Entropy invariants of generic actions

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Abstract

We show that the typical dynamical system sometimes begins to behave like a non-deterministic system with a small classical entropy, and this behavior lasts an extremely long time, until the system starts decreasing entropy. Then again it will become almost non-deterministic for a very very long time, but with more smaller classical entropy. Playing on this fact and considering sigma-compact families of measure-preserving zero-entropy transformations, for example, the rectangle exchange transformations, we choose the Kushnirenko entropy so that it is equal to zero for the transformations under consideration, but is infinite for the generic transformation.

The typical dynamical system is not mixing, but at some points in time it becomes like mixing, and such mixing intervals may last as long as the reader desires it and even longer. This fact was used in [17] to refine the theorem on the generic actions of lattices [23]. Now we would like to show that some powers of the generic transformation at very long time behave as a non-deterministic system.

In [6] it was proved the non-typicalness of the interval exchange transformations (IETs). In connection with this the following question arised: will the rectangle exchange transformation be non-typical? Little is known about the mixing properties of such systems, except that they have zero entropy[1]. We answer positively the question by use of special Kushnirenko’s invariants, which are infinite for the generic transformations and zero for the systems under consideration. In particular, we give the entropy proof of the mentioned result on IETs.

1 Generic properties of transformations

We fix a standard probability space \((X, \mathcal{B}, \mu)\) and consider the group of its automorphism \(Aut\), which is equipped by the Halmos complete metric \(\rho\). When

\[1\] J.-P. Thouvenot provided the author with an elegant proof of this and more general fact about the entropy of piecewise isometric transformations.
we traditionally write "the generic (typical) transformation is weakly mixing", we mean that there is a dense $G_\delta$-set consisting only of weakly mixing transformations. We say also that the set of all weakly mixing transformations is comeager. In fact there are generic properties, there are comeager sets, but there is no generic transformation (such a transformation that possesses all generic properties). For example, if $T$ is weakly mixing, then $T_{m_i} \to_w \Theta$, where $\Theta$ stands for the orthoprojection to the constants in $L_2(\mu)$. But this $T$ does not possess the following generic property: there is a subsequence $i_k$ such that $T_{m_{i_k}} \to I$. As the reader has already noted, we use the same notation for the transformation and the corresponding operator in $L_2(\mu)$.

1.1. To be weakly mixing and not to be mixing. These properties both are generic (P. Halmos, V. Rokhlin). To see this again let us fix an infinite set $M \subset \mathbb{N}$ and find a comeager set $Y \subset Aut$ such that for all $T \in Y$ one has

$$T_{m_i} \to \Theta, \ T_{m_i} \to I$$

for some sequences $\{m_i\}, \{n_i\} \subset M$. By this reason for any transformation $T$ the set of all transformations spectrally disjoint from $T$ is comeager.

Let us prove a bit more general assertion. An operator function $Q(T)$ is said admissible, if $Q(T) = a\Theta + \sum i a_i T^i$ and $a, a_i \geq 0$, $a + \sum i a_i = 1$.

Theorem 1.1. For an infinite set $M$ and any admissible functions $Q, R$ the set $\{T : T_{m} \to_w Q(T) \text{ as } m \in M, m \to \infty\}$ is meager,

the set $Y = \{T : T_{m} \to_w R(T) \text{ for some } \{m_k\} \subset M\}$ is comeager.

Corollary. For any transformation $T$ the set of all transformations spectrally disjoint from $T$ is comeager. So the orbit $\{RT R^{-1} : R \in Aut\}$ is meager [9] (see also [7]).

Proof. The first claim follows from the second. To get the latter we fix $\{J_q\}$, a dense family of transformations, then choose $M' \subset M$ and a transformation $S$ such that $S^m \to_w R(S), m \in M'$. It is not hard to construct such $S$ as a rank-one transformation. For any $n$ and $q$ we find $m = m(n, q)$ and a neighbourhood $U(n, q)$ of $J_q^{-1}S J_q$ such that the inequality

$$w(T_{m}, R(T)) < \frac{1}{n}$$

2
holds for all $T \in U(n, q)$. We get a dense $G_{\delta}$-set
\[ \bigcap_n \bigcup_q U(n, q) \subset Y. \]

There are many applications of such weak limits, see, for example, [14], [19], and [20], where the typicalness of the limits $a\Theta + (1 - a)I$ has been proven.

