SURGERY STABLE CURVATURE CONDITIONS

SEBASTIAN HOELZEL

Abstract. We give a simple criterion for whether a pointwise curvature condition is stable under surgery. Namely, a curvature condition $C$, which is understood to be an open, convex, $O(n)$-invariant cone in the space of algebraic curvature operators, is stable under surgeries of codimension at least $c$ provided it contains the curvature operator corresponding to $S^{c-1} \times \mathbb{R}^{n-c+1}$, $c \geq 3$.

This is used to generalize the well-known classification result of positive scalar curvature in the simply-connected case in the following way: Any simply-connected manifold $M^n$, $n \geq 5$, which is either spin with vanishing $\alpha$-invariant or else is non-spin admits for any $\epsilon > 0$ a metric such that the curvature operator satisfies $R > -\epsilon \|R\|$.  

1. Introduction

The aim of this work is to prove a sufficient criterium for a curvature condition to be stable under surgery and to exploit some of its consequences.

Given a smooth manifold $M^n$ with an $(n-k)$-dimensional sphere $S^{n-k}$ embedded with trivial normal bundle such that a tubular neighborhood of $S^{n-k}$ is diffeomorphic to $S^{n-k} \times D^k$, a surgery of codimension $k$ produces a new manifold via the following prescription:

$$
\chi(M^n, S^{n-k}) := \left[ M^n \setminus (S^{n-k} \times D^k) \right] \cup_{S^{n-k} \times S^{k-1}} \left[ D^{-k+1} \times S^{k-1} \right].
$$

Consider the vector space $C^B(\mathbb{R}^n)$ of algebraic curvature operators satisfying the Bianchi identity. A subset $C \subset C^B(\mathbb{R}^n)$ will be called a curvature condition if it is invariant under the natural $O(n)$-representation on $C^B(\mathbb{R}^n)$. We say that a Riemannian manifold $(M^n, g)$ satisfies $C$ provided for any linear isometry $\iota: \mathbb{R}^n \to T_pM$ the pullback $\iota^* R(p) \in C^B(\mathbb{R}^n)$ of the curvature operator $R(p) \in C_B(T_pM)$ of $(M, g)$ belongs to $C$.

This notion allows us to formulate

Theorem A. Let $C \subset C_B(\mathbb{R}^n)$ be an open, convex $O(n)$-invariant cone with $R_{S^{c-1} \times \mathbb{R}^{n-c+1}} \in C$, for some $c \in \{3, \ldots, n\}$. Suppose $(M^n, g)$ is a Riemannian manifold satisfying $C$. Then a manifold obtained from $M^n$ by performing surgery of codimension at least $c$ also admits a metric satisfying $C$.

Here, $R_{S^{c-1} \times \mathbb{R}^{n-c+1}} = \pi^\Lambda^2 \mathbb{R}^{c-1} : \Lambda^2 \mathbb{R}^n \to \Lambda^2 \mathbb{R}^n$ corresponds to the curvature operator of $S^{c-1} \times \mathbb{R}^{n-c+1}$ equipped with its canonical product metric.

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We proceed with a couple of examples, where the above theorem can be applied, which at the same time serve to illustrate the history of surgery theorems in Riemannian geometry.

(1) **Positive scalar curvature.** The set

\[ C_{\text{Scal}>0} := \{ R \in C_B(\mathbb{R}^n) \mid \text{tr}(R) > 0 \} \]

corresponds to the condition of positive scalar curvature in the usual sense. As it evidently contains \( R_{S^d \times \mathbb{R}^{n-d}} \), if \( d \geq 2 \), Theorem A yields stability of positive scalar curvature under surgery of codimension \( \geq 3 \), which was first obtained by Gromov and Lawson and, independently, Schoen and Yau (see \([\text{GL}80]\) and \([\text{SY}79]\), respectively). More precisely, the former proved the surgery stability for \( \text{Scal} > 0 \) as stated in Theorem A, whereas the latter deduced a slightly more general result as formulated in Theorem B below concerning this specific condition by the use of some singular partial differential equations.

(2) **Positive isotropic curvature.** Let \( \pi \subset \mathbb{R}^n \otimes \mathbb{C} = \mathbb{C}^n \) be a complex plane. The complex sectional curvature of \( \pi \) of an algebraic curvature tensor \( R \in C_B(\mathbb{R}^n) \) is given by

\[ \text{sec}(R)(\pi) := R_C(b_1, b_2, \bar{b}_2, \bar{b}_1), \]

where \( \{b_1, b_2\} \subset \pi \) is a unitary basis of \( \pi \) and \( R_C \) denotes the complex quadrilinear extension of \( R \). \( \pi \) is called isotropic, if for all \( v \in \pi \) we have \( (g_{\mathbb{R}^n})_C(v, v) = 0 \), where \( (g_{\mathbb{R}^n})_C \) denotes the complex bilinear extension of \( g_{\mathbb{R}^n} \).

Given a complex isotropic plane \( \pi \subset \mathbb{C}^n \), it is not hard to see that there exist orthonormal vectors \( e_1, \ldots, e_4 \in \mathbb{R}^{2n} \) such that \( \pi = \text{span}_\mathbb{C}\{e_1 + ie_2, e_3 + ie_4\} \). Then, using the shortcut notation \( R_{ijkl} := R(e_i, e_j, e_k, e_l) \), we get

\[ \text{sec}(R)(\pi) = R_{1331} + R_{1441} + R_{2332} + R_{2442} - 2R_{1234}. \]

It readily follows that \( R_{S^{n-1} \times \mathbb{R}} \) is contained in the set

\[ C_{\text{iso} > 0} := \{ R \in C_B(\mathbb{R}^n) \mid \text{sec}(R)(\pi) > 0 \text{ for all isotropic planes } \pi \subset \mathbb{C}^n \} \]

of operators with positive isotropic curvature. Thus Theorem A states the stability of the class of Riemannian manifolds with positive isotropic curvature under connected sum constructions, which recovers a theorem proved by Micallef and Wang (\([\text{MW}93]\) in 1993.

(3) **Positive \( p \)-curvature.** Consider

\[ C_{p>0} := \{ R \in C_B(\mathbb{R}^n) \mid s_p(R)(P) > 0 \text{ for any } p\text{-plane } P \subset \mathbb{R}^n \}, \]

where \( s_p(R)(P) := \sum_{j,k=p+1}^{n} R(e_j, e_k, e_k, e_j) \), with \( e_{p+1}, \ldots, e_n \) being an orthonormal basis of \( P^\perp \), is called the \( p \)-curvature of the plane \( P \) with respect to the operator \( R \). This is an open, convex, \( O(n) \)-invariant cone, and \( R_{S^d \times \mathbb{R}^{n-d}} \in C_{p>0} \), if and only if \( d \geq p+2 \), for this implies \( \dim (\mathbb{R}^d \cap P^\perp) \geq 2 \). Thus Theorem A gives stability under surgery of codimension \( \geq p+3 \) for \( C_{p>0} \), which was proved by Labbi in 1997 (see \([\text{La}97]\)) using the construction method employed in \([\text{GL}80]\).

(4) **Pointwise almost nonnegative sectional curvature.** In a quite similar fashion, Sung proved in 2004 (see \([\text{Su}04]\)) that, for \( \epsilon > 0 \), the condition given
by

$$\hat{C}_\epsilon := \{ R \in C_B(\mathbb{R}^n) \mid \sec(R) > -\epsilon \text{Scal}(R) \}$$

enjoys stability under surgery of codimension $\geq 3$, which is covered by Theorem A, as obviously $R^{S^2 \times \mathbb{R}^{n-2}} \in \hat{C}_\epsilon$.

$\hat{C}_\epsilon$ contains the cone of curvature operators with nonnegative sectional curvature and converges (in the pointed Gromov-Hausdorff sense) to this cone for $\epsilon \to 0$. Therefore the family $\hat{C}_\epsilon$ might reasonably be considered a condition of pointwise almost nonnegative sectional curvature.

(5) **Positive $S$-curvature.** By identifying $\bigwedge^2 \mathbb{R}^n$ with $\mathfrak{so}(n)$ and complexifying the latter to get $\mathfrak{so}(n, \mathbb{C})$, we can regard an operator $R \in C_B(\mathbb{R}^n)$ as an operator $R_C : \mathfrak{so}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C})$ through complex linear extension. For an $\text{Ad}_{SO(n,\mathbb{C})}$-invariant subset $S \subset \mathfrak{so}(n, \mathbb{C})$ the definition

$$C(S) := \{ R \in C_B(\mathbb{R}^n) \mid \langle R_C(X), \overline{X} \rangle_C > 0 \text{ for all } X \in S \}$$

yields an open, convex $O(n)$-invariant cone (which moreover turns out to be Ricci flow invariant; see [WT1]). Recently, it was proved in [GMS11] that $C(S)$ is stable under connected sum constructions, if $S$ does not contain any elements of the form $v \wedge w$, with $v \in \mathbb{R}^n$, $w \in \mathbb{C}^n$ and $(g_R)_C(v, w) = 0$. Since this latter condition is seen to be equivalent to the requirement $R^{S^{n-1} \times \mathbb{R}} \in C(S)$, this case also is covered by Theorem A. Furthermore, it was shown in [GMS11] that $C(S) \subset C_{iso > 0}$ and that $C(S_0) = C_{iso > 0}$ can be achieved by taking

$$S_0 = \{ X \in \mathfrak{so}(n, \mathbb{C}) \mid \text{rk}(X) = 2 \text{ and } X^2 = 0 \},$$

thus it is not surprising that the proof in [GMS11] consists in a generalization of the proof given in [MW93].

Theorem A follows from a more general result that we explain next. To fix notation, let $C \subset C_B(\mathbb{R}^n)$ be an open (and non-empty) curvature condition. We say $C$ satisfies an inner cone condition with respect to an operator $S \in C_B(\mathbb{R}^n) \setminus \{0\}$ if the following holds: For every $R \in C$ there is a $\rho = \rho(R) > 0$, depending continuously on $R$, such that

$$R + C_\rho := \{ R + T \mid T \in C_\rho \} \subset C,$$

where $C_\rho$ is an open, convex, $O(n)$-invariant cone containing $B_\rho(S)$.

Note that if $C \subset C_B(\mathbb{R}^n)$ is an open, $O(n)$-invariant convex cone and $S \in C$, then $C$ automatically satisfies an inner cone condition with respect to $S$.

**Theorem B.** Let $C \subset C_B(\mathbb{R}^n)$ be a curvature condition satisfying an inner cone condition with respect to $R^{S^{n-k-1} \times \mathbb{R}^{k+1}}$, for some $k \in \{0, \ldots, n-3\}$. Suppose $(M_i^n, g_{M_i})$, $i = 1, 2$, are two $n$-dimensional Riemannian manifolds satisfying $C$ and let $N_i^l \subset M_i^n$ be closed $l$-dimensional submanifolds of $M_i$, with $0 \leq l \leq k$.

If there is an isomorphism $\phi : \nu N_1 \to \nu N_2$ of the normal bundles of $N_i$ in $M_i$, then the joining of $M_1$ and $M_2$ along $\phi$ defined by

$$M_1 \#_\phi M_2 := (M_1 \setminus B_\epsilon(N_1)) \cup_{\overline{\phi}} (M_2 \setminus B_\epsilon(N_2))$$

also carries a metric satisfying $C$. 
Here, $\phi$ is given by $\exp \circ \phi \circ (\exp|_{\nu \mathbb{N}_i})^{-1} : \partial B_\epsilon(N_i) \cong \partial B_\epsilon(N_2)$ and $\epsilon > 0$ is meant to be chosen small enough such that the normal exponential mappings $\exp : \nu^{<\epsilon}N_i \rightarrow B_{2\epsilon}(N_i)$ are diffeomorphisms, $i = 1, 2$, where $\nu^{<\epsilon}N_i := \{\nu \in \nu N_i | \|\nu\| < r\}$.

This indeed implies Theorem A, for $\chi(M^n,S^{n-k}) = M^n\#(\bigoplus S^n_{k})$, with $\phi$ being the obvious isomorphism of the normal bundle of $S^{n-k} \subset M^n$ to the normal bundle of $S^{n-k} = S^n \cap (\mathbb{R}^{n-k+1} \times \{0\}) \subset S^n$.

Moreover, there are analogous results in the equivariant and conformally flat cases. These are outlined in sections 5 and 6, respectively.

Using the Ricci flow, Böhm and Wilking [BW08] proved that any Riemannian manifold with positive curvature operator is diffeomorphic to a spherical space form. Together with the work of Gallot and Meyer (cf. [Pe06]) this yields a complete understanding of manifolds with nonnegative curvature operator. More precisely, a closed, simply connected Riemannian manifold with nonnegative curvature operator consists of a Riemannian product of manifolds which either are diffeomorphic to spheres, isometric to compact symmetric spaces or are Kähler manifolds biholomorphic to complex projective spaces. In particular, the class of closed, simply connected manifolds of a given dimension admitting a metric with nonnegative curvature operator consists of finitely many diffeomorphism types.

In contrast to this, this rigidity result breaks down completely as soon as one tries to relax this curvature condition in the sense of Theorem C.

Let $C \subset \mathcal{C}_B(\mathbb{R}^n)$ be a curvature condition such that $C$ satisfies an inner cone condition with respect to any nonzero curvature operator with nonnegative eigenvalues (for instance, this holds, if $C$ is an open convex $O(n)$-invariant cone with $\{R \geq 0\} \setminus \{0\} \subset C$).

Suppose $M^n$, $n \geq 5$, is a closed, simply connected manifold. Then $M$ can be endowed with a metric satisfying $C$, if either $M$ is non-spin, or $M$ is spin and $\alpha(M) = 0$.

