Imprint of the equation of state of dense matter on gravitational waves emitted by oscillating neutron stars

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Abstract. A non radially oscillating star emits gravitational waves at frequencies and with damping times that depend on the equation of state of matter and on the dynamical processes that occur in the interior. In this talk we will discuss how the high density behaviour of matter affects the oscillation frequencies of neutron stars in different phases of their life.

1. Introduction
Since oscillating neutron stars are potential sources of gravitational waves (GW), during the past decades an extensive study has been performed in the literature to characterize the signals they may emit, providing information on peculiar features that could be used to devise appropriate detection strategies. The present generation of gravitational antennas, the resonant detectors like EXPLORER, NAUTILUS, AURIGA, MINIGRAIL, or the ground based interferometers, like VIRGO, LIGO, TAMA, GEO, do not have enough sensitivity to detect such signals, unless we are so lucky that the source is in our neighbourhood in the Galaxy; however, new and more sensitive detectors are under study that may be able to catch such signals, and therefore we may ask what kind of physics we could do using gravitational waves.

Stellar oscillations are a very well known phenomenon in astronomy, since they underline a variety of astrophysical processes. For instance, they are observed in the Sun, and the corresponding frequencies, measured with very high accuracy, are used in modern heliosismology to investigate the internal structure of our star. In addition, more than fifteen years ago it has been suggested that non-radial pulsations are at the origin of the drifting subpulses and micropulses detected in some radio sources, and of the quasi-periodic variability seen in some X-ray burst sources and in a number of bright X-ray sources [1]. Thus, due to their central role in astrophysics, oscillations of stars have been extensively studied both in the framework of the newtonian theory of gravity, and, starting in the late sixties, in general relativity [2] - [3], [5] - [8]. If the focus of our interest is on gravitational waves, our goal will be to compute the frequencies and the damping times of the modes of oscillations, and possibly to estimate the amount of energy that could be stored in different modes.

In general relativity non-radial pulsation modes are said "quasi-normal" because they are damped by the emission of gravitational waves, and the corresponding eigenfrequencies can be
found by solving the equations of non-radial stellar perturbations.

Quasi-normal modes (QNM) of neutron stars can be excited, for example, soon after birth, when the star just born in a gravitational collapse wildly oscillates; while the star is cooling down and contracting, a part of the residual mechanical energy will be dissipated in GW, with frequencies and damping times that depend on the very energetical physical process that occur in the interior, like those related to neutrino diffusion and thermalization processes. In section 2 we will discuss how the frequency and damping times of the QNM change during the first minute of life of a rapidly evolving protoneutron star, and give an example of what could be the waveform of the emitted gravitational signal. In section 3 we will see how rotation affects the mode frequencies and whether some modes may become unstable.

When the star has reached its final, cold configuration, further phenomena like starquakes, glitches, or a close interaction with a stellar companion may perturb the star, and set it into oscillations. Again, GW would be one of the channels through which energy would be dissipated, and the frequencies of the QNM would carry interesting information of the state of matter in the interior, which may reach densities larger than the equilibrium density of nuclear matter \( \rho_0 = 2.67 \times 10^{14} \text{ g/cm}^3 \). This is of particular interest, because neutron stars would give a unique opportunity to glimpse of the behavior of matter at supranuclear densities which cannot be reached in high energy experiments on Earth. In section 4 we will see how the QNM frequencies are affected by the equation of state (EOS) of matter at densities above \( \rho_0 \), further discussing whether the knowledge of these frequencies, possibly identified in a GW signal, would allow to discriminate among different EOS.

