2} = \frac{\eta v_0}{2 \epsilon^2 \Omega_0},
\]

(8)

and the coefficient of the \( x^{2\delta - 3} \) term on the right-hand side of equation (7) becomes \( v_0 \epsilon^{4/2R^2 \eta^3} \). Finally, we nondimensionalize the eigenfrequency \( \sigma \) by defining

\[
\hat{\sigma} = \frac{2\eta^3 R_0^2 \epsilon}{v_0^2}
\]

(9)

to obtain the final form of the twist equation

\[
x \frac{\partial^2 W}{\partial x^2} + \left( 2 - ix \right) \frac{\partial W}{\partial x} - i\hat{\sigma} x^{2\delta - 3} W = 0.
\]

(10)

For \( \delta = 3/2 \), equation (10) reduces to Kummer’s equation, as discussed in Paper I. (Note that the coefficient of \( W \) in eq. [9] of Paper I, \( 2\sigma/(\Gamma_0) \), is identical to \( \hat{\sigma} \) as defined here.) Equation (10) can thus be regarded as a modified Kummer’s equation. Unfortunately, this equation is, in general, analytically intractable, so that numerical solution is necessary. We discuss the asymptotic behavior of the solutions in Appendix A. For the physically relevant solutions (those that exhibit zero crossings), there are in all cases an infinite number of zero-crossing eigenfunctions. As we will see below, the special case \( \delta = 3/2 \) is degenerate; the real parts of the eigenvalues are all identical, with \( \text{Re}(\hat{\sigma}) = 1 \). For all the other values of \( \delta \) that we consider, the eigenvalues are distinct and each eigenfunction has at most one zero. In addition the real part of \( \hat{\sigma} \) must be greater than zero, which means that the direction of precession of the warp must be the same as the direction of disk rotation (i.e., prograde; see Appendix B). This is a simple consequence of the overall shape of the warp; it is easy to show that for a disk in which the gradient in the tilt \( \beta' > 0 \), the direction of precession due to the radiation torque is retrograde, and if \( \beta' < 0 \) the precession will be prograde. Since the solutions are constrained to return to the plane at the outer bound-
ary, $\beta$ is always negative at large radius, and this dominates the direction of the induced precession.

We have found one additional analytic solution to equation (10), for $\delta = 1$ (Appendix C). We discuss this and the numerical solutions to equation (10) for both steady state and unstable modes in the following section.

3. SOLUTIONS OF THE TWIST EQUATION

Real astrophysical accretion disks will generally be unwarped beyond some maximum radius $R_{\text{max}}$, either because the disk becomes optically thin to the incident or reemitted radiation (see the discussion in Paper I) so that the disk must eventually return to the initial plane, or because this is forced in some other manner by the physical outer boundary condition (e.g., if the accretion disk is fed by material from a companion star; see §4.2). We therefore consider solutions of the twist equation for which the disk returns to the original plane $Z = 0$ at some radius. This outer boundary condition is not rigorously correct, as we discuss in § 4, but it is in general an excellent approximation to the true outer boundary condition, and the zero-crossing eigenfunctions are very useful for understanding the behavior of the warping instability.

Equation (10) is a second-order differential equation and therefore requires two boundary conditions to specify the solution. In principle it could be solved as a two-point boundary value problem, except that the location of the outer boundary is not necessarily known a priori. It is computationally most convenient to solve it as an initial-value problem, making an initial guess for the value of $\delta$ and then iterating to find the solution that goes to zero. We separate equation (10) into a coupled pair of equations for the real and imaginary parts of $W$ and integrate outward from the origin using a fifth-order Runge-Kutta scheme (Press et al. 1992).

We require solutions that are regular at the origin. As $x \to 0$, the leading terms of equation (10) are

$$xW'' + 2W' = ix^3 - 2\delta \sigma W,$$

where primes now denote derivatives with respect to $x$. Adopting the boundary condition $W(0) = 1$, the leading behavior of equation (11) as $x \to 0$ gives

$$W' \sim \frac{i\delta}{5 - 2\delta} x^{3 - 2\delta}$$

as $x \to 0$.

Furthermore, we require that zero torque be exerted on the disk at the origin, which requires that $x^2W' \to 0$ as $x \to 0$. From equation (12), it is evident that this second boundary condition is satisfied for any $\delta < 5/2$.

Numerical solutions to the twist equation are presented below. However, we first note that there are simple scaling relations that are generic features of the instability.

1. The radiation torque per unit area is $\Gamma \sim (L/4\pi R^2 c)$ $\times R$, and the angular momentum density is $\sim \Omega R^2 \Sigma \sim \Omega R^2 M/3\pi v_1$, where $\Omega$ is the angular velocity at radius $R$ in the disk, $M$ is the mass accretion rate, and $v_1$ is the usual kinematic viscosity. The radiation torque timescale is thus given by

$$t_{\text{rad}} \sim \frac{\Omega R^3 M c^2}{L} = \frac{\Omega R^3}{c v_1} \epsilon^{-1},$$

which, for an accretion-fueled source, depends only on the accretion efficiency $\epsilon = L/Mc^2$ and not on the luminosity $L$ and the mass accretion rate individually (Paper I; Maloney & Begelman 1997a).

2. The viscous timescale is $t_{\text{visc}} \sim R/V_k = 2R^2/3v_1$, so the ratio of viscous to radiation torque timescales is given by

$$t_{\text{visc}} \sim \frac{\epsilon c}{\Omega R} \sim \epsilon \left(\frac{R}{R_g}\right)^{1/2}, \quad \epsilon = \frac{GM}{c^2},$$

which is independent of the form of the viscosity law and depends only on $\epsilon$ and the radius normalized to the gravitational radius. Thus the radiation torque always wins out at large radii, i.e., for $R > \epsilon^{-1} R_g$ but viscosity always dominates near the center. This is why the disk will be flat (but in general will have nonzero tilt) at small radii. Note that this equation also implies that $t_{\text{visc}}/t_{\text{rad}} \sim 1$ at $x = 1$.

3. As is apparent from equations (7)-(9), altering the values of $\eta$ and $\epsilon$ while keeping the ratio $\eta/\epsilon$ fixed will affect the growth rates, but not the shapes of the solutions, i.e., the value of $x$ is unaffected.

3.1. Steady State Solutions

We first discuss the steady state solutions to the twist equations, for which $\delta_i \equiv \text{Im}(\bar{\delta}) = 0$.

Figure 1 shows the location of the zero, $x_0$, for the first 10 zero-crossing eigenfunctions as a function of surface density index $\delta$. As noted earlier, for $\delta \neq 3/2$ the eigenfunction corresponding to each eigenvalue has only a single zero (Appendix B). We use “order” to specify how many eigenfunctions have their zero at values of $x$ smaller than or equal to the eigenfunction in question; thus the eigenfunction with the smallest value of $x_0$ is the first-order eigenfunction, that with the next smallest is the second-order eigenfunction, and so forth. The behavior of the higher order eigenfunctions is remarkably complex.

In Figure 2 we plot the normalized real eigenvalues, $\bar{\sigma}_i$, for the first 10 order eigenfunctions. The first-order eigenfunction has the largest eigenvalue for all values of $\delta$. The

![Fig. 1.—Location of the zero, $x_0$, for the first 10 zero-crossing eigenfunctions for the steady state ($\delta = 0$) solutions of the twist equation. The locations of the zeros are plotted as a function of the surface density power-law index $\delta$ for $-3/2 \leq \delta \leq 3/2$. The steps in $\delta$ are 0.05. The merging of eigenvalues that occurs for the higher order eigenfunctions at $\delta < 3/2$ is marked.](image)
degeneracy of the case $\delta = 3/2$ discussed in Paper I is immediately apparent, with $\tilde{\sigma}_i = 1$ for all the eigenfunctions. For $\delta \neq 3/2$, the degeneracy is lifted, with increasing separation of the eigenvalues as $\delta$ decreases. Closer examination shows that the behavior becomes quite complex for the higher order zeros with decreasing $\delta$, as merging of eigenvalues occurs.

The location of the first-order zero as a function of $\delta$ has an important physical significance. In real astrophysical disks, shadowing effects probably make the higher order eigenfunctions unimportant. Just as for the isothermal case, for the growing modes the location of the zero moves outward from the steady state value as the growth rate increases (see below); all the modes with zeros at smaller radius are damped. Thus the location of the first zero marks the critical boundary for disk stability. Accretion disks larger than

$$R_{cr} = \frac{1}{2} \left( \frac{\eta}{\epsilon} \right)^{2} x_{cr} R_{s}$$

(15)

are unstable to warping by radiation pressure, where the critical value $x_{cr}$ is equal to $2\pi$ for $\delta = 3/2$ and $x_{cr} \approx 4.891\pi$ for $\delta = -3/2$. Figure 1 shows that $x_{cr}$ increases smoothly as $\delta$ decreases. Disks with outer boundaries smaller than $R_{cr}$ are stable against radiation-driven warping. From equation (15), this critical radius scales as $\epsilon^{-2}$, so that accretion disks in low-efficiency systems will not be unstable unless the disks are extremely large. (Furthermore, the precession and growth timescales for the instability will be extremely long if $\epsilon$ is very small; cf. the discussion of eqs. [30] and [31] in § 5.)

