Solving Turán’s tetrahedron problem for the $\ell_2$-norm

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Abstract
Turán’s famous tetrahedron problem is to compute the Turán density of the tetrahedron $K^3_4$. This is equivalent to determining the maximum $\ell_1$-norm of the codegree vector of a $K^3_4$-free $n$-vertex 3-uniform hypergraph. We introduce a new way for measuring extremality of hypergraphs and determine asymptotically the extremal function of the tetrahedron in our notion. The codegree squared sum, $co_2(G)$, of a 3-uniform hypergraph $G$ is the sum of codegrees squared $d(x, y)^2$ over all pairs of vertices $xy$, or in other words, the square of the $\ell_2$-norm of the codegree vector of the pairs of vertices. We define $exco_2(n,H)$ to be the maximum $co_2(G)$ over all $H$-free $n$-vertex 3-uniform hypergraphs $G$. We use flag algebra computations to determine asymptotically the codegree squared extremal number for $K^3_4$ and $K^3_5$ and additionally prove stability results. In particular, we prove that the extremal $K^4_4$-free hypergraphs in $\ell_2$-norm have approximately the same structure as one of the conjectured extremal hypergraphs for Turán’s conjecture. Further, we prove several general properties about $exco_2(n,H)$ including the existence of a scaled limit, blow-up invariance and a supersaturation result.

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INTRODUCTION

For a $k$-uniform hypergraph $H$ (shortly $k$-graph), the Turán function (or extremal number) $\text{ex}(n,H)$ is the maximum number of edges in an $H$-free $n$-vertex $k$-uniform hypergraph. The graph case, $k=2$, is reasonably well-understood. The classical Erdős–Stone–Simonovits theorem [15, 17] determines asymptotically the extremal number for graphs with chromatic number at least three. However, for general $k$, the problem of determining the extremal function is much harder and widely open. Despite enormous efforts, our understanding of Turán functions is still limited. Even the extremal function of the tetrahedron $K_3^4$, the 3-graph on four vertices with four edges, is unknown. There are exponentially (in the number of vertices) many conjectured extremal hypergraphs which is believed to be the root of the difficulty of this problem. Brown [10], Kostochka [35], Fon-der-Flaass [23] and Frohmader [25] constructed families of $K_3^4$-free 3-graphs which they conjectured to be extremal. For an excellent survey on Turán functions of cliques, see [53] by Sidorenko.

Successively, the upper bound for extremal number of the tetrahedron has been improved by de Caen [13], Giraud (unpublished, see [11]), Chung and Lu [11], and finally Razborov [46] and Baber [2], both making use of Razborov’s flag algebra approach [45] (see also Baber and Talbot [3]). Another relevant result toward solving Turán’s tetrahedron problem is by Pikhurko [43]. Building on a result by Razborov [46], Pikhurko [43] determined the exact extremal hypergraph when the induced 4-vertex graph with one edge is forbidden in addition to the tetrahedron.

In this paper, we study a different notion of extremality and solve the tetrahedron problem asymptotically for this notion. It is interesting that the extremal $K_3^4$-free hypergraphs in $\ell_2$-norm have approximately the same structure as one of the conjectured extremal hypergraphs for Turán’s conjecture. For an integer $n$, denote by $[n]$ the set of the first $n$ integers. Given a set $A$ and an integer $k$, we write $(A)_k$ for the set of all subsets of $A$ of size $k$. Let $G$ be an $n$-vertex $k$-uniform hypergraph. For $T \subset V(G)$ with $|T| = k - 1$, we denote by $d_G(T)$ the codegree of $T$, that is, the number of edges in $G$ containing $T$. If the choice of $G$ is obvious, we will drop the index and just write $d(T)$. The codegree vector of $G$ is the vector

$$X \in \mathbb{Z}^{(V(G))_{k-1}}$$

where $X(v_1, v_2, ..., v_{k-1}) = d(v_1, v_2, ..., v_{k-1})$ for every $\{v_1, v_2, ..., v_{k-1}\} \in (V(G))_{k-1}$. The $\ell_1$-norm of the codegree vector, or to put it in other words, the sum of codegrees, is $k$ times the number of edges. Thus, Turán’s problem for $k$-graphs is equivalent to the question of finding the maximum $\ell_1$-norm for the codegree vector of $H$-free $k$-graphs. We propose to study this maximum with respect to other norms. A particular interesting case seems to be the $\ell_2$-norm of the codegree vector. We will refer to the square of the $\ell_2$-norm of the codegree vector as the codegree squared sum denoted by $\text{co}_2(G)$,

$$\text{co}_2(G) = \sum_{T \subset ([n])_{k-1}} d_G^2(T).$$

**Question 1.1.** Given a $k$-uniform hypergraph $H$, what is the maximum codegree squared sum a $k$-uniform $H$-free $n$-vertex hypergraph $G$ can have?

Many different types of extremality in hypergraphs have been studied:
The most related one is the minimum codegree-threshold. For a given $k$-graph, the minimum codegree-threshold is the largest minimum codegree an $n$-vertex $k$-graph can have without containing a copy of $H$. This problem has not even been solved for $H$ being the tetrahedron. For a collection of results on the minimum codegree-threshold, see [18–20, 38–42, 54].

Reiher, Rödl and Schacht [49, 50] introduced new variants of the Turán density, which ask for the maximum density for which an $H$-free hypergraph with a certain quasi-randomness property exists. Roughly speaking, a quasi-randomness property is a property which holds for the random hypergraph with high probability. Reiher, Rödl and Schacht [49] determined such a variant for the tetrahedron. In this paper, we solve asymptotically Question 1.1 for the tetrahedron. For a family $\mathcal{F}$ of $k$-uniform hypergraphs, we define $\text{exco}_2(n, \mathcal{F})$ to be the maximum codegree squared sum a $k$-uniform $n$-vertex $\mathcal{F}$-free hypergraph can have, and the codegree squared density $\sigma(\mathcal{F})$ to be its scaled limit, that is,

$$\text{exco}_2(n, \mathcal{F}) = \max_{G \text{ is an } n\text{-vertex } \mathcal{F}\text{-free } \text{k-uniform hypergraph}} \text{co}_2(G) \quad \text{and} \quad \sigma(\mathcal{F}) = \lim_{n \to \infty} \frac{\text{exco}_2(n, \mathcal{F})}{\binom{n}{k-1}(n-k+1)^2}. \quad (1)$$

We will observe in Proposition 1.8 that the limit in (1) exists. Denote by $K^3_4$ the complete 3-uniform hypergraph on 4 vertices. Our main result is that we determine the codegree squared density asymptotically for $K^3_4$ and $K^3_5$, respectively.

**Theorem 1.2.** We have

$$\sigma(K^3_4) = \frac{1}{3} \quad \text{and} \quad \sigma(K^3_5) = \frac{5}{8}.$$

Denote by $C_n$ the 3-uniform hypergraph on $n$ vertices with vertex set $V(C_n) = V_1 \cup V_2 \cup V_3$ such that $|V_i| - |V_j| \leq 1$ for $i \neq j$ and edge set

$$E(C_n) = \{abc : a \in V_1, b \in V_2, c \in V_3\} \cup \{abc : a, b \in V_1, c \in V_2\} \cup \{abc : a, b \in V_2, c \in V_3\} \cup \{abc : a, b \in V_3, c \in V_1\}.$$  

Further, denote by $B_n$ the balanced, complete, bipartite 3-uniform hypergraph on $n$ vertices, that is the hypergraph where the vertex set is partitioned into two sets $A, B$ such that $|A| - |B| \leq 1$ and the edge set is the set of triples intersecting both $A$ and $B$. See Figure 1 for an illustration of $C_n$ and $B_n$. The 3-graphs $C_n$ and $B_n$ are among the asymptotically extremal hypergraphs in $\ell_1$-norm for $K^3_4$ and $K^3_5$, respectively. We conjecture that $C_n$ and $B_n$ are the unique extremal hypergraphs in $\ell_2$-norm.

**Conjecture 1.3.** There exists $n_0$ such that for all $n \geq n_0$

$$\text{exco}_2(n, K^3_4) = \text{co}_2(C_n),$$

and $C_n$ is the unique $K^3_4$-free $n$-vertex 3-uniform hypergraph with codegree squared sum equal to $\text{exco}_2(n, K^3_4)$.

† This hypergraph is often referred to as Turán’s construction.
Note that Kostochka’s [35] result suggests that in the $\ell_1$-norm there are exponentially many extremal graphs, $C_n$ is one of them.

**Conjecture 1.4.** There exists $n_0$ such that for all $n \geq n_0$

$$\text{exco}_2(n, K^3_5) = \text{co}_2(B_n),$$

and $B_n$ is the unique $K^3_5$-free $n$-vertex 3-uniform hypergraph with codegree squared sum equal to $\text{exco}_2(n, K^3_5)$.

We believe that existing methods could prove these conjectures, though the potential proofs might be long and technical.

In Section 3.3, we observe that giving upper bounds on $\sigma(H)$ for some 3-graph $H$ is equivalent to giving upper bounds on a certain linear combination of densities of 4-vertex subgraphs in large $H$-free graphs, see (2). By now it is a standard technique in the field to use the computer-assisted method of flag algebras to prove such bounds. If one gets an asymptotically tight upper bound from a flag algebra computation, it is typically the case that there is an essentially unique stable extremal example and that one can extract a stability result from the flag algebra proof. This also happens for $K^3_4$ and $K^3_5$. For $\varepsilon > 0$, we say a given $n$-vertex 3-graph $H$ is $\varepsilon$-near to an $n$-vertex 3-graph $G$ if there exists a bijection $\phi : V(G) \to V(H)$ such that the number of 3-sets $xyz$ satisfying $xyz \in E(G), \phi(x)\phi(y)\phi(z) \notin E(H)$ or $xyz \notin E(G), \phi(x)\phi(y)\phi(z) \in E(H)$ is at most $\varepsilon |V(H)|^3$.

