UNIFORMITY FOR INTEGRAL POINTS ON SURFACES, POSITIVITY OF LOG COTANGENT SHEAVES, AND HYPERBOLICITY

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Abstract. We prove that the Lang-Vojta conjecture implies the number of stably integral points on curves of log general type, and surfaces of log general type with positive log cotangent sheaf are uniformly bounded. This generalizes work of Abramovich and Abramovich-Matsuki. In addition, we show that (1) all subvarieties of a surface with positive log cotangent bundle are of log general type, and that (2) smooth quasi-projective surfaces with positive and globally generated log cotangent have finitely many integral points, generalizing a theorem of Moriwaki.

1. Introduction

One of the most intriguing consequences of Lang’s Conjecture, proved in [CHM97], is that the number of rational points on curves of genus \(g \geq 2\) over a number field \(K\) is not only finite, but is also bounded by a constant that depends only on \(g\) and \(K\) (see Theorem 2.2). The original ideas of Caporaso, Harris, and Mazur, have since been extended and generalized leading to proofs that similar uniformity statements, conditional on the Lang Conjecture, hold in higher dimensions (see Section 2.1 for more details).

A natural question addressed in [Abr97b] is the following: does the Lang-Vojta Conjecture (Conjecture 2.3) imply similar uniformity statements for integral points? Abramovich showed this cannot hold unless one restricts the possible models used to define integral points. This led Abramovich to define the notion of \textit{stably integral points} (see Definition 4.1) to prove uniformity results, conditional on the Lang-Vojta Conjecture, for integral points on elliptic curves, and together with Matsuki in [AM01] for integral points on principally polarized abelian varieties (PPAVs). Roughly speaking, stably integral points are \((S, D)\)-integral points which remain integral after stable reduction.

The goal of this paper is to generalize uniformity statements for stably integral points to pairs of log general type. The first result we obtain is the generalization of the results of [Abr97b] to arbitrary stable pointed curves. Unless stated otherwise, \(K\) will denote a number field, and \(S\) a finite set of places of \(K\) containing the Archimedean ones.

\textbf{Theorem 1.1} (see Theorem 4.6). \textit{Assume the Lang-Vojta Conjecture. If \((C, D)\) is a pointed stable curve over \(K\), then the set of stably \(S\)-integral points on \(C\) is uniformly bounded.}

To obtain analogous results in higher dimensions one needs to tackle an extra problem: stable models do not exist due to the absence of a semistable reduction theorem over arbitrary Dedekind domains. As a result, there is no canonical choice of model for higher dimensional algebraic varieties.

Instead, using recent results on moduli of stable pairs (see Definition 3.1) the higher dimensional analogue of the moduli of stable pointed curves, we define “good” models which play the role of stable models for curves, and actually generalize them (see Section 5). This allows us to define a notion of \textit{moduli-stably-integral points} (\(ms\)-integral points, see Definition 5.3), integral points which are integral with respect to a fixed model of the moduli space.
The second result we obtain is that ms-integral points on families of stable pairs lie in a subscheme whose degree is uniformly bounded, which generalizes a theorem of Hassett [Has96, Theorem 6.2].

**Theorem 1.2** (see Corollary 6.2). Assume the Lang-Vojta Conjecture. Suppose that \( f : (X, D) \to B \) is a stable family over a smooth variety \( B \) with integral and openly log canonical (see Definition 3.7) general fiber over \( K \). For all \( b \in B(K) \), there exists a proper closed subscheme \( A_b \) containing all ms-integral points of \( X_b \) whose irreducible components have uniformly bounded degree.

There are two obstacles to obtaining uniformity. The first, which appears also in the case of rational points, is the presence of curves with non-positive Euler characteristic, which contain infinitely many integral points. One natural approach to circumvent this problem is by asking for positivity of the log cotangent sheaf. In particular, if a smooth proper variety has ample cotangent, then all subvarieties are of general type [Laz04, 6.3.28]. The extension of this property to the logarithmic setting is much more intricate. First, the log cotangent sheaf is never ample (see Proposition 7.6).

Instead, one can ask that the log cotangent sheaf is almost ample (see Definition 7.8) – asking that the sheaf is as positive as possible. For surfaces we prove the following.

**Proposition 1.3.** (see Proposition 7.10 and Corollary 7.13 for slc extension). Let \((X, D)\) be a log canonical surface pair with almost ample log cotangent. Then all pairs \((Y, E := Y \cap D)\) with \( Y \subset X \), such that \( Y \) is not contained in \( D \) are of log general type.

From here, the standard way to conclude uniformity from Theorem 1.2, is to run an induction argument once you answer “yes” to the following question, thus overcoming the second obstacle:

**Question 1.** Do ms-integral points satisfy the subvariety property – i.e. are ms-integral points for a pair \((X, D)\) lying on a pair \((Y, E := Y \cap D)\) with \( Y \subset X \) also ms-integral points for \((Y, E)\)?

The above question was answered affirmatively for abelian surfaces using Néron models [AM01]. Without having an explicit model to work with in higher dimensions, we reduce the question to a sufficient geometric criterion. In general, this is quite subtle. However, assuming positivity of the log cotangent sheaf, we are able to verify the subvariety property, thus proving uniformity.

**Theorem 1.4** (see Corollary 11.2). Assume the Lang-Vojta Conjecture. Let \((X, D)\) be a log canonical stable surface pair with good model \((\mathcal{X}, \mathcal{D})\) such that

1. \( D = \sum D_i \) is an effective \( \mathbb{Q} \)-Cartier divisor with \( K_X + D \) ample and
2. each fiber of \((\mathcal{X}, \mathcal{D})\) has almost ample log cotangent (see Definition 7.8),

then there exists a constant \( N = N(K, S, v) \) where \( v \) is the volume of \((X, D)\), such that the set of ms-integral points of \((X, D)\) has cardinality at most \( N \), i.e.

\[
\#(X \setminus D)({\mathcal{O}}_{K,S}^{ms}) \leq N = N(K, S, v)
\]

**Remark 1.5** (see Remark 7.7(2)). Assuming the Lang-Vojta conjecture, our methods give a proof for uniformity under any assumption that guarantees that all subvarieties are of log general type. We argue that asking for almost ample log cotangent is the most natural from a geometric standpoint.

It is also natural to ask whether positivity on the log cotangent sheaf implies finiteness statements for integral points. Indeed, Moriwaki proved that projective varieties \( X \) over a number field \( K \) with ample and globally generated \( \Omega^1_X \) have finitely many \( K \) points [Mor95, Theorem E]. We prove the following generalization.
**Theorem 1.6** (see Theorem 7.14). Let $V$ be a smooth quasi-projective surface with log smooth compatification $(X, D)$ over a number field $K$. If the log cotangent sheaf $\Omega^1_X(\log D)$ is globally generated and almost ample, then for any finite set of places $S$ the set of $S$-integral points $V(\mathcal{O}_{K, S})$ is finite.

Alternatively, Theorem 1.6 can also be seen as a consequence of the log cotangent being almost ample (using Proposition 1.3) and the following (see also [Mor95, Corollary C]).

**Theorem 1.7** (see Theorem 7.15). Let $V \cong (X \setminus D)$ be a log smooth surface over $K$. If the log cotangent sheaf $\Omega^1_X(\log D)$ is globally generated, then for any finite set of places $S$, every irreducible component of $V(\mathcal{O}_S)$ is geometrically irreducible and isomorphic to a semi-abelian variety.

Obstacles to proving many of the above theorems for $\dim(X) > 2$ stem from, e.g. the presence of singular subvarieties, and subtleties in defining their log cotangent sheaves (see Remark 7.11 and Remark 11.3).

In the process of proving uniformity, we stumbled upon the following example, which discusses hyperbolicity in families. In particular, positivity of the log cotangent sheaf on the normalization of every fiber is not enough to guarantee hyperbolicity is a closed condition.

**Remark 1.8.** (See Example 11.6) We show the existence of a stable family $(X, D) \to B$ where $B$ is a curve, the generic fiber $(X_\eta, D_\eta) \subset \mathbb{P}^3$ is a normal surface with almost ample log cotangent (and therefore hyperbolic), while the special fiber $(X_0, D_0)$, although having almost ample log cotangent on the normalization, contains a curve in the non-normal locus which is not of log general type.

One main ingredient in this paper, following the ideas of [CHM97], is a Fibered Power Theorem, proved in [AT16], which gives the analogue for pairs of the main Theorem of [Abr97a].

**Theorem 1.9** (see [AT16]). Let $(X, D) \to B$ be a stable family such that the general fiber is integral and openly log canonical (see Definition 3.7) over a smooth projective variety $B$. Then there exists an integer $n > 0$, a positive dimensional pair $(W, \Delta)$ openly of log general type (see Definition 3.2 and Remark 3.4), and a morphism $(X^n, D_n) \to (W, \Delta)$.

There are three appendices. In Appendix A, we define the stack of stable pairs over $\mathbb{Q}$. This is probably known, but we include it for lack of reference. In Appendix B we show there exists an almost ample log cotangent sheaf on the universal family of the moduli stack. Appendix C gives an alternative definition of $ms$-integral points that does not depend on the choice of models of stacks.

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In this paper, $K$ always denotes a number field.

2. Previous results

In this section we discuss previous results on uniformity of rational and integral points.
2.1. Uniformity for rational points. Faltings proved that for projective curves \( C \) over \( K \) of genus \( g = g(C) \geq 2 \), the set \( C(K) \) is finite [Fal83]. In higher dimensions there is a conjectural analogue:

**Conjecture 2.1** (Bombieri-Lang (surfaces), Lang (dim \( X > 2 \)), [Lan86a] and [Lan86b]). Let \( X \) be an algebraic variety defined over \( K \). If \( X \) is of general type, then the set \( X(K) \) is not Zariski-dense.

[CHM97] showed that Conjecture 2.1 implies that \( \# C(K) \) in Faltings’ Theorem is not only finite, but is also uniformly bounded by a constant \( N = N(g, K) \) that does not depend on the curve \( C \).

**Theorem 2.2.** [CHM97] Let \( K \) be a number field and \( g \geq 2 \) an integer. Assume Lang’s Conjecture. Then there exists a number \( B = B(K, g) \) such that for any smooth curve \( C \) defined over \( K \) of genus \( g \) the following holds: \( \# C(K) \leq B(g, K) \)

Pacelli [Pac97] (see also [Abr95]), proved that \( N \) only depends on \( g \) and \( \deg(K : \mathbb{Q}) \). More recently, cases of Theorem 2.2 have been proven unconditionally ([KRZB16], [Sto13] and [Paz15]) depending on the Mordell-Weil rank of the Jacobian of the curve and for [Paz15], on an assumption related to the Height Conjecture of Lang-Silverman. It has also been shown that families of curves of high genus with a uniformly bounded number of rational points in each fiber exist [DNP97].

Naïve translations fail in higher dimensions as subvarieties can contain infinitely many rational points. However, one can expect that after removing such subvarieties the number of rational points is bounded. Hassett proved that for surfaces of general type this follows from Conjecture 2.1, and that the set of rational points on surfaces of general type lie in a subscheme of uniformly bounded degree [Has96].

The idea behind both proofs is the following: consider a family \( f : X \to B \) whose general fibers are general type curves (resp. surfaces) over a base \( B \), the proof reduces to showing that the number of rational points in the fibers is uniformly bounded. If the total space of the family is itself a variety of general type, the Lang Conjecture trivially implies the uniformity statement. In general this is not the case, but if the family has maximal variation in moduli, then the dominant irreducible component of a high enough fibered power \( X^n_B \to B \) will be of general type. Conjecture 2.1 and an induction argument will then give uniformity. In general, it is always true that for \( n \) big enough \( X^n_B \) admits a dominant map to a variety of general type. From this, one can conclude the result in a similar fashion. This can be applied to a “global” family of curves to obtain the result of [CHM97].

The algebro-geometric result alluded to above is known as a fibered power theorem and was shown for curves in [CHM97], for surfaces [Has96] and in general by Abramovich [Abr97a]. The pairs analogue is Theorem 1.9 ([AT16]). In higher dimensions, similar uniformity statements hold conditionally on Lang’s Conjecture, and follow from the fibered power theorem under some additional hypotheses that take care of the presence of subvarieties that are not of general type ([AV96]).

2.2. Uniformity of Integral Points. The analogue of Faltings’ Theorem for quasi-projective curves is Siegel’s Theorem—every affine curve of positive Euler characteristic possesses a finite number of \( S \)-integral points. There is a conjectural generalization to higher dimensions, that extends Lang’s Conjecture 2.1 to the quasi-projective case:

**Conjecture 2.3** (Lang-Vojta). Let \( X \) be a quasi-projective variety and let \( \mathcal{X} \to \text{Spec} \mathcal{O}_{K,S} \) be a model over the \( S \)-integers. If \( X \) is openly of log general type, then \( \mathcal{X}(\mathcal{O}_{K,S}) \) is not Zariski dense.

A natural question, is whether Conjecture 2.3 implies a uniform bound on the set of \( S \)-integral points for quasi-projective curves openly of log general type (see Definition 3.2). Abramovich ([Abr97b, 0.3]) gave a counterexample: he constructed an elliptic curve, where the number of \( S \)-integral points in the complement of the origin grow arbitrarily when one suitably changes the model.
However, imposing minimality conditions on the model leads to statements similar to [CHM97]. In particular, if one considers stable models for the quasi-projective curves $E \setminus O_E$ over a number field, then the cardinality of the set of $S$-integral points of this model, called stably-integral points, is uniformly bounded, conditional on Conjecture 2.3. This was extended to PPAVs of dim 2 [AM01].

Both results rely on the existence of good models for elliptic curves and abelian varieties. While this can be extended to arbitrary stable curves, it is not clear how to define stable models in arbitrary dimensions outside of the abelian case. However, it was observed by Abramovich and Matzuki that stably integral points admit a nice moduli interpretation (see Section 5).

Unconditional results for uniformity of integral points in certain classes of curves, coming from Thue Equations, were proved in [LT02], given some bound on the Mordell-Weil rank of the Jacobian.

As Vojta’s Conjecture implies Lang’s Conjecture, and thus a uniform version of the conjecture, one can ask if Vojta’s conjecture implies a uniformity statement for heights. This was shown by Ih for curves [Ih02], and was generalized to certain families of hyperbolic varieties [AJ17].

3. Preliminaries and Notations

The ring of $S$-integers, i.e. the set $\{x \in K : \|x\|_v \leq 1, \forall v \notin S\}$ will be denoted by $O_{K,S}$.

**Definition 3.1.** Given an algebraic variety $X$ defined over $K$, a model of $X$ over $O_{K,S}$ is a separated scheme $X$ together with a flat map $X \to \text{Spec} O_S$ of finite type such that the generic fiber is isomorphic to $X$, i.e. $X \cong X \times_{\text{Spec} O_S} \text{Spec} K$.

Given a quasi-projective variety we will use the following definition for openly of log general type.

**Definition 3.2.** (see [AT16, Definition 1.3]) A quasi-projective variety $X$ is openly of log general type if there exists a desingularization $\tilde{X} \to X$ and a projective compactification $\tilde{X} \subset Y$ with $D = Y \setminus \tilde{X}$ a divisor of normal crossings, such that $\omega_Y(D)$ is big.

