Slipping and rolling on an inclined plane

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Abstract
In the first part of the paper, using a direct calculation two-dimensional motion of a particle sliding on an inclined plane is investigated for general values of friction coefficient (\(\mu\)). A parametric equation for the trajectory of the particle is also obtained. In the second part of the paper, the motion of a sphere on the inclined plane is studied. It is shown that the evolution equation for the contact point of a sliding sphere is similar to that of a point particle sliding on an inclined plane whose friction coefficient is \(\frac{2}{7}\mu\). If \(\mu > \frac{2}{7}\tan \theta\), for any arbitrary initial velocity and angular velocity, the sphere will roll on the inclined plane after some finite time. In other cases, it will slip on the inclined plane. In the case of rolling, the centre of the sphere moves on a parabola. Finally the velocity and angular velocity of the sphere are exactly computed.

1. Introduction

One of the standard problems in elementary mechanics is a particle sliding on an inclined plane. Usually it is assumed that the motion is one dimensional [1, 2]. However there are also some textbooks [3, 4] in which two-dimensional motion of a particle sliding on an inclined plane for a special choice of friction coefficient is considered. In a recent article [5], it is shown that there is an analogy between the curvilinear motion on an inclined plane and the pursuit problem.

In this paper, using a direct calculation two-dimensional motion of a particle on an inclined plane with angle \(\theta\) is studied. In section 2, sliding of a particle on an inclined plane is investigated for the general values of friction coefficient, \(\mu\). We obtain the particle’s velocity in terms of \(\varphi\), the slope of the particle’s trajectory. At first, general behaviour of the particle’s velocity at large times has been considered. It is shown that for \(\mu > \tan \theta\), after a finite time the particle’s velocity will vanish, and at this time \(\varphi\) is also equal to zero. For \(\mu \geq 1\), at large times the particle moves in a straight line. An exact calculation is also performed and a
relation between time, $t$, and $\phi$ is obtained. In section 3, the motion of a sphere on the inclined plane is studied. Depending on the friction coefficient and the initial velocity and angular velocity of the sphere, it may roll or slide on the inclined plane. The evolution equation for the contact point of a sliding sphere is obtained, and it is shown that it is similar to that of a point particle sliding on an inclined plane whose friction coefficient is $\frac{7}{2} \mu$. For $\mu > \frac{7}{2} \tan \theta$ depending on initial velocity, a sphere may initially slip but it will roll after some finite time. For $\mu \leq \frac{7}{2} \tan \theta$, it will slip forever. It is shown that when the sphere rolls on an inclined plane, generally the centre of the sphere moves on a parabola. Finally the velocity and angular velocity of the sphere are exactly computed.

2. Sliding a particle on an inclined plane

Let us consider an inclined plane with angle $\theta$. We study sliding a particle of mass $m$ on this plane. Here it is assumed that the friction coefficient, $\mu$, is constant and the problem is solved for general values of $\mu$. Newton’s equation of motion of the particle is

$$m \ddot{r} = mg \sin \theta \mathbf{i} - \mu mg \cos \theta \mathbf{e}_v,$$

where $\mathbf{e}_v$ is the unit vector in the direction of the particle’s velocity. See figure 1. Let us define $\lambda := \mu \cot \theta$, where $\phi$ is the slope of the particle’s trajectory at the point $(x, y)$. Then equation (1) is recast as

$$m \ddot{x} = mg \sin \theta (1 - \lambda \cos \phi), \quad m \ddot{y} = -\lambda mg \sin \theta \sin \phi. \quad (2)$$

Newton’s equation of motion of the particle tangential to its trajectory is

$$m \ddot{s} = mg \sin \theta (\cos \phi - \lambda), \quad (3)$$

where $s$ is the length parameter along the particle’s trajectory. Let us solve the problem for different values of $\lambda$. Note that $\lambda$ is a nonnegative parameter.

