NON-SQUEEZING PROPERTY OF CONTACT BALLS

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Abstract

In this paper we solve a contact non-squeezing conjecture proposed by Eliashberg, Kim and Polterovich. Let \( B_R \) be the open ball of radius \( R \) in \( \mathbb{R}^{2n} \) and let \( \mathbb{R}^{2n} \times S^1 \) be the prequantization space equipped with the standard contact structure. Following Tamarkin’s idea, we apply microlocal category methods to prove that if \( R \) and \( r \) satisfy \( 1 \leq \pi r^2 < \pi R^2 \), then it is impossible to squeeze the contact ball \( B_R \times S^1 \) into \( B_r \times S^1 \) via compactly supported contact isotopies.

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1. Introduction

1.1. Contact Non-Squeezability.

Since the time Gromov established the celebrated symplectic non-squeezing theorem, there have been attempts to find its analogue in contact topology. The problem is, as pointed out in [5] and [15], that the size of an open domain in a contact manifold cannot be recognized by contact embeddings. This is due to the conformal nature of contact structures. Consider the standard contact Euclidean space \( (\mathbb{R}^{2n+1}, dz - qdp) \), the scaling map \( (q, p, z) \mapsto (\lambda q, \lambda p, \lambda z) \) is a contactomorphism that squeezes any domain into a small neighborhood of the origin when the factor \( \lambda > 0 \) is taken small enough. To avoid the scaling effect, one considers the prequantization space of \( \mathbb{R}^{2n} \), that is \( \mathbb{R}^{2n} \times S^1 \) where \( S^1 = \mathbb{R}/\mathbb{Z} \), with the contact form \( dz + \frac{1}{2}(qdp - p dq) \). The contact ball of radius \( R \) is by definition the open domain \( B_R \times S^1 \). However, this is still not satisfactory. Let \( \mathbb{R}^{2n} \cong \mathbb{C}^n = \{\omega\} \). Pick a positive integer \( N \) and define functions \( \nu : \mathbb{C}^n \to \mathbb{R} \) and \( F_N : \mathbb{R}^{2n} \times S^1 \to \mathbb{R}^{2n} \times S^1 \) by formulas \( \nu(\omega) = \frac{1}{\sqrt{1 + N\pi |\omega|^2}} \) and \( F_N(\omega, z) = (\nu(\omega)) e^{2\pi i N z} \omega, z \). Then we have
Proposition 1.1. (c.f. [5]) \( F_N \) is a contactomorphism that maps \( B_R \times S^1 \) onto \( B_r \times S^1 \) with \( r = \frac{R}{1 + N} \). It turns out that when \( N \to \infty \), we can squeeze any domain into an arbitrary small neighborhood of \( \{0\} \times S^1 \) by a single contactomorphism.

In their pioneering paper [5], Eliashberg, Kim, and Polterovich settle the correct notion of contact squeezing, in the sense of the following:

Definition 1.2 (Eliashberg-Kim-Polterovich [5]). Let \( U_1, U_2 \) be open domains in a contact manifold \( V \). We say \( U_1 \) can be squeezed into \( U_2 \) if there exists a compactly supported contact isotopy \( \Phi_s : \overline{U}_1 \to V, s \in [0, 1], such that \( \Phi_0 = Id \), and \( \Phi_1(U_1) \subset U_2 \).

When \( V \) is the prequantization space \( \mathbb{R}^{2n} \times S^1 \), they address the question whether a contact ball of certain size can be squeezed into a smaller one, and then amazingly discover a squeezing and a non-squeezing results for different sizes respectively. The following is their non-squeezing theorem.

Theorem 1.3 (Eliashberg-Kim-Polterovich [5]). Assume that \( n \geq 2 \). Then for all \( 0 < r, R < \frac{1}{\sqrt{\pi}} \), the contact ball \( B_R \times S^1 \) can be squeezed into the smaller one \( B_r \times S^1 \).

The authors offer the following quasi-classical interpretation: think of the length of the circle \( S^1 \) to be the Planck constant. Let \( \mathbb{R}^n \to \mathbb{R} \) be the probability density function and \( \mathbb{R}^n \to S^1 \) be the phase function of an unit mass quantum particle at the initial time \( t = 0 \). The quantum motion of this particle in the presence of a potential function \( V \to \mathbb{R} \) under the Schrödinger equation:

\[
\begin{align*}
\frac{i}{2\pi} \frac{\partial \psi}{\partial t} &= \frac{-1}{8\pi} \Delta \psi + V(q) \\
\psi_0(q) &= \sqrt{\rho_0(q)} e^{2\pi i F_0(q)}.
\end{align*}
\]

The functions \( F_0 \) and \( V \) generate a Legendrian submanifold of \( \mathbb{R}^{2n} \times S^1 \):

\[
\mathcal{L}(F_0) = \{(q, p, z) | p = \frac{\partial F_0}{\partial q}, z = F_0(q)\}.
\]

On the other hand, the corresponding classical motion of this particle is described by the Hamiltonian flow on the classical phase space \( \mathbb{R}^{2n} \) which has a lift to a flow of contactomorphism \( \mathbb{R}^{2n} \times S^1 \xrightarrow{f_t} \mathbb{R}^{2n} \times S^1 \) of the prequantization space given by the following system:

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -\frac{\partial V}{\partial q} \\
\dot{z} &= \frac{1}{2}p^2 - V(q).
\end{align*}
\]

For time \( t \) small enough, the image \( f_t(\mathcal{L}(F_0)) \) can be written as a Legendrian submanifold \( \mathcal{L}(F_t) \), and the push-forward of density \( f_t^* (\pi^*(\rho dq)) \) can be written as \( \pi^*(\rho dq) \). Here \( \tau \) denotes the projection from \( \mathbb{R}^{2n} \times S^1 \) to \( \mathbb{R}^{2n} \). The corresponding wave function \( \sqrt{\rho(t)(q)} e^{2\pi i F_t(q)} \) is a quasi-classical approximate solution of the above Schrödinger equation. The balls \( B_R \) are in fact energy levels of the classical harmonic oscillator. The squeezing theorem says that at sub-quantum scale (this means that \( \pi R^2 \) is less than the Planck constant), level sets are indistinguishable. One can also say that the symplectic rigidity of the balls \( B_R \) in the classical phase space \( \mathbb{R}^{2n} \) is lost after
prequantization, i.e. after joining an extra $S^1$-variable.

However, the situation is totally different when it comes to large scale. They manage to prove a non-squeezing theorem when an integer can be inserted into the two sizes.

**Theorem 1.4** (Eliashberg-Kim-Polterovich [5]). *If there exists an integer $m \in [\pi r^2, \pi R^2]$, then $B_R \times S^1$ cannot be squeezed into $B_r \times S^1$.***

In [5], they develop the idea of orderability of contactomorphism group and apply this to the study of contact squeezability. For more detailed discussions about orderability we refer the reader to Giroux’s Bourbaki paper [7] as well as the work of Sandon [15][16] in which she uses generating functions to define a bi-invariant metric on the contactomorphism groups of $\mathbb{R}^{2n} \times S^1$ and recover the non-squeezing theorem of [5]. There is also work of Albers-Merry [1] to obtain similar results by using Rabinowitz Floer homology. Also see the work of Chernov and Nemirovski [3][4] for connection to Lorentz geometry.

The case $m < \pi r^2 < \pi R^2 < m + 1$ remains open in their paper [5]. This is covered by the main result of the present paper:

**Theorem 1.5.** *If $1 \leq \pi r^2 < \pi R^2$, then it is impossible to squeeze $B_R \times S^1$ into $B_r \times S^1$.***

1.2. **Microlocal Sheaf Formalism.**

The paper uses terminologies and results from the theory of algebraic microlocal analysis. The major machinery we rely on is about sheaves and their *microlocal singular supports* which are developed in [12]. Fix a ground field $K$ and let $D(X)$ denote the derived category of sheaves of $K$–vector spaces on a smooth manifold $X$. The notion of microlocal singular supports (micro-supports) which we denote by $SS$, is defined as follows:

**Definition 1.6.** ([12], Definition 5.1.2) *Let $\mathcal{F} \in D(X)$ and $p \in T^*X$, we say $p \not\in SS(\mathcal{F})$ if there exists an open neighborhood $U$ of $p$ such that for any $x_0 \in X$ and any real $C^1$-function $\phi$ defined in a neighborhood of $x_0$ such that $\phi(x_0) = 0$ and $d\phi(x_0) \in U$, we have

$$(RT\{x|\phi(x)>0\}(\mathcal{F}))_{x_0} \cong 0.$$*

It is proved in [12] that the set $SS(\mathcal{F})$ is always conic closed and involutive (also known as coisotropic) in $T^*X$ and this allows us to approximate a given lagrangian with those $SS(\mathcal{F})$ satisfying special requirements. In addition, they have certain functorial properties with respect to Grothendieck’s six operations. For convenience of the reader, these rules are listed in Appendix (see section 6). Recently, Guillermou [9] deduced a new proof of Gromov-Eliashberg $C^0$-rigidity theorem from the involutivity of micro-supports. See also the paper of Nadler and Zaslow [14] for its application to Fukaya category.

In [17], Tamarkin gives a new approach to symplectic non-displaceability problems based on microlocal sheaf theory. For a closed subset $A$ in $T^*X$ one might consider the category of
sheaves micro-supported in $A$, but in general the set $A$ is not conic. Tamarkin’s idea is to add an extra variable $R_t$ to $X$ and to work in the localized triangulated category $D_{>0}(X \times R_t) := D(X \times R_t)/D_{[k \leq 0]}(X \times R_t)$ where $k$ is the cotangent coordinate of $R_t$ and $D_{[k \leq 0]}(X \times R_t)$ denotes the full subcategory consists of objects $\mathcal{F}$ such that $SS(\mathcal{F}) \subset \{k \leq 0\}$. By conification along the extra variable, subsets in $T^*X$ can be lifted to conic ones in $T^*_R(X \times R_t)$, which allow us to define a full subcategory $D_A(X \times R_t)$ of objects in $D_{>0}(X \times R_t)$ micro-supported on the conification of $A$.

Tamarkin also proves a key theorem of Hamiltonian shifting which asserts that for a given compactly supported Hamiltonian isotopy $\Phi$ of $T^*X$, there exists a functor $\Psi$ from $D_A(X \times R_t)$ to $D_{\Phi(A)}(X \times R_t)$ such that any object $\mathcal{F}$ in $D_A(X \times R_t)$ is isomorphic to its image $\Psi(\mathcal{F})$ in $D_{>0}(X \times R_t)$ up to torsion objects. Let $T_c : t \mapsto t + c$ be the translation by $c$ units, the torsion object is defined to be the object $\mathcal{G}$ in $D_{>0}(X \times R_t)$ such that the natural morphism $\mathcal{G} \to T_{cs}\mathcal{G}$ becomes zero in $D_{>0}(X \times R_t)$ when the positive number $c$ is sufficiently large.

It turns out that contact manifolds and contact isotopies perfectly fit into Tamarkin’s conic symplectic framework. Moreover, the possibility of torsion objects is ruled out by the fact that the circle $S^1$ is an additive quotient of the extra variable $R$. In section 2 we transform the problem of contact isotopies to the level of conic $\mathbb{Z}$-equivariant Hamiltonian isotopies on a certain symplectic manifold.

Section 3 is devoted to construction of the projector. Let me explain what projector means through this paper. Given a category $\mathcal{D}$ and its full subcategories $\mathcal{C}_1$ and $\mathcal{C}_2$. We say that $\mathcal{C}_1$ is the left semi-orthogonal complement of $\mathcal{C}_2$ if $\mathcal{C}_1 = \{\mathcal{F} \in \mathcal{D} | \forall \mathcal{G} \in \mathcal{C}_2, \hom_{\mathcal{D}}(\mathcal{F}, \mathcal{G}) = 0\}$. A projector from $\mathcal{D}$ to $\mathcal{C}_1$ is a functor $f_1 : \mathcal{D} \to \mathcal{C}_1$ which is right adjoint to the embedding functor $\mathcal{C}_1 \hookrightarrow \mathcal{D}$. Similarly the projector $f_2 : \mathcal{D} \to \mathcal{C}_2$ is the left adjoint functor of $\mathcal{C}_2 \hookrightarrow \mathcal{D}$. In other words we have $\mathcal{C}_1 \cong \mathcal{D}/\mathcal{C}_2$ and $\mathcal{C}_2 \cong \mathcal{D}/\mathcal{C}_1$. Put $E = \mathbb{R}^n$. When $\mathcal{D} = D_{>0}(E \times R_t)$ and $\mathcal{C}_2$ is consisted of objects micro-supported outside the open contact ball (see 3.2), the projector $f_2$ is represented explicitly by a convolution kernel $Q \in D_{>0}(E \times E \times R_t)$. This means that for $\mathcal{F} \in \mathcal{D}$, we have $f_2(\mathcal{F}) \cong \mathcal{F} \bullet Q$. There also exist a kernel $P$ representing $f_1$ and a kernel $K_\Delta$ representing $Id_{\mathcal{D}}$. Moreover, the natural morphisms of functors $i_1f_1 \to Id_{\mathcal{D}} \to i_2f_2$ are represented by a distinguished triangle $P \to K_\Delta \to Q^+1$ in the triangulated category $D_{>0}(E \times E \times R_t)$.

In general these projectors are expected to shrink the possibilities of micro-supports in the cotangent space according to the type of our geometry problems. This allows us to write down the kernels of these projectors. By kernel we mean the sheaf that represents the prescribed functor by composition or convolution with it. The kernel associated to a projector here in this paper is in some sense unique. Tamarkin’s idea is to use $Rhom$ between these kernels to produce Hamiltonian invariants. In his paper [17], $Rhom$ realizes non-displaceability of two compact subsets in the symplectic manifold. As an example, he shows that in $\mathbb{CP}^n$ any two choices from the two subsets $R\mathbb{CP}^n$ and $\mathbb{T}^n$ (Clifford torus) are non-displaceable. One can also assign a Novikov ring action on those $Rhom$’s. We hope this formalism will inspire an alternative approach to Fukaya category in the future.

In section 4 we construct a family of contact invariants. Given any positive integer $N$ and certain open domains $U$ in $\mathbb{R}^{2n} \times S^1$ we define a complex $C_N(U)$ using $Rhom$ between the kernel that shrinks micro-supports into $U$ and the kernel of discretized circle $\mathbb{Z}/n\mathbb{Z}$. With the help of sheaf quantization techniques (see Kashiwara-Schapira-Guillermou [8][10]) we manage to prove
that these $C_N(U)$ are contact isotopy invariants (see \textbf{Theorem 4.7}). The more important thing is that the way we define $C_N(U)$ leads to a natural $\mathbb{Z}/N\mathbb{Z}$-action on $C_N(U)$ and makes $C_N(U)$ an object of the derived category of $K[\mathbb{Z}/N\mathbb{Z}]$-modules. Furthermore this makes $C_N(U)$ into an $\mathbb{Z}/N\mathbb{Z}$-equivariant object (see \textbf{4.2} for its motivation and explanation). This approach resembles the cyclic homology in a Hamiltonian fashion. We then move forward to the computation of these invariants when $U$ is a contact ball $B_R \times S^1$ using the explicit form of projectors constructed in the previous section.

In \textbf{section 5} we show that when $\pi r^2$ and $\pi R^2$ are greater than or equal to 1, one can select an appropriate integer $N$ which depends on both $r$ and $R$ such that the naturality of morphisms between the three invariants (i.e., $C_N(B_R \times S^1)$, $C_N(B_r \times S^1)$, $C_N(R^{2n} \times S^1)$) obtained from the presumed squeezability is violated by the graded algebra structure of the Yoneda product $\text{Ext}^\cdot_{K[\mathbb{Z}/N\mathbb{Z}]}(K; K)$ after we pass to the equivariant derived category $D_{\mathbb{Z}/N\mathbb{Z}}^+(pt)$. This destroys the existence of contact squeezing and proves the main theorem (\textbf{Theorem 1.5}).

Throughout this paper we are considering not only bounded derived categories of sheaves but also locally bounded ones. By locally bounded sheaf $\mathcal{F}$ we mean for any relative compact open subset $V$, the pull-back sheaf $\mathcal{F}|_V$ is cohomologically bounded and we are working in this full subcategory. The notion of micro-support is adopted. Also there is no \textit{a priori} notion of derived hom-functor for general triangulated categories, but in our paper all categories are obtained by localizations of derived categories of sheaves. So all $\text{Rhom}$-functors we implement here are inherited from the structure of cochain complexes. Due to the unboundedness issue it may be better to bear in mind the frameworks of the homotopy category of cochain complexes, differential graded categories and model structures. All tensor products $\otimes$, as well as external products $\boxtimes$, stand for left derived functors of usual tensor products of sheaves.

For more information about the interaction between sheaf theory and symplectic topology we would like to refer the reader to the excellent expository lecture by C. Viterbo [19] which can be found on his website. It also contains many useful discussions about generating function apparatus and their application to symplectic geometry. It is Sandon who [15] uses generating function techniques to define contact homology, by extending Traynor’s symplectic homology [18], for open domains in $\mathbb{R}^{2n} \times S^1$ and gives a new proof of non-squeezability result of Eliashberg-Kim-Polterovich (\textbf{Theorem 1.4}). It turns out that for open balls their symplectic/contact homology happen to coincide with our construction when forgetting $\mathbb{Z}/N\mathbb{Z}$-action. It will be very interesting if this coincidence holds for more general domains.

Finally we need to mention that during the preparation of this paper, the author is told by Eliashberg that Fraser [6] also proves the same non-squeezing theorem. In fact she has two proofs, one based on generating function techniques and another in the original context of [5]. All known approaches are using some sort of equivariant constructions inspired by Tamarkin several years ago.

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2. The Contact Manifold \( \mathbb{R}^{2n} \times S^1 \)

2.1. Contact Structure as Conic Symplectic Structure.

Let \( \mathbb{R}^{2n} \times \mathbb{R} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \{(q, p, z) | q, p \in \mathbb{R}^n, z \in \mathbb{R}\} \) be a contact manifold with the standard contact form \( \alpha = pdq + dz \). Note that the choice of 1-form \( \alpha \) in this paper is different from \( dz + \frac{1}{2} (q dq - p dp) \) in [5]. These two 1-forms are contactomorphic to each other. The advantage of choosing \( \alpha \) is to turn the contact manifold \( \mathbb{R}^{2n} \times \mathbb{R} \) into the \( \mathbb{R}_{>0} \)-quotient of its symplectization \( Y \) where

\[
Y = T^* \mathbb{R}^{2n} \times T^*_{>0} \mathbb{R} = \{(q, p, z, \zeta) | (q, p) \in T^* \mathbb{R}^n, (z, \zeta) \in T^* \mathbb{R}, \zeta > 0\}
\]

and the natural \( \mathbb{R}_{>0} \)-action given by \( F_\lambda(q, p, z, \zeta) = (q, \lambda p, z, \lambda \zeta) \).

