Reduction of the unbalance excited vibrations during the resonance passage of a rotor

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Abstract. Many rotor dynamic systems are faced with large oscillation amplitudes while passing through the resonance. The coupled differential equations for the position of the center of the shaft and the rotation angle describe the dynamic behaviour of the rotodynamic system. With an appropriate external excitation the amplitude of the oscillations can be reduced. An unbalance excited oscillator is used to study the effects of a modulated angular velocity, where the angular velocity is prescribed as linear increasing with a superimposed harmonic function. The resulting excitation force is expressed as a series of Bessel functions of the first kind. In order to achieve small amplitudes near the resonance frequency of the system, special conditions for the ratio of the modulation frequency and the angular acceleration and the argument of the Bessel function of the first kind with integer order zero are derived. These requirements are first developed for a linear oscillator with an excitation force. Due to the analogy of the solution of the equation of motion these solutions are then applied directly to an unbalance excited oscillator. The results show smaller displacement amplitudes compared to the case with only linear increasing angular velocity. For the resulting motion the required torque is computed.

1. Introduction
Vibration reduction is important to avoid high amplitudes near the resonance frequency of a system or during its passage through resonance. There are a lot of passive methods that can help to obtain higher comfort, better production processes or increased accuracy due to less vibrations near the resonance frequency. For the application of vibration isolation suitable elastic bearings may be used to separate the vibrating source from the surrounding environment or to isolate the dynamic system from disturbing vibrations, see [1]. These bearings need elastic and damping properties, that can be achieved for instance with adjustable springs and dampers, elastomeric bearings with or without hydrodynamic damping or controllable bearings. Another possibility is to modify or design the behavior of damping of a vibrating system, which means that some energy of the vibrating system is dissipated during every cycle, see [2]. In dynamic systems damping exists e.g. due to the internal damping or plasticity of a solid body, as viscous damping with tough liquids or gases or as friction damping. The application of a dynamic vibration absorber generates a counterforce due to a specific additional system. There are also methods for active vibration compensation that require sensors and control concepts, see [3]. For the transient case of a resonance passage of a vibrating system the transient solution has to be computed and analysed with respect to the maximum of the amplitudes. It is important to estimate the values of the maximum amplitudes as it has to be guaranteed that the amplitudes remain below a
certain value, e.g. in rotordynamic systems this condition prevents contact between stator and rotor. Only a few analytic solutions exist for such a case and some additional assumptions must be made about the performance of the passage through the resonance. As it is important to estimate the maximum of the vibration amplitude, some research studies have been performed in order to obtain suitable approximations. A first empirically developed approximation for the maximum resonance amplitude of a linear oscillator excited by a non-stationary force of unbalance is derived in [4]. In [5] the influence of the angular acceleration is studied and in [6] some results have been derived for vibrating machines. For a run-down phase the vibrations have been studied in [7]. In [8] the estimation of the maximum amplitude has been computed based on the analytical solutions for the passage through the resonance with a defined angular acceleration, where a rotordynamic system has been considered.

Contrary to the conventional vibration reduction methods and the estimation of maximum amplitudes of a dynamic system due to transient excitations, in this study the solution for a suitable modulation of the excitation frequency of linear dynamic systems is derived. The solution is analysed with respect to the influence of the modulation parameters on the maximum amplitudes, where some specific relations can be found. The analysis first starts with a linear oscillator driven by a periodic excitation force with a defined modulation. The solution is derived by a transformation of the excitation function involving a series expansion of periodic functions. For these excitation frequencies the analytical solution can be computed based on a series expansion. Based on the contribution of the different excitation frequencies contained in the series expansion important relations have been derived in order to get small amplitudes. The transient solution is computed by numerical integration and a parametric study shows the expected results of smaller vibration amplitudes. The derivations described above have been applied to a rotordynamic system to compute the amplitude of transient vibrations for the run-up of a rotor through the resonance. It is shown that for the run-up of a rotor through the resonance frequency an added modulation of the rotation speed can keep the vibration amplitudes very small and the relation between the parameters is the same also for this case.

