The Classical Inverse Problem for Multi-Particle Densities in the Canonical Ensemble Formulation

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Abstract: We provide sufficient conditions for the solution of the classical inverse problem in the canonical distribution for multi-particle densities. Specifically, we show that there exists a unique potential in the form of a sum of $m$-particle ($m \geq 2$) interactions producing a given $m$-particle density. Results are obtained for systems of single and several species of particles, with a weaker uniqueness statement for the latter. The validity of the multi-particle inverse conjecture is essential for the numerical simulations of matter using effective potentials derived from structural data. Such potentials are often employed in coarse-grained modeling. For the grand canonical distribution, the multi-particle inverse problem has been solved by Chayes and Chayes \cite{Chayes}. However, the setting of the canonical ensemble presents unique challenges arising from the impossibility of uncoupling interactions when the number of particles is fixed.

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1. Introduction

Consider a system of \( N \) identical particles with coordinates \( x_1, \ldots, x_N \) in some complete \( \sigma \)-finite measure space \((\Lambda; dx)\) with non-zero measure \( dx \). In the simplest case, \( \Lambda \subset \mathbb{R}^n, n = 1, 2, \text{ or } 3 \), \( x_i \) is the position of the \( i^{th} \) particle, and \( dx \) is the Lebesgue measure. However, the results here are not limited to this case and apply whenever the conditions stated in the paper are satisfied. Particularly, \((\Lambda; dx)\) may be momentum, phase, or any other space of internal variables. The completion of the product measure on \( \Lambda_k, 1 \leq k \leq N \), is denoted by \( d^k x \).

We begin with a somewhat formal description of the problem. The total potential of the system has the form \( W + U \), where \( W(x_1, \ldots, x_N) \) is a fixed scalar internal potential, and \( U \) is an additional internal or external potential. Since the particles in the system are identical, \( W \) and \( U \) are required to be symmetric functions.

For \( 1 \leq m \leq N \), the classical \( m \)-particle density in the canonical distribution is defined (up to a multiplicative constant) as \[ \rho^{(m)}(x_1, \ldots, x_m) = \frac{\int_{\Lambda^{N-m}} e^{-W-U} dx_{m+1} \cdots dx_N}{Z(U)}, \tag{1.1} \]

where \[ Z(U) = \int_{\Lambda^N} e^{-W-U} d^N x \tag{1.2} \]
is the canonical partition function. (The conditions ensuring that \( 0 < Z(U) < \infty \) will be specified later.) Throughout the paper the inverse temperature \( \beta \) is taken to be 1.

The \( m \)-particle inverse problem investigated here is whether, given a symmetric and positive function \( \rho^{(m)} \) on \( \Lambda^m \) with \( \int_{\Lambda^m} \rho^{(m)} d^m x = 1 \), there exists a
A unique potential $U$ of the form

$$U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m}) \text{ a.e.},$$

(1.3)

such that $\rho_{U}^{(m)} = \rho^{(m)}$ a.e. on $\Lambda^m$. In (1.3), $u$ is a symmetric function on $\Lambda^m$, so $U$ is sought as a sum of symmetric $m$-particle interactions. A generalization of this problem to systems of several species (mixtures) is considered in Section 6.

When $m = 1$ the inverse problem originates from density functional theory for inhomogeneous fluids [3]. It was solved for this case by Chayes et al. [4] Sections 2 and 8, for both the canonical and grand canonical ensembles. When $m \geq 2$, but only in the grand canonical ensemble, it was solved by Chayes and Chayes [1] when $\Lambda$ has finite $dx$ measure.

An immediate application of the multi-particle ($m \geq 2$) inverse problem is to the numerical modeling of liquid solutions and soft matter, particularly coarse-grained (CG) modeling. The effective ”particles” in a CG system are interaction sites that represent groups of atoms. The idea is to reduce the number of sites to render the simulation more efficient. In a typical modeling of a CG liquid, an all-atom simulation is run to obtain structural data on the CG sites, for example a set of radial distribution functions (RDFs). (RDF is a two-particle density in a homogenous and isotropic liquid, and so it depends only on the distance between the two particles [2]. The generalized RDFs, which are multi-particle densities depending on one or several parameters, can also be obtained [5].) The (generalized) RDFs are then ”inverted” to find (multi-) two-particle interactions between CG sites.

1 The uniqueness of $U$ implies uniqueness of $u$. See Section 4.
The existence and uniqueness of such interactions have been implicitly assumed by the long established numerical methods designed for calculating effective potentials from structural data, such as iterative Boltzmann inversion (IBI) and inverse Monte Carlo (IMC). (For the description of basic methods for deducing CG potentials and a survey of literature on the subject, see a recent review by Noid and numerous references therein.)

However, to the best of our knowledge, the inversion procedure has never been justified for the canonical ensemble, even though numerical simulations are often performed in this setting. Instead, it has been erroneously assumed that the conclusions of [1] are equally valid for the canonical distribution. This is a serious misconception that has been persisting in literature for decades. The setting of the grand canonical ensemble eliminates major difficulties that arise when the number $N$ of particles is fixed.

In this paper, we provide sufficient conditions for the existence and uniqueness of the solutions to the inverse problem in the canonical distribution when $m \leq N$. These results are summarized in Theorems 3.10 and 6.3. (Note, that for $m = N$, the solution is trivial. Namely, $U = -\log \rho^{(N)} - W + C$, where $C \in \mathbb{R}$ is any constant.) To the best of our knowledge, there is no mathematical treatment of the $m \geq 2$ case in that setting. The uniqueness -but not existence- was proved by Henderson [8] when $m = 2$.

This paper is organized as follows. In Section 3 we use, for the most part, a suitable modification of the arguments found in [4, Sections 2 and 6]. The material involved in Sections 4, 5, and most of Section 6 is purely measure-theoretical and apparently new. It deals with the matters specific to the canonical ensemble that can be completely avoided in the grand canonical ensemble setting.
Theorem 4.1 and Corollary 4.3 can be considered as general mathematical results which can probably find applications outside of the inverse problem. Theorem 6.4 and Corollary 6.5 are generalizations of weaker versions of Theorem 4.1 and Corollary 4.3 respectively to several species.

2. Preliminaries

In this section we introduce some terminology, state the assumptions, and prove some simple, but very important, facts that will be used repeatedly throughout the paper.

All functions considered take values in the extended real line $\mathbb{R}$, i.e. in $(-\infty, +\infty]$, unless specified otherwise.

The complement of a set $E \subset A^k$ is denoted $E^c = A^k \setminus E$, and $|E| = \int_E d^k x$ is the $d^k x$ measure of a measurable set $E \subset A^k$. “Almost everywhere” ("a.e.") is always understood relative to the measure $d^k x$, with $k$ obvious from the context, and the same is true regarding the measurability of functions. Subsets of $A^k$ of $d^k x$ measure 0 will be called null sets. Since reference will be very often made to the complements of the null sets, we will call such complements co-null. Thus, a property depending upon the points of $A^k$ holds a.e. if and only if it holds at the points of a co-null set of $A^k$. Note that such a subset is never empty since $A^k$ has strictly positive $d^k x$ measure from the assumption that $dx$ is not the 0 measure on $A$.

A set $S \subset A^k$ is called symmetric if $(x_1, ..., x_k) \in S$ implies that $(x_{\pi(1)}, ..., x_{\pi(k)}) \in S$ for every permutation $\pi$ of $\{1, ..., k\}$. A function $f$ on $A^k$ is called symmetric a.e. if the complement of the set $E := \{(x_1, ..., x_k) \in A^k : f(x_1, ..., x_k) =$
for every permutation \( \pi \) of \( \{1, ..., k\} \) has measure zero. Note that the set \( E \) (and \( E^c \)) is symmetric by definition.

For any set \( E \subset A^k \), we define its symmetrization by \( E^s := \bigcap \pi E^\pi \), where the intersection goes over all permutations \( \pi \) of \( \{1, ..., k\} \), and \( E^\pi := \{(x_{\pi(1)}, ..., x_{\pi(k)}) : (x_1, ..., x_k) \in E\} \). Obviously, \( E^s \subset E \) is a symmetric set, and \( E^s \) is co-null if and only if \( E \) is co-null.

**Lemma 2.1.** Let \( 1 \leq k \leq N \) be integers, and let \( T_k \) be a co-null subset of \( A^k \).
Then, the set \( T_N := \{(x_1, ..., x_N) \in A^N : (x_{j_1}, ..., x_{j_k}) \in T_k \text{ for every } 1 \leq j_1 < \cdots < j_k \leq N\} \) is co-null in \( A^N \).

**Proof.** \( T_N \) is the intersection of the sets \( \{(x_1, ..., x_N) \in A^N : (x_{j_1}, ..., x_{j_k}) \in T_k\} \)
for fixed \( 1 \leq j_1 < \cdots < j_k \leq N \). Every such set is the product of \( N - k \) copies of \( A \) and one copy of \( T_k \) and therefore co-null in \( A^N \). The conclusion follows from the fact that any finite intersection of co-null sets is co-null. \( \square \)

The next well-known result follows from the Fubini-Tonelli theorem for complete measures \([3, \text{Theorem 2.39}]\). Recall that a \((x_1, ..., x_k)\)-section of a set \( E \subset A^N \) is defined by

\[
E_{x_1, ..., x_k} := \{(x_{k+1}, ..., x_N) \in A^{N-k} : (x_1, ..., x_N) \in E\}. \tag{2.1}
\]

**Lemma 2.2.** Let \( E \subset A^N \) be measurable. Then, \( E_{x_1, ..., x_k} \subset A^{N-k} \) is measurable for almost all \((x_1, ..., x_k) \in A^k\). If also \( E \subset A^N \) is co-null, then \(|(E_{x_1, ..., x_k})^c| = 0\) for almost all \((x_1, ..., x_k) \in A^k\), i.e. \( E_{x_1, ..., x_k} \) is a.e. co-null in \( A^{N-k} \).

We shall assume that \( \rho(m) \) is the \( m \)-variable reduction to \( A^m \) of some a.e. positive and symmetric probability density \( P \) on \( A^N \), i.e. \( P > 0 \) and symmetric.
a.e., $\int_{\Lambda^N} P d^N x = 1$, and
\begin{align*}
\rho^{(m)}(x_1, \ldots, x_m) &= \int_{\Lambda^{N-m}} P(x_1, \ldots, x_N) dx_{m+1} \cdots dx_N \text{ a.e. on } \Lambda^m. \quad (2.2)
\end{align*}

It follows from the properties of $P$ that $\rho^{(m)}$ is a.e. positive and symmetric, $\int_{\Lambda^m} \rho^{(m)} d^m x = 1$, and if integration in (2.2) is performed with respect to any set of $N - m$ variables, the function so obtained is $\rho^{(m)}$ evaluated at the remaining $m$ variables. In fact, we have the following lemma (which is not immediately obvious because positivity and symmetry of $P$ hold a.e.):

**Lemma 2.3.** Let $P$ be an a.e. positive and symmetric probability density on $\Lambda^N$, and $\rho^{(m)}$ be its $m$-variable reduction. Then, there exists a co-null set $T_m \subset \Lambda^m$ such that for each $(x_1, \ldots, x_m) \in T_m$ and each permutation $\pi$ of $\{1, \ldots, m\}$,
\begin{align*}
\rho^{(m)}(x_{\pi(1)}, \ldots, x_{\pi(m)}) &= \rho^{(m)}(x_1, \ldots, x_m) > 0. \quad \text{Moreover, for every } 1 \leq i_1 < \cdots < i_m \leq N,
\end{align*}
\begin{align*}
\rho^{(m)}(x_1, \ldots, x_m) &= \int_{\Lambda^{N-m}} P(y_1, \ldots, y_N) dy_{i_1+1} \cdots \widehat{dy_{i_k}} \cdots dy_m \cdots dy_N, \quad (2.3)
\end{align*}
where $\widehat{dy_k}$ means that there is no integration over $dy_k$, and $(y_{i_1}, \ldots, y_{i_m}) = (x_1, \ldots, x_m)$.

**Proof.** Let $E \subset \Lambda^N$ be the (symmetric) co-null set on which $P > 0$ and symmetric. By Lemma 2.2 and the Fubini-Tonelli theorem there is a co-null set $T_m \subset \Lambda^m$ such that for every $(x_1, \ldots, x_m) \in T_m$, $|E_{x_1, \ldots, x_m}| = 0$ and $P(x_1, \ldots, x_m, \cdot) \in \mathcal{L}^{N-m}(\Lambda^{N-m}; d^{N-m} x)$. Therefore, for every $(x_1, \ldots, x_m) \in T_m$ and every permutation $\pi$ of $\{1, \ldots, m\}$:
\begin{align*}
0 < \rho^{(m)}(x_1, \ldots, x_m) &= \int_{E_{x_1, \ldots, x_m}} P(x_1, \ldots, x_m, x_{m+1}, \ldots, x_N) dx_{m+1} \cdots dx_N \\
&= \int_{E_{x_1, \ldots, x_m}} P(x_{\pi(1)}, \ldots, x_{\pi(m)}, x_{m+1}, \ldots, x_N) dx_{m+1} \cdots dx_N = \rho^{(m)}(x_{\pi(1)}, \ldots, x_{\pi(m)}).
\end{align*}
(2.4)
Equation (2.3) follows similarly. □

The fixed internal potential $W$ is assumed to be an a.e. finite and symmetric measurable function on $A^N$. Particularly, $|W^{-1}(\infty)| = 0$, which means that our conditions exclude hard core interactions. The absence of hard cores is also a main assumption in [4], but they were allowed in [1].

We will also require that

$$(W + \log P)_+ \in L^1(A^N; P d^N x).$$

The quantity $\int_{A^N} (W + \log P) P d^N x$ is an analogue (in the canonical ensemble) of one of the two key functionals employed in the density functional theory [3]. Therefore, the assumption (2.5) has some physical meaning. On these grounds a similar condition was imposed in [1].

### 3. Existence and Uniqueness

Similarly to [1,4], we use a variational method to prove the existence of solutions to the inverse problem. Consider the functional

$$F_P(V) := e^{-\int_{A^N} V P d^N x} Z(V),$$

with $Z$ as in (1.2), defined on a set

$$\mathcal{V}_P := \{ V \in L^1(A^N; P d^N x), e^{-V} \in L^1(A^N; e^{-W} d^N x),
V(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v(x_{i_1}, \ldots, x_{i_m}) \text{ a.e. on } A^N,
\text{ where } v \text{ is a symmetric real-valued measurable function on } A^m \}. \quad (3.2)$$

Note that $V \in \mathcal{V}_P$ is a.e. finite and symmetric by Corollary [4.3] (iii).
The appearance of a functional in the form of (3.1) in a variational proof of the inverse problem is not accidental. Disregarding the terms independent of $V$, $-\log F_P(V)$ is the Kullback and Leibler mean information for discrimination between two probability distributions [10]. As applied to (3.1), the mean information is a measure of overlap between the distributions characterized by probability densities $P$ and $e^{-W-V}/Z(V)$. This quantity is also often referred to as relative entropy [11]. Murtola et al. [12] were the first to notice the connection between the functionals used by Chayes et al. in [1,4] and the relative entropy considered by Shell in [11].

Before applying a variational procedure, we need to ensure that $\mathcal{V}_P \neq \emptyset$. If $\exp(-W) \in L^1(A^N; d^N \nu)$, then $\mathcal{V}_P \neq \emptyset$ because it contains all constant functions. (Particularly, $\mathcal{V}_P \neq \emptyset$ when $W = 0$ and $|A| < \infty$, which is usually the case in the CG simulations described in the introduction.) However, not to be limited by this option, we will assume, more generally, that $\mathcal{V}_P \neq \emptyset$.

In this section we will prove that there exists a unique maximizer (up to an additive constant) of $F_P$ on $\mathcal{V}_P$. Moreover, every such maximizer solves the inverse problem. This settles the question about the existence of solutions on $\mathcal{V}_P$. However, it is not clear why every solution of the inverse problem on $\mathcal{V}_P$ should be a maximizer of $F_P$, and so such a solution may not be unique. The uniqueness problem is more easily solved on a smaller set $\mathcal{V}_{\rho(m)} \subset \mathcal{V}_P$ (the inclusion will be
proven shortly) defined by

$$V_{\rho(m)} := \{ V : e^{-V} \in L^1(A^N; e^{-W}d^N x),$$

$$V(x_1, ..., x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v(x_{i_1}, ..., x_{i_m}) \text{ a.e. on } A^N,$$

where $v$ is a symmetric real-valued function on $A^m$, $v \in L^1(A^m; \rho^{(m)}d^m x)$.

(3.3)

Note that by Fubini-Tonelli theorem and Lemma 2.3:

$$\int_{A^N} V P d^N x = \sum_{1 \leq i_1, ..., i_m \leq N} \int_{A^m} \rho^{(m)}(x_{i_1}, ..., x_{i_m}) v(x_{i_1}, ..., x_{i_m}) dx_{i_1} \cdots dx_{i_m} = \binom{N}{m} \int_{A^m} \rho^{(m)} v d^m x \in \mathbb{R}. \quad (3.4)$$

Thus, $V_{\rho(m)} \subset V_P$. It is important to note that this inclusion is proper in general.

A counterexample is provided in the appendix.

We will see that (assuming $V_{\rho(m)} \neq \emptyset$) every maximizer of $F_P$ on $V_{\rho(m)}$ also solves the inverse problem, and if it exists, this maximizer is unique. Moreover, every solution of the inverse problem on $V_{\rho(m)}$ is a maximizer of $F_P$ on this set. Thus, the uniqueness of a solution on $V_{\rho(m)}$ (if one exists) is guaranteed.

However, it seems difficult to prove the existence of maximizers of $F_P$ on $V_{\rho(m)}$, and so this set may not contain solutions. Nevertheless, in Section 5, we will prove that $V_{\rho(m)} = V_P$ under an additional assumption about the probability density $P$. Thus, with this additional assumption, the inverse problem has a unique solution on $V_{\rho(m)} = V_P$. This result is summarized in Theorem 3.10.

The functional $F_P$ is well-defined on $V_P$ as a positive real number. Indeed, the numerator in (3.1) is finite and positive, and $Z(V) < \infty$ by definition of $V_P$. Also, $V \in L^1(A^N; Pd^N x)$, and so $V$ is a.e. finite (using $P > 0$ a.e.) Thus, $Z(V) > 0$ since $W$ is a.e. finite as well.
Remark 3.1. We mention here in passing that the requirement of \( W \) being finite a.e. is actually implied by our other assumptions. In fact, if \( W = \infty \) on a set of positive measure, then \( P > 0 \) a.e. implies that \( (W + \log P)_+ = \infty \) a.e. on this set, and therefore \( (W + \log P)_+ \notin L^1(\Lambda^N; Pd^N x) \). Also, if \( V \in \mathcal{V}_P \), then \( V \) is a.e. finite. Thus, if \( W = -\infty \) on a set of positive measure, then \( e^{-V} \notin L^1(\Lambda^N; e^{-W} d^N x) \), and so \( \mathcal{V}_P = \emptyset \).