In Figure 3 we plot the tilt $\beta$ as a function of radial variable $x$ for the first-order steady state solutions for several values of $\delta$. For clarity they have been plotted only to the zero of the mode. In all cases the maximum tilt is at the origin (note that $\beta_{max}$ is arbitrary and has been taken to be unity); as $\delta$ decreases from 3/2 the zero moves to larger $x$ and an increasing fraction of the disk has $\beta \approx \beta_{max}$.

The steady state modes have shapes that are time-independent in a frame rotating with angular velocity $\sigma_r$; physically, these are purely precessing modes. The frequency $\sigma_r$ is related to the dimensionless frequency $\tilde{\sigma}_r$ by

$$\sigma = \frac{v_0}{8R_{0}} \tilde{\sigma}_r .$$

(16)

Defining the viscous timescale as before as

$$t_{visc}(R) = \frac{2R^2}{3v_1},$$

(17)

we can then express $\sigma$ in terms of the viscous timescale at the critical radius,

$$\sigma = \frac{\eta x_{cr}^{4-2\delta}}{12t_{visc}(R_{cr})} \tilde{\sigma}_r,$$

(18)

where $x_{cr}$ is the corresponding value of $x$ (cf. eq. [15]). For $\delta = -3/2$, $x_{cr}^{-2\delta} \tilde{\sigma}_r = 176.8$, and for $\delta = +3/2$ this quantity equals $2\pi$, so that in terms of the viscous timescale at $R_{cr}$ the variation in $\sigma_r$ is only a factor of $\approx 28$ over the entire range of $\delta$. Alternatively, $\sigma$ can be expressed in terms of the viscous timescale at the outer edge of the disk (i.e., the location of the zero), with $x_{cr}$ replaced by $x_0$ and $R(x_0)$. [In fact, the variation in $\sigma_r$ is smaller than this because for a fixed value of the viscosity, the increase in $R_{cr}$ with decreasing $\delta$ causes $t_{visc}(R_{cr})$ to increase, offsetting the variation in $x_{cr}^{-2\delta} \tilde{\sigma}_r$; see § 5.]

3.2. Unstable Solutions

We now consider solutions that evolve with time, i.e., $\tilde{\sigma}_i \neq 0$. As for the isothermal disks analyzed in Paper I, the only modes that have zeros at radii smaller than $x_{cr}$, as discussed above, are damped, with $\tilde{\sigma}_i > 0$, and we do not consider them further.

For the case of an isothermal disk considered in Paper I, the zeros move steadily outward with increasing growth rate (i.e., increasing $-\tilde{\sigma}_i$), and there is no limit to the growth rate. For the nondegenerate cases considered here, the zeros also move outward with increasing $-\tilde{\sigma}_i$, as expected from the scalings discussed in § 3. However, since the degeneracy of the eigenvalues has been lifted, there is now a maximum growth rate for a given order mode. In particular, in Figure 4 we plot the location of the first-order zero, $x_0$ (as discussed

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**Fig. 2.**—Real eigenvalues for the same steady state eigenfunctions as in Fig. 1. Plotted are the real eigenvalues $\tilde{\sigma}_r$ for the first 10 zero-crossing eigenfunctions. The case $\delta = 3/2$ is obviously degenerate ($\tilde{\sigma}_i = 1$ for all order eigenfunctions). The first-order eigenfunction has the largest eigenvalue for all other values of $\delta$. The behavior of the higher order (more distant) zeros becomes very complex for smaller values of $\delta$, with merging of the eigenvalues, as seen in Fig. 1.

**Fig. 3.**—Magnitude of the tilt $\beta$ as a function of $x$ for the steady state eigenfunctions. Plotted is $\beta(x)$ for (left to right) $\delta = 1.5$ to $-1.5$ in steps of 0.25 in $\delta$. The tilt at the origin has been set to unity.
above this is probably the only physically relevant eigenfunction as a function of \(-\tilde{\sigma}_i\) for \(\delta = -3/2\) to 1.45 in steps of 0.05 in \(\delta\). The systematic behavior of the location of the zero is quite striking, and there is a clear transition across \(\delta = 1\). The maximum possible (normalized) growth rate \(-\tilde{\sigma}_i\) for the first-order zero steadily decreases as \(\delta\) decreases. But the location of the zero at this maximum growth rate, \(-\tilde{\sigma}_{i,\text{max}}\), moves to larger \(x\) as \(\delta\) decreases from 1.45 to 1.00, even though \(-\tilde{\sigma}_{i,\text{max}}\) systematically declines with decreasing \(\delta\).

The case \(\delta = 1\) is special. There is an analytic solution in this case, as discussed in Appendix C. The maximum growth rate is \(-\tilde{\sigma}_{i,\text{max}} = 0.25\); at this value of \(\tilde{\sigma}_i\), the zero has moved to \(x = \infty\). (The curve plotted in Fig. 4 ends at \(-\tilde{\sigma}_{i,\text{max}} = 0.249\).) This value of \(\delta\) is also special in that all of the eigenvalues merge as \(\tilde{\sigma}_i \to \tilde{\sigma}_{i,\text{max}}\), so that the zeros of all orders move to infinity at \(-\tilde{\sigma}_{i,\text{max}} = 0.25\). Figure 5a plots the location of the first three order zeros for \(\delta = 1\) as a function of \(-\tilde{\sigma}_i\), and Figure 5b shows the real parts of the corresponding eigenvalues.

For \(\delta < 1\), the maximum \(x\) value for the first-order zero decreases rapidly as \(\delta\) decreases, reaching a minimum at \(\delta \approx 0.15\), and then slowly increases as \(\delta\) approaches \(-3/2\) (see Fig. 4). The ratio \(x_0(\tilde{\sigma}_{i,\text{max}})/x_0(\tilde{\sigma}_i = 0)\) decreases as \(\delta\) decreases, so that the range of instability (out to the first zero) decreases with \(\delta\) in this range.

In Figure 6 we plot the real part of the eigenvalue, \(\tilde{\sigma}_r\), versus the imaginary part for the growing modes, for the same values of \(\delta\) as in Figure 4. In all cases \(\tilde{\sigma}_r\) shows a characteristic decrease as \(\tilde{\sigma}_{i,\text{max}}\) is approached. As in Figure 4, the curve for \(\delta = 1\) is plotted only to \(-\tilde{\sigma}_i = 0.249\); \(\tilde{\sigma}_r \to 0\) as \(-\tilde{\sigma}_i \to 0.25\). The magnitude of the drop in \(\tilde{\sigma}_r\) declines as \(\delta\) approaches \(-3/2\); for the isothermal case, \(\tilde{\sigma}_r = 1\), independent of \(\tilde{\sigma}_i\) (Paper I).

There is a dramatic change in the behavior of the unstable solutions across \(\delta = 1\). In Figure 7 we have plotted the shape of the eigenfunctions, \(i.e., \beta\) as a function of \(x\) for several values of \(\delta > 1\) for the fastest growing modes with \(\tilde{\sigma}_i = \tilde{\sigma}_{i,\text{max}}\). The eigenfunctions have been normalized so that the maximum value of \(\beta\) is unity; the normalization factor (i.e., \(1/\beta_{\text{max}}\)) is very small, of order \(10^{-6}\). The behavior of the eigenfunctions is very similar to the fast-growing modes of the isothermal disks discussed in Paper I: the warp has its maximum close to the outer boundary of the disk (assumed to be coincident with the zero), and the amplitude of the warp at this maximum is very large compared to that at \(x = 0\) (approximately \(10^6\) for the modes shown in Fig. 7). The shape of the fastest growing modes for \(\delta = 1\) is qualitatively similar to the \(\delta > 1\) modes; the only difference is that \(x_0\) approaches \(\infty\) (rather than a finite value) as \(\delta\) approaches \(\tilde{\sigma}_{i,\text{max}}\).