The above definition is independent of both the choice of desingularization and of the compactification. From both the viewpoint of birational geometry and of integral points on quasi projective varieties, it has become natural to consider pairs of a variety and a divisor.

**Definition 3.3.** A pair $(X,D)$ is the datum of a projective variety $X$ and a $Q$-divisor $D = \sum d_i D_i$ which is a linear combination of distinct prime divisors.

**Remark 3.4.** We will often say that a pair $(X,D)$, with $X$ a projective variety and $D$ a normal-crossings divisor is openly of log general type if the quasi-projective variety $X \setminus D$ is. In some applications we will require some conditions on the singularities of the pair.

**Definition 3.5.** A pair $(X,D)$ has log canonical singularities (or is lc) if $X$ is normal, $K_X + D$ is $Q$-Cartier, and there is a log resolution (see [KM98, Notation 0.4(10)]) $f : Y \to X$ such that

$$K_Y + \sum a_E E = f^*(K_X + D)$$

where all the $a_E \leq 1$ and the sum goes over all irreducible divisors on $Y$. The pair has canonical singularities if all $a_E \leq 0$.

**Definition 3.6.** A pair $(X,D)$ is openly canonical if $X \setminus D$ has canonical singularities.

**Definition 3.7.** An lc pair $(X,D)$ is openly log canonical if it is openly canonical.
Definition 3.8. Consider a pair \((X, D)\) with \(D\) a \(\mathbb{Q}\)-Cartier divisor over \(K\). A **model** for \((X, D)\) over \(\text{Spec} \mathcal{O}_K\) is a model \(\mathcal{X}\) of \(X\) together with an (effective) \(\mathbb{Q}\)-Cartier divisor \(\mathcal{D} \to \mathcal{X}\) whose restriction to the generic fiber is isomorphic to \(D\). In other words, a model for \((X, D)\) is the datum of a model \(\mathcal{X}\) of \(X\) and a compatible model \(\mathcal{D}\) of \(D\).

Models of pairs can be used to define integral points with respect to a divisor.

Definition 3.9. Consider a pair \((X, D)\), with \(D\) a Cartier divisor, and a model \((\mathcal{X}, \mathcal{D})\) over \(\text{Spec} \mathcal{O}_{K,S}\). An \((S, D)\)-integral point is a section \(P : \text{Spec} \mathcal{O}_{K,S} \to \mathcal{X}\) such that the support of \(P^* \mathcal{D}\) is contained in \(S\). An \(S\)-integral point of a quasi-projective variety \(X \setminus D\) is an \((S, D)\)-integral point for the pair \((X, D)\).

3.1. Stability. As mentioned in Section 2.2, in order to obtain uniformity results for integral points, it is necessary to restrict the possible models under consideration. We recall here some definitions that will be useful later. First we need a crucial definition.

Definition 3.10. A pair \((X, D = \sum d_i D_i)\) is **semi-log canonical (slc)** if \(X\) is reduced and \(S_2\), the divisor \(K_X + D\) is \(\mathbb{Q}\)-Cartier and the following hold:

1. \(X\) is Gorenstein in codimension one, and
2. if \(\nu : X^\nu \to X\) is the normalization, then the pair \((X^\nu, D^\nu + \Delta_0^\nu)\) is log canonical, where \(D^\nu\) denotes the preimage of \(D\) and \(\Delta_0^\nu\) denotes the preimage of the double locus \(\Delta_0\) on \(X^\nu\).

Slc pairs with ample log canonical sheaf generalize stable pointed curves to higher dimension.

Definition 3.11. A pair is **stable** if the \(\mathbb{Q}\)-Cartier line bundle \(\omega_X(D)\) is ample, and the pair is semi-log canonical. A **stable family** is a flat family \((X, D) \to B\) over a normal variety \(B\) such that

1. \(D\) avoids the generic and codimension one singular points of every fiber,
2. \(\omega_f(D)\) is \(\mathbb{Q}\)-Cartier, and
3. \((X_b, D_b)\) is a stable pair for all \(b \in B\).

We end this subsection by introducing notation for fiber powers of families of stable pairs.

Definition 3.12. Given a stable family \(\pi : (X, D) \to B\), denote by \((X^n_B, D_n)\) the \(n\)th **fibered power** of \(X\) over \(B\), where \(X^n_B\) is the (unique) irreducible component of the fiber power which dominates \(B\), and \(D_n := \pi^n_1 D + \cdots + \pi^n_i D\), where \(\pi_i : X^n_B \to X\) is the \(i\)-th projection.

Remark 3.13. The **moduli space of stable pairs** \(\mathcal{M}_\Gamma\) is constructed and proven to have projective coarse space in [KP17]. It requires a choice of invariants \(\Gamma = (n, v, J)\), where \(n\) is the dimension of the pairs, \(v\) is their volume, and \(I\) is a coefficient set satisfying the DCC condition.

4. Uniformity for log stable curves

Abramovich observed [Abr97b] that uniformity statements for integral points on curves cannot hold without any restrictions on the model (see [AM01, 0.3] - for an example and discussion), and instead one should consider stable models. Such a choice provides good models (see section 5 for a more general framework that extends to higher dimension) which possess “positivity” properties preventing the appearance of non-hyperbolic components in the model. As a result, such positivity does not allow the number of integral points to grow arbitrarily. Abramovich’s notion of stably integral points for the complement of the origin in an elliptic curve can be easily generalized to any stable pointed curve \((C, D)\) as follows.
Definition 4.1. Let \((C, D)\) be a stable pointed curve defined over \(K\). A point \(P \in (C \setminus D)(K)\) is called a stably \((S, D)\)-integral point if there exists a finite extension \(L \supset K\) and a stable model \((\mathcal{C}, \mathcal{D})\) over \(\text{Spec} \mathcal{O}_{L, S_L}\) such that \(P\) is a \((S_L, \mathcal{D})\)-integral for \(\mathcal{C}\). Equivalently \(P \in (C \setminus D)(\mathcal{O}_{L, S_L})\).

Given Definition 4.1 one can ask whether the same uniformity results proved in [Abr97b] hold more generally for a pair \((C, D)\) openly of log general type. To prove this we introduce the following:

Definition 4.2. Let \((C, D) \to B\) be a family of pointed stable curves over \(K\). Given a subset \(\mathcal{P} \subset C(K)\), denote by \(\mathcal{P}^n \subset C^n_B\) the \(n\)th fibered power of \(\mathcal{P}\) over \(B\). Then \(\mathcal{P}\) is \(n\)-correlated if there exists an \(n > 0\) such that \(\mathcal{P}^n\) is contained in a proper closed subset of \(C^n_B\).

The importance of \(n\)-correlated sets for uniformity questions is apparent in the following:

Lemma 4.3. Let \(C \to B\) a family of projective irreducible curves and let \(\mathcal{P}\) be an \(n\)-correlated subset of \(C(K)\). There exists a nonempty open set \(U \subset B\) and an \(N \in \mathbb{N}_{>0}\) such that for every \(b \in U\), \(\mathcal{P} \cap C_b \leq N\).

Proof. See [CHM97, Lemma 1.1], [Abr97b, Lemma 1], or [AM01, Lemma 1.1.2]. \(\square\)

In view of Lemma 4.3, in order to prove uniformity for stably integral points on curves openly of log general type, we need to prove that the set of stably integral points is \(n\)-correlated. We start by stating the following lemma, which will be important throughout.

Lemma 4.4. Let \((X, D)\) be a pair defined over \(K\) and let \(\phi : (X, D) \to (W, E)\) be a dominant morphism. Then, given a proper model \((X', D')\) over \(\mathcal{O}_{K, S}\), there exists \(S' \supset S\) and a model \((W, \mathcal{E})\) over \(\mathcal{O}_{K, S'}\) such that \(\phi\) extends to a map \((X, D)_{\text{Spec} \mathcal{O}_{K, S'}} \to (W, \mathcal{E})\).

Proof. The extension property follows from spreading out techniques, noting that the extensions to models of \(X\), \(W\), \(D\) and \(E\) are compatible after possibly enlarging \(S\). \(\square\)

Recall that a stable pair (Definition 3.11) in dimension one is a projective curve with at worst nodal singularities and a reduced divisor disjoint from the singular locus. Therefore we obtain:

Proposition 4.5. Let \(\pi : (C, D) \to B\) be a family of stable pointed curves with smooth general fiber and let \(\mathcal{P}\) be the set of stably integral points of the family. Then the Lang-Vojta Conjecture implies that \(\mathcal{P}\) is \(n\)-correlated for some \(n\) large enough.

Proof. We apply Theorem 1.9 to obtain a positive dimensional pair \((W, E)\) openly of log general type and a dominant morphism \((C^n, D_n) \to (W, E)\) which restricts to a regular map on the complement \(C^n \setminus D_n \to W \setminus E\). Applying Conjecture 2.3 to \((W, E)\) implies that there exists a proper closed subset \(V \subset W\) containing all the \((S, E)\)-integral points. By Lemma 4.4 there exists a proper closed subset \(F_n \subset C^n\) containing all the \((S, D_n)\)-integral points. Thus \(\mathcal{P}^n \subset F_n\). \(\square\)

Using Noetherian induction on the base we can prove the following:

Theorem 4.6. Assume the Lang-Vojta Conjecture. For all stable pointed curves \((C, D)\) defined over \(K\), the number of stably \(S\)-integral points on \(C\) is uniformly bounded.

Proof. Following [CHM97], we apply Proposition 4.5 to a “global” family of stable pointed curves as follows: let \(g\) be the genus of \(C\); by assumption we may assume that \(C\) is irreducible. The stability condition implies that \(2g - 2 + \text{deg} D\) is a positive integer. In particular, there exists \(l\), which does not depend on \(C\), such that each stable curve \(C'\) of genus \(g'\) and reduced divisor \(D'\) with \(g' = g\) and \(\text{deg} D' = \text{deg} D\), can be embedded in \(\mathbb{P}^N\) using the linear system \(|K_{C'}(D')|\). The theory of Hilbert
Schemes gives the existence of a family \( \pi : X \to B \) with \( \deg D \) sections defined over \( K \) such that given any pair \( (C', D') \) as before, there exists a \( K \)-rational point \( b \in B(K) \) such that \( X_b \cong (C', D') \). This family can be constructed by taking the closure of the locus of pluri-log canonical curves and its restriction to the universal family in the corresponding Hilbert scheme. By construction, the general fiber of \( X \to B \) is a smooth curve openly of log general type (in particular it is stable). Therefore, Proposition 4.5 applies, so the set of stably \( S \)-integral points of \( \pi : X \to B \) is \( n \)-correlated for \( n \) large enough. By Lemma 4.3, this implies the existence of an open subset \( U \subset B \) such that for every \( K \)-rational \( b \in U \), there exists a non-negative integer \( N = N(K, S, g, \deg D) \) such that

\[
X_b(O^\text{stably}_{K,S}) \leq N = N(K, S, g, \deg D).
\]

Finally one applies Noetherian induction on the dimension of \( B \setminus U \) to obtain a similar bound for all fibers in the family \( \pi : X \to B \). More explicitly one defines \( B_1 \) to be the union of all irreducible components of \( B \setminus U \) whose generic point is a smooth curve and considers the corresponding restricted family \( X_1 \to B_1 \). Applying Lemma 4.3 to this new family gives the existence of an open set \( U_2 \subset B_1 \) where the stably integral points of the fibers are uniformly bounded, possibly by a different constant \( N_1 \). This inductively gives a chain of base schemes \( B_i \) such that \( \dim B_i < \dim B_{i-1} \) and therefore stabilizes after a finite number of steps. For each \( B_i \) one has a uniform bound given by a constant \( N_i \) outside an open subset \( U_{i+1} \). Taking \( N \) to be the maximum of all \( N_i \) shows that the \( ms \)-integral points of any fiber are at most \( N \). Since the family has been chosen to be global, it follows that such a bound holds for all stable curves, and thus this proves the theorem. \( \square \)

5. Good models

In this section we extend the notion of stably integral points to higher dimensions. First we need to construct models that play the role of stable models, since the latter are not known to exist for \( \dim \geq 2 \). The main idea behind our construction is to fix models for the moduli stack of stable pairs and construct the models of the pairs as base changes of the models of the stacks, noting that models for stacks are completely analogous to those of varieties. The starting point is the following observation of Abramovich-Matsuki that gives a nice moduli interpretation of stably integral points.

**Proposition 5.1** ([AM01], Proposition 3.1.3). Let \( (A, \Theta) \) be a PPAV defined over \( K \) and let \( P \in (A \setminus \Theta)(K) \). Consider the associated moduli map \( P_m : \text{Spec} \ K \to (A, \Theta) \to (\overline{A}_{g,1}, \Theta) \). Then \( P \) is stably \( S \)-integral if and only if \( P_m \) is an \( S \)-integral point in \( \overline{A}_{g,1} \setminus \Theta \).

This implies that one way to characterize stably \( S \)-integral points is to look at their image in an appropriate moduli space and test their integrality with respect to a model of such a moduli stack.

Contrary to the moduli space of PPAVs, the moduli space of stable pairs has not been defined over \( \text{Spec} \ O_{K,S} \). We remedy this by fixing a model over a Dedekind domain and show that our results are independent of the choice of such a model.

5.1. **Construction of good models.** By Appendix A the moduli stack of stable pairs can be defined over \( \mathbb{Q} \). We now make a choice of models of such stacks as follows: choose a ring of integers
\(\mathcal{R}\) of a number field \(K_{\mathcal{R}}\) and models of \(\mathcal{M}_{\Gamma}, \mathcal{U}\) and \(\mathcal{D}\) over \(\mathcal{R}\) such that in the following diagram

\[
\begin{array}{ccc}
(U, D) & \rightarrow & (U, D) \\
\downarrow & & \downarrow \\
\mathcal{M}_{\Gamma} & \rightarrow & \mathfrak{M}_{\Gamma} \\
\downarrow & & \downarrow \\
\text{Spec } K_{\mathcal{R}} & \rightarrow & \text{Spec } \mathcal{R}
\end{array}
\]

\(U, D\) and \(\mathfrak{M}_{\Gamma}\) are proper stacks over \(\mathcal{R}\) that are models for \(U, D\) and \(\mathcal{M}_{\Gamma}\) over \(\mathcal{R}\). Note that the existence of the diagram follows from Section A and the definition of models. We can define models for stable pairs with respect to the choices of the models and of \(\mathcal{R}\) as follows:

**Definition 5.2.** Given a stable pair \((X, D)\) defined over \(K\), let \((\mathcal{X}, \mathcal{D})\) be any model over \(\text{Spec } \mathcal{O}_{K, S}\).