2. Linear oscillator with excitation force with modulated frequency
A linear oscillator with a mass \( m \) is considered in the following analysis, which is connected to the ground by a Kelvin-Voigt-Element with a spring \( c \) and a viscous damper \( d \). As a first step and for the sake of comparison the external harmonic excitation force \( F(t) = \hat{F} \sin(\omega t) \) is directly acting at the mass with the amplitude \( \hat{F} \) and the excitation frequency \( \omega \). The deflection \( x(t) \) of the mass is of interest and the equation of motion of the system is

\[
m \ddot{x}(t) + d \dot{x}(t) + c x(t) = F(t).
\]

(1)

The amplitude for the steady state solution of the deflection \( \hat{x} \) is given as a product of the corresponding static deflection \( \hat{x}_{\text{Stat}} \) and the transfer function \( V(\eta) \) and reads

\[
\hat{x} = \frac{\hat{F}}{c} \frac{1}{\sqrt{(1 - \eta^2)^2 + (2\zeta\eta)^2}} = \frac{\hat{F}}{c} V(\eta) = \hat{x}_{\text{Stat}} V(\eta).
\]

(2)

\( \eta = \frac{\omega}{\omega_0} \) is the dimensionless excitation frequency and the dimensionless damping is given by

\[
\zeta = \frac{d}{2m\omega_0}, \quad \omega_0 = \sqrt{\frac{c}{m}}
\]
denotes the natural frequency or eigenfrequency of the system. Eq. (2) is studied in the vibration reduction analysis for steady state conditions and serves as a reference solution here. The solution for transient vibrations is computed using the Duhamel convolution
integral where the solution due to the initial conditions \(x_0\) and \(\dot{x}_0\) has to be added

\[
x(t) = e^{-\zeta \omega_0 t} \left[ x_0 \cos(\omega_d t) + \frac{\dot{x}_0 + \zeta \omega_0 x_0}{\omega_d} \sin(\omega_d t) \right] + \frac{1}{m \omega_d} \int_0^t F(\tau) e^{-\zeta \omega_0 (t-\tau)} \sin(\omega_d (t-\tau)) d\tau.
\]

Here, the frequency of damped vibrations \(\omega_d = \omega_0 \sqrt{1 - \zeta^2}\) is used. Frequently during the run-up and run-down procedure the excitation frequency is not constant any more and passes through the resonance. To simulate such a process it is assumed that the excitation frequency is increased linearly with a given angular acceleration \(\alpha\) and the solution again can be computed using Eq. (3). Additionally a harmonic function is added to this basic linear increasing frequency

\[
\omega(t) = \dot{\varphi}(t) = \alpha t + \dot{\omega} \sin(\nu t).
\]

\(\dot{\omega}\) characterizes the deviation from the linear function and \(\nu\) is the frequency of the added modulation. The integration of Eq. (4) leads to

\[
\phi(t) = \frac{\alpha t^2}{2} - \frac{\dot{\omega}}{\nu} \cos(\nu t) + \varphi_0
\]

with \(\varphi_0\) considering the initial condition for the phase. The angle \(\phi(t)\) is the argument of the excitation force function and \(\varphi_0 = 0\) can be assumed zero in the following without loss of generality.

\[
F(t) = \hat{F} \sin \left( \alpha \frac{t^2}{2} - \eta \cos(\nu t) \right)
\]

The modulation index \(\eta = \frac{\dot{\omega}}{\nu}\) is used as a characteristic dimensionless parameter, where the amplitude \(\dot{\omega}\) of the modulation is divided by the modulation frequency \(\nu\). Using some mathematical transformations and manipulations, which are described in [9], a formulation is derived for the external excitation force which contains an infinite sum of products of Bessel functions of the first kind and harmonic functions with different frequencies

\[
F(t) = \hat{F} J_0(\eta) \sin(\alpha \frac{t^2}{2}) + \hat{F} \sum_{k=1}^{\infty} J_k(\eta) \left[ \sin \left( (\alpha \frac{t}{2} + k\nu)t - k\frac{\pi}{2} \right) + \sin \left( (\alpha \frac{t}{2} - k\nu)t - k\frac{\pi}{2} \right) \right].
\]

