Abstract: Let $\Omega$ be a Lipschitz bounded domain of $\mathbb{R}^N$, $N \geq 2$. The fractional Cheeger constant $h_s(\Omega)$, $0 < s < 1$, is defined by

$$h_s(\Omega) = \inf_{E \subset \Omega} \frac{P_s(E)}{|E|^s},$$

where $P_s(E) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{N+s}} \, dx \, dy$,

with $\chi_E$ denoting the characteristic function of the smooth subdomain $E$. The main purpose of this paper is to show that

$$\lim_{p \to 1^+} |\phi_p|_{L_i^1(\Omega)}^{1-p} = h_s(\Omega) = \lim_{p \to 1^+} |\phi_p|_{L_1^1(\Omega)}^{1-p},$$

where $\phi_p^s$ is the fractional $(s, p)$-torsion function of $\Omega$, that is, the solution of the Dirichlet problem for the fractional $p$-Laplacian: $-(\Delta)_p^s u = 1$ in $\Omega$, $u = 0$ in $\mathbb{R}^N \setminus \Omega$. For this, we derive suitable bounds for the first eigenvalue $\lambda_{1,p}^s(\Omega)$ of the fractional $p$-Laplacian operator in terms of $\phi_p^s$. We also show that $\phi_p^s$ minimizes the $(s, p)$-Gagliardo seminorm in $\mathbb{R}^N$, among the functions normalized by the $L^1$-norm.

Keywords: Fractional Cheeger Problem, Fractional $p$-Laplacian, Fractional Torsion Functions

MSC 2010: Primary 35P15, 35R11; secondary 47A75

1 Introduction

The Cheeger constant $h(\Omega)$ of a bounded domain $\Omega \subset \mathbb{R}^N (N > 1)$ is defined by

$$h(\Omega) = \inf_{E \subset \Omega} \frac{P(E)}{|E|},$$

where $E$ is a smooth subset of $\Omega$ and the nonnegative values $P(E)$ and $|E|$ denote, respectively, the distributional perimeter and the $N$-dimensional Lebesgue measure of $E$. A subset $E$ that minimizes the quotient is a Cheeger set of $\Omega$.

In [7] Kawohl and Fridman proved that

$$h(\Omega) = \lim_{p \to 1^-} \lambda_{1,p}(\Omega),$$
The first (fractional) eigenvalue $\lambda_{1,p}(\Omega)$ of $(-\Delta)_p^s$ is the least number $\lambda$ such that the problem
\[
\begin{cases}
(-\Delta)_p^s u = \lambda |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has a nontrivial weak solution (see [4, 8]). Its variational characterization is given by (see [1])
\[
\lambda_{1,p}(\Omega) := \min \{ [u]_{s,p}^p : u \in W_{0}^{s,p}(\Omega), |u|_p = 1 \},
\]
(1.2)
where
\[
[u]_{s,p} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right)^{\frac{1}{p}}.
\]
(1.3)
\[\text{is the (s, p)-seminorm of Gagliardo in } \mathbb{R}^N \text{ of a measurable function } u \text{ and } W_{0}^{s,p}(\Omega) \text{ is a suitable fractional Sobolev space defined in the sequel (see Definition 2.1).} \]
In [1] Brasco, Lindgren and Parini proved the $s$-Cheeger version of the result originally obtained by Kawohl and Friedman [7] for the Cheeger problem

$$ h_s(\Omega) = \lim_{p \to 1^+} \lambda^s_{1,p}(\Omega). \quad (1.4) $$

In this paper, by assuming that $\Omega$ is a Lipschitz bounded domain, we show, in the spirit of the paper [2], that the fractional version of the torsional creep problem

$$ \left\{ \begin{array}{ll}
(-\Delta)^s u = 1 & \text{in } \Omega, \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega
\end{array} \right. \quad (1.5) $$
is intrinsically connected to both the $s$-Cheeger problem and the first eigenproblem for the fractional Dirichlet $p$-Laplacian, as $p$ goes to 1.

