On Irreducible, Infinite, Non-affine Coxeter Groups

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Abstract

The following results are proved: The center of any finite index subgroup of an irreducible, infinite, non-affine Coxeter group is trivial; Any finite index subgroup of an irreducible, infinite, non-affine Coxeter group cannot be expressed as a product of two nontrivial subgroups. These two theorems imply a unique decomposition theorem for a class of Coxeter groups. We also obtain that the orbit of each element other than the identity under the conjugation action in an irreducible, infinite, non-affine Coxeter group is an infinite set. This implies that an irreducible, infinite Coxeter group is affine if and only if it contains an abelian subgroup of finite index.

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1 Introduction

A Coxeter system \((W, S)\) is a group \(W\) and a set \(S\) of generators such that \(W\) has a presentation of the form,

\[
W = \langle S | (st)^{m_{st}} = 1, \ s, t \in S \rangle,
\]

where \(m_{st} = m_{ts}\) is a positive integer or \(\infty\), and \(m_{st} = 1\) if and only if \(s = t\) (A relation \((st)^{\infty} = 1\) is interpreted as vacuous). \(W\) is called a Coxeter group. The cardinality \(|S|\) of \(S\) is called the rank of \(W\). We are mainly interested in Coxeter groups of finite rank. So, we assume \(|S|\) is finite in this paper.

For a Coxeter system \((W, S)\), its Coxeter graph is a graph with a vertex set \(S\), and with two vertices \(s \neq t\) joined by an edge whenever \(m_{st} \geq 3\). If \(m_{st} \geq 4\), the corresponding edge is labeled by \(m_{st}\). We say that a Coxeter group \((W, S)\) is irreducible if its Coxeter graph is connected.

Associated to a Coxeter group \((W, S)\), there is a symmetric bilinear form on a real vector space \(V\), having a basis \(\{\alpha_s | s \in S\}\) in one-to-one correspondence with \(S\). The bilinear form
(α, α) is defined by setting
\[
(\alpha_s, \alpha_t) = -\cos \frac{\pi}{m_{st}}.
\] (1.2)
The value on the right-hand side is interpreted to be −1 when \( m_{st} = \infty \).

A well-known fact is that a Coxeter group \( W \) is finite if and only if its bilinear form is positive definite. It is stated in \([2]\) (page 137) that an irreducible, infinite Coxeter group has a trivial center, and a proof using the canonical representations of a Coxeter group, developed by J. Tits (see \([2, 15]\)), is suggested.

If the bilinear form of an irreducible Coxeter group \((W, S)\) is positive semi-definite but not positive definite, then \( W = \mathbb{Z}^n \rtimes W_0 \), where \( W_0 \) is a finite Coxeter group and \( n = |S| - 1 \). We call \( W \) an irreducible, infinite, affine Coxeter group in this situation. A natural and interesting question, which was proposed to the author by M. Davis and T. Januszkiewicz, is to determine if the center of a finite index subgroup of an irreducible, infinite, non-affine Coxeter group is trivial. By “non-affine” we mean its bilinear form is neither positive definite nor positive semi-definite. The answer is yes.

**Theorem 1.1.** The center of any finite index subgroup of an irreducible, infinite, non-affine Coxeter group is trivial.

The solution of this question was inspired by a preprint \([18]\) of L. Paris. In \([15]\), by studying the essential elements (which will be defined in Section 3) of a Coxeter group, Paris obtained several interesting results on irreducible Coxeter groups. One of them is that any irreducible, infinite Coxeter group cannot be written as a product of two nontrivial subgroups.

The idea of studying essential elements (Krammer \([16]\), Paris \([15]\)) is important in the proof of Theorem 1.1. In addition, the author makes use of some arguments similar to those in the proofs of the Flat Torus Theorem and the Solvable Subgroup Theorem of CAT(0) spaces. For a detailed description of CAT(0) spaces, the reader is referred to \([3]\). We will explain briefly in Section 2 a geometric construction associated to a Coxeter system \((W, S)\) (see \([6, 7]\)), which yields a PE cell complex \( \Sigma = \Sigma(W, S) \) (here PE stands for “piecewise Euclidean”), now commonly called Davis complex. It is proved by G. Moussong \([17]\) that \( \Sigma \) is a CAT(0) space and \( W \) acts properly and cocompactly on \( \Sigma \) by isometries.

The proof of Theorem 1.1 also relies on the general theory of root system of a Coxeter group (see Bourbaki \([2]\), Deodhar \([10]\) and Krammer \([16]\)). Deodhar \([11]\) and M. Dyer \([12]\) independently proved a theorem, which says that any subgroup generated by a collection of reflections in a Coxeter group is a Coxeter group by itself. This theorem also plays an important role in the proof.

Using some arguments similar to the proof of Theorem 1.1, we obtain the following.

**Theorem 1.2.** Any finite index subgroup of an irreducible, infinite, non-affine Coxeter group cannot be expressed as a product of two nontrivial subgroups.
After proving these, the author discovered that in a revised version of [13], Paris extended his discussions to include the situation of Theorem 1.1 and 1.2 using purely algebraic arguments. It appears that we both realized the necessity of use of the reflection subgroup theorem obtained by Deodhar and Dyer to achieve this aim. A different proof of Theorem 1.2 can be found in de Cornulier and de la Harpe [5], where they mention that Theorem 1.1 can be obtained from a result in Benoist and de la Harpe [1].