**Theorem 1.5.** For every $\varepsilon > 0$, there exists $\delta > 0$ and $n_0$ such that for every $n > n_0$, if $G$ is a $K^3_4$-free 3-uniform hypergraph on $n$ vertices with

$$\text{co}_2(G) \geq \left(\frac{1}{3} - \delta\right)\frac{n^4}{2},$$

then $G$ is $\varepsilon$-near to $C_n$.

**Theorem 1.6.** For every $\varepsilon > 0$, there exists $\delta > 0$ and $n_0$ such that for every $n > n_0$, if $G$ is a $K^3_5$-free 3-uniform hypergraph on $n$ vertices with

$$\text{co}_2(G) \geq \left(\frac{5}{8} - \delta\right)\frac{n^4}{2},$$

then $G$ is $\varepsilon$-near to $B_n$. 
There is another $K_5^3$-free 3-graph [52] with the same edge density as $B_n$, namely $H_5$. The vertex set of $H_5$ is divided into four parts $A_1, A_2, A_3, A_4$ with $||A_j| - |A_i|| \leq 1$ for all $1 \leq i < j \leq 4$ and say a triple $e$ is not an edge of $H_5$ if and only if there is some $j$ ($1 \leq j \leq 4$) such that $|e \cap A_j| \geq 2$ and $|e \cap A_j| + |e \cap A_{j+1}| = 3$, where $A_5 = A_1$, see Figure 2 for an illustration of the complement of $H_5$. While $H_5$ is conjectured to be one of the asymptotically extremal hypergraphs in $\ell_1$-norm, it is not an extremal hypergraph in $\ell_2$-norm, because $B_n$ has an asymptotically higher codegree squared sum.

Besides giving asymptotic result for cliques, we prove an exact result for $F_{3,3}$. Denote by $F_{3,3}$ the 3-graph on six vertices with edge set $\{123, 145, 146, 156, 245, 246, 256, 345, 346, 356\}$, see Figure 2. We prove that the codegree squared extremal hypergraph of $F_{3,3}$ is the balanced, complete, bipartite hypergraph $B_n$. Keevash and Mubayi [33] and independently Goldwasser and Hansen [27] proved that $B_n$ is also extremal for the $\ell_1$-norm.

**Theorem 1.7.** There exists $n_0$ such that for all $n \geq n_0$

$$\text{exco}_2(n, F_{3,3}) = \text{co}_2(B_n).$$

Furthermore, $B_n$ is the unique $F_{3,3}$-free 3-uniform hypergraph $G$ on $n$ vertices satisfying

$$\text{co}_2(G) = \text{exco}_2(n, F_{3,3}).$$

We also prove some general results for $\sigma$. First, we prove that the limit in (1) exists.

**Proposition 1.8.** Let $F$ be a family of $k$-graphs. Then, $\frac{\text{exco}_2(n, F)}{(\frac{n}{k-1})(\frac{n}{k-1}+1)^2}$ is nonincreasing as $n$ increases. In particular, it tends to a limit $\sigma(F)$ as $n \to \infty$.

A classical result in extremal combinatorics is the supersaturation phenomenon, discovered by Erdős and Simonovits [16]. For hypergraphs, it states that when the edge density of a hypergraph $H$ exceeds the Turán density of a hypergraph $G$, then $H$ contains many copies of $G$. Proposition 1.9 shows that the same phenomenon holds for $\sigma$.

**Proposition 1.9.** Let $F$ be a $k$-graph on $f$ vertices. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, f) > 0$ and $n_0$ such that every $n$-vertex $k$-uniform hypergraph $G$ with $n > n_0$ and $\text{co}_2(G) > (\sigma(F) + \varepsilon)(\frac{n}{k-1})n^2$ contains at least $\delta \binom{n}{f}$ copies of $F$. 
Supersaturation has been used to show that blowing-up a \( k \)-graph does not change its Turán density \([16] \). We will use our Supersaturation result, Proposition 1.9, to show the same conclusion holds for \( \sigma \): Blowing-up a \( k \)-graph also does not change the codegree squared density.

For a \( k \)-graph \( H \) and \( t \in \mathbb{N} \), the blow-up \( H(t) \) of \( H \) is defined by replacing each vertex \( x \in V(H) \) by \( t \) vertices \( x^1, \ldots, x^t \) and each edge \( x_1 \cdots x_k \in E(H) \) by the \( t^k \) edges \( x_1^{a_1} \cdots x_k^{a_k} \) with \( 1 \leq a_1, \ldots, a_k \leq t \).

**Corollary 1.10.** Let \( H \) be a \( k \)-uniform hypergraph and \( t \in \mathbb{N} \). Then,

\[
\sigma(H) = \sigma(H(t)).
\]

Similarly to the Turán density \([14] \), the codegree squared density has a jump at 0, that is, it is strictly bounded away from 0. Note that this phenomenon does not happen for the minimum codegree threshold \([38] \).

**Proposition 1.11.** Let \( H \) be a \( k \)-uniform hypergraph. Then

(i) \( (\pi(H))^2 \leq \sigma(H) \leq \pi(H) \),

(ii) \( \sigma(H) = 0 \) or \( \sigma(H) \geq \frac{(k-1)!}{k^k} \).

Our paper is organized as follows. In Section 2, as a warm up, we determine the maximum \( \ell_2 \)-norm of cancelative\(^1 \) 3-graphs, which is an analogue of a classical result of Bollobás \([8] \). Next, in Section 3 we introduce terminology and give an overview of the tools we will be using. In Section 4, we present our general results on maximal codegree squared sums. Section 5 is dedicated to proving our main results on cliques, that is, proving Theorems 1.5 and 1.6. In Section 6, we present the proof of our exact result, Theorem 1.7.

In a follow-up paper \([4] \), we systematically study the codegree squared densities of several hypergraphs, including a longer discussion of related open problems.

## 2  FORBIDDING \( F_4 \) AND \( F_5 \)

In this section, we will provide an example of how a classical Turán-type result on the \( \ell_1 \)-norm can imply a result for the \( \ell_2 \)-norm. Denote by \( F_4 \) the 4-vertex 3-graph\(^\dagger \) with edge set \{123, 124, 234\} and \( F_5 \) the 5-vertex 3-graph with edge set \{123, 124, 345\}, see Figure 3. The 3-graphs which are \( F_4 \)- and \( F_5 \)-free are called cancelative hypergraphs. Denote by \( S_n \) the complete balanced 3-partite 3-graph on \( n \) vertices. This is the 3-graph with vertex partition \( A \cup B \cup C \) with part sizes \( |A| = \lfloor n/3 \rfloor \), \( |B| = \lfloor (n + 1)/3 \rfloor \) and \( |C| = \lfloor (n + 2)/3 \rfloor \), where triples \( abc \) are edges if and only if \( a, b \) and \( c \) are each from a different class. Bollobás \([8] \) proved that the \( n \)-vertex cancelative hypergraph with the most edges is \( S_n \). Using his result and a double counting argument, we show that \( S_n \) is also the largest cancelative hypergraph in the \( \ell_2 \)-norm.

**Theorem 2.1.** Let \( n \in \mathbb{N} \). We have

\[
\text{exco}_2(n, \{F_4, F_5\}) = \text{co}_2(S_n),
\]

\(^1\) A hypergraph is called cancelative if it is \( \{F_4, F_5\} \)-free. See Section 2 for the definition of \( F_4 \) and \( F_5 \).

\(^\dagger\) This hypergraph is also known as \( K_4^3 \).
and therefore also
\[ \sigma(\{F_4, F_5\}) = \frac{2}{27}. \]

The unique extremal hypergraph is \( S_n \).

**Proof.** Let \( G \) be an \( F_4 \)- and \( F_5 \)-free hypergraph with \( n \) vertices. For an edge \( e = xyz \in E(G) \), we define its weight \( w(e) = d(x, y) + d(x, z) + d(y, z) \). Then, \( w(e) \leq n \); otherwise \( G \) contains an \( F_4 \).

Bollobás [8] proved that \( |E(G)| \leq |E(S_n)| \) with equality if and only if \( G = S_n \). This allows us to conclude
\[
\text{co}_2(G) = \sum_{x,y \in \binom{[n]}{2}} d(x, y)^2 = \sum_{e \in E(G)} w(e) \leq n|E(G)| \leq n|E(S_n)| = \text{co}_2(S_n).
\]

Frankl and Füredi [24] proved that for \( F_5 \)-free 3-graphs, \( S_n \) is also the extremal example in the \( \ell_1 \)-norm when \( n \geq 3000 \). In a follow-up paper [4], we prove that for \( F_5 \)-free 3-graphs, \( S_n \) is also the extremal example in the \( \ell_2 \)-norm provided \( n \) is sufficiently large. However, this requires more work than the proof of Theorem 2.1 and it is not derived by just applying the corresponding Turán result.

## 3 | PRELIMINARIES

### 3.1 | Terminology and notation

Let \( H \) be a 3-uniform hypergraph, \( x \in V(H) \) and \( A, B \subseteq V(H) \) be disjoint sets.