2. Pulsation properties of a newly born, hot proto-neutron star

According to general relativity, non-radially pulsating stars emit GW at the characteristic frequencies and damping times of the quasi-normal modes. These modes are classified according to the nature of the restoring force that is prevailing inside the star: the \( g \)-modes, or gravity modes, if this force is due to an eulerian change of density, the \( p \)-modes, if it is due to pressure gradients. The two classes of modes occupy well defined regions of the frequency spectrum, and are separated by the frequency of the \( f \)-mode (fundamental mode). Numerical simulations show that most of the energy released in gravitational waves goes in the \( f \)-mode, which is characterized by a wavefunction with non-nodes inside the star. In addition, there exist modes that are purely gravitational, since they do not induce fluid motion, named \( s \)- and \( w \)-modes [7] - [9]. \( s \)-modes exist only for very compact stars and can have low frequency; \( w \)-modes are typically higher than the \( p \)-mode frequencies. The mode frequencies are ordered as follows

\[
\omega_{g_n} < \ldots < \omega_{g_1} < \omega_f < \omega_{p_1} < \ldots < \omega_{p_n} < \omega_{w_n} \ldots
\]

It should be mentioned that the \( g \)-modes are present in the frequency spectrum only if there are thermal or composition gradients inside the star. Indeed when a fluid element is displaced from equilibrium by an external perturbation, it acquires a radial acceleration which can be written as

\[
a = -\frac{e^{-\lambda/2}}{(\epsilon + p)\gamma p} \left( -\frac{dp}{dr} \right) S(r) \Delta r,
\]

where \( \epsilon \) and \( p \) are the energy density and the pressure, \( \Delta r \) is the radial displacement of the fluid element, \( \gamma = [(\epsilon + p)/p][\partial p/\partial \epsilon]_{s,Y_e} \) is the adiabatic index, with \( Y_e \) being the electron fraction, \(-e^\lambda \) is the \( g_{rr} \) component of the unperturbed metric tensor, and

\[
S(r) = \frac{dp}{dr} - \frac{\gamma p}{\epsilon + p} \frac{d\epsilon}{dr}
\]

is the relativistic Schwarzschild discriminant. If \( S(r) > 0 \), the displaced fluid element feels a force that tends to restore the initial equilibrium position, and a non-zero frequency spectrum
of stable $g$-modes appears; this happens when there are thermal or composition gradients in the stellar interior. Conversely, if $S(r) < 0$, the force tends to move the fluid element away from equilibrium and the star becomes unstable against convection. Finally, if $S(r) = 0$, all $g$-modes degenerate to zero frequency, as in the case of a zero temperature, chemically homogenous star.

In this section we shall see how the frequencies of the QNM of a proto-neutron star (PNS) that has just been formed in a gravitational core collapse change during the first minute of the stellar life.

The processes that precede the stellar birth can be very briefly summarized as follows. Immediately after the core bounce and the passage of the shock wave through the outer PNS mantle, the star is composed of a low entropy core of mass $\simeq 0.7 M_\odot$, in which neutrinos are trapped, surrounded by a low density, high entropy mantle. The mantle is accreting matter from the outer iron shell and it is rapidly losing energy due to electron captures and thermal neutrino emission. It extends up to the shock, which is temporarily stalled at a radius of about 200 km, until the supernova explosion lifts off the stellar envelope and, in a few tenths of seconds, due to extensive neutrino losses, the lepton pressure decreases and the mantle contracts. The radius of the PNS is now about 20 – 30 km. These initial phases of the PNS life are described by dynamical simulations [10] - [11].

The subsequent evolution of the star proceeds along less hectic path; indeed, it has been shown that at later times (i.e. for $t \gtrsim 0.1 - 0.2$ s) it can be described by a sequence of equilibrium configurations [12] - [14], that account for the following physical processes. Initially, neutrinos are trapped inside the star; they have high energy of the order of $\sim 200$ MeV, and diffuse from the core to he surface from where they escape as low-energy neutrinos ($10 - 20$ MeV). The diffusion process heats the star to temperatures up to 50 MeV, while the core entropy approximately doubles. In about 15 s the PNS becomes lepton-poor but it is still hot. The net number of neutrinos in its interior is low, but thermally produced neutrino pairs of all flavors are abundant and still dominate the emission. At the same time the star cools down, the average neutrino energy decreases, and the neutrino mean free path increases. After approximately 50 seconds the temperature has dropped to $\approx 10^{10}$ K, the mean free path becomes comparable to the stellar radius, and the star becomes transparent to neutrinos. By this time, the star has radiated almost all of its binding energy becoming what we call a neutron star.

