Fate of Inhomogeneity in Schwarzschild-deSitter Space-time

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Abstract

We investigate the global structure of the space time with a spherically symmetric inhomogeneity using a metric junction, and classify all possible types. We found that a motion with a negative gravitational mass is possible although the energy condition of the matter is not violated. Using the result, formation of black hole and worm hole during the inflationary era is discussed.

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1 Introduction

The inflationary scenario is a favorable model to explain the homogeneity and isotropy of the present universe. In this scenario, vacuum energy of the matter field plays a role of the cosmological constant, and the universe enters the phase of deSitter expansion. Initial inhomogeneity of the universe dumps due to the rapid cosmological expansion. To utilize these aspects of inflation, it is necessary to discuss whether the universe can enter the inflationary phase from the wide range of the initial condition. “Cosmic no hair conjecture” states that if a positive cosmological constant exists, all space-times approach deSitter space-time asymptotically. But it is difficult to prove and formulate this conjecture for general situation and we do not know whether it is true.

For spherical symmetric space-time with cosmological constants, it is shown that the space-time does not necessarily approaches deSitter but a black hole and a worm hole may be created\[1, 2\]. The global structure of the final space-time depends the scale and the amplitude of the initial inhomogeneity and this means that “no hair conjecture” is not necessarily established. But in a practical sense, a whole universe does not need to inflate and only a portion of our universe has to inflate(weak no hair conjecture)\[3\].

The problem is more complicated in the early stage of the universe because quantum effect of the matter becomes important. Even though the completely homogeneous universe is created at beginning, the inhomogeneity is continuously created by quantum fluctuation of the matter field in deSitter phase. Self-reproduction of the inflating region occurs and the universe becomes inhomogeneous on super large scale.

In this paper, we investigate the evolution of the inhomogeneity in the Schwarzschild-deSitter space time. As a source of inhomogeneity, we use a false vacuum bubble with thin wall approximation. This is the simplest model to represent the spherically symmetric inhomogeneity. Assuming that the inhomogeneity is created by quantum fluctuation of the scalar field in inflationary phase, we calculate the probability of black hole and worm hole formation. If the probability of black hole formation is too large, the universe becomes much inhomogeneous even though it started from a homogeneous initial condition.

2 Metric Junction

We assume that the inside space-time of the bubble is described by deSitter metric with cosmological constant $\Lambda_1$, the outside is described by Schwarzschild-deSitter metric with a gravitational mass $M$ and cosmological constant $\Lambda_2$. Using the static coordinate systems, they are written by

$$
\begin{align*}
 ds_{in}^2 &= -(1 - \chi_1^2 r^2)dt^2 + (1 - \chi_1^2 r^2)^{-1}dr^2 + r^2d\Omega^2, \\
 ds_{out}^2 &= -(1 - \frac{r_g}{r} - \chi_2^2 r^2)dt^2 + (1 - \frac{r_g}{r} - \chi_2^2 r^2)^{-1}dr^2 + r^2d\Omega^2,
\end{align*}
$$

where $\chi_1^2 = 8\pi G\Lambda_1/3$, $\chi_2^2 = 8\pi G\Lambda_2/3$ and $r_g = 2GM$. The motion of the bubble can be determined by the metric junction condition:

$$
K_j^i(in) - K_j^i(out) = 4\pi\sigma G\delta_j^i,
$$

1
where $K^i_j$ is the extrinsic curvature of $(2 + 1)$-dimensional hyper-surface swept out by the domain wall. $\sigma$ is the surface energy of the bubble and is a constant for the scalar field domain wall. $\theta-\theta$ component of this equation gives the equation of motion of the bubbles radius $r(\tau)$ where $\tau$ is the proper time on the wall. By introducing the dimension-less variables, our basic equation becomes
\begin{equation}
\left(\frac{dz}{d\tau}\right)^2 + V(z) = E, \tag{4}
\end{equation}
where
\begin{align*}
z &= r/r_0, \quad r_0^3 = 2G|M|/\chi^2_+, \quad \tau' = \chi^2_+/2\kappa, \quad \kappa = 4\pi G\sigma
\chi^2_+ &= \left[(\kappa^2 + \chi^2 - \chi^2_2)^2 + (2\kappa\chi_2)^2\right]^{1/2}, \quad V(z) = -\left(z - \frac{1}{z}\right)^2 - \gamma^2/z,
\gamma^2 &= 2 + 2\text{sgn}(M)(\kappa^2 + \chi^2 - \chi^2_2)/\chi^2_+,
E &= -\left(\frac{2\kappa}{\chi_+}\right)^2 (2G|M|\chi_+)^{-2/3}.
\end{align*}

