A point is normal for almost all maps $\beta x + \alpha \mod 1$ or generalized $\beta$-maps.

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Abstract

We consider the map $T_{\alpha,\beta}(x) := \beta x + \alpha \mod 1$, which admits a unique probability measure of maximal entropy $\mu_{\alpha,\beta}$. For $x \in [0, 1]$, we show that the orbit of $x$ is $\mu_{\alpha,\beta}$-normal for almost all $(\alpha, \beta) \in [0, 1) \times (1, \infty)$ (Lebesgue measure). Nevertheless we construct analytic curves in $[0, 1) \times (1, \infty)$ along them the orbit of $x = 0$ is at most at one point $\mu_{\alpha,\beta}$-normal. These curves are disjoint and they fill the set $[0, 1) \times (1, \infty)$. We also study the generalized $\beta$-maps (in particular the tent map). We show that the critical orbit $x = 1$ is normal with respect to the measure of maximal entropy for almost all $\beta$.

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1 Introduction

In this paper, we consider a dynamical system \((X, d, T)\) where \((X, d)\) is a compact metric space endowed with its Borel \(\sigma\)-algebra \(\mathcal{B}\) and \(T : X \to X\) is a measurable application. Let \(C(X)\) denote the set of all continuous functions from \(X\) into \(\mathbb{R}\). The set \(M(X)\) of all Borel probability measures is equipped with the weak*-topology. \(M(X, T) \subset M(X)\) is the subset of all \(T\)-invariant probability measures. For \(\mu \in M(X, T)\), let \(h(\mu)\) denote the measure-theoretic entropy of \(\mu\). For all \(x \in X\) and \(n \geq 1\), the empirical measure of order \(n\) at \(x\) is

\[
\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_x \circ T^{-i} \in M(X),
\]

where \(\delta_x\) is the Dirac mass at \(x\). Let \(V_T(x) \subset M(X, T)\) denote the set of all cluster points of \(\{\mathcal{E}_n(x)\}_{n \geq 1}\) in the weak*-topology.

**Definition 1.** Let \(\mu \in M(X, T)\) be an ergodic measure and \(x \in X\). The orbit of \(x\) under \(T\) is \(\mu\)-normal, if \(V_T(x) = \{\mu\}\), i.e. for all continuous \(f \in C(X)\), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int f \, d\mu.
\]

By the Birkhoff Ergodic Theorem, \(\mu\)-almost all points are \(\mu\)-normal, however it is difficult to identify a \(\mu\)-normal point. This paper is devoted to the study of the normality of orbits for piecewise monotone continuous applications of the interval. We consider a family \(\{T_\kappa\}_{\kappa \in K}\) of piecewise monotone continuous applications parameterized by a parameter \(\kappa \in K\), such that for all \(\kappa \in K\) there is a unique measure of maximal entropy \(\mu_\kappa\). In our case \(K\) is a subset of \(\mathbb{R}\) or \(\mathbb{R}^2\). For a given \(x \in X\), we estimate the Lebesgue measure of the subset of \(K\) such that the orbit of \(x\) under \(T_\kappa\) is \(\mu_\kappa\)-normal.

For example, let \(T_{\alpha, \beta} : [0, 1] \to [0, 1]\) be the piecewise monotone continuous application defined by \(T_{\alpha, \beta}(x) = \beta x + \alpha \mod 1\); here \(\kappa = (\alpha, \beta) \in [0, 1) \times (1, \infty)\). In [15], Parry constructed a \(T_{\alpha, \beta}\)-invariant probability measure \(\mu_{\alpha, \beta}\) absolutely continuous with respect to Lebesgue measure, which is the unique measure of maximal entropy. The main result of section 3 is Theorem 3 which shows that for all \(x \in [0, 1]\) the set

\[
\mathcal{N}(x) := \{(\alpha, \beta) \in [0, 1) \times (1, \infty) : \text{the orbit of } x \text{ under } T_{\alpha, \beta} \text{ is } \mu_{\alpha, \beta}\text{-normal}\}
\]

has full \(\lambda^2\)-measure, where \(\lambda^d\) is the \(d\)-dimensional Lebesgue measure. This is a generalization of a theorem of Schmeling in [19], where the case \(\alpha = 0\) and \(x = 1\) is studied. For the \(\beta\)-maps, the orbit of 1 plays a particular role, so the restriction to \(x = 1\) considered by Schmeling is natural. Similarly for \(T_{\alpha, \beta}\), the orbits of 0 and 1 are very important. In Theorem 4 we show that there exist curves in the plane \((\alpha, \beta)\) defined by \(\alpha = \alpha(\beta)\) along which the orbits of 0 or 1 are never \(\mu_{\alpha, \beta}\)-normal. The curve \(\alpha = 0\) is a trivial example of such a curve for the fixed point \(x = 0\). In section 4 we study the generalized \(\beta\)-maps introduced by Góra [9]. A generalized \(\beta\)-map is similar to a \(\beta\)-map, but each lap is replaced by an increasing or decreasing lap of constant slope \(\beta\) according to a sequence of signs. For a given class of generalized \(\beta\)-maps, there exists \(\beta_0\) such that for all \(\beta > \beta_0\), there is unique measure of maximal entropy \(\mu_\beta\) and the set

\[
\{\beta > \beta_0 : \text{the orbit of } 1 \text{ under } T_\beta \text{ is } \mu_\beta\text{-normal}\}
\]

has full \(\lambda^1\)-measure. Since the tent maps are generalized \(\beta\)-maps, we obtain an alternative proof of results of Bruin in [4].
2 Preliminaries

Let us define properly the coding for a piecewise monotone continuous application of the interval. The classical papers are [17, 15] and [12]. We consider the piecewise monotone continuous applications of the following type. Let \( k \geq 2 \) and \( 0 = a_0 < a_1 < \cdots < a_k = 1 \). We set \( \mathcal{A} := \{0, \ldots, k-1\} \), \( I_0 = [a_0, a_1) \), \( I_j = (a_j, a_{j+1}] \) for all \( j \in 1, \ldots, k-2 \), \( I_{k-1} = (a_{k-1}, a_k] \) and \( S_0 = \{a_j : j \in 1, \ldots, k-1\} \). For all \( j \in \mathcal{A} \), let \( f_j : I_j \to [0, 1] \) be a strictly monotone continuous map. A piecewise monotone continuous application \( T : [0, 1] \setminus S_0 \to [0, 1] \) is defined by

\[
T(x) = f_j(x) \quad \text{if } x \in I_j.
\]

We will state later in each specific case how to define \( T \) on \( S_0 \). We set \( X_0 = [0, 1] \) and for all \( n \geq 1 \)

\[
X_n = X_{n-1} \setminus S_{n-1} \quad \text{and} \quad S_n = \{x \in X_n : T^n(x) \in S_0\},
\]

so that \( T^n \) is well defined on \( X_n \). Finally we define \( S = \bigcup_{n \geq 0} S_n \) such that \( T^n(x) \) is well defined for all \( x \in [0, 1] \setminus S \) and all \( n \geq 0 \).

Let \( \mathcal{A} \) be endowed with the discrete topology and \( \Sigma_k = \mathcal{A}^\mathbb{Z}^+ \) be the product space. The elements of \( \Sigma_k \) are denoted by \( \underline{x} = x_0 x_1 \ldots \). A finite string \( \underline{w} = w_0 \ldots w_{n-1} \) with \( w_j \in \mathcal{A} \) is a word. The length of \( \underline{w} \) is \( |\underline{w}| = n \). There is a single word of length 0, the empty word \( \varepsilon \). The set of all words is \( \mathcal{A}^* \). For two words \( \underline{w}_1, \underline{w}_2 \), we write \( \underline{w}_1 \underline{w}_2 \) for the concatenation of the two words. For \( \underline{x} \in \Sigma_k \), let \( \underline{x}[i,j] = x_i \ldots x_{j-1} \) denote the word formed by the coordinates \( i \) to \( j-1 \) of \( \underline{x} \). For a word \( \underline{w} \in \mathcal{A}^* \) of length \( n \), the cylinder \( [\underline{w}] \) is the set

\[
[\underline{w}] := \{\underline{x} \in \Sigma_k : \underline{x}[0,n) = \underline{w}\}.
\]

