HUISKEN-YAU-TYPE UNIQUENESS FOR AREA-CONSTRAINED WILLMORE SPHERES

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Abstract. Let \((M, g)\) be a Riemannian three-manifold that is asymptotic to Schwarzschild. We study the existence of large area-constrained Willmore spheres \(\Sigma \subset M\) with non-negative Hawking mass and inner radius \(\rho\) dominated by the area radius \(\lambda\). If the scalar curvature of \((M, g)\) is non-negative, we show that no such surfaces with \(\log \lambda \ll \rho\) exist. This answers a question of G. Huisken.

1. Introduction

Let \((M, g)\) be a connected, complete Riemannian three-manifold. Let \(\Sigma \subset M\) be a closed, two-sided surface with area element \(d\mu\), outward normal \(\nu\), and mean curvature \(H\) with respect to \(\nu\). The Hawking mass of \(\Sigma\) is

\[ m_H(\Sigma) = \sqrt{\frac{\Sigma}{16 \pi}} \left( 1 - \frac{1}{16 \pi} \int_{\Sigma} H^2 d\mu \right). \]

In the case where \((M, g)\) arises as maximal initial data for the Einstein field equations, the Hawking mass has been proposed as a quasi-local measure for the strength of the gravitational field; see [13].

Recall that time-symmetric initial data for a Schwarzschild black hole with mass \(m > 0\) are given by

\[ \left\{ x \in \mathbb{R}^3 : |x| > \frac{m}{2} \right\}, \left( 1 + \frac{m}{2|x|} \right)^4 \bar{g} \]

where

\[ \bar{g} = \sum_{i=1}^{3} dx^i \otimes dx^i \]

is the Euclidean metric on \(\mathbb{R}^3\) and \(|x|\) denotes the Euclidean length of \(x \in \mathbb{R}^3\). A special class of general initial data consists of those asymptotic to Schwarzschild. Given a non-negative integer \(k\), we say that \((M, g)\) is \(C^k\)-asymptotic to Schwarzschild with mass \(m > 0\) if there is a non-empty compact set \(K \subset M\) such that the end \(M \setminus K\) is diffeomorphic to \(\{ x \in \mathbb{R}^3 : |x| > 1 \}\) and, in this special chart, there holds, as \(x \to \infty\),

\[ g = \left( 1 + \frac{m}{2|x|} \right)^4 \bar{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O(|x|^{-2-|J|}) \]

for every multi-index \(J\) with \(|J| \leq k\). Given \(r > 1\), we define \(B_r \subset M\) to be the compact domain whose boundary corresponds to \(S_r(0)\) in this chart. Moreover, given a closed, two-sided surface
$\Sigma \subset M$, we define the area-radius $\lambda(\Sigma) > 0$ and inner radius $\rho(\Sigma)$ of $\Sigma$ by
\begin{equation}
4 \pi \lambda(\Sigma)^2 = |\Sigma| \quad \text{and} \quad \rho(\Sigma) = \sup\{r > 1 : B_r \cap \Sigma = \emptyset\}.
\end{equation}

The Hawking mass $m_H(\Sigma)$ provides useful information on the strength of the gravitational field provided the surface $\Sigma$ is either a stable constant mean curvature sphere with large enclosed volume or, alternatively, an area-constrained Willmore sphere with large area; see also [1, p. 2348]. Stable constant mean curvature surfaces are stable critical points of the area functional under a volume constraint and therefore candidates to have least perimeter among surfaces of the same enclosed volume. S.-T. Yau and D. Christodoulou [6] have observed that the Hawking mass of stable constant mean curvature spheres is non-negative if $(M, g)$ has non-negative scalar curvature. Note that, in this context, the scalar curvature provides a lower bound for the energy density of the initial data set. Stable constant mean curvature spheres have since been studied extensively in the context of mathematical relativity; see for example the recent overview given in [10].

Recall that $\Sigma \subset M$ is an area-constrained Willmore surface if there is a number $\kappa \in \mathbb{R}$ such that
\begin{equation}
\Delta H + (|\dot{h}|^2 + \text{Ric}(\nu, \nu) + \kappa) H = 0.
\end{equation}
Here, $\Delta$ is the non-positive Laplace-Beltrami operator on $\Sigma$ with respect to the induced metric, $\dot{h}$ the traceless part of the second fundamental form $h$, and $\text{Ric}$ the Ricci curvature of $(M, g)$. Note that (3) is the Euler-Lagrange equation of the Willmore energy
\begin{equation}
\int_{\Sigma} H^2 \, d\mu
\end{equation}
with respect to an area constraint and $\kappa$ the corresponding Lagrange parameter. Area-constrained Willmore surfaces are therefore candidates to have largest Hawking mass among all closed surfaces of the same area. As observed in [11, p. 487], large area-constrained Willmore spheres capture information on the asymptotic distribution of scalar curvature that large stable constant mean curvature spheres are impervious to.

**Existence and uniqueness of large area-constrained Willmore surfaces.** In the recent papers [9,11], the first-named author and the second-named author have studied the existence, uniqueness, and physical properties of large area-constrained Willmore spheres. We recall the following result; see Figure 1.

**Theorem 1** ([11, Theorem 6]). Let $(M, g)$ be $C^4$-asymptotic to Schwarzschild with mass $m > 0$ and suppose that
\begin{equation}
\sum_{i=1}^{3} x^i \partial_i(|x|^2 R) \leq 0
\end{equation}
outside a compact set. There exists $\kappa_0 > 0$ and a family
\begin{equation}
\{\Sigma(\kappa) : \kappa \in (0, \kappa_0)\}
\end{equation}
of spheres $\Sigma(\kappa) \subset M$ where $\Sigma(\kappa)$ satisfies (3) with parameter $\kappa$. The family (6) sweeps out the complement of a compact set in $M$ and there holds $m_H(\Sigma(\kappa)) \geq 0$ for each $\kappa \in (0, \kappa_0)$.

Moreover, given $\delta > 0$, there exists $\lambda > 1$ and a compact set $K \subset M$ with the following property.

If $\Sigma \subset M \setminus K$ is an area-constrained Willmore sphere with $m_H(\Sigma) \geq 0$ and $|\Sigma| > 4\pi \lambda^2$, then either $\Sigma = \Sigma(\kappa)$ for some $\kappa \in (0, \kappa_0)$ or $\rho(\Sigma) < \delta \lambda(\Sigma)$.

Remark 2.

i) Note that (5) implies that $R \geq 0$ at infinity.

ii) T. Lamm, the third-named author, and the fourth-named author have previously proved the existence of an asymptotic foliation by large-area constrained Willmore spheres if $(M, g)$ is a so-called small perturbation of Schwarzschild; see [20, Theorem 1 and Theorem 2].

iii) In $\mathbb{R}^3$, round spheres are Willmore surfaces and the only closed surfaces with non-negative Hawking mass; see [24, (3)].

iv) S. Brendle [3] has shown that the spheres of symmetry are the only closed, embedded constant mean curvature surfaces in spatial Schwarzschild (1). It is not known if these are also the only area-constrained Willmore spheres; see also [17, Remark 1.5 and Theorem 1.6].

As discussed for example in the introduction of [9], the assumptions that the surfaces in consideration have non-negative Hawking mass, be large, and be disjoint from a certain bounded set appear to be essential for a characterization result such as Theorem 1 to hold. By contrast, as we explain below, we conjecture that the alternative $\rho(\Sigma) < \delta \lambda(\Sigma)$ in the conclusion of Theorem 1 does not actually arise. In fact, in this paper, we improve the uniqueness result in Theorem 1 by ruling out the existence of certain large area-constrained Willmore spheres whose respective inner radius is small compared to their area radius.

Theorem 3. Let $(M, g)$ be $C^4$-asymptotic to Schwarzschild and suppose that, as $x \to \infty$

\begin{equation}
R \geq -o(|x|^{-4}).
\end{equation}

There are $\delta > 0$ and $\lambda > 1$ with the following property.

There is no area-constrained Willmore sphere $\Sigma \subset M$ with

- $m_H(\Sigma) \geq 0$,
- $|\Sigma| > 4\pi \lambda^2$. 

Remark 4.

i) The conclusion of Theorem 3 can fail if the assumption (7) is dropped; see [11, Theorem 11].

ii) Note that, for every $s > 1$, $\log \lambda(\Sigma) < \lambda(\Sigma)^{1/s}$ provided that $\lambda(\Sigma)$ is sufficiently large. Analytically, the result established by G. Huisken and S.-T. Yau in [16, Theorem 5.1] on the uniqueness of large stable constant mean curvature spheres corresponds to the case where $1 < s < 2$.

iii) The assumption $\log \lambda(\Sigma) < \delta \rho(\Sigma)$ is essential to obtain (57) from (51) and seems to be optimal for the method employed in this paper. Specifically, note that (51) is the best possible estimate based on Lemma 8.

iv) The assumptions of Theorem 3 imply that $\Sigma \cap B_2 = \emptyset$. Note that, unlike in [16] and [22], we do not assume that $\Sigma$ encloses $B_2$.

Combining Theorem 3 with Theorem 1, we obtain the following corollary.

Corollary 5. Let $(M, g)$ be $C^4$-asymptotic to Schwarzschild. Suppose that

$$\sum_{i=1}^{3} x^i \partial_i (|x|^2 R) \leq 0$$

outside a compact set. Let $\{\Sigma_i\}_{i=1}^{\infty}$ be a sequence of area-constrained Willmore spheres $\Sigma_i \subset M$ with

- $m_H(\Sigma) \geq 0$,
- $\lim_{i \to \infty} \rho(\Sigma_i) = \infty$,
- $\lim_{i \to \infty} \lambda(\Sigma_i) = \infty$,
- $\Sigma_i$ is not part of the foliation (6).

There holds $\rho(\Sigma_i) = O(\log \lambda(\Sigma_i))$.

For Riemannian three-manifolds asymptotic to Schwarzschild and satisfying (5), Corollary 5 provides evidence that all area-constrained Willmore spheres with large inner radius, large area radius, and non-negative Hawking mass belong to the family (6). In addition, it stands to reason that, for each $\kappa \in (0, \kappa_0)$, the Hawking mass of $\Sigma(\kappa)$ is maximal among all spheres $\Sigma \subset M$ with $|\Sigma| = |\Sigma(\kappa)|$ provided that $\rho(\Sigma)$ is sufficiently large. We note that both of these conjectures are open, even in the case where $(M, g)$ is the spatial Schwarzschild manifold (1). By contrast, it is known that, for such Riemannian three-manifolds, stable constant mean curvature spheres with large area are the unique solutions of the isoperimetric problem for the volume they enclose; see the work of O. Chodosh and the first-named author [4] and of O. Chodosh, Y. Shi, H. Yu, and the first-named author [5].

Outline of related results. We say that a sequence $\{\Sigma_i\}_{i=1}^{\infty}$ of spheres $\Sigma_i \subset M$ with

$$\lim_{i \to \infty} \rho(\Sigma_i) = \infty \quad \text{and} \quad \rho(\Sigma_i) = o(\lambda(\Sigma_i))$$

(8)

is slowly divergent. As with large stable constant mean curvature spheres, a substantial obstacle towards establishing the uniqueness of large area-constrained Willmore spheres with non-negative
Figure 2. An illustration of a slowly divergent sequence \( \{ \Sigma_i \}_{i=1}^{\infty} \) of area-constrained Willmore spheres \( \Sigma_i \subset M \) on the scale of the area radius \( \lambda(\Sigma_i) \). The cross marks the origin in the asymptotically flat chart. Away from the origin, the surfaces \( \Sigma_i \) converge uniformly to a round sphere.

Hawking mass in Riemannian three-manifolds asymptotic to Schwarzschild is to rule out the possibility of a slowly divergent sequence \( \{ \Sigma_i \}_{i=1}^{\infty} \) of area-constrained Willmore spheres \( \Sigma_i \subset M \) with non-negative Hawking mass. The main difficulty in understanding the geometry of the spheres \( \Sigma_i \) owes to the fact that unrefined curvature estimates generally do not yield global analytic control. In fact, as \( i \to \infty \), there holds

\[
h(\Sigma_i) = O(\lambda(\Sigma_i)^{-1}) + O((\lambda(\Sigma_i)^{-1/2} + \rho(\Sigma_i)^{-1}) |x|^{-1});
\]

see Proposition 10. If for example \( \rho(\Sigma_i) = o(\lambda(\Sigma_i)^{1/2}) \), estimate (9) fails to bound the sequence \( \{ \lambda(\Sigma_i)^{-1} \Sigma_i \}_{i=1}^{\infty} \) in \( C^2 \). If for example \( \rho(\Sigma_i) = o(\log \lambda(\Sigma_i)) \), (9) even fails to bound the sequence \( \{ \lambda(\Sigma_i)^{-1} \Sigma_i \}_{i=1}^{\infty} \) in \( C^1 \); see Figure 2.

G. Huisken and S.-T. Yau [16] have shown that there are no slowly divergent sequences \( \{ \Sigma_i \}_{i=1}^{\infty} \) of stable constant mean curvature spheres \( \Sigma_i \subset M \) that enclose \( B_2 \) with \( \lambda(\Sigma_i) = O(\rho(\Sigma_i)^s) \) where \( 1 < s < 2 \). In this case, an estimate similar to (9) provides uniform estimates in \( C^2 \). These estimates are sufficient to conclude their argument based on analyzing a certain flux integral related to the variation of the area functional with respect to a translation. Following the same strategy, J. Qing and G. Tian [22] have shown that the assumption \( \lambda(\Sigma_i) = O(\rho(\Sigma_i)^s) \), \( 1 < s < 2 \), can be dropped.

To overcome the potential loss of \( C^1 \)-control, they carry out an asymptotic analysis based on the observation that the Gauss maps \( \{ \nu(\Sigma_i) \}_{i=1}^{\infty} \) form a sequence of almost harmonic maps. Finally, O. Chodosh and the first-named author [4] have shown that the assumption that \( \Sigma_i \) encloses \( B_2 \) can be dropped if the scalar curvature of \( (M, g) \) is non-negative. Their method is based on an analysis of the Hawking mass of \( \Sigma_i \). To obtain the required analytic control, they combine the Christodoulou-Yau estimate [6, p. 13]

\[
\frac{2}{3} \int_{\Sigma} (|\hat{h}|^2 + R) \, d\mu \leq 16 \pi - \int_{\Sigma} H^2 \, d\mu,
\]

valid for every stable constant mean curvature sphere \( \Sigma \subset M \), with global methods developed by G. Huisken and T. Ilmanen [15]. We also refer to the papers of L.-H. Huang [14], of S. Ma [21], and of the first-named author and second-named author [10] on slowly divergent sequences of large stable constant mean curvature spheres in general asymptotically flat Riemannian three-manifolds.
When studying slowly divergent sequences (8) of area-constrained Willmore spheres with non-negative Hawking mass, additional difficulties arise. On the one hand, the absence of an estimate comparable to (10) renders the curvature estimates for large area-constrained Willmore spheres less powerful than those for large stable constant mean curvature spheres. Moreover, the fourth-order nature of the area-constrained Willmore equation (3) poses additional analytical challenges. On the other hand, the variation of the Willmore energy (4) with respect to a translation is of a smaller scale than that of the area functional, at least when \( \Sigma \) encloses \( B_2 \). Consequently, more precise analytic control is needed. In fact, we are not aware of any previous positive results on the non-existence of slowly divergent sequences of area-constrained Willmore spheres with non-negative Hawking mass. By contrast, in [11, Theorem 11], the first-named author and the second-named author have shown that such sequences may exist if the scalar curvature of \( (M, g) \) is allowed to change sign.

**Outline of the proof of Theorem 3.** By scaling, we may assume that \( m = 2 \), that is,

\[
g = (1 + |x|^{-1})^4 \bar{g} + \sigma.
\]

We use a bar to indicate that a geometric quantity has been computed with respect to the Euclidean background metric \( \bar{g} \). Likewise, we use a tilde to indicate that the Schwarzschild metric

\[
\tilde{g} = (1 + |x|^{-1})^4 \bar{g}
\]

with mass \( m = 2 \) has been used in the computation.

