On spectral radii of unraveled balls

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Abstract

Given a graph \( G \), the unraveled ball of radius \( r \) centered at a vertex \( v \) is the ball of radius \( r \) centered at \( v \) in the universal cover of \( G \). We prove a lower bound on the maximum spectral radius of unraveled balls of fixed radius, and we show, among other things, that if the average degree of \( G \) after deleting any ball of radius \( r \) is at least \( d \) then its second largest eigenvalue is at least 

\[
2\sqrt{d-1} \cos\left(\frac{\pi}{r+1}\right).
\]

1 Introduction

The well-known result of Alon and Boppana [Nil91] states that for every \( d \)-regular graph \( G \) containing two edges at distance \( \geq 2r \), the second largest eigenvalue, denoted by \( \lambda_2(G) \), of the adjacency matrix of \( G \) satisfies:

\[
\lambda_2(G) \geq 2 \left( 1 - \frac{1}{r} \right) \sqrt{d-1} + \frac{1}{r}.
\]

Subsequently, Friedman [Fri93, Corollary 3.6] improved the above bound (see also [Nil04]): for every \( d \)-regular graph \( G \) with diameter \( \geq 2r \),

\[
\lambda_2(G) \geq 2 \left( 1 - \frac{\pi^2}{2r^2} + O\left(r^{-4}\right) \right) \sqrt{d-1}.
\]

All these proofs of the Alon–Boppana bound primarily relied on estimating the spectral radius of an induced subgraph on the vertices within certain distance from a given vertex or edge.

Definition 1. Given a graph \( G \) and a vertex \( v \), a ball of radius \( r \) centered at \( v \), denoted by \( G(v, r) \), is the induced subgraph of \( G \) on the vertices within distance \( r \) from \( v \).

The proof of Friedman uses the fact [Fri93, Lemma 3.3] that the spectral radius of \( G(v, r) \) is at least the spectral radius of the \( d \)-regular tree of depth \( r \). Note that the universal cover of a \( d \)-regular graph is the infinite \( d \)-regular tree. This motivates the following definition.

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Definition 2. Given a graph $G$, a walk $(v_0, v_1, \ldots)$ on $G$ is non-backtracking if $v_i \neq v_{i+2}$ for all $i$. For every vertex $v$, define the tree $\tilde{G}(v, r)$ as follows: the vertex set consists of all the non-backtracking walks on $G$ of length $\leq r$ starting at $v$, and two vertices are adjacent if one is a simple extension of another. In other words, $\tilde{G}(v, r)$ is the ball of radius $r$ centered at $v$ in the universal cover $\tilde{G}$ of $G$.

We call $\tilde{G}(v, r)$ the unraveled ball of radius $r$ centered at $v$, and we prove the following theorem on the spectral radii of unraveled balls. From now on, $\lambda_1(\cdot)$ denotes the spectral radius of a graph and $d(u)$ denotes the degree of $u$ in $G$.

Theorem 1. For any graph $G = (V, E)$ of minimum degree $\geq 2$ and $r \in \mathbb{N}$, there exists a vertex $v \in V$ such that

$$\lambda_1(\tilde{G}(v, r)) \geq \frac{1}{|E|} \sum_{u \in V} d(u) \sqrt{d(u)} - 1 \cdot \cos \left( \frac{\pi}{r+2} \right).$$

After presenting the proof of Theorem 1 in Section 2, we show in Section 3 a cheap lower bound on the spectral radius of the universal cover of a graph. We proceed in Section 4 and 5 to describe additional applications including an improvement to a result of Hoory [Hoo05]. The final section briefly discusses a potential extension to weighted graphs and its connection to the normalized Laplacian.

2 Proof of Theorem 1

The proof uses the old idea of constructing a test function by looking at non-backtracking walks (see e.g. [Chu16] and [ST18]). The innovation here is to weight the test function using the eigenvector of a path.

