Hyperbolic geometrical optics: Hyperbolic glass

Enrico De Micheli
IBF – Consiglio Nazionale delle Ricerche,
Via De Marini, 6 - 16149 Genova, Italy.

Irene Scorza
Dipartimento di Matematica - Università di Genova
Via Dodecaneso, 35 - 16146 Genova, Italy.

Giovanni Alberto Viano
Dipartimento di Fisica - Università di Genova
Istituto Nazionale di Fisica Nucleare - sez. di Genova
Via Dodecaneso, 33 - 16146 Genova, Italy.

We study the geometrical optics generated by a refractive index of the form $n(x, y) = 1/y$ ($y > 0$), where $y$ is the coordinate of the vertical axis in an orthogonal reference frame in $\mathbb{R}^2$. We thus obtain what we call “hyperbolic geometrical optics” since the ray trajectories are geodesics in the Poincaré-Lobachevsky half-plane $\mathbb{H}^2$. Then we prove that the constant phase surface are horocycles and obtain the horocyclic waves, which are closely related to the classical Poisson kernel and are the analogs of the Euclidean plane waves. By studying the transport equation in the Beltrami pseudosphere, we prove (i) the conservation of the flow in the entire strip $0 < y < 1$ in $\mathbb{H}^2$, which is the limited region of physical interest where the ray trajectories lie; (ii) the nonuniform distribution of the density of trajectories: the rays are indeed focused toward the horizontal $x$ axis, which is the boundary of $\mathbb{H}^2$. Finally the process of ray focusing and defocusing is analyzed in detail by means of the sine–Gordon equation.
I. INTRODUCTION

It is well known that the geometrical optics approximation of the wave equation is related to the asymptotic form of the integral representation of the field (if such exists), which is an exact solution of the wave problem. Suppose, for instance, that the field in a uniform medium can be written as an expansion in plane waves; the evaluation of this integral by the stationary phase method yields an asymptotic series. Then the leading term of this asymptotic expansion, which is composed by an amplitude and a phase, can be extracted to yield the approximation. The ray trajectories are the lines orthogonal to the constant phase surface and are described by the eikonal equation; the amplitude satisfies the transport equation, whose physical meaning is related to the conservation of the flow. In the simplest case of uniform medium, whose refractive index $n$ is a real constant, the rays are straight lines which are characterized by the following properties:

(i) They are geodesics of the Euclidean space.

(ii) Phase and amplitude are real–valued functions.

(iii) They can be derived by the Fermat’s principle.

Constrained by these properties the methods of geometrical optics are rather limited and fail to explain several phenomena as, for instance, the diffraction by a compact and opaque obstacle, that is the existence of non–null field in the geometrical shadow which, for this reason, is usually referred to as the classically (or geometrically) forbidden region.

In the decade 1950–1960 J. B. Keller\textsuperscript{1,2,3} wrote several papers where he introduced the so–called Geometrical Theory of Diffraction (GTD). The latter can be regarded as an extension of geometrical optics, which accounts for diffraction by introducing the diffracted rays in addition to the usual rays of geometrical optics. After these seminal works there has been a steady flow of papers addressing various aspects of the theory. On the one hand papers oriented to pure and applied electromagnetic theory, like radiation and scattering of waves, antenna design, waveguide theory and so on\textsuperscript{3}; on the other hand, a highly theoretical and mathematically sophisticated theory of propagation of singularities and diffraction of waves on manifolds\textsuperscript{4}. In spite of these efforts and a wide literature on these topics, not all the cases of interest have been studied. An example is what we could call the “hyperbolic geometrical optics”, that is the geometrical optics generated by the rays in the specific case of a refractive
index of the form \( n(x, y) = 1/y \ (y > 0) \), where \( y \) denotes a spatial coordinate, say vertical, in an appropriate orthogonal reference frame in \( \mathbb{R}^2 \). As far as we know, this problem has never been treated, except for some very marginal remarks (see, for instance, Ref. 5), in spite of its intrinsic geometrical interest and some possible applications to the physics of nonuniform optical fibers. It is precisely the main purpose of the present paper to fill this gap.

Let us return to Keller’s program of widening, from a geometrical viewpoint, the arena of Euclidean geometrical optics. With this in mind we adopt, first of all, the Jacobi’s form of the principle of least action (instead of Fermat’s), which concerns with the path of the system point rather than with its time evolution. More precisely, the Jacobi’s principle (generally applied in mechanics) can be formulated as follows: If there are no forces acting on the body, then the system travels along the shortest path length in the configuration space. Here we assume a wide extension of Jacobi’s principle, which can be formulated as follows: the geodesics associated with the Riemannian metric \( n(x, y) \sqrt{dx^2 + dy^2} \), i.e. the paths making the functional \( \int n(x, y) \sqrt{dx^2 + dy^2} \) stationary, are nicknamed rays. In other words, in place of Fermat’s principle which reads \( \delta \int_{P_0}^{P_1} dt = 0 \), where \( dt \) is the travel time measure, and \( P_0 \) and \( P_1 \) are prescribed starting and end points of the path, we write

\[
\delta \int_{P_0}^{P_1} n(x, y) \sqrt{dx^2 + dy^2} = 0, \tag{1}
\]

or, equivalently,

\[
\delta \int_{x_0}^{x_1} F(x, y, y') \, dx = 0, \quad \left( F(x, y, y') = n(x, y) \sqrt{1 + y'^2} \right), \tag{2}
\]

where \( y' = \tan \alpha, \alpha \) being the angle that the tangent to the curve \( y = y(x) \) forms with the \( x \) axis.

The simplest realization of this Jacobi’s principle consists in identifying \( n^2 \) with the Riemann metric tensor \( g_{ij} \), \textit{whenever this identification is admissible}. This identification requires great caution, indeed; the form \( g_{ij} dx^i dx^j \) must be symmetric and positive definite, and this poses a strict restriction. For instance, consider a refractive index (or, in mechanics, a potential) of the following form: \( n^2 = 1 - V/E \), where \( E \) is the energy of the incoming particle and \( V \) is the height of the potential, with \( V > E \) as in the case of the tunnel effect. In this situation the geometric interpretation of the trajectory as a real–valued geodesic in a Riemannian manifold is no longer possible. The only chance remains to extend the
admissible values of the phase to imaginary and/or complex values and, consequently, to speak of complex rays in the sense of Landau.

But let us return to the cases where this identification is admissible. As we already mentioned, it is obviously possible in the case of a uniform nonabsorbing medium: in this case we simply obtain a physical realization of Euclidean geometry. But it is also certainly admissible when the refractive index is of the form introduced above, i.e., \( n(x, y) = 1/y \) \((y > 0)\), where \( y \) denotes the coordinate of the vertical axis in an orthogonal reference frame in \( \mathbb{R}^2 \). In this case we are led to the Lobachevskian metric: \( ds^2 = (dx^2 + dy^2)/y^2 \). Then the rays are geodesics in the hyperbolic half-plane (Poincaré half-plane): i.e., Euclidean half-circles with centers on the \( x \) axis (horizontal axis), or Euclidean straight lines normal to the \( x \) axis. Let us recall that the refractive index \( n \) is defined as: \( n = c/v_{ph} \), where \( c \) is the light speed in vacuum, and \( v_{ph} \) is the phase velocity of radiation of a specific frequency in a specific material. Therefore \( n \geq 1 \), and in the case \( n(x, y) = 1/y \) only the strip \( 0 < y \leq 1 \) has physical interest; hence the actual rays will lie necessarily in this band. Accordingly, hereafter, the only optical paths considered will be the Euclidean half-circles with centers on the \( x \) axis and radius \( R \) bounded by \( 0 < R \leq 1 \).