1. \( L(x) \) denotes the link graph of \( x \), that is, the graph on \( V(H) \setminus \{x\} \) with \( ab \in E(L(x)) \) if and only if \( abx \in E(H) \).
2. \( L_A(x) = L(x)[A] \) denotes the induced link graph on \( A \).
3. \( L_{A,B}(x) \) denotes the subgraph of the link graph of \( x \) containing only edges between \( A \) and \( B \). This means \( V(L_{A,B}(x)) = V(H) \setminus \{x\} \) and \( ab \in E(L_{A,B}(x)) \) if and only if \( a \in A \), \( b \in B \) and \( abx \in E(H) \).
4. \( L^c_{A,B}(x) \) denotes the subgraph of the link graph of \( x \) containing only nonedges between \( A \) and \( B \). This means \( V(L_{A,B}(x)) = V(H) \setminus \{x\} \) and \( ab \in E(L^c_{A,B}(x)) \) if and only if \( a \in A \), \( b \in B \) and \( abx \notin E(H) \).
(5) $e(A, B)$ denotes the number of cross-edges between $A$ and $B$, this means
\[ e(A, B) := |\{xyz \in E(H) : x, y \in A, z \in B\}| + |\{xyz \in E(H) : x, y \in B, z \in A\}|. \]

(6) $e^c(A, B)$ denotes the number of missing cross-edges between $A$ and $B$, this means
\[ e^c(A, B) := \left(\frac{|A|}{2}\right)|B| + \left(\frac{|B|}{2}\right)|A| - e(A, B). \]

(7) For an edge $e = xyz \in E(H)$, we define its weight as
\[ w_H(e) = d(x, y) + d(x, z) + d(y, z). \]

3.2 Tool 1: Induced hypergraph removal Lemma

We will use the induced hypergraph removal lemma of Rödl and Schacht [51].

Definition 3.1. Let $\mathcal{P}$ be an arbitrary family of $k$-graphs and $\mathcal{P}$ be a family of $k$-graphs closed under relabeling of the vertices.

- $\text{Forb}_{\text{ind}}(\mathcal{P})$ denotes the family of all $k$-graphs $H$ which contain no induced copy of any member of $\mathcal{P}$.
- For a constant $\mu \geq 0$ we say a given $k$-graph $H$ is $\mu$-far from $\mathcal{P}$ if every $k$-graph $G$ on the same vertex set $V(H)$ with $|G_H| \leq \mu |V(H)|^k$ satisfies $G \notin \mathcal{P}$, where $G_H$ denotes the symmetric difference of the edge sets of $G$ and $H$. Otherwise we call $H$ $\mu$-near to $\mathcal{P}$.

Theorem 3.2 (Rödl, Schacht [51]). For every (possibly infinite) family $\mathcal{P}$ of $k$-graphs and every $\mu > 0$ there exist constants $c > 0, C > 0$, and $n_0 \in \mathbb{N}$ such that the following holds. Suppose $H$ is a $k$-graph on $n \geq n_0$ vertices. If for every $\ell = 1, \ldots, C$ and every $F \in \mathcal{P}$ on $\ell$ vertices, $H$ contains at most $cn^\ell$ induced copies of $F$, then $H$ is $\mu$-near to $\text{Forb}_{\text{ind}}(\mathcal{P})$.

3.3 Tool 2: Flag Algebras

In this section, we give an insight on how we apply Razborov’s flag algebra machinery [45] for calculating the codegree squared density. The main power of the machinery comes from the possibility of formulating a problem as a semidefinite program and using a computer to solve it.

The method can be applied in various settings such as graphs [28, 44], hypergraphs [3, 19], oriented graphs [29, 37], edge-colored graphs [5, 12], permutations [6, 55], discrete geometry [7, 36], or phylogenetic trees [1]. For a detailed explanation of the flag algebra method in the setting of 3-uniform hypergraphs, see [22]. Further, we recommend looking at the survey [47] and the expository note [48], both by Razborov. Here, we will focus on the problem formulation rather than a formal explanation of the general method.

Let $F$ be a fixed 3-graph. Let $\mathcal{P}$ denote the set of all $F$-free 3-graphs up to isomorphism. Denote by $\mathcal{P}_\ell$ all 3-graphs in $\mathcal{P}$ on $\ell$ vertices. For two 3-graphs $F_1$ and $F_2$, denote by $P(F_1, F_2)$ the probability that $|V(F_1)|$ vertices chosen uniformly at random from $V(F_2)$ induce a copy of $F_1$. A sequence
of 3-graphs \((G_n)_{n \geq 1}\) of increasing orders is convergent, if 
\[
\lim_{n \to \infty} P(H, G_n) \text{ exists for every } H \in \mathcal{P}.
\]
Note that if this limit exists, it is in \([0,1]\).

For readers familiar with flag algebras and its usual notation, for a convergent sequence \((G_n)_{n \geq 1}\) of \(n\)-vertex 3-graphs \(G_n\), we get

\[
\lim_{n \to \infty} \frac{\text{co2}(G_n)}{\binom{n}{2}(n-2)^2} = \left[ \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} \right]^2 = \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 + o(1),
\]

where \([\cdot]\) denotes the averaging operator and the terms on the right are interpreted as

\[
\lim_{n \to \infty} \frac{1}{6}P(K^3_4, G_n) + \frac{1}{2}P(K^3_{4^-}, G_n) + P(K^3_4, G_n),
\]

where \(K^3_4\) is the 3-graph with four vertices and two edges and \(K^3_{4^-}\) the 3-graph with four vertices and three edges, also known as \(F_4\). It is a routine application of flag algebras to find an upper bound on the right-hand side of (2).

For readers less familiar with flag algebras, the following paragraphs give a slightly less formal explanation of the problem formulation. Let \(G\) be a 3-graph. Let \(\vartheta\) be an injective function \(\{1, 2\} \to V(G)\). In other words, \(\vartheta\) labels two distinct vertices in \(G\). We call the pair \((G, \vartheta)\) a labeled 3-graph although only two vertices in \(G\) are labeled by \(\vartheta\).

Let \((H, \vartheta')\) and \((G, \vartheta)\) be two labeled 3-graphs. Let \(X\) be a subset of \(V(G) \setminus \text{Im } \vartheta\) of size \(|V(H)| - 2\) chosen uniformly at random. By \(P((H, \vartheta'), (G, \vartheta))\), we denote the probability that the labeled subgraph of \(G\) induced by \(X\) and the two labeled vertices, that is, \((G[X \cup \text{Im } \vartheta], \vartheta)\), is isomorphic to \((H, \vartheta')\), where the isomorphism maps \(\vartheta(i)\) to \(\vartheta'(i)\) for \(i \in \{1, 2\}\).

Let \(E\) be a labeled 3-graph consisting of three vertices, two of them labeled, and one edge containing all three vertices. Note that \(P(E, (G, \vartheta))(n - 2)\) is the codegree of \(\vartheta(1)\) and \(\vartheta(2)\) in a 3-graph \(G\). The square of the codegree of \(\vartheta(1)\) and \(\vartheta(2)\) is \(P(E, (G, \vartheta))(n - 2)^2\). One of the tricks in flag algebras is that calculating \(P(E, (G, \vartheta))^2\) in \(G\) of order \(n\) can be done within error \(O(1/n)\) by selecting two distinct vertices in addition to \(\vartheta(1)\) and \(\vartheta(2)\) and examining subgraphs on four vertices instead. In our case, it looks like the following, where \(P(H, (G, \vartheta))\) is depicted simply as \(H\).

\[
\left[ \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} \right]^2 = \left[ \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} \right] + \left[ \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} \right] + \left[ \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} \right] + \left[ \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}
\end{array} \right] + o(1)
\]

The next step is to sum over all possible choices for \(\vartheta\), there are \(n(n - 1)\) of them, and divide by 2 since the codegree squared sum is over unordered pairs of vertices, unlike \(\vartheta\). When summing over all possible \(\vartheta\), one could look at all subsets of vertices of size 4 of \(G\) and see what the probability is that randomly labeling two vertices among these four by \(\vartheta\) gives one of the labeled 3-graphs from the right-hand side of (3). This gives the coefficients on the right-hand side of (2).

We use flag algebras to prove Lemmas 5.1, 6.1, and 5.3. The calculations are computer assisted. We use CSDP [9] for finding numerical solutions of semidefinite programs and SageMath [56] for rounding the numerical solutions to exact ones. The files needed to perform the corresponding calculations are available at http://lidicky.name/pub/co2/.
4 | GENERAL RESULTS: PROOFS OF PROPOSITIONS 1.8, 1.9 AND 1.10

4.1 | The limit exists

Proof of Proposition 1.8. Let $n \geq k$ be a positive integer and let $G$ be an $F$-free $k$-graph on vertex set $[n]$ satisfying $\text{co}_2(G) = \text{exco}_2(n, F)$. Take $S$ to be a randomly chosen $(n-1)$-subset of $V(G)$. Now, we calculate the expectation of $\text{co}_2(G[S])$,

$$
E[\text{co}_2(G[S])] = \sum_{T \in \binom{[n]}{k-1}} E[1_{\{T \subset S\}}d^2_{G[S]}(T)] = \sum_{T \in \binom{[n]}{k-1}} \mathbb{P}(T \subset S)E[d^2_{G[S]}(T)|T \subset S]
$$

$$
= \sum_{T \in \binom{[n]}{k-1}} \frac{n-k}{n} \frac{(n-k+1)}{n} \frac{(n-k+2)}{n} \cdot \frac{(n-k+3)}{n} \text{co}_2(G).
$$

We used that $d_{G[S]}(T)$ conditioned on $T \subset S$ has hypergeometric distribution. By averaging, we conclude that there exists an $(n-1)$-vertex subset $S' \subset V(G)$ with $\text{co}_2(G[S']) \geq E[\text{co}_2(G[S])]$. Thus, we conclude that $G[S']$ is an $(n-1)$-vertex $k$-graph satisfying

$$
\text{co}_2(G[S']) \geq \left(\frac{n-k}{n} \right)^2 \text{co}_2(G).
$$

Therefore, since $G[S']$ is $F$-free,

$$
\text{exco}_2(n-1, F) \geq \text{co}_2(G[S']) \geq \text{co}_2(G) \geq \frac{\text{co}_2(n, F)}{(n-k+1)^2}\text{exco}_2(n, F).
$$

\[\Box\]

4.2 | Supersaturation

In this section, we prove Proposition 1.9. We will make use of the following tail bound on the hypergeometric distribution.