In a recent paper we have considered the part of the stellar evolution which can be described by the "quasi-stationary" approach, and we have computed, at different instant of time starting from $t \gtrsim 0.2$ s the QNM frequencies for different stellar models [15]. The models of evolving PNS we have used have been developed in [12, 14]. In these models the EOS of baryonic matter is a finite-temperature, field-theoretical model solved at the mean field level. Electrons and muons are included in the models as non interacting particles, being the contribution due to their interactions much smaller than that of the free Fermi gas. Neutrino transport is treated using the diffusion approximation (for further details see [15, 12, 14]). In this study, the star is considered as non rotating. The results are shown in Fig. 1 where we plot the frequencies (left panel) and damping times (right panel) of the fundamental mode and of the first $g$, $p$, $w$-modes of the evolving PNS as functions of the evolution time. The data refer to the evolutionary model labelled as model A in [15].

From Fig. 1 we see that during the first few tenths of seconds the frequency of all modes are smaller than those of the cold NS which forms at the end of the evolution, and cluster in a narrow region ($\nu \in [600,1500]$ Hz), beginning to differentiate after less than a second. This behaviour is more evident for the $p$- and $w$- modes, the frequency of which is much smaller than that of the corresponding modes of the final, cold neutron star.

It is known that in cold, chemically homogeneous stars the frequency of the $f$-mode scales as the square root of the average density; from Fig. 1 we see that while during the first second of life the star contracts and cools, the $f$-mode frequency remains nearly constant. In addition, due
Figure 1. The frequencies (left panel) and the damping times (right panel) of the fundamental mode, and of the first $g$-, $p$- and $w$-modes of an evolving proto–neutron star are plotted as functions of the time elapsed from the gravitational collapse, during the first 5 seconds of evolution. The data refer to model A of ref. [15]. The damping time of the first $w$–mode is not shown in the right panel because it is too small with respect to the scale of the figure ($\tau_{w1} = [1.4 - 2.1] \times 10^{-4}$).

To the strong thermal gradients that characterize this part of the life of the star, there exist high frequency $g$–modes ($> 500$ Hz). Fig. 1 also shows that after a few seconds the QNM frequencies tend to the typical values appropriate for cold NSs.

The evolution of the damping times of these modes is also interesting $^1$. In the right panel of Fig. 1, we see that at very early times ($0.2 \lesssim t \lesssim 5$ s) $\tau_f$ decreases rapidly from $\sim 10$ s to $\sim 2 - 3$ s. Conversely $\tau_p$ initially decreases and then grows, always remaining smaller than 5 s. The damping time of the first $g$–mode has a similar behaviour, but its growth is much stiffer and after one second is already greater than 10 s. These values have to be compared with those of the other dissipation processes that “compete” with the gravitational wave emission, i.e. with the timescales of neutrino viscosity, diffusivity, thermal conductivity, or thermodiffusion. As discussed in [15] a conservative estimate of the dissipative timescales associated to neutrino processes is $\tau_{diss} \approx 10 - 20$ seconds. From Fig. 1 we see that during the first few seconds of life both the $f$- and the first $p$–mode have damping times smaller than $\tau_{diss}$; therefore if there is energy initially stored in the $f$– and/or in the $p$–mode, it will be dissipated mainly in gravitational waves at the corresponding frequencies. Conversely, the $g$–mode may give a contribution to GW only during the first second of life of the PNS. The damping time of the first $w$–modes, $\tau_{w1}$, is not shown in the figure because it is much smaller than the others ($\tau_{w1} = [1.4 - 2.1] \times 10^{-4}$), and it appears to be basically unaffected by the evolution. Thus, if some energy is initially stored

$^1$ I would like to mention that in ref. [15] the values of the damping times plotted in Fig. 3 of that paper were erroneously multiplied by a factor $2\pi$. The misprint has been corrected in Fig. 1 of the present paper.
Figure 2. The strain amplitude of the gravitational signal produced by a proto–neutron star oscillating in the $f$–mode is plotted in the left panel as a function of the frequency. The energy stored into the mode is assumed to be $E_{GW} = 6 \times 10^{-9} M_{\odot} c^2$ and the source is in the Galaxy at a distance of 10 kpc from Earth. In the same figure we also plot the noise strain amplitude of the ground based interferometer VIRGO. On the right panel we plot a zoom of the same picture in the region where the gravitational signal is stronger into this, or in the higher order $w$–modes it will be emitted in gravitational waves. Furthermore, it is interesting to note that during the first second, $\tau_{g1}$ is much shorter than that expected from previous calculations for cold NSs[16].

