Locally-Adaptive Nonparametric Online Learning

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Abstract

One of the main strengths of online algorithms is their ability to adapt to arbitrary data sequences. This is especially important in nonparametric settings, where regret is measured against rich classes of comparator functions that are able to fit complex environments. Although such hard comparators and complex environments may exhibit local regularities, efficient algorithms whose performance can provably take advantage of these local patterns are hardly known. We fill this gap introducing efficient online algorithms (based on a single versatile master algorithm) that adapt to: (1) local Lipschitzness of the competitor function, (2) local metric dimension of the instance sequence, (3) local performance of the predictor across different regions of the instance space. Extending previous approaches, we design algorithms that dynamically grow hierarchical packings of the instance space, and whose prunings correspond to different “locality profiles” for the problem at hand. Using a technique based on tree experts, we simultaneously and efficiently compete against all such prunings, and prove regret bounds scaling with quantities associated with all three types of local regularities. When competing against “simple” locality profiles, our technique delivers regret bounds that are significantly better than those proven using the previous approach. On the other hand, the time dependence of our bounds is not worse than that obtained by ignoring any local regularities.

1 Introduction

In online convex optimization [Zinkevich, 2003, Hazan, 2016], a learner interacts with an unknown environment in a sequence of rounds. In the specific setting considered in this paper, at each round \( t = 1, 2, \ldots \) the learner observes an instance \( x_t \in \mathcal{X} \subset \mathbb{R}^d \) and outputs a prediction \( \hat{y}_t \) for the label \( y_t \in \mathcal{Y} \) associated with the instance. After predicting, the learner incurs the loss \( \ell_t(\hat{y}_t) \). We consider two basic learning problems: regression with square loss, where \( \mathcal{Y} \equiv [0, 1] \) and \( \ell_t(\hat{y}_t) = \frac{1}{2} (y_t - \hat{y}_t)^2 \), and binary classification with absolute loss, where \( \mathcal{Y} \equiv \{0, 1\} \) and \( \ell_t(\hat{y}_t) = |y_t - \hat{y}_t| \) (or, equivalently, \( \ell_t(\hat{y}_t) = \mathbb{P}(y_t \neq Y_t) \) for randomized predictions \( Y_t \) with \( \mathbb{P}(Y_t = 1) = \hat{y}_t \)). The performance of a learner is measured through the notion of regret, which is defined as the amount by which the cumulative loss of the learner predicting with \( \hat{y}_1, \hat{y}_2, \ldots \) exceeds the cumulative loss —on the same sequence of instances and labels— of any function \( f \) in a given reference class of functions \( \mathcal{F} \), namely

\[
R_T(f) = \sum_{t=1}^{T} \left( \ell_t(\hat{y}_t) - \ell_t(f(x_t)) \right) \quad \forall f \in \mathcal{F}.
\]

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In order to capture complex environments, we focus on nonparametric classes $\mathcal{F}$ containing Lipschitz functions $f : \mathcal{X} \to \mathcal{Y}$. The specific approach adopted in this paper is inspired by the simple and versatile algorithm of Hazan and Megiddo [2007], henceforth denoted with $\text{HM}$, achieving a regret bound of the form

$$R_T(f) = \begin{cases} \ln(T)(LT)^{d\pi^2} & \text{(square loss)} \\ L^{d\pi^2} T^{d\pi^2} & \text{(absolute loss)} \end{cases} \quad \forall f \in \mathcal{F}_L$$

for all given $L > 0$, where $\mathcal{F}_L$ is the class of $L$-Lipschitz functions $f : \mathcal{X} \to \mathcal{Y}$ such that

$$|f(x) - f(x')| \leq L \|x - x'\|$$

for all $x, x' \in \mathcal{X}^2$. Although Lipschitzness is a standard assumption in nonparametric learning, a function in $\mathcal{F}_L$ may alternate regions of low variation with regions of high variation. This implies that, if computed locally (i.e., on pairs $x, x'$ that belong to the same small region), the value of the smallest $L$ satisfying (3) would change significantly across these regions. If we knew in advance the local Lipschitzness profile, we could design algorithms that exploit this information to gain a better control on regret. Although asymptotic rates that improve on (2) can be obtained using different and more complicated algorithms, it is not clear whether these other algorithms can be made locally adaptive in a principled way as we do with $\text{HM}$.

**Local Lipschitzness.** Our first contribution is an algorithm for regression with square loss that competes against all functions in $\mathcal{F}_L$. However, unlike the regret bound (2) achieved by $\text{HM}$, the regret $R_T(f)$ of our algorithm depends in a detailed way on the local Lipschitzness profile of $f$. Our algorithm operates by sequentially constructing a $D$-level hierarchical packing $T$ of the instance space $\mathcal{X}$ with balls whose radius decreases with each level of the hierarchy. The $D$ levels are associated with local Lipschitz constants $L_1 < L_2 < \cdots < L_D = L$ provided as an input parameter to the algorithm.

![Figure 1: Matching functions to prunings. Profiles of local smoothness correspond to prunings so that smoother functions are matched to smaller prunings.](image)

If we view the hierarchical packing as a $D$-level tree whose nodes are the balls in the packing at each level, then the local Lipschitzness profile of a function $f$ translates into a pruning of this tree (this is visually

1. We use $f = O(g)$ to denote $f = O(g)$ and $f = \tilde{O}(g)$ to denote $f = \tilde{O}(g)$.

2. The bound for the square loss, which is not contained in [Hazan and Megiddo, 2007], can be proven with a straightforward extension of the analysis in that paper.
explained in Figure 1). By training a base predictor in each ball, we can use the leaves of a pruning \( E \) to approximate a function whose local Lipschitz profile “matches” \( E \). Namely, a function that satisfies (3) with \( L = L_k \) for all observed instances \( x, x' \) that belong to some leaf of \( E \) at level \( k \), for all levels \( k \) (since \( E \) is a pruning of the hierarchical packing \( T \), there is a one-to-one mapping between instances \( x_t \) and leaves of \( E \)). Because our algorithm is simultaneously competitive against all prunings, it is also competitive against all functions whose local Lipschitz profile —with respect to the instance sequence— is matched by some pruning. More specifically, we prove that for any \( f \in \mathcal{F}_L \) and for any pruning \( E \) matching \( f \) on the sequence \( x_1, \ldots, x_T \) of instances,

\[
R_T(f) \leq \mathbb{E} \left[ \left( \sum_{k=1}^{D} (L_k T_{E,k}) \frac{d}{d+1} \right) \frac{d}{d+1} \right] + \sum_{k=1}^{D} (L_k T_{E,k}) \frac{d}{d+1}
\]

(4)

where, from now on, \( T_{E,k} \) always denotes the total number of time steps \( t \) in which the current instance \( x_t \) belongs to a leaf at level \( k \) of the pruning \( E \). The expectation is with respect to the random variable \( K \) that takes value \( k \) with probability equal to the fraction of leaves of \( E \) at level \( k \). The first term in the right-hand side of (4) bounds the estimation error, and is large when most of the leaves of \( E \) reside at deep levels (i.e., \( f \) has just a few regions of low variation). The second term bounds the approximation error, and is large whenever most of the instances \( x_t \) belongs to leaves of \( E \) at deep levels.

In order to compare this bound to (2), consider \( L_k = 2^k \) with \( L = L_D = 2^D \). If \( f \) is matched by some pruning \( E \) such that most instances \( x_t \) belong to shallow leaves of \( E \), then our bound on \( R_T(f) \) becomes of order \( T^{d/(d+1)} \), as opposed to the bound of (2) which is of order \( (2^D T)^{d/(d+1)} \). On the other hand, for any \( f \in \mathcal{F}_L \) we have at least a pruning matching the function: the one whose leaves are all at the deepest level of the tree. In this case, our bound on \( R_T(f) \) becomes of order \( (2^D T)^{d/(d+1)} \), which is asymptotically equivalent to (2). This shows that, up to log factors, our bound is never worse than (2), and can be much better in certain cases.

Our locally adaptive approach can be generalized beyond Lipschitzness. Next, we present two additional contributions where we show that variants of our algorithm can be made adaptive with respect to different local properties of the problem.

**Local metric dimension.** It is well known that nonparametric regret bounds inevitably depend exponentially on the metric dimension of the set of data points [Hazan and Megiddo, 2007, Rakhlin et al., 2015]. Similarly to local Lipschitzness, we want to take advantage of cases in which most of the data points live on manifolds that locally have a low metric dimension. In order to achieve a dependence on the “local dimension profile” in the regret bound, we propose a slight modification of our algorithm, where each level \( k \) of the hierarchical packing is associated with a local dimension bound \( d_k \) such that \( d = d_1 > \cdots > d_D \). Note that —unlike the case of local Lipschitzness— the local dimension is decreasing as the tree gets deeper. Although this might seem counterintuitive, it is explained by the fact that higher-dimensional balls occupy a larger volume than lower-dimensional ones with the same radius, and so they occur at shallower levels in the hierarchical packing.

