BLOWUP BEHAVIOR OF THE KÄHLER-RICCI FLOW ON FANO MANIFOLDS

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ABSTRACT. We study the blowup behavior at infinity of the normalized Kähler-Ricci flow on a Fano manifold which does not admit Kähler-Einstein metrics. We prove an estimate for the Kähler potential away from a multiplier ideal subscheme, which implies that the volume forms along the flow converge to zero locally uniformly away from the same set. Similar results are also proved for Aubin’s continuity method.

1. INTRODUCTION

Let $X$ be a Fano manifold of complex dimension $n$, which is a compact complex manifold with positive first Chern class $c_1(X)$, and let $\omega_0$ be a Kähler metric on $X$ with $[\omega_0] = c_1(X) > 0$. Consider the normalized Kähler-Ricci flow, which is a flow of Kähler metrics $\omega_t$ in $c_1(X)$ which evolve by

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \omega_t$$

with initial condition $\omega_0$. Its fixed points are Kähler-Einstein (KE) metrics $\omega_{KE}$ which satisfy $\text{Ric}(\omega_{KE}) = \omega_{KE}$, and it is known [7, 16, 20, 28] that if $X$ admits a KE metric then the flow (1.1) converges smoothly to a (possibly different) KE metric. On the other hand not every Fano manifold admits a KE metric, and a celebrated conjecture of Yau [31] predicts that this happens precisely when $(X, K_X^{-1})$ is stable in a suitable algebro-geometric sense. The precise notion of stability is K-stability, introduced by Tian [25] and refined by Donaldson [9]. Solutions of this conjecture by Chen-Donaldson-Sun [3] and Tian [26] have appeared very recently.

We will also consider a different family of Kähler metrics $\tilde{\omega}_t$ in $c_1(X)$ which solve Aubin’s continuity method [1]

$$\text{Ric}(\tilde{\omega}_t) = t\tilde{\omega}_t + (1 - t)\omega_0,$$

with $t$ ranging in an interval inside $[0, 1]$. We have that $\tilde{\omega}_0 = \omega_0$, and if (1.2) is solvable up to $t = 1$, then $\tilde{\omega}_1$ is KE. On the other hand if no KE exists then (1.2) has a solution defined on a maximal interval $[0, R(X))$ where $R(X) \leq 1$.

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is an invariant of $X$ (independent of $\omega_0$) characterized by Székelyhidi [22] as the greatest lower bound for the Ricci curvature of metrics in $c_1(X)$.

In this note we consider a Fano manifold $X$ which does not admit a KE metric, and investigate the question of the behavior in this case of the Kähler-Ricci flow (1.1) as $t \to \infty$ or of the continuity method (1.2) as $t \to R(X)$. In several recent works this question has been studied by reparametrizing the evolving metrics by diffeomorphisms and studying the geometric limiting space [12, 18, 21, 27, 29]. The key point of this note is that we do not modify the evolving metrics by diffeomorphisms, but instead we want to understand the way in which they degenerate as tensors on the fixed complex manifold $X$.

To state our main result, let us introduce some notation. The Kähler-Ricci flow (1.1) is equivalent to a flow of Kähler potentials in the following way. We have that

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{\omega^n_t}{\omega^n_0} + \varphi_t - h_0, \quad \varphi_0 = c_0,$$

where $c_0$ is a suitable constant (defined in [16, (2.10)]) and where $h_0$ is the Ricci potential of $\omega_0$ (i.e. it satisfies $\text{Ric}(\omega_0) - \omega_0 = \sqrt{-1} \partial \bar{\partial} h_0$ and $\int_X (e^{h_0} - 1) \omega^n_0 = 0$). Let us rewrite (1.3) as the following complex Monge-Ampère equation

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n = e^{h_0 - \varphi_t + \varphi_t} \omega^n_0,$$

where here and henceforth, we’ll write $\varphi_t = \frac{\partial \varphi_t}{\partial t}$. The flow (1.3) has a global solution $\varphi_t, t \geq 0$ [2], and since $X$ does not admit KE metrics, we must have $\sup_{X \times [0,\infty]} \varphi_t = \infty$ (see e.g. [16]). From now on we fix a sequence of times $t_i \to \infty$ such that

$$\sup_{X \times [0,t_i]} \varphi_t = \sup_X \varphi_{t_i} \to \infty.$$

For simplicity we’ll write $\varphi_i = \varphi_{t_i}$ and $\omega_i = \omega_{t_i}$.