Having computed $\nu(t)$ and $\tau(t)$, we can model the gravitational signal that a PNS would emit while oscillating in one of these modes as a damped sinusoid

$$h(t) = A e^{(t_{arr} - t)/\tau} \sin \left[ 2\pi \nu \left( t - t_{arr} \right) \right], \quad (3)$$

where $t_{arr}$ is the arrival time, $\nu \equiv \nu(t)$ and $\tau \equiv \tau(t)$ are respectively the frequency and the damping time of the chosen mode, and $A$ is the unknown mode amplitude. The energy stored in the mode can be estimated by integrating over the surface and over the frequency the expression of the energy flux

$$\frac{dE_{mode}}{dSd\nu} = \frac{\pi}{2} \nu^2 |\tilde{h}(\nu)|^2. \quad (4)$$

If, as an example, we assume that the star is born in our Galaxy, and that an energy $\Delta E_f = 6 \cdot 10^{-9} M_{\odot} c^2$ is stored into the $f$–mode, using a matched filtering technique in the data analysis of the VIRGO interferometer the signal could be detectable with a signal to noise ratio $SNR = 3$. In the left panel of Fig. 2 we plot the strain amplitude (i.e. $\sqrt{\nu} h(\nu)$, where $h(\nu)$ is the Fourier transform of $h(t)$) associated to such signal, compared to the sensitivity curve of the VIRGO detector. In the left panel we plot a zoom of the same picture in the region where the signal is more significant: if one of such signals could be detected, we may be able to
reconstruct its structure and get interesting information on the early evolution of the emitting source.

3. The effect of rotation on the QNM frequencies

In section 2 we have studied the mode frequencies assuming that the star is non rotating. However, stars do rotate and it is interesting to see whether and how rotation modifies the picture described above. In particular, it is interesting to explore the possibility that some of the modes may become unstable due to the Chandrasekhar-Friedman-Schutz instability (CFS-instability). This instability was first discovered by Chandrasekhar for the \( m=2 \) bar mode of an incompressible MacLaurin spheroid [17], and was later shown to act in every rotating star by Friedman and Schutz [18],[19]. Very briefly, this instability works in the following way. In the limit of vanishing rotation, the \( l = \pm m \) modes \(^2\) are defined to be prograde (+) and retrograde (-), and have frequencies ±|\( \omega \)|, respectively. When the star oscillates, it emits GW and consequently the oscillations are damped and the amplitude of the oscillations decreases. The emitted wave removes positive angular momentum from the prograde modes and negative angular momentum from the retrograde ones. When the star rotates, the modes are dragged forwards; consequently, the absolute value of the frequency of a backwards moving mode will decrease and, if the rotation rate is sufficiently high, it will reach a neutral point where the mode frequency becomes zero; for higher rotation rates the frequency of such mode will be positive and the mode will be seen to move forwards with respect to an inertial observer at radial infinity. GWs from such a mode carry away positive angular momentum from the star, but the angular momentum of the mode, as seen by the star, is negative. Thus, the GW emission makes the angular momentum of the mode increasingly negative and the mode amplitude increases leading to an instability.

Previous studies have shown that the \( f \)-mode instability occurs at very high rotation rates, and that unless the temperature is very low viscous dissipation mechanisms contrast the instability making it uneffective [20] - [22]. It should be mentioned however that these studies have usually been done for cold stars and, except that in ref. [23] where a more realistic EOS has been considered, very simple polytropic EOS have been used to model the stellar matter.

The \( g \)-mode instability has been studied only for cold stars, i.e at the end of their evolution, in the Cowling approximation (i.e. neglecting the gravitational perturbations), and using the newtonian theory of stellar perturbations. In ref. [24] for example, a class of \( g \)-modes was considered for which the buoyancy is provided by the gradient of proton/neutron ratio in the stellar interior.

