The Bondi–Sachs metric at the vertex of a null cone: axially symmetric vacuum solutions

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**Abstract.** In the Bondi–Sachs formulation of General Relativity space-time is foliated via a family of null cones. If these null cones are defined such that their vertices are traced by a regular world-line then the metric tensor has to obey regularity conditions at the vertices. We explore these regularity conditions when the world line is a time-like geodesic. In particular, we solve the Einstein equations for the Bondi–Sachs metric near the vertices for axially symmetric vacuum space-times. The metric is calculated up to third order corrections with respect to a flat metric along the time-like geodesic, as this is the lowest order where non-linear coupling of the metric coefficients occurs. We also determine the boundary conditions of the metric to arbitrary order of these corrections when a linearized and axially symmetric vacuum space-time is assumed. In both cases we find that (i) the initial data on the null cone must have a very rigid angular structure for the vertex to be a regular point, and (ii) the initial data are determined by functions depending only on the time of a geodesic observer tracing the vertex. The latter functions can be prescribed freely, but if the vertex is assumed to be regular they must be finite and have finite derivatives along the geodesic.

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1. Introduction

The pioneering work of Bondi and Sachs led to a rigorous understanding of gravitational waves at large distances from a compact source (Bondi et al. 1962, Sachs 1962). They used a coordinate chart \((x^0, x^1, x^2, x^3) = (u, r, \theta, \phi)\) for the metric that was adapted to out-going null cones and where only one coordinate, the areal distance \(r\), varies along the out-going null geodesics generating these cones. The other coordinates are the ‘retarded time’ \(u\), and the standard spherical angles \(\theta\) and \(\phi\). Assuming a Minkowskian observer at infinitely far distances from an isolated gravitating object they showed that such a system can lose mass (energy) only via gravitational radiation. Their results were confirmed by Newman and Unti (1962) who used the Newman–Penrose formalism (Newman and Penrose 1962, 1963) to demonstrate the advantage of null foliations in General Relativity in order to characterize gravitational waves.

An observer at null infinity can only keep track of what is happening in the future or past of a source that emits gravitational waves. However, if one is interested in studying the emission process itself, e.g. by means of numerical simulations, one needs to consider conditions at the source. Tamburino and Winicour (1966) were the first to impose the boundary conditions for the gravitational field at a finite distance on the null cone, i.e. they placed a Minkowskian observer on a time-like world-tube of finite space-like radius, e.g. the radius of a star. This world-tube null formalism, where out-going null geodesics are attached to a world-tube, forms the basis of many numerical codes that are currently used to solve the Einstein equations in a domain extending from a finite-size world-tube to null infinity (see Winicour 2012a, for a recent review).

When the radius of the world-tube becomes zero, the world-tube degenerates into a world-line, i.e. into the time-like curve of an inertial observer tracing both the vertices of out-going null cones and the origin of the coordinate system. This so-called Fermi observer is a non-rotating observer in a rectangular Minkowskian coordinate system along the world-line. Isaacson et al. (1981) integrated the Einstein equations for the Bondi metric (Bondi 1960) when the world-tube had zero radius. The boundary conditions for the Bondi metric were chosen such that it approaches flat space values at the origin. To assure this behavior they restricted the fall-off of the metric functions towards \(r = 0\) to certain positive powers of \(r\). The work of Isaacson et al. motivated Gómez et al. (1994) and Siebel et al. (2002) to solve numerically the vacuum Einstein and the Einstein-fluid equations for the Bondi metric in axial symmetry from the vertex to null infinity. In their integration schemes, the boundary conditions for the Bondi metric included the lowest order curvature contribution. Both used a correct, but ad hoc, ansatz for the boundary conditions near the vertex, which they did not provide for all variables, however.

In this work we systematically investigate the boundary conditions of the Bondi–Sachs metric near the vertex. The vertex of a null cone is the focal point where all null geodesics generating the cone either converge to or emanate from. From the mathematical point of view, the null cone is not differentiable at its vertex, and consequently derivatives of the metric and therefore the curvature tensor cannot be calculated there. In particular, any tensor tensor that is expressed in terms of coordinates, like the Bondi–Sachs coordinates, adapted to the null cone cannot be expanded in terms of a Taylor series with respect to these coordinates at the vertex. Therefore, the boundary conditions for any fields expressed in Bondi–Sachs coordinates are a priori not known at the vertex. In principle these boundary conditions can

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be chosen freely. However, the vertex is the origin of an inertial observer along the geodesic, who implies regular boundary conditions. The assumption of such an observer at the vertex also requires that the vertex is a *regular point* for the considered tensor fields on the null cone (Dautcourt, 1967). A regular point $\mathcal{P}$ (the vertex) for a function $f$ (e.g. a metric component) on a open domain $\mathcal{W}$ (i.e. the null cone without the vertex) is defined as a point on the boundary $\partial \mathcal{W}$, where $f$ has a Taylor expansion based on $\mathcal{P}$ into $\mathcal{W}$ (Freitag and Busam, 2005). If the vertex is a regular point in the null cone, we will call it a *regular vertex*.

To find the proper boundary conditions of fields on a null cone at its vertex, we have to make additional assumptions:

(i) Assuming that the vertices of the null cones are traced by a world-line of a Fermi observer, we require this world line to be a time-like geodesic since then the acceleration of the curve vanishes, which greatly simplifies the analysis. This geodesic should be a *regular time-like geodesic*, i.e. its tangent vector should nowhere vanish along the curve.

(ii) The geodesic should be contained in a *convex normal neighborhood* (Hawking and Ellis, 1973). Such a neighborhood, $\Gamma$ say, has the property that two distinct points in $\Gamma$ can be connected by a unique geodesic and at any point, $\mathcal{P}$, in $\Gamma$ Riemann normal coordinates, $z^a$, can be established such that the metric in the neighborhood of $\mathcal{P}$ can be written as

$$g_{ab} = \eta_{ab} - \frac{1}{3} R_{acbd}(\mathcal{P}) z^c z^d + \ldots,$$

where $\eta_{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$ and $R_{acbd}(\mathcal{P})$ is the Riemann tensor evaluated at $\mathcal{P}$. In this regard, a metric is called a *regular metric* at a point $\mathcal{P} \in \Gamma$, if it is invertible at $\mathcal{P}$ and its determinant is finite and nonzero at $\mathcal{P}$.

These two assumptions are crucial to find the boundary conditions of the Bondi–Sachs metric at the vertex of the null cone. These two assumptions are necessary to find the boundary conditions. Assuming a regular metric of an inertial observer moving along a time-like geodesic, we will use coordinate transformations to find a Bondi–Sachs metric at the time-like geodesic. This Bondi–Sachs metric will not be regular at the vertex because the angular base vectors are not defined there. Nevertheless, by this procedure we will obtain the desired boundary conditions for the Bondi–Sachs metric corresponding to an observer in an inertial frame at the vertex, because the initial metric was a regular metric. To this end we will address four questions:

(I) What are the generic requirements of the boundary conditions of the Bondi–Sachs metric near the vertex of the null cone if the vertex is a regular point in the null cone?

(II) How can these boundary conditions be determined?

(III) What are the explicit boundary conditions for the Bondi–Sachs metric of an axially symmetric vacuum space-time?

(IV) What do the regular boundary conditions imply for the initial data on a null cone with a regular vertex for axially symmetric vacuum space-times?

To find the answers to these questions, we construct a Fermi normal coordinate system (Misner and Manasse, 1963) along the time-like geodesic. Based on this rectangular coordinate system, we then calculate a metric on a null cone where the radial coordinate is an affine parameter. This null metric is subsequently transformed
to a Bondi–Sachs metric, by changing the radial coordinate over to an areal distance coordinate. This has the advantage that the Einstein equation split into a hierarchy of hypersurface equations and evolution equations (Bondi et al., 1962, Winicour, 2012a) which simplifies their numerical treatment on a null cone. If an affine parameter is used instead, one of the hypersurface equations contains an additional time-derivative which is numerically more challenging (Winicour, 2012b). Alternatively, we could start with a regular metric along the time-like geodesic in deDonder coordinates as in Thorne (1986), Thorne and Hartle (1985), Zhang (1986) and calculating the boundary conditions for a Bondi–Sachs metric via an appropriate coordinate transformation. Since these authors derived their metric expansions for vacuum space-times, only, we do not use their expansions to answer question (I), since assuming a vacuum space-time restricts the generic properties of the boundary conditions (see Sec. 3.1). Second (and more importantly) starting from a Fermi normal system is simpler because the Fermi normal coordinates are geometrically adapted to the properties of the convex normal neighborhood. We will later compare our results with those of Zhang (1986), who obtains an expansion of the metric in the 3 + 1 formulation of space-time.

The behavior of arbitrary fields near the vertices of null cones was first discussed by Penrose (1963), who realized that these fields should be expanded in terms of normal coordinates defined at the point of origin of the expansion. His argument was formal providing only a sketch of the expansion. Friedrich (1986) considered the expansion for the case of the conformal vacuum Einstein equations, and, like Penrose, used the spin frame formalism to derive the connection and the curvature variables. As we are interested in the conditions at the vertex of the null cone in the Bondi–Sachs formulation, where the curvature is given as a function of the metric, the method of Penrose (1963) and Friedrich (1986) does not directly apply.

Ellis et al. (1985) discussed regularity conditions at the vertices of past null cones which are traced by a time-like geodesic of a cosmological (comoving) observer. Their work motivated Poisson and collaborators (Poisson, 2004, 2005, Poisson et al. 2006a, 2006b, 2010, 2011) to use light cone coordinates at a world-line to obtain the metric of a non-rotating black hole moving along the world line. As both Ellis et al. and Poisson and Vlasov used an affine parameter instead of an areal distance along along the null rays, they would have had to transform the former into the latter by an appropriate coordinate transformation to obtain the Bondi–Sachs metric.

In principle, we could have used the results of Ellis et al. and Poisson et al. as a starting point to answer questions (I)-(IV), but our aim is to present a complete and pedagogical derivation of the boundary conditions of a Bondi–Sachs metric at the vertex from a regular metric of an inertial observer along a time-like geodesic. With this in mind, we follow the approach of Ellis et al. (1985, App. A), but we provide additional pedagogical steps in the derivation and generalization of their results. Regarding Poisson and collaborators, we do not need the machinery of a bi-tensor theory or the Synge world function (Synge, 1960) to find a null metric with an affine parameter as radial coordinate. However, we will recover the results of Ellis et al. and Poisson et al. in our calculations.

Choquet–Bruhat et al. (2010, 2011) and Chruściel and Jezerski (2010) investigated the Cauchy problem of General Relativity for initial data given on the null cone, and Chruściel and Paetz (2012) took these studies up comparing the results with those of Friedrich (1986). They analyzed the ‘vertex problem’ raising questions about uniqueness and existence of solutions. Although these results are mathematically very interesting, they are too general to help in formulating boundary conditions
for a Bondi-Sachs metric, and in answering question (III) and (IV) above. We note that regularity issues at the origin of a curvi-linear coordinate system are also of importance in the 3+1 formulation of General Relativity. The first in-depth analysis of this problem for axially symmetric systems is due to Bardeen and Piran (1983).

The article is organized as follows: After defining our notation in the remainder of the introduction, we derive in Section 2 the limiting behavior of a Bondi–Sachs metric from a regular Fermi normal coordinate system, and answer question (I) concerning the generic properties of the Bondi–Sachs metric. In Section 3, we discuss and respond to question (II)-(III). In Section 4, we conclude with a summary and discussion of the results, and we respond to question (IV).

1.1. Notation

We assume the existence of a smooth, four-dimensional manifold $\mathcal{M}^4$ with a Lorentzian metric $g$ and signature $(-, +, +, +)$. The flat metric is always denoted with $\eta$. Greek indices $(\alpha, \beta, \ldots)$ and the small Latin indices $(a, b, \ldots, \Gamma)$ run from 0 to 3, while the small Latin indices $(i, j, k, \ldots)$ can assume the values 1, 2, or 3, respectively. Capital Latin indices $(A, B, C \ldots)$ label angular directions on a sphere and take the values 2 or 3 for the coordinates $(\theta, \phi)$ or $(y = -\cos \theta, \phi)$, respectively. The Kronecker symbol is $\delta_{ij} = \text{diag}(+, +, +, +).$ Unless stated otherwise, the Einstein summation convention holds. We use five distinct symbols for the four-dimensional space-time coordinates: (i) $y^\alpha$ are arbitrary coordinates; (ii) $z^a$ are Riemann normal coordinates; (iii) $y^a$ are Fermi normal coordinates; (iv) $\bar{x}^a$ are affine null coordinates; and (v) $x^a$ are Bondi–Sachs coordinates. Partial derivatives are denoted by a comma, and covariant derivatives by the $\nabla-$symbol. For the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$, the Riemann $R^\alpha_{\beta\mu\nu}$ and Ricci tensor $R_{\mu\nu}$, we follow the convention of Misner et al. (1972). Moreover, we use geometrized units where the speed of light and the gravitational constant are equal to unity. Associated Legendre polynomials of the first kind, $P_l^m(y)$, are defined as in Jackson (1999), i.e.

$$P_l^m(y) = \frac{(-1)^m}{2^l l!} (1 - y^2)^{m/2} \frac{d^{l+m}}{dy^{l+m}} (y^2 - 1)^l.$$

2. A Bondi–Sachs metric derived from a regular metric

2.1. A regular metric in Fermi normal coordinates

Let $(\mathcal{M}^4, g)$ be the four-dimensional space-time. Suppose $(\mathcal{M}^4, g)$ contains a simply convex normal neighborhood $\Gamma$ with a regular time-like geodesic $c(\tau)$, where $c^a$ is its unit, time-like tangent vector and $\tau$ the proper time. Since $\Gamma$ is a convex normal neighborhood and $c(\tau)$ is contained in $\Gamma$, a regular metric $g_{\alpha\beta}$ can be found along $c(\tau)$. Let $y^\alpha$ be a set of four-dimensional arbitrary coordinates along $c(\tau)$ such that the metric $g_{\alpha\beta}$ is regular and can be expanded along $c(\tau)$ like

$$g_{\alpha\beta}(y^\mu) = \left. g_{\alpha\beta} \right|_{c(\tau)} (y^\mu) + \left. g_{\alpha\beta,\kappa\lambda} \right|_{c(\tau)} y^{\kappa_1} + \frac{1}{2} \left. g_{\alpha\beta,\kappa_1\kappa_2} \right|_{c(\tau)} y^{\kappa_1} y^{\kappa_2} + \ldots \quad (2)$$

where the coefficients are evaluated along $c(\tau)$. Because of arguments given in section 2.3 below, it is sufficient to consider the expansion only up to the quadratic term.