On the other hand if we write $\tilde{\omega}_t = \omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_t$, then the continuity method (1.2) is equivalent to the complex Monge-Ampère equation

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_t)^n = e^{h_0 - t \tilde{\varphi}_t} \omega^n_0.$$

If $X$ does not admit KE metrics then a solution $\tilde{\varphi}_t$ exists for $t \in [0,R(X))$ with $0 < R(X) \leq 1$, and $\sup_X \tilde{\varphi}_t \to \infty$ as $t$ approaches $R(X)$. We then fix a sequence $t_i \in [0,R(X))$ with $t_i \to R(X)$ and write $\tilde{\varphi}_i = \tilde{\varphi}_{t_i}$ and $\tilde{\omega}_i = \tilde{\omega}_{t_i}$.

In [14], Nadel proved that there is a proper analytic subvariety $S \subset X$ (a suitable multiplier ideal subscheme [13]) such that the measures $\tilde{\omega}_i^n$ converge (as measures) to zero on compact subsets of $X \setminus S$. More recently, the same statement was proved for the measures $\omega_i^n$ along the Kähler-Ricci flow by Clarke-Rubinstein [5, Lemma 6.5] (see also [15] for a weaker statement). It
is natural to ask whether this convergence can be improved. In this note, we show that away from a possibly larger proper analytic subvariety the measures $\omega^n_i$ and $\tilde{\omega}_i^n$ converge to zero uniformly on compact sets. More precisely, we have:

**Theorem 1.1.** Assume that $X$ is a Fano manifold that does not admit a Kähler-Einstein metric, and let $\varphi_i, \omega_i$ be defined as above. Then for any $\varepsilon > 0$ there is a proper nonempty analytic subvariety $S_\varepsilon \subset X$ and a subsequence of $\varphi_i$ (still denoted by $\varphi_i$) such that given any compact set $K \subset X \setminus S_\varepsilon$ there is a constant $C$ that depends only on $K, \varepsilon, \omega_0$ such that for all $x \in K$ and for all $i$ we have

$$-(\varphi_i(x) + (1 - \varepsilon) \sup_X \varphi_i) \leq C.$$  

In particular, $\varphi_i$ goes to plus infinity locally uniformly outside $S_\varepsilon$, and the volume forms $\omega_0^n$ converge to zero in the same sense. Finally, the same properties hold for $\tilde{\varphi}_i$ and $\tilde{\omega}_i$ which solve Aubin’s continuity method.

Moreover we can identify the subvariety $S_\varepsilon$ as follows: from weak compactness of currents, there exists $\psi$ an $L^1$ function on $X$ which is $\omega_0$-plurisubharmonic, such that a subsequence of $\varphi_i - \sup M \varphi_i$ converges to $\psi$ in $L^1$. Then we have that

$$S_\varepsilon = V \left( I \left( \frac{C}{\varepsilon} \psi \right) \right),$$

where $C$ is a constant that depends only on $\omega_0$, and $I$ denotes the multiplier ideal sheaf. We note here that in the results of [14] and [5] the multiplier ideal sheaf that enters is $I(\gamma \psi)$, with $\frac{n}{n+1} < \gamma < 1$, which gives a smaller subvariety.

