CONSISTENT INTERACTIONS BETWEEN GAUGE FIELDS 
AND LOCAL BRST COHOMOLOGY: 
THE EXAMPLE OF YANG-MILLS MODELS∗

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ABSTRACT
Recent results on the cohomological reformulation of the problem of consistent interactions between gauge fields are illustrated in the case of the Yang-Mills models. By evaluating the local BRST cohomology through descent equation techniques, it is shown (i) that there is a unique local, Poincaré invariant cubic vertex for free gauge vector fields which preserves the number of gauge symmetries to first order in the coupling constant; and (ii) that consistency to second order in the coupling constant requires the structure constants appearing in the cubic vertex to fulfill the Jacobi identity. The known uniqueness of the Yang-Mills coupling is therefore rederived through cohomological arguments.

1. Introduction
Consider a free gauge theory with action
\[ (0) S_0[\varphi^i] \] and gauge symmetry given by
\[ \delta \varepsilon \varphi^i = R^{(0)}_{i\alpha} \varepsilon^{(0)}_{\alpha}, \quad \frac{\delta S_0}{\delta \varphi^i} R^{(0)}_{i\alpha} = 0. \] (1)
The gauge symmetry removes unphysical degrees of freedom which would otherwise either make the theory unstable or introduce negative norm states. The question investigated here is whether one can introduce couplings among the fields \( \varphi^i \) which fulfill the crucial physical requirement of preserving the number of gauge symmetries. Interactions fulfilling this condition will be said to be “consistent” [other consistency requirements such as causal physical propagation may have to be imposed, but this question will not be studied here].

In a recent paper, it has been shown that the problem of constructing consistent interactions can be naturally reformulated as a deformation problem of the solution

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of the master equation. This is so because the master equation contains all the information about the gauge structure of the theory. If a consistent interacting gauge theory can be constructed, then the solution \( S = S_0 + \text{"antifield contributions"} \) of the master equation for the free theory can be deformed into a solution \( S \),

\[
S = S_0 + g S_1 + g^2 S_2 + \ldots
\]

\((S, S) = 0, \quad (3)\)

of the master equation for the deformed theory which has the same spectrum of ghosts and antifields.

Eq. (3) can be analyzed order by order in the deformation parameter \( g \) (= coupling constant), leading to

\[
(0) S, (0) S = 0 \quad (4)
\]

\[
2(0) S, (1) S = 0 \quad (5)
\]

\[
2(0) S, (2) S + (1) S, (1) S = 0 \quad (6)
\]

\[
\ldots
\]

Eq. (4) is the master equation for the given free gauge theory and is thus satisfied. Eq. (5) requires \( (1) S \) to be a cocycle of the free BRST differential \( (0) s \equiv (, S) \). Only cohomologically non trivial solutions of (5) have to be considered, because BRST exact solutions \( (1) S = g(K, (1) S) \) correspond to interactions obtained from the free theory by making non-linear field redefinitions. Hence \( (1) S \) belongs to \( H^0(0) s \), the ghost number zero cohomological space of \( (0) s \). This space is known to be isomorphic with the space of physical observables of the free theory, i.e., the equivalence classes of on-shell gauge invariant functionals that coincides when the equations of motion of the free theory hold.

Turn now to eq. (6). Under very general regularity assumptions, the antibracket can be shown to be trivial in cohomology, i.e., the antibracket of two BRST closed functionals is BRST exact. This means that \( (1) S, (1) S \) is BRST exact, so that eq. (6) represents no obstruction to continuing the construction of the interacting action and that \( (2) S \) exists. Similarly, one finds that the higher order equations can also be satisfied and that the construction of an interacting theory starting from an element \( (1) S \in H^0(0) s \) is unobstructed.

However, when starting from a local free theory, one restricts the deformations \( S_1, S_2, \ldots \) to be local functionals as well. The above results do not take locality into account. If one imposes locality of the interacting action, the analysis gets much more involved because the antibracket of two local BRST cocycles need not be in general the BRST variation of a local functional. The obstructions that can arise have been illustrated in in the case of the abelian Chern-Simons models, where the local BRST cohomology can be completely analyzed because the equations of motion are trivial and there is no local physical degree of freedom. In this note,
we want to illustrate the usefulness of the cohomological techniques in the case of the deformation of free abelian vector fields. We prove the uniqueness of the Yang-Mills coupling, recovering thereby in a different fashion results of\textsuperscript{2,3,9,10}. The same techniques can be applied, in principle, to any other free gauge theory.