This connection will be developed in Section 2, where we introduce the $(s, p)$-torsion function of $\Omega$, that is, the weak solution $\phi^s_p$ of (1.5). We will derive the estimates

$$ \frac{1}{|\phi^s_p|_{p-1}} \leq \lambda^s_{1,p}(\Omega) \leq \left( \frac{|\Omega|}{|\phi^s_p|_1} \right)^{p-1} \quad (1.6) $$

and

$$ \frac{|\phi^s_p|_{p+\infty}}{|\phi^s_p|_1} \leq \frac{1}{|B_1|} \left( \frac{sp + N(p - 1)}{sp} \right)^{\frac{p(2N - p)}{sp}} \left( \frac{\lambda^s_{1,p}(\Omega)}{\lambda^s_{1,p}(B_1)} \right)^{\frac{N}{sp}} \quad (1.7) $$

where $B_1$ denotes the unit ball of $\mathbb{R}^N$.

Then, taking (1.4) into account, we will combine (1.6) with (1.7) in order to conclude the main result of this paper:

$$ \lim_{p \to 1^+} \frac{1}{|\phi^s_p|_{p-1}} = h_s(\Omega) = \lim_{p \to 1^+} \frac{1}{|\phi^s_p|_{p-1}}. $$

Moreover, in Section 2 we prove that $\phi^s_p$ minimizes, in $W_0^{s,p}(\Omega) \setminus \{0\}$, the Rayleigh quotient $[u]^p_{s,p}/|u|^p_1$. As an immediate consequence of this fact, we show that $\phi^s_p$ is a radial function when $\Omega$ is a ball.

## 2 The Main Results

From now on $\Omega$ denotes a Lipschitz bounded domain of $\mathbb{R}^N$, $N \geq 2$, and $0 < s < 1 < p < \frac{N}{s}$.

**Definition 2.1.** The Sobolev space $W_0^{s,p}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ with respect to the norm

$$ ||u|| := ||u||_{s,p} + ||u||_p, \quad (2.1) $$

where $||u||_{s,p}$ is defined by (1.3).

The functions in $W_0^{s,p}(\Omega)$ have a natural extension to $\mathbb{R}^N$ and, although $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the identity

$$ [u]^p_{s,p} = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy + 2 \int_{\mathbb{R}^N \setminus \Omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x)|^p}{|x - y|^{N+sp}} \, dx \, dy $$

shows dependence on values in $\mathbb{R}^N \setminus \Omega$.

It is worth mentioning that $W_0^{s,p}(\Omega)$ is a reflexive Banach space and that this space coincides with the closure of $C_0^{\infty}(\Omega)$ relative to the norm

$$ u \mapsto \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^\frac{1}{p} + ||u||_p, $$

if $\partial \Omega$ is Lipschitz (see [1, Proposition B.1]).
Moreover, thanks to the fractional Poincaré inequality (see [1, Lemma 2.4])
\[ |u|^p_p \leq C_{N,s,p,\Omega} |u|_{s,p}^p \quad \text{for all } u \in C_0^\infty(\Omega), \]
we have that \([\cdot]_{s,p}\) is also a norm in \(W_0^{s,p}(\Omega)\), equivalent to the norm defined in (2.1).

We refer the reader to [3] for fractional Sobolev spaces.

**Definition 2.2.** We say that a function \(u \in W_0^{s,p}(\Omega)\) is a weak solution of the fractional Dirichlet problem
\[
\begin{cases}
(-\Delta)^s_p u = f & \text{in } \Omega, \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
if
\[
\langle (-\Delta)^s_p u, \varphi \rangle = \int_\Omega f \varphi \, dx \quad \text{for all } \varphi \in W_0^{s,p}(\Omega),
\]
where
\[
\langle (-\Delta)^s_p u, \varphi \rangle := \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy,
\]
a notation that will be used from now on.

The existence of a weak solution of (2.2) when \(f \in L^1(\Omega)\) follows from direct minimization in \(W_0^{s,p}(\Omega)\) of the functional
\[
\frac{1}{p} |u|_{s,p}^p - \int_\Omega f(x) u(x) \, dx,
\]
whereas the uniqueness comes from, for instance, the comparison principle for the fractional \(p\)-Laplacian (see [8, Lemma 9]). The same principle shows that if \(f\) is nonnegative, then the weak solution \(u\) is nonnegative as well. When \(f \in L^{\infty}(\Omega)\) and \(\Omega\) is sufficiently smooth, say with boundary at least of class \(C^{1,1}\), the weak solutions are \(\alpha\)-Hölder continuous up to the boundary for some \(\alpha \in (0, 1)\), see [5].