$\theta-\theta$ component of the extrinsic curvature of the bubble interior and exterior are given by
\begin{align*}
\beta_{in} &= \left(\frac{G|M|}{r_0^2\kappa}\right) \frac{1}{z^2} \left[\text{sign}(M) - \left(\frac{\chi^2 - \chi^2_2 - \kappa^2}{\chi^2_+}\right)z^3\right], \tag{5}
\beta_{out} &= \left(\frac{G|M|}{r_0^2\kappa}\right) \frac{1}{z^2} \left[\text{sign}(M) - \left(\frac{\chi^2 - \chi^2_2 + \kappa^2}{\chi^2_+}\right)z^3\right]. \tag{6}
\end{align*}

The location of horizons in Schwarzschild-deSitter space is determined by the equation
\begin{equation}
1 - 2\frac{GM}{r} - \chi^2_2 r^2 = 0, \tag{7}
\end{equation}
and using dimension-less variables, it can be written
\begin{equation}
E = -\text{sign}(M) \left(\frac{4\kappa^2}{\chi^2_+}\right) \frac{1}{z} + \left(\frac{4\kappa^2\chi^2_2}{\chi^4_+}\right) z^2. \tag{8}
\end{equation}

The global structure of space-time is determined by solving 1-dimensional particle motion with the potential $V(z)$(Fig.1). The curve of horizon is tangent to the curve $V(z)$ at $z = z_s$. The sign of the extrinsic curvature $\beta_{out}$ changes at this point. The location of horizons is the intersection of $E = \text{const.}$ line and the horizon line. For $M > 0$, there are two horizons at most. As $|E|$ decreases, horizon disappears. For $M < 0$, one horizon always exits. There are three characteristic energy level that determines the behavior of the bubble. $E_1$ is the value that the horizon line becomes maximum(at $z = z_s$) and exists for the case of positive mass. $E_2$ is the maximum value of the potential $V(z)$(at $z = z_m$). $E_3$ is the value of the potential at which the extrinsic curvature $\beta_{out}$ changes its sign(at $z = z_s$).


Classification of the space-time

We can completely classify the global structure of the space-time. The results are shown in Fig.2 and Fig.3. Fig.2 is the parameter space $\left(2GM\chi_2, \chi_1/\chi_2\right)$. The horizontal axis ($\chi_1/\chi_2$) is the scale of the inhomogeneity and the vertical axis ($2GM\chi_2$) is the amplitude of the inhomogeneity. The parameter space is divided to regions R1-R14, and each region has the different global structure(Fig.3). As the Schwarzschild-deSitter side gives non-trivial global structure, we only draw this side. R1-R9 have the positive mass and R10-R14 have the negative mass. In R1 and R2, Schwarzschild horizon disappears and the space-time becomes deSitter like. R3 and R4 are also deSitter like space-time. R5 and R9 are a worm hole space-time. R6,R7,R8 correspond to a black hole space-time. From a viewpoint of “weak no hair conjecture”, a worm hole space time is “safe” because a bubble region does not meet a singularity in future. For the negative mass(R10-R14), the singularity becomes time like and only a deSitter horizon exits. Therefore space-times are all deSitter like. If we take the limit $\kappa \rightarrow 0$, the motion of the bubble becomes null and we get the same result of Maeda et al. [1]. In this limit, R6,R7,R9,R11 and R12 disappear.

We found that the motion with the negative gravitational mass is possible for the all value of $\chi_1/\chi_2$. Rewriting the junction condition, the gravitational mass is given by

$$2GM = \left(\chi_1^2 - \chi_2^2 - \kappa^2\right) r^3 + 2\kappa r^2 \text{sign}(\beta_{\text{out}}) \left(1 + r^2 - \chi_1^2 r^2\right)^{1/2}. \quad (9)$$

The first term is the volume energy(difference between bubble interior and exterior) and the second term is the surface energy of the bubble. For the monotonic type solutions(R1-R5,R13,R14), we can evaluate the above equation at $r = 0$:

$$2GM = 2\kappa \text{sign}(\beta_{\text{out}}) r^2 |\dot{r}|. \quad (10)$$

Therefore the sign of the mass is determined by the sign of the extrinsic curvature of the bubble exterior. The mass can become negative even if the energy condition of the matter is not violated($\kappa > 0$). For the bounce type solutions(R6-R12), we can evaluate the mass at the turning point $\dot{r} = 0$:

$$2GM = \left(\chi_1^2 - \chi_2^2 - \kappa^2\right) r^3 + 2\kappa r^2 \text{sign}(\beta_{\text{out}}) \left(1 - \chi_1^2 r^2\right)^{1/2}. \quad (11)$$

In this case, the sign of the mass is determined by the following characterics radius:

$$r_* = \frac{2\kappa}{\sqrt{4\kappa^2 \chi_1^2 + (\chi_1^2 - \chi_2^2 - \kappa^2)^2}}. \quad (12)$$

