Convergence of a continuous Galerkin method for mixed hyperbolic-parabolic systems

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Abstract. We study the numerical approximation by space-time finite element methods of a multi-physics system coupling hyperbolic elastodynamics with parabolic transport and modelling poro- and thermoelasticity. The equations are rewritten as a first-order system in time. Discretizations by continuous Galerkin methods in space and time with inf-sup stable pairs of finite elements for the spatial approximation of the unknowns are investigated. Optimal order error estimates of energy-type are proven. Superconvergence at the time nodes is addressed briefly. The error analysis can be extended to discontinuous and enriched Galerkin space discretizations. The error estimates are confirmed by numerical experiments.

Keywords. Poroelasticity, dynamic Biot model, thermoelasticity, space-time finite element approximation, continuous Galerkin method, error analysis.

1. Introduction

In this work we study the numerical approximation of the coupled equations

\[
\begin{align*}
\rho \frac{\partial^2}{\partial t^2} u - \nabla \cdot (C \varepsilon(u)) + \alpha \nabla p &= \rho f, \quad \text{in } \Omega \times (0,T], \tag{1.1a} \\
c_0 \frac{\partial}{\partial t} p + \alpha \nabla \cdot \partial_t u - \nabla \cdot (K \nabla p) &= g, \quad \text{in } \Omega \times (0,T], \tag{1.1b} \\
u(0) = u_0, \quad \partial_t u(0) = u_1, \quad p(0) = p_0, \quad \text{in } \Omega, \tag{1.1c} \\
u = 0, \quad p = 0, \quad \text{on } \partial \Omega \times (0,T]. \tag{1.1d}
\end{align*}
\]

Under the below made assumptions about the coefficients of (1.1), this is a system of mixed hyperbolic-parabolic type that is considered in the open Lipschitz bounded domain \( \Omega \subset \mathbb{R}^d \), with \( d \in \{2,3\} \), and the time interval \([0,T]\) with some final time \( T > 0 \). For simplicity, Dirichlet boundary conditions are prescribed here in (1.1d). Important applications of the model (1.1), that is studied as a prototype system, arise in poro- and thermoelasticity. In poroelasticity (cf. [54] and [13–15]), where Eqs. (1.1) are referred to as the dynamic Biot model, the system (1.1) is used to describe flow of a slightly compressible viscous fluid through a deformable porous matrix. The small deformations of the matrix

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are described by the Navier equations of linear elasticity, and the diffusive fluid flow is described by Duhamel’s equation. The unknowns are the effective solid phase displacement $u$ and the effective fluid pressure $p$. The quantity $\varepsilon(u) := (\nabla u + (\nabla u)^\top)/2$ denotes the symmetrized gradient or strain tensor. Further, $\rho$ is the effective mass density, $C$ is Gassmann’s fourth order effective elasticity tensor, $\alpha$ is Biot’s pressure-storage coupling tensor, $c_0$ is the specific storage coefficient and $K$ is the permeability field. For simplicity, the positive quantities $p > 0$, $\alpha > 0$ and $c_0 > 0$ are assumed to be constant in space and time. Moreover, the tensors $C$ and $K$ are assumed to be symmetric and positive definite (cf. (A.1) and (A.2)) and independent of the space and time variables as well. In thermoelasticity (cf. (11) and [15,11]), the system (1.1) describes the flow of heat through an elastic structure. In that context, $p$ denotes the temperature, $c_0$ is the specific heat of the medium, and $K$ is the conductivity. Then, the quantity $\alpha \nabla \cdot \partial_t u$ arises from the thermal stress in the structure, and the term $\alpha \nabla \cdot \partial_t u$ corresponds to the internal heating due to the dilation rate. For the sake of physical realism, the often used uncoupling assumption in which this term is deleted from the diffusion equation is not made here. Well-posedness of (1.1) is ensured. For completeness, the key result of existence and uniqueness of solutions to (1.1) is briefly summarized in the appendix. In this result of well-posedness, (1.1) is rewritten as a first-order system in time which also done below for its discretization. In order to enhance physical realism in poroelasticity, generalizations of the system (1.1) are presented in, e.g., [16,43].

The coupled hyperbolic-parabolic structure of the system (1.1) of partial differential equations adds an additional facet of complexity onto its numerical simulation. A natural and promising approach for the numerical approximation of coupled systems is given by the application of space-time finite element methods that are based on variational formulations in space and time. Therein, the discrete approximation for the numerical approximation of coupled systems is given by the application of space-time finite element methods. For this, the hyperbolic subproblem (1.1) is rewritten as a first-order system in time. In particular, continuous Galerkin methods provide energy conservative discretizations of wave equations (cf. [10, Sec. 6]), where the energy is measured by $E(t) := (\|\nabla u(t)\|^2_{L^2(G)} + \|\partial_t u(t)\|^2_{L^2(G)})^{1/2}$ in the scalar-valued case. Thus, continuous Galerkin methods offer the ability to preserve an essential property of solutions to the continuous problem on the discrete level. Here, the continuous Galerkin discretization is considered as a prototype scheme for microscopic families of space-time finite element methods. We refer to [51,12] for the construction of $C^k$-conforming variational time discretizations for some $k \geq 1$. In this work, we aim to elaborate the treatment of the coupling in (1.1) in the error analysis with the perspective of getting optimal order error estimates. We like to present our key arguments and not to overburden the error analysis with the additional terms arising in discontinuous space discretizations. The error analysis offers the potential and flexibility for its extension to spatial approximations by enriched Galerkin methods (cf. [35,40,57,69]) or discontinuous Galerkin approaches (cf., e.g., [9,20,21,23,25,50]). Also, for the application of discontinuous Galerkin space discretizations to the quasi-static Biot system, that differs from (1.1) by neglecting the acceleration term $\rho \partial_t^2 u$ in (1.1), we refer to [9,50]. Unsteady spatial approximations yield appreciable advantages, for instance, for the construction of iterative solver (cf., e.g., [36]) or the computation by post-processing of locally mass conservative (fluid) fluxes (cf. [10]) from the variable $p$ of (1.1). The latter is of importance if the system (1.1) is coupled further with the transport of species dissolved in the fluid. Discontinuous Galerkin time discretizations (cf. [33,58]) are not considered here due to their lack of energy conservation for second-order hyperbolic problems.

The numerical discretization of hyperbolic and parabolic equations by space-time finite have been studied over the past decades (cf., e.g., [5,22,23,31,55,53,55,56]). Further, strong relations and equivalences between continuous Galerkin times discretization schemes, collocation, and Runge–Kutta methods have been observed. In the literature, the relations are exploited in the formulation and analysis of the schemes. For the latter we particularly refer to [14]. In [2], Eq. (2.2)) nodal superconvergence properties of the cGP method are shown along such a line. Similarly, links between Runge–Kutta
type and Galerkin schemes are also identified and applied in [25,30,60]. Further, the approximation of the quasi-static Biot system has attracted researchers’ interest over the last years as well; cf., e.g., [11,44,58,40,62] and the references therein. Iterative splitting schemes based on space-time finite element discretizations of the subproblems of fluid flow and mechanical deformation are investigated in [11]. However, the approximation of multi-physics systems of mixed hyperbolic-parabolic type has rarely been investigated so far. In [19], the approximation of [11] by equal-order (i.e. non inf-sup stable) linear finite element methods in space and the Newmark finite difference approach in time is studied. Error estimates are proven for the second-order in time formulation of [11].

The coupling of (1.1a) and (1.1b) encounters new challenges for the error analysis of numerical schemes and shows a strong link to the mixed approximation by inf-sup stable pairs of finite elements of the Navier–Stokes system; cf. [32]. For this, we note that (1.1) yields a Stokes-type structure for the tuple \((\partial_t u, p)\) in the limit of vanishing coefficients \(c_0\) and \(K\) such that the well-known stability issues of mixed Stokes approximations emerge and argue either for inf-sup stable pairs of finite element spaces for \(u\) and \(p\) or for the stabilization of equal-order spatial discretizations. Here, we apply the first of the alternatives and use inf-sup stable pairs of finite element space for the spatial discretization.

For the approximation of the equations (1.1), rewritten as a first-order system in time with the additional variable \(v = \partial_t u\), by continuous finite element methods of piecewise polynomials of order \(k \geq 1\) in time and of order \(r \geq 1\) for \(p\) as well as of order \(r + 1\) for \(u\) and \(v\) in space we show in Thm. 4.8 that the discrete functions \(u_{\tau,h}, v_{\tau,h}\) and \(p_{\tau,h}\) satisfy the error inequality

\[
\max_{t \in [0,T]} \left\{ \| \nabla (u(t) - u_{\tau,h}(t)) \| + \| v(t) - v_{\tau,h}(t) \| + \| p(t) - p_{\tau,h}(t) \| \right\} \leq c(\tau^{k+1} + h^{r+1}).
\]  

(1.2)

The error estimate (1.2) is based on energy-type arguments applied to the coupled system (1.1). This bears out the quantities on left-hand side of (1.2) as the natural errors of the energy-type analysis. Thus, a control of the error in the energy quantity \(E(t) = (\| \nabla u(t) \|^2 + \| \partial_t u(t) \|^2)^{1/2}\) of the second-order hyperbolic equation and of the error in the magnitude \(\| p(t) \|\) of the unknown of the parabolic subproblem is obtained. Estimate (1.2) is of optimal order with respect to the error in the quantity \(E(t)\) and the pressure \(p\). A separation of the errors \(\| \nabla(u - u_{\tau,h}) \|\) and \(\| v - v_{\tau,h} \|\), offering the possibility to increase the spatial convergence order of \(\| v - v_{\tau,h} \|\) to \(h^{r+2}\), does not become feasible by our energy-type techniques, which is due to the fact that (1.1a) is rewritten as a first-order system in time. The resulting system demands for test functions that are adapted to its mixed structure of partial and ordinary differential equations and avoid terms in the error analysis that cannot be bounded from below by non-negative quantities. For a more refined discussion of the optimality of (1.2) we also refer to Rem. 4.9. A key feature of the proof of (1.2) is the appropriate definition of two interpolators in time for the vectorial unknowns \(u\) and \(v\). For this, we use the interpolators proposed in [34,35]. Moreover, the couplings terms in (1.3a) and (1.1b) are balanced carefully by the choice of suitable test functions along with the application of integration by parts for the time variable. Finally, by employing the techniques and results of [10], we obtain that for \(k \geq 2\) superconvergence in time at the discrete time nodes \(t_n\), given by

\[
\max_{n=1,\ldots,N} \left\{ \| \nabla(u(t_n) - u_{\tau,h}(t_n)) \| + \| v(t_n) - v_{\tau,h}(t_n) \| + \| p(t_n) - p_{\tau,h}(t_n) \| \right\} \leq c(\tau^{k+2} + h^{r+1}),
\]

(1.3)

is satisfied. The estimates (1.2) and (1.3) are illustrated by numerical experiments. The error analysis has the flexibility that is required to generalize the bounds (1.2) and (1.3) to enriched (or discontinuous) Galerkin approximations in space (cf. [35,40,62]) that we prefer for applications of practical interest.

This work is organized as follows. In Sec. 2 notations and auxiliary results are introduced. In Sec. 3 our continuous Galerkin approximation of (1.1) is presented. In Sec. 4 the error estimation is done and our main convergence result (1.2) is proven. In Sec. 5 superconvergence at the discrete time nodes for \(k \geq 2\) is reviewed. Finally, in Sec. 6 results of our numerical experiments are summarized.
2. Notations, finite element spaces and auxiliaries

2.1. Notations

In this work, standard notation is used. We denote by $H^m(\Omega)$ the Sobolev space of $L^2(\Omega)$ functions with weak derivatives up to order $m$ in $L^2(\Omega)$. We let $H^1_0(\Omega) = \{ u \in H^1(\Omega) \mid u = 0 \text{ on } \partial \Omega \}$. For short, we skip the domain $\Omega$ in the notation. Thus, we put $L^2 = L^2(\Omega)$, $H^m = H^m(\Omega)$ and $H^1_0 = H^1_0(\Omega)$. By $H^{-1} = H^{-1}(\Omega)$ we denote the dual space of $H^1_0$. For vector-valued functions we write those spaces bold. By $\langle \cdot, \cdot \rangle$ we define the $L^2$ inner product on the product space $(L^2)^2$. For the norms of the Sobolev spaces the notation is

$$\| \cdot \| := \| \cdot \|_{L^2}, \quad | \cdot |_m := \| \cdot \|_{H^m}, \quad \text{for } m \in \mathbb{N}_0, \quad (H^0 := L^2).$$

For a Banach space $B$ we let $L^2(0,T; B)$, $C([0,T]; B)$ and $C^m([0,T]; B)$, $m \in \mathbb{N}$, be the Bochner spaces of $B$-valued functions, equipped with their natural norms. Further, for a subinterval $J \subseteq [0,T]$, we will use the notations $L^2(J; B)$, $C^m(J; B)$ and $C^0(J; B) := C(J; B)$ for the corresponding Bochner spaces.

In what follows, the constant $c$ is generic and independent of the size of the space and time meshes. The value of $c$ can depend on norms of the solution to (1.1), the regularity of the space mesh, the polynomial degrees used for the space-time discretization and the data (including $\Omega$).

2.2. Finite element spaces

For the time discretization, we decompose the time interval $I = (0,T]$ into $N$ subintervals $I_n = (t_{n-1}, t_n]$, $n = 1, \ldots, N$, where $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ such that $I = \bigcup_{n=1}^N I_n$. We put $\tau := \max_{n=1,\ldots,N} \tau_n$ with $\tau_n = t_n - t_{n-1}$. Further, the set $\mathcal{M}_\tau := \{ I_1, \ldots, I_N \}$ of time intervals is called the time mesh. For a Banach space $B$ and any $k \in \mathbb{N}_0$, we let

$$\mathbb{P}_k(I_n; B) := \left\{ w_\tau : I_n \to B, \ w_\tau(t) = \sum_{j=0}^k W^j t^j \forall t \in I_n, \ W^j \in B \forall j \right\}.$$  \hfill (2.1)

For an integer $k \in \mathbb{N}$, we introduce the space

$$X^k_\tau(B) := \left\{ w_\tau \in C(\overline{I_n}; B) \mid w_\tau|_{I_n} \in \mathbb{P}_k(I_n; B) \forall I_n \in \mathcal{M}_\tau \right\}$$  \hfill (2.2)

of globally continuous in time functions and for an integer $l \in \mathbb{N}_0$ the space

$$Y^l_\tau(B) := \left\{ w_\tau \in L^2(I_n; B) \mid w_\tau|_{I_n} \in \mathbb{P}_l(I_n; B) \forall I_n \in \mathcal{M}_\tau \right\}$$  \hfill (2.3)

of global $L^2$-functions in time. For a function $w : I \to B$ that is piecewise continuous with respect to the time mesh $\mathcal{M}_\tau$, we define by

$$w(t^+_n) := \lim_{t \to t_n^+} w(t) \quad \text{and} \quad w(t^-_n) := \lim_{t \to t_n^-} w(t)$$

the one-sided limits of $w$. For brevity, we skip an upper index of the argument of $w$ in the second of the definitions, since by definition $I_n = (t_{n-1}, t_n]$ such that $w|_{I_n}(t_n)$ is well-defined.