The subsequent step in developing an optical geometry consists in finding the constant phase surfaces and, accordingly, describing the analog of the Euclidean plane wave. This problem will be solved in Sec. II, studying some geometrical properties of horocycles and introducing what we call horocyclic waves, which play in hyperbolic geometrical optics the same role as the plane waves do in the Euclidean one. At this point we have the main ingredients needed for writing the geometrical approximation of the wave function; what it is still missing is an analysis of the amplitude and of the related flux density. This latter problem can be analyzed at two different levels. First we prove that the flow of rays is conserved: once a pointlike source is fixed, no ray will be absorbed or created. This result will be proved in Sec. III. A more subtle question is the following: Is the flow of the ray trajectories homogeneous or do the rays focus? This issue, besides its intrinsic geometrical interest, could in our opinion be of some interest in possible applications to the propagation in optical fibers with non-uniform refractive index. This problem will be analyzed in detail in Sec. III. First we study the transport equation in the Beltrami pseudosphere, and prove that the flow of ray trajectories is not homogeneous, but there is a focusing of rays on the horizontal \( x \) axis. Glancing to possible applications to propagation in optical fibers
this result suggests a conjecture indicating a strong ray focusing along the fiber axis, when
the refractive index profile in the fiber is of hyperbolic type, instead of paraboliclike, as is
customary. Next, this problem will be reconsidered by studying the variation of the angle
that the tangent to the meridian of the Beltrami pseudosphere makes with the rotation
axis of this surface, which can be indeed represented as a surface of revolution generated
by a curve in \( \mathbb{R}^3 \). This leads to the sine–Gordon equation and provides a more precise
description of the ray focusing and defocusing processes. This analysis is necessarily local,
since the problem is worked out inside each horocycle; at the end of Sec. III we show how
to pass from a local description of the flow inside each horocycle to a global one.

Finally, in the Appendix, the geometric and algebraic ingredients which occur in Secs.
II and III will be given. This appendix is split in three parts: the first part is devoted
to the various models of hyperbolic geometry and to the conformal maps which allows the
transformation between them; in the second part we study the group \( SU(1, 1) \), which acts
transitively on the non–Euclidean disk, and prove some relationships connecting the spherical
functions to the horocyclic waves; the last part is devoted to the Beltrami pseudosphere.

II. THE FLOW IN THE STRIP \( 0 < y \leq 1 \)

A. Variational minimization of the Jacobi’s functional and the rays in hyperbolic
geometrical optics

Let us consider the upper half–plane model of the hyperbolic two–dimensional space \( \mathbb{H}^2 \):
i.e., \( U = \{ z = x + iy : y > 0 \} \) equipped with the metric \( d \) derived from the differential
\( ds = |dz|/\text{Im} \, z \) (see the Appendix). Then we apply the typical methods of variational
calculus to the Jacobi functional

\[
J = \int_{P_0}^{P_1} \sqrt{(dx)^2 + (dy)^2} / y,
\]

or, equivalently,

\[
J = \int_{x_0}^{x_1} \frac{\sqrt{1 + (y')^2}}{y} \, dx
\]

\((P_0 \text{ and } P_1 \text{ denote two points of the ambient space where light propagates}). \text{ First we prove }
the following proposition, which refers to the whole upper–half plane \( U \).
Proposition 1. (i) Let \( J \) be the following functional

\[
J = \int_{x_0}^{x_1} \frac{\sqrt{1 + (y')^2}}{y} \, dx,
\]
and let \( F \) denote the integrand of (3). The Euler–Lagrange equation for this functional reads

\[
yy'' + y'^2 + 1 = 0.
\]

(ii) The extremals of functional (3) are Euclidean half–circles with centers on the \( x \) axis, or Euclidean straight lines normal to the \( x \) axis lying in the half–plane \( y > 0 \). These are the geodesics in the hyperbolic geometry realized in the half–plane \( y > 0 \).

(iii) The Weierstrass condition for the functional (3) reads

\[
F_{y'y'} = \frac{1}{y(1 + y^2)^{3/2}} > 0 \quad (y > 0),
\]
and it is satisfied for any \( y' \).

(iv) There exists a field of extremals of functional (3), and the transversality condition becomes an orthogonality condition of these extremals to the curve \( \Phi(x,y) = \text{const.} \) (constant phase curve), which satisfies the following equation (eikonal equation):

\[
g^{ij} \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_j} = 1,
\]
where \( x_1 = x \), \( x_2 = y \), and \( g_{ij} \) is the metric tensor.

Proof. The proof makes use of standard procedures and can be found, for instance, in Ref. 5.

Remark. Let us recall once again that the domain of physical interest where the optical paths necessarily lie (in view of the fact that \( n \geq 1 \)) is the strip \( 0 < y \leq 1 \); therefore we shall consider only a subclass of the extremals of functional (3): i.e., the half–circles with centers on the \( x \) axis and radius bounded by \( 0 < R \leq 1 \).

B. Poisson–kernel and horocyclic waves

Let us now give a more precise formulation of the physical problem. Suppose that a pointlike source of light is pushed to \(-\infty\) on the \((x,y)\) plane. For the sake of simplicity, here we limit ourselves to the scalar representation of light, and phenomena associated with
polarization will not be considered. From Proposition 1 it follows that light rays are half-circles with centers on the $x$ axis. For several reasons which will appear clear in what follows, it is convenient to map conformally the half-plane $y > 0$ into the unit disk $|\zeta| < 1$, which amounts to pass from the Poincaré half-plane model $U$ to the Poincaré disk model $D$ (see the Appendix and Fig. 1). The appropriate conformal mapping is given by: $\zeta = i(z - i)/(z + i)$ ($z = x + iy; \zeta = \xi + i\eta$).

In the unit disk the light source will be located at $\zeta = i$. The band $y > 1$ will be mapped, in the $\zeta$–plane, into the disk tangent to the boundary $B$ of $D$ in $i$ with Euclidean radius $\frac{1}{2}$, and represents the forbidden region for the light rays. The circular arcs lying in the half-plane $y > 0$ and normal to the $x$ axis will be mapped, in the unit disk, into circular arcs perpendicular to the boundary $|\zeta| = 1$, which are precisely the geodesics of the hyperbolic geometry in the unit disk model.

From the transversality condition [see statement (iv) of Proposition 1], it follows that the constant phase curve is the curve that intersects orthogonally the extremals of functional (3): i.e., the geodesics. In the unit disk, parallel geodesics are geodesics corresponding to the same point $b = e^{i\phi}$ on the boundary $B$ of $D$. Therefore, in the physical problem being
treated, the circles tangent to the unit circle at the point $b$, which intersects orthogonally the pencil of \textit{parallel straight lines} (i.e., arcs of circle orthogonal to $B$) are the constant phase curves, they are a family of \textit{horocycles}, and are denoted by $H_b$.

We can now state the following proposition.

\textbf{Proposition 2.} (i) The Poisson kernel

$$P(\zeta, b) = \frac{1 - |\zeta|^2}{1 + |\zeta|^2 - 2|\zeta|\cos(\theta - \phi)} \quad (\zeta = |\zeta|e^{i\theta}; \ b = e^{i\phi}),$$

(7)

is constant on each horocycle $H_b$ with normal $b$.

(ii) The function

$$[P(\zeta, b)]^\nu = \left[\frac{1 - |\zeta|^2}{1 + |\zeta|^2 - 2|\zeta|\cos(\theta - \phi)}\right]^\nu \quad (\nu \in \mathbb{C}),$$

(8)

is an eigenfunction of the Laplace–Beltrami operator on the hyperbolic disk $D$ corresponding to the eigenvalue $\nu(\nu - 1)$.

(iii) The hyperbolic waves (horocyclic waves) are represented by the following expression:

$$e^{\nu(\zeta, b)} = \left[\frac{1 - |\zeta|^2}{1 + |\zeta|^2 - 2|\zeta|\cos(\theta - \phi)}\right]^\nu \quad (\nu \in \mathbb{C}),$$

(9)

where $\langle \zeta, b \rangle$ is the hyperbolic distance between the origin of $D$ and the horocycle of normal $b$ passing through $\zeta \in D$.