2. The Free Model

The action for several abelian vector fields in Minkowski space is given by

\[
S_0 = \frac{1}{4} \int d^4x \, k_{ab} F^{\mu\nu a} F_{\mu\nu}^b, \quad F^{\mu\nu a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a
\]

where \( k_{ab} \) is a non-degenerate, symmetric and constant matrix. This matrix must be positive definite in order for the physical Hamiltonian to be bounded from below and so we take \( k_{ab} = \delta_{ab} \). The equations of motion are \( \partial^\nu F_{\nu\mu}^a = 0 \). The minimal solution to the classical master equation is

\[
S = S_0 + \int d^3x \, A_\mu^a \partial_\mu C^a,
\]

leading to the BRST symmetry

\[
\mathcal{S} = \partial_\nu F_{\nu\mu}^b k_{ba} \frac{\partial}{\partial A_\mu^b} - \partial_\mu A_\mu^a \frac{\partial}{\partial C^a} + \partial_\mu C^a \frac{\partial}{\partial A_\mu^a}.
\]

One has \( \mathcal{S}^2 = 0 \), \([\mathcal{S}, \partial_\mu] = 0 \) and \( \mathcal{S} d + d \mathcal{S} = 0 \), where \( d \) is the exterior spacetime derivative.

3. Descent Equations and Construction of the Interaction Vertex

In this section, we determine all the local interaction vertices consistent to first order in the coupling constant \( g \), i.e., we analyze eq.\textsuperscript{5} with the further requirement that \( \mathcal{S} \) should be a local functional. Consistency to order \( g^2 \) (and higher) is analyzed next.

The condition \( \mathcal{S} = \int_M \mathcal{L} \in H^0(\mathcal{S}) \) implies for the integrand \( \mathcal{L} \)

\[
\mathcal{S} \mathcal{L} + da_{[3]} = 0
\]

Here, \( \mathcal{L} \) (respectively \( a_{[3]} \)) is a 4-form (respectively 3-form) valued polynomial in the fields, the antifields and a finite number of their derivatives. Furthermore, trivial interactions are eliminated by making the identification \( \mathcal{S} \mathcal{L} \sim \mathcal{L} + \mathcal{S} b_{[4]} + dc_{[3]} \), i.e., \( \mathcal{L} \in H^0(\mathcal{S})[d] \). Eq.\textsuperscript{10} gives rise to a set of descent equations because of the algebraic Poincaré lemma:\n
\[
\begin{align*}
(0) \mathcal{S} \mathcal{L} + da_{[3]} &= 0 \\
(0) a_{[3]} + da_{[2]} &= 0 \\
\vdots \\
(0) a_{[n]} &= 0
\end{align*}
\]
with $0 \leq b \leq 4$ and $a_{[4]} \equiv \mathcal{L}$. As in \([11]\) the analysis of eq.\((10)\) contains two steps:

(i) First, one determines all possible inequivalent last elements $a_{[b]}$ of the descent consistent with Poincaré covariance. To simplify the analysis, we impose the additional condition that the coupling constant $g$ has non positive dimension of length.

(ii) Second, one determines which last element $a_{[b]}$ can be lifted all the way up to the top of the ladder \([11-13]\) to yield a solution of eq.\((10)\). \([As it is known, not all solutions of \((13)\) come from a descent, there may be obstructions].\)

Both steps turn out to be immediate.

**Step 1:**

The last element $a_{[b]}$ of the descent has to be a non-trivial element of $H^g(0)$, with $g \geq 0$. Taking into account the triviality of the Koszul-Tate differential $\delta$ at positive antifield number and the results of \([12, 13, 14]\), one gets

$$a_{[b]} = a'_{[b]}([F^a_{\mu\nu}], C^a) + sc_{[b]}$$

where the notation $[\varphi^A]$ stands for $\varphi^A, \partial_\mu \varphi^A, \partial_\mu \partial_\nu \varphi^A,...$. The BRST exact term corresponds to trivial interactions and can be dropped. Translation invariance imposes that $a'_{[b]}$ should not depend on the coordinate $x^\mu$. If $\mathcal{L}$ is a cubic vertex, with ghost number zero and form degree 4, the descent can have length at most three. Furthermore, $a_{(D)}$ should have dimension (length) $-k$ with $k \leq D$ and be of ghost number $4 - D$. These requirements, together with the condition of Poincaré covariance, are easily seen to restrict non trivial descents to have length 2, with last element given by

$$a_{[2]} = -\frac{1}{48} f_{abc} F^a C^b C^c \quad \text{or} \quad a'_{[2]} = -\frac{1}{48} f_{abc} C^a C^b C^c.$$  

At this stage, $f_{abc}$ is required only to be antisymmetric in its last two indices.

