UNITED STATISTICAL ALGORITHMS, SMALL AND BIG DATA, FUTURE OF STATISTICIAN

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ABSTRACT

Role of big idea statisticians in future of Big Data Science. United Statistical Algorithms framework for comprehensive unification of traditional and novel statistical methods for modeling Small Data and Big Data, especially mixed data (discrete, continuous).

Goal: Model \((X,Y)\) by nonparametrically estimating conditional mean \(E[Y|X=x]\) and conditional quantile \(Q(u; Y|X=x)\). Modeling example data (Age,GAGurine). Notation population and sample distribution, quantile, mid-distribution, mid-quantile \(F(x; X), Q(u; X), F_{\text{mid}}(x; X),\) and \(Q_{\text{mid}}^u(u; X)\). Standardize \(Z(X) = (X - E[X])/\sigma(X), QI(X) = (X - MQ)/DQ,\) mid-quartile MQ, quantile deviation DQ, informative quantile \(QIQ(u; X) = QI(Q_{\text{mid}}^u(u; X); X)\).

Theorems: with probability 1, \(Q(F(X; X); X) = X, E[Y|X] = E[Y|F(X; X)]\). Corollary: Linear methods estimate \(E[Y|X] = \sum_j C_j T_j(X; X),\) custom score functions \(T_j(X; X)\) are functions of \(F_{\text{mid}}(X; X)\); for \(X\) continuous score Legendre polynomial function. Information measures dependence \((X,Y)\). LP comoments \(LP(j, k; X,Y)\) are covariances of \(T_j(X; X), T_k(Y; Y)\). Orthonormal series estimation comparison density, conditional comparison density, copula density.

Comparison probability, Bayes theorem, copula density. Two sample data modeling Combined mean, variance theorem. Apply to quickly derive normal parameters mean, variance conjugate prior Bayesian posterior update formulas. Correlation unification and extension traditional Student and Wilcoxon statistics test equality of distributions of two samples.

Keywords: Nonparametric high dimensional data modeling, mid-distribution, mid-quantile, comparison density, copula density, LP orthonormal score functions, LP moments, LP comoments, LPINFOR, correlation dependence measures, classification, logistic regression, unification statistical methods, analogies between analogies, quantile data analysis.
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1 BIG IDEA STATISTICIAN

Essays on the past, present, and future of Statistics (Davidian (2013), “Aren’t We Data Science?”) should be interpreted as about Future of Statisticians. Their ultimate goal (Wahba (2013), “Statistical Model Building, Machine Learning, and the Ah-Ha Moment”) is to provide advice about the Statistics skills that should be taught (in introductory and advanced courses) and also what (older popular) topics can be omitted. Wasserman (2013), “Rise of the Machines” opines that Statistics and Machine Learning do not differ in topics that are main tools (and big applicable ideas), including: likelihood, RKHS (reproducing kernel Hilbert spaces), classification, information measures of dependence, logistic regression, sparse regression, nonparametric regression, density estimation, model selection, Bayesian analysis. Parzen (1961) pioneered RKHS unification of small data (regression) and big data (time series)

To ensure a future for Statisticians we should be concerned what makes them uniquely useful (and employable) in era that many disciplines want to be Data Scientists (with a cookbook knowledge of statistical methods recipes, and not why they work, especially “analogies between analogies”). Modern statisticians envision their role (Irizarry (2013) “The Bright Future of Applied Statistics”) as VERY COLLABORATIVE APPLIED (mechanic) statisticians whose job is grant supported specific problem solving (parametric confirmatory rather than nonparametric exploratory). They emphasize (1) the science (understanding the scientific context and data collection), (2) computer mechanics (programming) required for real answers to scientific questions.

The continuing success of applied statisticians needs partnership with broad (big ideas) statisticians with knowledge of (and passion for) the BIG IDEAS of traditional and novel statistical methods provided by a comprehensible (and also comprehensive) unification of all of statistical methods (traditional and novel) applicable to modeling small and big data (including the different cultures of statistical science theory and applications (Breiman 2001)). The goal of this paper is a framework (with sketches of proofs) for applicable ideas of almost all of statistical modeling, based on research pioneered by Parzen (1979) and many papers, reports, and Ph.D, theses, especially Parzen (1992, 2004). We report a very important new development: extension to mixed (discrete, continuous) high dimensional data by Mukhopadhyay (2013) and Parzen and Mukhopadhyay (2012). Our framework for non-parametric statistical modeling provides statisticians with unique tools scalable for MAS-
SIVE DATA = samples of size n of p variables (discrete or continuous), where p can be massive and n small. While the theory is beautiful, its utility can only be demonstrated by its successful collaborative applications to real scientific problems.

2 QUANTILE, MID-DISTRIBUTION

2.1 THE GAG URINE PROBLEM

We provide an example of comprehensive data analysis and modeling: consider a sample, size \( n = 314 \), of \((X, Y)\) data (AGES, GAG) of GAG levels in urine of children. Scientific question: What are normal levels of GAG in children of each age 1-18? This data is popularized by \cite{Ripley2004} in the honor of David Coxs 80th birthday who discusses various model selection methods (polynomial, spline, local polynomials), which estimate nonparametrically conditional mean \( E[Y \mid X = x] \). Figure 1 plots our nonparametric estimate of conditional mean, and conditional quantile \( Q(u; Y \mid X = x) \) for \( u = .25, .75 \), is shown in Figure 2 which better answers the scientific question of normal levels at each age.