We say that \((\mathcal{X}, \mathcal{D})\) is a good model (with respect to the choices of the moduli stacks, the ring of integers \(\mathcal{R}\) and the models of the stacks) if there exists a number field \(L \supset K\) such that \(\mathcal{O}_{L, S_L} \supset \mathcal{R}\), where \(S_L\) is the set of places lying over \(S\), and

\[
(\mathcal{X}_{\mathcal{O}_{L, S_L}}, \mathcal{D}_{\mathcal{O}_{L, S_L}}) \simeq (\text{Spec } \mathcal{O}_{L, S_L} \times_{\mathcal{R}_S} U, \text{Spec } \mathcal{O}_{L, S_L} \times_{\mathcal{R}_S} \mathcal{D}),
\]

where \(\mathcal{X}_{\mathcal{O}_{L, S_L}}\) and \(\mathcal{D}_{\mathcal{O}_{L, S_L}}\) are the base change through \(\text{Spec } \mathcal{O}_{L, S_L} \rightarrow \text{Spec } \mathcal{O}_{K, S}\). We say \((\mathcal{X}, \mathcal{D})\) is defined over \(\mathcal{O}_{L, S_L}\) if there is a number field \(L\) such that the above isomorphism holds.

Good models play the role of stable models in dimension one and are the key ingredient in the definition of moduli stably integral points for stable pairs.

**Definition 5.3.** Let \((X, D)\) be a stable pair over \(K\). A rational point \(P\) is moduli stably \(S\)-integral (ms-integral for short when the reference to \(S\) is clear), if there exists a finite extension \(L \supset K\) and a good model \((\mathcal{X}, \mathcal{D})\) over \(\text{Spec } \mathcal{O}_{L, S_L}\) such that the image of the map:

\[
P : \text{Spec } L \rightarrow (X_L, D_L) \rightarrow (\mathcal{X}_{\mathcal{O}_{L, S_L}}, \mathcal{D}_{\mathcal{O}_{L, S_L}})
\]

is \((S_L, D_L)\)-integral in the good model \((\mathcal{X}_{\mathcal{O}_{L, S_L}}, \mathcal{D}_{\mathcal{O}_{L, S_L}})\). We will denote the set of all moduli stably \(S\)-integral points of \((X, D)\) as \(X(\mathcal{O}_{\mathcal{K}, S}^{\text{ms}})\).

**Remark 5.4.** For any two choices of models of the stacks over two different rings \(\mathcal{R}\) and \(\mathcal{R}'\), we can always find a ring of integers \(\mathfrak{B}\) containing \(\mathcal{R}\) and \(\mathcal{R}'\) such that the base change of any of the two models will define a model over \(\mathfrak{B}\) and any ms-integral point with respect to \(\mathcal{R}\) and \(\mathcal{R}'\) is integral with respect to \(\mathfrak{B}\). In particular the results of the following sections of this paper do not depend on the choice of the model, up to extending the ring of integers that we consider, and possibly adjusting the constants involved (see e.g. [Ryd15, Appendix B]).

The notion of ms-integral point extends the notion of stably integral points both for curves and PPAVs. In fact, the following more general observation holds.

**Proposition 5.5.** Let \(\mathfrak{X}\) be a proper Deligne-Mumford stack representing a functor of stable pairs and admitting a universal family \(U\) and universal divisor \(D\) such that \(\mathfrak{X}\) and \((U, D)\) have models
over a ring of integers \( R \). Suppose that \( O_{K,S} \supset R \), and let \((X,D)\) be a pair defined over \( K \) which admits the following diagram

\[
\begin{array}{ccc}
(X,D) & \rightarrow & (X,D) \\
\downarrow & & \downarrow \\
\text{Spec} K & \rightarrow & \text{Spec} O_{K,S} \rightarrow \mathfrak{X}
\end{array}
\]

where \((X,D) \simeq (\text{Spec} O_K \times_X \mathfrak{U}, \text{Spec} O_K \times_X \mathfrak{D})\) is a good model. If \( P \in (X \smallsetminus D)(K) \) is an \( ms \)-integral point of \( X \), then the image of \( P \) in \((\mathfrak{U}, \mathfrak{D})\) is \((\mathfrak{D}, O_{K,S})\)-integral.

**Proof.** \( P \) is \( ms \)-integral if it is integral in \((X,D)\), which is the base change of the universal family to the ring of integers \( O_{K,S} \). This implies that the image of \( P \) is integral in \((\mathfrak{U}, \mathfrak{D})\). \(\square\)

Thus \( ms \)-integral points agree with the corresponding notion in [Abr97b] and [AM01].

**Remark 5.6.** We see that moduli stably integral points in dimension one for the moduli functor of stable curves, \( R = \mathbb{Z} \) and \( \mathcal{M}_\Gamma \) the stack \( \overline{\mathcal{M}}_{g,n} \), are stably integral points for curves. In fact, given a stable curve \( \mathcal{C} \) over \( K \), if \( L \supset K \) is the finite extension where \( \mathcal{C} \) acquires stable reduction then

\[
\text{Spec} O_{L,S,L} \times_{\mathcal{M}_{g,n}} \mathfrak{U}
\]

is a stable model for \( \mathcal{C} \) and a good model according to Definition 5.2. Therefore moduli stably integral points for these choices coincide with stably integral points for curves. The same holds for the moduli functor of principally polarized quasi-abelian schemes (or abelic pairs à la Alexeev) [Ale99] and \( R = \mathbb{Z} \). In this case a “good” model is a stable quasi-abelian model and \( ms \)-integral points are stably integral points of [AM01].

We stress that whenever the existence of a moduli stack of stable pairs over any finitely generated \( \mathbb{Z} \) algebra with modular interpretation is known, the definitions above (and the consequential results to follow) will, in particular, be valid for this choice of model. Moreover, up to enlarging \( S \), the model of the universal family \((\mathfrak{U}, \mathfrak{D}) \to \mathcal{M}_\Gamma \) will be a stable family for an appropriate definition of singularities for fibers with residue field of characteristic \( p > 0 \).

In Appendix C we present an alternative approach that, although leading to weaker results, does not depend on the existence of such a model used in this section.

**6. Uniform bound for the degree of the subscheme containing \( ms \)-integral points**

Given the definition of \( ms \)-integral points (Definition 5.3), one could ask what kind of uniformity statements can follow from the Lang-Vojta Conjecture. In this section, we prove that for a stable family with nice singularities, \( ms \)-stably integral points lie on a subscheme whose degree is uniformly bounded. The following is inspired by Hassett [Has96].

**Theorem 6.1.** Assume the Lang-Vojta Conjecture. Let \((X,D) \to B\) be a stable family defined over \( K \) with integral and openly log canonical generic fiber over a smooth projective variety \( B \). There exists an open set \( U \subset B \) such that for all rational points \( b \) in \( U(K) \), there exists a proper subscheme \( A_b \) containing all \( ms \)-integral points of \( X_b \) such that, if \( N(b) \) is the sum of the degrees of the components of \( A_b \), then \( N(b) \) is uniformly bounded.

**Proof.** By Theorem 1.9 there exists a positive integer \( n \) and a positive dimensional pair \((W, \Delta)\) openly of log general type and a dominant morphism \( (X^n_B, D^n) \to (W, \Delta) \), which induces a regular
map $X^*_B \setminus D_n \to W \setminus \Delta$. Assuming the Lang-Vojta Conjecture for any model of $(W, \Delta)$, there exists a proper closed subvariety of $W \setminus \Delta$ containing all the $(\Delta, S)$-integral points.

Define $Z_n$ to be the preimage of this subvariety, which by definition and by Lemma 4.4 contains all $S$-integral points of $X^*_B$ and therefore the $ms$-integral points of the fibered power (note that by Proposition 4.5 of [AT16], the pair $(X^*_B, D_n)$ is a stable pair and by construction, the definition of $ms$-integral points are compatible with taking fiber powers). Define, by induction, closed subvarieties of $X^*_B$ for each $1 \leq j \leq n$ in the following way:

1. For each $j$, denote by $\pi_j : X^*_B \to X^*_{B,j-1}$ and by $\pi_{ij} : X^*_{B,j} \to X^*_B$ the projection morphisms;
2. For each $k = 1, \ldots, j$, denote by $\pi_{j,1}^k : X^*_{B,j} \to X$ the $k$-th projection;
3. For each $j$, denote by $Z_j$ the maximal closed subset of $X^*_B \setminus D_j$ such that
   \[ \pi^*_{n,j}(Z_j) \setminus \sum_{k>j} \pi^*_{k,1} D \subset Z_n; \]
4. For each $j$, denote by $U_j$ the complement of $Z_j$ in $X^*_B \setminus D_j$. By construction, $U_n$ does not contain any $ms$-integral points. Moreover, by maximality of $Z_j$ for each $j$, the preimage $\pi^{-1}_j(u)$ is not contained in $Z_j$ for every $u \in U_{j-1}$.

Note that by definition, for $k \leq j - 1$ we have that
\[ \pi^*_{j-1,1} \circ \pi_j = \pi^*_{j,1}, \]
and for each $j$ we have that
\[ D_j = \pi^*_{j,1} D_{j-1} + (\pi^*_{j,1})^* D. \]

Then one has that
\[ \pi^{-1}_j(Z_{j-1}) \setminus (\pi^*_{j,1})^* D \subset Z_j. \]

For every $u \in U_{j-1}$, its inverse image in $X^*_B$ intersects $Z_j$ in a proper subvariety (since $\pi^{-1}_j(u)$ is not contained in $Z_j$). Call $A_j$ this subvariety (which might not be of pure dimension) and let $d_j = \sum \deg(A_j)$. Let $N = \max_j d_j$ and $b$ be a $K$-rational point of $B$, then we claim $X_b(O^ms_{K,S})$ lies in a subvariety of degree $\leq N$. To prove the claim we define an index $\hat{j}$ as
\[ \hat{j} = \min\{ j : U_j \cap X^*_b(O^ms_{K,S}) = \emptyset \}. \]

Note that by construction $\hat{j} \leq n - 1$. Now pick a rational point $u$ in $U_{\hat{j}}$ which lies in $X^*_b(O^ms_{K,S})$; this in particular implies that $\pi^{-1}_j(u) = X_b$. By the above discussion $\pi^{-1}_j(u)(O^m_{K,S}) \subset A_j =: A_b$, which is a subvariety of degree $d_j \leq N$. This proves the claim, and thus the theorem.

**Corollary 6.2.** Assume the Lang-Vojta Conjecture. In the notation above, for every fiber $X_b$ the set of $ms$-integral points $X_b(O^ms_{K,S})$ is contained in a proper subscheme $A_b$. Moreover, the sum of the degrees of the irreducible components of $A_b$ are uniformly bounded.

**Proof.** For the family $(X, D) \to B$, Theorem 6.1 gives a uniform bound $N_0$ for the degree of the subscheme $A_b$ in all fibers $X_b$ with $b$ in an open subset of $B$. Consider the restricted family $(X_1, D_1) \to B_1$ where $B_1$ is the union of the irreducible components of $B$ with integral and openly log canonical generic fiber. Applying Theorem 6.1 gives a bound $N_1$ for $A_b$ in all fibers with $b$ in an open subset of $B_1$. Iterating this, we find integers $N_0, \ldots, N_m$ (finitely many by being Noetherian), such that for every fiber $X_b$, the sum of the degrees of the irreducible components of the proper subscheme $A_b$ is bounded by the max of the $N_i$. \qed
7. Positivity of the log cotangent sheaf

Given the degree bound obtained in Theorem 6.1 one would like to conclude, assuming the Lang-Vojta Conjecture, that \( ms \)-integral points in a stable pair satisfy some uniformity. However, one cannot expect a result as strong as Theorem 4.6, since a stable pair can contain curves with negative Euler characteristic (and thus contain infinitely many \( ms \)-integral points). Even if we are in a case where the \( ms \)-integral points are finite, there is still another problem to be tackled, namely the subvariety property mentioned in the introduction.

More precisely, given a stable pair \((X, D)\) over \(K\) and an \( ms \)-integral point \(P\), let \(Y \not\subset D\) be an irreducible subvariety, and let \((Y, D_Y)\) be the pair with \(D_Y = Y \cap D\) and \(P \in Y \setminus D_Y\).

**Definition 7.1.** We say that \( ms \)-integral points satisfy the subvariety property if \(P\) is an \( ms \)-integral point for \((Y, D_Y)\).

To prove that \(P\) is \( ms \)-integral for \((Y, D_Y)\), we have to exhibit a good model of the pair, possibly after extending \(K\). If \((Y, D_Y)\) is stable, we can construct a good model (up to extending \(K\)) using the moduli map (Definition 5.2). Since we do not have explicit models for \(X\) and \(Y\), we instead consider a stronger geometric condition that would imply the subvariety property.

A natural condition one hopes would solve this is to assume some notion of hyperbolicity. For rational points on a (normal) variety \(X\), i.e. \(D = \emptyset\), Abramovich-Voiculescu coined the term geometrically-mordellic to mean all subvarieties of \(X\) are of general type. This is implied by the geometric condition that the cotangent sheaf \(\Omega^1_X\) is ample when \(X\) is smooth (see [Laz04, 6.3.28]). We therefore study the generalization of this notion to pairs, namely we study the \( log \) cotangent sheaf.

In the following subsection we investigate positivity properties of the log cotangent sheaf, its consequences for the geometry of subvarieties, and its extensions to non-normal varieties. For the remainder of this paper, the focus is on surfaces (c.f. Remarks 7.11 and 11.3).

7.1. Definition of the log cotangent sheaf. We begin by collecting some facts about the log cotangent sheaf for log canonical pairs. Like the cotangent sheaf, it is not clear how to define the log cotangent sheaf for non-normal varieties which appear naturally when considering families of normal varieties. We avoid this by working with the log cotangent sheaf of any resolution. On the other hand, positivity properties of the log cotangent sheaf are more subtle than the cotangent sheaf, even for smooth varieties. In particular, one can show that the log cotangent sheaf is never ample (see Proposition 7.6 and [BD17]). However, one can still obtain some hyperbolicity properties assuming the log cotangent sheaf is, in a sense, as positive as possible (see Proposition 7.10).

We begin by recalling some facts about reflexive differentials following [GKKP11].

**Definition 7.2.** (see [GKKP11, Notation 2.16]) Suppose \((X, D)\) is a log canonical pair. Consider the open set \(U \subset X\) whose complement is the singular locus of \(X\). Denote by \(i : U \to X\) the inclusion. Then the sheaf of reflexive differentials \(\Omega^{[1]}_X(\log D) := i_*(\Omega^1_U(\log D))\).

This sheaf is reflexive and torsion free but does not need to be locally free. However, the following theorem asserts that any logarithmic 1-form defined on the smooth locus \(U \subset X\) can be extended to a logarithmic 1-form possibly with poles on the exceptional divisors on any resolution of singularities.

**Theorem 7.3.** [GKKP11, Theorem 1.5] Let \((X, D)\) be a log canonical pair, let \(\pi : \tilde{X} \to X\) denote a log resolution and let \(\tilde{D} = \) largest reduced divisor contained in \(\text{supp} \pi^{-1}(\text{non-klt locus})\). The sheaf of reflexive differentials is isomorphic to \(\pi_*(\Omega^{[1]}_X(\log \tilde{D}))\).
7.2. **Positivity properties of the log cotangent sheaf.** As mentioned above, the sheaf $\Omega^1_X(\log D)$ is in general only a coherent reflexive sheaf, therefore we recall here the notion of ampleness in this context borrowing ideas from both [Vie95] and [KP17].

