CUSPIDAL EDGES WITH THE SAME FIRST FUNDAMENTAL FORMS ALONG A KNOT

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Abstract. Letting $C$ be a compact $C^\omega$-curve embedded in $R^3$ ($C^\omega$ means real analyticity), we consider a $C^\omega$-cuspidal edge $f$ along $C$. When $C$ is non-closed, in the authors’ previous works, the local existence of three distinct cuspidal edges along $C$ whose first fundamental forms coincide with that of $f$ was shown, under a certain reasonable assumption on $f$. In this paper, if $C$ is closed, that is, $C$ is a knot, we show that there exist infinitely many cuspidal edges along $C$ having the same first fundamental form as that of $f$ such that their images are non-congruent to each other, in general.

Introduction

Let us first introduce some terminology. By ‘$C^r$-differentiability’ we mean $C^\infty$-differentiability if $r = \infty$ and real analyticity if $r = \omega$.

Let $f : U \to R^3$ be a $C^r$-map from a domain $U(\subset R^2)$ into the Euclidean 3-space $R^3$. A point $p \in U$ is called a cuspidal edge if there exist

- a local $C^r$-diffeomorphism $\varphi$ from a neighborhood $V(\subset U)$ of $p$ to a neighborhood of the origin of $R^2$, and
- a local $C^r$-diffeomorphism $\Phi$ from an open subset of $R^3$ containing $f(V)$ to a neighborhood of the origin of $R^3$

such that $\varphi(p) = (0, 0)$, $\Phi \circ f(p) = (0, 0, 0)$ and

$$\Phi \circ f \circ \varphi^{-1}(u, v) = f_C(u, v), \quad f_C(u, v) := (u^2, u^3, v).$$

The map $f_C$ is the standard cuspidal edge whose image is indicated in Figure 1 left.

![Figure 1: The standard cuspidal edge and two mutually congruent cuspidal edges along a helix](image)
We fix \( l > 0 \), and denote by \( I := [0, l] \) the closed interval and by \( S^1 := \mathbb{R}/l\mathbb{Z} \) the 1-dimensional torus of period \( l \). We set
\[
J := I \quad \text{or} \quad S^1,
\]
since we treat the bounded closed interval \( I \) and the one-dimensional torus \( S^1 \) uniformly. We then fix a \( C^r \)-embedded curve \( \gamma : J \to \mathbb{R}^3 \) with positive curvature function, and denote by \( C \) the image of \( \gamma \). For a positive number \( \varepsilon \), we set
\[
U_\varepsilon(J) := J \times (-\varepsilon, \varepsilon).
\]

**Definition 0.1.** We say that \( J_1 \) is a \( J \)-interval if \( J_1 := [0, a] \) when \( J = I \), and \( J_1 := \mathbb{R}/a\mathbb{Z} \) when \( J = S^1 \), where \( a > 0 \).

We fix a \( J \)-interval \( J_1 \). A \( C^r \)-cuspidal edge along \( C \) is a \( C^r \)-map \( \tilde{f} : U_\varepsilon(J_1) \to \mathbb{R}^3 \) such that
\[
\begin{align*}
\bullet & \quad J_1 \ni t \mapsto \tilde{f}(t, 0) \in \mathbb{R}^3 \quad \text{gives a parametrization of} \quad C, \\
\bullet & \quad (t, 0) \text{ is a cuspidal edge for each} \quad t \in J_1.
\end{align*}
\]
We denote by \( f \) the map germ along \( C \) induced by \( \tilde{f} \). For the sake of simplicity, we often identify \( f \) with \( \tilde{f} \), if it creates no confusion. (Later, we will give a special parametrization of \( f \) (cf. \( \text{[1]} \)). We denote by \( \mathcal{F}^r(C) \) the set of germs of \( C^r \)-cuspidal edges along \( C \).

**Definition 0.2.** Let \( g : U_{\varepsilon'}(J_2) \to \mathbb{R}^3 \) \( (\varepsilon' > 0) \) be a cuspidal edge along \( C \), where \( J_2 \) is a \( J \)-interval. Then \( g \) is said to be right equivalent to \( f \) if there exists a diffeomorphism \( \varphi \) from a neighborhood \( U_1(\subseteq U_\varepsilon(J_1)) \) of \( J_1 \times \{0\} \subseteq J_1 \times \mathbb{R} \) to a neighborhood \( U_2(\subseteq U_{\varepsilon'}(J_2)) \) of \( J_2 \times \{0\} \subseteq J_2 \times \mathbb{R} \) such that \( \varphi(J_1 \times \{0\}) = J_2 \times \{0\} \) and \( f = g \circ \varphi \) holds on \( U_1 \). We denote by \([f]\) the right equivalence class containing \( f \).

**Definition 0.3.** The cuspidal edge germ \( g \) is said to be isometric to \( f \in \mathcal{F}^r(C) \) if there exists a diffeomorphism \( \varphi \) defined on a neighborhood \( U_1(\subseteq U_\varepsilon(J_1)) \) of \( J_1 \times \{0\} \subseteq J_1 \times \mathbb{R} \) such that \( \varphi(J_1 \times \{0\}) = J_2 \times \{0\} \) and the pull-back metric \( \varphi^*ds_E^2 \) coincides with \( ds_f^2 \), where \( ds_f^2 \) (resp. \( ds_E^2 \)) is the first fundamental form of \( f \) (resp. \( g \)), that is, it is the pull-back of the Euclidean inner product on \( \mathbb{R}^3 \) by \( f \) (resp. \( g \)). We denote this relationship by \( g \sim f \). When \( g = f \), such a \( \varphi \) is called a symmetry of \( ds_f^2 \) if \( \varphi \) is not the identity map. Moreover, if
\[
\varphi(t, 0) = (t, 0) \quad (t \in J_1)
\]
holds, \( \varphi \) is said to be non-effective. Otherwise, \( \varphi \) is called an effective symmetry.