Lemma 3.1. The set \( \mathcal{V}_P \) is convex, and \( \log \mathcal{F}_P \) is concave on \( \mathcal{V}_P \). More precisely,

\[
\log \mathcal{F}_P(\lambda V_1 + (1 - \lambda)V_0) \geq \lambda \log \mathcal{F}_P(V_1) + (1 - \lambda) \log \mathcal{F}_P(V_0) \quad (3.5)
\]

for every \( \lambda \in (0, 1) \) and every \( V_0, V_1 \in \mathcal{V}_P \), with equality if and only if \( V_1 - V_0 \) is a constant a.e. In particular, if \( U_0, U_1 \in \mathcal{V}_P \) are two maximizers of \( \mathcal{F}_P \), then \( U_1 - U_0 \) is a constant a.e. The same statement holds true with \( \mathcal{V}_P \) substituted with \( \mathcal{V}_{\rho}^{(m)} \).

Proof. Let \( V_0, V_1 \in \mathcal{V}_P \) and \( \lambda \in (0, 1) \). Then \( \lambda V_1 + (1 - \lambda)V_0 \) is in \( L^1(\Lambda^N; Pd^N x) \) and has the "sum" structure required for membership in \( \mathcal{V}_P \). By Holder's inequality,

\[
\int_{\Lambda^N} e^{-\lambda V_1 - (1 - \lambda)V_0} e^{-W} d^N x \leq \|e^{-V_1} e^{-W}\|_1^\lambda \|e^{-V_0} e^{-W}\|_{1-(1-\lambda)} < \infty. \quad (3.6)
\]

Thus, \( \lambda V_1 + (1 - \lambda)V_0 \in \mathcal{V}_P \), and so the set \( \mathcal{V}_P \) is convex.

From (3.1), \( \log \mathcal{F}_P(V) = -\int_{\Lambda^N} VPD^N x - \log Z(V) \). Since the first term is linear in \( V \), inequality (3.5) holds if and only if

\[
\log Z(\lambda V_1 + (1 - \lambda)V_0) \leq \lambda \log Z(V_1) + (1 - \lambda) \log Z(V_0). \quad (3.5)
\]

But this inequality is equivalent to (3.6). Moreover, (3.5) is an equality if and only if \( V_1 - V_0 \) is a.e. a constant [9, Theorem 6.2]. (We have used the fact that measures \( d^N x \) and \( e^{-W} d^N x \) are absolutely continues with respect to each other.) The proof for \( \mathcal{V}_{\rho}^{(m)} \) is the same. \( \square \)
3.1. Existence and uniqueness of maximizers of $\mathcal{F}_P$ on $\mathcal{V}_P$. This subsection is devoted to the proof of the existence of maximizers of the functional $\mathcal{F}_P$ on the set $\mathcal{V}_P$, except for a single (crucial) issue resolved in Section 4. We begin with the proof that $\mathcal{F}_P$ is bounded.

Lemma 3.2. The functional $\mathcal{F}_P$ is bounded on $\mathcal{V}_P$. More precisely, $0 < \mathcal{F}_P(V) \leq e^{\int_{\Lambda_N} (W + \log P) d^N x}$ for every $V \in \mathcal{V}_P$.

Proof. By (2.5) and Jensen’s inequality for the measure $P d^N x$ [13, Theorem 3.3],

$$e^{-\int_{\Lambda_N} (V + (W + \log P)_+) d^N x} \leq \int_{\Lambda_N} e^{-V - (W + \log P)_+} P d^N x \leq \int_{\Lambda_N} e^{-V - W - \log P} P d^N x = \int_{\Lambda_N} e^{-V - W} d^N x. \ (3.7)$$

Thus, $0 \leq \mathcal{F}_P(V) \leq e^{\int_{\Lambda_N} (W + \log P)_+ P d^N x} < \infty$. □

From now on, we set $M := \sup_{V \in \mathcal{V}_P} \mathcal{F}_P(V) \in (0, \infty)$. (3.8)

Let $(V_n) \in \mathcal{V}_P$ be a maximizing sequence for $\mathcal{F}_P$, i.e. $\lim_{n \to \infty} \mathcal{F}_P(V_n) = M$. (Recall that, by definition of $\mathcal{V}_P$ in (3.2), $V_n(x_1, ..., x_N) = \sum_{1 \leq i_1 < ... < i_m \leq N} v_n(x_{i_1}, ..., x_{i_m})$ a.e., where $v_n$ is a symmetric real-valued measurable function on $A^m$.) If $V \in \mathcal{V}_P$ and $C \in \mathbb{R}$, then

$$V + C = \sum_{1 \leq i_1 < ... < i_m \leq N} \left[ v(x_{i_1}, ..., x_{i_m}) + \binom{N}{m}^{-1} C \right] \in \mathcal{V}_P, \ \text{and} \ \mathcal{F}_P(V + C) = \mathcal{F}_P(V).$$

Therefore, by adding a suitable constant to each $V_n$, it can be assumed that $Z(V_n) = 1$. Thus $1 = \int_{\Lambda_N} e^{-V_n} e^{-W} d^N x$, i.e. $e^{-V_n/2} \in L^2(\Lambda_N; e^{-W} d^N x)$, and $\|e^{-V_n/2}\|_{L^2(e^{-W} d^N x)} = 1$. Therefore, by reflexivity of $L^2(\Lambda_N; e^{-W} d^N x)$, there is a subsequence (still denoted by $(V_n)$) and $\Pi \in L^2(\Lambda_N; e^{-W} d^N x)$ such that $(e^{-V_n/2})$ converges weakly to $\Pi$ [14, Theorem 3.18].
Lemma 3.3. \(e^{-V_n/2} \rightarrow \Pi \) in \(L^2(A^N; e^{-W}d^N x)\). (In particular, \(\int_A \Pi^2 e^{-W} = 1\).)

Proof. For convenience, let us define \(\Pi_n := e^{-V_n/2}\). It suffices to show that
\[
1 = \lim_{n \to \infty} \|\Pi_n\|_{2,e^{-W}d^N x} = \|\Pi\|_{2,e^{-W}d^N x}.
\]
The weak convergence implies
\[
1 = \lim_{n \to \infty} \|\Pi_n\|_{2,e^{-W}d^N x} \geq \|\Pi\|_{2,e^{-W}d^N x}. \tag{3.9}
\]
Let \(\varepsilon > 0\) be fixed. There is \(n_0 \in \mathbb{N}\) such that for each \(n \geq n_0\),
\[
e^{-\int \Lambda N V_n P d^N x} = \mathcal{F}_P(V_n) > M(1 - \varepsilon). \tag{3.10}
\]
By Mazur’s theorem [15, Theorem 3.13], there is a sequence of convex combinations \((\tilde{\Pi}_n|n \geq n_0)\), i.e.
\[
\tilde{\Pi}_n = \sum_{k=n_0}^{n} \lambda_k^{(n)} \Pi_k, \quad \lambda_k^{(n)} \geq 0 \quad \forall \ n_0 \leq k \leq n, \quad \text{and} \quad \sum_{k=n_0}^{n} \lambda_k^{(n)} = 1, \tag{3.11}
\]
such that \(\lim_{n \to \infty} \|\tilde{\Pi}_n - \Pi\|_{2,e^{-W}d^N x} = 0\). For every \(n \geq n_0\) choose \(j_n, k_n \in \{n_0, ..., n\}\) such that \(\langle \Pi_{j_n}, \Pi_{k_n} \rangle \leq \langle \Pi_j, \Pi_k \rangle\) for every \(j, k \in \{n_0, ..., n\}\), where \(\langle \cdot, \cdot \rangle\) denotes the inner product of \(L^2(A^N; e^{-W}d^N x)\). Then,
\[
\|\tilde{\Pi}_n\|_{2,e^{-W}d^N x}^2 = \sum_{j,k=n_0}^{n} \lambda_j^{(n)} \lambda_k^{(n)} \langle \Pi_j, \Pi_k \rangle \geq \left( \sum_{j,k=n_0}^{n} \lambda_j^{(n)} \lambda_k^{(n)} \right) \langle \Pi_{j_n}, \Pi_{k_n} \rangle = \langle \Pi_{j_n}, \Pi_{k_n} \rangle. \tag{3.12}
\]
Since \(\langle \Pi_{j_n}, \Pi_{k_n} \rangle = \int_{A^N} e^{-\frac{V_{j_n} + V_{k_n}}{2} - W} d^N x\), (3.12) reads
\[
\|\tilde{\Pi}_n\|_{2,e^{-W}d^N x}^2 \geq Z(Y_n), \tag{3.13}
\]
where \(Y_n := \frac{V_{j_n} + V_{k_n}}{2} \in \mathcal{V}_P\) (because \(\mathcal{V}_P\) is convex by Lemma 3.1). Therefore, using (3.10),
\[
M \geq \frac{e^{-\int_{A^N} (V_{j_n} + V_{k_n})/2 P d^N x}}{Z(Y_n)} = \left( \frac{\mathcal{F}_P(V_{j_n}) \mathcal{F}_P(V_{k_n})}{Z(Y_n)} \right) \geq \frac{M(1 - \varepsilon)}{Z(Y_n)}. \tag{3.14}
\]
Inequalities (3.13) and (3.14) imply that \(\|\tilde{\Pi}_n\|^2_{2,e^{-W}d^N x} \geq Z(Y_n) > 1 - \varepsilon\). Therefore, using (3.9), \(1 \geq \|\Pi\|^2_{2,e^{-W}d^N x} - \lim_{n \to \infty} \|\tilde{\Pi}_n\|^2_{2,e^{-W}d^N x} \geq 1 - \varepsilon\) for any \(\varepsilon > 0\). Thus, \(\|\Pi\|^2_{2,e^{-W}d^N x} = 1\). \(\square\)

It follows from Lemma 3.3 (and the fact that \(d^N x\) and \(e^{-W}d^N x\) have the same zero measure sets) that there is a subsequence (still denoted \((V_n)\)) such that \(e^{-V_n/2} \to \Pi\) a.e. on \(\Lambda^N\). Let us define a co-null set \(\tilde{E} := \{(x_1, \ldots, x_N) \in \Lambda^N : \Pi(x_1, \ldots, x_N) \in \mathbb{R}, e^{-V_n(x_1, \ldots, x_N)/2} \to \Pi(x_1, \ldots, x_N)\}\). (3.15)

In particular, \(\Pi \in [0, \infty)\) on \(\tilde{E}\). Let us also define a function \(U\) on \(\Lambda^N\) as

\[
U(x) := \begin{cases} 
-2 \log \Pi(x) & \text{if } x \in \tilde{E} \cap \{y \in \Lambda^N : \Pi(y) > 0\}, \\
\infty & \text{if } x \in \tilde{E} \cap \{y \in \Lambda^N : \Pi(y) = 0\}, \\
0 & \text{if } x \in \tilde{E}^c.
\end{cases}
\]

(3.16)

It is easy to confirm that \(U\) is measurable, and \(V_n \to U\) on \(\tilde{E}\). We will prove that \(U \in \mathcal{V}_P\), and \(\mathcal{F}_P(U) = M\). We have

\[
1 = \int_{\Lambda^N} \Pi^2 e^{-W}d^N x = \int_{\tilde{E}} e^{-U} e^{-W}d^N x = \int_{\Lambda^N} e^{-U} e^{-W}d^N x. 
\]

(3.17)

Therefore, \(e^{-U} \in L^1(\Lambda^N; e^{-W}d^N x)\). It remains to prove that \(U \in L^1(\Lambda^N; Pd^N x)\), \(U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m})\) a.e. for some finite and symmetric measurable function \(u\) on \(\Lambda^m\) (i.e. \(U \in \mathcal{V}_P\), and \(\mathcal{F}_P(U) = M\).

**Lemma 3.4.** The function \(U\) defined in (3.16) satisfies \(U_- \in L^1(\Lambda^N; Pd^N x)\).

**Proof.** It follows from (3.17) that

\[
1 = \int_{\Lambda^N} e^{-U} e^{-W}d^N x = \int_{\Lambda^N} e^{-U} e^{-W - \log P}d^N x \geq \int_{\Lambda^N} e^{-U} (W + \log P) + Pd^N x.
\]

(3.18)
The relations \(e^t \geq t_+\) and \((-t)_+ = t_-\) for \(t \in \mathbb{R}\) together with (3.18) imply that
\[
\int_{\Lambda^N} (U + (W + \log P)_+)_- Pd^N x \leq 1.
\]
Particularly,
\[
(U + (W + \log P)_+)_- \in L^1(A^N; Pd^N x).
\]
Next, using \((t + s)_- \leq t_- + s_-\) for \(t \in \mathbb{R}\) and \(s \in \mathbb{R}\), with \(t = U + (W + \log P)_+\) and \(s = -(W + \log P)_+\), we obtain
\[
U_- \leq (U + (W + \log P)_+)_- + (W + \log P)_+.
\]
Therefore, by (2.5) and the above, \(U_- \in L^1(A^N; Pd^N x)\).

\begin{lemma}
The function \(U\) defined in (3.16) is in \(\mathcal{V}_P\). Moreover, \(F_P(U) = M\) with \(M\) from (3.8).
\end{lemma}

\begin{proof}
Since \(U_- \in L^1(A^N; Pd^N x)\) by Lemma (3.4) \(U \in L^1(A^N; Pd^N x)\) if and only if \(U_+ \in L^1(A^N; Pd^N x)\). This is proved below through an estimate that will also yield \(F_P(U) = M\) (after using Theorem 4.1 to show that \(U \in \mathcal{V}_P\)).

Let \(S \subset A^N\) be the co-null set on which \(W \in \mathbb{R}\) and \(P \in (0, \infty)\). For \(k, \ell \in \mathbb{N}\), define
\[
U^k := \min(U, k), \quad \phi^\ell := \chi_S \min(W + \log P, \ell).
\]
We have that
\[
0 \leq W + \log P - \phi^\ell \leq (W + \log P)_+ \quad \text{on } S
\]
because on this set \(W + \log P - \phi^\ell = 0\) when \(W + \log P \leq \ell\), and \(0 < W + \log P - \phi^\ell = (W + \log P)_+ - \ell < (W + \log P)_+\) when \(W + \log P > \ell\). Particularly, \(W + \log P - \phi^\ell \in L^1(A^N; Pd^N x)\) by (2.5). Moreover,
\[
\lim_{\ell \to \infty} \int_{\Lambda^N} (W + \log P - \phi^\ell) P = 0
\]
by dominated convergence. Since also \(U^k = -U_- + U^k_+ \in L^1(A^N; Pd^N x)\) by Lemma (3.4), it follows that \((U^k - V_n)/2 - W - \log P + \phi^\ell \in L^1(A^N; Pd^N x)\).
Therefore, by Jensen’s inequality
\[ e^{-\int_{\Lambda} \frac{V_n}{2} P dN x} e^{\int_{\Lambda} \left( \frac{U_k}{2} - W + \log P + \phi \right) P dN x} \leq \int_{\Lambda} e^{-\left( \frac{V_n - U_k}{2} + \phi \right) P dN x} e^{-W dN x}. \] (3.22)

Next, we will show that \( e^{\frac{U_k}{2} + \phi} \in L^2(\Lambda; e^{-W dN x}) \). Indeed, \( \phi \leq W + \log P \) on \( S \) implies \( e^{\phi - W} \leq P \) a.e., and therefore, \( e^{\phi - W} \in L^1(\Lambda; dN x) \). Also, \( \phi \leq \ell \), \( U_k \leq k \) imply \( U_k + 2\phi - W \leq k + \ell - W \). Therefore, \( e^{U_k + 2\phi - W} \leq e^{k + \ell - W} \). Equivalently,
\[ e^{\frac{U_k}{2} + \phi} \in L^2(\Lambda; e^{-W dN x}). \] (3.23)

With \( k, \ell \) being held fixed, let \( n \to \infty \). Then, the leftmost side of (3.22) converges to \( \sqrt{Me^{\int_{\Lambda} \frac{U_k}{2} P dN x}} e^{-\int_{\Lambda} (W + \log P + \phi) P dN x} \). By (3.23) and weak convergence, the rightmost side of (3.22) converges to
\[ \int_{\Lambda} e^{-\left( \frac{U_k}{2} + \phi \right) P dN x} e^{-W dN x} = \int_{\Lambda} e^{-\left( U - \frac{U_k}{2} + \phi \right) P dN x} \leq 1, \] (3.24)
where the last inequality follows by \( U - U_k \geq 0, W + \log P - \phi \geq 0 \), and \( \int_{\Lambda} P = 1 \). This yields
\[ \sqrt{Me^{\int_{\Lambda} U_k P dN x}} e^{-\int_{\Lambda} (W + \log P - \phi) P dN x} \leq 1. \] (3.25)

With \( k \) being held fixed, let \( \ell \to \infty \) in (3.25). Then, by (3.21), we obtain
\[ \sqrt{Me^{\int_{\Lambda} U_k P dN x}} \leq 1. \] Equivalently, using \( U_k = -U_+ + U_k^+ \),
\[ M \leq e^{-\int_{\Lambda} U_k P dN x} = e^{-\int_{\Lambda} U_k^+ P dN x} e^{\int_{\Lambda} U_+ P dN x}. \] (3.26)

Now, \( U_k^+ \) is increasing to \( U_+ \) pointwise. Thus, by monotone convergence,
\[ \lim_{k \to \infty} \int_{\Lambda} U_k^+ P dN x = \int_{\Lambda} U_+ P dN x. \] Taking \( k \to \infty \) in (3.26) results into
\[ 0 < M \leq e^{-\int_{\Lambda} U_+ P dN x} e^{\int_{\Lambda} U_+ P dN x}, \] (3.27)
which together with Lemma 3.4 shows that $U_+$ (and therefore $U$) belongs to $L^1(A^N; Pd^N x)$. In particular, since $P > 0$ a.e., $U$ is a.e. finite.