Assume that \( \{ \Sigma_i \}_{i=1}^{\infty} \) is a sequence of area-constrained Willmore spheres \( \Sigma_i \subset M \) with \( m_H(\Sigma_i) \geq 0 \) and

\[
\lim_{i \to \infty} \rho(\Sigma_i) = \infty, \quad \rho(\Sigma_i) = o(\lambda(\Sigma_i)), \quad \log \lambda(\Sigma_i) = o(\rho(\Sigma_i)).
\]

Let \( a \in \mathbb{R}^3 \). To prove Theorem 3, we expand the variation of the Willmore energy of \( \Sigma_i \) with respect to a translation in direction \( a \) given by

\[
0 = - \int_{\Sigma_i} g(a, \nu) \left[ \Delta H + (\|\mathring{h}\|^2 + \text{Ric}(\nu, \nu) + \kappa) H \right] d\mu.
\]

Contrary to the variation of the area functional with respect to a translation, we expect the right-hand side of (12) to be small independently of whether \( \Sigma_i \) encloses the origin or not; see (62). Consequently, precise analytic control is needed to expand the terms on the right-hand side of (12) with sufficient control on the error.

To this end, we first prove preliminary pointwise curvature estimates for large area-constrained Willmore spheres with non-negative Hawking mass; see Proposition 10. These estimates are based on an adaptation of the localized integral curvature estimates proved by E. Kuwert and R. Schätzle [19] for Willmore surfaces in \( \mathbb{R}^3 \) to the setting of large area-constrained Willmore spheres in Riemannian three-manifolds asymptotic to Schwarzschild, see Appendix A, and an \( L^2 \)-curvature estimate that builds on an argument given in [4, §4]. Combining Proposition 10 with (11), it follows that \( \Sigma_i \) is the radial graph of a function \( u_i \) over a large coordinate sphere \( S_i = S_{\lambda_i}(\lambda_i \xi_i) \) where \( \lambda_i > 1 \) and \( \xi_i \in \mathbb{R}^3 \); see Lemma 16. The resulting estimates in Lemma 19 below are still too coarse to expand (12). To overcome this, we first prove explicit estimates for the Laplace operator of a round sphere based on Green’s function methods; see Lemma 20. Second, we observe that the quotient of \( H(\Sigma_i) \)
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and the potential function of the spatial Schwarzschild manifold satisfies an equation slightly more useful than the area-constrained Willmore equation (76); see Lemma 77. We then use the explicit estimates for the Laplace operator to investigate this equation to obtain the sharp estimate

\[(13) \quad H(\Sigma_i) = (2 + o(1)) \lambda(\Sigma_i)^{-1} - 4 \lambda(\Sigma_i)^{-1} |x|^{-1} + o(\lambda(\Sigma_i)^{-1}\rho(\Sigma_i)^{-1}); \]

see Lemma 22. We note that this procedure requires assumption (11) in an essential way; see Remark 4.

With the estimate (13) at hand, we obtain that

\[0 = -\int_{\Sigma_i} g(\xi_i, \nu) \left[ \Delta H + (|\hat{h}|^2 + \text{Ric}(\nu, \nu) + \kappa) H \right] d\mu \]

\[= 8\pi \lambda(\Sigma_i)^{-1} \rho(\Sigma_i)^{-2} - \lambda(\Sigma_i)^{-1} \int_{\partial \Sigma_i} \bar{g}(\xi_i, \bar{\nu}) R d\bar{\mu} - o(\lambda(\Sigma_i)^{-1}\rho(\Sigma_i)^{-2}); \]

see (63). Using that \(R \geq -o(|x|^{-4})\), it follows that

\[0 \geq 8\pi \lambda(\Sigma_i)^{-1} \rho(\Sigma_i)^{-2} - o(\lambda(\Sigma_i)^{-1}\rho(\Sigma_i)^{-2}). \]

This is a contradiction for sufficiently large \(i\).

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2. Curvature estimates for large area-constrained Willmore spheres

We assume that \(g\) is a Riemannian metric on \(\mathbb{R}^3\) such that, as \(x \to \infty\),

\[g = (1 + |x|^{-1})^4 \bar{g} + \sigma \quad \text{where} \quad \partial_J \sigma + O(|x|^{-2-|J|}) \]

for every multi-index \(J\) with \(|J| \leq 4\).

Let \(\{\Sigma_i\}_{i=1}^{\infty}\) be a sequence of area-constrained Willmore spheres \(\Sigma_i \subset \mathbb{R}^3\) with

\[(14) \quad \int_{\Sigma_i} H^2 \, d\mu \leq 16\pi, \quad \lim_{i \to \infty} \rho(\Sigma_i) = \infty, \quad \rho(\Sigma_i) = O(\lambda(\Sigma_i)). \]

The goal of this section is to prove curvature estimates for \(\Sigma_i\). To this end, we combine the integral curvature estimates from Appendix A with the integral estimate on the second fundamental form proven in Lemma 8.

We abbreviate \(\rho_i = \rho(\Sigma_i)\) and \(\lambda_i = \lambda(\Sigma_i)\).
Lemma 6. As $i \to \infty$, there holds:

- $\int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu} = 16 \pi + O(\rho_i^{-1})$
- $\int_{\Sigma_i} |h|^2 \, d\mu = O(1)$
- $\int_{\Sigma_i} |\bar{h}|^2 \, d\bar{\mu} = O(1)$

Proof. Clearly,

$$\int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu} \geq 16 \pi.$$ 

Integrating the Gauss equation and using the Gauss-Bonnet theorem, we have

$$\int_{\Sigma_i} H^2 \, d\mu = 8 \pi + \int_{\Sigma_i} |h|^2 \, d\mu + 4 \int_{\Sigma_i} \left( \text{Rc}(\nu, \nu) - \frac{1}{2} R \right) \, d\mu.$$ 

Using Lemma 35 and (14), we obtain

$$\int_{\Sigma_i} |h|^2 \, d\mu \leq 8 \pi + O(\rho_i^{-1}) \int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu}.$$ 

By Lemma 42,

$$\int_{\Sigma_i} H^2 \, d\mu = \int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu} + O(|x|^{-1} |h|^2) \, d\mu + O(|x|^{-3}) \, d\bar{\mu}.$$ 

Using Lemma 35 and (15), we obtain

$$\int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu} \leq 16 \pi + O(\rho_i^{-1}) + O(\rho_i^{-1}) \int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu}.$$ 

The assertion follows from these estimates. \(\square\)

Lemma 7. There holds

$$\int_{\Sigma_i} (\bar{H} - 2 \lambda_i^{-1})^2 \, d\bar{\mu} = O(1) \int_{\Sigma_i} |\bar{h}|^2 \, d\bar{\mu} + O(\rho_i^{-2}).$$

Proof. By [8, (38)],

$$\int_{\Sigma_i} (\bar{H} - 2 \lambda(S_i)^{-1})^2 \, d\bar{\mu} = O(1) \int_{\Sigma_i} |\bar{h}|^2 \, d\bar{\mu}.$$ 

From Lemma 42 we obtain $\lambda_i = (1 + O(\rho_i^{-1})) \lambda(S_i)$ from which the assertion follows. \(\square\)

Lemma 8. There holds

$$\int_{\Sigma_i} |h - \lambda_i^{-1} g|_{\Sigma_i}^2 \, d\mu = O((\lambda_i^{-1/2} + \rho_i^{-1})^2).$$

Proof. Using Lemma 42, Lemma 35, and Lemma 6, we have

$$\int_{\Sigma_i} H^2 \, d\mu = \int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu} + O(\rho_i^{-2})$$

and

$$\int_{\Sigma_i} \bar{H}^2 \, d\bar{\mu} = \int_{\Sigma_i} \left[ \bar{H}^2 - 8 (1 + |x|^{-1})^{-1} |x|^{-3} \bar{g}(x, \nu) \bar{H} \right] \, d\bar{\mu} + O(\rho_i^{-2}).$$
By the Gauss-Bonnet theorem,

$$\int_{\Sigma_i} \tilde{H}^2 \, d\tilde{\mu} = 16 \pi + 2 \int_{\Sigma} |\hat{h}|_g^2 \, d\tilde{\mu}. $$

Moreover, by Lemma 35 and Lemma 7, we have

$$\left| \int_{\Sigma_i} (1 + |x|^{-1})^{-1} |x|^{-3} \tilde{g}(x, \tilde{\nu}) \tilde{H} \, d\tilde{\mu} \right| = \left| \int_{\Sigma_i} |x|^{-3} \tilde{g}(x, \tilde{\nu}) \tilde{H} \, d\tilde{\mu} \right| + O(\rho_i^{-2})$$

$$\leq 2 \lambda_i^{-1} \left| \int_{\Sigma_i} |x|^{-3} \tilde{g}(x, \tilde{\nu}) \, d\tilde{\mu} \right| + \frac{1}{8} \int_{\Sigma_i} |\hat{h}|_g^2 \, d\tilde{\mu} + O(\rho_i^{-2}).$$

Note that \(\bar{\text{div}}(|x|^{-3}x) = 0\). Using the divergence theorem, we find that

$$\int_{\Sigma_i} |x|^{-3} \tilde{g}(x, \tilde{\nu}) \, d\tilde{\mu} = 4 \pi$$

if \(\Sigma_i\) encloses \(B_2\) and

$$\int_{\Sigma_i} |x|^{-3} \tilde{g}(x, \tilde{\nu}) \, d\tilde{\mu} = 0$$

otherwise. In conjunction with (14), these estimates imply that

$$\int_{\Sigma_i} |\hat{h}|_g^2 \, d\tilde{\mu} = O(\lambda_i^{-1}) + O(\rho_i^{-2}).$$

Using this, Lemma 42, Lemma 35, and Lemma 6, we conclude that

$$\int_{\Sigma_i} |\hat{h}|^2 \, d\mu = \int_{\Sigma_i} |\hat{h}|_g^2 \, d\tilde{\mu} + O(\rho_i^{-1}) \int_{\Sigma_i} |\hat{h}|_g^2 \, d\tilde{\mu} + O(\rho_i^{-2}) = O(\lambda_i^{-1}) + O(\rho_i^{-2}).$$

Likewise, by Lemma 42, Lemma 7, Lemma 6, and Lemma 35, we have

$$\int_{\Sigma_i} (\hat{H}^2 - 2 \lambda_i^{-1})^2 \, d\mu \leq \int_{\Sigma_i} (\hat{H}^2 - 2 \lambda_i^{-1})^2 \, d\tilde{\mu} + O(\rho_i^{-2})$$

$$\leq O(1) \int_{\Sigma_i} |\hat{h}|^2 \, d\tilde{\mu} + O(\rho_i^{-2})$$

$$\leq O(\lambda_i^{-1}) + O(\rho_i^{-2}).$$

Using that

$$\int_{\Sigma_i} |h - \lambda_i^{-1} g|_{\Sigma_i}^2 \, d\mu = \int_{\Sigma_i} (\hat{H}^2 - 2 \lambda_i^{-1})^2 \, d\mu + \int_{\Sigma_i} |\hat{h}|^2 \, d\mu,$$

the assertion follows.

\[\square\]

**Lemma 9.** There holds \(\sup_{x \in \Sigma_i} |x| = O(\lambda_i)\).

**Proof.** The assertion follows from (73), (14), and Lemma 6. \[\square\]

For the proof of Proposition 10, note that, by Lemma 8, the sequence \(\{\Sigma_i\}_{i=1}^{\infty}\) satisfies assumption (64) of Appendix A.

**Proposition 10.** As \(i \to \infty\), there holds

\begin{equation}
\kappa(\Sigma_i) = O((\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-2})
\end{equation}
and

\begin{equation}
 h - \lambda_i^{-1} g|_{\Sigma_i} = O((\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-1}).
\end{equation}

In particular, $|\hat{h}|^2 = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 |x|^{-2})$.

**Proof.** We choose $\psi \in C^\infty(\mathbb{R})$ with
\begin{itemize}
  \item $0 \leq \psi \leq 1$,
  \item $\psi(s) = 1$ if $s \geq 1/4$,
  \item $\psi(s) = 0$ if $s \leq 1/8$.
\end{itemize}

We define $\gamma_i \in C^\infty(\mathbb{R}^3)$ by

\[
 \gamma_i(x) = \psi(\lambda_i^{-1} |x|).
\]

Using Lemma 8, we have

\[
 \int_{\Sigma_i} \gamma_i H \, d\mu = 2 \lambda_i^{-1} \int_{\Sigma_i} \gamma_i \, d\mu + o(\lambda_i).
\]

If $\rho_i \geq 1/4 \lambda_i$, then $\gamma_i(x) = 1$ for all $x \in \Sigma_i$. Consequently,

\[
 \int_{\Sigma_i} \gamma_i \, d\mu = 4 \pi \lambda_i^2.
\]

If $\rho_i \leq 1/4 \lambda_i$, then we choose $x_i \in \Sigma_i$ with $|x_i| = \rho_i$ and apply (72) with $r = \lambda_i/4 + \rho_i$. Using Lemma 6 and Lemma 42, we obtain

\[
 \int_{\Sigma_i} \gamma_i \, d\mu \geq 4 \pi \lambda_i^2 - (16 \pi + o(1)) \frac{3 + 2 \sqrt{2}}{16} (\lambda_i/4 + \rho_i)^2 \geq \frac{9 \pi}{4} \lambda_i^2
\]

for all $i$ sufficiently large.

Either way, it follows that

\begin{equation}
 4 \pi \lambda_i \leq \int_{\Sigma_i} \gamma_i H \, d\mu \leq \int_{\Sigma_i} |H| \, d\mu \leq 16 \pi \lambda_i
\end{equation}

for all sufficiently large $i$. Using (3), we have

\[
 -\kappa(\Sigma_i) \int_{\Sigma_i} \gamma_i H \, d\mu = \int_{\Sigma_i} (\Delta \gamma_i) H \, d\mu + \int_{\Sigma_i} \gamma_i H |\hat{h}|^2 \, d\mu + \int_{\Sigma_i} \gamma_i \, d\mu.
\]

Using (18), we have

\[
 |\kappa(\Sigma_i)| \int_{\Sigma_i} \gamma_i H \, d\mu \geq 4 \pi \lambda_i |\kappa(\Sigma_i)|.
\]

Note that

\[
 \nabla^2 \gamma_i = O(\lambda_i^{-1} |h|) + O(\lambda_i^{-2}).
\]

In conjunction with Lemma 8 and Lemma 6, we obtain that

\[
 \int_{\Sigma_i} (\Delta \gamma_i) H \, d\mu = \int_{\Sigma_i} (\Delta \gamma_i) (H - 2 \lambda_i^{-1}) \, d\mu = O((\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-1}).
\]

By Proposition 34 and Lemma 8, we have

\[
 |\hat{h}|^2 = O((\lambda_i^{-1/2} + \rho_i^{-1} \lambda_i^{-2}) + o(\kappa(\Sigma_i))
\]
on $\Sigma_i \cap \text{spt}(\gamma_i)$. In conjunction with (18), we obtain
\[
\int_{\Sigma_i} \gamma_i H |\hat{h}|^2 \, d\mu = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 \lambda_i^{-1}) + o(\kappa(\Sigma_i) \lambda_i).
\]

Likewise, (18) gives
\[
\int_{\Sigma_i} \gamma_i \text{Rc}(\nu, \nu) H \, d\mu = O(\lambda_i^{-2}).
\]
(16) follows from these estimates.

Using (16) and Proposition 34, we see that
\[
\int_{\Sigma_i} |\hat{h} - \lambda_i^{-1} g|_{\Sigma_i}^2 \, d\mu = O((\lambda_i^{-1/2} + \rho_i^{-1})^4)
\]
and
\[
\kappa(\Sigma_i)^2 \int_{\Sigma_i} |\hat{h} - \lambda_i^{-1} g|_{\Sigma_i}^2 \, d\mu = O((\lambda_i^{-1/2} + \rho_i^{-1})^4 \lambda_i^{-4}).
\]
Now, (17) follows from Proposition 34 using Lemma 9 and that $|x| \geq \rho_i$ for every $x \in \Sigma_i$. \(\square\)

**Proposition 11.** A subsequence of $\{\lambda_i^{-1} \Sigma_i\}_{i=1}^{\infty}$ converges to a round sphere in $C^2$ locally in $\mathbb{R}^3 \setminus \{0\}$.