For the space discretization, let $\mathcal{T}_h = \{ K \}$ be a family of shape-regular meshes of $\Omega$ consisting of quadrilateral or hexahedral elements $K$ with mesh size $h > 0$ that we use for our computations (cf. Sec. 6). Further, for any $r \in \mathbb{N}$ let $V^r_h$ be the finite element space that is built on the mesh of quadrilateral or hexahedral elements and is given by

$$V^r_h := \left\{ v_h \in C(\overline{\Omega}) \mid v_{hJK} \in Q_r(K) \forall K \in \mathcal{T}_h \right\} \cap H^1_0(\Omega),$$  \hfill (2.4)

where $Q_r(K)$ is the space defined by the reference mapping of polynomials on the reference element with maximum degree $r$ in each variable. For vector-valued functions we write the space bold.
2.3. **Auxiliaries: Quadrature formulas and interpolation operators in time**

We will need some quadrature formulas and interpolation and projection operators acting on the time variable. For the continuous finite element method time, a natural choice is to consider the \((k+1)\)-point Gauss–Lobatto quadrature formula on each time interval \(I_n = (t_{n-1}, t_n)\),

\[
Q_n(w) := \frac{\tau_n}{2} \sum_{\mu=0}^{k} \tilde{\omega}_\mu w|_{I_n}(t_{n,\mu}) \approx \int_{I_n} w(t) \, dt,
\]  

(2.5)

where \(t_{n,\mu} = T_n(\lambda_\mu)\), for \(\mu = 0, \ldots, k\), are the quadrature points on \(I_n\) and \(\tilde{\omega}_\mu\) the corresponding weights. Here, \(T_n(\lambda) := (t_{n-1}+t_n)/2 + (\tau_n/2)\) is the affine transformation from the reference interval \(\hat{I} = [-1, 1]\) to \(I_n\) and \(\lambda_\mu\), for \(\mu = 0, \ldots, k\), are the Gauss–Lobatto quadrature points on \(\hat{I}\). We note that for the Gauss–Lobatto formula the identities \(t_{n,0} = t_{n-1}\) and \(t_{n,k} = t_n\) are satisfied and that the values \(w|_{I_n}(t_{n,\mu})\) for \(\mu \in \{0, k\}\) denote the corresponding one-sided limits of values \(w(t)\) from the interior of \(I_n\) (cf. (2.2)). It is known that formula (2.5) is exact for all polynomials in \(\mathbb{P}_{2k-1}(I_n; \mathbb{R})\). For the Gauss–Lobatto quadrature points \(t_{n,\mu}\), with \(n = 1, \ldots, N\) and \(\mu = 0, \ldots, k\) we also define the global Lagrange interpolation operator \(I_\tau : C^0(\hat{T}; L^2) \to X^k(L^2)\) by means of

\[
I_\tau w(t_{n,\mu}) = w(t_{n,\mu}), \quad \mu = 0, \ldots, k, \quad n = 1, \ldots, N.
\]  

(2.6)

The \(k\)-point Gauss quadrature formula on \(I_n\) is denoted by

\[
Q^G_n(w) := \frac{\tau_n}{2} \sum_{\mu=1}^{k} \tilde{\omega}^G_\mu w(t_{n,\mu}) \approx \int_{I_n} w(t) \, dt,
\]  

(2.7)

where \(t^G_{n,\mu} = T_n(\lambda^G_\mu)\), for \(\mu = 1, \ldots, k\), are the Gauss quadrature points on \(I_n\) and \(\tilde{\omega}^G_\mu\) the corresponding weights, with \(\lambda^G_\mu\), for \(\mu = 1, \ldots, k\), being the Gauss quadrature points on \(\hat{I}\). Formula (2.7) is also exact for all polynomials in \(\mathbb{P}_{2k-1}(I_n; \mathbb{R})\). For \(n = 1, \ldots, N\), the local interpolant \(I^G_{\tau,n} : C^0(\hat{T}_n; L^2) \to \mathbb{P}_{k-1}(\hat{T}_n; L^2)\) is defined by means of

\[
I^G_{\tau,n} w(t^G_{n,\mu}) = w(t^G_{n,\mu}), \quad \mu = 1, \ldots, k.
\]  

(2.8)

Further, for a given function \(w \in L^2(I; B)\), we define the interpolate \(\Pi^{k-1}w \in \mathcal{Y}^{k-1}(B)\) such that its restriction \(\Pi^{k-1}w|_{I_n} \in \mathbb{P}_{k-1}(I_n; B)\), \(n = 1, \ldots, N\), is determined by local \(L^2\)-projection in time, i.e.

\[
\int_{I_n} (\Pi^{k-1}w - q) \, dt = \int_{I_n} (w - q) \, dt \quad \forall q \in \mathbb{P}_{k-1}(I_n; B).
\]  

(2.9)

**Remark 2.1** All operators, that act on the temporal variable only, are applied componentwise to a vector field \(F = (F_0, \ldots, F^d)\top\), for instance, \(I_\tau F = (I_\tau F_0, \ldots, I_\tau F_d)\top\). This is tacitly used below.

The following result (cf. [35 Eq. (2.6)] and [10 Lem. 4.5]) is proved easily.

**Lemma 2.2** Consider the Gauss quadrature formula (2.7). For all \(n = 1, \ldots, N\) there holds that

\[
\Pi^{k-1}_n w(t) = I^G_{\tau,n} w(t), \quad \forall t \in I_n,
\]  

(2.10a)

\[
\Pi^{k-1}_n w(t_{n,\mu}) = w(t^G_{n,\mu}), \quad \forall \mu = 1, \ldots, k,
\]  

(2.10b)

for all polynomials \(w \in \mathbb{P}_k(I_n; L^2)\).

Finally, we recall the following \(L^\infty\text{-}L^2\) inverse inequality; cf. [35 Eq. (2.5)].

**Lemma 2.3** For all \(n = 1, \ldots, N\) there holds that

\[
\|w\|_{L^\infty(I_n; \mathbb{R})} \leq c T_n^{-1/2} \|w\|_{L^2(I_n; \mathbb{R})}
\]  

(2.11)

for all polynomials \(w \in \mathbb{P}_k(I_n; \mathbb{R})\).
3. The fully discrete scheme and preparation for the error analysis

Here we propose our discretization of (1.1) by continuous finite element methods in time and space. For the discretization we rewrite Eq. (1.1a) as a first-order system in time (cf. (A.5)) such that time-discretizations for first-order systems of ordinary differential equations become applicable. This idea follows the proof of existence and uniqueness of solutions to (1.1) that is summarized in the appendix.

3.1. Bilinear forms and discrete operators

Here we introduce (bi-)linear forms for the discrete variational formulation and further operators related to the spatial discretization. For $u, v, \phi \in H_0^1$ and $p, \psi \in H_0^1$ we put

$$A(u, \phi) := \langle C\varepsilon(u), \varepsilon(\phi) \rangle, \quad B(p, \psi) := \langle K\nabla p, \nabla \psi \rangle, \quad C(v, \psi) := -\alpha \langle \nabla \cdot v, \psi \rangle,$$

$$F(\phi) := \langle \rho f, \phi \rangle, \quad G(\psi) := \langle g, \psi \rangle.$$

Firstly, we address the discretization of the hyperbolic equation (1.1a). By $L^2$-orthogonal projection onto $V^r_{h1}$ such that for $w \in L^2$, the identity

$$\langle P_h w, \phi_h \rangle = \langle w, \phi_h \rangle$$

is satisfied for all $\phi_h \in V^r_{h1}$. The operator $R_h : H_0^1 \rightarrow V^r_{h1}$ defines the elliptic projection onto $V^r_{h1}$ such that for $w \in H_0^1$, it holds that

$$\langle C\varepsilon(R_h w), \varepsilon(\phi_h) \rangle = \langle C\varepsilon(w), \varepsilon(\phi_h) \rangle \quad (3.1)$$

for all $\phi_h \in V^r_{h1}$. We let $A_h : H_0^1 \rightarrow V^r_{h1}$ be the discrete operator that is defined by

$$\langle A_h w, \phi_h \rangle = A(w, \phi_h) \quad (3.2)$$

for all $\phi_h \in V^r_{h1}$. Then, for $w \in H_0^1 \cap H^2$ it holds that

$$\langle A_h w, \phi_h \rangle = \langle C\varepsilon(w), \varepsilon(\phi_h) \rangle = \langle A w, \phi_h \rangle \quad (3.3)$$

for $\phi_h \in V^r_{h1}$, where $A : H_0^1 \rightarrow H^{-1}$ is defined by $\langle A w, \phi \rangle = A(w, \phi)$ for $\phi \in H_0^1$. Thus, $A_h w = P_h A w$ is satisfied for $w \in H_0^1 \cap H^2$.

Further, let $\mathcal{L}_h : H_0^1 \times L^2 \rightarrow V^r_{h1} \times V^r_{h1}$ be defined by

$$\mathcal{L}_h := \begin{pmatrix} 0 & -I \\ A_h & 0 \end{pmatrix}. \quad (3.4)$$

Then, for $U = (U_1, U_2) \in (H_0^1 \cap H^2) \times L^2$ we have that

$$\langle \mathcal{L}_h U, \Phi_h \rangle = \langle -U_2, \Phi_h^1 \rangle + \langle C\varepsilon(U_1), \varepsilon(\Phi_h^2) \rangle = \langle -U_2, \Phi_h^1 \rangle + \langle A U_1, \Phi_h^2 \rangle = \langle \mathcal{L} U, \Phi_h \rangle$$

for $\Phi_h = (\Phi_h^1, \Phi_h^2)^T \in V^r_{h1} \times V^r_{h1}$, where $\mathcal{L} : H_0^1 \times L^2 \rightarrow L^2 \times H^{-1}$, with $D(\mathcal{L}) = H_0^1 \times L^2$, is defined by

$$\mathcal{L} := \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix}. \quad (3.5)$$

Thus, the consistency of $\mathcal{L}_h$, $\mathcal{L}_h U = P_h \mathcal{L} U$, is satisfied on $(H_0^1 \cap H^2) \times L^2$.
Secondly, we address the discretization of the parabolic equation \([110]\). By \(P_h : L^2 \mapsto V_h^r\) we denote the \(L^2\)-orthogonal projection onto \(V_h^r\) such that, for \(w \in L^2\), the identity

\[
\langle P_h w, \psi_h \rangle = \langle w, \psi_h \rangle
\]

is satisfied for all \(\psi_h \in V_h^r\). The operator \(R_h : H_0^1 \mapsto V_h^r\) defines the elliptic projection onto \(V_h^r\) such that for \(w \in H_0^1\) it holds that

\[
\langle K \nabla R_h w, \nabla \psi_h \rangle = \langle K \nabla w, \nabla \psi_h \rangle
\]

for all \(\psi_h \in V_h^r\). Let \(B_h : H_0^1 \mapsto V_h^r\) be the discrete operator that is defined by

\[
\langle B_h w, \psi_h \rangle := B(w, \psi_h)
\]

for all \(\psi_h \in V_h\). Then, for \(w \in H_0^1 \cap H^2\) it holds that

\[
\langle B_h w, \psi_h \rangle = \langle K \nabla w, \nabla \psi_h \rangle = \langle B w, \psi_h \rangle
\]

for all \(\psi_h \in V_h^r\), where \(B : H_0^1 \mapsto H^{-1}\) is defined by \(\langle B w, \psi \rangle = B(w, \psi)\) for \(\psi \in H_0^1\). Thus, \(B_h w = P_h B w\) is satisfied for \(w \in H_0^1 \cap H^2\).

**Remark 3.1** We note that discrete functions of \(V_h^{r+1}\), with some \(r \in \mathbb{N}\), will be used for the approximation of the vectorial variable \(u\) of the hyperbolic equation and discrete functions of \(V_h^r\) for the approximation of the scalar variable \(p\) of the parabolic equation; cf. Subsec. 6.3. The projection and discrete differential operators are thus defined for finite element spaces of different polynomial degrees which is not shown explicitly by the notation of the operators.

### 3.2. Continuous Galerkin discretization

Here, we directly formulate our space-time finite element approximation of the system \([111]\). For the discretization in time, the continuous Galerkin method is applied; cf. \([8][10][11][22][55][53]\). Precisely, the time discretization is of Petrov–Galerkin type. For the discretization in space, a continuous finite element approach, based on inf-sup stable pairs of finite elements, is used as well. The latter is done in order to elucidate the key arguments of our error analysis and emphasize the control of the coupling mechanisms in the estimates. Generalizations of the error analysis to enriched or discontinuous Galerkin space discretizations that offer appreciable advantages over continuous ones appear feasible.

We make the following assumption about the discrete initial values \(u_{0,h}, v_{0,h} \in V_h^{r+1}\) and \(p_{0,h} \in V_h^r\).

**Assumption 3.2** Let \(u_{0,h}, v_{0,h} \in V_h^{r+1}\) and \(p_{0,h} \in V_h^r\) be chosen such the approximation properties

\[
\|\nabla (R_h u_0 - u_{0,h})\| \leq c h^{r+1} |u_0|_{r+2},
\]

\[
\|R_h u_1 - v_{0,h}\| \leq c h^{r+2} |u_1|_{r+2},
\]

\[
\|R_h p_0 - p_{0,h}\| \leq c h^{r+1} |p_0|_{r+1}
\]

are satisfied for initial values \(u_0, u_1 \in H_0^1 \cap H^{r+2}\) and \(p_0 \in H_0^1 \cap H^{r+1}\), where the elliptic projections \(R_h\) and \(R_h\) are defined by \([5.1]\) and \([5.6]\), respectively.

Here, we consider discretizing the system \([111]\) by the following sequence of variational problems.

**Problem 3.3** (Continuous Galerkin discretization: Variational form) Let \(k, r \geq 1\). For \(n = 1, \ldots, N\) and given \(u_{\tau,h,I_n^+}(t_{n-1}), v_{\tau,h,I_n^+}(t_{n-1}), p_{\tau,h,I_n^+}(t_{n-1})\) for \(n > 1\) and \(u_{\tau,h,I_n}(t_{n-1}) := u_{0,h}, v_{\tau,h,I_n}(t_{n-1}) := v_{0,h}, p_{\tau,h,I_n}(t_{n-1}) := p_{0,h}\) for \(n = 1\), find \(U_{\tau,h,I_n} = (u_{\tau,h,I_n}, v_{\tau,h,I_n})^\top \in (P_h(I_n; V_h^{r+1}))^d \times (P_h(I_n; V_h^r))^d\) satis...
By the exactness of the Gauss–Lobatto quadrature formula (2.5) for all polynomials in $P_2k-1(I_n;\mathbb{R})$ we can recover the variational problem (3.9) in the following numerically integrated form.

**Problem 3.5 (Continuous Galerkin discretization: Quadrature form)** Let $k,r \geq 1$. For $n = 1, \ldots, N$ and given $u_{\tau,h}|_{t_n}(t_{n-1}), v_{\tau,h}|_{t_n}(t_{n-1}), p_{\tau,h}|_{t_n}(t_{n-1})$ for $n > 1$ and $u_{\tau,h}|_{t_0}(t_0) := u_{0,0}$, $v_{\tau,h}|_{t_0}(t_0) := v_{0,0}$, $p_{\tau,h}|_{t_0}(t_0) := p_{0,0}$ for $n = 1$, find $u_{\tau,h}|_{t_n} \in (P_k(I_n;V_h^{r+1}))^d$, $v_{\tau,h}|_{t_n} \in (P_{k-1}(I_n;V_h^r))^d$, $p_{\tau,h}|_{t_n} \in P_{k-1}(I_n;V_h^r) \| \Phi_{\tau,h} \in \Phi_{\tau,h}^{r+1}$ such that $u_{\tau,h}|_{t_n}(t_n) = u_{\tau,h}|_{t_{n-1}}(t_{n-1})$, $v_{\tau,h}|_{t_n}(t_n) = v_{\tau,h}|_{t_{n-1}}(t_{n-1})$, $p_{\tau,h}|_{t_n}(t_n) = p_{\tau,h}|_{t_{n-1}}(t_{n-1})$ and

$$Q_n\left(\left\{\partial_t u_{\tau,h}, \Phi_{\tau,h}\right\} - \left\{v_{\tau,h}, \Phi_{\tau,h}\right\}\right) = 0,$$

$$Q_n\left(\left\{\partial_t v_{\tau,h}, X_{\tau,h}\right\} + A(u_{\tau,h}, X_{\tau,h}) + C(X_{\tau,h}, p_{\tau,h})\right) = Q_n\left(F(X_{\tau,h})\right),$$

$$Q_n\left(\left\{c_0 \partial_t p_{\tau,h}, \psi_{\tau,h}\right\} - C(\partial_t u_{\tau,h}, \psi_{\tau,h}) + B(p_{\tau,h}, \psi_{\tau,h})\right) = Q_n\left(G(\psi_{\tau,h})\right)$$

for $\Phi_{\tau,h} \in \Phi_{\tau,h}^{r+1}$, $X_{\tau,h} \in (P_{k-1}(I_n;V_h^r))^d$ and $\psi_{\tau,h} \in P_{k-1}(I_n;V_h^r)$.

