A Grassmann algebra approach to classifying real coboundary Lie bialgebras

J. de Lucas and D. Wysocki

Department of Mathematical Methods in Physics, University of Warsaw, ul. Pasteura 5, 02-093, Warsaw, Poland

Abstract

This work pioneers the systematic study and classification (up to Lie algebra automorphisms) of finite-dimensional coboundary Lie bialgebras through Grassmann algebras. Several mathematical structures on Lie algebras, e.g. Killing forms or root decompositions, are extended to the Grassmann algebras of Lie algebras. This simplifies the description of the procedures and tools appearing in the theory of Lie bialgebras and originates novel techniques for its study and classification up to Lie algebra automorphisms. As a particular case, the classification of real three-dimensional coboundary Lie bialgebras is retrieved.

Keywords: algebraic Schouten bracket, Grassmann algebra, Lie algebra, Lie bialgebra, g-invariant metric, Schouten-Nijenhuis bracket, Killing metric.

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1 Introduction

Lie bialgebras [13, 28, 30, 41] emerged from the study of integrable systems [17, 18]. Mathematically, they were introduced by V. Drinfeld [14, 15], one of the forefathers of the quantum group theory. In a nutshell, a Lie bialgebra consists of two Lie algebra structures, one on the linear space g and one on its dual g∗, that are compatible in a certain sense. Lie bialgebras play an important role, as being related to quantum universal enveloping algebras, function algebras, and other quantum groups [13]. To this respect, every Lie bialgebra gives rise to a quantum universal enveloping algebra whose first-order approximation is given by the Lie bialgebra itself [13].

Nowadays the research on Lie bialgebras and their quantizations is diversified, bringing attention of mathematicians and physicists alike. On the physical side, Lie bialgebras admit applications to quantum gravity, oriented towards constructions of non-commutative space-times [4, 5, 6, 34, 35]. Additionally, there are still interesting questions of purely mathematical nature. For instance, the classification of Lie bialgebras on a fixed Lie algebra g up to Lie algebra automorphisms is an unfinished task. Hitherto, mostly Lie bialgebras on two- and three-dimensional Lie algebras were obtained and classified [19, 22]. Specific types of Lie bialgebras, e.g. on semi-simple Lie algebras, have been also derived and studied [1, 8, 9, 32, 37, 38, 45]. Notwithstanding, the employed techniques do not seem effective enough to help analysing higher-dimensional Lie bialgebras. To mention a few, the computational approach [1, 19] makes the analysis rather involved, whereas other approaches, e.g. see [22], are rather ad-hoc and difficult to generalise.

The so-called coboundary Lie bialgebras represent one of the most interesting types of Lie bialgebras. These are Lie bialgebras determined through a so-called r-matrix. An r-matrix is here considered as a bi-vector of the Lie algebra where the Lie bialgebra is defined on. The r-matrices are solutions to the referred to as modified classical Yang-Baxter equations [13, 22].

This work aims at introducing novel procedures to classify and to investigate coboundary Lie bialgebra structures on a fixed finite-dimensional Lie algebra g up to Lie algebra automorphisms. The crux of our approach is the extension of the techniques from Lie algebra theory on g to its related Grassmann algebra Ag. This simplifies the characterization and analysis of the structures appearing in the classification (up to Lie algebra automorphisms) and derivation of Lie bialgebras on g.
Our novel techniques can be divided into three main interrelated topics:

- The Grassmann algebra $\Lambda g$ is endowed with a natural $g$-module structure [12, 29, 40] induced by the Lie algebra $g$ that allows us to simplify the analysis of the Lie algebra $g$ through the $g$-module structure on $\Lambda g$ and vice versa. This has applications to the description of the modified classical Yang-Baxter equations.

- The here introduced $g$-invariant multilinear maps on $g$-modules help us classifying Lie bialgebras on $g$ up to Lie algebra automorphisms. Remarkably, these maps need not be symmetric, they extend the Killing form on $g$ to $\Lambda g$, and describe other structures on Lie bialgebras, e.g. types of Poisson bivectors [25] or Casimir invariants [8], and other invariants on $g$-modules (cf. [40]) as particular cases.

- The root decomposition of a Lie algebra $g$ is extended to a grading on its Grassmann algebra, $\Lambda g$, that is compatible with the algebraic Schouten bracket on $\Lambda g$, namely the natural restriction of the Schouten-Nijenhuis bracket on a Lie group $G$ with Lie algebra $g$ to the Grassmann algebra $\Lambda g$ [44]. This provides information on the form of modified classical Yang–Baxter equations.

Previous developments are applied to the classification of coboundary Lie bialgebras up to Lie algebra automorphisms in an algorithmic way. Let us sketch this procedure. The $g$-module structure on $\Lambda g$ determines a subspace of invariant elements, $(\Lambda g)^g$, relative to the $g$-module structure. The space $\Lambda^m g$ is defined as the space of $m$-vectors of $g$ and $(\Lambda^m g)^g := \Lambda^m g \cap (\Lambda g)^g$. The spaces of $g$-invariant bi- and three-vectors of $g$, i.e. $(\Lambda^2 g)^g$ and $(\Lambda^3 g)^g$, are determined through the induced grading in $\Lambda g$ by a root decomposition in $g$ and other new findings detailed in Section 8 relating the structures of $g$, $\Lambda g$, and $(\Lambda g)^g$.

Once the above is accomplished, coboundary Lie bialgebras are studied. These Lie bialgebras can be determined through elements of $\Lambda^2 g$ satisfying the modified classical Yang-Baxter equations, whose form depends on the elements of $(\Lambda^3 g)^g$. Relevantly, the existence of $g$-invariant bilinear metrics on $\Lambda^3 g$ helps us to determine $(\Lambda^3 g)^g$. Since $r$-matrices differing in an element of $(\Lambda^2 g)^g$ give rise to the same coproduct (cf. [19]), the space of coboundary Lie bialgebras must be investigated through the space $\Lambda^2 g[f] := \Lambda^2 g/(\Lambda^2 g)^g$. The elements of $\Lambda g/(\Lambda g)^g$ are called reduced multivectors. It is shown that this space, along with the subspaces $\Lambda^m g/(\Lambda^m g)^g$ with $m \in \mathbb{Z}$, inherit a $g$-module structure from $\Lambda g$ along with other $g$-invariant structures, like $g$-invariant maps and induced algebraic Schouten brackets.

Next, $g$-invariant $k$-linear structures are employed to observe the equivalence up to inner automorphisms of the coboundary Lie bialgebras on $g$. This is more general than standard techniques based on Casimir elements [7]. It also enables us to describe more geometrically the problem of classification up to automorphisms of Lie bialgebras. The determination of automorphisms of Lie algebras is a complicated problem by itself, but it will be rather unnecessary in our approach. We generally restricted ourselves to studying the equivalence under inner Lie algebra automorphisms. Then, the determination of very few not inner Lie algebra automorphisms leads to obtaining the classification.

The classification of real three-dimensional coboundary Lie bialgebras up to Lie algebra automorphisms (see [10, 24] for related topics) is approached in an algorithmic way. Although this problem has been treated somewhere else in the literature [19, 22], we accomplish such a classification to illustrate our techniques, to fill in some gaps of previous works, and to give a new more geometrical approach. Our results are sketched in Figure 1, where all equivalent reduced $r$-matrices are colored in the same way.

The structure of the paper goes as follows. Section 2 surveys the main notions on Lie bialgebras and presents the notation to be used hereafter. Section 3 analyses $g$-modules, proposes new structures related to them, and gives several examples to be employed hereafter. Section 4 defines $g$-invariant maps and analyses its applications to Grassmann algebras. In particular, it provides methods to generate
such maps on subspaces of Grassmann algebras through ad-invariant maps on Lie algebras. Section 5 studies properties of Killing-type metrics, namely metrics on a Grassmann algebra $\mathfrak{g}$ whose definition is a generalization or extension, in the sense given in Section 4, of the standard Killing metric on $\mathfrak{g}$. The existence of $\mathfrak{g}$-invariant bilinear maps in $\mathfrak{g}$-modules is assessed in Section 6. Meanwhile, Section 7 proves that a root decomposition on a Lie algebra induce a new decomposition in its corresponding Grassmann algebra and the algebraic Schouten bracket respects this decomposition. The results of previous sections are employed in Section 8 to investigate the properties of $\mathfrak{g}$-invariant elements in $\Lambda\mathfrak{g}$ and to develop methods for their calculation. The problem of classification of coboundary Lie bialgebras is simplified in Section 9 to a certain quotient of their Grassmann algebras. Section 10 details several results on the existence of automorphisms of Lie algebras. Section 11 applies all previous methods to the classification problem up to Lie algebra automorphisms of three-dimensional coboundary Lie bialgebras. Finally, Section 12 resumes our achievements and sketches future lines of research.

2 Introduction

This section briefly surveys the theory of Lie bialgebras while establishing the notation to be used hereupon (see [13, 28] for details).

**Definition 2.1.** Let $\mathcal{V}^mM$ be the space of $m$-vector fields on a manifold $M$ and let $\mathcal{V}M := \bigoplus_{k\in\mathbb{Z}} \mathcal{V}^kM$. The Schouten-Nijenhuis bracket on $\mathcal{V}M$ is the unique bilinear map $[\cdot, \cdot]_S : \mathcal{V}M \times \mathcal{V}M \to \mathcal{V}M$ satisfying that $[f,g] = 0$ for arbitrary $f, g \in C^\infty(M)$, if $X$ is a vector field on $M$, then $[X, f]_S := Xf = -[f, X]_S$. 
and
\[
[X_1 \wedge \ldots \wedge X_s,Y_1 \wedge \ldots \wedge Y_l]_S := \sum_{i=1}^{s} \sum_{j=1}^{l} (-1)^{i+j} [X_i,Y_j] \wedge X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{Y}_j \wedge \ldots \wedge Y_l,
\]
where \(X_1,\ldots,X_s,Y_1,\ldots,Y_l\) are vector fields on \(M\), hatted vector fields are omitted in the above exterior product, and \([\cdot,\cdot]\) stands for the Lie bracket of vector fields.

Let \(X \in \mathcal{V}^k M\), \(Y \in \mathcal{V}^l M\), and \(Z \in \mathcal{V}^m M\). Then, the Schouten-Nijenhuis bracket satisfies the following properties \([33]\):

1. \([X,Y]_S = -(-1)^{(k-1)(l-1)} [Y,X]_S\),
2. \([X,Y \wedge Z]_S = [X,Y]_S \wedge Z + (-1)^{(k-1)} Y \wedge [X,Z]_S\),
3. \([X,[Y,Z]]_S = [[X,Y],Z]_S + (-1)^{(k-1)(l-1)} [Y,[X,Z]]_S\).

The second property above yields that the map \(Y \in \mathcal{VM} \mapsto [X,Y]_S \in \mathcal{VM}\) is a graded derivation on \(\mathcal{VM}\) of degree \(k - 1\) \([27]\). Moreover, the third property can be equivalently rewritten as
\[
(-1)^{(k-1)(m-1)} [X,[Y,Z]]_S + (-1)^{(l-1)(m-1)} [Y,[X,Z]]_S + (-1)^{(m-1)(l-1)} [Z,[X,Y]]_S = 0.
\]
The above equality is called the \emph{graded Jacobi identity} \([33]\).

If \(M = G\) is a Lie group, then Definition 2.1 ensures that the Schouten-Nijenhuis bracket of two left-invariant multivector fields on \(G\) is again a left-invariant multivector field on \(G\). If \(G\) has a Lie algebra \(\mathfrak{g}\), then the space of left-invariant multivector fields on \(G\) can be identified with \(\Lambda \mathfrak{g}\). Therefore, the Schouten-Nijenhuis bracket on \(\mathcal{VM}\) can be restricted to left-invariant multivector fields on \(G\) giving rise to a new bracket in the Grassmann algebra \(\Lambda \mathfrak{g}\) related to \(\mathfrak{g}\): the so-called \emph{algebraic Schouten bracket} \([44, \text{pg. 172}]\). For simplicity, the restriction to \(\Lambda \mathfrak{g}\) of the Schouten-Nijenhuis bracket on \(\mathcal{VM}\) will be also denoted by \([\cdot,\cdot]_S\).

In short, a Lie bialgebra (see \([13]\) for details) is a pair of Lie algebra structures on \(\mathfrak{g}\) and its dual \(\mathfrak{g}^*\) that are compatible in a certain sense. More specifically, one has the following definition. Hereafter we assume that all Lie algebras are always finite-dimensional.

**Definition 2.2.** A \emph{Lie bialgebra} is a pair \((\mathfrak{g},\delta)\), where \(\mathfrak{g}\) is a Lie algebra with a Lie bracket \([\cdot,\cdot]_{\mathfrak{g}}\) and \(\delta: \mathfrak{g} \to \Lambda^2 \mathfrak{g}\) is a linear map, called the \emph{cocommutator}, satisfying that:

1. the dual map \(\delta^*: \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*\) is a Lie bracket on \(\mathfrak{g}^*\),
2. the map \(\delta\) satisfies the condition
\[
\delta([v_1,v_2]_{\mathfrak{g}}) = [v_1,\delta(v_2)]_{\mathfrak{g}} + [\delta(v_1),v_2]_{\mathfrak{g}}, \quad \forall v_1, v_2 \in \mathfrak{g}.
\]

The condition (2.1) can be interpreted cohomologically via \emph{Chevalley-Eilenberg complexes}. In the sequel \(\text{End}(V)\) stands for the \(\mathbb{K}\)-algebra of endomorphisms on a \(\mathbb{K}\)-linear space \(V\). This space becomes a Lie algebra relative to the commutator of endomorphisms.

**Definition 2.3.** Let \(\mathfrak{g}\) be a Lie algebra over a field \(\mathbb{K}\) and let \(\rho: \mathfrak{g} \to \text{End}(V)\) be a Lie algebra morphism on a linear space \(V\). The \emph{Chevalley-Eilenberg complex} associated with \(\rho\) is a long exact sequence
\[
\text{Hom}(\mathbb{K}, V) \xrightarrow{d_V} \text{Hom}(\mathfrak{g}, V) \xrightarrow{d_V} \text{Hom}(\Lambda^2 \mathfrak{g}, V) \xrightarrow{d_V} \text{Hom}(\Lambda^3 \mathfrak{g}, V) \xrightarrow{d_V} \ldots,
\]
where \( \text{Hom}(\Lambda^k \mathfrak{g}, V) \) is the linear space of \( k \)-linear maps \( c_k : \Lambda^k \mathfrak{g} \rightarrow V \) and \( d_V : \text{Hom}(\Lambda^k \mathfrak{g}, V) \rightarrow \text{Hom}(\Lambda^{k+1} \mathfrak{g}, V) \), with \( k \in \mathbb{N} := \mathbb{N} \cup \{0\} \), is given by

\[
(d_V c_k)(v_1 \wedge \ldots \wedge v_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} \rho_{v_i} c_k(v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_{k+1}) + \sum_{i<j} (-1)^{i+j} c_k([v_i, v_j] \mathfrak{g} \wedge v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \hat{v}_j \wedge \ldots \wedge v_{k+1}).
\]

(2.2)

An element \( c \in \text{Hom}(\Lambda^k \mathfrak{g}, M) \) is called a \( k \)-cocycle if \( d_V c = 0 \). If there exists an element \( c' \in \text{Hom}(\Lambda^{k-1} \mathfrak{g}, V) \) such that \( d_V c' = c \), then \( c \) is a \( k \)-coboundary.

The main case of the Chevalley-Eilenberg cohomology to be studied next is given by \( V = \Lambda^2 \mathfrak{g} \) and the Lie algebra homomorphism \( \Lambda^2 \text{ad} : v \in \mathfrak{g} \mapsto [v, \cdot]_S \in \text{End}(\Lambda^2 \mathfrak{g}) \). In view of (2.2), the condition (2.1) in Definition 2.2 is equivalent to saying that \( \delta \) is a 1-cocycle, i.e. \( d_V \delta = 0 \), for the Chevalley-Eilenberg cohomology with \( V = \Lambda^2 \mathfrak{g} \). For this reason, it is called the \( 1 \)-cocycle condition. If \( \delta \) is exact, then \( \delta(x) = [x, r]_S \) for certain \( r \in \Lambda^2 \mathfrak{g} \). The Whitehead lemma [26] tells us that if \( \mathfrak{g} \) is semi-simple, then every closed \( \delta \) is exact.

**Definition 2.4.** A homomorphism of Lie bialgebras \((\mathfrak{g}, \delta_{\mathfrak{g}})\) and \((\mathfrak{h}, \delta_{\mathfrak{h}})\) is a homomorphism \( \phi : \mathfrak{g} \rightarrow \mathfrak{h} \) of Lie algebras such that

\[
(\phi \otimes \phi) \circ \delta_{\mathfrak{g}} = \delta_{\mathfrak{h}} \circ \phi.
\]

Let us introduce the main type of Lie bialgebras to be extensively analysed hereafter: the coboundary Lie bialgebras. They appear in many physical applications, e.g. in integrable systems [3, 11] and quantum gravity [35].

**Definition 2.5.** A coboundary Lie bialgebra is a Lie bialgebra \((\mathfrak{g}, \delta_{\mathfrak{g}})\) whose cocommutator \( \delta_{\mathfrak{g}} \) takes the form \( \delta_{\mathfrak{r}}(x) := [x, r]_S \) for a certain \( r \in \Lambda^2 \mathfrak{g} \) and every \( x \in \mathfrak{g} \). The element \( r \) is called an \( r \)-matrix.

A natural question is which properties should be satisfied by \( r \in \Lambda^2 \mathfrak{g} \) to give rise to a cocommutator. This is answered in Theorem 2.7, whose proof requires the following notions.

**Definition 2.6.** Let \( \text{ad}_{\mathfrak{g}}^{(m)} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^\otimes m) \) be the endomorphism of the form

\[
\text{ad}_{\mathfrak{g}}^{(m)} := \sum_{p=1}^{m} \text{id} \otimes \ldots \otimes \text{id} \otimes \text{id} \otimes \ldots \otimes \text{id}, \quad \forall v \in \mathfrak{g}.
\]

(2.3)

An element \( q \in \mathfrak{g}^\otimes m \) is called \( \mathfrak{g} \)-invariant if \( q \in \ker \text{ad}_{\mathfrak{g}}^{(m)} \) for all \( v \in \mathfrak{g} \). We denote the set of \( \mathfrak{g} \)-invariant elements of \( \mathfrak{g}^\otimes m \) by \( (\mathfrak{g}^\otimes m)^\mathfrak{g} \).

Each map \( \text{ad}_{\mathfrak{g}}^{(m)} \) can be restricted to the space \( \Lambda^m \mathfrak{g} \) of \( m \)-vectors associated with \( \mathfrak{g} \). Its restriction will be denoted by \( \Lambda^m \text{ad}_{\mathfrak{g}} \). Recall that the space of \( \mathfrak{g} \)-invariant \( m \)-vectors is denoted by \( (\Lambda^m \mathfrak{g})^\mathfrak{g} \).

**Theorem 2.7.** The map \( \delta_{\mathfrak{r}} : v \in \mathfrak{g} \mapsto [v, r]_S \in \Lambda^2 \mathfrak{g} \) defined by \( r \in \Lambda^2 \mathfrak{g} \) is a cocommutator if and only if

\[
[r, r]_S \in (\Lambda^3 \mathfrak{g})^\mathfrak{g}.
\]

(2.4)

The condition (2.4) is called the modified classical Yang-Baxter equation. The equation \([r, r]_S = 0\) is called the classical Yang-Baxter equation. If \([r, r]_S = 0\), the \( r \)-matrix \( r \) is called triangular.

The following proposition establishes when two \( r \)-matrices induce the same coproduct.
Proposition 2.1. Two $r$-matrices $r_1, r_2 \in \Lambda^2 g$ satisfy that $\delta r_1 = \delta r_2$ if and only if $r_1 - r_2 \in (\Lambda^2 g)\mathfrak{g}$.

Proof. Let $\delta_{r_i}$, with $i = 1, 2$, be the coproducts induced by $r_1, r_2$. Since $\delta r_1 = \delta r_2$, one obtains that $[v, r_1]_g = [v, r_2]_g$ for every $v \in g$. Hence, $r_1 - r_2 \in (\Lambda^2 g)\mathfrak{g}$. The converse is immediate.

Proposition 2.1 shows that what really matters to the determination of coboundary Lie bialgebras is not the determination of $r$-matrices, but the description of their equivalence classes in $\Lambda^2 g/(\Lambda^2 g)\mathfrak{g}$.

3 Structures on $g$-modules

Let us discuss the notion of $g$-modules [13] and some new related structures, which are necessary to our purposes. In the sequel, $GL(V)$ stands for the Lie group of automorphisms of the $\mathbb{K}$-linear space $V$.

Definition 3.1. A $g$-module is a pair $(V, \rho)$, where $V$ is a linear space over a field $\mathbb{K}$ and $\rho : v \in g \mapsto \rho_v \in \text{End}(V)$ is a Lie algebra morphism.

The $g$-module $(V, \rho)$ will be represented just by $V$ if $\rho$ is understood by context. In this case, $\rho_v(x)$ will be written simply as $\alpha x$ for any $v \in g$ and $x \in V$.

Example 3.1. Every Lie algebra $g$ induces a $g$-module $(g, \text{ad})$, where $\text{ad} : v \in g \mapsto [v, \cdot]_g \in \mathfrak{g}(g)$ is the adjoint representation of $g$ [20]. Then, each $[v, \cdot]_g$, with $v \in g$, is a derivation of the Lie algebra $g$ [20]. Hence, $\text{ad}$ can be considered as a mapping $\text{ad} : g \to \text{Der}(g)$, where $\text{Der}(g)$ is the Lie algebra of derivations on the Lie algebra $g$.

Example 3.2. Let $GL(g)$ and $\text{Aut}(g)$ be the groups of linear automorphisms and Lie algebra automorphisms of $g$, respectively. Assume $\mathfrak{g}(g)$ to be the space of endomorphisms in $g$. The group $\text{Aut}(g)$ can be characterized as the intersection of $GL(g)$ with $\Phi^{-1}(0)$ for $\Phi : T \in \mathfrak{g}(g) \mapsto [T(\cdot), T(\cdot)]_g - T[\cdot, \cdot]_g \in (g \otimes g)^* \otimes g$. Hence, $\text{Aut}(g)$ is a closed subgroup of the Lie group $GL(g)$ and it becomes a Lie group [31]. Let $\text{aut}(g)$ be its Lie algebra. The tangent map of the injection $\iota : \text{Aut}(g) \to GL(g)$ at the identity map $\text{id}_g$ on $g$ induces a Lie algebra morphism $\text{ad} : \text{aut}(g) \simeq T_{\text{id}_g} \text{Aut}(g) \to \mathfrak{g}(g) \simeq T_{\text{id}_g} GL(g)$. Then $(g, \text{ad})$ becomes an $\text{aut}(g)$-module.

Example 3.3. Every Grassmann algebra $\Lambda g$ admits naturally a $g$-module structure $(\Lambda g, \Lambda \text{ad})$, where $\Lambda \text{ad}$ is the Lie algebra morphism $\Lambda \text{ad} : v \in g \mapsto [v, \cdot]_g \in \text{End}(\Lambda g)$. Indeed, $\Lambda \text{ad}$ is a Lie algebra homomorphism due to the properties of the algebraic Schouten bracket [44].

Example 3.3 can be understood as a simple application of the corollary to the following proposition.

Proposition 3.1. Let $(V, \rho)$ be a $g$-module. The subspace $\Lambda^m V \subset AV$, with $m \in \mathbb{N}$, gives rise to a $g$-module of the form $(\Lambda^m V, \Lambda^m \rho)$, where $\Lambda^m \rho : v \in g \mapsto \Lambda^m \rho_v \in \text{End}(\Lambda^m V)$ is such that

$$\Lambda^m \rho_v := \rho_v \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes \rho_v, \quad \forall v \in g,$$

(3.1)

and $\text{id}$ is the identity on $V$.

Proof. Proving that $(\Lambda^m V, \Lambda^m \rho)$ is a $g$-module reduces to showing that $\Lambda^m \rho : v \in g \mapsto \Lambda^m \rho_v \in \text{End}(\Lambda^m V)$ is a well-defined homomorphism of Lie algebras. The definition (3.1) ensures that the image by $\Lambda^m \rho_v$ of an element of $\Lambda^m V$ is an element thereof. Hence, $\Lambda^m \rho_v$ is a well-defined linear endomorphism on $\Lambda^m V$. Moreover, the definition (3.1) also ensures the map $\Lambda^m \rho$ to be linear. It
is only left to prove that $\Lambda^m \rho_{[v_1,v_2]} = [\Lambda^m \rho_{v_1}, \Lambda^m \rho_{v_2}]_{\text{End}(\Lambda^m V)}$ for every $v_1, v_2 \in \mathfrak{g}$. As $(V, \rho)$ is a $\mathfrak{g}$-module and $\rho$ is therefore a Lie algebra morphism, it follows that

$$\Lambda^m \rho_{[v_1,v_2]} = \rho_{[v_1,v_2]} \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes \rho_{[v_1,v_2]}$$

$$= [\rho_{v_1}, \rho_{v_2}]_{\text{End}(V)} \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes [\rho_{v_1}, \rho_{v_2}]_{\text{End}(V)}. \quad (3.2)$$

The operators

$$\text{id} \otimes \ldots \otimes \rho_{v_1} \otimes \ldots \otimes \text{id} \quad \text{and} \quad \text{id} \otimes \ldots \otimes \rho_{v_2} \otimes \ldots \otimes \text{id}, \quad i \neq j,$$ 

commute for $i \neq j$ and every $v_1, v_2 \in \mathfrak{g}$. Thus,

$$[\Lambda^m \rho_{v_1}, \Lambda^m \rho_{v_2}]_{\text{End}(\Lambda^m V)} = [\rho_{v_1}, \rho_{v_2}]_{\text{End}(V)} \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes [\rho_{v_1}, \rho_{v_2}]_{\text{End}(V)} \quad (3.4)$$

for arbitrary $v_1, v_2 \in \mathfrak{g}$. The proposition follows by comparing (3.2) and (3.4).