Here, in the spin-case, the mapping $M \mapsto \alpha(M)$ is a homomorphism $\Omega^2_{_{\text{Spin}}} \rightarrow KO^{-*}(\text{pt})$, which coincides with a multiple of the $\hat{A}$-genus in dimensions $4k$.

This result was known in the case of positive scalar curvature by the combined work of Gromov-Lawson and Stolz (see [GL80], [St92]). By using the same methods, Sung proved Theorem C for the special case of the conditions $C_\epsilon$ mentioned above (see [Su04]). In fact, we show that these methods apply to the more general situation of Theorem C, where we employ a slightly generalized version regarding the vertical rescaling of a Riemannian submersion.

A particular case of Theorem C may be noted explicitly as

**Corollary D.** Let $M^n$ be as in Theorem C. Then for any $\epsilon > 0$ there exists a metric $g_\epsilon$ on $M$ such that the curvature operator $R = R_{(M^n,g_\epsilon)}$ fulfills

$$R > -\epsilon \|R\|.$$

Here, $\|R\|$ denotes the operator norm of $R$.

The paper is organized as follows. Theorems A and B are proved in section 2 and Section 3 is devoted to a simple submersion lemma which is used in section 4 to prove Theorem C. The last two sections deal with extensions of the surgery theorem to the equivariant and conformally flat cases, respectively.
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2. Proofs of Theorem A and Theorem B

The following theorem captures the main deformation procedure behind Theorem A and Theorem B.

**Theorem 2.1.** Let $C \subset C_B(\mathbb{R}^n)$ be a curvature condition satisfying an inner cone condition with respect to $R_{S^d \times \mathbb{R}^{n-d}}$. Let $(M^n, g_M)$ be a Riemannian manifold satisfying $C$ and suppose $N^k \subset M^n$ is a closed submanifold. Let $g_N$ be an arbitrary metric on $N^k$, $g_{vN}$ a vector bundle metric on $vN$ and $\nabla$ a connection on $vN$ being metric with respect to $g_{vN}$.

Then for $\rho > 0$ there is $\rho \in (0, \rho]$ such that for every $\rho \in (0, \rho]$ there exists a complete metric $g_D$ on the open manifold $D := M \setminus N$ with the following properties:

1. $g_D$ satisfies $C$.
2. $g_D$ coincides with $g_M$ on $M \setminus D(\rho)$, where $D(\rho) = \{x \in M \mid d_{g_M}(x, N) < \rho\}$.
3. In a neighborhood $U \subset M$ of $N$ the region $(U \setminus N, g_D)$ is isometric to $(\nu^* N \times (0, \infty), h_{|\nu^* N} + dt^2)$,

where $\nu^* N := \{v \in vN \mid g_{vN}(v, v) = \rho^2\}$ and $h$ is the connection metric determined by $g_N$, $g_{vN}$ and $\nabla$.

Theorems A and B follow immediately from Theorem 2.1 in conjunction with the following elementary property.

**Proposition 2.2.** Suppose $C \subset C_B(\mathbb{R}^n)$ satisfies an inner cone condition with respect to $R_{S^d \times \mathbb{R}^{n-d}}$, $2 \leq d \leq n - 1$. Then the same is true for $R_{S^d \times \mathbb{R}^{n-d-1}}$.

**Proof.** Suppose first that $C$ is an open, convex, $O(n)$-invariant cone with $R_d := R_{S^d \times \mathbb{R}^{n-d}} \in C$. The orthogonal projection $\pi : \mathbb{R}^n \to \mathbb{R}^{d+1}$ induces an inclusion $C_B(\mathbb{R}^{d+1}) \subset C_B(\mathbb{R}^n)$. Now, $R_d = \pi_{A_\infty}$ is contained in $C_B(\mathbb{R}^{d+1})$, and so is $A \ast R_d$, if $A \in O(d + 1) \subset O(n)$. This implies

$$ S := \int_{O(d+1)} A \ast R_d \ dm(A) \in C_B(\mathbb{R}^{d+1}). $$

Here, the standard Haar measure $m$ of $O(d + 1)$ is used with the normalization $\int_{O(d+1)} dm = 1$. Because $S \in C_B(\mathbb{R}^{d+1})$ is a fixed point of the representation of $O(d + 1)$, it follows easily that $S = \lambda R_{d+1}$ for some $\lambda > 0$, using the irreducible decomposition of this representation. Furthermore, $S$ is contained in the convex hull of the orbit $O(d + 1) \ast R_d$, because if $S \notin H$, we could find an $A_0 \in H$ such that $d(A_0, S) = \inf_{A \in H} d(A, S) =: d(H, S) > 0$ due to the compactness of $H$. Convexity of $H$ implies then $(S - A_0, A - A_0) \leq 0$ for each $A \in H$, which yields

$$ 0 < \langle S - A_0, S - A_0 \rangle = \int_{O(d+1)} \langle S - A_0, A - A_0 \rangle \ dm(A) \leq 0, $$

i.e. a contradiction. Hence convexity of $C$ gives us $\lambda R_{d+1} \in H \subset C$.

Now, suppose $C$ merely satisfies an inner cone condition as in the statement. We can find a sequence of compact subsets $K_j$, with $K_j \subset K_{j+1}$ and $C = \bigcup K_j$. By intersecting cones we get cones $C_j$ with $B_{\rho_j}(R_d) \subset C_j$, where $\rho_j := \min_{R \in K_j} \rho(R)$,
such that \( R + C_j \subset C \) for all \( R \in K_j \). The above argument gives us numbers \( \delta_j > 0 \) with \( B_{\delta_j}(R_{d+1}) \subset C_j \) for each \( j \). The function

\[
\hat{\delta}(R) := \max_{R \in K_j} \delta_j
\]

is positive on \( C \) and it is easy to see that we can construct a continuous function

\[
\delta : C \to \mathbb{R}^> 0
\]

with \( \delta \leq \hat{\delta} \) and \( B_{\delta(R)}(R + R_{d+1}) \subset R + C_j \subset C \) for all \( R \in K_j \subset C \). This shows that \( C \) satisfies an inner cone condition with respect to \( R_{d+1} \). \( \square \)

We now turn to the proof of Theorem 2.1.

### 2.1. Setup and curvature formulas

We are going to make use of a graph-like deformation procedure, which owes much to [GL80]. The general setup is explained next.

Without loss of generality \( \tau > 0 \) can be assumed to be small enough that \( \exp : \nu^< \mathbb{R} N \to M \) is an embedding. For some function \( \theta : [0, \infty) \to [0, \pi] \), the mappings

\[
(1) \quad r(s) := \tau - \int_0^s \cos \theta(u) \, du, \quad t(s) := \int_0^s \sin \theta(u) \, du,
\]

describe a curve \( \gamma(s) := (r(s), t(s)) \) in the \( (r,t) \)-space, parametrized by arc length. The angle between the tangent \( \gamma'(s) \) and \( -\partial_r \) is then given by \( \theta(s) \). The function \( \theta(s) \) will be chosen in such a way that

\[
(2) \quad t|_{[0, \tau]} \equiv 0 \text{ for some } \underline{s} > 0, \quad r'|_{[\tau, \infty]} \equiv 0 \text{ for some } \overline{\tau} > \underline{s} \text{ and } r > 0
\]

hold.

With the help of this ‘model curve’ we construct

\[
\gamma : \nu^1 N \times \mathbb{R}^> 0 \to M \times \mathbb{R}, \quad (\nu, s) \mapsto (\exp(r(s)\nu), t(s)),
\]

which we continue to call \( \gamma \). Using \( \gamma \) we deform the manifold \( (M, g_M) \) to a new manifold \( (D, g_D) \), defined by

\[
(3) \quad D := \{ \gamma(\nu, s) | \nu \in \nu^1 N, s \geq 0 \} \cup \{ (p, 0) \in M \times \mathbb{R} | d(p, N) \geq \tau \}
\]

and \( g_D \) being the induced metric. Because of (2), \( D \) will be indeed a smooth manifold diffeomorphic to \( M \setminus N \).

Our first task is to derive a useful formula for the curvature tensor of \( D \). In order to do so, we aim at reexpressing the second fundamental form of \( D \) in terms of known components.

The derivative of \( \gamma \) with respect to \( s \) will be denoted by \( \gamma'(\nu, s) = \frac{\partial}{\partial s}(\nu, s) \).

Observe that

\[
\gamma'(\nu, s) = d\exp(r(s)\nu)(r'(s)\nu) + t'(s) \partial_t|_{\gamma(\nu, s)}
\]

\[
(4) \quad = r'(s) \partial_r|_{\gamma(\nu, s)} + t'(s) \partial_t|_{\gamma(\nu, s)}
\]

\[
= -\cos \theta(s) \partial_r|_{\gamma(\nu, s)} + \sin \theta(s) \partial_t|_{\gamma(\nu, s)}.
\]

We choose a local normal vector field \( \mu(\nu, s) \) to \( D \) by rotating \( \gamma'(\nu, s) \) counterclockwise by \( \frac{\overline{\tau}}{2} \), thus getting

\[
\mu(\nu, s) := -r'(s) \partial_r|_{\gamma(\nu, s)} + t'(s) \partial_r|_{\gamma(\nu, s)}
\]

\[
(5) \quad = \cos \theta(s) \partial_r|_{\gamma(\nu, s)} + \sin \theta(s) \partial_t|_{\gamma(\nu, s)}
\]
In the following, we make use of the splitting of the tangential space of $D$ given by
\[
T_{\gamma(v, s)} D = \text{span} \{ \gamma'(v, s) \} \oplus T_{\exp(r(s)v)} T(r(s)),
\]
where $T(r) := \{ x \in M \mid d(x, N) = r \}$ denotes the distance tube of radius $r$ around $N$.

**Lemma 2.3.** The second fundamental form of $D \subset M \times \mathbb{R}$ with respect to the normal $\mu(v, s)$ at a point $\gamma(v, s)$, which is defined by $\Pi_D(v, w) = \langle \nabla^D_v \mu, w \rangle$, is given by
\[
\Pi_D(\gamma', \gamma') = -\theta'(s),
\]
\[
\Pi_D(\gamma', v) = 0,
\]
\[
\Pi_D(v, w) = \sin \theta(s) \Pi_{T(r(s))}(v, w).
\]

Here, $v, w \in T_{\exp(r(s)v)} T(r(s))$ and $\Pi_{T(r)}$ is the second fundamental form given by $\Pi_{T(r)}(X, Y) = \Pi^r_{T(r)}(X, Y) = -\langle \nabla_X Y, \nabla r \rangle$.

**Proof.** From (4) it follows
\[
\frac{\nabla^D \gamma'}{ds}(v, s) = \theta'(s) \left( \sin \theta(s) \partial_r|_{\gamma(v, s)} + \cos \theta(s) \partial_t|_{\gamma(v, s)} \right) = \theta'(s) \mu(v, s),
\]
because for a fixed $v \in \nu^1 N$ we have
\[
\frac{\nabla^D}{ds} \partial_t|_{\gamma(v, s)} = 0 = \frac{\nabla^D}{ds} \partial_r|_{\gamma(v, s)},
\]
since the two-dimensional submanifold $\{ (\exp(rv), t) \in M \times \mathbb{R} \mid r \in (0, \overline{r}), t \in \mathbb{R} \}$ is totally geodesic. Thus we obtain
\[
\Pi_D(\gamma'(v, s), \gamma'(v, s)) = -g_D \left( \frac{\nabla^D \gamma'}{ds}(v, s), \mu(v, s) \right) = -\theta'(s).
\]

For the next equation we compute
\[
\frac{\nabla^D \mu}{ds}(v, s) = \theta'(s) \left( -\sin \theta(s) \partial_t|_{\gamma(v, s)} + \cos \theta(s) \partial_r|_{\gamma(v, s)} \right) = -\theta'(s) \gamma'(v, s)
\]
and deduce
\[
\Pi_D(\gamma'(v, s), v) = g_D \left( \frac{\nabla^D \mu}{ds}(v, s), v \right) = 0.
\]

Finally, if we denote by $\tilde{v}$ and $\tilde{w}$ extensions of $v$ and $w$, respectively, to vector fields tangent to $M \subset M \times \mathbb{R}$, we get
\[
\Pi_D(v, w) = -g_D \left( \nabla^M \tilde{w}, \mu(v, s) \right)
= -g_M \left( \nabla^M \tilde{w}, \sin \theta(s) \partial_r|_{\gamma(v, s)} \right)
= \sin \theta(s) \Pi_{T(r(s))}(v, w).
\]

\[\square\]

We are going to compare the curvature tensor of $D$ at a point $\gamma(v, s)$ with the corresponding one of $M$ at $\exp(r(s)v)$. In order to do so, we introduce the following notation for pulling back a given curvature operator to $C_B(\mathbb{R}^n)$. 
\textbf{Definition 2.4.} Fix once and for all an ordered \( n \)-tuple \( E = (e_1, \ldots, e_n) \) of vectors \( e_i \in \mathbb{R}^n \) that form an orthonormal basis of \( \mathbb{R}^n \) and suppose an ordered \( n \)-tuple \( B = (b_1, \ldots, b_n) \) of orthonormal vectors \( b_j \in T_p M \) is given. Then the \( B \)-pullback of \( R_{(M,g)}(p) \in \mathcal{C}_B(T_p M) \) is given by \( \iota^* R_{(M,g)}(p) \in \mathcal{C}_B(\mathbb{R}^n) \), where \( \iota : \mathbb{R}^n \to T_p M \) is the linear map defined by \( e_i \mapsto b_i \) for \( i = 1, \ldots, n \).

Given a \( k \)-tuple \( B = (b_1, \ldots, b_k) \) and an \( l \)-tuple \( C = (c_1, \ldots, c_l) \), \( B + C \) will denote the \((k + l)\)-tuple \((b_1, \ldots, b_k, c_1, \ldots, c_l)\).