**Remark 3.6**

- Problem 3.6 or 3.4, respectively, yields a continuous in time solution on the interval $[0,T]$ such that

$$\left(u_{\tau,h}, v_{\tau,h}, p_{\tau,h}\right) \in (X^k(V_h^{r+1}))^d \times (X^k(V_h^r))^d \times X^k(V_h^r).$$

- A non-equal order spatial approximation of the unknowns $u$ and $p$ in the spaces $V_h^{r+1}$ and $V_h^r$, corresponding to Taylor–Hood elements, is applied here. The inf-sup (or LBB) stability condition is satisfied by this choice of spaces; cf. [14]. For vanishing coefficients $c_0 \to 0$ and $K \to 0$, a Stokes-type system structure is obtained in (1.1) for the variables $\partial_t u$ and $p$ such that the well-known stability issues of mixed approximations of the Stokes system emerge in the limit case of vanishing $c_0$ and $K$; cf. [14]. Therefore, equal order spatial discretizations do not become feasible without any additional stabilization of the discretization. For a more detailed discussion of stability properties of (spatial) discretizations for the quasi-static Biot system we also refer to, e.g., [14], [14], [14].

- In Problem 3.6, the Gauss-Lobatto quadrature formula is applied. This allows an efficient implementation of the continuity constraints at the discrete time nodes $t_n$, for $n = 0, \ldots, N-1$, in computer codes (cf. [14], [14]), and thus, is the most natural approach for the continuous Galerkin approximation in time. In the error analysis, the Gauss quadrature formula (2.7), that is also exact for all polynomials in $P_{2k-1}(I_n;\mathbb{R})$, is used as well, for instance, to exploit property (2.10b).
3.3. Preparation for the error analysis

Here we present some auxiliaries that will used below in the error analysis. Firstly, we introduce some special approximation \( w = (w_1, w_2) \) of the solution \((u, v)\), with \( v = \partial_t u \), that has been defined in [35].

**Definition 3.7 (Special approximation \((w_1, w_2)\) of \((u, v)\))** Let \( \partial_t u \in C^1(\mathcal{T}, H^1_0) \) be given. On the interval \( I_n := (t_{n-1}, t_n) \), for \( n = 1, \ldots, N \), we define that

\[
  w_1 := I_t \left( \int_{t_{n-1}}^t w_2(s) \, ds + R_h u(t_{n-1}) \right), \quad \text{where} \quad w_2 := I_t (R_h \partial_t u).
\]

(3.11)

Further, we put \( w_1(0) := R_h u(0) \).

We note that in Def. 3.7 we simply write \( w_j \), for \( j = 1, 2 \), instead of \( w_{j|I_n} \), and that the Lagrange interpolation operator \( I_t \) based on the Gauss-Lobatto quadrature points (cf. (2.6)) will act locally on \( I_n \) as \( I_t : C^n(I_n; B) \rightarrow \mathbb{P}_k(I_n; B) \), where \( B = V_{h}^{r+1} \) or \( B = H^1_0 \). The approximations \( w_j \in (\mathbb{P}_k(I_n; V_{h}^{r+1}))^d \), for \( j = 1, 2 \), satisfy the following variational equation (cf. [35, Lem. 3.1]).

**Lemma 3.8** For the approximations \( w_j \), with \( j = 1, 2 \), defined in Def. 3.7 there holds that

\[
  \int_{I_n} \langle \partial_t w_1, \phi_{t,h} \rangle \, dt = \int_{I_n} \langle w_2, \phi_{t,h} \rangle \, dt
\]

(3.12)

for all \( \phi_{t,h} \in (\mathbb{P}_k(I_n; V_{h}^{r+1}))^d \).

Further, we need the following auxiliary result for the error analysis.

**Lemma 3.9** For \( y_{t,h}, z_{t,h} \in (\mathbb{P}_k(I_n; V_{h}^{r+1}))^d \) let

\[
  \int_{I_n} \langle \partial_t y_{t,h}, \phi_{t,h} \rangle - \langle z_{t,h}, \phi_{t,h} \rangle \, dt = 0
\]

(3.13)

be satisfied for all \( \phi_{t,h} \in (\mathbb{P}_k(I_n; V_{h}^{r+1}))^d \). Then, there holds that

\[
  \partial_t y_{t,h}(t_{n,\mu}^G) = z_{t,h}(t_{n,\mu}^G)
\]

(3.14)

for \( \mu = 1, \ldots, k \), where \( \{t_{n,\mu}^G\}_{\mu=1}^k \) are the Gauss quadrature nodes (cf. 2.7) of the subinterval \( I_n \).

**Proof.** Let \( l \in \{1, \ldots, k\} \) be arbitrary but fixed and \( \phi_{t,h} \in (\mathbb{P}_k(I_n; V_{h}^{r+1}))^d \) be chosen as

\[
  \phi_{t,h}(t) := \zeta_n(t) \phi_h \quad \text{with} \quad \zeta_n(t) := \sum_{\nu=1}^h (t - t_{n,\nu}^G) \in \mathbb{P}_k(I_n; \mathbb{R}), \quad \phi_h \in V_{h}^{r+1},
\]

and the Gauss quadrature nodes \( t_{n,\mu}^G \), for \( \mu = 1, \ldots, k \); cf. 2.7. By the exactness of the Gauss quadrature formula (2.7) for all polynomials in \( \mathbb{P}_{2k-1}(I_n; \mathbb{R}) \) we deduce from (3.13) that

\[
  0 = \int_{I_n} \langle \partial_t y_{t,h}, \phi_{t,h} \rangle - \langle z_{t,h}, \phi_{t,h} \rangle \, dt
\]

\[
  = \frac{\tau_n}{2} \sum_{\mu=1}^k \frac{\hat{w}_\mu}{2} \langle \partial_t y_{t,h}(t_{n,\mu}^G), \phi_{t,h}(t_{n,\mu}^G) \rangle - \langle z_{t,h}(t_{n,\mu}^G), \phi_{t,h}(t_{n,\mu}^G) \rangle
\]

\[
  = \frac{\tau_n}{2} \sum_{\mu=1}^k \frac{\hat{w}_\mu}{2} \zeta_n(t_{n,\mu}^G) \langle \partial_t y_{t,h}(t_{n,\mu}^G), \phi_h \rangle - \langle z_{t,h}(t_{n,\mu}^G), \phi_h \rangle.
\]

Thus, we have that

\[
  \langle \partial_t y_{t,h}(t_{n,l}^G) - z_{t,h}(t_{n,l}^G), \phi_h \rangle = 0
\]

(3.15)

for all \( \phi_h \in V_{h}^{r+1} \). Choosing \( \phi_h = \partial_t u_{t,h}(t_{n,l}^G) - v_{t,h}(t_{n,l}^G) \) in (3.15), proves the assertion (3.14). \( \blacksquare \)
4. Error analysis

Here we derive error estimates for the scheme (3.9) or (3.10), respectively. In (1.1), let \( v := \partial_t u \). Let \((w_1, w_2)^\top\) be given by Def. 3.7. We put \( U = (u, v)^\top \) and \( U_{\tau,h} = (u_{\tau,h}, v_{\tau,h})^\top \). We split the error by

\[
U - U_{\tau,h} = \begin{pmatrix} u - u_{\tau,h} \\ v - v_{\tau,h} \end{pmatrix} = \begin{pmatrix} u - w_1 \\ v - w_2 \end{pmatrix} + \begin{pmatrix} w_1 - u_{\tau,h} \\ w_2 - v_{\tau,h} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} E_{1,\tau,h} \\ E_{2,\tau,h} \end{pmatrix} = \eta + E_{\tau,h}
\]

and

\[
p - p_{\tau,h} = p - I_\tau R_h p + I_\tau R_h p - p_{\tau,h} =: \omega + e_{\tau,h}.
\]

For some quantity \( Z = (Z_1, Z_2)^\top \in H^1_0 \times L^2 \) we define the norm

\[
\|Z\| := (|\nabla Z_1|^2 + |Z_2|^2)^{1/2}
\]

and the weighted (elastic) energy norm

\[
\|Z\|_c := \left( \frac{1}{2} (C \varepsilon(Z_1), \varepsilon(Z_1)) + \frac{\rho}{2} (Z_2, Z_2) \right)^{1/2}.
\]

By Korn’s inequality (cf. [23]) along with the positive definiteness of \( C \) (cf. Appendix A.1), these norms are equivalent in sense that for \( Z = (Z_1, Z_2)^\top \in H^1_0 \times L^2 \) there holds that

\[
c_1 \|Z\| \leq \|Z\|_c \leq c_2 \|Z\|
\]

with some positive constants \( c_1 \) and \( c_2 \). Finally, for some scalar-valued function \( q \in L^2 \) we define the weighted \( L^2 \)-norm

\[
\|q\|_c := \left( \frac{C_0}{2} (q, q) \right)^{1/2}.
\]

We start with providing estimates for the projection errors \( \eta \) and \( \omega \) of (4.1) and (4.2), respectively. For the Lagrange interpolation (2.3), \( s \in \{2, \infty\} \) and \( m \in \{0, 1\} \) we recall that (cf. [27])

\[
\|f - I_{s,m} f\|_{L^s(I_n, H^m)} \leq c r_{s,m}^k \|f\|_{L^s(I_n, H^m)}.
\]

For the elliptic projections (3.1) and (3.6) onto \( V_h^a \) and \( V_h^{a-1} \), respectively, we have that (cf., e.g., [17])

\[
\|p - R_h p\| + h |\nabla (p - R_h p)| \leq c h^{r+1} |p|_{r+1},
\]

\[
|v - R_h v| + h |\nabla (v - R_h v)| \leq c h^{r+2} |v|_{r+2}.
\]

**Lemma 4.1 (Estimates of \( \eta \))** For \( \eta = (u - w_1, v - w_2)^\top \) with \( v = \partial_t u \) and \( (w_1, w_2)^\top \), given by Def. 3.7, there holds that

\[
\|u - w_1\|_{L^s(I_n, L^2)} \leq c (r_{s,1}^k c_{r,1} + h r_{s,2}^k c_{r,2}^1),
\]

\[
\|v - w_2\|_{L^s(I_n, L^2)} \leq c (r_{s,1}^k c_{r,2} + h r_{s,2}^k c_{r,2}^2),
\]

\[
\|u - w_1\|_{L^s(I_n, H^1)} \leq c (r_{s,1}^k c_{r,3} + h r_{s,1}^k c_{r,3}^1),
\]

\[
\|R_h v - w_2\|_{L^s(I_n, H^1)} \leq c (r_{s,1}^k c_{r,4} + h r_{s,2}^k c_{r,4}^2),
\]

where the constants are given by

\[
C_{1,s}^{n,1} := \|\partial_t^{k+1} u\|_{L^2(I_n, L^2)} + C_{1,s}^{n,2},
\]

\[
C_{1,s}^{n,3} := \|\partial_t^{k+1} u\|_{L^2(I_n, H^1)} + n C_{1,s}^{n,4},
\]

\[
C_{1,s}^{n,1} := \|u\|_{L^2(I_n, H^{r+2})} + n C_{1,s}^{n,3},
\]

\[
C_{1,s}^{n,4} := \|\partial_t u\|_{L^2(I_n, H^{r+2})} + n \|\partial_t^2 u\|_{L^2(I_n, H^{r+2})}.
\]
Proof. For scalar-valued functions, estimates \((4.9a)\) and \((4.9b)\) are proved in \([35, \text{Lem. 3.3}]\) and \((4.9d)\) in \([10, \text{Appendix}]\). The estimates \((4.9c)\) hold similarly in the vector-valued case of Def. \(8.7\) \(\blacksquare\)

Next, we derive variational equations satisfied by the discretization errors \(E_{τ,h}\) and \(e_{τ,h}\).

**Lemma 4.2 (Variational equations for \(E_{τ,h}\) and \(e_{τ,h}\))** Let

\[
\begin{align*}
T_{I}^{n}_{v} := I_{τ} \int_{t_{n-1}}^{t_{n}} \partial_{t} v - I_{τ} \partial_{t} u \, ds, & \quad T_{II}^{n} := ρ \partial_{t}^{2} u - ρ \partial_{t} w_{2}, & \quad T_{III}^{n} := I_{τ} u - u, \quad (4.10a) \\
T_{IV}^{n} := ρ f - I_{τ} (ρ f), & \quad T_{V}^{n} := g - I_{τ} g, \quad (4.10b)
\end{align*}
\]

where \(I_{τ}\) is the Lagrange interpolation operator satisfying \((2.6)\). Then, for \(n = 1, \ldots, N\) the errors \(E_{τ,h|_{I_{n}}}\) and \(e_{τ,h|_{I_{n}}}\) of \((4.11)\) and \((4.12)\), respectively, satisfy the equations

\[
\begin{align*}
\int_{I_{n}} \langle D \partial_{t} E_{τ,h}, \Phi_{τ,h} \rangle + \langle L_{h} E_{τ,h}, \Phi_{τ,h} \rangle \, dt & = - \alpha \int_{I_{n}} \langle e_{τ,h}, \nabla \cdot \Phi_{τ,h}^{2} \rangle \, dt \\
& = \int_{I_{n}} \langle T_{IV}^{n}, \Phi_{τ,h}^{2} \rangle \, dt - \int_{I_{n}} \langle A_{h} T_{I}^{n}, \Phi_{τ,h}^{2} \rangle \, dt - \int_{I_{n}} \langle T_{II}^{n}, \Phi_{τ,h}^{2} \rangle \, dt \\
& \quad + \int_{I_{n}} \langle A_{h} T_{III}^{n}, \Phi_{τ,h}^{2} \rangle \, dt + \alpha \int_{I_{n}} \langle \omega, \nabla \cdot \Phi_{τ,h}^{2} \rangle \, dt,
\end{align*}
\]

\[
\begin{align*}
\int_{I_{n}} c_{0} \langle \partial_{t} e_{τ,h}, \psi_{τ,h} \rangle & + \langle B_{h} e_{τ,h}, \psi_{τ,h} \rangle \, dt + \gamma \int_{I_{n}} \langle \nabla \cdot \partial_{t} E_{τ,h}, \psi_{τ,h} \rangle \, dt \\
& = \int_{I_{n}} \langle T_{V}^{n}, \psi_{τ,h} \rangle - \int_{I_{n}} d_{0} \langle \partial_{t} \omega, \psi_{τ,h} \rangle \, dt \\
& \quad - \alpha \int_{I_{n}} \langle \nabla \cdot \partial_{t} \eta_{1}, \psi_{τ,h} \rangle \, dt - \int_{I_{n}} \langle K \nabla (p - I_{τ p}), \psi_{τ,h} \rangle \, dt
\end{align*}
\]

for all \(\Phi_{τ,h} \in (P_{k-1}(I_{n}; V_{h}^{n+1}))^{d} \times (P_{k-1}(I_{n}; V_{h}^{n+1}))^{d}\) and \(\psi_{τ,h} \in P_{k-1}(I_{n}; V_{h}^{n})\).

