Non universality of fluctuations of outlier eigenvectors for block diagonal deformations of Wigner matrices

Mireille Capitaine and Catherine Donati-Martin

Abstract. In this paper, we investigate the fluctuations of a unit eigenvector associated to an outlier in the spectrum of a spiked $N \times N$ complex Deformed Wigner matrix $M_N$. $M_N$ is defined as follows: $M_N = W_N/\sqrt{N} + A_N$ where $W_N$ is an $N \times N$ Hermitian Wigner matrix whose entries have a law $\mu$ satisfying a Poincaré inequality and the matrix $A_N$ is a block diagonal matrix, with an eigenvalue $\theta$ of multiplicity one, generating an outlier in the spectrum of $M_N$. We prove that the fluctuations of the norm of the projection of a unit eigenvector corresponding to the outlier of $M_N$ onto a unit eigenvector corresponding to $\theta$ are not universal. Indeed, we take away a fit approximation of its limit from this norm and prove the convergence to zero as $N$ goes to $\infty$ of the Lévy–Prohorov distance between this rescaled quantity and the convolution of $\mu$ and a centered Gaussian distribution (whose variance may depend depend upon $N$ and may not converge).

1. Introduction

To begin with, we introduce some notations.

- $M_N(\mathbb{C})$ is the set of $N \times N$ matrices with complex entries, $M_N^{sa}(\mathbb{C})$ the subset of self-adjoint elements of $M_N(\mathbb{C})$ and $I_N$ the identity matrix.
- $\text{Tr}_N$ denotes the trace and $\text{tr}_N = \frac{1}{N} \text{Tr}_N$ the normalized trace on $M_N(\mathbb{C})$. 

Received by the editors October 11th, 2019; accepted October 27th, 2020.

2010 Mathematics Subject Classification. 15A18, 15A52, 60F05, 60B20, 46L54.

Key words and phrases. Random matrices; Outliers; eigenvectors; Fluctuations; Nonuniversality, Free probability.
• \( \| \cdot \| \) denotes the operator norm on \( M_N(\mathbb{C}) \).

• For any \( X \in M_N^{sa}(\mathbb{C}) \), \( (\lambda_1(X),\ldots,\lambda_N(X)) \) denote the eigenvalues of \( X \) ranked in decreasing order and the empirical spectral measure of \( X \) is defined by

\[
\mu_X := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(X)}.
\]

• For a probability measure \( \tau \) on \( \mathbb{R} \), \( \text{supp}(\tau) \) denotes the support of \( \tau \) and \( g_{\tau} : z \in \mathbb{C} \setminus \text{supp}(\tau) \mapsto \int \frac{1}{z-x} d\tau(x) \) is the Stieltjes transform of \( \tau \).

• \( \delta_{LP} \) denotes the Lévy-Prohorov distance, which is a metric for the topology of the convergence in distribution.

1.1. Wigner matrices. Wigner matrices are complex Hermitian random matrices whose entries are independent (up to the symmetry condition). They were introduced by Wigner in the fifties, in connection with nuclear physics. Here, we will consider Hermitian Wigner matrices of the following form:

\[
X_N = \frac{1}{\sqrt{N}} W_N
\]

where \( W_N \) is an Hermitian matrix, \( \{W_{ii}, \sqrt{2}R W_{ij}, \sqrt{2}I W_{ij}\}_{1 \leq i \neq j} \) are independent identically distributed random variables with law \( \mu \), with mean zero and variance \( \sigma^2 \). If the entries are independent Gaussian variables, \( X_N =: X_N^G \) is a matrix from the Gaussian Unitary Ensemble (G.U.E.).

There is currently a quite precise knowledge of the asymptotic spectral properties (i.e. when the dimension of the matrix tends to infinity) of Wigner matrices. This understanding covers both the so-called global regime (asymptotic behavior of the spectral measure) and the local regime (asymptotic behavior of the extreme eigenvalues and eigenvectors, spacings...). Wigner proved that a precise description of the limiting spectrum of these matrices can be achieved.

**Theorem 1.1.** Wigner (1955, 1958)

\[
\mu_{X_N} \xrightarrow{w} \mu_{sc} \text{ a.s. when } N \to +\infty
\]

where

\[
\frac{d\mu_{sc}}{dx}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} 1_{[-2\sigma,2\sigma]}(x) \quad (1.1)
\]

is the so-called semi-circular distribution.

A priori, the convergence of the spectral measure does not prevent an asymptotically negligeable fraction of eigenvalues from going away from the limiting support (called outliers in the following). Actually, it turns out that Wigner matrices do not exhibit outliers.

**Theorem 1.2.** Bai and Yin (1988) Assume that the entries of \( W_N \) has finite fourth moment, then almost surely,

\[
\lambda_1(X_N) \to 2\sigma \text{ and } \lambda_N(X_N) \to -2\sigma \text{ when } N \to +\infty.
\]

In Tracy and Widom (1994), Tracy and Widom derived the limiting distribution (called the Tracy-Widom law) of the largest eigenvalue of a G.U.E. matrix.
Theorem 1.3. Let $q : \mathbb{R} \to \mathbb{R}$ be the unique solution of the differential equation
\[ q''(x) = xq(x) + 2q(x)^3 \]
such that $q(x) \sim_{x \to +\infty} \text{Ai}(x)$ where $\text{Ai}$ is the Airy function, unique solution on $\mathbb{R}$ of the differential equation $f''(x) = xf(x)$ satisfying $f(x) \sim (4\pi \sqrt{x})^{1/2} \exp(-2/3x^{3/2})$ at $+\infty$. Then
\[
\lim_{N \to +\infty} \mathbb{P} \left( \frac{N^{2/3}}{\sigma} \left( \frac{\lambda_1(X_N)}{\sqrt{N}} - 2\sigma \right) \leq s \right) = F_2(s),
\]
where $F_2(s) = \exp \left( -\int_{s}^{+\infty} (x-s)q^2(x)dx \right)$.

The first main step to prove the universality conjecture for fluctuations of the largest eigenvalue of Wigner matrices has been achieved by Soshnikov (1999); in Lee and Yin (2014), a necessary and sufficient condition on off-diagonal entries of the Wigner matrix is established for the distribution of the largest eigenvalue to weakly converge to the Tracy-Widom distribution. We also refer to these papers for references on investigations on edge universality.

In regards to eigenvectors, it is well known that the matrix whose columns are the eigenvectors of a G.U.E. matrix can be chosen to be distributed according to the Haar measure on the unitary group. In the non-Gaussian case, the exact distribution of the eigenvectors cannot be computed. However, the eigenvectors of general Wigner matrices have been the object of a growing interest and in several papers, a delocalization and universality property were shown for the eigenvectors of these standard models (see among others Bloemendal et al., 2014; Erdős et al., 2009a,b; Knowles and Yin, 2013; Tao and Vu, 2012 and references therein). Heuristically, delocalization for a random matrix means that its normalized eigenvectors look like the vectors uniformly distributed over the unit sphere. Let us state for instance the following sample result.

Theorem 1.4. (Isotropic delocalization, Theorem 2.16 from Bloemendal et al., 2014). Let $X_N$ be a $N \times N$ Wigner matrix satisfying some technical assumptions. Let $v(1), \ldots, v(N)$ denote the normalized eigenvectors of $X_N$. Then, for any $C_1 > 0$ and $0 < \epsilon < 1/2$, there exists $C_2 > 0$ such that
\[
\sup_{1 \leq i \leq N} |\langle v(i), u \rangle| \leq \frac{N^\epsilon}{\sqrt{N}},
\]
for any fixed unit vector $u \in \mathbb{C}^N$, with probability at least $1 - C_2 N^{-C_1}$.

1.2. Deformed Wigner matrices. Practical problems (in the theory of statistical learning, signal detection etc.) naturally lead to wonder about the spectrum reaction of a given random matrix after a deterministic perturbation. In those applications, the random matrix is the noise and the perturbed matrix is a noisy version of the information; the question is to know whether the observation of the spectral properties of the perturbed matrix can give access to significant parameters on the information. Theoretical results on these deformed random models may allow to establish statistical tests on these parameters. A typical illustration is the so-called BBP phenomenon (after Baik, Ben Arous, Péché) which put forward outliers (eigenvalues that move away from the rest of the spectrum) and their Gaussian fluctuations for spiked covariance matrices in Baik et al. (2005) and for low rank deformations of G.U.E. in Péché (2006).
In this paper, we consider additive perturbations of Wigner matrices. The pioneer works on additive deformations go back to Pastur (1972) for the behavior of the limiting spectral distribution and to Füredi and Komlós (1981) for the behavior of the largest eigenvalue.

We refer to Capitaine and Donati-Martin (2017) and the references therein for a survey on spectral properties of deformed random matrices.

The model studied is as follows:

\[ M_N := \frac{W_N}{\sqrt{N}} + A_N, \]

where

(W) \( W_N \) is a complex Wigner matrix, that is a \( N \times N \) random Hermitian matrix such that \( \{W_{ii}, \sqrt{2}R_{ij}, \sqrt{2}I_{ij}\}_{1 \leq i < j} \) are independent identically distributed random variables with law \( \mu \). We assume that \( \mu \) is a distribution with mean zero, variance \( \sigma^2 \), and satisfies a Poincaré inequality (see Appendix). Note that this condition implies that \( \mu \) has moments of any order (see Corollary 3.2 and Proposition 1.10 in Ledoux, 2001).

(A) \( A_N \) is a \( N \times N \) deterministic Hermitian matrix, whose spectral measure \( \mu_{A_N} \) converges to a compactly supported probability measure \( \nu \). We assume that \( A_N \) has a fixed number \( q \) of eigenvalues, not depending on \( N \), outside the support of \( \nu \) called spikes, whereas the distance of the other eigenvalues to the support of \( \nu \) goes to 0.

The empirical spectral distribution \( \mu_{M_N} \) converges a.s. towards the probability measure \( \lambda := \mu_{sc} \boxplus \nu \) where \( \mu_{sc} \) is the semicircular distribution with variance \( \sigma^2 \) and \( \boxplus \) denotes the free convolution, see Pastur (1972) (in this paper, the limiting distribution is given via a functional equation for its Stieltjes transform), Anderson et al. (2010, Theorem 5.4.5). We refer to Voiculescu et al. (1992); Mingo and Speicher (2017) for an introduction to free probability theory.

Concerning extremal eigenvalues, Capitaine et al. (2011) proved that the spikes of \( A_N \) can generate outliers for the limiting spectrum of \( M_N \), i.e. eigenvalues outside the support of the limiting distribution \( \lambda \). More precisely, Capitaine et al. (2011) proved the following (see Capitaine et al., 2011, Theorem 8.1 for a more general statement).

**Proposition 1.5.** Capitaine et al. (2011) Assume that a spike \( \theta \) with a fixed multiplicity \( k_0 \) in the spectrum of \( A_N \) satisfies:

\[ \theta \in \Theta_{\sigma, \nu} := \{u \in \mathbb{R} \setminus \text{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2}\}. \]

Denote by \( n_0 + 1, \ldots, n_0 + k_0 \) the descending ranks of \( \theta \) among the eigenvalues of \( A_N \). Then the \( k_0 \) eigenvalues \( \lambda_{n_0+i}(M_N), 1 \leq i \leq k_0 \) converge almost surely outside the support of \( \lambda \) towards \( \rho_\theta := \theta + \sigma^2 g_{\nu}(\theta) \). Moreover, these eigenvalues asymptotically separate from the rest of the spectrum since (with the conventions that \( \lambda_0(M_N) = +\infty \) and \( \lambda_{N+1}(M_N) = -\infty \)) there exists \( 0 < \delta_0 \) such that almost surely for all large \( N \),

\[ \lambda_{n_0}(M_N) > \rho_\theta + \delta_0 \text{ and } \lambda_{n_0+k_0+1}(M_N) < \rho_\theta - \delta_0. \]
Non universality of fluctuations of outlier eigenvectors

Note that Capitaine et al. (2011) assumes that the distribution $\mu$ is symmetric but this assumption can be removed. Indeed, this assumption is used for establishing Theorem 5.1 in Capitaine et al. (2011) that is now generalized by Theorem 1.1 in Belinschi and Capitaine (2017). Note also that Capitaine et al. (2011) assumes moreover that the support of $\nu$ has a finite number of connected components in order to prove Theorem 6.1 in Capitaine et al. (2011) but this assumption is removed in Theorem 2.3 in Capitaine and Péché (2016).

Remark 1.6. Note that

$$\{u \in \mathbb{R} \setminus \text{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2}\} = \{u \in \mathbb{R} \setminus \text{supp}(\nu), H'(u) > 0\}$$

and for any $\theta$ in this set, $\rho_\theta = \theta + \sigma^2 g_\theta(\theta) = H(\theta)$, where $H$ is defined by (2.14) (see Section 2.4 below).

It turns out that we can also describe the angle between the eigenvector associated to the outlier of $M_N$ and the corresponding eigenvector associated to the spike $\theta$. Capitaine (2013) (see also Capitaine and Donati-Martin, 2017) proved

**Proposition 1.7.** Capitaine (2013) We keep the notation and hypothesis of Proposition 1.5. Let $\xi$ be a unit eigenvector associated to one of the eigenvalues $(\lambda_{n+i}(M_N), 1 \leq i \leq k_0)$. Then, a.s.

$$\|P_{Ker(A_N-\theta I)}(\xi)\| \longrightarrow_{N \to +\infty} \tau(\theta) := 1 - \sigma^2 \int \frac{1}{(\theta-x)^2} d\nu(x), \quad (1.5)$$

where $P_E$ denotes the orthogonal projection onto any subspace $E$.

Note that fluctuations of outliers for deformed non-Gaussian Wigner matrices have been more extensively studied in the case of perturbations $A_N$ of fixed rank $r$. We emphasize that the limiting distribution in the CLT for outliers depends on the localisation/delocalisation of the eigenvector of the spike. Roughly speaking, in the delocalized case, the limiting distribution of the fluctuations of the corresponding outliers is Gaussian. In the localized case, the limiting distribution depends on the distribution $\mu$ of the entries and thus, this uncovers a non universality phenomenon. We refer to Capitaine et al. (2012) for these results.

We first recall the fluctuations of the largest eigenvalue $\lambda_1(M_N)$ when the matrix $A_N$ is a diagonal matrix of rank 1 in the localized case.

**Proposition 1.8.** Capitaine et al. (2009) Assume that $A_N = \text{diag}(\theta, 0, \ldots, 0)$ with $\theta > \sigma$. The fluctuations of $\lambda_1(M_N)$ around $\rho_0 = \theta + \frac{2^2}{\sigma^2}$ are given by

$$c_\theta \sqrt{N} (\lambda_1(M_N) - \rho_0) \xrightarrow{\text{law}}_{N \to \infty} \mu * N(0, v_0^2)$$

where $c_\theta = (1 - \frac{\sigma^2}{\theta^2})^{-1}$, $v_0^2 = \frac{1}{2} m_4 - \frac{3\sigma^4}{\theta^2}$ and $m_4$ denotes the fourth moment of $\mu$.

Capitaine and Péché (2016) proved a fluctuation result for any outlier of a full rank deformation of a G.U.E. matrix. Their result yields the following

**Proposition 1.9.** Capitaine and Péché (2016) Assume that $W_N$ is a G.U.E. matrix (that is $\mu = N(0, \sigma^2)$) and that $A_N$ is a diagonal matrix with a spike $\lambda_0(A_N) =$
\( \theta \in \Theta_{\sigma, \nu} \) of multiplicity one and limiting spectral distribution \( \nu \). The fluctuations of \( \lambda_{i_0}(M_N) \) around

\[
\rho_{\theta}^{(N)} = \theta + \sigma^{2} \frac{1}{N-1} \sum \frac{1}{\lambda_j(A_N) \neq \theta} \frac{1}{\theta - \lambda_j(A_N)}
\]

are given by\(^1\):

\[
c_{\theta, \nu} \sqrt{N} \left( \lambda_{i_0}(M_N) - \rho_{\theta}^{(N)} \right) \overset{\text{(law)}}{\rightarrow} N(0, \sigma_{\theta, \nu}^{2})
\]

where \( c_{\theta, \nu} = \left( 1 - \sigma^2 \int \frac{1}{(\theta - x)^2} d\nu(x) \right)^{-1} \) and

\[
\sigma_{\theta, \nu}^{2} = \sigma^2 \left( 1 - \sigma^2 \int \frac{1}{(\theta - x)^2} d\nu(x) \right)^{-1}.
\]

### 1.3. Main results.

In the following, we consider block diagonal perturbations. We focus on spikes of the perturbation with multiplicity one generating an outlier in the spectrum of the deformed Wigner model. Therefore, we shall consider the following assumption on \( A_N \) throughout the paper:

\( (A') \) \( A_N \) satisfies (A) and

\[ A_N = \text{diag}(A_p, A_{N-p}), \]

with \( A_{N-p} \) a \((N-p) \times (N-p)\) Hermitian matrix for some fixed integer \( p \), \( A_p = PDP^* \) is a fixed matrix (independent of \( N \)) where \( P \) a \( p \times p \) unitary matrix and \( D \) is a diagonal matrix.

