Higher Order Reentrant Post Modes in Cylindrical Cavities

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Reentrant cavities are microwave resonant devices employed in a number of different areas of physics. They are appealing due to their simple frequency tuning mechanism, which offers large tuning ranges. Reentrant cavities are, in essence, 3D lumped LC circuits consisting of a conducting central post embedded in a resonant cavity. The lowest order reentrant mode (which transforms from the \( \text{TM}_{010} \) mode) has been extensively studied in past publications. In this work we show the existence of higher order reentrant post modes (which transform from the \( \text{TM}_{01n} \) mode family). We characterize these new modes in terms of their frequency tuning, filling factors and quality factors, as well as discuss some possible applications of these modes in fundamental physics tests.

I. INTRODUCTION

The cylindrical reentrant cavity is a device that can provide high-Q microwave modes with large tuning ranges. It consists of a metal cavity with a conducting post or ring located centrally within the cavity. The gap between the top of this post or ring and the top of the cavity adjusts the mode frequency and at certain gap spacings traps the electric field within the gap. Such cavity designs have been extensively studied and allow for a standard reentrant mode tuning range on the order of GHz without the need for physically large cavities.¹⁻⁶

Since this structure was first investigated in connection with the development of klystrons, it has been widely used in the construction of microwave oscillators and particle accelerators,⁷⁻⁸ and is often chosen as a structure in the study of metamaterials.⁹⁻¹⁰ Some recent work has discussed potential applications in telecommunications,¹¹ and detection of gravitational waves,¹²⁻¹³ and dark matter.¹⁴ It is an interesting perspective to view the standard reentrant mode as a perturbed \( \text{TM}_{010} \) mode. The standard reentrant mode transforms into the empty cavity \( \text{TM}_{010} \) mode as the central post is removed from the cavity.¹⁵ In a similar way, there exist higher order reentrant modes, which can be viewed as perturbed \( \text{TM}_{01n} \) modes.

In this work we unequivocally demonstrate the existence of these higher order reentrant modes and characterize them in terms of their tuning range, and quality factor. Section II presents a theoretical study of these modes based on finite element analysis, and experimental data follows in Section III.

II. FINITE ELEMENT ANALYSIS OF THE REENTRANT CAVITY

![FIG. 1: Cylindrical cross-section of the cavity from the COMSOL model in the \( r-z \) plane (encompassing the origin) in cylindrical coordinates. Grey represents the empty cavity space, while the metallic cavity region is represented by grid-lined white area. The height of cavity is \( \sim 0.4 \) m and the diameter of the inner wall is 0.1337 m. The diameter of the central post is 0.024 m.](image)

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FIG. 2: $B_\phi$, $E_z$ and $E_r$ field components for the first higher order reentrant mode are shown from left to right, with a gap size of 1 cm.

We employed the COMSOL Multiphysics software to simulate a reentrant cavity structure. Since the single post reentrant cavity is an axisymmetric structure, it sufficed to utilize 2D modelling, with revolution around the z-axis. The structure of interest is shown in fig. 1.

We find that multiple higher order modes in the single post reentrant cavity follow similar mode structures to the standard reentrant mode. All of these modes have the same dominant field components: the axial electric field, $E_z$, which is located in the region directly above the post and below the top of the cavity, the azimuthal magnetic field $B_\phi$, which is separated into several rings surrounding the central post, in alternating phase, and a radial electric field, $E_r$, due to the gradient of $B_\phi$ in the $z$ direction. Fig. 2 shows the features of these modes.

The main feature of interest in the higher order modes is the azimuthal magnetic field, which alternates in direction between clockwise and counter-clockwise around the post. We name these higher order modes by their axial wave number, or the number of these “magnetic rings” minus one. Unlike the standard reentrant mode, the higher order modes store their electric energy both in the gap region and cylinder region. The former is determined by the strong $E_z$ field in the central post gap region, and the latter is determined by the radial electric field in the cylinder region which is strongest at the nodes of magnetic field between the “magnetic rings”.

