The polynomial error probability for LDPC codes

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(Dated: November 17, 2018)

We obtain exact expressions for the asymptotic behaviour of the average probability of the block decoding error for ensembles of regular low density parity check error correcting codes, by employing diagrammatic techniques. Furthermore, we show how imposing simple constraints on the code ensemble (that can be practically implemented in linear time), allows one to suppress the error probability for codes with more than 2 checks per bit, to an arbitrarily low power of \( N \). As such we provide a practical route to a (sub-optimal) expurgated ensemble.

PACS numbers: 89.70.+c, 75.10.Hk, 05.50.+q, 05.70.Fh, 89.70.+c

I. INTRODUCTION

Recent research in a cross-disciplinary field between the information theory (IT) and statistical mechanics (SM) revealed a great similarity between the low density parity check (LDPC) error correcting codes and systems of Ising spins (microscopic magnets) which interact with each other over random graphs\(^{1,2}\). On the basis of this similarity, notions and methods developed in SM were employed to analyse LDPC codes, which successfully clarified typical properties of these excellent codes when the code length \( N \) is sufficiently large\(^{3,4,5}\).

In general, an LDPC code is defined by a parity check matrix \( A \) which represents dependences between codeword bits and parity checks determined under certain constraints. This implies that the performance of LDPC codes, in particular, the probability of the block decoding error \( P_B(A) \) fluctuates depending on each realization of \( A \). Therefore, the average of the decoding error probability over a given ensemble \( P_B \) is frequently used for characterising the performance of LDPC code ensembles.

Detailed analysis in IT literature showed that \( P_B \) of naively constructed LDPC code ensembles is generally composed of two terms: the first term which depends exponentially on \( N \) represents the average performance of typical codes, whereas the second component scales polynomially with respect to \( N \) due to a polynomially small fraction of poor codes in the ensemble\(^{6,7}\). This means that even if the noise level of the communication channel is sufficiently low such that typical codes exhibit exponentially small decoding error probabilities (which is implicitly assumed throughout this paper), communication performance can still be very low exhibiting an \( O(1) \) decoding error probability with a polynomially small probability when codes are naively generated from the ensembles. As the typical behaviour has mainly been examined so far, the polynomial contribution from the atypical codes has not been sufficiently discussed in the SM approach. Although this slow decay in the error probability would not be observed for most codes in the ensemble, examining the causes of low error correction ability of the atypical poor codes is doubtlessly important both theoretically, and practically for constructing more reliable ensembles.

The purpose of this paper is to answer this demand from the side of SM. More specifically, we develop a scheme to directly assess the most dominant contribution of the poor codes in \( P_B \) on the basis of specific kinds of graph configuration utilising diagrammatic techniques. This significantly simplifies the evaluation procedure of \( P_B \) compared to the existing method\(^8\), and can be employed for a wider class of expurgated ensembles. Moreover, it provides insights that leads to a practical expurgation method that is also presented in this paper. Finally, the validity of the evaluation scheme and the efficacy of the proposed expurgation technique are computationally confirmed.

The paper is organised as follows:
- In the next section II we briefly review the general scenario of LDPC codes and introduce basic notions which are necessary for evaluating the error probability in the proceeding sections.
- In section III we introduce the various code ensembles we will work with, and different representations for a code construction.
- In section IV we link the probability of having a code with low minimal distance to the polynomial error probability, and we calculate the polynomial error probability by diagrammatic techniques for various code ensembles. As we can explicitly link it to the occurrence of short loops, the distribution of occurrence of such loops is also determined.
- In section V we present a practical linear time (in average) algorithm to remove short loops from a code construction, thus reducing the polynomial error probability to an arbitrarily low value. This is backed up by numerical simulations.
In order to reconstruct the original message, one has to obtain an estimate \( \hat{s} \) for the true noise \( \mathbf{n}^0 \). One major strategy for this is maximum likelihood (ML) decoding and is mainly focused on in this paper. It consists in the selection of that \( \hat{\mathbf{n}}^{ML} \) that minimises the number of non-zero elements (weight) \( w(\mathbf{n}) = \sum_{i=1}^{N} n_i \) satisfying the parity check equation \( \mathbf{z} = A\mathbf{n} \). Decoding failure occurs when \( \hat{\mathbf{n}}^{ML} \) differs from \( \mathbf{n}^0 \). The probability of this occurring is termed as the (block) decoding error probability \( P_D(A) \), which serves as a performance measure of the code specified by \( A \) given the (ML) decoding strategy.

It is worthwhile to mention that for any true noise vector \( \mathbf{n}^0 \), \( \mathbf{n} = \mathbf{n}^0 + \mathbf{x} \) (mod2) where \( \mathbf{x} \) is an arbitrary codeword vector for which \( A\mathbf{x} = 0 \) (mod 2) holds, also satisfies the parity check equation \( A\mathbf{n} = A\mathbf{n}^0 = \mathbf{z} \) (mod2). The set of indices of non-zero elements of \( \mathbf{x} \) is denoted by \( \mathcal{I}(\mathbf{x}) \). We denote the probability that a given decoding strategy (DS) will select \( \mathbf{n} = \mathbf{n}^0 + \mathbf{x} \) with \( w(\mathbf{x}) = w \) rather than \( \mathbf{n}^0 \), as \( P_{DS}(\epsilon|w) \). The ML decoding strategy fails in correctly estimating those noise vectors \( \mathbf{n}^0 \) for which less than half of \( n_i^{0|\mathcal{I}(\mathbf{x})} \) are zero, since the weight of \( \mathbf{n}^0 + \mathbf{x} \) (mod2) becomes smaller than \( w(\mathbf{n}^0) \). To noise vectors \( \mathbf{n}^0 \) for which exactly half of \( n_i^{0|\mathcal{I}(\mathbf{x})} \) are zero, we attribute an error 1/2, since the weight of \( \mathbf{n}^0 + \mathbf{x} \) (mod2) is equal to \( w(\mathbf{n}^0) \), such that

\[
P_{ML}(\epsilon|w) = \sum_{i=1}^{\text{int}(w-1)/2} \binom{w}{i} p^{w-i}(1-p)^i \left( \frac{1}{2} \left( \frac{w}{\mathcal{I}(\mathbf{x})} \right) (p(1-p))^{w/2} \right) \sim \mathcal{O}(p^{\frac{w}{\mathcal{I}(\mathbf{x})}})
\]

where \( \text{int}(x) \) is the integer part of \( x \) (for \( P(\epsilon|w) \) for other decoding schemes we refer to appendix D). The minimum of \( w(\mathbf{x}) \) under the constraints \( A\mathbf{x} = 0 \) (mod2) and \( \mathbf{x} \neq 0 \), is commonly known as the minimal distance of \( A \) and is here denoted as \( \mathcal{W}(A) \). It provides a lower bound for the decoding error probability of the ML scheme as \( P_D(A) > \mathcal{O}(p^{\text{int}(\mathcal{W}(A)/2)}) \).