Finally let us remark that we expect Theorem 1.1 to hold also when $\varepsilon = 0$, but our arguments below can only prove this when $n = 1$ (in which case the theorem is empty because there is just one Fano manifold, $\mathbb{C}P^1$, which does admit a KE metric). In fact more should be true: Tian’s conjectural “partial $C^0$ estimate” for the continuity method [26] roughly says that $-\tilde{\varphi}_i + \sup_X \tilde{\varphi}_i$ should blow up at most logarithmically as we approach a subvariety. The partial $C^0$ estimate was proved by Tian [24] for Kähler-Einstein Fano surfaces and more recently by Chen-Wang [4] for the Kähler-Ricci flow on Fano surfaces. Very recently it was proved by Donaldson-Sun [10] for Kähler-Einstein metrics on Fano manifolds, by Chen-Donaldson-Sun [3] and Tian [26] for conic Kähler-Einstein metrics and by Phong-Song-Sturm [17] for shrinking Kähler-Ricci solitons.

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2. Proof of the main theorem

Before we start the proof of the main theorem, we need to recall a few estimates which are known to hold along the Kähler-Ricci flow on Fano manifolds. The first one is the bound

\[ |\dot{\varphi}_t| \leq C, \]

which holds for all \( t \geq 0 \), and was proved by Perelman (see [20]). It uses crucially the choice of \( c_0 \) in (1.3) given by [16, (2.10)]. We will also need the following uniform Sobolev inequality [32, 33]

\[ \left( \int_X |f|^{2n} \omega^n \right)^{n-1} \leq C S \left( \int_X |\nabla f|^2 \omega_t^n + \int_X |f|^2 \omega_t^n \right), \]

which holds for all \( t \geq 0 \) and for all \( f \in C^\infty(X) \), for a constant \( C_S \) that depends only on \( \omega_0 \). The following Harnack inequality [19] will also be used

\[ -\inf_X \varphi_t \leq C + n \sup_X \varphi_t, \]

which again holds for all \( t \geq 0 \). Finally, we will use the following basic result:

**Proposition 2.1** (Tian [23]). Let \((X, \omega)\) be a compact Kähler manifold. For any fixed \( \lambda > 0 \) and for any sequence \( \varphi_i \) of Kähler potentials for \( \omega \), there exists a subsequence, still denoted by \( \varphi_i \), and a proper subvariety \( S \subset X \) such that for any \( p \in X \setminus S \) there exists \( r, C > 0 \) that depend only on \( \lambda, \omega \) and \( p \), such that

\[ \int_{B_\omega(p,r)} e^{-\lambda (\varphi_i - \sup_X \varphi_i)} \omega^n \leq C. \]

Also, the constants \( r, C \) are uniform when \( p \) ranges in a compact set of \( X \setminus S \).

**Proof of Theorem 1.1.** The starting point is the parabolic analogue of the Aubin-Yau’s \( C^2 \) estimate (see e.g. [2, 30]), which says that there exists a constant \( C \) that depend only on \( \omega_0 \) such that for all \( t \geq 0 \) we have

\[ \text{tr}_{\omega_0} \omega_t \leq C e^{C \varphi_t - (C+1) \inf_{X \times [0,t]} \varphi_s}. \]

Here and in the following we will denote by \( C \) a uniform positive constant which might change from line to line. Combining (2.12) with (2.10) and (1.5) we get

\[ \text{tr}_{\omega_0} \omega_t \leq C e^{C \sup_X \varphi_i + C \sup_{X \times [0,t]} \varphi_s} = C e^{C \sup_X \varphi_i}. \]
Notice that for any two Kähler metrics $\eta, \chi$ we always have that
\[
\text{tr}\eta \chi \leq \frac{1}{(n-1)!} (\text{tr}_\chi \eta)^{n-1} \chi^n
\]
and so in our case
\[
\text{tr}_\omega \omega_i \leq C e^{C \sup_X \varphi_i \omega_i^n / \omega_i^n}.
\]
Using the Monge–Ampère equation (1.14) and the estimate (2.8) we get
\[(2.13) \quad \text{tr}_\omega \omega_i \omega_0 \leq C e^{C \sup_X \varphi_i \omega_i^n / \omega_i^n},
\]
where $C_0$ and $D$ are uniform constants. An alternative derivation of (2.13) can be obtained by evolving the quantity $\log \text{tr}_\omega \omega_i \omega_0 - A \varphi_t$, with $A$ large, to obtain
\[
\text{tr}_\omega \omega_i \omega_0 \leq C e^{A (\varphi_t - \inf_{X \times [0,t]} \varphi_s)},
\]
and using again (2.10). As an aside, note that when $n = 1$ we can actually choose $D = 1$.