In Figure 8 we similarly plot the normalized eigenfunctions for the fastest growing modes for several values of \(\delta < 1\). The change in the nature of the eigenfunctions is marked; even for \(\delta = 0.95\), the amplitude of \(\beta\) at the maximum is only \(\sim 2.5\) times the value of \(\beta\) at \(x = 0\), \(\beta_0\), and as \(\delta\) decreases \(\beta_{\text{max}}/\beta_0\) declines toward unity. For \(\delta \lesssim 0.5\), the eigenfunctions converge to an essentially constant shape, with a plateau of constant \(\beta\) extending from the origin to \(x \sim 5\), followed by a decrease to the zero. This

![Fig. 4](image1.png)

**Fig. 4.**—Location of the zero for the (growing) time-dependent eigenfunctions. Plotted is \(x_0\) for the first-order growing modes as a function of the growth rate \(-\tilde{\sigma}_i\). From left to right, the curves are for \(\delta = -3/2\) to \(\delta = 1.45\) in steps of 0.05 in \(\delta\). The upper envelope delineates the maximum growth rates for the first-order modes. For \(\delta = 1\), \(x_0 \to \infty\) as \(-\tilde{\sigma}_i \to 0.25\); see Appendices B and C. The curve for this case has been plotted to \(-\tilde{\sigma}_i = 0.249\).

![Fig. 5a](image2.png)

**Fig. 5a**

![Fig. 5b](image3.png)

**Fig. 5b**

![Fig. 5](image4.png)
change in the behavior of the eigenfunctions across $\delta = 1$ has important implications, as we discuss below. Note also that this applies only to the unstable modes; the steady state solutions do not exhibit any change in behavior across $\delta = 1$, as is apparent from Figure 3.

4. OUTER BOUNDARY CONDITIONS

In § 3, we solved equation (10) with the requirement that the solution go to zero at the disk outer boundary, which is at some finite radius. Although this produces a well-defined set of solutions whose behavior is very useful for understanding the behavior of the instability, in real astrophysical disks the true outer boundary condition is likely to be different. Furthermore, the correct outer boundary condition for accretion disks around stellar-mass compact objects (neutron stars and black holes) in X-ray binaries will differ from that appropriate for accretion disks around massive black holes in active galactic nuclei. In this section we discuss the choice of outer boundary condition and how the solutions in these cases differ from the zero-crossing solutions discussed above.

4.1. Active Galactic Nuclei

It is expected that accretion disks will generally satisfy the requirement of optical thickness and will therefore be subject to the radiation-warping instability (see § 5 and Appendix D). However, accretion disks surrounding massive black holes in active galactic nuclei are likely to become optically thin (to the reemitted radiation) before the physical boundary of the accretion disk is reached. (This is true for $\delta > 0$, so that the disk surface density decreases with increasing radius; this is undoubtedly the case at the relevant radii of AGN accretion disks.) Once the disk becomes optically thin, the instability ceases to operate, as the reemitted radiation no longer exerts a torque on the disk (cf. Paper I). Therefore, equation (10) does not describe the behavior of the disk beyond the optically thin radius $R_{\text{min}}$. However, viscosity continues to operate, so that twist angular momentum generated by the instability can be transported beyond $R_{\text{min}}$. We therefore have to solve a pair of equations, with the solutions matched at $R_{\text{min}}$: interior to $R_{\text{min}}$, the equation governing the dynamics of the disk is equation (10), whereas exterior to $R_{\text{min}}$ the disk is described by equation (10) minus the torque term $-i\omega W'$. Both $W$ and $W'$ must be continuous at $R_{\text{min}}$; the outer disk solution must also obey $W \rightarrow 0$ as $R \rightarrow \infty$. In a real accretion disk, the radiation torque will presumably decline smoothly to zero rather than switch off abruptly, but as long as this occurs in a radial thickness $\Delta R \ll R$ we expect it to have little effect on the solutions.

For $\delta = 3/2$ and $\delta = 1$ the outer disk equation can be solved analytically; the solutions are modified Bessel functions ($K_1$ and $K_{1/2}$, respectively). In general it must be solved numerically. For a specified value of $R_{\text{thin}}$, we solve both the inner and outer disk equations and iterate to find the solution that satisfies the matching condition at $R_{\text{thin}}$. (In practical terms, the constraint that must be satisfied is that $\beta'$ be continuous at $R_{\text{thin}}$, since $\beta$ can be scaled arbitrarily and $\gamma$ does not enter into eq. [10], only $\gamma'$ and $\gamma''$; the value of $\gamma$ can always be matched by an arbitrary rotation of the outer disk solution.) Since the parameter space to be considered is very large, we consider only a few representative solutions for $\delta = 3/2$, 1.25, and 0.75; the behavior of solutions for $\delta < 0.75$ is very similar to that of the $\delta = 0.75$ solutions.

We first consider the modifications to the steady state, first-order solutions shown in Figure 3. We take $R_{\text{thin}}$ to be the zero of these modes and then find the solutions that satisfy the boundary conditions at $R_{\text{thin}}$. In Figure 9 we plot
the real versus the imaginary parts of the eigenvalues for these solutions. Not surprisingly, there is now a continuum of solutions rather than a single mode, since it is easier to satisfy the boundary condition of solutions rather than a single mode, since it is easier to these solutions. Not surprisingly, there is now a continuum of values for a given value of \( \delta \). In all cases there is a well-defined maximum (and minimum) growth rate at which the two values of \( \bar{\sigma}_i \) merge; the precession rate \( \bar{\sigma}_i \) corresponding to this fastest growing mode differs from that of the zero-crossing solution by \( \approx 2\%, 8.5\%, \) and \( 13\% \) for \( \delta = 1.5, 1.25, \) and \( 0.75 \), respectively.

In Figure 10 we plot the tilt \( \beta \), normalized to a maximum of unity as before, for the steady state zero-crossing modes (\textit{dotted lines}) and the corresponding fastest growing modes from Figure 9, with \( R_{\text{thin}} = x_0 \) (\textit{solid lines}). The behavior of the AGN modes is what one would expect: since the growth rates are nonzero, the maximum of the tilt has moved outward from the origin and the solutions decline smoothly to zero, with \( W' \to 0 \) as \( x \to \infty \). Since the differences in shape between the steady state mode and growing modes are minor for \( \delta \leq 0.75 \), as discussed in § 3, the differences between the zero-crossing modes and the optically thin boundary modes are minimal for the \( \delta = 0.75 \) case. We also note that beyond \( R_{\text{thin}} \) \( y' \) becomes negative (i.e., the line of nodes describes a retrograde spiral), since in the absence of radiation torque viscosity simply causes the line of nodes to unwind.

We now consider modifications to the rapidly growing modes. Figure 11 plots the precession rate versus the growth rate for rapidly growing modes for the three values of \( \delta \). For \( \delta = 1.25 \) and \( 0.75 \) the zero-crossing modes used to set the value of \( R_{\text{thin}} \) have growth rates close to the maximum for first-order modes. For \( \delta = 1.50 \) there is no maximum growth rate (owing to the degeneracy) and so the solution shown is simply a rapidly growing mode, with \( \bar{\sigma}_i = -5 \). The behavior of the continuum of modes that satisfies the boundary conditions is very similar to those shown in Figure 9, except that there are no damped modes in this case. Comparison with Figure 9 also shows that for \( \delta \neq 1.50 \), the width of the curve in the \( (\bar{\sigma}_n, \bar{\sigma}_i) \)-plane is significantly smaller for these fast-growing modes. The precession frequencies of the maximally growing modes are even closer to those of the zero-crossing modes than for those shown in Figure 9, differing by \( \approx 0.05\%, 1\% \), and \( 8\% \) in order of decreasing \( \delta \).

In Figure 12 we plot \( \beta \) as a function of \( x \), as in Figure 10, with the fastest growing modes shown as solid lines and the corresponding zero-crossing modes shown as dashed lines. The differences between the zero-crossing solutions and the \( x(R_{\text{thin}}) = x_0 \) modes is even smaller than in Figure 10; the peak in \( \beta(x) \) is displaced slightly outward and \( \beta \) declines smoothly to zero with increasing \( x \). For the \( \delta > 1 \) modes, which peak well away from the origin, the width of the peak in \( \beta(x) \) at half-maximum is negligibly different in the two cases. For \( \delta \leq 0.75 \), it is evident that the difference between the zero-crossing modes and the optically thin boundary modes is negligible for all growth rates.

