Inequivalent Quantization in the Skyrme Model

Hitoshi IKEMORI
Faculty of Economics, Shiga University, Hikone, Shiga 522-8522, Japan

Shinsaku KITAKADO
Department of Physics, Nagoya University, Nagoya 464-6801, Japan

Hajime NAKATANI
Chubu Polytechnic Center, Komaki, Aichi 485-0825, Japan

Hideharu OTSU
Faculty of Economics, Aichi University, Toyohashi, Aichi 441-8522, Japan

Toshiro SATO
Faculty of Political Science and Economics, Matsusaka University, Matsusaka, Mie 515-0044, Japan

Abstract

Quantum mechanics on manifolds is not unique and in general infinite number of inequivalent quantizations can be considered. They are specified by the induced spin and the induced gauge structures on the manifold. The configuration space of collective mode in the Skyrme model can be identified with $S^3$ and thus the quantization is not unique. This leads to the different predictions for the physical observables.

1ikemori@biwako.shiga-u.ac.jp
2kitakado@eken.phys.nagoya-u.ac.jp
3otsu@vega.aichi-u.ac.jp
4tsato@matsusaka-u.ac.jp
1 Introduction

Quantum mechanics on $S^n$ is not unique and in general infinite number of inequivalent quantizations can be considered. They are characterized by the induced spin $[1]$ and the induced gauge structure $[2, 3, 4]$. For example the induced gauge structure on $S^1$ is that of a vortex located at the center of $S^1 (r = 0)$, for $S^2$ the gauge structure is that of a magnetic monopole also located at the center of $S^2$ and furthermore for $S^3$ (and for $S^{2n-1}$, $n \geq 3$) the induced gauge field is that of meron or zero-size instanton (generalized meron or generalized zero-size instanton ) sitting at the center of $S^3 (S^{2n-1}) [5, 6]$, while for $S^4$ (and for $S^{2n}$, $n \geq 3$) the gauge field is that of instanton (generalized instanton) $[4, 7]$. Quantum mechanics on $S^n$ has been formulated in $[3]$ by extending the canonical commutation relations that are valid on the flat space, while the authors of $[4]$ have considered $S^n$ as a coset space $G/H$ and have obtained the effective Lagrangians that correspond to many inequivalent quantizations. Application of this idea to the physical models and extension to the field theory $[8, 9, 10, 11]$ are the interesting problems to be pursued.

In this article we consider the SU(2) Skyrme model $[12]$ as a concrete example, where in the semi-classical approximation the problem reduces to the quantum mechanics on $S^3$. Adkins, Nappi and Witten (ANW) $[13]$ have quantized the system and have shown that the model describes the static properties of the baryon within 30% accuracy. We show that the assignment of the baryonic states in this model is not unique and that ANW analysis corresponds to the simplest case of inequivalent quantizations on $S^3$. We shall apply the idea of quantum mechanics on $S^3$ to this problem and discuss the effects of induced spin as well as gauge structure to the physical quantities of this model.

Quantum mechanics on $S^3$ will be reviewed in Sec.2, and Sec.3 is a short review of SU(2) Skyrme model as developed by ANW. In Sec.4 we shall discuss the inequivalent quantizations in the Skyrme model. Sec.5 is devoted to summary and discussions.

2 Quantum mechanics on $S^3$

In this section we shortly recapitulate the quantum mechanics (QM) on $S^n$ $[3]$. It is described by the following fundamental algebra
\[ \left[ \hat{X}_\alpha, \hat{X}_\beta \right] = 0, \quad (2.1) \]
\[ \left[ \hat{X}_\lambda, \hat{G}_{\alpha \beta} \right] = i \left( \hat{X}_\alpha \delta_{\lambda \beta} - \hat{X}_\beta \delta_{\lambda \alpha} \right), \]
\[ \left[ \hat{G}_{\alpha \beta}, \hat{G}_{\lambda \mu} \right] = i \left( \delta_{\alpha \lambda} \hat{G}_{\beta \mu} - \delta_{\alpha \mu} \hat{G}_{\beta \lambda} + \delta_{\beta \mu} \hat{G}_{\alpha \lambda} - \delta_{\beta \lambda} \hat{G}_{\alpha \mu} \right), \]

where \( \hat{G}_{\alpha \beta} = -\hat{G}_{\beta \alpha} \) are the generators of SO\((n+1)\), and \( \hat{X}_\alpha \) the coordinates of \( S^n \) satisfying \( \sum_{\alpha=0}^{n+1} \hat{X}_\alpha^2 = r^2 \). It has been noted \[ \text{[3]} \] that there exist an infinite number of inequivalent representations of this algebra, that are characterized by the “induced spin” and “gauge structure”. This provides a strong contrast to QM on the flat space, where the irreducible representations of the canonical algebra

\[ \left[ \hat{X}_j, \hat{P}_k \right] = i \delta_{jk}, \quad (2.2) \]
\[ \left[ \hat{X}_j, \hat{X}_k \right] = \left[ \hat{P}_j, \hat{P}_k \right] = 0, \]

is essentially unique.