Assume that \( A_N \) has a spike of multiplicity one, which is an eigenvalue of \( A_p \), \( \theta = \lambda_{i_0}(A_N) \) for some \( i_0 \), satisfying (1.3). Without loss of generality, we can assume that \( \theta = D_{11} \).

We set

\[ W_N = \begin{pmatrix} W_p & Y^* \\ Y & W_{N-p} \end{pmatrix}, \]

where \( W_p \in M_p(\mathbb{C}), Y \in M_{(N-p) \times p}(\mathbb{C}) \) and \( W_{N-p} \in M_{N-p}(\mathbb{C}) \).

The main results of this paper are the following Theorems 1.11 and 1.12 on non universality of fluctuations of an eigenvector associated with such an outlier. But, sticking to the approach of Capitaine (2020), we first establish Theorem 1.10 below, which is an extension in the non-Gaussian case of Proposition 1.8 and Proposition 1.9, in the block diagonal case.

Let \( M_N \) be defined by (1.2) with assumptions (W) and (A'). By Proposition 1.5,

\[ \lambda_{i_0}(M_N) \overset{\rightarrow}{\rightarrow}_{N \to +\infty} \rho_{\theta} \ a.s.. \]  

(1.6)

Define

\[ \rho_N = \theta + \sigma^2 g_{\mu_{A_{N-p}}} (\theta). \]

(1.7)

Note that

\[ \rho_N \overset{\rightarrow}{\rightarrow}_{N \to +\infty} \rho_{\theta}. \]

(1.8)

\(^1\)They consider fluctuations around this point depending on \( N \) in order to not prescribe speed of convergence of \( \mu_{A_N} \) to \( \nu \).
Theorem 1.10. Let $M_N$ be defined by (1.2) with assumptions (W) and (A'). Define

$$C_p = \begin{cases} \text{com}(\theta I_p - A_p), & \text{if } \theta I_p - A_p \neq 0, \\ I_p, & \text{else} \end{cases}$$

(1.9)

where com(B) denotes the comatrix of a matrix B, and

$$c_p^{(1)} = \left(1 + \sigma^2 \int \frac{d\lambda(x)}{(\rho_i - x)^2}\right) \text{Tr}_p(C_p).$$

Then

$$d_{LP}(c_p^{(1)} \sqrt{N}(\lambda_{ij}(M_N) - \rho_N), \Phi_N) \xrightarrow{N \to \infty} 0$$

(1.10)

where $\Phi_N = \text{Tr}_p(C_p W_p) + Z_N$, $W_p$ is the $p \times p$ upper left corner of the Wigner matrix $W_N$, $Z_N$ is a Gaussian random variable, independent from $W_p$, with mean $0$ and variance $\nu_p(N)$, with

$$\nu_p(N) = \text{Tr}_p(C_p^2) \sigma^4 \int \frac{d\lambda(x)}{(\rho_i - x)^2} + \frac{1}{2} \left(m_4 - 3\sigma^4\right) \kappa_N \sum_{i=1}^p \left((C_p)_{ii}\right)^2,$$

$$\lambda = \mu_\text{as} \equiv \nu, \quad \kappa_N = \frac{1}{N^p} \sum_{i=1}^{N^p} ((\theta I_{N-p} - A_{N-p})^{-1})_{ii}^2.$$

In particular, if $A_{N-p}$ is diagonal, $c_p^{(1)} \sqrt{N}(\lambda_{ij}(M_N) - \rho_N)$ converges in distribution to $\text{Tr}_p(C_p W_p) + Z$ where $Z$ is a Gaussian random variable, independent from $W_p$, with mean $0$ and variance $\nu_p$, with

$$\nu_p = \text{Tr}_p(C_p^2) \sigma^4 \int \frac{d\lambda(x)}{(\rho_i - x)^2} + \frac{1}{2} \left(m_4 - 3\sigma^4\right) \int \frac{d\nu(x)}{(\theta - x)^2} \sum_{i=1}^p \left((C_p)_{ii}\right)^2.$$

The aim of Theorems 1.11 and 1.12 below is to study the fluctuations associated to the a.s. convergence given in Proposition 1.7 above, for block diagonal perturbations. We first state an approximation result in distribution, in the spirit of Najim and Yao (2016) for perturbations $A_N$ satisfying (A').

Theorem 1.11. Let $M_N$ be defined by (1.2) with assumptions (W) and (A'). Let $u_{i_0}$, resp. $v_{i_0}$ be a unit eigenvector associated to the spike $\theta$ of $A_N$, resp. the outlier $\lambda_{i_0}(M_N)$. Define $\tau_N(\theta)$ an approximation of $\tau(\theta)$ by

$$\tau_N(\theta) = 1 - \sigma^2 \int \frac{1}{(\theta - x)^2} d\mu_{A_{N-p}}(x).$$

(1.11)

Then

$$d_{LP}(\sqrt{N}(|u_{i_0}, v_{i_0}|^2 - \tau_N(\theta)), \Psi_N) \xrightarrow{N \to \infty} 0$$

where the r.v. $\Psi_N$ is given by

$$\Psi_N = (P^*(c_{\theta, \sigma} W_p + Z_{p,N}) P)_{11},$$

where

$$c_{\theta, \sigma} = \sigma^2 g''(\theta),$$

(1.12)

$W_p$ is the $p \times p$ upper left corner of the Wigner matrix $W_N$, $Z_{p,N}$ is a centered Gaussian Hermitian matrix of size $p$, independent from $W_p$, with independent entries (modulo the symmetry conditions). The diagonal coefficients are i.i.d. with variance

$$\sigma^4 B_{\theta, \nu} + \frac{1}{2} (m_4 - 3\sigma^4) A_{\theta, \nu,N}$$

(1.13)
where
\[ B_{\theta,\nu} = - \frac{1}{6} g''_{\nu}(\theta) - \frac{\sigma^2}{2} (g''_{\nu}(\theta))^2 \frac{1 + 2\sigma^2 g'_{\nu}(\theta)}{1 + \sigma^2 g'_{\nu}(\theta)}. \] (1.14)

\[ A_{\theta,\nu,N} = \frac{1}{N-p} \sum_{i=1}^{N-p} (\sigma^2 g''_{\nu}(\theta)[(\theta I_{N-p} - A_{N-p})^{-1}]_{ii} - (1 + \sigma^2 g'_{\nu}(\theta))[\theta I_{N-p} - A_{N-p}]^{-2})_{ii}), \] (1.15)

\[ m_4 \text{ is the fourth moment of } \mu, \text{ and } g_{\nu} \text{ is the Stieltjes transform of } \nu. \]

The off diagonal elements \(Z_{p,N}(i,j), i<j\) are iid complex Gaussian with distribution \(Z\) such that 
\[ \mathbb{E}(Z^2) = 0 \text{ and } \mathbb{E}(|Z|^2) = \sigma^4 B_{\theta,\nu}. \]

In the case where the matrix \(A_{N-p}\) is a diagonal matrix, the sequence \((A_{\theta,\nu,N})_N\) defined in (1.15) converges as \(N\) tends to infinity. This leads to the following fluctuations result:

**Theorem 1.12.** Let \(M_N\) be defined by (1.2) with assumptions (W), (A') and \(A_{N-p}\) diagonal.

Let \(u_{i_0}\), resp. \(v_{i_0}\) be a unit eigenvector associated to the spike \(\theta\) of \(A_N\), resp. to the outlier \(\lambda_{i_0}(M_N)\). Then,
\[ \sqrt{N}(|\langle u_{i_0}, v_{i_0} \rangle|^2 - \tau_N(\theta)) \xrightarrow{(law)} \left(P^*(c_{\theta,\sigma} W_p + Z_p)P \right)_{11}, \] (1.16)

where \(\tau_N(\theta)\) is defined by (1.11). \(W_p\) is the \(p \times p\) upper left corner of the Wigner matrix \(W_N\), \(Z_p\) is a centered Gaussian Hermitian matrix of size \(p\) independent from \(W_p\), with independent entries (modulo the symmetry condition). The diagonal coefficients are i.i.d. with variance
\[ \frac{1}{2}(m_4 - 3\sigma^4) A_{\theta,\nu} + \sigma^4 B_{\theta,\nu} \] (1.17)

and the off diagonal elements are iid complex Gaussian with distribution \(Z\) such that 
\[ \mathbb{E}(Z^2) = 0 \text{ and } \mathbb{E}(|Z|^2) = \sigma^4 B_{\theta,\nu} \]

where
\[ \begin{cases} 
\ c_{\theta,\nu} = \sigma^2 g''_{\nu}(\theta), \\
A_{\theta,\nu} = -\frac{1}{6} g''_{\nu}(\theta)(1 + \sigma^2 g'_{\nu}(\theta))^2 - 2\sigma^4 (g''_{\nu}(\theta))^2 g'_{\nu}(\theta) - \sigma^2 (g''_{\nu}(\theta))^2, \\
B_{\theta,\nu} = -\frac{1}{6} g''_{\nu}(\theta) - \sigma^2 (g''_{\nu}(\theta))^2 \frac{1 + 2\sigma^2 g'_{\nu}(\theta)}{1 + \sigma^2 g'_{\nu}(\theta)}. 
\end{cases} \] (1.18)

\[ m_4 \text{ is the fourth moment of } \mu \text{ and } g_{\nu} \text{ is the Stieltjes transform of } \nu. \]

In particular, we readily deduce the following corollary.

**Corollary 1.13.** Let \(M_N\) be defined by (1.2) with assumptions (W) and (A').

Assume moreover that \(A_N\) is a diagonal matrix \((p=1)\). Let \(u_{i_0}\), resp. \(v_{i_0}\) be a unit eigenvector associated to the spike \(\theta\) of \(A_N\), resp. the outlier \(\lambda_{i_0}(M_N)\). Then,
\[ \sqrt{N}(|\langle u_{i_0}, v_{i_0} \rangle|^2 - \tau_N(\theta)) \xrightarrow{(law)} c_{\theta,\nu} W_{11} + Z \] (1.19)

where \(Z\) is a centered Gaussian variable, independent from \(W_{11}\), with variance :
\[ \frac{1}{2}(m_4 - 3\sigma^4) A_{\theta,\nu} + \sigma^4 B_{\theta,\nu}. \] (1.20)

See Eq.(1.18) for the definitions of \(c_{\theta,\sigma}, A_{\theta,\nu}, B_{\theta,\nu}\).
Note that when the Wigner matrix is Gaussian, the choice of a diagonal matrix for $A_N$ is not a restriction, due to the unitary invariance of the G.U.E.

The proof of Theorem 1.11 relies upon a representation, through Helffer-Sjöstrand formula, of the variable $[\langle u_{i0}, v_{i0} \rangle]^2$ in terms of the $p \times p$-matrix valued process $\{G_p(z), z \in \mathbb{C}\setminus\mathbb{R}\}$ where $G_p(z)$ denotes the principal submatrix of size $p$ of the resolvent matrix $G(z) = (zI_N - M_N)^{-1}$. Then, the fluctuations of the process $\{G_p(z), z \in \mathbb{C}\setminus\mathbb{R}\}$ are analysed using Schur’s formula which enables to express $\{G_p(z), z \in \mathbb{C}\setminus\mathbb{R}\}$ in terms of random sesquilinear forms. This approach is described in Section 4 where we also prove Theorem 1.12. Section 2 gathers preliminary results used in the proof of the main results. Therein, first we recall classical algebraic identities, Helffer-Sjöstrand’s calculus, tightness criterion for random analytic process and some basic facts on free convolution with a semicircular distribution; later we establish some extension of central limit theorem for random quadratic forms and recall or deduce some results on deformed Wigner matrices from Belinschi and Capitaine (2017); Capitaine (2020). In Section 3, we first establish Theorem 1.10, sticking to the approach of Capitaine (2020). The last Section is an appendix reminding the reader about Poincaré inequality and concentration phenomenon.

For any integer number $k$, we will say that a matrix-valued function $f_N$ on $\mathbb{C}\setminus\mathbb{R}$ is $O\left(\frac{1}{N^k}\right)$ if there exists a polynomial $Q$ with nonnegative coefficients and an integer number $d$ such that for all large $N$, for any $z$ in $\mathbb{C}\setminus\mathbb{R}$,

$$\|f_N(z)\| \leq \frac{Q(|z|^{-1})(|z|+1)^d}{N^k}.$$ 

For a family of functions $f_N^{(i)}$, $i \in \{1, \ldots, N\}$, we will set $f_N^{(i)} = O\left(\frac{1}{N^k}\right)$ if for each $i$, $f_N^{(i)} = O\left(\frac{1}{N^k}\right)$ and moreover one can find a bound of the norm of each $f_N^{(i)}$ as above involving a common polynomial $Q$ and a common $d$, that is not depending on $i$.

For two sequences $(X_N)_N$ and $(Y_N)_N$ of random variables, $X_N = Y_N + o_p(1)$ means that $X_N - Y_N \rightarrow_{N \rightarrow \infty} 0$ in probability.

2. Preliminaries

2.1. Basic identities and inequalities. For a matrix $A \in M_N(\mathbb{C})$ and $I$ and $J$ non empty subsets of $\{1, \ldots, N\}$, we denote by $A_{I \times J}$ the submatrix of $A$ obtained by keeping the rows with indices $i \in I$ and columns with indices $j \in J$. We set $A_I := A_{I \times I}$.

**Proposition 2.1** (Schur inversion formula). Let $I$ be a non-empty subset of $\{1, \ldots, N\}$ and $A \in M_N(\mathbb{C})$ such that $A_I$ is invertible, then $A$ is invertible if and only if $A_{I^c} - A_{I^c \times I}A_I^{-1}A_{I \times I^c}$ is invertible, in which case the following formulas hold:

$$(A^{-1})_I = (A_I)^{-1} + (A_I)^{-1}A_{I \times I^c}(A_{I^c} - A_{I^c \times I}A_I^{-1}A_{I \times I^c})^{-1}A_{I^c \times I}(A_I)^{-1},$$

$$(A^{-1})_{I \times I^c} = -(A_I)^{-1}A_{I \times I^c}(A_{I^c} - A_{I^c \times I}A_I^{-1}A_{I \times I^c})^{-1},$$

$$(A^{-1})_{I^c \times I} = -(A_{I^c} - A_{I^c \times I}A_I^{-1}A_{I \times I^c})^{-1}A_{I \times I}(A_I)^{-1},$$

$$(A^{-1})_{I^c} = A_{I^c} - A_{I^c \times I}A_I^{-1}A_{I \times I^c}.$$
Lemma 2.2. For any matrix $B \in M_N(\mathbb{C})$ and for any fixed $k$, we have

$$\sum_{l=1}^{N} |B_{lk}|^2 \leq \|B\|^2$$  \hspace{1cm} (2.1)

(or equivalently

$$\sum_{l=1}^{N} |B_{kl}|^2 \leq \|B\|^2.$$

(2.2)

Therefore, we have

$$\frac{1}{N} \sum_{k,l=1}^{N} |B_{kl}|^2 \leq \|B\|^2.$$  \hspace{1cm} (2.3)

Proof: Note that

$$\sum_{l=1}^{N} |B_{lk}|^2 = \text{Tr}_N(BE_{kk}B^*)$$

$$= \text{Tr}_N(B^*BE_{kk})$$

$$\leq \|B\|^2 \text{Tr}_N(E_{kk}) = \|B\|^2.$$

Now, since

$$\sum_{l=1}^{N} |B_{kl}|^2 = \sum_{l=1}^{N} |B_{kl}|^2 = \sum_{l=1}^{N} |(B^*)_{lk}|^2 \text{ and } \|B^*\| = \|B\|,$$

(2.1) and (2.2) can be deduced from each other thanks to conjugate transposition. Finally (2.2) readily yields (2.3). \hfill \Box

Lemma 2.3 (Lemma A2 of Capitaine, 2020). Let $A$ and $H$ be $m \times m$ matrices such that, for some $K > 0$,

$$\|A\| \leq K, \|H\| \leq K.$$  \hspace{1cm} (2.4)

Then

$$\det(A + H) = \det(A) + \text{Tr}_m \left( \text{com}(A)H \right) + \epsilon,$$

where $\text{com}(A)$ denotes the comatrix of $A$, and there exists a constant $C_{m,K} > 0$, only depending on $m$ and $K$, such that $|\epsilon| \leq C_{m,K} \|H\|^2$.