At any point on $c(\tau)$ we choose an orthonormal tetrad $e_\alpha(\tau)$ which is parallel propagated along $c(\tau)$, i.e. $\nabla_c(\tau) e_\alpha = 0$. The time-like base vector of the coordinate system $e_0$ is tangent to $c(\tau)$, i.e. $e_0(\tau) = \partial/\partial \tau$. On any point of the geodesic $c(\tau)$, we
send out space-like geodesics \( b(\tau, n, \ell) \) which are parametrized by an affine parameter \( \ell \) and point into the direction \( n = n^i e_i(\tau) \), i.e. \( n^0 = 0 \) and \( n = \partial / \partial \ell|_{c(\tau)} \). The affine parameter \( \ell \) of the geodesics \( b(\tau, n, \ell) \) is defined to be zero along \( c(\tau) \). This specifies the coordinates \( y^a \) as Fermi normal coordinates \( y^a \) (Misner and Manasse, 1963, Ni and Li, 1979), which are given by

\[
y^0 := \tau , \quad y^i := n^i , \quad \ell(y^i) := \sqrt{\delta_{ij}y^j} ,
\]

(3)

where \( dn^i / d\tau = 0 \) and \( n^i = y^i / \ell \) are the direction cosines of the space-like geodesics \( b(\tau, n, \ell) \). Note that the three-dimensional coordinates \( y^i \) are Riemann normal coordinates \( \dagger \) for every value of \( \tau \) on \( c(\tau) \). Constructing the coordinates in that way assures that the metric along \( c(\tau) \) is the Minkowski metric, because the base vectors are orthonormal along \( c(\tau) \). In addition, it implies that the first order partial derivatives, \( g_{\alpha\beta\kappa\lambda} \), vanish along \( c(\tau) \), because of the parallel transport equation of \( n^i \) along \( c(\tau) \) and the geodesic equations of \( c(\tau) \) and \( b(\tau, n, \ell) \). The particular form of the second partial derivatives at the geodesic \( c(\tau) \) was first derived by Misner and Manasse (1963), and the corresponding metric reads

\[
\begin{align*}
g_{00}(y^a) &= -1 - R_{00ij}|_{c(\tau)}(\tau) y^i y^j + O([y^a]^3) , \\
g_{0k}(y^a) &= -\frac{2}{3} R_{0ijk}|_{c(\tau)}(\tau) y^i y^j + O([y^a]^3) , \\
g_{km}(y^a) &= \delta_{ij} - \frac{1}{3} R_{ikjm}|_{c(\tau)}(\tau) y^i y^j + O([y^a]^3) ,
\end{align*}
\]

(4)

where \( R_{abcd} \) are the Riemann normal components of a Fermi observer along the time-like geodesic \( c(\tau) \). These components are space-time invariants, i.e. knowing all twenty independent Riemann normal components along \( c(\tau) \) allows one to construct uniquely (up to spatial rotations) a metric up to quadratic terms of a power series expansion with respect to Fermi normal coordinates along the time-like geodesic in a sufficiently small neighborhood \( \Gamma \) of the geodesic.

2.2. An affine null metric

The null coordinate \( \bar{x}^0 := \bar{\tau}_w \) labels null cones whose vertices are along the time-like geodesic \( c(\tau) \), where \( \bar{\tau}_w \) is equal to the proper time \( \tau \) along the geodesic, i.e. \( \bar{\tau}_w|_{c(\tau)} = \tau \). For points not on \( c(\tau) \), \( \bar{\tau}_w \) is constant along parametrized null geodesics that emanate from \( c(\tau) \). We introduce the three spatial coordinates \( (\bar{x}^1, \bar{x}^2, \bar{x}^3) \). The radial coordinate \( \bar{x}^0 : = \ell \) is a positive affine parameter for null rays emanating from \( c(\tau) \), where \( \ell = 0 \). The two coordinates \( \bar{x}^A := (\bar{x}^2, \bar{x}^3) \), which are constant both along the time-like geodesic \( c(\tau) \) and the null geodesics, label the direction angles of the null rays. In the following, we will use the notation \( \Theta \) when we refer to an arbitrary point on \( c(\tau) \) that is also the vertex of an arbitrary null cone. For \( \ell \) to be an affine parameter, it has to obey the condition

\[
k^a \nabla_a \ell = 1 ,
\]

(5)

where \( \nabla_a \) is the covariant derivative with respect to the metric in Fermi normal coordinates. Equation (5) holds, if we define \( \ell \) as in (3) and choose the null vector \( k^a(y^b) := (1, y^j / \ell) \) in the Fermi normal coordinate system. Then \( \ell \) does not only

\[\dagger\] For a thorough discussion of Riemann normal coordinates, see e.g. Schouten (1954), Thomas (1991), and Iliev (2006).
measures the space-like distance from the geodesic \( c(\tau) \) in a Fermi frame, but it also
gives the positive affine distance between a vertex \( O \) and points on the respective null cone of \( O \). We note that an affine parameter along a curve is generally defined only up
to a constant. The choice \( \ell = 0 \) on \( c(\tau) \) implies \( b = 0 \). From \( a = \text{sign}(a/|a|) \) and the definition of \( \ell \) one sees that multiplying the Fermi normal coordinates \( y^i \) by \( |a| \) corresponds to scaling these coordinates, i.e. without loss of generality we can set \( |a| = 1 \). Hence, null rays \( k(\lambda) \) emanating from \( c(\tau) \) are most generally parametrized with the affine parameter \( \lambda = w\ell \), where \( w = \pm 1 \). We choose \( w = 1 \) for future-pointing null rays along \( c(\tau) \) and \( w = -1 \) for past-pointing ones by imposing the normalization condition
\[
\lim_{\ell \to 0} c_w k^\alpha = -w \tag{6}
\]
along \( c(\tau) \), where \( c^\alpha \) is the tangent vector of \( c(\tau) \).

A null geodesic \( k(\lambda) \) in the null-cone \( \tau_w = \text{const} \) emanating in \( \tilde{x}^A \)-direction,
admits an expansion in Fermi normal coordinates \( y^a \) with respect to the affine
parameter \( \lambda \) of the form (using \( \lambda = w\ell \) and \( w^2 = 1 \))
\[
y^a(\tilde{x}^\mu) = \tau_w \delta^a_{\mu} + \ell k^a \left[ y^b(\tilde{x}^A) \right] + \frac{\ell^2}{2!} \Gamma^a_{bc} \left[ y^b(\tilde{\tau}_w, \tilde{x}^A) \right] + \frac{w\ell^3}{3!} G^a \left[ y^b(\tilde{\tau}_w, \tilde{x}^A) \right] + \mathcal{O}(\ell^4), \tag{7}
\]
where the coefficient functions are evaluated along \( c(\tau) \). Setting \( \ell = 0 \) in (7) shows
that it describes the time-like geodesic \( c(\tau) \) with the tangent vector \( c^\alpha = \delta^\alpha_{\tau} \),
and that the tangent vector of the null geodesics \( k^a = (1/w)dy^a/d\ell|_{c(\tau)} \) does not
depend on \( \tau_w \), because the \( \tilde{x}^A \) are constant along \( c(\tau) \). If \( c(\tau) \) were no geodesic,
an additional dependence of \( \tau_w \), through the directional vector \( k^a \), must be taken into
account (Newman and Posadas, 1969). Hereafter, we use the null vector \( k^a \) in the form
\( k^a = (1, n^i(\tilde{x}^A)) \), where \( n^i \) is a three-dimensional unit vector being parametrized by
the angles \( \tilde{x}^A \).

The geodesic equations of the null rays \( k(w, \ell) \) read in Fermi normal coordinates
\[
\frac{d^2 y^a}{d\ell^2} = -\Gamma^a_{bc}(y^b) \frac{dy^b}{d\ell} \frac{dy^c}{d\ell}, \tag{8}
\]
and further differentiation with respect to \( \ell \) gives
\[
\frac{d^3 y^a}{d\ell^3} = -\Gamma^a_{bc,d}(y^b) \frac{dy^b}{d\ell} \frac{dy^c}{d\ell} \frac{dy^d}{d\ell}. \tag{9}
\]
Inserting (7) into (8) and (9), and equating the thus obtained relations along \( c(\tau) \) by
setting \( \ell = 0 \) leads to
\[
F^a(\tilde{\tau}_w, \tilde{x}^A) = 0, \quad G^a(\tilde{\tau}_w, \tilde{x}^A) = -\Gamma^a_{bc,d}(\tilde{\tau}_w)k^b(\tilde{x}^A)k^c(\tilde{x}^A)k^d(\tilde{x}^A). \]

In Fermi normal coordinates \( G^a \) vanishes, because the derivative \( \Gamma^a_{bc,d} \) can be expressed
as a linear function of the Riemann tensor \( R^a_{bc,d} \), which is antisymmetric in the last two
indices, and these two indices are contracted symmetrically. Hence, the parametric
representation of the null rays \( k(w, \ell) \) that emanate from the time-like geodesic \( c(\tau) \)
reads in the Fermi frame
\[
y^a(\tilde{x}^\alpha) = \tilde{\tau}_w \delta^a_{\tilde{\tau}_w} + \ell k^b(\tilde{x}^A) + \mathcal{O}(\ell^4). \]
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The metric in affine null coordinates, \( \bar{x}^\alpha \), is given by the coordinate transformation

\[
g_{\alpha\beta}(\bar{x}^\mu) = g_{ab}\left[ y^\mu(\bar{x}^\mu) \right] \frac{\partial y^a}{\partial \bar{x}^\alpha}(\bar{x}^\mu) \frac{\partial y^b}{\partial \bar{x}^\beta}(\bar{x}^\mu) ,
\]

and the Jacobian \( (\partial y^a/\partial \bar{x}^\alpha) \) between \( y^a \) and \( \bar{x}^\alpha \) by

\[
\frac{\partial y^a}{\partial \bar{x}^\alpha}(\bar{x}^\mu) = \delta^a_\tau + \frac{\ell^2}{2!} F^a_\tau(\bar{\tau}_w, \bar{x}^A) + \frac{w\ell^3}{3!} G^a_\tau(\bar{\tau}_w, \bar{x}^A) + O(\ell^4) ,
\]

\[
\frac{\partial y^a}{\partial \bar{\tau}_A}(\bar{x}^\mu) = wk^a(\bar{x}^A) + O(\ell^4) ,
\]

\[
\frac{\partial y^a}{\partial \bar{\tau}_A}(\bar{x}^\mu) = w\ell k^a(\bar{x}^B) + \frac{\ell^2}{2!} F^a_A(\bar{\tau}_w, \bar{x}^B) + \frac{w\ell^3}{3!} G^a_A(\bar{\tau}_w, \bar{x}^B) + O(\ell^4) ,
\]

which contains the derivatives of the coefficients \( F^a \) and \( G^a \) with respect to the null coordinates. Although these coefficients vanish at \( \ell = 0 \), their derivatives are not necessarily zero there, i.e. we have to calculate the derivatives first, and subsequently evaluate them in the limit \( \ell = 0 \). Using the first derivatives of the Christoffel symbols in a Fermi normal coordinate system as in Ni and Li (1979), we calculated the metric \( g_{\alpha\beta}(\bar{x}^\alpha) \) in affine null coordinates near the time-like geodesic \( c(\tau) \):

\[
g_{00}(\bar{\tau}_w, \ell, \bar{x}^A) = -1 - \ell^2 A(\bar{\tau}_w, \bar{x}^A) + O(\ell^3) ,
\]

\[
g_{0A}(\bar{\tau}_w, \ell, \bar{x}^A) = -\frac{2}{3} w\ell^3 B_A(\bar{\tau}_w, \bar{x}^A) + O(\ell^4) ,
\]

\[
g_{1\mu}(\bar{\tau}_w, \ell, \bar{x}^A) = -w\delta^\mu_\mu ,
\]

\[
g_{AB}(\bar{\tau}_w, \ell, \bar{x}^A) = \ell^2 q_{AB}(\bar{x}^C) - \frac{\ell^4}{3} S_{AB}(\bar{\tau}_w, \bar{x}^A) + O(\ell^5) ,
\]

where \( q_{AB}(\bar{x}^C) = \delta^i_\mu n^j_A n^k_B \) is a unit sphere metric around \( \partial \), because \( n^i(\bar{x}^A) \) is a unit vector labeling points on a sphere, and the coefficients

\[
A(\bar{\tau}_w, \bar{x}^C) = R_{0\alpha\beta\gamma}(\bar{\tau}_w) k^a(\bar{x}^C) k^b(\bar{x}^C) k^c(\bar{x}^C) ,
\]

\[
B_A(\bar{\tau}_w, \bar{x}^C) = R_{0\alpha\beta\gamma}(\bar{\tau}_w) k^a(\bar{x}^C) k^b(\bar{x}^C) k^c(\bar{x}^C) k^\gamma(\bar{x}^C) ,
\]

\[
S_{AB}(\bar{\tau}_w, \bar{x}^C) = R_{\alpha\beta\gamma\delta}(\bar{\tau}_w) k^a(\bar{x}^C) k^b(\bar{x}^C) k^c(\bar{x}^C) k^d(\bar{x}^C) ,
\]

depend on the Riemann normal components \( R_{\alpha\beta\gamma\delta}(\bar{\tau}_w) \) evaluated along the geodesic \( c(\tau) \), i.e. \( R_{\alpha\beta\gamma\delta}(\bar{\tau}_w) = R_{\alpha\beta\gamma\delta}(\bar{\tau}_w) \). We point out that the components \( g_{1\mu} \) are exact, and as they are equal to a constant, they do not require an order symbol. In particular, \( |g_{10}| = 1 \), because \( \ell \) is an affine parameter.

The components of the affine null metric (11a)-(11d) obtained by us agree with those in the literature (Ellis et al. 1985, Poisson et al. 2010). While Ellis et al. (1985) calculated a null metric at a past null cone only, we derived the metric for points both in a past and future null cone, which is achieved by setting \( w = 1 \) (future) or \( w = -1 \) (past) in our solution. The difference between our work and that of Poisson et al. (2010) concerns the computational approach of how to find the affine null metric along \( c(\tau) \). Poisson et al. (2010) use the Synge world function (Synge, 1960) to obtain the metric, whereas we use an explicit coordinate transformation based on a regular Fermi normal coordinate system. Both approaches are legitimate and it is a matter of taste which one prefers.
2.3. The Bondi–Sachs metric

Using the affine null metric $g_{\mu\nu}(\vec{x}^\alpha)$ derived in the previous subsection, we calculate a Bondi–Sachs metric and the corresponding Bondi–Sachs coordinates are labeled by $x^\alpha$. The null coordinate $x^0 = \tau_w$ and the angular coordinates $x^4$ are defined like their barred counterparts, $\bar{\tau}_w$ and $\bar{x}^A$, respectively. The radial coordinate $x^1 := r$ is a positive area distance coordinate, i.e. surfaces $d\tau_w = \text{const}$ and $dr = \text{const}$ have the area $4\pi r^2$. This also requires that the determinant of the 2-metric $g_{AB}$ divided by $r^4$ does not vary with time and radius.

The areal distance $r$ is defined by (Jordan et al. (1961), Sachs, 1961, Newman and Penrose, 1963)

$$k^\mu \nabla_\mu r = r \Theta(k) ,$$  

where $\Theta(k) := \frac{1}{2} \nabla_\mu k^\mu$ is the expansion rate of null rays with tangent vector $k^\mu$. Inserting the null vector $k^\mu(\vec{x}^\alpha) = w\delta^\mu_r$ into equation (13) we obtain

$$r(\vec{x}^\mu) = \ell \left( 1 - \frac{\ell^2}{3} S(\bar{\tau}_w, \bar{x}^A) + O(\ell^4) \right)^{1/4} , \quad S(\bar{\tau}_w, \bar{x}^A) := q^{AB}(\bar{x}^A)S_{AB}(\bar{\tau}_w, \bar{x}^A).$$  

The equation shows that the affine parameter is equal to the areal distance $r$ both for $|\ell| \approx 0$ and in a flat space-time. The latter holds, because for a flat metric $S$ vanishes due to its dependence on the Riemann tensor.

