Spin manipulation and decoherence in a quantum dot mediated by a synthetic spin–orbit coupling of broken $T$-symmetry

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Abstract

The electrical control of a spin qubit in a quantum dot (QD) relies on spin–orbit coupling (SOC), which could be either intrinsic to the underlying crystal lattice or heterostructure, or extrinsic via, for example, a micro-magnet. In experiments, micromagnets have been used as a synthetic SOC to enable strong coupling of a spin qubit in quantum dots with electric fields. Here we study theoretically the spin relaxation, pure dephasing, spin manipulation, and spin–photon coupling of an electron in a QD due to the synthetic SOC induced spin–orbit mixing. We find qualitative difference in the spin dynamics in the presence of a synthetic SOC compared with the case of the intrinsic SOC. Specifically, spin relaxation due to the synthetic SOC and deformation potential phonon emission (or Johnson noise) shows $B_0^3$ (or $B_0$) dependence with the magnetic field, which is in contrast with the $B_0^3$ (or $B_0^4$) dependence in the case of the intrinsic SOC. Moreover, charge noise induces fast spin dephasing to the first order of the synthetic SOC, which is in sharp contrast with the negligible spin pure dephasing in the case of the intrinsic SOC. These qualitative differences are attributed to the broken time-reversal symmetry ($T$-symmetry) of the synthetic SOC. An SOC with broken $T$-symmetry (such as the synthetic SOC from a micro-magnet) eliminates the ‘Van Vleck cancellation’ and causes a finite longitudinal spin–electric coupling that allows the longitudinal coupling between spin and electric field, and in turn allows spin pure dephasing. Finally, through proper choice of magnetic field orientation, the electric-dipole spin resonance via the synthetic SOC can be improved with potential applications in spin-based quantum computing.

1. Introduction

An electron spin qubit in a semiconductor quantum dot (QD) is a promising candidate for quantum information processing due to the long spin coherence time and possible scalability [1–6]. Exciting progress has been made in recent years on spin qubits, such as high fidelity spin manipulation in a Si QD [7, 8], strong coupling between a spin qubit in a Si/SiGe QD and a superconducting resonator [9–11], two-qubit CPHASE and CNOT gates based on the exchange interaction [12–14], and high fidelity two-qubit gates at an elevated 1.5 K temperature for potential integration of classical and quantum electronics [15, 16]. Indeed, up to nine controllable spin qubits in QDs have been demonstrated [17, 18].

A driving force behind many of the experimental achievements is the introduction of a micromagnet next to the QDs. The micromagnet creates an inhomogeneous magnetic field, which acts as a synthetic spin–orbit coupling (s-SOC) and allows electrical control of the spin qubit, leading to fast electric dipole...
spin resonance (EDSR) [7, 19–23] and strong spin–photon coupling [9, 10, 24, 25]. However, while the driving electric field could be applied externally or from a superconducting resonator [7, 9, 10, 19–24], it could also be from electrical noises [7, 26–29]. In other words, the micromagnet and the associated s-SOC open new spin decoherence channels. Various aspects of s-SOC-enabled decoherence have been explored previously, such as spin relaxation [19, 26, 27, 29] and dephasing [7, 28, 30], and effects of the magnetic noise from the micro-magnets [31, 32]. However, there is still a lack of theoretical understanding on how the apparent difference in the form between s-SOC and the intrinsic SOC (i-SOC) directly leads to wide ranging differences in both spin coherence and control in QD systems.

Time-reversal symmetry (T-symmetry) is one of the fundamental symmetries in quantum physics [33, 34]. In the literature, the irreversible dynamics due to dissipation is sometimes referred to as broken T-symmetry as well [35]. In this study, the T-symmetry we refer to is strictly in the quantum mechanical sense (an operator has T-symmetry if it is invariant under time-reversal transformation) [33, 34]. If an electronic Hamiltonian is T-symmetric, the electron spin states would be degenerate according to the Kramers theorem, and can only be lifted if a T-asymmetric term is introduced into the Hamiltonian [33]. Previously, the effect of broken T-symmetry of a uniform Zeeman Hamiltonian on spin relaxation and EDSR has been investigated [36]. It remains to be explored how the T-symmetry of SOC affects the spin dynamics, including the spin decoherence and spin manipulation.

In this work, we study theoretically the coupling between electric field and an electron spin in a QD due to the s-SOC induced spin–orbit mixing. We first introduce the Hamiltonian in section 2. Then, an effective spin Hamiltonian is obtained by performing an Schrieffer–Wolff (SW) transformation and the role of the T-symmetry of the SOC is discussed. Based on the effective Hamiltonian, we study explicitly spin relaxation, pure dephasing, EDSR, and spin–photon coupling. Furthermore, we connect time-reversal properties of the different SOC terms with how they would contribute to the low-energy Hamiltonian (the effective spin Hamiltonian) and the low-energy dynamics. When a general form of SOC is considered, we show how T-symmetry of SOC determines the important properties of a spin qubit. We draw the conclusion in the end.

2. System Hamiltonian

We consider an electron in a single gate-defined QD (see figure 1(a)). The QD confinement in the [001]-direction (defined as z-axis) is provided by the interface electric field, while the in-plane (i.e. xy plane) confinement is provided by top gates. A micromagnet (e.g. a cobalt magnet) is deposited over the QD and polarized by an applied magnetic field. We separate the total magnetic field into two parts \( \vec{B} = \vec{B}_0 + \vec{B}_1 \), where \( \vec{B}_0 \) (\( \vec{B}_1 \)) is the position-independent (position-dependent inhomogeneous) magnetic field. The Hamiltonian for this model system is thus

\[
H = H_Z + H_d + H_{SO} + V_{ext}(\vec{r}),
\]

where \( H_Z \) is the Zeeman Hamiltonian due to the position independent magnetic field \( \vec{B}_0 \), \( H_d \) is the orbital Hamiltonian, \( H_{SO} \) is the s-SOC, and \( V_{ext} \) is the potential due to the coupling to external electric field. The Zeeman Hamiltonian due to the position independent field is

\[
H_Z = \frac{1}{2} \vec{\mu}_B \vec{\sigma} \cdot \vec{B}_0,
\]

where \( g \) is the effective g-factor, and \( \vec{\sigma} \) is the Pauli operator for the electron spin.

\[
H_d = \frac{p^2}{2m^*} + \frac{1}{2} m^* \omega_d^2 \vec{r}^2
\]

is the usual electron 2D orbital Hamiltonian in a single QD, where \( \vec{r} = (x,y) \) and \( \vec{p} = -i \hbar \nabla + (e/c) \vec{A}(\vec{r}) \) are the in-plane 2D coordinate and kinetic momentum operators (\( e > 0 \)), and \( \omega_d \) is the characteristic frequency of the in-plane confinement. The out-of-plane dynamics is neglected due to the strong confinement at the interface. \( V_{ext}(\vec{r}) \) is the external electric potential, from sources such as electrical noises or a manipulation field.

In the presence of a micromagnet, the s-SOC term \( H_{SO} \) arises due to the Zeeman Hamiltonian of the position-dependent magnetic field \( \vec{B}_1 \). Keeping the lowest order position dependence,

\[
H_{SO} = \frac{1}{2} \vec{g}_B \vec{\sigma} \cdot \vec{b}_y x,
\]

where the inhomogeneity is assumed in the x-direction without loss of generality, and the gradient of the inhomogeneous magnetic field \( \vec{b}_1 \equiv \delta \vec{B}_1 / \delta x \bigg|_{x=0} \) becomes the coupling constant of the s-SOC. In this study, we assume the magnetic field to be uniform along the y-axis. In particular, the long edge of the micro-magnet is parallel to the y-axis and the external magnetic field is applied along the x-axis. As such, the y component of the magnetic field is zero. Consequently, the gradient \( \delta B_y / \delta x \) is also zero. Note that the QD is located in the heterostructure beneath the micromagnet. Although the \( \vec{B}_0 \) field is in-plane, the field \( \vec{B}_1 \) from the polarized micromagnet can have both longitudinal (x) and transverse (z) components at the location of the QD. Thus, the field gradient \( \vec{b}_1 = [b_{1x}, 0, b_{1z}] \) is chosen.
The i-SOC is always present in a host material no matter whether s-SOC is present [37]. When the $x$ and $y$ axes are along the [100] and [010] crystallographic directions, the i-SOC in a Si QD takes the form $H_{i-SOC} = -\alpha_i p_x \sigma_x + \alpha_i p_y \sigma_y$, where $\alpha_i = \alpha_D \pm \alpha_R$. In the absence of micromagnets, the i-SOC is the determining factor in spin relaxation and EDSR [36, 37]. In the presence of micromagnets, both the synthetic and the intrinsic SOC are present. However, the effect of the i-SOC can be omitted since it is relatively weak compared with that of the s-SOC [22].