(iv) The conical functions $\mathcal{P}_{-\frac{1}{2} + i\lambda}(\cosh r)$ (i.e., the first kind Legendre functions of index $(-\frac{1}{2} + i\lambda)$ ($\lambda \in \mathbb{R}$)) can be represented by

$$\mathcal{P}_{-\frac{1}{2} + i\lambda}(\cosh r) = \int_B e^{(\frac{1}{2} - i\lambda)\langle \zeta, b \rangle} \, db \quad (\lambda \in \mathbb{R}, B = \{ \zeta : |\zeta| = 1 \}),$$

(10)

and correspond to the fundamental series of the irreducible unitary representation of the group $SU(1, 1)$, which acts transitively on the hyperbolic disk $D$.

(v) The following equality holds:

$$\mathcal{P}_{-\frac{1}{2} + i\lambda}(\cosh r) = \mathcal{P}_{-\frac{1}{2} - i\lambda}(\cosh r) \quad (\lambda \in \mathbb{R}).$$

(11)

\textbf{Proof.} (i) The level lines of the Poisson kernel $P(\zeta, b)$ are the circles tangent to the unit circle at the point $b = e^{i\phi}$: i.e., the images of the horocycles $H_b$ with normal $b$ (see Ref. 9).

(ii) The Laplace–Beltrami operator $\Delta_D$ on the hyperbolic unit disk $D$ is given by

$$\Delta_D = \frac{1}{4} \left[1 - (\xi^2 + \eta^2)\right]^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right).$$

(12)
If $\nu \in \mathbb{C}$ is any complex number, a direct computation gives

$$\Delta_D P^{\nu}(\zeta, b) = \nu(\nu - 1)P^{\nu}(\zeta, b).$$

(iii) In the Euclidean case the function $x \rightarrow e^{ik(x, \omega)}$, where $k \in \mathbb{R}$, $\omega \in S^{(n-1)}$, $x \in \mathbb{R}^n$, represents a plane wave with normal $\omega$. It is indeed constant on each hyperplane perpendicular to $\omega$, and furthermore is an eigenfunction of the Laplacian on $\mathbb{R}^n$. The geometric analog of the plane wave in the case of the hyperbolic disk $D$ is the function represented by the equality (9) (see Ref. 11). In fact, it is an eigenfunction of the Laplace–Beltrami operator on $D$, as proved by statement (ii) [see Eq. (13)]. Further, putting $\theta = \phi$ in formulae (7) and (8), we have:

$$\ln \frac{1 - |\zeta|^2}{1 + |\zeta|^2 - 2|\zeta|} = \ln \frac{1 + |\zeta|}{1 - |\zeta|} = d(0, \zeta) = \langle |\zeta| e^{i\phi}, e^{i\phi} \rangle = \langle \zeta, b \rangle,$$

where $d(0, \zeta) = \ln[(1 + |\zeta|)/(1 - |\zeta|)]$ is the hyperbolic distance between the origin and the point $\zeta \in D$ (see the Appendix). Therefore, $\langle \zeta, b \rangle$ is the hyperbolic analog of $(x, \omega)$. In fact, in view of statement (i), $\langle \zeta, b \rangle$ is the distance between the origin and the horocycle of normal $b$ passing through $\zeta \in D$, assuming that the origin falls outside the horocycle; $\langle \zeta, b \rangle$ is positive if the origin is external to the horocycle, while it is negative ($\langle \zeta, b \rangle = \ln[(1 - |\zeta|)/(1 + |\zeta|)]$) if the origin is internal to the horocycle.

(iv) If we put: $\xi = \tanh(r/2) \cos \theta$, $\eta = \tanh(r/2) \sin \theta$, then $|\zeta| = \tanh(r/2)$. The Riemannian metric $ds^2 = [4(d\xi^2 + d\eta^2)/(1 - \xi^2 - \eta^2)^2]$ becomes $ds^2 = dr^2 + \sinh^2 r \, d\theta^2$. By the use of this substitution in the expression of the Poisson kernel (7) or (8), we have:

$$\int_B e^{\nu \langle \zeta, b \rangle} \, db = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{\cosh r + \sinh r \cos \phi} \right)^{\nu} \, d\phi = \mathcal{P}_{-\nu}(\cosh r) \quad (\nu \in \mathbb{C}),$$

where $B$ is the boundary of the hyperbolic disk $D$, and $\mathcal{P}_{-\nu}(\cosh r)$ are the first kind Legendre functions. Finally, setting $\nu = \frac{1}{2} - i\lambda$ ($\lambda \in \mathbb{R}$) we obtain the conical functions $\mathcal{P}_{-\frac{1}{2} + i\lambda}(\cosh r)$, which correspond to the fundamental series of the irreducible unitary representation of the group $SU(1, 1)$: i.e., the group of the matrices of the form:

$$\left( \begin{array}{cc} a & c \\ \bar{c} & \bar{a} \end{array} \right),$$
\[ |a|^2 - |c|^2 = 1; \ a, c \in \mathbb{C}, \text{ which acts as a group of isometries of the hyperbolic disk } D \text{ by means of the map } \]

\[ g(\zeta) = \frac{a\zeta + c}{\bar{c}\zeta + \bar{a}} \quad (\zeta \in D). \quad (17) \]

(v) Equality (11) is proved in the Appendix (see Proposition 6).

Remark. It is well known that the classical Fourier transform refers to the decomposition of a function, belonging to an appropriate space, into exponentials of the form \( e^{ikx} \) (\( k \) real), which can also be viewed as the irreducible unitary representation of the additive group of real numbers. Analogously, the exponentials \( e^{i(k, x)} \) are characters of the group \( \mathbb{R}^2 \). But the hyperbolic disk is not a group. Therefore a straightforward generalization of the exponential for \( D \) is not possible. Nevertheless, in view of the fact that the function \( \mathcal{P}_{-\nu}(\cosh r) \) corresponds to the fundamental series of the irreducible unitary representation of the group \( SU(1, 1) \) for \( \nu = \frac{1}{2} - i\lambda \), the exponential \( e^{\left(\frac{1}{2} - i\lambda\right)}(\zeta, b) \) (\( \lambda \in \mathbb{R} \)) represents the analog of the Euclidean exponential, and plays the same role in the hyperbolic Fourier analysis \( \Pi \).

C. Conservation of the flow

As already said in the Introduction, the ray trajectories are the lines orthogonal to the constant phase surface, and are described by the eikonal equation; moreover, \( \langle \zeta, b \rangle \) is the hyperbolic distance between the origin and the horocycle \( H_b \) of normal \( b \) passing through \( \zeta \). Therefore, in close analogy with the Euclidean optical geometry, and recalling that \( \mathcal{P}_{-\frac{1}{2} + i\lambda}(\cosh r) = \mathcal{P}_{-\frac{1}{2} - i\lambda}(\cosh r) \) (\( \lambda \in \mathbb{R} \)) [see statement (v) of Proposition 2], the expression of the analog of the Euclidean plane wave \( e^{ikx} \) (\( k \in \mathbb{R} \)) can be written as follows: \( e^{\left(\frac{1}{2} - i\lambda\right)}(\zeta, b) \) (\( \lambda \in \mathbb{R} \)). Thus the geometrical approximation of the wave function \( \psi \) can be obtained by multiplying \( e^{\left(\frac{1}{2} - i\lambda\right)}(\zeta, b) \) times a function which represents the amplitude. Then we can state the following proposition.

Proposition 3. The geometrical approximation of the wave function \( \psi \) reads:

\[ \psi(\zeta, \lambda, b) = A(\lambda)e^{\left(\frac{1}{2} - i\lambda\right)}(\zeta, b) \quad (\lambda \in \mathbb{R}, \zeta \in D, b \in B), \quad (18) \]

and the flow in the entire strip \( 0 < y \leq 1 \) is conserved.
Proof. Let $\sigma$ be the conformal map

$$z = \sigma(\zeta) = \frac{-i\zeta + i}{\zeta - 1},$$  \hspace{1cm} (19)$$
defined in the Appendix, that transfers the geometry of $D$ into $U$. Since $\sigma(0) = i$ and $\sigma(i) = \infty$, then the image by $\sigma$ of the horocycle $H_i$ passing through $\zeta = 0$ is the horizontal line $\tilde{H}_\infty = \{x + iy : y = 1\}$ in $U$ (the horocycles in the Poincaré half–plane will be hereafter denoted by $\tilde{H}_b$). The image by $\sigma$ of the horocycle $H_{\sigma^{-1}(b)}$ tangent to $H_i$ in $D$ is the horocycle $\tilde{H}_b$ of radius $\frac{1}{2}$ through $b \in \mathbb{R}$ and tangent to the horizontal line $\tilde{H}_\infty$ (in order to avoid proliferation of notations, we denote by the same letter $b$ both the points on the boundary $B$ of $D$ and the corresponding points belonging to the boundary of $\mathbb{H}^2$, i.e. belonging to $\mathbb{R}$).