The isometric relation defined above is an actual equivalence relation on \( \mathcal{F}^r(C) \), as well as being the right equivalence relation. The following assertion holds:

**Proposition 0.4.** Let \( f, g \in \mathcal{F}^r(C) \). If \( g \) is right equivalent to \( f \), then \( g \) is isometric to \( f \).

**Proof.** If \( g \) is right equivalent to \( f \), then there exists a local diffeomorphism \( \varphi \) such that \( f = g \circ \varphi \). If we denote by \( ds_E^2 \) the Euclidean inner product of \( \mathbb{R}^3 \), then we have that
\[
ds_f^2 = f^*ds_E^2 = (g \circ \varphi)^*ds_E^2 = \varphi^*(g^*ds_E^2) = \varphi^*ds_f^2,
\]
which proves the assertion. \( \square \)

We then define an “isomer” of \( f \in \mathcal{F}^r(C) \) as follows.

**Definition 0.5.** For a given \( f \in \mathcal{F}^r(C) \), a cuspidal edge \( g \in \mathcal{F}^r(C) \) is called an isomer of \( f \) (cf. \( \text{[2]} \)) if it satisfies the following conditions:

1. \( g \) is isometric to \( f \) (i.e. \( g \sim f \)), but
A subset $A$ of $\mathbb{R}^3$ is said to be congruent to a subset $B \subset \mathbb{R}^3$ if there exists an isometry $T$ in $\mathbb{R}^3$ such that $B = T(A)$. Moreover, we give the following definition:

**Definition 0.6.** The image of a germ $g \in F^r(C)$ is said to have the same image as a given germ $f \in F^r(C)$ if there exist open subsets $U_i$ ($i = 1, 2$) containing $J_i \times \{0\}$ such that $g(U_1) = f(U_2)$. On the other hand, $g$ is said to be congruent to $f$ if there exists an isometry $T$ of the Euclidean space $\mathbb{R}^3$ such that $T \circ g$ has the same image as $f$, as a map germ.

**Remark 0.7.** Here, we consider the case $J = I$, that is, $C$ is non-closed. Consider an “admissible” $C^\infty$-germ of a cuspidal edge $f$ (i.e. $f$ belongs to the class $F^\infty(C)$ defined in (1.4)). If the first fundamental form of $f$ has no effective symmetries, then there exist three distinct isomers $\tilde{f}, f^*, \tilde{f}^* \in F^\infty(C)$ such that (cf. [3])

- $\tilde{f}(t, v), f^*(t, v)$ and $\tilde{f}^*(t, v)$ have the same parameters as $f(t, v)$, and
- the coefficients of the first fundamental forms of $\tilde{f}, f^*, \tilde{f}^*$ with respect to the coordinate system $(t, v)$ coincide with those of $f$.

In fact,

- $\tilde{f}$ (called the dual of $f$) is the isomer whose cuspidal angle (cf. Definition 1.1) takes the opposite sign of that of $f$,
- $f^*$ (called the inverse of $f$) is the isomer which is obtained by reversing the orientation of the parametrization $\gamma$ of $C$. The sign of the cuspidal angle of $f^*$ takes the same sign as that of $f$,
- $\tilde{f}^*$ (called the inverse dual of $f$) is the dual of the inverse $f^*$,
- if $g$ is an isomer of $f$, then $g$ is right equivalent to one of $\tilde{f}, f^*, \tilde{f}^*$.

The four maps $f, \tilde{f}, f^*, \tilde{f}^*$ in $\mathbb{R}^3$ are non-congruent in general. However, if $C$ admits a symmetry, this is not true. For example, consider a helix $C_0$ and fix a point $P_0$ on $C_0$. Then there exists an orientation preserving isometry $T$ of $\mathbb{R}^3$ (which is not the identity map) satisfying $T(C_0) = C_0$ and $T(P_0) = P_0$. Then, we can construct a cuspidal edge along $C_0$ (cf. Remark 2.3) such that $g := T \circ f$ is an isomer of $f$.

From now on, we consider the case that $J = S^1$, that is, $C$ is a knot in $\mathbb{R}^3$, and show that each $f \in F^\infty(C)$ has infinitely many isomers which are mutually non-congruent, in general.

![Figure 2. Cuspidal edges along the curves $\gamma_1$ (left) and $\gamma_2$ (right) (cf. Example 0.6)](image_url)
1. Results

We set $J := S^1$ and consider the case that $C(=\gamma(S^1))$ is a closed $C^\alpha$-embedded curve with positive curvature function $\kappa(t)$. We let $\mathbf{n}(t)$ (resp. $\mathbf{b}(t)$) be the unit principal normal (resp. unit bi-normal) vector of $\gamma(t)$. We set

$$P_0 := \gamma(0),$$

which is considered as a base point of $C$. The parametrization $\gamma$ of $C$ gives an orientation of $C$. For this fixed base point $P_0$ and this fixed orientation of $C$, we would like to show that any cuspidal edge germ along $C$ can be uniquely represented using a normal form given as follows:

For sufficiently small $\varepsilon > 0$, consider a $C^\alpha$-map (called Fukui’s formula, cf. [2] and [3]) $f(t, v) ((t, v) \in U_\varepsilon(S^1))$ expressed by

$$(1.1) \quad f(t, v) := \gamma(t) + (A(t, v), B(t, v)) \begin{pmatrix} \cos \theta(t) & - \sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) \end{pmatrix} \begin{pmatrix} \mathbf{n}(t) \\ \mathbf{b}(t) \end{pmatrix}$$

such that

(a) $f(0, 0) = P_0,$

(b) $A(t, v), B(t, v)$ and $\theta(t)$ are $C^\alpha$-functions, and

(c) for each $t \in S^1$, $A_{vv}(t, 0), B_{vv}(t, 0)$ are not equal to zero.

In this setting, the angle $\theta(t)$ in $f$ is called the cuspidal angle at $\gamma(t)$.