**Proof.** Let \(v := \partial_{t} u\). Rewriting \((4.11a)\) as a first-order in time system, subtracting \((3.9)\) from the weak form of the resulting first-order-in-time continuous system and using the splitting \((4.11)\) and \((4.12)\) of the errors we get that

\[
\begin{align*}
\int_{I_{n}} \langle D \partial_{t} E_{τ,h}, \Phi_{τ,h} \rangle + \langle L_{h} E_{τ,h}, \Phi_{τ,h} \rangle & = - \alpha \langle e_{τ,h}, \nabla \cdot \Phi_{τ,h}^{2} \rangle \, dt \\
& = \int_{I_{n}} \langle F - I_{τ} F, \Phi_{τ,h} \rangle - \int_{I_{n}} \langle D \partial_{t} \eta_{1}, \Phi_{τ,h} \rangle + \langle L_{h} \eta_{1}, \Phi_{τ,h} \rangle - \alpha \langle \omega, \nabla \cdot \Phi_{τ,h}^{2} \rangle \, dt,
\end{align*}
\]

\[
\begin{align*}
\int_{I_{n}} c_{0} \langle \partial_{t} e_{τ,h}, \psi_{τ,h} \rangle & + \alpha \langle \nabla \cdot \partial_{t} E_{τ,h}, \psi_{τ,h} \rangle + \langle B_{h} e_{τ,h}, \psi_{τ,h} \rangle \, dt \\
& = \int_{I_{n}} \langle g - I_{τ} g, \psi_{τ,h} \rangle - \int_{I_{n}} c_{0} \langle \partial_{t} \omega, \psi_{τ,h} \rangle + \alpha \langle \nabla \cdot \partial_{t} \eta_{1}, \psi_{τ,h} \rangle + \langle B_{h} \omega, \psi_{τ,h} \rangle \, dt
\end{align*}
\]

for all \(\Phi_{τ,h} \in (P_{k-1}(I_{n}; V_{h}^{n+1}))^{d} \times (P_{k-1}(I_{n}; V_{h}^{n+1}))^{d}\) and \(\psi_{τ,h} \in P_{k-1}(I_{n}; V_{h}^{n})\).

Next, we rewrite some of the terms in \((4.12)\). First, from \((4.11)\) along with \((4.3)\) we find that

\[
\int_{I_{n}} \langle D \partial_{t} \eta_{1}, \Phi_{τ,h} \rangle + \langle L_{h} \eta_{1}, \Phi_{τ,h} \rangle \, dt = \int_{I_{n}} \langle \partial_{t} u - \partial_{t} w_{1} - v + w_{2}, \Phi_{τ,h}^{1} \rangle \, dt + \int_{I_{n}} \langle \rho \partial_{t} v - \rho \partial_{t} w_{2} + A_{h} (u - w_{1}), \Phi_{τ,h}^{2} \rangle \, dt.
\]

Recalling that \(v = \partial_{t} u\) and Lem. \(8.8\) we get for the first term on the right-hand side of \((4.13)\) that

\[
\int_{I_{n}} \langle \partial_{t} u - \partial_{t} w_{1} - v + w_{2}, \Phi_{τ,h}^{1} \rangle \, dt = 0
\]
for all \( \Phi_{\tau,h} \in (P_{k-1}(I_n; V_h^{r+1}))^d \). Let now

\[
  z(t) := \int_{t_{n-1}}^{t} w_2(s) \, ds + R_h u(t_{n-1}).
\]

(4.15)

Then, by definition we have that

\[
  w_{1|n} = I_{\tau} z.
\]

(4.16)

For the last term on the right-hand side of (4.11b) it holds by (3.6) that

Then, by definition we have that

\[
  \int_{I_n} \langle A_h w_1, \Phi_{\tau,h}^2 \rangle \, dt = \frac{\tau_n}{2} \sum_{\mu=0}^{k} \bar{\omega}_\mu \langle A_h z(t_{n,\mu}), \Phi_{\tau,h}^2(t_{n,\mu}) \rangle
\]

(4.17)

\[
  = \frac{\tau_n}{2} \sum_{\mu=0}^{k} \bar{\omega}_\mu \langle A_h \left( \int_{t_{n-1}}^{t_{n,\mu}} w_2(s) \, ds + R_h u(t_{n-1}) \right), \Phi_{\tau,h}^2(t_{n,\mu}) \rangle
\]

\[
  = - \int_{I_n} \{ A_h I_{\tau} \left( \int_{t_{n-1}}^{t} \partial_t u - I_{\tau} \partial_t u \, ds \right), \Phi_{\tau,h} \} \, dt + \int_{I_n} \langle A_h I_{\tau} u, \Phi_{\tau,h}^2 \rangle \, dt
\]

\[
  = - \int_{I_n} \{ A_h T_{I}^{\alpha}, \Phi_{\tau,h}^2 \} \, dt + \int_{I_n} \langle A_h u, \Phi_{\tau,h}^2 \rangle \, dt + \int_{I_n} \langle A_h T_{II}^{\alpha}, \Phi_{\tau,h}^2 \rangle \, dt
\]

(4.18)

with \( T_{I}^{\alpha} \) and \( T_{II}^{\alpha} \) being defined in (4.10a). Combining now (4.13) with (4.14) and (4.17) yields that

\[
  \int_{I_n} \langle D_{\partial_t \eta}, \Phi_{\tau,h} \rangle + \langle L_{h \eta}, \Phi_{\tau,h} \rangle \, dt = \int_{I_n} \langle A_h T_{I}^{\alpha}, \Phi_{\tau,h} \rangle \, dt
\]

\[
  + \int_{I_n} \langle T_{II}^{\alpha}, \Phi_{\tau,h} \rangle \, dt - \int_{I_n} \langle A_h T_{II}^{\alpha}, \Phi_{\tau,h} \rangle \, dt
\]

(4.18)

with \( T_{II}^{\alpha} \) being defined in (4.10a). Together, (4.12a) and (4.18) prove the assertion (4.11a).

For the last of the terms on the right-hand side of (4.11b) it holds by (3.6) that

\[
  \int_{I_n} \langle B_h \omega, \psi_{\tau,h} \rangle \, dt = \int_{I_n} \langle K \nabla(p - I_{\tau} R_h p), \nabla \psi_{\tau,h} \rangle \, dt
\]

\[
  = \int_{I_n} \langle K \nabla(p - I_{\tau} p), \nabla \psi_{\tau,h} \rangle \, dt + \int_{I_n} \langle K \nabla(I_{\tau} p - R_h I_{\tau} p), \nabla \psi_{\tau,h} \rangle \, dt
\]

(4.19)

\[
  = \int_{I_n} \langle K \nabla(p - I_{\tau} p), \nabla \psi_{\tau,h} \rangle \, dt
\]

for all \( \psi_{\tau,h} \in P_{k-1}(I_n; V_h^d) \). Together, (4.12b) and (4.19) prove the assertion (4.11b).

The following lemma provides estimates for the terms \( T_{I}, T_{II}^{\alpha}, T_{II}^{\alpha} \) and \( \partial_1 \eta_1 \) of (4.11).
Lemma 4.3 (Estimation of $T_i$, $T^0_i$, $T^1_{III}$, and $\partial_t \eta_1$) Let $n = 1, \ldots, N$. For $T_i^n$, $T^0_{III}$, and $T^1_{III}$, defined in (4.10a), and $\partial_t \eta_1$, defined in (4.1), there holds

$$\|AT^n_i\|_{L^2(I_n,L^2)} \leq c T^{k+1}_n \|A\partial_t^{k+1} u\|_{L^2(I_n,L^2)}, \quad \text{(4.20a)}$$

$$\int_{I_n} \langle T_i^n, \phi_{\tau,h} \rangle dt \leq c \left( \int_{I_n} \|\partial_t^{k+1} u\|_{L^2(I_n,L^2)} + h^{r+2} \|\partial_t^2 u\|_{L^2(I_n,H^{r+2})} \right) \|\phi_{\tau,h}\|_{L^2(I_n,L^2)}, \quad \text{(4.20b)}$$

$$\int_{I_n} \langle A_h T^0_{III}, \phi_{\tau,h} \rangle dt \leq c \|\partial_t^{k+1} u\|_{L^2(I_n,L^2)}, \quad \text{(4.20c)}$$

$$\int_{I_n} \langle \nabla \cdot \partial_t \eta_1, \psi_{\tau,h} \rangle dt \leq c \left( \int_{I_n} \|\partial_t^{k+1} u\|_{L^2(I_n,L^2)} + h^{r+1} \|\partial_t u\|_{L^2(I_n,H^{r+1})} \right) \|\psi_{\tau,h}\|_{L^2(I_n,L^2)}, \quad \text{(4.20d)}$$

$$\int_{I_n} \langle \partial_t \omega, \psi_{\tau,h} \rangle dt \leq c \left( \int_{I_n} \|\partial_t^{k+1} u\|_{L^2(I_n,L^2)} + h^{r+1} \|\partial_t p\|_{L^2(I_n,H^{r+1})} \right) \|\psi_{\tau,h}\|_{L^2(I_n,L^2)}, \quad \text{(4.20e)}$$

for $\phi_{\tau,h} \in (P_{k-1}(I_n; V_h^{r+1}))^d$ in (4.20b) and $\psi_{\tau,h} \in P_{k-1}(I_n; V_h^{r+1})$ in (4.20d).

**Proof.** The inequalities (4.20a) to (4.20c) can be proved along the lines of Lem. 3.3, Eqs. (3.12) to (3.14)] that are shown for scalar-valued functions. It remains to prove (4.20d) and (4.20e) for $\eta_1 = u - u_1$ and $\omega = p - I_r R_h p$. From the first of the definitions in (6.11) it follows that

$$\eta_1 = u - w_1 = u - I_r u + I_r u - I_r (R_h u) = I_r \int_{t_{n-1}}^{t} (w_2 - \partial_t R_h u) ds. \quad \text{(4.21)}$$

By (4.21) we then get that

$$\int_{I_n} \langle \nabla \cdot \partial_t \eta_1, \psi_{\tau,h} \rangle dt = \int_{I_n} \langle \nabla \cdot \partial_t (u - I_r u), \psi_{\tau,h} \rangle dt + \int_{I_n} \langle \nabla \cdot \partial_t I_r (u - R_h u), \psi_{\tau,h} \rangle dt$$

$$+ \int_{I_n} \langle \nabla \cdot \partial_t I_r \int_{t_{n-1}}^{t} (w_2 - \partial_t R_h u) ds, \psi_{\tau,h} \rangle dt =: \Gamma_1 + \Gamma_2 + \Gamma_3. \quad \text{(4.22)}$$

We start with estimating $\Gamma_1$. Firstly, let $k \geq 2$. Using integration by parts in time and recalling that the endpoints of $I_n$ are included in the set of Gauss–Lobatto quadrature points of $I_n$, we get that

$$\Gamma_1 = \int_{I_n} \langle \nabla \cdot \partial_t (u - I_r u), \psi_{\tau,h} \rangle dt = - \int_{I_n} \langle \nabla \cdot (u - I_r u), \partial_t \psi_{\tau,h} \rangle dt. \quad \text{(4.23)}$$

Let now $I^{k+1}_r$ denote the Lagrange interpolation operator at the $k + 2$ points of $T_n = \{t_{n-1}, t_n\}$ consisting of the $k + 1$ Gauss–Lobatto quadrature nodes $t_m$, $\mu = 0, \ldots, k$, and a further node in $(t_{n-1}, t_n)$ that is distinct from the previous ones. Then, $(I^{k+1}_r u) \partial_t \psi_{\tau,h}$ is a polynomial of degree $2k + 1$, such that

$$\int_{I_n} \langle \nabla \cdot (u - I_r u), \partial_t \psi_{\tau,h} \rangle dt = \int_{I_n} \langle \nabla \cdot (I^{k+1}_r u), \partial_t \psi_{\tau,h} \rangle dt. \quad \text{(4.24)}$$

Using integration by parts, the stability of the operator $I^{k+1}_r$ in the norm of $L^2(I_n; H^1)$, we have that

$$\|\Gamma_1\| \leq \int_{I_n} \langle \nabla \cdot \partial_t (u - I^{k+1}_r u), \psi_{\tau,h} \rangle dt$$

$$\leq \|\partial_t (u - I^{k+1}_r u)\|_{L^2(I_n; H^1)} \|\psi_{\tau,h}\|_{L^2(I_n,L^2)}$$

$$\leq c T^{k+1}_n \|\partial_t^{k+1} u\|_{L^2(I_n,L^2)} \|\psi_{\tau,h}\|_{L^2(I_n,L^2)} \quad \text{(4.24)}$$

For $k = 1$, we have that $\partial_t I_r u, \psi_{\tau,h} \in P_0(I_n; V_h^r)$ with $\partial_t I_r u = \{(u(t_n) - u(t_{n-1}))/\tau_n\}$. It follows that

$$\Gamma_1 = \langle \nabla \cdot \int_{I_n} (\partial_t u - \partial_t I_r u) dt, \psi_{\tau,h} \rangle = \langle \nabla \cdot (u(t_n) - u(t_{n-1}) - (u(t_n) - u(t_{n-1}))), \psi_{\tau,h} \rangle = 0. \quad \text{(4.24)}$$

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Next, we estimate $\Gamma_2$. For this we introduce the abbreviation $\xi := u - R_h u$. The Lagrange interpolant $I_T$ satisfies the stability results (cf. Eqs. (3.15) and (3.16))

$$
\|I_T w\|_{L^2(I_n; L^2)} \leq c \|w\|_{L^2(I_n; L^2)} + c T_n \|\partial_t w\|_{L^2(I_n; L^2)},
$$
(4.25a)

$$
\left\| \int_{I_{n-1}} w \, ds \right\|_{L^2(I_n; L^2)} \leq c T_n \|w\|_{L^2(I_n; L^2)}.
$$
(4.25b)

By the $H^1 - L^2$ inverse inequality $\|u\|_{{L^2(I_n; \mathbb{R})}} \leq c T_n^{-1} \|w\|_{L^2(I_n; \mathbb{R})}$, the stability results (4.25), the error estimate (4.8d) and viewing $\xi(t_{n-1}^*)$ as a function constant in time we find that

$$
|\Gamma_2| = \left| \int_{I_n} (\nabla \cdot \partial_t I_T \xi, \psi_{t,h}) \, dt \right|
= \left| \int_{I_n} (\nabla \cdot \partial_t I_T (\xi - \xi(t_{n-1}^*)), \psi_{t,h}) \, dt \right|
= \left| \int_{I_n} (\nabla \cdot \partial_t I_T \int_{I_{n-1}} \partial_t \xi \, ds, \psi_{t,h}) \, dt \right|
\leq c T_n^{-1} \|\int_{I_{n-1}} \nabla \cdot \partial_t \xi \, ds\|_{L^2(I_n; L^2)} \|\psi_{t,h}\|_{L^2(I_n; L^2)}
\leq c \|\partial_t u - R_h \partial_t u\|_{L^2(I_n; H^1)} \|\psi_{t,h}\|_{L^2(I_n; L^2)}
\leq c h^{r+1} \|\partial_t u\|_{L^2(I_n; H^{r+2})} \|\psi_{t,h}\|_{L^2(I_n; L^2)}.
$$
(4.26)

Finally, we estimate $\Gamma_3$. By the arguments of (4.26) it follows for $\Gamma_3$ that

$$
|\Gamma_3| = \left| \int_{I_n} (\nabla \cdot \partial_t I_T \int_{I_{n-1}} (w_2 - \partial_t R_h u) \, ds, \psi_{t,h}) \, dt \right|
\leq c \|w_2 - R_h \partial_t u\|_{L^2(I_n; H^1)} \|\psi_{t,h}\|_{L^2(I_n; L^2)}.
$$

Employing (4.9d) with $v = \partial_t u$, we obtain that

$$
|\Gamma_3| \leq c \left( T_n^{k+1} \|\partial_t u\|_{L^2(I_n; H^1)} + h^{r+1} \|\partial_t u\|_{L^2(I_n; H^{r+2})} \right) \|\psi_{t,h}\|_{L^2(I_n; L^2)}.
$$
(4.27)

Now, combining (4.22) with (4.24), (4.26) and (4.27) proves the assertion (4.20d). Estimate (4.20c) can be shown similarly to (4.20d) along the lines of (4.24) to (4.27).

