Continuous self-similar evaporation of a rotating cosmic string loop

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Abstract

A solution of the linearized Einstein and Nambu–Goto equations is constructed which describes the evaporation of a certain type of rotating cosmic string—the Allen–Casper–Ottewill loop—under the action of its own self-gravity. The solution evaporates self-similarly, and radiates away all its mass–energy and momentum in a finite time. Furthermore, the corresponding weak-field metric can be matched to a remnant Minkowski spacetime at all points on the future light cone of the final evaporation point of the loop.

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1. Introduction

Cosmic strings are thin filaments of topologically trapped Higgs field energy whose dynamics, in the zero-thickness limit, are controlled by the Nambu–Goto action. In this limit, cosmic strings are effectively line singularities which move under the action of their own elastic tension, and radiate gravitational energy and momentum in the process. They may have played a role in the formation of a large-scale structure in the early Universe (see [1] for a review), but their gravitational properties are still only poorly understood, and they consequently remain problematic objects for general relativity. In particular, a rigorous distributional description of line singularities has not yet been developed within the framework of general relativity, nor has it been demonstrated that zero-thickness cosmic strings can always be interpreted as the unique limits of well-behaved finite-thickness solutions [2].

At present, all known exact self-gravitating string solutions in a Minkowski background describe either infinite straight strings [3–6] or infinite strings interacting with plane-fronted gravitational waves [7, 8], and the only general result available on the dynamics of string loops in strong-field gravity is Hawking’s proof that a collapsing circular loop would form an event horizon after radiating away at most 29% of its original energy [9]. (In addition, the full 3-metric outside a circular loop at a moment of time symmetry has been constructed in [10].)
Given the complexity of the general strong-field problem, therefore, the most promising arena for the study of loop self-gravity would appear to be the weak-field limit. Fortunately, it can be shown that the self-gravity of a GUT string would almost everywhere be small enough to justify a weak-field treatment [11], although this treatment typically breaks down in the vicinity of certain common pathological features known as kinks and cusps.

In [12] (henceforth referred to as Paper I) it was shown that a certain type of rigidly rotating cosmic string loop—the Allen–Casper–Ottewill (ACO) solution [13]—evolves by self-similar shrinkage in the weak-field limit. This is the first explicit back-reaction calculation to have been published for a cosmic string loop, as all previous calculations have relied on numerical approximation [11]. However, the calculation relies on what might be called an adiabatic approximation, as the total flux of gravitational energy and momentum from the loop is integrated over a single-oscillation period of the loop’s unperturbed trajectory, and is shown to induce no changes other than a small decrease in the loop’s energy and length, plus a comparable rotational phase shift.

In this paper, I extend the analysis of Paper I by relaxing the adiabatic assumption and constructing a self-similar evaporating solution of the linearized-field equations which is identical to the Allen–Casper–Ottewill solution on spacelike sections and ultimately radiates away all its mass–energy. Although the trajectory of the string loop and the perturbations it induces in the background metric are only minimally different from the corresponding quantities in the adiabatic case, the evaporating solution represents an incremental advance in the understanding of loop self-gravity. In particular, the metric perturbations can be calculated at all points outside the string’s world sheet, and the perturbed (radiating) spacetime can be matched to a flat spacetime which occupies the causal future of the final point of evaporation of the loop. The naive expectation that the loop evaporates completely to leave behind a vacuum remnant spacetime, therefore, appears to be consistent at this level of approximation.

2. Preliminaries

The mathematical formalism used to describe zero-thickness cosmic strings has been discussed in some detail in Paper I, and in this section I will offer only a brief summary.

Throughout the paper, \( x^a \equiv [x^0, x^1, x^2, x^3] = [t, x, y, z] \) are local coordinates on the four-dimensional background spacetime \( (M, g_{ab}) \), and the metric tensor \( g_{ab} \) has signature \((+,-,-,-)\), so that timelike vectors have positive norm. The world sheet \( T \) of the string is the two-dimensional surface it traces out as it moves, and is described parametrically by a set of equations of the form \( x^a = X^a(\xi^A) \), where the parameters \( \xi^A \equiv (\xi^0, \xi^1) \) are gauge coordinates.

The intrinsic two-metric induced on \( T \) by a given choice of gauge coordinates is

\[
\gamma_{AB} = g_{ab} X_a^A X_b^B, \tag{2.1}
\]

where \( X_a^A \) is shorthand for \( \partial X^a/\partial \xi^A \). The intrinsic metric \( \gamma_{AB} \) is assumed to be non-degenerate with signature \((+,−)\) almost everywhere on \( T \). If \( \gamma \) denotes \( |\text{det}(\gamma_{AB})| \) the Nambu–Goto action [14, 15] has the form

\[
I = -\mu \int \gamma^{1/2} \, d^2 \xi, \tag{2.2}
\]

where \( \mu \) is the (constant) mass per unit length of the string.

The stress–energy tensor \( T^{ab} \) induced by a given trajectory is found by varying the Nambu–Goto action with respect to the background metric \( g_{ab} \), according to the standard
prescription $\delta I = - \frac{1}{2} \int T^{ab} \delta g_{ab} \sqrt{-g} \frac{1}{2} d^4 x$, where $g \equiv |\text{det}(g_{ab})|$. The equation for $\delta I$ inverts to give

$$T^{ab}(x^c) = - 2 g^{-1/2} \int (\delta L / \delta g_{ab}) \delta (4)(x^c - X_c) d^2 \zeta,$$

(2.3)

where $L \equiv - \frac{1}{2} \mu g^{1/2}$ is the Lagrangian density in (2.2), and $\delta L / \delta g_{ab} = - \frac{1}{2} \mu g^{1/2} \gamma^{AB} X_a^a X_b^b$, (2.4)

with $\gamma^{AB}$ being the inverse of $\gamma_{AB}$.

In Paper I, the gauge choice $\gamma_{AB} = \kappa \eta_{AB}$ was imposed from the outset, where $\eta_{AB} = \text{diag}(1, -1)$ is the Minkowski 2-tensor. However, it turns out that this choice is not as convenient when the string is evaporating (and the trajectory is no longer time periodic), and for this reason the more general versions of the string stress–energy and equation of motion will be retained in what follows.

The extrinsic curvature tensor of the world sheet $T$ has the general form

$$K^{abc} = p^a_{b} p^b_{c} \nabla_a p^m_{mc},$$

(2.5)

where $\nabla_a$ is the derivative operator associated with $g_{ab}$ and $p^a_{b} \equiv \gamma^{AB} X^a_A X^b_B$ is the projection tensor corresponding to $\gamma_{AB}$. In terms of the derivatives of the position function $X^a$ the curvature tensor $K^{abc}$ and its trace $K^c \equiv g_{ab} K^{abc}$ can be written as

$$K^{abc} = q^c_{a}(\gamma^{AC} \gamma^{BD} X^a_A X^b_B X^d_C D + p^m_{an} p^b_{mn} \Gamma^d_{mn})$$

(2.6)

and

$$K^c = q^c_{a}(\gamma^{CD} X^d_C D + p^m_{an} \Gamma^d_{mn})$$

(2.7)

with $q^a_{b} \equiv g^{ab} - p^a_{b}$ being the orthogonal complement of $p^a_{b}$ and $\Gamma^d_{bc}$ being the Christoffel symbol associated with $g_{ab}$. The equation of motion of the string is recovered by setting the functional derivative $\delta L / \delta X^a$ equal to zero, and reads simply

$$K^c = 0.$$  
(2.8)

In the absence of any sources of stress–energy other than the string itself, the system of equations for the metric and the string’s trajectory is closed by imposing the Einstein equation $G^{ab} = -8 \pi T^{ab}$, where $G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R$ is the Einstein tensor. Equations (2.3), (2.4) and (2.8), together with the Einstein equation, constitute the strong-field back-reaction problem. As was mentioned above, the only known solutions to the strong-field back-reaction problem describe either infinite straight strings (with or without an envelope of cylindrical gravitational waves [3–6]) or infinite strings interacting with plane-fronted gravitational waves (the so-called travelling-wave solutions [7, 8]). There are no known exact loop solutions in a vacuum background, and an analysis of loop evolution is currently feasible only at the level of the weak-field approximation.

In the weak-field formalism the field and trajectory variables are truncated at linear order in the mass per unit length $\mu$ of the string, which is assumed to be small compared to 1. This is certainly thought to be true of GUT strings, for which $\mu \sim 10^{-6}$. The underlying assumption is that the solution pair $(g_{ab}, X^a)$ to the strong-field back-reaction problem can be expanded as perturbative series of the form

$$g_{ab} = \sum_{k=0}^{\infty} \mu^k g^{(k)}_{ab} \quad \text{and} \quad X^a = \sum_{k=0}^{\infty} \mu^k X^{(k)}_a,$$

(2.9)

1 The convention adopted here for the Riemann tensor is that

$$R^{abcd} = \frac{1}{2} (g^{ac} g^{bd} - g^{ad} g^{bc} - g^{bc} g^{ad} + g^{bd} g^{ac}) \Gamma^e_{bd} \Gamma^d_{ae} - \Gamma^e_{cd} \Gamma^d_{ae}.$$

Also, geometrized units have been chosen, so that the gravitational constant $G$ and the speed of light $c$ are set to 1.
where the functions $g_{ab}^{(k)}$ and $X^a_{(k)}$ are independent of $\mu$. The weak-field back-reaction problem consists in setting $g_{ab}^{(0)}$ equal to $\eta_{ab}$, the Minkowski metric tensor, and solving the Einstein equation and equation of motion (2.8) at linear order in $\mu$ to obtain the metric perturbation $g_{ab}^{(1)}$ and the trajectory functions $X^a_{(0)}$ and $X^a_{(1)}$.

In what follows, the Minkowski 4-tensor will be taken to have its rigid form $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. In solving the weak-field problem, first a gauge must be chosen for the metric perturbation $g_{ab}^{(1)}$ and also for the functions $X^a_{(0)}$ and $X^a_{(1)}$. If $h_{ab} = \mu g_{ab}^{(1)}$, the standard harmonic gauge conditions read $h_{ab}^{\mu, \mu} = \frac{1}{2} h_{\mu, \mu}$, where $h = h_{\mu}^\mu$ and indices are everywhere lowered and raised using $\eta_{ab}$ and its inverse $g^{ab}$. The Einstein equation $G_{ab} = -8\pi T_{ab}$ then reduces to $\Box h_{ab} = -16\pi S_{ab}$, where $S_{ab} = T_{ab} - \frac{1}{2} \eta_{ab} T_\mu^\mu$ and $\Box \equiv \partial_\mu - \nabla^\mu$ is the flat-space d'Alembertian. With $x^a = [t, \mathbf{x}]^a$, the retarded solution for $h_{ab}$ is

$$h_{ab}(t, \mathbf{x}) = -4 \int \frac{S_{ab}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x', \quad (2.10)$$

where $t' = t - |\mathbf{x} - \mathbf{x}'|$ is the retarded time at the source point $\mathbf{x}'$. Since $g^{1/2} = 1$ to leading order in $h_{ab}$, equations (2.3) and (2.4) together give

$$S_{ab}(x^d) = \mu \int \gamma^{1/2} \gamma^{AB} \left( X^0_{(0a)} - \frac{1}{2} \eta_{ab} X^0_{(0)} \right) \delta^{(4)}(x^d - X^d) \, d^3\zeta, \quad (2.11)$$

with $\gamma^{1/2}$ and $\gamma^{AB}$ evaluated using $\eta_{ab}$ in place of $g_{ab}$ and $X^a_{(0)}$ in place of $X^a$.

Furthermore, if $g_{ab}^{(0)} = \eta_{ab}$, then $\Gamma^d_{mn} = 0$ to leading order, and at linear order in $h_{ab}$, the equation of motion $K^d = 0$ reads

$$q_d^c \left( \gamma^{CD} X^d_{(0)} \right) = -q^{d} X^c + X^c \left( h_{bd} - \frac{1}{2} h_{ab, d} \right), \quad (2.12)$$

where $q_d^c$ and $\gamma^{AC}$ are evaluated using $\eta_{ab}$ rather than $g_{ab}$ (but $X^a$ is retained in full). As was explained in Paper I, this equation is just the flat-space version of the Battye–Carter equation [16, 17].

If $X^a$ is now replaced by $X^a_{(0)} + \delta X^a$ where $\delta X^a = \mu X^a_{(1)}$, then, with $q_d^c$ and $\gamma^{AB}$ evaluated using $X^a_{(0)}$ in place of $X^a$, equation (2.12) splits into two parts:

$$q_d^c \gamma^{CD} X^d_{(0)} = 0 \quad (2.13)$$

and

$$q_d^c \gamma^{CD} \delta X^d_{(0)} = -2 q_{ab, c} \gamma^{BD} X^a_{(0)} \delta X^b + h_{bd} X^b_{(0)} \left( h_{a c, b} - \frac{1}{2} h_{ab, c} \right), \quad (2.14)$$

at zeroth order and first order in $\mu$ respectively. (Here, round brackets on indices denote symmetrization, so that $q_d^{(b) c} X^c_{(0)} \equiv \frac{1}{2} (q_d^{(b)} X^c_{(0)} + q_d^{(b)} X^c_{(0)})$.)

The weak-field back-reaction problem, in its most general form, therefore reduces to equations (2.10), (2.13) and (2.14) for $h_{ab}$, $X^a_{(0)}$ and $\delta X^a$. In order to specialize these equations to the problem in hand, it is necessary to first explain the choice that will be made for $X^a_{(0)}$, and the corresponding choice of gauge coordinates $\xi^0$ and $\xi^1$. This will be done in sections 3 and 4.
3. The Allen–Casper–Ottewill loop

The equation of motion at leading order (2.13) can be solved exactly by making a suitable choice of gauge coordinates. A common choice is the so-called standard gauge \( \xi^A = [\tau, \sigma]^A \), in which \( \gamma_{AB} \) is everywhere proportional to the Minkowski tensor \( \eta_{AB} = \text{diag}(1, -1) \). Since \( \gamma_{AB} = \eta_{ab} X_{(0)}^a \eta_{bc} X_{(0)}^c \), the geometric significance of this gauge choice is that the tangent vectors \( X_{(0),\tau} \) (which is timelike) and \( X_{(0),\sigma} \) (which is spacelike) are orthogonal and of equal magnitude. As was demonstrated in Paper I, the equation of motion (2.13) reduces in the standard gauge to the wave equation

\[
X_{(0),\tau\tau}^a - X_{(0),\sigma\sigma}^a = 0. \tag{3.1}
\]

If the subsidiary gauge alignment condition \( X_{(0)}^a(\tau, \sigma) = \tau \) is imposed then all loop solutions of (2.13) can be written in the form \( X_{(0)}^a(\tau, \sigma) = [\tau, X(\tau, \sigma)]^a \), where

\[
X(\tau, \sigma) = \left\{ \frac{1}{2} [a(\tau + \sigma) + b(\tau - \sigma)] \right\}
\]

for some choice of vector functions \( a \) and \( b \). The gauge constraints \( \gamma_{\tau\tau} + \gamma_{\sigma\sigma} = 0 \) and \( \gamma_{\tau\sigma} = 0 \) together imply that \( |a'|^2 = |b'|^2 = 1 \), where a prime denotes differentiation with respect to the relevant argument. If the loop is in its center-of-momentum frame then \( a \) and \( b \) are each periodic functions of their arguments with some parametric period \( L \), and since \( X(\tau + L/2, \sigma + L/2) = X(\tau, \sigma) \) for any values of \( \tau \) and \( \sigma \) when \( a \) and \( b \) are periodic, the fundamental oscillation period \( \tau_p \) of the loop is \( L/2 \). In the center-of-momentum frame, the total 4-momentum of the string loop on any surface of constant \( t = \tau \) is

\[
p^a = \mu \int_0^L X_{(0),\tau}^a \, d\sigma = \mu \int_0^L \left[ 1, \frac{1}{2} a' + \frac{1}{2} b' \right] \, d\sigma = \mu L[1, 0] \tag{3.3}
\]

and, in particular, the energy of the loop is \( E = \mu L \). The corresponding angular momentum of the loop is

\[
J = \frac{1}{4} \mu \int_0^L (a \times a' + b \times b') \, d\sigma. \tag{3.4}
\]

In the weak-field limit, the gravitational power \( P \) radiated by an oscillating compact source can be calculated from the standard expression for the power per unit solid angle in the wave zone (at distances large compared to the characteristic size \( L \) of the source):

\[
\frac{dP}{d\Omega} = \frac{\omega^2}{\pi} \sum_{m=1}^{\infty} m^2 \left[ \hat{T}^{ab} \hat{T}^{ab} - \frac{1}{2} \hat{T}^{a}_{a} \right],
\]

where \( \omega = 2\pi/\tau_p \) is the circular frequency of the source, and

\[
\hat{T}^{ab}(m, n) = t_p^{-1} \int_{0}^{t_p} \int_{R^3} T^{ab}(t, x) e^{im(t-nx)} \, dt \, d^3 x
\]

is the Fourier transform of its integrated stress–energy, with \( n \) being the unit vector in the direction of the field point.

For a string loop in a flat background in the standard gauge, the stress–energy tensor (2.3) becomes

\[
T^{ab}(x^\tau) = \mu \int (X_{(0),\tau}^a X_{(0),\tau}^b - X_{(0),\sigma}^a X_{(0),\sigma}^b) \delta^{(4)}(x^\tau - X_{(0)}^\tau) \, d\tau \, d\sigma,
\]

and the power per unit solid angle reduces to

\[
\frac{dP}{d\Omega} = 8\pi \mu^2 \sum_{m=1}^{\infty} m^2 \left[ (A \cdot A^*) (B \cdot B^*) + |A \cdot B^*|^2 - |A \cdot B|^2 \right], \tag{3.8}
\]
where

\[
A^e(m, n) = \frac{1}{L} \int_0^L e^{2\pi i n a(\sigma_\ast) / L} [1, a'(\sigma_\ast)]^\tau d\sigma_\ast \tag{3.9}
\]

and

\[
B^e(m, n) = \frac{1}{L} \int_0^L e^{2\pi i n b(\sigma_\ast) / L} [1, b'(\sigma_\ast)]^\tau d\sigma_\ast , \tag{3.10}
\]

with the gauge coordinates \( \sigma_\ast \) and \( \sigma_- \) being defined by \( \sigma_{\pm} \equiv \tau \pm \sigma \). (Here and elsewhere the dot product is taken with respect to the flat-space metric \( \eta_{ab} \)).

The radiative efficiency \( \gamma^0 \) of a string loop is defined to be \( \gamma^0 \equiv \mu^{-2} P \), where \( P = \int \frac{d\mu}{d\Omega} \) is the total power of the loop. The radiative efficiency has been calculated analytically for a large class of simple loop trajectories by Allen, Casper and Ottewill [13], and numerically for the loop by-products of string network simulations by Allen and Shellard [18] and Allen and Casper [19]. The simulation studies found that the value of \( \gamma^0 \) is typically of order 65–70, and seems to be bounded below by about 40. This observation is consistent with the results of the analytic study, which found that \( \gamma^0 \) has a minimum value of about 39.0025, and occurs if the mode functions have the form

\[
a(\sigma_\ast) = \left( (\sigma_\ast - \frac{1}{2}L) \hat{z} \quad \text{for} \quad -\frac{1}{2}L \leq \sigma_\ast \leq \frac{1}{2}L \right) \tag{3.11}
\]

and

\[
b(\sigma_-) = \frac{L}{2\pi} \left[ \cos(2\pi \sigma_- / L) \hat{x} + \sin(2\pi \sigma_- / L) \hat{y} \right] , \tag{3.12}
\]

with \( a \) being represented by its even periodic extension when \( \sigma_\ast \) lies outside \([-\frac{1}{2}L, \frac{1}{2}L]\).²

The corresponding loop, which I will henceforth refer to as the Allen–Casper–Ottewill or ACO loop, is rigidly rotating about the \( z \)-axis with an angular speed \( \omega = 4\pi / L \). The evolution of the loop is illustrated in figure 1, which shows the \( y-z \) projection of the loop at times \( \tau - \varepsilon = 0, L/16, L/8 \) and \( 3L/16 \) (top row) and \( \tau - \varepsilon = L/4, 5L/16, 3L/8 \) and \( 7L/16 \) (bottom row), where the time offset \( \varepsilon \) is 0.02L. The string has been artificially thickened for the sake of visibility, and the \( z \)-axis is also shown. The projections of the loop onto the \( x-y \) plane are circles of radius \( L/(4\pi) \). The points at the extreme top and bottom of the loop, where the helical segments meet and the modal tangent vector \( a' \) is discontinuous, are technically known as kinks. They trace out circles in the planes \( z = L/8 \) and \( z = -L/8 \). All other points on the loop trace out identical circles, although with varying phase lags. The net angular momentum of the loop is \( J = \frac{1}{\pi} L^2 \hat{z} \).

I should stress that it is not currently known whether the ACO solution has the lowest radiative efficiency of all possible string loops. Allen et al [13] showed only that the ACO solution minimizes \( \gamma^0 \) over a special class of solutions with the mode function \( a \) given by (3.11) and the mode function \( b \) confined to the \( x-y \) plane but otherwise arbitrary. Nonetheless, in Paper I the first-order equation of motion (2.14) was solved for a leading-order trajectory \( X^0 \) of ACO form, and it was shown that the net perturbation \( \delta X^0 \) induced in the string trajectory over a single-oscillation period \( \Delta \tau = t_p \) corresponds to no more than a change \( \Delta L = -\frac{1}{2} \gamma^0 / \mu L \approx -19.501 \mu L \) in the length and parametric period \( L \) of the loop, and a

² Strictly speaking, the \( a \) mode function was represented in both [13] and Paper I in the form

\[
a(\sigma_\ast) = \begin{cases} 
\sigma_\ast \hat{z} & \text{if} \quad 0 \leq \sigma_\ast < \frac{1}{2}L \\
(L - \sigma_\ast) \hat{z} & \text{if} \quad \frac{1}{2}L \leq \sigma_\ast < L 
\end{cases} 
\]

in which case the centroid of the loop lies at \((x, y, z) = (0, 0, \frac{1}{8} L)\). For the purposes of the analysis presented below, it is more convenient to translate the loop down the \( z \)-axis by an amount \( \frac{1}{8} L \) so that the centroid lies at the origin. The resulting mode function is then given by (3.11).
rotational phase shift with magnitude about $38.92 \mu$ acting to advance the overall pattern of
the loop. In other words, as it radiates the ACO loop retains its shape and, therefore, its
original radiative efficiency. If the ACO loop does indeed possess the lowest possible radiative
efficiency, it follows that it would be the longest lived loop in any ensemble of loops with the
same energy.

4. The trajectory of the evaporating ACO loop

Whether or not the ACO loop minimizes $\gamma^0$, the fact that $\gamma^0$ remains constant as the loop
evaporates is the key to a more extended description of the loop’s evolution. Over the course of
a double period $\Delta t = 2t_p = L$, the total energy $E = \mu L$ of the loop is reduced by an amount
$\Delta E = -P L = -\mu^2 \gamma^0 L$ and so $\Delta E / \Delta t = -\mu^2 \gamma^0$, or equivalently $\Delta L / \Delta t = -\mu \gamma^0$. Since
the loop is rotating uniformly, this relation is also true instantaneously, so that $dL/dt = -\mu \gamma^0$
and, therefore, $L(t) = L_0 - \mu \gamma^0 (t - t_0)$, where $L_0$ is the value of $L$ at some initial time $t_0$. In
particular, if the time coordinate is chosen so that $t_0 = 0$ then

$$L(t) = L_0 (1 - t/t_L) \tag{4.1}$$

for $t < t_L$, where $t_L \equiv L_0 / (\gamma^0 \mu)$ is the total future lifetime of the loop.

If the adiabatic calculation performed in Paper I accurately represents the evolution of the
ACO loop, it is expected that as it evaporates the loop retains the same spatial cross-section,
part from a uniform reduction in scale proportional to $L(t)$, and a rigid rotation about the
$z$-axis. What follows now is a heuristic argument leading to a plausible form for the trajectory
of the evaporating loop. It should be cautioned that the argument below relies implicitly on an
assumption that the background spacetime is flat, and consideration of the first-order equation of motion (2.14) quickly shows that this assumption is overly optimistic. Nonetheless, it is also clear on physical grounds what form the relativistic corrections to the trajectory should have, and the first-order equation of motion will be solved to give these corrections explicitly in section 6.

The most general form of the position function $X$ of the evaporating loop (with $\tau$ henceforth replaced by the time coordinate $t$, and $\xi$ a general parameter) is

$$X(t, \xi) = L(t)\bar{X}(t, \xi),$$

(4.2)

where $\bar{X}$ is rigidly rotating, so that if $t < t_L$ and $t^* < t_L$ are two different times then

$$\bar{X}(t, \xi) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{X}(t^*, \xi^*),$$

(4.3)

with $\phi(t^*, t)$ being a phase shift and $\xi^*(t, t^*, \xi)$ a reparametrization of $\xi$.

Here, if $t^*$ is fixed, $\bar{X}(t^*, \xi^*)$ is a reference function that at each value of $t$ must trace out the same loop in $\mathbb{R}^3$ as $\xi$ varies over its range. In particular, if $t \to t + \delta t$ then $\xi^*(t + \delta t, t^*, \xi + \delta \xi) = \xi^*(t, t^*, \xi)$ for some function $\delta \xi(\xi)$. That is

$$\partial_t \xi^* = \partial_\xi \xi^* = 0,$$

(4.4)
to linear order in $\delta t$ and $\delta \xi$, and since this is a homogeneous linear first-order PDE with characteristic $\xi = \alpha(t)$ for some function $\alpha$, it follows that $\xi^*$ can be any function of $\xi - \alpha(t)$.

Hence, if the fixed reference time $t^*$ is suppressed,

$$X(t, \xi) = L(t)\begin{bmatrix} \cos \phi(t) & -\sin \phi(t) & 0 \\ \sin \phi(t) & \cos \phi(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}[L(\phi(t)\bar{X}_0 + L(M\bar{X}_0 - \alpha'\bar{X}_0))],$$

(4.5)

for some choice of functions $\phi$, $\bar{X}_0$ and $\alpha$.

Furthermore, because the parametric period of a cosmic string loop scales with its length $L$, all timescales on the loop do likewise, and in particular the velocity field $X_t$ of the loop must at all times be rotating rigidly. Now,

$$X_t = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} [L'\bar{X}_0 + L(\phi'\bar{X}_0 + \alpha'\bar{X}_0)],$$

(4.6)

where

$$M = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin \phi & -\cos \phi \\ \cos \phi & -\sin \phi \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(4.7)

and the primes denote derivatives with respect to the relevant arguments.

If the vector functions $\bar{X}_0$, $M\bar{X}_0$ and $\bar{X}_0'$ are linearly independent, the requirement that $X_t$ be rigidly rotating constrains $L'$, $L\phi'$ and $L\alpha'$ to all be constant. The first condition, that $L'$ be constant, was derived independently at the beginning of this section, and is equivalent to requiring the radiative efficiency $\gamma^0$ to be constant. Since $L(t) = L_0(1 - t/t_L)$ for $t < t_L$, the remaining constraints can be integrated immediately to give

$$\phi(t) = -p \ln(1 - t/t_L) + \phi_0$$

and

$$\alpha(t) = -q \ln(1 - t/t_L) + \alpha_0,$$

(4.8)

where $p$, $q$, $\phi_0$ and $\alpha_0$ are constants.
By suitably rotating the $x$–$y$ coordinate axes and rezeroing the gauge coordinate $\xi$ it is always possible to set $\phi_0$ and $\omega_0$ to zero. Furthermore, if $\dot{\sigma} = p\xi/q$ then $X_0$ is a function of $\ddot{\sigma} + p \ln(1 - t/t_L)$ and so

$$X(t, \ddot{\sigma}) = L_0(1 - t/t_L) \begin{bmatrix} \cos \phi(t) & -\sin \phi(t) & 0 \\ \sin \phi(t) & \cos \phi(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} X_0(\ddot{\sigma} - \phi(t)), \quad (4.9)$$

with $\phi(t) = -p \ln(1 - t/t_L)$.

If this trajectory is to describe the evaporation of an ACO loop, the position function (4.9) and its time derivative should, to leading order in $\mu$, match the position function $X$ and velocity $X_t$ of the unperturbed ACO loop at time $t = 0$. The ACO loop is described by the mode functions (3.11) and (3.12), so its position function has the explicit form

$$X(t, \sigma) = \frac{1}{4\pi} L \left[ \cos(2\pi(t - \sigma)/L) \hat{x} + \sin(2\pi(t - \sigma)/L) \hat{y} \right] + \frac{1}{2} \left( |t + \sigma| - \frac{1}{4} L \right) \hat{z} \quad (4.10)$$

for $-\frac{1}{2} L \leq t + \sigma \leq \frac{1}{2} L$.

Here, the constant-scale factors $L_0$ and $L$ are taken to be identical. Since $\phi(0) = 0$, the position functions (4.9) and (4.10) will agree at $t = 0$ if

$$X_0(\ddot{\sigma}) = \frac{1}{4\pi} L \left[ \cos(2\pi\sigma/L) \hat{x} - \sin(2\pi\sigma/L) \hat{y} \right] + \frac{1}{2} \left( |\sigma| - \frac{1}{4} L \right) \hat{z}. \quad (4.11)$$

for $-\frac{1}{2} L \leq \sigma \leq \frac{1}{2} L$. Furthermore, $L'(t) = -L_0/t_L$ is of order $\mu$ and can be ignored when calculating $X_0$, while $\dot{\phi}(t) = p/t_L$ at $t = 0$. Equating the time derivatives of (4.9) and (4.10) at $t = 0$ then leads to the consistency conditions $\ddot{\sigma} = -4\pi\sigma/L$ and $p = 4\pi t_L/L_0 \equiv 4\pi/\gamma^0\mu$.

After substituting these conditions and equation (4.11) into (4.9), the position function of the evaporating loop becomes

$$X(t, \ddot{\sigma}) = \frac{1}{4\pi} L_0(1 - t/t_L)$$

$$\times \left\{ \cos \left[ \frac{1}{2} \left( \phi(t) + \ddot{\sigma} \right) \right] \hat{x} + \sin \left[ \frac{1}{2} (\phi(t) + \ddot{\sigma}) \right] \hat{y} + \frac{1}{2} \left( |\ddot{\sigma} - \pi| \right) \hat{z} \right\} \quad (4.12)$$

for $-2\pi \leq \phi(t) - \ddot{\sigma} \leq 2\pi$, where $\phi(t) = -(4\pi t_L/L_0) \ln(1 - t/t_L)$.