2.1. $\lambda = 0$

This case corresponds to a frictionless plane. The particle has a constant acceleration $g \sin \theta$, and the component of velocity along the $y$ axis, $v_y$, remains constant. Then the trajectory of the particle is generally a parabola.
2.2. $\lambda = 1$

This case has been partially studied in [3, 4]. Setting $\lambda = 1$, (2) and (3) are recast as
\begin{align*}
    m\ddot{x} &= mg \sin \theta (1 - \cos \phi), \\
    m\ddot{y} &= -mg \sin \theta \sin \phi, \tag{4} \\
    m\ddot{s} &= mg \sin \theta (\cos \phi - 1);
\end{align*}

it is seen that
\begin{equation}
    \ddot{s} + \ddot{x} = 0. \tag{5}
\end{equation}

Then $\dot{s} + \dot{x}$ should be constant. Assuming the initial velocity to be $v_0$ (see figure 1), then
\begin{equation}
    \dot{s} + \dot{x} = v_0 (1 + \cos \phi_0). \tag{6}
\end{equation}

Then using
\begin{align*}
    \dot{x} &= \dot{s} \cos \phi, \\
    \dot{y} &= \dot{s} \cos \phi,
\end{align*}

one arrives at
\begin{align*}
    \dot{s} &= \frac{v_0 (1 + \cos \phi_0)}{1 + \cos \phi}, \\
    \dot{x} &= \frac{v_0 \cos \phi (1 + \cos \phi_0)}{1 + \cos \phi}, \tag{8} \\
    \dot{y} &= \frac{v_0 \sin \phi (1 + \cos \phi_0)}{1 + \cos \phi}.
\end{align*}

As $\phi$ is the slope of particle’s trajectory at the point $(x, y)$, then $\tan \phi = \frac{\dot{y}}{\dot{x}}$, and
\begin{equation}
    \dot{\phi} \left(1 + \tan^2 \phi\right) = \frac{\ddot{y} \dot{x} - \ddot{x} \dot{y}}{\dot{x}^2}. \tag{9}
\end{equation}

Using (4) and (9), $\dot{\phi}$ can be obtained as
\begin{equation}
    \dot{\phi} = -\frac{g \sin \theta \sin \phi (1 + \cos \phi)}{v_0 (1 + \cos \phi_0)}. \tag{10}
\end{equation}

It is seen that for any $0 < \phi < \pi$, $\dot{\phi}$ is negative. So $\phi$ is a decreasing function of time. Let us consider its behaviour at large times,
\begin{equation}
    \dot{\phi} \approx -\frac{2g \sin \theta}{v_0 (1 + \cos \phi_0)} \phi, \quad \Rightarrow \quad \phi \propto e^{-\frac{2g \sin \theta \phi}{v_0 (1 + \cos \phi_0)}}. \tag{11}
\end{equation}

At large times $\phi$ goes to zero, the particle’s trajectory is a straight line and its velocity is
\begin{equation}
    \lim_{t \to \infty} \dot{s} = \lim_{t \to \infty} \dot{x} = \frac{v_0 (1 + \cos \phi_0)}{2}, \tag{12}
\end{equation}
\begin{equation}
    \lim_{t \to \infty} \dot{y} = 0.
\end{equation}

There is a maximum value for $y$,
\begin{align*}
    \int_0^{y_{\text{max}}} dy &= \frac{v_0^2 (1 + \cos \phi_0)^2}{g \sin \theta} \int_0^{\phi_0} \frac{d\phi}{(1 + \cos \phi)^2} \\
    &= \frac{v_0^2 (1 + \cos \phi_0)^2}{4g \sin \theta} \int_0^{\phi_0} d\phi \left(1 + \tan^2 \phi \right) + \tan^2 \phi \frac{\left(1 + \tan^2 \frac{\phi_0}{2}\right)}{2} \\
    y_{\text{max}} &= \frac{v_0^2 \sin \phi_0}{g \sin \theta} \left(1 + \frac{1}{3} \tan^2 \frac{\phi_0}{2}\right). \tag{13}
\end{align*}
2.3. \( \lambda \neq 1 \)