Gallager showed that for \( c \geq 3 \) the minimal distance grows as \( \mathcal{O}(N) \) for most codes characterised by the \((c,d)\)-constraints, which implies that the decoding error probability can decay exponentially fast with respect to \( N \) when \( p \) is sufficiently low. However, he also showed that the minimal distance and, more generally, the weights of certain codeword vectors become \( \mathcal{O}(1) \) for a polynomially small fraction of codes when the code ensemble is naively constructed, which implies that the average decoding error probability over the ensemble \( \mathcal{P}_B \) exhibits a slow polynomial decay with respect to \( N \) being dominated by the atypical poor codes. As Gallager did not examine the detailed properties of the poor codes, it was only recently that upper- and lower-bounds of \( \mathcal{P}_B \) were evaluated for several types of naively constructed LDPC code ensembles. However, the obtained bounds are still not tight in the prefactors. In addition, extending the analysis to other ensembles does not seem straightforward as the provided technique is rather complicated. The first purpose of this paper is to show that one can directly evaluate the leading contribution of \( \mathcal{P}_B \) by making a one-to-one connection between occurrence of low weights in codeword vectors and the presence of some dangerous finite diagram(s) (sub-graph(s)) in a graph representation of that code.

In order to suppress the influence of the atypical poor codes, Gallager proposed to work in an expurgated ensemble, where the codes with low minimal distance are somehow removed. However, a practical way to obtain the expurgated ensemble has not been provided so far. The second purpose of the current paper is to provide a (typically) linear-time practical algorithm for the expurgation and we numerically demonstrate its efficacy.
III. CODE ENSEMBLES AND REPRESENTATIONS OF CODE CONSTRUCTIONS

A. Code ensembles

As mentioned in the previous section, evaluating the distribution of low weights of the poor codes in a given ensemble becomes relevant for the current purposes. This distribution highly depends on the details of the definition of code ensembles. We here work on the following three ensembles defined on the basis of bipartite graphs (Fig. 1):

- **Miller-Burshtein (MB) ensemble:** Put $N$ vertices (message bits) and $L$ edges (parity checks) on the left and right, respectively. For each vertex and edge, we provide $c$ and $d$ arcs, respectively. In order to associate these, the arcs originating from the left are labelled from 1 to $Nc (= Ld)$, and similarly done for the right. A permutation $\pi$ is then uniformly drawn from the space of all permutations of $\{1, 2, \ldots, Nc\}$. Finally, we link the arc labelled $i$ on the left with the arc labelled $\pi_i$ on the right. This defines a code completely determining a specific dependence between message bits and parity checks. The uniform generation of $\pi_i$ characterises the ensemble. Note that in this way multiple links between a pair of vertex/edge are allowed.

- **No multiple links (NML) ensemble:** Multiple links in the bipartite graph possibly reduces the effective number of parity checks of the associated message bits, which may make the error correction ability weaker. The second ensemble is provided by expurgating graphs containing multiple links from the MB ensemble.

- **Minimum loop length $\ell$ (MLL-$\ell$) ensemble:** In the bipartite graph, length $\ell$ loops are defined as irreducible closed paths composed of $\ell$ different vertices and $\ell$ different edges [26]. As shown later, short loops become a cause of poor error correction ability, as they allow for a shorter minimal distance. Therefore, we construct code ensembles by completely expurgating graphs containing loops of length shorter than $\ell$ from the MB ensemble, and examine how well such expurgation contributes to the improvement of the average error correction capability. Note that the MB and NML ensemble correspond to the MLL-1 and MLL-2 ensemble, respectively.

B. Representations of code constructions

Although the ensembles above are constructed on the basis of bipartite graphs, for convenience we will also use other two representations:

- **Monopartite (hyper)-graph representation:** Each message bit is represented by a vertex. The vertices are connected by hyper-edges (each linking $d$ vertices), each vertex is involved $c$ times in an hyper-edge (see Fig. 2).

- **Matrix representation:** $L$ rows, $N$ columns, where $a_{ev}$ is the number of times vertex $v$ appears in edge $e$. For regular $(c, d, N)$ codes, the following constraints on $\{a_{ev}\}$ apply

$$
\begin{align*}
\sum_{v=1}^{N} a_{ev} &= d, & \forall e = 1, \ldots, L, \\
\sum_{e=1}^{L} a_{ev} &= c, & \forall v = 1, \ldots, N.
\end{align*}
$$

(3)
For clarity, we always use indices $v, w, .. = 1, .., N$ to indicate message bits (or vertices), and indices $e, f, .. = 1, .., L$ to indicate parity checks (or edges). Note that the parity check matrix of a given code is provided as $A = \{a_{ev}\}$ (mod 2). For the NML and MLL-$\ell \geq 3$ ensembles, the matrix elements are constrained to binary values as $a_{ev} = 0, 1$. Therefore, $\{a_{ev}\}$ itself represents the parity check matrix $A$ in these cases, which implies that every parity check is composed of $d$ bits and every bit is associated with $c$ checks. However, as $a_{ev}$ can take any integer from 0 to $c$ in the MB ensemble, it can occur that certain rows and/or columns of the parity check matrix $A$ are composed of only zero elements, which means that corresponding checks and/or bits do not contribute to the error correction mechanism. Note that in the matrix notation the exclusion of $l$-loops corresponds to the extra (redundant) constraints (additional to $a_{ev} = 0, 1$, and (3)):

$$\prod_{i=1}^{l} a_{v_i e_i} a_{v_{(i+1)\text{mod} l} e_i} = 0 \ \forall\{v_i, i = 1..l\}, \ \{e_i, i = 1..l\}. \quad (4)$$

where $\{v/e_i, i = 1..l\}$ is a group of $l$ different vertices/edges.

