CASTELNUOVO-MUMFORD REGULARITY OF DEFICIENCY MODULES

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Abstract. Let $d \in \mathbb{N}$ and let $M$ be a finitely generated graded module of dimension $\leq d$ over a Noetherian homogeneous ring $R$ with local Artinian base ring $R_0$. Let $\text{beg}(M)$, $\text{gendeg}(M)$ and $\text{reg}(M)$ respectively denote the beginning, the generating degree and the Castelnuovo-Mumford regularity of $M$. If $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, let $d^i_M(n)$ denote the $R_0$-length of the $n$-th graded component of the $i$-th $R_+$-transform module $D^i_{R_+}(M)$ of $M$ and let $K^i(M)$ denote the $i$-th deficiency module of $M$.

Our main result says, that $\text{reg}(K^i(M))$ is bounded in terms of $\text{beg}(M)$ and the "diagonal values" $d^i_M(-j)$ with $j = 0, \ldots, d-1$. As an application of this we get a number of further bounding results for $\text{reg}(K^i(M))$.

1. Introduction

This paper is motivated by a basic question of projective algebraic geometry, namely:

What bounds cohomology of a projective scheme?

The basic and initiating contributions to this theme are due to Mumford [21] and Kleiman [18] (see also [13]). The numerical invariant which plays a fundamental rôle in this context, is the Castelnuovo-Mumford regularity, which was introduced in [21]. Besides is foundational significance - in the theory of Hilbert schemes for example- this invariant is the basic measure of complexity in computational algebraic geometry(s. [1]). This double meaning of (Castelnuovo-Mumford) regularity made it to one of the most studied invariants of algebraic geometry. Notably a huge number of upper bounds for the regularity have been established. We mention only a few more recent references to such results, namely [1], [4], [7, 2], [10], [11], [12], [19], [22].

It is also known, that Castelnuovo-Mumford regularity is closely related to the boundedness - or finiteness- of cohomology at all. More precisely, the regularity of deficiency modules provides bounds for the so called cohomological postulation numbers, and thus furnishes a tool to attack the finiteness problem for (local) cohomology. This relation is investigated by Hoa-Hyry [17] and Hoa [16] in the case of graded ideals in a polynomial ring over a field. In [6] it was shown that for coherent sheaves over projective schemes over a field $K$, cohomology is bounded by the "cohomology diagonal". One challenge is to extend this later result to the case where the base field $K$ is replaced by an Artinian ring $R_0$ and hence to replace the
bounds given in [8] by "purely diagonal" ones. In the same spirit one could try to generalize the results of Hoa and Hoa-Hyry. This is what we shall do in the present paper.

Our basic result is a "diagonal bound" for the Castelnuovo-Mumford regularity of deficiency modules.

To formulate this result we introduce a few notations. By \( \mathbb{N}_0 \) we denote the set of all non-negative integers, by \( \mathbb{N} \) the set of all positive integers. Let \( R := \bigoplus_{n \geq 0} R_n \) be a Noetherian homogeneous ring with Artinian base ring \( R_0 \) and irrelevant ideal \( R_+ := \bigoplus_{n > 0} R_n \). Let \( M \) be a finitely generated graded \( R \)-module. For each \( i \in \mathbb{N}_0 \) consider the graded \( R \)-module \( D^i_{R_+}(M) \), where \( D^i_{R_+} \) denotes the \( i \)-th \( R_+ \)-transform functor, that is the \( i \)-th right derived functor of the \( R_+ \)-transform functor \( D_{R_+}(\bullet) := \lim \text{Hom}_R((R_+)^n, \bullet) \). In addition, for each \( n \in \mathbb{Z} \) let \( d^n_M(n) \) denote the (finite) \( R_0 \)-length of the \( n \)-th graded component of \( D^i_{R_+}(M) \). Moreover, let \( \text{beg}(M) \) and \( \text{reg}(M) \) respectively denote the beginning and the Castelnuovo-Mumford regularity of \( M \). If the (Artinian) base ring \( R_0 \) is local, let \( K^i(M) \) denote the \( i \)-th deficiency module of \( M \). Fix \( d \in \mathbb{N} \) and \( i \in \{0, \cdots, d\} \) and let \( \text{dim}(M) \leq d \). Then, the announced bounding result says (s. Theorem 3.6):

The beginning \( \text{beg}(M) \) of \( M \) and the cohomology diagonal \( (d^n_M(-i))_{i=0}^{d-1} \) of \( M \) give an upper bound for the regularity of \( K^i(M) \).

This leads to a further bounding result for \( \text{reg}(K^i(M)) \). To formulate it, let \( \text{reg}^2(M) \) denote the Castelnuovo-Mumford regularity of \( M \) at and above level 2 and let \( p_M \) denote the Hilbert polynomial of \( M \). Then (s. Theorem 4.2):

The invariant \( \text{reg}(K^i(M)) \) can be bounded in terms of the three invariants \( \text{beg}(M), \text{reg}^2(M) \) and \( p_M(\text{reg}^2(M)) \).

As a consequence we get (s. Corollary 4.4):

If \( \mathfrak{a} \subseteq R \) is a graded ideal, then \( \text{reg}(K^i(\mathfrak{a})) \) and \( \text{reg}(K^i(R/\mathfrak{a})) \) can be bounded in terms of \( \text{reg}^2(\mathfrak{a}), \text{length}(R_0), \text{reg}^1(R) \) and the number of generating one-forms of \( R \).

Applying this in the case where \( R = K[x_1, \cdots, x_d] \) is a polynomial ring over a field, we get an upper bound for \( \text{reg}(K^i(R/\mathfrak{a})) \) which depends only on \( d \) and \( \text{reg}^2(\mathfrak{a}) \). This is a (slightly improved) version of a corresponding result found in [17], which uses \( \text{reg}(\mathfrak{a}) \) instead of \( \text{reg}^2(\mathfrak{a}) \) as a bounding invariant.

As an application of Theorem 4.2 (cf. (1.2)) we prove a few more bounding results in the situation where \( R = R_0[x_1, \cdots, x_d] \) is a polynomial ring over a local Artinian ring \( R_0 \), namely (s. Corrolaries 4.6, 4.8, 4.13):

If \( U \neq 0 \) is a finitely generated graded \( R \)-module and \( M \subseteq U \) is a graded submodule, then \( \text{reg}(K^i(M)) \) and \( \text{reg}(K^i(U/M)) \) are bounded in terms of \( d, \text{length}(R_0), \text{beg}(U), \text{reg}(U) \), the number of generators of \( U \) and the generating degree \( \text{gendeg}(M) \) of \( M \).
If $F \rightarrow M$ is an epimorphism of graded $R$-modules such that $F$ is free and of finite rank, then $\text{reg}(K^i(M))$ is bounded in terms of $d$, length($R_0$), beg($F$), gendeg($F$), rank($F$) and gendeg(ker($p$)).

Let $U$ and $M$ be as above. Then $\text{reg}(K^i(M))$ and $\text{reg}(K^i(U/M))$ are bounded in terms of length($R_0$), beg($U$), reg($U$), the Hilbert polynomial $p_U$ of $U$ and the Hilbert polynomial $p_{U/M}$ of $U/M$.

For a fixed $i \in \mathbb{N}_0$ we consider the $i$-th cohomological Hilbert function of the second kind $d^i_M : \mathbb{Z} \rightarrow \mathbb{N}_0$ given by $n \mapsto d^i_M(n)$ and the corresponding $i$-th cohomological Hilbert polynomial $q^i_M \in \mathbb{Q}[x]$ so that $q^i_M(n) = d^i_M(n)$ for all $n \ll 0$. Based on these concepts we define the $i$-th cohomological postulation number of $M$ by:

$$\nu^i_M := \inf \{n \in \mathbb{Z} \mid q^i_M(n) \neq d^i_M(n)\}(\in \mathbb{Z} \cup \{\infty\}).$$

Now, let $d \in \mathbb{N}$ and let $\mathcal{D}^d$ be the class of all pairs $(R, M)$ in which $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is a Noetherian homogeneous ring with Artinian base ring $R_0$ and $M$ is a finitely generated graded $R$-module of dimension $\leq d$. As a first consequence of Theorem 3.6 we get, that for all pairs $(R, M) \in \mathcal{D}^d$ and all $i \in \{0, \cdots, d-1\}$ the cohomology diagonal $(d^i_M(-j))_{j=0}^{d-1}$ of $M$ bounds the $i$-th cohomological postulation number of $M$ (s. Theorem 5.3):

There is a function $E^i_d : \mathbb{N}_0^d \rightarrow \mathbb{Z}$ such that for all $x_0, \cdots, x_{d-1} \in \mathbb{N}_0$ and each pair $(R, M) \in \mathcal{D}^d$ such that $d^i_M(-j) \leq x_j$ for all $j \in \{0, \cdots, d-1\}$ we have

$$\nu^i_M \geq E^i_d(x_0, \cdots, x_n).$$

This is indeed a generalization of the main result of [6] which gives the same conclusion in the case where the base ring $R_0$ is a field. Moreover, in our present proof, the bounding function $E^i_d$ is defined much simpler than in [6].

