Fundamental Limits of Caching for Demand Privacy against Colluding Users

Qifa Yan Member, IEEE, and Daniela Tuninetti Senior Member, IEEE.

Abstract

This work investigates the problem of demand privacy against colluding users for shared-link coded caching systems, where no subset of users can learn any information about the demands of the remaining users. The notion of privacy used here is stronger than similar notions adopted in past work and is motivated by the practical need to insure privacy regardless of the file distribution. This paper provides both an achievable scheme, referred to as Linear Function Retrieval for Demand Privacy against Colluding Users (LFR-DPCU), and a novel information theoretic converse bound. By comparing the performance of the achievable scheme with the converse bound derived in this paper (for the small cache size regime) and existing converse bounds without privacy constraints, the communication load of LFR-DPCU turns out to be optimal to within a constant multiplicative gap in all parameter regimes. Numerical results show that LFR-DPCU outperforms known schemes based on the idea of virtual users, which also satisfy the stronger notion of user privacy adopted here, in some regime. Moreover, LFR-DPCU enjoys much lower subpacketization than known schemes based on virtual users.

Index Terms

Coded caching; colluding users; demand privacy; converse bound; linear function retrieval; sub-packetization;

I. INTRODUCTION

Coded caching is a promising technique to reduce network congestion during peak times. Consider a shared-link network consisting of a server having access to a library of $N$ files and being connected to $K$ users, each equipped with a local cache memory of size $M$ files. The network operates in two phases: the placement phase happens when the network is not congested, during which the server pushes some content into each user’s local cache without knowing their future demands; while the delivery phase happens at peak times, during which each user demands one file from the server and the server responds by sending a signal to satisfy the users’ demands. In coded caching, the communication load (or just load for short in the following) is reduced by creating multicast opportunities in the delivery phase by cleverly pushing content in the caches in the placement phase.

Information theoretic coded caching was first introduced in [1] by Maddha-Ali and Niesen (MAN). The MAN scheme was proved to achieve the optimal worst-case load among all uncoded placement schemes when $N \geq K$ in [2]. By removing redundant transmissions in the MAN scheme, the optimal load-memory tradeoff among all uncoded placement schemes was characterized for $N < K$ in [3]. Improved achievable loads by using coded placement were obtained in [4]–[6] and the original cut-set converse bound in [1] was improved upon in [7], [8]; we know however that uncoded placement is no more than a factor of 2 away from optimal [9].
Decoding the MAN multicast signals however requires global knowledge of the demanded files by the users, thus infringing the privacy of the users. Moreover, users may learn the content of files other than the requested one, thus threatening security. Information theoretic secure coded caching has been considered in [10], [11]; in [10], a wiretapper who observes the transmitted signal can not learn any information about the files; in [11], each user can not obtain any information on non-demanded files. Information theoretic demand-private coded caching was formalized in [12], where the aim is to guarantee that each user does not learn any information about the indices of the files demanded by the other users. Relevant for this work is a way to insure privacy informally referred to as scheme with virtual users, an idea that first appeared in [14] and was later analyzed in [12], [13]. The idea is that a private scheme for a system with \( K \) users and \( N \) files can be constructed from known non-private schemes [1], [3] for \( NK \) users and \( N \) files, that is, by introducing \( K(N-1) \) virtual users. In the placement phase of a virtual users scheme, the users choose their cache contents from the \( NK \) caches of the non-private scheme without replacement, privately and randomly; in the delivery phase, the demands of the \( K \) users are extended to demands for \( NK \) users (including \( K \) real users and \( N(K-1) \) virtual users) such that each file is demanded exactly \( K \) times. The server sends multicast signals to satisfy the extended demands of \( NK \) users according to the non-private coded caching scheme. Privacy of the real users is guaranteed since each real user can not distinguish the demands of real and virtual users. The general idea of transforming a non-private coded caching scheme for \( NK \) users to a private one for \( K \) users was further studied in [15], [16], and later extended to device-to-device network in [17], where a trusted server having no access to the file library coordinates the transmission among the users.

A. Paper Contributions

In this paper, we investigate the problem of demand privacy against colluding users (DPCU). In DPCU, privacy is guaranteed in the sense that any subset of colluding users, who may share their cache contents, can not learn any information about the indices of the files demanded by the remaining users. It was noted already in [12] that existing schemes that were not designed to fight colluding users are indeed private against colluding users as well. In this paper we further strengthen the privacy notion of [12] by imposing that a feasible scheme should work for any file realization. This notion is motivated by the fact that, in practical systems, the distribution of files are usually hard to characterize, or even known, they may not be identically and uniformly distributed, as many theoretical models assume.

The main contributions of this paper are as follows.

- We propose an achievable scheme referred to as linear function retrieval for demand privacy against colluding users (LFR-DPCU). LFR-DPCU is based on the idea of linear function retrieval in coded caching [18]. In [18], the optimal scheme for single file retrieval under uncoded placement in [3] was extended to the more general setup where each user can retrieve any linear combination of the files. The LFR-DPCU starts with the same file split and placement strategy as in MAN [11], [18]. In addition, each user privately caches a “key” that is formed as a random linear combination of the uncached subfiles under the MAN strategy. In the delivery phase, the server broadcasts multicast signals so that each user can decode a linear combination of the subfiles, which can be thought of as containing a desired subfile protected by the local private “key.”
- We derive an information theoretic converse bound for the novel privacy definition, which outperforms known bounds for the small memory regime when \( N > K \). The converse
is inspired by the approach in [16], which characterized the exact load-memory tradeoff for the case $N = K = 2$. In particular, we derive a lower bound on the sum of some entropies conditioned on known demands, which in turn provides a bound on a weighted sum of the load and the memory size. We combine different entropy functions by the using the fact that entropy function is submodular. We also leverage the fact that demands, in the conditioning of the entropy terms, can be changed to other demands in the process of combining conditional entropy terms, which enables us to tighten the bound as one can argue that “more files can be decoded after combining.” Changing the demands in the conditioning is possible because of the DPCU requirements.

• By using our novel converse bound with privacy combined with known bounds without privacy constraints [9], we show that the LFR-DPCU scheme is optimal to within a constant multiplicative gap in all parameter regimes. Numerical results indicate that LFR-DPCU outperforms the virtual users scheme of [13] when the library size $N$ is larger than $2K + 1$ and the memory size $M$ is smaller than $N - 1 - \frac{K}{2}$. Thus, even in the stronger sense of privacy used here, the virtual users scheme can be improved upon when $N$ is much larger than $K$.

It is also worth pointing out the superiority of LFR-DPCU compared to virtual users schemes in terms of subpacketization. It is well known that, with fixed number of files and cache size at each user, the subpacketization of MAN-type schemes increase exponentially with the number of users $K$ [19], [20]. This problem is even more prominent in virtual users schemes as they are obtained from non-private coded caching schemes with $KN$ users. Since LFR-DPCU does not introduce virtual users, the subpacketization does not increase compared to the (not private) MAN scheme. In other words, privacy need not come at the expense of subpacketization.

B. Paper Organization

The paper is organized as follows. Section II introduces the problem formulation and Section III presents the LFR-DPCU scheme. Section IV-A contains the derivation of our novel converse bound and shows that LFR-DPCU is order optimal. Section V presents some numerical results. Finally, Section VI concludes the paper. Some proofs can be found in Appendix.

C. Notation

In this paper, we use $\mathbb{R}_+$ to denote the set of non-negative real numbers. The notation $\mathbb{F}_2^n$ denotes the binary field. For a positive integer $n$, $\mathbb{F}_2^n$ is the $n$ dimensional vector space over the field $\mathbb{F}_2$, and $[n]$ is the set of the first $n$ positive integers $\{1, 2, \ldots, n\}$. For a sequence of variables $Z_1, Z_2, \ldots, Z_n$ and an index set $S \subseteq [n]$, we use the notation $Z_S \triangleq \{Z_i : i \in S\}$ and $Z_{[0]} \triangleq \emptyset$. If $A$ is a matrix with $n$ columns, $A_S$ denotes the submatrix that consists of columns in $S \subseteq [n]$. For integers $m, n$, we use $\binom{n}{m}$ to denote the binomial coefficient $\frac{n!}{m!(n-m)!}$, and adopt the convention $\binom{n}{m} = 0$ if $m > n$. The notation “$\oplus$” is used to denote the Exclusive OR (XOR) operation. We use $\Pr\{\cdot\}$ to denote the probability of an event. In the proofs of chain (in)equalities, we specify the relevant equations needed to justify the steps by the equation numbers on top of the (in)equality symbols.

II. System Model

Let $N, K, B$ be positive integers. An $(N, K)$ caching system consists of a server with $N$ files $W_1, W_2, \ldots, W_N$ and $K$ users $1, 2, \ldots, K$, where the server is connected to the users via an
error-free shared link. The $N$ files are independently and uniformly distributed over $\mathbb{F}_B^{2}$. Each user $k \in [K]$ has a memory of $MB$ bits for some $M \in [0, N]$. Since in the case $N = 1$ or $K = 1$, there is no problem of protecting privacy, we assume that $N \geq 2$ and $K \geq 2$ throughout this paper. The system operates in two phases as follows.