Proof of Theorem 1. Define $W_i$, for every $i \geq 1$, to be the set of all non-backtracking walks of length $i$ on $G$. Specifically $W_1$ is the set of directed edges of $G$. Define the forest $T$ as follows: the vertex set is $\bigcup_{i=1}^{r+1} W_i$ and two vertices are adjacent if and only if one is a simple extension of the other. For every $e = (v_0, v_1) \in W_1$, denote by $T_e$ the connected component of $T$ containing $e$. If one identifies every vertex $(v_0, v_1, \ldots, v_i)$ in $T_e$, where $i \in [r + 1]$, with the vertex $(v_1, \ldots, v_i)$ in $\tilde{G}(v_1, r)$, then $T_e$ becomes a subgraph of $\tilde{G}(v_1, r)$. By the monotonicity of spectral radius, $\lambda_1(\tilde{G}(v_1, r)) \geq \lambda_1(T_e)$. Because $\lambda_1(T) = \max \{ \lambda_1(T_e) : e \in W_1 \}$, there exists a vertex $v \in V$ such that $\lambda_1(\tilde{G}(v, r)) \geq \lambda_1(T)$. Set $\lambda = 2 \cos(\frac{\pi}{r+2})$. It suffices to prove

$$\lambda_1(T) \geq \lambda \cdot \sum_{u \in V} \frac{d(u)}{|W_1|} \sqrt{d(u)} - 1.$$
We observe that \( \lambda \in E \) since the minimum degree of \( G \) satisfies easily check that the uniform distribution on \( W \) to be the adjacency matrix of the forest \( T \). For any graph \( \tilde{G} \), Corollary 2.3 Spectral radius of the universal cover of length \( i \), which we denote by the random variables \( Y_i = (X_0, X_1, \ldots, X_i) \).

Recall that \( \lambda = 2 \cos(\frac{\pi}{r+2}) \) is the spectral radius of the path of length \( r \). Let \( (x_1, x_2, \ldots, x_{r+1}) \in \mathbb{R}^{r+1} \) be an eigenvector of the path associated with \( \lambda \). By the Rayleigh principle, we have

\[
\sum_{i=2}^{r+1} 2x_{i-1}x_i = \lambda \cdot \sum_{i=1}^{r+1} x_i^2
\]

(1)

Define the vector \( f : \bigcup_{i=1}^{r+1} W_i \to \mathbb{R} \) by \( f(w) = x_i \sqrt{\Pr(Y_i = w)} \) for \( w \in W_i \), and define the matrix \( A \) to be the adjacency matrix of the forest \( T \). For \( w = (v_0, v_1, \ldots, v_i) \), denote by \( w^- = (v_0, v_1, \ldots, v_{i-1}) \).

We observe that

\[
\langle f, f \rangle = \sum_{i=1}^{r+1} \sum_{w \in W_i} f(w)^2 = \sum_{i=1}^{r+1} \sum_{w \in W_i} x_i^2 \Pr(Y_i = w) = \sum_{i=1}^{r+1} x_i^2,
\]

(2)

\[
\langle f, Af \rangle = \sum_{i=2}^{r+1} \sum_{w \in W_i} 2f(w^-)f(w) = \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{w \in W_i} \sqrt{\Pr(Y_{i-1} = w^-)} \Pr(Y_i = w).
\]

(3)

By the Markov property, for every \( i \geq 2 \) and \( w = (v_0, v_1, \ldots, v_i) \in W_i \),

\[
\frac{\Pr(Y_i = w)}{\Pr(Y_{i-1} = w^-)} = \Pr(E_i = (v_{i-1}, v_i) \mid E_{i-1} = (v_{i-2}, v_{i-1})) = \frac{1}{d(v_{i-1}) - 1}.
\]

Thus the inner summation in the right hand side of (3) equals

\[
\sum_{w=(v_0,v_1,\ldots,v_i)\in W_i} \sqrt{d(v_{i-1}) - 1} \Pr(Y_i = w) = E \left[ \sqrt{d(X_{i-1}) - 1} \right] = \sum_{v \in V} \sqrt{d(v) - 1} \Pr(X_{i-1} = v). \quad (4)
\]