The manifold \( Y \) is a symplectic submanifold of the cotangent bundle with the standard symplectic form \( \omega = dp \wedge dq + d\zeta \wedge dz \), and it is easy to see that \( \omega \) satisfies the relation \( F_\lambda^* \omega = \lambda \omega \) which gives rise to the conic symplectic structure of \( Y \). Moreover, the infinitesimal \( \mathbb{R}_{>0} \)-action is given by the radial vector field \( \partial_\zeta = p \partial_p + \zeta \partial_z \), and we can see that the 1-form \( \alpha' := \iota_{\partial_\zeta} \omega = pdq + \zeta dz \) is the pull-back of the standard contact form \( \alpha \) of \( \mathbb{R}^{2n} \times \mathbb{R} \) via the quotient map \( Y \to \mathbb{R}^{2n} \times \mathbb{R} \) where \( (q, p, z, \zeta) \) is sent to \( (q, p/\zeta, z) \).

2.2. Lifting Contact Isotopy.

**Definition 2.1.** A map \( \Phi : \mathbb{R}^{2n} \times S^1 \to \mathbb{R}^{2n} \times S^1 \) is called a Hamiltonian contactomorphism if there exists a contact isotopy \( \{\Phi_t\}_{t \in I} : I \times \mathbb{R}^{2n} \times S^1 \to \mathbb{R}^{2n} \times S^1 \) such that each \( \Phi_t \) is a contactomorphism and \( \Phi_0 = Id, \Phi_1 = \Phi \). Here \( I \) is an interval containing \([0, 1]\). Furthermore, if there is a compact set \( K \) such that each \( \Phi_t \) is identity outside \( K \), we call \( \Phi \) a Hamiltonian contactomorphism with compact support.

Since the circle \( S^1 \) is the \( \mathbb{Z} \)-quotient of the real line \( \mathbb{R} \) by the shifting action \( T : z \mapsto z + 1 \), this isotopy can be lifted to \( \{\Phi_t\}_{t \in I} : I \times \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R} \) satisfying the following \( \mathbb{Z} \)-equivariant condition: \( \Phi_t \circ T = T \circ \Phi_t \).

We can then lift \( \{\Phi_t\}_{t \in I} \) again to obtain the conic Hamiltonian symplectic isotopy \( \{\Phi_s\}_{s \in I} : I \times Y \to Y \), satisfying \( \mathbb{R}_{>0} \)-homogeneity condition \( \Phi_s \circ F_\lambda = F_\lambda \circ \Phi_s \) and \( \mathbb{Z} \)-equivariant condition.
Triangulated Categories Associated to Contact Balls.

2.3. shape of micro-supports into account (see with the dual of supports along with the given Hamiltonian isotopy $S$ called the $T$ coincides the overall Lagrangian graph that there exists an essentially unique object $T$ individual graph can lift this family to a single Lagrangian submanifold of the symplectic manifold $E$. Let $D$ be the derived category of sheaves of vector spaces over $K$. Moreover, any inclusion $\{s\} \times X \times X \hookrightarrow I \times X \times X$ is non-characteristic for $\Lambda$, so that any individual graph $\Lambda_s$ of $\Phi_s$ can be reconstructed from $\Lambda$ via symplectic reduction with respect to $T^*_s I$. In Section 4 we will state a key theorem (Theorem 4.2) established in [8, Thm. 16.3] that there exists an essentially unique object $\mathcal{S} \in D(I \times X \times X)$ whose micro-support $SS(\mathcal{S})$ coincides the overall Lagrangian graph $\Lambda$ outside the zero-section and its pull-back to the initial time is isomorphic to the constant sheaf supported on the diagonal of $X$. This kind of sheaf $\mathcal{S}$ is called the sheaf quantization of $\{\Phi_s\}$ in [8] [10], in the sense that composing objects with kernel $\mathcal{S}$ gives rise to a family of endofunctors on the category of sheaves that transform their micro-supports along with the given Hamiltonian isotopy $\{\Phi_s\}$. In fact, by further kernel composition with the dual of $\mathcal{S}$ we can construct sheaf theoretic contact isotopy invariants which take the shape of micro-supports into account (see Proposition 4.5).

2.3. Triangulated Categories Associated to Contact Balls.

For any positive number $R$, denote the open ball of radius $R$ in $\mathbb{R}^{2n}$ by $B_R = \{(q,p)|q^2 + p^2 < R^2\}$. Let $U_R = B_R \times S^1$ be the contact ball in $\mathbb{R}^{2n} \times \mathbb{R}$. We can lift the contact ball $U_R$ to $\tilde{U}_R = B_R \times \mathbb{R}$ in the covering contact manifold $(\mathbb{R}^{2n} \times \mathbb{R}, \alpha)$. Define the conification of $U_R$ in $T^*_s X$ to be the inverse image of the homogeneous quotient map of $\tilde{U}_R$,

$$C_R := Cone(\tilde{U}_R) = \{(q,p,z,\zeta)|\zeta > 0, (q,p/\zeta, z) \in \tilde{U}_R\} \subset T^*_s X.$$

We want to describe the category of objects micro-supported on this ball in the following sense: Let $E := \{q \in \mathbb{R}^n\}$ and $X = E \times \mathbb{R}$. Fix a ground field $K$, let $D(X) = D(E \times \mathbb{R})$ denote the derived category of sheaves of vector spaces over $K$ on $X$. Let $D_{\leq 0}(X)$ be the full subcategory of $D(X)$, consisting of all sheaves whose microlocal singular support is within $\{\zeta \leq 0\}$. We define
a triangulated category \(\mathcal{D}_{>0}(X)\) to be the quotient [17]
\[
\mathcal{D}_{>0}(X) = \mathcal{D}_{>0}(E \times R) := D(X)/D_{\leq 0}(X).
\]

Let \(D_{Y \setminus C_R}(X) \subseteq \mathcal{D}_{>0}(X)\) be the full subcategory of all objects micro-supported on the conic closed set \(Y \setminus C_R\). We will see that the embedding functor \(D_{Y \setminus C_R}(X) \hookrightarrow \mathcal{D}_{>0}(X)\) has a left adjoint functor, hence it makes sense to write

**Definition 2.2.** Let \(\mathcal{D}_{C_R}(X) := \mathcal{D}_{>0}(X)/D_{Y \setminus C_R}(X)\) be the left semi-orthogonal complement of \(D_{Y \setminus C_R}(X)\).

In Section 3.5 we will see moreover this left adjoint functor \(\mathcal{D}_{>0}(X) \to D_{Y \setminus C_R}(X)\) is represented by a composition kernel \(\mathcal{Q}_R \in D(X \times X)\). Similarly, the projection functor \(\mathcal{D}_{>0}(X) \to \mathcal{D}_{C_R}(X)\) is given by another composition kernel \(\mathcal{R}_R \in D(X \times X)\). Let \(\tilde{\Delta} = \{q_1 = q_2, z_1 \geq z_2\}\) be the subset of \(X \times X\) then the constant sheaf \(K_{\tilde{\Delta}}\) becomes the composition kernel of the identity endofunctor of \(\mathcal{D}_{>0}(X)\).

The semi-orthogonal decomposition of \(\mathcal{D}_{>0}(X)\) is given by the fact that these three kernels fit into the following distinguished triangle in \(D(X \times X)\):
\[
\mathcal{Q}_R \to K_{\tilde{\Delta}} \to \mathcal{Q}_R \xrightarrow{+1}.
\]

3. Construction of the Projector

3.1. Basics of Convolution and Composition.

Let \(V\) be a vector space and \(M_1, M_2, M_3\) be manifolds. We now define a **convolution functor**:
\[
D(M_1 \times M_2 \times V) \times D(M_2 \times M_3 \times V) \to D(M_1 \times M_3 \times V)
\]
as follows. Let
\[
p_{ij} : M_1 \times M_2 \times M_3 \times V \times V \to M_i \times M_j \times V
\]
be the maps given by
\[
p_{12}(m_1, m_2, m_3, v_1, v_2) = (m_1, m_2, v_1), \quad p_{23}(m_1, m_2, m_3, v_1, v_2) = (m_2, m_3, v_2),
\]
and
\[
p_{13}(m_1, m_2, m_3, v_1, v_2) = (m_1, m_3, v_1 + v_2).
\]
Notice that \(p_{13}\) takes sum of the vector part.

**Definition 3.1.** For objects \(\mathcal{F} \in D(M_1 \times M_2 \times V)\) and \(\mathcal{G} \in D(M_2 \times M_3 \times V)\) we define their **convolution product** by
\[
\mathcal{F} \ast \mathcal{G} := R \mathcal{P}_{13}! (p_{12}^! \mathcal{F} \otimes p_{23}^! \mathcal{G}) \in D(M_1 \times M_3 \times V).
\]

For objects \(\mathcal{F} \in D(V)\) and \(\mathcal{G} \in D(M \times V)\), we identify \(V\) with \(pt \times pt \times V\) and \(M \times V\) with \(pt \times M \times V\) or \(M \times pt \times V\). Then we can define a commutative operation
\[
\mathcal{F} \ast \mathcal{G} := \mathcal{F} \ast pt \mathcal{G} \cong pt \mathcal{G} \ast \mathcal{F} \in D(M \times V).
\]

The following proposition allows us to work with categories of type \(\mathcal{D}\). This symbol stands for left semi-orthogonal pieces.
Proposition 3.2. (Proposition 2.2 of [17]) Take $V = \mathbb{R}$ and $\mathcal{F} \in D(M \times \mathbb{R})$ and let $K_{[0, \infty)} \to K_0$ be the restriction morphism in $D(\mathbb{R})$. Then $\mathcal{F} \in D_{>0}(M \times \mathbb{R})$ if and only if the natural morphism $\mathcal{F} \ast K_{[0, \infty)} \to \mathcal{F} \ast K_0 \simeq \mathcal{F}$ is an isomorphism.

Corollary 3.3. Let $\mathcal{F} \in D_{>0}(M_1 \times M_2 \times \mathbb{R})$ and $\mathcal{G} \in D(M_2 \times M_3 \times \mathbb{R})$. We then have $\mathcal{F} \ast \mathcal{G} \in D_{>0}(M_1 \times M_3 \times \mathbb{R})$.

Now let us introduce the concept of Fourier transform:

Definition 3.4. Let $E$ be a real vector space of dimension $n$. The Fourier transform with respect to $E = \{q\}$ is a functor $D_{>0}(E \times \mathbb{R}) \overset{\hat{\cdot}}{\rightarrow} D_{>0}(E^* \times \mathbb{R})$ given by

$$\mathcal{F} \overset{\hat{\cdot}}{\rightarrow} \hat{\mathcal{F}} := \mathcal{F} \ast K_{\{t + pq \geq 0\}} [n].$$

The Fourier transform functor has the following effect on the micro-supports. Let $E^* \times E \overset{\hat{\cdot}}{\rightarrow} E \times E^*$ be given by $(p, q) \overset{\hat{\cdot}}{\rightarrow} (-q, p)$.

Theorem 3.5. (Theorem 3.6 of [17]) $SS(\hat{\mathcal{F}}) = \hat{SS}(\mathcal{F})$ up to a subset of $\{k \leq 0\}$.

On the other hand, we have another operation called kernel composition. In contrast to the presence of vector summation in convolution, the notion of composition only involves projection maps of product manifolds. Let $N_1, N_2, N_3$ be manifolds accompanied with the projection maps $\pi_{i,j} : N_1 \times N_2 \times N_3 \to N_i \times N_j$ for $i < j$. We define the composition product by a functor

$$D(N_1 \times N_2) \times D(N_2 \times N_3) \xrightarrow{\circ} D(N_1 \times N_3).$$

Definition 3.6. Let $\mathcal{F} \in D(N_1 \times N_2)$ and $\mathcal{G} \in D(N_2 \times N_3)$ we set

$$\mathcal{F} \circ \mathcal{G} := \pi_{13}^{-1} \mathcal{F} \otimes \pi_{23}^{-1} \mathcal{G} \in D(N_1 \times N_3).$$

Throughout the paper we will need to switch between the notions of convolution and composition, so it is worth saying a few more about their relation. Following [11], assume that $\gamma$ is a convex closed proper cone in $V$ and we have a constant sheaf $K_\gamma \in D(V)$. Let $\gamma_l = \{(v_1, v_2) | v_1 - v_2 \in \gamma\}$ and $\gamma_r = \{(v_1, v_2) | v_2 - v_1 \in \gamma\}$. We then have two constant sheaves $K_{\gamma_l}, K_{\gamma_r} \in D(V_1 \times V_2)$, one translates the convolution to left composition and another to right composition.

Proposition 3.7. For any sheaf $\mathcal{F} \in D(M \times V)$, we have the following isomorphisms in $D(M \times V)$:

$$\mathcal{F} \ast K_\gamma \cong K_{\gamma_l} \circ \mathcal{F} \cong \mathcal{F} \circ K_{\gamma_r}.$$ 

The above relation can be understood in the sense of isomorphisms between endofunctors of $D(M \times V)$. In the following subsections, we will describe how to characterize semi-orthogonal projection functors between certain triangulated categories defined by microlocal conditions in terms of convolution and composition.

3.2. Quantizing Hamiltonian Actions.

Recall that $X = \{(q, z)\} = \mathbb{R}^n \times \mathbb{R} = E \times \mathbb{R}$. Before we construct the kernel of the projector $\mathcal{P}_R$ to $D_{C_k}(X)$ as an object of $D(X \times X)$, we should first consider its symplectic counterpart for open ball $B_R = \{q^2 + p^2 < R^2\}$ in $T^*E = \{(q, p) | q \in E, p \in E^*\}$. By introducing a new
real variable \( R = \{ t \} \) and its cotangent coordinates \( T^* R = \{(t, k)\} \), we can define the notion of sectional micro-supports as follows:

**Definition 3.8.** For object \( \mathcal{F} \in \mathcal{D}_{>0}(E \times \mathbb{R}) \) we define its sectional microlocal singular support \( \mu S(\mathcal{F}) \) by the following subset of \( T^* E \):

\[
\mu S(\mathcal{F}) := \{(q, p) \in T^* E | \exists t : (q, p, t, 1) \in SS(\mathcal{F}) \}.
\]

In other words, \( \mu S(\mathcal{F}) \) is the projection of the section of the conic closed set \( SS(\mathcal{F}) \) at the hyperplane \( \{k = 1\} \) on \( T^* E \). Now let \( D_{T^* E \setminus B}(E \times \mathbb{R}) \subset \mathcal{D}_{>0}(E \times \mathbb{R}) \) be the full subcategory of all objects \( \mathcal{F} \) whose sectional micro-supports \( \mu S(\mathcal{F}) \) sit outside of the open symplectic ball \( B_R \).

To be specific, for any \( \mathcal{F} \in D_{T^* E \setminus B}(E \times \mathbb{R}) \subset \mathcal{D}_{>0}(E \times \mathbb{R}) \), we should have \( \mu S(\mathcal{F}) \subset \{(q, p) \in T^* E | q^2 + p^2 > R^2\} \).

Let \( D_{T^* E \setminus B}(E \times \mathbb{R}) \) be the embedding functor. We show later in **Theorem 3.11** that this embedding functor has a left adjoint functor \( \mathcal{D}_{>0}(E \times \mathbb{R}) \to D_{T^* E \setminus B}(E \times \mathbb{R}) \) given by a convolution kernel \( Q_R \in D(E_1 \times E_2 \times \mathbb{R}) \). Put \( \Delta = \{(q_1, q_2, t) | q_1 = q_2, t \geq 0\} \subset E_1 \times E_2 \times \mathbb{R} \). The constant sheaf \( K_\Delta \in D(E_1 \times E_2 \times \mathbb{R}) \) stands for the convolution kernel of the identity endofunctor. Notice that we are using the additive structure of \( \mathbb{R} \) in convolution operation. Now consider the distinguished triangle in \( D(E_1 \times E_2 \times \mathbb{R}) \):

\[
\mathcal{P}_R \to K_\Delta \to Q_R \xrightarrow{+1}.
\]

Denote the left semi-orthogonal complement of \( D_{T^* E \setminus B}(E \times \mathbb{R}) \) in \( \mathcal{D}_{>0}(E \times \mathbb{R}) \) by \( \mathcal{D}_B(E \times \mathbb{R}) \). **Theorem 3.11** tells that \( \mathcal{D}_B(E \times \mathbb{R}) \cong \mathcal{D}_{>0}(E \times \mathbb{R}) / D_{T^* E \setminus B}(E \times \mathbb{R}) \). One can call \( \mathcal{D}_B(E \times \mathbb{R}) \) the category of objects micro-supported on the symplectic ball \( B_R \). It turns out that the semi-orthogonal projection functor \( \mathcal{D}_{>0}(E \times \mathbb{R}) \to \mathcal{D}_B(E \times \mathbb{R}) \) is given by convolution with \( \mathcal{P}_R \).

Let us start with geometric properties of the shape of the ball: rotational symmetry. Let \( H(q, p) = q^2 + p^2 \) be the distinguished Hamiltonian function of classical harmonic oscillator, then \( B_R \subset \{(q, p) \in T^* E | H(q, p) > R^2\} \). Consider the corresponding Hamiltonian equations:

\[
\frac{dq}{da} = \frac{\partial H}{\partial p} = 2p, \quad \frac{dp}{da} = -\frac{\partial H}{\partial q} = -2q.
\]

Its solution is given by Hamiltonian rotations \( \mathbb{H} : \mathbb{R} \to Sp(2n) \) such that for \( a \in \mathbb{R} \),

\[
\begin{pmatrix} q(a) \\ p(a) \end{pmatrix} = \mathbb{H}(a) \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} \cos(2a) & \sin(2a) \\ -\sin(2a) & \cos(2a) \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}.
\]

We have a unique lifting \( \overline{\mathbb{H}} : \mathbb{R} \to \widetilde{Sp}(2n) \) to the universal covering group of \( Sp(2n) \) such that \( \overline{\mathbb{H}}(0) = e \). Denote the image of \( \overline{\mathbb{H}} \) by \( G \). \( G \) is isomorphic to \( \mathbb{R} \).

For time \( a \in G \), put \( (q_1, p_1) = (q(0), p(0)) \) and \( (q_2, p_2) = (q(a), p(a)) \). Let us write down the change of variable formula given by the Hamiltonian rotation:

\[
(2) \quad q_2 = \frac{q_1 + \sin(2a)p_2}{\cos(2a)}, \quad p_1 = \frac{\sin(2a)q_1 + p_2}{\cos(2a)}
\]
By generating function we mean a function $S(q_1, q_2)$ satisfying equations
\[
\frac{\partial S}{\partial q_1} = -p_1, \quad \frac{\partial S}{\partial q_2} = p_2.
\]

By combining them with the explicit expression of the rotation $H(a)$ the above equations allow a solution (locally depends on $a$) up to a constant:
\[
(3) \quad S(q_1, q_2) = \frac{\cos(2a)}{2\sin(2a)}(q_1^2 + q_2^2) - \frac{q_1 q_2}{\sin(2a)}.
\]

Next, substitute the variable $q_2$ by $q_1 \cos(2a) + p_1 \sin(2a)$ and write $q_1 = q$ and $p_1 = p$ in the function $S(q, q_2)$, we get the local expression $S(q, p)$ appearing in our decomposition. Indeed, the globally defined generating function can be interpreted by the following action functional (see [13])
\[
S = \int_{\gamma} pdq - Hda
\]
where the path $\gamma$ is the integral curve of Hamiltonian equations, parametrized by $a$, starting from $(q_1, p_1)$ and ending at $(q_2, p_2)$. If the time interval is contained in $(-\frac{\pi}{2}, \frac{\pi}{2})$ we have the unique integral curve from $a = 0$. For time interval longer than this, we can divide it into subintervals so that each of them admits an unique integral curve, and take the sum of their integral values we obtain $S$.