Since $\Lambda V = \bigoplus_{m \in \mathbb{Z}} \Lambda^m V$, it is possible to extend all $\Lambda^m \rho_v$, for $v \in \mathfrak{g}$ and $m \in \mathbb{N}$, to an endomorphism on the whole $\Lambda V$. To this respect, it is convenient to define $\Lambda^m \rho_v : \Lambda \in \Lambda^m V \mapsto 0 \in \Lambda^m V$ for $m \leq 0$ and $v \in \mathfrak{g}$. Then, we set $\Lambda \rho_v := \bigoplus_{m \in \mathbb{Z}} \Lambda^m \rho_v$ for every $v \in \mathfrak{g}$. In other words, $\Lambda \rho_v$ is the unique linear morphism on $\Lambda V$ satisfying $\Lambda \rho_v |_{\Lambda^m V} = \Lambda^m \rho_v$ for every $m \in \mathbb{Z}$. The morphisms $\Lambda \rho_v$ give rise to the following immediate corollary.

**Corollary 3.1.** Every $\mathfrak{g}$-module $(V, \rho)$ induces a natural $\mathfrak{g}$-module $(\Lambda V, \Lambda \rho)$, where $\Lambda \rho : v \in \mathfrak{g} \mapsto \Lambda \rho_v \in \text{End}(\Lambda \rho)$.

We now aim to attach every $\mathfrak{g}$-module with a certain group to study its properties. This is accomplished in the following lemma, which is a crux for the study of Lie bialgebras.

**Lemma 3.2.** Let $(V, \rho)$ be a $\mathfrak{g}$-module and let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. If $\Phi : G \to GL(V)$ is a Lie group morphism making commutative the diagram

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\rho} & \mathfrak{gl}(V) \\
\exp G & \searrow & \exp \\
G & \xrightarrow{\Phi} & GL(V)
\end{array} \quad (3.5)
$$

where $\exp G$ and $\exp$ are exponential maps on $\mathfrak{g}$ and $\mathfrak{gl}(V)$ respectively, then $\Phi(G)$ is the Lie subgroup of $GL(V)$ generated by the elements $\exp(\rho(\mathfrak{g}))$.

**Proof.** Although $\exp : \mathfrak{g} \to G$ is not necessarily surjective (see [20, Ex. 8.40, pg. 118]), the elements of $\exp(\mathfrak{g})$ generate the connected component of $G$ containing its neutral element, i.e. the whole $G$ (see [20, pg. 116]). Since $\rho$ is a Lie algebra morphism, the elements of $\rho(\mathfrak{g})$ form a Lie subalgebra of $\mathfrak{gl}(V)$. As $\exp$ is a local diffeomorphism, it can be proved that the elements of $\exp(\rho(\mathfrak{g}))$ generate a connected Lie group $GL(\rho) \subset GL(V)$. Let us prove that $GL(\rho) = \Phi(G)$. The $GL(\rho)$ is generated by finite products of elements of $\exp(\rho(\mathfrak{g})) = \Phi(\exp G(\mathfrak{g}))$. Since $\Phi(G)$ is a group and $\exp(\rho(\mathfrak{g})) = \Phi(\exp G(\mathfrak{g})) \subset \Phi(G)$, one has that $GL(\rho) \subset \Phi(G)$. Conversely, since every element of $G$ is a product of elements of $\exp G(\mathfrak{g})$ (see [43, Theorem 3.2.1]), every element of $\Phi(G)$ is a product of elements of $\Phi(\exp G(\mathfrak{g})) = \exp(\rho(\mathfrak{g}))$. Thus, $\Phi(G) \subset GL(\rho)$ and $GL(\rho) = \Phi(G)$. Since $\Phi(G)$ is the image of a Lie group homomorphism, it becomes a Lie subgroup of $GL(V)$. \qed
Some remarks are in order. Since $GL(\rho)$ is generated by $\exp(\rho(g))$, the group $GL(\rho)$ is the smallest group containing $\exp(\rho(g))$. Consequently, this also implies that the space $\Phi(G)$ in (3.2) is independent of the chosen Lie group $G$ provided it is connected and its Lie algebra is isomorphic to $\mathfrak{g}$. Note that $\Phi(G)$ may not be an embedded submanifold of $GL(V)$: it is only a Lie subgroup in the sense that it is a subgroup and an immersed submanifold of $GL(V)$. If $\Phi$ is a closed map, then it can be ensured that $GL(\rho)$ is a Lie subgroup of $GL(V)$. These remarks motivate the following definition.

**Definition 3.3.** Given a $\mathfrak{g}$-module $(V, \rho)$, the Lie group of $(V, \rho)$, denoted by $GL(\rho)$, is the Lie subgroup of $GL(V)$ generated by the automorphisms on $V$ of the form $\exp(\rho(g))$.

The above definition can be rephrased by saying that $GL(\rho)$ is the smallest immersed Lie subgroup of $GL(V)$ containing the elements of $\Phi(\exp_{\mathfrak{c}}(\mathfrak{g}))$.

The correspondence between Lie algebras and simply connected Lie groups (the so-called *Lie algebra-Lie group correspondence*) allows us to attach every Lie algebra $\mathfrak{g}$ to a unique simply connected Lie group $\hat{G}$. Moreover, it is also possible to ensure that the Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ induces a unique Lie group homomorphism $\Phi : \hat{G} \rightarrow GL(V)$. Hence, every $\mathfrak{g}$-module $(V, \rho)$ induces a unique Lie group action $\Phi : \hat{G} \rightarrow GL(V)$ making the corresponding diagram (3.5) commutative. These facts are employed in the following propositions.

**Proposition 3.2.** Let $\text{Ad} : g \in G \mapsto \text{Ad}_g \in \text{Aut}(G)$ be the adjoint action of a connected Lie group $G$ with Lie algebra $\mathfrak{g}$. The Lie group of the $\mathfrak{g}$-module $(\mathfrak{g}, \text{ad})$ is isomorphic to the Lie group, $\text{Ad}(G)$, of the Lie algebra automorphisms $T_{\text{id}_G} \text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$.

**Proof.** Every Lie group $G$ admitting a Lie algebra $\mathfrak{g}$ is such that $\text{ad}$ is the tangent map to $\text{Ad}$ at the neutral element of $G$. Hence, one obtains the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{ad}} & \mathfrak{gl}(g) \\
\downarrow \exp_G & & \downarrow \exp \\
G & \xrightarrow{\text{Ad}} & GL(g)
\end{array}
\]

Applying the Lemma 3.2, we obtain the Lie group of the $\mathfrak{g}$-space $(\mathfrak{g}, \text{ad})$ is then given by $\text{Ad}(G)$.

Since $GL(\text{ad})$ is independent of the chosen connected Lie group $G$ provided its Lie algebra will be $\mathfrak{g}$, it makes sense to denote $GL(\text{ad})$ by $\text{Im}(\mathfrak{g})$.

**Proposition 3.3.** The Lie group of the $\mathfrak{aut}(\mathfrak{g})$-module $(\mathfrak{aut}(\mathfrak{g}), \tilde{\text{ad}})$ is given by the connected component, $\text{Aut}_{\mathfrak{c}}(\mathfrak{g})$, of the neutral element of $\text{Aut}(\mathfrak{g})$.

**Proof.** The injection $\iota : \text{Aut}_{\mathfrak{c}}(\mathfrak{g}) \hookrightarrow GL(\mathfrak{g})$ has a tangent map $\tilde{\text{ad}} : \mathfrak{aut}(\mathfrak{g}) \rightarrow \mathfrak{gl}(\mathfrak{g})$ at $\text{id}_\mathfrak{g} \in \text{Aut}_{\mathfrak{c}}(\mathfrak{g})$. This gives rise to the commutativity of the right-hand side and center part of the diagram below. Let $\tilde{\text{Aut}}(\mathfrak{g})$ be the simply connected Lie group associated with $\mathfrak{aut}(\mathfrak{g})$. The commutativity of the left-hand and center side of the diagram comes from the properties of $\text{Ad}$ for $\tilde{\text{Aut}}(\mathfrak{g})$ and $\tilde{\text{ad}}$.

\[
\begin{array}{ccc}
\mathfrak{aut}(\mathfrak{g}) & \xrightarrow{\tilde{\text{ad}}} & \mathfrak{gl}(\mathfrak{g}) \\
\downarrow \exp_{\mathfrak{aut}(\mathfrak{g})} & & \downarrow \exp \\
\tilde{\text{Aut}}(\mathfrak{g}) & \xrightarrow{\text{Ad}} & GL(\mathfrak{g}) \\
\downarrow \exp_{\tilde{\text{Aut}}(\mathfrak{g})} & & \downarrow \exp_{\text{Aut}_{\mathfrak{c}}(\mathfrak{g})} \\
\text{Aut}_{\mathfrak{c}}(\mathfrak{g}) & \xrightarrow{\tilde{\iota}} & \text{Aut}_{\mathfrak{c}}(\mathfrak{g})
\end{array}
\]

From the commutativity of the diagram and using Lemma 3.2, it follows that $GL(\tilde{\text{ad}}) = \tilde{\text{Aut}}(\mathfrak{g}) = \text{Aut}_{\mathfrak{c}}(\mathfrak{g})$. 

\[\square\]
The relevance of Propositions 3.2 and 3.3 relies in the fact that the Lie groups of the \( \mathfrak{g} \)-modules described in them are the Lie groups \( \text{Im}(\mathfrak{g}) \) and \( \text{Aut}_c(\mathfrak{g}) \). These Lie groups play a relevant role in the classification of Lie bialgebras up to Lie algebra automorphisms and previous constructions will be employed to study this problem.

As a \( \mathfrak{g} \)-module \((V,\rho)\) induces new ones \( (\Lambda^mV,\Lambda^m\rho)\), the \( GL(\rho) \) is related to \( GL(\Lambda^m\rho) \) as follows.

**Definition 3.4.** We write \( \Lambda^mT \), where \( T \in \text{Aut}(V) \) and \( m \in \mathbb{N} \), for the linear automorphism on \( \Lambda^mV \) given by \( \Lambda^mT := T \otimes \ldots \otimes T \) (\( m \) times). If \( m = 0 \), then \( \Lambda^0T \) is defined to be the identity on \( \mathbb{K} \). If \( m < 0 \), then \( \Lambda^mT \) is defined to be the zero linear morphism \( \Lambda^mV \rightarrow \Lambda^mV \). This allows us to define \( \Lambda T \) as the automorphism \( \Lambda T : \Lambda V \rightarrow \Lambda V \) of the form \( \Lambda T := \oplus_{m \in \mathbb{Z}} \Lambda^mT \).

**Proposition 3.4.** Given a \( \mathfrak{g} \)-module \((V,\rho)\), the induced \( \mathfrak{g} \)-module \( (\Lambda^mV,\Lambda^m\rho) \) is such that

\[
GL(\Lambda^m\rho) = \{ \Lambda^mT \mid T \in GL(\rho) \}.
\]

**Proof.** By definition, the group \( GL(\Lambda^m\rho) \) is generated by the composition of elements of \( \exp(\rho(\mathfrak{g})) \), namely

\[
\exp(\Lambda^m\rho_v) = \exp(\rho_v \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes \rho_v), \quad \forall v \in \mathfrak{g}.
\]

Since the operators (3.3) commute, one gets that

\[
\exp(\Lambda^m\rho_v) = \exp(\rho_v \otimes \text{id} \otimes \ldots \otimes \text{id}) \cdot \ldots \cdot \exp(\text{id} \otimes \ldots \otimes \text{id} \otimes \rho_v)
= \exp(\rho_v) \otimes \ldots \otimes \exp(\rho_v).
\]

Hence, \( GL(\Lambda^m\rho) \) is generated by the product of elements of the form \( T \otimes \ldots \otimes T \) (\( m \) times), where \( T \) is a product of operators \( \exp(\rho_v) \) with \( v \in \mathfrak{g} \). Since the \( \exp(\rho_v) \) generate \( GL(\rho) \), it follows that \( T \) is any element of \( GL(\rho) \). In consequence, the statement of the proposition follows. \( \square \)

### 4 The \( \mathfrak{g} \)-invariant maps on Grassmann algebras \( \Lambda \mathfrak{g} \)

This section extends standard notions on Lie algebras, like the ad-invariance, to \( \mathfrak{g} \)-modules. In particular, special attention is paid to the extension to Grassmann algebras of the form \( \Lambda \mathfrak{g} \). The main properties of the newly introduced structures will be analysed. Our findings will permit us to study Lie bialgebras in following sections.

**Definition 4.1.** A \( k \)-linear map \( b : V^\otimes k \rightarrow \mathbb{K} \) is called \( GL(\rho) \)-invariant relative to the \( \mathfrak{g} \)-module \((V,\rho)\) if

\[
b(Tx_1,\ldots,Tx_k) = b(x_1,\ldots,x_k), \quad \forall T \in GL(\rho), \quad \forall x_1,\ldots,x_k \in V.
\]

The following definition will be crucial for the study and classification of Lie bialgebras in following sections.

**Definition 4.2.** A \( k \)-linear map \( b : V^\otimes k \rightarrow \mathbb{K} \) is called \( \mathfrak{g} \)-invariant relative to the \( \mathfrak{g} \)-module \((V,\rho)\) if

\[
b(\rho_v(x_1),\ldots,x_k) + \ldots + b(x_1,\ldots,\rho_v(x_k)) = 0, \quad \forall v \in \mathfrak{g}, \quad \forall x_1,\ldots,x_k \in V.
\]

To simplify the notation, we hereafter assume that a metric is a symmetric (possibly degenerate) bilinear map.
Example 4.1. The Killing metric on a Lie algebra $\mathfrak{g}$, namely $\kappa_\mathfrak{g}(v_1, v_2) := \text{tr}(\text{ad}_{v_1} \circ \text{ad}_{v_2})$ with $v_1, v_2 \in \mathfrak{g}$ satisfies that $\kappa_\mathfrak{g}(\text{ad}_v v_1, v_2) + \kappa_\mathfrak{g}(v_1, \text{ad}_v v_2) = 0$ for all $v, v_1, v_2 \in \mathfrak{g}$ [20]. For this reason Killing metrics are called $\text{ad}$-invariant. In view of Definition 4.2, the Killing metric is $\mathfrak{g}$-invariant with respect to $(\mathfrak{g}, \text{ad})$. Thus, $\mathfrak{g}$-invariance can be interpreted as an extension of $\text{ad}$-invariance for Lie algebras to general $\mathfrak{g}$-modules.

As shown next, the invariance of a multilinear map on a $\mathfrak{g}$-module $V$ relative to the group of the $\mathfrak{g}$-module can be characterized by the $\mathfrak{g}$-invariance of the multilinear map.

Proposition 4.1. A $k$-linear map $b : V^\otimes k \to K$ is $GL(\rho)$-invariant relative to the $\mathfrak{g}$-module $(V, \rho)$ if and only if $b$ is $\mathfrak{g}$-invariant.

Proof. Assume first that $b$ is $GL(\rho)$-invariant. Let us prove that $b$ is $\mathfrak{g}$-invariant. Define $T := \exp(\rho_v) \in GL(\rho)$, with $v \in \mathfrak{g}$, and construct a curve $\gamma : t \in \mathbb{R} \mapsto \exp(t \rho_v) \in GL(\rho)$. From the $GL(\rho)$-invariance, one has that

$$b(\exp(t \rho_v) x_1, \ldots, \exp(t \rho_v) x_k) = b(x_1, \ldots, x_k) \quad (4.1)$$

for all $t \in \mathbb{R}$ and arbitrary $x_1, \ldots, x_k \in V$. Differentiating (4.1) at $t = 0$, one obtains

$$b(\rho_v(x_1), \ldots, x_k) + \ldots + b(x_1, \ldots, \rho_v(x_k)) = 0$$

and $b$ is $\mathfrak{g}$-invariant.

Conversely, if $b$ is assumed to be $\mathfrak{g}$-invariant, then for every $t \in \mathbb{R}$ and $x_1, \ldots, x_k \in V$ one has

$$\frac{d}{dt} b(\gamma(t)x_1, \ldots, \gamma(t)x_k) = b(\rho_v(\gamma(t)x_1), \ldots, \gamma(t)x_k) + \ldots + b(\gamma(t)x_1, \ldots, \rho_v(\gamma(t)x_k)) = 0.$$

It follows that

$$b(Tx_1, \ldots, Tx_k) = b(\gamma(1)x_1, \ldots, \gamma(1)x_k) = b(\gamma(0)x_1, \ldots, \gamma(0)x_k) = b(x_1, \ldots, x_k)$$

for every $t \in \mathbb{R}$ and $x_1, \ldots, x_k \in V$. As the previous equality is satisfied for an arbitrary $T = \exp(\rho_v)$ and such elements generate $GL(\rho)$, then $b$ becomes $GL(\rho)$-invariant. \hfill \Box

Subsequently, we assume that $\{v_1, \ldots, v_r\}$ is a basis of $\mathfrak{g}$ and define $v_J := v_{J(1)} \wedge \ldots \wedge v_{J(m)}$, where $J := (J(1), \ldots, J(m))$ with $J(1), \ldots, J(m) \in \mathbb{N}$, a multi-index of length $|J| = m$, the $S_m$ is the permutation group of $m$ elements, and $\text{sg}(\sigma_j)$ stands for the sign of the permutation $\sigma_j \in S_m$.

Theorem 4.3. Every $k$-linear map $b : V^\otimes k \to K$ that is $\mathfrak{g}$-invariant relative to the $\mathfrak{g}$-module $V$ induces a $k$-linear map, $b_{\Lambda V}$, on $\Lambda V$ that is $\mathfrak{g}$-invariant relative to the induced $\mathfrak{g}$-module $\Lambda V$ by imposing that

1. the spaces $\Lambda^m V$, with $m \in \mathbb{Z}$, are orthogonal between themselves relative to $b_{\Lambda V}$,

2. $b_{\Lambda V}(1, \ldots, 1) = 1$,

3. the restriction, $b_{\Lambda^m \mathfrak{g}}$, of $b$ to $\Lambda^m \mathfrak{g}$, with $m \in \mathbb{N}$, satisfies

$$b_{\Lambda^m \mathfrak{g}}(v_{J_1}, \ldots, v_{J_k}) := \sum_{\sigma_1, \ldots, \sigma_k \in S_m} \text{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^m b \left( v_{J_1(\sigma_r^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))} \right).$$

Proof. Since $1 \in \Lambda^0 V$, the decomposable elements $v_J = v_{J(1)} \wedge \ldots \wedge v_{J(m)}$ span $\Lambda \mathfrak{g}$, and $b_{\Lambda \mathfrak{g}}$ is $k$-linear, then the conditions 1,2, and 3 fix the value of $b_{\Lambda \mathfrak{g}}$ on the whole $\Lambda \mathfrak{g}$. Observe that the condition 3)
establishes a well-defined value of $b_{\Lambda g}$ independently of the representative for each $v_j$, with $s \in \overline{1, k}$. Indeed, defining $\sigma v_J := v_J(\sigma^{-1}(1)) \wedge \ldots \wedge v_J(\sigma^{-1}(m))$, we obtain

$$b_{\Lambda g}(\tilde{\sigma}_1 v_{j_1}, \ldots, \tilde{\sigma}_k v_{j_k}) = \sum_{\sigma_1, \ldots, \sigma_k \in S_m} \text{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b \left(v_{j_1}(\tilde{\sigma}_1^{-1}(r)), \ldots, v_{j_k}(\tilde{\sigma}_k^{-1}(r))\right).$$

Defining $\tilde{\sigma}_j := \tilde{\sigma}_j \cdot \sigma_j$, we obtain

$$b_{\Lambda g}(\tilde{\sigma}_1 v_{j_1}, \ldots, \tilde{\sigma}_k v_{j_k}) = \sum_{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k \in S_m} \text{sg}(\tilde{\sigma}_1 \ldots \tilde{\sigma}_k) \text{sg}(\tilde{\sigma}_1 \ldots \tilde{\sigma}_k) \frac{1}{m!} \prod_{r=1}^{m} b \left(v_{j_1}(\tilde{\sigma}_1^{-1}(r)), \ldots, v_{j_k}(\tilde{\sigma}_k^{-1}(r))\right)$$

$$= \text{sg}(\tilde{\sigma}_1 \ldots \tilde{\sigma}_k) b_{\Lambda g}(v_{j_1}, \ldots, v_{j_k}).$$

It is left to prove that $b_{\Lambda g}$ is $g$-invariant relative to the natural $g$-module structure on $\Lambda g$ induced by the standard $g$-module structure on $g$. By means of Proposition 4.1, the $g$-invariance of $b_{\Lambda g}$ can be inferred from its $GL(\Lambda g)$-invariance. In turn, this also reduces to the $GL(\Lambda^m g)$-invariance of the restrictions $b_{\Lambda^m g}$ for $m \in \mathbb{N}$. Using the $GL(\rho)$-invariance of $b$ and defining $e^{\rho v} := \exp(\rho v)$ for every $v \in g$, we get

$$b_{\Lambda g}(\Lambda^m e^{\rho v}(v_{j_1}), \ldots, \Lambda^m e^{\rho v}(v_{j_k})) = \sum_{\sigma_1, \ldots, \sigma_k \in S_m} \text{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b \left(e^{\rho v} v_{j_1}(\sigma_1^{-1}(r)), \ldots, e^{\rho v} v_{j_k}(\sigma_k^{-1}(r))\right)$$

$$= \sum_{\sigma_1, \ldots, \sigma_k \in S_m} \text{sg}(\sigma_1 \ldots \sigma_k) \frac{1}{m!} \prod_{r=1}^{m} b \left(v_{j_1}(\sigma_1^{-1}(r)), \ldots, v_{j_k}(\sigma_k^{-1}(r))\right)$$

$$= b_{\Lambda g}(v_{j_1}, \ldots, v_{j_k}).$$

Since the above holds for every $m \in \mathbb{N}$ and the invariance of $b_{\Lambda^0 V}$ is obvious, the map $b_{\Lambda g}$ is $GL(\Lambda g)$-invariant and Proposition 4.1 ensures that is $g$-invariant. $\square$

Since the Killing metric is $g$-invariant, it can be extended to each $\Lambda^m g$. The extensions of the Killing metric on $g$ to $\Lambda^2 g$ and $\Lambda^3 g$ are called the double and triple Killing metrics of $g$, respectively.

A simple consequence of Proposition 4.1 and the proof of the above theorem is given by the following corollary.

**Corollary 4.1.** Let $b$ be a $g$-invariant $k$-linear map on $V$ and let $T \in GL(\rho)$. Then, $b_{\Lambda g}$ is invariant with respect to $GL(\Lambda^m g)$, i.e.

$$b_{\Lambda g}(\Lambda^m T v, \ldots, \Lambda^m T v) = b_{\Lambda g}(v, \ldots, v).$$

Relevantly, the next proposition shows that certain extensions of a $g$-invariant metric on $g$ are trivial, and therefore useless.

**Proposition 4.2.** Let $b$ be a $g$-invariant $k$-linear map on $V$. The mapping $b_{\Lambda g}$ vanishes for all natural $m > 1$ and odd $k > 1$.

**Proof.** The key of the proof relies in proving that we can gather the summands appearing in $b_{\Lambda g}$ into families that sum up to zero. To do so, we first introduce the following equivalence relation on $S_m^k := \overline{S_m \times \ldots \times S_m}$ given by

$$\sigma_1, \ldots, \sigma_k \sim (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k) \iff \exists \sigma \in S_m : (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k) = (\sigma \sigma_1, \ldots, \sigma \sigma_k).$$
Let \([\{\sigma_1, \ldots, \sigma_k\}]\) be the equivalence class of elements of \(S^k_m\) related to \((\sigma_1, \ldots, \sigma_k)\) and let \(\mathcal{R}\) be the space of equivalence classes. If \([w]\) is an equivalence class of \(S^k_m\), the map \(b_{\Lambda=g}\) can be written as

\[
b_{\Lambda=g}(v_{j_1}, \ldots, v_{j_k}) := \sum_{[w] \in \mathcal{R}} \sum_{(\sigma_1, \ldots, \sigma_k) \in [w]} \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right).
\]

Since every equivalence class of \(\mathcal{R}\) is of the form

\[
[(\sigma_1, \ldots, \sigma_k)] = \{\sigma \sigma_1, \ldots, \sigma \sigma_k : \sigma \in S_m\},
\]

one has that

\[
b_{\Lambda=g}(v_{j_1}, \ldots, v_{j_k}) := \sum_{[(\sigma_1, \ldots, \sigma_k)] \in \mathcal{R}} \sum_{\sigma \in S_m} \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right).
\]

Let us show that the above sum vanishes for every equivalence class of \(\mathcal{R}\). First,

\[
\prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right) = \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right)
\]

Let us define \(\text{sg}(w) := \text{sg}(\sigma_1 \ldots \sigma_m)\). Then

\[
\prod_{r=1}^{m} \text{sg}(\sigma w) b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right) = \prod_{r=1}^{m} \text{sg}(\sigma w) b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right)
\]

and \(\text{sg}(\sigma w) = \text{sg}(\sigma)^k \text{sg}(w) = \text{sg}(\sigma) \text{sg}(w)\) since \(k\) is odd. Therefore,

\[
\text{sg}(\sigma)^k \text{sg}(w) \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right) = \text{sg}(\sigma) \text{sg}(w) \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right).
\]

Every equivalence class of \(\mathcal{R}\) has \(m!\) elements. All of them have the same absolute value. Half of them is odd and the other half is even. Hence,

\[
\sum_{\sigma \in S_m} \frac{1}{m!} \prod_{r=1}^{m} b\left(v_{J_1(\sigma_1^{-1}(r))}, \ldots, v_{J_k(\sigma_k^{-1}(r))}\right) = 0
\]

and \(b_{\Lambda=g} = 0\).