We will often use the following obvious facts that for any $z \in \mathbb{C} \setminus \mathbb{R}$, for any $N \times N$ Hermitian matrix $H$,

$$\| (zI_N - H)^{-1} \| \leq |3z|^{-1}$$

and for any probability measure $\nu$ on $\mathbb{R}$, the Stieltjes transform $g_\nu$ satisfies for any $z \in \mathbb{C} \setminus \mathbb{R}, 3zg_\nu(z) < 0$, $|g_\nu(z)| \leq |3z|^{-1}$, and for any $\alpha \geq 0$, any $x \in \mathbb{R},$

$$\frac{1}{|z - \alpha g_\nu(z) - x|} \leq |3z|^{-1}.$$  \hspace{1cm} (2.6)
2.2. Helffer-Sjöstrand’s calculus.

2.2.1. Helffer-Sjöstrand’s representation formula. We recall Helffer-Sjöstrand’s representation formula (see Benaych-Georges and Knowles, 2017, Proposition C.1): let $f \in C^{k+1}(\mathbb{R})$ with compact support and $M$ be a Hermitian matrix; we have

$$f(M) = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} F_k(f)(z) (M - z)^{-1} d^2z$$

where $d^2z$ denotes the Lebesgue measure on $\mathbb{C}$,

$$F_k(f)(x + iy) = \sum_{l=0}^{k} \frac{(iy)^l}{l!} f^{(l)}(x) \chi(y)$$

where $\chi : \mathbb{R} \to \mathbb{R}^+$ is a smooth compactly supported function such that $\chi \equiv 1$ in a neighborhood of $0$, and $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$.

The function $F_k(f)$ coincides with $f$ on the real axis and is an extension to the complex plane.

Moreover

$$\overline{\partial} F_k(f)(x + iy) = \frac{1}{2} \frac{(iy)^k}{k!} f^{(k+1)}(x) \chi(y) + i \sum_{l=0}^{k} \frac{(iy)^l}{l!} f^{(l)}(x) \chi'(y).$$

Thus, since $\chi \equiv 1$ in a neighborhood of $0$, we have that, in a neighborhood of the real axis,

$$\overline{\partial} F_k(f)(x + iy) = \frac{1}{2} \frac{(iy)^k}{k!} f^{(k+1)}(x) = O(|y|^k) \text{ as } y \to 0.$$  

2.2.2. Computation of Helffer-Sjöstrand’s integral.

**Proposition 2.4.** Let $h$ be a smooth function with compact support in $(\rho_0 - 2\delta, \rho_0 + 2\delta)$ and satisfying $h \equiv 1$ on $[\rho_0 - \delta, \rho_0 + \delta]$. Let $\chi$ be a compactly supported function on $(-L, L)$, and $\chi = 1$ around $0$. We denote by $D = (\rho_0 - 2\delta, \rho_0 + 2\delta) \times (-L, L)$. Let $\phi$ be a meromorphic function in $D$, with a pole in $\rho_0$. Then,

$$I(\phi) := \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} F_k(h)(z) \phi(z) d^2z = -\text{Res}(\phi, \rho_0)$$

where $F_k(h)$ is defined in (2.8). $\text{Res}(\phi, \rho_0)$ denotes the residue of the function $\phi$ at the point $\rho_0$.

**Proof:** Let $\epsilon$ small enough such that $F_k(h)(z) = 1$ for $z \in B(\rho_0, \epsilon)$. Set $D_\epsilon = D \setminus B(\rho_0, \epsilon)$. $\phi$ is holomorphic on $D_\epsilon$. Since $F_k(h)$ has compact support in $D$, we have,

$$0 = \int_{\partial D} F_k(h)(z) \phi(z) dz = \int_{\partial D_\epsilon} F_k(h)(z) \phi(z) dz + \int_{\partial B(\rho_0, \epsilon)} F_k(h)(z) \phi(z) dz$$

$$= 2i \int_{D_\epsilon} \overline{\partial} F_k(h)(z) \phi(z) d^2z + \int_{\partial B(\rho_0, \epsilon)} \phi(z) dz$$

where the first term is obtained by Green’s formula using that $\overline{\partial} \phi(z) = 0$ on $D_\epsilon$,

$$\int_{D_\epsilon} \overline{\partial} F_k(h)(z) \phi(z) d^2z \to 0$$

as $\epsilon \to 0$. Therefore,

$$\int_{D} \overline{\partial} F_k(h)(z) \phi(z) d^2z = -\pi I(\phi)$$

as $\epsilon \to 0$. Therefore,

$$\int_{D} \overline{\partial} F_k(h)(z) \phi(z) d^2z = -\pi I(\phi)$$

as $\epsilon \to 0$. Therefore,

$$\int_{D} \overline{\partial} F_k(h)(z) \phi(z) d^2z = -\pi I(\phi)$$

as $\epsilon \to 0$. Therefore,
and
\[ \int_{\partial B(\rho_0, r)} F_k(h)(z) \phi(z) dz = 2i\pi \text{Res}(\phi, \rho_0). \]

\[ \square \]

2.3. Tightness criterion for a sequence of random analytic processes. We recall here some results from Shirai (2012). Let \( D \subset \mathbb{C} \) be an open set in the complex plane. Denote by \( \mathcal{H}(D) \) the space of complex analytic functions in \( D \), endowed with the uniform topology on compact set. For \( f \in \mathcal{H}(D) \) and \( K \) a compact set of \( D \), we denote \( \|f\|_K = \sup_{z \in K} |f(z)| \). The space \( \mathcal{H}(D) \) is equipped with the (topological) Borel \( \sigma \)-field \( \mathcal{B}(\mathcal{H}(D)) \) and the set of probability measures on \( (\mathcal{H}(D); \mathcal{B}(\mathcal{H}(D))) \) is denoted by \( \mathcal{P}(\mathcal{H}(D)) \). By a random analytic function on \( D \) we mean an \( \mathcal{H}(D) \)-valued random variable on a probability space. The probability law of a random analytic function is uniquely determined by its finite dimensional distributions.

**Proposition 2.5** (Proposition 2.5. in Shirai, 2012). Let \( f_n \) be a sequence of random analytic functions in \( D \). If \( \|f_n\|_K \) is tight for any compact set \( K \), then \( \mathcal{L}(f_n) \) is tight in \( \mathcal{P}(\mathcal{H}(D)) \).

Using that, by Markov’s inequality, for any \( C > 0 \) and any \( r > 0 \),
\[ \mathbb{P}(\|f_n\|_K > C) \leq \frac{1}{C^r} \mathbb{E}(\|f_n\|_K^r), \tag{2.12} \]

the following lemma turns out to be useful to prove tightness results.

**Lemma 2.6** (lemma 2.6 of Shirai, 2012). For any compact set \( K \) in \( D \), there exists \( \delta > 0 \) such that
\[ \|f\|_K^r \leq (\pi\delta^2)^{-1} \int_{K_\delta} |f(z)|^r m(dz), \quad f \in \mathcal{H}(D), \]
for any \( r > 0 \), where \( K_\delta \subset D \) is the closure of the \( \delta \)-neighborhood of \( K \) and \( m \) denotes the Lebesgue measure.

2.4. Free convolution with a semicircular distribution. Let \( \tau \) be a probability measure on \( \mathbb{R} \). Its Stieltjes transform \( g_\tau : z \mapsto \int_{\mathbb{R}} \frac{1}{z-x} d\tau(x) \) is analytic on the complex upper half-plane \( \mathbb{C}^+ \). There exists a domain
\[ D_{\alpha, \beta} = \{u + iv \subset \mathbb{C}, |u| < \alpha, v > \beta \} \]
on which \( g_\tau \) is univalent. Let \( K_\tau \) be its inverse function, defined on \( g_\tau(D_{\alpha, \beta}) \), and
\[ R_\tau(z) = K_\tau(z) - \frac{1}{z}. \]

Given two probability measures \( \tau \) and \( \nu \), there exists a unique probability measure \( \lambda \) such that
\[ R_\lambda = R_\tau + R_\nu \]
on a domain where these functions are defined. The probability measure \( \lambda \) is called the free convolution of \( \tau \) and \( \nu \) and denoted by \( \tau \boxplus \nu \).

The free convolution of probability measures has an important property, called subordination, which can be stated as follows: let \( \tau \) and \( \nu \) be two probability measures on \( \mathbb{R} \); there exists an analytic map \( \omega_{\tau, \nu} : \mathbb{C}^+ \to \mathbb{C}^+ \) such that
\[ \forall z \in \mathbb{C}^+, \ g_{\tau \boxplus \nu}(z) = g_\nu(\omega_{\tau, \nu}(z)). \]
This phenomenon was first observed by D. Voiculescu under a genericity assumption in Voiculescu (1993), and then proved in generality in Biane (1998) Theorem 3.1. Later, a new proof of this result was given in Belinschi and Bercovici (2007), using a fixed point theorem for analytic self-maps of the upper half-plane.

In Biane (1997), P. Biane provides a deep study of the free convolution by a semicircular distribution. We first recall here some of his results that will be useful in our approach. Let \( \nu \) be a probability measure on \( \mathbb{R} \) and \( \mu_{sc} \) the semicircular distribution defined by (1.1). For any \( z \in \mathbb{C} \setminus \text{supp}(\mu_{sc} \boxplus \nu) \), the subordination function \( \omega_{\mu_{sc},\nu} \) is given by

\[
\omega_{\mu_{sc},\nu}(z) = z - \sigma^2 g_{\mu_{sc}}(z).
\]  

(2.13)

In the following, we will denote \( \omega_{\mu_{sc},\nu} \) by \( \omega \). For any \( x \in \mathbb{C} \setminus \text{supp}(\nu) \), define

\[
H(x) = x + \sigma^2 g_{\nu}(x).
\]  

(2.14)

We have for any \( x \in \mathbb{C} \setminus \text{supp}(\mu_{sc} \boxplus \nu) \),

\[
H(\omega(x)) = x.
\]  

(2.15)

More precisely, the following one to one correspondance holds:

\[
\mathbb{R} \setminus \text{supp}(\mu_{sc} \boxplus \nu) \xrightarrow{\omega} \{ u \in \mathbb{R} \setminus \text{supp} \nu, \int \frac{1}{(u-x)^2} d\nu(x) < \frac{1}{\sigma^2} \}. \]

(2.16)

Assume that \( A_N \) satisfies (A). Let \( \{ \theta_j, 1 \leq j \leq q \} \) be the spiked eigenvalue of \( A_N \) outside \( \text{supp}(\nu) \). Furthermore, for all \( \theta_j \in \Theta_{\sigma,\nu} \), where \( \Theta_{\sigma,\nu} \) is defined by (1.3), we set

\[
\rho_{\theta_j} := H(\theta_j) = \theta_j + \sigma^2 g_{\nu}(\theta_j)
\]  

(2.17)

which is outside the support of \( \mu_{sc} \boxplus \nu \) according to (2.16), and we define

\[
K_{\sigma,\nu}(\theta_1, \ldots, \theta_q) := \text{supp}(\mu_{sc} \boxplus \nu) \bigcup \{ \rho_{\theta_j}, \theta_j \in \Theta_{\sigma,\nu} \}.
\]

An important consequence of (2.16) is the following

**Proposition 2.7** (Theorem 2.3 in Capitaine and Péché, 2016). For any \( \delta > 0 \),

\[
\text{supp}(\mu_{sc} \boxplus \mu_{A_N}) \subset K_{\sigma,\nu}(\theta_1, \ldots, \theta_q) + (-\delta, \delta),
\]

when \( N \) is large enough.

2.5. Central limit theorem for processes of matrix valued random quadratic forms.

**Lemma 2.8** (Lemma 2.7 in Bai and Silverstein, 1998). There exists \( C > 0 \) such that for any \( N \times N \) deterministic matrix \( B = (b_{ij})_{1 \leq i,j \leq N} \), any random vectors

\[
Y = \begin{pmatrix}
y_1 \\
\vdots \\
y_N
\end{pmatrix}
\]

in \( \mathbb{C}^N \) with i.i.d. standardized entries ( \( E(y_i) = 0 \), \( E(|y_i|^2) = 1 \), \( E(y_i^2) = 0 \) ) such that \( \mathbb{E}(|y_i|^4) \leq \kappa \) and any independent copy \( X \) of \( Y \), one has

\[
\mathbb{E}((Y^*BY - \text{Tr}_N(B))^2) \leq C\kappa \text{Tr}_N(B^*B),
\]

\[
\mathbb{E}(|Y^*BX|^2) \leq C\kappa \text{Tr}_N(B^*B).
\]

In Najim and Yao (2016), the authors establish the following variation around the central limit theorem for martingales.
Lemma 2.9 (Lemma 5.6 in Najim and Yao, 2016). Suppose that for each \( n \) \((Y_{n,j}; 1 \leq j \leq r_n)\) is a \( \mathbb{C}^d \)-valued martingale difference sequence with respect to the increasing \( \sigma \)-field \( \mathcal{G}_{n,j}; 1 \leq j \leq r_n \) having second moments. Write
\[
Y^T_{nj} = (Y^1_{nj}, \ldots, Y^d_{nj}).
\]
Assume moreover that \((\Theta_n(k,l))_n\) and \((\tilde{\Theta}_n(k,l))_n\) are uniformly bounded sequences of complex numbers, for \( 1 \leq k, l \leq d \). If
\[
\sum_{j=1}^{r_n} \mathbb{E} \left( Y^k_{nj} Y^l_{nj} \ | \ \mathcal{G}_{n,j-1} \right) - \Theta_n(k,l) \xrightarrow{P} 0, \quad n \to +\infty
\]
(2.18)
and for each \( \epsilon > 0 \),
\[
\sum_{j=1}^{r_n} \mathbb{E} \left( \|Y_{nj}\|^2_1 \|Y_{nj}\| > \epsilon \right) \to_{n \to +\infty} 0,
\]
(2.19)
and for each \( \epsilon > 0 \),
\[
\sum_{j=1}^{r_n} \mathbb{E} \left( \|Y_{nj}\|^2_1 \|Y_{nj}\| > \epsilon \right) \to_{n \to +\infty} 0,
\]
(2.20)
where \( Z_n \) is a \( \mathbb{C}^d \)-valued centered Gaussian random vector with parameters
\[
\mathbb{E}(Z_n Z^*_n) = (\Theta_n(k,l))_{k,l} \text{ and } \mathbb{E}(Z_n Z^*_n) = (\tilde{\Theta}_n(k,l))_{k,l}.
\]

Following the lines of the proof of the central limit theorem for quadratic forms by Baik and Silverstein in the appendix of Capitaine et al. (2009) and using Lemma 2.9, we will establish the following extension.