Combining (2) and (3) gives
\[
\ddot{x} + \lambda \ddot{s} = g \sin \theta (1 - \lambda^2)
\]
(14)

or
\[
\ddot{x} + \lambda \ddot{s} = g \sin \theta (1 - \lambda^2) t + v_0 (\lambda + \cos \varphi_0),
\]
(15)

where we have used the boundary condition. Similar to the preceding case, one may obtain
\[
\dot{s} = \frac{g \sin \theta (1 - \lambda^2) t + v_0 (\lambda + \cos \varphi_0)}{\lambda + \cos \varphi},
\]
\[
\dot{x} = \frac{(g \sin \theta (1 - \lambda^2) t + v_0 (\lambda + \cos \varphi_0)) \cos \varphi}{\lambda + \cos \varphi},
\]
\[
\dot{y} = \frac{(g \sin \theta (1 - \lambda^2) t + v_0 (\lambda + \cos \varphi_0)) \sin \varphi}{\lambda + \cos \varphi},
\]
(16)

from which we obtain
\[
\dot{\varphi} = -\frac{g \sin \theta (\lambda + \cos \varphi) \sin \varphi}{g \sin \theta (1 - \lambda^2) t + v_0 (\lambda + \cos \varphi_0)},
\]
(17)

2.3.1. \( 0 < \lambda < 1 \). For special choice of initial conditions, the particle’s velocity may become zero, but the friction is not large enough to keep it at rest. At large times \( \varphi \) goes to zero. Let us consider its behaviour at large times, or small \( \varphi \):
\[
\dot{\varphi} \approx -\varphi \frac{(1 - \lambda)}{t}, \quad \Rightarrow \quad \varphi \propto t^{-\frac{1}{1 - \lambda}}.
\]
(18)

In both cases (\( \lambda = 1 \) and \( \lambda < 1 \)) at large times \( \varphi \to 0 \). In the previous case, it approaches zero exponentially, and in the latter case in the form of a power law. However, in both cases at large times the particle’s trajectory is a straight line. So, at large times for \( \mu < \tan \theta \), the particle goes down the inclined plane with a constant acceleration:
\[
\lim_{t \to \infty} \dot{s} \sim g \sin \theta (1 - \lambda) t,
\]
\[
\lim_{t \to \infty} \dot{x} \sim g \sin \theta (1 - \lambda) t,
\]
\[
\lim_{t \to \infty} \dot{y} = 0.
\]
(19)

2.3.2. \( \lambda > 1 \). In this case, the friction coefficient is larger than in previous cases, and the particle will finally be at rest. It is seen from (16) that \( \dot{s} \) will be zero at the time \( T \):
\[
T = \frac{v_0 (\lambda + \cos \varphi_0)}{g \sin \theta (\lambda^2 - 1)}.
\]
(20)

When the particle’s velocity vanishes because of friction, it will remain at rest. At the time \( t = T - \epsilon \),
\[
\dot{\varphi} = -\epsilon^{-1} \frac{g \sin \theta (\lambda + \cos \varphi) \sin \varphi}{g \sin \theta (\lambda^2 - 1)} \bigg|_{t=T-\epsilon}.
\]
(21)

So \( \varphi \) decreases rapidly until it reaches zero, and when the particle’s velocity approaches zero, its velocity is in the \( x \) direction.
2.4. Exact solution

We studied large time behaviour of the particle’s motion for different cases. Now let us perform an exact calculation. Using (17), one may arrive at

\[ -(1 - \lambda^2) \sin \varphi \, d\varphi = \frac{g \sin \varphi \theta(1 - \lambda^2) \, dr}{g \sin \theta(1 - \lambda^2)r + v_0(\lambda + \cos \varphi_0)}. \]

which can be written as

\[ \int_{\cos \varphi_0}^{\cos \varphi} d\cos \varphi' \left[ \frac{1}{\lambda + \cos \varphi'} + \frac{(1 - \lambda)}{2(1 - \cos \varphi')} - \frac{(1 + \lambda)}{2(1 + \cos \varphi')} \right] \]

\[ = \int_0^t g \sin \theta(1 - \lambda^2) \, dr' \]

Integrations can be done easily and give

\[ t = \frac{v_0(\lambda + \cos \varphi_0)}{g \sin \theta(1 - \lambda^2)} \left[ (\lambda + \cos \varphi_0) \sin \varphi_0 \cdot \left( \frac{\tan(\varphi/2)}{\tan(\varphi_0/2)} \right)^\lambda \right] - 1. \]

In the limiting case, \( \lambda = 1 \) changes to

\[ t = \frac{v_0(1 + \cos \varphi_0)}{g \sin \theta} \left[ \ln \left( \frac{\tan(\varphi/2)}{\tan(\varphi_0/2)} \right) + \frac{1}{1 + \cos \varphi} - \frac{1}{1 + \cos \varphi_0} \right]. \]

In figure 2, \( \varphi \) is drawn in terms of time \( t \), for two values of \( \lambda \) and five different values of \( \varphi_0 \). For \( \lambda = 2 \), \( \varphi \) goes to zero rapidly, but for \( \lambda = 0.5 \), it goes to zero asymptotically.