**Example 4.2.** Consider the Lie algebra \(\mathfrak{su}_2\) and its Killing form \(\kappa_{\mathfrak{su}_2}\), which is a \(\mathfrak{su}_2\)-invariant, bilinear, symmetric map on \(\mathfrak{su}_2\). Take a basis \(\{e_1, e_2, e_3\}\) of \(\mathfrak{su}_2\) satisfying

\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.
\]

Theorem 4.3 allows us to extend the Killing metric \(\kappa_{\mathfrak{su}_2}\) to the double Killing metric \(\kappa_{\Lambda^2\mathfrak{su}_2}\) and the triple Killing metric \(\kappa_{\Lambda^3\mathfrak{su}_2}\) on \(\Lambda^2\mathfrak{su}_2\) and \(\Lambda^3\mathfrak{su}_2\), respectively. Defining the bases \(\{e_1, e_2, e_3\}, \{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}, \{e_1 \wedge e_2 \wedge e_3\}\) for the spaces \(\Lambda^k\mathfrak{su}_2\) with \(k = 1, 2, 3\) respectively, we obtain that the matrix expressions for the previous metrics read

\[
[\kappa_{\mathfrak{su}_2}] := \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad [\kappa_{\Lambda^2\mathfrak{su}_2}] := \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad [\kappa_{\Lambda^3\mathfrak{su}_2}] := (-8).
\]
The previous example shows that the Killing metric and its extensions to $\Lambda^2\mathfrak{su}_2$ and $\Lambda^3\mathfrak{su}_2$ are simultaneously diagonal and non-degenerate. The corollary below provides an explanation of this fact.

**Corollary 4.2.** If $b$ is a symmetric $\mathfrak{g}$-invariant $k$-linear mapping on an $r$-dimensional Lie algebra $\mathfrak{g}$, then its extension $b_{\Lambda^g}$ to $\Lambda^g$ is symmetric. If $b$ is a metric diagonalizing in the basis $\{e_i\}_{i \in \mathcal{I}}$, then its extension to $\Lambda^m\mathfrak{g}$ takes a diagonal form in the basis $\{e_{J}\}_{|J|=m}$. Additionally, $b$ is non-degenerate if and only if $b_{\Lambda^g}$ is non-degenerate.\[ \Box \]

**Proof.** If $b$ is a symmetric $\mathfrak{g}$-invariant $k$-linear mapping on $\mathfrak{g}$, then Theorem 4.3 ensure that $b_{\Lambda^g}$ is symmetric on the elements of a basis $\nu_J$ of $\Lambda^g$. Indeed, the condition 3) guarantees the symmetry of $b_{\Lambda^m\mathfrak{g}}$ on decomposable elements of $\Lambda^m\mathfrak{g}$, $m \in \mathbb{N}$, whereas the condition 2) ensures the same for $m = 0$.

Since $b_{\Lambda^g}$ is additionally multilinear, it becomes symmetric on the whole $\Lambda^g$.

If $b$ is a metric, it can always be put into diagonal form in a certain basis $\{e_1, \ldots, e_r\}$ for $\mathfrak{g}$. This gives rise to a basis $\{e_J\}$ of $\Lambda^g$. Using the expression for $b_{\Lambda^g}$, this metric also becomes diagonal. The elements on the diagonal are of the form $\prod_{j=1}^{|J|} b(e_{J(j)}, e_{J(j)})$ for every multi-index $J$. Thus, $b$ is non-degenerate if and only if the induced symmetric metric $b_{\Lambda^m\mathfrak{g}}$ on each $\Lambda^m\mathfrak{g}$ is so as well. \[ \Box \]

**Example 4.3.** Consider the Lie algebra $\mathfrak{sl}_2$ of real $2 \times 2$ traceless matrices and a basis $\{e_1, e_2, e_3\}$ satisfying the commutation relations
\[ [e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3, \quad [e_2, e_3] = e_1. \] (4.3)

Choose also the induced bases $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ and $\{e_1 \wedge e_2 \wedge e_3\}$ in $\Lambda^2\mathfrak{sl}_2$ and $\Lambda^3\mathfrak{sl}_2$, respectively. In the above-mentioned bases in $\Lambda^2\mathfrak{sl}_2$ and $\Lambda^3\mathfrak{sl}_2$, the Killing metric and its extensions to $\Lambda^2\mathfrak{sl}_2$ and $\Lambda^3\mathfrak{sl}_2$, given by Theorem 4.3, read:

\[ [\kappa_{\mathfrak{sl}_2}] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad [\kappa_{\Lambda^2\mathfrak{sl}_2}] = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad [\kappa_{\Lambda^3\mathfrak{sl}_2}] = (-8). \] (4.4)

Since $\mathfrak{sl}_2$ is simple, the Cartan criterion states that its Killing metric is non-degenerate. In consequence, Corollary 4.2 ensures that the extended metrics on $\Lambda^2\mathfrak{sl}_2$ and $\Lambda^3\mathfrak{sl}_2$ must be also non-degenerate. This agrees with the explicit form of the extended metrics showed in (4.4).

## 5 Killing-type metrics

This section aims to describe the invariance properties of certain multilinear metrics on the spaces $\Lambda^m\mathfrak{g}$ of Killing type. The exposed methods allow us to obtain metrics invariant under the action of $\text{Aut}(\mathfrak{g})$, which will be of interest in the description of coboundary coproducts in Sections 7, 9, and 11.

**Proposition 5.1.** A Killing metric $\kappa_{\mathfrak{g}}$ on $\mathfrak{g}$ is $\text{aut}(\mathfrak{g})$-invariant.

**Proof.** In view of Proposition 4.1, the enunciate of this proposition amounts to proving that $\kappa_{\mathfrak{g}}$ is $GL(\text{ad})$-invariant, which in turn means that
\[ \kappa_{\Lambda^g}(\Lambda T \cdot, \Lambda T \cdot) = \kappa_{\Lambda^g}(\cdot, \cdot), \quad \forall T \in \text{Aut}_c(\mathfrak{g}), \]
where we recall that $\text{Aut}_c(\mathfrak{g})$ is the connected part of the neutral element of $\text{Aut}(\mathfrak{g})$. The Killing metric
is invariant relative to the action of Aut\((\mathfrak{g})\) [23]. If \(v_{J_1}, v_{J_2}\) are decomposable elements of \(\Lambda^m\mathfrak{g}\), then

\[
\kappa_{\Lambda \mathfrak{g}}(\Lambda^m T v_{J_1}, \Lambda^m T v_{J_2}) := \sum_{\sigma_1, \sigma_2 \in S_m} \text{sg}(\sigma_1 \sigma_2) \frac{1}{m!} \prod_{r=1}^{m} \kappa_\mathfrak{g} \left( T v_{J_1(\sigma_1^{-1}(r))}, T v_{J_2(\sigma_2^{-1}(r))} \right)
\]

\[
= \sum_{\sigma_1, \sigma_2 \in S_m} \text{sg}(\sigma_1 \sigma_2) \frac{1}{m!} \prod_{r=1}^{m} \kappa_\mathfrak{g} \left( v_{J_1(\sigma_1^{-1}(r))}, v_{J_2(\sigma_2^{-1}(r))} \right)
\]

\[
= \kappa_{\Lambda \mathfrak{g}}(v_{J_1}, v_{J_2}).
\]

Since \(\kappa_{\Lambda \mathfrak{g}}\) is bilinear and the above is satisfied for decomposable elements of \(\Lambda^m\mathfrak{g}\), which span \(\Lambda^m\mathfrak{g}\), the mapping \(\kappa_{\Lambda \mathfrak{g}}\) is invariant relative to the action of Aut\((\mathfrak{g})\) on \(\Lambda^m\mathfrak{g}\). Since this fact is true for every \(m\), the proposition follows. \(\square\)

Since \(\kappa_{\Lambda \mathfrak{g}}\) is invariant under the maps \(\Lambda T\) with \(T \in \text{Aut}(\mathfrak{g})\), it is therefore invariant under \(\Lambda T\) with \(T \in \text{Inn}(\mathfrak{g})\). In view of Proposition 4.1, the \(\kappa_{\Lambda \mathfrak{g}}\) is also \(\mathfrak{g}\)-invariant.

**Proposition 5.2.** The \(k\)-linear symmetric map on \(\mathfrak{g}\) given by

\[
b(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} \text{Tr}(\text{ad}_{v_{\sigma(1)}} \circ \ldots \circ \text{ad}_{v_{\sigma(k)}}), \quad \forall v_1, \ldots, v_k \in \mathfrak{g},
\]

is \(\text{aut}(\mathfrak{g})\)-invariant.

**Proof.** From Proposition 4.1, the mapping \(b\) is \(\text{aut}(\mathfrak{g})\)-invariant if and only if

\[
b(T v_1, \ldots, T v_k) = b(v_1, \ldots, v_k), \quad \forall T \in \text{Aut}(\mathfrak{g}), \quad \forall v_1, \ldots, v_k \in \mathfrak{g}.
\]

If \(T \in \text{Aut}(\mathfrak{g})\), then

\[
\text{ad}_T v_1 v_2 = [Tv_1, v_2] = [Tv_1, TT^{-1}v_2] = T[v_1, T^{-1}v_2] = T \circ \text{ad}_v \circ T^{-1} w, \quad \forall v_1, v_2 \in \mathfrak{g}.
\]

Therefore,

\[
b(T v_1, \ldots, T v_k) = \sum_{\sigma \in S_k} \text{Tr}(\text{ad}_{Tv_{\sigma(1)}} \circ \ldots \circ \text{ad}_{Tv_{\sigma(k)}})
\]

\[
= \sum_{\sigma \in S_k} \text{Tr}(T \circ \text{ad}_{v_{\sigma(1)}} \circ \ldots \circ \text{ad}_{v_{\sigma(k)}} \circ T^{-1}) = b(v_1, \ldots, v_k)
\]

for arbitrary \(v_1, \ldots, v_k\). \(\square\)

Following the idea of the proof of the latest proposition, we obtain the following corollary.

**Corollary 5.1.** The \(k\)-linear totally anti-symmetric map on \(\mathfrak{g}\) given by

\[
b(v_1, \ldots, v_k) := \sum_{\sigma \in S_k} \text{sg}(\sigma) \text{Tr}(\text{ad}_{v_{\sigma(1)}} \circ \ldots \circ \text{ad}_{v_{\sigma(k)}}), \quad \forall v_1, \ldots, v_k \in \mathfrak{g}
\]

is \(\text{aut}(\mathfrak{g})\)-invariant with respect to \(\widehat{\text{ad}}\).

Finally, let us prove that every Casimir element of order \(k\) gives rise to a \(\mathfrak{g}\)-invariant \(k\)-linear map on \(\mathfrak{g}\).
Definition 5.1. Let \( \{e_1, \ldots, e_r\} \) be a basis of \( \mathfrak{g} \). A polynomial Casimir element of order \( k \) for \( \mathfrak{g} \) is a symmetric element \( C \) of \( \mathfrak{g}^{\otimes k} \), let us say
\[
C := c^{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \quad i_1, \ldots, i_k \in \{1, \ldots, r\}
\]
with totally symmetric coefficients \( c^{i_1 \cdots i_k} \in \mathbb{K} \), satisfying that \( \text{ad}_v^{(k)} C = 0 \) for every \( v \in \mathfrak{g} \).

The following theorem shows that Casimir elements on Lie algebras give rise to \( \mathfrak{g} \)-invariant multilinear symmetric maps. To this concern, we recall that the Killing form \( \kappa \) on \( \mathfrak{g} \) defines a mapping \( \tilde{\kappa} : v \in \mathfrak{g} \mapsto \kappa_{\mathfrak{g}}(v, \cdot) \in \mathfrak{g}^* \) and there exists a natural isomorphism \( \mathfrak{g}^{\otimes k} \cong \left((\mathfrak{g}^*)^{\otimes k}\right)^* \).

Theorem 5.2. Every polynomial Casimir element \( C \) of order \( k \) on a Lie algebra \( \mathfrak{g} \) induces a \( \mathfrak{g} \)-invariant \( k \)-linear symmetric map \( b \) on \( \mathfrak{g} \) given by
\[
b(v_1, \ldots, v_k) := C(\tilde{\kappa}(v_1), \ldots, \tilde{\kappa}(v_k)), \quad \forall v_1, \ldots, v_k \in \mathfrak{g}.
\]

Proof. We have that
\[
\sum_{j=1}^{k} b(v_1, \ldots, \text{ad}_v v_j, \ldots, v_k) = \sum_{j=1}^{k} C(\tilde{\kappa}(v_1), \ldots, \tilde{\kappa}(\text{ad}_v v_j), \ldots, \tilde{\kappa}(v_k)).
\]
Since \( \kappa_{\mathfrak{g}} \) is \( \mathfrak{g} \)-invariant, one gets that
\[
[\text{ad}_v^* \circ \kappa_{\mathfrak{g}}(v_1)](v_2) = \tilde{\kappa}(v_1)(\text{ad}_v v_2) = \kappa_{\mathfrak{g}}(v_1, \text{ad}_v v_2) = -\kappa_{\mathfrak{g}}(\text{ad}_v v_1, v_2) = -[\tilde{\kappa} \circ \text{ad}_v](v_1)(v_2), \quad \forall v, v_1, v_2 \in \mathfrak{g}.
\]
Hence, \( \tilde{\kappa} \circ \text{ad}_v = -\text{ad}_v^* \circ \tilde{\kappa} \) for every \( v \in \mathfrak{g} \). As \( C \) is a Casimir element, one has that \( \text{ad}_v^{(k)} C = 0 \), which along with the above expression gives us
\[
\sum_{j=1}^{k} b(v_1, \ldots, \text{ad}_v v_j, \ldots, v_k) = \sum_{j=1}^{k} C(\tilde{\kappa}(v_1), \ldots, \text{ad}_v^* \tilde{\kappa}(v_j), \ldots, \tilde{\kappa}(v_k))
\]
\[
= -(\text{ad}_v^{(k)} C)(\tilde{\kappa}(v_1), \ldots, \tilde{\kappa}(v_k)) = 0, \quad \forall v_1, \ldots, v_k, v \in \mathfrak{g}.
\]
Therefore, \( b \) is \( \mathfrak{g} \)-invariant. \( \square \)

It stems from the above theorem that if \( \mathfrak{g} \) is semi-simple, then every \( \mathfrak{g} \)-invariant \( k \)-linear symmetric map gives rise to a Casimir element. Even in this latter case, \( \mathfrak{g} \)-invariant multilinear symmetric maps may be more versatile, as they allow for the use of richer techniques, e.g. quadratic forms, which enable us to understand geometrically Casimir elements.

6 On the existence of \( \mathfrak{g} \)-invariant bilinear maps

Although \( \mathfrak{g} \)-invariant maps can be of great utility, it may be difficult to calculate them straightforwardly when \( \mathfrak{g} \) is not a low-dimensional Lie algebra. Next, a series of observations allows us to simplify their calculation. As shown in Sections 3 and 4, our results will enable us to easily determine \( \mathfrak{g} \)-invariant metrics for two- and three-dimensional Lie algebras. Let us start our analysis by the following simple but useful observations.

Proposition 6.1. Let \( b \) be a \( \mathfrak{g} \)-invariant metric on a \( \mathfrak{g} \)-module \( V \). Then, \( b(vx, x) = 0 \) for every \( v \in \mathfrak{g} \) and \( x \in V \).
Proof. Using the \( \mathfrak{g} \)-invariance and symmetricity of \( b \), we get
\[
    b(vx, x) = -b(x, vx) = -b(vx, x), \quad \forall v \in \mathfrak{g}, \quad \forall x \in V.
\]
Therefore, \( b(vx, x) = 0 \) for every \( x \in V \) and \( v \in \mathfrak{g} \).

An anti-symmetric equivalent to the above proposition is given below.

**Proposition 6.2.** Let \( \omega \) be a \( \mathfrak{g} \)-invariant bilinear anti-symmetric map on \( \mathfrak{g} \) relative to the \( \mathfrak{g} \)-module \((\mathfrak{g}, \text{ad})\). Then \( \omega(\text{ad}_v(w), w) = 0 \) for every \( v, w \in \mathfrak{g} \).

*Proof.* From the \( \mathfrak{g} \)-invariance and anti-symmetricity of \( \omega \), it follows that
\[
    \omega(\text{ad}_v(v_2), v_2) = -\omega(\text{ad}_v(v_1), v_2) = \omega(v_1, \text{ad}_v(v_2)) = 0, \quad \forall v_1, v_2 \in \mathfrak{g}.
\]

**Proposition 6.3.** If \( b \) is a \( \mathfrak{g} \)-invariant bilinear map on \( V \) relative to a \( \mathfrak{g} \)-module \((V, \rho)\), then
\[
    b(\text{Im} \rho_v, \ker \rho_v) = 0, \quad \forall v \in \mathfrak{g}.
\]

*Proof.* Let \( v_1 \in \mathfrak{g} \). Every \( v_2 \in \text{Im} \rho_{v_1} \) can be written as \( v_2 := \rho_{v_1}(v_3) \) for a certain \( v_3 \in \mathfrak{g} \). Assume that \( v_4 \in \ker \rho_v \). As \( b \) is \( \mathfrak{g} \)-invariant and symmetric, it turns out that
\[
    b(v_2, v_4) = b(\rho_{v_1}(v_3), v_4) = -b(v_3, \rho_{v_1}(v_4)) = 0.
\]

The following proposition is a generalization of the one above.

**Proposition 6.4.** Let \( b : V \otimes V \to \mathbb{K} \) be a \( \mathfrak{g} \)-invariant bilinear map relative to the \( \mathfrak{g} \)-module \( V \). If \( W \) is a two-dimensional linear subspace of \( V \) satisfying that \( vW \subset W \) for any \( v \in \mathfrak{g} \), then for any linearly independent \( f_s, f_t \in V \) one has
\[
    \text{Tr}(\rho_v|W)b(f_s, f_t)f_t \wedge f_s = b(f_s, f_s)(vf_t) \wedge f_t + b(f_t, f_t)f_s \wedge (vf_s).
\]

*Proof.* Since \( vW \subset W \), there exists constants \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K} \) such that \( vf_s = \alpha_1 f_s + \alpha_2 f_t \) and \( vf_t = \beta_1 f_s + \beta_2 f_t \). The \( \mathfrak{g} \)-invariance of \( b \) ensures that
\[
    \alpha_1 b(f_s, f_t) + \alpha_2 b(f_t, f_t) = b(vf_s, f_t) = -b(f_s, vf_t) = -\beta_1 b(f_s, f_s) - \beta_2 b(f_s, f_t).
\]

After rearranging, the above expression gives the stated formula.

**Example 6.1.** Consider the three-dimensional Heisenberg Lie algebra \( \mathfrak{h} \). Take a basis \( \{e_1, e_2, e_3\} \) of \( \mathfrak{h} \) with commutation relations
\[
    [e_3, e_1] = 0, \quad [e_3, e_2] = 0, \quad [e_1, e_2] = e_3.
\]

Since this Lie algebra is nilpotent, its Killing form vanishes [23, pg. 480]. By Proposition 6.3, an \( \mathfrak{h} \)-invariant bilinear map \( \omega \) on \( \mathfrak{h} \) is associated in the given basis with the matrix
\[
    [\omega] := \begin{pmatrix}
        \alpha_1 & \alpha_2 & 0 \\
        \alpha_3 & \alpha_4 & 0 \\
        0 & 0 & 0
    \end{pmatrix}, \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}.
\]
In the induced bases \( \{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\} \) and \( \{e_1 \wedge e_2 \wedge e_3\} \) of \( \Lambda^2 \mathfrak{h} \) and \( \Lambda^3 \mathfrak{h} \), respectively, the extension of \( \omega \) to \( \Lambda^2 \mathfrak{h} \) and \( \Lambda^3 \mathfrak{h} \) given by Proposition 4.3 reads as
\[
[\omega_{\Lambda^2 \mathfrak{h}}] := \begin{pmatrix}
\alpha_1 \alpha_4 - \alpha_2 \alpha_3 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad [\omega_{\Lambda^3 \mathfrak{h}}] := (0).
\]

Using again Proposition 6.3, we can compute the general \( \mathfrak{h} \)-invariant map on \( \Lambda^2 \mathfrak{h} \)
\[
[\hat{\omega}_{\Lambda^2 \mathfrak{h}}] := \begin{pmatrix}
\beta_3 & \beta_2 & \beta_1 \\
\beta_4 & 0 & 0 \\
\beta_5 & 0 & 0
\end{pmatrix}, \quad \forall \beta_i \in \mathbb{R}.
\]

In a symmetric case, Proposition 6.1 implies \( \beta_1 = \beta_5 = 0 \) and \( \beta_2 = \beta_4 = 0 \). Meanwhile, in the anti-symmetric case, the following form
\[
[\hat{\omega}_{\Lambda^2 \mathfrak{h}}] := \begin{pmatrix}
0 & \beta_1 & 0 \\
-\beta_1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \forall \beta_1 \in \mathbb{R},
\]
can be proved to be \( \mathfrak{h} \)-invariant. Contrary to symmetric forms, antisymmetric ones are not exploited to classify Lie bialgebras. Nevertheless, we expect them to have a relevant geometrical meaning as well.

**Example 6.2.** Let us consider the Lie algebra \( \mathfrak{r}_{3,1} := \langle e_1, e_2, e_3 \rangle \) with commutation relations
\[
[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = 0.
\]
Propositions 6.1 and 6.3 give the following form on \( \Lambda^2 \mathfrak{r}_{3,1} \):
\[
[\omega_{\Lambda^2 \mathfrak{r}_{3,1}}] := \begin{pmatrix}
0 & a & 0 \\
0 & b & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \forall a, b \in \mathbb{R},
\]
in the basis \( \{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\} \) of \( \Lambda^2 \mathfrak{r}_{3,1} \). From Proposition 6.4 and assuming \( v := e_1 \), one gets that \( a = b = 0 \) and there are no \( \mathfrak{r}_{3,1} \)-invariant forms on \( \Lambda^2 \mathfrak{r}_{3,1} \).

## 7 Grading on Grassmann algebras

Let us show that a root decomposition for a Lie algebra \( \mathfrak{g} \) induces a new grading in the Grassmann algebra \( \Lambda \mathfrak{g} \) compatible with its algebraic Schouten bracket. This will be employed in following sections to study the structure and solutions to modified classical Yang-Baxter equations.

Let us assume that
\[
\mathfrak{g} = \bigoplus_{\alpha \in \Delta \setminus \{0\}} \mathfrak{g}^{(\alpha)} \oplus \mathfrak{h},
\]
where \( \Delta \subset \mathfrak{h}^* \) is the set of roots of \( \mathfrak{g} \) and \( \mathfrak{g}^{(\alpha)} \) is the root space associated with \( \alpha \in \Delta \), i.e. \([h, \mathfrak{g}^{(\alpha)}] = \alpha(h)\mathfrak{g}^{(\alpha)} \) for every \( h \in \mathfrak{h} \). Consequently \([\mathfrak{g}^{(\alpha)}, \mathfrak{g}^{(\beta)}] \subset \mathfrak{g}^{(\alpha+\beta)} \) with \( \mathfrak{g}^{(0)} = \mathfrak{h} \) for all \( \alpha, \beta \in \Delta \). Every semi-simple complex Lie algebra admits such a decomposition, called a root decomposition (cf. [36]).

The following theorem shows that a root decomposition on \( \mathfrak{g} \) can be extended to every \( \Lambda^m \mathfrak{g} \) and \( \Lambda \mathfrak{g} \). The induced decomposition on \( \Lambda \mathfrak{g} \) is then compatible with the algebraic Schouten bracket in a manner to be described next.
Definition 7.1. Let $\Delta$ be the root set for a Lie algebra $\mathfrak{g}$. We write $\Delta^m$ for the set of sums of $m$ elements of $\Delta$.

**Theorem 7.2.** Let $\mathfrak{g}$ admit a root decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \setminus \{0\}} \mathfrak{g}^{(\alpha)} \oplus \mathfrak{h}$. Then, there exists an induced decomposition

$$\Lambda^m \mathfrak{g} = \bigoplus_{\alpha \in \Delta^m} \Lambda^m \mathfrak{g}^{(\alpha)}, \quad \Lambda^m \mathfrak{g}^{(\alpha)} := \bigoplus_{\alpha_1 + \cdots + \alpha_m = \alpha} \mathfrak{g}^{(\alpha_1)} \wedge \cdots \wedge \mathfrak{g}^{(\alpha_m)},$$

such that $[h, \Lambda^m \mathfrak{g}^{(\alpha)}]_S = \alpha(h) \Lambda^m \mathfrak{g}^{(\alpha)}$ for every $h \in \mathfrak{h}$ and $\alpha \in \Delta^m$. This decomposition is compatible with the algebraic Schouten bracket, i.e.

$$[\Lambda^p \mathfrak{g}^{(\alpha)}, \Lambda^q \mathfrak{g}^{(\beta)}]_S \subset \Lambda^{p+q-1} \mathfrak{g}^{(\alpha + \beta)}, \quad \forall p, q \in \mathbb{Z}, \forall \alpha, \beta \in \Delta.$$

**Proof.** Let us show that $[h, w]_S = \alpha(h) w$ for any decomposable element $w \in \Lambda^m \mathfrak{g}^{(\alpha)}$ and $h \in \mathfrak{h}$. The space $\Lambda^m \mathfrak{g}^{(\alpha)}$ is spanned by the decomposable vectors $w := v_1 \wedge \cdots \wedge v_m$ for elements $v_j \in \mathfrak{g}^{(\alpha_j)}$, with $j = 1, \ldots, m$, and $\sum_{j=1}^m \alpha_j = \alpha$. Since $[h, ]_S$ is a derivation relative to the exterior product, it follows that

$$[h, w]_S = \sum_{j=1}^m \alpha_j(h) w = \alpha(h) w, \quad \forall h \in \mathfrak{h}.$$

As such decomposable elements $w$ span the whole $\Lambda^m \mathfrak{g}^{(\alpha)}$, the above implies that

$$[h, z]_S = \alpha(h) z, \quad \forall z \in \Lambda^m \mathfrak{g}^{(\alpha)}.$$

Next, we prove that $\Lambda^m \mathfrak{g}$ is the direct sum of subspaces $\Lambda^m \mathfrak{g}^{(\alpha)}$ for all possible $\alpha \in \Delta^m$. On the one hand, the latter subspaces have zero intersection for different values of $\alpha$:

$$z \in \Lambda^m \mathfrak{g}^{(\alpha)} \cap \Lambda^m \mathfrak{g}^{(\beta)} \Rightarrow [h, z]_S = \alpha(h) z = \beta(h) z = (\alpha - \beta)(h) z = 0,$$

If $\alpha \neq \beta$, then there exists $h \in \mathfrak{h}$ such that $(\alpha - \beta)(h) \neq 0$. Thus, $z = 0$ and $\Lambda^m \mathfrak{g}^{(\alpha)} \cap \Lambda^m \mathfrak{g}^{(\beta)} = \{0\}$ for $\alpha \neq \beta$.

