POINT-LIKE STRUCTURE IN STRINGS AND
NON-COMMUTATIVE GEOMETRY

by

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ABSTRACT

Dynamics of a free point particle on a multi world-line is presented and shown to reduce to that of a bosonic string theory at the appropriate limit. Other higher dimensional extended objects are argued to appear at other regions of the space of configurations of the theory.
1 Introduction

In the dramatic developments in the understanding of the strong coupling limit of string theory of the last two years, two ideas have resurfaced: Point-like structures in the small distance regime; and noncommutative aspects in these scales.

Point-like substructure of string theory has long been suspected and recently extensively studied [1, 2]. More directly relevant to the recent developments of string duality are the realization that D0-branes play an important role in string dynamics and in the presumably more fundamental M-theory [3, 4, 5]. In the distances shorter than string scale [6] there are signatures of the breakdown of the smooth structure of the world-sheet [4, 7]; higher dimensional D-branes have been argued to be bound states of D0-branes [8]; and it has been conjectured that M-theory is matrix model theory of D0-branes [9].

The noncommutative structures of space-time in the Planckian scales have been considered in various contexts; however, in relation to duality and D-branes of string theory, an explicit realization of a noncommutative space-time was given for the bound state of parallel D-branes [10]; and is an important ingredient in the matrix model proposed for M-theory [1].

Although this approach to noncommutative structure of space-time is distinct [3] from the celebrated mathematical formulation of noncommutative geometry by A. Connes [11] as e.g. applied to the standard model of electro-weak interactions (though may be dual to it), it is however tempting to apply Connes’ methods to problems of string theory. Also in the context of string field theory noncommutative geometry has been utilized [12]. Aside from the standard model, these methods have been used in a number of areas [13, 14].

As a first step in this project we try to show that the generalized geometrical concepts (such as manifold, differential structure, metric, distances,...) of noncommutative geometry can be used as a natural framework for studying substructure of strings. We consider a point particle on a specific multi-world-line manifold and formulate its dynamics in Connes’ framework. In a certain limit of a particular region of the space of configurations of the theory we recover the usual bosonic string theory. We argue that higher \((p)\) dimensional extended objects are also derivable from other regions of the configuration space.

We must emphasize that although the space-time coordinates of our particle are not strictly speaking noncommutative (they are diagonal matrices), yet by putting a noncommutative differential structure on the geometric space, we may find non-trivial results; it is the application of the general noncommutative framework of Connes to the simplest example of a free particle that is our main interest here which we hope to generalize to more complicated cases.

The advantages of our approach to discretization of strings are as follow:

1- It is manifestly relativistically covariant (no need to go to the light cone gauge).

2-There is no need for an ad hoc potential for the inter-particle interactions. The required potential is a consequence of the natural differential structure put on the geometric space.

3-The Virasoro constraints, which signal the enhancement of conformal symmetry in
the continuum limit of the discrete strings, follow directly from the formalism.

2 Formalism

To begin with let us recall the action of a free relativistic particle,

\[ S = -m \int d\tau \sqrt{-\dot{x}^2}, \quad (1) \]

where \( \tau \) is a label for the world-line. From the definition of momentum, \( p_\mu = \partial L/\partial \dot{x}^\mu \), the constraint,

\[ p^2 + m^2 = 0, \quad (2) \]

follows. The Hamiltonian of the system is identically zero,

\[ H_0 = p\dot{x} - L = 0. \quad (3) \]

For our purposes, it is convenient to linearize the action (1) by introducing a metric on the world-line and obtain,

\[ S = \frac{1}{2} \int e \, d\tau (e^{-2} \dot{x}^2 - m^2), \quad (4) \]

where \( e^{-2} \) is the single component of the metric \((g^{\tau\tau})\). The equation of motion for \( e \) yields,

\[ -\partial L/\partial e = e^{-2} \dot{x}^2 + m^2 = 0, \quad (5) \]

which is interpreted as a constraint.

Let us reformulate the above in terms of Connes’ noncommutative geometric (ncg) framework. In this framework a manifold, in this case the world-line of the particle, \((M)\) is replaced by an algebra \(A\), (in general the algebra of the smooth functions on the manifold, as it can be shown that the space of smooth functions on a manifold, uniquely determines the manifold). The differential structure is given by a Dirac operator \(D\) acting on a Hilbert space \(H\). In the simple case we choose it to be \(L^2(M)\), the set of all square integrable functions on \(M\), on which a representation of the algebra also acts. All the calculations are to be understood in the sense of the chosen Hilbert space. In the above case of an ordinary particle, the algebra is simply the algebra of functions on the real line,

\[ A = C^\infty(M), \quad (6) \]

and the natural Dirac operator is

\[ D = ie^{-1} \partial_\tau, \quad (7) \]