When \(f \equiv 1\) the Dirichlet problem (2.2) will be referred to as the \((s, p)\)-fractional torsional creep problem and its unique weak solution will be called \((s, p)\)-torsion function. Let us denote this function by \(\phi^s_p\). We have \(\phi^s_p \geq 0\) and
\[
\langle (-\Delta)^s_p \phi^s_p, \varphi \rangle = \int_\Omega \varphi \, dx \quad \text{for all } \varphi \in W_0^{s,p}(\Omega).
\]
In particular, by taking \(\varphi = \phi^s_p\) we obtain
\[
\langle (-\Delta)^s_p \phi^s_p, \phi^s_p \rangle = [\phi^s_p]_{s,p}^p = [\phi^s_p]_{1,s,p} = [\phi^s_p]_{1}.
\]

**Theorem 2.3.** We have
\[
\frac{1}{[\phi^s_p]_{1}^{p-1}} = \min \{ [v]_{s,p}^p : v \in W_0^{s,p}(\Omega), \, |v|_1 = 1 \} = \left[ \frac{\phi^s_p}{[\phi^s_p]_{1}} \right]_{s,p}.
\]
Moreover, \(\phi^s_p/[\phi^s_p]_{1}\) is the only nonnegative function attaining the minimum.

**Proof.** Since the functional \(v \mapsto |v|_1\) is not differentiable, we will first consider the minimization problem
\[
m := \inf \left\{ [v]_{s,p}^p : v \in W_0^{s,p}(\Omega), \, \int_\Omega v \, dx = 1 \right\}
\]
and show that it is uniquely solved by the positive function \(\phi^s_p/[\phi^s_p]_{1}\).

Thus, let us take a sequence \((u_n) \subset W_0^{s,p}(\Omega)\) such that
\[
\int_\Omega u_n \, dx = 1 \quad \text{and} \quad [u_n]_{s,p}^p \to m.
\]
We observe that the sequence \((u_n)\) is bounded in \(L^1(\Omega)\):
\[
[u_n]_{1}^p \leq |\Omega|^{p-1} |u_n|_{p}^p \leq |\Omega|^{p-1} \Lambda_{s,p}^{\infty}(\Omega)^{-1} [u_n]_{s,p}^p.
\]
Since \( W^{s,p}_0(\Omega) \) is reflexive, the \( L^1 \)-boundedness of \( (u_n) \) implies the existence of \( u \in W^{s,p}_0(\Omega) \) such that, up to a subsequence, \( u_n \rightharpoonup u \) (weak convergence) in \( W^{s,p}_0(\Omega) \) and \( u_n \rightarrow u \) in \( L^1(\Omega) \). The convergence in \( L^1(\Omega) \) implies that \( \int_\Omega u \, dx = 1 \), so that \( m \leq \|u\|_{s,p} \). On the other hand, the weak convergence guarantees that
\[
\|u\|_{s,p} + 1 = \|u\| \leq \lim\inf \|u_n\| = \lim\inf \|u_n\|_{s,p} + 1 = m^\frac{1}{s} + 1.
\]
It follows that \( m = \|u\|_{s,p} \), so that the infimum in (2.5) is attained by the weak limit \( u \).

By applying Lagrange multipliers, we infer the existence of a real number \( \lambda \) such that
\[
\langle (-\Delta)^s_p u, \varphi \rangle = \lambda \int_\Omega \varphi \, dx \quad \text{for all } \varphi \in W^{s,p}_0(\Omega).
\] (2.6)
Taking \( \varphi = u \), we conclude that \( \lambda = m > 0 \), since \( m = \|u\|_{s,p} \) and \( \int_\Omega u \, dx = 1 \). This fact and (2.6) imply that \( u \) is a weak solution of the Dirichlet problem
\[
\begin{aligned}
(-\Delta)^s_p u &= m \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.
\end{aligned}
\]