In what follows we shall be concerned with the case of \( S^3 \) only, the induced Lagrangian \[ \text{[4]} \] of which is given as

\[ L_{\text{induced}} = - \text{tr} \left( K h^{-1} \dot{h} \right) - \text{tr} \left( h K h^{-1} A_i(a) \right) \dot{a}_i, \quad (2.3) \]

where \( A_i \) \((i = 1, 2, 3)\) is the induced gauge field expressed in terms of the \( S^3 \)-variable \( a_i \). The variable \( h \), an element of SU\((2)_V \), expresses the induced spin \( S_i \) as

\[ S_i = - \text{tr} \left( T_i h K h^{-1} \right), \quad (2.4) \]

with

\[ h = h_0 - 2T_i h_i, \quad (2.5) \]
\[ T_i = \frac{1}{2i} \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}, \quad \sum_{\alpha=0}^{3} (h_\alpha)^2 = 1. \]

Here we utilize the coset space representation \( G/H \) for the manifold with the identification \( S^3 = \text{SO}(4)/\text{SO}(3) = (\text{SU}(2)_L \times \text{SU}(2)_R)/\text{SU}(2)_V \). \( K \) is chosen to be \( K = k T_3 \) with a constant \( k \). \( g \in G \) is given by

\[ g = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad (2.6) \]
where $B$ and $C$ are $2 \times 2$ SU(2) matrices.

Using the section $\sigma(a_i)$ of $G/H$ expressed as
\begin{equation}
\sigma(a) = gh^{-1},
\end{equation}
the induced gauge field ($H$-connection) $A_i$ is given by
\begin{equation}
A_i = \sum_{j=1}^{3} \text{tr} \left( \sigma^{-1}(a) \partial_{a_i} \sigma(a) T_j \right) T_j .
\end{equation}

The induced gauge field on $S^3$ is meron (anti-meron) or zero-size instanton (anti-instanton) located at the center of $S^3$ in $R^4$. The sections corresponding to the meron and instanton are respectively
\begin{equation}
\sigma = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad \sigma = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}
\end{equation}
with
\begin{equation}
A = a_0 + i\tau_k a_k, \quad \left( \sum_{\mu=0}^{3} (a_\mu)^2 = 1 \right).
\end{equation}

In terms of these the induced Lagrangian is
\begin{equation}
L_{\text{induced}} = -\text{tr} \left( K (\sigma h)^{-1} (\sigma h) \right).
\end{equation}

### 3 Skyrme Model

In this section we briefly review the SU(2) Skyrme model discussed in [13].

One starts with the Lagrangian
\begin{equation}
\mathcal{L} = \frac{F_\pi^2}{16} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) + \frac{1}{32e^2} \text{tr} \left[ (\partial_\mu U)^\dagger, (\partial_\nu U)^\dagger \right]^2 ,
\end{equation}
where $U$ is an SU(2) matrix transforming as $U \rightarrow AUB^{-1}$ under the chiral SU(2)$\times$SU(2), $F_\pi = 186$MeV is the pion decay constant. There are topological soliton solutions in the Lagrangian (3.1) and we identify the soliton number with the baryon number. To describe the static soliton, one starts with the ansatz
\begin{equation}
U(\vec{x}) = \exp \left( i F(r) \frac{\vec{r}}{r} \right) \equiv U_0(\vec{x}),
\end{equation}
with the boundary conditions
\[ F(r) = \pi \] at \( r = 0 \) and \( F(r) \to 0 \) as \( r \to \infty \).

\( F(r) \) is solved numerically by minimizing the static energy. As a semiclassical approximation to the Skyrme model, ANW consider

\[ U(\vec{x}, t) = A(t)U_0(\vec{x})A^\dagger(t), \quad (3.3) \]

and quantize the zero mode \( A(t) \). Substituting (3.3) into (3.1) and performing the space integral we get

\[ L = -M + \lambda \text{tr} \left( \partial_0 A \partial_0 A^\dagger \right), \quad (3.4) \]

where

\[ M = 36.5 \times \frac{F_\pi}{e} \quad (3.5) \]

is the static energy of the soliton (skyrmion) and

\[ \lambda = 50.9 \times \frac{2\pi}{3} \left( \frac{1}{e^3 F_\pi} \right) \quad (3.6) \]

As \( A(t) \) is an element of SU(2), (3.4) describes a system defined on \( S^3 \) (considered in section 2) and thus \( A(t) \) can be identified with \( A \) in (2.10). Isospin transformation of \( A(t) \) is expressed as \( A \to V A \). Space rotation, on the other hand, can be transferred to the spin transformation of \( A(t) \) and is expressed in terms of \( SU(2) \) matrix \( R \) as \( A \to A R \). Thus isospin and spin operators are obtained from the Noether currents as

\[ I_k = \frac{1}{2} i \left( a_0 \frac{\partial}{\partial a_k} - a_k \frac{\partial}{\partial a_0} - \varepsilon_{klm} a_l \frac{\partial}{\partial a_m} \right), \quad (3.7a) \]
\[ J_k = \frac{1}{2} i \left( a_k \frac{\partial}{\partial a_0} - a_0 \frac{\partial}{\partial a_k} - \varepsilon_{klm} a_l \frac{\partial}{\partial a_m} \right), \quad (3.7b) \]

where \( A = a_0 + i \tau_k a_k \) and \( \sum_{\mu=0}^{3} a_\mu^2 = 1 \). Note that \( I^2 = J^2 \).