Proposition 2.10. Let \((z_1, \ldots, z_q)\) be in \( I^q \), where \( I \) is a subset of \( \mathbb{C} \) such that \( \forall z \in I, \bar{z} \in I \). For any \( z \) in \( \{z_1, \ldots, z_q\} \), let \( B(z) = (b_{ij}(z)) \) be a \( N \times N \) matrix such that \( (B(z))^* = B(\bar{z}) \) and there exists a constant \( a > 0 \) (not depending on \( N \)) such that for any \( z \) in \( \{z_1, \ldots, z_q\} \), \( \|B(z)\| \leq a \). Let \( p \) be a fixed integer number and \( Y_N = (y_{ij})_{i \leq i \leq N, 1 \leq j \leq p} \) be a \( N \times p \) matrix which contains i.i.d. complex standardized entries with bounded fourth moment and such that \( \mathbb{E}y_{ij}^4 = 0 \). Set
\[
V_N = \left( (1/\sqrt{N}) (Y_N^* B(z_1) Y_N - \text{Tr}_N(B(z_1)) I_p), \ldots, (1/\sqrt{N}) (Y_N^* B(z_q) Y_N - \text{Tr}_N(B(z_q)) I_p) \right).
\]
If there exists uniformly bounded sequences \((f_N)_N\) and \((g_N)_N\) of functions on \( I^2 \) such that for any \( z_i, z_k \),
\[
\frac{1}{N} \sum_{i=1}^{N} (B(z_i))_{ii} (B(z_k))_{ii} = f_N(z_i, z_k) + o(1),
\]
and
\[
\text{tr}_N(B(z_i) B(z_k)) = g_N(z_i, z_k) + o(1),
\]
then \( d_{LP}(V_N, V_N) \to 0 \) where \( V_N = (\mathcal{G}_N(z_1), \ldots, \mathcal{G}_N(z_q)) \), \( \mathcal{G}_N \) is a centered Gaussian matrix valued process whose distribution is given as follows:
\begin{enumerate}
\item the processes \((\mathcal{G}_N(z))_{ij}\) for \( 1 \leq i \leq j \leq p \) are independent, and \( (\mathcal{G}_N(z))_{ji} = (\mathcal{G}_N(z))_{ij} \).
\end{enumerate}
2) For \( i \leq p \),
\[
\mathbb{E}(\mathcal{G}_N(z_l)_{ii}(\mathcal{G}_N(z_k))_{ii}) = (\mathbb{E}(|y_{l1}|^4 - 2)(f_N(z_l, z_k)) + g_N(z_l, z_k)
\]
and
\[
\mathbb{E}(\mathcal{G}_N(z_l)_{ij}(\mathcal{G}_N(z_k))_{ii}) = \mathbb{E}(\mathcal{G}_N(z_l)_{ij}(\mathcal{G}_N(z_2))_{ii})
\]
(2.21)
3) For \( 1 \leq i \neq j \leq p \),
\[
\mathbb{E}(\mathcal{G}_N(z_l)_{ij}(\mathcal{G}_N(z_k))_{ij}) = 0
\]
and
\[
\mathbb{E}(\mathcal{G}_N(z_l)_{ij}(\mathcal{G}_N(z_k))_{ij}) = g_N(z_l, z_k).
\]
(2.22)

Proof: First, for any \( B \) in \( \{B(z), z = z_1, \ldots, z_q\} \) and any \( 1 \leq s, t \leq p \), one can write \( (1/\sqrt{N})(Y_N^B BY_N - \text{Tr}(B)I_p)_{st} \) as a sum of martingale differences:

\[
(1/\sqrt{N})(Y_N^B BY_N - \text{Tr}(B)I_p)_{st} = (1/\sqrt{N})\sum_{i=1}^{N} \left( \bar{y}_{is}y_{it} - \delta_{st} \right) b_{ii} + \bar{y}_{is} \sum_{j<i} y_{jt}b_{ij} + \bar{y}_{is} \sum_{j>i} y_{jt}b_{ij}
\]

\[
= (1/\sqrt{N})\sum_{i=1}^{N} \left( \bar{y}_{is}y_{it} - \delta_{st} \right) b_{ii} + \bar{y}_{is} \sum_{j<i} y_{jt}b_{ij} + y_{it} \sum_{j<i} \bar{y}_{js}b_{ji}
\]

\[
= \sum_{i=1}^{N} (Z_i(B))_{st}
\]

where

\[
(Z_i(B))_{st} = (1/\sqrt{N}) \left( \bar{y}_{is}y_{it} - \delta_{st} \right) b_{ii} + \bar{y}_{is} \sum_{j<i} y_{jt}b_{ij} + y_{it} \sum_{j<i} \bar{y}_{js}b_{ji}
\]

Let \( \mathcal{F}_{N,i} \) be the \( \sigma \)-field generated by \( \{y_{1s}, \ldots, y_{is}, 1 \leq s \leq p\} \). Let also \( \mathbb{E}_t(\cdot) \) denote conditional expectation with respect to \( \mathcal{F}_{N,i} \). It is clear that for any \( B \) in \( \{B(z), z \in T\} \), \( Z_i(B) \in M_p(\mathbb{C}) \simeq \mathbb{C}^{p^2} \) is measurable with respect to \( \mathcal{F}_{N,i} \) and satisfies \( \mathbb{E}_{i-1}(Z_i(B)) = 0 \).

We will show that the conditions of Lemma 2.9 are met for the \( \mathbb{C}^{p^2} \)-valued martingale difference sequence \( Y_N^B = \{Z_i(B(z_1)), \ldots, Z_i(B(z_q))\} \).

Write \( (Z_i(B))_{st} = X_i^1 + X_i^2 \), with \( X_i^1 = (1/\sqrt{N})(\bar{y}_{is}y_{it} - \delta_{st})b_{ii} \). Then for \( \epsilon > 0 \),

\[
\sum_{i=1}^{N} \mathbb{E}(|X_i^1|^2 1_{(|X_i^1| \geq \epsilon)}) \leq a^2 \mathbb{E}(|\bar{y}_{is}y_{it} - \delta_{st}|^2 1_{(|\bar{y}_{is}y_{it} - \delta_{st}|^2 \geq \sqrt{N}/a)}) \to 0
\]
(2.23)
as \( N \to \infty \), by dominated convergence theorem.

We have

\[
\mathbb{E}[\sum_{j<i} |y_{jt}b_{ij}|^4
\]

\[
= \mathbb{E}[|y_{l1}|^4 \sum_{j<i} |b_{ij}|^4] + 2\mathbb{E}[\sum_{j<i} |b_{ij}|^2 |b_{ij}|^2] + \mathbb{E}[|y_{l1}|^2 \sum_{i,j} |b_{ij}|^2 |b_{ij}|^2]
\]

\[
\leq \mathbb{E}[|y_{l1}|^4 \mathbb{E}[\max_{i,j} |b_{ij}|^2 \max_{i,j} (BB^*)_{ii}]] + (2 + \mathbb{E}|y_{l1}|^2)\mathbb{E}[(BB^*)_{ii}^2]
\]

\[
\leq a^4 [\mathbb{E}|y_{l1}|^4 + 2 + \mathbb{E}|y_{l1}|^2]
\]
where the sum $\sum_i$ is over $\{j_1 < i, j_2 < i, j_1 \neq j_2\}$. Therefore $\mathbb{E}|X_2|^4 = o(N^{-1})$ so that for any $\epsilon > 0$,

$$\sum_{i=1}^{N} \mathbb{E}(|X_2|^2 1_{|X_2| \geq \epsilon}) \leq (1/\epsilon^2) \sum_{i=1}^{N} \mathbb{E}|X_2|^4 \to 0 \quad \text{as } N \to \infty. \quad (2.24)$$

Thus, by $(2.23)$, $(2.24)$ and $(A.4)$ in Capitaine et al. (2009), $\{Z_i(B)\}$ satisfies the Lindeberg condition of Lemma 2.9.

Now, we shall verify condition (2.19) of Lemma 2.9. We have for any $\|B\|$, $\|s\|$, $\|t\|, \|t^\prime\| \leq p$,

$$\sum_{i=1}^{N} \mathbb{E}_{i-1}(Z_i(B))_{st} Z_i(C)_{s^t t^\prime} \quad (2.25)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left\{ (\mathbb{E}|y_{11}|^4 - 1)\delta_{st}\delta_{ss^t}\delta_{tt^t} + \delta_{st}\delta_{tt^t}(1 - \delta_{ss^t})b_{ii}c_{ii} \right\} \quad (2.26)$$

$$+ \mathbb{E}(|y_{11}|^2 \bar{y}_{11}) \left[ \delta_{st}\sum_{j<i} y_{jt}c_{ij}b_{ii} + \delta_{s^t t^t}\sum_{j<i} y_{j}\bar{b}_{ij} \right]$$

$$+ \mathbb{E}(|y_{11}|^2 \bar{y}_{11}) \left[ \delta_{s^t t^t}\sum_{j<i} y_{ji}c_{ij}b_{ii} + \delta_{st}\sum_{j<i} \bar{y}_{ji}b_{ij} \right] \quad (2.27)$$

$$+ \delta_{s^t t^t}\sum_{j<i} \bar{y}_{ji}b_{ij} \sum_{j<i} y_{ji}c_{ij} + \delta_{st}\sum_{j<i} \bar{y}_{ji}c_{ij} \sum_{j<i} y_{ji}b_{ij} \right\}. \quad (2.28)$$

Let $B_L$ (resp. $B_U$) denote the strictly lower (resp. upper) triangular part of $B$. We have, using Cauchy-Schwarz’s inequality, that

$$\mathbb{E}[(1/N) \sum_{i=1}^{N} c_{ii} \sum_{j<i} y_{jt}b_{ij}]^2 = \mathbb{E}[(1/N) \sum_{j=1}^{N-1} y_{jt} \sum_{i>j} c_{ii}b_{ij}]^2$$

$$= (1/N^2) \mathbb{E}\left( \sum_{j=1}^{N-1} \sum_{i>j} c_{ii}b_{ij} \sum_{j>i} \bar{c}_{ij} \right)$$

$$= (1/N^2) \mathbb{E}\left( \sum_{i,j} c_{ii} \bar{c}_{ij} (B_L B_L^*)_{ij} \right)$$

$$\leq \mathbb{E}\left[ (\max_i |c_{ii}|)^2 (1/N) \sum_{i,j} |(B_L B_L^*)_{ij}|^2 \right]^{1/2}$$

$$\leq \mathbb{E}\left[ (\max_i |c_{ii}|)^2 (1/N) \text{Tr}((B_L B_L^*)^2) \right]^{1/2}$$

$$\leq \mathbb{E}\left[ (\max_i |c_{ii}|)^2 (1/\sqrt{N}) \|B_L\|^2 \right].$$

We apply the following bound (due to R. Mathias, see Mathias, 1993): $\|B_L\| \leq \gamma_N \|B\|$ where $\gamma_N = O(\ln N)$, and the bounds $\|B\| \leq a$, $\|C\| \leq a$, to conclude that

$$\frac{1}{N} \sum_{i=1}^{N} c_{ii} \sum_{j<i} y_{jt}b_{ij} \xrightarrow{P} 0, \quad N \to \infty.$$
Thus,
\[
\sum_{i=1}^{N} \mathbb{E}_{t-1}(Z_i(B))_s t Z_i(C)_{s't'}
\]
\[
= (1/N) \sum_{i=1}^{N} \left\{ (\mathbb{E}|y_{11}|^4 - 1)\delta_{st} \delta_{ss'} \delta_{s't'} + \delta_{st} \delta_{ss'} (1 - \delta_{s't'}) \right\} b_{ii} c_{ii}
\]
\[
+ \delta_{s't'} \frac{1}{N} \sum_{j<i} y_{ij} b_{ji} \sum_{j<i} y_{ji} c_{ij} + \delta_{st} \frac{1}{N} \sum_{j<i} y_{ji} c_{ij} b_{ij} + o_{F}(1)
\]
\[
= \left\{ (\mathbb{E}|y_{11}|^4 - 1)\delta_{st} \delta_{ss'} \delta_{s't'} + \delta_{st} \delta_{ss'} (1 - \delta_{s't'}) \right\} \frac{1}{N} \sum_{i=1}^{N} b_{ii} c_{ii}
\]
\[
+ \delta_{s't'} (1/N)(Y_N)^* B U C L (Y_N)_{t'} + \delta_{st} (1/N)(Y_N)^* C U B L (Y_N)_{t'} + o_{F}(1)
\]

Besides, from Lemma 2.8 we have
\[
\mathbb{E} \left| (1/N)((Y_N)^* B U C L (Y_N)_{t'} - \delta_{s't'} \text{Tr}(B U C L)) \right|^2 \leq (1/N^2) \mathbb{E}(\text{Tr}(C_L^t B_U^* B_U C_L))
\]
\[
\leq K \mathbb{E} \|B\|^2 \|C\|^2 \frac{\ln^4 N}{N} \to_{N \to \infty} 0.
\]

Hence,
\[
\sum_{i=1}^{N} \mathbb{E}_{t-1}(Z_i(B))_s t (Z_i(C))_{s't'}
\]
\[
= \left\{ (\mathbb{E}|y_{11}|^4 - 1)\delta_{st} \delta_{ss'} \delta_{s't'} + \delta_{st} \delta_{ss'} (1 - \delta_{s't'}) \right\} \frac{1}{N} \sum_{i=1}^{N} b_{ii} c_{ii}
\]
\[
+ \delta_{s't'} \frac{1}{N} \sum_{j<i} b_{ij} c_{ji} + \delta_{s't'} \delta_{ss'} \frac{1}{N} \sum_{j<i} b_{ij} c_{ij} + o_{F}(1)
\]
\[
= \left\{ (\mathbb{E}|y_{11}|^4 - 1)\delta_{st} \delta_{ss'} \delta_{s't'} + \delta_{st} \delta_{ss'} (1 - \delta_{s't'}) \right\} \frac{1}{N} \sum_{i=1}^{N} b_{ii} c_{ii}
\]
\[
+ \delta_{s't'} \delta_{s't'} \frac{1}{N} \text{Tr}(BC) - \delta_{s't'} \delta_{ss'} \frac{1}{N} \sum_{i} b_{ii} c_{ii} + o_{F}(1)
\]
\[
= \left\{ (\mathbb{E}|y_{11}|^4 - 1)\delta_{st} \delta_{ss'} \delta_{s't'} - \delta_{st} \delta_{ss'} \delta_{s't'} \delta_{ss'} \right\} \frac{1}{N} \sum_{i=1}^{N} b_{ii} c_{ii}
\]
\[
+ \delta_{s't'} \delta_{s't'} \text{Tr}(BC) + o_{F}(1).
\]

Proposition 2.10 readily follows. \(\square\)

2.6. Preliminary results on deformed Wigner matrices.

2.6.1. Preliminary results from Belinschi and Capitaine (2017). Note that, in Section 5 in Belinschi and Capitaine (2017), the authors consider for any fixed integer numbers \(m, r, t\) and any fixed \(m \times m\) Hermitian matrices \(\gamma, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_t\), the matrix model in \(M_m(\mathbb{C}) \otimes M_N(\mathbb{C}), \gamma \otimes I_N + \sum_{\nu=1}^{r} \alpha_\nu \otimes \frac{W_N^{(\nu)}}{\sqrt{N}} + \sum_{u=1}^{t} \beta_u \otimes A_N^{(u)}\) where the \(W_N^{(\nu)}\)’s are independent more general Wigner matrices than ours and
the $A^{(u)}_N$’s are deterministic matrices such that $\sup N \| A^{(u)}_N \| < \infty$. Therefore, the results therein apply to our model by choosing $m = 1$, $r = t = 1$ and $\alpha_1 = 1 = \beta_1$.

For any $z \in \mathbb{C} \setminus \mathbb{R}$, $G(z) = |G_{ij}(z)|_{1 \leq i,j \leq N} = (zI_N - M_N)^{-1}$ denotes the resolvant of $M_N$. Note that by (2.5),

$$ \| G(z) \| \leq |\Im z|^{-1}. \tag{2.28} $$

For any $z \in \mathbb{C} \setminus \mathbb{R}$, define $g_N(z) = \text{E}(\text{tr}_N G(z))$ and denote by $\tilde{g}_N(z)$ the Stieltjes transform of $\mu_{sc} \boxplus \mu_{A_N}$.

**Lemma 2.11.** For any $(i, j) \in \{1, \ldots, N\}^2$,

$$ \mathbb{E}(G_{ij}(z)) = \left[ ((z - \sigma^2\tilde{g}_N(z))I_N - A_N)^{-1} \right]_{ij} + O^{(u)} \left( \frac{1}{\sqrt{N}} \right). $$

**Proof:** According to Corollary 5.5 in Belinschi and Capitaine (2017), for any $(i, j) \in \{1, \ldots, N\}^2$,

$$ \begin{align*}
\mathbb{E}(G_{ij}(z)) &= (Y_N(z))_{ij} + \frac{(1 - \sqrt{-1})\kappa_3}{2\sqrt{2N\sqrt{N}}} \sum_{s,l=1}^{N} (Y_N(z))_{il} (Y_N(z))_{ss} (Y_N(z))_{ll} \mathbb{E}(G_{sj}(z)) \\
&\quad + O^{(u)} \left( \frac{1}{N} \right)
\end{align*} $$

where

$$ Y_N(z) = ((z - g_N(z))I_N - A_N)^{-1} $$

and $\kappa_3$ is the third classical cumulant of $\mu$. Note that $|\Im (z - g_N(z))| \geq |\Im z|$ so that, by (2.5)

$$ \| Y_N(z) \| \leq \frac{1}{|\Im z|}. \tag{2.29} $$

Now,

$$ \begin{align*}
\sum_{s,l=1}^{N} (Y_N(z))_{il} (Y_N(z))_{ss} (Y_N(z))_{ll} \mathbb{E}(G_{sj}(z)) \\
&\leq \sum_{s,l=1}^{N} |(Y_N(z))_{il}| |(Y_N(z))_{ss}| |(Y_N(z))_{ll}| |\mathbb{E}(G_{sj}(z))| \\
&\leq |\Im z|^{-2}N \left( \sum_{s=1}^{N} |\mathbb{E}(G_{sj}(z))|^2 \right)^{1/2} \left( \sum_{l=1}^{N} |(Y_N(z))_{jl}|^2 \right)^{1/2} \\
&\leq N|\Im z|^{-4}
\end{align*} $$

where we used Lemma 2.2, (2.29) and (2.28). Therefore

$$ \mathbb{E}(G_{ij}(z)) = (Y_N(z))_{ij} + O^{(u)} \left( \frac{1}{\sqrt{N}} \right). $$

Now, according to (5.56) in Belinschi and Capitaine (2017),

$$ \left\| Y_N(z) - \tilde{Y}_N(z) \right\| = O \left( \frac{1}{\sqrt{N}} \right) $$

where

$$ \tilde{Y}_N(z) = ((z - \tilde{g}_N(z))I_N - A_N)^{-1}. \tag{2.30} $$

Lemma 2.11 readily follows. \qed
Lemma 2.12. We have
\[
\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad g_N(z) = \tilde{g}_N(z) + O(\frac{1}{N}).
\]

Proof: According to Proposition 5.8 in Belinschi and Capitaine (2017), we have
\[
g_N(z) - \tilde{g}_N(z) = \left(1 - \tilde{g}_N'(z)\right) \tilde{L}_N(z) + O(\frac{1}{N\sqrt{N}})
\]
where
\[
\tilde{L}_N(z) = \frac{\kappa_4}{2N^3} \sum_{i,l=1}^{N} \left(\tilde{Y}_N(z)^2\right)_{il} \left[\left(\tilde{Y}_N(z)\right)_{ii}\right]^2 \left(\tilde{Y}_N(z)\right)_{il}
\]
and \(\tilde{Y}_N\) is defined by (2.30). (Note that in Proposition 5.8 in Belinschi and Capitaine, 2017, in full generality the \(\tilde{Y}_N(z)\)’s are \(m \times m\) matrices which a priori do not commute and \(\tilde{G}_N(zI_m)\) is a \(m \times m\) matrix too. But in the present paper, since \(m = 1\), \(\tilde{Y}_N(z)\)’s are scalar and obviously commute and \(\tilde{G}_N(zI_m) = \tilde{g}_N(z)\).) Note that, \(|\Im(z - \tilde{g}_N(z))| \geq |\Im(z)|\) so that, by (2.5),
\[
\|\tilde{Y}_N(z)\| \leq |\Im(z)|^{-1}.
\]
Lemma 2.12 readily follows by using Cauchy-Schwarz’s inequality and (2.3).

