ON THE GEOMETRIC AND TOPOLOGICAL RIGIDITY
OF HYPERBOLIC 3-MANIFOLDS

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0. Introduction

The main result of this paper asserts that if $N$ is a closed hyperbolic 3-manifold satisfying a certain geometric/topological insulator condition (Conjecture 0.6 says that it always happens), then $N$ is both topologically and geometrically rigid. As a special case we have

Theorem 0.1. Let $N$ be a closed hyperbolic 3-manifold containing an embedded hyperbolic tube of radius $(\log 3)/2 = 0.549306\ldots$ about a closed geodesic. Then

i) If $f : M \to N$ is a homotopy equivalence where $M$ is an irreducible 3-manifold, then $f$ is homotopic to a homeomorphism.

ii) If $f, g : M \to N$ are homotopic homeomorphisms, then $f$ is isotopic to $g$.

iii) The space of hyperbolic metrics on $N$ is path connected.

Remarks 0.2. i) If $M$ is hyperbolic, then conclusion i) follows from Mostow’s rigidity theorem [Mo]. Actually [Mo] implies that $f$ is homotopic to an isometry. If $N$ is instead Haken, then conclusions i)–ii) are due to Waldhausen [W]. If $N$ is Haken and hyperbolic, then conclusion iii) follows by combining [Mo] and [W]. Conclusions i), ii)–iii) can be viewed as a 2-fold generalization of [Mo]. See §5.

ii) There are only six known closed 3-manifolds which are probably hyperbolic and have a shortest geodesic $\delta$ with tube radius $(\delta) < (\log 3)/2$. The first, known as Vol 3, was found using Jeff Weeks’ tube radius/ortholength program [We]. Nathaniel Thurston very recently found 5 other such manifolds. All six of these manifolds appear to satisfy the insulator condition.

iii) An application of the hyperbolic law of cosines shows that if the shortest geodesic $\delta$ in $N$ has length $\geq 1.353$, then tube radius $(\delta) > (\log 3)/2$. See Remark 5.11.

iv) If $N$ has a geodesic $\delta$ of length $\leq 0.978$, then Meyerhoff’s tube radius formula [Me] implies that tube radius $(\delta) > (\log 3)/2$. See Remark 5.11. Combined with the work of Jorgensen [Gr], we obtain Corollary 5.12 which asserts that for any $n > 0$, all but finitely many hyperbolic 3-manifolds of volume $< n$ are both topologically and geometrically rigid.

v) Farrell and Jones [FJ] showed that if $f : M \to N$ is a homotopy equivalence between closed manifolds and $N$ is a hyperbolic manifold of dimension $\geq 5$, then $f$ is homotopic to a homeomorphism.
The theme of this paper is to abstract the ideas of the following example to the setting of homotopy hyperbolic 3-manifolds.

**Example 0.3.** Let $\delta$ be a simple closed geodesic in the hyperbolic 3-manifold $N$. $\delta$ lifts to a collection $\Delta = \{\delta_i\}$ of hyperbolic lines in $H^3$. To each pair $\delta_i, \delta_j$, there exists the midplane $D_{ij}$, i.e. the hyperbolic halfplane orthogonal to and cutting the middle of the orthocurve (i.e. the shortest line segment) between $\delta_i$ and $\delta_j$. Each $D_{ij}$ extends to a circle $\lambda_{ij}$ on $S^2_{\infty}$ containing $\partial \delta_i$ from $\partial \delta_j$. Now fix $i$. Let $H_{ij}$ be the closed $H^3$-halfspace bounded by $D_{ij}$ containing $\delta_i$. $W_i = \bigcap H_{ij} = D^2 \times \mathbb{R}$ is the Dirichlet torus domain associated to the geodesic $\delta_i$. $W_i$ projects to an open solid torus containing $\delta$ as its core. In fact $W = W_i/\langle \delta_i \rangle$ is a solid torus with boundary a finite union of totally geodesic polygons.

**Definition 0.4.** Let $G$ be a group of homeomorphisms of $S^2$ and $A = \{A_i\}$ a countable set of pairwise disjoint $G$-equivariant pairs of points of $S^2$, i.e. if $g \in G, A_i \in A$, then $g(A_i) \in A$. Let $\{\lambda_{ij}\}$ be a collection of smooth simple closed curves in $S^2$. $\{\lambda_{ij}\}$ is called a $(G,A)$ insulator family and each $\lambda_{ij}$ is an insulator if

i) **Separation:** If $i \neq j$, then $\lambda_{ij}$ separates $A_i$ from $A_j$.

ii) **Equivariance:** If $g \in G$, then $g(\lambda_{ij})$ is the curve associated to the pair $g(A_i), g(A_j)$. Also $\lambda_{ij} = \lambda_{ji}$.

iii) **Convexity:** To each $\lambda_{ij}$ there exist round circles respectively containing $A_i$ and $A_j$ and disjoint from $\lambda_{ij}$.

iv) **Local Finiteness:** For every $\epsilon > 0$ there exist only finitely many $\lambda_{ij}$ such that $i$ is fixed and $\text{diam}(\lambda_{ij}) > \epsilon$.

**Definition 0.5.** A $(G,A)$ insulator family is noncoalescable if it satisfies the following no trilinking property. For no $i$, does there exist $\lambda_{i1}, \lambda_{i2}, \lambda_{i3}$ whose union separates the points of $A_i$. See Figure 0.1. A hyperbolic 3-manifold satisfies the insulator condition if there exists a geodesic $\delta$ in $N$ and a $\{\lambda_{ij}\}$ noncoalescable insulator family. Here $\{\delta_i\}$ is the set of lifts of $\delta$ to $H^3$.

**Conjecture 0.6.** Every closed hyperbolic 3-manifold satisfies the insulator condition.

**Definition 0.7.** If $\delta$ is a geodesic in the hyperbolic 3-manifold $N$, then the insulator family $\{\lambda_{ij}\}$ constructed as in Example 0.3. is called the Dirichlet insulator family.

**Strong conjecture 0.8.** The Dirichlet insulator family associated to a shortest geodesic in a closed hyperbolic 3-manifold, is noncoalescable.

**Lemma 5.9.** If a geodesic $\delta$ has tube radius $> (\log 3)/2$, then its Dirichlet insulator family is noncoalescable.

**Theorem 0.9.** Let $f : M \to N$ be a homotopy equivalence, where $M$ is a closed, connected, irreducible 3-manifold and $N$ is a hyperbolic 3-manifold. If $N$ possesses a geodesic $\delta$ with a noncoalescable insulator family, then $f$ is homotopic to a homeomorphism.

**Outline of the proof.** Let $f : M \to N$ be a homotopy equivalence. $M$ and $N$ are covered by the same hyperbolic manifold $X$ [G3]. In §1 we show that we can assume
that the homotopy equivalence lifts and extends to a mapping \( \tilde{f} : B^3 \to B^3 \), where \( B^3 = H^3 \cup S^2_{\infty} \), so that \( \tilde{f} | S^2_{\infty} = \text{id} \). Furthermore the group actions on \( S^2_{\infty} \) defined by \( \pi_1(M) \) and \( \pi_1(N) \) are identical. In \( \S 2 \) we show that if there exists a simple geodesic \( \delta \) in \( N \) and a simple closed curve \( \gamma \) in \( M \) such that the \( B^3 \)-link \( \Delta \) is equivalent to the \( B^3 \)-link \( \Gamma \), then \( f \) is homotopic to a homeomorphism. Here \( \Delta \) is the preimage of \( \delta \) in \( H^3 \) extended to \( B^3 \) and \( \Gamma \) is defined similarly. These links are equivalent means that there exists a homeomorphism \( k : (B^3, \Gamma) \to (B^3, \Delta) \) so that \( k | S^2_{\infty} = \text{id} \). To prove that equivalent links imply topological rigidity we show that \( f \) can be homotoped to a map which sends \( N(\gamma) \) homeomorphically onto \( N(\delta) \) and restricts to a homotopy equivalence between \( M - \hat{N}(\gamma) \) and \( N - \hat{N}(\delta) \). By Waldhausen \([W]\), \( f \) is homotopic to a homeomorphism.

In \( \S 3 \) we establish technical lemmas about least area planes, discs and laminations in \( H^3 \). In particular we show that to each smooth simple closed curve \( \lambda_{ij} \) in \( S^2_{\infty} \), there exists a lamination \( \sigma_{ij} \) by injectively immersed least area (with respect to the metric induced by \( M \)) planes in \( H^3 \), with limit set \( \lambda_{ij} \) such that \( \sigma_{ij} \) lies in a fixed width hyperbolic regular neighborhood of the hyperbolic convex hull of \( \lambda_{ij} \). Here \( \{\lambda_{ij}\} \) is the \( (\pi_1(N), \{\partial \delta_i\}) \) and hence \( (\pi_1(M), \{\partial \delta_i\}) \) noncoalescable insulator family. Fix \( i \). Let \( H_{ij} \) be the \( H^3 \)-complementary region of \( \sigma_{ij} \) containing the ends of \( \delta_i \). In \( \S 4 \) we show that \( \bigcap H_{ij} \) contains a \( \hat{V}_i = D^2 \times \mathbb{R} \) which projects to an open solid torus in \( M \). (It is this step that requires the noncoalescable insulator hypothesis.) Define \( \gamma \) to be the core of this solid torus and \( \gamma_i \) the lift which lives in \( \hat{V}_i \). We show that the isotopy class of \( \gamma \) is independent of all choices, i.e. the metric on \( M \) and the choice of \( \{\sigma_{ij}\} \) for a fixed metric. Let \( \tau_0 \) be the link in \( X \) which is the preimage of \( \gamma \), so \( \{\tau_i\} \) is also the set of lifts of components of \( \tau_0 \) to \( H^3 \). The Riemannian metric \( \mu_0 \) on \( X \) induced from \( M \) and the hyperbolic metric \( \mu_1 \) on \( X \) induced from \( N \) are connected by a smooth path \( \mu_t \) of metrics. These metrics lift to \( \pi_1(X) \) equivariant metrics \( \hat{\mu}_t \) on \( H^3 \), so the above construction applied to the \( (\pi_1(X), \{\partial \delta_i\}) \) insulator family \( \{\lambda_{ij}\} \) with respect to the \( \hat{\mu}_t \) metric yields a link.
τ₁ in X. Since the isotopy class of τᵱ is independent of t, τ₀ is isotopic to τ₁, the
preimage of δ in X. We conclude that the B³-link Γ is equivalent to the B³-link Δ
and so by §2 f is homotopic to a homeomorphism.

\[ \text{Theorem 0.10. If } N \text{ is a closed hyperbolic 3-manifold possessing a geodesic } \delta \text{ with } \]
\[ \text{a noncoalescable insulator family, and } f : N \rightarrow N \text{ is a homeomorphism homotopic to } \]
\[ \text{id, then } f \text{ is isotopic to id.} \]

\[ \text{Idea of the proof. Let } \rho₀ \text{ denote the hyperbolic metric on } N. \text{ Let } \rho₁ \text{ be the pull back } \]
hyperbolic metric on N induced via } f, \text{ which we can assume is a diffeomorphism.}

\[ \text{These metrics are connected by a family } \rhoₜ \text{ of Riemannian metrics. As in the proof } \]
Theorem 0.9, to each } \rhoₜ, \text{ there is associated an oriented simple closed curve } \gammaₜ, \text{ where } \gamma₀ = \delta \text{ and } \gamma₁ = f⁻¹(δ) \text{ and that all of these } \gammaᵱ \text{ are isotopic. Therefore } f \text{ is}
\[ \text{isotopic to a map which fixes } \delta \text{ pointwise. A theorem of Siebenmann [BS] implies that } f \text{ is isotopic to id.} \]

\[ \text{Corollary 5.3. If } N \text{ satisfies the insulator condition, then } \text{Homeo}(N)/\text{Homeo}_0(N) \]
\[ = \text{Out}(π₁(N)) = \text{Isom}(N). \]

Section 5 contains the proofs of Theorem 0.1, Corollary 5.3 and some concluding results, remarks and conjectures.

1. Preliminary results and notation

From now on we will assume that all 3-manifolds in this paper are connected, irreducible, and orientable. Recall that by Remark 0.2 i), we can assume that M is non-Haken and hence orientable.

\[ \text{Proposition 1.1. If } f : M \rightarrow N \text{ is a homotopy equivalence, where } M \text{ is a closed } \]
3-manifold and N is a hyperbolic 3-manifold, then there exists a closed hyperbolic 3-manifold X and regular covering maps p₁ : X \rightarrow M, q₁ : X \rightarrow N such that f ∘ p₁ is homotopic to q₁. A lift } \tilde{f} : H³ \rightarrow H³ \text{ extends to id : } S²∞ \rightarrow S²∞. \text{ Furthermore the action of } π₁(M) \text{ on } H³ \text{ extends to a Möbius action on } S²∞ \text{ which is identical to the action of } π₁(N) \text{ on } S²∞. \]

\[ \text{Proof. By the Proof of Theorem 1.1 [G₃], there exists a hyperbolic 3-manifold } X \text{ and regular covering } \]
projections p₁ : X \rightarrow M, q₁ : X \rightarrow N such that f₁ : X \rightarrow X, \text{ a lift of } f \text{ to } X, \text{ is homotopic to a homeomorphism } h. \text{ By replacing } p₁ \text{ by } p₁ ∘ h⁻¹ \text{ and } f₁ \text{ by } f₁ ∘ h⁻¹, \text{ we can assume that } f₁ \text{ is homotopic to id. Let } x \in X. \text{ Fixing } m = p₁(x), n = q₁(x) \text{ and modifying } f \text{ via a homotopy we can assume that } f(m) = n \text{ and the homotopy } F₁ \text{ of id to } f₁ \text{ is rel } x. \]

Let } \pi : H³ \rightarrow X \text{ be the universal covering projection. Let } p : H³ \rightarrow M, q : H³ \rightarrow N \text{ be the induced coverings. If } g \in π₁(M, m), \text{ then let } g' \text{ denote } f_p(g) ∈ π₁(N, n). \text{ Fix } \tilde{e} ∈ π⁻¹(x). \text{ Let } g → g' \text{ denote the canonical 1-1 correspondence between } \pi₁(M, m) \text{ and points of } p⁻¹(m) \text{ such that } e → \tilde{e}, g' → \tilde{g}' \text{ has the similar meaning. The distance between points } \tilde{g}, \tilde{g}' \text{ of } H³ \text{ differs by a bounded amount, i.e. the diameter of } F₁ \text{ homotopy tracks. More generally the action of } π₁(M) \text{ on } \tilde{g} \text{ differs from the corresponding action of } π₁(N) \text{ on } \tilde{g}' \text{ by this same amount. Since the action of } π₁(M) \text{ on } H³ \text{ is approximated by the action on the points corresponding to } π₁(M), \text{ it follows that the actions of } π₁(M), π₁(N) \text{ on } S²∞ \text{ are identical. Here are more details.}

In what follows } dₚ \text{ (resp. } d_µ) \text{ will denote distance with respect to the hyperbolic metric } ρ \text{ (resp. the metric } µ \text{ induced from } M \text{) between points in } H³.
Claim. There exists \( k < \infty \) such that if \( g \in \pi_1(M, m) \) and \( y \in \mathbb{H}^3 \), then \( d_\rho(g(y), f_\#(g)(y)) \leq k \).

Proof of the Claim. Let \( \hat{F} \) denote the lift of \( F_1 \) to \( \mathbb{H}^3 \) and \( \hat{f} \) the lift of \( f \) to \( \mathbb{H}^3 \) such that \( \hat{F}(z, 0) = z \) and \( \hat{f}(z) = \hat{F}(z, 1) \).

The compactness of \( X \) implies the following facts i–iii) and the covering space theory implies iv–v).

i) There exists \( b_0 = \sup\{\text{diam}_\rho(\hat{F}(z \times I)) \mid z \in \mathbb{H}^3\} \). So \( z' = \hat{f}(z) \) implies \( d_\rho(z, z') \leq b_0 \).

ii) There exists \( b_1 \) such that for each \( y \in \mathbb{H}^3 \) there exists \( \tilde{r} \in p^{-1}(m) \) such that \( d_\rho(\tilde{r}, y) < b_1 \).

iii) There exists \( b_2 \) such that \( d_\rho(y_0, y_1) < b_1 \) implies \( d_\rho(y_0, y_1) < b_2 \).

iv) If \( g, r \in \pi_1(M) \), then \( \hat{f}(g(\tilde{r})) = g'(\tilde{r}') \), so \( d_\rho(g(\tilde{r}), g'(\tilde{r}')) \leq b_0 \). (Recall that \( g(\tilde{r}) = j(\tilde{r}), \) where \( j : [0, 1] \to M, [j] = g \ast r \in \pi_1(M, m) \), and \( j(0) = \tilde{e} \). Since \( [g' \ast r'] = [f_\#(g \ast r)] \), it is represented by \( f \circ j \), so \( g'(\tilde{r}') = f(\tilde{f}(\tilde{r})) \). Now apply i.)

v) The covering transformation associated to \( g \in \pi_1(M, m) \) (resp. \( g' \in \pi_1(N, n) \)) is an isometry in the \( \mu \) (resp. \( \rho \)) metric.

If \( y \in \mathbb{H}^3 \), then \( d_\rho(g(y), g'(y)) \leq d_\rho(g(y), g(\tilde{r}')) + d_\rho(g(\tilde{r}), g'(\tilde{r}')) + d_\rho(g'(\tilde{r}'), g'(y)) \leq b_0 + b_2 + d_\rho(\tilde{r}', y) \leq b_0 + b_2 + d_\rho(\tilde{r}', \tilde{r}) + d_\rho(\tilde{r}, y) \leq b_0 + b_2 + b_0 + b_2. \)

Since the actions of \( \pi_1(M), \pi_1(N) \) on \( \mathbb{H}^3 \) differ by a hyperbolically bounded amount, the action of \( \pi_1(M, m) \) extends to the same action on \( S_\infty^2 \) as that of \( \pi_1(N, n) \). I.e., for each \( g \in \pi_1(M) \) and \( y \in S_{\infty}^2, g(y) = g'(y) \). Also \( b_0 < \infty \) implies that \( \text{id} = \hat{f} : S_{\infty}^2 \to S_{\infty}^2 \).

Notation 1.2. \( M = \mathbb{H}^3 \backslash G, N = \mathbb{H}^3 \backslash H \), where \( \pi_1(M) = G \cong H = \pi_1(N), G \subset \text{Homeo}^+(\mathbb{B}^3), H \subset \text{Isom}^+(\mathbb{H}^3) \subset \text{Homeo}^+(\mathbb{B}^3) \), \( G \mid S_{\infty}^2 = H \mid S_{\infty}^2, G \) and \( H \) are naturally identified via the \( \pi_1 \)-isomorphism \( \phi = f_\# \). The space \( X \) and the maps \( p, q, p_1, q_1, \pi \) will be as in Proposition 1.1; \( \mu \) (resp. \( \rho \)) will denote the metric induced on \( X \) or \( \mathbb{H}^3 \) from \( M \) (resp. \( N \)). In particular \( \rho \) will always represent the hyperbolic metric. We abuse notation by letting \( H \) and \( G \) simultaneously denote the actions on \( \mathbb{H}^3, \mathbb{B}^3 \) or \( S_{\infty}^2 \).

Unless otherwise indicated we will assume that all maps on 3-manifolds without boundary are smooth.
Let \( \gamma \subset M \), a geodesic \( \delta \subset N \) and a homeomorphism \( k : (B^3, p^{-1}(\gamma)) \to (B^3, q^{-1}(\delta)) \) such that \( k | \partial B^3 = \text{id} \), then \( f \) is homotopic to a homeomorphism.