There is a one-to-one correspondence between the bipartite graph, and the matrix representation of the codes. Note however, that with each monopartite graph correspond a number of (identical up to permutation of the edges) of bipartite graph/matrix representations.

IV. DIAGRAMMATIC EVALUATION OF ERROR PROBABILITY

A. Low weights and most dangerous diagrams

Now, let us start to evaluate the error probability. For this, we first investigate necessary configurations in the bipartite graph representation for creating codeword vectors having a given weight.

Assume that a codeword vector $x$ which is characterised by $Ax = 0 \ (\text{mod} \ 2)$ has a weight $w(x) = n_v$. As addition of zero elements of $x$ does not change parity checks, we can focus on only the $n_v$ non-zero elements. Then, in order to satisfy the parity relation $Ax = 0 \ (\text{mod} \ 2)$, every edge associated with $n_v$ vertices corresponding to these non-zero elements must receive an even number of links from the $n_v$ vertices in the bipartite representation.

Let us now evaluate how frequently such configurations appear in the whole bipartite graph when a code is generated from a given ensemble. We refer to a sub-set of the $n_v$ vertices as $V_f$. Each vertex $v$ is directly linked to a subset $E(v)$ of the edges. We denote $E(V_f) = \bigcup_{v \in V_f} E(v)$, and $n_e = |E(V_f)|$. Then there are $cn_v$ links to be put between $V_f$ and $E(V_f)$. Note that there are exactly $c$ links arriving at each $v \in V_f$, and $d$ links arriving at each $e \in E(V_f)$. Each diagram consists in a combination $(V_f, E(V_f))$. For an admissible diagram, we have the extra condition that each $e \in E(V_f)$ receives an even number of links from $V_f$, such that the bits in $V_f$ can be collectively flipped preserving the parity relation $Ax = 0 \ (\text{mod} \ 2)$.

Note that we ignore the links arriving in $E(V_f)$ from outside the set $V_f$. For admissible diagrams, $n_e$ is limited from above by $\int(\frac{\pi}{2\pi})$, where $\int(x)$ is the integer part of $x$. A number $n_e$ of the links can be put freely, while the remaining $cn_v - n_e$ links all have to fall in the group $n_e$ (out of $L$), such that it can easily be seen that each of those (forced) links carries a probability $\sim N^{-1}$.

There are $\binom{N}{n_v} \simeq N^{n_v}/n_v!$ ways of picking $n_v(\ll N)$ out of $N$ vertices, such that each diagram consisting of $n_v$ vertices and $n_e$ edges carries a probability of occurrence proportional to $\sim N^{n_e-(n_v c - n_e)}$. 

\[ \text{FIG. 2: An example of a (small) regular hyper-graph with } (c, d, N) = (2, 3, 9) \]
This observation allows us to identify the “most dangerous” admissible diagrams as those with a probability of occurrence with the least negative power of \( N \), i.e. that combination of \((n_\ast^a, n_\ast^b)\) that maximises \( n_\ast = (c-1)n_\ast^a \). The collective contributions of all other diagrams are at least a factor \( \frac{1}{N} \) smaller, and therefore negligible. From this, it immediately follows that \( n_\ast \) must take its maximal value which is \( n_\ast^\ast = \text{int}(\frac{c}{2}) \), where \( n_\ast \) has to be minimised, compatible with the constraints of the code ensemble under consideration. Hence, the probability \( P_J = P_J(n_\ast^\ast) \) that a generated graph (code) contains a most dangerous diagram including \( n_\ast^\ast \) vertices, scales like

\[
P_J(n_\ast^\ast) \sim N^{n_\ast^\ast(1-\frac{\ell}{2})}.
\]  

with the constraint on \( n_\ast \) that \( \frac{cn_\ast^\ast}{2} \) is integer. Note that for all ensembles we consider in this paper \( n_\ast \sim O(1) \).

At this point, we make some important observations:

- Firstly, from eq. (3) it is easily seen that for \( c = 2 \), any diagram containing an equal number of vertices and edges scales like \( N^0 \). The number of diagrams contributing to \( P_B \) becomes infinite, and \( P_B \sim O(1) \). It was already recognised by Gallager [2] and [4] that regular \((2,d,N)\) codes have very bad decoding properties under the block error criterion, which is currently adopted. Therefore, in what follows, we will implicitly assume that \( 2 < c < d \).

- Secondly, as eq. (5) represents only the dependence on the code length \( N \), for an accurate evaluation of the asymptotic behaviour (for \( N \to \infty \)) of the error probability, we have to calculate the prefactor. Nevertheless, this kind of power counting is still highly useful because this directly identifies the major causes of the poor performance, and allows us to concentrate on only a few relevant diagrams for further calculation, ignoring innumerable other minor factors. This is more systematic and much easier to apply in various ensembles than the existing method of \( \ell \).

- Thirdly, once \( P_J \sim P_J(n_\ast^\ast) \) is accurately evaluated, the average block error probability \( \overline{P}_B \) for any decoding scheme \( DS \) and for sufficiently low flip rates \( p \) can be evaluated as

\[
\overline{P}_{BDS} \sim P_{BDS}(e|n_\ast^\ast)P_J,
\]

where e.g. for ML decoding \( P_{BML}(e|n_\ast^\ast) \) is given in [2] (the expressions for other decoding schemes can be found in appendix [3]).

- The NML and MLL-\( \ell \) ensembles are generated from the MB ensemble by expurgating specific kinds of codes, which slightly changes the distribution of codes such that the above argument based on free sampling of links in constructing graphs might not be valid. However, for \( n_\ast \sim O(1) \) the current evaluation still provides correct results for the leading contribution to \( P_J(n_\ast) \), since the influence of the expurgation procedure has a negligible effect on the leading contributions of the most dangerous diagrams.