Let us write down the local form of $S$ in $q, p$:
\[
S_a(q, p) = \frac{\sin(4a)}{4}(p^2 - q^2) - \sin^2(2a)pq.
\]

The above variables are actually vectors in $\mathbb{R}^n$ and here by writing their products we mean the inner products, for example $q_1 q_2 := \langle q_1, q_2 \rangle$. Let $\alpha := pdq$ be the Liouville 1-form on $T^*E$ and put $\eta = Hda$. Let $\mathcal{A}: G \times T^*E_1 \to T^*E_2$ be the corresponding Hamiltonian action via $\tilde{H}$. Taking differential of the integral interpretation $S = \int_{\gamma} pdq - Hda$ gives us the decomposition
\[
dS = \mathcal{A}^* \alpha - pdq - Hda = \mathcal{A}^* \alpha - \alpha - \eta.
\]

Then we have the following proposition:

**Proposition 3.9.** The function $G \times T^*E \xrightarrow{S} \mathbb{R}$ leads to the decomposition
\[
\mathcal{A}^* \alpha = \alpha + \eta + dS.
\]

It is also easy to verify that $\mathcal{A}^* \alpha - \alpha = (q^2 + p^2)da + dS$ explicitly.

With functions $H : G \times T^*E \to T^*G$ and $S : G \times T^*E \to \mathbb{R}$ we can perform the following Legendrian embedding:
\[
(4) \quad A : G \times T^*E \to T^*G \times T^*E_1 \times T^*E_2 \times \mathbb{R}
\]
\[
(a, x) \mapsto (-\eta(a, x), x^a, A_a(x), -S_a(x))
\]
here $x := (q_1, p_1)$ and its antipode $x^a := (q_1, -p_1)$ are in the cotangent space $T^*E_1$. The minus sign in 1-form $-\eta(a, x)$ stands for the element $(a, -H(x))$ of the cotangent space $T^*G$. From the above decomposition it is easy to see that the image of $A$ (still denoted by $A$) is a
Legendrian submanifold in \( T^*G \times T^*E_1 \times T^*E_2 \times \mathbb{R}_t \). Let \( L \) be the conification of this Legendrian in \( T^*G \times T^*E_1 \times T^*E_2 \times T^*\mathbb{R}_t \) by

\[
L = \text{Cone}(A) := \{(k\xi, t, k)| (\xi, t) \in A, k > 0\}
\]

then \( L \) becomes a conic Lagrangian submanifold in \( T^*G \times T^*E_1 \times T^*E_2 \times T^*\mathbb{R}_t \cong T^*(G \times E_1 \times E_2 \times \mathbb{R}_t) \).

**Proposition 3.10.** There exists an object \( S \) of \( D(G \times E_1 \times E_2 \times \mathbb{R}) \) such that \( S|_{0 \times E_1 \times E_2 \times \mathbb{R}} \cong K_\Delta \) where \( \Delta = \{(q_1, q_2, t)| q_1 = q_2, t \geq 0\} \) and \( SS(S) \cap T^*_{k>0}(G \times E_1 \times E_2 \times \mathbb{R}) \subset L \). The object \( S \) is called sheaf quantization of Hamiltonian rotations.

**Proof.** In fact, the general theory of the existence and the uniqueness of sheaf quantization of Hamiltonian symplectomorphisms has been well-established by **Proposition 3.2** and **Lemma 3.3** in the paper [10]. What we are going to do is to look for an incarnation of \( S \) for the sake of the computation of the contact invariant in the present paper.

For \( 0 < a < \frac{\pi}{2} \), let us first construct a constant sheaf in \( D_{>0}((0, \frac{\pi}{2}) \times E_1 \times E_2 \times \mathbb{R}) \) by

\[
S := K_{\{(0, q_1), q_2, t)| t + S_a(q_1, q_2) \geq 0\}}.
\]

It is easy to see that we have

\[
SS(S) = \{(kdS, k)| k \geq 0, t = -S_a(q_1, q_2)\} \bigcup \{k = 0\}.
\]

According to (4) one has \( SS(S) \cap T^*_{k>0} \subset L \).

Note that the function

\[
S_a(q_1, q_2) = \frac{\cos(2a)}{2 \sin(2a)}(q_1^2 + q_2^2) - \frac{q_1q_2}{\sin(2a)}
\]

is undefined at \( a = 0 \) and \( a = \frac{\pi}{2} \). To see the behavior as \( a \) approaches to \( 0 \) from the right, let us rewrite the generating function as follows:

\[
S_a(q_1, q_2) = \frac{\tan(a)}{2}(q_1^2 + q_2^2) + \frac{1}{2 \sin(2a)}(q_1 - q_2)^2.
\]

For those \( q_1 - q_2 \neq 0 \), the function \( S_a \) goes to the positive infinity as \( a \rightarrow 0^+ \). Hence the sheaf \( S \) has no support for \( \mathbb{R}_t \). On the other hand, for those \( q_1 - q_2 = 0 \), \( S_a \) goes to zero when \( a \rightarrow 0^+ \). In this case the support for \( \mathbb{R}_t \)-coordinate is \( t \geq 0 \). Thus by joining \( a = 0 \) to the domain of \( S \) we can define

(5)

\[
S|_{(0, \frac{\pi}{2}) \times E_1 \times E_2 \times \mathbb{R}} \cong K_{\{0 < a < \frac{\pi}{2}, t + S_a(q_1, q_2) \geq 0\}} \bigcup \{(0, q_1, q_2, t)| t \geq 0\} \bigcup \{a = \frac{\pi}{2}, q_1 + q_2 \neq 0\}.
\]

Let \( \Delta = \{q_1 = q_2, t \geq 0\} \subset E_1 \times E_2 \times \mathbb{R} \), we see that \( S|_{a=0} \cong K_\Delta \) is the convolution kernel of the identity functor of \( D_{>0}(E \times \mathbb{R}) \).

On the other hand, to see the behavior of \( S \) when \( a \) approaches to \( \pi/2 \), let us rewrite the generating function as the following form:

\[
S_a(q_1, q_2) = \frac{\cos(2a)}{2 \sin(2a)}(q_1^2 + q_2^2) - \frac{1}{\sin(2a)}q_1q_2 = \frac{q_1^2 + q_2^2}{2 \tan(a)} - \frac{(q_1 + q_2)^2}{2 \sin(2a)}.
\]

For those \( q_1 + q_2 \neq 0 \), when \( a \) approaches to \( \frac{\pi}{2} \) from the left, the function \( S_a(q_1, q_2) \) goes to the negative infinity, hence the sheaf is supported on the whole \( \mathbb{R}_t \). And for \( q_1 + q_2 = 0 \), \( S_a(q_1, q_2) \) goes to zero when \( a \) goes to \( \frac{\pi}{2} \). Thus by joining \( a = \frac{\pi}{2} \) we define

(6)

\[
S|_{(0, \pi/2) \times E_1 \times E_2 \times \mathbb{R}} \cong K_{\{0 < a < \frac{\pi}{2}, t + S_a(q_1, q_2) \geq 0\}} \bigcup \{(\frac{\pi}{2}, q_1, q_2, t)| t \geq 0\} \bigcup \{a = \frac{\pi}{2}, q_1 + q_2 \neq 0\}.
\]
This way one can even extend $\mathcal{S}$ to an object of $\mathcal{D}_{>0}(\mathbb{R}^2 \times E_1 \times E_2 \times \mathbb{R})$. However, due to the periodicity of the function $S_a$ this construction cannot be extended naively outside the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ since there will be multiple Hamiltonian flow trajectories from $q_1$ to $q_2$. Instead, having all data lifted to the universal covering $\hat{G}$, one convolutes local quantizations and constructs the global quantization $\hat{\mathcal{S}}$.

Note that we need $\mathcal{S}$ to act on the category $\mathcal{D}_{>0}(\mathbb{R}^2 \times \mathbb{R})$ by convolution, so one expects $\mathcal{S}|_{a_1+a_2} \cong \mathcal{S}|_{a_1} \bullet \mathcal{S}|_{a_2}$ for all $a_1, a_2$ in $\hat{G}$. Indeed, let $a_1, a_2$ in $\left(0, \frac{\pi}{2}\right)$ and write down their local expressions $\mathcal{S}|_{a_1} = \mathcal{K}_E((q_1, q_2, t)|t+S_{a_1}(q_1, q_2)\geq 0) \in \mathcal{D}_{>0}(\mathbb{R}^2 \times E_1 \times \mathbb{R})$ and $\mathcal{S}|_{a_2} = \mathcal{K}_E((q_2, q_3, t)|t+S_{a_2}(q_2, q_3)\geq 0) \in \mathcal{D}_{>0}(\mathbb{R}^2 \times E_3 \times \mathbb{R})$. Let $\rho$ be the projection along $E_2$. We then define stalkwisely

\begin{equation}
\mathcal{S}|_{a_1+a_2} = \rho_\ast \mathcal{K}_E((q_1, q_2, q_3, t)|t+S_{a_1}(q_1, q_2)+S_{a_2}(q_2, q_3)\geq 0).
\end{equation}

One extends this out by defining $\mathcal{S}|_{a_1+a_2+\cdots+a_m} = \mathcal{S}|_{a_1} \bullet \mathcal{S}|_{a_2} \bullet \cdots \bullet \mathcal{S}|_{a_m}$. The extension to $\mathcal{S}|_{a\leq 0}$ is similar. By the integral form of the action functional this extension $\mathcal{S}$ only depends on the sum $\Sigma a_j$ and it is easy to see $SS(\mathcal{S}) \cap T^*_k > 0 \subset L$.

\[\square\]

3.3. Objects Micro-Supported on a Symplectic Ball.

Once we obtain the sheaf quantization $\mathcal{S} \in \mathcal{D}_{>0}(\mathbb{R}^2 \times \mathbb{R})$, we can perform Fourier transform (Definition 3.4) on it with respect to $G$ (i.e. $a$-coordinate). Set

\[\hat{\mathcal{S}} := \mathcal{S} \bullet \mathcal{K}_E((a, b, t)|t+ab\geq 0)(1) \in \mathcal{D}_{>0}(\mathbb{R}^* \times \mathbb{R}).\]

Since $SS(\mathcal{S}) \subset L$, we have $SS(\hat{\mathcal{S}}) \subset \hat{L}$ where the hat $\hat{\wedge} : T^*G \to T^*G^*$ denotes the image under $(a, b) \mapsto (-b, a)$. Observe that for $\text{Cone}(\mathcal{A}) = L \subset T^*(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R})$ the map $\wedge$ operates on $\{(a, -H(x))\} \subset T^*G$ whenever there is a point $(a, x) \in \mathbb{R}^2 \times \mathbb{R}$ whose value $S(a, x)$ hits the last coordinate of $\mathcal{A}$. By (4) and Theorem 3.5 we get a useful estimation of our sectional micro-support:

\begin{equation}
\mu S(\hat{\mathcal{S}}) \subset \{(b, a, q_1, p_1, q_2, p_2)| (q_2, p_2) = A_0(q_1, p_1), b = H(q_1, p_1) = H(q_2, p_2)\}.
\end{equation}

The object $\hat{\mathcal{S}}$ will be our main ingredient to construct the desired projector. First, convolute $\hat{\mathcal{S}}$ with the constant sheaf $\mathcal{K}_R$:

\begin{equation}
\hat{\mathcal{S}} \circ \mathcal{K}_R = (\mathcal{S} \bullet \mathcal{K}_E((a, b, t)|t+ab\geq 0)(1)) \circ \mathcal{K}_R \cong \mathcal{S} \bullet (\mathcal{K}_E((a, b, t)|t+ab\geq 0) \circ \mathcal{K}_R)(1)
\end{equation}

\[\cong \mathcal{S} \bullet \mathcal{K}_E((a=0, t\geq 0) \cong \mathcal{S}|_{a=0} \cong \mathcal{K}_\Delta.\]

This gives us the kernel representing the identity endofunctor on $\mathcal{D}_{>0}(E \times \mathbb{R})$. Now let $\mathcal{K}_{b<R^2} \in D(G^*)$ be the constant sheaf supported on the open set $\{b < R^2\}$ and define

\begin{equation}
\mathcal{P}_R := \hat{\mathcal{S}} \circ \mathcal{K}_{b<R^2} \in \mathcal{D}_{>0}(E_1 \times E_2 \times \mathbb{R})
\end{equation}

and its counterpart

\begin{equation}
\mathcal{Q}_R := \hat{\mathcal{S}} \circ \mathcal{K}_{b>R^2} \in \mathcal{D}_{>0}(E_1 \times E_2 \times \mathbb{R})
\end{equation}
so that we can form the following distinguished triangle in \( D_{>0}(E_1 \times E_2 \times \mathbb{R}) \):

\[ \mathcal{P}_R \to \mathcal{K}_\Delta \to \mathcal{Q}_R \xrightarrow{+1} . \]

Recall that we define \( D_{T^*E \setminus B}(E \times \mathbb{R}) \) by the full subcategory of all objects micro-supported in the closed set \( T^*E \setminus B \) and \( \mathcal{D}_B(E \times \mathbb{R}) \) to be its left semi-orthogonal complement. One might think that \( \mathcal{D}_B(E \times \mathbb{R}) \) is some sort of subcategory of objects micro-supported on \( B_R \). The precise statement is included in the following theorem:

**Theorem 3.11.** Convolution with the distinguished triangle \( (\mathcal{P}_R \to \mathcal{K}_\Delta \to \mathcal{Q}_R \xrightarrow{+1}) \) over \( E \) gives a semi-orthogonal decomposition of the triangulated category \( D_{>0}(E \times \mathbb{R}) \) with respect to the null system \( D_{T^*E \setminus B}(E \times \mathbb{R}) \).

**Proof.** There are three major steps in constructing a semi-orthogonal decomposition of a given triangulated category. First, we identify a null system \( D_{T^*E \setminus B}(E \times \mathbb{R}) \) and its projector functor given by \( \mathcal{Q}_R \). Second, we identify its left semi-orthogonal complement \( \mathcal{D}_B(E \times \mathbb{R}) \) and the corresponding projector functor \( \mathcal{P}_R \). Finally we show that these two projectors, together with the identity endofunctor, give the desired decomposition of \( D_{>0}(E \times \mathbb{R}) \). We refer the reader to Section 6 (Appendix) for functorial properties of micro-supports which are used in the proof.

**Step 1.** Pick an arbitrary object \( \mathcal{G} \) of \( D(E_1 \times \mathbb{R}) \), we are going to show that \( \mathcal{G} \mathcal{E}_1 \mathcal{Q}_R \) is an object of \( D_{T^*E \setminus B}(E \times \mathbb{R}) \). Let us define the following notation of the maps:

\[
\begin{array}{ccc}
G^* \times E_1 \times E_2 \times \mathbb{R}_1 \times \mathbb{R}_2 & \xrightarrow{p_1} & E_1 \times \mathbb{R}_1 \\
& & \downarrow p \\
& & E_2 \times \mathbb{R} \\
& & \xleftarrow{q_2} G^* \times E_1 \times E_2 \times \mathbb{R}_2
\end{array}
\]

here \( p_1 \) and \( q_2 \) are projections, and \( p \) performs summation \( \mathbb{R}_1 \times \mathbb{R}_2 \to \mathbb{R} \) and projection for the rest of components. By (11) we have

\[
\mathcal{G} \mathcal{E}_1 \mathcal{Q}_R \cong \mathcal{G} \mathcal{E}_1 \tilde{S} \circ_{\mathbb{R}} \mathcal{K}_{b \geq \mathbb{R}^2} \]

\[
\cong R\mathcal{P}\mathcal{R}_p(p_1^{-1}\mathcal{G} \otimes q_2^{-1}\tilde{S} \otimes \mathcal{K}_{(\mathbb{R}^2,\infty) \times E_1 \times E_2 \times \mathbb{R}_1 \times \mathbb{R}_2}).
\]

Let us estimate the corresponding micro-supports. By Proposition 6.2 and (8) and Theorem 3.5 we have

\[
SS(p_1^{-1}\mathcal{G}) \subset \{ b, 0, q_1, k p_1, q_2, 0, t_1, k, t_2, 0 | k \geq 0 \}
\]

\[
SS(q_2^{-1}\tilde{S}) \subset \{ (b, k', \beta, q_1, k' p_1, q_2, k' p_2, t_1, 0, t_2, k') | k' \geq 0, (q_2, p_2') = A_\beta(q_1, p_1') = b \}.
\]

It is clear that

\[
SS(p_1^{-1}\mathcal{G}) \cap SS(q_2^{-1}\tilde{S}) \subset T_{G^* \times E_1 \times E_2 \times \mathbb{R}_1 \times \mathbb{R}_2}(G^* \times E_1 \times E_2 \times \mathbb{R}_1 \times \mathbb{R}_2),
\]

thus by Proposition 6.3(i) we have estimation

\[
SS(p_1^{-1}\mathcal{G} \otimes q_2^{-1}\tilde{S}) \subset SS(p_1^{-1}\mathcal{G}) + SS(q_2^{-1}\tilde{S})
\]

\[
\subset \{ (b, k', \beta, q_1, k p_1, k' p_1', q_2, k' p_2', t_1, k, t_2, k' | k, k' \geq 0, H(q_2, p_2') = b \}.
\]

On the other hand we know that

\[
SS(\mathcal{K}_{(\mathbb{R}^2,\infty) \times E_1 \times E_2 \times \mathbb{R}_1 \times \mathbb{R}_2}) = \{ (R^2, k') | k' \geq 0 \} \times T_{E_1 \times E_2 \times \mathbb{R}_1 \times \mathbb{R}_2}(E_1 \times E_2 \times \mathbb{R}_1 \times \mathbb{R}_2)
\]
\[ \bigcup \{(b,0)|b > R^2\} \times T_{E_1 \times E_2 \times R_1 \times R_2}^* (E_1 \times E_2 \times R_1 \times R_2) \].