To finish the proof, we need to show that $U \in V_P$. Because $V_n \to U$ on the co-null set $\tilde{E}$ defined in (3.15), it follows that the set $E = \{(x_1, \ldots, x_N) \in A^N : U(x_1, \ldots, x_N) - V_n(x_1, \ldots, x_N) \to U(x_1, \ldots, x_N)\}$ is also co-null. Since $V_n \to \chi_E U$ on $E$, Theorem 4.1 in the next section implies that there is a co-null set $E_m \subset A^m$ and a real-valued function $u$ on $A^m$ such that $v_n \to u$ on $E_m$. Let $\Omega = \{(x_1, \ldots, x_N) \in A^N : (x_{i_1}, \ldots, x_{i_m}) \in E_m \text{ for every } 1 \leq i_1 < \cdots < i_m \leq N\}$. The set $\Omega$ is co-null in $A^N$ by Lemma 2.1, and

$$U(x_1, \ldots, x_N) = \lim_{n \to \infty} \sum_{1 \leq i_1 < \cdots < i_m \leq N} v_n(x_{i_1}, \ldots, x_{i_m})$$

$$= \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m}), \quad \forall (x_1, \ldots, x_N) \in E \cap \Omega. \quad (3.28)$$

Particularly, $u$ is measurable by Corollary 4.3 (i) (or as an a.e. limit of a sequence of measurable functions $v_n$). Moreover, $u$ can be chosen to be symmetric on $A^m$. In fact, let $E^s_m \subset E$ be a symmetrization of $E_m$ defined in Section 2. Then, for every $(x_1, \ldots, x_m) \in E^s_m$ and every permutation $\pi$ of $\{1, \ldots, m\}$,

$$u(x_1, \ldots, x_m) = \lim_{n \to \infty} v_n(x_1, \ldots, x_m) = \lim_{n \to \infty} v_n(x_{\pi(1)}, \ldots, x_{\pi(m)}) = u(x_{\pi(1)}, \ldots, x_{\pi(m)}). \quad (3.29)$$

Substituting $u$ with $\chi_{E^s_m} u$ and using Lemma 2.1 and (3.28), we can assume that $u$ is a symmetric real-valued function on $A^m$, and

$$U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m}) \text{ a.e. Thus, } U \text{ has the sum structure required for membership in } V_P, \text{ and } U \in V_P. \text{ Then, (3.27) and (3.17) imply that } F_P(U) = e^{-\int_{A^N} U d^N x} = M. \quad \square$$
We can now finish the proof of the existence and uniqueness of maximizers of $\mathcal{F}_P$ on $\mathcal{V}_P$. In fact, all the steps of this proof are already completed, and we essentially just need to cite the previous results. For convenience, we repeat all the assumptions needed for the validity of the next theorem. Note that the symmetry of $P$ and $W$ has not been used yet, and will not be needed in the reminder of the proof of the existence and uniqueness of maximizers.

**Theorem 3.6.** Let $\rho^{(m)}$ be the $m$-variable reduction of an a.e. positive probability density $P$, i.e. $\rho^{(m)}$ is given by (2.2), and let $W$ be an a.e. finite measurable potential on $A^N$. Suppose that $(W + \log P)_+ \in L^1(A^N; P d^N x)$. If $\mathcal{V}_P \neq \emptyset$, there is $U \in \mathcal{V}_P$ (with $U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m})$ a.e.) such that $\mathcal{F}_P(V) \leq \mathcal{F}_P(U)$ for every $V \in \mathcal{V}_P$. Furthermore, if $U_1$ and $U_0$ are two maximizers of $\mathcal{F}_P$ on $\mathcal{V}_P$ (or on $\mathcal{V}_{\rho^{(m)}}$, if they exist), then $U_1 - U_0$ is a constant a.e., and the same is true for $u_1 - u_0$.

**Proof.** By Lemma 3.5 there is $U \in \mathcal{V}_P$ such that

$$\mathcal{F}_P(V) \leq \mathcal{F}_P(U) \text{ for every } V \in \mathcal{V}_P. \quad (3.30)$$

Suppose, there are $U_0, U_1 \in \mathcal{V}_P$ (or $\mathcal{V}_{\rho^{(m)}}$) satisfying (3.30). Then, by Lemma 3.1 there is $C \in \mathbb{R}$ such that $U_1 - U_0 = C$ a.e. on $A^N$. Therefore, by Corollary 4.3 (ii) in the next section, $u_1 - u_0 = (\frac{N}{m})^{-1} C$ a.e. on $A^m$. $\square$

### 3.2. Maximizers of $\mathcal{V}_P$ and solutions of the inverse problem

Now we are ready to prove the existence of solutions to the inverse problem.

**Theorem 3.7.** Let $P$ be an a.e. positive and symmetric probability density, and $W$ be measurable and a.e. finite and symmetric potential on $A^N$. Suppose that for some $U \in \mathcal{V}_P$, $\mathcal{F}_P(V) \leq \mathcal{F}_P(U)$ for every $V \in \mathcal{V}_P$. Then, $\rho_u^{(m)} = \rho^{(m)}$ a.e.
Remark 3.2. According to Theorem 3.7, every maximizer of $F_P(V)$ on $V_P$ is a solution. Since such maximizers exist by Theorem 3.6, the solutions of the inverse problem exist under the assumptions of Theorem 3.6 with the additional requirement for $P$ and $W$ to be a.e. symmetric.

Proof (of Theorem 3.7). Let $\xi \in L^\infty(\Lambda^m; d^m x)$ be a symmetric real-valued function. For every $(x_1, \ldots, x_N) \in \Lambda^N$, set
\[
\Xi(x_1, \ldots, x_N) := \sum_{1 \leq i_1 < \cdots < i_m \leq N} \xi(x_{i_1}, \ldots, x_{i_m}).
\]
(3.31)

By definition of the norm of $L^\infty(\Lambda^m; d^m x)$, there is a co-null set $T_m$ of $\Lambda^m$ such that $|\xi(x_1, \ldots, x_m)| \leq ||\xi||_{\infty, d^m x}$ for every $(x_1, \ldots, x_m) \in T_m$. By Lemma 2.1, $|\Xi(x_1, \ldots, x_N)| \leq \binom{N}{m} ||\xi||_{\infty, d^m x}$ for every $(x_1, \ldots, x_N)$ in some co-null set $T_N \subset \Lambda^N$, and so $\Xi \in L^\infty(\Lambda^N; d^N x)$.

For every $t \in \mathbb{R}$, $U + t\Xi \in V_P$, and therefore, $F_P(U + t\Xi) \leq F_P(U)$. The function $t \mapsto F_P(U + t\Xi)$ is smooth. This is obvious for the numerator in (3.1). The same property for the denominator follows by a theorem on differentiation of parameter-dependent integrals [9, Theorem 2.27]. ($\Xi \in L^\infty(\Lambda^N; d^N x)$ is used here.) Thus, $\frac{d}{dt}F_P(U + t\Xi)|_{t=0} = 0$. This gives
\[
F_P(U) \left[ \int_{\Lambda^N} e^{-U - W} \Xi d^N x \frac{Z(U)}{Z(\mathbb{R})} - \int_{\Lambda^N} \Xi P d^N x \right] = 0.
\]
(3.32)

By Lemma 2.3 (with $P$ and $\rho^{(m)}$ replaced by $e^{-U - W} / Z(\mathbb{R})$ and $\rho^{(m)}_U$ respectively when applied to the first term), (3.32) amounts to $\int_{\Lambda^m} \xi(\rho^{(m)}_U - \rho^{(m)}) d^m x = 0$. The same lemma implies that the (symmetric) set $S_m \subset \Lambda^m$ on which $\rho^{(m)}$ and $\rho^{(m)}_U$ are both finite and symmetric is co-null. Choosing $\xi = \chi_{S_m} \times \text{sign}(\rho^{(m)}_U - \rho^{(m)})$ (with sign(0) := 0), we obtain $\int_{\Lambda^m} |\rho^{(m)}_U - \rho^{(m)}| d^m x = 0$, and so $\rho^{(m)}_U = \rho^{(m)}$ a.e. on $\Lambda^m$. \qed
Remark 3.3. Theorem 3.7 is still true if $V_P$ is replaced by $V_{\rho(m)}$, as can be easily verified by simply repeating the arguments in the proof.

3.3. Uniqueness of the solutions to the inverse problem. It remains to resolve the problem of the uniqueness of solutions. So far, we have shown that solutions exist on $V_P$. However, as was mentioned in the introduction, they may not be unique. It will be proven in Section 5 that, under an additional assumption about $P$, the sets $V_P$ and $V_{\rho(m)}$ coincide. Thus, with this assumption, $V_{\rho(m)}$ contains solutions. According to the next theorem, every solution on $V_{\rho(m)}$ is a maximizer of $F_P$, and therefore is unique up to an additive constant by Theorem 3.6.

Theorem 3.8. Suppose that $U \in V_{\rho(m)}$, where

$U(x_1, ..., x_N) = \sum_{1 \leq i_1 < ... < i_m \leq N} u(x_{i_1}, ..., x_{i_m})$ a.e. Suppose also that $\rho_U^{(m)} = \rho^{(m)}$ a.e. Then, $F_P(V) \leq F_P(U)$ for every $V \in V_{\rho(m)}$. Consequently, if $U_0, U_1 \in V_{\rho(m)}$, and $\rho_{U_0}^{(m)} = \rho_{U_1}^{(m)} = \rho^{(m)}$ a.e., then $U_1 - U_0$ is a.e. a constant, and the same is true for $u_1 - u_0$.

Proof. The "Consequently" part of the conclusion follows directly from Theorem 3.6. To prove the rest, we need the following lemma:

Lemma 3.9. Let us call the functions in the form of (3.31) "admissible," for brevity. Then, $F_P(U + \Xi) \leq F_P(U)$ for every admissible $\Xi$.

Proof. Since, $\rho_{U_0}^{(m)} = \rho^{(m)}$ a.e., $\int_{\Lambda} \xi(\rho_U^{(m)} - \rho^{(m)})dm = 0$. Thus, $\frac{d}{dt} F_P(U + t\Xi)|_{t=0} = 0$ (by reversing the steps in the proof of Theorem 3.7). Moreover, by Lemma 3.4, $t \mapsto \log F_P(U + t\Xi)$ is a concave (smooth) function on $\mathbb{R}$. Therefore, $F_P(U + t\Xi)$ attains its maximum at $t = 0$. Taking $t = 1$, gives $F_P(U + \Xi) \leq F_P(U)$. $\square$
Next, we will show that for every $V \in V_{\rho(m)}$ and every $\varepsilon > 0$ there is an admissible $\Xi$ such that $F_P(V) \leq F_P(U + \Xi) + \varepsilon$, and so Lemma 3.9 implies that $F_P(V) \leq F_P(U)$ for every $V \in V_{\rho(m)}$.

Let $\varepsilon > 0$, and $V \in V_{\rho(m)}$, where $V(x_1, ..., x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v(x_{i_1}, ..., x_{i_m})$ a.e. Define $d_n = \max(v - u, -u)$, so $d_n$ is symmetric, bounded below, and $d_n + u \to v$ pointwise and in $L^1(A^m; \rho(m)dmx)$ (by dominated convergence, because $|d_n + u| \leq |v - u| + |u|$). Let $\Delta_n(x_1, ..., x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} d_n(x_{i_1}, ..., x_{i_m})$. Then, with the help of Lemma 2.3, we obtain

$$\int_{A^N} (U + \Delta_n)Pd^N x = \binom{N}{m} \int_{A^m} (u + d_n)\rho^m d^m x \to \binom{N}{m} \int_{A^m} v\rho^m d^m x = \int_{A^N} Vd^N x. \quad (3.33)$$

Since $e^{-U - \Delta_n}$ is increasing to $e^{-V}$ a.e., the monotone convergence gives

$$\int_{A^N} e^{-U - \Delta_n - W}d^N x \to \int_{A^N} e^{-V - W}d^N x. \quad (3.34)$$

By (3.33), (3.34) and (3.1), there is $n_0 \in \mathbb{N}$ such that $|F_P(V) - F_P(U + \Delta_{n_0})| < \varepsilon/2$.

Define $\xi_n = \min(d_{n_0}, u)$. Then, $\xi_n$ is symmetric, bounded, and $u + \xi_n \to u + d_{n_0}$ pointwise and in $L^1(A^m; \rho(m)dmx)$ (by dominated convergence, because $|u + \xi_n| \leq |u| + |v - u|$). Define $\Xi_n(x_1, ..., x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} \xi_n(x_{i_1}, ..., x_{i_m})$. Therefore, similarly to (3.33),

$$\int_{A^N} (U + \Xi_n)Pd^N x = \binom{N}{m} \int_{A^m} (u + \xi_n)\rho^m d^m x \to \binom{N}{m} \int_{A^m} (u + d_{n_0})\rho^m d^m x = \int_{A^N} (U + \Delta_{n_0})Pd^N x. \quad (3.35)$$
Because $e^{-U - \Xi_n} \to e^{-U - \Delta_{n_0}}$ a.e., and $e^{-U - \Xi_n} \leq e^{-U + (\Delta_n)^n_0} \in L^1(A^N; e^{-W}d^N x)$, the dominated convergence yields

$$\int_{A^N} e^{-U - \Xi_n} d^N x \to \int_{A^N} e^{-U - \Delta_{n_0}} d^N x. \quad (3.36)$$

By (3.35), (3.36), and (3.1), there is $n_1 \in \mathbb{N}$ such that $|\mathcal{F}_P(U + \Delta_{n_0}) - \mathcal{F}_P(U + \Xi_{n_1})| < \varepsilon/2$. Taking $\Xi := \Xi_{n_1}$, the triangle inequality gives $|\mathcal{F}_P(V) - \mathcal{F}_P(U + \Xi)| < \varepsilon$. 

**Remark 3.4.** The above argument will not work if it is only known that $V \in \mathcal{V}_P$ because $(\Delta_n)_+ \geq (V - U)_+$, and so $U + \Delta_n$ may fail to be in $L^1(A^N; P d^N x)$. If we were to define $\Delta_n = \max(V - U, -n)$, then $(\Delta_n)_+ = (V - U)_+$, but $\Delta_n$ would not have the required sum structure to construct $\Xi$.

**Remark 3.5.** The uniqueness of solutions on $\mathcal{V}_{\rho(m)}$ can also be proved by the argument used by Chayes et al. in Theorem 2.4 in [4], followed by Corollary 4.3 (ii). (The same argument has also been used by Henderson [8].) However, Theorem 3.8 gives us more information because $\mathcal{V}_{\rho(m)}$ may not contain maximizers, and therefore uniqueness of solutions on $\mathcal{V}_{\rho(m)}$ does not imply that every solution is a maximizer, as Theorem 3.8 asserts.

### 3.4. Existence and uniqueness of the solutions to the inverse problem.

As was mentioned earlier in this section, the inclusion $\mathcal{V}_{\rho(m)} \subset \mathcal{V}_P$ is proper in general. Theorem 5.1 gives us sufficient conditions on $P$ under which $\mathcal{V}_{\rho(m)} = \mathcal{V}_P$. Adding these conditions to the assumptions made previously, we are able to formulate the following theorem on the existence and uniqueness of solutions to the inverse problem. Since the solution for $m = N$ is trivial, we will assume that $m < N$. 
Theorem 3.10. Let $N \geq 2$, $1 \leq m \leq N - 1$ be integers, and $\rho^{(k)}$, $1 \leq k \leq m$, be a $k$-variable reduction of some a.e. positive and symmetric probability density $P$. That is, $\rho^{(k)}$ is defined by (2.2) with $m$ replaced by $k$. Suppose that for each $1 \leq k \leq m$, there is a set $A_{N-k} \subset \Lambda_{N-k}$ of positive measure such that for every $(x_{k+1}, \ldots, x_N) \in A_{N-k}$ there is a constant $C_{N-k}(x_{k+1}, \ldots, x_N) > 0$ satisfying
\[
P(\cdot, x_{k+1}, \ldots, x_N) \geq C_{N-k}(x_{k+1}, \ldots, x_N)\rho^{(k)}(\cdot) \text{ a.e. on } \Lambda^k.
\] (3.37)
Let $W$ be an a.e. finite and symmetric measurable potential on $\Lambda^N$. Then, $\mathcal{V}_P = \mathcal{V}_{\rho^{(m)}}$. If, in addition, $\mathcal{V}_P \neq \emptyset$ and $(W + \log P)_+ \in L^1(\Lambda^N; Pd^N x)$, there is $U \in \mathcal{V}_P$ (with $U = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m})$ a.e.) such that $\rho^{(m)}_U = \rho^{(m)}$ a.e., where $\rho^{(m)}_U$ is defined by (1.1) and (1.2). Moreover, if $U_0, U_1 \in \mathcal{V}_P$ and $\rho^{(m)}_{U_0} = \rho^{(m)}_{U_1} = \rho^{(m)}$ a.e., then $U_1 - U_0$ is a constant a.e., and the same is true for $u_1 - u_0$.

Proof. According to Theorems 3.6 and 3.7, there is a unique maximizer of $F_P$ on $\mathcal{V}_P$, and this maximizer is a solution of the inverse problem. Theorem 5.1 claims that if $P$ satisfies (3.37) then $\mathcal{V}_P = \mathcal{V}_{\rho^{(m)}}$. In its turn, Theorem 3.8 asserts that every solution of the inverse problem on $\mathcal{V}_{\rho^{(m)}}$ is a maximizer. Thus, uniqueness of solutions follows. □

4. Sum Structure and a.e. Convergence

In this section, we show, among other things, that the sum structure of the functions in the form of (1.3) is preserved by a.e. convergence, thereby completing the proof of Lemma 3.5 and Theorem 3.6. The matters discussed here are quite general, and may also be considered separately from the inverse problem.

\footnote{In many practical situations, (3.37) holds for arbitrarily close perturbations (in $L^1(\Lambda^N; d^N x)$) of any probability density $P$. See Remark 5.1}
**Theorem 4.1.** Let \(1 \leq m \leq N\) be integers, and \((u_n)\) be a sequence of real-valued functions on \(\Lambda^m\). Suppose that there is a real-valued function \(U\) on \(\Lambda^N\) such that

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} u_n(x_{i_1}, \ldots, x_{i_m}) \to U(x_1, \ldots, x_N) \quad \text{a.e.} \quad (4.1)
\]

Then, there exists a real-valued function \(u\) on \(\Lambda^m\) such that \(u_n \to u\) a.e.

**Remark 4.1.** Note, that the functions \(u_n\) and \(U\) are not assumed to be symmetric or measurable. Also, the conclusion is still true if \(U\) and \(u_n\) are a.e. finite. However, we chose them to be real-valued to avoid cluttering the proof with non-essential details.

**Proof (of Theorem 4.1).** If \(N = m\) then the statement is true with \(u = U\), so let us assume that \(1 \leq m \leq N - 1\) from now on. Let \(E \subset \Lambda^N\) be the co-null set on which \((4.1)\) holds.