**Placement Phase:** The server privately generates a random variable $P$ from some probability space $\mathcal{P}$. Then it fills the cache of each user $k \in [K]$ by using the cache function

$$\varphi_k : \mathcal{P} \times \mathbb{F}_B^{2N} \mapsto \mathbb{F}_2^{MB}.$$  

(1)

The content of user $k$ is denoted by

$$Z_k = \varphi_k(P, W_{[N]}), \quad \forall k \in [K].$$  

(2)

**Delivery Phase:** Each user $k \in [K]$ demands the file indexed by $D_k$, where $D_1, D_2, \ldots, D_K$ are independently distributed over $[N]$. The files $W_{[N]}$, the randomness $P$ and the demands $D_{[K]}$ are independent, that is,

$$H(D_{[K]}, P, W_{[N]}) = \sum_{k \in [K]} H(D_k) + H(P) + \sum_{n \in [N]} H(W_n).$$  

(3)

Given the demands, the server creates the signal $X$ by using the encoding function

$$\phi : \mathcal{P} \times [N]^K \times \mathbb{F}_B^{2N} \mapsto \mathbb{F}_2^{RB},$$  

(4)

for some $R \geq 0$, that is, the server transmits to the users via the shared link the signal

$$X = \phi(P, D_{[K]}, W_{[N]}).$$  

(5)

The quantity $R$ is called the worst-case load of the system.

Each user $k \in [K]$ must decode its demanded file $W_{D_k}$ by using $(X, Z_k)$, and privacy must be guaranteed against colluding users, that is, a working coded caching scheme for DPCU requires

[Correctness] \quad H(W_{D_k} \mid X, D_k, Z_k) = 0, \quad \forall k \in [K],  

(6)

[Privacy] \quad I(D_{[K]} \setminus S; X, D_S, Z_S \mid W_{[N]}) = 0, \quad \forall S \subseteq [K], S \neq \emptyset.  

(7)

**Definition 1.** A memory-load pair $(M, R) \in \mathbb{R}_{+}^{2}$ is said to be achievable if there exists a scheme such that all the above conditions are satisfied. The optimal worst-case load-memory tradeoff is defined as

$$R^*(M) = \liminf_{B \to +\infty} \{ R : (M, R) \text{ is achievable} \}.$$  

(8)

Our main objective is to characterize the optimal worst-case memory-load tradeoff $R^*(M)$ in Definition 1. By doing so we will also discuss the subpacketization of the proposed achievable scheme, where the subpacketization of a scheme is the minimum value of $B$ needed to realize the scheme.

**Remark 1.** The intuition behind the privacy guarantee in (7) is that, for any file realization $W_{[N]} = w_{[N]}$ and for any non-empty set of users $S \subseteq [K]$, the colluding users in $S$ can not learn any information on the demands of the other users. Thus, the privacy is guaranteed irrespective of

1In the results of this paper, the achievable scheme works for arbitrary distribution of $W_{[N]}$, but the converse relies on this assumption.

2We implicitly assume that $B$ is sufficient large so that $MB$ and $RB$ (see (4)) are integers.
file realizations and the subset of users participating the collusion. By the fact that the demands $D_{[K]}$ are independent of the files $W_{[N]}$ (see [5]), the privacy condition in (7) is equivalent to

$$I(D_{[K]} \setminus S; X, D_S, Z_S, W_{[N]}) = 0, \quad \forall S \subseteq [K], S \neq \emptyset. \quad (9)$$

**Remark 2.** There have been several different definitions of demand privacy in the literature. In [12], it assumed that the server can privately transmit to any subset of users by encryption using shared keys, thus the privacy condition is

$$[\text{Privacy in [12]}] \quad I(D_{[K]} \setminus \{k\}; \tilde{X}_k, D_k, Z_k) = 0,$$

where $\tilde{X}_k$ is the signal intended to be used for decoding by user $k \in [K]$. Other definitions include

$$[\text{Privacy in [15]}] \quad I(D_j; X, D_k, Z_k) = 0, \quad \forall (j, k) \in [K]^2 : j \neq k;$$

or

$$[\text{Privacy in [13, 16]}] \quad I(D_{[K]} \setminus \{k\}; X, D_k, Z_k) = 0, \quad \forall k \in [K],$$

where the latter is the strongest privacy condition used in past work and is obviously implied by our equivalent privacy condition in (9).

The definitions of privacy in this Remark involve one user at a time, that is, users are not assumed to be able to collude. Demand privacy against colluding users was first introduced in the device-to-device setup [17], where the privacy condition there was defined without conditioning on the library files $W_{[N]}$. The following Example 1 from [16] illustrates that the condition $I(D_{[K]} \setminus \{k\}; X, D_k, Z_k) = 0$ can not guarantee privacy for arbitrary distribution of $W_{[N]}$.

**Example 1 (A scheme from [16] for the case $N > K$).** Consider an $(N, K)$ caching system where $N > K$ and the files are uniformly and independently distributed over $\mathbb{F}_2^B$. Let $M \in [0, N]$.

In the placement phase, the server generates

$$P = (T_1, \ldots, T_K, S_1, \ldots, S_K, V_1, \ldots, V_K), \quad (10)$$

where $(T_1, \ldots, T_K)$ is a random permutation of $[K]$, which is uniformly drawn from all permutations of the set $[K]$; the $K$ identically and independently distributed (i.i.d.) random variables $S_1, \ldots, S_K$ are uniformly drawn from $[K]$; finally, $V_1, \ldots, V_K$ are i.i.d. random variables uniformly drawn from $\mathbb{F}_{2^{(1-M/N)B}}$. The three parts $T_{[K]}, S_{[K]}$ and $V_{[K]}$ are independently generated. Each file is split into two parts as $W_n = (W_n^{(c)}, W_n^{(u)}), n \in [N]$, where $W_n^{(c)}$ is of size $\frac{M}{N}B$ bits, and $W_n^{(u)}$ is of size $(1 - \frac{M}{N})B$ bits. Each user $k \in [K]$ caches $Z_k = (S_k, W_n^{(c)})$, where we refer to $S_k$ as the key of user $k \in [K]$.

In the delivery phase, for demands $D_1, \ldots, D_K$, the server first generates a sequence of numbers $J_1, \ldots, J_K$ inductively as follows

$$J_i = \begin{cases} J_j, & \text{if } D_i = D_j \text{ for some } j < i \\ T_i, & \text{if } D_i \neq D_j, \forall j < i \end{cases}, \quad (11)$$

and then sends a signal $X = (Q_{[K]}, Y_{[K]}),$ where $Q_{[K]}$ and $P_{[K]}$ are recursively generated as

$$Q_j = (J_j + S_j)_{(K)}, \quad \forall j \in [K]. \quad (12)$$

$$Y_j = \begin{cases} W^{(u)}_{D_i}, & \text{if } j = P_i \text{ for some } i \in [K] \\ V_j, & \text{otherwise} \end{cases}, \quad \forall j \in [K], \quad (13)$$
where \((\cdot)_{(K)}\) is the module operation defined as \((nK + j)_{(K)} = j\) for \(j = 1, 2, \ldots, K\) and any integer \(n\). The “side information” \(J_k\) records the position of the packet \(W_\text{D}_k^{(u)}\) in \(Y\). Since user \(k\) has the key \(S_k\), it can find the position of packet \(W_\text{D}_k^{(u)}\) from \(Q_k\), and hence it can decode its missing packet \(W_\text{D}_k^{(u)}\). Moreover, it was proved in [16] that \(I(D_{[K]\setminus\{k\}}, \{X, D_k, Z_k\}, 0) = 0\), where the proof relies on the fact that \((Y_1, \ldots, Y_K, W_1^{(c)}, \ldots, W_N^{(c)})\) is uniformly distributed over \(\mathbb{F}_2^{MB + K(1 - M/N)B}\), and is independent of \((D_{[K]}, Q_{[K]}, S_k, J_k)\) for each \(k \in [K]\).

Intuitively, the privacy guarantee relies on the fact that the user \(k\) can not distinguish if the signal \(Y_j\) is a random generated vector \(V_j\) or some partial file in \(W_\text{D}_k^{(u)}\), since both of them are i.i.d. uniformly distributed over \(\mathbb{F}_2^B\) and independent of the cached packets \(W_\text{D}_k^{(c)}\). In many practical applications, true file distributions maybe unavailable, or even they are available, the \(N\) files may have different distributions or not independent of the cached packets \(W_\text{D}_k^{(c)}\). In such cases, the user may infer some information on the demands. Obviously, the scheme does not satisfy our privacy condition in [7].

III. LINEAR FUNCTION RETRIEVAL FOR DEMAND PRIVACY AGAINST COLLUDING USERS

We propose here an achievable scheme, which we refer to as Linear Function Retrieval for Demand Privacy against Colluding Users (LFR-DPCU). We first state the result, then in Section III-A we provide an illustrative example to highlight the key ingredients of the scheme and finally in Section III-B we describe and analyze the general LFR-DPCU scheme.

Theorem 1. For an \((N, K)\) DPCU caching system, the lower convex envelop of the point \((M'_t, R'_t) \triangleq (0, N)\) and the following points is achievable

\[
(M_t, R_t) \triangleq \left(1 + \frac{t(N - 1)}{K}, \frac{K}{(t+1)} - \frac{K - \min\{N-1, K\}}{t+1} \right), \quad t \in [0 : K].
\]

Moreover, the point \((0, N)\) can be achieved with subpacketization \(1\), and the point \((M_t, R_t)\) can be achieved with subpacketization \(B_t \triangleq \binom{K}{t}\), \(t \in [0 : K]\).

Proof: For the point \((M, R) = (0, N)\), the server can trivially transmit all the \(N\) files to the users, obviously this scheme satisfies both the correctness and the privacy condition. For \(t = K\), the result is trivial, since all users can cache all the files. For \(t \in [0 : K - 1]\), we prove the theorem by analyzing the performance of the scheme in Section III-B. The other points on the lower convex envelope can be achieved by memory-sharing between those points.

Before we give the details of the scheme, we illustrate the idea behind the LFR-DPCU scheme through an example.

A. Example for \((N, K) = (3, 2)\)

Consider an \((N, K) = (3, 2)\) caching system with \(t = 1\). Let the three files be \(W_1, W_2, W_3\). Firstly, split each file into two equal-size packets, i.e., \(W_1 = \{W_{1,1}, W_{1,2}\}\), \(W_2 = \{W_{2,1}, W_{2,2}\}\) and \(W_3 = \{W_{3,1}, W_{3,2}\}\).