As an application of Theorem 5.3 we show, that there are only finitely many possible functions $d^i_M$ if the cohomology diagonal is fixed (s. Theorem 5.4):

Let $x_0, \cdots, x_{d-1} \in \mathbb{N}_0$. Then, the set of functions

$$\{d^i_M \mid i \in \mathbb{N}_0, (R, M) \in \mathcal{D}^d : d^i_M(-j) \leq x_j \text{ for } j = 0, \cdots, d-1\}$$

is finite.

2. Preliminaries

In this section we recall a few basic facts which shall be used later in our paper. We also prove a bounding result for the Castelnuovo-Mumford regularity of certain graded modules.
Notation 2.1. Throughout, let \( R = \bigoplus_{n \geq 0} R_n \) be a homogeneous Noetherian ring, so that \( R \) is positively graded, \( R_0 \) is Noetherian and \( R = R_0[l_0, \ldots, l_r] \) with finitely many elements \( l_0, \ldots, l_r \in R_1 \). Let \( R_+ \) denote the irrelevant ideal \( \bigoplus_{n > 0} R_n \) of \( R \).

Reminder 2.2. (Local cohomology and Castelnuovo-Mumford regularity) (A) Let \( i \in \mathbb{N}_0 := \{0, 1, 2, \cdots \} \). By \( H^i_{R_+}(\bullet) \) we denote the \( i \)-th local cohomology functor with respect to \( R_+ \). Moreover by \( D^i_{R_+}(\bullet) \) we denote the \( i \)-th right derived functor of the ideal transform functor \( D_{R_+}(\bullet) = \lim_{n \to \infty} ((R_+)^n, \bullet) \) with respect to \( R_+ \).

(B) Let \( M := \bigoplus_{n \in \mathbb{Z}} M_n \) be a graded \( R \)-module. Keep in mind that in this situation the \( R \)-modules \( H^i_{R_+}(M) \) and \( D^i_{R_+}(M) \) carry natural gradings. Moreover we then have a natural exact sequence of graded \( R \)-modules

\[
0 \to H^0_{R_+}(M) \to M \to D^0_{R_+}(M) \to H^1_{R_+}(M) \to 0
\]

and natural isomorphisms of graded \( R \)-modules

\[
D^i_{R_+}(M) \cong H^{i+1}_{R_+}(M) \quad \text{for all } i > 0.
\]

(C) If \( T \) is a graded \( R \)-module and \( n \in \mathbb{Z} \), we use \( T_n \) to denote the \( n \)-th graded component of \( T \). In particular, we define the beginning and the end of \( T \) respectively by

\[
(i) \quad \text{beg}(T) := \inf \{ n \in \mathbb{Z} | T_n \neq 0 \},
\]

\[
(ii) \quad \text{end}(T) := \sup \{ n \in \mathbb{Z} | T_n \neq 0 \}.
\]

with the standard convention that \( \inf \emptyset = \infty \) and \( \sup \emptyset = -\infty \).

(D) If the graded \( R \)-module \( M \) is finitely generated, the \( R_0 \)-modules \( H^i_{R_+}(M)_n \) are all finitely generated and vanish as well for all \( n \gg 0 \) as for all \( i \geq 0 \). So, we have

\[
-\infty \leq a_i(M) := \text{end}(H^i_{R_+}(M)) < \infty \quad \text{for all } i \geq 0
\]

with \( a_i(M) := -\infty \) for all \( i \geq 0 \).

If \( k \in \mathbb{N}_0 \), the Castelnuovo-Mumford regularity of \( M \) at and above level \( k \) is defined by

\[
(i) \quad \text{reg}^k(M) := \sup \{ a_i(M) + i | i \geq k \} \quad (< \infty),
\]

where as the Castelnuovo-Mumford regularity of \( M \) is defined by

\[
(ii) \quad \text{reg}(M) := \text{reg}^0(M).
\]

(E) If \( M \) is a graded \( R \)-module we denote the generating degree of \( M \) by \( \text{gendeg}(M) \), thus

\[
(i) \quad \text{gendeg}(M) = \inf \{ n \in \mathbb{Z} | M = \bigoplus_{m \leq n} M_m \}.
\]

Keep in mind the well known relation (s. [9, 15.3.1])
(ii) \( \text{gendeg}(M) \leq \text{reg}(M) \).

**Reminder 2.3. (Cohomological Hilbert functions)** (A) Let \( i \in \mathbb{N}_0 \) and assume that the base ring \( R_0 \) is Artinian. Let \( M \) be a finitely generated graded \( R \)-module. Then, the graded \( R \)-modules \( H^i_{R_+}(M) \) are Artinian (see [9, 7.1.4]). In particular for all \( i \in \mathbb{N}_0 \) and all \( n \in \mathbb{Z} \) we may define the non-negative integers

(i) \( h^i_M(n) := \text{length}_{R_0}(H^i_{R_+}(M)_n) \),

(ii) \( d^i_M(n) := \text{length}_{R_0}(D^i_{R_+}(M)_n) \),

Fix \( i \in \mathbb{N}_0 \). Then the functions

(iii) \( h^i_M : \mathbb{Z} \to \mathbb{N}_0, \ n \mapsto h^i_M(n) \),

(iv) \( d^i_M : \mathbb{Z} \to \mathbb{N}_0, \ n \mapsto d^i_M(n) \)

are called the \( i \)-th Cohomological Hilbert functions of the first respectively the second kind of \( M \).

(B) Let \( i \in \mathbb{N}_0 \) and let \( R \) and \( M \) be as in part (A). Then, there is a polynomial \( p^i_M \in \mathbb{Q}[x] \) of degree \(< i \) such that (see [9, 17.1.9])

(i) \( p^i_M(n) = h^i_M(n) \) for all \( n \ll 0 \);

(ii) \( \deg(p^i_M) \leq i - 1 \), with equality if \( i = \dim(M) \).

We call \( p^i_M \) the \( i \)-th Cohomological Hilbert polynomial of the first kind of \( M \). Now, clearly by the observation made in part (A) we also have polynomials \( q^i_M \in \mathbb{Q}[x] \) such that

(iii) \( q^i_M(n) = d^i_M(n) \) for all \( n \ll 0 \).

These are called the Hilbert polynomials of the second kind of \( M \). Observe that

(iv) \( q^i_M = p^{i+1}_M \) for all \( i \in \mathbb{N}_0 \).

Finally, for all \( i \in \mathbb{N}_0 \) we define the \( i \)-th cohomological postulation number of \( M \) as

(v) \( \nu^i_M := \inf\{n \in \mathbb{Z}| q^i_M(n) \neq d^i_M(n)\}(\in \mathbb{Z} \cup \{\infty\}) \).

Observe that these numbers \( \nu^i_M \) differ by 1 from the cohomological postulation numbers introduced in [8].

(C) Let \( R \) and \( M \) be as in part (A). By \( p_M \in \mathbb{Q}[x] \) we denote the Hilbert polynomial of \( M \).

By \( p(M) \) we denote the postulation number \( \sup\{n \in \mathbb{Z}| \text{length}_{R_0}(M_n) \neq p_M(n)\} \) of \( M \).

Keep in mind that according to the Serre formula we have (see [9, 17.1.6])
\[ p_M(n) = \sum_{i \geq 0} (-1)^i d_M^i(n) = \text{length}_{R_0}(M_n) - \sum_{j \geq 0} (-1)^j h_M^j(n). \]

**Reminder 2.4.** *(Filter regular linear forms)* (A) Let \( M \) be a finitely generated graded \( R \)-module and let \( x \in R_1 \). By \( \text{NZD}_R(M) \) resp. \( \text{ZD}_R(M) \) we denote the set of non-zerodivisors resp. of zero divisors of \( R \) with respect to \( M \).

The linear form \( x \in R_1 \) is said to be \((R_+^-)\) *filter regular with respect to* \( M \) if \( x \in \text{NZD}_R(M/\Gamma_{R_+}(M)) \).

(B) Finally if \( x \in R_1 \) is filter regular with respect to \( M \) then the graded short exact sequences
\[
0 \to (0 :_M x) \to M \to M/(0 :_M x) \to 0,
\]
\[
0 \to M/(0 :_M x)(-1) \to M \to M/xM \to 0
\]
imply
\[
\text{reg}^1(M) \leq \text{reg}(M/xM) \leq \text{reg}(M).
\]

The following result will play a crucial role in the proof of our bounding result for the regularity of deficiency modules.

**Proposition 2.5.** Assume that the base ring \( R_0 \) is Artinian. Let \( M \) be a finitely generated graded \( R \)-module, let \( x \in R_1 \) be filter regular with respect to \( M \) and let \( m \in \mathbb{Z} \) be such that \( \text{reg}(M/xM) \leq m \) and \( \text{gendeg}((0 :_M x)) \leq m \). Then
\[
\text{reg}(M) \leq m + h^0_M(m).
\]

**Proof.** By Reminder 2.4(B) we have \( \text{reg}^1(M) \leq \text{reg}(M/xM) \leq m \). So, it remains to show that
\[
a_0(M) = \text{end}(H^0_{R_+}(M)) \leq m + h^0_M(m).
\]

The short exact sequence of graded \( R \)-modules
\[
0 \to M/(0 :_M x)(-1) \xrightarrow{x} M \to M/xM \to 0
\]
induces exact sequences of \( R_0 \)-modules
\[
0 \to H^0_{R_+}(M/(0 :_M x))_n \to H^0_{R_+}(M)_n \to H^0_{R_+}(M/xM)_n \to H^1_{R_+}(M/(0 :_M x))_n
\]
for all \( n \in \mathbb{Z} \). As \( H^0_{R_+}(M/xM)_n = 0 \) for all \( n \geq m \), we thus get
\[
H^0_{R_+}(M/(0 :_M x))_n \cong H^0_{R_+}(M)_{n+1} \text{ for all } n \geq m.
\]

The short exact sequence of graded \( R \)-modules
\[
0 \to (0 :_M x) \to M \to M/(0 :_M x) \to 0
\]
together with the facts that \( H^0_{R_+}((0 :_M x)) = (0 :_M x) \) and \( H^1_{R_+}((0 :_M x)) = 0 \) induce short exact sequences of \( R_0 \)-modules
\[
0 \to (0 :_M x)_n \to H^0_{R_+}(M)_n \to H^0_{R_+}(M/(0 :_M x))_n \to 0
\]
for all $n \in \mathbb{Z}$.