Based on Theorem 1.1 and 1.2, we obtain the following result.

**Theorem 1.3.** If a group $G$ is a direct product of $n$ irreducible, infinite, non-affine Coxeter groups, then any finite index subgroup $H$ of $G$ has a trivial center, and $H$ can be expressed uniquely as a direct product of $m$ nontrivial subgroups of $H$ (up to the rearrangement of factors), where each factor cannot be further decomposed and $1 \leq m \leq n$.

There are examples, which will be explained in Section 4, showing that the situation $m < n$ can happen.

Using some arguments in the proof of Theorem 1.1, we obtain another result, which has an implication on the group ring $R[W]$, where $W$ is an irreducible, infinite, non-affine Coxeter group and $R$ is a commutative ring with identity. Here $R[W]$ is the free $R$-module generated by elements of $W$. Any element of $R[W]$ is of the form \( \sum_{w \in W} a(w)w \), where $a(w) \in R$ and $a(w) = 0$ for almost all $w$. The multiplication in $W$ extends uniquely to a $R$-bilinear product $R[W] \times R[W] \to R[W]$. This makes $R[W]$ a ring. First, we have the following.

**Theorem 1.4.** Given an irreducible, infinite, non-affine Coxeter group $(W, S)$ and any $w \neq 1$ in $W$, the cardinality of the set $\{gwg^{-1} | g \in W\}$ is infinite.

As a comparison, recall that an irreducible, infinite, affine Coxeter group $W$ has a decomposition $\mathbb{Z}^n \rtimes W_0$, where $W_0$ is a finite Coxeter group. In this situation, the cardinality of the set $\{gwg^{-1} | g \in W\}$ is finite for $w \in \mathbb{Z}^n$. In summary, Theorem 1.1, 1.2 and 1.4 illustrate the group theoretic differences between irreducible, infinite, non-affine Coxeter groups and irreducible, infinite, affine Coxeter groups, even though the classification between “affine” and “non-affine” is based on generators and relations and the associated bilinear form. Indeed, the following fact can be easily proved.

**Corollary 1.5.** An irreducible, infinite Coxeter group $W$ is affine if and only if it contains an abelian subgroup of finite index.

**Proof.** The “only if” part is obvious. For the “if” part, suppose that $A$ is a finite index abelian subgroup of the irreducible, infinite Coxeter group $W$. Since the number of distinct left cosets of $A$ in $W$ is the same as the number of distinct right cosets of $A$ in $W$, we assume that $W = \bigcup_{i=1}^{n} w_i A$, where $n$ is a positive integer, $\{w_1, w_2, \ldots, w_n\}$ is a designated finite subset of $W$. Given $a \neq 1 \in A$, we have $\{waw^{-1} | w \in W\} = \{w_1 aw_i^{-1} | i = 1, \ldots, n\}$. By Theorem 1.4, $W$ cannot be non-affine. Hence, $W$ is affine. \(\square\)
The corollary for the group ring $R[W]$ is as follows, the proof of which is left to the reader.

**Corollary 1.6.** Let $W$ be an irreducible, infinite, non-affine Coxeter group and $R$ be a commutative ring with identity. The center of $R[W]$ is $R$.

The paper is organized as follows. In Section 2, we state some basic facts about the combinatorial theory of a Coxeter group. Using these, the author gives another proof of the statement that an irreducible, infinite Coxeter group has a trivial center. This proof does not use the canonical representations of Coxeter groups and is of purely combinatorial nature.

Then we describe briefly the construction of the Davis complex of a Coxeter group. The root system of a Coxeter group is introduced in Section 3, where some key results from Krammer’s thesis are stated. These are important in our discussions. The proofs of these theorems are given in Section 4. The last section is an appendix, where the author supplies some missing arguments for an important statement (Theorem 3.6) in Section 3.

## 2 Basic Combinatorial Theory of Coxeter Groups

A Coxeter group may be characterized by some combinatorial conditions, which are stated below. For now, let $W$ be a group generated by a subset $S$ of involutions (elements of order 2). The length $l(w)$ or $l_S(w)$ of an element $w \in W$, with respect to $S$, is the smallest number $d$ such that $w = s_1 \cdots s_d$, with all $s_i \in S$. This expression for $w$ is called a reduced decomposition of $w$ if $d = l(w)$.

Consider the following conditions.

**D** Deletion Condition. If $w = s_1 \cdots s_d$ with $d > l(w)$, then there are indices $i < j$ such that $w = \hat{s}_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d$, where the hats indicate deleted letters.

**E** Exchange Condition. Given $w \in W$, $s \in S$, and any reduced decomposition $w = s_1 \cdots s_d$ of $w$, either $l(sw) = d+1$ or else there is an index $i$ such that $w = ss_1 \cdots \hat{s}_i \cdots s_d$.

**F** Folding Condition. Given $w \in W$ and $s, t \in S$ such that $l(sw) = l(w) + 1$ and $l(wt) = l(w) + 1$, either $l(swt) = l(w) + 2$ or else $sw = wt$.

The proof of the following theorem can be found in [2, 4] or [9].

**Theorem 2.1.** A group $W$ generated by a set $S$ of involutions gives a Coxeter system $(W, S)$ if and only if $W$ satisfies any one of the conditions (D), (E) and (F), with the length function $l(w) = l_S(w)$ defined as above.