Largely due to the radial electric field, higher order mode filling factors differ from the standard reentrant mode. This can be readily observed when the size of the gap becomes very small. Fig. 3 shows the $E_z$ filling factor of different modes, defined as

$$ FF = \frac{\int dV |E_z|^2}{\int dV |E_c|^2}. $$

(1)
FIG. 4: $B_\phi$ component transformation of the 1st higher order reentrant mode. From left to right: the first image represents the corresponding coaxial mode, the second and third images represent the purely reentrant mode. The fourth image shows the transition from the reentrant phase to the ‘pseudo-TM’ phase. The fifth image shows the ‘pseudo-TM’ phase of the 1st higher order mode. The sixth image shows the $TM_{011}$ mode in the cylindrical cavity. Redder regions represent regions of higher positively signed magnetic field, whilst bluer regions represent regions of higher negatively signed magnetic field. The ‘f’ and ‘FF’ values correspond to the frequency and $E_z$ filling factor respectively.

FIG. 5: $B_\phi$ component transformation of the 2nd higher order reentrant mode. From left to right: the first image represents the corresponding coaxial mode, the second image represents the purely reentrant mode. The third image shows the transition from the reentrant phase to the ‘pseudo-TM’ phase. The fourth and fifth images show the ‘pseudo-TM’ phase of the 2nd higher order mode, where some of the “magnetic rings” have moved from the post to the cavity. The sixth image shows the $TM_{012}$ mode in the cylindrical cavity. Redder regions represent regions of higher positively signed magnetic field, whilst bluer regions represent regions of higher negatively signed magnetic field. The ‘f’ and ‘FF’ values correspond to the frequency and $E_z$ filling factor respectively.
(a) Frequencies of multiple reentrant modes as a function of gap size. The modes are represented as detailed in the legend, where the “0th” order mode corresponds to the fundamental TM$_{010}$-like mode. Points above 2.2 GHz are suppressed. Simulated data begins with a gap size of 5mm in this run.

(b) Geometry factors of multiple reentrant modes as a function of gap size. The modes are represented as detailed in the legend, where the “0th” order mode corresponds to the fundamental TM$_{010}$-like mode. Points above 2.2 GHz are suppressed. Simulated data begins with a gap size of 5mm in this run.

FIG. 6

| Frequency (GHz) | Standard Mode | 1st higher order mode | 2nd higher order mode | 3rd higher order mode | 4th higher order mode |
|----------------|---------------|-----------------------|-----------------------|-----------------------|-----------------------|
| Initial        | 0             | 0.381                 | 0.761                 | 1.140                 | 1.520                 |
| Final          | 1.72          | 1.71                  | 1.73                  | 1.73                  | 1.81                  |
| Tuning         | 1.72          | 1.33                  | 0.967                 | 0.584                 | 0.313                 |

TABLE I: Tuning data for several reentrant modes. The initial frequencies are the frequencies of the modes when the gap size is zero (coaxial regime) where as the final frequencies are the mode frequencies at the end of each modes large tuning phase, when they begin to transition to an empty cavity mode.

Where $E_z$ is the electric field in the cavity. As a consequence of the $E_z$ filling factor of the fundamental TM$_{010}$ mode being unity, we can think of $E_z$ filling factors as normalization of mode field patterns relative to the fundamental. When the gap is of the order of mm, the capacitance of the structure formed by the two planes (the top of the post and the top of the cavity) becomes large, and the electric field in this area becomes stronger. In the standard reentrant, with a small $E_r$ electric field component, which itself is near the gap region, the $E_z$ filling factor grows larger at small gaps. However, because the filling factor is related to the integration volume, and due to the existence of larger radial electric field components in the larger non-gap region, the $E_z$ filling factors of higher order modes may first increase as the gap decreases but eventually approaches zero when the volume of the gap region becomes small.

Each higher order mode is slightly different, the first higher order mode reaches a maximum $E_z$ filling factor (about 0.2) when the gap is around 0.0015m, yet other modes’ $E_z$ filling factors continually decrease as the gap becomes smaller. Unlike the standard reentrant mode, these higher order mode frequencies do not tend to zero as the gap size tends to zero, as their effective capacitance does not tend to infinity due again to the radial electric field components. These modes transition to coaxial modes of fixed frequency when the gap size becomes zero.