The following choice of bases will prove useful: For each \( (\nu, r) \) we get a canonically defined subspace \( \mathcal{H}_{(\nu, r)} \subset \mathcal{T}_\nu M \cap \{ \partial_r \}^\perp \) by parallel translation of \( T_p N \) along \( t \mapsto \exp(t\nu) \), and an orthogonal complementary subspace \( \mathcal{V}_{(\nu, r)} \) such that
\[
T_{\exp(r\nu)} T(r) = \mathcal{V}_{(\nu, r)} \oplus \mathcal{H}_{(\nu, r)}.
\]
\( \nu := \bigcup_{\nu, r > 0} \mathcal{V}_{(\nu, r)} \) and \( \mathcal{H} := \bigcup_{\nu, r > 0} \mathcal{H}_{(\nu, r)} \) define smooth distributions on \( D(\mathcal{T}) \backslash N \).

For each \( (\nu, r) \) we choose an orthonormal basis \( V(\nu, r) = (v_1, \ldots, v_{n-k-1}) \) of \( \mathcal{V}_{(\nu, r)} \) and an orthonormal basis \( H(\nu, r) = (h_1, \ldots, h_k) \) of \( \mathcal{H}_{(\nu, r)} \) and consider the pullback of the following curvature operators to \( \mathcal{C}_B(\mathbb{R}^n) \). Set \( B(\nu, r) := V + H \). Then denote
\[
\begin{align*}
&\text{the } (B(\nu, r(s)) + (-\gamma'(\nu, s))) \text{-pullback of } R_{(D,gD)} \text{ by } \tilde{R}_D(\nu, s), \\
&\text{the } \left( B(\nu, r) + \left( \partial_t |_{\exp_r(\nu)} \right) \right) \text{-pullback of } R_{(M,g_M)} \text{ by } \tilde{R}_M(\nu, r) \text{ and} \\
&\text{the } (B(\nu, r) + (\partial_t)) \text{-pullback of } R_{(T(r) \times \mathbb{R}, g_M |_{r(t)} + dt^2)} \text{ by } \tilde{R}_T(\nu, r).
\end{align*}
\]

Note that the dependences of the pullback operators on the chosen bases are suppressed by the notation and that, in fact, the mappings \( (\nu, r) \mapsto \tilde{R}_D(\nu, r), \tilde{R}_M(\nu, r) \) are not even continuous, as in general there do not exist global sections of \( \mathcal{H} \) and \( \mathcal{V} \). However, any two choices of bases merely result in a change caused by a transformation by an element of \( O(n) \). This does not matter as long as all tensors are being pulled back by a consistent choice of bases as just described since we are dealing with \( O(n) \)-invariant subsets of \( \mathcal{C}_B(\mathbb{R}^n) \) and are only interested in whether or not those pullbacks are contained in these sets.

\textbf{Proposition 2.5.} With the above notation, the equation
\[
\tilde{R}_D(\nu, s) = \cos^2 \theta(s) \tilde{R}_M(\nu, r(s)) + \sin^2 \theta(s) \tilde{R}_T(\nu, r(s)) + E(\nu, s)
\]
holds on \( \nu^1 N \times (0, \infty) \). \( E : \nu^1 N \times (0, \infty) \to \mathcal{C}_B(\mathbb{R}^n) \) satisfies the estimate
\[
\|E(\nu, s)\| \leq C_1 (1 - \cos \theta(s)) C_2 \frac{\theta'(s) \sin \theta(s)}{r(s)}
\]
where the constants \( C_1, C_2 \) depend only on \( (D(\mathcal{T}), g_M) \) and \( N \).

In order to prove this, we will need the following fact about the second fundamental form of \( T(r) \) (cf. chapter 5 of \textbf{E87}).

\textbf{Lemma 2.6.} Let \( T(r) \subset M \) be a distance tube around a closed submanifold \( N^k \subset M^n \). Then there exists a mapping \( A : \nu^1 N \times (0, \mathcal{T}) \to S^2(T^* M), A(\nu, r) \in S^2(T_{\exp(r\nu)} M), \) such that
\[
\Pi_{T(r)}(v, w) = \frac{1}{r} \pi^T(\nu, w) + A(\nu, r)(v, w),
\]
where \( r \in (0, \pi) \), \( \nu \in \nu^1 N \), and \( v, w \in T_{\exp(r\nu)}M \cap \{ \nabla r \}^\perp \). (\( \pi_{V_{\nu,v}} \) denotes orthogonal projection onto \( V_{\nu,v} \subset T_{\exp(r\nu)}M \)). Moreover, we have \( \|A(\nu,r)\| \leq C \) for some constant and

\[
A(\nu,0)|_{T_x N \times T_x N} = \Pi^\nu_N, \quad A(\nu,0)|_{\nu_x N \times T_x N} = 0, \quad A(\nu,0)|_{\nu_x N \times \nu_x N} = 0,
\]

if \( \nu \in \nu_x N \).

**Proof.** Consider \( h := \frac{1}{2}r^2 \), where \( r(p) = d(p, N) \). Since we have \( h = \frac{1}{2} \sum_{j=0}^n (x^j)^2 \) in Fermi coordinates, where \( \{ x^1 = \ldots = x^k = 0 \} \subset N \), this is a smooth function on \( D(\tilde{r}) \) and has vanishing differential on \( N \). Thus, the hessian of \( h \) is readily computed to be \( H^h(x) = \pi^\nu_{\nu_x N} \) at \( x \in N \). Moreover, the corresponding coordinate expressions immediately show

\[
H^h \circ \exp(r\nu) = \pi^\nu_{\exp(r\nu)} + r\hat{A}(\nu, r),
\]

where \( \hat{V}_{\exp(r\nu)} \), given by parallel transport of \( \nu_x N \) along \( t \mapsto \exp(t\nu) \), with \( \nu \in \nu^1 N \), defines a smooth distribution on \( D(\tilde{r}) \) and \( \hat{A} \) is a bounded continuous map \( \nu^1 N \times [0, \pi) \to S^2(T^* M) \). Recalling that the second fundamental form of \( T(r) \) is given by \( \Pi_{T(r)} = H^r \) and

\[
H^h = dr^2 + r H^r,
\]

the first claims of the lemma follow by restricting the above equation to \( \{ \partial_t \}^\perp \) and observing that \( V_{(\nu, r)} = \hat{V}_{\exp(r\nu)} \cap \{ \partial_t \}^\perp \).

For the last claim, let \( v \in \nu^1 N \cap \{ \nu \}^\perp \), \( h \in T^1 N \), and consider Jacobi fields \( J_v \) and \( J_h \) along \( t \mapsto \exp(t\nu) \) with initial conditions \( J_v(0) = 0 \), \( J'_v(0) = v \) and \( J_h(0) = h \), \( J'_h(0) = \Pi^\nu_N(h, h) \in T_x N \), respectively. Recall that \( \Pi_{T(r)}(J, J) = \langle J', J \rangle \), then

\[
A(\nu,0)(h, h) = \lim_{t \to 0} A(\nu, t)(J_h(t), J_h(t)) = \lim_{t \to 0} \langle J'_h(t), J_h(t) \rangle = \frac{\pi^\nu_{\nu_x N}(J_h(t), J_h(t))}{t},
\]

which equals \( \Pi^\nu_N(h, h) \), since \( \| \pi^\nu_{\nu_x N}(J_h(t)) \| = O(t^2) \). Observing that \( \frac{J_h(t)}{\|J_h(t)\|} \to v \), we derive

\[
A(\nu,0)(h, v) = \lim_{t \to 0} \left( \frac{J'_h(t)}{\|J_h(t)\|} \right) \frac{1}{t} \pi^\nu_{\nu_x N} \left( \frac{J_h(t)}{\|J_h(t)\|} \right) = 0.
\]

Finally, we have

\[
A(\nu,0)(v, v) = \lim_{t \to 0} \frac{\langle J'_h(t), J_h(t) \rangle}{\langle J_h(t), J_h(t) \rangle} - \frac{1}{t}
\]

which delivers the sought conclusion by repeated application of l’Hospital’s rule. \( \square \)

**Proof of Proposition 2.3** By restricting all arguments to \( T_{\exp(r\nu)(s)} T(r) \subset T_{\gamma(\nu, s)} D \) we obtain, using the Gauss equation twice and lemma 2.3

\[
R_D|_{\gamma(\nu, s)} = (R_M \times \mathbb{R} + \Pi_D \wedge \Pi_D)|_{\gamma(\nu, s)}
\]

\[
= (R_M + \sin^2 \theta(s) \Pi_{T(r)} \wedge \Pi_{T(r)})|_{\exp(r\nu)(s)}
\]

\[
= (\cos^2 \theta(s) R_M + \sin^2 \theta(s) R_D)|_{\exp(r\nu)(s)}.
\]

By switching to the algebraic identifications made above, we thus immediately get

\[
\tilde{R}_D(\nu, s) = \left( \cos^2 \theta(s) R_M + \sin^2 \theta(s) R_D \right)(\nu, r(s)),
\]

where both sides are to be restricted to \( \mathbb{R}^{n-1} \times \{ 0 \} \subset \mathbb{R}^n \).
Next, using (1) and the Gauß equation we compute, with \( v_i \in T_{e^p(r(s)\nu)T(r)} \subset T_{\gamma(\nu,s)}D \),
\[
R_D|_{\gamma(\nu,s)}(v_1, \gamma', \gamma, v_2) = (R_M \times \Pi_D \wedge \Pi_D)|_{\gamma(\nu,s)}(v_1, \gamma', \gamma, v_2)
= \cos^2 \theta(s) R_M|_{e^p(r(s)\nu)}(v_1, \partial_r, \partial_r, v_2)
- \theta'(s) \sin \theta(s) \Pi_T|_{e^p(r(s)\nu)}(v_1, v_2),
\]
since \( \Pi_D(v_1, \gamma') = 0 \). Expressed in the algebraic setting, this simply means, with \( \tilde{v}_1, \tilde{v}_2 \in \mathbb{R}^{n-1} \subset \mathbb{R}^n \),
\[
\hat{R}_D(\nu, s)(\tilde{v}_1, e_n, e_n, \tilde{v}_2) = \cos^2 \theta(s) \hat{R}_M(\nu, r(s))(\tilde{v}_1, e_n, e_n, \tilde{v}_2) \\
+ \theta'(s) \sin \theta(s) \left( \frac{1}{r(s)} \pi^{2}_{\mathbb{R}^{n-k-1}}(\tilde{v}_1, \tilde{v}_2) + O(1) \right),
\]
Finally, computations done in the same manner yield
\[
\hat{R}_D(\nu, s)(\tilde{v}_1, \tilde{v}_2, v_3, e_n) = - R_D|_{\gamma(\nu,s)}(v_1, v_2, v_3, \gamma')
= - (R_M \times \Pi_D \wedge \Pi_D)|_{\gamma(\nu,s)}(v_1, v_2, v_3, \gamma')
= \cos \theta(s) R_M|_{e^p(r(s)\nu)}(v_1, v_2, v_3, \partial_r)
- (\Pi_D(v_1, \gamma')) \Pi_D(v_2, v_3) - \Pi_D(v_1, v_3) \Pi_D(v_2, \gamma'))
= (\cos^2 \theta(s) + \cos \theta(s)(1 - \cos \theta(s)))
\cdot \hat{R}_M(\nu, r(s))(\tilde{v}_1, \tilde{v}_2, v_3, e_n).
\]
This finishes the proof. \( \Box \)

The occurrence of \( \hat{R}_T \) in the above formula will play a key role, because of the following theorem.

**Theorem 2.7.** Let \( C \subset C_B(\mathbb{R}^n) \) be a curvature condition satisfying an inner cone condition with respect to \( R_{S^0} \times \mathbb{R}^{k+1} \). Let \( N^k \subset M^n \) be a closed submanifold of the Riemannian manifold \( (M^n, g_M) \).

Then there exists \( r_* > 0 \) such that for all \( r \in (0, r_*) \) the Riemannian manifold \( \left( T(r) \times \mathbb{R}, g_T(r) + g_{\mathbb{R}} \right) \) satisfies \( C \). Moreover, there exists \( L > 0 \) such that for a \((V + H + (\partial_t))\)-pullback \( \hat{R}_T(\nu, r) \) of \( R_{(T(r) \times \mathbb{R}, g_M|_{T(r)} + dt^2)} \) we have
\[
\hat{R}_T(\nu, r) \in B_{\frac{L}{2}} \left( R_{S^0 \times \mathbb{R}^{k+1}} \right) \subset C.
\]

Of course, \( r_* \) is meant to be chosen so small that \( T(r) \) is an embedded submanifold of \( M \) for all \( r < r_* \).