2.2 MID-DISTRIBUTION TRANSFORM

To model relations (dependence) of joint variables \((X, Y)\) the question of transforming the variables can be avoided by mid-distribution rank transforms \( F_{\text{mid}}(X; X), F_{\text{mid}}(Y; Y) \) where the mid-distribution of \( X \) is defined \cite{Parzen1983, Eubank1987} as

\[
    F_{\text{mid}}(x; X) = F(x; X) - .5p(x; X), \quad p(x; X) = \Pr[X = x], \quad F(x; X) = \Pr[X \leq x]. \tag{2.1}
\]

Sample mid-distribution is computed by mid-rank algorithm \texttt{rank(X)} in R: \( \tilde{F}_{\text{mid}}(x; X) = (\text{rank}(X) - .5)/n \), where \( n \) is sample size.

Non-parametric modeling of \((X, Y)\) is based on custom score functions \( T_j(X; X) \), orthonormal basis of copula density whose coefficients are LP comoments. Their important role in our mid-distribution rank based algorithms, derives from the following FUNDAMENTAL CONDITIONING THEOREM:

\[
    \mathbb{E}[Y \mid X] = \mathbb{E}[Y \mid F_{\text{mid}}(X; X)] \quad \text{with probability} \; 1. \tag{2.2}
\]

Proof follows from fact that a function \( h(X) = hQ(F(X)) \), with probability 1, defining \( hQ(u) = h(Q(u; X)) \), where \( Q(u; X) \) is quantile function and \( Q(F(X; X); X) = X \) with probability 1.
Figure 1: The estimated conditional mean curves are shown.
2.3 ALGORITHM

Show that conditional expectation \( \mathbb{E}[g(Y)|X] - \mathbb{E}[g(Y)] \) may be approximated by linear regression methods by \( \sum_j C_j T_j(X; X) \) with coefficients \( C_j = \mathbb{E}[g(Y) T_j(X; X)] \) for selected score functions. This is implemented in Figure 1 plot of nonparametric regression of GAG urine on AGE.

To look at the data we recommend plot three scatter diagrams: \((X, Y), (F_{mid}(X; X), Y), \) and \((F_{mid}(X; X), F_{mid}(Y; Y))\), each has a correlation - shown in Figure 3. To measure dependence of \( X \) and \( Y \), calculate the following correlations (at least four versions):

\[
\text{Figure 2: "Normal" GAG concentration Band.}
\]
Figure 3: Three Scatter Plots.

- Pearson \( R(X, Y) = \text{Cor}(X, Y) = \mathbb{E}[Z(X)Z(Y)], Z(X) = (X - \mathbb{E}[X]) / \sigma(X), \sigma^2(X) = \text{Var}[X] \).

- Spearman \( R(F^{\text{mid}}(X; X), F^{\text{mid}}(X; X)) = \mathbb{E}[T_1(X; X)T_1(Y; Y)], T_1(X; X) = Z[F^{\text{mid}}(X; X)] \)

- Gini (two types) \( R(X, F^{\text{mid}}(Y; Y)), R(F^{\text{mid}}(X; X), Y) = \mathbb{E}[T_1(X; X)Z(Y)] \).

2.4 DATA WITH TIES

Our definition of Spearman correlation is important because it works for discrete data and data with ties; applied statisticians can implement it in \( \text{R} \) as the Pearson correlation of mid-distribution transformed \( X \) and \( Y \). When \( X \) is 0, 1 valued \( Z(X) = Z(F^{\text{mid}}(X; X)) = \).
\( T_1(X; X) \). We show below that therefore there are two correlations: (1) Pearson, equivalent to Student t test of equality of means of two samples; (2) Spearman, equivalent to Wilcoxon nonparametric test of equality of two sample distributions. This is an example of unification of small data and big data parametric and non-parametric statistical methods.

### 2.5 LP COMOMENTS

Dependence of \( X \) and \( Y \), measured in general by information measures, is estimated by higher order correlations, called LP comoments, computed by taking covariance of higher order score functions \( T_j(X; X) \) and \( T_k(Y; Y) \), introduced below.

### 2.6 QUANTILE

Distribution of \( X \) is modeled by Quantile function \( Q(u; X) \), \( 0 < u < 1 \), inverse of distribution function, defined as smallest \( x \) that \( F(x; X) \geq u \).

To Simulate \( X \) use THEOREM: In distribution \( X = Q(U; X) \) where \( U \) is Uniform(0, 1).

A quick proof follows from THEOREM [Parzen 1979]: If \( g(x) \) is quantile like function (non-decreasing, left continuous), then \( g(X) \) has quantile \( Q(u; g(X)) = g(Q(u; X)) \). Note Uniform(0, 1) \( U \) has quantile \( Q(u; U) = u \). Location scale parameter model \( Q(u; X) = \mu + \sigma Q_0(u) \) has internal representation [Parzen 2008]

\[
X = \mu + \sigma X_0, \quad Q_0 = Q(u; X_0).
\] (2.3)

When \( X_0 \) is Normal(0, 1) we denote it by \( Z \). From identical distribution of \( X \) and \( Q(U; X) \) one can compute, and estimate, mean \( E[X] \), variance \( \text{Var}[X] \) by mean and variance of \( Q(U; X) \).

### 2.7 COMPARISON DENSITY, SKEW-G MODEL

To fit distributions to data our framework prefers to approach it via comparison density series estimation using score functions \( T_j(X; X) \). Related concepts are relative density or grade density [Handcock and Morris 1999], and density ratio estimation in machine learning [Sugiyama et al. 2012].

To estimate probability density \( f(x; X) \) of \( X \) continuous, or to simulate a sample from \( F(x; X) \) choose parametric model \( G(x) \) with quantile \( Q_G(u) \), estimate Comparison Distribution \( D(u; G, F) = F(Q_G(u); X), 0 < u < 1 \), Comparison Density \( d(u) = d(u; G, F) = f(Q_G(u); X)/g(Q_G(u)) \).
A model for unknown $f(x; X)$ is $f(x; X) = g(x)d(G(x))$, called a SKEW-G model. If $d(u)$ has upper bound $C$ one can simulate $X$ from $F(x; X)$ by $X$ from $G(x)$, which one accepts if $(1/C)d(G(X)) > U$, Uniform(0, 1). Important diagnostic tool is graph of $D(u; G, F)$, called a P-P plot [Parzen 1993]; it plots $(G(x), F(x))$.