In the placement phase, the server first generates two vectors \(p_1 = (p_{1,1}, p_{1,2}, p_{1,3})^\top\) and \(p_2 = (p_{2,1}, p_{2,2}, p_{2,3})^\top\), which are uniformly and independently drawn from \(\{(1, 0, 0)^\top, (0, 1, 0)^\top, (0, 0, 1)^\top, (1, 1, 1)^\top\}\), i.e., the set of binary vectors of length 3 whose Hamming weight is odd. Then the server generates two keys \(S_1\) and \(S_2\), one for each user, as follows

\[
S_1 = p_{1,1}W_{1,2} \oplus p_{1,2}W_{2,2} \oplus p_{1,3}W_{3,2},
\]

\[
S_2 = p_{2,1}W_{1,2} \oplus p_{2,2}W_{2,2} \oplus p_{2,3}W_{3,2}.
\]
\[ S_2 = p_{2,1}W_{1,1} \oplus p_{2,2}W_{2,1} \oplus p_{2,3}W_{3,1}. \]  

The contents of the caches are given by
\[
Z_1 = \{W_{1,1}, W_{2,1}, W_{3,1}, S_1\},
\]
\[
Z_2 = \{W_{1,2}, W_{2,2}, W_{3,2}, S_2\}.
\]

In the delivery phase, assume user 1 demands \( W_1 \) and and user 2 demands \( W_2 \). In the non-private MAN scheme, the server sends the signal \( W_{1,2} \oplus W_{2,1} \). In the LFR-DPCU scheme, the server sends
\[
X = (Q, Y),
\]
\[
Q = \begin{bmatrix} p_{1,1} \oplus 1 & p_{2,1} \\ p_{1,2} & p_{2,2} \oplus 1 \\ p_{1,3} & p_{2,3} \end{bmatrix},
\]
\[
Y = (W_{1,2} \oplus S_1) \oplus (W_{2,1} \oplus S_2).
\]

Notice that
\[
W_{1,2} \oplus S_1 = (p_{1,1} \oplus 1)W_{1,2} \oplus p_{1,1}W_{2,2} \oplus p_{1,3}W_{3,2},
\]
\[
W_{2,1} \oplus S_2 = p_{2,1}W_{1,1} \oplus (p_{2,2} \oplus 1)W_{2,1} \oplus p_{2,3}W_{3,1}.
\]

By (17) and (22) (resp. (18) and (23)), user 1 (resp. 2) can compute \( W_{2,1} \oplus S_2 \) (resp. \( W_{1,2} \oplus S_1 \)) by using the second (resp. first) column of \( Q \) and the contents of its cache. Hence user 1 and 2 decode their un-cached packet respectively. The privacy is guaranteed since each user does not know the key of the other user, and the two columns of \( Q \) are uninformly and independently distributed over \( \{(0,0,0)^T, (1,1,0)^T, (1,0,1)^T, (0,1,1)^T\} \), i.e., the set of binary vectors of length 3 whose Hamming weight is even.

Notice that each user caches 4 packets, each of size \( \frac{B}{2} \) bits. In the signal \( X \), the main payload \( Y \) is a coded packet of length \( \frac{B}{2} \) and the matrix \( Q \) can be sent in \( H(Q) = 2 \log_2 4 = 4 \) bits, which does not scale with \( B \). Thus, the scheme achieves the memory-load pair \( (M, R) = (2, \frac{1}{2}) \).

**Remark 3.** It can be observed from the example in this Section that, compared to the non-private MAN scheme [1], the file are partitioned in the same way. The placement phase is similar to MAN; in addition, each user also caches a random linear combination of the uncached packets under the MAN placement, which is used as a key. The placement is thus not uncoded. In the delivery phase, the server broadcasts a coded signal so that each user can decode a linear combination of files as per the scheme in [18]. The linear combination is designed such that each user can decode its demanded file with its cached key. Recall that in an \( (N, K) \) non-private caching system [3], the optimal worst-case load with uncoded placement is given by the lower convex envelope of the points \( (\tilde{M}_t, \tilde{R}_t) \), where
\[
\tilde{M}_t = \frac{tn}{K}, \quad \tilde{R}_t = \frac{K - \min\{N, K\}}{t + 1}, \quad t \in [0 : K].
\]

By comparing (14) with (24), it can be observed that the cache size here is larger than in the MAN scheme (by \( 1 - \frac{t}{K} \)), and the privacy is guaranteed with same or even better load (as (14) is a function of \( N - 1 \) while (24) of \( N \)). The additional cache size is used to cache a key.

We are now ready to describe the general LFR-DPCU scheme.
B. The LFR-DPCU Scheme

Let $t \in [0 : K]$

$$\Omega_t \triangleq \{ \mathcal{T} : \mathcal{T} \subseteq [K], |\mathcal{T}| = t \}.$$  (25)

For fixed $t \in [0 : K - 1]$, the system operates as follows.

**Placement Phase:** The server partitions the file $W_n$ into $\binom{K}{t}$ equal-size packets denoted as

$$W_n = \{ W_{n, \mathcal{T}} : \mathcal{T} \in \Omega_t, \quad \forall \ n \in [N] \}.  \quad (26)$$

The server uniformly and independently generates $K$ vectors $p_1, \ldots, p_K$ from the set of all vectors in $\mathbb{F}_2^N$ with odd Hamming weights, i.e.,

$$p_k \triangleq (p_{k,1}, \ldots, p_{k,N})^\top \sim \text{Unif}\left\{ (x_1, \ldots, x_N)^\top \in \mathbb{F}_2^N : \bigoplus_{n\in[N]} x_n = 1 \right\}, \quad \forall \ k \in [K].  \quad (27)$$

The server fills the cache of user $k \in [K]$ as

$$Z_k = \{ W_{n, \mathcal{T}} : \mathcal{T} \in \Omega_t, k \in \mathcal{T}, n \in [N] \} \cup \left\{ \bigoplus_{n\in[N]} p_{k,n} \cdot W_{n, \mathcal{T}} : \mathcal{T} \in \Omega_t, k / \notin \mathcal{T} \right\}.  \quad (28a)$$

The random variable $P$ is given in the form of the $N \times K$ matrix

$$P = [p_1, p_2, \ldots, p_K].  \quad (29)$$

**Delivery Phase:** Let $e_1, \ldots, e_N$ be the standard unit vectors in $\mathbb{F}_2^N$, i.e., $e_n$ is the vector such that the $n$-th entry is one and all other entries are zeros. After receiving the users’ demands $D_{[K]}$, the server generates the $N \times K$ matrix

$$Q = [q_1, \ldots, q_K]  \quad (30)$$

as follows

$$q_k = p_k \oplus e_{D_k} = (q_{k,1}, q_{k,2}, \ldots, q_{k,N})^\top, \quad \forall \ k \in [K],  \quad (31)$$

where the XOR operation is performed element-wise.

Denote the rank of $Q$ over the binary field $\mathbb{F}_2$ by $L = \text{rank}_2(Q)$. Let $\mathcal{L}$ be a fixed subset of $[K]$ of size $L$ such that $\text{rank}_2(Q_{\mathcal{L}}) = \text{rank}_2(Q) = L$. The set $\mathcal{L}$ can be arbitrary chosen given $Q$. Define

$$Y_S \triangleq \bigoplus_{j \in S} \bigoplus_{n\in[N]} q_{k,n} \cdot W_{n, S \setminus \{j\}}, \quad \forall \ S \in \Omega_{t+1}.  \quad (32)$$

The server transmits the signal

$$X = (\mathcal{L}, Q, Y)  \quad (33)$$

to all the users, where

$$Y \triangleq \{ Y_S : S \in \Omega_{t+1}, S \cap \mathcal{L} \neq \emptyset \}.  \quad (34)$$

**Proof of Correctness:** By (28a), each user $k \in [K]$ needs to decode its demanded packets that were not cached. For each packet $W_{D_k, \mathcal{T}}$ such that $k / \notin \mathcal{T}$, user $k$ can decode $W_{D_k, \mathcal{T}}$ from the
signal $Y_{T \cup \{k\}}$, since by (31) and (32), we can express

$$Y_{T \cup \{k\}} = W_{D_k,T}$$  \hspace{1cm} (35a)

$$\oplus \bigoplus_{n \in [N]} p_{k,n} \cdot W_{n,T}$$  \hspace{1cm} (35b)

$$\oplus \bigoplus_{j \in T} \bigoplus_{n \in [N]} q_{j,n} \cdot W_{n,T \cup \{k\} \setminus \{j\}},$$  \hspace{1cm} (35c)

where the term in (35b) is a cached key (see (28b)), while all packets and all coefficients in (35c) are cached (see (28a)) or sent via $Q$ (see (33)). Thus, $W_{D_k,T}$ in (35a) can be decoded from $Y_{T \cup \{k\}}$.

We still need to prove that each user can obtain all the signals

$$\{Y_S : S \in \Omega_{t+1}, S \cap \mathcal{L} = 0\},$$  \hspace{1cm} (36)

which are not included in $Y$ (see (34)). We note that the signals $Y_S$ in (32) in the main payload are exactly the same as in the non-private case where each user $k$ demands the following linear combination of the files [18]

$$\bigoplus_{n \in [N]} q_{k,n} \cdot W_n.$$  \hspace{1cm} (37)

It has been proved in [18] that, the signals in (36) can be obtained by linear combinations of those in (34). For completeness, we briefly describe how to do so.