Given a Coxeter system $(W, S)$, for each subset $T$ of $S$, let $W_T$ be the subgroup generated by $T$. Call it a special subgroup of $W$. Then any element $w \in W$ can be expressed as $w = w_0a$ where $a \in W_T$ and $w_0$ is the shortest element in the left coset $wW_T$. $w_0$ is characterized by the property $l(w_0t) = l(w_0) + 1$ for any $t \in T$ and is unique in $wW_T$. We say $w_0$ is $(\emptyset, T)$-reduced in this situation. It is clear this type of decomposition for $w$ is unique and $w_0$ satisfies that $l(w_0b) = l(w_0) + l(b)$ for any $b \in W_T$. Similar discussions for right cosets give a “right-hand version” of the decomposition and the definition of $(T, \emptyset)$-reduced elements.
For \( w \in W \), define a subset \( \text{In}(w) \) of \( S \) by
\[
\text{In}(w) = \{ s \in S \mid l(ws) = l(w) - 1 \},
\]
and put
\[
\text{Out}(w) = S - \text{In}(w).
\]

We collect some basic facts of finite special subgroups of a Coxeter group.

**Lemma 2.2.** Suppose \( W_T \) is a finite subgroup, where \( T \subset S \). Then there is a unique element \( w_T \) in \( W_T \) of longest length. Moreover, the following statements are true.

1. \( w_T \) is an involution.
2. For any \( x \in W_T \), \( x = w_T \) if and only if \( \text{In}(x) = T \).
3. For any \( x \in W_T \), \( l(w_T x) = l(w_T) - l(x) \).

This lemma is taken from exercises in Chapter 4 of [2]. The proof of this lemma and the following two lemmas can be found in Chapter 3 of Davis [9]. For readers’ convenience, we include the proof of Lemma 2.4.

**Lemma 2.3.** (8) For any \( x \in W \), \( W_{\text{In}(x)} \) is a finite subgroup.

**Lemma 2.4.** (8) If \( W_T \) is a finite subgroup of \( W \) and \( w_T \) is the longest element in \( W_T \), then for \( s \in S - T \), \( sw_T = w_T s \) if and only if \( m_{st} = 2 \) for all \( t \in T \).

**Proof.** If \( m_{st} = 2 \), then \( s \) and \( t \) commute. Hence, if \( m_{st} = 2 \) for all \( t \in T \), then \( s \) and \( w_T \) commute.

Conversely, suppose \( s \) and \( w_T \) commute, where \( s \notin T \). Then \( l(w_T s) = l(w_T) + 1 \), so \( s \in \text{In}(w_T s) \). Since \( w_T s = sw_T \), \( T \subset \text{In}(w_T s) \). Therefore, \( \text{In}(w_T s) = T \cup \{ s \}, w_{T \cup \{ s \}} = w_T s \). We want to show that \( m_{st} = 2 \) for all \( t \in T \). Suppose, to the contrary, that \( m_{st} > 2 \), for some \( t \in T \). Then \( l(sts) = 3, l((w_T s)(sts)) = l(w_T) + 1 - 3 = l(w_T) - 2 \) by Lemma 2.2. On the other hand, \( l((w_T s)(sts)) = l(w_T t) + 1 = l(w_T) \), a contradiction. Hence, the conclusion of Lemma 2.4 holds.

Now we prove the claim.

**Proposition 2.5.** The center of an irreducible, infinite Coxeter group \( (W, S) \) is trivial.

**Proof.** If \( w \neq 1 \) is in the center of \( W \), then \( ws = sw \) for any \( s \in S \). Put \( S_1 = \text{In}(w) \) and \( S_2 = \text{Out}(w) \). Then \( S_1 \neq \emptyset \). Write \( w = w_0 a \) with \( a \in W_{S_1} \) and \( w_0 \) being \( (\emptyset, S_1) \) reduced. Notice that \( l(w) - 1 = l(w_0 s) = l(w_0 a s) \) for any \( s \in S_1 \), it follows that \( l(as) = l(a) - 1 \) for all \( s \in S_1 \). By Lemma 2.2 and Lemma 2.3, \( a \) is the (unique) longest element in the finite subgroup \( W_{S_1} \), and \( a^2 = 1 \).
Now, continue our discussions and consider the “right-hand version” of the above-mentioned decomposition of \( w \). Since \( w \) is in the center of \( W \), we have \( w = aw_1 \), where \( w_1 \) is \((S_1, \emptyset)\) reduced, \( a \) is the longest element in \( W_{S_1} \). Hence, \( w_0 = wa = aw = w_1, w = aw_0 = w_0a \).

Notice that for any \( t \in S_2 \), \( l(wt) = l(w) + 1 \), it follows that \( l(w_0t) = l(w_0) + 1 \), since otherwise we would have \( l(wt) \leq l(a) + l(w_0t) \leq l(w) - 1 \), contradicting the definition of \( S_2 \). Therefore, \( w_0 \) is \((\emptyset, S_2)\)-reduced, and hence is \((\emptyset, S)\)-reduced. This implies that \( w_0 = 1 \) and \( w = a \), i.e., \( w = w_{S_1} \). So, \( w_{S_1} \) commutes with every element in \( S_2 = S - S_1 \). Now, Lemma 2.4 implies that \( m_{st} = 2 \) for any \( s \in S_1, t \in S_2 \). The irreducibility of \( W \) implies \( S_2 = \emptyset \) and hence \( W \) is a finite Coxeter group, a contradiction. This finishes the proof of Proposition 2.5.