By uniqueness, we have \( u = m^{\frac{1}{s}} \phi^s_p \geq 0 \). Since \( \int_\Omega u \, dx = 1 \), we conclude that
\[
m = \frac{1}{|\phi^s_p|_{p,1}} \quad \text{and} \quad u = \frac{\phi^s_p}{|\phi^s_p|_{1}}.
\]
We remark that
\[
\frac{1}{|\phi^s_p|_{p,1}} \leq |v|_{s,p} \leq |v|_{s,p}
\]
for every \( v \in W^{s,p}_0(\Omega) \) such that \( |v|_1 = 1 \). This finishes the proof since
\[
\frac{1}{|\phi^s_p|_{p,1}} = \left[ \frac{\phi^s_p}{|\phi^s_p|_{1}} \right]_{s,p} \quad \text{and} \quad \left[ \frac{\phi^s_p}{|\phi^s_p|_{1}} \right]_1 = 1.
\]

The next result recovers [5, Lemma 4.1].

**Corollary 2.4.** The \((s, p)\)-torsion function is radial when \( \Omega \) is a ball.

**Proof.** Let \((\phi^s_p)^* \in W^{s,p}_0(\Omega)\) be the Schwarz symmetrization of \( \phi^s_p \), that is, the radially decreasing function such that
\[
\{ \phi^s_p > t \}^* = \{ (\phi^s_p)^* > t \} , \quad t > 0,
\]
where, for any \( D \subset \mathbb{R}^N \), \( D^* \) stands for the \( N \)-dimensional ball with the same volume of \( D \).

It is well known that \((\phi^s_p)^* \geq 0\), \((\phi^s_p)^* s,p \leq |\phi^s_p|_{s,p}\) and \((\phi^s_p)^* |1 = |\phi^s_p|_1\). Therefore, \((\phi^s_p)^* \leq (\phi^s_p)^* |1\) attains the minimum in (2.4) and, by uniqueness, we have \((\phi^s_p)^* = \phi^s_p\).

It is also well known that the first eigenfunctions of the fractional \( p \)-Laplacian belong to \( L^{\infty}(\Omega) \) and are either positive or negative almost everywhere in \( \Omega \). Moreover, they are scalar multiple of each other. So, let us denote by \( e^s_p \) the positive and \( L^{\infty} \)-normalized first eigenfunction. It follows that \( |e^s_p|_{L^{\infty}} = 1 \) and
\[
\begin{aligned}
\{ (-\Delta)^s_p e^s_p = \lambda_{1,p}(\Omega) (e^s_p)^{p-1} \} &\quad \text{in } \Omega, \\
e^s_p &= 0 \quad \text{on } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]
meaning that
\[
\langle (-\Delta)^s_p e^s_p, \varphi \rangle = \lambda_{1,p}(\Omega) \int_\Omega (e^s_p)^{p-1} \varphi \, dx \quad \text{for all } \varphi \in W^{s,p}_0(\Omega).
\] (2.7)
Of course, by taking \( \varphi = e^s_p \) in (2.7) we obtain
\[
\langle (-\Delta)^s_p e^s_p, e^s_p \rangle = |e^s_p|^s_{s,p} = \lambda_{1,p}(\Omega) |e^s_p|^p_{p}.
\]
As mentioned in the introduction, \( \lambda^s_{1,p}(\Omega) \) is variationally characterized by (1.2).
Proposition 2.5. Let $u \in W^{1,p}_0(\Omega)$ be the weak solution of (2.2) with $f \in L^\infty(\Omega) \setminus \{0\}$. Then
\[
|f|_{\infty}^{\frac{1}{p-1}} u \leq \phi_p^s \text{ a.e. in } \Omega
\]
and
\[
\lambda_1^{s,p}(\Omega) \leq |f|_{\infty}^{\frac{1}{p-1}} \left(\frac{|\Omega|}{|u|_1} \right)^{p-1}.
\]

Proof. Since $u$ and $\phi_p^s$ are both equal zero in $\mathbb{R}^N \setminus \Omega$ and
\[
\langle (-\Delta)^{\gamma}_p f | f \in L^\infty(\Omega) \setminus \{0\}angle
\]
holds for every nonnegative $\varphi \in W^{1,p}_0(\Omega)$, inequality (2.8) follows from the comparison principle (see [8, Lemma 9]).