The family \( \{[\underline{w}] : \underline{w} \in \mathcal{A}^*\} \) is a base for the topology and a semi-algebra generating the Borel \( \sigma \)-algebra. For all \( \beta > 1 \), there exists a metric \( d_\beta \) compatible with the topology defined by

\[
d_\beta(\underline{x}, \underline{x}') := \begin{cases} 0 & \text{if } \underline{x} = \underline{x}' \\ \beta^{-\min\{n \geq 0 : \underline{x}_n \neq \underline{x}'_n\}} & \text{otherwise.} \end{cases}
\]

The left shift map \( \sigma : \Sigma_k \to \Sigma_k \) is defined by

\[
\sigma(\underline{x}) = x_1 x_2 \ldots.
\]

It is a continuous map. We define a total order on \( \Sigma_k \) denoted by \( < \). We set

\[
\delta(j) = \begin{cases} +1 & \text{if } f_j \text{ is increasing} \\ -1 & \text{if } f_j \text{ is decreasing} \end{cases}
\]

and for word \( \underline{w} \)

\[
\delta(\underline{w}) = \begin{cases} 1 & \text{if } \underline{w} = \varepsilon \\ \delta(w_0) \cdots \delta(w_{n-1}) & \text{otherwise.} \end{cases}
\]

Let \( \underline{x} \neq \underline{x}' \in \Sigma_k \) and define \( n = \min\{j \geq 0 : x_j \neq x'_j\} \), then

\[
\underline{x} < \underline{x}' \iff \begin{cases} x_n < x'_n & \text{if } \delta(\underline{x}[0,n)) = +1 \\ x_n > x'_n & \text{if } \delta(\underline{x}[0,n)) = -1. \end{cases}
\]

When all maps \( f_j \) are increasing, this is the lexicographic order.

We define the coding map \( \mathbf{i} : [0, 1] \setminus S \to \Sigma_k \) by

\[
\mathbf{i}(x) := \mathbf{i}_0(x) \mathbf{i}_1(x) \ldots \quad \text{with } \mathbf{i}_n(x) = j \iff T^n(x) \in I_j.
\]
The coding map \( i \) is left undefined on \( S \). Henceforth we suppose that \( T \) is such that \( i \) is injective. A sufficient condition for the injectivity of the coding is the existence of \( \lambda > 1 \) such that \( |f'_j(x)| \geq \lambda \) for all \( x \in I_j \) and all \( j \in A \), see \([15]\). This condition is satisfied in all cases considered in the paper. The coding map is order preserving, i.e., for all \( x, x' \in [0,1] \setminus S \)

\[
x < x' \Rightarrow i(x) < i(x').
\]

Define \( \Sigma_T := \overline{i([0,1])} \). We introduce now the \( \varphi \)-expansion as defined by Parry. For all \( j \in A \), let \( \varphi^j : [j, j + 1) \to [a_j, a_{j+1}] \) be the unique monotone extension of \( f^{-1}_j : (c, d) \to (a_j, a_{j+1}) \) where \( (c, d) := f_j((a_j, a_{j+1})) \). The map \( \varphi : \Sigma_k \to [0,1] \) is defined by

\[
\varphi(x) = \lim_{n \to \infty} \varphi^{x_0}(x_0 + \varphi^{x_1}(x_1 + \cdots + \varphi^{x_n}(x_n))).
\]

Parry proved that this limit exists if \( i \) is injective. The map \( \varphi \) is order preserving. Moreover \( \varphi|_{[0,1]} = i^{-1} \) and for all \( n \geq 0 \) and all \( x \in [0,1] \setminus S \)

\[
T^n(x) = \varphi \circ \sigma^n \circ i(x).
\]

If the coding map is injective, one can show that the map \( \varphi \) is continuous (see Theorem 2.3 in \([8]\)). Using the continuity and the monotonicity of \( \varphi \), we have \( \varphi(\Sigma_T) = [0,1] \). Remark that there is in general no extension of \( i \) on \([0,1] \) such that equation (2) is valid on \([0,1] \). For all \( j \in A \), define

\[
\overline{u}^j := \lim_{x \to a_j} i(x) \quad \text{and} \quad \underline{v}^j := \lim_{x \to a_{j+1}} i(x) \quad \text{with} \quad x \in [0,1] \setminus S.
\]

The strings \( \overline{u}^j \) and \( \underline{v}^j \) are called critical orbits and (see for instance \([12]\))

\[
\Sigma_T = \{ x \in \Sigma_k : \overline{u}^x \preceq \sigma^n x \preceq \underline{v}^x \quad \forall n \geq 0 \}.
\]

Moreover the critical orbits \( \overline{u}^j, \underline{v}^j \) satisfy for all \( j \in A \)

\[
\begin{cases}
\overline{u}^n \preceq \sigma^n \overline{u}^j \preceq \underline{v}^n \preceq \sigma^n \underline{v}^j \preceq \underline{v}^n & \forall n \geq 0.
\end{cases}
\]

Let us recall the construction of the Hausdorff dimension. Let \((X, d)\) be a metric space and \( E \subset X \). Let \( D_\varepsilon(E) \) be the set of all finite or countable cover of \( E \) with sets of diameter smaller then \( \varepsilon \). For all \( s \geq 0 \), define

\[
H_\varepsilon(E, s) := \inf \{ \sum_{B \in C} (\text{diam } B)^s : C \in D_\varepsilon(E) \}
\]

and the \( s \)-Hausdorff measure of \( E \), \( H(E, s) := \lim_{\varepsilon \to 0} H_\varepsilon(E, s) \). The Hausdorff dimension of \( E \) is

\[
\dim_H E := \inf \{ s \geq 0 : H(E, s) = 0 \}.
\]

In \([1]\), Bowen introduced a definition of the topological entropy of non compact set for a continuous dynamical system on a metric space. We recall this definition. Let \((X, d)\) be a metric space, \( T : X \to X \) a continuous application. For \( n \geq 1 \), \( \varepsilon > 0 \) and \( x \in X \), let

\[
B_n(x, \varepsilon) = \{ y \in X : d(T^j(x), T^j(y)) < \varepsilon \ \forall j = 0, \ldots, n-1 \}.
\]

For \( E \subset X \), such that \( T(E) \subset E \), let \( G_n(E, \varepsilon) \) be the set of all finite or countable covers of \( E \) with Bowen’s balls \( B_m(x, \varepsilon) \) for \( m \geq n \). For all \( s \geq 0 \), define

\[
C_n(E, \varepsilon, s) := \inf \{ \sum_{B_m(x, \varepsilon) \in C} e^{-ms} : C \in G_n(x, \varepsilon) \}.
\]
and \(C(E, \varepsilon, s) := \lim_{n \to \infty} C_n(E, \varepsilon, s)\). Now, let
\[
h_{\text{top}}(E, \varepsilon) := \inf\{s \geq 0 : C(E, \varepsilon, s) = 0\}
\]
and finally \(h_{\text{top}}(E) = \lim_{\varepsilon \to 0} h_{\text{top}}(E, \varepsilon)\) (this last limit increase to \(h_{\text{top}}(E)\)). There is an evident similarity of this definition with the Hausdorff dimension. This similarity is the key of the next lemma.

**Lemma 1.** For \(\beta > 1\), consider the dynamical system \((\Sigma_k, d_\beta, \sigma)\). Let \(E \subset \Sigma_k\) be such that \(\sigma(E) \subset E\), then
\[
\dim_H E \leq \frac{h_{\text{top}}(E)}{\log \beta}.
\]

**Proof:** Let \(\varepsilon \in (0, 1), s \geq 0, n \geq 0\) and \(C \in \mathcal{G}_n(E, \varepsilon)\). Since \(\text{diam}(B_m(x, \varepsilon)) \leq \varepsilon \beta^{-m+1} \leq \varepsilon \beta^{-n+1}\) for all \(B_m(x, \varepsilon) \in C\), \(C\) is a cover of \(E\) with sets of diameter smaller than \(\varepsilon \beta^{-n+1}\). Moreover
\[
\sum_{B_m(x, \varepsilon) \in C} \text{diam}(B_m(x, \varepsilon)) \leq \varepsilon \beta^{-n+1} \sum_{B_m(x, \varepsilon) \in C} e^{-ms}.
\]
Thus \(H_\delta(E, \frac{s}{\log \beta}) \leq (\varepsilon \beta)^{\frac{s}{\log \beta}} C(E, \varepsilon, s)\) with \(\delta = \varepsilon \beta^{-n+1}\). Taking the limit \(n \to \infty\), we obtain
\[
H(E, \frac{s}{\log \beta}) \leq (\varepsilon \beta)^{\frac{s}{\log \beta}} C(E, \varepsilon, s).
\]
If \(s > h_{\text{top}}(E, \varepsilon)\), then \(H(E, \frac{s}{\log \beta}) = 0\) and \(\frac{s}{\log \beta} \geq \dim_H E\). This is true for all \(s > h_{\text{top}}(E, \varepsilon)\), thus
\[
\dim_H E \leq \frac{h_{\text{top}}(E, \varepsilon)}{\log \beta} \leq \frac{h_{\text{top}}(E)}{\log \beta}. \quad \square
\]

The next lemma is a classical result about the Hausdorff dimension, it is Proposition 2.3 in [6].