Next, we prove a stability estimate for the error $\|E_{t,h}(t_n)\|_c^2 + \|e_{t,h}(t_n)\|_c^2$.

**Lemma 4.4 (Stability estimate)** Let $n = 1, \ldots, N$ and

$$
\delta_n := \langle \omega(t_n), \nabla \cdot \mathbf{E}_{t,h}(t_n) \rangle \quad \text{and} \quad \delta_{n-1} := \langle \omega(t_{n-1}^*), \nabla \cdot \mathbf{E}_{t,h}(t_{n-1}^*) \rangle,
$$
(4.28)

where the errors $E_{t,h}$, $e_{t,h}$ and $\omega$ are defined in (4.11) and (4.2), respectively. Then, there holds that

$$
\|E_{t,h}(t_n)\|_c^2 + \|e_{t,h}(t_n)\|_c^2 \leq \|E_{t,h}(t_{n-1}^*)\|_c^2 + \|e_{t,h}(t_{n-1}^*)\|_c^2 + \delta_n - \delta_{n-1}
+ c \|E_{t,h}\|_{L^2(I_n; L^2)}^2 + c \|e_{t,h}\|_{L^2(I_n; L^2)}^2
+ C_T r^{2(k+1)} (\mathcal{E}_{k+1}^1)^2 + c h^{2(r+1)} (\mathcal{E}_{r+1}^1)^2 + c h^{2(r+2)} (\mathcal{E}_{r+2}^1)^2
$$
(4.29)

with $\mathcal{E}_{k+1}^1 := \mathcal{E}_{k+1}^l + \mathcal{E}_{k+1}^r + \mathcal{E}_{k+1}^n + \mathcal{E}_F^T + \mathcal{E}_G^T + \mathcal{E}_W^T + \mathcal{E}_P^T$, $\mathcal{E}_x^1 := \mathcal{E}_{x,1}^l + \mathcal{E}_{x,1}^r + \mathcal{E}_{x,1}^n$, $\mathcal{E}_n^1 := \mathcal{E}_{n,1}^l + \mathcal{E}_{n,1}^r + \mathcal{E}_{n,1}^n$, where

$$
\mathcal{E}_{k+1}^l := \|\partial_t^k u\|_{L^2(I_n; H^2)}, \quad \mathcal{E}_{k+1}^r := \|\partial_t^{k+1} u\|_{L^2(I_n; L^2)}, \quad \mathcal{E}_{k+1}^n := \|\partial_t^{k+1} \mathbf{E}\|_{L^2(I_n; H^2)}, \quad \mathcal{E}_{x,1}^l := \|\partial_t^k u\|_{L^2(I_n; H^2)},
$$
$$
\mathcal{E}_{x,1}^r := \|\partial_t^{k+1} u\|_{L^2(I_n; L^2)}, \quad \mathcal{E}_x^l := \|\partial_t^k \mathbf{F}\|_{L^2(I_n; L^2)}, \quad \mathcal{E}_x^r := \|\partial_t^{k+1} \mathbf{F}\|_{L^2(I_n; L^2)},
$$
$$
\mathcal{E}_n^l := \|\partial_t^{k+2} \mathbf{E}\|_{L^2(I_n; H^2)}, \quad \mathcal{E}_n^r := \|\partial_t^{k+2} \mathbf{E}\|_{L^2(I_n; L^2)}, \quad \mathcal{E}_n^n := \|\partial_t^{k+2} \mathbf{E}\|_{L^2(I_n; H^2)}.
$$
Proof. In (4.11), we choose the test functions

$$\Phi_{\tau,h} = \begin{pmatrix} \Pi_{k-1}^{G} & 0 \\ 0 & \Pi_{k}^{G} \end{pmatrix} \begin{pmatrix} A_h E_{\tau,h}^1 \\ 0 \end{pmatrix}$$

and

$$\psi_{\tau,h} = \Pi_{k-1}^{G} e_{\tau,h}.$$ (4.30)

Firstly, we address some of the terms in (4.11a) for the test function $\Phi_{\tau,h}$ of (4.30). By the exactness of the Gauss quadrature formula (2.7) for all polynomials in $P_{k-1}$ and the symmetry of $\Lambda$, we deduce that

$$\int_{I_n} \left\{ \frac{I_d}{0} \rho \lambda_d \right\} \left( \Pi_{k-1}^{G} A_h E_{\tau,h}^1 \right) \left( \Pi_{k}^{G} E_{\tau,h}^2 \right) dt = \frac{\tau_n}{2} \sum_{\mu=1}^{k} \omega_{G} \left( \Pi_{k-1}^{G} A_h E_{\tau,h}^1 \left( t_{n,\mu}^G \right) \right) \left( \Pi_{k}^{G} E_{\tau,h}^2 \left( t_{n,\mu}^G \right) \right)$$

$$= \int_{I_n} \left\{ \frac{\partial_t E_{\tau,h}^1 (\tau, u)}{\partial_t E_{\tau,h}^2 (\tau, u)} A_h \right\} dt$$

$$= \left\| \left( E_{\tau,h} (t_n) \right) \right\|_{2}^{2} - \left\| E_{\tau,h} (t_{n-1}) \right\|_{2}^{2}.$$ (4.31)

Further, by (3.3), the exactness of the Gauss quadrature formula (2.7) for all polynomials in $P_{2k-1}$, Lem. (2.2) and the symmetry of $A_h$ we have that

$$\int_{I_n} \left\{ - E_{\tau,h}^1 \right\} \left( \Pi_{k-1}^{G} A_h E_{\tau,h}^1 \right) \left( \Pi_{k}^{G} E_{\tau,h}^2 \right) dt = \frac{\tau_n}{2} \sum_{\mu=1}^{k} \omega_{G} \left( - E_{\tau,h}^1 \left( t_{n,\mu}^G \right) \right) \left( A_h E_{\tau,h}^1 \left( t_{n,\mu}^G \right) \right) = 0.$$ (4.32)

Now, we recall (4.31) defining $(E_{\tau,h}^1, E_{\tau,h}^2)$. The pair $(w_1, w_2)$ satisfies (3.12) and $(u_{\tau,h}, v_{\tau,h})$ fulfills the first of the identities in (3.3a) or (4.10a), respectively. Therefore, Lem. (5.9) can be applied to $(w_1, w_2)$ and $(u_{\tau,h}, v_{\tau,h})$ and the conclusion (5.14) holds for both pairs of functions. This implies that

$$(E_{\tau,h}^1 (t_{n,\mu}^G), E_{\tau,h}^2 (t_{n,\mu}^G)) = (w_2 (t_{n,\mu}^G), v_{\tau,h} (t_{n,\mu}^G)) = \partial_t w_1 (t_{n,\mu}^G) - \partial_t u_{\tau,h} (t_{n,\mu}^G) = \partial_t E_{\tau,h}^1 (t_{n,\mu}^G)$$ (4.33)

for $\mu = 1, \ldots, k$. Using this along with (4.31), it follows that

$$\int_{I_n} \left\{ e_{\tau,h}, \nabla \cdot \Pi_{k-1}^{G} E_{\tau,h}^2 \right\} dt = \frac{\tau_n}{2} \sum_{\mu=1}^{k} \omega_{G} e_{\tau,h} (t_{n,\mu}^G), \nabla \cdot E_{\tau,h}^2 (t_{n,\mu}^G))$$

$$= \frac{\tau_n}{2} \sum_{\mu=1}^{k} \omega_{G} e_{\tau,h} (t_{n,\mu}^G), \nabla \cdot E_{\tau,h}^2 (t_{n,\mu}^G)) = \int_{I_n} \left\{ e_{\tau,h}, \nabla \cdot E_{\tau,h}^1 \right\} dt.$$ (4.34)

By the same arguments and using that $\partial_t E_{\tau,h}^1 (P_{k-1} (I_n; V_{h}^{1/r}))$, we have that

$$\int_{I_n} \left\{ \omega, \nabla \cdot \Pi_{k-1}^{G} E_{\tau,h}^2 \right\} dt \int_{I_n} \left\{ \Pi_{k-1}^{G} \omega, \nabla \cdot \Pi_{k-1}^{G} E_{\tau,h}^2 \right\} dt = \frac{\tau_n}{2} \sum_{\mu=1}^{k} \omega_{G} \left( \Pi_{k-1}^{G} \omega (t_{n,\mu}^G), \nabla \cdot E_{\tau,h}^2 (t_{n,\mu}^G) \right)$$

$$= \int_{I_n} \left\{ \Pi_{k-1}^{G} \omega, \nabla \cdot E_{\tau,h}^1 \right\} dt = \int_{I_n} \left\{ \omega, \nabla \cdot E_{\tau,h}^1 \right\} dt.$$ (4.35)

Applying integration by parts (for the time variable) to the last term in (4.32), we get that

$$\int_{I_n} \left\{ \omega, \nabla \cdot \Pi_{k-1}^{G} E_{\tau,h}^2 \right\} dt = - \int_{I_n} \left\{ \partial_t \omega, \nabla \cdot E_{\tau,h}^1 \right\} dt$$

$$+ \left( \omega (t_n), \nabla \cdot E_{\tau,h}^1 (t_n) \right) - \left( \omega (t_{n-1}), \nabla \cdot E_{\tau,h}^1 (t_{n-1}) \right).$$ (4.36)
Secondly, we address some terms of (4.11b) for $\psi_{\tau,h}$ given by (4.30). Similarly to (4.31), we get that
\[
c_0 \int_{I_n} \langle \partial_t e_{\tau,h}, \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt = c_0 \frac{\tau_n}{2} \sum_{\mu=1}^{k} \langle \partial_t e_{\tau,h}(t_{n,\mu}^0), e_{\tau,h}(t_{n,\mu}^0) \rangle
\]
\[
= c_0 \int_{I_n} \langle \partial_t e_{\tau,h}, e_{\tau,h} \rangle \,dt = \|e_{\tau,h}(t_n)\|_c^2 - \|e_{\tau,h}(t_{n-1})\|_c^2.
\]
(4.37)

Further, it holds that
\[
\int_{I_n} \langle B_h e_{\tau,h}, \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt = \int_{I_n} \langle B_h \Pi_{\tau}^{k-1} e_{\tau,h}, \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt.
\]
(4.38)

Now, adding the equations (4.11a) and (4.11b) for the test functions (4.30) and using (4.31) to (4.38) we obtain that
\[
\|E_{\tau,h}(t_n)\|_c^2 + \|e_{\tau,h}(t_n)\|_c^2 + \int_{I_n} \langle B_h \Pi_{\tau}^{k-1} e_{\tau,h}, \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt = \|E_{\tau,h}(t_{n-1})\|_c^2 + \|e_{\tau,h}(t_{n-1})\|_c^2
\]
\[
+ \alpha \langle (\omega(t_n), \nabla \cdot \Pi_{\tau}^1 e_{\tau,h}(t_n)) - (\omega(t_{n-1}), \nabla \cdot \Pi_{\tau}^1 e_{\tau,h}(t_{n-1})) \rangle
\]
\[
+ \int_{I_n} \langle T_{IV}^n, \Pi_{\tau}^{k-1} E_{\tau,h}^2 \rangle \,dt - \int_{I_n} \langle A_h T_{IIV}^n, \Pi_{\tau}^{k-1} E_{\tau,h}^2 \rangle \,dt - \int_{I_n} \langle T_{IV}^n, \Pi_{\tau}^{k-1} E_{\tau,h}^2 \rangle \,dt
\]
(4.39)
\[
+ \int_{I_n} \langle A_h T_{IIIV}^n, \Pi_{\tau}^{k-1} E_{\tau,h}^2 \rangle \,dt - \alpha \int_{I_n} \langle \partial_t \omega \cdot \nabla \cdot \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt + \int_{I_n} \langle T_{IV}^n, \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt
\]
\[
- c_0 \int_{I_n} \langle \partial_t \omega, \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt - \alpha \int_{I_n} \langle \nabla \cdot \partial_t \eta_{1}, \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt - \int_{I_n} \langle K \nabla(p - I_t p), \Pi_{\tau}^{k-1} e_{\tau,h} \rangle \,dt.
\]

By the assumption (A.2) of the positive-definiteness of $K$, the inequalities of Cauchy–Schwarz and Cauchy–Young, identity (3.3) and integration by parts, to the last of the terms in (4.39), we conclude from (4.39) that
\[
\|E_{\tau,h}(t_n)\|_c^2 + \|e_{\tau,h}(t_n)\|_c^2 + \int_{I_n} \|\nabla \Pi_{\tau}^{k-1} e_{\tau,h}\|_c^2 \,dt \leq \|E_{\tau,h}(t_{n-1})\|_c^2 + \|e_{\tau,h}(t_{n-1})\|_c^2
\]
\[
+ \delta_n - \delta_{n-1} + c \|E_{\tau,h}\|_c^2 \|L^2(I_n;L^2)\|^2 + c \|e_{\tau,h}\|_c^2 \|L^2(I_n;L^2)\|^2 + c \|A T_{IIV}^n\|_c^2 \|L^2(I_n;L^2)\|^2
\]
\[
+ c \int_{I_n} \|T_{IIV}^n\|_c^2 \|L^2(I_n;L^2)\|^2 + c \|T_{IV}^n\|_c^2 \|L^2(I_n;L^2)\|^2 + c \|T_{IV}^n\|_c^2 \|L^2(I_n;L^2)\|^2
\]
(4.40)

where $\delta_n$ and $\delta_{n-1}$ are defined in (4.28). Combining (4.40) with Lem. 4.3 and the bounds (4.7) and (4.8a) proves the assertion (4.29) of this lemma. ■

Next we estimate the right-hand side term $\|E_{\tau,h}\|_{L^2(I_n;L^2)} + \|e_{\tau,h}\|_{L^2(I_n;L^2)}^2$ in (4.26).

**Lemma 4.5 (Estimate of $\|E_{\tau,h}\|_{L^2(I_n;L^2)}^2 + \|e_{\tau,h}\|_{L^2(I_n;L^2)}^2$)** Let $n = 1, \ldots, N$. For the errors $E_{\tau,h}$ and $e_{\tau,h}$, defined in (4.11) and (4.12), there holds that
\[
\|E_{\tau,h}\|_{L^2(I_n;L^2)}^2 + \|e_{\tau,h}\|_{L^2(I_n;L^2)}^2 \leq c T_n \left( \|E_{\tau,h}(t_{n-1})\|_{L^2(I_n;L^2)}^2 + \|e_{\tau,h}(t_{n-1})\|_{L^2(I_n;L^2)}^2 \right)
\]
\[
+ c r_n \left( \sum_{i=1}^{2(k+1)} \left( c_{\xi_{i}}^1 \right)^2 + h^2(r+1) \left( E_{\xi_{1}}^{0,1} + E_{\xi_{2}}^{0,2} \right)^2 + h^2(r+2) \left( E_{\xi_{3}}^{0,3} \right)^2 \right),
\]
(4.41)

where $E_{\xi_{1}}^{0,1}$, $E_{\xi_{2}}^{0,1}$ and $E_{\xi_{3}}^{0,2}$ are defined in Lem. 4.4 and $E_{\xi_{3}}^{0,3} := \|p\|_{L^\infty(I_n;H^{r+1})}$.  

