ON THE CAUCHY PROBLEM FOR STOCHASTIC INTEGRODIFFERENTIAL PARABOLIC EQUATIONS IN THE SCALE OF $L^p$-SPACES OF GENERALIZED SMOOTHNESS

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Abstract. Stochastic parabolic integro-differential problem is considered in the whole space. By verifying stochastic Hörmander condition, the existence and uniqueness is proved in $L_p$-spaces of functions whose regularity is defined by a scalable Levy measure. Some rough probability density function estimates of the associated Levy process are used as well.

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1. Introduction

Let $\sigma \in (0, 2)$ and $\mathfrak{M}_\sigma$ be the class of all nonnegative measures $\pi$ on $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ such that $\int |y|^2 \wedge 1 \, d\pi < \infty$ and

$$\sigma = \inf \left\{ \alpha < 2 : \int_{|y| \leq 1} |y|^\alpha \, d\pi < \infty \right\}.$$  

In addition, we assume that for $\pi \in \mathfrak{M}_\sigma$,

$$\int_{|y| > 1} |y| \, d\pi < \infty \text{ if } \sigma \in (1, 2),$$

$$\int_{R < |y| \leq R'} y \, d\pi = 0 \text{ if } \sigma = 1 \text{ for all } 0 < R < R' < \infty.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration of $\sigma$–algebras on $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ satisfying the usual conditions. Let $\mathcal{R}(\mathbb{F})$ be the progressive $\sigma$–algebra on $[0, \infty) \times \Omega$. Let $(U, \mathcal{U}, \Pi)$ be a measurable space with $\sigma$–finite measure $\Pi, \mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$. Let $p(dt, dz)$ be $\mathbb{F}$–adapted point measures on $([0, \infty) \times U, \mathcal{B}(\{0, \infty\}) \otimes \mathcal{U})$ with compensator $\Pi (dv) dt$. We denote the martingale measure $q(dt, dz) = p(dt, dz) - \Pi (dz) dt$.

In this paper we consider the parabolic Cauchy problem

$$\begin{align*}
du(t, x) &= [Lu(t, x) - \lambda u(t, x) + f(t, x)] \, dt + \int_U \Phi(t, x, z) \, q(dt, dz), \\
u(0, x) &= g(x), t \geq 0, x \in \mathbb{R}^d,
\end{align*}$$

with $\lambda \geq 0$ and integro-differential operator

$$L \varphi(x) = L^\pi \varphi(x) = \int [\varphi(x + y) - \varphi(x) - \chi_\sigma(y) \varphi(x) \cdot \nabla \varphi(x)] \pi(dy), \varphi \in C_0^\infty(\mathbb{R}^d),$$

where $\pi \in \mathfrak{M}_\sigma, \chi_\sigma(y) = 0$ if $\sigma \in [0, 1), \chi_\sigma(y) = 1_{\{|y| \leq 1\}}(y)$ if $\sigma = 1$ and $\chi_\sigma(y) = 1$ if $\sigma \in (1, 2)$. The symbol of $L$ is

$$\psi(\xi) = \psi^\pi(\xi) = \int \left[e^{i2\pi \xi \cdot y} - 1 - i2\pi \chi_\sigma(y) \xi \cdot y\right] \pi(dy), \xi \in \mathbb{R}^d.$$

Note that $\pi(dy) = dy/|y|^{d+\sigma} \in \mathfrak{M}_\sigma$ and, in this case, $L = L^\pi = c(\sigma, d)(-\Delta)^{\sigma/2}$, where $(-\Delta)^{\sigma/2}$ is a fractional Laplacian. The equation (1.1) is forward Kolmogorov equation for the Levy process associated to $\psi^\pi$. We assume that $g, f$ and $\Phi$ are resp. $\mathcal{F}_0 \otimes \mathcal{B}(\mathbb{R}^d), \mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d), \Phi$ is $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$–measurable.

Let $\mu \in \mathfrak{M}_\sigma$ and

$$c_1 |\psi^\mu(\xi)| \leq |\psi^\pi(\xi)| \leq c_2 |\psi^\mu(\xi)|, \xi \in \mathbb{R}^d.$$
for some $0 < c_1 \leq c_2$. Given $\mu \in \mathfrak{N}, p \in [1, \infty), s \in \mathbb{R}$, we denote $H^s_p(E) = H^s_p(E)$ the closure in $L^p(E)$ of $C^\infty_0(E)$ with respect to the norm

$$|f|_{H^s_p(E)} = |\mathcal{F}^{-1} (1 - \text{Re} \psi^{\mu})^s \mathcal{F}f|_{L^p(E)},$$

where $\mathcal{F}$ is the Fourier transform in space variable. In this paper, under certain “scalability” and nondegeneracy assumptions (see assumptions $B\sigma\mu$, $L\sigma\mu\lambda$ below), we prove the existence and uniqueness of solutions to (1.1) in the scale of spaces $\mathbb{H}^{\mu,s}_p(E)$. Moreover,

$$\|u\|_{\mathbb{H}^{\mu,s}_p(E)} \leq C \left[ |f|_{H^{s-1}_p(E)} + |g|_{H^{s-\frac{1}{2}}_p(\mathbb{R}^d)} + |\Phi|_{H^{s-\frac{1}{2}}_p(E)} + |\Phi|_{H^{s-\frac{1}{2}}_{p,p,p}(E)} \right]$$

if $p \geq 2$, where $\mathbb{H}^{s}_{p,pp}$ is the Besov “counterpart” of $\mathbb{H}^{s}_p$. This paper is a continuation of [13] and [14], where (1.1) with $\Phi = 0$ was considered. Since the symbol $\psi^{\mu}(\xi)$ is not smooth in $\xi$, the standard Fourier multiplier results do not apply in this case. In order to prove the estimate involving $\Phi$ in (1.3), we follow the idea of [3], by applying a version of Calderon-Zygmund theorem by associating to $L^\pi$ a family of balls and verifying for it the stochastic Hörmander condition (see Theorem 14 in Appendix). As an example, we consider $\pi \in \mathfrak{N}$ defined in radial and angular coordinates $r = |y|, w = y/r$, as

$$\pi(\Gamma) = \int_0^\infty \int_{|w|=1} \chi_{\Gamma} \left( r w \right) a \left( r, w \right) j \left( r \right) \frac{d |w|}{\sin \sigma} S \left( dw \right) dr,$$

where $S \left( dw \right)$ is a finite measure on the unit sphere on $\mathbb{R}^d$. In [19], (1.1) with $g = 0$ and $\Phi = 0$, was considered, with $\pi$ in the form (1.4) with $a = 1, j \left( r \right) = r^{-d-\sigma}$, and such that

$$\int_0^\infty \int_{|w|=1} \chi_{\Gamma} \left( r w \right) r^{-1-\sigma} \rho_0 \left( w \right) S \left( dw \right) dr \leq \pi(\Gamma) = \int_0^\infty \int_{|w|=1} \chi_{\Gamma} \left( r w \right) r^{-1-\sigma} a \left( r, w \right) S \left( dw \right) dr$$

$$\leq \int_0^\infty \int_{|w|=1} \chi_{\Gamma} \left( r w \right) r^{-1-\sigma} S \left( dw \right) dr, \Gamma \in \mathcal{B} \left( \mathbb{R}^d_0 \right),$$

and (1.2) holds with $\psi^{\mu}(\xi) = |\xi|^\sigma, \xi \in \mathbb{R}^d$. In this case, $H^\mu_\sigma(E) = H^\sigma_\mu(E)$ is the standard fractional Sobolev space. The solution estimate (1.3) for (1.1) was derived in [19], using $L^\infty$-$\text{BMO}$ type estimate. In [7], an elliptic problem in the whole space with $L^\pi$ was studied for $\pi$ in the form (1.4) with $S \left( dw \right) = dw$ being a Lebesgue measure on the unit sphere in $\mathbb{R}^d$, with $0 < c_1 \leq a \leq c_2$, and a set of technical assumptions on $j \left( r \right)$. A sharp function estimate based on the solution Hölder norm estimate (following the idea in [3]) was used in [7].

The paper is organized as follows. In Section 2, the main theorem is stated, and some examples of the form (1.4) are considered. In Section 3, auxiliary results on approximation of input functions and some probability
density estimates are presented. In section 4, the main result is proved. In Appendix, stochastic integrals driven by jump measures are constructed and Hörmander condition discussed.

2. Notation, Function spaces main results and examples

2.1. Notation. The following notation will be used in the paper.

Let \( N = \{1, 2, \ldots\} \), \( N_0 = \{0, 1, \ldots\} \), \( \mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\} \). If \( x, y \in \mathbb{R}^d \), we write

\[
x \cdot y = \sum_{i=1}^{d} x_i y_i, \quad |x| = \sqrt{x \cdot x}.
\]

We denote by \( C_0^\infty(\mathbb{R}^d) \) the set of all infinitely differentiable functions on \( \mathbb{R}^d \) with compact support.

We denote the partial derivatives in \( x \) of a function \( u(t,x) \) on \( \mathbb{R}^d+1 \) by \( \partial_i u = \partial u / \partial x_i, \) \( \partial_{ij} u = \partial^2 u / \partial x_i \partial x_j, \) etc.; \( Du = \nabla u = (\partial_1 u, \ldots, \partial_d u) \) denotes the gradient of \( u \) with respect to \( x \); for a multiindex \( \gamma \in N^d_0 \) we denote

\[
D_{\gamma} x u(t,x) = \partial^n u(t,x),\quad \partial^\gamma u(t,x) = \partial x_{\gamma_1} \ldots \partial x_{\gamma_d} u(t,x).
\]

For \( \alpha \in (0, 2] \) and a function \( u(t,x) \) on \( \mathbb{R}^d+1 \), we write

\[
\partial^\alpha u(t,x) = -F^{-1}(|\xi|^\alpha Fu(t,\xi))(x),
\]

where

\[
F h(t,\xi) = \hat{h}(\xi) = \int_{\mathbb{R}^d} e^{-i2\pi \xi \cdot x} h(t,x) dx, \quad F^{-1} h(t,\xi) = \int_{\mathbb{R}^d} e^{i2\pi \xi \cdot x} h(t,\xi) d\xi.
\]

For \( \mu \in \mathbb{A}^\sigma \), we denote \( Z^\mu_t, t \geq 0 \), the Levy process associated to \( L^\mu \), i.e., \( Z^\mu \) is cadlag with independent increments and its characteristic function

\[
E e^{i2\pi \xi \cdot Z^\mu_t} = \exp \{ \psi^\mu(\xi) t \}, \xi \in \mathbb{R}^d, t \geq 0.
\]

The letters \( C = C(\cdot, \ldots, \cdot) \) and \( c = c(\cdot, \ldots, \cdot) \) denote constants depending only on quantities appearing in parentheses. In a given context the same letter will (generally) be used to denote different constants depending on the same set of arguments.

2.2. Function Spaces. Let \( S(\mathbb{R}^d) \) be the Schwartz space of real-valued rapidly decreasing functions. Let \( V \) be a Banach space with norm \( |\cdot|_V \). The space of \( V \)-valued tempered distribution we denote by \( S'(\mathbb{R}^d, V) \) (\( f \in S'(\mathbb{R}^d, V) \) is a continuous \( V \)-valued linear functional on \( S(\mathbb{R}^d) \)). If \( V = \mathbb{R} \), we write \( S'(\mathbb{R}^d, V) = S'(\mathbb{R}^d) \) and denote by \( \langle \cdot, \cdot \rangle \) the duality between \( S(\mathbb{R}^d) \) and \( S'(\mathbb{R}^d) \).

For a \( V \)-valued measurable function \( h \) on \( \mathbb{R}^d \) and \( p \geq 1 \) we denote

\[
|h|^p_{V,p} = \int_{\mathbb{R}^d} |h(x)|_V^p dx.
\]
We fix \( \mu \in \mathfrak{M} \). Obviously, \( \Re \psi^\mu = \psi^{\mu_{\text{sym}}} \), where

\[
\mu_{\text{sym}}(dy) = \frac{1}{2} [\mu(dy) + \mu(-dy)].
\]

Let

\[
J_v = J_\mu v = (I - L^{\mu_{\text{sym}}})v = v - L^{\mu_{\text{sym}}}v, v \in \mathcal{S}\left(\mathbb{R}^d, V\right).
\]

For \( s \in \mathbb{R} \) set

\[
J^s v = (I - L^{\mu_{\text{sym}}})^s v = \mathcal{F}^{-1}[(1 - \psi^{\mu_{\text{sym}}})^s \hat{v}], v \in \mathcal{S}\left(\mathbb{R}^d, V\right).
\]

Further, for a characterization of our function spaces we will use the following construction (see \[2\]). We fix a continuous function \( \mu \in \mathfrak{M} \) such that \( \lim_{R \to 0} \mu(R) = 0, \lim_{R \to \infty} \mu(R) = \infty \). Assume there is a nondecreasing continuous function \( l(\varepsilon), \varepsilon > 0 \), such that \( \lim_{\varepsilon \to 0} l(\varepsilon) = 0 \) and

\[
\kappa(\varepsilon r) \leq l(\varepsilon) \kappa(r), r > 0, \varepsilon > 0.
\]

We say \( \kappa \) is a scaling function and call \( l(\varepsilon), \varepsilon > 0 \), a scaling factor of \( \kappa \). Fix an integer \( N > 1 \) so that \( l(N^{-1}) < 1 \).

**Remark 1.** For an integer \( N > 1 \) there exists a function \( \phi = \phi^N \in C_0^\infty(\mathbb{R}^d) \) (see Lemma 6.1.7 in \[2\]), such that \( \text{supp} \phi = \{ \xi : \frac{1}{N} \leq \|\xi\| \leq N \} \), \( \phi(\xi) > 0 \) if \( N^{-1} < \|\xi\| < N \) and

\[
\sum_{j=-\infty}^{\infty} \phi(N^{-j} \xi) = 1 \quad \text{if} \; \phi \neq 0.
\]

Let

\[
(2.1) \quad \tilde{\phi}(\xi) = \phi(\xi) + \phi(\xi) + \phi(N^{-1} \xi), \xi \in \mathbb{R}^d.
\]
Note that \( \text{supp} \, \tilde{\phi} \subseteq \{ N^{-2} \leq |\xi| \leq N^{2} \} \) and \( \tilde{\phi} \phi = \phi \). Let \( \varphi_k = \varphi_k^N = \mathcal{F}^{-1} \phi \left( N^{-k} \cdot \right), k \geq 1, \) and \( \varphi_0 = \varphi_0^N \in \mathcal{S} \left( \mathbb{R}^d \right) \) is defined as

\[
\varphi_0 = \mathcal{F}^{-1} \left[ 1 - \sum_{k=1}^{\infty} \phi \left( N^{-k} \cdot \right) \right].
\]

Let \( \phi_0 (\xi) = \mathcal{F} \varphi_0 (\xi), \tilde{\phi}_0 (\xi) = \mathcal{F} \varphi_0 (\xi) + \mathcal{F} \varphi_1 (\xi), \xi \in \mathbb{R}^d, \tilde{\varphi} = \mathcal{F}^{-1} \tilde{\phi}, \varphi = \mathcal{F}^{-1} \varphi, \) and

\[
\tilde{\varphi}_k = \sum_{l=-1}^{1} \varphi_{k+l}, k \geq 1, \tilde{\varphi}_0 = \varphi_0 + \varphi_1
\]

that is

\[
\mathcal{F} \tilde{\varphi}_k = \phi \left( N^{-k+1} \xi \right) + \phi \left( N^{-k} \xi \right) + \phi \left( N^{-k-1} \xi \right)
\]

\[
= \tilde{\phi} \left( N^{-k} \xi \right), \xi \in \mathbb{R}^d, k \geq 1.
\]

Note that \( \varphi_k = \tilde{\varphi}_k \ast \varphi_k, k \geq 0. \) Obviously, \( f = \sum_{k=0}^{\infty} f \ast \varphi_k \) in \( \mathcal{S}' \left( \mathbb{R}^d \right) \) for \( f \in \mathcal{S} \left( \mathbb{R}^d \right) \).