Equation (4.12) suggests that a convenient choice of gauge coordinates $(\xi^0, \xi^1) = (u, v)$ for the trajectory of the evaporating ACO loop is

$$u = \frac{1}{2} (\phi(t) - \ddot{\sigma}) \quad \text{and} \quad v = \frac{1}{2} (\phi(t) + \ddot{\sigma}). \quad (4.13)$$

Then $1 - t/t_L = e^{-\kappa\mu(u+v)}$, where $\kappa\mu \equiv (4\pi t_L/L_0)^{-1}$ or equivalently $\kappa = \gamma^0/(4\pi)$. This, in turn, means that the time coordinate $t$ on the loop can be written as $t = t_L [1 - e^{-\kappa\mu(u+v)}]$, and the world sheet $T$ of the evaporating loop can be represented by the 4-function

$$X^a(u, v) = \left[ t_L, 0 \right]^a + \frac{1}{4\pi} L_0 e^{-\kappa\mu(u+v)} \left[ -\frac{1}{\kappa\mu}, r(u, v) \right]^a, \quad (4.14)$$

where

$$r(u, v) = \cos(v) \hat{x} + \sin(v) \hat{y} + \left( |u| - \frac{1}{2} \pi \right) \hat{z} \quad (4.15)$$

for $-\pi \leq u \leq \pi$, and $t_L = \frac{1}{4\pi\gamma^0} L_0$.

Strictly speaking, the vector function $r$ appearing in the 4-function (4.14) is the $2\pi$-periodic extension (in $u$) of the function defined in (4.15). The gauge coordinate pairs $(u, v)$ and $(u + 2\pi k, v - 2\pi k)$, therefore, correspond to the same spacetime point for any integer $k$. However, for the sake of simplicity the range of the parameter $u$ will henceforth be restricted
to \((-\pi, \pi]\). The final evaporation point \(x^a = [t_L, 0]^a\) of the loop then corresponds to the limit \(v \to \infty\). Note that in this limit the scale factor \(L(t) \equiv L_0 e^{-\kappa \mu(u+v)}\) goes to zero, while the angular speed \(dv/dt\) of the loop diverges and \(r(u, v)\) is undefined (although the loop’s 4-velocity \(X^a_\tau\), being scale-free, remains subluminal everywhere).

If \(|\kappa \mu(u+v)| \ll 1\), the loop’s position function \((4.14)\) can be expanded in powers of \(\mu\) to give

\[
X^{a}_{(0)}(u, v) = \frac{1}{4\pi} L_0[u + v, r(u, v)]^a \quad (4.16)
\]

and

\[
\delta X^a(u, v) = -\frac{1}{4\pi} L_0 \kappa \mu \left[ \frac{1}{2} (u + v)^2, (u + v)r(u, v) \right]^a. \quad (4.17)
\]

It is these functions that need to satisfy the zeroth- and first-order equations of motion \((2.13)\) and \((2.14)\). Note that the leading-order position function \((4.16)\) is effectively generated by an expansion of \(X^a\) about the curve \(u + v = 0\) on the loop’s world sheet, or equivalently about the \(t = 0\) spacelike cross-section.

The position function \(X^a\) can be expanded about other cross-sections of constant \(t < t_L\) by writing \(v = v_s + \tilde{v}\) where \(v_s\) is any fixed real number. Then equation \((4.16)\) becomes

\[
X^a(u, \tilde{v}) = [t_L, 0]^a + \frac{1}{4\pi} L_s e^{-\kappa \mu(u+v)} \left[ -\frac{1}{\kappa \mu}, r(u, v_s + \tilde{v}) \right]^a, \quad (4.18)
\]

with \(L_s \equiv L_0 e^{-\kappa \mu v_s}\), and if \(|\kappa \mu(u+\tilde{v})| \ll 1\) the zeroth-order position function is

\[
X^{a}_{(0)}(u, \tilde{v}) = [t_s, 0]^a + \frac{1}{4\pi} L_s[u + \tilde{v}, r(u, v_s + \tilde{v})]^a, \quad (4.19)
\]

where \(t_s \equiv \frac{1}{2\kappa \mu} L_0(1-e^{-\kappa \mu v_s})\). The analogue of equation \((4.17)\) for \(\delta X^a\) is found by replacing \(L_0\) with \(L_s\), \(u + v\) with \(u + \tilde{v}\), and \(r(u, v)\) with \(r(u, v_s + \tilde{v})\). Clearly, any two expansions of this type will be related by a time translation, a constant dilation and a rotation in the \(x-y\) plane, and if any of the expansions satisfies the equations of motion \((2.13)\) and \((2.14)\) then they all will.

The equations of motion will be considered more closely in section 6. It will come as no surprise that the zeroth-order position function \((4.16)\) does satisfy the zeroth-order equation of motion \((2.13)\), which reads simply \(X^a_{(0),uv} = 0\). Before considering the first-order equation of motion \((2.14)\), however, it is necessary to calculate the metric perturbations \(h_{ab}\), which depend only on \(X^a_{(0)}\). This will be done in section 5.

After a long calculation described in detail in section 6, it turns out that the first-order position function \(\delta X^a\) given by \((4.17)\) does not satisfy the first-order equation of motion \((2.14)\). The reason for this is not difficult to appreciate. As was mentioned at the beginning of this section, the calculation leading to the evaporating-loop trajectory \((4.14)\) was predicated on the assumption that the background metric is flat. It is evident from \((2.14)\), however, that the first-order equation of motion involves the first-order metric perturbation \(h_{ab}\). Once the background metric \(g_{ab}\) ceases to be flat, the spacetime coordinates \([t, x, y, z]\) no longer measure proper times or proper lengths. In particular, the self-similar shrinkage driven by the scale factor \(L_0 e^{-\kappa \mu(u+v)}\) may proceed at different rates in different directions at various points on the world sheet \(T\).

This possibility is easily accounted for by adding scale corrections, of order \(\mu\), to the components of the evaporating loop’s position function \((4.14)\). The form of these scale corrections is, of course, constrained by the equation of motion \((2.14)\), but even so there remains considerable freedom in the choice of the correction functions, corresponding not only to possible transformations (at order \(\mu\)) of the gauge coordinates \((u, v)\) and the spacetime
coordinates \([t, x, y, z]\), but also to the propagation of free vibrations (with amplitude \(\mu\)) around the loop.

In what follows, the freedom in the choice of gauge coordinates will be partially fixed by taking the position vector to have the form

\[
X^a(u, v) = [t_L, 0]^a + \frac{1}{4\pi} L_0 e^{-\kappa \mu \mu^a(u+v)} \left[ -\frac{1}{\kappa \mu} e^{\kappa^2 F(u,v)} \bar{r}(u, v) \right]^a,
\]

(4.20)

where now

\[
\bar{r}(u, v) = e^{\mu G(u,v)} \left[ \cos(v) \hat{x} + \sin(v) \hat{y} \right] + \left[ \left| u \right| \frac{1}{2\pi} + \mu A(u, v) \right] \hat{z}
\]

(4.21)

for \(-\pi \leq u \leq \pi\). Here, the correction functions \(F, G\) and \(A\) are assumed to be continuous and \(2\pi\)-periodic in both \(u\) and \(v\), so that \((u, v)\) and \((u + 2\pi, v - 2\pi)\) again correspond to the same spacetime point, and all three functions remain small compared to \(\mu^{-1}\) as \(u + v \to \infty\). The assumption that the trajectory is dilated by the same factor \(e^{\mu G}\) in the \(x\) and \(y\) directions effectively removes any freedom to redefine the gauge coordinate \(v\).

The presence of free vibrations, which break the self-similarity of the world sheet, can be removed by requiring the intrinsic 2-metric \(\delta X_a(u, v)\) to have the strictly self-similar form

\[
\gamma_{AB}(u, v) = \left( \frac{1}{4\pi} L_0 \right)^2 e^{-2\mu(u+v)} \overline{\gamma}_{AB}(u)
\]

(4.22)

to linear order in \(\mu\), with \(\overline{\gamma}_{AB}\) being a 2-metric to be determined. This constraint will be considered in more detail in section 6.2.

Given (4.20), the leading-order position function \(X^a_{(0)}\) near \(t = 0\) retains the form (4.16), while the first-order position function becomes

\[
\delta X^a(u, v) = \frac{1}{4\pi} L_0 \kappa \mu \left[ \frac{1}{2} (u + v)^2, (u + v) \bar{r}(u, v) \right]^a
\]

\[
+ \frac{1}{4\pi} L_0 \mu \left[ -F(u, v), G(u, v) \left[ \cos(v) \hat{x} + \sin(v) \hat{y} \right] + A(u, v) \hat{z} \right]^a.
\]

(4.23)

The actual form of the correction functions \(F, G\) and \(A\) will be determined once the first-order equation of motion (2.14) has been solved, and additional gauge conditions have been imposed.

Two remarks about the final form (4.20) of the loop trajectory are in order here. The first is that although (4.20) was developed to describe an evaporating ACO loop over the time interval \(0 \leq t < t_L\) (or equivalently \(0 \leq u + v < \infty\)), it can without difficulty be assumed to extend over the entire interval \(-\infty < t < t_L\) (or \(-\infty < u + v < \infty\)). The trajectory then describes a loop that has been evaporating ‘eternally’, with its energy and length scaling (to leading order in \(\mu\)) as \(1 - t/t_L\). This is obviously just a mathematical idealization, but it does side step the need to construct initial data for the metric perturbations \(h_{ab}\) at \(t = 0\).

The second point is that the value of the constant \(\kappa\) appearing in the corrected trajectory (4.20) is initially undetermined. Although it is to be expected in view of the results of Paper I that \(\kappa \approx 39.0025/(4\pi) \approx 3.1037\), no particular value of \(\kappa\) is assumed in the analysis that follows. It is only after the first-order equation of motion is solved that it becomes evident that the periodicity constraint on \(G\) forces \(\kappa\) to take on its expected value.

---

5 To see this, suppose that the term \(G(u, v) \left[ \cos(v) \hat{x} + \sin(v) \hat{y} \right]\) in equation (4.23) has the more general form \(G(u, v) \left[ \cos(v) \hat{x} + H \sin(v) \hat{y} \right]\). Then if \(v\) is replaced by a new coordinate \(v'\) defined by \(v = v' + \mu V (u, v')\) for some function \(V\), the correction term \(G \cos(v') \hat{x} + H \sin(v') \hat{y}\) becomes \((G \cos v' - V \sin v') \hat{x} + (H \sin v' + V \cos v') \hat{y}\) to leading order in \(\mu\). The unique gauge choice \(V = (G - H) \sin v' \cos v'\) then reduces this term to the isotropic form \(G' \left[ \cos(v') \hat{x} + \sin(v') \hat{y} \right]\), with \(G' = G \cos^2 v' + H \sin^2 v'\).
A schematic representation of the evaporation of the loop is shown in figure 2. The thickened line corresponds to the outer envelope of the loop, whose radius shrinks to zero at the final evaporation point \( x^a = [t_L, 0]^a \). Also shown are the future light cones \( F_0 \) and \( F_L \) of the origin and the evaporation point respectively. Note that, because of the first-order corrections to the background metric and the loop trajectory \( X^a \), the envelope of the loop and the light cones depicted in figure 2 should deviate from straight lines by terms of order \( \mu \).

Now that the choice of the evaporating-loop trajectory (4.20) has been explained in some depth, the remainder of the paper is a straightforward application of the solution methods discussed in section 2. The metric perturbations \( h_{ab} \) generated by \( X^a_{(0)} \) are calculated in section 5, the equation of motion for the perturbation functions \( \delta X^a \) is constructed and solved in section 6, and the metric perturbations on the future light cone \( F_L \) of the final evaporation point are shown to match onto an empty, flat spacetime in section 7.

5. The metric perturbations

5.1. Rotating self-similarity of the metric perturbations

The metric perturbations \( h_{ab} \) at any field point \( x^a \equiv [t, x]^a \) away from the world sheet of the evaporating loop can be calculated from equation (2.10). Because the analysis of section 7 depends on an understanding of the behavior of \( h_{ab} \) arbitrarily close to the light cone \( F_L \), it will be necessary to consider source points on the evaporating loop close to the final evaporation point, where \( u + v \to \infty \), and so it cannot be assumed that \( |\kappa \mu (u + v)| \ll 1 \). The scale factor \( L_0 e^{-\kappa \mu (u + v)} \) will, therefore, be retained in full in this section, although the correction functions \( F, G \) and \( A \), which are bounded and appear only at order \( \mu \) in (4.20), will be ignored. In other words, the trajectory will in this section be assumed to have its flat-space form (4.14).

The first step in the calculation is to write down the leading-order expressions for the intrinsic 2-metric \( \gamma_{AB} \) and the source function \( S_{ab} \). Since

\[
X^a_{,\alpha} = \frac{1}{4\pi} L_0 e^{-\kappa \mu (u+v)} [1, \ \text{sgn}(u)\hat{z}]^a
\]

and

\[
X^a_{,\nu} = \frac{1}{4\pi} L_0 e^{-\kappa \mu (u+v)} [1, -\sin(\nu)\hat{x} + \cos(\nu)\hat{y}]^a
\]
to leading order in $\mu$, it follows that the intrinsic 2-metric $\gamma_{AB} = X_A \cdot X_B$ on the world sheet has components

$$\gamma_{uu} = \gamma_{vv} = 0 \quad \text{and} \quad \gamma_{uv} = \left(\frac{1}{4\pi} L_0^4\right)^2 e^{-2\mu(u+v)} \quad \text{(5.3)}$$

The source function defined in equation (2.11) can be written as

$$S_{ab}(x^d) = \mu \int \Psi_{ab}(u,v) \delta^d(x^d - X^d) \, du \, dv,$$  \quad \text{(5.5)}

where

$$\Psi_{ab} \equiv \gamma^{1/2} \gamma_{AB} \left(\eta_{ab} x^a X^b - \frac{1}{2} \gamma_{ab} X^a X^b\right),$$

(5.6) evaluated again at leading order in $\mu$. Note here that $\gamma^{AB} x^a X^b$ is just the worldsheet projection tensor $p_{ab}$, which in a matrix form is

$$p_{ab} = \begin{bmatrix}
2 & \sin v & -\cos v & -s \\
\sin v & 0 & 0 & -s \sin v \\
-\cos v & 0 & 0 & s \cos v \\
-s & -s \sin v & s \cos v & 0 \\
\end{bmatrix}_{ab},$$

(5.7) where $s = \text{sgn}(u)$. The function $\Psi_{ab}$ is, therefore, equal to $-\gamma^{1/2} q_{ab}$, where $q_{ab} \equiv \eta_{ab} - p_{ab}$ was introduced in section 2 as the orthogonal complement of $p_{ab}$. Hence,

$$\Psi_{ab} = \left(\frac{1}{4\pi} L_0^4\right)^2 e^{-2\mu(u+v)} \bar{\Psi}_{ab}$$

with

$$\bar{\Psi}_{ab}(s,v) \equiv \begin{bmatrix}
1 & \sin v & -\cos v & -s \\
\sin v & 1 & 0 & -s \sin v \\
-\cos v & 0 & 1 & s \cos v \\
-s & -s \sin v & s \cos v & 1 \\
\end{bmatrix}_{ab}.$$  \quad \text{(5.8)}

Given (5.5), the integral expression (2.10) for $h_{ab}$ reads

$$h_{ab}(t,x) = -4\mu \int \frac{d^4 x'}{|x - x'|} \int \Psi_{ab}(u,v) \delta^d(x^d - X^d) \, du \, dv,$$  \quad \text{(5.9)}

where the source point is $x'^d = [t', x']^d$, with $t' \equiv t - |x - x'|$ being the retarded time. The distributional factor $\delta^d(x^d - X^d)$ in (5.9) can be integrated out by first transforming from $x'^d = [u,v]$ to $u' \equiv [t', x']^d = X^d(u,v)$, with $v, t$ and $x$ held constant. The Jacobian of this transformation is

$$|\partial y'^d/\partial x^b| = |\partial x^b/\partial y'^d|^{-1} = |X^0_{,u} - n \cdot X_{,u}|^{-1},$$  \quad \text{(5.10)}

where $n = (x - x')/(|x - x'|)$ is the unit vector from the source point $x'$ to the field point $x$.

On integrating over $x'^d$, $t'$ is everywhere replaced by $X^0 \equiv t [1 - e^{-\mu(u+v)}]$, and $x'$ by $X = \frac{1}{2\pi} L_0 e^{-\mu(u+v)} r(u,v)$, while $u$ becomes an implicit function of $v, t$ and $x$ through the equation $X^0(u,v) = t - |x - X(u,v)|$. The integral for $h_{ab}$, therefore, reads

$$h_{ab}(t,x) = -4\mu \int \|x - X|X^0_{,u} - (x - X) \cdot X_{,u}|^{-1} \Psi_{ab}(u,v) \, dv,$$  \quad \text{(5.11)}

where, as before, $X^0_{,u} = \frac{1}{2\pi} L_0 e^{-\mu(u+v)}$ and $X_{,u} = \frac{1}{2\pi} L_0 e^{-\mu(u+v)} s \dot{x}$ to leading order in $\mu$.

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4 Note that, at a physical level, the equation $t' = t - |x - x'|$ imposes the requirement that the source point $[t', x']$ lies on the backwards light cone of the field point $[t, x]$ to leading order in $\mu$. 

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Evaluation of integral (5.11) is simplified by the fact that \( h_{ab} \) satisfies a very elegant self-similarity relation. If \([t, x]\) is any field point with \( t - t_L \ll |x| \), where \( t - t_L = |x| \) is just the equation, at leading order, of the future light cone \( F_L \) of the final evaporation point, first define
\[
\psi(t, x) = -(\kappa \mu)^{-1} \ln(|x|-t+t_L)/t_L, \tag{5.12}
\]
then consider a second field point \([\bar{t}, \bar{x}]\) with
\[
\bar{t} \equiv e^{\epsilon \mu \psi}|x| \quad \text{and} \quad \bar{x} \equiv e^{\epsilon \mu \psi} R(\psi)x, \tag{5.13}
\]
where
\[
R(\theta) \equiv \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.14}
\]
is the standard rotation matrix about the \( z \)-axis. Note that \([\bar{t}, \bar{x}]\) lies on the future light cone \( F_0 \) of the origin, as \( \bar{t} \equiv |\bar{x}| \).

Now, the equation \( t - X^0(u, v) = |x - X(u, v)| \) which specifies \( u \) as a function of \( v \) in integral (5.11)—or equivalently on the backwards light cone of the original field point \([t, x]\)—reads explicitly
\[
t - t_L [1 - e^{-\kappa \mu(u+v)}] = |x - 1/4\pi L_0 e^{-\kappa \mu(u+v)} r(u, v)|. \tag{5.15}
\]
Multiplying this equation by \( e^{\epsilon \mu \psi(t, x)} \) gives
\[
(t - t_L) e^{\epsilon \mu \psi} + t_L e^{-\kappa \mu(u+v-\psi)} = |e^{\epsilon \mu \psi} x - 1/4\pi L_0 e^{-\kappa \mu(u+v-\psi)} r(u, v)|, \tag{5.16}
\]
where \( (t - t_L) e^{\epsilon \mu \psi} = e^{\epsilon \mu \psi}|x| - t_L \) and
\[
\begin{align*}
\hat{r}(u, v) & \equiv \cos(v)\hat{x} + \sin(v)\hat{y} + (|u|-\frac{1}{2}\pi)\hat{z} \\
& = R(\psi)\left[ \cos(v - \psi)\hat{x} + \sin(v - \psi)\hat{y} + (|u|-\frac{1}{2}\pi)\hat{z} \right] \\
& \equiv R(\psi) R(u, v - \psi).
\end{align*} \tag{5.17}
\]
Since \(|v| = |R(-\psi)v|\) for any 3-vector \( v \), and \( R(-\psi) R(\psi) = I \), equation (5.16) can be rewritten as
\[
\bar{t} - t_L [1 - e^{-\kappa \mu(u+v-\psi)}] = \bar{x} - 1/4\pi L_0 e^{-\kappa \mu(u+v-\psi)} R(\psi)(u, v - \psi), \tag{5.18}
\]
or equivalently \( \bar{t} - X^0(u, v - \psi) = |\bar{x} - X(u, v - \psi)| \).

Hence, if \( u = U(v) \) is the equation of the curve of intersection of the world sheet \( T \) with the backwards light cone of a given field point \([t, x]\), then the equation for \( u \) on the backwards light cone of the corresponding image point \([\bar{t}, \bar{x}]\) on \( F_0 \) is \( u = U(v - \psi(t, x)) \). Similarly, the integrand in equation (5.11) for \( h_{ab} \), which is equivalent to
\[
\frac{1}{4\pi} L_0 e^{-\kappa \mu \psi|v|} |x - X(u, v)| = s |x - X(u, v)| \cdot \hat{z}^{-1} \Psi_{ab} (s, v), \tag{5.19}
\]
can be re-expressed as
\[
\frac{1}{4\pi} L_0 e^{-\kappa \mu \psi|v|} |x - X(u, v)| \cdot \hat{z} = [e^{\epsilon \mu \psi} R(-\psi) x - X(u, v - \psi)] \cdot \hat{z}. \tag{5.19}
\]
5 Note here that
\[
e^{\epsilon \mu \psi}|x - X(u, v)| \cdot \hat{z} = [e^{\epsilon \mu \psi} R(-\psi) x - X(u, v - \psi)] \cdot \hat{z}
\]
for the simple reason that the \( z \)-component of \( x \) is unaffected by the rotation matrix \( R \), while the \( z \)-component of \( X \) is
\[
\frac{1}{4\pi} L_0 e^{-\kappa \mu \psi|v|} (|u|-\frac{1}{2}\pi). 
\]
\[
1 \frac{1}{4\pi} L_0 e^{-x\mu(u+v-\psi)\parallel x - X(u, v - \psi) - s[x - X(u, v - \psi) \cdot \hat{z}]^{-1}}
\times \tilde{h}_{cd}(s, v)
\]
\[
= \frac{1}{4\pi} L_0 e^{-x\mu(u+v-\psi)\parallel x - X(u, v - \psi) - s[x - X(u, v - \psi) \cdot \hat{z}]^{-1}}
\times R_c^d(\psi)\tilde{h}_{cd}(s, v - \psi)R^d_b(\psi)
\]
where
\[
R_c^d(\psi) \equiv (1 \oplus R(\psi))^d_c = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \psi & -\sin \psi & 0 \\
0 & \sin \psi & \cos \psi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\] (5.21)

Comparison of equations (5.19) and (5.20), together with the fact that if \( u = U(v) \) on the backwards light cone of \([t, x] \) then \( u = U(v - \psi) \) on the backwards light cone of \([-t, \bar{x}] \), indicates that
\[
h_{ab}(t, x) = R_c^d(\psi) h_{cd}(-t, \bar{x}) R^d_b(\psi),
\] (5.22)
where \( \psi(t, x) \) is given by equation (5.12). The geometric meaning of this relation is illustrated in figure 3. To calculate \( h_{ab} \) at a field point \([t, x] \) in the past of \( F_L \), simply dilate the point by a factor \( e^{x\psi} \) through \([tL, 0]\) and rotate it by an angle \(-\psi \) in the \( x-y \) plane to map it to the image point \([\bar{t}, \bar{x}] \) on \( F_0 \). Then calculate \( h_{ab} \) at \([\bar{t}, \bar{x}] \) and rotate each of the components of the resulting tensor by an angle \( \psi \) in the \( x-y \) plane to give \( h_{ab} \) at \([t, x] \). From a computational viewpoint, equation (5.22) allows us to calculate \( h_{ab} \) at any point away from the world sheet of the loop from a knowledge of \( h_{ab} \) on the null surface \( F_0 \) alone. Furthermore, it points to the existence of a symmetry in the background metric induced by the self-similarity of the loop’s trajectory. This symmetry can be better appreciated by writing components (5.13) of the image point explicitly in terms of \( t \) and \( x \):
\[
\tilde{t} \equiv tL(1 + |w|) |w|^{-1} \quad \text{and} \quad \tilde{x} \equiv tL(1 + |w|) R(-\psi)w,
\] (5.23)
where
\[
w \equiv x/(tL - t).
\] (5.24)
If it were not for the presence of the rotation operators in (5.22) and (5.23), which depend on $\psi$ rather than $w$, the metric $n_{ab} + h_{ab}$ would be a function of $w$ alone and so would be strictly self-similar. As it is, the rotation operators act to enforce a slightly more complicated symmetry in the metric which could perhaps be termed 'rotating self-similarity'.

5.2. Explicit form of the metric perturbations on $F_0$

Suppose now that $|\vec{x}|, \vec{x}$ is a general point on the future light cone $F_0$ of the origin. Equation (5.15) for $u$ as a function of $v$ on the backwards light cone of $|\vec{x}|, \vec{x}$ reads

$$|\vec{x}| - t_L \left[1 - e^{-\kappa \mu(u+v)}\right] = |\vec{x} - \frac{1}{4\pi} L_0 e^{-\kappa \mu(u+v)} r(u,v)|.$$

Application of the triangle inequality to the norm on the right-hand side of this equation then gives

$$t_L \left|1 - e^{-\kappa \mu(u+v)}\right| \leq \frac{1}{4\pi} L_0 e^{-\kappa \mu(u+v)} \left[1 + \left(|u| - \frac{1}{2}\pi \right)^2 \right]^{1/2},$$

as $|r(u,v)| = \left[1 + \left(|u| - \frac{1}{2}\pi \right)^2 \right]^{1/2}$. Since $t_L = \frac{1}{4\pi \kappa \mu} L_0$ and the maximum value of $\left[1 + \left(|u| - \frac{1}{2}\pi \right)^2 \right]^{1/2}$ for $u \in (-\pi, \pi]$ is $(1 + \frac{1}{4}\pi^2)^{1/2} \approx 1.8621$, this inequality can be re-expressed as

$$|\kappa \mu(u+v) - 1| \leq \left(1 + \frac{1}{4}\pi^2 \right)^{1/2}. \kappa \mu.$$ (5.27)

It is, therefore, evident that $|\kappa \mu(u+v)|$ will remain small at all points on the backwards light cone of $|\vec{x}|, \vec{x}$ provided that $\kappa \mu$ itself is sufficiently small. For example, $|\kappa \mu(u+v)|$ will be nowhere no greater than $10^{-2}$ if $\ln\left[1 - \left(1 + \frac{1}{4}\pi^2 \right)^{1/2} \kappa \mu \right] \geq -10^{-2}$, and this, in turn will be true if $\kappa \mu \lesssim 5.34 \times 10^{-3}$. Given that we expect that $\kappa \approx 3.1037$ and $\mu$ will be of order $10^{-6}$ or smaller for a GUT string, the assumption that $|\kappa \mu(u+v)|$ remains small is not an unreasonable one.

Expanding $e^{-\kappa \mu(u+v)}$ in powers of $\mu$ in (5.25) and retaining only the leading-order terms gives the following equation for $u$ as a function of $v$:

$$T - (u + v) = \left[(X - \cos v)^2 + (Y - \sin v)^2 + \left[Z - (|u| - \frac{1}{2}\pi)\right]^2 \right]^{1/2},$$

where

$$(X, Y, Z) \equiv 4\pi L_0^{-1} (\vec{x}, \vec{y}, \vec{z}) \quad \text{and} \quad T \equiv (X^2 + Y^2 + Z^2)^{1/2}.$$ (5.29)

Equation (5.28) can easily be solved to give an explicit formula for $u$, but it turns out that in calculating $h_{ab}$ it is more useful to regard $u$ as the independent variable.

Similarly, for small values of $|\kappa \mu(u+v)|$, integrand (5.19) in formula (5.11) for $h_{ab}$ at $|\vec{x}|, \vec{x}$ becomes

$$\left[\left[(X - \cos v)^2 + (Y - \sin v)^2 + \left[Z - (|u| - \frac{1}{2}\pi)\right]^2 \right]^{1/2} - s[Z - (|u| - \frac{1}{2}\pi)]\right]^{-1} \tilde{q}_{ab}(s,v).$$ (5.30)

Note that the denominator in this expression is manifestly positive or zero. In view of equation (5.28) and the fact that $s|u| = u$, the denominator can be expressed in the simpler form

$$\left[\left[(X - \cos v)^2 + (Y - \sin v)^2 + \left[Z - (|u| - \frac{1}{2}\pi)\right]^2 \right]^{1/2} - s[Z - (|u| - \frac{1}{2}\pi)]\right]
= T - s\left(Z + \frac{1}{2}\pi\right) - v,$$ (5.31)
and (5.11) reduces to

$$h_{ab}(\sqrt{s}, \sqrt{\bar{s}}) = -4\mu \int_{V_0}^{V_1} \int_{-\infty}^{\infty} \left[ T - s \left( Z + \frac{1}{2}\pi \right) - v \right]^{-1} \Psi_{ab}(s, v) \, dv. \quad (5.32)$$

All that remains now is to determine the limits of integration. The entire curve of intersection $\Gamma$ of the world sheet with the backwards light cone of $[\sqrt{s}, \sqrt{\bar{s}}]$ will be traced out as $u$ varies from $-\pi$ to $\pi$. Integral (5.32), therefore, breaks into two parts: one with $s = -1$ and $u \in [-\pi, 0]$, and one with $s = +1$ and $u \in [0, \pi]$. The corresponding values of $v$, which I will denote as $V_k \equiv V_{u=k\pi}$ for $k = -1, 0$ and $1$, are roots of the light-cone equation (5.28):

$$T - (k\pi + V_k) = \left[ (X - \cos V_k)^2 + (Y - \sin V_k)^2 + \left( Z - (|k| - \frac{1}{2})\pi \right)^2 \right]^{1/2}. \quad (5.33)$$

Although it is not possible to solve for $V_k$ explicitly except in certain special cases, it is clear by inspection that $V_{-1} = V_{1} + 2\pi$. Furthermore, the implicit differentiation of (5.28) with $[T, X, Y, Z]$ constant indicates that

$$\frac{dv}{du} = -\frac{T - s(Z + \frac{1}{2}\pi) - v}{T - (u + v) + X \sin v - Y \cos v}. \quad (5.34)$$

Some of the mathematical consequences of equations (5.33) and (5.34) are examined in appendix A. In particular, it is shown that the values of $V_0$ and $V_1$ are bounded on $F_0$, and that $v$ is a monotonic decreasing function of $u$. The last property follows partly from the fact that the numerator in (5.34) is positive or zero from (5.31), while the denominator is positive or zero as a consequence of (5.28) and the identity

$$(X - \cos v)^2 + (Y - \sin v)^2 = (X \sin v - Y \cos v)^2 + (X \cos v + Y \sin v - 1)^2. \quad (5.35)$$

However, cases where either the numerator or denominator is zero for some value of $u$ need to be considered separately, and it is shown in appendices A.2 and A.3 that the corresponding spacetime points have interesting physical interpretations. Perhaps the most important result is that the spacetime derivatives of $h_{ab}$ are undefined at all points with the property that the denominator in (5.34) is zero at one of the extreme values $u = 0$ or $\pm\pi$.

