Evaluation of ground state entanglement in spin systems with the random phase approximation

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(Dated: September 22, 2010)

We discuss a general treatment based on the mean field plus random phase approximation (RPA) for the evaluation of subsystem entropies and negativities in ground states of spin systems. The approach leads to a tractable general method, becoming straightforward in translationally invariant arrays. The method is examined in arrays of arbitrary spin with $XYZ$ couplings of general range in a uniform transverse field, where the RPA around both the normal and parity breaking mean field state, together with parity restoration effects, are discussed in detail. In the case of a uniformly connected $XYZ$ array of arbitrary size, the method is shown to provide simple analytic expressions for the entanglement entropy of any global bipartition, as well as for the negativity between any two subsystems, which become exact for large spin. The limit case of a spin $s$ pair is also discussed.

PACS numbers: 03.67.Mn, 03.65.Ud, 75.10.Jm

I. INTRODUCTION

The study of entanglement constitutes one of the most active and challenging research areas, being of central interest in the fields of quantum information [1] and many-body physics [2]. The concept of entanglement has provided a new perspective for analyzing quantum correlations and quantum critical phenomena in many particle systems, leading to fundamental results and new insights in the field [2,3]. Nonetheless, the evaluation of entanglement in general strongly interacting many-body systems remains a difficult task, particularly in systems with long range interactions, high connectivity and large dimensionality, where usual treatments such as Quantum MonteCarlo [6], DMRG [7] or matrix product states [8] become more involved or difficult to implement. In previous works [6,10] we have applied a general mean field plus RPA treatment to the evaluation of pairwise entanglement (i.e., that between two elementary components) in spin systems at zero and finite temperature. The approach was able to capture the main features of the entanglement between two spins in arrays with $XY$ and $XYZ$ couplings of different ranges, including the prediction of full range pairwise entanglement in the vicinity of the factorizing field [10,12]. The accuracy of the approach was shown to increase with the interaction range or connectivity.

The aim of the present work is to examine the capability of the previous method for predicting, in the ground state of spin systems, the entanglement properties of general subsystems. We will focus on the entanglement entropy of arbitrary bipartitions of the whole system, as well as on the negativity between any two subsystems, not necessarily complementary, where the rest of the spins play the role of an environment and entanglement can no longer be measured through the subsystem entropy. Other measures, like the negativity (an entanglement monotone computable for general mixed states [13,14]) have to be employed. This type of entanglement has recently received special attention [15,17] since its behavior can differ from that of global bipartitions. We will show that the present approximation provides a general tractable scheme for evaluating these quantities, becoming analytic in translationally invariant systems.

In Section II we present the general RPA formalism, describing the RPA spin state, the associated bosonic estimation of subsystem entropies and negativities, the implementation in translationally invariant systems and the application to a general spin $s$ array with $XYZ$ couplings of arbitrary range in a transverse magnetic field. Symmetry restoration effects in the case of parity-breaking mean fields are also discussed. As illustration, we derive in sec. III results for a spin $s$ pair and for a fully connected finite spin $s$ array, where RPA is able to provide simple full analytic expressions for subsystem entropies and negativities, which represent the exact large spin limit at any fixed size. Appendix A discusses the equivalence between the spin and the bosonic RPA treatments, whereas appendix B contains details of the analytic results of sec. III. Conclusions are drawn in IV.

II. FORMALISM

A. RPA for spin systems at $T = 0$

We will consider a general finite system of spins $s_i = (s_{ix}, s_{iy}, s_{iz})$, connected through general quadratic couplings and immersed in a magnetic field, not necessarily uniform. The corresponding Hamiltonian is

$$H = \sum_{i,\mu} B^{\mu s_i} - \frac{1}{2} \sum_{i\neq j,\mu,\nu} J^{\mu s_i s_j s_{j'}},$$

where $\mu = x, y, z$ and $B^{\mu s_i}$ are the field components at site $i$. Ising, $XY$, $XYZ$ ($J^{\mu s_i s_j} = \delta^{\mu s} J_{s_i s_j}$) as well as Dzyaloshinskii-Moriya ($J^{\mu s_i s_j} = -J^{\mu s_i s_{j'}}$) couplings of arbitrary range are particular cases of Eq. (1).

The first step in the RPA [18] is to determine the mean
field ground state, i.e., the separable state

$$|0\rangle \equiv \otimes_{i=1}^n |i_0\rangle = |0_1 \ldots 0_n\rangle$$

with the lowest energy $$\langle H \rangle_0 = \langle 0 | H | 0 \rangle$$, given by

$$\langle H \rangle_0 = \sum_{i,\mu} B_{\mu i} \langle s_{\mu i}| - \frac{1}{2} \sum_{i,j,\mu,\nu} J_{\mu i,\nu j} \langle s_{\mu i}| s_{\nu j} \rangle_0$$

(2)

where $$\langle s_{\mu i}|_0 = \langle 0| s_{\mu i}|0\rangle$$. Each local state $$|i\rangle$$ can be determined self-consistently as the lowest eigenstate of the local mean field Hamiltonian

$$h_i = \sum_{\mu} \frac{\partial \langle H \rangle_0}{\partial (s_{\mu i})} s_{\mu i} = \lambda^i \cdot s_i ,$$

(3)

being then the state with maximum spin $$s_i$$ directed along $$-\lambda^i$$ (a local coherent state). This leads to the self-consistent equations

$$\lambda^i = B_{\mu i} - \sum_{j \neq i, \nu} J_{\mu i, \nu j} \langle s_{\mu i}| s_{\nu j} \rangle_0 , \quad \langle s_{\mu i}| = -s_i \delta_{\mu 0}$$

(4)

where $$\lambda^i = [\lambda^i]$$, Eq. [4], can be solved iteratively starting from an initial guess for $$|0\rangle$$ or $$\lambda^i$$, although other procedures (like the gradient method) can be employed. Eq. [2] becomes then $$\langle H \rangle_0 = \frac{1}{2} \sum_i (\lambda^i + B^i) \cdot \langle s_{\mu i}|_0$$.

Since the form [11] is valid for any choice of the local axes, it is now convenient to choose $$z_i$$ along $$\lambda^i$$, such that $$\langle s_{\mu i}|_0 = -s_i \delta_{\mu 0}$$ and $$\lambda^\mu = \lambda^i \delta^{\mu i}$$, with $$\lambda^0 > 0$$. The second step in the RPA is the approximate bosonization

$$s_{i+} \to \sqrt{2s_i} b_{i+}^\dagger , \quad s_{i-} \to -\sqrt{2s_i} b_{i-} , \quad s_{iz} \to -s_{i} + b_{i+}^\dagger b_{i-}^\dagger ,$$

(5)

where $$s_{i\pm} = s_{iz} \pm is_{iy}$$ and $$b_{i+}^\dagger$$, $$b_{i-}$$ are considered standard boson operators ($$[b_{i+}^\dagger, b_{i-}] = \delta_{ij}$$, $$[b_{i+}^\dagger, b_{j+}] = [b_{i-}, b_{j-}] = 0$$), with $$|0\rangle \to |0_b\rangle$$ their vacuum. This bosonization is in agreement with that implied by the path integral formalism of [3, 10] for $$T \to 0$$, and preserves two of the exact spin commutators exactly ($$\{s_{iz}, s_{iz}^\dagger\} = \pm b_{i+}^\dagger b_{i+}$$), the remaining one preserved as vacuum average ($$\langle [s_{iz}, s_{iz}^\dagger]\rangle_0 = 2s_i \delta_{ij}$$). It coincides with the Holstein-Primakoff and other exact bosonizations [18, 21] up to zeroth order in $$s_i^{-1}$$.