To prove the theorems stated in the introduction, we need the fact that a Coxeter group acts properly and cocompactly on a CAT(0) space. Here we give a brief description of the Davis complex. For a more complete account of it, the reader is referred to [6] [7] [8].

Let \((W, S)\) be a Coxeter system. We define a poset, denoted \( S_f(W, S) \) (or simply \( S_f \)), by
\[
S_f = \{ T | T \subseteq S \text{ and } W_T \text{ is finite} \}.
\]
This poset is partially ordered by inclusion. It is clear that \( S_f - \{\emptyset\} \) is isomorphic to the poset of simplices of an abstract simplicial complex, which is denoted by \( N(W, S) \) (or simply \( N \)). \( N \) is called the nerve of \((W, S)\).

**Theorem 2.6.** (Gromov, Moussong [7] [17]). Associated to a Coxeter system \((W, S)\), there is a PE cell complex \( \Sigma(W, S) \) (or simply \( \Sigma \)) with the following properties.

1. The poset of cells in \( \Sigma \) is the poset of cosets
\[
WS_f = \coprod_{T \in S_f} W/W_T.
\]
2. \( W \) acts by isometries on \( \Sigma \) with finite stabilizers and with compact quotient.
3. Each cell in \( \Sigma \) is simple (so that the link of each vertex is a simplicial cell complex).
In fact, this complex is just \( N(W, S) \).
4. \( \Sigma \) is CAT(0).

### 3 Root System and Essential Elements of a Coxeter Group

Recall from the introduction that for a Coxeter system \((W, S)\), there is a symmetric bilinear form \(( , \) \) on a real vector space \( V \), having a basis \( \{\alpha_s | s \in S\} \) in one-to-one correspondence with \( S \).

Now, for each \( s \in S \), define a linear transformation \( \sigma_s : V \to V \) by \( \sigma_s \lambda = \lambda - 2(\alpha_s, \lambda)\alpha_s \). Then \( \sigma_s \) is a linear reflection. It has order 2 and fixes the hyperplane \( H_s = \{ \delta \in V | (\delta, \alpha_s) = 0 \} \) pointwise, and \( \sigma_s \alpha_s = -\alpha_s \). We have the following theorem (see [10], [15]).
Theorem 3.1. There is a unique homomorphism \( \sigma : W \to GL(V) \) sending \( s \) to \( \sigma_s \). This homomorphism is a faithful representation of \( W \) and the group \( \sigma(W) \) preserves the bilinear form. Moreover, for each pair \( s, t \in S \), the order of \( st \) in \( W \) is precisely \( m_{st} \).

From now on, we write \( w(\alpha) \) for \( \sigma(w)(\alpha) \), when \( \alpha \in V \) and \( w \in W \).

The root system \( \Phi \) of \( W \), is defined to be the collection of all vectors \( w(\alpha_s) \), where \( w \in W \) and \( s \in S \). An important fact about the root system is that any root \( \alpha \in \Phi \) can be expressed as

\[
\alpha = \sum_{s \in S} c_s \alpha_s,
\]

where all the coefficients \( c_s \geq 0 \) (we call \( \alpha \) a positive root and write \( \alpha > 0 \)), or all the coefficients \( c_s \leq 0 \) (call \( \alpha \) a negative root and write \( \alpha < 0 \)). Write \( \Phi^+ \) and \( \Phi^- \) for the respective sets of positive and negative roots. Then \( \Phi^+ \cap \Phi^- = \emptyset \), \( \Phi^+ \cup \Phi^- = \Phi \) and \( \Phi^- = -\Phi^+ \). The map from \( \Phi \) to \( R = \{wtw^{-1} | w \in W, t \in S\} \) (the set of reflections in \( W \)) given by \( \alpha = w(\alpha_s) \mapsto wsw^{-1} \) is well-defined and restricts to a bijection from \( \Phi^+ \) (\( \Phi^- \)) to \( R \), and \( \sigma(wsw^{-1}) = t_\alpha \), where \( t_\alpha \) is the linear reflection given by \( t_\alpha \lambda = \lambda - 2(\alpha, \lambda)\alpha \). The following fact is important when discussing root systems.

Proposition 3.2. ([10], [15]) Let \( w \in W, \alpha \in \Phi^+ \). Then \( l(wt_\alpha) > l(w) \) if and only if \( w(\alpha) > 0 \).

The statements in the remaining part of this section are mostly due to Krammer [16], as revised by Paris [18]. Let \( u, v \in W \) and \( \alpha \in \Phi \). We say that \( \alpha \) separates \( u \) and \( v \) if \( \alpha u \in \Phi^c \) and \( \alpha v \in \Phi^- \), where \( c \in \{+,-\} \). Let \( w \in W \) and \( \alpha \in \Phi \). We say that \( \alpha \) is \( w \)-periodic if there is some positive integer \( m \) such that \( w^m\alpha = \alpha \).

Lemma 3.3. ([18]) Let \( w \in W \) and \( \alpha \in \Phi \). Then exactly one of the following holds.

1. \( \alpha \) is \( w \)-periodic.
2. \( \alpha \) is not \( w \)-periodic, and the set \( \{m \in \mathbb{Z}| \alpha \) separates \( w^m \) and \( w^{m+1}\} \) is finite and has an even cardinality.
3. \( \alpha \) is not \( w \)-periodic, and the set \( \{m \in \mathbb{Z}| \alpha \) separates \( w^m \) and \( w^{m+1}\} \) is finite and has an odd cardinality.