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Proof. Firstly, we consider (4.11a). Let $E_{\tau,h} = (E_{\tau,h}^1, E_{\tau,h}^2)^T$, defined in (4.1), be represented by

$$E_{\tau,h}^m(t) = \sum_{j=0}^{k} E_{n,j}^m \phi_{n,j}(t), \quad \text{for } t \in I_n, \ m \in \{1,2\},$$

(4.42)

where $E_{n,j}^m \in V_{h}^{r+1}$, for $j = 0, \ldots, k$, and $\phi_{n,j} \in P_k(I_n; \mathbb{R})$, for $j = 0, \ldots, k$, are the Lagrange interpolants with respect to $t_{n-1}$ and the Gauss quadrature nodes $t_{n,1}^G, \ldots, t_{n,k}^G \in (t_{n-1}, t_n)$ of (2.7). Then, it holds that $E_{n,0}^m = E_{\tau,h}^m(t_{n-1}^-)$. In (4.11a), we choose the test function

$$\Phi_{\tau,h}(t) = \sum_{i=1}^{k} (i_i^G)^{-1/2} \left( A_h \tilde{E}_{n,i}^1 \right) \psi_{n,i}(t),$$

(4.43)

where $\tilde{E}_{n,i}^m = (i_i^G)^{-1/2} E_{n,i}^m$, for $m \in \{1,2\}$ and $i = 1, \ldots, k$, and $\psi_{n,i} \in P_{k-1}(I_n; \mathbb{R})$, for $i = 1, \ldots, k$, are the Lagrange interpolants with respect to the Gauss quadrature nodes $t_{n,1}^G, \ldots, t_{n,k}^G \in (t_{n-1}, t_n)$ of (2.7). In (4.43), the quantities $i_i^G$, for $i = 1, \ldots, k$, denote the quadrature nodes of the Gauss formula (2.7) on the reference interval $J$. Using the evaluation (4.42), for the test function (4.43) it follows that

$$\int_{I_n} \left\| \begin{pmatrix} 0 & -I \\ A_h & 0 \end{pmatrix} \left( E_{\tau,h}^1, E_{\tau,h}^2 \right) \right\| \left( \Phi_{\tau,h}^1 \right) dt$$

$$= \int_{I_n} \left\| \begin{pmatrix} 0 & -I \\ A_h & 0 \end{pmatrix} \left( E_{\tau,h}^1, E_{\tau,h}^2 \right), \sum_{i=1}^{k} (i_i^G)^{-1/2} \left( A_h \tilde{E}_{n,i}^1 \right) \psi_{n,i} \right\| dt$$

$$= \frac{\tau_n}{2} \sum_{\mu=1}^{k} \omega_\mu \left( i_\mu^G \right)^{-1} \left( -E_{\tau,h}^1, \psi_{n,i} \right)$$

(4.44)

where the symmetry of $A_h$ has been used in the last identity. By the expansion (4.43) along with the observation that $E_{\tau,h}^m(t_{n-1}^-) = E_{n,0}^m$, for $m \in \{1,2\}$, we have that

$$Q_n := \int_{I_n} \left\| \begin{pmatrix} I_d & 0 \\ 0 & \rho I_d \end{pmatrix} \left( \partial_t E_{\tau,h}^1, \partial_t E_{\tau,h}^2 \right) \right\| \left( \Phi_{\tau,h}^1 \right) dt$$

$$= \int_{I_n} \left\| \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \left( \partial_t E_{\tau,h}^1, \partial_t E_{\tau,h}^2 \right), \sum_{i=1}^{k} (i_i^G)^{-1/2} \left( A_h \tilde{E}_{n,i}^1 \right) \psi_{n,i} \right\| dt$$

(4.45)

$$= \sum_{i,j=1}^{k} \tilde{m}_{ij} \left( (A_h \tilde{E}_{n,j}^1, E_{n,i}) + \rho (E_{n,j}^2, E_{n,i}) \right)$$

$$+ \sum_{i=1}^{k} m_0 (i_i^G)^{-1/2} \left( (A_h \tilde{E}_{n,i}^1, E_{n,i}) + \rho (E_{n,i}^2, \tilde{E}_{n,i}^1) \right).$$

where the matrix $M = (m_{ij})_{i,j=1,...,k}$ and vector $m_0 = (m_0)_{i=1,...,k}$ are defined by

$$m_{ij} := \int_{I_n} \phi'_{n,j}(t) \psi_{n,i}(t) dt, \quad \text{for } i \in \{1, \ldots, k\}, \ j \in \{1, \ldots, k\},$$

$$m_{0i} := \int_{I_n} \phi'_{n,0}(t) \psi_{n,i}(t) dt, \quad \text{for } i \in \{1, \ldots, k\},$$

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and the matrix $\overline{M} = (\overline{m}_{ij})_{i,j=1,...,k}$ is given by
\[ \overline{M} := D^{-1/2} MD^{1/2} , \quad \text{with} \quad D = \text{diag}(\tilde{r}_1^G, \ldots, \tilde{r}_k^G). \]

By the positivity of $\overline{M}$ (cf. [35, Lem. 2.1]) we then have that
\[ Q_n \geq c \sum_{j=1}^k \|\overline{E}_{n,j}\|^2 - c \left( \sum_{j=1}^k \|\overline{E}_{n,j}\|^2 \right)^{1/2} \|E_{\tau,h}(t_{n-1}^\star)\| \geq c \sum_{j=1}^k \|\overline{E}_{n,j}\|^2 - c \|E_{\tau,h}(t_{n-1}^\star)\|^2 . \] (4.46)

By the equivalence of $\sum_{j=1}^k \|\overline{E}_{n,j}\|$ and $\sum_{j=1}^k \|E_{n,j}\|$ along with the equivalence (cf. [35, Eq. (2.4)])
\[ c_1 \tau_n \sum_{j=0}^k \|E_{n,j}\|^2 \leq \|E_{\tau,h}\|^2_{L^2(I_n,L^2)} \leq c_2 \tau_n \sum_{j=0}^k \|E_{n,j}\|^2 , \]
we conclude from (4.45) to (4.47) that
\[ \tau_n Q_n \geq c \|E_{\tau,h}\|^2_{L^2(I_n,L^2)} - c \tau_n \|E_{\tau,h}(t_{n-1}^\star)\|^2 . \] (4.48)

Next, we address the last term on the left-hand side of (4.11a) for the test function $\psi_{n,i}$. Similarly to (4.12), for the error $e_{\tau,h}$ we use the representation
\[ e_{\tau,h}(t) = \sum_{j=0}^k e_{n,j} \psi_{n,j}(t), \quad \text{for} \ t \in I_n , \]
where $e_{n,j} \in V_h^n$, for $j = 0, \ldots, k$. Further we put $\tilde{c}_{n,i} := (\tilde{r}_i^G)^{-1/2} e_{n,i}$, for $i = 1, \ldots, k$. Using (4.43) and (4.49) along with (4.33) and recalling that $\psi_{n,i} \in P_{k-1}(I_n; \mathbb{R})$ in (4.33), it follows that
\[ \int_{I_n} \left( e_{\tau,h}, \nabla \cdot \Phi_{\tau,h}^2 \right) dt = \int_{I_n} \left( e_{\tau,h}, \nabla \cdot \sum_{i=1}^k (\tilde{r}_i^G)^{-1/2} E_{n,i}^2 \psi_{n,i} \right) dt \\
= \frac{\tau_n}{2} \sum_{\mu=1}^k \omega_{\mu}^G \left( e_{\tau,h}(t_{n,\mu}), \nabla \cdot \sum_{i=1}^k (\tilde{r}_i^G)^{-1/2} E_{n,i}^2 \psi_{n,i}(t_{n,\mu}) \right) \\
= \frac{\tau_n}{2} \sum_{\mu=1}^k \omega_{\mu}^G (\tilde{r}_i^G)^{-1} (e_{n,\mu}, \nabla \cdot E_{\tau,h}^2(t_{n,\mu})) \\
= \frac{\tau_n}{2} \sum_{\mu=1}^k \omega_{\mu}^G (\tilde{r}_i^G)^{-1} (e_{n,\mu}, \nabla \cdot \partial_t E_{\tau,h}^1(t_{n,\mu})) \\
= \int_{I_n} \left( \nabla \cdot \partial_t E_{\tau,h}^1, \sum_{i=1}^k (\tilde{r}_i^G)^{-1/2} \psi_{n,i} \right) dt . \] (4.50)

Finally, we address the last term on the right-hand side of (4.11a) with (4.43). Similarly to (4.50), using (4.43) with $\psi_{n,i} \in P_{k-1}(I_n; \mathbb{R})$ and employing (4.33), we find that
\[ R_n := \int_{I_n} \left( \omega, \nabla \cdot \Phi_{\tau,h}^2 \right) dt = \int_{I_n} \left( \Pi_{k-1}^-, \omega, \nabla \cdot \sum_{i=1}^k (\tilde{r}_i^G)^{-1/2} E_{n,i}^2 \psi_{n,i} \right) dt \\
= \frac{\tau_n}{2} \sum_{\mu=1}^k \omega_{\mu}^G (\tilde{r}_i^G)^{-1} (\Pi_{k-1}^-, \omega(t_{n,\mu}), \nabla \cdot E_{\tau,h}^2(t_{n,\mu})) \\
= \frac{\tau_n}{2} \sum_{\mu=1}^k \omega_{\mu}^G \left( \sum_{i=1}^k (\tilde{r}_i^G)^{-1} \Pi_{k-1}^-, \omega(t_{n,i}), \psi_{n,i}(t_{n,\mu}), \nabla \cdot \partial_t E_{\tau,h}^1(t_{n,\mu}) \right) \\
= \int_{I_n} \left( \sum_{i=1}^k (\tilde{r}_i^G)^{-1} \Pi_{k-1}^-, \omega(t_{n,i}), \nabla \cdot \partial_t E_{\tau,h}^1 \right) dt . \] (4.51)
From (4.51) along with \(\int_{\Omega} \phi_{n,i}^2 \, dt \leq c_{\tau,n} \) and the inequality of Cauchy–Young we get that
\[
\tau_n R_n \leq c_{\tau,n} \max_{i=1,\ldots,k} \left\{ \| \Pi_{\tau}^{-1} \omega(t_{n,i}^G) \|_{L^2(I_n,\mathbb{R})} \right\} \| \partial_t \nabla \cdot \mathbf{E}_{\tau,h}^1 \|_{L^2(I_n,\mathbb{R})}
\]
\[
\leq c_{\tau,n} \max_{i=1,\ldots,k} \left\{ \| \Pi_{\tau}^{-1} \omega(t_{n,i}^G) \|_{L^2(I_n,\mathbb{R})} \right\} ^2 + \varepsilon c_{\tau,n}^2 \| \partial_t \nabla \cdot \mathbf{E}_{\tau,h}^1 \|_{L^2(I_n,\mathbb{R})}^2
\]
with a sufficiently small constant \(\varepsilon > 0\). The \(L^\infty - L^2\) inverse relation (2.11), the error estimate (4.84) for the elliptic projection \(R_h\) in \(\omega = p - R_h p\) and the \(H^1 - L^2\) inverse inequality then imply that
\[
\tau_n R_n \leq \varepsilon \| \Pi_{\tau}^{-1} w \|_{L^2(I_n,\mathbb{R})}^2 + \varepsilon \| \mathbf{E}_{\tau,h} \|_{L^2(I_n,\mathbb{R})}^2
\]
\[
\leq c_{\tau,n} \| w \|_{L^\infty(I_n,\mathbb{R})}^2 + \varepsilon \| \mathbf{E}_{\tau,h} \|_{L^2(I_n,\mathbb{R})}^2
\]
\[
\leq c_{\tau,n} h^{2(r+1)} \| p \|_{L^2(I_n,\mathbb{H}^{r-1})}^2 + \varepsilon \| \mathbf{E}_{\tau,h} \|_{L^2(I_n,\mathbb{R})}^2. \tag{4.52}
\]
For a suitable choice of \(\varepsilon\), the second term on right-hand side of (4.52) can be absorbed by the left-hand side of (1.11). The remaining terms on the right-hand side of (1.11) can be treated as before in Lem. 4.4.

Now, we consider (4.11b). We choose the test function
\[
\psi_{\tau,h}(t) = \sum_{i=1}^k (t_i^G)^{-1/2} \bar{c}_{n,i} \psi_{n,i}(t). \tag{4.53}
\]
By arguments similarly to (4.45) to (4.48) and with (4.49), we then have that
\[
S_n := \int_{t_n} \left( \partial_t e_{\tau,h} \psi_{\tau,h} \right) \, dt = \int_{t_n} \left( \partial_t e_{\tau,h} \sum_{i=1}^k (t_i^G)^{-1/2} \bar{c}_{n,i} \psi_{n,i} \right) \, dt
\]
\[
= \frac{\tau_n}{2} \sum_{i=1}^k \bar{\omega}_\mu \left( \partial_t e_{\tau,h}(t_{n,\mu}^G) \right) \sum_{i=1}^k (t_i^G)^{-1/2} \bar{c}_{n,i} \psi_{n,i}(t_{n,\mu}^G) \, dt
\]
\[
= \sum_{i,j=1}^k \bar{m}_{ij} (\bar{c}_{n,j}, \bar{c}_{n,i}) + \sum_{i=1}^k \bar{m}_{0i} (t_i^G)^{-1/2} \left( e_{\tau,h}(t_{n-1}^*), \bar{c}_{n,i} \right)
\]
\[
\geq c \sum_{j=1}^k \| \bar{c}_{n,j} \|_{L^2(I_n,\mathbb{R})}^2 - c \| e_{\tau,h}(t_{n-1}^*) \|^2. \tag{4.54}
\]
Similarly to (4.45), we conclude from (4.54) that
\[
\tau_n S_n \geq c \| e_{\tau,h} \|_{L^2(I_n,\mathbb{R})}^2 - c \| e_{\tau,h}(t_{n-1}^*) \|^2. \tag{4.55}
\]
Further, we obtain by (4.49) along with the positive definiteness (1.2) of \(K\) that
\[
\int_{t_n} \left( B_h e_{\tau,h} \psi_{\tau,h} \right) \, dt = \int_{t_n} \left( B_h e_{\tau,h} \sum_{i=1}^k (t_i^G)^{-1/2} \bar{c}_{n,i} \psi_{n,i} \right) \, dt
\]
\[
= \frac{\tau_n}{2} \sum_{\mu=1}^k \bar{\omega}_\mu \left( B_h e_{\tau,h}(t_{n,\mu}^G) \right) \sum_{i=1}^k (t_i^G)^{-1/2} \bar{c}_{n,i} \psi_{n,i}(t_{n,\mu}^G) \tag{4.56}
\]
\[
= \frac{\tau_n}{2} \sum_{\mu=1}^k \bar{\omega}_\mu (t_{n,\mu})^{-1} \left( B_h e_{\tau,h}(t_{n,\mu}^G), e_{\tau,h}(t_{n,\mu}^G) \right) \geq 0.
\]
The terms on the right-hand side of (1.11b) can be treated as before in Lem. 4.4.
Finally, we sum up the error equations (4.11a) and (4.11b) for the test functions (4.43) and (4.53). After summation, we use (4.44), (4.45), (4.48), (4.49), (4.51), and (4.52) along with (4.54), (4.55) and (4.56). The remaining terms are treated as before in the proof of Lem. 4.4. By (4.50), the terms \( \int_{\Omega} \{ e_{\tau,h} \nabla \Phi_{\tau,h} \} \, dt \) and \( \alpha \int_{\Omega} \{ \nabla \cdot \mathbf{E}_{\tau,h} \} \, dt \) cancel out for the test functions (4.43) and (4.53).

By Lem. (4.4) and employing the inequality of Cauchy–Young, we then conclude the assertion (4.41) of this lemma.

It remains to estimate \( \| E_{\tau,h}(t^{n-1}_n) \|_{L^2} + \| e_{\tau,h}(t^{n-1}_n) \|_{L^2} \), arising on the right-hand side of (4.29).