The wave functions for baryons, i.e., the eigenstates of \((I, I_3, J, J_3)\) are

\[ |p \uparrow\rangle = \frac{1}{\pi} (a_1 + ia_2), \quad |p \downarrow\rangle = -\frac{i}{\pi} (a_0 - i a_3), \quad (3.8) \]
\[ |n \uparrow\rangle = \frac{i}{\pi} (a_0 + i a_3), \quad |n \downarrow\rangle = -\frac{1}{\pi} (a_1 - i a_2), \quad (3.8) \]

\[ |\Delta^{++}, s_z = \frac{3}{2}\rangle = \frac{\sqrt{2}}{\pi} (a_1 + i a_2)^3, \]
\[ |\Delta^{+}, s_z = \frac{1}{2}\rangle = -\frac{\sqrt{2}}{\pi} (a_1 + i a_2) \left\{ 1 - 3 \left( a_0^2 + a_3^2 \right) \right\}, \ldots. \]
From (3.4) the Hamiltonian is

\[ H = M + \frac{1}{2\lambda} J^2 \]  
\[ = M + \frac{1}{4\lambda} (I^2 + J^2) \]  

and the masses of nucleon and delta are

\[ M_N = M + \frac{1}{2\lambda} \cdot \frac{3}{4}, \quad M_\Delta = M + \frac{1}{2\lambda} \cdot \frac{15}{4}. \]  

Using the experimental values for \( M_N \) and \( M_\Delta \) one has

\[ F_\pi = 129\text{MeV}, \quad e = 5.45. \]  

The magnetic moments of these states are obtained as follows. The isoscalar and the isovector parts of the magnetic moments are respectively

\[ (\mu_{I=0})_i \equiv \frac{1}{2} \int \varepsilon_{ijk} x_j B_k d^3x, \]  
\[ (\mu_{I=1})_i \equiv \frac{1}{2} \int \varepsilon_{ijk} x_j V^3_k d^3x, \]  

Here

\[ B_i = i\varepsilon_{ijk} \sin^2 \frac{F}{2\pi} \frac{F'}{r} F' \hat{x}_k \text{tr} \left\{ (\partial_0 A^\dagger) A \tau_j \right\} \]  

is the space component of the baryon number (topological number) current and \( V^3_i \) is the space component of the isovector current which satisfies the relation

\[ \int d\Omega x_k V^a_i = \frac{1}{3} i\pi \Lambda \text{tr} \left( \tau_k \tau_i A^\dagger A^\tau^a \right). \]  

To calculate \( \mu_{I=0} \), one substitutes (3.13) into (3.12a), then carrying out the space integral one is lead to

\[ (\mu_{I=0})_3 = i \frac{0.09}{2\pi} \frac{e}{F_\pi} \lambda \langle p \uparrow| \text{tr} \left( \partial_0 A^\dagger A \tau_3 \right) |p \uparrow \rangle \]  
\[ = \frac{0.09}{2\pi} \frac{e}{F_\pi} \langle p \uparrow| J_3 |p \uparrow \rangle \]  
\[ = \frac{0.09}{2\pi} \frac{e}{F_\pi} \times \frac{1}{2} \]  
\[ = 3.0 \times 10^{-4}. \]
Here we used the relation
\[ \lambda \text{tr} \left( \partial_0 A^\dagger A \tau_3 \right) = -i J_3 . \] (3.16)

Similarly, calculation of \( \mu_{I=1} \) is reduced to that of \( \langle p \uparrow | \text{tr} \left( \tau_3 A^\dagger \tau_3 A \right) | p \uparrow \rangle \) and using
\[ \langle N' | \text{tr} \left( \tau_i A^\dagger \tau_j A \right) | N \rangle = -\frac{2}{3} \langle N' | \sigma_i \tau_j | N \rangle , \] (3.17)

which is valid for the nucleonic states, one finally has
\[ (\mu_{I=1})_3 = -50.9 \times \frac{\pi}{3} \left( \frac{1}{e^3 F_\pi} \right) \langle p \uparrow | \text{tr} \left( \tau_3 A^\dagger \tau_3 A \right) | p \uparrow \rangle \] (3.18)
\[ = -50.9 \times \frac{\pi}{3} \left( \frac{1}{e^3 F_\pi} \right) \left( -\frac{2}{3} \right) \langle p \uparrow | \sigma_3 \tau_3 | p \uparrow \rangle \]
\[ = \frac{2}{9} \frac{50.9}{e^3 F_\pi} \]
\[ = 1.7 \times 10^{-3} . \]

From (3.15) and (3.18) the magnetic moments of proton and neutron in terms of the Bohr magneton are \( \mu_p = 1.87 \) and \( \mu_n = -1.31 \) respectively.