**Case 1:** \(m = 1\). By Lemma 2.2 there is a co-null set \(\tilde{S}_{N-1} \subset \Lambda^{N-1}\) such that for each \((\tilde{x}_2, \ldots, \tilde{x}_N) \in \tilde{S}_{N-1}\) the set \(E_{\tilde{x}_2, \ldots, \tilde{x}_N} := \{x \in A : (x, \tilde{x}_2, \ldots, \tilde{x}_N) \in E\}\) is co-null in \(A\). From now on, \((\tilde{x}_2, \ldots, \tilde{x}_N) \in \tilde{S}_{N-1}\) is chosen and fixed for the rest of the proof. For each \(x \in E_{\tilde{x}_2, \ldots, \tilde{x}_N}\),

\[
u_n(x) + u_n(\tilde{x}_2) + \cdots + u_n(\tilde{x}_N) \to U(x, \tilde{x}_2, \ldots, \tilde{x}_N). \quad (4.2)
\]

Thus, if \((x_1, \ldots, x_N) \in E_{\tilde{x}_2, \ldots, \tilde{x}_N}^N\), \((4.2)\) implies

\[
u_n(x_j) + u_n(\tilde{x}_2) + \cdots + u_n(\tilde{x}_N) \to U(x_j, \tilde{x}_2, \ldots, \tilde{x}_N), \quad 1 \leq j \leq N, \quad (4.3)
\]

and so, by addition

\[
\sum_{j=1}^N \nu_n(x_j) + N \sum_{j=2}^N u_n(\tilde{x}_j) \to \sum_{j=1}^N U(x_j, \tilde{x}_2, \ldots, \tilde{x}_N). \quad (4.4)
\]
Let \((x_1, \ldots, x_N) \in E\) be chosen and fixed for the rest of the proof. Then, both (4.4) and (4.1) hold for this point. Therefore,

\[ \sum_{j=2}^{N} u_n(\tilde{x}_j) \to C \in \mathbb{R}, \]  

where

\[ C = \frac{1}{N} \left[ \sum_{j=1}^{N} U(x_j, \tilde{x}_2, \ldots, \tilde{x}_N) - U(x_1, \ldots, x_N) \right]. \]

Thus, by (4.2), \(u_n(x) \to u(x)\) for every \(x \in E\), where

\[ u(x) := U(x, \tilde{x}_2, \ldots, \tilde{x}_N) - C. \]

**Case 2**: \(2 \leq m \leq N - 1\). The proof for this case proceeds by induction on \(m\).

Let \(2 \leq m \leq N - 1\), and assume that the statement of the theorem is true for \(m - 1\). The following technical definition will greatly simplify the exposition of arguments. For \(1 \leq \ell \leq k \leq N\), define an operator \(\hat{P}^k_\ell\) transforming a function \(f : A^\ell \to \mathbb{R}\) into a function \(\hat{P}^k_\ell f : A^k \to \mathbb{R}\) by

\[ \left( \hat{P}^k_\ell f \right)(x_1, \ldots, x_k) := \sum_{1 \leq i_1 < \cdots < i_\ell \leq k} f(x_{i_1}, \ldots, x_{i_\ell}). \]

We will make use of the following lemma:

**Lemma 4.2.** Let \(2 \leq m \leq L\), \(1 \leq K \leq m\) be integers, \(\{f^{(m-k)} : 1 \leq k \leq m-1\}\) be real-valued functions on \(A^{m-k}\), and \(f^{(0)}\) be a constant. Define a function \(g\) on \(A^{m-1}\) by

\[ g(x_1, \ldots, x_{m-1}) = \sum_{k=1}^{K} \frac{1}{K} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq m-1} f^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}}). \]

Then,

\[ \left( \hat{P}^L_{m-1} g \right)(x_1, \ldots, x_L) = \frac{1}{L - m + 1} \sum_{k=1}^{K} \binom{L - m + k}{k} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq L} f^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}}). \]
It is understood that in (4.9) and (4.10) the inner sum is equal to \( f(0) \) when \( k = m \).

**Proof.** On the left hand side of (4.10), all possible sets of \( m - 1 \) distinct numbers are selected out of \( \{1, \ldots, L\} \). For fixed \( 1 \leq k \leq m - 1 \) and \( 1 \leq i_1 < \cdots < i_{m-k} \leq L \), there are \( \binom{L-m+k}{k-1} \) ways to choose the remaining \( k - 1 \) indexes, and so the term \( f^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}}) \) will repeat \( \binom{L-m+k}{k-1} \) times, while the constant term \( f^{(0)} \) \( (k = m) \) will repeat \( \binom{L-m+1}{k} \) times. Therefore,

\[
\left( \hat{P}_{m-1}^L \right)(x_1, \ldots, x_L) = \sum_{k=1}^{K} \frac{1}{L-m+1} \binom{L-m+k}{k-1} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq L} f^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}}) = \frac{1}{L-m+1} \sum_{k=1}^{K} \binom{L-m+k}{k} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq L} f^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}}). \tag{4.11}
\]

By Lemma 2.2, there is a co-null set \( \tilde{S}_{N-m} \subset A^{N-m} \) such that for each \( (\tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \tilde{S}_{N-m} \) the set \( E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} := \{ (x_1, \ldots, x_m) \in A^m : (x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in E \} \) is co-null in \( A^m \). From now on, \( (\tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \tilde{S}_{N-m} \) is chosen and fixed for the rest of the proof. For each \( n \in \mathbb{N} \) and each \( 1 \leq k \leq \min(m-1, N-m) \), let us define a function \( v_n^{(m-k)} \) on \( A^{m-k} \) as

\[
v_n^{(m-k)}(x_1, \ldots, x_{m-k}) = \sum_{m+1 \leq j_1 < \cdots < j_k \leq N} u_n(x_1, \ldots, x_{m-k}, \tilde{x}_{j_1}, \ldots, \tilde{x}_{j_k}), \tag{4.12}
\]

and let

\[
v_n^{(0)} = \sum_{m+1 \leq j_1 < \cdots < j_m \leq N} u_n(\tilde{x}_{j_1}, \ldots, \tilde{x}_{j_m}). \tag{4.13}
\]
Then, for each \((x_1, \ldots, x_m) \in E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}\),

\[
\left( \hat{P}_m^n u_n \right)(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) = u_n(x_1, \ldots, x_m) + \sum_{k=1}^{K} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq m} v_n^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}}) \rightarrow U(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N),
\]

(4.14)

where \(K = \min(m, N-m)\), and the inner sum on the right hand side of the equality is equal to \(v^{(0)}_n\) when \(k = m\). The set \(\Omega := \{(x_1, \ldots, x_N) \in \Lambda^N : (x_{i_1}, \ldots, x_{i_m}) \in E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} \text{ for every } 1 \leq i_1 < \cdots i_m \leq N\}\) is co-null in \(\Lambda^N\) by Lemma 2.1. If \((x_1, \ldots, x_N) \in \Omega\), (4.14) implies that

\[
\left( \hat{P}_m^n u_n \right)(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \rightarrow U(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \mathbb{R},
\]

\[1 \leq i_1 < \cdots < i_m \leq N. \quad (4.15)\]

Adding (4.15) over all \(\binom{N}{m}\) selections of \(m\) distinct numbers out of \(\{1, \ldots, N\}\) and using (4.14) we obtain

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} \left( \hat{P}_m^n u_n \right)(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u_n(x_{i_1}, \ldots, x_{i_m}) + \sum_{k=1}^{K} \binom{N-m+k}{k} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq N} v_n^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}}) \rightarrow \sum_{1 \leq i_1 < \cdots < i_m \leq N} U(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \quad (4.16)
\]

for every \((x_1, \ldots, x_N) \in \Omega\). The equality in (4.16) is valid because for fixed \(1 \leq k \leq K\) and \(1 \leq i_1 < \cdots < i_{m-k} \leq N\), there are \(\binom{N-m+k}{k}\) ways to choose the remaining \(k\) indexes, and so the term \(v_n^{(m-k)}(x_{i_1}, \ldots, x_{i_{m-k}})\) will appear \(\binom{N-m+k}{k}\) times as different sets of \(m\) distinct numbers are selected out of \(\{1, \ldots, N\}\) on the left hand side of (4.16).
For each \( n \in \mathbb{N} \), let us define a function \( \omega_n \) on \( A^{m-1} \) as

\[
\omega_n(x_1, \ldots, x_{m-1}) = \sum_{k=1}^{K} \frac{1}{k} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq m-1} t_n^{(m-k)}(x_1, \ldots, x_{i_{m-k}}).
\] (4.17)

Then, using Lemma 4.2 with \( L = m \) and \( L = N \) respectively, (4.14) and (4.16) can be rewritten as

\[
u_n(x_1, \ldots, x_m) + \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq m} \omega_n(x_{i_1}, \ldots, x_{i_{m-1}}) \rightarrow U(x_1, \ldots, x_m, \tilde{x}_m+1, \ldots, \tilde{x}_N)
\] (4.18)

for every \((x_1, \ldots, x_m) \in E_{\tilde{x}_m+1, \ldots, \tilde{x}_N} \), and

\[
\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} u_n(x_{i_1}, \ldots, x_{i_{m-1}}) + (N-m+1) \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega_n(x_{i_1}, \ldots, x_{i_{m-1}}) \rightarrow \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} U(x_{i_1}, \ldots, x_{i_{m-1}}, \tilde{x}_m+1, \ldots, \tilde{x}_N)
\] (4.19)

for every \((x_1, \ldots, x_N) \in \Omega \). Because both (4.19) and (4.1) hold on \( \Omega \cap E \),

\[
\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega_n(x_{i_1}, \ldots, x_{i_{m-1}}) \rightarrow \frac{1}{N-m+1} \left[ \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} U(x_{i_1}, \ldots, x_{i_{m-1}}, \tilde{x}_m+1, \ldots, \tilde{x}_N) - U(x_1, \ldots, x_N) \right]
\] (4.20)

for every \((x_1, \ldots, x_N) \in \Omega \cap E \). Then, the induction hypothesis implies that there is a co-null set \( T_{m-1} \subset A^{m-1} \) and a real-valued function \( \omega \) on \( A^{m-1} \) such that \( \omega_n(x_1, \ldots, x_{m-1}) \rightarrow \omega(x_1, \ldots, x_{m-1}) \) for every \((x_1, \ldots, x_{m-1}) \in T_{m-1} \). Let \( T_m := \{(x_1, \ldots, x_m) \in A^m : (x_{i_1}, \ldots, x_{i_{m-1}}) \in T_{m-1} \text{ for every } 1 \leq i_1 < \cdots < i_{m-1} \leq m \} \).
\[ i_{m-1} \leq m \}. \text{ By Lemma 2.1 the set } T_m \text{ is co-null in } A^m, \text{ and by (4.18), } u_n \to u \text{ on } E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} \cap T_m, \text{ where}\]

\[ u(x_1, \ldots, x_m) := \]

\[ U(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) - \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq m} \omega(x_{i_1}, \ldots, x_{i_{m-1}}). \quad \square \] (4.21)

**Corollary 4.3.** Let \( 1 \leq m \leq N \) be integers, \( u \) be a real-valued function on \( A^m \), and \( U \) be a real-valued function on \( A^N \). Suppose that

\[ U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m}) \text{ a.e.} \quad (4.22) \]

Then,

(i) \( U \) is measurable if and only if \( u \) is measurable.

(ii) \( U = 0 \) a.e. if and only if \( u = 0 \) a.e.

(iii) \( U \) is symmetric a.e. if and only if \( u \) is symmetric a.e.

**Remark 4.2.** The conclusions are still true if \( U \) and \( u \) are a.e. finite.

**Proof (of Corollary 4.3).** Since the statement is true for \( N = m \), let us assume that \( 1 \leq m \leq N - 1 \).

(i) (a) Suppose that \( u \) is measurable. By (4.22),

\[ U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} U_{i_1, \ldots, i_m}(x_1, \ldots, x_N) \text{ a.e.} \quad (4.23) \]

where \( U_{i_1, \ldots, i_m}(x_1, \ldots, x_N) := u(x_{i_1}, \ldots, x_{i_m}) \) defines a measurable function on \( A^N \). Thus, \( U \) is measurable by the completeness of measure \( d^N x \).

(i) (b) Suppose that \( U \) is measurable. Let \( E \subset A^N \) be the co-null set on which (4.22) holds.

**Case 1:** \( m = 1 \). By the Fubini-Tonelli theorem (and Lemma 2.2 in particular), there is a co-null set \( \tilde{S}_{N-1} \in A^{N-1} \) such that for every \((\tilde{x}_2, \ldots, \tilde{x}_N) \in \tilde{S}_{N-1} \) the
set \( E_{\tilde{x}_2, \ldots, \tilde{x}_N} := \{ x \in A : (x, \tilde{x}_2, \ldots, \tilde{x}_N) \in E \} \) is co-null in \( A \) and \( U(\cdot, \tilde{x}_2, \ldots, \tilde{x}_N) \) is measurable. Let us choose and fix \((\tilde{x}_2, \ldots, \tilde{x}_N) \in \tilde{S}_{N-1}\). Since

\[
\sum_{i=1}^{N} u(x_i) = U(x_1, \ldots, x_N) - \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq m} \omega(x_{i_1}, \ldots, x_{i_{m-1}}) \tag{4.24}
\]

for each \( x \in E_{\tilde{x}_2, \ldots, \tilde{x}_N} \), \( u \) is measurable by the completeness of measure \( dx \).

**Case 2**: \( 2 \leq m \leq N-1 \). As in the proof of Theorem 4.1 we will use induction on \( m \). Let \( 2 \leq m \leq N-1 \), and assume that the "only if" statement of (i) is true for \( m-1 \). By the Fubini-Tonelli theorem and Lemma 2.2 there is a co-null set \( \tilde{S}_{N-m} \subset A^{N-m} \) such that for every \((\tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \tilde{S}_{N-m}\) the set \( E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} := \{ (x_1, \ldots, x_m) \in A^m : (x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in E \} \) is co-null in \( A^m \) and \( U(\cdot, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \) is measurable. Let us choose and fix \((\tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \tilde{S}_{N-m}\). By the proof of Theorem 4.1 there is a real-valued function \( \omega \) on \( A^{m-1} \) such that

\[
u(x_1, \ldots, x_m) = U(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) - \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq m} \omega(x_{i_1}, \ldots, x_{i_{m-1}}) \tag{4.25}\]

for every \((x_1, \ldots, x_m) \in E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}\) (cf. (4.18)). Moreover, there is a co-null set \( E_N \subset A^N \) on which

\[
\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega(x_{i_1}, \ldots, x_{i_{m-1}}) = \frac{1}{N - m + 1} \left[ \sum_{1 \leq i_1 < \cdots < i_{m} \leq N} U(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) - U(x_1, \ldots, x_N) \right] \tag{4.26}\]

(cf. (4.20)). The function on the right hand side of (4.26) is measurable by (i) (a). Thus, the induction hypothesis implies that \( \omega \) is measurable. By (i) (a)
again, the rightmost term of (4.25) is a measurable function on $A^m$, and so $u$ is measurable by the completeness of measure $d^m x$.

(ii) The "if" statement of (ii) can be easily proved using Lemma 2.1. Suppose that

$$ U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m}) = 0 \quad \text{a.e.}, \quad (4.27) $$

and let $E \subset A^N$ be the co-null set on which (4.27) holds.

**Case 1:** $m = 1$. By Lemma 2.2, there is a co-null set $\tilde{S}_{N-1} \subset A^{N-1}$ such that for each $(\tilde{x}_2, \ldots, \tilde{x}_N) \in \tilde{S}_{N-1}$ the set $E_{\tilde{x}_2, \ldots, \tilde{x}_N} := \{ x \in A : (x, \tilde{x}_2, \ldots, \tilde{x}_N) \in E \}$ is co-null in $A$. From now on, $(\tilde{x}_2, \ldots, \tilde{x}_N) \in \tilde{S}_{N-1}$ is chosen and fixed. For each $x \in E_{\tilde{x}_2, \ldots, \tilde{x}_N}$,

$$ u(x) + u(\tilde{x}_2) + \cdots + u(\tilde{x}_N) = 0. \quad (4.28) $$

Thus, if $(x_1, \ldots, x_N) \in E_{\tilde{x}_2, \ldots, \tilde{x}_N}$, (4.28) implies that

$$ u(x_j) + u(\tilde{x}_2) + \cdots + u(\tilde{x}_N) = 0, \quad 1 \leq j \leq N, \quad (4.29) $$

and so, by addition

$$ \sum_{j=1}^{N} u(x_j) + N \sum_{j=2}^{N} u(\tilde{x}_j) = 0. \quad (4.30) $$

Let $(x_1, \ldots, x_N) \in E_{\tilde{x}_2, \ldots, \tilde{x}_N} \cap E$. Then, for this point both (4.30) and (4.27) hold, and therefore $\sum_{j=2}^{N} u(\tilde{x}_j) = 0$. Thus, by (4.28), $u(x) = 0$ for every $x \in E_{\tilde{x}_2, \ldots, \tilde{x}_N}$.

**Case 2:** $2 \leq m \leq N - 1$. The proof proceeds by induction on $m$. Let $2 \leq m \leq N - 1$, and assume that the "only if" statement of (ii) is true for $m-1$. By Lemma 2.2, there is a co-null set $\tilde{S}_{N-m} \subset A^{N-m}$ such that for each $(\tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \tilde{S}_{N-m}$ the set $E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} := \{ (x_1, \ldots, x_m) \in A^m : (x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in E \}$ is co-null in $A^m$. From now on, $(\tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \tilde{S}_{N-m}$ is chosen and fixed. Let $\Omega = \{ (x_1, \ldots, x_N) \in A^N : (x_{i_1}, \ldots, x_{i_m}) \in E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} \text{ for every } 1 \leq i_1 < \cdots i_m \leq N \}$. 


By repeating the arguments in the proof of Theorem 4.1, we conclude that there is a real-valued function $\omega$ on $\Lambda^{m-1}$ such that

$$u(x_1, \ldots, x_m) + \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq m} \omega(x_{i_1}, \ldots, x_{i_{m-1}}) = 0 \quad (4.31)$$

for every $(x_1, \ldots, x_m) \in E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}$, and

$$\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega(x_{i_1}, \ldots, x_{i_{m-1}}) = 0 \quad (4.32)$$

for every $(x_1, \ldots, x_N) \in \Omega \cap E$. Since $\Omega \cap E \subset \Lambda^N$ is co-null, therefore by the induction hypothesis there is a co-null set $T_{m-1} \subset \Lambda^{m-1}$ such that $\omega(x_1, \ldots, x_{m-1}) = 0$ for every $(x_1, \ldots, x_{m-1}) \in T_{m-1}$ for every $1 \leq i_1 < \cdots < i_{m-1} \leq m$. By Lemma 2.1, the set $T_m$ is co-null in $\Lambda^m$. Moreover, by (4.31), $u(x_1, \ldots, x_m) = 0$ for every $(x_1, \ldots, x_m) \in E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} \cap T_m$.