So, for all $n \geq m$ we get an exact sequence of $R_0$-modules

$$0 \longrightarrow (0 :_M x)_n \longrightarrow H^0_{R_+} (M)_n \longrightarrow H^0_{R_+} (M)_{n+1} \longrightarrow 0.$$  

To prove our claim, we may assume that $a_0(M) > m$. As $\text{end}((0 :_M x)) = a_0(M)$ and $\text{gendeg}((0 :_M x)) \leq m$ it follows that $(0 :_M x)_n \neq 0$ for all integers $n$ with $m \leq n \leq a_0(M)$. Hence, for all these $n$, the homomorphism $\pi_n$ is surjective but not injective, so that $h^0_M(n) > h^0_M(n+1)$. Therefore, for $n \geq m$ the function $n \mapsto h^0_M(n)$ is strictly decreasing until it reaches the value 0. Thus $h^0_M(n) = 0$ for all $n > m + h^0_M(m)$, and this proves our claim. 

□

We now recall a few basic facts about deficiency modules and graded local duality.

**Reminder 2.6. (Deficiency modules and local duality)**  
(A) We assume that the base ring $R_0$ is Artinian and local with maximal ideal $m_0$. As $R_0$ is complete it is a homomorphic image of a complete regular ring $A_0$. Factoring out an appropriate system of parameters of $A_0$ we thus may write $R_0$ as a homomorphic image of a local Artinian Gorenstein ring $(S_0, n_0)$. Let $d'$ be the minimal number of generators of the $R_0$-module $R_1$ and consider the polynomial ring $S := S_0[x_1, \ldots, x_{d'}]$. Then, we have a surjective homomorphism $S \rightarrow R$ of graded rings.

For all $i \in \mathbb{N}_0$ and all finitely generated graded $R$-modules, the $i$-th deficiency module of $M$ is defined as the finitely generated graded $R$-module (cf [23, Section 3.1] for the corresponding concept for a local Noetherian ring $R$ which is a homomorphic image of a local Gorenstein ring $S$.)

(i) $K^i(M) := \text{Ext}^d_{S}(-i)(M, S(-d')).$

The module

(ii) $K(M) := K^\dim(M)(M)$

is called the canonical module of $M$.

(B) Keep the previous notations and hypotheses. Let $E_0$ denote the injective envelope of the $R_0$-module $R_0/m_0$. Then, by Graded Matlis Duality and the Graded Local Duality Theorem (s. [9, 13.4.5] for example) we have

$$\text{length}_{R_0}(K^i(M)_n) = h^i_M(-n)$$

for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.

(C) As an easy consequence of the last observation we now get the following relations for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$:
(i) \( d_M^i(n) = \text{length}_{R_0}(K^{i+1}(M)_n) \), if \( i > 0 \) and \( d_M^0(n) \geq \text{length}_{R_0}(K^1(M)_n) \) with equality if \( n < \text{beg}(M) \); 

(ii) \( p_M^i(n) = p_{K^i(M)}(-n) \); 

(iii) \( q_M^i(n) = p_{K^{i+1}(M)}(-n) \); 

(iv) \( a_i(M) = -\text{beg}(K^i(M)) \); 

(v) \( \text{end}(K^i(M)) = -\text{beg}(H^i_{R_1}(M)) \); 

(vi) \( \nu_M^i = -p(K^{i+1}(M)) \).

\[ \sum_{l=0}^{i-j} \binom{i-j}{l} d_M^l(l-i) \]

3. Regularity of Deficiency Modules

We keep the notations introduced in Section 2. Throughout this section we assume in addition that the Noetherian homogenous ring \( R = \bigoplus_{n \geq 0} R_n \) has Artinian local base ring \( (R_0, m_0) \).

The aim of the present section is to show that the Castelnuovo-Mumford regularity of the deficiency modules \( K^i(M) \) of the finitely generated graded \( R \)-module \( M \) is bounded in terms of the beginning \( \text{beg}(M) \) of \( M \) and the "cohomology diagonal" \( (d_M^i(-i))_{i=0}^{\text{dim}(M)-1} \) of \( M \).

We first prove three auxiliary results.

**Lemma 3.1.** \( \text{depth}(K^{\text{dim}(M)}(M)) \geq \min\{2, \text{dim}(M)\} \).

**Proof.** In the notation introduced in Reminder (2.6) we have \( K(M)_m \cong K(M_m) \). As \( M \) and \( K(M) \) are finitely generated graded \( R \)-modules we have \( \text{dim}(M) = \text{dim}(M_m) \) and \( \text{depth}(K(M)) = \text{depth}(K(M_m)) \). Now, we conclude by [23, Lemma 3.1.1(C)]. \( \square \)

**Lemma 3.2.** Let \( x \in R_1 \) be filter regular with respect to \( M \) and the modules \( K^j(M) \). Then, there are short exact sequences of graded \( R \)-modules
\[ 0 \longrightarrow (K^{i+1}(M)/xK^{i+1}(M))(+1) \longrightarrow K^i(M/xM) \longrightarrow (0 :_{K^i(M)} x) \longrightarrow 0 \]

**Proof.** In the local case, this result is shown in [24, Proposition 2.4]. In our graded situation, one may conclude in the same way. \( \square \)

**Lemma 3.3.** Let \( i \in \mathbb{N}_0 \) and \( n \geq i \). Then
\[ \text{length}_{R_0}(K^{i+1}(M)_n) \leq \sum_{j=0}^{i} \binom{n-j-1}{i-j} \left[ \sum_{l=0}^{i-j} \binom{i-j}{l} d_M^l(l-i) \right]. \]
Remark 3.5. (A) Let \( d \in \mathbb{N}_0 \) and \( i \in \{0, \cdots, d\} \) we define the functions
\[
F^i_d : \mathbb{N}_0^d \times \mathbb{Z} \to \mathbb{Z}
\]
as follows: In the case \( i = 0 \) we simply set
\[
(i) \quad F^0_d(x_0, \cdots, x_{d-1}, y) := -y.
\]
Concerning the case \( i = 1 \) we set
\[
(ii) \quad F^1_1(x_0, y) := 1 - y
\]
and
\[
(iii) \quad F^1_d(x_0, \cdots, x_{d-1}, y) := \max\{0, 1 - y\} + \sum_{i=0}^{d-2} \binom{d-1}{i} x_{d-i-2}, \text{ for } d \geq 2.
\]
In the case \( i = d = 2 \) we define
\[
(iv) \quad F^2_2(x_0, x_1, y) := F^1_2(x_0, x_1, y) + 2.
\]
If \( d \geq 3 \) and \( 2 \leq i \leq d-1 \) and under the assumption that \( F^{i-1}_{d-1}, F^i_{d-1} \) and \( F^{i-1}_d \) are already defined, we first set
\[
(v) \quad m_i := \max\{F^{i-1}_{d-1}(x_0 + x_1, \cdots, x_{d-2} + x_{d-1}, y), F^i_{d-1}(x_0, \cdots, x_{d-1}, y) + 1\} + 1,
\]
\[
(vi) \quad n_i := F^i_{d-1}(x_0 + x_1, \cdots, x_{d-2} + x_{d-1}, y),
\]
\[
(vii) \quad t_i := \max\{m_i, n_i\}.
\]
Then, using this notation we define
\[
(viii) \quad F^i_d(x_0, \cdots, x_{d-1}, y) := t_i + \sum_{j=0}^{i-1} \binom{i-j-1}{i-j} \Delta_{ij},
\]
where \( \Delta_{ij} = \sum_{l=0}^{i-j-1} \binom{i-j-1}{l} x_{i-l-1} \).
Finally, assuming that \( d \geq 3 \) and that \( F^{d-1}_{d-1} \) and \( F^{d-1}_d \) are already defined, we set
\[
(ix) \quad F^d_d(x_0, \cdots, x_{d-1}, y) :=
\]
\[
\max\{F^{d-1}_{d-1}(x_0 + x_1, \cdots, x_{d-2} + x_{d-1}, y), F^d_{d-1}(x_0, \cdots, x_{d-1}, y) + 1\} + 1.
\]

Remark 3.5. (A) Let \( d \in \mathbb{N}_0 \) and \( i \in \{0, \cdots, d\} \). Let \((x_0, \cdots, x_{d-1}, y), (x'_0, \cdots, x'_{d-1}, y') \in \mathbb{N}_0^d \times \mathbb{Z}\) such that
\[
x_i \leq x'_i \text{ for all } i \in \{0, \cdots, d-1\} \text{ and } y' \leq y.
\]
Then it follows easily by induction on \( i \) and \( d \) that
\[
F^i_d(x_0, \cdots, x_{d-1}, y) \leq F^i_d(x'_0, \cdots, x'_{d-1}, y').
\]
(B) It also follows by induction on \( i \), that the auxiliary numbers \( m_i \) and \( t_i \) of Definition 3.4 all satisfy the inequality \( \min\{m_i, t_i\} \geq i \).