Let \( s \in \mathbb{R} \) and \( p, q \geq 1. \) For \( \mu \in \mathcal{A}^\sigma \), we introduce the Besov space \( B^s_{pq} = B^s_{pq} \left( \mathbb{R}^d, V \right) \) as the closure of \( \mathcal{S} \left( \mathbb{R}^d, V \right) \) in the norm

\[
|v|_{B^s_{pq} \left( \mathbb{R}^d, V \right)} = |v|_{B^s_{pq} \left( \mathbb{R}^d, V \right)} = \left( \sum_{j=0}^{\infty} |J^s \varphi_j \ast v|^q_{L^p(\mathbb{R}^d, V)} \right)^{1/q},
\]

where \( J = J_{\mu} = I - L^{\mu_{sym}}. \)

We introduce the corresponding spaces of generalized functions on \( E = [0, T] \times \mathbb{R}^d. \) The spaces \( B^s_{pq} \left( E, V \right) \) (resp. \( H^s_{pq} \left( E, V \right) \)) consist of all measurable \( B^s_{pq} \left( \mathbb{R}^d, V \right) \) (resp. \( H^s_{pq} \left( \mathbb{R}^d, V \right) \)) -valued functions \( f \) on \( [0, T] \) with finite corresponding norms:

\[
|f|_{B^s_{pq} \left( E, V \right)} = |f|_{B^s_{pq} \left( E, V \right)} = \left( \int_0^T |f(t, \cdot)|^q_{B^s_{pq} \left( \mathbb{R}^d, V \right)} dt \right)^{1/q},
\]

\[
(2.2) \, |f|_{H^s_{pq} \left( E, V \right)} = |f|_{H^s_{pq} \left( E, V \right)} = \left( \int_0^T |f(t, \cdot)|^p_{H^s_{pq} \left( \mathbb{R}^d, V \right)} dt \right)^{1/p}.
\]

Similarly we introduce the corresponding spaces of random generalized functions.

Let \( \Omega, \mathcal{F}, (\mathbb{F}) \) be a complete probability spaces with a filtration of \( \sigma\)–algebras \( \mathbb{F} = (\mathcal{F}_t) \) satisfying the usual conditions. Let \( \mathcal{R}(\mathbb{F}) \) be the progressive \( \sigma\)–algebra on \( [0, \infty) \times \Omega. \)

The spaces \( \mathbb{B}^s_{pq} \left( \mathbb{R}^d, V \right) \) and \( \mathbb{H}^s_{pq} \left( \mathbb{R}^d, V \right) \) consists of all \( \mathcal{F}\)–measurable random functions \( f \) with values in \( B^s_{pq} \left( \mathbb{R}^d, V \right) \) and \( H^s_{pq} \left( \mathbb{R}^d, V \right) \) with finite norms

\[
|f|_{\mathbb{B}^s_{pq} \left( \mathbb{R}^d, V \right)} = \left\{ \mathbb{E} \left| f \right|_{B^s_{pq} \left( \mathbb{R}^d, V \right)}^p \right\}^{1/p},
\]

\[
|f|_{\mathbb{H}^s_{pq} \left( \mathbb{R}^d, V \right)} = \left\{ \mathbb{E} \left| f \right|_{H^s_{pq} \left( \mathbb{R}^d, V \right)}^p \right\}^{1/p}.
\]
and
\[
|f|_{B_{pp}^t(\mathbb{R}^d,V)} = \left\{ E\left| f\right|^p_{H_p^t(\mathbb{R}^d,V)} \right\}^{1/p}.
\]

The spaces $B_{pp}^t(E,V)$ and $H_p^t(E,V)$ consist of all $\mathcal{R}(\mathbb{F})$-measurable random functions with values in $B_{pp}^t(E,V)$ and $H_p^t(E,V)$ with finite norms
\[
|f|_{B_{pp}^t(E,V)} = \left\{ E\left| f\right|^p_{B_{pp}^t(E,V)} \right\}^{1/p}
\]
and
\[
|f|_{H_p^t(E,V)} = \left\{ E\left| f\right|^p_{H_p^t(E,V)} \right\}^{1/p}.
\]

If $V_r = L_r(U,\mathcal{U},\Pi)$, $r \geq 1$, the space of $r$-integrable measurable functions on $U$, and $V_0 = \mathbb{R}$, we write
\[
B_{r,pp}^t(A) = B_{pp}^t(A,V), \quad B_{r,pp}^s(A) = B_{pp}^s(A,V),
\]
\[
H_{r,p}^s(A) = H_{p}^s(A,V), \quad H_{r,p}^s(A) = H_{r,p}^s(A,V),
\]
and
\[
L_{r,p}(A) = H_{r,p}^0(A), \quad \mathbb{L}_{r,p}(A) = \mathbb{H}_{r,p}^0(A),
\]
where $A = \mathbb{R}^d$ or $E$. For scalar functions we drop $V$ in the notation of function spaces.

2.3. Main Results. We introduce an auxiliary Levy measure $\mu^0$ on $\mathbb{R}_0^d$ such that the following assumption holds.

**Assumption A**. Let $\mu^0 \in \mathfrak{A} = \cup_{\sigma \in (0,2)} \mathfrak{A}^\sigma$, $\chi_{\{|y| \leq 1\}} \mu^0(dy) = \mu^0(dy)$, and
\[
\int |y|^2 \mu^0(dy) + \int |\xi|^4 \left[ 1 + \lambda(\xi) \right]^{d+3} \exp \{-\psi_0(\xi)\} d\xi \leq n_0,
\]
where
\[
\psi_0(\xi) = \int_{|y| \leq 1} [1 - \cos (2\pi \xi \cdot y)] \mu^0(dy),
\]
\[
\lambda(\xi) = \int_{|y| \leq 1} \chi_\sigma(y) |y| \left[ |\xi| |y| \right] \mu^0(dy), \quad \xi \in \mathbb{R}^d.
\]
In addition, we assume that for any $\xi \in S_{d-1} = \{ \xi \in \mathbb{R}^d : |\xi| = 1 \}$,
\[
\int_{|y| \leq 1} |\xi \cdot y|^2 \mu^0(dy) \geq c_1 > 0.
\]
For $\pi \in \mathfrak{A} = \cup_{\sigma \in (0,2)} \mathfrak{A}^\sigma$ and $R > 0$, we denote
\[
\pi_R(\Gamma) = \int \chi_{\Gamma}(y/R) \pi(dy), \quad \Gamma \in \mathcal{B}\left(\mathbb{R}_0^d\right).
\]

**Definition 1.** We say that a continuous function $\kappa : (0,\infty) \to (0,\infty)$ is a scaling function if $\lim_{R \to 0} \kappa(R) = 0$, $\lim_{R \to \infty} \kappa(R) = \infty$ and there is a nondecreasing continuous function $l(\varepsilon), \varepsilon > 0$, such that $\lim_{\varepsilon \to 0} l(\varepsilon) = 0$ and
\[
\kappa(\varepsilon r) \leq l(\varepsilon) \kappa(r), \quad r > 0, \varepsilon > 0.
\]
We call $l(\varepsilon), \varepsilon > 0$, a scaling factor of $\kappa$.

For a scaling function $\kappa$ with a scaling factor $l$ and $\pi \in \mathcal{A}$ we introduce the following assumptions.

**D($\kappa, l$).** For every $R > 0$,
\[ \bar{\pi}_R(dy) = \kappa(R) \pi_R(dy) \geq 1_{\{|y| \leq 1\}} \mu^0(dy), \]
with $\mu^0 = \mu^{0, \pi}$ satisfying Assumption $A_0(\sigma)$. If $\sigma = 1$ we, in addition assume that $\int_{R < |y| \leq R'} y \mu^0(dy) = 0$ for any $0 < R < R' \leq 1$. Here $\pi_R(dy) = \kappa(R) \pi_R(dy)$.

**B($\kappa, l$).** There exist $\alpha_1$ and $\alpha_2$ and a constant $N_0 > 0$ such that
\[ \int_{|z| \leq 1} |z|^{\alpha_1} \bar{\pi}_R(dz) + \int_{|z| > 1} |z|^{\alpha_2} \bar{\pi}_R(dz) \leq N_0 \forall R > 0, \]
where $\alpha_1, \alpha_2 \in (0, 1]$ if $\sigma \in (0, 1)$; $\alpha_1, \alpha_2 \in (1, 2]$ if $\sigma \in (1, 2)$; $\alpha_1 \in (1, 2]$ and $\alpha_2 \in (0, 1)$ if $\sigma = 1$.

The main result for (1.1) is the following statement.

**Theorem 1.** Let $\pi, \mu \in \mathcal{A}, p \in (1, \infty), s \in \mathbb{R}$. Assume there is a scaling function $\kappa$ with a scaling factor $l$ such that $D(\kappa, l)$ and $B(\kappa, l)$ hold for both, $\pi$ and $\mu$. Assume
\[ \int_1^\infty \frac{dt}{t \gamma(t)^{1 + \alpha_2}} < \infty, \]
and there are $\beta_0 < \alpha_2$ and $\beta_1, \beta_2 > 0$ such that
\[ \int_0^1 \gamma(t)^{-\beta_1} dt + \int_0^1 l(t)^{\beta_2} \frac{dt}{t} + \int_1^\infty \frac{1}{\gamma(t)^{\beta_0}} \frac{dt}{t} < \infty \text{ if } p > 2, \]
where $\gamma(t) = \inf \{r : l(r) \geq t\}, t > 0$.

Then for each $f \in H^s_{p, \mu^s}(E), g \in H^s_{p, pp} \left( \mathbb{R}^d \right), \Phi \in H^s_{p, pp} \left( E \right) \cap H^s_{2, p} \left( \mathbb{R}^d \right)$ if $p \in (2, \infty)$ and $\Phi \in H^s_{p, pp} \left( E \right) \cap H^s_{2, p} \left( \mathbb{R}^d \right)$ if $p \in (1, 2)$, there is a unique $u \in H^s_{p, \mu^s}(E)$ solving (1.1). Moreover, there is $C = C(d, p, \kappa, l, n_0, d_0, c_1)$ such that for $p \in (2, \infty),
\begin{align*}
|L^\mu u|_{H^s_{p, \mu^s}(E)} & \leq C \left[ |f|_{H^s_{p, \mu^s}(E)} + |g|_{H^s_{p, pp} \left( \mathbb{R}^d \right)} + \left| \Phi \right|_{H^s_{p, pp} \left( \mathbb{R}^d \right)} + \left| \Phi \right|_{H^s_{2, p} \left( \mathbb{R}^d \right)} \right], \\
|u|_{H^s_{p, \mu^s}(E)} & \leq C \left[ \rho_\lambda |f|_{H^s_{p, \mu^s}(E)} + \rho_\lambda^{1/p} |g|_{H^s_{p, \mu^s}(\mathbb{R}^d)} + \rho_\lambda^{1/p} |\Phi|_{H^s_{p, p} \left( \mathbb{R}^d \right)} + \rho_\lambda^{1/2} |\Phi|_{H^s_{2, p} \left( \mathbb{R}^d \right)} \right],
\end{align*}
and for $p \in (1, 2),
\begin{align*}
|L^\mu u|_{H^s_{p, \mu^s}(E)} & \leq C \left[ |f|_{H^s_{p, \mu^s}(E)} + |g|_{H^s_{p, pp} \left( \mathbb{R}^d \right)} + \left| \Phi \right|_{H^s_{p, pp} \left( \mathbb{R}^d \right)} + \left| \Phi \right|_{H^s_{2, p} \left( \mathbb{R}^d \right)} \right], \\
|u|_{H^s_{p, \mu^s}(E)} & \leq C \left[ \rho_\lambda |f|_{H^s_{p, \mu^s}(E)} + \rho_\lambda^{1/p} |g|_{H^s_{p, \mu^s}(\mathbb{R}^d)} + \rho_\lambda^{1/p} |\Phi|_{H^s_{p, p} \left( \mathbb{R}^d \right)} + \rho_\lambda^{1/p} |\Phi|_{H^s_{2, p} \left( \mathbb{R}^d \right)} \right].
\end{align*}
where \( \rho_\lambda = \frac{1}{\lambda} \land T \).

**Remark 2.**

1. Assumptions \( \mathcal{D}(\kappa, l), \mathcal{B}(\kappa, l) \) hold for both, \( \pi, \mu, \) means that \( \kappa, l, \) and the parameters \( \alpha_1, \alpha_2, n_0, c_1, N_0 \) are the same (\( \mu^0 \) could be different).

2. For every \( \varepsilon > 0, B_{\mu, N; s, \varepsilon} (\mathbb{R}^d) \) is continuously embedded into \( H_{\mu, s} (\mathbb{R}^d), p > 1; \) for \( p \geq 2, H_{\mu, s} (\mathbb{R}^d) \) is continuously embedded into \( B_{\mu, N; s} (\mathbb{R}^d) \).

2.4. **Examples.** Let \( \Lambda (dt) \) be a measure on \((0, \infty)\) such that \( \int_0^\infty (1 \land t) \Lambda (dt) < \infty \), and let

\[
\phi (r) = \int_0^\infty (1 - e^{-rt}) \Lambda (dt), \; r \geq 0,
\]

be a Bernstein function (see [9], [7]). Let \( \rho \) be a Bernstein function (see [9], [7]). Let

\[
j (r) = \int_0^\infty (4\pi t)^{-d/2} \exp \left( -\frac{r^2}{4t} \right) \Lambda (dt), \; r > 0.
\]

We consider \( \pi \in \mathfrak{A} = \bigcup_{\sigma \in (0, 2)} \mathfrak{A}^\sigma \) defined in radial and angular coordinates

\[
r = |y|, \; w = y/r, \; as
\]

\[
(2.3) \quad \pi (r) = \int_0^\infty \int_{|w|=1} \chi_r (rw) a (r, w) j (r) r^{d-1} S (dw) dr, \; \Gamma \in \mathcal{B} (\mathbb{R}_0^d),
\]

where \( S (dw) \) is a finite measure on the unite sphere on \( \mathbb{R}^d \). If \( S (dw) = dw \) is the Lebesgue measure on the unit sphere, then

\[
\pi (r) = \pi^{Ja} (r) = \int_{\mathbb{R}^d} \chi_r (y) a (|y|, y/|y|) J (y) dy, \; \Gamma \in \mathcal{B} (\mathbb{R}_0^d),
\]

where \( J (y) = j (|y|), \; y \in \mathbb{R}^d \). Let \( \mu = \pi^{J, 1} \), i.e.,

\[
(2.4) \quad \mu (r) = \int_{\mathbb{R}^d} \chi_r (y) J (y) dy, \; \Gamma \in \mathcal{B} (\mathbb{R}_0^d).
\]

We assume

**H.** (i) There is \( N > 0 \) so that

\[
N^{-1} \phi (r^{-2}) r^{-d} \leq j (r) \leq N \phi (r^{-2}) r^{-d}, \; r > 0.
\]

(ii) There are \( 0 < \delta_1 \leq \delta_2 \leq 1 \) and \( N > 0 \) so that for \( 0 < r \leq R \)

\[
N^{-1} \left( \frac{R}{r} \right)^{\delta_1} \leq \frac{\phi (R)}{\phi (r)} \leq N \left( \frac{R}{r} \right)^{\delta_2}.
\]

**G.** There is \( \rho_0 (w) \geq 0, \; |w| = 1, \) such that \( \rho_0 (w) \leq a (r, w) \leq 1, \; r > 0, \; |w| = 1, \) and for all \( |\xi| = 1, \)

\[
\int_{|w|=1} |\xi \cdot w|^2 \rho_0 (w) S (dw) \geq c > 0
\]

for some \( c > 0 \).

For example, in [9] and [7] among others the following specific Bernstein functions satisfying \( \mathcal{H} \) are listed:

(0) \( \phi (r) = \sum_{i=1}^n r^{\alpha_i}, \; \alpha_i \in (0, 1), \; i = 1, \ldots, n; \)

(1) \( \phi (r) = (r + r^\alpha)^\beta, \; \alpha, \beta \in (0, 1); \)
(2) \( \phi (r) = r^\alpha (\ln (1 + r) )^\beta , \alpha \in (0, 1), \beta \in (0, 1 - \alpha) \);
(3) \( \phi (r) = [\ln (\cosh \sqrt{T})]^{\alpha}, \alpha \in (0, 1) \).