Let $\Gamma_B$ be the family of subsets $\mathcal{V} \subseteq \mathcal{B}$ such that $|\mathcal{V}| = \text{rank}_2(Q_{\mathcal{V}}) = L$ for any $\mathcal{B} \subseteq [K]$. Then the following lemma was proved in [18].

**Lemma 1** ([18] Lemma 1). Let $t \in [K]$ such that $t + 1 + L \leq K$, for any $\mathcal{B} \subseteq [K]$ such that $|\mathcal{B}| = t + 1 + L$, consider any $n \in [N], \mathcal{T} \subseteq \mathcal{B}$ such that $|\mathcal{T}| = t$, the number of $\mathcal{V} \in \Gamma_B$ such that the packet $W_{n,T}$ shows up in the signal $Y_{\mathcal{B} \setminus \mathcal{V}}$ is even.

By Lemma 1 let $\mathcal{B} = \mathcal{L} \cup \mathcal{S}$, then

$$\bigoplus_{\mathcal{V} \in \Gamma_{\mathcal{S} \cup \mathcal{L}}} Y_{\mathcal{S} \cup \mathcal{L} \setminus \mathcal{V}} = 0,$$  \hspace{1cm} (38)

and therefore, $Y_S$ can be obtained as

$$Y_S = \bigoplus_{\mathcal{V} \in \Gamma_{\mathcal{S} \cup \mathcal{L}}, \mathcal{V} \neq \mathcal{L}} Y_{\mathcal{S} \cup \mathcal{L} \setminus \mathcal{V}}.$$  \hspace{1cm} (39)

**Proof of Privacy:** We prove the scheme satisfy the equivalent condition in (9). In fact, for any $\mathcal{S} \subseteq [K]$,

$$I(D_{[K]\setminus\mathcal{S}}; X, D_{\mathcal{S}}, Z_{\mathcal{S}}, W_{[N]})$$  \hspace{1cm} (40a)

$$I(D_{[K]\setminus\mathcal{S}}; L, Q, Y, D_{\mathcal{S}}, Z_{\mathcal{S}} W_{[N]})$$  \hspace{1cm} (40b)

$$I(D_{[K]\setminus\mathcal{S}}; L, Q, D_{\mathcal{S}}, Z_{\mathcal{S}}, W_{[N]})$$  \hspace{1cm} (40c)

$$I(D_{[K]\setminus\mathcal{S}}; L, Q, D_{\mathcal{S}}, Z_{\mathcal{S}}, p_{\mathcal{S}}, W_{[N]})$$  \hspace{1cm} (40d)

$$I(D_{[K]\setminus\mathcal{S}}; D_{\mathcal{S}}, p_{\mathcal{S}}, W_{[N]}) + I(D_{[K]\setminus\mathcal{S}}; Q | D_{\mathcal{S}}, p_{\mathcal{S}}, W_{[N]})$$  \hspace{1cm} (40e)
\[ I(D_{[K]}; L | Q, D_S, p_S, W_{[N]}) \]
\[ = I(D_{[K]}; Q | D_S, p_S, W_{[N]}) \]
\[ I(D_{[K]}; q_{[K]} | D_S, p_S, W_{[N]}) \]
\[ I(D_{[K]}; q_{[K]} | D_S, p_S, W_{[N]}) \]
\[ = H(q_{[K]} | D_S, p_S, W_{[N]}) - H(q_{[K]} | D_{[K]}, p_S, W_{[N]}) \]
\[ = H(q_{[K]} | D_S, p_S, W_{[N]}) - H(p_{[K]} | D_{[K]}, p_S, W_{[N]}) \]
\[ = H(q_{[K]} | D_S, p_S, W_{[N]}) - H(p_{[K]} | D_{[K]}, p_S, W_{[N]}) \]
\[ = 0, \]

where in (40g) we used (3) and the fact that \( (D_{[K]}, p_S, W_{[N]}) \rightarrow Q \rightarrow L \) forms a Markov chain; (40k) holds because \( q_{[K]} \) are independent of \( (D_S, p_S, W_{[N]}) \) and \( q_{[K]} \) determines each other given \( D_{[K]} \) and \( p_{[K]} \); and (40m) holds because \( q_1, \ldots, q_K \) are uniformly distributed over all even Hamming weight vectors of \( \mathbb{F}_2^N \) and \( p_1, \ldots, p_K \) are uniformly distributed over all odd Hamming weight vectors of \( \mathbb{F}_2^N \) by construction.

**Performance:** By (26), each file is equally split into \( \left( \begin{array}{c} K \\ t \end{array} \right) \) packets, each of size \( B / \left( \begin{array}{c} K \\ t \end{array} \right) \) bits, thus the subpacketization is \( \left( \begin{array}{c} K \\ t \end{array} \right) \). Moreover, by (28), the number of packets cached by each user is \( N\left( \begin{array}{c} K-1 \\ t-1 \end{array} \right) + \left( \begin{array}{c} K-1 \\ t \end{array} \right) \), thus the memory size is

\[
M_t = \frac{1}{B} \cdot \left( N\left( \begin{array}{c} K-1 \\ t-1 \end{array} \right) + \left( \begin{array}{c} K-1 \\ t \end{array} \right) \right) \cdot \frac{B}{\left( \begin{array}{c} K \\ t \end{array} \right)}
\]

\[ = \frac{K + t(N-1)}{K}, \]

By (32) and (34), the main payload \( Y \) contains \( \left( \begin{array}{c} K \nonumber \vphantom{b} \\ t+1 \end{array} \right) - \left( \begin{array}{c} K-\text{rank}_2(Q) \nonumber \vphantom{b} \\ t+1 \end{array} \right) \) packets, each of size \( B / \left( \begin{array}{c} K \nonumber \vphantom{b} \\ t \end{array} \right) \) bits. Notice that the set \( L \) and the matrix \( Q \) can be sent in \( K \) and \( NK \) bits, respectively. By (27) and (31), \( q_1, q_2, \ldots, q_K \) are uniformly and independently distributed over an \( N-1 \) dimensional subspace of \( \mathbb{F}_2^N \). Thus, the worst-case is \( \text{rank}_2(Q) = \min\{N-1, K\} \). Therefore, the scheme achieves the worst-case load

\[
R_t = \liminf_{B \to \infty} \frac{1}{B} \left( \left( \left( \begin{array}{c} K \nonumber \vphantom{b} \\ t+1 \end{array} \right) - \left( \begin{array}{c} K-\min\{N-1, K\} \nonumber \vphantom{b} \\ t+1 \end{array} \right) \right) \cdot \frac{B}{\left( \begin{array}{c} K \nonumber \vphantom{b} \\ t \end{array} \right)} + K + NK \right)
\]

\[
= \frac{K}{\left( \begin{array}{c} K \\ t+1 \end{array} \right)} - \frac{K-\min\{N-1, K\}}{\left( \begin{array}{c} K \\ t \end{array} \right)}.
\]

This concludes the description of the general LFR-DPCU scheme.

**Remark 4.** The “single file” retrieval setup analyzed so far can be extended to the “linear function” retrieval setup studied in [18]. In this more general setup, the files \( W_1, \ldots, W_N \) are uniformly distributed over \( \mathbb{F}_q^B \), for some prime power \( q \), and each demand \( D_k \) takes the form of \( d_k = (d_{k,1}, \ldots, d_{k,N}) \in \mathbb{F}_q^N \), which means that user \( k \in [K] \) wants to retrieve the linear combination of the files

\[
W_{D_k} \triangleq \bigoplus_{n \in [N]} d_{k,n} \cdot W_n,
\]

under the correctness constrain in (6) and the privacy constrain in (7), where the operations “\( \bigoplus \)” and “\( \cdot \)” here denotes the addition and scaler-vector multiplication over the finite field \( \mathbb{F}_q^B \).
The LFR-DPCU scheme can be adapted to linear function retrieval as follows:
1) The range of the independent and uniformly distributed vectors \( p_1, \ldots, p_K \) in (27) is now \( \mathbb{F}_q^N \).
2) The vector \( e_{D_k} \) in (31) is replaced by \( d_k \).
3) \( \text{rank}_2(Q) \) becomes \( \text{rank}_q(Q) \), the rank of the matrix \( Q \) over the field \( \mathbb{F}_q \) when choosing the set \( L \).
4) The correctness and privacy can be verified by defining

\[
W_{D_k, T} \triangleq \bigoplus_{n \in [N]} d_{k,n} \cdot W_{n, T}, \quad \forall T \in \Omega_t, \quad \forall k \in [K],
\]

and by following the same lines of the proofs (note that Lemma 1 has been generalized to the case where the files are over \( \mathbb{F}_q \) in [13]). The performance analysis follows the same lines except that the rank of matrix \( Q \) is given by \( \text{rank}_q(Q) \triangleq \min \{ N, K \} \) since the columns of \( Q \) are independent and uniformly distributed over \( \mathbb{F}_q^N \) in this case, thus the worst-case load in (42) is replaced by

\[
R^*_t \triangleq \left( \frac{K}{t+1} \right) - \left( \frac{K - \min \{ N, K \} - 1}{t+1} \right). \tag{45}
\]

IV. Converse and Optimality

In this section, we first derive a converse bound in Section [IV-A] and then that the LFR-DPCU scheme is order optimal in Section [IV-B]

A. Converse Bound

In order to establish our converse, we need the following three lemmas. The proof of Lemma 5 and 4 are deferred to Appendix [A] and [B] respectively.