**Proof.** Lemma 2.6 implies
\[
\Pi_{T(r)} \wedge \Pi_{T(r)} = \frac{1}{r^2} \pi^2_{\mathbb{V}} \wedge \pi^2_{\mathbb{V}} + \frac{2}{r} \pi^2_{\mathbb{V}} \wedge A(\nu, r) + O(1)
\]
Hence, together with the Gauß equation
\[
R_{T(r)} = R_{(M, g_M)|_{T_T(r)}} + \Pi_{T(r)} \wedge \Pi_{T(r)}
\]
we get
\[
\hat{R}_T(\nu, r) = \frac{1}{r^2} R_{S^0 \times \mathbb{R}^{k+1}} + E(\nu, r),
\]
upon pulling back to \( \mathbb{R}^n \), with some tensor \( E(\nu, r) \in C_B(\mathbb{R}^n) \) satisfying \( \|E(\nu, r)\| \leq L r^{-1} \), where \( L \) does not depend on the choice of pullback.
There exists an $O(n)$-invariant open cone $\tilde{C} \subset C_B(\mathbb{R}^n)$ with $S := R_{S^{n-k-1} \times \mathbb{R}^{k+1}} \in \tilde{C}$ and a compact set $K \subset C_B(\mathbb{R}^n)$ such that $\tilde{C}\setminus K \subset C$, i.e. $C$ contains a truncated cone in whose interior $\lambda S$ can be found, for all $\lambda > 0$ sufficiently large. Indeed, take an arbitrary $R \in C$. Then $R + C_{\rho} \subset C$ implies $B_{\mu}\rho \subset C$ for $\mu > 0$. Therefore we can find a $\lambda_0 > 0$ such that $\lambda S \subset C$ for $\lambda \geq \lambda_0$. Due to the inner cone condition, we then have $S + C_{\rho'} \subset C$ for some $\rho' > 0$, and we can find an open cone $C'$ such that $C' \setminus (S + C_{\rho'})$ is bounded. Then $\tilde{C} = O(n) \ast C'$ has the desired properties.

Now, given $\rho > 0$ with $B_{\rho} (R_{S^{n-k-1} \times \mathbb{R}^{k+1}}) \subset \tilde{C}$ and it follows that

$$B_{\rho'} (R_{S^{n-k-1} (r) \times \mathbb{R}^{k+1}}) \subset \tilde{C},$$

for if $S \in B_{\rho'} (R_{S^{n-k-1} (r) \times \mathbb{R}^{k+1}})$, then we have $\|r^2 S - R_{S^{n-k-1} \times \mathbb{R}^{k+1}}\| < \rho$, i.e. $r^2 S \in B_{\rho} (R_{S^{n-k-1} \times \mathbb{R}^{k+1}}) \subset \tilde{C}$. Since $\tilde{C}$ is a cone, we have $S \in \tilde{C}$. Moreover, for some $\tilde{r} > 0$, even

$$(8) \quad B_{\tilde{r}} (R_{S^{n-k-1} (r) \times \mathbb{R}^{k+1}}) \subset \tilde{C}\setminus K \subset C$$

holds, if $r \in (0, \tilde{r})$.

Then, for $r \in (0, r_{\ast})$, $r_{\ast} := \min \{\tilde{r}, \frac{\rho}{\tilde{r}}\}$, the inequality $\frac{\rho}{\tilde{r}} < \tilde{r}$ holds, hence (7) implies

$$\tilde{R}_T (\nu, r) \in B_{\frac{\rho}{\tilde{r}}} (R_{S^{n-k-1} (r) \times \mathbb{R}^{k+1}}) \subset B_{\frac{\rho}{\tilde{r}}} (R_{S^{n-k-1} (r) \times \mathbb{R}^{k+1}}) \subset C.$$

This completes the proof.

Using entirely analogous arguments we can also prove the following variant.

**Proposition 2.8.** Let $C \subset C_B(\mathbb{R}^n)$ be a curvature condition satisfying an inner cone condition with respect to $R_{S^{n-k-1} \times \mathbb{R}^{k+1}}$. Let $N^{k+1} \subset M^{n+1}$ be a compact submanifold of a Riemannian manifold $(M^{n+1}, g_M)$, with totally geodesic boundary $\partial M$, $\partial N \subset \partial M$ and $v_q N \subset T_q \partial M$ for $q \in \partial N$, such that the normal exponential map is defined on $\nu^q N$, for some $\epsilon > 0$.

Then there exists $r_{\ast} \in (0, \epsilon)$ such that for all $r \in (0, r_{\ast})$ the Riemannian manifold $(T(r), g_T(r))$ satisfies $C$.

### 2.2. Construction of the deformed metric.

In this section we describe a construction of the new metric $g_D$. This consists mainly in carefully prescribing the angular function $\theta(s)$ which needs to be done in a way maintaining the given curvature condition. To that end, we first choose appropriate radius parameters which ensure the validity of the estimates given above. The construction itself will be subdivided into three steps.

In summary, the goal of the first two steps consists in constructing a monotone increasing function $\theta : [0, \infty) \rightarrow [0, \frac{\pi}{2}]$, which starts out of zero and reaches $\frac{\pi}{2}$ in finite time, whereas the third step is necessary to smooth out the metric which at that time will already be given as the product of the metric induced by a distance tube and a real line.

By restricting our attention to the compact region

$$D(\mathbb{T}) = \{p \in M \mid d(p, N) \leq \mathbb{T}\}$$

we can find a compact $O(n)$-invariant set $K \subset C$ such that $(D(\mathbb{T}), g_M)$ satisfies $K$. Then the inner cone condition with respect to $R_{S^{n-k-1} \times \mathbb{R}^{k+1}}$ implies the existence...
of a number \( \rho > 0 \) such that \( R + C_\rho \subset C \) for all \( R \in K \), where \( C_\rho \) contains \( B_\rho (R_{n-k-1} \times R^{k+1}) \). This in turn implies

\[
B_{\frac{\epsilon}{2}} (R + R_{n-k-1}(\lambda) \times R^{k+1}) \subset C
\]

for \( \lambda > 0 \).

The starting radius of the bending process \( r_S \) is chosen in order to fulfill

\[
r_S < \min \left\{ 1, r_s, \frac{\rho}{4L} \sin 2^\frac{1}{2}, \frac{\sup_{D(\overline{\tau})} \| R_M \| + C_1}{\rho}, \frac{1}{4} \right\}.
\]

Here, the constants \( L \) and \( r_s \) are those given by Theorem 2.7, \( C_1 \) by Proposition 2.8.

2.3. **Step 1: Initial bending.** This first step consists in a slight increase of the bending angle, beginning from zero and reaching an arbitrary small, but positive angle.

**Lemma 2.9.** There exists \( s_0 > 0 \) and a non-decreasing smooth function \( \theta : [0, s_0] \to [0, \theta_0] \) with \( \theta \) being constant in neighborhoods of 0 and \( s_0 \) with values 0 and \( \theta_0 > 0 \), respectively, such that \( \overline{R}_D(\nu, s) \in C \) and \( r(s) > 0 \) for \( s \in [0, s_0] \).

**Proof.** There exists \( \epsilon > 0 \) such that for all \( r \in (\overline{\tau}, \overline{\tau}) \) and \( \nu \in \nu^1 N \) we have

\[
B_{\frac{\epsilon}{2}} (\overline{R}_M(\nu, r)) \subset C.
\]

Next we choose \( \theta_0 \in (0, \overline{\tau}) \) so small that the following conditions hold:

\[
\sin^2 \theta_0 \left( \sup_{p \in S} \| R_M(p) \| + \sup_{p \in S} \| R_T(d(p, \overline{\tau})) \| \right) < \frac{\epsilon}{2},
\]

\[
(1 - \cos \theta_0) C_1 + \frac{2 \sin \theta_0}{r_S} C_2 < \frac{\epsilon}{2}.
\]

We set \( s_0 : = \overline{\tau} - \frac{r_S}{\theta_0} \) and prescribe \( \theta(s) \) on the initial interval \( [0, s_0] \) via

\[
\theta(s) := \begin{cases} 
0, & \text{for } s \in [0, \overline{\tau} - r_S], \\
\theta_0, & \text{for } s \in [\overline{\tau} - \frac{3}{4} r_S, s_0],
\end{cases}
\]

such that \( \theta'(s) \in [0, 1] \). Now, with the help of Proposition 2.9 and (12) we see that for \( s \in [0, s_0] \)

\[
\left\| \overline{R}_D(\nu, s) - \overline{R}_M(\nu, r(s)) \right\| \leq \sin^2 \theta(s) \left( \| R_M \| + \| R_T(r(s)) \| \right) \exp(\frac{\theta(s)}{C_1})
+ \cos \theta(s) (1 - \cos \theta(s)) C_1 + \frac{\theta'(s) \sin \theta(s)}{r(s)} C_2 < \epsilon.
\]

In the last inequality we used that \( r(s) > \overline{\tau} - s_0 = \frac{r_S}{\theta_0} \) for \( s \in [0, s_0] \), which follows from (11) and the construction of \( \theta \). The upper inequality combined with (11) implies \( \overline{R}_D(\nu, s) \in C \) for \( s \in [0, s_0], \nu \in \nu^1 N \).
2.4. Step 2: Inductive increasing of the bending angle. In order to extend \( \theta \), while keeping \( \tilde{R}_D \) in \( C \) we use the decomposition given by Proposition \( 2.5 \) and the fact that \( C \) satisfies the appropriate inner cone condition.

**Lemma 2.10.** There exists \( r^* \in (0, \pi) \) such that for every \( r \in (0, r^*) \) there is an extension of \( \theta \) to a smooth non-decreasing function \( \theta : [0, \infty) \to [0, \pi] \) such that \( \tilde{R}_D(\nu, s) \in C \), \( r(s) > 0 \) for \( s \geq 0 \) as well as \( \theta|_{\pi, \infty} = \pi \) and \( r|_{\pi, \infty} = r \) for some \( \pi > 0 \) big enough.

**Proof.** In fact, because of \( 9 \) it will be sufficient to show

\[
\tilde{R}_D(\nu, s) \in B_{\rho^2 \sin^2(\theta(\phi))} \left( \tilde{R}_M(\nu, r(s)) + R_{S_{n-k-1}(\frac{r(s)}{\sin(\theta(s))} \times \mathbb{R}^{k+1})} \right)
\]

for maintaining \( \tilde{R}_D \in C \), which of course is equivalent to

\[
\left\| \tilde{R}_D(\nu, s) - \left( \tilde{R}_M(\nu, r(s)) + R_{S_{n-k-1}(\frac{r(s)}{\sin(\theta(s))} \times \mathbb{R}^{k+1})} \right) \right\| < \rho^2 \frac{\sin^2(\theta(s))}{r(s)^2}.
\]

Therefore, we can estimate employing Proposition \( 2.5 \)

\[
\left\| \tilde{R}_D(\nu, s) - \left( \tilde{R}_M(\nu, r(s)) + R_{S_{n-k-1}(\frac{r(s)}{\sin(\theta(s))} \times \mathbb{R}^{k+1})} \right) \right\|
\leq \sin^2(\theta(s)) \left( \left\| \tilde{R}_M(\nu, r(s)) \right\| + \left\| \tilde{R}_T(\nu, r(s)) - R_{S_{n-k-1}(r(s)) \times \mathbb{R}^{k+1}} \right\| \right) + \|E(\nu, s)\|
\leq \sin^2(\theta(s)) \left( \sup_{p \in D(\pi)} \|R_M(p)\| + \frac{L}{r(s)} \right) + \cos(\theta(s)) (1 - \cos(\theta(s))) C_1
\]

\[
+ \frac{\theta'(s) \sin(\theta(s))}{r(s)} C_2,
\]

where we used Theorem \( 2.7 \) in the last inequality. Due to \( 10 \), we readily estimate

\[
\sin^2(\theta(s)) \sup_{p \in D(\pi)} \|R_M(p)\| + \cos(\theta(s)) (1 - \cos(\theta(s))) C_1
\]

\[
\leq \sin^2(\theta(s)) \left( \sup_{p \in D(\pi)} \|R_M(p)\| + C_1 \right) < \sin^2(\theta(s)) \frac{\rho}{4r(s)^2}
\]

and

\[
\frac{L}{r(s)} < \frac{\rho}{4r(s)^2}.
\]

Hence \( 13 \) holds, provided

\[
\frac{\theta'(s) \sin(\theta(s))}{r(s)} C_2 \leq \sin^2(\theta(s)) \frac{\rho}{2r(s)^2},
\]

which is equivalent to

\[
\theta'(s) \leq \frac{\rho}{2C_2} \frac{\sin(\theta(s))}{r(s)}.
\]

The remaining task is to extend the function \( \theta \) from \([0, s_0] \), to the interval \([0, \infty) \) while maintaining inequality \( 14 \), the condition \( r(s) > 0 \) and \( \theta|_{\pi, \infty} = \pi \) for some \( \pi > 0 \) big enough.
One possible way of doing this is to prescribe this extension of \( \theta \) inductively as follows (but cf. also [RS01]). Suppose \( \theta \) is already defined on \([0, s_l]\). Set
\[
\theta_t := \theta(s_l) \in \left[ \theta_0, \frac{\pi}{2} \right], \quad r_t := r(s_l) > 0
\]
(if \( \theta_t = \frac{\pi}{2} \), we are done) and define \( s_{l+1} := s_l + \frac{r_t}{2} \). We construct a smooth function \( \eta_t : \mathbb{R} \to [0, \frac{\rho}{4C_2} \sin \theta_t / r_t] \) matching the following requirements:
\[
\eta_t \equiv \begin{cases} 
0, & \text{on } \left[ s_l, s_l + \frac{r_t}{8} \right], \\
\frac{\rho}{4C_2} \sin \theta_t / r_t, & \text{on } \left[ s_l + \frac{r_t}{8}, s_{l+1} - \frac{r_t}{8} \right], \\
0, & \text{on } \left[ s_{l+1} - \frac{r_t}{8}, s_{l+1} \right].
\end{cases}
\]

Using this bump function we extend \( \theta \) smoothly to the interval \([0, s_{l+1}]\) by setting for \( s \in (s_l, s_{l+1}) \)
\[
\theta(s) := \theta(s_l) + \int_{s_l}^{s} \eta(u) \, du.
\]
Then, because of (1),  
\[
r_t \geq r(s) \geq \frac{r_t}{2} > 0
\]
and inequality (14) is fulfilled, since
\[
\theta'(s) \leq \frac{\rho}{4C_2} \frac{\sin \theta_t}{r_t} < \frac{\rho}{2C_2} \frac{\sin \theta(s)}{r(s)}.
\]
Furthermore, we obtain the decisive estimate
\[
\theta_{l+1} - \theta_l := \theta(s_{l+1}) - \theta(s_l) \geq \int_{s_l + \frac{r_t}{8}}^{s_{l+1} - \frac{r_t}{8}} \eta(u) \, du \geq \frac{\rho}{16C_2} \sin \theta_0.
\]

Therefore the amount of growth of \( \theta \) in the interval \([s_l, s_{l+1}]\) is bounded from below independently of its length. This shows that the target value \( \theta = \frac{\pi}{2} \) can be reached by finitely many, say \( m \), such bends. Of course, for the last bend the function \( \eta_m \) has to be adjusted slightly in order to avoid values above \( \frac{\pi}{2} \). Moreover, any \( r < r^* := r(s_m) \) can be achieved by inserting a straight line segment before performing the last bending step.