It is also easy to see from (12) and (13) we have
\[
SS(p_1^{-1} \mathcal{G} \otimes q_2^{-1} \hat{\mathcal{S}}) \cap SS(K_{[R^2, \infty]} \times E_1 \times E_2 \times R_1 \times R_2)^a \subset T_{G^* \times E_1 \times E_2 \times R_1 \times R_2}^* (G^* \times E_1 \times E_2 \times R_1 \times R_2).
\]

Let us denote
\[ \mathcal{H} := p_1^{-1} \mathcal{G} \otimes q_2^{-1} \hat{\mathcal{S}} \otimes K_{[R^2, \infty]} \times E_1 \times E_2 \times R_1 \times R_2. \]

Apply Proposition 6.3(i) on (12)(13) we get the following estimate
\[
(14) \quad SS(\mathcal{H}) \subset SS(p_1^{-1} \mathcal{G} \otimes q_2^{-1} \hat{\mathcal{S}}) + SS(K_{[R^2, \infty]} \times E_1 \times E_2 \times R_1 \times R_2)
\]
\[
\subset \{(R^2, k'', k', q_1, k'p_1, q_2, k'p_2, t_1, k, t_2, k')|k, k', k'' \geq 0, H(q_1, p_1') = R^2\}
\]
\[
\cup \{(b, k', q_1, k'p_1, q_2, k'p_2, t_1, k, t_2, k')|k, k' \geq 0, H(q_1, p_1') = b > R^2\}.
\]

Before qualifying the members of the set \(SS(Rp_1\mathcal{H})\), let us factor the map \(p\) into a summation map \(\sigma\) followed by a projection \(\pi\):
\[
p: G^* \times E_1 \times E_2 \times R_1 \times R_2 \xrightarrow{\sigma} G^* \times E_1 \times E_2 \times R_1 \times R_2 \xrightarrow{\pi} E_2 \times R_2,
\]
where \(\sigma: (t_1, t_2) \mapsto (t_1, t_1 + t_2)\) and the rest of them are projections. Then apply Proposition 6.1 on (14) we have
\[
(15) \quad SS(R\sigma_1\mathcal{H}) \subset \{(R^2, k'', k', q_1, k'p_1, q_2, k'p_2, t_1, k - k', t_2, k')|k, k', k'' \geq 0, H(q_1, p_1') = R^2\}
\]
\[
\cup \{(b, k', q_1, k'p_1, q_2, k'p_2, t_1, k - k', t_2, k')|k, k' \geq 0, H(q_1, p_1') = b > R^2\}.
\]

Denote the above union of sets by \(Z\), and define
\[
\rho: T^*(G^* \times E_1 \times E_2 \times R_1 \times R_2) \to T^*(E_2 \times R_2) \times G^{**} \times E_1^* \times R_1^*
\]
to be the projection along \(G^* \times E_1 \times R_1\), and let
\[
i: T^*(E_2 \times R_2) \to T^*(E_2 \times R_2) \times G^{**} \times E_1^* \times R_1^*
\]
be the embedding into the zero of cotangent components \(G^{**} \times E_1^* \times R_1^*\). For any vector \(u = (q_2, p, t_2, 1) \in SS(R\pi_1(R\sigma_1\mathcal{H})) \subset T^*(E_2 \times R_2)\), apply Proposition 6.4 on (15) we have the following relation
\[
(-, 0, -0, q_2, p, -0, t_2, 1) = i(u) \in \overline{\rho(SS(R\sigma_1\mathcal{H}))} \subset \rho(Z)
\]
where \(\overline{\rho(Z)}\) means the closure of \(\rho(Z)\) and hyphens denote the arguments. Hence there are sequences \(\{k\}, \{k'\}\) in terms of the corresponding coordinates of \(Z\) satisfying the limit conditions \(\{k'\} \to 1\) and \(\{k - k'\} \to 0\) and \(\{k'p_2\} \to p\), respectively. This means that both \(\{k\}\) and \(\{k'\}\) approach to 1, and then \(\{p_2\}\) approaches to \(p\).

Since for any \(p_2' \in \{p_2\}\) we have \(H(q_2, p_2') \geq R^2\), we must have its limit satisfying \(H(q_2, p) \geq R^2\). Hence the object \(R\pi_1R\sigma_1\mathcal{H}\) must be a member of the category \(D_{T^*E_1 \setminus B}(E_2 \times R_2)\). So we finish the Step 1 by concluding that
\[
\mathcal{G} \bullet_{E_1} Q_R \cong Rp_1\mathcal{H} \cong R\pi_1R\sigma_1\mathcal{H} \in D_{T^*E_1 \setminus B}(E_2 \times R_2).
\]

Step 2. We need to prove that convolution with \(\mathcal{P}_R\) projects onto the left semi-orthogonal complement of \(D_{T^*E_1 \setminus B}(E \times R)\), which means for any object \(\mathcal{F}\) in \(D_{T^*E_1 \setminus B}(E \times R)\) and object \(\mathcal{G}\) in \(D_{>0}(E \times R)\), we have \(\text{Rhom}(\mathcal{G} \bullet_{E_1} \mathcal{P}_R, \mathcal{F}) \cong 0\).
Let us define a bunch of maps in the following: let
\[ p_1 : G^* \times E_1 \times E_2 \times R_1 \times R_2 \xrightarrow{q_1} G^* \times E_1 \xrightarrow{\pi_1} E_1 \times R_1 \]
be successive projections onto the domain of \( \mathcal{G} \), and
\[ p : G^* \times E_1 \times E_2 \times R_1 \times R_2 \xrightarrow{s} G^* \times E_2 \times R \xrightarrow{\pi_2} E_2 \times R \]
onto the domain of \( \mathcal{F} \). Here the map \( s \) performs summation \( R_1 \times R_2 \to R \), and for the rest of the components \( s \) is projection. \( \pi_2 \) is projection along \( G^* \).

Next, we want to describe the restriction to \( (-\infty, R^2) \). Let
\[ (-\infty, R^2) \times E_1 \times E_2 \times R_1 \times R_2 \xrightarrow{j_1} G^* \times E_1 \times E_2 \times R_1 \times R_2 \]
\[ (-\infty, R^2) \times E_1 \times R_1 \xrightarrow{j_2} G^* \times E_1 \times R_1 \]
\[ (-\infty, R^2) \times E_2 \times R \xrightarrow{j_3} G^* \times E_2 \times R \]
be open embeddings of \( (-\infty, R^2) \) into \( G^* \). Also define the projection map \( G^* \times E_1 \times E_2 \times R_1 \times R_2 \xrightarrow{\pi_2} G^* \times E_1 \times E_2 \times R_2 \) onto the domain of \( \mathcal{S} \).

With the above maps and (10) we can unwrap \( \text{Rhom}(\mathcal{G} \bullet \mathcal{P}_R, \mathcal{F}) \) as the following:

\[ \text{Rhom}(\mathcal{G} \bullet \mathcal{P}_R, \mathcal{F}) \]
\[ \cong \text{Rhom}(\mathcal{G} \bullet (\mathcal{S} \circ b_{b < R^2}); \mathcal{F}) \]
\[ \cong \text{Rhom}((\mathcal{G} \bullet \mathcal{S}) \circ b_{b < R^2}; \mathcal{F}) \]
\[ \cong \text{Rhom}(R\pi_2 Rj_2 j_2^{-1}(Rs_1(p_1^{-1}\mathcal{G} \otimes q_2^{-1}\mathcal{S})); \mathcal{F}) \]
\[ \cong \text{Rhom}(R\pi_2 Rj_2 R\pi_1 j_1^{-1}(p_1^{-1}\mathcal{G} \otimes q_2^{-1}\mathcal{S})); \mathcal{F}) \]
\[ \cong \text{Rhom}(R\pi_2 R\pi_1 j_1^{-1}(q_1^{-1}\pi_1^{-1}\mathcal{G} \otimes q_2^{-1}\mathcal{S})); \mathcal{F}) \]
\[ \cong \text{Rhom}(R\pi_1 ((Rj_1 q_1^{-1} \pi_1^{-1}\mathcal{G} \circ q_2^{-1}\mathcal{S}); \mathcal{F}) \]
\[ \cong \text{Rhom}(R\pi_1 ((Rj_1 q_1^{-1} \pi_1^{-1}\mathcal{G} \circ q_2^{-1}\mathcal{S}); \mathcal{F}) \]
\[ \cong \text{Rhom}(Rj_1 q_1^{-1} \pi_1^{-1}\mathcal{G}; \text{Rhom}(q_2^{-1}\mathcal{S}; p^1 \mathcal{F})) \]
Denote the embedding to the zero section of the second cotangent component. We know that 

\[ Rq \]

(20)

\[ Rhom(q_2^{-1} \tilde{S}; p^1 \mathcal{F}) \]

Denote the object \( Rhom(q_2^{-1} \tilde{S}; p^1 \mathcal{F}) \) by \( \mathcal{K} \), then the above sequence of adjunctions tells us

\[ \text{(16)} \]

\[ Rhom(\mathcal{G} \bullet P_R, \mathcal{F}) \cong Rhom(j_1^{-1} \pi_1^{-1} \mathcal{G}; Rq_1 \ast j^1 \text{Rhom}(q_2^{-1} \tilde{S}; p^1 \mathcal{F})) \]

Let us estimate the corresponding micro-supports with the help of (8) Proposition 6.2:

\[ (17) \]

\[ SS(q_2^{-1} \tilde{S}) \subset \{(b, k, \beta, q_1, k, q_2, k, p_2, t_1, 0, t_2, k)|k \geq 0, (q_2, p_2) = A_\beta(q_1, p_1), H(q_1, p_1) = b\}. \]

Recall that from \( (18) \) is easy to see that 

\[ \text{Proposition } 6.3(ii) \]

\[ SS(\mathcal{K}) \subset \{(b, 0, q_1, 0, q_2, k, p_2, t_1, k', t_2, k')|k' > 0, H(q_2, p_2) \geq R^2\} \cup \{k' = 0\}. \]

From (17)(18) it is easy to see that

\[ SS(q_2^{-1} \tilde{S}) \cap SS(p^1 \mathcal{F}) \subset T_G^* \times E_1 \times E_2 \times R_1 \times R_2 (G^* \times E_1 \times E_2 \times R_1 \times R_2), \]

so by Proposition 6.3(ii) we have estimation

\[ SS(Rhom(q_2^{-1} \tilde{S}, p^1 \mathcal{F})) \subset SS(q_2^{-1} \tilde{S})^a + SS(p^1 \mathcal{F}) \]

\[ \subset \{(b, -k, \beta, q_1, k, q_2, k, p_2 - k, p_2, t_1, k', t' - k)|k \geq 0, k' > 0, H(q_1, p_1) = b, H(q_2, p_2) \geq R^2\} \]

\[ \cup T^*(G^* \times E_2 \times R_1 \times R_2) \times \{(0, -k) \in R^* \times R^*|k \geq 0\}. \]

Denote the above union of sets by \( Z \), then the above inequality translates to 

\[ SS(\mathcal{K}) \subset Z. \]

On the other hand, let us define

\[ \rho : T^*((-\infty, R^2) \times E_1 \times E_2 \times R_1 \times R_2) \rightarrow T^*((-\infty, R^2) \times E_1 \times R_1) \]

to be the projection along \( E_2 \times R_2 \), and let 

\[ i : T^*(G^* \times E_1 \times R_1) \hookrightarrow T^*(G^* \times E_1 \times R_1) \times E_2^* \times R_2^* \]

be the embedding to the zero section of the second cotangent component. Now we assume that in \( T^*((-\infty, R^2) \times E_1 \times R_1) \) there exists a vector

\[ v = (b, \beta, q_1, p, t_1, 1) \in SS(Rq_1, j^{-1} \mathcal{K}). \]

By Proposition 6.4, the image \( i(v) = (b, \beta, q_1, p, -0, t_1, 1, -0) \) must lie in the closure of \( \rho(SS(j^{-1} \mathcal{K}, p^1 \mathcal{F})) \) and hence in the closure of \( \rho(Z) \). This means that there exist sequences \( \{p_2\}, \{p_2'\}, \{k\}, \{k'\} \) of cotangent coordinates of \( Z \) such that \( \{k'p_2' - kp_2\} \rightarrow 0 \) and \( \{k\} \rightarrow 1 \) and \( \{k' - k\} \rightarrow 0 \). Thus both \( \{k'\} \) and \( \{k\} \) approach to 1 and since \( \{p_2\} \) ranges in the compact set \( H(q_2, p_2) = b \) then both \( \{p_2\} \) and \( \{p_2'\} \) have the same finite limit. However, from \( b \in (-\infty, R^2) \)

we know that \( H(q_2, p_2) = b < R^2 = H(q_2, p_2') \), which leads to a contradiction.

The above discussion shows that

\[ (19) \]

\[ Rq_1 \ast j^{-1} \mathcal{K} \in D_{\leq 0}(\mathcal{G} \bullet P_R, \mathcal{F}) \]

On the other hand since \( \mathcal{G} \in D_{>0}(E_1 \times R_1) \) we have

\[ (20) \]

\[ j_1^{-1} \pi_1^{-1} \mathcal{G} \in D_{>0}(\mathcal{G} \bullet P_R, \mathcal{F}). \]
Then from (16)(19)(20) we can conclude that
\[ \text{Rhom}(\mathcal{G} \bullet \mathcal{P}_R; \mathcal{F}) \cong \text{Rhom}(j_1^{-1} \pi_1^{-1} \mathcal{G}; Rq_1 \ast j_1^{-1} \mathcal{H}) \cong 0. \]

Hence \( \mathcal{P}_R \) is the convolution kernel of the projector onto the left semi-orthogonal complement of \( D_{T^*E \setminus B}(E \times \mathbb{R}) \).

**Step 3.** We focus on the triangulated structure of \( D_{>0}(E \times \mathbb{R}) \). Since \( D_{T^*E \setminus B}(E \times \mathbb{R}) \) is defined by the full subcategory of objects whose sectional micro-supports are outside \( B_R \), then for any object \( \mathcal{F} \) it is clear that \( \mathcal{F} \in D_{T^*E \setminus B}(E \times \mathbb{R}) \) if and only if \( \mathcal{F}[1] \in D_{T^*E \setminus B}(E \times \mathbb{R}) \). Moreover, for any distinguished triangle in \( D_{>0}(E \times \mathbb{R}) \):
\[ \mathcal{F} \to \mathcal{G} \to \mathcal{H} \xrightarrow{+1} \]
we always have the estimation
\[ SS(\mathcal{G}) \subset SS(\mathcal{F}) \bigcup SS(\mathcal{H}). \]

It implies that if both \( \mathcal{F} \) and \( \mathcal{H} \) live in \( D_{T^*E \setminus B}(E \times \mathbb{R}) \), then so does \( \mathcal{G} \). Also, \( 0 \in D_{T^*E \setminus B}(E \times \mathbb{R}) \), hence \( D_{T^*E \setminus B}(E \times \mathbb{R}) \) forms a null system in \( D_{>0}(E \times \mathbb{R}) \). Furthermore, for any \( \mathcal{F} \) and \( \mathcal{G} \) in \( D_{>0}(E \times \mathbb{R}) \), we have the relation
\[ SS(\mathcal{F} \bigoplus \mathcal{G}) = SS(\mathcal{F}) \bigcup SS(\mathcal{G}), \]
thus the triangulated subcategory \( D_{T^*E \setminus B}(E \times \mathbb{R}) \) is **thick**: condition \( \mathcal{F} \bigoplus \mathcal{G} \in D_{T^*E \setminus B}(E \times \mathbb{R}) \) implies that both \( \mathcal{F} \) and \( \mathcal{G} \) live in \( D_{T^*E \setminus B}(E \times \mathbb{R}) \). Let us denote its left semi-orthogonal complement by \( D_B(E \times \mathbb{R}) \). Now for any object \( \mathcal{F} \in D_{>0}(E \times \mathbb{R}) \) we have a decomposition in terms of the following distinguished triangle:
\[ \mathcal{F} \bullet \mathcal{P}_R \to \mathcal{F} \to \mathcal{F} \bullet \mathcal{Q}_R \xrightarrow{+1} \]
where \( \mathcal{F} \bullet \mathcal{P}_R \in D_B(E \times \mathbb{R}) \) and \( \mathcal{F} \bullet \mathcal{Q}_R \in D_{T^*E \setminus B}(E \times \mathbb{R}) \). This means that the embedding functor \( D_{T^*E \setminus B}(E \times \mathbb{R}) \hookrightarrow D_{>0}(E \times \mathbb{R}) \) admits a right adjoint and there is an equivalence \( D_{T^*E \setminus B}(E \times \mathbb{R}) \cong D_{>0}(E \times \mathbb{R})/D_B(E \times \mathbb{R}) \). It is equivalent to say that \( D_B(E \times \mathbb{R}) \hookrightarrow D_{>0}(E \times \mathbb{R}) \) admits a left adjoint functor, and we have the following equivalence of triangulated categories:
\[ D_B(E \times \mathbb{R}) \cong D_{>0}(E \times \mathbb{R})/D_{T^*E \setminus B}(E \times \mathbb{R}). \]

\( \square \)

### 3.4. Interlude: the geometry of \( \mathcal{P}_R \).

In this subsection we would like to mention the periodic structure of the projector \( \mathcal{P}_R \). First, let us unwrap \( \mathcal{P}_R \) in terms of the sheaf quantization \( \mathcal{S} \) and its convolutions (or compositions). By (10) we have
\[ \mathcal{P}_R \overset{(21)}{=} \hat{\mathcal{S}} \circ \mathcal{K}_{b < R^2} \]
\[ = \left( \mathcal{S} \circ \mathcal{K}_{\{t + ab \geq 0\}}[1] \right) \circ \mathcal{K}_{b < R^2} \]
\[ \cong \mathcal{S} \circ \left( \mathcal{K}_{\{t + ab \geq 0\}} \circ \mathcal{K}_{b < R^2} \right)[1] \]
\[ \cong S \bullet K_{((a,t)|a \leq 0, t+aR^2 \geq 0)}. \]

Define \( \mathcal{G} : a \mapsto a - \frac{\pi}{2} \) to be the translation along \( G \) and \( \mathcal{T} : t \mapsto t - \frac{\pi}{2} R^2 \) the translation along \( \mathbb{R}_t \). We have \( \mathcal{G}_* S \cong K_A[-n] \circ S \), here \( A = \{(q_1, q_2)\} \) is the anti diagonal in \( E \times E \). Moreover, \( \mathcal{G} \) and \( \mathcal{T} \) interact in the following way:

\[
\mathcal{P}_R \cong S \bullet K_{((a,t)|a \leq 0, t+aR^2 \geq 0)} \cong \mathcal{G}_* S \bullet \mathcal{G}_* K_{((a,t)|a \leq 0, t+aR^2 \geq 0)} \cong K_A[-n] \circ S \bullet K_{a \leq -\frac{\pi}{2}, t+aR^2 \geq 0} \cong K_A[-n] \circ S \bullet K_{\mathcal{T}^{-1}(a \leq -\frac{\pi}{2}, t+aR^2 \geq 0)} \cong (K_A[-n] \circ \mathcal{T}^{-1} K_{a \leq -\frac{\pi}{2}, t+aR^2 \geq 0}).
\]

After applying \( \mathcal{T} \) on both sides and composing with \( K_A \) it becomes

\[
K_A \circ E \mathcal{T}(\mathcal{P}_R) \cong S \bullet K_{a \leq -\frac{\pi}{2}, t+aR^2 \geq 0}[-n].
\]

On the other hand, from the closed embedding \( \{a \leq 0\} \hookrightarrow \{a \leq \frac{\pi}{2}\} \) there is a restriction morphism

\[
\gamma : S \bullet K_{a \leq \frac{\pi}{2}, t+aR^2 \geq 0} \rightarrow S \bullet K_{a \leq 0, t+aR^2 \geq 0}
\]

which is exactly the morphism

\[
\gamma : K_A[n] \circ E \mathcal{T}(\mathcal{P}_R) \rightarrow \mathcal{P}_R.
\]

Let \( \Gamma \) be the co-cone of \( \gamma \). By co-cone we mean that \( \Gamma \) can be embedded into the distinguished triangle

\[
\Gamma \rightarrow K_A[n] \circ E \mathcal{T}(\mathcal{P}_R) \rightarrow \mathcal{P}_R \xrightarrow{+1},
\]

thus by (22) we have

\[
\Gamma \cong S \bullet K_{0 < a \leq \frac{\pi}{2}, t+aR^2 \geq 0}.
\]

The object \( \Gamma \) is a constant sheaf concentrated in degree 0 as follows. For \( 0 < a < \frac{\pi}{2} \), let

\[
f(a) = -S_a(q_1, q_2) - aR^2.
\]

The function \( f(a) \) has two critical points (or one with multiplicity two) \( a_1 \) and \( a_2 \). We assume that \( f(a_1) \geq f(a_2) \). Let \( d = (q_1q_2)^2 - R^2(q_1^2 + q_2^2 - R^2) \) and \( D = \{(q_1, q_2)|d \geq 0\} \).