$T$-symmetry generally plays an important role in spin dynamics. Here, we explore the connection between the time-reversal properties of the SOC terms in the Hamiltonian and the low-energy effective spin Hamiltonian (and thus the low-energy spin dynamics). For simplicity, we consider an in-plane magnetic field, where the vector potential $\vec{A}$ is chosen such that it does not contribute to the 2D momentum operator $\vec{p}$. The Zeeman term $H_Z$ breaks $T$-symmetry since the $\vec{\sigma}$ operator is odd under time-reversal, $T_\vec{\sigma} T_{\vec{\sigma}}^{-1} = -\vec{\sigma}$. $H_d$ and $V_{\text{ext}}$ preserves $T$-symmetry since they contain either the coordinate operator $r_i$ (even under time reversal) or even power of the momentum operator $p_i$ (odd under time-reversal). To be specific, we denote the Hamiltonian of s-SOC and i-SOC as $H_{s-SOC}$ and $H_{i-SOC}$. Given that $H_{s-SOC} \sim p_i \sigma_j$ and $\vec{p}$ is odd under time reversal, therefore, $H_{i-SOC}$ is time-reversal symmetric (TRS). In contrast, since $H_{s-SOC} \sim r_i \sigma_j$, $H_{i-SOC}$ is time-reversal asymmetric (TRA). The $T$-symmetry of SOC plays an important role in determining properties of a spin qubit as shown below.

### 3. Effective Hamiltonian

A SW transformation can be performed to decouple the spin and orbital dynamics, so that we can focus on the spin degree of freedom. In general, the energy scale of SOC is much smaller than the orbital and Zeeman energy in a QD. Thus, we perform an SW transformation, $H_{\text{eff}} = e^tHe^{-t}$, to eliminate $H_{SO}$ in the leading order by requiring that $[H_d + H_Z, S] = H_{SO}$ [36, 37, 42, 43]. The generator $S$ of the SW transformation can be written as [43]

$$S = \sum_{m=0}^{\infty} (L_Z L_d^{-1})^m L_d^{-1} H_{SO},$$

where the super-operators $L_d$ and $L_Z$ are defined such that $L_d O = [H_d, O]$ and $L_Z O = [H_Z, O]$ for any given operator $O$. Once $S$ is given, an effective spin Hamiltonian $H_{\text{eff}} = H_Z + [S, V_{\text{ext}}]$ can be obtained.

![Figure 1](link)

**Figure 1.** Schematic diagrams. (a) Schematics of an electron spin qubit in a gate-defined QD next to a micromagnet. An external magnetic field is applied along the $z$-axis and polarizes the micromagnet. $Z$-axis is defined along the spin quantization direction. Spin relaxation happens in the presence of s-SOC from the inhomogeneous magnetic field and electrical noises, such as $1/f$ charge noise, phonons, or photon fluctuations. (b) Schematics of the energy levels in a QD. The parabolic curve represents the confinement potential. The two solid lines are orbital states, and two dashed lines represents the lifted spin degeneracies in a transverse and longitudinal field gradients. The s-SOC from $\vec{B}_s$ hybridizes the spin states and results in the spin–electric-field coupling.
To perform the SW transformation, an explicit form of the generator $S$ needs to be obtained. From the model Hamiltonian and the definition of $L_d$, we have $L_d^{-1}p_x = \frac{\hbar}{2\Omega}x$ and $L_d^{-1}x = \frac{\hbar}{2\omega_c} (p_x + m' \omega_c y)$, where $\omega_c = eB_0/(m' c)$ is the cyclotron frequency. To the lowest order of $|H_{SO}|/E_d$ and $E_d/E_H$, the generator $S$ is given by

$$S \approx L_d^{-1} H_{SOC} = \vec{\sigma} \cdot \vec{\eta},$$

(4)

where $\vec{\eta} = \frac{-2\omega_c}{2\hbar \omega_c} (p_x + m' \omega_c y)$. Then, the external electric potential gives rise to a term $\langle \psi | [S, V_{ext}(\vec{r})] | \psi \rangle = -i e \hbar \vec{\nabla}_S \cdot \vec{\nabla} V_{ext} \equiv \frac{1}{2} g \mu_B \vec{\sigma} \cdot \vec{\Omega}$, where the electron is assumed to be in the ground orbital state $|\psi\rangle$ and $\vec{\Omega}$ is an effective magnetic field proportional to the electric field.

After performing the SW transformation, the effective spin Hamiltonian is

$$H_{eff} = \frac{1}{2} g \mu_B \vec{\sigma} \cdot (\vec{B}_0 + \vec{\Omega}),$$

(5)

where $\vec{\Omega} = -\vec{B}_0 \partial_\vec{x} V_{ext}/(m' \omega_c^2)$ is generated from the electric potential through the QD displacement $\delta_x = -\partial_\vec{x} V_{ext}/(m' \omega_c^2)$ and the s-SOC.

The effective Hamiltonian exhibits the following properties. First, the effective magnetic field $\vec{\Omega}$ from the electric field can have a longitudinal component parallel to the constant magnetic field $\vec{B}_0$. If $V_{ext}$ is a potential from electrical noise, the corresponding effective magnetic noise $\vec{\Omega}$ would in general lead to both spin relaxation and pure dephasing (see figure 1(b)) unless a specific design (such as design of the micromagnet) eliminates the longitudinal coupling to electric noise. Second, $\vec{\Omega}$ is independent of $B_0$. Both are different from the case of the i-SOC, where the effective noise magnitude is linearly proportional to $B_0$, and induces only spin relaxation [37, 42].

The properties of the effective Hamiltonian is strongly influenced by the property of the SOC under time-reversal. Note that the generator $S = L_d^{-1} H_{i-SOC}$ can be written formally as $S = f(x, p) \sigma_j$, where the spin operator $\sigma_j$ is TRA. Thus, if the ground state $\psi$ is TRS, i.e. $T_R \psi(\vec{r}) \equiv \psi^*(\vec{r}) = \psi(\vec{r})$ [33], then $f(x, p)$ that contains orbital operators must have TRS, so that the Hamiltonian $\langle \psi | [S, V_{ext}(\vec{r})] | \psi \rangle = \langle \psi | [f(x, p) V_{ext}(\vec{r})] | \psi \rangle$ is non-vanishing. (Note that $\langle \psi | [S, V_{ext}(\vec{r})] | \psi \rangle$ is still an operator instead of an expectation value since $|\psi\rangle$ contains only the orbital degree of freedom. The essence of SW transformation is to decouple the low-energy part of the Hilbert space from the rest for a given Hamiltonian, and retain the degree of freedom of our interest.). Given that $\vec{\sigma}$ is TRA, $S = L_d^{-1} H_{i-SOC} = f(x, p) \sigma_j$ must contain TRA terms to have a non-vanishing effective magnetic field $\vec{\Omega}$.

For the i-SOC, $S \approx L_d^{-1} H_{i-SOC} \propto i \vec{\zeta}(x, y) \cdot \vec{\sigma}$, where $\vec{\zeta}(x, y)$ is a function of position operators [37, 44]. Thus, $L_d^{-1} H_{i-SOC}$ is TRS,

$$T_R L_d^{-1} H_{i-SOC} T_R^{-1} = L_d^{-1} H_{i-SOC}.$$  

(6)

where $T_R$ is the time-reversal operator (recall $T_R^2 T_R^{-1} = T_R$, $T_R T_R^{-1} = \tau$, and $T_R T_R^2 = -\tau$). In contrast, for the s-SOC, $L_d^{-1} H_{s-SOC} = \vec{\sigma} \cdot \vec{\eta}$ contains both TRS and TRA terms, thus,

$$T_R L_d^{-1} H_{s-SOC} T_R^{-1} \neq L_d^{-1} H_{s-SOC}.$$  

(7)

Therefore, the first order term $L_d^{-1} H_{i-SOC}$ in $S$ is allowed for the s-SOC, while it is forbidden for the i-SOC. We emphasize here that the SW transformation does not change the symmetry of the Hamiltonian, and the total Hamiltonian always breaks $T$-symmetry due to the presence of a magnetic field. However, each of the terms such as $H_{SO}$, $H_T$, or $H_d$ contributes to the effective spin Hamiltonian after the SW transformation will depend on the symmetry of each individual term.

Below, spin relaxation, pure dephasing, EDSR, and spin-photon coupling are studied in the presence of s-SOC based on the effective spin Hamiltonian. Afterward, we examine a general connection between the $T$-symmetry of SOC and the longitudinal coupling between spin and electric field (and thus spin dephasing) without assuming a specific form of SOC.