- Finally, we note that this analysis essentially follows the weight enumerator formalism [5, 6], which can be regarded as a certain type of the annealed approximation [7, 8]. Although we have argued that such formalism is not capable of accurately evaluating the performance of typical codes that decays exponentially with respect to \( N \) [11], the current scheme correctly assesses the leading contribution of the average error probability \( \overline{P}_B \) of the above ensembles, which scales polynomially with respect to \( N \) being dominated by atypically poor codes. This is because the probability of occurrence scales like \( N^{-\alpha} \) (with \( \alpha \geq 1 \)) for all admissible diagrams. Therefore, we can safely ignore the possibility that more than one such diagram occurs in the same graph (as illustrated in fig. 4), and to leading order for \( P_J \), we can simply add the contributions of all most dangerous diagrams. This is not so for the typical case analysis (with \( n_\ast \sim O(N) \)), where exponentially rare codes may have an exponentially large contribution. In order to avoid this kind of over counting, a quenched magnetisation enumerator based treatment is then more suitable [11]. Note furthermore that in the SM treatment of typical codes, the polynomial error probability is hidden in the ferro-magnetic solution (since \( \frac{w(n_\ast)}{N} \sim \frac{w(n_\ast|n_\ast \text{ flipped})}{N} \) when \( n_\ast \sim O(N) \)), and is therefore easily overlooked.

**B. Evaluation of error probability for various ensembles**

Once the notion of most dangerous diagrams is introduced, the asymptotic behaviour of the probability \( P_J \) for sufficiently low flip rates \( p \) is easily evaluated for various ensembles.

- **MB ensemble**: For the MB ensemble, multiple links between a pair of vertex/edge are allowed, which forces us to make a distinction between even and odd \( c \). For even \( c \) the minimal admissible \( n_\ast \) is 1, such that the error probability will scale like \( N^{1-\frac{\ell}{2}} \), while that for odd \( c \) is 2 which provides a faster scaling \( N^{2-\ell} \).

- For \( c \) even, the most dangerous diagram is given in Fig.9 and the probability \( P_J \sim P_J(n_\ast = 1) \) is given by (an explanation of the diagrams, and how there multiplicity is obtained can be found in appendix A)

\[
P_J(\ell = 1, c \text{ even}) \sim N^{1-\frac{\ell}{2}} (1 - R)^{-\frac{\ell}{2}} \left( \frac{(d-1)}{d} \right)^{\frac{\ell}{2}} \frac{c!}{2^\ell (\frac{\ell}{2})!}.
\]  

(7)
Note that with \( x \simeq y \), we indicate that \( x = y \left( 1 + \frac{2(N)}{N} \right) \).

- For \( c \) odd the minimal admissible \( n_v \) is 2, such that the error probability will scale like \( N^{2-c} \). The most dangerous diagrams are given in Fig. 3 (with \( k = 0, 1, \ldots, \text{int}(\frac{c}{2}) \)), and the probability \( P_f \simeq P_f(n_v = 2) \) is given by (for details see appendix A)

\[
P_f(\ell = 1, c \text{ odd}) \simeq N^{2-c} (1 - R)^{-c} \left( \frac{(d-1)}{d} \right)^{c} \frac{e!}{2} \left[ \sum_{k=0}^{\text{int}(c/2)} \frac{e!}{2^k k!^2 (c-2k)!} \right]
\]

One can check that the values (7) and (8), which we believe to be exact (not bounds), satisfy the bounds given in (6).

- **NML ensemble**: In the NML ensemble, multiple links are not allowed. In this ensemble, even and odd \( c \) can be treated on the same footing. In both cases, the minimal admissible \( n_v = 2 \), such that the error probability will scale like \( N^{2-c} \). The most dangerous diagram is given in Fig. 3 (note that in this case only \( k = 0 \) is allowed). The probability \( P_f \simeq P_f(n_v = 2) \) is given by (for details see appendix A)

\[
P_f(\ell = 2) \simeq N^{2-c} (1 - R)^{-c} \left( \frac{(d-1)}{d} \right)^{c} \frac{e!}{2} \]

- **MLL-3 ensemble**: In the MLL-3 ensemble, neither multiple links nor loops of length 2 are allowed. Note that this also implies that pairs of vertices may not appear more than once together in a parity check, such that all parity checks are different, making each monopartite graph correspond to exactly \( L! \) bipartite graphs/matrices. In the monopartite graph any pair of vertices is now connected by at most one (hyper-)edge. One can easily convince oneself that the minimal \( n_v = c + 1 \) (each \( v \) needs \( c \) other vertices to connect to), and the most dangerous diagram is
given in Fig. 4. The dominant part of $P_f \simeq P_f(c+1)$ is given by (for details see appendix A)

$$P_f(\ell = 3) \simeq N^{c+1 - \frac{c+1}{2} (1 - R)^{\frac{c+1}{2}}} \frac{(d-1)^{\frac{c+1}{2}}}{(c+1)!} \frac{e^{c+1}}{(c+1)!}$$

(10)

• MLL-\(\ell\) ensemble: In the general MLL-\(\ell\) ensemble, neither multiple links nor loops of length \(l < \ell\) are allowed. In general, identifying the most dangerous diagram(s) and especially calculating their combinatorial prefactor becomes increasingly difficult with increasing \(\ell\), but we can still find the scaling of \(P_f\) relatively easily by power counting. To this purpose it is more convenient to use the monopartite graph representation. In Fig. 5 we observe that for \(\ell = 3, 4\) and 5, the minimal \(n_v\) is given by \(c+1, 2c\) and \(2c + c(c-1) = c(c+1)\) respectively, such that using (5) we obtain

$$P_f(\ell = 3) \sim N^{c+1(1-\frac{1}{2})} \quad P_f(\ell = 4) \sim N^{2c(1-\frac{1}{2})} \quad P_f(\ell = 5) \sim N^{c(c+1)(1-\frac{1}{2})}$$

(11)