Using (16) and (24), one can obtain the velocity components

\[ \dot{x} = v_0 \sin \varphi_0 \cot \varphi \left( \frac{\tan(\varphi/2)}{\tan(\varphi_0/2)} \right)^\lambda, \]

\[ \dot{y} = v_0 \sin \varphi_0 \left( \frac{\tan(\varphi/2)}{\tan(\varphi_0/2)} \right)^\lambda. \]
Figure 3. Trajectories of three projectiles with the same velocity $\mathbf{v}_0 = -i + 2j$ (m s$^{-1}$) on an inclined plane with angle $\theta = \pi/3$, from time $t = 0$ s to 0.5 s for different values of $\lambda$.

Now a parametric equation for the trajectory of the particle can be obtained. It is easy to obtain

$$\frac{dx}{d\varphi} = -\frac{v_0^2 \sin^2 \varphi_0}{g \sin \theta (\tan(\varphi_0/2))^{2\lambda}} \left( \frac{1}{\sin^2 \varphi} \right),$$

$$\frac{dy}{d\varphi} = -\frac{v_0^2 \sin^2 \varphi_0}{g \sin \theta (\tan(\varphi_0/2))^{2\lambda}} \left( \frac{1}{\sin^2 \varphi} \right),$$

which can be integrated and leads to a parametric equation for the trajectory of the particle,

$$x = -\frac{v_0^2 \sin^2 \varphi_0}{4g \sin \theta (\tan(\varphi_0/2))^{2\lambda}} \left\{ \left( \frac{(\tan(\varphi/2))^{2\lambda-2}}{2\lambda - 2} - \frac{(\tan(\varphi/2))^{2\lambda+2}}{2\lambda + 2} \right) - [\varphi \to \varphi_0] \right\},$$

$$y = -\frac{v_0^2 \sin^2 \varphi_0}{2g \sin \theta (\tan(\varphi_0/2))^{2\lambda}} \left\{ \left( \frac{(\tan(\varphi/2))^{2\lambda-1}}{2\lambda - 1} + \frac{(\tan(\varphi/2))^{2\lambda+1}}{2\lambda + 1} \right) - [\varphi \to \varphi_0] \right\}.$$

The equation of the trajectory of the particle in the case $\lambda = 1$ is

$$x = \frac{v_0^2 (1 + \cos \varphi_0)^2}{16g \sin \theta} \left[ \tan^4 \frac{\varphi_0}{2} - \tan^4 \frac{\varphi_0}{2} - 4 \ln \left( \frac{\tan(\varphi/2)}{\tan(\varphi_0/2)} \right) \right],$$

$$y = \frac{v_0^2 (1 + \cos \varphi_0)^2}{6g \sin \theta} \left[ 3 \tan^2 \frac{\varphi_0}{2} + \tan^3 \frac{\varphi_0}{2} - 3 \tan \frac{\varphi}{2} - \tan^3 \frac{\varphi}{2} \right].$$

In figure 3 the trajectories of three projectiles with the same velocity are drawn from time $t = 0$ s to 0.5 s for different values of $\lambda$. It is based on numerical calculation. All three have the same initial velocity $\mathbf{v}_0 = -i + 2j$ (m s$^{-1}$), and the angle of the inclined plane is $\theta = \pi/3$.