**Definition 7.4.** (See [KP17, Definition 3.7]) Let $F$ be a coherent sheaf on a normal and reduced quasi-projective variety $X$ and let $\mathcal{H}$ be an ample line bundle on $X$.

1. We say that $F$ is *ample* if there exists a positive integer $a > 0$ such that the sheaf $\text{Sym}^a F \otimes \mathcal{H}^{-1}$ is globally generated.
2. We say that $F$ is *big* if there exists a positive integer $a > 0$ such that the sheaf $\text{Sym}^a F \otimes \mathcal{H}^{-1}$ is generically globally generated.

**Remark 7.5.** These definitions are independent of the choice of $\mathcal{H}$ (see [Vie95, Lemma 2.14.a]).

As mentioned earlier, the log cotangent sheaf is never ample (see [BD17, Section 2.3]).

**Proposition 7.6.** Let $(X, D)$ be a log smooth pair. The sheaf $\Omega^1_X(\log D)$ is never ample.

**Proof.** Suppose that $\Omega^1_X(\log D)$ were ample. Consider the following exact sequence (see [EV92, Proposition 2.3]):

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \bigoplus_{i \in I} \mathcal{O}_{D_i} \to 0.$$  

Consider the restriction of this sequence to a component $D_i \subseteq D$, and tensor the above sequence with $\mathcal{O}_{D_i}$ to obtain a surjection:

$$A \to \mathcal{O}_{D_i} \oplus Q \to 0,$$

where $A$ is an ample sheaf (the restriction of an ample sheaf), and $Q$ is a torsion sheaf supported at $D_i \cap D_j \neq \emptyset$ for all $j \in I$ such that $i \neq j$. However, since $\mathcal{O}_{D_i} \oplus Q$ is not ample, there cannot exist such a surjection from an ample sheaf, and so $\Omega^1_X(\log D)$ can never be ample.

In light of the above, we are led to a definition that captures the strongest positivity assumption one can make on $\Omega^1_X(\log D)$, even for non-normal $X$. Given an slc pair, after normalizing, we arrive at a lc pair $(X', D' \supseteq (\log D_{\text{slc}}))$. A natural condition is thus to assume positivity of $\Omega^1_{X'}(\log (D' + \Delta_{\text{slc}}^{\text{dlt}}))$.

**Remark 7.7.**

1. Since we want to understand the behavior of subvarieties in $(X, D)$, we will want to restrict the log cotangent sheaf to subvarieties. Unfortunately, the restriction of $\Omega^1_{X'}(\log (D' + \Delta_{\text{slc}}^{\text{dlt}}))$ to subvarieties passing through strictly lc singularities is not well behaved (see [GKKP11, Example 3.2]). Since the normalization of an slc pair may have strictly lc singularities, we need to assume something slightly stronger. We assume that $\Omega^1_{X'}(\log \overline{D})$ is positive, where $\pi : \overline{X'} \to X'$ is the minimal resolution, and $\overline{D} = \pi^{-1}(D' + \Delta_{\text{slc}}^{\text{dlt}})$. We do note, however, that if $(X', D' + \Delta_{\text{slc}}^{\text{dlt}})$ is a dlt pair, nothing new is assumed (see Remark 7.9).

2. The following Definition 7.8 is a natural assumption in light of both Proposition 7.6, as well as the aforementioned work of Greb-Kebekus-Kovács-Peternell (esp. [GKKP11, Example 3.2]), to exclude the existence of subvarieties which are not of log general type in a stable pair $(X, D)$. It has the advantage of being geometric in nature and natural when considering stable pairs and their moduli. We note that there are examples of moduli spaces of stable pairs for which each object does not contain subvarieties of log general type, e.g. the moduli space of pointed stable curves, and the moduli space of semiabelic pairs ([Ale99]). In particular, *any positivity condition which guarantees the non-existence of subvarieties of log general type will allow us to prove uniformity*. 


Definition 7.8. Let \((X, D)\) be a log canonical surface pair, and let \(\pi : \overline{X} \to X\) be the minimal resolution. Let \(\overline{D} = \pi^*(D)\), and let \(E_\pi\) be the exceptional divisors of \(\pi\). We say that the log cotangent sheaf of \((X, D)\) is \textit{almost ample} if:

1. \(\Omega^{[1]}_X(\log \overline{D})\) is big, and
2. \(B_+(\Omega^{[1]}_X(\log \overline{D})) \subseteq \text{Supp}(\overline{D} + E_\pi)\).

If \((X, D)\) is an slc pair, then we say the log cotangent sheaf of \((X, D)\) is almost ample if the log cotangent sheaf of the normalization \((X^\nu, D^\nu + \Delta^\nu_{\text{dl}})\) is almost ample.

Remark 7.9.

1. When \(X\) is smooth our notion coincides with almost ample as in [BD17, Definition 2.1].
2. [BD17] shows that for smooth projective \(X\), there always exists a \(D\) so that \(\Omega^1_X(\log D)\) is almost ample. In particular, one expects many moduli spaces of stable pairs in which (at least) the smooth objects have almost ample log cotangent.
3. Condition (2) solve the problem that the pullback of reflexive differentials under the resolution will acquire log poles along exceptional divisors coming from strictly lc singularities.
4. The presence of the exceptional divisors in condition (2) is necessary even when \(D = \emptyset\): e.g. [Laz04, 6.3.28] that deduce the absence of subvarieties of general type on a smooth variety \(X\) when \(\Omega^1_X\) is ample does not generalize when \(X\) has strictly lc singularities.
5. When \(D = \emptyset\), “almost ample” (log) cotangent implies that \(X\) is \textit{geometrically mordellic} in the sense of Abramovich-Voicu [AV96].
6. For dlt pairs Definition 7.8 is equivalent to requiring that the log cotangent sheaf is big and its base locus is contained in \(D\).
7. Requiring both conditions on the \textit{minimal resolution} is equivalent to requiring it on any resolution (see Corollary B.13).
8. One can make the same definition for arbitrary dimensions replacing \(\overline{X} \to X\) with a suitable dlt modification.

We will prove in the following proposition that, if the log cotangent of a log canonical pair \((X, D)\) is almost ample in the sense of Definition 7.8, then every pair \((Y, E)\) with \(Y \subset X\) not contained in \(D\) is of log general type. This generalizes [Laz04, 6.3.28] to the pairs setting.

Proposition 7.10. Let \((X, D)\) be a log canonical surface pair over \(K\). If \((X, D)\) has almost ample log cotangent then all pairs \((Y, E)\) where \(E := (Y \cap D)_{\text{red}}\) with \(Y \subset X\) irreducible and not contained in \(D\) are of log general type.

Proof. We follow the notation of Definition 7.8. By assumption, \(\Omega^{[1]}_X(\log \overline{D})\) is big, and so its restriction to \(Y := \pi^{-1}Y\), which is not contained in \(\overline{D}\), is still big. However, since \(Y\) is a curve, big is equivalent to ample. Consider the normalization \(\phi : Y^\nu \to Y\), and denote by \(E^\nu\) the divisor \(\phi^{-1}(E) \cup (\text{exceptional set of } \phi)\), where \(E = \pi^{-1}E\). Since \(\Omega^{[1]}_X(\log \overline{D})\) is ample, its pullback \(\phi^*(\Omega^{[1]}_X(\log \overline{D})|_Y)\) is big. Considering the generically surjective map (see [GKKP, Theorem 4.3])

\[\phi^*(\Omega^{[1]}_X(\log \overline{D})|_Y) \to \Omega^{[1]}_{Y^\nu}(E^\nu) = \mathcal{O}_{Y^\nu}(K_{Y^\nu} + E^\nu)\]

proves that \(K_{Y^\nu} + E^\nu\) is big and hence \((Y, E)\) is of log general type. \(\square\)

Remark 7.11. The above proof does not generalize to \(\dim(X) > 2\), even for smooth \(X\). Various issues exist when considering the log cotangent sheaves of singular subvarieties that are not curves (see e.g. Remark 7.9).
Remark 7.12. If \((X, D)\) is a dlt surface pair, then the proof and conclusion of Proposition 7.10 are true under the weaker assumption that \(\pi^*(\Omega^1_X(\log D))\) is big and \(\mathcal{B}_+(\pi^*(\Omega^1_X(\log D)) \subseteq \text{Supp}(D + E_\pi).\) See also Remark 7.9 (3).

This result extends to semi-log canonical pairs as follows.

Corollary 7.13. If \((X, D)\) is an slc surface pair with almost ample log cotangent, then any irreducible subvariety of \(X\) not contained in either \(D\) or \(\text{Sing}(X)\) is of log general type.

7.3. Generalizations of Moriwaki’s results. In [Mor95], Moriwaki proved that for smooth projective varieties over number fields \(K\) with globally generated cotangent bundle, every irreducible component of \(\overline{X}(K)\) is geometrically irreducible and isomorphic to an abelian variety. Moreover, if in addition the cotangent bundle is ample then there are only finitely many \(K\) points. We stated that these results generalize for integral points on log smooth surfaces by replacing “ample cotangent” with “almost ample log cotangent”. The proofs are essentially the same as those in Moriwaki’s paper with a few changes, and so in this section we give a sketch of them. Let \(V\) denote a smooth quasi-projective surface with log smooth completion \((X, D)\), let \(\mathcal{A}_V\) denote the quasi-Albanese variety, and let \(\alpha : V \to \mathcal{A}_V\) denote the quasi-Albanese morphism (see [Fuj, Section 2.7]).

Theorem 7.14. Let \(V\) be a smooth quasi-projective surface with log smooth completion \((X, D)\) over a number field \(K\). If the log cotangent sheaf \(\Omega^1_X(\log D)\) is globally generated and almost ample, then for any finite set of places \(S\) the set of \(S\)-integral points \(V(\mathcal{O}_K, S)\) is finite.

Sketch of proof. Assume that \(Y\) is an irreducible component of \(\overline{V(\mathcal{O}_S)}\) with \(\dim Y \geq 1\) and completion \((\overline{Y}, \mathcal{E})\). Since \(\Omega^1_X(\log D)\) is almost ample and globally generated, its restriction to \(\overline{Y}\) is as well. By [Mor95, Lemma 2.3], \(\dim(\mathcal{A}_Y) \geq 2 \dim Y\). In particular, if \(\dim(Y) = 1\) then \(L = \mu^*(\omega_{\overline{Y}}(\mathcal{E}))/\mu^*(\omega_{\overline{Y}}(\mathcal{E}))_{\text{tors}}\) is big where \(\mu : Y' \to \overline{Y}\) is a resolution. Otherwise, \(\dim(Y) = 2\) and \(L = \mathcal{O}_P(1)\) is big where \(P = \text{Proj}(\Omega^1_X(\log D))\). The rest of the proof of [Mor95, Lemma 2.3] holds verbatim. Finally, by [Mor95, Theorem 1.1], \(Y(\mathcal{O}_S)\) is not dense in \(Y\), which is a contradiction. Indeed, Moriwaki’s proof works by replacing \(\alpha\) by the quasi-Albanese morphism (see [Fuj]) and by replacing Faltings’ theorem ([Mor95, Theorem A]) by Vojta’s theorems ([Voj96, Voj99]).

Theorem 7.15. Let \(V \cong (X \setminus D)\) be a smooth surface over \(K\). If the log cotangent sheaf \(\Omega^1_X(\log D)\) is globally generated, then for any finite set of places \(S\), every irreducible component of \(\overline{V(\mathcal{O}_S)}\) is geometrically irreducible and isomorphic to a semiabelian variety.

Sketch of proof. Since \(\Omega^1_X(\log D)\) is assumed to be globally generated, its restriction to \(V\), namely \(\Omega^1_V\) is as well, so that there is a surjection \(H^0(V, \Omega^1_V) \otimes \mathcal{O}_V \to \Omega^1_V\). By [Fuj, Lemma 3.12], \(H^0(V, \Omega^1_V) \otimes \mathcal{O}_{\mathcal{A}_V} \cong \Omega^1_{\mathcal{A}_V}\), so pulling back by \(\alpha : V \to \mathcal{A}_V\) gives a surjection \(\alpha^*(\Omega^1_{\mathcal{A}_V}) \to \Omega^1_V\). Therefore, similarly to [Mor95, Theorem B], every irreducible component of \(\overline{V(\mathcal{O}_S)}\) is isomorphic to a semi-abelian variety. Indeed, Vojta’s Theorem ([Voj96, Voj99]) implies that every irreducible component of a semi-abelian variety containing infinitely many integral points is a translate of a semi-abelian subvariety. Therefore the same argument as in Moriwaki’s proof applies since smooth étale covers of semi-abelian varieties are semi-abelian varieties by [Fuj, Theorem 4.2].

Again, the obstacles to proving the above two theorems for \(\dim(X) > 2\) stem from singular subvarieties and trouble working with their log cotangent sheaves (see Remark 11.3).

In the following sections, we will show that \(ms\)-integral points behave well for subvarieties of pairs with positivity assumptions on the log cotangent sheaf. To obtain uniformity results we need to show that these positivity conditions can be defined at the level of the moduli stack. More
precisely, we prove the existence of the desired sheaf on the a resolution of the universal family over a stratification of the moduli space of stable pairs. This is carried out in Appendix B.

8. \textit{ms}-INTEGRAL POINTS AND SUBVARIETIES

In Section 6 we showed that \textit{ms}-integral points lie in a subscheme whose degree is uniformly bounded (see Theorem 6.1). To conclude uniformity, one would hope to use an induction argument, by proving that \textit{ms}-integral points that lie on a curve are stably integral for that curve, and then apply Theorem 4.6. The purpose of this section is to show that the only obstruction to such an argument holding is the existence of contractible components of curves (see Proposition 8.3). We will then use this result in Section 10 to show that the \textit{subvariety property} does hold under a positivity assumption on the log cotangent sheaf.

Let \((X, D)\) be a stable pair of dimension two and let \(Y\) be an irreducible subvariety such that \(Y \not\subset D\). We begin by studying the behaviour of \textit{ms}-integral points lying on \(Y\) when the pair \((Y, D_Y)\) is stable where \(D_Y = (D \cap Y)\); in particular \(D_Y\) is a reduced divisor. We want to show that an \textit{ms}-integral point \(P\) that lies in \(Y\) is a stably integral point for \((Y, D_Y)\). This amounts to exhibiting a stable model \((\mathcal{Y}^s, D_Y^s)\) of \((Y, D)\) where \(P\) is integral. Since \(P\) is \textit{ms}-integral in \((X, D)\) we are given a natural model of the two dimensional pair – the good model \((X, \mathcal{D})\) where \(P\) is integral. This defines a proper model of \((Y, D)\) as follows:

\textbf{Definition 8.1.} Let \((\mathcal{X}, \mathcal{D})\) be a good model of a log canonical surface pair \((X, D)\) and let \(Y \subset X\) be a proper irreducible curve. If \((Y, D_Y)\) is a stable pair, where \(D_Y = (D \cap Y)\), then we call the closure of \((Y, D)\) in \((\mathcal{X}, \mathcal{D})\) the \textit{induced model} of \(Y\) inside \((\mathcal{X}, \mathcal{D})\) and we denote it by \((\mathcal{Y}, \mathcal{D}_Y)\).