**Definition 2.2.** A \( B^3 \)-link is a collection \( \{\alpha_i\} \) of properly embedded arcs in \( B^3 \) whose restriction to \( H^3 \) is locally finite. Two \( B^3 \)-links are said to be equivalent if there exists a homeomorphism \( k : B^3 \to B^3 \) taking one link to the other such that \( k | S^2_{\infty} = \text{id} \).

**Remarks.** i) Each component of \( p^{-1}(\gamma) \) extends to a properly embedded arc \( \gamma_i \) in \( B^3 \). We will abuse notation by also referring to \( \gamma_i \) as an arc in \( H^3 \). Proposition 2.1 says that if the \( B^3 \)-link \( \{p^{-1}(\gamma)\} \) is equivalent to the link \( \{q^{-1}(\delta)\} \), then \( f \) is homotopic to a homeomorphism.

ii) The \( H^3 \)-link determined by \( \delta \) is the trivial link on infinitely many components. Thus Proposition 2.1 crucially depends on the condition \( k | \partial B^3 = \text{id} \).

**Notation 2.3.** Let \( \Gamma = p^{-1}(\gamma) \) with components \( \{\gamma_i\} \), \( \Delta = q^{-1}(\delta) \) with components \( \{\delta_i\} \), \( V = p^{-1}(N(\gamma)) \) with components \( \{V_i\} = \{N(\gamma_i)\} \), \( W = q^{-1}(N(\delta)) \) with components \( \{W_i\} = \{N(\delta_i)\} \). Indices are chosen so that \( i = j \) if and only if \( \partial \gamma_i = \partial \delta_i \). Recall that if \( g \in G \), then the associated map \( g : B^3 \to B^3 \) has two fixed points, one attracting and one repelling. Thus the correspondence of \( \gamma_i \) to \( \delta_i \) is determined by \( \phi \).

**Idea of the proof.** We show that \( f \) is homotopic to a map \( h : (M, M - \tilde{N}(\gamma), N(\gamma)) \to (N, N - \tilde{N}(\delta), N(\delta)) \) which restricts to a homeomorphism on \( N(\gamma) \) and a \( \pi_1 \)-injective map on \( M - \tilde{N}(\gamma) \). The map \( f \) is homotopic to a homeomorphism, since \( [W] \) implies that \( h \) is homotopic to a homeomorphism rel \( N(\gamma) \). Given a handle structure on \( M \) which contains \( N(\gamma) \) as a 1-handle we use \( k \) to define \( h \) so that the other handles miss \( \tilde{N}(\delta) \), approximately by considering \( g \circ \alpha \circ p^{-1} \) restricted to a handle. There is never a problem with the 1-handles. The difficulty is that the image of the 2-handles might cross \( \delta \), e.g. consider the case that \( \delta, \gamma \) are distinct knots in \( S^3 = M = N \). In our setting this difficulty vanishes if for each \( g \in G, \phi(g) \circ k \circ g^{-1} \circ k^{-1} | H^3 - \tilde{N}(\Delta) \) is homotopic to \( \text{id} \) rel \( \partial N(\Delta) \). We show that \( k \) can be chosen so that this holds.

The following result follows by the usual isotopy/unicity of regular neighborhood arguments.

**Lemma 2.4.** Under the hypothesis of Proposition 2.1, \( k : B^3 \to B^3 \) can be chosen so that for each \( g \in G, k \circ g \) \( | V \cup S^2_{\infty} = \phi(g) \circ k \) \( | V \cup S^2_{\infty} \).
Remark 2.5. Replace the covering projection \( p : \mathbb{H}^3 \to M \) by \( p \circ k^{-1} : \mathbb{H}^3 \to M \).

The effect on covering transformations is to replace each \( g \in G \) by \( k \circ g \circ k^{-1} \). We now assume that \( V = W, k = \text{id} \) and \( g \mid V \cup S^2_\infty = \phi(g) \mid W \cup S^2_\infty \).

**Definition 2.6.** (Interesting self-maps of \( \mathbb{B}^3 \))

For any \( \delta_0 \cup S^2_\infty \) pointwise, parameterize \( \mathbb{H}^3 \) by \( (r, \theta) \), \( r \geq 0 \), \( \theta \in S^1 \), \( s \in \mathbb{R} \). Here \( \delta_0 \) consists of those points with parameters \( \{(0,0,s) \mid s \in \mathbb{R}\} \), the \( s \) parameter denoting parametrization by arc length. The plane \( P_t \) orthogonal to \( \delta_0 \) at \( t \) has \( s \equiv t \) and is parametrized by polar coordinates via the \((r,\theta)\) coordinates. Finally the parametrization of \( \mathbb{H}^3 \) is chosen so that the \((r,\theta)\) parameters are preserved under pure translation along \( \delta_0 \). Call \( 0 \in \delta_0 \), the point with \( s \)-parameter equal to 0. Assume that \( W_0 = \{(r,\theta,s) \mid r \leq c\} \).

Let \( \alpha, \beta : [0, \infty) \to [0,1] \) be smooth maps such that \( \alpha \mid [0,c] = 1 \), \( \alpha \mid [2c, \infty] = 0 \), \( \beta \mid [0,2c] = 0 \), and \( \beta \mid [3c, \infty] = 1 \).

For \( a \neq 0 \), define \( T_a : \mathbb{B}^3 \to \mathbb{B}^3 \) by \( T_a \mid S^2_\infty = \text{id}, T_a (r,\theta,s) = (r,\theta + \alpha(r)2\pi s/a,s) \).

Define \( R : \mathbb{B}^3 \to \mathbb{B}^3 \) by \( R \mid S^2_\infty = \text{id} \), and \( R(r,\theta,s) = (r,\theta + \beta(r)2\pi s,a) \).

Define a *twist* to be a map conjugate to \( T_a \), and a *roll* a map conjugate to \( R \). In particular a twist or roll about axis \( W_j \), should be viewed as being supported very close to \( W_j \). See Figure 2.1. Figure 2.2 suggests the twist-image of a standardly embedded disc in \( \mathbb{B}^3 \). \(^\text{Figure 2.1}\)

**Lemma 2.7.**

a) If \( h : \mathbb{B}^3 \to \mathbb{B}^3 \) is such that \( h \mid W_0 \cup S^2_\infty = \text{id} \), then \( h \simeq R^n \) rel \( W_0 \cup S^2_\infty \) for some \( n \).

b) If \( W' = \{W_1, \ldots, W_r\} \) is a finite subset of \( W \), and \( h : \mathbb{B}^3 \to \mathbb{B}^3 \) a map such that \( h \mid S^2_\infty \cup W' = \text{id} \), then \( h \) is homotopic rel \( S^2_\infty \cup W' \) to a composit of rollings about elements of \( W' \).

c) If \( h : (\mathbb{B}^3, W_0) \to (\mathbb{B}^3, W_0) \) is a homeomorphism, then \( h \circ R^n \simeq R^n \circ h \) rel \( W_0 \cup S^2_\infty \).

d) If \( h \) is a hyperbolic isometry fixing \( W_0 \) with real length \( a \neq 0 \), then \( T_a^{-n} \circ h^{-1} \circ T_a^n \circ h \simeq R^{-n} \) rel \( W_0 \cup S^2_\infty \).

**Proof.** a) The value of \( n \) is determined by the class \( h((r,0,0) \mid r \in \mathbb{R}) \) \( \in \pi_1(\mathbb{B}^3, W_0 \cup S^2_\infty) \). For the correct \( n \), the straight line homotopy of \( R^{-n} \circ h \) to

\[ \text{Figure 2.1} \]
Figure 2.2. $\sigma$ bounds an embedded disc in $\mathbb{B}^3 - \hat{W}$

id in the appropriately parametrized universal cover of $\mathbb{B}^3 - \hat{W}_0$ projects to a homotopy of $R^{-n} \circ h$ to id.

b) Let $\{A_k\}$ be a set of annuli properly embedded in $\mathbb{B}^3 - \hat{W}'$ which cut off the ends of $\{W_i\}$. There are two $A_k$'s for each $W_i$. If $\mathbb{B}^3$ is the unit ball in $\mathbb{R}^3$, and $e$ is an end of $W_i$, then the $A_k$ associated to $e$ is the intersection of a very small Euclidean sphere centered at $e$, with $\mathbb{B}^3 - \hat{W}_i$. After a preliminary homotopy we can assume that $h$ fixes each $A_k$ setwise. After precomposing $h$ by rollings the resulting function also called $h$ can be homotoped to fix the $A_k$'s pointwise.

Each $A_k$ together with a meridian disc of $W_i$ cuts off a halfball $B_i$ containing no other $A_j$'s. Homotop $h \mid B_i$ to id rel $\partial B_i \cup W_i$. As in the first paragraph, this homotopy can be expressed as the projection of a straight line homotopy associated to a lift of $h$ to an appropriately parametrized covering of $\hat{B} - W_i$. The manifold $H = \mathbb{B}^3 - (\bigcup \hat{B}_i \cup \hat{W}')$ is a handlebody, for a finite set of hyperbolic geodesics in $\mathbb{B}^3$ are unlinked. Since $h \mid \partial H = \text{id}$, $h \mid H$ is isotopic to id rel $\partial H$, e.g., by [W].

c) The action of $R^n \circ h \circ R^{-n} \circ h^{-1}$ on $\pi_1(\mathbb{B}^3, W_0 \cup S^2_\infty)$ is trivial.

d) \[ T_a^{-n} \circ h^{-1} \circ T_a^n \circ h(r, 0, 0) = T_a^{-n} \circ h^{-1} \circ T_a^n(r, b, a) \]
\[ = T_a^{-n} \circ h^{-1}(r, b + n\alpha(2\pi/a, a, a) \]
\[ = T_a^{-n}(r, n\alpha(2\pi, 0) = (r, n\alpha(r)2\pi, 0). \]

Similarly $T_a^{-n} \circ h^{-1} \circ T_a^n \circ h \mid W_0 = \text{id}$. Now apply a).

Proposition 2.8. If $f : (\mathbb{B}^3, \mathbb{B}^3 - \hat{W}, W) \to (\mathbb{B}^3, \mathbb{B}^3 - \hat{W}, W)$ is a map such that $f \mid S^2_\infty \cup W = \text{id}$, and for each $i$, $f : (\mathbb{B}^3, \mathbb{B}^3 - \hat{W}_i, W_i) \to (\mathbb{B}^3, \mathbb{B}^3 - \hat{W}_i, W_i)$ is homotopic to id rel $W_i \cup S^2_\infty$, then $f$ is homotopic to id rel $W \cup S^2_\infty$.

Proof. If $W'$ is a finite subset of components of $W$, then $f : (\mathbb{B}^3, \mathbb{B}^3 - \hat{W}', W') \to (\mathbb{B}^3, \mathbb{B}^3 - \hat{W}', W')$ is homotopic to id rel $W'$, for by Lemma 2.7 b) $f$ is homotopic
to a composite of $n_3$-rolls supported near components $W_i$ of $W'$. By hypothesis each $n_3=0$.

Next we show that if $\alpha$ is a properly embedded closed interval in $\mathbf{H}^3 - \hat{\omega}$, then $f|\alpha \simeq \text{id}$ in $\mathbf{B}^3 - \hat{\omega}$, rel $W \cup S^2_\infty$. Let $B \subset \mathbf{H}^3$ be a round ball centered at 0, transverse to $\Delta$ containing $\alpha$ and $f(\alpha)$. Let $W'$ be the components of $W$ whose cores hit $B$. By viewing $W$ as a thin hyperbolic neighborhood of $\Delta$, we can assume that $W'$ is exactly the set of components of $W$ which hit $B$, and $W'$ is transverse to $B$ with each component hitting $\partial B$ in meridional 2-discs. Now $(\mathbf{H}^3 - \hat{B}, (\mathbf{H}^3 - \hat{B}) \cap W')$ is homeomorphic to $(S^2 \times [0,1], J)$, where $J$ is a neighborhood of vertical arcs in $S^2 \times [0,1]$, and under this identification $S^2 \times 0 = \partial B$. By the first paragraph $f|\alpha \simeq \text{id} \mid \alpha$ in $\mathbf{B}^3 - \hat{\omega}$ rel $\partial \alpha$. By composing this homotopy with the retraction of $\mathbf{H}^3$ onto $B$, which retracts each $(0,1)$ fibre to 0, we see that $f|\alpha \simeq \text{id} \mid \alpha$ rel $\partial \alpha$ in $\mathbf{B}^3 - \hat{\omega}$.

Next consider any handle decomposition of $\mathbf{H}^3 - \hat{\omega}$ by handles of bounded hyperbolic diameter. The previous paragraph really showed that if $h$ is any 1-handle, then $f|h$ is homotopic to $\text{id} \mid \partial h$ (i.e. the attaching region to $W$) via a homotopy supported in $B \cap \mathbf{H}^3 - \hat{\omega}$, where $B$ is any closed smooth convex region of $\mathbf{H}^3$ containing both $h$ and $f(h)$. Thus $f$ can be homotoped to $f_i$, where $f_i | (W \cup 1$-handles) = $\text{id}$ via a homotopy restricting the trace of each 1-handle $h$ to a neighborhood of the convex hull of $h \cup f(h)$. In a similar way homotope $f_i$ to $\text{id}$ on the 2- and 3-handles.

\begin{remark}
In what follows we will be only using the homotopy described in the first two paragraphs of the proof of Proposition 2.8.
\end{remark}

\begin{lemma}
$k$ can be chosen so that for every $\gamma_i \in \Gamma$ and every $g \in G$,
\begin{enumerate}[(*)]
\item $k \circ g \simeq \phi(g) \circ k \text{ rel } V_i \cup S^2_\infty$.
\end{enumerate}
\end{lemma}

\begin{proof}
Let $g_0 \in G$ be a generator of $\text{Stab}(\gamma_0)$. Suppose that $\text{Re}(\text{length}(\gamma_0))=a$. We first show how to choose $k$ so that (*) holds for $g=g_0$ and $V_i = V_0$. We let $g'$ denote $\phi(g)$. By Lemma 2.7, $g_0 \simeq g'_0 \circ R^m$. Replace $k$ (which is id, by Remark 2.5) by $T^m_\alpha$ near $V_0$ and conjugates of $T^m_\alpha$ near each $V_i$ (so that Lemma 2.4 still holds). With respect to this new $k$, $k^{-1} \circ g'^{-1} \circ k \circ g \simeq (T^{-n}_a \circ g_0^{-1} \circ T^n_a \circ g_0 \simeq T^{-n}_a \circ g'_0^{-1} \circ T^n_a \circ g'_0 \circ R^n \simeq R^{-n} \circ R^n = \text{id}$. All homotopies taken rel $V_0 \cup S^2_\infty$.

We show that (*) holds for $g \in \text{Stab}(\gamma_i) \text{ and } V_i$. If $g(\gamma_i) = \gamma_i$, then $g = g_i^{-1} \circ g_0^\circ g_i$ for some $g_i$ taking $\gamma_i$ to $\gamma_0$. For some $m, k \simeq g'^{-1}\circ k \circ g_i \circ R^m \text{ rel } V_i \cup S^2_\infty$, where $R^m$ is an $m$-roll about $V_i$. Therefore $g' \circ k \simeq g' \circ g_i^{-1} \circ k \circ g_i \circ R^m = g_i^{-1} \circ k \circ g_i \circ R^m \simeq k \circ R^{-m} \circ R^m \simeq k \circ R^m \text{ rel } V_i \cup S^2_\infty.\]

For each $i$, pick $g_i \in G$ such that $g_i(\gamma_i) = \gamma_0$. Let $n_i$ be such that $g_i^{-1} \circ k \circ g_i \simeq R^{n_i} \circ k \text{ rel } V_i \cup S^2_\infty$. Again $R^{n_i}$ denotes an $n_i$-roll about $V_i$. The value $n_i$ is a function of $i$ rather than $g_i$, for if $g_i(\gamma_i) = \gamma_0$, then $g = g'_0 \circ g_i$ and $g'^{-1} \circ k \circ g = g_i^{-1} \circ g'_0^{-1} \circ k \circ g'_0 \circ g_i \simeq g_i^{-1} \circ k \circ g_i \simeq R^{n_i} \circ k$. For each $i$, replace $k$ near $V_i$ by $R^{n_i} \circ k$. By Lemma 2.7, this change does not effect the validity of (*) for $g,V_i$ where $g \in \text{Stab}(V_i)$. However, (*) now holds for $g, V_i$ where $g(V_i) = V_0$ or $g(V_0) = V_i$. This in turn implies that (*) holds in general. \end{proof}
Lemma 2.10. Let \( r_g = \phi(g) \circ k \circ g^{-1} \circ k^{-1} \) (resp. \( l_g = g \circ k^{-1} \circ \phi(g^{-1}) \circ k) \). If \( \alpha \) is a properly embedded closed interval in \( H^3 - W \) (resp. \( H^3 - V \)), then \( r_g | _{\partial V} \alpha \simeq \text{id} \) in \( H^3 - \omega \), rel \( \partial V \) (resp. \( l_g | _{\partial W} \alpha \simeq \text{id} \) in \( H^3 - \omega \), rel \( \partial W \)).

Proof of Lemma 2.10. By Lemma 2.7 b), for each \( i \), \( r_g \) is homotopic rel \( W_i \cup \Sigma^3_{\infty} \) to an \( n_i \)-roll supported near \( W_i \). By Lemma 2.9 \( n_i = 0 \). Now apply Proposition 2.8. The argument for \( l_g \) is similar.

Proof of Proposition 2.1. We construct maps \( f : (M, M - \tilde{N}(\gamma), N(\gamma)) \to (N, N - N(\delta), N(\delta)), g : (N, N - N(\delta), N(\delta)) \to (M, M - \tilde{N}(\gamma), N(\gamma)) \) such that \( g \circ f | M^1 = \text{id} \). Here \( M \) has a handlebody structure with a unique 0-handle and \( M^1 \) is the union of the 0- and 1-handles. Also \( N(\gamma) \subset M \) is the union of the 0-handle and a single 1-handle. It will then follow that degree \( f = 1 \), \( f | N(\gamma) \) is a homeomorphism onto \( N(\delta) \) and \( (g \circ f)_# : \pi_1(M - \tilde{N}(\gamma)) \to \pi_1(M - N(\gamma)) \) is \( \pi_1 \)-injective and restricts to a homeomorphism on \( \partial N(\gamma) \). Waldhausen [W] implies that \( f | M - N(\gamma) \) is homotopic to a homeomorphism rel boundary and hence \( f \) is homotopic to a homeomorphism.