In ref. [25] we have studied how the frequency of the \( f \)-and \( g \)-modes of the PNS considered in previous section change as a function of the angular velocity as the star evolves. We solve the equations of the perturbations of a slowly rotating star in general relativity, in the Cowling approximation. This approximation is known to reproduce with a good accuracy the \( g \)-mode frequency, in which we are primarily interested, because the gravitational perturbation induced by these modes is much smaller than that of the \( f \)-mode. The reason why we are mainly interested in \( g \)-modes is that during the first minute of the PNS life they have frequencies that are higher than those of the \( g \)-modes of a cold star, but smaller than the frequencies of the other modes; thus, they may become unstable for smaller values of the angular velocity. The results of our study are summarized in Fig. 3 and Fig. 4 where we plot, respectively, the frequencies of the \( f \)-mode, and those of the \( g_1 \)- and \( g_2 \)- modes as a function of the rotation parameter \( \varepsilon = \Omega / \Omega_K \).

Here \( \Omega \) is the star rotation rate, and \( \Omega_K = \sqrt{\frac{M}{R^3}} \). The mode frequencies are plotted for selected values of time elapsed from the stellar birth, ranging within [0.5 – 40] s, where the most interesting

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\(^2\) \( l \) and \( m \) are the harmonic indices associated to the expansion of the perturbed tensors in tensorial spherical harmonics.
Figure 3. The frequency of the fundamental mode of the evolving proto-neutron star is plotted as a function of the rotational parameter $\varepsilon = \Omega/\Omega_K$, for assigned values of the time elapsed from the gravitational collapse. We see that as the time increases, the frequency increases and tends to that of the cold neutron star which forms at the end of the evolutionary process. Reminding that the onset of the CFS instability occurs when the mode frequency becomes zero, we see that the $f$-mode would become unstable only for extremely high values of the rotational parameter, as it is for cold stars.

phases of the cooling process occur. From Fig. 3 we see that, for any value of the rotation rate, as time grows the $f$-mode frequency increases approaching that of the cold neutron star which forms at the end of the evolution. Conversely, Fig. 4 shows that the frequency of the $g$-modes initially decreases, reaches a minimum for $t_{ev} \simeq 12$ s, and subsequently smoothly increases. This behaviour is due to the fact that $g$-modes are associated to entropy and composition gradients, and whereas during the first 10-12 seconds the dynamical evolution of the star is dominated by strong entropy gradients that progressively smooth out, after $\sim 12$ s the entropy becomes nearly constant throughout the star and $g$-modes due to composition gradients take over.

The onset of the CFS instability is signaled by the vanishing of the mode frequency for some value of the angular velocity (neutral point). From Fig. 3 and 4 we see that while the $f$-mode does not become unstable during the first minute of the PNS life, both the $g_1$- and the $g_2$- modes do become unstable. The $g_1$- frequency remains positive during the first second, but at later times it vanishes for very low values of the rotation rate. For instance, at $t_{ev} = 3$ s it crosses the zero axis for $\Omega = 0.17 \Omega_K$, even though its value for the corresponding nonrotating star is still quite high, $\nu_{g_1} = 486$ Hz. The behaviour of the $g_2$- mode is similar but being the frequency lower, the instability sets in at even lower rotation rates. (These data refer to the model of evolving proto-neutron star, labelled as model A in [15], but similar results are obtained for the second model considered there).

A mode instability is physically significant if its growth time is sufficiently small with respect to the timescales typical of the stellar dynamics; in which case the instability has sufficient time to grow before other processes damp it out or the structure of the evolving star changes. In [25] we give an order of magnitude estimate of the growth time of the $g$-modes by computing the energy $E$ associated with a given mode in Newtonian approximation as in [24, 27], and the gravitational luminosity $\frac{dE}{dt}$ using a multipole expansion as in [28]. The growth time associated
Figure 4. The frequency of the $g_1$- and $g_2$- modes of the proto-neutron star are plotted, as in figure 3, as functions of the rotational parameter for assigned values of the time elapsed from the gravitational collapse. Unlike the $f$-mode, as the time increases the frequency of the $g$-modes decreases, reaches a minimum at about $t_{ev} = 12$ s and then slightly increases (see text). We see that for both modes the CFS instability sets in at values of the rotational parameter much lower that that needed for the $f$-mode.