We say that a pruning of the tree associated with the packing matches a sequence \( x_1, \ldots, x_T \) of instances if the number of leaves of the pruning at each level \( k \) is \( O((L T)^{d_k/(1+d_k)}) \). For regression with square loss we can prove that, for any \( f \in \mathcal{F}_L \) and for any pruning \( E \) matching \( x_1, \ldots, x_T \), this modified algorithm achieves regret

\[
R_T(f) \leq \mathbb{E} \left[ \left( \sum_{k=1}^{D} (L T_{E,k}) \frac{d_k}{1+d_k} \right) \frac{d_k}{1+d_k} \right] + \sum_{k=1}^{D} (L T_{E,k}) \frac{d_k}{1+d_k}
\]

(5)
where, as before, the expectation is with respect to the random variable \( K \) that takes value \( k \) with probability equal to the fraction of leaves of \( E \) at level \( k \). If most \( x_t \) lie in a low-dimensional manifold of \( \mathcal{X} \), so that \( x_1, \ldots, x_T \) is matched by some pruning \( E \) with deeper leaves, we obtain a regret of order \( (L T)^{d_D/(1+d_D)} \).

This is nearly a parametric rate whenever \( d_D < d \). In the worst case, when all instances are concentrated at the top level of the tree, we still recover (2).

**Local loss bounds.** Whereas the local Lipschitz profile measures a property of a function with respect to an instance sequence, and the local dimension profile measures a property of the instance sequence, we now consider the local loss profile, which measures a property of a local online learner with respect to a sequence of examples \((x_t, y_t)\). The local loss profile describes how the cumulative loss of the local predictor changes across different regions of the instance space. To this end, we introduce the functions \( \tau_k \), which upper bound the total loss incurred by our local predictors sitting on nodes at level \( k \). We can use the local predictors on the leaves of a pruning \( E \) to predict a sequence of examples whose local loss profile matches that of \( E \). Namely, such that the online local learners run on the subsequence of examples \((x_t, y_t)\) that belong to leaves at level \( k \) of \( E \) incur a total loss bounded by \( \tau_k(T_{E,k}) \), for all levels \( k \). In order to take advantage of good local loss profiles, we focus on losses —such as the absolute loss— for which we can prove “first-order” regret bounds that scale with the loss of the expert against which the regret is measured. For the absolute loss, the algorithm we consider attains regret

\[
R_T(f) = \mathbb{E} \left[ (L \tau_K(T))^\frac{d}{d+1} \right] + \sum_{k=1}^{D} \mathbb{E} \left[ (L \tau_k(T_{E,k}))^\frac{d}{d+1} \right] + \sqrt{\mathbb{E} \left[ (L \tau_K(T))^\frac{d}{d+1} \right]} \sum_{k=1}^{D} \tau_k(T_{E,k})
\]

for any \( f \in \mathcal{F}_L \), where —as before— the expectation is with respect to the random variable \( K \) that takes value \( k \) with probability equal to the fraction of leaves of \( E \) at level \( k \). For concreteness, set \( \tau_k(n) = n^{\frac{1}{d+1-k}} \), so that deeper levels \( k \) correspond to loss rates that grow faster with time. When \( E \) has shallow leaves and \( T_{E,k} \) is negligible for \( k > 1 \), the regret becomes of order \( (L T)^{\frac{d}{d+1}} \), which has significantly better dependence on \( T \) than \( L^{\frac{d}{d+1}} T^{\frac{d+1}{d+2}} \) achieved by \textsc{HM}. Note that we have a pruning matching all sequences: the one whose leaves are all at the deepest level of the tree. Indeed, \( \tau_D(n) = n \) is a trivial upper bound on the absolute loss of any online local learner. In this case, our bound on \( R_T(f) \) becomes of order \( (L T)^{\frac{d+1}{d+2}} \), which is asymptotically equivalent in \( T \) compared to (2). Note that our dependence on the Lipschitz constant is slightly worse than (2). This happens because we have to pay an additive constant regret term in each ball which is unavoidable in any first-order regret bounds.

**Intuition about the proof.** Hazan and Megiddo [2007] prove (2) using a greedy construction of a ball packing of the instance space, where each ball hosts a local online learner, and the label for a new instance is predicted by the learner in the nearest ball. Balls shrink at a polynomial rate in time, and a new ball is allocated whenever an instance falls outside the current packing. The algorithms we present here generalize this approach to a hierarchical construction of packings at multiple levels. Each ball at a given level contains a lower-level packing using balls of smaller radius, and we view this nested structure of packings as a tree. Radii are now tuned not only with respect to time, but also with respect to the level \( k \), where the dependence on \( k \) is characterized by the specific locality setting (i.e., local smoothness, local dimension, or local losses). The main novelty of our proof is in the fact that we analyze \textsc{HM} in a level-wise manner, while simultaneously competing against the best pruning over the entire hierarchy. Our approach is adaptive because regret now depends on both the number of leaves of the best pruning and the number of observations made by the pruning
at each level. In other words if the best pruning has no leaves at a particular level, or is active for a few time steps at that level, then the algorithm will seldom use the local predictors hosted at that level.

Our main algorithmic technology is the sleeping experts framework of Freund et al. [1997], where each node of the tree is treated as an expert predicting with the learner hosted in the associated ball, and active (non-sleeping) experts in a given time step are those along the root-to-leaf path associated with the current instance. For regression with square loss we use exponential weights (up to re-normalization due to active experts). For classification with absolute loss, we avoid the tuning problem by resorting to the parameter-free algorithm AdaNormalHedge of Luo and Schapire [2015]. This makes our approach computationally efficient: despite the exponential number of experts in the comparison class we only pay in the regret a factor corresponding to the depth of the tree.

2 Definitions

Throughout the paper, we assume instances \( x_t \) have a bounded norm, \( \|x_t\| \leq 1 \), so that \( \mathcal{X} \) is the unit ball with center in 0. We use \( B(z,r) \) to denote the ball of center \( z \in \mathbb{R}^d \) and radius \( r > 0 \), and we write \( B(r) \) instead of \( B(0,r) \).

**Definition 1** (Coverings and packings). An \( \varepsilon \)-cover of a set \( \mathcal{X}_0 \subseteq \mathcal{X} \) is a subset \( \{x'_1, \ldots, x'_n\} \subseteq \mathcal{X}_0 \) such that for each \( x \in \mathcal{X}_0 \) there exists \( i \in \{1, \ldots, n\} \) such that \( \|x - x'_i\| \leq \varepsilon \). An \( \varepsilon \)-packing of a set \( \mathcal{X}_0 \subseteq \mathcal{X} \) is a subset \( \{x'_1, \ldots, x'_m\} \subseteq \mathcal{X}_0 \) such that for any distinct \( i, j \in \{1, \ldots, m\} \), we have \( \|x'_i - x'_j\| > \varepsilon \).

**Definition 2** (Metric dimension). A set \( \mathcal{X} \) has metric dimension \( d \) if there exists \(^3\) \( C > 0 \) such that, for all \( \varepsilon > 0 \), \( \mathcal{X} \) has an \( \varepsilon \)-cover of size at most \( C \varepsilon^{-d} \).

In this paper we consider the following online learning protocol with oblivious adversary. Given an unknown sequence \( (x_1, y_1), (x_2, y_2), \ldots \in \mathcal{X} \times \mathcal{Y} \) of instances and labels, for every round \( t = 1, 2, \ldots \)

1. The environment reveals the instance \( x_t \in \mathcal{X} \).
2. The learner selects an action \( \hat{y}_t \in \mathcal{Y} \) and incurs the loss \( \ell(\hat{y}_t, y_t) \).
3. The learner observes \( y_t \).

In the rest of the paper, we use \( \ell_t(\hat{y}_t) \) as an abbreviation for \( \ell(\hat{y}_t, y_t) \).

### 2.1 Hierarchical packings, trees, and prunings

A *pruning* of a rooted tree is the tree obtained after the application of zero or more replace operations, where each replace operation deletes the subtree rooted at an internal node without deleting the node itself (which becomes a leaf).

Recall that our algorithms work by sequentially building a hierarchical packing of the instance sequence. This tree-like structure is defined as follows.

**Definition 3** (Hierarchical packing). A hierarchical packing of depth \( D \) of an instance sequence \( \sigma_T = (x_1, \ldots, x_T) \) is a sequence of nonempty subsets \( S_1 \subseteq \cdots \subseteq S_D \subseteq \{1, \ldots, T\} \) and radii \( \varepsilon_1 > \cdots > \varepsilon_D > 0 \) satisfying the following properties. For each level \( k = 1, \ldots, D \):

1. the set \( S_k \) is a \( \varepsilon_k \)-packing of the elements of \( \sigma_T \) with balls \( \{B(x_s, \varepsilon_k)\}_{s \in S_k} \)
2. for all \( t = 1, \ldots, T \) either \( t \in S_k \) or \( x_t \in B(x_s, \varepsilon_k) \) for some \( s \in S_k \).
3. if \( t \in S_{k+1} \setminus S_k \), then there exists \( s \in S_k \) such that \( x_t \in B(x_s, \varepsilon_k) \).