The third step is to replace Eq. [4] in the original Hamiltonian [11], neglecting all cubic and quartic terms in $$b_{i+}^\dagger$$. This leads to the quadratic boson Hamiltonian

$$H^b = \langle H \rangle_0 + \sum_i \lambda^i b_{i+}^\dagger b_{i-} - \sum_{i \neq j} \Delta_{ij} b_{i+}^\dagger b_{j-} + \frac{1}{2}(\Delta_{ij} b_{i+}^\dagger b_{j+}^\dagger + h.c.)$$

(6)

$$= \langle 0 | H | 0 \rangle - \frac{1}{2} \sum_i \lambda^i + \frac{1}{2} Z^\dagger H Z ,$$

$$Z = \begin{pmatrix} b_{i+}^\dagger \\ b_{i-}^\dagger \end{pmatrix} , \quad H = \begin{pmatrix} \Lambda - \Delta + & -\Delta - \\
-\Delta + & \Lambda + \Delta \end{pmatrix} , \quad \Delta_{ij} = \frac{1}{2} \sqrt{s_{i}s_{j}} \left[ J^{ixj} + J^{jyix} - i(J^{jyix} + J^{ixjy}) \right] ,$$

(7)

(8)

where $$Z^\dagger = (b_{i+}^\dagger, b_{i-}^\dagger)$$ and $$\Lambda_{ij} = \lambda^i \delta^{ij}$$. The choice of the mean field axes for the bosonization [5] ensures that no linear terms in $$b_{i+}^\dagger$$ appear in $$H^b$$, reflecting the stability of the mean field state $$|0\rangle$$ with respect to one site excitations.

The last step is the diagonalization of the bosonic quadratic form [6], which is always possible when the hermitian matrix $$H$$ in (7) is positive definite, i.e., when $$|0\rangle$$ is a stable vacuum [18]. $$H^b$$ can then be rewritten as

$$H^b = \langle H \rangle_0 + \sum_{\alpha} \omega^\alpha b_{i+}^\dagger b_{i-}^\dagger + \frac{1}{2}(\omega^\alpha - \lambda^\alpha) ,$$

(9)

where $$\lambda^\alpha$$ stands for $$\lambda^i$$, $$\omega^\alpha$$ are the symplectic eigenvalues of $$H$$, i.e., the positive eigenvalues of the matrix

$$M^\alpha = \begin{pmatrix} \Lambda - \Delta + & -\Delta - \\
-\Delta + & \Lambda + \Delta \end{pmatrix} , \quad M = \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}$$

(10)

whose eigenvalues come in pairs of opposite sign (and which is diagonalizable with real non-zero eigenvalues when $$H$$ is positive definite), and $$b_{i+}^\dagger$$, $$b_{i-}^\dagger$$ are “collective” boson operators related to the local ones by a Bogoliubov transformation $$Z = WZ'$$, i.e.,

$$\begin{pmatrix} b_{i+}^\dagger \\ b_{i-}^\dagger \end{pmatrix} = W_{\alpha} \begin{pmatrix} b'_{i+}^\dagger \\ b'_{i-}^\dagger \end{pmatrix} , \quad W = \begin{pmatrix} U & V \\
V & U \end{pmatrix}$$

(11)

with $$(\psi'_\alpha)$$ and $$(\phi'_\alpha)$$ the eigenvectors of $$M^\alpha$$ associated with the eigenvalues $$\omega^\alpha$$ and $$-\omega^\alpha$$ respectively (such that $$W^{-1}M^\alpha W = M$$, with $$\Omega_{\alpha\alpha'} = |\omega^\alpha|\delta_{\alpha\alpha'}$$). In order to preserve the boson commutation relations, which can be cast as $$ZZ^\dagger - [(Z^\dagger)^{tr}Z^{tr}]^{tr} = M$$, $$W$$ should satisfy

$$WGW^\dagger = M$$

(12)

which implies also $$W^\dagger MW = M$$ and hence $$W^\dagger HW = \Omega$$. This entails $$U^\dagger V - V^{tr}U = 0$$, $$U^\dagger U - V^{tr}V = I$$, which are the natural orthogonality relations fulfilled by the eigenvectors of (10) with normalization $$(\psi')_\alpha^\dagger M(\psi')_\alpha = 1$$.

The RPA matrix (7) is of dimension $$2n \times 2n$$, with $$n$$ the number of spins. RPA involves then an exponential reduction in the dimension (from $$(2s + 1)^n$$ to $$2n$$ or $$n$$ identical spins). Moreover, in a translationally invariant system (sec. 14), it can be further reduced to $$n \times n$$ matrices, becoming then fully analytic.

B. The RPA ground state

The vacuum of the new bosons $$b'$$ ($$b'_0|0'_b\rangle = 0$$) is [18]

$$|0'_b\rangle = C_b |\prod_{i,j} \sum Z^{ij} b_{i+}^\dagger b_{j-}^\dagger|0_b\rangle , \quad Z = V U^{-1} ,$$

(13)

where $$C_b = \langle 0|0'_b\rangle = \text{Det}(U)^{-1/2}$$ is a normalization factor and $$Z$$ a symmetric matrix. The associated RPA spin state can then be defined as

$$|0_{\text{RPA}}\rangle = C_s |\prod_{i,j} \frac{1}{2} \sum Z^{ij} s_{i+} s_{j+}|0\rangle .$$

(14)
The expectation values generated by \([14]\) will be close to those obtained with the mapping \([15]\), coinciding exactly up to second order in \(V\) (Appendix A). In contrast with \(|0\rangle\), the state \(|14\rangle\) is entangled (unless \(V \neq 0\)).

Let us note that for the quadratic Hamiltonian \(|16\):

i) \(|0_{\text{RPA}}\rangle = |0\rangle\) if and only if \(|0\rangle\) is an exact eigenstate of \(H\), since \(H^b\) contains the exact matrix elements connecting \(|0\rangle\) with the rest of the Hilbert space:

\[
H|0\rangle = \langle H|0\rangle - \frac{1}{2} \sum_{i,j} \Delta_{ij}|1_i1_j\rangle,
\]

where \(|1_i1_j\rangle = \frac{\delta_{ij} + \rho_{ij}}{\sqrt{\lambda_i + \lambda_j}}|0\rangle\) and we have used the mean field condition \(|1_i|H|0\rangle = \langle 1_i|h_i|0\rangle|0\rangle = 0\) (Eqs. \([19]-[21]\)). Thus, if \(|0_{\text{RPA}}\rangle = |0\rangle\), \(Z = 0\) and hence \(V = 0\) in \(W\), implying \(\Delta = 0\). \(|0\rangle\) is then an exact eigenstate by Eq. \([15]\). Conversely, if \(|0\rangle\) is an exact eigenstate, it is a solution of the mean field equations leading to \(\Delta = 0\), implying \(|0_{\text{RPA}}\rangle = |0\rangle\) (although \(\Delta = \mu \neq \lambda^0\)). In particular, when \(H\) has an exactly separable ground state \(|0\rangle\) (e.g., at the factorizing field \([12]\)), \(|0_{\text{RPA}}\rangle = |0\rangle\).

ii) \(|0_{\text{RPA}}\rangle\) is always exact for sufficiently strong fields \(|B| \gg J\). In this limit \(|0\rangle\) is the state with all spins \(s_i\) fully aligned along \(-B^i\) plus small corrections \((\lambda^i \approx B^i + s_iJ^B|B^i|)\). Up to first order in \(\Delta^+ / \lambda\), Eqs. \([10]-[13]\) lead then to \(Z^ij \approx V_{ij} \approx \frac{\Delta_{ij}}{\lambda_i + \lambda_j}|1_i1_j\rangle\), entailing

\[
|0_{\text{RPA}}\rangle \approx |0\rangle + \sum_{i<j} \frac{\Delta_{ij}}{\lambda_i + \lambda_j}|1_i1_j\rangle,
\]

which, by Eq. \([15]\), is just the first order expansion (in \(\Delta / \Lambda\)) of the exact ground state.

In the case of a symmetry-breaking mean field, the RPA spin state allows to implement the necessary rotations for symmetry restoration: The exact ground state will actually be close to the superposition with the correct symmetry of the degenerate RPA ground states (rather than to a particular RPA state), as will be discussed in Sec. \([16]\) in the context of parity-breaking. This restoration enlarges considerably the capabilities of the RPA.

C. Bosonic evaluation of subsystem entropy and negativity

The direct evaluation of many-body correlations and entanglement measures from the RPA spin state \([14]\) is in general difficult. However, the values of these quantities in the associated bosonic vacuum \([13]\), which will be close to those obtained from \([14]\), can be straightforwardly evaluated using the general gaussian state formalism \([23]-[26]\). The reduced density matrix of any subsystem is just a gaussian state, i.e., a canonical thermal state of an effective quadratic bosonic Hamiltonian, since Wick’s theorem holds for the evaluation of the mean value of any observable, and in particular those of the subsystem. We may then evaluate its entropy and other invariants through standard expressions for independent boson systems.