**Proof.** Let $\hat{\Sigma}_i = \lambda_i^{-1} \Sigma_i$. By Proposition 10 and Lemma 42, we have
\[
\bar{h}(\hat{\Sigma}_i) - \bar{g}|_{\hat{\Sigma}_i} = o(1)
\]
locally uniformly in $\mathbb{R}^3 \setminus \{0\}$. Let $x_i \in \hat{\Sigma}_i$ with
\[
|x_i| = \sup\{|y| : y \in \hat{\Sigma}_i\}.
\]
By Lemma 9, there is $x \in \mathbb{R}^3$ such that, passing to a subsequence,
\[
\lim_{i \to \infty} x_i = x.
\]
By (72) and Lemma 6, $x \neq 0$. Given $\delta \in (0, 1/2)$, let $\hat{\Sigma}_i^\delta$ be the connected component of $\hat{\Sigma}_i \setminus B_\delta(0)$ containing $x_i$. Using (19), it follows that $\hat{\Sigma}_i^\delta$ converges to $S_1((1-|x|^{-1}) x) \setminus B_\delta(0)$ in $C^2$. In particular,
\[
\int_{\hat{\Sigma}_i^\delta} \bar{H}^2 \, d\bar{\mu} \geq 16 \pi - 4 \pi \delta^2 - o(1).
\]
If $\hat{\Sigma}_i \setminus B_{2\delta}(0)$ has more than one component for infinitely many $i$, we may apply the same argument to the second component to conclude that
\[
\liminf_{i \to \infty} \int_{\hat{\Sigma}_i \setminus B_\delta(0)} \bar{H}^2 \, d\bar{\mu} \geq 32 \pi - 8 \pi \delta^2.
\]
This estimate is incompatible with Lemma 6.

The assertion now follows from taking a suitable diagonal subsequence. \(\square\)

**Proposition 12.** Suppose that $\rho_i = o(\lambda_i)$. A subsequence of $\{\rho_i^{-1} \Sigma_i\}_{i=1}^{\infty}$ converges to a flat plane with unit distance to the origin in $C^2$ locally in $\mathbb{R}^3$ as $i \to \infty$.

**Proof.** Let $\hat{\Sigma}_i = \rho_i^{-1} \Sigma_i$. By Proposition 10 and Lemma 42, we have
\[
\bar{h}(\hat{\Sigma}_i) = o(1)
\]
locally uniformly in $\mathbb{R}^3$. Let $x_i \in \hat{\Sigma}_i$ with $|x_i| = 1$. Given $r > 1$, let $\hat{\Sigma}^r_i$ be the connected component of $\hat{\Sigma}_i \cap B_r(0)$ containing $x_i$. Using (20), it follows that, passing to a subsequence, $\hat{\Sigma}^r_i$ converges to a bounded subset of a flat plane with unit distance to the origin in $C^2$. If $\hat{\Sigma}_i \cap B_r(0)$ has more than one connected component for infinitely many $i$, then, passing to a further subsequence, $\hat{\Sigma}_i \cap B_{2r}(0)$ has a second component that passes through $B_r(0)$ and converges to a bounded subset of a flat plane. In particular,
\[
\liminf_{i \to \infty} r^{-2} |\hat{\Sigma}_i \cap B_{2r}(0)| \bar{g} > 4\pi.
\]
Applying (74) with $x = x_i$ and letting $t \to \infty$, we conclude that
\[
\liminf_{i \to \infty} \int_{\hat{\Sigma}_i} \bar{H}^2 \, d\mu > 16\pi.
\]
This estimate is incompatible with Lemma 6.

The assertion now follows from taking a suitable diagonal subsequence. □

**Proposition 13.** Suppose that $\rho_i = o(\lambda_i)$ and that $\{x_i\}_{i=1}^\infty$ is a sequence of points $x_i \in \Sigma_i$ with $\rho_i = o(|x_i|)$ and $x_i = o(\lambda_i)$. A subsequence of $|x_i|^{-1} \Sigma_i$ converges to a flat plane passing through the origin in $C^2$ locally in $\mathbb{R}^3 \setminus \{0\}$ as $i \to \infty$.

*Proof.* The proof is similar to that of Proposition 12. We omit the formal modifications. □

**Corollary 14.** There holds
\[
|h - \lambda_i^{-1} g|_{\Sigma_i} + |x| |\nabla h| = O((\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-1}).
\]
Likewise,
\[
|\bar{h} - \lambda_i^{-1} \bar{g}|_{\Sigma_i} + |x| |\bar{\nabla} \bar{h}|_{\bar{g}} = O((\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-1}).
\]

*Proof.* Let $x_i \in \Sigma_i$ be such that
\[
|x_i|^2 |(\bar{\nabla} h)(x_i)| = \sup_{x \in \Sigma_i} |x|^2 |\bar{\nabla} h|.
\]
By Lemma 9, $x_i = O(\lambda_i)$. Using either Proposition 11 if $\lambda_i = O(|x_i|)$, Proposition 12 if $x_i = O(\rho_i)$, or Proposition 13 if $x_i = o(\lambda_i)$ and $\rho_i = o(|x_i|)$, it follows that $|x_i|^{-1} (\Sigma_i \cap B_{3|x_i|/4}(x_i))$ converges either to a subset of a round sphere or to a bounded subset of a flat plane in $C^2$. In particular, the geometry of $|x_i|^{-1} (\Sigma_i \cap B_{3|x_i|/4}(x_i))$ is uniformly bounded.

By Lemma 42 and Proposition 10,
\[
\Delta H = \bar{\Delta} H + O(|x|^{-1} |\bar{\nabla}^2 H|) + O(|x|^{-2} |\bar{\nabla} H|).
\]
In conjunction with the area-constrained Willmore equation (3) and Proposition 10, we conclude that
\[
\bar{\Delta} H + O(|x|^{-1} |\bar{\nabla}^2 H|) + O((|x|^{-2} |\bar{\nabla} H|) = O((\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-2} H).
\]
By interior $L^4$-estimates as in [12, Theorem 9.11] and the Sobolev embedding theorem,

$$|x_i|^{5/2} \left( \int_{\Sigma_i \cap B_{x_i/2}(x_i)} |\nabla^2 H|_{\bar{g}}^4 \, d\bar{\mu} \right)^{1/4} + |x_i|^2 |(\bar{\nabla} H)(x_i)| \bar{g}$$

$$= O(|x_i|^{1/2}) \left( \int_{\Sigma_i \cap B_{x_i/4}(x_i)} (H - 2 \lambda_i^{-1})^4 \, d\bar{\mu} \right)^{1/4}$$

$$+ O(\lambda_i^{-1/2} + \rho_i^{-1}) |x_i|^{1/2} \left( \int_{\Sigma_i \cap B_{x_i/4}(x_i)} H^4 \, d\bar{\mu} \right)^{1/4}$$

$$= O(\lambda_i^{-1/2} + \rho_i^{-1}).$$

We have used Proposition 10 in the last equation.

Applying the same argument to (66), using also (21), we conclude that

$$|x_i|^2 |(\bar{\nabla} h)(x_i)| = O(\lambda_i^{-1/2} + \rho_i^{-1}).$$

The assertion follows from this and Lemma 42. □

3. Asymptotic analysis of large area-constrained Willmore spheres

We assume that $g$ is a Riemannian metric on $\mathbb{R}^3$ such that, as $x \to \infty$,

$$g = (1 + |x|^{-1})^4 \bar{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O(|x|^{-2-|J|})$$

for every multi-index $J$ with $|J| \leq 4$.

Let $\xi \in \mathbb{R}^3$ and $\lambda > 0$. Given $u \in C^\infty(S_\lambda(\lambda \xi))$, we define the map

$$\Phi_{\xi,\lambda}^u : S_\lambda(\lambda \xi) \to \mathbb{R}^3 \quad \text{given by} \quad \Phi_{\xi,\lambda}^u(x) = x + u(x) (\lambda^{-1} x - \xi).$$

We denote by

$$\Sigma_{\xi,\lambda}(u) = \Phi_{\xi,\lambda}^u(S_\lambda(\lambda \xi))$$

the Euclidean graph of $u$ over $S_\lambda(\lambda \xi)$. We tacitly identify functions defined on $\Sigma_{\xi,\lambda}(u)$ with functions defined on $S_\lambda(\lambda \xi)$ by precomposition with $\Phi_{\xi,\lambda}^u$.

We consider a sequence $\{\Sigma_i\}_{i=1}^\infty$ of area-constrained Willmore spheres $\Sigma_i \subset \mathbb{R}^3$ with

$$m_H(\Sigma_i) \geq 0, \quad \lim_{i \to \infty} \rho(\Sigma_i) = \infty, \quad \rho(\Sigma_i) = o(\rho(\Sigma_i)).$$

We assume that, as $i \to \infty$,

$$\log \lambda(\Sigma_i) = o(\rho(\Sigma_i)).$$

The goal of this section is to study the shape of $\Sigma_i$ as $i \to \infty$. More precisely, we show that $\Sigma_i$ is a graph over a nearby coordinate sphere, provided $i$ is sufficiently large.

We abbreviate $\rho_i = \rho(\Sigma_i)$ and $\lambda_i = \lambda(\Sigma_i)$.

Passing to a subsequence, we may assume that either $\Sigma_i$ encloses $B_2$ for every $i$ or that the bounded region enclosed by $\Sigma_i$ is disjoint from $B_2$ for every $i$. Let $x_i \in \Sigma_i \cap S_{\rho_i}(0)$. Passing to a further subsequence if necessary, we may assume that there is $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ such that

$$\lim_{i \to \infty} \rho_i^{-1} x_i = -\xi.$$
Lemma 15. If $\Sigma_i$ encloses $B_2$ for every $i$, the surfaces $\lambda_i^{-1}\Sigma_i$ converge to $S_1(\xi)$ in $C^1$. If the bounded region enclosed by $\Sigma_i$ is disjoint from $B_2$ for every $i$, the surfaces $\lambda_i^{-1}\Sigma_i$ converge to $S_1(-\xi)$ in $C^1$.

Proof. This is similar to an argument given in [10]. We repeat the argument for the reader’s convenience.

We first assume that $\Sigma_i$ encloses $B_2$ for every $i$.

We may assume that $\xi = e_3$. Let $a_i \in \mathbb{R}^3$ with $|a_i| = 1$ and $a_i \perp x_i$, $e_3$. Let $R_i \in SO(3)$ be the unique rotation with $R(a_i) = a_i$ and $R(x_i) = |x_i| e_3$. By (25), $\lim_{i \to \infty} R_i = Id$.

Let $\gamma_i > 0$ be the largest radius such that there is a smooth function $u_i : \{y \in \mathbb{R}^2 : |y| \leq \gamma_i\} \to \mathbb{R}$ with
\[
\begin{align*}
&\circ |(\nabla u_i)(y)| \leq 1 \\
&\circ (y, \rho_i + u_i(y)) \in R_i(\Sigma_i)
\end{align*}
\]
for all $y \in \mathbb{R}^2$ with $|y| \leq \gamma_i$. Clearly, $\gamma_i > 0$, $(\nabla u_i)(0) = 0$, and $u_i(0) = 0$. It follows that
\[
|y| + \rho_i \leq 3 |(y, \rho_i + u_i(y))| \leq 6 (|y| + \rho_i)
\]
and
\[
|((\nabla^2 u_i)(y)| \leq 8 |\tilde{h}(R_i(\Sigma_i))((y, \rho_i + u_i(y)))|
\]
for every $y \in \mathbb{R}^2$ with $|y| \leq \gamma_i$. Moreover, by Corollary 14,
\[
\tilde{h}(R_i(\Sigma_i)) = \lambda_i^{-1} \tilde{g}|_{R_i(\Sigma_i)} + O(\lambda_i^{-1/2} + \rho_i^{-1} |x|^{-1}).
\]
Combining (27), (28), and (26), we have
\[
|((\nabla^2 u_i)|_y| \leq 16 \lambda_i^{-1} + O((\lambda_i^{-1/2} + \rho_i^{-1}) (|y| + \rho_i)^{-1}).
\]
Integrating and using (26), Lemma 9, (23), and (24),
\[
|((\nabla u_i)|_y| \leq 16 |y| \lambda_i^{-1} + O(\log(\rho_i^{-1} \lambda_i) (\lambda_i^{-1/2} + \rho_i^{-1})) = 16 |y| \lambda_i^{-1} + o(1).
\]
It follows that $32 \gamma_i \geq \lambda_i$ for all $i$ sufficiently large. (29) also shows that, given $\varepsilon > 0$, there is $\delta > 0$ such that
\[
|\tilde{v}(R_i(\Sigma_i)) - e_3| \leq \varepsilon \quad \text{on} \quad \{(y, \rho_i + u_i(y)) : y \in \mathbb{R}^2 \text{ with } \lambda_i^{-1}|y| \leq \delta\}.
\]
According to Proposition 11, $\lambda_i^{-1} R_i(\Sigma_i)$ converges to $S_1(\tilde{\xi})$ in $C^2$ locally in $\mathbb{R}^3 \setminus \{0\}$ where $\tilde{\xi} \in \mathbb{R}^3$.

The preceding argument shows that $\tilde{\xi} = \xi$ and that the convergence is in $C^1$ in $\mathbb{R}^3$.

This finishes the proof in the case where each $\Sigma_i$ encloses $B_2$. The case where $B_2$ is disjoint from the bounded region enclosed by $\Sigma_i$ for every $i$ requires only formal modifications. \hfill $\square$

If $\Sigma_i$ encloses $B_2$, we define
\[
\xi_i = (\lambda_i^{-1} - \rho_i^{-1}) x_i.
\]
If the bounded region enclosed by $\Sigma_i$ is disjoint from $B_2$, we define
\[
\xi_i = (\lambda_i^{-1} + \rho_i^{-1}) x_i.
\]
Note that, in either case,
\[
|1 - |\xi_i|| = \lambda_i^{-1} \rho_i \quad \text{and} \quad x_i = \lambda_i (1 - |\xi_i|^{-1}) \xi_i \in S_{\lambda_i}(\lambda_i \xi_i);
\]
see Figure 3.

The following lemma is an immediate consequence of Lemma 15.

**Lemma 16.** For all sufficiently large \(i\), there are \(u_i \in C^\infty(S_{\lambda_i}(\lambda_i, \xi_i))\) with the following properties.

- \(\Sigma_i = \Sigma_{\xi_i, \lambda_i}(u_i)\)
- \(u_i(x_i) = 0\)
- \((\bar{\nabla} u_i)(x_i) = 0\)
- \(\bar{\nabla} u_i = o(1)\)

We abbreviate \(S_i = S_{\lambda_i}(\lambda_i, \xi_i)\) and \(\Phi_i = \Phi_{\xi_i, \lambda_i}^{u_i}\).

**Remark 17.** It follows from Lemma 16 that \(u_i(x) = o(|x|)\) and \(\Phi_i(x) = x + o(|x|)\).

To proceed, we need the following technical lemma.