We already saw that the horocycle $\tilde{H}_b$ of normal $b$ is perpendicular to each geodesic starting from $b$. To calculate the amplitude of the wave function, we must see how many geodesics perpendicular to $\tilde{H}_b$ intersect $\tilde{H}_b$, with the additional condition that these geodesics belong to the band $0 < y \leq 1$. This corresponds to find the amount of normal vectors at $\tilde{H}_b$, with unit norm, that are tangent vectors of geodesics in the band $0 < y \leq 1$.

In general, if $b$ is a point in $\mathbb{R} \cup \{\infty\}$ and $T_1U$ is the unit tangent bundle of $U$, then the horocycle flow $h_{j,b} : T_1U \rightarrow T_1U$ is the flow which slides the inward normal vectors to each $\tilde{H}_b$ to the right along $\tilde{H}_b$ at unit speed. To find the equation of the flow $h_{j,b}$, first we consider the flow $h_{j,\infty}$ of geodesics perpendicular to the horocycle $\tilde{H}_\infty$ of normal $\infty$. Then we choose a transformation $M_b$ which maps the horocycle $\tilde{H}_\infty$ into the horocycle $\tilde{H}_b$. In particular, the map $M_b$ transfers the flow $h_{j,\infty}$ into the flow $h_{j,b}$.

From the definition,

$$h_{j,\infty}(v_i) = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} v_i,$$  \hspace{1cm} (20)$$
where $v_i$ denotes the unit vector vertically upwards based at $i \in U$. This is because in the simplest case of horocycle flow $h_{j,\infty}$, the geodesics perpendicular to $\tilde{H}_\infty$ are vertical lines and the isometry sending one vertical line into another vertical line is the horizontal translation. Therefore, the horocycle flow along $\tilde{H}_\infty$ is simply the horizontal translation.

Let us now consider the transformation $M_b$ such that $M_b(\infty) = b$. Then, the horocycle
flow $h_{j,b}$ along $\tilde{H}_b$ is the image of $h_{j,\infty}$ by $M_b$, hence

$$h_{j,b}(v) = M_b \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} v. \quad (21)$$

It is clear from the definition that the amount of geodesics in the flow $h_{j,b}$ does not depend on the radius of the horocycle. Given two different points $b_1$ and $b_2$ in the boundary of the hyperbolic plane, then the composition of $M_{b_1}$ and $M_{b_2}^{-1}$ sends the point $b_2$ in $b_1$. Moreover, $M_{b_1} \circ M_{b_2}^{-1}$ sends the horocycle flow $h_{j,b_2}$ into the horocycle flow $h_{j,b_1}$. This proves that the amplitude of the wave does not depend on $b$ and $\zeta$.

Using Proposition 2, we obtain that there exists a function $A(\lambda)$ independent of $\zeta$ and $b$ such that Eq. (18) is satisfied, and the conservation of the flow along the entire strip $0 < y \leq 1$ is proved.

**Remark.** It is interesting to compare the propagation of light in vacuum with that within the strip $0 < y \leq 1$ belonging to $\mathbb{H}^2$. In vacuum each ray cuts orthogonally all the constant phase planes: i.e., each ray emerging from a plane cuts orthogonally all the other parallel planes. In $\mathbb{H}^2$ propagation proceeds in a completely different form. Take two horocycles lying in the strip $0 < y \leq 1$, and tangent at the point $z = (1 + i)/2$: the first horocycle, denoted by $\tilde{H}_0$, has normal $b_0 = 0$; the second one, denoted by $\tilde{H}_1$, has normal $b_1 = 1$. Only one geodesic, denoted $\gamma_t$, lying in $\tilde{H}_0$, cuts orthogonally $\tilde{H}_1$; it emerges from $b_0 = 0$ and ends at $b_1 = 1$. All the geodesics $\gamma_>$, emerging from $b_0 = 0$ and lying in $\tilde{H}_0$ above $\gamma_t$, cut orthogonally horocycles $\tilde{H}_b$ with $b > 1$; the geodesics $\gamma_<$, emerging from $b_0 = 0$ and lying in $\tilde{H}_0$ below $\gamma_t$, cut orthogonally horocycles $\tilde{H}_b$ with $b < 1$. However, the density of the flow of geodesics entering orthogonally each horocycle equals the density of the flow of geodesics exiting orthogonally the same horocycle.

**III. TRANSPORT EQUATION AND DISTRIBUTION OF THE DENSITY OF TRAJECTORIES**

**A. Transport equation in the Beltrami pseudosphere**

Working out the problem in the space $\mathbb{H}^2$ allows us to describe each trajectory as a geodesic in the Poincaré plane (or disk), but this setting is not appropriate for describing
the evolution of a bunch of trajectories. Hereafter we will switch to a representation more suitable for an effective characterization of the amplitude factor in the geometrical approximation of the field. To this aim, let us first recall the following well-known negative result due to Hilbert: there is no regular smooth immersion \( X : \mathbb{H}^2 \to \mathbb{R}^3 \). However, one can look for a local immersion \( X : \mathcal{U} \to \mathbb{R}^3 \), where \( X \) is a continuous differentiable function, and \( \mathcal{U} \subset \mathbb{H}^2 \) is an open subset. We keep for \( \mathcal{U} \) an open horocycle based at \( b \). This local immersion can be realized by means of the Beltrami pseudosphere, denoted hereafter by \( P_b \) (see the Appendix and Fig. 2). In fact, let us consider in the hyperbolic disk \( D \) an infinite strip lying between two parallel straight lines emerging from the source point located on the absolute at \( \zeta = -i \). Then we take on these parallel geodesics a pair of points \( A_0 \) and \( B_0 \), lying on a horocycle of normal \( b_0 = e^{-ir/2} = -i \) and cutting orthogonally these straight lines; \( A_0 \) and \( B_0 \) are spaced at distance of \( 2\pi \). One is then led to consider the domain \( (-i, A_0, B_0) \). The Beltrami surface cut along any of its generators can be isometrically mapped into the domain \( (-i, A_0, B_0) \) (see Ref. 14). On a Lobachevskian plane there always exists reflection (i.e., a hyperbolic isometry) about an arbitrary straight line; in particular, reflecting the strip \( (-i, A_0, B_0) \) about the straight line \( (-i, A_0) \) we obtain a new strip isometric to the initial one and realized as a cut of the Beltrami surface in \( \mathbb{R}^3 \). Reflecting then this new strip \( (-i, A_1, A_0) \) (the segment \( A_1 A_0 \) has length \( 2\pi \)) about the straight line \( (-i, A_1) \) we obtain the strip \( (-i, A_2, A_1) \) with the same properties. Exactly the same procedure can be repeated on the other side of \( (-i, A_0, B_0) \), leading to \( (-i, B_2, B_1) \). We thus obtain strips of the form \( (-i, A_k, A_{k-1}) \) and \( (-i, B_k, B_{k-1}) \) (\( 1 \leq k < \infty \)); all segments \( (A_k, A_{k-1}) \) and \( (B_k, B_{k-1}) \) have the same length \( 2\pi \). Working with the same procedure we can now construct the map of the open horocycle \( H_{b_0} \), tangent at the boundary to the forbidden region (this latter represented by the horocycle \( H_i \) of normal \( i \) and passing through the origin), into a Beltrami funnel, such that each strip of the type \( (-i, A_k, A_{k-1}), (-i, A_0, B_0), \) and \( (-i, B_k, B_{k-1}), \) \( (1 \leq k < \infty) \) (referred, now, to the horocycle \( H_{b_0} \)), is mapped isometrically into the Beltrami surface, the horocycle \( H_{b_0} \) being wound infinitely many times into the Beltrami surface\(^{\ref{14}}\) (see Fig. 2). We can repeat the same procedure for each point \( b \in B \), since there is a rotation (i.e., a hyperbolic isometry) sending each \( b \in B \) onto \( b_0 \).