We define a coordinate transformation between the affine null coordinates $\vec{x}^\mu$ and the Bondi–Sachs coordinates $x^\mu$ according to

$$\tau_w = \bar{\tau}_w , \quad r(\vec{x}^\mu) = \ell - \frac{\ell^3}{12} S(\bar{\tau}_w, \bar{x}^A) + O(\ell^4) , \quad x^A = \bar{x}^A$$

Applying this coordinate transformation to the metric components (11a) - (11d) results in the following Bondi–Sachs metric near the geodesic c(τ):

$$g_{00}(x^\mu) = -1 - r^2 A(\tau_w, x^C) + O(r^3) , \quad (16a)$$

$$g_{0A}(x^\mu) = -\frac{2}{3} w r^3 B_A(\tau_w, x^C) + O(r^4) , \quad (16b)$$

$$g_{1\mu}(x^\mu) = -w \left[ 1 + \frac{r^2}{4} S(\tau_w, x^C) + O(r^3) \right] \delta^\mu_0 , \quad (16c)$$

$$g_{AB}(x^\mu) = r^2 q_{AB}(x^C) - \frac{r^4}{2} \left[ S_{AB}(\tau_w, x^C) - \frac{1}{2} q_{AB}(x^C)S(\tau_w, x^C) \right] + O(r^5) . \quad (16d)$$

Note that the above metric is indeed of the Bondi–Sachs type, because the determinant of $g_{AB}/r^2$ is a function of $x^A$, only.

Traditionally, the Bondi–Sachs metric is written in the form (Bondi 1960, Sachs, 1962)

$$ds^2 = -e^{2\Phi+4\beta} d\tau_w^2 - 2we^{2\beta} d\tau_w dr + r^2 h_{AB} \left( dx^A - U^A d\tau_w \right) \left( dx^B - U^B d\tau_w \right) , \quad (17)$$

where the determinant of the 2-metric, $\det(h_{AB})$, is a function of $x^A$ only. Because of this restriction, the tensor $h_{AB}$ possesses only two degrees of freedom. Using the standard spherical coordinates as angular coordinates, $x^A = (\theta, \phi)$, allows us to write $h_{AB}$ in the form (van der Burg, 1966)

$$h_{AB} dx^A dx^B = e^{2\gamma} \cosh(2\delta) d\theta^2 + 2 \sin \theta \sinh(2\delta) d\theta d\phi + e^{-2\gamma} \cosh(2\delta) \sin^2 \theta d\phi^2 . \quad (18)$$

\(\uparrow\) (1) Bondi and Sachs, and also most numerical relativists (see Winicour (2012) for a review) use retarded time $u := \tau_+ \uparrow$ as time coordinate; (2) for computational convenience, we renamed the original metric function $V/r$ as $\exp(2\Phi + 2\beta)$ in the metric component $g_{00}(x^\alpha)$. 

\(\uparrow\)
By linearizing $h_{AB}$ with respect to $\gamma$ and $\delta$, it can be seen that $\gamma$ and $\delta$ correspond to the two components of a two-dimensional transverse and trace-less tensor, e.g. the tensor $\chi_{AB}$ in Chuâśiel et al. (1998, 2002). According to (17) the Bondi–Sachs metric $g_{\alpha\beta}(x^\mu)$ is not defined at $r = 0$, because the four-dimensional volume element $\sqrt{-g}$ vanishes there.

For convenience we introduce the quantity $\mathcal{F} \in \{\gamma, \delta, \beta, \Phi, U^A\}$ to abbreviate the set of the six functions describing the Bondi–Sachs metric. To find the behavior of $\mathcal{F}$ at the vertex, we assume that $\mathcal{F}$ can formally be expanded into a power series at $r = 0$, i.e.

$$\mathcal{F}(x^\mu) = \mathcal{F}^{(0)}(\tau_w, x^A) + r\mathcal{F}^{(1)}(\tau_w, x^A) + \frac{r^2}{2!}\mathcal{F}^{(2)}(\tau_w, x^A) + \ldots ,$$  \hspace{1cm} (19)

where the coefficient functions $\mathcal{F}^{(n)}(\tau_w, x^A)$ are evaluated along the geodesic $c(\tau)$. Inserting the expansion of each element of $\mathcal{F}$ into (17) and (18) results in a series expansion for every component of the Bondi–Sachs metric in terms of the elements of $\mathcal{F}$. A comparison of these series expansions with the expression given in equations (16a) - (16d) yields the following relations between $\mathcal{F}$ and the contracted Riemann normal components $A, B_A$ and $S_{AB}$:

1. $\gamma(x^\alpha) = -\frac{r^2}{12}\frac{S_{\theta\theta}(\tau_w, x^A)}{\sin^2 \theta} - \frac{r^2}{6}\frac{S_{\phi\phi}(\tau_w, x^A)}{\sin \theta} + O(r^3)$, \hspace{1cm} (20a)
2. $\delta(x^\alpha) = -\frac{r^2}{6}\frac{S_{\phi\phi}(\tau_w, x^A)}{\sin \theta} + O(r^3)$, \hspace{1cm} (20b)
3. $\beta(x^\alpha) = \frac{r^2}{8}S(\tau_w, x^A) + O(r^3)$, \hspace{1cm} (20c)
4. $U^A(x^\alpha) = -\frac{2}{3}r\epsilon^{AC}B_C(\tau_w, x^D) + O(r^2)$, \hspace{1cm} (20d)
5. $\Phi(x^\alpha) = \frac{r^2}{2}[A(\tau_w, x^D) - \frac{1}{2}S(\tau_w, x^D)] + O(r^3)$ \hspace{1cm} (20e)

Equations (20a)-(20e) show how the traditionally-used Bondi–Sachs metric functions $\mathcal{F}$ behave to lowest order near the vertex of the null cone, when the vertex coincides with the origin of a Fermi normal coordinate system along $c(\tau)$. Hereafter, these lowest order corrections are also referred to as first-order corrections, $\mathcal{C}_1$, of the Bondi–Sachs metric with respect to the flat metric at the vertex along the timelike geodesic or briefly first-order corrections of the Bondi-Sachs metric. These first order corrections $\mathcal{C}_1$ arise at $O(r)$ for $U^A$ and at $O(r^2)$ for $\beta, \gamma, \delta$, and $\Phi$, where as higher-order correction coefficients $C_n$ are expected at $O(r^n)$ for $U^A$ and at $O(r^{n+1})$ for $\beta, \gamma, \delta$, and $\Phi$ respectively.

Since equations (20a)-(20e) were derived form a regular metric at the vertex and the vertices were assumed to be on a timeline geodesic, we deduce (without solving the Einstein equations) the following regularity requirements on a power series expansion of the traditionally-used functions $\mathcal{F}$, which assure that the Bondi–Sachs metric is regular at its origin:

1. The power series of the metric functions $\mathcal{F}$ in $r$ must start at $r = 0$ with a certain positive power of $r$, i.e. $\gamma, \delta, \beta, \Phi$ are of $O(r^2)$ and $U^A$ of $O(r)$.
2. The angles $x^A$ must parameterize topological spheres centered at the vertex and the radial coefficients of $\mathcal{F}$ must show a specific angular behavior determined by
the contractions of the Riemann normal components \( \parallel \) with the tangent \( e^\alpha \), the null vector \( k^\alpha \), and \( k^\alpha_A \) along \( e(\tau) \).

In Sec. 3.2.1, we will show that the topological requirement is crucial for regularity, since it is also possible to have non-regular solutions at the vertex, when this assumption is not imposed.

(iii) The radial expansion of \( \mathcal{F} \) have specific numerical factors in the radial expansion coefficients, for example the factor \( 1/12 \) in (20a).

Two additional important properties of the functions \( \mathcal{F} \) at the vertex cannot be inferred from the first-order corrections in (16a)-(16d), because they manifest themselves only at second- or third-order deviations from the flat metric at \( r = 0 \). While the first-order corrections originate from the second derivative of the Fermi metric at the origin through a coordinate transformation, the second and third-order corrections result from the third and fourth derivative of the Fermi metric. To learn more about the two missing properties we consider a qualitative argument, which also avoid tedious calculations of the corresponding coordinate transformations. We recall that in Fermi normal coordinates the Christoffel symbols vanish along \( e(\tau) \), and that in any coordinate system, where the Christoffel symbols vanish, the following correspondence can be made between third and fourth partial derivatives of the metric and the Riemann tensor (Schouten, 1954):

\[
\partial^3 g = L \left[ \nabla (\text{Riem}) \right], \tag{21a}
\]

\[
\partial^4 g = L \left[ \nabla^2 (\text{Riem}), (\text{Riem})^2 \right], \tag{21b}
\]

where \( \partial^n g \) is the \( n^{th} \)-partial derivative of the metric, \( L(\cdot) \) a linear functional of its arguments, \( (\text{Riem}) \) the Riemann normal components, and \( \nabla^n \) the \( n^{th} \)-covariant derivative, respectively. The explicit dependence between the third and fourth-order partial derivatives of the metric and the Riemann tensor in Fermi normal coordinates is given in Ni and Li (1979), Dolgov et al. (1983), and Ishii et al. (2005).

To calculate the Bondi–Sachs metric including the second and third-order corrections, the corresponding affine null metric must be determined first. Following and extending schematically the procedure of section 2.2 to higher order in the approximation reveals that the affine null metric involves time derivatives of the Riemann normal components at the vertices on \( e(\tau) \). These derivatives arise because covariant derivatives of the Riemann tensor in the Jacobian are contracted, like \( e.g. \ n^i(x^A)\nabla_i A \). Starting at second-order corrections, these time derivatives show up in hierarchical manner, \( i.e. \) the second-order corrections contain first-order time derivatives of first-order corrections to the flat metric at the vertex, like \( e.g. \ A_{\tau_w} \). Similarly, the third-order corrections contain second-order time derivatives of first-order corrections, like \( e.g. \ A_{\tau_w \tau_a} \). Moreover, there are first-order time derivatives of second-order corrections in the third-order correction coefficients, like \( e.g. \) the time derivative of the second-order correction \( n^i(x^A)\nabla_i A \).

At this stage, it is instructive to compare where time derivatives of the Riemann normal components occur in the metric expansion using different coordinate conditions. In table 1, we list the occurrence of the contracted Riemann normal components \( \parallel \) with the tangent \( e^\alpha \), the null vector \( k^\alpha \), and \( k^\alpha_A \) along \( e(\tau) \). Physically speaking, this requirement means that if the space-time \( (\mathbb{M}^4, g) \) is curved, the path of a null ray emanating from the vertex is affected in the neighborhood of the vertex \( (i.e. \) the Fermi observer) by the curvature (the Riemann tensor and its covariant derivatives) at the vertex.
The Bondi–Sachs metric at the vertex of a null cone

Table 1. Occurrence of the contracted Riemann normal component \( \mathcal{R}_{00} n^i n^j \), their contracted spatial covariant derivatives and their respective time derivative along the time-like geodesic in the metric component \( g_{00} \) in affine null coordinates, Fermi normal coordinates and de Donder coordinates. The order of the correction coefficient \( \mathcal{C}_n \) corresponds to the power of \( r^{n+1} \) in the radial expansion of \( g_{00} \).

| order of correction coefficient | affine null coordinates | Fermi normal coordinates | deDonder coordinates |
|-------------------------------|------------------------|------------------------|----------------------|
| \( \mathcal{C}_1 \)           | \( \mathcal{R}_{0j0n} n^n \) | \( \mathcal{R}_{0j0n} n^n \) | \( \mathcal{R}_{0j0n} n^n \) |
| \( \mathcal{C}_2 \)           | \( \mathcal{R}_{0j0,k} n^n n^n \) | \( \mathcal{R}_{0j0,k} n^n n^n \) | \( \mathcal{R}_{0j0,k} n^n n^n \) |
| \( \mathcal{C}_3 \)           | \( \mathcal{R}_{0j0,kl} n^n n^n n^n \) | \( \mathcal{R}_{0j0,kl} n^n n^n n^n \) | \( \mathcal{R}_{0j0,kl} n^n n^n n^n \) |

\[ \partial (\mathcal{R}_{0j0n} n^n) / \partial \tau_w \]
\[ \partial^2 (\mathcal{R}_{0j0n} n^n) / \partial \tau_w^2 \]

The components \( \mathcal{R}_{0j0} \) and their spatial covariant derivatives in the first three correction coefficients of \( g_{00} \) to a flat metric on the geodesic for an expansion in Fermi-normal coordinates, in deDonder coordinates and in affine null coordinates, respectively. The behavior for the Fermi normal coordinates can be taken from Ni and Li (1979), Dolgov et al. (1983), or Ishii et al. (2005), and the one for the deDonder coordinates from Zhang (1986, eq. 3.26). In Fermi normal coordinates there are no time derivatives of the Riemann normal components or their spatial derivative, whereas in deDonder coordinates the lowest (second) order time derivative occurs in the third order correction coefficient. In affine null coordinates, however, we recognize a hierarchical order of the time derivative of the Riemann normal components and their spatial covariant derivatives.

Following the same line of arguments as for the time derivatives, one can show that the third-order correction coefficients depend on the square of the Riemann normal components. This behavior of the affine null metric near the vertex was also observed by Poisson and Vlasov (2010), who derived an affine null metric for vacuum spacetimes. Since the Bondi–Sachs metric and the affine null metric are related by a transformation of the radial coordinate only, everything that we said above about the affine null metric applies to the functions \( \mathcal{F} \), too.

As normal coordinates exist along \( c(\tau) \) and as a power series of a metric in normal coordinates can be transformed into a power series of a metric in Bondi-Sachs coordinates, regularity implies the following restrictions on the power series expansion coefficients of \( \mathcal{F} \) at the vertex:

(iv) The correction coefficient \( \mathcal{C}_n \), \( n > 1 \) depends on the time derivatives of order \( (n-k) \) of the correction coefficient \( \mathcal{C}_k \), \( 1 \leq k < n \) and these time derivatives must be finite.

\( \mathcal{F} \) In deDonder coordinate, these time derivatives also occur in hierarchical order, i.e. the \( \mathcal{C}_{2k} \) correction coefficients of the metric along the geodesic depends on the \( (2k)^{th} \) time derivative of the \( k^{th} \) covariant derivative of \( \mathcal{R}_{0j0} \).
(v) Non-linear coupling occurs at lowest order near the vertex in the third-order correction coefficient of $F$, i.e. at $O(r^4)$ for $\gamma, \delta, \beta, \Phi$, and at $O(r^3)$ for $U^A$.