Since the minimum degree of \( G \) is \( \geq 2 \), the Markov chain has no absorbing states. Moreover, one can easily check that the uniform distribution on \( W_1 \) is a stationary distribution of the Markov chain, that is, \( \Pr(E_i = e) = 1/|W_1| \) for all \( i \geq 1 \) and \( e \in W_1 \). Thus \( \Pr(X_{i-1} = v) = d(v)/|W_1| \) for all \( i \geq 2 \) and \( v \in V \). Plugging this into (4), we can simplify (3) to

\[
\langle f, Af \rangle = \sum_{i=2}^{r+1} 2x_{i-1}x_i \sum_{v \in V(\tilde{G})} \frac{d(v)}{|W_1|} \sqrt{d(v) - 1}.
\]

Finally we combine with (1) and (2), and the Rayleigh principle \( \lambda_1(T) \geq \langle f, Af \rangle / \langle f, f \rangle \).

\[
\square
\]

3 Spectral radius of the universal cover

Since \( \tilde{G}(v, r) \) is an induced subgraph of \( \tilde{G} \), the monotonicity of spectral radius implies immediately a lower bound on \( \lambda_1(\tilde{G}) \) by letting \( r \) go to infinity in Theorem 1.

**Corollary 2.** For any graph \( G = (V, E) \) of minimum degree \( \geq 2 \), the spectral radius of its universal cover satisfies

\[
\lambda_1(\tilde{G}) \geq \frac{1}{|E|} \sum_{u \in V} d(u) \sqrt{d(u) - 1}.
\]

(5)
Remark 1. By the inequality of arithmetic and geometric means, the right hand side of (1) satisfies

\[ \frac{1}{|E|} \sum_{u \in V} d(u) \sqrt{d(u)} - 1 = 2 \cdot \frac{\sum_{u \in V} d(u) \sqrt{d(u)} - 1}{\sum_{u \in V} d(u)} \geq 2 \prod_{u \in V} \left( \sqrt{d(u)} - 1 \right) \frac{d(u)}{\sum_{v \in V} d(v)} , \]

which recovers the lower bound on \( \lambda_1(\tilde{G}) \) in [Hoo05, Theorem 1].

4 Maximum spectral radius of balls

The following result, which is essentially due to Mohar [Moh10, Theorem 2.2], connects the spectral radii of a ball and its corresponding unraveled ball.

Lemma 3. For every vertex \( v \) of a graph \( G \) and \( r \in \mathbb{N} \), \( \lambda_1(G(v, r)) \geq \lambda_1(\tilde{G}(v, r)) \).

To prove Lemma 3 we need the following simple fact. For the sake of completeness we include the short proof in Appendix A.

Lemma 4. For every connected graph \( G = (V, E) \) and every vertex \( v \in V \), \( \lambda_1(G) = \limsup \sqrt[k]{s_k(v)} \), where \( s_k(v) \) is the number of closed walks of length \( k \) starting at \( v \) in \( G \). In fact, \( \lambda_1(G) = \lim \sqrt[k]{s_{2k}(v)} \).

Proof of Lemma 4. Recall that a vertex of \( \tilde{G}(v, r) \) is a non-backtracking walk of length \( \leq r \) starting at \( v \). Denote the non-backtracking walk of length \( 0 \) starting at \( v \) by \( w := (v) \). For every \( k \), we naturally map a closed walk \( w = w_1, w_2, \ldots, w_k = w \) of length \( k \) in \( \tilde{G}(v, r) \) to a closed walk \( v = v_1, v_2, \ldots, v_k = v \) of length \( k \) in \( G(v, r) \), where \( v_j \) is the terminal vertex of \( w_j \) for \( j \in [k] \). One can show that this map is injective, and so the number of closed walks of length \( k \) starting at \( v \) in \( G(v, r) \) is at least the number of closed walks of length \( k \) starting at \( w \) in \( \tilde{G}(v, r) \). Lemma 4 thus implies that \( \lambda_1(G(v, r)) \geq \lambda_1(\tilde{G}(v, r)) \).