ANW have calculated also other physical quantities like \( g_A \), \( g_{\pi N} \) and \( g_{\pi N \Delta} \). Their conclusion is that the model describes the reality within about 30%.

\section{Inequivalent Quantizations}

As we saw in the previous section the configuration space of the semiclassical approximation to the Skyrme model is \( S^3 \). The question of inequivalent quantizations, discussed in sec.2, arises here. It is clear that the ANW analysis of the previous section corresponds to the trivial quantization with the “induced spin” \( S = 0 \) and induced gauge field \( A_\mu = 0 \). In this section we shall discuss the possibility of the other non-trivial quantizations with non-zero “induced spin” and induced gauge field and examine the physical effects to the results of the Skyrme model. In the following we shall consider the cases when the induced gauge field configuration is 1) that of meron and 2) that of zero-size instanton.
4.1 Meron case

The section $\sigma(a)$ for the case of meron configuration is

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a_0 + i\tau_k a_k \end{pmatrix} \quad (4.1)$$

As quantum mechanics on $S^3$ we introduce the “induced spin” and the induced gauge field by substituting (4.1) into (2.8) and (2.3). Then the resulting effective Lagrangian is

$$L = L_0 + L_{\text{induced}} \quad (4.2)$$

$$= -M + \lambda \text{tr} (\partial_0 A_0 A^1) - \text{tr} (K h^{-1} h) + B_\mu \dot{a}_\mu ,$$

where

$$B_0 = S_i a_i, \quad B_i = -S_i a_0 - \varepsilon_{ijk} S_j a_k \quad (4.3)$$

and $S_i$ is the “induced spin” operator (2.4). As the dynamical variables we consider $h_\mu$ and $a_\mu$. Since the non-trivial quantization has been already taken into account in (4.2), we can carry out the constrained quantization $\text{a la}$ Dirac (see Appendix). We obtain the following commutation relations

$$[\hat{a}_\mu, \hat{a}_\nu] = 0 \quad (4.4)$$

$$[\hat{a}_\mu, \hat{\pi}_\nu] = i (\delta_{\mu\nu} - \hat{a}_\mu \hat{a}_\nu) ,$$

$$[\hat{\pi}_\mu, \hat{\pi}_\nu] = i (\hat{a}_\nu \hat{a}_\mu - \hat{a}_\mu \hat{a}_\nu) ,$$

$$[\hat{S}_i, \hat{S}_j] = i \varepsilon_{ijk} \hat{S}_k ,$$

$$[\hat{a}_\mu, \hat{S}_i] = [\hat{\pi}_\mu, \hat{S}_i] = 0 .$$

$\hat{\pi}_\mu$ is canonical conjugate of $\hat{a}_\mu$ and in the $\hat{a}_\mu$-diagonal representation is given as

$$\hat{\pi}_\mu = -i \left( \frac{\partial}{\partial a_\mu} - a_\mu a_\nu \frac{\partial}{\partial a_\nu} \right) . \quad (4.5)$$

Although the expression (4.4) is identical with the one given in ANW, from

$$\pi_\mu = \frac{\partial L}{\partial \dot{a}_\mu} = 4\lambda \dot{a}_\mu + B_\mu \quad (4.6)$$

8
we have
\[ \dot{a}_\mu = \frac{1}{4\lambda} (\pi_\mu - \mathcal{B}_\mu), \]
which depends on the “induced spin” and induced gauge field in contrast to ANW.
From
\[ L_{\text{induced}} = -\text{tr} \left( K (\sigma h)^{-1} (\sigma h) \right), \]
isospin and spin transformations are respectively
\[ A \to VA, \]
\[ \begin{cases} 
A \to AR \\
h \to \begin{pmatrix} R^{-1} & 0 \\
0 & R^{-1} \end{pmatrix} h.
\end{cases} \]
Thus using the Noether currents isospin and spin operators are
\[ I_i = I_i^{\text{ANW}}, \]
\[ J_i = J_i^{\text{ANW}} + S_i, \]
where \( I_i^{\text{ANW}} \) and \( J_i^{\text{ANW}} \) are the expressions of ANW given in (3.7a) and (3.7b).
As is seen, spin \( J_i \) depends on the “induced spin” \( S_i \), thus the eigen states \((I, J)\) are fixed according to the representations of \( S \). The case of ANW corresponds to \( S = 0 \). The only other possibility that can describe \( N=\left(\frac{3}{2}, \frac{3}{2}\right) \) and \( \Delta = \left(\frac{3}{2}, \frac{3}{2}\right) \) is the case of \( S = 1 \). This case is capable of describing the following states:
\[ (I, J) = \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{5}{2}\right), \cdots \]
In what follows we shall be concerned with the effects of the “induced spin” and the induced gauge fields on the Skyrme model, identifying the \( N \) and \( \Delta \) with the \( \left(\frac{1}{2}, \frac{1}{2}\right) \) and \( \left(\frac{3}{2}, \frac{3}{2}\right) \) states in \( S = 1 \).
Baryonic wave functions in our case can be expressed in terms of those of ANW as
\[ |p \uparrow\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} |p \downarrow\rangle \\ - |p \uparrow\rangle \\ 0 \end{pmatrix}, \]
\[ |p \downarrow\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ |p \downarrow\rangle \\ -\sqrt{2} |p \uparrow\rangle \end{pmatrix}, \]
\[ |n \uparrow\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} |n \downarrow\rangle \\ - |n \uparrow\rangle \\ 0 \end{pmatrix}, \]
\[ |n \downarrow\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ |n \downarrow\rangle \\ -\sqrt{2} |n \uparrow\rangle \end{pmatrix}, \]
\[ 9 \]
\[ \left| \Delta^{++}, s_z = \frac{3}{2} \right\rangle = \frac{1}{\sqrt{5}} \left( -\sqrt{2} \left| \Delta^{++}, s_z = \frac{1}{2} \right\rangle + \frac{1}{\sqrt{3}} \left| \Delta^{++}, s_z = \frac{3}{2} \right\rangle \right), \quad \cdots, \]