**Lemma 18.** Let \(c \geq 1\) and \(\beta : [0, 1] \to \mathbb{R}\) be a non-negative, measurable function with

\[
\int_0^1 \beta(s) \, ds \leq \frac{1}{16} c^{-2} (1 + 2c)^{-1} \exp(-2c).
\]

Suppose that \(\alpha : [0, 1] \to \mathbb{R}\) be a differentiable function with absolutely continuous derivative such that \(\alpha(0) = \alpha'(0) = 0\) and

\[
|\alpha''| \leq c^2 |\alpha| + c^2 (\alpha')^2 + \beta
\]

almost everywhere. Then

\[
|\alpha'| \leq 4 (1 + 2c) \exp(2c) \int_0^1 \beta(s) \, ds.
\]

**Proof.** We will assume that \(\beta > 0\). The general case follows by approximation.

Let \(\omega : [0, 1] \to \mathbb{R}\) be given by

\[
\omega(t) = 4 t \exp(2ct) \int_0^1 \beta(s) \, ds.
\]
We claim that $|\alpha'(t)| < \omega'(t)$ on $[0, 1]$. To see this, note that

\begin{equation}
4 \int_{0}^{1} \beta(s) \, ds < \omega' \leq 4 (1 + 2c) \exp(2c) \int_{0}^{1} \beta(s) \, ds \leq \frac{1}{4} c^{-2}
\end{equation}

and

\begin{equation}
0 \leq 4 c^2 \omega < \omega''.
\end{equation}

Suppose that there is $t_0 \in (0, 1]$ such that $|\alpha'(t_0)| = \omega'(t_0)$ and $|\alpha'(t)| < \omega'(t)$ on $[0, t_0)$. It follows that $|\alpha(t)| < \omega(t)$ on $[0, t_0)$. Consequently,

\begin{equation}
\omega'(t_0) = |\alpha'(t_0)| \leq \int_{0}^{t_0} |\alpha''(s)| \, ds \leq c^2 \int_{0}^{t_0} \omega(s) \, ds + c^2 \int_{0}^{t_0} \omega'(s)^2 \, ds + \int_{0}^{1} \beta(s) \, ds.
\end{equation}

By (31), $\omega'(t) \leq \omega'(t_0)$ on $[0, t_0)$. Using this, (30), and (31), we have

\[ c^2 \int_{0}^{t_0} \omega(s) \, ds + c^2 \int_{0}^{t_0} \omega'(s)^2 \, ds \leq \frac{1}{4} \int_{0}^{t_0} \omega''(s) \, ds + \frac{1}{4} \int_{0}^{t_0} \omega'(s) \, ds < \frac{1}{2} \omega'(t_0). \]

In conjunction with (32), we conclude that

\[ \omega'(t_0) \leq 2 \int_{0}^{1} \beta(s) \, ds. \]

This is incompatible with (30).

It follows that $|\alpha'(t)| < \omega'(t)$ on $[0, 1]$. The assertion now follows from (30).

\begin{lemma}
There holds

\[ |x|^{-1} |u_i| + |\nabla u_i| + |x| |\nabla^2 u_i| = O((\log(\rho_i^{-1} \lambda_i))^{1/2} (\lambda_i^{-1/2} + \rho_i^{-1})). \]

\end{lemma}

\begin{proof}
Let $z_i \in S_i$ and $\gamma_i : [0, 1] \to S_i$ be a minimizing geodesic with respect to $\bar{g}$ such that $\gamma_i(0) = x_i$ and $\gamma_i(1) = z_i$. Note that

\begin{equation}
|\gamma_i| \leq \pi \lambda_i.
\end{equation}

Given an integer $\ell$, let

\[ S_{i, \ell} = \{ z \in S_i : 2^{\ell-1} \rho_i \leq |z| < 2^\ell \rho_i \}. \]

Note that

\[ S_i \cap B_{|z|/2}(z) \subset S_{i, \ell-1} \cup S_{i, \ell} \cup S_{i, \ell+1} \]

for every $z \in S_{i, \ell}$ and that

\[ \int_{\gamma \cap S_{i, \ell}} |z|^{-1} \, d\bar{\mu} = O(1) \]

uniformly for all $\ell$.

Let $z \in S_i$. By Lemma 42 and Proposition 10, we have

\[ \hat{h}(\Sigma_i)(\Phi_i(z)) = h(\Sigma_i)(\Phi_i(z)) + O(|(\Phi_i(z))|^{-1} |h(\Sigma_i)(\Phi_i(z))|) + O(|\Phi_i(z)|^{-2}) \]

\[ = h(\Sigma_i)(\Phi_i(z)) + O(|\Phi_i(z)|^{-2}) \]

and

\[ \bar{g}|_{\Sigma_i} = g|_{\Sigma_i} + O(|\Phi_i(z)|^{-1}). \]
By Lemma 16, 
\begin{equation}
(h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}) (\Phi_i(z)) \\
= O(|\Phi_i(z)|^{-1}) \left( \int_{\Sigma \cap B_i(\gamma_i)} |h - \lambda_i^{-1} g|_{\Sigma_i}|^2 \, d\mu \right)^{1/2} \\
+ O(|\Phi_i(z)|^{-2}) + O((\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-1}).
\end{equation}
Using Remark 17 and Lemma 43, we conclude that
\begin{equation}
(\bar{\nabla}^2 u_i)(z) = O(|z|^{-1}) \left( \int_{\gamma \cap S_i(z)/2} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \, d\mu \right)^{1/2} \\
+ O(|z|^{-2}) + O((\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-1}) + O(\lambda_i^{-2} |u_i(z)|) + O(\lambda_i^{-1} |\bar{\nabla} u_i(z)|^2).
\end{equation}
We have
\begin{equation}
\int_\gamma |z|^{-2} + \int_\gamma (\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-1} = O(\rho_i^{-1}) + O(\lambda_i^{-1/2} + \rho_i^{-1}) = O(\lambda_i^{-1/2} + \rho_i^{-1}).
\end{equation}
Let $k_i = [\log(2) - \log(\rho_i^{-1} \lambda_i)]$ and note that $k_i = O(\log(\rho_i^{-1} \lambda_i))$. We have
\begin{equation}
\begin{array}{l}
\int_\gamma |z|^{-1} \left( \int_{\gamma \cap S_i(z)/2} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \, d\mu \right)^{1/2} \\
\quad = O(1) \sum_{\ell=1}^{k_i} \int_{\gamma \cap S_i,\ell} |z|^{-1} \, d\mu(z) \left( \int_{S_i,\ell-1 \cup S_i,\ell \cup S_i,\ell+1} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \, d\mu \right)^{1/2} \\
\quad = O(\sqrt{k_i}) \left( \sum_{\ell=1}^{k_i} \int_{S_i,\ell-1 \cup S_i,\ell \cup S_i,\ell+1} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \, d\mu \right)^{1/2} \\
\quad = O((\log(\rho_i^{-1} \lambda_i))^{1/2}) \left( \int_{\Sigma_i} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \, d\mu \right)^{1/2} \\
\quad = O((\log(\rho_i^{-1} \lambda_i))^{1/2} (\lambda_i^{-1/2} + \rho_i^{-1})).
\end{array}
\end{equation}
We have used Lemma 8 in the last equation.

Let $c \geq 1$ and $\alpha_i, \beta_i : [0, 1] \to \mathbb{R}$ be given by
\begin{equation}
\alpha_i(s) = \int_0^s |(\bar{\nabla} u_i)(\gamma(t))| \, dt
\end{equation}
and
\begin{equation}
\beta_i(s) = c |\dot{\gamma}_i(s)| |\dot{\gamma}_i(s)|^{-1} \left( \int_{\gamma \cap B_i(\gamma_i(s))/2} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \, d\mu \right)^{1/2} \\
+ c |\dot{\gamma}_i(s)| |\dot{\gamma}_i(s)|^{-2} + c |\dot{\gamma}_i(s)| (\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-1}.
\end{equation}
By Lemma 16, $|u_i(\gamma_i(s))| \leq |\dot{\gamma}_i(s)| \alpha_i(s)$ for all $s \in [0, 1]$. Moreover, whenever $\alpha''_i(s)$ exists, there holds $|\alpha''_i(s)| \leq |\dot{\gamma}_i(s)| \left( |(\bar{\nabla}^2 u_i)(\gamma_i(s))| \right)$. In conjunction with (33) and (34), we obtain
\begin{equation}
|\alpha''_i| \leq c^2 |\alpha_i| + c^2 (\alpha'_i)^2 + \beta_i.
\end{equation}
Lemma 20. There is a constant \( c > 0 \) with the following property. Let \( \xi \in \mathbb{R}^3 \) and \( \lambda > 0 \). Suppose that \( u, f \in \Lambda_0(S_0(\lambda \xi)) \perp \) are such that \( \Delta u = f \). Then

\[
\sup_{x \in S_0(\lambda \xi)} |x| |\nabla u(x)| \leq c \left( \int_{S_0(\lambda \xi)} |f| \, d\bar{\mu} + \sup_{x \in S_0(\lambda \xi)} |x|^2 |f| \right).
\]
Figure 4. An illustration of the partition (41) for \(|\xi| \approx 1\) and \(|x| \approx |1 - \xi|\). The cross marks the origin of \(\mathbb{R}^3\). The gradient of the Green's function \(G\) is large within the part of \(S_1(\xi)\) illustrated by the black line while \(f\) may be large within the part of \(S_1(\xi)\) illustrated by the dashed, gray line.

Proof. By scaling, we may assume that \(\lambda = 1\) and

\[
\int_{S_1(\xi)} |f| \, d\bar{\mu} + \sup_{x \in S_1(\xi)} |x|^2 |f| = 1.
\]

Recall from, e.g., [2, §A.1], that the Green’s function of \(\bar{\Lambda} : \Lambda_0(S_1(0))^{\perp} \rightarrow \Lambda_0(S_1(0))^{\perp}\) is given by

\[
G(x, y) = \frac{1}{2\pi} \log |x - y|.
\]

It follows that

\[
(\bar{\nabla} u)(x) = \int_{S_1(\xi)} (\bar{\nabla} G)(x, y) \, f(y) \, d\bar{\mu}(y)
\]

where differentiation is with respect to \(x\). Note that

\[
(\bar{\nabla} G)(x, y) = O(1) |x - y|^{-1}
\]

for all \(x, y \in \mathbb{R}^3\) with \(x \neq y\).

Let \(x \in S_1(\xi)\). We may assume that \(x \neq 0\). We estimate the integral (40) over the regions

\[
\{y \in S_1(\xi) : 2|y - x| \geq |x|\} \quad \text{and} \quad \{y \in S_1(\xi) : 2|y - x| \leq |x|\}
\]

separately; see Figure 4. We have

\[
\int_{\{y \in S_1(\xi) : 2|y - x| \geq |x|\}} |x - y|^{-1} |f| \, d\bar{\mu}(y) \leq O(1) |x|^{-1} \int_{S_1(\xi)} |f| \, d\bar{\mu}(y) \leq O(1) |x|^{-1}.
\]

Likewise,

\[
\int_{\{y \in S_1(\xi) : 2|y - x| \leq |x|\}} |x - y|^{-1} |f| \, d\bar{\mu}(y) \leq \int_{\{y \in S_1(\xi) : 2|y - x| \leq |x|\}} |x - y|^{-1} |y|^{-2} \, d\bar{\mu}(y)
\]

\[
\leq O(1) |x|^{-2} \int_{\{y \in S_1(\xi) : 2|y - x| \leq |x|\}} |x - y|^{-1} \, d\bar{\mu}(y)
\]
Lemma 22. As \( \lambda \to \infty \), there holds
\[
\text{proj}_{\Lambda_0(S_i)} H(\Sigma_i) = \frac{1}{4\pi} \lambda_i^{-2} \int_{S_i} H(\Sigma_i) \, d\bar{\mu} = 2 \lambda_i^{-1} + o(\lambda_i^{-1})
\]
and
\[
\kappa(\Sigma_i) = o(\lambda_i^{-2} \rho_i^{-1}).
\]
Moreover,
\[
\text{proj}_{\Lambda_0(S_i)}^1 H(\Sigma_i) = -4 \lambda_i^{-1} |x|^{-1} + o(\lambda_i^{-1} \rho_i^{-1}).
\]

Proof. By Proposition 10 and Remark 17,
\[
H(\Sigma_i) = 2 \lambda_i^{-1} + O((\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-1}).
\]

Using Lemma 37, we have
\[
\int_{S_i} H(\Sigma_i) \, d\bar{\mu} = 8 \pi \lambda_i + O(\lambda_i^{-1/2} + \rho_i^{-1}) \int_{S_i} |x|^{-1} \, d\bar{\mu} = 8 \pi \lambda_i + O((\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i)
\]
and
\[
\int_{S_i} |H(\Sigma_i)| \, d\bar{\mu} = O(\lambda_i).
\]
Consequently,
\[
\text{proj}_{\Lambda_0(S_i)} H(\Sigma_i) = 2 \lambda_i^{-1} + O((\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-1}).
\]

We define the function \( F_i : \Sigma_i \to \mathbb{R} \) by \( F_i = N^{-1} H(\Sigma_i) \) where
\[
N : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R} \quad \text{is given by} \quad N(x) = (1 + |x|^{-1})^{-1} (1 - |x|^{-1})
\]
is the potential function of Schwarzschild; see (77). By Remark 17,
\[
F_i = (1 + 2 |x|^{-1} + o(|x|^{-1})) H(\Sigma_i).
\]
It follows that
\[
\int_{S_i} F_i \, d\bar{\mu} = \int_{S_i} H(\Sigma_i) \, d\bar{\mu} + o(1) \int_{S_i} |H(\Sigma_i)| \, d\bar{\mu} = \int_{S_i} H(\Sigma_i) \, d\bar{\mu} + o(\lambda_i).
\]
In conjunction with (44), we obtain
\begin{equation}
\text{proj}_{\lambda_0(S_i)} F_i = 2 \lambda_i^{-1} + o(\lambda_i^{-1}).
\end{equation}

By Lemma 45, Lemma 40, Lemma 42, and Proposition 10, we have
\begin{equation}
\Delta_{\Sigma_i} F_i = - (|\hat{h}(\Sigma_i)|^2 + \kappa(\Sigma_i)) F_i
\end{equation}
\begin{equation}
+ O(|x|^{-1} + \lambda_i^{-1} |x|^{-2} + (\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-3}) |F_i| + O(|x|^{-3}) |x| |\nabla F_i|.
\end{equation}

Using Lemma 42, Proposition 10, and Remark 17, we have
\begin{equation}
\bar{\Delta}_{\Sigma_i} F_i = (1 + O(|x|^{-1})) \Delta_{\Sigma_i} F_i + O(|x|^{-3}) (|x| |\nabla F_i| + |x|^2 |\nabla^2 F_i|).
\end{equation}

In conjunction with (47) and Proposition 10, we conclude that
\begin{equation}
\bar{\Delta}_{\Sigma_i} F_i = - (|\hat{h}(\Sigma_i)|^2 + \kappa(\Sigma_i)) F_i
\end{equation}
\begin{equation}
+ O(|x|^{-1} + \lambda_i^{-1} |x|^{-2} + (\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-3}) |F_i| + O(|x|^{-3}) (|x| |\nabla F_i| + |x|^2 |\nabla^2 F_i|).
\end{equation}

Using Lemma 44, Lemma 19, and Lemma 9, we have
\begin{equation}
\Delta_{\Sigma_i} F_i = (1 + o(1)) \Delta_{\Sigma_i} F_i + O(\log(\rho_i^{-1} \lambda_i) (\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-2}) (|x| |\nabla F_i| + |x|^2 |\nabla^2 F_i|).
\end{equation}

In conjunction with (48), (38), and (39), we obtain
\begin{equation}
\Delta_{\Sigma_i} F_i = O(|\hat{h}(\Sigma_i)|^2 + |\kappa(\Sigma_i)|) |F_i|
\end{equation}
\begin{equation}
+ O(\lambda_i^{-1} |x|^{-2} + (\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-3}) |F_i|
\end{equation}
\begin{equation}
+ O((\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-2}) (|x| |\nabla F_i| + |x|^2 |\nabla^2 F_i|).
\end{equation}

According to Proposition 10 and Remark 17,
\begin{equation}
\sup_{x \in S_i} |x|^2 (|\hat{h}(\Sigma_i)|^2 + |\kappa(\Sigma_i)| + \lambda_i^{-1} |x|^{-2} + (\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-3}) = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)|).
\end{equation}

Using Lemma 42, Remark 17, and Lemma 16, we have
\begin{equation}
d\mu(\Sigma_i) = (1 + o(1)) d\bar{\mu}(\Sigma_i) = (1 + o(1)) d\bar{\mu}(S_i).
\end{equation}