Finally, we extend \( \theta \) to \([0, \infty)\) by setting \( \theta \equiv \frac{\pi}{2} \) on \([s_m, \infty)\).

\[\square\]

2.5. **Step 3: Smoothing of the end.**

**Lemma 2.11.** There exists \( r^* > 0 \) such that for any \( r \in (0, r^*) \) there exists a metric \( g(t) + dt^2 \) on \( T(r) \times [t(\overline{r}), \infty) \) which coincides with the induced metric of \( M \times \mathbb{R} \) for \( t \in [t(\overline{r}), t(\overline{r}) + 1] \), which equals \( h|_{\nu^* N} + dt^2 \) for \( t \geq t(\overline{r}) + 2 \) and such that \( C \) is satisfied.

Here, \( h \) is the connection metric given by the prescribed metrics on \( N \), \( \nu^* N \) and the connection \( \nabla \).

**Proof.** On \( B = D^n(\overline{r}) \) we construct a one-parameter family \( g(t) \), \( t \in [0, 1] \), with
\[
g(t) = \begin{cases} 
g_M|_B, & \text{for } t \in [0, \frac{1}{2}], \\
h|_B, & \text{for } t \in \left[ \frac{1}{2}, 1 \right].
\end{cases}
\]

We apply Proposition 2.8 to \( N \times [0, 1] \subset B \times [0, 1] \), the latter being equipped with the metric \( \tilde{g} := g(t) + dt^2 \), obtaining a constant \( r^{**} \) such that for \( r \in (0, r^{**}) \) the
$n$-dimensional distance tube $\tilde{T}(r) \subset B \times [0, 1]$ with the induced metric satisfies $C$. Now, for $r$ small enough we can find $\epsilon > 0$ such that

$$\left( \tilde{T}(r) \cap (B \times [0, \epsilon)), \tilde{g}|_{\tilde{T}(r) \cap (B \times [0, \epsilon))} \right) = (T(r) \times [0, \epsilon), g_B)$$

and

$$\left( \tilde{T}(r) \cap (B \times (1 - \epsilon, 1]), \tilde{g}|_{\tilde{T}(r) \cap (B \times (1 - \epsilon, 1))} \right) = \left( T(r) \times (1 - \epsilon, 1], h|_{T(r)} + dt^2 \right).$$

By extending the metric constantly for $t \geq 1$ and relabelling, the claim follows. \hfill \Box

Therefore, if we choose as the final radius in the bending process in Lemma 2.10 a radius $r$ that fulfills $0 < r < r^1 := \min\{r^*, r^{**}\}$, we can simply replace the metric on $D \cap (M \times [t(\pi), \infty)) = T(r) \times [t(\pi), \infty)$ by the one constructed in Lemma 2.11. This completes the proof of Theorem 2.1.

### 3. Vertical rescaling of Riemannian submersions

In this section, we consider Riemannian submersions $\pi : (M^n, g_M) \to (B^{n-k}, g_B)$ of closed Riemannian manifolds and show that the total space $M^n$ with a vertically rescaled metric $g_M'$ satisfies a given curvature condition, provided the fibers do so in an appropriate way. To define $g_M'$, recall that the fiber $F(b) := \pi^{-1} \{\{b\}\}$ over a point $b \in B$ is naturally an embedded submanifold of $M$ and will henceforth be endowed with the induced metric, denoted by $g_{F(b)}$. This implies the existence of a smooth orthogonal splitting of the tangent bundle $TM$, given pointwise by

$$T^v_p M := T_p F(\pi(p)), \quad T^h_p M := T_p F(\pi(p))^\perp.$$ 

The corresponding orthogonal projections are smooth maps and will be denoted by $w \mapsto w^v$ and $w \mapsto w^h$, respectively.

Now, by shrinking the metric $g_M$ in the direction of the fibers we get a new metric $g_M'$, $t > 0$, defined by

$$g_M'(w_1, w_2) := t^2 g_M(w_1^v, w_2^v) + g_M(w_1^h, w_2^h)$$

$$= t^2 g_{F(\pi(p))}(w_1^v, w_2^v) + \pi^* g_{B}(w_1, w_2),$$

for $w_1, w_2 \in T_p M$, $p \in M$. With respect to this deformed metric the map $\pi : (M^n, g_M') \to (B^{n-k}, g_B)$ continues to be a Riemannian submersion with the same decomposition $TM = T^v M \oplus T^h M$ of the tangent bundle into a vertical and horizontal subbundle.

**Theorem 3.1.** Let $C \subset C_B(\mathbb{R}^n)$ be a curvature condition. Let $\pi : (M^n, g_M) \to (B^{n-k}, g_B)$ be a Riemannian submersion, $M^n$ and $B^{n-k}$ being closed manifolds. If $C$ satisfies an inner cone condition with respect to any curvature operator corresponding to

$$R_{(F(b) \times \mathbb{R}^{n-k}, g_{F(b)} + g_{B^{n-k}})}(p),$$

with $b \in B$, $p \in F(b)$, then there is $t_\ast > 0$ such that for each $t \in (0, t_\ast)$ the Riemannian manifold $(M, g_M')$ satisfies $C$.

**Remark 3.2.** If $C$ happens to be a convex cone, the given condition simplifies to $\tilde{R}_{F(b) \times \mathbb{R}^{n-k}}(p) \in C$. 

For the proof of Theorem 3.1 recall that the behavior of a Riemannian submersion is determined by two tensorial invariants of type (2,1) which are given by
\[
T_XY = (\nabla_X Y^V)^H + (\nabla_X Y^H)^V,
\]
\[
A_XY = (\nabla_X Y)^H + (\nabla_X Y^H)^V,
\]
where \(X, Y \in \Gamma(TM)\). The tensor \(T\) essentially describes the second fundamental form of the fibers, whereas the tensor \(A\) serves as an obstruction to the integrability of the horizontal distribution \(T^h M\).

Now, using the well-known O’Neill formulas and the variational behavior of \(A\) and \(T\) with respect to vertically rescaled metrics (see [Be76], Theorem 9.28, and Lemma 9.69, respectively) we derive the following set of equations.

\textbf{Lemma 3.3.} Let \(v_1, \ldots, v_4 \in T_p^h M\) and \(h_1, \ldots, h_4 \in T_p^h M\), \(b = \pi(p)\). Then the \((4,0)\)-curvature tensor of \((M^n, g_M)\) is given by
\[
R^t_M(v_1, v_2, v_3, v_4) = t^2 R^t_{F(b)}(v_1, v_2, v_3, v_4)
\]
\[
- t^4 (g_M(T_{v_2} v_3, T_{v_1} v_4) - g_M(T_{v_1} v_3, T_{v_2} v_4))
\]
\[
R^t_M(v_1, v_2, v_3, h_1) = t^2 R^t_M(v_1, v_2, v_3, h_1)
\]
\[
- (t^2 - t^4)(g_M(T_{v_1} v_3, A_{h_1} v_2) + g_M(T_{v_2} v_3, A_{h_1} v_1))
\]
\[
R^t_M(v_1, v_2, h_1, v_2) = t^2 R^t_M(v_1, v_2, h_1, v_2)
\]
\[
+ (t^2 - t^4)(g_M(A_{h_1} v_2, A_{h_1} v_1) - g_M(A_{h_1} v_1, A_{h_1} v_2))
\]
\[
R^t_M(h_1, v_1, v_2, v_2) = t^2 R^t_M(h_1, v_1, v_2, v_2) + (t^2 - t^4)g_M(A_{h_1} v_2, A_{h_1} v_1)
\]
\[
R^t_M(h_1, h_2, h_3, v_1) = t^2 R^t_M(h_1, h_2, h_3, v_1)
\]
\[
R^t_M(h_1, h_2, h_3, h_4) = t^2 R^t_M(h_1, h_2, h_3, h_4) + (1 - t^2)(\pi^* R_B)(h_1, h_2, h_3, h_4).
\]

\textbf{Proof of Theorem 3.1.} Let \(U \subset M\) be an open subset such that there exists an \(g_M\)-orthonormal frame \(V(q) + H(q)\), with \(V(q) = (v_1(q), \ldots, v_k(q)), H(q) = (h_1(q), \ldots, h_{n-k}(q))\), \(v, h_j \in \Gamma(U, TM)\) with \(v(q) \in T_p^h M\) and \(h_j(q) \in T_p^h M\), \(q \in U\). By setting \(V^t(q) := (v^t_1(q), \ldots, v^t_k(q)), v^t_j := \frac{1}{t}v_j\), we obtain a corresponding frame on \(U\) with respect to the metric \(g_M^t\).

Let \(\tilde{R}^t_M(q)\) denote the \((V^t(q) + H(q))\)-pullback of \(R_{(M,g^t)}(q)\) as in Definition 2.4 and denote by \(\tilde{R}^t_F(q)\) the \((V^t(q) + (e_{k+1}, \ldots, e_n))\)-pullback of the curvature tensor corresponding to the manifold \((F(\pi(q)) \times \mathbb{R}^{n-k}, t^2 g_{F(\pi(q))} + g_{\mathbb{R}^{n-k}})\).

Now, Lemma 3.3 shows that \(\tilde{R}^t_M\) can be decomposed as
\[
\tilde{R}^t_M = \tilde{R}^t_F + E^t,
\]
with \(E^t : U \to C_B(\mathbb{R}^n)\) collecting the remaining terms. By assumption we can find an \(O(n)\)-invariant cone \(\hat{C}\) as in the proof of Theorem 2.5 such that \(\hat{C} \setminus K \subset C\) and that for some \(p \in U\) and \(\epsilon > 0\) we have \(B_{\epsilon}(\tilde{R}^t_F(p)) \subset \hat{C}\). \(\hat{C}\) being a cone, we get \(B_{\epsilon}^{\hat{C}}(\tilde{R}^t_F(p)) \subset \hat{C}\) for \(t > 0\) and then \(B_{\epsilon}^{\hat{C}}(\tilde{R}^t_F(p)) \subset \hat{C} \setminus K\) for some \(t_p > 0\). By shrinking \(U\), if necessary, we find \(\delta > 0\) such that
\[
B_\delta^{\hat{C}}(\tilde{R}^t_F(q)) \subset C
\]
for all \( q \in U \) and \( t \in (0, t_p) \). Thus, it is enough to show the existence of some \( t_* \in (0, t_p] \) with

\[
\|E^t(q)\| < \frac{\delta}{t^2}
\]

for \( q \in U \) and \( t \in (0, t_*) \). Because \( M \) can be covered by finitely many neighborhoods \( U \), this will finish the proof.

Now, to prove (16), a case by case study of the equations of Lemma 3.3 yields, with \( i, j, m, l \in \{1, \ldots, k\} \), \( r, s, u, v \in \{1, \ldots, n - k\} \),

\[
E^t(e_i, e_j, e_m, e_l) = \left( R^t_M - R^t_P \right) (e_i, e_j, e_m, e_l) = (t^{-4} R^t_M - t^{-2} R^t_F(\pi(p))) (v_i, v_j, v_m, v_l),
\]

\[
E^t(e_i, e_j, e_k+r, e_{k+s}) = R^t_M (v_i, v_j, h_r, h_s),
\]

\[
E^t(e_{k+r}, e_i, e_{k+s}, e_j) = R^t_M (h_r, v_i, h_s, v_j) + (1 - t^2) g_M (A, h_r, v_i, A, h_s, v_j).
\]

Thus, we can find a constant \( C > 0 \) such that \( \|E^t(q)\| \leq \frac{C}{t} \) for all \( q \in U \), which evidently implies (16) for some \( t_* \).

\[\Box\]

4. Proof of Theorem C

We consider first the case that \( M \) is spin. Stolz proved that the vanishing of the \( \alpha \)-invariant implies that \( M \) is spin cobordant to the total space \( N^o \) of a fiber bundle with fiber \( \mathbb{H}P^2 \) (cf. Theorem B of [St92]). Now, \( \mathbb{H}P^2 \times \mathbb{R}^{n-8} \) satisfies \( C \) by assumption, hence - using Theorem [3.1] - \( N \) can be equipped with a metric satisfying \( C \). Of course, \( M \) might be nullcobordant, in which case it is obviously coborbdant to the \( n \)-dimensional sphere \( S^n \). By a theorem in [GL80], \( M \) can be obtained from \( N \) (or \( S^n \)) by surgeries of codimension at least 3. This shows the claim in the case of \( M \) being spin.

In the non-spin case, it was also observed by Gromov and Lawson that two simply connected manifolds which are oriented cobordant can in fact be obtained from one another by surgeries of codimension at least 3. Therefore it suffices to give a list of generators of the oriented cobordism ring \( \Omega^*_O \), all of which carry a metric satisfying \( C \), if their dimension matches \( n \). In the following, we will see that - again - the list proposed by Gromov and Lawson for the case of positive scalar curvature suffices for our purposes (see [GL80] for more details).