If $r > r_*$, $\text{sign}(M) = \text{sign}(\chi_1^2 - \chi_2^2 - \kappa^2)$(R7-R9,R10,R11) and the sign of the mass is equal to the sign of the volume energy of the bubble. If $r < r_*$, $\text{sign}(M) = \text{sign}(\beta_{\text{out}})$(R6,R12)

For negative mass solutions, the singularity becomes time-like and is not hidden by the event horizon. But if we use the spatially flat time slice to foliate this space-time, this naked singularity does not appear to our universe.
4 Inhomogeneity during the Inflation

In the inflationary era, the inhomogeneity of the space-time is generated by the quantum fluctuation of the inflaton field. If the universe is created completely homogeneous and isotropic at beginning, later evolution is not necessarily homogeneous because of continuous generation of the quantum fluctuation. We discuss the probability of a black hole and a worm hole formation during the inflation. If too many black holes are created by quantum fluctuation, the final space-time becomes much inhomogeneous and the inflation will not succeed. We can estimate the probability of black hole and worm hole formation within linear perturbation using the result of the previous section (Fig. 2).

The energy density is dominated by the potential energy $V = \frac{1}{2}m^2 \phi^2$ of the inflaton field and the Hubble parameter is $\chi_0^2 = \frac{8\pi}{(3m_{pl}^2)}V(\phi_0)$. Let $\delta \phi_1, \delta \phi_2$ be the fluctuation of the inflaton field interior and exterior of the bubble, respectively. $\delta \phi$ is Gaussian random field with the average $< \delta \phi > = 0$ and the dispersion $< \delta \phi^2 > = \chi^4/m^2$. The size of the bubble is given by the horizon scale $L = \chi^{-1}_0$. The ratio of Hubble parameters is given by

$$\frac{\chi_1}{\chi_2} = \frac{\phi_0 + \delta \phi_1}{\phi_0 + \delta \phi_2} = 1 + \delta_1 - \delta_2,$$

(13)

where $\delta = \delta \phi/\phi$. The surface energy density is estimated to be $\kappa = 4\pi G\sigma \sim \Lambda_0/m_{pl}^2 \chi_0^{-1} \sim \chi_0$, and the velocity of the bubble is $\dot{r} \sim L\chi^{-1}_0 = 1$. The mass excess due to the fluctuation of the inflaton field becomes

$$2GM\chi_0 = 2(\delta_1 - \delta_2) + 2\text{sign}(\beta_{out})(1 - \delta_1).$$

(14)

The probability distribution of $\delta$ is given by

$$P(\delta_1, \delta_2) = N \exp \left( -\frac{\delta_1^2}{2<\delta_1^2>} - \frac{\delta_2^2}{2<\delta_2^2>} \right),$$

(15)

where $< \delta^2 > = \chi_0^2/m_{pl}^2$. Combining eq.(13) and eq.(14), the probability of black hole and worm hole formation for a given energy scale is obtained by monte carlo calculation:

| Energy Scale | BH | WH | deSitter |
|--------------|----|----|----------|
| $10^{19}$GeV | 4.68% | 0.15% | 95.17% |
| $10^{18}$GeV | 0.05% | 0.00% | 99.97% |

We can say that almost all universe becomes deSitter like. The probability of worm hole and black hole formation increases as the energy scale grows. This indicates that space-time foam structure is realized at Planck energy scale. Although the probability of black hole formation is not so small, the characteristic mass of a black hole is small ($\sim 10^{-8}$kg) at Planck scale. It evaporate soon and does not affect later evolution of the universe.

5 Summary

We analyzed the motion of a false vacuum bubble in Schwarzschild-deSitter space-time and obtained all possible type of motion. The result is classified in the parameter space
\( (\chi_1/\chi_2, 2GM\chi_2) \) which characterizes the inhomogeneity. Provided that the initial condition is given by random Gaussian quantum fluctuation, we estimate the probability of black hole and worm hole creation.

Our analysis here is limited to spherically symmetric case. But the spatial pattern of high energy density regions by quantum fluctuation does not necessarily have spherical shapes even though the fluctuation is treated as perturbation. Therefore more realistic treatment without imposing spherical symmetry is necessary to understand correct picture of worm hole and black hole formation via quantum fluctuation. This is our next problem [5].

Acknowledgments

We would like to thank S. Konno and Prof. Tomimatsu for valuable discussions.

References

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Figure Captions

**Fig.1** The shape of the potential for the wall motion. For positive mass, the wall intersects two times with horizon line. For negative mass, the wall crosses horizon only once.

**Fig.2** Classification of the type of space-time in parameter space \((\chi_1/\chi_2, 2GM\chi_2)\). The region between two dashed curves corresponds to bounce type solution. In the limit \(\kappa \to 0\), regions R6, R7, R8, R11 and R12 disappear.

**Fig.3** Trajectories of the wall in conformal diagram of space-time. deSitter space is attached to the left side of each trajectory. Fig.3a is the case of positive mass and Fig.3b is the case of negative mass.