ITERATIVE DIFFERENTIAL EMBEDDING PROBLEMS IN
POSITIVE CHARACTERISTIC

STEFAN ERNST

Abstract. In this paper, we prove that every iterative differential emb- edding problem over an algebraic function field in positive characteristic with an algebraically closed field of constants has a proper solution.

1. Introduction

The inverse problem of differential Galois theory asks which linear algebraic groups occur as Galois groups of a Picard-Vessiot extension $E/F$, for a given differential field $F$. We say that a linear algebraic group $G$ is realizable over $F$, if there exists a Picard-Vessiot extension $E/F$, such that the Galois group $\text{Gal}(E/F)$ is isomorphic to $G(K)$, where $K$ is the field of constants of $F$. In [MvdP03], B.H. Matzat and M. van der Put showed that every reduced linear algebraic group is realizable over an algebraic function field of positive characteristic with an algebraically closed field of constants.

An embedding problem is an intensification of the inverse problem: Given an epimorphism of linear algebraic groups $\beta: \tilde{G} \to G$ and a realization $E/F$ of $G$, does there exist a realization $\tilde{E}/F$ of $\tilde{G}$, such that $E$ is a subfield of $\tilde{E}$? If we can give a positive answer to this question, we say the embedding problem has a proper solution. If only a subgroup of $\tilde{G}$ is realizable in this way, we just say that the embedding problem is solvable.

In usual Galois theory the study of embedding problems offers a great deal of information regarding the structure of the absolute Galois group $G_F := \text{Gal}(F_{\text{sep}}/F)$ (here $F$ denotes an arbitrary field). For example, the group $G_F$ is projective, if and only if all embedding problems over $F$ are solvable. Further by Iwasawa’s Freiheitssatz [MM99, Theorem IV.1.12], the group $G_F$ is free (of countable infinite rank) if and only if all embedding problems over $F$ have a proper solution.

Now let $F$ again be an iterative differential field. The Picard-Vessiot extensions of $F$ form a neutral Tannakian category $\mathbf{T}$, with fibre functor $\omega: \mathbf{T} \to \text{Vect}_K$ (for a suitable field $K$). Then by the Main Theorem of Tannakian formalism, the functor $\text{Aut}^\circ(\omega)$ of $K$-algebras is representable by an affine group scheme $\pi_1(\mathbf{T})$. This group scheme is called the Tannakian fundamental group of $\mathbf{T}$. For a different fibre functor of $\mathbf{T}$, we get a fundamental group which is isomorphic to $\pi_1(\mathbf{T})$ by an inner automorphism, so the fundamental group is independent of the choice of the fibre functor. This fundamental group is an analogon of the absolute Galois group in usual Galois theory, for example the point group of this group scheme can be understood as the automorphism group of a "universal" Picard-Vessiot extension and is a prolinear group (the projective limit of linear algebraic groups). In this way the solvability of embedding problems is a measure of the freeness of $\pi_1(\mathbf{T})$.

In this paper we only consider differential fields with an algebraically closed field of constants. The most important example for us is $F = K(t)$, which is the rational function field in one variable.

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Differential Galois theory in positive characteristic is explained in detail in the preprint [Mat01] from B.H. Matzat. From there we follow the idea laid out in the dissertation of T. Oberlies ([Obe03]), who studied embedding problems in characteristic zero. At the end of this article we obtain the following result:

**Theorem:** Let $F$ be an algebraic function field in one variable over an algebraically closed field $K$ of positive characteristic. Then every ID-embedding problem in $\text{AffGr}_K^\text{red}$ over $F$ with reduced kernel has a proper solution.

Here $\text{AffGr}_K^\text{red}$ denotes the category of reduced linear algebraic groups which are defined over $K$. It is an interesting open question whether this theorem holds in characteristic zero.

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2. **Basics and Notation**

First we give a short introduction into differential Galois theory in positive characteristic, which is based on the preprint [Mat01]. So for more details and explicit proofs the reader is referred to [Mat01].

Throughout this article, we assume that all rings and fields have characteristic $p > 0$. Further we assume that $K$ is an algebraically closed field. In the situation of positive characteristic one is confronted with the problem that the usual differentiation, extended to some transcendental extension of a differential field, causes new constants. This problem can be attacked by using iterative derivations, which were introduced for the first time by H. Hasse and F.K. Schmidt [HS37]. For this purpose let $R$ be a commutative ring. A family $\partial^* = (\partial^{(k)})_{k \in \mathbb{N}}$ of maps $\partial^{(k)}: R \to R$ is called an **iterative derivation** of $R$ if

$$
\partial^{(0)} = \text{id}_R,
\partial^{(k)}(a \cdot b) = \sum_{i+j=k} \partial^{(i)}(a)\partial^{(j)}(b),
\partial^{(k)}(a + b) = \partial^{(k)}(a) + \partial^{(k)}(b),
\partial^{(i)} \circ \partial^{(j)} = \partial^{(i+j)}
$$

for all $a, b \in R$ and all $i, j, k \in \mathbb{N}$. The tuple $(R, \partial^*)$ is called an **iterative differential ring** (ID-ring). An element $c \in R$ is a **(differential) constant**, if all its iterative derivatives vanish, i.e., $\partial^{(k)}(c) = 0$ for all $k \geq 1$. The set of all constants of $R$ is denoted by $C(R)$.

We observe that $(\partial^{(k)})^p = 0$ for $k > 0$ (this phenomenon is called the **trivial p-curvature**). Moreover, the iterative derivation $\partial^*$ is already determined by all $\partial^{(k)}$, where $k$ is a $p$-power.

An **iterative differential field** (ID-field) $F$ is an ID-ring which is a field. In this case the set of constants $C(F)$ is also a field. The most important example of an ID-field is the field of rational functions $K(t)$ together with the iterative derivation $\partial^{(k)}(t^n) = \binom{n}{k}t^{n-k}$, which will be denoted by $\partial^*_t$. Thus the field of constants is $K$.

Contrary to ordinary differentiation in positive characteristic we do not obtain new constants here.

Let $(R, \partial^*_R)$ be an ID-ring and set $R_i := \bigcap_{j < i} \ker(\partial^{(j)}_R)$. Then $(R_i, (\partial^{(k)p)}_R)_{k \in \mathbb{N}}$ is an ID-ring (see [Mat01], Corollary 1.8).

Next we extend the concept of iterative derivations to modules. For this let $(R, \partial^*_R)$ be an ID-ring and $M$ be an $R$-module. A family $\partial^*_M = (\partial^{(k)}_M)_{k \in \mathbb{N}}$ of maps $\partial^{(k)}_M: M \to M$ satisfying

1. $\partial^{(0)}_M = \text{id}_M$, 
2. $\partial^{(k)}_M(a \cdot b) = \sum_{i+j=k} \partial^{(i)}_M(a)\partial^{(j)}_M(b)$, 
3. $\partial^{(k)}_M(a + b) = \partial^{(k)}_M(a) + \partial^{(k)}_M(b)$, 
4. $\partial^{(i)}_M \circ \partial^{(j)}_M = \partial^{(i+j)}_M$
\[ \partial_M^{(0)} = \text{id}_M, \]
\[ \partial_M^{(1)}(a \cdot x) = \sum_{i+j=k} \partial_R^{(i)}(a) \partial_M^{(j)}(x), \quad \partial_M^{(k)}(x + y) = \partial_M^{(k)}(x) + \partial_M^{(k)}(y), \quad \partial_M^{(k)} \circ \partial_M^{(l)} = \partial_M^{(i+j)}, \]

for all \( x, y \in M \) and all \( a \in R \) is called an \textit{iterative derivation} of \( M \) and the tuple \((M, \partial_M)\) is called an \textit{iterative differential module} (ID-module). The \( C(R) \)-module

\[ V_M := \bigcap_{k \in \mathbb{N}^+} \ker(\partial_M^{(k)}) \]

is called the \textit{solution space} of \( M \). We call \( M \) a \textit{trivial iterative differential module} if \( M \cong V_M \otimes_{C(R)} R \).

Let \((M, \partial_M^*)\) and \((N, \partial_N^*)\) be two ID-modules, and let \( \varphi \in \text{Hom}_R(M, N) \). Then \( \varphi \) is called an \textit{iterative differential homomorphism} (ID-homomorphism), if

\[ \varphi \circ \partial_M^{(k)} = \partial_N^{(k)} \circ \varphi \]

for all \( k \in \mathbb{N} \). Further, we denote by \( \text{IDMod}_R \) the category of all finitely generated iterative differential modules over \( R \) with ID-homomorphisms as morphisms.

If \( F \) is an ID-field with algebraically closed field of constants \( K \), then by [Mat01], Remark 2.6, the category \( \text{IDMod}_F \) together with the forgetful functor

\[ \omega : \text{IDMod}_F \to \text{Vect}_K, \quad (M, \partial_M^*) \mapsto V_M \otimes_F E \]

to the category of \( K \)-vector spaces as fibre functor is a \textit{neutral Tannakian category}, for a suitable extension \( E/F \) (see [Del90] for a definition and properties).

By [Mat01], Theorem 2.8, we get the following connection between ID-modules and projective systems.

**Theorem 1.** Let \((F, \partial_F^*)\) be an ID-field of characteristic \( p > 0 \). Then the category \( \text{IDProj}_F \) of projective systems \((\tilde{N}_l, \psi_l)\) with the properties

1. \( \tilde{N}_l\) is an \( F_l \)-vector space of finite dimension and \( \psi_l \) is \( F_{l+1} \)-linear
2. each \( \tilde{\psi}_l \) uniquely extends to an \( F_l \)-isomorphism \( \tilde{\psi}_l : \tilde{N}_{l+1} \otimes_{F_{l+1}} F_l \to \tilde{N}_l \)

is equivalent to the category \( \text{IDMod}_F \).

Where \( F_l := \bigcap_{j<l} \ker(\partial_F^{(j)}) \), as above.

Let \( F \) be an ID-field with field of constants \( K \) and let \( M \in \text{IDMod}_F \) be an \( n \)-dimensional ID-module, then we get an associated \textit{iterative differential equation} (IDE)

\[ \partial^{(p)}(y) = A_l y \]

with \( A_l = \partial_F^{(p)}(D_0 \cdots D_l)(D_0 \cdots D_l)^{-1} \), where \( D_l \in \text{GL}_n(F_l) \) are the matrices of \( \psi_l \) with respect to a basis \( B_l \) of \( M_l \). Let \((R, \partial_R^*)\) be an ID-ring with \( R \geq F \) and such that \( \partial_R^* \) extends \( \partial_F^* \), i.e., \( \partial_R^*|_F = \partial_F^* \). A matrix \( Y \in \text{GL}_n(R) \) is called a \textit{fundamental solution matrix} for the IDE \((*)\) if \( \partial^{(p)}(Y) = A_l Y \) for all \( l \in \mathbb{N} \).