**Lemma 2.** Let \((X, d), (X', d')\) be two metric spaces and \(\rho : X \to X'\) be an \(\alpha\)-Hölder continuous application with \(\alpha \in (0, 1]\). Let \(E \subset X\), then
\[
\dim_H \rho(E) \leq \frac{\dim_H E}{\alpha}.
\]

Finally we report Theorem 4.1 from [13]. This theorem is used to estimate the topological entropy of sets we are interested in.

**Theorem 1.** Let \((X, d, T)\) be a continuous dynamical system and \(F \subset M(X, T)\) be a closed subset. Define
\[
G := \{x \in X : V_T(x) \cap F \neq \emptyset\}.
\]
Then
\[
h_{\text{top}}(G) \leq \sup_{\nu \in F} h(\nu).
\]

### 3 Normality for the maps \(\beta x + \alpha \mod 1\)

In this section, we study the piecewise monotone continuous applications \(T_{\alpha, \beta}\) defined by \(T_{\alpha, \beta}(x) = \beta x + \alpha \mod 1\) with \(\beta > 1\) and \(\alpha \in [0, 1)\). These maps were studied by Parry in [15] as a generalization.
of the $\beta$-maps. In his paper Parry constructed a $T_{\alpha,\beta}$-invariant probability measure $\mu_{\alpha,\beta}$, which is absolutely continuous with respect to Lebesgue measure. Its density is

$$h_{\alpha,\beta}(x) := \frac{d\mu_{\alpha,\beta}}{d\lambda}(x) = \frac{1}{N_{\alpha,\beta}} \sum_{n \geq 0} 1_{x<T_{\alpha,\beta}(1)} - \sum_{n \geq 0} 1_{x<T_{\alpha,\beta}(0)},$$

with $N_{\alpha,\beta}$ the normalization factor. In [10], Halfin proved that $h_{\alpha,\beta}(x)$ is nonnegative for all $x \in [0, 1]$. Set $k := [\alpha + \beta]$ and let $i^{\alpha,\beta}$ denote the coding map under $T_{\alpha,\beta}$, $\varphi^{\alpha,\beta}$ the corresponding $\varphi$-expansion, $\Sigma_{\alpha,\beta} := \Sigma_{T_{\alpha,\beta}}$, $a^{\alpha,\beta} := \lim_{x \uparrow 0} i^{\alpha,\beta}(x)$ and $\bar{a}^{\alpha,\beta} := \lim_{x \downarrow 1} i^{\alpha,\beta}(x)$. We specify how $T_{\alpha,\beta}$ is defined at the discontinuity points. We choose to define $T_{\alpha,\beta}$ by right-continuity at $a_j \in S_0$. Doing this we can also extend the definition of the coding map $i^{\alpha,\beta}$ using the disjoint intervals $[a_j, a_{j+1})$ for $j \in \mathbb{A}$, so that $i^{\alpha,\beta}$ is now defined for all $x \in [0, 1)$ [1]. We can show that $a^{\alpha,\beta} = \bar{a}^{\alpha,\beta}(0)$ and equation (2) is true for all $x \in [0, 1)$. It is easy to check that formula (3) becomes

$$\Sigma_{\alpha,\beta} = \{x \in \Sigma_k : u^{\alpha,\beta}_n \preceq \sigma^n x \preceq v^{\alpha,\beta} \quad \forall n \geq 0\}$$

and inequations (4) become

$$\left\{
\begin{array}{l}
u^{\alpha,\beta} \preceq \sigma^n \nu^{\alpha,\beta} \preceq \bar{v}^{\alpha,\beta} \\
\bar{v}^{\alpha,\beta} \preceq \sigma^n \bar{v}^{\alpha,\beta} \preceq v^{\alpha,\beta}
\end{array}
\right. \quad \forall n \geq 0.$$

It is known that the dynamical system $(\Sigma_{\alpha,\beta}, \sigma)$ has topological entropy $\log \beta$. Moreover, Hofbauer showed in [13] that it has a unique measure of maximal entropy $\hat{\mu}_{\alpha,\beta}$, $\mu_{\alpha,\beta} = \hat{\mu}_{\alpha,\beta} \circ (\varphi^{\alpha,\beta})^{-1}$ and $\mu_{\alpha,\beta}$ is the unique measure of maximal entropy for $T_{\alpha,\beta}$. In view of (6) and (7), for a couple $(u, v) \in \Sigma_k^2$ satisfying

$$\left\{
\begin{array}{l}
u \preceq \sigma^n u \preceq v \\
u \preceq \sigma^n v \preceq v
\end{array}
\right. \quad \forall n \geq 0,$$

we define the shift space

$$\Sigma_{u,v} := \{x \in \Sigma_k : u \preceq \sigma^n x \preceq v \quad \forall n \geq 0\}.$$ 

We give now a lemma and a proposition which are the keys of the main theorem of this section. In the lemma, we show that for given $x$ and $\alpha$, there is exponential separation between the orbits of $x$ under the two different dynamical systems $T_{\alpha,\beta_1}$ and $T_{\alpha,\beta_2}$. The proposition asserts that the topological entropy of $\Sigma_{u,v}$ depends continuously on the the critical orbits $u$ and $v$.

**Lemma 3.** Let $x \in [0, 1]$, $\alpha \in [0, 1]$ and $1 < \beta_1 \leq \beta_2$. Define $l = \min\{n \geq 0 : i^1_n(x) \neq i^2_n(x)\}$ with $i^j(x) = i^{\alpha,\beta_j}$ for $j = 1, 2$. If $x \neq 0$, then

$$\beta_2 - \beta_1 \leq \frac{\beta_2^{-l+1}}{x}.$$

If $x = 0$ and $\alpha \neq 0$, then

$$\beta_2 - \beta_1 \leq \frac{\beta_2^{-l+2}}{\alpha}.$$

**Proof:** Let $\delta := \beta_2 - \beta_1 \geq 0$. We prove by induction that for all $m \geq 1$, $i^1_{[0,m]}(x) = i^2_{[0,m]}(x)$ implies

$$T^m_2(x) - T^m_1(x) \geq \beta_2^{m-1} \delta x,$$

This convention differs from that made in the previous section; however it is the most convenient choice when all $f_j$ are increasing.
where \( T_i = T_{\alpha, \beta} \). For \( m = 1 \),

\[
T_2(x) - T_1(x) = \beta_2 x + \alpha - i_0^2(x) - (\beta_1 x + \alpha - i_0^1(x)) = \delta x.
\]

Suppose that this is true for \( m \), then \( i_{[0,m+1]}^1 = i_{[0,m+1]}^2 \) implies

\[
T_{2}^{m+1}(x) - T_{1}^{m+1}(x) = \beta_2 T_{2}^{m}(x) + \alpha - i_{m}^2(x) - (\beta_1 T_{1}^{m}(x) + \alpha - i_{m}^1(x)) = \beta_2 (T_{2}^{m}(x) - T_{1}^{m}(x)) + \delta T_{1}^{m}(x) \geq \beta_2^m \delta x.
\]

On the other hand, \( 1 \geq T_{2}^{m}(x) - T_{1}^{m}(x) \geq \beta_2^{m-1} \delta x \). Thus \( \delta \leq \frac{\beta_2^{-m+1}}{\beta_2} \) for all \( m \) such that \( i_{[0,m]}^1 = i_{[0,m]}^2 \).