In conjunction with Lemma 8 and Lemma 37, we see that
\begin{equation}
\int_{S_i} (|\hat{h}(\Sigma_i)|^2 + |\kappa(\Sigma_i)| + \lambda_i^{-1} |x|^{-2} + (\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-3}) d\bar{\mu}
\end{equation}
\begin{equation}
= O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)| + \log(\rho_i^{-1} \lambda_i) \lambda_i^{-1}).
\end{equation}

Likewise,
\begin{equation}
\int_{S_i} (\lambda_i^{-1/2} + \rho_i^{-1}) |x|^{-2} d\bar{\mu} = O(\log(\rho_i^{-1} \lambda_i) (\lambda_i^{-1/2} + \rho_i^{-1})).
\end{equation}

Using (49) and Lemma 20, we conclude that
\begin{equation}
\sup_{x \in S_i} |x|^2 (|\nabla F_i| = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)| + \log(\rho_i^{-1} \lambda_i) \lambda_i^{-1}) \sup_{x \in S_i} |F_i|
\end{equation}
\begin{equation}
+ O(\log(\rho_i^{-1} \lambda_i) (\lambda_i^{-1/2} + \rho_i^{-1})) (\sup_{x \in S_i} |x| |\nabla F_i| + \sup_{x \in S_i} |x|^2 |\nabla^2 F_i|).
\end{equation}
By standard elliptic theory,
\begin{equation}
\sup_{x \in S_i} |x|^2 |\nabla^2 F_i| = O(1) \sup_{x \in S_i} |x| |\nabla F_i| + O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)|) \sup_{x \in S_i} |F_i|; 
\end{equation}
see Remark 23. Using (38) and (39) and absorbing, we conclude that
\begin{equation}
\sup_{x \in S_i} |x| |\nabla F_i| + \sup_{x \in S_i} |x|^2 |\nabla F_i|^2 = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)|) \sup_{x \in S_i} |F_i|. 
\end{equation}
Note that there is \(z \in S_i\) with \(\text{proj}_{\Lambda_0(S_i)^\perp} F_i(z) = 0\). Integrating, we find
\begin{equation}
\sup_{x \in S_i} |\text{proj}_{\Lambda_0(S_i)^\perp} F_i| = O(\log(\rho_i^{-1} \lambda_i)) \sup_{x \in S_i} |x| |\nabla F_i|.
\end{equation}
Using (16), (38), and (39), we have
\begin{equation}
(\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)| + \log(\rho_i^{-1} \lambda_i) \lambda_i^{-1} = o((\log(\rho_i^{-1} \lambda_i))^{-1}).
\end{equation}
Returning to (53) and absorbing, we obtain
\begin{equation}
(\log(\rho_i^{-1} \lambda_i))^{-1} \sup_{x \in S_i} |\text{proj}_{\Lambda_0(S_i)^\perp} F_i| + \sup_{x \in S_i} |x| |\nabla F_i| + \sup_{x \in S_i} |x|^2 |\nabla F_i|^2
\end{equation}
\begin{equation}
= O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)| + \log(\rho_i^{-1} \lambda_i) \lambda_i^{-1}) \sup_{x \in S_i} |\text{proj}_{\Lambda_0(S_i)} F_i|.
\end{equation}
\begin{equation}
= O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)| + \log(\rho_i^{-1} \lambda_i) \lambda_i^{-1}) \lambda_i^{-1}.
\end{equation}
We have used (46) in the last equation. In particular, using (16),
\begin{equation}
F_i = \text{proj}_{\Lambda_0(S_i)} F_i + \text{proj}_{\Lambda_0(S_i)^\perp} F_i = O(\lambda_i^{-1}).
\end{equation}
Using (48), (54), (55), (38), and (39), we have
\begin{equation}
\tilde{\Delta}_{\Sigma_i} F_i = -\kappa(\Sigma_i) F_i + O(\lambda_i |x|^{-3} |\kappa(\Sigma_i)|) + O(\lambda_i^{-1} |\tilde{h}(\Sigma_i)|^2)
\end{equation}
\begin{equation}
+ O(\lambda_i^{-2} |x|^{-2}) + O((\lambda_i^{-1/2} + \rho_i^{-1}) \lambda_i^{-1} |x|^{-3}).
\end{equation}
Integrating and using (46), (50), Lemma 37, and Lemma 8, we obtain
\begin{equation}
\kappa(\Sigma_i) = O(\lambda_i^{-2} \rho_i^{-2}) + O(\log(\rho_i^{-1} \lambda_i) \lambda_i^{-3}).
\end{equation}
Returning to (54), we conclude that
\begin{equation}
\text{proj}_{\Lambda_0(S_i)^\perp} F_i = o(\lambda_i^{-1} \rho_i^{-1}).
\end{equation}
Moreover, by (56), \(\kappa(\Sigma_i) = o(\lambda_i^{-2} \rho_i^{-1})\). We have used (38) and (39) in both of these estimates.
By (45), (55), and (57), we have
\begin{equation}
\text{proj}_{\Lambda_0(S_i)^\perp} H(\Sigma_i) = \text{proj}_{\Lambda_0(S_i)^\perp} F_i + \text{proj}_{\Lambda_0(S_i)^\perp} O(\rho_i^{-1}) H(\Sigma_i)
\end{equation}
\begin{equation}
= o(\lambda_i^{-1} \rho_i^{-1}) + O(\rho_i^{-1}) \sup_{x \in S_i} |H(\Sigma_i)|
\end{equation}
\begin{equation}
= o(\lambda_i^{-1} \rho_i^{-1}) + O(\rho_i^{-1}) \sup_{x \in S_i} |F_i|
\end{equation}
\begin{equation}
= o(\lambda_i^{-1}).
\end{equation}
In conjunction with (44) and (45), we obtain
\[ F_i = H(\Sigma_i) + 4|x|^{-1} \lambda_i^{-1} + o(\rho_i^{-1} \lambda_i^{-1}). \]

By Lemma 37,
\[ \text{proj}_{\Lambda_0(S_i)} |x|^{-1} = O(\lambda_i^{-1}) = o(\rho_i^{-1}). \]

Using (57), we conclude that
\[ \text{proj}_{\Lambda_0(S_i)} H(\Sigma_i) = -4|x|^{-1} \lambda_i^{-1} + o(\lambda_i^{-1} \rho_i^{-1}). \]

The assertion follows. \(\square\)

**Remark 23.** We provide additional details on how to obtain (52).

Let \( z_i \in S_i \) and \( a_i \in \mathbb{T}_z S_i \) with \( |a_i| = 1 \). The estimates below are independent of these choices.

Let \( X_i = \frac{a_i}{|a_i|} \). By interior \( L^4 \)-estimates as in [12, Theorem 9.11] and the Sobolev embedding theorem, using also Lemma 42, Lemma 44, Proposition 10, and Lemma 19, we have
\[ |z_i|^2 |a_i \cdot (\nabla^2 F_i)(z_i)| = O(1) \sup_{x \in S_i} |x| |\nabla F_i| + O(|z_i|^{5/2}) \left( \int_{S_i \cap B_{|z_i|/2}(z_i)} (\Delta \nabla_{X_i} F_i)^4 d\mu \right)^{1/4}. \]

Note that
\[ \Delta \nabla_{X_i} F_i = \nabla_{X_i} \Delta F_i + K^{\Sigma} g(X_i, \nabla F_i) + 2g(\nabla X_i, \nabla^2 F_i) + g(\text{tr} \nabla^2 X_i, \nabla F_i) \]
where \( K^{\Sigma} \) is the Gauss curvature of \( \Sigma \). Using this, Lemma 45, (37), (16), and Corollary 14, we obtain
\[ \Delta \nabla_{X_i} F_i = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 |x|^{-3}) |F_i| + O(|x|^{-3}) \left( |x| |\nabla F_i| + |x|^2 |\nabla^2 F_i| \right). \]

Consequently,
\[ |z_i|^{5/2} \left( \int_{S_i \cap B_{|z_i|/2}(z_i)} (\Delta \nabla_{X_i} F_i)^4 d\mu \right)^{1/4} = O((\lambda_i^{-1/2} + \rho_i^{-1})^2) \sup_{x \in S_i} |F_i| + O(1) \sup_{x \in S_i} |x| |\nabla F_i| \]
\[ + O(|z_i|^{3/2}) \left( \int_{S_i \cap B_{|z_i|/2}(z_i)} |\nabla^2 F_i|^4 d\mu \right)^{1/4}. \]

Finally, by (47) and Proposition 10,
\[ \Delta F_i = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)|) |x|^{-2} |F_i| + O(|x|^{-3}) |x| |\nabla F_i|. \]

By interior \( L^4 \)-estimates, we conclude that
\[ |z_i|^{3/2} \left( \int_{S_i \cap B_{|z_i|/2}(z_i)} |\nabla^2 F_i|^4 d\mu \right)^{1/4} = O(1) \sup_{x \in S_i} |x| |\nabla F_i| + O((\lambda_i^{-1/2} + \rho_i^{-1})^2 + \lambda_i^2 |\kappa(\Sigma_i)|) \sup_{x \in S_i} |F_i|. \]

5. Variations of the Willmore energy by translations

We assume that \( g \) is a Riemannian metric on \( \mathbb{R}^3 \) such that, as \( x \to \infty \),
\[ g = (1 + |x|^{-1})^4 \bar{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O(|x|^{-2-|J|}). \]

(58)
Lemma 24. There holds

\[(61)\]

\[\lim_{i \to \infty} \rho(\Sigma_i) = \infty, \quad \rho(\Sigma_i) = o(\lambda(\Sigma_i))\]

and assume that, as \(i \to \infty\),

\[(60)\]

\[\log \lambda(\Sigma_i) = o(\rho(\Sigma_i)).\]

As before, we abbreviate \(\lambda_i = \lambda(\Sigma_i)\) and \(\rho_i = \rho(\Sigma_i)\). Recall from Lemma 16 that, for all \(i\) large, \(\Sigma_i = \Sigma{\xi_i, \lambda_i}(u_i)\) is the Euclidean graph of a function \(u_i\) over the sphere \(S_i = S_{\lambda_i}(\lambda_i \xi_i)\) where \(\xi_i \in \mathbb{R}^3\) satisfies

\[(61)\]

\[|1 - |\xi_i|| = \lambda_i^{-1} \rho_i.\]

In this section, we compute an expansion of the variation of the Willmore energy of \(\Sigma_i\) with respect to a translation in direction \(\xi_i\).

For the statement of the next lemma, we define the form

\[\zeta_i : \Gamma(T\Sigma_i) \times \Gamma(T\Sigma_i) \to C^\infty(\mathbb{R}) \text{ given by } \zeta_i(X, Y) = g(D_X \xi_i, Y).\]

**Lemma 24.** There holds

\[\int_{\Sigma_i} g(\xi_i, \nu) \left[ \Delta H + |\hat{h}|^2 H + \text{Ric}(\nu, \nu) H \right] d\mu = \int_{\Sigma_i} [g(\text{tr}_{\Sigma_i} D^2 \xi_i, \nu) + \text{Ric}(\xi_i, \nu)] H d\mu + \frac{1}{2} \int_{\Sigma_i} [\text{div}_{\Sigma_i} \xi_i - 2 g(D_{\nu} \xi_i, \nu)] H^2 d\mu + 2 \int_{\Sigma_i} g(\zeta_i, \hat{h}) H^2 d\mu.\]

**Proof.** Note that

\[\Delta(g(\xi_i, \nu)) = g(\text{tr}_{\Sigma_i} D^2 \xi_i, \nu) - g(D_{\nu} \xi_i, \nu) H + 2 g(\zeta_i, h) + (\text{div}_{\Sigma_i} h)(\xi_i^T) - g(\xi_i, \nu) |\hat{h}|^2.\]

Integrating by parts and using the trace of the Gauss-Codazzi equation,

\[\text{div}_{\Sigma_i} h = \nabla H + \nu \circ \text{Rc},\]

we obtain

\[\int_{\Sigma_i} g(\xi_i, \nu) \Delta H d\mu = \int_{\Sigma_i} [g(\text{tr}_{\Sigma_i} D^2 \xi_i, \nu) H - g(D_{\nu} \xi_i, \nu) H^2 + 2 g(\zeta_i, h) H + H g(\xi_i, \nabla H) + \text{Ric}(\xi_i^T, \nu) H - g(\xi_i, \nu) |\hat{h}|^2 H] d\mu.\]

Note that

\[\int_{\Sigma_i} H g(\xi_i, \nabla H) d\mu = \frac{1}{2} \int_{\Sigma_i} g(\xi_i, \nabla H^2) d\mu = \frac{1}{2} \int_{\Sigma_i} g(\xi_i, \nu) H^3 - (\text{div}_{\Sigma_i} \xi_i) H^2 d\mu\]

where we have integrated by parts in the second equality. Using the decomposition

\[h = \frac{1}{2} H g|_{\Sigma_i} + \hat{h},\]

we see that

\[2 g(\zeta_i, h) = (\text{div}_{\Sigma_i} \xi_i) H + 2 g(\zeta_i, \hat{h}).\]
and

$$2 g(\xi_i, \nu) |h|^2 = g(\xi_i, \nu) H^2 + 2 g(\xi_i, \nu) |\hat{h}|^2.$$  

Using that $\xi_i = \xi_i^\top + \xi_i^\perp$, we obtain

$$\text{Ric}(\xi_i^\top, \nu) = \text{Ric}(\xi_i, \nu) - g(\xi_i, \nu) \text{Ric}(\nu, \nu).$$

The assertion follows from these identities.

For the proof of Lemma 25 below, let $e_1, e_2, e_3$ be the standard basis of $\mathbb{R}^3$.

**Lemma 25.** As $i \to \infty$, there holds

$$\int_{\Sigma_i} \left[ g(\text{tr}_{\Sigma_i} D^2 \xi_i, \nu) + \text{Ric}(\xi_i, \nu) \right] H \, d\mu = -8 \pi \lambda_i^{-1} \rho_i^{-2} + \lambda_i^{-1} \int_{S_i} \bar{g}(\xi_i, \bar{\nu}) R \, d\bar{\mu} + o(\lambda_i^{-1} \rho_i^{-2}).$$

**Proof.** Using Lemma 40, Lemma 41, and Lemma 42, we have

$$\left[ \bar{g}(\text{tr}_{\Sigma_i} D^2 \xi_i, \bar{\nu}) + \text{Ric}(\xi_i, \bar{\nu}) \right] d\bar{\mu} = \left[ 4 \, |x|^{-3} \bar{g}(\xi_i, \bar{\nu}) - 12 \, |x|^{-5} \bar{g}(x, \xi_i) \bar{g}(x, \bar{\nu}) + 4 \, |x|^{-4} \bar{g}(\xi_i, \bar{\nu}) - 8 \, |x|^{-6} \bar{g}(x, \xi_i) \bar{g}(x, \bar{\nu}) + O(|x|^{-5}) \right] d\bar{\mu}.$$

Using also Lemma 39, we conclude that

$$\left[ g(\text{tr}_{\Sigma_i} D^2 \xi_i, \nu) + \text{Ric}(\xi_i, \nu) \right] d\mu = \left[ 4 \, |x|^{-3} \bar{g}(\xi_i, \bar{\nu}) - 12 \, |x|^{-5} \bar{g}(x, \xi_i) \bar{g}(x, \bar{\nu}) + 4 \, |x|^{-4} \bar{g}(\xi_i, \bar{\nu}) - 8 \, |x|^{-6} \bar{g}(x, \xi_i) \bar{g}(x, \bar{\nu}) + \frac{1}{2} \sum_{j=1}^3 [(D^2_{\xi_i, e_j} \sigma)(e_j, \bar{\nu}) - (D^2_{\xi_i, \bar{\nu}} \sigma)(e_j, e_j)] + \frac{1}{2} \sum_{j=1}^3 (D^2_{\xi_i, e_j} \sigma)(e_j, \bar{\nu}) - (D^2_{\xi_i, \bar{\nu}} \sigma)(\bar{\nu}, \bar{\nu}) + O(|x|^{-5}) \right] d\bar{\mu}.$$