For \(\ell \geq 6\), even finding the most dangerous diagram and thus power counting becomes quite difficult to do by hand,

but we can still easily upper bound the power by the following observation: from fig. 6 we observe that for a given minimal allowed loop length \(\ell\) the minimum number of generations without links between them starting from any vertex \(v\) is given by \(\text{int}(\frac{\ell-1}{2})\). Therefore the minimal \(n_v\) is lower bounded by

$$n_v \geq 1 + c \sum_{k=0}^{\text{int}(\frac{\ell-1}{2})} (c-1)^k = 1 + c \left( \frac{(c-1)\text{int}(\frac{\ell-1}{2}) - 1}{(c-1) - 1} \right),$$

(12)

which implies that \(P_f\) can be upper bounded as

$$P_f(\ell) \leq O\left(N^{1+c\left(\frac{(c-1)\text{int}(\frac{\ell-1}{2})}{c-1} - 1\right)}\right).$$

(13)

In fig. 7 we have plotted the frequencies of occurrence of dangerous diagrams that scale like \(N^{-1}\). We have randomly generated \(10^6\) code realisations (for \(N \approx 50, 100, 200, 400\) i.e. the nearest integer for which \(L = cN/d\) is also integer), and have plotted both the total frequency (multiplied by \(N^\ell\)) of occurrence of a diagram (dashed lines), and the frequency that a graph contains the diagram at least once (full lines). Note that in the limit \(N \to \infty\), both coincide, illustrating the fact that we can safely ignore the possibility that more than one such diagram occurs in the same graph. We observe that the extrapolations \(1/N \to 0\) are all in full accordance with the theoretical predictions.

All this clearly illustrates how the exclusion of short loops reduces \(P_f\), and thus through (6) the polynomial error probability probability. Furthermore, from figs. 3-6, it is clear that all most dangerous diagrams contain short loops. Knowledge of the distribution of the number of short loops in the various ensembles is therefore relevant for our current purposes, and we analyse the distribution of the number of \(\ell\)-loops in the next subsection.
FIG. 6: All loops of length \( l < \ell \) are removed. The minimal size of the last generation can not be less than \( c(c-1)^{\text{int}((\ell-1)/2)} \) without generating loops of length < \( \ell \).

FIG. 7: Frequencies (multiplied with \( N \)) of occurrence of dangerous diagrams that scale like \( N^{-1} \). Left: \( c = 3, d = 4, 6 \) with \( k = 0 \) both with (\( \square \)) and without (+) removing 1-loops, and with \( k = 1 \) (\( \times \)). Right: \( c = 4, d = 2 \ldots 6 \).

C. The distribution of the number of \( \ell \)-loops

In this subsection we investigate the distribution \( P_\ell(k) \) of the number \( k \) of \( \ell \)-loops for the various code ensembles. Note that we only consider irreducible \( \ell \)-loops, in the sense that they are not combinations of shorter loops (i.e. they do not visit the same vertex or edge twice). Note that an \( \ell \)-loop in the monopartite graph corresponds to a 2\( \ell \)-cycle in the bipartite graph representation \( \square \). The number of irreducible \( \ell \)-loops in a random regular \( (c, d, N) \) graph (with \( N \to \infty \)) has the following distribution:

\[
P_\ell(k) = P(\# \ell \text{-loops} = k) \simeq \frac{\lambda_\ell^k}{k!} \exp(-\lambda_\ell), \quad \lambda_\ell \equiv \frac{(c-1)^{\ell}(d-1)^{\ell}}{2\ell}
\]  

(14)

For the derivation of this result we refer to appendix A. From (14) we observe that the average number of short loops increases rapidly with \( c \) and \( d \). Furthermore we note the symmetry between \( c \) and \( d \), which reflects the edge/vertex duality which is typical for loops.

As explained in appendix A, the constraint that no loops of length \( l < \ell \) are present in the graph, has no influence in the leading order to the diagrams for loops of length \( \geq \ell \).
We denote the number of codes in the ensemble where the minimum loop length is $\ell$ (i.e., loops of length $l < \ell$ have been removed) by $N_\ell(c, d, N)$, such that the size of the original (MB) ensemble with all regular $(c, d, N)$ codes is denoted by $N_1(c, d, N)$. From (14) it follows that the size of $N_\ell(c, d, N)$ is given by

$$
N_\ell(c, d, N) = \exp\left(-\sum_{l=1}^{\ell-1} \lambda_l\right) N_1(c, d, N) = \exp\left(-\sum_{l=1}^{\ell-1} \frac{(c-1)^l(d-1)^l}{2l}\right) N_1(c, d, N) \tag{15}
$$

The reduction factor $\exp\left(-\sum_{l=1}^{\ell-1} \lambda_l\right)$ is $O(1)$, for any finite loop length $\ell$. Since the number of non-equivalent codes in the original ensemble $N_1(c, d, N) \sim \frac{(cN)!}{(c!)^d L!}$, the final ensemble $N_\ell(c, d, N)$ is still very big, but clearly smaller than Gallager’s ideally expurgated ensemble which has a reduction factor of just $\frac{1}{2}$.

Note that the presence of (short) loops in a code does not only adversely affect the polynomial error probability, but also the success rate for practical decoding algorithms such as belief propagation.

V. PRACTICAL LINEAR ALGORITHM TO THE $\ell$-LOOP EXPURGATED ENSEMBLE

In this section we propose a linear time (in $N$) algorithm that generates codes and removes loops up to arbitrary length (the combination $(c, d, N)$ permitting). We also present simulation results, which corroborate our assumptions about the validity of the diagrammatic approach as presented in this paper. Finally we give some practical limits and guidelines for code-ensembles with large but finite $N$.

1. Generating a random regular $(c, d, N)$ code:

The algorithm to generate a random regular $(c, d, N)$ code consists in the following steps:

1. make a list of available vertices $A_v$, of initial length $N_{av} = cN$, where each vertex appears exactly $c$ times
2. for each of the $L = \frac{cN}{d}$ parity checks, $d$ times:
   
   (a) randomly pick a vertex from $A_v$,
   (b) remove it from the list
   (c) $N_{av} = N_{av} - 1$. 

Note that in the process we keep construct the lists:

1. $EV[v][i], v = 1..N, i = 1..c$ containing the edges each vertex $v$ is involved in,
2. $VE[e][j], e = 1..L, j = 1..d$ containing the vertices each edge $e$ involves.

It is clear that this algorithm is linear in $N$. 