If \( x = 0 \), then \( T_1(x) = T_2(x) = \alpha \) and we can apply the first statement to \( y = \alpha > 0 \). \( \square \)

**Proposition 1.** Let \((\nu, \nu'), (\nu', \nu'') \in \Sigma_k^2 \) satisfy (5). Let \( L \in \mathbb{N} \) and suppose that \( \nu, \nu' \) have a common prefix of length larger than \( L \) and \( \nu', \nu'' \) have a common prefix of length larger than \( L \). Then for all \( \delta > 0 \) there exists \( L(\delta) \) such that for any \( L \geq L(\delta) \),

\[
|h_{\text{top}}(\Sigma_{\nu', \nu''}) - h_{\text{top}}(\Sigma_{\nu, \nu'})| \leq \delta.
\]

This proposition is a stronger reformulation of Proposition 9.3.15 in [2]. It follows from the proof of Proposition 9.3.15 given in this book, except that the argument at the very end of the proof is incomplete; but it is completed in [5]. Now we can state our first theorem and his corollary about the normality of orbits under \( T_{\alpha, \beta} \). The proof of the theorem is inspired by the proof of Theorem C in [19], where the case \( x = 1 \) and \( \alpha = 0 \) is considered.

**Theorem 2.** Let \( x \in [0, 1) \) and \( \alpha \in [0, 1) \) excepted \((x, \alpha) = (0, 0)\). Then the set

\[
\{\beta > 1 : \text{the orbit of } i^{\alpha, \beta}(x) \text{ under } \sigma \text{ is } \hat{\mu}_{\alpha, \beta}\text{-normal}\}
\]

has full \( \lambda \)-measure.

**Corollary 1.** Let \( x \in [0, 1) \) and \( \alpha \in [0, 1) \) excepted \((x, \alpha) = (0, 0)\). Then the set

\[
\{\beta > 1 : \text{the orbit of } x \text{ under } T_{\alpha, \beta} \text{ is } \mu_{\alpha, \beta}\text{-normal}\}
\]

has full \( \lambda \)-measure.

Remark that the theorem and its corollary may also be formulated for \( x \in (0, 1] \) using a left-continuous extension of \( T_{\alpha, \beta} \) on \([0, 1)\) and a coding \( i^{\alpha, \beta}_{*, \alpha, \beta} \) defined using intervals \((a_j, a_{j+1}]\) for all \( j \in \mathbb{A} \).

**Proof of the theorem:** We briefly sketch the proof. We use the uniqueness of the measure of maximal entropy \( \hat{\mu}_{\alpha, \beta} \): for \( x \in \Sigma_{\alpha, \beta} \) not \( \hat{\mu}_{\alpha, \beta}\)-normal, there exists \( \nu \in V_{\alpha}(x) \) such that \( h(\nu) < h(\hat{\mu}_{\alpha, \beta}) = \log \beta \).

The main idea is to imbed \( \{i^{\alpha, \beta}(x) : \beta \in [\beta_1, \beta_2]\} \) in a shift space \( \Sigma^* := \Sigma_{u^*, \nu^*}^\ast \) with \( u^* \) and \( v^* \) well chosen. Writing \( D^* \subset \Sigma^* \) for the range of the imbedding, we estimate the Hausdorff dimension of the subset of \( D^* \) corresponding to points \( i^{\alpha, \beta}(x) \) which are not \( \hat{\mu}_{\alpha, \beta}\)-normal. Then we estimate the coefficient of Hölder continuity of the application \( \rho_\ast \) defined as the inverse of the imbedding. This gives us an estimate of the Hausdorff dimension of the non \( \hat{\mu}_{\alpha, \beta}\)-normal points in the interval \( [\beta_1, \beta_2] \).

To obtain uniform estimates, we restrict our proof to the interval \([\beta, \bar{\beta}]\) with \( 1 < \beta < \bar{\beta} < \infty \). This is sufficient, since there exist a countable cover of \((1, \infty)\) with such intervals. Let \( k := [\alpha + \beta] \) and \( \Omega := \{\beta \in [\beta, \bar{\beta}] : i^{\alpha, \beta}(x) \text{ is not } \hat{\mu}_{\alpha, \beta}\text{-normal}\} \). For \( \beta \in \Omega \), we have \( V_{\alpha}(i^{\alpha, \beta}(x)) \neq \{\hat{\mu}_{\alpha, \beta}\} \). Since \( \hat{\mu}_{\alpha, \beta} \) is the unique measure of maximal entropy \( \log \beta \), there exist \( N \in \mathbb{N} \) and \( \nu \in V_{\alpha}(i^{\alpha, \beta}(x)) \) such that \( h(\nu) < (1 - 1/N) \log \beta \). Setting

\[
\Omega_N := \{\beta \in [\beta, \bar{\beta}] : \exists \nu \in V_{\alpha}(i^{\alpha, \beta}(x)) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta\},
\]

where \( F_{0} = T_{\alpha, \beta} \). For \( m = 1 \),

\[
T_{2}(x) - T_{1}(x) = \beta_{2} x + \alpha - i_{0}^2(x) - (\beta_{1} x + \alpha - i_{0}^1(x)) = \delta x.
\]
we have $\Omega = \bigcup_{N \geq 1} \Omega_N$. We will prove that $\dim H \Omega_N < 1$, so that $\lambda(\Omega_N) = 0$ for all $N \geq 1$.

For $N \in \mathbb{N}$ fixed, define $\varepsilon := \frac{\beta \log \beta}{N \log \beta} > 0$ and $\delta := \log (1 + \varepsilon/\beta)$. Choose $L \geq L(\delta)$ (Proposition 1).

Consider the family of subsets of $[\beta, \beta']$ of the following type

$$J(w, w') = \{ \beta \in [\beta, \beta'] : w_{\alpha, \beta}(0, L) = w_{\alpha, \beta}'(0, L) = w' \}$$

where $w, w'$ are two words of length $L$. $J(w, w')$ is either empty or it is an interval, since the applications $\beta \mapsto w_{\alpha, \beta}$ and $\beta \mapsto w_{\beta, \beta'}$ are both monotone increasing. Moreover, $[\beta, \beta'] = \bigcup_{w, w'} J(w, w')$ where the union is finite, since the set of words of length $L$ in $A^*$ has finite cardinality. We want to work with closed intervals, thus we cover the non-closed $J(w, w')$ with countably many closed intervals if necessary. For example, if $J(w, w') = (a, b)$, we write $J(w, w') = \bigcup_{n \geq 1} [a + 1/m, b]$. We prove that $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$ where $\beta_1 < \beta_2$ are such that $w_{\alpha, \beta_1}(0, L) = w_{\alpha, \beta_1}'(0, L)$ and $w_{\alpha, \beta_2}(0, L) = w_{\alpha, \beta_2}'(0, L)$.

Let $w^i = w_{\alpha, \beta_i}$ and $w^j = w_{\beta, \beta_j}$. Using (7) and the monotonicity of $\beta \mapsto w_{\alpha, \beta}$ and $\beta \mapsto w_{\beta, \beta}$, we have

$$\forall n \geq 0.$$ 

Hence the couple $(w^1, w^2)$ satisfy (8) and we set $\Sigma^* := \Sigma^{w^1, w^2}$ and

$$D^* := \{ \bar{z} \in \Sigma^* : \exists \beta \in [\beta_1, \beta_2] \text{ s.t. } \bar{z} = i_{\alpha, \beta}(x) \}.$$ 

We define an application $\rho_* : D^* \rightarrow [\beta_1, \beta_2]$ by $\rho_* (\bar{z}) = \beta$ if and only if $i_{\alpha, \beta}(x) = \bar{z}$. This application is well defined: by definition of $D^*$, for all $\bar{z} \in D^*$ there exists a $\beta$ such that $\bar{z} = i_{\alpha, \beta}(x)$; moreover this $\beta$ is unique, since by Lemma 8 $\beta \mapsto i_{\alpha, \beta}(x)$ is strictly increasing. On the other hand, for all $\beta \in [\beta_1, \beta_2]$, we have from (6)

$$\forall n \geq 0,$n

whence $i_{\alpha, \beta}(x) \in \Sigma^*$ and $\rho_* : D^* \rightarrow [\beta_1, \beta_2]$ is surjective. Let $\beta_* := h_{top}(\Sigma^*)$; then by Proposition 1

$$\beta_* - \beta_1 \leq e^{\log \beta_1 \log \varepsilon} \log \beta_1.$$ (10)