We shall combine Lemma 3 and Theorem 4 to provide a lower bound on the maximum spectral radius of balls in Lemma 6, which slightly strengthens [JP17, Lemma 12]. We need the following fact.

Theorem 5 (Theorem 2 of Collatz and Sinogowitz [CS57]; Theorem 3 of Lovász and Pelikán [LP73]). If \( G \) is a tree of \( n \) vertices then \( \lambda_1(P_n) \leq \lambda_1(G) \), where \( P_n \) is the path with \( n \) vertices.

Lemma 6. For any graph \( G = (V, E) \) of average degree \( d \geq 1 \) and \( r \in \mathbb{N} \), there exists \( v \in V \) such that \( \lambda_1(G(v, r)) \geq 2\sqrt{d-1} \cos\left(\frac{r}{2} \right) \).

Proof. If \( G \) has more than one connected component, we shall just prove for one of the connected components with average degree \( \geq d \). Hereafter, we assume that \( G \) is connected.

Case 1 \( d < 2 \): For a connected graph, having an average degree \( < 2 \) is the same as being a tree. Pick any \( v \in V \). If \( G(v, r) = G \), then \( \lambda_1(G(v, r)) \geq d = (d-1) + 1 \geq 2\sqrt{d-1} \). Otherwise \( G(v, r) \) is a tree of \( \geq r + 1 \) vertices, and \( \lambda_1(G(v, r)) \geq \lambda_1(P_{r+1}) = 2\cos\left(\frac{r}{2}\right) \) by Theorem 5.
Case $d \geq 2$: Since removing leaf vertices from a graph of average degree $d \geq 2$ cannot decrease its average degree, without loss of generality, we may assume that the minimum degree of $G$ is $\geq 2$. By Lemma 3 and Theorem 1 there exists a vertex $v \in V$ such that

$$
\lambda_1(G(v, r)) \geq \lambda_1(\tilde{G}(v, r)) \geq \frac{1}{|E|} \sum_{u \in V} d(u) \sqrt{d(u) - 1} \cdot \cos \left( \frac{\pi}{r + 2} \right).
$$

A straightforward calculation can verify that the function $x \mapsto x \sqrt{x - 1}$ is convex for $x \geq 2$. It follows from the Jensen’s inequality that the right hand side of the above is at least

$$
\frac{1}{|E|} |V| \sqrt{d - 1} \cdot \cos \left( \frac{\pi}{r + 2} \right) = 2 \sqrt{d - 1} \cos \left( \frac{\pi}{r + 2} \right).
$$

5 Second largest eigenvalue

It is natural to generalize the Alon–Boppana bound to graphs that may not be regular. It is conceivable that for any sequence of graphs $G_i$ with average degree $\geq d$ and growing diameter, $\lim \inf \lambda_2(G_i) \geq 2 \sqrt{d - 1}$. However, Hoory constructed in [Hoo05] a counterexample to such a statement. In his construction, the average degree drops drastically after deleting a ball of radius 1. Hoory then extended the Alon–Boppana bound to graphs that have a robust average degree.

Definition 3. A graph has an $r$-robust average degree $\geq d$ if the average degree of the graph is $\geq d$ after deleting any ball of radius $r$.

Theorem 7 (Theorem 3 of Hoory [Hoo05]). Given a real number $d \geq 2$ and a natural number $r \geq 2$, for any graph $G$ that has an $r$-robust average degree $\geq d$, its second largest eigenvalue in absolute value satisfies:

$$
\max \{ \lambda_2(G), \lambda_{-1}(G) \} \geq 2 \left( 1 - c \cdot \frac{\log r}{r} \right) \sqrt{d - 1},
$$

where $\lambda_{-1}(G)$ denotes the smallest eigenvalue of $G$, and $c$ is an absolute constant.