---

FIG. 8: The dominant diagram for $\ell$-loops.
2. Finding loops of length $\ell$ in the code:

We now describe the algorithm to detect (and store) all $\ell(\geq 2)$-loops in the graph:

1. we consider all the vertices as a possible starting point $v_0$ of the loop
2. given a starting point $v_0$ grow a walk of length $\ell$. Each growing step consists in:
   (a) take $e_l$ from $EV[v_l]$ and check conditions for valid step
   (b) take $v_{l+1}$ from $VE[e_l]$ and check conditions for valid step
   (c) if all conditions are satisfied goto next step
      else if possible goto the next $e_l \in EV[v_l]$ or $v_{l+1} \in VE[e_l]$
      else go to previous step
3. finally check whether the end point of the loop $v_\ell = v_0$, if so store the loop i.e. $L_\ell = \{(v_l, e_l), l = 0..\ell - 1\}$

The conditions for a valid step are the following:

1. (a) $e_l \neq e_i$, ($i = 0..l-1$) for $\ell > 2$.
   (b) $e_l > e_0$ for $\ell = 2$.
2. (a) $v_l > v_0$, $v_l \neq v_i$ ($i = 1..l-1$) for $l = 1..\ell-1$, $\ell \geq 2$.
   (b) $v_l > v_1$ for $l = \ell-1$, $\ell > 2$.
   (c) $v_l = v_0$ for $l = \ell$.

Note that the conditions $v_l > v_0$ fix the starting point of the loop, while the conditions $v_{l-1} > v_1$ for $\ell > 2$, and $e_1 > e_0$ for $\ell = 2$, fix its orientation. This has a double advantage: it avoids over counting, and reduces execution time by early stopping of the growing process.

For 1-loops ($2$-cycles), for all $v = 1..N$ we simply look in $EV[v]$ for double links to the same edge, i.e. $EV[v][i] = EV[v][j]$. Imposing that $i < j$, then avoids double counting.

Since each vertex is connected to $c$ edges, and each edge is connected to $d$ vertices, the number of operations to check whether any vertex $v$ is involved in a loop, remains $O(1)$ (compared to $N \rightarrow \infty$). As we have to check this for all vertices, the loop finding stage of the algorithm is linear in $N$.

3. Removing loops of length $\ell$ from the code:

We start by detecting and removing the smallest loops and then work our way towards longer loops. Assuming that all shorter loops have been successfully removed, and having found and listed all the loops of length $\ell$, the procedure for removing them is very simple. For all stored $\ell$-loops $L_{\ell}$:

1. randomly pick a vertex/edge $(v_p, e_p)$ combination from $L_{\ell} = \{(v_l, e_l), l = 0..\ell-1\}$
2. swap it with a random other vertex $v_s$ in a random other edge $e_s$.
3. for $v_p$ and $v_s$ check whether they are now involved in a $l$-loop with $l \leq \ell$.
   (a) if so undo the swap and goto 1.
   (b) else accept the swap, the loop is removed.

This procedure of removing loops takes typically $O(1)$ operations. The typical number of loops of each length $\ell$ is $O(1)$. For each loop we only have to swap one vertex/edge combination to remove it, the checks that the swap is valid take $O(1)$ operations, and we typically need only $O(1)$ swap-trials to get an acceptable swap.

Although the algorithm is linear in $N$, the number of operations needed to detect and remove loops of length $\ell$ in a $(c,d)$ code, grows very rapidly with $c,d$ and even exponentially with $\ell$. Furthermore, we note that only in the $N \rightarrow \infty$ limit all short loops can be removed. In practice, for large but finite $N$ and given $(c,d)$, the maximum loop length $\ell$ is clearly limited. A rough estimate for this limit is given by

$$\max\left( c \frac{1 - (c-1)\frac{\ell\ell}{1 - (c-1)}}{(1 - (c-1))'} d \frac{1 - (d-1)\frac{\ell\ell}{1 - (d-1)}}{(1 - (d-1))'} \right) \sim N \quad (16)$$

because $v$ (resp. $e$) is not allowed to be its own $1,2,..\ell$-th nearest neighbour (see fig. 6). Hence, loops of logarithmic length in $N$ can not be avoided. For practical $(c,d,N)$, however, this loop length is reached rather quickly. Therefore, we have built in the possibility for the algorithm to stop trying to remove a given loop when after $\text{max-swap}$ trials, no suitable swap has been obtained. By choosing $\text{max-swap}$ sufficiently large, the maximum removable loop length is
easily detected. In practice we find that for all loop lengths that can be removed, we typically need 1, and occasionally 2 trial swaps per loop.

In Fig. 9 we show the distribution of loops over the \((c, d, N)\) ensemble, for \((c, d) = (3, 4)\), for \(N = 10^4\) and averaged over \(10^4\) codes, up to loops of length \(\ell = 4\) (corresponding to length 8 cycles in the IT terminology [24]), before and after removal of shorter loops. In general we observe that the Poisson distribution with \(\lambda\) given in (14) fits the simulations very well, for all \(\ell\) not exceeding the maximal removable loop length, while it breaks down above that.

![Distribution of loops](image)

Note that, in principle, this method also could be used to obtain Gallager’s ideally expurgated ensemble. We would start by finding and removing the most dangerous diagrams, and then move on to the next generation of most dangerous diagrams, and so on. However, the next most dangerous diagrams are obtained by adding additional vertices (and necessary edges) and/or by removing edges from the current most dangerous diagrams. One can easily convince oneself that the number of next most dangerous diagrams soon becomes enormous. In addition for each generation we only reduce the polynomial error probability by a factor \(N^{-1}\). Therefore, although in principle possible, this method is not practical, and we have opted for the removal of loops. The fact that we only have to look for one type of diagram (i.e. loops), and the fact that we expurgate many entire generations of next most dangerous diagrams in one go, makes the cost of over-expurgating the ensemble a small one to pay.