The requirements $(i)-(v)$ state the general properties of the boundary conditions for the Bondi–Sachs metric functions $F$, when the vertex is a regular point in the null cone which is traced by a time-like geodesic. As these properties have been derived without the solution of the Einstein Equations (they can be considered as “kinematical conditions”), they depend on the solution of Einstein equations (which is a “dynamical condition”).

We summarize the general requirements on the boundary conditions of the functions $F$ at $O$ in the following Vertex Lemma:

*Let the Bondi–Sachs metric functions, $F$, be represented by power series expansions with respect to the areal distance coordinate $r$ at the vertex of a null cone on a time-like geodesic. If the coefficients of this radial expansion obey

1. the general properties $(i)-(v)$,
2. and the Einstein equations,

then this power series expansion of $F$ can be used to formulate boundary conditions for the metric functions $F$ at the vertex of the null cone, and the vertex is a regular point in the null cone.*

In the next section, we explicitly calculate the boundary conditions for $F$ at the vertex for axisymmetric vacuum space-times. We choose these space-times, to complete and extend the boundary conditions as used by Gomez et al. (1994) and Siebel et al. (2002), and to recover and justify their ad hoc assumptions on the boundary condition employed in their numerical algorithms. In addition, axisymmetric space-times are the simplest ones to show all the relevant addressed in the vertex lemma.

**3. Solutions for axially symmetric space-times**

A four-dimensional axially symmetric space-time ($M^4, g$) is a space-time containing a time-like 2-surface $A$ consisting of points that are invariant under the action of a one-parametric, cyclic group $G$ that is isomorphic to SO(2) (Carter, 1970, Stephani et al. 2003, and references therein). In particular, since the metric is invariant under the action of $G$, i.e. there exists a Killing vector field $\xi(\phi)$ in ($M^4, g$), where $\phi$ is the parameter of the group $G$. The vector field $\xi(\phi)$ is constant along the orbits of the group action, and the Lie derivative $\mathcal{L}_{\xi(\phi)}g_{\mu\nu}$ vanishes along the curves generated by $\xi(\phi)$. It can be shown (Carot, 2000, and references therein) that the 2-surface $A$, the so-called axis of symmetry, is auto-parallel. Since ($M^4, g$) is endowed with a metric, the fact that $A$ is auto-parallel is equivalent to $A$ being totally geodesic (Spivak, 1999). As $A$ is time-like, the axis of symmetry contains a family of time-like geodesics. Since these geodesics distinguish themselves from other geodesics in $\mathcal{M}^4$ due to their axial symmetry, we call them axial geodesics. They are the natural choice to trace the origin (vertex) of a Bondi–Sachs coordinate system in an axially symmetric space-time.

Hereafter, let $c(\tau)$ be an axial geodesic that is normalized in such a way that $\tau$ is the proper time. Given an orthonormal tetrad $e_\mu(\tau)$ along $c(\tau)$, we define a

$^+$ Spherically symmetric space-times are of no use here, because they possess no angular structure, i.e. property (ii) of the vertex lemma cannot be demonstrated.
Fermi normal coordinate system \(y^a\) along \(c(\tau)\), where \(\partial/\partial \tau\) is tangent to \(c(\tau)\) and \(\tau\) is the time coordinate. We further choose three mutually orthogonal space-like vectors \(\partial/\partial y^i\) at every point on \(c(\tau)\), which are parallel transported along the axial geodesic. Of these three vectors, the two vectors \(\partial/\partial y^1\) and \(\partial/\partial y^2\) are normal to the time-like 2-surface \(\mathcal{A}\), while the vector \(\partial/\partial y^3\) is tangent to it. When \(y^0 = \text{const}\) and \(y^1 = \text{const}\), the coordinates \(y^1\) and \(y^2\) label points of Killing orbits of \(G\) in \(\mathcal{M}\) whose fixed-points are located on the axial geodesic \(c(\tau)\). When instead \(y^0 = \text{const}\) and \(y^1 = 0 = y^2\) holds, the coordinate \(y^3\) labels points on the symmetry axis \(\mathcal{A}\). In the Fermi normal coordinate system introduced above the Killing vector has the form (Carot, 2000)

\[
\xi^a(y^b) = y^2 \delta^a_1 - y^1 \delta^a_2 \quad .
\]

In the following, we will present two approaches to determine the behavior of the Bondi–Sachs metric functions \(\mathcal{F}\) near \(c(\tau)\). In the first approach, in Section 3.1, we solve the Einstein equations in the Fermi normal coordinate system to obtain the Riemann normal components, which are then used to calculate the Bondi–Sachs metric functions from the expressions (12a) – (12e). In the second approach, in Section 3.2, we expand the metric functions \(\mathcal{F}\) into a power series obeying both axial symmetry and the limiting behavior of \(\mathcal{F}\) near \(r = 0\) as given in (20a) - (20e). Subsequently, we solve the vacuum Einstein equations in Bondi–Sachs coordinates for the power series coefficients.

### 3.1. Lowest order non-trivial boundary conditions for the Bondi–Sachs metric derived directly from the Fermi metric

The vacuum Einstein equations are \(\mathcal{R}_{\mu\nu} = 0\). In axial symmetry, every tensor \(T\) has to obey the Killing condition, i.e. the Lie derivative \(\mathcal{L}_\xi T\) has to vanish. If we apply this condition to the Riemann normal components \(\mathcal{R}_{abcd}\) in a Fermi normal coordinate system using the Killing vector (22), we find the following relations among the non-zero components of \(\mathcal{R}_{abcd}\):

\[
\mathcal{R}_{0101} = \mathcal{R}_{0202} \ , \mathcal{R}_{0113} = \mathcal{R}_{0223} \ , \mathcal{R}_{0312} = -2\mathcal{R}_{0123} \ , \mathcal{R}_{1313} = \mathcal{R}_{2323} \ , \mathcal{R}_{0303} \ , \mathcal{R}_{1212}.
\]

(23)

For the further discussion we conveniently combine these non-zero components into a set \(\mathcal{I} \in \{A, B, C, D, E, F\}\) with

\[
A := \mathcal{R}_{0101} \ , \ B := \mathcal{R}_{0303} \ , \ C := \mathcal{R}_{0113} \ , \ D := \mathcal{R}_{0123} \ , \ E := \mathcal{R}_{1212} \ , \ F := \mathcal{R}_{1313} \quad .
\]

(24)

The Ricci tensor reads in Fermi normal coordinates

\[
\mathcal{R}_{\alpha\beta}(y^a) = \left(\begin{array}{cccc}
-2A - B & 0 & 0 & -2C \\
0 & A - \frac{1}{2}(E + F) & 0 & 0 \\
-2C & 0 & A - \frac{1}{2}(E + F) & 0 \\
0 & 0 & 0 & B - F
\end{array}\right)_{\mathcal{I}(c(\tau))} + \mathcal{O}(y^a) \quad ,
\]

(25)

where the functions \(\mathcal{I}\) are evaluated along the axial geodesic \(c(\tau)\). We note that the function \(D|_{c(\tau)}\) is not determined by the Ricci tensor. Hence, it is not determined by the vacuum Einstein equations in Fermi normal coordinates, and thus can be prescribed freely at the vertices of the null cones. Solving the vacuum Einstein equations for the zeroth-order coefficient of the expansion (25) results in the solution

\[
B|_{c(\tau)} = -2A|_{c(\tau)} \ , \ C|_{c(\tau)} = 0 \ , \ E|_{c(\tau)} = 4A|_{c(\tau)} \ , \ F|_{c(\tau)} = -2A|_{c(\tau)} \quad ,
\]

(26)
which depends only on the function $A|_{c(\tau)}$ freely specifiable along $c(\tau)$.

Next we set $A|_{c(\tau)}(\tau_w) := 6\tilde{\gamma}_2(\tau_w)$ and $D|_{c(\tau)}(\tau_w) := 6\tilde{\delta}_2(\tau_w)$, and calculate the Bondi–Sachs metric up to the first-order correction to the flat metric for an axially symmetric vacuum space-time using (12a) – (12c) and (20a) – (20e). We find

$$\gamma(x^\alpha) = r^2\tilde{\gamma}_2(\tau_w)P_2^2(y) + O(r^3),$$

$$\beta(x^\alpha) = O(r^3),$$

$$U^\theta(x^\alpha) = -4r\tilde{\gamma}_2(\tau_w)P_2^1(y) + O(r^2),$$

$$\Phi(x^\alpha) = -6r\tilde{\gamma}_2(\tau_w)P_2^0(y) + O(r^3),$$

and

$$\delta(x^\alpha) = r^2\tilde{\delta}_2(\tau_w)P_2^2(y) + O(r^3),$$

$$U^\phi(x^\alpha) = -4r\tilde{\delta}_2(\tau_w)\left(\frac{P_2^1(y)}{\sin\theta}\right) + O(r^2),$$

where $y := -\cos\theta$ and $P_m^l(y)$ are the associated Legendre polynomials of the first kind. The functions $\tilde{\gamma}_2$ and $\tilde{\delta}_2$ determine, respectively, the electric and magnetic part of the Weyl tensor at the vertex. Moreover, they also define the lowest order curvature contributions in the norm and twist of the Killing vector w.r.t. their flat space values at the vertex, respectively.

Equations (27a)-(27f) give the lowest non-trivial order boundary conditions for $\mathcal{F}$ at a regular vertex for axially symmetric vacuum space-times. Equation (27b) shows that $\beta$ does not behave as $O(r^2)$ as expected by property (i) of the vertex lemma. This also demonstrates that if we had started with the expansions of Zhang (1986) to find the generic properties of the Bondi–Sachs metric at the vertex, the most general limiting behavior of $\beta$ would be wrong. In fact it can be shown that for axially symmetric perfect fluid space-times $\beta$ is of $O(r^2)$ at the vertex.

In principle, it is possible to calculate also the next order correction coefficients with the above approach, but this involves contractions of the covariant derivatives of the Riemann normal components over five and more indices with the null vector $k^a$ and its angular derivatives, e.g. the $O(r^3)$ coefficient of $\gamma$ depends on $\langle\nabla_{\phi}R_{bcde}\rangle k^a k^b k^c k^d k^e$. Thus, we describe in section 3.2 another approach that avoids the tedious calculation of these contractions, and also is easily extendable to higher order.

### 3.2. Regular boundary conditions for the Bondi–Sachs metric derived from a power series of $\mathcal{F}$

To find axially symmetric solutions for the Bondi–Sachs metric (17), we transform the Killing vector (22) to Bondi–Sachs coordinates $x^\alpha$, i.e. $\xi^\alpha(x^\alpha) = \delta^\alpha_\phi$. The Killing equations $\mathcal{L}_\xi g_{\alpha\beta}(x^\mu) = 0$ imply that the Bondi–Sachs metric functions $\mathcal{F}$ do not depend on the coordinate $\phi$. To facilitate the computations, we employ the coordinate transformation $y = -\cos\theta$ in the metric (17). Points on the axis of symmetry, i.e. the poles, are given by $y = \pm 1$. We further define the auxiliary function $s(y) := \sqrt{1 - y^2}$, and anticipating from (20a) - (20e) we assume the following expansions for the metric functions $\mathcal{F}$:

$$\gamma(\tau_w, r, y) = \sum_{n=1}^N (wr)^{n+1}\gamma_{n+1}(\tau_w, y) + O\left([wr]^{N+2}\right),$$

$$\beta(\tau_w, r, y) = O\left([wr]^{N+2}\right),$$

$$U^\theta(\tau_w, r, y) = -4r\tilde{\gamma}_2(\tau_w)P_2^1(y) + O(r^2),$$

$$\Phi(\tau_w, r, y) = -6r\tilde{\gamma}_2(\tau_w)P_2^0(y) + O(r^3),$$

$$\delta(\tau_w, r, y) = r^2\tilde{\delta}_2(\tau_w)P_2^2(y) + O(r^3),$$

$$U^\phi(\tau_w, r, y) = -4r\tilde{\delta}_2(\tau_w)\left(\frac{P_2^1(y)}{\sin\theta}\right) + O(r^2).$$
\begin{align*}
\delta(\tau_{w}, r, y) &= \sum_{n=1}^{N} (wr)^{n+1} \delta_{n+1}(\tau_{w}, y) + O[(wr)^{N+2}], \\
\beta(\tau_{w}, r, y) &= \sum_{n=1}^{N} (wr)^{n+1} \beta_{n+1}(\tau_{w}, y) + O[(wr)^{N+2}], \\
\Phi(\tau_{w}, r, y) &= \sum_{n=1}^{N} (wr)^{n+1} \Phi_{n+1}(\tau_{w}, y) + O[(wr)^{N+2}], \\
U^{A}(\tau_{w}, r, y) &= \sum_{n=1}^{N} (wr)^{n} U^{A}_{n}(\tau_{w}, y) + O[(wr)^{N+1}],
\end{align*}

where the set of expansion coefficients \( C_{n} := \{\gamma_{n+1}, \delta_{n+1}, \beta_{n+1}, \Phi_{n+1}, U^{A}_{n}\} \) is calculated at \( r = 0 \), \( N \) is the order up to which the field. Note, this expansion only respects property (i) of the vertex lemma, and we do not impose other restrictions at this stage. After calculating the vacuum Einstein equations for the coefficients \( C_{n} \), we end up with coupled partial differential equations for the \( C_{n} \) with respect to \( u \) and \( y \). We solve these equations in general, and then restrict their solution to be regular, i.e. finite, at the boundaries. This will provide us with the boundary conditions for the Bondi-Sachs metric at a regular vertex \( O \). These regular boundary conditions at \( O \) will then also comply with the general properties stated in the vertex lemma. In the following, we refer to \( C_{n} \) as the \( n \)-th-order correction of the Bondi–Sachs metric functions \( \mathcal{F} \) with respect to a flat metric at \( O \) or simply the \( n \)-th-order correction of \( \mathcal{F} \).

In Section 3.2.1 and 3.2.2, we solve the vacuum Einstein equations, which can be grouped in Bondi–Sachs coordinates into six so-called main equations, three supplementary equations, and one trivial equation, respectively (Bondi et al. 1962, Sachs, 1962, van der Burg, 1966, Winicour, 2012). The six main equation can be split further into two evolution equations for the transverse-traceless part of the 2-metric \( h_{AB}(\gamma, \delta) \), and four hyper-surface equations for the variables \( \beta, U^{A}, \) and \( \Phi \). Furthermore, from the twice contracted Bianchi identities follows the lemma (Bondi et al. 1962, Sachs, 1962, Tamburino and Winicour, 1966): If the main equations hold on one null cone and if the optical expansion rate of the null rays does not vanish on this cone then the trivial equation is fulfilled algebraically and the supplementary equations hold if they are fulfilled at one radius \( r \). Consequently, we need to consider only the main equation to find the solution of the Einstein equations at the vertex \( O \). The supplementary equations then provide a check of this solution. The Ricci tensor components and the quantities derived from it that appear in the supplementary, hyper-surface and evolution equations determining the corrections \( C_{n} \) are given in Appendix A.