The ring \( R \) is called an \textit{iterative Picard-Vessiot ring} (IPV-ring) for the IDE \((*)\) if it satisfies the following conditions:

1. \( R \) is a simple ID-ring (i.e., contains no nontrivial proper ID-ideals),
2. the IDE has a fundamental solution matrix \( Y \) with coefficients in \( R \),
3. \( R \) is generated over \( F \) by the coefficients of \( Y \) and \( \det(Y)^{-1} \).

For each ID-module \( M \) there exists an IPV-ring \( R \), which is unique up to ID-isomorphisms. Each IPV-ring is an integral domain and we can build the quotient field \( \text{Quot}(R) = E \) (IPV-field), which contains no new constants, i.e., \( C(E) = K \).
The group $\text{Gal}_{ID}(E/F) := \text{Aut}_{ID}(E/F)$ of all ID-automorphisms is called the iterative differential Galois group of the extension $E/F$. This group is isomorphic to the $K$-rational points of an affine group scheme, which is defined over $K$. If $E/F$ is a separable extension, then $\text{Gal}_{ID}(E/F)$ is a reduced affine group scheme and the converse holds. In this article we consider mostly reduced affine group schemes, which are equivalent (as a category) to the reduced linear algebraic groups. Therefore $\text{Gal}_{ID}(E/F) \leq \text{GL}_{n,K}$ and we denote by $\text{AffGr}^\text{red}_K$ the category of all reduced linear algebraic groups, which are defined over $K$, and with homomorphisms of reduced affine group schemes as morphisms.

By the following theorem we see that the iterative differential Galois theory is a generalization of the usual Galois theory (see [Mat01], Theorem 3.12).

**Theorem 2.** Let $F/K$ be an algebraic function field in one variable over $K$. Then the following are equivalent:

1. $E/F$ is a finite Galois extension with group $G$.
2. $E/F$ is an iterative Picard-Vessiot extension with finite differential Galois group $G$.

Now we can formulate the main theorem of differential Galois theory in positive characteristic.

**Theorem 3 (Galois Correspondence).** Let $(F,\partial^*_F)$ be an ID-field with algebraically closed field of constants $K$ and let $M \in \text{IDMod}_F$. Let $E/F$ be an IPV-extension for $M$ and let $G \leq \text{GL}_{n,K}$ be a reduced linear algebraic group such that $G(K) = \text{Gal}_{ID}(E/F)$.

Then

1. there is an antiisomorphism of the lattices
   $H := \{H | H \leq G \text{ is a Zariski closed reduced linear algebra group}\}$,
   and
   $L := \{L | L \text{ is an ID-field } F \leq L \leq E, \text{ such that } E/L \text{ is separable}\}$,
   given by $\Psi : H \to L, H \mapsto E^H(K)$ and $\Psi^{-1} : L \to H, L \mapsto H$, where $H(K) := \text{Gal}_{ID}(E/L)$;

2. if $H \triangleleft G$ is a Zariski closed reduced normal subgroup, then $L := E^H(K)$ is an IPV-extension of $F$ with Galois group $(G/H)(K)$.

**Proof.** [Mat01], Theorem 4.7. \hfill $\square$

After the basics of the differential Galois theory, we need some definitions about morphisms in $\text{AffGr}^\text{red}_K$.

**Definition 4.** Let $\beta : \tilde{G} \to G$ be an epimorphism in $\text{AffGr}^\text{red}_K$.

- $\beta$ is called **split**, if there exists a monomorphism $\sigma : \tilde{G} \hookrightarrow \tilde{G}$, such that $\beta \circ \sigma = \text{id}_{\tilde{G}}$. In this case we say that $\sigma$ is a **homomorphic section** for $\beta$.
- $\beta$ is called **direct split**, if $\beta$ is split and $\sigma(G)$ is a normal subgroup of $\tilde{G}$ (that means $\tilde{G} \cong \ker(\beta) \times \sigma(G)$).
- We say that $\beta$ is a **Frattini-epimorphism**, if $\tilde{G}$ is the only reduced closed supplement of $\ker(\beta)$ in $\tilde{G}$, i.e., any $U$ in $\tilde{G}$ which is reduced, closed, and satisfies $\ker(\beta) \cdot U = \tilde{G}$ already equals $\tilde{G}$.

The following theorem of A. Borel and J.P. Serre is very important for considering non-connected groups.

**Theorem 5.** Let $G$ be a reduced affine group scheme. Then there exists a finite supplement $H$ for the connected component $G^o$ in $G$, i.e., $G = G^o \cdot H$. 
ITERATIVE DIFFERENTIAL EMBEDDING PROBLEMS IN POSITIVE CHARACTERISTIC 5

Proof. [BS65], Lemme 5.11. \qed

Thus any reduced linear algebraic group $G$ is a quotient of a semidirect product $G^0 \rtimes H$. Therefore it is advisable to consider semidirect products.

**Definition 6.** Let $H$ be a finite group.

- A reduced, connected linear algebraic group $G^0$ together with a group homomorphism $H \to \text{Aut}(G^0)$, is called an $H$-group.
- Let $\tilde{G}^0$ and $G^0$ be $H$-groups. Let $\beta : \tilde{G}^0 \to G^0$ be a morphism in $\text{AffGr}_K^{\text{red}}$. We call $\beta$ an $H$-morphism, if $\beta$ commutes with the action of $H$.
- We say an $H$-epimorphism is $H$-split, if there exists a homomorphic section which is an $H$-morphism.

For any $H$-group $G^0$ we can build the semidirect product $G^0 \rtimes H$. Every $H$-epimorphism $\beta : \tilde{G}^0 \to G^0$ can be uniquely extended to an epimorphism $\beta : G^0 \rtimes H \to G^0 \rtimes H$, such that $\beta|_H = \text{id}_H$.

Such epimorphisms are very important for us, so we give them their own name.

**Definition 7.** Let $H$ be a finite group. Let $\tilde{G}^0$ and $G^0$ be two $H$-groups. An epimorphism $\beta : \tilde{G}^0 \rtimes H \to G^0 \rtimes H$ is called $H$-rigid, if $\beta|_H = \text{id}_H$.

Then we say $\beta$ is $H$-split if there exists a homomorphic section which is the identity on $H$.

We call $\beta$ subdirect $H$-split, if $\beta$ is $H$-split and $\tilde{G} \cong \ker(\beta) \times G$.

Let $\beta : \tilde{G}^0 \rtimes H \to G^0 \rtimes H$ be an $H$-rigid epimorphism. Then the restriction $\beta^\circ := \beta|_{\tilde{G}^0} : \tilde{G}^0 \to G^0$ is an $H$-epimorphism, in particular we call $\beta^\circ$ the connected component of $\beta$. Note that $\beta$ is $H$-split if and only if $\beta^\circ$ is $H$-split.

3. ID-EMBEDDING PROBLEMS

**Definition 8.** Let $F$ be an ID-field with field of constants $K$, let $E/F$ be an IPV-extension with Galois group $\tilde{G}(K) \cong \text{Gal}_D(E/F)$, and let $\beta : \tilde{G} \to G$ be an epimorphism in $\text{AffGr}_K^{\text{red}}$. The corresponding iterative differential embedding problem (ID-embedding problem) asks for the existence of an IPV-extension $\tilde{E}/F$ and a monomorphism $\tilde{\alpha}$ which maps $\text{Gal}_D(\tilde{E}/F)$ onto a closed subgroup of $\tilde{G}(K)$, such that the diagram

\[ \begin{array}{ccc}
\text{Gal}_D(\tilde{E}/F) & \xrightarrow{\text{res}} & \text{Gal}_D(E/F) \\
\alpha \downarrow & & \cong \alpha \\
1 & \xrightarrow{\beta} & \tilde{G}(K) \end{array} \]

commutes. We denote such an ID-embedding problem by $E(\alpha, \beta)$. The kernel $A(K)$ of the map $\beta$ is also called the kernel of $E(\alpha, \beta)$. We say that $E(\alpha, \beta)$ is split, if $\beta$ is split (respectively direct split, $H$-split, subdirect $H$-split and Frattini). We call $\tilde{\alpha}$ a solution of $E(\alpha, \beta)$. Further we say that $\tilde{\alpha}$ is a proper solution if $\tilde{\alpha}$ is an isomorphism.

Observe that by definition the affine group schemes $\tilde{G}$ and $G$ are reduced, whereas the kernel $A$ need not be reduced.

In order to solve such an ID-embedding problem we will make a decomposition into ID-embedding problems which are easier to solve. More precisely, a decomposition of an ID-embedding problem means a decomposition of the underlying epimorphism $\beta$ in the following sense:

**Definition 9.** Let $\mathcal{M}$ be a finite set of epimorphisms in $\text{AffGr}_K^{\text{red}}$. The set $\tilde{\mathcal{M}}$ is called an elementary decomposition if it is formed in the following way:
(1) Replace \( \beta_2 \in M \) by epimorphisms \( \beta_1 \) and \( \beta_3 \) in \( \text{AffGr}^{\text{red}}_K \), so that an epimorphism \( \beta_3 \) in \( \text{AffGr}^{\text{red}}_K \) exists with \( \beta_2 = \beta_1 \circ \beta_3 \).

Let \( \beta: \tilde{G} \to G \) be an epimorphism in \( \text{AffGr}^{\text{red}}_K \). Then \( \beta \) is called decomposable into \( \beta_1, \ldots, \beta_n \), if there exist sets \( M_i \) (1 \( \leq \) \( i \) \( \leq \) \( m \)), with \( M_1 = \{ \beta \} \) and \( M_m = \{ \beta_1, \ldots, \beta_n \} \), such that \( M_{i+1} \) is an elementary decomposition of \( M_i \) for all \( i \).