Lemma 4.1 (For example [30], p. 29). Let $\beta, \lambda > 0$ with $\beta + \lambda < 1$. Suppose that $X \subseteq [n]$ and $|X| \geq (\beta + \lambda)n$. Then

$$
\left| \left\{ S \in \binom{[n]}{m} : |S \cap X| \leq \beta m \right\} \right| \leq \binom{n}{m} e^{-\frac{\lambda^2 m}{2(\beta + \lambda)}} \leq \binom{n}{m} e^{-\lambda^2 m/3}.
$$

Mubayi and Zhao [41] used Lemma 4.1 to prove a supersaturation result for the minimum codegree threshold. We adapt their proof to our setting.
Lemma 4.2. Let $\alpha > 0$, $\varepsilon > 0$ and $k \geq 3$. Then there exists $m_0$ such that the following holds. If $n \geq m \geq m_0$ and $G$ is a $k$-graph on $[n]$ with $\co_2(G) \geq (\alpha + \varepsilon)\binom{n}{k-1}(n - k + 1)^2$, then the number of $m$-sets $S$ satisfying $\co_2(G[S]) > \alpha\binom{m}{k-1}(m - k + 1)^2$ is at least $\frac{\varepsilon}{4}\binom{n}{m}$.

Proof. Given a $(k - 1)$-element set $T \subset [n]$, we call an $m$-set $S$ with $T \subset S \subset [n]$ bad for $T$ if $|d(T) \cap S| \leq \left(\frac{d(T)}{n-k+1} - \frac{\varepsilon}{6}\right)(m - k + 1)$. An $m$-set is bad if it is bad for some $T$. Otherwise, it is good. We will show that there are only few bad sets. Denote by $\Phi$ the number of bad $m$-sets, and let $\Phi_T$ be the number of $m$-sets that are bad for $T$. Then, by applying Lemma 4.1 with $\beta = \frac{d(T)}{n-k+1} - \frac{\varepsilon}{6}$ and $\lambda = \frac{\varepsilon}{7}$, we get

$$
\Phi \leq \sum_{T \in [n]_{k-1}} \Phi_T = \sum_{T \in [n]_{k-1}} \left| \left\{ S' \in \binom{[n] \setminus T}{m-k+1} : |d(T) \cap S'| \leq \left(\frac{d(T)}{n-k+1} - \frac{\varepsilon}{6}\right)(m - k + 1) \right\} \right|
\leq \sum_{T \in [n]_{k-1}} \binom{n-k+1}{m-k+1} \exp\left(-\frac{\varepsilon^2(m-k+1)}{147}\right) \leq \binom{n}{k-1} \binom{n-k+1}{m-k+1} \exp\left(-\frac{\varepsilon^2(m-k+1)}{147}\right)
= \binom{n}{m} \binom{m}{k-1} \exp\left(-\frac{\varepsilon^2(m-k+1)}{147}\right) \leq \frac{\varepsilon}{4}\binom{n}{m},
$$

where the last inequality holds for $m$ large enough. So the number of bad $m$-sets is at most $\frac{\varepsilon}{4}\binom{n}{m}$. Now let $\ell\left(\frac{n}{m}\right)$ be the number of $m$-sets $S$ satisfying

$$
\sum_{T \in [n]_{k-1}} d_G^2(T) \geq \left(\alpha + \frac{\varepsilon}{2}\right)\binom{m}{k-1}(n - k + 1)^2.
$$

(4)

On one side

$$
\sum_{|S|=m} \sum_{T \in [n]_{k-1}} d_G^2(T) = \binom{n-k+1}{m-k+1} \co_2(G) = \binom{n-k+1}{m-k+1} \binom{n}{k-1}(n - k + 1)^2(\alpha + \varepsilon).
$$

On the other side,

$$
\sum_{|S|=m} \sum_{T \in [n]_{k-1}} d_G^2(T) \leq \left(\alpha + \frac{\varepsilon}{2}\right)\binom{m}{k-1}(n - k + 1)^2\binom{n}{m} + \ell\left(\frac{n}{m}\right)\binom{m}{k-1}(n - k + 1)^2\binom{n}{m}
= \left(\alpha + \frac{\varepsilon}{2} + \ell\right)\binom{m}{k-1}(n - k + 1)^2\binom{n}{m}.
$$

By this double counting argument, we conclude $\ell \geq \varepsilon/2$. Since the number of bad $m$-sets is at most $\frac{\varepsilon}{4}\binom{n}{m}$, there are at least $\frac{\varepsilon}{4}\binom{n}{m}$ good $m$-sets satisfying (4). All of these $m$-sets satisfy

$$
\co_2(G[S]) = \sum_{T \in [n]_{k-1}} d_G^2(T) \geq \sum_{T \in [n]_{k-1}} \left(\left(\frac{d_G(T)}{n-k+1} - \frac{\varepsilon}{6}\right)(m - k + 1)\right)^2.
$$
SOLVING TURÁN’S TETRAHEDRON PROBLEM FOR THE $\ell_2^2$-NORM

\[
= \frac{(m-k+1)^2}{(n-k+1)^2} \sum_{T \in \binom{\mathcal{S}_{k-1}}{2}} \left( d_G(T) - \frac{\varepsilon}{6}(n-k+1)^2 \right)^2
\]

\[
\geq \frac{(m-k+1)^2}{(n-k+1)^2} \sum_{T \in \binom{\mathcal{S}_{k-1}}{2}} \left( d_G^2(T) - \frac{\varepsilon}{3}(n-k+1)^2 \right)
\]

\[
\geq \frac{(m-k+1)^2}{(n-k+1)^2} \left( \left( \alpha + \frac{\varepsilon}{2} \right) \binom{m}{k-1} (n-k+1)^2 - \binom{m}{k-1} \frac{\varepsilon}{3} (n-k+1)^2 \right)
\]

\[
> \alpha \binom{m}{k-1} (m-k+1)^2,
\]

proving the statement of this lemma. \qed

Proof of Proposition 1.9. This proof follows Erdős and Simonovits’s proof [16] of the supersaturation result for the Turán density.

Let $F$ be a $k$-graph on $f$ vertices, $\varepsilon > 0$ and $G$ be an $n$-vertex $k$-graph satisfying $\text{co}_2(G) > (\sigma(F) + \varepsilon)(\frac{n}{k-1})n^2$ for $n$ large enough. By Lemma 4.2, there exists an $m_0$ such that for $m \geq m_0$ the number of $m$-sets $S$ satisfying $\text{co}_2(G[S]) > (\sigma(F) + \varepsilon/2)(\frac{m}{k-1})(m-k+1)^2$ is at least $\frac{\varepsilon}{8} \binom{n}{m}$. There exists some fixed $m_1 \geq m_0$ such that $\text{exco}_2(m_1, F) \leq (\sigma(F) + \varepsilon/2)(\binom{m_1}{k-1})(m_1-k+1)^2$. Thus, there are at least $\frac{\varepsilon}{8} \binom{n}{m_1}$ $m_1$-sets $S$ such that $G[S]$ contains $F$. Each copy of $F$ may be counted at most $\binom{n-f}{m_1-f}$ times. Therefore, the number of copies for $F$ is at least

\[
\frac{\varepsilon}{8} \binom{n}{m_1} = \delta \binom{n}{f},
\]

for $\delta = \frac{\varepsilon}{8 \binom{m_1}{f}}$. \qed

4.3 Proof of Corollary 1.10 and Proposition 1.11

Now we use a standard argument to show that blowing-up a $k$-graph does not change the codegree squared density. We will follow the proof of the analogous Turán result given in [31].

Proof of Corollary 1.10. Since $H \subset H(t)$, $\text{exco}_2(n, H(t)) \leq \text{exco}_2(n, H)$ holds trivially. Thus, $\sigma(H(t)) \leq \sigma(H)$.

For the other direction, let $\varepsilon > 0$ and $G$ be an $n$-vertex $k$-uniform hypergraph satisfying $\text{co}_2(G)/((\frac{n}{k-1})(n-k+1)^2) > \sigma(H) + \varepsilon$. Then, by Proposition 1.9, $G$ contains at least $\delta \binom{n}{v(H)}$ copies of $H$ for $\delta = \delta(\varepsilon, k) > 0$. We create an auxiliary $v(H)$-graph $F$ on the vertex set $V(G)$. A $v(H)$-set $A \subset V(G)$ is an edge in $F$ if and only if $G[A]$ contains a copy of $H$. The auxiliary hypergraph $F$ has density at least $\delta/v(H)!$. Thus, as it is well-known [14], for any $t' > 0$ as long as $n$ is large enough, $F$ contains a copy of $K_{v(H)}^{t'}$, the complete $v(H)$-partite $v(H)$-graph with $t'$ vertices in each part. We choose $t'$ large enough such that the following is true. We color each edge of $K_{v(H)}^{t'}$ by one of $v(H)!$ colors, depending on which of the $v(H)!$ orders the vertices of $H$ are
mapped to in the corresponding copy of $H$ in $G$. By a classical result in Ramsey theory (for a density version see [14]), there is a monochromatic copy of $K^H(H)(t)$, which contains a copy of $H(t)$ in $G$. We conclude $\sigma(H(t)) \leq \sigma(H) + \varepsilon$ for all $\varepsilon > 0$. □

Proof of Proposition 1.11. Let $H$ be a $k$-graph. For any $k$-graph $G$, we have by the Cauchy–Schwarz inequality

$$\text{co}_2(G) = \sum_{T \in \binom{[n]}{k-1}} d_G(T)^2 \geq \left( \frac{\sum_{T \in \binom{[n]}{k-1}} d_G(T)}{\binom{n}{k-1}} \right)^2 = \left( \frac{|E(G)|}{\binom{n}{k-1}} \right)^2.$$

Applying this for an $H$-free hypergraph $G$, and scaling, we obtain $\sigma(H) \geq \pi(H)^2$. For $\sigma(H) \leq \pi(H)$, we use

$$\text{co}_2(G) = \sum_{T \in \binom{[n]}{k-1}} d_G(T)^2 = \sum_{e \in E(G)} w_G(e) \leq kn|E(G)|,$$

where $w_G(e) := \sum_{T \in \binom{e}{k-1}} d_G(T)$. After scaling this implies $\sigma(H) \leq \pi(H)$, completing the proof of part (i).