3.2.1. Boundary conditions depending on \( C_{1}, C_{2} \) and \( C_{3} \) According to property (v) of the vertex lemma and the expansion of a metric in normal coordinates, see e.g. relation (21b), we expect the non-linear coupling of the \( C_{n} \)-coefficients to happen at lowest order in the \( C_{3} \)-coefficients. Indeed, using (A.7), \( R_{rr} = 0 \) and omitting terms \( O(r^{3}) \) gives

\begin{align*}
\beta_{2} &= 0 \ , \\
\beta_{3} &= 0 \ .
\end{align*}
\[ \beta_4 = \frac{1}{2} \left( (\gamma_2)^2 + (\delta_2)^2 \right). \]  

Equation (29c) shows that the \( \mathcal{C}_3 \) correction of \( \beta \) depends quadratically on the solution of the \( \mathcal{C}_1 \) correction.

To find the solutions for the \( \mathcal{C}_n, n \in \{1, 2, 3\} \), coefficients we solve first for \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) setting the constant and linear coefficients of the \( r \)-series in (A.7) - (A.12) equal to zero. We next insert the solution of the \( \mathcal{C}_1 \)-correction into the quadratic coefficients of the \( r \)-series of the Ricci tensor components in (A.7) - (A.12) and solve for the \( \mathcal{C}_3 \)-coefficient.

Inserting the solution \( \beta_2 = 0 = \beta_3 \) into (A.8) - (A.11) and utilizing \( R_{\alpha \beta} = 0 \) we find

\[ 0 = 2U^\gamma_1 + 2(s^2 \gamma_2)_,y \left[ 5U^\gamma_2 + 3(s^2 \gamma_3)_,y \right] r w^{-1} + \mathcal{O}(r^2), \]  

\[ 0 = 2U^\phi_1 + 2s^{-2}(s^2 \delta_2)_,y + \left[ 5U^\phi_2 + 3s^{-2}(s^2 \delta_3)_,y \right] r w^{-1} + \mathcal{O}(r^2), \]  

\[ 0 = -12 \Phi_2 + 5U^\gamma_1 + 2s^{-2}(s^4 \gamma_2)_,y - 4 \gamma_2 
+ \left\{ -16 \Phi_3 + 6U^\gamma_2 + 2s^{-2}(s^4 \gamma_3)_,y - 4 \gamma_3 \right\} w r + \mathcal{O}(r^2), \]  

\[ 0 = -12 \gamma_2 + 3s^2(s^{-2} U^\gamma_1)_,y + \left[ 12 \gamma_2,\tau_w - 24 \gamma_3 + 4s^2(s^{-2} U^\gamma_2)_,y \right] w r + \mathcal{O}(r^2), \]  

\[ 0 = 12 \delta_2 - 3s^2 U^\phi_1 + \left( -12 \delta_2,\tau_w + 24 \delta_3 - 4s^2 U^\phi_2 \right) w r + \mathcal{O}(r^2). \]

Combining algebraically the zeroth-order term in the \( r \)-expansion of (30a) and (30d) gives

\[ 0 = s^2 \gamma_2,\tau_w - 2y \gamma_2, y + \left( 6 - \frac{4}{s^2} \right) \gamma_2, \]  

while the same procedure yields for the first-order terms

\[ 0 = s^2 \gamma_3,\tau_y - 2y \gamma_3, y + \left( 12 - \frac{4}{s^2} \right) \gamma_3 - 5 \gamma_2,\tau_w , \]

and leads to similar equations as (31) and (32) when applied to (30b) and (30e). \( \gamma_2 \) being replaced by \( \delta_2 \), and \( \gamma_3 \) by \( \delta_3 \), respectively. Hence solving (31) and (32) does not only give \( \gamma_2 \) and \( \gamma_3 \), but also provides the structure of a solution of \( \delta_2 \) and \( \delta_3 \).

Inserting the ansatz \( \gamma_2(\tau_w, y) = \tilde{\gamma}_2(\tau_w) S(y) \) in (31) yields

\[ 0 = \tilde{\gamma}_2(\tau_w) \left[ (1 - y^2) \frac{d^2 S}{dy^2} - 2y \frac{dS}{dy} + \left( 6 - \frac{4}{1 - y^2} \right) S \right]. \]

Since \( \tilde{\gamma}_2 \) is an arbitrary function, the term in the parenthesis has to vanish, which also provides us with an associated Legendre differential equation for \( S(y) \) of the form

\[ 0 = (1 - y^2) \frac{d^2 S}{dy^2} - 2y \frac{dS}{dy} + \left[ l(l + 1) - \frac{m^2}{1 - y^2} \right] S, \]

with \( l = 2 \) and \( m = 2 \). This equation holds for the Legendre polynomials of the first kind, \( P_l^m(y) \), and of the second kind, \( Q_l^m(y) \), respectively. Hence the most general solution of (34) involves two integration constants \( S_1 \) and \( S_2 \), i.e.

\[ S(y) = S_1 P_l^m(y) + S_2 Q_l^m(y). \]

Since the associated Legendre polynomials of the second kind, \( Q_l^m(y) \), are singular at the poles \( y = \pm 1 \), we set \( S_2 = 0 \). Hereafter, we will always set integration constants connected to Legendre polynomials of the second kind equal to zero. Thereby we
guarantee regular solutions at the poles and point (ii) of the vertex lemma, since the $P^m_1(y)$ are well defined for all angles $y \in [-1, 1]$. A regular solution of (31) assuming separation of variables is

$$
\gamma_2(\tau_w, y) = \gamma_2(\tau_w)P^2_1(y) ,
$$

(35)

where we absorbed the integration constant $S_1$ into the arbitrary function $\gamma_2(\tau_w)$. To solve (32), we make the ansatz

$$
\gamma_3(\tau_w, y) = \sum_{k=2}^{\infty} \gamma_3^k(\tau_w)P^2_k(y) ,
$$

(36)

where the coefficients $\gamma_3^k(\tau_w)$ depend on the time at the vertex and are determined by the differential equation (32). Inserting (36) into (32), reveals that $\gamma_3(\tau_w)$ is freely specifiable function, $\gamma_3^2(\tau_w) = 5/6(d\gamma_2/d\tau_w)$, and all remaining coefficients $\gamma_3^k(\tau_w)$ must be zero. Therefore, the solution for $\gamma_3$ is

$$
\gamma_3(\tau_w, y) = \gamma_3(\tau_w)P^2_3(y) + \frac{5}{6} \left( \frac{d\gamma_2}{d\tau_w}(\tau_w) \right) P^2_2(y) .
$$

(37)

As already mentioned above, we can derive similar partial differential equations as (31) and (32) for $\delta_2$ and $\delta_3$, i.e. we may assume a solution for $\delta_2$ and $\delta_3$ as

$$
\delta_2(\tau_w, y) = \delta_2(\tau_w)P^2_2(y) ,
$$

(38)

$$
\delta_3(\tau_w, y) = \delta_3(\tau_w)P^2_3(y) + \frac{5}{6} \left( \frac{d\delta_2}{d\tau_w}(\tau_w) \right) P^2_2(y) ,
$$

(39)

where both $\delta_2(\tau_w)$ and $\delta_3(\tau_w)$ are arbitrary functions of $\tau_w$.

To solve for $U_1^\tau$ and $U_2^\tau$ we insert the solutions for $\gamma_2$, $\gamma_3$, $\delta_2$, and $\delta_3$ into (30a) and (30b) and solve in the constant and linear term of the $r$-series for the respective variable. This yields

$$
U_1^\tau(\tau_w, y) = -4\gamma_2(\tau_w)P^1_1(y)s(y) ,
$$

(40)

$$
U_2^\tau(\tau_w, y) = -4\delta_2(\tau_w)P^1_1(y)s^{-1}(y) ,
$$

(41)

$$
U_2^\tau(\tau_w, y) = - \left( 6\gamma_3(\tau_w)P^1_3(y) + 2 \left( \frac{d\gamma_2}{d\tau_w}(\tau_w) \right) P^1_2(y) \right) s(y) ,
$$

(42)

$$
U_2^\tau(\tau_w, y) = - \left( 6\delta_3(\tau_w)P^1_3(y) + 2 \left( \frac{d\delta_2}{d\tau_w}(\tau_w) \right) P^1_2(y) \right) s^{-1}(y) .
$$

(43)

The solutions for $\Phi_2$ and $\Phi_3$ are found inserting the solution for $\gamma_2$, $\gamma_3$, $U_1^\tau$, and $U_2^\tau$ into Eq. (30c) and solving the constant and linear term of the $r$-series for $\Phi_2$ and $\Phi_3$, respectively. This gives

$$
\Phi_2(\tau_w, y) = -6\gamma_2(\tau_w)P^0_2(y) ,
$$

(44)

$$
\Phi_3(\tau_w, y) = -12\gamma_3(\tau_w)P^0_3(y) - 2 \left( \frac{d\gamma_2}{d\tau_w}(\tau_w) \right) P^0_2(y) .
$$

(45)

Having obtained the complete solutions for the coefficients $\gamma_2$ in (29a), (35), (38), (40), (41), and (44), and $\gamma_3$ in (29b), (37), (39), (42), (43) and (45), we now determine the $\gamma_3$-coefficients of the metric function $\mathcal{F}$. We insert the solutions for $\gamma_2$ and $\delta_2$ into Eq. (29c), which gives

$$
\beta_4(\tau_w, y) = \frac{1}{2} \left( \gamma_2(\tau_w)^2 + \delta_2(\tau_w)^2 \right) \left[ P^2_2(y) \right] ,
$$

(46)
This equation shows that the solution can be expressed again as a linear combination of associated Legendre polynomials. We insert the remaining $c_2$ coefficients into (A.8), and then set these equations equal to zero. Considering only the quadratic coefficients results in the following equations:

\begin{align}
0 &= 9U_3^{y} + 4(s^2 \gamma_4)_y + 324yds^4 \left[ (\gamma_2)^2 + (\delta_2)^2 \right], \quad (47a) \\
0 &= 9U_3^{y} + 4s^2 (s^2 \delta_4)_y, \quad (47b) \\
0 &= -20 \Phi_4 + 2s^2 (s^4 \gamma_4)_y - 4\gamma_4 + 7U_3^{y} - 9(100y^4 - 32y^2 + 12)(\gamma_2)^2 \\
&\quad + 72(10y^4 - 11y^2 + 1)(\delta_2)^2, \quad (47c) \\
0 &= -40\gamma_4 + 5s^2 (s^{-2}U_3^{y})_y + 40\gamma_2,_{\tau_{w},\tau_{w}} + 240ys^2 \gamma_3,_{\tau_{w}} + 540s^4 \left[ (\gamma_2)^2 + (\delta_2)^2 \right] \\
&\quad - 720s^2 \gamma_2^2, \quad (47d) \\
0 &= 40\delta_4 - 5s^2 U_3^{y} - 40s^2 \delta_{2,\tau_{w},\tau_{w}} - 240ys^2 \delta_{3,\tau_{w}} + 720s^2 \delta_2 \gamma_2. \quad (47e)
\end{align}

Combining (47a) with (47d), and (47b) with (47e) yields

\begin{align}
0 &= s^2 \gamma_{4,yy} - 2y\gamma_{4,y} + \left( \frac{20}{s^2} - \frac{4}{y} \right) \gamma_4 - 18s^2 \left\{ \gamma_{2,\tau_{w},\tau_{w}} + 6y\gamma_3,_{\tau_{w}} - 9 \left( (\gamma_2)^2 - (\delta_2)^2 \right) \right\}, \quad (48) \\
0 &= s^2 \delta_{4,yy} - 2y\delta_{4,y} + \left( \frac{20}{s^2} - \frac{4}{y} \right) \delta_4 - 18s^2 \left\{ \delta_{2,\tau_{w},\tau_{w}} + 6y\delta_3,_{\tau_{w}} - 18 \gamma_2 \delta_2 \right\}. \quad (49)
\end{align}

Since the square of an associated Legendre polynomial can be expressed by a linear combination of Legendre polynomials with the same degree of $m$ but varying $l$, we assume that both $\gamma_4$ and $\delta_4$ obey an expansion in terms of $P^2_4(y)$ as

\begin{align}
\gamma_4(\tau_{w}, y) &= \sum_{l=2}^{\infty} \gamma_4^l(\tau_{w})P^2_4(y), \quad \delta_4(\tau_{w}, y) = \sum_{l=2}^{\infty} \delta_4^l(\tau_{w})P^2_4(y).
\end{align}

Inserting this ansatz into (48) and (49) provides us with the solutions for the coefficients $\gamma_4^l$ and $\delta_4^l$, respectively,

\begin{align}
\gamma_4(\tau_{w}, y) &= \gamma_4(\tau_{w})P^2_4(y) + \frac{9}{10} \left[ \frac{d\gamma_3}{d\tau_{w}}(\tau_{w}) \right] P^3_3(y) \\
&\quad + \left( \frac{3}{7} \frac{d^2\gamma_2(\tau_{w})}{d\tau_{w}^2} \right) - \frac{27}{7} \left\{ \left[ \gamma_2(\tau_{w}) \right]^2 - \left[ \delta_2(\tau_{w}) \right]^2 \right\} \right] P^2_2(y), \quad (50) \\
\delta_4(\tau_{w}, y) &= \delta_4(\tau_{w})P^2_4(y) + \frac{9}{10} \left[ \frac{d\delta_3}{d\tau_{w}}(\tau_{w}) \right] P^3_3(y) \\
&\quad + \left( \frac{3}{7} \frac{d^2\delta_2(\tau_{w})}{d\tau_{w}^2} \right) - \frac{54}{7} \gamma_2(\tau_{w}) \delta_2(\tau_{w}) \right] P^2_2(y), \quad (51)
\end{align}

where $\gamma_4(\tau_{w}) := \gamma_4^1(\tau_{w})$ and $\delta_4(\tau_{w}) := \delta_4^1(\tau_{w})$ are freely specifiable functions. While (47a), (47b), and (47c) provide the solutions for $U_3^{y}$ and $\Phi_4$, respectively:

\begin{align}
U_3^{y}(\tau_{w}, y) &= - \left\{ \gamma_4(\tau_{w}) + \frac{72}{35} \gamma_2(\tau_{w}) + \frac{72}{35} \delta_2(\tau_{w}) \right\} P^4_4(y) + 4 \left[ \frac{d\gamma_3}{d\tau_{w}}(\tau_{w}) \right] P^3_3(y) \\
&\quad + \frac{16}{21} \frac{d^2\gamma_2(\tau_{w})}{d\tau_{w}^2} \right\} P^1_2(y) s(y), \quad (52) \\
U_3^{\varphi}(\tau_{w}, y) &= - \left\{ \gamma_4(\tau_{w}) + 4 \left[ \frac{d\gamma_3}{d\tau_{w}}(\tau_{w}) \right] P^3_3(y) \right\}.
\end{align}
The Bondi–Sachs metric at the vertex of a null cone

\[ + \left[ \frac{16}{21} \frac{d^2 \tilde{\gamma}_2}{d^2 \tau_w} - \frac{96}{7} \tilde{\gamma}_2(\tau_w) \tilde{\delta}_2(\tau_w) \right] P_2^1(y) \right] s^{-1}(y) \]  

\[ \Phi_4(\tau_w, y) = - \left\{ 20 \tilde{\gamma}_4(\tau_w) + \frac{864}{35} [\tilde{\gamma}_2(\tau_w)]^2 + \frac{216}{35} [\tilde{\delta}_2(\tau_w)]^2 \right\} P_4^0(y) - 6 \left[ \frac{d \tilde{\gamma}_3}{d \tau_w}(\tau_w) \right] P_3^0(y) \]

\[ - \left[ \frac{4}{7} \frac{d \tilde{\gamma}_2}{d \tau_w}(\tau_w) - \frac{24}{7} \{ [\tilde{\gamma}_2(\tau_w)]^2 + [\tilde{\delta}_2(\tau_w)]^2 \} \right] P_2^0(y) \]

\[ - \frac{48}{5} [\tilde{\gamma}_2(\tau_w)]^2 - \frac{12}{5} [\tilde{\delta}_2(\tau_w)]^2 \]  \hspace{1cm} (54)

Equations (46), (50) - (54) constitute a solution for the \( C_3 \)-coefficients of \( \mathcal{F} \).