1.2. Asymmetry. If $T$ and $T^{-1}$ are not isomorphic, we call $T$ asymmetric. The first example of such $T$ has been presented by N.Anzai. A weakly mixing asymmetric transformation was appeared in [15], the typicalness of this property has been established in [9]. Later it was appeared the following more special property (see [18]).

**Theorem 1.2.** For some sequences $m_i, n_i$ there is a transformation $T$ with the property:
\[
\mu(A \cap T^{m_i}A \cap T^{n_i}A) \to (\mu(A) + 2\mu(A)^3)/3,
\]
\[
\mu(A \cap T^{-m_i}A \cap T^{-n_i}A) \to \mu(A)^2.
\]
The set of transformations $S$ with the same property for subsequences of the sequence $i$ is comeager. All such $S$ are asymmetric.

1.3. Refined typicalness. The work [12] on the roots of the generic transformation stimulated the discovery of new facts in ergodic theory of generic measure-preserving systems. A generic transformation has many roots, it is a finite group extension [2]. At the same time it is a relative weakly mixing extension of some nontrivial factor [8]. Generic transformations are embedded in a flow [16], even in weakly mixing $\mathbb{R}^n$-actions [23]. The centralizer of the generic transformation has a reach structure, which contains a free action of the infinite-dimensional torus [21].

The theory of the generic group actions is not quite a thing in itself, sometimes it turns out to be useful for applications. O. Ageev used nontrivial generic arguments to solve the homogeneous spectrum problem in the class of the weakly mixing transformations, and later S. Tikhonov did the same in case of mixing.
2 Non-typical transformations

Let $K$ be a compact in $(Aut, \rho)$ family of transformations. Is it true that the orbit of $K$, i.e.

$$K^{Aut} = \{STS^{-1} : T \in K, S \in Aut\},$$

is meager, in other words, out of it there is a dense $G_\delta$-set?

In [19] the following partial result is appeared.

**Theorem 2.1.** Let $K \subset Aut$ be compact set in $(Aut, \rho)$ and for some $r > 0$ for all transformations $T \in K$ and any positive integer $n$, there exists $m > n$ such that $w(T^m, \Theta) > r$, where $w$ is a metric defining the weak operator topology. Then $K^{Aut}$ is meager.

Proof. Let $\min(T, j)$ be the minimal number among those $m > j$ for which $w(T^m, \Theta) > r$. Since $K$ is compact, $\min(T, j)$ is a bounded function on $K$. We denote by $M(j)$ the maximum value of $\min(T, j)$ and consider the sets

$$F_j = \{j, j+1, \ldots, M(j)\}.$$ 

It was proved in [17] that there exists comeager set $Y$ such that for every $S \in Y$ there is a sequence $j(k)$ such that

$$w(S^m, \Theta) \to 0 \text{ as } m \in F_{j(k)}, m \to \infty.$$ 

This transformation $S$ does not belong to $K$, otherwise

$$\exists m \in F_{j(k)} \text{ dist}(S^m, \Theta) > r,$$

but this is forbidden for the transformations from $Y$. Since $Y^{Aut} \cap K = \phi$, we get $Y \cap K^{Aut} = \phi$.

The interval exchange transformations (as we fix a number of intervals) satisfy the conditions of Theorem 2.1. Thus, we have a simpler proof of the result [6] about non-typicalness of IETs. Our proof in shorter, since we did not care about estimating the numbers $M(j)$. The partial rigidity of the IETs (see [10]) provides the condition $w(T^m, \Theta) > r$.

The remarks kindly sent by Benjamin Weiss suggested to the author the following version of Theorem 2.1.