\(^3\)Note that \( C \) is exactly quantifiable for various metrics [Clarkson, 2006].
Any such hierarchical packing can be viewed as a rooted tree $\mathcal{T}$ (conventionally, the root of the tree is the unit ball $\mathcal{X}$) defined by the parent function, where $x_s = \text{PARENT}(x_t)$ if and only if $x_t \in B(x_s, \varepsilon_k)$ for $s \in S_k$, $t \in S_{k+1}$ and $k = 1, \ldots, D - 1$ —see Figure 2.

Given an instance sequence $\sigma_T$, let $\mathcal{T}_D(\sigma_T)$ be the family of all trees $\mathcal{T}$ of depth $D$ generated from $\sigma_T$ by choosing the $\varepsilon_k$-packings at each level in all possible ways. Given $\mathcal{T}$ and its pruning $E$, we use $\text{LEAVES}_k(\mathcal{T}, E)$ to denote the subset of $S_k$ containing the nodes of $\mathcal{T}$ that correspond to leaves of $E$ —see Figure 3. When $\mathcal{T}$ is clear from the context, we abbreviate $\text{LEAVES}_k(\mathcal{T}, E)$ with $E_k$. For any fixed $\mathcal{T} \in \mathcal{T}_D(\sigma_T)$ let also $|E| = |E_1| + \cdots + |E_D|$ be the number of leaves in $E$.

### 3 Related Work

In nonparametric prediction, a classical topic in statistics, one is interested in predicting well compared to the best function in a large class, which typically includes all functions that have certain regularities. In online learning, nonparametric prediction was studied by Vovk [2006a,b, 2007], who analyzed the regret of algorithms against Lipschitz function classes with bounded metric entropy. Rakhlin and Sridharan [2014] later used a non-constructive argument establishing minimax regret rates $T^{(d-1)/d}$ (when $d > 2$) for both square and absolute loss. Inspired by their work, Gaillard and Gerchinovitz [2015] devised the first online algorithms for nonparametric regression enjoying minimax regret. A computationally efficient variant of their algorithm, with running time $O(\text{poly}(T))$, relies on a nested covering of a function class, where —roughly speaking— functions are approximated by an aggregation of indicator functions at different levels of a cover. In this work we employ a nested packing approach, which bears a superficial resemblance to the construction of Gaillard and Gerchinovitz [2015] and to the analysis technique of Rakhlin and Sridharan [2014]. However, the crucial difference is that we hierarchically cover the input space, rather than the function class, and use
local no-regret learners within each element of the cover. Our algorithm is conceptually similar to the one of Hazan and Megiddo [2007], however their space packing can be viewed as a “flat” version of the one proposed here, while their analysis only holds for a known time horizon (which is later improved till unknown one by Kpotufe and Orabona [2013]).

Our algorithms adapt to the regularity of the problem in an online fashion using the tree-expert variant of the prediction with expert advice setting —see also [Cesa-Bianchi and Lugosi, 2006]. In this setting, originally introduced by Helmbold and Schapire [1997], there is a tree-expert for each pruning of a complete tree with a given branching factor. Although the number of such prunings is exponential, predictions and updates can be performed in time linear in the tree depth \( D \) using the context tree algorithm of Willems et al. [1995]. In this work we consider a conceptually simpler version, which relies on sleeping experts [Freund et al., 1997], where each node of a tree is associated with an expert, and on each round only \( D \) experts are awake. The goal is to compete against the best pruning in hindsight, which typically requires knowledge of the pruning size for tuning purposes. In case of prediction with absolute loss, we avoid the tuning problem by exploiting a parameter-free algorithm of Luo and Schapire [2015].

Local adaptivity to regularities of a competitor, as discussed in the current paper, can be also viewed as automatic parameter tuning through hierarchical expert advice. A similar idea, albeit without the use of a hierarchy, was explored by van Erven and Koolen [2016] for automatic step size tuning in online convex optimization —see [Orabona and Pál, 2016] for a detailed discussion on the topic. Finally, the idea to exploit a variant of a context tree for nonlinear classification was also explored in neural network learning by Veness et al. [2019], where —roughly speaking— context trees are used to combine randomly initialized halfspaces. While standard results in statistics assume some form of a uniform regularity of an optimal function (such as Lipschitzness or Hölder continuity), several works have investigated nonparametric regression under local smoothness assumptions. For instance, Mammen and van de Geer [1997] considered a one-dimensional nonparametric regression problem with a fixed design, where the regression function belongs to the class of \( k \)-times weakly differentiable functions with bounded total variation. They proposed and analyzed locally-adaptive regression splines, where an estimator is a variant of Regularized Least Squares (RLS) with a total variation penalty, and showed minimax optimal rates with exponential dependence in \( k \). A similar direction was also pursued by Tibshirani [2014] through trend filtering. He proposed a less computationally intensive algorithm with comparable rates. Unlike these works, here we address local Lipschitzness in general metric spaces without any statistical assumptions.

Adaptivity of \( k \)-NN regression and kernel regression to the local effective dimension of the stochastic data-generating process was studied by Kpotufe [2011], Kpotufe and Garg [2013], however they considered a notion of locality different from the one studied here. The idea of adaptivity to the global effective dimension, combined with the packing construction of Hazan and Megiddo [2007] in the online setting, were proposed by Kpotufe and Orabona [2013]. Kuzborskij and Cesa-Bianchi [2017] investigated a stronger form of adaptivity to the dimension in nonparametric online learning, which is related to the recovering of the subspace where the target function is smoother. In online convex optimization, adaptivity to the global Lipschitz constant of the loss function was recently proposed by Mhammedi et al. [2019].

Finally, related ideas of hierarchical covering were also explored in the global optimization literature [Munos, 2011], where adaptivity to local regularities is exploited for the search of critical points.

4 Description of the algorithm

Recall that we identify a hierarchical packing \( S_1, \ldots, S_D \) with a tree \( T \) whose nodes correspond to the elements of the packing. Our algorithm predicts using a hierarchical packing \( T \) evolving with time, and
competes against the best pruning of the tree corresponding to the final hierarchical packing. A local online learner is associated with each node of \( T \) except for the root. When a new instance \( x_t \) is observed, it is matched with the closest center \( x_s \in S_k \) at each level \( k \), until a leaf is reached. The local learners associated with these closest centers output predictions, which are then aggregated using an algorithm for prediction with expert advice, where the local learner at each node is viewed as an expert. Since only a fraction of experts (i.e., those associated with the closest centers, which form a path in a tree) are active at any given round, this can be viewed as an instance of the “sleeping experts” framework of Freund et al. [1997]. In the regression case, since the square loss is exp-concave for bounded predictions, we can directly apply the results of Freund et al. [1997]. In the classification case, we use instead the parameter-free approach of Luo and Schapire [2015].

One might wonder if a similar algorithm can be formulated without dynamically evolving packing by constructing a fixed partition of the instance space ahead of time. Such algorithm would be inferior to ours since it would be competitive only for a known time horizon (unless one would use a cumbersome doubling trick or resort to a non-trivial tree-growing extension of the algorithm). In addition, identifying an element in such partition is straightforward for \( L_\infty \) metric, while it would be computationally non-trivial for an arbitrary metric. On the other hand, the dynamic algorithm presented here works with any metric and enjoys local adaptivity on the induced metric space.

Algorithm 1 contains the pseudocode for the case of exp-concave loss functions. The algorithm invokes two subroutines propagate and update. The former collects the predictions of the local learners along the path of active experts corresponding to an incoming instance, the latter updates these learners. We use \( \pi_t \) to denote the root-to-leaf path in \( T \) of active experts associated with the current instance \( x_t \). The subroutine propagate finds in each level \( k \) the center closest to \( x_t \). Then, the path \( \pi \) of active experts associated with these centers and the vector \( \hat{y}_t \) of their predictions are returned to the algorithm (line 4). The sum \( W_{t-1} \) of the current weight \( w_{v,t-1} \) of each active expert on the path \( \pi_t \) is computed in line 5, where \( v \subseteq \pi_t \) is used to denote a node in \( T \) whose path is a prefix of \( \pi_t \). This sum is used to compute the aggregated prediction on
After observing the true label \(y_t\) (line 7), the subroutine \texttt{update} updates the active experts. Finally, the weights of the active experts are updated (lines 9 and 11).

We now describe concrete implementations of \texttt{propagate} and \texttt{update} which will be used in Section 5. For simplicity, we assume that all variables of the meta-algorithm which are not explicitly given as input values are visible to both procedures.