Let us formalize the previous scheme. We will use a generalized contraction matrix formalism, equivalent to that based on covariance matrices \([25]-[26]\), which is more natural for the present RPA formulation. In the new vacuum \(|0_B\rangle\), \((b^\dagger_A b^\dagger_A)\omega = (b^\dagger_A b^\dagger_A)\omega = 0\), implying

\[
F_{ij} = \langle b^\dagger_1 b_0 \rangle = \langle V^V \rangle_{ij},
\]

\[
G_{ij} = \langle b_0 b_0 \rangle = \langle V^{V_{ij}} \rangle.
\]

Eqs. \([23]-[27]\) determine the basic RPA spin averages and correlations, i.e., \(\langle s_{i\alpha} s_{j\beta} \rangle = \delta_{\alpha\beta}(F_{ij} - s_i)\) and, for \(i \neq j\),

\[
\langle s_{i+} s_{j-} \rangle = 2s_i s_j F_{ji}, \langle s_{i-} s_{j-} \rangle = 2s_i s_j G_{ji},
\]

with \(\langle s_{i+} s_{j+} \rangle = 0\), which coincide exactly with the averages derived from \([14]\) up to second order in \(V\), i.e., first order in the average occupation \(V^V\) (normally very small outside critical regions). Through the use of Wick’s theorem, we also obtain \(\langle s_{i\alpha} s_{j\alpha} \rangle = \langle s_{i\alpha} \rangle (s_{j\alpha} + |F_{ij}|^2 + |G_{ij}|^2)\) for \(i \neq j\).

We may now define the generalized contraction matrix

\[
D \equiv \begin{pmatrix} Z & W \end{pmatrix} = \begin{pmatrix} F & G \\ G & I + F \end{pmatrix},
\]

which exhibits the correct transformation rule under Bogoliubov transformations: If \(Z = WZ'\), then

\[
D = WD'W^\dagger.
\]

with \(D' = (Z' Z'^\dagger)\omega - M\). Eq. \([17]\) can in fact be written in the form \([20]\) if \(W\) is the diagonalizing Bogoliubov matrix \([11]\) and \(D'\) the vacuum density \((F' = G' = 0)\). We may then also obtain \(W\) and \(D'\) through the symplectic diagonalization of \(D\), i.e., through the diagonalization of

\[
D M = \begin{pmatrix} F & -G \\ G & I - F \end{pmatrix},
\]

such that \(W^{-1} D M W = D' M\), with \(D'\) diagonal.

Let us consider now a subsystem \(A\) of \(m < n\) sites. It will be characterized by a truncated contraction matrix

\[
D_A = \begin{pmatrix} Z_A & W_A \end{pmatrix} = \begin{pmatrix} F_A & G_A \\ G_A & I + F_A \end{pmatrix},
\]

where \(Z_A\) contains just the bosons of sites in \(A\). A symplectic diagonalization of \(D_A\) will lead to

\[
D_A = W_A D_A' W_A^\dagger, \quad D_A' = \begin{pmatrix} f_A & 0 \\ 0 & I + f_A \end{pmatrix},
\]

where \(f_A^\dagger f_A' = f_A^\dagger f_A' = 0\) with \(f_A^\dagger f_A' \geq 0\) \((D_A M_A\) has eigenvalues \(f_A^\dagger f_A' = -1 - f_A^\dagger f_A'\) and \(W_A M_A W_A^\dagger = M_A\), with \(Z_A = W_A Z_A'\). The entanglement between \(A\) and its complement \(\bar{A}\) is then given by the associated bosonic entropy,

\[
S_{\bar{A}}(\omega_A) = -\text{Tr} \rho_A^b \log \rho_A^b
\]

\[
= -\sum_{\alpha} f_A^\dagger f_A' \log f_A^\dagger f_A' - (1 + f_A^\dagger f_A') \log (1 + f_A^\dagger f_A')(25)
\]
Here $\rho_A^\dagger = \text{Tr}_A |0\rangle_1 \langle 0|_1$ is the bosonic reduced density of subsystem $A$, which can be explicitly written as

$$\rho_A^\dagger = C \exp[-\frac{1}{2} Z_A^2] \mathcal{H}_A Z_A] = C \exp[-\sum_\alpha \omega_\alpha A b_\alpha A^\dagger b_\alpha A]$$

where $C = \prod_\alpha (1 + f_\alpha)$ and $\mathcal{H}_A$, $\mathcal{D}_A$ are related by

$$\mathcal{D}_A M_A = [\exp(M_A \mathcal{H}_A) - I]^{-1}. \quad (27)$$

Here $\mathcal{H}_A$ represents an effective “Hamiltonian” matrix for subsystem $A$ with symplectic eigenvalues $\omega_\alpha A$ such that $f_\alpha A = (e^{-\omega_\alpha A} - 1)^{-1}$ (and hence $1 - f_\alpha A = (e^{-\omega_\alpha A} - 1)^{-1}$). Eq. (26) leads then to the contraction matrix (22), and hence to the same expectation values as the full vacuum $|0\rangle_1 \langle 0|_1$ for any operator of subsystem $A$.

Eq. (26) provides a tractable RPA estimation of the entanglement entropy of any subsystem. It is shown in the Appendix A that a direct spin evaluation of the subsystem entropy based on the RPA state (14) coincides with (26) up to second order in $V$.

On the other hand, the internal entanglement of subsystem $A$ with respect to a partition $(B, C)$ of $A$ (where the complement $A$ plays the role of an environment) can be measured through the corresponding negativity (13), defined as minus the sum of the negative eigenvalues of the partial transpose $\rho_A^{BC}$ of $\rho_A$:

$$N_{BC} = \frac{1}{2} \text{Tr} |\rho_A^{BC}| - 1. \quad (28)$$

Expectation values with respect to $(\rho_A^{BC})^{BC}$ of an observable $O_A^{BC}$ correspond to those of the partial transpose $(O_A^{BC})^{BC}$ with respect to $\rho_A$. This implies the replacements $F_{ij} \leftrightarrow G_{ij}$, $F_{j'i'} \leftrightarrow F_{ij}$, $G_{j'i'} \leftrightarrow G_{ij}$, in the contraction matrix for $j, j' \in C$, $i \in B$, leading to a matrix $\mathcal{D}_A$ with symplectic eigenvalues $f_\alpha A$. The latter can now be negative. We may then still write $(\rho_A^{BC})^{BC}$ as in Eq. (26) in terms of an effective matrix $\mathcal{H}_A$ with symplectic eigenvalues $\omega_\alpha A$ such that $f_\alpha A = (e^{-\omega_\alpha A} - 1)^{-1}$.

Since the trace remains unchanged $\text{Tr} (\rho_A^{BC})^{BC} = 1$, $|e^{-\omega_\alpha A}| < 1$, implying $f_\alpha A > -1/2$. A negative $f_\alpha A > -1/2$ corresponds to $e^{-\omega_\alpha A} < 0$ and hence to a non-positive $(\rho_A^{BC})^{BC}$, indicating an entangled $\rho_A^{BC}$ with respect to this bipartition. We then obtain, noting that $(1 + e^{-\omega_\alpha A})^{-1} = (1 + f_\alpha A^2)/(1 + 2 f_\alpha A)$, the final result

$$\text{Tr} |(\rho_A^{BC})^{BC}| = \prod_{f_\alpha A < 0} \frac{1}{1 + 2 f_\alpha A}, \quad (29)$$

which allows the evaluation of the negativity (28). Negativities obtained from the spin density matrices coincide with this result up to first order in $V$ (Appendix A).