**Theorem 2.2.** The set of the transformations that are disjoint from all ergodic IETs is comeager. In short, $\{\text{IETs}\}^\perp$ is comeager.
Proof. Let $T$ be generic and $S$ be an ergodic IET. We can find a sequence $m_k$ such that

$$T^{m_k} \to \Theta, \quad S^{m_k} \to aI + (1 - a)P,$$

where $P$ is some Markov operator commuting with $S$. The sequence $m_r$ is said mixing for $T$ and partially rigid for $S$. We find such partially rigid sequences $m_k$ within the above mixing sets $F_{j(k)}$ corresponding to the generic transformations $T$ (see the proof of Theorem 2.1). The disjointness follows easy from the above conditions. Let $J$ be Markov operator such that $SJ = JT$. Then we have

$$S^{m_j}J = JT^{m_j}, \quad S^{m_k}J = JT^{m_k},$$

$$aJ + (1 - a)PJ = J\Theta = \Theta.$$

Since $\Theta$ is an extreme point in the set of all Markov operators that intertwine ergodic $S$ with weakly mixing $T$, we get

$$J = \Theta.$$

Thus, $S$ and $T$ are disjoint in sense of Furstenberg.

However, we cannot use similar arguments for rectangle exchange transformations. The mixing properties of the latters have not been studied. For this reason, we will use the following entropy arguments.

3 Entropy invariants

Among the wide variety of the entropy notions (see, for example, [4], [11], [22], [24]) we consider slightly modified invariants of Kushnirenko [13].

Let $P = \{P_j\}$ be a sequence of finite subsets of a countable infinite group $G$. We suppose that $|P_j| \to \infty$. For a measure-preserving $G$-action $T = \{T_g\}$ we define

$$h_j(T, \xi) = \frac{1}{|P_j|} H \left( \bigvee_{p \in P_j} T_p \xi \right),$$

$$h_P(T, \xi) = \limsup_j h_j(T, \xi),$$

$$h_P(T) = \sup_\xi h_P(T, \xi),$$

5
\[ h^\inf_P(T) = \sup_\xi \liminf_j h_j(T, \xi), \]

where \( \xi \) is a finite measurable partition of \( X \).

3.1. Upper \( P \)-entropy. Without getting carried away with an overly general situation, let’s consider below a rather ascetic case of progressions: \( G = Z, P_j = \{ j, 2j, \ldots, L(j)j \} \), for some sequence \( L(j) \to \infty \).

**Theorem 3.1.** The class \( \{ S : h_P(S) = \infty \} \) is comeager.

Proof. Let \( \{ J_q \}, q \in \mathbb{N} \), be dense in \( Aut \), and \( T \) be a Bernoulli transformation, so are \( T_q = J_q^{-1} T J_q \). The set \( \{ T_q \} \) is dense in \( Aut \). We fix a dense collection of partitions \( \xi_i \) (in fact the density is not nessesary).

For any \( n, q \) there is \( j = j(n, q) \) such that for all \( i \leq n \)

\[ h_j(T_q, \xi_i) = \frac{1}{L_j} H(\bigvee_{n=1}^{L(j)} T_q^{n_j} \xi_i) > H(\xi_i) - \frac{1}{n}. \]  

Indeed, \( T_q \) is Bernoulli, we find a partition \( \xi \) which is close to \( \xi_i \) (\( i \) is fixed) and for some \( M(i, q, n, ) \) the partitions \( T_q^{n_j} \xi \) are independent for all \( n \) as \( j > M(i, n, q) \). This implies (n) for sufficiently large \( j \).

There is a neighbourhood \( U(n, q) \) of the transformation \( T_q \) such that all \( S \in U(n, q) \) satisfy the same inequality

\[ h_j(S, \xi_i) > H(\xi_i) - \frac{1}{n}. \]

We consider

\[ W = \bigcap_n \bigcup_q U(n, q), \]

which is a dense \( G_\delta \)-set. If \( S \in W \), then for any \( n \) there is \( q(n) \) such that the inequality

\[ h_{j(n, q(n))}(S, \xi_i) > H(\xi_i) - \frac{1}{n} \]

holds for all \( i \leq n \). This implies \( h_P(S) = \infty \). Thus, the family \( \{ S : h_P(S) = \infty \} \) contains the comeager set \( W \).