The next proposition shows how a decomposition of the underlying epimorphism affects the ID-embedding problem:

**Proposition 10.** Let the following commutative diagram of epimorphisms in \( \text{AffGr}^{\text{red}}_K \) be given.

\[
\begin{array}{ccc}
G_2 & \xrightarrow{\beta_3} & G_3 \\
\beta_2 & \searrow & \swarrow \\
G_1 & \xrightarrow{\beta_1} & G_3
\end{array}
\]

(1) Let \( \tilde{\alpha}: \text{Gal}_{\text{ID}}(\tilde{E}/F) \to G_2(K) \) be a proper solution of the ID-embedding problem \( \mathcal{E}(\alpha, \beta_1) \) and let \( \overline{\alpha}: \text{Gal}_{\text{ID}}(\bar{E}/F) \to G_1(K) \) be a (proper) solution of the ID-embedding problem \( \mathcal{E}(\overline{\alpha}, \beta_3) \).

Then \( \overline{\alpha} \) is also a (proper) solution of the ID-embedding problem \( \mathcal{E}(\alpha, \beta_2) \).

(2) If conversely \( \overline{\alpha}: \text{Gal}_{\text{ID}}(\bar{E}/F) \to G_1(K) \) is a proper solution of the ID-embedding problem \( \mathcal{E}(\alpha, \beta_2) \) and

\[
\tilde{E} := \bar{E}^{\text{ker}(\beta_3)},
\]

then there exists exactly one monomorphism \( \tilde{\alpha}: \text{Gal}_{\text{ID}}(\tilde{E}/F) \to G_2(K) \), such that

\[
\tilde{\alpha} \circ \text{res} = \beta_3 \circ \overline{\alpha}.
\]

In addition \( \tilde{\alpha} \) is a proper solution of the ID-embedding problem \( \mathcal{E}(\alpha, \beta_1) \).

**Proof.** (1) By the assumptions we have the following commutative diagram, which proves the claim:

\[
\begin{array}{ccc}
\text{Gal}_{\text{ID}}(\tilde{E}/F) & \xrightarrow{\text{res}} & \text{Gal}_{\text{ID}}(\bar{E}/F) \\
\downarrow{\text{res}} & & \downarrow{\text{res}} \\
\tilde{G}_1(K) & \xrightarrow{\beta_3} & G_2(K) \\
\uparrow{\overline{\alpha}} & & \uparrow{\text{res}} \\
\text{Gal}_{\text{ID}}(\bar{E}/F) & \xrightarrow{\text{res}} & \text{Gal}_{\text{ID}}(E/F) \\
\downarrow{\overline{\alpha}} & & \downarrow{\text{res}} \\
G_1(K) & \xrightarrow{\beta_2} & G_3(K) \\
\end{array}
\]

(2) We define \( \tilde{\alpha} \) by equation (*)). Then \( \tilde{\alpha} \) is well-defined, because \( \beta_3 \) and the restriction map have the same kernel. Since the restriction map is surjective, \( \tilde{\alpha} \) is unique. It is obvious that \( \tilde{\alpha} \) is a proper solution of \( \mathcal{E}(\alpha, \beta_1) \).

By the second part of this proposition we can reduce the ID-embedding problem \( \mathcal{E}(\alpha, \beta_1) \) to the ID-embedding problem \( \mathcal{E}(\alpha, \beta_2) \). The first part provides a decomposition of the ID-embedding problem \( \mathcal{E}(\alpha, \beta_2) \) into the ID-embedding problems \( \mathcal{E}(\alpha, \beta_1) \) and \( \mathcal{E}(\overline{\alpha}, \beta_3) \). But in this case the ID-embedding problem \( \mathcal{E}(\overline{\alpha}, \beta_3) \) depends on the selected proper solution from \( \mathcal{E}(\alpha, \beta_1) \). This problem can be attacked by the following definition.
Definition 11. An epimorphism in $\text{AffGr}^\text{red}_K$ is called an embedding epimorphism (over $F$) if all corresponding ID-embedding problems over $F$ have a proper solution.

The next proposition is a consequence of Proposition 10.

Proposition 12. Let $\beta: \tilde{G} \to G$ be an epimorphism in $\text{AffGr}^\text{red}_K$. If $\beta$ is decomposable into the embedding epimorphisms $\beta_1, \ldots, \beta_n$, then $\beta$ is an embedding epimorphism.

4. Decomposition of ID-Embedding Problems

All results of this section are taken from [Obe03]. We give the proofs only for completeness, since the work of T. Oberlies [Obe03] is up to now only published in German.

The first lemma gives us the background to use the theory of algebraic groups to decompose an epimorphism. More precisely, a “decomposition” of the kernel induces a decomposition of the epimorphism.

Lemma 13. Let $\beta: \tilde{G} \to G$ be an epimorphism in $\text{AffGr}^\text{red}_K$ with kernel $A$. Let $V$ be a closed subgroup of $A$ which is normal in $\tilde{G}$. Then $\beta$ is decomposable into the epimorphism $\beta_1: \tilde{G} \to \tilde{G}/V$ and the epimorphism $\beta_2: \tilde{G}/V \to G$.

If in addition $\beta$ is split, then $\beta_2$ is also split.

Proof. See Definition 9. □

The next lemma shows us that an epimorphism can be decomposed into epimorphisms of extreme cases.

Lemma 14. Let $\beta: \tilde{G} \to G$ be an epimorphism in $\text{AffGr}^\text{red}_K$. Then $\beta$ is decomposable into a Frattini-epimorphism and a split epimorphism (with the same kernel as $\beta$).

Proof. The reduced subgroups of $\tilde{G}$ correspond to the radical ideals of $K[\tilde{G}]$, the coordinate ring of $\tilde{G}$. Since $K[\tilde{G}]$ is Noetherian, there exists a minimal, reduced, closed supplement $U$ for $A$ in $\tilde{G}$. Therefore $\tilde{G} = U \cdot A$ and the following commutative diagram arises

\[
\begin{array}{ccccccc}
1 & \to & A & \to & A \times U & \to & U & \to & 1 \\
& & \uparrow_{\cong} & & \uparrow_{\psi} & & \uparrow_{\beta|_U} & & \uparrow_{\beta|_U} \\
1 & \to & A & \to & \tilde{G} & \to & G & \to & 1,
\end{array}
\]

where $\psi$ maps $(a, u)$ to the product $a \cdot u$ in $\tilde{G}$. By Definition 9 we see that $\beta$ is decomposable into $\beta|_U \circ \text{pr}_U$, which is decomposable into $\beta|_U$ and $\text{pr}_U$. Since $U$ is minimal it follows that $\beta|_U$ is a Frattini-epimorphism. □

Frattini ID-embedding problems and split ID-embedding problems are extreme cases in the following sense: Every split ID-embedding problem is solvable by definition. But it is questionable whether the solution is proper. For a Frattini ID-embedding problem the existence of a solution is not obvious. But if a solution exists, it is obvious by definition that this solution is proper (see Proposition 34). For our further proceeding we need the following classification of Frattini-epimorphisms with finite kernel.

Lemma 15. Let $\beta: \tilde{G} \to G$ be an epimorphism in $\text{AffGr}^\text{red}_K$ with finite kernel $A$. Let $\kappa: \tilde{G} \to \tilde{G}/\tilde{G}^o$ be the canonical epimorphism and let $\Phi(\tilde{G}/\tilde{G}^o)$ the Frattini-subgroup of the finite group $\tilde{G}/\tilde{G}^o$. Then the following statements are equivalent:
Next we will apply Lemma 15 to
\[ \mu \]
and therefore \( \mu \) is a Frattini-epimorphism. Since, if \( W \) is a proper subgroup of \( H \) with \( W \cdot (H \cap G^\circ) = H \), then \( G = W \cdot G^\circ \), which is a contradiction to the minimality of \( H \).

By applying Lemma 16 to the epimorphism \( H \to H/H \cap G^\circ \), we see that \( H \cap G^\circ \) lies in the Frattini-subgroup \( \Phi(H) \) of \( H \).

Next we will apply Lemma 15 to \( \mu \). In this case the canonical epimorphism \( \kappa \) is the projection \( G^\circ \times H \to H \). So we have
\[ \ker(\mu) = \{(g^{-1}, g) | g \in H \cap G^\circ\} \]
and therefore \( \kappa(\ker(\mu)) = H \cap G^\circ \leq \Phi(H) \) holds. \( \square \)

**Definition 17.** We call such a Frattini-epimorphism \( \mu : G^\circ \times H \to G^\circ \cdot H \), \( (g, h) \mapsto g \cdot h \), an epimorphism of type \( \mu \).

**Lemma 18.** Let \( \beta : A \times G \to G \) be an epimorphism in \( \text{AffGr}^\text{red}_K \) with reduced, connected kernel \( A \). Then there exists a finite subgroup \( H \) of \( G \), such that \( \beta \) is decomposable into a Frattini-epimorphism \( \mu \) with finite kernel and an \( H \)-split, \( H \)-rigid epimorphism \( \overline{\beta} \) with kernel \( A \).

**Proof.** Let \( H \) be a finite subgroup of \( G \) with the properties of Lemma 16. Therefore \( \mu : G^\circ \times H \to G^\circ \cdot H \), \( (g, h) \mapsto g \cdot h \), is a Frattini-epimorphism. The epimorphism \( \beta(A \times G^\circ) : A \times G^\circ \to G^\circ \) is an \( H \)-epimorphism, since \( H \) acts via conjugation on \( A \) and \( G^\circ \). Therefore, we consider the corresponding \( H \)-rigid, \( H \)-split epimorphism
\[ \overline{\beta} : (A \times G^\circ) \times H \to G^\circ \times G. \]

By defining the epimorphism
\[ \text{id}, \mu : (A \times G^\circ) \times H \to A \times H, \]
\[ ((a, g), h) \mapsto (a, g \cdot h), \]
we obtain the following commutative diagram
\[
\begin{array}{ccc}
A \times G & \xrightarrow{\beta} & G \\
\text{id}, \mu \downarrow & & \downarrow \mu \\
(A \times G^\circ) \times H & \xrightarrow{\overline{\beta}} & G^\circ \times H.
\end{array}
\]

Which proves the claim. \( \square \)
Lemma 19. Let $\beta: \tilde{G} \to G$ be a Frattini-epimorphism in $\text{AffGr}_K^{\text{red}}$ with reduced kernel $A$. Then there exists a finite subgroup $H$ of $\tilde{G}$, such that $\beta$ is decomposable into a Frattini-epimorphism $\mu$ with finite kernel and an $H$-rigid Frattini-epimorphism $\beta$ with kernel $A$.