2.8 MID-QUANTILE FUNCTION

To define quartile and median of $X$ we define its mid-quantile $Q^{\text{mid}}(u; X)$, $0 < u < 1$, which is always a continuous function. For discrete $X$, with probable values $x_j$, true for a sample quantile function, construct mid-quantile $Q^{\text{mid}}(u; X)$, $0 < u < 1$, by connecting linearly $(F^{\text{mid}}(x_j; X), x_j)$. For $X$ continuous define $Q^{\text{mid}}(u; X) = Q(u; X)$. Define quartiles $Q_1, Q_3$, and median $Q_2$ by $Q_1 = Q^{\text{mid}}(.25; X), Q_2 = Q^{\text{mid}}(.5; X), Q_3 = Q^{\text{mid}}(.75; X)$. Mid-quartile $MQ = .5(Q_1 + Q_3)$, quartile deviation $DQ = 2(Q_3 - Q_1)$. Large sample theory of mid-quantile given in Ma, Genton, and Parzen (2011).

2.9 INFORMATIVE QUANTILE

Distribution symmetry and tails (long, medium, short) can be identified for practical purposes from the plot of informative quantile function $Q^{\text{IQ}}(u; X) = Q^{\text{I}}(Q^{\text{mid}}(u; X)), Q^{\text{I}}(X) = (X - MQ)/DQ$. Interpretation for data modeling [Parzen 2004] best taught from a portfolio of data examples [Gupta and Parzen 2004]. Figure 4 plots informative quantile of GAG urine; one learns its distribution is not symmetric, short left tail, long right tail.

2.10 GENERAL QUANTILE THEOREM

With probability 1, $Q(F(X; X); X) = X$. For an idea of proof see [Shorack 2000, page 113). COROLLARY: Conditional quantile is given by

$$Q[v; Y|X] = Q[Q(v; F(Y; Y)|X); Y],$$  \hspace{1cm} (2.4)

which estimated by separately estimating $Q(u; Y)$ and $Q(v; F(Y; Y)|X)$, noted by Parzen (2004).

2.11 DISTRIBUTION TRANSFORMATION TO UNIFORM

We apply THEOREM: When $X$ is continuous, $F(Q(u; X); X) = u$ for all $u$. COROLLARY: $F[Q(u; X); X]Q'(u; X)) = 1$; Parzen (1979) calls $fQ(u; X) = f(Q(u; X); X)$ density quantile, $Q'(u; X)$ quantile density, $hQ(u; X) = fQ(u; X)/(1 - u)$ hazard density quantile.
Figure 4: Histogram and QIQ plot for the GAG Urine is showed.

2.12 TRANSFORMATION TO UNIFORM CRITERION FOR A DISTRIBUTION G TO FIT DATA

Probability integral (rank) transform $F(X; X)$ equals in distribution a Uniform(0, 1) random variable $U$. A continuous distribution $G(x)$ is considered a model for continuous $X$ if “approximately” $G(X) = U$ in distribution.

THEOREM: Under the assumption $G$ is the true distribution, the functional limit theorem says $\sqrt{n}[\tilde{F}(Q_G(u); X) - u], 0 < u < 1$, converges in distribution to Brownian Bridge $B(u)$ whose RKHS norm squared $\|h\|^2 = \int_0^1 |h'(u)|^2 \, du$. Therefore a model fitting criterion is not usual goodness of fit distances from $u$ of the distribution function of $G(X)$, but is an information distance between 1 and the density of $G(X)$ (this insight can motivate maximum likelihood estimation of the parameters of a parametric model).

2.13 MID-DISTRIBUTION VERSION CENTRAL LIMIT THEOREM

Applicable probability theory taught in introductory statistics courses should discuss Central Limit Theorem: If $S$ is sum of many independent random variables then $S$ is approximately equal in distribution to $E[S] + \sigma[S]Z$, $Z$ denotes Normal(0, 1). In many applications $S$ is discrete; then $F_{\text{mid}}(x; S) = F(x; E[S] + \sigma[S]Z)$ is more accurate approximation.
3 ORTHONORMAL SERIES COMPARISON DENSITY ESTIMATION

The distribution of $G(X)$ when $F$ is the true distribution is denoted $D(u; G, F) = F(Q_G(u); X)$, called comparison distribution, with comparison density $d(u; G, F) = f(Q_G(u); X)/g(Q_G(u))$.

An estimator $\hat{d}(u)$ leads to an estimator

$$\hat{f}(x; X) = g(x)\hat{d}(G(x)),$$

(3.1)

called SKEW G model. Estimation of density $d(u)$ has many approaches, and an enormous literature. Orthogonal series approaches usually suggest that there is no natural choice of basis functions. We argue that a natural choice is orthonormal shifted Legendre polynomials on interval $[0, 1]$, denoted $\text{Leg}_j(u)$. Note $\text{Leg}_0(u) = 1, \text{Leg}_1(u) = \sqrt{12}(u - .5)$. When using orthonormal series estimators we have two approaches: L2 estimators not guaranteed non-negative but still applicable; MaxEnt exponential model estimators the gold standard. They have formulas:

$$d(u) - 1 = \sum_j C_j \text{Leg}_j(u),$$

(3.2)

$$\log d(u) = \theta_0 + \sum_j \theta_j \text{Leg}_j(u)$$

(3.3)

For MaxEnt density estimators we have estimating equations for parameters $\text{Mukhopadhyay} (2013)$. For L2 density estimators we have explicit formula for parameters $C_j$:

$$C_j = \int_0^1 d(u) \text{Leg}_j(u) \, du = \mathbb{E}[\text{Leg}_j(G(X))].$$

(3.4)