Given that $V_{-1} = V_{1} + 2\pi \geq V_0 \geq V_1$, the equation for the metric perturbations becomes

$$h_{ab}(\sqrt{s}, \sqrt{\bar{s}}) = -4\mu \int_{V_0}^{V_1} \int_{-\infty}^{\infty} \left[ T + \left( Z + \frac{1}{2}\pi \right) - v \right]^{-1} \Psi_{ab}(-1, v) \, dv$$

$$- 4\mu \int_{V_1}^{V_0} \int_{-\infty}^{\infty} \left[ T - \left( Z + \frac{1}{2}\pi \right) - v \right]^{-1} \Psi_{ab}(+1, v) \, dv. \quad (5.36)$$

After referring back to definition (5.8) of $\Psi_{ab}$, it is readily seen that

$$h_{ab}(\sqrt{s}, \sqrt{\bar{s}}) = 4\mu \begin{bmatrix} I_+ + I_- & S_+ + S_- & -C_+ - C_- & I_+ - I_- \\ S_+ + S_- & I_+ + I_- & 0 & S_+ - S_- \\ -C_+ - C_- & 0 & I_+ + I_- & -C_+ + C_- \\ I_+ - I_- & S_+ - S_- & -C_+ + C_- & I_+ + I_- \end{bmatrix}_{ab}, \quad (5.37)$$

where

$$I_+ = \ln[(\chi_+ - V_1)/(\chi_+ - V_0)], \quad I_- = \ln[(\chi_- - V_0)/(\chi_- - V_1)]. \quad (5.38)$$

$$\begin{bmatrix} S_+ \\ C_+ \end{bmatrix} = \begin{bmatrix} -\cos \chi_+ & \sin \chi_+ \\ \sin \chi_+ & \cos \chi_+ \end{bmatrix} \begin{bmatrix} \text{Si}(\chi_+ - V_1 - 2\pi) - \text{Si}(\chi_+ - V_0) \\ \text{Ci}(\chi_+ - V_1 - 2\pi) - \text{Ci}(\chi_+ - V_0) \end{bmatrix}, \quad (5.39)$$

and

$$\begin{bmatrix} S_- \\ C_- \end{bmatrix} = \begin{bmatrix} -\cos \chi_- & \sin \chi_- \\ \sin \chi_- & \cos \chi_- \end{bmatrix} \begin{bmatrix} \text{Si}(\chi_- - V_0) - \text{Si}(\chi_- - V_1) \\ \text{Ci}(\chi_- - V_0) - \text{Ci}(\chi_- - V_1) \end{bmatrix}. \quad (5.40)$$
with \( \chi \equiv T \pm (Z + \frac{x}{2}) \) and \( \text{Si}(x) = \int_0^x w^{-1} \sin w \, dw \) and \( \text{Ci}(x) = -\int_x^\infty w^{-1} \cos w \, dw \) being the usual sine and cosine integrals.

One case in which it is possible to solve explicitly for \( V_0 \) and \( V_1 \), and so to write explicit formulae for the metric perturbations \( h_{ab} \), occurs when the image point \([|x|, x]\) lies on the \( \xi \)-axis. Then \( X = Y = 0 \) and \( T = |Z| \), and equation (5.33) reduces to

\[
V_k = |Z|^{-k\pi} - \left[ 1 + \left( Z - \frac{|Z| - \frac{1}{2})\pi \right)^2 \right]^{1/2}, \quad (5.41)
\]

In particular,

\[
V_0 = |Z|^{-2\pi} - \left[ 1 + \left( Z - \frac{\pi}{2} \right)^2 \right]^{1/2} \quad \text{and} \quad V_1 = |Z|^{-\pi} - \left[ 1 + \left( Z - \frac{\pi}{2} \right)^2 \right]^{1/2}, \quad (5.42)
\]

while \( \chi_\pm = |Z| \pm \left( Z + \frac{\pi}{2} \right) \).

Hence,

\[
(X_+ - V_1 - 2\pi)/(X_+ - V_0) = (X_+ - V_0)/(X_- - V_1) = \left\{ \begin{array}{l}
Z - \frac{\pi}{2} + \left[ 1 + \left( Z - \frac{\pi}{2} \right)^2 \right]^{1/2} \\
-Z - \frac{\pi}{2} + \left[ 1 + \left( Z + \frac{\pi}{2} \right)^2 \right]^{1/2} 
\end{array} \right\} \left\{ \begin{array}{l}
-Z - \frac{\pi}{2} + \left[ 1 + \left( Z + \frac{\pi}{2} \right)^2 \right]^{1/2} \\
-Z - \frac{\pi}{2} + \left[ 1 + \left( Z - \frac{\pi}{2} \right)^2 \right]^{1/2} 
\end{array} \right\}, \quad (5.43)
\]

and \( I_- = I_+ \). This means that \( h_{zz} = 0 \), and that

\[
h_{xx} = h_{yy} = h_{zz} = 8\mu \ln \left\{ \begin{array}{l}
Z - \frac{\pi}{2} + \left[ 1 + \left( Z - \frac{\pi}{2} \right)^2 \right]^{1/2} \\
-Z - \frac{\pi}{2} + \left[ 1 + \left( Z + \frac{\pi}{2} \right)^2 \right]^{1/2} 
\end{array} \right\} \left\{ \begin{array}{l}
-Z - \frac{\pi}{2} + \left[ 1 + \left( Z + \frac{\pi}{2} \right)^2 \right]^{1/2} \\
-Z - \frac{\pi}{2} + \left[ 1 + \left( Z - \frac{\pi}{2} \right)^2 \right]^{1/2} 
\end{array} \right\}, \quad (5.44)
\]

increases monotonically from approximately \(-19.734\mu \) (at \( Z = 0 \)) to 0 (as \( |Z| \to \infty \)). In fact, for large values of \( |Z| \),

\[
h_{xx} = h_{yy} = h_{zz} = -8\pi \mu / |Z| + O(|Z|^{-2}). \quad (5.45)
\]

The variation of \( h_{11}/(4\mu) \) with \( Z \) is plotted in figure 4(a).

Explicit formulae for the remaining nonzero metric perturbations, \( h_{1x} = 4\mu(S_+ + S_-), h_{1y} = -4\mu(C_+ + C_-) \), \( h_{1z} = 4\mu(S_+ - S_-) \) and \( h_{2y} = -4\mu(C_+ - C_-) \) can be found by substituting expressions (5.42) into (5.39) and (5.40). However, there is little to be gained from writing these formulae out in full. It is easily seen that if \( Z \) is replaced by \(-Z \) then \( S_+ \leftrightarrow -S_- \) and \( C_+ \leftrightarrow -C_- \), and so \( h_{1x} \to -h_{1x}, h_{1y} \to -h_{1y}, h_{1z} \to h_{1z} \) and \( h_{2y} \to h_{2y} \). This set of symmetries is to be expected on physical grounds, as the ACO loop is invariant under a complete spatial inversion \( x \to -x, y \to -y \) and \( z \to -z \), and therefore in this special case the space–space components of \( h_{ab} \) should be even functions of \( Z \), and the time–space components odd functions of \( Z \).

The variations of \( h_{1x}/(4\mu), h_{1y}/(4\mu), h_{1z}/(4\mu) \) and \( h_{2y}/(4\mu) \) are plotted in figures 4(b) to 4(e) respectively. At \( Z = 0 \), the perturbations \( h_{1x} \) and \( h_{1y} \) are, of course, both zero, while \( h_{1z} \approx -5.308\mu \) and \( h_{2y} \approx 12.395\mu \). All the perturbations tend to zero as \( |Z| \to \infty \), with

\[
h_{1x} = 4\pi \mu / Z + O(|Z|^{-3}), \quad (5.46)
\]

\[
h_{1y} = -4\pi \mu / (|Z||Z|) + O(|Z|^{-4}), \quad (5.47)
\]

\[
h_{1z} = -4\pi \mu / |Z| + O(|Z|^{-3}) \quad (5.48)
\]

and

\[
h_{2y} = \frac{13}{6} \pi \mu / Z^4 + O(|Z|^{-6}). \quad (5.49)
\]
6. Solving the string equations of motion

6.1. Near-field form of the metric perturbations

For the purposes of setting out the first-order equation of motion (2.14), it is necessary to generate expressions for the metric perturbations $h_{ab}$ in the neighborhood of the world sheet of the evaporating loop. As was mentioned in section 4, when doing this it is sufficient to restrict attention to points near the spacelike surface $t = 0$ only. So consider a field point $x^a = [t, x]^a$ of the form $X^a(u, v) + \delta x^a$, where $X^a(u, v)$ is a fixed point on the flat-space trajectory (4.14) with $|\kappa \mu (u + v)| \ll 1$, and the components of the displacement $\delta x^a$ are all of the order of some small parameter.

A first point to note is that if $X^a(\bar{u}, \bar{v})$ is any source point on the loop’s trajectory on the backwards light cone of $X^a(u, v)$ then $|\kappa \mu (\bar{u} + \bar{v})|$ will also be small compared to 1 for
reasonable values of $\kappa \mu$. This follows from equation (5.15), which in the current situation reads

$$t_L[1 - e^{-\kappa \mu(u+v)}] - t_L[1 - e^{-\kappa \mu(\bar{u}+\bar{v})}] = \left| \frac{1}{4\pi} L_0 e^{-\kappa \mu(u+v)} r(u, v) - \frac{1}{4\pi} L_0 e^{-\kappa \mu(\bar{u}+\bar{v})} r(\bar{u}, \bar{v}) \right|.$$  

(6.1)

An application of the triangle inequality gives

$$t_L |e^{-\kappa \mu(\bar{u}+\bar{v})} - e^{-\kappa \mu(u+v)}| \leq \frac{1}{4\pi} L_0 e^{-\kappa \mu(u+v)} |r(u, v)| + \frac{1}{4\pi} L_0 e^{-\kappa \mu(\bar{u}+\bar{v})} |r(\bar{u}, \bar{v})|,$$

or equivalently (in parallel with the derivation of (5.27)),

$$|1 - e^{\kappa \mu(u+v)-\kappa \mu(\bar{u}+\bar{v})}| \leq (1 + e^{\kappa \mu(\bar{u}+\bar{v})-\kappa \mu(u+v)})(1 + \frac{1}{2\pi^2})^{1/2} \kappa \mu.$$  

(6.3)

Hence,

$$|\kappa \mu(\bar{u}+\bar{v}) - \kappa \mu(u+v)| \leq \ln \left[ \frac{1 + (1 + \frac{1}{2\pi^2})^{1/2} \kappa \mu}{1 - (1 + \frac{1}{2\pi^2})^{1/2} \kappa \mu} \right].$$  

(6.4)

and so, for example, $|\kappa \mu(\bar{u}+\bar{v})|$ will be no greater than $10^{-3}$ everywhere if $\kappa \mu \lesssim 2.69 \times 10^{-3}$.

Since $|\kappa \mu(\bar{u}+\bar{v})|$ is guaranteed to be small, the equations (5.33) for $V_k$ and (5.37)–(5.40) for $h_{ab}$ continue to apply in this case, with

$$T = u + v + 4\pi L_0^{-1} \delta t \quad \text{and} \quad (X, Y, Z)^T = r(u, v) + 4\pi L_0^{-1} \delta x,$$

(6.5)

where as before $r(u, v) = \cos(v) \hat{x} + \sin(v) \hat{y} + (|u|\frac{1}{2}\pi) \hat{z}$. In particular, if $\delta x^a = 0$ the equation for $V_k$ becomes

$$u + v - (k\pi + V_k) = [2 - 2\cos(v - V_k)] + ([u] - [k]\pi^2)^{1/2},$$

(6.6)

and, given that $u$ is restricted to the range ($-\pi, \pi$), the trivial solution $V_k = v$ holds in two cases: (i) if $u > 0$ and $k = 0$; and (ii) if $u < 0$ and $k = -1$.

In the analysis that follows it will be assumed that $u > 0$. Some remarks about the behavior of $h_{ab}$ near points $X^a(u, v)$ with $u < 0$ will be made towards the end of this section. When $u > 0$, the $z$-component of the field point satisfies $Z + \frac{1}{2}\pi = u + 4\pi L_0^{-1} \delta z$, and so

$$\chi_+ \equiv v + (u \pm u) + 4\pi L_0^{-1}(\delta t \pm \delta z).$$  

(6.7)

Furthermore, if $\delta x^a = 0$ the limits of integration in (5.36) are given by $V_0 = v$ and $V_1 = v - W(u)$, where $W$ is the unique root of the equation

$$u - \pi + W = [2 - 2\cos W + (u - \pi)^2]^{1/2}.$$  

(6.8)

Note that because $V_0 \geq V_1$ it follows that $W \geq 0$, while it is clear from (6.8) that $W = 0$ only at the kink point $u = \pi$. For future reference, it is useful to rearrange (6.8) in the form

$$u - \pi = W^{-1}(1 - \cos W) - \frac{1}{W}.$$  

(6.9)

(Incidentally, one way to verify that (6.8) has a unique solution $W$ is to note that the function on the right-hand side of (6.9) is strictly decreasing.)

Implicit differentiation of the defining equation (5.33) for $V_k$ gives

$$\partial T V_k = H_k^{-1}[T - (k\pi + V_k)]$$

and

$$(\partial X V_k, \partial Y V_k, \partial Z V_k) = H_k^{-1}\left( \cos V_k - X, \sin V_k - Y, (|k|\frac{1}{2}\pi - Z) \right).$$  

(6.10)

with

$$H_k \equiv T - (k\pi + V_k) + X \sin V_k - Y \cos V_k.$$  

(6.11)

Note that if $\delta x^a = 0$ then $H_0 = u$ and $H_1 = u - \pi + W - \sin W$. 

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Hence, to linear order in $\delta x^a$,

$$V_0 = v + 4\pi L_0^{-1} (\delta t - \delta z)$$

and

$$V_1 = v - W + 4\pi L_0^{-1} \delta V,$$  \hspace{1cm} (6.12)

where

$$\delta V \equiv H^{-1} [(u - \pi + W) \delta t + \{\cos(v - W) - \cos v\} \delta x$$

$$+ [\sin(v - W) - \sin v] \delta y + (\pi - u) \delta z].$$  \hspace{1cm} (6.13)

Unfortunately, the linear expansion (6.12) for $V_0$ is not sharp enough to resolve the singular behavior of $h_{ab}$ near the world sheet, as $\chi_- = v + 4\pi L_0^{-1} (\delta t - \delta z)$ and so $\chi_- - V_0 = 0$ at this order.

At quadratic order in $\delta x^a$ it turns out that

$$V_0 = v + 4\pi L_0^{-1} (\delta t - \delta z) - \frac{1}{2} u^{-1} (4\pi L_0^{-1})^2 Q_+, $$

where

$$Q_+ \equiv (\delta x)^2 + (\delta y)^2 + (\delta t - \delta z)(\delta t - \delta z + 2\delta x \sin v - 2\delta y \cos v)$$

$$\equiv \Psi_{ab}(+1, v) \delta x^a \delta x^b. $$  \hspace{1cm} (6.15)

Since $\Psi_{ab}$ is proportional to the orthogonal complement $q_{ab}$ of the projection tensor $p_{ab}$, the quadratic form $Q_+$ is zero if and only if $\delta x^a$ lies in the tangent plane to the world sheet at $X^a(u, v)$. It should, therefore, be understood that in evaluating $h_{ab}$ and its derivatives the limit $\delta x^a \to 0$ is taken only from directions outside the world sheet.

The algebraic computations whose results appear throughout the rest of this section are long and very mechanical, and their details have been relegated to appendix B. Neglecting contributions that tend to zero in the limit as $\delta x^a \to 0$, the metric perturbations have the near-field form

$$ (4\mu)^{-1} h_{ab} = \left[ \ln \left( \frac{(4\pi L_0^{-1})^2 Q_+}{\Psi_{ab}(+1, v) \Phi_{ab}(+1, v) + (\gamma E - Ci W) \Phi_{ab}(+1, v)} \right) - \ln(2u) \right] \Phi_{ab}(+1, v) + \left( \gamma E - Ci W \right) \Phi_{ab}(+1, v)$$

$$+ (Si W) \Omega_{ab}(+1, v) + K(u) \Phi_{ab}(-1, 2u + v) - \Sigma(u) \Omega_{ab}(-1, 2u + v)$$

$$- \ln(W) \Pi_{ab}(+1) + \left[ \ln(W + 2u - 2\pi) - \ln(2u) \right] \Pi_{ab}(-1),$$  \hspace{1cm} (6.16)

where $\gamma_E \approx 0.5772$ is Euler’s constant,

$$\Phi_{ab}(s, v) \equiv \left[ \begin{array}{ccc} 0 & \sin v & -\cos v \\
\sin v & 0 & 0 \\
-\cos v & 0 & \sin v \\
0 & -\sin v & s \cos v \\
0 & s \sin v & 0 \end{array} \right]_{ab},$$  \hspace{1cm} (6.17)

$$\Omega_{ab}(s, v) \equiv \partial_v \Phi_{ab},$$  \hspace{1cm} (6.18)

and

$$\Pi_{ab}(s) \equiv \left[ \begin{array}{ccc} 1 & 0 & 0 & -s \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-s & 0 & 0 & 1 \end{array} \right]_{ab}.$$  \hspace{1cm} (6.19)

Note, in particular, that $\Psi_{ab}(s, v) = \Phi_{ab}(s, v) + \Pi_{ab}(s)$. Also, the functions $K$ and $\Sigma$ are shorthand for

$$K(u) \equiv Ci(W + 2u - 2\pi) - Ci(2u) \quad \text{and} \quad \Sigma(u) \equiv Si(W + 2u - 2\pi) - Si(2u).$$  \hspace{1cm} (6.20)
Similarly, the leading-order contributions to the spacetime derivatives of \( h_{ab} \) are

\[
(4\mu)^{-1} \left( \frac{1}{4\pi} L_0 \right) \partial_t h_{ab} = \left( \frac{1}{4\pi} L_0 \right) (\partial_t \Lambda) \Psi_{ab}(+1, v) + (\delta t - \delta \xi)(\partial_t \Lambda) \Omega_{ab}(+1, v) \\
+ [\Lambda + \gamma_\xi - C W - \ln(2\mu)]\Omega_{ab}(+1, v) - (S_i W)\Phi_{ab}(+1, v) \\
+ K(u)\Omega_{ab}(-1, 2u + v) + \Sigma(u)\Phi_{ab}(-1, 2u + v) + \frac{\sin W}{u - \pi + W - \sin W} \\
\times \left[ \frac{1}{W} \Psi_{ab}(+1, v - W) - \frac{1}{W + 2u - 2\pi} \Psi_{ab}(-1, v - W) \right].
\]

(6.21)

\[
(4\mu)^{-1} \left( \frac{1}{4\pi} L_0 \right) \partial_v h_{ab} = \left( \frac{1}{4\pi} L_0 \right) (\partial_v \Lambda) \Psi_{ab}(+1, v) \\
+ (\delta t - \delta \xi)(\partial_v \Lambda) \Omega_{ab}(+1, v) + \frac{\cos(v - W) - \cos v}{u - \pi + W - \sin W} \\
\times \left[ \frac{1}{W} \Psi_{ab}(+1, v - W) - \frac{1}{W + 2u - 2\pi} \Psi_{ab}(-1, v - W) \right].
\]

(6.22)

\[
(4\mu)^{-1} \left( \frac{1}{4\pi} L_0 \right) \partial_u h_{ab} = \left( \frac{1}{4\pi} L_0 \right) (\partial_u \Lambda) \Psi_{ab}(+1, v) \\
+ (\delta t - \delta \xi)(\partial_u \Lambda) \Omega_{ab}(+1, v) + \frac{\sin(v - W) - \sin v}{u - \pi + W - \sin W} \\
\times \left[ \frac{1}{W} \Psi_{ab}(+1, v - W) - \frac{1}{W + 2u - 2\pi} \Psi_{ab}(-1, v - W) \right].
\]

(6.23)

and

\[
(4\mu)^{-1} \left( \frac{1}{4\pi} L_0 \right) \partial_\mu h_{ab} = \left( \frac{1}{4\pi} L_0 \right) (\partial_\mu \Lambda) \Psi_{ab}(+1, v) + (\delta t - \delta \xi)(\partial_\mu \Lambda) \Omega_{ab}(+1, v) \\
- [\Lambda + \gamma_\xi - C W - \ln(2\mu)]\Omega_{ab}(+1, v) - (S_i W)\Phi_{ab}(+1, v) \\
+ K(u)\Omega_{ab}(-1, 2u + v) + \Sigma(u)\Phi_{ab}(-1, 2u + v) - \frac{\sin W}{u - \pi + W - \sin W} \\
\times \left[ W - \sin W \Psi_{ab}(+1, v - W) \\
\frac{2u - 2\pi + W - \sin W}{W + 2u - 2\pi} \Psi_{ab}(-1, v - W) \right].
\]

(6.24)

where

\[ \Lambda \equiv \ln \left[ (4\pi L_0^{-1})^2 Q_\ast \right]. \]

(6.25)

The expressions listed here for \( h_{ab} \) and its derivatives neglect additional terms of order \( \delta x^a \ln \left[ (4\pi L_0^{-1})^2 Q_\ast \right] \) or higher, which are zero in the limit \( \delta x^a \to 0 \), provided of course that this limit is taken from directions outside the tangent plane to the world sheet at \( X^a(u, v) \). However, the expressions also contain terms of order \( \ln \left[ (4\pi L_0^{-1})^2 Q_\ast \right] \) and \( \delta x^a \partial_\mu \ln \left[ (4\pi L_0^{-1})^2 Q_\ast \right] \), which are undefined in the limit as \( \delta x^a \to 0 \). The presence of the singular term \( 4\mu \ln \left[ (4\pi L_0^{-1})^2 Q_\ast \right] \Psi_{ab}(+1, v) \) in \( h_{ab} \) is a reflection of the conical singularity that is expected to mark the location of any zero-thickness cosmic string, as has been explained in more detail in Paper I. Fortunately, none of the singular terms makes a contribution to the first-order equation of motion, as will be seen shortly.

Another important point to note is that equations (6.16) and (6.21)–(6.24) are valid only near points \( X^a(u, v) \) with \( u \in (0, \pi) \), which lie on one of the two smooth segments of the
loop shown in figure 1. The metric perturbations near points on the second segment, for which \( u \in (0, \pi) \), can be calculated by making use of the identities \( I_\nu(u - \pi, v) = I_{-\nu}(u, v) \) and \( I_{-\nu}(u - \pi, v) = I_\nu(u, v) \) for \( u \in (0, \pi) \), and the analogous identities satisfied by the pairs \((S_\nu, S_-)\) and \((C_\nu, C_-)\). In view of equation (5.37) for \( h_{ab} \), it follows that all the components of \( h_{ab} \), except \( h_{xz}, h_{yz} \), and \( h_{xz} \), are half-range periodic in \( u \) near the world sheet (that is, \( h_{ab}(u - \pi, v) = h_{ab}(u, v) \)), while the components \( h_{xz}, h_{yz} \), and \( h_{xz} \), are half-range anti-periodic (that is, \( h_{ab}(u - \pi, v) = -h_{ab}(u, v) \)). This behavior was previously mentioned of the tangential projections \( h_{ab}X_{(0),A}^a \) \( X_{(0),B}^b \) (which are all half-range periodic in \( u \)) in Paper I. Although Paper I deals with a loop that is strictly stationary, the statements it makes about the local behavior of \( h_{ab} \) are still applicable here, as the local effects of loop evaporation first appear at order \( \mu^2 \) in \( g_{ab} \).

6.2. The equations of motion

It is now possible to write down and verify or solve the equations of motion (2.13) and (2.14). Since the only nonzero component of the 2-metric \( \gamma^{AB} \) is \( \gamma^{uv} = \left( \frac{1}{2} L_0 \right)^{-2} \) (to leading order in \( \mu \)), the zeroth-order equation of motion \( q_{d}^{\ell} \gamma^{CD} X_{(0),cD}^d = 0 \) reads simply \( q_{d}^{\ell} \gamma^{CD} X_{(0),cD}^d = 0 \). It is clear by inspection of (4.16) that \( X_{(0),uv}^d = 0 \), and so (2.13) is automatically satisfied. The second term in the first-order equation of motion (2.14), which is proportional to \( \gamma^{CD} X_{(0),cD}^d \), is consequently also zero.

The details of the calculation of the remaining terms in (2.14) can be found in appendix C. Because all the terms in the first-order equation of motion are projected onto \( q_{d}^{\ell} \), and so lie in the subspace orthogonal to the tangent plane at \( X^a(u, v) \), the equation contains only two independent components. In what follows, it proves to be convenient to decompose each term in the directions of the vectors

\[
M^a(s, v) \equiv [1, -\sin(v)\hat{x} + \cos(v)\hat{y} + s\hat{z}]^a \quad \text{and} \quad N^a(v) \equiv [0, \cos(v)\hat{x} + \sin(v)\hat{y}]^a,
\]

(6.26)

which span the orthogonal subspace.

The first and third terms in (2.14), which depend on the perturbation \( \delta X^a \) and, therefore, on the correction functions \( F, G \) and \( A \), are

\[
q_{d}^{\ell} \gamma^{CD} \delta X_{(0),cD}^d = 2\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} (F_{uv} + sA_{uv} + G_{uv} - \kappa) M^c + 2\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} G_{uv} N^c
\]

and

\[
-2q_{d}^{\ell} \gamma^{AC} \gamma^{BD} \eta_{ab} X_{(0),A}^a \delta X_{(0),B}^b X_{(0),cD}^d = 2\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \left[ \kappa s (|u| - \frac{1}{2}\pi) - (F + sA_u) \right] N^c.
\]

(6.28)

The remaining terms depend on \( h_{ab} \) and its derivatives, which were evaluated earlier in the case \( u \in (0, \pi) \) only. For positive values of \( u \) (or equivalently, for \( s = +1 \)) it turns out that

\[
-q_{d}^{\ell} \gamma^{AC} \gamma^{BD} h_{ab} X_{(0),A}^a X_{(0),B}^b X_{(0),cD}^d = 16\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \ln(W + 2u - 2\pi) - \ln(2u)] N^c,
\]

(6.29)

\[
-q_{d}^{\ell} \gamma^{AB} X_{(0),A}^a X_{(0),B}^b h_{bc} u = 16\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \Sigma(u)[-2(\cos 2u)M^c + (\sin 2u)N^c]
\]

\[
+ 16\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} K(u)[(2\sin 2u)M^c + (\cos 2u)N^c]
\]
\[ +4\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \frac{1}{u - \pi + W - \sin W} \left[ (1 - \cos W)M^c - (\sin W)N^c \right] \]
\[ +4\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \frac{1}{u - \pi + W - \sin W} \left[ 1 - \cos W \right] \]
\[ \times [3(1 - \cos W)M^c - (\sin W)N^c] \quad (6.30) \]

and
\[ \frac{1}{2} q^{cd} \gamma^{AB} X_{(0)A}^a X_{(0)B}^b h_{ab,cd} = -16\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \left[ K(u) \sin 2u - \Sigma(u) \cos 2u \right] M^c \]
\[ -8\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \frac{1}{u - \pi + W - \sin W} \left[ W + 2u - 2\pi - \sin W \right] \]
\[ -4\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \frac{1}{u - \pi + W - \sin W} \left[ \sin W - 2\cos W \right] \quad (6.31) \]

Now, it was shown in Paper I that \( h_{ab} X_{(0)A}^a X_{(0)B}^b \) is half-range periodic in \( u \), and it is easily seen that taking the inner product with \( q^{cd} \gamma^{AB} \gamma^{CD} X_{(0)A}^a X_{(0)B}^b \) preserves this symmetry. Hence, the functional dependence of the term on the left of (6.29) for \( u \in (-\pi, 0) \) is the half-range periodic extension of the term on the right. Similarly, the sum of the terms in (6.30) and (6.31)—which is to say, the term on the right-hand side of the first-order equation of motion (2.14)—is just \( 2\alpha^u \), where \( \alpha^u \) is the acceleration vector introduced in Paper I. It was demonstrated in Paper I that the \( r \)-, \( x \)- and \( y \)-components of \( \alpha^u \) are half-range periodic in \( u \), while the \( z \)-component is half-range anti-periodic, and this is also true of the vectors \( M^c \) and \( N^c \). Hence, the first-order equation of motion requires the sum of (6.27) and (6.28) to have the same symmetries, and since the term \( k/s \) in (6.28) is half-range periodic this means that \( F + sA \) and \( G \) must be half-range periodic as well. These properties can also be predicted from the geometry of the problem. The two segments of the evaporating loop should share the same geometry, so the correction functions \( F \), \( G \) and \( A \) are expected to have the same symmetries as the corresponding components of the original ACO loop. That is, \( F \) and \( G \) should be half-range periodic in \( u \) as the \( r \)-, \( x \)- and \( y \)-components of the ACO loop are, while \( A \) should be half-range anti-periodic as the \( z \)-component \( |u| = \frac{1}{2} \pi \) is.

On setting \( s = +1 \) in the first two terms (6.27) and (6.28), the equation of motion for \( u \in (0, \pi) \) reduces to two partial differential equations:

\[ F_{uv} + A_{uv} + G_u - \kappa = 8[K(u) \sin 2u - \Sigma(u) \cos 2u] \]
\[ +4\mu \frac{1}{u - \pi + W - \sin W} \left[ W + 2u - 2\pi - 2\sin W \right] \]
\[ \quad (6.32) \]

and

\[ G_{uv} + \kappa \left( \frac{1}{2} \pi \right) - (F_u + A_u) + 8[\ln(W + 2u - 2\pi) - \ln(2u)] \]
\[ = 8[K(u) \cos 2u + \Sigma(u) \sin 2u] \]
\[ -4\mu \frac{1}{u - \pi + W - \sin W} \left[ \sin W - 2\cos W \right] \cos W \] \quad (6.33)

which appear as the coefficients of \( M^c \) and \( N^c \) respectively.