VI. SUMMARY

In summary, we have developed a method to directly evaluate the asymptotic behaviour of the average probability with respect to the block decoding error for various types of low density parity check code ensembles using diagrammatic techniques. The method makes it possible to accurately assess the leading contribution with respect to the codeword length \(N\) of the average error probability which originates from a polynomially small fraction of poor codes in the ensemble, by identifying the most dangerous admissible diagrams in a given ensemble by a power counting scheme. The most dangerous diagrams are combinations of specific types of multiple closed paths (loops) in the bipartite graph representation of codes, and allow for codewords with low weights. The contribution of a diagram to the error probability becomes larger as the size of the diagram is smaller, which implies that one can reduce the average error probability by excluding all codes that contain any loops shorter than a given threshold \(\ell\). We have theoretically clarified how well such a sub-optimal expurgation scheme improves the asymptotic behaviour. We have also provided a practical algorithm which can be carried out typically in a linear scale of \(N\) for creating such sub-optimally expurgated ensembles. The numerical experiments utilising the provided algorithm have verified the validity of the theoretical predictions.

The current approach is relatively easy to adapt to irregularly constructed codes [12, 13], codes over the extended fields [14, 15], other noise channels [14, 17], and other code constructions such as the MN codes [18]. Work in that direction is currently underway.

Acknowledgements Kind hospitality at the Tokyo Institute of Technology (JvM) and support by Grants-in-aid, MEXT, Japan, No.
consider all possible sub-groups of \( n \) vertices and \( n \) edges, with another one not from within the diagram. Since the probability for each link to be present at least 2 links arriving to each of the nodes from within the diagram. Suppose now that we replace a single node they have (many) vertices and/or edges in common. It then suffices to calculate the probability of occurrence of a single diagram, and to count how many times such a diagram could occur in the graph, in order to extract its overall expectation, allowing us to calculate all quantities that depend on it.

To illustrate this, consider the following scenario. All diagrams we consider, consist of \( n_v \) vertices and \( n_e \) edges, with at least 2 links arriving to each of the nodes from within the diagram. Suppose now that we replace a single node (vertex or edge), with another one not from within the diagram. Since the probability for each link to be present is \( \geq \frac{1}{N} \), there are at least a 4 link difference between the diagrams, thus making the correlation between them negligible to leading order.

The rules for calculating the combinatorial pre-factor of the diagram are easily described as follows:

- Consider all possible sub-groups of \( n_v \) vertices and \( n_e \) edges.
- Calculate the probability \( P_g \) that a given group of \( n_v \) vertices and \( n_e \) edges forms the diagram we’re interested in. Since we assume that (to leading order in \( N \)) these probabilities are independent for all groups, we just have to multiply \( P_g \) with all the possible ways of picking \( n_v \) vertices and \( n_e \) edges from the graph (i.e. \( \binom{N}{n_v} \left( \frac{n_e}{L} \right) \)).

Combined this leads to the following simple recipe for the calculation: the contribution of a diagram to \( P_f \):

- for each vertex from which \( x \) links depart, add a factor \( N c!/(c-x)! \).
- for each edge from which \( x \) links depart, add a factor \( L d!/(d-x)! \).

APPENDIX A:

Diagrams are finite sub-graphs. Provided that the graph is large and provided that the correlations between the different diagrams is not too strong, we can treat them as effectively independent to leading order in \( N \), even when they have (many) vertices and/or edges in common. It then suffices to calculate the probability of occurrence of a single diagram, and to count how any times such a diagram could occur in the graph, in order to extract its overall expectation, allowing us to calculate all quantities that depend on it.

To illustrate this, consider the following scenario. All diagrams we consider, consist of \( n_v \) vertices and \( n_e \) edges, with at least 2 links arriving to each of the nodes from within the diagram. Suppose now that we replace a single node (vertex or edge), with another one not from within the diagram. Since the probability for each link to be present is \( \geq \frac{1}{N} \), there are at least a 4 link difference between the diagrams, thus making the correlation between them negligible to leading order.

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Combined this leads to the following simple recipe for the calculation: the contribution of a diagram to \( P_f \):

- for each vertex from which \( x \) links depart, add a factor \( N c!/(c-x)! \).
- for each edge from which \( x \) links depart, add a factor \( L d!/(d-x)! \).
for each link, add a factor \( \frac{1}{c} \).
- divide by the number of symmetries, i.e. the number of permutations of vertices, edges or links, that lead to the same diagram.

### a. calculation of the diagrams in fig.3

We calculate the probability \( P_f(1, \frac{c}{2}) \) that a combination of 1 vertex \( v \), and \( \frac{c}{2} \) edges forms the left diagram fig.3 in the following steps:
- 1 vertex with \( c \) links (\( Nc! \)).
- \( \frac{c}{2} \) edges with 2 links (\( (Ld(d-1))^{\frac{c}{2}} \)).
- \( c \) links (\( \left( \frac{1}{cN} \right)^{c} \)).
- symmetry: \( \frac{c}{2} \) double links (2\( \frac{c}{2} \)).
- symmetry: permutation of the edges (\( \left( \frac{c}{2} \right)! \)).

So, combined we have that

\[
P_f(1, \frac{c}{2}) \simeq \frac{c!}{2^\frac{c}{2} \left( \frac{c}{2} \right)!} \left( d(d-1)/2 \right)^{\frac{c}{2}} \frac{NL^{\frac{c}{2}}}{(cN)^{\frac{c}{2}}} , \tag{A1}
\]

and after some reworking we obtain (7).

We calculate the probability \( P_{2,c,\kappa} \) that a combination of 2 vertices, and \( c \) edges forms the right diagram in fig.3 with \( k = \kappa \) in the following steps:
- 2 vertices with \( c \) links (\( (Nc!)^2 \)).
- \( c \) edges with 2 links (\( (Ld(d-1))^c \)).
- 2\( c \) links (\( \left( \frac{1}{cN} \right)^{2c} \)).
- symmetry: permute edges in groups of \( \kappa \) (\( \kappa! \)).
- symmetry: permute edges in group of \( c - 2\kappa \) (\( (c - 2\kappa)! \)).
- symmetry: \( 2k \) double links (2\( \frac{2k}{2} \)).
- symmetry: simultaneously permute the vertices and the groups of \( \kappa \) edges (2).

So, combined we have that

\[
P_f(2, c, k) \simeq \frac{c!}{2} \left[ \frac{c!}{2^{2k}k!(c - 2k)!} \right] \left( d(d-1)/2 \right)^{\frac{c}{2}} \frac{N c^{\frac{c}{2}} L^{\frac{c}{2}}}{(cN)^{\frac{c}{2}}} , \tag{A2}
\]

and after some reworking we obtain (8) and (9).