On the other hand, a basis of $\Lambda^m \mathfrak{g}$ is given by elements $v_J$, $|J| = m$, with the $v_{J(i)}$ for $i \in \{1, \ldots, m\}$ belonging to a basis adapted to a root decomposition of $\mathfrak{g}$. Every $v_J$ belongs to only one $\Lambda^m \mathfrak{g}^{(\alpha)}$. The elements $v_J$ belonging to the same $\Lambda^m \mathfrak{g}^{(\alpha)}$ give rise to a basis of this space. Hence, $\Lambda^m \mathfrak{g} = \bigoplus_{\alpha \in \Delta^m} \Lambda^m \mathfrak{g}^{(\alpha)}$.

Finally, it is left to prove that the decomposition of $\Lambda \mathfrak{g}$ is compatible with the algebraic Schouten bracket. Let us take elements $w_p \in \Lambda^p \mathfrak{g}^{(\alpha)}$, $w_q \in \Lambda^q \mathfrak{g}^{(\beta)}$. If $h \in \mathfrak{h}$, then

$$[h, [w_p, w_q]]_S = [[h, w_p], w_q]_S + [w_p, [h, w_q]]_S = \alpha(h)[w_p, w_q]_S + \beta(h)[w_p, w_q]_S = (\alpha + \beta)(h)[w_p, w_q]_S.$$

Hence, $[w_p, w_q]_S \subset \Lambda^{p+q-1} \mathfrak{g}^{(\alpha + \beta)}$. \qed

**Example 7.1.** Let us study the Lie algebra $\mathfrak{so}(2, 2) \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ by means of Theorem 7.2. This Lie algebra admits a basis $\{e_-, e_0, e_+ \oplus f_-, f_0, f_+\}$, where $\{e_-, e_0, e_+\}$ and $\{f_-, f_0, f_+\}$ are bases of each copy of $\mathfrak{sl}_2$ within $\mathfrak{so}(2, 2)$. The Lie algebra $\mathfrak{so}(2, 2)$ admits a root decomposition given in Figure 2. Hence, $\mathfrak{h} = \{f_0, e_0\}$. We write $\{e^i, f^i\}$ for the dual basis in $\mathfrak{h}^*$ of $\{e_0, f_0\}$.

Root spaces are denoted by the eigenvalues $(i,j)$ relative to the basis $\{e_0, f_0\}$ of $\mathfrak{h}$. The spaces $\Lambda^2 \mathfrak{so}(2, 2)$ and $\Lambda^3 \mathfrak{so}(2, 2)$ admit decompositions depicted in Figures 3 and 4, respectively. The bases for their root spaces are described exhaustively in Table 1.
belongs to it. Hence, it is a sub-
several interesting cases, to determine the whole \( \Lambda_m^g \) for a characterization of \( \Lambda_m^g \). The space
in the analysis of Lie bialgebras and \( \text{Aut}(\mathfrak{g}) \) [13].

The first part of this section concerns the study of \( \Lambda_m^g \) for Lie algebras with a root decomposition
given by the grading introduced in Section 7. The second part is focused upon the study of \( \mathfrak{g} \)-invariant
elements of the Grassmann algebra related to nilpotent Lie algebras.

Let us begin with a trivial interesting fact.

**Proposition 8.1.** The space \( (\Lambda \mathfrak{g})^\theta \) is a \( K \)-algebra relative to the exterior product.

**Proof.** The space \( (\Lambda \mathfrak{g})^\theta \) can be characterized as the subspace of \( \Lambda \mathfrak{g} \) given by the intersection of the
kernels of derivations of the form \( [v, \cdot]^\theta \), with \( v \in \mathfrak{g} \), relative to the exterior product, i.e. \( (\Lambda \mathfrak{g})^\theta = \bigcap_{v \in \mathfrak{g}} \ker [v, \cdot]^\theta \). As a consequence, \( (\Lambda \mathfrak{g})^\theta \) is a linear space and the exterior product of elements of \( (\Lambda \mathfrak{g})^\theta \)
belongs to it. Hence, it is a sub-\( K \)-algebra of \( \Lambda \mathfrak{g} \) relative to the exterior product.

Proposition 8.1 ensures that the exterior product of elements of \( (\Lambda \mathfrak{m} \mathfrak{g})^\theta, \mathfrak{z}(\mathfrak{g}) \), which belong to
\( (\Lambda \mathfrak{g})^\theta \), give rise to new elements of \( \Lambda \mathfrak{g} \), e.g. \( \Lambda \mathfrak{m} \mathfrak{z}(\mathfrak{g}) \subseteq (\Lambda \mathfrak{m} \mathfrak{g})^\theta \). Although these results do not allow
for a characterization of \( (\Lambda \mathfrak{m} \mathfrak{g})^\theta \), they will be general enough to obtain many of its elements and, in
several interesting cases, to determine the whole \( (\Lambda \mathfrak{m} \mathfrak{g})^\theta \).

**Proposition 8.2.** Each space \( (\Lambda \mathfrak{m} \mathfrak{g})^\theta \) is \( \text{ad}-\)invariant.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( j \backslash i \) & -1 & 0 & 1 & -1 & 0 & 1 \\
\hline
-1 & e_- \wedge f_- & e_0 \wedge f_0 & e_+ \wedge f_- & e_- \wedge e_0 \wedge f_- & e_0 \wedge f_- \wedge f_0 & e_0 \wedge e_+ \wedge f_- \\
0 & e_- \wedge e_0, & e_- \wedge f_0 & e_0 \wedge e_+ & e_- \wedge e_0 \wedge f_0 & e_0 \wedge e_+ \wedge f_0 & e_0 \wedge e_0 \wedge f_0 \\
& e_- \wedge f_0 & e_0 \wedge e_+ & e_0 \wedge f_0 & e_- \wedge e_0 \wedge f_0 & e_0 \wedge e_0 \wedge f_0 & e_0 \wedge e_0 \wedge f_0 \\
1 & e_- \wedge f_+ & e_0 \wedge f_+ & e_+ \wedge f_+ & e_- \wedge e_0 \wedge f_+ & e_0 \wedge f_+ \wedge f_0 & e_0 \wedge e_+ \wedge f_+ \\
& e_0 \wedge f_0 \wedge f_+ & e_- \wedge e_0 \wedge f_+ & e_0 \wedge e_0 \wedge f_0 \wedge f_+ & e_- \wedge e_0 \wedge f_0 \wedge f_+ & e_0 \wedge e_0 \wedge f_0 \wedge f_+ \\
\hline
\end{tabular}
\caption{Bases for the subspaces \( \Lambda^2 \mathfrak{so}(2, 2)^{(i,j)} \) (light gray) and \( \Lambda^3 \mathfrak{so}(2, 2)^{(i,j)} \) (dark gray).}
\end{table}
Proof. In virtue of Proposition 4.1, the claim of this proposition amounts to the fact that \((\Lambda^n g)^g\) is invariant under automorphisms of the form \(\Lambda^m T\) with \(T \in \text{Aut}_c(g)\), where \(\text{Aut}_c(g)\) is the connected part of the neutral element of \(\text{Aut}(g)\). To prove the latter, observe that if \(w \in (\Lambda^n g)^g\) and \(T \in \text{Aut}(g)\), then
\[
[v, \Lambda^m T w]|_S = [TT^{-1}v, \Lambda^m T w]|_S = \Lambda^m T[T^{-1}v, w]|_S = 0, \quad \forall v \in g.
\]
In consequence \(\Lambda^m T w \in (\Lambda^n g)^g\) for every \(T \in \text{Aut}(g)\) and \(w \in (\Lambda^n g)^g\). Hence, \((\Lambda^n g)^g\) is invariant relative to the action of \(\text{Aut}_c(g)\) on it. \(\square\)

In order to develop the following results, it is mandatory to introduce the following relevant notion.

**Definition 8.1.** If \(g\) is a Lie algebra, a **traceless ideal** of \(g\) is an ideal \(h \subset g\) satisfying that the restriction of each \(\text{ad}_v\), with \(v \in g\), to \(h\), say \(\text{ad}_v|h\), is traceless. In particular, \(g\) is called **unimodular** if \(g\) is a traceless ideal of \(g\), namely the adjoint representation of \(g\) is of the form \(\text{ad} : g \to \mathfrak{sl}(g)\).

Unimodular Lie algebras have been applied to several different mathematical and physical problems [2, 39], which motivates its study. There exist several conditions ensuring that a Lie algebra is unimodular [2, 21, 23]. It can be proved that a Lie group admits a right-invariant Haar measure if and only if its Lie algebra is unimodular [23]. This allows us to determine many conditions for the unimodularity of a Lie algebra, namely the Lie algebra of abelian, compact, semi-simple, or nilpotent groups are unimodular [7, 23]. In our work, one of the reasons to study unimodular Lie algebras and their traceless ideals is given by the following proposition.

**Proposition 8.3.** Every traceless ideal \(h\) of a solvable Lie algebra \(g\) induces a subspace \(\Lambda^\dim h h \subset (\Lambda^\dim h g)^g\).

**Proof.** The ideal \(h\) induces a one-dimensional space \(\Lambda^\dim h h\). Let \(r \in \Lambda^\dim h h\). Since \(h\) is a traceless ideal by assumption, \([v, r]|_S = \text{Tr}(\text{ad}_v|h)r = 0\) for every \(v \in g\). Hence, \(r \in (\Lambda^m g)^g\). \(\square\)

Proposition 8.3 is also interesting due to fact that it implicitly says that aforesaid ideals can be determined by decomposable elements of \((\Lambda^\dim h g)^g\), which in turn can be obtained by a family of equations involving the algebraic Schouten bracket. This approach enables us an alternative manner to determine traceless ideals in \(g\) instead of looking for them straightforwardly in \(g\) as standardly done in the literature.

The following theorem gives a relevant clue to determine the \(g\)-invariant elements in \(\Lambda g\).

**Theorem 8.2.** Every non-zero decomposable \(\Omega \in (\Lambda g)^g\) induces a unique ideal \(h \subset g\) such that \(\langle \Omega \rangle = \Lambda^\dim h h\) and every \(\text{ad}_v\), with \(v \in g\), acts tracelessly on \(h\). In turn such a non-zero-dimensional ideal \(h\) gives rise to a subspace \(\Lambda^\dim h h \subset (\Lambda g)^g\) of decomposable elements of \(\Lambda g\).

**Proof.** If one has a non-zero decomposable element \(\Omega \in (\Lambda g)^g\), then \(\Omega = v_1 \wedge \ldots \wedge v_m\) for some linearly independent elements \(v_1, \ldots, v_m \in g\). This gives rise to a unique subspace \(h := \langle v_1, \ldots, v_m \rangle \subset g\) that is independent of the chosen \(v_1, \ldots, v_m\). Moreover, \(\langle \Omega \rangle = \Lambda^\dim h h\). Since \([v, \Omega]|_S = 0\) for every \(v \in g\), it follows that \(h\) is an ideal of \(g\). Let us prove this fact. If \(\widehat{\Omega} : g^* \to \Lambda g\) is given by \(\widehat{\Omega}(\beta) := \iota_\beta \Omega\) for any \(\beta \in g^*, \) then \(\ker \widehat{\Omega} = h^g\). Assuming that \(v \in g\) and \(\theta \in h^o\), we obtain
\[
\iota_{\text{ad}_{\theta}^*} \theta \Omega = \text{ad}_{\theta}^* \theta(v_1) \wedge \ldots \wedge v_m + \ldots + (-1)^m v_1 \wedge \ldots \wedge \text{ad}_{\theta}^* \theta(v_m)
= \theta([v, v_1]) \wedge \ldots \wedge v_m + \ldots + (-1)^m v_1 \wedge \ldots \wedge \theta([v, v_m])
= \iota_{\theta}([v, v_1]) \wedge \ldots \wedge v_m + \ldots + v_1 \wedge \ldots \wedge [v, v_m]
= \iota_{\theta}[v, \Omega]|_S.
\]

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That is, \( t_{ad^*} \theta \Omega = t_\theta [v, \Omega]_S \) for every \( v \in \mathfrak{g} \) and \( \theta \in \mathfrak{h}^\circ \). Then \( ad^* \theta \in \mathfrak{h}^\circ \) for every \( \theta \in \mathfrak{h}^\circ \). Consequently, \( ad_\mathfrak{h} \subset \mathfrak{h} \) for every \( v \in \mathfrak{g} \) and \( \mathfrak{h} \) is an ideal of \( \mathfrak{g} \). In other words, \( \Lambda^{\dim \mathfrak{h}} \mathfrak{h} \subset (\Lambda \mathfrak{g})^\theta \).

Conversely, if \( \mathfrak{h} \) is a non-zero ideal, then \( \Lambda^{\dim \mathfrak{h}} \mathfrak{h} \) is one-dimensional and it admits a basis, \( \Omega \), given by the exterior product of the elements of a basis of \( \mathfrak{h} \). Since every \( ad_\mathfrak{h} \), with \( v \in \mathfrak{g} \), acts on \( \mathfrak{h} \) tracelessly by assumption, \( [v, \Omega]_S = (\text{Tr} \ ad_\mathfrak{h})\Omega = 0 \) and \( \Omega \in (\Lambda^{\dim \mathfrak{h}} \mathfrak{g})^\theta \).

\[ \square \]

**Corollary 8.1.** There exists a one-to-one correspondence between one-dimensional subspaces of decomposable \( \mathfrak{g} \)-invariant elements of \( \Lambda \mathfrak{g} \) and ideals of \( \mathfrak{g} \) where \( \mathfrak{g} \) acts tracelessly.

**Proof.** Let \( D_\mathfrak{g} \) be the set of one-dimensional subspaces of decomposable elements of \( \Lambda \mathfrak{g} \) and let \( \text{Tr}(\mathfrak{g}) \) be the space of non-zero ideals of \( \mathfrak{g} \) where \( \mathfrak{g} \) acts tracelessly. Define the mapping

\[
\phi : D_\mathfrak{g} \to \text{Tr}(\mathfrak{g})
\]

\[
(\Omega) \mapsto \mathfrak{h}_\Omega,
\]

where \( \Omega \) is a \( \mathfrak{g} \)-invariant decomposable element of \( \Lambda \mathfrak{g} \) and \( \mathfrak{h}_\Omega \) is the unique element of \( \text{Tr}(\mathfrak{g}) \) induced by \( \langle \Omega \rangle \) in virtue of Theorem 8.2. The above map is well defined as it does not depend on the element \( \Omega \) spanning the space \( \langle \Omega \rangle \). In turn, \( \Lambda^{\dim \mathfrak{h}_\Omega} \mathfrak{h}_\Omega = \langle \Omega \rangle \). Hence, \( \phi \) has a right inverse. Additionally, an ideal \( \mathfrak{h} \) gives rise in view of Theorem 8.2 to an element of \( \Lambda^{\dim \mathfrak{h}} \mathfrak{h} \) that is \( \mathfrak{g} \)-invariant. In turn this element is related to \( \mathfrak{h} \). Therefore \( \phi \) has a left inverse. This gives the searched isomorphism. \[ \square \]

On the other hand, the aforesaid ideal \( \mathfrak{h} \) in Theorem 8.2 induced by a non-zero \( \mathfrak{g} \)-invariant decomposable multivector acts on itself by the adjoint action tracelessly, i.e. it is unimodular.

The following proposition will be a crux to determine equivalent solutions to modified classical Yang-Baxter equations.

**Proposition 8.4.** The dimension of the orbit of the \( \text{Inn}(\mathfrak{g}) \)-action on \( \Lambda^m \mathfrak{g} \) through \( w \in \Lambda^m \mathfrak{g} \) is \( \dim \text{Im} \Theta^m_w \), with \( \Theta^m_w : v \in \mathfrak{g} \mapsto [v, w]_S \in \Lambda^m \mathfrak{g} \).

**Proof.** Since elements of \( G \) act on \( \mathfrak{g} \) by inner automorphisms, the action of \( G \) on each homogeneous layer of \( \Lambda \mathfrak{g} \) is given by \( g \cdot w := \Lambda^m \text{Ad}_g w \). Define \( \exp(tv) = gv, g_1 := g \) for \( v \in \mathfrak{g} \). Obviously \( \dim G \cdot w = \dim G - \dim G_w \), where \( G_w \) is the isotropy group of the point \( w \in \Lambda^m \mathfrak{g} \). The Lie algebra \( \mathfrak{g}_w \) of \( G_w \) is given by the elements \( v \) of \( \mathfrak{g} \) such that

\[
\left. \frac{d}{dt} \right|_{t=0} \Lambda^m \text{Ad}_g (w) = [v, w]_S = 0 \iff v \in \ker \Theta^m_w.
\]

Hence, the dimension of the orbit is given by \( \dim \mathfrak{g} - \dim \mathfrak{g}_w = \dim \text{Im} \Theta^m_w \). \[ \square \]

### 8.1 Lie algebras with a root decomposition

The grading on \( \Lambda^m \mathfrak{g} \) introduced in Section 7 helps us in characterizing spaces \( (\Lambda^m \mathfrak{g})^\theta \). This is very relevant due to the fact that \( (\Lambda^2 \mathfrak{g})^\theta \) allows us to determine the modified classical Yang-Baxter equations leading to the characterization of coboundary Lie bialgebras [13]. Additionally, \( r \)-matrices differing in an element of \( (\Lambda^2 \mathfrak{g})^\theta \) give rise to the same coproduct (see Proposition 2.1).

The first consequence of the grading introduced in Section 7 is the following proposition, whose proof is immediate.

**Proposition 8.5.** The space \( (\Lambda^m \mathfrak{g})^\theta \) is contained in \( (\Lambda^m \mathfrak{g})^{(0)} \).
As a result of Proposition 8.5, it is only needed to look for elements of \((\Lambda^m g)^g\) within \((\Lambda^m g)^{(0)}\), which is much easier than determining the elements \((\Lambda^m g)^g\) within \(\Lambda^m g\), whose dimension is large in general. Additionally, \((\Lambda^m g)^{(0)}\) can be obtained by means of the root decomposition of \(g\) as shown in the previous section.

Apart from the above finding, it is interesting to analyse the behaviour of the aforesaid spaces relative to the \(g\)-invariant metrics on \(\Delta g\). On the one hand, this illustrates the structure of \(\Lambda^m g\) and provides us with new results that facilitate finding the elements of \((\Lambda^m g)^g\). To show these points, we start by proving the following result.

**Theorem 8.3.** Let \(b\) be a \(g\)-invariant bilinear symmetric map on \(g\). Then, \(b_{\Lambda^m g}\) satisfies that

\[
b_{\Lambda^m g}(v_\alpha, v_\beta) = 0, \quad \forall v_\alpha \in \Lambda^m g^{(\alpha)} , \quad \forall v_\beta \in \Lambda^m g^{(\beta)}
\]

for \(\alpha + \beta \neq 0\).

**Proof.** Consider an element \(h \in \mathfrak{h}\). Since \(b_{\Lambda^m g}\) is \(g\)-invariant, one gets

\[
b_{\Lambda^m g}([h, v_\alpha])_S, v_\beta) = - b_{\Lambda^m g}(v_\alpha, [h, v_\beta])_S, \quad \forall v_\alpha \in \Lambda^p g^{(\alpha)}, \quad \forall v_\beta \in \Lambda^q g^{(\beta)}.
\]

Hence,

\[(\alpha + \beta)(h)b_{\Lambda^m g}(v_\alpha, v_\beta) = 0.\]

If \(\alpha \neq \beta\), then there exists \(h \in \mathfrak{h}\) such that \((\alpha + \beta)(h) \neq 0\) and (8.1) follows. \(\square\)

**Example 8.1.** Let us illustrate Theorem 8.3 for \(\Lambda^2 \mathfrak{so}(2, 2)\). Since \(\mathfrak{sl}_2\) is semi-simple [36], the Cartan criterion ensures that its Killing form is non-degenerate. Therefore, the Killing form \(\kappa_{\mathfrak{so}(2, 2)}\) can be defined as standardly. A simple calculation in the basis \(\{e_-, e_0, e_+, f_0, f_+\}\) of \(\mathfrak{so}(2, 2)\) used in Example 7.1 allows us to ensure that:

\[
\kappa_{\mathfrak{so}(2, 2)}(e_0, e_0) = \kappa_{\mathfrak{so}(2, 2)}(f_0, f_0) = 2, \quad \kappa_{\mathfrak{so}(2, 2)}(e_-, e_+) = \kappa_{\mathfrak{so}(2, 2)}(f_-, f_+) = 2.
\]

The previous calculation enables us to determine an orthogonal basis of \(\Lambda^2 \mathfrak{so}(2, 2)\) relative to the induced form \(\kappa_{\Lambda^2 \mathfrak{so}(2, 2)}\):

\[
\{e_- \wedge f_-, e_+ \wedge f_-, e_0 \wedge f_0, e_0 \wedge f_-, f_- \wedge f_0, f_- \wedge f_+, e_+ \wedge f_+, e_- \wedge f_0, e_- \wedge e_+, e_+ \wedge f_+\}.
\]

One easily sees that this basis satisfies the orthogonality relations determined by Theorem 8.3.

**Corollary 8.2.** If \(g\) is a semi-simple Lie algebra and \(b\) is a non-degenerate bilinear symmetric form on \(g\), then the restriction of \(b_{\Lambda^m g}\) to \((\Lambda^m g)^{(0)}\), \(\Lambda^m g^{(\alpha)} \oplus \Lambda^m g^{(-\alpha)}\) is non-degenerate. If \(b\) is positive-definite or negative-definite, then the restriction of \(b_{\Lambda^m g}\) to \((\Lambda^m g)^g\) is also non-degenerate.

**Proof.** Let us prove both results by reduction to contradiction.

Assume that the restriction of \(b_{\Lambda^m g}\) to \((\Lambda^m g)^{(0)}\) is degenerate. Then, there exists an element of \((\Lambda^m g)^{(0)}\) perpendicular (with respect to \(b_{\Lambda^m g}\)) to every element of this space. In view of Theorem 8.3, this element is perpendicular to the whole \(\Lambda^m g\). This implies that \(b_{\Lambda^m g}\) is degenerate, but this goes against Corollary 4.2. Hence, \(b_{\Lambda^m g}\) is non-degenerate on \((\Lambda^m g)^{(0)}\).

Similarly if an element of \(\Lambda^m g^{(\alpha)} \oplus \Lambda^m g^{(-\alpha)}\) is orthogonal to \(\Lambda^m g^{(\alpha)} \oplus \Lambda^m g^{(-\alpha)}\), then it stems from Theorem 8.3 that such an element is orthogonal to the whole \(\Lambda^m g\), which is a contradiction to our initial hypothesis concerning the non-degeneracy of \(b_{\Lambda^m g}\).
Finally, let us suppose that the restriction of $b_{\Lambda^m g}$ to $(\Lambda^m g)^g$ is degenerate. In virtue of previous paragraphs, $b_{\Lambda^m g}$ is non-degenerate on $(\Lambda^m g)^{(0)}$. Since $b$ is definite, then $b_{\Lambda^m g}$ is definite (see Proposition 4.2) on $(\Lambda^m g)^{(0)}$ and the orthogonal to $(\Lambda^m g)^g$ within this space is also a complementary subspace. Hence, if an element is perpendicular to the whole $(\Lambda^m g)^g$, it will also be perpendicular to the whole $(\Lambda^m g)^{(0)}$ and hence to $\Lambda^m g$. This is a contradiction and it follows that $b_{\Lambda^m g}$ is non-degenerate on $(\Lambda^m g)^g$.

\[\textbf{Example 8.2.}\] Let us illustrate Corollary 8.2 by studying the Lie algebra $\mathfrak{so}(2, 2)$. Let us remind (for more details see Example 7.1) that

$$\Lambda^3 \mathfrak{so}(2, 2)^{(0)} = \langle e_- \wedge e_0 \wedge e_+, e_- \wedge e_+ \wedge f_0, e_0 \wedge f_+ \wedge f_-, f_- \wedge f_0 \wedge f_+ \rangle.$$ 

Since the above was proved to be an orthogonal basis, it follows that the metric $\kappa_{\Lambda^3 \mathfrak{so}(2, 2)}$ is non-degenerate as claimed in Corollary 8.2.

Theorem 8.3 and Corollary 8.2 simplify the determination of elements of $(\Lambda^3 g)^g$. Assume for instance the case of $b$ being positive (or negative) definite. If $w \in (\Lambda^3 g)^g$, then the orthogonal $W$ to $w$ within $(\Lambda^3 g)^{(0)}$ contains a set of elements that along with $w$ span the whole $(\Lambda^3 g)^g$. In other words, it is enough to look for the remaining elements of $(\Lambda^3 g)^g$ within $W$.

### 8.2 Nilpotent Lie algebras

The main feature of nilpotent Lie algebras is that they possess a flag of ideals, called the lower central series of $g$, defined recurrently as $\mathfrak{g}_{(s)} := [\mathfrak{g}, \mathfrak{g}_{(s-1)}]$ for $s \in \mathbb{N}$ with $\mathfrak{g}_{(0)} := \mathfrak{g}$. Then,

$$\mathfrak{g} \supseteq \mathfrak{g}(1) \supset \cdots \supset \mathfrak{g}_{(p-1)} \supset \mathfrak{g}_{(p)} = \{0\}.$$ 

This flag enables us to describe a number of properties of the $g$-invariant elements of $g$. First, the nilpotency of a Lie algebra allows for the characterization of decomposable elements of $\Lambda^m g$ by means of the Lie subalgebras of $g$. This is accomplished in the next proposition.

\[\textbf{Proposition 8.6.}\] If $g$ is a nilpotent Lie algebra, then every non-zero decomposable element of $(\Lambda g)^g$ expands the space $\Lambda^{d \dim h} h$ of a non-zero nilpotent ideal $h$ of $g$.

\[\textbf{Proof.}\] Theorem 8.2 shows that every non-zero decomposable element of $(\Lambda g)^g$ gives rise to a non-zero ideal $h \subseteq g$. Since $g$ is nilpotent, $h$ is nilpotent as well.