In order to prove (2.9), we use (1.2) and Hölder’s inequality:
\[
\lambda_1^{s,p}(\Omega) \leq \frac{|u|^p_{L^p|f|_{\infty}^{\frac{1}{p-1}}} = \frac{\int_\Omega f \varphi \, dx}{|u|^p_{L^p|f|_{\infty}^{\frac{1}{p-1}}} = \frac{\int_\Omega |\varphi(x)| \, dx}{|u|^p_{L^p|f|_{\infty}^{\frac{1}{p-1}}} = \frac{\int_\Omega f \varphi \, dx}{|u|^p_{L^p|f|_{\infty}^{\frac{1}{p-1}}} = \frac{\int_\Omega f \varphi \, dx}{\int_\Omega \left(\frac{|\Omega|}{|u|_1} \right)^{p-1} = |f|_{\infty}^{\frac{1}{p-1}} \left(\frac{|\Omega|}{|u|_1} \right)^{p-1}.
\]

Corollary 2.6. We have
\[
e_p^s \leq \lambda_1^{s,p}(\Omega) \leq |f|_{\infty}^{\frac{1}{p-1}} \phi_p^s \text{ a.e. in } \Omega
\]
and
\[
\frac{1}{|\phi_p^s|_{L^p|f|_{\infty}^{\frac{1}{p-1}}} \leq \lambda_1^{s,p}(\Omega) \leq |f|_{\infty}^{\frac{1}{p-1}} |\phi_p^s|_{L^p|f|_{\infty}^{\frac{1}{p-1}}}.
\]

Proof. Taking $u = e_p^s$ and $f = \lambda_1^{s,p}(\Omega)(e_p^s)^{p-1}$ in (2.8), we readily obtain (2.10). Hence, passing to maxima, we arrive at the first inequality in (2.11). The second inequality follows from (2.9) with $u = \phi_p^s$ and $f \equiv 1$. 

We would like to emphasize the following consequence of (2.10): $\phi_p^s > 0$ almost everywhere in $\Omega$.

A Faber–Krahn inequality also holds true for the first fractional eigenvalue.

Lemma 2.7 ([1, Theorem 3.5]). Let $p > 1$ and $s \in (0, 1)$. For every bounded domain $D \subset \mathbb{R}^N$, we have
\[
|B_1| \leq \lambda_1^{s,p}(B_1) = |B| \leq \lambda_1^{s,p}(B) \leq |D| \leq \lambda_1^{s,p}(D),
\]
where $B$ is any $N$-dimensional ball and $B_1$ denotes the unit ball of $\mathbb{R}^N$.

Remark 2.8. Since $h_s(D) = \lim_{r \to 1^+} \lambda_1^{s,p}(D)$, one has, immediately,
\[
|B_1| \leq h_s(B) = |B| \leq h_s(B) \leq |D| \leq h_s(D).
\]
The next estimate is obtained by applying standard set-level techniques; however, the bounds obtained are adequate to study the asymptotic behavior as $p \to 1^+$.

Proposition 2.9. Let $u \in W^{1,p}_0(\Omega) \setminus \{0\}$ be a nonnegative weak solution of (2.2) with $f \in L^\infty(\Omega)$. Then $u \in L^\infty(\Omega)$ and
\[
|u|_{\infty} \leq \frac{1}{|B_1|} \left(\frac{sp + N(s - 1)}{sp} \right)^{\frac{p-1}{p}} \left(\frac{|f|_{\infty}}{\lambda_1^{s,p}(B_1)|u|_{L^p|f|_{\infty}^{\frac{1}{p-1}}}} \right)^{\frac{1}{p}}.
\]

Proof. For each $k > 0$, we set
\[
A_k = \{x \in \Omega : u(x) > k\}.
\]
Since $u \in W^{s,p}_0(\Omega)$ and $u \geq 0$ in $\Omega$, the function
\[
(u - k)^+ = \max\{u - k, 0\} = \begin{cases} u - k & \text{if } u > k, \\ 0 & \text{if } u \leq k \end{cases}
\]
belong to $W^{s,p}_0(\Omega)$. Therefore, choosing $\varphi = (u - k)^+$ in (2.3), we obtain
\[
\langle (-\Delta)^{\gamma}_p u, (u - k)^+ \rangle = \int_{A_k} f(x)(u - k) \, dx.
\]
It is not difficult to check that
\[(u - k)^{p}_s, p \leq \langle (-\Delta)^{s} u, (u - k)^{p} \rangle.
\]
Thus, we have
\[\int_{A_k} (u - k)^{p} \leq \int_{A_k} (u - k)^{p} \leq |f|_{\infty} \int_{A_k} (u - k)^{p} \, dx. \tag{2.14}\]