Let $A > 0$ be a constant, to be determined later, and compute
\[
(2.14) \quad \Delta \omega_i e^{-A \varphi_i} \geq -A e^{-A \varphi_i} \Delta \omega_i \varphi_i \geq -n A e^{-A \varphi_i}.
\]

**Proposition 2.2.** For any $x \in X$ and $0 < r < \text{diam}(X, \omega_i)/2$ there is a constant $C$ that depends only on $A, \omega_0$ such that
\[
(2.15) \quad \sup_{B_{\omega_i}(x,r/2)} e^{-A \varphi_i} \leq \frac{C}{r^{2n}} \int_{B_{\omega_i}(x,r)} e^{-A \varphi_i} \omega_i^n,
\]
holds for all $i$.

**Proof.** We apply the method of Moser iteration to the inequality (2.14). The method is standard, except for the fact that $r$ could be bigger than the injectivity radius of $\omega_i$ and so the balls $B_{\omega_i}(x, r)$ need not be diffeomorphic to Euclidean balls, and in fact might not even be smooth domains. From now on let $i$ be fixed, and fix two positive numbers $\rho < R < \text{diam}(X, \omega_i)/2$. Then let $\eta$ be a cutoff function of the form $\eta(y) = \psi(\text{dist}_{\omega_i}(y, x))$ where $\psi$ is a smooth nonincreasing function from $\mathbb{R}$ to $\mathbb{R}$ such that $\psi(y) = 1$ for $y \leq \rho$, $\psi(y) = 0$ for $y \geq (R + \rho)/2$ and
\[
\sup_{\mathbb{R}} |\psi'| \leq \frac{4}{R - \rho}.
\]
Then $\eta$ is Lipschitz, equal to 1 on $B_{\omega_i}(x, \rho)$, supported inside $B_{\omega_i}(x, R)$ and satisfies
\[
|\nabla \eta|_{\omega_i} \leq \frac{4}{R - \rho}
\]
almost everywhere. For simplicity of notation we will let \( f = e^{-A\varphi} \) and suppress all references to the metric \( \omega_i \). So we can write (2.14) as
\[
(2.16) \quad \Delta f \geq -nAf.
\]
Then for any \( p \geq 2 \) we compute
\[
\int_{B(x,R)} \eta^2 |\nabla (f^{p/2})|^2 = \frac{p^2}{4(p-1)} \int_{B(x,R)} \eta^2 \langle \nabla (f^{p-1}), \nabla f \rangle.
\]
At this point we want to integrate by parts, and we can do this because of the following argument: we can exhaust \( B(x,R) \) with an increasing sequence of subdomains \( B_j, j = 1, 2, \ldots \), that have smooth boundary. Then we can apply Stokes' Theorem to each \( B_j \), and when \( j \) is sufficiently large \( \eta \) will vanish on \( \partial B_j \) so we get
\[
\int_{B_j} \eta^2 \langle \nabla (f^{p-1}), \nabla f \rangle = -2 \int_{B_j} \eta f^{p-1} \langle \nabla \eta, \nabla f \rangle - \int_{B_j} \eta^2 f^{p-1} \Delta f.
\]
Then we can let \( j \) go to infinity and by dominated convergence the integrals on \( B_j \) converge to the same integrals on \( B(x,R) \). Thus
\[
\int_{B(x,R)} \eta^2 |\nabla (f^{p/2})|^2 = -2 \int_{B(x,R)} \eta f^{p-1} \langle \nabla \eta, \nabla f \rangle + \eta \Delta f.
\]
Using (2.16) and the Cauchy-Schwarz and Young inequalities we have that
\[
\int_{B(x,R)} \eta^2 |\nabla (f^{p/2})|^2 \leq Cp \int_{B(x,R)} \eta^2 f^p + p \int_{B(x,R)} f^p |\nabla \eta|^2
\]
\[
+ \frac{p}{4} \int_{B(x,R)} \eta^2 f^{p-2} |\nabla f|^2,
\]
where \( C \) depends only on \( A, n \).