3. The motion of a sphere on an inclined plane

In this section, we study the motion of a sphere with the radius $R$ and mass $m$ on an inclined plane. Depending on the friction coefficient and the initial velocity and angular velocity of the
sphere, it may roll or slide on the inclined plane. Newton’s equation of motion for the sphere is

\[ m \ddot{\mathbf{r}}_{\text{cm}} = mg \sin \theta \mathbf{i} + \mathbf{f} \]
\[ I \ddot{\mathbf{\Omega}} = \mathbf{R} \times \mathbf{f}, \tag{30} \]

where \( \mathbf{R} = -R \mathbf{k} \), \( I = 2mR^2/5 \) is the moment of inertia of the sphere with respect to its centre, \( \mathbf{r}_{\text{cm}} \) is the radius vector from the origin to the centre of mass, and \( \mathbf{\Omega} \) is the angular velocity of the sphere. Let us first consider the rolling of the sphere.

3.1. Rolling a sphere on an inclined plane

The rolling constraint demands the velocity of the contact point of the sphere with the inclined plane, \( A \), to be zero. Then

\[ \mathbf{v}_A = \dot{\mathbf{r}}_{\text{cm}} + \mathbf{\Omega} \times \mathbf{R} = 0. \tag{31} \]

Differentiating the above equation with respect to time and using (30), one obtains

\[ \mathbf{f} = -\frac{I}{R^2} \ddot{\mathbf{r}}_{\text{cm}} \]
\[ \ddot{\mathbf{r}}_{\text{cm}} = \frac{5g}{7} \mathbf{i}, \]
\[ \mathbf{f} = -\frac{2mg}{7} \sin \theta \mathbf{i}. \tag{33} \]

The sphere rolls on the inclined plane if \( f \leq \mu mg \cos \theta \). Then the rolling occurs if \( \mu \geq \frac{2}{7} \tan \theta \). If the sphere rolls on the inclined plane, friction will be a constant force. Then for arbitrary initial velocity and angular velocity, the trajectory of sphere’s centre is generally a parabola.

3.2. Sliding a sphere on an inclined plane

If the sphere slips, then the velocity of the contact point \( A \) is not zero, and the friction is

\[ \mathbf{f} = -\mu mg \cos \theta \mathbf{e}_A \]
\[ = -\mu mg \cos \theta (\cos \varphi \mathbf{i} + \sin \varphi \mathbf{j}), \tag{34} \]

where \( \mathbf{e}_A = \frac{\mathbf{v}_A}{v_A} \) is the unit vector along the velocity of the contact point, and \( \varphi \) is the angle between \( \mathbf{v}_A \) and \( \mathbf{i} \). Using (30) the time evolution equation of \( \mathbf{v}_A \) is

\[ m \ddot{\mathbf{v}}_A = mg \sin \theta \mathbf{i} - \frac{2}{7} \mu mg \cos \theta \mathbf{e}_A, \tag{35} \]

As it is seen, the evolution equation for the velocity of the contact point of a sliding sphere with friction coefficient \( \mu \) is exactly the same as that of the velocity of a point particle sliding on an inclined plane whose friction coefficient is \( \frac{2}{7} \mu \). Compare (35) with (1). So it is not necessary to solve the equation for \( \mathbf{v}_A \), and all the previous results can be used only by replacing \( \lambda \) with \( \lambda_s := \frac{2}{7} \mu \cot \theta \), e.g. \( \mathbf{v}_A \) can be obtained through (26)

\[ v_{Ax} = v_{0Ax} \sin \varphi \cot \varphi \left( \frac{\tan (\varphi/2)}{\tan (\varphi_0/2)} \right)^{\lambda_s}, \]
\[ v_{Ay} = v_{0Ay} \sin \varphi \left( \frac{\tan (\varphi/2)}{\tan (\varphi_0/2)} \right)^{\lambda_s}, \tag{36} \]

where \( v_{0Ax} \) and \( v_{0Ay} \) are the components of the initial velocity of the contact point. It should be noted that this is not enough to know \( \mathbf{v}_{\text{cm}} \) and \( \mathbf{\Omega} \).
Similar to the sliding particle, three cases may occur:

3.2.1. $\lambda_s > 1 (\mu > \frac{2}{7} \tan \theta)$. If $\lambda_s > 1 (\mu > \frac{2}{7} \tan \theta)$, as shown in the previous section after a finite time, $T_r$,

$$T_r = \frac{v_{A0}(\lambda_s + \cos \varphi_0)}{g \sin \theta (\lambda_s^2 - 1)}.$$

$v_A$ will become equal to zero. When the rolling constraint holds true, the sphere will roll. At the time $T_r$, friction is along the $x$ direction and is a constant force. So the sphere rolls down the inclined plane, and its centre moves on a parabola.