Lemma 8.7 in Belinschi and Capitaine (2017) implies in particular the following variance estimates.

Lemma 2.13.
\[
\text{Var}(G_{ij}(z)) = O(1/N).
\]

Lemma 2.14.
\[
\text{Var}(\text{tr}_N G(z)) = O(1/N^2)
\]

The following result is a corollary of Theorem 1.1 in Belinschi and Capitaine (2017).

Proposition 2.15 (Theorem 1.1 in Belinschi and Capitaine, 2017). Let \([b; c]\) be a real interval such that there exists \(\delta > 0\) such that, for any large \(N\), \([b + \delta; c + \delta]\) lies outside the support of \(\mu_{sc} \oplus \mu_{AN}\). Then, almost surely, for all large \(N\), there is no eigenvalue of \(M_N\) in \([b; c]\).
2.6.2. Quantitative asymptotic freeness. Let $\epsilon_0 > 0$ be fixed such that $d(\rho_0, \text{supp}(\mu_{sc} \boxplus \nu) \cup \{\rho_0, \theta_j \neq \theta\}) > \epsilon_0$ and $d(\theta, \text{supp}(\nu) \cup \{\theta_j, \theta_j \neq \theta\}) > \epsilon_0$. Let $\Omega_N$ be the event on which there is no eigenvalue of $W_N^{-2} + A_{N-p}$ in $[\rho_0 - \epsilon_0; \rho_0 + \epsilon_0]$, $\lambda_0(M_N)$ is the unique eigenvalue of $M_N$ in $[\rho_0 - \epsilon_0/2; \rho_0 + \epsilon_0/2]$ and $\|W_N\| \leq 3$. Propositions 2.7, 2.15 applied to $M_{N-p}$, Proposition 1.5 and Bai-Yin’s theorem lead that

$$
\lim_{N \to +\infty} \mathbb{1}_{\Omega_N} = 1, \text{ a.s.} \tag{2.31}
$$

Denoting by $\hat{G}_{N-p}$ the resolvent of the lower right submatrix of size $N - p$ of $M_N$ and $\rho_N$ being defined by (1.7), we have the following

**Proposition 2.16.**

$$
\sqrt{N} \left\{ \text{tr}_{N-p} \hat{G}_{N-p}(\rho_N) \mathbb{1}_{\Omega_N} - \int \frac{d\mu_{sc} \boxplus \mu_{A_{N-p}}(x)}{(\rho_N - x)} \right\}
$$
goes to zero in probability.

**Proof:** We stick to the proof of Proposition 5.5 in Capitaine (2020). Using Proposition 2.7, for $N$ large enough,

$$
d(\rho_N, \text{supp}(\mu_{sc} \boxplus \mu_{A_{N-p}})) > \epsilon_0/2 \tag{2.32}
$$

and on $\Omega_N$, $d \left( \{\rho_N, \lambda_0(M_N)\}, \text{spect} \left( W_N^{-2} + A_{N-p} \right) \right) > \epsilon_0/2$, so that

$$
\left\| \hat{G}_{N-p}(\rho_N) \right\| \leq \frac{2}{\epsilon_0}, \left\| \hat{G}_{N-p}(\lambda_0(M_N)) \right\| \leq \frac{2}{\epsilon_0}. \tag{2.33}
$$

Moreover, there exists $K > 0$ such that for any $x \in \text{supp}(\mu_{sc} \boxplus \mu_{A_{N-p}})$,

$$
|\rho_N - x| \leq K \text{ and on } \Omega_N, \left\| \rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right\| \leq K.
$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function with support in $\{\epsilon_0/4 \leq |x| \leq 2K\}$ and such that $g \equiv 1$ on $\{\epsilon_0/2 \leq |x| \leq K\}$. $f : x \mapsto \frac{g(x)}{x}$ is a $C^\infty$ function with compact support. Note that

$$
\int \frac{d\mu_{sc} \boxplus \mu_{A_{N-p}}(x)}{(\rho_N - x)} = \int f(\rho_N - x) d\mu_{sc} \boxplus \mu_{A_{N-p}}(x) \tag{2.34}
$$

and on $\Omega_N$, $\hat{G}_{N-p}(\rho_N) = f \left( \rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right)$. \tag{2.35}

According to Lemma 2.12, for any $z \in \mathbb{C} \setminus \mathbb{R},$

$$
\sqrt{N} \text{tr}_{N-p} \mathbb{E} \left[ \hat{G}_{N-p}(\rho_N - z) \right] = \sqrt{N} \frac{d\mu_{sc} \boxplus \mu_{A_{N-p}}(x)}{(\rho_N - z - x)} + o(z)(1), \tag{2.36}
$$

where there exist polynomials $Q_1$ and $Q_2$ with non negative coefficients and $(d, k) \in \mathbb{N}^2$ such that

$$
\|o(z)(1)\| \leq \frac{Q_1(|z|^{-1})(|z| + 1)^d}{\sqrt{N}} \leq \frac{1}{\sqrt{N}} \frac{Q_2(|z|)(|z| + 1)^d}{|z|^k}. \tag{2.37}
$$

Therefore, by Helffer-Sjöstrand functional calculus (see Section 2.2),

$$
\sqrt{N} \text{tr}_{N-p} \mathbb{E} \left[ f \left( \rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right) \right]
$$
It readily follows that

\[
\sqrt{N} \int \frac{1}{\pi} \int_{C \setminus \mathbb{R}} \partial F_k(f(z)) \sqrt{N} \text{tr}_{N-p} \mathbb{E} \left[ \hat{G}_{N-p} ( \rho_N - z ) \right] d^2 z
\]

and

\[
\sqrt{N} \int f \left( (\rho_N - x) \right) d\mu_{sc} \oplus \mu_{A_{N-p}}(x) = \frac{1}{\pi} \int_{C \setminus \mathbb{R}} \partial F_k(f(z)) \sqrt{N} \frac{d\mu_{sc} \oplus \mu_{A_{N-p}}(x)}{(\rho_N - z - x)} d^2 z.
\]

Hence, using (2.36) and (2.34), we can deduce that

\[
\sqrt{N} \int f \left( \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right) = \frac{1}{\pi} \int_{C \setminus \mathbb{R}} \partial F_k(f(z)) \sqrt{N} \frac{1}{(\rho_N - x)} d^2 z.
\]

Thus, using (2.37) and (2.10),

\[
\left\| \frac{1}{\pi} \int_{C \setminus \mathbb{R}} \partial F_k(f(z)) \right\| \leq \frac{C}{\sqrt{N}}.
\]

Thus,

\[
\sqrt{N} \left\{ \mathbb{E} \text{tr}_{N-p} \left( f \left( \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right) \right) - \int \frac{d\mu_{sc} \oplus \mu_{A_{N-p}}(x)}{(\rho_N - x)} \right\} \rightarrow N \rightarrow \infty 0.
\]

Define \( k : M_{N-p}^o ( \mathbb{C} ) \rightarrow \mathbb{C} \) by

\[
k(X) = \text{tr}_{N-p} \left[ f \left( \frac{W_{N-p}}{\sqrt{N}} - A_{N-p} \right) \right].
\]

Applying Poincaré inequality, we get that

\[
\mathbb{E} \left( \left| k \left( \frac{W_{N-p}}{\sqrt{N}} \right) - \mathbb{E}(k(\frac{W_{N-p}}{\sqrt{N}})) \right|^2 \right) \leq \frac{C}{N} \mathbb{E} \left( \left\| \text{grad} \left( \frac{W_{N-p}}{\sqrt{N}} \right) \right\|^2 \right),
\]

with

\[
\left\| \text{grad} k(X) \right\|^2 = \sup_{w \in S_1(M_{N-p}^o(\mathbb{C} ))} \left| \frac{d}{dt} k(X + tw) \right|_{t=0} \right|^2.
\]

Since \( f \) is a Lipschitz function on \( \mathbb{R} \) with Lipschitz constant \( C_L \), its extension on Hermitian matrices is \( C_L \)-Lipschitz with respect to the norm \( \| M \|_e = (\text{Tr}_{N-p} M^2)^{1/2} \). Therefore,

\[
\sup_{w \in S_1(M_{N-p}^o(\mathbb{C} ))} \left| \frac{d}{dt} k(X + tw) \right|_{t=0} \right|^2 \leq \frac{C}{N},
\]

and then

\[
\mathbb{E} \left( \left| \sqrt{N} \left\{ k \left( \frac{W_{N-p}}{\sqrt{N}} \right) - \mathbb{E} \left( k \left( \frac{W_{N-p}}{\sqrt{N}} \right) \right) \right\} \right|^2 \right) \leq \frac{C}{N}.
\]

It readily follows that

\[
\sqrt{N} \text{tr}_{N-p} \left\{ f(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p}) - \mathbb{E} \left( f(\rho_N I_{N-p} - \frac{W_{N-p}}{\sqrt{N}} - A_{N-p}) \right) \right\} = o_p(1)
\]

Proposition 2.16 follows from (2.35), (2.38), (2.39) and (2.31).
3. Proof of Theorem 1.10

The approach to prove (1.10) is the one of Capitaine (2020). On $\Omega_N$, defined at the beginning of Section 2.6.2, we have by Proposition 2.1,
\[ \det(X_p(N)) = 0, \]  
(3.1)
where
\[ X_p(N) = \lambda_{i_0}(M_N)I_p - \frac{W_p}{\sqrt{N}} - A_p - \frac{1}{N}Y^*\hat{G}_{N-p}(\lambda_{i_0}(M_N))Y, \]
and $\hat{G}_{N-p}$ is the resolvent of $\frac{W_{N-p}}{\sqrt{N}} + A_{N-p}$. Let $\rho_N$ be as defined by (1.7). Using the identity
\[ \hat{G}_{N-p}(\rho_N) - \hat{G}_{N-p}(\lambda_{i_0}(M_N)) = (\lambda_{i_0}(M_N) - \rho_N)\hat{G}_{N-p}(\rho_N)\hat{G}_{N-p}(\lambda_{i_0}(M_N)), \]
we have
\[ X_p(N) = H_p(N) + X_p^{(0)}, \]
where
\[ X_p^{(0)} = \theta I_p - A_p, \]
\[ H_p(N) = (\lambda_{i_0}(M_N) - \rho_N)I_p - \Delta_1(N) - \Delta_2(N) \]
\[ + (\lambda_{i_0}(M_N) - \rho_N)r_1(N) - \frac{W_p}{\sqrt{N}} - (\lambda_{i_0}(M_N) - \rho_N)^2r_2(N) \]
with
\[ r_1(N) = \frac{1}{N}Y^*\hat{G}_{N-p}(\rho_N)^2I_{\Omega_N}Y, \]
\[ r_2(N) = \frac{1}{N}Y^*\hat{G}_{N-p}(\rho_N)^2\hat{G}_{N-p}(\lambda_{i_0}(M_N))I_{\Omega_N}Y, \]
\[ \Delta_1(N) = \frac{1}{N}Y^*\hat{G}_{N-p}(\rho_N)I_{\Omega_N}Y - \frac{\sigma^2}{\sqrt{N}}\operatorname{Tr}_{N-p}\left(\hat{G}_{N-p}(\rho_N)I_{\Omega_N}\right), \]
\[ \Delta_2(N) = \frac{1}{N}\operatorname{Tr}_{N-p}\left(\hat{G}_{N-p}(\rho_N)I_{\Omega_N}\right) - \int \frac{d\mu_{sc} \oplus \mu_{A_{N-p}}(x)}{(\rho_N - x)}. \]
First, by Lemma 2.8 we have that
\[ r_1(N) - \sigma^2\frac{1}{N}\operatorname{Tr}_{N-p}(\hat{G}_{N-p}(\rho_N)^2)I_{\Omega_N}I_p = o_p(1). \]
By (2.31), (2.35), (1.8), (2.32) and asymptotic freeness of $\frac{W_{N-p}}{\sqrt{N}}$ and $A_{N-p}$ (see Theorem 5.4.5 Anderson et al., 2010),
\[ \frac{1}{N}\operatorname{Tr}_{N-p}(\hat{G}_{N-p}(\rho_N)^2)I_{\Omega_N} \xrightarrow{N \to \infty} \int \frac{d\lambda(x)}{(\rho_N - x)^2} \text{ almost surely.} \]  
(3.2)
Therefore,
\[ r_1(N) \xrightarrow{p} N \to \infty \sigma^2 \int \frac{d\lambda(x)}{(\rho_N - x)^2}I_p. \]  
(3.3)
Now on $\Omega_N$, using (2.33),
\[ ||r_2(N)|| \leq ||\hat{G}_{N-p}(\rho_N)I_{\Omega_N}||^2 \ ||\hat{G}_{N-p}(\lambda_{i_0}(M_N))I_{\Omega_N}\| ||Y||^2 \leq 9 \left(\frac{2}{\epsilon_0}\right)^2. \]  
(3.4)
By Lemma 2.8,
\[ \Delta_1(N) = o_p(1). \]  
(3.5)
Lemma 3.1.

\[
\frac{1}{N-p} \sum_{j=1}^{N-p} \left[ \tilde{G}_{N-p}(\rho_N)_{jj} \right]^2 \mathbf{1}_{\Omega_N} = \frac{1}{N-p} \sum_{j=1}^{N-p} \left[ (\theta - A_{N-p})^{-1} \right]_{jj}^2 + o_p(1).
\]

Proof: By (4.29) which will be proved below, for any \( r > 0 \),

\[
\frac{1}{N-p} \sum_{j=1}^{N-p} \left[ \tilde{G}_{N-p}(\rho_0 + \frac{i}{r})_{jj} \right]^2 \mathbf{1}_{\Omega_N}
\]

= \frac{1}{N-p} \sum_{j=1}^{N-p} \left[ (\omega(\rho_0 + \frac{i}{r}) - A_{N-p})^{-1} \right]_{jj}^2 + o_p(1),

with \( \omega \) defined by (2.13). Now, using resolvent identity, one can easily obtain that there exists some constant \( C(\epsilon_0) \) such that for any \( r > 0 \) and any \( N \),

\[
\left| \frac{1}{N-p} \sum_{j=1}^{N-p} \left[ \tilde{G}_{N-p}(\rho_N)_{jj} \right]^2 \mathbf{1}_{\Omega_N} - \frac{1}{N-p} \sum_{j=1}^{N-p} \left[ \tilde{G}_{N-p}(\rho_0 + \frac{i}{r})_{jj} \right]^2 \mathbf{1}_{\Omega_N} \right|
\]

\[
\leq C(\epsilon_0) \left( \frac{1}{r} + \rho_N - \rho \right).
\]

Moreover, \( \omega(\rho_0) = \theta \) so that for all large \( N \), \( d(\omega(\rho_0), \text{supp}(\mu_{A_{N-p}})) > \epsilon_0/2 \).