### 4. Spin relaxation

Spin relaxation time gives the upper limit of spin coherence time. It is important to understand the spin relaxation in a QD mediated by the s-SOC. Although the different magnetic field dependences of spin relaxation for the s-SOC and i-SOC can be inferred from the presence or absence of Van Vleck cancellation, other dependences, such as on the QD size and the magnetic field gradients, are not as obvious. Moreover, the magnetic field dependence also depends on the detailed spectral density of phonon and Johnson noise. Therefore, here we work out an explicit theory for spin relaxation in a QD mediated by the synthetic SOC.
Suppose the direction of magnetic field $\vec{B}_0$ (assumed along the x-axis in this work) is defined as the new Z-axis, while X- and Y-axis are orthogonal to the Z-axis (see figure 1(b)), the relaxation rate is then given by $1/T_1 = S_{g} g_\text{S} \mu_B B_0$ [37, 42], where $S_{g}(\omega)$ is the power spectral density of the magnetic noise in the ith direction, and $\omega Z = g_\mu_B B_0 / h$ is the Larmor frequency. In other words, spin relaxation is determined by the transverse magnetic noise $\Omega Z = -b_{iZ} \partial Z / \left( m^* \omega Z^2 \right)$, and the relaxation rate is

$$1/T_1 = \left[ g_\mu_B b_{iZ} / (2 \hbar m^* \omega Z^2) \right]^2 S_{g}(\omega Z),$$

where $S_{g}(\omega)$ is the spectral density of the force correlation of the noise (see appendix A) [42, 45]. When $E_Z > E_d$, there could be an additional spin relaxation channel via an intermediate state, which flips the spin and orbital states simultaneously, followed by a direct orbital relaxation [46, 47]. However, in the limit of $E_Z \ll E_d$, as assumed in deriving our effective Hamiltonian, this spin relaxation mechanism is absent.

Qualitatively, spin relaxation depends on the transverse field gradient, with $1/T_1 \propto b_{iZ}^2 = (\partial B_{iZ} / \partial x)^2$. It also has a strong dependence on the QD confinement, $1/T_1 \propto 1/\omega Z^3$, and is thus suppressed in a smaller QD. Lastly, the dependence of spin relaxation on the magnetic field $B_0$ is determined by $S_{g}(\omega Z)$ (neglecting the weak dependence of $b_{iZ}$ on $B_0$ when the micromagnet is saturated), whose form is given in appendix A [42, 45].

We next evaluate the spin relaxation rate. We choose $g = 2$, $m^* = 0.19 m_0$, and $\hbar \omega Z = 8$ meV for the effective g-factor, the effective mass, and the orbital confinement of an electron in a silicon QD [46]. For the SOC constants, we choose the Rashba constant as $\alpha = 10$ m s$^{-1}$ (within the range of values used in the literature), and the Dresselhaus constant is set to zero for rough estimation [46, 48]. We choose $b_{iZ} = b_{iZ} = 0.1$ mT nm$^{-1}$ for the transverse and longitudinal field gradient [7]. We take $v_1 = 5900$ m s$^{-1}$ and $v_2 = v_3 = 3750$ m s$^{-1}$ for the speed of the different acoustic phonon branches, $\rho_s = 2200$ kg m$^{-3}$ for the mass density, $\Xi_D = 5.0$ eV and $\Xi_u = 8.77$ eV for the dilation and shear deformation potential constants [45]. The electron temperature is set to be zero for simplicity. We choose the amplitude $E_Z = 5 \mu$V for the 1/f charge noise, the cutoff frequency $\omega_0 = 1$ s$^{-1}$, and length scale $l_0 = 100$ nm. In silicon QD, the valley physics can modify spin relaxation at low magnetic field via the spin–valley mixing [45, 46, 49, 50]. Here, we assume the Zeeman splitting is away from the valley splitting so that the spin–valley mixing is suppressed and we focus on the relaxation due to the intra-valley spin–orbit mixing.

Figure 2(a) shows the spin relaxation rate as a function of the magnetic field $B_0$ due to deformation potential coupling to phonon in silicon or dipole coupling to Johnson noise or 1/f charge noise. As mentioned above we focus on the intra-valley spin–orbit mixing. Spin relaxation due to phonon emission mediated by s-SOC shows a $B_0^5$ dependence (inferred from analytical expression and the numerical values) before it saturates due to the bottleneck effect [45] at high fields, $1/T_1 \propto B_0^5$, in contrast to the $B_0^3$ dependence in the case of the i-SOC [37, 51], where the extra $B_0^2$ dependence is due to the T-symmetry of the i-SOC that leads to the Van Vleck cancellation. For the same reason, spin relaxation due to Johnson noise mediated by s-SOC shows a linear $B_0$ dependence, $1/T_1 \propto B_0$, in contrast to the $B_0^3$ dependence in the case of the i-SOC [42]. Similarly, 1/f charge noise leads to a $1/B_0$ dependence for the spin relaxation rate, as compared to a linear $B_0$ dependence in the case of i-SOC [42]. Consequently, at low B-field spin relaxation is dominated by 1/f charge noise and Johnson noise, while at higher B-field it is dominated by phonon emission, as shown in figure 2. Numerically, spin relaxation rate grows from 0.01 s$^{-1}$ to 1000 s$^{-1}$ as $B_0$ increases from 0.1 T to 10 T.

The spin relaxation channel discussed here can be relevant in a silicon QD experiment considering that i-SOC is generally weak in Si. For comparison, let us consider the case of an electron spin in a silicon QD without a micromagnet. The spin relaxation is mostly due to phonon and Johnson noise mediated by the i-SOC induced spin–valley or spin–orbit mixing [45]: at low magnetic field, spin relaxation is dominated by the spin–valley mixing, where $1/T_1 \sim 1$ to 1000 s$^{-1}$. When Zeeman splitting is less than the orbital splitting but larger than the valley splitting, $E_{VS} < E_Z < E_d$ (requires $B_0 > B_V = E_{VS} / (g_\mu_B) \sim 1$ T if the valley splitting $E_{VS} = 0.1$ meV), spin relaxation via spin–valley mixing is strongly suppressed, and spin relaxation is dominated by spin–orbit mixing (corresponding to the case of i-SOC + ph in figure 2(a)). Note that although the orbital splitting is generally large, spin–orbital mixing could still dominate over spin–valley mixing because of the small dipole moment between valley states (typically around 1 nm) [46].

The numerical estimate in figure 2(a) for spin relaxation due to phonon noise via s-SOC can be dominant in a silicon QD when $E_{VS} < E_Z < E_d$, where the spin relaxation shows $B_0^5$ dependence. Recent experiments in a single Si QD with micromagnets show that spin relaxation in the high field limit indeed has a $B_0^5$ dependence [26, 27], deviating from the $B_0^3$ dependence normally observed in silicon without micromagnets [52–54], but consistent with our theoretical results here.
At magnetic fields when $B_0 < B_{VS}$, experimental results also show the difference between the s-SOC and i-SOC \([26, 27]\), consistent with our theory. However, due to the generally small valley splitting in these experiments, valley effects have to be included \([45]\) to explicitly make the comparison. A recent study indicates that the results, such as the $B_0$ and $b_{ul}$ dependences of spin relaxation, related to the broken $T$-symmetry of SOC are still valid for spin–valley mixing without assuming the $T$-symmetry of the orbital ground state \([55]\). For simplicity, we focus on the spin–orbital mixing in this study, although the results based on the symmetry argument can be generalized to cases such as spin–valley mixing. Furthermore, in a device with small orbital splitting (large QD) and large valley splitting, $E_d < E_{VS}$, where the spin–valley relaxation is suppressed and the spin–orbit mixing is dominant, our result also becomes relevant at low $B$ fields.

5. Pure dephasing

As discussed above, the broken $T$-symmetry leads to possible longitudinal spin–electric coupling and thus spin dephasing at the lowest order of s-SOC, which is forbidden with i-SOC \([37, 42]\). Such charge noise induced dephasing has indeed been measured and discussed in references \([7, 28]\), where it has been demonstrated that the $1/f$ charge noise induced spin pure dephasing is the dominant dephasing channel in the presence of s-SOC in their experiments. Below we examine explicitly the dependence of the spin pure dephasing on system parameters such as the QD size and the field gradient, and give some numerical results.

Pure dephasing is determined by the spectral density $S_{ZZ}(\omega)$ of the effective longitudinal magnetic noise \([56, 57]\). Suppose the spectral density of the electrical potential fluctuation of $1/f$ charge noise is $S_{1/f} = A/\omega$, where $A$ characterizes the noise strength. For the spin qubit, we then have $S_{ZZ}(\omega) = A_{eff}/\omega$, where $\sqrt{A_{eff}} = \sqrt{A} [g\mu_B b_{ul} (1/2\hbar m^* \omega_0^2 b_{ul})]$ is the amplitude of the effective magnetic noise, and the length $l_0$ converts the electric potential to the field strength of the charge noise.