Erdős [14] proved that the Turán density of a $k$-partite $k$-graph is 0. In this case, the codegree squared density is also 0 by part (i). If $H$ is not $k$-partite, then the complete $k$-partite hypergraph is $H$-free providing a construction for lower bounds. Hence, as it was observed by Erdős [14], the Turán density of $H$ is at least $k!/k^k$. Similarly, we get $\sigma(H) \geq (k-1)/k^k$. □

5 CLIQUES

In this section, we will prove Theorems 1.5 and 1.6.

5.1 Proof of Theorem 1.5

Flag algebras give us the following results for $K^3_4$.

Lemma 5.1. For all $\varepsilon > 0$, there exists $\delta > 0$ and $n_0$ such that for all $n \geq n_0$: if $G$ is a $K^3_4$-free 3-uniform graph on $n$ vertices with $\text{co}_2(G) \geq (1 - \delta)^2n^3/2$, then the densities of all 3-graphs on 4, 5 and 6 vertices in $G$ that are not contained in $C_n$ are at most $\varepsilon$. Additionally,

$$\sigma(K^3_4) = \frac{1}{3}.$$

The flag algebra calculation proving Lemma 5.1 is computer assisted. The calculation is available at http://lidicky.name/pub/co2/. For proving Theorem 1.5, we will make use of the following stability result due to Pikhurko [43].

Theorem 5.2 (Pikhurko [43]). For every $\varepsilon > 0$, there exists $\delta > 0$ and $n_0$ such that for every $n \geq n_0$, if $G$ is a $K^3_4$-free 3-uniform hypergraph on $n$ vertices not spanning exactly one edge on four vertices
and with

\[ e(G) \geq \left( \frac{5}{9} - \delta \right) \binom{n}{3}, \]

then \( G \) is \( \epsilon \)-near to \( C_n \).

**Proof of Theorem 1.5.** Let \( \epsilon > 0 \) be fixed. We choose \( n_0 \) sufficiently large for the following proof to work. We will choose constants

\[ 1 \gg \epsilon \gg \delta \gg \delta_2 \gg \delta_1 \gg \delta \gg 0 \]

in order from left to right where each constant is a sufficiently small positive number depending only on the previous ones. Let \( G \) be a \( K_4^3 \)-free 3-uniform hypergraph on \( n \geq n_0 \) vertices with

\[ \text{co}_2(G) \geq \left( \frac{1}{3} - \delta \right) \frac{n^4}{2}. \]

By applying Lemma 5.1, we get that the density of the 4-vertex 3-graph with exactly one edge in \( G \) is at most \( \delta_1 \). Now, we apply the induced hypergraph removal lemma, Theorem 3.2, to obtain \( G' \) where \( G' \) is \( \delta_2 \)-near to \( G \), and \( G' \) is \( K_4^3 \)-free and does not induce exactly one edge on four vertices. We have

\[ \text{co}_2(G') \geq \text{co}_2(G) - 6\delta_2 n^4 \geq \left( \frac{1}{3} - \delta \right) \frac{n^4}{2} - 6\delta_2 n^4 \geq (1 - 37\delta_2) \frac{1}{6} n^4, \]

where the first inequality holds because when one edge is removed from a 3-uniform hypergraph, then the codegree squared sum can go down by at most \( 6n \). By a result of Falgas-Ravry and Vaughan [21, Theorem 4], \( P(K_4^3, G') \leq 16/27 + o(1) \). Let \( x \in [0,1] \) such that \( P(K_4^3, G') = 16/27(1 - x) + o(1) \). By (2) and the fact that \( G' \) is \( K_4^3 \)-free, we have

\[ \frac{1}{3} (1 - 37\delta_2) \leq \frac{\text{co}_2(G')}{\binom{n}{2} (n-2)^2} = \frac{1}{6} P(K_4^3, G') + \frac{1}{2} P(K_4^3, G') \leq \frac{1}{6} P(K_4^3, G') + \frac{8}{27} (1 - x) + \delta_2. \]

Thus,

\[ P(K_4^3, G') \geq \frac{2 + 16x}{9} - 80\delta_2. \]  \hspace{1cm} (5)

Since \( G' \) does not contain a 4-set spanning exactly 1 or 4 edges, a result of Razborov [46] says

\[ \left| \frac{E(G')}{{n \choose 3}} \right| \leq \frac{5}{9} + o(1). \]  \hspace{1cm} (6)

The edge density can also expressed as

\[ \frac{|E(G')|}{{n \choose 3}} = \frac{1}{2} P(K_4^3, G') + \frac{3}{4} P(K_4^3, G') + o(1). \]  \hspace{1cm} (7)
By combining (5) and (7), we get

\[
\frac{|E(G')|}{\binom{n}{3}} \geq \frac{1}{2} P(K_{4}^{3}, G') + \frac{3}{4} P(K_{4}^{3-}, G') - \delta_2 \geq \frac{5 + 4x}{9} - 41\delta_2.
\]

This implies \(x \leq 100\delta_2\). Thus, by Pikhurko’s stability theorem (Theorem 5.2), \(G'\) is \(\delta_3\)-near to \(C_n\). Since \(G'\) is \(\delta_2\)-near to \(G\), we conclude that \(G\) is \(\varepsilon\)-near to \(C_n\). \(\square\)

### 5.2 Proof of Theorem 1.6

Flag algebras give us the following for \(K_5^3\).

**Lemma 5.3.** For all \(\varepsilon > 0\), there exists \(\delta > 0\) and \(n_0\) such that for all \(n \geq n_0\): if \(G\) is a \(K_5^3\)-free 3-uniform graph on \(n\) vertices with \(\text{co}_2(G) \geq (1 - \delta)^{\frac{2}{5}} n^4 / 2\), then the densities of all 3-graphs on 4, 5 and 6 vertices in \(G\) that are not contained in \(B_n\) are at most \(\varepsilon\). In particular,

\[
\sigma(K_5^3) = \frac{5}{8}.
\]

Again, the flag algebra calculation proving Lemma 5.3 is computer assisted and available at http://lidicky.name/pub/co2/. We use this result to prove Theorem 1.6.

**Proof of Theorem 1.6.** Let \(\varepsilon > 0\). During the proof we will use the following constants:

\[
1 \gg \varepsilon \gg \delta_2 \gg \delta_1 \gg \delta \gg 0.
\]

The constants are chosen in this order and each constant is a sufficiently small positive number depending only on the previous ones. Apply Lemma 5.3 and get \(\delta = \delta(\delta_1) > 0\) such that for all \(n\) large enough: if \(G\) is a \(K_5^3\)-free 3-uniform graph on \(n\) vertices with \(\text{co}_2(G) \geq (1 - \delta)^{\frac{2}{5}} n^4 / 2\), then the densities of all 3-graphs on 4, 5 and 6 vertices in \(G\) that are not contained in \(B_n\) are at most \(\delta_1\).

Now, apply the induced hypergraph removal lemma Theorem 3.2 to obtain \(G'\) where \(G'\) is \(\delta_2\)-near to \(G\), and \(G'\) contains only those induced subgraphs on 4, 5 or 6 vertices which appear as induced subgraphs in \(B_n\). Note that

\[
\text{co}_2(G') \geq \text{co}_2(G') - 6\delta_2 n^4 \geq (1 - \delta)\frac{5}{8} n^4 - 6\delta_2 n^4 \geq (1 - 20\delta_2)\frac{5}{8} n^4 / 2,
\]

because when one edge is removed the codegree squared sum can go down by at most \(6n\). Next we show that \(G'\) has to have the same structure as \(B_n\). We say that a 3-graph \(H\) is 2-colorable, if there is a partition of the vertex set \(V(H) = V_1 \cup V_2\) such that \(V_1\) and \(V_2\) are independent sets in \(H\).

**Claim 5.4.** \(G'\) is 2-colorable.

**Proof.** Take an arbitrary nonedge \(abc\) in \(G'\). For \(0 \leq i \leq 4\), define \(A_i\) to be the set of vertices \(v \in V(G) \setminus \{a, b, c\}\) such that \(G'\) induces \(i\) edges on \(\{a, b, c, v\}\). Then, \(A_1 = A_2 = \emptyset\) because on four
vertices there are either 0, 3 or 4 edges in $B_n$, hence in $G'$ as well. Further $A_4 = \emptyset$, because $abc$ is a nonedge. Clearly, $A_0$ is an independent set, because if there is an edge $v_1v_2v_3$ in $G'[A_0]$, then the induced graph of $G'$ on $\{a, b, c, v_1, v_2, v_3\}$ spans a forbidden subgraph, that is, a hypergraph which is not an induced subhypergraph of $B_n$. Similarly, $A_3$ is an independent set, otherwise $G'$ were to contain a copy of $F_{3,3}$, which is not an induced subhypergraph of $B_n$. Let $A' = A_0 \cup \{a, b, c\}$. Then $V(G') = A_3 \cup A'$ and $A'$ also forms an independent set. To observe the second statement, let $v_1, v_2, v_3$ be three vertices in $A_0$. The number of edges induced on $\{v_1, v_2, v_3, a, b, c\}$ is at most nine, because every edge need to be incident to exactly two vertices of $\{a, b, c\}$ by the definition of $A_0$. However, 6-vertex induced subgraphs of $B_n$ have either 0, 10, 16, or 18 edges. We conclude that $\{v_1, v_2, v_3, a, b, c\}$ induces no edge in $G'$. Thus, $A'$ is also an independent set in $G'$ and therefore $G'$ is 2-colorable.