Now, choose \( r_0 \) large enough such that for all large \( N \), \( \forall r \geq r_0 \), \( d(\omega(\rho_0 + \frac{i}{r}), \text{supp}(\mu_{A_{N-p}})) > \epsilon_0/4 \). Using resolvent identity, one can easily obtain that there exists some constant \( C(\epsilon_0) \) such that for all large \( N \), \( \forall r \geq r_0 \),

\[
\left| \frac{1}{N-p} \sum_{j=1}^{N-p} \left[ (\omega(\rho_0 + \frac{i}{r}) - A_{N-p})^{-1} \right]_{jj}^2 - \frac{1}{N-p} \sum_{j=1}^{N-p} \left[ (\omega(\rho_0) - A_{N-p})^{-1} \right]_{jj}^2 \right|
\]

\[
\leq C(\epsilon_0)/r.
\]

Lemma 3.1 follows by letting \( N \) go to infinity and then \( r \) go to infinity. \( \square \)

(3.2), Lemma 3.1 and Proposition 2.10 yield that

\[
\sqrt{N} \Delta_1(N)^2 = o_p(1), \quad (3.6)
\]

Now, one can prove that by Proposition 2.16, we have

\[
\sqrt{N} \Delta_2(N) = o_p(1). \quad (3.7)
\]

Thus (1.6), (1.8), (3.3), (3.4), (3.5) and (3.7) yield that

\[
H_p(N) = o_p(1). \quad (3.8)
\]

Therefore, according to Lemma 2.3 (using (3.8)) and (3.1), with a probability going to one as \( N \) goes to infinity,

\[
0 = \text{det} X_p(N) = \text{det}(X_p^{(0)} + H_p(N)) = \text{det}(X_p^{(0)}) + \text{Tr}_p \left[ B_{X_p^{(0)}} H_p(N) \right] + \epsilon_N
\]

\[
= \text{Tr}_p \left[ B_{X_p^{(0)}} H_p(N) \right] + \epsilon_N,
\]
where 
\[ B_{X_p^{(o)}} = \text{com}(X_p^{(o)}), \]
\[ \epsilon_N = O(H_p(N))^2. \]
Thus, using (1.6), (1.8), (3.3), (3.4), (3.7) and (3.6),
\[ \sqrt{N}\epsilon_N = o_p(\sqrt{N}(\lambda_{i_0}(M_N) - \rho_N)) + o_p(1). \]
Hence, with a probability going to one as \( N \) goes to infinity,
\[ \sqrt{N}(\lambda_{i_0}(M_N) - \rho_N) \left[ Tr_p B_{X_p^{(o)}} (I_p + r_1(N)) + o_p(1) \right] = Tr_p \left[ B_{X_p^{(o)}} \left( \sqrt{N}\Delta_1(N) + W_p \right) \right] + o_p(1). \]
(1.10) readily follows from (3.2), Lemma 3.1, Proposition 2.10, the independence of \( \Delta_1(N) \) and \( W_p \). When \( A_{N-p} \) is diagonal, the result follows using (4.29).

4. Proofs of Theorems 1.11 and 1.12

Let \( M_N \) be defined by (1.2) with assumptions (W) and (A'). We denote by \( \lambda_{i_0}(A_N) \), resp. \( \lambda_{i_0}(M_N) \), the eigenvalues of \( A_N \), resp. \( M_N \) and \( u_i \), resp. \( v_i \) the normalized associated eigenvectors. According to assumption (A), there exists \( \delta > 0 \) such that for all large \( N \), the distance from \( \theta \) to the rest of the spectrum (that is the other eigenvalues of \( A_N \) except \( \theta \)) is greater than \( \delta \). Moreover, from Proposition 1.5, we know that a.s.
\[ \lambda_{i_0}(M_N) \rightarrow_{N \rightarrow +\infty} \rho_\theta = \theta + \sigma^2 g_\nu(\theta). \]
and there exists \( \delta_0 > 0 \) such that almost surely for all large \( N \), the distance from \( \rho_\theta \) to the rest of the spectrum (that is the other eigenvalues of \( M_N \) except \( \lambda_{i_0}(M_N) \)) is greater than \( \delta_0 \).

Throughout this section, \( h \) is a smooth function with support in \( ]\rho_\theta - \delta_0/2; \rho_\theta + \delta_0/2[ \) which is equal to 1 near \( \rho_\theta \).

4.1. Representation in terms of resolvent. The aim of this section is to be brought back to the study of the fluctuations of the \( p \times p \)-matrix valued process \( \{G_p(z), z \in \mathbb{C} \setminus \mathbb{R}\} \) where \( G_p(z) \) denotes the principal submatrix of size \( p \) of the resolvant matrix \( G(z) = (zI_N - M_N)^{-1} \).

**Proposition 4.1.** Almost surely, for all large \( N \),
\[ \sqrt{N}(|\langle u_{i_0}, v_{i_0} \rangle|^2 - \tau_N(\theta)) = -\frac{1}{\pi} \int_{\mathbb{C}} \hat{\partial}F_k(h)(z) \left( P^* \sqrt{N} (G_p(z) - \Lambda_p(z)) P \right)_{11} \, d^2z, \]
where \( \tau_N(\theta) \) is defined by (1.11),
\[ \Lambda_p(z) = (zI_p - A_p - \sigma^2 \tilde{g}_{N-p}(z) I_p)^{-1} \]
and \( \tilde{g}_{N-p}(z) \) is the Stieltjes transform of \( \mu_{sc} \boxplus \mu_{AN-p} \).

Proposition 4.1 readily follows from the two preliminaries Lemmas 4.2 and 4.3.
Lemma 4.2. Almost surely, for all large $N$, for any integer number $k$,

$$
|\langle u_i, v_i \rangle|^2 = -\frac{1}{\pi} \int_C \bar{\partial} F_k(h)(z) (P^* G_p(z) P)_{11} d^2z,
$$

(4.3)

where $F_k(h)$ is defined by (2.8).

Proof: Let $f$ be any smooth function with support in $|\theta - \delta/2; \theta + \delta/2|$ which is equal to 1 near $\theta$. From the formula

$$
\text{Tr}_N(h(M_N)f(AN)) = \sum_{i,j=1}^{N} h(\lambda_i(M_N)) f(\lambda_j(AN)) |\langle u_j, v_i \rangle|^2,
$$

(4.4)

we easily deduce that, almost surely, for all large $N$,

$$
|\langle u_i, v_i \rangle|^2 = \sum_{i,j=1}^{p} (P^*)_{i,j} h(M_N)_{ij} P_{j1}.
$$

(4.5)

By Helffer-Sjöstrand’s representation formula (2.7), we can write $h(M_N)_{ij}$ as

$$
h(M_N)_{ij} = -\frac{1}{\pi} \int_C \bar{\partial} F_k(h)(z) G_{ij}(z) d^2z,
$$

so that

$$
\sum_{i,j=1}^{p} (P^*)_{i,j} h(M_N)_{ij} P_{j1} = -\frac{1}{\pi} \int_C \bar{\partial} F_k(h)(z) (P^* G_p(z) P)_{11} d^2z.
$$

(4.6)

Lemma 4.2 follows from (4.5) and (4.6).

\[\square\]

Lemma 4.3. For $N$ large enough,

$$
\tau_N(\theta) = 1 - \sigma^2 \int \frac{1}{(\theta - x)^2} d\mu_{AN-p}(x)
$$

$$
= -\frac{1}{\pi} \int_C \bar{\partial} F_k(h)(z) \left( (P^* (zI_p - A_p - \sigma^2 \tilde{g}_{N-p}(z) I_p) - 1) P \right)_{11} d^2z
$$

where $\tilde{g}_{N-p}(z)$ is the Stieltjes transform of $\mu_{sc} \boxplus \mu_{AN-p}$.

Proof: Note that

$$
(P^* (zI_p - A_p - \sigma^2 \tilde{g}_{N-p}(z) I_p) - 1)_{11} = \frac{1}{z - \sigma^2 \tilde{g}_{N-p}(z) - \theta}.
$$

Let us define for any $z \in \mathbb{C} \setminus \text{supp}(\mu_{AN-p})$,

$$
H_{N-p}(z) = z + \sigma^2 \tilde{g}_{AN-p}(z)
$$

(4.7)

and for any $z \in \mathbb{C} \setminus \text{supp}(\mu_{sc} \boxplus \mu_{AN-p})$,

$$
\omega_{N-p}(z) = z - \sigma^2 g_{\mu_{sc} \boxplus \mu_{AN-p}}(z) = z - \sigma^2 \tilde{g}_{N-p}(z).
$$

(4.8)

Note that, for any $z \in \mathbb{C} \setminus \mathbb{R}$, $|3\omega_{N-p}(z)| \geq |3z| > 0$. Moreover, according to (2.16), the following one to one correspondance holds:

$$
\mathbb{R} \setminus \text{supp}(\mu_{sc} \boxplus \mu_{AN-p}) \leftrightarrow \{ u \in \mathbb{R} \setminus \text{supp}(\mu_{AN-p}), \int \frac{1}{(u - x)^2} d\mu_{AN-p}(x) < \frac{1}{\sigma^2} \}
$$
and for any \( x \in \mathbb{C} \setminus \text{supp}(\mu_\omega \boxplus \mu_\nu) \), \( H_{N-p}(\omega_{N-p}(x)) = x \). Hence \( \rho_N = H_{N-p}(\theta) = \theta + \sigma^2 \mu_{A_{N-p}}(\theta) \) is the single pole of \( \frac{1}{z-\sigma^2 g_{N-p}(z)-\theta} \) in \( \mathbb{C} \). Therefore, (1.8) and Proposition 2.4 (used with \( \phi(z) = \frac{1}{\omega_{N-p}(z-\theta)} \)) imply that

\[
-\frac{1}{\pi} \int_{\mathbb{C}} \partial F_k(h)(z) \frac{1}{z - \sigma^2 \tilde{g}_{N-p}(z) - \theta} d^2z = \frac{1}{\omega_{N-p}(\rho_N)} = H'_{N-p}(\theta) = \tau_N(\theta). \tag{4.9}
\]

We now consider the process

\[
\xi_N(z) = \left( P^* \sqrt{N} (G_p(z) - \Lambda_p(z)) P \right)_{11}, \tag{4.10}
\]

where \( \Lambda_p \) is defined by (4.2).

4.2. Tightness of the sequence of processes \( \{\xi_N\}_N \).

**Proposition 4.4.** \( \xi_N \) is tight on \( \mathcal{H}(\mathbb{C} \setminus \mathbb{R}) \).

**Proof:** \( \xi_N : z \mapsto \left( P^* \sqrt{N} (G_p(z) - \Lambda_p(z)) P \right)_{11} \) is analytic on \( \mathbb{C} \setminus \mathbb{R} \). Let \( K \) be a compact set in \( \mathbb{C} \setminus \mathbb{R} \). According to Lemma 2.6, there exists \( \delta > 0 \) such that \( \bar{K}_\delta \subset \mathbb{C} \setminus \mathbb{R} \) and for any \( r > 0 \),

\[
\|\xi_N\|_K^r \leq (\pi \delta^2)^{-1} \int_{\bar{K}_\delta} |\xi_N(z)|^r m(dz).
\]

Therefore

\[
\mathbb{E}(\|\xi_N\|_K^r) \leq (\pi \delta^2)^{-1} \int_{\bar{K}_\delta} \mathbb{E}(|\xi_N(z)|^r) m(dz) \tag{4.11}
\]

\[
\leq (\pi \delta^2)^{-1} \sup_{z \in \bar{K}_\delta} \mathbb{E}(|\xi_N(z)|^r) m(\bar{K}_\delta). \tag{4.12}
\]

In order to prove the tightness of \( \xi_N \), using (2.12) and (4.12), we are going to show that, for any compact set \( K \subset \mathbb{C} \setminus \mathbb{R} \), there exists a constant \( C' > 0 \) such that for all large \( N \),

\[
\sup_{z \in K} \mathbb{E}\left(|\xi_N(z)|^2\right) < C'. \tag{4.13}
\]

We have for any \( z_1 \) and \( z_2 \) in \( \mathbb{C} \setminus \mathbb{R} \),

\[
\mathbb{E}(\xi_N(z_1)\xi_N(z_2)) = N \sum_{i,j,u,v=1}^{p} P^*_{i1} P^*_{j1} P^*_{iu} P^*_{vu} \mathbb{E}\left( (G_{ij}(z_1) - (\Lambda_p(z_1))_{ij}) (G_{uv}(z_2) - (\Lambda_p(z_2))_{uv}) \right),
\]

and

\[
\mathbb{E}\left( (G_{ij}(z_1) - (\Lambda_p(z_1))_{ij}) (G_{uv}(z_2) - (\Lambda_p(z_2))_{uv}) \right)
= \mathbb{E}\left( (G_{ij}(z_1) - \mathbb{E}(G_{ij}(z_1))) (G_{uv}(z_2) - \mathbb{E}(G_{uv}(z_2))) \right)
+ [\mathbb{E}(G_{ij}(z_1)) - (\Lambda_p(z_1))_{ij}] [\mathbb{E}(G_{uv}(z_2)) - (\Lambda_p(z_2))_{uv}].
\]

According to Lemma 2.11 and the block diagonal structure of \( A_N \), we have for any \( l = 1, 2 \),

\[
\mathbb{E}(G_p(z_l)) = ((z_l - \sigma^2 \tilde{g}_N(z_l)) I_p - A_p)^{-1} + O(1/\sqrt{N}), \tag{4.14}
\]
where \( \tilde{g}_N \) is the Stieltjes transform of \( \mu_{A_N} \oplus \mu_{sc} \). First set
\[
\tilde{G}_{N-p}(w) = \left( zI_{N-p} - \frac{W_{N-p}}{\sqrt{N-p}} - A_{N-p} \right)^{-1}.
\]
By (2.5), we have
\[
\| \tilde{G}_{N-p}(z) \| \leq |\Im z|^{-1}. \tag{4.15}
\]
Note that
\[
\hat{G}_{N-p}(z) = \tilde{G}_{N-p}(z) + p \left( \frac{zI_N - p - W_{N-p}}{\sqrt{N-p}} \right) \left( I_{N-p} - \tilde{G}_{N-p}(z) (zI_{N-p} - A_{N-p}) \right) \tilde{G}_{N-p}(z). \tag{4.16}
\]
Now, Lemma 2.12 yields that
\[
\tilde{g}_N(z) = \mathbb{E} (\text{tr}_N G(z)) + O(1/N)
\]
and
\[
\tilde{g}_{N-p}(z) = \mathbb{E} (\text{tr}_{N-p} \tilde{G}_{N-p}(z)) + O(1/N),
\]
and then, using (4.16), that
\[
\tilde{g}_{N-p}(z) = \mathbb{E} (\text{tr}_{N-p} \hat{G}_{N-p}(z)) + O(1/N). \tag{4.17}
\]
Since by (A.1.12) in Bai and Silverstein (2010), we have:
\[
\mathbb{E} (\text{tr}_N G(z)) = \mathbb{E} (\text{tr}_{N-p} \hat{G}_{N-p}(z)) + O(1/N),
\]
we can deduce that \( \tilde{g}_N(z) = \tilde{g}_{N-p}(z) + O(1/N) \), and thus that, for any \( 1 \leq i, j \leq p \),
\[
\mathbb{E}(G_{ij}(z_i)) = (A_p(z_i))_{ij} + O(1/\sqrt{N}).
\]
Moreover, by Lemma 2.13,
\[
\text{Var}(G_{ij}(z_i)) = O(1/N).
\]
It readily follows that there exist polynomials \( P_1 \) and \( P_2 \) with nonnegative coefficients such that
\[
\mathbb{E}(\xi_N(z_1)\xi_N(z_2)) \leq P_1 \left( |\Im z_1|^{-1} \right) P_2 \left( |\Im z_2|^{-1} \right)
\]
and then there exists some polynomial \( P_3 \) with nonnegative coefficients such that for all large \( N \) and all \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[
\mathbb{E} \left( |\xi_N(z)|^2 \right) \leq P_3 \left( |\Im z|^{-1} \right). \tag{4.18}
\]
This implies (4.13). Therefore, for any compact subset \( K \) in \( \mathbb{C} \setminus \mathbb{R} \), there exists a constant \( C > 0 \) such that for all large \( N \), \( \mathbb{E} \left( \|\xi_N(z)\|_K^2 \right) < C \) and the tightness of \( \xi_N \) in \( \mathcal{H}(\mathbb{C} \setminus \mathbb{R}) \) follows from (2.12) and Proposition 2.5. \( \square \)
4.3. Finite dimensional distributions of $\xi_N$. Set

$$\nabla_N(z) = \sqrt{N} \left( G_p(z) - \Lambda_p(z) \right).$$

We use Proposition 2.1 for the inversion of

$$zI_N - M_N = \begin{pmatrix} zI_p - \frac{1}{\sqrt{N}} W_p - A_p & -\frac{1}{\sqrt{N}} Y^* \hat{G}_{N-p}(z) Y \\ -\frac{1}{\sqrt{N}} Y & zI_{N-p} - M_{N-p} \end{pmatrix}$$

where $M_{N-p}$ is the lower right submatrix of size $N - p$ of $M_N$, leading to:

$$G_p(z) = (zI_p - \frac{1}{\sqrt{N}} W_p - A_p - \frac{1}{N} Y^* \hat{G}_{N-p}(z) Y)^{-1} \quad (4.19)$$

where $\hat{G}_{N-p}$ is the resolvent of $M_{N-p}$. Thus,

$$\nabla_N(z) = G_p(z)(W_p + \sqrt{N}(\frac{1}{N} Y^* \hat{G}_{N-p}(z) Y - \sigma^2 \tilde{g}_{N-p}(z) I_p)) \Lambda_p(z). \quad (4.20)$$