A common use of a reentrant cavity structure is to utilize the transformation process from empty cavity modes to reentrant modes in order to achieve high frequency tuning ranges. We find that then nth order reentrant mode arises from the transition of a TM$_{01n}$ mode in an empty cylindrical cavity.

Figs. 4 and 5 show the transformation process of the 1st and 2nd higher order modes. As we can see in the figures, because TM$_{011}$ and TM$_{012}$ exhibit periodic change in $\phi$ direction magnetic field, the transformation process is complex. As the post moves out of the cavity the top “magnetic ring” will transform into the upper lobe of the TM$_{01n}$ empty cylinder mode. As the post continues to move the other rings do the same. The higher order modes undergoes multiple transition phases, from coaxial to reentrant, and from reentrant to empty cylinder via a ‘pseudo-TM’ phase where some of the “magnetic rings” have become empty cavity lobes. Each mode transitions slightly differently, due to the number of “magnetic rings” and lobes in the empty cavity mode. These transition phases affect frequency tuning.

We have calculated the resonant mode properties as the gap size changes from 0.1mm to ~40 cm. Fig 6b shows the eigenfrequencies of different reentrant modes. Table I lists tuning ranges of the standard reentrant mode.
and some high order modes first phases. As mentioned, higher order modes go through multiple phases, with the first phase, the purely reentrant phase, exhibiting the largest tuning range. After this point the modes become quite crowded in frequency space, as they are nearly at the frequencies of the corresponding empty cavity TM modes. By monitoring different modes, it is possible tune through different frequency ranges at the same time, which would be beneficial in many fundamental physics tests such as searches for dark matter and gravitational waves.

We have also characterized these modes in terms of their geometry factor, which is directly proportional to mode quality factor

$$G = \frac{\omega \mu_0 \int |\vec{H}|^2 \, dV_c}{\int |\vec{H}|^2 \, dS_c}$$

$$Q_{cav} = \frac{G}{R_s}. \quad (2)$$

Where $R_s$ is the surface resistivity of the metal. Fig. 6b shows the geometry factors as a function of gap size for various reentrant modes. It is obvious that the geometric factor of higher order modes is much higher than that of the standard mode. High-Q modes are highly sought after in fundamental physics tests, which makes these modes promising candidates. One potential application of this class of modes is as a type of receiver known as a haloscope, to search for axion dark matter. In halocope searches, one of the most critical parameters is the axion electromagnetic form factor, which defines the overlap between a cavity mode electromagnetic field, and the electromagnetic field generated by the axion in the presence of an external magnetic field. This factor is entirely mode dependent, and thus a good measure of the suitability of a given mode for axion haloscopes. Previously, electromagnetic form factors and axion sensitivity of the standard, lowest order reentrant mode have been computed, fig. 7 shows the electromagnetic form factors for the first few higher order reentrant modes as a function of gap size. Furthermore, it can be shown that $C^2V^2G$, where $C$ is the form factor, $V$ is the volume of the resonator and $G$ is the geometry factor, should be employed as a figure of merit in axion haloscope design. $C^2V^2G$ is explicitly, generally defined as

$$C = \frac{\int dV_c E_c \cdot \vec{z}^2}{2 V \int dV_c \epsilon_r |B_c|} + \frac{\omega_0^2}{2 V \int dV_c \mu_r |B_c|}$$ \quad (3)

For arbitrary dielectric and magnetic materials. We present $C^2V^2G$ products as a function of gap size in figure 8. This is interesting, as we can see that the most axion sensitive mode changes as the gap size changes. This could be of interest in axion haloscope searches, as it would open up the possibility of scanning multiple frequency ranges in a single post sweep. It would be possible to begin the search in one range, tracking the lowest order mode, and then move to higher order modes as they become more sensitive, creating a wider feasible scanning range within the same cavity.

III. EXPERIMENTAL DATA

The properties of the resonant modes were measured with a vector network analyzer to acquire the complex value of $S21$ in transmission. Two loop probes were used for the measurement. One probe was located near the base of the central post through a 1-mm-diameter hole in the lid of the cavity, whilst the other was inserted through a hole on the side wall. Both of these probes were used to couple to the $B_\phi$ field component. Fig. 9 is
FIG. 9: A cross-section of the cavity used for the experiment. The critical components are labelled, whilst the small gap region between the post and the lid referred to in the text is highlighted by the small red arrows. This is the gap which we suggest is responsible for much of the unaccounted for losses. Dimensions are exaggerated for clarity.