At a schematic level, these equations read

\[ F_{uv} + A_{uv} + G_u = P_1(u) \quad \text{and} \quad G_{uv} - F_u - A_u = P_2(u), \] \quad (6.34)
where the functions $P_1$ and $P_2$ represent all the remaining terms. Taking the $v$ derivative of each equation in turn and adding or subtracting the other leads to two forced simple harmonic equations:
\[ G_{uv} + G_u = P_1(u) \quad \text{and} \quad (F_u + A_u)_{vv} + F_u + A_u = -P_2(u), \] (6.35)
whose general solutions are
\[ G = c_1(v) + a(u) \cos v + b(u) \sin v + \int P_1(u) \, du \] (6.36)
and
\[ F + A = c_2(v) + b(u) \cos v - a(u) \sin v - \int P_2(u) \, du, \] (6.37)
where $c_1$ and $c_2$ are arbitrary continuous $2\pi$-periodic functions of $v$, and $a$ and $b$ are the restrictions to $(0, \pi)$ of arbitrary half-range periodic $C^1$ functions of $u$.

The physical significance of the arbitrary homogeneous contributions to (6.36) and (6.37) can be understood by referring to the intrinsic $2$-metric $\gamma_{AB}$. The components of $\gamma_{AB}$ are expanded to order $\mu$ in appendix D.1, where it is shown that $\gamma_{AB}$ takes on the self-similar form (4.22) at this order if and only if
\[ F_{uv} + A_{uv} = 0, \quad F_{uv} + F_{vu} + A_{vv} = 0 \quad \text{and} \quad G_v + F_{vv} = 0. \] (6.38)

The first of these constraints reduces the first equation in (6.34) to $G_u = P_1(u)$, and so from (6.36) $a$ and $b$ are both zero. The trigonometric functions of $v$ in (6.36) and (6.37), therefore, represent free vibrations of the loop with amplitudes of order $\mu$, which break the self-similarity of the evaporating loop and, if allowed to remain, would contribute to the metric tensor $g_{ab}$ at order $\mu^2$.

The second part of constraints (6.38) now reduces to $F_{uv} = -c_2''(v)$, and since $F$ is by assumption continuous and half-range periodic in $u$ it follows that $c_2' = 0$ and, therefore, that $c_2$ is a constant (and can be absorbed into $\int P_2(u) \, du$). Consequently, $F_{uv} = 0$, and $F$ has the generic form $F_1(u) + F_2(v)$ for some choice of continuous and suitably periodic functions $F_1$ and $F_2$. Furthermore, the third constraint now becomes $G_v = -F_2''$, and so from (6.36) $c_1(v) = -F_2''$.

The possibility of further constraining $F$, $G$ and $A$ arises from a consideration of transformations of the spacetime coordinates $x^a = [t, x, y, z]^0$ of the form $x^a \rightarrow \tilde{x}^a$ where
\[ \tilde{x}^a = x^a + \mu y^a(x^b). \] (6.39)

The only transformations of this type that are admissible are those that preserve the form of the metric tensor $g_{ab} = \eta_{ab} + \mu h_{ab}$ at order $\mu$. Since
\[ \tilde{g}_{ab} = \eta_{ab} + \mu h_{ab} + \mu \eta_{ca} \partial_b y^c + \mu \eta_{cb} \partial_a y^c \] (6.40)
at linear order in $\mu$, admissible transformations satisfy the condition $\eta_{ca} \partial_b y^c = 0$ and so are generators of the Poincaré transformations (that is, combinations of boosts, rotations and translations).

Under such a transformation, the position vector $X^a$ of the loop is replaced by
\[ \tilde{X}^a = X'^a_{(0)} + \delta X^a + \mu Y^a(u, v), \] (6.41)
where $X'^a_{(0)}$ and $\delta X^a$ are given by (C.1) and (C.3) respectively, and
\[ Y^a(u, v) \equiv \frac{1}{4\pi} L_0[-\delta F(u, v), \delta G(u, v)[\cos(v)\delta + \sin(v)\delta^\prime] + \delta A(u, v)\delta^\prime] \] (6.42)
for some choice of functions $\delta F$, $\delta G$ and $\delta A$.

6 Note that the $x$- and $y$-components of $Y^a$ have the forms shown so as to preserve the isotropic gauge condition discussed in section 4, following equation (4.20).
It is only possible to impose the admissibility condition $\eta_{\epsilon,\alpha} (\partial_\beta) y^\epsilon = 0$ on $Y^\alpha$ if the derivatives are taken tangent to the world sheet of the loop, and so $Y^\alpha$ must satisfy the equations

\[ X_{(0),u} \cdot Y_u = 0, \quad X_{(0),v} \cdot Y_v = 0 \quad \text{and} \quad X_{(0),u} \cdot Y_v + X_{(0),v} \cdot Y_u = 0. \tag{6.43} \]

In terms of $\delta F, \delta G$ and $\delta A$ these equations read, for $u \in (0, \pi)$,

\[ \delta F_u + \delta A_u = 0 \quad \text{and} \quad \delta G + \delta F_v = 0 \quad \text{and} \quad \delta F_v + \delta A_v + \delta F_u = 0. \tag{6.44} \]

The choice $\delta F = -F_2(v), \delta G = F'_2(v)$ and $\delta A = F_2(v)$ satisfies all three equations and specifies a spacetime transformation which reduces $F$ and $A$ to functions of $u$ only, and sets $c_1 = 0$ in (6.36).

In a suitable set of spacetime coordinates, therefore, the correction functions become

\[ G = \int P_1(u) \, du \quad \text{and} \quad F + A = -\int P_2(u) \, du, \tag{6.45} \]

with $F$ being a continuous and half-range periodic function of $u$ alone. In fact, $F$ is effectively an arbitrary function of $u$, as it is always possible to define a transformation $u \rightarrow u' = u + \mu U(u)$ of the gauge coordinate $u$, where $U$ is continuous and half-range periodic but otherwise arbitrary, under which $X^a \rightarrow X^a + \mu U X^a_{(0)}$, and so $F \rightarrow F - U$ and $A \rightarrow A + U$ for $u \in (0, \pi)$, but $G$ remains unchanged.

6.3. Solving for $F$, $G$ and $A$

Equation (6.9) specifying $u = \pi$ as a function of $W$ can be used to eliminate $u$ from the terms in (6.32) and (6.33) which mix both $u$ and $W$. The resulting expressions for $P_1$ and $P_2$ read

\[ P_1(u) = \kappa + 8[ K(u) \sin 2u - \Sigma(u) \cos 2u ] + 4(1 - \cos W - W \sin W) / J(W) \]

and

\[ P_2(u) = -\kappa(u - \pi/2) - 8[ \ln(2W^{-1}(1 - \cos W)) - \ln(2u) ] + 8[ K(u) \cos 2u + \Sigma(u) \sin 2u ] - 4(\sin W - W \cos W) / J(W), \tag{6.47} \]

where

\[ J(W) \equiv W^{-1}(1 - \cos W) + \frac{1}{W} W - \sin W \tag{6.48} \]

and now

\[ K(u) = \text{Ci}[2W^{-1}(1 - \cos W)] - \text{Ci}(2u) \quad \text{and} \quad \Sigma(u) = \text{Si}[2W^{-1}(1 - \cos W)] - \text{Si}(2u). \tag{6.49} \]

Furthermore, the derivative of (6.9) reads $du = -W^{-1} J(W) \, dW$, and so

\[ G = \kappa u + 8 \int [ K(u) \sin 2u - \Sigma(u) \cos 2u ] \, du - 4 \int W^{-1}(1 - \cos W - W \sin W) \, dW \]

\[ = \kappa u - 4[ K(u) \cos 2u + \Sigma(u) \sin 2u ] + 4 \ln(1 - \cos W) - 4 \ln(uW) + C_1 \tag{6.50} \]

and

\[ F + A = \frac{1}{2}\kappa u(u - \pi) + 8 \int [ \ln(2W^{-1}(1 - \cos W)) - \ln(2u) ] \, du \]

\[ - 8 \int [ K(u) \cos 2u + \Sigma(u) \sin 2u ] \, du - 4 \int W^{-1}(\sin W - W \cos W) \, dW \]

\[ = \frac{1}{2}\kappa u(u - \pi) - 8W^{-1}[ 1 + \ln W - \ln(1 - \cos W) ](1 - \cos W) \]

\[ - 4W(2 - \ln W) - 8(\ln u - u) - 4[ K(u) \sin 2u - \Sigma(u) \cos 2u ] \]

\[ - 4 \int_{2\pi}^{W} \ln(1 - \cos \omega) \, d\omega + C_2. \tag{6.51} \]
Here, the integral $\int \ln(1 - \cos \omega) \, d\omega$ unfortunately cannot be expressed in a closed form as a combination of standard functions. (The lower bound in the integral has been set to $\omega = 2\pi$, because $W \to 2\pi^-$ as $u \to 0^+$.) Equations (6.50) and (6.51) can, of course, be verified directly by differentiation. However, in tracing out the original calculations—that is, in integrating $P_1$ and $P_2$—it proves to be most efficient to integrate the terms $K(u)\sin 2u - \Sigma(u)\cos 2u$ in (6.50) and $\ln(1 - \cos W)$ and $\Sigma(u)\sin 2u + K(u)\cos 2u$ in (6.51) by parts with respect to $u$ first, then integrate the remainders with respect to $W$.

Turning to the initial conditions satisfied by $G$ and $F + A$, it should be recalled that $G$ is continuous and half-range periodic in $u$ and so $G(0) = G(\pi)$. Similarly, $F$ is half-range periodic in $u$ and so $F(0) = F(\pi)$. By contrast, $A$ is continuous and half-range anti-periodic in $u$, and therefore $A(0) = -A(\pi)$. (Note that the combination $F + sA$, which appears in the leading terms (6.27) and (6.28) in the first-order equation of motion, is half-range periodic for $u \in [-\pi, \pi]$ as required. However, it is not necessarily continuous at $u = 0$.)

Now, the $u \to 0^+$ lower limit of equation (6.51) reads

$$F(0) + A(0) = -16\pi + 8\pi\ln 2\pi + C_2,$$

while the $u \to \pi^-$ upper limit reads

$$F(\pi) + A(\pi) = 8\pi - 8\pi\ln(2\pi) - 4\text{Si}(2\pi) + C_2.$$  

(6.53)

(Note, in particular, that $\int_{2\pi}^{\pi} \ln(1 - \cos \omega) \, d\omega = 2\pi\ln 2$ at the upper limit.) Subtracting (6.53) from (6.52), therefore, gives

$$2A(0) \equiv A(0) - A(\pi) = -24\pi + 16\pi\ln 2\pi + 4\text{Si}(2\pi)$$

and so

$$A(0) = -12\pi + 8\pi\ln 2\pi + 2\text{Si}(2\pi) \approx 11.328.$$  

(6.54)

Similarly, addition of (6.52) and (6.53) gives

$$2F(0) \equiv F(0) + F(\pi) = -8\pi - 4\text{Si}(2\pi) + 2C_2$$

and so

$$F(0) = -4\pi - 2\text{Si}(2\pi) + C_2 \approx -15.403 + C_2.$$  

(6.55)

(6.56)

(6.57)

The constant of integration $C_2$ in (6.57) represents a vestige of coordinate freedom in the specification of the correction function $F$ that, in view of (6.55), is not present in the function $A$. In fact, different values of $C_2$ correspond to the same physical configuration of the evaporating loop, as the value of $C_2$ can be changed at will by making a suitable Poincaré transformation (6.41). This follows from the fact that the choice $\delta G = \delta A = 0$ and $\delta F$ any arbitrary constant (representing a translation in the coordinate $t$ at order $\mu$) automatically satisfies the Poincaré constraints (6.44). One natural gauge choice for $F$ is to set $F(0) = 0$, in which case $C_2 = 4\pi + 2\text{Si}(2\pi)$.

Consider now equation (6.50) for $G$. Taking the lower limit $u \to 0^+$ (or equivalently $W \to 2\pi^-$) in this equation gives

$$G(0) = C_1,$$

while taking the upper limit $u \to \pi^-$ (equivalently $W \to 0^+$) gives

$$G(\pi) = \pi \kappa + 4\text{Ci}(2\pi) - 4\ln 2\pi - 4\gamma_E + C_1.$$  

(6.58)

(6.59)

So it follows that $G(0) = G(\pi)$ if and only if

$$\kappa = [4\ln 2\pi + 4\gamma_E - 4\text{Ci}(2\pi)]/\pi \approx 3.1037.$$  

(6.60)
That is, iff $\kappa = \gamma^0/(4\pi)$ with $\gamma^0 \approx 39.0025$ the radiative efficiency of the ACO loop, as expected.

Unlike $C_2$, the integration constant $C_1$ in (6.50) does have an invariant physical significance, and it cannot be arbitrarily changed by making a Poincaré or gauge transformation without affecting the form of $F$. In fact, different values of $C_1$ correspond to different values for the length of the evaporating loop as measured along the curves of the constant-scale factor $e^{-\kappa\mu(u+v)}$, as is explained in more detail in appendix D.2. If the length $L_{csf}$ of the loop along such a curve is assumed to be independent of $\mu$ (with $L_{csf} = \sqrt{2}L_0$ along the curve $u+v = 0$) then

$$C_1 = -28 - 6\gamma^0 + 10\ln 2\pi + 6\text{Ci}(2\pi) + 4\pi^{-1}\text{Si}(2\pi) + 4\pi^{-1}\text{Si}(4\pi) \approx -9.5143 \quad (6.61)$$

With $C_1$ assigned this value, the correction function $G$, which is the half-range periodic extension of the function on the right-hand side of (6.50), is plotted for $u \in [-\pi, \pi]$ in figure 5(a). The half-range periodic extension to $[-\pi, \pi]$ of the function $F + sA$ defined by equation (6.51) with $C_2 = 4\pi + 2\text{Si}(2\pi)$ is plotted against $u$ in figure 5(b). As mentioned at the end of section 6.2, the functions $F$ and $A$ are not separately uniquely determined, as different choices of $F$ correspond to different choices of the gauge coordinate $u$. However, the requirement that $A(0) = -A(\pi) = \lambda$ (with $\lambda \equiv -12\pi + 8\pi \ln 2\pi + 2\text{Si}(2\pi)$) does place a restriction on the choice of $F$. One natural gauge choice is to set $A$ proportional to the $z$-component $|u| - \frac{1}{2}\pi$ of the unperturbed position function. Then

$$A(u) = -\frac{2\lambda}{\pi} \left(|u| - \frac{1}{2}\pi\right) \quad (6.62)$$
is continuous and half-range anti-periodic on \([-\pi, \pi]\) with \(A(0) = \lambda\), while from (6.51) \(F\) is the half-range periodic extension to \([-\pi, \pi]\) of
\[
F(u) = \frac{1}{2} \kappa u(u - \pi) + \frac{2\lambda}{\pi} \left( u - \frac{1}{2}\pi \right) - 8W^{-1}[1 + \ln W - \ln(1 - \cos W)](1 - \cos W) \\
- 4W(2 - \ln W) - 8u \ln u - u - 4[K(u) \sin 2u - \Sigma(u) \cos 2u] \\
- 4 \int_{2\pi}^{w} \ln(1 - \cos \omega) d\omega + 4\pi + 2\text{Si}(2\pi).
\]

The correction functions \(A\) and \(F\) in this gauge are plotted against \(u\) in figures 5(c) and (d) respectively.

To summarize, the position function (4.20) with \(F, G\) and \(A\) suitable extensions of (6.63), (6.50) and (6.62) is a solution of the first-order string equation of motion (2.14) that reduces to the ACO solution in the limit as \(\mu \to 0\). Furthermore, if the integration constant \(C_1\) is chosen to take value (6.61) then the solution is the unique self-similar loop trajectory (up to gauge and Poincaré transformations) with the property that the length of the loop retains its ACO value \(\frac{1}{\sqrt{2}}L_0\) along the curve \(u + \nu = 0\) (which to leading order in \(\mu\) corresponds to the spacelike slice \(t = 0\)).

7. Matching the solution to the Minkowski vacuum

By prescription, the evaporating ACO loop radiates away the last of its energy at \(t = t_L\), the moment when it shrinks to a point at the origin. It is, therefore, to be expected that the metric induced by the loop reduces to the Minkowski vacuum at all points in the chronological future of the evaporation point \(x^a = [t_L, 0]^a\). If the evaporation point is denoted by \(p_L\), the boundary of the chronological future \(I^+(p_L)\) is the future light cone \(F_L\) of \(p_L\), plus \(p_L\) itself. The weak-field metric will match smoothly onto Minkowski spacetime in \(I^+(p_L)\) if an appropriate set of junction conditions is satisfied on \(F_L\). I will show explicitly that these junction conditions are indeed satisfied later in this section. (Note, however, that the weak-field metric remains singular in every neighborhood of \(p_L\), and cannot be matched onto Minkowski spacetime at this one point.)

7.1. The surface \(t - t_L = |x|\) is null to order \(\mu\)

Before proceeding, it is important to first establish that the surface \(F_L\) is, to order \(\mu\), just the surface \(t - t_L = |x|\), which is, of course, null with respect to the unperturbed metric tensor \(\eta_{ab}\).

To show that the surface \(t - t_L = |x|\) is also null with respect to the first-order metric tensor \(\eta_{ab} + h_{ab}\), it is sufficient to show that \(h_{ab}(t, x) \to 0\) as the field point \(x^a = [t, x]^a\) approaches the surface. Suppose then that \(\varepsilon\) is a number in \((0, 1)\), and consider the set of field points \([t, x]\) lying to the past of the surface \(t - t_L = |x|\) and satisfying the conditions
\[
|x| - t + t_L = \varepsilon t_L \quad \text{and} \quad [(t - t_L)^2 + |x|^2]^{1/2} > \varepsilon^{1/2}t_L.
\]

The first condition implies that \(\varepsilon\) is a dimensionless measure of the distance from the point \([t, x]\) to the surface \(t - t_L = |x|\) in the background Minkowski manifold. In fact, it is easily seen that the Euclidean distance from the point to the surface is just \(\frac{1}{\sqrt{2}}\varepsilon t_L\). The second condition ensures that the field point avoids a certain neighborhood of the evaporation point \([t_L, 0]\). (The scaling factor \(\varepsilon^{1/2}\) is chosen here because the exclusion of a neighborhood about \([t_L, 0]\) with the smaller radius \(\varepsilon t_L\) does not lead to a useful bound on the components of \(h_{ab}\), as will be seen shortly.) It is evident that as \(\varepsilon \to 0\) the points satisfying (7.1) include all points in some timelike past neighborhood of every point on the surface \(t - t_L = |x|\) except
the evaporation point. Eliminating $t$ from conditions (7.1) yields one non-trivial bound on $|x|$ if $\varepsilon < 1$, namely

$$|x| > \frac{1}{2}[\varepsilon + (2\varepsilon - \varepsilon^2)^{1/2}]t_L,$$

(7.2)

which, in turn, implies that $|x| > \frac{1}{2\varepsilon} e^{1/2} t_L$ (as can readily be verified by dividing the expression on the right of (7.2) by $\varepsilon^{1/2} t_L$ and showing that the minimum value of the resulting function is $1/\sqrt{2}$).

Now, the corresponding image point $[\bar{t}, \bar{x}]$ on the future light cone $F_0$ of the origin has coordinates

$$\bar{t} = e^{\varepsilon \mu \psi} |x| \quad \text{and} \quad \bar{x} = e^{\varepsilon \mu \psi} R(-\psi)x,$$

(7.3)

where

$$e^{\varepsilon \mu \psi} \equiv [(|x| - \varepsilon t_L)/t_L]^{-1} = \varepsilon^{-1}$$

(7.4)

and $R$ is the rotation matrix defined in (5.14). So, in particular,

$$\bar{t} = \varepsilon^{-1} |x| > \frac{1}{\sqrt{2}} \varepsilon^{-1/2} t_L,$$

(7.5)

while, of course, $|\bar{x}| = \bar{t}$.

Note here that $\bar{t} \to \infty$ as $\varepsilon \to 0$. Also, according to (5.22),

$$h_{ab}(\bar{t}, x) = R^a_b(\psi) h_{cd}(\bar{t}, \bar{x}) R^c_d(\psi),$$

(7.6)

where the components of the rotation operators $R^a_b$ are all bounded functions of $\psi$. Hence, to prove that $h_{ab}(\bar{t}, x) \to 0$ as $\varepsilon \to 0$ in conditions (7.1), it is sufficient to show that $|h_{ab}(\bar{t}, x)|$ is bounded above on $F_0$ by a function of $\bar{t}$ which goes to zero (uniformly with respect to $\bar{x}$) as $\bar{t} \to \infty$. It should also be clear that a proof of the last statement would not be sufficient if the factor $e^{1/2}$ in the second condition in (7.1) were replaced by $\varepsilon$ or a multiple of $\varepsilon$. For then (7.2) would be replaced by an inequality of the form $|x| > c \varepsilon$ for some constant $c$, and the image point $[\bar{t}, \bar{x}]$ would be placed under the much weaker constraint $\bar{t} > c$, which does not guarantee that $h_{ab}(\bar{t}, x) \to 0$ as $\varepsilon \to 0$.

Now, given that the components of $\bar{\Psi}_{ab}$ in equation (5.36) for $h_{ab}(\bar{t}, \bar{x})$ all satisfy $|\Psi_{ab}(s, v)| \leq 1$, it follows that

$$|h_{ab}(\bar{t}, \bar{x})| \leq 4\mu \int_{V_1}^{V_2} |T + (Z + \frac{1}{2} \pi) - v|^{-1} dv + 4\mu \int_{V_1}^{V_2} [T - (Z + \frac{1}{2} \pi) - v]^{-1} dv = 4\mu |I_1| + 4\mu |I_2|,$$

(7.7)

where $[T, X, Y, Z] = 4\pi L_0^{-1} [\bar{t}, \bar{x}]$ as before. So $|h_{ab}(\bar{t}, \bar{x})|$ will be bounded above by a function which goes uniformly to zero as $\bar{t} \to \infty$ if the same is true, separately, of $|I_1|$ and $|I_2|$.

Consider then

$$|I_2| = \int_{V_1}^{V_2} [E(1, v)]^{-1} dv, \quad \text{where} \quad E(s, v) = T - s (Z + \frac{1}{2} \pi) - v.$$

(7.8)

Since $E$ is positive semi-definite (see appendix A.2), there are two possible cases that need to be considered: either (i) $E(1, v) = 0$ for some $v \in [V_1, V_2]$, or (ii) $E(1, v) > 0$ for all $v \in [V_1, V_2]$.

Case (i) has been treated in some detail in appendix A.2, where it is shown that $E(1, v) = 0$ for some $v$ only if the point $(X, Y)$ lies on the unit circle (so that $T = (1 + Z^2)^{1/2}$ and $Z > -\frac{1}{2} \pi$). Furthermore, if $Z > \frac{1}{2} \pi$ (and so $T > (1 + \frac{1}{2} \pi)^{1/2} \approx 1.8621$, which will henceforth be assumed) then the values of $V_0$ and $V_1$ coincide, and according to (A.9),

$$|I_2| = \ln \left[1 + \pi/(Z - \frac{1}{2} \pi)\right].$$

(7.9)
In view of the inequality $\ln(1 + x) \leq x$, it follows that
\[ |L_-| \leq \pi / (Z - \frac{1}{2}\pi) \equiv \pi / \left[ (T^2 - 1)^{1/2} - \frac{1}{2}\pi \right], \tag{7.10} \]
where the function of $T$ appearing on the right is bounded above by $8\pi / T$ if $T \geq 8\pi / \sqrt{3} (4\pi + 8\sqrt{16\pi^2 + 63}) \approx 2.0869$. (The significance of the factor $8\pi$ will become apparent shortly.) So, in this case,
\[ |L_-| \leq 8\pi / T \quad \text{if} \quad T \geq 2.0869. \tag{7.11} \]

In case (ii), $E(1, v) > 0$ for all $v \in [V_1, V_0]$. Since $E$ is a linear decreasing function of $v$, there exists a number $V_e$ in $[V_1, V_0]$ with the property that $E(1, v) \leq \frac{1}{2}T$ for all $v \in (V_1, V_e)$ and $0 < E(1, v) < \frac{1}{2}T$ for all $v \in (V_e, V_0)$. An immediate consequence is that
\[ |L_-| = \int_{V_1}^{V_e} [E(1, v)]^{-1} dv + \int_{V_e}^{V_0} [E(1, v)]^{-1} dv, \tag{7.12} \]
where
\[ \int_{V_1}^{V_e} [E(1, v)]^{-1} dv < 2(V_e - V_1) / T \leq 4\pi / T, \tag{7.13} \]
as $V_e - V_1 \leq V_0 - V_1 \leq 2\pi$.

The integral over the remaining segment $(V_e, V_0)$ can be bounded above by using equation (5.34) to replace $v$ with $u$ as the integration variable, so that
\[ \int_{V_e}^{V_0} [E(1, v)]^{-1} dv = \int_0^{u(V_e)} [H(v)]^{-1} du, \tag{7.14} \]
where $H(v) = T - (u + v) + X \sin v - Y \cos v$ is a second positive semi-definite function, examined in some detail in appendix A.3. In particular, it is evident from the first line of (A.10) that
\[ H(v) \geq |Z - (u|v| - \frac{1}{2}\pi)| \geq Z - \frac{1}{2}\pi \tag{7.15} \]
if $u$ is in $[0, \pi]$ and $Z > \frac{1}{2}\pi$. Also, the condition $E(1, v) \leq \frac{1}{2}T$ corresponds to the inequality
\[ Z > \frac{1}{2}T - \frac{1}{2}\pi - v \geq \frac{1}{2}T - \frac{3}{2}\pi - \left( 1 + \frac{1}{4}\pi^2 \right)^{1/2}. \tag{7.16} \]

Here $v$ has been replaced by its maximum possible value, $\pi + (1 + \frac{1}{4}\pi^2)^{1/2}$ (see appendix A.1). In particular, the condition $Z > \frac{1}{2}\pi$ will be satisfied if $T \geq 16.2906$. Hence, if the latter condition is assumed,
\[ \int_0^{u(V_e)} [H(v)]^{-1} du < u(V_e) \left[ \frac{1}{2}T - 2\pi - \left( 1 + \frac{1}{4}\pi^2 \right)^{1/2} \right]^{-1}, \tag{7.17} \]
where the maximum possible value of $u(V_e)$ is $\pi$. The function of $T$ appearing on the right of (7.17) is, therefore, bounded above by $4\pi / T$ if $T \geq 8\pi + 2\sqrt{4 + \pi^2} \approx 32.5811$.

In summary, inequalities (7.13) and (A.10) together imply that
\[ |L_-| \leq 8\pi / T \quad \text{if} \quad T \geq 32.5811 \tag{7.18} \]
in case (ii), and so in view of (7.11), $|L_-|$ is bounded above by a function that goes to zero uniformly as $T \to \infty$, as required. Note that the factor of $8\pi$ in the bounding function arises as a consequence of choosing $V_e$ to separate the domains $E \leq \frac{1}{2}T$. The factor $1/2$ is to some extent arbitrary here, as any factor less than 1 will induce similar bounds on $|L_-|$. However,\[ \text{The value of } V_e \text{ will, of course, depend on } [T, X, Y, Z]. \text{ Also, } V_e \text{ may coincide with either } V_0 \text{ or } V_1, \text{ in which case one of the two domains specified for } v \text{ will be empty.} \]
a factor 1 or greater is not viable, as it generates no useful bound on $Z$, and hence no useful bound on integral (7.14).

The proof that the second contribution $|I_2| \to |h_{ab}(\bar{t}, \bar{x})|$ is also uniformly bounded above by $8\pi / T$ for sufficiently large values of $T$ is formally identical to the proof just given, provided that $|I_2|$ is everywhere replaced with $|I_2|$, $E_{1}$ $v$ with $E_{1} (1, v)$, $Z$ with $-Z$, $V_0$ with $V_1 + 2\pi$ and $V_1$ with $V_0$. Also, the limit $u = 0$ in the integral on the right of (7.14) should there and below be replaced with $u = -\pi$, and the bound $u(V_0) \leq \pi$ with $u(V_4) \leq 0$.

In conclusion, therefore, $|h_{ab}(\bar{t}, \bar{x})| \leq 64\pi \mu / T \equiv 16L_0 / f$ for sufficiently large values of $\bar{t}$, and so $h_{ab}(t, x) \to 0$ as $\epsilon \to 0$ in conditions (7.1). It follows that the future light cone $F_L$ of the final evaporation point is, to order $\mu$, just the surface $t - t_L = |x|$.

7.2. Checking the junction conditions

The junction conditions that apply when matching two spacetime regions across a null hypersurface $\mathcal{N}$ were first usefully formulated by Barrabès and Israel [20] and have been handily summarized by Poisson in [21]. The first step in the Barrabès–Israel method is to parametrize the null hypersurface in the form $\sigma = \sigma_{AB}$, where $\sigma$ is an arbitrary coordinate along the generators of the hypersurface, and each generator is labeled by the two remaining coordinates $\theta^A$ (for $A = 1, 2$). This parametrization can, in turn, be used to construct an orthonull basis $\{\mathbf{k}^a, \mathbf{e}_1^a, \mathbf{e}_2^a, \mathbf{N}^a\}$ at each point on $\mathcal{N}$, with

$$k^a = \partial \sigma^a / \partial r, \quad e^a_A = \partial \sigma^a / \partial \theta^A$$

and the remaining (null) basis field $\mathbf{N}^a$ uniquely determined by the constraints

$$g_{ab} N^a k^b = 1 \quad \text{and} \quad g_{ab} N^a e^b_A = 0. \quad (7.19)$$

The hypersurface then has an associated mass density $\rho$, current density $j^A$ and isotropic pressure $p$ given by

$$\rho = -\frac{1}{8\pi} \sigma^{AB} [C_{AB}], \quad j^A = \frac{1}{8\pi} \sigma^{AB} [C_{AB}] \quad \text{and} \quad p = -\frac{1}{8\pi} [C_{33}], \quad (7.21)$$

where $\sigma_{AB} = g_{ab} e^a_A e^b_B$ and $C_{AB} = e^a_A e^b_B \nabla_b N_a$, with $\alpha, \beta$ running from 1 to 3 and $e^a_3 \equiv k^a$. Also, $[C_{ab}]$ denotes $C_{ab}(\mathcal{N}^+) - C_{ab}(\mathcal{N}^-)$, where $\mathcal{N}^-$ and $\mathcal{N}^+$ are the (timelike) past- and future-facing sides of $\mathcal{N}$ respectively. (Note that $C_{ab}$ typically has different values on the two sides of $\mathcal{N}$, because the limiting values of the covariant derivatives $\nabla_b N_a$ are different, even though the metric tensor $g_{ab}$ is assumed here to be continuous across $\mathcal{N}$.) The two spacetime regions can be matched across $\mathcal{N}$ without invoking an extraneous source of stress–energy on $\mathcal{N}$ if $\rho$, $j^A$ and $p$ are all zero.