where \(|p\uparrow\rangle, |n\uparrow\rangle, \cdots\) are given in (3.8). The Hamiltonian
\[ H = M + \frac{1}{8\lambda} (\pi_{\mu} - B_{\mu})^2 \]
expressed in terms of \(J_i = J_i^{ANW} + S_i\) and \(I_i\) is
\[ H = M + \frac{1}{4\lambda} \left( I^2 + J^2 - \frac{1}{2} S^2 \right), \]
thus the baryon mass does have the effect of the “induced spin”. From (4.13) we have
\[ H_N = M + \frac{1}{2\lambda} \cdot \frac{1}{4}, \]
\[ H_\Delta = M + \frac{1}{2\lambda} \cdot \frac{13}{4}, \]
and using the experimental N and \(\Delta\) masses we obtain
\[ F_\pi = 135 \text{ MeV} \quad (\text{ANW's value } F_\pi = 129 \text{ MeV}), \]
\[ e = 5.37 \quad (\text{ANW's value } e = 5.45). \]
Note that the value of \(F_\pi e^3\) is the same as that of ANW.

Next we examine the magnetic moments. As baryon number current and vector current are the same as in ANW, calculation of \((\mu_{I=0})_3\) and \((\mu_{I=1})_3\) reduces to that of the matrix elements \(\langle p \uparrow | \lambda \text{tr} (\partial_{\mu} A^{\dagger} A_{\tau_3}) | p \uparrow \rangle\), \(\langle p \uparrow | \text{tr} (\tau_3 A^{\dagger} A_{\tau_3}) | p \uparrow \rangle\). Using (4.7), (4.5) and (4.9b) we have
\[ \text{tr} (\partial_{\mu} A^{\dagger} A_{\tau_3}) = 2i (-a_0 \hat{a}_3 + a_3 \hat{a}_0 - a_1 \hat{a}_2 + a_2 \hat{a}_1) \]
\[ = -\frac{i}{\lambda} \left( J_3^{ANW} + \frac{1}{2} S_3 \right), \]
consequently
\[ \langle p \uparrow | \lambda \text{tr} (\partial_{\mu} A^{\dagger} A_{\tau_3}) | p \uparrow \rangle \]
\[ = -i \frac{1}{\sqrt{3}} \left( \sqrt{2} \langle p \downarrow | - \langle p \uparrow |, 0 \right) \left( J_3^{ANW} + \frac{1}{2} S_3 \right) \frac{1}{\sqrt{3}} \left( \sqrt{2} |p \downarrow\rangle - |p \uparrow\rangle \right) \]
\[ = -i \frac{1}{3} \left\{ 2 \langle p \downarrow | J_3^{ANW} | p \downarrow \rangle + \langle p \uparrow | J_3^{ANW} | p \uparrow \rangle + \langle p \downarrow | p \downarrow \rangle \right\} \]
\[ = -\frac{i}{6}. \]
Thus

\[
(\mu_{I=0})_3 = \frac{i}{2\pi} \frac{e}{F_\pi} \lambda \langle \langle p \uparrow | \text{tr} \left( \partial_0 A^\dagger A \tau_3 \right) | p \uparrow \rangle \rangle \]

(4.17)

\[
= \frac{0.09}{2\pi} \frac{e}{F_\pi} \times \frac{1}{6}
\]

\[
= \frac{0.09}{2\pi} \left( \frac{e}{F_\pi} \right)^{\text{ANW}} \times 0.95 \times \frac{1}{6}
\]

\[
= (\mu_{I=0})_3^{\text{ANW}} \times 0.32.
\]