**Lemma 2** (Submodularity of Entropy Functions [21]). Let \( \mathcal{X} \) be a set of random variables. For any \( \mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{X} \),

\[
H(\mathcal{X}_1) + H(\mathcal{X}_2) \geq H(\mathcal{X}_1 \cup \mathcal{X}_2) + H(\mathcal{X}_1 \cap \mathcal{X}_2). \tag{46}
\]

**Lemma 3.** For an \((N, K)\) DPCU caching system, assume \((M, R) \in \mathbb{R}_+^2 \) is achievable; then, for any \( \ell \in [N] \) and \( b \in [0 : \min \{ \ell, K \}] \), and for sufficiently large \( B \), the following holds

\[
(R + \ell \cdot M) B \geq H(W_{B_t}, X, Z_{A_b} \mid D_{A_b} = d_{A_b}) \tag{47}
\]

for any \( B_\ell \subseteq [N] \), \( A_b \subseteq [K] \) such that \( \left| B_\ell \right| = \ell \), \( \left| A_b \right| = b \), and \( d_{A_b} = \{ d_k : k \in A_b \} \) is any \( b \) distinct indices in \( B_\ell \).

**Lemma 4.** For fixed \( \ell \in [N - 1] \), define \( b_\ell \triangleq \min \{ \ell, K - 1 \} \), which satisfies \( b_\ell \leq \min \{ \ell, K \} \) and \( b_\ell + 1 = \min \{ \ell + 1, K \} \leq K \). For any \( a \in [\ell : N] \) define

\[
h_\ell(a) \triangleq \sum_{j=0}^{b_\ell - 1} H(W_{[a]}, X, Z_{[j]} \mid D_{[j]} = [j]) + H(W_{[a]}, Z_{[b_\ell + 1]}).
\]

Under linear combination file retrieval, we need to select columns of \( P \) uniformly and independent over all vectors of \( \mathbb{F}_q^N \). In this case, we can not constrain them to something equivalent to “odd weight vectors” in the binary case because the demand \( D_k \) may takes all vectors over \( \mathbb{F}_q^N \), so the query space (range of columns of \( Q \)) needs at least \( \mathbb{F}_q^N \) distinct vectors. Therefore, in this case, the worst-case load is slightly increased.
Then for any $a \in [\ell : N - 1]$, the following holds

$$h_\ell(a) + H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell]) \geq h_\ell(a + 1) + H(W_{[\ell]}).$$

(49)

We are now ready to present our novel converse result.

**Theorem 2.** For an $(N, K)$ DPCU caching system, for any $M \in [0, N]$, the optimal memory-load tradeoff $R^*(M)$ satisfies

$$R^*(M) \geq \max_{\ell \in [N]} \left\{ \ell + \frac{\min\{\ell + 1, K\} \cdot (N - \ell \cdot M)}{N - \ell + \min\{\ell + 1, K\}} \right\}.$$  

(50)

**Proof:** Notice that, for fixed $\ell \in [N - 1]$ and for sufficiently large $B$, we have

$$\sum_{j=0}^{b_\ell-2} H(W_{[\ell]}, X, Z_{[j]} | D_{[j]} = [j]) + H(W_{[\ell]}, X, Z_{[b_\ell - 1]}, Z_{[b_\ell + 1]} | D_{[b_\ell - 1]} = [b_\ell - 1], D_{[b_\ell + 1]} = \ell)$$

$$+ (N - \ell + 1) H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell])$$

(51c)

$$\sum_{j=0}^{b_\ell-2} H(W_{[\ell]}, X, Z_{[j]} | D_{[j]} = [j]) + H(W_{[\ell]}, X, Z_{[b_\ell - 1]}, Z_{[b_\ell + 1]} | D_{[b_\ell]} = [b_\ell], D_{[b_\ell + 1]} = \ell)$$

$$+ H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell], D_{[b_\ell + 1]} = \ell) + (N - \ell) H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell])$$

(51d)

$$\sum_{j=0}^{b_\ell-2} H(W_{[\ell]}, X, Z_{[j]} | D_{[j]} = [j]) + H(W_{[\ell]}, X, Z_{[b_\ell - 1]} | D_{[b_\ell]} = [b_\ell], D_{[b_\ell + 1]} = \ell)$$

$$+ H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell], D_{[b_\ell + 1]} = \ell) + (N - \ell) H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell])$$

(51e)

$$\sum_{j=0}^{b_\ell-2} H(W_{[\ell]}, X, Z_{[j]} | D_{[j]} = [j]) + H(W_{[\ell]}, X, Z_{[b_\ell - 1]} | D_{[b_\ell]} = [b_\ell], D_{[b_\ell + 1]} = \ell)$$

$$+ H(W_{[\ell]}, Z_{[b_\ell + 1]} | D_{[b_\ell]} = [b_\ell], D_{[b_\ell + 1]} = \ell) + (N - \ell) H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell])$$

(51f)

$$h_\ell(\ell) + (N - \ell) H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell])$$

(51g)

$$h_\ell(\ell + 1) + (N - \ell - 1) H(W_{[\ell]}, X, Z_{[b_\ell]} | D_{[b_\ell]} = [b_\ell]) + H(W_{[\ell]})$$

(51i)

$$:$$

$$h_\ell(N) + (N - \ell) H(W_{[\ell]})$$

(51j)

$$\sum_{j=0}^{b_\ell-1} H(W_{[N]}, X, Z_{[j]} | D_{[j]} = [j]) + H(W_{[N]}, Z_{[b_\ell + 1]} + (N - \ell) H(W_{[\ell]})$$

(51k)

$$\sum_{j=0}^{b_\ell-1} H(W_{[N]} | D_{[j]} = [j]) + H(W_{[N]}) + (N - \ell) H(W_{[\ell]})$$

(51l)
\begin{align}
&= \sum_{j=0}^{b_\ell - 1} H(W[N]) + H(W[N]) + (N-\ell)H(W[\ell]) \\
&= (b_\ell + 1)H(W[N]) + (N-\ell)H(W[\ell])
\end{align}

where, to get from (51i) to (51j), we recursively applied \(a = \ell, \ell + 1, \ldots, N-1\). Finally, since the files are uniformly distributed over \(\mathbb{F}_2^B\), we have \(H(W[N]) = NB\) and thus

\[R \geq \ell + \min\{\ell + 1, K\} \cdot (N-\ell) - \ell \cdot M\]

holds for \(\ell \in [N-1]\). Moreover, let \(\ell = N\), \(B_\ell = [N]\) and \(A_b = \emptyset\) in (47) to obtain

\[(R + NM)B \geq H(W[N], X) \geq H(W[N]) = NB.
\]

Thus the inequality (52) also holds for \(\ell = N\). Therefore, we proved (50).

**B. Optimality of the LFR-DPCU Scheme**

For clarity, we use \(R(M)\) to denote the achievable load of the LFR-DPCU scheme in Section III-B, i.e., the lower convex envelope of the points defined in Theorem 1. The proof of the following theorem is deferred to Appendix C.

**Theorem 3.** For an \((N, K)\) DPCU caching system, the ratio of the achieved communication loads of LFR-DPCU scheme \(R(M)\) and the optimal communication load \(R^*(M)\) is upper bounded by

\[
\frac{R(M)}{R^*(M)} \leq \begin{cases} 
2, & \text{if } 0 \leq M \leq 1, N \geq 2K \\
2, & \text{if } 0 \leq M \leq \frac{K}{2}, N < 2K \\
4, & \text{if } \frac{1}{2} \leq M \leq 1, N < 2K \\
4, & \text{if } 1 \leq M \leq N, N \geq \frac{2K(K+1)}{4} \\
4.0177, & \text{if } 1 \leq M \leq N, K < N < \frac{2K(K+1)}{4} \\
5.4606, & \text{if } 1 \leq M \leq N, N \leq K \
\end{cases}
\]

**Remark 5.** It was showed in [13] that the load achieved by the virtual users scheme in [13] is optimal to within a multiplicative factor of 8 if \(N \leq K\), or of 4 if \(N > K\) and \(M \geq \frac{K}{2}\), thus leaving open the regime \(N > K, M < \frac{N}{K}\). Here, we show that the LRF-DPCU scheme is order optimal in all regimes under the DPCU requirement, due to the new converse we derived in Theorem 2.

**Remark 6.** The gap in Theorem 3 is obtained under the privacy definition in (7). It is worthy pointing out that, at \(M = 0\), the best known converse under other the privacy definitions in Remark 1 is \(\min\{N, K\}\). It was noticed in [17] that the virtual users scheme [13] satisfies the privacy against user colluding. Both LFR-DPCU and virtual users schemes achieve the load \(R = N\) at \(M = 0\). Thus, at \(M = 0\), both LFR-DPCU and virtual users schemes achieve optimal load under privacy definition (7), but unbounded gap when \(K > N\) under the other privacy definitions in Remark 1.

**V. PERFORMANCE COMPARISON AND NUMERICAL RESULTS**

In this section, we numerically compare the performance of the LFR-DPCU scheme with known schemes in [13] in terms of memory-load tradeoff and subpacketizations. As mentioned
in Remark 6, the virtual users scheme in [13] in fact satisfies the DPCU condition (7). The performances of non-private scheme in [3] are also presented for reference.

For clarity, we list in Table I the performance of the corner points of various schemes where, in term of an integer parameter $t$ whose range are in the second row, with the memory size $M$ one achieves load $R$ and subpacketization $B$. We compare the schemes in the two regimes $N \leq K$ and $N > K$, where we choose parameters $(N, K) = (10, 30)$ and $(N, K) = (30, 10)$. For both cases, we plot two figures showing

(a) the load-memory tradeoff curves of the three schemes and the lower bounds in Theorem 2 and [9];

(b) the logarithm of subpacketization, i.e., $\log_{10} B$ as a function of memory size $M$ for the corner points.

The comparisons for $N \leq K$ and $N > K$ are presented in Fig. 1 and Fig. 2 respectively. By comparing LFR-DPCU with the virtual users schemes, we note that

1) case $N \leq K$ (Fig. 1): the LFR-DPCU achieves worse load-memory tradeoff than the virtual users scheme, which can approach almost the same performance as the non-private scheme. However, the LFR-DPCU could maintain a similar subpacketization order as the non-private scheme, while the virtual users scheme has a significantly larger subpacketization.