The map \( k \) induces a homeomorphism \( f : N(\gamma) \to N(\delta) \). We extend this map to \( M \) as follows. By restricting the size of \( V \), we can assume that \( k \circ g | N(V) = \phi(g) \circ k | N(V) \), where \( N(V) \) is an \( H^3 \) regular neighborhood of \( V \). Give \( M \) a handlebody decomposition with a unique 0-handle \( \alpha \) and 1-handles \( \{ B_0, \ldots, B_r \} \) such that \( A \cup B_0 = N(\gamma) \). If \( i > 0 \), let \( \tilde{B}_i \) be a lift of \( B_i \) to \( H^3 \) and define \( f | B_i = q \circ k \circ p_i \) with the range restricted to \( \tilde{B}_i \). By perturbing the handle structure slightly we can assume that \( f | M^1 \) is an embedding. Let \( C_j \) be a 2-handle, \( \tilde{C}_j \) any lift to \( H^3 \). Define \( f_j = q \circ k \circ p_j \), where \( p_j : C_j \to \tilde{C}_j \) is given by \( \beta^{-1} \). Let \( c_j = C_j \cap M^1 \). This annulus is a union of solid squares which alternately lie in 0 and 1-handles. By Lemma 2.4 if \( s \) is a square of \( c_j \) lying in \( A \) or \( B_0 \), then \( f_j | s = f | s \), however that conclusion may be false for squares in \( B_i, i > 0 \). See Figure 2.3. To define the desired extension of \( f \) to \( C_j \), i.e. so that \( f(C_j) \subset N - \tilde{N}(\delta) \), it suffices to show that for each square \( s \), \( f | s \simeq f_j | s \) rel \( s \cap A \) via a homotopy in \( H^3 - W \). \( s = \alpha \times I \) where \( \alpha \) is an embedded path in \( B_i \) with endpoints in \( A \). \( \alpha \) has two lifts, \( \alpha_i \) determined by \( \tilde{B}_i \) and \( \alpha_j \) determined by \( \tilde{C}_j \). For some \( g \in G, \alpha_j = g(\alpha_i) \). By Lemma 2.10 it follows that when restricted to \( \alpha \), \( f_j = q \circ k \circ p_j \simeq q \circ r(g) \circ k \circ p_j = q \circ \phi(g) \circ k \circ g^{-1} \circ k^{-1} \circ k \circ p_j = q \circ \phi(g) \circ k \circ p_i = q \circ k \circ p_i = f_i \) rel \( \partial \alpha \).

The map \( f \) extends across the 3-handle of \( M \) since \( \pi_2(N - \tilde{N}(\delta)) = 1 \). Construct a handle decomposition of \( N \) with a single 0-handle \( E \) and 1-handles \( \{ F_0, \ldots, F_s \} \) such that \( E = f(A) \) and for \( i \leq r \), \( F_i = f(B_i) \). Construct \( g : (N, N - \tilde{N}(\delta), N(\delta)) \to (M, M - \tilde{N}(\gamma), N(\gamma)) \) in a manner similar to the construction of \( f \), taking care to use the lifts \( k(\tilde{B}_i) \) for \( F_i, i \leq r \). By construction \( g \circ f | M^1 = \text{id} \). 

The methods of this section lead to an elementary proof of the following Proposition 2.11, an unpublished circa 1981 theorem of Siebenmann [BS] communicated to the author by Francis Bonahon. Siebenmann’s proof employed Thurston’s geometrization theorem, [Th], Mostow’s rigidity theorem [Mo], Waldhausen’s isotopy...
Theorem [W] and the following theorem of Neumann [Ne]: \( \text{id} \) is the only periodic homeomorphism of \( B^1 \) which fixes \( S^2 \) pointwise. Here is the idea. \( N - \delta \) has a hyperbolic structure [Th], hence \( f \mid N - \delta \) is homotopic [Mo], hence isotopic [W] to an isometry, which is necessarily periodic. A lift \( \tilde{f} \) extends to a periodic map of \( B^3 \) pointwise fixing \( S^2_{\infty} \), so \( f = \text{id} \) by [Ne].

**Proposition 2.11.** If \( f : N \to N \) is a homeomorphism homotopic to \( \text{id} \) and \( f \mid \delta = \text{id} \), where \( N \) is a closed hyperbolic 3-manifold and \( \delta \) is a simple closed geodesic in \( N \), then \( f \) is isotopic to \( \text{id} \).

**Proof.** After a preliminary isotopy we can assume that \( f \) is smooth by [Mu], [Ki].

We will show that either \( N \) is Haken or \( f \) is isotopic to a map \( g \) such that \( g \mid N(\delta) = \text{id} \) and \( g \mid N - \bar{N}(\delta) \) is homotopic to \( \text{id} \) rel \( \partial N(\delta) \). In either case \( f \) is isotopic to \( \text{id} \) by [W].
We can assume that either $N$ is Haken or $f \mid N(\delta) = \text{id}$. In fact, a standard argument shows that $f \mid N(\delta)$ is isotopic to Dehn twists about the meridian of $N(\delta)$. For homological reasons there exists an essential simple closed curve $\alpha \subset \partial N(\delta)$, unique up to isotopy, such that some power is homologically trivial in $N - \tilde{N}(\delta)$. Therefore, $f \mid N(\delta)$ has nontrivial Dehn twisting only if $\alpha$ is a meridian, which implies that $H_2(N) \neq 0$, which implies that $N$ is Haken.

Let $W = \bigcup W_i = q^{-1}(N(\delta))$. Since $f$ fixes $N(\delta)$ pointwise and $f$ is homotopic to id, there exists a lift $\hat{f}$ such that $\hat{f} \mid W = \text{id}$. By applying Lemma 2.7 a) and isotoping $f$ to achieve a rolling of $N(\delta)$, we can assume that for each $i$, $\hat{f} : (B^3, B^3 - \tilde{W}_i, W_i) \to (B^3, B^3 - \tilde{W}_i, W_i)$ is homotopic to id rel $W_i \cup S^2_{\infty}$. Construct a relative handle decomposition of $N - \tilde{N}(\delta)$. Let $h$ be a 1-handle and $\tilde{h}$ a lift to $H^3$. By Proposition 2.8 $\hat{f} \mid \tilde{h}$ is homotopic to id rel $\tilde{h} \cap W$, via a homotopy which does not cross $W$. Project this homotopy into $N$ to obtain a homotopy of $f \mid h$ to id| h which does not cross $N(\delta)$. Construct a similar homotopy on each 1-handle. The vanishing of $\pi_2, \pi_3$ on $N - \tilde{N}(\delta)$ allows us to extend the homotopy over the 2- and 3-handles. Therefore $f \mid N - \tilde{N}(\delta) \simeq \text{id} \mid \partial N(\delta)$. By [W] $f$ is isotopic to id rel $N(\delta)$. 

\section{Minimal surface lemmas}

The main results of this section, Propositions 3.9 and 3.10, assert that given a Riemannian metric $r$ on $H^3$ arising from the closed hyperbolic 3-manifold $X$ and a simple closed curve $\lambda$ in $S^2_{\infty}$, there exists a lamination $\sigma$ by $r$-least area planes in $H^3$ which spans $\lambda$. A sequence of such laminations, with underlying metrics $s_i \to s$, converges to a spanning lamination by $s$-least area planes.

In the setting $r$ is the hyperbolic metric, Anderson [A] showed “$\lambda$ bounds a least area properly embedded plane”. Our $\sigma$ will be the limit of a sequence of compact $r$-least area discs in $H^3$ whose boundaries approach $\lambda$. Unlike [A] the restriction of our sequence to a fixed compact region of $H^3$ may not have uniformly bounded area. As a consequence the leaves of our laminations may not be properly embedded. See Conjecture 3.12.

All the background needed to read this section is contained in pp. 89–99 of Joel Hass’ and Peter Scott’s sharp and to the point paper [HS]. Our arguments rely on the foundational results of [Mor], [MSY] and [S].

\textbf{Definition 3.1.} Let $\rho, p, q, M, X, B^3, S^2_{\infty}$, etc. be as in Notation 1.2. If $E \subset B^3$, then $C(E)$ denotes its hyperbolic convex hull. We abuse notation by letting a Riemannian metric on $M$ or $X$ also denote the induced metric on $X$ or $H^3$. An immersed disc with boundary $\gamma$ is a least area disc if it is least area among all immersed discs with boundary $\gamma$. An injectively immersed plane is a least area plane if each compact subdisc is a least area disc.

A codimension-$k$ lamination $\sigma$ in the $n$-manifold $Y$ is a codimension-$k$ foliated closed subset of $Y$, i.e. $Y$ is covered by charts of the form $R^{n-k} \times R^k$ and $\sigma \mid R^{n-k} \times R^k$ is the product lamination on $R^{n-k} \times C$, where $C$ a closed subset of $R^k$. Here and later we abuse notation by letting $\sigma$ also denote the underlying space of its lamination, i.e. the points of $Y$ which lie in leaves of $\sigma$. Laminations in this paper will be codimension-1 in manifolds of dimension 2 or 3.
A complementary region \( J \) is a component of \( Y - \sigma \). Given a Riemannian metric on \( Y \), \( J \) has an induced path metric, the distance between two points being the infimum of lengths of paths in \( J \) connecting them. A closed complementary region is the metric completion of a complementary region with the induced path metric. As a manifold with boundary, a closed complementary region is independent of metric.

**Definition 3.2.** The sequence \( \{S_i\} \) of embedded surfaces or laminations in a Riemannian manifold \( Y \) converges to the lamination \( \sigma \) if

i) \( \sigma = \{ x = \text{Lim}_{n \to \infty} x_i \mid x_i \in S_i \text{ and } \{x_i\} \text{ a convergent sequence in } Y \} \);

ii) \( \sigma = \{ x = \text{Lim}_{n \to \infty} x_{n_i} \mid \{n_i\} \text{ an increasing sequence in } \mathbb{N}, x_{n_i} \in S_{n_i} \text{ and } \{x_{n_i}\} \text{ a convergent sequence in } Y \} \text{ def } \text{Lim}\{S_i\}, \)

Let \( t_i \) be a \([0, 1]\)-parameter family of Riemannian metrics on \( \mathbb{H}^3 \) obtained by lifting a \([0, 1]\)-parameter family on a closed hyperbolic manifold \( X \). If \( S_i \) is a sequence of embedded least area discs in \( \mathbb{H}^3 \) with the \( r\)-metric, where \( \partial S_i \to \infty \), then after passing to a subsequence \( \{S_i\} \) converges to a (possibly empty) lamination by \( r \)-least area planes.

**Lemma 3.3** (convergence of least area discs). i) Let \( r \) be a Riemannian metric on \( \mathbb{H}^3 \) which is the lift of a metric on a closed hyperbolic manifold \( X \). If \( \{S_i\} \) is a sequence of embedded least area discs in \( \mathbb{H}^3 \) with the \( r \)-metric, where \( \partial S_i \to \infty \), then after passing to a subsequence \( \{S_i\} \) converges to a (possibly empty) lamination by \( r \)-least area planes.

**Proof.** We first give the proof of i).

**Step 1.** After passing to a subsequence \( \text{Lim}\{S_i\} = \{ x = \text{Lim}_{i \to \infty} x_i \mid x_i \in S_i \text{ and } \{x_i\} \text{ a convergent sequence in } \mathbb{H}^3 \} = Z \), a closed subset of \( \mathbb{H}^3 \).

**Proof of Step 1.** For each \( j \) subdivide \( \mathbb{H}^3 \) into a finite number of closed regions, such that the \( j + 1 \)st subdivision subdivides the \( j \)th one and such that for each closed ball \( B \) in \( \mathbb{H}^3 \), the mesh of these subdivisions restricted to \( B \) converges to 0. Choose a subsequence of \( \{S_i\} \) so that if \( i \geq j \) and \( S_i \) hits a region of the \( j \)th subdivision, then so does \( S_{k_i} \), if \( k > i \).

**Step 2.** Let \( \{z_j\} \) be a countable dense subset of \( Z \). There exists \( \epsilon > 0 \) such that after passing to a subsequence of \( \{S_j\} \) the following holds. For each \( i \) there exists a sequence of embedded discs \( D_i \subset S_j \) which converges to a smoothly embedded least area disc \( D_i \) such that \( z_i \in D_i \) and \( \partial D_i \cap B_\epsilon(z_i) = \emptyset \).

**Proof of Step 2.** The compactness of \( X \) and Theorem 3 of [S] imply that there exist \( n, \epsilon > 0 \) such that if \( x \in \mathbb{H}^3 \) and if \( S \) is an embedded \( r \)-least area compact disc such that \( \partial S \cap B_\epsilon(n \epsilon, x) = \emptyset \), then after deleting isolated points \( S \cap B_\epsilon(2 \epsilon, x) \) is a union of properly embedded discs of bounded second fundamental form. (Informally, Schoen’s local theorem asserts that a least area surface restricted to a sufficiently small ball \( B \) does not bend very much provided that the boundary is sufficiently far from \( B \). The bound on bending depends only on the local curvature tensor. Since
the $r$-metric on $\mathbb{H}^3$ is induced from a closed manifold, we can make the above global statement.) By reducing the size of $\epsilon$, if necessary, we can assume that all closed balls of $r$-radius $\alpha$, $\alpha < n \epsilon$, are $B^3$'s with strictly convex boundary. Fix $i$. For $j$ sufficiently large let $D_{ij}$ be a component of $S_i \cap B_r(2\epsilon, z_i)$ such that $d(z_i, D_{ij}) \to 0$. Since the $D_{ij}$'s are $r$-least area, they have area bounded above by $\text{Sup} 1/2\{\text{Area } \partial B_r(2\epsilon, x) \mid x \in \mathbb{H}^3\}$. By Lemma 3.3 [HS] after passing to a subsequence and restricting to $B_r(\epsilon, z_i)$ the $D_{ij}$'s converge (in the sense of Definition 3.2) to the desired $D_i$. Since this is true for each $i$, the usual diagonal subsequence argument completes the proof of Step 2.

**Step 3.** There exists a lamination $\sigma$ with underlying space $Z$, such that each $D_i$ is contained in a leaf. Furthermore $\{S_i\}$ converges to $\sigma$.

**Proof of Step 3.** By Step 1, i) of Definition 3.2 holds. By Step 2, for each $i$, $D_i \subset Z$. If $x \in D_i \cap D_j$, then $D_i$ and $D_j$ coincide in a neighborhood of $x$. Otherwise being minimal surfaces, $D_i$ and $D_j$ would cross transversely at some point close to $x$ (e.g. Lemma 3.6 [HS]), which would imply that $S_i$ was not embedded for $k$ sufficiently large. If $z \in Z$, then the argument of Step 2 shows that there exists a convergent sequence $\{D_{iz}\} \to D_z$, where $D_{iz}$ is a subdisc of some $D_j$, $z \in D_z$ and $\partial D_z \cap B_r(\epsilon, z) = \emptyset$. Again since the $D_i$'s pairwise either locally coincide or are disjoint, $D_z$ is uniquely determined in an $\epsilon$-neighborhood of $z$. Thus $Z = \bigcup_{z \in Z} D_z$. Using the $D_z$'s to define a topology on $Z$, it follows that connected components are leaves of a lamination $\sigma$ with underlying space $Z$. The uniqueness of $D_z$ in $B_r(\epsilon, z)$ implies that near $z$ leaves of $\sigma$ are graphs of functions over $D_z$ and that $\{S_i\}$ converges to $\sigma$.

Since $\{S_i\}$ converges to $\sigma$, we obtain

**Step 4.** If $g : D \to L$ is an immersion of a disc into a leaf $L$ of $\sigma$, then for all $i$ sufficiently large there exists an immersion $g_i : D \to S_i$ such that $g_i \to g$ in the $C^\infty$ topology.

**Step 5.** Each leaf $L$ of $\sigma$ is a least area plane.

**Proof of Step 5.** Let $\tau$ be an essential simple closed curve in $L$ and $A \subset L$ a thin (e.g. $< 5\epsilon$) regular neighborhood of $\tau$. Let $B \subset \mathbb{H}^3$ be a 3-ball transverse to $\bigcup S_i$ such that $A \subset \overset{\circ}{B}$. Let $g : D \to L$ be an isometric immersion of a disc such that $g(D) = A$ and $\text{Area}(D) > \text{Area}(\partial B)$. (Think of $D$ as being a long thin rectangle.) By Step 4, for $i$ sufficiently large, $g$ is closely approximated by an isometric immersion of a 2-disc, i.e. $g_i : D_i \to S_i$ and $\text{Area}(D_i) > \text{Area}(\partial B)$. For $i$ sufficiently large $g_i(D_i)$ is an annulus which closely approximates $A$. Otherwise $g_i(D_i)$ is an embedded disc which spirals around and closely approximates $A$. This contradicts the fact that if $B$ is a ball and $\partial S_i \cap B = \emptyset$, then $\text{Area}_\epsilon(P) \leq 1/2\text{Area}_\epsilon(\partial B)$, where $P$ is a component of $S_i \cap B$. Thus for each sufficiently large $i$, there exists an embedded simple closed curve $\tau_i \subset S_i$ such that $\{\tau_i\}$ converges to $\tau$. Each $\tau_i$ bounds a disc $E_i \subset S_i$ of uniformly bounded area. The sequence of discs $\{E_i\}$ converges to a disc in $L$ bounded by $\tau$ via arguments similar to those of the proof of Step 3. Thus $L$ is simply connected. $L$ is not a sphere else for $i$ sufficiently large each $S_i$ would be a sphere. Since each embedded subdisc of $L$ is the limit of least area discs by Step 4, each embedded subdisc of $L$ is least area and hence $L$ is a least area plane.
Proof of ii). The proof of ii) follows exactly as the proof of i). Perhaps only Step 2 requires some clarification. Again by [S], there exists an $\epsilon > 0$ independent of $x$, such that if $j$ is sufficiently large (so that $t_j$ is very close to $t$ and $\partial S_j \cap B_r(\epsilon, x) = \emptyset$), then each component of $S_j \cap B_r(2\epsilon, x)$ is a properly embedded disc of uniformly bounded second fundamental form. Similarly given $\delta > 0$, then for $j$ sufficiently large $\text{Area}_{r_j}(D_{t_j})$ is bounded above by $1/2\text{Area}_{r_j}(\partial(B_r(2\epsilon, z_j))) + \delta$ and hence the $D_{t_j}$’s can be parametrized to have uniformly bounded energy with respect to the $r_j$-metric. These are the facts needed to invoke the proof of Lemma 3.3 [HS].

Remark. The lemma could have been stated in more generality by allowing each $S_i$ to be a finite union of pairwise disjoint least area discs such that $\partial S_i \to \infty$.

**Definition 3.4.** A lamination $\sigma$ which is a limit as in Lemma 3.3 of a sequence of embedded least area discs $\{S_i\}$ (or more generally a lamination by finite unions of pairwise disjoint least area discs) such that $\partial S_i \to \infty$ will be called a $D^2$-limit lamination. The $D^2$-limit lamination $\sigma$ spans the simple closed curve $\tau \subset S^{2}_{\infty}$, if there exists $\epsilon > 0$ such that $\sigma \subset N_{\rho}(\epsilon, C(\tau))$ and the components of $S^{2}_{\infty} \setminus \tau$ lie in different components of $B^{3} \setminus \sigma$. Recall that $\rho$ denotes the hyperbolic metric.

The following standard result records all the other elementary least area surface facts needed in this section. Most of these observations were made either implicitly or explicitly in the proof of Lemma 3.3. For convenience we record several $D^2$-limit laminations facts too.

**Lemma 3.5.** Let $X$ be a closed hyperbolic $3$-manifold. Let $r_1, t \in [0, 1]$ be a family of metrics on $H^3$ induced from a $1$-parameter family of Riemannian metrics on $X$.

i) For each $t \in [0, 1]$, the $r_1$-area differs infinitesimally from the $\rho$-area by bounded factors $1/c_{2}, c_{2}$, where $c_{2}$ is independent of $t$.

ii) There exist constants $c_{0}, c_{1}$ such that if $P$ is a least area disc or plane in $H^3$ with the $r_1$-metric, $y \in P$ and $B = B_{\rho}(c_{0}, y) \subset H^3$ is such that $B \cap \partial P = \emptyset$, then $\text{Area}_{\rho}(P \cap B) > c_{1}$.

iii) If $P$ is an $r_1$-least area disc, $y \in P$ and $d_{\rho}(y, \partial P) > 3c_{0}$, then $d_{\rho}(y, \partial P) < 3c_{0}\text{Area}_{\rho}(P)/c_{1}$.

iv) If $\sigma$ is a $D^2$-limit lamination, then $\sigma$ has no holonomy.

v) Let $W$ be a smooth compact codimension-$0$ submanifold of $H^3$ transverse to the $D^2$-limit lamination $\sigma$. Then $\sigma | W$ is a finite union of product laminations. I.e. there exist finitely many pairwise disjoint submanifolds $W_{i}$ of $W$ of the form $F_{i} \times I$, where $F_{i}$ is a compact surface, $(F_{i} \times I) \cap \partial W = (\partial F_{i}) \times I$ and $\sigma | F_{i} \times I$ is the product lamination $F_{i} \times C_{i}$, where $C_{i} \subset I$ is compact.