Proof. Let $H$ be a finite subgroup of $\tilde{G}$ with the properties of Lemma 16. Therefore $\tilde{\mu}: \tilde{G}^o \times H \to \tilde{G}, (g, h) \mapsto g \cdot h$ is a Frattini-epimorphism. Since $\beta(\tilde{G}^o) = G^o$, the group $G^o$ is an $H$-group via $h \cdot \beta(g) := \beta(hgh^{-1})$. So the restriction $\beta|_{\tilde{G}^o}$ is an $H$-epimorphism and induces the $H$-rigid epimorphism $\beta: \tilde{G}^o \times H \to G^o \times H$. By defining the epimorphism $\mu: G^o \times H \to G, (g, h) \mapsto g \cdot \beta(h)$, we obtain the following commutative diagram

\[\begin{array}{ccc}
A^o & \longrightarrow & \tilde{G} \\
\downarrow \beta & & \downarrow \mu \\
A^o \times H & \longrightarrow & G^o \times H.
\end{array}\]

Which proves the claim. \qed

Now we have all parts together to prove a helpful decomposition of epimorphisms.

Proposition 20. Let $\beta: \tilde{G} \to G$ be an epimorphism in $\text{AffGr}_K^{\text{red}}$ with reduced kernel $A$. Then $\beta$ is decomposable into

1. epimorphisms with finite kernel,
2. Frattini-epimorphisms, which are $H$-rigid with respect to a finite group $H$,
3. epimorphisms with reduced, connected, semi-simple, centerless kernel, which are $H$-rigid, $H$-split with respect to a finite group $H$,
4. epimorphisms with reduced, torus kernel, which are $H$-rigid, $H$-split with respect to a finite group $H$,
5. epimorphisms with minimal, reduced, connected, unipotent kernel, which are $H$-rigid, $H$-split with respect to a finite group $H$.

Proof. We prove the proposition in nine steps:

1. The epimorphism $\beta$ is decomposable into an epimorphism with reduced, connected kernel and an epimorphism with finite kernel:
   Since $A^o$ is a reduced, characteristic subgroup of $A$ it is normal in $\tilde{G}$. With Lemma 13 the claim follows.

2. Let $\beta$ be an epimorphism with reduced, connected kernel. Then $\beta$ is decomposable into an epimorphism with reduced, connected, semi-simple kernel and an epimorphism with reduced, connected, solvable kernel:
   The radical $R(A)$ is a reduced, connected, characteristic subgroup of $A$ and hence normal in $\tilde{G}$. By Lemma 13 the claim follows.

3. Let $\beta$ be an epimorphism with reduced, connected, semi-simple kernel $A$. Then $\beta$ is decomposable into an epimorphism with reduced, abelian kernel and an epimorphism with reduced, connected, semi-simple, centerless kernel:
   The center $Z(A)$ is a reduced, characteristic subgroup of $A$ and hence normal in $\tilde{G}$. Again by Lemma 13 the claim follows.

4. Let $\beta$ be an epimorphism with reduced, connected, solvable kernel. Then $\beta$ is decomposable into epimorphisms with reduced, connected, abelian kernel:
   If $A$ is abelian, there is nothing to do. Let $A$ not be abelian, then the dimension of the commutator group $[A, A]$ is strictly smaller than the dimension of $A$. Since $[A, A]$ is a reduced, characteristic subgroup of $A$ it is normal in $\tilde{G}$. With Lemma 13...
\[10\] STEFAN ERNST

\(\beta\) is decomposable into a reduced epimorphism with reduced, connected, abelian kernel and an epimorphism with reduced, connected, solvable kernel \([A, A]\). Iteration of this process yields the result.

5. Let \(\beta\) be an epimorphism with reduced, connected, abelian kernel. Then \(\beta\) is decomposable into an epimorphism with reduced, connected, unipotent kernel and an epimorphism with reduced torus kernel:

\[A, A\]

Iteration of this process yields the result.

6. Let \(\beta\) be an epimorphism with reduced, connected, unipotent kernel. Then \(\beta\) is decomposable into epimorphisms with minimal, reduced, connected, unipotent kernel:

Use induction on the dimension of the kernel together with Lemma 13.

7. Let \(\beta\) be an epimorphism with reduced, connected kernel. Then \(\beta\) is decomposable into a Frattini-epimorphism and a split epimorphism with the same kernel like \(\beta\):

See Lemma 14.

8. Let \(\beta\) be a split epimorphism with reduced, connected kernel. Then \(\beta\) is decomposable into an epimorphism with finite kernel and an \(H\)-rigid, \(H\)-split epimorphism with the same kernel like \(\beta\):

See Lemma 18.

9. Let \(\beta\) be a Frattini-epimorphism with reduced kernel. Then \(\beta\) is decomposable into an epimorphism with finite kernel and an \(H\)-rigid, Frattini-epimorphism with the same kernel like \(\beta\):

See Lemma 19.

\[\square\]

5. H-EFFECTIVE EMBEDDING PROBLEMS

Before we start to solve the ID-embedding problems, we take a look at the Galois theory of the semidirect product \(G \rtimes H\), where \(H\) is a finite group.

**Definition 21.** Let \(F\) be an ID-field and \(E/F\) be an IPV-extension, which is defined by matrices \(D_l \in GL_n(F_l)\) and Galois group \(G(K) \cong Gal_{ID}(E/F)\) (the matrices \(D_l\) were defined in section 2). Then we call \(E/F\) effective, if \(D_l \in G(F_l)\) for all \(l \in \mathbb{N}\).

**Remark 22.** If \(E/F\) is effective, then one can show that the Galois group \(Gal_{ID}(E/F)\) is necessarily connected.

Now we show that for an important class of fields the converse holds.

**Remark 23.** A field \(F\) has cohomological dimension \(\leq 1\) (\(cd(F) \leq 1\)) if \(F\) is the only central division algebra over \(F\). For more information about the cohomological dimension see [Ser97]. Important examples of fields of cohomological dimension \(\leq 1\) are \(K\{t\}, K((t))\) and algebraic extensions, where \(K\) is an algebraically closed field (see [Ser97], II.3.3).  

**Theorem 24.** Let \(F\) be an ID-field with \(cd(F) \leq 1\) and with algebraically closed field of constants \(K\). Let \(H \leq GL_n, K\) be a reduced, connected linear algebraic group. Let \(R/F\) be an IPV-ring and assume that the Galois group \(G(K) \cong Gal_{ID}(R/F)\) is connected. Suppose that the defining matrices \(D_l\) satisfy \(D_l \in H(F_l)\). Then there exist matrices \(C_l \in H(F_l)\) such that \(D_l := C_l D_l C_{l+1}^{-1} \in G(F_l)\).

**Proof.** [Mat01], Theorem 5.9. \[\square\]

**Remark 25.** Let \(L/F\) be a finite IPV-extension. Let \(H\) be a finite group and let \(\overline{\alpha}\colon Gal(L/F) \to H\) be an isomorphism of groups. Since any element in \(Gal(L/F)\)
is an ID-automorphism, the map \( \text{Gal}(L/F) \xrightarrow{\text{res}} \text{Gal}(L_1/F_1) \) is an isomorphism of groups. Then there exists exactly one isomorphism \( \sigma : \text{Gal}(L_1/F_1) \to H \), such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Gal}(L/F) & \xrightarrow{\sigma} & H \\
\text{res} \downarrow & & \downarrow \chi \\
\text{Gal}(L_1/F_1) & \xrightarrow{\sigma} & H
\end{array}
\]

**Proposition 26.** Let \( F \) be an ID-field with field of constants \( K \) and \( \text{cd}(F) \leq 1 \). Let \( G := G^\circ \times H \leq \text{GL}_n(K) \), with regular homomorphic section \( \sigma : H \to G(K) \). Further let \( E/F \) be an IPV-extension, with \( \alpha : \text{Gal}(E/F) \xrightarrow{\cong} G(K) \) and fundamental solution matrices \( \tilde{Y}_l \in \text{GL}_n(E_l) \).

1. Let \( L := E^{\sigma}(K) \). Then \( L/F \) is a finite IPV-extension with Galois group isomorphic to \( H \) via \( \sigma \) and for all \( l \in \mathbb{N} \) there exist elements \( Z_l \in \text{GL}_n(L_1) \) satisfying \( \eta(Z_l) = Z_l C_\eta \) for all \( \eta \in H \cong \text{Gal}(L/F) \), where \( C_\eta = (\sigma \circ \overline{\sigma})(\eta) \).
2. \( E/L \) is an effective IPV-extension with Galois group isomorphic to \( G^\circ \langle K \rangle \) and fundamental solution matrices \( Y_l := Z_l^{-1} \tilde{Y}_l \), such that \( \epsilon(Y_l) = Y_l C_\epsilon \) for all \( \epsilon \in \text{Gal}(E/L) \) and \( \overline{\eta}(Y_l) = C_{\overline{\eta}}^{-1} Y_l C_\eta \) for all \( \eta \in \text{Gal}(L/F) \) (where \( C_\epsilon = \alpha(\epsilon) \) and \( \overline{\eta} := (\alpha^{-1} \circ \sigma \circ \overline{\sigma})(\eta) \)).

**Proof.** (1) Since \( \tilde{Y}_l \in \text{GL}_n(E_l) \) are fundamental solution matrices, we have that \( \gamma(Y_l) = Y_l C_\gamma \) for all \( \gamma \in \text{Gal}(E/F) \). By Hilbert's Theorem 90 ([Ser97], III.1, Lemma 1), there exists an element \( Z_0 \in \text{GL}_n(L) \) with \( \eta(Z_0) = Z_0 C_\eta \) for all \( \eta \in \text{Gal}(L/F) \). For any \( \eta \in \text{Gal}(L/F(Z_0)) \) the equation \( (\sigma \circ \overline{\sigma})(\eta) = C_\eta = Z_0^{-1} \eta(Z_0) = 1 \) holds and so \( \eta = \text{id}_L \), hence \( L = F(Z_0) \). Again by Hilbert's Theorem 90, there exists an element \( Z_1 \in \text{GL}_n(L_1) \) such that \( \eta(Z_1) = Z_1 C_\eta \) for all \( \eta \in \text{Gal}(L_1/F_1) \cong \text{Gal}(L/F) \). Iteration of this process yields the result.