The argument in the Bessel functions \(\eta = \frac{\dot{\omega}}{\nu}\) is the previously defined modulation index. After inserting this excitation force function into the equation of motion (1), the transient solution can be computed resulting in an analytic expression using the convolution integral (3). Analysing the series expansion representation of the excitation it is possible to discuss and find suitable parameters and their relation. The modulation index \(\eta\) is the argument of the Bessel functions, which determines the amplitude by multiplication with the sine-function. In a first step the modulation index \(\eta\) is chosen to be constant and will be considered as a time dependent function \(\eta(t)\) in a second step. The frequency \(\nu\) of the added function in Eq. (6) defines the distance between the sideband amplitudes in the spectrum after the modifications, which can be seen in Eq. (7). The angular acceleration \(\alpha\) is assumed to be constant and defines the increase of the angular velocity. As shown in Fig. 1 the Bessel functions are zero for different modulation indices \(\eta\). The Bessel function of the first kind of order zero has the first root at \(J_0(2.4048) = 0\) and that of higher order have small amplitudes at the position \(\eta = 2.4048\). The next roots of
the Bessel function of order zero are \( J_0(5.5201) = 0 \), \( J_0(8.6537) = 0 \), \( J_0(11.7915) = 0 \), etc. The further roots can be approximated by adding \( \pi \) to the preceding value, see [10]. From Eq. (7) and Fig. 1 it can be seen that for the limit value of the modulation index \( \eta = 0 \) the value of the Bessel function is \( J_0(0) = 1 \) and all the other Bessel functions are \( J_k(0) = 0 \), so that the sum in Eq. (7) vanishes. However for \( \eta = 0 \) the added modulation vanishes.

In the second step instead of a constant modulation index \( \eta \) a piecewise linear function is defined between \( t_1 \) and \( t_3 \) as follows

\[
\eta(t) = \begin{cases} 
0 & \text{for } t \leq t_1, \\
2.4048 \left( \frac{t - t_1}{t_2 - t_1} \right) & \text{for } t_1 < t \leq t_2, \\
2.4048 - \frac{2.4048}{t_3 - t_2} (t - t_2) & \text{for } t_2 < t \leq t_3, \\
0 & \text{for } t > t_3.
\end{cases}
\] (8)

This strategy allows a further modification of the external excitation force. \( \eta(t) \) is defined to be non-zero between the sideband frequencies and reaches \( \eta = 2.4048 \) at \( t_2 \). The times \( t_1, t_2 \) and \( t_3 \) are chosen in correspondence to \( \omega(t) = \alpha t \). With the natural frequency \( \omega_0 \) of the system \( t_2 = \omega_0/\alpha \) and the upper and lower sideband frequencies lead to \( t_{1,3} = (\omega_0 \pm \nu)/\alpha \). In Fig. 1 the Bessel functions are shown over the modulation index \( \eta \). The amplitude \( J_0(\eta) \) of the sine-function, which defines the linear rising angular velocity in Eq. (7), has a value of 1 most of the time when using \( \eta(t) \) from Eq. (8). Only for a short time-span around the resonance frequency the other amplitudes become different from zero and \( J_0(\eta) \) is decreased reaching the value of zero at \( \omega_0 \). Due to the parameter \( \nu \) the arguments in Eq. (7) are different from the resonance frequency as shown in Fig. 2, where the discrete amplitudes of \( J_0, J_1 \) and \( J_2 \) are drawn over the angular velocity in different colors.