In the case of a global bipartition $(A, \bar{A})$, $N_{A\bar{A}}$ becomes a function of the reduced density $\rho_A$, namely

$$N_{A\bar{A}} = \frac{1}{2} \text{Tr} |\rho_A| (0|_1 \langle 0|_1 + 1) - 1 = \frac{1}{2} (|\text{Tr} \sqrt{\rho_A}|^2 - 1). \quad (30)$$

In a boson system, this implies that $N_{A\bar{A}}$, a limit case of Eqs. (25)--(29), can be also expressed just in terms of the symplectic eigenvalues $f_\alpha A^2$ of the contraction matrix $\mathcal{D}_A$:

$$N_{A\bar{A}} = \frac{1}{2} \prod_{\alpha} \left( (\sqrt{f_\alpha A} + \sqrt{1 + f_\alpha A^2})^2 - 1 \right). \quad (31)$$

D. Translationally invariant systems

The only quantities required in the bosonic RPA scheme are, therefore, the basic contractions (17). Their evaluation becomes remarkably simple in translationally invariant systems, either in one or $d$ dimensions, i.e., systems with a common spin $s_i = s$ in a uniform field $B^i = B$ with couplings dependent just on separation:

$$J^{\mu \nu} = J^{\mu \nu}(i - j), \quad (32)$$

where $J^{\mu \nu}(l) = J^{\mu \nu}(-l)$, and $J^{\mu \nu}(-l) = J^{\mu \nu}(n - l)$ in a finite cyclic chain or system (in $d$ dimensions, $i, j, l, n$ stand for $d$-dimensional vectors). We will also assume a uniform mean field $\lambda = \lambda$, which should then satisfy

$$\lambda^\mu = B^\mu - \sum_\nu J^\mu_0 (s_\nu)_0, \quad J^\mu_0 \equiv \sum_l J^{\mu \nu}(l), \quad (33)$$

with $\langle s \rangle_0 = -s\lambda/\lambda$ (Eq. (4)). The uniform mean field is thus determined just by the total strengths $J^\mu_0$.

Choosing again the $z$ axis in the direction of $\lambda$, such that $\langle s^{\mu \nu} \rangle = -s \delta_{\mu z}$ and $B^\mu + s J^{\mu \nu}_0 = \lambda \delta_{\mu z}$, with $\lambda > 0$, the bosonized Hamiltonian will have the form (10) with couplings $\Delta_j^\nu = \Delta_j (i - j)$. By means of a discrete Fourier transform of the boson operators, we can rewrite it as

$$H_B = \langle H \rangle_0 + \sum_k (\lambda - \Delta_k^b) b_k^\dagger b_k - \frac{1}{2} (\Delta_k^b b_k^\dagger b_{-k}^\dagger + h.c), \quad (34)$$

$$\Delta_k^b = \frac{\sum_{k=1}^{n-1} e^{2\pi ik/n} \Delta_k(l)}{l=0} \quad (35)$$

where $k = 0, \ldots, n - 1$ and $b_k = \sqrt{\lambda} \sum_{j=1}^n e^{2\pi ij/n} b_j$ are boson operators in momentum space, with $b_{-k} = b_{n-k}$. Diagonalization of (34) is straightforward and leads to

$$H^b = \langle H \rangle_0 + \sum_k \omega_k^b b_k^\dagger b_k + \frac{1}{2} (\omega_k - \lambda + \Delta_k^b), \quad (36)$$

where $\omega_k = \omega_k^b + \Delta_k^b = \Delta_k^b - \Delta_k^b$ is the renormalized frequency $b_k^\dagger b_k^\dagger b_{-k}^\dagger + h.c$ and $\Delta_k^b = \sqrt{(\lambda - \Delta_k^b)^2 - |\omega_k|^2}$, $\Delta_k^b = \Delta_k^b + \Delta_k^b$.

$$u_k = \frac{\lambda - \Delta_k^b + \omega_k}{2 \Delta_k^b}, \quad v_k = \Delta_k^b \frac{\lambda - \Delta_k^b - \omega_k}{2 \Delta_k^b} (37)$$

with $\Delta_k^b = \frac{1}{2} (\Delta_k^b + \Delta_k^b)$, $|u_k|^2 - |v_k|^2 = 1$, and $u_k = u_{-k}$, $v_k = v_{-k}$. All $\omega_k$ should be real and positive for a stable mean field, implying the stability conditions

$$0 \leq |\Delta_k^b| < \lambda - \Delta_k^b, \quad k = 0, \ldots, n - 1. \quad (39)$$
We can now obtain the basic contractions explicitly,
\[ \langle b_k^\dagger b_{k'} \rangle_{\psi'} = \delta_{kk'} |v_k^2|, \quad \langle b_k b_{-k'} \rangle_{\psi'} = \delta_{kk'} u_k v_k = \frac{\Delta_k}{2\alpha_k^2} \]
which lead finally to (Eq. \ref{eq:17})
\[ F_{ij} = F(i-j) = \frac{1}{n} \sum_k e^{-i2\pi k(i-j)/n} |v_k^2|, \quad (41a) \]
\[ G_{ij} = G(i-j) = \frac{1}{n} \sum_k e^{-i2\pi k(i-j)/n} u_k v_k. \quad (41b) \]
For strong fields \(|B|\) such that \(\lambda > |\Delta_\pm|\), \(u_k v_k \approx \frac{1}{2}\Delta_k^2/\lambda\) and \(|v_k^2| \approx \frac{1}{4}|\Delta_k|^2/\lambda^2\). The RPA vacuum \[ \langle 0 \rangle = C_b \exp \left[ \sum_{i,j} Z(i-j) b_i^\dagger b_j \right] \langle 0 \rangle, \tag{42} \]
where \(C_b = \prod_k u_k^{-1/2}\) and \(Z(l) = \frac{1}{\lambda} \sum_k e^{-i2\pi k/n} v_k^2\).

Thus, these systems allow an analytic evaluation of the contractions \[ \langle \psi \rangle. \] Both the mean field equations \[ \text{and the RPA Hamiltonian (41)} \] become independent of the common spin \(s\) after a rescaling \(J^{\mu\nu}(l) \rightarrow J^{\mu\nu}(l)/s\), which we will adopt in what follows and which indicates that RPA is describing the large spin limit of the system, as is apparent from Eq. \[ \text{(46).} \]

E. XYZ systems

Let us now examine in more detail the previous formalism in a translationally invariant spin \(s\) array with XYZ couplings of arbitrary range in a uniform transverse field:
\[ H = B \sum_i s_{iz} - \frac{1}{2} \sum_{i\neq j \mu=x,y,z} J_{\mu}(i-j)s_{i\mu}s_{j\mu}. \tag{43} \]
Eq. \[ \text{(43)} \] commutes with the \(S_z\) spin parity,
\[ [H, P_z] = 0, \quad P_z = \exp[i\pi \sum_i (s_{iz} - s)], \]
for any value of its parameters, such that the exact ground state in a finite array will always have a definite parity outside degeneracy points. We will focus here on the ferromagnetic type case where \(J_x(l) \geq 0 \forall l\) with
\[ |J_y(l)| \leq J_x(l), \tag{44} \]
which exhibits a normal and parity breaking phase at the mean field level.

1. RPA around the normal state

For the Hamiltonian \[ \text{(43),} \] the state \(\langle 0 \rangle\) with all spins fully aligned along the \(-z\) axis is always a solution of the mean field equation \[ \text{(33),} \] being the lowest solution for a sufficiently strong field \(B\). It leads to \(\lambda = \lambda \delta_{xz}\), with
\[ \lambda = |B| + J_0^0 > 0, \quad J^0_z = \sum_i J_z(l). \tag{45} \]
All previous equations can then be directly applied. Now \(\Delta_\pm(l) = \frac{J_\mu(l) + J_\nu(l)}{2} = \Delta_\pm(-l), \) implying \(\Delta_\pm = \Delta_\pm^k\) and
\[ \omega^k = \sqrt{(\lambda - J^0_x)(\lambda - J^0_y)}, \tag{46} \]
where \(J^k = \sum_\mu e^{i2\pi \mu k/n} J_\mu(l)\). This solution is therefore stable provided \(J_\mu^k \leq \lambda \forall k\) and \(x, y, \) i.e. for \(|B|\) above a certain critical field \(B_c\). In the case \[ \text{(41)}, \] the strongest condition is obtained for \(k = 0\), i.e.,
\[ |B| > B_c \equiv J_0^0 - J^0_z. \tag{47} \]