All the assumptions of Theorem 1 hold under \( H, G \). Indeed, \( H \) implies that there are \( 0 < c \leq C \) so that
\[
cr^{-d - 2\delta_1} \leq j (r) \leq C r^{-d - 2\delta_2}, r \leq 1,
\]
\[
2 r^{-d - 2\delta_2} \leq j (r) \leq C r^{-d - 2\delta_1}, r > 1.
\]
Hence \( 2\delta_1 \leq \sigma \leq 2\delta_2 \). In this case \( \kappa (R) = j (R)^{-1} R^{-d}, R > 0 \), is a scaling function, and \( \kappa (\varepsilon R) \leq l (\varepsilon) \kappa (R), \varepsilon, R > 0 \), with
\[
l (\varepsilon) = \begin{cases} C_1 \varepsilon^{2 \delta_1} & \text{if } \varepsilon \leq 1, \\
C_1 \varepsilon^{2 \delta_2} & \text{if } \varepsilon > 1
\end{cases}
\]
for some \( C_1 > 0 \). Hence
\[
\gamma (t) = l^{-1} (t) = \begin{cases} C_1^{-1/2 \delta_1} t^{1/2 \delta_1} & \text{if } t \leq C_1, \\
C_1^{-1/2 \delta_2} t^{1/2 \delta_2} & \text{if } t > C_1.
\end{cases}
\]
We see easily that \( \alpha_1 \) is any number \( > 2\delta_2 \) and \( \alpha_2 \) is any number \( < 2\delta_1 \). The measure \( \mu^0 \) for \( \pi \) is
\[
\mu^0 (dy) = \mu^{0, \pi} (dy) = c_1 \int \chi_{dy} (rw) \chi_{\{r \leq 1\}} r^{-1 - 2\delta_1} \rho_0 (w) S (dw) dr;
\]
and \( \mu^0 \) for \( \mu \) is
\[
\mu^0 (dy) = \mu^{0, \mu} (dy) = c_1' \int \chi_{dy} (rw) \chi_{\{r \leq 1\}} r^{-1 - 2\delta_1} dw dr
\]
with some \( c_1, c_1' \). Integrability conditions easily follow.

3. Auxiliary results

3.1. Approximation of input functions. Let \( V_r = L_r (U, U, \Pi), r \geq 1 \), the space of \( r \)-integrable measurable functions on \( U \), and \( V_0 = \mathbb{R} \). For brevity of notation we write
\[
B_{r,pp}^s (A) = B_{pp}^s (A; V_r), \quad \mathbb{B}_{r,pp}^s (A) = \mathbb{B}_{pp}^s (A; V_r),
\]
\[
H_{r,p}^s (A) = H_p^s (A; V_r), \quad \mathbb{H}_{r,p}^s (A) = \mathbb{H}_{p}^s (A; V_r),
\]
\[
L_{r,p} (A) = H_{r,p}^0 (A), \quad L_{r,p} (A) = \mathbb{H}_{r,p}^0 (A),
\]
where \( A = \mathbb{R}^d \) or \( E \). We use the following equivalent norms of Besov spaces \( B_{r,pp}^s (\mathbb{R}^d), r = 0, p \) (see [14])
\[
|v|_{B_{r,pp}^s (\mathbb{R}^d)} = \left( \sum_{j=0}^{\infty} \kappa (N-j)^{-sp} \int |\varphi_j * v|_{L_r^p}^p dx \right)^{1/p},
\]
where $\varphi_j = \varphi_j^N, j \geq 0$, is the system of functions defined in Remark 1. The equivalent norms of $H_{r,p}^s (\mathbb{R}^d)$, $r = 0, 2$, (see [14]) are defined by

$$
(3.1) \quad |v|_{H_{r,p}^s (\mathbb{R}^d)} = |v|_{H_{r,p}^s, N_s (\mathbb{R}^d)} = \left( \sum_{j=0}^{\infty} \kappa (N^{-j})^{-s} \varphi_j * v \right)^{1/2}_{L_p (\mathbb{R}^d)}.
$$

We define the equivalent norms of functions on $E$ as well:

$$
|v|_{\tilde{B}_{r,pp}^s (E)} = \left( \int_0^T |v (t)|_{\tilde{B}_{r,pp}^p (\mathbb{R}^d)}^p dt \right)^{1/p}, \quad |v|_{\tilde{H}_{r,p}^s (E)} = |v|_{\tilde{H}_{r,p}^s, N_s (E)} = \left( \int_0^T |v (t)|_{\tilde{H}_{r,pp}^p (\mathbb{R}^d)}^p dt \right)^{1/p}.
$$

For $D = D_r (A) = B_{r,pp}^s (A)$ or $H_{r,p}^s (A), A = \mathbb{R}^d, E$, we consider corresponding equivalent norms on random function spaces $D = D_r = \mathbb{R}^s_{r,pp} (A)$ or $\mathbb{H}^s_{r,p} (A) :

$$
|v|_D = \left\{ \mathbb{E} \left( |v|_D^p \right) \right\}^{1/p}.
$$

Let $U_n \in \mathcal{U}, U_n \subseteq U_{n+1}, n \geq 1, \cup U_n = U$ and $\pi (U_n) < \infty, n \geq 1$. We denote by $\mathcal{C}_{r,p}^\infty (E), 1 \leq p < \infty$, the space of all $\mathcal{R} (\mathbb{F}) \otimes \mathcal{B} (\mathbb{R}^d)$ -measurable $V_r$ -valued random functions $\Phi$ on $E$ such that for every $\gamma \in \mathbb{N}_0^d,

$$
\mathbb{E} \int_0^T \sup_{x \in \mathbb{R}^d} |D^\gamma \Phi (t,x)|_{V_r}^p dt + \mathbb{E} \left( |D^\gamma \Phi|_{L_p (E; V_r)}^p \right) < \infty,
$$

and $\Phi = \Phi \chi_{U_n}$ for some $n$ if $r = 2, p$. Similarly we define the space $\mathcal{C}_{r,p}^\infty (\mathbb{R}^d)$ by replacing $\mathcal{R} (\mathbb{F})$ and $E$ by $\mathcal{F}$ and $\mathbb{R}^d$ respectively in the definition of $\mathcal{C}_{r,p}^\infty (E)$.

**Lemma 1.** Let $D (\kappa, l)$ and $B (\kappa, l)$ hold for $\mu \in \mathbb{A}$ with scaling function $\kappa$ and scaling factor $l$. Let $U_n \in \mathcal{U}, U_n \subseteq U_{n+1}, n \geq 1, \cup U_n = U$ and $\pi (U_n) < \infty, n \geq 1$. Let $s \in \mathbb{R}, p \in (1, \infty), \Phi \in \mathbb{D}_{r,p}$, where $\mathbb{D}_{r,p} = \mathbb{D}_{r,p} (A) = \mathbb{H}_{r,pp}^s (A)$ with $r = 0, p$, or $\mathbb{D}_{r,p} = \mathbb{H}_{r,p}^s (A)$ with $r = 0, 2, A = \mathbb{R}^d$ or $E$. For $\Phi \in \mathbb{D}_{r,p}$ we set

$$
\Phi_n = \sum_{j=0}^{n} \Phi \ast \varphi_j \chi_{U_n}, \quad \text{if } r = 2, p, \quad \Phi_n = \sum_{j=0}^{n} \Phi \ast \varphi_j, \quad \text{if } r = 0.
$$

Then there is $C > 0$ so that

$$
|\Phi_n|_{\mathbb{D}_{r,p}} \leq C |\Phi|_{\mathbb{D}_{r,p}}, \quad \Phi \in \mathbb{D}_{r,p}, n \geq 1,
$$

and $|\Phi_n - \Phi|_{\mathbb{D}_{r,p}} \to 0$ as $n \to \infty$. Moreover, for $r = 0, 2, p$, every $n$ and multiindex $\gamma \in \mathbb{N}_0^d,

$$
\mathbb{E} \int_0^T \sup_{x \in \mathbb{R}^d} |D^\gamma \Phi_n|_{V_r}^p dt + |D^\gamma \Phi_n|_{L_p (E)}^p < \infty \text{ if } A = E,
$$

$$
\mathbb{E} \left( \sup_{x \in \mathbb{R}^d} |D^\gamma \Phi_n|_{V_r}^p \right) + |D^\gamma \Phi_n|_{L_p (\mathbb{R}^d)}^p < \infty \text{ if } A = \mathbb{R}^d,
$$

where
Proof. Let $\Phi_n = \Phi_{X_{U_n}}, n \geq 1$. Since

$$\varphi_k = \sum_{l=-1}^{1} \varphi_{k+l} \ast \varphi_k, k \geq 1, \varphi_0 = (\varphi_0 + \varphi_1) \ast \varphi_0,$$

we have for $n > 1$,

$$\left(\Phi_n - \Phi_n\right) \ast \varphi_k = 0, k < n,$$
$$\left(\Phi_n - \Phi_n\right) \ast \varphi_k = \left(\Phi_n \ast \varphi_{k-1} + \Phi_n \ast \varphi_k + \Phi_n \ast \varphi_{k+1}\right) \ast \varphi_k, k > n + 1,$$
$$\left(\Phi_n - \Phi_n\right) \ast \varphi_n = \left(\Phi_n \ast \varphi_{n+1}\right) \ast \varphi_n,$$
$$\left(\Phi_n - \Phi_n\right) \ast \varphi_{n+1} = \left(\Phi_n \ast \varphi_{n+1} + \Phi_n \ast \varphi_{n+2}\right) \ast \varphi_{n+1}.$$

Let $V_r = L_r(U, \mathcal{U}, \Pi), r = 2, p$. By Corollary 2 in [14], there is a constant $C$ independent of $\Phi \in \mathbb{H}_{2,p}^X(E)$ so that

$$\left|\left(\sum_{j=0}^{\infty} \kappa \left(N^{-j}\right)^{-s} \left(\Phi_n - \Phi_n\right) \ast \varphi_j \right)^{1/2}\right|_{L_p(E)} \leq C \left(\sum_{j=n}^{\infty} \left(\kappa \left(N^{-j}\right)^{-s} \Phi \ast \varphi_j \right)^{1/2}\right)_{L_p(E)} \to 0$$

as $n \to \infty$. Obviously

$$\left|\left(\Phi_n - \Phi_n\right) \ast \varphi_j\right|_{L_{r,p}(E)} \leq C \sum_{k=j-1}^{j+1} \left|\Phi_n \ast \varphi_k\right|_{L_{r,p}(E)}, j \geq n,$$

$$\left|\left(\Phi_n - \Phi_n\right) \ast \varphi_j\right|_{L_{r,p}(E)} = 0, j < n, r = 0, p,$$

and

$$|\Phi_n|_{D_{r,p}(E)} \leq C |\Phi|_{D_{r,p}(E)}, \Phi \in D_{r,p}, n \geq 1, r = 0, 2, p.$$

Thus $|\Phi_n - \Phi|_{D_{r,p}(E)} \to 0$ as $n \to \infty, r = 0, 2, p$.

Let $r = 0, 2, p, \Phi \in D_{r,p}(E)$. Obviously, for any $k \geq 0$,

$$\mathbb{E} \int_E |\Phi \ast \varphi_k|^p_{V_r} \, dx dt < \infty,$$

where $r = 0, 2, p$ with $V_0 = \mathbb{R}$. Since for any multiindex $\gamma$,

$$\Phi \ast \varphi_k = \Phi \ast \varphi_k \ast \varphi_k, D^\gamma \Phi \ast \varphi_k = \Phi \ast \varphi_k \ast D^\gamma \varphi_k,$$
and $\mathbb{P}$-a.s. for all $s, x$, with $\frac{1}{q} + \frac{1}{p} = 1$,

$$|D^γ \Phi \ast \varphi_k (s, x)|_{V_r} \leq \int |\Phi \ast \varphi_k (s, x - y)|_{V_r} |D^γ \varphi_k (y)| \, dy,$$

$$\sup_x |D^γ \Phi \ast \varphi_k (s, x)|_{V_r} \leq \left( \int |\Phi \ast \varphi_k (s, \cdot)|_{V_r}^p \, dx \right)^{1/p} |D^γ \varphi_k|_{L_q (\mathbb{R}^d)},$$

we have for any multiindex $γ$,

$$\sup_x |D^γ \Phi \ast \varphi_k|_{L_r, p (E)} < \infty,$$

and

$$\mathbb{E} \int_0^T \sup_x |D^γ \Phi \ast \varphi_k|_{V_r} \, dt < \infty, r = 0, 2, p.$$  

The proof for the case of $A = \mathbb{R}^d$ is a repeat with obvious changes. The statement follows. □

**Corollary 1.** The space $C_0^\infty (\mathbb{R}^d, V_r)$ of $V_r$-valued infinitely differentiable functions with compact support is dense in $D_r (\mathbb{R}^d)$, $r = 0, 2, p$.

**Proof.** In the view of Lemma 1 it suffices to show that for any $V = V_r$-valued function $v$ such that for all multiindex $γ \in \mathbb{N}_0^d$,

$$\sup_x |D^γ v (x)|_{V_r} + |D^γ v|_{L_p (\mathbb{R}^d, V_r)} < \infty$$

there exists $v_n \in C_0^\infty (\mathbb{R}^d, V_r)$ so that $v_n \to v$ in $D_r (\mathbb{R}^d)$. Let $g \in C_0^\infty (\mathbb{R}^d)$ with $0 \leq g (x) \leq 1, x \in \mathbb{R}^d$, $g (x) = 1$ for $|x| \leq 1$, and $g (x) = 0$ for $|x| \geq 2$. Let

$$v_n (x) := v (x) g (x/n) , x \in \mathbb{R}^d.$$

Obviously $v_n \in C_0^\infty (\mathbb{R}^d, V_r)$, and for any multiindex $β$,

$$D^β v_n (x) = D^β v (x/n) + \sum_{\beta_1 + \beta_2 = \beta, |\beta_2| \geq 1} n^{-|\beta_2|} |D^β_1 v (x) \left( D^β g \right) (x/n), x \in \mathbb{R}^d,$$

$$\left| D^β v_n \right|_{L_p (\mathbb{R}^d, V_r)} \leq C (|\beta|) \sup_{|\beta'| \leq |\beta|} \left| D^β v \right|_{L_p (\mathbb{R}^d, V_r)},$$

and $\left| D^β v_n - D^β v \right|_{L_p (\mathbb{R}^d, V_r)} \to 0$. Since for any multiindex $β$ we have $\int g^β \varphi_j (y) \, dy = 0$, it follows for $m > 0, j \geq 1$, by Taylor remainder theorem, for $x \in \mathbb{R}^d$, 

$$\int g^β \varphi_j (y) \, dy = 0,$$

and $\left| D^β v_n - D^β v \right|_{L_p (\mathbb{R}^d, V_r)} \to 0$.
\[ v_n \ast \varphi_j (x) = \int \varphi_j (y) \left\{ v_n (x - y) - \sum_{\beta : |\beta| \leq m} \frac{D^\beta v_n (x)}{\beta!} (-y)^\beta \right\} dy \]
\[ = \int \varphi_j (y) \sum_{\beta : |\beta| = m + 1} \int_0^1 \frac{(1-t)^{m+1}}{(m+1)!} \left( D^\beta v_n \right) (x - ty) (-y)^\beta dt dy \]
\[ = N^{-j(m+1)} \sum_{\beta : |\beta| = m + 1} \int \varphi (y) \int_0^1 \frac{(1-t)^{m+1}}{(m+1)!} \left( D^\beta v_n \right) (x - tN^{-j} y) dt (-y)^\beta dy. \]

By Lemma 6 of [14], there exists \( \sigma' \) such that \( \kappa \left( N^{-j} \right)^{-s} \leq N^{j\sigma'} \). Let \( m > 1 \) be such that \( t = N^{\sigma'} N^{-m} < 1 \). Hence there is a constant \( C = C (m) \) (independent of \( n \)) so that
\[ \kappa \left( N^{-j} \right)^{-s} \left| v_n \ast \varphi_j \right|_{L_p (\mathbb{R}^d; \nu^2)} \leq C (m) t^j, \quad j \geq 0. \]

Now, for any \( k \geq 0 \),
\[ \left| \sum_{j=k}^{\infty} \left| \kappa \left( N^{-j} \right)^{-s} v_n \ast \varphi_j (x) \right|_2 \right|_{L_p (\mathbb{R}^d)} \leq \sum_{j=k}^{\infty} \left| \kappa \left( N^{-j} \right)^{-s} v_n \ast \varphi_j (x) \right|_{L_p (\mathbb{R}^d; \nu^2)} \leq C (m) \sum_{j=k}^{\infty} t^j. \]

Since the same estimate holds for \( v \),
\[ \left| \sum_{j=k}^{\infty} \left| \kappa \left( N^{-j} \right)^{-s} v \ast \varphi_j (x) \right|_2 \right|_{L_p (\mathbb{R}^d)} \leq C (m) \sum_{j=k}^{\infty} t^j, \]
and
\[ \left| \sum_{j<k} \left| \kappa \left( N^{-j} \right)^{-s} (v - v_n) \ast \varphi_j (x) \right|_2 \right|_{L_p (\mathbb{R}^d)} \rightarrow 0, \]
it follows that
\[ \left| \sum_j \left| \kappa \left( N^{-j} \right)^{-s} (v - v_n) \ast \varphi_j (x) \right|_2 \right|_{L_p (\mathbb{R}^d)} \rightarrow 0 \]
as \( n \rightarrow \infty \).
Likewise, for \( r = 0, p, \)

\[
\lim_{n \to \infty} |v_n - v|_{B^s_{r,pp}} = \lim_{n \to \infty} \left( \sum_{j=0}^{\infty} \left| \kappa \left( N^{-j} \right) ^{-s} (v_n - v) * \varphi_j \right|_{L_p(\mathbb{R}^d, V_r)} \right)^{1/p} = 0.
\]

An obvious consequence of Lemma 1 (the form of the approximating sequence is identical for different \( V \)) is the following

**Lemma 2.** Let \( p \geq 1 \) and \( s, s' \in \mathbb{R} \). Then the set \( \tilde{C}_p(E) \) is a dense subset in \( \mathbb{B}^{s'}_{pp}(E) \), \( \tilde{C}_p(E) \) is a dense subset of \( \mathbb{B}^{s'}_{pp}(\mathbb{R}^d) \), and \( \tilde{C}_p^{\infty}(E) \) is dense in \( \mathbb{H}^{r}_{pp}(E), r = 0, 2 \). Moreover, the set \( \tilde{C}_p^{\infty}(E) \cap \tilde{C}_p^{\infty}(E) \) is a dense subset of \( \mathbb{B}^{s'}_{pp}(E) \cap \mathbb{H}^{r}_{pp}(E) \).