The ring \( \Omega^*_O \) modulo torsion is generated by complex projective spaces \( \mathbb{C}P^k \) and Milnor manifolds \( H_{k,m} \) given as hypersurfaces of degree \( (1, 1) \) in \( \mathbb{C}P^k \times \mathbb{C}P^m \), \( k \leq m \). The former obviously carry a metric satisfying \( C \), and so do the latter, by application of Theorem [3.1] as above. In order to see this, recall that \( H_{k,m} \) can be
defined as

\[ H_{k,m} := \left\{ ([w_0, \ldots, w_k], [z_0, \ldots, z_m]) \in \mathbb{CP}^k \times \mathbb{CP}^m \mid \sum_{j=1}^k w_j z_j = 0 \right\}. \]

Together with the projection \( H \rightarrow \mathbb{CP}^k \) onto the first factor and using the induced metric, \( H \) can be easily endowed with a metric turning the projection into a Riemannian submersion, with fibers being isometric to \( \mathbb{CP}^{m-1} \).

Generators of the torsion of \( \Omega^*_{\text{SO}} \) consist of two types of manifolds. The first type is a so-called Dold manifold \( D_{k,m} \) defined by

\[ D_{k,m} := \left( S^k \times \mathbb{CP}^m \right) / \mathbb{Z}_2, \]

where the \( \mathbb{Z}_2 \)-action is given by \( (p, [z]) \mapsto (-p, [\bar{z}]) \). The obvious metric on this manifold is non-flat and has non-negative curvature operator, hence it satisfies \( C \), possibly after rescaling.

The second type can be constructed as follows. Define \( P_{k,m} := (D_{k,m} \times S^1)/\mathbb{Z}_2 \) with \( \mathbb{Z}_2 \)-action the map \( ([p, [z]], \phi) \mapsto ([r(p), [z]], -\phi), r : S^k \rightarrow S^k \) being a reflection about a hyperplane. With the help of the induced projection \( \psi_{k,m} : P_{k,m} \rightarrow S^1 \), the second type of torsion generators is constructed as

\[ V := \{(x_1, \ldots, x_l) \in P_{k_1, m_1} \times \cdots \times P_{k_l, m_l} \mid \psi_{k_1, m_1}(x_1) \cdots \psi_{k_l, m_l}(x_l) = 1\}. \]

The map \( (x_1, \ldots, x_l) \mapsto (\psi_1(x_1), \ldots, \psi_l(x_l)) \), where \( \psi_j = \psi_{k_j, m_j} \), defines a submersion \( \pi : V \rightarrow T^{l-1} := \left\{ (t_1, \ldots, t_l) \in (S^1)^l \mid t_1 \cdots t_l = 1 \right\} \). Since \( P_{k,m} \) is locally isometric to \( D_{k,m} \times \mathbb{R} \), \( V \) is locally isometric to a product of \( l \) Dold manifolds and \( \mathbb{R}^{l-1} \) and thus satisfies \( C \), again possibly after rescaling. This completes the proof of Theorem C.

In order to deduce Corollary D, we have to consider the condition

\[ C_\epsilon := \{ R \in C_B(\mathbb{R}^n) \mid R > -\epsilon \| R \| \}, \]

for a given \( \epsilon > 0 \). \( C_\epsilon \) is not a convex condition for small \( \epsilon > 0 \), nevertheless we have

**Proposition 4.1.** The curvature condition \( C_\epsilon \) satisfies an inner cone condition with respect to any \( 0 \neq S \in C_B(\mathbb{R}^n) \) with nonnegative eigenvalues. In particular, \( C_\epsilon \) is stable under surgeries of codimension at least 3.

**Proof.** We show that \( C_\epsilon \) satisfies an inner cone condition with respect to \( S \), where we assume w.l.o.g. \( \| S \| = 1 \). Fix \( R \in C_\epsilon \), then \( R + \epsilon \| R \| > 0 \) for some \( 0 < \epsilon' < \epsilon \).

We will establish the existence of a \( \delta = \delta(R) \) such that \( B_{t\delta}(R + tS) \subset C_\epsilon \) for all \( t \geq 0 \). Equivalently, we claim that for \( T \in C_B(\mathbb{R}^n) \) with \( \| T \| < \delta \) and \( \omega \in \wedge^2 \mathbb{R}^n \) with \( \| \omega \| = 1 \) the function

\[ f(t) := \langle (R + t(S + T))(\omega), \omega \rangle + \epsilon \| (R + t(S + T))\| \]

is positive. Now, for \( 0 \leq t \leq t_0 := (1 + \epsilon) \| R \| \) we estimate

\[ f(t) \geq \langle R(\omega), \omega \rangle - t\delta + \epsilon (\| R + tS \| - t\delta) \]

\[ \geq \langle R(\omega), \omega \rangle + (\epsilon - \delta(1 + \epsilon)^2) \| R \| \]

\[ \geq \langle R(\omega), \omega \rangle + \epsilon' \| R \| \]

\[ > 0, \]
provided $\delta < \frac{\epsilon}{(1 + \epsilon)}$; we additionally used the inequality $\|R + tS\| \geq \|R\|$. Similarly, in order to arrive at an analogous estimate for $t \geq t_0$, we first choose $\omega_0 \in \bigwedge^2 \mathbb{R}^n$, $\|\omega_0\| = 1$, with $(S(\omega_0), \omega_0) = 1$. Then

$$\|R + tS + tT\| \geq \langle R(\omega_0), \omega_0 \rangle + (1 - \delta) t \geq -\epsilon \|R\| + (1 - \delta)t,$$

and hence

$$f(t) \geq \langle R(\omega), \omega \rangle - \delta t + \epsilon ((1 - \delta)t - \epsilon \|R\|) = \langle R(\omega), \omega \rangle - \epsilon^2 \|R\| + (\epsilon - \delta(1 + \epsilon))t.$$

For $\delta < \frac{\epsilon}{1 + \epsilon}$ the coefficient in front of $t$ is positive, thus we obtain using $t \geq t_0 = (1 + \epsilon) \|R\|$

$$f(t) \geq \langle R(\omega), \omega \rangle + (\epsilon - \delta(1 + \epsilon)^2) \|R\| \geq \langle R(\omega), \omega \rangle + \epsilon' \|R\| > 0,$$

as above. This finishes the proof. \qed

Putting these things together, Corollary D follows.

5. Equivariant Surgery

There exist corresponding equivariant versions of the above surgery and gluing theorems A and B.

**Theorem 5.1.** Let $C, N^1_i \subset M^\nu_i$ and $\phi$ be given as in Theorem B. Additionally, suppose $G$ is a compact Lie group acting isometrically on $M^\nu_i$, $i = 1, 2$, so that $\gamma(N_i) = N_i$ for all $\gamma \in G$ and such that $\phi$ is $G$-equivariant with respect to the induced actions of $G$ on $\nu N_i$.

Then, $M_1 \#_\phi M_2$, the joining of $M_1$ with $M_2$ along $\phi$, carries a metric satisfying $C$ on which $G$ acts isometrically.

$G$-equivariant surgery is a special case of the joining of two $G$-manifold via a $G$-equivariant vector bundle isomorphism. More precisely, we have

**Definition 5.2.** Suppose there are a closed subgroup $H \subset G$ and orthogonal representations of $H$ on $\mathbb{R}^c$ and $\mathbb{R}^{d+1}$, which induce actions on $S^d \subset \mathbb{R}^{d+1}$ and $S^{d+c} \subset \mathbb{R}^{d+1} \times \mathbb{R}^c$. Let $N$ be a submanifold of an $n$-dimensional manifold $M^\nu$, $N$ being $G$-equivariantly diffeomorphic to $G \times_H S^d$. If there is a $G$-equivariant isomorphism of the normal bundles of $N$ and $G \times_H S^d \subset G \times_H S^{d+c}$, then $G$-equivariant surgery of dimension $d$ and codimension $c$, denoted $\chi_G(M, N)$, is given by joining of $G \times_H S^{d+c}$ and $M$ along this isomorphism.

Since the normal bundle of $G \times_H S^d$ in $G \times_H S^{d+c}$ is equivariantly diffeomorphic to $G \times_H (S^d \times \mathbb{R}^c)$, the submanifold $N \subset M$ as above is required to admit a tubular neighborhood which is equivariantly diffeomorphic to $G \times_H (S^d \times D^c)$ for some open ball $D^c \subset \mathbb{R}^c$. This leads to a somewhat more common (but equivalent) way of stating equivariant surgery: After removing a region like $G \times_H (S^d \times D^c)$, a new region like $G \times_H (\tilde{D}^{d+1} \times S^{c-1})$ is pasted in along the common boundary:

$$\chi_G(M, N) = [M \setminus G \times_H (S^d \times D^c)] \cup_{G \times_H (S^d \times S^{c-1})} [G \times_H (\tilde{D}^{d+1} \times S^{c-1})].$$

Since $G \times_H S^{c+d}$ is naturally a fiber bundle over $G/H$ with fibers diffeomorphic to $S^{c+d}$, the construction outlined in Theorem 3.1 yields a $G$-invariant metric such that $C$ as above is satisfied. We therefore deduce from Theorem 5.1 the following
Theorem 5.3. Let $C \subset C_B(\mathbb{R}^n)$ be a curvature condition satisfying an inner cone condition with respect to $R_{c-1} \times \mathbb{R}^{n-c+1}$, $c \in \{3, \ldots, n\}$. Let $(M^n, g_M)$ be a Riemannian manifold satisfying $C$ and suppose there is a compact Lie group $G$ acting isometrically on $M$.

Then a manifold obtained from $M^n$ by performing $G$-equivariant surgery of codimension at least $c$ also admits a metric satisfying $C$.

Remark 5.4. The validity of Theorem 5.3 in the case of positive scalar curvature was first observed in [BBS3] (cf. also [Ha8]).

The proof of Theorem 5.1 consists of an application of the following equivariant analogue of Theorem 2.1.

Theorem 5.5. Additionally to the assumptions of Theorem 2.1 let $G$ be a compact Lie group which acts isometrically on $(M, g_M)$, leaving $N$ invariant. Furthermore, suppose that $G$ acts isometrically on $(N, g_N)$ as well as on $(\nu N, g_{\nu N})$ by the induced action and that the connection $\nabla$ commutes with this action.

Then, for $\tau > 0$ there is $\mathcal{E} \in (0, \tau)$ such that for every $r \in (0, \mathcal{E})$ there exists a complete metric $g_D$ on the open manifold $D := M \setminus N$ with the following properties:

1. $g_D$ satisfies $C$.
2. $g_D$ coincides with $g_M$ on $M \setminus D(\tau)$, where $D(\tau) = \{x \in M \mid d_M(x, N) < \tau\}$.
3. In a neighborhood $U \subset M$ of $N$ the region $(U \setminus N, g_D)$ is isometric to $(\nu^r N \times (0, \infty), h|_{\nu^r N} + dt^2)$, where $h$ is the connection metric determined by $g_N$, $g_{\nu N}$ and $\nabla$.
4. $G$ acts isometrically on $(D, g_D)$. Moreover, for any $t_0 \in (0, \infty)$, the submanifold $\nu^r N \times \{t_0\} \subset (U \setminus N, g_D)$ is invariant under the $G$-action.

The proof of Theorem 2.1 given in section 2 carries over to the equivariant case almost verbatim, due to the following observation.

Proposition 5.6. Let $(D, g_D) \subset (M \times \mathbb{R}, g_M + dt^2)$ be given by (3) with $\gamma(\nu, s)$ being constructed as in the first two steps of the proof of Theorem 2.1 in section 2. Suppose $f : (M, g_M) \rightarrow (M, g_M)$ is an isometry with $f(N) = N$. Then, there is a unique isometric extension $\bar{f} : (D, g_D) \rightarrow (D, g_D)$ coinciding with $f$ on $M \setminus D(\tau)$.

Proof. Consider $\bar{f} : M \times \mathbb{R} \rightarrow M \times \mathbb{R}, (p, t) \mapsto (f(p), t)$. Since $f(N) = N$, $f$ maps geodesics which start orthogonal to $N$ to geodesics of the same type, thus we have $\bar{f}(\gamma(\nu, s)) = \gamma(df(\nu), s)$. This immediately implies $\bar{f}(D) = D$ by construction of $D$. Since $\bar{f}$ is an isometry, so is its restriction $\bar{f} := \bar{f}|_D$. \qed

Thus the first two construction steps can be reused, only the blending step needs to be adjusted in order to ensure that the induced action of $G$ be isometric. This is accomplished by averaging the family $g(t)$ of metrics in step 3 which hence is being replaced by

$$\bar{g}(t) = \int_G \gamma^* g(t) \, dm(\gamma).$$

This averaged metric $\bar{g}(t)$ coincides with $g(t)$ where it is already $G$-invariant, namely for $t \in [0, \varepsilon]$ as well as for $t \in [1 - \varepsilon, 1]$. The former holds because the metric equals the induced metric of a distance tube around $N$, the latter holds since the connection metric $h$ is made up of equivariant pieces. Indeed, if $g \in G$, then $g$ acts on $\nu N$ via the differential $dg : \nu N \rightarrow \nu N$. Then the differential of $dg$ maps the
horizontal distribution $\mathcal{H}_\nu \subset T_\nu \nu N$, which is determined by the connection, onto $\mathcal{H}_{dg(\nu)}$, since $\nabla$ commutes with $G$. Now, $dg$ preserves both the fiber metric on the horizontal distribution as well as the one given on the vertical distribution, which implies the preservation of $h$ under the action of $G$.

This completes the proof of Theorem 5.5.

6. Surgery of conformally flat manifolds

A similar surgery result holds for the class of conformally flat manifolds, but - in contrast to the other cases - here only 0-surgery, i.e. connected sum constructions, can be expected, since $S^{n-1} \times \mathbb{R}^l$, $l > 0$, $n \geq 3$, is conformally flat precisely for $l = 1$.