2) case $N > K$ (Fig. 2): the LFR-DPCU outperforms the virtual users scheme in both load-memory tradeoff and subpacketization.

We explain intuitively the results in Fig. 1(a) and 2(a) as follows. Both LFR-DPCU and the virtual users schemes are based on the MAN uncoded placement scheme. In LFR-DPCU, in addition each user caches some random linear combinations of uncached MAN subfiles. This negative effect on load-memory tradeoff becomes less significant when the number of files $N$ becomes large. In the virtual users scheme, the server creates multicast signals to satisfy $NK$ users, which includes $K$ real users and $N(K − 1)$ virtual users, thus some multicast signals are only useful for virtual users, which increases the load compared to the non-private scheme. This negative effect on load-memory tradeoff becomes more significant when $N$ becomes large.

The regime where LFR-DPCU outperforms the virtual users scheme can be found by the following observations:

1) Both schemes achieve the point $(0, N)$, and their slopes at $M = 0$ are $-\max \{N - K, \frac{2N + 1 - K}{K + 1}\}$ and $\frac{-N + 1}{2}$ respectively. When $N = 2K + 1$, the two slopes are equal, so $2K + 1$ is the threshold of $N$ such that the LFR-DPCU scheme outperforms the virtual users scheme when $M$ is close to 0.

| Scheme | LFR-DPCU | Virtual users [13] | Non-private [3] |
|--------|----------|-------------------|-----------------|
| $t$    | $0 \leq t \leq K$ | $0 \leq t \leq KN$ | $0 \leq t \leq K$ |
| $M$    | $1 + \frac{t(N-1)}{K}$ | $\frac{tN}{K}$ | $\frac{tN}{K}$ |
| $R$    | $(\frac{K}{t}) - (\frac{K - \min\{K, N-1\}}{t+1})$ | $(\frac{KN}{t}) - (\frac{(K-1)N}{t+1})$ | $(\frac{K}{t}) - (\frac{K - \min\{N,K\}}{t+1})$ |
| $B$    | $(\frac{K}{t})$ | $(\frac{KN}{t})$ | $(\frac{K}{t})$ |

The comparisons for $N \leq K$ and $N > K$ are presented in Fig. 1 and Fig. 2 respectively. By comparing LFR-DPCU with the virtual users schemes, we note that

1) case $N \leq K$ (Fig. 1): the LFR-DPCU achieves worse load-memory tradeoff than the virtual users scheme, which can approach almost the same performance as the non-private scheme. However, the LFR-DPCU could maintain a similar subpacketization order as the non-private scheme, while the virtual users scheme has a significantly larger subpacketization.

2) case $N > K$ (Fig. 2): the LFR-DPCU outperforms the virtual users scheme in both load-memory tradeoff and subpacketization.

We explain intuitively the results in Fig. 1(a) and 2(a) as follows. Both LFR-DPCU and the virtual users schemes are based on the MAN uncoded placement scheme. In LFR-DPCU, in addition each user caches some random linear combinations of uncached MAN subfiles. This negative effect on load-memory tradeoff becomes less significant when the number of files $N$ becomes large. In the virtual users scheme, the server creates multicast signals to satisfy $NK$ users, which includes $K$ real users and $N(K − 1)$ virtual users, thus some multicast signals are only useful for virtual users, which increases the load compared to the non-private scheme. This negative effect on load-memory tradeoff becomes more significant when $N$ becomes large.

The regime where LFR-DPCU outperforms the virtual users scheme can be found by the following observations:

1) Both schemes achieve the point $(0, N)$, and their slopes at $M = 0$ are $-\max \{N - K, \frac{2N + 1 - K}{K + 1}\}$ and $\frac{-N + 1}{2}$ respectively. When $N = 2K + 1$, the two slopes are equal, so $2K + 1$ is the threshold of $N$ such that the LFR-DPCU scheme outperforms the virtual users scheme when $M$ is close to 0.
2) It was proved in [13] that for $M \geq N - \frac{1}{K}$, the virtual users scheme achieves the cut-set bound $1 - \frac{M}{N}$, while LFR-DPCU achieves $1 - \frac{M-1}{N-1}$ when $M \geq N - \frac{N-1}{K}$. Thus, the virtual users scheme eventually outperforms the LFR-DPCU scheme when $M$ increases to $N$. The load of virtual user scheme is given by $\frac{K(N-M)}{KM+1}$ for $M \in \{N - \frac{i}{K} : i \in [0 : N - 1]\}$. Therefore, if $N \geq K+2$ and $M = N - 1 - \frac{N-1}{K}$, the loads of the two schemes are equal. So, $M = N - 1 - \frac{1}{K}$ is the threshold that the virtual users scheme outperforms the LFR-DPCU scheme.

These observations together with extensive numerical results indicate that when $N > 2K+1$ and $0 < M < N - 1 - \frac{1}{K}$, the LFR-DPCU scheme outperforms the virtual user scheme. Notice that for the regime $N - 1 - \frac{1}{K} \leq M \leq N$, the multiplicative gap of LFR-DPCU scheme compared to the cut-set bound $1 - \frac{M}{N}$ is $\frac{N}{N-1}$, which is very close to one for large $N$.

It is worthy pointing out that, the LFR-DPCU scheme is within the multiplicative gap of the optimal load-memory tradeoff by Theorem 3 even in the regime $N \leq K$. From Fig. 1(b) and 2(b), the LFR-DPCU has much lower subpacketization in all parameter regimes, since LFR-DPCU scheme is designed to satisfy $K$ users, instead of the $NK$ users in virtual users schemes.

From Fig. 1(a) and 2(a) the bound derived in Theorem 2 outperforms the existing bound on an interval beginning with $M = 0$ in the case $N > K$. From Theorem 3, we know that the lower bound enable us to bound the performance of LFR-DPCU scheme with a constant multiplicative gap over the interval $[0, 1]$.

VI. CONCLUSIONS

In this paper, we investigated a shared-link caching system where the demands of users must be protected against any subset of colluding users. The proposed LFR-DPCU scheme is proved to be order optimal in all parameter regimes. The LFR-DPCU scheme outperforms existing virtual users schemes in some parameter regime and has much lower subpacketization.
Fig. 1: Performance comparison for an \((N, K) = (10, 30)\) system.
Fig. 2: Performance comparison for an \((N, K) = (30, 10)\) system.
Appendix

A. Proof of Lemma 3

We first prove that the conclusion holds for the case \( b = \min \{ \ell, K \} \) by induction on \( \ell \).

For \( \ell = b = 1 \), let \( B_1 = \{ n \} \) and \( A_1 = \{ k \} \) for some \( n \in [N] \) and \( k \in [K] \).

\[
H(W_n, X, Z_k \mid D_k = n) \\
\geq H(X, Z_k \mid D_k = n) \\
\leq H(X \mid D_k = n) + H(Z_k \mid D_k = n) \\
= H(X) + H(Z_k) \\
\leq (R + M)B.
\]

Thus, the conclusion holds for \( \ell = 1 \).

Now, assume that the conclusion holds for \( \ell \) where \( \ell \in [N - 1] \). Consider the case \( \ell + 1 \), let \( b' = \min \{ \ell + 1, K \} \). Let \( B_{\ell+1} \) and \( A_{b'} \) be any subset of \([N]\) and \([K]\) with cardinalities \( \ell + 1 \) and \( b' \) respectively. Let \( d_{A_{b'}} \) be the demands of users in \( A_{b'} \), which can be any distinct demands in \( B_{\ell+1} \). We have

1) if \( \ell < K \), then \( b' = b + 1 = \ell + 1 \), pick any \( k \in A_{b+1} \).

\[
(R + (\ell + 1) \cdot M)B \\
= (R + \ell \cdot M)B + MB \\
\geq H(W_{B_{\ell+1} \setminus \{ d_k \}}, X, Z_{A_{b+1} \setminus \{ k \}} \mid D_{A_{b+1} \setminus \{ k \}} = d_{A_{b+1} \setminus \{ k \}}) + H(Z_k) \\
\geq H(W_{B_{\ell+1} \setminus \{ d_k \}}, X, Z_{A_{b+1} \setminus \{ k \}} \mid D_{A_{b+1}} = d_{A_{b+1}}) + H(Z_k \mid D_{A_{b+1}} = d_{A_{b+1}}) \\
= H(W_{B_{\ell+1} \setminus \{ d_k \}}, X, Z_{A_{b+1} \setminus \{ k \}} \mid D_{A_{b+1}} = d_{A_{b+1}}),
\]

where (62) follows from the induction assumption.

2) if \( \ell \geq K \), then \( b' = b = K < \ell + 1 \) and \( A_{b'} = A_b = [K] \). Pick any \( n \in B_{\ell+1} \setminus d_{[K]} \) and \( k \in [K] \), let \( d'_{[K]} = \{ d'_j : j \in [K] \} \) be another demand of users such that

\[
d'_j = \begin{cases} 
  d_j, & \text{if } j \neq k \\
  n, & \text{if } j = k
\end{cases}
\]

Then,

\[
(R + (\ell + 1) \cdot M)B \\
= (R + \ell \cdot M)B + MB \\
\geq H(W_{B_{\ell+1} \setminus \{ d_k \}}, X, Z_{[K]} \mid D_{[K]} = d'_{[K]}) + H(Z_k) \\
\geq H(W_{B_{\ell+1} \setminus \{ d_k \}}, X, Z_{[K]} \mid D_{[K]} = d_{[K]}) + H(Z_k \mid D_{[K]} = d_{[K]}) \\
= H(W_{B_{\ell+1} \setminus \{ d_k \}}, X, Z_{[K]} \mid D_{[K]} = d_{[K]}),
\]

where (62) follows from the induction assumption.
Therefore, the conclusion holds for $b = \min\{\ell, K\}$.