2. RPA around the parity breaking state

For \(|B| \leq B_c\), the normal state becomes unstable: the lowest normal RPA frequency \(\omega^k\) vanishes for \(|B| \rightarrow B_c\) and becomes imaginary for \(|B| < B_c\). The mean field for \(|B| < B_c\) corresponds instead to a parity-breaking state with all spins aligned along an axis in the \(xz\) plane forming an angle \(\theta\) with the \(z\) axis:
\[ \langle 0 \rangle \rightarrow \langle \theta \rangle \equiv \langle \theta_1 \ldots \theta_n \rangle, \quad |\theta_j\rangle = \exp[-i\delta s_{jy}][0_j]. \tag{48} \]
This leads to \(\langle s_j \rangle_0 = -s(\sin \theta, 0, 0\cos \theta) = -s\lambda/\lambda, \) with
\[ \lambda = J^0_x, \quad \cos \theta = B/B_c, \tag{49} \]
as determined by \[ \text{(33)} \]. We should now express the original spin operators in terms of the rotated operators, i.e.,
\[ s_{ix} = s_{ix'} \cos \theta + s_{iz'} \sin \theta, \quad s_{iz} = s_{iz'} \cos \theta - s_{ix'} \sin \theta \tag{50} \]
with \(s_{iy} = s_{iy'}\). The RPA around this state amounts therefore to the replacements
\[ \lambda \rightarrow J^0_x, \quad J^k_x \rightarrow J^k_x = J^k_x \cos^2 \theta + J^k_y \sin^2 \theta, \tag{51} \]
in Eq. \[ \text{(46)}, \] with \(J^k_y\) unchanged and \(\Delta_\pm = \frac{1}{2}(J^k_x \pm J^k_y). \) Correlations \(\langle s_{i\mu} s_{j\nu} \rangle_{\text{RPA}}\) of rotated spin operators have the same previous expressions \[ \text{(17)}, \] whereas those of the original operators must be obtained using Eqs. \[ \text{(50)}. \] It should be remarked, however, that in a finite system, the associated RPA spin state will no longer be a good approximation to the actual ground state due to parity breaking. Parity restoration, at least approximately, must be implemented before obtaining final results. We will not discuss here the case of a continuous broken symmetry (arising for instance in the \(XXZ\) case), which can be treated through the RPA formalism of ref. \[ \text{(9)}. \]
3. Definite Parity RPA ground states

Since \( [H, P_z] = 0 \), the parity breaking mean field state \(|\Theta\rangle\) is degenerate: Both \(|\Theta\rangle\) and \(|-\Theta\rangle = P_z|\Theta\rangle\) are mean field ground states. In order to describe the definite parity ground states, the correct RPA ground state should be taken as the definite parity combinations

\[
|\Theta_{\text{RPA}}^\pm \rangle = \frac{|\Theta_{\text{RPA}}\rangle \pm -|\Theta_{\text{RPA}}\rangle}{\sqrt{2}(1 \pm -|\Theta_{\text{RPA}}\rangle|\Theta_{\text{RPA}}\rangle)},
\]

(52)

where \(|\pm \Theta_{\text{RPA}}\rangle\) are the RPA states around each mean field. The overlap \(<-\Theta_{\text{RPA}}|\Theta_{\text{RPA}}\rangle = <\Theta_{\text{RPA}}|P_z|\Theta_{\text{RPA}}\rangle\) is proportional to the overlap between the two mean fields,

\[
<-\Theta|\Theta = \cos^{2ns}(B/B_c)^{2ns},
\]

(53)

which is small except for \(B \rightarrow B_c\) or small \(ns\).

Neglecting the previous overlap, Eq. (52) will lead to reduced densities

\[
\rho_{\pm}^A = \frac{1}{2} [\rho_A(\Theta) + \rho_A(-\Theta)]
\]

(54)

provided the complementary overlap \(<-\Theta_{\text{RPA}}^\pm |\Theta_{\text{RPA}}^\mp \rangle \propto (B/B_c)^{2(n-n_s)}\) can also be neglected. Here \(\rho_A(\Theta)\) are the reduced spin densities determined by each RPA state, given up to \(O(V^2)\) by the expressions of Appendix A.

The restoration (53) is essential to achieve a good description of the actual subsystem entropy, although its main effect for a not too small subsystem \(A\) is actually quite simple: If the product \(\rho_A(\Theta)\rho_A(-\Theta) \propto (B/B_c)^{2n_s}\) can be neglected, Eq. (52) can be considered as the sum of two densities with orthogonal support and identical distributions, leading to

\[
S(\rho_A^\pm) = S(\rho_A(\Theta)) + 1,
\]

(55)

where \(S(\rho_A(\Theta))\) can be evaluated through the boson approximation (26). Under the same assumptions, the effect on the global negativity (20) is just

\[
N_{AA}(\rho_A^\pm) = 2N_{AA}(\rho_A(\Theta)) + \frac{1}{2},
\]

(56)

as \(\text{Tr} \rho_A^\pm \approx \sqrt{2} \text{Tr} \rho_A(\Theta)\), while the subsystem negativity \(N_{BC}(\rho_A^\pm)\) is just \(N_{BC}(\rho_A^\pm) \approx N_{BC}(\rho_A(\Theta))\).

When the product \(\rho_A(\Theta)\rho_A(-\Theta)\) cannot be neglected (as in a subsystem of two spins), we should in principle construct the spin density (53). This can be done by rotating \(\rho_A(\Theta)\) (Eq. 12 in the mean field frame) to the original \(z\) axis and removing all parity breaking elements (which is the final effect of Eq. (54)). For instance, the reduced two-spin density for \(s = 1/2\) has the blocked form (12) in the standard basis of \(s_{ix}, s_{iy}\) eigenstates in the normal phase as well as in the parity breaking phase after parity restoration (12). The final effect on \(S(\rho_A)\) is the replacement of the term \(+1\) in (55) by the entropy of the reduced mean field mixture \(-\sum_{q = \pm} q \log_2 q\), with \(q_{\pm} = \frac{1}{2}(1 \pm (B/B_c)^{2n_s})\), plus small RPA corrections.

While \(\rho_A^\pm\) are both identical in the approximation (54), the actual \(\rho_A^\pm\) in a small system will depend on parity. The correct parity in such a case should be chosen as that leading to the lowest energy \(E_{\text{RPA}}^\pm = \langle \Theta_{\text{RPA}}^\pm |H|\Theta_{\text{RPA}}\rangle\).

4. Factorizing Field

The explicit value of the basic RPA couplings \(\Delta_{ij}^k\) in the parity breaking phase are, using Eqs. (51)–(19),

\[
\Delta_{ik}^k = \frac{1}{2} [(J_x^k - J_y^k)(B/B_c)^2 + J_z^k \pm J_y^k]
\]

(57)

In the case of a common anisotropy, such that the ratio

\[
\chi = \frac{J_y(l) - J_x(l)}{J_x(l) - J_z(l)}
\]

(58)

is independent of the separation \(l\), we have \(J_y^k - J_x^k = \chi(J_x^k - J_z^k)\) and hence \(\Delta_{ik}^k = \frac{1}{2}(J_x^k - J_z^k) [(B/B_c)^2 - \chi]\). It is then seen that if \(\chi \in [0,1]\), \(\Delta_{ik}^k = 0 \forall k\) when

\[
|B| = B_s \equiv B_c \sqrt{\chi}
\]

(59)

with all \(\Delta_{ik}^k\) changing sign at \(|B| = B_s\). Here \(B_s\) is the factorizing field (2, 12, 22, 24): At \(B = B_s\) the parity breaking mean field state becomes an exact ground state, since the RPA corrections vanish (sec. 11B). This effect is independent of the number of spins \(n\) (as long as \(\chi\) is constant) and spin \(s\) (with the present scaling). Nonetheless, the actual side limits at \(B = B_s\) will be given by the definite parity states (52), which are still entangled. As a consequence, the subsystem entropy \(S(\rho_A)\) and the negativity \(N_{AA}\) will actually approach a finite value for \(B \rightarrow B_s\) (1 and 1/2 respectively in the approximation (56)–(58)), while the entanglement between two spins will reach there infinite range (11, 12). Note finally that at \(B = B_s\), \(\Delta_{ik}^k = J_y^k\) and hence,

\[
\omega^k = J_x^0 - J_y^k
\]

(60)