**Definition 1.1.** A map $f(t, v)$ satisfying (a), (b) and (c) is called a normal form of the cuspidal edge along $C$ with respect to the base point $P_0$ if

(d) $t$ is an arc-length parameter of $C$,

(e) for each $t \in S^1$, $v$ is the normalized half-arc-length parameter of the sectional cusp $v \mapsto (A(t, v), B(t, v))$ (see [3] Appendix A) for the definition of normalized half-arc-length parameters), that is, there exists a $C^\alpha$-function $m(t, v)$ satisfying $m(t, 0) \neq 0$ such that $A(t, v)$ and $B(t, v)$ have the following expressions:

$$A(t, v), B(t, v) := \int_0^v w \left( \cos \lambda(t, w), \sin \lambda(t, w) \right) dw, \quad \lambda(t, v) := \int_0^v m(t, w)dw.$$

The function $m(t, v)$ is called the extended half-cuspidal curvature function (cf. [3]).

In this situation, the singular curvature $\kappa_s(t)$ and the limiting normal curvature $\kappa_v(t)$ (cf. [3]) along the singular set of $f \in \mathcal{F}^\alpha(C)$ are given by (cf. [3])

$$(1.2) \quad \kappa_s(t) = \kappa(t) \cos \theta(t), \quad \kappa_v(t) = \kappa(t) \sin \theta(t).$$

The following assertion holds:

**Proposition 1.2.** For each $f \in \mathcal{F}^\alpha(C)$, there exists a unique normal form $\hat{f} \in \mathcal{F}^\alpha(C)$ with respect to the base point $f(0, 0)$ such that

- $[f] = [\hat{f}]$, and
- the orientation of $C$ given by the parametrization $t \mapsto \hat{f}(t, 0)$ coincides with that induced by the parametrization $t \mapsto f(t, 0)$.

**Proof.** The uniqueness of such an $\hat{f}$ follows from the fact that $\hat{f}(0, 0) = f(0, 0)$ and $\hat{f}(t, 0) = \gamma(t)$, since $t$ is an arc-length parameter of $\gamma$ and $v$ is the normalized half-arc-length parameter of the sectional cusps of $\hat{f}$. \hfill \Box

We prepare a lemma:

**Lemma 1.3.** Let $f, g \in \mathcal{F}^\alpha(C)$ be two normal forms of cuspidal edges along $C$. If the image of $f$ coincides with that of $g$ and $f(t, 0) = g(t, 0)$ holds for $t \in S^1$, then either $f(t, v) = g(t, v)$ or $f(t, v) = g(t, -v)$ holds.
Proof. Suppose that the images of the two maps coincide. Since \( f \) and \( g \) are written in normal forms, the fact that \( t \) is an arc-length parameter and \( v \) is a half-arc-length parameter implies
\[
g(t, v) = f(\sigma t + a, \sigma' v) \quad (t \in S^1),
\]
where \( a \in S^1 \) and \( \sigma, \sigma' \in \{1, -1\} \). Since \( f(t, 0) = g(t, 0) \) holds for \( t \in S^1 \), we have \( \sigma = 1 \) and \( a = 0 \), proving the assertion. \( \square \)

As a consequence, the following assertion holds:

**Proposition 1.4.** Let \( f, g \in F^r(C) \) be two cuspidal edge germs. Then \( g \) is right equivalent to \( f \) if and only if \( g \) has the same image as \( f \).

Proof. The “if”-part is obvious. So it is sufficient to show the “only if”-part. By Proposition 1.2, there exist normal forms \( f \) and \( \hat{g} \) of cuspidal edges along \( C \) such that
\[
[f] = [\hat{f}], \quad [g] = [\hat{g}], \quad \hat{f}(0, 0) = \hat{g}(0, 0).
\]
We suppose that \( g \) has the same image as \( f \). Then \( \hat{g} \) also has the same image as \( \hat{f} \). Replacing \( \hat{g}(t, v) \) by \( \hat{g}(-t, v) \) if necessary, we may assume that \( \hat{g}(t, 0) = \hat{f}(t, 0) \) for \( t \in S^1 \). Thus, by Lemma 1.3, we have \( \hat{g}(t, v) = \hat{f}(t, \pm v) \), which implies \([g] = [f] \). \( \square \)

**Corollary 1.5.** Let \( f, g \in F^r(C) \) be two cuspidal edge germs. If \( g \) is congruent to \( f \), then there exists an isometry \( T \) of \( \mathbb{R}^3 \) such that \( T \circ g \) is right equivalent to \( f \).

Proof. Suppose that \( g \) is congruent to \( f \). Then, there exists an isometry \( T \) of \( \mathbb{R}^3 \) such that \( T \circ g \) has the same image as \( f \). By Proposition 1.4, we can conclude that \( T \circ g \) is right equivalent to \( f \). \( \square \)

We fix \( f \in F^r(C) \). Then there exists a normal form \( \hat{f} \) of the cuspidal edge along \( C \) such that \([\hat{f}] = [f] \). The expression \( \hat{f}(t, v) \) means that \( t \) is the arc-length parameter of \( C \) and \( v \) is the normalized half-arc-length parameter of the sectional cusps. Let \( \kappa_s : S^1 \to \mathbb{R} \) be the singular curvature function of \( \hat{f} \) along \( C \). By (1.2), \( |\kappa_s(t)| \leq \kappa(t) \) holds, and \( \kappa_s(t) \) depends only on the first fundamental form of \( f \) (cf. (2.1) and (3)). We then consider the condition
\[
(1.3) \quad \max_{t \in S^1} |\kappa_s(t)| < \min_{t \in S^1} \kappa(t),
\]
and define the subclass
\[
F^*_r(C) := \{ f \in F^r(C) ; \text{ \( f \) satisfies (1.3) } \}
\]
of \( F^r(C) \). A germ of a cuspidal edge \( f \in F^r(C) \) is called admissible if it belongs to this subclass \( F^*_r(C) \). For \( f \in F^*_r(C) \), we may assume that its cuspidal angle \( \theta(t) \) satisfies
\[
(1.5) \quad 0 < |\theta(t)| < \pi \quad (t \in S^1).
\]
It should be remarked that cuspidal edges with constant Gaussian curvature satisfy \( \kappa_s = 0 \), that is, \( |\kappa_s| = \kappa \) on \( S^1 \). In particular, such surfaces do not belong to \( F^*_r(C) \). (If \( f \) is of constant Gaussian curvature, then \( \kappa_s \) vanishes identically. For such a case, see (1).)