**Step 2:**

One can easily lift $a_{[2]}$ and $a'_{[2]}$ once. This yields

$$a_{[3]} = -\frac{1}{24} f_{abc} F^a A^b C^c, \quad a'_{[3]} = -\frac{1}{24} f_{abc} * F^a A^b C^c + f_{abc} A^d C^b C^c.$$  

To lift $a_{[2]}$ or $a'_{[2]}$ once more requires $f_{abc}$ to be completely antisymmetric. Otherwise $\mathcal{L}$ does not exist. But if $f_{abc}$ is completely antisymmetric, then $a_{[2]}$ is BRST exact modulo $d$,

$$a_{[2]} = \frac{1}{48} f_{abc} A^a A^b A^c + d(-\frac{1}{48} f_{abc} A^a A^b C^c),$$

and can be discarded. Only $a'_{[2]}$ defines non-trivial interactions, with $\mathcal{L}$ explicitly given by

$$\mathcal{L} = f_{abc} \left( \frac{1}{2} F^{\mu \nu a} A^b A^c - A^* A^d A^b C^c - \frac{1}{2} C^d k^c A^b C^c + \frac{1}{2} C^d k^d C^b C^c \right) d^4 x.$$  

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\(4\) Consistent interactions and local BRST cohomology . . .
The antifield independent part\(^a\) of \(\hat{\mathcal{L}}\) is the standard Yang-Mills cubic vertex except that, so far, \(f_{abc}\) is not yet constrained to fulfill the Jacobi identity. The term linear in \(A^\mu_a\) determines the modified gauge transformations to order \(g\). Finally, the term linear in \(C^\alpha_a\) shows that the modified gauge transformations close to order \(g\) with structure constants equal to \(f^a_{bc}\).

4. Jacobi Identity

The existence of \(\hat{\mathcal{L}}\) is equivalent to the consistency of the interaction up to order \(g\). The interaction is then consistent also to order \(g^2\) if and only if \(\hat{\mathcal{L}}\) exists, i.e., if and only if one can satisfy eq. (6) above with \(\hat{\mathcal{L}} = \int \hat{\mathcal{L}}\) a local functional. This is where the Jacobi identity comes in. Indeed, in order to satisfy eq. (6) in the space of local functionals, the integrand of \((\hat{\mathcal{S}}, \hat{\mathcal{S}})\), which is

\[
2(-A^\mu_a f^a_{cc} C^c + F^\mu_a A^\nu_c f_{acc} + \partial_\nu (A^\mu_a A^c_\nu f_{ace}) (-A^b_\mu C^d f^e_{bd}) + (f^{ac} C^c + A^\mu_a A^\nu_c f_{ace}) f^e_{bd} C^d C^d \tag{19}
\]

(up to a total divergence) must be equal to \(-\frac{(\hat{\mathcal{S}}, \hat{\mathcal{S}})}{2} + \partial_\mu j^\mu\).

The term \(\frac{1}{2} f^{ac} C^c f_{bd} C^b C^d\) cannot be of that form, and hence must vanish. This is the case if and only if the structure constants satisfy the Jacobi identity. The terms of antifield number 1 then also vanish and \(\hat{\mathcal{L}}\) can be taken to be \(-\frac{1}{4} A^\mu_a A^\nu_c f^e_{ac} A^\mu_b A^d f_{ebd}\), the well-known Yang-Mills quartic coupling. The higher order equations are then satisfied with \(\hat{\mathcal{L}} = \hat{\mathcal{L}} = \ldots = 0\). The construction yields therefore the standard non-abelian Yang-Mills models\(^b\) as unique renormalisable, Poincaré covariant deformations of several abelian vector fields.

5. Conclusions

In this paper, we have demonstrated the uniqueness of the Yang-Mills vertex. We believe that our approach is more interesting than the result itself, which has been indeed already derived in the literature\(^b\). We have shown how the existence of consistent couplings can be reformulated in terms of various cohomologies, namely, the cohomology of the free BRST differential \(\hat{\mathcal{S}}\), the cohomology of \(d\), and the cohomology of \((\hat{\mathcal{S}})\) modulo \(d\), for which various calculational tools have been developed in the context of the algebraic study of anomalies.

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\(^a\)As it is well known, all the information about a BRST cocycle \(A\) is contained in its antifield independent part \(A_0\), of which \(A\) is merely a “BRST invariant extension”. This is a standard result of Homological Perturbation Theory (see e.g.\(^b\)).

\(^b\)It is easy to see that there are no quartic or higher order vertices satisfying the above requirements.
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