The magnetic noise amplitude $\sqrt{A_{eff}}$ depends on the longitudinal field gradient and the orbital splitting as $\sqrt{A_{eff}} \propto b_{ul}/\omega_0^2$, consistent with the intuitive argument given in the supplementary material in reference \([7]\). We emphasize here that this intuition works only in the case for the s-SOC, where the $T$-symmetry is broken, but does not work in the case for the i-SOC \([42]\). Due to the initial Gaussian time dependence for the off-diagonal density matrix element, the spin pure dephasing rate $1/T_\varphi \propto \sqrt{A_{eff}}$ \([57]\). Thus the pure dephasing rate is determined by the amplitude of the $1/f$ charge noise, the magnitude of the longitudinal field gradient, and QD confinement as $1/T_\varphi \propto \frac{b_{ul}\sqrt{\omega_0}}{\omega_0^2}$.

Figure 2(b) shows the spin dephasing rate with or without echo as a function of $b_{ul}$ due to $1/f$ charge noise. The dephasing rate is extracted numerically from the dynamics as shown in the inset (calculated similarly as in reference \([57]\)). Quantitatively, the dephasing rate goes from 8000 s$^{-1}$ to 90000 s$^{-1}$ (or from 2300 s$^{-1}$ to 30000 s$^{-1}$ with spin echo) as the magnetic field gradient $b_{ul}$ increases from 0.1 mT nm$^{-1}$ to 1 mT nm$^{-1}$. The pure dephasing rate can vary if the amplitude of charge noise or orbital confinement varies (here, the dephasing rate at $b_{ul} = 0.5$ mT nm$^{-1}$ is $1/T_\varphi \sim 10^4$ s$^{-1}$ corresponding to the value observed in reference \([7]\)).
The pure dephasing due to charge noise can be compared with other possible dephasing mechanisms. For an electron spin in a QD without a micromagnet, nuclear spin noise is a major source for spin dephasing, with spin dephasing time measured to be 360 ns (1/T2* ≈ 2.7 × 10^6 s⁻¹) in natural silicon, and as long as 120 μs (1/T2* ≈ 8 × 10^1 s⁻¹) in isotopically enriched 28Si [12]. Thus, our numerical estimate indicates that dephasing due to charge noise via s-SOC can be dominant when b1z is bigger than 0.1 mT nm⁻¹ (also dependent on the charge noise amplitude) in an isotopically enriched silicon QD.

To suppress pure dephasing from charge noise and improve spin coherence in the presence of s-SOC, a straightforward approach is to reduce longitudinal field gradient b1z. The components of the field gradient have to satisfy Maxwell’s equations \( \nabla \cdot \vec{B} = 0 \) and \( \nabla \times \vec{B} = 0 \). Therefore, in addition to \( b_{1z} = \partial B_z / \partial x \) and \( b_{1y} = \partial B_y / \partial x \), the gradient \( \partial B_z / \partial z \) and \( \partial B_y / \partial z \) would have to be finite as well (all other gradients can be zero with translational symmetry along y). We emphasize that, to optimize the operation of the spin qubit system, we need to consider field gradient and confinement together. We neglected the effects of the gradient \( \partial B_z / \partial z \) and \( \partial B_y / \partial z \) in our case because the orbital motion along z axis is suppressed due to the strong vertical confinement. Thus, only gradients \( b_{1y} \) and \( b_{1x} \) have to be taken into account. In this case, if we can adjust the quantization field \( \vec{B}_0 \), such that \( \vec{B}_0 \) is perpendicular to the vector \( \partial \vec{B} / \partial x = (b_{1y}, 0, b_{1z}) \), then the induced effective magnetic field becomes perpendicular to the quantization axis. Consequently, the pure dephasing from charge noise can be minimized, while the transverse magnetic field is maximized for EDSR.

6. Optimization of spin control using direction of magnetic field

The discussion above shows that it is possible to optimize spin manipulation by adjusting the direction of the applied magnetic field, which we explore in this section.

In a conventional spin resonance experiment, a constant magnetic field establishes the quantization axis of the spins, while a small transverse AC magnetic field is used to flip the spins. The same principle is behind EDSR experiments in the context of s-SOC [7, 19, 36, 58–61]. Here we consider an oscillating electric field \( \vec{E}(t) \) that is applied along the x-axis, \( \vec{E}(t) = \vec{E}_0 \cos(\omega_0 t + \phi_0) \), where \( \vec{E}_0 \) is the field magnitude, and \( \omega_0 \) and \( \phi_0 \) are the frequency and the phase of the field. Driven by this electric field, the electron spin experiences an effective oscillating magnetic field via the s-SOC. The driven spin Hamiltonian takes the form

\[
H_0 = \frac{\omega_0}{2} \sigma_z + (\Omega_{0x} \sigma_x + \Omega_{0y} \sigma_y) \cos(\omega_0 t + \phi_0),
\]

where \( \Omega_{0x} = -g_\mu_B b_1 b_{1y} 2 m^2 \omega_0^2 / e Ek_{\text{max}} \) and \( \Omega_{0y} = -g_\mu_B b_1 b_{1x} 2 m^2 \omega_0^2 / e Ek_{\text{max}} \) are the maximum transverse and longitudinal magnetic field the spin experiences. Due to the breaking of T-symmetry by s-SOC, the effective field \( \Omega_{0z} \) does not depend on the magnitude \( B_0 \) [19], contrary to the i-SOC-mediated EDSR [36]. Therein lies an advantage for s-SOC enabled EDSR: a lower magnetic field can be applied to suppress the spin relaxation without sacrificing the speed of EDSR. Moreover, besides the transverse term \( \Omega_{0x} \), there is in general also a longitudinal term \( \Omega_{0y} \). In this case, the choice of the direction of magnetic field can simultaneously enhance the EDSR while also suppress the charge-noise-induced spin dephasing mediated by s-SOC, such that the quality of the single-qubit gates based on EDSR can be optimized.

Here, we explore the explicit dependence of the Rabi frequency and quality factor of EDSR on the orientation of the spin quantization axis determined by \( \vec{B}_0 \). We assume that the micromagnet is fully polarized, so that the field gradient is insensitive to a change of direction of the applied magnetic field. The orientation of \( \vec{B}_0 \) can be modified by changing the direction of the applied magnetic field [29]. To evaluate the spin decoherence, a background dephasing 1/T2 is 10³ s⁻¹ is assumed from other dephasing mechanisms such as nuclear spins, corresponding to the case of an isotopically purified silicon QD [12]. The total dephasing is then estimated as 1/T2′ = 1/T2 + 1/Tc (contribution from T1 process can be omitted since it is much slower than pure dephasing at low magnetic fields). For the s-SOC induced EDSR, we choose similar parameters as above, and choose the maximum electric field \( E_{\text{max}} = 6 \times 10^5 \) V m⁻¹ for the microwave driving.

As illustrated in figure 3, as the quantization axis goes away from \( \theta = 0 \) (out-of-plane), Rabi frequency is enhanced, while the pure dephasing rate is suppressed simultaneously. This favorable and opposite trend, and especially the strong suppression of spin pure dephasing, means that the quality factor \( Q = \omega R / T_\phi / \pi \) for the spin control can be enhanced by nearly an order of magnitude, as shown in the lower panel of figure 3.
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been demonstrated experimentally in a Si double QD with a micromagnet [9, 10]. Note that to achieve the i-SOC or s-SOC, has become the only realistic approach to reach the strong-coupling limit, and has recently been demonstrated experimentally in a Si double QD with a micromagnet [9, 10]. With magnetic coupling too weak to be useful, spin–photon coupling via the cavity electric field, assisted by device (such as a large MOS QD [46]), our theory would be applicable. Although qualitatively the magnetic superconductivity of the cavity electrodes. When the valley splitting is larger than the orbital splitting in the spin–photon strong coupling, the applied magnetic field should be small so that it does not destroy cancellation, it is still necessary to have an explicit theory of the spin–photon coupling in a QD mediated by the synthetic SOC to understand the QD size dependence and field gradient dependence of the coupling strength. It is also an important task to evaluate whether and how strong coupling limit can be reached below, we examine the properties of spin–photon coupling between a spin qubit and a superconducting resonator.