For an explicit equation of the immersion \( X \), the reader is referred to Ref. 15.

In general, the Laplace–Beltrami operator \( \Delta_M \) on a two-dimensional Riemannian mani-
fold $M$ with metric tensor $g_{ij}$ ($g = |\det(g_{ij})|$, $g^{ij} = g^{-1}_{ij}$) is defined as follows:

$$\Delta_M = \frac{1}{\sqrt{g}} \left[ \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{2} g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right) \right].$$ \hspace{1cm} (22)

In the specific case of the hyperbolic metric associated with the refractive index $n(y) = 1/y$ (see the Appendix), the Laplace–Beltrami operator reads:

$$\Delta_H = \frac{1}{n^2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$ \hspace{1cm} (23)

We then have the following proposition.

**Proposition 4.** (i) The Helmholtz equation reads

$$\Delta_H \psi + k_H^2 \psi = 0,$$ \hspace{1cm} (24)
where $k_{2n}^2 = \lambda^2 + \frac{1}{4} \ (\lambda \in \mathbb{R})$.

(ii) The geometrical approximation of the wave function $\psi$ (for $|\lambda| \to \infty$), written in terms of the Beltrami coordinates (see the Appendix), reads

$$\psi_{\pm} (\lambda, u) = C(\lambda) e^{u/2} e^{i\lambda u} \quad (\lambda \in \mathbb{R}; u \geq 0).$$

(25)

Proof. (i) Let us consider the horocyclic waves which generate the conical functions $P_{-\frac{1}{2} \pm i\lambda} (\cosh r)$, corresponding to the irreducible unitary representation of the $SU(1, 1)$ group, which acts transitively on the hyperbolic disk $D$. This amounts to put in the exponent $\nu \in \mathbb{C}$ of the Poisson kernel: $\nu = \frac{1}{2} \pm i\lambda \ (\lambda \in \mathbb{R})$. Accordingly, the horocyclic waves read $e^{(\frac{1}{2} \pm i\lambda)(\zeta, b)}$ [see statements (iv) and (v) of Proposition 2]. From statement (ii) of Proposition 2 and Eq. (13) we get:

$$\Delta H e^{(\frac{1}{2} \pm i\lambda)(\zeta, b)} = -k_{2n}^2 H e^{(\frac{1}{2} \pm i\lambda)(\zeta, b)},$$

(26)

where $k_{2n}^2 = \lambda^2 + \frac{1}{4} \ (\lambda \in \mathbb{R})$. Next, proceeding in close analogy with the Euclidean case, where the Euclidean plane wave plays the role of the horocyclic wave, we obtain Eq. (21).

(ii) Let us now go back to the mapping of the horocycle into the Beltrami funnel (without a cut) in $\mathbb{R}^3$, illustrated above. Next, we apply the Laplace–Beltrami operator to the wave function $\psi$, supposed to belong to $C^\infty (B)$ ($B$ denoting the Beltrami pseudosphere); in $x_i \ (i = 1, 2)$ stand for the Beltrami coordinates $u, v$. Recall that the first fundamental form in Beltrami coordinates reads [see part (C) of the Appendix]:

$$I = du^2 + e^{-2u} dv^2 \quad (u \geq 0).$$

(27)

Accordingly, we have $g_{11} = 1, \ g_{22} = e^{-2u}, \ g_{12} = g_{21} = 0, \ g = |\det (g_{ij})| = e^{-2u}, \ g^{ij} = g^{-1}_{ij}$. Thus, we are led to the following equation:

$$\Delta_B \psi + k_{2n}^2 \psi = 0,$$

(28)

where $\Delta_B$ is the Laplace–Beltrami operator, referred to the Beltrami pseudosphere. In this equation, we pass from the coordinates $(\xi, \eta)$ of the hyperbolic disk $D$ to the Beltrami coordinates $(u, v)$ of the Beltrami pseudosphere. We illustrate with more details this passage. First we embed an open horocycle $H_b$ of normal $b$ and tangent to the forbidden region (represented by the horocycle $H_i$ passing through the origin of $D$ and with normal $i$; see Fig. 1) into a Beltrami pseudosphere. Notice that in the present analysis, as well as in
Proposition 3, and in strict analogy with the classical Euclidean procedure, we consider the distance from the origin of the hyperbolic disk $D$ (rather than from the point source located at $\zeta = i$) to the horocycle $H_{\zeta, b}$ (inside $H_b$) of normal $b$ passing through a point $\zeta$. Thus we have

$$\langle \zeta, b \rangle = d(0, H_b) + d(H_b, H_{\zeta, b}) := d_b + d(H_b, H_{\zeta, b}). \quad (29)$$

When we embed $H_b$ into a Beltrami pseudosphere, the distance $d(H_b, H_{\zeta, b})$ between horocycles corresponds to the distance between different parallels $u = \text{const.}$ inside the pseudosphere. Since $H_b$ is fixed, then $d_b$ is fixed too. Then, following the standard method of stationary phase, we look now for a solution of equation (28), of the following form:

$$\psi(\lambda, x) = \int A(x, \ell) e^{\left(\frac{1}{2} - i\lambda\right)\langle \zeta, b \rangle} d\ell = e^{\left(\frac{1}{2} - i\lambda\right)d_b} \int A(x, \ell) e^{\left(\frac{1}{2} - i\lambda\right)d(H_b, H_{\zeta, b})} d\ell$$

$$= C(\lambda) \int A(x, \ell) e^{\left(\frac{1}{2} - i\lambda\right)\Phi(x, \ell)} d\ell, \quad (30)$$

where $x = (x_1, x_2)$, $x_1 = u, x_2 = v$; $\ell$ is the pathlength inside the pseudosphere to the point of coordinate $x$, $\Phi(x, \ell)$ denotes the phase (recall the statements of Proposition 2).

The r.h.s. of equality (30) is an integral of oscillating type. The principal contribution to $\psi(\lambda, x)$, as $|\lambda| \to +\infty$, corresponds to the stationary point of $\Phi$, in the neighborhood of which the exponential ceases to oscillate rapidly. These stationary points can be obtained from the equation $\partial \Phi/\partial \ell = 0$ (provided that $\partial^2 \Phi/\partial \ell^2 \neq 0$). If the condition $\partial \Phi/\partial \ell = 0$ is satisfied by a unique value $\ell_0$ of $\ell$, corresponding to the unique ray trajectory (geodesic) passing across the point of coordinates $(u, v)$, we say that $\Phi$ has a critical nondegenerate point at $\ell = \ell_0$. Moreover, recalling that the manifolds with nonpositive curvature do not have conjugate points, we can state that all the critical points of $\Phi$ are nondegenerate. Then, by applying the Morse lemma on the representation of the functions all of whose critical points are nondegenerate, we obtain the following asymptotic evaluation of integral (30):

$$\psi(\lambda, x) = C(\lambda) e^{\frac{1}{2} \Phi(x, \ell_0)} e^{-i\lambda \Phi(x, \ell_0)} \sum_{m=0}^{\infty} \frac{A_m(x)}{(i\lambda)^m}. \quad (31)$$