The solution for the corrections \( C_2 \) and \( C_3 \) demonstrates points (iii), (iv) and (v) of the vertex lemma, as the expansion coefficients contain specific numerical factors (point (iii)). Time derivatives (point (iv)) of the lower order coefficients \( C_1 \) and \( C_2 \) occur hierarchically ordered, i.e. \( C_2 \) depends on the first time derivative of \( C_1 \) whereas \( C_3 \) depends on the second time derivative of \( C_1 \) and on the first time derivative of \( C_2 \). The non-linear coupling (point (v)) in the correction coefficients occurs at lowest order in \( C_3 \), since \( C_3 \) depends on \( C_1^2 \). If one uses the solution for \( C_n, n = 1, 2, 3 \) as boundary conditions for the Bondi–Sachs metric in numerical simulations for axially symmetric space, the free functions \( \tilde{\gamma}_2 \) and \( \tilde{\delta}_2 \) must be at least twice differentiable with finite derivatives, \( \tilde{\gamma}_3 \) and \( \tilde{\delta}_3 \) must be differentiable with finite derivatives, whereas \( \tilde{\gamma}_4 \) and \( \tilde{\delta}_4 \) must be continuous. Thus, requiring the vertex to be a regular point in the null cone rigidly fixes the boundary conditions for the metric at the vertex. The only freedom left is the choice of the functions \( \tilde{\gamma}_n \) and \( \tilde{\delta}_n \), \( n \in \{1, 2, 3\} \).

3.2.2. Boundary conditions that are linear in \( C_n \) and valid for \( n \geq 1 \) To investigate the boundary conditions of \( \mathcal{F} \) further, we consider (28a) - (28c) for an arbitrary value of \( N \). Since the metric functions \( \mathcal{F} \) are power series in \( r \) with coefficients \( C_n \), the corresponding Ricci tensor can be written as

\[ \mathcal{R}_{\mu\nu}(\tau_w, r, y) = \sum_{n=1}^{N} r^n \mathcal{R}_{\mu\nu}^{(n)}(\tau_w, y) \]  \hspace{1cm} (55)

i.e. as a power series in \( r \), too. As the coefficients \( \mathcal{R}_{\mu\nu}^{(n)} \) will, in general, not depend linearly on \( C_n \), we linearize them with respect to \( C_n \) to firstly simplify the calculations, and secondly to find the qualitative dependence of the hierarchical order of the time derivatives addressed in property (v) of the vertex lemma. The linearized Ricci tensor components for the main equations of the vacuum Einstein equations are given in Appendix A.2. Introducing the operator

\[ L_4(f) := (1 - y^2) f_{yy} - 2yf_y + \left[l(l + 1) - \frac{4}{1 - y^2}\right] f \]  \hspace{1cm} (55)

for a function \( f \) that depends on \( y \), allows us to write some of the upcoming equations in a more compact way. In particular, if \( f(y) = P_l^2(y) \), we find

\[ L_4\left[ P_j^2(y) \right] = b^j_l P_j^2(y) \]  \hspace{1cm} (56)

where there is no summation performed over \( j \) on the right hand side of the first expression. Note that the associated Legendre polynomial \( P_j^2(y) \) commutes * with the operator \( L_4 \), because \( b^j_l = 0 \).

* A similar relation as (56) could be derived for the Legendre polynomials of the second kind, \( Q_j^2(y) \), but we do not investigate this further here as these polynomials are not defined at \( y = \pm 1 \).
From (A.13) and the vacuum Einstein equation \( R^{(n)}_{\nu\tau} = 0 \), we deduce
\[
\beta_{n+1}(\tau_w, r, y) = 0 , \quad \forall \ n \geq 1 .
\] (57)
Inserting the solution for \( \beta_n \) into (A.14) - (A.20) and combining \( R^{(1)}_{rA} = 0 \) from (A.14) and (A.15) with \( R^{(2)}_{(\gamma)} = 0 \) and \( R^{(2)}_{(\delta)} = 0 \) from (A.17) and (A.18), respectively, yields
\[ 0 = L_2(\gamma_2) , \quad 0 = L_2(\delta_2) . \] (58)

Since the associated Legendre polynomial \( P^2_2(y) \) commutes with \( L_2 \), regular solutions of (58) for \( y \in [-1, 1] \) may be assumed to have the form
\[
\gamma_2(\tau_w, y) = \gamma^2_2(\tau_w) P^2_2(y) , \quad \delta_2(\tau_w, y) = \delta^2_2(\tau_w) P^2_2(y) .
\] (59)
where \( \gamma^2_2(\tau_w) \) and \( \delta^2_2(\tau_w) \) are arbitrary functions depending on the time \( \tau_w \) along the geodesic \( c(\tau) \). Given the solutions for \( \gamma_2 \) and \( \delta_2 \), and from \( R^{(1)}_{rA} = 0 \), we derive
\[
U^1_1(\tau_w, y) = -4\gamma^2_2(\tau_w) P^2_2(y) s(y) , \quad U^1_2(\tau_w, y) = -4\delta^2_2(\tau_w) P^2_1(y) s^{-1}(y) .
\] (60)
Setting \( R^{(2)}_{(2L)} \) in (A.16) equal to zero and inserting \( \gamma_2 \) and \( U^1_1 \) leads to
\[
\Phi^2_2(\tau_w, y) = -6\gamma^2_2(\tau_w) P^2_2(y) ,
\] (61)
which shows that the solution for the lowest order correction \( \gamma_1 \) obtained with the linear approach is equivalent to that of the generic approach in the previous section.

To find the solutions for \( \gamma_2, \gamma_3, \ldots, \gamma_n, n > 1 \), we assume hereafter \( n > 1 \). Combining \( R^{(n)}_{(A)} = 0 \) from Eqs. (A.14) and (A.15) with \( R^{(n+1)}_{(\gamma)} = 0 \) and \( R^{(n+1)}_{(\delta)} = 0 \) from Eqs. (A.19) and (A.20), respectively, gives rise to two equations that determine the solutions for \( \gamma_{n+1} \) and \( \delta_{n+1} \):
\[
0 = L_{n+1}(\gamma_{n+1}) + a_{n+1} \gamma_{n+1} y , \quad 0 = L_{n+1}(\delta_{n+1}) + a_{n+1} \delta_{n+1} y \] (62)
where
\[
a_n := -\frac{2(n + 2)(n - 1)}{(n + 1)} .
\] (63)
The expressions in (62) show that the solutions for \( \gamma_n \) and \( \delta_n \) obey the same type of equation, i.e. knowing one of the solutions gives us the form of the other solution, too.

We treat the solutions for \( \gamma_n \) and \( \delta_n \) together by defining
\[
I^A_k := \gamma_k \delta^A_y + \delta_k \delta^A_\phi
\] with \( k \geq 2 \), whereby (62) becomes
\[
0 = L_{n+1}(I^A_{n+1}) + a_{n+1} I^A_{n+1} y .
\] (64)
Making the ansatz
\[
I^A_{n+1}(\tau_w, y) = \sum_{k=2}^{n+1} I^A_{n+1,k}(\tau_w) P^2_{k}(y) ,
\] (65)
which we insert into (64) and obtain for \( n = 2, 3, \) and \( 4 \)
\[
0 = L_3(I^A_3) + a_3 I^A_{2,\tau_w} = b_3^3 I^A_{3,3} P^2_3 + \left( b^3_{3} \frac{dI^A_{3,2}}{d\tau_w} + a_3 \frac{dI^A_{3,2}}{d\tau_w} \right) P^2_2 , \] (66a)
\[
0 = L_4(I^A_4) + a_4 I^A_{3,\tau_w} = b^4_4 I^A_{4,4} P^2_4 + \left( b^4_4 \frac{dI^A_{4,3}}{d\tau_w} + a_4 \frac{dI^A_{4,3}}{d\tau_w} \right) P^2_3 + \left( b^4_3 \frac{dI^A_{4,2}}{d\tau_w} + a_4 \frac{dI^A_{4,2}}{d\tau_w} \right) P^2_2 , \] (66b)
\[
0 = L_5(I^A_5) + a_5 I^A_{4,\tau_w} = b^5_5 I^A_{5,5} P^2_5 + \left( b^5_5 \frac{dI^A_{5,4}}{d\tau_w} + a_5 \frac{dI^A_{5,4}}{d\tau_w} \right) P^2_4 + \left( b^5_4 \frac{dI^A_{5,3}}{d\tau_w} + a_5 \frac{dI^A_{5,3}}{d\tau_w} \right) P^2_3 + \left( b^5_3 \frac{dI^A_{5,2}}{d\tau_w} + a_5 \frac{dI^A_{5,2}}{d\tau_w} \right) P^2_2 . \] (66c)
Since the associated Legendre polynomials depend on the arbitrary angle and are non-zero in general, the coefficients of the spectral series in terms of $P_l^3$ must vanish in order to fulfill (66a) - (66c). Since the diagonal coefficients $b_i$ are equal to zero, the functions $\bar I^A_{(l)}(\tau_w)$ can be chosen arbitrarily. For the other functions, we find

$$\bar I^A_{3,2} = - \left[ \frac{a_3}{b_3^2} \right] \frac{d}{d\tau_w} \bar I^A_{2,2},$$

$$\bar I^A_{4,2} = \left[ \frac{a_3 a_4}{b_3^2 b_4^2} \right] \frac{d^2}{d\tau_w^2} \bar I^A_{2,2} - \left[ \frac{a_4}{b_4^4} \right] \frac{d}{d\tau_w} \bar I^A_{3,3},$$

$$\bar I^A_{5,2} = - \left[ \frac{a_3}{b_3^2 b_4^2 b_5^2} \right] \frac{d^3}{d\tau_w^3} \bar I^A_{2,2} - \left[ \frac{a_4}{b_4^4 b_5^4} \right] \frac{d^2}{d\tau_w^2} \bar I^A_{3,3} - \left[ \frac{a_5}{b_5^8} \right] \frac{d}{d\tau_w} \bar I^A_{4,4},$$

(67a)

(67b)

(67c)

Defining $J^A_n(\tau_w) := \check \gamma_n(\tau_w)\delta^A_y + \check \delta_n(\tau_w)\delta^A_\varphi$ with $n \geq 2$, where $\check \gamma_n(\tau_w)$ and $\check \delta_n(\tau_w)$ are arbitrary functions of $\tau_w$, we deduce from the recursive behavior of $\bar I^A_{n+k}(\tau_w)$ in (67a) - (67c) the general form of the time dependent coefficients of $I^A_{n+1}(\tau_w, y)$ as

$$\bar I^A_{n+1,l}(\tau_w) = \begin{cases} J^A_{n+1}(\tau_w) & \text{for } l = (n+1), \\ c^{n+1}_l \int d\tau_w J^A_{l+1}(\tau_w) & \text{for } 2 \leq l \leq n, \end{cases}$$

(68)

where

$$c^n_k := (-1)^{n+k} \frac{\prod_{i=3}^{n} a_i \prod_{i=3}^{k} b_i}{\prod_{i=3}^{n} a_i \prod_{i=3}^{k} b_i} = \frac{\prod_{s=k+1}^{n} \left( \frac{2(s+2)(s-1)}{s+1} \right)}{\prod_{t=k+1}^{n} \left( t(t+1) - k(k+1) \right)}.$$  

(69)

The last term in the right hand side of (69) can be expressed by factorials using the computer algebra program Maple and some properties of the Gamma function,

$$c^n_k = \frac{2^{n-k-1}k(n+2)(2k+2)(n-1)!}{(k+2)!(n+k+1)!(n-k)!}.$$  

(70)

We find the solution for $U^A_n$ with $n > 1$ by inserting $I^A_n$ into $R^{(n)}_{r,A} = 0$, which leads to

$$U^A_n(\tau_w, y) = - \frac{2(n+1)}{n(n+3)} \sum_{l=2}^{n+1} \bar I^A_{n+1,l}(\tau_w) \left[ (l+2)(l-1) \right] P_l^y(y)q^A(y),$$

(71)

where there is no summation performed over the index $A$ on the right hand side, and $q^A := s\delta^A_y + s^{-1}\delta^A_\varphi$. Setting $R^{(n)}_{(2D)}$ in (A.16) for $n > 1$ equal to zero and inserting $I^y_{n+1}$ and $U^y_{n+1}$ gives rise to the solution (for $n > 1$)

$$\Phi_{n+1}(\tau_w, y) = - \sum_{l=2}^{n+1} \left[ \frac{l(l+1)(l+2)(l-1)}{n(n+3)} \right] P_l^y y_{n+1,l}(\tau_w) P_l^0(y).$$  

(72)

In summary, the axially symmetric vacuum solution for $\mathcal{C}_n$ with arbitrary $n > 1$ is given by $\beta_{n+1}$ in (57), the coefficients $I^A_{n+1,l}$ in (68) for the power series (65), $U^A_n$ in (71), and $\Phi_{n+1}$ in (72), respectively. For this solution each set of coefficients $\mathcal{C}_n$ is determined by two functions $\check \gamma_{n+1}(\tau_w)$ and $\check \delta_{n+1}(\tau_w)$, which can be prescribed freely along the geodesic $c(\tau)$.

\footnote{Denoting with $\Gamma(n)$ the Gamma function, we use $\Gamma(n+1) = n!$ and $\Gamma(n+1/2) = 2^{n}\pi^{1/2}(2n)!/n!$.}
4. Summary and discussion

We studied the boundary conditions of the Bondi–Sachs metric functions $F \in \{\gamma, \delta, \beta, U^A, \Phi\}$ at the vertices of null cones assuming that these vertices are traced by a time-like geodesic $c(\tau)$ and that the observer emitting the null rays from this geodesic is an inertial observer. General requirements of these boundary conditions were found in three major calculational steps after assuming $c(\tau)$ was contained in a convex normal neighborhood: In the first step, we constructed a metric in Fermi normal coordinates along $c(\tau)$. In the second step, we defined affine null coordinates at $c(\tau)$, where the radial coordinate is an affine parameter, and transformed the metric from Fermi normal coordinates to an affine null metric, which agrees with that of Ellis et al. (1985), Poisson (2004, 2005) and Poisson et al. (2006a, 2006b, 2010, 2011). In the third step, we calculated a Bondi–Sachs metric along $c(\tau)$, by changing in the affine null metric the radial coordinate over to an areal distance coordinate $r$ that defines the radial coordinate in the Bondi–Sachs metric. The boundary conditions of $F$ at the vertex were found as Taylor series in terms of $r$ with respect to a flat metric along $c(\tau)$ where the expansion coefficients of the series are called correction coefficients $C_n$.