(2) By defining \( Y_l := Z_l^{-1} \tilde{Y}_l \), we get that \( E = L(Y_0) \) and \( E/L \) is effective, because \( \text{cd}(L) \leq 1 \) (see Theorem 24). Let \( D_l \) be matrices which define the IPV-extension \( E/F \), i.e. \( \tilde{Y}_l = D_l \tilde{Y}_l \). Therefore \( Y_{l+1} = Z_{l+1}^{-1} \tilde{Y}_{l+1} = Z_{l+1}^{-1} D_l^{-1} \tilde{Y}_l = Z_{l+1}^{-1} D_l^{-1} Z_l Y_l \) holds and \( D_l := Z_{l+1}^{-1} D_l^{-1} Z_l \) are matrices associated to the fundamental solution matrices \( Y_l \). Further we have \( \epsilon(Y_l) = Z_l^{-1} \epsilon(Y_l) = Z^{-1} Y_l C_\epsilon = Y_l C_\epsilon \) for all \( \epsilon \in \text{Gal}(E/L) \) and \( \overline{\eta}(Y_l) = \eta(Z_l^{-1}) \overline{\eta}(Y_l) = C_{\overline{\eta}}^{-1} Z_l^{-1} \tilde{Y}_l C_\eta = C_{\eta}^{-1} Y_l C_\eta \) for all \( \eta \in \text{Gal}(L/F) \). \( \square \)

**Definition 27.** Let \( H \) be a reduced linear algebraic group. Let \( L/F \) be an IPV-extension and let \( \sigma : \text{Gal}(L/F) \to H(K) \) be an isomorphism of groups. Let \( G := G^\circ \times H \leq \text{GL}_n(K) \), with regular homomorphic section \( \sigma : H \to G(K) \). We call matrices \( D_l \in G^\circ(L_l) \) \( H \)-equivariant (via \( \alpha \)) if for all \( \eta \in H \)

\[
\eta(D_l) = \chi(\eta)^{-1} D_l \chi(\eta)
\]

holds for all \( l \in \mathbb{N} \). Here the action of \( \eta \) on the left-hand side is the (coefficient-wise) Galois action, while on the right-hand side \( \eta \) via the monomorphism \( \chi := \sigma \circ \overline{\sigma} : \text{Gal}(L/F) \to H(K) \to (G^\circ \times H)(K) \), constructed as an element of \( (G \times H)(K) \), which acts via conjugation on \( G(F_l) \).

**Theorem 28.** Let \( H \) be a finite group and let \( G := G^\circ \times H \leq \text{GL}_n(K) \), with regular homomorphic section \( \sigma : H \to G(K) \). Further, let \( F \) be an ID-field with field of constants \( K \) and \( \text{cd}(F) \leq 1 \).
(1) Let $L/F$ be a finite IPV-extension with Galois group isomorphic to $H$ via $\sigma$. Let
\[ \chi := \sigma \circ \tau : \text{Gal}_D(L/F) \to \sigma(H) \leq \text{GL}_n(K), \quad \eta \mapsto C_\eta \]
be the composite isomorphism. Then for all $l \in \mathbb{N}$ there exist elements $Z_l \in \text{GL}_n(L_l)$ satisfying $\eta(Z_l) = Z_l C_\eta$ for all $\eta \in H \cong \text{Gal}_D(L/F)$ and $C_\eta \in \text{GL}_n(F_1)$ such that $Z_{l+1} = C_1^{-1}Z_l$. Moreover, $L = F(Z)$ with $Z := Z_0$. In other words, $Z$ is a fundamental solution matrix for the extension $L/F$ on which the Galois group $\text{Gal}_D(L/F)$ acts via $\tau$.

(2) Let $E/L$ be an IPV-extension with Galois group isomorphic to $G^\circ(K)$ via an isomorphism
\[ \alpha_L : \text{Gal}_D(E/L) \to G^\circ(K) \cong G(K), \quad \epsilon \mapsto C_\epsilon. \]
Then there exist elements $Y_l \in G(E_l)$ satisfying $\epsilon(Y_l) = Y_l C_\epsilon$ for all $\epsilon \in \text{Gal}_D(E/L)$ and $D_l \in G^\circ(L_l)$ such that $Y_{l+1} = D_l^{-1} Y_l$. Moreover $E = L(Y)$, with $Y := Y_0$. In other words, $Y$ is a fundamental solution matrix for the extension $E/L$ on which the Galois group $\text{Gal}_D(E/L)$ acts via $\alpha_L$.

(3) Suppose in addition that $D_l$ are $H$-equivariant via $\tau$: $\eta(D_l) = C_1^{-1}D_l C_\eta$ for all $l \in \mathbb{N}, \eta \in H$.

Then $E/F$ is an IPV-extension with Galois group isomorphic to $G(K)$ and $\tilde{Y} := \text{ZY}$ is a fundamental solution matrix of this extension which satisfies $\tilde{Y}_{l+1} = \tilde{D}_l^{-1} \tilde{Y}_l \in \text{GL}_n(E_l)$ for all $l \in \mathbb{N}$, where $\tilde{D}_l := Z_l D_l Z_l^{-1} \in \text{GL}_n(F_1)$.

Further, the isomorphism $\alpha_L$ of part 2. can be extended to an isomorphism
\[ \alpha : \text{Gal}_D(E/F) \to G(K) \quad \text{with} \quad \text{res} \circ \alpha = \beta \circ \alpha. \]

Proof. [Mat01], Theorem 8.2. \hfill \Box

Remark 29. With assumptions as in Theorem 28 we get the following ID-embedding problem
\[
\begin{array}{ccc}
\text{Gal}_D(E/F) & \xrightarrow{\text{res}} & \text{Gal}_D(L/F) \\
\downarrow{\alpha} & & \downarrow{\cong} \\
G^\circ(K) & \xrightarrow{\chi} & G^\circ(K) \rtimes H \\
\end{array}
\]
One of the key steps in the proof of Theorem 28 is that each $\gamma \in \text{Gal}_D(E/F)$ can be decomposed into $\gamma = \epsilon \circ \eta$, where $\epsilon := \gamma|_{\text{Gal}_D(E/L)}$ and $\eta := \tilde{\alpha}^{-1} \circ \chi \circ \text{res}_L(\gamma)$. In particular, every element of $\text{Gal}_D(L/F)$ can be extended (via the $H$-equivariance) to an element of $\text{Gal}_D(E/F)$ (see proof of [Mat01], Theorem 8.2).

Remark 30. With notations as in Theorem 28 $H$ acts on $G^\circ(L)$ via
\[ \eta * g_l := \chi(\eta)\eta(g_l)\chi(\eta)^{-1}, \quad g_l \in G^\circ(L_l), \eta \in \text{Gal}_D(L/F). \]
Therefore the $H$-equivariance condition may be reformulated as an invariance condition:
\[ g_l = \eta * g_l \quad \text{for all} \quad \eta \in \text{Gal}_D(L/F) \quad (g_l \in G^\circ(L_l)). \]
The homomorphism $\chi$ defines an element $\chi$ in $H^1(\text{Gal}_D(L/F), G^\circ(L) \rtimes H)$. Then there is a canonical map $G^\circ(L_l) \rtimes H \to \text{Aut}(G^\circ(L_l) \rtimes H), g_l \mapsto (h \mapsto g_l h g_l^{-1})$. The induced map on cohomology maps $\chi$ to an element
\[ \text{Int}(\chi) \in H^1(\text{Gal}_D(L/F), \text{Aut}_L(G \rtimes H)). \] Any automorphism of $G^\circ(L) \rtimes H$ stabilizes the connected component, i.e., we obtain an element in $H^1(\text{Gal}_D(L/F), \text{Aut}_L(G^\circ))$,
which is again denoted by $\chi$. We may also define a twisted action as above on the coordinate ring $L[\mathcal{G}]$ by

$$(\eta \ast q)(g_t) = \eta(q)(\chi(\eta)^{-1}g_t\chi(\eta)), \quad q \in L[\mathcal{G}], g_t \in \mathcal{G}(L_t),$$

where $\eta(q)$ denotes the Galois action on the coefficients of $q$. Note that this $\ast$-action is semilinear and thus defines an $L/F$-form $\mathcal{G}_0^\circ$ of $\mathcal{G}$, on which the $\ast$-action is the Galois action (see also [Spr98], 12.3.7).

By Remark 25 we can analogous define an $L/F_1$-form $\mathcal{G}_X$ of $\mathcal{G}$ (where $\chi_1 = \sigma \circ \overline{\eta}$). But since $\text{Gal}(L_1/F_1)$ and $\text{Gal}(L/F)$ are natural isomorphic, in the following we write the form $\mathcal{G}_X$ in place of the family of forms $(\mathcal{G}_X^\circ)_H$.

Now we can extend the concept of effectivity to the semidirect product $\mathcal{G} \rtimes H$.

**Definition 31.** Let $H$ be a finite group and let $\mathcal{G} := \mathcal{G} \rtimes H \leq \text{GL}_{n,K}$. Let $E/F$ be an IPV-extension and let $L$ be the algebraic closure of $F$ in $E$. The monomorphism $\alpha : \text{Gal}(E/F) \to \mathcal{G}(K)$ is called $H$-effective, if the following conditions hold:

1. There exist matrices $Z_t \in \text{GL}_n(L_t)$, such that $\iota_{Z_0} : \text{Gal}(L/F) \to H, \eta \mapsto C_\eta := Z_0\eta(Z_0)^{-1}$ is an isomorphism.
2. There exist matrices $D_t \in \mathcal{G}(L_t)$, which are $H$-equivariant via $\iota_{Z_0}$, with fundamental solution matrices $Y_t \in \mathcal{G}(E_t)$.
3. For $\tilde{Y}_t := Z_tY_t \in \text{GL}_n(E_t)$, $\alpha = \iota_{\tilde{Y}_0}$ holds.

For fields of cohomological dimension $\leq 1$ we get an analogous result as for connected groups.

**Theorem 32.** Let $F$ be an ID-field with $\text{cd}(F) \leq 1$ and with algebraically closed field of constants $K$. Let $E/F$ be an IPV-extension. Let $H$ be a finite group and let $\mathcal{G} := \mathcal{G} \rtimes H \leq \text{GL}_{n,K}$. Then every isomorphism $\alpha : \text{Gal}(E/F) \to \mathcal{G}(K)$ is $H$-effective.

**Proof.** The first part of Definition 31 follows by Proposition 26 (1). By Proposition 26 (2) we have $\tilde{Y}_t(Y_t) = C_\eta^{-1} Y_t C_\eta$ for all $\eta \in \text{Gal}(L/F)$ and $Y_{t+1} = D_\eta^{-1} Y_t$. Hence the desired property $\eta(D_t) = \eta(Y_t)\eta(Y_{t+1})^{-1} = C_\eta^{-1} D_\eta C_\eta$ holds. Finally, by the inclusion

$$(\mathcal{G} \rtimes H)(K) \xrightarrow{\alpha^{-1}} \text{Gal}(E/F) \xrightarrow{\iota_{\tilde{Y}_0}} \text{GL}_n(K),$$

we can assume that $\iota_{\tilde{Y}_0} = \alpha$ holds.