The induced model \((\mathcal{Y}, \mathcal{D}_Y)\) has stable generic fiber but in general might not be semi-stable. However the stable reduction theorem gives maps

\[
(\mathcal{Y}, \mathcal{D}_Y) \leftarrow (\mathcal{Y}^{ss}, \mathcal{D}_Y^{ss}) \xrightarrow{\phi} (\mathcal{Y}^s, \mathcal{D}_Y^s)
\]

where \((\mathcal{Y}^{ss}, \mathcal{D}_Y^{ss})\) is the semistable reduction of \((\mathcal{Y}, \mathcal{D}_Y)\), and all the maps are defined possibly in some finite extension of the base rings. Since the semistable reduction map is a composition of blow-ups, normalization, and base change under ramified covers we obtain the following.

\textbf{Lemma 8.2.} Let \((X, D)\) be a log canonical stable surface pair defined over \(K\), and let \(P\) be an \textit{ms}-integral point with respect to a good model \((\mathcal{X}, \mathcal{D})\) over \(\text{Spec} \mathcal{O}_{K,S}\). Let \(Y\) be an irreducible subvariety not contained in \(D\) such that \((Y, D_Y)\) is a stable pointed curve. Then if \(P \in Y\), (the image of) \(P\) is integral in the semistable model \((\mathcal{Y}^{ss}, \mathcal{D}_Y^{ss})\) of the induced model \((\mathcal{Y}, \mathcal{D}_Y)\).

\textit{Proof.} The proof follows by observing that the semistable reduction map sends \(\mathcal{D}_Y^{ss}\) to \(\mathcal{D}\). \hfill \Box

\textbf{Proposition 8.3.} Let \(\phi: (\mathcal{Y}^{ss}, \mathcal{D}_Y^{ss}) \rightarrow (\mathcal{Y}^s, \mathcal{D}_Y^s)\) be the stable reduction of the semi-stable model. If an \textit{ms}-integral point \(P\) of \((X, D)\) lies in \((Y, D_Y)\) and no irreducible component of a fiber of \((\mathcal{Y}^{ss}, \mathcal{D}_Y^{ss})\) containing the image of \(P\) is contracted by \(\phi\), then \(P\) remains integral in \((\mathcal{Y}^s, \mathcal{D}_Y^s)\).

\textit{Proof.} We will prove that the result holds under the weaker hypothesis that in each fiber of the induced model \((\mathcal{Y}, \mathcal{D}_Y)\), the image of \(P\) does not lie in a contractible component in a chain of
rational curves containing a marked point. Consider the following diagram

\[
\begin{array}{c}
(Y, D) \\ \\
\sigma \\
\downarrow \\
(Y^{ss}, D_Y^{ss}) \\ \\
\phi \\
\downarrow \\
(Y^s, D_Y^s)
\end{array}
\]

By Lemma 8.2, \( P \) is integral in the semistable model \( Y^{ss} \); we need to prove that the image of the contraction morphism \( \phi(P) \) does not reduce to \( D^s \) over any prime \( p \notin S^{ss} \). Since this is local, we can fix a prime \( p \) and work over the completion of \( O_{K^{ss}, S^{ss}} \) at \( p \). The corresponding morphism, which we denote by \( \phi_p \), is an isomorphism over the generic fiber \( Y^{ss}_p \) and a contraction on the special fiber \( Y^s_p \). We have to prove that \( \phi_p \) does not specialize to a point lying in \( D^s_p \) in the fiber \( Y^s_p \).

We argue by contradiction: assume that \( \phi(P) \) intersects \( D^s_Y \) in the fiber \( Y^s_p \). Since \( \phi_p \) is the stable reduction of the fiber \( Y^{ss}_p \) (see [ACG11, Chapter X]) we can assume that \( Y^{ss}_p \) was not already stable, otherwise \( \phi_p \) would be an isomorphism, and we are assuming that \( P \) is integral in the semistable model. Therefore, the fiber \( Y^{ss}_p \) is semistable but not stable and the map \( \phi_p \) is a contraction of an exceptional chain \( \Gamma \), i.e., a chain of rational curves which meets the rest of \( Y_p \) in at most two points (see the proof of [Liu02, Theorem 10.3.34 (b)]).

We are then reduced to the case in which the reduction of \( P \) in \( Y^s_p \) lies in \( \Gamma \). By assumption, \( P \) specializes to a point of \( D_Y^s \), so there exists at least one component of \( \Gamma \) containing at least one marked point, i.e., a point of \( D^{ss}_Y \). We will show that this cannot happen.

Recall that by hypothesis, the image of \( P \) in the fiber \( Y_q \) of the model \( Y \), where \( q \) is the prime lying under \( p \), was not in any exceptional chain which meets the rest of \( Y_q \) in at most two points and contains a marked point. By Lemma 8.2, this implies that the same holds true in the semi-stable model. Therefore no component of \( \Gamma \) can contain a marked point, exhibiting the contradiction. This proves that \( \phi_p(\Gamma) = \phi_p(\bar{P}) \) is disjoint from \( D^s_Y \) and thus proves the proposition.

Lemma 8.2 shows that any integral point \( P : \text{Spec} O_{K,S} \rightarrow (Y, D_Y) \) remains integral in the semistable model. However, the image of \( P \) under \( \phi \) might intersect \( D^s_Y \) in some fiber, so that \( P \) is not integral in the stable model. To ensure that this will not happen, we will assume the log cotangent sheaf is almost ample.

By Corollary 7.13 the almost ampleness assumption ensures that all irreducible components of all fibers of \((Y, D)\) are of log general type, as long as they are not contained in \( D \) or the double locus of an slc fiber. Since the stable reduction morphism contracts only components that are not of log general type, the almost ampleness assumption, combined with Proposition 8.3, implies that \( \phi(P) \) remains integral in \((Y^s)\). This shows that \( ms \)-integral points lying in stable subpairs outside the double locus are stably integral in a stable pair of dimension two with almost ample log cotangent.

In particular, applying Proposition 8.3 with a positivity assumption on the log cotangent of each fiber allows us to state the following Corollary.

**Corollary 8.4.** Suppose that \((X, D)\) is a log canonical stable surface pair with good model \((\mathcal{X}, \mathcal{D})\) such that each fiber has almost ample log cotangent. Let \((Y, D_Y)\) be the induced model of a stable curve \((\mathcal{Y}, \mathcal{D}_Y)\). If no fiber of \((Y, D_Y)\) lies in the double locus of a fiber of \((X, D)\), then \( ms \)-integral points of \((X, D)\) lying on \((Y, D_Y)\) are stably integral for \( Y \), i.e., they satisfy the subvariety property.

**Proof.** Proposition 8.3 ensures all fibers of \((Y, D_Y)\) are log general type, as we assumed no fiber lies in the double locus of a fiber of \((\mathcal{X}, \mathcal{D})\). Given any \( ms \)-integral point \( P \) of \((X, D)\) lying on \((Y, D_Y)\), this implies that the image of \( P \) in the semistable model \((Y^{ss}, D_Y^{ss})\) either lies in an exceptional
curve not containing points of $\mathcal{D}_Y^\circ$, or in a component of log general type. In particular either the component is not contracted by stable reduction, or it is contracted to a point not in $\mathcal{D}_Y^\circ$. \qed

9. \textsc{ms-integral points in singular curves}

Corollary 8.4 implies that \textit{ms}-integral points lying on a stable pair \((Y, D_Y)\) satisfy the subvariety property provided that no component of a fiber of its induced model \((\mathcal{X}, D_X)\) (see Definition 8.1) lie in the double locus of a fiber of \((\mathcal{X}, D)\). However, the \textit{ms}-integral points might lie in non stable curves, (e.g. curves with worse than nodal singularities), in which the generic fiber of the induced model is not semi-stable and therefore we cannot apply stable reduction. Instead, we consider a stable map \(Y' \to Y \hookrightarrow X\), and study how the \textit{ms}-integral points on \(Y'\) relate to those on \(Y \subset X\).

9.1. \textbf{Singular curves}. Given a log canonical stable surface \((X, D)\), let \(Y \subset X\) be an irreducible curve not contained in \(D\) such that the pair \((Y, D_Y)\) is not stable. Therefore either \(Y\) is singular, or \(D_Y\) is non-reduced. If \(Y\) is singular, but \(D_Y\) is reduced, then we take the normalization \(Y' \to Y\) to obtain a stable map \(Y' \to X\), and proceed with Proposition 9.4. However, if \(D_Y\) is nonreduced, then we give a construction of a stable map \(f : Y' \to X\) where \(Y'\) is a pre-stable curve, \(f\) is a stable map, and the number of marked points in \(Y'\) equals deg\((D_Y) = \text{deg}(D \cap Y)\). In particular, we \textit{cannot} simply use the normalization as the degree equality will not be satisfied. This last condition will be necessary for studying curves that degenerate into the double locus of a degeneration of \(X\) (see Section 10 and Proposition 10.4). We begin by giving an example that clarifies our motivation.

Recall that a \textit{stable map} (see e.g. [Abr08, Section 1.3]) is a map from a marked nodal connected projective curve to a projective variety such that the nodes are disjoint from the markings, and the group of automorphisms fixing the markings for a component contracted by the map is finite.

\textbf{Example 9.1.} Let \((X, D)\) be a log smooth stable surface with \(D\) smooth and irreducible, and let \(Y\) be a non-singular curve in \(X\). Suppose that \(\text{Supp}(Y \cap D) = \{P\}\) and that the multiplicity of intersection is 2. In this case \(D_Y = D \cap Y = 2P\). One natural way to construct a stable map \(f : Y' \to X\) is to consider \(Y' = Y\) where \(f\) is the inclusion. However in this case the number of marked points on \(Y'\) will be 1, while \(\text{deg} D_Y = \text{deg}(D \cap Y) = 2\). On the other hand one can consider the surface pair \((X, D + Y)\) and the map \(\pi : \tilde{X} \to (X, D + Y)\), a compositions of two blow-ups centered at \(P\) which gives a log-resolution of the pair. Then \(\tilde{Y} = \pi^* Y = Y_1 + E_1 + 2E_2\) is a non-reduced but nodal curve and \(\tilde{Y} \cap \pi^{-1} D\) is transverse; it is a point in \(E_2\). We can take finite covers \(t : \tilde{X} \to \tilde{X}\) branched along components of \(\tilde{Y}\) and take the (normalization of the) fibered product \(\tilde{Y} = \tilde{X} \times_{\tilde{X}} \tilde{Y}\), which also gives a map \(h : \tilde{Y} \to X\). In this case, we take a single degree 2 cover \(\tilde{X} \to \tilde{X}\) branched over \(Y \cup E_1\). We then obtain a pre-stable curve \(\tilde{Y} \subset \tilde{X}\) whose markings \(D_{\tilde{Y}}\) are given by the preimages of \(D_Y\), which are 2 points. Finally we can contract the non-stable components of the map \(h\) – namely the preimage of \(E_1\) in \(\tilde{Y}\) – and we get a curve \(Y'\) which has two irreducible components, one isomorphic to \(Y\) and the other a genus 1 component \(E_2'\) containing the markings. Since we have contracted only unstable components for the map \(h\) we get an induced map \(f : Y' \to X\) which is stable and the number of markings of \(Y'\) is precisely \text{deg} \(D_Y\).

\textbf{Construction 9.2.} Let \(\pi : \tilde{X} \to (X, D + Y)\) be a log resolution of the pair \((X, D + Y)\) and consider the curve \(\tilde{Y} = \pi^* Y\) (see Remark 9.3). By construction \(\tilde{Y}_{\text{red}}\) is a nodal curve. Consider the divisor \(D_{\tilde{Y}}\) on \(\tilde{Y}\) defined as \(D_{\tilde{Y}} = \tilde{Y} \cap \pi^{-1} D\). Consider a composition of finite covers \(t : \tilde{X} \to \tilde{X}\) branched along components of \(\tilde{Y}\) such that the fibered product \(\tilde{Y} = \tilde{X} \times_{\tilde{X}} \tilde{Y}\) is a pre-stable curve. It comes with a map \(h : \tilde{Y} \to X\), and markings \(D_{\tilde{Y}}\) that are the preimages of \(D_{\tilde{Y}}\) under the cover.
Contracting any unstable components relative to the morphism \( h : Y \to X \) gives a curve \( Y' \) with a map \( \mu : Y' \to Y' \), and a stable map \( f : Y' \to X \) whose image is \( Y \), with markings given by the images \( D_{Y'} = \mu(D_Y) \). By construction \( \deg(D_{Y'}) = \deg(D_Y) \).

**Remark 9.3.** Note that in the case in which \( Y \) is not a \( \mathbb{Q} \)-Cartier divisor, we can consider a minimal \( \mathbb{Q} \)-factorialization \( q : X' \to X \) (see e.g. [Kol13, Corollary 1.36]) and replace \( Y \) by its strict transform in \( X' \): this will not change the intersection \( Y \cap D \) locally around \( D \) since \((X, D)\) is log canonical.

We first show that the stable map \( f \) allows us to construct a nice semistable model of \( Y' \).

**Proposition 9.4.** The stable map \( f \) induces a semistable model \( Y' \) of \( Y' \) with a map \( F : Y' \to (\mathcal{X}, \mathcal{D}) \). Moreover, if each fiber of a good model of the log canonical stable surface \((X, D)\) has almost ample log cotangent then every positive dimensional component of every fiber of the stable reduction map \( Y' \to Y'' \) is mapped to \( D \) or the double locus of any fiber of \((\mathcal{X}, \mathcal{D})\) by \( F \).

**Proof.** Let \( g \) and \( n \) denote the genus and the number of markings of \( Y' \) respectively, and let \( \beta \) be the class in \( H_2(\mathcal{X}, \mathbb{Z}) \) of the closure of \( Y \) in the good model of \((X, D)\). Then by [AO01], the stack of stable maps \( \mathcal{M}_{g,n}(\mathcal{X}, \beta) \) is a proper Artin stack, and so the map \( f \) admits a closure \( F : Y' \to \mathcal{X} \) as a stable map over \( \mathcal{O}_{K, S} \) (after possibly extending \( K \) and \( S \)). The map \( F \) can be thought of a family of stable maps whose generic fiber is \( f \). By definition, the curves appearing as fibers of \( Y' \) are connected, projective, and at worst nodal. Therefore this is actually a semistable model for \( Y' \).