(C) Let \( s, d \in \mathbb{N} \) with \( s \leq d \) and let \( i \in \mathbb{N}_0 \) with \( i \leq s \). Moreover, let \( (x_0, \cdots, x_{s-1}, y) \in \mathbb{N}^s \times \mathbb{Z} \). We then easily obtain by induction on \( i \), that

\[
F_s^i(x_0, \cdots, x_{s-1}, y) \leq F_d^i(x_0, \cdots, x_{s-1}, 0, \cdots, 0, y).
\]

Now we are ready to state the main result of the present section.

**Theorem 3.6.** Let \( d \in \mathbb{N}, i \in \{0, \cdots, d\} \) and let \( M \) be a finitely generated graded \( R \)-module such that \( \dim(M) = d \). Then

\[
\text{reg}(K^i(M)) \leq F_d^i(d_0^0M(0), d_1^1(-1), \cdots, d_{d-1}^{d-1}(1-d), \text{beg}(M)).
\]

**Proof.** We shall proceed by induction on \( i \) and \( d \). As \( \dim(K^0(M)) \leq 0 \) and in view of Reminder 2.6(C)(v) we first have

\[
\text{reg}(K^0(M)) = \text{end}(K^0(M)) = -\text{beg}(H_{R_+}^0(M)) \leq -\text{beg}(M) = F_d^0(d_0^0(0), \cdots, d_{d-1}^{d-1}(1-d), \text{beg}(M)).
\]

This proves the case where \( i = 0 \).

So, let \( i > 0 \). We may assume that \( R_0/m_0 \) is infinite. In addition, we may replace \( M \) by \( M/H_{R_+}^0(M) \) and hence assume that \( \text{depth}(M) > 0 \).

Let \( x \in R_1 \) be a filter regular element with respect to \( M \) and all the modules \( K^j(M) \). Observe that \( x \in \text{NZD}(M) \). By Lemma 3.2 we have the exact sequences of graded \( R \)-modules

\[
0 \longrightarrow (K^{j+1}(M)/xK^{j+1}(M))(+1) \longrightarrow K^j(M/xM) \longrightarrow (0 :_{K^j(M)} x) \longrightarrow 0 \tag{1}
\]

for all \( j \in \mathbb{N}_0 \).

Since \( \text{depth}(M) > 0 \) we have \( K^0(M) = 0 \). So, the sequence (1) yields an isomorphism of graded \( R \)-modules

\[
(K^1(M)/xK^1(M))(+1) \cong K^0(M/xM). \tag{2}
\]

As \( \dim(K^0(M/xM)) \leq 0 \) the isomorphism (2) and Reminder 2.6(C)(v) imply

\[
\text{reg}(K^1(M)/xK^1(M)) = \text{reg}(K^0(M/xM)) + 1 = \text{end}(K^0(M/xM)) + 1
\]

\[
= 1 - \text{beg}(H_{R_+}^0(M/xM)) \leq 1 - \text{beg}(M/xM) \leq 1 - \text{beg}(M).
\]

Therefore,

\[
\text{reg}(K^1(M)/xK^1(M)) \leq 1 - \text{beg}(M). \tag{3}
\]
Assume first that $d = \dim(M) = 1$. Then, by Lemma 3.1 we have $\text{depth}(K^1(M)) \geq \min\{2, \dim(M)\} = 1$, whence $\text{reg}(K^1(M)) = \text{reg}^1(K^1(M))$. It follows that (cf. Reminder 2.4(B))

$$\text{reg}(K^1(M)) \leq \text{reg}(K^1(M)/xK^1(M)) \leq 1 - \text{beg}(M) = F_1^1(d_M^0(0), \text{beg}(M)).$$

This proves our result if $d = 1$.

So, from now on we assume that $d \geq 2$. We first focus to the case $i = 1$ and consider the exact sequence (1) for $j = 1$, hence

$$0 \longrightarrow (K^2(M)/xK^2(M))(+1) \longrightarrow K^1(M/xM) \longrightarrow (0:_{K^1(M)} x) \longrightarrow 0. \quad (4)$$

If $d = \dim(M) = 2$, we have $\dim(M/xM) = 1$, and so by the case $d = 1$ we get

$$\text{reg}(K^1(M/xM)) \leq 1 - \text{beg}(M/xM) \leq 1 - \text{beg}(M).$$

From (4) and Reminder 2.2(E)(ii) it follows that

$$\text{gendeg}((0:_{K^1(M)} x)) \leq \text{reg}(K^1(M/xM)) \leq 1 - \text{beg}(M).$$

Set $m_0 := 1 - \text{beg}(M)$. If $m_0 \leq 0$, by the inequality (3), Proposition 2.5 (applied with $m = 0$) and Reminder 2.6(C)(i) we obtain

$$\text{reg}(K^1(M)) \leq 0 + h_{K^1(M)}^0(0) \leq \text{length}(K^1(M)_0) = d_M^0(0).$$

If $m_0 > 0$ we have $d_M^0(-m_0) \leq d_M^0(0)$. So, by (3), Proposition 2.5 and Reminder 2.6(C)(i) we get

$$\text{reg}(K^1(M)) \leq m_0 + h_{K^1(M)}^0(m_0) \leq m_0 + \text{length}(K^1(M)_{m_0}) = 1 - \text{beg}(M) + d_M^0(-m_0) \leq 1 - \text{beg}(M) + d_M^0(0).$$

So, (cf. Definition 3.4(iii))

$$\text{reg}(K^1(M)) \leq \max\{d_M^0(0), 1 - \text{beg}(M) + d_M^0(0)\} \leq \max\{0, 1 - \text{beg}(M)\} + d_M^0(0) = F_2^1(d_M^0(0), d_M^1(-1), \text{beg}(M)).$$

This proves the case $d = 2, i = 1$. 
If \( d \geq 3 \), by induction on \( d \), we have (cf. Definition 3.4(iii))

\[
\text{reg}(K^1(M/xM)) \leq F^1_{d-1}(d^0_{M/xM}(0), \ldots, d^{d-2}_{M/xM}(2-d), \text{beg}(M/xM))
\]

\[
= \max\{0, 1 - \text{beg}(M/xM)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} d^{d-i-3}_{M/xM}(i + 3 - d)
\]

\[
\leq \max\{0, 1 - \text{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} [d^{d-i-3}_{M}(i + 3 - d) + d^{d-i-2}_{M}(i + 2 - d)].
\]

Set

\[
t_0 := \max\{0, 1 - \text{beg}(M)\} + \sum_{i=0}^{d-3} \binom{d-2}{i} [d^{d-i-3}_{M}(i + 3 - d) + d^{d-i-2}_{M}(i + 2 - d)].
\]

By the exact sequence (4) and Reminder 2.2(E)(ii) we now get

\[
\text{gendeg}((0 : K^1(M) x)) \leq \text{reg}(K^1(M/xM)) \leq t_0.
\]

By (3) we also have \( \text{reg}(K^1(M)/xK^1(M)) \leq t_0 \). As \( t_0 \geq 0 \), we have \( d^0_{M}(-t_0) \leq d^0_{M}(0) \). So, by Proposition 2.5 and Reminder 2.6(C)(i) we obtain

\[
\text{reg}(K^1(M)) \leq t_0 + h^0_{K^1(M)}(t_0) \leq t_0 + \text{length}(K^1(M)) + t_0 + d^0_{M}(-t_0) \leq t_0 + d^0_{M}(0)
\]

\[
= \max\{0, 1 - \text{beg}(M)\} + \sum_{i=0}^{d-2} \binom{d-1}{i} d^{d-i-2}_{M}(i + 2 - d).
\]

From this we conclude that (cf. Definition 3.4(iii))

\[
\text{reg}(K^1(M)) \leq F^1_{d}(d^0_{M}(0), \ldots, d^{d-1}_{M}(1-d), \text{beg}(M)).
\]

So, we have done the case \( i = 1 \) for all \( d \in \mathbb{N} \).

We thus attack now the case with \( i \geq 2 \). First, let \( d = 2 \). Then, in view of the sequence (4), by the fact that \( x \) is filter regular with respect to \( K^1(M) \) and by what we have already shown in the cases \( d \in \{1, 2\} \) and \( i = 1 \), we get

\[
\text{reg}(K^2(M)/xK^2(M)) \leq \max\{\text{reg}(K^1(M/xM)), \text{reg}((0 : K^1(M) x)) + 1\} + 1
\]

\[
\leq \max\{\text{reg}(K^1(M/xM)), \text{reg}(K^1(M)) + 1\} + 1
\]

\[
\leq \max\{\text{reg}(K^1(M/xM)), \text{reg}(K^1(M)) + 1\} + 1
\]

\[
\leq \max\{1 - \text{beg}(M), \max\{0, 1 - \text{beg}(M)\} + d^0_{M}(0)\} + 1
\]

\[
\leq \max\{0, 1 - \text{beg}(M)\} + d^0_{M}(0) + 2.
\]
As \( \text{depth}(K^2(M)) \geq \min\{2, \dim(M)\} \) (s. Lemma 3.1) we have \( \text{depth}(K^2(M)) = 2 \), thus \( \text{reg}(K^2(M)) = \text{reg}^1(K^2(M)) \). Hence (cf. Reminder 2.4(B) and Definition 3.4(iii),(iv))

\[
\text{reg}(K^2(M)) \leq \text{reg}(K^2(M)/xK^2(M)) \\
\leq \max \{0, 1 - \text{beg}(M)\} + d_M^0(0) + 2 \\
= F_2^2(d_M^0(0), d_M^1(-1), \text{beg}(M)).
\]

This completes the case \( d = 2 \). So, let \( d > 2 \).