here \( e \) is the one dimensional position dependent Dirac \(\gamma\)-matrix ( obeying the Clifford algebra \([e^{-1}, e^{-1}]_+ = 2e^{-2} = 2g^{\tau\tau}\)). The coordinates of a particle are functions (scalars) in the sense of world-line manifold; thus they belong to the algebra \(A\). So we introduce
the $x^\mu(\tau)$’s as coordinates, also as representation of $\mathcal{A}$ which are operators on the Hilbert space $\mathcal{H}$. Then, the action (4) in the language of noncommutative geometry becomes,

$$S = \kappa \text{Tr}_\omega \left( \left[ D, x^\mu \right] \left[ D, x^\mu \right]^* - m^2 \right) \mid D \mid^{-1},$$

(8)

where $\kappa$ is a constant and $\text{Tr}_\omega$ is the Dixmier trace defined as below for a (compact) operator $T$

$$\text{Tr}_\omega(T) = \lim_{N \to \infty} \frac{1}{\log N} \sum_{j=1}^{N} (t_1 + t_2 + \ldots + t_j)$$

and $t$’s are absolute values of the eigenvalues of $T$ arranged in a decreasing sequence. One can show that (8) is equivalent to (4) \cite{11, 13}.

We will now present our generalisation of the free particle which as we will show is capable of describing discretized extended objects, in particular discretized strings. We take the more general algebra

$$\mathcal{A} = C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M}) \oplus \ldots \oplus C^\infty(\mathcal{M}) = C^\infty(\mathcal{M}) \otimes \mathbb{Z}_N$$

(9)

which is geometrically equivalent \cite{3} with $N$ copies of the manifold $\mathcal{M}$ (in our case world-line manifold). For its representation we take,

$$X^\mu = \text{diag} \left( x^\mu_1(\tau), x^\mu_2(\tau), \ldots, x^\mu_N(\tau) \right)$$

(10)

What is nontrivial is the choice of the Dirac operator which is an $N \times N$ hermitian matrix. At a particular region of the possible ‘configurations’ of this matrix we take \cite{4, 14}.

$$D = \begin{pmatrix}
    i e_1^{-1} \partial_\tau & \Phi / \sqrt{2} & 0 & \ldots & 0 \\
    \Phi / \sqrt{2} & i e_2^{-1} \partial_\tau & \Phi / \sqrt{2} & 0 & \ldots & 0 \\
    0 & \Phi / \sqrt{2} & i e_3^{-1} \partial_\tau & \Phi / \sqrt{2} & 0 & \ldots & 0 \\
    \begin{array}{c}
    \vdots
    \end{array} \\
    0 & \ldots & 0 & \Phi / \sqrt{2} & i e_N^{-1} \partial_\tau
\end{pmatrix}$$

(11)

or,

$$D_{ij} = i e_i^{-1} \partial_\tau \delta_{ij} + \frac{\Phi}{\sqrt{2}} (\delta_{i,j-1} + \delta_{i,j+1}),$$

(12)

where $\Phi$ is a constant. Defining a generalized determinant of the metric \cite{14} as,

$$e = \text{diag} \left( e_1, e_2, \ldots, e_N \right),$$

(13)

one can write the action in the form similar to that of the standard free particle eq.(8)

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3 More precisely they are topologically equivalent, so one can, as we will do, put different metrics on them.

4 A more general form of Dirac operator in 1-dimension is $D = e^{-1} i \partial_\tau + \Phi(\tau)$ where $e^{-1}$ is the inverse of the matrix (13) and $\Phi(\tau)$ is a hermitian $N \times N$ matrix.
\[ S = \frac{1}{2} \int d\tau \text{tr} \left( e ([D, X^\mu][D, X_\mu]^* - m^2 1_N) \right), \quad (14) \]

or explicitly as,

\[ S = \frac{1}{2} \int d\tau \sum_{j=1}^{N} \left( e_j^{-1} \dot{x}_j^2 - e_j \frac{\Phi^2}{2} \left( x_{j,j-1}^2 + x_{j,j+1}^2 + m^2 \right) \right) \quad (15) \]

where \( x_{k,l} = x_k - x_l \).

Let us now restrict ourselves to the case of open chains and we use following boundary conditions (the same which one put for open strings),

\[ x_0 - x_1 = x_N - x_{N+1} = 0; \quad (16) \]

closed chains can be similarly treated.