4.2. X-Ray Binaries

In X-ray binary systems the situation is rather different. It is likely that the accretion disk remains optically thick to the physical outer boundary of the disk. However, the disk will again not obey equation (10) throughout because the assumption of a power-law surface density cannot hold for the entire disk. What we have taken to be the outer boundary of the disk in the zero-crossing solutions corresponds to the circularization radius \( R_{\text{cirl}} \) in a real X-ray binary system.
The disk will actually extend to the tidal truncation radius $R_{\text{out}}$, which is larger by a factor of 2–3 for reasonable mass ratios (e.g., Frank et al. 1992, p. 83). In this outer disk region, $R_{\text{circ}} < R \leq R_{\text{out}}$, the radial velocity $V_R = 0$, since there is no flux of mass through this region of the disk, only angular momentum. Conservation of angular momentum then requires that

$$v_1 \Sigma = (v_1 \Sigma)_{R_{\text{circ}}} \left(\frac{R_{\text{circ}}}{R}\right)^{1/2}.$$  \hfill (19)

To determine the surface density distribution in the outer disk thus requires an assumption about the viscosity. We assume that $v_1 = v_{\text{out}}$ is constant in the outer disk; in fact, since $R_{\text{out}}$ is only $\sim 2–3R_{\text{circ}}$, the results are entirely insensitive to this assumption unless $v$ is an extremely strong function of radius in the outer disk. The term in brackets on the left-hand side of equation (1) is then

$$V_R - v_1 \Omega - \frac{1}{2} v_2 \left(\Sigma R^2 \Omega\right) = \frac{1}{2} v_{\text{out}}^2 \left(3 - \eta\right).$$  \hfill (20)

Fourier transforming with respect to time and linearizing as before, the twist equation becomes

$$\eta W'' + \left[\frac{2(\eta - 3)}{\eta} - \frac{2i\Gamma_{\text{out}}}{v_{\text{out}}}\right] W' - \frac{2i\sigma W}{v_{\text{out}}} = 0,$$  \hfill (21)

where $\Gamma_{\text{out}}$, defined as in equation (5), is constant since $\Sigma R^2 \Omega$ is also constant under the assumption that $v$ is independent of the radius in the outer disk. This does not include the torque due to the companion star, which can be of vital importance for the disk modes in X-ray binary systems; as shown in Maloney & Begelman (1997b), the companion torque allows retrograde modes as well as prograde modes to exist. Since the orbital period is short compared to the viscous timescale in the disk, we average over azimuth and keep only the leading (quadrupole) term in the companion torque. This contributes a term $-i\omega_0 (R/R_0)^{3/2} W$ to the left-hand side of equation (21), where $\omega_0$ is the quadrupole precession frequency at the fiducial radius $R_0$ (Maloney & Begelman 1997b).

By continuity, the viscosity $v_{\text{out}}$ and the radiation torque parameter $\Gamma_{\text{out}}$ in the outer disk are given by

$$v_{\text{out}} \equiv v_1(R_{\text{circ}}) = v_0(R_{\text{circ}}/R_0)^{4/3},$$  \hfill (22)

$$\Gamma_{\text{out}} \equiv \Gamma_0 R_{\text{circ}}^{1/2}.$$  \hfill (23)

Transforming to radius variable $x$ and nondimensionalizing both $\sigma$ and $\omega_0$ by $2\eta^3 R_0^2/v_0 e^3$ (cf. eq. [9]) the linearized twist equation for the outer disk becomes

$$x W'' + \left[\frac{2(\eta - 3)}{\eta} - 1\right] W' - \frac{ix^2}{x_{\text{circ}}} W' = \frac{ix^3}{x_{\text{circ}}^3} \left(\tilde{\sigma} + \tilde{\omega}_0 x^3\right) W.$$  \hfill (24)

For typical neutron star X-ray binary parameters, $x_{\text{circ}} \approx 30$ (assuming $e \approx 0.1$) and $x_{\text{out}} \approx (3)^{1/2} x_{\text{circ}}$.

The value of $W$ at the outer boundary is arbitrary, but we require that $W' = 0$ at $R_{\text{out}}$; this implies that, in the absence of tidally induced precession, there is no torque acting on the outer disk boundary. As in the case of the AGN (optically thin) boundary conditions discussed above, the key question is how the outer disk solutions couple to the inner disk, where equation (10) is valid. In Appendix E, we show that at $R_{\text{circ}}$, the disk must satisfy a jump condition,* given by

$$R_{\text{circ}} \Delta W' = \frac{3}{\eta},$$  \hfill (25)

where $\Delta W' = W'_+ - W'_-$ is the jump in $W'$ at $R_{\text{circ}}$ (i.e., the difference in the values of $W'$ at radii infinitesimally larger and smaller than $R_{\text{circ}}$). This result was first derived by J. E. Pringle (1997, private communication). Note that equation

* This jump condition is derived by assuming that mass and angular momentum are added via the accretion stream at a fixed radius. It assumes no viscosity discontinuities due to the stream and hence is itself an idealization. On the other hand, assuming the warp goes to zero at $R_{\text{circ}}$ is in some sense the opposite limit, as it implicitly posits that the effect of viscosity is to pin the disk to the orbital plane at the circularization radius. That these two extreme views yield comparable results is an indication of the insensitivity of our results to the exact choice of boundary conditions.

**Fig. 11.**—Same as Fig. 9 but with $R_{\text{thin}}$ set equal to the zero for rapidly growing zero-crossing modes rather than steady state zero-crossing modes.

**Fig. 12.**—Same as Fig. 10 but with $R_{\text{thin}}$ set equal to the zero for rapidly growing zero-crossing modes rather than steady state zero-crossing modes.
(25) indicates that $W_+^\prime$ is larger than $W_-^\prime$; thus if $W_+^\prime < 0$, then $W_-^\prime$, the gradient of $W$ just interior to $R_{\text{circ}}$, must be steeper (more negative).

The outer disk equation can be solved numerically to find the appropriate boundary condition at $R_{\text{circ}}$ and then determine the matching inner disk solution, as in § 4.1. However, we can show more directly that the solutions will always be close to the zero-crossing solutions. It is straightforward to find the gradient of the asymptotic solution for the outer disk in the usual WKB approximation (e.g., Bender & Orszag 1978) that $W = e^\phi$ (see also Appendix A)

$$\frac{W^\prime}{W} = \frac{(5 + i\chi x)}{2} \pm \frac{i}{2} \sqrt{\frac{x^3}{25} - i x^3 \left[ \frac{10}{x_{\text{circ}}} + 4 x_{\text{circ}}^2 - 24(\tilde{\sigma} + \tilde{\omega}_0 x^3) \right]}^{1/2},$$

(26)

where $\chi \equiv x/x_{\text{circ}} \geq 1$. Comparison with numerical solutions of the outer disk equation shows that equation (26) (with choice of minus sign) is generally accurate to 10%, with the worst error being 15%–20% (the latter being the case for some of the prograde disk modes, which tend to have smaller values of $x_{\text{circ}}$ than the retrograde modes); equation (26) always underestimates the magnitude of the gradient. [Note that eq. (26) does not predict $W^\prime \rightarrow 0$ as $x \rightarrow x_{\text{out}}$, since the assumption that $W^\prime \ll (S)^2$ breaks down as $x$ approaches $x_{\text{out}}$.] Since we are chiefly interested in equation (26) for deriving the jump condition at the circularization radius, we take the limit $x \rightarrow x_{\text{circ}}$, so that $\chi \rightarrow 1$

$$\frac{W^\prime}{W} = \frac{(5 + i\chi_{\text{circ}} x)}{2} \pm \frac{i}{2} \left[ x_{\text{circ}}^3 - 25 - i \left[ 10 x_{\text{circ}} + 4 x_{\text{circ}}^2 - 24(\tilde{\sigma} + \tilde{\omega}_0 x_{\text{circ}}^3) \right] \right]^{1/2}.$$  

(27)

In Figure 13 we plot the value of the gradient in the disk tilt $xW^\prime/W$ just outside the circularization radius for all of the prograde and retrograde warp models presented in Maloney & Begelman (1997a, 1997b). The solid portions of the curves indicate retrograde modes. In all cases the gradient is negative and generally large, especially for the retrograde modes; to satisfy the jump condition the gradient just inside $R_{\text{circ}}$ must be even steeper (note that $xW^\prime/W = 6/\eta$). Because the gradient is always negative and steep where the inner disk solution patches on to the outer disk solution, the former must have a shape close to that of the zero-crossing mode with a zero at $R_{\text{circ}}$. Moreover, the tilt of the outer disk solution always goes rapidly to zero for $R > R_{\text{circ}}$. For example, Figure 14 plots the tilt of the outer disk solutions as a function of radius for the most rapidly precessing steady state ($\tilde{\sigma}_i = 0$) solutions, $\tilde{\delta} = 0.75$ and 1.25. Even for the most slowly declining solution, the amplitude of the tilt at $x_{\text{out}}$ is less than 1% of the value at $x_{\text{circ}}$.