Conversely, if $g$ is a nilpotent Lie algebra, then every element $v \in g$ is such that $\text{ad}_v$ is nilpotent on $g$. If $h$ is an ideal, then $\text{ad}_v$ is also nilpotent in $h$. Then, Theorem 8.2 ensures that $\Lambda^{d \dim h} h \subseteq (\Lambda g)^g$ is generated by a decomposable element.

\[\textbf{Proposition 8.7.}\] If $\dim \mathfrak{j}(g) = 1$, then $\mathfrak{j}(g) \wedge \mathfrak{g}_{(p-2)} \subset (\Lambda^2 g)^g$.

\[\textbf{Proof.}\] From the definition of the algebraic Schouten bracket

$$[v, a \wedge b]_S = [v, a] \wedge b + a \wedge [v, b], \quad \forall a, b, v \in g. \tag{8.2}$$

It stems from $\{0\} \neq \mathfrak{g}_{(p-1)} \subset \mathfrak{j}(g)$ and $\dim \mathfrak{j}(g) = 1$ that $\mathfrak{g}_{(p-1)} = \mathfrak{j}(g)$. Hence, $[\mathfrak{g}, \mathfrak{g}_{(p-2)}] = \mathfrak{g}_{(p-1)} = \mathfrak{j}(g)$. Let us assume that $a \in \mathfrak{j}(g)$ and $b \in \mathfrak{g}_{(p-2)}$. Then $[v, b] \in \mathfrak{j}(g)$ and $a \wedge [v, b] = 0$, since $\dim \mathfrak{j}(g) = 1$. As $[v, a] = 0$ and using (8.2), we obtain that $a \wedge b \in (\Lambda^2 g)^g$. \hfill \Box

\[\textbf{Proposition 8.8.}\] For a nilpotent Lie algebra $g$: 

\[\]
1. If \( \dim \mathfrak{z}(\mathfrak{g}) = 2 \), then \( \Lambda^2 \mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}(p-2) \subset (\Lambda^3 \mathfrak{g})^g \).

2. If \( \dim \mathfrak{z}(\mathfrak{g}) = 1 \) and \( \dim \mathfrak{g}(p-2) > 1 \), then \( \mathfrak{z}(\mathfrak{g}) \wedge \Lambda^2 \mathfrak{g}(p-2) \subset (\Lambda^3 \mathfrak{g})^g \).

3. If \( \dim \mathfrak{z}(\mathfrak{g}) = 1 \) and \( \dim \mathfrak{g}(p-2) = 1 \), then \( \mathfrak{z}(\mathfrak{g}) \wedge \mathfrak{g}(p-2) \wedge \mathfrak{g}(p-3) \subset (\Lambda^3 \mathfrak{g})^g \).

**Proof.** Let us remind first that

\[
[v, a \wedge b \wedge c]_S = [v, a] \wedge b \wedge c + a \wedge [v, b] \wedge c + a \wedge b \wedge [v, c], \quad \forall a, b, c, v \in \mathfrak{g}.
\]

Let us prove 1), 2), and 3) by verifying the given inclusions on decomposable elements. The general case follows from it. To prove the first case consider \( a, b \in \mathfrak{z}(\mathfrak{g}) \) and \( c \in \mathfrak{g}(p-2) \). Then \( [v, a \wedge b \wedge c]_S = a \wedge b \wedge c \). Since \( [v, \mathfrak{g}(p-2)] \subset \mathfrak{g}(p-1) \subset \mathfrak{z}(\mathfrak{g}) \) and \( \dim \mathfrak{z}(\mathfrak{g}) = 2 \), it follows that \( a \wedge b \wedge [v, c] \in \Lambda^3 \mathfrak{z}(\mathfrak{g}) = \{0\} \) and \( a \wedge b \wedge c \in (\Lambda^3 \mathfrak{g})^g \).

To prove the second formula, assume \( a \in \mathfrak{z}(\mathfrak{g}) \) and \( b, c \in \mathfrak{g}(p-2) \). Then, \( [v, a \wedge b \wedge c]_S = a \wedge (v, b) \wedge c \). Obviously \( [v, b], [v, c] \in \mathfrak{z}(\mathfrak{g}) \) and \( [v, a \wedge b \wedge c]_S = a \wedge (v, b) \wedge c \). Since \( \dim \mathfrak{z}(\mathfrak{g}) = 1, \), \( \Lambda^2 \mathfrak{z}(\mathfrak{g}) = \{0\} \) and \( a \wedge b \wedge c \in (\Lambda^3 \mathfrak{g})^g \).

The third case is similar to previous ones. Assume \( a \in \mathfrak{g}(p-1) = \mathfrak{z}(\mathfrak{g}), b \in \mathfrak{g}(p-2), c \in \mathfrak{g}(p-3) \). Then, using the assumptions on the dimensions of \( \mathfrak{z}(\mathfrak{g}), \mathfrak{g}(p-2), \mathfrak{g}(p-3) \), we obtain

\[
[v, a \wedge b \wedge c]_S = [v, a] \wedge b \wedge c \quad \forall a, b, c, v \in \mathfrak{g}.
\]

In view of the expression for \( [v, a \wedge b \wedge c]_S \) and previous relations, it follows that \( a \wedge b \wedge c \subset (\Lambda^3 \mathfrak{g})^g \). \( \square \)

### 8.3 Semidirect sum of Lie algebras

This section concerns the analysis of \( \mathfrak{g} \)-invariant multivectors for the semidirect sum of Lie algebras. In particular, the following result allows us to obtain several \( \mathfrak{g} \)-invariant bivectors.

**Proposition 8.9.** Let \( \mathfrak{g} = \mathfrak{g}_1 \oplus_{\mathfrak{h}} \mathfrak{h} \) be a semi-direct sum of Lie algebras \( \mathfrak{g}_1 \) and \( \mathfrak{h} \), where \( \mathfrak{h} \) is nilpotent and \( \varphi : \mathfrak{g}_1 \to \mathfrak{sl}(\mathfrak{h}) \) is a Lie algebra homomorphism. Then,

\[
(\Lambda^\dim \mathfrak{h})^\mathfrak{h} \subset (\Lambda^\dim \mathfrak{h})^\mathfrak{g}.
\]

**Proof.** If \( \mathfrak{h} = 0 \), then the proposition follows trivially. Otherwise, recall that every \( \Omega \in \Lambda^\dim \mathfrak{h}\{0\} \) is an exterior product of the elements of a basis of \( \mathfrak{h} \). Let us choose a basis \( \{v_1, \ldots, v_r\} \) of \( \mathfrak{h} \) and let us write \( \Omega = v_1 \wedge \ldots \wedge v_r \). The Lie bracket in \( \mathfrak{g} \) is given by

\[
[(x, y), (x', y')]_\mathfrak{g} := ([x, x']_{\mathfrak{g}_1}, [y, y']_{\mathfrak{h}} + \varphi_x(y') - \varphi_{x'}(y), \quad \forall (x, y), (x', y') \in \mathfrak{g}.
\]

Therefore,

\[
[z, \Omega]_S = [z, v_1]_\mathfrak{g} \wedge v_2 \wedge \ldots \wedge v_r \wedge \ldots + v_1 \wedge \ldots \wedge [z, v_r]_\mathfrak{g}, \quad \forall z \in \mathfrak{g}.
\]

Since every \( z \in \mathfrak{g} \) can be written as \( z = (x, y) \), where \( x \in \mathfrak{g}_1, y \in \mathfrak{h} \), one gets

\[
[z, v_i]_\mathfrak{g} = [(x, y), (0, v_i)]_\mathfrak{g} = (0, [y, v_i]_{\mathfrak{h}} + \varphi_x(v_i)), \quad \forall v_i \in \mathfrak{h}.
\]

It stems from the \( \mathfrak{h} \)-invariance of \( \Omega \) that

\[
[z, \Omega]_S = (\varphi_x(v_1) + [y, v_1]_{\mathfrak{h}}) \wedge v_2 \wedge \ldots \wedge v_k + \ldots + v_1 \wedge \ldots \wedge (\varphi_x(v_r) + [y, v_r]_{\mathfrak{h}})
\]

\[
= (\text{Tr} \varphi_x + \text{Tr} \text{ad}_y^{(x)}) v_1 \wedge \ldots \wedge v_r = 0,
\]

since elements of \( \mathfrak{sl}(\mathfrak{h}) \) are traceless and \( \mathfrak{h} \) is nilpotent. \( \square \)
9 Reduced modified classical Yang-Baxter equations

Recall that an \( r \in \Lambda^2g \) satisfies the modified classical Yang-Baxter equation if and only if \([r,r]_S \in (\Lambda^3g)g\). If the dimension of \( g \) is higher than three, this equation is computationally very complicated to solve. The next results show a simplification of the modified classical Yang-Baxter equation which mainly concerns not semi-simple Lie algebras \( g \). This is done by replacing \( \Lambda^2g \) with a quotient whose classes give rise to the same coproduct.

**Definition 9.1.** The elements of \( \Lambda^m g/(\Lambda^m g)g \) for \( m \in \mathbb{Z} \) are called reduced \( m \)-vectors.

**Proposition 9.1.** Let \( \pi_p : w_p \in \Lambda^p g \mapsto [w_p] \in \Lambda^p g/(\Lambda^p g)g \), with \( p \in \mathbb{Z} \), be the projection onto the space of reduced \( p \)-vectors. The algebraic Schouten bracket induces a new bracket on

\[
\Lambda_R g := \bigoplus_{p \in \mathbb{Z}} \Lambda_R^p g, \quad \Lambda^p_R g := \Lambda^p g/(\Lambda^p g)g, \quad (9.1)
\]

of the form

\[
[[w_p],[w_q]]_R := [[w_p,w_q]]_S, \quad \forall w_p \in \Lambda^p g, \quad \forall w_q \in \Lambda^q g.
\]

This new bracket, called the reduced Schouten bracket, induces a graded algebra on \( \Lambda_R g \) compatible with \([\cdot,\cdot]_R\) in such a way that \( \pi = \bigoplus_{p \in \mathbb{Z}} \pi_p \) is a homomorphism of graded algebras, i.e., \([\pi(a),\pi(b)]_R = \pi([a,b]_S)\) for arbitrary \( a,b \in \Lambda g \) and \( \pi(\Lambda^p g) \subseteq \Lambda^p_R g \) for any \( p \in \mathbb{Z} \). Then, \( r \in \Lambda^2 g \) is an \( r \)-matrix if and only if \([\pi(r),\pi(r)]_R = 0\).

**Proof.** Let us show that (9.2) is well defined, i.e. its value does not depend on the representative of \([w_p]\) and \([w_q]\). If \([w_p] = [\tilde{w}_p]\) and \([w_q] = [\tilde{w}_q]\) for \( w_p, \tilde{w}_p \in \Lambda^p g \) and \( w_q, \tilde{w}_q \in \Lambda^q g \), then

\[
[[w_p],[w_q]]_R = [\tilde{w}_p,\tilde{w}_q]_S = [[w_p - \tilde{w}_p + w_q - \tilde{w}_q]]_S,
\]

since \( w_p - \tilde{w}_p, w_q - \tilde{w}_q \in (\Lambda g)_S \) and \( ([\Lambda^p g]_S,\Lambda^q g)_S = 0 \), it follows that

\[
[[w_p],[w_q]]_R = [[\tilde{w}_p],[\tilde{w}_q]]_R =: [\tilde{w}_p],[\tilde{w}_q]]_R.
\]

To prove that \( \Lambda_R g \) is a graded algebra relative to the reduced bracket, it is enough to see that the reduced bracket (9.2) satisfies that \( [\Lambda^p_R g,\Lambda^q_R g]_R \subset \Lambda^{p+q-1}_R g \) and

1. \([[[w_p],[w_q]]_R] = -((-1)^{(p-1)(q-1)})[[w_q],[w_p]]_R\),
2. \((-1)^{(p-1)(r-1)}[[w_p],[w_q]]_R + (-1)^{(q-1)(p-1)}[[w_q],[w_r]]_R + (-1)^{(r-1)(q-1)}[[w_r],[w_p]]_R = 0\)

for all \( w_p \in \Lambda^p g, w_q \in \Lambda^q g, w_r \in \Lambda^r g \). \( \Box \)

Let us recall that a coproduct \( \delta \) can be considered as an element of the Chevalley-Eilenberg cohomology of \( g \) taking values in \( \Lambda^2 g \). More specifically, \( \delta \in g^* \otimes \Lambda^2 g \). It is known that \( \delta \) is closed relative to the above-mentioned cohomology [13, 28]. The following proposition shows that there exists a simpler cohomology on \( g^* \otimes \Lambda^2_R g \) allowing us to study the coproducts. In fact, \( \Lambda^2_R g \) is a \( g \)-module as shown in the following lemma.

**Lemma 9.2.** The pair \( \Lambda^m_R g, \sigma : v \in g \mapsto [[v],\cdot]]_R \in \text{End}(\Lambda^m_R g) \) is a \( g \)-module and \( \Psi : T \in \text{Aut}(g) \mapsto [\Lambda^m T] \in \text{GL}(\Lambda^m_R g) \) with

\[
[\Lambda^m T][[w]] = [\Lambda^m T(w)], \quad \forall w \in \Lambda^m g,
\]

is a Lie group action.
Proof. Let us first show that \((\Lambda^p_R \mathfrak{g}, \sigma)\) is a \(\mathfrak{g}\)-module. The mapping \(\sigma\) is well defined and it is a Lie algebra homomorphism because the property 2) of the reduced bracket in the proof of Proposition 9.1 for \(p = 1\), and arbitrary \(q, r\) reduces to
\[
[[w_1], [[w_q], [w_r]]]_R + [[w_q], [[w_r], [w_1]]]_R + (-1)^{(r-1)(q-1)}[[w_r], [[w_1], [w_q]]]_R = 0
\]
and therefore
\[
\sigma(w_1)([[w_q], [w_r]]_R) = [\sigma(w_1)([[w_q], [w_r]]_R) + [[w_q], \sigma(w_1)([w_r])]_R.
\]
This implies that \(\sigma\) is a derivation relative to \([\cdot, \cdot]_R\) and, in consequence, \(\sigma\) is a Lie algebra homomorphism. Hence, \((\Lambda^p_R \mathfrak{g}, \sigma)\) is a \(\mathfrak{g}\)-module.

Let us now prove that \(\text{Aut}(\mathfrak{g})\) gives rise to a well-defined Lie group action on \(\Lambda^m_R \mathfrak{g}\). In this respect, it is only necessary to verify that the mapping \(\Psi(T)\) must be unambiguous. This amounts to proving that if \(w, w' \in [w]\), then \([\Lambda^m T([w])] = \Lambda^m T([w'])\). Note that
\[
\Lambda^m T(w) = \Lambda^m T(w - w + w') = \Lambda^m T(w - w') + \Lambda^m T(w').
\]
Since \(\Lambda^m T(\Lambda^m \mathfrak{g}) \subset (\Lambda^m \mathfrak{g})^\mathfrak{g}\) in virtue of Proposition 8.2, one has that the above expression amounts to the fact that \([\Lambda^m T([w])] = \Lambda^m T([w'])\). Therefore,
\[
[\Lambda^m T([w])] = [\Lambda^m T(w)] = [\Lambda^m T(w')] = [\Lambda^m T([w'])
\]
and \([\Lambda^m T]\) is well-defined.

As a first outcome of Lemma 9.2, the following proposition allows us to obtain \(\mathfrak{g}\)-invariant maps on \(\Lambda^p_R \mathfrak{g}\) out of certain \(\mathfrak{g}\)-invariant maps on \(\Lambda^m \mathfrak{g}\).

Proposition 9.2. If \(b: \Lambda^m \mathfrak{g} \to \mathbb{K}\) is a symmetric or anti-symmetric \(\mathfrak{g}\)-invariant \(k\)-linear map and its kernel contains \((\Lambda^m \mathfrak{g})^\mathfrak{g}\), then the \(k\)-linear map \(b_R\) on \(\Lambda^p_R \mathfrak{g}\) given by
\[
b_R([w_1], \ldots, [w_k]) := b(w_1, \ldots, w_k), \quad \forall w_1, \ldots, w_k \in \Lambda^m \mathfrak{g}, \quad (9.3)
\]
is \(\mathfrak{g}\)-invariant as well.

Proof. The map (9.3) is well defined because if \(w'_i \in [w_i]\) for \(i \in \overline{1, k}\), then the fact that the kernel of \(b\) contains \((\Lambda^m \mathfrak{g})^\mathfrak{g}\) leads to
\[
b_R([w_1], \ldots, [w_k]) = b(w_1, \ldots, w_k) = b(w_1 - w'_1 + w'_1, \ldots, w_k) = b(w'_1, w_2, \ldots, w_k).
\]
Similarly, it follows that
\[
b_R([w_1], \ldots, [w_k]) = b(w'_1, \ldots, w'_{k-1}, w_k) = b(w'_1, \ldots, w_k - w'_k + w'_k) = b(w'_1, \ldots, w'_k) = b_R([w'_1], \ldots, [w'_k])
\]
and the value of \(b_R\) does not depend on the representative of each particular equivalence class of \(\Lambda^p_R \mathfrak{g}\).

The \(\mathfrak{g}\)-invariance of \(b_R\) stems from the following relations
\[
b_R(\sigma(v)[w_1], \ldots, [w_k]) + \ldots + b_R([w_1], \ldots, \sigma(v)[w_k]) = b([v, w_1]_S, \ldots, w_k) + \ldots + b(w_1, \ldots, [v, w_k]_S) = 0.
\]

In view of Lemma 9.2, one can construct the associated Chevalley-Eilenberg cohomology of \(\mathfrak{g}\) with values in \(\Lambda^2_R \mathfrak{g}\).
Proposition 9.3. There is a natural cohomology complex on the spaces $\Lambda^i g^* \otimes \Lambda^2_R g$ making the following diagram commutative:

$$
\begin{array}{c}
\mathbb{K} \otimes \Lambda^2 g \xrightarrow{d} g^* \otimes \Lambda^2 g \xrightarrow{d} \Lambda^2 g^* \otimes \Lambda^2 g \xrightarrow{d} \Lambda^3 g^* \otimes \Lambda^2 g \xrightarrow{d} \ldots \\
\downarrow \text{id} \otimes \pi_2 \downarrow \text{id} \otimes \pi_2 \downarrow \text{id} \otimes \pi_2 \downarrow \text{id} \otimes \pi_2 \\
\mathbb{K} \otimes \Lambda^2_R g \xrightarrow{d_R} g^* \otimes \Lambda^2_R g \xrightarrow{d_R} \Lambda^2_R g^* \otimes \Lambda^2_R g \xrightarrow{d_R} \Lambda^3_R g^* \otimes \Lambda^2_R g \xrightarrow{d_R} \ldots 
\end{array}
$$

Proof. It is enough to see that if $w \in \Lambda^2 g$ and $\theta \in \Lambda^m g^*$, then

$$
d_R(\theta \otimes [w])(v_1, \ldots, v_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} \theta(v_1, \ldots, \hat{v}_i, \ldots, v_{k+1}) \otimes [[v_1, [w]]_R \\
+ \sum_{p,q=1}^{k+1} (-1)^{p+q} \theta([v_p, v_q], v_1, \ldots, \hat{v}_p, \ldots, \hat{v}_q, \ldots, v_{k+1}) \otimes [w],
$$

where the hatted elements are dropped, is $(m+1)$-linear and anti-symmetric. Moreover, $d_R(\theta \otimes [r]) = [d(\theta \otimes r)]$, the commutativity of the diagram is straightforward and $d_R^2 = 0$ stems immediately from the fact that $d^2 = 0$. \hfill \Box

10 On the automorphisms of a Lie algebra

This section addresses the investigation of the group of automorphisms of a Lie algebra and their relations to $\text{Aut}(g)$-invariant metrics on $Ag$. This will be a key to accomplish the classification of solvable Lie bialgebras in Section 11.

As shown in Example 3.2, the space $\text{Aut}(g)$ is a Lie group. This allows us to easily characterize its Lie algebra, $\text{aut}(g)$, as follows.

Proposition 10.1. The Lie algebra $\text{aut}(g)$ is isomorphic to the Lie algebra $\text{der}(g)$ of derivations of $g$.

Proof. The Lie algebra, $\text{aut}(g)$, is spanned by the tangent vectors to curves passing by the neutral element of $\text{Aut}(g)$. Consider a curve $\gamma : t \in \mathbb{R} \mapsto T_t \in \text{Aut}(g)$ within $\text{Aut}(g)$ passing through the neutral element for $t = 0$. Since the family $\{T_t\}_{t \in \mathbb{R}}$ is contained in $\text{Aut}(g)$, one has

$$[T_t(v_1), T_t(v_2)] - T_t[v_1, v_2] = 0 \Rightarrow \left. \frac{d}{dt} \right|_{t=0} ([T_t(v_1), T_t(v_2)] - T_t[v_1, v_2]) = 0, \quad \forall v_1, v_2 \in g.
$$

If we define $D := \left. \frac{d}{dt} \right|_{t=0} T_t \in \text{aut}(g)$, then

$$D([v_1, v_2]) = [D(v_1), v_2] + [v_1, D(v_2)], \quad \forall v_1, v_2 \in g. \quad (10.1)
$$

In other words, $D$ is a derivation of $g$. Conversely, every derivation $D : g \rightarrow g$ induces by exponentiation a curve $\gamma : t \in \mathbb{R} \mapsto T_t := \exp(tD) \in GL(g)$. Since $T_0 = \text{Id}$ and $D$ is a derivation, it follows that $T_t \in \text{Aut}(g)$ for every $t \in \mathbb{R}$. \hfill \Box

Derivations of $g$ can be obtained by determining the linear endomorphisms $T \in \mathfrak{gl}(g)$ satisfying (10.1). As (10.1) is a linear system of equations, it can be solved by using computer programs of mathematical manipulation even for relatively high-dimensional Lie algebras.
Proposition 10.1 also provides information about the connected part of the neutral element of \( \text{Aut}(g) \), namely \( \text{Aut}_c(g) \). Unfortunately, \( \text{Aut}(g) \) need not be connected and the determination of its different connected parts can be tricky.

To illustrate the above claim, consider again the Lie algebra \( \mathfrak{sl}_2 \) analysed in Example 4.3. The Killing metric on \( \mathfrak{sl}_2 \) given by (4.4) in the basis (4.3) is indefinite and non-degenerate with signature (2, 1). The quadratic function on \( \mathfrak{sl}_2 \) induced by this Killing metric is given by \( f(xe_1 + ye_2 + z e_3) = 2x^2 + 4yz \). The surfaces, \( S_k \), where \( f \) takes a constant value \( k \) are given implicitly by \( 2x^2 + 4yz = k \).

If \( k < 0 \) such surfaces are two-sheeted hyperboloids: one of them with elements of \( z > 0 \) and the other one with elements of \( z < 0 \). It is known that the space \( \text{Aut}(\mathfrak{sl}_2) \) consists of isometries of the Killing metric. Therefore, \( \text{Aut}(\mathfrak{sl}_2) \) is contained in the Lie group \( O(2,1) \) of isometries of the Killing metric on \( \mathfrak{sl}_2 \). The group \( O(2,1) \) has four non-connected parts and two of them, the \( O(2,1) \), having determinant one. The connected part of the neutral element leaves invariant the elements of each component of the two-sheet hyperboloid, while the second one does not. The element \( T \in \text{Aut}(\mathfrak{sl}_2) \), written in a basis (4.3) as

\[
[T] := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},
\]

does not preserve the sign of the coordinate \( z \). Consequently, it does not belong to the connected part of the neutral element of \( SO(2,1) \). Therefore \( T \notin \text{Aut}_c(\mathfrak{sl}_2) \) and \( \text{Aut}(\mathfrak{sl}_2) \) is not connected. Since \( \text{Inn}(\mathfrak{g}) \) is connected, \( \text{Inn}(\mathfrak{sl}_2) \neq \text{Aut}(\mathfrak{sl}_2) \). This shows that the assumption \( \text{Inn}(\mathfrak{sl}_2) = \text{Aut}(\mathfrak{sl}_2) \), made in [19], is incorrect.

The above example illustrates that \( \text{Aut}(\mathfrak{g}) \)-invariant metrics on a Lie algebra \( \mathfrak{g} \) can unveil important properties concerning \( \text{Aut}(\mathfrak{g}) \). These metrics are easier to obtain than the whole group \( \text{Aut}(\mathfrak{g}) \). It will be shown in Section 11 that this will frequently be enough to characterize coboundary real three-dimensional Lie bialgebras.

To use the above fact in practical applications is convenient to enunciate the following result.

**Theorem 10.1.** If \( b \) is a \( k \)-linear map on \( \mathfrak{g} \) and invariant under \( \text{Aut}(\mathfrak{g}) \), then its extensions to \( \Lambda^m \mathfrak{g} \) are invariant under the action of \( \text{Aut}(\mathfrak{g}) \).

The crux to us is that if \( b \) is a \( k \)-linear symmetric metric on \( \mathfrak{g} \) invariant relative to \( \text{Aut}(\mathfrak{g}) \), then the spaces \( S_k \) where the associated polynomial for an extension \( b_{\Lambda^m \mathfrak{g}} \) to \( \Lambda^m \mathfrak{g} \), namely

\[
p(v) := b^{k\text{-times}}_{\Lambda^m \mathfrak{g}}(v, \ldots, v), \quad \forall v \in \Lambda^m \mathfrak{g},
\]

takes a constant value \( k \) are invariant under the action of \( \text{Aut}(\mathfrak{g}) \) on \( \Lambda^m \mathfrak{g} \). The orbits of \( \text{Aut}(\mathfrak{g}) \) on \( \Lambda^m \mathfrak{g} \) need not be connected, but they must be contained in a single \( S_k \). Using that \( \text{Inn}(\mathfrak{g}) \) can be easily obtained and it gives information on the connected component of \( \text{Aut}(\mathfrak{g}) \), the action of the whole \( \text{Aut}(\mathfrak{g}) \) can be obtained by searching elements connecting the different orbits of \( \text{Inn}(\mathfrak{g}) \) within the same \( S_k \). This process will be illustrated in Section 11.