Model selection of AIC (or BIC) type choose significant coefficients $C_j$ and diagnose if distribution $G$ fits sample of variable $X$ by criterion how close to 0 is

$$\int_0^1 |d(u) - 1|^2 \, du = \sum_j |\mathbb{E}[\text{Leg}_j(G(X))]|^2$$

(3.5)

LP MOMENTS: Diagnostics of distribution of $X$ are provided by L moments

$$\text{LLeg}(j; X) = \mathbb{E}[Z(X) \text{Leg}_j(F\text{mid}(X; X))]$$

(3.6)

similar to concept L moments introduced by $\text{Hosking} (1990)$ for $X$ continuous. We give a definition, called LP moments, applicable to continuous or discrete data:

$$\text{LP}(j; X) = \mathbb{E}[Z(X)T_j(X; X)],$$

(3.7)
$T_j(X; X)$ are custom score functions to be constructed. The discrete case LP definition is used to define a sample estimator of the continuous case LLeg. Interpret moments $\text{LP}(j; X)$ by smallest order $m$ such that $\sum_{j=1}^{m} |\text{LP}(j; X)|^2 > .95$. If $m > 1$, conclude data may be non-normal, long tailed, non-symmetric. Note $|\text{LLeg}(1; \text{Normal})|^2 = 3/\pi = .954$, a famous constant in non-parametric statistical theory equal to efficiency of Wilcoxon statistic when testing equality of Normal distributions.

We apply this diagnosis to GAG variable. The first five LP moments for the GAG is as follows:

$$\text{LP}[\text{GAG}] = [0.90, 0.32, 0.21, 0.11, 0.12],$$

which gives the LP tail-index $m = 3$.

Shapiro Wilk test of normality tests if

$$\text{LHermite}(1; X) = E[Z(X)Q(F_{mid}(X; X); \text{Normal}(0, 1))]$$

equals 1. (3.9)

This criterion is the ratio of two estimators of standard deviation; one may prefer to conduct the test by distance of logarithm from 0 using empirical rule $-\log \text{LHermite}(1; X) > 1/n$ for significance at .05 level [Parzen, 1991].

### 4 COMPARISON PROBABILITY, BAYES THEOREM, COPULA DENSITY

Bayes theorem for events $A, B$ can be stated in terms of COMPARISON PROBABILITY

$$\text{ComPr}[A|B] = \frac{\text{Pr}[A|B]}{\text{Pr}[A]} = \frac{\text{Pr}[B|A]}{\text{Pr}[B]} = \text{ComPr}[B|A].$$

Joint distribution of mixed $X, Y$ ($X$ continuous, $Y$ discrete) is provided by either side of identity

PRE-BAYES THEOREM: $\text{Pr}[Y = y | f(x; X) = y] = f(x; X) \text{Pr}[Y = y | X = x]$

BAYES THEOREM FOR RANDOM VARIABLES ($X, Y$) DISCRETE OR CONTINUOUS:

$$\text{ComPr}[Y = y | X = x] = \frac{\text{Pr}[Y = y | X = x] \text{Pr}[Y = y]}{\text{Pr}[Y = y]} = \frac{f(x; X | Y = y)}{f(x; X)} = \text{ComPr}[X = x | Y = y]$$

COPULA DENSITY: Copula density function of mixed variables $X, Y$ is defined for $0 < u, v < 1$
\[ \text{cop}(u, v; X, Y) = \text{ComPr}[Y = Q(v; Y)|X = Q(u; X)] = \text{ComPr}[X = Q(u; X)|Y = Q(v; Y)] \]
\[ = d[v; Y|X = Q(u; X)] = d[u; X|Y = Q(v; Y)]. \quad (4.3) \]

When \( X, Y \) are both continuous or both discrete, the copula density is the joint probability density (mass function) divided by the product of marginal probability densities (mass functions).

**EMPIRICAL COPULA DENSITY:** When \( X, Y \) continuous copula density is joint density of rank transforms \( F(X; X), F(Y; Y) \), estimated by sample mid-distribution transforms \( \tilde{F}^{\text{mid}}(X; X), \tilde{F}^{\text{mid}}(Y; Y) \)

**MULTI-DIMENSIONAL COPULA DENSITY:** The joint probability distribution of a vector \((X_1, \ldots, X_r)\) is described by marginal distributions and joint copula density \( \text{cop}(u_1, \ldots, u_r; X_1, \ldots, X_r) \) equal
\[
\prod_{k=2}^{r} d[u_k; X_k, X_k|X_1 = Q(u_1; X_1), \ldots, X_{k-1} = Q(u_{k-1}; X_{k-1})]. \quad (4.4)
\]

An indirect method of nonparametric regression estimation of \( E[Y|X] \) derives from the THEOREM:
\[
E[Y|X = Q(u; X)] = \int_0^1 Q(v; Y)d[v; Y, Y|X = Q(u; X)] \, dv. \quad (4.5)
\]

### 5 LP CORRELATIONS, LEGENDRE POLYNOMIAL SCORE FUNCTIONS, CUSTOM SCORE FUNCTIONS

To unify methods for discrete and continuous random variables custom construct score functions \( T_j(X; X) \), orthonormal functions of \( F^{\text{mid}}(X; X) \), by Gram Schmidt orthonormalization of the powers of \( T_1(X; X) = Z(F^{\text{mid}}(X; X)) \). Legendre polynomial like score functions on \( 0 < u < 1 \) are constructed \( S_j(u; X) = T_j(Q(u; X); X) \). For \( X \) continuous, \( S_j(u; X) = \text{Leg}_j(u), T_j(X; X) = \text{Leg}_j[F^{\text{mid}}(X; X)] \).