3.2. Representation of fractional operator and some density estimates. We will use repeatedly the following representation of the fractional operator. Let \( \mu \in \mathfrak{M}_{sym} = \{ \eta \in \mathfrak{M} : \eta \) is symmetric, \( \eta = \eta_{sym} \} \), then for \( \delta \in (0, 1) \) and \( f \in S(\mathbb{R}^d) \), we have

\[
- (\psi^\mu(\xi))^{\delta} \hat{f}(\xi) = c_3 \int_{\mathbb{R}^d} t^{-\delta} \left[ \exp(\psi^\mu(\xi) t) - 1 \right] \frac{dt}{t} \hat{f}(\xi), \xi \in \mathbb{R}^d,
\]

and

\[
(3.2) \quad L^{\mu, \delta} f(x) = \mathcal{F}^{-1} \left[ - (\psi^\mu)^{\delta} \hat{f} \right](x)
= c_3 \mathbb{E} \int_{0}^{\infty} t^{-\delta} \left[ f(x + Z_t^\mu) - f(x) \right] \frac{dt}{t}, x \in \mathbb{R}^d.
\]

**Lemma 3.** Let \( \mu \in \mathfrak{M}_{sym}, \delta \in (0, 1) \).

a) For any \( p \geq 1, \epsilon > 0 \) there is \( C \) so that

\[
\left| L^{\mu, \delta} f \right|_{L_p(\mathbb{R}^d)} \leq \epsilon |f|_{L_p(\mathbb{R}^d)} + C |f|_{L_p(\mathbb{R}^d)}, f \in S(\mathbb{R}^d).
\]

b) Let \( D(\kappa, l) \) and \( B(\kappa, l) \) hold for \( \pi \in \mathfrak{M} \) with scaling function \( \kappa \) and scaling factor \( l \), and

\[
\int_{|y| \leq 1} |y|^{\alpha_1} d\mu_R + \int_{|y| > 1} |y|^{\alpha_2} d\mu_R \leq M, R > 0
\]

\((\alpha_1, \alpha_2 \) are exponents in \( B(\kappa, l) \)). Let \( p^R(t, x) = p^\pi R(t, x), x \in \mathbb{R}^d \), be the pdf of \( Z_t^\pi, t > 0, R > 0 \). Then for each \( \beta \in (0, 2) \) there is \( C = C(\kappa, l, N_0, \beta) \) \((N_0 \) is a constant in \( B(\kappa, l) \)) so that for \( |k| \leq 2 \),

\[
\int \left( 1 + |x|^{\beta} \right) \left| D^k L^{\mu, \delta} p^R(1, x) \right| dx \leq CM,
\]

\[
\int \left( 1 + |x|^{\beta} \right) \left| L^{\mu, \delta} L^{\pi, \delta} p^R(1, x) \right| dx \leq CM.
\]
Lemma 4. Let function \( f \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d \), by Ito formula and (3.2),

\[
L^{\mu;\delta} f(x) = c \mathbb{E} \int_0^t t^{-\delta} \int_0^t L^\mu f(x + Z^\mu_t) \, dt \, \frac{dt}{t} + c \mathbb{E} \int_0^\infty t^{-\delta} \left[ f(x + Z^\mu_t) - f(x) \right] \, dt \frac{dt}{t}.
\]

The statement a) follows by Minkowski inequality.

By (3.3), for any \( L \) where \( a > 0, f \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d \), by Ito formula and (3.2),

\[
\int \left(1 + |x|^{\beta} \right) \left| L^{\mu;\delta} D^k p^R(1,x) \right| \, dx 
\leq C \left( \int_0^1 t^{-\delta} \int_0^t \left(1 + |x|^{\beta} \right) \left| L^\mu D^k p^R(1,x + Z^\mu_t) \right| \, dx \, dr \frac{dt}{t} + C \int_0^\infty t^{-\delta} \int_0^t \mathbb{E} \left( |Z^\mu_t|^{\beta} \right) \, dx \, dt \frac{dt}{t}. \]

So, \( 1 + a > 0, f \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d \), by Ito formula and (3.2),

\[
\int \left(1 + |x|^{\beta} \right) \left| L^{\mu;\delta} D^k p^R(1,x) \right| \, dx 
\leq C, \quad \text{and}
\]

\[
A_1 \leq C \int_0^1 t^{-\delta} \int_0^t \left(1 + |x|^{\beta} \right) \left| L^\mu D^k p^R(1,x) \right| \, dx \, dt + C \int_0^\infty t^{-\delta} \mathbb{E} \left( |Z^\mu_t|^{\beta} \right) \, dx \, dt \frac{dt}{t} 
\leq C, \quad \text{and}
\]

\[
A_2 \leq C \int_1^\infty t^{-\delta} \int_0^t \left(1 + |x|^{\beta} \right) \left| D^k p^R(1,x) \right| \, dx \, dt + C \int_1^\infty t^{-\delta} \mathbb{E} \left( |Z^\mu_t|^{\beta} \right) \, dx \, dt \frac{dt}{t} 
\leq C \left(1 + \int_1^\infty t^{-\delta} dt \right) \frac{dt}{t} \leq C.
\]

Similarly, the second inequality of part b) is proved.

\( \square \)

Let \( \mathfrak{A}^\sigma_{\text{sign}} = \mathfrak{A}^\sigma - \mathfrak{A}^\sigma = \{ \eta - \rho : \eta, \rho \in \mathfrak{A}^\sigma \} \).

Lemma 4. Let \( \delta \in (0,1), D(\kappa,1) \) and \( B(\kappa,1) \) hold for \( \pi \in \mathfrak{A}^\sigma \) with scaling function \( \kappa \) and scaling factor 1. Let \( \mu \in \mathfrak{A}^\sigma_{\text{sym}} \) and \( \eta \in \mathfrak{A}^\sigma_{\text{sign}} \). Then

\[
p^\pi(t,x) = a(t)^{-d} p^\pi_a(t) \left( 1, xa(t)^{-1} \right), x \in \mathbb{R}^d, t > 0,
\]

\[
L^{\mu;\delta} p^\pi(t,x) = \frac{1}{t^\delta} a(t)^{-d} (L^{\mu;\delta} p^\pi_a(t)) \left( 1, xa(t)^{-1} \right), x \in \mathbb{R}^d, t > 0,
\]

\[
L^\eta L^{\mu;\delta} p^\pi(t,x) = \frac{1}{t^{\delta + 1}} a(t)^{-d} (L^{\mu;\delta} p^\pi_a(t)) \left( 1, xa(t)^{-1} \right), x \in \mathbb{R}^d, t > 0,
\]

where \( a(t) = \inf \{ r \geq 0 : \kappa(r) \geq t \}, t > 0 \).
Proof. Indeed, by Lemma 5 in \[13\], for each \( t > 0 \) and \( r > 0 \), the density \( p^{\tilde{\alpha}_{a(t)}}(r,x) \), \( x \in \mathbb{R}^d \), is 4 times continuously differentiable in \( x \) bounded and integrable. Obviously,

\[
\exp \{ \psi^n (\xi) t \} = \exp \{ \psi^{\tilde{\alpha}_{a(t)}}(a(t) \xi) \}, \quad t > 0, \xi \in \mathbb{R}^d,
\]

and

\[
(-\psi^\mu (\xi))^\delta \exp \{ \psi^n (\xi) t \} = \frac{1}{t^\delta} \left(-\psi^{\tilde{\alpha}_{a(t)}}(a(t) \xi)\right)^\delta \exp \{ \psi^{\tilde{\alpha}_{a(t)}}(a(t) \xi) \}, \quad t > 0, \xi \in \mathbb{R}^d.
\]

We derive the first two equalities by taking Fourier inverse. Similarly, the third equality can be derived. The claim follows. \( \square \)

**Lemma 5.** Let \( \delta \in (0,1) \), \( D(\kappa, l) \) and \( B(\kappa, l) \) hold for \( \pi \in \mathcal{A}^\sigma \) with scaling function \( \kappa \) and scaling factor \( l \). Let \( \mu \in \mathcal{A}^\sigma_{sym} \). Assume

\[
\int_{|y| \leq 1} |y|^\alpha_1 d\tilde{\mu}_R + \int_{|y| > 1} |y|^\alpha_2 d\tilde{\mu}_R \leq M, \quad R > 0
\]

(\( \alpha_1, \alpha_2 \) are exponents in \( B(\kappa, l) \)). Then there exists \( C = C(\kappa, l, N_0) > 0 \) such that for \( |k| \leq 2, \beta \in [0, \delta \alpha_2] \),

\[
\int_{|x| > c} \left| L^{\mu, \delta} D^k p^\pi (t, x) \right| dx \leq CM^{-\delta a(t)}(t)^{\beta - |k|} c^{-\beta},
\]

\[
\int_{|x| > c} \left| L^{\mu, \delta} D^k p^{\tilde{\alpha}_{a(t)}}(t, x) \right| dx \leq CM^{-\delta (\beta - |k|)},
\]

with \( a(t) = \inf \{ r > 0 : \kappa(r) \geq t \} \), \( t > 0 \). Recall \( \alpha_1, \alpha_2 \in (0,1] \) if \( \sigma \in (0,1) \); \( \alpha_1, \alpha_2 \in (1,2] \) if \( \sigma \in (1,2) \) and \( \alpha_2 \in (0,1), \alpha_1 \in (1,2) \) if \( \sigma = 1 \). \( \square \)

**Proof.** Indeed, by Lemma \[14\] Chebyshev inequality, and Lemma \[23\] for \( |k| \leq 2, \beta \in [0, \delta \alpha_2] \),

\[
\int_{|x| > c} \left| L^{\mu, \delta} D^k p^\pi (t, x) \right| dx
= \frac{1}{t^\delta} a(t)^{-d-k} \int_{|x| > c} \left| L^{\tilde{\alpha}_{a(t)}} D^k p^{\tilde{\alpha}_{a(t)}} \left( 1, \frac{x}{a(t)} \right) \right| dx
\leq \frac{a(t)^{\beta - k} c^{-\beta}}{t^\delta} \int_{|x| > c} \left| L^{\tilde{\alpha}_{a(t)}} D^k p^{\tilde{\alpha}_{a(t)}}(1, x) \right| dx \leq CM a(t)^{\beta - k} c^{-\beta}.
\]

Similarly, we derive the second estimate. \( \square \)

**Lemma 6.** Let \( D(\kappa, l) \) and \( B(\kappa, l) \) hold for \( \pi \in \mathcal{A}^\sigma \) with scaling function \( \kappa \) and scaling factor \( l \). Let \( \mu \in \mathcal{A}^\sigma_{sym} \). Assume

\[
\int_{|y| \leq 1} |y|^\alpha_1 d\tilde{\mu}_R + \int_{|y| > 1} |y|^\alpha_2 d\tilde{\mu}_R \leq M, \quad R > 0
\]

(\( \alpha_1, \alpha_2 \) are exponents in \( B(\kappa, l) \)). Then for \( \delta \in (0,1) \),

\[
\int_{|x| > c} \left| L^{\mu, \delta} D^k p^\pi (t, x) \right| dx \leq CM^{-\delta a(t)}(t)^{\beta - |k|} c^{-\beta},
\]

\[
\int_{|x| > c} \left| L^{\mu, \delta} D^k p^{\tilde{\alpha}_{a(t)}}(t, x) \right| dx \leq CM^{-\delta (\beta - |k|)}.
\]
a) There exists $C = C(\kappa, l, N_0) > 0$ such that

$$
\int_{\mathbb{R}^d} \left| L^{\mu^\delta} p^\pi(t, x - y) - L^{\mu^\delta} p^\pi(t, x) \right| \, dx \leq CM \frac{|y|}{t^{\beta a}(t)}, \quad t > 0, y \in \mathbb{R}^d,
$$

where $a(t) = \inf \{ r : \kappa(r) \geq t \}, t > 0$.

b) There is a constant $C = C(\kappa, l, N_0)$ such that

$$(3.4) \quad \int_{2a}^\infty \left( \int |L^{\mu^\delta} p^\pi(t - s, x) - L^{\mu^\delta} p^\pi(t, x)| \, dx \right)^2 \, dt$$

$$
\leq CM, \quad |s| \leq a < \infty.
$$

**Proof.** By Lemma $4$ and Lemma $3$

$$
\int_{\mathbb{R}^d} \left| L^{\mu^\delta} p^\pi(t, x - y) - L^{\mu^\delta} p^\pi(t, x) \right| \, dx
$$

$$
= \frac{1}{t^\delta} \int \left| L^{\mu^\delta} p^\pi(1, x - \frac{y}{a(t)}) - L^{\mu^\delta} p^\pi(1, x) \right| \, dx
$$

$$
\leq \frac{1}{t^\delta} \int_0^1 \left| \nabla L^{\mu^\delta} p^\pi(1, x - s \frac{y}{a(t)}) \right| \frac{|y|}{a(t)} \, dxds
$$

$$
\leq C \frac{|y|}{t^{\beta a}(t)} \int |L^{\mu^\delta} \nabla p^\pi(1, x)| \, dx \leq CM \frac{|y|}{t^{\beta a}(t)}.
$$

Similarly, we derive the estimate $(3.4)$. By Lemma $4$ and Lemma $3$

$$
\int_{2a}^\infty \left( \int \left| L^{\mu^\delta} p^\pi(t - s, x) - L^{\mu^\delta} p^\pi(t, x) \right| \, dx \right)^2 \, dt
$$

$$
\leq |s|^2 \int_{2a}^\infty \left( \int_0^1 \left| L^{\mu^\delta} L^{\mu^\delta} p^\pi(t - rs, x) \right| \, dxdr \right)^2 \, dt
$$

$$
\leq C |s|^2 \int_{2a}^\infty \left( \int_0^1 \frac{dr}{(t - rs)^{1 + \frac{\gamma}{2}}} \right)^2 \, dt \leq C \int_{2a}^\infty \frac{1}{(t - s)^{\frac{1}{2}}} - \frac{1}{t^{\frac{1}{2}}} \, dt
$$

$$
\leq C |s| \int_{2a}^\infty \frac{dt}{(t - s) t} = C \int_{2a}^\infty \frac{1}{t} dt \leq C.
$$

4. PROOF OF THE MAIN THEOREM

We split the proof into several steps. First we derive the existence of smooth solutions for the equation with smooth input functions. Then we prove the main estimate for them by verifying Hörmander condition. At the end we extend the estimates and regularity result for general input functions.
4.1. Existence and uniqueness of solution for smooth input functions. Let \( \pi \in \mathfrak{S} \), and \( Z_t = Z_t^\pi, t \geq 0 \), be the Levy process associated to it. Let \( P_t(dy) \) be the distribution of \( Z_t^\pi, t > 0 \), and for a measurable \( f \geq 0 \),

\[
T_t f(x) = \int f(x + y) P_t(dy), (t, x) \in E.
\]

For the representation of the solution to \( (1.1) \) we will use the following operators:

\[
T_t^\lambda g(x) = e^{-\lambda t} \int g(x + y) P_t(dy), (t, x) \in E, \quad g \in \tilde{C}_0^\infty(\mathbb{R}^d), p > 1,
\]

\[
R_\lambda f(t, x) = \int_0^t e^{-\lambda(t-s)} \int f(s, x + y) P_{t-s}(dy) ds, (t, x) \in E, \quad f \in \tilde{C}_0^\infty(E), p > 1,
\]

and

\[
\tilde{R}_\lambda \Phi(t, x) = \int_0^t e^{-\lambda(t-s)} \int \Phi(s, x + y, z) P_{t-s}(dy) q(ds, dz), (t, x) \in E,
\]

\[
\Phi \in \tilde{C}_{2,p}(E) \cap \tilde{C}_{p,p}(E) \text{ if } p \geq 2,
\]

\[
\Phi \in \tilde{C}_{p,p}(E) \text{ if } p \in (1, 2).
\]

First we present some simple estimates of \( T_t^\lambda g, R_\lambda f, \tilde{R}_\lambda \Phi \).