**Lemma 4.6 (Estimate of \( \| E_{\tau,h}(t^{n-1}_n) \|_{L^2} + \| e_{\tau,h}(t^{n-1}_n) \|_{L^2} \))** Let \( n = 2, \ldots, N \). For the errors \( E_{\tau,h} \) and \( e_{\tau,h} \), defined in (4.1) and (4.2), there holds that

\[
\| E_{\tau,h}(t^{n-1}_n) \|_{L^2}^2 + \| e_{\tau,h}(t^{n-1}_n) \|_{L^2}^2 \leq (1 + \tau_{n-1}) (\| E_{\tau,h}(t^{n-1}_n) \|_{L^2}^2 + \| e_{\tau,h}(t^{n-1}_n) \|_{L^2}^2) + c \tau_{n-1}^{2(k+1)} (E_t^{n-1,2})^2,
\]

where \( E_t^{n-1,2} := \| \partial_t^{k+2} u \|_{L^2(t^{n-1}_n, H^1)} \).

**Proof.** For brevity, we study \( \| E_{\tau,h}(t^{n-1}_n) \|_{L^2}^2 + \| e_{\tau,h}(t^{n-1}_n) \|_{L^2}^2 \) of (4.41) only. The corresponding estimate for (4.29) can be done along the same lines. Firstly, from (4.11) and (4.11), the continuity constraint imposed on \( v_{\tau,h} \) in Problem 3.3 or 3.5, respectively, and the assumption that \( u \) is sufficient regular we get that

\[
E_{\tau,h}^2(t^{n-1}_n) = |w_2(t^{n-1}_n) - v_{\tau,h}(t^{n-1}_n)| = |R_h \partial_t u(t^{n-1}_n) - v_{\tau,h}(t^{n-1}_n)|
\]

\[
= |R_h \partial_t u(t^{n-1}_n) - v_{\tau,h}(t^{n-1}_n)| = |E_{\tau,h}^2(t^{n-1}_n)|.
\]

For (4.10), we recall the notation that \( v_{\tau,h}(t^{n-1}_n) = \lim_{t \to t^{n-1}_n} v_{\tau,h}(t) \) and \( v_{\tau,h}(t^{n-1}_n) = v_{\tau,h}(t^{n-1}_n) \).

Secondly, by the continuity constraint imposed on \( u_{\tau,h} \) in Problem 3.3 or 3.5, respectively, we have that

\[
E_{\tau,h}^2(t^{n-1}_n) = w_1(t^{n-1}_n) - u_{\tau,h}(t^{n-1}_n) = (w_1(t^{n-1}_n) - w_1(t^{n-1}_n)) + E_{\tau,h}^2(t^{n-1}_n).
\]

Then, by the triangle inequality we can conclude that

\[
(C \varepsilon(E_{\tau,h}^2(t^{n-1}_n)), \varepsilon(E_{\tau,h}^2(t^{n-1}_n)))^{1/2} \leq C \| \nabla(w_1(t^{n-1}_n) - w_1(t^{n-1}_n)) \|
\]

\[
+ (C \varepsilon(E_{\tau,h}^2(t^{n-1}_n)), \varepsilon(E_{\tau,h}^2(t^{n-1}_n)))^{1/2}.
\]

By (4.11) and (4.7) there holds that

\[
\| \nabla(w_1(t^{n-1}_n) - w_1(t^{n-1}_n)) \| \leq C \| \nabla(w_1(t^{n-1}_n) - R_h u(t^{n-1}_n)) \|
\]

\[
+ \| \nabla R_h u(t^{n-1}_n) - \int_{t^{n-2}_n}^{t^{n-1}_n} I_\tau(\partial_t u) \, dt - u(t^{n-2}_n) \| \leq C \tau_{n-1}^{1/2} \| \partial_t^{k+2} u \|_{L^2(t^{n-1}_n, H^1)}.
\]

Thirdly, since \( e_{\tau,h} \in X^k_t(V_h^t) \subset C([0,T]; V_h^t) \) we have that

\[
\| e_{\tau,h}(t^{n-1}_n) \| = \| e_{\tau,h}(t^{n-1}_n) \|.
\]

Combining (4.58) to (4.61) and applying the arithmetic and geometric mean inequality proves the assertion (4.57).

The term \( \| E_{\tau,h}(t^{n-1}_n) \|_{L^2}^2 + \| e_{\tau,h}(t^{n-1}_n) \|_{L^2}^2 \), arising on the right-hand side of (4.41), can be estimated by (4.57) along with (4.54). Finally, we address the term \( \delta_n - \delta_{n-1} \) of (4.29).
Lemma 4.7 Let $\delta_n$ and $\delta_{n-1}^+$ be defined by (4.28). For $n = 2, \ldots, N$ there holds that

$$\delta_n - \delta_{n-1}^+ \leq \delta_n - \delta_{n-1} + c\tau_{n-1}^{2(k+1)} (E_{t}^{n-1,3})^2 + c\tau_{n-1} h^{2(r+1)} (E_{x}^{n-1,3})^2,$$

where $E_{t}^{n-1,3} := \| \partial_t^{k+2} u \|^{2}_{L^{\infty}(I_{n-1}, H^1)}$ and $E_{x}^{n-1,3}$ is defined by Lem. 4.7. For $n = 1$ there holds that

$$|\delta_1 - \delta_0^-| \leq ch^{2(r+1)} (\|p_0\|_{r+1}^2 + \|p(t_1)\|_{r+1}^2 + \|u_0\|_{r+2}^2) + \varepsilon \|E_{\tau,h}(t_1)\|_e$$

for a (sufficiently small) constant $\varepsilon > 0$.

Proof. By definition (4.28) of $\delta_n$ and $\delta_{n-1}^+$ along with (4.11), (4.2), (3.11), the interpolation property (2.6) of $I_\tau$, the continuity of $u_{\tau,h}$ and the approximation properties (4.7) and (4.8a) we have for $n = 2, \ldots, N$ that

$$\delta_{n-1} = (p(t_{n-1}) - R_h p(t_{n-1}), \nabla \cdot (R_h u(t_{n-1}) - u_{\tau,h}(t_{n-1})))
= (p(t_{n-1}) - R_h p(t_{n-1}), \nabla \cdot (w_1(t_{n-1}) - u_{\tau,h}(t_{n-1})))
+ (p(t_{n-1}) - R_h p(t_{n-1}), \nabla \cdot (R_h u(t_{n-1}) - w_1(t_{n-1})))$$

$$= \delta_{n-1} + \int_{t_{n-2}}^{t_{n-1}} I_\tau(t) \nabla \cdot (R_h u(t_{n-1}) - w_1(t_{n-1})) dt$$

$$= \delta_{n-1} + \int_{t_{n-2}}^{t_{n-1}} I_\tau(t) \nabla \cdot (R_h u(t_{n-1}) - w_1(t_{n-1})) dt = \delta_{n-1}^+ + \varepsilon_{n-1},$$

such that

$$\delta_n - \delta_{n-1}^+ = \delta_n - \delta_{n-1} - \varepsilon_{n-1},$$

where

$$|\varepsilon_{n-1}| \leq c\tau_{n-1} h^{r+1} E_{t}^{n-1,3} + \varepsilon_{n-1}$$

$$\leq c\tau_{n-1}^{2(k+1)} (E_{t}^{n-1,3})^2 + c\tau_{n-1} h^{2(r+1)} (E_{x}^{n-1,3})^2.$$

Now, the assertion (4.62) is a direct consequence of (4.64) and (4.65).

For $n = 1$, there holds by (4.28), (4.11), (4.2) and (3.11) along with the Assumption 3.2 that

$$\delta_{0}^+ = (p_0 - R_h p_0, \nabla \cdot (R_h u_0 - u_{0,h})) \leq ch^{2(r+1)} (\|p_0\|_{r+1}^2 + \|u_0\|_{r+2}^2).$$

Further, by the inequalities of Cauchy–Schwarz and Cauchy–Young along with (4.8a) we have that

$$\delta_1 = (p(t_1) - R_h p(t_1), \nabla \cdot E_{\tau,h}(t_1)) \leq ch^{2(r+1)} \|p(t_1)\|_{r+1}^2 + \varepsilon \|E_{\tau,h}(t_1)\|_e^2$$

with $\varepsilon > 0$. By (4.65) and the triangle inequality we get (4.63) from (4.66) and (4.67). \(\square\)

Theorem 4.8 (Main convergence result) For the approximation $(u_{\tau,h}, v_{\tau,h}, p_{\tau,h})$ defined by Problem 3.8 or 3.9, respectively, of the sufficiently regular solution $(u, v, p)$ with $v = \partial_t u$ to (1.1) there holds that

$$\|\nabla (u(t) - u_{\tau,h}(t))\| + \|v(t) - v_{\tau,h}(t)\| + |p(t) - p_{\tau,h}(t)| \leq c\tau^{k+1} + ch^{r+1}, \text{ for } t \in I.$$
Proof. Combining the estimates (4.29) and (4.41) and recalling the norm equivalence (4.5) yields that

\[
\begin{align*}
\|E_{\tau,h}(t_n)\|_p^2 + \|e_{\tau,h}(t_n)\|_p^2 &\leq \delta_n - \delta_{n-1} + (1 + c\tau_n) \left( \|E_{\tau,h}(t_{n-1})\|_p^2 + \|e_{\tau,h}(t_{n-1})\|_p^2 \right) \\
&+ c \left( \tau_n^{2(k+1)} (\xi_{t_{n-1}}^1)^2 + c\tau_n^{-2(1)} (\xi_{x}^1)^2 + c\tau_n^{2(r+2)} (\xi_{x}^{n-3})^2 \right)
\end{align*}
\]

(4.69)

for \( n = 1, \ldots, N \). Employing now (4.57) and (4.62) in (4.69), implies that

\[
\begin{align*}
\|E_{\tau,h}(t_n)\|_p^2 + \|e_{\tau,h}(t_n)\|_p^2 &\leq \delta_n - \delta_{n-1} + (1 + c\tau_n)(1 + \tau_{n-1}) \left( \|E_{\tau,h}(t_{n-1})\|_p^2 + \|e_{\tau,h}(t_{n-1})\|_p^2 \right) \\
&+ c \left( \tau_n^{2(k+1)} (\xi_{t_{n-1}}^1)^2 + h^{2(r+1)} (\xi_{x}^1)^2 + c\tau_n^{2(r+2)} (\xi_{x}^{n-3})^2 \right)
\end{align*}
\]

(4.70)

for \( n = 2, \ldots, N \). It remains to consider the case that \( n = 1 \). By Problem 3.3 we have that \( U_{\tau,h} \in (C([0,T];V^{(r+1)}))^{2d} \) and \( p_{\tau,h} \in C([0,T];V_{n}^{1}) \). By (4.11) we have that \( w_1(t_0) = R_h u_0 \) and \( w_2(t_0) = R_h u_1 \). Thus, for \( \|E_{\tau,h}(t_0)\|_p \) and \( \|e_{\tau,h}(t_0)\|_p \) it follows under the Assumption 3.4 that

\[
\begin{align*}
\|E_{\tau,h}(t_0)\|_p^2 + \|e_{\tau,h}(t_0)\|_p^2 &\leq c\|\nabla (R_h u_0 - u_{0,0})\|^2 + c\|R_h u_1 - v_{0,h}\|^2 \\
&+ c\|R_h p_0 - p_{0,h}\|^2 \leq ch^{2(r+1)}
\end{align*}
\]

(4.71)

Employing (4.71) and (4.63) in (4.69), we obtain that, for sufficiently regular solutions \((u,p)\) \( \text{\cite{11}}, \)

\[
\|E_{\tau,h}(t_1)\|_p^2 + \|e_{\tau,h}(t_1)\|_p^2 \leq c\tau_1^{2(k+1)} + ch^{2(r+1)}.
\]

(4.72)

Next, we introduce the abbreviation that

\[
A_n := \|E_{\tau,h}(t_n)\|_p^2 + \|e_{\tau,h}(t_n)\|_p^2, \quad \text{for \( n = 0, \ldots, N \).}
\]

(4.73)

Then, we recover (4.70) as

\[
A_n \leq \delta_n - \delta_{n-1} + (1 + c\tau_n) (1 + \tau_{n-1}) A_{n-1} + c\tau_n^{2(k+1)} \left( (\xi_{t_{n-1}}^1)^2 + (\xi_{x}^{n-2})^2 \right) \\
+ c\tau_n^{2(r+1)} \left( (\xi_{x}^{n-2})^2 + (\xi_{x}^1)^2 \right) + c\tau_n \tau_{n-1}^{2(k+1)} (\xi_{t_{n-1}}^{n-3})^2 + c\tau_n h^{2(r+1)} \left( (\xi_{x}^{n-3})^2 + (\xi_{x}^{n-3})^2 \right)
\]

(4.74)

for \( n = 2, \ldots, N \). From (4.72) we have that

\[
A_1 \leq c\tau_1^{2(k+1)} + c_2 h^{2(r+1)}.
\]

(4.75)

Now, we apply the discrete Gronwall inequality \( \text{\cite{21} Lem. 1.4.2} \) to (4.74) and (4.75). For this, we change the index \( n \) to \( m \) in (4.74) and sum up the resulting inequality from \( m = 2 \) to \( m = n \). This yields that

\[
A_n \leq |\delta_1| + |\delta_n| + \sum_{m=2}^{n} (c\tau_m + \tau_{m-1} + c\tau_m \tau_{m-1}) A_{m-1} + \left( \tau_1^{2(k+1)} + h^{2(r+1)} \right) (M_n + N_n),
\]

(4.76)

where by the definition of \( \xi_{t_{m-1}}^{n-1} \) and \( \xi_{x}^{n-i} \), for \( i \in \{1,2,3\} \), there holds that

\[
M_n := \sum_{m=1}^{n} \left( (\xi_{t_{m-1}}^{n-2})^2 + (\xi_{x}^{n-2})^2 + (\xi_{x}^{n-1})^2 \right) \leq c < \infty,
\]

(4.77a)

\[
N_n := \sum_{m=1}^{n} \tau_m \left( (\xi_{t_{m-1}}^{n})^2 + (\xi_{x}^{n})^2 \right) \leq c < \infty
\]

(4.77b)

for sufficiently regular solutions \((u,p)\) to the system \( \text{\cite{11}} \) and \( n = 1, \ldots, N \). We have that

\[
\prod_{j=1}^{n-1} (1 + c\tau_j) \leq c^{\tau T}.
\]

(4.78)
Combining (4.67) and (4.75) yields that
\[ |δ_1| ≤ cτ_1(2^{k+1}) + ch^{2(r+1)}. \] (4.79)
From the definitions (4.28), (4.2) and (4.73) we conclude by the inequalities of Cauchy–Schwarz and Cauchy–Young and (4.8a) that, for some sufficiently small \( ε > 0 \), there holds that
\[ |δ_n| = (ω(t_n), \nabla \cdot E_{τ,h}(t_n)) ≤ ch^{2(r+1)} + ε A_n \] (4.80)
The Gronwall argument, along with (4.77) to (4.80) and Assumption 3.2 then implies that
\[ \|E_{τ,h}(t_n)\|^2 + \|e_{τ,h}(t_n)\|^2 ≤ cτ_1(2^{k+1}) + ch^{2(r+1)}, \ ] \text{for } n = 0, \ldots, N, \] (4.81)
where \( τ = \max_{n=1, \ldots, N} τ_n \); cf. Subsec. 2.2. By (4.41), (4.57), (4.81) and (4.5) we then get that
\[ \|E_{τ,h}\|^2_{L^2(I_τ;L^2)} + \|e_{τ,h}\|^2_{L^2(I_τ;L^2)} ≤ cτ_1(2^{k+1}) + ch^{2(r+1)} \] (4.82)
for \( n = 2, \ldots, N \). For \( n = 1 \), estimate (4.82) follows from (4.41) along with (4.71) and (4.5). By the \( L^∞ - L^2 \) inverse relation (2.11) we conclude from (4.82) that
\[ \|E_{τ,h}(t)\|^2 + \|e_{τ,h}(t)\|^2 ≤ cτ_1(2^{k+1}) + ch^{2(r+1)}, \ ] \text{for } t \in [0, T]. \] (4.83)
Finally, applying the triangle inequality to the splitting (4.1) and (4.2) of the errors in \( u, v \) and \( p \) and employing the estimates (4.3) proves the assertion (4.68). For this, we note that (4.91) holds analogously for \( ω \) in (4.2); cf. [35 Eq. (3.20)].