**Theorem 6.1.** Let $C \subset C_B(\mathbb{R}^n)$ be a curvature condition satisfying an inner cone condition with respect to $R_{S^{n-1} \times \mathbb{R}}$. Assume that $C$ is star-shaped with respect to 0. Suppose $(M^i, g_{M_i})$, $i = 1, 2$, are $n$-dimensional conformally flat Riemannian manifolds satisfying $C$. Then the connected sum $M_1 \# M_2$ also admits a conformally flat metric satisfying $C$.

**Remark 6.2.** The validity of Theorem 6.1 in the case of positive scalar curvature was noticed in [SY79]. Conceivably, Theorem 6.1 follows directly from

**Theorem 6.3.** Let $C \subset C_B(\mathbb{R}^n)$ be a curvature condition satisfying an inner cone condition with respect to $R_{S^{n-1} \times \mathbb{R}}$ and suppose $C$ is star-shaped with respect to the origin. Let $(M^n, g)$ be a conformally flat manifold satisfying $C$. Given $p \in M$ and any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\gamma \in (0, \delta)$ there exists a positive function $\sigma \in C^\infty(M\setminus\{p\})$ which defines a complete metric $g_D := \sigma^2 g$ on the open manifold $D := M\setminus\{p\}$ with the following properties:

1. $(D, g_D)$ is conformally flat and satisfies $C$.
2. $g_D$ coincides with $g$ on $M\setminus D(\epsilon)$, where $D(\epsilon) = \{x \in M \mid d^g(p, x) < \epsilon\}$.
3. There exists a neighborhood $U \subset D(\epsilon)$ of $p$ such that $(U\setminus\{p\}, g_D)$ is isometric to $(S^{n-1}(\gamma) \times (0, \infty))$ with $S^{n-1}(\gamma)$ being equipped with the standard metric of constant curvature $\gamma^{-2}$.

The proof of this theorem is based on an extension of a method developed in [MW93]. Before going into the details of the construction, it is useful to collect some formulas which show the influence of a conformal deformation on the curvature of a Riemannian manifold.

**Lemma 6.4.** Let $\sigma$ denote a positive, smooth function on a Riemannian manifold $(M^n, g)$.

1. The gradient of a function $f \in C^\infty(M)$ with respect to $\sigma^2 g$ is given by $\nabla_{\sigma^2 g} f = \sigma^{-2} \nabla g f$.
2. The Levi-Civita connections are related by $\nabla_{\sigma^2 g} Y = \nabla g Y + \sigma^{-1} (d\sigma(X)Y + d\sigma(Y)X - g(Y, X)\nabla g \sigma)$
3. The $(2, 0)$-hessian of a function $f \in C^\infty(M)$ is given by $H_{\sigma^2 g}^2(f) = H^g(f) + \sigma^{-1} (d\sigma(\nabla g f) g - d\sigma \odot df)$, where $d\sigma \odot df := d\sigma \odot df + df \otimes d\sigma$. 

(4) In case $\sigma = w \circ f$, where $w \in C^\infty(\mathbb{R})$ and $f \in C^\infty(M)$, the $(4,0)$-curvature tensor is given by

$$R_{(M,\sigma^2g)} = \sigma^2 \left[ R_{(M,g)} - \left( \frac{w'}{w} \right) \circ f \cdot g \wedge \left( 2 \mathcal{H}^g(f) + \left( \frac{w'}{w} \right) \circ f \cdot \| df \|^2_g g \right) - 2 \left( \frac{w''}{w} - 2 \left( \frac{w'}{w} \right)^2 \right) \circ f \cdot (g \wedge df^2) \right].$$

Now we turn to the construction of the function $\sigma$ as described in Theorem 6.3. First of all, due to the conformal flatness we are able to choose a function $v \in C^\infty(V)$ in a neighborhood $V \subset B_\epsilon(p)$ of $p$ such that $v^2 g_M$ is flat. By possibly rescaling $v$ and shrinking $V$, we can suppose that $v(p) = 1$, $2^{-1} \leq v \leq 2$, and that $V$ is a diffeomorphic image of some open ball $B_\epsilon(0) = \{ \nu \in T_p M \mid g(\nu,\nu) < \epsilon' \}$ under the mapping $\exp := \exp_{g}^v$. Let $r(x) := d_{g}^{v^2}(p,x)$ denote the distance function to $p$ with respect to this flat metric. Let $\phi : \mathbb{R} \to [0,1]$ be a smooth cut-off function satisfying $\phi|_{[0,\frac{1}{2}]} \equiv 1$ and $\phi|_{[1,\infty)} \equiv 0$. Given $\lambda > 0$, which will be determined in due course, we define

$$\phi_\lambda(x) := \phi \left( \frac{r(x)}{\lambda} \right)$$
on V. Using this rescaled cut-off function we set

$$v_\lambda := \sqrt{\phi_\lambda v^2 + (1 - \phi_\lambda)} = \sqrt{1 + \phi_\lambda (v^2 - 1)}$$

Then the estimate $2^{-1} \leq v_\lambda \leq 2$ continues to hold. We set $q := \frac{\lambda}{v_\lambda}$, and get $4^{-1} \leq q \leq 4$.

Furthermore, as in [MW93], we consider the differential equation

$$(17) \quad \frac{u'(r)}{u(r)} = -\frac{\alpha(r)}{r},$$

with $\alpha : [0,\infty) \to [0,1]$ yet to be determined such that a solution $u : [0,\infty) \to [1,\infty)$ can be constructed matching the following conditions:

1. $u \equiv 1$ on $[r_0,\infty)$ for some $r_0 > 0$.
2. $u(r) = \frac{\gamma}{r}$ for all $r \leq \lambda$, where $\gamma$ is given as in the statement of Theorem 6.3.
3. $(D,g_D) := (M \setminus \{p\}, \sigma^2 g)$, satisfies $C$. Here, $\sigma \in C^\infty(M \setminus \{p\})$ is given by $\sigma = w_\lambda$ where both $u$ and $v_\lambda$ are defined and by 1 otherwise, which yields a smooth function by construction.

The purposes served by the two complementary deformations is to first address (via $u$) the problem of deforming the distance spheres of varying radii $r$ with respect to the flat metric $v^2 g$ around $p \in M$ towards a uniform radius $\gamma$ (still with respect to the metric $v^2 g$), which will make the new end of the manifold roughly look like a cylinder, and, secondly, to blend (using $v_\lambda$) the original metric smoothly into the aforementioned flat metric in order to make the end look exactly like a cylinder.

Once $u$ and $v_\lambda$ are determined, Theorem 6.3 will be proven. Indeed, $[1]$ and $[2]$ of Theorem 6.3 hold true obviously; for the validity of $[3]$ observe that the metric $g_D$ on $\{r < \frac{\lambda}{2}\}$ has the form

$$\sigma^2 g = \frac{\gamma^2}{r^2} v^2 g = \frac{\gamma^2}{r^2} \left( dv^2 + r^2 g_{S^{n-1}} \right) = ds^2 + g_{S^{n-1}(\gamma)},$$
where \( s := \gamma \log(r) \in (-\infty, \gamma \log(\frac{1}{r})) \). This also implies the completeness of \( g_D \).

Again, before the various parameters in the bending process can be specified, a close inspection and decomposition of the involved curvature tensor is necessary. To this end, we choose for any \( \nu \in T^1_p M = \{ \nu \in T_p M \mid g(\nu, \nu) = 1 \} \) and \( r > 0 \) an orthonormal basis \( B(\nu, r) = (b_1, \ldots, b_n) \), \( b_i \in T_{exp(\nu r)} M \), with \( b_n = d\exp(\nu r)(\nu) = \nabla^v g r \). Using the notation \( w \cdot B(\nu, r) := (w(\exp(\nu r))b_1, \ldots, w(\exp(\nu r))b_n) \) for a function \( w \), we denote using the notation of Definition 2.4

\[
\begin{align*}
\text{the } (uq)^{-1} \cdot B(\nu, r) & - \text{pullback of } R_{(D, v^2)g} \text{ by } \tilde{R}_D(\nu, r), \\
\text{the } v \cdot B(\nu, r) & - \text{pullback of } R_{(M, g)} \text{ by } \tilde{R}_M(\nu, r) \text{ and} \\
\text{the } q^{-1} \cdot B(\nu, r) & - \text{pullback of } R_{(M, v^2)g} \text{ by } \tilde{R}_M^q(\nu, r). 
\end{align*}
\]

Now we can state

**Proposition 6.5.** Given the identifications made above, the identity

\[
R_D(\nu, r) = u^{-2} \tilde{R}_M^q(\nu, r) + (uq)^{-2} \left[ \frac{\alpha(r)}{r^2} R_{S^{-2} \mathbb{R}} + \frac{2\alpha'(r)}{r} g_{\mathbb{R}^n} \wedge \left( e^h_n \otimes e_n^h \right) + \frac{2\alpha(r)}{r} E^\lambda(\nu, r) \right]
\]

holds for \((\nu, r) \in T^1_p M \times (0, e')\), where \( E^\lambda : T^1_p M \times (0, e') \to C_B(\mathbb{R}^n) \) has a bounded image, i.e.

\[
\| E^\lambda(\nu, r) \| \leq C_1
\]

for some constant \( C_1 \) that does not depend on \( \lambda \).

Notice that \( \tilde{R}_M^q = \tilde{R}_M \) for \( r \geq \lambda \).

**Proof.** Using Lemma 6.4(3) we first obtain

\[
R_{(D, v^2)g} = u^2 \left[ R_{(M, v^2)g} - \frac{u'}{u} v^2_g \wedge \left( 2 H^v g(r) + \frac{u'}{u} \|dr\|^2 v^2_g \right) \right] - 2 \left( \frac{u''}{u} - 2 \left( \frac{u'}{u} \right)^2 \right) v^2_g \wedge dr^2.
\]

Since \( \|dr\|^2 v^2_g = q^{-2} \) and

\[
H^v g(r) = \frac{1}{r} v^2_g + q^{-1} (dq(\partial_r) v^2_g - dq \odot dr),
\]

where we used the orthogonal decomposition \( v^2 g = dq^2 + v^2 g_r \) and Lemma 6.4(3) \((\partial_r := \nabla^v g r \text{ is used here and henceforth as an abbreviation})\), we get

\[
\begin{align*}
\frac{u'}{u} v^2_g \wedge \left( 2 H^v g(r) + \frac{u'}{u} \|dr\|^2 v^2_g \right) &= v^2_g \wedge \left( -\frac{2\alpha}{r^2} v^2_g + \frac{\alpha^2}{r^2} v^2 g \right) - \frac{2\alpha}{r} q^{-1} v^2_g \wedge (\partial_r q v^2_g - dq \odot dr) \\
&= \frac{\alpha(2 - \alpha)}{r^2} v^2_g \wedge v^2_g \wedge dr^2 + \frac{2\alpha(\alpha - 1)}{r^2} v^2 g_r \wedge dr^2 \\
&= -\frac{2\alpha}{r} q^{-1} v^2_g \wedge (\partial_r q v^2_g - dq \odot dr)
\end{align*}
\]

(19)
and

\[ 2 \left( \frac{u''}{u} - 2 \left( \frac{u'}{u} \right)^2 \right) v^2 g \wedge dq = - \left( \frac{2\alpha'}{r} + \frac{2\alpha(\alpha - 1)}{r^2} \right) q^2 g \wedge dq \wedge dr^2 \]

Combining (19) and (20), we get

\[ R_{(D, u^\nu, v^2 g)} = u^2 \left[ R_{(M, \nu^2 g)} + \frac{\alpha(2 - \alpha)}{r^2} v^2 g \wedge dq + \frac{2\alpha'}{r} q^2 g \wedge dq \wedge dr^2 \right. \]

\[ \left. + \frac{2\alpha}{r} q^{-1} v^2 g \wedge (\partial_r q v^2 g - dq \wedge dq) \right]. \]

Switching to the algebraic pullback requires multiplying with \((uq)^{-4}\), replacing \(v^2 g \wedge dq\) by \(R_{S^o M, R} \wedge dq\) by \(e_n^\nu\) and \(v^2 g\) by \(g_{\nu^2 g}\) and hence directly leads to (13), where the error term is given explicitly by

\[ E^\lambda(\nu, r) := \left[ q^{-1} g_{\nu^2 g} \wedge \left( \partial_r q g_{\nu^2 g} - dq \wedge e_n^\nu \right) \right] \circ \exp_{p}^v g(rv). \]

Since \( \|A \wedge B\| \leq c \|A\| \|B\| \) for selfadjoint operators \(A, B\), and \( \frac{1}{4} \leq q \leq 4 \), in order to get the desired bound on \( \|E^\lambda\| \), we are left with finding a suitable bound of \( \|dq\|_{\nu, v^2 g} \) which does not depend on \(\lambda\). Recall that \(q = \frac{2\alpha}{v} = \sqrt{v^{-2} + \phi\lambda(1 - v^{-2})}\). We compute

\[ dq(X) = \frac{1}{2q} \left( (1 - \phi\lambda) X(v^{-2}) + X(\phi\lambda)(1 - v^{-2}) \right). \]

Since \( |\partial_r \phi\lambda| \leq \frac{c}{v} \) for some \(c\) and, similarly, \( |1 - v^{-2}| \leq Cr\), we readily obtain

\[ |\partial_r (\phi\lambda)(1 - v^{-2})| \leq \begin{cases} cC, & \text{if } r < \lambda, \\ 0, & \text{if } r \geq \lambda, \end{cases} \]

for \(\phi\lambda \equiv 0\), if \(r \geq \lambda\). Because of \(X(\phi\lambda) = 0\), if \(X \perp \partial_r\), this yields

\[ \|dq\|_{\nu, v^2 g} \leq \frac{1}{8} \left( \|dv^{-2}\|_{\nu, v^2 g} + cC \right) \]

on \( \{ x \mid dv^{-2}(x, p) < \epsilon' \} \). This upper bound is evidently independent of \(\lambda\). \(\square\)

6.1. Construction of the deforming functions. Before the deforming functions can be described, some parameters - which solely depend on the geometry of \((M, g)\) around \(p\) - have to be fixed. The various choices being made will become clear during the course of the proof.