**FIGURE 5** CUSTOM SCORE FUNCTIONS \( S_j(u; \text{AGE}), j = 1, 2, 3, 4 \) have shapes (linear, quadratic, cubic, quadratic) similar to Legendre polynomial score functions

Model \((X, Y)\) diagnostics are LP moments and LP comoments (extending Serfling and Xiao)
Figure 5: The shapes of the first four score functions are shown for GAG data.

We compute the LP comoment matrix for the pair (AGE,GAG)

$$LP(\text{Age}, \text{GAG}) = \begin{bmatrix} -0.910 & -0.010 & 0.009 & 0.037 \\ 0.032 & 0.716 & -0.074 & 0.031 \\ 0.068 & 0.019 & -0.587 & 0.120 \\ -0.048 & -0.094 & -0.071 & 0.421 \end{bmatrix}$$

One can show that LP comoments are L2 orthonormal coefficients of copula density function.
Figure 6: The Nonparametric Copula Density Estimate.

given by

\[
\text{cop}(u,v;X,Y) - 1 = \sum_{j,k} \text{LP}[j,k;X,Y] S_j(u;X) S_k(v;Y).
\]  

(5.5)

This gives us a strategy to estimate the copula density nonparametrically utilizing the LP comoment matrix computed in (5.4), displayed in Figure 6.

The copula estimation also provides estimators of conditional density of \(Y\) given \(X = Q(u;X)\), and therefore by accept-reject simulation we generate samples from the conditional distribution \(f(y;Y|X = Q(u;X))\), shown in Figure 7 for \(u = .05, .25, .75, .95\). It is interesting to note the appearances of bimodality at the lower and the upper most extreme quantiles, which might have some biological relevance. It is also evident from the figure that the classical location-scale shift regression model is inappropriate for this example, which ne-
Figure 7: The Nonparametric conditional distributions.

Cessitates to go beyond the conditional conditional mean description for modeling the GAG data. Our conditional quantile curves (will be shown next) gives much complete picture of the effect of AGE on GAG level, which can tackle the non-Gaussian heavy tailed response (3.7).

From the simulated samples from the conditional distribution we estimation the conditional quantiles \( Q(v; Y|X = Q(u; X)) \), which is the ultimate solution to the problem of how the distribution of \( Y \) depends on the value of \( X \). On the scatter diagram of \( (X, Y) \) data plot \( Q(v; Y|X = x) = Q[v; Y|F(X; X) = F(x; X)] \) for \( v = .05, .25, .5, .75, .95 \). Figure 8
plots conditional quartiles for (AGE,GAG) computed from conditional comparison density $d(v; GAG, GAG|AGE)$ for median, quartile ages.

Information measures (Kullback-Leibler, Renyi, entropy, mutual information) of dependence measure the distance from 1 of $\text{cop}(u, v; X, Y)$ provided by integrals of $\log \text{cop}(u, v; X, Y)$, $|\text{cop}(u, v; X, Y) - 1|^2$. Important diagnostic is $\text{LPINFOR}(X, Y)$ estimated by sum of squares of model selected $\text{LP}(j, k; X, Y)$ comoments, denoted by bold symbols in (5.4). For (AGE,GAG) pair

$$\text{LPINFOR}(\text{Age}, \text{GAG}) = (-0.91)^2 + (0.716)^2 + (-0.587)^2 + (0.421)^2 = 1.863. \quad (5.6)$$

For $X, Y$ discrete the traditional Chi-square statistic is a “raw nonparametric” information measure, which we interpret by finding an approximately equal “smooth” information measure with far fewer degrees of freedom because it is the sum of squares of only a few data-driven LP moments, which is $\text{LPINFOR}(X, Y)$ for (X discrete, Y discrete).

### 6 TWO SAMPLE DATA MODELING

Our unification of small and big data starts with the fundamental (widely applicable) TWO SAMPLE data modeling problem, especially the traditional Student t test for the hypothesis $H_0$ of the equality of the populations means of two populations, and the nonparametric Wilcoxon rank statistic.

**STEP 0.** DATA. We have independent samples (observations, data) denoted $Y(t; 1), t = 1,, n_1$, and $Y(t; 2), t = 1,, n_2$. Define $n = n_1 + n_2$.

**STEP 1.** $(X,Y)$ DATA, SCATTER DIAGRAM PLOT. Combine two samples to form combined sample. Represent the two sample data as observations on joint variables $Y$ and $X$ where $X$ equals 1 or 2, for the population from which a $Y$ value is observed. Our observations are denoted $(X(t), Y(t)), t = 1, n$ where for $t = 1,, n_1 : X(t) = 1$ and $Y(t) = Y(t; 1)$; for $t = n_1 + 1,, n: X(t) = 2, Y(t) = Y(t - n_1; 2)$. The important step of looking at the data is achieved by a scatter diagram on the $(x, y)$ plane of the two dimensional points $(X(t), Y(t))$. The statistical method of regression fits a straight line to these points which can be interpreted to provide traditional two sample data analysis.

**STEP 2.** SAMPLE MEANS, POPULATION VARIANCES OF SAMPLE MEANS. Each population (indexed by $X$) has sample mean defined for $k = 1, 2$ as a conditional mean

$$M_k = M(Y|X = k) = (1/n_k) \sum_{t=1}^{n_k} Y(t; k) \quad (6.1)$$
A traditional approach to statistical learning states the statistical problem: “learn” from data the population conditional expectations $E[Y|X = k]$. The pooled sample is interpreted as observations of a variable $Y$ with unconditional population mean denoted $E[Y]$. Define population variance of $Y$ by $\text{Var}[Y] = E[(Y - E[Y])^2]$.