Claim 5.5. We have $|E(G')| \geq (1 - 2\sqrt{\delta_2})\frac{n^3}{8}$.

Proof. By Claim 5.4, $G'$ is 2-colorable and we can partition the vertex set $V(G') = A \cup B$ such that $A$ and $B$ are independent sets. Let $a \in [0, 1]$ such that $|A| = an$ and $|B| = (1 - a)n$. We have

$$
(1 - 20\delta_2)\frac{5}{8}n^4 \leq \co_2(G') \leq \left(\frac{a^2}{2}(1 - a)^2 + \frac{(1 - a)^2}{2}a^2 + a(1 - a)\right)n^4 \leq \frac{5}{4}a(1 - a)n^4.
$$

Thus, $4a(1 - a) \geq 1 - 20\delta_2$. We conclude $1/2 - 3\sqrt{\delta_2} \leq a \leq 1/2 + 3\sqrt{\delta_2}$, otherwise

$$
4a(1 - a) < 4\left(\frac{1}{2} - 3\sqrt{\delta_2}\right)\left(\frac{1}{2} + 3\sqrt{\delta_2}\right) = 1 - 36\delta_2,
$$

a contradiction. For every edge $e \in E(G')$, we have $w_{G'}(e) \leq (5/2 + 3\sqrt{\delta_2})n$. Therefore,

$$(1 - 20\delta_2)\frac{5}{8}n^4 \leq \co_2(G') = \sum_{e \in E(G')} w_{G'}(e) \leq |E(G')|(\frac{5}{2} + 3\sqrt{\delta_2})n.
$$

Thus,

$$
|E(G')| \geq \frac{(1 - 20\delta_2)}{1 + \frac{6}{5}\sqrt{\delta_2}}\frac{n^3}{8} \geq (1 - 2\sqrt{\delta_2})\frac{n^3}{8}.
$$

The 3-graph $G$ is $\delta_2$-near to $G'$. By Claims 5.4 and 5.5, $G'$ is $\varepsilon/2$-near to $B_n$. Therefore we can conclude that $G$ is $\delta_2 + \varepsilon/2 \leq \varepsilon$-near to $B_n$.

5.3 Discussion on cliques

Keevash and Mubayi [31] constructed the following family of 3-graphs obtaining the best-known lower bound for the Turán density of cliques. Denote by $D_k$ the family of directed graphs on $k - 1$ vertices that are unions of vertex-disjoint directed cycles. Cycles of length two are allowed, but loops are not. Let $D \in D_k$ and $V = [n] = V_1 \cup \ldots \cup V_{k - 1}$ be a vertex partition with class sizes as balanced as possible, that is $||V_i| - |V_j|| \leq 1$ for all $i \neq j$. Denote by $G(D)$ the 3-graph on $V$ where...
a triple is a nonedge if and only if it is contained in some $V_i$ or if it has two vertices in $V_i$ and one vertex in $V_j$ where $(i, j)$ is an arc of $D$. The 3-graph $G(D)$ is $K_3^2$-free and has edge density $1 - (2/t)^2 + o(1)$. While all directed graphs $D \in \mathcal{D}_k$ give the same edge density for $G(D)$, up to isomorphism there is only one $D$ maximizing the codegree squared sum $\text{co}_2(G(D))$. Let $D^*_k \in \mathcal{D}_k$ be the directed graph on $k - 1$ vertices $v_1, \ldots, v_{k-1}$ such that if $k$ odd, then

$$(v_i v_{i+1}), (v_{i+1} v_i) \in E(D^*_k) \quad \text{for all odd } i,$$

and if $k$ even, then

$$(v_i v_{i+1}), (v_{i+1} v_i) \in E(D^*_k) \quad \text{for all odd } i \leq k - 5$$

and

$$(v_{k-3} v_{k-2}), (v_{k-2} v_{k-1}), (v_{k-1} v_{k-3}) \in E(D^*_k).$$

Note that $D^*_k$ is maximizing the number of directed cycles. The 3-graph $G(D^*_4)$ is isomorphic to $C_n$ and $G(D^*_5)$ is isomorphic to $B_n$. See Figure 4 for a drawing of $D^*_7$, $D^*_8$ and the complements $\overline{G(D^*_7)}$ and $\overline{G(D^*_8)}$ of $G(D^*_7)$ and $G(D^*_8)$, respectively. Next, we observe that among all directed graphs $D \in \mathcal{D}_k$, $D^*_k$ maximizes the codegree squared sum of $G(D)$.

For a function $f : X \to \mathbb{R}$, and $S \subseteq X$, define

$$\arg \max_{x \in S} f(x) := \{ x \in S : f(s) \leq f(x) \text{ for all } s \in S \}.$$
Lemma 5.6. Let \( k \geq 4 \). For \( n \) sufficiently large, \( D^*_k \) is isomorphic to any directed graph in

\[
\arg \max_{D \in D_k} \text{co}_2(G(D)).
\]

Proof. Let \( D \in \arg \max_{D \in D} \text{co}_2(G(D)) \). Suppose for contradiction that \( D \) contains a directed cycle \( v_1, v_2, \ldots, v_{\ell} \) of length \( \ell \geq 4 \). Construct a directed graph \( D' \) by replacing that \( \ell \)-cycle with an \((\ell - 2)\)-cycle \( v_1, v_4, \ldots, v_{\ell-2} \) and a 2-cycle \( v_2, v_3 \). Let \( V_1, V_2, \ldots, V_\ell \) be the corresponding classes in \( G \). The only pairs of vertices \( x, y \) for which the codegree changes by more than \( O(1) \) are described in the following.

1. For \( x \in V_1, y \in V_2 \), \( d(x, y) \) increased from \( n - n/(k-1) + O(1) \) to \( n + O(1) \).
2. For \( x \in V_3, y \in V_4 \), \( d(x, y) \) increased from \( n - n/(k-1) + O(1) \) to \( n + O(1) \).
3. For \( x \in V_2, y \in V_3 \), \( d(x, y) \) decreased from \( n - n/(k-1) + O(1) \) to \( n - 2n/(k-1) + O(1) \) if \( \ell = 4 \) or from \( n + O(1) \) to \( n - n/(k-1) + O(1) \) if \( \ell > 4 \).

Thus, if \( \ell = 4 \)

\[
\text{co}_2(G(D')) - \text{co}_2(G(D)) \geq O(1) + \frac{n^4}{(k-1)^2} \left( 2 - 4 \left( 1 - \frac{1}{k-1} \right)^2 + 2 \left( 1 - \frac{2}{k-1} \right)^2 \right) > 0,
\]

and if \( \ell > 4 \)

\[
\text{co}_2(G(D')) - \text{co}_2(G(D)) \geq O(1) + \frac{n^4}{(k-1)^2} \left( 1 - 2 \left( 1 - \frac{1}{k-1} \right)^2 + \left( 1 - \frac{2}{k-1} \right)^2 \right) > 0,
\]

a contradiction. Therefore, \( D \) contains no cycle of length at least 4. Next, toward a contradiction, suppose that \( D \) contains at least two cycles of length 3. Let \( v_1, v_2, v_3 \) and \( v_4, v_5, v_6 \) be the vertices of two 3-cycles. Let \( D' \) be the directed graph constructed from \( D \) by replacing those two 3-cycles with three 2-cycles \( v_1, v_2 \) and \( v_3, v_4 \) and \( v_5, v_6 \). Performing a similar analysis to the one above, we get that

\[
\text{co}_2(G(D')) - \text{co}_2(G(D)) = O(1) + \frac{n^4}{(k-1)^2} \left( 3 + 3 \left( 1 - \frac{2}{k-1} \right)^2 - 6 \left( 1 - \frac{1}{k-1} \right)^2 \right) > 0,
\]

a contradiction. Thus, we can conclude that \( D \) contains at most one 3-cycle. Hence, \( D \) is isomorphic to \( D^*_k \). \( \square \)

The directed graph \( D^*_k \) contains a 3-cycle if and only if \( k \) is odd. Based on Lemma 5.6 it seems reasonable to conjecture that in the case when \( k \) is odd the hypergraph \( G(D^*_k) \) could be an asymptotically extremal example in the \( \ell_2 \)-norm.

Question 5.7. Let \( k \geq 7 \) odd and \( \ell = (k - 1)/2 \). Is

\[
\sigma(k^3_k) = \lim_{n \to \infty} \frac{\text{co}_2(G(D^*_k))}{\binom{n}{2}(n-2)^2} = 1 - \frac{2}{\ell^2} + \frac{1}{\ell^3} ?
\]
The situation is slightly different for even \( k \). In this case, it is better to consider an unbalanced version of \( G(D_k^*) \) with parts of \( G(D_k^*) \) corresponding to the unique 3-cycle receiving different weights to the parts involved in 2-cycles. Denote by \( G^*(D_k^*) \) the 3-graph with the largest codegree squared sum among the following 3-graphs \( G \). Partition the vertex set of \( G \) into \([n] = V_1 \cup ... \cup V_{k-1}, \) where the class sizes are balanced as follow:

- \(||V_i| - |V_j|| \leq 1\) for all \( i \neq j \) with \( i, j \leq k - 4 \).
- \(||V_i| - |V_j|| \leq 1\) for all \( i \neq j \) with \( k - 3 \leq i, j \leq k - 1 \).