(iii) (a) Let $S_m \subset A^m$ be the (symmetric) co-null set on which $u$ is symmetric, $E \subset \Lambda^N$ be the co-null set on which (4.22) holds, and $E^s \subset E$ be the symmetrization of $E$ defined in Section 2. The sets $E^s$ and $\Omega := \{(x_1, \ldots, x_N) \in \Lambda^N : (x_{i_1}, \ldots, x_{i_m}) \in S_m, \forall 1 \leq i_1 < \cdots < i_m \leq N\}$ are both co-null and symmetric. Moreover, for every $(x_1, \ldots, x_N) \in E^s \cap \Omega$ and every permutation $\pi$ of $\{1, \ldots, N\}$, we have that

$$U(x_{\pi(1)}, \ldots, x_{\pi(N)}) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{\pi(i_1)}, \ldots, x_{\pi(i_m)}) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m}) = U(x_1, \ldots, x_N). \quad (4.33)$$

(iii) (b) Let $S \subset \Lambda^N$ be the (symmetric) co-null set on which $U$ is symmetric, and $E \subset \Lambda^N$ be the co-null set on which (4.22) holds. Let $A := E^s \cap S$, where $E^s \subset E$ is the symmetrization of $E$ mentioned in (iii) (a). The "only if" statement of (iii) holds trivially for $m = 1$. As previously, we will use induction on
Let $2 \leq m \leq N - 1$, and assume that the "only if" statement of (iii) holds for $m - 1$. By Lemma 2.2 there is a co-null set $\tilde{S}_{N-m} \subset A^{N-m}$ such that for every $(\tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in \tilde{S}_{N-m}$ the set

$$A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} := \{(x_1, \ldots, x_m) \in A^m : (x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in A\}$$

is co-null in $A^m$. Symmetry of $A$ implies symmetry of $A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}$. From now on $(\tilde{x}_{m+1}, \ldots, \tilde{x}_N)$ is chosen and fixed. Let $\Omega := \{(x_1, \ldots, x_N) \in A^N : (x_{i_1}, \ldots, x_{i_m}) \in A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}, \forall 1 \leq i_1 < \cdots < i_m \leq N\}$. Then, $\Omega$ is co-null and symmetric in $A^N$. By the proof of Theorem 4.1, there is a real-valued function $\omega$ on $A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}$ such that (4.25) holds for every $(x_1, \ldots, x_m) \in A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}$, and (4.26) holds for every $(x_1, \ldots, x_N) \in \Omega \cap A$. Furthermore, $U(\cdot, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)$ is symmetric on $A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}$. (In fact, for every $(x_1, \ldots, x_m) \in A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}$, $(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \in A \subset S$. Therefore, $U(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) = U(x_{\pi(1)}, \ldots, x_{\pi(m)}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)$ for every permutation $\pi$ of $\{1, \ldots, m\}$.) Thus, the function on the right hand side of (4.26) is symmetric on $\Omega \cap A$, and so $\omega$ is symmetric a.e. on $A^{m-1}$ by the induction hypothesis. This together with (iii) (a) implies that the second term on the right hand side of (4.25) is symmetric a.e. on $A^m$. Let $S_m$ be the co-null (symmetric) set on which $\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq m} \omega(x_{i_1}, \ldots, x_{i_{m-1}})$ is symmetric. Then, $u$ is symmetric on $A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N} \cap S_m$ by (4.25).

**5. Sum Structure and Integrability**

In this section we focus on certain integrability properties of functions in the form of (1.3). Specifically, we introduce a condition on $P$ under which $U \in L^1(A^N; Pd^N x)$ if and only if $u \in L^1(A^m; \rho^{(m)} d^m x)$. Particularly, if $P$ satisfies this condition, then $\mathcal{V}_P = \mathcal{V}_{\rho^{(m)}}$. This fact is used in the proof of Theorem 3.10.
For convenience, it will be assumed that \( m < N \), since the conclusions hold trivially for \( m = N \).

**Theorem 5.1.** Let \( N \geq 2 \) and \( 1 \leq m \leq N - 1 \) be integers, \( u \) be a real-valued function on \( \Lambda^m \), and \( U \) be a real-valued function on \( \Lambda^N \). Suppose that

\[
U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u(x_{i_1}, \ldots, x_{i_m}) \text{ a.e.} \quad (5.1)
\]

For every \( 1 \leq k \leq m \), let \( \rho^{(k)} \) be a \( k \)-variable reduction of some a.e. symmetric probability density \( P \geq 0 \). That is

\[
\rho^{(k)}(x_1, \ldots, x_k) = \int_{\Lambda^{N-k}} P(x_{k+1}, \ldots, x_N) dx_{k+1} \cdots dx_N \text{ a.e. on } \Lambda^k. \quad (5.2)
\]

Then,

(i) \( \rho^{(k)}(x_1, \ldots, x_k) \) is measurable by Corollary 4.3 (i).

Indeed, if \( u \in L^1(\Lambda^m; \rho^{(m)}d^m x) \), then \( U \) is measurable by Corollary 4.3 (i).

Moreover,

\[
\int_{\Lambda^N} |U|P d^N x \leq \sum_{1 \leq i_1 < \cdots < i_m \leq N} \int_{\Lambda^N} |u(x_{i_1}, \ldots, x_{i_m})|P(x_{i_1}, \ldots, x_N) d^N x
\]

\[
= \binom{N}{m} \int_{\Lambda^m} |u|\rho^{(m)} d^m x, \quad (5.4)
\]

where the equality follows by Lemma 2.3 and the Fubini-Tonelli theorem.
(ii) Let us assume that $U \in L^1(A^N; Pd^N x)$. Then $u$ is measurable by Corollary 4.3 (i). To show that it follows that $u \in L^1(A^m; \rho^m d^m x)$, we will follow closely the line of arguments used in the proof of Theorem 4.1. Let $E \subset A^N$ be the co-null set on which (5.5) holds.

**Case 1:** $m = 1$. By the Fubini-Tonelli theorem (and Lemma 2.2 in particular), there is a co-null set $\tilde{S}_{N-1} \subset A^{N-1}$ such that for every $(\tilde{x}_2, ..., \tilde{x}_N) \in \tilde{S}_{N-1}$ the set $E_{\tilde{x}_2, ..., \tilde{x}_N} := \{ x \in A : (x, \tilde{x}_2, ..., \tilde{x}_N) \in E \}$ is co-null in $A$, $P(\cdot, \tilde{x}_2, ..., \tilde{x}_N) \in L^1(A; dx)$, and $P(\cdot, \tilde{x}_2, ..., \tilde{x}_N)U(\cdot, \tilde{x}_2, ..., \tilde{x}_N) \in L^1(A; dx)$. Then, $\tilde{S}_{N-1} \cap A_{N-1}$ has a positive measure, and so is not empty. Let $(\tilde{x}_2, ..., \tilde{x}_N) \in \tilde{S}_{N-1} \cap A_{N-1}$ be chosen and fixed for the rest of the proof. Since for each $x \in E_{\tilde{x}_2, ..., \tilde{x}_N}$,

$$u(x) + u(\tilde{x}_2) + \cdots + u(\tilde{x}_N) = U(x, \tilde{x}_2, ..., \tilde{x}_N),$$

(5.5)

it follows that

$$\int_A |u(x)| \rho(1)(x) dx \leq C_{N-1}^{-1}(\tilde{x}_2, ..., \tilde{x}_N) \int_A P(x, \tilde{x}_2, ..., \tilde{x}_N)|u(x)| dx$$

$$\leq C_{N-1}^{-1}(\tilde{x}_2, ..., \tilde{x}_N) \times$$

$$\left( \int_A P(x, \tilde{x}_2, ..., \tilde{x}_N)|U(x, \tilde{x}_2, ..., \tilde{x}_N)| dx + \rho^{(N-1)}(\tilde{x}_2, ..., \tilde{x}_N) \sum_{j=2}^N |u(\tilde{x}_j)| \right) < \infty.$$  

(5.6)

**Case 2:** $2 \leq m \leq N - 1$. The proof proceeds by induction on $m$. Let $2 \leq m \leq N - 1$, and assume that (ii) is true for $m - 1$. By the Fubini-Tonelli theorem, there is a co-null set $\tilde{S}_{N-m} \in A^{N-m}$ such that for every $(\tilde{x}_{m+1}, ..., \tilde{x}_N) \in \tilde{S}_{N-m}$ the set $E_{\tilde{x}_{m+1}, ..., \tilde{x}_N} := \{ (x_1, ..., x_m, \tilde{x}_{m+1}, ..., \tilde{x}_N) \in E \}$ is co-null in $A^m$, $P(\cdot, \tilde{x}_{m+1}, ..., \tilde{x}_N) \in L^1(A^m; d^m x)$, and $P(\cdot, \tilde{x}_{m+1}, ..., \tilde{x}_N)U(\cdot, \tilde{x}_{m+1}, ..., \tilde{x}_N) \in L^1(A^m; d^m x)$. Then, $\tilde{S}_{N-m} \cap A_{N-m}$ has a positive measure, and so is not empty. Let $(\tilde{x}_{m+1}, ..., \tilde{x}_N) \in \tilde{S}_{N-m} \cap A_{N-m}$ be chosen and fixed. By the arguments in
the proof of Theorem 4.1, there is a real-valued function \( \omega \) on \( A^{m-1} \) such that (4.25) holds for every \((x_1, \ldots, x_m) \in E_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}\). Moreover, (4.26) holds on some co-null set \( E_N \subset A^N \). The first term inside the square brackets in (4.26) can be written as a sum of measurable functions. Namely,

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} U(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} U_{i_1, \ldots, i_m}(x_1, \ldots, x_N),
\]

(5.7)

where \( U_{i_1, \ldots, i_m}(x_1, \ldots, x_N) := U(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \) for every \( 1 \leq i_1 < \cdots < i_m \leq N \). Moreover, for every \( 1 \leq i_1 < \cdots < i_m \leq N \),

\[
\int_{A^N} |U_{i_1, \ldots, i_m}| P d^N x = \int_{A^m} |U(\cdot, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)| \rho^{(m)}(\cdot) d^m x \leq \\
[C_{N-m}(\tilde{x}_{m+1}, \ldots, \tilde{x}_N)]^{-1} \int_{A^m} |U(\cdot, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)| P(\cdot, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) d^m x < \infty,
\]

(5.8)

where the equality follows by the Fubini-Tonelli theorem and Lemma 2.3. Therefore, the right hand side of (4.26) defines a function in \( L^1(A^N; P d^N x) \), and so, by the induction hypothesis, \( \omega \in L^1(A^{m-1}; \rho^{(m-1)} d^{m-1} x) \). By (i), with \( P \) replaced by \( \rho^{(m)} \), \( \rho^{(m)} \) replaced by \( \rho^{(m-1)} \), and \( u \) replaced by \( \omega \), the second term on the right hand side of (4.27) belongs to \( L^1(A^m; \rho^{(m)} d^m x) \). Since, the same is true for \( U(\cdot, \tilde{x}_{m+1}, \ldots, \tilde{x}_N) \) by (5.3), it follows that \( u \in L^1(A^{m}; \rho^{(m)} d^m x) \). \( \square \)

**Remark 5.1.** If \( A \) has finite \( dx \) measure and contains subsets of arbitrarily small positive measure, then any a.e. symmetric probability density \( P \) can be approximated in \( L^1(A^N; d^N x) \) by an a.e. symmetric probability density satisfying (5.8).

In fact, let \( N \geq 2 \) and \( 1 \leq m \leq N - 1 \) be integers, and assume first that \( P \) is essentially bounded. If the above conditions on \( (A; dx) \) are true, then for each
\[ \varepsilon > 0, \text{ there is } A_{\varepsilon} \subset A \text{ with } |A_{\varepsilon}| < \varepsilon. \]

Let
\[ E_\varepsilon := \bigcup_{k=1}^{m} \bigcup_{1 \leq i_1 < \cdots < i_{N-k} \leq N} \{(x_1, \ldots, x_N) \in A^N : (x_{i_1}, \ldots, x_{i_{N-k}}) \in A^{N-k}_\varepsilon \} \subset A^N. \]

(5.9)

Then \( E_\varepsilon \) is measurable and symmetric. Moreover, \(|E_\varepsilon| \leq \sum_{k=1}^{m} \binom{N}{N-k}_\varepsilon^{N-k}|A|^k\), and so \(|E_\varepsilon| \to 0\) as \( \varepsilon \to 0^+ \). Define \( P_\varepsilon = a_\varepsilon \) on \( E_\varepsilon \), and \( P_\varepsilon = a_\varepsilon P \) on \( A^N \setminus E_\varepsilon \), where \( a_\varepsilon = \left( |E_\varepsilon| + \int_{A^N \setminus E_\varepsilon} P \right)^{-1} \) is the normalization constant. Then \( P_\varepsilon \) is an a.e. symmetric probability density, and \( a_\varepsilon \to 1 \) as \( \varepsilon \to 0^+ \). Since \( 0 < a_\varepsilon \leq 2 \) for \( \varepsilon \) small enough, we have
\[
\int_{A^N} |P - P_\varepsilon| d^N x = \int_{E_\varepsilon} |P - a_\varepsilon| d^N x + \int_{A^N \setminus E_\varepsilon} |1 - a_\varepsilon| d^N x 
\leq (||P||_\infty + 2)|E_\varepsilon| + (1 - a_\varepsilon) \to 0 \text{ as } \varepsilon \to 0^+. \tag{5.10}
\]

Now, we will show that \( P_\varepsilon \) satisfies (5.3) for small enough \( \varepsilon \) (such that \( 0 < a_\varepsilon \leq 2 \)). Indeed, \((x_1, \ldots, x_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_N) \in E_\varepsilon \) for every \( 1 \leq k \leq m \), every \((\tilde{x}_{k+1}, \ldots, \tilde{x}_N) \in A^{N-k}_\varepsilon \), and every \((x_1, \ldots, x_k) \in A^k \). Therefore,
\[
\rho^{(k)}_\varepsilon(x_1, \ldots, x_k) = \int_{A^{N-k}_\varepsilon} P_\varepsilon(x_1, \ldots, x_k) dx_{k+1} \cdots dx_N 
\leq 2(1 + ||P||_\infty)|A|^{N-k} \]
\[
\quad \times ||P||_{\infty} |A|^{N-k} a_\varepsilon = C_{N-k}^{-1} P_\varepsilon(x_1, \ldots, x_k, \tilde{x}_{k+1}, \ldots, \tilde{x}_N), \tag{5.11}
\]
where \( C_{N-k} = a_\varepsilon/(2(1 + ||P||_\infty)|A|^{N-k}) \).

Suppose that \( P \) is not essentially bounded. Then, for each \( n \in \mathbb{N} \), define \( P_n = a_n P \) on \( \{P \leq n\} := \{(x_1, \ldots, x_N) \in A^N : P(x_1, \ldots, x_N) \leq n \} \), and \( P_n = a_n \) on \( \{P > n\} := A^N \setminus \{P \leq n\} \), where \( a_n = \left( |\{P > n\}| + \int_{\{P \leq n\}} P \right)^{-1} \) is the normalization constant. Then, \( P_n \) is a bounded a.e. symmetric probability density. By Chebyshev’s inequality [9] Theorem 6.17, \(|\{P > n\}| \leq \frac{1}{n} \to 0 \) as \( n \to \infty \), and \( \int_{\{P \leq n\}} P \to 1 \) by monotone convergence. Therefore, \( a_n \to 1 \) as \( n \to \infty \).
Moreover, $0 = |\{P = \infty\}| = \cap_{n=1}^{\infty} \{P > n\}$ implies that $\int_{\{P>n\}} Pd^N x \to 0$. Thus,

$$\int_{A^N} |P - P_n| d^N x = \int_{\{P>n\}} |P - a_n| d^N x + \int_{\{P\leq n\}} P|1 - a_n| d^N x \leq \int_{\{P>n\}} Pd^N x + a_n|\{P > n\}| + (1 - a_n) \to 0 \text{ as } n \to \infty. \quad (5.12)$$

Therefore, $P$ can be approximated in $L^1(A^N; d^N x)$ by $P_n$, and then $P_n$ can be approximated in $L^1(A^N; d^N x)$ by $P_{\varepsilon}$ using the previous argument.

6. Generalization to Several Species

Consider a system consisting of $N_1$ particles of type 1, $N_2$ particles of type 2,..., $N_L$ particles of type $L$, so that $\sum_{\ell=1}^{L} N_\ell = N$. The interaction potentials and densities in such a system have to depend on the composition of particles. They also have to be symmetric with respect to the interchange of particles of the same type. To describe these properties precisely, we need to introduce a number of technical definitions.

Let us start by defining $L + 1$ integers: $M_0 = 0$ and $M_\ell = \sum_{t=1}^{\ell} N_\ell$ for every $1 \leq \ell \leq L$, so that $M_\ell$ is the total number of particles having types 1 through $\ell$. Next, for every $1 \leq k \leq N$, we define a set of $k$-tuples of natural numbers by

$$A_k := \{ (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k : 1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq L, \text{ and } \alpha_{a-1} < \alpha_a = \alpha_p = \alpha_b < \alpha_{b+1} \text{ implies } b - a + 1 \leq N_{\alpha_p} \forall 1 \leq p \leq k, \text{ where } \alpha_0 := 0 \text{ and } \alpha_{k+1} := L + 1 \}. \quad (6.1)$$

The second condition on $(\alpha_1, \ldots, \alpha_k)$ precludes the number $\alpha_p$ to repeat more than $N_{\alpha_p}$ times for every $1 \leq p \leq k$. In what follows, the index $(\alpha_1, \ldots, \alpha_k)$ will correspond to a particular composition of $k$ particles selected out of $L$ types.
such that no more than \( N_\ell \) particles of type \( \ell \) are selected for each \( 1 \leq \ell \leq L \).

For example, if \( L = N_1 = 2 \) and \( N_2 = 3 \), then \( A_3 = \{(1,1,2), (1,2,2), (2,2,2)\} \), where \( (1,1,2) \) denotes the composition of two particles of type one and one particle of type two, and so on.

For every \( 1 \leq k \leq N \) and every \((\alpha_1, \ldots, \alpha_k) \in A_k\), let us fix a corresponding set of \( k \) integers \( 1 \leq i_1 < \cdots < i_k \leq N \) by

\[
i_1 = M_{\alpha_1-1} + 1, \\
i_p = \begin{cases} i_{p-1} + 1 & \text{if } \alpha_p = \alpha_{p-1}, \\ M_{\alpha_p-1} + 1 & \text{if } \alpha_p > \alpha_{p-1} \end{cases} \tag{6.2}
\]

for \( 2 \leq p \leq k \).

If \((\alpha_1, \ldots, \alpha_k)\) corresponds to a certain composition of \( k \) particles, then \((i_1, \ldots, i_k)\) determines a particular way to choose a set of \( k \) particles out of \( N \) having this composition. Namely, if we were to choose \( b \) particles of type \( \ell \), then particles \( M_{\ell-1} + 1, \ldots, M_{\ell-1} + b \) (i.e., the first \( b \) particles of type \( \ell \)) will be selected.

Using these integers, we define a linear transformation \( T^{\alpha_1: \alpha_N}(x_1, \ldots, x_N) = (y_1, \ldots, y_N) \) on \( \Lambda^N \) by

\[
y_p = x_p \quad \forall 1 \leq p \leq k, \\
y_i = \begin{cases} x_{i+k-p} & \text{if } i_p < i < i_{p+1} \text{ for some } 1 \leq p \leq k-1, \\ x_{i+k} & \text{if } 1 \leq i < i_1, \\ x_i & \text{if } i_k < i \leq N. \end{cases} \tag{6.3}
\]

In other words, \( T^{\alpha_1: \alpha_N} \) puts \( x_1, \ldots, x_k \) into \( i_1^{th}, \ldots, i_k^{th} \) places, while \( x_{k+1}, \ldots, x_N \) occupy the remaining places in increasing order. For instance, if \( L = N_1 = N_2 = \)
2, then

\[ T^{11}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4), \]
\[ T^{12}(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2, x_4), \]
\[ T^{22}(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1, x_2). \]

(6.4)

We will also make use of the following sets of permutations:

\[ B := \{ \pi \text{ is a permutation of } \{1, \ldots, N\} \text{ satisfying} \]
\[ M_{\ell-1} + 1 \leq i \leq M_{\ell} \implies M_{\ell-1} + 1 \leq \pi(i) \leq M_{\ell} \quad \forall 1 \leq i \leq N \}. \]

(6.5)

If \{1, \ldots, N\} enumerates \(N\) particles of our system, then \(B\) is the set of all permutations of \{1, \ldots, N\} that interchange particles of the same species.