**Proposition 3.12.** Let \( \Sigma = \{(q_1, q_2, t)|(q_1, q_2) \in D, f(a_2) \leq t < f(a_1)\} \) be the subset of \( E_1 \times E_2 \times \mathbb{R} \). We have \( \Gamma \cong K_\Sigma \).

**Proof.** According to the construction of the sheaf quantization object \( S(\text{Proposition 3.10}) \) we know from (6) there is

\[
S_{([0, \pi/2] \times E \times E \times \mathbb{R})} \cong K_{(0 < a < \frac{\pi}{2}, t+S_a(q_1, q_2) \geq 0)} \cup \{(\frac{\pi}{2}, q_a, t)|t \geq 0\} \cup \{a=\frac{\pi}{2}, q_a + q_2 \neq 0\}.
\]

Now let \( \pi_1 : G \times E \times E \times \mathbb{R}_1 \times \mathbb{R}_2 \rightarrow G \times \mathbb{R}_1 \) and \( \pi_2 : G \times E \times E \times \mathbb{R}_1 \times \mathbb{R}_2 \rightarrow G \times E \times E \times \mathbb{R}_2 \) be the projection maps, and let \( \pi : G \times E \times E \times \mathbb{R}_1 \times \mathbb{R}_2 \rightarrow G \times E \times E \times \mathbb{R}_t \) be the summation
We fix the order by setting the discriminant, then only when it is always a positive number. So whenever it possesses a real solution we arrive to the quadratic equation in the Hamilton-Jacobi equation for the function $f(a) = -S_a(q_1, q_2) - aR^2$ on $0 < a < \frac{\pi}{2}$. The function $f$ goes to the negative infinity when $a$ goes to $\frac{\pi}{2}$, and $f$ goes to the positive infinity when $a$ goes to 0 while maintaining $q_1 \neq q_2$. So it is clear that the object $R\pi K_W$, at any given $q_1 \pm q_2 \neq 0$, is only possibly supported on those $t$ which sit between the critical values of $f(a)$. The corresponding critical points satisfy $\frac{\partial f}{\partial a} = 0$ and are exactly the solutions of the Hamilton-Jacobi equation for the function $H(q, p) = q^2 + p^2$ being fixed to $R^2$. It means that there exists $p_1$ and $p_2$ satisfying $H(q_1, p_1) = R^2$ and $(p_1, p_2) = H(a)(q_1, p_1)$. From the equations (2)

$$p_1 = \frac{q_2 - q_1 \cos(2a)}{\sin(2a)}, \quad R^2 = q_1^2 + p_1^2$$

we arrive to the quadratic equation in $\cos(2a)$:

$$R^2 \cos^2(2a) - 2q_1q_2 \cos(2a) + (q_1^2 + q_2^2 - R^2) = 0.$$ 

We see that for $|\xi| > 1$,

$$R^2 \xi^2 - 2q_1q_2 \xi + (q_1^2 + q_2^2 - R^2) = (\xi q_1 - q_2)^2 + (R^2 - q_1^2)(\xi^2 - 1)$$

is always a positive number. So whenever it possesses a real solution $\xi$, $\xi$ must be in $[-1, 1]$, to which we can assign with a cosine value $\xi = \cos(2a)$. Let $d = (q_1q_2)^2 - R^2(q_1^2 + q_2^2 - R^2)$ be its discriminant, then only when $d > 0$ we have two distinct solutions $\cos(2a_1)$ and $\cos(2a_2)$. Here we fix the order by setting $0 \leq a_1 \leq a_2 \leq \frac{\pi}{2}$. According to the boundary behavior of $S_a$ when $a$ approaches 0 and $\frac{\pi}{2}$ respectively, it is always true that $f(a_2) < f(a_1)$. Set $D = \{(q_1, q_2)|d \geq 0\}$, so we know that in the case $(q_1, q_2) \in \text{int}(D)$, the object $R\pi K_W$ is supported by

$$\{(q_1, q_2, t)|(q_1, q_2) \in \text{int}(D), f(a_2) \leq t < f(a_1)\}.$$ 

Case 2. For $q_1 = q_2 = q$, the function $S_a(q_1, q_2)$ becomes $-\tan(a)q^2$, and it is easy to see that $f(a) = \tan(a)q^2 - aR^2$ has only one critical point $a_2$ satisfying

$$\cos(a_2) = \frac{|q|}{R}.$$ 

Since $f(a)$ goes to 0 as $a$ goes to 0, and $f(a)$ goes to the infinity as $a$ goes to $\frac{\pi}{2}$, the object $R\pi K_W$ has support on

$$\{(q, q, t)|f(a_2) \leq t < 0\}.$$
**Case 3.** \( q_1 = -q_2 := q \), the function \( S_a(q_1, q_2) \) becomes \( \frac{q^2}{\tan(a)} \). Let us consider the function \( f(a) = -\frac{q^2}{\tan(a)} - aR^2 \) where \( a \) ranges over \((0, \frac{\pi}{2}]\). Solving the critical point equation \( \frac{\partial f}{\partial a} = 0 \) we get the unique solution \( a_1 \in (0, \frac{\pi}{2}] \) satisfying

\[
\sin(a_1) = \frac{|q|}{R}.
\]

Since \( f(a) \) goes to negative infinity as \( a \) approaches to 0 and goes to \(-\frac{\pi}{2}R^2\) as \( a \) goes to \( \frac{\pi}{2} \), the object \( RpKW \) has support on the set

\[
\{(q, -q, t)| -\frac{\pi}{2}R^2 \leq t < f(a_1)\}.
\]

These three cases are depicted in the following graph. The horizontal X-axis represents \( a \), and the vertical Y-axis parametrizes \( t \).

Notes that in **Case 3** the function \( f(a) \) can be defined at \( a = \frac{\pi}{2} \), where the fiber of \( p \) has a closed end sitting at the rightmost red vertical line. While in other cases we have open ends at both 0 and \( \frac{\pi}{2} \), so the cohomology of compact support only counts for \( t \) values of convex part. It is clear that **Case 2** and **Case 3** are actually continuous degenerations of **Case 1** to the left and right respectively. Thus we can synthesize definitions of \( f(a) \) and \( a_1, a_2 \) in all cases and by the definition of \( \Sigma = \{(q_1, q_2, t)| (q_1, q_2) \in D, f(a_2) \leq t < f(a_1)\} \), we have the following isomorphisms:

\[
\Gamma \cong RpKW \cong K\Sigma.
\]

When \( E \) is one dimensional, we can visualize \( \Sigma \) in the following pictures. The vertical axis represents \( t \)-coordinate. The top is open while the bottom boundary surface is closed.

Recall the distinguished triangle (23)

\[
\Gamma \to K_A[n] \circ \mathcal{I}(P_R) \to P_R \Rightarrow 1.
\]

The building blocks of \( P_R \) consist of copies of \( \Gamma \) as follows. Let \( \Gamma_1 = \mathcal{I}^{-1}\Gamma \) and for \( j \geq 2 \) let \( \Gamma_{j+1} = \mathcal{I}^{-1}K_A[-n] \circ \Gamma_j \). Each constant sheaf \( \Gamma_{j+1} \) is obtained from the previous constant sheaf \( \Gamma_j \) by lifting its \( t \)-coordinate by \( \frac{\pi}{2}R^2 \) and twisting \( E_1 \times E_2 \) part and then shift its cohomological.
degree by $n$. The support of $P_R$ is a stack of the supports of those building blocks $\Gamma_j$’s. In fact one can glue all $\Gamma_j$ (concentrated at degree $jn$) to obtain the projector $P_R$ in the sense of homotopy colimit.

Let us mention a few more words about the micro-support of $\Gamma$. $SS(\Gamma)$ has nontrivial cotangent fibers only for those $t$ hitting the critical value of $f(a)$. From the expression $f(a) = -S_a(q_1, q_2) - aR^2$ we have

$$df = -dS - d(aR^2) = -A^\ast\alpha + \alpha + H da - R^2 da$$

$$= p_1 dq_1 - p_2 dq_2 + (H - R^2) da = \frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \frac{\partial f}{\partial a} da.$$

After evaluating at the critical points of $f(a)$, we get $H = R^2$, and hence

$$q_i^2 + \left(\frac{\partial f}{\partial q_2}\right)^2 = q_i^2 + p_i^2 = H = R^2.$$

This also allows one to show the projector property (Theorem 3.11) by concretely considering each blocks and their gluing. Notice that the upper cap can be deformed to the lower bottom by some non-characteristic deformations.

3.5. Objects Micro-Supported on a Contact Ball.

Now we can define the contact projector $\mathcal{P}_R$ by lifting the symplectic framework. Recall that by Theorem 3.11 there exists a symplectic projector which is represented by the object $\mathcal{P}_R$ (10) of $D_{>0}(E_1 \times E_2 \times \mathbb{R}_t)$, in the sense that for any object $\mathcal{F}$ of $D_{>0}(E_1 \times \mathbb{R}_t)$ we have

$$\mathcal{F} \bullet \mathcal{P}_R \in D_B(E_2 \times \mathbb{R}_t).$$

Consider the difference map $\delta : \mathbb{R}_1 \times \mathbb{R}_2 \to \mathbb{R}_t, (z_1, z_2) \mapsto z_2 - z_1$. $\delta$ lifts kernel convolutions to compositions. Let us define

$$\mathcal{P}_R := \delta^{-1}\mathcal{P}_R \in D(E_1 \times \mathbb{R}_1 \times E_2 \times \mathbb{R}_2) = D(X_1 \times X_2).$$

**Proposition 3.13.** $\mathcal{P}_R$ is the composition kernel of the projector from $D_{>0}(X)$ to $D_{C_R}(X)$ (see Definition 2.2).
Proof. For any object $F \in D_{>0}(X_1) = D_{>0}(E_1 \times R_1)$ we have the following isomorphisms by Proposition 3.7:

$$F \circ P_R = F \circ (\delta^{-1}P_R) \cong F \circ P_R.$$

This shows that composition with $P_R$ sends objects in $D_{>0}(X)$ to objects in $D_{C_R}(X)$. On the other hand, let $Q_R$ be the pull-back $\delta^{-1}Q_R$ and $\tilde{\Delta}$ be the set $\{(q_1, z_1, q_2, z_2) | q_1 = q_2, z_2 \geq z_1\}$ in $X_1 \times X_2$. Then by applying the functor $\delta^{-1}$ on this distinguished triangle

$$P_R \rightarrow K_{\Delta} \rightarrow Q_R \rightarrow,$$

we get the following distinguished triangle:

$$P_R \rightarrow K_{\tilde{\Delta}} \rightarrow Q_R \rightarrow.$$

Thus the object $P_R$ serves as projector from $D_{>0}(X)$ to $D_{C_R}(X)$. $\square$

In fact, $P_R$ lives in $D_{<0,>0}(X_1 \times X_2)$. The triangulated category $D_{<0,>0}(X_1 \times X_2)$ can be identified with the quotient of $D(X_1 \times X_2)$ by the full subcategory of objects micro-supported on the closed cone $T^*(R_1 \times R_2) \setminus \{\zeta_1 < 0, \zeta_2 > 0\}$. In other words, an object $G$ lives in $D_{<0,>0}(X_1 \times X_2)$ if and only if the natural morphism $G \rightarrow G \circ K_{\{z_1 \leq 0, z_2 \geq 0\}}$ is an isomorphism. This is analogue to Proposition 3.2 and can be generalized to closed proper cones in the cotangent fibers.

We would also like to remark that the composition with an object of $D_{<0,>0}(X \times X)$ gives rise to an endofunctor of the category $D_{>0}(X)$. One might think symbolically that the kernels in $D_{<0,>0}$ eats the input $> 0$ with its $< 0$ and leaves another $> 0$ as the output.

4. Contact Isotopy Invariants

4.1. Admissible Open Subsets in the Prequantization Space.

In this section we define a family of contact isotopy invariants for certain open subsets $U$ in the prequantization space $R^{2n} \times S^1$. First, lift $U$ to an open set $\tilde{U}$ in $R^{2n} \times R_2$. Here $\tilde{U}$ is periodic in the sense that it satisfies $T\tilde{U} = \tilde{U}$ where $T : z \mapsto z + 1$. Let $X$ denote $R^n \times R_z$ and denote the cotangent coordinate of $z$ by $\zeta$. Consider the conification of $\tilde{U}$ in $T_{>0}^*(X)$, namely the conic set $C_U := \{(q, p, z, \zeta) | \zeta > 0, (q, p / \zeta, z) \in \tilde{U}\}$.

$$C_U \subset T_{>0}^*(X) \quad \downarrow \quad \downarrow$$

$$\quad \tilde{U} \subset R^{2n} \times R \quad . \quad \downarrow \quad \downarrow$$

$$\quad U \subset R^{2n} \times S^1$$

Let $D_{Y \setminus C_U}(X) \rightarrow D_{>0}(X)$ be the full subcategory of objects micro-supported outside $C_U$. 
**Definition 4.1.** We call the open set $U$ **admissible** if the above embedding has a left adjoint functor, and if there exists a projector kernel $\mathcal{P}_U \in D(X \times X)$ whose composition with $D_{>0}(X)$ projects to the quotient category $D_{>0}(X)/D_{Y \setminus C_U}(X) \cong D_U(X)$.

**Example.** The contact ball $B_R \times S^1$ is admissible by Proposition 3.13. The associated projector kernel is $\mathcal{P}_R$.

We need to show that the admissibility of an open subset is preserved under Hamiltonian contactomorphism. Let $\Phi$ be a Hamiltonian contactomorphism with compact support. This means that $\Phi$ is given by a contact isotopy (with compact support) $\Phi_s: \mathbb{R}^{2n} \times S^1 \to \mathbb{R}^{2n} \times S^1$ where $s \in I$ and $I$ is an interval containing $[0, 1]$, $\Phi_0 = Id$ and $\Phi_1 = \Phi$. Recall that in Section 2.3 we lift those $\Phi_s$ as in the following diagram:

$$
\begin{array}{c}
C_U \subset T_{>0}^*(X) \xrightarrow{\Phi} T_{>0}^*(X) \\
\downarrow \quad \downarrow \\
\bar{U} \subset \mathbb{R}^{2n} \times \mathbb{R} \xrightarrow{\Phi_s} \mathbb{R}^{2n} \times \mathbb{R} \\
\downarrow \quad \downarrow \\
U \subset \mathbb{R}^{2n} \times S^1 \xrightarrow{\Phi_s} \mathbb{R}^{2n} \times S^1
\end{array}
$$

So the lifting of $\Phi$ becomes a homogeneous Hamiltonian symplectomorphism of $T_{>0}^*(X)$. To see how $\Phi$ comes into play in the triangulated category of sheaves on $T_{>0}^*(X)$, we need to look for its sheaf quantization. Recall that we have defined the overall Lagrangian graph (1)

$$
\Lambda = \{(s, -h_s(\Phi_s(y)), y^a, \Phi_s(y)) | s \in I, y \in \mathbb{R}^n\} \subset T^*(I \times X \times X).
$$

Roughly speaking, the sheaf quantization of $\Phi$ is a sheaf micro-supported in $\Lambda$. In the papers of Guillermou-Kashiwara-Schapira [10] and Guillermou [8], the existence and uniqueness of sheaf quantization have been well-established. We restate their result here:

**Theorem 4.2.** ([10] Proposition 3.2) There exists an unique locally bounded sheaf $\mathcal{I}$ in $D(I \times X \times X)$ satisfying

1. $SS(\mathcal{I}) \subset \Lambda \cup T_{I \times X \times X}^*(I \times X \times X)$,

2. $\forall s \in I$, $\mathcal{I}_s \circ \mathcal{I}_s^{-1} \cong \mathcal{I}_s^{-1} \circ \mathcal{I}_s \cong \mathcal{I}_0 \cong K_{\Delta X}$.

Here $\mathcal{I}_s$ denotes the pull-back of $\mathcal{I}$ to $\{s\} \times X \times X$ and $\mathcal{I}_s^{-1}$ is defined to be the object $v^{-1}\text{Rhom}(\mathcal{I}_s; \omega_X \boxtimes K_X)$, $v: (x, y) \mapsto (y, x)$. In [8] (Theorem 16.3) it is proved that we can make $SS(\mathcal{I}) = \Lambda$ outside the zero-section.

**Remark 4.3.** In Guillermou [8] the term sheaf quantization is named by the fact that, outside the zero-section we have $SS(\mathcal{I} \circ \mathcal{I}_s) = \Phi_s (SS(\mathcal{I}))$ for any sheaf $\mathcal{I}$ on $X$.

**Remark 4.4.** The sheaf $\mathcal{I}$ has a representative in the left semi-orthogonal piece $\mathcal{D}_{<0, \geq 0}(I \times X \times X)$.

With this sheaf quantization $\mathcal{I}$ we can characterize admissibility under Hamiltonian contactomorphism $\Phi$.

**Proposition 4.5.** If $U$ is admissible then $\Phi(U)$ is admissible as well. In fact, for any $s \in I$ we can set $\mathcal{P}_{\Phi_s(U)} \cong \mathcal{I}_s^{-1} \circ \mathcal{P}_U \circ \mathcal{I}_s$. 