For the case $b < \min\{\ell, K\}$, let $\tilde{A} \subseteq \{K\}$ be a subset of size $\min\{\ell, K\}$ such that $A_b \subseteq \tilde{A}$ and $d = \{d_j \mid j \in A_b\} \cup \{d_j \mid j \in \tilde{A} \setminus A_b\}$ be a distinct demand of users in $\tilde{A}$ such that

1) it has consistent demands on the users in $A_b$;
2) the extended demands $\{d_j \mid j \in \tilde{A} \setminus A_b\}$.

Then by the conclusion for the case $b = \min\{\ell, K\}$, for sufficiently large $B$,

\[
(R + \ell \cdot M) B \geq H(W_{B\ell}, X, Z_{A\ell} | D_{A\ell} = d_{\tilde{A}}) \geq H(W_{B\ell}, X, Z_{A_b} | D_{A_b} = d_{A_b}) \geq H(W_{B\ell}, X, Z_{A_b} | D_{A_b} = d_{A_b}).
\]

\[
(R + \ell \cdot M) B \geq H(W_{B\ell}, X, Z_{A\ell} | D_{A\ell} = d_{\tilde{A}}) \geq H(W_{B\ell}, X, Z_{A_b} | D_{A_b} = d_{A_b}) \geq H(W_{B\ell}, X, Z_{A_b} | D_{A_b} = d_{A_b}).
\]

B. Proof of Lemma 2

By the definition of $h_\ell(a)$ in (48), we have

\[
h_\ell(a) + H(W_{[\ell]}, X, Z_{[b\ell]} | D_{[b\ell]} = [b\ell]) = \sum_{j=0}^{b\ell-1} H(W_{[\ell]}, X, Z_{[j]} | D_{[j]} = [j]) + H(W_{[\ell]}, Z_{[b\ell+1]} | D_{[b\ell]} = [b\ell], D_{b\ell+1} = a + 1)
\]

\[
+ H(W_{[\ell]}, X, Z_{[b\ell+1]} | D_{[b\ell]} = [b\ell], D_{b\ell+1} = a + 1)
\]

\[
+ H(W_{[\ell]}, X, Z_{[b\ell+1]} | D_{[b\ell]} = [b\ell], D_{b\ell+1} = a + 1)
\]

\[
+ H(W_{[\ell]}, X, Z_{[b\ell+1]} | D_{[b\ell]} = [b\ell], D_{b\ell+1} = a + 1)
\]

Notice that, for any $j \in [0: b\ell - 1]$, we have

\[
H(W_{[\ell]}, X, Z_{[j]} | D_{[j]} = j) + H(W_{[\ell]}, Z_{[j+1]} | D_{[j]} = j, D_{j+1} = a + 1) + H(W_{[\ell]}, Z_{[j+1]} | D_{[j]} = j, D_{j+1} = a + 1)
\]

\[
H(W_{[\ell]}, X, Z_{[j]} | D_{[j]} = j) + H(W_{[\ell]}, Z_{[j+1]} | D_{[j]} = j, D_{j+1} = a + 1) + H(W_{[\ell]}, Z_{[j+1]} | D_{[j]} = j, D_{j+1} = a + 1)
\]
\[ H(W[a], X, Z[j+1] \mid D[j] = j, D_{j+1} = a + 1) + H(W[e], Z[j] \mid D[j] = j, D_{j+1} = a + 1) \] (85)

\[ H(W[a+1], X, Z[j+1] \mid D[j] = j, D_{j+1} = a + 1) + H(W[e], Z[j] \mid D[j] = j, D_{j+1} = a + 1) \] (86)

\[ H(W[a+1], X, Z[j] \mid D[j] = j, D_{j+1} = a + 1) + H(W[e], Z[j] \mid D[j] = j, D_{j+1} = a + 1) \] (87)

\[ H(W[a+1], X, Z[j] \mid D[j] = j) + H(W[e], Z[j]) \] (88)

Thus, we can continue with (82) as

\[ h_\ell(a) + H(W[\ell], X, Z[b_\ell] \mid D[b_\ell] = [b_\ell]) \] (89)

\[ \geq \sum_{j=0}^{b_\ell-2} H(W[a], X, Z[j] \mid D[j] = [j]) + H(W[\ell], Z[b_{\ell-1}]) \]

\[ + H(W[a+1], X, Z[b_{\ell-1}] \mid D[b_{\ell-1}] = [b_{\ell-1} - 1]) + H(W[a+1], Z[b_{\ell+1}]) \] (90)

\[ \vdots \]

\[ \geq H(W[\ell]) + \sum_{j=0}^{b_{\ell-1}} H(W[a+1], X, Z[b_{\ell-1}] \mid D[b_{\ell-1}] = [b_{\ell-1} - 1]) + H(W[a+1], Z[b_{\ell+1}]) \] (92)

\[ = h_\ell(a + 1) + H(W[\ell]), \] (93)

where in (91), we recursively apply (88) to \( j = b_\ell - 2, b - 3, \ldots, 0 \) sequentially.

\[ C. \text{ Proof of Theorem 3} \]

We bound \( \frac{R(M)}{R^*(M)} \) for \( 0 \leq M \leq 1 \) and \( 1 \leq M \leq N \) separately.

1) Case \( 0 \leq M \leq 1 \): For clarity, we denote the function in the braces of (50) by \( f(M, \ell) \), i.e.,

\[ f(M, \ell) \triangleq \ell + \frac{\min\{\ell + 1, K\} \cdot (N - \ell)}{N - \ell + \min\{\ell + 1, K\}} - \ell \cdot M, \quad \forall \ell \in [N], M \in [0, N]. \] (94)

We further discuss in two subcases \( N \geq 2K \) and \( N < 2K \).

1) If \( N \geq 2K \), \( R(M) \) is upper bounded by the line segment connecting \((M', R'_0) = (0, N)\) and \((M_0, R_0) = (1, K)\), i.e.,

\[ R(M) \leq L_1(M) \triangleq N - (N - K)M, \quad M \in [0, 1]. \] (95)

We further discuss in two sub-cases, i.e., \( 0 \leq M \leq 1 - \frac{K}{N} \) and \( 1 - \frac{K}{N} \leq M \leq 1 \).

a) If \( 0 \leq M \leq 1 - \frac{K}{N} \),

\[ \frac{R(M)}{R^*(M)} \overset{50, 55}{\leq} \frac{L_1(M)}{f(M, N)} \] (96)

\[ \leq \frac{L_1(1 - \frac{K}{N})}{f(1 - \frac{K}{N}, N)} \] (97)

\[ = 2 - \frac{K}{N} \] (98)

\[ < 2, \] (99)

where we utilized the fact that \( \frac{L_1(M)}{f(M, N)} \) increases with \( M \) over \([0, 1 - \frac{K}{N}]\).
b) If \( 1 - \frac{K}{N} \leq M \leq 1 \),
\[
\frac{R(M)}{R^*(M)} \leq \frac{L_1(M)}{f(M, K)} \leq \max_{M \in \{1 - \frac{K}{N}, 1\}} \left\{ \frac{L_1(M)}{f(M, K)} \right\} \leq 2.
\]

where (101) holds because for any fixed \((N, K)\) such that \( N \geq 2K \), the linear fractional function \( \frac{L_1(M)}{f(M, K)} \) is either increasing or decreasing in \( M \) over \([1 - \frac{K}{N}, 1]\), and (104) follows from the fact \( \frac{K}{N} \leq \frac{1}{2} \).

2) If \( N < 2K \), \( R(M) \) is upper bounded by the line segment connecting \((M', R'_0) = (0, N)\) and \((M_1, R_1) = (1, \min\{N - 1, K\})\), i.e.,
\[
R(M) \leq L_2(M) \triangleq N - \max\{1, N - K\} \cdot M, \quad M \in [0, 1].
\]

We further discuss in two sub-cases, i.e., \( 0 \leq M \leq \frac{1}{2} \) and \( \frac{1}{2} < M \leq 1 \).

a) If \( 0 \leq M < \frac{1}{2} \),
\[
\frac{R(M)}{R^*(M)} \leq \frac{L_2(M)}{f(M, N)} \leq \frac{L_2(\frac{1}{2})}{f(\frac{1}{2}, N)} = 2 \min \left\{ 1 - \frac{1}{2N}, \frac{N + K}{2N} \right\} < 2.
\]

where (107) holds because the function \( \frac{L_2(M)}{f(M, N)} \) increases with \( M \) over \([0, \frac{1}{2}]\).

b) If \( \frac{1}{2} < M \leq 1 \),
\[
\frac{R(M)}{R^*(M)} \leq \frac{L_2(M)}{f(M, \lceil N/2 \rceil)} \leq \frac{N - \max\{1, N - K\} \cdot M}{\lceil N/2 \rceil + \left( \lceil N/2 \rceil + 1 \right) \left( N - \lceil N/2 \rceil - \lfloor N/2 \rfloor \right) / N + \lfloor N/2 \rfloor} \leq 4,
\]

where in (114), we used the fact \( \frac{N}{2} \geq \lfloor N/2 \rfloor \geq \frac{N}{2} - \frac{1}{2} \).
2) Case $1 \leq M \leq N$: Denote the optimal centralized and decentralized load for an $(N, K)$ caching system with memory size $M$ at each user under uncoded placement without privacy constraint by $r_C(M)$ and $r_D(M)$ respectively. By the results of [3], $r_C(M)$ is the lower convex envelope of the points $\{(M_t, R_t) : t \in [0 : K]\}$ in (24), and $r_D(M)$ is given by

$$r_D(M) \triangleq \frac{N - M}{M} \left(1 - \left(1 - \frac{M}{N}\right)^{\min(N,K)}\right), \quad \forall M \in [0, N].$$

Denote the optimal load for an $(N, K)$ caching system with memory size $M$ at each user without privacy constraint by $r^\ast(M)$. By the results in [9],

$$r_C(M) \leq r_D(M) \leq \begin{cases} 2.00884 \cdot r^\ast(M), & \text{if } N < \frac{K(K+1)}{2} \\ 2 \cdot r^\ast(M), & \text{if } N \geq \frac{K(K+1)}{2} \end{cases}.$$  \hspace{1cm} (116)

Notice that the optimal load for DPCU is no less than that without privacy constraint, i.e.,

$$r^\ast(M) \leq R^\ast(M), \quad \forall M \in [0, N].$$

We bound $\frac{R(M)}{R^\ast(M)}$ for the subcases $N > K$ and $N \leq K$ separately.