In the case of the evaporating ACO loop, $\mathcal{N}$ is the null hypersurface $F_L$ defined by $t - t_L = |x|$ and $g_{ab} = n_{ab}$ there. An obvious choice of parametrization is

$$\sigma = (r, \theta^1, \theta^2) = (|t_L, \mathbf{0}|^a + [r, r \sin \theta^1 \cos \theta^2, r \sin \theta^1 \sin \theta^2, r \cos \theta^1]^a). \quad (7.22)$$

The corresponding orthonull basis fields are

$$k^a = [1, \sin \theta^1 \cos \theta^2, \sin \theta^1 \sin \theta^2, \cos \theta^1]^a, \quad (7.23)$$

$$e^0_A = [0, r \cos \theta^1 \cos \theta^2, r \cos \theta^1 \sin \theta^2, -r \sin \theta^1]^a, \quad (7.24)$$

$$e^1_A = [0, -r \sin \theta^1 \sin \theta^2, r \sin \theta^1 \cos \theta^2, 0]^a \quad (7.25)$$

and

$$\mathbf{N}^a = \frac{1}{2} [1, -\sin \theta^1 \cos \theta^2, -\sin \theta^1 \sin \theta^2, -\cos \theta^1]^a. \quad (7.26)$$
In particular, the line element $\sigma_{AB} \, d\theta^A \, d\theta^B$ intrinsic to the hypersurface is just that of a spatial 2-sphere:

$$\sigma_{AB} = -r^2 \text{diag}(1, \sin^2 \theta^1) \quad \text{and} \quad \sigma^{AB} = -r^{-2} \text{diag}(1, \csc^2 \theta^1). \quad (7.27)$$

To linear order in the derivatives of $h_{ab}$, the transverse curvature of $F_L$ is

$$C_{ab} = e^a_{\sigma} e^b_{\tau} \partial_\sigma N_\tau - \frac{1}{2} e^a_{\sigma} e^b_{\tau} N^\rho (\partial_\rho h_{ab} + \partial_a h_{bc} - \partial_b h_{ac}), \quad (7.28)$$

where the first term $e^a_{\sigma} e^b_{\tau} \partial_\sigma N_\tau$ is identical on the two sides of $F_L$, and the remaining terms are, of course, zero on the Minkowski side $F_0$. Because $h_{ab} = 0$ on $F_0$, the derivatives $e^a_{\sigma} \partial_\sigma h_{bc}$ tangent to $F_0$ are all zero, but the limiting values of the transverse derivatives $N^\rho \partial_\rho h_{ab}$ are not necessarily zero there. So

$$[C_{ab}] = -\frac{1}{2} e^a_{\sigma} e^b_{\tau} N^\rho \partial_\rho h_{ab} |_{F_0}. \quad (7.29)$$

Now, the derivatives of $h_{ab}$ can be evaluated formally by using (5.22) to write

$$\partial_c h_{ab}(\bar{t}, \bar{x}) = \left[ \frac{d}{d\hat{\psi}} \bar{R}_b^p (\bar{t}, \bar{x}) \bar{R}_a^q (\bar{t}, \bar{x}) \right] \partial_c \hat{\psi} + \left( \partial \hat{\chi} / \partial x^e \right) \bar{R}_b^o \partial_\beta h_{pq} (\bar{t}, \bar{x}) \bar{R}_a^p, \quad (7.30)$$

where $\partial_c \hat{\psi}$ and $\partial \hat{\chi} / \partial x^e$ can, in turn, be calculated from (5.12) and (5.13). However, as is shown in Appendix A.3, the derivatives $\partial_\beta h_{pq}$ diverge at image points that lie in the beaming directions of the two kinks, and the loci of such points extend to future null infinity on $F_0$. Although points of this type have $\bar{z} = \pm r L_0$, and so (since $\bar{z} = e^{-x^2/2}$) the only points on $F_L$ which are limit points of the singularities of $\partial_\beta h_{pq}$ lie on the plane $z = 0$, this feature of the metric makes (7.30) less than ideal for calculating $[C_{ab}]$. Instead, it is more convenient when calculating $[C_{ab}]$ to use the formal definition of the derivative

$$\lim_{||\delta y|| \to 0} \||\delta y||^{-1} [f(y^a + \delta y^a) - f(y^a) - \delta y^a \partial_c f] = 0, \quad (7.31)$$

where $f$ denotes any of the relevant combinations of the components of $h_{ab}$, and $|| \cdot ||$ is the Euclidean norm. This approach has the advantage of demonstrating that the relevant combinations of the metric derivatives do, in fact, exist at all points on the null boundary (except at $p_L$ itself).

Consider then a fixed point $y^a = [t_L + r, r \bar{F}]^a$ on $F_L \setminus \{p_L\}$ (with $\bar{F} = (\hat{x}, \hat{y}, \hat{z})$ and $|\bar{F}| = 1$), and let $x^a = y^a + \delta y^a$ be a field point at a Euclidean distance $\sigma^a t_L$ from (and in the timelike past of) $y^a$, so that $\delta y^a = \sigma t_L [\delta \hat{x}, \delta \hat{y}]^a$, where $\delta \bar{F} \equiv (\delta \hat{x}, \delta \hat{y}, \delta \hat{z})$ and $||\delta \bar{F}, \delta \bar{F}|| = 1$. If $\varepsilon$ denotes $e^{-x^2/2}$ (as in the previous section) then

$$\varepsilon = |t_L^{-1} |\bar{F} + \sigma \delta \bar{F}|^{-1} r - \sigma \delta \bar{t} = (\bar{t} \cdot \delta \bar{F} - \delta \bar{t}) \sigma + O(\sigma^2). \quad (7.32)$$

It is easily seen that $\varepsilon \leq \sqrt{2} \sigma$ and that $\varepsilon > 0$, when $x^a$ lies in the past of $y^a$. The image point $\bar{x}'$ corresponding to $x^a$ has components

$$\bar{t}' = \varepsilon^{-1} |r \bar{F} + \sigma t_L \delta \bar{F}|, \quad \hat{x}' = \varepsilon^{-1} [r \hat{t} + \sigma t_L \delta \hat{t}] \cos \psi + \sin \psi, \quad \bar{y}' = \varepsilon^{-1} [\bar{y} + \sigma t_L \delta \bar{y}] \cos \psi - \sigma t_L \delta \hat{t} \sin \psi \quad \text{and} \quad \varepsilon = e^{-1} (r \hat{t} + \sigma t_L \delta \bar{t}). \quad (7.33)$$

If the non-dimensionalized components $[T, X, Y, Z] = 4 \pi L_0^{-1} [\bar{t}, \hat{x}, \bar{y}, \hat{z}]$ are substituted into equation (5.33) for $V_k$ then, after squaring both sides and rearranging, the equation reads (for $k = 0$ or 1):

$$|r \bar{F} + \sigma t_L \delta \bar{F}| (V_k + k \pi) - r \bar{F} \cdot X_k = \sigma t_L \left[ \delta \bar{t} \cdot X_k + \frac{1}{2} k \mu (\varepsilon / \sigma)((V_k + k \pi)^2 - (1 + \pi^2/4)) \right],$$

(7.34)
where the factor of $\kappa \mu$ arises because $\frac{1}{4\pi} L_0 = \kappa \mu t_L$, and

$$X_k \equiv \left( \cos (V_k + \psi), \sin (V_k + \psi), (k - \frac{1}{2})\pi \right).$$

(7.35)

Equation (7.34) suggests that, for small values of $\sigma$, the root $V_k$ can be approximated by the solution of the equation $V_k + k\pi = \hat{r} \cdot X_k$, which I will denote by $V_k^\infty$. In explicit form, the equations for $V_0^\infty$ read

$$V_0^\infty = \hat{x} \cos \left(V_0^\infty + \psi\right) + \hat{y} \sin \left(V_0^\infty + \psi\right) - \frac{\pi}{2} \hat{z}$$

(7.36)

and

$$V_1^\infty + \pi = \hat{x} \cos \left(V_1^\infty + \psi\right) + \hat{y} \sin \left(V_1^\infty + \psi\right) + \frac{\pi}{2} \hat{z}.$$ 

(7.37)

Note, however, that $V_0^\infty$ are typically still functions of $\sigma$, as $\psi = - (\kappa \mu)^{-1} \ln \varepsilon$ increases without bound, and in particular $\lim_{\sigma \to 0} V_0^\infty$ does not exist unless $\hat{x} = 0$.

Although $\lim_{\sigma \to 0} V_1^\infty$ does not generally exist, it can be shown that $\lim_{\sigma \to 0} |V_k - V_1^\infty| = 0$, a fact that will prove important in establishing that certain combinations of the metric components are differentiable on $F_k$. To show that $\lim_{\sigma \to 0} |V_k - V_1^\infty| = 0$, first rearrange (7.34) to give

$$r(V_k + k\pi - \hat{r} \cdot X_k) = \sigma \omega L \left[ \delta \hat{r} \cdot X_k + \frac{1}{2} \kappa \mu \varepsilon \Theta(\sigma) \right] \\
+ \left( r - |r\hat{r} + \sigma \omega L \delta \hat{r}| \right) (V_k + k\pi) \\
\equiv \Theta(\sigma).$$

(7.38)

Since $|V_k + k\pi|$ is bounded above (see appendix A.1), it is clear that the function $\Theta(\sigma)$ on the right-hand side of this equation tends to zero as $\sigma \to 0$. The root $V_k^\infty$ satisfies the same equation (7.38) with $\Theta(\sigma) = 0$.

Subtracting the equation for $V_k^\infty$ from (7.38) gives

$$V_k - V_k^\infty - \hat{x} \left[ \cos (V_k + \psi) - \cos (V_k^\infty + \psi) \right] = \hat{y} \left[ \sin (V_k + \psi) - \sin (V_k^\infty + \psi) \right] = r^{-1} \Theta(\sigma).$$

In view of the identities $\cos 2A - \cos 2B = 2 \sin(B - A) \sin(A + B)$ and $\sin 2A - \sin 2B = 2 \sin(A - B) \cos(A + B)$, this equation becomes

$$V_k - V_k^\infty + 2 \hat{x} \sin \left( \frac{1}{2} V_k + \frac{1}{2} V_k^\infty + \psi \right) \\
- \hat{y} \cos \left( \frac{1}{2} V_k + \frac{1}{2} V_k^\infty + \psi \right) \sin \left[ \frac{1}{2} (V_k - V_k^\infty) \right] = r^{-1} \Theta(\sigma),$$

(7.40)

and so

$$|V_k - V_k^\infty| - 2 (\hat{x}^2 + \hat{y}^2)^{1/2} \left| \sin \left( \frac{1}{2} (V_k - V_k^\infty) \right) \right| \leq r^{-1} |\Theta(\sigma)|.$$ 

(7.41)

Since $(\hat{x}^2 + \hat{y}^2)^{1/2} \leq 1$ and $|w| - 2 \sin \left( \frac{1}{2} w \right) \geq \frac{1}{2} |w|^3$ for all real $w$, it follows that

$$\frac{1}{4\pi} |V_k - V_k^\infty| \leq r^{-1} |\Theta(\sigma)|$$

(7.42)

and, therefore, $\lim_{\sigma \to 0} |V_k - V_k^\infty| = 0$ as claimed. Furthermore, because $\Theta(\sigma)$ goes to zero linearly in $\sigma$, $|V_k - V_k^\infty|$ is at worst of order $\sigma^{1/3}$ for small values of $\sigma$.

The asymptotic forms of the components of $h_{ab}$ can now be developed by first substituting the expressions

$$\chi_\pm = 4\pi L_0^{-1} \epsilon^{-1} (t \pm z) \pm \frac{\pi}{2}$$

(7.43)
into (5.38)–(5.40) to give
\[
I_+ = \frac{1}{4\pi L_0\theta} \frac{V_0 - V_1 - 2\pi}{t + z} + O(\epsilon^2), \quad I_- = \frac{1}{4\pi L_0\theta} \frac{V_1 - V_0}{t - z} + O(\epsilon^2),
\]
\[
S_\pm = \pm \frac{1}{4\pi L_0\theta} \frac{\cos V_1 - \cos V_0}{t \pm z} + O(\epsilon^2) \quad \text{and} \quad C_\pm = \pm \frac{1}{4\pi L_0\theta} \frac{\sin V_0 - \sin V_1}{t \pm z} + O(\epsilon^2)
\]
\[
(7.44)
\]
(assuming for the moment that \( t \pm z \neq 0 \)).

With \( t = |r\hat{r} + \sigma t_\delta \delta \hat{r}| \) and \( z = r\hat{z} + \sigma t_\delta \delta \hat{z} \), equation (5.37) generates the following expansion for the components of \( h_{ab} \):
\[
h_{ab}(t, x) = R_c^\delta(\psi)h_{cd}(|x|, x)R_d^\delta(\psi)
\]
\[
= \frac{2L_0\mu\epsilon}{\pi r(1 - \hat{z}^2)} \times \left[ \begin{array}{ccc}
\Delta V \hat{z} - (1 - \hat{z})\pi & (c_0 - c_1)\hat{z} & (s_0 - s_1)\hat{z} \\
(c_0 - c_1)\hat{z} & \Delta V \hat{z} - (1 - \hat{z})\pi & 0 \\
(s_0 - s_1)\hat{z} & 0 & \Delta V \hat{z} - (1 - \hat{z})\pi
\end{array} \right]_{ab}
\]
\[
+ O(\sigma^2),
\]
\[
(7.45)
\]
where
\[
c_0 = \cos(V_0 + \psi), \quad s_0 = \sin(V_0 + \psi) \quad \text{and} \quad \Delta V = V_1 - V_0.
\]

In terms of the components of \( \hat{r} \), the relevant orthonormal basis vectors are
\[
k_0 = [1, \hat{x}, \hat{y}, \hat{z}]^a, \quad e_1^a = r(1 - \hat{z}^2)^{-1/2}[0, \hat{x}, \hat{y}, -\hat{x}^2 + \hat{y}^2]
\]
\[
\text{and} \quad e_2^a = r[0, -\hat{y}, \hat{x}, 0]^a,
\]
\[
(7.46)
\]
and so
\[
k^a k^b h_{ab} = -4(L_0/r)\mu\epsilon + O(\sigma^2), \quad k^a e_2^b h_{ab} = O(\sigma^2)
\]
\[
k^a e_1^b h_{ab} = \frac{2}{\pi(1 - \hat{z}^2)^{3/2}}L_0\mu\epsilon (V_1 + \pi - \hat{r} \cdot \mathbf{X}_1 - V_0 + \hat{r} \cdot \mathbf{X}_0) + O(\sigma^2),
\]
\[
(7.47)
\]
and
\[
\sigma^{AB} e_A^a e_B^b h_{ab} = -r^{-2}[e_1^a e_1^b h_{ab} + (1 - \hat{z}^2)^{-1} e_2^a e_2^b h_{ab}]
\]
\[
= -\frac{4}{\pi(1 - \hat{z}^2)}(L_0/r)\mu\epsilon (V_1 + \pi - \hat{r} \cdot \mathbf{X}_1 - V_0 + \hat{r} \cdot \mathbf{X}_0)\hat{z}
\]
\[
- \pi(1 - \hat{z}^2) + O(\sigma^2).
\]
\[
(7.50)
\]
Now, given that \( \lim_{\sigma \to 0} |V_k - V_k^\infty| = 0 \) and \( c_k \) and \( s_k \) are continuous functions of \( V_k \),
\[
\lim_{\sigma \to 0} (V_1 + \pi - \hat{r} \cdot \mathbf{X}_1) = \lim_{\sigma \to 0} \left[ V_1^\infty + \pi - \hat{x} \cos (V_1^\infty + \psi) - \hat{y} \sin (V_1^\infty + \psi) - \frac{\pi \hat{z}}{2} \hat{z} \right] = 0,
\]
\[
(7.51)
\]
and similarly \( \lim_{\sigma \to 0} (V_0 - \hat{r} \cdot \mathbf{X}_0) = 0 \). Hence, in view of expansion (7.32),
\[
\lim_{\sigma \to 0} \sigma^{-1} k^a k^b h_{ab} = -4(L_0/r)\mu(\hat{r} \cdot \delta \hat{r} - \delta \hat{t}), \quad \lim_{\sigma \to 0} \sigma^{-1} k^a e_1^b h_{ab} = 0
\]
\[
\lim_{\sigma \to 0} \sigma^{-1} k^a e_2^b h_{ab} = 0 \quad \text{and} \quad \lim_{\sigma \to 0} \sigma^{AB} e_A^a e_B^b h_{ab} = 4(L_0/r)\mu(\hat{r} \cdot \delta \hat{r} - \delta \hat{t}).
\]
\[
(7.52)
\]
Since $\delta y^\alpha = \sigma_t t_L [\delta \hat{t}, \delta \hat{p}^\alpha]$, it follows from (7.31) that $k^a k^b h_{ab}$, $k^a e_b h_{ab}$ and $\sigma^{AB} e^a A e_b h_{ab}$ are all differentiable at each point on the null boundary $F_L$, with $\hat{z}^2 \neq 1$, and that

$$k^a k^b \partial_a h_{ab} = -\sigma^{AB} e^a A e_b \partial_a h_{ab} = 4(L_0/r)\hat{t}_L^{-1} \mu [1, -\hat{p}], \quad \text{and} \quad k^a e_b \partial_a h_{ab} = 0$$

there. Given that $N^c[1, -\hat{p}] = 1$, it seems, therefore, that

$$[C_{33}] = -\sigma^{AB}[C_{AB}] = -2(L_0/r)\hat{t}_L^{-1} \mu \quad \text{and} \quad \sigma^{AB}[C_{AB}] = 0,$$  

and hence that matching the evaporating ACO spacetime to the Minkowski spacetime across $F_L$ induces a boundary layer with a nonzero density $\rho$ and isotropic pressure $p$. However, it should be recalled that $L_0\hat{t}_L^{-1} = 4\pi \kappa \mu$, and so the apparent density and pressure of this boundary layer,

$$\rho = -p = -\kappa \mu z/r,$$  

are quadratic in $\mu$. \(^8\) Thus, $\rho$ and $p$ are zero at the level of the first-order approximation used throughout this paper, and the evaporating ACO spacetime can be matched to the Minkowski spacetime without inducing a boundary layer at this level of approximation.

All that remains now is to show that the same conclusion holds at all points on $F_L$ with $\hat{z}^2 = 1$ (that is, on the $z$-axis). In this case, $\hat{x} = \hat{y} = 0$ and $V_k$ tends to the limiting value

$$V_k^\infty = -k \pi + (k - \frac{1}{2})\pi \hat{z}.$$  

(7.56)

Also, it is convenient to use a slightly different asymptotic form for the spatial components of the field point $x^a$, namely

$$x = \frac{1}{4\pi} L_0 x^a \delta, \quad y = \frac{1}{4\pi} L_0 y^a \delta \quad \text{and} \quad z = r(\hat{z} + \frac{1}{4\pi} L_0 \chi^a) \delta,$$  

(7.57)

where $x^a, y^a$ and $z^a$ are all of order $\delta^0$. Then

$$X = x^a \cos \psi + y^a \sin \psi, \quad Y = y^a \cos \psi - x^a \sin \psi \quad \text{and} \quad Z = 4\pi \delta L_0^{-1} \hat{r} \hat{z} + z^a.$$  

(7.58)

and the analogue of (7.43) becomes

$$\chi = 4\pi r L_0^{-1} (1 + \hat{z}) \epsilon^{-1} + z^a \hat{z} (1 + \hat{z}) \pm \frac{1}{2} \pi + \frac{1}{8\pi} L_0 r^{-1}(X^2 + Y^2) \epsilon$$

$$- \frac{1}{2} \left( \frac{1}{4\pi} L_0 \right)^2 r^{-2} z^a (X^2 + Y^2) \epsilon^2$$  

(7.59)

to order $\epsilon^2$, while the analogue of equation (7.38) for $V_k$ can be used to calculate the asymptotic expansion

$$\chi = V_k = 4\pi r L_0^{-1} (1 + \hat{z}) \epsilon^{-1} + z^a \hat{z} (1 + \hat{z}) \pm \frac{1}{2} \pi + \frac{1}{8\pi} L_0 r^{-1}(X^2 + Y^2) \epsilon$$

$$+ \frac{1}{8\pi} L_0 r^{-1}(X^2 + Y^2 \hat{z} + 1) \epsilon + \frac{1}{2} \left( \frac{1}{4\pi} L_0 \right)^2 r^{-2} X(2Y + \hat{z}) \epsilon^2$$

$$+ \frac{1}{2} \left( \frac{1}{4\pi} L_0 \right)^2 r^{-2} \hat{z} \left( \left[ \hat{k} - \frac{1}{2} \right] \pi - z^a \right)(X^2 + Y^2 + 1) \epsilon^2$$  

(7.60)

(plus terms of order $\epsilon^3$).

The expansions need to be developed to such a high order because if $\hat{z} = 1$ (resp. $\hat{z} = -1$), then the terms appearing at order $\epsilon$ and $\epsilon^2$ make no contribution to $L_-, S_-$ or $C_-$ (resp. $L_+, S_+$).

\(^8\) Incidentally, it is to be expected that neither $\rho$ nor $p$ depend separately on $L_0$ or $t_L$, as the values of the latter will vary as the zero point of $t$ changes, but the stress–energy content of $F_L$ should remain invariant under time translations.
or \( C_\ast \), and the terms of order \( \epsilon \) plus the terms proportional to \( (k - \frac{1}{2}) \) at order \( \epsilon^2 \) dominate the asymptotic behavior. Otherwise (if \( \hat{z} = -1 \), resp. \( \hat{z} = 1 \)), it is the terms at order \( \epsilon^{-1} \) and \( \epsilon^0 \) that dominate. A straightforward but lengthy calculation shows that

\[
I_+ = I_- = -\frac{1}{2}(L_0/r)\epsilon, \quad S_+ = -\frac{1}{2}(1 - \hat{z})(L_0/r)\epsilon, \quad S_- = \frac{1}{2}(1 + \hat{z})(L_0/r)\epsilon \quad \text{and} \quad C_+ = C_- = 0 \tag{7.61}
\]

to order \( \epsilon \), and hence from (5.37) that

\[
h_{ab}(t, x) = (L_0/r)\mu\epsilon \begin{bmatrix} -2 & \hat{z}\cos\psi & \hat{z}\sin\psi & 0 \\ \hat{z}\cos\psi & -2 & 0 & -\cos\psi \\ \hat{z}\sin\psi & 0 & -2 & -\sin\psi \\ 0 & -\cos\psi & -\sin\psi & -2 \end{bmatrix} \delta_{ab} + O(\epsilon^2). \tag{7.62}
\]

Furthermore, when \( \hat{z}^2 = 1 \) the spherical coordinates \( (r, \theta^1, \theta^2) \) parametrizing the null surface \( F_L \) are singular, as \( \sigma_{\alpha\beta} \) is degenerate. A more suitable parametrization of \( F_L \) in this case would align the poles of the spherical coordinate system with (say) the \( x \)-axis. Then \( k^\alpha = [1, 0, 0, \hat{z}]^\alpha \), \( e^\alpha_1 = r[0, 0, -1, 0] \) and \( e^\alpha_2 = r[0, \hat{z}, 0, 0]^\alpha \), \( \sigma^{AB} = -r^2\delta^{AB} \). With this choice of orthonull basis, it is easily checked that

\[
k^a k^b h_{ab} = -4(L_0/r)\mu\epsilon + O(\epsilon^2), \quad k^a e^b_1 h_{ab} = O(\epsilon^2), \quad k^a e^b_2 h_{ab} = O(\epsilon^2) \tag{7.64}
\]

and

\[
\sigma^{AB} e^a_A e^b_B h_{ab} = 4(L_0/r)\mu\epsilon + O(\epsilon^2), \tag{7.65}
\]

just as in the case \( \hat{z}^2 \neq 1 \) considered earlier. So it can be seen that the null boundary \( F_L \backslash \{p_L\} \) separating the evaporating ACO spacetime from the relic Minkowski spacetime contains no stress–energy to first order in \( \mu \).

8. Dynamical stability of the ACO loop

Realistic cosmic string loops would not, of course, have the exact shape of an ACO loop and would not evaporate in a strictly self-similar manner. It is, therefore, important to be able to estimate the impact that deviations from the ACO loop trajectory would have on the evaporation process. A full analysis of the stability of the ACO loop in the presence of back reaction is beyond the scope of this paper (not least because the dynamical consequences of the self-gravity of a string loop in the strong-field limit are not yet understood), and in this section, I will limit myself to some observations about the dynamical stability of the flat-space ACO solution.

The trajectory of the flat-space ACO loop is fully described by the position vector \( X(\tau, \sigma) \) defined in (3.2), where the mode functions \( a \) and \( b \) are specified by equations (3.11) and (3.12). For the purposes of the perturbation analysis of this section, I will refer to the unperturbed mode functions as \( a_0 \) and \( b_0 \), so that

\[
a_0(\sigma_+) = (|\sigma_+| - \frac{1}{2}L)\hat{z} \quad \text{for} \quad -\frac{1}{2}L \leq \sigma_+ \leq \frac{1}{2}L \tag{8.1}
\]

and

\[
b_0(\sigma_-) = \frac{L}{2\pi} [\cos(2\pi\sigma_-/L)\hat{x} + \sin(2\pi\sigma_-/L)\hat{y}], \tag{8.2}
\]

where \( \sigma_\pm = \tau \pm \sigma \). (Strictly speaking, of course, \( a_0 \) is the even periodic extension of the function on the right of (8.1).)
A generic perturbation of the ACO solution has mode functions of the form
\[ a(\sigma_+) = a_0(\sigma_+) + \delta a(\sigma_+) \quad \text{and} \quad b(\sigma_-) = b_0(\sigma_-) + \delta b(\sigma_-), \quad (8.3) \]
where \( \delta a \) and \( \delta b \) are continuous functions of their arguments, and are both small, in the sense that \( |\delta a| \ll L \) and \( |\delta b| \ll L \) everywhere. Without loss of generality it can be assumed that the perturbed solution has the same parametric period, \( L \), as the unperturbed ACO loop, and so \( \delta a \) and \( \delta b \) are periodic functions with period \( L \). The only other constraints on \( \delta a \) and \( \delta b \) are the gauge conditions \( |a|^2 = |b|^2 = 1 \), which to linear order in the perturbations require that \( a_0' \cdot a' = 0 \) and \( b_0' \cdot b' = 0 \) at all points where these derivatives exist.

Note, in particular, that the derivatives \( \delta a' \) and \( \delta b' \) need not be continuous, nor is it necessarily true that either \( |\delta a'| \) or \( |\delta b'| \) is small. A simple way to visualize the effect of a perturbation is to plot the unit vectors \( a' \) and \( b' \) on the surface of a unit sphere, an approach first popularized by Kibble and Turok [22]. Figure 6 contains three plots of this type. The first represents the unperturbed ACO loop, and shows the \( b' \) curve following the equator of the sphere, while \( a' = \pm \hat{z} \) is concentrated at just two points, the north and south poles. In the second plot, the perturbations \( \delta a' \) and \( \delta b' \) are assumed to be both continuous and small, with the result that the \( a' \) and \( b' \) curves depart only minimally from the unperturbed curves. The \( a' \) curve, in particular, is understood to jump discontinuously from the small circle around the south pole to the small circle around the north pole when \( \sigma_+ \approx 0 \) then back again when \( \sigma_+ \approx \pi \), so the two kinks are preserved under perturbations of this form.

The third plot in figure 6 shows the effect of a perturbation in which \( \delta a' \) is continuous but no longer small. In fact, \( |\delta a'| \) is assumed to be of order unity for a narrow range of values of the parameter \( \sigma_+ \) centered on \( \sigma_+ \approx 0 \) and \( \sigma_+ \approx \pi \), creating two smooth segments on the \( a' \) curve linking the north and south poles. This has the effect of removing the discontinuity in the \( a' \) curve, and thus smoothing the two kinks, but an unavoidable consequence is that the \( a' \) and \( b' \) curves now cross at two points, indicating that a pair of cusps will momentarily form during each oscillation of the perturbed loop.

So although the unperturbed ACO loop does not support cusps, it is evident that there exist solutions of the flat-space equation of motion arbitrarily close to the ACO loop in \( a-b \) trajectory space that do support cusps. The presence of cusps complicates the back-reaction problem considerably. Because the bulk velocity of a string loop momentarily reaches the speed of light \( c \) at a cusp, a non-negligible fraction of the loop’s total energy can be concentrated there. A simple kinematic argument ([23], section 6.2) suggests that the total energy \( M_r \) inside a sphere of radius \( r \) centered on a cusp scales as \( r^{1/2} \), and so the gravitational potential \( M_r/r \) diverges as \( r^{-1/2} \). This, in turn, is an indication that the effects of gravitational back reaction at a cusp cannot be modeled accurately at the level of the weak-field approximation\(^9\).

\(^9\) By contrast, because the bulk Lorentz factor of a string is bounded at a kink, the energy \( M_r \) inside a sphere of radius \( r \) centered on a kink typically scales as \( r \), and the gravitational potential \( M_r/r \) has no problematic divergences. The breakdown of the weak-field approximation at a kink is due not to a local spike in the energy content of the string, but due to the fact that the tangent vector \( X_{\chi_0,\omega} \) is discontinuous (see section 9).
Any cusp forming on a perturbed solution that is close to the ACO loop will necessarily be very narrow, as the energy in such a cusp will be proportional to the range of the parameter \( \sigma_\ast \), covered by the segment linking the two poles on the Kibble–Turok sphere. What effect back reaction would have on the cusps is still an open question. It is possible that back reaction would act to narrow the cusps even further, thus driving the perturbed solution closer to the unperturbed ACO loop. The ACO loop would then be dynamically stable.

Alternatively, the cusps could broaden in the presence of back reaction and ultimately expand to the size of ordinary macrocusps, which typically contain a fraction of order \( \mu \) of the total energy of the string loop ([23], section 6.2). In this case, it seems likely that the ACO loop would act to narrow the cusps even further, thus driving the perturbed solution closer to the unperturbed ACO loop. The ACO loop would then be dynamically stable.

A third possibility is that cusps on perturbed solutions close to the ACO loop are unstable to gravitational collapse, with consequences that have not yet been explored and can at present only be imagined.

Another potential source of dynamical instability is loop self-intersection. String loops that intersect themselves are believed to split into two daughter loops with high probability ([23], section 2.7). The ACO loop itself has no self-intersections, but there do exist trajectories that intersect themselves. A perturbed trajectory that diverges rapidly from the ACO loop will have an increased radiative efficiency and will evaporate more quickly, whereas perturbations that remain close to the ACO loop will survive longer.