The leading term of expansion (31) reads

$$\psi(\lambda, x) = C(\lambda)A_0(x) e^{\frac{1}{2} \Phi(x, \ell_0)} e^{-i\lambda \Phi(x, \ell_0)}, \quad (32)$$

where

$$A_0(x) = A(x, \ell_0) \left( \left| \frac{\partial^2 \Phi}{\partial \ell^2} \right|^{-1/2} \right)_{\ell=\ell_0} \exp \left[ i \frac{\pi}{4} \text{sgn} \left( \frac{\partial^2 \Phi}{\partial \ell^2} \right) \right]_{\ell=\ell_0}. \quad (33)$$
For simplicity, in the following we shall write the leading term of expansion (31) as: 
\[ C(\lambda)A(x)e^{\left(\frac{1}{2} - i\lambda\right)\Phi(x)} \], dropping the zero subscripts. Substituting this expression into Eq. (28), collecting powers of \((i\lambda)\) and, finally, equating to zero their coefficients, two equations are obtained: the eikonal (or Hamilton–Jacobi) equation
\[ g^{ij}\frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_j} = 1, \] (34)
and the transport equation
\[ \frac{1}{\sqrt{g}} \sum_{j=1}^{2} \frac{\partial}{\partial x_i} \left[ \sqrt{g}A^2 e^\Phi \sum_{j=1}^{2} g^{ij} \frac{\partial \Phi}{\partial x_j} \right] = 0. \] (35)

Let us note that in the present problem the wave functions are radial, in view of the fact that we are considering a family of horocycles having all the same normal \(b\). Therefore, \(\psi(\lambda, x)\) [where \(x \equiv (u, v)\), \(u, v\) being the Beltrami coordinates, see the Appendix] does not depend on \(v\). Then, Eq. (34) becomes:
\[ \left(\frac{d\Phi}{du}\right)^2 = 1, \] (36)
which gives the following expression of the phase: \(\Phi^{(\pm)} = \pm u + c\) \((c = \text{const.})\). Proceeding analogously with Eq. (35), we have:
\[ \frac{d}{du} \left( A^2 e^{\pm u} e^{-u} \frac{d\Phi^{(\pm)}}{du} \right) = 0. \] (37)

Substituting in the leading term (32) the expressions of \(\Phi\) and \(A\), which derive from (36) and (37), the rhs of (25) follows.

\[ \square \]

B. Sine–Gordon equation and the flow of trajectories

The analysis of propagation in the Beltrami pseudosphere has allowed us to study the distribution of the density of trajectories and, accordingly, the ray focusing along the horizontal axis. Another parameter, related once again to the Beltrami pseudosphere, and whose characterization is relevant in our description of the flow of trajectories, is the angle \(\varphi\) that the tangent to the meridian (in the Beltrami pseudosphere) makes with the \(z\) axis [see section (C) of the Appendix].
In the Appendix, the Beltrami pseudosphere is described in terms of the Beltrami coordinates $u$ and $v$. Here we choose another parameterization $(p, q)$ by setting

\[ dp = -\csc \varphi \, d\varphi, \quad (38a) \]
\[ dq = -dv. \quad (38b) \]

Integrating $(38a)$, we obtain:

\[ \varphi = 2 \tan^{-1} \left( e^{-p} \right), \quad (39) \]

and

\[ \frac{d\varphi}{dp} = -\sin \varphi = \frac{1}{\cosh p}. \quad (40) \]

Next, substituting in formula $(A.31)$ of the Appendix (with $\rho = -1$) $\varphi$ and $v$ in terms of the parameters $p$ and $q$, the position vector $r$ of the pseudosphere can be rewritten as follows:

\[ r(p, q) = \begin{pmatrix} -\frac{1}{\cosh p} \cos q \\ \frac{1}{\cosh p} \sin q \\ p - \tanh p \end{pmatrix}, \quad (41) \]

where the downward vertex of the pseudosphere corresponds to $p \to -\infty$, while the rim corresponds to $p = 0$. The parameters $(p, q)$ can be related to the arc lengths $(\alpha, \beta)$ along asymptotic lines as follows:

\[ p = \alpha + \beta, \quad q = \alpha - \beta. \quad (42) \]

Setting $\omega = 2\varphi$, the first fundamental form reads

\[ I = d\alpha^2 + 2 \cos \omega d\alpha \, d\beta + d\beta^2, \quad (43) \]

which can be easily derived from the expression $(A.32)$ of the Appendix through the formulae $(38a)$ and $(42)$ (recalling that $\rho^2 = 1$). Moreover, from Gauss’ and Weingarten’s equations, it follows that $\omega$, which is the angle between the asymptotic lines, satisfies the classical sine–Gordon equation, which reads

\[ \frac{\partial^2 \omega}{\partial \alpha \partial \beta} = \sin \omega. \quad (44) \]

This latter equation rewritten in terms of the coordinates $p, q$ becomes

\[ \frac{\partial^2 \varphi}{\partial p^2} - \frac{\partial^2 \varphi}{\partial q^2} = \sin \varphi \cos \varphi. \quad (45) \]

We can finally state the following proposition:
Proposition 5. (i) The angle $\varphi$, that the tangent to the meridian of the pseudosphere makes with the $z$ axis (i.e., the rotation axis), is represented by the following formula:

$$\varphi = 2 \tan^{-1} (e^{-p}),$$

which is the so-called “one-soliton” solution of the sine-Gordon equation (45).

(ii) The angle $\varphi$ varies from $\varphi = \frac{\pi}{2}$ to $\varphi = \pi$ in the process of focusing, and from $\varphi = \pi$ to $\varphi = \frac{\pi}{2}$ in the process of defocusing.

Proof. (i) The proof of this statement follows easily by direct calculation.

(ii) The downward vertical $z$ axis of the pseudosphere we are considering is negatively oriented; then varying $p$ from 0 to $-\infty$ [in equation (46)], $\varphi$ varies from $\varphi = \frac{\pi}{2}$ to $\varphi = \pi$ (focusing). Next varying $p$ from $-\infty$ to 0, $\varphi$ varies from $\varphi = \pi$ to $\varphi = \frac{\pi}{2}$ (defocusing).

So far in this section we have considered a local description of the flow in order to show that the density of the flow is not homogeneously distributed. We now want to recover a global description in the entire strip $0 < y \leq 1$. But we have already remarked that the map of a horocycle into a pseudosphere excludes the boundary of the horocycle. Therefore this global description cannot be reached by considering only horocycles lying in the strip $0 < y \leq 1$ and tangent to the line $y = 1$. Recall, however, that while the physical geodesics, which lie within the strip $0 < y \leq 1$ in $\mathbb{H}^2$, cannot enter the forbidden region $y > 1$, this is not the case for horocycles, provided we limit ourselves to consider in these horocycles (i.e., those entering the forbidden region) those segments of geodesics which lie in the strip $0 < y \leq 1$. In view of these considerations it is sufficient to consider horocycles $\tilde{H}'_b \supset \tilde{H}_b$ ($\tilde{H}'_b$ entering the forbidden region), and, accordingly, the maps $X'_b$ embedding $\tilde{H}'_b$ into a Beltrami pseudosphere $P'_b$. We can thus obtain a sequence of Beltrami pseudospheres $P'_b$ which, by varying $b$, allows us to connect in $\mathbb{R}^3$ the geodesics lying in the strip $0 < y \leq 1$. We thus pass from a local to a global description.

IV. CONCLUSIONS AND DISCUSSION

In this paper we have presented the geometrical optics generated by a refractive index of hyperbolic type. The ray trajectories are geodesics in the Poincaré–Lobachevsky half-plane, and the horocyclic waves, which are related to the Poisson kernel, represent the analogs of the Euclidean plane waves. We thus obtain two main results:
(a) The flow in the entire strip $0 < y \leq 1$ is conserved (see Proposition 3);

(b) inside each horocycle (by embedding horocycles in Beltrami pseudospheres) the ray focusing on each point $b$ of the horizontal $x$ axis: i.e., toward the boundary of $\mathbb{H}^2$, is shown.