Regularity at the vertex implied the following five general requirements on the power series of $F$:

(i) The power series of $F$ must start at $r = 0$ with a certain positive power of $r$, in general $\gamma, \delta, \beta, \Phi$ are of $O(r^2)$ and $U^A$ of $O(r)$.††

(ii) The coefficients $C_n$ have a rigid angular structure that is given by polynomials of harmonic and regular functions on 2-spheres centered on the vertices.

(iii) The polynomial coefficients of the harmonic base functions of $C_n$ carry strict numerical factors depending on the physical problem under consideration.

(iv) The correction coefficient $C_n, (n > 1)$ depends on the time derivatives of order $(n - k)$ of the correction coefficient $C_k, (1 \leq k < n)$ and these time derivatives must be finite.

(v) Non-linear coupling occurs at lowest order near the vertex in the third-order correction coefficient of $F$, i.e. at $O(r^4)$ for $\gamma, \delta, \beta, \Phi$, and at $O(r^3)$ for $U^A$.

Requirements (i) – (v) on the boundary conditions of $F$ are kinematical regularity conditions, as they are derived from a regular metric along the geodesic tracing the vertices. The Einstein equations are not used in this establishing these conditions. The requirement that the $C_n$ have to obey Einsteins equations, this is a dynamical condition.

We summarized these general requirements on the boundary conditions of $F$ in a Vertex Lemma, which also answers question (I) from the introduction:

Let the Bondi–Sachs metric functions, $F$, be represented by power series expansions with respect to the areal distance coordinate $r$ at the vertex of a null cone on a time-like geodesic. If the coefficients of this radial expansion obey

††If the curve tracing the vertices is no geodesic but a general time-like curve, with a tangent vector $u^\alpha$, the lowest order terms depend differently on $r$, and one has to consider the acceleration, $a^\mu := u^\alpha \nabla_\alpha u^\mu$, of the curve. Poisson et al. (2011) consider an affine null metric along such a general time-like curve. Transforming this metric to a Bondi–Sachs metric tells one how property (i) should be changed in that case. However, with our assumption of axial symmetry, it is natural to assume that the curve is a geodesic on the axis of symmetry.
(1) the general properties (i) – (v),
(2) and the Einstein equations,
then this power series expansion of $\mathcal{F}$ can be used to formulate boundary conditions for the metric functions $\mathcal{F}$ at the vertex of the null cone, and the vertex is a regular point in the null cone.

We investigated the implication of the vertex lemma for axially axisymmetric vacuum space-times, which are the simplest space-times that allow us to demonstrate its relevant features. We also corrected and generalized the boundary conditions for such space times existing in the literature (Gomez et al., 1994, Siebel, et al., 2001). Since axially symmetric space-times contain a designated class of observers defined by time-like geodesics on the axis of symmetry, we introduce a Fermi normal coordinate system with respect to one of these axial geodesics $c(\tau)$.

We then proposed two approaches to answer question (II) of the introduction, i.e. how one can determine the boundary conditions of the Bondi–Sachs metric at the vertex.

In the first one, we calculated the Riemann normal components in the Fermi normal coordinate system, and contracted these components with a null vector at $c(\tau)$, its angular derivatives, and the tangent vector of $c(\tau)$. These contracted Riemann normal component provided first-order correction coefficients of the Bondi–Sachs metric with respect to a flat metric at the vertex. Although this approach works well for the first-order corrections, it is unsuited to find the higher order correction coefficients, because one would have to perform tedious full contractions of covariant derivatives of the Riemann normal components, and more importantly, because the approach itself cannot easily be extended to higher order.

In the second approach we proposed the usage of a power series expansion of the Bondi–Sachs variables $\mathcal{F}$, where the functions are expanded with respect to positive powers $r$, the coefficients depending on the time at the vertex and the direction angle of the emanating null ray. The lowest powers of $r$ in these series were chosen to agree with those given in property (i) of the vertex lemma. We then showed that the other properties of the vertex lemma (ii – v) follow when integrating the Einstein equations, considering only the $C_1^-, C_2^-$, and $C_3^-$-correction coefficients.

Considering property (ii) of the vertex lemma, we solved the partial differential equations for the $C_n$, and that they can in principle depend on the associated Legendre polynomials of first and second kind, $P_l^m(y)$ and $Q_l^m(y)$, respectively. Since the $Q_l^m(y)$ are singular at the poles, $y = \pm 1$, we rejected them as possible solution. This is in agreement with the ‘near-roundness’ condition of Choquet-Bruhat et al. (2010). The polynomials $P_l^m(y)$ are the harmonic and regular functions mentioned in property (ii) of the vertex lemma. We further note that the polynomials, $Q_l^m(y)$, were not encountered in the first approach where the Bondi–Sachs metric was determined directly from the Fermi metric, because the regularity at the poles is already assured by choosing the coordinate frame to be regular, i.e. choosing a Fermi normal coordinate system along $c(\tau)$.

We also confirmed properties (iii) and (iv) of the vertex lemma, by exploiting the fact that the spectral expansion of $C_n$, $n > 1$, contains time derivatives of the indicated hierarchy (see table 2) and carries numerical factors following from the solution of the Einstein equations.

Finally, we proved property (v) of the vertex lemma by showing that the solution for the $C_3$ coefficient depends linearly on the square of $C_1$. 

The Bondi–Sachs metric at the vertex of a null cone
Thus, they provide the answer to question (III) raised in the Introduction. The second conditions for the Bondi–Sachs metric with a regular vertex along an axially symmetric vacuum. These six free functions correspond to the time-dependent functions along \( c(\tau) \) which can be prescribed freely, i.e., both functions are not constrained by the vacuum Einstein equations. If these two free functions are \((3-n)\) times differentiable with finite derivatives along \( c(\tau) \), the solutions of \( \mathcal{C}_1 \), \( \mathcal{C}_2 \) and \( \mathcal{C}_3 \) provide the explicit boundary conditions for the Bondi–Sachs metric with a regular vertex along \( c(\tau) \) and for an axially symmetric vacuum. These six free functions correspond to the \( l = 2, 3, 4 \)- multipoles of the electric and magnetic part of the Weyl tensor evaluated at the vertex in agreement with the findings of Thorne(1980), Hartle & Thorne (1985) and Zhang (1986), who found the metric expansions of a vacuum space-time along a timeline geodesic in deDonder coordinates. The boundary conditions found in the second approach approximate a regular vertex up to third-order corrections in Fermi normal coordinates and incorporate the non-linear coupling due to curvature at lowest order. Thus, they provide the answer to question (III) raised in the Introduction.

As an example for such non-linear boundary conditions let us consider the following six functions

\[
\tilde{\gamma}_2(\tau_w) = K \quad \tilde{\gamma}_3(\tau_w) = 0 \quad \tilde{\gamma}_4(\tau_w) = \frac{9}{35} K^2 \quad \tilde{\delta}_2(\tau_w) = \tilde{\delta}_3(\tau_w) = \tilde{\delta}_4(\tau_w) = 0 \quad (73)
\]

where \( K \geq 0 \). Inserting these functions into the radial expansion coefficients in the solution for the \( \mathcal{C}_n \), \( n \in \{1, 2, 3\} \) corrections in section 3.2.1, and comparing the thus obtained expressions with the radial expansion of the static, axially-symmetric vacuum solution, so-called SIMPLE (see Bičák et al. 1983, Gómez, et al. 1994 or Appendix B), in (B.7) - (B.12) shows that they agree. Hence, putting the functions (73) into our general solution for \( \mathcal{C}_1 \), \( \mathcal{C}_2 \) and \( \mathcal{C}_3 \) approximates SIMPLE in a sufficiently small neighborhood of the vertex. Moreover, using the Bondi–Sachs metric at the vertex determined by our first approach, we can identify \( K \) with the Riemann normal component \( 6 \times R_{1313} \), which gives rise to the \((l = 2)\)- multipole of the electric part of the Weyl tensor.

By linearizing the vacuum Einstein equations with respect to the coefficients \( \mathcal{C}_n \), \( 1 \leq n \leq N \), of \( \mathcal{F} \), we found how the hierarchical pattern of the time derivatives can be determined for arbitrary \( N \), which provided us with the complete boundary conditions at a regular vertex for the Bondi–Sachs metric in linearized gravity and for axially symmetric vacuum space-times. The solution depends on \( 2N \) free functions, \( \tilde{\gamma}_n \) and \( \tilde{\delta}_n \), along the geodesic \( c(\tau) \). Regularity of the vertices along \( c(\tau) \) is guaranteed, if the two free functions in the coefficient \( \mathcal{C}_n \) are at least \((N - n)\) times differentiable with finite derivatives along \( c(\tau) \). These \( 2N \) functions also determine the complete axially, symmetric initial data on the null cone for the linearized Bondi–Sachs metric.

| correction coefficient | free functions | time derivatives |
|------------------------|----------------|------------------|
| \( \mathcal{C}_1 \)   | \( \tilde{\gamma}_2(\tau_w), \tilde{\delta}_2(\tau_w) \) | \( \emptyset \) |
| \( \mathcal{C}_2 \)   | \( \tilde{\gamma}_3(\tau_w), \tilde{\delta}_3(\tau_w) \) | \( \tilde{\gamma}_2(\tau_w), \tilde{\delta}_2(\tau_w) \) |
| \( \mathcal{C}_3 \)   | \( \tilde{\gamma}_4(\tau_w), \tilde{\delta}_4(\tau_w) \) | \( \tilde{\gamma}_2(\tau_w), \tilde{\delta}_2(\tau_w) \) |

For an axially symmetric vacuum space-time the solutions for each of the \( \mathcal{C}_1 \), \( \mathcal{C}_2 \), and \( \mathcal{C}_3 \) – coefficients depend on two time-dependent functions along \( c(\tau) \) which can be prescribed freely, i.e., both functions are not constrained by the vacuum Einstein equations.

Table 2. The hierarchical dependence of the time derivatives of the free functions arising in the solution of the correction coefficients \( \mathcal{C}_1 \), \( \mathcal{C}_2 \), and \( \mathcal{C}_3 \), where the time derivatives with respect to \( \tau_w \) are denoted by overdots.
in vacuo. We claim that these $2N$ functions also determine the complete non-linear initial data on the cone to arbitrary order of $N$. Up to $N = 3$, this claim is backed up by our calculations. For $N > 3$, we learn from the coupling of the Riemann normal components in the Taylor series of normal coordinates at a space-time point (Schouten, 1954, Thomas, 1991) that the non-linear coupling in a higher order coefficient occurs only between Riemann normal components and covariant derivatives determining lower order coefficients. Hence, our claim yields the answer to question (IV) raised in the introduction. We formulate it as a Conjecture:

Suppose $(\mathcal{M}^4, g)$ is an axially symmetric vacuum space-time, $c(\tau)$ a time-like geodesic on the axis of symmetry $\mathcal{A}$, $\mathcal{K}_w$ is a future ($w = 1$) or past ($w = -1$) null cone with its vertex $\Sigma$ on $c(\tau)$, $g(x^\alpha)$ is a Bondi-Sachs metric on $\mathcal{K}_w$ with respect to coordinates $x^\alpha = (\tau_w, r, y, \phi)$, and its six metric functions $F$ are represented by the finite power series (28a)-(28e) in terms of the areal distance $r$ with coefficients $C_n$, $1 < n \leq N$ evaluated at $\mathcal{K}_w$. Then the initial data on $\mathcal{K}_w$ for $F$ are given by $2N$ functions along $c(\tau)$, which can be chosen freely. Regularity at $\mathcal{K}_w$ requires that $C_n$ must be decomposed by a spectral series of associated Legendre polynomial $P_{n+1}^m(y)$ with respect to the angular variable $y$ and that the two free functions in the coefficient $C_n$ are at least $(N - n)$ times differentiable and the derivatives of these functions are finite along $c(\tau)$.

The main consequence of the conjecture concerns the arbitrariness of the vacuum initial data that can be imposed on a null cone in Bondi-Sachs coordinates. If the null cone is assumed to have a vertex and if this vertex is assumed to be a regular point in the null cone, then one cannot impose any data. Instead the data have to obey a regular angular structure as it is provided by the solution of the Einstein equations for the coefficients of a Taylor series expansion of the metric at the vertex. The only freedom one has left in the choice of the data are free functions on the geodesic tracing the vertex. These free functions determine the highest $l$—multipole of the Legendre basis $P_l^m$ in each power of $r$ in the Taylor series of the initial data at the vertex. They correspond to the $l$—multipoles of the axisymmetric parts of the electric and magnetic parts of the Weyl tensor evaluated along the time-like geodesic.

Our study raises further questions which should be investigated such as: How many free functions determine the initial data on a null cone with a regular vertex in space-times with no symmetry? What are the properties these free functions have to obey so that the initial data on the null cone describe an asymptotically flat space-time? What is the structure of the Bondi–Sachs metric at the vertex, if matter is assumed at the vertex and in its neighborhood? We plan to address the last question in a future publication.