**Example 1.6.** Consider the following \( 2\pi \)-periodic curves giving a series of torus knots \( (n := 2m - 1, m = 1, 2, 3, \ldots) \):
\[
(1.6) \quad \gamma_m(t) := \left( 2 + \cos nt \cos 2t, (2 + \cos nt) \sin 2t, \sin nt \right) \quad (t \in \mathbb{R}),
\]
and denote by \( C_m \) their images. The curve \( C_2 \) gives a trefoil knot. The cuspidal edges \( f_m \) for \( m = 1, 2 \) along \( C_m \) obtained from (1.1) by substituting
\[
A(t, v) = t^2, \quad B(t, v) = t^3, \quad \theta = \pi/4
\]
are indicated in Figure 2. Each of $f_m$ ($m \geq 1$) belongs to the class $F^*_r(C_m)$ as a map germ, since we have chosen $\theta$ so that $0 < \theta < \pi/2$.

**Definition 1.7.** We say that $C$ has a symmetry if there exists an isometry $T$ of the Euclidean space $\mathbb{R}^3$ such that $T(C) = C$ and $T$ is not the identity map. On the other hand, a $C^\infty$-function $\mu : S^3(= \mathbb{R}/(\mathbb{Z})) \to \mathbb{R}$ is said to have a symmetry if there exists a constant $c \in (0, l)$ or $c' \in [0, l)$ such that

$$\mu(t) = \mu(t + c) \quad \text{or} \quad \mu(t) = \mu(c' - t)$$

holds for $t \in \mathbb{R}$.

If $C$ has a symmetry $T$, and if there is a point $P \in C$ such that $T(P) \neq P$, then $\kappa(t)$ also admits a symmetry. Our main result is as follows:

**Theorem 1.8.** Let $C$ be the image of a closed $C^\omega$-curve embedded in $\mathbb{R}^3$, and let $g$ be a cuspidal edge germ belonging to $F^*_r(C)$. Then there are four continuous 1-parameter families of real analytic cuspidal edges $\{f_p\}_{P \in C}$ ($i = 1, 2, 3, 4$) which satisfy the following properties:

(i) Each $f^*_p$ ($i \in \{1, 2, 3, 4\}$, $P \in C$) belongs to $F^*_r(C)$ and is isometric to $g$.

Moreover, there exist $t_0 \in \{1, 2, 3, 4\}$ and $P_0 \in C$ such that $f^*_{P_0}$ is right equivalent to $g$.

(ii) If $h \in F^*_r(C)$ is an isomer of $g$, then $h$ is right equivalent to a cuspidal edge germ belonging to one of these four families.

(iii) Suppose that $C$ is not a circle. If the first fundamental form $ds^2$ of $g$ admits at most finitely many effective symmetries (in particular, this assumption follows if the singular curvature function $\kappa_s$ of $g$ is not constant, see Remark 1.10), then for each choice of $f^*_P$ ($i \in \{1, 2, 3, 4\}$, $P \in C$),

$$\Lambda^*_P := \{(j, Q) \in \{1, 2, 3, 4\} \times C : f^*_Q \text{ is congruent to } f^*_P\}$$

is a finite set. In particular, there are uncountably many mutually non-congruent isomers of $g$.

(iv) Suppose $C$ has no symmetries. If $ds^2$ does not admit any effective symmetries (in particular, this assumption follows if $\kappa_s$ has no symmetries, see Remark 1.10), then the set $\Lambda^*_P$ is a one-point-set for each $(i, P) \in \{1, 2, 3, 4\} \times C$.

**Remark 1.9.** To show the existence of $f$ as an isomer of $f \in F^*_r(C)$, the assumption (1.5) is needed to apply the Cauchy-Kowalevski theorem (see [3, Theorem 1]). (On the other hand, if (1.5) fails, one can find cuspidal edges whose isomers do not exist, see [3, Corollary 4.13] for details.) However, to construct infinitely many isomers of $f \in F^*_r(C)$, the condition (1.5) is not sufficient, and we need to assume that $f$ must belong to the class $F^*_r(C)$, see (2.6) in the proof of Theorem 1.8 below.

**Remark 1.10.** We may assume that the initially given $g \in F^*_r(C)$ is a normal form. Suppose that $ds^2$ admits an effective symmetry $\varphi$. (Later, we will show that any symmetry of $ds^2$ is effective, see Corollary 1.2.) Then it induces a symmetry of the singular curvature function $\kappa_s$ of $g$. Hence,

(1) the conclusion of (iv) follows if $\kappa_s$ has no symmetries.

Moreover,

(2) the conclusion of (iii) follows if $\kappa_s$ is non-constant.

In fact, if $ds^2$ admits infinitely many distinct effective symmetries, then they give infinitely many symmetries of $\kappa_s$. Since $\kappa_s$ is real analytic, it must be constant.
To construct \( f_0 \) along the knot \( C \), the real analyticity of \( f \) and the condition (1.3) are required, because we need to apply the Cauchy-Kowalevski theorem inductively (cf. Lemma 2.1).