Let us now provide hints to characterize automorphisms for Lie algebras. More specifically, let us analyse properties of \( \Lambda^2 T \) for every \( T \in \text{Aut}(\mathfrak{g}) \).

In the case of complex simple or semi-simple Lie algebras, the space of Lie algebra automorphisms is determined by the inner automorphisms of the Lie algebra, which already had a nice characterization in this work, and the Dynkin diagram [23, 26]. Meanwhile, automorphisms of general Lie algebras cannot be determined so easily. In particular, we focus upon automorphisms of solvable and nilpotent Lie algebras, which are by far the most complicated ones to be determined.
Consider for instance a solvable or nilpotent Lie algebra \( \mathfrak{g} \). Hence we have the derived and lower central series defined recurrently as

\[
\mathfrak{g}^{(p)} := [\mathfrak{g}^{(p-1)}, \mathfrak{g}^{(p-1)}], \quad \mathfrak{g}(p) := [\mathfrak{g}, \mathfrak{g}(p-1)], \quad \mathfrak{g}^{(0)} := \mathfrak{g}, \quad \forall p \in \mathbb{N}.
\]

The spaces \( \mathfrak{g}^{(p)} \) and \( \mathfrak{g}(p) \) are ideals [23]. Moreover, if \( T \in \text{Aut}(\mathfrak{g}) \), then \( T\mathfrak{g} = \mathfrak{g} \) and \( T\mathfrak{g}(p) = [T\mathfrak{g}, T\mathfrak{g}(p-1)] = [\mathfrak{g}, T\mathfrak{g}(p-1)] \). By induction, \( T\mathfrak{g}(p) = \mathfrak{g}(p) \) for \( p \in \mathbb{N} \cup \{0\} \). A similar result applies to derived series.

Given a solvable Lie algebra, its automorphisms leave invariant the elementary sequence

\[
\mathfrak{s}_{pq} := \mathfrak{g}(p) \wedge \mathfrak{g}(q), \quad p \leq q, \quad p, q \in \mathbb{N} \cup \{0\}.
\]

If \( \mathfrak{s}_{pq} \neq 0 \), then \( \mathfrak{s}_{pq} \supset \mathfrak{s}_{lm} \) if and only if \( p \leq l \) and \( q \leq m \). Similar results apply to the spaces

\[
\mathfrak{s}^{pq} := \mathfrak{g}^{(p)} \wedge \mathfrak{g}^{(q)}, \quad p \leq q,
\]

for \( p, q \in \mathbb{N} \cup \{0\} \). Taking into account above relations, the form of \( \Lambda^2 T \) can be estimated.

11 Study of real three-dimensional coboundary Lie bialgebras

This section exploits the techniques devised in previous sections to analyse and to classify, up to Lie algebra automorphisms, coboundary real three-dimensional Lie bialgebras. Instead of accomplishing their classification by deriving all their automorphisms, which is complicated (cf. [19]), our techniques focus upon the classification up to inner Lie algebra automorphisms, which can be accomplished rather easily in an algorithmic way. Next, the derivation of a few simple outer Lie algebra automorphisms leads to the final classification.

Our results retrieve in a more geometric manner findings described in [19, 22]. We solve minor gaps in the previous literature, and we provide a new unifying simplified approach. On the other hand, this section illustrates our previous methods, and it paves the way for determining further useful structures in the classification and investigation of Lie bialgebras. In fact, findings of previous sections stem from our attempt of improving previous classifications of three-dimensional real Lie bialgebras. To this concern, this case is interesting because it is simple enough to obtain the classification in several ways while showing general properties of higher-dimensional Lie bialgebras.

11.1 General properties

Before addressing the study of each particular three-dimensional real coboundary Lie bialgebra, we prove a few results concerning the characterization of the subspaces \((\Lambda^m \mathfrak{g})^\theta\) for a general real three-dimensional Lie algebra \( \mathfrak{g} \). If not otherwise stated, we assume henceforth that all Lie algebras are three-dimensional. If \( \{e_1, \ldots, e_r\} \) is a basis of \( \mathfrak{g} \), then we define \( e_{i_1 \ldots i_r} := e_{i_1} \wedge \ldots \wedge e_{i_r} \), with \( i_1, \ldots, i_r \in \overline{1,r} \), to shorten the notation.

**Lemma 11.1.** Every \( w \in \Lambda^2 \mathfrak{g} \) is decomposable and induces an even-dimensional subspace \( E_w := \{\iota_\theta w : \theta \in \mathfrak{g}^*\} \). The subspaces of the form \( E_w \) are the only even-dimensional ideals of \( \mathfrak{g} \) on which every ad-\( v \), with \( v \in \mathfrak{g} \), acts tracelessly.

**Proof.** Let us prove the first part of the lemma. It is a well-known fact that every \( w \in \Lambda^2 \mathfrak{g} \) induces a unique mapping \( \tilde{w} : \theta \in \mathfrak{g}^* \mapsto \iota_{\theta w} \in \mathfrak{g} \) whose image, \( \text{Im} \tilde{w} = E_w \), is even-dimensional. Then \( E_w \) can be zero- or two-dimensional. If \( E_w \) is zero-dimensional, then \( \tilde{w} = 0 \) and \( w = 0 \) is decomposable. If \( E_w \) is two-dimensional, then one finds that \( w = v_1 \wedge v_2 \), where \( \{v_1, v_2\} \) is a certain basis of \( E_w \).
Let us prove the second part of the lemma. We examine separately the cases when $E_0$ is two- and zero-dimensional. The zero $0 \in \Lambda^2 g$ induces a zero-dimensional ideal $E_0$ where the $\text{ad}_v$ with $v \in g$ act tracelessly. Conversely, a zero-dimensional space $h = \{0\}$ is an ideal where all $\text{ad}_v$ with $v \in g$ act tracelessly, and $h = E_0$.

Every non-zero $w \in (\Lambda^2 g)^g$ is decomposable and, in virtue of Theorem 8.2, it induces a unique ideal $h \subset g$ where every $\text{ad}_v$, with $v \in g$, acts tracelessly and $\langle w \rangle = \Lambda^\dim h$. As $E_w$ satisfies that $\langle w \rangle = \Lambda^\dim E_w$, it follows that $E_w = h$. The converse stems immediately from Theorem 8.2.

**Proposition 11.1.** A three-dimensional Lie algebra $g$ satisfies that $\Lambda^3 g$ admits a $g$-invariant metric if and only if $\Lambda^3 g = (\Lambda^3 g)^g$.

**Proof.** Let $\{x, y, z\}$ be a basis for $g$. Then $\Lambda^3 g = \langle x \wedge y \wedge z \rangle$. If $\Lambda^3 g$ allows for a $g$-invariant metric $g_3$, then one has for a non-zero element $\theta \in \Lambda^3 g$:

$$g_3([v, \theta]s, \theta) + g_3(\theta, [v, \theta]s) = 2g_3(\theta, [v, \theta]s) = 0, \quad \forall v \in g \implies [v, \theta]S = 0, \quad \forall v \in g.$$

Hence, $\Lambda^3 g = (\Lambda^3 g)^g$. The converse is immediate.

It follows from the former proposition that a three-dimensional Lie algebra $g$ does not admit a $g$-invariant metric if and only if $0 = (\Lambda^3 g)^g$. Since $[v, \theta]S = \text{Tr}(\text{ad}_v)\theta$ for every $v \in g$ and $\theta \in \Lambda^3 g$, one has that $(\Lambda^3 g)^g \neq 0$ if and only if the adjoint representation of $g$ is traceless. Recall that a Lie algebra satisfying that the adjoint representation is traceless is called *unimodular*. This motivates the following corollary.

**Corollary 11.1.** Let $g$ be a three-dimensional Lie algebra. Then, $\Lambda^3 g = (\Lambda^3 g)^g$ if and only if $g$ is unimodular.

**Proposition 11.2.** Let $g$ be a unimodular three-dimensional Lie algebra. Every non-zero element $w \in (\Lambda^2 g)^g$ determines a two-dimensional ideal $E_w := \{\iota_\theta w : \theta \in g^*\}$ containing $[g, g]$.

**Proof.** Since $g$ is unimodular, Corollary 11.1 ensures that $\Lambda^3 g = (\Lambda^3 g)^g$. If $w \in (\Lambda^2 g)^g \setminus \{0\}$ and $v \in g$, then $w \wedge v \in (\Lambda^3 g)^g$. Hence,

$$0 = [z, w \wedge v]S = [z, w]S \wedge v + w \wedge [z, v] = w \wedge [z, v], \quad \forall z \in g.$$

As $w \neq 0$ by assumption, one has that $[z, v] \in E_w$ for arbitrary $z, v \in g$. In other words, $[g, g] \subset E_w$. 

**Corollary 11.2.** Every perfect three-dimensional Lie algebra $g$ satisfies that $(\Lambda^2 g)^g = 0$.

**Proof.** A perfect Lie algebra is a Lie algebra satisfying that $[g, g] = g$. By Proposition 11.2 every non-zero element $w \in (\Lambda^2 g)^g$ induces a two-dimensional space $E_w$ containing $[g, g] = g$, which is therefore three-dimensional. Therefore, $(\Lambda^2 g)^g = \{0\}$.

Finally, let us provide some results concerning the description of Lie algebra automorphisms that will be useful in the study of three-dimensional solvable Lie bialgebras. In particular, we want to know enough about the Lie algebra automorphisms of a Lie algebra $g$ so as to classify Lie bialgebras on $g$ without working out the explicit form of all $\text{Aut}(g)$.

**Proposition 11.3.** Let $g$ be a three-dimensional Lie algebra with a non-zero Killing metric and a two-dimensional abelian subalgebra $g(1) := [g, g]$ contained in the kernel of the Killing metric. If $v \notin g(1)$, then every $T$ leaves invariant the set of eigenvectors of $\text{ad}_v|_{g(1)}$. 

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Proof. Let us prove that if \( T \in \text{Aut}(g) \), then \( Tv \in v + g_{(1)} \) or \( Tv \in -v + g_{(1)} \) for every \( v \notin g_{(1)} \). In this respect, since \( T \in \text{Aut}(g) \), one has that

\[
\kappa_g(Tv,Tv) = \kappa_g(v,v), \quad \forall v \in g.
\]

Since \( v \) and \( g_{(1)} \) generate \( g \), we can write \( Tv = \lambda v + h \) for an \( h \in g_{(1)} \) and \( \lambda \in \mathbb{K} \). As \( g_{(1)} \) is contained in the radical of the Killing metric, \( \kappa_g \), of \( g \) by assumption, one has

\[
\kappa_g(Tv,Tv) = \lambda^2 \kappa_g(v,v), \quad \forall v \in g.
\]

The fact that \( \kappa_g \) is not zero identically by assumption ensures that \( \lambda \in \{ \pm 1 \} \). Hence, \( Tv \in v + g_{(1)} \) or \( Tv \in -v + g_{(1)} \). Since \( g_{(1)} \) is a two-dimensional abelian ideal of \( g \) and it is invariant under automorphisms of \( g \), one obtains that

\[
\text{ad}_v|_{g(1)} = \pm \text{ad}_Tv|_{g(1)} \implies \text{ad}_v|_{g(1)} = \pm T|_{g(1)} \circ \text{ad}_v|_{g(1)} \circ T^{-1}|_{g(1)}.
\]

In consequence, if \( e \) is an eigenvector of \( \text{ad}_v|_{g(1)} \), then \( T|_{g(1)} e \) is a new eigenvector of \( \text{ad}_v|_{g(1)} \). \( \Box \)

Proposition 11.3 can be modified to give a very accurate form of \( T|_{g(1)} \). For instance, if \( \text{ad}_v|_{g(1)} \) has two eigenvectors \( e_1, e_2 \) with different eigenvalues \( \lambda_1, \lambda_2 \) satisfying that \( \lambda_1 + \lambda_2 = 0 \) and \( Tv \in -v + g_{(1)} \), then \( T|_{g(1)} \) must be an anti-diagonal matrix in the basis \( \{ e_1, e_2 \} \). If \( \lambda_1 + \lambda_2 \neq 0 \) and \( \lambda_1 \neq \lambda_2 \), then \( Tv \in v + g_{(1)} \) and \( T|_{g(1)} \) is diagonal. Several variations of this reasoning can be applied, e.g. to the case when \( \text{ad}_v|_{g(1)} \) is triangular.

**Proposition 11.4.** Let \( g \) be a three-dimensional real Lie algebra and let \( \Omega \in (\Lambda^3 g^*) \backslash \{0\} \). Assume that \( \Upsilon : \Lambda^2 g \ni r \mapsto \Omega([r,r]_S) \in \mathbb{R} \) is a semi-definite function and it does not vanish identically. Then, every automorphism of \( g \) has positive determinant.

**Proof.** Since \( g \) is three-dimensional, \( \Omega \) is a basis of \( \Lambda^3 g^* \) and there exists a one-element dual basis \( \theta \in \Lambda^3 g \). As \( \Omega([r,r]_S) = \Upsilon(r) \), then \( [r,r]_S = \Upsilon(r) \theta \). Since \( \Upsilon \) is not identically zero, there exists an \( r \in \Lambda^2 g \) such that \( [r,r]_S = \Upsilon(r) \theta \neq 0 \). If \( T \in \text{Aut}(g) \), then

\[
\Upsilon(r) \det(T) \theta = \det(T) [r,r]_S = \Lambda^3 T[r,r]_S = [\Lambda^2 Tr, \Lambda^2 Tr]_S = \Upsilon(\Lambda^2 Tr) \theta.
\]

Hence, \( \Upsilon(r) \det(T) = \Upsilon(\Lambda^2 Tr) \neq 0 \). Due to the semi-definiteness of \( \Upsilon \), both sides of the equality must have the same sign and \( \det(T) > 0 \). \( \Box \)

### 11.2 Simple Lie algebras

This subsection aims to determine and to analyse all coboundary Lie bialgebra structures on three-dimensional real simple Lie algebras. There exists only two such Lie algebras: \( su_2 \) and \( sl_2 \) [42]. As commented at the beginning of Section 11, we first study the classification problem up to inner automorphisms, and then simple remarks allow us to accomplish the classification up to automorphisms.

#### 11.2.1 Lie bialgebras on \( sl_2 \)

Let us consider again the basis \( \{ e_1, e_2, e_3 \} \) of \( sl_2 \) adapted to its natural root decomposition given in (4.3). Choose also the induced bases \( \{ e_{12}, e_{13}, e_{23} \} \) and \( \{ e_{123} \} \) in \( \Lambda^2 sl_2 \) and \( \Lambda^3 sl_2 \), respectively. Since every \( \text{ad}_{e_i} \), with \( v \in sl_2 \), is traceless, the Lie algebra \( sl_2 \) is unimodular and Corollary 11.1 ensures that \( (\Lambda^3 sl_2)^{sl_2} = \Lambda^3 sl_2 \). Consequently, every \( r \in \Lambda^2 sl_2 \) is a solution to the modified classical Yang-Baxter equation on \( \Lambda^2 sl_2 \).
Let us analyse \((\Lambda^2\mathfrak{sl}_2)^{\mathfrak{sl}_2}\), which allows us, in virtue of Proposition 2.1, to determine those \(r \in \Lambda^2\mathfrak{sl}_2\) giving rise to different coproducts \(\delta_r\). Since \([\mathfrak{sl}_2, \mathfrak{sl}_2] = \mathfrak{sl}_2\), Corollary 11.2 guarantees that \((\Lambda^2\mathfrak{sl}_2)^{\mathfrak{sl}_2} = 0\). Alternatively, this fact can be retrieved by recalling that the root decomposition for \(\mathfrak{sl}_2\) (see Figure 5) induces a root decomposition on \(\Lambda^2\mathfrak{sl}_2\) and \(\Lambda^3\mathfrak{sl}_2\) (see Section 7 and Figures 6 and 7).

In view of Proposition 8.5 and Figure 6, it follows that \((\Lambda^2\mathfrak{sl}_2)^{\mathfrak{sl}_2} \subset (\Lambda^2\mathfrak{sl}_2)^{(0)} = \langle e_{23} \rangle\). It is therefore simple to see that \((\Lambda^2\mathfrak{sl}_2)^{\mathfrak{sl}_2} = 0\). In consequence, every \(r \in \Lambda^2\mathfrak{sl}_2\), which is always a solution to the modified classical Yang-Baxter equations, induces a different coproduct \(\delta_r(\cdot) := [\cdot, r]_{\mathfrak{s}_2}\).

Now, we tackle the problem of determining equivalence classes of coboundary coproducts on \(\mathfrak{sl}_2\) up to inner Lie algebra automorphisms of \(\mathfrak{sl}_2\) via \(\mathfrak{sl}_2\)-invariant metrics. In the above-mentioned bases of \(\Lambda^2\mathfrak{sl}_2\) and \(\Lambda^3\mathfrak{sl}_2\), the Killing metric and its extensions to \(\Lambda^2\mathfrak{sl}_2\) and \(\Lambda^3\mathfrak{sl}_2\), given by Theorem 4.3, take the form (4.4). It is worth recalling that since \(\mathfrak{sl}_2\) is simple, it admits a non-degenerate Killing metric. Therefore, Corollary 4.2 ensures that the extended metrics on \(\Lambda^2\mathfrak{sl}_2\) and \(\Lambda^3\mathfrak{sl}_2\) must be also non-degenerate. This agrees with the explicit form of extended metrics showed in (4.4). Moreover, the existence of the \(\mathfrak{sl}_2\)-invariant metric \(\kappa_{\Lambda^2\mathfrak{sl}_2}\) can be also explained in view of Proposition 11.1, the fact that \(\mathfrak{sl}_2\) is unimodular, and Figure 7.

A simple calculation and Proposition 8.4 ensure that the dimension of the orbits of the action of \(\text{Inn}(\mathfrak{sl}_2)\) on \(\Lambda^2\mathfrak{sl}_2\) is given by \(\dim \Theta_w^2 = 2\) for \(w \in \Lambda^2\mathfrak{sl}_2 \setminus \{0\}\) and \(\dim \Theta_w^3 = 0\) otherwise. In view of this and since \(\text{Inn}(\mathfrak{sl}_2)\) is connected, the orbits of the action of \(\text{Inn}(\mathfrak{sl}_2)\) on \(\Lambda^2\mathfrak{sl}_2\) are two- or zero-dimensional connected immersed submanifolds. Each orbit must be contained in a connected submanifold of a region \(S_k\), where the quadratic function \(f_{\Lambda^2\mathfrak{sl}_2} : \Lambda^2\mathfrak{sl}_2 \to \mathbb{R}\) related to the \(\mathfrak{sl}_2\)-invariant metric \(\kappa_{\Lambda^2\mathfrak{sl}_2}\) takes a constant value \(k\). Each \(S_k\) must be the union of orbits of the action of \(\text{Inn}(\mathfrak{sl}_2)\) and on each maximal submanifold of \(S_k\) such orbits are open and closed. Therefore, each orbit of \(\text{Inn}(\mathfrak{sl}_2)\) represents a maximal (in the sense of inclusion) connected submanifold within \(S_k\).

In coordinates, \(r = x e_{12} + y e_{13} + z e_{23}\) and \(f_{\Lambda^2\mathfrak{sl}_2}(r) := \kappa_{\Lambda^2\mathfrak{sl}_2}(r, r) = 8xy - 4z^2\). Then, \(f_{\Lambda^2\mathfrak{sl}_2}\) admits three types of spaces \(S_k\) according to the sign of \(k\). If \(k < 0\), then \(S_k\) is a one-sheeted hyperboloid; \(S_0\) consists of two cones, one opposite to the other, and the origin on \(\Lambda^2\mathfrak{sl}_2\); meanwhile \(S_k\) for \(k > 0\) is a two-sheeted hyperboloid (see Figure 8).

Consequently, there are five inequivalent classes of \(r\)-matrices on \(\mathfrak{sl}_2\), relative to the action of \(\text{Inn}(\mathfrak{sl}_2)\) (cf. [19]). The representatives of each class are \(r_0 = 0\), \(r = ae_{23}\), with \(a > 0\) (one-sheeted hyperboloids), \(r = a(e_{12} + e_{13})\), with \(a \in \mathbb{R} \setminus \{0\}\), (two-sheeted hyperboloids), and \(r = \pm e_{12}\) (cones).

The orbits of the action of \(\text{Aut}(\mathfrak{sl}_2)\) on \(\Lambda^2\mathfrak{sl}_2\) can be immediately obtained from the above characterisation. All derivations of \(\mathfrak{sl}_2\) are of the form \(\text{ad}_v\), for a certain \(v \in \mathfrak{sl}_2\) [26]. In view of Proposition 10.1, \(\text{inn}(\mathfrak{sl}_2) = \text{der}(\mathfrak{sl}_2) = \text{aut}(\mathfrak{sl}_2)\). Hence, \(\text{Inn}(\mathfrak{sl}_2)\) is the connected component of \(\text{Aut}(\mathfrak{sl}_2)\) corresponding to the neutral element, each orbit of the action of \(\text{Aut}(\mathfrak{sl}_2)\) in \(\Lambda^2\mathfrak{sl}_2\) is the sum of one or
several orbits of the action of \text{Inn}(\mathfrak{s}l_2). As the Killing metric is invariant under the action of \text{Aut}(\mathfrak{s}l_2), i.e. it is \text{GL}(\mathfrak{ad})-invariant, Corollary 4.1 ensures that the extended metric \(\kappa_{\mathfrak{su}^2}\) is invariant under the extended action of \text{Aut}(\mathfrak{s}l_2) on \(\Lambda^2\mathfrak{s}l_2\) and each one of its orbits must be contained in a single \(\mathcal{S}_k\).

The automorphism \(T\) of \(\mathfrak{s}l_2\) given by

\[
T(e_1) := e_1, \quad T(e_2) := -e_2, \quad T(e_3) := -e_3
\]

can be extended to \(\Lambda^2 T\) giving rise to a map

\[
\Lambda^2 T(e_{12}) = -e_{12}, \quad \Lambda^2 T(e_{13}) = -e_{13}, \quad \Lambda^2 T(e_{23}) = e_{23}.
\]

The map \(\Lambda^2 T\) connects the two-sheeted hyperboloids within \(\mathcal{S}_k\) for each fixed \(k > 0\). It also maps the two surfaces contained in the cone \(\mathcal{S}_0\). Therefore, we have three types of \(r\)-matrices up to the action of \text{Aut}(\mathfrak{s}l_2).

Our result agrees with the findings given in [22], but they do not match the work [19]. This is due to the fact that Farinati and coworkers assume that \text{Inn}(\mathfrak{s}l_2) = \text{Aut}(\mathfrak{s}l_2) (see [19, p. 56]), which was proved to be wrong.

**11.2.2 Lie bialgebras on \(\mathfrak{su}_2\)**

Let us analyse \(\mathfrak{su}_2\). Consider a basis \(\{e_1, e_2, e_3\}\) of \(\mathfrak{su}_2\) such that

\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.
\]

To study the Lie bialgebras of \(\mathfrak{su}_2\), it is relevant to analyse \((\Lambda^2 \mathfrak{su}_2)_{\mathfrak{su}_2}\) and \((\Lambda^3 \mathfrak{su}_2)_{\mathfrak{su}_2}\). Since the adjoint representation of \(\mathfrak{su}_2\) is traceless, Proposition 11.1 shows that \((\Lambda^3 \mathfrak{su}_2)_{\mathfrak{su}_2} = \Lambda^3 \mathfrak{su}_2\) and every \(r \in \Lambda^2 \mathfrak{su}_2\) satisfies the modified classical Yang-Baxter equation on \(\Lambda^2 \mathfrak{su}_2\). Meanwhile, \(\mathfrak{su}_2\) is simple, thus \([\mathfrak{su}_2, \mathfrak{su}_2] = \mathfrak{su}_2\) and Corollary 11.2 entails that \((\Lambda^2 \mathfrak{su}_2)_{\mathfrak{su}_2} = \{0\}\). Hence, every \(r \in \Lambda^2 \mathfrak{su}_2\) induces a different coproduct and the classification of coboundary coproducts of \(\mathfrak{su}_2\) up to its Lie algebra automorphisms amounts to the classification of their corresponding \(r\)-matrices.

Let us study the equivalence of \(r\)-matrices under inner automorphisms by using \(\mathfrak{su}_2\)-invariant metrics on \(\Lambda^2 \mathfrak{su}_2\), \(\Lambda^3 \mathfrak{su}_2\), and \(\Lambda^3 \mathfrak{su}_2\). The Killing metric of \(\mathfrak{su}_2\) and its extensions are given by the matrices

\[
[k_{\mathfrak{su}_2}] = -2 \mathbb{I}_{3 \times 3}, \quad [k_{\Lambda^2 \mathfrak{su}_2}] = 4 \mathbb{I}_{3 \times 3}, \quad [k_{\Lambda^3 \mathfrak{su}_2}] = -8 \mathbb{I}_{1 \times 1}
\]

in the bases induced by \(\{e_1, e_2, e_3\}\) in \(\Lambda^2 \mathfrak{su}_2\) and \(\Lambda^3 \mathfrak{su}_2\), namely \(\{e_{12}, e_{13}, e_{23}\}\) and \(\{e_{123}\}\) respectively. It is worth noting that the existence of an \(\mathfrak{su}_2\)-invariant metric on \(\Lambda^2 \mathfrak{su}_2\) also implies, in view of Proposition 11.1, that \((\Lambda^3 \mathfrak{su}_2)_{\mathfrak{su}_2} = \Lambda^3 \mathfrak{su}_2\) and every \(r \in \Lambda^2 \mathfrak{su}_2\) satisfies the modified classical Yang-Baxter equation, giving rise to a coboundary coproduct.