Let us compute the coefficient of Hölder continuity of $\rho_* : (D^*, d_{\beta_*}) \rightarrow [\beta_1, \beta_2]$. Let $\bar{z} \neq \bar{z}' \in D^*$ and $n = \min \{ l \geq 0 : z_l \neq z'_l \}$, then $d_{\beta_*}(\bar{z}, \bar{z}') = \beta_*^{-n}$. By Lemma 8 there exists $C$ such that

$$|\rho_*(\bar{z}) - \rho_*(\bar{z}')| \leq C \rho_*(\bar{z})^{-n} \leq C \beta_*^{-n} = C(d_{\beta_*}(\bar{z}, \bar{z}'))^{\log \beta_* \log \beta_1}.$$ 

We may choose $C$ independently of $\beta$, since we work on the compact interval $[\beta_1, \beta_2] \subset (1, \infty)$. By equation (10) and the choice of $\varepsilon$, we have

$$\beta_* - \beta_1 \leq \frac{\beta \log \beta}{2N - 1} \Rightarrow \beta_* - \beta_1 \leq \frac{\beta_1 \log \beta_1}{2N - 1}$$

$$\Rightarrow 1 + \frac{\beta_* - \beta_1}{\beta_1} \leq 1 + \frac{1}{2N - 1}$$

$$\Rightarrow \log \beta_1 + \frac{\beta_* - \beta_1}{\beta_1} \leq \frac{2N}{2N - 1}$$

$$\Rightarrow \frac{\log \beta_1}{\log \beta_*} \geq \frac{\log \beta_1}{\log \beta_1 + \frac{\beta_* - \beta_1}{\beta_1}} \geq 1 - \frac{1}{2N}.$$ 

In last line, we use the concavity of the logarithm, so the first order Taylor development is an upper estimate. Thus $\rho_*$ has Hölder-exponent $1 - \frac{1}{2N}$. 

8
Define
\[ G^*_N := \{ z \in \Sigma^* : \exists \nu \in V_\sigma(z) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta_* \}. \]

Let \( \beta \in \Omega_N \cap [\beta_1, \beta_2] \). Then there exists \( \nu \in V_\sigma(\hat{i}^{\alpha, \beta}(x)) \) such that
\[ h(\nu) < (1 - 1/N) \log \beta \leq (1 - 1/N) \log \beta_* . \]

Since \( \hat{i}^{\alpha, \beta}(x) \in D^* \subset \Sigma^* \), we have \( \hat{i}^{\alpha, \beta}(x) \in G^*_N \). Using the surjectivity of \( \rho_* \), we obtain \( \Omega_N \cap [\beta_1, \beta_2] \subset \rho_*(G^*_N \cap D^*) \). We claim that \( h_{\text{top}}(G^*_N) \leq (1 - 1/N) \log \beta_* \). This implies, using Lemmas 2 and 1,
\[ \dim_H(\Omega_N \cap [\beta_1, \beta_2]) \leq \dim_H \rho_*(G^*_N \cap D^*) \leq \frac{\dim_H G^*_N}{1 - \frac{1}{2N}} \leq \frac{h_{\text{top}}(G^*_N)}{(1 - \frac{1}{2N}) \log \beta_*} \leq \frac{1 - \frac{1}{N}}{1 - \frac{1}{2N}} < 1. \]

Thus \( \lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0 \).

It remains to prove \( h_{\text{top}}(G^*_N) \leq (1 - 1/N) \log \beta_* \). Recall that \( h(\nu) = \lim_n \frac{1}{n} H_n(\nu) \), where \( H_n(\nu) \) is the entropy of \( \nu \) with respect to the algebra of cylinder sets of length \( n \) and that \( \frac{1}{n} H_n(\nu) \) is decreasing. For all \( m \geq 1 \), we set
\[
\begin{align*}
F^*_N(m) & := \{ \nu \in M(\Sigma^*, \sigma) : \frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_* \} \\
G^*_N(m) & := \{ z \in \Sigma^* : V_\sigma(z) \cap F^*_N(m) \neq \emptyset \}.
\end{align*}
\]

Let \( z \in G^*_N \), then there exists \( \nu \in V_\sigma(z) \) such that \( h(\nu) < (1 - 1/N) \log \beta_* \). Since \( \frac{1}{m} H_m(\nu) \downarrow h(\nu) \), there exists \( m \geq 1 \) such that \( \frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_* \); whence \( \nu \in F^*_N(m) \) and \( z \in G^*_N(m) \). This implies \( G^*_N \subset \bigcup_{m \geq 1} G^*_N(m) \). Since \( H_m(\cdot) \) is continuous, \( F^*_N(m) \) is closed for all \( m \geq 1 \). Finally we obtain using Theorem 1
\[ h_{\text{top}}(G^*_N) = \sup_m h_{\text{top}}(G^*_N(m)) \leq \sup_m \sup_{\nu \in F^*_N(m)} h(\nu) \leq \sup_m \sup_{\nu \in F^*_N(m)} \frac{1}{m} H_m(\nu) \leq (1 - 1/N) \log \beta_* . \]

**Proof of the Corollary:** Let \( \beta > 1 \) be such that the orbit of \( \hat{i}^{\alpha, \beta}(x) \) under \( \sigma \) is \( \hat{\mu}_{\alpha, \beta} \)-normal. Let \( f \in C([0, 1]) \), then \( \hat{f} : \Sigma_{\alpha, \beta} \to \mathbb{R} \) defined by \( \hat{f} := f \circ \varphi^{\alpha, \beta} \) is continuous, since \( \varphi^{\alpha, \beta} \) is continuous. Using \( \mu_{\alpha, \beta} := \hat{\mu}_{\alpha, \beta} \circ (\varphi^{\alpha, \beta})^{-1} \), we have
\[
\begin{align*}
\int_{[0,1]} f \, d\mu_{\alpha, \beta} & = \int_{\Sigma_{\alpha, \beta}} \hat{f} \, d\hat{\mu}_{\alpha, \beta} = \lim_{n \to \infty} \sum_{i=0}^{n-1} \hat{f}(\sigma^i \hat{i}^{\alpha, \beta}(x)) \\
& = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(\varphi^{\alpha, \beta}(\sigma^i \hat{i}^{\alpha, \beta}(x))) = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(T_{\alpha, \beta}(x)).
\end{align*}
\]

The second equality comes from the \( \hat{\mu}_{\alpha, \beta} \)-normality of the orbit of \( \hat{i}^{\alpha, \beta}(x) \) under \( \sigma \), the last one is (2) which is true for all \( x \in [0, 1] \) with our convention for the extension of \( T_{\alpha, \beta} \) and \( \hat{i}^{\alpha, \beta} \) on \([0, 1] \). \( \square \)

The next step is to consider the question of \( \mu_{\alpha, \beta} \)-normality in the whole plane \((\alpha, \beta)\) instead of working with \( \alpha \) fixed. Define \( \mathcal{R} := [0, 1] \times (1, \infty) \).

**Theorem 3.** For all \( x \in [0, 1] \), the set
\[ \mathcal{N}(x) := \{ (\alpha, \beta) \in \mathcal{R} : \text{the orbit of } x \text{ under } T_{\alpha, \beta} \text{ is } \mu_{\alpha, \beta} \text{-normal} \} \]
has full \( \lambda^2 \)-measure.
Proof: We have only to prove that $\mathcal{N}(x)$ is measurable and to apply Fubini's Theorem and Corollary 1. The first step is to prove that for all $x \in [0, 1)$ and all $n \geq 0$, the applications $(\alpha, \beta) \mapsto i_{\alpha, \beta}(x)$ and $(\alpha, \beta) \mapsto T_{\alpha, \beta}^n(x)$ are measurable. First remark that for all $n \geq 1$

$$T_{\alpha, \beta}^n(x) = \beta^n x + \alpha \frac{\beta^n - 1}{\beta - 1} - \sum_{j=0}^{n-1} i_{\alpha, \beta}^j(x) \beta^{n-j-1}. \quad (11)$$

The proof by induction is immediate. To prove that $(\alpha, \beta) \mapsto i_{\alpha, \beta}(x)$ is measurable, it is enough to prove that for all $n \geq 0$ and for all words $w \in \mathcal{A}^*$ of length $n$

$$\{(\alpha, \beta) \in \mathcal{R} : i_{\alpha, \beta}^n(x) = w\}$$

is measurable, since the $\sigma$-algebra on $\Sigma_k$ is generated by the cylinders. This set is the subset of $\mathbb{R}^2$ such that

$$\begin{cases} 
\beta > 1 \\
0 \leq \alpha < 1 \\
w_j < \beta T_{\alpha, \beta}^j(x) + \alpha \leq w_j + 1 \quad \forall 0 \leq j < n
\end{cases}$$