The connection between hyperbolic geometry and optics of spatially nonuniform media can be made even tighter. In fact, it can be shown that the transfer matrix associated with lossless layered optical media is an element of the group $SU(1, 1)$, no matter how complicated the stepwise profile of the refractive index might be\textsuperscript{16,17,18}. Therefore, the action of any lossless optical multilayer can be regarded as a Möbius transformation on the unit disk\textsuperscript{16}, and therefore the natural geometric environment for these physical systems is the hyperbolic one. From this point of view, the geometrical optics description of light propagation in the “hyperbolic glass” discussed so far can be regarded as the study (in a spatially continuum setting) of the particular case of special interest, in which only motions along hyperbolic geodesics are allowed.

Appendix

(A) Let us consider the upper half–plane model of the hyperbolic two–dimensional space $U = \{z = x + iy : y > 0\}$. Then the boundary $\partial U$ of $U$ is the real axis and infinity. On $U$ we can define a metric $d$ derived from the differential $ds = |dz|/\text{Im } z$ where $dz$ is the standard Euclidean metric. The geodesics of $U$ are vertical half–lines and Euclidean semicircles with center on the real axis. The group of the orientation preserving isometries of $(U, d)$ is the Möbius group $\text{PSL}_2(\mathbb{R})$, that is the group of the $2 \times 2$ matrices of real coefficients with determinant 1. The action on $U \cup \partial U$ is defined as $\gamma(z) = (az + b)/(cz + d)$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$.

Another model of the hyperbolic two–dimensional space is the Poincaré disk $D = \{\zeta = \xi + i\eta : \xi^2 + \eta^2 < 1\}$. The map

$$\zeta = i \frac{z - i}{z + i},$$

(A.1)

{with inverse $z = -i[(\zeta + i)/(\zeta - i)]$} transfers the geometry of $U$ into the geometry of $D$. In particular, the metric on $D$ is given by the differential

$$d\zeta = \frac{2}{1 - |\zeta|^2} d_E \zeta,$$

(A.2)
where \(d_E\) is the standard Euclidean metric. In this model, the geodesics are circular arcs perpendicular to the boundary \(|\zeta| = 1\). Moreover, the group of the orientation preserving isometries of the Poincaré disk \(D\) is the group \(SU(1, 1)\) of the maps of the following form:

\[
\left\{ \frac{a\zeta + \pi}{c\zeta + \pi} : |a|^2 - |c|^2 = 1 \right\}.
\]

(A.3)

(B) The group \(G = SU(1, 1)\) admits two subgroups relevant for our analysis:

1. The subgroup \(K\) of rotations

\[
k_\theta = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \quad (0 \leq \theta < 4\pi).
\]

2. The subgroup \(A\) of matrices

\[
a_r = \begin{pmatrix} \cosh \frac{r}{2} & \sinh \frac{r}{2} \\ \sinh \frac{r}{2} & \cosh \frac{r}{2} \end{pmatrix} \quad (r \in \mathbb{R}).
\]

Let \(A^+\) denote the set \(a_r\) with \(r \geq 0\). Then the following decomposition holds.

**Cartan decomposition**

Any element \(g \in G\) can be decomposed as follows:

\[
g = k_\theta a_r k_\phi \quad (r \geq 0, 0 \leq \theta < 4\pi, 0 \leq \phi < 2\pi),
\]

(A.4)

i.e., \(G = KA^+K\). The decomposition is unique if \(g \notin K\).

We now introduce the so-called spherical functions \(\Phi_\nu(g)\) on \(G/K\) \([g \in G = SU(1, 1), \nu \in \mathbb{C}]\), which are defined as follows:

**Definition 1.** The spherical functions on \(G/K\) \([G = SU(1, 1), K = SO(2)]\) are defined as follows:

\[
\Phi_\nu(g) = \int_B \left| \frac{d(g^{-1} \cdot b)}{db} \right|^{\nu} db \quad [g \in SU(1, 1), \nu \in \mathbb{C}],
\]

(A.5)

where \(B\) is the boundary of the hyperbolic disk \(D\) (i.e., \(B = \{\zeta : |\zeta| = 1\}\)).

We can then prove the following proposition.

**Proposition 6.** The functions \(\Phi_\nu(g)\) satisfy the following properties:

(i)

\[
\Phi_\nu(g) = \mathcal{P}_{-\nu}(\cosh r) \quad [g \in SU(1, 1), \nu \in \mathbb{C}],
\]

(A.6)
where $\mathcal{P}_-\nu(\cdot)$ are the first kind Legendre functions.

(ii) \[
\Delta_D \Phi_\nu(g) = \nu(\nu - 1) \Phi_\nu(g), \quad [g \in SU(1, 1), \nu \in \mathbb{C}], \tag{A.7}
\]
where $\Delta_D$ is the hyperbolic Laplace–Beltrami operator.

(iii) \[
\mathcal{P}_{-\frac{1}{2} + i\lambda} (\cosh r) = \mathcal{P}_{-\frac{1}{2} - i\lambda} (\cosh r) \quad (\lambda \in \mathbb{R}), \tag{A.8}
\]

Proof. Let us consider the following integral \[ \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\phi}) \, d\phi \quad [f \in L^1(B)], \]
and evaluate how the Lebesgue measure \( \frac{1}{2\pi} \, d\phi \) changes when an element \( g \in SU(1, 1) \) acts on \( B \) (boundary of \( D \)). Recalling that the action of \( g \) is \( g \cdot \zeta = (a\zeta + c)/(\bar{c}\zeta + \bar{a}) \) \( (\zeta \in D) \) (see Proposition 2), we have \( g \cdot e^{i\phi} = e^{i\chi} = (ae^{i\phi} + c)/(\bar{c}e^{i\phi} + \bar{a}) \). We thus have\[ e^{i\phi} = \frac{|e^{i\chi} - a|^2}{|\bar{c}e^{i\chi} - 1|^2} = \left(1 - \frac{|c|^2}{|a|^2}\right) \frac{|\bar{c}e^{i\chi} - 1|^2}{|\bar{c}e^{i\chi} - 1|^2}, \tag{A.9}\]
since \( |a|^2 - |c|^2 = 1 \). Let us now note that \( g \cdot 0 = c/\bar{a} \); on the other hand, in view of the Cartan decomposition, we have:

\[ g \cdot 0 = k_\theta a_r \cdot 0 = e^{i\theta} \tanh \left(\frac{r}{2}\right) = |\zeta|e^{i\theta}. \tag{A.10}\]

Therefore \( 1 - |c|^2/|a|^2 = 1 - |\zeta|^2 \), and

\[ \left| \frac{\bar{c}e^{i\chi} - 1}{a} \right|^2 = \frac{1}{1 - |\zeta|e^{i(\chi - \theta)}} = \frac{1}{1 + |\zeta|^2 - 2|\zeta| \cos(\chi - \theta)}. \tag{A.11}\]

We thus have \[ \frac{1}{2\pi} \int_0^{2\pi} f(g \cdot e^{i\phi}) \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\chi}) \left|\frac{d\phi}{d\chi}\right| \, d\chi, \tag{A.12}\]
and, in view of formulae (A.9), (A.10), (A.11), (A.12), we get

\[ P(g \cdot 0, b) = \frac{1 - |\zeta|^2}{1 + |\zeta|^2 - 2|\zeta| \cos(\chi - \theta)} = \left| \frac{d(g^{-1} \cdot b)}{db} \right|. \tag{A.13}\]

Recalling Definition 17 we can write

\[ \Phi_\nu(g) = \int_B \left| \frac{d(g^{-1} \cdot b)}{db} \right|^\nu \, db = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1 - |\zeta|^2}{1 + |\zeta|^2 - 2|\zeta| \cos \phi} \right)^\nu \, d\phi. \tag{A.14}\]

Finally, writing \( \tanh(r/2) \) in place of \( |\zeta| \) [see formula (A.10)], we obtain

\[ \Phi_\nu(g) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(\cosh r + \sinh r \cos \phi)^\nu} \, d\phi = \mathcal{P}_{-\nu}(\cosh r), \tag{A.15}\]
where the last equality follows from the integral representation of the first kind Legendre functions. Formula (A.15) proves statement (i).