**STEP 3.** Mean Variance Big Idea

Fundamental formulas (linking conditional and unconditional means and variances) From properties of conditional expectation one can prove (Parzen, 1962)

**COMBINED MEAN VARIANCE THEOREM:** $E[Y] = E[E[Y|X]], \text{Var}[Y] = E[\text{Var}[Y|X]] + \text{Var}[E[Y|X]]$.

When $X$ is two valued $0, 1$, conditional and unconditional mean and variance are related

\[
E[Y] = \text{Pr}[X = 0]E[Y|X = 0] + \text{Pr}[X = 1]E[Y|X = 1], \quad (6.2)
\]

\[
\text{Var}[Y] = \text{Pr}[X = 0]\text{Var}[Y|X = 0] + \text{Pr}[X = 1]\text{Var}[Y|X = 1] + \\
\quad \text{Pr}[X = 0]\text{Pr}[X = 1](E[Y|X = 1] - E[Y|X = 0])^2. \quad (6.3)
\]

A proof is given below for sample means and variances.

**STEP 4.** REALISTIC STATISTIC TO TEST EQUALITY OF MEANS OF TWO SAMPLES. To test the null hypothesis $H_0$ that two population means are equal one can justify (from various principles of statistical inference) test statistic the difference of sample means

\[
\text{MDIFF} = M(Y|X = 2) - M(Y|X = 1) = M_2 - M_1 \quad (6.4)
\]

To interpret the observed value of MDIFF frequentist (Neyman Pearson) statistical inference first solves the sampling distribution problem: find exactly or approximately (for large samples) the sampling distribution of MDIFF. Under the null hypothesis $H_0$ the test statistic MDIFF has zero population mean and population variance (by the law from probability theory that the variance of a sum or difference of independent random variables is the sum of their variances) we can show

THEOREM : $\text{Var}[M_2 - M_1] = (1/n_1)\text{Var}[Y|X = 1] + (1/n_2)\text{Var}[Y|X = 2].$ \quad (6.5)

To continue the calculation of the variance of MDIFF one has a choice of assumptions (equal or unequal) about the population variance of $Y$ given $X = k$.

**STEP 5A.** UNEQUAL VARIANCE: Unequal variances of the two samples is the more realistic assumption, which we treat either by classical Bayesian analysis (posterior distribution of population mean given data), or by confidence quantile analysis (thinking Bayesian,
computing frequentist), with quantile approach advocated by \cite{Parzen2008, Parzen2013} while discussing the paper by \cite{XieSingh2013}. To compute the confidence distribution of population mean $\mathbb{E}[Y|X=k]$ given the data we derive an internal representation of the symbolic random variable $(\mathbb{E}[Y|X=k]|data)$ which we learn from inverting sampling distribution of sample mean $M_k$ with internal representation

$$M_k = \mathbb{E}[Y|X=k] + \sigma(M_k)Z, \ Z \text{ is Normal}(0,1).$$

Detailed practical formulas for two sample mean confidence quantiles are given by \cite{Parzen2008}.

**STEP 5B** EQUAL VARIANCE: We discuss the easier theory of the two sample mean problem assuming equal conditional variances $\text{Var}[Y|X=1] = \text{Var}[Y|X=2] = \text{Veq}$. THEOREM: Under assumption of equal variance Veq of two samples

$$\text{Var}[M_2 - M_1] = \text{Veq}(1/n_1 + 1/n_2) = \text{Veq}(n/n_1n_2)$$

**STEP 6:** ESTIMATED POPULATION VARIANCE OF DIFFERENCE OF SAMPLE MEANS IN EQUAL VARIANCE CASE. Notation for sample probabilities of $X = k$; define $\text{Pr}[X = k] = \tau_k = n_k/n$. Our notation $\tau$ is chosen to think of index $t$ as a time variable; sample is observed sequentially divided into a beginning sample and an ending sample (when $\tau_1$ is unknown estimating it is called change analysis or change point analysis, \cite{Parzen1992}).

THEOREM: variance of MDIFF, $\text{Var}[M_2 - M_1] = \text{Veq}\frac{\tau_1 \tau_2}{n}$

TRADITIONAL STUDENT TEST STATISTIC FOR EQUALITY OF MEANS:

$$T = (M_2 - M_1)\sqrt{(n-2)\frac{\tau_1 \tau_2}{\text{Veq}}}$$

(6.8)

When Veq is estimated and Y is assumed to be normally distributed the small sample sampling distribution of $T$ is Student’s distribution with $n - 2$ degrees of freedom.

**7 ESTIMATING POPULATION VARIANCE, SEQUENTIAL, BAYESIAN**

**STEP 1:** SAMPLE QUANTILE, SAMPLE VARIANCE ONE SAMPLE VARIABLE $Y$.

When one observes a sample $Y(t), t = 1, n$, of a variable $Y$ sample mean $M(Y)$ can be computed by the definition $M(Y) = n^{-1}\sum_{t=1}^{n} Y(t)$.  


An equivalent formula for computing $M(Y)$ is to determine the unique distinct values $y_1 < \cdots < y_r$ in the sample, compute sample probabilities (called sample probability mass function) $p(y_j; Y) = \text{Fraction } Y \text{ sample equal to } y_j$;

**THEOREM:** $M(Y) = \sum_{j=1}^{r} y_j p(y_j; Y)$

Sample Quantile function $Q(u; Y)$ of $Y$ provides definition of sample mean as area under a curve (and a computation sorting before adding). For $0 < u < 1$ define $Q(u; Y) = y_j$ on successive subintervals of length $p(y_j; Y)$.

**THEOREM:** $M(Y) = \int_{0}^{1} Q(u; Y) \, du$.

Example: The $X$ sample has distinct values 1,2; $p(1; X) = \tau_1$, $p(2; X) = \tau_2$, $M(X) = \tau_1 + 2\tau_2 = 1 + \tau_2$. Note $M(X - 1) = \tau_2$, $\Pr[(X - 1) = 1] = \tau_2$.