The last term in (2.17) is equal to \( \frac{1}{p} \int_{B(x,R)} \eta^2 |\nabla (f^{p/2})|^2 \) and so can be absorbed in the left hand side. We thus get
\[
\int_{B(x,R)} \eta^2 |\nabla (f^{p/2})|^2 \leq Cp \int_{B(x,R)} \eta^2 f^p + \frac{Cp}{(R - \rho)^2} \int_{B(x,R)} f^p
\]
\[
\leq \frac{Cp}{(R - \rho)^2} \int_{B(x,R)} f^p,
\]
as long as \( R - \rho \) is small. Now we use the Cauchy-Schwarz and Young inequalities again to bound
\[
\int_{B(x,R)} |\nabla (f^{p/2})|^2 \leq 2 \int_{B(x,R)} \eta^2 |\nabla (f^{p/2})|^2 + 2 \int_{B(x,R)} f^p |\nabla \eta|^2
\]
\[
\leq 2 \int_{B(x,R)} \eta^2 |\nabla (f^{p/2})|^2 + \frac{C}{(R - \rho)^2} \int_{B(x,R)} f^p.
\]
Combining (2.18) and (2.19) we have
\[ \int_{B(x,R)} |\nabla(\eta f^{p/2})|^2 \leq \frac{C_p}{(R-\rho)^2} \int_{B(x,R)} f^p. \]
This together with the Sobolev inequality (2.9) gives
\[ \left( \int_{B(x,R)} \eta^{2\beta} f^{p^{1/p}} \right)^{1/\beta} \leq \frac{C}{(R-\rho)^2} \int_{B(x,R)} f^p, \]
where we write \( \beta = n/(n-1) > 1 \). Raising this to the \( 1/p \) gives
\[ \left( \int_{B(x,\rho)} f^{p\beta} \right)^{1/p} \leq \frac{C^{1/p} p^{1/p}}{(R-\rho)^{2/p}} \left( \int_{B(x,R)} f^p \right)^{1/p}. \]
For each \( j \geq 0 \) we now set \( p_j = 2\beta^j \) and \( R_j = \rho + \frac{R-\rho}{2^j} \). Setting \( p = p_j, R = R_j, \rho = R_{j+1} \) in (2.21) and iterating (notice that \( (R_{j+1} - R_j) = (R-\rho)2^{-j-1} \) is small) we easily get
\[ \sup_{B(x,\rho)} f \leq \frac{C}{(R-\rho)^n} \left( \int_{B(x,R)} f^2 \right)^{1/2} \leq \frac{C}{(R-\rho)^n} \left( \sup_{B(x,R)} f \right)^{1/2} \left( \int_{B(x,R)} f \right)^{1/2}. \]
Using Young’s inequality we see that
\[ \sup_{B(x,\rho)} f \leq \frac{1}{2} \sup_{B(x,R)} f + \frac{C}{(R-\rho)^{2n}} \int_{B(x,R)} f. \]
From here a standard iteration argument (see e.g. [11] Lemma 3.4) implies that
\[ \sup_{B(x,\rho)} f \leq \frac{C}{(R-\rho)^{2n}} \int_{B(x,R)} f, \]
and finally setting \( \rho = r/2, R = r \) we get (2.15).