III. APPLICATION

A. Spin s pair

As a first example, let us consider a system of two spins \(s\) coupled through the Hamiltonian (45). We can obviously always set here \(J_y \geq |J_y|\) (Eq. (41)), since the sign of \(J_x\) can be changed by a \(\pi\)-rotation around the \(z\) axis of one of the spins (and we can always set \(J_x \geq |J_y|\) by a proper choice of axes). The Fourier transform of \(J_y(l) = \delta_{ij}J_y\) reduces here to \(J_y(l) = (-)^k J_{y_i}\), \(k = 0, 1\), leading to an attractive and a repulsive normal mode:

\[
\omega_0 = \sqrt{(\lambda - J_x)(\lambda - J_y)}, \quad \omega_1 = \sqrt{(\lambda + J_x)(\lambda + J_y)}.
\]

The contractions (41) become \(F_{ij} = \frac{\lambda^2 - \Delta^2}{4\omega_0} - \frac{\lambda^2 + \Delta^2}{4\omega_1}(1 - 2\delta_{ij}) - \frac{1}{4}\delta_{ij}\), \(G_{ij} = \frac{\Delta^2}{4\omega_0} + \frac{\Delta^2}{4\omega_1}(1 - 2\delta_{ij})\), where \(\Delta \rightarrow -\Delta\)
for the average local occupation $f$, approach those of RPA as $s$ increases, differences for not too small $s$ arising just for $B$ close to $B_c$. At the factorizing field $B_s \approx B_c/\sqrt{2}$, $S_E = 1$ while $N_{12} = 1/2$.

\[ \frac{1}{2}(J_x \pm J_y) \] and replacements \[ (61) \] are to be applied for $|B| < B_c$. The ensuing entanglement entropy of the pair in the bosonic approximation \[ (62) \] is just

\[ S(\rho_1) = -f \log_2 f + (1 + f) \log_2 (1 + f) + \delta, \quad (61) \]

\[ f = \frac{1}{2} \left( \sqrt{1 + \frac{\lambda^2 - \omega^2}{\omega_0 \omega_1}} - 1 \right), \quad \omega = \frac{\omega_0 + \omega_1}{2} \quad (62) \]

where $f = \sqrt{(F_{11} + \frac{1}{2})^2 - (G_{11})^2} - \frac{1}{2}$ is the positive symplectic eigenvalue of the $2 \times 2$ contraction matrix for one spin and $\delta = 0$ (1) for $|B| > B_c$ ($< B_c$) in the approximation \[ (65) \] valid for $(B/B_c)^{2s} \ll 1$. For small $f$, we may just use $S(\rho_1) \approx f(\log_2 e - \log_2 f)$, with $f \approx F_{11}$, in agreement with the results of Appendix A.

Thus, at the RPA level entanglement is determined by the average local occupation $f$ and driven by the ratio $\frac{\lambda^2 - \omega^2}{\omega_0 \omega_1}$, which is small away from $B_c$ and vanishes at $B = B_c$ (where $\omega = \lambda = J_0$ by Eq. \[ (60) \], and hence $f = 0$). For $|B| \gg B_c$, $f \approx (J_x-J_y)^2$, while in the vicinity of $B_s$, $f \propto (B - B_s)^2$. For $B \to B_c$, $f \approx \frac{1}{2} \sqrt{\frac{\lambda^2 - \omega^2}{\omega_0 \omega_1}} \propto |B - B_c|^{-1/4}$, with $S(\rho_1) \approx \log_2 f e_c$.

The bosonic RPA negativity \[ (28)-(29) \] becomes

\[ N_{12} = \frac{-\tilde{f}}{1 + 2f} = f + \sqrt{f(f+1)} \quad (63) \]

where $\tilde{f} = f - \sqrt{f(f+1)}$ is the negative symplectic eigenvalue of the $4 \times 4$ contraction matrix. Correction

\[ (64) \] (for $s = 1/2$, $s = 10$, $s = 50$, and $s = 100$) for the $XY$ coupling with $J_y/J_x = 0.5$. The exact entanglement entropy $S_{E} = S(\rho_1)$ (top) and negativity (bottom) for different values of the spin $s$, and the bosonic RPA results, Eqs. \[ (61), (63) \] are depicted. The exact results approach those of RPA as $s$ increases, differences for not too small $s$ arising just for $B$ close to $B_c$. At the factorizing field $B_s \approx B_c/\sqrt{2}$, $S_E = 1$ while $N_{12} = 1/2$.

\[ |N_{12} \rightarrow 2N_{12} + \frac{1}{2}| \] should be applied for $|B| < B_c$. For small $f$, we have simply $N_{12} \approx -\tilde{f} \approx \sqrt{f}$. This will lead to a slope discontinuity of $N_{12}$ at the factorizing field $B_s$ (see Fig. 1), as $f$ vanishes there quadratically ($N_{12} - \frac{1}{2} \propto |B - B_c|$ for $B \approx B_c$). On the other hand, for $f \to \infty (|B| \to B_c)$, $\tilde{f} \to -\frac{1}{2}$, with $\tilde{f} \approx -\frac{1}{2} + \frac{1}{\sqrt{f}}$ and $N_{12} \approx 2f$. Both $S(\rho_1)$ and $N_{12}$ are concave increasing functions of $f$ and measure the entanglement of the pair.

Comparison with exact numerical results, obtained through the diagonalization of $H$ (a $(2s+1)^2 \times (2s+1)^2$ matrix), are shown in Fig. 1 for the $XY$ case ($J_y = 0$) with anisotropy $\chi = J_y/J_x = 0.5$. Exact results are seen to rapidly approach the RPA values \[ (61)-(64) \] as the spin $s$ increases, the discrepancy for finite $s$ arising just in the vicinity of $B_c$, or for very small $s$, i.e., where tunneling effects arising from the non-zero overlap \[ (55) \] between the degenerate parity breaking states become appreciable.

Nonetheless, this overlap can be taken into account using the full definite parity RPA spin state \[ (62) \] (dashed-dotted line), compared with the bosonic RPA result \[ (61) \] and the exact value, for $s = 10$ at the same parameters of fig. 1. The result from the RPA spin state improves the bosonic RPA for $B$ just below $B_c$. Right: The average local boson occupation $f$, Eq. \[ (62) \], which is small away from $B_c$, and the negative eigenvalue $\tilde{f}$ of the partial transpose of the contraction matrix ($\tilde{f} \approx \sqrt{f}$ for small $f$). Bottom: Left: RPA energies $\omega_0, \omega_1$, together with the mean field energy $\lambda$ and the mean RPA energy $\omega$ appearing in \[ (62) \]. Right: The quantities $Z_k = v_k/u_k$ for $k = 0, 1$, which determine the RPA state \[ (12) \] and vanish at the factorizing field $B_s$.
away from $B_c$, all bosonic RPA results can be reproduced by the spin densities of Appendix A, with $\hat{f} \approx \sqrt{f}$. In the bottom panels we depict the RPA energies $\omega_0, \omega_1$ and the RPA state coefficients $Z_k \equiv v_k/u_k$ used in Eq. (12). Although $\omega_0$ vanishes at $B_c$, the difference $\lambda - \omega_0$ responsible for entanglement, remains everywhere quite small. Both $Z_k$ vanish and change sign at the factorizing field $B_s$, indicating a qualitative change in the type of correlations at this point: Entanglement between two spins $1/2$ is well known to change from antiparallel to parallel (in the original frame) at $B_s$ [12], an effect arising within the RPA from this sign change.