Using (11), this system of inequations can be rewritten

$$\begin{cases} 
\beta > 1 \\
0 \leq \alpha < 1 \\
\alpha > \frac{\beta - 1}{\beta^{j+1}} \left( \sum_{i=0}^{j} w_i \beta^{j-i} - \beta^{j+1} x \right) \quad \forall 0 \leq j < n \\
\alpha \leq \frac{\beta - 1}{\beta^{j+1}} \left( 1 + \sum_{i=0}^{j} w_i \beta^{j-i} - \beta^{j+1} x \right) \quad \forall 0 \leq j < n
\end{cases}$$

From this, the measurability of $i_{\alpha, \beta}$ follows. If $(\alpha, \beta) \mapsto i_{\alpha, \beta}(x)$ is measurable, then by formula (11), $(\alpha, \beta) \mapsto T_{\alpha, \beta}^n(x)$ is clearly measurable for all $n \geq 0$. Then for all $f \in C([0, 1])$ and all $n \geq 1$, the application $(\alpha, \beta) \mapsto S_n(f) := \frac{1}{n} \sum_{i=0}^{n-1} f(T_{\alpha, \beta}^i(x))$ is measurable and consequently

$$\{(\alpha, \beta) : \lim_{n \to \infty} S_n(f) \text{ exists} \}$$

is a measurable set.

On the other hand, if $f \in C([0, 1])$, then $(\alpha, \beta) \mapsto \int f d\mu_{\alpha, \beta}$ is measurable. Indeed

$$\int f d\mu_{\alpha, \beta} = \int f h_{\alpha, \beta} d\lambda$$

and in view of equation (11) and the measurability of $(\alpha, \beta) \mapsto T_{\alpha, \beta}(x)$, the application $(\alpha, \beta) \mapsto h_{\alpha, \beta}$ is clearly measurable. Therefore

$$\{(\alpha, \beta) : \lim_{n \to \infty} S_n(f) = \int f d\mu_{\alpha, \beta} \}$$

is measurable for all $f \in C([0, 1])$. Let $\{f_m\}_{m \in \mathbb{N}} \subset C([0, 1])$ be countable subset which is dense with respect to the uniform convergence. Then setting

$$D_m := \{(\alpha, \beta) \in \mathcal{R} : \lim_{n \to \infty} S_n(f_m) = \int f_m d\mu_{\alpha, \beta} \},$$

we have $\mathcal{N}(x) = \bigcap_{m \in \mathbb{N}} D_m$, whence it is a measurable set. □

We have shown that for a given $x \in [0, 1)$, the orbit of $x$ under $T_{\alpha, \beta}$ is $\mu_{\alpha, \beta}$-normal for almost all $(\alpha, \beta)$. The orbits of 0 and 1 are of particular interest (see equations (5) or (9)). Now we show that
by any point \((\alpha_0, \beta_0)\), there passes a curve defined by \(\alpha = \alpha(\beta)\) such that the orbit of 0 under \(T_{\alpha(\beta), \beta}\) is \(\mu_{\alpha(\beta), \beta}\)-normal for at most one \(\beta\). A trivial example of such a curve is \(\alpha = 0\), since \(x = 0\) is a fixed point. The idea is to consider curves along which the coding of 0 is constant, ie to define \(\alpha(\beta)\) such that \(\mu_{\alpha(\beta), \beta}\) is constant. The results below depend on reference \([8]\), where we solve the following inverse problem: given \(u\) and \(v\) verifying \((8)\), can we find \(\alpha, \beta\) such that \(u = u^{\alpha, \beta}\) and \(v = v^{\alpha, \beta}\) ?

Let

\[ U := \{u : \exists (\alpha, \beta) \in \mathcal{R} \text{ s.t. } u = u^{\alpha, \beta}\}. \]

We define an equivalence relation in \(\mathcal{R}\) by

\[ (\alpha, \beta) \sim (\alpha', \beta') \iff u^{\alpha, \beta} = u^{\alpha', \beta'}. \]

An equivalence class is denoted by \([u]\). The next lemma describes \([u]\).

**Lemma 4.** Let \(u \in U\) and set

\[ \alpha(\beta) = (\beta - 1) \sum_{j \geq 0} \frac{u_j}{\beta_j+1}. \]

Then there exists \(\beta_u \geq 1\) such that

\[ [u] = \{ (\alpha(\beta), \beta) : \beta \in I_u \} \]

with \(I_u = (\beta_u, \infty)\) or \(I_u = [\beta_u, \infty)\).

**Proof:** If \(u = 000\ldots\), then the statement is trivially true with \(\alpha(\beta) \equiv 0\) and \(\beta_u = 1\). Suppose \(u \neq 000\ldots\). First we prove that

\[ (\alpha, \beta) \sim (\alpha', \beta) \implies \alpha = \alpha' \]

then

\[ (\alpha, \beta) \in [u] \implies (\alpha(\beta'), \beta') \in [u] \quad \forall \beta' \geq \beta. \]

Let \((\alpha, \beta) \in [u]\). Using \((2)\), we have \(\varphi^{\alpha, \beta}(\sigma u) = T_{\alpha, \beta}(0) = \alpha\). Since the map \(\alpha \mapsto \varphi^{\alpha, \beta}(\sigma u) - \alpha\) is continuous and strictly decreasing (Lemmas 3.5 and 3.6 in \([8]\)), the first statement is true. Let \(\beta' > \beta\). By Corollary 3.1 in \([8]\), we have that \(\varphi^{\alpha, \beta}(\sigma u) > \varphi^{\alpha, \beta'}(\sigma u)\). Therefore there exists a unique \(\alpha' < \alpha\) such that \(\varphi^{\alpha', \beta'}(\sigma u) = \alpha'\). We prove that \(u^{\alpha', \beta'} = u\). By point 1 of Proposition 2.5 in \([8]\), we have \(u \leq u^{\alpha', \beta'}\). By Proposition 3.3 in \([8]\), we have

\[ h_{\text{top}}(\Sigma_{u^{\alpha', \beta'}}) = h_{\text{top}}(\Sigma_{u^{\alpha', \beta}}) = \log \beta'. \]

Since \(\Sigma_{\alpha, \beta} = \Sigma_{u^{\alpha, \beta}}\) and \(\beta' > \beta\), we must have \(u^{\alpha, \beta} < u^{\alpha', \beta'}\). Therefore

\[ \begin{cases} u \leq \sigma u < u^{\alpha, \beta} < u^{\alpha', \beta'} \quad \forall n \geq 0, \\ u \leq u^{\alpha', \beta'} < \sigma u < u^{\alpha, \beta} \leq u^{\alpha', \beta'} \end{cases} \]

are the inequalities (4.1) in \([8]\) for the pair \((u, u^{\alpha', \beta'})\). We can apply Proposition 3.2 and Theorem 4.1 in \([8]\) to this pair and get \(u = u^{\alpha', \beta'}\). It remains to show that \(\alpha' = \alpha(\beta')\). Following the definition of the \(\varphi\)-expansion of Rényi, we have for all \(x \in [0, 1)\) and all \(n \geq 0\)

\[ x = \sum_{j=0}^{n-1} \frac{i_j^{\alpha, \beta}(x) - \alpha}{\beta_j+1} + T_{\alpha, \beta}(x) \frac{x^n}{\beta^n}. \]

11
Since $T_{\alpha, \beta}^n(x) \in [0, 1)$, for all $\beta > 1$ we find an explicit expression for $\varphi_{\alpha, \beta}$ on $\Sigma_{\alpha, \beta}$

$$x = \sum_{j \geq 0} \frac{i_j^{\alpha, \beta}(x) - \alpha}{\beta^{j+1}}.$$  

In particular, applying this equation to $x = 0$, we have for all $(\alpha, \beta) \in \mathcal{R}$

$$\alpha = (\beta - 1) \sum_{j \geq 0} \frac{u_j^{\alpha, \beta}}{\beta^{j+1}}.$$  

Since for all $\beta > \beta_u$, we have $u \in \Sigma_{\alpha, \beta}$, this complete the proof. □

For each $u \in U$, the equivalence class $[u]$ defines an analytic curve in $\mathcal{R}$, which is strictly monotone decreasing (excepted for $u = 000 \ldots$),

$$[u] = \{(\alpha, \beta) : \alpha = (\beta - 1) \sum_{j \geq 0} \frac{u_j}{\beta^{j+1}}, \beta \in I_u\}.$$  

There curves are disjoint two by two and their union is $\mathcal{R}$.