We now consider \(k > 0\) such that \(|A_k| > 0\). In order to estimate \([u - k]^p_s, p\) from below, let us fix a ball \(B \subset \mathbb{R}^N\) and apply Lemma 2.7 to obtain
\[|B_1|^{\frac{sp}{Np - p(p - 1)}} |A_k|^{\frac{sp}{Np - p(p - 1)}} \leq \lambda_1^{\frac{sp}{Np - p(p - 1)}}(A_k) \leq \frac{[u - k]^p_s, p}{\int_{A_k} (u - k)^{p} \, dx}.
\]

Hence, Hölder’s inequality yields
\[
\left( \int_{A_k} (u - k) \, dx \right)^{p - 1} \leq \left( \int_{A_k} (u - k)^{p} \, dx \right)^{p - 1} \leq |A_k|^{p - 1} \int_{A_k} ([u - k]^p_s, p) \, dx,
\]
which yields
\[
\left( \int_{A_k} (u - k) \, dx \right)^{p - 1} \leq \frac{|f|_{\infty} |A_k|^{\frac{sp}{Np - p(p - 1)}}}{|B_1|^{\frac{sp}{Np - p(p - 1)}} \lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)},
\]
and so
\[\left( \int_{A_k} (u - k) \, dx \right)^{\frac{Np}{sp} \left( \frac{p - 1}{Np - p(p - 1)} \right)} \leq \frac{|f|_{\infty} |A_k|^{\frac{sp}{Np - p(p - 1)}}}{|B_1|^{\frac{sp}{Np - p(p - 1)}} \lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)}.
\]

Define
\[g(k) := \int_{A_k} (u - k) \, dx = \int_{\frac{\lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)}}^{\infty} |A_k| \, dt,
\]
the last equality being a consequence of Cavalieri’s principle. Combining the definition of \(g(k)\) with (2.15), we have
\[|g(k)|^{\frac{Np}{sp} \left( \frac{p - 1}{Np - p(p - 1)} \right)} \leq \frac{|f|_{\infty} |A_k|^{\frac{sp}{Np - p(p - 1)}}}{|B_1|^{\frac{sp}{Np - p(p - 1)}} \lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)} g(k).
\]
Therefore,
\[1 \leq \frac{|f|_{\infty} |A_k|^{\frac{sp}{Np - p(p - 1)}}}{|B_1|^{\frac{sp}{Np - p(p - 1)}} \lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)} g(k)^{- \frac{Np}{sp} \left( \frac{p - 1}{Np - p(p - 1)} \right)} g(k).
\]
Integration of (2.16) from 0 to \(k\) produces
\[k \leq \left( \frac{sp + N(p - 1)}{sp} \right) \left( \frac{|f|_{\infty} |A_k|^{\frac{sp}{Np - p(p - 1)}}}{|B_1|^{\frac{sp}{Np - p(p - 1)}} \lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)} \right)^{\frac{Np}{sp} \left( \frac{p - 1}{Np - p(p - 1)} \right)} \left[ g(0)^{\frac{sp}{sp + N(p - 1)}} - g(k)^{\frac{sp}{sp + N(p - 1)}} \right]
\]
\[\leq \left( \frac{sp + N(p - 1)}{sp} \right) \left( \frac{|f|_{\infty} |A_k|^{\frac{sp}{Np - p(p - 1)}}}{|B_1|^{\frac{sp}{Np - p(p - 1)}} \lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)} \right)^{\frac{Np}{sp} \left( \frac{p - 1}{Np - p(p - 1)} \right)} |u|^{\frac{sp}{sp + N(p - 1)}},
\]
since \(g(k) \geq 0\) and \(g(0) = |u|_1\).