In particular the leaves of $\sigma | W$ have uniformly bounded area and the leaves of $\sigma | \partial W$ are simple closed curves of uniformly bounded length.

vi) Each leaf $L$ of a $D^2$-limit lamination $\sigma$ has an exhaustion by compact discs $P_{i}$, such that $\partial P_{i} \to \infty$. Furthermore if $\sigma = \text{Lim}\{S_{i}\}$, then for each $i$, there exist least area discs $\{E_{i, j}\}$ converging to $P_{i}$, such that $E_{i, j} \subset S_{j}$.

vii) If, for $i = 1, 2$, $L_{i}$ is a leaf of the $D^2$-limit lamination $\sigma_{i}$, then no component of $L_{1} \cap L_{2}$ contains a simple closed curve.

viii) If, for $i = 1, 2$, $\sigma_{i}$ is a $D^2$-limit lamination spanning $\lambda_{i} \subset S^{2}_{\infty}$ and $\sigma_{1} \cap \sigma_{2} \neq \emptyset$, then $\lambda_{1} \cap \lambda_{2} \neq \emptyset$.

ix) If $\sigma$ is a codimension-$1$ lamination in the $3$-manifold $Y$, then there exists a possibly empty, nowhere dense sublamination $\kappa$ such that each closed complementary region of $\sigma$ is a closed complementary region of $\kappa$. 
x) If $\kappa$ is a nowhere dense lamination in the 3-manifold $Y$ and $W$ is a compact codimension-0 submanifold of $Y$, then $W$ can be isotoped slightly to be transverse to $\kappa$.

Proof. i) The metrics on $H^3$ arise from a $[0,1]$-family of metrics on a closed manifold.

ii) follows from i) and the monotonicity formula (e.g. Lemma 2.3 [HS]).

iii) Apply ii).

iv) Since each leaf is simply connected, this follows from the Reeb stability theorem applied to laminations. E.g. see [GO].

v) We first show that each leaf of $\sigma | W$ is compact. Let $B$ be a ball such that $W \subset \bar{B}$ and $B$ is transverse to $\sigma$ except possibly at finitely many points. At these points the tangencies should be Morse like. If a leaf $P$ of $\sigma | B$ was noncompact, then $P$ would pass through a lamination chart in $B$ infinitely often and so $P$ would have infinite area. By Step 4 of the proof of Lemma 3.3, compact regions of $P$ are closely approximated by compact regions of $S_j$ for $j$ sufficiently large. This contradicts the fact that the $r_i$-area of components of $S_j \cap \bar{B}$ is bounded above by $c_2 \text{Area}_p(\partial B)$. Since each leaf of $\sigma | W$ is contained in a leaf of $\sigma | B$, the leaves of $\sigma | W$ are compact and of uniformly bounded area.

By the Reeb stability theorem and iv) each leaf $F$ of $\sigma | W$ has a neighborhood $W_i \subset W$ such that $\sigma | W_i$ is a product lamination. Conclusion v) now follows from the compactness of $\sigma | W$. The condition $C_i \subset \tilde{I}$ follows if one uses maximal product laminations. In reality, by ix)–x), our laminations will never be locally dense, so the condition $C_i \subset \tilde{I}$ is essentially automatic.

vi) Fix $x \in \tilde{L}$. Using the proof of v) construct $P_{j}, j \in \mathbb{N}$ so that for each $i, P_i \subset P_{i+1}$ and $\partial P_i \subset \tilde{L} \cap B_p(i, x)$. The second part follows from Step 4 of the proof of Lemma 3.3.

vii) If such a curve exists, then apply the Meeks-Yau exchange roundoff trick to show that one of $L_1, L_2$ is not least area.

viii) Proof by contradiction. If $x \in \sigma_1 \cap \sigma_2$, then let $B$ be a ball such that $x \in B$ and $\partial B \cap N_p(e_1, C(\lambda_1)) \cap N_p(e_2, C(\lambda_2)) = \emptyset$. The $e_i > 0$ are chosen to have the property that, for $i = 1, 2$, $\sigma_i \subset N_p(e_i, C(\lambda_i))$. If for $i = 1, 2$ there exists a leaf $L_i$ of $\sigma_i | B$ such that $x \in L_i$, then $L_1 \cap L_2$ contains a circle of intersection, for each $L_i$ is compact. (Recall 2.6 [HS].) This contradicts vii).

ix) Take $\kappa$ to be the closure of all the boundary leaves of $\sigma$. This lemma allows us to avoid some technicalities in the very unlikely event that a lamination arising from Propositions 3.9 and 3.10 is somewhere locally dense. I.e. we can treat the lamination more like a properly embedded surface than like a foliation.

x) Use general position.

\[\square\]

**Definition 3.6.** Let $\alpha$ be an unknotted simple closed curve in $H^3$ with the $r$-metric. Change the $r$-metric of $U = H^3 - \hat{\mathcal{N}}(\alpha)$ by one which coincides with $r$ away from a very small neighborhood of $\partial U$ and which gives $U$ a strictly convex boundary. It follows by [MSY] that an essential simple closed curve on $\partial \mathcal{N}(\alpha)$, also called $\alpha$, bounds a properly embedded disc $D \subset U$, least area among all immersed discs $E \subset U$ with $\partial E \subset \partial U$ and $\partial E$ essential in $\partial U$. Call a disc that arises from this construction a relatively least area disc in $H^3$. 

Lemma 3.7. Let \( r_i \) be a \([0,1]\)-parameter family of Riemannian metrics on \( \mathbb{H}^3 \) obtained by lifting a \([0,1]\)-parameter family on a closed hyperbolic manifold \( X \). There exists \( \varepsilon > 0 \) such that if \( S \) is a relatively least area disc in \( \mathbb{H}^3 \) with the \( r_i \)-metric, then \( S \subset N_\varepsilon(e,C(\partial S)) \).

The proof we give is a technically simpler version of the following more concisely stated outline. Either Lemma 3.7 holds or by applying Lemma 3.3 to a sequence of discs we obtain an embedded least area plane lying in a horoball based at a point in \( S^2_\infty \). Such a plane can be chosen to be disjoint from all its translates under \( G \). The projection to \( X \) is a leaf of an essential lamination \( \kappa \) by least area planes. By Imanishi (see [G2]) only the 3-torus has an essential lamination by planes.

Proof of Lemma 3.7. Step 1. There exists an \( r \)-least area plane \( \tilde{L} \) which is a leaf of a \( D^2 \)-limit lamination and which lies in a horoball of \( \mathbb{H}^3 \).

Proof of Step 1. Suppose that for each \( i \), there exists a relatively \( r_i \)-least area disc \( D'_i \) such that \( D'_i \not\subset N_\varepsilon(i,C(\partial D'_i)) \). Let \( z_i \in D'_i \) be a point farthest from \( C(\partial D'_i) \). A covering transformation of \( \tilde{q} : \mathbb{H}^3 \rightarrow X \) is an isometry in both the \( r_i \) and hyperbolic metrics. Therefore by replacing each \( D'_i \) by a covering translate and passing to a subsequence, we can assume that the \( z_i \) converge to a fixed \( z \in \mathbb{H}^3 \). After reparametrizing \( \mathbb{H}^3 \) and using the unit disc model, we can assume that \( z = 0 \). By passing to another subsequence we can assume that \( \text{Lim}\{C(\partial D'_i)\} = w \subset S^2_\infty \). Conclude that \( \text{Lim}\{D'_i\} \subset H \), the horoball which contains both 0 and \( w \). [Note that if \( y \in \mathbb{H}^3 - H \) and \( t \in \mathbb{H}^3 \subset \mathbb{B}^3 \) is sufficiently Euclidian close to \( w \), then \( d_p(0,t) < d_p(y,t) \).]

Cut down the size of the \( D'_i \) and pass to a subsequence to obtain a new sequence as above, called \( \{D_i\} \), such that for each \( i \), \( D_i \) is a \( r_i \)-least area disc (i.e. rather than just relatively least area). To prove this use the following observations. For \( N \subset \mathbb{N} \) let \( B(N) \) denote \( B_\varepsilon(N,0) \) perturbed slightly to be transverse to \( \bigcup D'_i \). If \( \tau \) is a component of \( D'_i \cap \partial B(N) \) and \( i \) is sufficiently large, then the subdisc \( E \) of \( D'_i \) bounded by \( \tau \) is an \( r_i \)-least area disc. Otherwise, since \( \text{Area}_{r_i}(E) < 1/2\text{Area}_{r_i}(\partial B(N)) \), any least area disc \( F \) with \( \partial F = \partial E \) (which exists by [Mor] (see [HS])) must be somewhat close to 0 by Lemma 3.5 iii) and hence be disjoint from \( \partial D'_i \) for \( i \) sufficiently large. Since \( D'_i \) is relatively least area, \( F \) and \( E \) have the same \( r_i \)-area and hence \( E \) is \( r_i \)-least area. Finally observe that

\[
\text{Lim} d_p(0,C(\partial B(N) \cap H)) \rightarrow \infty.
\]

Apply Lemma 3.3 ii) to pass to a subsequence of the \( D_i \) and obtain the \( D^2 \)-limit lamination \( \sigma \), each of whose leaves is an \( r \)-least area plane. Let \( \tilde{L} \) be the leaf which contains 0. By Lemma 3.5 vi) \( \tilde{L} \) is a union of embedded nested area discs whose boundaries go to infinity. Replace the old sequence of discs by this sequence, also denoted \( \{D_i\} \).

Step 2. Let \( G_X \) denote the group of covering translations of \( \mathbb{H}^3 \) associated to \( X \). There exists an \( r \)-least area plane \( \tilde{Q} \) such that for each \( g \in G_X \), either \( g(\tilde{Q}) = \tilde{Q} \) or \( g(\tilde{Q}) \cap \tilde{Q} = \emptyset \). Furthermore either \( g(\tilde{Q}) \cap \tilde{L} = \emptyset \) or \( g(\tilde{Q}) = \tilde{L} \).

Proof of Step 2. If \( w \) is not the fixed point of any element of \( G_X \), then \( \tilde{L} \) is the desired \( \tilde{Q} \), otherwise there exists \( g \in G_X \) such that \( g \neq \text{id} \) and \( g(\tilde{L}) \cap \tilde{L} \neq \{\emptyset, \tilde{L}\} \). Since \( g(w) \neq w \), there exists some \( i \) such that \( g(D_i) \cap D_i \neq \emptyset \) but \( g(\partial D_i) \cap (\partial D_i) = \emptyset \). This leads to a contradiction by the exchange roundoff trick.
The other possibility is that $w$ is the fixed point of some primitive element $f$ of $G_X$. We find $Q$ as follows. Let $A_f$ denote the hyperbolic axis of $f$. There does not exist $N > 0$ such that $\hat{L} \subset N \pi(N, A_f)$. Otherwise, for any $t \in A_f$, each component of $H_t \cap \hat{L}$ would have area bounded by $c_2 \cdot \text{(area of the hyperbolic disc of radius } N)$. Here $H_t$ is the $B^3$ halfspace disjoint from $w$ and bounded by the hyperbolic plane orthogonal to $A_f$ at $t$. This contradicts Lemma 3.5 iii), for $t$ close to $w$.

Let $\{y_i\}$ be a sequence of points in $\hat{L}$ such that $d(y_i, A_f) > i$. Let $g_i \in G_X$ be such that $g_i(y_i) = v_i$ lies in a fixed $X$-fundamental domain $V$ in $H^3$. By passing to a subsequence we can assume that $v_i \to v \in H^3$ and $g_i(w) \to w'$. By passing to another subsequence we can assume that $i \neq j$ implies that $w_i \overset{\text{def}}{=} g_i(w) \neq g_j(w) \overset{\text{def}}{=} w_j$. Suppose on the contrary that for all $i, j, g_i(w) = g_j(w)$. Then $g_i(w) = g_j(w) \implies g_j^{-1} \circ g_i(w) = w \implies g_j^{-1} \circ g_i = f^{n_i} \implies g_i = g_j \circ f^{n_i}$. Now $g_i(y_i) \subset V \implies y_i \in g_i^{-1}(V) = f^{-n_i} \circ g_j^{-1}(V) \implies d(y_i, A_f) \leq \max\{d(g_i^{-1}(z), A_f) \mid z \in V\}$. The finiteness of the latter contradicts the choice of $y_i$, for $i$ large.

Let $\tilde{Q}$ be a least area plane passing through $v$, obtained by applying Lemma 3.3 to the sequence $g_i(\hat{L}) = \hat{L}_i$, or more precisely to $\{g_i(D_{n_i})\}$, where $\{n_i\}$ is a sufficiently fast growing sequence. There exists no $h \in G_X$ such that $h(\tilde{Q}) \cap \tilde{Q} \neq \{\emptyset, \tilde{Q}\}$; else for sufficiently large $i, j$, $h(\tilde{L}_j) \cap \tilde{L}_i \neq \emptyset$. Therefore there exists $i, j$ such that $h(\tilde{L}_j) \cap \tilde{L}_i \neq \emptyset$ and $w_i \neq h(w_j)$. This implies that $g_i^{-1} \circ h \circ g_j(\tilde{L}) \cap \tilde{L} \neq \emptyset$ and $g_i^{-1} \circ h \circ g_j(w) \neq w$, which is a contradiction. A similar argument shows that $h(\tilde{L}) \cap \tilde{Q} \in \{\emptyset, \tilde{Q}\}$.

Step 3. There exists an $r$-least area properly embedded plane $\hat{P}$ contained in a horoball in $H^3$ such that for each $g \in G_X, g(\hat{P}) = \hat{P}$ or $g(\hat{P}) \cap \hat{P} = \emptyset$. If $\pi : H^3 \to X$ is the covering projection, then $\pi(\hat{P})$ projects to a leaf $P$ of an essential lamination $\kappa$ in $X$. Finally the leaves of $\kappa$ lift to $r$-least area planes in $H^3$ and each leaf of $\kappa$ is dense in $\kappa$.

Proof of Step 3. Let $\lambda$ be the lamination in $X$ obtained by taking the closure of the injectively immersed surface $Q$ which is the projection of $\tilde{Q}$. We show that $\lambda$ is essential by showing that each leaf is incompressible and end incompressible [GO]. Each leaf $Q_\alpha$ of $\lambda$ lifts to a surface $\hat{Q}_\alpha$ in $H^3$ which is a limit of translates of subdiscs of $\tilde{Q}$, hence $\hat{Q}_\alpha$ is a leaf of a $D^2$-limit lamination and hence is a least area plane, so $Q_\alpha$ is incompressible. An end compression of $Q_\alpha$ would imply the existence of a monogon in $H^3$ connecting two very close together subdiscs of $\tilde{Q}$ of very much larger area, contradicting the fact that $\hat{Q}_\alpha$ is least area as in Figure 4 of [HS].

Let $\kappa$ be a nontrivial sublamination of $\lambda$ such that each leaf of $\kappa$ is dense in $\kappa$.

The lift $\tilde{\kappa}$ of $\kappa$ to $H^3$ is a sublamination of the lamination which is the closure of all the $G_X$-translates of $\tilde{Q}$. Since $\hat{L}$ is either disjoint from $\tilde{\kappa}$ or a leaf of $\tilde{\kappa}$, it follows that $L = \pi(\hat{L})$ is either a leaf of $\kappa$ or disjoint from $\kappa$. By construction $\kappa \subset \hat{L}$ since $\tilde{Q}$ is in the closure of $G_X(\hat{L})$.

If $\hat{L}$ is a leaf of $\tilde{\kappa}$, then Step 3 holds with $\hat{P} = \hat{L}$. In that case since $\hat{L}$ is the lift of a leaf of an essential lamination, it follows by [GO] that $\hat{L}$ is properly embedded in $H^3$.

Next consider the case that $L \subset J$ is a closed complementary region of $\kappa$. The proof of Step 3 follows from the following
Claim. \( J = \hat{D}^2 \times I \) and \( L \) is homotopic to \( \hat{D} \times 1/2 \) via a homotopy in \( J \) in which
points of \( L \) are moved by homotopy tracks of uniformly bounded length.

Proof of the Claim. As in [GO] \( J \) is of the form \( A \cup Z \), where each component of \( A \) is an \( I \)-bundle over a noncompact surface, \( Z \) is a connected compact 3-manifold and \( A \cap Z \) is a union of annuli. Since \( X \) is of finite volume, by taking \( Z \) to be sufficiently big (by reducing the size of \( A \)) we can assume that the \( I \)-fibres are very short \( \rho \)-geodesic arcs nearly orthogonal to \( \partial J \). Recall that by [S] \( \hat{L} \) and hence \( L \) have bounded second fundamental form. This implies that if the \( I \)-fibres are sufficiently short, then they must be transverse to \( L \). Thus we can assume that \( L \) is transverse to the \( I \)-fibres of \( A \).

Assume \( A \neq \emptyset \). If \( E \) is a vertical annulus in \( A \), i.e. a union of \( I \)-fibres, then either \( E \) spans a \( D^2 \times I \subset J \) or \( E \cap L = \emptyset \). Otherwise \( E \) lifts to an \( I \times \mathbb{R} \) whose core \( \alpha \) is properly homotopic (by the previous paragraph) to a curve lying in \( \hat{L} \), contradicting Step 1, for \( \alpha \) has distinct endpoints in \( S^2_{\infty} \). Since \( \kappa \subset \hat{L} \), it follows that some component \( A_1 \) of \( A \) and hence each component of \( A_1 \cap Z \) nontrivially intersect \( L \) and hence \( A_1 = A \) and \( J \) is obtained by attaching 2-handles to \( A \) along \( A \cap Z \). Since each vertical annulus in \( A \) bounds a \( D^2 \times I \), it follows that \( J = \hat{D}^2 \times I \).

Since \( J \) is simply connected, it lifts to \( \mathbb{H}^3 \) and hence \( L \) is embedded in \( J \) since \( \hat{L} \) is embedded in \( \mathbb{H}^3 \). Therefore if \( E \subset A \) is a vertical annulus, then \( E \cap L \) is a union of embedded circles. Each such circle bounds a disc in \( L \) which is isotopic rel boundary to a horizontal disc in the associated \( D^2 \times I \). If \( P \) is a component of \( \partial J \), then vertical projection of \( L \cap A \) to \( P \cap A \) extends to an immersion of \( L \) to \( P \). \( P \) being simply connected implies that this is in fact a diffeomorphism. Again as in [GO] each lift of \( P \) is properly embedded.

If \( A = \emptyset \), derive a contradiction as follows. In this case \( \kappa \) is a closed \( \pi_1 \)-injective surface \( S_0 \). Consider an incompressible surface \( S_1 \) in \( X \) split open along \( S_0 \) which nontrivially intersects \( S_0 \) and consider \( L \cap S_1 \) to argue that the limit set of \( \hat{L} \) consists of more than a point.

Since each leaf of \( \kappa \) is dense in \( \kappa \) the above argument shows that \( \kappa \) has no closed leaves.

Step 4. Proof of Lemma 3.7.