Since the curvature function of the curve \( \gamma_m \) \((m \geq 1)\) given in (1.4) is non-constant, the cuspidal edges \( f_m \) as in Example 1.9 satisfy the condition (iii) in Theorem 1.8 and each \( f_m \) has infinitely many isomers. On the other hand, \( f_m \) does not satisfy the condition (iv), since \( C \) admits a symmetry. We can show the existence of an example satisfying (iv), as follows:

**Example 1.11.** We consider a closed convex \( C^\infty \)-regular curve \( C \), lying in the 2-plane \( \mathbb{R}^2 := \{ (x, y, 0) \in \mathbb{R}^3 ; x, y \in \mathbb{R} \} \). We can choose \( C \) so that it has no symmetries as a plane curve. Considering the approximation of \( C \) by Fourier series, by Lemma A.4 in the appendix, we may assume that \( C \) is real analytic. Let \( \pi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2 \) be the stereographic projection. We let \( \gamma(t) \) \((0 \leq t \leq l)\) be the arc-length parameterization of \( C \) and set

\[
\tilde{\gamma}_u(t) := \frac{\pi^{-1}(u \gamma(t)) + (0,0,1)}{2u},
\]

which is a real analytic 1-parameter deformation of \( \tilde{\gamma}_0 := \gamma \) for sufficiently small \(|u|\). We denote by \( \tilde{C}_u \) the image of \( \tilde{\gamma}_u(t) \). Then \( \tilde{C}_0 = C \). Since the length \( L(u) \) of \( \tilde{C}_0 \) depends real analytically on \( u \), it is a real analytic function of \( u \). So we set

\[
\gamma_u(t) := \frac{l}{L(u)} \tilde{\gamma}_u(t),
\]

which gives a 1-parameter family of \( C^\infty \)-embedded space curves of period \( l \) satisfying \( \gamma_0 = \gamma \). Since \( C \) has no symmetries, the curvature function \( \kappa_u \) of \( \gamma_u \) also has no symmetries for sufficiently small \(|u|\) (cf. Lemma A.4). Since \( \gamma_u(u \neq 0) \) lies on a sphere, and its curvature function as a space curve is not constant, it cannot be a part of any circle, and so its torsion function never vanishes identically. In particular, it does not lie in any plane. As a consequence, the image of \( \gamma_u(u \neq 0) \) has no symmetries. We fix \( u \), and parametrize \( \gamma_u \) by arc-length. Then by Fukui’s formula (cf. (1.3)), we can construct a \( C^\infty \)-cuspidal edge \( f_u \) with constant cuspidal angle \( \theta_0 \in (0, \pi/2) \) and with \((A(t,v),B(t,v)) = (t^2,t^3)\). Since the singular curvature function \( \kappa_u \cos \theta_0 \) of \( f_u \) along the \( t \)-axis has no symmetries, part (1) of Remark 1.10 implies that \( f_u \) satisfies (iv) of Theorem 1.8.

2. Proof of Theorem 1.8

A positive semi-definite \( C^\infty \)-metric \( ds^2 = Edt^2 + 2Fdt dv + Gdv^2 \) defined on \( U_{\varepsilon}(S^1) \) is called a periodic Kossowski metric (cf. [4] or [5]) if it satisfies the following:

\begin{itemize}
  \item[(a)] \( F(t,0) = G(t,0) = 0 \), \( E_v(t,0) = 2F_t(t,0) \) and \( G_t(t,0) = G_v(t,0) = 0 \),
  \item[(b)] there exists a \( C^\infty \)-function \( \lambda \) defined on \( U_{\varepsilon}(S^1) \) satisfying \( EG - F^2 = \lambda^2 \), \( \lambda(t,0) = 0 \) and \( \lambda_v(t,0) \neq 0 \) for each \( t \in S^1 \).
\end{itemize}

Under the assumption that \( \lambda_v(t,0) > 0 \), the singular curvature \( \kappa_v \) of \( ds^2 \) is defined by (cf. [3] Remark 3.5)]

\[
\kappa_v(t) := \frac{-F_v(t,0)E_t(t,0) + 2E(t,0)F_{tv}(t,0) - E(t,0)E_{vv}(t,0)}{2E^{3/2}(t,0)\lambda_v(t,0)}
\]

for each \( t \in S^1 \). If \( ds^2 \) is the first fundamental form of \( f \in \mathcal{F}^\infty(C) \), it is a periodic Kossowski metric (cf. [3] Lemma 2.9)), and the singular curvature of \( f \) has the above expression.
Lemma 2.1. Let \( \gamma(t) \) (\( t \in S^1 \)) be a closed \( C^\omega \)-curve embedded in \( \mathbb{R}^3 \) parametrized by arc-length whose curvature function \( \kappa(t) \) is positive everywhere. Let \( ds^2 \) be a periodic Kossowski metric on \( U_\delta(S^1) \) satisfying

\[
E(t, 0) = 1, \quad F(t, 0) = G(t, 0) = 0 \quad (t \in S^1).
\]

Suppose that the singular curvature \( \kappa_s \) along the singular curve \( S^1 \ni t \mapsto (t, 0) \in U_\varepsilon(S^1) \) satisfies

\[
(2.2) \quad |\kappa_s(t)| < \kappa(t) \quad (t \in S^1).
\]

Then there exists a cuspidal edge \( f_+ \) (resp. \( f_- \)) along \( C := \gamma(S^1) \) satisfying

1. \( f_+ \) (resp. \( f_- \)) is defined on \( U_\delta(S^1) \) for some \( \delta \in (0, \varepsilon) \),
2. \( f_+(t, 0) = \gamma(t) \) (resp. \( f_-(t, 0) = \gamma(t) \)) for each \( t \in S^1 \),
3. the first fundamental form of \( f_+ \) (resp. \( f_- \)) is \( ds^2 \),
4. the limiting normal curvature of \( f_+ \) (resp. \( f_- \)) is positive-valued (resp. negative-valued) and is equal to

\[
\sqrt{\kappa(t)^2 - \kappa_s(t)^2}, \quad (\text{resp. } -\sqrt{\kappa(t)^2 - \kappa_s(t)^2}),
\]

and

5. \( f_+ \) and \( f_- \) belong to \( \mathcal{F}^\omega(C) \).

Moreover, if there exists a cuspidal edge \( g \) defined on an open subset \( V(\subset U_\varepsilon(S^1)) \) containing \( S^1 \times \{0\} \) such that \( g(t, 0) = \gamma(t) \) holds for \( t \in S^1 \) and the first fundamental form of \( g \) is \( ds^2 \), then \( g \) is right equivalent to \( f_+ \) or \( f_- \).