By the divergence theorem,

$$(62) \quad \int_{\Sigma_i} \left[ |x|^{-3} \bar{g}(\xi_i, \bar{\nu}) - 3 \, |x|^{-5} \bar{g}(x, \xi_i) \bar{g}(x, \bar{\nu}) \right] d\bar{\mu} = 0.$$
Note that this holds independently of whether $\Sigma_i$ encloses the origin or not. In conjunction with Lemma 16, Lemma 22, and Lemma 35, we conclude that
\[
\int_{\Sigma_i} \left[ g (\text{tr}_{\Sigma_i} D^2 \xi_i, \nu) + \text{Ric}(\xi_i, \nu) \right] H \, d\mu
= -8 \lambda_i^{-1} \int_{S_i} \left[ |x|^{-4} \tilde{g}(\xi_i, \tilde{\nu}) - 4 |x|^{-6} \tilde{g}(x, \xi_i) \tilde{g}(x, \tilde{\nu}) \right] d\tilde{\mu}
+ \lambda_i^{-1} \int_{S_i} \sum_{j=1}^{3} \left[ (\tilde{D}^2_{\xi_i, e_j}) \nu (e_j, \tilde{\nu}) - (\tilde{D}^2_{\xi_i, \rho \sigma}) (e_j, e_j) \right] d\tilde{\mu}
+ \lambda_i^{-1} \int_{S_i} \sum_{j=1}^{3} \left[ (\tilde{D}^2_{\xi_i, e_j}) \nu (e_j, \tilde{\nu}) - (\tilde{D}^2_{\xi_i, \rho \sigma}) (\tilde{\nu}, \tilde{\nu}) \right] d\tilde{\mu}
+ o(\lambda_i^{-1} \rho_i^{-2}).
\]

Using Lemma 38 and Lemma 37, we have
\[
\lambda_i^{-1} \int_{S_i} \left[ |x|^{-4} \tilde{g}(\xi_i, \tilde{\nu}) - 4 |x|^{-6} \tilde{g}(x, \xi_i) \tilde{g}(x, \tilde{\nu}) \right] d\tilde{\mu}
= \pi \lambda_i^{-3} (1 - |\xi_i|)^{-2} + o(\lambda_i^{-3} (1 - |\xi_i|)^{-2}).
\]

Using (61), we conclude that
\[
\lambda_i^{-1} \int_{S_i} \left[ |x|^{-4} \tilde{g}(\xi_i, \tilde{\nu}) - 4 |x|^{-6} \tilde{g}(x, \xi_i) \tilde{g}(x, \tilde{\nu}) \right] d\tilde{\mu} = \pi \lambda_i^{-1} \rho_i^{-2} + o(\lambda_i^{-1} \rho_i^{-2}).
\]

On the one hand, applying the divergence theorem, commuting derivatives, and applying the divergence theorem again, we obtain
\[
\int_{S_i} \sum_{j=1}^{3} \left[ (\tilde{D}^2_{\xi_i, e_j}) \nu (e_j, \tilde{\nu}) - (\tilde{D}^2_{\xi_i, \rho \sigma}) (e_j, e_j) \right] d\tilde{\mu} = \int_{S_i} [\tilde{\text{div}} \tilde{\text{div}} \sigma - \tilde{\Delta} \tilde{\text{tr}} \sigma] \tilde{g}(\xi_i, \tilde{\nu}) \, d\tilde{\mu}.
\]

On the other hand, note that
\[
\tilde{\text{div}}_{S_i} (\nu \cdot \tilde{D}_{\xi_i} \sigma) = \sum_{j=1}^{3} (\tilde{D}^2_{\xi_i, e_j}) \nu (e_j, \tilde{\nu}) - (\tilde{D}^2_{\xi_i, \rho \sigma}) (\tilde{\nu}, e_j) + \lambda_i^{-1} \tilde{D}_{\xi_i} \tilde{\text{tr}} \sigma - 3 \lambda_i^{-1} (\tilde{D}^2_{\xi_i, \sigma}) (\tilde{\nu}, \tilde{\nu}).
\]

Consequently,
\[
\int_{S_i} \left[ \sum_{j=1}^{3} (\tilde{D}^2_{\xi_i, e_j}) \nu (e_j, \tilde{\nu}) - (\tilde{D}^2_{\xi_i, \rho \sigma}) (\tilde{\nu}, e_j) \right] \, d\tilde{\mu} = O(\lambda_i^{-1} \rho_i^{-1}).
\]

In conjunction with Lemma 40, Lemma 35, and (59), we conclude that
\[
\lambda_i^{-1} \int_{S_i} \left[ \sum_{j=1}^{3} \left[ (\tilde{D}^2_{\xi_i, e_j}) \nu (e_j, \tilde{\nu}) - (\tilde{D}^2_{\xi_i, \rho \sigma}) (e_j, e_j) \right] + \sum_{j=1}^{3} (\tilde{D}^2_{\xi_i, e_j}) \nu (e_j, \tilde{\nu}) - (\tilde{D}^2_{\xi_i, \rho \sigma}) (\tilde{\nu}, \tilde{\nu}) \right] d\tilde{\mu}
= \lambda_i^{-1} \int_{S_i} \tilde{g}(\xi_i, \tilde{\nu}) \, R \, d\tilde{\mu} + o(\lambda_i^{-1} \rho_i^{-2}).
\]

The assertion follows from these estimates. \(\square\)

For the proof of Lemma 26, recall that a tilde indicates that a geometric quantity is computed with respect to the Schwarzschild background metric with mass 2.
Lemma 26. As \( i \to \infty \), there holds
\[
\int_{\Sigma_i} g(\zeta, \hat{h}) \, d\mu = o(\lambda_i^{-1} \rho_i^{-2})
\]
and
\[
\int_{\Sigma_i} [\text{div}_{\Sigma_i} \xi_i - 2 g(D_\nu \xi_i, \nu)] \, d\mu = o(\lambda_i^{-1} \rho_i^{-2}).
\]

Proof. Using (58), Lemma 39, Lemma 42, Proposition 10, Lemma 22, and Lemma 35, we have
\[
\int_{\Sigma_i} g(\zeta, \hat{h}) \, d\mu = 2 \lambda_i^{-1} \int_{\Sigma_i} \tilde{g}(\zeta, \hat{h}) \, d\tilde{\mu} + o(\lambda_i^{-1} \rho_i^{-2}).
\]

Similarly, using also (59),
\[
\int_{\Sigma_i} [\text{div}_{\Sigma_i} \xi_i - 2 g(D_\nu \xi_i, \nu)] \, d\mu = 4 \lambda_i^{-2} \int_{\Sigma_i} \tilde{d}\text{div}_{\Sigma_i} \xi_i - 2 \tilde{g}(\tilde{D}_\nu \xi_i, \tilde{\nu}) \, d\tilde{\mu} + o(\lambda_i^{-1} \rho_i^{-2}).
\]

By Lemma 39,
\[
\tilde{g}(\tilde{\zeta}, \hat{h}) = 2 (1 + |x|^{-1})^{-1} |x|^{-3} \tilde{g}(x, \xi) \tilde{g}|_{\Sigma_i}.
\]

Consequently, \( \tilde{g}(\tilde{\zeta}, \hat{h}) = 0 \). Using Lemma 39 again, we have
\[
\text{div}_{\Sigma_i} \xi_i = 2 \tilde{g}(\tilde{D}_\nu \xi_i, \tilde{\nu}) = 4 (1 + |x|^{-1})^{-1} \tilde{g}(x, \xi_i).
\]

The assertion follows. \( \Box \)

Lemma 27. There holds
\[
\kappa(\Sigma_i) \int_{\Sigma_i} g(\xi_i, \nu) \, d\mu = o(\lambda_i^{-1} \rho_i^{-2}).
\]

Proof. Using Lemma 22, Lemma 42, and (58), we have
\[
\int_{\Sigma_i} g(\xi_i, \nu) \, d\mu = \text{proj}_{\Lambda_{\alpha}(S_i)} H(\Sigma_i) \int_{\Sigma_i} \tilde{g}(\xi, \tilde{\nu}) \, d\tilde{\mu} + O(\lambda_i^{-1}) \int_{\Sigma_i} |x|^{-1} \, d\tilde{\mu}.
\]

Note that
\[
\int_{\Sigma_i} |x|^{-1} \, d\tilde{\mu} = O(\lambda_i^2 \rho_i^{-1}).
\]

By the divergence theorem,
\[
\int_{\Sigma_i} \tilde{g}(\xi_i, \tilde{\nu}) \, d\tilde{\mu} = 0.
\]

The assertion follows in conjunction with Lemma 22. \( \Box \)

6. Proof of Theorem 3

Suppose, for a contradiction, that there exists a sequence \( \{\Sigma_i\}_{i=1}^\infty \) of area-constrained Willmore spheres \( \Sigma_i \subset M \) such that (59) and (60) hold. Assembling Lemma 24, Lemma 25, Lemma 26, and Lemma 27, we have
\[
0 = -\int_{\Sigma_i} g(\xi_i, \nu) \left[ \Delta H + |\hat{h}|^2 + \text{Ric}(\nu, \nu) + \kappa(\Sigma_i) \right] \, d\mu
\]
\[
= 8 \pi \lambda_i^{-1} \rho_i^{-2} - \lambda_i^{-1} \int_{S_i} \tilde{g}(\xi, \tilde{\nu}) R \, d\tilde{\mu} + o(\lambda_i^{-1} \rho_i^{-2}).
\]
Using $R \geq -o(|x|^{-4})$ and Lemma 35, we find that
\[- \int_{S_i} \tilde{g}(\xi, \tilde{\nu}) R \, d\tilde{\mu} \geq - \int_{\{x \in S_i : \tilde{g}(\xi, \tilde{\nu}) \geq 0\}} \tilde{g}(\xi, \tilde{\nu}) R \, d\tilde{\mu} - o(\rho_i^{-2}).\]
Moreover, using (58) and (59),
\[\int_{\{x \in S_i : \tilde{g}(\xi, \tilde{\nu}) \geq 0\}} \tilde{g}(\xi, \tilde{\nu}) R \, d\tilde{\mu} = O(\lambda_i^{-2}) = o(\rho_i^{-2}).\]
These estimates are incompatible with (63).

A. Integral curvature estimates

In [19], E. Kuwert and R. Schätzle have established integral curvature estimates for Euclidean Willmore surfaces whose traceless second fundamental form is small in $L^2$. In this section, we adapt their method to establish integral curvature estimates for large area-constrained Willmore spheres in Riemannian three-manifolds which are asymptotic to Schwarzschild whose curvature is small in $L^2$.

In short, we use integration by parts, the area-constrained Willmore equation, and the $L^2$-estimate (64) to prove local $W^{2,2}$-bounds for the second fundamental form $h$. In conjunction with the Sobolev inequality, we obtain an $L^\infty$-estimate for $h$. Compared to [19], additional curvature terms owing to the non-flat background need to be addressed.

We assume that $g$ is a Riemannian metric on $\mathbb{R}^3$ such that, as $x \to \infty$,
\[g = (1 + |x|^{-1})^4 \tilde{g} + \sigma \quad \text{where} \quad \partial_J \sigma = O(|x|^{-2-|J|})\]
for every multi-index $J$ with $|J| \leq 4$.

Let $\{\Sigma_i\}_{i=1}^\infty$ be a sequence of spheres $\Sigma_i \subset \mathbb{R}^3$ which satisfy the area-constrained Willmore equation (3) with
\[\lim_{i \to \infty} \rho(\Sigma_i) = \infty, \quad \rho(\Sigma_i) = O(\lambda(\Sigma_i)),\]
and
\[\int_{\Sigma_i} |h - \lambda(\Sigma_i)^{-1} g_{\Sigma_i}|^2 d\mu = o(1).\]
Here, $\lambda(\Sigma_i)$ and $\rho(\Sigma_i)$ are the area radius and inner radius of $\Sigma_i$ defined in (2). We abbreviate $\rho_i = \rho(\Sigma_i)$ and $\lambda_i = \lambda(\Sigma_i)$.

Let $X, Y, Z$ be vector fields tangent to $\Sigma$ and $Rm$ the Riemann curvature tensor of $(M, g)$. We recall the Gauss-Codazzi equation
\[(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) + Rm(X, Y, \nu, Z)\]
and its trace
\[\operatorname{div}_{\Sigma_i} h = \nabla H + \nu \cdot Rc.\]
For the statement of the following Simons-type identities, note that the contraction of the divergence is with respect to the first entry. Given a covariant tensor $T$, we follow the convention that

$$ X_j (\nabla T) = (\nabla_X T). $$

**Lemma 28** ([18, Lemma 3.2]). There holds, on $\Sigma_i$,

\begin{align}
\Delta \hat{h} &= \nabla^2 \hat{h} + \frac{1}{2} H^2 \hat{h} + \hat{h} \ast \hat{h} \ast \hat{h} + O(|x|^{-3} |h|) + O(|x|^{-4}), \\
\text{div}_{\Sigma_i} \nabla^2 H &= \nabla \Delta H + \frac{1}{4} H^2 \nabla H + \hat{h} \ast \nabla \hat{h} + O(|x|^{-3} |\nabla H|), \\
\text{div}_{\Sigma_i} \nabla^2 \hat{h} &= \nabla \Delta \hat{h} + h \ast \hat{h} \ast \hat{h} \\
&\quad + O(|x|^{-3} |\hat{h}| |h|) + O(|x|^{-4} |\hat{h}|) + O(|x|^{-3} |\nabla \hat{h}|).
\end{align}

Let $\psi \in C^\infty(\mathbb{R})$ with

- $0 \leq \psi \leq 1$,
- $\psi(1) = 1$,
- $\psi(s) = 0$ if $s < 3/4$ or $s > 5/4$,
- $|\psi'| \leq \frac{9}{2}$.

We fix $x \in \mathbb{R}^3$ with $x \neq 0$ and define $\eta \in C^\infty(\mathbb{R}^3)$ by

$$ \eta(z) = \psi(|z| |x|^{-1}). $$

Note that

$$ |D\eta| \leq 5 |x|^{-1}. $$

Moreover, by Lemma 42, (72), and Lemma 6, we have, uniformly for all $x \in \Sigma_i$,

$$ |\Sigma_i \cap \text{spt}(\eta)| = O(|x|^2). $$

The following lemma is an adaptation of [19, Lemma 2.2].