Let \( \sigma : Y' \to Y'' \) be the stable reduction map. We have to prove that any component contracted by \( \sigma \) is sent to \( D \) or to the double locus of a fiber of \((\mathcal{X}, \mathcal{D})\) through \( F \). Assume that there exists a prime \( \mathfrak{p} \) not in \( S \), and a irreducible component \( Z \) of \( Y'_p = Y' \times_{\text{Spec} \mathcal{O}_{K, S}} \mathbb{F}_p \) which makes \( Y'_p \) non-stable as a curve. This implies in particular, that \( 2g(Z) - 2 + n_p \leq 0 \), where \( n_p \) is the number of special points in \( Z \). Since the map \( F \) is stable, this implies that \( F \) is not constant on \( Z \), and therefore \( F \) restricted to \( Z \) gives a finite map \( F : Z \to F(Z) \). In particular \( F(Z) \) is irreducible. However, by Corollary 7.13, every irreducible curve in any fiber of the good model \((\mathcal{X}, \mathcal{D})\) not contained in \( D \) or the double locus of any non-normal fiber is of log general type. Since \( F(Z) \) is not of log general type, this implies that \( F(Z) \) is contained in \( D \) or the double locus of a fiber. \(\square\)

The above proposition implies that the study of ms-integral points on singular curves on \( X \) can be reduced to the study of integral points on the semistable model of the normalization endowed with the stable map to the good model. In analogy with Definition 8.1 we give the following:

**Definition 9.5.** Let \((\mathcal{X}, \mathcal{D})\) be a good model of a log canonical surface pair \((X, D)\) and let \( Y \subset X \) be a proper irreducible curve not contained in \( D \). We call the model of the map \( f : Y' \to X \) of Proposition 9.4 the **induced model** of \( Y \) and denote it by \((Y', \mathcal{D}_{Y'})\), or \( Y' \). It comes endowed with a stable map \( F : Y' \to (\mathcal{X}, \mathcal{D}) \).

**Lemma 9.6.** Let \( P : \text{Spec} \mathcal{O}_{K, S} \to (\mathcal{X}, \mathcal{D}) \) be an ms-integral point of \( X \) lying in the smooth locus of a non-stable curve \( Y \subset X \). Let \( F : Y' \to (\mathcal{X}, \mathcal{D}) \) be the induced model of \( Y \) and assume that no fiber of \( \mathcal{Y}' \) is mapped to the double locus of a fiber of \((\mathcal{X}, \mathcal{D})\). Then \( P \) lifts as an integral point on \( \mathcal{Y}' \) and remains integral in the stable model \( \mathcal{Y}'' \) of \( \mathcal{Y}' \). In particular, any ms-integral point lying in \( Y \) is stably integral in the normalization \( Y' \), i.e. ms-integral points satisfy the subvariety property.

**Proof.** Since \( P \) is a smooth point of \( Y \), the point \( P \) automatically lifts to a point of \( Y' \). Since the model \( \mathcal{Y}' \) is semistable, the point \( P \) remains integral in the stable model if and only if it does not hit any component of \( \mathcal{Y}' \) that gets contracted under stable reduction that contains a marked point. By Proposition 9.4, any such component is mapped either to \( D \) or to the double locus of a fiber.
of \((X, D)\) by \(F\). Since \(P\) was \(ms\)-integral in \(X\) the lift of \(P\) on \(\mathcal{Y}'\) cannot intersect any component mapped to \(D\). On the other hand, by hypothesis there is no component of \(\mathcal{Y}'\) mapped to the double locus of a fiber of \((X, D)\). This implies that the image of \(P\) in the stable model of \(Y'\) is integral. □

**Remark 9.7.** By definition of a stable map, in the semistable model of the normalization any lifting of an \(ms\)-integral cannot hit any vertical component that is mapped to \(D\) outside \(S\), since this will contradict the fact that the point is integral in the good model \((X, D)\).

### 10. Subvariety Property for Surfaces

By the work of Sections 8 and 9, we see that we would be able to conclude uniformity (under a positivity assumption on the log cotangent) if no component of a fiber of the induced model \((\mathcal{Y}, D_Y) \subset (X, D)\) which lies in the double locus of a fiber of \((X, D)\) is contractible, and we will show this in Proposition 10.3 and 10.4. We first recall a property of the double locus \(\Delta_{dl}\) of an slc pair.

**Proposition 10.1.** [Kol13, Sec. 5.2] The morphism \(\pi : \Delta_{dl}^{\nu} \to \Delta_{dl}\) is generically finite of degree two, ramified at the pinch points.

**Proposition 10.2.** Let \((X, D)\) be an slc surface pair. If \(D \cap \Delta_{dl} \neq \emptyset\), then the intersection must be a nodal point of \(D\).

**Proof.** There is a Galois involution \(\tau : \Delta_{dl}^{\nu} \to \Delta_{dl}\), from the normalization of \(\Delta_{dl}^{\nu}\) to the normalization of \(\Delta_{dl}\) (see [Kol13, Section 5.1]). By [Kol13, Proposition 5.12] the different \((\nabla_{\Delta_{dl}}^{\nu}, \text{Diff}_{\Delta_{dl}}^{\nu})D^{\nu}\), where \(\nabla_{\Delta_{dl}}^{\nu}\) is the normalization of \(\Delta_{dl}^{\nu}\), is \(\tau\)-invariant. In particular, \(\text{Supp}(\text{Diff}_{\Delta_{dl}}^{\nu}) = \text{Supp}(D^{\nu} \cap \Delta_{dl}^{\nu})\) by [Kol13, Proposition 4.5]. Since \((X, D)\) is slc, the intersection points \(D^{\nu} \cap \Delta_{dl}^{\nu}\) cannot contain the preimage of a pinch point. Therefore, for any point \(p \in D \cap \Delta_{dl}\), the set \(\{D^{\nu} \cap \Delta_{dl}^{\nu}\}\) contains the whole fiber \(\nu^{-1}(p)\), which consists of two points by Proposition 10.1. Therefore \(p \in D\) is a node. □

**Proposition 10.3.** Let \((X, D)\) be a good model of a log canonical stable surface pair such that every fiber has almost ample log cotangent. Let \((Y, E)\) be a stable curve inside the generic fiber of \((X, D)\), and let \((\mathcal{Y}, D_Y)\) be its induced model (see Definition 8.1). Then every positive dimensional component of every fiber of the stable reduction is not mapped to \(D\).

**Proof.** This is true for any component outside the double locus of every fiber of \((X, D)\) by Corollary 7.13. Let \(Z\) be a component of a fiber of \((\mathcal{Y}, D_Y)\) which is contained in the double locus of a fiber of \((X, D)\). If \(Z \cap D = \emptyset\) the result is clear. Otherwise, by Proposition 10.2 all of the points \(Z \cap D\) are nodes of \(D\). The strict transform of \(Z\) in \((\mathcal{Y}^{ss}, D_{\mathcal{Y}}^{ss})\) (obtained by blowing up these nodes) is thus disjoint from \(D\), and therefore it is either not contracted, or it is contracted to a point not contained in \(D\). □

**Proposition 10.4.** Let \((X, D)\) be a good model of a log canonical stable surface pair such that every fiber has almost ample log cotangent. Let \((Y, E)\) be a unstable curve inside the generic fiber of \((X, D)\), and let \((\mathcal{Y}', D_{\mathcal{Y}}')\) be its induced model (see Definition 9.5). Then every positive dimensional component of every fiber of the stable reduction map \(\mathcal{Y}' \to Y''\) does not contain any marked points.

**Proof.** For components of \(\mathcal{Y}'\) not mapped to the double locus of any fiber of \((X, D)\) the result was proven in Proposition 9.4. Suppose \(Z\) is a component of a fiber \(\mathcal{Y}'_{\mathcal{Y}}\) contracted by stable reduction which is mapped to the double locus of a fiber of \((X, D)\). Since \(F\) is a stable map, the restriction \(F|_Z\) is a finite map. Since \(Z\) gets contracted by stable reduction, we may assume that \(g(Z) = 0\). By flatness we can further assume that \(\mathcal{Y}'_{\mathcal{Y}}\) is not irreducible so that \(Z\) has at least one special point...
to the curves gives a bound on the maximum number with Propositions implies that every and Lemma this number in addition to our special point which is not a marked point, finishing the proof. \[\square\]

**Corollary 10.5.** The ms-integral points on log canonical stable surface pairs with good model such that every fiber has almost ample log cotangent satisfy the subvariety property.

*Proof.* Combine the results of Corollary 8.4 and Lemma 9.6 with Propositions 10.3 and 10.4. \[\square\]

## 11. Uniformity and hyperbolicity

We begin by proving uniformity results for surfaces.

**Theorem 11.1.** Assume the Lang-Vojta Conjecture. Let \((X, D) \rightarrow B\) be a stable family of surface pairs over \(K\) with integral and openly log canonical general fiber over a smooth projective variety \(B\). If the log cotangent of each fiber is almost ample, then the cardinality of the set of ms-integral points of \((X_b, D_b)\) is uniformly bounded for all \(b\) in \(B(K)\).

*Proof.* By Theorem 6.1 there exists a closed subset \(A_b\) containing all the ms-integral points, for every fiber \((X_b, D_b)\) and every rational point \(b \in U \subset B\), where \(U\) is an open subset of \(B\). Moreover \(d_b = \deg A_b\) is uniformly bounded by a constant \(N\) (depending on the invariants defining the moduli space of stable pairs where the fibers of the family lie). We can write \(A_b = A_{0,b} \cup A_{1,b} = A_0 \cup A_1\) where \(A_0\) is the part of pure dimension zero and \(A_1\) is the part of pure dimension one. Since \(\deg A_b = d_b < N\) it follows that \(A_0\) contains at most \(N\) points.

The dimension one part \(A_1\), consists of irreducible curves \(\{C_i\}_{i \in I} \subset A_b\) for a finite set \(I\) whose cardinality is uniformly bounded by \(N\). We restrict our attention to the subset \(J \subset I\) of curves \(C_j\) which are not contained in \(D_b\); note that the curves \(C_i\) with \(i \in I \setminus J\) do not contain any ms-integral points since they are contained in \(D_b\).

For every \(j \in J\), we let \(E_j = (C_j \cap D_b)\). Then Corollary 10.5 implies that every ms-integral point of \((X_b, D_b)\) lying on \((C_j, E_j)\) is a stably-integral point. Moreover, since \(N\) is uniformly bounded, there are only finitely many choices for the genus of the \(C_j\) the degree of \(E_j\), and the number of singular points of \(C_j\). Applying Theorem 4.6 to the curves gives a bound on the maximum number of ms-integral points in the union of \(\{C_j\}\) for \(j \in J\), which does not depend on \(b\).

Therefore the total number of ms-integral points in \(A_b\) is uniformly bounded for every \(b \in U\).

To obtain the result for all fibers we apply Noetherian induction on the base: let \(N_0\) be the bound obtained above for the cardinality of the set of ms-integral points on every fiber \(X_b\) with \(b\) lying in \(U \subset B\). Let \(B_1\) be the union of all irreducible components of \(B \setminus U\) whose generic point is openly log-canonical and consider the restricted family \((X_1, D_1) \rightarrow B_1\). Applying the procedure described in this proof to this new family gives a bound \(N_1\) for the fiber lying in an open set \(U_1 \subset B_1\). This process gives a sequence of \(B_i\) which by Noetherian induction, since \(\dim B_i > \dim B_{i+1}\), exhausts all points of \(B\) corresponding to log-canonical stable pairs in a finite number of steps. Then \(N = \max N_i\) is a uniform bound for the cardinality of the set of ms-integral points on every fiber. \[\square\]

**Corollary 11.2.** Assume the Lang-Vojta Conjecture. Let \((X, D)\) be a log canonical stable surface pair with \(D\) a \(\mathbb{Q}\)-Cartier divisor and good model \((\mathcal{X}, \mathcal{D}) \rightarrow B\). Suppose that each fiber of \((\mathcal{X}, \mathcal{D})\) has almost ample log cotangent. Then there exists a constant \(N = N(K, S, v)\) where \(v\) is the volume of \((X, D)\), such that the set of ms-integral points of \((X, D)\) has cardinality at most \(N\), i.e.

\[\#(X \setminus D)(\mathcal{O}_{K, S}^{\text{ms}}) \leq N = N(K, S, v)\]
Remark 11.3. Corollary 11.2 can be extended to higher dimensions considering the set of ms-integral points lying on stable subvarieties. However in that case the conclusion is too weak, since on a variety of \( \dim > 1 \) there is a dense collection of non-stable subvarieties.

Proof. It is enough to apply Theorem 11.1 to the tautological family of the moduli space \( \mathcal{M}_F \) of stable pairs of dimension 2, volume \( v \) and coefficient set \( I = \{1\} \) (see [KP17]).

Remark 11.4. If we weaken Definition 7.8 to requiring positivity on the normalization instead of a resolution, we can obtain uniformity results (assuming Lang-Vojta) outside of a finite set of curves: namely those whose induced model contains irreducible component of fibers that pass through any strictly lc points (see Remark 7.7). Such a result is analogous to [Has96, Theorem 6.2].

11.1. Hyperbolicity. In this section we give an example that shows that hyperbolicity is not a closed condition, even under the strong assumption of almost ample log cotangent in the resolution.

Proposition 11.5. Let \( C \subset \mathbb{P}^3 \) be a curve. If there exists a surface \( X \subset \mathbb{P}^3 \) of sufficiently high degree such that \( C \subset X \), then there exists a smoothing of \( X \) in \( \mathbb{P}^3 \) that contains a smoothing of \( C \).

Proof. Let \( T = \text{Spec} \mathcal{R} \) with \( \mathcal{R} \) a DVR and \( C_T \) be a smoothing of \( C \) in \( \mathbb{P}^3 \) such that \( C \cong C_0 \). Consider the sequence of sheaves over \( T \)

\[ 0 \to I_{C_T} \to \mathcal{O}_{\mathbb{P}^3_T} \to \mathcal{O}_{C_T} \to 0. \]

Twisting by \( \mathcal{O}(d) \), for any \( d \), we have

\[ 0 \to I_{C_T}(d) \to \mathcal{O}_{\mathbb{P}^3_T}(d) \to \mathcal{O}_{C_T}(d) \to 0. \]

Pushing forward along \( \pi : \mathbb{P}^3_T \to T \), we have

\[ 0 \to \pi_*I_{C_T}(d) \to \pi_*\mathcal{O}_{\mathbb{P}^3_T}(d) \to \pi_*\mathcal{O}_{C_T}(d) \to R^1\pi_*I_{C_T}(d), \]

but for all \( d \gg 0 \) we have \( H^1(\mathbb{P}^3_T, I_{C_T}(d)) = 0 \), hence by Cohomology and Base Change [Har77, Theorem 12.11], \( R^1\pi_*I_{C_T}(d) = 0 \). Therefore we get an exact sequence

\[ 0 \to \pi_*I_{C_T}(d) \to \pi_*\mathcal{O}_{\mathbb{P}^3_T}(d) \to \pi_*\mathcal{O}_{C_T}(d) \to 0, \]

which shows that \( \pi_*I_{C_T}(d) \) is a vector bundle over \( T \). Let \( X \) be a surface with \( C \subset X \), considered as an element of \( H^0(\mathbb{P}^3_0, I_{C_0}(d)) \). Let \( \sigma \) be a general section of \( \pi_*I_{C_T}(d) \) such that \( \sigma_0 \cong X \). To show that a general \( \sigma \) gives a smoothing of \( X \), it suffices to show that the general surface \( X_t \) containing \( C_t \) is smooth. However, \( C_t \) is a smooth curve, so there exists a surface \( X_t \) containing \( C_t \) and the general such \( S_t \) is smooth (see e.g. [BN11, Theorem 1.1, Remarks 1.2(a)]).