**B. Fully connected spin system**

Let us now consider a fully and uniformly connected XYZ array of $n$ spins, where

$$J_\mu(l) = (1 - \delta_{l0})J_\mu/(n-1),$$

in [13]. This scaling ensures a finite intensive energy $\langle H \rangle/n$ for large $n$ and finite $J_\mu$. Entanglement properties of this well-known model [18, 27] for $s = 1/2$ in the large $n$ limit have been previously analyzed [28], including recently Holstein-Primakoff based bosonization [16, 21, 29, 30]. Direct application of the present RPA formalism will be here shown to yield full analytic expressions for any size $n$ and spin $s$. The present treatment does not exactly coincide with that of refs. [16, 21], since the absence of self-interacting terms $s_{i\mu}s_{j\nu}$ (non-trivial for $s > 1/2$) is here exactly taken into account and leads to repulsive RPA corrections ($\omega_1$), non-zero for finite $n$. The Fourier transform of $\langle H \rangle$ is $J^0_\mu = J_\mu$ and $J^i_\mu = -J_\mu/(n-1)$ for $k = 1, \ldots, n-1$, leading again to two distinct RPA energies: One associated with a fundamental attractive mode ($\omega_0$) and $n-1$ degenerate weak repulsive modes $\omega_k = \omega_1, k \neq 0$, which just add a small repulsive correction accounting for the absence of self-energy terms:

$$\omega_0 = \sqrt{(\lambda - J_x)(\lambda - J_y)}, \quad \omega_1 = \sqrt{(\lambda - J_x)(\lambda + J_y)}$$

where the replacements [51] are to be used for $B < B_c$. The ensuing contractions [51] become obviously independent of separation for $i \neq j$:

$$F_{ij} = \frac{1}{2n}\left(\frac{\lambda - \Delta^0}{\omega_0} - \frac{\lambda - \Delta^1}{\omega_1}(1-n\delta_{ij})\right) - \frac{1}{2}\delta_{ij},$$

$$G_{ij} = \frac{1}{2n}\left(\frac{\lambda - \Delta^0}{\omega_0} - \frac{\lambda - \Delta^1}{\omega_1}(1-n\delta_{ij})\right),$$

and imply that for any bipartition $(L, n-L)$, the entanglement entropy $S(\rho_L)$ will depend just on $L$. Moreover, there is again a single non-zero eigenvalue $f_L$ of the reduced matrix $D_L$ of $L$ spins for any $L$ (see Appendix B), such that in the bosonic approximation [26, 55],

$$S(\rho_L) = -f_L \log_2 f_L + (1 + f_L) \log_2(1 + f_L) + \delta,$$

$$f_L = \frac{1}{2}\sqrt{1 + 2\alpha_L \Delta - 1}, \quad \alpha_L = (n-L)/n^2$$

where $\delta = 0(1)$ for $|B| < B_c$ ($\lambda (B/B_c)^2 s \ll 1$) and

$$\gamma = \frac{n^2(\lambda^2 - \omega^2)}{2(n-1)\omega_0\omega_1}, \quad \omega = \frac{(n-1)\omega_0}{n}$$

For $n = 2$ we recover Eqs. (51, 52), while for large $n$, $\Delta \approx \frac{\lambda - \Delta^0}{\omega_0} - 1$. Entanglement is then driven again by the ratio $\frac{\lambda^2 - \omega^2}{\omega_0^2}$, which is small away from $B_c$ and vanishes at $B_c$. For small $\Delta$, $f_L \approx \frac{1}{2}\alpha_L \Delta$, with $\Delta \approx \frac{1}{2}(\frac{\lambda - \Delta^0}{\omega_0} - 1)^2$ for $|B| \gg B_c$ and $\Delta \approx (B-B_c)^2$ in the vicinity of $B_c$. For $B \rightarrow B_c$, $f_L \propto \sqrt{c}\gamma_{L}(B-B_c)^{-1/4}$ and $S(\rho_L) \approx \log_2 f_L$.

The bosonic negativity of a bipartition $(m, L-m)$ of a subsystem of $L \leq n$ spins can again be explicitly obtained, since there is also a single negative eigenvalue $\tilde{f}_{Lm}$ of the partial transpose of the contraction matrix (see appendix B):

$$N_{m,L-m} = \frac{-\tilde{f}_{Lm}}{1 + 2\tilde{f}_{Lm}},$$

$$\tilde{f}_{Lm} = \frac{1}{2}\sqrt{1 + \gamma_{Lm}\Delta - \sqrt{8\beta_{Lm}\Delta + \gamma_{Lm}^2 \Delta^2 - 7\gamma_{Lm}^2 \Delta^2}}$$

$$\gamma_{Lm} = \alpha_L + 4\beta_{Lm}, \quad \beta_{Lm} = m(L-m)/n^2.$$  (70)

For a global partition $(L = n)$, $\alpha_n = 0$ while $\beta_{nm} = \alpha_m$, and $\tilde{f}_{nn} = f_L = \sqrt{f_L(f_L+1)}$, with $N_{nn} = f_L + \sqrt{f_L(f_L+1)}$, as in Eq. (63). In general, for small $\Delta$,

$$\tilde{f}_{Lm} \approx -\sqrt{\gamma_{Lm}} \approx \frac{\gamma_{Lm}}{\Delta} \approx \frac{\beta_{Lm}}{\alpha_L} \Delta$$

such that for strong fields, $\tilde{f}_{Lm} \approx \sqrt{\gamma_{Lm}(B-B_c)}$, while for $B$ close to $B_c$, $\tilde{f}_{Lm} \propto \sqrt{\beta_{Lm}(B-B_c)}$. On the other
for $B < B_c$ and large $L$ and using Eq. (54) for $L = 2$.

IV. DISCUSSION

We have shown that the mean field plus RPA method is able to provide, through the bosonic representation, a general tractable method for estimating, in the ground state of general spin arrays, the entanglement entropy of any bipartition of the whole system as well as the negativity associated with any bipartition of any subsystem. The approach becomes fully analytic in systems with translational invariance, where no numerical diagonalization is required for obtaining the basic contraction matrices.

The bosonic treatment provides essentially the exact behavior of the system in the large spin limit. Finite spin corrections can be taken into account through the corresponding RPA spin state, which allows in particular to implement the non-negligible symmetry restoration effects in the case of the parity-breaking mean field, but which otherwise yields result which are in full agreement with the bosonic treatment at first order in the average local boson occupation. The latter is normally very low away from critical regions.

Through direct application of the present method, simple analytic expressions for the entanglement entropy and negativities for a spin-$s$ pair and for a fully connected array of $n$ spins $s$ in a uniform field, have been straightforwardly obtained, which depend explicitly on the RPA energies. The agreement with exact numerical results is confirmed to improve as the spin $s$ increases at fixed size, and in the fully connected case also as $n$ increases at fixed $s$, differences being in fact negligible away from the critical region for not too small $s$ or size.

An important general prediction that arises from the present treatment is that entanglement from elementary excitations approaches a non-vanishing spin independent limit as the spin increases. An RPA quantum regime, characterized by weak entanglement, emerges then between strictly classical and strongly quantum regimes.

The authors acknowledge support from CIC (RR) and CONICET (JMM,NC) of Argentina.

Appendix A: RPA spin densities

We will here construct the spin density matrices compatible with the RPA spin state (13) and the contractions (17) up to second order in $V$, i.e., first order in the average occupation $VV^\dagger$ (implying zero or one boson per site). At this order, $F \approx GG^\dagger$ (Eqs. (17)) and the support of $\rho = |0_{\text{RPA}}\rangle \langle 0_{\text{RPA}}|$ is just the subspace spanned by the mean field state $|0\rangle$ plus the two site excitations $|1_1\rangle$ (Eq. (16)), leading to

$$\rho \approx \left( GG^\dagger \begin{pmatrix} G & G^\dagger \end{pmatrix} \begin{pmatrix} 1 - G^\dagger G \\ 0 \end{pmatrix} \right)$$  (A1)
where $G$ denotes a column matrix of elements $G_{ij}$, $i < j$. At this order, $\rho^2 = \rho$. The ensuing reduced density matrix $\rho_A = \text{Tr}_A \rho$ of a subsystem $A$ of $L$ spins becomes

$$
\rho_A \approx \begin{pmatrix} G_A G_A^d & 0 & G_A \\ 0 & F_A - G_A G_A^d & 0 \\ G_A^d & 0 & 1 - \text{tr} F_A + G_A^d G_A \end{pmatrix}
$$

where $F_A$, $G_A$ are the $L \times L$ contraction matrices of subsystem $A$ and $G_A$ the concomitant column vector (of length $L(L-1)/2$). The central block contains the one-site elements $|1\rangle \langle 1|$ arising from the partial trace of $GG^d$. Here we have used the approximate identity $\sum_{i \in A} G_{ik} G_{kj} \approx F_{ij} - \sum_{k \in A} G_{ik} G_{jk}$ for $i,j \in A$ (and neglected diagonal elements $G_{ii}$, of higher order due to the absence of self-energy terms), which allows to write $\rho_A$ entirely in terms of local contractions. Eq. (A2) is then in agreement with direct state tomography at this order (for $i,j,k,l \in A$, $(b_ib_l^\dagger \prod_{i \neq j} (1 - b_ib_j^\dagger)b_{lj}^\dagger \rho) \approx (F_A - G_A G_A^d)_{ij}, (b_ib_l^\dagger b_ib_j^\dagger \rho) \approx G_A G_A G_{ij}$). Up to $O(V^2)$, $\rho_A$ is a positive matrix with $\text{Tr} \rho_A = 1$, but is no longer pure.