![Figure 1. Bessel functions \( J_k(\eta) \) of the first kind \((k = 0...10)\).](image1)

![Figure 2. Discrete amplitudes of \( J_0, J_1 \) and \( J_2 \) over the angular velocity.](image2)

**Table 1.** Parameters for the force excited harmonic oscillator.

| Mass    | Spring stiffness | Damping factor | Amplitude | Resonance frequency |
|---------|------------------|---------------|-----------|--------------------|
| \( m = 1 \) kg | \( c = 900 \) N/m | \( d = 0.01 \) kg/s | \( A = 1 \) N | \( \omega_0 = 30 \) 1/s |

With the defined function and assumptions from an analysis with various parameters the best values for the parameters \( \alpha \) and \( \nu \) can be found. Based on the desired and available conditions the
acceleration $\alpha$ can be defined arbitrarily. A reasonable relation $\nu(\alpha)$ between the two parameters is needed. The amplitude of the deflection is small if the time-span $T_\alpha = t_3 - t_1$ of the linear rise and fall off is equal to the period $T_\nu$ according to the modulation frequency $\nu$. The definition of $T_\alpha$ results to

$$T_\alpha = \frac{\omega_0 + \nu}{\alpha} - \frac{\omega_0 - \nu}{\alpha} = 2 \frac{\nu}{\alpha}$$

and the period of the sine-function is

$$T_\nu = \frac{2 \pi}{\nu}.$$  
(10)

With the further assumption that these characteristic time-scales should be equal, the relation of the parameters follows to

$$\frac{\nu^2}{\alpha} = \pi \quad \text{or} \quad \nu = \sqrt{\frac{\pi}{\alpha}}.$$  
(11)

Another important feature is the value of the phase of the sinusoidal function. It is significant that the negative amplitude of the sine-function comes first and starts at $t_1$. This leads to the function

$$\omega(t) = \alpha t + \hat{\omega} \sin [\nu (t - t_1) - \pi]$$

for the angular velocity. For the computation of an example a value of $\alpha = 0.5 \, 1/s^2$ is assumed and the conditions for $\eta(t)$ corresponding to Eq. (8) and Eq. (11) are used. The excitation frequency $\omega(t)$ is according to Eq. (12), where $\hat{\omega} = \nu \cdot \eta$ and the parameters from Tab. 1 are taken. The numerical solution for the deflection $x(t)$ of the force driven harmonic oscillator can be calculated and is shown in Fig. 5 as the black line.

Fig. 3 shows the piecewise linear function according to Eq. (8) that leads to a $\hat{\omega}$, which is only different from zero in the time-span between $t_1$ and $t_3$. The angular velocity is shown in Fig. 4, where the function $\omega(t)$ rises linearly before $t_1$ and after $t_3$. The small difference to the linear function of the angular velocity has a strong impact to the deflection $x(t)$ depicted in Fig. 5, where the solution for the passage through the resonance with a constant acceleration results in higher vibration amplitudes (gray line) than for the case with the derived parameters for the modulation (black line). In Fig. 6 the amplitude spectrum of the excitation force is displayed. It can be seen that the minimum of the amplitude is at the resonance frequency $f_0 = \omega_0/(2\pi) = 4.77 \, 1/s$. 

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**Figure 3.** Piecewise linear function $\eta(t)$ from Eq. (8).

**Figure 4.** Angular velocity from Eq. (12).
3. Unbalance excited oscillator with modulated rotation speed

In the next step the derived results are applied to a rotor-dynamic system, where the solution for a passage through the resonance is computed for a Laval-rotor. The equation of motion of the mechanical model

\[(m_1 + m_e) \ddot{y}(t) + d \dot{y}(t) + c y(t) = F_Z \sin \varphi(t) - F_a \cos \varphi(t) = \ddot{R}_e\]  

describes the behaviour of the dynamic system shown in Fig. 7 in y-direction with the force \(F_Z = m_e \omega^2 r_e\) in radial direction and \(F_a = m_e \dot{\omega} r_e\) in circumferential direction. \(m_1\) is the mass of the rotating disc, \(c\) the stiffness of the shaft, \(d\) a viscous damping parameter and \(m_e\) is the assumed unbalance mass at the radius \(r_e\). These two forces can also be written as \(R_e = -m_e r_e \sin \varphi(t)\), see Eq. (13). The application of the balance of angular momentum gives

\[(J_1 + m_e r_e^2) \ddot{\varphi}(t) = M(t) - m_e \ddot{y}(t) r_e \cos \varphi(t)\]  

which contains the moment of inertia \(J_1 = \frac{1}{2} m_1 r^2\) of the rotating disc and the driving torque \(M(t)\).