In view of Proposition 8.4 and the fact that \text{Inn}(\mathfrak{su}_2) is connected, the orbits of the action of \text{Inn}(\mathfrak{su}_2) on \(\Lambda^2 \mathfrak{su}_2\) have a dimension given by \(\text{Im} \Theta^2_w\). This value is two for \(w \in \Lambda^2 \mathfrak{su}_2 \setminus \{0\}\) and zero otherwise.
The orbits of the action of $\text{Inn}(\mathfrak{su}_2)$ on $\Lambda^2\mathfrak{su}_2$ are connected immersed submanifolds contained in the spaces $S_k$ given by the points in $\Lambda^2\mathfrak{su}_2$ where the quadratic function $f_{\Lambda^2\mathfrak{su}_2}(r) := \kappa_{\Lambda^2\mathfrak{su}_2}(r, r) = 4(x^2 + y^2 + z^2)$ takes a constant value $k$. Since the orbits of $\text{Inn}(\mathfrak{su}_2)$ must be open relative to the topology of each $S_k$, which are connected, each orbit of $\text{Inn}(\mathfrak{su}_2)$ must be the whole $S_k$ for each $k \geq 0$. Hence, non-equivalent $r \in \Lambda^2\mathfrak{su}_2$, with respect to the action of $\text{Inn}(\mathfrak{su}_2)$, are given by elements $r$ with different modulus, e.g. $r_a = ae_{12}$, with $a \geq 0$. Since the orbits of the action of $\text{Aut}(\mathfrak{su}_2)$ on $\Lambda^2\mathfrak{su}_2$ are given by the sum of orbits of $\text{Inn}(\mathfrak{su}_2)$ and they are contained in the surfaces $S_k$, one gets that the orbits of the action of $\text{Aut}(\mathfrak{su}_2)$ in $\Lambda^2\mathfrak{su}_2$ are indeed the spheres $S_k$ with $k > 0$ and the point $k = 0$.

### 11.3 Nilpotent Lie algebras: The 3D-Heisenberg Lie algebra

Let us consider the three-dimensional (3D) Heisenberg algebra [19], namely the Lie algebra $\mathfrak{h} := \langle e_1, e_2, e_3 \rangle$, where the basis $\{e_1, e_2, e_3\}$ satisfy the commutation relations

$$[e_3, e_1] = 0, \quad [e_3, e_2] = 0, \quad [e_1, e_2] = e_3.$$ 

This is the only, up to a Lie algebra isomorphism, three-dimensional nilpotent Lie algebra [42].

Let us analyse $(\Lambda^2\mathfrak{h})^h$ and $(\Lambda^3\mathfrak{h})^h$. Since $\mathfrak{h}$ is nilpotent, the Engel’s theorem [23] ensures that the adjoint representation of $\mathfrak{h}$ is traceless and $\mathfrak{h}$ is unimodular. Then, Corollary 11.1 ensures that $(\Lambda^3\mathfrak{h})^h = \Lambda^3\mathfrak{h}$ and every $r \in \Lambda^2\mathfrak{h}$ satisfies the modified classical Yang-Baxter equation. In view of Proposition 11.2 and Lemma 11.1, every non-zero element of $(\Lambda^2\mathfrak{h})^h$ induces a two-dimensional space containing $[\mathfrak{h}, \mathfrak{h}] = \langle e_3 \rangle$. This means that if $w \in (\Lambda^2\mathfrak{h})^h$, then $w = e_3 \wedge a$ for some $a \in \mathfrak{h}$. Since $\mathfrak{h}$ is nilpotent, the Engel’s theorem ensures again that $\mathfrak{h}$ must act tracelessly on each two-dimensional ideal $(a, e_3)$. This easily gives that $a$ is any element of $\mathfrak{g}$ linearly independent relative to $e_3$. Hence, $(\Lambda^3\mathfrak{h})^h = \langle e_{13}, e_{23} \rangle$ and

$$\Lambda^2\mathfrak{h} = \frac{\Lambda^2\mathfrak{h}}{(\Lambda^2\mathfrak{h})^h} \simeq \langle e_{12} \rangle.$$ 

Alternatively, one can obtain the same result by means of the methods developed in Section 8.2, which are based upon the fact that $\mathfrak{h}$ is nilpotent. As a consequence, the 3D-Heisenberg Lie algebra admits a lower central series

$$\mathfrak{h}(0) = \langle e_1, e_2, e_3 \rangle \supset \mathfrak{h}(1) = \langle e_3 \rangle = \mathfrak{z}(\mathfrak{h}) \supset \mathfrak{h}(2) := \{0\},$$

and, in view of Proposition 8.7 and since dim $\mathfrak{z}(\mathfrak{h}) = 1$, one obtains that $\mathfrak{h}(0) \cap \mathfrak{h}(1) = \{e_{13}, e_{23}\} \subseteq (\Lambda^2\mathfrak{h})^h$. Since $\Lambda^2\mathfrak{h} \neq (\Lambda^2\mathfrak{h})^h$, it is straightforward that

$$(\Lambda^2\mathfrak{h})^h = \langle e_{31}, e_{32} \rangle = \mathfrak{z}(\mathfrak{h}) \wedge \mathfrak{h}.$$ 

The figure below depicts the equivalence classes of $\Lambda^2\mathfrak{h}$ in $\Lambda^2\mathfrak{h}$.
Figure 10: Representative orbits of the action of \( \text{Inn}(\mathfrak{h}) \) on \( \Lambda^2 \mathfrak{h} \).

Recall that Proposition 2.1 ensures that elements belonging to the same class of \( \Lambda^2 \mathfrak{h} \) give rise to the same coproduct. Since we are interested in classifying coboundary coproducts, their analysis can be reduced to studying reduced \( r \)-matrices in \( \Lambda^2 \mathfrak{h} \). Additionally, it is mandatory to determine which of these equivalence classes are equivalent under the action of \( \text{Inn}(\mathfrak{h}) \). To accomplish this task, we study the \( \mathfrak{h} \)-invariant metrics on \( \Lambda^2 \mathfrak{h} \) giving rise to an \( \mathfrak{h} \)-invariant metric on \( \Lambda^2 \mathfrak{h} \). As \( \mathfrak{h} \) is nilpotent, its Killing form vanishes [20]. Nevertheless, \( \mathfrak{h} \) does admit symmetric and anti-symmetric bilinear \( \mathfrak{h} \)-invariant maps (see Example 6.1), which can be extended to \( \Lambda^2 \mathfrak{h} \) and \( \Lambda^3 \mathfrak{h} \) by using Theorem 4.3.

In the bases \( \{e_{12}, e_{13}, e_{23}\} \) in \( \Lambda^2 \mathfrak{h} \) and \( \{e_{123}\} \) in \( \Lambda^3 \mathfrak{h} \), all referred \( \mathfrak{h} \)-invariant symmetric bilinear maps read (cf. Example 6.1)

\[
[b^{(s)}]_\mathfrak{h} := \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [b^{(s)}]_{\Lambda^2 \mathfrak{h}} := \begin{pmatrix} \alpha_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [b^{(s)}]_{\Lambda^3 \mathfrak{h}} := (\alpha_5), \quad \forall \alpha_i \in \mathbb{R}.
\]

Recall that \( \Lambda^2 \mathfrak{h} = \Lambda^2 \mathfrak{h}/(\Lambda^2 \mathfrak{h})^b \) is one-dimensional. It is possible to define an \( \mathfrak{h} \)-invariant form \( b_R \) on \( \Lambda^2 \mathfrak{h} \) by using the above results. The \( \mathfrak{h} \)-invariant map \( b^{(s)}_{\Lambda^2 \mathfrak{h}} \) is such that \((\Lambda^2 \mathfrak{h})^b\) is contained in its radical. Proposition 9.2 allows us to restrict \( b^{(s)}_{\Lambda^2 \mathfrak{h}} \) to \( \Lambda^2 \mathfrak{h} \), which gives rise to an \( \mathfrak{h} \)-invariant metric \( b_R := b^{(s)}_{\Lambda^2 \mathfrak{h}}(w_1, w_2) \) for all \( w_1, w_2 \in \Lambda^2 \mathfrak{h} \).

However, \( \mathfrak{h} \) admits the automorphisms \( T_{\alpha} \), with \( \alpha \in \mathbb{R} \setminus \{0\} \), given by

\[
T_{\alpha}(e_1) := \alpha e_1, \quad T(e_2) := e_2, \quad T(e_3) := \alpha e_3.
\]

Therefore, \( \Lambda^2 \mathfrak{h}_{\alpha}(e_{12}) = \alpha e_{12} \) for any \( \alpha \neq 0 \). Since \( (\Lambda^2 \mathfrak{h})^b \) is invariant under the action of \( \text{Aut}(\mathfrak{h}) \), it has sense to consider the induced action of \( \Lambda^2 T \) on \( \Lambda^2 \mathfrak{h} \). Since such a restriction does not respect the aforesaid metric on \( \Lambda^2 \mathfrak{h} \), the automorphisms \( T_{\alpha} \) are not inner. Then, the induced action of \( \text{Aut}(\mathfrak{h}) \) on \( \Lambda^2 \mathfrak{h} \) has two orbits \([0]\) and \([e_{12}]\). Thus, we have only one class of non-zero coboundary coproducts can be represented by the \( r \)-matrix \( r := e_{12} \). The space of coboundary coproducts is depicted in Figure 11.

**Figure 11:** Orbits of \( \text{Aut}(\mathfrak{h}) \) in \( \Lambda^2 \mathfrak{h} \).

### 11.4 Solvable non-nilpotent Lie algebras

There exist six classes of solvable but not nilpotent three-dimensional real Lie algebras [42]. The following subsections aim at classifying all Lie bialgebras on them.
11.4.1 The Lie algebra $\mathfrak{t}'_{3,0}$

Consider the Lie algebra $\mathfrak{t}'_{3,0} := \langle e_1, e_2, e_3 \rangle$, whose basis $\{e_1, e_2, e_3\}$ has commutation relations

\[
[e_1, e_2] = -e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = 0.
\]

Let us analyse $(\Lambda^2 \mathfrak{t}'_{3,0})^{t_{3,0}}$ and $(\Lambda^3 \mathfrak{t}'_{3,0})^{t_{3,0}}$. Since $\mathfrak{t}'_{3,0}$ is unimodular, Corollary 11.1 ensures that $(\Lambda^3 \mathfrak{t}'_{3,0})^{t_{3,0}} = \Lambda^3 \mathfrak{t}'_{3,0}$ and every $r \in \Lambda^2 \mathfrak{t}'_{3,0}$ satisfies the corresponding modified classical Yang-Baxter equation. Concerning $(\Lambda^2 \mathfrak{t}'_{3,0})^{t_{3,0}}$, Proposition 11.2 shows that every non-zero element of $(\Lambda^2 \mathfrak{t}'_{3,0})^{t_{3,0}}$ induces a two-dimensional ideal containing $[t_3, 0] = \langle e_2, e_3 \rangle$, where $t'_{3,0}$ acts tracelessly and vice versa. This means that $(\Lambda^2 \mathfrak{t}'_{3,0})^{t_{3,0}} \subset \langle e_{23} \rangle$. Then, it is simple to verify that $(\Lambda^2 \mathfrak{t}'_{3,0})^{t_{3,0}} = \langle e_{23} \rangle$ and

\[
\Lambda^2 \mathfrak{t}'_{3,0} = \frac{\Lambda^2 \mathfrak{t}'_{3,0}}{(\Lambda^2 \mathfrak{t}'_{3,0})^{t_{3,0}}} \simeq \langle e_{12}, e_{13} \rangle.
\]

In view of Proposition 2.1, different equivalence classes of $\Lambda^2 \mathfrak{t}'_{3,0}$ generate different coproducts. Let us classify the non-equivalent (up to inner Lie algebra automorphisms of $\mathfrak{t}'_{3,0}$) coboundary coproducts on $\mathfrak{t}'_{3,0}$ by using $\mathfrak{t}'_{3,0}$-invariant metrics on $\Lambda^2 \mathfrak{t}'_{3,0}$. As in Section 11.3, this is accomplished by determining $\mathfrak{t}'_{3,0}$-invariant metrics on $\Lambda^2 \mathfrak{t}'_{3,0}$ that give rise to a $\mathfrak{t}'_{3,0}$-invariant metric on $\Lambda^2 R \mathfrak{t}'_{3,0}$. To this respect, the results given next

\[
\text{Im } \Lambda^2 \text{ad}_{e_1} = \langle e_{13}, e_{12} \rangle, \quad \text{ker } \Lambda^2 \text{ad}_{e_1} = \langle e_{23} \rangle, \quad \text{Im } \Lambda^2 \text{ad}_{e_2} = \langle e_{23} \rangle, \quad \text{ker } \Lambda^2 \text{ad}_{e_2} = \langle e_{13}, e_{23} \rangle, \quad \text{Im } \Lambda^2 \text{ad}_{e_3} = \langle e_{23} \rangle, \quad \text{ker } \Lambda^2 \text{ad}_{e_3} = \langle e_{12}, e_{23} \rangle
\]

and

\[
b_{\Lambda^2 \mathfrak{t}'_{3,0}}(e_{12}, e_{12}) = b_{\Lambda^2 \mathfrak{t}'_{3,0}}([e_1, e_{13}], e_{12}) = b_{\Lambda^2 \mathfrak{t}'_{3,0}}(e_{13}, [e_1, e_{12}], e_{12}) = -b_{\Lambda^2 \mathfrak{t}'_{3,0}}(e_{13}, e_{12}),
b_{\Lambda^2 \mathfrak{t}'_{3,0}}(e_{13}, e_{12}) = -b_{\Lambda^2 \mathfrak{t}'_{3,0}}([e_1, e_{13}], e_{12}) = b_{\Lambda^2 \mathfrak{t}'_{3,0}}(e_{13}, [e_1, e_{12}], e_{12}) = -b_{\Lambda^2 \mathfrak{t}'_{3,0}}(e_{12}, e_{13})
\]

allow us to use Propositions 6.1 and 6.4 to obtain that every $\mathfrak{t}'_{3,0}$-invariant metric on $\Lambda^2 \mathfrak{t}'_{3,0}$ must match the form

\[
[b_{\Lambda^2 \mathfrak{t}'_{3,0}}] = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R}.
\]

A straightforward calculation shows that this metric is $\mathfrak{t}'_{3,0}$-invariant for every $a \in \mathbb{R}$. Hence, these are the only $\mathfrak{t}'_{3,0}$-invariant metrics on $\Lambda^2 \mathfrak{t}'_{3,0}$. Since their kernel contains $\Lambda^2 \mathfrak{t}'_{3,0}$, Proposition 9.2 allows us to construct out of them $\mathfrak{t}'_{3,0}$-invariant metrics on $\Lambda^2 R \mathfrak{t}'_{3,0}$.

Alternatively, let us discuss directly the existence of $\mathfrak{t}'_{3,0}$-invariant metrics on $\Lambda^2 R \mathfrak{t}'_{3,0}$. In the basis $\{\langle e_{12}, e_{13} \rangle\}$ of $\Lambda^2 R \mathfrak{t}'_{3,0}$, its general form reads

\[
[b_{\Lambda^2 R \mathfrak{t}'_{3,0}}] = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}, \quad a_i \in \mathbb{R}.
\]

If $\Lambda^2 \text{ad} : v \in \mathfrak{g} \mapsto \langle v, \cdot \rangle \in \text{End}(\Lambda^2 \mathfrak{g})$, where $\langle \cdot, \cdot \rangle$ is the bracket on $\Lambda^2 \mathfrak{g}$ induced by the algebraic bracket on $\Lambda^2 \mathfrak{g}$, then

\[
\text{Im } \Lambda^2 \text{ad}_{e_1} = \langle \langle e_{13}, [e_{12}] \rangle \rangle, \quad \ker \Lambda^2 \text{ad}_{e_1} = \langle \langle 0 \rangle \rangle, \quad \text{Im } \Lambda^2 \text{ad}_{e_2} = \langle \langle 0 \rangle \rangle, \quad \ker \Lambda^2 \text{ad}_{e_2} = \langle \langle e_{13} \rangle \rangle, \quad \text{Im } \Lambda^2 \text{ad}_{e_3} = \langle \langle e_{23} \rangle \rangle = \langle \langle 0 \rangle \rangle, \quad \ker \Lambda^2 \text{ad}_{e_3} = \langle \langle e_{12} \rangle \rangle
\]

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and
\[ b_{\Lambda^2\mathfrak{r}_L}^R([e_{12}],[e_{12}]) = b_{\Lambda^2\mathfrak{r}_L}^R([[e_1],[e_{13}]], [e_{12}]) = -b_{\Lambda^2\mathfrak{r}_L}^R([e_{13}],[e_{12}]) = b_{\Lambda^2\mathfrak{r}_L}^R([e_{12}],[e_{13}]). \]

Therefore,
\[ [b_{\Lambda^2\mathfrak{r}_L}^R] = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & a_1 \end{pmatrix}, \quad \forall a_1 \in \mathbb{R}. \]

It is seen that \( b_{\Lambda^2\mathfrak{r}_L} \) reduces to \( b_{\Lambda^2\mathfrak{r}_L}^R \), since \( e_{23} \in (\Lambda^2\mathfrak{r}_{3,0})^\vee \) is in the kernel of \( b_{\Lambda^2\mathfrak{r}_L}^R \).

Since all elements of \( \Lambda^2\mathfrak{r}_{3,0} \) give rise to a coboundary coproduct, their study can be reduced to studying the equivalence classes of \( \Lambda^2\mathfrak{r}_{3,0}^\vee \). Let us study the equivalence of reduced \( r \)-matrices up to inner automorphisms of \( \mathfrak{r}_{3,0} \). The equivalence classes in \( \Lambda^2\mathfrak{r}_{3,0} \) can be written as \( x[e_{12}] + y[e_{13}] \) in the basis \( \{[e_{12}],[e_{13}]\} \). It follows that \( b_{\Lambda^2\mathfrak{r}_{3,0}^\vee}([r],[r]) = a(x^2 + y^2) \). The image of \( \Theta_r^2 \) is one-dimensional for \( x^2 + y^2 \neq 0 \) and zero-dimensional otherwise. In consequence, the orbits in \( \Lambda^2\mathfrak{r}_{3,0} \) relative to the action of \( \text{Aut}(\mathfrak{r}_{3,0}) \) are circles with radius \( a \) and there exists a nontrivial family of \( r \)-matrices \( r = ae_{12} \), \( a \in \mathbb{R}^+ \), giving rise to different non-zero coproducts, whose coproducts are not equivalent up to inner automorphisms of \( \mathfrak{r}_{3,0} \). Figure 12 depicts certain orbits of the action of \( \text{Inn}(\mathfrak{r}_{3,0}) \) on \( \Lambda^2\mathfrak{r}_{3,0} \).

Let us classify coboundary coproducts up to Lie algebra automorphisms of \( \mathfrak{r}_{3,0} \). Consider the automorphisms \( T_\alpha \in \text{Aut}(\mathfrak{r}_{3,0}) \), with \( \alpha \in \mathbb{R}\setminus\{0\} \), satisfying
\[ T_\alpha(e_1) := e_1, \quad T_\alpha(e_2) := \alpha e_2, \quad T_\alpha(e_3) := \alpha e_3. \]

These automorphisms induce elements \( \Lambda^2T_\alpha \in GL(\Lambda^2\mathfrak{r}_{3,0}) \) such that \( \Lambda^2T_\alpha(e_{12}) = \alpha e_{12} \). In turn, these automorphisms induce isomorphisms \( \Lambda^2T_\alpha \) on \( \Lambda^2\mathfrak{r}_{3,0} \). The \( \Lambda^2T_\alpha \) map the circles with different positive radius among themselves. Hence, their sum forms the only orbit of \( \text{Aut}(\mathfrak{r}_{3,0}^\vee) \) on \( \Lambda^2\mathfrak{r}_{3,0}^\vee \) related to a non-zero coboundary coproduct. Hence, there is only one non-zero coboundary coproduct, up to the action of \( \text{Aut}(\mathfrak{r}_{3,0}) \), induced by an \( r \)-matrix \( r = e_{12} \). Figure 13 represents the orbits of the action of \( \text{Aut}(\mathfrak{r}_{3,0}^\vee) \) on \( \Lambda^2\mathfrak{r}_{3,0}^\vee \). This matches the results in [19].

11.4.2 The Lie algebra \( \mathfrak{r}_{3,-1} \)

Consider now the Lie algebra \( \mathfrak{r}_{3,-1} := \langle e_1, e_2, e_3 \rangle \), where \( \{e_1, e_2, e_3\} \) is a basis satisfying the commutation relations
\[ [e_1, e_2] = e_2, \quad [e_1, e_3] = -e_3, \quad [e_2, e_3] = 0. \]
Let us analyse the elements of \((\Lambda^2 \tau_{3,-1})^{\tau_{3,-1}}\) and \((\Lambda^3 \tau_{3,-1})^{\tau_{3,-1}}\). Since \(\tau_{3,-1}\) is unimodular, Corollary 11.1 ensures that \((\Lambda^3 \tau_{3,-1})^{\tau_{3,-1}} = \Lambda^3 \tau_{3,-1}\). Hence, every \(r \in \Lambda^2 \tau_{3,-1}\) is a solution to the modified classical Yang-Baxter equation on \(\Lambda^2 \tau_{3,-1}\). Moreover, Proposition 11.2 ensures that every non-zero element of \((\Lambda^2 \tau_{3,-1})^{\tau_{3,-1}}\) must generate an ideal of \(\tau_{3,-1}\) containing \([\tau_{3,-1}, \tau_{3,-1}] = \langle e_2, e_3 \rangle\). This shows that \((\Lambda^2 \tau_{3,-1})^{\tau_{3,-1}} \subset \langle e_{23} \rangle\). Since \(e_{23}\) is \(\tau_{3,-1}\)-invariant, it follows that \((\Lambda^2 \tau_{3,-1})^{\tau_{3,-1}} = \langle e_{23} \rangle\) and \(\Lambda^2 \tau_{3,-1} \simeq \langle e_{12}, e_{13} \rangle\).

Let us determine the \(\tau_{3,-1}\)-invariant symmetric bilinear maps on \(\Lambda^2 \tau_{3,-1}\) and \(\Lambda^3 \tau_{3,-1}\). To this respect, observe that ker \(\Lambda^2 \operatorname{ad} e_1 = \langle e_{23} \rangle\), \(\operatorname{Im} \Lambda^2 \operatorname{ad} e_1 = \langle e_{12}, e_{13} \rangle\), ker \(\Lambda^2 \operatorname{ad} e_2 = \langle e_{12}, e_{23} \rangle\), \(\operatorname{Im} \Lambda^2 \operatorname{ad} e_2 = \langle e_{12}, e_{23} \rangle\).

Then, Propositions 6.1 and 6.3 show that \(\tau_{3,-1}\)-invariant metrics on \(\Lambda^2 \tau_{3,-1}\) must be of the form

\[
[b_{\Lambda^2 \tau_{3,-1}}] = \begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

for certain values of \(\beta \in \mathbb{R}\). A simple calculation leads to proving that \(b_{\Lambda^2 \tau_{3,-1}}\) is indeed \(\tau_{3,-1}\)-invariant for every value of \(\beta \in \mathbb{R}\). Since we are interested in determining the surfaces where the quadratic function associated with \(b_{\Lambda^2 \tau_{3,-1}}\), or its reduction to \(\Lambda^2_R \tau_{3,-1}\), is constant, it is enough to our purposes to assume \(\beta = 1\). Since \((\Lambda^2 \tau_{3,-1})^{\tau_{3,-1}}\) is contained in the kernel of \(b_{\Lambda^2 \tau_{3,-1}}\), the metric \(b_{\Lambda^2 \tau_{3,-1}}\) gives rise to a metric on \(\Lambda^2_R \tau_{3,-1}\) in the natural way.

Analogously, we can obtain straightforwardly an \(\tau_{3,-1}\)-invariant metric \(b^R_{\Lambda^2 \tau_{3,-1}}\) on \(\Lambda^2_R \tau_{3,-1}\). Define \(\Lambda^2_R \operatorname{ad} : v \in \tau_{3,-1} \mapsto [v, \cdot]_R \in \operatorname{End}(\Lambda^2_R \tau_{3,-1})\). In the basis \(\{[e_{12}], [e_{13}]\}\) of \(\Lambda^2_R \tau_{3,-1}\), one gets

\[
b^R_{\Lambda^2 \tau_{3,-1}}([e_{12}], [e_{12}]) = b^R_{\Lambda^2 \tau_{3,-1}}([e_{12}], [e_{12}]), b^R_{\Lambda^2 \tau_{3,-1}}([e_{12}], [e_{13}]) = b^R_{\Lambda^2 \tau_{3,-1}}([e_{12}], [e_{13}]).
\]

In consequence, an \(\tau_{3,-1}\)-invariant metric on \(\Lambda^2_R \tau_{3,-1}\) must take the form

\[
[b^R_{\Lambda^2 \tau_{3,-1}}] = \begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

for certain values of \(\beta \in \mathbb{R}\). These are indeed the \(\tau_{3,-1}\)-invariant metrics on \(\Lambda^2_R \tau_{3,-1}\) given by \(b_{\Lambda^2 \tau_{3,-1}}\) via Proposition 9.2. Let \(\{x, y\}\) be the coordinates associated with the basis \(\{[e_{12}], [e_{13}]\}\) of \(\Lambda^2_R \tau_{3,-1}\). Then \(r_R = x[e_{12}] + y[e_{13}]\) and the quadratic function related to \(b^R_{\Lambda^2 \tau_{3,-1}}\) reads for \(\beta = 1\) as \(f^R_{\Lambda^2 \tau_{3,-1}}(r_R) = 2xy\), which can be considered as a quadratic function on the coordinates \(x, y\) of the quotient \(\Lambda^2_R \tau_{3,-1}\). The image of \(\Theta^2_R\) is one-dimensional for \(x^2 + y^2 \neq 0\) and zero-dimensional otherwise.

Figure 14: Representative orbits of the action of \(\operatorname{Inn}(\tau_{3,-1})\) on \(\Lambda^2_R \tau_{3,-1}\).