**Definition 33.** With notation as in Definition 8 we call an ID-embedding problem $\mathcal{E}(\alpha, \beta)$ $H$-effective, if $\alpha$ is $H$-effective. Further we say the solution is $H$-effective, if $\tilde{\alpha}$ is $H$-effective.

### 6. Frattini-Embedding Problems

**Proposition 34.** Let $F$ be an ID-field with field of constants $K$. Then every solution of a Frattini ID-embedding problem in $\text{AffGr}_K^{\text{red}}$ is proper.

**Proof.** [Mat01], Proposition 5.15.

**Theorem 35.** Let $F$ be an ID-field with $\text{cd}(F) \leq 1$. Then every $H$-rigid Frattini ID-embedding problem in $\text{AffGr}_K^{\text{red}}$ over $F$ has an $H$-effective, proper solution.
Proof. By Theorem \[\text{32}\] we can assume that the ID-embedding problem is $H$-effective. Let

$$\begin{array}{c}
\text{Gal}_\text{ID}(E/F) \\
\downarrow^{\alpha}
\end{array} \xrightarrow{\text{iso}} \begin{array}{c}
\text{Gal}_\text{ID}(L/F)
\end{array}$$

be an $H$-rigid, $H$-effective Frattini ID-embedding problem, with regular homomorphic section $\sigma: H \to G^\circ(K) \rtimes H$. For the semidirect product $G^\circ(K) \rtimes H$ we use the notation as in Theorem \[\text{28}\]. Since $E/F$ is $H$-effective, the IPV-extension $E/L$ is defined by matrices $D_l \in G^\circ(L_1)$, where $L := E^\circ(K)$. Let $\chi := \sigma \circ \chi : \text{Gal}_\text{ID}(L/F) \to H \to G^\circ(K) \rtimes H$, then by Remark \[\text{30}\] there exists an $L$-form $G^\circ$ of $G^\circ$ defined over $F$ such that the action given by

$$\eta \ast A_l = \chi(\eta)\eta(A_l)\chi(\eta)^{-1}, \quad A_l \in G^\circ(L_1), \eta \in \text{Gal}_\text{ID}(L/F)$$

is the Galois action on $G^\circ(L)$. We may view the $F_1$-rational points of $G^\circ$ as lying inside $G^\circ(L_1)$, invariant under the action described above. In this formulation, $D_l$ satisfies the equivariance condition if and only if $D_l \in G^\circ(L_1)$ (see again Remark \[\text{30}\]).

We also can consider the map $\tilde{\chi} := \tilde{\sigma} \circ \tilde{\chi} : \text{Gal}_\text{ID}(L/F) \to H \to \tilde{G}^\circ(K) \rtimes H$, where $\tilde{\sigma}: H \to \tilde{G}^\circ(K) \rtimes H$ is a regular homomorphic section and $\tilde{\beta}|_H = \text{id}_H$. Therefore there exists an $L$-form $\tilde{G}^\circ$ of $\tilde{G}^\circ$ defined over $F$ with the same $*$-action as above. The epimorphism $\beta: \tilde{G}^\circ(K) \rtimes H \to G^\circ(K) \rtimes H$ induces an epimorphism of $L$-forms $\tilde{\beta}: \tilde{G}^\circ \to G^\circ$, because $\tilde{\beta} \circ \tilde{\chi} = \chi \circ \beta$ and $\beta|_H = \text{id}_H$. Given that $D_l$ satisfies the equivariance condition we have $D_l \in G^\circ(F_1)$. Choose preimages $\tilde{D}_l \in \tilde{G}^\circ(F_1)$. Let $\tilde{E}/L$ be an IPV-extension defined by the matrices $\tilde{D}_l$, then by \[\text{Mat01}, \text{Theorem 5.12}\], $\tilde{E} \geq E$ up to an ID-isomorphism. Further, by \[\text{Mat01}, \text{Theorem 5.1}\], there exists a monomorphism $\tilde{\alpha}: \text{Gal}_\text{ID}(\tilde{E}/L) \to \tilde{G}^\circ(K)$ which is a solution of the corresponding connected ID-embedding problem $\tilde{E}(\tilde{\beta}^\circ, \tilde{\alpha}^\circ)$, where $\tilde{\beta}^\circ := \beta|_{\tilde{G}^\circ}$ and $\tilde{\alpha}^\circ := \alpha|_{\text{Gal}_\text{ID}(E/L)}$. By Proposition \[\text{34}\] $\tilde{\alpha}^\circ$ is proper, so $\text{Gal}_\text{ID}(\tilde{E}/L) \cong \tilde{G}^\circ(K)$. Since $\tilde{D}_l \in \tilde{G}^\circ(F_1)$ we have that $\tilde{D}_l$ are $H$-equivariant. Hence by Theorem \[\text{28}\], $\tilde{E}/F$ is an IPV-extension with monomorphism $\tilde{\alpha}: \text{Gal}_\text{ID}(\tilde{E}/F) \to \tilde{G}^\circ(K) \rtimes H$, and $\tilde{\alpha}|_{\text{Gal}_\text{ID}(E/F)} = \alpha$. Since $\tilde{\alpha}^\circ$ is an isomorphism, the map $\tilde{\alpha}$ is also an isomorphism and therefore a $H$-effective, proper solution of the initial embedding problem. \[\square\]

7. ID-Embedding Problems with Finite Kernel

By an easy calculation we obtain the following useful lemma.

**Lemma 36.** Let $E/F$, $\tilde{E}/F$ and $E \cap \tilde{E}/F$ be IPV-extensions. Then the canonical map $\text{Gal}_\text{ID}(E \cap \tilde{E}/F) \to \text{Gal}_\text{ID}(E/F) \times_{\text{Gal}_\text{ID}(E \cap E/F)} \text{Gal}_\text{ID}(\tilde{E}/F)$ is an isomorphism of groups. Here the product on the right hand side is the fibre product.

**Theorem 37.** Let $F$ be an algebraic function field in one variable over $K$. Then every ID-embedding problem in $\text{AffGr}_{K}$ over $F$ with finite kernel has a proper solution.

**Proof.** Let the following ID-embedding problem with finite kernel be given:

$$\begin{array}{c}
1 \xrightarrow{\alpha} A(K) \xrightarrow{\beta} \tilde{G}(K) \xrightarrow{\beta} G(K) \xrightarrow{\beta} 1.
\end{array}$$
If $\tilde{G}$ is connected, this ID-embedding problem is a Frattini ID-embedding problem by Lemma 15. Moreover this embedding problem is $H$-rigid where in this situation the group $H$ is trivial. Otherwise if $\tilde{G}$ is non-connected, $\beta$ is decomposable into a split epimorphism and a Frattini-epimorphism (see Lemma 14). Further by Lemma 19 this Frattini-epimorphism is decomposable into a Frattini-epimorphism of type $\mu$ (see Definition 17) and an $H$-rigid Frattini-epimorphism. So we have to consider three cases:

1. $\beta$ is an $H$-rigid Frattini-epimorphism:
   
   Since $\text{cd}(F) \leq 1$ these ID-embedding problems have an $H$-effective, proper solution by Proposition 35.

2. $\beta$ is a Frattini-epimorphism of type $\mu$:
   
   Let
   
   \[
   \begin{align*}
   \text{Gal}_{\text{ID}}(E/F) & \cong \alpha \\
   1 & \longrightarrow G^\circ(K) \cap H \longrightarrow G^\circ(K) \times H \longrightarrow G^\circ(K) \cdot H \longrightarrow 1
   \end{align*}
   \]

   be an ID-embedding problem of type $\mu$. Since $G^\circ$ is a normal subgroup of $G^\circ \cdot H$, there exists an IPV-extension $L/F$, such that $\text{Gal}_{\text{ID}}(E/L) \cong G^\circ(K)$. Therefore, $\text{Gal}_{\text{ID}}(L/F) \cong H/(H \cap G^\circ(K))$ and this leads to a finite ID-embedding problem of the form

   \[
   \begin{align*}
   \text{Gal}_{\text{ID}}(\tilde{L}/F) & \overset{\text{res}}{\longrightarrow} \text{Gal}_{\text{ID}}(L/F) \\
   1 & \longrightarrow G^\circ(K) \cap H \longrightarrow H \longrightarrow H/(H \cap G^\circ(K)) \longrightarrow 1
   \end{align*}
   \]

   Since $\pi_{1,\text{alg}}(F)$ is free ([MM99], Corollary V.2.11), this embedding problem has a proper solution $\tilde{\beta}$: $\text{Gal}_{\text{ID}}(\tilde{L}/F) \cong H$, where without loss of generality it can be assumed that $\tilde{L} \cap E = L$. We take the composite $\tilde{E} := E \cdot \tilde{L}$, and $\tilde{E}/F$ is an IPV-extension. By Lemma 36 we see that $\text{Gal}_{\text{ID}}(\tilde{E}/F) \cong \text{Gal}_{\text{ID}}(E/F) \times_{\text{Gal}_{\text{ID}}(L/F)} \text{Gal}_{\text{ID}}(\tilde{L}/F)$ and hence

   \[
   \begin{align*}
   \text{Gal}_{\text{ID}}(\tilde{E}/F) & \cong \text{Gal}_{\text{ID}}(E/F) \times_{\text{Gal}_{\text{ID}}(L/F)} \text{Gal}_{\text{ID}}(\tilde{L}/F) \\
   & \cong G^\circ(K) \cdot H \times_{H/(H \cap G^\circ(K))} H \cong G^\circ(K) \times H
   \end{align*}
   \]

   is a proper solution of the initial ID-embedding problem.

3. $\beta$ is a split epimorphism:

   Thanks to case 2, we can assume that $G \cong G^\circ \times H$ with finite group $H$. Therefore we have to solve the following split ID-embedding problem:

   \[
   (*) \quad \begin{align*}
   \text{Gal}_{\text{ID}}(E/F) & \cong \alpha \\
   1 & \longrightarrow A \longrightarrow A \times (G^\circ \times H)(K) \longrightarrow (G^\circ \times H)(K) \longrightarrow 1
   \end{align*}
   \]

   Since $A$ is a finite group, $G^\circ$ acts trivially on $A$. So $H$ acts on $A$ and this induces the following finite, split ID-embedding problem:

   \[
   \begin{align*}
   \text{Gal}_{\text{ID}}(L/F) & \cong \alpha \\
   1 & \longrightarrow A \longrightarrow A \times H \longrightarrow H \longrightarrow 1
   \end{align*}
   \]
Since \( \pi^\text{alg}_1(F) \) is free (see [MIM99], Corollary V.2.11). Therefore, we can take the composite \( \hat{E} := E \cdot \hat{L} \), and

\[
\text{Gal}_{\text{ID}}(\hat{E}/F) \cong G^\circ(K) \rtimes (A \rtimes H) \cong (G^\circ(K) \times A) \rtimes H \cong (A \times G^\circ(K)) \rtimes H
\]

is a proper solution of the ID-embedding problem \((\ast)\). \( \square \)

8. ID-Embedding Problems with Unipotent Kernel

The next theorem is an extension of Theorem 28.