Proof. We can find a partition \( 0 = t_0 < t_1 < \cdots < t_n = l \) for \( S^1 = \mathbb{R}/\mathbb{Z} \) such that there exist local coordinate systems \((U_i; x_i, y_i) \) (\( i = 1, \ldots, n \)) of \( U_\varepsilon(S^1) \) containing \([t_{i-1}, t_i] \times \{0\} \), where \( n \) is a certain positive integer. For each \( i \in \{1, \ldots, n\} \), the metric has the expression \( ds^2 = E_i(dx_i)^2 + G_i(dy_i)^2 \) on \( U_i \). Since \( ds^2 \) satisfies \( (2.2) \), we can apply [3, Theorem 3.9] by setting \( U := U_i \) for each \( i \), and obtain a map \( g_{+i} : U_i \to \mathbb{R}^2 \) (resp. \( g_{-i} : U_i \to \mathbb{R}^2 \)) \( i = 1, \ldots, n \) satisfying (2) and (3) on \( U_i \), and the limiting normal curvature of \( g_{\pm i} \) is equal to

\[
\sqrt{\kappa(t)^2 - \kappa_s(t)^2}, \quad (\text{resp. } -\sqrt{\kappa(t)^2 - \kappa_s(t)^2}).
\]

In other words, the cuspidal angles \( \theta_{\pm i} \) of \( g_{\pm i} \) satisfy (cf. \( (1.2) \) and also \( (3.6) \))

\[
(2.3) \quad \kappa(t)\sin \theta_{\pm i}(t) = \pm \sqrt{\kappa(t)^2 - \kappa_s(t)^2}, \quad \theta_{-i}(t) = -\theta_{+i}(t).
\]

Since conditions (2) and (3) do not depend on coordinates, the uniqueness of such a pair of maps yields that \( g_{\pm i} = g_{\pm i-1} \) holds on \( U_i \cap U_{i-1} \). So we obtain a map \( f_+ \) (resp. \( f_- \)) defined on \( U_\delta([0, l]) \) for a certain \( \delta \in (0, \varepsilon) \) such that each \( g_{\pm i} \) (resp. \( g_{-i} \)) coincides with \( f_+ \) (resp. \( f_- \)) on \( U_\delta([0, l]) \cap U_i \) for \( i = 1, \ldots, n \). In particular, the cuspidal angle functions \( \theta_{\pm i}(t) \) of \( f_{\pm} \) satisfy (cf. \( (1.3) \) and \( (1.5) \))

\[
\theta_{-i}(t) = -\theta_{+i}(t), \quad \kappa(t)\sin \theta_{+i}(t) = \kappa_s(t), \quad \kappa(t)\sin \theta_{+i}(t) = \sqrt{\kappa(t)^2 - \kappa_s(t)^2}
\]

for \( t \in S^1 \). Then, we obtain \( f_+ = g_{+1} \) and \( f_- = g_{-1} \) on \( U_n \cap U_1 \). In fact, if not, we have \( g_{\pm n} = g_{\pm 1} \) and the function \( \theta_{i}(t) \) takes different signs at \( t = 0 \) and \( t = l \). Then by the continuity of \( \theta_{i}(t) \), it has a zero on \( [0, l] \), a contradiction. So \( f_{\pm} \) are \( l \)-periodic. Moreover, \( f_+ \) and \( f_- \) belong to \( \mathcal{F}^\omega(C) \), because of \( (1.3) \). The last statement of Lemma 2.1 follows from the last assertion of [3, Theorem 3.9].

As an application of this lemma, we can prove the following important conclusion:

Corollary 2.2. Let \( f \in \mathcal{F}^\omega(C) \). If the first fundamental form \( ds_f^2 \) of \( f \) has a symmetry \( \varphi \), then \( \varphi \) is effective.
Suppose that $g$ is a non-effective symmetry of $ds_f^2$. Then $\varphi(t, 0) = (t, 0)$ holds. If we set $g = f \circ \varphi$, then $f(t, 0) = g(t, 0)$ and $g$ has the same first fundamental form as $f$. Applying Lemma 2.1 to $ds_f^2$, we obtain two cuspidal edges $f_{\pm}$ whose first fundamental forms are $ds^2$ such that $f_{\pm}(t, 0) = f(t, 0)$. Then the last assertion of Lemma 2.1 yields that either $f = f_+ = f_-$ holds. Moreover, applying the last assertion of Lemma 2.1 again, we can conclude that $g$ coincides with $f_+$ or $f_-$. Since $g$ has the same image as $f$, the cuspidal angle of $g$ at each point of $C$ coincides with that of $f$. Thus, $g$ must coincide with $f$, which implies $f \circ \varphi = f$. Since $f$ is a cuspidal edge, it is an injective map. So $\varphi$ must be the identity map. □

**Proof of Theorem 1.8.** Without loss of generality, we may assume that $g(t, v)$ itself is expressed in a normal form and that $\gamma(t)$ is parametrized by arc-length. Replacing $\gamma(t)$ by $\gamma(\sigma t + b)$ for suitable $\sigma \in \{1, -1\}$ and $b \in S^1$, we may assume that

\[ g(t, 0) = \gamma(t) \quad (t \in S^1). \]

We denote by $ds^2$ the first fundamental form of $g$. We now construct the four families $f^i_{\gamma(a)}$ for each $a \in [0, 1]$. Since $f$ satisfies (1.3), it holds that

\[ \kappa_s(t) \leq \max_{u \in S^1} |\kappa_s(u)| < \min_{u \in S^1} \kappa(u) \leq \kappa(\sigma t + a) \quad (\sigma \in \{1, -1\}) \]

for $t \in S^1$. In particular, we can apply Lemma 2.1 for the closed $C^\infty$-curves

\[ t \mapsto \gamma(t) \quad \text{and} \quad t \mapsto \gamma(-t) \quad (a \in [0, 1]). \]

Thus, we obtain four isomers $f^i_{\gamma(a)}$ associated with $g$ ($i = 1, 2, 3, 4$) such that