Proof of Step 4. Note that \( \hat{P} \) could have been chosen so that \( w \in S^2_{\infty} \) is the base-point of the horoball containing \( \hat{P} \). If \( B \) is the region in \( \mathbb{H}^3 \) bounded by \( \hat{P} \) such that \( B \cap S^2_{\infty} = w \), then \( G_B = \{ g \in G_X \mid g(B) \cap B \neq \emptyset \} \) is a subgroup of the stabilizer \( G_w \) of \( w \). Since \( G_w \) is generated by \( f \), \( G_B \) is generated by \( f^n \) for some \( n \in \mathbb{Z} \). First suppose that \( G_B \neq \mathrm{id} \). We can assume that \( f^n(B - w) \subset \hat{B} \). Since \( \hat{P} \) is proper, each \( z \in \hat{P} \) has a neighborhood \( W \subset \mathbb{H}^3 \) such that \( W \cap (f^n(P) \cup f^{-n}(P)) = \emptyset \) and hence \( \{ g \in G_X \mid g(\hat{P}) \cap W \neq \emptyset \} = \mathrm{id} \). This implies that \( P \) is isolated, contradicting the fact that each leaf of \( \kappa \) is dense and \( \kappa \) has no closed leaves. Finally consider the case \( G_B = \mathrm{id} \). In this case \( \hat{B} \cap \kappa = \emptyset \), otherwise \( P \) is dense in \( \kappa \) implies that some covering translate of \( \hat{P} \) lies in \( \hat{B} \). Let \( I \) be an \( I \)-fibre of \( A \) and let \( \hat{I} \) be the lift which intersects \( \hat{P} \). Since \( P \) is nonisolated, \( \hat{I} \subset B \), with one endpoint \( i \in \hat{B} \). We obtain the contradiction \( \kappa \cap \hat{B} \neq \emptyset \).
Lemma 3.8 (Convex hull facts). Let \( \tau \) be a smooth simple closed curve in \( S^2_\infty \) and \( k > 0 \). Then

i) \( N_\rho(k,C(\tau)) \) is a convex, smooth, properly embedded \( D^2 \times I \).

ii) The product structure can be chosen so that for every \( \epsilon > 0 \), \( \{x \in D^2 \mid \text{length}_\rho(x \times I) > 2k + \epsilon\} \) is bounded.

iii) If \( \tau_\epsilon \subset B^3 \) is a Euclidean \( \epsilon \)-neighborhood of \( \tau \), then \( N_\rho(k,C(\tau)) = \bigcap_{\epsilon > 0} N_\rho(k,C(\tau_\epsilon)) \).

**Proof.** A proof of smoothness of \( \partial N(k,C(\tau)) \) due to Bowditch can be found in [EM, p. 119]. The remainder is an elementary exercise in hyperbolic geometry. Use the fact that if \( x \in C(\tau) \cap H^3 \) and \( x \) is Euclidean very close to \( \tau \), then in the visual sphere of \( x \), \( \tau \) approximates a great circle.

Proposition 3.9 (Least area spanning laminations exist). Let \( \tau \) be a smooth simple closed curve in \( S^2_\infty \) and \( r \) a Riemannian metric on \( H^3 \) induced from a metric on a closed hyperbolic 3-manifold. Then there exists a \( D^2 \)-limit lamination \( \sigma \subset H^3 \) by \( r \)-least area planes spanning \( \tau \). Furthermore there exists \( \epsilon > 0 \), which depends only on \( r \) (and hence independent of \( \tau \)), such that if \( \sigma \) is any spanning lamination by \( r \)-least area planes, then \( \sigma \subset N_\rho(e,C(\tau)) \).

**Proof.** Let \( \epsilon > 0 \) be as in Lemma 3.7. Let \( \omega \) be a properly embedded path in \( B^3 \) connecting points in distinct components of \( S^2_\infty - \tau \).

**Step 1.** There exists a sequence of relatively least area discs \( \{E_i\} \) such that for each \( i \), \( E_i \subset N_\rho(2\epsilon,C(\tau)) \), \( \partial E_i \to \infty \), and \( |\langle E_i, \omega \rangle| \neq 0 \). Here \( \langle , \rangle \) denotes oriented intersection number.

**Proof of Step 1.** By Lemma 3.8 \( N_\rho(e,C(\tau)) \) has an exhaustion by regions \( R_0 \subset R_1 \subset \cdots \), where \( R_i = D^2 \times I \) and \( \partial R_i = (\partial D^2) \times I \). Here \( \partial R_i \) denotes the relative boundary of \( R_i \). Choose \( R_0 \) such that \( \omega \cap N_\rho(e,C(\tau)) \subset R_0 \). The disc \( E_i \) is obtained by applying the procedure of Definition 3.6 to an essential circle in \( \delta R_i \). By Lemma 3.7 \( E_i \subset N_\rho(2\epsilon,C(\tau)) \).

**Step 2.** There exists a sequence \( \{D_i\} \) of least area discs such that for each \( i \), \( D_i \subset N_\rho(2\epsilon,C(\tau)) \), \( \partial D_i \to \infty \), and \( |\langle D_i, \omega \rangle| \neq 0 \).

**Proof of Step 2.** Apply a procedure, similar to the one of the second paragraph of the proof of Lemma 3.7, to obtain the sequence \( \{D_i\} \) from the sequence \( \{E_i\} \).

**Step 3.** After passing to a subsequence, \( \{D_i\} \) converges to a lamination \( \sigma \) by \( r \)-least area planes which spans \( \tau \).

**Proof of Step 3.** Let \( \sigma \) be a \( D^2 \)-limit lamination obtained by applying Lemma 3.3 to \( \{D_i\} \). We need to show that each component of \( S^2_\infty - \tau \) lies in a different complementary region of \( \sigma \), the other conditions being self evident. If \( \omega_1 \subset B^3 - \sigma \) is a properly embedded path connecting these two components, then since \( \omega_1 \cap N_\rho(2\epsilon,C(\tau)) \) is compact and disjoint from \( \sigma \), it follows that for \( i \) sufficiently large \( D_i \cap \omega_1 = \emptyset \). This contradicts the fact that for \( i \) sufficiently large, \( |\langle \omega, D_i \rangle| = |\langle \omega_1, D_i \rangle| \).

**Step 4.** If \( \sigma \) spans \( \tau \), then \( \sigma \subset N_\rho(e,C(\tau)) \).
Proof of Step 4. If \( L \) is a leaf of \( \sigma \), then by Lemma 3.5 vi) \( L \) has an exhaustion by compact discs \( P_i \) such that \( \partial P_i \to \infty \). Hence for each \( \epsilon > 0 \), there exists \( N_\epsilon \) such that if \( i > N_\epsilon \), then \( \partial P_i \subset N_\epsilon(\epsilon, \tau) \subset B^3 \), where \( E \) denotes Euclidean metric. Therefore by Lemmas 3.7 and 3.8 iv), \( L \subset N_\epsilon(\epsilon, C(\tau)) \).

**Proposition 3.10** (Convergence of spanning laminations). Let \( r_i, t \in [0,1] \) be a smooth family of Riemannian metrics on \( H^3 \) induced from Riemannian metrics on a closed hyperbolic 3-manifold and let \( \{t_i\} \) be a sequence in \([0,1]\) such that \( \lim t_i = t \). Let \( r_i \) (resp. \( r \)) denote the \( r_i \) (resp. \( r \)) metric. Let \( \tau \) be a smooth simple closed curve in \( S^2_\infty \). If \( \{\sigma_i\} \) is a sequence of \( D^2 \)-limit laminations by \( r_i \)-least area planes spanning \( \tau \), then after passing to a subsequence \( \{\sigma_i\} \) converges to a \( D^2 \)-limit lamination \( \sigma \) by \( r \)-least area planes which spans \( \tau \).

**Proof.** Let \( \epsilon > 0 \) be as in Lemma 3.7 for the metrics \( r_s, s \in [0,1] \). The proof of Lemma 3.3 works equally well for sequences of \( D^2 \)-limit laminations as it does for sequences of least area discs. In fact suppose that for each \( i, D_i \) is chosen to be a finite union of discs embedded in leaves of \( \sigma_i \) which are \( 2^{-i} \) dense in \( \sigma \cap B_p(i,0) \) (i.e. \( \sigma_i \cap B_p(i,0) \subset N_{\frac{1}{2^i}}(2^{-i}, D_i \cap B_p(i,0)) \)) and \( \partial D_i \to \infty \). Then the subsequence \( \{D_{i_n}\} \) of \( \{D_i\} \) converges to the lamination \( \sigma \) if and only if the subsequence \( \{\sigma_{i_n}\} \) of \( \{\sigma_i\} \) converges to the lamination \( \sigma \).

Now suppose that \( \{\sigma_i\} \) converges to the \( D^2 \)-limit lamination \( \sigma \). Again, we need to show that each component of \( S^2_\infty - \tau \) lies in a different complementary region of \( \sigma \), the other conditions being self evident. Let \( R_i \) be as in the proof of Step 1 of Proposition 3.9. After a small perturbation assume further that \( R_i \) is transverse to \( \bigcup \sigma_i \). Since \( \sigma_i \) separates the components of \( S^2_\infty - \tau \), some leaf \( \zeta_i \) of \( \sigma_i \cap \delta R_i \) is essential in \( \delta R_i \). \( \zeta_i \) bounds a disc \( F_i \subset \sigma_i \). Thus \( |\langle F_i, \omega \rangle| = 1 \). As in the proof of the previous lemma, a subsequence of \( \{F_i\} \) limits on a lamination \( \sigma' \) which separates the components of \( S^2_\infty - \tau \). Since \( \sigma' \) is a sublamination of \( \sigma \), \( \sigma \) separates the components of \( S^2_\infty - \tau \).

**Corollary 3.11.** A limit of \( D^2 \)-limit laminations is a \( D^2 \)-limit lamination.

**Conjecture 3.12.** Let \( r \) be a Riemannian metric on \( H^3 \) induced from a Riemannian metric on a closed hyperbolic 3-manifold. If \( \lambda \) is a smooth simple closed curve in \( S^2_\infty \), then \( \lambda \) spans a properly embedded \( r \)-least area plane in \( H^3 \).

**Remarks 3.13.** i) Freedman and He [FH] have constructed an example of a non-properly-embedded plane which is least area with respect to the hyperbolic metric on \( H^3 \).

   ii) A result similar to Proposition 3.9 can be found in [L2]. The lamination that arises there is properly embedded but not necessarily by planes. On the other hand, it is a theorem about all dimensions and codimensions and requires only that the Riemannian metric induce a topological metric Lipshitz equivalent to the hyperbolic metric. I suspect that under this weaker hypothesis on the Riemannian metric, the planes that span \( \rho \subset S^2_\infty \) need not be properly embedded. One can deduce an independent proof of Lemma 3.7 from the proof of Theorem 2 of [L1].

**Remark 3.14.** In a natural way the results of this section generalize to 3-manifolds with negatively curved fundamental group.
4. Constructing the tubes

Lemma 4.1. Let $\mu$ be a Riemannian metric on $\mathbb{H}^3$ induced from a closed hyperbolic 3-manifold. Let $\{\sigma_i\}$ be a locally finite set of $D^2$-limit laminations of $\mathbb{H}^3$ by $\mu$-least area planes. If $J$ is a component of $\mathbb{H}^3 - \bigcup \sigma_i$, then

i) $\pi_1(J) = 1$.

ii) If $J^*$ is the metric completion of $J$ with the induced path metric, then $J^*$ is a manifold with boundary.

iii) The natural map $\Pi : J^* \to \mathbb{H}^3$ is an injective immersion.

Proof of i). Let $\tau$ be a closed curve in $J$ and $f : D^2 \to \mathbb{H}^3$ be transverse to each $\sigma_i$ such that $f \mid \partial D^2 = \tau$. Then $f(D^2) \cap \sigma_i = \emptyset$ for all but finitely many $i$, say $i = 1, \ldots, n$. Assume that $f$ is chosen to minimize $n$. If $n = 0$, then $\tau$ is homotopically trivial in $J$. If $n > 0$ we obtain a contradiction as follows. By Lemma 3.5 $\sigma$ has no holonomy, so $f^{-1}(\sigma_i)$ is a lamination by circles so there is a finite number of outermost circles $\tau_1, \ldots, \tau_m$ of $f^{-1}(\sigma_n)$ bounding discs in $D$ whose union contains $f^{-1}(\sigma_n)$. Each $\tau_i$ maps to an immersed curve $\alpha_r$ in a leaf $L_i$ of $\sigma_n$. $L_i$ is a least area plane and $\alpha_r$ is disjoint from $\sigma_k$ for $k > n$ imply that $\alpha_r$ is homotopically trivial in $L_i$ via a homotopy disjoint from $\sigma_k, k > n$. In fact, the outermost component of $\partial N(\alpha_r) \subset L_i$ bounds an embedded disc $E$ whose boundary is disjoint from $\sigma_k, k > n$, hence $E$ is disjoint from $\sigma_k, k > n$. Being outermost, $f(\tau_i)$ lies on the boundary of a closed complementary region of $\sigma_n$. By replacing the image of the subdisc of $D$ bounded by each $\tau_s$ with a disc close to but disjoint from $\sigma_n$, we obtain a new immersed disc spanning $\tau$ intersecting at most $\sigma_i, 1 \leq i \leq n - 1$.

Proof of ii). By Lemma 3.5 ix) we can assume that each $\sigma_i$ is nowhere dense. Given $x \in \bigcup \sigma_i$, there exists a short geodesic arc $\alpha$ passing through $x$ and transverse to $\bigcup \sigma_i$ with $\partial \alpha \cap (\bigcup \sigma_i) = \emptyset$. Let $\tilde{\sigma}_i$ denote $\sigma_i \mid D^2 \times I$, the $D^2 \times I$ being a regular neighborhood of $\alpha$ where $0 \times 1/2 = x$. If the $D^2$ factor is sufficiently small, then each leaf of any $\tilde{\sigma}_i$ is the graph of a function $g : D^2 \to I$ and $\tilde{\sigma}_i$ is a product lamination. Also the projection of the intersection of any two leaves $L_i \cap \tilde{\sigma}_i, L_j \cap \tilde{\sigma}_j$ into the $D^2$ factor is either empty or a smooth properly embedded arc or $n$ properly embedded arcs which intersect at a single point in $\bar{D}^2$, the arcs having distinct slopes at the common point. The latter occurs if $L_i$ and $L_j$ are tangent and uses the normal form theorem for tangencies between least area surfaces, e.g., Lemma 2.6 [HS]. By local finiteness and reindexing we assume that $D^2 \times I$ intersects only $\sigma_i, 1 \leq i \leq q$.

Let $K$ be a component of $D^2 \times I - \bigcup \sigma_i$. To prove ii) it suffices to show that $\Pi \mid K^*$ is injective and $\Pi(K^*)$ is a manifold with boundary. For each $i \leq q$ there exist leaves $A_i, B_i$ of $\tilde{\sigma}_i$ such that $K$ lies in the complementary region of $\tilde{\sigma}_i$ defined by points lying above $A_i$ and below $B_i$, though possibly one of $A_i, B_i = \emptyset$. The region above the leaves $A_1, \ldots, A_q$ (resp. below $B_1, \ldots, B_q$) is the region above (resp. below) the graph of a function $A : D^2 \to I$ (resp. $B : D^2 \to I$). So $A$ (resp. $B$) is the maximum (resp. minimum) of a finite set of smooth functions. Therefore each component $K$ of $D^2 \times I - \bigcup \sigma_i$ either lies between the graph of two functions defined over an open subset $U$ of $D^2$, or lies either above or below the graph of a function on $D^2$. The proof of ii) in the latter case is clear. To show that $\Pi \mid K^*$ is injective and $\Pi(K^*)$ is a manifold with boundary it suffices to show that $U$ is a manifold with boundary. The projection $c_{ij}$ of $A_i \cap B_j$ into $\bar{D}^2$ has a natural
normal orientation, i.e. the normal points into the side where the region below $B_j$ and above $A_i$ is nontrivial. $\bar{U}$ is the closure of a connected region defined by the $c_{ij}$, with all normals pointing in. Since $L_i \cap L_j$ contains no embedded circles, the region $U$ is a disc. It is routine to check that $\bar{U}$ is topologically a closed disc.

**Proof of iii).** Proving that $\Pi$ is an injective immersion reduces to showing that if the discs $U_1$ and $U_2$ in $\mathcal{D}^2$ are defined by the same leaves $B_1, \ldots, B_q$ and $A_1, \ldots, A_q$, and $\bar{U}_1 \cap \bar{U}_2 \neq \emptyset$, then the regions $K_1$ and $K_2$ associated to them lie in distinct path components of $\mathcal{H}^3 - \bigcup \sigma_i$. If not, then one could pass to a minimal example of the saddle or spike type which are described and dispatched in the following paragraphs.

**Saddle example.** $\bigcup \sigma_i = \sigma_1 \cup \sigma_2$ and $\bar{U}_1 \cap \bar{U}_2$ correspond to a saddle tangency between leaves $A_1 \subset \sigma_1$ and $B_2 \subset \sigma_2$, where $B_1 = A_2 = \emptyset$. See Figure 4.3(b). A down isotopy of $\sigma_1$ near the saddle creates a non-simply-connected component of $\mathcal{H}^3 - \sigma'_1 \cup \sigma_2$, where $\sigma'_1$ denotes the isotoped $\sigma_1$. The intersection of a leaf $A$ of $\sigma'_1$ and a leaf $B$ of $\sigma_2$ must contain a simple closed curve; else one could argue as in the proof of i) to conclude that each component of $\mathcal{H}^3 - (\sigma'_1 \cup \sigma_2)$ was simply connected. Since $\sigma_1$ (resp. $\sigma_2$) is isolated above (resp. below) $A_1$ (resp. $B_2$), the only possibility is that $A$ is the isotoped $A_1$ and $B = B_2$. This implies that $A_1 \cap B_2$ contains a simple closed curve, again contradicting Lemma 3.5 vii).

**Spike example.** Here $\bigcup \sigma_i = \sigma_1 \cup \sigma_2 \cup \sigma_3$; $A_1, B_2, B_3 \neq \emptyset$ and the lines $A_1 \cap B_2, A_1 \cap B_3$ intersect tangentially at a single point $x \in \bar{U}_1 \cap \bar{U}_2$. See Figure 4.1. A small down isotopy of $\sigma_1$ creates a non-simply-connected component of $\mathcal{H}^3 - \sigma'_1 \cup \sigma_2 \cup \sigma_3$ where $\sigma'_1$ denotes the isotoped $\sigma_1$. On the other hand, since $\sigma_1$ is transverse to $\sigma_2 \cup \sigma_3$ near $x$, $\sigma'_1 \cap \sigma_j$ contains a simple closed curve if and only if $\sigma_1 \cap \sigma_j$ contains a simple closed curve, for $j = \{2, 3\}$. Thus one obtains a contradiction as in the saddle example.