And what about the group actions? The above proof is valid for the following generalization.
Theorem 3.1.1. For a given countable infinite group $G$, let $Q$ be a sequence of the collections $Q_j = \{q_j(1), q_j(2), \ldots, q_j(L(j))\} \subset G$, $L(j) \to \infty$, such that for any finite $F \subset G$ for all large $j$ and all $m, n, m \neq n \leq L(j)$ the product $q_j(m)^{-1}q_j(n)$ is outside $F$.

Then the set of actions $\{T : h_Q(T) = \infty\}$ is comeager in the space of all $G$-actions.

3.2. Compact sets without of generic entropy properties. Recall that $K^{Aut} = \{J^{-1}SJ : S \in K, J \in Aut\}$ (the orbit of $K$). The class of zero entropy transformations is denoted by $E_0$.

Theorem 3.2. If $K \subset E_0$ is a compact set in $(Aut, \rho)$, then $K^{Aut}$ is meager.

Remark. To prove that the set $K^{Aut}$ is meager we need to place it in the comeager set $E_0$. Why an inessential, meager set $Aut \setminus E_0$ significantly interferes with the proof that $C^{Aut}$ is meager for an arbitrary compact set $C$?

Proof. We fix a dense collection of partitions $\xi_i$. If $h(S) = 0$, then for any $i$

$$h(S^j, \xi) = \lim_{L \to \infty} \frac{1}{L} H \left( \bigvee_{p=1}^{L} S^j \xi_i \right) = 0.$$ 

Given $S \in K$ and $j$, we find a progression sequence $P(S) = \{P_j(S)\}$, $P_j(S) = \{j, 2j, \ldots, LS(j)j\}$ such that

$$\frac{1}{|P_j(S)|} H \left( \bigvee_{p \in P_j(S)} S^p \xi_i \right) < \frac{1}{j}$$

holds for all $i < j$.

Since $K$ is a compact set, using the progression structure of our sets $P_j(S)$, we can find a sequence $L(j)$ such that $L(j) > LS(j)$ for all $S \in K$. For the corresponding progression sequence $P$ we get $h_P(S) = 0$.

Since the $\{T : h_P(T) = \infty\}$ is invariant by the conjugations, we obtain

$$\{T : h_P(T) = \infty\} \cap K^{Aut} = \phi,$$

thus, $K^{Aut}$ is meager.
**Corollary.** The generic transformation is not an exchange rectangle transformation.

Indeed, the set of all exchanges of $n$ rectangles within a fixed rectangle $X$ is compact. Each of them has zero entropy, what we can explain by use of Jean-Paul’s hint. Let $\xi$ be a rectangle partition, then the sum of the boundary measures of atoms in the partition $\xi^N = \bigvee_{n=0}^{N-1} T^n \xi$ grows linearly. This implies that the entropy of $\xi^N$ grows very slowly and $H(\xi^N)/N \to 0$. It is clear that this method works in a much more general situation. Now we apply Theorem 3.2 and get the above corollary.

**Remark.** J. Chaika pointed us to the result [5] on topological zero entropy of piecewise isometric transformations. In fact we get the non-typicalness of several natural classes of dynamical systems: for example, all explicite classes of rank one transformations, or $\sigma$-compact classes of special flows over rotations. This shows that the ergodists prefer to work with non-typical systems, the exceptions are those cases when they write articles about generic transformations, including this note.

### 3.3. Lower $P$-entropy of the generic transformation.

**Theorem 3.3.** For some progression sequence $P$ the class $\{S : h_{P}^{inf}(S) = 0\}$ contains a dense $G_\delta$ set.

Proof. As in the proof of Theorem 3.1 we consider again the partitions $\xi_i$, the family $\{J_q\}$, the dense set $T_q = J_q^{-1} T J_q$ for some ergodic $T$ of zero entropy. For any $n, q$ there is $j(n, q)$ such that for all $i \leq n$

$$h_{j(n, q)}(T_q, \xi_i) < \frac{1}{n},$$

and all $S$ from some neighbourhood $U(n, q)$ of $T_q$ satisfy the same inequality. We get that $\cap_n \cup_q U(n, q)$ is a dense $G_\delta$-subset of the class $\{S : h_{P}^{inf}(S) = 0\}$. Q.E.D.