Proof. First, by Definition 4.1 for any \( \mathcal{F} \in \mathcal{D}_{>0}(X) \) we have \( \mathcal{F} \circ \mathcal{U} \in D_{Y \setminus C_U}(X) \). By further composition with \( \mathcal{J}_s \) and Remark 4.3 we have
\[
\mathcal{F} \circ (\mathcal{U} \circ \mathcal{J}_s) \cong (\mathcal{F} \circ \mathcal{U}) \circ \mathcal{J}_s \in D_{Y \setminus \Phi_s(C_U)}(X).
\]

Since composition with \( \mathcal{J}_s^{-1} \) presents an auto-equivalence of \( \mathcal{D}_{>0}(X) \), we have the composed functor
\[
\mathcal{D}_{>0}(X) \xrightarrow{\mathcal{J}_s^{-1}} \mathcal{D}_{>0}(X) \xrightarrow{(\mathcal{U} \circ \mathcal{J}_s)} \mathcal{D}_{Y \setminus \Phi_s(C_U)}(X).
\]

Second, for any object \( \mathcal{F} \in \mathcal{D}_{>0}(X) \) and \( \mathcal{G} \in D_{Y \setminus \Phi_s(C_U)}(X) \), again by Remark 4.3 we have \( \mathcal{G} \circ \mathcal{J}_s^{-1} \in D_{Y \setminus C_U}(X) \). So
\[
\text{Rhom}(\mathcal{F} \circ (\mathcal{U} \circ \mathcal{J}_s); \mathcal{G}) \\
\cong \text{Rhom}((\mathcal{F} \circ \mathcal{U}) \circ \mathcal{J}_s; \mathcal{G} \circ (\mathcal{J}_s^{-1} \circ \mathcal{J}_s)) \\
\cong \text{Rhom}((\mathcal{F} \circ \mathcal{U}) \circ \mathcal{J}_s; (\mathcal{G} \circ \mathcal{J}_s^{-1}) \circ \mathcal{J}_s) \\
\overset{\mathcal{J}_s^{-1}}{\cong} \text{Rhom}(\mathcal{F} \circ \mathcal{U} ; \mathcal{G} \circ \mathcal{J}_s^{-1}).
\]

On the other hand we have quasi-inverse morphism:
\[
\text{Rhom}(\mathcal{F} \circ \mathcal{U}; \mathcal{G} \circ \mathcal{J}_s^{-1}) \overset{\mathcal{J}_s}{\cong} \text{Rhom}((\mathcal{F} \circ \mathcal{U}) \circ \mathcal{J}_s; (\mathcal{G} \circ \mathcal{J}_s^{-1}) \circ \mathcal{J}_s).
\]

Thus by the projector property of \( \mathcal{P}_U \) we get
\[
\text{Rhom}(\mathcal{F} \circ (\mathcal{U} \circ \mathcal{J}_s); \mathcal{G}) \cong \text{Rhom}(\mathcal{F} \circ \mathcal{U} ; \mathcal{G} \circ \mathcal{J}_s^{-1}) \cong 0.
\]

So we obtain a functor to the left semi-orthogonal complement
\[
\mathcal{D}_{>0}(X) \xrightarrow{\mathcal{J}_s^{-1}} \mathcal{D}_{>0}(X) \xrightarrow{(\mathcal{U} \circ \mathcal{J}_s)} \mathcal{D}_{Y \setminus \Phi_s(C_U)}(X).
\]

Similarly, the composition with \( \mathcal{J}_s^{-1} \circ \mathcal{U} \circ \mathcal{J}_s \) gives a functor to the right semi-orthogonal complement
\[
\mathcal{D}_{>0}(X) \rightarrow (D_{\Phi_s(C_U)}(X))^\perp.
\]

Finally, from the distinguished triangle with respect to \( U \)
\[
\mathcal{P}_U \rightarrow K_\Delta \rightarrow \mathcal{Q}_U \xrightarrow{+1}.
\]

It is easy to see that after conjugation we have another triangle:
\[
\mathcal{J}_s^{-1} \circ \mathcal{P}_U \circ \mathcal{J}_s \rightarrow (\mathcal{J}_s^{-1} \circ K_\Delta \circ \mathcal{J}_s \cong K_\Delta) \rightarrow \mathcal{J}_s^{-1} \circ \mathcal{Q}_U \circ \mathcal{J}_s \xrightarrow{+1}.
\]

As a result, the kernel \( \mathcal{P}_{\Phi_s(U)} := \mathcal{J}_s^{-1} \circ \mathcal{P}_U \circ \mathcal{J}_s \) represents the projector with respect to the (hence admissible) open set \( \Phi_s(U) \). Here is an illustration diagram
\[
\begin{array}{ccc}
\mathcal{D}_{>0}(X) & \xrightarrow{\mathcal{J}_s} & \mathcal{D}_{>0}(X) \\
\mathcal{P}_U & \xrightarrow{+1} & \mathcal{P}_{\Phi_s(U)} \\
\mathcal{D}_{C_U}(X) & \xrightarrow{\mathcal{J}_s} & \mathcal{D}_{\Phi_s(C_U)}(X) \cong \mathcal{D}_{C_{\Phi_s(U)}}(X).
\end{array}
\]

□
4.2. Cyclic Actions and Contact Invariants.

Pick a positive integer $N$ and consider the cyclic action on the space $(X \times X)^N$,

$$\sigma : (x_1, x_2, \cdots, x_{2N-1}, x_{2N}) \mapsto (x_{2N}, x_1, x_2, \cdots, x_{2N-1}).$$

**Assumption.** From now on, we set the integer $N$ to be a prime number and let the ground field $K$ be the finite field of $N$ elements.

This subsection is aimed to stress the necessity of cyclic group actions associated to $\sigma : (x_1, \cdots, x_{2N-1}, x_{2N}) \mapsto (x_{2N}, x_1, \cdots, x_{2N-1})$. Let me explain the motivation here. In order to homogenize contact isotopy $\{\Phi_s\}$ on $\mathbb{R}^{2n} \times S^1$, we lift the admissible open set $U \subset \mathbb{R}^{2n} \times S^1$ to its covering $\tilde{U} \subset \mathbb{R}^{2n} \times \mathbb{R}$ so that it becomes a quotient of $T^*_\zeta(X)$ where $X$ denotes $\mathbb{R}^{2n} \times \mathbb{R}_z$.

Then the machinery of micro-support comes into play and defines the projector $\mathcal{P}_U \in D(X \times X)$.

What has been done so far only allows $\mathcal{P}_U$ to see the real line $\mathbb{R}_z$ but not the circle $S^1$. The idea is to pick a large enough integer $N$ to approximate $S^1$ by the finite cyclic group $\mathbb{Z}/N\mathbb{Z}$. Here the infinitesimal circle action is approximated by the cyclic generator $\sigma^2$ of $\mathbb{Z}/N\mathbb{Z}$.

To be precise we need to adopt the terminology of *equivariant derived category*. For the general theory of equivariant derived categories we refer the reader to the book by Bernstein and Lunts [2]. Let $G$ be a group acting on a topological space $Z$. Consider the following diagram of spaces

$$\begin{array}{ccc}
G \times G \times Z & \xrightarrow{d_0} & G \times Z \\
& d_1 \searrow & \downarrow s_0 \\
& d_2 & \downarrow d_1
\end{array}$$

A $G$-equivariant sheaf on $Z$ is a pair $(\mathcal{F}, \tau)$ where $\mathcal{F} \in \text{Sh}(Z)$ and $\tau : d_1^* \mathcal{F} \cong d_0^* \mathcal{F}$ is an isomorphism satisfying the cocycle conditions $d_2^* \tau \circ d_1^* \tau = d_1^* \tau$ and $s_0^* \tau = \text{id}_{\mathcal{F}}$. Equivariant sheaves form an abelian category $\text{Sh}_G(Z)$. Now let $G$ be the cyclic group $\mathbb{Z}/N\mathbb{Z}$ acting on the space $Z = (X \times X)^N$. Since $\mathbb{Z}/N\mathbb{Z}$ is finite, it follows that $\text{Sh}_{\mathbb{Z}/N\mathbb{Z}}((X \times X)^N)$ has enough injectives and its derived category is equivalent to the bounded below $\mathbb{Z}/N\mathbb{Z}$-equivariant derived category $D^+_\mathbb{Z}/N\mathbb{Z}((X \times X)^N)$.

Let $\mathcal{F}$ be the constant sheaf supported on the shifted diagonal, namely

$$\mathcal{F} := \mathbb{K}_{q_1 = q_2, z_1 = z_2 = 1} \in D(X \times X).$$

Consider the object $\sigma_* \mathcal{F}^{\mathbb{Z}/N}$. It is just the constant sheaf supported on the set $\{q_{2j} = q_{2j+1}, z_{2j} = z_{2j+1} = 1, j \in \mathbb{Z}/N\mathbb{Z}\}$, which is obviously an $\mathbb{Z}/N\mathbb{Z}$-equivariant sheaf. On the other hand, the object $\mathcal{P}_U$ is a complex (no matter which complex we choose from its class) on which the group $\mathbb{Z}/N\mathbb{Z}$ acts trivially, thus the object $\mathcal{P}_U^{\mathbb{Z}/N}$ can be represented by a complex consists of objects of $\text{Sh}_{\mathbb{Z}/N\mathbb{Z}}((X \times X)^N)$.

We now formulate the invariant as follows:

**Definition 4.6.** Given an admissible open set $U$ and a prime integer $N$. The contact isotopy invariant of $U$ is an object of $D^+_\mathbb{Z}/N\mathbb{Z}(pt)$ defined by

$$\mathcal{C}_N(U) := \text{Rhom}_{(X \times X)^N}(\mathcal{P}_U^{\mathbb{Z}/N}; \sigma_* \mathcal{F}^{\mathbb{Z}/N}).$$
Here the functor $\text{Rhom}$ is taken over the category $D^+_{\mathbb{Z}/\mathbb{N}Z}((X \times X)^N)$. Let $B(\mathbb{Z}/\mathbb{N}Z)$ be the classifying space of the group $\mathbb{Z}/\mathbb{N}Z$. The equivariant derived category $D^+_\mathbb{N}Z(pt)$ is equivalent to the full subcategory of $D^+(B(\mathbb{Z}/\mathbb{N}Z))$ consisting of all complexes with constant cohomology sheaves of $K$-vector spaces. One may also define $C_N(U)$ as an object of the derived category of the modules over the group algebra $K[\mathbb{Z}/\mathbb{N}Z]$ and consider its "quotient" in $D^+_\mathbb{N}Z(pt)$.

We claim that $C_N$ is indeed a contact isotopy invariant for admissible open subsets of $\mathbb{R}^{2n} \times S^1$. Now we can state the invariance property under contact isotopies.

**Theorem 4.7.** (1) For any admissible open set $U$ and any $s \in I$ there is an isomorphism between invariants $C_N(U) \cong C_N(\Phi_s(U))$, and (2) any embedding $V \hookrightarrow U$ of one admissible into another induces an morphism $C_N(U) \xrightarrow{\iota^*} C_N(V)$ naturally.

**Proof.** **Proof of (1).** We first investigate the relationship between external tensor product and composition operation. Let $\mathcal{F}$ and $\mathcal{G}$ be objects in $D(X_1 \times X_2)$. For $1 \leq j \leq N$, let $p_j : X_{2j-1} \times X_j \times X_{2j} \to X_{2j-1} \times X_j' \times X_{2j}$ and $r_j : X_{2j-1} \times X_j' \times X_{2j} \to X_{2j-1} \times X_{2j}$ be projections and let $\pi_j : \Pi^N_{j=1}(X_{2j-1} \times X_{2j}) \to X_{2j-1} \times X_{2j}$ be projection to the $j$-th factor. Notes that all notations $X_j$ and $X_j'$ are just labels identical to the same space $X$.

We have

$$\left(\mathcal{F} \otimes \mathcal{G}\right)_{\mathbb{N}Z} = \bigotimes_{j=1}^N \pi_j^{-1}(p_j^{-1} \mathcal{F} \otimes q_j^{-1} \mathcal{G})$$

$$\cong \left(\prod_{j=1}^N p_j \right)^{-1} \mathcal{F}_{\mathbb{N}Z} \otimes \left(\prod_{j=1}^N q_j \right)^{-1} \mathcal{G}_{\mathbb{N}Z} \cong \mathcal{F}_{\mathbb{N}Z} \circ \mathcal{G}_{\mathbb{N}Z}.$$
which reads

\[ \mathcal{C}_N(\Phi_s(U)) \cong \mathcal{C}_N(U). \]

**Proof of (2).** Let \( V \hookrightarrow U \) be two admissible open subsets in \( \mathbb{R}^{2n} \times S^1 \). Consider the distinguished triangle in \( D_{<0,>0}(X \times X) \):

\[ \mathcal{P}_U \to K_\Delta \to \mathcal{Q}_U \to +1 \]

and apply \( \circ \mathcal{P}_V \) on it, we get

\[ \mathcal{P}_U \circ \mathcal{P}_V \to \mathcal{P}_V \to \mathcal{Q}_U \circ \mathcal{P}_V \to +1 \].

Since \( V \subset U \), we have \( D_{Y \setminus C_U}(X) \subset D_{Y \setminus C_V}(X) \). Hence \( \mathcal{Q}_U \circ \mathcal{P}_V \cong 0 \), and the above distinguished triangle reads

(27) \[ \mathcal{P}_U \circ \mathcal{P}_V \cong \mathcal{P}_V. \]

On the other hand, by admissibility of \( V \) we have the morphism \( \mathcal{P}_V \to K_\Delta \). After composing with \( \mathcal{P}_U \) we get

(28) \[ \mathcal{P}_U \circ \mathcal{P}_V \to \mathcal{P}_U. \]

By (27)(28) there is a morphism \( i : \mathcal{P}_V \cong \mathcal{P}_U \circ \mathcal{P}_V \to \mathcal{P}_U \). It is clear that this morphism extends to \( \mathcal{P}_V^\Sigma N \to \mathcal{P}_U^\Sigma N \), hence gives rise to a morphism in \( D_{z/NX}^+(X \times X)^N) \):

\[ \mathcal{C}_N(U) = \text{Rhom}(\mathcal{P}_U^\Sigma N; \sigma_* \mathcal{F}^\Sigma N) \to \text{Rhom}(\mathcal{P}_V^\Sigma N; \sigma_* \mathcal{F}^\Sigma N) = \mathcal{C}_N(V). \]

Naturality comes from the following commutative diagram between projectors: let \( W \hookrightarrow V \hookrightarrow U \) be admissible open subsets,

\[
\begin{array}{c}
\mathcal{P}_W \cong \mathcal{P}_V \circ \mathcal{P}_W \to \mathcal{P}_V \cong \mathcal{P}_U \circ \mathcal{P}_V \to \mathcal{P}_U \\
\downarrow \cong \downarrow \cong \downarrow i \circ j \downarrow \cong \downarrow \cong \downarrow i \circ j \circ \mathcal{P}_W \\
(\mathcal{P}_U \circ \mathcal{P}_V) \circ \mathcal{P}_W \cong \mathcal{P}_U \circ (\mathcal{P}_V \circ \mathcal{P}_W) \cong \mathcal{P}_U \circ \mathcal{P}_W.
\end{array}
\]

We obtain the desired commutative diagram

\[ \mathcal{C}_N(U) \to \mathcal{C}_N(V) \]

\[ \to \mathcal{C}_N(W). \]

\[ \square \]

When \( U \) is the contact ball \( B_R \times S^1 \), the corresponding projector \( \mathcal{P}_R \) encodes Hamiltonian rotations. We will see later that it gives a good enough approximation to the Hamiltonian loop space in \( \mathbb{R}^n \) after we unwrap the invariant \( \mathcal{C}_N(B_R \times S^1) \). Here the cyclic action shifts the time parameter of the loop.
4.3. Invariants for Contact Balls.

In this subsection we make an computational approach to the contact isotopy invariants $C_N(U)$ for $U = \mathbb{R}^{2n} \times S^1$ and $U = B_R \times S^1$. Let $\delta : \Pi_{j=1}^N E_{2j-1} \times E_{2j} \times \mathbb{R}_{2j-1} \times \mathbb{R}_{2j} \rightarrow \Pi_{j=1}^N E_{2j-1} \times E_{2j} \times \mathbb{R}_{t_j}$ be the difference map on each pair $(z_{2j-1}, z_{2j})$. Let us begin with
\[ \sigma_* \mathcal{F}^{\otimes N} = K_{(q_{2j} = q_{2j+1}, z_{2j} - z_{2j+1} = 1, j \in \mathbb{Z}/\mathbb{N})}. \]

We introduce new variables $t_j = z_{2j} - z_{2j-1}$. If $z_{2j} - z_{2j+1} = 1$ for each $j$, we have $\Sigma t_j = \Sigma(z_{2j} - z_{2j+1}) = N$. Conversely, any $t_j$ satisfying $\Sigma t_j = N$ can be rewritten in the form $t_j = z_{2j} - z_{2j-1}$ such that $z_{2j} - z_{2j+1} = 1$ for each $j$. Hence we have the following sequence of isomorphisms in $D((E \times E \times \mathbb{R})^N)$ after applying $\delta$ on $\sigma_* \mathcal{F}^{\otimes N}$:
\[ \tilde{\delta}_* \sigma_* \mathcal{F}^{\otimes N} = K_{(q_{2j} = q_{2j+1}, \Sigma t_j = N, j \in \mathbb{Z}/\mathbb{N})} = \tilde{\Delta}_* K_{E^N \times \{\Sigma t_j = N\}} = \tilde{\Delta}_* \pi_{\Sigma}^{-1} K_{E^N \times \{t = N\}} \cong \tilde{\Delta}_* \pi_{\Sigma}^{-1} K_{E^N \times \{t = N\}}[1 - N] = \Delta_* \pi_{\Sigma}^{-1} K_{t = N}[1 - N] \cong \Delta_* \pi_{\Sigma}^{-1} K_{t = N}[1 - N], \]

Here $\pi : E^N \times \mathbb{R} \rightarrow \mathbb{R}$ is the projection along $E^N$, $\pi_{\Sigma} : E^N \times \mathbb{R}^N \rightarrow E^N \times \mathbb{R}$ takes summation of all $t_j$ to the new variable $t$, and $\Delta$ is the (shifted)diagonal embedding of $E^N$ into $E^{2N}$ defined by $E_j \xrightarrow{\Delta} E_{2j-2} \times E_{2j-1}$ for each $j \in \mathbb{Z}/\mathbb{N}$.

Therefore for $U = \mathbb{R}^{2n} \times S^1$ we have

**Lemma 4.8.** $C_N(\mathbb{R}^{2n} \times S^1) \cong K[(1+n)(1-N)]$ in $D(K[\mathbb{Z}/\mathbb{N}])$. Here $K[\mathbb{Z}/\mathbb{N}]$ acts on $K$ trivially.

**Proof.** By (29), $C_N(\mathbb{R}^{2n} \times S^1) = \text{Rhom}(K_\Delta^{\otimes N}; \sigma_* \mathcal{F}^{\otimes N}) \cong \text{Rhom}(\delta^{-1} K_\Delta^{\otimes N}; \sigma_* \mathcal{F}^{\otimes N})$
\[ \cong \text{Rhom}(K_\Delta^{\otimes N}; \tilde{\delta}_* \sigma_* \mathcal{F}^{\otimes N}) \cong \text{Rhom}(K_\Delta^{\otimes N}; \Delta_* \pi_{\Sigma}^{-1} K_{t = N}[1 - (n + 1)N]) \cong \text{Rhom}(R_{\pi_1} R_{\pi_2} \Delta^{-1}(P_R^{\otimes N}); K_{t = N}[1 - (n + 1)N]) \cong K[(1+n)(1-N)]. \]

It is easy to see that the cyclic group acts trivially on both $K_{t \geq 0}[-n]$ and $K_{t = N}[1 - (n + 1)N]$ and hence gives trivial action on $C_N(\mathbb{R}^{2n} \times S^1)$.