We now apply Proposition 2.1 with \( \lambda = A + 1 \) and get an analytic subvariety \( S \) with the property that given any compact set \( K \subset X \setminus S \) there exists \( r_0 > 0 \) such that for any \( x \in K \) we have that \( B_{r_0}(x) \subset X \setminus S \) and
\[ \int_{B_{r_0}(x)} e^{-(A+1)(\varphi_1 - \sup_X \varphi_1)n} \omega_0^n \leq C. \]
holds for all $i$. We now let $r_i = r_0(C_0e^{D_{\sup X} \phi_i})^{-1/2}$, so that (2.13) implies that
\begin{equation}
B_{\omega_i}(x, r_i) \subset B_{\omega_0}(x, r_0),
\end{equation}
so in particular $r_i < \text{diam}(X, \omega_i)/2$. Therefore we can apply Proposition 2.2, and combining (2.15), (2.22) and (2.23) we obtain
\begin{align*}
e^{-A\phi_i(x)} &\leq \sup_{B_{\omega_i}(x, r_i/2)} e^{-A\phi_i} \\
&\leq \frac{C}{r_i^{2n}} \int_{B_{\omega_i}(x, r_i)} e^{-A\phi_i} \omega_i^n \\
&\leq Ce^{nD_{\sup X} \phi_i} \int_{B_{\omega_0}(x, r_0)} e^{-(A+1)\phi_i} \omega_0^n \\
&= Ce^{(nD - A - 1)_{\sup X} \phi_i} \int_{B_{\omega_0}(x, r_0)} e^{-(A+1)(\phi_i - \sup_X \phi_i)} \omega_0^n \\
&\leq Ce^{(nD - A - 1)_{\sup X} \phi_i},
\end{align*}
where $C$ depends only on given data and $A$. Taking log gives
\begin{align*}
-A\phi_i(x) + (A - nD + 1) \sup_X \phi_i &\leq C.
\end{align*}
We now let $A = \frac{nD}{\varepsilon}$ and divide by $A$ (keeping in mind that $\sup_X \phi_i > 0$), and obtain the desired bound
\begin{align*}
-\phi_i(x) + (1 - \varepsilon) \sup_X \phi_i &\leq C,
\end{align*}
where $C$ does not depend on $i$ or on $x \in K \subset X \setminus S$, and where the subvariety $S = S_{\varepsilon}$ now depends on $\varepsilon$. This, together with (2.8), immediately implies that the volume form $\omega_i^n = \omega_0^n e^{h_0 - \phi_i + \dot{\phi}_i}$ goes to zero locally uniformly on $X \setminus S$.

We now identify the subvariety $S_{\varepsilon}$: from its definition, that is from Proposition 2.1, we see that $S_{\varepsilon}$ is equal to the multiplier ideal subscheme of the sequence $\phi_i$ with exponent $(nD/\varepsilon + 1)$ as defined by Nadel in [13]. But the main Theorem in [8] then shows that this is the same as the multiplier ideal subscheme defined by $(nD/\varepsilon + 1)\psi$ where $\psi$ is a weak limit of $\phi_i - \sup_M \phi_i$.

The same proof as above goes through with minimal changes in the case of Aubin’s continuity method, that is for the functions $\tilde{\phi}_i$ and the metrics $\tilde{\omega}_i$. We just need to justify why estimates analogous to (2.9) and (2.10) hold. To see these, note that the metrics $\tilde{\omega}_i$ satisfy $\text{Ric}(\tilde{\omega}_i) \geq C^{-1}\tilde{\omega}_i > 0$, so the Bonnet-Myers theorem gives us the estimate $\text{diam}(X, \tilde{\omega}_i) \leq C$, which together with the fact that their volume is fixed allows us to apply a result
of Croke [6] which gives a uniform Sobolev inequality of the form (2.9) for the metrics $\tilde{\omega}_i$. The Harnack inequality (2.10) in this case is proved in [23].

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