A self-intersection occurs on a general flat-space loop when

\[
\mathbf{a}(\sigma_\ast + \Delta) - \mathbf{a}(\sigma_\ast) = \mathbf{b}(\sigma_\ast) - \mathbf{b}(\sigma_\ast - \Delta)
\]  

(8.4)

for some values of \( \sigma_\ast \), \( \sigma_\ast \) and \( \Delta \), with \( \Delta \in (0, L/2) \) ([23], section 3.8). If \( \mathbf{a} \) and \( \mathbf{b} \) are decomposed in accordance with (8.3), the x- and y-components of equation (8.4) read

\[
\cos(2\pi \sigma_\ast/L) - \cos[2\pi (\sigma_\ast - \Delta)/L] = 2\pi L^{-1}(\delta_\Delta \mathbf{a} - \delta_\Delta \mathbf{b}) \cdot \mathbf{\hat{x}}
\]  

(8.5)

and

\[
\sin(2\pi \sigma_\ast/L) - \sin[2\pi (\sigma_\ast - \Delta)/L] = 2\pi L^{-1}(\delta_\Delta \mathbf{a} - \delta_\Delta \mathbf{b}) \cdot \mathbf{\hat{y}}
\]  

(8.6)

respectively, where

\[
\delta_\Delta \mathbf{a} \equiv \delta \mathbf{a}(\sigma_\ast + \Delta) - \delta \mathbf{a}(\sigma_\ast) \quad \text{and} \quad \delta_\Delta \mathbf{b} \equiv \delta \mathbf{b}(\sigma_\ast) - \delta \mathbf{b}(\sigma_\ast - \Delta).
\]  

(8.7)

Equations (8.5) and (8.6) can be solved to give

\[
\cos(2\pi \sigma_\ast/L) = \pi L^{-1}(\delta_\Delta \mathbf{a} - \delta_\Delta \mathbf{b}) \cdot \left[ \mathbf{\hat{x}} + \frac{\sin(2\pi \Delta/L)}{1 - \cos(2\pi \Delta/L)} \mathbf{\hat{y}} \right]
\]  

(8.8)

and

\[
\sin(2\pi \sigma_\ast/L) = \pi L^{-1}(\delta_\Delta \mathbf{a} - \delta_\Delta \mathbf{b}) \cdot \left[ \mathbf{\hat{y}} - \frac{\sin(2\pi \Delta/L)}{1 - \cos(2\pi \Delta/L)} \mathbf{\hat{x}} \right].
\]  

(8.9)

In particular,

\[
1 - \cos(2\pi \Delta/L) = 2\pi^2 L^{-2}[(\delta_\Delta \mathbf{a} - \delta_\Delta \mathbf{b}) \cdot \mathbf{\hat{x}}]^2 + [(\delta_\Delta \mathbf{a} - \delta_\Delta \mathbf{b}) \cdot \mathbf{\hat{y}}]^2,
\]  

(8.10)

and so \( \Delta \) must be of order \( |\delta_\Delta \mathbf{a} - \delta_\Delta \mathbf{b}| \) or smaller. For small perturbations of the ACO loop, therefore, self-intersections can only occur with \( \Delta \ll L \).
Moreover, for small values of $\Delta$, equations (8.5) and (8.6) reduce to

\[
\Delta^{-1}(\delta a - \delta b) \cdot \mathbf{e} \approx -\sin(2\pi \sigma_\pm / L) \quad \text{and} \quad \Delta^{-1}(\delta a - \delta b) \cdot \hat{y} \approx \cos(2\pi \sigma_\pm / L)
\]

(8.11)

respectively, and can be satisfied only if either $\delta a'$ or $\delta b'$ is discontinuous or of order unity at the self-intersection point $(\sigma_+, \sigma_-) \approx (\sigma_+ + \Delta, \sigma_- - \Delta)$. Hence, at least one of the $a'$ and $b'$ curves will deviate significantly from the corresponding unperturbed ACO curve on the Kibble–Turok sphere.

The picture that emerges, therefore, is that a flat-space solution close to the ACO loop can intersect itself only by pinching off a small daughter loop with parametric period $\Delta \ll L$. A solution that does just this is easily constructed by choosing (at time $t = 0$) a small segment of the unperturbed ACO loop with a length of order $\Delta$ and deforming it so that it crosses itself. But the same perturbation procedure could be applied to any flat-space loop solution, so self-intersections of this type are in a sense rather trivial.

Furthermore, the larger of the two daughter loops created by the self-intersection will be little changed from the original parent loop. It will have a marginally smaller parametric length $L - \Delta$, and in place of the self-intersection there will be two narrow kinks traveling in opposite directions around the loop. Analytic studies of kinks on long strings strongly suggest that gravitational back reaction will suppress a small-scale structure on a string with a characteristic length $r$ on a timescale $\Delta t \sim r/(8\pi^2 \mu)$ ([23], section 6.7), and this estimate has been confirmed in numerical simulations of planar loops carrying up to 32 kinks by Quashnock and Spergel [11]. In the case of a kink formed by the pinching off of a daughter loop with a parametric period $\Delta$ the characteristic length $r$ will be of order $\Delta$, and the decay time $\Delta t$ for the kink will be much smaller than the radiative lifetime $t_L = L/(4\pi \kappa \mu)$ of the ACO loop. So it is to be expected that the two narrow kinks created by the self-intersection will quickly dissipate, leaving behind a daughter loop that very closely resembles the ACO loop.

To summarize, there is as yet no reason to believe that the ACO solution is dynamically unstable, although a definitive statement on the matter will depend on the results of full back-reaction stability analyses, both at the weak-field level (without cusps) and in the strong-field limit (with cusps). Nonetheless, it seems safe to conclude that the effect of possible self-intersections on the stability of the ACO loop is negligible. Furthermore, even if it were to turn out that the ACO loop is unstable to certain mechanisms (such as cusp formation), this would not in itself invalidate the importance of the self-similar evaporating ACO solution described above, as the longest lived loops in any ensemble of loops with the same initial energy would be those that most closely resemble the ACO loop—on the presumption, of course, that the ACO solution is unique in having the lowest radiative efficiency of any string loop.

I should add that at present there is an ongoing debate about the nature of the scaling peaks in the production spectrum of cosmic string loops, fueled by the conflicting results of recent numerical [24, 25] and analytic [26, 27] studies of the primordial string network. However, this debate has no direct bearing on the stability or evolution of the ACO loop.

9. Conclusions

In this paper, I have constructed a solution of the linearized Einstein and Nambu–Goto equations, describing a cosmic string loop that evaporates self-similarly under the action of its own self-gravity, eventually radiating away all its energy and momentum to leave a remnant flat spacetime, and whose spacelike cross-sections are identical (to leading order in the string’s mass per unit length $\mu$) to those of the Allen–Casper–Ottewill loop.
From a purely physical viewpoint, the ACO loop is one of the most important flat-space cosmic string solutions, as it is long-lived and approximates well the lowest efficiency daughter loops observed in low-resolution string network simulations [18, 19]10. The fact that its evolution and ultimate evaporation are analytically tractable is an added bonus and makes the ACO loop an object of potentially great interest to mathematical relativity.

However, much work needs to be done before it could be claimed with any confidence that the evaporating ACO loop can be described by a self-consistent solution of the Einstein equations. A first step in this direction would be to construct a solution to the strong-field back-reaction problem that reduces to the bare ACO loop in the limit $\mu \to 0$, or at the very least, to demonstrate that such a solution exists. A second desideratum would be to find a solution of the Abelian Higgs field equations that can act as a suitable source for the evaporating ACO metric. Neither task is likely to be easy.

One question that arises naturally in this context is whether the self-similarity of the linearized solution would be preserved in the corresponding fully nonlinear solution (if indeed such a solution exists). This question remains an open one, but it can be given a more rigorous formulation as follows. If the original Minkowski coordinates $x^a = [t, \mathbf{x}]^a$ describing the linearized solution are written in terms of the similarity coordinates $[\psi, \bar{\mathbf{x}}]$, where $\psi$ and $\bar{\mathbf{x}}$ are defined by equations (5.12) and (5.13) respectively, then it can be shown that the vector field $k^a \equiv \partial x^a / \partial \psi = [0, -y, x, 0]^a - \kappa \mu [t - t_L, x, y, z]^a$ (9.1) satisfies the equation

\[
(\eta^{cd}(h_{cd} + \dot{h}_{cd})) \partial_j k^c + \frac{1}{2} k^c \partial_i h_{ab} = -\kappa \mu \eta_{ab} \tag{9.2}
\]

to linear order in $\mu$. That is, $k^a$ satisfies the linearized version of the equation $\nabla_a k_b = -\kappa \mu g_{ab}$ for a conformal Killing vector field.

The corresponding nonlinear solution will, therefore, preserve self-similarity if it admits a vector field $k^a$, satisfying the exact conformal Killing equation $\nabla_a k_b = -\kappa \mu g_{ab}$, in which case it is always possible to find coordinates $[\psi, \bar{\mathbf{x}}]$ in terms of which the metric tensor has the conformal form

\[
g_{ab}(\psi, \bar{\mathbf{x}}) = e^{-2\kappa \mu \psi} p_{ab}(\bar{\mathbf{x}}). \tag{9.3}
\]

(The conformal Killing field in this case is $k^a = \delta^a_\psi$.) So the question of whether self-similarity would be preserved by a fully nonlinear solution can be recast as a question about the existence of solutions to the strong-field back-reaction problem with the conformal form (9.3).

Another of the more problematic aspects of the ACO loop is the behavior of the gravitational field near the two kink points. The discontinuity in the null tangent vector $X_{(0)}^a$ at the kinks generates two apparent anomalies in the weak-field analysis described in section 6. The first is that the term $X_{(0),CD}^a$ appearing in the first-order equation of motion (6.28) is undefined at the kinks, and so it cannot meaningfully be said that the equation of motion is satisfied at these points. The second anomaly is that the metric perturbations $h_{ab}$

10 More recent high-resolution simulations indicate that string loops, when they first form in an evolved string network, have an approximately fractal structure, with a fractal dimension approaching 2 on large scales [28–30]. The ACO loop, therefore, approximates realistic incipient string loops less closely than was first believed. Nonetheless, it remains true that the ACO loop has the lowest known radiative efficiency of any string loop. Furthermore, as was explained in section 8, there are good theoretical reasons for believing that radiative back reaction acts to preferentially eliminate a small-scale structure from a loop on timescales much shorter than the lifetime of the loop (although the principal decay channel for the structure on the very smallest scales would probably be particle production [28]). Hence, it seems likely that any fractal loop similar in shape to an ACO loop on large scales would radiate away much of its small-scale structure and quickly come to resemble the ACO loop on all scales. It has also been suggested [26, 27] that loop fragmentation will quickly act to suppress a fractal structure, by triggering a cascade of daughter loops with progressively fewer cusps and kinks.
fail to be differentiable on a manifold of points extending from the trajectories of the kinks out to future null infinity, as is explained in more detail in appendix A.3.

The breakdown in the differentiability of $h_{ab}$ poses no intractable difficulty for the linearized analysis of section 6, because the ill-defined terms make no overall contribution to the *linearized* Einstein tensor. However, the fully nonlinear Einstein tensor contains products of terms of the form $\delta g_{ab}$, and in the generic case it is expected that a distributional interpretation of the Einstein equation is possible only if the metric derivatives $\partial_{c}g_{ab}$ are locally square integrable [2]. So the breakdown of differentiability in the beaming directions of the kinks may bode ill for the prospects of a strong-field solution.

A further problem is that the non-differentiability of $h_{ab}$ is superimposed on the conical singularity marking the string world sheet at the kink points themselves. The singularity in $h_{ab}$ at the kinks was examined briefly in Paper I, and is far more pathological than the simple logarithmic singularity that appears at all other points on the string world sheet. An appreciation for the nature of the singularity can be gained by repeating the analysis performed earlier, in section 6.2, to calculate the near-field metric perturbations at a kink point.

If the field point $x^{a} = \frac{1}{2\pi} L_{0}[T, X, Y, Z]^{a}$ is chosen to be close to the lower of the kink points (at $u = 0$) near the spacelike surface $t = 0$, so that

$$[T, X, Y, Z]^{a} = \left[ v, \cos v, \sin v, -\frac{1}{2}\pi \right] + 4\pi L_{0}^{-1}\delta x^{a}$$

for some $v \in (-\pi, \pi)$, then it can be shown that $\chi_{\pm} = v + 4\pi L_{0}^{-1}(\delta t \pm \delta z)$, and that

$$\chi_{+} - V_{1} = 2\pi \left( 4\pi L_{0}^{-1} \right)^{2} Q_{-}, \quad \chi_{-} - V_{1} = 2\pi - 8\pi L_{0}^{-1}\delta z$$

and

$$\chi_{\pm} - V_{0} = 4\pi L_{0}^{-1} \left( \frac{\eta_{ab}\delta x^{a}\delta x^{b}}{\delta \pi \pm \delta x \sin v - \delta y \cos v} \right)$$

(9.5)

to first or leading order in $\delta x^{a}$, where

$$Q_{-} = (\delta x)^{2} + (\delta y)^{2} + (\delta t + \delta z)(\delta t + \delta z + 2\delta x \sin v - 2\delta y \cos v) = \Psi_{ab}(1, v)\delta x^{a}\delta x^{b}.$$  

(9.6)

All four of the functions in (9.5) appear as arguments of the logarithmic and sine and cosine integral functions contributing to the terms $I_{\pm}, S_{\pm},$ and $C_{\pm}$ in (5.37). For points on the future light cone of the kink, where $\eta_{ab}\delta x^{a}\delta x^{b} = 0$, expansions (9.5) simplify considerably, as $\chi_{\pm} - V_{0}$ then becomes $4\pi L_{0}^{-1}(\delta t \pm \delta z)$, while $Q_{-} = 2(\delta t + \delta z)(\delta t + \delta z + 2\delta x \sin v - 2\delta y \cos v)$. (Furthermore, the function $\delta t + \delta x \sin v - \delta y \cos v$ appearing in $Q_{-}$ is zero if and only if the field point $x^{a} + \delta x^{a}$ lies in the beaming direction of the kink; see appendix A.3.)

However, for the purposes of calculating the near-field derivatives $\partial_{a}h_{ab}$, it is necessary to retain the term $\eta_{ab}\delta x^{a}\delta x^{b}$ in full, with the result that many of the derivatives are undefined in the limit as $\delta x^{a} \to 0$. The near-field expansion of $h_{ab}$ appearing in the footnote in section 4.2 of Paper I is as complicated as it is, because the ‘off-shell’ contributions have been included so as to give a hint of the complexity of the derivatives.

It should be clear from the foregoing remarks that the breakdown of the field equations at the kinks is not strictly speaking a strong-field effect, but stems rather from the zero-thickness assumption underpinning the Nambu–Goto action, which imposes an infinite world sheet curvature at the kinks. It seems unlikely that the singularity at the kinks would be resolved in a fully nonlinear solution of the Einstein equations, and I suspect that the kinks can be smoothed only by the inclusion of a finite-thickness field-theoretic source. However, the fact—mentioned in the footnote in section 8—that the Newtonian gravitational potential
the vector \((X, Y, Z)\) corresponding to the spatial components \(\mathbf{x} = \frac{1}{4\pi} L_0(X, Y, Z)\) of the image point is represented in standard spherical polar coordinates

\[
(X, Y, Z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta),
\]

then since \(T = r\) equation (5.33) for the roots \(V_k\) specialized to \(V_1\) reads

\[
V_1 + \pi = r - \left[ (r \sin \theta \cos \phi - \cos V_1)^2 + (r \sin \theta \sin \phi - \sin V_1)^2 + (r \cos \theta - \frac{1}{4\pi})^2 \right]^{1/2}. \tag{A.2}
\]

Differentiating the expression inside the square root with respect to \(V_1\) indicates that \(V_1 + \pi\) takes on an extremal value whenever \(\sin(V_1 - \phi) = 0\) unless \(r \sin \theta = 0\), in which case equation (A.2) solves explicitly to give

\[
V_1 + \pi = r - \left[ 1 + (r \mp \frac{1}{4\pi})^2 \right]^{1/2}, \tag{A.3}
\]

with the sign choice corresponding to \(\cos \theta = \pm 1\) if \(r \neq 0\). The case \(r \sin \theta = 0\) is, of course, equivalent to \(X = Y = 0\) and is discussed in more detail at the end of section 5.2. It is easily verified that, in this case, \(V_1 + \pi\) is a strictly monotonically increasing function of \(r\), and so lies in the range \([- (1 + \frac{1}{4\pi^2})^{1/2}, \frac{1}{2}\pi]\) if \(\cos \theta = +1\) and in \([- (1 + \frac{1}{4\pi^2})^{1/2}, \frac{1}{2}\pi]\) if \(\cos \theta = -1\).

If \(\sin(V_1 - \phi) \neq 0\), then the right-hand side of (A.2) is extremized if either \((\cos V_1, \sin V_1) = (\cos \phi, \sin \phi)\) or \((\cos V_1, \sin V_1) = (- \cos \phi, - \sin \phi)\). It follows from (A.2) that

\[
r - \left[ (r \sin \theta + 1)^2 + (r \cos \theta - \frac{1}{4\pi})^2 \right]^{1/2} \leq V_1 + \pi \leq r
\]

\[
- \left[ (r \sin \theta - 1)^2 + (r \cos \theta - \frac{1}{4\pi})^2 \right]^{1/2}. \tag{A.3}
\]

Again, differentiation with respect to \(r\) of the lower and upper bounding functions in (A.3) is sufficient to show that both are monotonically increasing functions of \(r\) and furthermore the left-hand function is strictly monotonic unless \(\tan \theta = -2/\pi\), while the right-hand function is strictly monotonic unless \(\tan \theta = 2/\pi\). From this fact, it is readily seen that

\[
- (1 + \frac{1}{4\pi^2})^{1/2} \leq V_1 + \pi \leq (1 + \frac{1}{4\pi^2})^{1/2} \tag{A.4}
\]

the lower bound being achieved when \(r = 0\) or \(\tan \theta = -2/\pi\), and the upper bound when \(\tan \theta = 2/\pi\) and \(r \geq (1 + \frac{1}{4\pi^2})^{1/2}\). Since the root \(V_0\) lies in the interval \([V_1, V_1 + 2\pi]\), it is evident that \(V_0\) is also bounded:

\[
- (1 + \frac{1}{4\pi^2})^{1/2} - \pi \leq V_0 \leq (1 + \frac{1}{4\pi^2})^{1/2} + \pi \approx 5.00369. \tag{A.5}
\]
The fact that \( V_0 \) and \( V_1 \) are bounded should come as no surprise, as \( V_0 \) and \( V_1 + \pi \) are the values of the scaled time coordinate \( \bar{T} \) at the points of intersection of the backwards light cone of the image point \([\bar{x}], \bar{y}]\) with the trajectories of the lower \((u = 0)\) and upper \((u = \pm \pi)\) kink points respectively.

**A.2. The zeroes of** \( T = s(Z + \frac{1}{2}\pi) - v \)**

According to (5.34), the function \( v(u) \) describing the curve of intersection \( \Gamma \) of the world sheet with the backwards light cone of \([\bar{x}], \bar{y}]\) \( = \frac{1}{2\pi} L_0[\bar{T}, \bar{X}, \bar{Y}, \bar{Z}] \) has a possible stationary point whenever \( E = T - s(Z + \frac{1}{2}\pi) - v = 0 \). (Recall here that \( s \equiv \text{sgn}(u) \), or equivalently \( s \) is the sign of \( \bar{z} \cdot \bar{X}_u \) on the segment of the string loop at which the intersection point occurs.) It is important to be able to locate those points \([\bar{x}], \bar{y}]\) on the surface \( F_0 \) at which \( E \) goes to zero for some value of \( v \) in \([V_1, V_1 + 2\pi]\), because \( E \) appears as the denominator in the integrals contributing to \( h_{ab}(\bar{x}, \bar{y}) \) in (5.36), and so the components of \( h_{ab} \) are potentially divergent at such a point.

As mentioned in section 5.2, \( E \) is positive semi-definite by virtue of equation (5.31), which can be rewritten in the form
\[
T - s(Z + \frac{1}{2}\pi) - v = u - s(Z + \frac{1}{2}\pi) + [(X - \cos v)^2 + (Y - \sin v)^2 + [Z - (|u| - \frac{1}{2}\pi)]^2]^{1/2} \geq u - s(Z + \frac{1}{2}\pi) + [Z - (|u| - \frac{1}{2}\pi)],
\]
(A.6)
Here, equality occurs if and only if \( X = \cos v \) and \( Y = \sin v \), while the function on the third line is zero if and only if \( u \leq s(Z + \frac{1}{2}\pi) \).

Suppose, therefore, that the point \((X, Y)\) lies on the unit circle, so that \((X, Y) = (\cos v \bar{v}, \sin v \bar{v})\) for some angle \( v \bar{v} \). Then \( T - s(Z + \frac{1}{2}\pi) - v \bar{v} \) will be zero whenever
\[
Z = \lambda - \frac{1}{2}\pi \quad \text{and} \quad T = v \bar{v} + s \lambda,
\]
(A.7)
where \( \lambda \) is any number satisfying \( \lambda \geq u \) if \( u > 0 \), or \( \lambda \leq -u \) if \( u < 0 \). (Note that if the null constraint \( T = (X^2 + Y^2 + Z^2)^{1/2} \) is also imposed then \( v \bar{v} = \left[1 + \left(\lambda - \frac{1}{2}\pi\right)^2\right]^{1/2} - s \lambda \), and for each value of \( s \) there is a one-parameter family of spacetime points on \( F_0 \) for which \( E = 0 \) at some point on \( \Gamma \).)

If \( 0 \leq \lambda \leq \pi \), then the spacetime point \( \frac{1}{2\pi} L_0[\bar{T}, \bar{X}, \bar{Y}, \bar{Z}] \) coincides with the point \( X^\alpha(u, \bar{v}) \equiv X^\alpha(s \lambda, \bar{v}) \) on the world sheet. It follows from the results of the previous paragraph that \( E \) will be zero for all values of \( u \) in the range \([0, \lambda]\) if \( s = 1 \), and all values of \( u \) in the range \([\lambda, \pi]\) if \( s = -1 \). This, in turn, means that the curve \( \Gamma \) includes a horizontal segment with \( v \equiv v \bar{v} \) over the ranges of \( u \) indicated.

More important for the purposes of calculating \( h_{ab}(\bar{x}, \bar{y}) \) are image points that lie off the world sheet. These have \(|Z| > \frac{1}{2}\pi\), and so either \( \lambda > \pi \) if \( s = 1 \) or \( \lambda < 0 \) if \( s = -1 \). In geometric terms, such points either lie on the upwards \((Z > \frac{1}{2}\pi)\) extension of the helical segment of the ACO loop with \( u > 0 \), or on the downwards \((Z < -\frac{1}{2}\pi)\) extension of the helical segment with \( u < 0 \). Furthermore, it is evident from the preceding analysis that in this case \( E \) will be zero on the curve \( \Gamma \) for all values of \( u \in [0, \pi] \) if \( s = 1 \), and all values of \( u \) in \([-\pi, 0]\) if \( s = -1 \), and so \( \Gamma \) includes a straight-line segment with \( v \equiv v \bar{v} \) over one of these ranges. An immediate consequence is that \( V_0 = V_1 \equiv v \bar{v} \) if \( s = 1 \), and \( V_0 = V_1 + 2\pi \equiv v \bar{v} \) if \( s = -1 \). (This straight-line segment corresponds to the 'trivial solution' \( V_k = v \) mentioned in section 6.1.)

It is evident now that if the image point \([\bar{x}], \bar{y}]\) lies on one of the two helical extensions of the loop, but not on the loop itself, then the corresponding integral in equation (5.36) for
The problematic integrals can be reformulated by making use of equation (5.34) for $du/du$ to replace $v$ with $u$ as the variable of integration. Furthermore, since $(X, Y) = (\cos v_F, \sin v_F)$, the denominator $T - (u + v) + X \sin v - Y \cos v$ becomes simply $T - u - v_F$, where $v_F = \chi_\pm \equiv T \pm (Z + 2\pi)$ if $s = \mp 1$. Hence,

\[
\int_{V_0}^{V_1+2\pi} \left[ T + \left( Z + \frac{1}{2} \pi \right) - v \right]^{-1} \Psi_{ab}(-1, v) \, dv \quad \text{is replaced by} \quad \int_{-\pi}^{0} \left[ - \left( Z + \frac{1}{2} \pi \right) - u \right]^{-1} \Psi_{ab}(-1, \chi_+) \, du
\]

and

\[
\int_{V_1}^{V_0} \left[ T - \left( Z + \frac{1}{2} \pi \right) - v \right]^{-1} \Psi_{ab}(+1, v) \, dv \quad \text{is replaced by} \quad \int_{0}^{\pi} \left( Z + \frac{1}{2} \pi - u \right)^{-1} \Psi_{ab}(+1, \chi_-) \, du.
\]

The new integrals are easily evaluated, giving

\[ I_s = -\ln \left[ \left( \frac{1}{2} \pi - Z \right) / \left( -\frac{1}{2} \pi - Z \right) \right], \quad S_s = I_s \sin(\chi_s), \quad C_s = I_s \cos(\chi_s) \quad (A.8) \]

(if $s = -1$) and

\[ I_s = -\ln \left[ \left( Z + \frac{1}{2} \pi \right) / \left( Z - \frac{1}{2} \pi \right) \right], \quad S_s = I_s \sin(\chi_s), \quad C_s = I_s \cos(\chi_s) \quad (A.9) \]

(if $s = 1$) in place of the corresponding terms in (5.38)–(5.40).

A.3. The zeroes of $T - (u + v) + X \sin v - Y \cos v$

A second consequence of (5.34) is that the graph of $v$ against $u$ on the intersection curve $\Gamma$ typically has a point of vertical inflection whenever $H \equiv T - (u + v) + X \sin v - Y \cos v = 0$. As was indicated in section 5.2, the factor $H$ is positive semi-definite, because

\[
T - (u + v) + X \sin v - Y \cos v = \left[ (X \sin v - Y \cos v)^2 + (X \cos v + Y \sin v - 1)^2 \right]^{1/2} + X \sin v - Y \cos v
\]

\[
\geq |X \sin v - Y \cos v| + X \sin v - Y \cos v,
\]

with equality occurring if and only if $X \cos v + Y \sin v = 1$ and $Z = |u| - \frac{1}{2} \pi$. In particular, it follows that spacetime points with $H = 0$ somewhere on $\Gamma$ have $|Z| \leq \frac{1}{2} \pi$. It was seen earlier that spacetime points outside the world sheet with $E = 0$ somewhere on $\Gamma$ have $|Z| > \frac{1}{2} \pi$, so the two classes of points are disjoint outside the world sheet.

Suppose now that $H = 0$ at the point $(u, v) = (v_F, v_F)$ on $\Gamma$. The equations $H = 0$ and $X \cos v + Y \sin v = 1$ (with $X \sin v - Y \cos v \leq 0$) can then be solved exactly to give

\[ T = u_F + v_F + (R^2 - 1)^{1/2}, \quad X = \cos v_F = (R^2 - 1)^{1/2} \sin v_F \quad \text{and} \]

\[ Y = \sin v_F + (R^2 - 1)^{1/2} \cos v_F, \]

while, of course, $Z = |v_F| - \frac{1}{2} \pi$. The value of $R \geq 1$ is fixed by the null constraint $T = (X^2 + Y^2 + Z^2)^{1/2}$, which reads explicitly

\[ u_F + v_F = \left[ R^2 + \left( |v_F| - \frac{1}{2} \pi \right)^2 \right]^{1/2} - (R^2 - 1)^{1/2}. \]

So there exists a two-parameter family of spacetime points with the property that $H = 0$ somewhere on $\Gamma$. 

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\[ v(\mu_0 + \delta \mu) - v_0 = -(6\delta \mu)^{1/3} \]  
\((A.13)\)
to leading order in \(\delta \mu\). Hence, \(v\) remains a monotonic decreasing function of \(\mu\) even in the singular case where \(H = 0\) somewhere on \(\Gamma\).

The importance of the zeroes of \(H\) lies in the fact that the spacetime derivatives \(\partial_v h_{ab}\) of the metric perturbations \(h_{ab}(\vec{t}, \vec{x})\) are potentially divergent at any image point \([\vec{t}, \vec{x}] \equiv \frac{1}{\pi} L_0[T, X, Y, Z]\) with the property that \(H = 0\) when \(v = V_0\) or \(V_1\). This can be seen by first differentiating equation (5.36) for \(h_{ab}\) and then integrating by parts to give

\[
\partial_v h_{ab} = -4\mu(\partial_v \chi_+ + \partial_v \chi_-) \int_{V_0 = 2\pi}^{V_0 = 2\pi} (\chi_+ - v)^{-1} \Omega_{ab}(-1, v) \, dv \]
\[ - 4\mu(\partial_v \chi_+ - \chi_+ + 2\pi) \Omega_{ab}(-1, v) \, dv 
+ 4\mu(\partial_v \chi_+ - \chi_+ + 2\pi) \Omega_{ab}(1, v) \, dv 
- 4\mu(\partial_v \chi_+ - \chi_+ - 2\pi) \Omega_{ab}(1, v) \, dv 
+ 4\mu(\partial_v \chi_+ - \chi_+ + 2\pi) \Omega_{ab}(1, v) \, dv 
- 4\mu(\partial_v \chi_+ - \chi_+ + 2\pi) \Omega_{ab}(1, v) \, dv 
+ 4\mu(\partial_v \chi_+ - \chi_+ + 2\pi) \Omega_{ab}(1, v) \, dv

where again \(\chi_+ = T + (Z + \frac{1}{2})\) and \(\Omega_{ab}(s, v) = \partial_v \Psi_{ab}(s, v)\) as defined in section 6.1.