To enhance the cavity electric field, we assume that two gates of the QD are connected to the electrodes of a superconducting resonator [9, 10, 64], so that the electric field \( E = -\partial_z V_{\text{ext}}/|e| \) in the effective spin Hamiltonian is given by the electric field \( E_{\text{sc}} \) across the QD due to the voltage difference of the center pin and ground of the resonator. The voltage operator in the resonator is given by \( V_{\text{sc}} = V_{\text{qg}}(a^+ + a) \), where \( a \) is the photon annihilation operator in the single-mode superconducting resonator. The voltage \( V_{\text{qg}} = \omega_0 \sqrt{\hbar Z_0} \) is the voltage amplitude due to the zero-point fluctuation in the resonator, where \( \omega_0 \) is the resonator frequency and \( Z_0 \) is the characteristic impedance of the resonator. Therefore, the electric field operator becomes \( E_{\text{sc}} = E_{\text{qg}}(a^+ + a) \), where \( E_{\text{qg}} = V_{\text{qg}}/d_0 \) and \( d_0 \) (about the QD size) is the length for the voltage drop. The spin–photon coupling Hamiltonian thus takes the form

\[
H_{\text{ph}} = g_{\text{ph}}(\sigma_+ a + a^+ \sigma_-) + g_{\text{ph}}(\sigma Z(a + a^+)
\]

where \( \sigma_{\pm} = \sigma_x \pm i \sigma_y \) is the spin creation or annihilation operator. The strength for the transverse and the longitudinal spin–photon coupling are given by \( g_{\text{ph}} = -\frac{\omega_0}{2m_e^2} \omega_0 \hbar Z_0 / d_0 \), and \( g_{\text{ph}} = -\frac{\omega_0}{2m_e^2} \omega_0 \hbar Z_0 / d_0 \).

In comparison, for i-SOC and at low magnetic field (\( g_{\mu_B} B_0 \ll \hbar \omega_d \)), \( g_{\text{soc,loc}} = 0 \), and \( g_{\text{soc,loc}} = -\frac{\omega_0}{\lambda_m m_e^2 \omega_d d_0} \).

Figure 3. Upper panel: Rabi frequency (black solid) and pure dephasing rate (blue dashed) of a spin qubit versus the polar angle \( \theta \) of \( \vec{B} \) when \( h_0 = h_0 = 0.5 \text{ mT} \text{ nm}^{-1} \). A background dephasing \( 1/T_\text{ph} = 10^4 \text{ s}^{-1} \) is assumed from other dephasing mechanisms such as nuclear spins, corresponding to the case of an isotopically purified silicon QD. As the quantization axis goes away from \( \vec{B} = 0 \) (out-of-plane), Rabi frequency is enhanced while the dephasing rate is suppressed. Lower panel: quality factor \( Q = \omega_0 T_\text{qg}/\pi \) of the spin control versus the angle \( \theta \). Because of the out-of-phase variation of Rabi frequency and suppression of dephasing rate, the quality factor can be enhanced by almost an order of magnitude at the optimal angles.

7. S-SOC induced spin–photon coupling

Spin qubit communication and coupling via a cavity could be a key ingredient in a scalable quantum computer, and have generated extensive theoretical and experimental explorations [9, 10, 24, 62–64]. With magnetic coupling too weak to be useful, spin–photon coupling via the cavity electric field, assisted by i-SOC or s-SOC, has become the only realistic approach to reach the strong-coupling limit, and has recently been demonstrated experimentally in a Si double QD with a micromagnet [9, 10]. Note that to achieve the spin–photon strong coupling, the applied magnetic field should be small so that it does not destroy superconductivity of the cavity electrodes. When the valley splitting is larger than the orbital splitting in the device (such as a large MOS QD [46]), our theory would be applicable. Although qualitatively the magnetic field dependences of spin–photon coupling for the s-SOC and i-SOC can be inferred from the Van Vleck cancellation, it is still necessary to have an explicit theory of the spin–photon coupling in a QD mediated by the synthetic SOC to understand the QD size dependence and field gradient dependence of the coupling strength. It is also an important task to evaluate whether and how strong coupling limit can be reached below, we examine the properties of spin–photon coupling between a spin qubit and a superconducting resonator.

To enhance the cavity electric field, we assume that two gates of the QD are connected to the electrodes of a superconducting resonator [9, 10, 64], so that the electric field \( E = -\partial_z V_{\text{ext}}/|e| \) in the effective spin Hamiltonian is given by the electric field \( E_{\text{sc}} \) across the QD due to the voltage difference of the center pin and ground of the resonator. The voltage operator in the resonator is given by \( V_{\text{sc}} = V_{\text{qg}}(a^+ + a) \), where \( a \) is the photon annihilation operator in the single-mode superconducting resonator. The voltage \( V_{\text{qg}} = \omega_0 \sqrt{\hbar Z_0} \) is the voltage amplitude due to the zero-point fluctuation in the resonator, where \( \omega_0 \) is the resonator frequency and \( Z_0 \) is the characteristic impedance of the resonator. Therefore, the electric field operator becomes \( E_{\text{sc}} = E_{\text{qg}}(a^+ + a) \), where \( E_{\text{qg}} = V_{\text{qg}}/d_0 \) and \( d_0 \) (about the QD size) is the length for the voltage drop. The spin–photon coupling Hamiltonian thus takes the form

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\]

where \( \sigma_{\pm} = \sigma_x \pm i \sigma_y \) is the spin creation or annihilation operator. The strength for the transverse and the longitudinal spin–photon coupling are given by \( g_{\text{ph}} = -\frac{\omega_0}{2m_e^2} \omega_0 \hbar Z_0 / d_0 \), and \( g_{\text{ph}} = -\frac{\omega_0}{2m_e^2} \omega_0 \hbar Z_0 / d_0 \).

In comparison, for i-SOC and at low magnetic field (\( g_{\mu_B} B_0 \ll \hbar \omega_d \)), \( g_{\text{soc,loc}} = 0 \), and \( g_{\text{soc,loc}} = -\frac{\omega_0}{\lambda_m m_e^2 \omega_d d_0} \).
\(\omega_0 \sqrt{\hbar Z_0}\), where \(\lambda_{\omega_0} = \hbar/(m^* \alpha)\) is the effective length of i-SOC, with \(\alpha\) being the Rashba SOC constant [42].

The spin–photon coupling strength \(g_{s,t}\) and \(g_{s,l}\) have linearly dependence on the resonator frequency \(\omega_0\), and \(\sqrt{Z_0}\) dependence with the resonator characteristic impedance, and have no dependence on the magnitude of magnetic field \(B_0\). The coupling strengths also have strong dependence on the size of the QD, with \(g_{s,t} \sim 1/(\omega_0 t_0) \ll T_0^3\), where \(T_{QD}\) is the size of the QD. Lastly, \(g_{s,l}\) (\(g_{s,t}\)) has linear dependence on the transverse (longitudinal) field gradient, with \(g_{s,t} \sim b_{\Omega}(g_{s,t} \sim b_{\Omega})\).

Figure 4 shows the spin–photon transverse coupling strength as a function of the magnetic field \(B_0\). The characteristic impedance is chosen as \(Z_0 = 50 \Omega\), and the spin and the resonator are assumed to be on resonance, i.e. \(\omega_0 = \omega_2\). The coupling strength \(g_{s,t}\) shows a linear \(B_0\) dependence (note the photon energy \(\hbar \omega_0\) increases with \(B_0\)), in contrast to the \(B_0^2\) dependence for \(g_{s,t,\text{iSOC}}\). The extra \(B_0\) dependence is due to the \(T\)-symmetry of the i-SOC, consistent with the results of the Van Vleck cancellation. Indeed, the ratio of the two transverse spin–photon coupling strength satisfies

\[
\frac{g_{s,t}}{g_{s,t,\text{iSOC}}} = \frac{b_{\Omega}(\lambda_{\omega_0})}{B_0},
\]

which is independent of the QD and cavity parameters such as \(\omega_d\) and \(Z_0\). With \(b_{\Omega}\) fixed by the fabrication and \(\lambda_{\omega_0}\) fixed by the material and the interface electric field, \(B_0\) becomes an important indicator of the coupling strengths. Specifically, \(g_{s,t}\) can be much stronger than \(g_{s,t,\text{iSOC}}\) at lower magnetic fields. For example, when \(E_d = 8\) meV, \(g_{s,t}\) is about \(5 \times 10^4\) s\(^{-1}\) at \(B_0 = 0.1\) T, and increases linearly as \(B_0\) increases, while \(g_{s,t,\text{iSOC}}\) grows from \(3 \times 10^7\) s\(^{-1}\) to \(3 \times 10^9\) s\(^{-1}\) as \(B_0\) increases from 0.1 T to 3 T. The result suggests that spin–photon strong coupling is possible with s-SOC at low magnetic fields even in a single QD, which is an important consideration when integrating a spin system with a superconducting resonator. Moreover, \(g_{s,t}\) has a weak dependence on the QD confinement in the form of \(1/\omega_0^2\). Thus when the QD is larger, for example when \(\hbar \omega_d = 1\) meV, the coupling strength increases to \(10^6\) s\(^{-1}\) at \(B_0 = 0.1\) T (about 20 times faster than the case when \(\hbar \omega_d = 8\) meV. Here, \(d_0 \sim \sqrt{\hbar/(m \omega_d)}\) is also increased for a larger dot.) In short, benchmarked against spin dephasing rate, the strong coupling limit can be achieved more easily at lower magnetic field with s-SOC than with i-SOC. Moreover, if the photon decay rate in a resonator is on the order of 100 kHz or smaller [9], it is possible to reach the strong coupling limit between a spin in a big dot and a photon in a resonator at low magnetic fields.