4. The QNM frequencies of cold, old neutron stars

After some time from its birth in a stellar collapse, the neutron star finally cools down and adjusts itself into a final equilibrium configuration. At this stage, densities can be reached in the core that exceed the equilibrium density of nuclear matter $\rho_0$, and at such densities matter must be described by taking into account the fundamental theory of strong interactions, quantum chromo dynamic (QCD). Due to the complexity of this theory, the EOS available in the literature which describe strongly interacting matter have been obtained within models, based on the theoretical knowledge of the underlying dynamics and constrained, as much as possible,
by empirical data. It should be stressed that in the core of a neutron star, the main contribution
to pressure, which comes from neutrons, cannot be associated only to Pauli’s exclusion principle
because at these densities we can no longer treat neutrons as non interacting particles. Indeed,
if we would restrict to Pauli’s contribution, we would find that the maximum mass of a NS is
0.7 \( M_\odot \) which, according to astronomical observations, is far too low. Therefore, depending on
the particular way we choose to model this interaction, we shall have a different composition,
forming heavy baryons, quarks etc.

In recent years, a number of new EOS have been proposed to describe matter at supranuclear
densities \( \rho > \rho_0 \), some of them allowing for the formation of a core of strange baryons and/or
deconfined quarks, or for the appearance of a Bose condensate. Since different composition leads
to different QNM frequencies, it is interesting to see whether the differences are such that would
allow to discriminate among different EOS. To this purpose, in ref. [26] we have considered
models of neutron stars formed of an outer crust, an inner crust and a core, each shell being
described by a non viscous fluid which obeys equations of state appropriate to describe different
density regions. To model the outer crust, \( 10^7 \lesssim \rho \lesssim 4 \cdot 10^{11} \) g/cm\(^3\), we use the Baym-Pethick-
Sutherland (BPS) EOS [29], and for the inner crust \( 4 \cdot 10^{11} \lesssim \rho \lesssim 2 \cdot 10^{14} \) g/cm\(^3\), we use the
Pethick-Ravenhall-Lorenz (PRL) EOS [30].

For the inner core, \( \rho > 2 \cdot 10^{14} \) g/cm\(^3\) we use recent EOS which model hadronic interactions
in different ways leading to different composition and dynamics. We label them as follows.

- **APR2**. Matter consists of neutrons, protons, electrons and muons in weak equilibrium. The
  EOS is obtained within the nuclear many body theory, with a Hamiltonian including two-
  and three- nucleon interaction terms. The two-nucleon term is the Argonne \( v_{18} \) potential
  [31], the three-nucleon term is the Urbana IX potential [32]. The many-body Schrödinger
equation is solved using a variational approach [33, 34]. The calculations include relativistic
corrections to the two-nucleon potential, arising from the boost to a frame in which the total
momentum of the interacting pair is nonvanishing. These corrections are necessary to use
phenomenological potentials, describing interactions between nucleons in their center of
mass frame, in a locally inertial frame associated with the star. The maximum mass for this
EOS is \( M_{\text{max}} = 2.202M_\odot \).

- **APRB200, APRB120**. At densities lower than \( \sim 4\rho_0 \) matter is described by the EOS
  APR2 ; at higher density matter is composed of deconfined quarks described within the
  MIT bag model. Quark matter consists of massless up and down quarks and massive
  strange quarks, with \( m_s = 150 \) MeV, in weak equilibrium. The color coupling constant
  is set to \( \alpha_s = 0.5 \) and one-gluon-exchange interactions between quarks of the same flavor
  are taken into account at first order in \( \alpha_s \). The value of the bag constant is 200 and
  120 MeV/fm\(^3\) in the APRB200 and APRB120 model, respectively. The phase transition
  is described requiring the fulfillment of Gibbs conditions, leading to the formation of a
  mixed phase, and neglecting surface and Coulomb effects [33, 35]. The maximum mass is
  \( M_{\text{max}} = 2.029M_\odot \) for APRB200 , and \( M_{\text{max}} = 1.894M_\odot \) for APRB120 .