Changing the outer boundary condition does not allow a range of solutions to exist, unlike the optically thin boundary condition discussed above. The relevant physical question then becomes, "What is the nature of the unstable mode for a given value of $x_{\text{circ}}$ and $\tilde{\omega}_0$?" In Figure 15 we show the precession and growth rates for the unstable modes as a function of $\tilde{\omega}_0$ for a fixed value of $x_{\text{circ}} = 30$. The upper half of the figure shows the precession rate, and the lower half shows the growth rate. As expected, at small values of the quadrupole torque parameter the unstable mode is prograde. With increasing $\tilde{\omega}_0$, the unstable mode switches to retrograde precession and the growth rate starts to decline, eventually going to zero. This is easy to understand physically: for a fixed value of $x_{\text{circ}}$, the effect of increasing $\tilde{\omega}_0$ is to make the external torque more important at $x_{\text{circ}}$, until eventually this dominates over the radiation torque in controlling the precession and forces the warping mode to become retrograde. At very large values of the external torque, the viscous torques produced by the driven differential precession become too strong compared to the radiation-pressure torques, and the instability is switched off; thus the growth rate goes to zero as $\tilde{\omega}_0$ approaches this critical value. 

![Figure 13](image1.png)

**Fig. 13.**—Gradient in the disk tilt $xW^\prime/W$ just outside the circularization radius for all of the prograde and retrograde warp models presented in Maloney & Begelman (1997b). This has been derived by means of the asymptotic solution to the outer disk equation, given by eq. (27). The solid lines indicate the solution regimes where the precession direction is retrograde; the prograde solutions are $\tilde{\delta} = 0$, $\delta = 0.75$, $\tilde{\delta} = 0$ (short-dashed line); $\tilde{\delta} = 0.75$, maximum $-\tilde{\sigma}_i$ (long-dashed line); $\delta = 1.25$, maximum $-\tilde{\sigma}_i$ (dot-dashed line). Note that in all cases the gradient is large and negative.

![Figure 14](image2.png)

**Fig. 14.**—Tilt $\theta$ of the outer disk solutions for the most rapidly precessing steady state warp modes found by Maloney & Begelman (1997b), $\tilde{\delta} = 0.75$ and 1.25. The retrograde solutions are shown by the solid ($\tilde{\delta} = 1.25$) and dotted ($\tilde{\delta} = 0.75$) lines, and the prograde solutions are shown by the short-dashed ($\tilde{\delta} = 1.25$) and long-dashed ($\tilde{\delta} = 0.75$) lines.
There is actually a second branch of retrograde solutions displayed in Figure 15 (dashed curve); these have comparable growth rates to the modes just discussed but precession rates about an order of magnitude larger. This second branch consists of modes for which the gradient in the tilt is positive (and generally large) at the circularization radius, unlike the solutions displayed in Figures 13 and 14. For these solutions the warp actually peaks in the outer disk at \( R > R_{\text{circ}} \) and then declines rapidly toward zero. It is not clear that these solutions will ever occur in real astrophysical disks, as our treatment of the outer disk \( (R_{\text{circ}} \leq R \leq R_{\text{warp}}) \) is much more of an idealization than that of the inner disk.

5. DISCUSSION

Earlier work on the radiation-driven warping instability discovered by Pringle (P96; Paper I) considered only the isothermal, \( \delta = 3/2 \) case. In this paper we have considered more general power-law disk density distributions, from the isothermal disk to \( \delta = -3/2 \), corresponding to a radiation-pressure–supported disk; this spans the range that is likely to be relevant to astrophysical disks. Although the shapes of the eigenfunctions do change with decreasing \( \delta \), the most important features of the instability are generic. Most importantly, the instability exists over the entire range of surface density index that we have considered, and the critical radius above which disks are unstable to radiation-driven warping changes only by a factor of \( \approx 6 \) from \( \delta = 3/2 \) to \( \delta = -3/2 \). Similarly, the growth and precession rates (in dimensional units) do not depend strongly on \( \delta \) (see the discussion after eq. [18] and below). Evaluating equation (15) for the critical radius,

\[
R_{\text{cr}} = (5.9 \times 10^8 \text{ to } 3.5 \times 10^9 \frac{\eta}{\varepsilon_{0.1}} \left( \frac{M}{M_\odot} \right)^2 \text{ cm ,}
\]

\[
= (5.9 \times 10^{16} \text{ to } 3.5 \times 10^{17} \frac{\eta}{\varepsilon_{0.1}} \left( \frac{M}{10^9 M_\odot} \right)^2 \text{ cm ,}
\]

where the range in numerical values is for \( \delta = 3/2 \) to \( \delta = -3/2 \) and \( \varepsilon = 0.1 \). The only warping modes with zeros at \( R < R_{\text{cr}} \) are damped, so that disks that are smaller than \( R_{\text{cr}} \) are stable against warping. In consequence of the \( \varepsilon^{-2} \) scaling of \( R_{\text{cr}} \), accretion disks in systems with very low radiative efficiency will not be unstable to radiation-driven warping unless they are implausibly large. For this reason, this mechanism cannot provide an explanation for the warp in the thin maser disk of NGC 4258 (e.g., Miyoshi et al. 1995; Herrnstein et al. 1998) if the inner disk is advection-dominated with \( \varepsilon \sim 10^{-3} \) (Lasota et al. 1996; see the discussion in Paper I), since the maser disk would be far too small for instability in this case. This also indicates that radiation-driven warping generally will not be important in cataclysmic variables or protostellar disks dominated by accretion-powered luminosity, since the radiative efficiency is limited to small values as the stellar surfaces are at \( R_* \gg R_c \) (but see Armitage & Pringle 1997 for a discussion of the possible action of the instability in the protostellar case).

To evaluate the typical precession timescales, we need to evaluate the viscous inflow timescale at \( R_{\text{cr}} \). Letting \( v_1 = a_c H_c \), where \( c_s \) is the isothermal sound speed and \( H \) is the scale height, we can write the viscous timescale as

\[
t_{\text{visc}} \sim \frac{2}{3} \frac{R}{V_\phi} \frac{\alpha}{\varepsilon} \left( \frac{H}{R} \right)^{-2},
\]

where \( V_\phi \) is the rotational velocity (assumed to be Keplerian) and \( H/R \) is evaluated at the radius in question. Since \( t_{\text{visc}} \propto R^{3/2} \) and \( R_{\text{cr}} / R_0 = x_{\text{cr}} \),

\[
t_{\text{visc}}(R_{\text{cr}}) \sim \frac{1}{3} \left( \frac{\eta}{\varepsilon} \right)^3 \frac{R_{\text{cr}}}{\alpha c_s} x_{\text{cr}}^{3/2} \left( \frac{H}{R} \right)^{-2},
\]

where \( H/R \) is now evaluated at \( R_{\text{cr}} \). Taking the precession timescale \( t_{\text{prec}} = 2\pi/\sigma_\nu \), where \( \sigma_\nu \) is given by equation (18), and evaluating the constants, we find

\[
t_{\text{prec}} \sim 12 \frac{\eta^2}{\varepsilon_{0.1}} \frac{M/M_\odot}{\alpha_{0.1}} \left( \frac{H/R}{0.01} \right)^{-2} \text{ days}
\]

with only weak dependence on \( \delta \); the numerical coefficients only vary by a factor of 2 over the whole range of \( \delta \). Thus the precession timescales for X-ray binary systems (the only systems in which precession can actually be observed) are expected to be of the order of weeks to months.

This is of course the precession timescale for the steady state modes from linear theory. As discussed in § 3, real disks will ordinarily be unwarped beyond some maximum radius, either the physical edge of the disk or where the disk becomes optically thin. This outer boundary, which will not in general correspond to the critical radius, will determine the warp growth rate. We expect that the warp will eventually saturate at some amplitude (but see Pringle 1997). Assuming that the disk does reach a steady state, what will the precession rate be? There is reason to suspect it may not be very different from the linear theory result. Figure 6 shows that, except for growth rates very close to the maximum, the real part of the eigenvalue \( i \sigma_\nu \), i.e., the precession rate, is nearly independent of the growth rate. In the isothermal case, in fact, \( \sigma_\nu \) is independent of \( \sigma_\eta \). This suggests that, however different modes may couple in reaching the final state, the precession rate will be similar to the linear steady state result.

Implicit throughout this paper has been the assumption
that the disks are optically thick to both absorption and reemission, so that they are subject to the radiation-driven warping instability. This requirement imposes a minimum mass accretion rate that must be exceeded for the disk to be optically thick. In Appendix D, we derive this critical mass accretion rate for three different possible sources of opacity in astrophysical disks (electron scattering, dust absorption, and Kramer’s opacity) and show that it does not in general place any significant limitations on occurrence of the instability.

As discussed in § 3.2, there is one very important systematic change in the nature of the instability with δ. The difference in the behavior of the growing modes for δ ≥ 1 and δ < 1 is of fundamental importance for the evolution of disks warped by radiation pressure. For δ ≥ 1, the fast-growing modes all have their maximum warp (i.e., tilt β) close to the outer edge of the disk, and the amplitude βmax is much greater than β0, the tilt at the origin. This immediately implies that the warp must reach the nonlinear regime when the tilt at small radius is negligible. In this case the evolution of the disk at radii interior to the warp maximum is almost certainly driven by the nonlinear evolution of the outer warp (e.g., Pringle 1997), so that nonlinear effects must be important if the warp extends to the disk inner edge.