Beside the transverse coupling, s-SOC also allows longitudinal coupling between a spin qubit and a superconducting resonator. The longitudinal coupling \(g_{s,l}\) is in general finite in contrast to the vanishing \(g_{s,l,\text{iSOC}}\) in the case of the i-SOC. The magnitude of \(g_{s,l}/g_{s,t}\) depends on the ratio of longitudinal and transverse gradient of magnetic field, with \(g_{s,l}/g_{s,t} = b_{\Omega}/b_{\Omega}\). To make use of the longitudinal component, the spin quantization axis can also be adjusted to enhance the coupling strength of the s-SOC induced spin–photon coupling and suppress simultaneously the spin pure dephasing from charge noise.
8. General consequence of the $T$-symmetry of SOC on spin dynamics

So far we have explicitly studied properties of an electron spin in a QD mediated by the s-SOC. Here, we generalize the above results, and show the impact of the $T$-symmetry of $HSO$ on the spin dynamics without assuming a specific form of SOC or confinement potential. Although it is well-known that broken $T$-symmetry is required for spin–electric coupling (to break the Kramers degeneracy), the exact consequence of the $T$-symmetry of each term (such as $H_z$ or SOC) in the Hamiltonian is still important to explore. The impact of broken $T$-symmetry of $H_z$ on spin dynamics has been studied in the literature [36, 65, 66], thus our focus here is the impact of broken $T$-symmetry of $HSO$.

Recall that the effective spin Hamiltonian is obtained by performing the SW transformation, which eliminates the $HSO$ to the first order. From equation (3), for any SOC, the generator $S$ can be formally rewritten as $S = \tilde{r} \cdot \tilde{\sigma}$, where $\tilde{r}$ contains the orbital operators. Suppose the ground orbital state is TRS, since $V_{ext}(\tilde{r})$ is also TRS, the matrix element $\langle \psi | [\tilde{f}, V_{ext}(\tilde{r})] | \psi \rangle$ would be finite only when $\tilde{f}$ is also TRS. (Note if $V_{ext}$ is time-dependent, one can rewrite $V_{ext} = -\tilde{\sigma} \cdot \tilde{E}(t)$, where $\tilde{r}$ is the electron coordinate operator, and $\tilde{E}$ is a $c$-number representing the applied (external) time-dependent electric field. Since both $| \psi \rangle$ and $\tilde{r}$ are TRS, $\langle \psi | [\tilde{f}, V_{ext}] | \psi \rangle = -e \sum_i (\tilde{\psi} | \tilde{f}, r_i | \tilde{\psi}) E_i(t)$ would again be finite only when $\tilde{f}$ is TRS.) Given that spin operator $\tilde{\sigma}$ is TRA and $S = \tilde{r} \cdot \tilde{\sigma}$, only the TRA terms in $S$ could contribute to an effective magnetic field $\tilde{\Omega}$ (arising from the external electric potential).

For the TRA s-SOC, the first term $L_z^{-1}HSO$ in $S$ is TRA, which is allowed. On the other hand, for the TRS i-SOC, the first term $L_z^{-1}HSO$ in $S$ is TRS, which is forbidden. The lowest order contribution would come from the next order term $L_z L_z^{-2}H_{i-SOC}$, which is TRA and is allowed (note $L_z$ involves Zeeman term $H_z$ and breaks $T$-symmetry). Clearly, the different $T$-symmetries of the s-SOC and the i-SOC ensures that their contribution to the spin Hamiltonian are of different orders in $E_z/E_{\text{th}}$ and leads to qualitatively different results.

More specifically, for i-SOC, the lowest-order contribution is from the second term $L_z L_z^{-2}HSO$ in $S$, which is linearly proportional to $B_0$. This leads to an extra $B_0^2$ dependence in the spin relaxation rate (shown below), underlying the so-called Van Vleck cancellation [65, 66]. Moreover, given that all the higher-order terms in $S$ contains the $L_z$ operator, and the property $L_z(\tilde{\epsilon} \cdot \tilde{\sigma}) \propto (\tilde{B}_0 \times \tilde{\epsilon}) \cdot \tilde{\sigma}$ is satisfied by any vector $\tilde{\epsilon}$, the resulting effective spin–electric coupling $[S, V_{ext}] \propto (\tilde{B}_0 \times [\tilde{\epsilon}, V_{ext}]) \cdot \tilde{\sigma}$ contains only transverse coupling, and the effective field sensed by the spin is always transverse. In short, the conservation of the $T$-symmetry of the i-SOC results in the vanishing longitudinal effective magnetic field to any order in the perturbative expansion of equation (3), and spin dephasing via SOC with $T$-symmetry would vanish to the first order of SOC and all orders of Zeeman interaction. This effect can be termed as cancellation of ‘longitudinal spin–electric coupling’. In comparison, the breaking of $T$-symmetry by the s-SOC means that the generator $S$ could contain a term $L_z^{-1}HSO$ that is independent of $L_z$ operator, therefore removing the condition for the Van Vleck cancellation and moreover preserves ‘longitudinal spin–electric coupling’. In other words, for s-SOC, the effective magnetic field could be independent of $B_0$, and the longitudinal effective magnetic field is allowed.

The analysis here does not assume any specific form of the SOC and $H_d$ other than their $T$-symmetry and the $T$-symmetry of the ground orbital state. It is thus generally applicable to other physical systems as long as the SOC is smaller in magnitude than the orbital and Zeeman splitting. For example, the results can be used for an electron spin qubit in a double QD with or without a micro-magnet, or an electron spin at donor(s) with a Coulombic confinement. Moreover, the theory could also be applicable to other forms of SOC.

Note that it is well known that $T$-symmetry of the i-SOC results in a linear magnetic field dependence for the transverse spin–electric coupling at the lowest order, leading to extra powers in the magnetic field dependence of spin relaxation (also known as the Van Vleck cancellation) [65, 66]. In addition, previous studies have also shown an absence of pure spin dephasing through the i-SOC in a QD [36, 37]. However, a systematic, explicit, and model-independent discussion on how $T$-symmetry relates to all the properties of a spin qubit with respect to SOC, whether intrinsic or extrinsic, through both transversal and longitudinal spin–electric coupling, is still absent in the literature. For example, in reference [36], the absence of spin–electric coupling at zero magnetic fields has been discussed as a consequence of $T$-symmetry of the total Hamiltonian at zero magnetic field, i.e. the second term $L_z L_z^{-2}HSO$ vanishes at zero magnetic fields; While the absence of the first term $L_z^{-1}HSO$ was obtained as a consequence of a specific model Hamiltonian. In contrast, here we show explicitly that the absence of the first term $L_z^{-1}HSO$ (and the absence of longitudinal spin–electric coupling) in the case of the i-SOC is a direct consequence of the $T$-symmetry of the i-SOC; And the first term $L_z^{-1}HSO$ can contribute a longitudinal effective field in the case of the s-SOC of broken $T$-symmetry. Another example is the experimental demonstration of spin pure dephasing that arises from the longitudinal magnetic field gradient and charge noise [7, 28], where the role of $T$-symmetry
were not discussed. On the other hand, our result on the connection of the longitudinal spin–electric coupling to the broken T-symmetry of SOC is quite general, and is applicable to different forms of SOC and orbital Hamiltonians. In short, we show explicitly the relation between the longitudinal spin–electric coupling (arising from the first term $L_z H_{SO}$) and the breaking of T-symmetry of the $H_{SO}$. (In comparison, the Van Vleck cancellation studied the relation between a finite (transverse) spin–electric coupling and the breaking of T-symmetry of the total Hamiltonian by the term $H_Z$.)

9. Discussion

Our results in this study clearly demonstrate that spin decoherence, including relaxation and dephasing, can be modified strongly by the symmetry property of the interaction Hamiltonian under time-reversal operation. Conversely, the B-field dependence of spin relaxation or spin dephasing represents a hallmark to characterize symmetry properties of the SOC Hamiltonian in a system, which may serve as a tool to investigate the possible origin of decoherence in the system.