We say that \( \alpha \) is \( w \)-even in Case 2, and \( w \)-odd in Case 3.

Lemma 3.4. ([18]) Let \( \alpha \in \Phi, w \in W, \) and \( p \in \mathbb{N}, p \geq 1 \). Then

1. \( \alpha \) is \( w \)-periodic if and only if \( \alpha \) is \( w^p \)-periodic.
2. \( \alpha \) is \( w \)-even (resp., \( w \)-odd) if and only if \( \alpha \) is \( w^p \)-even (resp., \( w \)-odd).

A subgroup \( G \) is called a parabolic subgroup of a Coxeter group \((W, S)\) if \( G = xW_Ix^{-1} \) for some \( x \in W \) and \( I \subset S \). An element \( w \in W \) is called an essential element if it does not lie in any proper parabolic subgroup of \( W \).
Following Krammer [16], we define the parabolic closure \( \text{Pc}(A) \) of a subset \( A \) of a Coxeter group \( W \) to be the intersection of all parabolic subgroups containing \( A \). In [19], the author gives a proof of the conclusion that \( \text{Pc}(A) \) is a parabolic subgroup of \( W \). Using this terminology, an element \( w \in W \) is essential if and only if \( \text{Pc}(w) = W \).

Paris shows the existence of essential elements in [18].

**Proposition 3.5.** (Paris [18]) Given a Coxeter group \((W, S)\), where \( S = \{s_1, s_2, \cdots, s_n\} \), then \( c = s_n \cdots s_2 s_1 \) (which is called a Coxeter element) is an essential element of \( W \).

In [18] Paris attributes the following result to Krammer [16].

**Theorem 3.6.** For an irreducible, infinite Coxeter group \((W, S)\), an element \( w \in W \) is essential if and only if \( W \) is generated by the set \( \{t_\alpha | \alpha \in \Phi^+ \text{ and } \alpha \text{ is } w\text{-odd}\} \).

To the author’s understanding, this statement does not appear in Krammer’s thesis [16], but it can be proved using some results that Krammer has established. The author will give the proof of this theorem in the Appendix.

The next result is obvious from the discussions in the preceding paragraphs.

**Corollary 3.7.** (18) Assume that \( W \) is an irreducible, infinite Coxeter group. Let \( w \in W \) and \( p \) be a positive integer. Then \( w \) is essential if and only if \( w^p \) is essential.

The following theorem, which appears in Krammer [16] (page 69), is very important to us.

**Theorem 3.8.** Assume that \( W \) is an irreducible, infinite, non-affine Coxeter group. Let \( w \in W \) be an essential element. Then \( \langle w \rangle = \{w^m | m \in \mathbb{Z}\} \) is a finite index subgroup of the centralizer \( C(w) \) of \( w \) in \( W \).

Now we come to the proofs of the theorems stated in the Introduction.

### 4 Proofs of the Theorems

We begin with a lemma.

**Lemma 4.1.** If an irreducible Coxeter group \((W, S)\) contains an infinite cyclic subgroup \( \langle x \rangle \) as a finite index subgroup, then \( W \cong D_\infty \), the infinite dihedral group.

**Proof.** The key point is that \( W \) acts on the \( \text{CAT}(0) \) space \( \Sigma = \Sigma(W, S) \) properly and cocompactly. What follows is similar to the proof of the Flat Torus Theorem for \( \text{CAT}(0) \) spaces (see page 246 in [3]). Since \( |W : \langle x \rangle| \) is finite, there is a positive integer \( k \) such that \( \langle x^k \rangle \) is a normal subgroup. It is a finite index subgroup of \( W \). \( \text{Min}((x^k)) \) is a non-empty closed subspace of \( \Sigma \) isometric to one of the form \( Y \times \mathbb{R} \), where \( Y \) is a closed convex subspace of \( \Sigma \), \( \mathbb{R} \) is the set of real numbers (For any group \( \Gamma \) acting on a \( \text{CAT}(0) \) space by isometries, \( \text{Min}(\Gamma) \) is a certain subspace as defined on page 229 in [3]). \( x^k \) acts as a nontrivial translation on the
a CAT(0) space, then every abelian subgroup of \( \Gamma \) (see page 247 in [3]) states that if a group \( G \) is finitely generated and hence, is finite. A result of Tits states that \( G \subset Z \) (the infinite dihedral group generated by \( g \)).

Since \( S \) is an essential element of \( G \), \( G \) is contained in the centralizer \( C \) of \( S \). Therefore, in this case, the parabolic Coxeter group is contained in a finite parabolic subgroup \( G/S \) of \( G \). By properties of CAT(0) spaces, the fixed point set of \( C \) is finite by Theorem 3.8. Therefore, \( |W : W_1| \) is finite. Now, Lemma 4.1 implies that \( W \) is finite unless \( W = W_{\{s_1,s_2\}} = D_\infty \), because of the following result proved by Deodhar [10]: For an irreducible, infinite Coxeter group \((W,S)\) and any proper subset \( J \) of \( S \), \( |W : W_J| \) is infinite (This Deodhar’s Theorem was discovered again in T. Hosaka [13], using different methods).

Proof of Theorem 1.1. Assume that \((W,S)\) is an irreducible, infinite, non-affine Coxeter group and \( G \) is a finite index subgroup of \( W \). Let \( Z(G) \) be the center of \( G \). The proof is divided into two cases.