Now we give an example that shows that in a stably family \( (X, D) \to B \) with almost ample log cotangent there might exist families of curves that contain components of a fiber that are not of log general type. This example works whenever the double locus contains at least two components, with at least one being rational.

Example 11.6. Let \( C = C_0 \cup C_g \subset \mathbb{P}^3 \) be the union of a line \( C_0 \) and a curve \( C_g \) of genus \( g \) such that \( C_0 \cap C_g \) is a single point \( p \). For example, one could take \( C_g \) to be a high genus curve in a hyperplane section \( H \subset \mathbb{P}^3 \) and \( C_0 \) to be a line transverse to \( H \).

We can find a semismooth surface \( X \) containing \( C \) such that \( C_0 \) is contained in the double locus \( \Delta_{dd} \subset X \). First, take any smooth surface \( X_1 \) of sufficiently high degree containing \( C_g \) (e.g. see [BN11]) and any surface \( X_2 \) containing \( C_0 \). If \( X_3 \) is a sufficiently general surface containing \( C_0 \), the surface \( X = X_1 \cup X_2 \cup X_3 \) is semismooth, contains \( C \), and has \( C_0 \) contained in the double locus.
Applying Proposition 11.5, we can find a smoothing $\mathcal{X}_T$ of $X_0 \cong X$ in $\mathbb{P}^3$ that is simultaneously a smoothing of $C = C_0$ to a genus $g$ curve $C_t$. If $D \subset \mathbb{P}^3$ is a general hyperplane section, the family $(\mathcal{X}_T, D \times T)$ is a family of semismooth pairs such that each fiber $(\mathcal{X}_{t \neq 0}, D)$ has almost ample log cotangent and the central fiber $(\mathcal{X}_0, D)$ has almost ample log cotangent. However, the family of curves $(C_T, (C_T \cap D \times T))$ is not stable. In the central fiber $C_0$, the rational component $C_q$ meets $C_0$ in a single point $p$ and $C_0 \cap D$ is a double point $q$. Therefore, in stable reduction we blow up $q$ and the strict transform of $C_0$ is contracted.

**Remark 11.7.** Example 11.6 shows the existence of a stable family $(X, D) \to B$ over a curve, whose generic fiber $(X_q, D_q)$ is normal with almost ample log cotangent, and therefore hyperbolic. However, the special fiber $(X_0, D_0)$, although having almost ample log cotangent in the normalization, contains a rational curve in the double locus which only meets $D_0$ at a single (nodal) point. Therefore, the special fiber contains a curve which is not of log general type.

**Appendix A. Stack of stable pairs over $\mathbb{Q}$**

Suppose that we fix any moduli functor $\mathfrak{F}$ of stable log varieties of fixed dimension, volume and coefficient set as in [KP17]; Let $(U, D) \to M_\Gamma$ be the universal family, universal divisor and moduli stack representing the functor $\mathfrak{F}$. Before dealing with models we first need to prove that $M_\Gamma$ can be defined over $\mathbb{Q}$, and is still a Deligne-Mumford stack with projective coarse moduli space.

The main theorem of [KP17] proves that the moduli stack $M_\Gamma$ exists over an algebraically closed field of characteristic 0. Here we show the existence and projectivity of the moduli space over $\mathbb{Q}$. Throughout this section, we denote the Deligne-Mumford moduli stack over $\overline{\mathbb{Q}}$, by $\overline{M}_\Gamma$.

**Theorem A.1.** The moduli stack of stable pairs over $\mathbb{Q}$ is Deligne-Mumford.

**Proof.** Let $S = \text{Spec} \overline{\mathbb{Q}}$ and let $(S_\alpha, u_{\alpha, \beta})$ be a projective system of schemes so that $S = \varprojlim S_\alpha$ and all $S_\alpha = \text{Spec} K_\alpha$, where $K_\alpha$ is a finite extension of $\mathbb{Q}$. By [KP17, Proposition 5.11], the stack $M_{\Gamma, \overline{\mathbb{Q}}}$ of stable pairs is a Deligne-Mumford stack of finite type over $S$. By [Ols06, Proposition 2.2], there exists a Deligne-Mumford stack of finite type $M_{\Gamma, S_\alpha}$ over some $S_\alpha$ and an isomorphism $M_{\Gamma, S_\alpha} \cong M_{\Gamma, S_\alpha} \times S_\alpha \text{Spec} \overline{\mathbb{Q}}$ (if necessary, replace $S_\alpha$ so that $M_{\Gamma, S_\alpha}$ is a moduli stack).

Since $M_{\Gamma, S_\alpha}$ is Deligne-Mumford, there is a surjective étale morphism $u : U \to M_{\Gamma, S_\alpha}$ where $U$ is a scheme. Define $M_{\Gamma, S_\alpha}$ to be the moduli stack of stable pairs over $\mathbb{Q}$. We now wish to prove that $M_{\Gamma, S_\alpha}$ is a Deligne-Mumford stack. To do so, consider the surjective morphism $\psi : U \to M_{\Gamma, S_\alpha} \to M_{\Gamma, \overline{\mathbb{Q}}}$. By [Sta15, Tag05UL], $M_{\Gamma, \overline{\mathbb{Q}}}$ is a Deligne-Mumford stack if the morphism $\psi$ is surjective, smooth, and representable by algebraic spaces.

Then the morphism $\psi$ is smooth and surjective, since the morphism $u$ is smooth and surjective, and the morphism $\phi$ is étale and surjective since the map $S_\alpha \to \text{Spec} \mathbb{Q}$ is. To show representability of $\psi$, it suffices to check representability of $\phi$ as we know that the map $u$ is representable. Consider the following diagram, where $T$ is an algebraic space, and both squares are fiber product diagrams.
To show that \( \phi \) is representable, we must show that the fiber product \( \tilde{T} \) is an algebraic space whenever \( T \) is. Since both squares are fiber products, \( \tilde{T} \cong S_\alpha \times_{\text{Spec} \mathbb{Q}} T \) is also a fiber product, and is thus an algebraic space since it is the base change of an algebraic space by a field extension. \( \Box \)

For Appendix C, where we give an alternative approach to models of stable pairs, we need to appeal to the projective coarse moduli space over \( \mathbb{Q} \), so we show its projectivity.

**Theorem A.2.** The coarse moduli space \( M_{\Gamma, \mathbb{Q}} \) is a projective variety.

**Proof.** By [KP17, Corollary 6.3], the coarse moduli space \( M_{\Gamma, \mathbb{P}} \) is a projective variety. Let \( L \) denote the ample line bundle on \( M_{\Gamma, \mathbb{P}} \). While \( L \) may not be defined over \( M_{\Gamma, \mathbb{S}_\alpha} \), by [Ols06, Proposition 2.2], we obtain some \( S_{\beta} \) so that \( L \to M_{\Gamma, \mathbb{P}} \) is defined over \( S_{\beta} \), where \( S_{\beta} \) again corresponds to a finite extension of \( \mathbb{Q} \). Taking an algebraic extension containing the fields corresponding to both \( S_{\alpha} \) and \( S_{\beta} \), call the corresponding affine scheme \( S_{\gamma} \), we obtain a stack \( M_{\Gamma, S_{\gamma}} \) where the ample line bundle \( L \) is defined. Now we wish to show that \( L \) is an ample line bundle exhibiting the coarse moduli space \( M_{\Gamma, S_{\gamma}} \) as a projective variety. There Serre criterion for ampleness tells us that the higher cohomology vanishes on \( M_{\Gamma, S_{\gamma}} \) for a high enough power of \( L \). By [Con, Proposition 2.1], there is an isomorphism between the higher cohomology groups of \( M_{\Gamma, \mathbb{P}} \) and \( M_{\Gamma, S_{\gamma}} \), and thus shows that ampleness of the line bundle descends to \( M_{\Gamma, S_{\gamma}} \). As a result, the coarse moduli space \( M_{\Gamma, S_{\gamma}} \) is a projective variety. Using Galois descent, we show that projectivity descends to \( M_{\Gamma, \mathbb{Q}} \).

Suppose \( M_{\Gamma, S_{\gamma}} \) is defined over a field \( K_{\gamma} \). Let \( G \) be the finite Galois group corresponding to the finite extension \( K_{\gamma}/\mathbb{Q} \). Then \( \tilde{L} = (\otimes_{g \in G} L^g) \) gives a Galois invariant line bundle on \( M_{\Gamma, S_{\gamma}} \), where \( L^g \) denotes the pullback of the line bundle \( L \) through the isomorphisms induced by \( g \in G \). Since \( L^g \) are ample line bundles for all \( g \in G \), we see that the line bundle \( \tilde{L} \) is a Galois invariant ample line bundle. Moreover, by Galois descent, the Galois invariant line bundle \( \tilde{L} \) is pulled back from a line bundle \( L' \) on \( M_{\Gamma, \mathbb{Q}} \). Since the morphism \( M_{\Gamma, S_{\gamma}} \to M_{\Gamma, \mathbb{Q}} \) is finite, the line bundle \( L' \) is ample. This line bundle thus gives the desired projectivity of the coarse moduli space of \( M_{\Gamma, \mathbb{Q}} \). \( \Box \)

**Appendix B. Sheaves on the universal family**

The goal of this section is to show that an almost ample log cotangent sheaf can be defined on the level of the universal family over the moduli stack of stable pairs.

**B.1. Simultaneous normalizations.** We begin by finding a sheaf on the normalization of fibers of the universal family, and to do so we use simultaneous normalizations of Chiang-Hsieh and Lipman [CHL06] or Kollár [Kol11]. We recall the definition of a simultaneous normalization.

**Definition B.1.** Let \( f : X \to B \) be a morphism. A simultaneous normalization of \( f \) is a morphism \( \overline{\nu} : \overline{X}' \to X \) such that:

1. \( \overline{\nu} \) is finite and an isomorphism at the generic points of the fibers of \( f \), and
2. \( \overline{f} := f \circ \overline{\nu} : \overline{X}' \to B \) is flat with geometrically normal fibers.

**Theorem B.2.** [Kol11, Theorem 12] Let \( B \) be semi-normal, and let \( f : X \to B \) be a projective morphism with generically reduced fibers of \( \dim X_b = n \). The following are equivalent:

1. \( X \) has a simultaneous normalization \( \nu : \overline{X}' \to X \)
2. The Hilbert polynomial of the normalization of the fibers \( \chi(X_B, \mathcal{O}(tH)) \) is locally constant.

**Remark B.3.** If \( B \) is normal, then the total space \( \overline{X}' \) provided by the simultaneous normalization (if it exists) coincides with the normalization of the total space ([CHL06, Theorem 2.3]).
We need a simultaneous normalization as above, but in the pairs setting. The only complication is to determine what divisor to choose on $\overline{X'}$ which corresponds to the double locus on fibers.

**Definition B.4.** Let $X$ be a reduced scheme and let $\nu : X' \to X$ be its normalization. The conductor ideal sheaf is the annihilator sheaf $\text{Ann}_{O_X}(\nu_* O_{X'}/O_X)$, is the largest ideal sheaf on $X$ that is also an ideal sheaf on $X'$.

**Remark B.5.** If $X$ is slc, then the conductor ideal sheaf of $X'$ corresponds to a divisor (see [KSS10, Remark 4.5]). We will denote the divisor corresponding to the conductor by $C$.

**Remark B.6.** We now discuss the extension of Theorem B.2 for pairs. Given a stable family $(X, D) \to B$ with log canonical general fiber over a semi-normal base $B$, if there exists a simultaneous normalization $\pi : \overline{X'} \to X$, then the total space is given by $(\overline{X'}, \overline{D'} + \Delta_{\text{dl}})$. Moreover, if the base $S$ is normal, then this total space is the normalization of $(X, D)$. In particular, the divisor $\overline{\Delta}_{\text{dl}}$, which cuts out the locus corresponding to the double locus of the normalization, corresponds to the double locus of the normalization of each fiber.

**B.2. Simultaneous resolution.** To utilize some results of [GKKP11], we will need to ensure that we work with sheaves on pairs $(X, D)$ whose singularities (away from $D$) are at worst klt, and in particular have no strictly lc points. As a result, it is not enough to work with the normalization, and we thus pass to a simultaneous resolution.

**Theorem B.7.** [KM08, Theorem 7.68] Let $S$ be a scheme over $\mathbb{C}$ and $g : Y \to S$ a flat and proper morphism whose fibers are surfaces with isolated singularities only. For every $s \in S$ let $Y_s \to Y$ denote the minimal resolution of the fiber. Then $g$ has a simultaneous resolution after a finite and surjective base change $S' \to S$ if and only if $s \mapsto (K_Y^2)$ is locally constant.

**Remark B.8.** Since we will apply this theorem to a simultaneous normalization, we are guaranteed to have families whose fibers have only isolated singularities, allowing us to apply the above result.

**B.3. Universal family construction.**

**Remark B.9.** We sketch the approach to stratifying $(\mathcal{U}, \mathcal{D}) \to \mathcal{M}_\Gamma$. Let $(\mathcal{U}, \mathcal{D}) \to \mathcal{M}_\Gamma$ be models of the universal family and modulil space. We stratify so that:

1. Each strata $\mathcal{M}_{\Gamma,b}$ satisfies Theorem B.2 (i.e. locally constant Hilbert polynomial), giving a simultaneous normalization $(\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_{\Gamma,b}$.
2. Each strata is normal, to apply Remark B.3, and then take a flattening stratification to ensure that the conductor divisor (Remark B.5) is flat over the base.
3. Each strata satisfies Theorem B.7 (locally constant $K^2$), giving a simultaneous resolution.
4. The simultaneous resolution is functorial (which exists by e.g. [AW97]).
5. The resolution of the model of the universal family $(\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_{\Gamma,b}$ is compatible with the resolution of $(\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_b$, after removing finitely many primes.

**Remark B.10.** We need Remark B.9 (4) to guarantee that the resolution of the charts of the Deligne-Mumford stack $\mathcal{U}_b$ also gives a resolution of the stack.

First we prove existence of a quasicoherent sheaf on each strata $(\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_b$, and then show it also descends to a quasicoherent sheaf on any model $(\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_{\Gamma,b}$. Fix a strata $(\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_b$ of the moduli stack of stable pairs with fixed invariants $(\mathcal{U}, \mathcal{D}) \to \mathcal{M}_\Gamma$.

**Lemma B.11.** There exists a coherent sheaf $\mathcal{F}_b$ on $(\mathcal{U}_b, \mathcal{D}_b)$, up to finite base change, such that the restriction of $\mathcal{F}_b$ to any fiber is isomorphic to the log cotangent sheaf.
Proof. By Remark B.9 we can assume $\mathcal{M}_b$ is normal. Since $U_B$ and $\mathcal{M}_B$ are Deligne-Mumford stacks, there exist surjective étale maps from schemes making the follow commute:

$$
\begin{aligned}
V_{ib} & \longrightarrow U_b \\
& \downarrow \\
V_{\mathcal{M}_b} & \longrightarrow \mathcal{M}_b
\end{aligned}
$$

Since $\mathcal{M}_b$ is normal, by Remark B.3 the simultaneous normalization $\nu$ of $V_{ib} \to V_{\mathcal{M}_b}$ is the normalization. Denote the new total space by $V_{ib}'$. By [AB16, Lemma A.4], $V_{ib}'$ provides an atlas for a normal Deligne-Mumford stack $U_b'$. Furthermore, obtain a universal divisor $D_b'$, which is $D_b' + I_b'$, where $I_b'$ denotes the divisor corresponding to the conductor ideal sheaf.