For every \(1 \leq k \leq m\) and every \((\alpha_1, \ldots, \alpha_k) \in A_k\), let

\[ B^{\alpha_1 \cdots \alpha_k} := \{ \pi \text{ is a permutation of } \{1, \ldots, k\} \text{ satisfying} \]
\[ \alpha_{a-1} < \alpha_a = \alpha_i = \alpha_b < \alpha_{b+1} \implies a \leq \pi(i) \leq b \quad \forall 1 \leq i \leq k \]
\[ \text{where } \alpha_0 := 0 \text{ and } \alpha_{k+1} := L \}. \]

(6.6)

If \{1, \ldots, k\} enumerates \(k\) particles, the first one having the type \(\alpha_1\), the second having the type \(\alpha_2\) and so on, then \(B^{\alpha_1 \cdots \alpha_k}\) is the set of all permutations of \{1, \ldots, k\} that interchange particles of the same species.

From now on, a set \(S \subset A^N\) will be called \(B\)-symmetric if \((x_1, \ldots, x_N) \in S\) implies that \((x_{\pi(1)}, \ldots, x_{\pi(N)}) \in S\) for every permutation \(\pi \in B\). A function \(f\) on \(A^N\) will be called \(B\)-symmetric on a \((B\text{-symmetric})\) set \(S\) if \(f(x_1, \ldots, x_N) = f(x_{\pi(1)}, \ldots, x_{\pi(N)})\) for every \((x_1, \ldots, x_N) \in S\) and every permutation \(\pi \in B\). Similar definitions will be used for \(B^{\alpha_1 \cdots \alpha_k}\)-symmetric sets and functions.

Finally, let \(\alpha : \{1, \ldots, N\} \to \{1, \ldots, L\}\) be a function determined by

\[ \alpha(i) = \ell \quad \text{if} \quad M_{\ell-1} + 1 \leq i \leq M_{\ell} \quad \forall 1 \leq i \leq N. \]

(6.7)
If \( \{1, \ldots, N\} \) enumerates \( N \) particles of our system, then \( \alpha(i) \) is the type of \( i^{th} \) particle.

As before, the total potential of the system has the form \( W + U \), where \( W \) is a fixed internal potential assumed to be measurable, a.e. finite function on \( \Lambda^N \). Naturally, \( W \) has to be a.e. symmetric with respect to the interchange of particles of the same type. In other words, \( W \) is a.e. \( B \)-symmetric. For example, if \( L = N_1 = N_2 = 2 \), then \( W(x_1, x_2, x_3, x_4) = W(x_2, x_1, x_3, x_4) \) a.e., but \( W(x_1, x_2, x_3, x_4) \neq W(x_1, x_4, x_3, x_2) \) in general.

Suppose that for \( 1 \leq m \leq N \), we are given a set of functions \( \{\rho^{\alpha_1 \ldots \alpha_m} : (\alpha_1, \ldots, \alpha_m) \in A_m\} \) on \( \Lambda^m \). It is also known that for every \( (\alpha_1, \ldots, \alpha_m) \in A_m \), \( \rho^{\alpha_1 \ldots \alpha_m} \) is an \( m \)-variable reduction to \( \Lambda^m \) of a.e. positive and \( B \)-symmetric probability density \( P \) on \( \Lambda^N \). That is

\[
\rho^{\alpha_1 \ldots \alpha_m}(x_1, \ldots, x_m) = \int_{\Lambda^{N-m}} P(T^{\alpha_1 \ldots \alpha_m}(x_1, \ldots, x_N)) dx_{m+1} \cdots dx_N \quad \text{a.e.,} \tag{6.8}
\]

where the transformation \( T^{\alpha_1 \ldots \alpha_m} \) is determined by (6.2) and (6.3). For example, if \( L = N_1 = N_2 = 2 \), then

\[
\begin{align*}
\rho^{11}(x_1, x_2) &= \int_{A^2} P(x_1, x_2, x_3, x_4) dx_3 dx_4, \\
\rho^{12}(x_1, x_2) &= \int_{A^2} P(x_1, x_3, x_2, x_4) dx_3 dx_4, \\
\rho^{22}(x_1, x_2) &= \int_{A^2} P(x_3, x_4, x_1, x_2) dx_3 dx_4.
\end{align*}
\tag{6.9}
\]

By the arguments analogous to the proof of Lemma 2.3, \( \rho^{\alpha_1 \ldots \alpha_m} \) is seen to be a.e. positive and symmetric with respect to the interchange of particles of the same species. That is, \( \rho^{\alpha_1 \ldots \alpha_m} \) is a.e. \( B^{\alpha_1 \ldots \alpha_m} \)-symmetric. Moreover, the transformation \( T^{\alpha_1 \ldots \alpha_m} \) in (6.8) can be defined using any \( m \) integers satisfying \( M_{\alpha_{i_1} - 1} + 1 \leq i_1 < \cdots < i_m \leq M_{\alpha_m} \) and \( M_{\alpha_p - 1} + 1 \leq i_p \leq M_{\alpha_p} \) for every...
\[ 1 \leq p \leq m. \] (Particularly, using (6.2) with \( k = m \).) For instance, taking the earlier example,

\[ \rho^{12}(x_1, x_2) = \int_{A^2} P(x_1, x_3, x_2, x_4) dx_3 dx_4 = \int_{A^2} P(x_3, x_1, x_2, x_4) dx_3 dx_4. \] (6.10)

For \( 1 \leq m \leq N \) and a measurable potential \( U \), we define a set of classical \( m \)-particle densities \( \{ \rho^{\alpha_1 \cdots \alpha_m}_U : (\alpha_1, \cdots, \alpha_m) \in \mathcal{A}_m \} \) by

\[ \rho^{\alpha_1 \cdots \alpha_m}_U(x_1, \ldots, x_m) = \int_{\Lambda^{N-m}} e^{-[W+U](T^{\alpha_1 \cdots \alpha_m}(x_1, \ldots, x_N))} dx_{m+1} \cdots dx_N / Z(U), \] (6.11)

where

\[ Z(U) = \int_{\Lambda^N} e^{-W-U} d^N x. \] (6.12)

Whenever \( 0 < Z(U) < \infty \).

6.1. Existence and uniqueness. Similarly to the single species case, maximizers of a certain functional turn out to be solutions of the inverse problem. A suitable functional here is

\[ \mathfrak{F}_P(V) := e^{-\int_{\Lambda^N} V d^N x} / Z(V), \] (6.14)

\[^3\] Unlike the case of a single species, the uniqueness of \( U \) does not imply the uniqueness of the set \( \{ u^{\alpha_1 \cdots \alpha_m} : (\alpha_1, \cdots, \alpha_m) \in \mathcal{A}_m \} \). See Subsection 6.2.
defined on a (convex) set of functions

\[ \mathcal{D}_P := \{ V \in L^1(A^N; Pd^N x), e^{-V} \in L^1(A^N; e^{-W} d^N x), \]
\[ V(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \text{ a.e.}, \]
\[ \text{where } v^{\alpha_1 \cdots \alpha_m} \text{ is a } B^{\alpha_1 \cdots \alpha_m}\text{-symmetric real-valued measurable function on } A^m \text{ for every } (\alpha_1, \ldots, \alpha_m) \in A_m. \] (6.15)

We will also need a smaller set \( \mathcal{D}_{\rho(m)} \subset \mathcal{D}_P \) defined by

\[ \mathcal{D}_{\rho(m)} := \{ V : e^{-V} \in L^1(A^N; e^{-W} d^N x), \]
\[ V(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \text{ a.e.}, \]
\[ \text{where } v^{\alpha_1 \cdots \alpha_m} \text{ is a } B^{\alpha_1 \cdots \alpha_m}\text{-symmetric real-valued function on } A^m, \]
\[ v \in L^1(A^m; \rho^{\alpha_1 \cdots \alpha_m} d^m x) \text{ for every } (\alpha_1, \ldots, \alpha_m) \in A_m. \] (6.16)

We will assume that (2.5) holds, and that \( \mathcal{D}_P \neq \emptyset \). (When \( e^{-W} \in L^1(A^N; d^N x) \), \( \mathcal{D}_P \) contains all constant functions.) The functional \( \Im_P \) is well defined on \( \mathcal{D}_P \) as a positive real number.

The next two theorems assert the existence of solutions to the inverse problem for several species under the conditions stated above.

**Theorem 6.1.** Let \( \{ \rho^{\alpha_1 \cdots \alpha_m} : (\alpha_1, \ldots, \alpha_m) \in A_m \} \) be a set of \( m \)-variable reductions of an a.e. positive probability density \( P \). That is, \( \rho^{\alpha_1 \cdots \alpha_m} \) is given by (6.5). Let \( W \) be an a.e. finite measurable potential on \( A^N \). Suppose that \( (W + \log P)_+ \in L^1(A^N; P d^N x) \) and \( \mathcal{D}_P \neq \emptyset \). Then, there is \( U \in \mathcal{D}_P \) such that \( \Im_P(V) \leq \Im_P(U) \) for every \( V \in \mathcal{D}_P \). Furthermore, if \( U_1 \) and \( U_0 \) are two maximizers of \( \Im_P \) on \( \mathcal{D}_P \) (or on \( \mathcal{D}_{\rho(m)} \), if they exist), then \( U_1 - U_0 \) is a.e. a constant.
Proof. This theorem can be established by exactly the same arguments that lead to Theorem 3.6 with one notable exception. Namely, the fact that \( U \) has the required sum structure for membership in \( \mathcal{D}_P \) is verified by referring to Theorem 6.4 and Corollary 6.5 which are proved in the next subsection. □

**Theorem 6.2.** Let \( P \) be an a.e. positive and \( \mathcal{B} \)-symmetric probability density, and \( W \) be an a.e. finite and \( \mathcal{B} \)-symmetric measurable potential on \( \Lambda^N \). Suppose that for some \( U \in \mathcal{D}_P \), \( \mathcal{I}_P(V) \leq \mathcal{I}_P(U) \) for every \( V \in \mathcal{D}_P \). Then, \( \rho^{\alpha_1 \ldots \alpha_m} = \rho^{\alpha_1 \ldots \alpha_m} \) a.e. for every \( (\alpha_1, \ldots, \alpha_m) \in A_m \).

**Proof.** For every \( (\alpha_1, \ldots, \alpha_m) \in A_m \), let \( \xi^{\alpha_1 \ldots \alpha_m} \in L^\infty(A^m; d^m x) \) be a \( \mathcal{B}^{\alpha_1 \ldots \alpha_m} \)-symmetric real-valued function. For every \( (x_1, \ldots, x_N) \in \Lambda^N \), set

\[
\Xi(x_1, \ldots, x_N) := \sum_{1 \leq i_1 < \cdots < i_m \leq N} \xi^{\alpha(i_1) \ldots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}). \tag{6.17}
\]

Then, \( |\Xi(x_1, \ldots, x_N)| \leq \sum_{1 \leq i_1 < \cdots < i_m \leq N} ||\xi^{\alpha(i_1) \ldots \alpha(i_m)}||_{\infty} \) a.e., and so \( \Xi \in L^\infty(A^N; d^N x) \).

For every \( t \in \mathbb{R} \), \( U + t\Xi \in \mathcal{D}_P \) and, therefore, \( \mathcal{I}_P(U + t\Xi) \leq \mathcal{I}_P(U) \). This implies that \( \frac{d}{dt}\mathcal{I}_P(U + t\Xi)|_{t=0} = 0 \) because the function \( t \mapsto \mathcal{I}_P(U + t\Xi) \) is smooth. Equivalently,

\[
\mathcal{I}_P(U) \left[ \frac{\int_{A^N} e^{-U-W} \Xi d^N x}{Z(U)} - \int_{A^N} \Xi P d^N x \right] = 0. \tag{6.18}
\]

By the properties of \( \rho^{\alpha_1 \ldots \alpha_m} \) (which are also applicable to \( \rho^{\alpha_1 \ldots \alpha_m}_U \)) discussed right after (6.9), (6.18) amounts to

\[
\sum_{(\alpha_1, \ldots, \alpha_m) \in A_m} C^{\alpha_1 \ldots \alpha_m} \int_{A^m} \xi^{\alpha_1 \ldots \alpha_m}(\rho^{\alpha_1 \ldots \alpha_m}_U - \rho^{\alpha_1 \ldots \alpha_m}) d^m x = 0, \tag{6.19}
\]

where \( C^{\alpha_1 \ldots \alpha_m} \) is a positive constant. By the same properties, for every \( (\alpha_1, \ldots, \alpha_m) \in A_m \), the \( (\mathcal{B}^{\alpha_1 \ldots \alpha_m} \)-symmetric) set \( S_m^{\alpha_1 \ldots \alpha_m} \subset A^m \) on which \( \rho^{\alpha_1 \ldots \alpha_m} \) and \( \rho^{\alpha_1 \ldots \alpha_m}_U \) are both finite and \( \mathcal{B}^{\alpha_1 \ldots \alpha_m} \)-symmetric is co-null. Choosing \( \xi^{\alpha_1 \ldots \alpha_m} = \chi_{S_m^{\alpha_1 \ldots \alpha_m}} \times \chi_{\Lambda^N} \), where \( \chi \) is the characteristic function of \( S_m^{\alpha_1 \ldots \alpha_m} \times \Lambda^N \) and \( \Omega \) the domain of \( \mathcal{I}_P \), we have

\[
\int_{\Omega} e^{-U-W} \chi_{S_m^{\alpha_1 \ldots \alpha_m}}(x) \chi_{\Lambda^N}(y) d^N x = 0.
\]
sign($\rho^{\alpha_1 \cdots \alpha_m} - \rho^{\alpha_1 \cdots \alpha_m}$) (with sign(0) := 0), we obtain $\int_{A_m} |\rho_U^{\alpha_1 \cdots \alpha_m} - \rho^{\alpha_1 \cdots \alpha_m}| \, dm \, x = 0$, and so $\rho_U^{\alpha_1 \cdots \alpha_m} = \rho^{\alpha_1 \cdots \alpha_m}$ a.e. for every $(\alpha_1, \ldots, \alpha_m) \in A_m$. \qed

The solutions to the inverse problem on $D_P$ do not have to be unique. However, if $U \in D_{\rho^{(m)}}$ and $\rho_U^{\alpha_1 \cdots \alpha_m} = \rho^{\alpha_1 \cdots \alpha_m}$ a.e. for every $(\alpha_1, ..., \alpha_m) \in A_m$, then $\Im_P(V) \leq \Im_P(U)$ for every $V \in D_{\rho^{(m)}}$. In other words, every solution of the inverse problem on $D_{\rho^{(m)}}$ is a maximizer. This fact can be verified by a straight-forward modification of the proof of Theorem 3.8. But then, Theorem 6.1 implies that solutions of the inverse problem on $D_{\rho^{(m)}}$ are unique, in a sense that if $U_0$ and $U_1$ are in $D_{\rho^{(m)}}$, and $\rho_{U_0}^{\alpha_1 \cdots \alpha_m} = \rho_{U_1}^{\alpha_1 \cdots \alpha_m} = \rho^{\alpha_1 \cdots \alpha_m}$ a.e. for every $(\alpha_1, ..., \alpha_m) \in A_m$, then $U_1 - U_0$ is a.e. a constant. Therefore, the existence and uniqueness of solutions are guaranteed when $D_{\rho^{(m)}} = D_P$.

Theorem 6.6 in the next subsection provides sufficient conditions on $P$ under which $D_P = D_{\rho^{(m)}}$. Adding these conditions to the assumptions stated in Theorems 6.1 and 6.2 we can formulate the following theorem on the existence and uniqueness of solutions to the inverse problem in the case of several particle species. Since the solution is trivial for $m = N$, we will assume that $m < N$.

**Theorem 6.3.** Let $N \geq 2$, $2 \leq L \leq N$, $1 \leq m \leq N - 1$, and $N_1, ..., N_L$ be integers such that $\sum_{\ell=1}^L N_\ell = N$. For every $1 \leq k \leq m$ and every $(\alpha_1, ..., \alpha_k) \in A_k$, let $\rho^{\alpha_1 \cdots \alpha_k}$ be a $k$-variable reduction of some a.e. positive and $B$-symmetric probability density $P$. This means that $\rho^{\alpha_1 \cdots \alpha_k}$ is given by (6.8) with $m$ replaced by $k$. Suppose that, for each $1 \leq k \leq m$ and each $(\alpha_1, ..., \alpha_k) \in A_k$, there is a set $A_{N-k}^{\alpha_1 \cdots \alpha_k} \subset A^{N-k}$ of positive measure such that for every $(x_{k+1}, ..., x_N) \in A_{N-k}^{\alpha_1 \cdots \alpha_k}$...
there is a constant \( C_{N-k}^{\alpha_1 \cdots \alpha_k}(x_{k+1}, \ldots, x_N) > 0 \) satisfying

\[
P(T^{\alpha_1 \cdots \alpha_k}(\cdot, x_{k+1}, \ldots, x_N)) \geq C_{N-k}^{\alpha_1 \cdots \alpha_k}(x_{k+1}, \ldots, x_N)\rho^{\alpha_1 \cdots \alpha_k}(\cdot) \quad \text{a.e. on } \Lambda^k. \tag{6.20}
\]

Let \( W \) be an a.e. finite and \( \mathcal{B} \)-symmetric measurable potential on \( \Lambda^N \). Then, \( \mathcal{D}_P = \mathcal{D}_{\rho^{(m)}} \). If, in addition, \( \mathcal{D}_P \neq \emptyset \) and \( (W + \log P)_+ \in L^1(\Lambda^N; Pd^N x) \), there is \( U \in \mathcal{D}_P \) such that \( \rho_U^{\alpha_1 \cdots \alpha_m} = \rho^{\alpha_1 \cdots \alpha_m} \) a.e. for every \( (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m \), where \( \rho_U^{\alpha_1 \cdots \alpha_m} \) is defined by (6.11) and (6.12). Moreover, if \( U_0, U_1 \in \mathcal{D}_P \) and \( \rho_U^{\alpha_1 \cdots \alpha_m} = \rho_{U_1}^{\alpha_1 \cdots \alpha_m} = \rho^{\alpha_1 \cdots \alpha_m} \) a.e. for every \( (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m \), then \( U_1 - U_0 \) is a.e. a constant.

6.2. Properties of a.e. convergent sums. The material presented in this subsection is a generalization of the results developed in Sections 4 and 5. Particularly, we show that the sum structure of functions in the form of (6.13) is preserved by a.e. convergence. We also prove some other properties of such functions. These results are used in the proofs of Theorems 6.1 and 6.3. Throughout this subsection we will use the definitions made at the beginning of Section 6.