Similarly the isovector part is

\[
(\mu_{I=1})_3 = -50.9 \times \frac{\pi}{3} \left( \frac{1}{e^3 F_\pi} \right) \langle \langle p \uparrow | \text{tr} \left( \tau_3 A^\dagger A \tau_3 \right) | p \uparrow \rangle \rangle \]

(4.18)

\[
= -50.9 \times \frac{\pi}{3} \left( \frac{1}{e^3 F_\pi} \right)
\]

\[
\times \left( -\frac{2}{3} \right) \frac{1}{\sqrt{3}} \left( \sqrt{2} \langle p \downarrow |, - \langle p \uparrow |, 0 \rangle \right) \left( \sigma_3 \tau_3 \right) \frac{1}{\sqrt{3}} \left( \begin{array}{c} \sqrt{2} | p \downarrow \rangle \\ - | p \uparrow \rangle \end{array} \right)
\]

\[
= \frac{2}{9} \frac{50.9}{e^3 F_\pi} \left\{ 2 \langle p \downarrow | \sigma_3 \tau_3 | p \downarrow \rangle + \langle p \uparrow | \sigma_3 \tau_3 | p \uparrow \rangle \right\}
\]

\[
= \frac{2}{9} \frac{50.9}{e^3 F_\pi} \left( -\frac{1}{3} \right)
\]

\[
= (\mu_{I=1})_3^{\text{ANW}} \times \left( -\frac{1}{3} \right).
\]

As a result, the values of \((\mu_{I=0})_3\) and \((\mu_{I=1})_3\) differ from those of ANW by the factors 0.32 and \(-\frac{1}{3}\).

### 4.2 Instanton case

The section \(\sigma(a)\) for the zero-size instanton configuration is

\[
\sigma = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} a_0 + i\tau_k a_k & 0 \\ 0 & a_0 + i\tau_k a_k \end{pmatrix}.
\]

The effective Lagrangian is given as

\[
L = L_0 + L_{\text{induced}}
\]

\[
= -M + \lambda \text{tr} \left( \partial_0 A \partial_0 A^\dagger \right) - \text{tr} \left( K h^{-1} \dot{h} \right) + \tilde{B}_\mu \dot{a}_\mu,
\]

(4.19)
where $\tilde{B}_\mu = 2B_\mu$.

Commutation relations, isospin and spin operators, baryon wave functions are the same as in the meron case, the only change being

\[
\dot{a}_\mu = \frac{1}{4\lambda} \left( \pi_\mu - \tilde{B}_\mu \right) = \frac{1}{4\lambda} (\pi_\mu - 2B_\mu) \quad (4.20)
\]

i.e., the contribution from the “induced spin” and the induced gauge field to $\dot{a}_\mu$ is different by a factor 2.

The Hamiltonian expressed in terms of $J_3$ and $I_i$ is

\[
H = M + \frac{1}{8\lambda} \left( \pi_\mu - \tilde{B}_\mu \right)^2 \quad (4.21)
\]

\[
= M + \frac{1}{8\lambda} (\pi_\mu - 2B_\mu)^2 \\
= M + \frac{1}{4\lambda} (I^2 + J^2)
\]

which unlike the case of meron has no contribution from the “induced spin”. $N$ and $\Delta$ masses follow from (4.21)

\[
H_N = M + \frac{1}{2\lambda} \cdot \frac{3}{4}, \quad (4.22)
\]

\[
H_\Delta = M + \frac{1}{2\lambda} \cdot \frac{15}{4},
\]

and if we use the experimental values as the input, we obtain

\[
F_\pi = 129 \text{ MeV}, \quad e = 5.45
\]

in agreement with ANW.

From (4.20) we have

\[
\text{tr} \left( \partial_0 A^\dagger A \tau_3 \right) = 2i \left( -a_0 \dot{a}_3 + a_3 \dot{a}_0 - a_1 \dot{a}_2 + a_2 \dot{a}_1 \right) \quad (4.23)
\]

\[
= -\frac{i}{\lambda} (J_3^{\text{ANW}} + S_3) \\
= -\frac{i}{\lambda} J_3,
\]

and the isoscalar part of magnetic moments is

\[
(\mu_{I=0})_3 = i \frac{0.09}{2\pi} \frac{e}{F_\pi} \lambda \langle \langle p \uparrow \| \text{tr} \left( \partial_0 A^\dagger A \tau_3 \right) \| p \uparrow \rangle \rangle \\
= \frac{0.09}{2\pi} \frac{e}{F_\pi} \langle \langle p \uparrow \| J_3 \| p \uparrow \rangle \rangle \quad (4.24)
\]

\[
= 3.0 \times 10^{-4},
\]

which is identical to that of ANW. On the other hand $(\mu_{I=1})_3$ is the same as that of the meron case (4.18).