1. each $f^i_{\gamma(a)}$ belongs to $F^\infty(C)$ whose first fundamental form is $ds^2$,
2. $f^i_{\gamma(a)}(t, 0) = (t + a) (t \in S^1)$ holds for $j = 1, 2$, and $f^k_{\gamma(a)}(t, 0) = (t - a)$ ($t \in S^1)$ holds for $k = 3, 4$, and
3. the limiting normal curvature of $f^i_{\gamma(a)}$ ($i = 1, 2, 3, 4$) is equal to

\[ \sigma_i^2 \sqrt{\kappa(\sigma t + a)^2 - \kappa_s(t)^2}, \]

where $\kappa_s$ is the singular curvature of $g$ along $\gamma$ and

\[ \sigma_i := \begin{cases} 1 & \text{if } i = 1, 2, \\ -1 & \text{if } i = 3, 4 \end{cases} \quad \text{and} \quad \sigma'_i := \begin{cases} 1 & \text{if } i = 1, 3, \\ -1 & \text{if } i = 2, 4. \end{cases} \]

**Remark 2.3.** In the above construction, each $f^i_{\gamma(a)}$ might not be expressed as a normal form. We give here such an example. We consider the helix (cf. [3] Example 5.4)

\[ \gamma(t) := \left( \cos \left( \frac{t}{\sqrt{2}} \right), \sin \left( \frac{t}{\sqrt{2}} \right), \frac{t}{\sqrt{2}} \right) \]

parametrized by arc-length defined on a certain bounded closed interval containing $t = 0$. We set

\[ (A(v), B(v)) := \int_0^v w(\cos w, \sin w)dw = (v \sin v + \cos v - 1, \sin v - v \cos v), \]

and fix a constant $\theta \in (0, \pi/2)$. Then the map $f(t, v)$ induced by Fukui’s formula (1.1) gives a normal form of a cuspidal edge along the helix. As shown in [3] Example 5.4, we can express the first fundamental form $ds_f^2$ of $f$ in the following form

\[ ds_f^2 = E(v)dt^2 + 2F(v)dtdv + G(v)dv^2, \]

where $F(v) = v(v - \sin v)/2$, $G(v) = v^2$, and $E(v)$ is a certain positive valued $C^\infty$-function of $t$ which depends on $\theta$ (one can compute $E(v)$ explicitly using the formula given in [3] Proposition 4.8].)
Also, as shown in [3] Example 5.4, an isomer (the dual) $\tilde{f}$ of $f$ can be written as (the figures of the images of $f$ and $\tilde{f}$ are indicated in Figure 1, right)

$$\tilde{f} = T \circ f \circ \varphi,$$

where $T$ is the 180°-rotation with respect to the principal normal line at $\gamma(0)$ of the helix $\gamma$, and $\varphi$ is an effective symmetry of $ds^2$ fixing $(0,0)$. (The inverse and the inverse dual of $f$ are given by $f_* = T \circ f$ and $\tilde{f}_* = T \circ \tilde{f}$, respectively.) Such a symmetry $\varphi$ must be uniquely determined by [3] Proposition 3.15. If $\tilde{f}(t, v)$ is also a normal form, then the map $\varphi$ must have the expression $\varphi(t, v) = (-t, v)$. However, this contradicts the fact that $\varphi(x, y) = (-x, y)$ holds for the local coordinate system at $(0, 0)$ given by (cf. [3] (5.2))

$$x(t, v) := t + \int_0^v \frac{w(w - \sin w) - (w - \sin w)^2}{2E(w)} dw, \quad y(t, v) := \int_0^v \sqrt{\frac{4E(w) - (w - \sin w)^2}{4E(w)}} dw.$$

We return to the proof of Theorem 1.8 and prove (i). By (1), each $f^i_{\gamma(a)} (i = 1, 2, 3, 4, a \in S^1)$ has the same first fundamental form as $g$. In particular, the singular curvature function of $f^1_{\gamma(a)}$ coincides with that of $g$. Thus, $f^i_{\gamma(a)}$ satisfies \cite{13} and belongs to $F^\omega(C)$. By \cite{24} and (3), we have $g = f^4_{\gamma(0)}$ (resp. $g = f^2_{\gamma(0)}$) if the cuspidal angle of $g$ is positive (resp. negative), proving all assertions in (i).

We next prove (ii): Suppose that $h$ is an isomer of $g$. Then, there exists a local diffeomorphism $\varphi$ such that $\varphi^*ds^2_h = ds^2$, where $ds^2_h$ is the first fundamental form of $h$. Then $h \circ \varphi$ has the same first fundamental form as $g$. Since $t \mapsto h \circ \varphi(t, 0)$ gives an arc-length parametrization of $C$, we can write

$$h \circ \varphi(t, 0) = \gamma(\sigma_1 t + b)$$

for some $\sigma_1 \in \{1, -1\}$ and $b \in S^1$. By the last statement of Lemma 2.1 $h \circ \varphi$ coincides with $f^j_{\gamma(b)}$ for some $j \in \{1, 2, 3, 4\}$, proving (ii).

We then prove (iii): Let $f_n := f^n_{\gamma(a_n)} (n = 1, 2, \ldots)$ be mutually distinct isomers of $g$ which are congruent to each other, where $f_n \in \{1, 2, 3, 4\}$ and $a_n \in [0, 1)$. Replacing $\{f_n\}$ by a suitable subsequence if necessary, we may assume that the sequence $\{a_n\}$ consists of distinct values. By Corollary 1.5 there exist an isometry $T_n$ of $R^3$ and a local diffeomorphism $\varphi_n$ such that

\begin{equation}
(2.8) \quad f_n = T_n \circ f_1 \circ \varphi_n
\end{equation}

holds (i.e. $T_1$ and $\varphi_1$ are identity maps), which implies $\varphi_n^*ds^2 = ds^2$. By Corollary 2.2 $\varphi_n$ is effective unless it is the identity map. Since $ds^2$ admits only finitely many effective symmetries, we may assume that $\varphi := \varphi_n$ does not depend on $n$. Then we have

$$T_n^{-1} \circ f_n = f_1 \circ \varphi = T_2^{-1} \circ f_2$$

for $n \geq 3$. Substituting $v = 0$ and using the fact that $f_n = f^n_{\gamma(a_n)}$, we have