**STEP 2:** SAMPLE VARIANCE AND ADJUSTED VARIANCE: Sample variance of $Y$ is defined 

$$ \text{Var}[Y] = M[(Y - M(Y))^2] = \int_{0}^{1} [Q(u, Y) - M(Y)]^2 \, du. \quad (7.1) $$

Example: Verify that $\text{Var}(X) = \tau_1 \tau_2$

ADJUSTED VARIANCE: Many textbooks of statistics define sample variance by a definition which we call adjusted variance, defined $\text{VarAdj}(Y) = [n/(n - 1)] \text{Var}[Y]$.

When applied to $\text{Var}[X]$, this definition is not useful (although many computer packages mistakenly compute it). Our definition of sample variance leads to simpler formulas in applications. At the end of the analysis we will compute the same test $T$ statistics as are obtained using the adjusted variance concept by applying a factor $n - 1$ where traditional textbooks apply a factor $n$.

**STEP 3** UNIFYING FORMULAS! MEAN AND VARIANCE OF COMBINED SAMPLE:

When we observe two samples $(X, Y), X = 1 \text{ or } 2$, each sample has sample mean $M_k = M(Y|X = k)$ and sample variance $V_k = \text{Var}(Y|X = k)$.

The estimator of $V_{eq}$, denoted $V_{pool}$, is defined (more simply than in standard textbooks!)

$$ V_{pool} = \tau_1 V_1 + \tau_2 V_2 \quad (7.2) $$

The combined sample, composed of both observed samples, has sample mean $M(Y)$ and sample variance $\text{Var}[Y]$ which we want to compute from our knowledge of $M_1, M_2, V_1, V_2, \tau_1$.

Big Theorem: **FUNDAMENTAL FORMULA FOR MEAN AND VARIANCE OF COM-
\[ M = M(Y) = \tau_1 M_1 + \tau_2 M_2 = M_1 + \tau_2(M_2 - M_1) \quad (7.3) \]
\[ V = \text{Var}[Y] = \text{Var}_{\text{pool}} + (\tau_1 \tau_2)(M_2 - M_1)^2 \quad (7.4) \]

**PROOF:** First note that,
\[ nM(Y) = \sum_{t=1}^{n_1} Y(t; 1) + \sum_{t=1}^{n_2} Y(t; 2) = n_1 M_1 + n_2 M_2. \]

Now the total variance can be written as,
\[ n \text{Var}(Y) = \sum_{t=1}^{n_1} (Y(t; 1) - M(Y))^2 + \sum_{t=1}^{n_2} (Y(t; 2) - M(Y))^2 \]
\[ = n_1 V_1 + n_1 \tau_2^2 (M_1 - M_2)^2 + n_2 V_2 + n_2 \tau_1^2 (M_1 - M_2)^2. \]

Verify that \( \tau_1 \tau_2^2 + \tau_2 \tau_1^2 = \tau_1 \tau_2 \) to complete proof.

**STEP 4. RECURSIVE COMPUTATION MEAN VARIANCE COMBINED SAMPLE:**
Compute mean \( M_n(Y) \) and variance \( \text{Var}_n(Y) \) of sample of size \( n \) from mean \( M_{n-1}(Y) \) and variance \( V_{n-1}(Y) \) of first sample of size \( n - 1 \) and second sample consisting only of \( Y(n) \).
Note \( \tau_1 = (n - 1)/n, \tau_2 = 1/n. \)
\[ M_n(Y) = M_{n-1}(Y) + (1/n)(Y(n) - M_{n-1}(Y)) \quad (7.5) \]
\[ V_n(Y) = [(n - 1)/n]V_{n-1}(Y) + [(n - 1)/n^2](Y(n) - M_{n-1}(Y))^2 \quad (7.6) \]

Verify squariance \( nV_n(Y) \) can be represented as sum of squares of innovations \( Y_k - M_{k-1}(Y) \):
\[ nV_n(Y) = \sum_{k=2}^{n} (Y(k) - M_{k-1}(Y))^2(k - 1)/k \quad (7.7) \]

**STEP 5. BAYESIAN ESTIMATION MEAN VARIANCE NORMAL DATA CONJUGATE PRIOR:** Our formulas for mean and variance of combined sample can be applied to remembering update formulas \cite{Gelman} for Bayesian estimation of mean and variance of a normal sample, that are stated as parameter update formulas usually derived by extensive algebra. Prior distribution of population mean and variance can be interpreted as a first sample with sample size \( n_1 \), mean \( M_1 \), variance \( V_1 \). Observed sample is regarded as second sample with size \( n_2 \), sample mean \( M_2 \), sample variance \( V_2 \). We calculate formulas for posterior distribution of parameters by regarding it as combined sample of size \( n \), mean \( M \), variance \( V \).
8 CORRELATION UNIFICATION OF TRADITIONAL STATISTICS TO TEST $H_0$ EQUALITY OF TWO SAMPLE POPULATION MEANS

From statistics $M_1, M_2, M, V_1, V_2, V_{\text{pool}}, V$ compute

\[ R^2 = \tau_1 \tau_2 (M_2 - M_1)^2 / V \]  
\[ 1 - R^2 = V_{\text{pool}} / V \]  
\[ T^2 = R^2 / (1 - R^2) = \tau_1 \tau_2 (M_2 - M_1)^2 / V_{\text{pool}} \]  
\[ R^2 = T^2 / (1 + T^2) \]

Our statistics omit a multiplication factor based on pooled sample size $n$. We write the traditional Student test statistic for $H_0$ as $\sqrt{n - 2}T$. Its sampling distribution is Student distribution with $n - 2$ degrees of freedom when observations $Y$ are from Normal distribution.