From formula (A.15) it follows that the spherical functions $\Phi_\nu(g)$ are bi--invariant; indeed, using the Cartan decomposition, we have

$$\Phi_\nu(g) = \Phi_\nu(k_\theta a_r k_\phi) = \Phi_\nu(a_r) = \mathcal{P}_{-\nu}(\cosh r).$$  \hspace{1cm} (A.16)

Furthermore, we can prove that [see formulae (A.13) and (A.14)]:

$$\Phi_\nu(E) = \int_B [P(E \cdot 0, b)]^\nu \, db = 1 = \mathcal{P}_{-\nu}(1).$$ \hspace{1cm} (A.17)

Let us now return to statement (ii) of Proposition 2, and recall that $P_\nu(\zeta, b)$ is an eigenfunction of the hyperbolic Laplace–Beltrami operator with eigenvalue $\nu(\nu - 1)$. We then consider an integral of the following form:

$$\int_B [P(g \cdot 0, b)]^\nu \, db = \Phi_\nu(g).$$ \hspace{1cm} (A.18)

It can be proved that this integral superposition is still an eigenfunction of the hyperbolic Laplace–Beltrami operator $\Delta_D$, having $\nu(\nu - 1)$ as eigenvalue. In particular, it follows that $\Phi_\nu(g)$ and $\Phi_{1-\nu}(g)$ [$g \in SU(1, 1)$] are both eigenfunctions of $\Delta_D$ with the same eigenvalue, and therefore they coincide. We can thus state that $\mathcal{P}_{-\frac{1}{2}+i\lambda}(\cosh r) = \mathcal{P}_{-\frac{1}{2}-i\lambda}(\cosh r)$; statements (ii) and (iii) are thus proved.

Finally, from the representation of the Poisson kernel in terms of horocyclic waves, we have

$$\mathcal{P}_{-\frac{1}{2}+i\lambda}(\cosh r) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{\cosh r + \sinh r \cos \phi} \right)^{1/2-i\lambda} \, d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1 - |\zeta|^2}{1 + |\zeta|^2 - 2|\zeta| \cos \phi} \right)^{1/2-i\lambda} \, d\phi = \int_B e^{(\frac{1}{2}-i\lambda)(\zeta,b)} \, db \quad (\lambda \in \mathbb{R}, \, \zeta \in D).$$ \hspace{1cm} (A.19)

(C) In Sec. III we have seen that it is possible to have a local isometric immersion from an horocycle to $\mathbb{R}^3$. The image of this immersion is a Beltrami pseudosphere. Now, we describe in more details the equation of the pseudosphere and its coordinates.

First, we want to recover the pseudosphere as a surface of revolution of a curve in $\mathbb{R}^3$. Let us recall that, in general, the position vector $\mathbf{r}$ of the surface of revolution generated by
the rotation of a plane curve \( z = \Phi(r) \) about the \( z \) axis is given by:

\[
\mathbf{r} = \begin{pmatrix}
    r \cos v \\
    r \sin v \\
    \Phi(r)
\end{pmatrix},
\]  

(A.20)

where \( v \) varies between 0 and \( 2\pi \). Here the circles \( r = \text{const.} \) are the parallels and the curves \( v = \text{const.} \) are the meridians. The first fundamental form associated with the surface (A.20) is given by

\[
I = \left\{ 1 + \left[ \Phi'(r) \right]^2 \right\} dr^2 + r^2 dv^2.
\]

(A.21)

We now rewrite the form (A.21) as follows:

\[
I = du^2 + r^2 dv^2,
\]

(A.22)

where

\[
du = \sqrt{1 + \left[ \Phi'(r) \right]^2} dr; \quad r = r(u).
\]

(A.23)

From the general Gauss’ theory of surfaces, we have that the total curvature is given by

\[
K = -\frac{1}{r} \frac{d^2 r}{du^2},
\]

(A.24)

whence the general pseudospherical surface of revolution with \( K = -1/\rho^2 \) adopts the form

\[
r(u) = c_1 \cosh \frac{u}{\rho} + c_2 \sinh \frac{u}{\rho}.
\]

(A.25)

In the case \( c_1 = c_2 = c \), which corresponds to a parabolic pseudospherical surface of revolution, the meridians are given by

\[
r(u) = ce^{u/\rho},
\]

(A.26)

while

\[
z = \Phi(r) = \int \sqrt{1 - \left( \frac{c}{\rho} \right)^2 e^{2u/\rho} } \, du.
\]

(A.27)

Then, the first fundamental form, with \( c = 1 \), has the following expression:

\[
I = du^2 + e^{2u/\rho} \, dv^2.
\]

(A.28)

The coordinates \( u, v \) are called \textit{Beltrami coordinates}.

The substitution

\[
\sin \varphi = \frac{c}{\rho} e^{u/\rho}
\]

(A.29)
in \((A.27)\) yields
\[
 z = \rho \left( \cos \varphi + \ln \left| \tan \frac{\varphi}{2} \right| \right)
\]  

\((A.30)\)

From formulae \((A.20)\), \((A.26)\), \((A.29)\) and \((A.30)\) we obtain
\[
 r = \begin{pmatrix}
 \rho \sin \varphi \cos v \\
 \rho \sin \varphi \sin v \\
 \rho \left( \cos \varphi + \ln \left| \tan \frac{\varphi}{2} \right| \right)
\end{pmatrix}
\]  

\((A.31)\)

and the first fundamental form (in terms of \(\varphi\) and \(v\)) is
\[
 I = \rho^2 \cot^2 \varphi \, d\varphi^2 + \rho^2 \sin^2 \varphi \, dv^2.
\]  

\((A.32)\)

Equation \((A.31)\) is the parametric form of the parabolic pseudosphere, seen as surface of revolution about the \(z\) axis of the curve called \textit{tractrix}, which satisfies the following property: the length of the tangent from the point where it touches the curve to the point where it intersects the \(z\) axis is constant and equal to \(|\rho|\); \(\varphi\) is the angle that the tangent to the meridian makes with the \(z\) axis. The angle \(\varphi\) varies between 0 and \(\pi\) so that, keeping \(\rho = 1\), the related parabolic pseudosphere has vertices at \(z = +\infty\) (corresponding to \(\varphi = \pi\)) and at \(z = -\infty\) (corresponding to \(\varphi = 0\)) and rim at \(z = 0\) (corresponding to \(\varphi = \frac{\pi}{2}\)). The curve is continuous and regular except at the point \(z = 0\), which is a cusp point. Choosing \(\rho = -1\) and varying \(\varphi\) from 0 to \(\pi\), we shall have the upward vertical \(z\) axis positively oriented (\(\varphi\) varying from 0 to \(\frac{\pi}{2}\)), and the downward vertical \(z\) axis negatively oriented (\(\varphi\) varying from \(\frac{\pi}{2}\) to \(\pi\)). In accordance with Hilbert’s theorem, for which it is impossible to embed the entire hyperbolic disk onto \(\mathbb{R}^3\), and since we want the immersion from the horocycle to the pseudosphere to be regular, then the image must be contained either in the downward component or in the upward component of the pseudosphere; thus it does not contain the cuspidal rim. Accordingly, taking \(\rho = -1\) once and for all, the first form, written in Beltrami coordinates, reads
\[
 I = du^2 + \left( e^{-u} dv \right)^2 \quad (u \geq 0),
\]  

\((A.33)\)

and the tractrix is:
\[
 x = -\sin \varphi,
 z = - \left( \cos \varphi + \ln \left| \tan \frac{\varphi}{2} \right| \right).
\]  

\((A.34)\)

Then, in Sec. \(\text{III} \text{A}\) we use the form \((A.33)\) with \(u \geq 0\), varying \(\varphi\) from \(\frac{\pi}{2}\) to 0 and, accordingly, \(z \geq 0\) (see Fig. \(\text{2}\)). In Sec. \(\text{III} \text{B}\) where the coordinate \(u\) does not enter the
game and we have chosen the coordinates $p$ and $q$, it is convenient to vary $\varphi$ from $\frac{\pi}{2}$ to $\pi$ and, in accordance, taking the downward vertical axis negatively oriented, i.e., $z \leq 0$. 
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