We summarize all the results in table 1.
|                | meron | instanton |
|----------------|-------|-----------|
| **isospin** | $I_i^{\text{ANW}}$ | $I_i^{\text{ANW}}$ |
| **spin**     | $J_i^{\text{ANW}} + S_i$ | $J_i^{\text{ANW}} + S_i$ |
| **Hamiltonian $H$** | $\frac{1}{4\lambda} (I^2 + J^2 - \frac{1}{2} S^2)$ | $\frac{1}{4\lambda} (I^2 + J^2)$ |
| $F_\pi / F_\pi^{\text{ANW}}$ | 1.05 | 1 |
| $e / e^{\text{ANW}}$ | 0.99 | 1 |
| $\mu_{I=0}/\mu_{I=0}^{\text{ANW}}$ | 0.32 | 1 |
| $\mu_{I=1}/\mu_{I=1}^{\text{ANW}}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |

Table 1: Comparison of the meron and the instanton cases with ANW.

## 5 Summary and discussions

SU(2) Skyrme model reduces in the semiclassical approximation to the $S^3$ system. We have applied to this system the idea of inequivalent quantizations of quantum mechanics on $S^3$. Inequivalent quantizations are specified by the “induced spin”; $S = 0, \frac{1}{2}, 1, \cdots$ and by the induced gauge fields; trivial $A_\mu = 0$, meron (ME), zero-size instanton (INS), etc. The baryonic states $N = (\frac{1}{2}, \frac{1}{2})$ and $\Delta = (\frac{3}{2}, \frac{3}{2})$ are present only in $S=0$ and $S = 1$ cases. The $S = 0$ case corresponds to ANW analysis. The $S = 1$ case has been discussed in Sec 4. We compare the cases $S = 0($ANW$)$, $S = 1($ME$)$ and $S = 1($INS$)$ in the following. In $S = 1$ case the “induced spin” contributes to the spin operator of the skyrmion. The Hamiltonian, on the other hand, has the same expression for $S = 0$ and $S = 1($INS$)$, and there appears $S$ dependence for $S = 1($ME$)$. Using the experimental values for $M_N$ and $M_\Delta$ we obtain the same values for $F_\pi$, $e$ and the isoscalar part of the nucleon magnetic moment in the cases $S = 0$ and $S = 1($INS$)$, while in the case $S = 1($ME$)$ these values get modified. The isovector part of the nucleon magnetic moment, on the other hand, gets modified in the cases $S = 1($ME$)$ and $S = 1($INS$)$, and has the same values different from the case of $S = 0($ANW$)$.

Thus we see that for SU(2) Skyrme model to describe the baryonic states, three cases $S = 0($ANW$)$, $S = 1($ME$)$ and $S = 1($INS$)$ of quantization are possible. When compared with experiment, $S = 0($ANW$)$ case realizes the reality best. However, from the viewpoint of the quantum mechanics on $S^3$, these three cases are on the same footing and there is no way of determining which quantization has to be adopted.

Since the Skyrme model stems from the nonlinear sigma model, it would
be interesting to trace the field theoretical origin of $L_{\text{induced}}$. An obvious candidate is the Wess-Zumino term which is also related to topology of the field configuration space. The field theoretical origin of $L_{\text{induced}}$ is important also for the argument related to $g_A$, because we need to consider the axial vector current of the model. This problem is left for future investigations.

Appendix

In this Appendix, we derive the commutation relations (4.4), using the Dirac’s quantization method for the system (4.2).

Rewriting the Lagrangian (4.2) as

$$L = -M + 2\lambda (\dot{a}_\mu)^2 + \mathcal{H}_\mu \dot{h}_\mu + \mathcal{B}_\mu \dot{a}_\mu,$$

where $\mathcal{H}_\mu = (2h_3, 2h_2, -2h_1, -2h_0)$ and $\mathcal{B}_\mu$ has appeared in (4.3), we treat $a_\mu$ and $h_\mu$ as the dynamical variables with the constraints

$$\chi_1 = a_\mu^2 - 1 \approx 0,$$

$$\chi_3 = h_\mu^2 - 1 \approx 0.$$

Conjugate momenta for $a_\mu$ and $h_\mu$ are defined by

$$\Pi_\mu = \frac{\partial L}{\partial \dot{a}_\mu} = 4\lambda \dot{a}_\mu + \mathcal{B}_\mu,$$

$$\Pi^h_\mu = \frac{\partial L}{\partial \dot{h}_\mu} = \mathcal{H}_\mu,$$

respectively. (A.3b) gives the primary constraints

$$\chi_{4+\mu} = \Pi^h_\mu - \mathcal{H}_\mu \approx 0.$$

Hamiltonian is given by

$$H_0 = \pi_\mu \dot{a}_\mu + \pi^h_\mu \dot{h}_\mu - L = M + \frac{1}{8\lambda} (\pi_\mu - \mathcal{B}_\mu)^2.$$

Then consistency condition

$$\dot{\chi}_1 = \{\chi_1, H\} \approx 0$$

leads to the secondary constraint

$$\chi_2 = a_\mu \pi_\mu \approx 0.$$
No further constraints follow from other consistency conditions.