For δ < 1, the behavior is qualitatively different, as βmax/β0 is always of order unity. In this regime, nonlinearity will be important only if the warp has grown out of the linear regime at the origin. Furthermore, because the shapes of the growing warps in these two regimes are so dissimilar, the effects of shadowing of the central source by the warping of the disk will be very different. These distinctions are liable to be crucial for X-ray binaries such as SS 433 and Her X-1, which show evidence for a global precessing warp.

One final point regarding X-ray binary systems must be mentioned. In one of the best studied systems, Her X-1, the direction of precession of the warp is inferred to be retrograde with respect to the direction of rotation (e.g., Gerend & Boynton 1976) and this has also been suggested for SS 433 (Leibowitz 1984; Brinkmann, Kawai, & Matsuoka 1989). As shown in Appendix B, in the absence of external torques the direction of precession of the warp must be prograde. However, the qualification on this statement is extremely important; as pointed out in § 4.2 and discussed in detail by Maloney & Begelman (1997b), including the quadrupole torque from a companion star allows retrograde as well as prograde solutions to exist.

The zero-crossing outer boundary condition that we have imposed will not be strictly correct in real astrophysical disks. However, as discussed in § 4, the solutions that obey the likely realistic outer boundary conditions—the optically thin outer boundary for accretion disks in active galactic nuclei and a flat outer boundary for disks in X-ray binaries—are in all important respects similar to the zero-crossing solutions.

Radiation-driven warping and precession offers a robust mechanism for producing tilted, precessing accretion disks, in accreting binary systems such as Her X-1 and SS 433, and in active galactic nuclei. Because radiation-driven warping is an inherently global mechanism, it avoids the difficulties inherent in other proposed mechanisms for producing warping and precession, e.g., communicating a single precession frequency through a fluid, differentially rotating disk. This mechanism can thus explain the simultaneous precession of inner disks (as evidenced by the jets of SS 433 and the pulse profile variations of Her X-1) and outer disks (as required to match the periodicities in X-ray flux and disk emission in these same objects).

A full understanding of the nature of the radiation-driven warping instability will require nonlinear simulations of the type presented in Pringle (1997), which will not only allow for inclusion of the nonlinear terms but also inherently nonlinear effects such as shadowing. This will be the subject of future work.

We have greatly benefited from discussions with and comments from Jim Pringle and Phil Armitage, who also kindly provided results in advance of publication. We are especially grateful to Jim Pringle for his insightful comments regarding the choice of outer boundary conditions. We would also like to thank the referee for helpful comments on the paper. P. R. M. was supported by the NASA Long Term Space Astrophysics Program under grant NAGW-4454. M. C. B. acknowledges support from NSF grant AST-9529175. The research of P. R. M. and M. C. B. was supported by NASA grant NAG5-4061 under the Astrophysical Theory Program. M. A. N. was supported by the NASA LTSA Program under grant NAG5-3225.

APPENDIX A

ASYMPTOTIC SOLUTIONS TO THE TWIST EQUATION

In this appendix we examine the behavior of the twist equation at large x and show that, unlike the δ = 3/2 case studied in Paper I, the solutions for W always diverge as x → ∞. We also show from the asymptotic solutions that the eigenfunctions for the case δ = 1 always diverge as the growth rate −δi → 0.25, independent of the real eigenvalue δr, as is seen in the numerical solutions (§ 3.2).

We write the twist equation (10) as

\[ xW'' + (2 - ix)W' - i\beta x^\beta W = 0, \]  

(A1)

where primes denote derivatives with respect to x and \( \mu \equiv 3 - 2\delta \geq 0 \) for \( \delta \leq 3/2 \); recall that x is real and \( \delta \) is complex. For any value of \( \mu \), this equation has an irregular singular point at infinity. Assuming that \( W = e^{\delta} \) (e.g., Bender & Orszag 1978), we obtain

\[ xS'' + x(S')^2 + (2 - ix)S' - i\beta x^\beta = 0. \]  

(A2)
We further assume that \( S'' \ll (S')^2 \). As \( x \to \infty \), the \( 2S' \) term can be neglected compared to \( -ixS' \); however, we retain it for reasons that will become obvious below. Equation (A2) is then

\[
(S')^2 + \left( \frac{2}{x} - i \right) S' - i \sigma x = 0 ,
\]

where \( \kappa \equiv \mu - 1 \geq -1 \). The solution to this equation is

\[
S' \sim \frac{1}{2} \left[ \left( i - \frac{2}{x} \right) \pm \left( -\frac{4i}{x} - 1 + 4i\sigma x^2 \right)^{1/2} \right] .
\]

This equation has qualitatively different behavior at large \( x \) for \( \kappa < 0 \) and \( \kappa \geq 0 \), corresponding to \( \delta > 1 \) and \( \delta \leq 1 \), respectively. (This difference in behavior is of course also seen in the numerical solutions, as discussed in § 3.) If \( \kappa < 0 \), then all the \( x \) terms in the square root are small compared to unity, and expansion of the square root gives the two solutions

\[
S' \sim i - \frac{2}{x} + \sigma x^2 , \quad -\sigma x^2 , \quad -1 \leq \kappa < 0 .
\]

[Note that the validity of the assumption that \( S'' \ll (S')^2 \) requires that \( \sigma x^2 \gg \kappa/x \); this obviously breaks down as \( \kappa \to -1 \).] Integrating and exponentiating then gives the two solutions for \( W \)

\[
W_* \sim e^{ixx^2 - 2e^{(\sigma/\mu)x^2}} , \quad W_- \sim e^{-e^{(\sigma/\mu)x^2}} , \quad 0 < \mu < 1
\]

\[
W_* \sim e^{ixx^2 - 2} , \quad W_- \sim x^{\sigma - 2} , \quad \mu = 0 .
\]

The general solution will be a linear combination of \( W_* \) and \( W_- \). Equation (A7) is just the asymptotic behavior of the Kummer functions \( M(a, b, x) \) (cf. Paper 1; Abramowitz & Stegun 1964, eq. [13.5.1]), with \( a = \sigma \) and \( b = 2 \). Equation (A7) shows that for \( \mu = 0 \) we can impose the condition \( W \to 0 \) as \( x \to \infty \) provided that \( 0 < \sigma < 2 \). However, equation (A6) shows that it is not possible to impose this condition for \( 0 < \mu < 1 \) if \( \sigma \neq 0 \); one solution for \( W \) always diverges as \( x \to \infty \). Thus, unlike the degenerate \( \delta = 3/2 \) solutions, \( W \) always diverges as \( x \to \infty \), for any value of \( \sigma \).

For \( \kappa = 0 \), corresponding to \( \delta = 1 \), the \( \sigma \) term under the square root in equation (A4) is independent of \( x \). Since \( |4i\sigma| \) is not necessarily small compared to unity, we expand the square root as

\[
\sqrt{1 - 4i\sigma} = \omega^{1/2} \left( i - \frac{2}{x} \right)
\]

with \( \omega \equiv 1 - 4i\sigma \). We thus obtain

\[
S_\pm \sim \frac{ix}{2} - \ln x \pm \omega^{1/2} \left( i - \frac{1}{\omega} \ln x \right) .
\]

The first term inside the parentheses on the right-hand side of (A9) will cause \( W \) to diverge as \( x \to \infty \), as exp \( \left[ \text{Im} (\omega^{1/2})x \right] \). Since \( \text{Re} (\omega^{1/2}) \to 0 \) as \( \sigma \to -0.25 \), the second term will cause \( W \) to diverge as \( \sigma \to -0.25 \), independent of \( \sigma \); see also Appendix C). This divergence is reflected in the behavior of the zero-crossing solutions; see § 3.2 and Figures 5a and 5b.

Finally, consider the case \( \kappa > 0 \). In this case, at large \( x \) the \( x^\lambda \) term under the square root of equation (A4) dominates, and so

\[
S_\pm \sim \frac{ix}{2} - \ln x \pm \frac{(i\sigma)^{1/2}}{\lambda} x^\lambda , \quad \lambda \equiv 1 + \kappa/2 = (\mu + 1)/2 , \quad \lambda > 1 \text{ for } \delta < 1 .
\]

For any \( \sigma \not= 0 \), the third term causes \( W \) to diverge exponentially as \( x \to \infty \), as for the other solutions for \( \delta < 3/2 \).