When $U$ is the contact ball $B_R \times S^1$ we replace $K_\Delta$ by $\mathcal{P}_R$ and the same adjunctions gives
\[ C_N(B_R \times S^1) = \text{Rhom}(\mathcal{P}_R^{\otimes N}; \sigma_* \mathcal{F}^{\otimes N}) \cong \text{Rhom}(R_{\pi_1} R_{\pi_2} \Delta^{-1}(P_R^{\otimes N}); K_{t = N}[1 - (n + 1)N]). \]

Define
\[ \mathcal{F}_R := R_{\pi_1} R_{\pi_2} \Delta^{-1}(P_R^{\otimes N}) \]
(30)
then we can rewrite $\mathcal{C}_N(B_R \times S^1) \cong \text{Rhom}(\mathcal{F}_R; \mathbb{K}_{t=N}[1 - (n + 1)N])$. Now we are going to unravel the stalks of $\mathcal{F}_R$ by computing cohomology with compact supports of certain free loop space with an cyclic action on it.

**Lemma 4.9.** Given $T \geq 0$, choose an integer $M$ such that $M > \frac{4T}{\pi N R^2}$. For $A \in G$ satisfying $0 \leq -A \leq \frac{T}{NR^2}$ and $q_{j,k} \in E^{NM}$, define the quadratic form $\Omega(A) = \sum_{j=1}^{N} \sum_{k=1}^{M} S_{A/M}(q_{j,k}, q_{j,k+1})$. Define $\mathcal{W} = \{(A, \{q_{j,k}\}) | \Omega(A) \geq 0\}$ a subset of $G \times E^{NM}$. Denote $\rho : G \times E^{NM} \to G$ the projection onto $G$ and let $\mathcal{E} = R\rho \mathcal{K}_\mathcal{W}$. One has

$$\mathcal{F}_R|_T \cong R\Gamma_c(\{0 \leq -A \leq \frac{T}{NR^2}\}; \mathcal{E}).$$

The cyclic action on the set $\mathcal{W}$ is given by $q_{1,j} \mapsto q_{1,j-1}$. The cyclic action on $\mathcal{E}$ is induced from $\mathcal{W}$ by computing the equivariant cohomology of the pre-image of $\rho$.

**Proof.** First, let us unravel $\mathcal{P}_R$ in terms of the sheaf quantization $\mathcal{S}$ and its convolutions (or compositions). By (10) we have

$$\mathcal{P}_R := \hat{S} \circ \mathbb{K}_{\{b < R^2\}} = (\mathcal{S} \bullet \mathbb{K}_{\{t + ab \geq 0\}, [1]} \circ \mathbb{K}_{\{b < R^2\}} \cong \mathcal{S} \bullet (\mathbb{K}_{\{t + ab \geq 0\}, b} \circ \mathbb{K}_{\{b < R^2\}})[1] \cong \mathcal{S} \bullet \mathbb{K}_{\{(a,t) | a \leq 0, t + ab \geq 0\}}.$$

To see $\mathcal{P}_R^{\otimes N}$, for each $j \in \mathbb{Z}/N\mathbb{Z}$ let $\pi_j$ be the projection from $(E \times E)^N$ to $E_{2j-1} \times E_{2j}$ and let $\mathcal{S}_j \in \mathcal{D}_{>0}(G \times E_{2j-1} \times E_{2j} \times \mathbb{R}_{t_j})$ be the sheaf quantization of Hamiltonian rotations. Also let $\pi_G$ be the projection along $G^N$ and $\bar{s}$ takes addition $\mathbb{R}_t^N \times \mathbb{R}_t^N \to \mathbb{R}_t^N$. We have the following isomorphisms:

$$\mathcal{P}_R^{\otimes N} \cong (\otimes_{j=1}^{N} \pi_j^{-1} \mathcal{S}_j) \bullet \mathbb{K}_{\{\forall j, -t_j/R^2 \leq a_j \leq 0\}} \cong R\pi_G R\bar{s}_{\otimes}[(\otimes_{j=1}^{N} \pi_j^{-1} \mathcal{S}_j) \otimes \mathbb{K}_{\{\forall j, -t_j/R^2 \leq a_j \leq 0\}}].$$

Furthermore let $\mathbb{R}_t \times \mathbb{R}_t \overset{\delta}{\to} \mathbb{R}_t$ denote the summation. Therefore we can write $\mathcal{F}_R$ (30) as

$$\mathcal{F}_R := R\pi_{\Sigma} \mathcal{F}_R \Delta^{-1}(\mathcal{P}_R^{\otimes N}) \cong R\pi_{\Sigma} R\pi_G \mathcal{F}_{\bar{s}}[(\mathcal{P}_R^{\otimes N}) \Delta^{-1}(\otimes_{j=1}^{N} \pi_j^{-1} \mathcal{S}_j)] \otimes \mathbb{K}_{\{\forall j, a_j \leq 0, t \geq -\sum_{j} a_j R^2\}}.$$

Let me describe the object $R\pi_{\Sigma} \Delta^{-1}(\mathcal{P}_R^{\otimes N})$. Given any tuple $\bar{a} = (a_1, \ldots, a_j, \ldots, a_N)$ in $G^N$ in which each $a_j$ is non-positive, there exists an positive integer $M$ such that for any $j$ we have $-\frac{\pi}{M} \leq \frac{a_j}{M} \leq 0$. We can divide $[a_j, 0]$ into subintervals of length less than $\frac{a_j}{M}$ by setting $[a_j, 0] = \bigcup_{k=1}^{M} \left[\frac{k}{M} a_j, \frac{k+1}{M} a_j\right]$. Recall that on each subinterval $[\frac{k}{M} a_j, \frac{k+1}{M} a_j]$ the object $\mathcal{S}_j$ has the expression in terms of the following local generating function (3)

$$S_0(q, q') = \frac{1}{2 \tan(2a)}(q^2 + q'^2) - \frac{1}{\sin(2a)}qq'.$$

Iterated convolutions of (7) from 1 to $M$ gives us

$$\mathcal{S}_j|_{a_j} = R\rho_j \mathbb{K}_{\{(q_{j,1}, \ldots, q_{j,k}, \ldots, q_{j,M+1}, t_j)| t_j + \sum_{k=1}^{M} S_{a_j/M}(q_{j,k}, q_{j,k+1}) \geq 0\}}$$

here $\rho_j$ denotes the projection: $(q_{j,1}, \ldots, q_{j,k}, \ldots, q_{j,M+1}, t_j) \mapsto (q_{j,1}, q_{j,1}, q_{j,1}, t_j)$, and we identify the variables $q_{j,1} = q_{2j-1}$ and $q_{j,M+1} = q_{2j}$ such that $\mathcal{S}_j|_{a_j} \in \mathcal{D}_{>0}(E_{2j-1} \times E_{2j} \times \mathbb{R}_{t_j})$. 

\[\text{Eq. (32)}\]
Observe that the morphism $\tilde{\Delta}^{-1}$ gives further identification $q_{2j} = q_{2j+1}$ (i.e., $q_{j,M+1} = q_{j+1,1}$) for all $j \in \mathbb{Z}/N\mathbb{Z}$. Denote

$$W(\bar{a}) = \{(\{q_{j,k}\}, t) | t + \sum_{j=1}^{N} \sum_{k=1}^{M} S_{a_{j}/M}(q_{j,k}; q_{j,k+1}) \geq 0\}$$

the subset of $E^{NM} \times \mathbb{R}_t$, and $\rho_E$ denotes the projection: $E^{NM} \to E^N$ mapping from $\{q_{j,k}\}$ to $\{q_j\}$. We see that the functor $R\pi_{\Sigma!}$ is given by $\sum_{j=1}^{N}$ in the definition of $W(\bar{a})$. Therefore by (32) and (33) we get

$$R\pi_{\Sigma!}(\tilde{\Delta}^{-1} \bigotimes_{j=1}^{N} \pi_j^{-1}(S_j|_{a_j})) \cong R\rho_E!K_W(\bar{a}).$$

Each configuration $\{q_{j,k}\}$ gives rise to a Hamiltonian broken loop trajectory in $E$ with $NM$ nodes, as shown on the left ($M = N = 3$).

$W$ presents further conditions regarding action integrals on those loops. This allows us to compute cohomology with compact support of such loop space.

Therefore pointwisely $R\pi_{\Sigma!}(\tilde{\Delta}^{-1} \bigotimes_{j=1}^{N} \pi_j^{-1}S_j)$ describes the concatenation of sheaf quantization of Hamiltonian rotations. Again by the argument around (7) the sheaf quantization should only depend on $\sum a_j$. Namely, let $\alpha_\Sigma : G^N \to G$ be the summation $\bar{a} \mapsto \Sigma a_j$, then the object $(R\pi_{\Sigma!}(\tilde{\Delta}^{-1} \bigotimes_{j=1}^{N} \pi_j^{-1}S_j))$ is a pull-back by $\alpha_\Sigma$. So there is an object $\mathcal{G} \in D_{\mathbb{Z}/N\mathbb{Z}}(G \times E^N \times \mathbb{R}_t)$ such that

$$R\pi_{\Sigma!}(\tilde{\Delta}^{-1} \bigotimes_{j=1}^{N} \pi_j^{-1}S_j) \cong \alpha^{-1}_\Sigma \mathcal{G}.$$ 

We introduce a new variable $A := \Sigma a_j/N$. Denote the subset

$$W(A) = \{t + \sum_{j=1}^{N} \sum_{k=1}^{M} S_{A/M}(q_{j,k}; q_{j,k+1}) \geq 0\} \subset E^{NM} \times \mathbb{R}.$$ 

By (34) and (35), pointwisely we have

$$\mathcal{G}|_A \cong R\rho_E!K_W(A).$$
Let $\pi_A$ be the projection along $A$. We can decompose the projection $\pi_G$ into the composition of $\pi_A$ and $\alpha_\Sigma$. Plug those expressions into $\mathcal{F}_R$ (31) we get

$$
\mathcal{F}_R \cong R\pi_1 R\pi_G R\alpha_\Sigma [\alpha_\Sigma^{-1}(G) \boxtimes K_{\{q_j, 0 \leq t \leq -\Sigma a_j R^2\}}]
$$

$$
\cong R\pi_1 R\pi_A R\alpha_\Sigma [\alpha_\Sigma^{-1}(G) \boxtimes K_{\{q_j, 0 \leq t \leq -\Sigma a_j R^2\}}]
$$

$$
\cong R\pi_1 R\pi_A R\alpha_\Sigma [\alpha_\Sigma^{-1}(G) \boxtimes K_{\{q_j, 0 \leq t \leq -\Sigma a_j R^2\}}]
$$

$$
\cong R\pi_1 R\pi_A R\alpha_\Sigma [\alpha_\Sigma^{-1}(G) \boxtimes K_{\{q_j, 0 \leq t \leq -\Sigma a_j R^2\}}].
$$

To unwrap $\mathcal{F}_R$ to more explicit form, we write $\mathcal{G} \in D_{/Z}(E \times E^N \times \mathbb{R})$ and $K_{\{t \leq -NAR^2 \geq 0\}} \in \mathcal{D}_{>0}(E \times E \times \mathbb{R})$ and the summation $s : R_1 \times R_2 \rightarrow R_4$. Consider the projection along $E^N$ by $\pi_E$ : $G \times E^N \times \mathbb{R} \rightarrow G \times \mathbb{R}$, we claim there exist $\mathcal{E} \in D_{/Z}(G)$ and $K \in \mathcal{D}_{>0}(\mathbb{R})$ such that

$$
R\pi_E \mathcal{G} \cong \mathcal{E} \boxtimes K.
$$

To be more precise, let us fix $A = \Sigma a_j/N$ and recall (36) that we have $\mathcal{G}|_A \cong R\rho_E K_{W(A)} \in \mathcal{D}_{>0}(E^N \times \mathbb{R})$ where $W(A) = \{t_1 + \sum_{j=1}^N \sum_{k=1}^M S_{A/M}(q_{j,k}, q_{j,k+1}) \geq 0\} \subset E^N \times \mathbb{R}$ and $\rho_E$ denotes the projection: $E^M \rightarrow E$ mapping from $\{q_{j,k}\}$ to $\{q_j\}$. Let

$$
\Omega(A) = \sum_{j=1}^N \sum_{k=1}^M S_{A/M}(q_{j,k}, q_{j,k+1})
$$

be the homogeneous quadratic form on $E^N$, we see that the only critical value of $-\Omega(A)$ is zero. Hence $R\pi_1 \mathcal{G}|_A \cong R\pi_1 R\pi_E \mathcal{K}_{W(A)}$ is supported by points greater or equal to zero, which is independent of the choice of $A$. This shows the existence of the factor $\mathcal{E}$, and we can set another factor $K$ to be the constant sheaf $K_{\{t \leq 0\}}$.

Let $\rho_t : G \times E^N \times \mathbb{R} \xrightarrow{\rho_E} G \times E^N \times \mathbb{R} \xrightarrow{\pi_E} G \times \mathbb{R}$. Since (36) $\mathcal{G}|_A \cong R\rho_E K_{W(A)} \cong R\pi_1 R\pi_E \mathcal{K}_{W(A)}$, pointwisely we have

$$
\mathcal{E}|_A \cong R\pi_1 R\pi_E K_{W(A)}|_{t = 0} = R\rho_t K_{W(A)}|_{t = 0}.
$$

Denote $\rho = \rho_t|_{t = 0} : G \times E^N \rightarrow G$. Define $W = \{\{A, q_{j,k}\}|\Omega(A) \geq 0\}$ a subset of $G \times E^N$. Then we have

$$
\mathcal{E} \cong R\rho K_{W}.
$$

Let us continue on $\mathcal{F}_R$. Recall that $R\pi_1 \mathcal{G} \cong \mathcal{E} \boxtimes K_{\{t \geq 0\}}$. Therefore by (37) and (38) in $\mathcal{D}_{>0}(\mathbb{R})$ we have

$$
\mathcal{F}_R \cong R\pi_1 R\pi_A R\alpha_\Sigma [\mathcal{G} \boxtimes K_{\{t \geq -NAR^2 \geq 0\}}] \cong R\pi_A [\mathcal{E} \boxtimes K_{\{t \geq -NAR^2 \geq 0\}}].
$$

Therefore for any given $t = T \geq 0$ we get (here $R\pi$ computes $\mathbb{Z}/\mathbb{Z}_N$-equivariant cohomology)

$$
\mathcal{F}_R|_T \cong R\pi_{\alpha}(\{0 \leq -A \leq \frac{T}{NR^2}\}; \mathcal{E}).
$$

When $T \geq 0$ is given, there is a uniform integer $M$ such that $0 \leq -\frac{A}{M} < \frac{T}{4R^2}$, namely we can pick any integer $M > \frac{4T}{\pi N R^2}$. \qed
Notice that from $\mathcal{E} \cong R \rho R K W$ we have

\begin{equation}
\mathcal{F}_{R^{|T|}} \cong \mathcal{F}_{\rho}(\{0 \leq -A \leq \frac{T}{NR^2}\}; \mathcal{E}) \cong \mathcal{F}_{\rho}(\{0 \leq -A \leq \frac{T}{NR^2}\}) \cap \mathcal{W}).
\end{equation}

This way we see that the invariant $\mathcal{C}_{N}(B_{R} \times S^{1}) \cong \text{Rhom}(\mathcal{F}_{R}; K_{t=N}[1-(n+1)N])$ is given by computing compactly supported equivariant cohomology of the space $\rho^{-1}(\{0 \leq -A \leq \frac{T}{NR^2}\}) \cap \mathcal{W}$. Moreover, we have

**Lemma 4.10.** $\mathcal{F}_{R^{|T|}}$ is the compactly supported equivariant cohomology of the space $R^{D+1}$ where $R^{D+1} \hookrightarrow E^{NM}$ and $D = 2n[\frac{T}{NR^2}] - n - 1$. The cyclic group action on $R^{D+1}$ is induced from the action on $E^{NM} = E \times C^{N_{M-1}/n}$ by complex multiplication by the exponents of $\omega^{M}$ on the complex coordinates respectively. Here $\omega$ stands for the primitive $MN$-th root of unity.

**Proof.** Relabel the index $\{q_{j,k}\} = \{q_{\ell}\} = \tilde{q}$ in lexicographical order and $\ell \in \mathbb{Z}/MN\mathbb{Z}$. The condition $\Omega(A) \geq 0$ on $\mathcal{W}$ translates to (39) $\sum_{\ell} S_{A/M}(q_{\ell}) \geq 0$. In other words, the points $(A, \tilde{q}) \in \mathcal{W}$ are characterized by the inequality:

\[
\cos(\frac{2A}{M})(\sum_{\ell} q_{\ell}^{2}) \leq (\sum_{\ell} q_{\ell} q_{\ell+1}).
\]

By the assumption of $M$, which is $|2A/M| < \pi/2$, on $\mathcal{W}$ we have

\[
0 \leq \cos(\frac{2A}{M})(\sum_{\ell} q_{\ell}^{2}) \leq (\sum_{\ell} q_{\ell} q_{\ell+1}).
\]

When $\tilde{q}$ has identical components, for $A$ the condition (40) $\rho^{-1}(\{0 \leq -A \leq \frac{T}{NR^2}\}) \cap \mathcal{W}$ becomes the closed interval $0 \leq -A \leq \frac{T}{NR^2}$. On the other hand, by the fact $|\sum_{\ell} q_{\ell} q_{\ell+1}| \leq 1$, other $\tilde{q}$ give another closed interval constraint on $A$:

\[
\arccos(\frac{\sum_{\ell} q_{\ell} q_{\ell+1}}{\sum_{\ell} q_{\ell}^{2}}) \leq -\frac{2A}{M} \leq \frac{2T}{MN R^2} \leq \frac{\pi}{2}.
\]

In any of the above cases, the constraint on $A$ has compactly supported cohomology $(\cdots \rightarrow K \rightarrow \cdots)$ where $K$ is located at degree 0.