- **BBS1**. Matter composition is the same as in the APR2 model. The EOS is obtained
  within the non relativistic nuclear many body theory, using a slightly different Hamiltonian,
  including the Argonne \( v_{18} \) two-nucleon potential and the Urbana VII three-nucleon potential
  [36]. The ground state energy is calculated using G-matrix perturbation theory [37]. The
  maximum mass is \( M_{\text{max}} = 2.014M_\odot \).

- **BBS2**. Matter consists of nucleons leptons and strange heavy baryons (\( \Sigma^- \) and \( \Lambda^0 \)).
  Nucleon interactions are described as in BBS1 . Hyperon-nucleon interactions are described
  using the potential of ref. [38], while hyperon-hyperon interactions are neglected altogether.
  The binding energy is obtained from G-matrix perturbation theory [37]. The maximum
  mass is \( M_{\text{max}} = 1.218M_\odot \).
Figure 5. The frequency of the fundamental mode is plotted in the left panel as a function of the square root of the average density for the different EOS considered in this paper. We also plot the fit given by Andersson and Kokkotas (AK-fit) and our fit (New fit). The new fit is systematically lower (about 100 Hz) than the old one. The damping time of the fundamental mode is plotted in the right panel as a function of the compactness $M/R$. In this case the AK-fit and our fit, plotted respectively as a continuous and a dashed line, do not show significant differences.

- G240. Matter composition includes leptons and the complete octet of baryons (nucleons, $\Sigma^{0,\pm}$, $\Lambda^0$ and $\Xi^{\pm}$). Hadron dynamics is described in terms of exchange of one scalar and two vector mesons. The EOS is obtained within the mean field approximation [39]. The maximum mass is $M_{max} = 1.553M_\odot$.

In addition to the above models we have considered two models of strange stars, the existence of which is predicted as a consequence of the hypothesis, suggested by Bodmer [40] and Witten [41], that the ground state of strongly interacting matter consist of up, down and strange quarks. In the limit in which the mass of the strange quark can be neglected, the density of quarks of the three flavors is the same and charge neutrality is guaranteed even in absence of leptons. To gauge the difference between this exotic scenario and the more conventional ones, based mostly on hadronic degrees of freedom, we have calculated the EOS of strange quark matter within the MIT bag model, setting all quark masses and the color coupling constant to zero and choosing $B = 110$ MeV/fm$^3$. The models denoted SS1 and SS2 correspond to a quark star with a crust and to a bare quark star.

Using the above EOS to describe the matter in the crust and in the stellar core, we have constructed models of NS by integrating the relativistic equations of stellar structure for assigned values of the gravitational mass, ranging from 1.2 $M_\odot$ to the maximum mass allowed by the selected EOS. Then the equations of stellar perturbations for non rotating stars have been integrated to find the QNM frequencies and damping times. Some of the results of these calculations are shown in Fig. 5 and 6. In the left panel of Fig. 5 the frequency of the f-mode
Figure 6. The frequency of the $f$-mode (left) and of the first $p$-mode (right) are plotted as functions of the stellar mass for the different EOS and stellar models (see text).

is plotted as a function of the square root of the average density $(M/R^3)^{1/2}$ for the considered EOS; the corresponding damping times are plotted in the right panel as functions of $M/R$. $M$ and $R$ are the stellar mass and radius, respectively. These data can be fitted by the following expressions.

For the $f$-mode frequency:

$$
\nu_f = a + b \sqrt{\frac{M}{R^3}}, \quad a = 0.79 \pm 0.09, \quad b = 33 \pm 2
$$

where $a$ is given in kHz and $b$ in km · kHz.

For the $f$-mode damping time

$$
\tau_f = \frac{R^4}{cM^3} \left[ a + b \frac{M}{R} \right]^{-1},
$$

$$
a = [8.7 \pm 0.2] \cdot 10^{-2}, \quad b = -0.271 \pm 0.009.
$$

In these fits frequencies are expressed in kHz, masses and radii in km, damping times in s and $c = 3 \cdot 10^5$ km/s.