3.2.2. $\lambda_s = 1 (\mu = \frac{2}{7} \tan \theta)$. Using our previous results on a sliding particle, at large times $v_A$ approaches a constant value

$$v_A \to \frac{v_{A0}(1 + \cos \varphi_0)}{2}.$$ 

so the sphere slides forever on the plane. At large times, friction is along the $x$ direction and is a constant force. In this case, at large times $\dot{r}_A = 0$. Then at large times, the acceleration of the centre of mass is

$$\ddot{v}_{cm} = \frac{5g}{7} \sin \theta i.$$ 

(39)

3.2.3. $\lambda_s < 1 (\mu < \frac{2}{7} \tan \theta)$. This case corresponds to $\lambda < 1$ in the previous section. The sphere slides forever on the plane, and at large times $v_A$ is along the $x$ direction. So at large times, friction is along the $x$ direction and is a constant force. Then the acceleration of the centre of mass is

$$\ddot{r}_{cm} = g \sin \theta (1 - \mu \cot \theta)i.$$ 

(40)

3.3. Exact solution

Let us solve exactly Newton’s equation. Integrating (30) gives

$$v_{cm} = v_{bcm} + Ig \sin \theta t + \frac{1}{m} \int_0^t \, dt \, f$$

$$\Omega = \Omega_0 + \frac{R}{I} \times \int_0^t \, dt \, f.$$ 

To know $v_{cm}$ and $\Omega$, $f = -\mu mg \cos \theta(i \cos \varphi + j \sin \varphi)$ should be integrated. Two main integrals should be calculated,

$$\int_0^t \, dt \, \sin \varphi = \int_0^t \, \frac{d\varphi}{\dot{\varphi}} \sin \varphi = -\frac{v_{0A} \sin \varphi_0}{g \sin \theta (\tan(\varphi_0/2))^2} \int_{\varphi_0}^\varphi \frac{d\varphi (\tan(\varphi/2))^\lambda}{\sin \varphi},$$

$$= \frac{v_{0A} \sin \varphi_0}{\lambda g \sin \theta} \left[ 1 - \left( \frac{\tan(\varphi/2)}{\tan(\varphi_0/2)} \right)^\lambda \right]$$ 

(42)

$$\int_0^t \, dt \, \cos \varphi = -\frac{v_{0A} \sin \varphi_0}{g \sin \theta (\tan(\varphi_0/2))^2} \int_{\varphi_0}^\varphi \frac{d\varphi \cos \varphi (\tan(\varphi/2))^{\lambda}}{\sin^2 \varphi},$$

$$= -\frac{v_{0A} \sin \varphi_0}{2g \sin \theta (\tan(\varphi_0/2))^2} \left\{ \left[ \frac{(\tan(\varphi/2))^{\lambda - 1}}{\lambda - 1} - \frac{(\tan(\varphi/2))^{\lambda + 1}}{\lambda + 1} \right] - [\varphi \to \varphi_0] \right\}.$$ 

(43)
Substituting both integrals, \( v_{cm} \) and \( \Omega \) can be obtained. Let us consider the case \( \lambda_s > 1 \) as an example and compute the velocity and angular velocity of the sphere when it starts to roll. At that time, \( v_A \) and \( \varphi \) are equal to zero. It can easily be shown that

\[
\begin{align*}
\int_0^{T_r} dr \sin \varphi &= \frac{v_{0A} \sin \varphi_0}{\lambda_s g \sin \theta} \\
\int_0^{T_r} dr \cos \varphi &= \frac{v_{0A} (\lambda_s \cos \varphi_0 + 1)}{g \sin \theta (\lambda_s^2 - 1)} \\
\int_0^{T_r} dr f &= -mv_{0A} \mu \cot \theta \left[ \frac{1}{\lambda_s^2 - 1} + \frac{j \sin \varphi_0}{\lambda_s} \right].
\end{align*}
\]

(44)

After the time \( T_r \), the sphere will roll and its centre move on a parabola.

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