The inner cone condition of \(C\) with respect to \(R_{\nu, x^2 g}\) implies the existence of a \(\rho > 0\) so that

\[ B_{\lambda r} \left( \hat{R}_M(\nu, r) + \lambda R_{\nu, x^2 g} \right) \subset C \]

for all \((\nu, r) \in T_{\nu}^o M \times (0, \epsilon')\), \(\lambda > 0\). We require the starting radius of the bending process \(r_0\) to obey \(r_0 < \min \left\{ \frac{\rho}{4C_1}, \epsilon' \right\}\), where \(C_1\) is the constant given by Proposition 6.5.
6.2. Step 1: Initial bending. The aim of the initial bending is to prescribe \( \alpha \) on a (tiny) interval \([r_1, r_0]\), which will affect the geometry of the manifold in an annular region \( \{ r_1 \leq r(x) \leq r_0 \} \subset V \) in such a way as to maintain \( \tilde{R}_D \in C \) while making \( \alpha \) positive at \( r_1 \). That \( (M, g) \) satisfies \( C \) allows us to find \( \epsilon > 0 \) such that for any isometry \( \iota : \mathbb{R}^n \rightarrow T_xM, x \in V \), we have

\[
B_{\epsilon} \left( \iota^* R_{(M,g)}(x) \right) \subset C.
\]

Using the star-shapedness of \( C \) it therefore suffices to achieve

\[
\tilde{R}_D \in B_{\frac{1}{u^2}} \left( \frac{1}{u^2} \tilde{R}_M \right),
\]

which in turn holds provided the right side of

\[
\left\| \tilde{R}_D - \frac{1}{u^2} \tilde{R}_M \right\| \leq (uq)^{-2} \left[ \frac{\alpha(2 - \alpha)}{r^2} + 2|\alpha'| + \frac{2}{r} \right] C
\]

\[
\leq \frac{32}{u^2} \left[ \frac{\alpha}{r^2} + \frac{|\alpha'|}{r_1} + \frac{\alpha}{r_1} \right] C,
\]

where \( C := \max \{ \| R_{S^{n-1} \times \mathbb{R}} \|, \| g_{R^n} \wedge e_n^r \wedge e_n^b \|, C_1 \} \), is smaller than \( \frac{\alpha}{r} \). This is obviously true, if \( \alpha \leq \tau \) for some small \( \tau > 0 \) and if a suitable bound for \( \alpha' \) holds on \([r_1, r_0]\). Now \( \alpha : [r_1, r_0] \rightarrow [0, \tau] \) can easily be prescribed such that \( \alpha \) is constant near \( r_1 \) and \( r_0 \), being constant with values \( \tau \) and \( 0 \), respectively, in these neighborhoods.

6.3. Step 2: Main bending. The next step is to produce an extension of \( \alpha \) to an interval \([r_2, r_1]\), \( r_2 > \lambda \), such that \( \alpha \) equals 1 on a neighborhood around \( r_2 \) while maintaining \( \tilde{R}_D(\nu, r) \in C \) for these \( r \in [r_2, r_1] \), \( \nu \in T^1_0 M \). We derive the following sufficient condition for this:

**Lemma 6.6.** There exists \( c > 0 \), which does not depend on \( \lambda \), such that if \( \alpha : (\lambda, r_1] \rightarrow [0, 1], \alpha'(r) \leq 0, \) fulfills

\[
(22) \quad \alpha'(r) + c \frac{\alpha(r)(2 - \alpha(r))}{r} \geq 0,
\]

then \( \tilde{R}_D(\nu, r) \in C \)

**Proof.** Due to (21) and the star-shapedness of \( C \), the relation \( \tilde{R}_D(\nu, r) \in C \) holds, if

\[
(23) \quad \tilde{R}_D(\nu, r) \in B_{\rho \frac{\alpha(2 - \alpha)}{r^2}(uq)^{-2}} \left( \frac{1}{u^2} S \right),
\]

where \( S := \tilde{R}_M(\nu, r) + \frac{\alpha(2 - \alpha)}{r^2} R_{S^{n-1} \times \mathbb{R}} \). By Proposition 6.5 we have

\[
\left\| \tilde{R}_D(\nu, r) - \frac{1}{u^2} S \right\| \leq (uq)^{-2} \left[ \frac{2\alpha'}{r} \right] \left\| g_{R^n} \wedge e_n^r \wedge e_n^b \right\| + \frac{2\alpha}{r} \left\| E^\lambda \right\|.
\]

Hence condition (23) will hold, if the latter expression is smaller than

\[
\rho \frac{\alpha(2 - \alpha)}{r^2}(uq)^{-2},
\]

which in turn is satisfied, if

\[
(24) \quad \left\| E^\lambda \right\| < \frac{\rho}{4r}
\]
and
\[ |\alpha'| \leq \frac{\rho}{4} \left\| g_{R_n} \land e_n^b \otimes e_n^b \right\|^{-1} \frac{\alpha(2 - \alpha)}{r} \]
are true. (24) holds because of \( r < r_0 < \frac{\rho}{4} \). If we choose \( c < \frac{\rho}{4} \), (22) implies (25), which proves the lemma.

□

So we have to solve a differential equation:

Lemma 6.7. Given \( \tau \in (0, 1] \), there exists \( r_2 \in (0, r_1) \) and a monotone, non-increasing \( \alpha : (0, r_1) \rightarrow [\tau, 1] \) fulfilling (22) such that \( \alpha \) is constant on a neighborhood of \( r_1 \), with value \( \tau \), and \( \alpha \big|_{[0, r_2]} = 1 \). Moreover, if \( \alpha \) is extended to an interval \( (0, r_0] \) as described above, there exists \( \delta > 0 \) such that for any \( \gamma \in (0, \delta) \) the function \( \alpha \) (and \( r_2 \)) can be chosen in such a way as to ensure that the solution \( u \) of (17) is given by

\[ u(r) = \frac{\gamma}{r} \]
for \( r < r_2 \).

Note that \( r_2 \) depends only on \( c \) and \( \tau \).

Proof. We use the same transformation that was used in [MW93] to tackle the corresponding problem there. So set \( r(s) := r_1 e^{-s} \) and consider \( \beta(s) := \alpha(r(s)) \). Then the existence of \( \alpha \) as claimed in the statement is easily seen to be equivalent to the existence of a non-decreasing function \( \beta : [0, s_2] \rightarrow [\tau, 1] \) satisfying

(26) \[ \beta'(s) \leq \beta(s)(2 - \beta(s)), \]
\( \beta \) being constant on a neighborhood of 0 with value \( \tau \), and \( \beta \big|_{[s_2, \infty)} = 1 \), where \( s_2 := s(r_2) \).

The construction of \( \beta \) is easily accomplished: Take any non-decreasing function \( \beta \) matching the named side conditions together with the grow restriction \( \beta'(s) \leq \tau(2 - \tau) \). Then (25) automatically holds simply because \( x \mapsto x(2 - x) \), restricted to \( x \in [\tau, 1] \), has a minimum at \( x = \tau \).

For the second statement observe that, since \( u(r_0) = 1 \), we have

\[ u(r) = \exp \left( - \int_{r_0}^{r} \frac{\alpha(t)}{t} dt \right). \]

In particular for \( r < r_2 \)

\[ u(r) = \frac{r_2}{r} \exp \left( \int_{r_2}^{r_1} \frac{\alpha(t)}{t} dt \right) \exp \left( \int_{r_1}^{r_0} \frac{\alpha(t)}{t} dt \right) \]
The identity \( \int_{r_2}^{r_1} \frac{\alpha(t)}{t} dt = \frac{1}{c} \int_{0}^{s_2} \beta(s) ds \) yields

\[ \gamma = \exp \left( \int_{r_2}^{r_1} \frac{\alpha(t)}{t} dt \right) r_1 \exp \left( \frac{1}{c} \int_{0}^{s_2} (\beta(s) - 1) ds \right) \]
Thus, by suitably arranging \( \beta \) (and \( s_2 = s(r_2) \)) any desired value for \( \gamma \) in a range \( (0, \delta) \), \( \delta := r_1 \exp \left( \int_{r_1}^{r_0} \frac{\alpha(t)}{t} dt \right) \), can be achieved. □
6.4. Step 3: Smoothing of the end. In the final step the yet unspecified parameter $\lambda \in (0, r_2)$ has to be chosen.

**Lemma 6.8.** There exists a constant $C_2 > 0$ depending only on $(M, g)$, $V$ and $p$ so that

$$\|\hat{R}_M^\lambda (v, r)\| \leq \frac{C_2}{\lambda}$$

for $\lambda < r_0$.

**Proof.** We only need to consider the region $\{ r \leq \lambda \}$, because we have $R_{(M, v^2 g)} = R_{(M, g)}$ outside this region. Applying Theorem 1.159b of [Besse87] we get

$$R_{(M, v^2 g)} = -q^2 \left[ 2v^2 g \wedge \left( \partial^2 v^2 g (\log(q)) - \left( d \log(q) \right)^2 + \frac{1}{2} \| d \log(q) \|_{v^2 g}^2 v^2 g \right) \right]$$

This formula together with the rule $\| A \wedge B \| \leq c \| A \| \| B \|$ for selfadjoint operators $A, B$ immediately yields the desired result once we have the estimates

$$\| dv_{\lambda} \|_{v^2 g} = O(1), \quad \| \partial^2 v_{\lambda} \|_{v^2 g} = O(\lambda^{-1}).$$

The first estimate was already derived in the proof of Proposition 6.5. For the second one we make use of the identity $H^2 g (v_{\lambda})(X, Y) = X (Y v_{\lambda}) - \nabla^2 g Y (v_{\lambda})$ and compute

$$\partial^2 v_{\lambda} = -\frac{1}{2 v_{\lambda}^2} \partial_r v_{\lambda} ((\partial_r \phi_{\lambda}) (v^2 - 1) + \phi_{\lambda} \partial_r (v^2 - 1)) + \frac{1}{2 v_{\lambda}^2} \left( \partial^2 \phi_{\lambda} (v^2 - 1) + 2 \partial_r \phi_{\lambda} \partial_r (v^2 - 1) + \phi_{\lambda} \partial_r^2 (v^2 - 1) \right).$$

Because of $v^2 - 1 = O(r)$, $r \leq \lambda$ and $\partial_r^k \phi_{\lambda} = O(\lambda^{-k})$ we get

$$\partial^2 v_{\lambda} = O(\lambda^{-1}).$$

Similarly, if $X \perp \partial_r$, by recalling that $\phi_{\lambda}$ is a radial function we obtain

$$\partial_r (X v_{\lambda}) = -\frac{1}{2 v_{\lambda}^2} \partial_r v_{\lambda} \left( \phi_{\lambda} X (v^2 - 1) \right) + \frac{1}{2 v_{\lambda}^2} \left( \partial_r \phi_{\lambda} X (v^2 - 1) + \phi_{\lambda} \partial_r X (v^2 - 1) \right).$$

This gives us $\partial_r (X v_{\lambda}) = O(\lambda^{-1})$ by the same reasoning. Finally, for $X, Y \perp \partial_r$, we easily get $X (Y v_{\lambda}) = O(1)$. This proves the hessian estimate in (27). \qed

For $r \leq r_2$ Proposition 6.3 yields

$$\tilde{R}_D (v, r) = u^{-2} \hat{R}_M^\lambda (v, r) + (u q)^2 \left[ \frac{1}{r^2} R_{S^{n-1} \times \mathbb{R}} + 2 \frac{r}{v} E_{\lambda} (v, r) \right],$$

since $\alpha_{[\theta, r_2]} = 1$. For $r \in [\lambda, r_2]$, $\tilde{R}_D \in C$ continues to hold by Lemma 6.6. For $r < \lambda$, again it is enough to guarantee the inclusion condition

$$\tilde{R}_D (v, r) \in B_{\frac{1}{u^2} (u q)^{-2}} \left( \frac{1}{u^2} \left( \hat{R}_M (v, r) + \frac{1}{r^2 q^2} R_{S^{n-1} \times \mathbb{R}} \right) \right)$$

in order to keep the curvature tensor inside $C$. Thus by Lemma 6.8 the analogous estimate reads

$$\left\| \tilde{R}_D (v, r) - \frac{1}{u^2} \left( \hat{R}_M + \frac{1}{r^2 q^2} R_{S^{n-1} \times \mathbb{R}} \right) \right\| \leq \frac{1}{u^2} \left( \left\| \hat{R}_M \right\| + \frac{2 C_1}{q^2 r} + \frac{C_2}{\lambda} \right).$$
This is smaller than $\rho_{\left(\frac{1}{ruq}\right)}$ provided

$$\lambda < \min \left\{ r_2, \rho^{\frac{1}{4}} \left( 48 \max \left\| \bar{R}_M \right\| \right)^{-\frac{1}{2}} \frac{\rho}{6C_1}, \frac{\rho}{48C_2} \right\}.$$  

Choosing $\lambda$ such that these constraints are met is possible since none of the expressions of the right side depend on $\lambda$. By doing so, the third step of the deformation construction is completed, and so is the proof of Theorem 6.3.

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**Mathematisches Institut, WWU Münster, Germany**

E-mail address: sebastian.hoelzel@uni-muenster.de