The equation of motion for \( e \)'s is interpreted as constraints,

\[ T_j \equiv -\partial L/\partial e_j = e_j^{-2} \dot{x}_j^2 + \frac{\Phi^2}{2} x_{j,j-1}^2 + \frac{\Phi^2}{2} x_{j,j+1}^2 + m^2 = 0. \quad (17) \]

By solving for \( e \)'s and inserting in the action we find the Nambu-Goto like action,

\[ S = -\int d\tau \sum_{j=1}^{N} \sqrt{-\dot{x}_j^2 \left( \frac{\Phi^2}{2} \left( x_{j,j-1}^2 + x_{j,j+1}^2 \right) + m^2 \right)}. \quad (18) \]

The momenta,

\[ p_{j\mu} \equiv \partial L/\partial \dot{x}_j^\mu = \frac{\dot{x}_j \sqrt{\frac{\Phi^2}{2} \left( x_{j,j-1}^2 + x_{j,j+1}^2 \right) + m^2}}{\sqrt{-\dot{x}_j^2}}, \quad (19) \]

now satisfy the constraints,

\[ p_j^2 + \frac{\Phi^2}{2} \left( x_{j,j-1}^2 + x_{j,j+1}^2 \right) + m^2 = 0, \quad (20) \]

which are the same as in eq.(17). The Hamiltonian of the system is again identically zero; we therefore follow Dirac’s recipe for a constrained system. First we introduce a set of functions \( \lambda_n \) as a linear combination of constraints \( T_j \) of eq.(17). For convenience we set \( m = 0 \),

\[ \lambda_n = (1/N) \sum_{j=1}^{N} \cos \left( \frac{n\pi (j - 1/2)}{N} \right) T_j, \quad (21) \]

\[ 0 \leq n \leq N - 1. \]

Because of the vanishing of the Hamiltonian, the Poisson brackets are satisfied and we only have to ensure closure of the algebra of constraints. Now the total Hamiltonian is

\[ H = H_0 + v_n \lambda_n + u_m \eta_m, \quad (22) \]
where \(\eta's\) are secondary constraints and \(v's\) and \(u's\) are arbitrary constants which we choose them as follows:

\[
u_n = \frac{1}{2} \delta_{n0};
\]

so we have

\[
H = \frac{1}{2} \lambda_0 = \frac{1}{2} \sum_{j=1}^{N} (p_j^2 + \Phi^2 x_{j-1,j}^2),
\]

The equations of motion,

\[
\dot{x}_j^\mu = [H, x_j^\mu]_{PB},
\]

\[
\dot{p}_j^\mu = [H, p_j^\mu]_{PB};
\]

lead to,

\[
\ddot{x}_j^\mu + \Phi^2 (2x_j^\mu - x_{j+1}^\mu - x_{j-1}^\mu) = 0,
\]

with the open boundary conditions, \(x_0 = x_1, x_{N+1} = x_N\).

The solution is,

\[
x_j^\mu(\tau) = x_0^\mu + \frac{V_0^\mu}{N+1} + \frac{i}{\sqrt{N+1}} \sum_{n \neq 0} \int \frac{1}{\omega_n} \alpha_n^\mu e^{-i\omega_n \tau} \cos\left(\frac{n\pi(j-1/2)}{N}\right),
\]

\(1 \leq j \leq N\)

and satisfying \([x_j^\mu, p_k^\nu]_{PB} = -\delta_{jk}\eta^{\mu\nu}\) yeilds the Poisson brackets,

\[
[x_0^\mu, \alpha_n^\nu]_{PB} = i\delta_{m,-n}\eta^{\mu\nu}\omega_n,
\]

\[
[x_0^\mu, V_0^n]_{PB} = -\eta^{\mu\nu},
\]

\[
[x_0^\mu, \alpha_n^\nu]_{PB} = [\alpha_n^\mu, x_0^n]_{PB} = 0,
\]

\[
\omega_n = 2\Phi \sin\left(\frac{n\pi}{2N}\right).
\]

Then,

\[
\lambda_n = \left(1 + \cos\left(\frac{n\pi}{2N}\right)\right) \sum_{m} (\alpha_m, \alpha_{m-n} + \alpha_m, \alpha_{m+n})
\]

\[
+ \left(1 - \cos\left(\frac{n\pi}{2N}\right)\right) \sum_{m} (\alpha_m, \alpha_{m-n} + \alpha_m, \alpha_{m+n})
\]

\(0 \leq n \leq N - 1\)

The closure of the constraints for finite \(N\) is a complicated problem. We will instead study a simple limit. As is well known \([11, 13]\), in ncg the inverse of the off diagonal elements of the Dirac operator, \(\text{note that their dimensions are length}^{-1}\), estimate the distance between the individual parts of the geometric space, which in our case becomes
the distance between the nearby world-line manifolds. We can therefore define the following as the continuum limit (joining many world-line manifolds to produce a higher dimensional manifold),

\[ N \to \infty, \quad \Phi \to \infty, \quad \Phi/N = 1/\pi \]