By induction on \( d \) and in view of Remark 3.5(A) we have

\[
\text{reg}(K^k(M/xM)) \leq F_{d-1}^k(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-2}(2 - d) + d_M^{d-1}(1 - d), \text{beg}(M)),
\]

for \( 0 \leq k \leq d - 1 \).

Therefore

\[
\text{reg}(K^k(M/xM)) \leq F_{d-1}^k(d_M^0(0) + d_M^1(-1), \ldots, d_M^{d-2}(2 - d) + d_M^{d-1}(1 - d), \text{beg}(M))
\]

for all \( k \in \{0, \ldots, d-1\} \).

We first assume that \( 2 \leq i \leq d - 1 \). Then, by induction on \( i \) we have

\[
\text{reg}(K^{i-1}(M)) \leq F_{d}^{i-1}(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1-d), \text{beg}(M)).
\]

If we apply the exact sequence (1) with \( j = i - 1 \) and keep in mind that \( x \) is filter regular with respect to \( K^{i-1}(M) \) we thus get by (5) and (6):

\[
\text{reg}(K^i(M)/xK^i(M)) \leq \max\{\text{reg}(K^{i-1}(M/xM)), \text{reg}((0 : K^{i-1}(M) \cdot x) + 1\} + 1 \\
\leq \max\{\text{reg}(K^{i-1}(M/xM)), \text{reg}(K^{i-1}(M)) + 1\} + 1 \\
\leq \max\{F_{d-1}^{i-1}(d_M^0(0) + d_M^1(-1), \ldots, d_M^{d-2}(2 - d) + d_M^{d-1}(1 - d), \text{beg}(M)), \\
F_{d}^{i-1}(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1 - d), \text{beg}(M)) + 1\} + 1.
\]

If we apply the sequence (1) with \( j = i \), we obtain

\[
\text{gengdeg}((0 : K^i(M) \cdot x)) \leq \text{reg}(K^i(M/xM)).
\]

According to (5) we have the inequality

\[
\text{reg}(K^i(M/xM)) \leq F_{d-1}^i(d_M^0(0) + d_M^1(-1), \ldots, d_M^{d-2}(2 - d) + d_M^{d-1}(1 - d), \text{beg}(M)).
\]

Set

\[
m_i := \max\{F_{d-1}^{i-1}(d_M^0(0) + d_M^1(-1), \ldots, d_M^{d-2}(2 - d) + d_M^{d-1}(1 - d), \text{beg}(M)), \\
F_{d}^{i-1}(d_M^0(0), d_M^1(-1), \ldots, d_M^{d-1}(1 - d), \text{beg}(M)) + 1\} + 1,
\]

\[
n_i := F_{d-1}^{i}(d_M^0(0) + d_M^1(-1), \ldots, d_M^{d-2}(2 - d) + d_M^{d-1}(1 - d), \text{beg}(M)),
\]

and

\[
t_i := \max\{m_i, n_i\}.
\]
Note that by Remark 3.5 (B) we have $t_i \geq i$. Hence, by Proposition 2.5 and Lemma 3.3
\[
\reg(K^i(M)) \leq t_i + h^0_{K^i(M)}(t_i) \\
\leq t_i + \length(K^i(M)) \\
\leq t_i + \sum_{j=0}^{i-1} \binom{t_i - j - 1}{i - j - 1} \left( \sum_{l=0}^{i-j-1} \binom{i-j-1}{l} d^{i-l-1}_M(l+i+1) \right).
\]
Thus, we obtain (cf. Definition 3.4(viii))
\[
\reg(K^i(M)) \leq F^i_d(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \beg(M)).
\]
This completes the case where $i \leq d - 1$. It thus remains to treat the cases with $i = d > 2$.

Now, by Lemma 3.1 we have $\depth(K^d(M)) \geq 2$. So, again by Reminder 2.4(B) and by use of the sequence (1) we get
\[
\reg(K^d(M)) \leq \reg(K^d(M)/xK^d(M)) \\
\leq \max\{\reg(K^{d-1}(M/xM)), \reg((0 :_{K^{d-1}(M)} x)) + 1\} + 1 \\
\leq \max\{\reg(K^{d-1}(M/xM)), \reg(K^{d-1}(M)) + 1\} + 1.
\]
By induction and Remark 3.5(A) it holds
\[
\reg(K^{d-1}(M/xM)) \leq F^{d-1}_d(d^0_{M/xM}(0), d^1_{M/xM}(-1), \ldots, d^{d-2}_M(2-d), \beg(M/xM)) \\
\leq F^{d-1}_d(d^0_M(0) + d^1_M(-1), d^1_M(-1) + d^2_M(-2), \ldots, d^{d-2}_M(2-d) + d^{d-1}_M(1-d), \beg(M)).
\]
By the case $i = d - 1$ we have
\[
\reg(K^{d-1}(M)) \leq F^{d-1}_d(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \beg(M)).
\]
This implies that (cf. Definition 3.4(ix))
\[
\reg(K^d(M)) \leq \max\{F^{d-1}_d(d^0_M(0) + d^1_M(-1), \ldots, d^{d-2}_M(2-d) + d^{d-1}_M(1-d), \beg(M)), \\
F^{d-1}_d(d^0_M(0), \ldots, d^{d-1}_M(1-d), \beg(M)) + 1\} + 1 \\
= F^d_d(d^0_M(0), \ldots, d^{d-1}_M(1-d), \beg(M)).
\]
So, finally we may conclude that
\[
\reg(K^i(M)) \leq F^i_d(d^0_M(0), d^1_M(-1), \ldots, d^{d-1}_M(1-d), \beg(M)),
\]
for all $d \in \mathbb{N}$ and all $i \in \{0, \ldots, d\}$.

\textbf{Corollary 3.7.} Let $d \in \mathbb{N}$, $i \in \{0, \ldots, d\}$, $(x_0, \ldots, x_{d-1}, y) \in \mathbb{N}_0^d \times \mathbb{Z}$ and let $M$ be a finitely generated graded $R$-module such that $\dim(M) \leq d$, $d^i_M(-j) \leq x_j$ for all $j \in \{0, \ldots, d-1\}$ and $\beg(M) \geq y$. Then
\[
\reg(K^i(M)) \leq F^i_d(x_0, \ldots, x_{d-1}, y).
\]
Proof. If \( M = 0 \), we have \( K^i(M) = 0 \) and so our claim is obvious.

If \( \dim(M) = 0 \), we have \( K^i(M) = 0 \) for all \( i > 0 \) and \( \dim(K^0(M)) \leq 0 \) so that (s. Reminder 2.6(C)(v))

\[
\text{reg}(K^0(M)) = \text{end}(K^0(M)) = -\text{beg}(H^0_{R+}(M)) = -\text{beg}(M) \leq -y = F^0(x_0, \cdots, x_{d-1}, y).
\]

So, it remains to show our claim if \( \dim(M) > 0 \). But now, we may conclude by Theorem 3.6 and Remark 3.5(A), (C). \( \square \)

4. Bounding \( \text{reg}(K^i(M)) \) in terms of \( \text{reg}^2(M) \)

We keep the notations introduced in section 3. In particular we always assume that the homogeneous Noetherian ring \( R = \bigoplus_{n \geq 0} R_n \) has Artinian local base ring \( (R_0, m_0) \). We have seen in the previous section, that the Castelnuovo-Mumford regularity of the deficiency modules \( K^i(M) \) of a finitely generated graded \( R \)-module \( M \) is bounded in terms of the invariants \( d^i_M(-j) \) \((j = 0, \cdots, \dim(M) - 1)\) and \( \text{beg}(M) \). We shall use this result in order to bound the numbers \( \text{reg}(K^i(M)) \) in terms of the Castelnuovo-Mumford regularity of \( M \). This idea is inspired by Hoa-Hyry [17] who gave similar results for graded ideals in a polynomial ring over a field.

As an application we shall derive a number of further bounds on the invariants \( \text{reg}(K^i(M)) \).

**Definition 4.1.** Let \( d \in \mathbb{N} \) and \( i \in \{0, \cdots, d\} \). We define a bounding function

\[
G_d^i : \mathbb{N}_0 \times \mathbb{Z}^2 \rightarrow \mathbb{Z}
\]

by

\[
G_d^i(u, v, w) := F_d^i(u, 0, \cdots, 0, v - w) - w.
\]

Now, we are ready to give a first result of the announced type. It says that the numbers \( \text{reg}(K^i(M)) \) find upper bounds in terms of \( \text{reg}^2(M) \) and the Hilbert polynomial of \( M \).