Again, a triple is a nonedge in \( G^*(D_k^*) \) if and only if it is contained in some \( V_i \) or if it has two vertices in \( V_i \) and one vertex in \( V_j \) where \((i, j)\) is an arc of \( D_k^* \).

**Question 5.8.** Let \( k \geq 6 \) even. Is

\[
\sigma(K_k^3) = \lim_{n \to \infty} \frac{\text{co}_2(G^*(D_k^*))}{\binom{n}{2}(n - 2)^2}.
\]

**6 | PROOF OF THEOREM 1.7**

In this section, we prove Theorem 1.7, that is, we determine the codegree squared extremal number of \( F_{3,3} \). Flag algebras give us the following corresponding asymptotical result and also a weak stability version.

**Lemma 6.1.** For all \( \varepsilon > 0 \), there exists \( \delta > 0 \) and \( n_0 \) such that for all \( n \geq n_0 \): if \( G \) is an \( F_{3,3} \)-free 3-uniform graph on \( n \) vertices with \( \text{co}_2(G) \geq (1 - \delta) \frac{5}{8} n^4/2 \), then the densities of all 3-graphs on 4, 5 and 6 vertices in \( G \) that are not contained in \( B_n \) are at most \( \varepsilon \). Additionally,

\[
\sigma(F_{3,3}) = \frac{5}{8}.
\]

This result implies the following stability theorem.

**Theorem 6.2.** For every \( \varepsilon > 0 \) there is \( \delta > 0 \) and \( n_0 \) such that if \( G \) is an \( F_{3,3} \)-free 3-uniform hypergraph on \( n \geq n_0 \) vertices with \( \text{co}_2(G) \geq (1 - \delta) \frac{5}{8} n^4/2 \), then we can partition \( V(G) \) as \( A \cup B \) such that \( e(A) + e(B) \leq \varepsilon n^3 \) and \( e(A, B) \geq \frac{1}{8} n^3 - \varepsilon n^3 \).

**Proof.** The proof is the same as the proof of Theorem 1.6, except instead of applying Lemma 5.3 we apply Lemma 6.1. \qed

We now determine the exact extremal number by using the stability result, Theorem 6.2, and a standard cleaning technique, see, for example, [26, 32, 34, 43]. To do so, we will first prove the statement under an additional universal minimum-degree-type assumption.

**Theorem 6.3.** There exists \( n_0 \) such that for all \( n \geq n_0 \) the following holds. Let \( G \) be an \( F_{3,3} \)-free \( n \)-vertex 3-graph such that

\[
q(x) := \sum_{y \neq x} d(x, y)^2 + 2 \sum_{[v, w] \in E(U(x))} d(v, w) \geq \frac{5}{4} n^3 - 6n^2 =: d(n)
\]
for all $x \in V(G)$. Then,

$$\text{co}_2(G) \leq \text{co}_2(B_n) = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right)^2 + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor (n-2)^2.$$

Furthermore, $B_n$ is the unique such 3-graph $G$ satisfying $\text{co}_2(G) = \text{exco}_2(n, F_{3,3})$.

**Proof.** Let $G$ be a 3-uniform $F_{3,3}$-free hypergraph which has a codegree squared sum at least $\text{co}_2(G) \geq \text{co}_2(B_n)$ and satisfies (8). Choose $\varepsilon = 10^{-10}$ and apply Theorem 6.2. We get a vertex partition $A \cup B$ with $\epsilon_e(A) + \epsilon_e(B) \leq 1$ and $\epsilon_e^c(A, B) \leq \varepsilon n^3$. Among all such partitions choose one which minimizes $\epsilon_e(A) + \epsilon_e(B)$. We can assume that $|L_B(x)| \geq |L_A(x)|$ for all $x \in A$ and $|L_A(x)| \geq |L_B(x)|$ for all $x \in B$, as otherwise we could switch a vertex from one class to the other class and strictly decrease both $\epsilon_e(A) + \epsilon_e(B)$ and $\epsilon_e^c(A, B)$, a contradiction. This is not possible, because we chose $A$ and $B$ minimizing $\epsilon_e(A) + \epsilon_e(B)$. We start by making an observation about the class sizes. □

**Claim 6.4.** We have

$$|A| - \frac{n}{2} \leq 2\sqrt{\varepsilon}n \quad \text{and} \quad |B| - \frac{n}{2} \leq 2\sqrt{\varepsilon}n.$$

**Proof.** Assume $|A| < n/2 - 2\sqrt{\varepsilon}n$. Then, we have

$$e(A, B) \leq \left(\frac{|A|}{2}\right)|B| + |A|\left(\frac{|B|}{2}\right) \leq \frac{1}{2}|A|(n - |A|)n$$

$$< \frac{1}{8} (n^2 - 2\sqrt{\varepsilon}n) (n^2 + 2\sqrt{\varepsilon}n) n < \frac{1}{8} n^3 - \varepsilon n^3,$$

a contradiction. Thus, $|A| \geq n/2 - 2\sqrt{\varepsilon}n$. Similarly, we get $|B| \geq n/2 - 2\sqrt{\varepsilon}n$. □

Define *junk* sets $J_A, J_B$ to be the sets of vertices which are not typical, that is,

$$J_A := \{ x \in A : |L_{A, B}(x)| \geq \sqrt{\varepsilon}n^2 \} \cup \{ x \in A : |L_A(x)| \geq \sqrt{\varepsilon}n^2 \},$$

$$J_B := \{ x \in B : |L_{A, B}(x)| \geq \sqrt{\varepsilon}n^2 \} \cup \{ x \in B : |L_B(x)| \geq \sqrt{\varepsilon}n^2 \}.$$

These junk sets need to be small.

**Claim 6.5.** We have $|J_A|, |J_B| \leq 5\sqrt{\varepsilon}n$.

**Proof.** Toward contradiction assume that $|J_A| > 5\sqrt{\varepsilon}n$. Then the number of vertices $x \in J_A$ satisfying $|L_{A, B}(x)| \geq \sqrt{\varepsilon}n^2$ is at least $2\sqrt{\varepsilon}n$ or the number of vertices $x \in J_A$ satisfying $|L_A(x)| \geq \sqrt{\varepsilon}n^2$ is at least $3\sqrt{\varepsilon}n$. If the first case holds, then we get $\epsilon_e^c(A, B) > \varepsilon n^3$. In the second case, we have $\epsilon_e(A) > \varepsilon n^3$. Both are in contradiction with the choice of the partition $A \cup B$. Thus, $|J_A| \leq 5\sqrt{\varepsilon}n$.

The second statement of this claim, $|J_B| \leq 5\sqrt{\varepsilon}n$, follows by a similar argument. □
**Claim 6.6.** \( A \setminus J_A \) and \( B \setminus J_B \) are independent sets.

**Proof.** If there is an edge \( a_1a_2a_3 \) with \( a_1, a_2, a_3 \in A \setminus J_A \), since all its vertices satisfy \( |L^c_B(a_i)| \leq \sqrt{\varepsilon n^2} \), we can find a triangle in \( L_B(a_1) \cap L_B(a_2) \cap L_B(a_3) \), call its vertices \( b_1, b_2, b_3 \). However, now \( \{b_1, b_2, b_3, a_1, a_2, a_3\} \) spans an \( F_{3,3} \) in \( G \), a contradiction. A similar proof gives that \( B \setminus J_B \) is an independent set. \( \Box \)

**Claim 6.7.** There is no edge \( a_1a_2a_3 \) with \( a_1 \in J_A \), \( a_2, a_3 \in A \setminus J_A \) or with \( a_1 \in J_B \), \( a_2, a_3 \in B \setminus J_B \).

**Proof.** Let \( a_1a_2a_3 \) be an edge with \( a_1 \in J_A \), \( a_2, a_3 \in A \setminus J_A \). We show that \( q(a_1) < d(n) \), to get a contradiction with (8). Let \( M_2 \), for \( i = 2, 3 \), be the set of nonedges in \( L_B(a_i) \) and \( L_A, B(a_i) \). Set \( K = L(a_1) - M_2 - M_3 \). Since \( |M_2|, |M_3| \leq 2 \sqrt{\varepsilon n^2} \), we have \( \|E(K)\| \geq |L(a_1)| - 4 \sqrt{\varepsilon n^2} \). Let

\[
\Delta = \max_{x \in A \setminus \{a_1, a_2, a_3\}} \frac{|N_K(x) \cap B|}{n}
\]

be the maximum size of a neighborhood in the graph \( K \) in \( B \) of a vertex in \( A \), scaled by \( n \). We have \( 0 \leq \Delta \leq |B|/n \leq 1/2 + \sqrt{\varepsilon} \). Let \( z \in A \setminus \{a_1, a_2, a_3\} \) such that \( |N_K(z) \cap B| = \Delta n \). Observe that \( N_K(z) \cap B \) is an independent set in \( K \), otherwise if \( v, w \in N_K(z) \cap B \) with \( vw \in E(K) \), then \( \{v, w, z, a_1, a_2, a_3\} \) spans an \( F_{3,3} \) in \( G \). Now,

\[
\sum_{x \in V \setminus \{a_1\}} d(a_1, x)^2 = \sum_{x \in V \setminus \{a_1\}} \text{deg}_{L(a_1)}(x)^2 \leq 16 \sqrt{\varepsilon n^2} + \sum_{x \in V(K)} \text{deg}_K(x)^2, \tag{9}
\]

because for each edge removed from the link graph \( L(a_i) \) the degree squared sum can go down by at most \( 4n \). Now, we bound the sum on the right-hand side of (9) from above. For \( x \in A \), \( \text{deg}_K(x) \leq |A| + \Delta n \) and for \( x \in N(z) \cap B \), \( \text{deg}_K(x) \leq n - \Delta n \). Thus, we get

\[
\sum_{x \in V \setminus \{a_1\}} d(a_1, x)^2 \leq 16 \sqrt{\varepsilon n^2} + |A|(|A| + \Delta n)^2 + \Delta n(n - \Delta n)^2 = n^3 \left( \frac{5}{8} + \Delta - \frac{3}{2} \Delta^2 + \Delta^3 + 25 \sqrt{\varepsilon} \right). \tag{10}
\]