Our theory on the consequences of the T-symmetry is valid when $|H_{SOC}| \ll E_Z \ll E_d$, which is satisfied in many experiments, with the purpose of protecting the spin qubit from electrical noise by reducing the spin–orbit mixing. While Van Vleck cancellation is long known to be related to the T-symmetry, we study explicitly not only individual phenomenon such as spin relaxation, EDSR, and spin–photon coupling mediated by the s-SOC, but also their common link to T-symmetry, such that our studies can be extended to arbitrary forms of SOC. Such a unified general understanding of spin relaxation, pure dephasing, and spin control, and their connection to the T-symmetry enables us to predict properties in many other physical systems.

Consider an example where $H_{SO} \sim \sigma_x p_y$. Without explicit calculations, our theory indicates that if there is an out-of-plane component of $\vec{B}_0$, $H_{SO} \sim \sigma_x p_y$ breaks the T-symmetry due to the presence of the vector potential in the kinetic momentum operator $p_y$. Consequently, longitudinal spin–electric coupling would be allowed. When $\vec{B}_0$ is in-plane, on the other hand, spin dephasing will be suppressed due to the ‘spin–orbit field cancellation’ (electron motion is assumed to be 2D when vertical confinement is strong), which is non-trivial but consistent with the explicit calculations in the literature [36].

The consistency of our theory with previous results in the literature sets an example of the predicting power and generality of the theory. For example, the theory can be applied to an electron spin in a double or triple QD with different forms of SOC. It can also be applied to the case of an electron spin qubit in a hybrid donor-QD [67] or a double donor system with different numbers of nuclei in each donor, where the hyperfine coupling difference could provide a longitudinal magnetic field gradient. Furthermore, in a double QD, a position dependent g-factor of an electron spin could arise due to the position dependent SOC (from inhomogeneous electric field) [40, 41], and the g-factor difference in a double QD also results in an effective SOC that breaks the T-symmetry.

10. Conclusion

In conclusion, we have studied spin relaxation, pure dephasing, EDSR, and spin–photon coupling in a single QD via an s-SOC generated by a micromagnet. Analysis based on T-symmetry suggests that longitudinal effective field via a T-symmetric SOC, such as the intrinsic Dresselhaus and Rashba SOC in semiconductor heterostructures, would vanish to the first order of SOC and all orders of Zeeman interaction. The absence of such a longitudinal field means the vanishing of pure dephasing for the electron spin. On the other hand, we find that the s-SOC breaks the T-symmetry, resulting in an effective magnetic field different from the case of the i-SOC. In particular, the transverse effective field does not depend on the applied field $B_0$, while the longitudinal effective field is allowed by the s-SOC. We further study explicitly spin relaxation, pure dephasing, EDSR, and spin–photon coupling and compared with the case of the i-SOC. Moreover, we show that s-SOC mediated EDSR could be optimized through the orientation of the applied magnetic field. Overall, our results clarify the decoherence and control properties of a spin qubit in the presence of an s-SOC in a single QD, and reveal the underlying general connection between the symmetry of the SOC and the spin properties. These understandings could contribute significantly to our effort in building a scalable semiconductor quantum computer.
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**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

**Appendix A. Spectral density of electrical noise**

In the section, we present the detailed derivation of the spectral density of phonon noise and Johnson noise.

**Electron–phonon interaction**—The electron–phonon interaction $H_{EP}$ is one of the noise source provides the energy dissipation. In silicon QD, we have [68]

$$H_{EP} = \sum_{\vec{q}} f(\vec{q}) e^{i\vec{q}\cdot\vec{r}^e} M^{(2)}_{\vec{q}} (b^\dagger_{\vec{q}} + b_{\vec{q}}),$$  \hfill (A1)

$$M^{(2)}_{\vec{q}} = i \sqrt{\hbar/(2\rho_{\omega q})} \hat{q} \Xi^{(2)}_{\vec{q}},$$  \hfill (A2)

$$\Xi^{(2)}_{\vec{q}} = \vec{\varphi}(0) \cdot (\Xi_0 1 + \Xi_{\alpha} \mathcal{U}^{(2)}_{\vec{q}}) \cdot \hat{q},$$  \hfill (A3)

where $b^\dagger_{\vec{q}} (b_{\vec{q}})$ is the creation (annihilation) operator of a phonon with wave vector $\vec{q}$ and branch-index $j$, $j = 1$ (longitudinal mode), $t_1$, or $t_2$ (transverse modes). The function $f(\vec{q}) = \langle \psi_{\vec{q}}(z) | e^{i\vec{q}\cdot\vec{r}} | \psi_{\vec{q}}(z) \rangle \sim e^{-\vec{q}^2 d^2/4}$ equals unity for $|\vec{q}| < d^{-1}$, and vanishes for $|\vec{q}| \gg d^{-1}$, where $d$ is the characteristic size of the quantum well along the $z$ axis, $\rho_c$ is the sample density, $v_j$ is phonon velocity, $\vec{\varphi}(0)$ and $\vec{q}$ are unit vectors of phonon polarization and wave vector, $\Xi_0$ and $\Xi_{\alpha}$ are the dilution and uniaxial shear deformation potential constants. Each component of $\Xi^{(2)}_{\vec{q}}$ can be evaluated as $\Xi_{\vec{q}} = \Xi_0 + \Xi_{\alpha} \cos^2 \theta$ (TA), $\Xi_{\beta_0} = 0$, and $\Xi_{\beta_2} = \Xi_{\alpha} \cos \theta \sin \theta$ (LA), where $\theta$ is polar angle of phonon wave-vector with respect to the growth direction (crystal axis $[001]$).

**Spectral density of phonon noise**—The correlation of the electric force due to phonons, $F = -eE(\vec{r}) = -\nabla U_{\text{ph}}(\vec{r})$, is thus given by (x component),

$$\langle F_x(0)F_x(t) \rangle = \sum_{\vec{q}} \frac{|f(\vec{q})|^2}{2\rho_{\omega q}/\hbar} \frac{\hbar^2 e^{i\vec{q}\cdot\vec{r}^e} |\vec{q}\Xi_{\vec{q}}|^2}{|b_{\vec{q}} h_{\vec{q}} e^{i\omega_{\vec{q}} t} + b^\dagger_{\vec{q}} h_{\vec{q}} e^{-i\omega_{\vec{q}} t}|}. \hfill (A4)$$

We consider the adiabatic condition, where the energy scale of the noise is much less than the dot confinement energy $E_d = \hbar\omega_d$ and the valley splitting, so that the electron orbital state stays in the instantaneous ground state $\psi(\vec{r}) = \text{exp} \left(-r^2 / 2\lambda^2\right) / \lambda \sqrt{\pi}$, where $\lambda^2 = h^{-1} \sqrt{(m^*\omega_d)^2 + (eB_z/2c)^2}$ is the effective radius. Then, we simplify the exponential terms $e^{i\vec{q}\cdot\vec{r}}$ by its mean-field value $e^{-\vec{q}^2 r^2/4}$.

The spectrum of the phonon noise in the $x$-direction is

$$S_{FF}(\omega) = |f_j(\omega, \theta)|^2,$$

where $N_\omega = [\exp(\hbar\omega/k_B T) - 1]^{-1}$ is the phonon excitation number and the cutoff function $f_j(\omega, \theta) = \left| f_j(\omega \cos \theta / v_j) \right|^2 e^{-\omega^2 \lambda^2 \sin^2 \theta / 2c^2}$ is due to the suppression of the matrix element for the electron–phonon interaction in a large QD. For the noise in the $y$ and $z$ direction, we have $S_{FF,y}(\omega) = S_{FF,\omega}(\omega)$ and

$$S_{FF,z}(\omega) = \sum \frac{\hbar\omega^2 (2N_\omega + 1)}{8\pi^2 \rho_c v_j^2} \int_0^{\pi/2} d\theta \sin \theta \cos^2 \theta f_j(\omega, \theta).$$

If the dipole approximation $e^{i\vec{q}\cdot\vec{r}} \approx 1 + i\vec{q}\cdot\vec{r}$ is employed (valid when the magnetic field is weak), then, we have $f_j(\omega, \theta) = 1$. Furthermore, the temperature $T$ of the lattice vibration is normally very low ($T < 1$ K), so
that $2N_e + 1 = \coth(h\omega/2k_B T) \approx 1$, in which case the spectrum of phonon noise shows a nice $\omega^5$

dependence.

**Interaction of an electron to Johnson noise or 1/f charge noise**—For the Johnson noise or 1/f charge
noise, the photon wave vector is small due to the fast speed of light. Thus, the dipole approximation is
applicable,

$$V_{\text{ex}}(\vec{r}) = -\vec{F}(t),$$  \hspace{1cm} (A5)

where $\vec{F}(t)$ is the random force due to the Johnson noise or 1/f charge noise.