Let \(c\) denote, just for a moment, the right-hand side of the latter inequality. We have proved that \(k \leq c\) whenever \(|A_k| > 0\). Since \(c\) does not depend on \(k\), this implies that \(|A_k| = 0\) for every \(k > c\), thus allowing us to conclude that \(u \in L^{\infty}(\Omega)\) and also that \(|u|_{\infty} \leq c\). So,
\[|u|_{\infty} \leq \left( \frac{sp + N(p - 1)}{sp} \right) \left( \frac{|f|_{\infty} |A_k|^{\frac{sp}{Np - p(p - 1)}}}{|B_1|^{\frac{sp}{Np - p(p - 1)}} \lambda_1^{\frac{sp}{Np - p(p - 1)}}(B_1)} \right)^{\frac{Np}{sp} \left( \frac{p - 1}{Np - p(p - 1)} \right)} |u|^{\frac{sp}{sp + N(p - 1)}}.
\]
or, equivalently,
\[ |u|_{\infty}^{\frac{n(p-1)}{sp}} \leq \left( \frac{sp + N(p-1)}{sp} \right)^{\frac{sp+n(p-1)}{sp}} \left( \frac{|f|_{\infty}}{|B_1|^{\frac{sp}{p}} \lambda_p^{s}(B_1)} \right)^{\frac{n}{p}} |u|_{1}, \]
from which (2.13) follows. 

**Corollary 2.10.** The \((s, p)\)-torsion function \(\phi_p^s\) belongs to \(L^{\infty}(\Omega)\) and, in addition,
\[ \frac{1}{|\Omega|} \leq \frac{1}{|\phi_p^s|_{\infty}} \leq \frac{1}{|B_1|} \left( \frac{sp + N(p-1)}{sp} \right)^{\frac{sp+n(p-1)}{sp}} \left( \frac{1}{\lambda_p^{s}(B_1)} \right)^{\frac{n}{p}}. \] (2.17)

**Proof.** The first inequality is obvious. Proposition 2.9 with \(u = \phi_p^s\) and \(f \equiv 1\) yields
\[ \frac{|\phi_p^s|_{\infty}}{|\phi_p^s|_{1}} \leq \frac{1}{|B_1|} \left( \frac{sp + N(p-1)}{sp} \right)^{\frac{sp+n(p-1)}{sp}} \left( \frac{1}{\lambda_p^{s}(B_1)} \right)^{\frac{n}{p}}. \]
Now, the second inequality in (2.17) follows from the first inequality in (2.11).

**Theorem 2.11.** One has
\[ \lim_{p \to 1} \frac{1}{|\phi_p^s|_{\infty}^{p-1}} = h_s(\Omega) = \lim_{p \to 1} \frac{1}{|\phi_p^s|_{1}^{p-1}}. \]

**Proof.** Taking (1.4) into account, we have
\[ \lim_{p \to 1} \frac{\lambda_p^{s}(\Omega)}{\lambda_p^{s}(B_1)} = h_s(\Omega) \quad \in (0, \infty). \]
Hence, from (2.17) it follows that
\[ \lim_{p \to 1} \left( \frac{|\phi_p^s|_{\infty}}{|\phi_p^s|_{1}} \right)^{p-1} = 1. \]
Thus, by making \(p\) go to 1 in (2.11), we have
\[ \lim_{p \to 1} \lambda_p^{s}(\Omega) \leq \lim_{p \to 1} \frac{|\Omega|^{p-1}}{|\phi_p^s|_{1}^{p-1}} = \lim_{p \to 1} \frac{1}{|\phi_p^s|_{1}^{p-1}} = \lim_{p \to 1} \left( \frac{|\phi_p^s|_{\infty}}{|\phi_p^s|_{1}} \right)^{p-1} \lim_{p \to 1} \frac{1}{|\phi_p^s|_{\infty}^{p-1}} = \lim_{p \to 1} \lambda_p^{s}(\Omega). \]
Since \(\lim_{p \to 1} \lambda_p^{s}(\Omega) = h_s(\Omega)\), we are done.