Furthermore, we can give an upper bound for the second summand in \( q(a_1) \):

\[
2 \sum_{\{x, y\} \in E(L(a_1))} d(x, y) \leq 8 \sqrt{\varepsilon n^2} + 2 \sum_{\{x, y\} \in E(K)} d(x, y), \tag{11}
\]

where we used that for each edge removed from \( G \), the sum on the left-hand side in (11) is lowered by at most \( n \). Now, we will give an upper bound for the right-hand side of (11). For edges \( xy \in E(K[A]) \) not incident to \( J_A \), we have \( d_G(x, y) \leq |J_A| + |B| \) because by Claim 6.6 they have...
no neighbor in \( A \setminus J_A \). Similarly, for edges \( xy \in E(K[B]) \) not incident to \( J_B \) we have \( d_G(x, y) \leq |J_B| + |A| \). For all other edges \( xy \in E(K) \), we will use the trivial bound \( d_G(x, y) \leq n \). We have

\[
2 \sum_{\{x, y\} \in E(L(a_1))} d(x, y) \leq \sqrt{\varepsilon}n^3 + 2e(K[A, B])n + e(K[A])(|J_A| + |B|) + |J_A||A|n + e(K[B])(|J_B| + |B|) + |J_B||B|n. \tag{12}
\]

By the choice of our partition, we have \( |L_A(x_1)| \leq |L_B(x_1)| \) and thus \( e(K[A]) \leq e(K[B]) + 4\sqrt{\varepsilon}n^2 \). Therefore, by upper bounding the right-hand side in (12), we get

\[
2 \sum_{\{x, y\} \in E(L(a_1))} d(x, y) \leq 2\left( \Delta n^2 |A| + 2e(K[B])\left( 7\sqrt{\varepsilon}n + \frac{n}{2} \right) + 18\sqrt{\varepsilon}n^3 \right)
\leq 2n^3 \left( \frac{\Delta}{2} + \frac{e(G[B])}{n^2} + 30\sqrt{\varepsilon} \right)
\leq 2n^3 \left( \frac{\Delta}{2} + \Delta \left( \frac{|B|}{n} - \Delta \right) + \frac{1}{4} \left( \frac{|B|}{n} - \Delta \right)^2 + 30\sqrt{\varepsilon} \right)
\leq 2n^3 \left( \frac{\Delta}{2} + \Delta \left( \frac{1}{2} - \Delta \right) + \frac{1}{4} \left( \frac{1}{2} - \Delta \right)^2 + 40\sqrt{\varepsilon} \right)
\leq n^3 \left( -\frac{3}{2}\Delta^2 + \frac{3}{2}\Delta + \frac{1}{8} + 80\sqrt{\varepsilon} \right), \tag{13}
\]

where we used that \( e(K[B]) \leq \Delta n(|B| - \Delta n) + \frac{(|B| - \Delta n)^2}{4} \), because \( K[B] \) contains an independent set of size \( \Delta n \) and is triangle-free. Now, we can combine (10) and (13) to upper bound \( q(a_1) \).

\[
q(a_1) \leq n^3 \left( \frac{5}{8} + \frac{\Delta}{2} - \frac{3}{2}\Delta^2 + \Delta^3 + 25\sqrt{\varepsilon} \right) + n^3 \left( -\frac{3}{2}\Delta^2 + \frac{3}{2}\Delta + \frac{1}{8} + 80\sqrt{\varepsilon} \right)
= n^3 \left( \Delta^3 - 3\Delta^2 + 2\Delta + \frac{3}{4} + 105\sqrt{\varepsilon} \right) \leq \left( \frac{2}{3\sqrt{3}} + \frac{3}{4} + 105\sqrt{\varepsilon} \right)n^3 < \frac{5}{4}n^3 - 6n^2,
\]

contradicting (8). In the second-to-last inequality, we used that the polynomial \( \Delta^3 - 3\Delta^2 + 2\Delta \) has its maximum in \([0,1]\) at \( \Delta = 1 - \frac{1}{\sqrt{3}} \). \( \Box \)

Now, we can make use of Claim 6.7 to show that there is no edge inside \( A \), respectively, inside \( B \).

**Claim 6.8.** \( A \) and \( B \) are independent sets.

**Proof.** Let \( \{a_1, a_2, a_3\} \subset A \) span an edge. Again, \( L_B(a_1) \cap L_B(a_2) \cap L_B(a_3) \) is triangle-free. Thus, \( |L_B(a_1) \cap L_B(a_2) \cap L_B(a_3)| \leq |B|^2/4 \). By the pigeonhole principle, we may assume without loss of generality that \( |L_B(a_1)| \leq 5|B|^2/12 \). Furthermore, by Claims 6.6 and 6.7, \( |L_A(a_1)| \leq |J_A||A| \leq 5\sqrt{\varepsilon}n^2 \). Again, our strategy will be to give an upper bound on \( q(a_1) \). Let \( L \) be the graph obtained
from \(L(a_1)\) by removing all edges inside \(A\).

\[
\sum_{x \in V \setminus \{a_1\}} d(a_1, x)^2 = \sum_{x \in V \setminus \{a_1\}} \deg_{L(a_1)}(x)^2 \leq 20\sqrt{\varepsilon} n^3 + \sum_{x \in V(L)} \deg_L(x)^2 \\
\leq 20\sqrt{\varepsilon} n^3 + |B| n^2 + |A||B|^2 \leq n^3 \left( \frac{5}{8} + 30\sqrt{\varepsilon} \right).
\] (14)

Furthermore,

\[
2 \sum_{\{x,y\} \in E(L(a_1))} d(x, y) \leq 10\sqrt{\varepsilon} n^3 + 2 \sum_{xy \in E(L)} d(x, y) \\
\leq 2 \left( \frac{5}{12} |B|^2 (|A| + |J_B|) + 5\sqrt{\varepsilon} n^3 + |A||B| n \right) \\
\leq 2n^3 \left( \frac{5}{96} + 20\sqrt{\varepsilon} + \frac{1}{4} \right) = n^3 \left( \frac{29}{48} + 40\sqrt{\varepsilon} \right).
\] (15)

Thus, by combining (14) and (15), we give an upper bound on \(q(a_1)\),

\[
q(a_1) \leq \left( \frac{5}{8} + 30\sqrt{\varepsilon} \right)n^3 + n^3 \left( \frac{29}{48} + 40\sqrt{\varepsilon} \right) = n^3 \left( \frac{59}{48} + 70\sqrt{\varepsilon} \right) < \frac{5}{4}n^3 - 6n^2,
\]

contradicting (8). Therefore \(A\) is an independent set. By a similar argument, \(B\) is also an independent set. \(\square\)

By Claim 6.8, \(G\) is 2-colorable. Since among all 2-colorable 3-graphs \(B_n\) has the largest codegree squared sum, we conclude \(c^2_2(G) \leq c^2_2(B_n)\). This completes the proof of Theorem 6.3.

We now complete the proof of Theorem 6.3 by showing that imposing the additional assumption (8) is not more restrictive.

**Proof of Theorem 1.7.** Let \(G\) be an \(n\)-vertex 3-uniform \(F_{3,3}\)-free hypergraph which has a codegree squared sum at least \(c^2_2(G) \geq c^2_2(B_n)\). Set \(d(n) = 5/4 n^3 - 6n^2\) and note that \(c^2_2(B_n) - c^2_2(B_{n-1}) > d(n) + 1\). We claim that we can assume that every vertex \(x \in V(G)\) satisfies (8). Otherwise, we can remove a vertex \(x \in V(G)\) where \(q(x) < d(n)\) to get \(G_{n-1}\) with \(c^2_2(G_{n-1}) \geq c^2_2(B_n) - d(n) \geq c^2_2(B_{n-1}) + 1\). By repeating this process as long as possible, we obtain a sequence of hypergraphs \(G_m\) on \(m\) vertices with \(c^2_2(G_m) \geq c^2_2(B_m) + n - m\), where \(G_m\) is the hypergraph obtained from \(G_{m+1}\) by deleting a vertex \(x \in V(G_m)\) where \(q(x) \leq d(m + 1)\). We cannot continue until we reach a hypergraph on \(n_0 = n^{1/4}\) vertices, as then \(c^2_2(G_{n_0}) > n - n_0 > \binom{n_0}{2} (n_0 - 2)^2\) which is impossible. Therefore, the process stops at some \(n'\) where \(n \geq n' > n_0\) and we obtain the corresponding hypergraph \(G_{n'}\) satisfying \(q(x) \leq d(n')\) for all \(x \in V(G_{n'})\) and \(c^2_2(G_{n'}) \geq c^2_2(B_{n'})\) (with strict inequality if \(n > n'\)). Hence, we can assume that \(G\) satisfies \(q(x) \geq d(n')\) for all \(x \in V(G_{n'})\). Applying Theorem 6.3 finishes the proof. \(\square\)

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SOLVING TURÁN’S TETRAHEDRON PROBLEM FOR THE $\ell_2$-NORM

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