1) If $N > K$, we bound $\frac{R(M)}{R^\ast(M)}$ for the sub-cases $(N, K) = (3, 2)$ and $(N, K) \neq (3, 2)$ separately.

a) If $(N, K) = (3, 2)$, the lower convex envelope of the points $(M_0, R_0) = (0, 3)$, $(M_0, R_0) = (1, 2)$, $(M_1, R_1) = (2, 1)$ and $(M_2, R_2) = (3, 0)$ is given by

$$R(M) = \begin{cases} 3 - \frac{5}{4}M, & \text{if } 0 \leq M \leq 2 \\ \frac{3}{2} - \frac{1}{2}M, & \text{if } 2 \leq M \leq 3 \end{cases}.$$ \hspace{1cm} (118)

By Theorem 2 and the cut set bound $NR + M \geq N$ (see [1]), we have $R^\ast(M) \geq 1 - \frac{1}{3}M$, thus

i) If $1 \leq M \leq 2$,

$$\frac{R(M)}{R^\ast(M)} \leq \frac{3 - \frac{5}{4}M}{1 - \frac{1}{3}M} \leq \frac{21}{8} < 4.$$ \hspace{1cm} (119)

ii) For $2 \leq M < 3$,

$$\frac{R(M)}{R^\ast(M)} \leq \frac{\frac{3}{2} - \frac{1}{2}M}{1 - \frac{1}{3}M} = \frac{3}{2} < 4.$$ \hspace{1cm} (120)

b) If $(N, K) \neq (3, 2)$, we prove the following inequality in Appendix D

$$\frac{R(M)}{r_C(M)} \leq 2, \quad \forall 1 \leq M \leq N.$$ \hspace{1cm} (121)

Therefore,

$$\frac{R(M)}{R^\ast(M)} = \frac{R(M)}{r_C(M)} \cdot \frac{r_C(M)}{r^\ast(M)} \cdot \frac{r^\ast(M)}{R^\ast(M)}$$ \hspace{1cm} (122)
If \( N \leq K \), we prove the following inequality in Appendix D:

\[
\frac{R(M)}{r_D(M)} \leq e, \quad \forall M \in [1, N].
\]  

Therefore,

\[
\frac{R(M)}{r_C(M, N, K)} \leq 2, \quad 1 \leq M \leq N.
\]  

2) If \( N \leq K \), we prove the following inequality in Appendix D:

\[
\frac{R(M)}{r_D(M)} \leq e, \quad \forall M \in [1, N].
\]  

D. Proof of Two Inequalities

For notational clarity, in this subsection, we will use \( r_C(M, N, K) \) and \( r_D(M, N, K) \) to denote the optimal load of an \((N, K)\) caching system without privacy constraint. We aim to prove the inequalities (121) and (125), i.e.,

**Lemma 5.** For an \((N, K)\) system,

1) If \( N > K \) and \((N, K) \neq (2, 3)\),

\[
\frac{R(M)}{r_C(M, N, K)} \leq 2, \quad 1 \leq M \leq N.
\]  

2) If \( N \leq K \),

\[
\frac{R(M)}{r_D(M, N, K)} \leq e, \quad 1 \leq M \leq N.
\]  

**Proof:** By Theorem 1 for \( M \geq 1 \), \( R(M) \) is upper bounded by the lower convex envelope of the points \( \{(M_t, R_t) : t \in [0 : K]\} \) in (14), which is exactly \( r_C(M - 1, N - 1, K) \) by (24). Thus, by (116), for \( 1 \leq M \leq N \),

\[
R(M) \leq r_C(M - 1, N - 1, K) \leq r_D(M - 1, N - 1, K).
\]  

Therefore,

1) If \( N > K \) and \((N, K) \neq (3, 2)\), since \( R(M) \) is convex in \( M \), and \( r_C(M, N, K) \) is piecewise linear in \( M \) with corner points such that \( M \in \{1\} \cup \left\{ \frac{tN}{K} : t \in [K]\right\} \) over \([1, N]\), it suffices to prove (129) for the cases \( M \in \{1\} \cup \left\{ \frac{tN}{K} : t = 1, 2, \ldots, K\right\} \).

a) If \( M = 1 \), let \( \theta = 1 - \frac{K}{N} \), then \( M = 1 = \theta \cdot 0 + (1 - \theta) \frac{N}{K} \), thus

\[
r_C(1, N, K) = \theta \cdot r_C(0, N, K) + (1 - \theta) \cdot r_C\left(\frac{N}{K}, N, K\right)
\]  

\[
= \left(1 - \frac{K}{N}\right)K + \frac{K(K - 1)}{2N}.
\]  

\[
\leq 2 \cdot \frac{r_C(M)}{r_C(M, N, K)} \leq 2
\]

\[
\leq 4, \quad \text{if } N \geq \frac{K(K+1)}{2}
\]

\[
4.0177, \quad \text{if } K < N < \frac{K(K+1)}{2}
\]

\[
< 5.4606.
\]  

\[
< 5.4606.
\]  

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(116)
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(117)
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(131)
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(132)
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\[
(133)
\]
\[ R(1) \geq K, \quad (134) \]

where we used the fact \( N \geq K + 1 \). Thus,

\[ \frac{R(1)}{r_C(1, N, K)} \leq \frac{r_C(0, N-1, K)}{r_C(1, N, K)} \leq 2. \quad (136) \]

b) If \( M = \frac{N}{K} \) and \( N = K + 1 \), \( R(M) \) is upper bounded by the line segment connecting \((M'_0, R'_0) = (0, N)\) and \((M_1, R_1) = (\frac{K+1}{2}, \frac{K}{2})\), i.e.,

\[ R(M) \leq L_3(M) \triangleq N - \frac{K(2N + 1 - K)}{2(K + N - 1)} M, \quad \forall M \in \left[0, \frac{K + N - 1}{K}\right]. \quad (137) \]

Thus,

\[ \frac{R(\frac{N}{K})}{r_C(\frac{N}{K}, N, K)} \leq \frac{N - \frac{K(2N + 1 - K)}{2(K + N - 1)} \cdot \frac{N}{K}}{\frac{K-1}{2}} = \frac{3(K+1)}{2K} \leq 2, \quad (138) \]

where we used the fact \( K \geq 3 \) since \((N, K) \neq (3, 2)\).

c) If \( M = \frac{tN}{K} \), where \( t \geq 2 \) or \( N \geq K + 2 \) hold, we have

\[ M - 1 = \theta \left( \frac{(t-1)(N-1)}{K} \right) + (1 - \theta) \left( \frac{t(N-1)}{K} \right), \quad (139) \]

where \( \theta = \frac{K-t}{N-1} < 1 \). Thus,

\[ \frac{R(M)}{r_C(M, N, K)} \leq \frac{r_C(M-1, N-1, K)}{r_C(M, N, K)} \leq \theta \cdot r_C\left(\frac{(t-1)(N-1)}{K}, N-1, K\right) + (1 - \theta) \cdot r_C\left(\frac{t(N-1)}{K}, N-1, K\right) \]

\[ = \frac{K-t}{N-1} \cdot \frac{K-t+1}{t} + \frac{N-1-K+t}{N-1} \cdot \frac{K-1}{t+1} \]

\[ = 1 + \frac{K+1}{(N-1)t} \]

\[ \leq 2, \quad (144) \]

where in the last step, we used the fact \( t \geq 2 \) or \( N \geq K + 2 \).

2) If \( N \leq K \), let \( q \triangleq 1 - \frac{M}{N} \in \left[0, 1 - \frac{M}{N}\right] \), then \( 1 - \frac{M-1}{N-1} = \frac{N}{N-1} q \). Hence,

\[ \frac{R(M)}{r_D(M, N, K)} \leq \frac{r_D(M-1, N-1, K)}{r_D(M, N, K)} \leq \frac{N - \frac{M}{N-1} \cdot \left(1 - \left(1 - \frac{M-1}{N-1}\right)^{N-1}\right)}{N-1} \]

\[ \leq \frac{N}{N-1} \cdot \frac{M}{N-1} \cdot \frac{N}{1 - \left(1 - \frac{M}{N}\right)^N} \quad (147) \]
\[
\frac{N}{N-1} \cdot \frac{1 - q}{1 - \frac{N}{N-1}q} \cdot \frac{1 - \left(\frac{N}{N-1}\right)^{N-1}q^{N-1}}{1 - q^N} = \frac{N}{N-1} \cdot \frac{1 + \frac{N}{N-1}q + \ldots + \left(\frac{N}{N-1}\right)^{N-2}q^{N-2}}{1 + q + \ldots + q^{N-1}} \leq \left(\frac{N}{N-1}\right)^{N-1}\frac{1 + q + \ldots + q^{N-2}}{1 + q + \ldots + q^{N-1}} \leq \left(1 + \frac{1}{N-1}\right)^{N-1} \leq e. \tag{152}
\]

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