We define the matrix $C_{ij}$ $(i, j = 1, \cdots, 7)$ by $C_{ij} \equiv \{\chi_i, \chi_j\}$ in terms of Poisson bracket $\{\cdot, \cdot\}$. Then, $\det C_{ij} = 0$ shows that there exists first class constraint among the above seven constraints $\chi_i$ $(i = 1, \cdots, 7)$.

In order to construct the Dirac bracket, we introduce one gauge fixing condition

$$\chi_8 = h_0 - 1 \approx 0 .$$

Then, all the constraints $\chi_{\alpha}$ $(\alpha = 1, \cdots, 8)$ become second class; $\det C_{\alpha\beta} = \det \{\chi_\alpha, \chi_\beta\} \neq 0$ $(\alpha, \beta = 1, \cdots, 8)$,

$$C_{\alpha\beta} \equiv \{\chi_\alpha, \chi_\beta\} = \begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2h_0 & 2h_1 & 2h_2 & 2h_3 & 0 \\
0 & 0 & -2h_0 & 0 & 0 & 0 & -4k & -1 \\
0 & 0 & -2h_1 & 0 & 0 & -4k & 0 & 0 \\
0 & 0 & -2h_2 & 0 & 4k & 0 & 0 & 0 \\
0 & 0 & -2h_3 & 4k & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix} . \quad (A.9)$$

Inverse of $C_{\alpha\beta}$ is

$$C^{-1}_{\alpha\beta} = \begin{pmatrix}
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4k} & \frac{1}{4h_3} & \frac{1}{4h_3} \\
0 & 0 & 0 & 0 & -\frac{1}{4k} & 0 & \frac{1}{4h_3} & -\frac{1}{4h_3} \\
0 & 0 & \frac{1}{2h_3} & 0 & \frac{1}{4h_3} & 0 & \frac{1}{4h_3} & -\frac{1}{4h_3} \\
0 & 0 & -\frac{1}{2} & -1 & -\frac{1}{h_3} & -\frac{1}{h_3} & \frac{h_3}{h_3} & 0 \\
0 & 0 & -\frac{1}{2h_3} & -1 & -\frac{1}{4h_3} & -\frac{1}{4h_3} & \frac{h_3}{h_3} & 0 \\
\end{pmatrix} . \quad (A.10)$$

The definition of Dirac bracket

$$\{A, B\}_D \equiv \{A, B\} - \{A, \chi_\alpha\} C^{-1}_{\alpha\beta} \{\chi_\beta, B\} \quad (A.11)$$

gives the relations

$$\{a_\mu, a_\nu\}_D = 0 , \quad (A.12)$$
$$\{a_\mu, \pi_\nu\}_D = \delta_\mu^{\nu} - a_\mu a_\nu ,$$
$$\{\pi_\mu, \pi_\nu\}_D = -a_\mu \pi_\nu + a_\nu \pi_\mu ,$$
$$\{S_i, S_j\}_D = \varepsilon_{ijk} S_k ,$$
$$\{a_\mu, S_i\}_D = \{\pi_\mu, S_i\}_D = 0 .$$

15
where \( S_i \equiv -\text{tr}(T_i K \hbar^{-1}) \) is the “induced spin” variable. From (A.12), commutation relations for the system (4.2) become (4.4).

Acknowledgments We would like to thank K.Hasebe for discussions and comments.

References

[1] G. W. Mackey, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, 1968), Lectures given at Scuola Normale, Pisa, Italy, 1967.

[2] N. P. Landsman and N. Linden, Nucl. Phys. **B365**, 121 (1991).

[3] Y. Ohnuki and S. Kitakado, J. Math. Phys. **34**, 2827 (1993).

[4] D. McMullan and I. Tsutsui, Ann. Phys. **237**, 269 (1995), [hep-th/9308027](#).

[5] H. Ikemori, S. Kitakado, H. Nakatani, H. Otsu, and T. Sato, Mod. Phys. Lett. **A13**, 15 (1998), [hep-th/9710209](#).

[6] M. Hirayama, H.-M. Zhang, and T. Hamada, Prog. Theor. Phys. **97**, 679 (1997), [hep-th/9612173](#).

[7] K. Fujii, S. Kitakado, and Y. Ohnuki, Mod. Phys. Lett. **A10**, 867 (1995).

[8] S. Tanimura, Phys. Lett. **B340**, 57 (1994), [hep-th/9408090](#).

[9] Y. Igarashi, S. Kitakado, H. Otsu, and T. Sato, Mod. Phys. Lett. **A12**, 57 (1997).

[10] H. Kobayashi, I. Tsutsui, and S. Tanimura, Nucl. Phys. B**514**, 667 (1998), [hep-th/9705183](#).

[11] H. Miyazaki and I. Tsutsui, (1997), [hep-th/9706167](#).

[12] T. H. R. Skyrme, Proc. Roy. Soc. Lond. A**247**, 260 (1958).

[13] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. B**228**, 552 (1983).