**Subroutine 2** \texttt{propagate}.

**Require:** instance \(x_t \in \mathcal{X}\), time step index \(t\)

1: \(B_{\text{PARENT}} \leftarrow \mathcal{X}\) \hspace{1cm} ▷ Start from root
2: for depth \(k = 1, \ldots, D\) do
3: \hspace{1cm} if \(S_k \equiv \emptyset\) then
4: \hspace{2cm} \(S_k \leftarrow \{t\}\) \hspace{1cm} ▷ Create initial ball at depth \(k\)
5: \hspace{2cm} Create predictor at \(x_t\)
6: \hspace{1cm} end if
7: \hspace{1cm} \(s \leftarrow \arg \min_{i \in S_k} \|x_i - x_t\|\) \hspace{1cm} ▷ Find active expert at level \(k\)
8: \hspace{1cm} \(\pi_k \leftarrow s\) \hspace{1cm} ▷ Add index of active expert to path
9: \hspace{1cm} \(\hat{y}_{s,t} \leftarrow\) prediction of active expert \hspace{1cm} ▷ Add prediction to prediction vector
10: \hspace{1cm} \(\varepsilon \leftarrow \rho(k, t)\) \hspace{1cm} ▷ Get current radius
11: \hspace{1cm} \(B_{\text{PARENT}} \leftarrow B(x_s, \varepsilon)\) \hspace{1cm} ▷ Set ball of active expert as current element in the packing
12: end for

**Ensure:** path \(\pi\) of active experts and vector \(\hat{y}\) of active expert predictions

The subroutine \texttt{propagate} finds in a tree \(T\) the path of active experts associated with an instance \(x_t\). When invoked at time \(t = 1\), the tree is created as a list of nested balls with common center \(x_1\) and radii \(\varepsilon_{k,1}\) for \(k = 1, \ldots, D\) (lines 4–5). For all \(t > 1\), starting from the root node set as parent node (line 1), the procedure finds in each level \(k\) the center \(x_s\) closest to the current instance \(x_t\) among those centers which belong to the parent node (line 7). Note that the parent node is a ball and therefore there is at least one center in \(B_{\text{PARENT}}\).

The active expert indices are collected in a vector \(\pi\), while their predictions are stored in a vector \(\hat{y}\) and then aggregated using Algorithm 1. We use \(T_i\) to denote the subset of time steps on which the expert at node \(i\) is active. These are the \(t \in \{1, \ldots, T\}\) such that \(i\) occurs in \(\pi_t\).

**Subroutine 3** \texttt{update}.

**Require:** Path \(\pi\) of active experts, example \((x_t, y_t)\), time step \(t\)

1: for depth \(k = 1, \ldots, D\) do
2: \hspace{1cm} \(\varepsilon \leftarrow \rho(k, t)\) \hspace{1cm} ▷ Get current radius
3: \hspace{1cm} \(s \leftarrow \pi_k\) \hspace{1cm} ▷ Get next active expert in path \(\pi\)
4: \hspace{1cm} if \(\|x_t - x_s\| \leq \varepsilon\) then
5: \hspace{2cm} Update active expert \(s\) using \((x_t, y_t)\)
6: \hspace{1cm} else
7: \hspace{2cm} \(S_k \leftarrow S_k \cup \{t\}\) \hspace{1cm} ▷ Add new center to level \(k\)
8: \hspace{2cm} Create predictor at \(x_t\) and initialize it with \((x_t, y_t)\)
9: \hspace{1cm} end if
10: end for

The subroutine \texttt{update} checks whether the current instance belongs to the each ball that host an active
expert listed in $\pi$. If $x_t$ belongs to the active ball at level $k$, then $(x_t, y_t)$ is used to update the expert (line 5). If $x_t$ is outside of the active ball at level $k$, then a new ball with center $x_t$ is created in the packing at that level. Then, a new predictor is created associated with that ball and initialized using the current example $(x_t, y_t)$ (line 8).

5 Nonparametric regression with local Lipschitzness

We first consider the case of local Lipschitz bounds for regression with square loss $\ell_t(\hat{y}) = \frac{1}{2} (y_t - \hat{y})^2$, where $y_t \in [0, 1]$ for all $t \geq 1$. Here we use Follow-the-Leader (FTL) as local online predictor. As explained in the introduction, we need to match prunings to functions with certain local Lipschitz profiles. This is implemented by the following definition.

**Definition 4** (Functions admissible with respect to a pruning). Given $0 < L_1 < \cdots < L_D$, a hierarchical packing $T \in T_D(\sigma_T)$ of an instance sequence $\sigma_T$, and a time-dependent radius tuning function $\rho$, we define the set of admissible functions with respect to a pruning $E$ of $T$ by $F(E, T) \equiv \{ f : \mathcal{X} \to [0, 1] \mid \forall x \in B(x_i, \rho(k, t)), \forall i \in \text{LEAVES}_k(T, E) \}
\leq L_k \rho(k, t), \quad k = 1, \ldots, D, \quad t = 1, \ldots, T\}.$

Equipped with this definition, we establish a regret bound with respect to admissible functions. Recall that $T_{E, k}$ is the total number of time steps $t$ in which the current instance $x_t$ belongs to a leaf at level $k$ of the pruning $E$.

**Theorem 1.** Given $0 < L_1 < \cdots < L_D$, suppose that Algorithm 1 using Subroutines 2 and 3 is run for $T$ rounds with radius tuning function $\rho(k, t) = (L_k t)^{-\frac{1}{d+1}}$, and let $T$ be the resulting hierarchical packing. Then, for all prunings $E$ of $T$ the regret satisfies

$$R_T(f) \overset{\mathbb{E}}{=} \mathbb{E}\left[ L_K^{\frac{d}{d+1}} \right] T^{\frac{d}{d+1}} + \sum_{k=1}^{D} (L_k T_{E, k})^{\frac{d}{d+1}} \forall f \in F(E, T).$$

The expectation is understood with respect to the random variable $K$ that takes value $k$ with probability equal to the fraction of leaves of $E$ at level $k$.

Since $T$ is the hierarchical packing generated by Algorithm 1, the prunings $E$ and the admissible functions $F(E, T)$ depend on the algorithm through $T$. Similar remarks hold for our results in Sections 6 and 7.

6 Nonparametric regression with local dimension

In this section we look at a different notion of adaptivity, namely we demonstrate that Algorithm 1 is also capable of adapting to the local dimension of the data sequence. We consider a decreasing sequence $d = d_1 > \cdots > d_D$ of local dimension bounds, where $d_k$ is assigned to the level $k$ of the hierarchical packing maintained by Algorithm 1. We also make a small modification to update (Subroutine 3). Namely, we add a new center at level $k$ only if the designated size of the packing (which depends on the local dimension bound) has not been exceeded. The modified subroutine is updateDim (Algorithm 4).

Since the local dimension assumption is made on the instance sequence rather than on the function class, in this scenario we may afford to compete against the class $F_L$ of all $L$-Lipschitz functions, while we restrict
Subroutine 4 updateDim.

Require: Path \( \pi \) of active experts, example \((x_t, y_t)\), time step \( t \), \( C \) (see Def. 2)

1: for depth \( k = 1, \ldots, D \) do
2: \[ \varepsilon \leftarrow \rho(k, t) \] \( \triangleright \) Get current radius
3: \[ s \leftarrow \pi_k \] \( \triangleright \) Get next active expert in path \( \pi \)
4: if \( \|x_t - x_s\| \leq \varepsilon \) then
5: Update active expert \( s \) using \((x_t, y_t)\)
6: else if \( |S_k| \leq C (2/\varepsilon)^{dk} \) then
7: \[ S_k \leftarrow S_k \cup \{t\} \] \( \triangleright \) Restrict packing size at each level
8: Create predictor at \( x_t \) and initialize it with \((x_t, y_t)\)
9: end if
10: end for

We prove the following regret bound.

**Theorem 2.** Given \( d = d_1 > \cdots > d_D \), suppose that Algorithm 1 using Subroutines 2 and 4 is run for \( T \) rounds with radius tuning function \( \rho(k, t) = (L t)^{-\frac{1}{1+dk}} \), and let \( T \) the resulting hierarchical packing. Then, for all prunings \( E \in \mathcal{E}_{\text{dim}}(T) \) the regret satisfies

\[
R_T(f) \leq \mathbb{E}\left[ (L T)^{\frac{d_k}{1+dk}} \right] + \sum_{k=1}^{D} (L T_{E,k})^{\frac{d_k}{1+dk}} \quad \forall f \in \mathcal{F}_L.
\] (8)

### 7 Nonparametric classification with local losses

The third notion of adaptivity we study is with respect to the loss of the local learners in each node of a hierarchical packing. The local loss profile is parameterized with respect to a sequence \( \tau_1, \ldots, \tau_D \) of nonnegative and nondecreasing \( \tau_k : \{1, \ldots, T\} \to \mathbb{R} \) such that each \( \tau_k \) bounds the total loss of all local learners at level \( k \) of the hierarchical packing. In order to achieve better regrets when the data sequence can be predicted well by local learners in a shallow pruning we assume \( \tau_1(n) < \cdots < \tau_D(n) = n \) for all \( n = 1, \ldots, T \), where the choice of \( \tau_D(n) = n \) allows us to fall back to the standard regret bounds if the data sequence is hard to predict.