**Lemma 4.5.** Define for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\hat{\nabla}_N(z) = (zI_p - A_p - \sigma^2 g(z) I_p)^{-1}(W_p + Q_N(z))(zI_p - A_p - \sigma^2 g(z) I_p)^{-1}, \quad (4.21)$$

where $Q_N(z)$ is the following matrix of size $p$:

$$Q_N(z) = \frac{1}{\sqrt{N}} (Y^* \hat{G}_{N-p}(z) Y - \sigma^2 \text{Tr}_{N-p}(\hat{G}_{N-p}(z)) I_p), z \in \mathbb{C} \setminus \mathbb{R} \quad (4.22)$$

For any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\nabla_N(z) - \hat{\nabla}_N(z) \xrightarrow{N \to \infty} 0. \quad (4.23)$$

**Proof:** Obviously, we have

$$\Lambda_p(z) \xrightarrow{N \to \infty} (zI_p - A_p - \sigma^2 g(z) I_p)^{-1}, \quad (4.24)$$

and (4.14) and Lemma 2.13 yield that

$$G_p(z) \xrightarrow{p} (zI_p - A_p - \sigma^2 g(z) I_p)^{-1}. \quad (4.25)$$

We write

$$\sqrt{N}(\frac{1}{N} Y^* \hat{G}_{N-p}(z) Y - \sigma^2 \tilde{g}_{N-p}(z) I_p)$$

$$= \frac{1}{\sqrt{N}} (Y^* \hat{G}_{N-p}(z) Y - \sigma^2 \text{Tr}_{N-p}(\hat{G}_{N-p}(z)) I_p)$$

$$+ \sqrt{N} \sigma^2 (\text{tr}_{N-p}(\hat{G}_{N-p}(z)) I_p - \tilde{g}_{N-p}(z) I_p). \quad (4.26)$$

(4.17) and Lemma 2.14 yield that

$$\sqrt{N} \sigma^2 (\text{tr}_{N-p}(\hat{G}_{N-p}(z)) I_p - \tilde{g}_{N-p}(z) I_p) \xrightarrow{N \to \infty} 0. \quad (4.27)$$

Lemma 4.5 readily follows from (4.20), (4.25), (4.23), (4.24), (4.26) and the tightness of $Q_N(z)$ (which readily follows from Lemma 2.8), by using Slutsky’s theorem and classical operations on convergence in probability.

We now state an approximation result in distribution for the finite dimensional distributions of the process $\{Q_N(z), z \in \mathbb{C} \setminus \mathbb{R}\}$. 
Lemma 4.6. Let \((z_1, \ldots, z_q)\) be in \((\mathbb{C} \setminus \mathbb{R})^q\). Set \(V_N = (Q_N(z_1), \ldots, Q_N(z_q))\). Then, under the assumptions of Theorem 1.11
\[
d_{LP}(V_N, (G_N(z_1), \ldots, G_N(z_q))) \to 0
\]
where \(G_N\) is a centered matrix valued Gaussian process whose distribution is given as follows:
1) the processes \(((G_N)_{ij}(z))_z\) for \(1 \leq i \leq j \leq p\) are independent, and \((G_N)_{ji}(z) = (G_N)_{ij}^*(z)\).
2) For \(i \leq p\),
\[
E((G_N)_{ii}(z_k)(G_N)_{ii}(z_l)) = (m_4 - 3\sigma^4) \sum_{i=1}^{N-p} \left( (z_k - \sigma^2 g(z_k) - A_{N-p})^{-1} \right)_{ii} \left( (z_l - \sigma^2 g(z_l) - A_{N-p})^{-1} \right)_{ii}
+ \sigma^4 \int \frac{1}{(z_k - x)(z_l - x)} d\lambda(x)
\]
(4.27)
3) For \(1 \leq i \neq j \leq p\),
\[
E((G_N)_{ij}(z_k)(G_N)_{ij}(z_l)) = 0
\]
and
\[
E((G_N)_{ij}(z_k)(G_N)_{ij}(z_l)) = \sigma^4 \int \frac{1}{(z_k - x)(z_l - x)} d\lambda(x).
\]
Proof: We apply Proposition 2.10 to the matrices \(B(z) = \hat{G}_{N-p}(z), z \in \mathbb{C} \setminus \mathbb{R}\). Note that these matrices are random but they are independent of the \(N \times p\) matrix \(Y\). In order to conclude, we need to show that
\[
I_N := \frac{1}{N-p} \sum_{i=1}^{N-p} ((\hat{G}_{N-p}(z_1))_{ii}(\hat{G}_{N-p}(z_2))_{ii}) = f_N(z_1, z_2) + o(1)
\]
(4.29)
with
\[
f_N(z_1, z_2) = \frac{1}{N-p} \sum_{i=1}^{N-p} ((z_k - \sigma^2 g(z_k) - A_{N-p})^{-1})_{ii} ((z_l - \sigma^2 g(z_l) - A_{N-p})^{-1})_{ii}
\]
and
\[
J_N := \text{tr}_{N-p}(\hat{G}_{N-p}(z_1)\hat{G}_{N-p}(z_2)) \xrightarrow{p} \int \frac{1}{(z_1 - x)(z_2 - x)} d\lambda(x).
\]
The second convergence follows from the convergence of \(\mu_{M_{N-p}}\) towards \(\lambda\). For the first one, Lemma 2.11 and (4.16) yield that, for \(k \leq N - p\),
\[
E((\hat{G}_{N-p}(z))_{kk}) = \left[ (z - \sigma^2 \hat{g}_{N-p}(z)) I_{N-p} - A_{N-p} \right]_{kk}^{-1} + O\left( \frac{1}{\sqrt{N}} \right)
= \left[ (z - \sigma^2 g(z)) I_{N-p} - A_{N-p} \right]_{kk}^{-1} + o^{(w)}(1).
\]
Thus, using Lemma 2.13, we can deduce that
\[
E((\hat{G}_{N-p}(z_1))_{kk}(\hat{G}_{N-p}(z_2))_{kk})
= [(z_1 - \sigma^2 g(z_1) - A_{N-p})^{-1}]_{kk} [(z_2 - \sigma^2 g(z_2) - A_{N-p})^{-1}]_{kk} + o^{(w)}(1)
\]
(4.30)
From Lemma 5.1 in the Appendix, using that \( f_N(W) = \frac{1}{N-p} \sum_{i=1}^{N-p} [(z_1 - W - A_{N-p})^{-1}]_{ii} [(z_2 - W - A_{N-p})^{-1}]_{ii} \) is Lipschitz with constant \(|\Im(z_1)|^{-2} |\Im(z_2)|^{-2}|\), we can deduce that

\[
\frac{1}{N-p} \sum_{i=1}^{N-p} (\hat{G}_{N-p}(z_1))_{ii} (\hat{G}_{N-p}(z_2))_{ii}
= \frac{1}{N-p} \sum_{i=1}^{N-p} \mathbb{E} \left( (\hat{G}_{N-p}(z_1))_{ii} (\hat{G}_{N-p}(z_2))_{ii} \right) + o_p(1).
\]

We also use to obtain (4.28) from Proposition 2.10 that for \( \sigma^2 = 1 \),

\[
\mathbb{E}(|y_{11}|^4) - 2 = \frac{1}{2}(m_4 - 3),
\]

where we recall that \( \mu \) is the distribution of \( \sqrt{2} \Re y_{11} \) and \( \sqrt{2} \Im y_{11} \).

**Corollary 4.7.** Under the assumptions of Theorem 1.11, for any \( z_1, \ldots, z_q \) be in \( (\mathbb{C} \setminus \mathbb{R})^q \),

\[
d_{LP}((\xi_N(z_1), \ldots, \xi_N(z_q)), (\mathcal{T}_N(z_1), \ldots, \mathcal{T}_N(z_q))) \to 0
\]

where \( \xi_N \) is the process defined by (4.10) and

\[
\mathcal{T}_N(z) = (z - \theta - \sigma^2 g(z))^{-2} \left( P^*(W_p + \mathcal{G}_N(z))P \right)_{11}
\]

\( \mathcal{G}_N \) being a centered Gaussian matrix valued process independent from \( W_p \) whose distribution is described in Lemma 4.6.

### 4.4. Fluctuations of the eigenvector.

Recall from Proposition 4.1 that, for any \( k \in \mathbb{N}^* \), \( \Phi_N \) defined as

\[
\sqrt{N}(|\langle u_k, v_{ik} \rangle|^2 - \tau_N(\theta))
\]

has the representation

\[
\Phi_N = -\frac{1}{\pi} \int_{\mathbb{C}} \partial F_k(h)(z) \xi_N(z) d^2 z.
\]

We follow the proof of Lemma 6.3 in Najim and Yao (2016) based upon the following estimates, from (4.18):

\[
\sup_N \mathbb{E}(|\xi_N(z)|) \leq P_3(|\Im(z)|^{-1}),
\]

and from Lemma 4.6 and (2.6), \( (\mathcal{T}_N \) being defined by (4.31))

\[
\sup_N \mathbb{E}(|\mathcal{T}_N(z)|) \leq P_5(|\Im(z)|^{-1}),
\]

where \( P_3, P_4 \) and \( P_5 \) are some polynomial with nonnegative coefficients. Hence in the following, in (4.32), we choose \( k \) greater than the degrees of \( P_3, P_4 \) and \( P_5 \).

**Proposition 4.8.** Under the assumptions of Theorem 1.11, \( d_{LP}(\Phi_N, \tilde{\Phi}_N) \to 0 \)

where \( \tilde{\Phi}_N \) is given by

\[
\tilde{\Phi}_N = (P^*(c_{\theta, \nu}W_p + Z_{p,N})P)_{11},
\]

where \( W_p \) is a Wigner matrix of size \( p \), \( Z_{p,N} \) is a centered Gaussian Hermitian matrix of size \( p \) with independent entries (modulo the symmetry condition); the diagonal coefficients are iid with variance

\[
\frac{1}{2}(m_4 - 3\sigma^4)A_{\theta, \nu} + \sigma^4 B_{\theta, \nu}.
\]
and the off diagonal elements are i.i.d. complex Gaussian with distribution $Z$ such that $\mathbb{E}(Z^2) = 0$ and $\mathbb{E}(|Z|^2) = \sigma^4 B_{\theta,\nu}$.

See Eq.(1.18) for the definitions of $c_{\theta,\sigma}$, $B_{\theta,\nu}$ and (1.15) for the definition of $A_{\theta,\nu,N}$.

Proof: For the reader’s convenience, we repeat here the strategy of Lemma 6.3 in Najim and Yao (2016). For any $0 < \epsilon < 1$ small enough such that $\chi \equiv 1$ on $]-\epsilon;\epsilon[$, define

$$ D_\epsilon = \{z \in \mathbb{C}, |\Im(z)| \geq \epsilon\}. $$

Set

$$ U_N = \int_{\mathbb{C}} \overline{\partial F_k}(h)(z)\xi_N(z) d^2z, \quad U_N^\epsilon = \int_{D_\epsilon} \overline{\partial F_k}(h)(z)\xi_N(z) d^2z $$

and

$$ V_N = \int_{\mathbb{C}} \overline{\partial F_k}(h)(z)T_N(z) d^2z, \quad V_N^\epsilon = \int_{D_\epsilon} \overline{\partial F_k}(h)(z)T_N(z) d^2z, $$

where $T_N$ is defined by (4.31). Let $f$ be a bounded continuous complex function on $\mathbb{C}$.

We have

$$ |\mathbb{E}(f(U_N)) - \mathbb{E}(f(V_N))| \leq |\mathbb{E}(f(U_N)) - \mathbb{E}(f(U_N^\epsilon))| + |\mathbb{E}(f(U_N^\epsilon)) - \mathbb{E}(f(V_N^\epsilon))| + |\mathbb{E}(f(V_N^\epsilon)) - \mathbb{E}(f(V_N))| $$

Let $\delta > 0$.

i) For any $\eta > 0$ and $K > 0$, we have

$$ |\mathbb{E}(f(U_N)) - \mathbb{E}(f(U_N^\epsilon))| \leq \left| \mathbb{E}(f(U_N) - f(U_N^\epsilon)) 1_{U_N - U_N^\epsilon > \eta} \right| + \left| \mathbb{E}(f(U_N) - f(U_N^\epsilon)) 1_{U_N - U_N^\epsilon \leq \eta, |U_N| \vee |U_N^\epsilon| > K} \right| + \left| \mathbb{E}(f(U_N) - f(U_N^\epsilon)) 1_{U_N - U_N^\epsilon \leq \eta, |U_N| \vee |U_N^\epsilon| \leq K} \right| $$

(4.36) (4.37) and (4.38) readily yield that for any $\epsilon > 0$, for any $N$,

$$ \mathbb{E}(|U_N|) \vee \mathbb{E}(|U_N^\epsilon|) \leq \int_{\mathbb{C}} |\overline{\partial F_k}(h)(z)|\xi_N(z)| d^2z \leq C, $$

and therefore

$$ \mathbb{P}(|U_N| \vee |U_N^\epsilon| > K) \leq \mathbb{P}(|U_N| > K) + \mathbb{P}(|U_N^\epsilon| > K) \leq \frac{2}{K} \mathbb{E}(|U_N|) \vee \mathbb{E}(|U_N^\epsilon|) \leq \frac{C}{K}. $$

Thus, we can choose $K$ such that, for any $\epsilon > 0$, any $\eta > 0$ and any $N$, the RHS in (4.37) is smaller than $\delta$. Now, since $f$ is uniformly continuous on $\{z \in \mathbb{C}, |\Im(z)| \leq K\}$, one can choose $\eta$ small enough such that, for any $\epsilon > 0$ and any $N$, (4.38)
are smaller that $\delta$. Finally, by using (4.33) and (4.39), for any $\epsilon$ small enough, for any $N$,

$$\mathbb{E} |U_N - U_N'| = \mathbb{E} \left| \int_{\{z, |\Im(z)| < \epsilon\}} \overline{\partial F_k(h)(z)} \xi_N(z) d^2z \right| \leq C \epsilon, \tag{4.40}$$

and then

$$\mathbb{P} (|U_N - U_N'| > \eta) \leq \frac{C}{\eta} \epsilon.$$  

The term $|\mathbb{E}(f(V_N')) - \mathbb{E}(f(V_N))|$ is treated in the same way, using (4.34).

ii) Note that since $h$ and $\chi$ are compactly supported, $\int_{D_\epsilon} \overline{\partial F_k(h)(z)} \xi_N(z) d^2z$ may be seen as an integral on a fixed compact set $K_\epsilon \subset D_\epsilon$. Proposition 4.4 readily yields that $\xi_N$ is tight on the set $C(K_\epsilon)$ of complex continuous functions on $K_\epsilon$. Thus, using Corollary 4.7, since $f \mapsto \int_{D_\epsilon} \overline{\partial F_k(h)(z)} f(z) d^2z$ is continuous on $C(K_\epsilon)$, it remains to prove the tightness of the process $\{T_N(.)\}$ on any compact set $K$ in $\{z, |z| \geq \epsilon\}$ to deduce from Lemma 5.7 in Najim and Yao (2016) that:

$$|\mathbb{E}(f(U_N')) - \mathbb{E}(f(V_N'))| \rightarrow_{N \rightarrow +\infty} 0.$$  

We postpone the proof of the tightness of the process $\{T_N(.)\}$ (see Lemma 4.9 below) Thus, up to the proof of Lemma 4.9, the convergence to 0 of $d_{L^2}(\Phi_n, -\frac{1}{2} \int_{C} \overline{\partial F_k(h)}(z) T_N(z) d^2z)$ follows.

It remains to compute $c_{\theta,\nu}, B_{\theta,\nu}, A_{\theta,\nu,N}$. The computation of $c_{\theta,\nu}$ follows from Proposition 2.4:

$$c_{\theta,\nu} = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial F_k(h)(z)} \frac{1}{(z - \sigma^2 g(z) - \theta)^2} d^2z = \text{Res}(\frac{1}{(z - \sigma^2 g(z) - \theta)^2}, \rho_\theta).$$

A straightforward computation gives

$$\text{Res}(\frac{1}{(z - \sigma^2 g(z) - \theta)^2}, \rho_\theta) = -\frac{\omega''(\rho_\theta)}{(\omega'(\rho_\theta))^3} = H''(\theta)$$

where $\omega(z) = z - \sigma^2 g(z)$ ($g := g_\lambda$) and $H(z) = z + \sigma^2 g_\nu(z)$.