It is noticeable that Theorem 7 may not be optimal in comparison to the Alon–Boppana bound. The left hand side of (6) should be simply $\lambda_2(G)$, and inside the right hand side $c \cdot \frac{\log r}{r}$ could be improved to $c \cdot \frac{1}{r}$. We prove that this is indeed the case.

Theorem 8. Given a real number $d \geq 1$ and a natural number $r \geq 1$, if a graph $G$ has an $r$-robust average degree $\geq d$, then

$$
\lambda_2(G) \geq 2 \sqrt{d - 1} \cos \left( \frac{\pi}{r + 1} \right).
$$

Proof. After deleting an arbitrary ball of radius $r$, as the average degree is $\geq d$, by Lemma 6 and the monotonicity of spectral radius, there is $v_1 \in V$ such that the spectral radius of $G_1 := G(v_1, r - 1)$ is at least $2 \sqrt{d - 1} \cos (\frac{\pi}{r + 1}) =: \lambda_*$. Let $G' = (V', E')$ be the graph after deleting the ball of radius $r$ centered at $v_1$ from $G$. Repeating this argument, we can find $v_2 \in V'$ such that the spectra radius of
$G_2 := G'(v_2, r - 1)$ is at least $\lambda_*$. For $i = 1, 2$, let $A_i$ be the adjacency matrix of $G_i$ and let $f_i$ be the eigenvector of $A_i$ associated with $\lambda_1(G_i)$.

Denote the adjacency matrix of $G$ by $A$. Choose scalars $c_1, c_2$, not all zero, such that the vector $f : V \to \mathbb{R}$, defined by

$$f(u) = \begin{cases} c_1 f_1(u) & \text{if } u \in V(G_1); \\ c_2 f_2(u) & \text{if } u \in V(G_2); \\ 0 & \text{otherwise,} \end{cases}$$

is perpendicular to an eigenvector of $A$ associated with $\lambda_1(G)$. If $u \in V(G_1)$, that is $u$ is within distance $r - 1$ from $v_1$, and $v$ is adjacent to $u$, then $v$ is within distance $r$ from $v_1$, hence $v \notin V(G_2)$. In other words, $\{u, v\} \notin E$ for all $u \in V(G_1)$ and $v \in V(G_2)$. Thus we obtain a Rayleigh quotient from which $\lambda_2(G) \geq \lambda_*$ follows:

$$\frac{\langle f, Af \rangle}{\langle f, f \rangle} = \frac{c_1^2 \langle f_1, A_1 f_1 \rangle + c_2^2 \langle f_2, A_2 f_2 \rangle}{c_1^2 \langle f_1, f_1 \rangle + c_2^2 \langle f_2, f_2 \rangle} = \frac{c_1^2 \lambda_1(G_1) \langle f_1, f_1 \rangle + c_2^2 \lambda_1(G_2) \langle f_1, f_1 \rangle}{c_1^2 \langle f_1, f_1 \rangle + c_2^2 \langle f_2, f_2 \rangle} \geq \lambda_*.$$

### 6 Concluding remarks

The normalized Laplacian $N$ of $G = (V, E)$ is defined by $N = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$, where $D$ is the diagonal degree matrix with $D_{v,v} = d(v)$ for all $v \in V$ and $A$ is the adjacency matrix. The second smallest eigenvalue of $N$, denoted by $\mu_2(G)$, is tightly connected to various expansion properties of $G$ (see, for example, [Chu16, Section 2]).