Then eq. (30) becomes (for large \( N \) and long wavelengths with respect to \( \Phi^{-1}, n \ll N \))

\[ \lambda_n = L_n + L_{-n}, \quad (31) \]

\[ 0 \leq n \]

where L’s are Virasoro constraints \( (\alpha_0 \sim V_0) \). When we take the Poisson bracket of \( \lambda_n, (n \neq 0) \) with \( \lambda_0 \), we find the secondary constraints,

\[ \eta_n = [\lambda_n, \lambda_0]_{PB} = i n (L_n - L_{-n}). \quad (32) \]

Combining (31), (32) we find all the Virasoro constraints as the constraints of our system,

\[ L_n = 0, \quad \forall n \in \mathbb{Z}. \quad (33) \]

which obey the classical algebra

\[ [L_m, L_n]_{PB} = i (m - n) L_{m+n} \]

The continuum limit of (27), where the term in the parenthesis divided by \( \Phi^{-2} \) acts as the second derivatives, is

\[ \ddot{x}_\mu - x_\mu'' = 0, \quad (34) \]

and the corresponding action is:

\[ S = 1/2 \int (\dot{x}^2 - x'^2) \, d\tau d\sigma, \quad (35) \]

and because \( N.\Phi^{-1} = \pi, \sigma \in [0, \pi] \).

The action (35) with the Virasoro constraints, which appeared as the natural constraints of the dynamical system, is equivalent to the classical Nambu string.

In the above description we have not specified the nature of the target space as the intrinsic differential structure we have built on the world-line does not require any further specification than that the 'coordinates' \( x_\mu \) must represent the algebra. We think this is an advantage of the noncommutative formulation. Yet in the spirit of the original application of these methods to the standard model [13], we may interpret the formalism as a multi-layered world-line or multi-layered target space. The supersymmetric string is also easily derivable from a similar noncommutative supersymmetric free particle and will be presented elsewhere.

This formalism also allows description of higher \( (p) \) dimensional extended objects. In fact the Dirac operator of the theory has a large number of possible configurations over
which one has to sum in a quantum path integral, of which that of eq.(12) which led to
the string is only one example.

To get other higher \((p)\) dimensional extended objects we must allow for other non-zero
entries in the matrix for the Dirac operator. Depending on which entries are set equal to
zero different objects, including \(p\) dimensional extended objects appear at large \(N\)
limit. For example an appropriate\(^5\) set of non-zero off diagonal constant matrix elements of \(D\)
results in the action for a \(p\) dimensional extended object,

\[
S = \frac{1}{2} \int dt \sum_{j_1, \ldots, j_p=1}^{N_1, \ldots, N_p} \left( e_{j_1 \ldots j_p}^{-1} \dot{x}_{j_1 \ldots j_p}^2 - e_{j_1 \ldots j_p} \frac{\Phi^2}{2} \sum_{i=1}^p (x_{j_i,j_i+1}^2 + x_{j_i,j_i-1}^2) \right),
\]

where \(x_{j_i,j_i \pm 1} = x_{j_1 \ldots j_i \pm 1 \ldots j_p} - x_{j_1 \ldots j_i \ldots j_p}\) and \(N = N_1 N_2 \ldots N_p\). Here \(j_1 \ldots j_p\) are an appropriate
one-to-one map from \(\mathbb{Z}\) to \(\mathbb{Z}^p\). Again equations of motion of \(e_{j_1 \ldots j_p}\)'s give \(N\) constraints \(T_{j_1 \ldots j_p}\).

Its continuum limit is,

\[
S = \frac{1}{2} \int (e^{-1} \dot{x}^2 - e \sum_{i=1}^p x_i^2) \, d\tau d^p \sigma,
\]

where \(x_i = \partial x / \partial \sigma_i\).

Equation of motion of \(e\) is:

\[
- \partial L / \partial e = e^{-2} \dot{x}^2 + \sum_{i=1}^p x_i^2 = p^2 + \sum_{i=1}^p x_i^2 = 0,
\]

and one can define \(\lambda\)'s as below:

\[
\lambda_{m_1 \ldots m_p} = (1/\pi)^{p} \int_0^\pi d^p \sigma \left( p^2 + \sum_{i=1}^p x_i^2 \right) \cos(m_1 \sigma_1) \ldots \cos(m_p \sigma_p),
\]

\[m_i \in \mathbb{Z}^+ \cup \{0\},\]

Again the Hamiltonian can be chosen as a linear combination of \(\lambda\)'s and we take:

\[
H = 1/2 \lambda_{0 \ldots 0},
\]

which leads to the equation of motion,

\[
\ddot{x}^\mu - \sum_{i=1}^p x^\mu_{i,i} = 0
\]

As in the case of strings, closure of the constraints algebra will give new constraints
which must be added to the primary ones for the specification of the physical states [15].

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\(^5\)In general allowing the element \(D_{jk}\) of the Dirac operator to be non-zero gives a link between the
sites \(j\) and \(k\) of the lattice.
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