**Remark 4.2.** The following observations follow from the proof of Lemma 4.1.

i) If $K_1$ and $K_2$ are distinct connected complementary regions of $D^2 \times I - \bigcup \sigma_i$ which limit on $x$, then one of the following three situations occur, up to reversing the parametrization on $I$.

a) There exists $j$ such that $K_1$ and $K_2$ lie in distinct complementary regions of $\sigma_j \mid D^2 \times I$. In this case $\sigma_j \mid D^2 \times I$ has an isolated leaf separating $K_1$ and $K_2$. Therefore $K_1$, $K_2$ lie in distinct complementary regions of $\mathcal{H}^3 - \sigma_j$.

b) There exist $j, k$, leaves $A_j \subset \sigma_j \mid D^2 \times I, B_k \subset \sigma_k \mid D^2 \times I$, such that $A_j$ and $B_k$ have a saddle tangency at $x$, $K_1, K_2$ lie above $A_j$ and below $B_k$, and $K_1, K_2$ lie in distinct components of $D^2 \times I - (\sigma_j \cup \sigma_k)$.

c) There exist $j, k, l$ leaves $A_j \subset \sigma_j \mid D^2 \times I, B_k \subset \sigma_k \mid D^2 \times I, B_l \subset \sigma_l \mid D^2 \times I$ such that $K_1, K_2$ lie above $A_j$ and below $B_k, B_l$. Finally $K_1, K_2$ lie in different components of $\mathcal{H}^3 - \sigma_j \cup \sigma_k \cup \sigma_l$.

ii) If $J_1, \ldots, J_n$ are finitely many components of $\mathcal{H}^3 - (\sigma_j \cup \sigma_k \cup \sigma_l)$ such that for each $r \neq s$ each point of $\Pi(J'_s) \cap \Pi(J'_r)$ occurs at a saddle or spike as in i), then $\bigcup \Pi(J'_s)$ is simply connected. Otherwise one obtains a contradiction as in the proof of Lemma 4.1.
Lemma 4.3. Let $x, y \in S^2_\infty$, $r$ a Riemannian metric on $H^3$ induced from the closed 3-manifold $M$ or $X$, and $\lambda_1, \ldots, \lambda_m$ smooth simple closed curves in $S^2_\infty - \{x, y\}$ such that no $\lambda_i$ separates $x$ from $y$. For each $i$, let $\sigma_i$ be a lamination by $r$-least area planes which spans $\lambda_i$ and let $H_i$ be the complementary region of $B^3 - \sigma_i$ which contains $x, y$. Then either

i) $x, y$ lie in the same component $H$ of $\bigcap_{i=1}^m H_i$, or

ii) there exist $\lambda_i, \lambda_j, \lambda_k$ such that $\lambda_i \cup \lambda_j \cup \lambda_k$ separate $x$ from $y$ in $S^2_\infty$.

Proof. If $m \leq 3$ and ii) does not hold, then $x, y$ lie in the same component of $S^2_\infty - \bigcup \lambda_i$, so i) holds. Assuming inductively that the lemma is true for $n < m$, we will establish it for cardinality $m$. Therefore either ii) holds or

\[(\ast) \quad \text{for every } j \leq m, \ x \text{ and } y \text{ lie in the same component of } \bigcap_{i \neq j} H_i.\]

We show that if $(\ast)$ holds, then either i) holds or for each $j, k$, $\lambda_j \cap \lambda_k \neq \emptyset$. Let $\tau_j \subset \bigcap_{i \neq j} H_i$ (resp. $\tau_k \subset \bigcap_{i \neq k} H_i$) be a path from $x$ to $y$ transverse to $\sigma_j$ (resp. $\sigma_k$). By Lemma 4.1 i), there exists $h : I \times I \to \bigcap_{i \notin \{j, k\}} H_i$, a homotopy from $\tau_j$ to $\tau_k$, which is transverse to both $\sigma_j$ and $\sigma_k$. Either $\sigma_k \cap \sigma_j \neq \emptyset$ and hence $\tau_k \cap \tau_j \neq \emptyset$ (by Lemma 3.5 viii) or $h^{-1}(\sigma_k \cup \sigma_j)$ is a lamination by circles and arcs. Each arc has both endpoints on one of $I \times 0$ or $I \times 1$. Thus $\tau_k \cap \tau_j = \emptyset$ implies that there exists an embedded path from $0 \times I$ to $1 \times I$ disjoint from $h^{-1}(\sigma_k \cup \sigma_j)$ and hence conclusion i) holds.
Either i) holds or there exists a minimal \( s > 0 \) and a reordering of the \( \lambda_i \) so that \( \lambda_1 \cup \cdots \cup \lambda_s \) separate \( x \) from \( y \) in \( S^2 \). Minimality implies that some component \( \tau \) of \( \lambda = (\lambda_1 \cup \cdots \cup \lambda_{s-1}) \) has the property that \( \tau \cup \lambda \) separate \( x, y \) and there exists a path \( \alpha \subset S^2 \) from \( x \) to \( y \) such that \( \alpha \) intersects \( \tau \cup \lambda \) transversely exactly once in \( \tau \). The closure of \( \tau \) has endpoints on \( \lambda_i \) and \( \lambda_j \), where \( i, j \leq s - 1 \). Since \( \lambda_i \cap \lambda_j \neq \emptyset \), there exists a simple closed curve \( \beta \subset \lambda_i \cup \lambda_j \cup \tau \) such that \( \tau \subset \beta \). Therefore \( \lambda_i \cup \lambda_j \cup \lambda_s \) separate \( x \) from \( y \).

\[ \square \]

**Definition 4.4.** If \( Y = D^2 \times S^1 \) or \( \tilde{D}^2 \times S^1 \), then a core of \( Y \) is a curve of the form \( z \times S^1 \), where \( z \in \tilde{D}^2 \).

**Lemma 4.5.** i) Cores of solid tori are unique up to isotopy.

ii) If \( h : Y_2 \to Y_1 \) is a covering map between solid tori, then \( c \) is a core of \( Y_1 \) if and only if \( h^{-1}(c) \) is a core of \( Y_2 \).

iii) If \( Y_1 \) and \( Y_2 \) are \( \tilde{D}^2 \times S^1 \)'s, such that \( c_1 \subset Y_2 \subset Y_1 \), where \( c_1 \) is a core of \( Y_1 \), then \( c_1 \) is a core of \( Y_2 \).

iv) Let \( \{ D_i \} \) be a locally finite collection of pairwise disjoint properly embedded planes in \( Y_2 = \tilde{D}^2 \times S^1 \). If \( Y_1 \) is a \( \tilde{D}^2 \times S^1 \) component of \( Y_2 \setminus \bigcup D_i \), then any core of \( Y_1 \) is a core of \( Y_2 \).

v) Let \( Y_2, Y_3 \) be \( \tilde{D}^2 \times S^1 \)'s embedded in the 3-manifold \( Y \) and let \( c_1 \) be a core of \( Y_3 \). If \( c_1 \subset Y_2 \) and there exists an embedded 2-disc \( E \subset \partial Y_3 \) such that \( Y_2 \cap \partial Y_3 \subset E \), then \( c_1 \) is a core of \( Y_2 \).

**Proof.** We first establish versions of i)–iii) in the case of closed solid tori.

ic) Here is a hint to this well known result. Let \( c_1 \) and \( c_2 \) be cores of \( Y \), the \( c_2 \) being an \( S^1 \) fibre of \( Y = D^2 \times S^1 \). First isotope \( c_1 \) to be transverse to this \( D^2 \)-fibration. Then isotope \( c_1 \) to \( c_2 \). The first isotopy follows from the existence of an embedded annulus connecting \( c \) to a simple closed curve \( d \) in \( \partial D^2 \times S^1 \) and the isotopy classification of simple closed curves on the torus.

iiic) The product structure of \( Y_1 \) lifts to a product structure on \( Y_2 \), so a core of \( Y_1 \) lifts to a core of \( Y_2 \). Conversely suppose that \( h^{-1}(c) = d \) is a core of \( Y_2 \). Let \( R \) be a connected embedded orientable surface of maximal Euler characteristic with two boundary components, one boundary component \( c \) and one boundary component on \( \partial Y_1 \). The curve \( c \) is necessarily a generator of \( \pi_1(Y_1) \) so \( R \) exists for homological reasons. The surface \( R \) lifts up to \( \tilde{R} \) in \( Y_2 \). Each primitive element of \( H_2(Y_2 - N(d), \partial Y_2 \cup \partial N(d)) \) and in particular \( [\tilde{R}] \) is represented by an annulus. By [G1, Corollary 6.13] \( \deg(h)(R) = \chi(\tilde{R}) = 0 \), therefore \( R \) is an annulus and \( c \) is a core of \( Y_1 \).

iic) Here we are assuming that \( Y_1, Y_2 \) are \( D^2 \times S^1 \)'s and \( c_1 \subset Y_2 \subset Y_2 \subset Y_1 \) and \( c_1 \) is a core of \( Y_1 \). \( \partial Y_2 \) is incompressible in \( N(Y_1) - \tilde{N}(c) = T^2 \times I \), so by [W], \( \partial Y_2 \) is isotopic in \( N(Y_1) - \tilde{N}(c) \) to \( T^2 \times 1/2 \). Therefore \( Y_2 \) has a product structure which restricts to a product structure on \( N(c) \).

We now prove i)–iv) for open solid tori.

iii) For \( i = 1, 2 \), let \( c_i \) be a core of \( Y_i \). Let \( N_2 \subset Y_2 \) be a large regular neighborhood of \( c_2 \) such that \( c_1 \subset N_2 \). Let \( N_1 \subset Y_1 \) be a large regular neighborhood of \( c_1 \) such that \( N_2 \subset N_1 \). Then iic) applied to the inclusions \( c_1 \subset N_2 \subset N_1 \) yields that \( c_1 \) is a core of \( N_2 \). By ic) \( c_1 \) is isotopic to \( c_2 \) and hence \( c_1 \) is a core of \( Y_2 \).
i) Apply iii).

ii) Suppose \( c \subset Y_1 \) lifts to a core \( d \subset Y_2 \) and let \( N_1 \) be a regular neighborhood of a core of \( Y_1 \) such that \( d \subset \text{int}(\tilde{N}_1) \) where \( \tilde{N}_1 \) is the lift of \( N_1 \) to \( Y_2 \). By iii) \( d \) is a core of \( \tilde{N}_1 \) and so \( c \) is a core of \( N_1 \) and hence \( Y_1 \) by iiic).

iv) Let \( N_2 \) be a large standardly embedded \( D^2 \times S^1 \subset Y_2 \) such that \( c_1 \subset N_2 \) and \( N_2 \) is transverse to \( \bigcup D_j \). Since \( c_1 \) is homotopically nontrivial in \( Y_2 \), any disc component of \( D_j \cap N_2 \) must separate off a ball in \( N_2 \) disjoint from \( c_1 \). On the other hand, a subdisc \( D \) of \( D_j \) with \( \partial D \subset N_2 \), \( \tilde{D} \cap N_2 = \emptyset \) together with a subdisc of \( \partial N_2 \) bounds a ball in \( Y_2 - \tilde{N}_2 \). Therefore the usual innermost disc (in \( D_j \)) and an isotopy argument allows us to assume, after isotopy of \( N_2 \), that \( c_1 \subset N_2 \subset Y_1 \). Now apply iii).

v) By doing a finite sequence of compressions and 2-handle attachments to \( Y_2 \) in a small neighborhood of \( E \), we obtain a new manifold \( W \) such that \( c_1 \subset W \subset Y_3 \).

Topologically \( W \) must be a \( D^2 \times S^1 \), possibly with some balls removed, otherwise one obtains a \( \pi_1 \) contradiction. The contradiction depends on whether \( |\pi_1(W)| < \infty \) or \( W = D^2 \times S^1 - 3 \)-balls. Therefore every compression or 2-handle attachment was trivial. Since \( c_1 \) is a core of \( W \) if and only if \( c_1 \) is a core of \( Y_2 \), v) follows from iii). \( \square \)

If \( \delta \subset H^3 \) is a geodesic, then \( S^2_\infty - \partial \delta \) is naturally parametrized by \( S^1 \times \mathbb{R} \), where each \( x \times \mathbb{R} \) lies in the ideal boundary of a hyperbolic half-plane bounded by \( \delta \), and the \( \mathbb{R} \) parameter is given by the hyperbolic nearest point projection of \( B^3 - \partial \delta \) to \( \delta \).

**Definition 4.6.** If \( R \subset S^2_\infty - \partial \delta \), then define \( \delta \text{-visual angle}(R) = \inf\{\theta_2 - \theta_1 \mod 2\pi \mid R \subset [\theta_1, \theta_2] \times \mathbb{R}\} \in [0, 2\pi] \). The possible choice of 0 or 2\( \pi \) is made in the obvious manner.

**Lemma 4.7.** If \( P \subset H^3 \) is a hyperbolic plane with ideal boundary \( \lambda \) and \( P \cap \delta = \emptyset \), then \( \delta \text{-visual angle}(\lambda) = 2\sin^{-1}(1/\cosh(d)) \), where \( d \) is the hyperbolic distance between \( \delta \) and \( P \).

**Proof.** Let \( \tau \) be the orthogonal geodesic segment between \( \delta \) and \( P, x = \delta \cap \tau \) and \( y = P \cap \tau \). Let \( Q \) be the hyperbolic plane orthogonal to \( \delta \) containing \( \tau \) and \( \sigma = Q \cap P \). The \( H^2 \) visual angle of \( \sigma \) viewed from \( x \in Q \) is equal to the \( \delta \)-visual angle of \( \lambda \). Now apply the formula (e.g. \( [F] \), p. 92) \( \sin(\alpha) \cosh(d) = 1 \) associated to the right angle triangle \( xyz \), where \( z \) is an endpoint of \( \sigma \), \( d = d_p(x, y) \) and \( \alpha \) is the angle \( zxy \). \( \square \)

**Corollary 4.8.** If \( P \subset H^3 \) is a geodesic plane with ideal boundary \( \lambda \) and \( d_p(\delta, P) = (\log(3))/2 = .549306 \ldots \), then \( \delta \text{-visual angle}(\lambda) = 2\pi/3 \).

Mark Culler told me that \( .549306 \ldots = (\log(3))/2 \).

**Proofs of Theorems 0.9 and 0.10.** By hypothesis there exists a \( (\pi_1(N), \{\partial c_\delta\}) \) non-coalescable insulator family \( \{\lambda_{jk}\} \), where \( \delta \) is a closed geodesic in \( N \) and \( \{\delta_j\} = q^{-1}(\delta) \). To prove Theorem 0.9 it suffices by Proposition 2.1 to find a simple closed curve \( \gamma \) in \( M \), such that the \( B^3\text{-link} p^{-1}(\gamma) = \Gamma \) is isotopic rel \( S^2_\infty \) to the \( B^3\text{-link} q^{-1}(\delta) = \Delta \). In the context of Theorem 0.10 \( M = N \) and \( f \) is a homeomorphism homotopic to id. To prove Theorem 0.10 it suffices by Proposition 2.11 to show that \( f^{-1}(\delta) \) is isotopic to \( \delta \).
Our terminology will follow that of Notation 1.2. In particular $G$ denotes the action of $\pi_1(N), \pi_1(M)$ on $S\subset \mathbb{H}_\infty$ as well as the action of $\pi_1(M)$ on $B^3$.

**Step 1.** We can assume that, with only finitely many $G$-orbits of exceptions, each $\lambda_{ij}$ is the ideal boundary of the midplane (see Example 0.3) $D_{ij}$ between $\delta_i$ and $\delta_j$. If $E_{ij}$ denotes the component of $S_{ij}^2 - \lambda_{ij}$ which does not contain $\partial \delta_i$, then $S_{ij}^2 - \partial \delta_i = \bigcup_j E_{ij}$.

**Proof of Step 1.** By convexity and local finiteness there exists $\beta < \pi$ such that for each $j$, $\delta_i$-visual angle($\lambda_{ij}$) $< \beta$. Let $\alpha = \min\{2\pi - 2\beta, 2\pi/3\}$ and let $d = \cosh^{-1}(1/\sin(\alpha/2))$. Define a new $(\pi_1(N), \{\partial \delta_i\})$ insulator family by the rule

$$
\lambda_{ij} = \begin{cases} 
\lambda_{ij}, & d_\rho(\delta_i, \delta_j) \leq d, \\
\partial D_{ij}, & \text{otherwise.}
\end{cases}
$$

Using Lemma 4.7 and the choice of $d$ it follows that this family satisfies the no-trilinking condition of Definition 0.3 hence is noncoalescable. The first part of Step 1 is established by replacing $\{\lambda_{ij}\}$ by $\{\lambda_{ij}\}$.

Let $x \in S_{ij}^2 - \partial \delta_i$. Let $\tau$ be a geodesic from $x$ to $\delta_i$, which is orthogonal to $\lambda_i$. An extended hyperbolic plane $P \subset B^3$ disjoint from $\delta_i \cup x$ separates $\lambda_i$ from $x$ if and only if $P \cap \tau \neq \emptyset$. If $C = 2(d(\text{diam}_\rho(X)))$, then there exists a $\delta_i$ such that $d_\rho(\delta_i, \tau) < C$, but $d_\rho(\delta_i, \delta_j) > \max(10C, d)$. The midplane $D_{ij}$ between $\delta_i$ and $\delta_j$ crosses $\tau$ and hence $\partial D_{ij}$ separates $x$ from $\partial \delta_i$. By definition $\partial D_{ij} = \lambda_{ij}$. \hfill $\square$

From now on $i$ will denote a fixed integer and $g \in G$ will denote a fixed generator of $\text{Stab}(\partial \delta_i) = \langle g \rangle$. By the equivariance and local finiteness properties of insulator families, there exists an integer $n > 0$ such that $g^n \in \pi_1(X)$ and for all $j$ and all $r \neq 0$, $g^n(\lambda_{ij}) \cap \lambda_{ij} = \emptyset$. By Step 1, the compactness of $(S_{ij}^2 - \partial \delta_i)/\langle g^n \rangle$ and the $G$-equivariance of insulator families, there exists only finitely many outermost $\langle g^n \rangle$-orbits of $\{\lambda_{ij}, \lambda_{ij} \ldots \}$. $\lambda_{ij}$ is outermost means that there exists no $E_{ik}$ such that $E_{ij} \subset \hat{E}_{ik}$. From now on $n$ will be the integer determined as above.

Fix a Riemannian metric $\mu$ on $M$ and let $\mu$ denote the induced metric on $H^3$. For each $j, k$, let $\sigma_{jk}$ be a laminating spanning $\lambda_{jk}$ by $\mu$-least area planes. The $\sigma_{jk}$ should be chosen $G$-equivariantly, i.e. $h(\lambda_{jk}) = \lambda_{rs}$ implies $h(\sigma_{jk}) = \sigma_{rs}$ and $\sigma_{kj} = \sigma_{jk}$. Let $H_{jk}$ denote the $H^3$-complementary region of $\sigma_{jk}$ which contains the ends of $\delta_j$. Let $H_j = \bigcap_k H_{jk}$.

Reorder the $\delta_i$'s so that $\{\lambda_{i1}, \ldots, \lambda_{im}\}$ denote representatives of the outermost $\langle g^n \rangle$-orbits of $\lambda_{ij}$'s. By Lemma 3.5 viii) it follows that $H_i = \bigcap_{r=1}^m \bigcap_{r \in \mathbb{Z}} g^{rn}(H_{ij})$. By equivariance, $h \in G, h(\delta_i) = \delta_j$ implies that $h(H_i) = H_j$.

**Step 2.** Establish the following properties of the $H_j$.

i) There exists $a > 0$ such that $H_i \subset N_p(a, \delta_i)$.

ii) $H_i \cap H_j \neq \emptyset$ if and only if $\delta_i = \delta_j$.

iii) If $h \in G$, then $h(H_i) \cap H_i \neq \emptyset$ if and only if $h = g^k$ for some $k$.

iv) If $J$ is a component of $H_i$, then for each $x \in J$, there exists a standard $D^2 \times I$ neighborhood of $x$ as in the proof of Lemma 4.1 with the following additional properties. $J \cap D^2 \times I$ is the region which lies above the leaves $A_1, \ldots, A_q$ and below the leaves $B_1, \ldots, B_q$, notation as in the proof of Lemma 4.1 ii). Both $J \cap D^2 \times I$ and $\hat{J} \cap (\partial D^2) \times I$ are connected. Finally if $C \neq D$ where $C, D \in \{A_1, \ldots, A_q, B_1, \ldots, B_q\}$ then $\partial C$ is transverse to $\partial D$ in $\partial D^2 \times I$. 


v) If $J$ is a component of $H_i$ and $J^*$ is the metric completion of $J$ with the induced path metric, and $\Pi : J^* \to H^3$ is the natural map, then $\Pi$ is an embedding (rather than injective immersion) of $J^*$ onto $J \subset H^3$.

vi) If $J$ is a component of $H_i$ and $h \in (g)$, then $h(J) \cap J \in \{J, \emptyset\}$.

vii) There exists $N_j$ such that every $x \in H^3$ lies in the closure of at most one component of $H_i$ having $\rho$-diameter $\geq N_1$. Each bounded component of $H_i$ has $\rho$-diameter $\leq N_1$.