$$\gamma(\sigma_{n_1} t + a_n) = f_n(t, 0) = T_n \circ T_2^{-1} \circ f_2(t, 0) = T_n \circ T_2^{-1} \circ \gamma(\sigma_{n_2} t + a_2),$$

where $\sigma_k (k = 1, 2, 3, \ldots)$ are defined in [27]. In particular,

$$\kappa(\sigma_{n_1} t + a_n) = \kappa(\sigma_{n_2} t + a_2), \quad \sigma_{n_1}^\tau(\sigma_{n_1} t + a_n) = \tau(\sigma_{n_2} t + a_2)$$

hold, where (“det” denotes the determinant of square matrices)

$$\sigma_{n}^\tau := \det(T_n \circ T_2^{-1}) \in \{1, -1\}$$

and $\tau(t)$ is the torsion function of $\gamma(t)$. Substituting $t = 0$, we have

$$\kappa(a_n) = \kappa(a_2), \quad \sigma_{n}^\tau(a_n) = \tau(a_2).$$
Since the sequence \( \{a_n\} \) takes distinct values, this accumulates to a value \( a_\infty \in S^1 \). Since \( \kappa(t) \) and \( \tau(t) \) are real analytic functions, they must be constant. Since \( C \) is a knot, it must be a circle lying in a plane, a contradiction.

Finally, we show (iv): We fix \( f_0 := f_{\gamma(a_i)}^j (i \in \{1, 2, 3, 4\}) \), where \( a \in [0, l] \). Suppose that \( f_1 = f_{\gamma(b)}^j \) \((\gamma \neq (a, b))\) is congruent to \( f_0 \), where \( j \in \{1, 2, 3, 4\} \) and \( b \in [0, l] \). By Corollary 2.2 there exist an isometry \( T \) of \( R^3 \) and a local diffeomorphism \( \varphi \) such that \( f_1(t, \tau) = T \circ f_0 \circ \varphi \). Since \( C \) has no symmetries, \( T \) must be the identity map. Moreover, since \( f_1 \) and \( f_0 \) have the same first fundamental form, \( \varphi^*ds^2 = ds^2 \) holds. Since \( ds^2 \) does not admit any effective symmetries, Corollary 2.2 yields that \( \varphi \) is the identity map. Hence \( f_1 = f_0 \), and \( \Lambda_{\gamma(a)}^i (a \in [0, l]) \) is a one-point set.

**Remark 2.4.** When \( J = I \), that is, \( C \) is non-closed, the authors showed in [3] that for each germ of a cuspidal edge \( f \) satisfying

\[
(2.9) \quad \max_{t \in I} |\kappa_s(t)| < \min_{t \in I} \kappa(t),
\]

there exist three isomers \( \hat{f}, \tilde{f}, \text{and } \ddot{f} \) of \( f \) (see Remark 0.7). If we set \( I_a := [a - \delta, a + \delta] \) for sufficiently small \( \delta > 0 \), the restrictions \( (f^j_{\gamma(a)})(j = 2, 3, 4) \) of \( f^j_{\gamma(a)} \) (constructed in the above proof) to the interval \( I_a \subset S^1 \) coincide with these three isomers (cf. (2.7)).

**Appendix A. A Property of Non-symmetric Functions**

We prove the following assertion:

**Lemma A.1.** Let \( \{\mu_s(t)\}_{s \in [0, l]} \) be a continuous one-parameter family of \( C^\infty \)-functions on \( R \) satisfying \( \mu_s(t + l) = \mu_s(t) \) for each \( s \). If \( \mu_0 \) has no symmetries (cf. Definition 1.7), then \( \mu_s \) also has no symmetries for sufficiently small \( s(> 0) \).

**Proof.** If the assertion fails, then there exists a monotone decreasing sequence \( \{s_n\} \) converging to 0 such that \( \mu_{s_n} := \mu_{s_n} \) has a certain symmetry, that is, there exist a constant \( c_n \in [0, l] \) and a sign \( \sigma_n \in \{1, -1\} \) such that

\[
(\text{A.1}) \quad \mu_{s_n}(\sigma_n t + c_n) = \mu_{s_n}(t).
\]

Since \( \sigma_n = \pm 1 \), by replacing \( \{s_n\} \) by some subsequence if necessary, we may assume that \( \sigma := \sigma_n \) does not depend on \( n \). Since \( R/\mathbb{Z} \) is compact, replacing \( \{s_n\} \) by some subsequence if necessary, we may assume that \( c_n \) converges to \( c_0 \). Then taking the limit \( n \to \infty \), we have \( \mu_0(\sigma t + c_0) = \mu_0(t) \). Since \( \mu_0 \) is non-symmetric, we have \( \sigma = 1 \) and \( c_0 \in [0, l] \). Then, we may assume that \( c_0 = 0 \) without loss of generality. If \( c_n \) is an irrational number, then \( \{\sigma m + c_n\}_{m \in \mathbb{Z}} \) is dense in \( S^1 \), and (A.1) yields that \( \mu_n \) is a constant function. Since \( \mu_0 \) has no symmetries, we may assume that \( c_n \) is a rational number for sufficiently large \( n \), and we can write \( c_n := q_n/p_n \), where \( p_n \) and \( q_n \) are relatively prime integers. Then there is a pair \( (a, b) \) of integers such that \( ap_n + bq_n = 1 \) and

\[
(\text{A.2}) \quad \mu_n(t) \equiv \mu_n(t + bc_n) = \mu_n(t + b) = \mu_n(t + 1 - \frac{ap_n}{p_n}) = \mu_n(t + \frac{1}{p_n}).
\]

Fix an irrational number \( x_0 \in (0, 1) \). Then there exist integers \( r_n \) \((n = 1, 2, 3, \ldots)\) so that \( x_n := r_n/p_n \) converges to \( x_0 \). Since \( \mu_n(t) = \mu_n(t + x_n) \) by (A.2), taking the limit as \( n \to \infty \), we have \( \mu_0(t) = \mu_0(t + x_0) \), contradicting the assumption that \( \mu_0(t) \) has no symmetries. \( \square \)

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