**Lemma 7.** The following estimates hold for any multiindex \( \gamma \):

(i) \( \mathbb{P} \)-a.s.

\[
|D^\gamma T^\lambda g|_{L_p(E)} \leq \rho_\lambda \rho_\mathbb{P} |D^\gamma g|_{L_p(\mathbb{R}^d)}, \quad g \in \tilde{C}_0^\infty(\mathbb{R}^d), p \geq 1,
\]

\[
|D^\gamma R_\lambda f|_{L_p(E)} \leq \rho_\lambda |D^\gamma f|_{L_p(E)}, \quad f \in \tilde{C}_0^\infty(E), p \geq 1,
\]

and

\[
|D^\gamma R_\lambda f(t, \cdot)|_{L_p(\mathbb{R}^d)} \leq \int_0^t |D^\gamma f(s, \cdot)|_{L_p(\mathbb{R}^d)} ds, t \geq 0,
\]

\[
|T_t^\lambda g|_{L_p(\mathbb{R}^d)} \leq e^{-\lambda t} |g|_{L_p(\mathbb{R}^d)}, t \geq 0, p \geq 1;
\]

(ii) For each \( p \geq 2 \),

\[
|D^\gamma \tilde{R}_\lambda \Phi|_{L_p(E)}^p \leq C \rho_\mathbb{P} \rho_\lambda \int_0^T |D^\gamma \Phi(s, \cdot)|_{L_{2,p}(\mathbb{R}^d)}^p ds + \rho_\lambda |D^\gamma \Phi|_{L_{p,p}(E)}^p,
\]

\[
\Phi \in \tilde{C}_{2,p}(E) \cap \tilde{C}_{p,p}(E),
\]

and for each \( p \in (1, 2) \),

\[
|D^\gamma \tilde{R}_\lambda \Phi|_{L_p(E)}^p \leq C \rho_\lambda |D^\gamma \Phi|_{L_{p,p}(E)}^p, \quad \Phi \in \tilde{C}_{p,p}(E),
\]

\[
(1.1)
\]
where \( \rho_\lambda = T \wedge \frac{1}{\lambda} \). Moreover,

\[
\left| D^\gamma \tilde{R}_\lambda \Phi (t, \cdot) \right|_{L_p(R^d)}^p \\
\leq C \left\{ E \left[ \left( \int_0^t |D^\gamma \Phi (s, \cdot)|^2_{L_2(R^d)} ds \right)^{p/2} \right] + E \int_0^t |D^\gamma \Phi (s, \cdot)|^p_{L_{p,p}(R^d)} ds \right\},
\]

if \( p \geq 2 \), and

\[
\left| D^\gamma \tilde{R}_\lambda \Phi (t, \cdot) \right|_{L_p(R^d)}^p \leq C E \int_0^t |D^\gamma \Phi (s, \cdot)|^p_{L_{p,p}(R^d)} ds, t > 0,
\]

if \( p \in (1, 2) \).

**Proof.** The estimates (i) follow from Lemma 15 in [14] and Lemma 8 in [13]. Let \( p \geq 2, \Phi \in \mathcal{C}^\infty_{\text{loc}} (E) \cap \mathcal{C}^\infty_{p,p} (E) \). Recall \( \Phi = \Phi_{|U_n} \) for some \( U_n \in \mathcal{U} \) with \( \pi (U_n) < \infty \). Obviously, for any multiindex \( \gamma, (t, x) \in E \),

\[
D^\gamma \tilde{R}_\lambda \Phi (t, x) = \int_0^t e^{-\lambda(t-s)} \int_U D^\gamma \Phi (s, x+y, z) P_{t-s} (dy) q (ds, dz)
\]

\[
= \int_0^t e^{-\lambda(t-s)} \int_U D^\gamma \Phi (s, x+y, z) P_{t-s} (dy) p (ds, dz)
\]

\[
- \int_0^t e^{-\lambda(t-s)} \int_U D^\gamma \Phi (s, x+y, z) P_{t-s} (dy) \pi (dz) ds.
\]

By Kunita’s inequality (see [12, 16]), for \( t > 0 \),

\[
E \int |D^\gamma \tilde{R}_\lambda \Phi (t, x)|^p dx \\
\leq C E \left( \int_0^t e^{-2\lambda(t-s)} \left( \int_U |D^\gamma \Phi (s, x+y, z)|^2 \pi (dz) \right) P_{t-s} (dy) ds \right)^{p/2} dx \\
+ C E \int_0^t e^{-\lambda(t-s)} |T_{t-s} D^\gamma \Phi (s, x, z)|^p \pi (dz) ds dx
\]

\[
= B (t) + D (t).
\]

By Fubini theorem and Minkowski inequality,

\[
D (t) \leq C E \int_0^t e^{-\lambda(t-s)} |D^\gamma \Phi (s, \cdot)|^p_{L_{p,p}(R^d)} ds, t > 0,
\]

and

\[
B (t) \leq C E \left[ \left( \int_0^t e^{-2\lambda(t-s)} |D^\gamma \Phi (s, \cdot)|^2_{L_2(R^d)} ds \right)^{p/2} \right]
\]

\[
= C E \left[ \left( \int_0^t e^{-2\lambda(t-s)} |D^\gamma \Phi (s, \cdot)|^2_{L_2(R^d)} ds \right)^{p/2} \right]
\]

\[
\leq C \left( \frac{1}{\lambda} \right)^{p/2} E \left[ \int_0^t 2\lambda e^{-2\lambda(t-s)} |D^\gamma \Phi (s, \cdot)|^p_{L_{2,p}(R^d)} ds \right].
\]
Now,
\[
\int_0^T D(t) \, dt \leq C p \lambda E \int_0^T |D^\gamma \Phi (s, \cdot)|_L_{p,p}(\mathbb{R}^d) \, ds
\]
and
\[
\int_0^T B(t) \, dt \leq C p \lambda E \int_0^T |D^\gamma \Phi (s, \cdot)|_L_{2,p}(\mathbb{R}^d) \, ds.
\]

Similarly we consider the case \( p \in (1, 2) \). □

**Lemma 8.** For \( \mu \in \mathfrak{M} \), let \( f \in \tilde{C}_{0,p}^\infty (E), g \in \tilde{C}_{p,p}^\infty (\mathbb{R}^d), \Phi \in \tilde{C}_{2,p}^\infty (E) \cap \tilde{C}_{p,p}^\infty (E) \) for \( p \in [2, \infty) \) and \( \Phi \in \tilde{C}_{2,p}^\infty (E) \) for \( p \in (1, 2) \), then there is unique \( u \in \tilde{C}_{0,p}^\infty (E) \) solving (1.1). Moreover,

\[
u (t, x) = T_t \gamma (x) + R_x f (t, x) + \tilde{R}_x \Phi (t, x), (t, x) \in E,
\]
and \( u_1 (t, x) = T_t \gamma (x), (t, x) \in E, \) solves (1.1) with \( f = 0, \Phi = 0, u_2 = R_x f \) solves (1.1) with \( g = 0, \Phi = 0, \) and \( u_3 = \tilde{R}_x \Phi \) solves (1.1) with \( g = 0, f = 0. \)

**Proof.** Uniqueness is a simple repeat of the proof of Lemma 8 in [13]. We prove that \( u_1, u_2 \) solve the corresponding equations by repeating the proofs of Lemma 8 in [13] and Lemma 15 in [14]. Let \( \Phi \in \tilde{C}_{2,p}^\infty (E) \cap \tilde{C}_{p,p}^\infty (E) \) if \( p \in [2, \infty) \) or \( \Phi \in \tilde{C}_{p,p}^\infty (E) \) if \( p \in (1, 2) \). Recall that \( \Phi = \Phi \chi_{U_n} \) for some \( U_n \in \mathcal{U} \) with \( \pi (U_n) < \infty \). Let \( v = u_3 = \tilde{R}_x \Phi \). A simple application of Ito formula and Fubini theorem show that \( \mathbb{P} \)-a.s.

\[
v (t, x) = \int_0^t e^{-\lambda (t-s)} \int_U \Phi (s, x + y, z) P_{t-s} (dy) q (ds, dz)
\]
\[
= \int_0^t \int_U \Phi (s, x, z) q (ds, dz) + \int_0^t \int_s^t e^{-\lambda (r-s)} ×
\times \int_U \left[ L^\pi \Phi (s, x + y, z) - \lambda \Phi (s, x + y, z) \right] P_{t-s} (dy) dq (ds, dz)
\]
\[
= \int_0^t \int_U \Phi (s, x, z) q (ds, dz) + \int_0^t [L^\pi v (s, x) - \lambda v (s, x)] ds, (t, x) \in E.
\]

□

5. **Main estimate**

Let

\[G^\lambda_{s,t} (x) = \exp (\lambda (t-s)) p^\pi (t-s, x), 0 < s < t, x \in \mathbb{R}^d,\]
where \( \pi^* (dy) = \pi (dy), \) and

\[u (t, x) = \int_0^t \int_U G^\lambda_{s,t} * \Phi (s, x, \nu) q (ds, d\nu), (t, x) \in E.\]

The main estimate for the solution with smooth input functions is the following statement.
Lemma 9. Let $\pi, \mu \in \mathfrak{M}^\sigma$. Assume there is a scaling function $\kappa$ with a scaling factor $l$ such that $D(\kappa, l)$ and $\beta(\kappa, l)$ hold for both, $\pi$ and $\mu$. Let $\Phi \in \tilde{C}_2^\infty(E) \cap \tilde{C}^\infty(E)$, for $p \in [2, \infty)$ and $\Phi \in \tilde{C}_p^\infty(E)$ for $p \in (1, 2)$. Assume

$$\int_1^\infty \frac{1}{\gamma(t)^{\beta_0}} \frac{dt}{t} < \infty$$

for some $\beta_0 < \alpha_2$. Then

$$\|L^\mu u\|_{L_p(E)} \leq C \left( \|L^\mu \frac{\Phi}{\tilde{C}_2, p(E)} \|_{L_2, p(E)} + \|\Phi\|_{1, \frac{1}{p}, p(E)} \right), \; p \in [2, \infty)$$

$$\|L^\mu u\|_{L_p(E)} \leq C \|\Phi\|_{1, \frac{1}{p}, p(E)}, \; p \in (1, 2),$$

where $C = C(\kappa, l, p, d)$.

5.0.1. Proof of Lemma 9. Let

$$G_{s,t}^\lambda(x) = \exp(-\lambda(t-s)) p^{\pi^*(t-s,x)} \chi_{[\varepsilon, \infty]}(t-s), 0 < s < t, x \in \mathbb{R}^d,$$

where $\pi^*(dy) = \pi(-dy)$. Denote for $\varepsilon > 0,$

$$Q(t,x) = \mathbb{E} \left[ \left| \int_0^t \int_{\mathbb{R}^d} \tilde{\Phi}_\varepsilon(s,x,\nu) q(ds, d\nu), (t,x) \right|_{L_p(E)} \right],$$

$$\tilde{\Phi}_\varepsilon(s,x,\nu) = \mathbb{E} \left[ \left| \int \left( L^\mu G_{s,t}^{\lambda, \varepsilon} \right)(x-y) \Phi(s,y,\nu) dy, (s,x) \right|_{L_p(E)} \right].$$

Obviously, with $K_\lambda^\varepsilon(t,x) = e^{-\lambda t} L^{\pi^*} p^{\pi^*(t,x)} \chi_{[\varepsilon, \infty]}(t), t > 0, x \in \mathbb{R}^d,$ we have

\begin{equation}
\mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \int \left( L^\mu G_{s,t}^{\lambda, \varepsilon} \right)(x-y) \Phi(s,y,\nu) dy ds d\nu \right]_{L_p(E)}^p
= \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \int L^{\mu,1/2} G_{s,t}^{\lambda, \varepsilon} (x-y) L^{\mu,1/2} \Phi(s,y,\nu) dy ds d\nu \right]_{L_p(E)}^p
= \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \int K_\lambda^\varepsilon(t-s,x-y) L^{\mu,1/2} \Phi(s,y,\nu) dy ds d\nu \right]_{L_p(E)}^p.
\end{equation}

If $2 \leq p < \infty$, then

$$\mathbb{E} \int_0^T |Q(t,\cdot)|_{L_p(\mathbb{R}^d)}^p dt$$

$$\leq \mathbb{E} \left\{ \int_0^T \left[ \int_0^t \int_{\mathbb{R}^d} \int \left( \tilde{\Phi}_\varepsilon(s,\cdot,\nu) \right)^2 \Pi(d\nu) ds \right]^{1/2} dt \right\}^{1/2}$$

$$+ \mathbb{E} \left\{ \int_0^T \int_{\mathbb{R}^d} \int \left| \tilde{\Phi}_\varepsilon(s,\cdot,\nu) \right|_{L_p(\mathbb{R}^d)}^p \Pi(d\nu) ds dt \right\} = C (\mathbb{E}I_1 + \mathbb{E}I_2).$$
If $1 < p < 2$, then by Lemma 9 (see also Remark 1 therein) in [17],

$$
E \int_0^T |Q(t, \cdot)|_{L_p^p(\mathbb{R}^d)}^p dt \leq C E I_2.
$$

Estimate of $E I_2$. Let $B_t^\lambda g(x) = e^{-\lambda t} E g(x + Z_t^\epsilon), (t, x) \in E, g \in \tilde{C}^\infty_{0,p} (\mathbb{R}^d)$. Then

$$
I_2 = \int_0^T \int_0^t \int_U |\hat{\Phi}_\epsilon(s, \cdot, \nu)|_{L_p^p(\mathbb{R}^d)}^p (dv) dstdt
= \int_0^T \int_0^t \int_U \left( (L^\mu G_{s, t}^{\lambda, \epsilon}) \ast \Phi (s, \cdot, \nu) \right)_{L_p^p(\mathbb{R}^d)}^p (dv) dstdt
\leq \int_0^T \int_0^t \int_U L^\mu B_{t-s}^\lambda \Phi (s, \cdot, \nu)_{L_p^p(\mathbb{R}^d)}^p (dv) dstdt
= \int_U \int_0^T \int_s^T \left| L^\mu B_{t-s}^\lambda \Phi (s, \cdot, \nu) \right|_{L_p^p(\mathbb{R}^d)}^p dtds (dv)
$$

It follows from Proposition 1 (see section 4.4 as well) of [13] that for $p > 1$,

$$
E I_2 \leq E \int_U \int_0^T \int_s^T \left| L^\mu B_{t-s}^\lambda \Phi (s, \cdot, \nu) \right|_{L_p^p(\mathbb{R}^d)}^p (dv) dstds (dv)
\leq C E \int_U \int_0^T \sum_{j=0}^\infty (N^{-j})^{-(1-1/p)} \left| \Phi (s, \cdot, \nu) \ast \varphi_j \right|_{L_p^p(\mathbb{R}^d)}^p (dv) ds (dv)
= C |\Phi|_{L_p^p, N^{-1-1/p}(E)}^p.
$$

Estimate of $E I_1$. It is enough to show that

$$
I = \int \int \left\{ \int \left| K_\lambda^s (t-s, \cdot) \ast \Phi (s, \cdot) (x) \right|_{L_2^2}^2 ds \right\}^{\frac{p}{2}} dx dt
\leq C \int \int \left( \left| \Phi (t, x) \right|_{L_2^2}^2 \right)^{p/2} dx dt, \ \Phi \in \tilde{C}^\infty_{2,p} (\mathbb{R}^{d+1}).
$$

where $V_2 = L_2 (U, U, \Pi)$, $C$ is independent of $\epsilon$ and $\Phi$.