Whereas in Sections 5 and 7, where we consider regression with the square loss, here we work with binary classification with absolute loss \( \ell_t(\hat{y}_t) = |\hat{y}_t - y_t| \), which — unlike the square loss—is not exp-concave. As we explained in Section 1, using losses that are not exp-concave is motivated by the presence of first-order regret bounds, which allow us to take advantage of good local loss profiles. While the exp-concavity of the square loss dispensed us from the need of tuning Algorithm 1 using properties of the pruning, here we circumvent the tuning issue by replacing Algorithm 1 with the parameter-free Algorithm 5 (stated in
Appendix A), which is based on the AdaNormalHedge algorithm of Luo and Schapire [2015]. Instead of the standard exponential weights on which the updates of Algorithm 1 are based, AdaNormalHedge performs update using the function

\[
\psi(r, c) = \frac{1}{2} \left( \exp \left( \frac{(r + 1)^2}{3(c + 1)} \right) - \exp \left( \frac{(r - 1)^2}{3(c + 1)} \right) \right).
\]

As online local learners we use self-confident Weighted Majority [Cesa-Bianchi and Lugosi, 2006, Exercise 2.10] with two constant experts predicting 0 and 1. In the following, we denote by \( \Lambda_{i,T} \) the cumulative loss of a local learner at node \( i \) over the time steps \( T \) when the expert is active. Similarly to the previous section, we compete against the class \( \mathcal{F}_L \) of all Lipschitz functions, and introduce the following constraint on the prunings

\[
\mathcal{E}_{\text{loss}}(\mathcal{T}) = \left\{ E \in \mathcal{T} : \sum_{i \in \text{LEAVES}_E(\mathcal{T}, E)} \Lambda_{i,T} \leq \tau_k(T_{E,k}), \ k = 1, \ldots, D \right\}.
\]  

(9)

If \( E \in \mathcal{E}_{\text{loss}}(\mathcal{T}) \) then the total loss of all the leaves at a particular level behaves in accordance with \( \tau_1, \ldots, \tau_D \).

Theorem 3. Suppose that the Algorithm 5 runs self-confident weighted majority at each node with radius tuning function

\[
\rho(k, t) = (L \tau_k(t))^{-\frac{1}{2+d}}
\]

and let \( \mathcal{T} \) the resulting hierarchical packing. Then for all pruning \( E \in \mathcal{E}_{\text{loss}}(\mathcal{T}) \) the regret satisfies \( \forall f \in \mathcal{F}_L: \)

\[
R_T(f) \preceq E \left[ (L \tau_K(T))^\frac{d}{2+d} \right] + \sum_{k=1}^{D} (L \tau_k(T_{E,k}))^\frac{1+d}{2+d} + \sqrt{E \left[ (L \tau_K(T))^\frac{d}{2+d} \right] \sum_{k=1}^{D} \tau_k(T_{E,k})}.
\]

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A Algorithm for nonparametric classification with local losses

Algorithm 5 Locally Adaptive Online Learning (AdaNormalHedge style)

Require: Depth parameter $D$, radius tuning function $\rho : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$

1: $S_1 \leftarrow \emptyset, \ldots, S_D \leftarrow \emptyset$ \hspace{1cm} $\triangleright$ Centers at each level
2: for each round $t = 1, 2, \ldots$ do
3: \hspace{1cm} Receive $x_t$ \hspace{1cm} $\triangleright$ Prediction
4: \hspace{1cm} $(\pi_t, \hat{y}_t) \leftarrow \text{propagate}(x_t, t)$ \hspace{1cm} $\triangleright$ Algorithm 2
5: \hspace{1cm} for each $v \subseteq \pi_t$ do
6: \hspace{2cm} if $t = 1$ then
7: \hspace{3cm} $w_{v,t} \leftarrow \psi(0,0)$
8: \hspace{2cm} else
9: \hspace{3cm} $w_{v,t} \leftarrow \psi(\bar{r}_{v,t-1}, C_{v,t-1})$
10: \hspace{2cm} end if
11: \hspace{1cm} end for
12: \hspace{1cm} Predict $\hat{y}_t \leftarrow \frac{1}{Z_t} \sum_{v \subseteq \pi_t} w_{v,t} \hat{y}_{v,t}$ where $Z_t = \sum_{v \subseteq \pi_t} w_{v,t}$ \hspace{1cm} $\triangleright$ Update
13: \hspace{1cm} Observe $y_t$
14: \hspace{1cm} update$(\pi_t, x_t, y_t)$ \hspace{1cm} $\triangleright$ Algorithm 3
15: \hspace{1cm} $\ell_t \leftarrow \sum_{v \subseteq \pi_t} w_{v,t} \ell_t(\hat{y}_{v,t})$
16: \hspace{1cm} for each $v \subseteq \pi_t$ do
17: \hspace{2cm} $r_{v,t} \leftarrow \ell_t - \ell_t(\hat{y}_{v,t}), \quad \bar{r}_{v,t} \leftarrow \bar{r}_{v,t-1} + r_{v,t}, \quad C_{v,t} \leftarrow C_{v,t-1} + |r_{v,t}|$
18: \hspace{1cm} end for
19: end for

B Learning with expert advice over trees

In order to prove the regret bounds in our locally-adaptive learning setting, we start by deriving bounds for prediction with expert advice when the competitor class is all the prunings of a tree whose each node hosts an
expert, a framework initially investigated by Helmbold and Schapire [1997]. Our analysis uses the sleeping experts setting of Freund et al. [1997], in which only a subset $E_t$ of the node experts are active at each time step $t$. In our locally-adaptive setting, the set of active experts at time $t$ corresponds to the active root-to-leaf path $\pi_t$ selected by the current instance $x_t$ —see Section 4. The inactive experts at time $t$ neither output predictions nor get updated. The prediction of a pruning $E$ at time $t$, denoted with $f_{E,t}$ is the prediction $\hat{y}_{i,t}$ of the node expert corresponding to the unique leaf $i$ of $E$ on $\pi_t$.

**Algorithm 6** Learning over trees through sleeping experts

**Require:** Tree $T$ and initial weights for each node of the tree

1. for each round $t = 1, 2, \ldots$ do
2. Observe predictions of active experts $E_t$ (corresponding to a root-to-leaf path in the tree)
3. Predict $\hat{y}_t$ and observe $y_t$
4. Update the weight of each active expert
5. end for

Next, we consider two algorithms for the problem of prediction with expert advice over trees. In order to be simultaneously competitive with all prunings, we need algorithms that do not require tuning of their parameters depending on the specific pruning against which the regret is measured. In case of exp-concave losses (like the square loss) tuning is not required and Hedge-style algorithms work well. In case of generic convex losses, we use the more complex parameterless algorithm AdaNormalHedge.

We start by recalling the algorithm for learning with sleeping experts and the basic regret bound of Freund et al. [1997]. The sleeping experts setting assumes a set of $M$ experts without any special structure. At every time step $t$ only an adversarially chosen subset $E_t$ of the experts provides predictions and gets updated —see Algorithm 7. The regret bound is parameterized in terms of the relative entropy $\text{KL}(u \parallel w_1)$ between the initial of distribution over experts $w_1$ and any target distribution $u$. The following theorem states a slightly more general bound that holds for any $\eta$-exp-concave loss function (for completeness, the proof is given in Appendix A).

**Theorem 4** ([Freund et al., 1997]). If Algorithm 7 is run on any sequence $\ell_1, \ldots, \ell_T$ of $\eta$-exp-concave loss functions, then for any sequence $E_1, \ldots, E_T \subseteq \{1, \ldots, M\}$ of awake experts and for any distribution $u$ over $\{1, \ldots, M\}$, the following holds

$$\sum_{t=1}^T U_t \ell_t(\hat{y}_t) - \sum_{t=1}^T \sum_{i \in E_t} u_i \ell_t(\hat{y}_{i,t}) \leq \frac{1}{\eta} \text{KL} \left( u \parallel \frac{w_1}{\|w_1\|_1} \right)$$

(10)
where $U_t = \sum_{i \in E_t} u_i$.

By taking $w_1$ to be uniform over the experts, the above theorem implies a bound with a $\ln M$ factor. However, since we predict and perform updates only with respect to awake experts, this can be improved to $\ln M_T$, where $M_T$ is the number of distinct experts ever awake throughout the $T$ time steps. The following lemma (whose proof is deferred to Appendix A) formally states this fact.

Fix a sequence $E_1, \ldots, E_T \subseteq \{1, \ldots, M\}$ of awake experts such that $|E_1 \cup \cdots \cup E_T| = M_T$. Let the uniform distribution supported over the awake experts, denoted with $w_1^E$, be defined by $w_{i,1}^E = 1/M_T$ if $i \in E_1 \cup \cdots \cup E_T$ and 0 otherwise.

**Lemma 1.** Suppose Algorithm 7 is run with initial weights $w_{i,1} = 1$ for $i = 1, \ldots, M$ and with a sequence $E_1, \ldots, E_T \subseteq \{1, \ldots, M\}$ of awake experts. Then the regret of the algorithm initialized with $w_1$ matches the regret of the algorithm initialized with $w_1^E$.

We use Theorem 4 and Lemma 1 to derive a regret bound for Algorithm 6 when predictions and updates are provided by Algorithm 7. The same regret bound can be achieved through the analysis of [Mourtada and Maillard, 2017, Theorem 3], albeit their proof follows a different argument.