The two integrals on the right-hand side of (A.14) converge because all the components of \(\Omega_{ab}\) are identical to components of \(\Psi_{ab}\), and it was demonstrated in the previous subsection that integrals of this type will converge unless the field point lies on the world sheet itself. Also, the derivatives \(\partial_v \Psi_{ab}\) of \(\Psi_{ab}\) are everywhere well defined. However, the terms involving \(\partial_v V_1\) and \(\partial_v V_0\) in (A.14) are potentially problematic. The implicit differentiation of equation (5.28) defining \(v\) gives

\[
\partial_v v = [T - (u + v) + X \sin v - Y \cos v]^{-1} 4\pi L_0^{-1} [T - (u + v), -(X - \cos v), -(Y - \sin v), -Z - (|u| - \frac{1}{2})\pi],
\]
\((A.15)\)
where the denominator is, of course, just \(H\). The derivative \(\partial_v V_k\), therefore, typically diverges if \(H = 0\) at \(v = V_k\) (and in fact the components of the 4-vector multiplying \(H^{-1}\) in (A.15) will all vanish only if the point \(\frac{1}{\pi} L_0[T, X, Y, Z]\) lies on the world sheet).\(^{11}\)

In fact, if \(k = 0\) or 1, and the image point \(\vec{x}^a = \frac{1}{\pi} L_0[T, X, Y, Z]^a\) has the property that \(X \cos V_k + Y \sin V_k = 1\), \(T = k\pi + V_k - X \sin V_k + Y \cos V_k\) and \(Z = (|k| - \frac{1}{2})\pi\),
\((A.16)\)

\(\text{with} \ X \sin V_k - Y \cos V_k \leq 0\) then the perturbative solution of equation (5.33) for \(V_k\) at a nearby point \(\vec{x}^a + \delta \vec{x}^a\) reads

\[
V_k(\vec{x}^a + \delta \vec{x}^a) - V_k(\vec{x}^0) = 6^{1/3}(\delta T + \delta X \sin V_k - \delta Y \cos V_k)^{1/3}
+ \frac{1}{2}(X \sin V_k - Y \cos V_k)^{-1} 6^{1/3}(\delta T + \delta X \sin V_k - \delta Y \cos V_k)^{1/3},
\]
\((A.17)\)
plus terms linear in \(\delta \vec{x}^a\).

\(^{11}\) It might be supposed from (A.14) that \(\partial_v h_{ab}\) is also undefined whenever \(\chi_+ = V_1\), with \(V = V_1, V_0\) or \(V_{-1}\). That this is not so (at least for image points outside the world sheet) can be seen by replacing \(v\) with \(u\) as the integration variable in equation (5.36) for \(h_{ab}\) to give

\[
\int_{-\infty}^{\infty} H^{-1} \Psi_{ab}(-1, v) \, dv - 4\mu \int_{-\infty}^{\infty} H^{-1} \Psi_{ab}(1, v) \, dv.
\]
It is clear that \(\partial_v h_{ab}\) involves only terms of the form \(\Omega_{ab} \partial_v v\) and \(H^{-1} \partial_v H\), where \(\partial_v H\) is a linear combination of \(\partial_v T, \partial_v X, \partial_v Y\) and \(\partial_v v\), and according to (A.15) \(\partial_v v\) is undefined only when \(H = 0\).
Substitution of expansion (A.17) into equation (5.36) for $h_{ab}$ then gives

$$h_{1z}(\bar{x}^a + \delta \bar{x}^a) - h_{1z}(\bar{x}^a) = -8\tilde{\delta} \mu \left( \delta_k^{1/3} + \frac{1}{4} \delta_k^{2/3} \right) + O(\delta \bar{x}),$$  \hspace{1cm} (A.18)

$$h_{1z}(\bar{x}^a + \delta \bar{x}^a) - h_{1z}(\bar{x}^a) = -8\tilde{\delta} \mu \left[ \delta_k^{1/3} \sin V_k + \frac{1}{4} \delta_k^{2/3} \left( \sin V_k + 2D_k \cos V_k \right) \right] + O(\delta \bar{x}),$$  \hspace{1cm} (A.19)

and

$$h_{1z}(\bar{x}^a + \delta \bar{x}^a) - h_{1z}(\bar{x}^a) = 8\tilde{\delta} \mu \left[ \delta_k^{1/3} \cos V_k + \frac{1}{4} \delta_k^{2/3} \left( \cos V_k - 2D_k \sin V_k \right) \right] + O(\delta \bar{x}).$$  \hspace{1cm} (A.20)

where

$$\delta_k \equiv 6D_k^{-3} (\delta T + \delta X \sin \bar{V}_k - \delta Y \cos \bar{V}_k),$$  \hspace{1cm} (A.21)

$$D_k \equiv -(X \cos \bar{V}_k - Y \sin \bar{V}_k) \geq 0$$  \hspace{1cm} (A.22)

and $\tilde{\delta} \equiv \text{sgn}(Z) \equiv \text{sgn}(k - \frac{1}{2})$, while all other components of $h_{ab}(\bar{x}^a + \delta \bar{x}^a) - h_{ab}(\bar{x}^a)$ are of linear order or higher in the components of $\delta \bar{x}$.\footnote{Incidentally, the factor $D_k^{-3}$ appearing in the equation for $\delta_k$ is guaranteed to be well defined if the field point lies away from the string world sheet, because if $X \sin \bar{V}_k - Y \cos \bar{V}_k = 0$, then the previous constraints (A.16) imply that $X = \cos \bar{V}_k, \quad Y = \sin \bar{V}_k, \quad T = k\pi + \frac{1}{2} \pi $ and so the field point is identical to the point $(u, v) = (k\pi, \bar{V}_k)$ on the world sheet.}

In summary, if the field point $\bar{x}^a$ satisfies constraints (A.16) for either $k = 0$ or $k = 1$ then the derivatives of the metric components $h_{1z}$, $h_{zz}$ and $h_{zz}$ with respect to $\bar{t}$, $\bar{x}$ and $\bar{y}$ are all undefined there. Field points of this type have $\tilde{\delta} = \pm \frac{1}{2} L_0$ and so have the same altitude as one of the kink points on the string. The failure of $h_{ab}$ to be differentiable at such points, therefore, has no effect on the analysis of the first-order string equation of motion in section 6, as the equation of motion is there developed and solved only for points on the string away from the kinks. However, the singularities in the derivatives of $h_{ab}$ do potentially affect the matching of the weak-field solution to the Minkowski vacuum across the null surface $F_L$. Fortunately, the relevant combinations of the components of $h_{ab}$ are all differentiable on $F_L$, as is shown in detail in section 7.2.

It might seem puzzling that the weak-field metric induced by the ACO loop is singular on a manifold of points that extends away from the world sheet. However, this pathological behavior has a simple physical explanation. It is well known ([23], section 6.3) that a generic kink on a zero-thickness string emits a narrow beam of gravitational radiation in the direction of its motion. Although the metric derivative is singular on the locus of the beam, the singularity is nonetheless relatively benign, as the geodesic equations for a free particle remain integrable across the beam.

In the case of the ACO loop, the kink points correspond to $u = k\pi$ for $k = 0$ or 1 and so trace out the two families of spacetime points

$$X^a_a(0) = \frac{1}{4\pi} L_0 \left[ k\pi + v, \cos v, \sin v, \left( |k| - \frac{1}{2} \right) \pi \right]^a.$$  \hspace{1cm} (A.23)

The instantaneous 4-velocities of the kink points are, therefore,

$$\partial_v X^a_a(0) = \frac{1}{4\pi} L_0 [1, -\sin v, \cos v, 0]^a,$$  \hspace{1cm} (A.24)

and the beams, which trace out null curves with tangent vector $\partial_v X^a_a(0)$, can be represented parametrically in the form

$$[T, X, Y, Z] = [k\pi + v, \cos v, \sin v, (|k| - \frac{1}{2}) \pi] + \lambda [1, -\sin v, \cos v, 0],$$  \hspace{1cm} (A.25)
with \( \lambda \) an affine parameter. Hence \( Z = (|k| - \frac{1}{2}) \pi \) and, after eliminating \( \lambda \),
\[
T - (k \pi + v) + X \sin v - Y \cos v = 0.
\] (A.26)

The last equation is just the constraint \( H = 0 \) with \( u = k \pi \). So a spacetime point lying off the world sheet satisfies constraints (A.16) if and only if it lies in the beaming direction of a kink.

**Appendix B. Calculating \( h_{ab} \) and its derivatives**

With \( \chi_{\pm} \equiv v + (u \pm u) + 4\pi L_0^{-1} (\delta t \pm \delta z) \) and
\[
\begin{align*}
\chi_+ - V_0 &= 2u + 8\pi L_0^{-1} \delta z, \\
\chi_+ - V_1 &= W + 2u + 4\pi L_0^{-1} (\delta t + \delta z - \delta V) \\
\chi_- - V_0 &= \frac{1}{2} u^{-1} (4\pi L_0^{-1})^2 Q_+ \\
\chi_- - V_1 &= W + 4\pi L_0^{-1} (\delta t - \delta z - \delta V),
\end{align*}
\] (B.1)

where \( \delta V \) is of linear order in \( \delta x^a \) and is defined in (6.13), while \( Q_+ \) is of quadratic order and is defined in (6.15), expressions (5.38)–(5.40) become, for \( u > 0 \) and \( \delta x^a \) small,
\[
\begin{align*}
I_+ &= \ln (W + 2u - 2\pi) - \ln (2u) + 4\pi L_0^{-1} \left( \frac{\delta t + \delta z - \delta V}{W + 2u - 2\pi} - \frac{\delta z}{u} \right), \\
I_- &= \ln \left( \left(4\pi L_0^{-1}\right)^2 Q_+ \right) - \ln (2u W) - 4\pi L_0^{-1} \left( \frac{\delta t - \delta z - \delta V}{W} \right).
\end{align*}
\] (B.2)

\[
\begin{align*}
\left[ S_+ \right] &= \left[ -\cos(2u + v) \quad \sin(2u + v) \right] \left[ \frac{\text{Si}(W + 2u - 2\pi) - \text{Si}(2u)}{\text{Ci}(W + 2u - 2\pi) - \text{Ci}(2u)} \right] \\
&\quad + 4\pi L_0^{-1} \left[ \frac{\sin(2u + v)}{\cos(2u + v)} \quad \frac{\text{Ci}(W + 2u - 2\pi) - \text{Ci}(2u)}{\text{Si}(W + 2u - 2\pi) - \text{Si}(2u)} \right] \left( \delta t + \delta z \right) \\
&\quad - 4\pi L_0^{-1} u^{-1} \left[ \frac{\sin v}{\cos v} \right] \delta z + 4\pi L_0^{-1} (W + 2u - 2\pi)^{-1} \left( \frac{\sin(v - W)}{\cos(v - W)} \right) \left( \delta t + \delta z - \delta V \right)
\end{align*}
\] (B.4)

and
\[
\begin{align*}
\left[ S_- \right] &= \left[ \ln \left( \left(4\pi L_0^{-1}\right)^2 Q_+ \right) + \gamma_E - \ln (2u) \right] \left[ \frac{\sin v}{\cos v} \right] - \left[ \frac{-\cos v}{\sin v} \quad \frac{\sin v}{\cos v} \right] \left[ \frac{\text{Si} W}{\text{Ci} W} \right] \\
&\quad + 4\pi L_0^{-1} \left[ \ln \left( \left(4\pi L_0^{-1}\right)^2 Q_+ \right) + \gamma_E - \ln (2u) \right] \left[ \frac{\cos v}{-\sin v} \right] \left( \delta t - \delta z \right) \\
&\quad - 4\pi L_0^{-1} W^{-1} \left[ \frac{\sin(v - W)}{\cos(v - W)} \right] \left( \delta t - \delta z - \delta V \right)
\end{align*}
\] (B.5)

Note here that \( \text{Ci} \cdot x = \ln x + \gamma_E + O(x^2) \) for small values of \( x \), where \( \gamma_E \approx 0.5772 \) is Euler’s constant, while \( \text{Si} \cdot x = x + O(x^3) \).

If the terms which tend to zero as \( \delta x^a \to 0 \) in (B.2)–(B.5) are neglected, and the resulting leading-order expansions substituted into (5.37), the metric perturbations \( h_{ab} \) take on the form shown in equation (6.16).

Also, given that
\[
\begin{align*}
\partial(\delta V)/\partial(\delta x^a) &= (u - \pi + W - \sin W)^{-1} [u - \pi + W, \{\cos(v - W) - \cos v\}] \hat{x} \\
&\quad + [\sin(v - W) - \sin v] \hat{y} + (\pi - u) \hat{z}
\end{align*}
\] (B.6)
the terms in the small-distance expansions of the derivatives of $I_\pm, S_\pm$ and $C_\pm$ which do not vanish in the limit as $\delta x^a \to 0$ can be read off directly from equations (B.2)–(B.5):

$$\partial_0 I_+ = 4\pi \frac{1}{L^0} \frac{1}{W(u - \pi + W - \sin W)} D_a - 4\pi \frac{1}{L^0} u^{-1} [0, \hat{z}]_a$$  \hspace{1cm} (B.7)

$$\partial_0 I_- = \partial_0 \Lambda + 4\pi \frac{1}{L^0} \frac{1}{W(u - \pi + W - \sin W)} E_a$$  \hspace{1cm} (B.8)

$$\begin{bmatrix} \partial_a S_+ \\ \partial_a C_+ \end{bmatrix} = 4\pi \frac{1}{L^0} \left[ \sin(2u + v) \cos(2u + v) \right] \begin{bmatrix} \sin(W + 2\pi - 2\pi) - \sin(2u) \\ \cos(W + 2\pi - 2\pi) - \cos(2u) \end{bmatrix} [1, \hat{z}]_a$$

$$- 4\pi \frac{1}{L^0} u^{-1} \left[ \begin{array}{c} \sin v \\ \cos v \end{array} \right] [0, \hat{z}]_a + 4\pi \frac{1}{L^0} \frac{1}{W(u - \pi + W - \sin W)}$$

$$\times \left[ \begin{array}{c} \sin(v - W) \\ \cos(v - W) \end{array} \right] D_a$$  \hspace{1cm} (B.9)

$$\begin{bmatrix} \partial_a S_- \\ \partial_a C_- \end{bmatrix} = \partial_a \Lambda \left[ \begin{array}{c} \sin v \\ \cos v \end{array} \right] + 4\pi \frac{1}{L^0} \left[ 0 + \gamma_{E} - \ln(2\pi) \right] \begin{bmatrix} \cos v \\ -\sin v \end{bmatrix} [1, -\hat{z}]_a$$

$$+ 4\pi \frac{1}{L^0} \partial_a \Lambda \begin{bmatrix} \cos v \\ -\sin v \end{bmatrix} (\delta t - \delta z) - 4\pi \frac{1}{L^0} \begin{bmatrix} \sin v \\ \cos v \end{bmatrix} \begin{bmatrix} \sin(W + 2\pi - 2\pi) - \sin(2u) \\ \cos(W + 2\pi - 2\pi) - \cos(2u) \end{bmatrix} [1, -\hat{z}]_a$$

$$+ 4\pi \frac{1}{L^0} \frac{1}{W(u - \pi + W - \sin W)} \begin{bmatrix} \sin(v - W) \\ \cos(v - W) \end{bmatrix} E_a.$$  \hspace{1cm} (B.10)

where $\Lambda = \ln \left[ (4\pi \frac{1}{L^0})^2 Q_+ \right]$.  \hspace{1cm} (C.1)

$$D_a \equiv [-\sin W, -\{\cos(v - W) - \cos v\} \hat{x} - \{\sin(v - W) - \sin v\} \hat{y}$$

$$+ (2u - 2\pi + W - \sin W) \hat{z}]_a$$  \hspace{1cm} (B.11)

and $E_a \equiv [\sin W, \{\cos(v - W) - \cos v\} \hat{x} + \{\sin(v - W) - \sin v\} \hat{y} + (W - \sin W) \hat{z}]_a$.  \hspace{1cm} (B.12)

Again, expressions (6.21)–(6.24) for the spacetime derivatives of $h_{ab}$ at points on the null surface $F_0$ and close to the world sheet of the loop follow directly by substituting the leading-order expansions (B.7)–(B.10) into (5.37).

### Appendix C. Deriving the equations of motion

If the position function $X^a$ of the evaporating loop is assumed to have the form (4.20) then

$$X^a_{(0)}(u, v) = \frac{1}{4\pi} L_0[u + v, r(u, v)]^a,$$  \hspace{1cm} (C.1)

with

$$r(u, v) = \cos(v) \hat{x} + \sin(v) \hat{y} + (|u| - \frac{1}{2} \pi) \hat{z},$$  \hspace{1cm} (C.2)

and the first-order perturbation $\delta X^a$ is as given in (4.23), namely

$$\delta X^a(u, v) = -\frac{1}{4\pi} L_0 \kappa \left[ \frac{1}{2} (u + v)^2, (u + v) r(u, v) \right]^a$$

$$+ \frac{1}{4\pi} L_0 \mu \left[ -F, G \{\cos(v) \hat{x} + \sin(v) \hat{y} \} + A \hat{z}]^a. \hspace{1cm} (C.3)$$
The first derivatives of $X^a(0)$ are, therefore,
\[ X^a_{(0),u} = \frac{1}{4\pi} L_0[1, s\hat{z}]^u \] and \[ X^a_{(0),v} = \frac{1}{4\pi} L_0[1, -\sin(v)\hat{x} + \cos(v)\hat{y}]^v, \] (C.4)
with $s = \text{sgn}(u)$, while the only nonzero component of $X^a_{(0), A\hat{B}}$ is\(^\text{13}\):
\[ X^a_{(0), u\hat{v}} = \frac{1}{4\pi} L_0[0, -\cos(v)\hat{x} - \sin(v)\hat{y}]^v. \] (C.5)

Similarly, the first derivatives of $\delta X^a$ are
\[ \delta X^a_{,u} = -\frac{1}{4\pi} L_0\kappa \mu [u + v, \mathbf{r}(u, v) + s(u + v)\hat{z}]^u \]
\[ + \frac{1}{4\pi} L_0\mu [-F_u, G_u[\cos(v)\hat{x} + \sin(v)\hat{y}] + A_u\hat{z}]^u, \] (C.6)
\[ \delta X^a_{,v} = -\frac{1}{4\pi} L_0\kappa \mu [u + v, \mathbf{r}(u, v)]^u \]
\[ + \frac{1}{4\pi} L_0\mu [-F_v, G_v[\cos(v)\hat{x} + \sin(v)\hat{y}] + A_v\hat{z}]^v \] (C.7)
and the only second derivative of $\delta X^a$ contributing to the first-order equation of motion is
\[ \delta X^a_{,uv} = -\frac{1}{4\pi} L_0\kappa \mu [1, s\hat{z}]^u \]
\[ + \frac{1}{4\pi} L_0\mu [G_u - \kappa] [0, -\sin(v)\hat{x} + \cos(v)\hat{y}]^v \]
\[ + \frac{1}{4\pi} L_0\mu [-F_{uv}, G_{uv}[\cos(v)\hat{x} + \sin(v)\hat{y}] + A_{uv}\hat{z}]^v. \] (C.8)

The four-dimensional tangent space at the point $x^a = X^a(u, v)$ is spanned by the vectors $X^a_{(0),u}$ and $X^a_{(0),v}$, plus $M^a(s, v) = [1, -\sin(v)\hat{x} + \cos(v)\hat{y} + s\hat{z}]^a$ and $N^a(v) = [0, \cos(v)\hat{x} + \sin(v)\hat{y}]^a$. (C.9)

In calculating the various contributions to the first-order equation of motion (2.14), it is useful to note that all the terms are transversed with $q_{d}^{\hat{a}}$, where $q_{ab} = -
\Psi_{ab}(s, v)$ is the projection operator orthogonal to the world sheet at $X^a(u, v)$, so that
\[ q_{d}^{\hat{a}}X_{(0),u}^{d} = q_{d}^{\hat{a}}X_{(0),v}^{d} = 0, \quad q_{d}^{\hat{a}}M^d = M^c \quad \text{and} \quad q_{d}^{\hat{a}}N^d = N^c. \] (C.10)

Other helpful identities involving $q_{d}^{\hat{a}}$ include
\[ q_{d}^{\hat{a}}[0, -\sin(v)\hat{x} + \cos(v)\hat{y}]^d = M^c, \quad q_{d}^{\hat{a}}[1, 0]^d = -M^c \quad \text{and} \quad q_{d}^{\hat{a}}[0, s\hat{z}]^d = M^c. \] (C.11)

Thus, recalling that the only nonzero component of $\gamma^{AB}$ (at zeroth order in $\mu$) is $\gamma^{uv} = (\frac{1}{4\pi} L_0)^{-2}$, the first term in (2.14) is
\[ q_{d}^{\hat{a}}\gamma^{CD}\delta X^{d}_{,C\hat{D},uv} = 2\left(\frac{1}{4\pi} L_0\right)^{-2} q_{d}^{\hat{a}}\delta X^{d}_{,uv} \]
\[ = 2\mu \left(\frac{1}{4\pi} L_0\right)^{-1} (F_{uv} + sA_{uv} + G_u - \kappa)M^c + 2\mu \left(\frac{1}{4\pi} L_0\right)^{-1} G_{uv}N^c. \] (C.12)

\(^\text{13}\) The claim that $X^a_{(0),uv} = 0$ is true away from the kink points at $u = 0$ and $u = \pi$, but it can be seen from the expression for $X^a_{(0),uv}$ that the $c$-component of $X^a_{(0),uv}$ contains a distributional term at the kinks. The possibility of pathological behavior at the kinks will be ignored here, as the ramifications of the singularities at the kinks have already been discussed in Paper I. It seems unlikely that the dynamical role of the kinks can be clarified without developing a solution of the Abelian Higgs field equations as a finite-thickness source for the ACO loop.
The second term in (2.14) is identically zero by virtue of the zeroth-order equation of motion, while the third term is

\[
-2q_d^\nu \gamma^{AC} \gamma^{BD} \eta_{ab} X_{(0),A}^u \delta X_{(0),B}^u X_{(0),C,D} = -2 \left( \frac{1}{4\pi L_0} \right)^4 q_d^\nu X_{(0),v}^u \left( \eta_{ab} X_{(0),u}^u \delta X_{(0),a}^a \right)
\]

\[= 2\mu \left( \frac{1}{4\pi L_0} \right)^{-1} \left[ \kappa s \left( |u| - \frac{1}{2}\pi \right) - (F_u + sA_u) \right] N^c. \quad (C.13)\]

The remaining terms in (2.14) are more complicated, as they involve \( h_{ab} \) or its derivatives. However, evaluation of these terms is simplified by making use of the following table, which represents the inner products of the terms in the first column with those in the first row in the form \((k_1, k_2, k_3, k_4)\), which in turn is shorthand for \(k_1 X_{(0),a}^a + k_2 X_{(0),b}^b + k_3 M_a + k_4 N_a\). Also, \(c_2\) and \(s_2\) stand in for \(\cos 2\mu\) and \(\sin 2\mu\) respectively, while \(c_W\) and \(s_W\) are shorthand for \(\cos W\) and \(\sin W\).

| \(X_{(0),a}^a\) | \(X_{(0),v}^v\) |
|---|---|
| \(\Psi_{ab}(s, v)\) | 0 | 0 |
| \(\Psi_{ab}(-s, v)\) | (0, 4, -2, 0) | 0 |
| \(\Phi_{ab}(s, v)\) | 0 | (-2, 0, 1, 0) |
| \(\Phi_{ab}(-s, 2u + v)\) | (-2, 0, 2, 0)c_2 + (0, 0, 0, -2)s_2 | (-2, -2, 3, 0)c_2 + (0, 0, 0, -1)s_2 |
| \(\Omega_{ab}(s, v)\) | 0 | (0, 0, 0, -1) |
| \(\Omega_{ab}(-s, 2u + v)\) | (2, 0, -2, 0)c_2 + (0, 0, 0, -2)s_2 | (2, -2, -3, 0)c_2 + (0, 0, 0, -1)s_2 |
| \(\Pi_{ab}(s)\) | 0 | (2, 0, -1, 0) |
| \(\Pi_{ab}(-s)\) | (2, 4, -2, 0) | (2, 2, -3, 0) |
| \(\Psi_{ab}(s, v - W)\) | 0 | (2, 0, -1, 0)(1 - c_W) + (0, 0, 0, 1)s_W |
| \(\Psi_{ab}(-s, v - W)\) | (2(1 - c_W), 4, -2(2 - c_W), 2s_W) | (2, 2, -3, 0)(1 - c_W) + (0, 0, 0, 1)s_W |

Furthermore, the vectors \(X_{(0),a}^a\) and \(X_{(0),v}^v\) are both null and orthogonal to \(M^a\) and \(N^a\), with \(X_{(0),a}^a = (\frac{1}{4\pi L_0})^2\).

It should, of course, be remembered that the expansions (6.16) and (6.21)–(6.24) were developed for the case \(u > 0\) only, and so \(s = +1\) in the expressions in this table. Because the term involving \(\ln \left[ \left( \frac{4\pi L_0}{\Omega} \right)^3 \Phi_s \right]\) in the equation for \(h_{ab}\) is proportional to \(\Psi_{ab}(+1, v)\), the term containing \(h_{ab} X_{(0),A}^a X_{(0),B}^b\) in (2.14) is free of logarithmic singularities. Similarly, the singular terms proportional to \(\Psi_{ab}(+1, v)\) in (6.21)–(6.24) do not contribute to the term \(X_{(0),B}^b \left( h_{bd,a} - \Psi_{bd,a} \right)\) on the right-hand side of (2.14). However, there remain singular terms proportional to \(\Omega_{ab}(+1, v)\) in the derivatives of \(h_{ab}\) whose contributions to the first-order equation of motion need to be examined more carefully.

Now, the fourth term in the equation of motion is

\[-q_d^\nu \gamma^{AC} \gamma^{BD} h_{ab} X_{(0),A}^a X_{(0),B}^b X_{(0),C,D}^D = -\left( \frac{1}{4\pi L_0} \right)^4 q_d^\nu X_{(0),v}^u \left( h_{ab} X_{(0),u}^u \delta X_{(0),a}^a \right) \quad (C.14)\]

where the only nonzero term in \(h_{ab} X_{(0),u}^u \delta X_{(0),a}^a\) is proportional to the coefficient of \(\Pi_{ab}(-1)\):

\[h_{ab} X_{(0),u}^u \delta X_{(0),a}^a = 16\mu \left( \frac{1}{4\pi L_0} \right)^2 [\ln(W + 2u - 2\pi) - \ln(2u)], \quad (C.15)\]

and so

\[-q_d^\nu \gamma^{AC} \gamma^{BD} h_{ab} X_{(0),A}^a X_{(0),B}^b X_{(0),C,D} = 16\mu \left( \frac{1}{4\pi L_0} \right)^{-1} [\ln(W + 2u - 2\pi) - \ln(2u)] N^c. \quad (C.16)\]
Furthermore, the right-hand expression in (2.14) can be broken into two parts:

\[-q^c d \gamma^{AB} X^a_{(0),A} X^b_{(0),B} \hbar \delta_{bd,a} = - \left( \frac{1}{4\pi L_0} \right)^{-2} q^c d \left( X^a_{(0),u} X^b_{(0),v} + X^a_{(0),v} X^b_{(0),u} \right) \hbar \delta_{bd,a} \]

and

\[\frac{1}{2} q^c d \gamma^{AB} X^a_{(0),A} X^b_{(0),B} \hbar \delta_{ab,d} = \left( \frac{1}{4\pi L_0} \right)^{-2} q^c d X^a_{(0),u} X^b_{(0),v} \hbar \delta_{ab,d} .\]

Here, \( q^c d X^a_{(0),u} X^b_{(0),v} \hbar \delta_{bd,a} \) involves terms of the form

\[-q^c d X^b_{(0),v} \partial_v h_{bd} = -4\mu (Si W) M^c + 4\mu (\delta t - \delta z)(\partial_t, \Lambda) N^c + 4\mu [\Lambda + \gamma_E - Ci W - \ln(2u)] N^c + 4\mu \Sigma(u) [- (3 \cos 2u) M^c + (\sin 2u) N^c]
+ 4\mu K(u) [(3 \sin 2u) M^c + (\cos 2u) N^c]
+ 4\mu sin W \left\{ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c] \right\}
+ 4\mu sin W \left\{ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c] \right\}
- 4\mu sin W \left\{ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c] \right\} .\]

and

\[-q^c d X^b_{(0),v} \partial_v h_{bd} = -4\mu (Si W) M^c + 4\mu (\delta t - \delta z)(\partial_t, \Lambda) N^c - 4\mu [\Lambda + \gamma_E - Ci W
- \ln(2u)] N^c + 4\mu \Sigma(u) [- (3 \cos 2u) M^c + (\sin 2u) N^c]
+ 4\mu K(u) [(3 \sin 2u) M^c + (\cos 2u) N^c]
+ 4\mu W - \sin W \left\{ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c] \right\}
+ 4\mu W - \sin W \left\{ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c] \right\}
- 4\mu W - \sin W \left\{ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c] \right\} .\]

So

\[- \left( \frac{1}{4\pi L_0} \right)^{-2} q^c d X^a_{(0),u} X^b_{(0),v} \hbar \delta_{bd,a} = - \left( \frac{1}{4\pi L_0} \right)^{-1} q^c d X^b_{(0),v} \left( \partial_v h_{bd} + \partial_t h_{bd} \right)\]

\[= 4\mu \left( \frac{1}{4\pi L_0} \right)^{-1} [(\delta t - \delta z)(\partial_t, \Lambda) + (\partial_z, \Lambda)] N^c
+ 2\Sigma(u) [- (3 \cos 2u) M^c + (\sin 2u) N^c]
+ 2K(u) [(3 \sin 2u) M^c + (\cos 2u) N^c]
+ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c]
+ \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c]
- \frac{1}{u - \pi + W - \sin W} [(1 - \cos W) M^c - (\sin W) N^c] .\]

where

\[(\partial_t, \Lambda) + (\partial_z, \Lambda) = Q_+^{-1} \partial_t (\bar{\Psi}_{ab} \delta x^a \delta x^b) + Q_+^{-1} \partial_z (\bar{\Psi}_{ab} \delta x^a \delta x^b)
= 2Q_+^{-1} (\bar{\Psi}_{ab} + \bar{\Psi}_{ba}) \delta x^a = 0,\]

with \( \bar{\Psi}_{ab} \) here denoting \( \Psi_{ab} (+1, v) \).
Similarly, $q^{cd} X_{(0),v}^a X_{(0),a}^b h_{bd,a}$ is a sum of three terms proportional to

$$-q^{cd} X_{(0),a}^b \partial_x h_{bd} = 4\mu \Sigma(u) [-(2 \cos 2u)M^c + (2 \sin 2u)N^c]$$

$$+ 4\mu K(u) [(2 \sin 2u)M^c + (2 \cos 2u)N^c]$$

$$- 8\mu \left(\frac{1}{u - \pi + W - \sin W W + 2u - 2\pi}\right)^{(2 - \cos W)M^c - (\sin W)N^c},$$

(C.23)

$$-q^{cd} X_{(0),v}^a \partial_x h_{bd} = -8\mu \cos(v - W) - \cos v \left(\frac{1}{u - \pi + W - \sin W W + 2u - 2\pi}\right)^{(2 - \cos W)M^c - (\sin W)N^c}$$

(C.24)

$$-q^{cd} X_{(0),a}^b \partial_x h_{bd} = -8\mu \sin(v - W) - \sin v \left(\frac{1}{u - \pi + W - \sin W W + 2u - 2\pi}\right)^{(2 - \cos W)M^c - (\sin W)N^c}.$$  

(C.25)

So

$$- \left(\frac{1}{4\pi L_0}\right)^2 q^{cd} X_{(0),v}^a X_{(0),a}^b h_{bd,c}$$

$$= - \left(\frac{1}{4\pi L_0}\right)^{-1} q^{cd} X_{(0),v}^a \partial_x h_{bd} - (\sin v)\partial_x h_{bd} + (\cos v)\partial_x h_{bd}$$

$$= 4\mu \left(\frac{1}{4\pi L_0}\right)^{-1} \Sigma(u) [-(2 \cos 2u)M^c + (2 \sin 2u)N^c]$$

$$+ 4\mu \left(\frac{1}{4\pi L_0}\right)^{-1} K(u) [(2 \sin 2u)M^c + (2 \cos 2u)N^c].$$

(C.26)

Adding together (C.21) and (C.26) gives equation (6.30).