In [1], Brasco, Lindgren and Parini also proved that
\[ h_s(\Omega) = \inf \{ |v|_{s,1}^{p} : v \in W_{0}^{s,1}(\Omega), |v|_{1} = 1 \}. \]
Since \(W_{0}^{s,1}(\Omega)\) is not reflexive, they were able to prove that the minimum \(h_s(\Omega)\) is attained on the larger Sobolev space
\[ \mathcal{W}_{0}^{s,1}(\Omega) := \{ v \in L^{1}(\Omega) : |v|_{s,1}^{p} < \infty, u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}. \]
For completeness, we state the following result on the behavior of the \(L^{1}\)-normalized family \(\{\phi_p^s/|\phi_p^s|_{1}\}\) as \(p \to 1\). It corresponds to [1, Theorem 7.2], which was proved for the family \(\{e_p^s/|e_p^s|_{1}\}\). Its proof is similar and thus omitted.
Theorem 2.12. Let $u_p := \phi_p^p/|\phi_p^p|_1$. There exists a sequence $(p_n)$ such that $p_n \to 1^+$ and $u_{p_n} \to u$ in $L^q(\Omega)$ for every $q < \infty$. The limit function $u$ is a solution of the minimization problem

$$h_s(\Omega) = \min_{v \in \mathcal{W}_0^s(\Omega)} \{[v]_{s, 1} : u \geq 0, |u|_1 = 1\}.$$ 

Moreover, $u \in L^\infty(\Omega)$ and

$$\frac{1}{|\Omega|} \leq |u|_\infty \leq \frac{1}{|B_1|} \left( \frac{h_s(\Omega)}{h_s(B_1)} \right)^\frac{n}{q} \tag{2.18}.$$

The upper bound that appears in the statement of [1, Theorem 7.2] is

$$\frac{|B|^{\frac{n}{q}}}{P_s(B)} \left( \frac{h_s(\Omega)}{h_s(B_1)} \right)^\frac{n}{q}. \tag{2.19}$$

However, it is very simple to check, by applying (2.12), that it is equal to the upper bound in (2.18).

We remark that, once obtained the convergence in $L^q(\Omega)$ stated above, the upper bound in (2.18) follows from (2.17). Indeed, since

$$|u|_q = \lim_{n \to \infty} |u_{p_n}|_q \leq |\Omega|^{\frac{1}{q}} \lim_{n \to \infty} |u_{p_n}|_\infty,$$

(2.17) implies that

$$|\Omega|^{-\frac{1}{q}} |u|_q \leq \lim_{n \to \infty} |u_{p_n}|_\infty \leq \frac{1}{|B_1|} \left( \frac{h_s(\Omega)}{h_s(B_1)} \right)^\frac{n}{q}.$$ 

Hence, the upper bound in (2.18) follows, since $|u|_\infty = \lim_{q \to \infty} |\Omega|^{-\frac{1}{q}} |u|_q$.

The lower bound in (2.18), which does not appear in the statement of [1, Theorem 7.2], follows by taking $q = 1$, since

$$1 = \lim_{n \to \infty} |u_{p_n}|_1 = |u|_1 \leq |u|_\infty |\Omega|.$$ 

It is interesting to note that, as it happens with the standard $p$-torsion functions, $|u|_\infty = |\Omega|^{-1}$ when $\Omega$ is a ball. In fact, in this case (2.12) yields

$$\frac{1}{|B_1|} \left( \frac{h_s(\Omega)}{h_s(B_1)} \right)^\frac{n}{q} = \frac{1}{|\Omega|}.$$

Funding: The authors acknowledge the support of CNPq-Brazil and FAPEMIG.

References

[1] L. Brasco, E. Lindgren and E. Parini, The fractional Cheeger problem, Interfaces Free Bound. 16 (2014), 419–458.
[2] H. Bueno and G. Ercole, Solutions of the Cheeger problem via torsion functions, J. Math. Anal. Appl. 381 (2011), 263–279.
[3] R. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), no. 5, 521–573.
[4] G. Franzina and G. Palatucci, Fractional $p$-eigenvalue, Riv. Math. Univ. Parma (N.S.) 5 (2014), no. 5, 373–386.
[5] A. Iannizzotto, S. Mosconi and M. Squassina, Global Hölder regularity for the fractional $p$-Laplacian, Rev. Mat. Iberoam., to appear.
[6] B. Kawohl, On a family of torsional creep problems, J. Reine Angew. Math. 410 (1990), 1–22.
[7] B. Kawohl and V. Fridman, Isoperimetric estimates for the first eigenvalue of the $p$-Laplace operator and the Cheeger constant, Comment Math. Univ. Carolin. 44 (2003), 659–667.
[8] E. Lindgren and P. Lindqvist, Fractional eigenvalues, Calc. Var. Partial Differential Equations 49 (2014), no. 1–2, 795–826.