### APPENDIX B

**PROOF OF THE DISTINCTNESS OF THE EIGENVALUES**

We separate the twist equation into real and imaginary parts

\[
x^\mu \beta'' + x^\mu \beta' (1 - \gamma') + 2 \beta' = -x^\mu \beta \sigma , \quad (B1)
\]

\[
x^\mu \gamma'' + \frac{x^\mu}{\beta} \beta' (2\gamma' - 1) + 2 \gamma' = x^\mu \sigma , \quad (B2)
\]

where \( \mu \equiv 3 - 2\delta \) and primes denote derivatives with respect to \( x \). Equation (B2) shows that a necessary condition for an eigenvalue to occur is \( 2\gamma' - 1 = 0 \) (i.e., \( \gamma' \to 1/2 \) as \( \beta \to 0 \)). Multiplying equation (B2) by \( 2x^\mu \beta^2 \) and rearranging, we obtain

\[
[x^2 \beta^2 (2\gamma' - 1)]' = 2x^\mu \beta^2 (x^\mu \sigma - 1) .
\]
Integration then gives
\[ x^2 \beta^2 (2 \gamma' - 1) = 2 \int_0^x \beta^2 x (x^b \sigma_r - 1) dx. \] (B4)

Consider first the case \( \mu > 0 (\delta < 3/2) \). Changing variables to \( \omega = \sigma_r x^a \), we rewrite equation (B4) as
\[ (2 \gamma' - 1) = \frac{2}{\mu \omega^{3/2}} \int_0^x (\omega - 1) \omega^{3/2} - 1 \beta^2 d\omega. \] (B5)

The integral on the right-hand side of (B5) is monotonic; hence, there can be at most one zero for a given \( \sigma_r \). Note also that \( \sigma_r \) must be greater than zero for there to be an eigenvalue, i.e., the precession of the warp must be prograde (in the same direction as the rotation of the disk).

The change of variables to \( \omega \) is not valid for \( \mu = 0 (\delta = 3/2) \). In this case equation (B3) is simply
\[ [x^2 \beta^2 (2 \gamma' - 1)]' = 2x \beta^2 (\sigma_r - 1). \] (B6)

If \( \sigma_r = 1 \), the right-hand side is identically zero, and so
\[ \gamma' = \frac{1}{2} + \frac{C}{2x^2 \beta^2}; \] (B7)
where \( C \) is a constant. The boundary condition \( \gamma' \to \sigma_r / 2 \) as \( x \to 0 \) for \( \delta = 3/2 \) (cf. eq. [12]) requires that \( C = 0 \), and thus for \( \sigma_r = 1 \), \( \gamma' = \frac{1}{2} \) for all \( x \), as can be seen from the Kummer \( M \) functions with \( a = \sigma_r = 1 \) (Paper I).

If \( \sigma_r \neq 1 \), we integrate equation (B6) to obtain
\[ (2 \gamma' - 1) = \frac{2}{x^2 \beta^2} (\sigma_r - 1) \int_0^x \beta^2 x dx. \] (B8)

This equation shows that there are no zeros for \( \sigma_r \neq 1 \), as the integral on the right-hand side is positive definite. Instead we simply have \( \gamma' > \frac{1}{2}, \sigma_r > 1, \gamma' < \frac{1}{2}, \sigma_r < 1 \).

### APPENDIX C

**SOLUTION FOR THE CASE \( \delta = 1 \)**

The twist equation (10) can be rewritten as
\[ \frac{\partial}{\partial x} \left( x^2 e^{-ix} \frac{\partial W}{\partial x} \right) + x^2 e^{-ix} (ix^2 - 2i \sigma W) = 0, \] (C1)
which for \( \delta = 1 \) simplifies to
\[ \frac{\partial}{\partial x} \left( x^2 e^{-ix} \frac{\partial W}{\partial x} \right) + x^2 e^{-ix} (i \sigma W) = 0. \] (C2)

We define the new function \( \phi \) by
\[ W = x^{-1} e^{ix/2} \phi, \] (C3)
so that
\[ \frac{\partial W}{\partial x} = x^{-1} e^{ix/2} \frac{\partial \phi}{\partial x} + x^{-1} e^{ix/2} \phi \left( \frac{i}{2} - x^{-1} \right). \] (C4)

Then the twist equation (C2) can be rewritten as
\[ \frac{\partial^2 \phi}{\partial x^2} + \left( \frac{1}{4} - i \sigma \right) + \frac{i}{x} \phi = 0. \] (C5)

If we now define \( b^2 = -(1 - 4i \sigma)^{-1} \) and a new radial variable \( y = x / b \), equation (C5) becomes
\[ \frac{\partial^2 \phi}{\partial y^2} + \left( -\frac{1}{4} + \frac{ib}{y} \right) \phi = 0. \] (C6)
This is Whittaker’s equation (e.g., Abramowitz & Stegun 1964, equation [13.1.31]),
\[
\frac{\partial^2 \omega}{\partial z^2} + \left[ -\frac{1}{4} + \frac{\kappa}{z} + \frac{(1/4 - \mu^2)}{z^2} \right] \omega = 0,
\]
where the Whittaker functions \( \omega = M_{\kappa,\mu} \) are related to the Kummer functions \( M(a, b, z) \) by
\[
M_{\kappa,\mu} = e^{-z/2} z^{\mu + 1/2} M\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z\right),
\]
where in our case \( \mu = \frac{1}{2}, \kappa = ib \). In terms of \( W \) and \( x \), this solution is
\[
W(x) = e^{-(1-i)b/2b} M\left(1 - ib, 2, \frac{x}{b}\right).
\]

**APPENDIX D**

**CRITICAL ACCRETION RATE**

For an accretion-fueled source, neither the growth/precession rates nor the critical radius for instability explicitly depend on the mass accretion rate, but only on the radiative efficiency \( \epsilon \equiv L/Mc^2 \). However, the requirement that the disk be optically thick to absorption and reemission of the incident flux does impose a minimum value for the surface density \( \Sigma \propto \dot{M}/x \) in the usual \( x \) viscosity formalism.

We define a critical mass accretion rate \( \dot{M}_{cr} \) such that the disk is optically thick at the critical radius for instability, i.e., the surface density \( \Sigma = \Sigma_{cr} \) there. We consider three possible sources of opacity.

1. **Dust opacity.**—This will dominate if the gas is molecular or atomic with \( T \lesssim 10^4 \) K. (The dust will be considerably cooler than the gas.) In this case the Rosseland mean opacity coefficient \( \kappa_R = 0.7, 2.7, 5.0, \) and 7.5 cm\(^2\) g\(^{-1}\) for dust temperatures \( T_d = 50, 100, 300, \) and 600 K, respectively (Pollack et al. 1994; this assumes solar neighborhood abundances and depletion patterns). The corresponding critical surface densities are
\[
\Sigma_{cr}(\text{dust}) \approx 4.3 \times 10^{23} \kappa_R^{-1} \text{ cm}^{-2} \approx 6 \times 10^{22} - 6 \times 10^{23} \text{ cm}^{-2}
\]
for \( T_d = 50-600 \) K.

2. **Electron scattering.**—Ignoring Klein-Nishina corrections, in this case \( \kappa_R \approx 0.35 \) cm\(^2\) g\(^{-1}\), and so
\[
\Sigma_{cr} \approx 1.2 \times 10^{24} \text{ cm}^{-2}
\]
are evaluated at \( \Sigma_{cr} \) by.

3. **Kramer’s (bound-free and free-free) opacity for ionized gas.**—From Schwarzschild (1958) the Rosseland mean opacity in this case is approximately \( \kappa_R \approx 3 \times 10^{23} \rho T^{-7/2} \) cm\(^2\) g\(^{-1}\) for electron temperatures \( T \gtrsim 10^6 \) K, which gives a critical density
\[
\Sigma_{cr} \approx 6 \times 10^{23} n_H^{-1} T^{7/2} \text{ cm}^{-2}.
\]
The numerical coefficient is for \( T \sim 10^6 \) K; it slowly decreases with increasing \( T \).