**Theorem 5.** Suppose that Algorithm 6 is run using predictions and updates provided by Algorithm 7. Then, for any sequence $\ell_1, \ldots, \ell_T$ of $\eta$-exp-concave losses and for any pruning $E$ of the input tree $T$,

$$\sum_{t=1}^T (\ell_t(\hat{y}_t) - \ell_t(f_{E,t})) \leq \frac{|E|}{\eta} \ln \frac{M_T}{|E|}.$$  

**Proof.** Let $u$ be the uniform distribution over the $|E|$ terminal nodes of $E$. At each round, exactly one terminal node of $E$ is in the active path of $T$. Therefore $\ell_t(f_{E,t}) = \sum_{i \in E_t} u_i \ell_t(\hat{y}_{i,t})$, and also $U_t = \frac{1}{|E|}$ for all $t$ because only one expert in $E_t$ is awake in the support of $u$. Now note that although the algorithm is actually initialized with $w_{i,1} = 1$, Lemma 1 shows that the regret remains the same if we assume the algorithm is initialized with $w_1^E$. The choice of the competitor $u$ gives us $\text{KL}(u \mid \mid w_1^E) = \ln (M_T/|E|)$. By applying Theorem 4 we finally get

$$\sum_{t=1}^T U_t \ell_t(\hat{y}_t) - \sum_{t=1}^T \sum_{i \in E_t} u_i \ell_t(\hat{y}_{i,t})$$

$$= \frac{1}{|E|} \sum_{t=1}^T (\ell_t(\hat{y}_t) - \ell_t(f_{E,t}))$$

(only one expert awake in the active path)

$$\leq \frac{1}{\eta} \ln \frac{M_T}{|E|},$$

concluding the proof.  

In case of general convex losses, we simply apply the following theorem where $\Lambda_E = \ell_1(f_{E,1}) + \cdots + \ell_T(f_{E,T})$ is the cumulative loss of pruning $E$.

**Theorem 6** (Section 6 in [Luo and Schapire, 2015]). Suppose that Algorithm 6 is run using predictions and updates provided by AdaNormalHedge. Then, for any sequence $\ell_1, \ldots, \ell_T$ of convex losses and for any pruning $E$ of the input tree $T$,

$$\sum_{t=1}^T (\ell_t(\hat{y}_t) - \ell_t(f_{E,t})) \overset{\mathcal{O}}{=} \sqrt{|E| \Lambda_E \ln \frac{M_T}{|E|}}.$$
C Proofs for nonparametric prediction

We start by proving a master regret bound that can be specialized to various settings of interest. Recall that the prediction of a pruning $E$ at time $t$ is $f_{E,t} = \hat{y}_{t,i}$, where $\hat{y}_{t,i}$ is the prediction of the node expert sitting at the unique leaf $i$ of the pruning $E$ on the active path $\pi_t$. Recall also that $x_i$ is the center of the ball in the hierarchical packing corresponding to node $i$ in the tree. As in our locally-adaptive setting node experts are local learners, $\hat{y}_{t,i}$ should be viewed as the prediction of the local online learning algorithm sitting at node $i$ of the tree. Let $T_i$ be the subset of time steps when $i$ is on the active path $\pi_t$. We now introduce the definitions of regret for the tree expert

$$R_{\text{tree}}(E) = \sum_{t=1}^{T} \left( \ell_t(\hat{y}_t) - \ell_t(f_{E,t}) \right)$$

and for node expert $i$

$$R_{\text{loc}}(i) = \sum_{t \in T_i} \left( \ell_t(\hat{y}_{t,i}) - \ell_t(y^*_i) \right)$$

where $\mathcal{H}$ is either $[0,1]$ (regression with square loss) or $\{0,1\}$ (classification with absolute loss), and

$$y^*_i = \arg \min_{y \in \mathcal{H}} \sum_{t \in T_i} \ell_t(y) .$$

Note that, for all $f : \mathcal{X} \to [0,1]$ and for $y^*_i$ defined as above,

$$\sum_{t \in T_i} \left( \ell_t(y^*_i) - \ell_t(f(x_i)) \right) \leq 0 . \quad (11)$$

Lemma 2. Suppose that Algorithm 1 (or, equivalently, Algorithm 5) is run on a sequence $\ell_1, \ldots, \ell_T$ of convex and $L'$-Lipschitz losses and let $T$ be the resulting hierarchical packing. Then for any pruning $E$ of $T$ and for any $f : \mathcal{X} \to \mathcal{Y}$,

$$R_T(f) \leq R_{\text{tree}}(E) + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} R_{i,T}^\text{loc} + L' \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} |f(x_i) - f(x_t)| .$$

Proof. We decompose regret into two terms: one capturing the regret of the algorithm with respect to a pruning $E$, and one capturing the regret of $E$ against the competitor $f$,

$$R_T(f) = \sum_{t=1}^{T} \left( \ell_t(\hat{y}_t) - \ell_t(f(x_t)) \right) = R_{\text{tree}}(E) + \sum_{t=1}^{T} \left( \ell_t(f_{E,t}) - \ell_t(f(x_t)) \right) .$$

We now split the second term into estimation and approximation error. Define the prediction of a local learner
at node $i$ and time step $t$ as $\hat{y}_{i,t}$.

\[ \sum_{t=1}^{T} \left( \ell_t(f_{E,t}) - \ell_t(f(x_t)) \right) = \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \left( \ell_t(\hat{y}_{i,t}) - \ell_t(y^*_i) \right) \]

\[ = \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \left( \ell_t(y^*_i) - \ell_t(f(x_t)) \right) \]

\[ + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \left( \ell_t(f(x_i)) - \ell_t(f(x_t)) \right) \]

\[ \leq D \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} |f(x_i) - f(x_t)| \]

using (11) and the fact that $\ell_t$ is $L'$-Lipschitz. Combining terms completes the proof.

The next key lemma bounds the number of leaves in a pruning $E$ for different settings of the ball radius function.

**Lemma 3.** For any instance sequence $\sigma_T$, for any $T \in \mathcal{T}_D(\sigma_T)$, and for any pruning $E$ of $T$, let the random variable $K$ be such that $P(K = k) = \frac{|E_k|}{|E|}$ for $k = 1, \ldots, D$. Then the following statements hold for each $k$,

\[ |E| \leq 2^d \mathbb{E} \left[ L_K^{d+1} \right] T^{\frac{d}{1+d}} \quad \text{for} \quad \varepsilon_{k,t} = (L_k t)^{-\frac{1}{1+d}} \quad \text{(Local Lipschitzness)} \]

\[ |E| \leq \mathbb{E} \left[ 2^{dK}(LT)^{dK} \right] \quad \text{for} \quad \varepsilon_{k,t} = (L t)^{-\frac{1}{1+dK}} \quad \text{(Local dimension)} \]

\[ |E| \leq 2^d \mathbb{E} \left[ (L \tau_K(T))^{d+1} \right] \quad \text{for} \quad \varepsilon_{k,t} = (L \tau_K(t))^{-\frac{1}{1+d}} \quad \text{(Local losses)} \]

**Proof.** We first recall that leaves of a pruning $E$ correspond to balls in a $\varepsilon_{k,T}/2$-packing. Thus, to give a bound on the number of leaves at level $k$, that is $|E_k|$, we estimate the size of the packing formed at level $k$. However, instead of directly bounding size of the packing, we use a more careful volumetric argument. In particular, at level $k$ we only pack the volume that is not occupied yet by previous levels —this helps to avoid gross overestimates, since we take into account the fact that we can only pack a limited volume. Denote volume of a set in an Euclidean space by $\text{vol}(\cdot)$, and let $\text{pack}_k$ stand for the collection of balls at level $k$ of the packing.
**Local Lipschitzness.** Pick any \( k = 1, \ldots, D \). Recalling that \( X \) is the unit ball,

\[
|E_k| \leq \frac{\text{vol}(X) - \text{vol}\left(\bigcup_{s=1}^{k-1} \text{pack}_s\right)}{\text{vol}(B(\varepsilon_k, T))} = 1 - \sum_{s=1}^{k-1} \frac{|E_s| \left(\frac{\varepsilon_s}{2}\right)^d}{\left(\frac{\varepsilon_k}{2}\right)^d}
\]

\[
= 2^d \left( L_k(T) \frac{d}{1+d} - \sum_{s=1}^{k-1} |E_s| \left( \frac{L_k}{L_s} \right)^{\frac{d}{1+d}} \right)
\]

(\text{using the definition of } \varepsilon_{k,t}.)

Dividing both sides by \( L_k^{\frac{d}{1+d}} \) we get

\[
\sum_{s=1}^{k} \frac{|E_s|}{L_s^{\frac{d}{1+d}}} \leq 2^d T^{\frac{d}{1+d}}
\]

Since \( k \) is chosen arbitrarily, we can set \( k = D \) and write

\[
\sum_{s=1}^{D} \frac{|E_s|}{L_s^{\frac{d}{1+d}}} \leq 2^d T^{\frac{d}{1+d}}
\]

or, equivalently,

\[
1 \leq \left( \sum_{s=1}^{D} \frac{|E_s|}{L_s^{\frac{d}{1+d}}} \right)^{-1} 2^d T^{\frac{d}{1+d}}.
\]

Multiplying both sides by \( |E| \) gives

\[
|E| \leq \left( \sum_{s=1}^{D} \frac{|E_s|/|E|}{L_s^{\frac{d}{1+d}}} \right)^{-1} 2^d T^{\frac{d}{1+d}}.
\]

Now observe that the factor in the right-hand side is a weighted harmonic mean with weights \( \frac{|E_s|}{|E|}, \ldots, \frac{|E_D|}{|E|} \). Therefore the HM-GM-AM inequality implies that

\[
|E| \leq 2^d \mathbb{E} \left[ L_K^{\frac{d}{1+d}} \right] T^{\frac{d}{1+d}}
\]

where the expectation is with respect to \( \mathbb{P}(K = k) = \frac{|E_k|}{|E|} \). This proves the first statement.