\textbf{Proof of Step 2.} i) Consider a fundamental domain $F$ of $(B^3 - \partial \delta_1)/(g^n)$. $F \cap S^2_{\infty}$ is covered by a finite number of $E_{ij}$, hence a $B^3$-neighborhood $N$ of $F \cap S^2_{\infty}$ is covered by a finite number of $H_{ij}$, where $H_{ij}$ is the component of $B^3 - N_{\rho}(e, C(\lambda_{ij}))$ which does not contain $\partial \delta_1$. By Proposition 3.9 $H_{ij} \cap \sigma_{ij} = \emptyset$, so $\tilde{H}_{ij} \cap H_i = \emptyset$. Choose $a$ to be sufficiently large so that $F - N \subset N_{\rho}(a, \delta_i)$. Using the equivariance of insulator families and the fact that $g^n$ is an isometry in both the $\rho$ and $\mu$ metrics, it follows.

ii) If $i \neq j$, then $\lambda_{ij}$ separates $\partial \delta_1$ from $\partial \delta_j$. By definition of spanning lamination, the complementary regions of $\sigma_{ij}$ which contain $\partial \delta_i$ and $\partial \delta_j$ are distinct and hence $H_i \cap H_j \subset H_{ij} \cap H_{ji} = \emptyset$.

iii) The $\sigma_{ij}$ were chosen to be $G$-equivariant, hence $h(\partial \delta_i) = \partial \delta_j$ implies that $h(H_i) = H_j$. Conversely if $h(\partial \delta_i) \neq \partial \delta_j$, so by ii) $h(H_i) \cap H_i = \emptyset$.

iv) By Lemma 3.5 (ix) we can assume that each $\sigma_{ij}$ is nowhere dense. Let $x \in H^3$. Here we are identifying, via Lemma 4.1, $J^*$ with the region $\Pi(J^*) \subset H^3$. Let $x \in \sigma_{ij} \cap \partial J^*$. Let $B \subset H^3$ be a large ball such that $x \in B$, $B$ is transverse to $\cup_j \sigma_{ij}$, and $N_{\rho}(a, \delta_i) \cap \partial B \cap N_{\rho}(e, C(\lambda_{ij})) = \emptyset$. Let $L$ be the leaf of $\sigma_{ij} | B$ which contains $x$. By Lemma 3.5 (v) $L$ is compact and has a neighborhood of the form $L \times I$ such that $\sigma_{ij} \mid L \times I = L \times C$, with the product lamination, where $C$ is compact. (It is this technical point which allows us to treat $\sigma_{ij}$ as though it is a proper plane.) Thus if $y, z \in H_i \cap L \times I$ and $y, z$ lie in distinct complementary regions of $\sigma_{ij} \mid L \times I$, then $y$ and $z$ lie in distinct components of $H_i$ for $\partial L \times I \cap H_i = \emptyset$. Therefore any sufficiently small $D^2 \times I$ neighborhood of $x$ satisfies the second sentence of iv).

By making the $D^2 \times I$ sufficiently small, both $J \cap D^2 \times I$ and $J \cap \partial D^2 \times I$ are connected. See Remark 4.2. By making $\partial D^2$ transverse to the various $C \cap D$ arcs of intersection as well as choosing $\partial D^2$ to avoid the finitely many tangencies among the leaves $\{A_1, \ldots, A_q, B_1, \ldots, B_q\}$, $\partial D^2 \times I$ has the desired transversality property.

v) This follows by iv) and Lemma 4.1.

vi) Let $D^2 \times I$ be a small neighborhood of $x \in h(J) \cap \tilde{J}$ as in iv), i.e. such that each of $h(J)$, $J$ intercepts $D^2 \times I$ in a single component. By the proof of iv) $h(J) \cap D^2 \times I$ is not separated from $J \cap D^2 \times I$ by a leaf of some $\sigma_{ij} \mid D^2 \times I$.

Therefore, by Remark 4.2 i), if $x \in h(J) \cap J \neq J$, then $\tilde{J}, h(J)$ meet at a spike or saddle point created by the laminations $\sigma_{ij}, 1 \leq s \leq k \in \{2, 3\}$. If $J_r$ is the component of $H^3 - (\bigcup_{s=1}^k \sigma_{ij})$ containing $h^r(J)$, then for some finite $t$, $\bigcup_{r=1}^t \Pi(J^*_r)$ is connected but not simply connected. A nontrivial cycle passing through $x$ is obtained as follows. First chain together a path, through $x$ and all the $\Pi(J^*_r)$, from $\Pi(J^*_1)$ to $\Pi(J^*_t)$. This path (which has endpoints near $\partial \delta_i$) together with a path “near” $S^2_{\infty}$ and disjoint from $\bigcup_{s=1}^k \lambda_{ij}$ yields the desired cycle, contradicting Remark 4.2 ii). This uses the fact that $\bigcup_{s=1}^k \lambda_{ij}$ does not separate $\partial \delta_i$ in $S^2_{\infty}$. 
Lemma 4.1. If \( x \) projects to \( y \in \delta_i \), via orthogonal projection, then \( \bigcup_{k=1}^{k} \sigma_{ij} \) projects into \( B_{\rho}(N_0, y) \cap \delta_i \), for each of these \( k \) laminations have the point \( x \) in common. If for \( r = 1, 2 \), \( \text{diam}(J_r) > N_4 \), then since \( J_r \subset N_\alpha(a, \delta_i) \) the orthogonal projection of \( J_r \) is not contained in \( B_{\rho}(N_0, y) \cap \delta_i \). Therefore \( J_1, J_2 \) lie in the same component of \( H^3 - \bigcup_{k=1}^{k} \sigma_{ij} \) and one constructs the desired cycle.

By iv), any \( x \in H^3 \) has a standard \( D^2 \times I \) neighborhood whose boundary intersects at most finitely many components of \( H_i \). This together with Step 2 i) implies that given \( \epsilon > 0 \), then modulo the action of \( \langle g^\epsilon \rangle \) which acts isometrically in both the \( \mu \) and \( \rho \) metrics, there are only finitely many components of \( H_i \) with \( \rho \)-diameter \( > \epsilon \). Therefore there exists an \( N_2 \) such that the \( \rho \)-diameters of bounded \( H_i \) components are uniformly bounded by \( N_2 \). Finally take \( N_1 = \max(N_2, N_4) \).

**Step 3.** Let \( P' : B^3 - \partial \delta_i \to D^2 \times S^1 \) be the quotient map under the action of \( \langle g \rangle \). Then \( P'(H_i) \) is the union of open balls and exactly one open solid torus, whose closures are respectively closed balls and one solid torus \( T \). Therefore \( H_i \subset H^3 \) is a union of uniformly bounded open balls and exactly one component \( V_i \) whose \( H^3 \)-closure is a \( D^2 \times R \) whose ends limit on \( \partial \delta_i \). Finally \( p(V_i) = V \) is a \( \hat{D}^2 \times S^1 \), where \( p : H^3 \to M \) is the universal covering map.

**Proof of Step 3.** Each component \( Z \) of \( P'(H_i) \) has \( \pi_1(Z) \in \{1, Z\} \) since it is covered by a simply connected component of \( H_i \), by Lemma 4.1, with covering translations contained in \( \langle g \rangle \). \( Z \) is a compact manifold with boundary by Lemma 4.1 and v)–vii) of Step 2. \( Z \) is irreducible since it is covered by an irreducible manifold [MSY]. Therefore \( Z \) is a closed ball or solid torus.

We show that there exists some \( \hat{D}^2 \times S^1 \) component of \( P'(H_i) \). Parametrize \( B^3 - \partial \delta_i \) by \( D^2 \times R \) so that \( g \) acts by \( (x, t) \to (x, t+1) \). If \( P'(H_i) \) contains no \( \hat{D}^2 \times S^1 \) component, then by Step 2 vii) the components of \( H_i \) have uniformly bounded \( \rho \)-diameter. Hence there exists an integer \( N_3 > 0 \) such that if \( \hat{Z} \) is a component of \( H_i \) and \( \hat{Z} \cap D^2 \times 0 \neq \emptyset \), then \( \hat{Z} \subset D^2 \times (-N_3, N_3) \). By Step 1 and Lemma 3.5 viii), \( H_i \cap (D^2 \times [-N_3, N_3]) = Z_\alpha \cap (D^2 \times [-N_3, N_3]) \) where \( Z_\alpha = \bigcap_{j_k \in \alpha} H_{ij_k} \) for some finite set \( \alpha = \{j_1, \ldots, j_r\} \). By Lemma 4.3 some component \( Z_\beta \) of \( Z_\alpha \) contains an embedded path \( \tau \) connecting the points of \( \partial \delta_i \). This implies that some component of \( H_i \) nontrivially intersects \( D^2 \times (-N_3) \) and \( D^2 \times N_3 \) and hence \( D^2 \times 0 \), which is a contradiction.

Suppose that \( P'(H_i) \) had two solid tori \( T_1 \) and \( T_2 \). Their cores \( c_1, c_2 \) would lift to paths \( \hat{c}_1, \hat{c}_2 \) between the elements of \( \partial \delta_i \) such that for some component \( S \) of \( Z_\alpha \cap D^2 \times 0 \), \( \langle S, \hat{c}_1 \rangle = 1 \) and \( S \cap \hat{c}_2 = \emptyset \). Again \( \langle, \rangle \) denotes the algebraic intersection number. Since the ends of \( \hat{c}_1 \) and \( \hat{c}_2 \) lie in a neighborhood of \( \partial \delta_i \), contained in \( Z_\alpha \), the ends can be truncated and fused to create a closed curve \( \gamma \) in \( Z_\alpha \) such that \( \langle \tau, S \rangle = 1 \), contradicting the simple connectivity of \( Z_\alpha \) established in Lemma 4.1.
Definition. The solid torus \( V \) constructed above is said to arise from the insulator construction.

Let \( P : \mathbb{B}^3 - \partial \delta_i \to Y = D^2 \times S^1 \) be the quotient map given by the action of \( \langle g^n \rangle \). Let \( \gamma \) be a core of \( V = p(\hat{V}_j) \). (Recall Notation 1.2.) For each \( j \) let \( \gamma_j \) denote the lift of \( \gamma \) to \( \hat{V}_j \) extended to be a properly embedded arc in \( \mathbb{B}^3 \).

Step 4. The isotopy class of \( \gamma \) and hence the \( \mathbb{B}^3 \)-link \( \{ \gamma_j \} = \Gamma \) is independent of the choice of \( \{ \sigma_{ij} \} \).

Proof of Step 4. Let \( \{ \sigma'_{ij} \} \) be another collection of laminations spanning \( \{ \lambda_{ij} \} \), with associated regions \( \{ H'_i \} \), \( \{ H'_i \} \). Let \( H''_{ij} = H_{ij} \cap H'_{ij} \) and \( H''_{ij} = \bigcap_i H''_{ij} \). The arguments of Steps 1–3 show that \( H_i, H'_i, H''_{ij} \) each contain a unique unbounded \( \tilde{D}^2 \times \mathbb{R} \) respectively called \( \tilde{V}_i, \tilde{V}'_i, \tilde{V}''_i \), which project respectively to open solid tori \( V, V', V'' \) in \( M \) such that \( V'' \subset V \) and \( V'' \subset V' \).

To prove Step 4 we will pass to the \( n \)-fold cyclic covering space \( Y_0 \subset Y \) of \( V \) and there find \( Y_m \subset Y_{m-1} \subset \cdots \subset Y_0 \) which are respectively open solid tori with a common core. \( Y_m \) will be the \( n \)-fold cyclic covering space of \( V'' \). Since a core \( c \) of \( V'' \) lifts to a core of \( Y_m \), which is a core of \( Y_0 \), this curve \( c \) when viewed in \( V \) is therefore a core by Lemma 4.5 ii). A similar argument shows that any core of \( V'' \) is a core of \( V' \). Therefore \( V \) and \( V' \) have common cores. Since cores are unique up to isotopy (Lemma 4.5 i)), Step 4 is established.

Recall that \( \lambda_{i1}, \ldots, \lambda_{in} \) are representatives of the distinct outermost \( \langle g^n \rangle \)-orbits of \( \{ \lambda_{ij} \} \) and that \( n \) was chosen so that \( r \neq 0 \) implies that for all \( k \), \( g^n(\lambda_{ik}) \cap \lambda_{ik} = \emptyset \).

For \( 1 \leq j \leq m \) let \( \kappa_j = \bigcup_{r=-\infty}^{\infty} g^n(\sigma'_{ij}) \) and \( H''_{ij} = \bigcap_{r=-\infty}^{\infty} g^n(\tilde{H}'_{ij}) \). Again by Lemma 3.5 viii) \( H'_i = \bigcap_{j=1}^{m} H''_{ij} \). For \( 1 \leq t \leq m \), let \( W_t = \bigcap_{j=1}^{t} H''_{ij} \cap H_i \), so \( W_m = H''_i \). Define \( W_0 = \emptyset \). As in the proof of Steps 1–3, each \( W_j \) contains a unique \( \tilde{D}^2 \times \mathbb{R} \) component \( \tilde{V}^j \) which projects via \( P \) to a \( \tilde{D}^2 \times S^1 \) called \( Y_j \). Step 4 will follow from the following claim, for when combined with Lemma 4.5 iii) it asserts that for all \( j \), any core of \( Y_{j+1} \) is a core of \( Y_j \), and hence any core of \( Y_m \) is a core of \( Y_0 \).

Claim. Each leaf \( L_o \) of \( P(\kappa_{j+1}) \mid Y_j \) is a properly embedded separating disc, exactly one complementary component being a \( \tilde{B}^3 \). The collection \( \{ L_{ij} \} \) of outermost such discs is locally finite in \( Y_j \). (Outermost means, maximal in the partial order defined by inclusion of complementary \( \tilde{B}^3 \)-s.) Finally \( Y_{j+1} = \bigcap C_\beta \), where \( C_\beta \) is the non-simply-connected component of \( Y_j - D_\beta \) and \( D_\beta \) is an outermost disc.

Proof of the Claim. By Lemma 3.5 vi) and Step 2 i) each leaf \( L \) of \( \kappa_{j+1} = \kappa_{j+1} \mid \tilde{V}^j \) is a properly embedded planar surface. By Lemma 3.5 vii) \( L \) must be a disc. It follows from the choice of \( n \) that for each \( j \), \( P \mid \lambda_{ij} \to P(\lambda_{ij}) \) and hence \( P \mid \sigma_{ij} \to P(\sigma_{ij}) \) were embeddings. Arguing as in the proofs of Steps 1)–3), it follows that the \( \mathbb{H}^3 \)-closures of the components of \( \tilde{V}^j \) consist of a \( \tilde{B}^3 \) called \( B_L \) and a properly embedded \( D^2 \times R \). Finally by the choice of \( n \), the complementary regions of \( P(L) \subset Y_j \) consist of an \( \tilde{D}^2 \times S^1 \) and a \( \tilde{B}^3 = P(\tilde{B}_L) = P(\bigcup_{r=-\infty}^{\infty} (g^n(\tilde{B}_L))) \).

By Lemma 3.5 v) \( \tilde{V}^j \) is covered by a locally finite \( \langle g^n \rangle \)-equivariant collection of lamination charts \( \{ U_o \} \) such that each leaf \( L \) of \( \tilde{V}^j \) passes through \( U_o \) at most once. To see this, apply Lemma 3.5 v) where \( W \) is a large ball transverse to \( \kappa_{j+1} \) such that \( \partial W \cap \kappa_{j+1} \cap \tilde{V}^j = \emptyset \) and argue as in the proof of Step 2 iv). Since at
most 2 outermost leaves can pass through a given \( U_{\alpha} \), local finiteness of outermost leaves is established.

Again by the choice of \( n \), if \( L \) and \( \hat{L} \) are distinct leaves of \( \hat{\kappa}_{j+1} \), then they bound balls \( B_{L}, B_{\hat{L}} \subset V_{j}^{i} \) which are either disjoint or nested. Thus \( \hat{V}_{i}^{j+1} = \bigcap (\hat{V}_{j}^{i} - B_{L}) \), the intersection taken with respect to outermost leaves of \( \hat{\kappa}_{j+1} \). By projecting into \( P(\mathbb{B}^{3} - \partial \delta_{i}) \) the last assertion follows. \( \Box \)

**Remark.** An isotopy of \( \gamma \) to a core \( \gamma' \) of \( V' \) can be expressed as the composition of two isotopies, the first (resp. second) of which is supported in \( V \) (resp. \( V' \)).

**Step 5.** The isotopy classes of \( \gamma \subset M \) and the \( \mathbb{B}^{3} \)-link \( \Gamma \) are independent of the choice of metric on \( M \).

**Proof.** Since the space of Riemannian metrics is path connected, it suffices to show that if \( \mu_{s} \) is a \([0,1]\)-family of smooth metrics and \( \gamma_{s} \) is a core curve arising from the insulator construction with respect to \( \mu_{s} \), then the isotopy class of \( \gamma_{s} \subset M \) is locally constant as a function of \( s \). (For fixed \( s \) the isotopy class of \( \gamma_{s} \) is well defined by Step 4.) Suppose that there exists a sequence \( t_{k} \to t \) such that for each \( k \), \( \gamma_{t_{k}} \) is not isotopic to \( \gamma_{t} \). Let \( \sigma_{ij}^{k} \) be a laminations spanning \( \lambda_{ij} \) by \( \mu_{t_{k}} \)-least area planes. By Proposition 3.10 after passing to a subsequence we can assume that for \( j = 1, \ldots, m \), \( \sigma_{ij}^{k} \to \sigma_{ij} \), where \( \sigma_{ij}^{k} \) (resp. \( \sigma_{ij} \)) is a laminations spanning \( \lambda_{ij} \) by \( \mu_{t} \)-least area (resp. \( \mu_{t} \)-least area) planes. Using \( \{ \sigma_{ij}^{k} \} \) (resp. \( \{ \sigma_{ij} \} \)) the insulator construction associates to each \( t_{k} \) (resp. \( t \)) a \( \hat{V}_{i}^{k} = \hat{D}^{2} \times \mathbb{R} \) (resp. \( \hat{V}_{i} \)) which covers \( V^{k} \subset M \) (resp. \( V \)), where \( V^{k}, V \) are \( \hat{D}^{2} \times S^{1} \)-s. Let \( T^{k} \) (resp. \( T \)) denote the \( D^{2} \times S^{1} \) whose interior is \( P(\hat{V}_{i}^{k}) \) (resp. \( P(\hat{V}_{i}) \)).

Let \( \hat{\gamma} \) be the lift of \( \gamma_{t} \) to \( Y \). To complete the proof of Step 5 it suffices to show that for \( k \) sufficiently large, \( \hat{\gamma} \) is a core of \( T^{k} \). By Lemma 4.5 ii) for \( k \) sufficiently large \( \gamma_{t} \) would be a core of \( V^{k} \). By Lemma 4.5 i) cores are unique up to isotopy, so we conclude that \( \gamma_{t_{k}} \) is isotopic to \( \gamma_{t} \), a contradiction.