### b. calculation of the diagram in fig.4

We calculate the probability \( P_f(c+1, \frac{c(c+1)}{2}) \) that a combination of \( c+1 \) vertices, and \( \frac{c(c+1)}{2} \) edges forms diagram in fig.4 in the following steps:
- \( c+1 \) vertices with \( c \) links (\( (Nc)^{c+1} \)).
- \( \frac{c(c+1)}{2} \) edges with 2 links (\( (Ld(d-1))^{\frac{c(c+1)}{2}} \)).
- \( c(c+1) \) links (\( \left( \frac{1}{cN} \right)^{c(c+1)} \)).
- symmetry: permute vertices (\( (c+1)! \)).
- symmetry: permute edges (\( \left( \frac{c(c+1)}{2} \right)! \)).

So, combined we have that

\[
P_f(c+1, \frac{c(c+1)}{2}) \simeq \frac{c^{c+1}}{(c+1)!} \left( \frac{c^{c+1}}{2} \right)! \left( d(d-1) \right)^{\frac{c(c+1)}{2}} \frac{N c^{c+1} L^{\frac{c(c+1)}{2}}}{(cN)^{c(c+1)}}, \tag{A3}
\]

and after some reworking we obtain (10).
c. **distribution of the number of loops of length \( \ell \)**

The probability \( P_\ell(k) \) that there are \( k \) loops of length \( \ell \) (i.e. including \( \ell \) vertices and edges of the bipartite graph), can be calculated from diagram fig.8. By power counting it is easily checked that the probability for any loop length \( \ell \) to occur is \( O(1) \). Therefore, we adapt a slightly different strategy compared to the diagrams above, (this also illustrates where some of the rules of our recipe originate from)

First we calculate the probability \( P_{\ell,g} \) that a given group of \( \ell \) vertices (and \( \ell \) edges) forms a “true” \( \ell \)-loop in the following steps:

- We order the \( \ell \) vertices into a ring \( (\frac{\ell!}{2}) \) ways.
- For each pair of consecutive vertices we pick on of the edges to connect to both \((\ell)!\) ways).
- For each vertex choose a link to each edge it is connected to \((c(c-1)!)\) ways).
- For each edge choose a link to each vertex it is connected to \((d(d-1))\) ways).
- The probability that a chosen left and right link are connected is given by \( \frac{1}{cN} \).

So, combined we have that

\[
P_{\ell,g} \approx \frac{\ell^2}{2\ell} (c(c-1)d(d-1))^\ell \left( \frac{1}{cN} \right)^{2\ell} \quad (A4)
\]

There are \( \binom{N}{\ell} \) ways to pick the vertices, and \( \binom{\ell}{\ell} \) ways to pick the edges.

We want exactly \( k \) of these to form a loop, and \( \binom{N}{\ell} \binom{\ell}{\ell} - k \) of these not to form an \( \ell \)-loop, therefore:

\[
P_\ell(k) = \binom{N}{\ell} \binom{\ell}{\ell} P_{\ell,g}^k (1 - P_{\ell,g})^{\binom{N}{\ell} \binom{\ell}{\ell} - k} \approx \frac{k^k}{k!} \exp(-\lambda_k) \quad (A5)
\]

Note that the exclusion (or not) of shorter loops, has no influence on the leading order of \( P_\ell(k) \), since the probability of having a short-cut i.e. another edge that connects 2 vertices from within the group of \( \ell \) (or vertex that connects 2 edges from within the group of \( \ell \)), requires 2 extra links to be present which adds a factor \( \approx \frac{1}{(cN)^2} \) to the probability \( P_{\ell,g} \), and is therefore negligible.

**APPENDIX B:**

As shown, for a given code ensemble, the probability \( P_f \) that a finite group of \( n_v \) bits can be collectively flipped, is completely dominated by sub-sets of size \( n_v^* \), such that \( P_f \equiv P_f(n_v^*) \). From this we can then determine the polynomial error probability \( P_B \), which depends on the decoding scheme employed. Here, we concentrate on the BSC\((p,1-p)\) for the following decoding schemes:

1. **ML decoding** [7]: Since this decoding scheme selects the code word with the lowest weight, an error occurs when the \( n_v^* \) collectively flipped bits have a lower weight than the original ones. When the \( n_v^* \) collectively flipped bits have an equal weight to the original ones, we declare an error with probability \( \frac{1}{2} \), such that one immediately obtains [2].

2. **MPM decoding** [19, 20, 21]: This decoding scheme selects the code word that maximizes the marginal posterior, and minimizes the bit error rate (or in a statistical physics framework that minimizes the free energy at the Nishimori temperature [22]).

   Effectively this attributes a posterior probability \( \exp(\beta F w(n))/Z \) to each codeword \( n \), where \( \beta F = \ln \left( \frac{1}{1-p} \right) \), and where \( Z = \sum_n \exp(\beta F w(n)) \), with \( \sum_n \) being the sum over all code words. Since we assume that we are in the decodable region, we have that \( Z \simeq \exp(\beta F w(n_0)) + \exp(\beta F w(n_f)) \) with \( n_f \) being \( n_0 \) with \( n_v^* \) bits flipped. Hence, by selecting the solution with the maximal marginal posterior probability we obtain that \( P_{MPM}(e|n_v^*) = P_{ML}(e|n_v^*) \) as given in [2].
3. Typical set decoding (TS) [8,23]: This decoding scheme randomly selects a code word from the typical set. We declare an error when a noise different from \( n_0 \) is selected. Hence the error probability is given by \( \frac{n_{ts} - 1}{n_{ts}} \), where \( n_{ts} \) is the number of code words in the typical set. Since we are in the decodable region, for \( n_{ts}^* \sim \mathcal{O}(1) \), the original and the flipped code word are both (and the only) codewords in the typical set, such that

\[
P_{TS}(e|n_{ts}^*) = \frac{1}{2}.
\] (B1)

Note that TS decoding has an inferior performance for \( P_B \) compared to ML and MPM decoding.