In this example the differential equations Eq. (13) and Eq. (14) are coupled and can not be solved independently. For this reason, it is decided to predefined the angular velocity \(\omega(t)\) according to Eq. (12) and then to calculate \(\varphi(t) = \int \omega(t) dt\) for Eq. (13). Furthermore \(\alpha = 0.75 \, \text{1/s}^2\) is used and the modified frequency \(\nu = \sqrt{\pi \alpha}\), the characteristic times \(t_{1,3} = (\omega_0 \pm \nu) / \alpha\) and \(t_2 = \omega_0 / \alpha\).
Table 2. Parameters for the Laval-rotor with unbalance excitation.

| Mass   | Radius | Spring stiffness | Damping factor | Resonance frequency |
|--------|--------|------------------|----------------|--------------------|
| $m_1$  | 10 kg  | $r = 0.25 \text{ m}$ | $d = 0.1 \text{ kg/s}$ | $\omega_0 = \sqrt{\frac{c}{(m_1 + m_e)}} = 31.59 \text{ 1/s}$ |
| $m_e$  | 0.02 kg| $r_e = 0.2 \text{ m}$ |                |                    |

for the external excitation are calculated. $\eta(t)$ again is given by Eq. (8). With these given conditions Eq. (13) can be solved and the resulting deflection $y(t)$ for the rotordynamic system with the parameters of Tab. 2 is shown in Fig. 8 for the modulated run-up in black colour. The gray line displays the higher deflection amplitudes for the case of a linear increasing angular velocity only. The resulting torque is shown in Fig. 9, which is the torque $M(t)$ necessary to fulfill the mentioned function $\omega(t)$ from Eq. (12) and the conditions associated with this function. The torque is calculated using Eq. (14), which requires the solution for $y(t)$ using the predefined $\varphi(t)$. For the given example a torque of 0.235 Nm is needed most of the time and the added modulation results in a short peak with a maximum value of about 2 Nm.

Figure 8. Deflection $y(t)$ of the center of the shaft for the cases of linear and modulated run-up for $\alpha = 0.75 \text{ 1/s}^2$.

Figure 9. Required torque $M(t)$ for the case of the modulated run-up.

Figure 10. Deflection $y(t)$ of the center of the shaft for modulated run-up with $\alpha = 0.75 \text{ 1/s}^2$ and a linear run-up with $\alpha = 6.38 \text{ 1/s}^2$.

Figure 11. Angular velocity $\omega(t)$ for the linear run-up with $\alpha = 6.38 \text{ 1/s}^2$ and $M = 2 \text{ Nm}$.
For sake of comparison in an additional computation the maximum torque of 2 Nm has been used for the whole run-up of the rotating system. This corresponds to an angular acceleration of \( \alpha = 6.38 \frac{1}{s^2} \). The coupled differential equations are solved for the constant torque. The deflection \( y(t) \) due to that faster passage through the resonance is shown in Fig. 10 in gray colour together with the above described modulated run-up in black colour. Fig. 11 shows the rotation speed \( \dot{\varphi}(t) = \omega(t) \) of the solution for the case with a constant torque of 2 Nm.

4. Conclusion
For a linear oscillator and a rotordynamic system the solution for the deflection has been computed for a passage through the resonance. A modified external excitation force has been considered with a constant angular acceleration and an added modulated frequency. The solutions have been analysed and compared. After some transformations and manipulations of the excitation force function the parameters and their relations are defined within a series expansion. For a reduction of the amplitude of vibrations during a passage through the resonance specific values of the parameters and their relations have been derived. For this case the modulation index for small vibration amplitudes corresponds to the root of the Bessel function of first kind and order zero. A reduction of the vibration amplitude during run-up has been shown using the computed results when the suitable modulation index has been used. Furthermore it can be concluded that the method with a piecewise linear modulation index \( \eta(t) \) and the required conditions for the modulation frequency result in a passage through the resonance with a much smaller maximum amplitude of vibrations. The maximum amplitude of the deflection with the presented method is also smaller than the amplitude which results for a higher constant torque which is equal to the necessary maximum torque for the modulated run-up.

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