**Theorem 4.** Let $(\alpha, \beta) \in \mathcal{R}$, $u = \underline{u}^{\alpha, \beta}$ and define $\alpha(\beta)$ and $\beta_u$ as in Lemma 4. Then for all $\beta > \beta_u$, the orbit of $x = 0$ under $T_{\alpha(\beta), \beta}$ is not $\mu_{\alpha(\beta), \beta}$-normal.

**Proof:** Let $\hat{\nu} \in M(\Sigma_k, \sigma)$ (with $k$ large enough) be a cluster point of $\{E_n(u)\}_{n \geq 1}$. By Lemma 4 $\underline{u}^{\alpha(\beta), \beta} = \underline{u}$ for any $\beta > \beta_u$. Therefore

$$h(\hat{\nu}) \leq h_{\text{top}}(\Sigma_{\alpha(\beta), \beta}) = \log \beta \quad \forall \beta > \beta_u$$

and $\hat{\nu}$ is not a measure of maximal entropy, as well as $\nu_{\beta} := \hat{\nu} \circ (\varphi_{\alpha(\beta), \beta})^{-1}$ [12], for all $\beta > \beta_u$. □

Recall that

$$\mathcal{N}(0) = \{(\alpha, \beta) \in \mathcal{R} : \text{the orbit of 0 under } T_{\alpha, \beta} \text{ is } \mu_{\alpha, \beta}\text{-normal}\}.$$  

By Theorem 3, $\mathcal{N}(0)$ has full Lebesgue measure. On the other hand, by Theorem 4, we can decompose $\mathcal{R}$ into a family of disjoint analytic curves such that each curve meets $\mathcal{N}(0)$ in at most one point. This situation is very similar to the one presented in [14] by Milnor following an idea of Katok.

### 4 Normality in generalized $\beta$-maps

In this section, we consider another class of piecewise monotone continuous applications, the generalized $\beta$-maps. Introduced by Góra in [9], they have only one critical orbit like $\beta$-maps, but they admit increasing and decreasing laps. A family $\{T_\beta\}_{\beta > 1}$ of generalized $\beta$-maps is defined by $k \geq 2$ and a sequence $s = (s_n)_{0 \leq n < k}$ with $s_i \in \{-1, 1\}$. For any $\beta \in (k - 1, k]$, let $a_j = j/\beta$ for $j = 0, \ldots, k - 1$ and $a_k = 1$. Then for all $j = 0, \ldots, k - 1$, the map $f_j = I_j \to [0, 1]$ is defined by

$$f_j(x) := \begin{cases} 
\beta x \mod 1 & \text{if } s_j = +1 \\
1 - (\beta x \mod 1) & \text{if } s_j = -1.
\end{cases}$$

In particular when $s = (1, -1)$, then $T_\beta$ is a tent map. Here we left the map undefined on $a_j$ for $j = 1, \ldots, k - 1$.

Góra constructed the unique measure $\mu_\beta$ absolutely continuous with respect to Lebesgue measure (Theorem 6 and Proposition 8 in [9]). Using the same argument as Hofbauer in [11], we deduce that
a measure of maximal entropy is always absolutely continuous with respect to Lebesgue measure, hence the measure $\mu_\beta$ is the unique measure of maximal entropy. Let $k = \lceil \beta \rceil$ and let us denote $i^\beta$ for the coding map under $T_\beta$, $\varphi^\beta := (i^\beta)^{-1}$ for the inverse of the coding map, $\Sigma_\beta := \Sigma_{T_\beta}$ and $\eta^\beta := \lim_{x \to 1} i^\beta(x)$. Now it is easy to check that formula (3) becomes

$$\Sigma_\beta = \{ x \in \Sigma_k : \sigma^n x \preceq \eta^\beta \quad \forall n \geq 0 \}$$

and inequalities (11) become

$$\sigma^n \eta^\beta \preceq \eta^\beta \quad \forall n \geq 0.$$  

(13)

It is known that the dynamical system $(\Sigma_\beta, \sigma)$ has topological entropy $\log \beta$ and, by general works of Hofbauer in [12], it has a unique measure of maximal entropy $\hat{\mu}_\beta$ such that $\mu_\beta = \hat{\mu}_\beta \circ (\varphi^\beta)^{-1}$.

As in the previous section, we state two lemmas which we need for the proof of the main theorem of this section. We study the normality only of $x = 1$, so these lemmas are formulated only for $x = 1$. Let $S_n(\beta) \equiv S_n$ and $S(\beta) \equiv S$ be defined by (1).

**Lemma 5.** For any family of generalized $\beta$-maps defined by $(s_n)_{0 \leq n < k}$, the set $\{ \beta \in (k-1, k] : 1 \in S(\beta) \}$ is countable.

**Proof:** For a fixed $n \geq 1$, we study the map $\beta \mapsto T^n_\beta(1)$. This map is well defined everywhere in $(k-1, k]$ excepted for finitely many points and it is continuous on each interval where it is well defined. Indeed this is true for $n = 1$. Suppose it is true for $n$, then $T^{n+1}_\beta(1)$ is well defined and continuous wherever $T^n_\beta(1)$ is well defined and continuous, excepted when $T^n_\beta(1) \in S_0(\beta)$. By the induction hypothesis, there exists a finite family of disjoint open intervals $J_i$ and continuous functions $g_i : J_i \to [0, 1]$ such that $(k-1, k] \setminus (\bigcup J_i)$ is finite and

$$T^n_\beta(x) = g_i(\beta) \quad \text{if } \beta \in J_i.$$

Then

$$\{ \beta \in (k-1, k] : T^n_\beta(1) \text{ is well defined and } T^n_\beta(1) \in S_0(\beta) \} = \bigcup_{i,j} \{ \beta \in J_i : g_i(\beta) = \frac{j}{\beta} \}.$$

We claim that $\{ \beta \in J_i : g_i(\beta) = \frac{j}{\beta} \}$ has finitely many points. From the form of the map $T_\beta$, it follows immediately that each $g_i(\beta)$ is a polynomial of degree $n$. Since $\beta > 1$,

$$g_i(\beta) = \frac{j}{\beta} \iff \beta g_i(\beta) - j = 0.$$

This polynomial equation has at most $n+1$ roots. In fact, using the monotonicity of the map $\beta \mapsto \eta^\beta$, we can prove that this set has at most one point. The lemma follows, since $S(\beta) = \bigcup_{n \geq 0} S_n(\beta)$. \square

**Lemma 6.** Consider a family $\{T_\beta\}_{\beta > 1}$ of generalized $\beta$-maps defined by a sequence $s = (s_n)_{n \geq 0}$. Let $1 < \beta_1 \leq \beta_2$ and define $l = \min\{n \geq 0 : \eta_j^1 \neq \eta_j^2 \}$ with $\eta_j^1 = \eta_j^2$, for $j = 1, 2$. If $k \geq 3$, for all $\beta_0 > 2$, there exists $K$ such that $\beta_1 \geq \beta_0$ implies

$$\beta_2 - \beta_1 \leq K\beta_2^{-l}.$$

If $s = (+1, +1)$, then

$$\beta_2 - \beta_1 \leq \beta_2^{-l+1}.$$

If $s = (+1, -1)$ or $(-1, +1)$, then for all $\beta_0 > 1$, there exists $K$ such that $\beta_1 \geq \beta_0$ implies

$$\beta_2 - \beta_1 \leq K\beta_2^{-l}.$$

If $s = (-1, -1)$, then there exists $\beta_0 > 1$ and $K$ such that $\beta_1 \geq \beta_0$ implies

$$\beta_2 - \beta_1 \leq K\beta_2^{-l}.$$
The proof is very similar to the proof of Brucks and Misiurewicz for Proposition 1 of \cite{3}, see also Lemma 23 of Sands in \cite{18}.