**Theorem 6.4.** For each \( (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m \), let \( (u_n^{\alpha_1 \cdots \alpha_m}) \) be a sequence of real-valued functions on \( \Lambda^m \). Suppose that there is a real-valued function \( U \) on \( \Lambda^N \) such that

\[
U_n(x_1, \ldots, x_N) := \sum_{1 \leq i_1 < \cdots < i_m \leq N} u_n^{\alpha_{i_1} \cdots \alpha_{i_m}}(x_{i_1}, \ldots, x_{i_m}) \to U(x_1, \ldots, x_N) \quad \text{a.e.} \tag{6.21}
\]
Then, there are real-valued functions \( \{u^{\alpha_1 \cdots \alpha_m} : (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m\} \) on \( \Lambda^m \) such that

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) = U(x_1, \ldots, x_N) \text{ a.e.} \tag{6.22}
\]

**Remark 6.1.** Note that the conclusion here is weaker than in Theorem 4.1 because we do not prove that \( u^{\alpha_1 \cdots \alpha_m}_n \to u^{\alpha_1 \cdots \alpha_m} \) a.e.

**Proof (of Theorem 6.4).** Let \( E \subset \Lambda^N \) be the co-null set on which \( (6.21) \) holds.

**Case 1: \( m = 1 \).** In this case \( (6.21) \) reads

\[
\sum_{\ell=1}^L \sum_{M_{\ell-1}+1 \leq i \leq M_{\ell}} u^{\ell}_n(x_i) \to U(x_1, \ldots, x_N). \tag{6.23}
\]

By Lemma 2.2 for every \( 1 \leq \ell \leq L \), there is a co-null set \( S^\ell_{N-1} \subset \Lambda^{N-1} \) such that for each \( (\tilde{x}_2, \ldots, \tilde{x}_N) \in S^\ell_{N-1} \) the set

\[
E^\ell_{x_2, \ldots, \tilde{x}_N} := \{ x \in \Lambda : (\tilde{x}_2, \ldots, \tilde{x}_{M_{\ell-1}+1}, x, \tilde{x}_{M_{\ell}-1+2}, \ldots, \tilde{x}_N) \in E \} \text{ is co-null in } \Lambda.
\]

Because the set \( \bigcap_{\ell=1}^L S^\ell_{N-1} \subset \Lambda \) is co-null, it is not empty. Let us fix \( (\tilde{x}_2, \ldots, \tilde{x}_N) \in \bigcap_{\ell=1}^L S^\ell_{N-1} \). For each \( x \in E^\ell_{x_2, \ldots, \tilde{x}_N} \)

\[
u_n^1(\tilde{x}_2) + \cdots + u_n^{\ell-1}(\tilde{x}_{M_{\ell-1}+1}) + u_n^\ell(x) + u_n^\ell(\tilde{x}_{M_{\ell-1}+2}) + \cdots + u_n^L(\tilde{x}_N) \to U(\tilde{x}_2, \ldots, \tilde{x}_{M_{\ell-1}+1}, x, \tilde{x}_{M_{\ell}-1+2}, \ldots, \tilde{x}_N). \tag{6.24}
\]

Thus, if \( (x_1, \ldots, x_N) \in \Omega := \prod_{\ell=1}^L \prod_{M_{\ell-1}+1 \leq i \leq M_{\ell}} E^\ell_{x_{\ell+2}, \ldots, \tilde{x}_N} \), \( (6.24) \) implies that

\[
u_n^1(\tilde{x}_2) + \cdots + u_n^{\ell-1}(\tilde{x}_{M_{\ell-1}+1}) + u_n^\ell(x_i) + u_n^\ell(\tilde{x}_{M_{\ell-1}+2}) + \cdots + u_n^L(\tilde{x}_N) \to U(\tilde{x}_2, \ldots, \tilde{x}_{M_{\ell-1}+1}, x_i, \tilde{x}_{M_{\ell}-1+2}, \ldots, \tilde{x}_N). \tag{6.25}
\]
for every $1 \leq \ell \leq L$ and every $M_{\ell-1} + 1 \leq i \leq M_{\ell}$. Adding (6.25) over $1 \leq i \leq N$ yields

$$
\sum_{\ell=1}^{L} \sum_{M_{\ell-1}+1 \leq i \leq M_{\ell}} u_{\alpha}(x_{i}) + C_{n} \rightarrow 
\sum_{\ell=1}^{L} \sum_{M_{\ell-1}+1 \leq i \leq M_{\ell}} U(\tilde{x}_{2}, \ldots, \tilde{x}_{M_{\ell-1}+1}, x_{i}, \tilde{x}_{M_{\ell-1}+2}, \ldots, \tilde{x}_{N}).
$$

(6.26)

for every $(x_{1}, \ldots, x_{N}) \in \Omega$, where $C_{n}$ is a constant independent of $(x_{1}, \ldots, x_{N})$.

The set $\Omega \cap E$ is not empty, as co-null. Let us fix $(y_{1}, \ldots, y_{N}) \in \Omega \cap E$. For this point both (6.23) and (6.26) hold. Therefore,

$$
C_{n} \rightarrow C := 
\sum_{k=1}^{L} \sum_{M_{\ell-1}+1 \leq i \leq M_{\ell}} U(\tilde{x}_{2}, \ldots, \tilde{x}_{M_{\ell-1}+1}, y_{i}, \tilde{x}_{M_{\ell-1}+2}, \ldots, \tilde{x}_{N}) - U(y_{1}, \ldots, y_{N}).
$$

(6.27)

But then, (6.26) yields

$$
\sum_{\ell=1}^{L} \sum_{M_{\ell-1}+1 \leq i \leq M_{\ell}} u_{\alpha}(x_{i}) \rightarrow U(x_{1}, \ldots, x_{N}) = \sum_{\ell=1}^{L} \sum_{M_{\ell-1}+1 \leq i \leq M_{\ell}} u_{\ell}(x_{i})
$$

(6.28)

for every $(x_{1}, \ldots, x_{N}) \in \Omega \cap E$, where

$$
u_{\ell}(x) := U(\tilde{x}_{2}, \ldots, \tilde{x}_{M_{\ell-1}+1}, x, \tilde{x}_{M_{\ell-1}+2}, \ldots, \tilde{x}_{N}) - \frac{C}{N}
$$

(6.29)

for every $1 \leq \ell \leq L$.

Case 2: $2 \leq m \leq N - 1$. For the ensuing argument, it is convenient to start by rewriting (6.21) as

$$
U_{\alpha}(x_{1}, \ldots, x_{N}) =
\sum_{(\alpha_{1}, \ldots, \alpha_{m}) \in A_{m}} \sum_{M_{\alpha_{1}-1} + 1 \leq i_{1} < \cdots < i_{m} \leq M_{\alpha_{m}}} u_{\alpha_{1},\ldots,\alpha_{m}}(x_{i_{1}}, \ldots, x_{i_{m}})
$$

$$
\rightarrow U(x_{1}, \ldots, x_{N}) \text{ on } E.
$$

(6.30)
In (6.30), the first sum goes over all possible compositions of \( m \) particles, while the second sum goes over all possible sets of \( m \) particles selected out of \( N \) and having the composition determined by index \((\alpha_1, ..., \alpha_m)\).

By Lemma 2.2 for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m\), there is a co-null set \( \tilde{S}^{\alpha_1, ..., \alpha_m}_{N-m} \subset A^{N-m} \) such that for each \((\tilde{x}_{m+1}, ..., \tilde{x}_N) \in \tilde{S}^{\alpha_1, ..., \alpha_m}_{N-m}\) the set \( E^{\alpha_1, ..., \alpha_m}_{\tilde{x}_{m+1}, ..., \tilde{x}_N} := \{ (x_1, ..., x_m) \in A^m : T^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m, \tilde{x}_{m+1}, ..., \tilde{x}_N) \in E \} \) is co-null in \( A^m \).

Let us choose and fix \((\tilde{x}_{m+1}, ..., \tilde{x}_N) \in \bigcap_{1 \leq \alpha_1 \leq ..., \leq \alpha_m \leq L} \tilde{S}^{\alpha_1, ..., \alpha_m}_{N-m}. \) By (6.30), for every \((x_1, ..., x_m) \in E^{\alpha_1, ..., \alpha_m}_{\tilde{x}_{m+1}, ..., \tilde{x}_N}\)

\[
\begin{align*}
U_n(T^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m, \tilde{x}_{m+1}, ..., \tilde{x}_N)) &= u_n^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m) + \\
F_n^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m) &\rightarrow \psi^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m) := U(T^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m, \tilde{x}_{m+1}, ..., \tilde{x}_N)),
\end{align*}
\]

where

\[
F_n^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m) := \\
U_n(T^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m, \tilde{x}_{m+1}, ..., \tilde{x}_N)) - u_n^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m). \quad (6.32)
\]

For every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m\), let

\[
\Omega^{\alpha_1, ..., \alpha_m} := \{ (x_1, ..., x_N) \in A^N : (x_{i_1}, ..., x_{i_m}) \in E^{\alpha_1, ..., \alpha_m}_{\tilde{x}_{m+1}, ..., \tilde{x}_N} \}
\]

for every \(M_{\alpha_1-1} + 1 \leq i_1 < \cdots < i_m \leq M_{\alpha_m}\)

\[
\text{such that } M_{\alpha_k-1} + 1 \leq i_k \leq M_{\alpha_k} \forall 1 \leq k \leq m. \quad (6.33)
\]

If \((x_1, ..., x_N) \in \Omega := \bigcap_{1 \leq \alpha_1 \leq ..., \leq \alpha_m \leq L} \Omega^{\alpha_1, ..., \alpha_m}\), then (6.31) implies that

\[
u_n^{\alpha_1, ..., \alpha_m}(x_{i_1}, ..., x_{i_m}) + F_n^{\alpha_1, ..., \alpha_m}(x_{i_1}, ..., x_{i_m}) \rightarrow \psi^{\alpha_1, ..., \alpha_m}(x_{i_1}, ..., x_{i_m}) \quad (6.34)
\]

for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m\) and every \(M_{\alpha_1-1} + 1 \leq i_1 < \cdots < i_m \leq M_{\alpha_m}\) such that \(M_{\alpha_k-1} + 1 \leq i_k \leq M_{\alpha_k} \forall 1 \leq k \leq m. \) Adding (6.34) over \(\binom{N}{m}\) selections
of \( m \) distinct integers out of \( \{1, \ldots, N\} \) (or, equivalently, performing the double summation, as in (6.30)) yields

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} \left[ u_n^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) + F_n^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \right] \\
\rightarrow \sum_{1 \leq i_1 < \cdots < i_m \leq N} \varphi^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \quad (6.35)
\]

for every \((x_1, \ldots, x_N) \in \Omega\). We can rewrite

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} F_n^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) = \\
\sum_{k=1}^{K} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq N} v_n^{\alpha(i_1) \cdots \alpha(i_{m-k})}(x_{i_1}, \ldots, x_{i_{m-k}}), \quad (6.36)
\]

where, \( K = \min(m - 1, N - m) \), and \( v_n^{\alpha_1 \cdots \alpha_{m-k}} \) is a real-valued function on \( \mathcal{A}^{m-k} \) for every \( n \in \mathbb{N} \), every \( 1 \leq k \leq m - 1 \), and every \((\alpha_1, \ldots, \alpha_{m-k}) \in \mathcal{A}^{m-k}\).

**Remark 6.2.** The general form of \( v_n^{\alpha_1 \cdots \alpha_{m-k}} \) is prohibitively complex to write down explicitly, but it is not difficult to see how these functions are constructed by considering a simple example. If \( m = L = N_1 = N_2 = 2 \), then using (6.4) and the definition of \( F_n^{ij} \) in (6.32) for \( 1 \leq i \leq j \leq 2 \), we find that

\[
F_n^{11}(x_1, x_2) + \sum_{1 \leq i \leq 2} F_n^{12}(x_i, x_j) + F_n^{22}(x_3, x_4) = \sum_{i=1}^{2} v_n^1(x_i) + \sum_{j=3}^{4} v_n^2(x_j), \\
3 \leq j \leq 4
\]

(6.37)

were

\[
v_n^1(x) = u_n^{12}(x, \bar{x}_3) + 3u_n^{12}(x, \bar{x}_4) + 2u_n^{11}(x, \bar{x}_3) + C_n/4,
\]

\[
v_n^2(x) = u_n^{12}(\bar{x}_4, x) + 3u_n^{12}(\bar{x}_3, x) + 2u_n^{22}(x, \bar{x}_4) + C_n/4,
\]

(6.38)

and \( C_n = u_n^{11}(\bar{x}_3, \bar{x}_4) + 4u_n^{12}(\bar{x}_3, \bar{x}_4) + u_n^{22}(\bar{x}_3, \bar{x}_4) \).
Next, for every \( n \in \mathbb{N} \) and every \((\alpha_1, \ldots, \alpha_{m-1}) \in \mathcal{A}_{m-1}\), we define a real-valued function \( \omega^{\alpha_1 \ldots \alpha_{m-1}}_n \) on \( \Lambda_{m-1} \) by

\[
\omega^{\alpha_1 \ldots \alpha_{m-1}}_n(x_1, \ldots, x_{m-1}) := \sum_{k=1}^K \binom{N - m + k}{k - 1}^{-1} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq m-1} v^{\alpha_{i_1} \ldots \alpha_{i_{m-k}}}_n(x_{i_1}, \ldots, x_{i_{m-k}}). \tag{6.39}
\]

Then,

\[
\sum_{k=1}^K \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq N} v^{\alpha_{i_1} \ldots \alpha_{i_{m-k}}}_n(x_{i_1}, \ldots, x_{i_{m-k}}) = \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega^{\alpha_{i_1} \ldots \alpha_{i_{m-1}}}_n(x_{i_1}, \ldots, x_{i_{m-1}}). \tag{6.40}
\]

Indeed, for fixed \( 1 \leq k \leq m - 1 \) and \( 1 \leq i_1 < \cdots < i_{m-k} \leq N \), there are \( \binom{N - m + k}{k - 1} \) ways to choose the remaining \( k - 1 \) integers. Thus, the term \( v^{\alpha_{i_1} \ldots \alpha_{i_{m-k}}}_n(x_{i_1}, \ldots, x_{i_{m-k}}) \) will repeat \( \binom{N - m + k}{k - 1} \) times as different sets of \( m - 1 \) distinct numbers are selected out of \( \{1, \ldots, N\} \) on the right hand side of (6.40).

Substituting (6.40) in (6.36) and (6.36) in (6.35) gives:

\[
\sum_{1 \leq i_1 < \cdots < i_{m} \leq N} \omega^{\alpha_{i_1} \ldots \alpha_{i_{m}}}_n(x_{i_1}, \ldots, x_{i_{m}}) = \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega^{\alpha_{i_1} \ldots \alpha_{i_{m-1}}}_n(x_{i_1}, \ldots, x_{i_{m-1}}) \rightarrow \sum_{1 \leq i_1 < \cdots < i_{m} \leq N} \varphi^{\alpha_{i_1} \ldots \alpha_{i_{m}}}(x_{i_1}, \ldots, x_{i_{m}}) \tag{6.41}
\]

for every \((x_1, \ldots, x_N) \in \Omega\). If \((x_1, \ldots, x_N) \in \Omega \cap E\), then for this point both (6.41) and (6.21) hold. Therefore,

\[
\sum_{1 \leq i_1 < \cdots < i_{m} \leq N} \omega^{\alpha_{i_1} \ldots \alpha_{i_{m}}}_n(x_{i_1}, \ldots, x_{i_{m-1}}) \rightarrow \sum_{1 \leq i_1 < \cdots < i_{m} \leq N} \varphi^{\alpha_{i_1} \ldots \alpha_{i_{m}}}(x_{i_1}, \ldots, x_{i_{m}}) - U(x_1, \ldots, x_N). \tag{6.42}
\]
for every \((x_1, \ldots, x_N) \in \Omega \cap E\). But then, the induction hypothesis implies that there are real-valued functions \(\{\omega^{\alpha_1, \ldots, \alpha_m-1} : (\alpha_1, \ldots, \alpha_{m-1}) \in \mathcal{A}_{m-1}\}\) on \(A^{m-1}\) such that
\[
\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega^{\alpha(i_1) \cdots \alpha(i_{m-1})}(x_{i_1}, \ldots, x_{i_{m-1}}) 
\rightarrow \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega^{\alpha(i_1) \cdots \alpha(i_{m-1})}(x_{i_1}, \ldots, x_{i_{m-1}}) \quad \text{a.e. (6.43)}
\]

For every \((\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m\), let \(u^{\alpha_1, \ldots, \alpha_m} \) be a function on \(A^m\) defined by
\[
u^{\alpha_1, \ldots, \alpha_m}(x_1, \ldots, x_m) := \phi^{\alpha_1, \ldots, \alpha_m}(x_1, \ldots, x_m) -
(N - m + 1)^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq m} \omega^{\alpha_1, \ldots, \alpha_{m-1}}(x_{i_1}, \ldots, x_{i_{m-1}}). \quad (6.44)
\]

Arguments similar to the proof of (6.40) show that
\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} \nu^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) -
\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \omega^{\alpha(i_1) \cdots \alpha(i_{m-1})}(x_{i_1}, \ldots, x_{i_{m-1}}) =
\sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}). \quad (6.45)
\]

Therefore, if \(T \subset A^N\) denotes the co-null set on which (6.43) holds, then (6.41) implies that
\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \rightarrow U(x_1, \ldots, x_N)
= \sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \quad (6.46)
\]
for every \((x_1, \ldots, x_N) \in \Omega \cap T \cap E\). □

**Corollary 6.5.** Let \(\{u^{\alpha_1, \ldots, \alpha_m} : (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m\}\) be a set of real-valued functions on \(A^m\), and let \(U\) be a real-valued function on \(A^N\). Suppose that
\[
U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \quad \text{a.e. (6.47)}
\]
Then,

(i) (a) If \( u^{\alpha_1 \cdots \alpha_m} \) is measurable for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m \) then \( U \) is measurable.

(i) (b) If \( U \) is measurable then there exists a set of measurable functions \( \{v^{\alpha_1 \cdots \alpha_m} : (\alpha_1, ..., \alpha_m) \in \mathcal{A}_m \} \) on \( \Lambda^m \) such that

\[
U(x_1, ..., x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, ..., x_{i_m}) \quad a.e. \quad (6.48)
\]

(ii) If \( u^{\alpha_1 \cdots \alpha_m} = 0 \) a.e. for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m \) then \( U = 0 \) a.e.

(iii) (a) If \( u^{\alpha_1 \cdots \alpha_m} \) is \( \mathcal{B}^{\alpha_1 \cdots \alpha_m} \)-symmetric a.e. for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m \), then \( U \) is \( \mathcal{B} \)-symmetric a.e.