For $p = 2$, by Plancherel’s theorem and Fubini’s theorem, denoting $\hat{\Phi} = \mathcal{F} \Phi$,

$$
I = \int_{\mathbb{R}^{d+1}} \int \left| L^\mu \frac{1}{2} G_{s, t}^{\lambda, \epsilon} \ast \Phi (s, x) \right|_{L_2^2}^2 ds dx dt
= \int_0^T \int_0^{t-\epsilon} \int_U \exp \{2 (\psi^\epsilon (\xi) - \lambda) (t-s) \} \left| \psi^\mu (\xi) \right| \left| \mathcal{F} \Phi (s, \xi, z) \right|_{L_2^2}^2 \Pi (dz) dstd\xi dt
\leq C \int_0^T \int \int_U \left| \mathcal{F} \Phi (s, \xi, z) \right|_{L_2^2}^2 \Pi (dz) dstd\xi dx = C \int_0^T \int \int_U \left| \Phi (s, x, z) \right|_{L_2^2}^2 \Pi (dz) dstdx
$$
Hence (5.2) follows for \( p = 2 \).

Next we prove (5.2) for \( p > 2 \). According to Lemma 14 (see Appendix), it is sufficient to show that there exists \( C_0 > 0 \) such that for all \( |s| \leq \kappa (\delta), |y| \leq \delta, \delta > 0, \) we have

\[
I = \int \chi_{Q_{C_0 \delta}(0)} |K^\xi(t - s, x - y) - K^\xi(t, x)| \, dx \, dt \leq N,
\]

where \( Q_{C_0 \delta}(0) = (-\kappa(C_0 \delta), \kappa(C_0 \delta)) \times \{ x : |x| < C_0 \delta \} \).

Verification of Hörmander condition (5.3). Let

\[
a(r) = \inf \{ t : \kappa(t) \geq r \}, r > 0,
\]

\[
a^{-1}(s) = \inf \{ t : a(t) \geq s \}, s > 0,
\]

\[
\gamma(t) = \inf \{ r : l(r) \geq t \}, t > 0.
\]

It follows from Lemma 9 in [14] that

\[
a^{-1}(r) = \sup_{s \leq r} \kappa(s) \leq l(1) \kappa(r), r > 0,
\]

\[
a^{-1}(r \varepsilon) \leq l(\varepsilon) a^{-1}(r), \varepsilon, r > 0.
\]

and

\[
a(\varepsilon r) \geq a(r) \gamma(\varepsilon), r, \varepsilon > 0.
\]

In particular, \( \gamma(\varepsilon) \leq a(\varepsilon) a(1)^{-1} \), and

\[
a(r) a(r') \leq \gamma \left( \frac{r'}{r} \right)^{-1}, r, r' > 0.
\]

Let \( C_0 > 3 \) and \( 3l(1) l(C_0^{-1}) < 1 \). Now, we follow the splitting in [8]. Let

\[
I = \int_{-\infty}^{2|s|} \left[ \int \ldots \right]^2 dt + \int_{2|s|}^{\infty} \left[ \int \ldots \right]^2 dt = I_1 + I_2.
\]

Since \( \kappa(C_0 \delta) > 3\kappa(\delta), \delta > 0, \) it follows by Lemma 9 in [14] (see (5.4), and Lemma 5 with \( k_0 = C_0 - 1 \) and \( \beta \in (\frac{\beta_0}{2}, \frac{\beta}{2}) \),

\[
|I_1| \leq C \int_0^{3|s|} \left[ \int_{|x| > k_0 a(|s|)} \left| L^{\mu + \frac{2}{3} \varepsilon^*} (t, x) \right| \, dx \right]^2 dt
\]

\[
\leq C \int_0^{3|s|} \left( t^{-\frac{2}{3} a(t)} (k_0 a(|s|))^{-\beta} \right)^2 dt \leq C \int_0^{3|s|} \left( \frac{a(t)}{a(s)} \right)^{2\beta} \frac{dt}{t}
\]

\[
\leq C \int_0^{3|s|} \frac{1}{\gamma(t)^{2\beta}} \frac{dt}{t} \leq C \int_{1/3}^{\infty} \frac{1}{\gamma(t)^{2\beta}} \frac{dt}{t}
\]
Now,

\[ |\mathcal{I}_2| \leq 2 \int_2^\infty \left( \int_{2|s|} \left| \chi_{Q_{C_0\delta}(0)} (t-s,x-y) - L^{\mu_1} \chi_{\{y,0\}} (t-s,x) \right| \, dx \right)^2 \, dt + \int_2^\infty \left\{ \int_{2|s|} \left| \chi_{Q_{C_0\delta}(0)} (t-s) L^{\mu_1} \chi_{\{y,0\}} (t-s,x) \right| \, dx \right\}^2 \, dt \]

We split the estimate of \( \mathcal{I}_{2,1} \) into two cases.

Case 1. Assume \( |y| \leq a(2|s|) \). Then by Lemma 6(b),

\[ \mathcal{I}_{2,1} \leq C \int_2^\infty \frac{|y|^2}{(t-s)a(t-s)^2} \, dt \leq C |y|^2 a(2|s|)^{-2} \int_2^\infty \frac{a(2|s|)^2}{a(t-s)^2} (t-s)^{-1} \, dt \leq C \int_2^\infty \frac{\gamma(t-s)^{-2}}{a(t-s)^2} (t-s)^{-1} \, dt \leq C \int_2^\infty \frac{\gamma(t-s)^{-2}}{a(t-s)^2} (t-s)^{-1} \, dt \leq C \int_1^\infty \frac{\gamma(r)^{-2}}{r} \, dr . \]

Case 2. Assume \( |y| > a(2|s|) \) i.e. \( \delta \geq |y| > a(2|s|) \) and \( a^{-1}(\delta) \geq a^{-1}(|y|) \geq 2|s| \). We split

\[ \mathcal{I}_{2,1} = \int_2^{|2|s|+a^{-1}(|y|)} \left[ \int \ldots \right]^2 + \int_2^{|2|s|+a^{-1}(|y|)} \left( \int \ldots \right)^2 = \mathcal{I}_{2,1,1} + \mathcal{I}_{2,1,2} . \]

If \( 2|s| \leq t \leq 2|s| + a^{-1}(|y|) \), then \( 0 \leq t \leq 3a^{-1}(\delta) \leq 3l(1) \kappa(\delta) \leq \kappa(C_0\delta) \). Hence, \( |x| > C_0\delta \geq a(2|s|) + |y| \) and

\[ |x-y| \geq (C_0-1) \delta = k_0 \delta \geq \frac{k_0}{2} \left[ a(2|s|) + |y| \right] \geq a(2|s|) + |y| \text{ if } (t,x) \notin Q_{C_0\delta}(0) . \]

Also,

\[ 2 \geq \frac{2|s| + a^{-1}(|y|)}{2|s| + a^{-1}(|y|) - s} \geq \frac{2}{3} \]

and by Lemma 9 of [4],

\[ \frac{a(2|s| + a^{-1}(|y|))}{a(2|s|) + |y|} \leq \frac{a(2a^{-1}(|y|))}{a(2|s|) + |y|} \leq \gamma(2^{-1})^{-1} \frac{a(a^{-1}(|y|))}{a(2|s|) + |y|} \leq \gamma(2^{-1})^{-1} . \]
Hence, with \(\beta \in \left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)\), by Lemma 9 in [14] (see (5.4)), Lemma 5, (5.5) and (5.6),

\[
I_{2,1,1} \leq C \int_{2|s|}^{2|s|+a^{-1}(|y|)} \left[ \frac{C}{(a(2|s|)+|y|)^{2\beta}} \int_{2|s|}^{2|s|+a^{-1}(|y|)} \frac{a(t-s)^{2\beta}}{a(2|s|+a^{-1}(|y|))^{2\beta}} \right] dt
\]

Then by Lemma (6a) and (5.6),

\[
I_{2,1,2} \leq C \int_{2|s|+a^{-1}(|y|)}^{\infty} \left[ \frac{C}{(a(2|s|)+|y|)^{2\beta}} \int_{2|s|+a^{-1}(|y|)}^{\infty} \frac{a(t-s)^{2\beta}}{a(2|s|+a^{-1}(|y|))^{2\beta}} \right] dt
\]

because

\[
\frac{|y|}{a(2|s|+a^{-1}(|y|))} \leq \frac{a(a^{-1}(|y|))+}{a(2|s|+a^{-1}(|y|))} \leq 1.
\]

Hence, \(I_{2,1} \leq C\). Since

\[
I_{2,2} \leq \int_{2|s|}^{\infty} \left[ \frac{C}{(a(2|s|)+|y|)^{2\beta}} \int_{2|s|+a^{-1}(|y|)}^{\infty} \frac{a(t-s)^{2\beta}}{a(2|s|+a^{-1}(|y|))^{2\beta}} \right] dt
\]

it follows by Lemma (6b) that

\[
I_{2,2,1} \leq C.
\]
Clearly, $I_{2.2.2} \leq C$ as well. Hence (5.3) and (5.2) are proved. Combining the estimates of $I_1$ and $I_2$, we deduce that the claim of Lemma 9 holds.

5.1. Proof of Theorem 1. We finish the proof of Theorem 1 in a standard way. Since by Proposition 1 and 2 of [14], $J^t_\mu : H^\mu_{p,p}(R^d, l_2) \to H^\mu_{p,p}([0, T] \times R^d, l_2)$ and $J^s_\mu : H^\mu_{p,p}([0, T] \times R^d, l_2) \to H^\mu_{p,p}([0, T] \times R^d, l_2)$ is an isomorphism for any $s, t \in R$, it is enough to derive the statement for $s = 0$. Let $f \in L_p(E), g \in H^\mu_{p,p}(R^d)$, and

$$\Phi \in H^\mu_{p,p}(E) \cap H^\mu_{2,p}(E) \text{ if } p \geq 2,$$

$$\Phi \in H^\mu_{p,p}(E) \text{ if } p \in (1, 2).$$

According to Lemma 2 there are sequences $f_n \in C^\infty_{0,p}(E), g_n \in C^\infty_{0,p}(R^d), \Phi_n \in C^\infty_{2,p}(E) \cap C^\infty_{p,p}(E)$ if $p \geq 2$, and $\Phi_n \in C^\infty_{p,p}(E)$ if $p \in (1, 2)$, such that

$$f_n \to f \text{ in } L_p(E), g_n \to g \text{ in } H^\mu_{p,p}(R^d),$$

and

$$\Phi_n \to \Phi \text{ in } H^\mu_{p,p}(E) \cap H^\mu_{2,p}(E) \text{ if } p \geq 2,$$

$$\Phi_n \to \Phi \text{ in } H^\mu_{p,p}(E) \text{ if } p \in (1, 2).$$

For each $n$, there is unique $u_n \in C^\infty_{0,p}(E)$ solving (1.1). Hence for $u_{n,m} = u_n - u_m$, we have

$$\partial_t u_{n,m} = (L^\pi - \lambda) u_{n,m} + f_n - f_m + \int_U (\Phi_n - \Phi_m) q (dt, d\nu),$$

$$u_{n,m} (0, x) = g_n (x) - g_m (x), x \in R^d.$$

By Lemma 8 (4.15), (4.20) in [14], and Lemma 9 for $p \geq 2$,

$$|L^\mu u_{n,m}|_{L_p(E)} \leq C \left[ |f_n - f_m|_{L_p(E)} + |g_n - g_m|_{H^\mu_{p,p}(R^d)} \right]$$

$$+ |\Phi_n - \Phi_m|_{H^\mu_{p,p}(E)} + |\Phi_n - \Phi_m|_{H^\mu_{2,p}(E)},$$

and

$$|L^\mu u_{n,m}|_{L_p(E)} \leq C \left[ |f_n - f_m|_{L_p(E)} + |g_n - g_m|_{H^\mu_{p,p}(R^d)} \right]$$

$$+ |\Phi_n - \Phi_m|_{H^\mu_{p,p}(E)}$$

if $p \in (1, 2)$.

By Lemma 7

$$|u_{n,m}|_{L_p(E)} \leq C [\rho_\lambda |f_n - f_m|_{L_p(E)} + \rho_\lambda^{1/p} |g_n - g_m|_{L_p(R^d)}$$

$$+ \rho_\lambda^{1/p} |\Phi_n - \Phi_m|_{L_p,p(E)} + \rho_\lambda^{1/2} |\Phi_n - \Phi_m|_{L_2,p(E)}]$$

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if \( p \geq 2 \), and
\[
|u_{n,m}|_{L^p(E)} \leq C \rho_\lambda |f_n - f_m|_{L^p(E)} + \rho_\lambda^{1/p} |g_n - g_m|_{L^p(\mathbb{R}^d)} + \rho_\lambda^{1/p} |\Phi_n - \Phi_m|_{L^p,\rho(E)}
\]
if \( p \in (1, 2) \). Hence there is \( u \in \mathbb{H}^{\mu_1}_p (E) \) so that \( u_n \to u \) in \( \mathbb{H}^{\mu_1}_p (E) \). Moreover, by Lemma \( \ref{lem:7} \)
\begin{equation}
\sup_{t \leq T} |u_n (t) - u (t)|_{L^p(\mathbb{R}^d)} \to 0,
\end{equation}
and \( u \) is \( L^p (\mathbb{R}^d) \)-valued continuous. According to Lemma \( \ref{lem:14} \) of \( \cite{14} \),
\begin{equation}
|L^\sigma f|_{L^p(E)} \leq C |L^\mu f|_{L^p(E)} \cdot f \in \tilde{C}^\infty_{0,p} (E).
\end{equation}

By Lemma \( \ref{lem:10} \) (see Appendix) and Remark \( \ref{rem:2} \)
\begin{equation}
\sup_{t \leq T} \left| \int_0^t \int_U \Phi_n q (ds, dz) - \int_0^t \Phi q (ds, dz) \right|_{L^p(\mathbb{R}^d)} \to 0
\end{equation}
as \( n \to \infty \) in probability.

Hence (see \( \ref{eq:5.7} \)-\( \ref{eq:5.9} \)) we can pass to the limit in the equation
\begin{equation}
u_n (t) = g_n + \int_0^t L^\sigma u_n (s) \lambda u_n (s) + f_n (s) ds + \int_0^t \Phi_n q (ds, dz), 0 \leq t \leq T.
\end{equation}

Obviously, \( \ref{eq:5.10} \) holds for \( u, g \) and \( f, \Phi \). We proved the existence part of Theorem \( \ref{thm:1} \).

Uniqueness. Assume \( u_1, u_2 \in \mathbb{H}^{\mu_1}_p (E) \) solve \( \ref{eq:1.1} \). Then \( u = u_1 - u_2 \in \mathbb{H}^{\mu_1}_p (E) \) solves \( \ref{eq:1.1} \) with \( f = 0, g = 0, \Phi = 0 \). Thus the uniqueness follows from the uniqueness of a deterministic equation (see \( \cite{14} \)).

Theorem \( \ref{thm:1} \) is proved.

6. Appendix

6.1. Stochastic Integral. We discuss here the definition of stochastic integrals with respect to a martingale measure. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a filtration of \( \sigma \)-algebras on \( \mathcal{F} = (\mathcal{F}_t, t \geq 0) \) satisfying the usual conditions. Let \((U, \mathcal{U}, \Pi)\) be a measurable space with \( \sigma \)-finite measure \( \Pi \), \( \mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\} \). Let \( p(dt, d\nu) \) be \( \mathcal{F} \)-adapted point measures on \(([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{U})\) with compensator \( \Pi (d\nu) dt \). We denote the martingale measure \( q(dt, d\nu) = p(dt, d\nu) - \Pi (d\nu) dt \).