**Theorem 4.2.** Let \( p \in \mathbb{N}_0, d \in \mathbb{N}, i \in \{0, \cdots, d\}, b, r \in \mathbb{Z} \) and let \( M \) be a finitely generated graded \( R \)-module with \( \dim(M) \leq d \), \( \text{beg}(M) \geq b \), \( \text{reg}^2(M) \leq r \) and \( p_M(r) \leq p \). Then

\[
\text{reg}(K^i(M)) \leq G_d^i(p, b, r).
\]

**Proof.** Observe that \( \text{beg}(M(r)) \geq b - r \). On use of Corollary 3.7 we now get

\[
\text{reg}(K^i(M)) + r = \text{reg}(K^i(M)(-r)) = \text{reg}(K^i(M(r)))
\]

\[
\leq F_d^i(d^0_{M(r)}(0), d^0_{M(r)}(-1), \cdots, d^{d-1}_{M(r)}(1 - d), b - r)
\]

\[
= F_d^i(d^0_{M(r)}, d^1_{M(r)}(r - 1), \cdots, d^{d-1}_{M(r)}(r + 1 - d), b - r).
\]

For all \( j \in \mathbb{N} \) we have \( d^j_M(r - j) = h^{j+1}_M(r - j) = 0 \), so that

\[
d^1_M(r - 1) = \cdots = d^{d-1}_M(r + 1 - d) = 0.
\]
In addition $d_i^j_M(r) = h_i^{j+1}(r) = 0$ for all $j \in \mathbb{N}$, which implies that $d_0^0_M(r) = p_M(r) \leq p$ (s. Reminder 2.3(C)). In view of Remark 3.5(A) the above inequality now induces
\[
\text{reg}(K^i(M)) + r \leq F_d^i(p, 0, \cdots, 0, b - r)
\]
and this proves our claim. \qed

Bearing in mind possible application to Hilbert schemes for example one could ask for bounds which apply uniformly to all graded submodules $M$ of a given finitely generated graded $R$-module $U$ and depend only on basic invariants of $M$.

Our next result gives such a bound which depends only on $\text{reg}^2(M)$ and the Hilbert polynomial $p_U$ of the ambient module $U$.

**Corollary 4.3.** Let $p, d, i, b$ and $r$ be as in Theorem 4.2. Let $U$ be a finitely generated and graded $R$-module such that $\dim(U) \leq d$, $\text{beg}(U) \geq b$, $\text{reg}^2(U) \leq r$ and $p_U(r) \leq p$. Then, for each graded submodule $M \subseteq U$ such that $\text{reg}^2(M) \leq r$ we have
\[
\max\{\text{reg}(K^i(M)), \text{reg}(K^i(U/M))\} \leq G_d^i(p, b, r).
\]

**Proof.** Let $M$ be as above, so that $\text{reg}^2(M) \leq r$. Then, the short exact sequence
\[
0 \longrightarrow M \longrightarrow U \longrightarrow U/M \longrightarrow 0
\]
implies that $\text{reg}^2(U/M) \leq r$. Now, as previously we get on use of Reminder 2.3(C)
\[
d_0^0_M(r) = p_M(r), d_0^0_U(r) = p_U(r), d_0^0_{U/M}(r) = p_{U/M}(r).
\]
As $D_{R_1}(M)_r \cong H_{R_1}^2(M)_r = 0$ the sequence (1) implies
\[
d_0^0_M(r) + d_0^0_{U/M}(r) = d_0^0_U(r).
\]
In view of the equalities (2) we thus get
\[
p_M(r), p_{U/M}(r) \leq p.
\]
As $\dim(M), \dim(U/M) \leq d$ and $\text{beg}(M), \text{beg}(U/M) \geq b$ we now get the requested inequalities by Theorem 4.2. \qed

Corollary 4.3 immediately implies a bounding result which is of the type given by Hoa-Hyry [17].

**Corollary 4.4.** Let $d, m, r \in \mathbb{N}$, $i \in \{0, \cdots, d\}$ and assume that $\dim(R) \leq d$, $\text{reg}^1(R) \leq r$ and $\dim_{R_0/\mathfrak{m}_0}(R_1/\mathfrak{m}_0R_1) \leq m$. Let
\[
\gamma := G_d^i\left(\binom{m + r - 1}{r - 1}\right)\text{length}(R_0), 0, r).
\]
Then, for each graded ideal $\mathfrak{a} \subseteq R$ with $\text{reg}^2(\mathfrak{a}) \leq r$ we have
\[
\max\{\text{reg}(K^i(\mathfrak{a})), \text{reg}(K^i(R/\mathfrak{a}))\} \leq \gamma.
\]
Proof. Let $x_1, \ldots, x_m$ be indeterminates. Then, there is a surjective homomorphism of graded $R_0$-algebras $R_0[x_1, \ldots, x_m] \twoheadrightarrow R$, so that $p_R(r) \leq \binom{m+r-1}{r-1} \text{length}(R_0)$.

As $\text{beg}(R) = 0$ we now conclude by corollary 4.3. \qed

Remark 4.5. If $d \geq 2$ and $R = K[x_1, \ldots, x_d]$ is a standard graded polynomial ring over a field $K$ and $a \subseteq R$ is a graded ideal with $\text{reg}_2(a) \leq r$, the previous result shows that

$$\text{reg}(K^i(R/a)) \leq G_d^i\left(\frac{d+r-1}{r-1}, 0, r\right).$$

This inequality bounds $\text{reg}(K^i(R/a))$ in terms of $\text{reg}_2(a)$. So, our result in a certain way improves [17, Theorem 14], which bounds $\text{reg}(K^i(R/a))$ only in terms of $\text{reg}(a) = \text{reg}_1(a)$. On the other hand we do not insist that our bound is sharper from the numerical point of view.

Recently, "almost sharp" bounds on the Castelnuovo-Mumford regularity in terms of the generating degree have been given by Caviglia- Sbarra [11], Chardin-Fall-Nagel [12] and [4]. Combining these with the previous results of the present section, we get another type of bounding results for the Castelnuovo-Mumford regularity of deficiency modules. Here, we restrict ourselves to give two such bounds which hold over polynomial rings, as the corresponding statements get comparatively simple in this case.

Corollary 4.6. Let $d, m \in \mathbb{N}$, let $i \in \{0, \cdots, d\}$, let $b, r \in \mathbb{Z}$, let $R = R_0[x_1, \cdots, x_d]$ be a standard graded polynomial ring and let $U \not= 0$ be a graded $R$-module which is generated by $m$ homogeneous elements and satisfies $\text{beg}(U) = b$ and $\text{reg}(U) < r$.

Set

$$\varrho := [r + (m + 1) \text{length}(R_0) - b]^{2d-1},$$

$$\pi := m\left(\frac{d + \varrho - 1}{\varrho - 1}\right) \text{length}(R_0) \text{ and}$$

$$\delta := G_d^i(\pi, b, \varrho + b).$$

Then, for each graded submodule $M \subseteq U$ with $\text{gendeg}(M) \leq r$ we have

$$\max\{\text{reg}(K^i(M)), \text{reg}(K^i(U/M))\} < \delta.$$ 

Proof. Let $U = \sum_{i=1}^{m} Ru_i$ with $u_i \in U_{n_i}$ and $b = n_1 \leq n_2 \leq \cdots \leq n_m = \text{gendeg}(U) \leq \text{reg}(U) < r$.

As $r - b > 0$ we have $r < \varrho + b$, whence $\text{reg}(U) < \varrho + b$. Therefore by Reminder 2.3(C) we obtain $p_U(\varrho + b) = \text{length}(U_{\varrho+b})$. As there is an epimorphism of graded $R$-modules

$$\bigoplus_{i=1}^{m} R(-n_i) \twoheadrightarrow U$$
we thus obtain
\[
p_U(\rho + b) \leq \sum_{i=1}^{m} \left( \frac{d + \rho + b - n_i - 1}{\rho + b - n_i - 1} \right) \text{length}(R_0)
\]
\[
\leq m \left( \frac{d + \rho - 1}{\rho - 1} \right) \text{length}(R_0) = \pi.
\]
Finally, by [4, Proposition 6.1] we have \(\text{reg}(M) \leq \rho + b\) for each graded submodule \(M \subseteq U\) with \(\text{gendeg}(M) \leq r\). Now we conclude by corollary 4.3. \(\square\)

**Remark 4.7.** Let \(d, i > 1\) and \(R\) be as in Corollary 4.6 and let \(a \subset R\) be a graded ideal of positive height. Let
\[
r := [\text{gendeg}(a)(1 + \text{length}(R_0))]^{2d-2},
\]
\[
\gamma := G_i^d\left(\left(\frac{d + r - 1}{r - 1}\right) \text{length}(R_0), 0, r\right).
\]
Then, combining [4, Corollary (5.7)(b)] with Corollary 4.4 we get
\[
\max\{\text{reg}(K^i(a)), \text{reg}(K^i(R/a))\} < \gamma.
\]
For more involved but sharper bounds of the same type one should combine the bounds given in [12] with Corollary 4.3.

Our next bound is in the spirit of the classical ”problem of finitely many steps” (cf. [15], [14]): it bounds \(\text{reg}(K^i(M))\) in terms of the discrete data of a minimal free presentation of \(M\). Again we content ourselves to give a bounding result which is comparatively simple and concerns only the case where \(R\) is a polynomial ring.