On the other hand, to unwrap the condition $\Omega(A) \geq 0$ of $\mathcal{W}$ we need to diagonalize the matrix of quadratic form $\Omega(A)(39)$. Let $\omega = \exp(\frac{2\pi i}{MN})$ be the generator of $MN$-th roots of unity. Here we can take $MN$ to be an odd integer. The set of eigenvalues of $\Omega(A)$ is

\[
\{\lambda_{\ell} = \cot(2A/M) - \frac{1}{2} \csc(2A/M)(\omega^{\ell} + \omega^{NM-\ell})|\ell \in \mathbb{Z}/MN\mathbb{Z}\}
\]

or

\[
\{\lambda_{\ell} = \cot(2A/M) - \csc(2A/M) \cos(\frac{2\pi}{MN})|\ell \in \mathbb{Z}/MN\mathbb{Z}\}.
\]

Among them the eigenvalue $\lambda_{0}$ corresponds to eigenvector $(1, 1, \cdots, 1)$. For each $\ell \neq 0$, set $\tilde{\theta}_{\ell} = (1, \omega^{\ell}, \omega^{2\ell}, \cdots)$ and denote its real and imaginary part by $\tilde{\theta}_{\ell} = \tilde{R}_{\ell} + i \tilde{I}_{\ell}$. Then the eigenspace of $\lambda_{\ell} = \lambda_{MN-\ell}$ is orthogonally spanned by $\tilde{R}_{\ell}$ and $\tilde{I}_{\ell}$. Let $\xi_{0} = \langle(1, 1, \cdots, 1), \tilde{q}\rangle$ and $u_{\ell} = \langle\tilde{R}_{\ell}, \tilde{q}\rangle$ and $v_{\ell} = \langle\tilde{I}_{\ell}, \tilde{q}\rangle$. Therefore the condition $\Omega(A) \geq 0$ becomes

\[
\lambda_{0} \xi_{0}^{2} + \sum_{\ell=1}^{(MN-1)/2} \lambda_{\ell}(u_{\ell}^{2} + v_{\ell}^{2}) \geq 0.
\]
Hence the constraint (40) \( \rho^{-1}(\{0 \leq -A \leq \frac{T}{NR^2}\}) \cap W \) is homeomorphic to a Euclidean space \( \mathbb{R}^{D+1} \). Here \( D+1 \) stands for homogeneity of the quadratic form \( \Omega \). Thus we can write
\[
\mathcal{F}_R|_T \cong R\Gamma_c(\rho^{-1}(\{0 \leq -A \leq \frac{T}{NR^2}\}) \cap W) \cong R\Gamma_c(\mathbb{R}^{D+1}).
\]

The generator of the cyclic action is given by \( q \ell \mapsto q_{-M} \) and induces
\[
\sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} q_{\ell} \omega^{\ell} \mapsto \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} q_{-M} \omega^{\ell} = \omega^{M} \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} q_{\ell} \omega^{\ell}.
\]
Therefore the cyclic action on \( E^{NM} \) is generated by
\[
u_{\ell} + iv_{\ell} \mapsto \omega^M(\nu_{\ell} + iv_{\ell}).
\]
To compute \( D \), let \( I \) be the number of positive eigenvalues of \( \Omega \) among \( \ell = 1, \ldots, \frac{MN-1}{2} \). After projecting along \( A \) we can replace \( -A \) by \( \frac{T}{NR^2} \), then \( I \) becomes the number of solutions of \( \ell \) of
\[
\lambda_{\ell} = \cot\left(-\frac{2T}{MN R^2}\right) - \csc\left(-\frac{2T}{MN R^2}\right) \cos\left(\frac{2\pi}{MN} \ell\right) > 0.
\]
Therefore \( I = \left\lceil \frac{T}{\pi R^2} \right\rceil - 1 \). On the other hand, from the fact \( \mathcal{F}_R|_T \cong R\Gamma_c(\mathbb{R}^{D+1}) \) we know that \( D + 1 = n + 2nI \). Here doubling \( I \) comes from pairing up \( u_{\ell} \) and \( v_{\ell} \), and \( n = \dim(E) \). Therefore
\[
D = n + 2nI - 1 = 2n\left\lceil \frac{T}{\pi R^2} \right\rceil - n - 1.
\]

The cyclic group acts on \( \mathcal{F}_R|_T \) by its action on \( \mathbb{R}^{D+1} \). In this case one may take \( T = N \) and write \( C_N(B_R \times S^1) \cong \text{Rhom}(\mathbb{K}_{t=N}[-D-1]; \mathbb{K}_{t=N}[1 - (n + 1)N]) \cong \mathbb{K}[D + 2 - (n + 1)N] \) in \( D(\mathbb{K}[\mathbb{Z}/N\mathbb{Z}]) \).

**Corollary 4.11.** \( \mathcal{F}_\infty|_T \cong R\Gamma_c(\mathbb{K}[\mathbb{Z}/N\mathbb{Z}]) \)-module.

**Proof.** Also note that \( \mathbb{R}^{2n} \times S^1 \) corresponds to the limit \( R \to \infty \). In this case we must have \( A = 0 \) and the inequality \( \left| \sum_\ell q_{\ell} \hat{q}_{\ell} \right| \leq 1 \) should hold its equality and forces \( \hat{q} \) to have identical components parametrized by \( E \). Therefore the cyclic group acts trivially on it and as a \( \mathbb{K}[\mathbb{Z}/N\mathbb{Z}] \)-module we get
\[
\mathcal{F}_\infty|_T \cong R\Gamma_c(\{ A = 0 \}; \mathcal{E}) \cong R\Gamma_c(\mathcal{E}) \cong \mathbb{K}[-n].
\]

This recovers **Lemma 4.8**.

### 4.4. The Cones of Embeddings.

For the purpose of testing squeezability in the ambient space \( \mathbb{R}^{2n} \times S^1 \), instead of individual \( C_N(B_R \times S^1) \) we need to focus on the cone of the morphism \( C_N(\mathbb{R}^{2n} \times S^1) \xrightarrow{J_R} C_N(B_R \times S^1) \). By the definition (30) of \( \mathcal{F}_R \) this is the cone of
\[
\text{Rhom}(\mathcal{F}_\infty; \mathbb{K}_{t=N}[1 - (n + 1)N]) \to \text{Rhom}(\mathcal{F}_R; \mathbb{K}_{t=N}[1 - (n + 1)N]).
\]
Proposition 4.12. Let $\mathcal{L}_R$ be the cocone of the morphism $\mathcal{F}_R \to \mathcal{F}_\infty$ (which means that $\mathcal{L}_R[1]$ is the cone). Then in $D(K^{[\mathbb{Z}/n\mathbb{Z}]})$ one has $\mathcal{L}_R \cong R\Gamma_c(S^{D-n} \times E \times \mathbb{R}_{>0})$. Here $D = 2n\lceil \frac{T}{\pi R^2} \rceil - n + 1$. The cyclic group acts on $S^{D-n}$ by complex multiplication (41).

Proof. Lemma 4.9, and Corollary 4.11 tell us

$$\mathcal{L}_R|_T \cong R\Gamma_c(\{0 < -A \leq \frac{T}{NR^2}\}; \mathcal{E}) \cong R\Gamma_c(\rho^{-1}(\{0 < -A \leq \frac{T}{NR^2}\}) \cap W).$$

Following Lemma 4.10 it is easy to see that the constraint set is homeomorphic to $R^{D+1}$ with all diagonals ($\mathbf{q}, \mathbf{q}, \mathbf{q}, \cdots, \mathbf{q}$) removed. This is homeomorphic to $S^{D-n} \times E \times \mathbb{R}_{>0}$.

Recall that to $E^{NM-1}$ we assign complex coordinate $u_\ell + iv_\ell \in \mathbb{C}^n$ for $\ell = 1, \cdots, \frac{MN-1}{2}$. The $\mathbb{Z}/n\mathbb{Z}$ group action on $S^{D-n}$ is given by (41) $u_\ell + iv_\ell \mapsto \omega^\ell(u_\ell + iv_\ell)$ for those $\ell$ corresponding to positive eigenvalues. The $\mathbb{Z}/n\mathbb{Z}$ actions are trivial on both $E$ and $R_{>0}$ parts.

Recall that $C_N(B_R \times S^1) \cong \text{Rhom}(\mathcal{F}_R; K_{N=M}[1 - (n+1)N])$. We should take $T = N$ so that in this case we have $D = 2n\lceil \frac{N^2}{\pi NR^2} \rceil - n + 1$. In the rest of the paper we are only interested in the case $\pi R^2 \geq 1$. Then the initial assumption of $M > \frac{4T}{\pi NR^2}$ in Lemma 4.9 can be simplified to $M > \frac{4T}{\pi NR^2}$.

Assumption. We assume $T = M = N > 3$ is a prime number.

Recall that $\mathcal{I}$ is the number of positive eigenvalues $\lambda_\ell$ among $\ell = 1, \cdots, \frac{N^2-1}{2}$. Let $\zeta = \omega^M = \omega^N$ be the primitive $N$-th root of the unity. The cyclic action on $\mathcal{L}_R|_{T=N}$ is given by (41) $u_\ell + iv_\ell \mapsto \zeta^\ell(u_\ell + iv_\ell)$ for $\ell = 1, \cdots, \mathcal{I}$. Therefore by (43) we have

Corollary 4.13. If $\pi R^2 \geq 1$ then the group $\mathbb{Z}/n\mathbb{Z}$ acts freely on the sphere $S^{D-n}$.

Proof. This is because $\pi R^2 \geq 1$ implies $\mathcal{I} = \lceil \frac{N^2}{\pi R^2} \rceil - 1 \leq N - 1$. □

Recall that $\mathcal{L}_R|_{T=N}$ computes the compactly supported equivariant cohomology of $S^{D-n} \times E \times \mathbb{R}_{>0}$. From the above corollary, we see the $\mathbb{Z}/n\mathbb{Z}$-quotient of this sphere $S^{D-n} = S^{2n\mathcal{I}}$ is the lens space $L(N; 1, \cdots, 1, 2, \cdots, 2, \cdots, \mathcal{I}, \cdots, \mathcal{I})$. Note that the group $\mathbb{Z}/n\mathbb{Z}$ acts trivially on the $E \times \mathbb{R}_{>0}$ part. So we have

Corollary 4.14. Assume that $\pi R^2 \geq 1$. In the equivariant derived category $\mathcal{L}_R|_{T=N}[n + 1]$ becomes the cohomology of $(D-n)$-dimensional lens space.

We will apply this fact to the proof of non-squeezability.

5. Proof of Non-Squeezability

Now given $1 \leq \pi r^2 < \pi R^2$. We are going to prove contact non-squeezability by contradiction. Assume that $B_R \times S^1$ can be squeezed into $B_R \times S^1$ by $\Phi$, and $\Phi$ can be joint to the identity via compactly supported contact isotopy.
By Theorem 4.7 on this sequence of open embeddings $\Phi(B_R \times S^1) \xrightarrow{j^*} B_r \times S^1 \xrightarrow{j_r^*} \mathbb{R}^{2n} \times S^1$ we have the following commutative diagram for induced morphisms:

$$
\begin{array}{ccc}
C_n(\Phi(B_R \times S^1)) & \xrightarrow{j^*} & C_n(B_r \times S^1) \\
\cong & & \leftarrow C_n(\mathbb{R}^{2n} \times S^1) \\
& \xleftarrow{j_r^*} & \end{array}
$$

Recall Proposition 4.12 that there is a distinguished triangle

$$
C_n(\mathbb{R}^{2n} \times S^1) \xrightarrow{j_r^*} C_n(B_r \times S^1) \rightarrow \text{Rhom}(\mathcal{L}_R; K_{t=N}[1-(n+1)N]) \xrightarrow{+1} .
$$

Before working in the equivariant setting, let us shift $\mathcal{L}_R$ by $n+1$ to insure they all live in non-negative degrees:

$$
C_n(\mathbb{R}^{2n} \times S^1) \xrightarrow{j_r^*} C_n(B_r \times S^1) \rightarrow \text{Rhom}(\mathcal{L}_R[n+1]; K_{t=N}[n+2-(n+1)N]) \xrightarrow{+1} .
$$

We need to consider the cyclic action on the cone of the morphism $j_R^*$. In the category of $D(K[\mathbb{Z}/N\mathbb{Z}])$ one has $C_n(\mathbb{R}^{2n} \times S^1) \cong K[D+1-(n+1)N]$ and $C_n(B_r \times S^1) \cong K[D+2-(n+1)D]$. The group algebra $K[\mathbb{Z}/N\mathbb{Z}]$ acts on the former $K$ by zero. To distinguish, we write $C_n(\mathbb{R}^{2n} \times S^1) \cong K_1[D+1-(n+1)N]$ and $C_n(B_r \times S^1) \cong K_2[D+2-(n+1)D]$ where $K[\mathbb{Z}/N\mathbb{Z}]$ acts trivially on $K_1$. We write $j_R^* \in \text{Rhom}(K[\mathbb{Z}/N\mathbb{Z}])$, $K_2)$ $[D-n+1]$. $N$ is an odd prime and $K$ is the finite field of $N$ elements.

In order to discuss the equivariant cohomology of $cone(j_R^*)$, let us pass to the cohomology of $\text{Rhom}(K[\mathbb{Z}/N\mathbb{Z}])$. It is the Yoneda product $Ext^\bullet_K(\mathbb{Z}/N\mathbb{Z})$. The product of cohomology classes $[j^*] \circ [j_r^*] = [j_R^*]$ is given by concatenation

$$
Ext^\bullet(K_2) \times Ext^\bullet(K_1; K_2) \rightarrow Ext^{\bullet+n}(K_1; K_2).
$$

It turns out that the size of the contact ball has a crucial cohomological interpretation using this $Ext^\bullet$ structure:

**Lemma 5.1.** If $\pi R^2 \geq 1$ then $[j_R^*]$ is nonzero in $Ext^{D-n+1}_K(K_1; K_2)$.

**Proof.** If the cohomology class of $j_R^*$ becomes zero in $Ext^{D-n+1}_K(K_1; K_2)$, then up to a degree shifting $cone(j_R^*)$ is quasi-isomorphic to $K_1 \bigoplus K_2[D-n]$ in the derived category $D(K[\mathbb{Z}/N\mathbb{Z}])$. It turns out that in $D^{+}_{\mathbb{Z}/N\mathbb{Z}}(pt)$ up to a degree shifting $cone(j_R^*)$ is just the equivariant cohomology of the disjoint union of a point $\{\ast\}$ and a $\mathbb{R}^{D-n}$. Since $\mathbb{Z}/N\mathbb{Z}$-action fixed $\{\ast\}$, we see that in $D^{+}_{\mathbb{Z}/N\mathbb{Z}}(pt)$ the object $cone(j_R^*)$ has unbounded cohomology.

On the other hand, we have $cone(j_R^*) \cong \text{Rhom}(\mathcal{L}_R[n+1]; K_{t=N}[n+2-(n+1)N])$. Corollary 4.14 tells that the equivariant cohomology of $\mathcal{L}_R[T=N][n+1]$ is the cohomology of a $(D-n)$-dimensional lens space. We see that in $D^{+}_{\mathbb{Z}/N\mathbb{Z}}(pt)$ the object $cone(j_R^*)$ should have bounded cohomology.

We arrive at the contradiction by computing the equivariant cohomology of $cone(j_R^*)$ in two different ways. Therefore $j_R^*$ does pass to a nonzero class in $Ext^\bullet_K(K_1; K_2)$. And it makes sense to write $\deg([j_R^*]) = D-n+1$.

$\square$

The non-squeezing property of contact balls in large scale is followed by the degree counting.
Theorem 5.2. There is no such \( j^* \) making the diagram commutative.

Proof. For the diagram to commute we have \( j^*_R = j^* \circ j^*_r \). By Lemma 5.1 we have \( \deg([j^*_R]) = \deg([j^*]) + \deg([j^*_r]) \). It follows from Proposition 4.12 that

\[
2n\left\lfloor \frac{N}{\pi R^2} \right\rfloor - 2n + 2 = \deg([j^*_R]) \geq \deg([j^*_r]) = 2n\left\lfloor \frac{N}{\pi r^2} \right\rfloor - 2n + 2.
\]

On the other hand, according to the assumption \( \pi r^2 < \pi R^2 \), we can pick large enough prime number \( N \) (thanks to Euclid) such that

\[
\left\lfloor \frac{N}{\pi R^2} \right\rfloor < \left\lfloor \frac{N}{\pi r^2} \right\rfloor,
\]

which leads to a contradiction. □

As a corollary we conclude that

Theorem 1.5 If \( 1 \leq \pi r^2 < \pi R^2 \), then it is impossible to squeeze \( B_R \times S^1 \) into \( B_r \times S^1 \).

6. Appendix

A bunch of relations between subsets are extensively applied in estimating microlocal singular supports of certain sheaves. Their explanations and proofs can be found in several references (see [10][11][12][17]). Here we make a short list of those which take roles in Section 3.

Let \( Y \xrightarrow{f} X \) be morphism of manifolds. We have natural morphisms

\[
T^*X \xrightarrow{f^*} T^*X \times_X Y \xrightarrow{f^!} T^*Y.
\]

Microlocal singular supports under Grothendieck’s six functors (\( Rf_*, Rf_!, f^{-1}, f^!, \otimes, \text{Rhom} \)) are bounded by the original ones, via \( f^! \) and \( f_\pi \), in the sense of the following three propositions:

Proposition 6.1. ([12], Proposition 5.4.4) Let \( \mathcal{G} \in D(Y) \) and assume \( f \) is proper on \( \text{supp}(\mathcal{G}) \), then

\[
SS(Rf_*\mathcal{G}) = SS(Rf_!\mathcal{G}) \subset f_\pi((f^!)^{-1}(SS(\mathcal{G}))).
\]

Proposition 6.2. ([12], Proposition 5.4.13) Let \( \mathcal{F} \in D(X) \) and assume \( f \) is non-characteristic for \( SS(\mathcal{F}) \), i.e., \( f^{-1}_\pi(SS(\mathcal{F})) \cap T^*_X X \subset Y \times_X T^*_X X \). Then

(i) \( SS(f^{-1}\mathcal{F}) \subset (f^!)(f^{-1}_\pi(SS(\mathcal{F}))) \).

(ii) \( f^{-1}\mathcal{F} \otimes \omega_{Y/X} \cong f^!\mathcal{F} \).

Proposition 6.3. ([12], Proposition 5.4.14) Let \( \mathcal{F}, \mathcal{G} \in D(X) \). Then

(i) \( SS(\mathcal{F}) \cap SS(\mathcal{G})^a \subset T^*_X X \Rightarrow SS(\mathcal{F} \otimes \mathcal{G}) \subset SS(\mathcal{F}) + SS(\mathcal{G}) \).

(ii) \( SS(\mathcal{F}) \cap SS(\mathcal{G}) \subset T^*_X X \Rightarrow SS(\text{Rhom}(\mathcal{F}; \mathcal{G})) \subset SS(\mathcal{F})^a + SS(\mathcal{G}) \).
In addition, we need a more accurate control of microlocal singular supports under projection maps. Let $E$ be a nontrivial finite-dimensional real vector space and $p : X \times E \to X$ be the projection map. If an object $\mathcal{F}$ is nonsingular in a neighborhood of $\{(x, \omega)\} \times T^*_E E$ in $T^*(X \times E)$, then its projection along $E$ will remain nonsingular at $(x, \omega)$.

**Proposition 6.4.** ([17], Corollary 3.4) Let $\kappa : T^* X \times E \times E^* \to T^* X \times E^*$ be the projection and let $i : T^* X \to T^* X \times E^*$ be the closed embedding $(x, \omega) \mapsto (x, \omega, 0)$ of zero-section. Then for $\mathcal{F} \in D(X \times E)$ we have

$$SS(Rp_! \mathcal{F}), SS(Rp_* \mathcal{F}) \subset i^{-1} \kappa(\overline{SS(\mathcal{F})}).$$

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