**Lemma 29.** There holds, uniformly for all $x \in \Sigma_i$,

$$ \int_{\Sigma_i} \eta^2 |\nabla \hat{h}|^2 \, d\mu \leq \frac{1}{2} \int_{\Sigma_i} \eta^2 H^2 |\hat{h}|^2 \, d\mu + O(1) \int_{\Sigma_i} \eta^2 |\hat{h}|^4 \, d\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} |h - \lambda_i^{-1} g|_{\Sigma_i}^2 \, d\mu \\
&\quad + O(|x|^{-4}) + O(\kappa(\Sigma_i)^2 |x|^2). $$

**Proof.** We multiply (66) by $\eta^2 \hat{h}$ and integrate by parts. Using (65), (70), and (71), we obtain

$$ \int_{\Sigma_i} \eta^2 \left[ |\nabla \hat{h}|^2 + \frac{1}{2} H^2 |\hat{h}|^2 \right] \, d\mu $$

$$ = \frac{1}{2} \int_{\Sigma_i} \eta^2 |\nabla H|^2 \, d\mu + O(|x|^{-1}) \int_{\Sigma_i} \eta |\hat{h}| |\nabla \hat{h}| \, d\mu + O(1) \int_{\Sigma_i} \eta^2 |\hat{h}|^4 \, d\mu \\
&\quad + O(|x|^{-4}) \int_{\Sigma_i} \eta |\hat{h}| \, d\mu + O(|x|^{-3}) \int_{\Sigma_i} \eta^2 |\hat{h}| \, d\mu \\
&\quad + O(|x|^{-3}) \int_{\Sigma_i} \eta^2 |\nabla \hat{h}| \, d\mu + O(|x|^{-6}) \int_{\Sigma_i} \eta^2 \, d\mu.$$
Note that
\[ O(|x|^{-1}) \int_{\Sigma_i} \eta |\hat{h}| |\nabla \hat{h}| \, d\mu + O(|x|^{-4}) \int_{\Sigma_i} \eta |\hat{h}| \, d\mu + O(|x|^{-3}) \int_{\Sigma_i} \eta^2 |\nabla \hat{h}| \, d\mu \]
\[ \leq \frac{1}{2} \int_{\Sigma_i} \eta^2 |\nabla \hat{h}|^2 \, d\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 \, d\mu + O(|x|^{-6}) \int_{\Sigma_i} \eta^2 \, d\mu. \]
Likewise, using Lemma 6,
\[ O(|x|^{-3}) \int_{\Sigma_i} \eta^2 |\hat{h}| \, d\mu \leq O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 \, d\mu + O(|x|^{-4}). \]
Integrating by parts we have
\[ \int_{\Sigma_i} \eta^2 |\nabla H|^2 \, d\mu = - \int_{\Sigma_i} \eta^2 (H - 2 \lambda_i^{-1}) \Delta H \, d\mu - 2 \int_{\Sigma_i} \eta (H - 2 \lambda_i^{-1}) g(\nabla \eta, \nabla H) \, d\mu. \]
Using (70), we estimate
\[ -2 \int_{\Sigma_i} \eta (H - 2 \lambda_i^{-1}) g(\nabla \eta, \nabla H) \, d\mu \leq \frac{1}{2} \int_{\Sigma_i} \eta^2 |\nabla H|^2 \, d\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} (H - 2 \lambda_i^{-1})^2 \, d\mu. \]
Using (3), we have
\[ - \int_{\Sigma_i} \eta^2 (H - 2 \lambda_i^{-1}) \Delta H \, d\mu \]
\[ = \int_{\Sigma_i} \eta^2 (H - 2 \lambda_i^{-1}) H |\hat{h}|^2 \, d\mu + O(|x|^{-3} + |\kappa(\Sigma_i)|) \int_{\Sigma_i} \eta^2 |H - 2 \lambda_i^{-1}| |H| \, d\mu. \]
Note that
\[ \int_{\Sigma_i} \eta^2 (H - 2 \lambda_i^{-1}) H |\hat{h}|^2 \, d\mu = \int_{\Sigma_i} \eta^2 H^2 |\hat{h}|^2 \, d\mu + \int_{\Sigma_i} \eta^2 [\lambda_i^{-1} (H - 2 \lambda_i^{-1}) + 2 \lambda_i^{-2}] |\hat{h}|^2 \, d\mu \]
and
\[ \int_{\Sigma_i} \eta^2 [\lambda_i^{-1} (H - 2 \lambda_i^{-1}) + 2 \lambda_i^{-2}] |\hat{h}|^2 \, d\mu \]
\[ \leq O(1) \int_{\Sigma_i} \eta^2 |\hat{h}|^4 \, d\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 \, d\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} (H - 2 \lambda_i^{-1})^2 \, d\mu. \]
Moreover, using (64),
\[ O(|x|^{-3} + |\kappa(\Sigma_i)|) \int_{\Sigma_i} \eta^2 |H - 2 \lambda_i^{-1}| |H| \, d\mu \]
\[ \leq O(|x|^{-4}) + O(\kappa(\Sigma_i)^2 |x|^2) + O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} (H - 2 \lambda_i^{-1})^2 \, d\mu. \]
The assertion follows from these estimates and (72).

\[ \square \]

**Corollary 30.** There holds, uniformly for all \( x \in \Sigma_i \),
\[ \int_{\Sigma_i} \eta^2 |\nabla H|^2 \, d\mu \leq 4 \int_{\Sigma_i} \eta^2 H^2 |\hat{h}|^2 \, d\mu + O(1) \int_{\Sigma_i} \eta^2 |\hat{h}|^4 \, d\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \text{spt}(\eta)} |h - \lambda_i^{-1} g|_{\Sigma_i}^2 \, d\mu \]
\[ + O(|x|^{-4}) + O(\kappa(\Sigma_i)^2 |x|^2). \]

**Proof.** This follows from Lemma 29 and (65).

\[ \square \]
The next two lemmas follow [19, Lemma 2.3].

**Lemma 31.** There holds, uniformly for all $x \in \Sigma_i$,

\[
\int_{\Sigma_i} \eta^4 H^2 |\nabla \hat{h}|^2 \, d\mu + \int_{\Sigma_i} \eta^4 H^4 |\hat{h}|^2 \, d\mu \\
\leq 2 \int_{\Sigma_i} \eta^4 H^2 |\nabla H|^2 \, d\mu + O(1) \int_{\Sigma_i} \eta^4 H^4 |\hat{h}|^4 \, d\mu \\
+ O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 |\nabla \hat{h}|^2 \, d\mu + O(|x|^{-4}) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 \, d\mu + O(|x|^{-6}).
\]

**Proof.** We multiply (66) by $\eta^4 H^2 \hat{h}$ and integrate by parts. Using (65) and (70), we have

\[
\int_{\Sigma_i} \eta^4 H^2 |\nabla \hat{h}|^2 \, d\mu + \frac{1}{2} \int_{\Sigma} \eta^4 H^4 |\hat{h}|^2 \, d\mu \\
\leq \frac{1}{2} \int_{\Sigma} \eta^4 H^2 |\nabla H|^2 \, d\mu + O(1) \int_{\Sigma_i} \eta^4 H^4 |\hat{h}|^4 \, d\mu \\
+ O(|x|^{-1}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| |\nabla \hat{h}| \, d\mu + O(|x|^{-1}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| |\nabla H| \, d\mu \\
+ O(1) \int_{\Sigma_i} \eta^4 |H| |\hat{h}| |\nabla H| |\nabla \hat{h}| \, d\mu + O(1) \int_{\Sigma_i} \eta^4 |H| |\hat{h}| |\nabla H|^2 \, d\mu \\
+ O(|x|^{-3}) \int_{\Sigma_i} \eta^4 H^2 |\nabla H| \, d\mu + O(|x|^{-3}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| |h| \, d\mu \\
+ O(|x|^{-4}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| \, d\mu.
\]

Note that

\[
O(|x|^{-1}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| |\nabla \hat{h}| \, d\mu + O(|x|^{-1}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| |\nabla H| \, d\mu \\
\leq \frac{1}{16} \int_{\Sigma_i} \eta^4 H^2 |\nabla H|^2 \, d\mu + \frac{1}{16} \int_{\Sigma_i} \eta^4 H^2 |\nabla \hat{h}|^2 \, d\mu \\
+ \frac{1}{16} \int_{\Sigma_i} \eta^4 H^4 |\hat{h}|^2 \, d\mu + O(|x|^{-4}) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 \, d\mu.
\]

Likewise,

\[
O(1) \int_{\Sigma_i} \eta^4 |H| |\hat{h}| |\nabla H| |\nabla \hat{h}| \, d\mu \leq \frac{1}{32} \int_{\Sigma_i} \eta^4 H^2 |\nabla H|^2 \, d\mu + O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 |\nabla \hat{h}|^2 \, d\mu.
\]

Moreover, using (65),

\[
O(1) \int_{\Sigma_i} \eta^4 |H| |\hat{h}| |\nabla H|^2 \, d\mu \\
\leq \frac{1}{32} \int_{\Sigma_i} \eta^4 H^2 |\nabla H|^2 \, d\mu + O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 |\nabla \hat{h}|^2 \, d\mu + O(|x|^{-6}) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 \, d\mu.
\]

Finally, using (64),

\[
O(|x|^{-3}) \int_{\Sigma_i} \eta^4 H^2 |\nabla H| \, d\mu + O(|x|^{-3}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| |h| \, d\mu + O(|x|^{-4}) \int_{\Sigma_i} \eta^4 H^2 |\hat{h}| \, d\mu
\]
The assertion follows from these estimates, Corollary 30, Lemma 31, and (65).

Note that $Z_{\text{Lemma 33}}$.

Proof. We multiply (67) by $\eta^4 \nabla H$ and integrate by parts. Using (65) and (70), we obtain

$$\int_{\Sigma_i} \eta^4 \left[ |\nabla^2 H|^2 + \frac{1}{4} H^2 |\nabla H|^2 \right] \, d\mu 
\leq \int_{\Sigma_i} \eta^4 (\Delta H)^2 \, d\mu + 40 |x|^{-1} \int_{\Sigma_i} \eta^3 |\nabla H| |\nabla^2 H| \, d\mu + O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 |\nabla \hat{h}|^2 \, d\mu 
+ O(|x|^{-6}) \int_{\Sigma_i} \eta^4 |\nabla H|^2 \, d\mu.$$

Using (3) and (64), we conclude that

$$\int_{\Sigma_i} \eta^4 \left[ |\nabla^2 H|^2 + \frac{1}{4} H^2 |\nabla H|^2 \right] \, d\mu 
\leq \int_{\Sigma} \eta^4 H^2 |\hat{h}|^4 \, d\mu + \frac{1}{2} \int_{\Sigma_i} \eta^4 |\nabla^2 H|^2 \, d\mu + 10^3 |x|^{-2} \int_{\Sigma_i} \eta^4 |\nabla H|^2 \, d\mu 
+ O(|x|^{-6}) + O(\kappa(\Sigma_i)^2) + O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 |\nabla \hat{h}|^2 \, d\mu + O(|x|^{-6}) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 \, d\mu.$$

Note that

$$\int_{\Sigma} \eta^4 H^4 |\hat{h}| \, d\mu \leq \frac{1}{32} \int_{\Sigma} \eta^4 H^4 |\hat{h}|^2 \, d\mu + O(1) \int_{\Sigma} \eta^4 |\hat{h}|^6 \, d\mu$$

and

$$\int_{\Sigma_i} \eta^2 H^2 |\hat{h}|^2 \, d\mu \leq \frac{1}{128} 10^{-3} |x|^2 \int_{\Sigma_i} \eta^4 H^4 |\hat{h}|^2 \, d\mu + O(|x|^{-2}) \int_{\Sigma_i(\text{spt(}\eta))} |\hat{h}|^2 \, d\mu.$$

The assertion follows from these estimates, Corollary 30, Lemma 31, and (65).

The following lemma is an adaptation of [19, Proposition 2.4].

Lemma 33. There holds, uniformly for all $x \in \Sigma_i$,

$$\int_{\Sigma_i} \eta^4 \left[ |\nabla^2 h|^2 + |h|^2 |\nabla h|^2 + |h|^4 |\hat{h}|^2 \right] \, d\mu 
\leq O(|x|^{-4}) \int_{\Sigma_i(\text{spt(}\eta))} |h - \lambda_i^{-1} g| \, d\mu + O(|x|^{-6}) + O(\kappa(\Sigma_i)^2).$$
Proof. We multiply (68) by $\eta^4 \nabla^2 \hat{h}$ and integrate by parts. Using Lemma 6 and (70), we find that

$$\int_{\Sigma_i} \eta^4 |\nabla^2 \hat{h}|^2 d\mu \leq \int_{\Sigma_i} \eta^4 |\Delta \hat{h}|^2 d\mu + \frac{1}{4} \int_{\Sigma_i} \eta^4 |\nabla^2 \hat{h}|^2 d\mu + O(|x|^{-2}) \int_{\Sigma_i} |\nabla \hat{h}|^2 d\mu + O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 |\nabla \hat{h}|^2 d\mu + O(|x|^{-6}).$$

Using (66) and Lemma 6, we obtain

$$\int_{\Sigma_i} \eta^4 |\Delta \hat{h}|^2 d\mu \leq \int_{\Sigma_i} \eta^4 |\nabla^2 H|^2 d\mu + O(1) \int_{\Sigma_i} \eta^4 H^4 |\hat{h}|^2 d\mu + O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^4 d\mu + O(|x|^{-6}) + O(|x|^{-8}) \int_{\Sigma_i} \eta^2 d\mu.$$

Assembling these estimates and using Lemma 32, Lemma 29, and (71), we have

$$\int_{\Sigma_i} \eta^4 \left[ |\nabla^2 \hat{h}|^2 + |\hat{h}|^2 |\nabla \hat{h}|^2 + |\hat{h}|^4 |\hat{h}|^2 \right] d\mu \leq O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^2 |\nabla \hat{h}|^2 d\mu + O(1) \int_{\Sigma_i} \eta^4 |\hat{h}|^6 d\mu + O(|x|^{-4}) \int_{\Sigma_i \cap \text{spt}(\eta)} |h - \lambda^{-1}_i g|_{\Sigma_i}^2 d\mu + O(\kappa(\Sigma_i)^2) + O(|x|^{-6}).$$

The argument now concludes as in [19, Lemma 2.5 and Proposition 2.6], using (64) and the Michael-Simon Sobolev inequality in the form [16, Proposition 5.4]. \qed

**Proposition 34.** There holds, uniformly for all $x \in \Sigma_i$,

$$|h - \lambda^{-1}_i g|_{\Sigma_i}^4 = O(|x|^{-4}) \left( \int_{\Sigma_i \cap B_{|x|/4}(x)} |h - \lambda^{-1}_i g|_{\Sigma_i}^2 d\mu \right)^2$$

$$+ O(|x|^{-8}) + O(\kappa(\Sigma_i)^2) \int_{\Sigma_i \cap B_{|x|/4}(x)} |h - \lambda^{-1}_i g|_{\Sigma_i}^2 d\mu.$$

**Proof.** Repeating the argument that led to [19, Lemma 2.8] using [16, Proposition 5.4], we find that

$$|\eta^2 \hat{h}|_{L^\infty(\Sigma_i)} \leq O(1) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 d\mu \left[ \int_{\Sigma_i} \eta^8 \left[ |\nabla^2 \hat{h}|^2 + H^4 |\hat{h}|^2 \right] d\mu + |x|^{-4} \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 d\mu \right]$$

and

$$|\eta^2 (H - 2 \lambda^{-1}_i)|_{L^\infty(\Sigma_i)}^4 \leq O(1) \int_{\Sigma_i \cap \text{spt}(\eta)} |\hat{h}|^2 d\mu \left[ \int_{\Sigma_i} \eta^8 \left[ |\nabla^2 H|^2 + H^4 (H - 2 \lambda^{-1}_i)^2 \right] d\mu + |x|^{-4} \int_{\Sigma_i \cap \text{spt}(\eta)} (H - 2 \lambda^{-1}_i)^2 d\mu \right].$$

By (64),

$$\int_{\Sigma_i} \eta^8 H^4 (H - 2 \lambda^{-1}_i)^2 d\mu \leq o(1) |\eta^2 (H - 2 \lambda^{-1}_i)|_{L^\infty(\Sigma_i)}^4 + O(\lambda^{-4}_i) \int_{\Sigma_i \cap \text{spt}(\eta)} (H - 2 \lambda^{-1}_i)^2 d\mu.$$  

By (69),

$$\text{spt}(\eta) \subset \cap B_{|x|/4}(x).$$

The assertion follows from these estimates and Lemma 33. \qed
B. Surfaces with Bounded Euclidean Willmore Energy

In this section, we recall estimates for closed surfaces in $\mathbb{R}^3$ in terms of a bound on their Euclidean Willmore energy.

**Lemma 35** ([16, Lemma 5.2]). For every $q > 2$ there is a constant $c(q) > 0$ such that for every closed surface $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$,

$$\rho(\Sigma)^{q-2} \int_{\Sigma} |x|^{-q} \, d\mu \leq c(q) \int_{\Sigma} \tilde{H}^2 \, d\tilde{\mu}.$$  

The estimates in the following lemma are stated in [23] except for the explicit constants. We revisit the proof in [23] and compute explicit constants below.