**Proof:** Let \( \delta := \beta_2 - \beta_1 \geq 0 \) and denote \( T_j = T_{\beta_j} \) and \( \beta^i = \beta_j \) for \( j = 1, 2 \). Let \( a_1, a_2 \in [0, 1] \) such that \( r := \beta^1(a_1) = \beta^2(a_2) \). Considering four cases according to the signs of \( a_2 - a_1 \) and \( s_r \), we have

\[
|T_2(a_2) - T_1(a_1)| \geq \beta_2|a_2 - a_1| - \delta.
\]

Applying \( n \) times this formula, we find that \( \beta_{[0,n]}(a_1) = \beta_{[0,n]}(a_2) \) implies

\[
|T^n_2(a_2) - T^n_1(a_1)| \geq \beta_2^n \left( |a_2 - a_1| - \frac{\delta}{\beta_2 - 1} \right).
\]

Consider the case \( k \geq 3 \). Then \( a_i = T_i(1) \) for \( i = 1, 2 \) are such that \( |a_2 - a_1| = \delta > \frac{\delta}{\beta_2 - 1} \geq \delta \). Using \( |T^n_2(a_2) - T^n_1(a_1)| \leq 1 \), we conclude that for all \( \beta_0 \leq \beta_1 \leq \beta_2 \), if \( \eta^1_{[0,n]}(\beta_0) = \eta^2_{[0,n]}(\beta_1) \) then

\[
\delta \leq \frac{\beta_0 - 1}{\beta_0 - 2} \beta_2^{-n+1}.
\]

For the case \( s = (+1, +1) \), we can apply Lemma \cite{3} with \( \alpha = 0 \) and \( x = 1 \). The case \( s = (+1, -1) \) or \( (-1, +1) \) is considered in Lemma 23 of \cite{18}.

For the case \( s = (-1, -1) \): for a fixed \( n \), we want to find \( \beta_0 \) such that for all \( \beta_0 \leq \beta_1 \leq \beta_2 \) we have

\[
|T^n_2(1) - T^n_1(1)| > \frac{\delta}{\beta_2 - 1}.
\]

Then we conclude as in the case \( k \geq 3 \). The formula \((14)\) is true, if \( \frac{\partial}{\partial \beta} T^n_\beta(1) > \frac{1}{\beta - 1} \) for all \( \beta \geq \beta_0 \). When \( n \) increases, \( \beta_0 \) decreases. With \( n = 3 \), we have \( \beta_0 \approx 1.53 \). \( \square \)

In the tent map case, the separation of orbits is proved for \( \beta \in (\sqrt{2}, 2] \) and then extended arbitrarily near \( \beta_0 = 1 \) using the renormalization. In the case \( s = (-1, -1) \), there is no such argument and we are forced to increase \( n \) to obtain a lower bound \( \beta_0 \). With the help of a computer, we obtain \( \beta_0 \approx 1.27 \) for \( n = 12 \). For more details, see \cite{7}.

Now we turn to the question of normality for generalized \( \beta \)-maps. The structure of the proof is very similar to the proof of Theorem \cite{2} and Corollary \cite{1}.

**Theorem 5.** Consider a family \( \{T_\beta\}_{k-1 \leq \beta \leq k} \) of generalized \( \beta \)-maps defined by a sequence \( s = (s_n)_{0 \leq n < k} \). Let \( \beta_0 \) be defined as in Lemma \cite{7} according to \( s \). Then the set

\[
\{ \beta > \beta_0 : \text{the orbit of } \eta^\beta \text{ under } \sigma \text{ is } \tilde{\mu}_\beta \text{-normal} \}
\]

has full \( \lambda \)-measure.

**Corollary 2.** Consider a family \( \{T_\beta\}_{\beta \geq 1} \) of generalized \( \beta \)-maps defined by a sequence \( s = (s_n)_{n \geq 0} \). Let \( \beta_0 \) be defined as in Lemma \cite{7} according to \( s \). Then the set

\[
\{ \beta > \beta_0 : \text{the orbit of } 1 \text{ under } T_\beta \text{ is } \mu_\beta \text{-normal} \}
\]

has full \( \lambda \)-measure.

**Proof of Theorem:** Let

\[
B_0 := \{ \beta \in (\beta_0, \infty) : 1 \notin S(\beta) \}.
\]

From Lemma \cite{5} this subset has full Lebesgue measure. To obtain uniform estimates, we restrict our proof to the interval \([\beta, \bar{\beta}]\) with \( \beta_0 < \beta < \bar{\beta} < \infty \). Let \( k := |\bar{\beta}| \) and \( \Omega := \{ \beta \in [\beta, \bar{\beta}] \cap B_0 : \eta^\beta \text{ is not } \tilde{\mu}_\beta \text{-normal} \} \). As before, setting

\[
\Omega_N := \{ \beta \in [\beta, \bar{\beta}] \cap B_0 : \exists \nu \in V_\sigma(\eta^\beta) \text{ s.t. } h(\nu) < (1 - 1/N) \log \beta \},
\]

14
we have $\Omega = \bigcup_{N \geq 1} \Omega_N$. We prove that $\dim_H \Omega_N < 1$. For $N \in \mathbb{N}$ fixed, define $\varepsilon := \frac{\beta \log \beta}{2N-1} > 0$ and $L$ such that $\eta_{\beta}^{\beta}([0,L]) = \eta_{\beta'}^{\beta'}([0,L])$ implies $|\beta - \beta'| \leq \varepsilon$ (see Lemma 6). Consider the family of subsets of $[\beta, \beta']$ of the following type

$$J(w) = \{ \beta \in [\beta, \beta'] : \eta_{\beta}^{\beta}([0,L]) = w \}$$

where $w$ is a word of length $L$. $J(w)$ is either empty or it is an interval. We cover the non-closed $J(w)$ with countably many closed intervals if necessary. We prove that $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$ where $\beta_1 < \beta_2$ are such that $\eta_{\beta_1}^{\beta_1}([0,L]) = \eta_{\beta_2}^{\beta_2}([0,L])$.

Let $\eta^j = \eta_j^\beta$. Let

$$D^* := \{ z \in \Sigma_{\eta^j} : \exists \beta \in [\beta_1, \beta_2] \cap B_0 \ s.t. \ z = \eta^\beta \}.$$ 

Define $\rho_* : D^* \to [\beta_1, \beta_2] \cap B_0$ by $\rho_*(z) = \beta \iff \eta^\beta = z$. As before, from formula (12) and strict monotonicity of $\beta \mapsto \eta^\beta$, we deduce that $\rho_*$ is well defined and surjective. We compute the coefficient of Hölder continuity of $\rho_* : (D^*, d_{\beta_*}) \to [\beta_1, \beta_2]$. Let $z \neq z' \in D^*$ and $n = \min\{l \geq 0 : z_l \neq z'_l\}$, then $d_{\beta_*}(z,z') = \beta_*^{-n}$. By Lemma 6 there exists $C$ such that

$$|\rho_*(z) - \rho_*(z')| \leq C \rho_*(z)^{-n} \leq C \beta_*^{-n} \rho_*(z)^{\log \beta_1 / \log \beta_*}.$$ 

By the choice of $L$ and $\varepsilon$, we have

$$\frac{\log \beta_1}{\log \beta_*} \geq 1 - \frac{1}{2N},$$

thus $\rho_*$ has Hölder-exponent of continuity $1 - \frac{1}{2N}$. Define

$$G^*_{\beta_*} := \{ z \in \Sigma^* : \exists \nu \in V_{\rho_*}(z) \ s.t. \ h(\nu) < (1 - 1/N) \log \beta_* \}.$$ 

As before, we have $\Omega_N \cap [\beta_1, \beta_2] \subset \rho_*(G^*_{\beta_*} \cap D^*)$ and $h_{\operatorname{top}}(G^*_{\beta_*}) \leq (1 - 1/N) \log \beta_*$. Finally $\dim_H(\Omega_N \cap [\beta_1, \beta_2]) < 1$ and $\lambda(\Omega_N \cap [\beta_1, \beta_2]) = 0$. \qed

**Proof of the Corollary:** The proof is similar to the proof of Corollary 1. Equation (2) is true, since we work on $B_0$. \qed

In particular, when we consider the tent map ($s = (1,-1)$), we recover the main Theorem of Bruin in [4]. We do not state this theorem for all $x \in [0,1]$ as for the map $T_{\alpha, \beta}$, because we do not have an equivalent of Lemma 6 for all $x \in [0,1]$. This is the unique missing step of the proof.

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