**Theorem 38.** Let \( F \) be an ID-field with field of constants \( K \) and \( \text{cd}(F) \leq 1 \). Let \( H \) be a finite group, \( G^\circ \) an \( H \)-group with semidirect product \( G := G^\circ \rtimes H \in \text{AffGr}_F^\text{red} \) and let \( U \in \text{AffGr}_F^\text{red} \) be a connected group with semidirect product \( \hat{G} := U \rtimes G \in \text{AffGr}_K^\text{red} \).

1. Let \( E/F \) be an IPV-extension with isomorphism \( \alpha : \text{Gal}_{\text{ID}}(E/F) \cong G(K) \), \( \gamma \mapsto C_\gamma \) and let \( L := E^{G^\circ(K)} \). Then for all \( l \in \mathbb{N} \) there exist matrices \( Z_l \in \text{GL}_n(L_l) \) and \( Y_l \in G^\circ(E_l) \) such that \( Y_l = Z_l Y_l \) satisfying \( E = F(Y_l) \) and \( \gamma(Y_l) = \tilde{Y}_l C_\gamma \) for all \( \gamma \in \text{Gal}_{\text{ID}}(E/F) \).
2. Let \( U_l \in \mathcal{U}(E_l) \) be \( G^\circ \rtimes H \)-equivariant via \( \alpha \), i.e. \( \gamma(U_l) = C_\gamma^{-1} U_l C_\gamma \) for all \( \gamma \in \text{Gal}(E/F) \). Then \( U_l \) define an IPV-extension \( \tilde{E}/E \) with fundamental solution matrices \( X_l \in \mathcal{U}(\tilde{E}_l) \) and monomorphism \( \tilde{\alpha}_E : \text{Gal}_{\text{ID}}(\tilde{E}/E) \to \mathcal{U}(K) \).
3. Further \( \tilde{E}/F \) is an IPV-extension with fundamental solution matrices \( \tilde{X}_l = \tilde{Y}_l X_l \) and monomorphism \( \tilde{\alpha} : \text{Gal}_{\text{ID}}(\tilde{E}/F) \to \hat{G}(K) \), such that \( \tilde{\alpha}|_{\text{Gal}(E/F)} = \alpha \) and \( \tilde{\alpha}|_{\text{Gal}(\tilde{E}/\tilde{E})} = \hat{\alpha}_E \).

*Proof.* With notation as in Theorem 28 we consider the following ID-embedding problem:

\[
\begin{array}{cccccc}
\text{Gal}_{\text{ID}}(\tilde{E}/F) & \overset{\text{res}_{\tilde{E}/E}}{\longrightarrow} & \text{Gal}_{\text{ID}}(E/F) & \overset{\text{res}_{E/F}}{\longrightarrow} & \text{Gal}_{\text{ID}}(L/F) \\
\overset{\bar{\gamma}}{\longrightarrow} & \overset{\bar{\beta}}{\longrightarrow} & \overset{\bar{\alpha}}{\longrightarrow} & \text{Gal}_{\text{ID}}(\mathcal{U}(K) \rtimes (G^\circ(K) \rtimes H)) \overset{\sigma}{\longrightarrow} & \text{Gal}_{\text{ID}}(\mathcal{U}(K) \rtimes H) & \overset{\sigma}{\longrightarrow} & \mathcal{U}(K) \\
1 & \overset{\sim}{\longrightarrow} & \mathcal{U}(K) \rtimes (G^\circ(K) \rtimes H) & \overset{\sim}{\longrightarrow} & \mathcal{U}(K) \rtimes H & \overset{\sim}{\longrightarrow} & 1.
\end{array}
\]

Remark that the upper row in this diagram is not an exact sequence.

1. The first claim follows by Theorem 28. For the second claim we decompose \( \gamma \in \text{Gal}(E/F) \) into \( \epsilon \circ \eta \), as in Remark 29. Then \( \gamma(Y_l) = \eta(Z_l) \epsilon(\eta(Y_l)) = Z_l C_\eta \epsilon(C_\eta^{-1} Y_l C_\eta) = Z_l C_\epsilon Y_l C_\eta = \tilde{Y}_l C_\gamma \).
2. Since \( U_l \in \mathcal{U}(E_l) \), the IPV-extension \( \tilde{E}/E \) has fundamental solution matrices \( X_l \in \mathcal{U}(\tilde{E}_l) \) with \( X_{l+1} = U_l X_l \) and hence we get a monomorphism \( \tilde{\alpha}_E : \text{Gal}_{\text{ID}}(\tilde{E}/E) \to \mathcal{U}(K) \).
3. With notation as in Theorem 28 we have that \( Y_{l+1} = D_l^{-1} Y_l \). Let \( \tilde{U}_l := Y_l U_l Y_{l+1}^{-1} \), then \( \gamma(\tilde{U}_l) = (Y_l U_l Y_{l+1}^{-1}) Y_l C_\gamma C_\gamma^{-1} U_l C_\gamma C_\gamma^{-1} Y_{l+1}^{-1} = \tilde{U}_l \) for all \( \gamma \in \text{Gal}(E/L) \) and therefore \( \tilde{U}_l \in \tilde{G}(L_l) \). A quick calculation shows that \( \tilde{X}_l := Y_l X_l \in (\mathcal{U}(\tilde{G})(\tilde{E}_l)) \) are fundamental solution matrices for \( \tilde{U}_l \). Given that \( \beta(\tilde{U}_l) = D_l \), we get by [Mat01], Theorem 5.12, that \( \tilde{E} \geq L(\tilde{X}) \geq E \) up to an ID-isomorphism, where \( \tilde{X} := \tilde{X}_0 \).

Since \( X_l = Y_l^{-1} \tilde{X}_l \), we have \( L(\tilde{X}) = E(\tilde{X}) = \tilde{E} \).
Let \( \bar{\eta} := \alpha^{-1} \circ \chi(\eta) \in \Gal_{\ID}(E/F) \) be a preimage of \( \eta \), then for all \( \eta \in \Gal_{\ID}(L/F) \) the following equation holds
\[
\eta(\bar{U}_i) = \bar{\eta}(\bar{U}_i) = \bar{\eta}(Y_i u_i Y_{i+1}^{-1}) = C_{\eta}^{-1} Y_i C_{\eta} C_{\gamma}^{-1} U_i C_{\eta} C_{\gamma}^{-1} Y_{i+1}^{-1} C_{\eta} = C_{\eta}^{-1} Y_i U_i Y_{i+1}^{-1} C_{\eta} = C_{\eta}^{-1} \bar{U}_i C_{\eta}.
\]
That is \( \bar{U}_i \) are \( H \)-equivariant and so the claim follows by Theorem 28. \( \square \)

**Lemma 39.** With assumptions as in Theorem 38 we have: The matrices \( U_i \in \Gal(E/F) \)-equivariant if and only if \( U_i \in \bar{Y}_i^{-1} \Upsilon_{\chi}(F_i) \bar{Y}_i \).

Proof. Let \( U_i \in \Upsilon(E_i) \) be \( \Gal(E/F) \)-equivariant and let \( \bar{U}_i := \bar{Y}_i U_i \bar{Y}_i^{-1} \in \Upsilon(E_i) \).

By the notations from Theorem 38 (i), we have that \( \bar{Y}_i = Z_i^{-1} Y_i \) and for all \( \gamma = \epsilon \circ \eta \in \Gal(E/F) \) the following equation holds:
\[
\gamma(\bar{U}_i) = \gamma(Z_i^{-1} Y_i U_i Y_{i+1}^{-1} Z_i) = \eta(Z_i^{-1} Y_i C_{\gamma}^{-1} U_i C_{\gamma}^{-1} Y_{i+1}^{-1} \eta(Z_i)) = C_{\eta}^{-1} Z_i^{-1} Y_i U_i Y_{i+1}^{-1} Z_i C_{\eta} = C_{\eta}^{-1} \bar{U}_i C_{\eta} = \eta(\bar{U}_i) \in \Upsilon(L_i).
\]
Therefore the matrices \( U_i \) are \( \Gal(E/F) \)-equivariant if and only if \( \bar{U}_i \in \Upsilon_{\chi}(F_i) \).

By [Roe07], Theorem 9.11, every “connected” ID-embedding problems with unipotent kernel has a proper solution. With the work we have done above, we can extend this theorem to the case of \( H \)-rigid ID-embedding problems. The proof is up to some small details the same as in [Roe07].

**Theorem 40.** Let \( F \) be an ID-field. Then every \( H \)-rigid, \( H \)-split, \( H \)-effective ID-embedding problem in \( \AffGr^\res_K \) over \( F \) with minimal, reduced, connected, unipotent kernel has an \( H \)-effective, proper solution.

Proof. Let
\[
\begin{array}{ccc}
\Gal_{\ID}(\bar{E}/F) \xrightarrow{\res} & \Gal_{\ID}(E/F)^\res \\
\bar{\sigma} \downarrow & \downarrow \cong \\
\Upsilon(K) \xrightarrow{\Upsilon(K) \times (\mathcal{G}^\varnothing(K) \times H)} \mathcal{G}^\varnothing(K) \times H \xrightarrow{\bar{\sigma}} 1.
\end{array}
\]
be an \( H \)-rigid, \( H \)-split, \( H \)-effective ID-embedding problem over \( F \) with minimal, reduced, connected, unipotent kernel. We use the notation as in Theorem 38.

The group \( \mathcal{G}^\varnothing \times H \) acts on \( \Upsilon \) and the center \( Z(\Upsilon) \) is a non-trivial, characteristic subgroup of \( \Upsilon \), so it is \( \mathcal{G}^\varnothing \times H \)-invariant and a normal subgroup of \( \Upsilon \). Since \( \Upsilon \) is unipotent, \( Z(\Upsilon) \) is nontrivial and therefore by minimality of \( \Upsilon \), we get \( Z(\Upsilon) = \Upsilon \), i.e. \( \Upsilon \) is abelian.