The various terms contributing to the second part of the last term in (2.14) are

$$X_{(0),v}^a X_{(0),a}^b \partial_x h_{ab} = 8\mu \left(\frac{1}{4\pi L_0}\right) \left[ K(u) \sin 2u - \Sigma(u) \cos 2u \right]$$

$$- 8\mu \left(\frac{1}{4\pi L_0}\right) \left(\frac{\sin W}{u - \pi + W - \sin W W + 2u - 2\pi}\right)$$

(C.27)

$$X_{(0),v}^a X_{(0),a}^b \partial_x h_{ab} = -8\mu \left(\frac{1}{4\pi L_0}\right) \left[ \cos(v - W) - \cos v \left(\frac{1}{u - \pi + W - \sin W W + 2u - 2\pi}\right) \right]$$

(C.28)

$$X_{(0),v}^a X_{(0),a}^b \partial_x h_{ab} = -8\mu \left(\frac{1}{4\pi L_0}\right) \left[ \sin(v - W) - \sin v \left(\frac{1}{u - \pi + W - \sin W W + 2u - 2\pi}\right) \right]$$

(C.29)

and

$$X_{(0),v}^a X_{(0),a}^b \partial_x h_{ab} = 8\mu \left(\frac{1}{4\pi L_0}\right) \left[ K(u) \sin 2u - \Sigma(u) \cos 2u \right]$$

$$+ 8\mu \left(\frac{1}{4\pi L_0}\right) \left(\frac{2u - 2\pi + W - \sin W W + 2u - 2\pi}{u - \pi + W - \sin W W + 2u - 2\pi}\right).$$

(C.30)

Now, if $V_d$ is any contravariant vector and $s = +1$, then

$$q^{cd} V_d = -[V_r - (\sin v)V_x + (\cos v)V_y + V_z] M^c - [(\cos v)V_x + (\sin v)V_y] N^c.$$  

(C.31)

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Hence, the second part of the last term in (2.14) becomes
\[
\left( \frac{1}{4\pi} L_0 \right)^{-2} q^{cd} X_{(0),u}^a X_{(0),v}^b h_{ab,d} = -16\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} [K(u) \sin 2u - \Sigma(u) \cos 2u] M^c
\]
\[
- 8\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \frac{W + 2u - 2\pi - \sin W}{u - \pi + \sin W} \frac{W + 2u - 2\pi}{1 - \cos W} M^c
\]
\[
- 8\mu \left( \frac{1}{4\pi} L_0 \right)^{-1} \frac{1 - \cos W}{u - \pi + \sin W} \frac{1 - \cos W}{W + 2u - 2\pi} N^c.
\]  

(C.32)

Appendix D. The intrinsic 2-metric and the length of the evaporating ACO loop

D.1. The intrinsic 2-metric $\gamma_{AB}$

To linear order in $\mu$, the intrinsic 2-metric $\gamma_{AB} = g_{ab} X^a \cdot A X^b \cdot B$ consists of three basic terms:
\[
\gamma_{AB} = X_{(0)\cdot A} \cdot X_{(0)\cdot B} + 2X_{(0)\cdot A} \cdot \delta X_{\cdot B} + h_{ab} X_{(0)\cdot X}^a X_{(0)\cdot B}^b,
\]  

(D.1)

with $X_{(0)\cdot}$ and $\delta X$ as given in (C.1) and (C.3). In particular, $X_{(0)\cdot u} \cdot X_{(0)\cdot u} = X_{(0)\cdot u} \cdot X_{(0)\cdot w} = 0$ and $X_{(0)\cdot u} \cdot X_{(0)\cdot v} = \left( \frac{1}{4\pi} L_0 \right)^2$, while
\[
2X_{(0)\cdot u} \cdot \delta X_{\cdot u} = 2\mu \left( \frac{1}{4\pi} L_0 \right)^2 \left[ \kappa \left( u - \frac{1}{2}\pi \right) - (F_u + A_u) \right],
\]  

(D.2)

\[
2X_{(0)\cdot u} \cdot \delta X_{\cdot v} = \mu \left( \frac{1}{4\pi} L_0 \right)^2 \left[ \kappa \left( u - \frac{1}{2}\pi \right) - 2\kappa (u + v) - F_u - (F_v + A_v) \right]
\]  

(D.3)

and
\[
2X_{(0)\cdot v} \cdot \delta X_{\cdot v} = -2\mu \left( \frac{1}{4\pi} L_0 \right)^2 (G + F_v)
\]  

(D.4)

for $u > 0$.

One of the components of the term proportional to $h_{ab}$, namely
\[
h_{ab} X_{(0)\cdot u}^a X_{(0)\cdot v}^b = 16\mu \left( \frac{1}{4\pi} L_0 \right)^2 \left[ \ln(W + 2u - 2\pi) - \ln(2u) \right]
\]  

(D.5)

has already been evaluated in appendix C. The other two components are
\[
h_{ab} X_{(0)\cdot u}^a X_{(0)\cdot v}^b = -8\mu \left( \frac{1}{4\pi} L_0 \right)^2 \left[ K(u) \cos 2u + \Sigma(u) \sin 2u \right.
\]
\[
\left. - \ln(W + 2u - 2\pi) - \ln(2u) \right].
\]  

(D.6)

and
\[
h_{ab} X_{(0)\cdot v}^a X_{(0)\cdot v}^b = -8\mu \left( \frac{1}{4\pi} L_0 \right)^2 \left[ \gamma_{E} - Ci W + K(u) \cos 2u + \Sigma(u) \sin 2u + \ln W \right.
\]
\[
\left. - \ln(W + 2u - 2\pi) - \ln(2u) \right].
\]  

(D.7)

So
\[
\gamma_{uv} = \left( \frac{1}{4\pi} L_0 \right)^2 e^{-2\mu (u+v)} \left[ -2\mu \left( F_u + A_u - \kappa \left( u - \frac{1}{2}\pi \right) \right) \right.
\]
\[
+ 16\mu \left[ \ln(W + 2u - 2\pi) - \ln(2u) \right].
\]  

(D.8)
\[
\gamma_{uv} = \left( \frac{1}{4\pi} L_0 \right)^2 e^{-2\kappa \mu (u+v)} \left[ 1 - \mu \left( F_u + F_v + A_v - \kappa \left( u - \frac{1}{2}\pi \right) \right) \right] \\
- \left( - \ln(W + 2u - 2\pi) + \ln(2u) + K(u) \cos 2u + \Sigma(u) \sin 2u \right) \\
(\text{D.9})
\]

and
\[
\gamma_{vv} = \left( \frac{1}{4\pi} L_0 \right)^2 e^{-2\kappa \mu (u+v)} \left[ -2\mu (G + F_v) \right] \\
- \left( \gamma_{vE} - C \right) \left[ W - \ln(W + 2u - 2\pi) \right] \\
+ \ln(2u) + K(u) \cos 2u + \Sigma(u) \sin 2u] \\
(\text{D.10})
\]

to linear order in \(\mu\), provided that \(\kappa \mu |u + v| \ll 1\). Here, the factor of \(e^{-2\kappa \mu (u+v)}\) inserted into \(\gamma_{uv}\) absorbs the term \(-2\kappa (u+v)\) in \(\text{(D.3)}\), while multiplying \(\gamma_{ha}\) and \(\gamma_{vv}\) by \(e^{-2\kappa \mu (u+v)}\) commits an error of order \(\mu^2\).

Since \(W\) is a function of \(u\) alone by virtue of \(\text{(6.9)}\), the requirement that \(\gamma_{AB}\) take on the self-similar form \(\left( \frac{1}{4\pi} L_0 \right)^2 e^{-2\kappa \mu (u+v)} \gamma_{AB}(u)\) to linear order in \(\mu\), as explained in section 4, places three constraints on the correction functions \(F, G\) and \(A\), namely:
\[
F_{uv} + A_{uv} = 0, \quad F_{a\nu} + F_{\nu v} + A_{v v} = 0 \quad \text{and} \quad G_v + F_v = 0. \\
(\text{D.11})
\]

It should be noted that although equations \(\text{(D.8)}\)–\(\text{(D.10)}\) were derived on the assumption that \(\kappa \mu |u + v| \ll 1\), they do in fact hold for all values of \(u + v\). This can be seen by first taking the corrected position vector \(\text{(4.20)}\) and using this to calculate \(X_A \cdot X_B\), with the multiplicative scale factor \(e^{-2\kappa \mu (u+v)}\) retained but all other terms truncated at order \(\mu\). This recovers the contribution of \(X_0(0) \cdot X_A(0) + 2X_0(A) \cdot X_B(0)\) to \(\text{(D.8)}\)–\(\text{(D.10)}\) as shown.

The contribution of the remaining term \(h_{ab}X_a(0) \cdot X_B(0)\) to \(\gamma_{AB}\), where \(X_a(0)\) is given by \(\text{(5.1)}\), can be evaluated by referring back to the discussion of rotating self-similarity in section 5.1. As was demonstrated there, the value of \(h_{ab}\) at an arbitrary point
\[
x^a = X^a(0, u, v) \equiv \frac{1}{4\pi} L_0 \left[ (1 - e^{-\kappa \mu (u+v)}) e^{-\kappa \mu (u+v)} \mathbf{r}(u, v) \right]^a \\
(\text{D.12})
\]
on the world sheet is related to its value \(\tilde{h}_{ab}\) at the image point
\[
\tilde{x}^a = \frac{1}{4\pi} L_0 e^{\kappa \mu \psi - \kappa \mu (u+v)} \left[ |\mathbf{r}(u, v)|, R(-\psi) \mathbf{r}(u, v) \right]^a \\
(\text{D.13})
on the future light cone \(F_0\) of the origin through the formula
\[
h_{ab} = R_a^c(\psi) \bar{h}_{cd} R_d^b(\psi), \\
(\text{D.14})
\]
where, from \(\text{(5.13)}\),
\[
\psi = u + v - (\kappa \mu)^{-1} \ln(1 + \kappa \mu |\mathbf{r}(u, v)|). \\
(\text{D.15})
\]

Here, the image point
\[
\tilde{x}^a = \frac{1}{4\pi} L_0 \left[ (1 + \kappa \mu |\mathbf{r}(u, v)|)^{-1} \left| \mathbf{r}(u, v) \right|, R(-\psi) \mathbf{r}(u, v) \right]^a \\
(\text{D.16})
\]
is just the point \(X^a(0, \bar{u}, \bar{v})\) on the world sheet with \(\bar{u} = u\) and \(\bar{v} = v - \psi \equiv (\kappa \mu)^{-1} \ln(1 + \kappa \mu |\mathbf{r}(u, v)|) - u\), and it was shown in section 5.2 that \(\kappa \mu |\bar{u} + \bar{v}| \ll 1\) for reasonable values of \(\mu\). So
\[
X^a(0, A) = X^a(0, A) (\bar{u}, \bar{v} + \psi) = e^{-\kappa \mu (u+v)} R_a^c(\psi) X^c(0, A) (\bar{u}, \bar{v}) \\
(\text{D.17})
\]
leading to order in \(\mu\), and, therefore,
\[
h_{ab} X^a(0, A) X^b(0, B) \big|_{(u, v)} = e^{-2\kappa \mu (u+v)} h_{ab} X^a(0, A) X^b(0, B) \big|_{(\bar{u}, \bar{v})}. \\
(\text{D.18})
\]
That is, to linear order in \(\mu\) the term \(h_{ab} X^a(0, A) X^b(0, B)\) at a general point on the world sheet differs from its value at the image point \(\tilde{x}^a\) only by a multiplicative factor of \(e^{-2\kappa \mu (u+v)}\), and equations \(\text{(D.8)}\)–\(\text{(D.10)}\) are recovered in full.
D.2. The length of the loop as measured along curves of constant scale factor

Equations (D.8)–(D.10) can be used to calculate the length of the evaporating loop as measured along any curve of constant scale factor $e^{-\kappa \mu (u+v)}$. The length $L_{csf}$ of the loop along the curve $u+v = u_* + v_*$ is given by

$$L_{csf} = \int_0^{\pi} \left(2\gamma_{uv} - \gamma_{uu} - \gamma_{vv}\right)^{1/2} \left|v = u_* + v_* - u\right| du,$$

where, to linear order in $\mu$,

$$(2\gamma_{uv} - \gamma_{uu} - \gamma_{vv})^{1/2} = \sqrt{\frac{1}{4\pi} L_0} e^{-\kappa \mu (u+v)} \left[1 + \frac{1}{2} \mu (G + A_u) + 2\mu \left(\gamma_E - Ci W \right.ight.$$

$$\left. + \ln(2uW) - \ln(W + 2u - 2\pi) - K(u) \cos 2u - \Sigma(u) \sin 2u\right].$$

Hence, in view of expression (6.50) for $G$:

$$L_{csf} = 2\sqrt{\frac{1}{4\pi} L_0} e^{-\kappa \mu (u_* + v_*)} \int_0^{\pi} \left[1 + \frac{1}{2} \mu (\kappa u + C_1 + A_u) + 2\mu (C_i W - Ci W - 2K(u) \cos 2u - 2\Sigma(u) \sin 2u\right] du,$$

where

$$\int_0^{\pi} [\ln W - Ci W - 2K(u) \cos 2u - 2\Sigma(u) \sin 2u] du$$

$$= W^{-1}(2 + \ln W)(1 - \cos W) - \frac{1}{2} W(1 + \ln W) - \frac{1}{2} W^{-1}(1 - \cos 2W)$$

$$- \frac{3}{2} \sin W - (u - \pi) Ci W - Si W$$

$$+ Si(2W) - [K(u) \sin 2u - \Sigma(u) \cos 2u]$$

and so

$$\int_0^{\pi} [\ln W - Ci W - 2K(u) \cos 2u - 2\Sigma(u) \sin 2u] du = \pi \ln 2\pi + \pi - \pi Ci(2\pi) - Si(4\pi).$$

Furthermore, it follows from (6.60) and (6.55) that

$$\int_0^{\pi} \kappa u du = \frac{1}{2} \mu \kappa \pi^2 = 2\pi \ln 2\pi + 2\pi \gamma_E - 2\pi Ci(2\pi)]$$

and

$$\int_0^{\pi} A_u du = A(\pi) - A(0) = 24\pi - 16\pi \ln 2\pi - 4 Si(2\pi).$$

Substituting (D.23), (D.24) and (D.25) into (D.21), therefore, gives

$$L_{csf} = \left(\frac{1}{\sqrt{2}} L_0\right) e^{-\kappa \mu (u_* + v_*)} \left[1 + \frac{1}{2} \mu \left(C_1 + 28 + 6\gamma_E - 10 \ln 2\pi - 6 Ci(2\pi) \right.$$

$$\left. - 4\pi^{-1} Si(2\pi) - 4\pi^{-1} Si(4\pi)\right].$$

Note here that $L_{csf} \to \frac{1}{\sqrt{2}} L_0$ in the flat-space limit $\mu \to 0$. This is to be expected, as the curves of constant scale factor are curves of constant $t$ in this limit, and the speed $|\partial_t X|$ is equal to $1/\sqrt{2}$ at all points on the unperturbed ACO loop (except at the kink points at...
Another measure of the length of the loop that is potentially of interest here is the \(D.3\) invariant length of the evaporating loop \(k\) from the reference point by a term of order \(\mu\) unless \(C_1\) takes on the particular value

\[
C_1 = -28 - 6\gamma_E + 10\ln 2\pi + 6\text{Ci}(2\pi) + 4\pi^{-1}\text{Si}(2\pi) + 4\pi^{-1}\text{Si}(4\pi) \approx -9.5143. \quad (D.27)
\]

There is, of course, no necessary reason to choose this value for \(C_1\), but it bears repeating that different values of \(C_1\) will lead to different ‘dressed’ values of \(L_{csf}\) along any given curve of constant \(u + v\).

**D.3. Invariant length of the evaporating loop**

Another measure of the length of the loop that is potentially of interest here is the invariant length \(L_1\), which is a gauge-invariant quantity defined at each point \((u_0, v_0)\) on the world sheet by

\[
L_1(u_0, v_0) = 2\left(\int_{D(u_0, v_0)} \gamma^{1/2} \, du \, dv\right)^{1/2}, \quad (D.28)
\]

where \(D(u_0, v_0)\) is the closure of the subset of the world sheet that is causally disconnected from the reference point \((u_0, v_0)\) \([31]\). Calculating \(L_1\) to order \(\mu\) at a general point on the trajectory \((4.20)\) of the evaporating ACO loop is a lengthy process whose details are of marginal importance, so I will offer only an abbreviated summary here.

In the flat-space limit \(\mu = 0\), the domain \(D(u_0, v_0)\) is bounded by the null curves \(u = u_0\) and \(v = v_0\), which, in view of the identification of \((u, v)\) with \((u + 2\pi k, v - 2\pi k)\) for all integers \(k\), intersect at the points \((u_0, v_0 + 2\pi) \equiv (u_0 + 2\pi, v_0)\) and \((u_0, v_0 - 2\pi) \equiv (u_0 - 2\pi, v_0)\). The boundary of \(D(u_0, v_0)\) is, therefore, completed by the curves

\[
v = u + 2\pi - u_0 + v_0 \quad \text{and} \quad v = u - 2\pi - u_0 + v_0 \quad (D.29)
\]

and

\[
\lim_{\mu \to 0} L_1(u_0, v_0) = 2 \left(\int_{u_0 - 2\pi}^{u_0} \int_{v_0}^{v_0 + 2\pi - u_0 + v_0} \left(\frac{1}{4\pi} L_0\right)^2 \, dv \, du + \int_{u_0}^{u_0 + 2\pi} \int_{v_0 - 2\pi + u_0 - v_0}^{v_0} \left(\frac{1}{4\pi} L_0\right)^2 \, dv \, du\right)^{1/2} = L_0. \quad (D.30)
\]

In order to calculate \(L_1\) when \(\mu\) is nonzero, it is useful to first transform to a set of gauge coordinates \((u', v')\) with the property that the curves \(u' = u_0\) and \(v' = v_0\) are (to order \(\mu\)) null curves through the reference point \((u, v) = (u_0, v_0)\). A suitable choice is

\[
u' = u - \mu(v - v_0)G(u) \quad \text{and} \quad v' = v - \mu F(u; u_0), \quad (D.31)
\]

where

\[
G(u) \equiv G(u) + 4\gamma_E - \text{Ci} W + \ln W - \ln(W + 2u - 2\pi) + \ln(2u) + K(u) \cos 2u + \Sigma(u) \sin 2u
\]

\[
= \kappa u + 4\gamma_E - \text{Ci} W + \ln W + C_1 \quad (D.32)
\]

and

\[
F(u; u_0) \equiv \int_{u_0}^{u} \left[\left\{F_u + A_u - \kappa \left(u - \frac{1}{2}\pi\right)\right\} - 8[\ln(W + 2u - 2\pi) - \ln(2u)]\right] \, du. \quad (D.33)
\]

\(u = 0 \text{ and } \pi, \text{ where it is undefined). The measured length } L_{csf} = \frac{1}{\sqrt{2}} L_0 \text{ is, therefore, just the Lorentz-contracted fundamental length } L_0.\)
In the primed gauge the 2-metric components (D.8)–(D.10) become \( \gamma_{uu'} = \gamma_{vv'} = 0 \) and
\[
\gamma^{1/2} \equiv \gamma_{uu'} = \left( \frac{1}{4\pi L_0} \right)^2 e^{-2\mu(u+v)} \left[ 1 - \mu \left( F_u - \kappa \left( u - \frac{1}{2}\pi \right) \right) + \mu (v - \nu_s) \partial_u G(u) \right]
\]
\[= -8\mu \left[ - \ln(W + 2u - 2\pi) + \ln(2u) + K(u) \cos 2u + \Sigma(u) \sin 2u \right] \]
(D.34)
to linear order in \( \mu \).

Recalling again that the point \((u, v)\) is identified with \((u \pm 2\pi, v \mp 2\pi)\), the curves \(u' = u\) and \(v' = v\) intersect at the points
\[
(u, v) = (u_\ast - \Delta u, v_\ast + 2\pi - \Delta v) \equiv (u_\ast + 2\pi - \Delta u, v_\ast - \Delta v) \quad \text{(D.35)}
\]
and
\[
(u, v) = (u_\ast + \Delta u, v_\ast - 2\pi + \Delta v) \equiv (u_\ast - 2\pi + \Delta u, v_\ast + \Delta v), \quad \text{(D.36)}
\]
where
\[
\Delta u \equiv -2\pi \mu G(u_\ast)
\]
(D.37)
and
\[
\Delta v \equiv -\mu \mathcal{F}(u_\ast + 2\pi; u_\ast)
\]
\[= -2\mu \int_0^\pi \left[ \left\{ F_u + A_u - \kappa (u - \frac{1}{2}\pi) \right\} - 8\left[ \ln(W + 2u - 2\pi) - \ln(2u) \right] \right] du
\]
\[= - \left[ 16\pi - 8 \sin(2\pi) \right] \mu \approx -38.92\mu. \quad \text{(D.38)}
\]
The deviation of \(v\) by the amount \(\pm 38.92\mu\) from its flat-space values \(v_\ast\) and \(v_\ast \pm 2\pi\) at the four intersection points accounts for the rotational phase shift of the same magnitude, mentioned at the end of section 3, that is induced by gravitational back reaction on the ACO loop over the course of a single-oscillation period.

The boundary of \(\mathcal{D}(u_\ast, v_\ast)\) is completed by the straight line segments (in \((u, v)\) coordinates) joining \((u_\ast - \Delta u, v_\ast + 2\pi - \Delta v)\) to \((u_\ast - 2\pi + \Delta u, v_\ast + \Delta v)\), and \((u_\ast + 2\pi - \Delta u, v_\ast - \Delta v)\) to \((u_\ast + \Delta u, v_\ast - 2\pi + \Delta v)\). In the primed gauge, these segments become \(v' = v_\ast + (u' - u_\ast + 2\pi)[1 - \pi^{-1} \Delta v + \pi^{-1} \Delta u + \mu G(u')] + \Delta v = - \mu \mathcal{F}(u'; u_\ast)\) (D.39)
and
\[
v' = v_\ast + (u' - u_\ast - 2\pi)[1 - \pi^{-1} \Delta v + \pi^{-1} \Delta u + \mu G(u')] - \Delta v + \Delta u - \mu \mathcal{F}(u'; u_\ast)
\]
(D.40)
to first order in \(\mu\).

Equation (D.28), therefore, becomes
\[
L_1(u_\ast, v_\ast) = \left( \int_{u_\ast - 2\pi + \Delta u}^{u_\ast} \int_{v_\ast}^{v_1(u')} \gamma^{1/2} dv' \, du' + \int_{u_\ast}^{u_\ast + 2\pi - \Delta u} \int_{v_\ast}^{v_2(u')} \gamma^{1/2} dv' \, du' \right)^{1/2}, \quad \text{(D.41)}
\]
where \(v_1\) and \(v_2\) are the functions on the right-hand sides of (D.39) and (D.40) respectively, and the form of \(\gamma^{1/2}\) is given in (D.34), with \(u\) and \(v\) everywhere replaced by \(u'\) and \(v'\). If the scale factor \(e^{-2\mu(u+v)}\) in (D.34) is expanded as \(e^{-2\mu(u'+v')}[1 - \mu(u' - u_\ast + v' - v_\ast)]\) then the integrals inside the square root in (D.41) can be broken into two parts:
\[
\left( \frac{1}{4\pi L_0} \right)^2 e^{-2\mu(u_\ast + v_\ast)} \left( \int_{u_\ast - 2\pi + \Delta u}^{u_\ast} [v_1(u') - v_\ast] \, du' + \int_{u_\ast}^{u_\ast + 2\pi - \Delta u} [v_\ast - v_2(u')] \, du' \right) \quad \text{(D.42)}
\]

14 Strictly speaking, the functions of \(u\) on the right-hand side of (D.32), (D.33) and (D.34) should be the half-range periodic extensions of the expressions shown. However, all three equations are accurate if \(u\) and \(u_\ast\) both lie in \([0, \pi]\).
and
\[ \left( \frac{1}{4\pi} L_0 \right)^2 e^{-2\mu (u_s + v_s)} \left( \int_{u_s - 2\pi v_s}^{u_s} \int_{v_s}^{u_s + 2\pi - u_s + v_s} \mu \Delta \, dv' \, du' + \int_{u_s}^{u_s + 2\pi} \int_{v_s}^{u_s - 2\pi - u_s + v_s} \mu \Delta \, dv' \, du' \right), \]

where
\[ \Delta \equiv -2(u' - u_s + v' - v_s) - \left\{ F_{u'} - \kappa \left( u' - \frac{1}{2} \pi \right) \right\} + (v' - v_s) \delta_{u'} \mathcal{G}(u') - 8\ln(W + 2u' - 2\pi) + \ln(2u') + \mathcal{K}(u') \cos 2u' + \Sigma(u') \sin 2u'. \]

Now, it is easily seen that the contribution of the term \(-2(u' - u_s + v' - v_s)\) in \(\Delta\) to the integrals in (D.43) is zero, and since the remaining terms in \(\Delta\) are all half-range periodic functions of \(u'\) it turns out that
\[ \int_{u_s - 2\pi v_s}^{u_s} \int_{v_s}^{u_s + 2\pi - u_s + v_s} \mu \Delta \, dv' \, du' + \int_{u_s}^{u_s + 2\pi} \int_{v_s}^{u_s - 2\pi - u_s + v_s} \mu \Delta \, dv' \, du' = 2\pi \mu \int_{u_s}^{u_s + 2\pi} (u' - u_s - \pi) \delta_{u'} \mathcal{G}(u') \, du - 4\pi \mu \int_0^\pi \left\{ F_{u'} - \kappa \left( u - \frac{1}{2} \pi \right) \right\} \, du \]
\[ - 32\pi \mu \int_0^\pi \left\{ \ln(W + 2u' - 2\pi) + \ln(2u') \right\} \, du + \mathcal{K}(u') \cos 2u' + \Sigma(u') \sin 2u' \, du'. \]

Furthermore, the contribution of the second integral here is zero, as \(F(\pi) = F(0)\) and \(u - \frac{1}{2} \pi\) integrates to zero.

Turning to the integrals in (D.42), the terms in \(v_1(u') - v_s\) and \(v_s - v_2(u')\) of order \(\mu^0\) contribute \(4\pi^2\) to the integrals as expected, while the terms proportional to \(\Delta u\) and \(\Delta v\) integrate to give zero to order \(\mu\). Also, since \(\mathcal{G}\) is half-range periodic in \(u'\), and \(\mu \mathcal{F}(u'; u_s) - \mu \mathcal{F}(u' - 2\pi; u_s) = -\Delta v\) from (D.33) and (D.38), it turns out that
\[ \int_{u_s - 2\pi v_s}^{u_s} \left[ v_1(u') - v_s \right] \, du' + \int_{u_s}^{u_s + 2\pi - \Delta u} \left[ v_s - v_2(u') \right] \, du' = 4\pi^2 + 2\pi \mu \int_{u_s}^{u_s + 2\pi} \mathcal{G}(u') \, du' - 2\pi \Delta v. \]

Combining (D.45) and (D.46) gives
\[ \int_{u_s - 2\pi v_s}^{u_s} \int_{v_s}^{v_1(u')} \gamma^{1/2} \, dv' \, du' + \int_{u_s}^{u_s + 2\pi - \Delta u} \int_{v_s}^{v_2(u')} \gamma^{1/2} \, dv' \, du' = 4\pi^2 + 4\pi^2 \mu \mathcal{G}(u_s) + 2\pi \mu [16\pi - 8 \text{Si}(2\pi)] \]
\[ - 32\pi \mu \int_0^\pi \left\{ \ln(W + 2u' - 2\pi) + \ln(2u') \right\} \, du + \mathcal{K}(u') \cos 2u' + \Sigma(u') \sin 2u' \, du', \]

where the value of the integral in the final term is \(2\pi \ln 2\pi + \text{Si}(2\pi) - 3\pi\).

So, to first order in \(\mu\), the invariant length of the loop is
\[ L_1(u_s, v_s) = 2 \left( \frac{1}{4\pi} L_0 \right) e^{-\kappa \mu (u_s + v_s)} \left\{ -16\pi \mu [-8\pi + 4\pi \ln 2\pi + 3 \text{Si}(2\pi)]^{1/2} \right\} \]
\[ = L_0 e^{-\kappa \mu (u_s + v_s)} \left\{ 1 + \mu \left[ \frac{1}{2} \kappa u_s + 2 (\gamma_E - \text{Ci} W_s + \ln W_s) + C_1 \right] \right\} \]
\[ - 2\mu [-8 + 4 \ln 2\pi + 3\pi^{-1} \text{Si}(2\pi)] \]

(D.48)
for $u_+ \in (0, \pi)$. For values of $u_+$ outside this range, the term multiplying the scale factor $e^{-\kappa u_+ (u_+, v_+)}$ should be replaced by its half-range periodic extension. Furthermore the $u_+$-dependent part of this term, $\frac{1}{2}x_0 u_+ + 2(y_E - Ci \omega_+ W_+ + ln W_+)$, tends to $2(y_E - Ci 2\pi + ln 2\pi)$ as $u_+ \to 0^+$, and to $\frac{1}{2}x_0 \pi$ as $u_+ \to \pi^-$. In view of value (6.60) deduced for $\kappa$ in section 6.3, the two limits are the same, and so $L_4$ is continuous for all values of $u_+$ and $v_+$. 

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