We prove the following based on Lemma 12 from \( \cite{15} \).

Lemma 10. Let \( s \in \mathbb{R}, \Phi \in \mathbb{H}^{s}_2 (E) \cap \mathbb{H}^{s}_{p,p} (E) \) for \( p \in [2, \infty) \) and \( \Phi \in \mathbb{H}^{s}_{p,p} (E) \) for \( p \in [1, 2) \). There is a unique cadlag \( H^{s}_p (\mathbb{R}^d) \)-valued process
\[
M(t) = \int_0^t \int \Phi (r, x, \nu) q(dr, d\nu), 0 \leq t \leq T, x \in \mathbb{R}^d,
\]
such that for every \( \varphi \in \mathcal{S}(\mathbb{R}^d) \),

\[
\langle M(t), \varphi \rangle = \int_0^t \int \left( \int J^s \Phi(r, \cdot, \nu) J^{-s} \varphi dx \right) q(dr, d\nu), \quad 0 \leq t \leq T.
\]

Moreover, there is a constant \( C \) independent of \( \Phi \) such that

\[
\mathbb{E} \sup_{t \leq T} \left| \int_0^t \int \Phi(r, \cdot, \nu) q(dr, d\nu) \right|_{H^s_p(\mathbb{R}^d)} \leq C \sum_{j=2}^{\infty} |\Phi|_{H^s_{j,p}(E)}, \quad p \geq 2,
\]

\[
\mathbb{E} \sup_{t \leq T} \left| \int_0^t \int \Phi(r, \cdot, \nu) q(dr, d\nu) \right|_{H^s_p(\mathbb{R}^d)} \leq C |\Phi|_{H^s_{p,p}(E)}, \quad p \in (1, 2).
\]

**Proof.** According to Proposition 2 in [13], it is enough to consider the case \( s = 0 \). Let \( \Phi_n \) be a sequence defined in Lemma 1 that approximates \( \Phi \). Note first that by Lemma 9 for all \( x \),

\[
\mathbb{E} \int_0^T \sup_{x} \int |D^r \Phi_n(r, x, \nu)|^p \Pi(d\nu) dr < \infty.
\]

Recall for each \( n \), we have \( \Phi_n = \Phi_n \chi_{U_k} \) for some \( U_k \in \mathcal{U} \) with \( \Pi(U_k) < \infty \). Consequently, we define for each \( x \in \mathbb{R}^d \) and \( \mathbb{P} \)-a.s. for all \( (t, x) \in E \),

\[
M_n(t, x) = \int_0^t \int \Phi_n(r, x, z) q(dr, dz)
\]

\[
= \int_0^t \int \Phi_n(r, x, z) p(dr, dz) - \int_0^t \int \Phi_n(r, x, z) \Pi(dz) dr.
\]

Obviously, \( M^n(t, x) \) is cadlag in \( t \) and infinitely differentiable in \( x \). Obviously, \( M_n(t, \cdot) \) is \( \mathbb{L}_p(\mathbb{R}^d) \)-valued cadlag and, according to [16], there is a constant \( C \) independent of \( \Phi_n \) such that

\[
\mathbb{E} \sup_{t \leq T} |M_n(t)|^p_{L_p(\mathbb{R}^d)} \leq C \mathbb{E} \sum_{j=2}^{\infty} |\Phi_n|^p_{L_{j,p}(E)}, \quad 2 \leq p < \infty.
\]

By Lemma 9 of [17],

\[
\mathbb{E} \sup_{t \leq T} |M_n(t)|^p_{L_p(\mathbb{R}^d)} \leq C \mathbb{E} |\Phi_n|^p_{L_{p,p}(E)}, \quad 1 < p < 2.
\]

In addition, by Fubini theorem, \( \mathbb{P} \)-a.s. for \( 0 \leq t \leq T, \varphi \in \mathcal{S}(\mathbb{R}^d) \),

\[
\langle M_n(t), \varphi \rangle = \int M_n(t, x) \varphi(x) dx
\]

\[
= \int_0^t \int \left( \int \Phi_n(r, x, \nu) \varphi(x) dx \right) q(dr, d\nu).
\]
By Lemma 11
\[ \mathbb{E} \sum_{j=2,p} |\Phi_n - \Phi_{L,j,p}^p|_{L_p}(\mathbb{R}^d) \to 0, \quad 2 \leq p, \]
\[ \mathbb{E} |\Phi_n - \Phi_{L,p}^p|_{L_p(E)} \to 0, \quad p \in (1, 2). \]

Similarly, for each \( p \in [2, \infty), \)
\[ \mathbb{E} \sup_{t \leq T} |M_n(t) - M_m(t)|^p_{L_p}(\mathbb{R}^d) \leq CE \sum_{j=2,p} |\Phi_n - \Phi_{L,j,p}^p|_{L_p(E)} \to 0, \]
and for each \( p \in (1, 2) \)
\[ \mathbb{E} \sup_{t \leq T} |M_n(t) - M_m(t)|^p_{L_p}(\mathbb{R}^d) \leq CE |\Phi_n - \Phi_{L,p}^p|_{L_p(E)} \to 0, \]

as \( n, m \to \infty. \) Therefore there is an adapted càdlàg \( L_p(\mathbb{R}^d) \)-valued process \( M(t) \) so that
\[ \mathbb{E} \sup_{t \leq T} |M_n(t) - M(t)|^p_{L_p}(\mathbb{R}^d) \to 0 \]
as \( n \to \infty. \) Passing to the limit as \( n \to \infty \) in (6.6), (6.7) and (6.8) we derive (6.2), (6.1). Henceforth, we define \( \int_0^T \int \Phi(r, x, \nu) \eta(dr, d\nu) \) to be \( M(t) \) in this lemma. \( \square \)

6.2. Maximal and sharp functions, Hörmander condition. Given a function \( \kappa : (0, \infty) \to (0, \infty), \) consider the collection \( \mathcal{Q} \) of sets \( Q_\delta = Q_\delta(t, x) = (t - \kappa(\delta), t + \kappa(\delta)) \times B_\delta(x), (t, x) \in \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{d+1}, \delta > 0, \)
where \( B_\delta(x) \) is the standard open ball of radius \( \delta \) centered at \( x. \) The volume \( |Q_\delta(t, x)| = c_0 \kappa(\delta) \delta^d. \) We will need the following assumptions.

A1. \( \kappa \) is continuous, \( \lim_{\delta \to 0} \kappa(\delta) = 0 \) and \( \lim_{\delta \to \infty} \kappa(\delta) = \infty. \)

A2. There is a nondecreasing continuous function \( l(\varepsilon), \varepsilon > 0, \) such that \( \lim_{\varepsilon \to 0} l(\varepsilon) = 0 \) and
\[ \kappa(\varepsilon r) \leq l(\varepsilon) \kappa(r), r > 0, \varepsilon > 0. \]

Since \( Q_\delta(t, x) \) not exactly increases in \( \delta, \) we present the basic estimates involving maximal functions based on the system \( \mathcal{Q} = \{Q_\delta\}. \)

We state the following engulfing property from [13].

Lemma 11. Let A2 hold. If \( Q_\delta(t, x) \cap Q_{\delta'}(r, z) \neq \emptyset \) with \( \delta' \leq \delta, \) then there is \( K_0 \geq 3 \) such that \( Q_{K_0 \delta}(t, x) \) contains both, \( Q_\delta(t, x) \) and \( Q_{\delta'}(r, z) \), and
\[ |Q_\delta(t, x)| \leq |Q_{K_0 \delta}(t, x)| \leq K_0^d l(K_0) |Q_\delta(t, x)|. \]

6.2.1. Maximal and sharp functions. Following [18], for a locally integrable function \( f(t, x) \) on \( \mathbb{R}^{d+1} \) we define
\[ (A_\delta f)(t, x) = \frac{1}{|Q_\delta(t, x)|} \int_{Q_\delta(t, x)} f(s, y) ds dy, (t, x) \in \mathbb{R} \times \mathbb{R}^d, \delta > 0 \]
and the maximal function of \( f \) by
\[ \mathcal{M} f(t, x) = \sup_{\delta > 0} (A_\delta |f|)(t, x), (t, x) \in \mathbb{R}^{d+1}. \]
We use collection \( Q \) to define a larger, noncentered maximal function of \( f \), as
\[
\tilde{M}f(t, x) = \sup_{(t, x) \in Q} \frac{1}{|Q|} \int_Q |f(s, y)| \, ds \, dy, \quad (t, x) \in \mathbb{R}^{d+1},
\]
where \( \sup \) is taken over all \( Q \in Q \) that contain \((t, x)\).

**Remark 3.** It is shown in [13] that if \( A2 \) hold then there exists \( K_0 > 0 \) such that for a locally integrable \( f \) on \( \mathbb{R}^{d+1} \),
\[
\mathcal{M}f \leq \tilde{M}f \leq \frac{1}{K_0^d} \mathcal{M}f.
\]

The following result is proved in [13].

**Theorem 2.** Let \( A2 \) hold and \( f \) be a measurable function on \( \mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d \).

(a) If \( f \in L_p, 1 \leq p \leq \infty \), then \( \mathcal{M}f \) is finite a.e.

(b) If \( f \in L_1 \), then for every \( \alpha > 0 \),
\[
|\{\mathcal{M}f(t, x) > \alpha\}| \leq \frac{c}{\alpha} \int |f| \, dt \, dx.
\]

(c) If \( f \in L_p, 1 < p \leq \infty \), then \( \mathcal{M}f \in L_p \) and
\[
|\mathcal{M}f|_{L_p} \leq N_p |f|_{L_p},
\]
where \( N_p \) depends only on \( p, l \) and \( K_0 \).

Calderon-Zygmund decomposition, sharp functions. Assume \( A1, A2 \) hold. Let \( F \subseteq \mathbb{R} \times \mathbb{R}^d \) be closed and \( O = F^c = \mathbb{R}^{d+1} \setminus F \). For \((t, x) \in O\), let
\[
D(t, x) = \inf \{\delta > 0 : Q_\delta(t, x) \cap F \neq \emptyset\}.
\]

In [13], the following statement is proved.

**Lemma 12.** (Lemma 15 in [13]) Assume \( A1, A2 \) hold. Given a closed nonempty \( F \), there are sequences \( Q^k, Q^{*k} \) and \( Q^{**k} \) in \( Q \) having the same center but with radius expanded by the same factor \( c_1^{**} > c_1^* > c_1 \) so that \( Q^k \subseteq Q^{*k} \subseteq Q^{**k} \) (all of them are of the form \( Q_{bD(t_k, x_k)}(t_k, x_k) \) with \( b = c_1, c_1^*, c_1^{**} \) correspondingly) and

(a) the sets \( Q^k \) are disjoint.

(b) \( \cup_k Q^k = O = F^c \).

(c) \( Q^{**k} \cap F \neq \emptyset \) for each \( k \).

**Remark 4.** Assume \( A1, A2 \) hold and \( Q^k \subseteq Q^{*k} \subseteq Q^{**k} \) be the sequences in \( Q \) from Lemma 12. It is easy to find a sequence of disjoint measurable sets \( C^k \) so that \( Q^k \subseteq C^k \subseteq Q^{*k} \) and \( \cup_k C^k = O \). For example (see Remark, p. 15, in [13]),
\[
C^k = Q^{*k} \cap (\bigcup_{j < k} C^j)^c \cap (\bigcup_{j > k} Q^j)^c.
\]

We have the following Calderon-Zygmund decomposition for \( Q \).
Theorem 3. (Theorem 4 in \cite{13}) Assume A1, A2 hold. Let \( f \in L^1(\mathbb{R} \times \mathbb{R}^d) \), \( \alpha > 0 \) and \( O_{\alpha} = \{ \widehat{M}f > \alpha \} \). Consider the sets \( Q^k \subseteq C^k \subseteq Q^{*k} \subseteq O \) of Lemma \ref{lemma} and Remark \ref{remark} associated to \( O_{\alpha} \).

There is a decomposition \( f = g + b \) with

\[
g(x) = \begin{cases} f(x) & \text{if } x \notin O_{\alpha}, \\ \frac{1}{|C^k|} \int_{C^k} f & \text{if } x \in C^k, k \geq 1, \end{cases}
\]

and with \( b = \sum_k b_k \), where

\[
b_k = \chi_{C^k} \left[ f(x) - \frac{1}{|C^k|} \int_{C^k} f \right], \quad k \geq 1,
\]

(note \( C^k \) are disjoint, \( \bigcup_k C^k = O_{\alpha} \)). Also,

(i) \( |g(x)| \leq c\alpha \) for a.e. \( x \).

(ii) \( \text{support}(b_k) \subseteq Q^{*k}, \quad \hat{b}_k = 0 \) and \( |\hat{b}_k| \leq c\alpha |Q^{*k}| \).

(iii) \( \sum_k |Q^{*k}| \leq \frac{\pi}{\alpha} \int |f| \).

The \( L^p \) norm of \( f \) can be controlled by its sharp function as well.

Definition 2. Given a locally integrable \( f \) on \( \mathbb{R} \times \mathbb{R}^d \), we define its sharp function as

\[
f^\sharp(t,x) = \sup_{\delta} \frac{1}{|Q_\delta(t,x)|} \int_{Q_\delta(t,x)} |f(s,y) - f_{Q_\delta(t,x)}| dsdy.
\]

or

\[
f^\sharp(t,x) = \sup_{(t,x) \in Q} \frac{1}{|Q|} \int_{Q} |f(s,y) - f_Q| dsdy,
\]

where

\[
f_Q = \frac{1}{Q} \int_{Q} f dm,
\]

and sup is taken over all \( Q \in \mathcal{Q} \) containing \( (t,x) \).

Obviously,

\[
f^\sharp(t,x) \leq 2\widehat{M}f(t,x), \quad f^\sharp \leq 2Mf.
\]

Remark 5. Let \( f \in L^1_{\text{loc}}, \, B \subseteq \mathbb{R}^d \) be a bounded measurable subset. Then for any constant \( C \),

\[
\frac{1}{|B|} \int_B \int_B |f(s,y) - f(t,z)| dsdy \leq \frac{1}{|B|} \int_B \int_B |f(s,y) - C| dsdy + \frac{1}{|B|^2} \int_B \int_B |C - f(t,z)| dsdy
\]

\[
\leq \frac{2}{|B|} \int_B |f(t,x) - C| dx.
\]
Hence
\[
\frac{1}{|B|} \int_B |f - f_B| \leq \frac{1}{|B|^2} \int_B \int_B |f(s, y) - f(t, z)| \, dt \, dz \, ds \, dy
\leq 2 \frac{1}{|B|} \int_B |f - f_B|
\]

As a consequence of (6.8), the following holds.

Remark 6. 1. As in the case of maximal functions, \( f \natural(t, x) \leq 2 f \natural(t, x) \leq 2 \left( \frac{K_d}{K_0^d(K_0)} \right)^2 f_\natural(t, x) \) or
\[
\frac{1}{2} f_\natural \leq f \natural \leq 2 \left( \frac{K_d}{K_0^d(K_0)} \right)^2 f_\natural.
\]

2. Since \( ||a| - |b|| \leq |a - b| \), it follows by (6.8) that \( (|f|)^\natural \leq 2 f \natural, (|f|)^\# \leq 2 f \# \).

Lemma 13. (cf. Lemma 9, p.101, in [11]) Let \( \lambda = \frac{1}{2c} \) (c is from Theorem 3), \( f \in L^1 \) and \( \alpha > 0 \). Then
\[
(6.9) \quad |\{|f| > \alpha\}| \leq \frac{4}{\alpha} \int \chi_{\{\widetilde{M}f > \lambda \alpha\}} f_\natural.
\]

If \( f \geq 0 \), we \( 4/\alpha \) can be replaced by \( 2/\alpha \).

Proof. Assume \( f \geq 0 \). Apply Theorem 3 with \( \alpha \) replaced by \( \lambda \alpha = \alpha/2c \), i.e. \( \lambda = 1/(2c) \). We have \( f = g + b \) with \( |g| \leq c \lambda \alpha = \alpha/2 \) a.e. and
\[
b = \sum_k b_k = \sum_k \chi_{C_k} \left[ f(x) - \frac{1}{|C|} \int_{C_k} f \right].
\]

Recall \( O = \{ \widetilde{M}f > \lambda \alpha \} = \cup_k C_k \) and \( C_k \) are disjoint. Since \( \{|f| > \alpha\} \subseteq \{|b| > \alpha/2\} \), (6.9) follows.