From the definition of \( R_0 \) (cf. eq. [8]) we have
\[
\frac{R_{cr}}{R_0} = x_{cr}^2
\]
and so the requirement that the disk be optically thick at the critical radius then becomes
\[
\Sigma_{cr} = \Sigma_0 x_{cr}^{-2b},
\]
which can be written as a constraint on \( \Sigma_0 = \Sigma(R_0) \)
\[
\Sigma_0 = \Sigma_{cr} x_{cr}^{2b}.
\]
The surface density \( \Sigma_0 = \dot{M}/3\pi \nu_0 \), which in the \( x \) viscosity prescription is \( \dot{M}/3\pi x(c_s H)_0 \), where the isothermal sound speed \( c_s \) and the disk scale height \( H \) are evaluated at \( R_0 \). For a thin disk with negligible self-gravity, \( H \approx R_c/V_{\text{orb}} \). From the definition of \( R_c, V_{\text{orb}}(R_0) = (\epsilon/\eta)c \). To adequate accuracy we can take the sound speed to be \( c_s \approx 13T_4^{1/2} \) km s\(^{-1}\), where the gas temperature \( T = 10^4 T_4 \) K, and so
\[
\Sigma_0 = 3.8 \times 10^{-3} \frac{\dot{M}}{x} (T_c R_c)^{-1} \left(\frac{\epsilon}{\eta}\right)^{3/2},
\]
where \( R_c \) is in units of centimeters. If we write \( \dot{M} \) in terms of the Eddington accretion rate
\[
\dot{M}_E \equiv \frac{L_E}{c_s^2} = 4.7 \times 10^{11} R_s \text{ g s}^{-1},
\]
then

\[ \Sigma_0 = 7.7 \times 10^{32} \frac{\dot{m}}{\alpha} T_4^{-1} \left( \frac{\epsilon}{\eta} \right)^3 \text{cm}^{-2}, \]  
\[ \text{(D9)} \]

where \( \dot{m} \equiv \dot{M}/M_\odot \). Equating this to \( \Sigma_{cr} x_{cr}^{2\delta} \), we finally obtain an expression for the critical mass accretion rate

\[ \frac{\dot{m}}{\alpha} = 1.3 \times 10^{-10} \frac{\Sigma_{cr}}{10^{33} \text{cm}^{-2}} T_4 \left( \frac{\eta}{\epsilon} \right)^{3/2} x_{cr}^{2\delta}. \]  
\[ \text{(D10)} \]

Equation (D10) is simply applicable for the case of dust or electron-scattering opacity. However, for Kramer’s opacity, we need to know the density as well as the column density. Solving for the density at and proceeding as before, we obtain a slightly different equation for the critical accretion rate

\[ \frac{\dot{m}}{\alpha} = 7.6 \times 10^{-11} T_6^{3/2} R_s^{1/2} \left( \frac{\eta}{\epsilon} \right)^{9/2} x_{cr}^{2\delta}, \]  
\[ \text{(D11)} \]

where \( T_6 = T/10^6 \text{ K}, \) a more realistic value for the Kramer’s opacity case.

Consider first equation (D10), for the case of dust and Thompson opacity.

1. The dust temperature is likely to be a few hundred K when this is the dominant source of opacity, so that \( \Sigma_{cr} \approx 10^{23} \text{ cm}^{-2} \), and \( T_6 \approx 0.1-1 \). For \( \delta = 3/2 \) this is

\[ \frac{\dot{m}}{\alpha} = 3.2 \times 10^{-5} T_6 \frac{\eta^3}{\epsilon_{0.1}} \]  
\[ \text{(D12)} \]

where \( x_{cr} = 2\pi \) and we have normalized \( \epsilon = 0.1\epsilon_{0.1} \). For \( \delta = 1 \), the numerical coefficient is reduced by a factor of 5. This does not impose a serious constraint on disk stability; of course, \( \dot{m}_{cr} \propto x_r^{2\delta} \), so for \( \delta > 0 \), opacity effects become more stringent if the disk is to be optically thick at \( R \gg R_{cr} (x \gg x_{cr}) \).

2. For electron scattering \( \Sigma_{cr} \approx 1.2 \times 10^{24} \text{ cm}^{-2} \) and \( T \gtrsim 10^6 \text{ K} \), so

\[ \frac{\dot{m}}{\alpha} = 1.6 \times 10^{-4} T_6 \frac{\eta^3}{\epsilon_{0.1}} x_{cr}^{2\delta}. \]  
\[ \text{(D13)} \]

This requirement is most stringent for \( \delta > 0 \), as \( x_{cr}^{2\delta} \) ranges from 1 to \( (2\pi)^3 \approx 248 \) for \( 0 \leq \delta \leq 3/2 \), although an isothermal disk is unlikely to be a good assumption in the electron-scattering regime.

Now consider equation (D11) for Kramer’s opacity. In this case \( T_6 \gtrsim 1 \), and substituting for \( R_s \) in equation (D11)

\[ \frac{\dot{m}}{\alpha} = 1.3 \times 10^{-3} T_6 \left( \frac{M}{M_\odot} \right)^{1/2} \frac{\eta^{9/2}}{\epsilon_{0.1}^{7/4}} x_{cr}^{2\delta}, \]  
\[ \text{(D14)} \]

where \( M \) is the mass of the central object. For active galactic nuclei this can be a significant constraint if \( \delta > 0 \), since \( M \approx 10^6-10^9 M_\odot \). However, \( T_6 < 1 \) for accretion disks around AGNs at the radii of interest for warping, so this is probably irrelevant.

APPENDIX E

DISK OUTER BOUNDARY JUMP CONDITION IN X-RAY BINARIES

We start with the basic conservation equations from Pringle (1992), his equations (2.1) and (2.2). We assume that matter is injected into the disk at \( R_{circ} \) and that \( V_R \) is zero for \( R_{circ} \leq R \leq R_{out} \). Integrating the mass and angular momentum conservation equations across \( R_{circ} \) gives the jump conditions

\[ \Sigma V_R \bigg|_+ = \frac{\dot{M}}{R_{circ}} \]  
\[ \text{(E1)} \]

\[ \Sigma \left( V_R - \frac{v_1}{\Omega} \bigg) \bigg|_+ = \frac{1}{2} \Omega v_2 \Sigma \frac{\partial l}{\partial R} \bigg|_+ = \frac{\dot{M}}{R_{circ}} \frac{\hat{l}}{M_{circ}}, \]  
\[ \text{(E2)} \]

where \( \hat{l} \) is the unit tilt vector giving the direction of angular momentum at \( R \), as before, and \( \hat{l} \) is the unit vector normal to the equatorial plane of the binary

\[ \hat{l} = (0,0,1) \]  
\[ \text{(E3)} \]
in our usual Cartesian coordinate system. Plus and minus signs denote quantities evaluated at radii just outside and inside of \( R_{\text{circ}} \), respectively. We assume that the disk is continuous, so that \( I_+ = I_- = I \), but that the derivative \( \partial l/\partial R \) may be discontinuous across the boundary. Furthermore, since there may also be a discontinuity in the surface density, we assume that

\[
(\Sigma v_1)_+ = (\Sigma v_1)_- \left( \frac{\Sigma_+}{\Sigma_-} \right) = (\Sigma v_1)_- y. \tag{E4}
\]

Substituting from equation (E1) into equation (E2) and rearranging, we have:

\[
(y - 1) \frac{\Omega}{\Omega} I + \frac{\eta}{2} \left[ y \left( \frac{\partial l}{\partial R} \right)_+ - \left( \frac{\partial l}{\partial R} \right)_- \right] = \frac{M}{R_{\text{circ}}(\Sigma v_1)_-} (I - I_0). \tag{E5}
\]

Given \( \beta_- \), \( \gamma_- \), and \( (\Sigma v_1)_- \), equation (E5) gives the three conditions necessary to determine \( y, \beta_+, \) and \( \gamma_+ \). For example, to find \( y \) we take the dot product of equation (E5) with \( I \), since

\[
I \cdot \left( \frac{\partial l}{\partial R} \right)_+ = I \cdot \left( \frac{\partial l}{\partial R} \right)_- = 0, \tag{E6}
\]

we have

\[
(y - 1) \frac{\Omega}{\Omega} = \frac{M}{R_{\text{circ}}(\Sigma v_1)_-} (1 - \cos \beta), \tag{E7}
\]

which gives the useful result that in the small angle (\( \beta \ll 1 \)) limit, \( y = 1 + O(\beta^2) \), so that the surface density is continuous in this regime. Similarly, in this limit the \( Z \) component of equation (E5) is also \( O(\beta^2) \), and so we neglect it. In terms of \( W \equiv \beta e^y \), equation (E5) in the linear (small \( \beta \)) regime then becomes

\[
\frac{\Delta W'}{W_{\text{circ}}} = \frac{2M}{\eta R_{\text{circ}}(\Sigma v_1)_-}. \tag{E8}
\]

Since in the small angle limit we also have

\[
\frac{M}{R_{\text{circ}}} = -(\Sigma V_\phi)_- = -(\Sigma v_1)_- \left( \frac{\Omega}{\Omega} \right) = \frac{3}{2} \left( \Sigma v_1 \right)_- \tag{E9}
\]

(assuming a Keplerian rotation curve), we finally obtain

\[
R_{\text{circ}} \frac{\Delta W'}{W_{\text{circ}}} = \frac{3}{\eta} \tag{E10}
\]

which is the desired jump condition. Equation (E10) was first derived by J. E. Pringle (1997, private communication).

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