CORRELATION INTERPRETATION OF TRADITIONAL TEST STATISTICS. The least squares straight line to the scatter diagram $(X(t), Y(t))$ has equation

\[ Y(t) - M(Y) = R\sqrt{V/\tau_1 \tau_2} (X(t) - M(X)) \]

equivalently $Z(Y(t)) = RZ(X(t))$. Recall $Z(Y(t)) = (Y(t) - M(Y)) / \sigma(Y(t))$, $Z(X(t)) = (X(t) - M(X)) / \sigma(X(t))$.

The important concept of correlation coefficient $R = \text{Cor}(X(t), Y(t))$ is defined

\[ R = \text{Cor}(X, Y) = M(Z(X(t))Z(Y(t))) = M((Y - M(Y))(X - M(X)) / \sqrt{V(Y)V(X)} \].

(8.6)

THEOREM: When $X$ is $0-1$ valued, computation of correlation is equivalent to computation of conditional mean of $Z(Y)$ given $X = 1$:

\[ \text{Cor}(X, Y) = M(Z(Y)|X = 1) \sqrt{\text{odds}(\text{Pr}[X = 1])} \]

(8.7)

Define for a probability $p$, $\text{odds}(p) = p/(1 - p)$.

THEOREM Traditional Student t statistic $T$ to test equality of two means of populations indexed by $X = 0, 1$ is up to a factor $\sqrt{n - 2}$ equivalent to $R/\sqrt{1 - R^2}$ where $R = \text{Cor}(X, Y)$, $\tau = \text{Pr}[X = 1]$, $M_1 = M(Y|X = 1), M_0 = M(Y|X = 0), M = M(Y)$ the pooled sample...
mean, and

\[
R = M(Z(Y)|X = 1)\sqrt{\text{odds}(\tau)} = (M_1 - M)/\sigma(Y)\sqrt{\text{odds}(\tau)}
\]

\[
= (M_1 - M_0)\sqrt{\tau(1 - \tau)/V}
\]  

(8.8)

Verify \( T = (M_1 - M_0)\sqrt{(1 - \tau)/V_{\text{pool}}}, \ V_{\text{pool}} = V\sqrt{1 - R^2}. \)

9 NONPARAMETRIC LINEAR RANK WILCOXON COMPARISON TWO POPULATIONS

Nonparametric rank Wilcoxon method tests equality of two populations by computing conditional mean in sample \( X = 1 \) of the ranks \( F_{\text{mid}}(Y; Y) \) in the pooled sample.

THEOREM [Parzen (2004)]: \( M = M(F_{\text{mid}}(Y; Y)) = .5; \ V = \text{Var}[F_{\text{mid}}(Y; Y)] = (1/12)(1 - \sum_j \text{Pr}(Y = y_j|^3)). \) Statistic equivalent to traditional Wilcoxon statistic

\[
W = (M_1 - .5)\sqrt{(1/1 - \tau)V} = \mathbb{E}[Z(F_{\text{mid}}(Y; Y)|X = 1)]\sqrt{\text{odds}(\text{Pr}[X = 1])}
\]

\[
= \mathbb{E}[Z(F_{\text{mid}}(Y; Y))]Z(F_{\text{mid}}(X; X)) = \text{LP}(1, 1; X, Y).
\]  

(9.1)

where is \( M_1 = M(F_{\text{mid}}(Y; Y)|X = 1). \) Asymptotic sampling distribution of \( \sqrt{n} W \) under null hypothesis \( H_0 \) is Normal(0, 1) [Alexander, 1989]. For small values of \( n \) one may prefer factor \( \sqrt{n - 1} \) or an approximation by a hypergeometric distribution.

DEFINITION: High order Wilcoxon statistics are LP comoments of high order score functions \( T_k(Y; Y): \)

\[
\text{LP}(1, k; X, Y) = \mathbb{E}[T_1(X; X)T_k(Y; Y)] = \sqrt{\text{odds}(\text{Pr}[X = 1])}\mathbb{E}[T_k(Y; Y)|X = 1]
\]  

(9.2)

From LP comoments one can compute coefficients \( C_k \) used to form orthonormal score series estimators of comparison density;

\[
C_k = \mathbb{E}[T_k(Y; Y)|X = 1] = \int_0^1 S_k(v; Y) d(v; Y, Y|X = 1) \ dv.
\]  

(9.3)

ALGORITHM Data driven orthonormal score function series estimator comparison density \( d(v) = d(v; Y, Y|X = 1) \) computed by AIC type model selection of coefficients \( C_k \) in smooth conditional comparison density estimator

\[
\hat{d}(u) = 1 + \sum_k C_k S_k(v; Y)
\]  

(9.4)
CLASSIFICATION: Classify population $X$ associated with observed value $Y$ by estimating

$$\Pr[X = 1|Y = Q(v; Y)]/\Pr[X = 1] = d(v; Y, Y|X = 1) \quad (9.5)$$

LOGISTIC REGRESSION: Our framework provides approach to identifying significant score functions to fit logistic regression models as an alternative to using parameter estimates to identify significant variables in the model. Using LP comoments identify score functions $T_k(y; Y)$ for logistic regression model

$$\log \text{odds } \Pr[X = 1|Y = y] = \sum \beta_k T_k(y; Y) \quad (9.6)$$

Logistic regression software provide alternative algorithms to estimation of comparison density.

HIGH DIMENSIONAL DATA MODELING A high dimensional classification estimates

$$\Pr[\text{class of observation}|\text{values of many features}]$$

To account for dependence in the features our theory starts with a Master Equation involving high dimensional copula functions whose practical estimation is implemented on real data in each application. To reduce computational problem of high dimensions we propose a Markovian approach which orders features $X_1, \ldots, X_r$ so that their dependence is Markovian - tree graphical model, which will be generalized to other structures subsequently.

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Figure 8: The Nonparametric conditional quantile curves.