**Corollary 4.8.** Let \(d, m \in \mathbb{N}\), let \(i \in \{0, \ldots, d\}\), let \(R = R_0[x_1, \ldots, x_d]\) be a standard graded polynomial ring, let \(p : F \rightarrow N\) be an epimorphism of finitely generated graded \(R\)-modules such that \(F\) is free of rank \(m > 0\).

Set \(b := \text{beg}(F)\) and \(r := \max\{\text{gendeg}(F) + 1, \text{gendeg}(\text{ker}(p))\}\) and define \(\delta\) as in Corollary 4.6. Then
\[
\text{reg}(K^i(N)) < \delta.
\]

**Proof.** Apply Corollary 4.6 with \(U = F\) and with \(\text{ker}(p)\) instead of \(M\). \(\square\)

Our last application is a bound in the spirit of Mumford’s classical result [21] which uses the Hilbert coefficients as key bounding invariants. To formulate our result we first introduce a few notations.

**Reminder 4.9.** (Hilbert coefficients) (A) Let \(d \in \mathbb{N}\) and let \(\underline{e} := (e_0, \ldots, e_{d-1}) \in \mathbb{Z}^d \setminus \{0\}\).

We introduce the polynomial
\[
(i) \quad p_\underline{e}(x) := \sum_{i=0}^{d-1} (-1)^i e_i \binom{x+d-i-1}{d-i-1} \in \mathbb{Q}[x]
\]
which satisfies
(ii) \( \deg(p_e) = d - 1 - \min\{i | e_i \neq 0\} \).

(B) If \( M \) is a finitely generated graded \( R \)-module of dimension \( d \), we define the Hilbert coefficients \( e_i(M) \) of \( M \) for \( i = 0, \cdots, d - 1 \) such that

(i) \( p_M(x) = p_{(e_0(M),\cdots,e_{d-1}(M))}(x) \).

In particular \( e_0(M) \in \mathbb{N} \) is the Hilbert-Serre multiplicity of \( M \). In addition we set:

(ii) \( e_i(M) := 0 \) for all \( i \in \mathbb{Z}\{0,\cdots, d - 1\} \).

**Notation 4.10.** Let \( m, d \in \mathbb{N} \) with \( d > 1 \). We define a numerical function \( H^m_d : \mathbb{Z}^d \to \mathbb{Z} \), recursively on \( d \), as follows (cf. Reminder 4.9(A)(i))

(i) \( H^m_d(e_0, e_1) := 1 - p_{(e_0,e_1)}(-1) \).

If \( d > 2 \) and the function \( H^m_{d-1} \) has already been defined, let \( \underline{e} := (e_0,\cdots, e_{d-1}) \in \mathbb{Z}^d \), set

(ii) \( \underline{e}' := (e_0,\cdots, e_{d-2}), \ f := H^m_{d-1}(\underline{e}') \),

and define (cf. Reminder 4.9(A)(i))

(iii) \( H^m_d(\underline{e}) := \text{length}(R_0)m\left(\frac{d+3}{d-1}\right) - p_{\underline{e}}(f - 2) + f \),

with the convention that \( \binom{t}{d-1} := 0 \) for all \( t < d - 1 \).

**Remark 4.11.** Let \( m, d \in \mathbb{N} \) be with \( d > 1 \) and set \( \underline{0} := (0,\cdots, 0) \). Then in the notation of [9, 17.2.4], we have

\[
H^m_d = F^{(d)}_{\underline{0}}.
\]

The next result is of preliminary nature and extends [9, 17.2.7] which at its turn generalizes Mumford bounding result (s. [21, pg.101]).

**Proposition 4.12.** Let \( d, m \in \mathbb{N} \) with \( d > 1 \), let \( r \in \mathbb{Z} \) and let \( U \) be a finitely generated graded \( R \)-module with \( \dim(U) = d, \ \reg(U) \leq r \) and \( \dim_{R_0/m_0}(U/m_0U_r) \leq m \). Let \( M \subseteq U \) be a graded submodule. Then, setting \( L := U/M, \ h := d - \dim(L) \) and

\[
t := H^m_d(m \text{length}(R_0) - (-1)^h e_{-h}(L(r)), (-1)^h e_{1-h}(L(r)), \cdots, (-1)^h e_{d-1-h}(L(r))),
\]

we have

(a) \( \reg^1(L) \leq \max\{0, t - 1\} + r \);

(b) \( \reg^2(M) \leq \max\{1, t\} + r \).
Proof. If $M$ is $R_+$-torsion, we have $\operatorname{reg}^1(L) = \operatorname{reg}^1(U) \leq r$ and $\operatorname{reg}^2(M) = -\infty$ so that our claim is obvious. Therefore we may assume that $M$ is not $R_+$-torsion.

We may assume that $R_0/m_0$ is infinite. We may in addition replace $R$ by $R/(0 :_R U)$ and hence assume that $\dim(R) = d$. We now find elements $a_1, \ldots, a_d \in R_1$ which form a system of parameters for $R$. In particular $R$ is a finite integral extension of $R_0[a_1, \ldots, a_d]$. Consider the polynomial ring $R_0[x_1, \ldots, x_d]$ and the homomorphism of $R_0$-algebras $f : R_0[x_1, \ldots, x_d] \to R$ given by $x_i \mapsto a_i$ for $i = 1, \ldots, d$. Then, $M$ is a finitely generated graded module over $R_0[x_1, \ldots, x_d]$ and $\sqrt{R_+} = \sqrt{(x_1, \ldots, x_d)}R$.

So, the numerical invariants of $U$ and $M$ which occur in our statement do not change if we consider $U$ and $M$ as $R_0[x_1, \ldots, x_d]$-modules by means of $f$. Therefore, we may assume that $R = R_0[x_1, \ldots, x_d]$. Now, we have $\operatorname{gendeg}(U(r)) \leq \operatorname{reg}(U(r)) \leq 0$ and $\dim_{R_0/m_0}(U(r)/m_0U(r)_0) \leq m$. This implies that the $R$-module $U(r)_{\geq 0}$ is generated by (at most) $m$ homogeneous elements of degree 0. Therefore we have an epimorphism of graded $R$-modules

$$R^\oplus m \xrightarrow{g} U(r)_{\geq 0}.$$  

Let $N := g^{-1}(M(r)_{\geq 0})$. As $M$ is not $R_+$-torsion we have $M(r)_{\geq 0} \neq 0$ and hence $N \neq 0$. As $N \subseteq R^\oplus m$ and by our choice of $R$ we thus have $\dim(N) = d$. Now, the isomorphism of graded $R$-modules $R^\oplus m/N \cong (L(r))_{\geq 0}$ implies

$$m \text{length}(R_0) \left( \frac{x + d - 1}{d - 1} \right) - \sum_{i=0}^{d-1} (-1)^i e_i(N) \left( \frac{x + d - i - 1}{d - i - 1} \right) \quad \text{for all } i \in \{1, \ldots, d-1\}.$$  

Therefore

$$e_0(N) = m \text{length}(R_0) - (-1)^h e_{-h}(L(r))$$

and

$$e_i(N) = (-1)^h e_{-h}(L(r)) \quad \text{for all } i \in \{1, \ldots, d-1\}.$$  

So, according to [9, 17.2.7] and Remark 4.11 we obtain

$$\operatorname{reg}^2(N) \leq F_0^{(d)}(e_0(N), \ldots, e_{d-1}(N)) \quad \text{for all } i \in \{1, \ldots, d-1\}.$$  

Now, the short exact sequence of graded $R$-modules

$$0 \to N \to R^\oplus m \to (U(r)/M(r))_{\geq 0} \to 0$$

implies $\operatorname{reg}^1((U(r)/M(r))_{\geq 0}) \leq \max\{0, t-1\}$, whence $\operatorname{reg}^1(U(r)/M(r)) \leq \max\{0, t-1\}$, so that finally

$$\operatorname{reg}^1(L) = \operatorname{reg}^1(U(r)/M(r)) + r \leq \max\{0, t-1\} + r.$$
and
\[ \operatorname{reg}^2(M) = \operatorname{reg}^2(M(r)) + r \leq \max \{ \operatorname{reg}^2(U(r)), \operatorname{reg}^1(U(r)/M(r)) + 1 \} + r \]
\[ \leq \max \{ 0, \max \{ 0, t - 1 \} + 1 \} + r \leq \max \{ 1, t \} + r. \]

This proves our claim.

Now, we may bound the Castelnuovo-Mumford regularity of deficiency modules as follows:

**Corollary 4.13.** Let the notations and hypothesis be as in Proposition 4.12. In addition let \( b \in \mathbb{Z} \) and \( p \in \mathbb{N}_0 \) such that \( \operatorname{beg}(U) \geq b \) and \( \operatorname{pu}(r) \leq p \).

Then, for all \( i \in \{ 0, \cdots, d \} \) we have
\[ \max \{ \operatorname{reg}(K^i(M)), \operatorname{reg}(K^i(U/M)) \} \leq G^i_d(p, b, \max \{ 1, t \} + r). \]

**Proof.** This is clear by Corollary 4.3 and Proposition 4.12(b).

Applying this to the "classical" situation of [21] where \( M = a \) is a graded ideal of a polynomial ring we finally can say

**Corollary 4.14.** Let \( R = R_0[x_1, \cdots, x_d] \) be a standard graded polynomial ring with \( d > 1 \) and let \( a \subseteq R \) be a graded ideal. Set \( h := \operatorname{height}(a) \) and
\[ t := H^1_d(\ell(R_0)) - (-1)^{h}e_{-h}(R/a), (-1)^{h}e_{1-h}(R/a), \cdots, (-1)^{h}e_{d-1-h}(R/a). \]

Then, for all \( i \in \{ 0, \cdots, d \} \) we have
\[ \max \{ \operatorname{reg}(K^i(a)), \operatorname{reg}(K^i(R/a)) \} \leq G^i_d(1, 0, \max \{ 1, t \}). \]

**Proof.** Choose \( U := R, M := a, m = 1, r = 0, b = 0, p = 1. \) Observe also that \( d - \dim(R/a) = h \) and apply Corollary 4.13.