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Appendix A. Tensor components for the Einstein equations

In the following, we give the components of $\mathcal{R}_{\alpha \beta}$ that are used in Sect. 3 to calculate the solution of the vacuum Einstein equations up to third-order corrections and for the solution of the $n^{th}$-order correction coefficients of linearized vacuum Einstein equations. We also introduce the following notations

\begin{align}
\mathcal{R}_{(2D)}^{(xy)} := g^{AB}R_{AB} \quad , \\
\mathcal{R}_{(\gamma)} := \left[s^{2}c^{-2\gamma}R_{yy} + s^{-2}c^{2\gamma}R_{\phi \phi}\right] \cosh^{-1}(2\delta) \quad , \\
\mathcal{R}_{(\delta)} := \left[s^{2}c^{-2\gamma}R_{yy} - s^{-2}c^{2\gamma}R_{\phi \phi}\right] \sinh(2\delta) - 2\cosh(2\delta)R_{y \phi} \quad .
\end{align}

Appendix A.1. Ricci Tensor Components for the Third-Order Correction Coefficients

The Ricci tensor components for the supplementary conditions are

\begin{align}
R_{\tau \tau}^{(xy)} = \left[s^{2}(\Phi_{2} + 2\beta_{2})_{y}\right]_{y} + 6(\Phi_{2} + 2\beta_{2}) + \left\{s^{2}(\Phi_{3} + 2\beta_{3})_{y}\right\}_{y} + 12(\Phi_{3} + 2\beta_{3}) \\
- 4\beta_{2} + 2\Phi_{2,\tau} - U_{1,y}^{(y)} \right\} w r \left\{s^{2}(\Phi_{4} + 2\beta_{4})_{y}\right\}_{y} + 20(\Phi_{4} + 2\beta_{4}) \\
+ 2\Phi_{3,\tau} - 6\beta_{3,\tau} - U_{2,yy}^{(y)} + (2\Phi_{2} + 4\beta_{2} - 2\gamma_{2}) \left[s^{2}(\Phi_{2} + 2\beta_{2})_{y}\right]_{y} \\
+ \left\{4(yU_{1}^{(y)}U_{1,y}^{(y)} - 2(\Phi_{2} + 2\beta_{2})\gamma_{2} + 2(\Phi_{2})^{2} + 6(\beta_{2})^{2} + 6\Phi_{2,\gamma_{2}}\beta_{2,y} \\
+ \left(\frac{9}{2}(U_{1}^{(y)})^{2}\right)s^{2} - \left[U_{1,y}^{(y)}U_{1,y}^{(y)} - \frac{1}{2}(U_{1,y}^{(y)})^{2}\right]s^{2} - (\Phi_{2} + U_{1,yy}^{(y)} + 8\beta_{2})U_{1}^{(y)} \\
+ (9\beta_{2} - 2U_{1,y}^{(y)})\Phi_{2} + 32(\Phi_{2})^{2} + 2(\beta_{2})^{2} - 2yU_{1,y}^{(y)} + \frac{13}{2}U_{1}^{(y)}s^{2} - 2U_{1,y}^{(y)} \\
- 4\beta_{2}U_{1,y}^{(y)} - (U_{1,y}^{(y)})^{2} - 2s^{-4}U_{1,y}^{(y)}^{2}\right\} r^{2} + \mathcal{O}(r^{3}) \quad .
\end{align}

\begin{align}
r^{-1}s^{2}\mathcal{R}_{\tau \tau}^{(xy)} = \left[2(\beta_{2} + \Phi_{2})_{y}g_{yy} + 3U_{1,y}^{(y)}\right] \right\} + \left\{2(3\beta_{3} + \Phi_{3}) - \beta_{2,\tau_{w}}\right\}_{y} s^{2} + 6U_{1}^{(y)} \\
+ (s^{2}\gamma_{2,\tau_{w}},y - \frac{1}{2}U_{1,\tau_{w}}^{(y)}\right\} r \left\{4(\Phi_{4} + \beta_{4}) + \beta_{3,\tau_{w}}\right\}_{y} s^{2} + (s^{2}\gamma_{3,\tau_{w}},y \\
+ (s^{2}\beta_{2,\gamma_{2},y}g_{U_{1,y}^{(y)}} + \left[2\Phi_{2} + 3\beta_{2} - \gamma_{2})(\Phi_{2} + \beta_{2}),y + 4(3U_{1}^{(y)} - yU_{1,y}^{(y)})\right]s^{2} \\
+ \left[\delta_{2}U_{1,y}^{(y)} - 2U_{1,y}^{(y)}\right]s^{4} + \left(10\Phi_{2} + 2\beta_{2} + 14\gamma_{2} - \frac{11}{2}U_{1,y}^{(y)}\right)U_{1}^{(y)} + 10U_{3}^{(y)} \\
- U_{1,\tau_{w}}^{(y)} - 4s^{-2}(U_{1}^{(y)})^{2}\right\} wr^{2} + \mathcal{O}(r^{3}) \quad .
\end{align}

\begin{align}
r^{-1}s^{2}\mathcal{R}_{\phi \tau_{w}}^{(xy)} = \left[\frac{1}{2}s^{2}(s^{2}U_{1,y}^{(y)}),y + 2U_{1,y}^{(y)}\right] \right\} + \left\{\frac{1}{2}(s^{2}U_{2,y}^{(y)})_{y} + (s^{2}\delta_{2,\tau_{w}},y \right\} s^{2} + 5U_{2}^{(y)} \\
- \frac{1}{2}U_{1,\tau_{w}}^{(y)} \right\} + \left\{\frac{1}{2}(s^{2}U_{3,y}^{(y)}),y + (s^{2}\delta_{2,\tau_{w}},y - (s^{2}\gamma_{2},yU_{1,y}^{(y)} - 2(3U_{1}^{(y)} - yU_{1,y}^{(y)})\right\} s^{2} \\
+ 9U_{3}^{(y)} - U_{2,\tau_{w}}^{(y)} - 2yU_{1,y}^{(y)}\beta_{2,y} - 2s^{2}U_{1,y}^{(y)}\gamma_{2,\tau_{w}} + \left(2\beta_{2} - \frac{3}{2}U_{1,y}^{(y)} - 10\gamma_{2} + 10\Phi_{2}\right)U_{1}^{(y)}
\end{align}
The Bondi–Sachs metric at the vertex of a null cone

\[ + \left( \delta_{2,yy} - 2U_{1,y}^\phi \right) U_1^\gamma + \left[ U_{1,yy}^\gamma - 4(\Phi_2 + \beta_2),y \right] \delta_2 + 2 \left[ 2yU_1^\phi + 7\delta_2 \right] s^{-2}U_1^y + 2\delta_{2,},y U_{1,y}^\gamma \right) + \mathcal{O}(r^3) \]  

(A.6)

The Ricci tensor components for the hyper-surface equations are

\[ \mathcal{R}_{rr} = 8\beta_2 + 12\beta_3 wr + \left[ 16\beta_4 - 8(\gamma_2^2 + \delta_2^2) \right] r^2 + \mathcal{O}(r^3) , \]  

(A.7)

\[ r^{-1}s^2\mathcal{R}_{yy} = 2U_1^y + 2(s^2_2),y + \left[ 5U_2^y - s^2_3, y + 3(s^2_3),y \right] rw^{-1} + \left[ 9U_3^y - 2s^2_5, y + 4(s^2_2), y + (6U_1^\phi - 4\delta_2, y)s^2_2 \right] \delta_2 + 6(\gamma_2 - \beta_2)U_1^\phi r^2 + \mathcal{O}(r^3) , \]  

(A.8)

\[ r^{-1}s^{-2}\mathcal{R}_{y\phi} = 2U_1^\phi + 2s^{-2}(s^2_2), y + \left[ 5U_1^\phi + 3s^{-2}(s^2_2), y \right] rw^{-1} + \left[ 9U_3^\phi + 4s^{-2}(s^2_4), y + 6s^{-2}U_1^\phi \beta_2 - 2\left( \gamma_2 + \beta_2 \right)U_1^\phi \right] r^2 + \mathcal{O}(r^3) , \]  

(A.9)

\[ \mathcal{R}_{(2D)} = -12\Phi_2 + 5U_1^y - 2(\beta_2, y s^2_2), y - 8\beta_2 + 2s^{-2}(s^4_\gamma \gamma_2, y) - 4\gamma_2 + \left[ -16\Phi_3 + 6U_2^y - 2(\beta_3,y s^2_2), y - 12\beta_3 + 2s^{-2}(s^4_\gamma \gamma_3, y) - 4\gamma_3 \right] wr + \left\{ -20\Phi_4 + 7U_3^y - 2(\beta_4,y s^2_2), y - 16\beta_4 + 2s^{-2}(s^4_\gamma \gamma_4, y), y - 4\gamma_4 - 4s^2(\gamma_2^2 + \delta_2^2) + \frac{1}{2} \left( \frac{U_1^y}{s} \right)^2 - \frac{1}{2} \left( sU_1^\phi \right)^2 - 4s^{-2}(s^4_\delta \delta_2, y) \delta_2 - 2s^2(\beta_2, y - 2\gamma_2), \beta_2, y + 4 \left( \left( s^2_2 \beta_2, y, y - s^{-2}(s^4_\gamma \gamma_2, y) \right) \right) \gamma_2 + 4(\gamma_2^2 + \delta_2^2) - 2(8\Phi_2 + 5U_1^\phi), \beta_2 \right\} r^2 + \mathcal{O}(r^3) , \]  

(A.10)

and for the evolution equations

\[ r^{-2}\mathcal{R}_{(\gamma)} = -12\gamma_2 + 2s^2_\beta_2, yy + 3s^2_2(s^2_2 U_1^y), y + \left[ 12\gamma_2, r, w - 24\gamma_3 - 2s^2_\beta_3, yy \right] + 4s^2(s^2_2 U_1^y), y \right) wr + \left[ 16\gamma_3, r, w - 40\gamma_4 + 5s^2(s^2_2 U_3^y), y - 2s^2_\beta_4, yy \right] - 6s^2_\beta_2(s^2_2 U_1^y), y + 4 \left( U_1^y, y - 10\Phi_2 - 4\beta_2 \right) \gamma_2 + 14U_1^y \gamma_2, y - \frac{1}{2} \left( \frac{U_1^y}{s} \right)^2 + \left[ 4\gamma_2, \beta_2, yy - 2(\beta_2, y s^2_2), \beta_2, y + 14\delta_2, s^2_2, \beta_2, \beta_2, y + \frac{1}{2} \left( \frac{U_1^\phi}{s} \right)^2 - 2 \right] r^2 + \mathcal{O}(r^3) , \]  

(A.11)

\[ r^{-2}\mathcal{R}_{(\delta)} = 12\delta_2 - 3s^2_2 U_1^\phi + \left( -12\delta_2, r, w + 24\delta_3 - 4s^2_2 U_2^\phi \right) wr + \left\{ -16\delta_3, r, w + 40\delta_4 - 5s^2_2 U_3^\phi, y + \left[ 6(\beta_2 + 14\gamma_2), U_1^\phi, y - 4\beta_2, y, y \right] s^2 + (U_1^\phi - 14\delta_2, y), U_1^y + 4(\beta_2 - U_1^y, y + 10\Phi_2) \right\} \delta_2 \right] r^2 + \mathcal{O}(r^3) . \]  

(A.12)
Appendix A.2. The linearized Ricci tensor for the $\mathcal{C}_n$ correction coefficients

The Ricci quantities for the main equations that are linearized with respect to $\mathcal{C}_n$ are assumed to be of the form

$$\mathcal{R}_{\mu\nu} = \sum_{n=0}^{N} (\mathcal{R}(n)) \mathcal{R}_{\mu\nu}^{(n)} , \quad \mathcal{R}_{(\cdot)} = \sum_{n=0}^{N} (\mathcal{R}(n)) \mathcal{R}_{(\cdot)}^{(n)} ,$$

where $\mathcal{R}_{(\cdot)} \in \{ \mathcal{R}(\gamma), \mathcal{R}(\delta), \mathcal{R}_{(2D)} \}$. The relevant non-zero coefficients for the hypersurface equations read

$$\mathcal{R}_{(\gamma)}^{(n)} = (4n + 8)\beta_{n+2} : n \geq 0 , \quad (A.13)$$

$$s^2 \mathcal{R}_{(\gamma)}^{(n)} = \frac{1}{2} n(n + 3)U_n^y + (1 - n)\beta_{n+1,y} + (n + 1) \left( s^2 \gamma_{n+1} \right)_{,y} : n \geq 1 , \quad (A.14)$$

$$s^{-2} \mathcal{R}_{(\phi)}^{(n)} = \frac{1}{2} n(n + 3)U_n^\phi + s^{-2}(n + 1) \left( s^2 \delta_{n+1} \right)_{,y} : n \geq 1 , \quad (A.15)$$

$$\mathcal{R}_{(2D)}^{(n)} = -4(n + 3)\Phi_{n+2} - 4(n + 2)\beta_{n+2} + (n + 5)U_n^{y,y} - 2 \left( s^2 \beta_{n+2,y} \right)_{,y} + 2s^{-2} \left( s^4 \gamma_{n+2,y} \right)_{,y} - 4\gamma_{n+2} : n \geq 0 , \quad (A.16)$$

and the evolution equations for $\gamma$ and $\delta$ are given by

$$\mathcal{R}_{(\gamma)}^{(2)} = -12\gamma_2 - 2s^2\beta_{2,y} + 3s^2 \left( s^{-2}U_1^y \right)_{,y} , \quad (A.17)$$

$$\mathcal{R}_{(\delta)}^{(2)} = 12\delta_2 - 3s^2U_1^\phi , \quad (A.18)$$

and by

$$\mathcal{R}_{(\gamma)}^{(n+1)} = 2(n + 1) \left[ 2\gamma_{n+1} - (n + 2)\gamma_{n+1} \right] + s^2(n + 2) \left( s^{-2}U_n^y \right)_{,y} - 2s^2\beta_{n+1,y} , \quad (A.19)$$

$$\mathcal{R}_{(\delta)}^{(n+1)} = 2(n + 1) \left[ (n + 2)\delta_{n+1} - 2\delta_{n+1} \right] - s^2(n + 2)U_n^\phi . \quad (A.20)$$

for $n > 1$, respectively.

Appendix B. The vacuum solution SIMPLE

The static vacuum solution SIMPLE is the only known explicit non-linear and axially symmetric solution of the Einstein equations in Bondi coordinates (Bičák et al. 1983). SIMPLE is boost and rotation symmetric (Bičák et al. 1984), and depends on one free parameter $a$. The solution given in Gómez et al. (1994) reads in our coordinates $x^a = (r_w, r, y, \phi)$ and nomenclature of variables

$$\Sigma(a, r, y) = \sqrt{1 + a^2 r^2 s^2(y)} \quad (B.1)$$

$$\gamma(r, y) = \ln(1 + \Sigma(a, r, y)) - \ln(2) , \quad (B.2)$$

$$\beta(r, y) = \ln \left[ 1 + \Sigma(a, r, y) \right] - \frac{1}{2} \ln \Sigma(a, r, y) - \ln 2 , \quad (B.3)$$

$$U^y(r, y) = \frac{a^2 y s^2(y)}{\Sigma(a, r, y)} , \quad (B.4)$$

$$\Phi(r, y) = \frac{1}{2} \ln \left[ 2a^2 r^2 s^2(y) - a^2 r^2 + 1 \right] - \ln \left[ 1 + \Sigma(a, r, y) \right] + \ln 2 , \quad (B.5)$$

$$\delta(r, y) = U^\phi(r, y) = 0 . \quad (B.6)$$
The Bondi–Sachs metric at the vertex of a null cone

Defining \( a = (12K)^{1/2} \) and expanding the Bondi functions of SIMPLE near \( r = 0 \) gives

\[
\gamma(r, y) = r^2 KP_2^2(y) + \left[ -\frac{27}{2} K^2 P_2^2(y) + \frac{9}{35} K^2 P_4^2(y) \right] r^4 + \mathcal{O}(r^6), \tag{B.7}
\]

\[
\delta(r, y) = 0, \tag{B.8}
\]

\[
\beta(r, y) = \left[ \frac{9}{7} K^2 P_2^2(y) - \frac{3}{35} K^2 P_4^2(y) \right] r^4 + \mathcal{O}(r^6), \tag{B.9}
\]

\[
U^y(r, y) = -4rKs(y)P_1^1(y) + \left[ \frac{96}{7} K^2 P_2^1(y) - \frac{144}{35} K^2 P_4^1(y) \right] s(y)r^3 + \mathcal{O}(r^5), \tag{B.10}
\]

\[
U^\phi(r, y) = 0, \tag{B.11}
\]

\[
\Phi(r, y) = -6r^2 KP_2^0(y) + \left[ -\frac{48}{5} K^2 + \frac{24}{7} K^2 P_4^0(y) - \frac{1044}{35} K^2 P_4^0(y) \right] r^4 + \mathcal{O}(r^6), \tag{B.12}
\]

where we expressed powers of \( y \) in terms of associated Legendre polynomials.

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