From Lemma 4.6 and using Fubini theorem, the diagonal entries of $Z_N$ have variance equal to

$$\frac{1}{2} (m_4 - 3 \sigma^4) A_{\theta,\nu,N} + \sigma^4 B_{\theta,\nu}$$

and the off diagonal entries of $Z_N$ have variance equal to $\sigma^4 B_{\theta,\nu}$ where

$$A_{\theta,\nu,N} = \frac{1}{N - p} \sum_{i=1}^{N-p} \left( \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial F_k(h)(z)} \frac{[(z - \sigma^2 g(z) - A_{N-p})^{-1}]_i}{(z - \sigma^2 g(z) - \theta)^2} d^2z \right)^2 d\nu(x)$$

$$B_{\theta,\nu} = \int_{\mathbb{R}} \left( \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial F_k(h)(z)} \frac{1}{(z - \sigma^2 g(z) - \theta)^2 (z - x)} d^2z \right)^2 d\lambda(x).$$

The functions $\phi_i(z) = \frac{[(z - \sigma^2 g(z) - A_{N-p})^{-1}]_i}{(z - \sigma^2 g(z) - \theta)^2 (z - x)}$ for $i \leq N - p$ and $\phi_{\nu}(z) = \frac{1}{(z - \sigma^2 g(z) - \theta)^2 (z - x)}$ for $x \in \text{supp}(\lambda)$ satisfy the hypothesis of Proposition 2.4.
Non universality of fluctuations of outlier eigenvectors

Straightforward computations lead to:

\[ A_{\theta,\nu,N} = \frac{1}{N-p} \sum_{i=1}^{N-p} \left( \frac{\text{Res} \left( \left( (z - \sigma^2 g(z) - A_{N-p})^{-1} \right)_{ii}, \rho_\theta \right) }{(z - \sigma^2 g(z) - \theta)^2} \right) \]

\[ = \frac{1}{N-p} \sum_{i=1}^{N-p} \left( H''(\theta) \left[ (\theta I_{N-p} - A_{N-p})^{-1} \right]_{ii} - H'(\theta) \left[ (\theta I_{N-p} - A_{N-p})^{-2} \right]_{ii} \right) \]

\[ = \frac{1}{N-p} \sum_{i=1}^{N-p} \left( (\sigma^2 g''(\theta)) \left[ (\theta I_{N-p} - A_{N-p})^{-1} \right]_{ii} - (1 + \sigma^2 g''(\theta)) \left[ (\theta I_{N-p} - A_{N-p})^{-2} \right]_{ii} \right) \]

and

\[ B_{\theta,\nu} = \int_{\mathbb{R}} \left( \text{Res} \left( \frac{1}{(z - \sigma^2 g(z) - \theta)^2(z-x)^2} \right) \rho_\theta \right) d\lambda(x) \]

\[ = -\frac{1}{6} g''''(\theta) - \frac{\sigma^2}{2} (g''(\theta)) 1 + 2\sigma^2 g''(\theta) \frac{1 + \sigma^2 g''(\theta)}{1 + \sigma^2 g''(\theta)}. \]

\[ \square \]

We now prove the following Lemma, used in the proof of the above Proposition.

**Lemma 4.9.** Let \( K \) be a compact subset in \( \{ z, |z| \geq \epsilon \} \), for some \( \epsilon > 0 \). The process \( \{ T_N(z) \} \) defined in (4.31) is a tight sequence on \( K \), more precisely,

\[ \sup_{z_1, z_2 \in K, n \in \mathbb{N}} \frac{\mathbb{E}((T_N(z_1) - T_N(z_2))^2)}{|z_1 - z_2|^2} < \infty. \]  

(4.41)

**Proof:** From Lemma 4.6,

\[ \mathbb{E}((G_N)_{ij}(z_1) - (G_N)_{ij}(z_2))^2) = \delta_{ij} \frac{1}{2} (m_4 - 3\sigma^4) \times \]

\[ \frac{1}{N-p} \sum_{i=1}^{N-p} \left( \left| \left( (z_1 - \sigma^2 g(z_1) - A_{N-p})^{-1} \right)_{ii} - \left( (z_2 - \sigma^2 g(z_2) - A_{N-p})^{-1} \right)_{ii} \right|^2 \right) \]

\[ + \sigma^4 \int \left| \frac{1}{z_1 - x} - \frac{1}{z_2 - x} \right|^2 d\lambda(x). \]

From the resolvent identity,

\[ (z_1 - \sigma^2 g(z_1) - A_{N-p})^{-1} = (z_2 - \sigma^2 g(z_2) - A_{N-p})^{-1} \]

\[ = (z_2 - z_1 - \sigma^2 (g(z_2) - g(z_1))) (z_1 - \sigma^2 g(z_1) - A_{N-p})^{-1} (z_2 - \sigma^2 g(z_2) - A_{N-p})^{-1}, \]

thus,

\[ \left| \left( (z_1 - \sigma^2 g(z_1) - A_{N-p})^{-1} \right)_{ii} - \left( (z_2 - \sigma^2 g(z_2) - A_{N-p})^{-1} \right)_{ii} \right| \]

\[ \leq \frac{1}{\epsilon^2} (1 + \frac{\sigma^2}{\epsilon^2}) |z_1 - z_2| \]

and thus,

\[ \frac{1}{N-p} \sum_{i=1}^{N-p} \left( \left| \left( (z_1 - \sigma^2 g(z_1) - A_{N-p})^{-1} \right)_{ii} - \left( (z_2 - \sigma^2 g(z_2) - A_{N-p})^{-1} \right)_{ii} \right|^2 \right) \]

\[ \leq \frac{1}{\epsilon^2} (1 + \frac{\sigma^2}{\epsilon^2}) |z_1 - z_2|^2. \]
Since moreover
\[ \int \left| \frac{1}{z_1 - x} - \frac{1}{z_2 - x} \right|^2 d\lambda(x) \leq |3z_1|^2 |z_2|^2 |z_1 - z_2|^2 \leq \frac{1}{\varepsilon^4} |z_1 - z_2|^2 \]

(4.41) readily follows. The tightness follows from Kolmogorov’s criterion (see Billingsley, 1999).

Proposition 4.8 and Proposition 4.1 readily yield Theorem 1.11.

4.5. Proof of Theorem 1.12. Theorem 1.12 follows from Theorem 1.11 once we proved that \( A_{\theta,\nu,N} \) converge to \( A_{\theta,\nu} \).

**Lemma 4.10.** Assume that the matrix \( A_N \) satisfies (A') with \( A_{N-p} \) diagonal. Then, the sequence \( (A_{\theta,\nu,N})_N \) defined by (1.15) converges to \( A_{\theta,\nu} \) defined by (1.18).

**Proof:** Denote by \( d_i \) the eigenvalues of \( A_{N-p} \).

\[
A_{\theta,\nu,N} = \frac{1}{N-p} \sum_{i=1}^{N-p} \left( H''(\theta)(\theta - d_i)^{-1} - H'(\theta)(\theta - d_i)^{-2} \right)^2
\]

\[
\rightarrow_{N\rightarrow\infty} (H''(\theta))^2 \int \frac{1}{(\theta - x)^2} d\nu(x) - 2H'(\theta)H''(\theta) \int \frac{1}{(\theta - x)^3} d\nu(x)
\]

\[
+ (H'(\theta))^2 \int \frac{1}{(\theta - x)^3} d\nu(x)
\]

\[
= -(H''(\theta))^2 g''_\nu(\theta) - H'(\theta)H''(\theta) g'''_\nu(\theta) - \frac{1}{6} (H'')^2 g''''_\nu(\theta)
\]

Using, \( H'(\theta) = 1 + \sigma^2 g''_\nu(\theta) \) and \( H''(\theta) = \sigma^2 g''_\nu(\theta) \), we obtain the formula for \( A_{\theta,\nu} \) given in (1.18).

5. Appendix: Poincaré inequality and concentration phenomenon

A probability \( \mu \) satisfies a Poincaré inequality if for any \( C^1 \) function \( f : \mathbb{R} \rightarrow \mathbb{C} \) such that \( f \) and \( f' \) are in \( L^2(\mu) \),

\[
\mathcal{V}(f) \leq C_{PI} \int |f'|^2 d\mu,
\]

with \( \mathcal{V}(f) = \int |f - f d\mu|^2 d\mu \).

If the law of a random variable \( X \) satisfies the Poincaré inequality with constant \( C_{PI} \) then, for any fixed \( \alpha \neq 0 \), the law of \( \alpha X \) satisfies the Poincaré inequality with constant \( \alpha^2 C_{PI} \).

Assume that probability measures \( \mu_1, \ldots, \mu_M \) on \( \mathbb{R} \) satisfy the Poincaré inequality with constant \( C_{PI}(1), \ldots, C_{PI}(M) \) respectively. Then the product measure \( \mu_1 \otimes \cdots \otimes \mu_M \) on \( \mathbb{R}^M \) satisfies the Poincaré inequality with constant \( C_{PI}^* = \max_{i \in \{1, \ldots, M\}} C_{PI}(i) \) in the sense that for any differentiable function \( f \) such that \( f \) and its gradient \( \text{grad} f \) are in \( L^2(\mu_1 \otimes \cdots \otimes \mu_M) \),

\[
\mathcal{V}(f) \leq C_{PI}^* \int \|\text{grad} f\|_2^2 d\mu_1 \otimes \cdots \otimes d\mu_M
\]

with \( \mathcal{V}(f) = \int |f - f d\mu_1 \otimes \cdots \otimes \mu_M|^2 d\mu_1 \otimes \cdots \otimes d\mu_M \).
Lemma 5.1 (Lemma 4.4.3 and Exercise 4.4.5 in Anderson et al., 2010 or Chapter 3 in Ledoux, 2001). Let $\mathbb{P}$ be a probability measure on $\mathbb{R}^M$ which satisfies a Poincaré inequality with constant $C_{P1}$. Then there exists $K_1 > 0$ and $K_2 > 0$ such that, for any Lipschitz function $F$ on $\mathbb{R}^M$ with Lipschitz constant $|F|_{Lip}$,

$$\forall \epsilon > 0, \mathbb{P}(|F - E_{\mathbb{P}}(F)| > \epsilon) \leq K_1 \exp \left(-\frac{\epsilon}{K_2 \sqrt{C_{P1} |F|_{Lip}}} \right).$$

Acknowledgements

The authors want to thank an anonymous referee who encouraged them to establish more general results and to provide a better readability, which led to an overall improvement of the paper.

References

Anderson, G. W., Guionnet, A., and Zeitouni, O. An introduction to random matrices, volume 118 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2010). ISBN 978-0-521-19452-5. MR2760897.

Bai, Z. D. and Silverstein, J. W. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. Ann. Probab., 26 (1), 316–345 (1998). MR1617051.

Bai, Z. D. and Silverstein, J. W. Spectral analysis of large dimensional random matrices. Springer Series in Statistics. Springer, New York, second edition (2010). ISBN 978-1-4419-0660-1. MR2567175.

Bai, Z. D. and Yin, Y. Q. Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. Ann. Probab., 16 (4), 1729–1741 (1988). MR958213.

Baik, J., Ben Arous, G., and Péché, S. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. Ann. Probab., 33 (5), 1643–1697 (2005). MR2165575.

Belinschi, S. T. and Bercovici, H. A new approach to subordination results in free probability. J. Anal. Math., 101, 357–365 (2007). MR2346550.

Belinschi, S. T. and Capitaine, M. Spectral properties of polynomials in independent Wigner and deterministic matrices. J. Funct. Anal., 273 (12), 3901–3963 (2017). MR3711884.

Benaych-Georges, F. and Knowles, A. Lectures on the local semicircle law for Wigner matrices, in Advanced topics in random matrices., volume 53 of Panoramas et Synthèses. Société Mathématique de France, Paris (2017). ISBN 978-2-8669-850-3.

Biane, P. On the free convolution with a semi-circular distribution. Indiana Univ. Math. J., 46 (3), 705–718 (1997). MR1488333.

Biane, P. Processes with free increments. Math. Z., 227 (1), 143–174 (1998). MR1605393.

Billingsley, P. Convergence of probability measures. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition (1999). ISBN 0-471-19745-9. MR1700749.

Bloemendal, A., Erdős, L., Knowles, A., Yau, H.-T., and Yin, J. Isotropic local laws for sample covariance and generalized Wigner matrices. Electron. J. Probab., 19, no. 33, 53 (2014). MR3183577.
Capitaine, M. Additive/multiplicative free subordination property and limiting eigenvectors of spiked additive deformations of Wigner matrices and spiked sample covariance matrices. *J. Theoret. Probab.*, 26 (3), 595–648 (2013). MR3090543.

Capitaine, M. Nonuniversality of fluctuations of outliers for Hermitian polynomials in a complex Wigner matrix and a spiked diagonal matrix. *Random Matrices Theory Appl.*, 9 (4), 2050013, 42 (2020). MR4133069.

Capitaine, M. and Donati-Martin, C. Spectrum of deformed random matrices and free probability. In *Advanced topics in random matrices*, volume 53 of Panor. Synthèses, pp. 151–190. Soc. Math. France, Paris (2017). MR3792626.

Capitaine, M., Donati-Martin, C., and Féral, D. The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations. *Ann. Probab.*, 37 (1), 1–47 (2009). MR2489158.

Capitaine, M., Donati-Martin, C., and Féral, D. Central limit theorems for eigenvalues of deformations of Wigner matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48 (1), 107–133 (2012). MR2919200.

Capitaine, M., Donati-Martin, C., Féral, D., and Février, M. Free convolution with a semicircular distribution and eigenvalues of spiked deformations of Wigner matrices. *Electron. J. Probab.*, 16, no. 64, 1750–1792 (2011). MR2835253.

Erdős, L., Schlein, B., and Yau, H.-T. Local semicircle law and complete delocalization for Wigner random matrices. *Comm. Math. Phys.*, 287 (2), 641–655 (2009a). MR2481753.

Erdős, L., Schlein, B., and Yau, H.-T. Semicircle law on short scales and delocalization of eigenvectors for Wigner random matrices. *Ann. Probab.*, 37 (3), 815–852 (2009b). MR2537522.

Füredi, Z. and Komlós, J. The eigenvalues of random symmetric matrices. *Combinatorica*, 1 (3), 233–241 (1981). MR637828.

Knowles, A. and Yin, J. Eigenvector distribution of Wigner matrices. *Probab. Theory Related Fields*, 155 (3-4), 543–582 (2013). MR3034787.

Ledoux, M. *The concentration of measure phenomenon*, volume 89 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI (2001). ISBN 0-8218-2864-9. MR1849347.

Lee, J. O. and Yin, J. A necessary and sufficient condition for edge universality of Wigner matrices. *Duke Math. J.*, 163 (1), 117–173 (2014). MR3161313.

Mathias, R. The Hadamard operator norm of a circulant and applications. *SIAM J. Matrix Anal. Appl.*, 14 (4), 1152–1167 (1993). MR1238930.

Mingo, J. A. and Speicher, R. *Free probability and random matrices*, volume 35 of *Fields Institute Monographs*. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON (2017). ISBN 978-1-4939-6941-8; 978-1-4939-6942-5. MR3585560.

Najim, J. and Yao, J. Gaussian fluctuations for linear spectral statistics of large random covariance matrices. *Ann. Appl. Probab.*, 26 (3), 1837–1887 (2016). MR3513608.

Pastur, L. A. The spectrum of random matrices. *Teoret. Mat. Fiz.*, 10 (1), 102–112 (1972). MR475502.
Péché, S. The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Related Fields*, **134** (1), 127–173 (2006). MR2221787.

Shirai, T. Limit theorems for random analytic functions and their zeros. In *Functions in number theory and their probabilistic aspects*, RIMS Kōkyūroku Bessatsu, B34, pp. 335–359. Res. Inst. Math. Sci. (RIMS), Kyoto (2012). MR3014854.

Soshnikov, A. Universality at the edge of the spectrum in Wigner random matrices. *Comm. Math. Phys.*, **207** (3), 697–733 (1999). MR1727234.

Tao, T. and Vu, V. Random matrices: universal properties of eigenvectors. *Random Matrices Theory Appl.*, **1** (1), 1150001, 27 (2012). MR2930379.

Tracy, C. A. and Widom, H. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.*, **159** (1), 151–174 (1994). MR1257246.

Voiculescu, D. V. The analogues of entropy and of Fisher’s information measure in free probability theory. I. *Comm. Math. Phys.*, **155** (1), 71–92 (1993). MR1228526.

Voiculescu, D. V., Dykema, K. J., and Nica, A. *Free random variables. A non-commutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups*, volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI (1992). ISBN 0-8218-6999-X. MR1217253.

Wigner, E. P. Characteristic vectors of bordered matrices with infinite dimensions. *Ann. of Math. (2)*, **62**, 548–564 (1955). MR77805.

Wigner, E. P. On the distribution of the roots of certain symmetric matrices. *Ann. of Math. (2)*, **67**, 325–327 (1958). MR95527.