**Spectral density of Johnson noise**—The spectrum of Johnson noise is given by [45]

$$S_V(\omega) = 2\xi\omega^2f_c(\omega)(\omega/2k_B T),$$  \hspace{1cm} (A6)

where $S_V$ is the spectrum of electrical voltage $S_V(\omega) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \bar{V}(0,t) \overline{V(t)} \cos(\omega t) \, dt$, $\xi = R/R_k$ is a
dimensionless constant, $R_k = h/e^2 = 26 \, k\Omega$ is the quantum resistance, and $R$ is the resistance of the circuit.

$f_c(\omega) = 1/[1 + (\omega/\omega_R)^2]$ is a natural cutoff function for Johnson noise, where $\omega_R = 1/RC$ is the cutoff
frequency, and $C$ is capacitors in parallel with the resistance $R$.

The Johnson noise of the circuits outside the dilution refrigerator is generally well-filtered. Thus we
consider only Johnson noise of the low-temperature circuit inside a dilution refrigerator. The corresponding
spectrum of electrical force is $S_{\text{FFJ}}(\omega) = S_V(\omega)/\xi l_0^2$, where $i = X, Y, \text{ or } Z$, and $l_0$ is the length scale of the QD.

**References**

[1] Loss D and DiVincenzo D P 1998 Phys. Rev. A 57 120
[2] Petta J R, Johnson A C, Taylor J M, Laird E A, Yacoby A, Lukin M D, Marcus C M, Hanson M P and Gossard A C 2005 Science 309 2180
[3] Hanson R, Kouwenhoven L P, Petta J R, Tarucha S and Vandersypen L M K 2007 Rev. Mod. Phys. 79 1217
[4] Morton J J L, McCarney D R, Eerola T M and Lyon S A 2011 Nature 479 345
[5] Zwanenburg F A, Dzurak A S, Morello A, Simmons M Y, Hollenberg L C M, Klimov Y P, Rogge S, Coppersmith S N and Eriksson M A 2013 Rev. Mod. Phys. 85 961
[6] Kloeffel C and Loss D 2013 Annu. Rev. Condens. Matter Phys. 4 51
[7] Yoneda J et al 2018 Nat. Nanotechnol. 13 102
[8] Leon R C et al 2020 Nat. Commun 11 1
[9] Mi X, Benito M, Putz S, Zajac D M, Taylor J M, Burkard G and Petta J R 2018 Nature 555 599
[10] Samkhadarz N, Zheng G, Kalhor N, Brousse D, Sammak A, Mendes U C, Blais A, Scappucci G and Vandersypen L M K 2018 Science 359 1123
[11] Landig A J et al 2018 Nature 560 179
[12] Veldhorst M et al 2015 Nature 526 410
[13] Zajac D M, Sigillito A J, Russ M, Borjans F, Taylor J M, Burkard G and Petta J R 2018 Science 359 439
[14] Watson T F et al 2018 Nature 555 633–7
[15] Yang C H et al 2020 Nature 580 350
[16] Petit L, Eenkin H G J, Russ M, Lawrie W I L, Hendrickx N W, Philips S G J, Clarke J S, Vandersypen L M K and Veldhorst M 2020 Nature 580 355
[17] Zajac D M, Hazard T M, Mi X, Wang K and Petta J R 2015 Appl. Phys. Lett. 106 223507
[18] Mills A R, Zajac D M, Gullans M J, Schupp F J, Hazard T M and Petta J R 2018 arXiv:1809.03976 [cond-mat, physicsquant-ph]
[19] Tokura Y, van der Wiel W G, Obata T and Tarucha S 2006 Phys. Rev. Lett. 96 047202
[20] Pioro-Ladrière M, Tokura Y, Obata T, Kubo T and Tarucha S 2007 Appl. Phys. Lett. 90 024105
[21] Pioro-Ladrière M, Obata T, Tokura Y, Shin Y-S, Kubo T, Yoshida K, Tanigawa S and Tanaka S 2008 Nature Phys. 4 776
[22] Kawakami E et al 2014 Nat. Nanotechnol. 9 666
[23] Wu X et al 2014 Proc. Natl. Acad. Sci. 111 11393
[24] Hu X, Liu Y-x. and Nori F 2012 Phys. Rev. B 86 035314
[25] Benito M, Petta J R and Burkard G 2019 Phys. Rev. B 100 081412(R)
[26] Borjans F, Zajac D M, Hazard T M and Petta J R 2019 Phys. Rev. Appl. 11 044063
[27] Hoffmann A et al 2020 Phys. Rev. Appl. 13 034068
[28] Struck T et al 2020 npj Quantum Inf. 6 1
[29] Zhang X et al 2020 Phys. Rev. Lett. 124 257701
[30] Li R 2019 Phys. Scr. 94 085808
[31] Neumann R and Schreiber L R 2015 J. Appl. Phys. 117 193903
[32] Kha A, Joynt R and Culcer D 2015 Appl. Phys. Lett. 107 172101
[33] Sakurai J J 1994 Modern Quantum Mechanics (Reading, MA: Addison-Wesley Developers Press)
[34] Dresselhaus M S, Dresselhaus G and Jorio A 2008 Group Theory: Application to the Physics of Condensed Matter (Berlin: Springer)
[35] Feng E H and Crooks G K 2008 Phys. Rev. Lett. 101 090602
[36] Golovach N, Borhani M and Loss D 2006 Phys. Rev. B 74 165319
[37] Golovach N, Khaetskii A and Loss D 2004 Phys. Rev. Lett. 93 016601
[38] Prada M, Klimov G and Joynt R 2011 New J. Phys. 13 013009
[39] Fordou R et al 2018 npj Quantum Inf. 4 26
[40] Jock R M et al 2018 Nat. Commun. 9 1768
[41] Tanatt V et al 2019 Phys. Rev. X 9 021028
[42] Huang P and Hu X 2014 Phys. Rev. B 89 195302
[43] Borhani M, Golovach N and Loss D 2006 Phys. Rev. B 73 155311
[44] Huang P and Hu X 2013 Phys. Rev. B 88 075301
[45] Huang P and Hu X 2014 Phys. Rev. B 90 235315
[46] Yang C H et al 2013 Nat. Commun. 4 2069
[47] Srinivasa V, Nowack K C, Shafiei M, Vandersypen L M K and Taylor J M 2013 Phys. Rev. Lett. 110 196803
[48] Tahan C and Joynt R 2005 Phys. Rev. B 71 075315
[49] Tahan C and Joynt R 2014 Phys. Rev. B 89 075302
[50] Hao X, Ruskov R, Xiao M, Tahan C and Jiang H 2014 Nat. Commun. 5 3860
[51] Khaetskii A V and Nazarov Y V 2001 Phys. Rev. B 64 125316
[52] Amasha S, MacLean K, Radu I P, Zumbuhl D M, Kastner M A, Hanson M P and Gossard A C 2008 Phys. Rev. Lett. 100 046803
[53] Hayes R R et al 2009 arXiv:0908.0173 [cond-mat]
[54] Xiao M, House M G and Jiang H W 2010 Phys. Rev. Lett. 104 096801
[55] Huang P and Hu X 2020 arXiv:2010.14844
[56] Hu X and Das Sarma S 2006 Phys. Rev. Lett. 96 100501
[57] Huang P, Zimmerman N M and Bryant G W 2018 npj Quantum Inf. 4 62
[58] Rasha E I and Efros A L 2003 Phys. Rev. Lett. 91 126405
[59] Flindt C, Sørensen A S and Flensberg K 2006 Phys. Rev. Lett. 97 240501
[60] Nowack K C, Koppens F H L, Nazarov Y V and Vandersypen L M K 2007 Science 318 1430
[61] Rasha E I 2008 Phys. Rev. B 78 195302
[62] Imamoglu A, Awschalom D D, Burkard G, DiVincenzo D P, Loss D, Sherwin M and Small A 1999 Phys. Rev. Lett. 83 4204
[63] Trif M, Troiani F, Stepanenko D and Loss D 2008 Phys. Rev. Lett. 101 217201
[64] Borjans F, Croot X G, Mi X, Guillan M J and Petta J R 2020 Nature 577 195
[65] Van Vleck J H 1940 Phys. Rev. 57 426
[66] Orbach R 1961 Proc. Phys. Soc. 77 821
[67] Tosi G, Mohiyaddin F A, Schmitt V, Tenberg S, Rahman R, Klimeck G and Morello A 2017 Nat. Commun. 8 450
[68] Yu P and Cardona M 2010 Fundamentals of Semiconductors: Physics and Materials Properties 4th edn (Berlin: Springer)