The reason why we compute these fits is the following. In 1998, extending a previous work of Lindblom and Detweiler [42], Andersson and Kokkotas computed the frequencies of the $f$-mode, of the first $p$-mode and of the first polar $w$-mode [43] of a non rotating NS for several EOS for superdense matter available at that time, the most recent of which was that obtained by Wiringa, Fiks & Fabrocini in 1988 [45]. They fitted the data with appropriate functions of the radius and the mass, and showed that these empirical relations could be used to constraint these parameters [43, 44]. In ref. [26] we have extended their work, including more modes and considering new equations of state. Thus, it is interesting to compare the fit we obtain in (6)
and (7) with their fit. For this reason in Fig. 5 we plot the fit (6) as a thick solid line, and the fit given in [43], which is based on the EOSs considered in that paper, as a dashed line labelled as ‘AK-fit’. From Fig. 5 we see that our new fit for $\nu_f$ is systematically lower than the AK fit by about 100 Hz; this basically shows that the new EOS are, on average, less compressible (i.e. stiffer) than the old ones. Conversely, eq. (7) is very similar to the fit found in [43].

Since the mass of a NS is the only observable on which we might have reliable information, it is also interesting to see whether the knowledge of the mode frequencies and of the stellar mass can help in discriminating among different EOSs. To this purpose in Fig. 6 we plot, respectively, $\nu_f$ (left panel) and $\nu_p$ (right panel) as a function of the mass, for all EOS and all stellar models.

From this figure we see that the presence of quark matter in the star inner core (EOS APRB200 and APRB120) does not affect the frequencies of these modes. Conversely, the BBS1 and APR2 equation of state, based on similar dynamical models, yield appreciably different f-mode frequencies. This is due to the effect of the relativistic corrections included in APR2 and to different treatments of three-nucleon interactions. While the variational approach of ref. [33], used to derive APR2 naturally allows for inclusion of the three-nucleon potential, in G-matrix perturbation theory, used to derive BBS1, this potential has to be replaced with an effective two-nucleon potential $\tilde{V}_{ij}$, obtained by averaging over the position of the third particle [46].

When hyperons are present in the core, as in the BBS2 model, the resulting EOS is softer, and stable NS configurations of very low mass are allowed to exist. As a consequence of the softening of the EOS, the corresponding f-mode frequency is significantly higher than those obtained with the other EOS. So much higher, in fact, that its detection would provide evidence of the presence of hyperons in the NS core.

It is also interesting to compare the f-mode frequencies corresponding to models BBS2 and G240, as they both predict the occurrence of heavy strange baryons but are obtained from different theoretical approaches, based on different descriptions of the underlying dynamics. The behavior of $\nu_f$ displayed in Fig. 6 directly reflects the relations between mass and central density obtained from the two EOS, larger frequencies being always associated with larger densities. For example, the NS configurations of mass $\sim 1.2 M_\odot$ obtained from the G240 and BBS2 have central densities $\sim 7 \cdot 10^{14}$ g/cm$^3$ and $\sim 2.5 \cdot 10^{15}$ g/cm$^3$, respectively. On the other hand, the G240 model requires a central density of $\sim 2.5 \cdot 10^{15}$ g/cm$^3$ to reach a mass of $\sim 1.55 M_\odot$ and a consequent $\nu_f$ equal to that of the BBS2 model.

Strange stars models, SS1 and SS2 also correspond to values of $\nu_f$ and $\nu_p$ well above those obtained from the other models. The peculiar properties of these stars largely depend upon the self-bound nature of strange quark matter.

5. Concluding Remarks
According to general relativity, a star excited by an external or internal non radial perturbation oscillates in its quasi-normal modes emitting gravitational waves. We have seen that the frequencies of oscillations and the damping times of the modes carry the imprint of the equation of state of matter in the interior, and consequently of the nuclear and thermodynamical processes that occur during the star evolution.

The present generation of resonant and ground based interferometers will probably not be able to catch the whisper of a newly born neutron star or of a "glitching" star, but future detectors will, opening the era of gravitational wave asteroseismology.

6. References

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