**Lemma 36** ([23, Lemma 1.1 and (1.3)]). Let $\Sigma \subset \mathbb{R}^3$ be a closed surface. Given $x \in \Sigma$ and $r > 0$, we have

$$r^{-2} |\Sigma \cap B_r(x)| \leq \frac{3 + 2\sqrt{2}}{16} \int_{\Sigma} \tilde{H}^2 \, d\tilde{\mu}.$$  

Moreover

$$\sup \{ |y - z|^2 : y, z \in \Sigma \} \leq \frac{172}{25} \frac{3^4}{\pi^2} |\Sigma| g \int_{\Sigma} \tilde{H}^2 \, d\tilde{\mu}.$$  

**Proof.** Let $x \in \Sigma$. Recall from [23, (1.2)] that, for all $0 < r \leq t$,

$$r^{-2} |\Sigma \cap B_r(x)|_g$$

$$\leq t^{-2} |\Sigma \cap B_t(x)|_g + \frac{1}{16} \int_{\Sigma \cap B_t(x)} \tilde{H}^2 \, d\tilde{\mu}$$

$$+ \frac{1}{2} t^{-2} \int_{\Sigma \cap B_t(x)} \tilde{H} \, g(z - x, \tilde{\nu}) \, d\tilde{\mu}(z) - \frac{1}{2} t^{-2} \int_{\Sigma \cap B_t(x)} \tilde{H} \, g(z - x, \nu) \, d\tilde{\mu}(z).$$

To show (73), we obtain, using the estimates

$$\left| \int_{\Sigma \cap B_t(x)} \tilde{H} \, g(z - x, \tilde{\nu}) \, d\tilde{\mu}(z) \right| \leq \frac{3}{8} t^2 \int_{\Sigma \cap B_t(x)} \tilde{H}^2 \, d\tilde{\mu} + \frac{2}{3} |\Sigma \cap B_t(x)|$$

and

$$\left| \int_{\Sigma \cap B_t(x)} \tilde{H} \, g(z - x, \nu) \, d\tilde{\mu}(z) \right| \leq \frac{1}{4} t^2 \int_{\Sigma \cap B_t(x)} \tilde{H}^2 \, d\tilde{\mu} + |\Sigma \cap B_t(x)|,$$

that

$$r^{-2} |\Sigma \cap B_r(x)|_g \leq \frac{9}{4} \left( t^{-2} |\Sigma \cap B_t(x)|_g + \frac{1}{4} \int_{\Sigma \cap B_t(x)} \tilde{H}^2 \, d\tilde{\mu} \right).$$

Revisiting the proof of [23, Lemma 1.1] and using [23, (1.3)] with the explicit constant $C = \frac{9}{4}$ computed in (75), we obtain (73).

To obtain (72), we let $t \to \infty$ in (74) and estimate

$$r^{-2} |\Sigma \cap B_r(x)|_g \leq \frac{1}{16} \int_{\Sigma} \tilde{H}^2 \, d\tilde{\mu} - \frac{1}{2} r^{-2} \int_{\Sigma \cap B_r(x)} \tilde{H} \, g(z - x, \tilde{\nu}) \, d\tilde{\mu}(z)$$

$$\leq \frac{1}{16} \frac{\sqrt{2}}{\sqrt{2} - 1} \int_{\Sigma} \tilde{H}^2 + (\sqrt{2} - 1) r^{-2} |\Sigma \cap B_r(x)|_g.$$
C. Geometric identities on round spheres

**Lemma 37.** Let $\xi \in \mathbb{R}^3$. The following hold if $|\xi| < 1$.

- $\int_{S_1(\xi)} |x|^{-1} \, d\bar{\mu} = 4 \pi$
- $\int_{S_1(\xi)} |x|^{-3} \, d\bar{\mu} = 4 \pi (1 - |\xi|^2)^{-1}$
- $\int_{S_1(\xi)} |x|^{-5} \, d\bar{\mu} = \frac{4 \pi}{3} (3 + |\xi|^2) (1 - |\xi|^2)^{-3}$

The following hold if $|\xi| > 1$.

- $\int_{S_1(\xi)} |x|^{-1} \, d\bar{\mu} = 4 \pi |\xi|^{-1}$
- $\int_{S_1(\xi)} |x|^{-3} \, d\bar{\mu} = 4 \pi |\xi|^{-1} (|\xi|^2 - 1)^{-1}$
- $\int_{S_1(\xi)} |x|^{-5} \, d\bar{\mu} = \frac{4 \pi}{3} |\xi|^{-1} (1 + 3 |\xi|^2) (|\xi|^2 - 1)^{-3}$

The following hold if $|\xi| \neq 0, 1$.

- $\int_{S_1(\xi)} |x|^{-2} \, d\bar{\mu} = 2 \pi |\xi|^{-1} \log \frac{1 + |\xi|}{|1 - |\xi||}$
- $\int_{S_1(\xi)} |x|^{-4} \, d\bar{\mu} = 4 \pi (1 - |\xi|^2)^{-2}$
- $\int_{S_1(\xi)} |x|^{-6} \, d\bar{\mu} = 4 \pi (1 + |\xi|^2) (1 - |\xi|^2)^{-4}$

**Lemma 38.** Let $\xi \in \mathbb{R}^3$. The following identities hold on $S_1(\xi)$.

- $2 \bar{g}(x, \bar{\nu}) = |x|^2 + 1 - |\xi|^2$
- $2 \bar{g}(x, \xi) = |x|^2 + |\xi|^2 - 1$
- $2 \bar{g}(\xi, \bar{\nu}) = |x|^2 - 1 - |\xi|^2$

D. Geometric expansions for perturbations of the Euclidean metric

In this section, we collect some expansions that relate geometric quantities computed with respect to different background metrics.

We assume that $g$ is a Riemannian metric on $\mathbb{R}^3$ such that, as $x \to \infty$,

$$g = (1 + |x|^{-1})^4 \bar{g} + \sigma$$

where $\sigma = O(|x|^{-2-|J|})$

for every multi-index $J$ with $|J| \leq 4$. We denote by $\bar{g} = (1 + |x|^{-1})^4 \bar{g}$ the Schwarzschild metric of mass $m = 2$. We use a bar for geometric quantities pertaining to $\bar{g}$ and a tilde for quantities pertaining to $\bar{g}$.

Recall that $e_1, e_2, e_3$ denotes the standard basis of $\mathbb{R}^3$. 
Lemma 39. Let $\xi \in \mathbb{R}^3$ and $i, j \in \{1, 2, 3\}$. There holds
\[\tilde{D}_{e_i}\xi = 2 (1 + |x|^{-1})^{-1} |x|^{-3} (\tilde{g}(\xi, e_i) x - \tilde{g}(\xi, x) e_i)\xi .\]
Moreover, as $x \to \infty$,
\[D_{e_i}\xi - \tilde{D}_{e_i}\xi = O(|x|^{-3}),\]
\[D_{e_i,e_j}\xi - \tilde{D}_{e_i,e_j}\xi = \frac{1}{2} \sum_{k=1}^{3} \left[ (\tilde{D}_{e_i,e_j}^2 \sigma)(e_k, e_j) + (\tilde{D}_{e_k,e_j}^2 \sigma)(e_i, e_j) - (\tilde{D}_{e_i,e_k}^2 \sigma)(e_j, e_j) \right] e_k + O(|x|^{-5}).\]

Lemma 40 (Lemma 37). There holds
\[\tilde{\text{Ric}}(e_i, e_j) = 2 (1 + |x|^{-1})^{-2} |x|^{-3} [\tilde{g}(e_i, e_j) - 3 |x|^{-2} \tilde{g}(e_i, x) \tilde{g}(e_j, x)].\]
Moreover, as $x \to \infty$,
\[\text{Ric}(e_i, e_j) - \tilde{\text{Ric}}(e_i, e_j) = \frac{1}{2} \sum_{k=1}^{3} \left[ (\tilde{D}_{e_k,e_i}^2 \sigma)(e_k, e_j) + (\tilde{D}_{e_k,e_j}^2 \sigma)(e_i, e_j) - (\tilde{D}_{e_i,e_k}^2 \sigma)(e_j, e_j) \right] e_k + O(|x|^{-5})\]
and
\[R = \sum_{i,j=1}^{3} \left[ (\tilde{D}_{e_i,e_j}^2 \sigma)(e_i, e_j) - (\tilde{D}_{e_i,e_i}^2 \sigma)(e_j, e_j) \right] + O(|x|^{-5}).\]

Lemma 41. Let $\xi \in \mathbb{R}^3$. There holds, as $x \to \infty$,
\[\tilde{g}(\tilde{D}_{e_1,e_1}^2 \xi + \tilde{D}_{e_2,e_2}^2 \xi, e_3) = \text{Ric}(\xi, e_3) + 4 |x|^{-4} \tilde{g}(\xi, e_3) - 8 |x|^{-6} \tilde{g}(\xi, x) \tilde{g}(e_3, x) + O(|x|^{-5}).\]
Proof. This follows from Lemma 39 and Lemma 40. \hfill \Box

Lemma 42. Let $\{\Sigma_i\}_{i=1}^{\infty}$ be a sequence of surfaces $\Sigma_i \subset \mathbb{R}^3$ such that $\lim_{i \to \infty} \rho(\Sigma_i) = \infty$. The following expansions hold.
\begin{itemize}
  \item $\tilde{\nu} = (1 + |x|^{-1})^{-2} \tilde{\nu}$
  \item $\tilde{H} = (1 + |x|^{-1})^{-2} \tilde{H} - 4 (1 + |x|^{-1})^{-3} |x|^{-3} \tilde{g}(x, \tilde{\nu})$
  \item $\tilde{\bar{h}} = (1 + |x|^{-1})^{-2} \tilde{\bar{h}}$
  \item $\tilde{\nabla} h = \nabla h + O(|x|^{-3}) + O(|x|^{-2} \tilde{h}) + O(|x|^{-1} |\nabla \tilde{h}|)$
  \item $d\tilde{\mu} = (1 + |x|^{-1})^4 d\tilde{\mu}$
  \item $\nu = \tilde{\nu} + O(|x|^{-2})$
  \item $H = \tilde{H} + O(|x|^{-3}) + O(|x|^{-2} \tilde{h})$
  \item $\tilde{\bar{h}} = \tilde{\bar{h}} + O(|x|^{-3}) + O(|x|^{-2} \tilde{\bar{h}})$
  \item $\nabla h = \nabla \tilde{h} + O(|x|^{-4}) + O(|x|^{-3} \tilde{h}) + O(|x|^{-2} |\nabla \tilde{h}|)$
  \item $d\mu = [1 + O(|x|^{-2})] d\tilde{\mu}$
\end{itemize}
Moreover, if \( \{u_i\}_{i=1}^{\infty} \) is a sequence of functions \( u_i \in C^\infty(\Sigma_i) \), then
\[
\begin{align*}
\tilde{\nabla} u_i &= (1 + |x|^{-1})^{-2} \tilde{\nabla} u_i, \\
\tilde{\Delta} u_i &= (1 + |x|^{-1})^{-4} \tilde{\Delta} u_i,
\end{align*}
\]
and
\[
\begin{align*}
\nabla u_i &= \tilde{\nabla} u_i + O(|x|^{-2} |\tilde{\nabla} u_i|), \\
\Delta u_i &= \tilde{\Delta} u_i + O(|x|^{-2} |\tilde{\nabla}^2 u_i|) + O(|x|^{-3} |\tilde{\nabla} u_i|) + O(|x|^{-2} |\tilde{h}||\tilde{\nabla} u_i|).
\end{align*}
\]

E. Geometric expansions for graphs over Euclidean spheres

In this section, we collect some geometric identities for graphs over Euclidean spheres.

Let \( \xi \in \mathbb{R}^3 \), \( \lambda > 0 \), and \( u \in C^\infty(S_\lambda(\lambda \xi)) \). Recall from (22) that, \( \Sigma_{\xi,\lambda}(u) \) denotes the Euclidean graph of \( u \) over \( S_\lambda(\lambda \xi) \).

**Lemma 43.** The following identities hold.

\[
\begin{align*}
\bar{g}|_{\Sigma_{\xi,\lambda}(u)} &= (1 + \lambda^{-1} u)^2 \bar{g}|_{S_\lambda(\lambda \xi)} + du \otimes du, \\
\bar{g}|_{\Sigma}^{-1} &= (1 + \lambda^{-1} u)^{-2} \left[(\bar{g}|_{S_\lambda(\lambda \xi)})^{-1} - ((1 + \lambda^{-1} u)^2 + |\tilde{\nabla} u|^2)^{-1} \tilde{\nabla} u \otimes \tilde{\nabla} u\right], \\
\bar{\nu}(\Sigma_{\xi,\lambda}(u)) &= ((1 + \lambda^{-1} u)^2 + |\tilde{\nabla} u|^2)^{-1/2} ((1 + \lambda^{-1} u) \bar{\nu}(S_\lambda(\lambda \xi)) - \tilde{\nabla} u), \\
\bar{h}(\Sigma_{\xi,\lambda}(u)) &= ((1 + \lambda^{-1} u)^2 + |\tilde{\nabla} u|^2)^{-1/2} \left((\lambda^{-1} + (1 + \lambda^{-1} u)^2) \bar{g}|_{S_\lambda(\lambda \xi)} + 2 \lambda^{-1} du \otimes du - (1 + \lambda^{-1} u) \tilde{\nabla}^2 u\right)
\end{align*}
\]

**Lemma 44.** Suppose that
\[
\lambda^{-1} |u| + |\tilde{\nabla} u| \leq 1. \tag{76}
\]
There holds, for all \( f \in C^\infty(S_\lambda(\lambda \xi)) \),
\[
\tilde{\Delta}_{\Sigma_{\xi,\lambda}(u)} f = (1 - 2 \lambda^{-1} u) \tilde{\Delta}_{S_\lambda(\lambda \xi)} f + O(|\tilde{\nabla} f| |\tilde{\nabla} u| (\lambda^{-2} |u| + \lambda^{-1} |\tilde{\nabla} u| + |\tilde{\nabla}^2 u|)) + O(|\tilde{\nabla}^2 f| (\lambda^{-2} u^2 + |\tilde{\nabla} u|^2)).
\]

F. The potential function

In this section, we collect some facts about the potential function of the spatial Schwarzschild manifold.

We assume that \( g \) is a Riemannian metric on \( \mathbb{R}^3 \). We denote by \( \bar{g} = (1+|x|^{-1})^4 \bar{g} \) the Schwarzschild metric of mass \( m = 2 \). We use a bar for geometric quantities pertaining to \( \bar{g} \) and a tilde for quantities pertaining to \( \bar{g} \).

Recall from [7, §2] that the potential function \( N : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R} \) of the spatial Schwarzschild manifold is given by
\[
\bar{N}(x) = (1 + |x|^{-1})^{-1} (1 - |x|^{-1}). \tag{77}
\]
Moreover, recall that \( \bar{N} \) satisfies the static metric equation
\[
\bar{D}^2 \bar{N} = \bar{N} \bar{R}c. \tag{78}
\]
Let $\Sigma \subset M$ be a closed, two-sided surface with outward normal $\nu$, mean curvature $H$ with respect to $\nu$, second fundamental form $h$, and non-positive Laplace-Beltrami operator $\Delta$ such that
\[
\Delta H + (|\hat{h}|^2 + \text{Rc}(\nu, \nu) + \kappa) H = 0
\]
for some $\kappa \in \mathbb{R}$.

**Lemma 45.** Let $F : \Sigma \to \mathbb{R}$ be given by $F = N^{-1} H(\Sigma)$ and suppose that $X \in \Gamma(T\Sigma)$. There holds
\[
\Delta F = - (|\hat{h}|^2 + \kappa + \text{Rc}(\nu, \nu) - \tilde{\text{Rc}}(\tilde{\nu}, \tilde{\nu}) + N^{-1} \tilde{\Delta} N - N^{-1} \tilde{\Delta} N
- \tilde{g}(\tilde{\nu}, \tilde{D}N \tilde{H}) F - 2 N^{-1} g(\nabla F, \nabla N).
\]

**Proof.** Note that
\[
\Delta F = - N^{-1} F \Delta N + 2 N^{-2} F g(\nabla N, \nabla N) - 2 N^{-2} g(\nabla H, \nabla N) + N^{-1} \Delta H.
\]
Using (78) and that $\tilde{R} = 0$, we obtain
\[
\tilde{\Delta} N = - N^{-1} \tilde{\text{Rc}}(\tilde{\nu}, \tilde{\nu}) - \tilde{g}(\tilde{\nu}, \tilde{D}N \tilde{H}) \tilde{H}.
\]
Clearly,
\[
\nabla H = N \nabla F + F \nabla N.
\]
The assertion follows from these identities. $\square$

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