**Local dimension.** Using again the volumetric argument and the appropriate definition of \( \varepsilon_{k,t} \)

\[
|E_k| \leq 2^{d_k} \left( \sum_{s=1}^{k-1} \frac{|E_s| \varepsilon_{s,T}^{d_k}}{\varepsilon_{k,T}^{d_k}} \right)^{\frac{d_k}{1+d_k}} - \sum_{s=1}^{k-1} |E_s| (L T)^{\frac{d_k}{1+d_k}} - \frac{d_k}{1+d_k}.
\]

Dividing both sides by \( (L T)^{\frac{d_k}{1+d_k}} \) and rearranging gives

\[
|E| \leq 2^{d_k} \left( \sum_{s=1}^{D} \frac{|E_s|/|E|}{(L T)^{\frac{d_k}{1+d_k}}} \right)^{-1}.
\]
Once again, observing that the factor in the right-hand side is a weighted harmonic mean with weights $|E_1|, \ldots, |E_D|/|E|$, by the HM-GM-AM inequality we get

$$|E| \leq \mathbb{E} \left[ 2^d k (L T)^d \frac{d\eta}{2K} \right]$$

where the expectation is with respect to $\mathbb{P}(K = k) = \frac{|E_k|}{|E|}$.

**Local losses.** Using once more the volumetric argument and the appropriate definition of $\varepsilon_{k,t}$,

$$|E_k| \leq 2^d - \sum_{s=1}^{k-1} \frac{|E_s|\varepsilon_{s,T}}{d_{k,T}} = 2^d (L \tau_k(T)) \frac{d}{2+\eta} - \sum_{s=1}^{k-1} |E_s| \left( \frac{\tau_k(T)}{\tau_s(T)} \right)^{\frac{d}{2+\eta}}.$$  

Dividing both sides by $(L \tau_k(T)) \frac{d}{2+\eta}$ and multiplying by $|E|$ we get

$$|E| \leq 2^d \left( \sum_{s=1}^{D} \frac{|E_s|/|E|}{(L \tau_s(T)) \frac{d}{2+\eta}} \right)^{-1} \leq 2^d \mathbb{E} \left[ (L \tau_K(T)) \frac{d}{2+\eta} \right]$$

where —as before— the expectation is with respect to $\mathbb{P}(K = k) = \frac{|E_k|}{|E|}$. The proof is concluded.  

### C.1 Proof of Theorem 1

We start from Lemma 2 with the square loss $\ell_t(y) = \frac{1}{2} (y - y_t)^2$ and $Y \equiv \mathcal{H} \equiv [0, 1]$. As $\ell_t$ is $\eta$-exp-concave for $\eta \leq \frac{1}{2}$ and 1-Lipschitz in $[0, 1]$, we can apply Theorem 5 with $L' = 1$. This gives us

$$R_T(f) \leq R_T^{\text{tree}}(E) + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} R_{i,T}^{\text{loc}} + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} |f(x_i) - f(x_t)|.$$  

Using Theorem 5 combined with $M_T \leq DT$, and then using the first statement of Lemma 3, we get that

$$R_T^{\text{tree}}(E) \overset{\circ}{=} |E| \overset{\circ}{=} \mathbb{E} \left[ L^d_{\frac{d}{2+\eta}} \right] T^{-\frac{d}{2+\eta}}.$$  

**Bounding the estimation error.** Using the regret bound of Follow the Leader (FTL) with respect to the square loss [Cesa-Bianchi and Lugosi, 2006, p. 43], we get

$$\sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} R_{i,T}^{\text{loc}} \leq 8 \ln(eT) |E| \leq 2^{3+d} \ln(eT) \mathbb{E} \left[ \frac{L^d_{\frac{d}{2+\eta}}}{} \right] T^{-\frac{d}{2+\eta}}$$

where we used Lemma 3 to obtain the second inequality.
Bounding the approximation error. By hypothesis, \( f \in \mathcal{F}(E, T) \). Using Definition 4 and the fact that at time \( t \) ball radii at depth \( k \) are \( \varepsilon_{k,t} \),

\[
\sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \left| f(x_i) - f(x_t) \right| \leq \sum_{k=1}^{D} L_k \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \varepsilon_{k,t}
\]

\[
\leq \sum_{k=1}^{D} L_k \sum_{i \in \text{LEAVES}_k(E)} \sum_{t=1}^{|T_i|} \varepsilon_{k,t}
\]

\[
= \sum_{k=1}^{D} L_k^{d_{1+d}} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t=1}^{|T_i|} t^{-\frac{1}{1+d}}
\]

\[
\leq \sum_{k=1}^{D} L_k^{d_{1+d}} \int_0^{T_{E,k}} \tau^{-\frac{1}{1+d}} d\tau
\]

\[
\leq 2 \sum_{k=1}^{D} (L_k T_{E,k})^{\frac{d}{1+d}}.
\]

Combining the bound on \( R^\text{tree}_T(E) \) with the bounds on the estimation and approximation errors, we get that

\[
R_T(f) \overset{\tilde{\circ}}{=} \mathbb{E} \left[ L_K^{\frac{d}{1+d}} T^{\frac{d}{1+d}} + \sum_{k=1}^{D} (L_k T_{E,k})^{\frac{d}{1+d}} \right] \quad \forall f \in \mathcal{F}(E, T) \quad (12)
\]

which completes the proof.

C.2 Proof of Theorem 2

Similarly to the proof of Theorem 1, we use the properties of the square loss and Lemma 2. This gives us

\[
R_T(f) \leq R^\text{tree}_T(E) + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} R^\text{loc}_{i,T} + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \left| f(x_i) - f(x_t) \right|.
\]

Using Theorem 5 combined with \( M_T \leq DT \) (the largest number of traversed distinct paths), and then using Lemma 3 (second statement), we get that

\[
R^\text{tree}_T(E) \overset{\tilde{\circ}}{=} |E| \overset{\tilde{\circ}}{=} \mathbb{E} \left[ (L T)^{\frac{d}{1+d}} \right].
\]

Bounding the estimation error. Using —as before— the regret bound of FTL with respect to the square loss we immediately get

\[
\sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} R^\text{loc}_{i,T} \leq 8 \ln(eT)|E| \leq 8 \ln(eT) \mathbb{E} \left[ (L T)^{\frac{d}{1+d}} \right]
\]

where the last inequality uses Lemma 3.
Bounding the approximation error. For all $f \in \mathcal{F}_L$ and for all $E \in \mathcal{E}_{\dim}(\mathcal{T})$, since at time $t$ the ball radii at depth $k$ are $\varepsilon_{k,t}$,

$$
\sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} |f(x_i) - f(x_t)| \leq L \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \varepsilon_{k,t} \tag{13}
$$

$$
\leq L \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t=1}^{|T_i|} \varepsilon_{k,t} \tag{14}
$$

$$
\leq \sum_{k=1}^{D} L^{1-\frac{1}{1+\sigma_k}} \int_{0}^{T_E,k} \frac{1}{\tau^{1+\sigma_k}} d\tau \tag{15}
$$

$$
\leq 2 \sum_{k=1}^{D} (LT_{E,k})^{\frac{d_k}{2+\sigma_k}}. \tag{16}
$$

Combining the bound on $R_{\text{tree}}(E)$ with the bounds on the estimation and approximation errors, we get that

$$
R_T(f) \leq \tilde{\Omega} \left[ (L T)^{\frac{d_k}{2+\sigma_k}} \right] + \sum_{k=1}^{D} (LT_{E,k})^{\frac{d_k}{2+\sigma_k}} \quad \forall f \in \mathcal{F}_L. \tag{17}
$$

The proof is complete.

C.3 Proof of Theorem 3

Here we use the 1-Lipschitz absolute loss function $\ell_t(y) = |y - y_t|$ and run self-confident Exponentially Weighted Average (EWA) [Auer et al., 2002] at every node of the tree with $\mathcal{H} \equiv \{0, 1\}$. Lemma 2 gives us the decomposition

$$
R_T(f) \leq R_{\text{tree}}(E) + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} R_{i,T}^{\text{loc}} + \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} |f(x_i) - f(x_t)|.
$$

Theorem 6 gives us

$$
R_{\text{tree}}(f_E) = \tilde{\Omega} \left[ |E| \Lambda_E \ln \left( \frac{M_T}{|E|} \right) \right].
$$

Using once more $M_T \leq DT$, the fact that any pruning $E$ has at least one leaf, and Lemma 3 (third statement), we get

$$
1 \leq |E| \leq 2^d \mathbb{E} \left[ (L \tau_K(T))^\frac{d}{2+\sigma_k} \right].
$$

Recall that $\hat{y}_{i,t}$ is the output at time $t$ of the local predictor at node $i$. By definition of $\tau_k$,

$$
\Lambda_E = \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \ell_t(\hat{y}_{i,t}) \leq \sum_{k=1}^{D} \tau_k(T_{E,k}).
$$

This gives us

$$
R_{\text{tree}}(E) = \tilde{\Omega} \left[ \sum_{k=1}^{D} \tau_k(T_{E,k}) \mathbb{E} \left[ (L \tau_K(T))^\frac{d}{2+\sigma_k} \right] \right].
$$
Bounding the estimation error. Let the cumulative loss of the best expert for and node $i$ be defined by

$$\Lambda_{i,T}^* = \sum_{t \in T} \ell_t(y_i^*)$$

where $y_i^* = \arg\min_{y \in \{0,1\}} \sum_{t \in T} \ell_t(y)$.