(iii) (b) If \( U \) is \( \mathcal{B} \)-symmetric a.e., then there exist \( \mathcal{B}^{\alpha_1 \cdots \alpha_m} \)-symmetric functions \( \{v^{\alpha_1 \cdots \alpha_m} : (\alpha_1, ..., \alpha_m) \in \mathcal{A}_m \} \) on \( \Lambda^m \), such that (6.48) holds.

**Remark 6.3.** Note that this corollary is weaker than Corollary 4.3. First of all, (ii) here is lacking the "only if" part. This is evident from a simple example. Let \( N = 2 \) and \( m = N_1 = N_2 = 1 \). Then, \( u^1(x_1) + u^2(x_2) = 0 \) on \( A^2 \) if \( u^1 \equiv c \) and \( u^2 \equiv -c \) for any \( c \in \mathbb{R} \). In the same spirit, we do not prove in (i) (b) and (iii) (b) here that \( u^{\alpha_1 \cdots \alpha_m} \) itself is measurable or \( \mathcal{B}^{\alpha_1 \cdots \alpha_m} \)-symmetric a.e.

**Proof (of Corollary 6.5).** Since the arguments are more involved, but still similar to the verification of Corollary 4.3, only the proof of (iii) will be shown.

(iii) (a) Suppose that \( u^{\alpha_1 \cdots \alpha_m} \) is a.e. \( \mathcal{B}^{\alpha_1 \cdots \alpha_m} \)-symmetric for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m \). This means that for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m \), there is a co-null set \( S^{\alpha_1 \cdots \alpha_m} \) such that \((x_1, ..., x_m) \in S^{\alpha_1 \cdots \alpha_m} \) implies \((x_{\pi(1)}, ..., x_{\pi(m)}) \in S^{\alpha_1 \cdots \alpha_m} \) and \( u^{\alpha_1 \cdots \alpha_m}(x_1, ..., x_m) = u^{\alpha_1 \cdots \alpha_m}(x_{\pi(1)}, ..., x_{\pi(m)}) \) for every permutation \( \pi \in \mathcal{B}^{\alpha_1 \cdots \alpha_m} \).

Let \( E \subset \Lambda^N \) be the co-null set on which (6.47) holds. For every \( \pi \in \mathcal{B} \), define a set \( E^\pi := \{(x_{\pi(1)}, ..., x_{\pi(m)}) \in \Lambda^m : (x_1, ..., x_m) \in E \} \). It is easy to verify
that \( E^B := \cap_{\pi \in \mathcal{B}} E^\pi \subset E \) is co-null and \( \mathcal{B} \)-symmetric. (We will refer to \( E^B \) and similar such sets as "symmetrized.") For every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m\), let

\[
\Omega^{\alpha_1, ..., \alpha_m} := \{(x_1, ..., x_N) \in A^N : (x_{i_1}, ..., x_{i_m}) \in S^{\alpha_1, ..., \alpha_m} \}
\]

for every \( M_{\alpha_1-1} + 1 \leq i_1 < \cdots < i_m \leq M_{\alpha_m} \) such that \( M_{\alpha_k-1} + 1 \leq i_k \leq M_{\alpha_k} \forall 1 \leq k \leq m \} \).

Then, \( \Omega := \cap_{(\alpha_1, ..., \alpha_m) \in \mathcal{A}_m} \Omega^{\alpha_1, ..., \alpha_m} \) is co-null and \( \mathcal{B} \)-symmetric. Moreover, for every \((x_1, ..., x_N) \in \Omega \cap E^B\) and every permutation \( \pi \in \mathcal{B} \)

\[
U(\pi(1), ..., \pi(N)) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha_1(i_1) ... \alpha_m(i_m)}(x_{\pi(i_1)}, ..., x_{\pi(i_m)})
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha_1(i_1) ... \alpha_m(i_m)}(x_{i_1}, ..., x_{i_m}). \quad (6.50)
\]

(iii) (b) Let \( S \) be the co-null set on which \( U \) is \( \mathcal{B} \)-symmetric. That is, for every \((x_1, ..., x_N) \in S \) and every \( \pi \in \mathcal{B} \), \((x_{\pi(1)}, ..., x_{\pi(N)}) \in \mathcal{B} \) and \( U(x_{\pi(1)}, ..., x_{\pi(N)}) = U(x_1, ..., x_N) \). Let \( E \) be the co-null set on which \( (6.47) \) holds, and \( A := E^B \cap S \), where \( E^B \subset E \) is the symmetrized co-null set defined in the proof of (iii) (a).

Particularly, \( E^B \) is \( \mathcal{B} \)-symmetric. The statement of (iii) (b) holds trivially for \( m = 1 \). Let \( 2 \leq m \leq N - 1 \) and assume that the statement holds for \( m - 1 \). We will prove that it then holds for \( m \), and so (iii) (b) will follow by induction.

By Lemma 2.2 for every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m\), there is a co-null set \( \tilde{S}_{N-m}^{\alpha_1, ..., \alpha_m} \subset A^{N-m} \) such that for every \((\tilde{x}_{m+1}, ..., \tilde{x}_N) \in \tilde{S}_{N-m}^{\alpha_1, ..., \alpha_m} \) the set \( A_{\tilde{x}_{m+1}, ..., \tilde{x}_N}^{\alpha_1, ..., \alpha_m} := \{(x_1, ..., x_m) \in A^m : T^{\alpha_1, ..., \alpha_m}(x_1, ..., x_m, \tilde{x}_{m+1}, ..., \tilde{x}_N) \in A \} \) is co-null in \( A^m \). As can be readily verified, the set \( A_{\tilde{x}_{m+1}, ..., \tilde{x}_N}^{\alpha_1, ..., \alpha_m} \) is \( \mathcal{B}^{\alpha_1, ..., \alpha_m} \)-symmetric. Let us choose and fix \((\tilde{x}_{m+1}, ..., \tilde{x}_N) \in \cap_{(\alpha_1, ..., \alpha_m) \in \mathcal{A}_m} \tilde{S}_{N-m}^{\alpha_1, ..., \alpha_m} \). For every \((\alpha_1, ..., \alpha_m) \in \mathcal{A}_m\),
let

\[ \Omega^{\alpha_1 \cdots \alpha_m} := \{(x_1, \ldots, x_N) \in A^N : (x_{i_1}, \ldots, x_{i_m}) \in A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}^{\alpha_1 \cdots \alpha_m} \} \]

for every \( M_{\alpha_1 - 1} + 1 \leq i_1 < \cdots < i_m \leq M_{\alpha_m} \) such that \( M_{\alpha_p - 1} + 1 \leq i_p \leq M_{\alpha_p} \forall 1 \leq p \leq m \). \hspace{1cm} (6.51)

It follows that the set \( \Omega := \bigcap_{(\alpha_1, \ldots, \alpha_m) \in A_m} \Omega^{\alpha_1 \cdots \alpha_m} \) is co-null and \( B \)-symmetric.

By the proof of Theorem 6.4, there are real-valued functions \( \{\omega^{\alpha_1 \cdots \alpha_m-1} : (\alpha_1, \ldots, \alpha_{m-1}) \in A_{m-1}^m \} \) on \( A_{m-1}^m \) such that

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} \omega^{\alpha_1 \cdots \alpha_{m-1}}(x_{i_1}, \ldots, x_{i_m-1}) + \sum_{1 \leq i_1 < \cdots < i_m \leq N} U(T^{\alpha_1 \cdots \alpha_{m-1}}(x_{i_1}, \ldots, x_{i_m-1}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)) \hspace{1cm} (6.52)
\]

for every \((x_1, \ldots, x_N) \in \Omega \) (cf. (6.41)). For each \((x_1, \ldots, x_N) \in \Omega \cap A\), both (6.52) and (6.47) hold. Therefore,

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} \omega^{\alpha_1 \cdots \alpha_{m-1}}(x_{i_1}, \ldots, x_{i_{m-1}}) + \sum_{1 \leq i_1 < \cdots < i_m \leq N} U(T^{\alpha_1 \cdots \alpha_{m-1}}(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)) - U(x_1, \ldots, x_N). \hspace{1cm} (6.53)
\]

for every \((x_1, \ldots, x_N) \in \Omega \cap A\). Obviously, \( U \) is \( B \)-symmetric on \( A \). Moreover,

\[
U(T^{\alpha_1 \cdots \alpha_m}(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)) = U(T^{\alpha_1 \cdots \alpha_m}(x_{\pi(1)}, \ldots, x_{\pi(m)}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)) \hspace{1cm} (6.54)
\]

for every \((x_1, \ldots, x_m) \in A_{\tilde{x}_{m+1}, \ldots, \tilde{x}_N}^{\alpha_1 \cdots \alpha_m} \) and every \( \pi \in B^{\alpha_1 \cdots \alpha_m} \). This implies that the first term on the right hand side of (6.53) is \( B \)-symmetric on \( \Omega \). Thus, the right
hand side of (6.53) is $B$-symmetric on $\Omega \cap A$. Therefore, by the induction hypothesis, there are real-valued functions \( \{ \phi^{\alpha_1\cdots\alpha_{m-1}} : (\alpha_1, ..., \alpha_{m-1}) \in A_{m-1} \} \) on $A^m$ such that $\phi^{\alpha_1\cdots\alpha_{m-1}}$ is $B^{\alpha_1\cdots\alpha_{m-1}}$-symmetric a.e. for every $(\alpha_1, ..., \alpha_{m-1}) \in A_{m-1}$, and

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} \omega^{(i_1)\cdots(i_{m-1})}(x_{i_1}, \ldots, x_{i_{m-1}}) = \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \phi^{(i_1)\cdots(i_{m-1})}(x_{i_1}, \ldots, x_{i_{m-1}}) \quad \text{a.e. (6.55)}
\]

By symmetrizing, we can assume that (6.55) holds on some co-null and $B$-symmetric set $T \subset \Lambda^N$.

For every $(\alpha_1, ..., \alpha_m) \in A_m$, let us define a function $v^{\alpha_1\cdots\alpha_m}$ on $A^m$ by

\[
v^{\alpha_1\cdots\alpha_m}(x_1, \ldots, x_m) := U(T^{\alpha_1\cdots\alpha_m}(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)) -
(N - m + 1)^{-1} \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq m} \phi^{\alpha_1\cdots\alpha_{i_{m-1}}}(x_{i_1}, \ldots, x_{i_{m-1}}). \quad (6.56)
\]

By (6.54), $U(T^{\alpha_1\cdots\alpha_m}(x_1, \ldots, x_m, \tilde{x}_{m+1}, \ldots, \tilde{x}_N))$ is $B^{\alpha_1\cdots\alpha_m}$-symmetric on $A^{\alpha_1\cdots\alpha_m}_{x_{m+1}, \ldots, \tilde{x}_N}$, while the second term on the right hand side of (6.56) is $B^{\alpha_1\cdots\alpha_m}$-symmetric a.e. by (iii) (a). Thus, the function $v^{\alpha_1\cdots\alpha_m}$ is $B^{\alpha_1\cdots\alpha_m}$-symmetric a.e. for every $(\alpha_1, ..., \alpha_m) \in A_m$. Moreover, as was shown in the proof of Theorem 6.4 (cf. (6.54)),

\[
\sum_{1 \leq i_1 < \cdots < i_m \leq N} U(T^{(i_1)\cdots(i_m)}(x_{i_1}, \ldots, x_{i_m}, \tilde{x}_{m+1}, \ldots, \tilde{x}_N)) -
\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq N} \phi^{(i_1)\cdots(i_{m-1})}(x_{i_1}, \ldots, x_{i_{m-1}}) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v^{(i_1)\cdots(i_m)}(x_{i_1}, \ldots, x_{i_m}). \quad (6.57)
\]
This, together with (6.52) and (6.55), imply that

\[ U(x_1, ..., x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, ..., x_{i_m}) \quad \text{on } E^B \cap \Omega \cap T. \quad \square \]

(6.58)

The next theorem is a generalization of a weaker version of Theorem 5.1 to several species. It can be established by using the proof of Theorem 6.4. Since all the techniques needed for the proof can be found in Section 5 and the earlier part of this section, it will be omitted. It is assumed that \( 1 \leq m \leq N - 1 \) because conclusions hold trivially for \( m = N \).

**Theorem 6.6.** Let \( \{ u^{\alpha_1 \cdots \alpha_m} : (\alpha_1, ..., \alpha_m) \in A_m \} \) be a set of real-valued functions on \( A^m \), and \( U \) be a real-valued function on \( A^N \). Suppose that

\[ U(x_1, ..., x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} u^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, ..., x_{i_m}) \quad \text{a.e.} \quad (6.59) \]

For every \( 1 \leq k \leq m \) and every \( (\alpha_1, ..., \alpha_k) \in A_k \), let \( \rho^{\alpha_1 \cdots \alpha_k} \) be a \( k \)-variable reduction of some \( \mathcal{B} \)-symmetric probability density \( P \geq 0 \). This means that \( \rho^{\alpha_1 \cdots \alpha_k} \) is given by (6.8) with \( m \) replaced by \( k \). Then,

(i) \( u^{\alpha_1 \cdots \alpha_m} \in L^1(A^m; \rho^{\alpha_1 \cdots \alpha_m} d^m x) \) for every \( (\alpha_1, ..., \alpha_m) \in A_m \) implies that \( U \in L^1(A^N; P d^N x) \).

(ii) Suppose, in addition, that for each \( 1 \leq k \leq m \) and each \( (\alpha_1, ..., \alpha_k) \in A_k \) there is a set \( A^\alpha_{N-k} \subset A^{N-k} \) of positive measure such that for every \( (x_{k+1}, ..., x_N) \in A^\alpha_{N-k} \) there is a constant \( C^\alpha_{N-k}(x_{k+1}, ..., x_N) > 0 \) satisfying

\[ P(T^{\alpha_1 \cdots \alpha_k}(\cdot, x_{k+1}, ..., x_N)) \geq C^\alpha_{N-k}(x_{k+1}, ..., x_N) \rho^{\alpha_1 \cdots \alpha_k}(\cdot) \quad \text{a.e. on } A^k. \]

(6.60)
Then, if \( U \in L^1(\Lambda^N; Pd^N x) \), there exist functions \( \{v^{\alpha_1 \cdots \alpha_m} : (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m\} \) on \( \Lambda^m \) such that \( v^{\alpha_1 \cdots \alpha_m} \in L^1(\Lambda^m; \rho^{\alpha_1 \cdots \alpha_m} d^m x) \) for every \( (\alpha_1, \ldots, \alpha_m) \in \mathcal{A}_m \), and

\[
U(x_1, \ldots, x_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} v^{\alpha(i_1) \cdots \alpha(i_m)}(x_{i_1}, \ldots, x_{i_m}) \text{ a.e.} \tag{6.61}
\]

**Remark 6.4.** If \( \Lambda \) has finite \( dx \) measure and contains subsets of arbitrarily small positive measure, then any a.e. \( \mathcal{B} \)-symmetric probability density \( P \) can be approximated in \( L^1(\Lambda^N; d^N x) \) by an a.e. \( \mathcal{B} \)-symmetric probability density satisfying (6.60). The proof of this fact follows from Remark 5.1 with virtually no modification.

**APPENDIX**

In this appendix we give an example of \( \mathcal{V}_P \neq \mathcal{V}_{\rho(m)} \) with all the assumptions on \( P \) and \( W \) stated in Theorems 3.6 and 3.7 satisfied.

Let \( N = 2, m = 1, \Lambda = (0, \infty) \), and \( dx \) be the Lebesgue measure. Set \( I_i := (i - 1, i] \) and \( Q_{ij} := I_i \times I_j \) for every \( i, j \in \mathbb{N} \). We define \( u \) on \( (0, \infty) \) by

\[
u := (-1)^i \text{ on } I_i \tag{A1}
\]

and \( P \) on \( (0, \infty)^2 \) by

\[
P = p_{ij} / \left[ \sum_{i,j=1}^{\infty} p_{ij} \right] \text{ on } Q_{ij}, \text{ where } p_{ij} := \begin{cases} (i + j)^{-2} & \text{if } |i - j| = 1, \\ (i + j)^{-4} & \text{if } |i - j| \neq 1. \end{cases} \tag{A2}
\]

The sum in the square brackets converges. In fact,

\[
\sum_{i,j=1}^{\infty} p_{ij} = \sum_{|i-j|=1} (i + j)^{-2} + \sum_{|i-j| \neq 1} (i + j)^{-4}. \tag{A3}
\]

\(^4\) This example was created by Patrick J. Rabier.
The first term on the far-most right is equal to
\[
\frac{1}{9} + \sum_{i=2}^{\infty} \left( (2i - 1)^{-2} + (2i + 1)^{-2} \right) = 2 \sum_{i=1}^{\infty} (2i + 1)^{-2} < \infty. \quad (A4)
\]

The second term can be estimated as
\[
\sum_{|i-j|\neq 1} (i+j)^{-4} \leq \sum_{i,j=1}^{\infty} (i+j)^{-4} = \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} k^{-4} = \sum_{k=2}^{\infty} (k-1)k^{-4} < \infty. \quad (A5)
\]

Thus, \( P \) is a well-defined positive and symmetric probability density on \((0, \infty)^2\).

We also define \( W = -U - \log P \), with \( U(x_1, x_2) = u(x_1) + u(x_2) \) for every \((x_1, x_2) \in (0, \infty)^2\). Obviously, \( W \) is finite and symmetric everywhere on \((0, \infty)^2\).

Since \(|u(i) + u(j)| = 1\) when \(|i - j| = 1\), we have
\[
\int_{A^2} |U| P d^2x = \sum_{|i-j|=1} (i+j)^{-2} + \sum_{|i-j|\neq 1} |(-1)^i i + (-1)^j j|(i+j)^{-4}. \quad (A6)
\]

As before, the first term on the right is equal to \( 2 \sum_{i=1}^{\infty} (2i + 1)^{-2} < \infty \). The second term is also finite by the estimate similar to (A5). Namely,
\[
\sum_{|i-j|\neq 1} |(-1)^i i + (-1)^j j|(i+j)^{-4} \leq \sum_{i,j=1}^{\infty} (i+j)^{-3} = \sum_{k=2}^{\infty} (k-1)k^{-3} < \infty. \quad (A7)
\]

Thus \( U \in L^1(A^2; P d^2x) \). Now, it is easy to show that \( e^{-U} \in L^1(A^2; e^{-W} d^2x) \), and (2.5) holds. Indeed, \( \int_{A^2} e^{-W} e^{-U} d^2x = \int_{A^2} P d^2x = 1 \), and \( \int_{A^2} (W+\log P)_+ d^2x = \int_{A^2} (U_-) P d^2x < \infty \). It remains to prove that \( u \notin L^1(A; \rho^{(1)} dx) \). For every \( x \in (i-1, i], i \geq 2 \), \( \rho^{(1)}(x) = \sum_{j=1}^{\infty} p_{ij} \geq (2i - 1)^{-2} + (2i + 1)^{-2} \geq 2(2i + 1)^{-2} \).

Therefore, \( \int_{A} \rho^{(1)} |u| dx \geq 2 \sum_{i=2}^{\infty} i(2i + 1)^{-2} = \infty \). The proof is complete.

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