Case 1. \( Z(G) \) contains an element \( x \) of infinite order. By Proposition 6.5, we can pick an essential element \( w \) of \( W \). Since the number of cosets \( \{Gw^m|m \in Z\} \) is finite, there is a positive integer \( q \) such that \( w^q \in G \). By Corollary 6.7, \( w^q \) is essential in \( W \). Since \( x \) is in the centralizer \( C(w^q) \) of \( w^q \), it is contained in the centralizer \( C(x) \) of \( x \). By considering the collection of cosets \( \{x^m|m \in Z\} \) of \( W/w^q \), we know that \( x^q \in (w^q) \). Hence, \( |W : (w^q)| \) is finite, so, \( |W : W_{\{s_1,s_2\}}| \) is finite.

Case 2. \( Z(G) \) is a torsion subgroup, i.e., every element in \( Z(G) \) has a finite order. A preliminary result used in the proof of the Solvable Subgroup Theorem for CAT(0) spaces (see page 247 in [3]) states that if a group \( \Gamma \) acts properly and cocompactly by isometries on a CAT(0) space, then every abelian subgroup of \( \Gamma \) is finitely generated. It follows that \( Z(G) \) is finitely generated and hence, is finite. A result of Tits states that any finite subgroup of a Coxeter group is contained in a finite parabolic subgroup. Therefore, in this case, the parabolic closure \( Pc(Z(G)) \) of \( Z(G) \) is a finite parabolic subgroup of \( W \). Without loss of generality, we may assume that \( Pc(Z(G)) \) is a finite special parabolic subgroup \( W_K \), where \( K \subset S \).

Since \( Z(G) \) is normal in \( G \), \( gW_Kg^{-1} \) is a parabolic subgroup containing \( Z(G) \) for any \( g \in G \). By the uniqueness of the parabolic closure (or by the discussion of the rank of the intersection of two parabolic subgroups in [19]), we have \( gW_Kg^{-1} = W_K \) and hence, \( G \subset N(W_K) \) (the normalizer of \( W_K \) in \( W \)). Therefore, \( |W : N(W_K)| \) is finite. This implies that the set \( R_1 = \{wtw^{-1}|t \in K, w \in W\} \) is finite. Now, consider the reflection subgroup \( W_1 \) of \( W \) generated by \( R_1 \). \( W_1 \) is a Coxeter group by [11] or [12], with a set \( S_1 \) of distinguished
generators, where \( S_1 \subseteq \bigcup_{w_1 \in W_1} w_1 R_1 w_1^{-1} \) (It is clear that \( \bigcup_{w_1 \in W_1} w_1 R_1 w_1^{-1} = R_1 \) in the present situation). Hence, the set of reflections in \( W_1 \), which by definition is \( \{w_1 t_1 w_1^{-1} | t_1 \in S_1, w_1 \in W_1\} \subseteq R_1 \), is finite. Therefore, \( W_1 \) is a finite Coxeter group. Suppose that \( \text{Pc}(W_1) = yW_L y^{-1} \), where \( y \in W \) and \( L \) is a proper subset of \( S \). Since \( W_1 \) is normal in \( W \), \( yW_L y^{-1} \) is a (proper) normal subgroup of \( W \). This is impossible, due to the result proved by Paris [13]: any proper nontrivial special subgroup of an irreducible Coxeter group is not normal. In the current situation, we need to replace the distinguished set \( S \) of generators by \( ySy^{-1} \).

In conclusion, \( Z(G) = \{1\} \). This finishes the proof of Theorem 1.1.

**Remark.** Paris obtains the above mentioned result based on R. Howlett’s description of the normalizer of a special subgroup of a Coxeter group in [13], where Howlett discusses the situation of finite Coxeter groups. In fact, Howlett’s description is valid in general. The proof is just a slight modification of that for finite Coxeter groups.

The proof of Theorem 1.2 is similar.

**Proof of Theorem 1.2.** Let \( G \) be a finite index subgroup of an irreducible, infinite, non-affine Coxeter group. Suppose that \( G = A \times B \), where \( A \) and \( B \) are nontrivial subgroups. Pick an essential element \( w \) in \( W \). Following the idea in the proof of Theorem 1.1, we know that there is a positive integer \( p \) such that \( w^p \in G \). Now \( w^p \) is essential and \( w^p = ab \) for some \( a \in A \) and \( b \in B \). At least one of them, say, \( a \) has an infinite order, because \( w^p \) has an infinite order, and \( a \) and \( b \) commute. Notice that since \( aw^p = w^pa \), \( a \) is in the centralizer \( C(w^p) \) of \( w^p \). By considering the collection of cosets \( \{(w^p)am^{-1} | m \in Z\} \), we conclude that there is a positive integer \( q \) such that \( a^q \in \langle w^p \rangle \). So, \( a \) is an essential element. Since each element of \( B \) commutes with \( a, B \subseteq C(a) \), the centralizer of \( a \). Since the collection of cosets \( \{(a)m | m \in B\} \) is finite and \( A \cap B = \{1\} \), we know that \( B \) is finite. Without loss of generality, we can assume that \( \text{Pc}(B) = W_1 \), a finite parabolic subgroup. Since \( B \) is normal in \( G \), the uniqueness of parabolic closure implies that \( G \subseteq N(W_1) \). So, \( |W : N(W_1)| \) is finite. The arguments in Case 2 of the proof of Theorem 1.1 now apply to derive a contradiction. Hence, the conclusion of Theorem 1.2 holds.