Then, [Cesa-Bianchi and Lugosi, 2006, Exercise 2.11] implies that for a positive constant $c$ (independent of the number of experts and $\Lambda_{i,T}^*$), $R_{i,T}^{\text{loc}} \leq 2\sqrt{2 \ln(2) \Lambda_{i,T}^* + c \ln(2)}$. We can thus write

$$\sum_{k=1}^D \sum_{i \in \text{LEAVES}_k(E)} R_{i,T}^{\text{loc}} \leq \sum_{k=1}^D \sum_{i \in \text{LEAVES}_k(E)} \left(2\sqrt{2 \ln(2) \Lambda_{i,T}^* + c \ln(2)} \right)$$

$$\leq 2\sqrt{2 \ln(2)} \sum_{k=1}^D |E_k| \sum_{i \in \text{LEAVES}_k(E)} \Lambda_{i,T}^* + c \ln(2) |E|$$

$$\leq 2\sqrt{2 \ln(2)} \sum_{k=1}^D |E_k| \tau_k(T_{E,k}) + c \ln(2) |E|$$

since, according to the definition of $\tau_k$,

$$\sum_{i \in \text{LEAVES}_k(E)} \Lambda_{i,T}^* \leq \tau_k(T_{E,k}).$$

Next, using the Cauchy-Schwartz inequality,

$$\sum_{k=1}^D \sqrt{|E_k| \tau_k(T_{E,k})} \leq \sqrt{\sum_{k=1}^D |E_k| \sum_{k=1}^D \tau_k(T_{E,k})} \leq \sqrt{\left( \sum_{k=1}^D \tau_k(T_{E,k}) \right) \mathbb{E} \left[ (L_{\tau_K(T)})^{\frac{d}{\pi \alpha}} \right]}$$

where the last inequality is a consequence of Lemma 3 (third statement). This gives us the following bound on the estimation error

$$\sum_{k=1}^D \sum_{i \in \text{LEAVES}_k(E)} R_{i,T}^{\text{loc}} \leq \sqrt{\left( \sum_{k=1}^D \tau_k(T_{E,k}) \right) \mathbb{E} \left[ (L_{\tau_K(T)})^{\frac{d}{\pi \alpha}} \right]} + \mathbb{E} \left[ (L_{\tau_K(T)})^{\frac{d}{\pi \alpha}} \right].$$

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Bounding the approximation error. Since we are competing against the class of $L$-Lipschitz functions,

$$
\sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} |f(x_i) - f(x_t)| \leq L \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t \in T_i} \varepsilon_{k,t}
$$

$$
\leq L \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t=1}^{|T_i|} \varepsilon_{k,t}
$$

$$
= L^{1 - \frac{1}{2+d}} \sum_{k=1}^{D} \sum_{i \in \text{LEAVES}_k(E)} \sum_{t=1}^{|T_i|} \tau_k(t) \varepsilon_{k,t}
$$

$$
\leq L^{1 + \frac{d}{2+d}} \sum_{k=1}^{D} \int_0^{f_{k}(T_{E,k})} \theta^{-\frac{1}{2+d}} d\theta \quad \text{(since $\tau_k$ is non-decreasing)}
$$

Combining all terms together, the final regret bound is

$$
R_T(f) = \tilde{O}\left(\sum_{k=1}^{D} \tau_k(T_{E,k}) \mathbb{E}\left[ (L \tau_k(T))^\frac{d}{2+d} \right] + \mathbb{E}\left[ (L \tau_k(T))^\frac{d}{2+d} \right] + \sum_{k=1}^{D} \left( L \tau_k(T_{E,k}) \right)^{\frac{1+d}{2+d}} \right).$
$$

A Additional Proofs

Proof of Theorem 4. Recall that $\eta$-exp-concavity of $\ell_t$ implies that $e^{-\eta \ell_t(x)}$ is concave for all $x$. Observe that the relative entropy satisfies

$$
\text{KL}(u \mid \mid w_t) - \text{KL}(u \mid \mid w_{t+1})
$$

$$
= \sum_{i=1}^{M} u_i \ln \frac{w_{i,t+1}}{w_{i,t}}
$$

$$
= \sum_{i \in \mathcal{E}_t} u_i \ln \frac{w_{i,t+1}}{w_{i,t}}
$$

$$
= -\eta \sum_{i \in \mathcal{E}_t} u_i \ell_t(\mu_{i,t}) - U_t \ln \sum_{j \in \mathcal{E}_t} w_{j,t} e^{-\eta \ell_t(\hat{y}_{j,t})} \quad \text{(update step in Alg. 7)}
$$

$$
\geq -\eta \sum_{i \in \mathcal{E}_t} u_i \ell_t(\hat{y}_{i,t}) + \eta U_t \ell_t \left( \frac{\sum_{j \in \mathcal{E}_t} w_{j,t} \hat{y}_{j,t}}{\sum_{j \in \mathcal{E}_t} w_{j,t}} \right) \quad \text{(exp-concavity and Jensen’s)}
$$

$$
= -\eta \sum_{i \in \mathcal{E}_t} u_i \ell_t(\hat{y}_{i,t}) + \eta U_t \ell_t(\hat{y}_t)
$$

Summing both sides over $t = 1, \ldots, T$ we get

$$
\text{KL}(u \mid \mid w_1) \geq \text{KL}(u \mid \mid w_1) - \text{KL}(u \mid \mid w_T) = -\eta \sum_{t=1}^{T} \sum_{i \in \mathcal{E}_t} u_i \ell_t(\hat{y}_{i,t}) + \eta \sum_{t=1}^{T} U_t \ell_t(\hat{y}_t).
$$

The proof is now complete. \qed
Proof of Lemma 1. The proof exploits the fact that whenever the weights are initialized uniformly over a subset of the experts, the sequence of predictions remains the same as if the weights were initialized uniformly over all experts. In particular, we show that the predictions obtained assuming weights are initialized with \( w_{i,1} = 1/M_T \) for \( i \in E_1 \cup \cdots \cup E_T \) with \( |E_1 \cup \cdots \cup E_T| = M_T \) are the same as the predictions obtained with \( w_{i,1} = 1 \) for all \( i \). We use an inductive argument to prove that the factor \( 1/M_T \) introduced by the initialization \( w_{i,1} = 1/M_T \) is preserved after each update. Fix a round \( t > 1 \) and assume that all \( w_{i,t−1} \) contain the initialization factor \( 1/M_T \). Split the set of awake experts into observed ones \( E_o^t \subseteq E_1 \cup \cdots \cup E_{t−1} \) (that is experts which were awake at least once before), and unobserved ones \( E_u^t \equiv E_t \setminus E_o^t \). Clearly \( w_{i,t} = 1/M_T \) for every \( i \in E_u^t \), as they were never updated. For \( i \in E_o^t \), the update rule

\[
w_{i,t} = \frac{w_{i,t−1} e^{−η\ell_{i,t−1}}}{\sum_{j \in E_{t−1}} w_{j,t−1} e^{−η\ell_{j,t−1}}} \sum_{j \in E_{t−1}} w_{j,t−1}
\]

shows that the initialization factors that occur in the terms \( w_{j,t−1} \) contained in the two sums cancel out, whereas the one contained in \( w_{i,t−1} \) remains unchanged.

We can now write the prediction at round \( t \) as

\[
\hat{y}_t = \frac{\sum_{i \in E_t} w_{i,t} \hat{y}_{i,t}}{\sum_{i \in E_t} w_{i,t}} = \frac{\sum_{i \in E_o^t} w_{i,t} \hat{y}_{i,t} + \sum_{i \in E_u^t} w_{i,1} \hat{y}_{i,t}}{\sum_{i \in E_o^t} w_{i,t} + \sum_{i \in E_u^t} w_{i,1}} = \frac{M_T \sum_{i \in E_o^t} w_{i,t} \hat{y}_{i,t} + \sum_{i \in E_u^t} \hat{y}_{i,t}}{M_T \sum_{i \in E_o^t} w_{i,t} + |E_u^t|} = \frac{\sum_{i \in E_t} w_{i,t}' \hat{y}_{i,t}}{\sum_{i \in E_t} w_{i,t}'}
\]

where in the last step we canceled the initialization factor \( 1/M_T \) from \( w_{i,t} \) and introduced \( w_{i,t}' \) which differs from \( w_{i,t} \) only due to the initialization \( w_{i,1}' = 1 \). This completes the proof.

□