Hence, the representatives of the action of \(\operatorname{Inn}(\tau_{3,-1})\) on \(\Lambda^2_R \tau_{3,-1}\) have the form presented on Figure 14. The representatives of inequivalent reduced \(r\)-matrices, up to the action of inner Lie algebra automorphisms, are given by:

\[
\begin{align*}
r^{(\pm, \pm)} &= a(\pm [e_{12}] \pm [e_{13}]), & r^{(\pm)}_2 &= \pm b[e_{12}], & r^{(\pm)}_3 &= \pm b[e_{13}], & r_0 &= [e_{23}], & \forall a, b > 0.
\end{align*}
\]
The Lie algebra \( \mathfrak{r}_{3,-1} \) satisfies the conditions given in Proposition 11.3. Hence, all automorphisms of \( \mathfrak{r}_{3,-1} \) must match one of the following automorphisms

\[
T_{\alpha,\beta}(e_1) := e_1, \quad T_{\alpha,\beta}(e_2) := \alpha e_2, \quad T_{\alpha,\beta}(e_3) := \beta e_3, \quad \forall \alpha, \beta \in \mathbb{R} \setminus \{0\},
\]

\[
T'_{\alpha,\beta}(e_1) := -e_1, \quad T'_{\alpha,\beta}(e_2) := \alpha e_3, \quad T'_{\alpha,\beta}(e_3) := \beta e_2, \quad \forall \alpha, \beta \in \mathbb{R} \setminus \{0\}.
\]

The extensions \( \Lambda^2 T_{\alpha,\beta} \) and \( \Lambda^2 T'_{\alpha,\beta} \) satisfy

\[
\Lambda^2 T_{\alpha,\beta}(e_{12}) = \alpha e_{12}, \quad \Lambda^2 T_{\alpha,\beta}(e_{13}) = \beta e_{13}, \quad \Lambda^2 T_{\alpha,\beta}(e_{23}) = \alpha \beta e_{23}, \quad \forall \alpha, \beta \in \mathbb{R} \setminus \{0\},
\]

\[
\Lambda^2 T'_{\alpha,\beta}(e_{12}) = -\alpha e_{13}, \quad \Lambda^2 T'_{\alpha,\beta}(e_{13}) = -\beta e_{12}, \quad \Lambda^2 T'_{\alpha,\beta}(e_{23}) = -\alpha \beta e_{23}, \quad \forall \alpha, \beta \in \mathbb{R} \setminus \{0\}.
\]

In turn, the above automorphisms \( \Lambda^2 \mathfrak{r}_{3,-1} \) induce new automorphisms on \( \Lambda^2_{\mathfrak{r}} \mathfrak{r}_{3,-1} \). The above transformations do not preserve the connected components of the regions \( S_k \) where the function \( f^R_{\mathfrak{r}}(r) \) takes a constant value equal to \( k \). As a consequence, \( T_{\alpha,\beta} \) and \( T'_{\alpha,\beta} \) do not induce new automorphisms and the non-equivalent non-zero coboundary coproducts on \( \mathfrak{r}_{3,-1} \), relative to the action of \( \text{Aut}(\mathfrak{r}_{3,-1}) \), are induced by the \( r \)-matrices: \( r = e_{12}, r' = e_{12} - e_{13} \). Indeed, recall that \( r_0 = e_{23} \) gives rise to a zero coboundary coproduct. The figure aside depicts the action of \( \text{Aut}(\mathfrak{r}_{3,-1}) \) on \( \Lambda^2_{\mathfrak{r}} \mathfrak{r}_{3,-1} \).

**Figure 15:** The three orbits of the action of \( \text{Aut}(\mathfrak{r}_{3,-1}) \) on \( \Lambda^2_{\mathfrak{r}} \mathfrak{r}_{3,-1} \).

### 11.4.3 The Lie algebra \( \mathfrak{r}_{3,1} \)

Consider the Lie algebra \( \mathfrak{r}_{3,1} := \langle e_1, e_2, e_3 \rangle \), where the elements of the basis \( \{e_1, e_2, e_3\} \) satisfy the commutation relations

\[
[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = 0.
\]

As previously, let us study the space \( (\Lambda^2 \mathfrak{r}_{3,1})^{\mathfrak{r}_{3,1}} \). The Lie algebra \( \mathfrak{r}_{3,1} \) is not unimodular (notice that \( \text{Tr} \ ad_{e_1} = 2 \)). Due to this and Corollary 11.1, one has that \( (\Lambda^2 \mathfrak{r}_{3,1})^{\mathfrak{r}_{3,1}} = \{0\} \). Therefore, the determination of \( r \)-matrices demands solving the corresponding modified classical Yang–Baxter equation. In the coordinates \( \{x, y, z\} \) corresponding to the basis \( \{e_{12}, e_{13}, e_{23}\} \) of \( \Lambda^2 \mathfrak{r}_{3,1} \), one has \( r = xe_{12} + ye_{13} + ze_{23} \) and it turns out that

\[
[r, r]_S = 0
\]

for every \( r \in \Lambda^2 \mathfrak{r}_{3,1} \). Hence, every element of \( \Lambda^2 \mathfrak{r}_{3,1} \) is an \( r \)-matrix giving rise to a coboundary coproduct.
By now Proposition 11.2 ensures that every $r \in (\Lambda^2\mathfrak{t}_{3,1})^{r_{3,1}}$ induces, when considered as a linear function $\tilde{r} : \theta \in \mathfrak{t}_{3,1}^* \mapsto \iota_\theta r \in \mathfrak{t}_{3,1}$, an image that must contain $[\mathfrak{t}_{3,1}, \mathfrak{t}_{3,1}] = \langle e_2, e_3 \rangle$. Hence, $(\Lambda^2\mathfrak{t}_{3,1})^{r_{3,1}} \subset \langle e_{23} \rangle$ and, as $e_{23}$ is not $\mathfrak{t}_{3,1}$-invariant, $(\Lambda^2\mathfrak{t}_{3,1})^{r_{3,1}} = \{0\}$. In consequence, every $r$-matrix induces a different coproduct. The fundamental vector fields of the action of $\text{Inn}(\mathfrak{t}_{3,1})$ on $\Lambda^2\mathfrak{t}_{3,1}$ are spanned by

$$X_1 := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}; \quad X_2 := y \frac{\partial}{\partial y}, \quad X_3 := z \frac{\partial}{\partial z}.$$ 

They generate an integrable two-dimensional distribution off the line $x = y = 0$ with integrals given by semi-planes of the form given in the figure aside. The line $x = y = 0$ can be also divided into three orbits of the action of $\text{Inn}(\mathfrak{t}_{3,1})$ consisting of the points with the same sign of $z$. Obviously, the $r = 0$ is an orbit of the action of $\text{Aut}(\mathfrak{t}_{3,1})$ on $\Lambda^2\mathfrak{t}_{3,1}$.

Moreover, the Lie algebra automorphisms $T_{\alpha,\beta,\gamma,\delta}$ given by

$$T_{\alpha,\beta,\gamma,\delta}(e_1) := e_1, \quad T_{\alpha,\beta,\gamma,\delta}(e_2) := \alpha e_2 + \beta e_3,$$

$$T_{\alpha,\beta,\gamma,\delta}(e_3) = \gamma e_2 + \delta e_3, \quad \alpha \delta - \beta \gamma \neq 0,$$

Then, the $\Lambda^2T_{\alpha,\beta,\gamma,\delta}$ connects different semi-planes in $\Lambda^2\mathfrak{t}_{3,1}$. Moreover, the above automorphisms connect the parts $z > 0$ and $z < 0$ of the line $x = y = 0$. Hence, there exist two non-zero non-equivalent coboundaries induced by the $r$-matrices $r_1 = e_{13}$ and $r_2 = e_{23}$.

11.4.4 The Lie algebra $\mathfrak{t}_3$

Let us consider the Lie algebra $\mathfrak{t}_3 := \langle e_1, e_2, e_3 \rangle$ with a basis $\{e_1, e_2, e_3\}$ satisfying the commutation relations

$$[e_3, e_1] = e_1, \quad [e_3, e_2] = e_1 + e_2, \quad [e_1, e_2] = 0.$$ 

Since $\mathfrak{t}_3$ is not unimodular, the description of coboundary Lie bialgebras on $\mathfrak{t}_3$ demands the solution of its associated modified classical Yang-Baxter equation. Since the space of solutions to this equation, let us say $Y\!G$, is invariant under the action of $\text{Aut}(\mathfrak{t}_3)$, the classification of such Lie bialgebras can be reduced to the study of equivalent $r$-matrices in $Y\!G$.

Let $\{x, y, z\}$ be the coordinates on $\Lambda^2\mathfrak{t}_3$ induced by the basis $\{e_{12}, e_{13}, e_{23}\}$. The modified classical Yang-Baxter equation, where $r = xe_{12} + ye_{13} + ze_{23}$, reads

$$[r, r]_S = -2z^2 e_{123}.$$ 

Hence, $Y\!G = \langle e_{12}, e_{13} \rangle$ is presented in Figure 18.
Let us determine the space \((\Lambda^2 r_3)^{t_3}\) so as to know whether different \(r\)-matrices induce different coboundary coproducts. Proposition 11.2 states that a non-zero element of \((\Lambda^2 r_3)^{t_3}\) gives rise to a two-dimensional subalgebra containing \([t_3, t_3] = \langle e_1, e_2 \rangle\). Hence, \((\Lambda^2 r_3)^{t_3} \subset \langle e_{12} \rangle\). As \(e_{12}\) is not \(t_3\)-invariant, one gets that \((\Lambda^2 r_3)^{t_3} = \{0\}\). In consequence, every \(r\)-matrix induces a different coboundary coproduct.

A long but simple calculation shows that \(\Lambda^2 r_3\) admits no \(t_3\)-invariant metrics. Nevertheless, one can still classify \(r\)-matrices up to the action of \(\text{Inn}(r_3)\).

Let us determine coboundary coproducts up to Lie algebra inner automorphisms. This demands to consider the action of \(\text{Inn}(r_3)\) on \(\Lambda^2 r_3\). The classification up to \(\text{Aut}(r_3)\) will follow immediately. The fundamental vector fields of the action of \(\text{Inn}(r_3)\) on \(\Lambda^2 r_3\) are spanned by

\[
X_1 := z \frac{\partial}{\partial x}, \quad X_2 := (-y + z) \frac{\partial}{\partial x}, \quad X_3 := 2x \frac{\partial}{\partial x} + (y + z) \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}.
\]

The restriction of the above vector fields to \(Y_B\) reads

\[
X_1|_{Y_B} = 0, \quad X_2|_{Y_B} = -y \frac{\partial}{\partial x}, \quad X_3|_{Y_B} = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.
\]

The latter vector fields span the tangent space to \(Y_B\) when \(y \neq 0\) and they span \(\langle \partial/\partial x \rangle\) for \(y = 0\) and \(x \neq 0\). In consequence, there exist five orbits of \(\text{Inn}(r_3)\) depicted in Figure 18.

Let us accomplish the classification of coboundary coproducts up to the action of elements of \(\text{Aut}(r_3)\) on \(YG \subset \Lambda^2 r_3\). Since \(r_3\) obeys the assumptions of Proposition 11.3, each \(T \in \text{Aut}(r_3)\) must respect the eigenvectors of \(ad_{e_3}\). Moreover, since \([r, r]|_S\) satisfies the condition in Proposition 11.4 and making a slight modification of Proposition 11.3, one has that there exists a family of automorphisms

\[
T_{\alpha, \beta}(e_3) = e_3, \quad T_{\alpha, \beta}(e_1) = \alpha e_1, \quad T_{\alpha, \beta}(e_2) = \alpha e_2 + \beta e_1, \quad \forall \alpha \in \mathbb{R}\backslash\{0\}, \beta \in \mathbb{R}.
\]

Therefore

\[
\Lambda^2 T_{\alpha, \beta}(e_{12}) = \alpha^2 e_{12}, \quad \Lambda^2 T_{\alpha, \beta}(e_{13}) = \alpha e_{13}, \quad \Lambda^2 T_{\alpha, \beta}(e_{23}) = \alpha e_{23} + \beta e_{13}, \quad \forall \alpha \in \mathbb{R}\backslash\{0\}, \beta \in \mathbb{R}.
\]

It was proven in Section 10 that every element of \([t_3, t_3] = \langle e_1, e_2 \rangle\) is invariant under the action of \(\text{Aut}(r_3)\). Thus, the subspace \(\langle e_{12} \rangle\) of \(\Lambda^2 r_3\) is invariant under the action of \(\text{Aut}(r_3)\) on \(\Lambda^2 r_3\). In view of the previous automorphisms, it follows that \((e_{12})\) has three orbits: given by the zero and the orbits of \(\pm e_{12}\). Since this subspace belongs to \(Y_B\), it becomes clear that \(\pm e_{12}\) are \(r\)-matrices giving rise to non-zero coproducts. Since there exist automorphisms on \(g\) inverting the coordinate \(y\) and leaving \(x\) invariant, then such automorphisms show that there exists only one equivalence class of non-zero solutions in \(Y_B\) out of \(y = z = 0\) given by \(r_1 = e_{13}\). Hence, we have the equivalence class of \(r\)-matrices:

\[
r_0 = 0, \quad r_\pm = \pm e_{12}, \quad r = e_{13},
\]

as depicted in color in Figure 19.
11.4.5 The Lie algebra $\mathfrak{r}_{3,\lambda}$ ($\lambda \in (-1,1)$)

Let us consider the Lie algebra $\mathfrak{r}_{3,\lambda} := \langle e_1, e_2, e_3 \rangle$, whose basis $\{e_1, e_2, e_3\}$ satisfies the following commutation relations:

\[
[e_3, e_1] = e_1, \quad [e_3, e_2] = \lambda e_2, \quad [e_1, e_2] = 0, \quad \lambda \in (-1,1).
\]

Since $\mathfrak{r}_{3,\lambda}$ is not unimodular, the determination of coboundary Lie bialgebras on $\mathfrak{r}_{3,\lambda}$ requires to solve the corresponding modified classical Yang–Baxter equation. In the basis $\{e_{12}, e_{13}, e_{23}\}$ of $\Lambda^2\mathfrak{r}_{3,\lambda}$, an element $r$ can be written as $r = xe_{12} + ye_{13} + ze_{23}$ and the modified classical Yang-Baxter equation reads

\[
[r, r]_S = 2(\lambda - 1)yz e_{123}.
\]

Hence, the space of $r$-matrices, $YB$, can be divided into the plane with $y = 0$ and the plane of points with $z = 0$.

Let us determine $(\Lambda^2\mathfrak{r}_{3,\lambda})^{t_{3,\lambda}}$ to study whether different $r$-matrices give rise to different coboundary coproducts. Assume first that $\lambda \neq 0$. Every non-zero element of $(\Lambda^2\mathfrak{r}_{3,\lambda})^{t_{3,\lambda}}$ must give rise, in the standard way, to a two-dimensional subspace of $\mathfrak{r}_{3,\lambda}$ containing $[t_{3,\lambda}, t_{3,\lambda}] = \langle e_1, e_2 \rangle$. Hence, $(\Lambda^2\mathfrak{r}_{3,\lambda})^{t_{3,\lambda}} \subset \langle e_{12} \rangle$. Since $e_{12}$ is not $t_{3,\lambda}$-invariant for any $\lambda \in (-1,1)$, it follows that $(\Lambda^2\mathfrak{r}_{3,\lambda})^{t_{3,\lambda}} = \{0\}$ and every $r$-matrix gives rise to a different coproduct. Let us now consider the case $\lambda = 0$. To find elements of $(\Lambda^2\mathfrak{r}_{3,0})^{t_{3,0}}$, we employ Proposition 11.2. Since $[t_{3,0}, t_{3,0}] = \langle e_1 \rangle$, an element $w \in (\Lambda^2\mathfrak{r}_{3,0})^{t_{3,0}}$ is of the form $w = e_1 \wedge v, v \in \mathfrak{r}_{3,0}$. By Lemma 11.1, the adjoint action of $\mathfrak{r}_{3,0}$ on the ideal $\langle e_1, v \rangle$ must be traceless. Since this condition cannot be satisfied for $\text{ad}_{e_3}$, it follows that $(\Lambda^2\mathfrak{r}_{3,0})^{t_{3,0}} = \{0\}$.

As standard, we now accomplish the classification of the coboundary coproducts up to inner automorphisms of $\mathfrak{r}_{3,\lambda}$. Since $(\Lambda^2\mathfrak{r}_{3,0})^{t_{3,0}} = 0$, this demands to obtain classes of solutions of the modified classical Yang-Baxter equations in three subcases: a) $y = 0$ with $z \neq 0$, denoted by $YB_1$; b) $z = 0$ with $y \neq 0$, denoted by $YB_2$, and c) the line $y = z = 0$ denoted by $YB_3$.

The desired classification can be achieved by analyzing the fundamental vector fields of the action of $\text{Inn}(\mathfrak{r}_{3,\lambda})$ on $\Lambda^2\mathfrak{r}_{3,\lambda}$. These are spanned by

\[
X_1 := z \frac{\partial}{\partial x}, \quad X_2 := -\lambda y \frac{\partial}{\partial x}, \quad X_3 := (1 + \lambda)z \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \lambda z^2 \frac{\partial}{\partial z}.
\]

Let us consider $YB_3$. The points for $y = z = 0$ and $x \neq 0$ are such that the distribution generated by the fundamental vector field of the action of $\text{Inn}(\mathfrak{r}_{3,\lambda})$ is one-dimensional. Meanwhile, the distribution has rank zero at $x = y = z = 0$. Hence, $YB_3$ is divided into three orbits for points $(x, 0, 0)$ with $x > 0$, $x < 0$, and $x = 0$.

In the case $YB_1$, one has $y = 0$ and $z \neq 0$. Then

\[
Z_1|_{YB_1} = z \frac{\partial}{\partial x}, \quad Z_2|_{YB_1} = (1 + \lambda)z \frac{\partial}{\partial x} + \lambda z^2 \frac{\partial}{\partial z}
\]

span the tangent space to $YB_1$. Hence, this gives rise to two orbits of $YB_1$ for its points with $z < 0$ and $z > 0$, respectively.
If we restrict our problem to \( z = 0 \) and \( y \neq 0 \), i.e. \( YB_2 \), then the fundamental vector fields of the action of \( \text{Inn}(r_{3,0}) \) now read

\[
Z_1|_{YB_2} = 0, \quad Z_2|_{YB_2} = -\lambda y \frac{\partial}{\partial x}, \quad Z_3|_{YB_2} = (1 + \lambda)x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.
\]

These vector fields span the tangent space to \( YB_2 \) for \( y \neq 0 \). In consequence, we have two orbits of points with \( y > 0 \) and \( y < 0 \), correspondingly. Previous results are summarised in the Figure 20.

It is now time to determine the classification of coboundary coproducts up to the action of \( \text{Aut}(r_{3,\lambda}) \). Since the space \([r_{3,\lambda}, r_{3,\lambda}] = \langle e_1, e_2 \rangle\) is invariant under automorphisms of \( r_{3,\lambda} \), the space \( \langle e_{12} \rangle \) is also invariant relative to the action of elements of \( \text{Aut}(r_{3,\lambda}) \).

Moreover, the automorphisms of the form

\[
T_{\alpha,\beta}(e_1) = \beta e_1, \quad T_{\alpha,\beta}(e_2) = \alpha e_2, \quad T_{\alpha,\beta}(e_3) = e_3, \quad \forall \alpha \in \mathbb{R}\setminus\{0\}
\]

are such that the induced \( \Lambda^2 T_{\alpha,\beta} \) enable us to obtain that \( \langle e_{12} \rangle \) has only two equivalence classes: 0 and \( e_{12} \). This finishes the study of solutions with \( y = z = 0 \).

Meanwhile, the \( \Lambda^2 T_{\alpha,\beta} \) change the sign of \( y \) and \( z \). This maps the two semiplane orbits for \( \text{Inn}(r_{3,\lambda}) \) for the \( r \)-matrices with \( z = 0 \) and \( y = 0 \). Therefore, we get three classes of inequivalent coboundary coproducts (up to the action of \( \text{Aut}(r_{3,\lambda}) \)) induced by the \( r \)-matrices: \( r_0 = e_{12}, r_y = e_{23}, \) and \( r_z = e_{13} \). This is depicted in Figure 21.

Let us now tackle the case \( \lambda = 0 \). The corresponding Lie algebra is denoted by \( r_{3,0} \). The analysis of solutions to the modified classical Yang-Baxter equation for aforesaid subcases a) and b) goes similarly as in the previous case. The fundamental vector fields of the action of \( \text{Inn}(r_{3,0}) \) read

\[
Z_1 := z \frac{\partial}{\partial x}, \quad Z_2 := 0, \quad Z_3 := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.
\]

If \( z = y = 0 \), the previous vector fields span a one-dimensional distribution for \( x \neq 0 \) and zero-dimensional for \( x = 0 \). We obtain therefore three orbits gathering those points with \( z = y = 0 \) and equal sign of \( x \).

Restricting to the case \( y = 0, z \neq 0 \), we get

\[
Z_1|_{YB_1} = z \frac{\partial}{\partial x}, \quad Z_2|_{YB_1} = x \frac{\partial}{\partial x},
\]

which span \( \langle \partial/\partial x \rangle \). Thus, the orbits of the action of \( \text{Inn}(r_{3,0}) \) on this space are lines \( (x, 0, z_0) \) with a constant value \( z_0 \neq 0 \). Restricting to the case \( z = 0, y \neq 0 \), we get a unique vector field

\[
Z_2|_{YB_1} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},
\]

which span \( \langle x\partial/\partial x + y\partial/\partial y \rangle \). Thus, the orbits of the action of \( \text{Inn}(r_{3,0}) \) on this space are lines \( (\mu x, \mu y, 0) \) with \( \mu > 0 \).

The automorphisms \( T_{\alpha,\beta,\gamma} \) of the form

\[
T_{\alpha,\beta,\gamma}(e_1) := \alpha e_1 + \gamma e_2, \quad T_{\alpha,\beta,\gamma}(e_2) := \beta e_2, \quad T_{\alpha,\beta,\gamma}(e_3) := e_3,
\]

Figure 21: Orbits of the action of \( \text{Aut}(r_{3,\lambda}) \) on \( YB \subset \Lambda^2 r_{3,\lambda} \).
with $\alpha, \beta \in \mathbb{R}\backslash\{0\}$ and $\gamma \in \mathbb{R}$, are such that $\Lambda^2 T_{\alpha,\beta,\gamma}$ identify the lines $(x, 0, z_0)$ and $(x_0, y_0, 0)$ with different $z_0 \neq 0$ and $x_0, y_0$ among themselves. In consequence, we get two $r$-matrices $r_y := e_{13}$ and $e_{23}$. If $z = y = 0$, the automorphisms $\Lambda^2 T_{\alpha,\beta,\gamma}$ map points with positive and negative values of $x$.

We get three classes of inequivalent non-zero coboundary coproducts up to Lie algebra automorphisms of $\mathfrak{t}_{3,\lambda}$ induced by the $r$-matrices given by the non-zero $r$-matrices $r_0 = e_{12}$, $r_y = e_{23}$ and $r_z = e_{13}$, as shown in Figure 21.

11.4.6 The Lie algebra $\mathfrak{t}_{3,\lambda}'(\lambda > 0)$

Let us consider the Lie algebra $\mathfrak{t}_{3,\lambda}' := \langle e_1, e_2, e_3 \rangle$, where $\{e_1, e_2, e_3\}$ is a basis thereof with commutation relations

$[e_3, e_1] = \lambda e_1 - e_2, \quad [e_3, e_2] = \lambda e_2 + e_1, \quad [e_1, e_2] = 0, \quad \lambda > 0$.

This Lie algebra is not unimodular and we cannot ensure straightforwardly that every element of $\Lambda^2 \mathfrak{t}_{3,\lambda}'$ is an $r$-matrix. The corresponding modified classical Yang-Baxter equations read

$[r, r]_S = -2(y^2 + z^2)e_{123}$.

Hence, the only solutions have $y = z = 0$. We call this space of solutions $YG$.

The space $(\Lambda^2 \mathfrak{t}_{3,\lambda}')_{\mathfrak{t}_{3,\lambda}}$ is such that every element must induce in the standard way a subspace of $\mathfrak{t}_{3,\lambda}'$, containing $[\mathfrak{t}_{3,\lambda}', \mathfrak{t}_{3,\lambda}] = \langle e_1, e_2 \rangle$. In consequence, one gets that $(\Lambda^2 \mathfrak{t}_{3,\lambda}')_{\mathfrak{t}_{3,\lambda}} \subset \langle e_{12} \rangle$. Since $\lambda \neq 0$, then $(\Lambda^2 \mathfrak{t}_{3,\lambda}')_{\mathfrak{t}_{3,\lambda}} = 0$.

The invariance of the Killing metric of $\mathfrak{t}_{3,\lambda}'$ under automorphisms and the Lie algebra structure show that Proposition 11.3 applies and all automorphisms have the form

$T(e_1) := \alpha e_1, \quad T(e_2) := \alpha e_2, \quad T(e_3) := e_3, \quad \alpha \in \mathbb{R}\backslash\{0\}$,

is such that $\Lambda^2 T(e_{12}) = \alpha^2 e_{12}$. Therefore we obtain three coboundary coproducts invariant under the action of $\text{Aut}(\mathfrak{t}_{3,\lambda}')$ given by the $r$-matrices $\pm e_{12}$ and 0. The result is summarised in Figure 22 aside.

12 Conclusion and outlook

As a main achievement, this work has extended methods from Lie algebra theory, like root decompositions and $g$-invariant maps, to the realm of Grassmann algebras. This allowed for a deeper understanding of Lie bialgebras, their $g$-invariant multivectors, classical modified Yang-Baxter equations, and their classification up to Lie algebra automorphisms. This was partially due to the fact that objects appearing in Lie algebras and Lie bialgebras admit a simpler description in terms of entities in their related Grassmann algebras. Our achievements have been applied to the problem of classification of coboundary Lie algebras in general and the three-dimensional real case has been studied in detail. Our approach simplifies needed calculations to accomplish the classification up to Lie algebra automorphisms of coboundary real Lie bialgebras. For instance, we may skip the determination of all automorphisms of the underlying Lie algebra as in the previous literature [19].
Our techniques can be applied to the classification problem of higher-order Lie bialgebras and their applications to physics. This will be the goal of future works. Additionally, we are interested in studying techniques of quantization of the obtained Lie bialgebras as well as the study of topological techniques on the diagrams on $A^2_\mathbb{R}g$ to study the properties of the associated Drinfeld doubles and the induced Lie algebra structures on $g^*$.

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