In the context of the normalized Laplacian, the Alon–Boppana bound says that for a $d$-regular graph $G$ with diameter $\geq 2r$, $\mu_2(G) \leq 1 - \frac{2\sqrt{d-1}}{d} \left(1 - \frac{\sqrt{d}}{2d} + O(r^{-4})\right)$. Young [You11, Section 3] refuted the natural generalization by showing an infinite family of graphs $G_1, G_2, \ldots$ with common average degree $d$ and growing diameter and some fixed $\varepsilon > 0$ such that $\mu_2(G_i) \geq 1 - \frac{2\sqrt{d-1}}{d} + \varepsilon$ for all $i$. He also proved an upper bound of the form $\mu_2(G) \leq 1 - \frac{2\sqrt{d-1}}{d} \left(1 - c \cdot \frac{\ln k}{k}\right)$, where $d$ is the average degree, $d$ is the second order average degree and $k$ is the normalized Laplacian eigenradius (see [You11, Theorem 6]). Recently Chung proved, under some technical assumptions on $G$, another upper bound of the form $\mu_2(G) \leq 1 - \sigma(G)(1 - \frac{\varepsilon}{k})$, where $\sigma(G) := 2 \sum_{u \in V} d(u) \sqrt{d(u) - 1}/\sum_{u \in V} d(u)^2$ and $k$ is the diameter of $G$ (see [Chu16, Theorem 9]).

Observe that the matrix $D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$ in the definition of $N$ can be seen as a graph $G$ with the weight $d(u)^{-\frac{1}{2}} d(v)^{-\frac{1}{2}}$ assigned to each edge $\{u, v\}$. Moreover, the second largest eigenvalue of this weighted graph is equal to $1 - \mu_2(G)$. Based on these two observations, the author believes that the machinery developed in this paper can be generalized to weighted graphs to provide a better upper bound on $\mu_2(G)$, possibly under fewer assumptions on the graph.
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A Spectral radius and closed walks

Proof of Lemma 4. Let \( v = v_1, v_2, \ldots, v_n \) be the vertices of \( G \), and let \( \lambda_1(G) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) be the eigenvalues of the adjacency matrix \( A \) of \( G \). An elementary graph theoretic interpretation identifies the trace of \( A^k \) as the number of closed walks of length \( k \) in \( G \). But a standard matrix result equates \( \text{tr} A^k \) to the \( k \)th moment of \( A \) defined as \( \sum_{i=1}^{n} \lambda_i^k \). Thus, we have found the following:

\[
\sum_{i=1}^{n} \lambda_i^k = \text{tr} A^k = \sum_{i=1}^{n} s_k(v_i) =: s_k. \tag{7}
\]

Observe that \( s_k \) is always a natural number. We see from (7) that \( \lambda_1 \geq |\lambda_i(G)| \) for all \( i \in [n] \), hence

\[
\lambda_1(G) = \lim sup \sqrt[k]{s_k}. \tag{8}
\]

For every \( i \in [n] \), let \( k_i \) be the distance from \( v \) to \( v_i \). By prepending the walk from \( v \) to \( v_i \) of length \( k_i \) and appending the reverse, we extend a closed walk of length \( k \) starting at \( v_i \) to one of length \( k + 2k_i \) starting at \( v \), and we obtain that \( s_k(v_i) \leq s_{k+2k_i}(v) \). Similarly by appending a closed walk of length 2 starting at \( v \), we extend a closed walk of length \( k \) starting at \( v \) to one of length \( k + 2 \), and we obtain that \( s_k(v) \leq s_{k+2}(v) \) for all \( k \in \mathbb{N} \). Thus

\[
s_k(v) \leq s_k = \sum_{i=1}^{n} s_k(v_i) \leq \sum_{i=1}^{n} s_{k+2k_i}(v) \leq n \cdot s_{k+2k^*}(v),
\]

where \( k^* = \max \{k_1, k_2, \ldots, k_n\} \). In view of (8), we get that \( \lambda_1(G) = \lim sup \sqrt[k]{s_k(v)} \). Lastly, note that (8) can be made more precise as \( \lambda_1(G) = \lim 2\sqrt[k]{s_k(v)} \) to obtain \( \lambda_1(G) = \lim 2\sqrt[k]{s_{2k}(v)} \). \( \square \)