By Lemma 4.5 v) it suffices to find a regular neighborhood \( N(T) \) of \( T \), and a disc \( E \subset \partial N(T) \) such that for \( k \) sufficiently large \( \hat{\gamma} \subset T^{k} \) and \( \partial N(T) \cap T^{k} \subset E \). Reconcile our notation here with that of Lemma 4.5 by letting \( c_{1} = \hat{\gamma}, Y_{2} = T^{k}, Y_{3} = N(T), Y = Y \) and \( E = E \).

Let \( \gamma_{\hat{\omega}} \) be the lift of \( \hat{\gamma} \) to \( \hat{V}_{i} \). For fixed \( j \) and for \( k \) sufficiently large \( \gamma_{\hat{\omega}} \cap \sigma_{ij}^{k} = \emptyset \); else by Definition 3.2 \( \gamma_{\hat{\omega}} \cap \sigma_{ij} \neq \emptyset \), a contradiction. The finiteness of \( m \) (recall Step 2) and the \( \langle g^{n} \rangle \)-invariance of \( \gamma_{\hat{\omega}} \) imply that for \( k \) sufficiently large \( \gamma_{\hat{\omega}} \subset \hat{V}_{i}^{k} \) and hence \( \hat{\gamma} \subset T^{k} \).

We now find the desired \( N(T) \) and \( E \). Let \( N_{1}(T) \subset Y \) be a regular neighborhood of \( T \) transverse to \( P(\bigcup_{j} \sigma_{ij}) \).

**Claim 1.** i) There exist properly embedded compact separating surfaces \( S_{1}, \ldots, S_{m} \subset N_{1}(T) \) such that for \( p \leq m \) there exists a closed complementary region \( J_{p} \subset N_{1}(T) \) of \( S_{p} \) such that \( T \subset \bigcap_{p=1}^{m} J_{p} = J \). Each \( S_{p} \) is a (possibly disconnected) surface lying in leaves of \( P(\sigma_{ip}) \) and \( \hat{\omega} \cap P(\sigma_{ip}) = \emptyset \).

ii) Fix \( \epsilon > 0 \). For \( k \) sufficiently large, there exist properly embedded compact separating surfaces \( S_{1}^{k}, \ldots, S_{m}^{k} \subset N_{1}(T) \) such that for \( p \leq m \) there exists a closed complementary region \( J_{p}^{k} \subset N_{1}(T) \) of \( S_{p}^{k} \) such that \( T^{k} \subset \bigcap_{p=1}^{m} J_{p}^{k} = J^{k} \). Each \( S_{p}^{k} \) is a (possibly disconnected) surface lying in leaves of \( P(\sigma_{ip}^{k}) \). Finally for every \( j, S_{j}^{k} \)
is $\epsilon$-close to $S_j$ in the $C^2$ topology and $J^k$ is $\epsilon$-close to $J$ with respect to Hausdorff distance.

Proof of Claim 1. i) Fix $j \in \{1, \ldots, m\}$. Let $\tilde{N}_1(T)$ be the lift of $N_1(T)$ to $H^3$ which contains $V_i$. Using Lemma 3.5 v), the closure of the component of $H_{ij} \cap \tilde{N}_1(T)$ which contains $\tilde{V}_i$ intersects $\sigma_{ij} \cap \tilde{N}_1(T)$ in a compact surface $L_j$ lying in leaves of $\sigma_{ij}$. Part i) follows by taking $S_j = P(L_j)$ and $J_j$ the closed complementary region containing $\gamma$. 

ii) Since $\sigma_{ij}^k \to \sigma_{ij}$ and $\tilde{N}_1(T) \cap \sigma_{ij}$ is compact, it follows that for $k$ sufficiently large $\sigma_{ij}^k | \tilde{N}_1(T)$ very closely approximates $\sigma_{ij} | \tilde{N}_1(T)$. In particular $\sigma_{ij} | \tilde{N}_1(T)$ consists of finitely many families of parallel compact surfaces, so $\sigma_{ij}^k | \tilde{N}_1(T)$ consists of parallel families which approximate in a bijective fashion the $\sigma_{ij}$ families. In fact by Definition 2.2 for $k$ sufficiently large, each $x \in \sigma_{ij}^k \cap \tilde{N}_1(T)$ must lie very close to a $\sigma_{ij}$ family. Conversely a $\sigma_{ij}$ family is a product lamination of the form $F \times C \subset F \times I, C$ compact, and the proof of Lemma 3.3 shows that for $k$ sufficiently large $\sigma_{ij}^k \cap F \times I \subset F \times \bar{I}$ and each connected leaf is a compact surface transverse to the $I$-fibres. Therefore $\sigma_{ij}^k | F \times I$ is isotopic to a product lamination of the form $F \times C', C' \neq \emptyset$. Hence for $k$ sufficiently large, the component of $H_{ij}^k \cap \tilde{N}_1(T)$ which contains $\tilde{V}_i^k$, intersects $\sigma_{ij}^k \cap \tilde{N}_1(T)$ in a compact surface $L_j^k$ which is $\epsilon$-close to $L_j$. Part ii) follows by taking $S_j^k = P(L_j^k)$ and $J_j^k$ the closed complementary region containing $T^k$. By construction for $k$ sufficiently large, then for every $j$, $J_j^k$ is $\epsilon$-close to $J_j$ and $J^k$ is $\epsilon$-close to $J$. 

Claim 2. There exist an $\epsilon > 0$, a neighborhood $N(T)$ of $T$ and an embedded disc $E \subset \partial N(T)$ such that if $x \in \partial N(T) - E$, then there exists $j$ such that $d_\rho(x, J_j) > \epsilon$.

Proof of Claim 2. Let $S = \bigcup_{j=1}^m S_j$. Each $x \in \partial T$ has a $D^2 \times I \subset N_1(T)$ neighborhood satisfying the conclusion of Step 2 iv). In our context $S_j \cap D^2 \times I = A_j \cup B_j$, where at most one of $A_j, B_j$ is nonempty. The last conclusion of Step 2 iv) implies that associated to $D^2 \times I \cap \partial T$ is a small annulus $F_x \subset (\partial D^2) \times I$ such that if $F_x = S^1 \times I$, then $F_x \cap \partial T = S^1 \times 0$, and if $y \in S^1 \times t, t > 0$, then there exists $j$ such that $d_\rho(y, J_j) > 0$. See Figure 4.2. From a finite collection of curves of the form $(\partial D^2) \times I \cap \partial T$, we obtain a 1-complex $C \subset \partial T$ whose complement is a union of $\tilde{D}^2$’s. By the usual transversality arguments there exists a regular neighborhood $N(T) \subset N_1(T)$, a regular neighborhood $N(C) \subset \partial T$ and an $\epsilon > 0$ such that if $y \in N(C)$, then there exists $j$ such that $d_\rho(y \times 1, J_j) > \epsilon$. Here $N(T) - \tilde{\bar{\partial}}(\partial T)$ is identified with $\bar{\partial} T \times I$ (resp. $\partial T \times 0$). Finally let $E \subset \partial N(T)$ be a 2-disc such that $\partial N(T) - (N(C) \times 1) \subset E$. 

To complete the proof of Step 5, let $N(T)$ and $E$ be as in Claim 2. By Claim 1, if $k$ sufficiently large, then for each $y \in \partial N(T) - \tilde{\bar{\partial}}$ there exists $j$ such that $d_\rho(y, J_j^k) > \epsilon/2$. Therefore $(\partial N(T) - \tilde{\bar{\partial}}) \cap T^k \subset (\partial N(T) - \tilde{\bar{\partial}}) \cap J^k = \emptyset$. 

Remark. i) Near saddle tangencies $T^k$ may spill way out of $T$. In Figure 4.3 $\tilde{V}_i^k$ (resp. $\tilde{V}_i$) is locally defined by two leaves, whose $\tilde{V}_i^k$ (resp $\tilde{V}_i$) sides are indicated by arrows. $T^k$ may also spill out near spikes of $T$. Compare Figure 4.1. 

ii) The isotopy from $\gamma_t$ to $\gamma_{t_{\epsilon}}$ was supported in $V^k$. 

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iii) One could alternatively show that $\hat{\gamma}$ is a core of $T^k$ by first showing that $\hat{\gamma}$ is a core of $Y$ and then invoking Lemma 4.5 iii) with respect to the inclusion $\hat{\gamma} \subset T^k \subset Y$.

Step 6. Steps 1–5 applied to the $(\pi_1(X), \{\partial\delta_i\})$ insulator family $\{\lambda_{ij}\}$ yields the isotopy class of the link $\Gamma_1 = p_1^{-1}(\gamma) \subset X$ and the isotopy class of the $B^3$-link
These classes are independent of the metric on $X$ and the choice of spanning laminations.

Remark. We leave it to the reader to check that the analogues of Steps 1–5 work in the context of the $(\pi_1(X), \{\partial \delta_i\})$ insulator family $\{\lambda_{ij}\}$. In particular Steps 1–3 show that a metric $\mu$ on $X$ and a collection $\{\sigma_{i\mu}^{\lambda_{ij}}\}$ of $\mu$-least area laminations spanning $\{\lambda_{ij}\}$ give rise to a finite set $V_{\mu^\alpha}$ of pairwise disjoint $D^2 \times S^1$'s in $X$. Each isotopy required in the proofs of Steps 4–5 can be expressed as the composition of isotopies each of which is supported in some $V_{\mu^\alpha}$. Each component of $V_{\mu^\alpha}$ supports the isotopy restricted to exactly one component of the link. See the remarks following Steps 4–5.

Step 7. (Proof of Theorem 0.9.) Using the hyperbolic metric on $H^3$, the $(\pi_1(X), \{\partial \delta_i\})$ insulator family $\{\lambda_{ij}\}$ yields the $B^3$-link $\Delta$. The $B^3$-link $\Gamma$ is isotopic to the $B^3$-link $\Delta$.

Proof of Step 7. By the convexity property of Definition 0.4 there exists, for each $j$, a hyperbolic half-plane $P_{ij}$ separating $\partial \delta_i$ from $\lambda_{ij}$. Thus $P_{ij}$ separates $\delta_i$ from any $\rho$-least area lamination $\sigma_{ij}$ which spans $\lambda_{ij}$. Applying the insulator construction to the $(\pi_1(X), \{\partial \delta_i\})$ insulator family $\{\lambda_{ij}\}$ using the hyperbolic metric on $H^3$, we get for each $j$: $\delta_j \subset \tilde{V}_j$ and $\Delta_1 = \pi(\Delta) = q_1^{-1}(\delta_j)$ are cores of the tori $\pi(\tilde{V}_j)$. To see that $\pi(\delta_j)$ is a core of $\pi(\tilde{V}_j)$, consider $P_1 : (H^3 - \partial \delta_j) \to (H^3 - \partial \delta_j)/\langle g_j \rangle = Y_j = \hat{D}^2 \times S^1$, where $g_j$ generates $\text{Stab}_{\pi_1(X)}(\delta_j)$. Apply Lemma 4.5 iii) to $P(\delta_j) \subset P(\tilde{V}_j) \subset Y_j$. By Step 6, $\Delta_1$ is isotopic to $\Gamma_1 = p_1^{-1}(\gamma)$. Lift this isotopy to $H^3$ to complete the proof of Theorem 0.9.

Step 8. (Proof of Theorem 0.10.) The curve $f^{-1}(\delta)$ is isotopic to $\delta$.

Proof of Step 8. In the context of Theorem 0.10 $M = N$ and $f$ is a diffeomorphism (after a preliminary isotopy) homotopic to $\text{id}$. $M$ has two Riemannian metrics, the given hyperbolic metric $\rho$ and the pullback metric $f^*(\rho)$. The construction of

\[ H_i \neq \emptyset \]

\[ H_i = \emptyset \]

With the hyperbolic metric

With the $\mu$-metric

Figure 4.4
Steps 1–5 using the metric $\rho$ yields the geodesic $\delta$ and using the metric $f^*(\rho)$ yields the curve $f^{-1}(\delta)$. Since the isotopy class is independent of metric, it follows that $f^{-1}(\delta)$ is isotopic to $\delta$.

\begin{proof}

Remark 4.9. (Why a coalescable insulator is bad.) It is possible that the $H_1$ resulting from the construction applied to a coalescable insulator family would contain no unbounded $\mathbb{D}^2 \times \mathbb{R}$ component, i.e. Step 3 fails. Perhaps even $H_i = \emptyset$. It is conceivable that, as in [GS], using the wrong metric some $\rho$-totally geodesic plane $P$ transverse to $\epsilon_i$ may be disjoint from $H_i$. See Figure 4.4.

5. Applications and concluding remarks

Mostow’s Rigidity Theorem [Mo]. If $f : M \to N$ is a homotopy equivalence between closed hyperbolic manifolds of dimension $> 2$, then $f$ is homotopic to an isometry.

Remark 5.1. (What Mostow does not say.) If $\rho_0$ is a hyperbolic metric on $N$, then associated to a nontrivial element $\alpha$ of $\pi_1(N)$, there exists a unique geodesic $\delta \subset N$ which is freely homotopic to $\alpha$. Mostow does not rule out the possibility that with respect to a different hyperbolic metric $\rho_1$, the geodesic $\delta'$ associated to $\alpha$ would lie in a different isotopy class than $\delta$. What Mostow does assert is that there exists a diffeomorphism $f : N \to N$, homotopic to $\alpha$, such that $f(\delta) = \delta'$.

If $N$ satisfies the insulator condition, then this diffeomorphism is isotopic to $\alpha$, by Theorem 0.10, so we obtain

Theorem 5.2. Let $N$ be a closed hyperbolic 3-manifold satisfying the insulator condition. If $\rho_0$ and $\rho_1$ are hyperbolic metrics on $N$, then there exists a diffeomorphism $f : N \to N$ isotopic to $\alpha$ such that $f^*(\rho_1) = \rho_0$. In particular the space of hyperbolic metrics on $N$ is path connected.

Remark. So for manifolds satisfying the insulator condition, hyperbolic structures are unique up to isotopy.

Corollary 5.3. Let $N$ be a closed orientable 3-manifold satisfying the insulator condition. Then $\text{Homeo}(N)/\text{Homeo}_0(N) = \text{Out}(\pi_1(N)) = \text{Isom}(N)$. (\text{Homeo}_0(N) is the group of homeomorphisms isotopic to id.)

Proof. The second equality follows from Mostow. Let $\mathcal{H} : \text{Homeo}(M)/\text{Homeo}_0(N) \to \text{Out}(\pi_1(N))$ be the map induced by the action of $\pi_1(N)$. Since $N$ is a $k(\pi, 1)$, it follows that $\mathcal{H}([h]) = \mathcal{H}([g])$ if and only if $h$ is homotopic to $g$. By Mostow $\mathcal{H}$ is surjective and by Theorem 0.10 $\mathcal{H}$ is injective. $\text{Homeo}(N)/\text{Homeo}_0(N)$ is often called the mapping class group of $N$.

Definition 5.4. i) Call a finite set of pairwise disjoint simple closed curves in $N$ a homotopy essential link if each component is homotopically nontrivial in $N$ and no two components lie in the same $\mathbb{Z}$ subgroup of $\pi_1(N)$.

ii) If $q : H^3 \to N$ is the universal covering projection of a hyperbolic 3-manifold $N$, then $q^{-1}$ induces the map $Q : \{\text{isotopy classes of homotopy essential links in } N\} \to \{\text{isotopy classes of } B^3\text{-links}\}$. If $q_X : X \to N$ is a finite covering map, then $q_X^{-1}$ induces the map $Q_X : \{\text{isotopy classes of homotopy essential links in } N\} \to \{\text{isotopy classes of homotopy essential links in } X\}$.

Recall that isotopies of $B^3\text{-links}$ are required to fix $S^2_\infty$ pointwise.

Conjecture 5.5A. $Q$ is injective.
Conjecture 5.5B. $Q_X$ is injective.

Corollary 5.6. Let $\delta$ be a simple closed geodesic and $\delta'$ a simple closed curve in $N$ such that $Q(\delta) = Q(\delta')$. Then $\delta$ is isotopic to $\delta'$ if $N$ satisfies the insulator condition.

Proof. Apply Proposition 2.1 to find a homeomorphism $f : N \to N$ such that $f(\delta) = \delta'$. By construction $\tilde{f}$ fixes $S_{\infty}^2$ pointwise, so $f$ is homotopic to id. Now apply Theorem 0.10.

Remark 5.7. By Corollary 5.6 if $N$ satisfies the insulator condition, then Conjecture 5.5A is true for geodesics. Also if $X$ satisfies the insulator condition, then Conjecture 5.5B implies Conjecture 5.5A for geodesics.

Definition 5.8. If $\delta$ is a closed geodesic in the hyperbolic 3-manifold $N$, then the tube radius of $\delta = \text{Sup} \{ \text{radii of embedded hyperbolic tubes about } \delta \} = 1/2 \min \{ d(\delta_i, \delta_j) \mid \delta_i, \delta_j \text{ are distinct preimages of } \delta \text{ in } H^3 \}$.

Lemma 5.9. If the hyperbolic manifold $N$ has a geodesic $\delta$ with tube radius $> (\log 3)/2$, then the Dirichlet insulator family associated to $\delta$ is noncoalescable. In particular $N$ satisfies the insulator condition.

Proof. By Corollary 4.8, if tube radius($\delta$) $> (\log 3)/2$ and $\delta_i$ is a lift of $\delta$, then $\delta_i$-visual angle($\lambda_{ij}$) $< 2\pi/3$ where $\lambda_{ij}$ is the ideal boundary of the Dirichlet midplane between $\delta_i$ and $\delta_j$. Therefore for no $j, k, l$ does $\lambda_{ij} \cup \lambda_{ik} \cup \lambda_{il}$ separate $\partial \delta_i$.

Corollary 5.10. Each hyperbolic 3-manifold $N$ is covered by a hyperbolic 3-manifold $X$ satisfying the insulator condition.

Proof. By [G3] for every $\epsilon > 0$, there exists a finite regular cover $X$ of $N$ such that tube radius($\delta$) $> \epsilon$, where $\delta$ is a shortest geodesic in $X$.

Proof of Theorem 0.1. Combine Theorems 0.9, 0.10, and 5.2 with Lemma 5.9.

Remark 5.11. i) Via a technique called fudging, the inequality given in Lemma 5.9 can be improved at least to an equality. The idea is that a small perturbation of a just barely coalescable Dirichlet insulator would create a noncoalescable insulator family.

ii) An application of the hyperbolic law of cosines shows that if the shortest geodesic $\delta$ in $N$ has length $L \geq 1.353$, then tube radius($\delta$) $> (\log 3)/2$. One finds a right triangle as in Figure 5.1.

iii) Applying the Meyerhoff tube radius formula, i.e. Corollary of [Me, §3] with $n \leq 8$, we conclude that if $N$ has a geodesic $\delta$ of length $\leq .0978$, then tube radius($\delta$) $> (\log 3)/2$. Combined with the work of Jorgenson [Gr], this shows that for $n > 0$ there are at most finitely many hyperbolic 3-manifolds of volume $< n$ which can fail to satisfy the insulator condition.

Corollary 5.12. For any $n > 0$, all but at most finitely many closed hyperbolic 3-manifolds of volume $< n$ are both geometrically and topologically rigid.

Remark 5.13. A 3-manifold $N$ is topologically rigid means that any homotopy equivalence between $N$ and an irreducible 3-manifold is homotopic to a homeomorphism. The hyperbolic 3-manifold $N$ is geometrically rigid means that its hyperbolic metric is unique up to isotopy.
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