J.-P. Thouvenot drew our attention to the preprint [1], emphasizing some analogy of the results on entropy typicalness that were independently obtained by T. Adams and the author.
4 Closing remarks

Of course, many questions arise in connection with topological analogs and the actions of various groups, but we will pay attention to the following problems.

4.1. The space of mixing actions. Bashtanov’s result [3] together with the well-known facts shows that the generic mixing transformation has no factor and commutes only with its powers. So for the space $Mix$ there are no analogues of the mentioned sophisticated results by Ageev, Glasner-Weiss, King, de la Rue-de Sam Lazaro, and Eremenko-Stepin.

Recall that $Mix$ for $\mathbb{Z}$-actions is equipped by Alpern-Tikhonov’s metric $d = \rho + m$, where $m(S, T) = \sup_{k \in \mathbb{Z}} w(S^k, T^k)$ for a fixed metric $w$ which defines the weak operator topology. The space $Mix$ is complete. Indeed, the Halmos metric $\rho$ is so, then from the conditions $\rho(T_i, T) \to 0$, $m(T_i, T_j) \to 0$ as $i, j \to \infty$, and $T_i \in Mix$ it follows that $T \in Mix$ and $m(T_i, T) \to 0$, thus, $d(T_i, T) \to 0$.

Theorem 4.1. By the same reason the statements of Theorems 3.1, 3.2, 3.3 are valid for the space $Mix$.

4.2. Asymmetry in Mix. Let $T \in E_0$ be multiple mixing. May it possesses the following asymmetry property? For some sequences $N_k, m_k, n_k \to \infty$ and any generating partition $\xi$ one has

$$\frac{H(\xi^{N_k} \vee T^{m_k} \xi^{N_k} \vee T^{n_k} \xi^{N_k})}{H(\xi^{N_k})} \to a,$$

and for $b \neq a$

$$\frac{H(\xi^{N_k} \vee T^{-m_k} \xi^{N_k} \vee T^{-n_k} \xi^{N_k})}{H(\xi^{N_k})} \to b,$$

where $\xi^N$ denotes the partition $\bigvee_{n=0}^{N-1} T^n \xi$. This property (or another one of a similar nature) seems to be generic in $Mix$. It implies that $T$ is not isomorphic to $T^{-1}$.

4.3. $\{2^n\}$-entropy. Classical Kushnirenko’s $\{2^n\}$-entropy of the horocycle flow is finite [13], but it is infinite for the typical transformation. This fact contrasts with the usual impression that the generic measure-preserving transformation has very weak mixing properties, while the horocyclic flow has Lebesgue spectrum and possesses multiple mixing property.
Let us consider the sequence $A = \{A_j\}$, where $A_j = \{2^j, 2^{j+1}, \ldots, 2^{j^2}\}$.

**Theorem 4.2.** The family $\{S : h_A(S) = \infty\}$ contains a dense $G_δ$ set.

**Corollary.** Kushnirenko’s $\{2^n\}$-entropy of the generic transformation is infinite.

In the proof of Theorem 3.1 we replace $P_j$ by $A_j$. Q.E.D.

**4.4. Zero $P$-entropy factor.** A number of questions arise in connection with our attempt to understand the entropy properties of a generic transformation. Does the generic $T$ have completely positive $P$-entropy?

In connection with the results of Ageev and Glasner-Weiss on factors of the generic transformation, it would be more interesting to have in the typical situation ($h_P(T) = \infty$) the existence of a nontrivial factor with zero $h_P$-entropy.

**4.5. Attractive transformations.** Taking a rare opportunity, the author draws the readers’ attention to an unusual problem in the theory of typical transformations. The question is inspired by the study of analogs of Gordin’s homoclinic group.

Let’s call a transformation $T$ attractive, if for some transformation $S \neq T$ and some sequence $m_k$ the condition

$$T^{-m_k}ST^{m_k} \to T$$

is satisfied.

Is the attractiveness generic?

In conclusion, we only recall that the Bernoulli actions are very attractive. Therefore, in the key of our note, it would be natural to ask, does the typical transformation inherit this property?

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