Theorem 4. (Fefferman/Stein) Let \( f \in L^p(\mathbb{R} \times \mathbb{R}^d), p \in (1, \infty) \). Then
\[
|f|_{L^p} \leq N \left| f_\natural \right|_{L^p}.
\]

Proof. Indeed, using Lemma 13, (6.9), Theorem 2 and Hölder inequality, for \( f \in L^1 \) we have
\[
|f|^p_{L^p} = \int_0^{\infty} \left| \left\{ |f| > \alpha^{1/p} \right\} \right| \, d\alpha \leq 4 \int_0^{\infty} \alpha^{-1/p} \int \chi_{\{\widetilde{M}f > \lambda \alpha^{1/p}\}} f_\natural \, d\alpha
\leq c \int \int (\widetilde{M}f)^p \alpha^{-p} \, d\alpha \, f_\natural = c \int (\widetilde{M}f)^p f_\natural \leq N \left| \widetilde{M}f \right|_{L^p}^{p-1} \left| f_\natural \right|_{L^p}.
\]

If in addition \( f \in L^p \), then we are done. If only \( f \in L^p \), then we take a sequence \( f_n \in L^1 \cap L^p \) converging to \( f \) in \( L^p \) and notice that
\[
f_n^2 \leq (f - f_n)^2 + f^2
\]
and
\[ |(f - f_n)^2|_{L^p} \leq 2 |\mathcal{M}(f - f_n)|_{L^p} \leq C |f - f_n|_{L^p}. \]

\[ \square \]

6.2.2. \textit{Hörmander Condition and } L^p\textit{-estimate.} Let \( V \) be a separable Hilbert space, let \( f \) be a measurable \( V \)-valued function on \( \mathbb{R}^{d+1} \), define an operator \( \mathcal{G} \) by
\[ (\mathcal{G} f)(t, x) = \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}} K(t, x, s, y) f(s, y) dy \left\| \mathcal{V} \right\| ds \right]^{1/2}, (t, x) \in \mathbb{R}^{d+1} \]

Let \( K \) be a measurable function and for almost all \( (t, x) \in \mathbb{R}^{d+1} \) the function \( K(t, x, \cdot) f \) is integrable for all \( f \in C_0^\infty(\mathbb{R}^{d+1}, V) \).
We assume that \( \mathcal{G} \) is bounded on \( L^2 \), i.e.,
\[ |\mathcal{G} f|_{L^2} \leq M_0 |f|_{L^2(\mathbb{R}^{d+1}; V)}, f \in L_2(\mathbb{R}^{d+1}; V) \]

In [10] and [16], an \( L^p \)-estimate for \( \mathcal{G} f \) was derived by estimating directly its sharp function \( (\mathcal{G} f)^\# \). It was shown in [8], that \( (\mathcal{G} f)^\# \)-estimate follows by verifying a Hörmander condition. We adjust [8] to our setting, and show that the sharp function estimate follows form the following Hörmander condition: there are constants \( C_0 > 1, M_1 > 0 \) so that for any \( Q_\delta(t, x) \in Q \),
\[ \left( \int_{Q_\delta(t, x)} \chi_{Q_\delta(t, x)} \left| K(t, x, r, y) - K(\bar{t}, \bar{x}, r, y) \right| dy \right)^2 dr \leq M_1, \quad \forall (\bar{t}, \bar{x}) \in Q_\delta(t, x). \]

\textbf{Lemma 14.} Let \( A_1, A_2, (6.11), (6.12) \) hold. Then \( \mathcal{G} \) is bounded in \( L^p \)-norm on \( L^2 \cap L^p \) if \( p > 2 \). More precisely
\[ |\mathcal{G} f|_{L^p} \leq A_p |f|_{L^p(\mathbb{R}^{d+1}; V)}, \quad \forall f \in L_2(\mathbb{R}^{d+1}; V) \cap L_p(\mathbb{R}^{d+1}; V), \quad p > 2, \]

where \( A_p \) depends only on the constants \( M_1, M_0 \) and \( p \).

For convenience we denote,
\[ \mathcal{K} f(t, x, s) = \int_{\mathbb{R}^d} K(t, x, s, y) f(s, y) dy, \]
and
\[ \mathcal{G} f(t, x, s, y) = \left[ \int \mathcal{K} f(t, x, r) - \mathcal{K}(s, y, r) \right]_V^2 dr \right]^{1/2}. \]

For the proof of Lemma 14, we will need some auxiliary results.
Lemma 15. Let \((t_1, x_1) \in Q_\delta (t_0, x_0)\). Then for any \(f_1, f_2 \in L_2 (\mathbb{R}^{d+1}; V)\),
\[
\int_{Q_\delta (t_0, x_0)} \int_{Q_\delta (t_0, x_0)} |\mathcal{G} (f_1 + f_2) (t, x) - \mathcal{G} (f_1 + f_2) (s, y)| \, dt \, dx \, ds \, dy
\]
\[
\leq 2M (\mathcal{G} f_1) (t_1, x_1) + \int_{Q_\delta (t_0, x_0)} \int_{Q_\delta (t_0, x_0)} G f_2 (t, s, x, y) \, dt \, dx \, ds \, dy
\]
Proof. Set \(f = f_1 + f_2\), and let \((t, x), (s, y) \in Q_\delta (t_0, x_0)\). Then
\[
|\mathcal{G} f (t, x) - \mathcal{G} f (s, y)|
= \left[ \int |\mathcal{K} f (t, x, r) |^2_V \, dr \right]^{1/2} - \left[ \int |\mathcal{K} f (s, y, r) |^2_V \, dr \right]^{1/2}
\]
\[
\leq \left[ \int |\mathcal{K} f (t, x, r) - \mathcal{K} f (s, y, r) |^2_V \, dr \right]^{1/2}
\]
\[
\leq \left[ \int |\mathcal{K} f_1 (t, x, r) |^2_V \, dr \right]^{1/2} + \left[ \int |\mathcal{K} f_1 (s, y, r) |^2_V \, dr \right]^{1/2}
\]
\[
+ \left[ \int |\mathcal{K} f_2 (t, x, r) - \mathcal{K} f_2 (s, y, r) |^2_V \, dr \right]^{1/2}
\]
Taking average on \(Q_\delta (t_0, x_0)\), the result follows. \(\square\)

Lemma 16. Suppose \((6.11)\) holds, \(f\) belongs to \(L_2 (\mathbb{R}^{d+1}; V) \cap L_\infty (\mathbb{R}^{d+1}; V)\) and vanishes outside of \(Q_\gamma \delta \ (t_0, x_0)\), \(\gamma > 0\). Then
\[
A := \int_{Q_\delta (t_0, x_0)} \int_{Q_\delta (t_0, x_0)} G f (t, x, s, y) \, dt \, dx \, ds \, dy
\]
where \(C = C (d, \gamma, M_0)\).
Proof. Obviously, \(G f (t, x, s, y) \leq G f (t, x) + G f (s, y)\) for any \((t, x), (s, y) \in Q_\delta (t_0, x_0)\). Hence, \(A \leq 2 \int_{Q_\delta (t_0, x_0)} G f (t, x) \, dt \, dx\).
Applying Hölder’s inequality and using \((6.11)\),
\[
\int_{Q_\delta (t_0, x_0)} G f (t, x) \, dt \, dx
\leq \frac{1}{|Q_\delta (t_0, x_0)|^{1/2}} \left[ \int |G f (t, x)|^2 \, dt \, dx \right]^{1/2}
\leq \frac{M_0}{|Q_\delta (t_0, x_0)|^{1/2}} \left[ \int |f (t, x) |^2_V \, dt \, dx \right]^{1/2}
\leq M_0 \left[ |Q_\gamma \delta (t_0, x_0)|^{1/2} |f|_{L_\infty (\mathbb{R}^{d+1}; V)} \right]^{1/2} \leq M_0 \left[ \gamma^d l (\gamma) \right]^{1/2} |f|_{L_\infty (\mathbb{R}^{d+1}; V)}.
\]
\(\square\)

Lemma 17. Let \(b \geq 2, l (b^{-1}) \leq 1/2, \gamma \geq bC_0 + 1\) and \(l (\gamma^{-1})^{-1} = l (bC_0) + 1\). Suppose that \(f \in L_\infty (\mathbb{R}^{d+1}; V)\) vanishes on \(Q_\gamma \delta \ (t_0, x_0)\) and \((6.12)\) hold.
Then,
\[
\int_{Q_{\delta}(t_0,x_0)} \int_{Q_{\delta}(t_0,x_0)} Gf(t,x,s,y) \, dt \, ds \, dy \leq C |f|_{L_\infty (\mathbb{R}^{d+1,V})}
\]
where \( C = C (M_1) \).

**Proof.** If \((t,x), (s,y) \in Q_\delta(t_0,x_0)\), then \(|x - y| < 2\delta \leq b\delta\), and
\[
|t - s| < 2\kappa (\delta) \leq 2l \left( \frac{1}{b} \right) \kappa (b\delta) \leq \kappa (b\delta),
\]
i.e. \((t,x) \in Q_{b\delta}(s,y)\). If \((r,z) \in Q_{r\delta}(t_0,x_0)^c\) then either \(|z - x_0| \geq \gamma \delta\) or \(|r - t_0| \geq \kappa (\gamma \delta)\).

If \(|z - x_0| \geq \gamma \delta\) then \(|z - y| \geq |z - x_0| - |y - x_0| \geq \gamma \delta - \delta \geq bC_0\delta\). On the other hand, if \(|r - t_0| \geq \kappa (\gamma \delta)\) then
\[
|r - s| \geq |r - t_0| - |t_0 - s| \geq \kappa (\gamma \delta) - \kappa (\delta) \geq \kappa (\delta) \left[ l (\gamma^{-1} - 1) \right] \geq \kappa (\delta) l (bC_0) \geq \kappa (bC_0 \delta).
\]

Both cases imply that \((r, z) \in Q_{C_0b\delta}(s, y)^c\), and by the stochastic Hörmander condition \(6.12\)
\[
|Gf(t,x,y)|^2 \leq \int \left[ \int_{\mathbb{R}^d} |K(t,x,r,z) - K(s,y,r,z)| \, |f(r,z)|_V \, dz \right]^2 dr
\]
\[
\leq |f|_{L_\infty (\mathbb{R}^{d+1,V})}^2 \int \left[ \int_{Q_{C_0b\delta}^c(s,y)^c} |K(t,x,r,z) - K(s,y,r,z)| \, dz \right]^2 dr
\]
\[
\leq M_1 |f|_{L_\infty (\mathbb{R}^{d+1,V})}^2.
\]

Therefore,
\[
\int_{Q_{\delta}(t_0,x_0)} \int_{Q_{\delta}(t_0,x_0)} Gf(t,x,s,y) \, dt \, ds \, dy \leq M_1^{1/2} |f|_{L_\infty (E; V)}
\]

**Lemma 18.** Let \( f_1 \in L_2 (\mathbb{R}^{d+1}; V), f_2 \in L_2 (\mathbb{R}^{d+1}; V) \cap L_\infty (\mathbb{R}^{d+1}; V) \) and suppose that \(6.11\) and \(6.12\) hold. Then for any \( X_0 = (t_0, x_0) \in \mathbb{R}^{d+1}, \)
\[
|G (f_1 + f_2)|^2 (t_0, x_0) \leq 2\hat{M} (G f_1) (t_0, x_0) + C |f_2|_{L_\infty (\mathbb{R}^{d+1}; V)}
\]
where \( C = C (d, M_0, M_1, C_0) \).

**Proof.** By Lemma \(15\) for any \((t_0, x_0) \in \mathbb{R}^{d+1}, \)
\[
\int_{Q_{\delta}(t_0,x_0)} \int_{Q_{\delta}(t_0,x_0)} |G (f_1 + f_2) (t, x) - G (f_1 + f_2) (s, y)| \, dt \, ds \, dy
\]
\[
\leq 2\hat{M} (G f_1) (t_0, x_0) + \int_{Q_{\delta}(t_0,x_0)} \int_{Q_{\delta}(t_0,x_0)} G f_2 (t, s, x, y) \, dt \, ds \, dy.
\]
Moreover, defining \( f_{2,1}(t, x) := f_2(t, x) \chi_{Q_{\delta}(t_0, x_0)}(t, x) \) and \( f_{2,2}(t, x) := f_2(t, x) - f_{2,1}(t, x) \), we have

\[
\int_{Q_{\delta}(t_0, x_0)} \int_{Q_{\delta}(t_0, x_0)} G f_{2} t d t d x d s d y \\
\leq \int_{Q_{\delta}(t_0, x_0)} \int_{Q_{\delta}(t_0, x_0)} G f_{2,1} t d t d x d s d y + \int_{Q_{\delta}(t_0, x_0)} \int_{Q_{\delta}(t_0, x_0)} G f_{2,2} t d t d x d s d y.
\]

We obtain the results by \((6.8)\), and Lemmas \(16\) and \(17\) and taking \(\gamma\) satisfying the assumptions of Lemma \(17\).

Proof of Lemma \(14\) Let \( p > 2, f \in L_2(\mathbb{R}^{d+1}; V) \cap L_p(\mathbb{R}^{d+1}; V) \). For \( \lambda > 0, \delta > 0 \), we define \( f = f_{1,\lambda} + f_{2,\lambda} \) with

\[
f_{1,\lambda} = f 1|f|_V > \delta \lambda, f_{2,\lambda} = f 1|f|_V \leq \lambda \lambda.
\]

By Lemma \(18\)

\[
|\mathcal{G}(f)|^2(t, x) \leq 2 \tilde{M}(\mathcal{G} f_{1,\lambda})(t, x) + C |f_{2,\lambda}|_{L_\infty(E, V)} \\
\leq 2 \tilde{M}(\mathcal{G} f_{1,\lambda})(t, x) + C \delta \lambda, (t, x) \in \mathbb{R}^{d+1},
\]

where \( C \) is a constant in Lemma \(18\) (independent of \( \delta \) and \( \lambda, f \)). Fix \( \delta > 0 \) so that \( C \delta < \frac{1}{2} \). Then the above inequality implies that

\[
\mathcal{G}(f)^2(t, x) \leq 2 \tilde{M}(\mathcal{G} f_{1,\lambda})(t, x) + \lambda/2, (t, x) \in \mathbb{R}^{d+1}.
\]

Since \( \{ \lambda \leq \mathcal{G}(f)^2 \} \subseteq \{ \lambda \leq 4 \tilde{M}(\mathcal{G} f_{1,\lambda}) \} \), it follows by Theorem 2 and \((6.11)\),

\[
\left| \mathcal{G}(f)^2 \right|^p_{L_p(\mathbb{R}^{d+1})} \\
= p \int_0^\infty \lambda^{p-1} \left| \lambda \leq \mathcal{G}(f)^2 \right| d\lambda \leq p \int_0^\infty \lambda^{p-1} \left| \lambda \leq 4 \tilde{M}(\mathcal{G} f_{1,\lambda}) \right| d\lambda \\
\leq C \int_0^\infty \lambda^{p-3} \int \tilde{M}(\mathcal{G} f_{1,\lambda})(t, x)^2 d t d x d \lambda \leq C \int_0^\infty \lambda^{p-3} \int |\mathcal{G} f_{1,\lambda}(t, x)|^2 d t d x d \lambda \\
\leq C \int_0^\infty \lambda^{p-3} \int |f_{1,\lambda}(t, x)|_{V}^2 d t d x d \lambda \leq C \int_0^\infty \lambda^{p-3} \int \{|f(t, x)|_V > \delta \lambda\} |f(t, x)|^2_{V} d t d x \\
\leq C \int \left( \int_0^{|f(t, x)|_V / \delta} \lambda^{p-3} d \lambda \right) |f(t, x)|^2_{V} d t d x = C \int \frac{|f(t, x)|^p_{V}}{\delta^{p-2} (p - 2)} |f(t, x)|^2_{V} d t d x \\
= C |f|^p_{L_p(\mathbb{R}^{d+1}; V)}.
\]

The proof is completed by Theorem 4.
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