Further if \( \mathcal{A} \) is a non-connected, \( \mathcal{G}^\varnothing \times H \)-invariant, normal subgroup of \( \Upsilon \), then \( \mathcal{A} \) is finite, because its identity component \( \mathcal{A}^0 \) is also \( \mathcal{G}^\varnothing \times H \)-invariant and normal in \( \Upsilon \) and hence trivial by minimality of \( \Upsilon \).

By Theorem 38, every sequence \( U_i \in \Upsilon(E_i)^{\mathcal{G}^\varnothing \times H} := \{ U_i \in \Upsilon(E_i)| U_i \text{ are } \mathcal{G}^\varnothing \times H\text{-equivariant}\} \in \mathbb{N} \) defines an IPV-extension \( \bar{E}/E \) with \( \Gal_{\ID}(\bar{E}/E) \leq \Upsilon(K) \leq (\mathcal{G}^\varnothing(K) \times H) \). Since \( \Upsilon \) is minimal, we obtain that \( \Gal_{\ID}(\bar{E}/E) \) is finite or \( \Gal_{\ID}(\bar{E}/E) = \Upsilon(K) \). So we have to show that there exists a sequence \( U_i \in \Upsilon(E_i)^{\mathcal{G}^\varnothing \times H} \) such that \( \bar{E}/E \) is not finite.

By Lemma 39, every sequence \( \bar{U}_i \in \Upsilon_{\chi}(F_i) \) define fundamental solution matrices \( \bar{X}_i \) with \( \bar{X}_{i+1} = \bar{U}_i \bar{X}_i^{-1} \). Two such sequences \( (\bar{U}_i)_{i \in \mathbb{N}}, (\bar{U}_i')_{i \in \mathbb{N}} \) have fundamental solution matrices \( \bar{X}_i, \bar{X}_i' \) which are ID-isomorphic over \( F \) if and only if
Thus for dimensional reasons, there exists a sequence $\mathcal{U}(F_{i+1})$. Therefore we have a one-to-one correspondence between ID-isomorphism classes of those fundamental solution matrices and the infinite product

$\lim_{i \to \infty} (\mathcal{U}_K(F_i))/\mathcal{U}_K(F_i)) = \prod_{i \in \mathbb{N}} \mathcal{U}_K(F_i)/\mathcal{U}_K(F_{i+1}).$

Since $\mathcal{U} \cong \mathcal{G}^m_n$ and $H^1(\text{Gal}_{ID}(L/F), \mathcal{G}^m_n) = 1$ (see [Spr98], Example 12.3.5), by [Spr98], Proposition 12.3.2, we obtain that $\mathcal{U}_K(F_i) \cong \mathcal{U}(F_i)$ as $F_i$-vector spaces. Thus $\mathcal{U}_K(F_i)/\mathcal{U}_K(F_{i+1})$ is a $K$-vector space with $\dim_K(\mathcal{U}_K(F_i)/\mathcal{U}_K(F_{i+1})) \geq \dim_K(F_i/F_{i+1}) \geq 2$. Hence the dimension of the infinite product as $K$-vector space is uncountable ($\geq 2^{\aleph_0}$).

Those fundamental solution matrices whose IPV-extension is finite are given by maximal ideals in the ring $E[\tilde{X}_{ij}, \det(\tilde{X})^{-1}]$. Every maximal ideal is given by $n^2$ polynomials, thus the $E$-vector space $V$ of $n^2$-tuples of polynomials gives an upper bound to the number of those fundamental solution matrices with finite IPV-extension. But since $V$ is an $E$-vector space of countable dimension and $E$ is a $K$-vector space of countable dimension, $V$ is a $K$-vector space of countable dimension.

Thus for dimensional reasons, there exists a sequence $U_i \in \mathcal{U}(E_i)$ with IPV-extension $E_i$, such that $\text{Gal}_{ID}(E_i/F_i) \cong \mathcal{U}(K) \times \mathcal{G}^\circ(K) \times H$ and $E_i/F$ is $H$-effective by construction.

9. The Embedding Theorem

The next lemma is taken from [Obe03].

**Lemma 41.** Let $H$ be a finite group and $\tilde{G} := \mathcal{G}^\circ \times H$, $\mathcal{G} := \mathcal{G}^\circ \times H \in \text{AffGr}^\text{red}_{K}$, with $H$-rigid, $H$-split epimorphism $\beta: \mathcal{G}^\circ \times H \to \mathcal{G}^\circ \times H$.

1. If $\ker(\beta)$ is a reduced torus $T$, then $\beta$ is subdirect $H$-split, i.e., $\tilde{G}^\circ \cong T \times \mathcal{G}^\circ$.
2. If $\ker(\beta)$ is a reduced, semi-simple, centerless group $A$, then $\beta$ is subdirect $H$-split, i.e., $\tilde{G}^\circ \cong A \times \mathcal{G}^\circ$.

**Proof.** 1. Since $\tilde{G} = N_{\tilde{G}}(T)$, we have $\tilde{G}^\circ = N_{\tilde{G}}(T)^{\circ} = C_{\tilde{G}}(T)^{\circ}$, by [Spr98], Corollary 3.2.9 and therefore $T \leq Z(\tilde{G}^\circ)$. Then via the homomorphic section $\tilde{G}^\circ$ will be a subgroup of $\tilde{G}$ and hence $[T, \tilde{G}^\circ] = 1$.

2. Immediately from the assumptions we obtain $A \cap C_{\tilde{G}^\circ}(A) = Z(A) = 1$. The group $\tilde{G}^\circ$ acts on $A$ via conjugation as automorphism and $A \cdot C_{\tilde{G}^\circ}(A)$ acts also on $A$ via conjugation as inner automorphism, so $\tilde{G}^\circ/A \cdot C_{\tilde{G}^\circ}(A) \leq \text{Aut}(A)/\text{Inn}(A)$ is a finite group by [Hum08], Theorem 27.4. Given that $\tilde{G}^\circ$ is connected, we obtain $A \cdot C_{\tilde{G}^\circ}(A) = \tilde{G}^\circ = A \times C_{\tilde{G}^\circ}(A)$. Since $\beta$ is $H$-rigid, $H$ acts on $C_{\tilde{G}^\circ}(A)$ via conjugation in the same way as on $\tilde{G}^\circ$, hence $\sigma: \tilde{G}^\circ \times H \to C_{\tilde{G}^\circ}(A) \times H$ is a regular, $H$-rigid homomorphic section.

**Proposition 42.** Let $F$ be a field and $H$ be a finite group. Then every $H$-rigid, subdirect $H$-split ID-embedding problem in $\text{AffGr}^\text{red}_{K}$ over $F$ with reduced, connected kernel has a proper solution.

**Proof.** By Theorem 32 we can assume that all such embedding problems are $H$-effective. Let

$$\text{Gal}_{ID}(E/F) \xrightarrow{\beta} \mathcal{G}^\circ(K) \times H \xrightarrow{\gamma} \mathcal{G}^\circ(K) \times H$$

\[ \begin{array}{c}
1 \longrightarrow A(K) \longrightarrow (A \times \mathcal{G}^\circ)(K) \times H \xrightarrow{\beta} \mathcal{G}^\circ(K) \times H \longrightarrow 1
\end{array} \]
be an $H$-rigid, subdirect $H$-split, $H$-effective ID-embedding problem with connected kernel and let $L := E G^r(K)$. By the assumptions $H$ acts on $A$ and therefore $H$ acts also on $A^r$, where $r \in \mathbb{N}$. With $\alpha \colon \text{Gal}(L/F) \to H$ and $\beta : A^r \times H \to H$ (projection on the second factor), we get an ID-embedding problem $E(\alpha, \beta)$ with connected kernel and finite cokernel. Such an ID-embedding problem has a proper solution by [Mat01], Corollary 8.9 and if we choose $r > \dim_K(G^K)$, there exists a fixed field $\tilde{L}$ of $A^r(G^K) \leq A^r(K)$, with $\tilde{L} \cap E = L$. Hence $\text{Gal}(\tilde{L}/F) \cong (A(K) \times H) \times_H G(K) \cong (A \times G^r)(K) \times H$.

Theorem 43. Let $F$ be an algebraic function field in one variable over an algebraically closed field $K$ of positive characteristic. Then every ID-embedding problem in $\text{AffGr}_K^{\text{red}}$ over $F$, with reduced kernel has a proper solution.

Proof. By Propositions 12 and 20 we have to show that ID-embedding problems of types (1) - (5) have a proper solution. For type (1) see Theorem 37, type (2) is done by Theorem 35, types (3) and (4) are completed by Lemma 41 together with Proposition 42, and type (5) follows by Theorem 40.

From the above theorem we obtain a result about the structure of the Tannakian fundamental group for $\text{IDMod}_F$.

Definition 44. A prolinear group $\pi$ over $K$ is called reduced free, if every ID-embedding problem

\[
1 \overset{\alpha}{\longrightarrow} A(K) \overset{\beta}{\longrightarrow} \tilde{G}(K) \overset{\pi}{\longrightarrow} G(K) \overset{\tilde{\alpha}}{\longrightarrow} 1,
\]

with $A, \tilde{G}, G \in \text{AffGr}_K^{\text{red}}$ and epimorphisms $\tilde{\alpha}, \alpha, \beta$, has a proper solution.

In usual Galois theory by Iwasawa’s Freiheitssatz [MM99], Theorem IV.1.12, the absolute Galois group $\text{Gal}(F^{\text{sep}}/F)$ is free (of countable infinite rank) if and only if all embedding problems over $F$ have a proper solution. With the above definition, the main result, Theorem 43, can be seen as a generalization of this result (remember that the usual Galois theory is a special case of the iterative differential Galois theory; see Theorem 2).

Remark 45. Let $F$ be an algebraic function field in one variable over an algebraically closed field $K$ of positive characteristic. Let $\pi_1(\text{IDMod}_F)$ be the Tannakian fundamental group for $\text{IDMod}_F$. Then $\pi_1(\text{IDMod}_F)$ is an analogon to the absolute Galois group, as in the introduction explained and therefore by Theorem 43 $\pi_1(\text{IDMod}_F)$ is a reduced free group scheme.

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Stefan Ernst, RWTH Aachen, Lehrstuhl A für Mathematik, Templergraben 55, 52062 Aachen, Germany

E-mail address: stefan.ernst@matha.rwth-aachen.de