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5. **Bounding Cohomological Postulation Numbers**

In [6, Theorem 4.6] it is shown that the cohomological postulation numbers of a projective scheme \( X \) over a field \( K \) with respect to a coherent sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \) are bounded by the cohomology diagonal \( (h^i(X, \mathcal{F}(-i)))_{i=0}^{\dim(\mathcal{F})} \) of \( \mathcal{F} \). On use of Theorem 3.6 this "purely diagonal bound" now can be generalized to the case where the base field \( K \) is replaced by an arbitrary Artinian ring. To do so, we first introduce some appropriate notions.

**Definition 5.1.** For \( d \in \mathbb{N} \) and \( i \in \{ 0, \cdots, d - 1 \} \) we define the bounding function
\[ E^i_d : \mathbb{N}_0^d \rightarrow \mathbb{Z} \]
by
\[ E^i_d(x_0, \cdots, x_{d-1}) := -F^{i+1}_d(x_0, \cdots, x_{d-1}, 0), \]
where \( F^{i+1}_d \) is defined according to Definition 3.4.
Definition 5.2. Let $d \in \mathbb{N}$. By $\mathcal{D}^d$ we denote the class of all pairs $(R, M)$ in which $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is a Noetherian homogenous ring with Artinian base ring $R_0$ and $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a finitely generated graded $R$-module with dim$(M) \leq d$.

Now, we are ready to state the announced "purely diagonal" bounding result as follows:

Theorem 5.3. Let $d \in \mathbb{N}$, let $x_0, \cdots, x_{d-1} \in \mathbb{N}_0$ and let $(R, M) \in \mathcal{D}^d$ such that $d^i_M(-j) \leq x_j$ for all $j \in \{0, \cdots, d-1\}$. Then for all $i \in \{0, \cdots, d-1\}$ we have

$$
\nu^i_M \geq E^i_d(x_0, \cdots, x_{d-1}).
$$

Proof. On use of standard reduction arguments and the monotonicity statement of Remark 3.5(A) we can restrict ourselves to the case where the Artini an base ring $R_0$ is local. Consider the graded submodule $N := M_{\geq 0} = \bigoplus_{n \geq 0} M_n$ of $M$. As the module $M/N$ is $R_+$-torsion, the graded short exact sequence $0 \to N \to M \to M/N \to 0$ yields isomorphisms of graded $R$-modules $D^i_{R_+}(M) \cong D^i_{R_+}(N)$ and hence equalities $d^i_M = d^i_N$ for all $j \in \mathbb{N}_0$. These allow to replace $M$ by $N$ and hence to assume that $\text{beg}(M) \geq 0$.

Now, on use of Corollary 3.7 and Reminders 2.6(C)(vi) and 2.3(C) we get

$$
\nu^i_M = -p(K^{i+1}(M)) \geq -\text{reg}(K^{i+1}(M)) \geq -E^i_d(x_0, \cdots, x_{d-1}, 0) = E^i_d(x_0, \cdots, x_{d-1}).
$$

As a consequence of Theorem 5.3 we get the following finiteness result which is shown in [6] for the special case of homogeneous rings $R$ whose base rings $R_0$ are field.

Theorem 5.4. Let $d \in \mathbb{N}$ and let $x_0, \cdots, x_{d-1} \in \mathbb{N}_0$. Then, the set of cohomological Hilbert functions

$$
\{d^i_M \mid i \in \mathbb{N}_0; (R, M) \in \mathcal{D}^d; d^i_M(-j) \leq x_j \text{ for } j = 0, \cdots, d-1\}
$$

is finite.

Proof. First, we set

$$
\mathcal{D} := \{(R, M) \in \mathcal{D}^d \mid d^i_M(-j) \leq x_j \text{ for } j = 0, \cdots, d-1\}.
$$

As $d^i_M \equiv 0$ if $(R, M) \in \mathcal{D}^d$ and $i \geq d$, it suffices to show that the set

$$
\{d^i_M \mid i < d, (R, M) \in \mathcal{D}\}
$$

is finite.

According to [8, Lemma 4.2] we have

$$
d^i_M(n) \leq \sum_{j=0}^{i} \left(\begin{array}{c} -n-j-1 \\ i-j \end{array}\right) \left[\sum_{l=0}^{i-j} \left(\begin{array}{c} i-j \\ l \end{array}\right) x_{i-l}\right]
$$

for all $i \in \mathbb{N}_0$, all $n \leq -i$ and all $(R, M) \in \mathcal{D}$. According to Theorem 5.3 there is some integer $c \leq -d+1$ such that $\nu^i_M > c$ for all $(R, M) \in \mathcal{D}$ and all $i < d$. So, using the notation of Reminder 2.3(B) we have $q^i_M(n) = d^i_M(n)$ for all $i < d$ and all $n \leq c$. 

As \( \deg(q^i_M) \leq i \) (s. Reminder 2.3 (B)(ii),(iv)) it follows from (1) that the set
\[
\{q^i_M \mid i < d, (R, M) \in \mathcal{D}\}
\]
is finite. Consequently, the set
\[
\{d^i_M(n) \mid i < d, n \leq c, (R, M) \in \mathcal{D}\}
\]
is finite, too. So, in view of (1) the set
\[
\{d^i_M(n) \mid i < d, n \leq -i, (R, M) \in \mathcal{D}\}
\]
must be finite. It thus remains to show that for each \( i < d \) the set
\[
S_i := \{d^i_M(n) \mid n \geq -i, (R, M) \in \mathcal{D}\}
\]
is finite. To this end, we fix \( i \in \{1, \ldots, d-1\} \). According to [7, Corollary (3.11)] there are two integers \( \alpha, \beta \) such that
\[
d^i_M(n) \leq \alpha \quad \text{for all } n \geq -i \quad \text{and all } (R, M) \in \mathcal{D},
\]
\[
\text{reg}^2(M) \leq \beta \quad \text{for all } (R, M) \in \mathcal{D}. \tag{2}
\]
The inequality (3) implies that \( d^i_M(n) = 0 \) for all \( n \geq \beta - i + 1 \) and hence by (2) the set \( S_i \) is finite.

It remains to show that the set \( S_0 \) is finite.

To do so, we write \( \overline{M} := D_{R_+}(M)_{\geq 0} \) for all pairs \( (R, M) \in \mathcal{D} \). As \( (D_{R_+}(M)/\overline{M})_{\geq 0} = 0 \), \( H^k_{R_+}(D_{R_+}(M)) = 0 \) for \( k = 0, 1 \) and \( D^j_{R_+}(D_{R_+}(M)) \cong D^j_{R_+}(M) \) for all \( j \in \mathbb{N}_0 \) we get \( \Gamma_{R_+}(\overline{M}) = 0 \), \( \text{end}(H^1_{R_+}(M)) < 0 \) and \( d^i_M \equiv d^i_M \) for all \( j \in \mathbb{N}_0 \) and for all \( (R, M) \in \mathcal{D} \). In particular \( (R, \overline{M}) \in \mathcal{D} \) for all \( (R, M) \in \mathcal{D} \). So, writing
\[
\overline{\mathcal{D}} := \{(R, M) \in \mathcal{D} | \Gamma_{R_+}(M) = 0, \text{end}(H^1_{R_+}(M)) < 0\}
\]
it suffices to show that the set
\[
\overline{S}_0 = \{d^0_M(n) | n \geq 0, (R, M) \in \overline{\mathcal{D}}\}
\]
is finite.

If \( (R, M) \in \overline{\mathcal{D}} \) we conclude by statement (3) that \( p(M) \leq \text{reg}(M) = \text{reg}^1(M) = \max\{\text{end}(H^1_{R_+}(M)) + 1, \text{reg}^2(M)\} \leq \max\{0, \beta\} := \beta' \). As \( \deg(p_M) < d \) it follows by statement (2) that the set of Hilbert polynomials \( \{p_M | (R, M) \in \overline{\mathcal{D}}\} \) is finite. Consequently, the set \( \{d^0_M(n) | n > \beta', (R, M) \in \overline{\mathcal{D}}\} \) is finite. Another use of statement (2) now implies the finiteness of \( \overline{S}_0 \). \( \square \)

**Corollary 5.5.** Let the notations be as in Theorem 5.4 and Reminder 2.3. Then the sets of polynomials
\[
\{q^i_M \mid i \in \mathbb{N}_0; (R, M) \in \mathcal{D}^d; d^i_M(-j) \leq x_j \quad \text{for } j = 0, \ldots, d-1\},
\]
\[
\{p_M \mid (R, M) \in \mathcal{D}^d; d^i_M(-j) \leq x_j \quad \text{for } j = 0, \ldots, d-1\}
\]
are finite.
Proof. This is clear by Theorem 5.4. □

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