We perform a simultaneous resolution (which is functorial by Remark B.9 (4)), possibly up to finite base change, using Theorem B.7. In particular, call the new total space $\nabla_{ib}$. We will abuse notation and use the same notation, even if we possibly had to base change. Note that this provides an atlas for a smooth Deligne-Mumford stack $\mathcal{U}_b$ (see Remark B.9(4) and e.g. [AW97]).

Since the map $\nabla_{ib} \to \mathcal{U}_b$ is faithfully flat, by [Vis89, 7.18], to define a sheaf $\mathcal{F}_b$ on $\mathcal{U}_b$ is equivalent to giving a sheaf over $\nabla_{ib}$ with descent data. The scheme $\nabla_{ib}$ comes equipped with a coherent sheaf, namely the relative log cotangent with respect to the map $\nabla_{ib} \to V_{\mathcal{M}_b}$. Denoting this sheaf by $\mathcal{G}_b$, we want to exhibit descent data yielding the the desired sheaf on $(\mathcal{U}_b, \overline{\mathcal{D}}_b)$. We have a presentation

$$
\begin{aligned}
V_R & \overset{p_1}{\underset{p_2}{\longrightarrow}} \nabla_{ib} \longrightarrow \mathcal{U}_b
\end{aligned}
$$

This gives two sheaves on $V_R$, namely $p_1^*\mathcal{G}_b$ and $p_2^*\mathcal{G}_b$. The sheaf $\mathcal{G}_b$ satisfies the universal property of the log cotangent sheaf (i.e. it is universal among derivations with simple poles on the divisor), implying the existence of an isomorphism $\tau : p_1^*\mathcal{G}_b \to p_2^*\mathcal{G}_b$. In particular $\tau$ is a gluing datum for $\mathcal{G}_b$.

Note that we have the following diagram.

$$
\begin{aligned}
V := V_R \times_{\nabla_{ib}} V_R & \overset{p_1}{\underset{p_2}{\longrightarrow}} \nabla_{ib} \longrightarrow \mathcal{U}_b \\
& \downarrow \\
& \downarrow
\end{aligned}
$$

To show that $\tau$ is a descent datum, we need to show that the fiber product of the three pullbacks of $\tau$ through the naturally defined maps $V \to V_R$ satisfy a cocycle relation. This follows from the universal property of the log cotangent ([Vis89, 7.20: (ii)]). Thus there is a sheaf $\mathcal{F}_b$ on $(\mathcal{U}_b, \overline{\mathcal{D}}_b)$ that coincides fiberwise with the log cotangent sheaf of the corresponding stable pair. Taking the reflexive hull of $\mathcal{F}_b$ gives the desired sheaf on $(\mathcal{U}_b, \overline{\mathcal{D}}_b)$. $\square$

The existence of the sheaf $\mathcal{F}_b$ on the resolution of the universal family over any fixed strata implies that there is a choice of a finite extension $S' \supset S$ such that the log cotangent sheaf of every fiber of the fixed model of this universal family is almost ample.

Lemma B.12. Suppose that every stable pair of dimension two with fixed invariants $\Gamma$ defined over $K$ has almost ample log cotangent. Then there is a finite set of places $S$ and a sheaf $\mathcal{F}_b$, possibly up to finite base change, of $(\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_b, \Gamma$ with fixed invariants $\Gamma$ over $\mathcal{O}_{K,S}$ which is relatively almost ample away from $S$.

Proof. As above, let $\mathcal{M}_b$ be a (normal) strata of the moduli stack of stable pairs, let $(\mathcal{U}_b, \mathcal{D}_b)$ be the universal family, and let $(\mathcal{U}_b, \overline{\mathcal{D}}_b)$ be the resolution outlined above (possibly after finite base change). By Lemma B.11, the family $(\mathcal{U}_b, \overline{\mathcal{D}}_b) \to \mathcal{M}_b$ comes with a coherent sheaf, the log cotangent sheaf
\( \mathcal{F}_b \), which by assumption is almost ample. We need to show that this sheaf extends to an almost ample coherent sheaf on any model. Consider the following presentation,

\[
\begin{array}{ccc}
R_{\mathcal{M}_b} & R_{\mathcal{M}_b} \\
\downarrow & \downarrow \\
V_{\mathcal{M}_b} & V_{\mathcal{M}_b} \\
\downarrow & \downarrow \\
\mathcal{U}_b & \mathcal{U}_b \\
\downarrow & \downarrow \\
\mathcal{M}_b & \mathcal{M}_{\Gamma,b}
\end{array}
\]

and recall by Remark B.9, that we stratified so that the resolution defined above \((\mathcal{U}_b, \mathcal{D}_b) \to \mathcal{M}_b\) was compatible with the resolution of the model \((\mathcal{M}_b, \mathcal{D}_b) \to \mathcal{M}_{\Gamma,b}\). By construction of \( \mathcal{F}_b \) in Lemma B.11, we have an almost ample coherent sheaf, which we call \( F_b \), on the atlas \( V_{\mathcal{M}_b} \) and therefore, by pushing forward, we have a coherent sheaf on \( V_{\mathcal{M}_b} \). The two pullbacks of this sheaf to the presentation \( R_{\mathcal{U}_b} \) are compatible since they agree on the generic fiber and therefore give a descent datum for the sheaf. This implies that the sheaf \( \mathcal{F}_b \) extends over an open subset of the base \( \mathcal{M}_{\Gamma,b} \).

Recall that the sheaf \( \mathcal{F}_b \) is almost ample on every fiber of \((\mathcal{U}_b, \mathcal{D}_b)\) (possibly up to finite base change), and that the model of the resolution is the resolution of the model (see Remark B.9 (5)). Hence, after a finite extension \( S' \supset S \) if necessary (but noting that this extension does not depend on \((X, D)\)), we can assume that outside \( S' \), the log cotangent sheaf extends to a relatively almost ample sheaf \( \mathcal{F}_b \) over any resolution of any model of the universal family.

\section*{Appendix C. Coarse moduli-stably integral points}

In this appendix we show how to define \( ms \)-integral points without appealing to good models (see Section 5). The definition we state here is weaker, in the sense that it relies on models of the coarse moduli space of stable pairs instead of models of the stack. However, it has the advantage that such models can be proven to exist unconditionally. We start by recalling the following definition:

\begin{definition}[DR73] A coarse moduli space for a stack \( \mathcal{M}_\Gamma \) over a base scheme \( S \) is an algebraic space \( [\mathcal{M}_\Gamma] \) over \( S \) with a \( S \)-morphism \( \phi_{[\mathcal{M}_\Gamma]} : \mathcal{M}_\Gamma \to [\mathcal{M}_\Gamma] \) such that:

\begin{enumerate}
\item Every morphism \( \mathcal{M}_\Gamma \to X \), for an algebraic space \( X \), factors uniquely through \( \phi_{[\mathcal{M}_\Gamma]} \);
\end{enumerate}
\end{definition}
(2) For every geometric point \( \bar{s} \in S(\mathbb{k}) \), \( \pi \) induces a bijection between isomorphism classes of \( \mathcal{M}_\Gamma \) over \( \bar{s} \) and \([\mathcal{M}_\Gamma(\bar{s})]\).

Given a stable family \((X, D) \to B\) with fixed volume, dimension and coefficient set (as in [KP17]), any moduli functor \( \mathfrak{F} \) for which \((X, D)\) is an object of \( \mathfrak{F}(B) \) is proper, and any algebraic space which is a coarse moduli space for the functor is a projective variety ([KP17, Theorem 1.1]). In particular, if we fix one of such functor (for example the one described in [KP17, Definition 5.6], which in addition is a Deligne-Mumford-stack of finite type over any algebraic closed field of char 0), then the corresponding moduli stack of stable pairs \( \mathcal{M}_\Gamma \), for fixed geometric invariants, is projective and possesses a universal family \((\mathcal{U}, \mathcal{D})\) with the property that the following diagram commutes:

\[
\begin{array}{ccc}
(X, D) & \longrightarrow & (\mathcal{U}, \mathcal{D}) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \mathcal{M}_\Gamma
\end{array}
\]

By Theorems A.1 and A.2 the above diagram holds over \( \mathbb{Q} \), and therefore over any number field, and the corresponding coarse moduli spaces are projective over the same ground field. It is not known in general whether the moduli stacks \( (\mathcal{U}, \mathcal{D}) \to \mathcal{M}_\Gamma \) possess models over Dedekind domains which are moduli stacks for stable pairs over such domains (although it is possible to find specific models of the such stacks using limit methods for algebraic stacks, see [Ols06] or [Ryd15]). This would make the previous diagram hold when the family \((X, D) \to B\) has a model over the ring of integers of a number field. However, allowing finite extensions of \( S \), one can identify specific models for the coarse moduli spaces that depend only on the stacks, the number field, and the set of places.

**Theorem C.2.** Given a number field \( K \) and a finite set of places \( S \) there exists \( S^m \supset S \) such that the coarse moduli spaces \([\mathcal{M}_\Gamma, \mathcal{U}], \mathcal{D}\) admit proper models over \( \text{Spec} \mathcal{O}_{K, S^m} \). Moreover, the set \( S^m \) depends only on the (models of the) underlying projective varieties.

**Proof.** By definition the stack \( \mathcal{M}_\Gamma \) has an underlying coarse moduli space which is a projective variety, and therefore it is the locus of the common zeroes of finitely many polynomials \( g_1, \ldots, g_n \) with coefficients in \( K \). It then follows that since the number of overall coefficients are finite, there exists a (minimal) set of places \( T \) such that the \( g_i \) are polynomials with \( T \)-integral coefficients. For such \( T \), the coarse moduli space \([\mathcal{M}_\Gamma]\) has a model: \([\mathcal{M}_\Gamma]^m \to \mathcal{O}_{K,T}\). Since the model is projective by construction, it is also proper. Applying the same procedure to \( \mathcal{U} \) and \( \mathcal{D} \), one finds a (possibly larger) set of places \( T \) such that \(([\mathcal{U}],[\mathcal{D}]) \to [\mathcal{M}_\Gamma]\) have a model \(([\mathcal{U}]^m, [\mathcal{D}]^m) \to [\mathcal{M}_\Gamma]^m \to \mathcal{O}_{K,T}\). Define \( S^m = S \cup T \); by definition all the coarse moduli spaces have models over \( \mathcal{O}_{K,S^m} \) and, at most after enlarging the set \( S^m \), the moduli map \((\mathcal{U}, \mathcal{D}) \to \mathcal{M}_\Gamma\) extends to a map on the models. \( \square \)

Given the model defined in Theorem C.2, we can define coarse moduli stably \( S \)-integral points with respect to the choice of the model we made.

**Definition C.3.** Let \( P \) be a \( K \)-rational point of a stable pair \((X, D)\) defined over \( K \), and let \( \overline{P} \) denote the corresponding maps to the moduli space

\[
\begin{array}{ccc}
\overline{P} : \text{Spec} K & \longrightarrow & (X, D) \\
& \downarrow & \downarrow \\
\text{Spec} K & \longrightarrow & \mathcal{M}_\Gamma
\end{array}
\]
We say that $P$ is coarse moduli-stably $S$-integral over $K$, or $cms$-integral, if the image is $(S^m, D)$ integral in $U$. We denote by $X(O^{cms}_{K,S})$ the set of all $cms$ $S$-integral points over $K$.

Note that the previous definition depends on the choice of models made in Theorem C.2. This would be canonical if the existence of a moduli stack $M_\Gamma$ over $\text{Spec} O_{K,S}$ was known.

The definition of $cms$-integral points depend on models of the coarse moduli space. As we will use results comparing integral points of fibered powers of families of stable pairs with integral points of the pair itself, we need to ensure that Definition C.3 is compatible with fibered powers. In general fibered powers do not commute with the formation of coarse moduli spaces. However a weaker property of base change holds: for any map of schemes $S' \to S$, the following map $\iota : [M_\Gamma \otimes_S S'] \to [M_\Gamma] \otimes_S S'$ is universally injective. Recall that a universally injective morphism is a morphism that is injective on $K$-valued points for every field $K$.

We want to compare the coarse moduli space of the fiber product $[M_\Gamma \times M_\Gamma]$ with the fiber product of the coarse moduli space $[M_\Gamma] \times [M_\Gamma]$ (every product is over the fixed base, $\text{Spec} K$) in the case in which the coarse moduli space $[M_\Gamma]$ is a scheme. There is a map between the fibered product of the stack to the fibered power of the coarse moduli space given by composition as follows:

$$\alpha : M_\Gamma \times M_\Gamma \to M_\Gamma \times [M_\Gamma] \to [M_\Gamma] \times [M_\Gamma] \to [M_\Gamma] \times [M_\Gamma].$$

By the first property of coarse moduli spaces, this defines a map: $\beta_2 : [M_\Gamma \times M_\Gamma] \to [M_\Gamma] \times [M_\Gamma]$, which, inductively, defines a map $\beta_n : [M_\Gamma^n] \to [M_\Gamma]^{n!}$.

However this map is not injective in general, since the group of automorphisms of the fiber product is, in general, strictly contained in the product of the automorphism group. In particular, if (étale) locally over the coarse moduli space, $M_\Gamma$ is presented locally as $X/G$, $M_\Gamma \times M_\Gamma$ is presented by the quotient of $X \times X$ by a finite group $G'$, which in general will be a subgroup of the $\text{Aut}(X \times X)$. On the other hand, the coarse moduli space $[M_\Gamma] \times [M_\Gamma]$ will look locally like $X/G \times X/G$. Since $G' \subset G \times G$ this gives locally a finite map $\beta : [M_\Gamma \times M_\Gamma] \to [M_\Gamma] \times [M_\Gamma]$.

**Definition C.4.** Let $(X, D)$ be a stable pair defined over a $K$, let $O_{K,S}$ be the ring of $S$-integers, and let $n$ be a positive integer. A rational point $P \in (X^n, D_n)$ is coarse moduli stably $(S, D_n)$-integral if $\beta(P)$ is $(S^m, D)$ integral in the model of the coarse moduli space given by the fiber product of the models defined in Theorem C.2.

Explicitly, this happens if the pair over $\text{Spec} K$ comes with a commutative diagram

$$\begin{array}{ccc}
(X, D) & \longrightarrow & (U, D) \\
\downarrow & & \downarrow \\
\text{Spec} K & \longrightarrow & M_\Gamma
\end{array}$$

for a specific moduli stack $M_\Gamma$. Let $[U]^m$ be the model of the coarse moduli space $[U]$ over $O_{S^m,K}$: then $([U]^m)^n$ is a model of $[U]^n$. One can prove Theorem 6.1 using $cms$-integral points instead of $ms$-integral points. However, to prove uniformity, one would need to extend the subvariety property to $cms$-integral points, which a priori is harder to achieve since there is no good model to refer to.

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