FIG. 10: Experimental and theoretical frequency data of the standard and first higher order reentrant mode. The average error is below 3%. The modes are represented as detailed in the legend, where the "0th" order mode corresponds to the fundamental TM\textsubscript{010}-like mode.

the device used for the measurement, which consists of an empty cylindrical cavity, a conducting central post, and a micrometer for the movement of the post. The setup was used to track the frequency of the standard and first higher order mode, and the results are displayed in fig. 10. On average, the experimental mode frequencies differed from the theory by less than 3%.

Mode quality factor was also measured via insertion of two weakly coupled magnetic loop probes into the cavity. The measured and predicted unloaded quality factors for each mode are shown in fig. 11. The simulated values assume a conductivity of $2 \times 10^6$ S/m, which is within the range of values for brass alloys. Brass conductivities are typically on the order of $10^6 - 10^7$ S/m, with variation accounted for by different compositions and manufacturing processes. Initial Q values were significantly lower, this was partially mitigated by polishing the interior surfaces, which resulted in a factor of 2 increase in Q. Further increases in Q were attained by electrically connecting the tuning post to the cavity outer walls. Despite this, there are still losses from a number of factors, including the small gap between the post and the cavity lid, the lack of an RF choke and several other loss mechanisms.

We may better model the Q of a real resonator as

\[
\frac{1}{Q_0} = \frac{1}{Q_{\text{cav}}} + \frac{1}{Q_{\text{other}}} 
\]

Where $Q_0$ is the measured unloaded quality factor, $Q_{\text{cav}}$ is the predicted quality factor from geometry factors as per eq. 2 and $Q_{\text{other}}$ is a measure of the other loss mechanisms. We can measure $Q_{\text{other}}$ by comparing our computed $Q_{\text{cav}}$ with the measured unloaded quality factors. Data for $Q_{\text{other}}$ is also presented in fig. 11.

It is important to note that there are many possible combinations of $Q_{\text{cav}}$ and $Q_{\text{other}}$ which could lead to the same value for measured $Q_0$. We present $Q_{\text{other}}$ values based on the assumptions outlined above regarding surface resistance of the walls of the resonator. It is indeed possible that the surface resistance is different from that presented and that our findings are in closer agreement, but the pattern of the measurements indicates some separate loss mechanism that changes as a function of frequency. We observe good agreement between theory and experiment for small and large gap sizes, however in the case of both modes there is large disagreement for medium gap sizes. We attribute this to some spurious resonant effect, potentially as a result of a highly localized resonance in the small gap between the tuning post and the cavity lid. The high level of agreement between the frequencies found from finite element modelling and the frequencies found experimentally leads us to conclude that we are tracking the correct modes, and that some spurious loss mechanisms, such as gaps around probe holes, and the gap shown in fig. 9 are responsible.

IV. SUMMARY AND CONCLUSION

In summary, we have demonstrated the existence of higher order reentrant modes with higher frequencies
than the fundamental in the single-post cylindrical reentrant cavity. We have verified our finite-element modelling by comparison with experimental data, and the agreement is good. We find that the higher order reentrant modes arise from perturbed $TM_{01n}$ modes in empty cylinders, and finally convert to coaxial modes when the gap size becomes zero. The higher order reentrant modes have higher geometry factors when compared to the standard reentrant mode. We have found and characterized the different phases that exist in the transformation process. These modes have possible applications in a number of fundamental physics tests, such as gravity wave and dark matter detection experiments. Application of these modes to an axion halo search is discussed, with sensitivity figures of merit presented. Furthermore, it is worth considering that an understanding of these reentrant modes affords us a deeper understanding of the highly localized “gap modes” present in many experiments. For example, it has been discussed that axion halo experiments utilizing tuning rods to adjust mode frequency can have their sensitivity degraded by the presence of highly localized modes which arise due to the small gap between the end of the tuning rod and the walls of the cavity, which must exist for real tuning rods.\cite{29}

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