**Proof of Theorem 1.3.** Assume that

\[
G = W_1 \times W_2 \times \cdots \times W_n \tag{4.1}
\]

where each \( W_i \) is an irreducible, infinite, non-affine Coxeter group. Let \( H \) be a finite index subgroup of \( G \). Denote by \( p_i \) the projection \( p_i : G \to W_i \) of \( G \) to the \( i \)th factor \( W_i \). Let \( c = \prod_{i=1}^{n} c_i \in Z(H) \), the center of \( H \), where \( c_i \in p_i(H) \subseteq W_i \). Pick an arbitrary \( h = \prod_{i=1}^{n} h_i \in H \) with \( h_i \in p_i(H) \). Then \( ch = hc \). This implies that \( \prod_{i=1}^{n} (c_i h_i) = \prod_{i=1}^{n} (h_i c_i) \), so, \( c_i h_i = h_i c_i \) for any \( i \). Notice that since \( h_i \in p_i(H) \) is arbitrary, \( c_i \in Z(p_i(H)) \). Since \( W_i = p_i(G) \) and
Lemma 4.2. Assume that each factor cannot be further decomposed. Let \( q_i : H \to K_i \) be the projection of \( H \) onto its \( i \)th factor in the decomposition \((4.3)\). We know that \( K_i = q_i(H) = q_i(H_1)q_i(H_2) \cdots q_i(H_m) \). And, notice that \( q_i(H_k) \cap \left( \prod_{j \neq k} q_i(H_j) \right) = \{1\} \), because this intersection is contained in the center \( Z(K_i) \) of \( K_i \), and \( Z(K_i) \subset Z(H) \), while the latter is trivial by the first part of this theorem, we have the following,

\[
K_i = q_i(H_1) \times \cdots \times q_i(H_m).
\]
By the assumption that $K_i$ cannot be further decomposed, $K_i = q_i(H_{\phi(i)})$ for some $\phi(i)$, and $q_i(H_j)$ is trivial for $j \neq \phi(i)$. So, $\phi$ defines a map from $\{1, 2, \ldots, r\}$ to $\{1, 2, \ldots, m\}$, and it is surjective because, for any $j \in \{1, 2, \ldots, m\}$, there is an $i \in \{1, 2, \ldots, r\}$ such that the restriction $q_i|_{H_j}$ is nontrivial. Hence $m \leq r$. Similar discussions also yield $r \leq m$. Therefore, $m = r$ and $\phi$ is a bijection. After re-indexing, we may assume $\phi = \text{id}$ (the identity map). This means that $H_i \subset K_i$ and the restriction $q_i|_{H_i}$ is indeed the inclusion $H_i \hookrightarrow K_i$. Since

$$H = H_1 \times \cdots \times H_m = K_1 \times \cdots \times K_m,$$

we know that $H_i = K_i$, for $i = 1, \ldots, m$. This is the claimed unique decomposition of $H$ and $m$ is determined uniquely by $H$. The proof of Theorem 1.3 is completed.

The situation that $m < n$ may happen. To illustrate this, we need the following lemma.

**Lemma 4.3.** Let $W = W_1 \times W_2$, where $W_1$ and $W_2$ are irreducible, infinite, non-affine Coxeter groups. If $H$ is a finite index subgroup of $W$ and $H = H_1 \times H_2$, where $H_i \neq \{1\}$, $i = 1, 2$, then after re-indexing, $H_i \subset W_i$.

**Proof.** As in the proof of Theorem 1.3, let $p_i$ be the projection $p_i : W \to W_i$. It follows that $p_1(H)$ is a finite index subgroup of $W_1$ and $p_1(H) = p_1(H_1) \times p_1(H_2)$ by repeating the arguments in the proof of Theorem 1.3. In this product decomposition, only one factor, say, $p_1(H_1)$ is nontrivial by Theorem 1.2. So, $p_1(H) = p_1(H_1)$ and $p_1(H_2) = \{1\}$. This implies that $H_2 \subset W_2$. So, $p_2(H_2) \neq \{1\}$. By Theorem 1.2, $p_2(H_1) = \{1\}$, $H_1 \subset W_1$.  

In the following two examples, let $(W_i, S_i)$ be an irreducible, infinite, non-affine Coxeter group, $i = 1, 2$. Suppose that $W = W_1 \times W_2$ and $S = S_1 \cup S_2$.

**Example 4.4.** (M. Davis) Suppose that $f_i : W_i \to G$ ($i = 1, 2$) is a surjective homomorphism from $W_i$ to a finite group $G$. Let $H = \{(w_1, w_2) \in W | f_1(w_1) = f_2(w_2)\}$. Then $|W : H| = |G|$, $p_1(H) = W_1$ and $p_2(H) = W_2$, where $p_i$ is the projection defined in Lemma 4.3. It follows from Lemma 4.3 that $H$ cannot be further decomposed.

**Example 4.5.** Let $\phi : W \to D_1$ be the surjective homomorphism such that $\phi(s) = -1$ for any $s \in S$, where $D_1 = \{-1, 1\}$. Let $H = \ker(\phi)$. Then $|W : H| = 2$. If $H$ had a decomposition $H = H_1 \times H_2$, $H_i \neq \{1\}$, then one of the factors, say, $H_1$ would be $W_1$ because of Lemma 4.3 and Lemma 4.2 and $|W : H| = 2$. This would imply that $\phi(s_1) = 1$ for any $s_1 \in S_1$, a contradiction. Hence, $H = \ker(\phi)$ cannot be further decomposed.