Its entropy $S(\rho_A) = -\text{Tr} \rho_A \log_2 \rho_A$ is determined, at this order, by the central block $\rho^2_A = F_A - G_A G_A^d$, $S(\rho_A) \approx \text{tr} \rho_A (\log_2 \rho - \log_2 \rho^2_A)$, which coincides with Eq. (25) up to second order in $V$ (at this order $f^2_A$ coincides with the eigenvalues of $\rho^2_A$ and Eq. (25) becomes $\approx \sum f_A (\log_2 \rho - \log_2 f_A^2)$).

On the other hand, the leading term in the negativity arising from a bipartition $(B,C)$ of $A$ is of first order in $V$ and is just the sum of the singular values of the submatrix $G_{BC}$ (of elements $G_{ij}$, $i \in B, j \in C$), whence $N_{BC} \approx \text{tr} [G_{BC} G_{BC}^d]^{1/2}$. At this order, the negative symplectic eigenvalues $f^2_A$ in (29) are again minus the singular values of $G_{BC}$, while Eq. (28) becomes $N_{BC} \approx -\sum f_A$, leading again to the previous result.

Let us finally notice that Eq. (A2) always commutes with the $S_z$ parity (along the mean field axis) of subsystem $A$, i.e., $[\rho_A, F_A] = 0, F_A = \exp[\pi i \sum_{i \in A} (s_{zi} - s_i)]$. In the case of two spins $i,j$, $G_A$ has length 1 and Eq. (A2) is just a $4 \times 4$ blocked matrix, while in the case of a single spin $i$, $G_A$ has length 0 and Eq. (A2) becomes just $\rho_i \approx F_{ii} |i\rangle \langle i| + (1 - F_{ii}) |0\rangle \langle 0|$. 

### Appendix B: Fully connected system

In the fully connected $XY$ spin system, the contractions (B1) are of the form $F_{ij} = F_0 \delta_{ij} + F_1$, $G_{ij} = G_0 \delta_{ij} + G_1$, with $F_0, F_1, G_0, G_1$ real. The ensuing contraction matrix $D_L$ for a subsystem of $L$ spins will then have symplectic eigenvalues (see also (20))

$$f_L = \sqrt{(F_0 + LF_1 + \frac{1}{2})^2 - (G_0 + LG_1)^2} - \frac{1}{2}$$

$$f_0 = \sqrt{(F_0 + \frac{1}{2})^2 - G_0^2} - \frac{1}{2}$$

plus their partners $1 + f_L, 1 + f_0$, where $f_L$ is non-degenerate while $f_0$ has $L-1$ degeneracy. Eqs. (B1)–(B2) can be obtained either by a Fourier transform of the local operators or by noticing that the $L \times L$ contraction matrix $F_L$ can be written as $F_L = F_0 I_L + F_1 1_L 1_L^\dagger$ (and similarly for $G_L$), with $I_L$ the $L \times L$ identity and $1_L$ a column $L \times 1$ vector with unit elements. $F_L$ and $G_L$ will then be diagonal in the same local basis with eigenvalues $F_0 + LF_1$ and $F_0$ $(L-1)$ degenerate and similarly for $G_L$, which leads to Eqs. (B1)–(B2). In the case of a global vacuum, $f_0 = 0$ (since for $L = n$, we should have $f_{L=n} = f_0 = 0$), implying a single positive eigenvalue $f_L$ for any $L < n$. Eq. (B1) leads then to Eq. (67).

For evaluating the negativity $N_{mp}$ of a bipartition $(m, p)$ of a subsystem of $L$ spins $(m + p = L)$, we may first note that $F_L$ will be composed of blocks $F_{mm} = F_0 I_m + F_1 1_m 1_m^\dagger, F_{mp} = F_11_m 1_p^\dagger, F_{pp} = F_0 I_p + F_1 1_p 1_p^\dagger$, and similarly for $G_L$. A local transformation allows then to write $F_L$ as a direct sum of a $(L - 2) \times (L - 2)$ diagonal block $F_0 I_{L-2}$ plus the block $F_0 I_2 + F_1 (\sqrt{\text{mp}})$, and similarly for $G_L$. The ensuing partially transposed contraction matrix will then have symplectic eigenvalues $f_0 = f_0$ (Eq. (B2)), $L - 2$ degenerate (with $f_0 = 0$ for a global vacuum) and

$$f_{LM}^\pm = \frac{1}{2} \sqrt{\text{Tr} A^2 \pm \sqrt{\text{Tr} A^2}^2 - 16 \text{det} A - \frac{1}{2}}$$

together with their partners $1 + f_0, 1 + f_{LM}^\pm$, where $A = (A_{FG} - A_{FG}^\dagger)$ is a $4 \times 4$ matrix with blocks $A_{FG} = (\frac{1}{2} + F_0 I_2 + (\sqrt{\text{mp}}) I_2 F_1)$ and similarly for $A_{FG}$. Here $f_{LM}^\pm > 0$ but $f_{LM}^- < 0$. The latter is the single negative symplectic eigenvalue given in Eq. (70).

[1] M.A. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge Univ. Press (2000).
[2] L. Amico, R. Fazio, A. Osterloh and V. Vedral, Rev. Mod. Phys. 80, 516 (2008).
[3] J. Eisert, M. Cramer, M.B. Plenio, Rev. Mod. Phys. 82, 277 (2010).
[4] T.J. Osborne, M.A. Nielsen, Phys. Rev. A 66, 032110 (2002).
[5] G. Vidal, J.I. Latorre, E. Rico and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
A 78, 042319 (2008).
[11] L. Amico et al, Phys. Rev. A 74, 022322 (2006); F. Baron et al, J. Phys. A 40, 9845 (2007).
[12] R. Rossignoli, N. Canosa, and J.M. Matera, Phys. Rev. A 77, 052322 (2008).
[13] G. Vidal and R.F. Werner, Phys. Rev. A 65, 032314 (2002).
[14] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A 58, 883 (1998). K. Zyczkowski, ibid. 60, 3496 (1999).
[15] H. Wichterich, J. Molina-Vilaplana, and S. Bose, Phys. Rev. A 80, 010304(R) (2009).
[16] H. Wichterich, J. Vidal, and S. Bose Phys. Rev. A 81, 032311 (2010).
[17] N. Canosa and R. Rossignoli, Phys. Rev. A 73, 022347 (2006); R. Rossignoli and N. Canosa, Phys. Rev. A 72, 012335 (2005).
[18] Peter Ring and Peter Schuck, The Nuclear Many-Body Problem, Springer-Verlag (1980).
[19] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).
[20] A. Klein and E.R. Marshalek, Rev. Mod. Phys. 63, 375 (1991).
[21] J. Vidal, S. Dusuel, and T. Barthel, J. Stat. Mech. (2007) P01015
[22] J. Kurmann, H. Thomas, and G. Müller, Physica A 112, 235 (1982).
[23] S.M. Giampaolo, G. Adesso, and F. Illuminati, Phys. Rev. Lett. 100, 197201 (2008); Phys. Rev. B 79, 224434 (2009).
[24] R. Rossignoli, N. Canosa, and J.M. Matera, Phys. Rev. A 80, 062325 (2009).
[25] K. Audenaert, J. Eisert, M.B. Plenio, R.F. Werner, Phys. Rev. A 66, 042327 (2002); M. Cramer, J. Eisert, M.B. Plenio, J. Dreißig, Phys. Rev. A 73, 012309 (2006).
[26] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A 70, 022318 (2004); A. Serafini, G. Adesso, and F. Illuminati, Phys. Rev. A 71, 032349 (2005); G. Adesso and F. Illuminati, Phys. Rev. A 78, 042310 (2008).
[27] H.J. Lipkin, N. Meshkov, and A.J. Glick, Nucl. Phys. 62, 188 (1965).
[28] J.I. Latorre, R. Orús, E. Rico, J. Vidal, Phys. Rev. A 71, 064101 (2005).
[29] T. Barthel, S. Dusuel, and J. Vidal, Phys. Rev. Lett. 97, 220402 (2006).
[30] S. Dusuel and J. Vidal, Phys. Rev. B 71, 224420 (2005).