Remark 4.9
- We note that the constant of the error estimate (4.68) depends in particular on the norms of the continuous solution that are induced by Lem. 2.1, Lem. 4.4 and Lem. 4.5 to Lem. 4.7. Thereby, the tacitly assumed regularity of the continuous solution becomes obvious.
- For arbitrary \( t \in I \), estimate (4.68) is of optimal order with respect to the time and space discretization, if the approximation error is measured by \( \|p(t) - p_{τ,h}(t)\| \) and the energy error \( \|\nabla (u(t) - u_{τ,h}(t))\| + \|v(t) - v_{τ,h}(t)\| \), with \( v = \partial_τ u \).
- From (4.68), an error estimate for \( \|u(t) - u_{τ,h}(t)\| \) can be obtained by the Poincaré inequality. However, the resulting estimate for \( \|u(t) - u_{τ,h}\| \), as well as the estimate of \( \|v(t) - v_{τ,h}\| \) in (4.68), are of suboptimal order with respect to the space discretization only. This is due to the coupling structure of the system (1.1), the energy-type arguments of the error analysis bounding the combined quantity \( \|\nabla (u(t) - u_{τ,h}(t))\| + \|v(t) - v_{τ,h}(t)\| \) and, finally, the non-equal order approximation of \( u \) and \( p \) by inf-sup stable pairs of finite element spaces. Similar observations regarding the coupling of the errors in the approximation of the unknowns are well-known from the discretization of the Navier–Stokes equations by inf-sup stable pairs of finite element spaces. In Sec. 6 the convergence rates of the error estimate (4.68) are confirmed by our numerical experiments.
- In [25], the convergence of a continuous Galerkin method for a scalar-valued nonlinear wave equation in \( u \) is studied. Optimal order \( L^2 \)-error estimates, for the quantities \( u \) and \( v = \partial_τ u \), are proven. A key ingredient of this optimality is the special choice of the initial values, which is in contrast to our more general one given by Assumption 3.3. Compared to the purely hyperbolic case of [25], in our analysis the projection error that is induced by the coupling term \( α∇ \cdot \partial_τ u \) in (1.11) implies the loss of one order of accuracy for the spatial discretization of the overall system such that the result of [25] regarding the \( L^2 \)-error convergence of \( u \) and \( v \) cannot be transferred directly to the system (1.1). Optimal order estimates for \( \|u - u_{τ,h}\| \) and \( \|v - v_{τ,h}\| \) might require proper decoupling techniques for the subproblems of (1.11) which has to be left as a work for the future.
5. Superconvergence at discrete time nodes

For $k \geq 2$, the continuous Galerkin approximation in time is known to be superconvergent at the discrete time nodes $t_n$, for $n = 1, \ldots, N$, with an improved order of convergence of $r^{k+2}$. In [10], this is proved for the continuous Galerkin approximation of the scalar-valued wave equation by a computationally cheap post-processing technique. The post-processed discrete solution is built of piecewise polynomials in time of order $k+1$ and coincides with the continuous Galerkin approximation with piecewise polynomials of order $k$ in all $k+1$ Gauss–Lobatto quadrature nodes of the subinterval $I_n$. Superconvergence at the discrete time nodes $t_n$ is then implied by the post-processed discrete solution. For further details of the post-processing, proof of its convergence and numerical experiments we refer to [10]. We conjecture that the superconvergence of the continuous Galerkin time discretization at the discrete time nodes $t_n$ also holds for the approximation (5.3) or (5.10), respectively, of the coupled system (1.1). This is summarized in Cor. 5.1 and confirmed by our numerical experiment presented in Sec. 6. The proof of Cor. 5.1 can be done along the lines of [10] by using its post-processing technique.

Corollary 5.1 Let $k \geq 2$. For the approximation $(u_{r,h}, v_{r,h}, p_{r,h})$ defined by (5.3) or (5.10), respectively, of the solution $(u, v, p)$ with $v = \partial_t u$ to (1.1) there holds at the discrete time nodes $t_n$, for $n = 1, \ldots, N$, that

$$\|\nabla(u(t_n) - u_{r,h}(t_n))\| + \|v(t_n) - v_{r,h}(t_n)\| + \|p(t_n) - p_{r,h}(t_n)\| \leq cr^{k+2} + ch^{r+1}. \quad (5.1)$$

6. Numerical convergence test

Here we present the results of our performed numerical experiments in order to confirm Thm. 4.8. The implementation of the numerical scheme was done in the high-performance DTM++/biot-allard frontend solver (cf. [30] [33]) for the deal.II library [2]. We study (6.1) for the prescribed solution

$$u(x,t) = \phi(x,t)I_2 \quad \text{and} \quad p(x,t) = \phi(x,t) \quad \text{with} \quad \phi(x,t) = \sin(\omega_1 t^2) \sin(\omega_2 x_1) \sin(\omega_2 x_2) \quad (6.1)$$

for $\Omega = (0,1)^2$ and $I = (0,T)$. We put $\rho = 1.0$, $\alpha = 0.9$, $c_0 = 0.01$ and $K = I_2$ with the identity matrix $I_2 \in \mathbb{R}^{2,2}$. For the fourth-order elasticity tensor $C$, isotropic material properties with Young’s modulus $E = 100$ and Poisson’s ratio $\nu = 0.35$ are assumed.

We study the space-time convergence behavior to confirm the error bounds (4.68) and (5.1). For this, the domain $\Omega$ is decomposed into a sequence of successively refined meshes $\Omega^l$, with $l = 0, \ldots, 3$, of quadrilateral finite elements. The step size of the coarsest time mesh is given in Table 6.1 and 6.2, respectively. In the computations, the spatial and temporal mesh sizes are successively refined by a factor of two in each refinement step. For the finite element spaces we choose the polynomial orders $k = 2$ and $r = 2$ such that discrete solutions $u_{r,h}, v_{r} \in (X_2^V(K_h))$ and $p_{r,h} \in X_2^P(K_h)$ are obtained; cf. (2.2) and (2.4). This amounts to a piecewise quadratic approximation in time of $u, v$ and $p$, a piecewise cubic approximation of $u$ and $v$ in space and a piecewise quadratic discretization of $p$ in space.

The calculated errors and corresponding experimental orders of convergence are summarized in Table 6.1 and 6.2. The norm of $C(\bar{T};L^2)$ is approximated numerically by

$$\|w\|_{C(L^2)} = \max\{\|w(t_m)\| \mid m = 1, \ldots, M\} \quad (6.2)$$

for equidistant time points $t_m \in \bar{T}$, $m = 1, \ldots, M$, and $M = 10N$. Table 6.1 confirms our main convergence result (4.68). In Table 6.2 superconvergence at the discrete time nodes (cf. Cor. 5.1) is clearly observed for the approximation of the variable $v$ by its space-time convergence of fourth order. The approximation of $\nabla u$ and $p$ is dominated by the spatial error that is of third order accuracy only. Increasing the polynomial order in space from $r = 2$ to $r = 3$ would lead to superconvergence of the discretization of $\nabla u$ and $p$ as well. However, this is not done here due to the fact that the solutions and errors are computed
by a single computing process only and the restricted memory for this serial approach. In general, our software is able to compute problems in parallel with distributed memory. Moreover, for the wave equation, superconvergence is proved and validated numerically in [110]. A future enhancement of the applied iterative algebraic solver (cf. [36,38]) for the fully discrete system (6.10) will be the application of our geometric multigrid preconditioner, developed for the Navier–Stokes equations and studied in [3].

| $\tau$ | $h$ | $|\nabla(u - u_{\tau,h})|_{C(L^2)}$ EOC | $|v - v_{\tau,h}|_{C(L^2)}$ EOC | $|p - p_{\tau,h}|_{C(L^2)}$ EOC |
|------|-----|-----------------|-----------------|-----------------|
| $\tau_0/2^0$ $h_0/2^0$ | 7.7627e-02 | – | 1.0581e-02 | – |
| $\tau_0/2^1$ $h_0/2^1$ | 9.6428e-03 | 3.01 | 7.3938e-04 | 3.84 |
| $\tau_0/2^2$ $h_0/2^2$ | 1.2006e-03 | 3.01 | 1.0014e-04 | 2.88 |
| $\tau_0/2^3$ $h_0/2^3$ | 1.4988e-04 | 3.00 | 2.3149e-05 | 2.11 |

Table 6.1.: Errors and EOC for (6.1) with $k = r = 2$ and numbers $h_0 = 1/(2\sqrt{2})$, $\tau_0 = 0.01$, $T = 1.0$. 

| $\tau$ | $h$ | $|\nabla(u - u_{\tau,h})|_{L_2}$ EOC | $|v - v_{\tau,h}|_{L_2}$ EOC | $|p - p_{\tau,h}|_{L_2}$ EOC |
|------|-----|-----------------|-----------------|-----------------|
| $\tau_0/2^0$ $h_0/2^0$ | 1.2039e-03 | – | 1.0436e-03 | – |
| $\tau_0/2^1$ $h_0/2^1$ | 1.5114e-04 | 2.99 | 6.3300e-05 | 4.04 |
| $\tau_0/2^2$ $h_0/2^2$ | 1.8852e-05 | 3.00 | 3.9751e-06 | 3.99 |
| $\tau_0/2^3$ $h_0/2^3$ | 2.3539e-06 | 3.00 | 2.4973e-07 | 3.99 |

Table 6.2.: Errors and EOC for (6.1) with $\omega_1 = \pi$, $\omega_2 = 2\pi$, norm $\|w\|_{L_2} := \max_{n=1,...,N} \|w(t_n)\|_{L^2}$, polynomial degrees $k = r = 2$ and numbers $h_0 = 1/\sqrt{2}$, $\tau_0 = 0.005$, $T = 0.1$.

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Appendix

A. Well-posedness of the continuous problem

For the sake of completeness and self-containment, we briefly recall the well-posedness of the continuous system of equations \((1.1)\). For further details we refer to [31]. The result of existence and uniqueness of solutions to \((1.1)\) is based on rewriting \((1.1a)\) as a first-order system in time. This is also done for the discretization of the time variable. Here, the parameters \(\rho\), \(\alpha\) and \(c_0\) in \((1.1)\) are assumed to be positive constants. The constant fourth order tensor \(C = (C_{ijkl})_{ijkl}\) is assumed to be positive definite such that

\[
\exists k_0 > 0 \forall \xi_{ij} = \xi_{ij} \in \mathbb{R} : \quad \xi_{ij} C_{ijkl} \xi_{kl} \geq k_0 \sum_{j,k=1}^{d} \left| \xi_{jk} \right|^2 ,
\]

(A.1)

where the Einstein summation convention (i.e. repeated indices indicate summation) is used. The constant matrix \(K = (K_{ij})_{ij}\) is supposed to be positive definite as well such that

\[
\exists k_1 > 0 \forall \xi_i \in \mathbb{R} : \quad \xi_i K_{ij} \xi_j \geq k_1 \sum_{i=1}^{d} \left| \xi_i \right|^2 .
\]

(A.2)
Firstly, we rewrite the equations (1.1a) and (1.1b), by using Voigt’s notation (cf. [26, 61]), as
\[
\rho \partial_t^2 u - D^2 S \partial_t u + D^1 p = \rho f, \quad \text{in } \Omega \times (0,T],
\]
(A.3a)
\[
c_{00} \partial_t p + \Gamma^\top D^1 \partial_t u - \nabla^\top K \nabla p = g, \quad \text{in } \Omega \times (0,T],
\]
(A.3b)
where the generalized gradient \(D\), the matrix \(S\) and the vector \(\Gamma\) are defined by
\[
D = \begin{pmatrix}
\partial_1 & 0 & 0 \\
0 & \partial_2 & 0 \\
0 & 0 & \partial_3 \\
\partial_3 & 0 & \partial_1 \\
\partial_2 & \partial_1 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1144} & C_{1155} \\
C_{2222} & C_{2233} & C_{2244} & C_{2255} & C_{2266} \\
C_{3333} & C_{3344} & C_{3355} & C_{3366} & C_{3377} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
0 \\
0 \\
0 \\
\alpha \\
\alpha
\end{pmatrix}
\]
for \(d = 3\). Their definition for \(d = 2\) is skipped here; cf. [31]. By \(\nabla\) we denote the usual gradient vector.

Secondly, we reformulate (A.3) along with (1.1c), (1.1e) as an evolution problem in \(H := L^2(\Omega)\). Introducing the solution vector \(V\), the matrix of coefficients \(Q\) and the differential operator \(N\) by
\[
V = \begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
\partial_t u \\
p
\end{pmatrix}, \quad Q := \begin{pmatrix}
S^{-1} & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & c_0
\end{pmatrix}, \quad N := \begin{pmatrix}
0 & D & 0 \\
0 & D^\top & 0 \\
\Gamma & D^\top & -\nabla^\top K \nabla
\end{pmatrix},
\]
we recover Eqs. (A.3) and (1.1c) as
\[
\partial_t V + Q^{-1} N V = f, \quad V(0) = V_0,
\]
(A.4)
where \(F := (0, f, g/c_0)^\top\) and \(V_0 := (SDu_0, u_1, p_0)^\top\). We define the operator \(A : D(A) \subset H \to A\) by
\[
AV := Q^{-1} NV,
\]
where
\[
D(A) := \{V \in H \mid V_2 \in H^1_0(\Omega), V_3 \in H^1_0(\Omega), N V \in H\}
= \{V \in H \mid V \in D(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega), \nabla^\top K \nabla V_3 \in L^2(\Omega)\},
\]
\[
D(\Omega) := \{W \in L^2(\Omega) \mid D W \in L^2(\Omega)\}.
\]
By these definitions we find the following evolution form of (A.3) and (1.1c).

**Problem A.1 (Evolution problem)** Let \(V_0 \in D(A)\) and \(F \in C^1((0,T]; H)\) be given. Find \(V \in C^1([0,T]; H) \cap C([0,T]; D(A))\), such that
\[
\partial_t V(t) + AV(t) = F(t), \quad V(0) = V_0.
\]
(A.5)
For Problem A.1 there holds the following result (cf. [31]).

**Theorem A.2 (Existence and uniqueness of solutions to Problem A.1)** \(A\) is a closed operator with dense domain in \(H\) and \(-A\) is the generator of a contraction semigroup \(\{T(t)\}_{t \in [0,T]}\), \(T(t) = e^{-At}\). For \(V_0 \in D(A)\) and \(F \in C^1((0,T]; H)\) there exists a unique solution \(V\) to Problem A.1. For \(t \in [0,T]\), the solution \(V\) can be represented as
\[
V(t) = T(t)V_0 + \int_0^t T(t-s)F(s) \, ds.
\]
(A.6)

**Remark A.3** As usual, the domain \(D(A)\) can be characterized further in terms of Sobolev spaces by using regularity theory for elliptic differential operators (cf., e.g., [26]). For more regular data \(V_0\) and \(F\), higher regularity of the solution \(V\) can then be deduced from formula (A.6) by standard arguments of semigroup theory (cf., e.g., [47]), i.e. by finding a maximum value of \(\beta \in \mathbb{R}\) such that \(V \in D(A^\beta)\) is satisfied and, then, characterizing \(D(A^\beta)\) in terms of Sobolev spaces.