**Proof of Theorem 1.4.** Let $w \neq 1$ be an element of $W$. We want to prove that $|W : C(w)| = \infty$. The proof is divided into two cases.

Case 1. The order of $w$ is finite. In this case, the parabolic closure $\text{Pc}(\langle w \rangle)$ of $\langle w \rangle$ is a finite parabolic subgroup. Without loss of generality, assume that $\text{Pc}(\langle w \rangle) = W_K$, where $K \subset S$. The uniqueness of the parabolic closure and the fact that $gwg^{-1} = w$ for any $g \in C(w)$ imply
that \( gW_Kg^{-1} = W_K \) for \( g \in C(w) \). Hence, \( C(w) \subset N(W_K) \). The discussion of Case 2 in the proof of Theorem 1.1 yields that \( [W : N(W_K)] = \infty \) when \( W \) is an irreducible, infinite Coxeter group and \( W_K \) is a nontrivial finite special subgroup. Therefore, in this situation, \( [W : C(w)] = \infty \).

Case 2. The order of \( w \) is infinite. Suppose that \( [W : C(w)] < \infty \). Pick an essential element \( x \in W \). Since the number of cosets \( \{C(w)x^k | k \in \mathbb{Z} \} \) is finite, there is a positive integer \( m \) such that \( x^m \in C(w) \). By Corollary 5.7 \( x^m \) is essential. Notice that \( w \in C(x^m) \) and the number of cosets \( \{C(x^m)t^l | l \in \mathbb{Z} \} \) in \( C(x^m) \) is finite because of Theorem 3.8, we conclude that there is a positive integer \( n \) such that \( w^n \in (x^m) \). Now, Corollary 5.7 implies that \( w^n \) is essential, and hence, so is \( w \). Then, by Theorem 3.8, we have \( [W : \langle w \rangle] < \infty \). It follows from Lemma 4.1 that \( W = D_\infty \), contradicting the assumption that \( W \) is non-affine.

In conclusion, \( [W : C(w)] = \infty \) for \( w \neq 1 \). The conclusion of the theorem follows immediately.

5 Appendix

In this section we summarize some results from Krammer [16] and supply the missing arguments for the proof of Theorem 3.6. Recall that we assume that the Coxeter system \((W, S)\) has a finite rank. It has a faithful representation as described in Section 3.

A \( w \)-periodic root \( \alpha \) is called \( w \)-critical (\[16\] page 53) if it satisfies the condition: The bilinear form (defined in the Introduction) is positive definite on \( \text{Span}\{\langle w \rangle \alpha \} \) and \( \sum_{k=1}^{\infty} w^k \alpha = 0 \), where \( k \) is the smallest positive integer with \( w^k \alpha = \alpha \).

We need the following results.

**Lemma 5.1.** (\[16\] page 54) The subgroup generated by \( \{t_\alpha | \alpha \text{ is } w \text{-critical}\} \) is a finite subgroup of \( W \).

**Theorem 5.2.** (\[16\] page 56) The parabolic closure \( Pc(w) \) of \( w \) equals the subgroup of \( W \) generated by the reflections \( \{t_\alpha | \alpha \text{ is } w \text{-critical or } w \text{-odd}\} \).

Let \( W_1 \) be the subgroup generated by \( \{t_\alpha | \alpha \text{ is } w \text{-odd}\} \) and \( \Phi_1 = \{w_1(\alpha) | w_1 \in W_1, \alpha \text{ is } w \text{-odd}\} \).

**Lemma 5.3.** (\[16\] page 56) Let \( \alpha \) be \( w \)-periodic, then either \( (\alpha, \beta) = 0 \) for any \( \beta \in \Phi_1 \), or \( \alpha \in \Phi_1 \).

**Lemma 5.4.** (\[15\] page 131) Assume that \((W, S)\) is irreducible, then any proper \( W \)-invariant subspace is contained in the radical \( V^\perp \) (of the bilinear form), i.e., \( V^\perp = \{v \in V | (v, \alpha_s) = 0 \text{ for any } s \in S\} \).

**Proof of Theorem 5.6.** Proposition 3.5 states that essential elements exist in a Coxeter group. Now, assume that \((W, S)\) is an irreducible, infinite Coxeter group. For any essential element
$w$ of $W$, $Pc(w) = W$. It follows from Lemma 5.1 and Theorem 5.2 that $\Phi_1$ is nonempty. Let $V_0$ be the subspace spanned by those $w$-critical roots $\alpha$ satisfying the condition that $(\alpha, \beta) = 0$ for any $\beta \in \Phi_1$. Then Theorem 5.2 and Lemma 6.3 imply that $V_0$ is a $W$-invariant subspace. Obviously, $V_0 \neq V$, since otherwise, $(\beta, \beta) = 0$ for $\beta \in \Phi_1$, which is absurd. It follows from Lemma 5.4 that $V_0 \subset V^\perp$. This is impossible unless $V_0 = 0$, since any critical root $\alpha \in V_0$ satisfies $(\alpha, \alpha) = 1$. Therefore, if $w \in W$ is an essential element, $W = Pc(w)$ is generated by $w$-odd reflections. The converse is clear from Theorem 6.2.

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