Abstract In this paper, we study the further improvements of the reverse Young and Heinz inequalities for the wider range of \( v \), namely \( v \in \mathbb{R} \). These modified inequalities are used to establish corresponding operator inequalities on a Hilbert space.

Keywords Young’s inequality · Heinz inequality · Operator inequality

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1 Introduction

The weighted arithmetic-geometric mean inequality, which is also called Young’s inequality, states

\[(1 - v)a + vb \geq a^{1-v}b^v\]
for \( a, b \geq 0 \) and \( v \in [0, 1] \). If \( v = \frac{1}{2} \), we obtain the arithmetic-geometric mean inequality
\[
\frac{a + b}{2} \geq \sqrt{ab}.
\]
The Heinz means, introduced in [3], are defined by
\[
H_v(a, b) = \frac{a^{1-v}b^v + a^v b^{1-v}}{2}
\]
for \( a, b \geq 0 \) and \( v \in [0, 1] \). It is easy to see that
\[
\sqrt{ab} \leq H_v(a, b) \leq \frac{a + b}{2},
\]
which is called Heinz inequality.

Let \( B(H) \) denote the \( C^* \)-algebra of all bounded linear operators on a complex Hilbert space \( H \). A self-adjoint operator \( A \in B(H) \) is called positive, and we write \( A \geq 0 \) if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in H \). The set of all positive operators is denoted by \( B^+(H) \). The set of all invertible operators in \( B^+(H) \) is denoted by \( B^{++}(H) \). We say \( A \geq B \) if \( A - B \geq 0 \).

Let \( A, B \in B^{++}(H) \) and \( v \in [0, 1] \). The \( v \)-weighted operator geometric mean of \( A \) and \( B \), denoted by \( A^\sharp_v B \), is defined as
\[
A^\sharp_v B = A^{\frac{v}{2}} \left( A^{-\frac{v}{2}} B A^{\frac{v}{2}} \right)^v A^{\frac{v}{2}}
\]
and the \( v \)-weighted operator arithmetic mean of \( A \) and \( B \), denoted by \( A^\nabla_v B \), is
\[
A^\nabla_v B = (1 - v)A + vB.
\]
When \( v = \frac{1}{2} \), \( A^\sharp_\frac{1}{2} B \) and \( A^\nabla_\frac{1}{2} B \) are called operator geometric mean and operator arithmetic mean, and denoted by \( A^\sharp B \) and \( A^\nabla B \), respectively [19]. For positive operators \( A, B \) and \( v \in [0, 1] \), we have [10, Definition 5.2]
\[
A^\sharp_v B = B^\sharp_{1-v} A.
\]
It is well known that if \( A, B \in B^{++}(H) \) and \( v \in [0, 1] \), then [8,9]
\[
A^\nabla_v B \geq A^\sharp_v B,
\]
which is the operator version of the scalar Young’s inequality. An operator version of Heinz means was introduced in [3] by
\[
H_v(A, B) = \frac{A^\sharp_v B + A^\sharp_{1-v} B}{2},
\]
where \( v \in [0, 1] \). In particular \( H_1(A, B) = H_0(A, B) = A^\nabla B \). It is easy to see that the Heinz operator means interpolate between the arithmetic and geometric operator means
\[
A^\sharp B \leq H_v(A, B) \leq A^\nabla B,
\]
which is called the Heinz operator inequality \[14,15\]. We note that we use in Sect. 3 the following notations

\[ A^\nu B \equiv A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\nu A^{\frac{1}{2}}, \quad \hat{H}_v(A, B) \equiv \frac{A^1_v B + A^{1-v}_v B}{2}, \]

for all \( v \in \mathbb{R} \) including the range \( v \notin [0,1] \).

Improvements of Young and Heinz inequalities and their reverses have been done for the weight \( v \in [0,1] \) by many researchers. We refer the reader to \[1,2,4,5,6,7,11,12,13,14,15,16,17,18,20,21,22,23,24\] as a sample of the extensive use of Young and Heinz inequalities. One of the first refinements is as follows in Lemma 1 which one positive term was added to the right-hand side of the Young’s inequality.

**Lemma 1** ([16]) Let \( a, b \geq 0 \) and \( v \in [0,1] \). Then

\[(1 - v)a + vb \geq a^{1-v}b^v + r_0 \left( \sqrt{a} - \sqrt{b} \right)^2, \]

where \( r_0 = \min\{v, 1-v\} \).

However, in the recent paper [24], Zhao and Wu provided two refining terms of Young’s inequality in the following way.

**Lemma 2** ([24]) Let \( a, b \geq 0 \) and \( v \in [0,1] \).

(i) If \( v \in [0,\frac{1}{2}] \), then

\[(1 - v)a + vb \geq a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt{a} - \sqrt{ab})^2, \]

(ii) If \( v \in [\frac{1}{2},1] \), then

\[(1 - v)a + vb \geq a^{1-v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt{b} - \sqrt{ab})^2, \]

where \( r = \min\{v, 1-v\} \) and \( r_0 = \min\{2r, 1-2r\} \).

In the same paper, the following refined reverse versions have been proved too.

**Lemma 3** ([24, Lemma 2]) Let \( a, b \geq 0 \) and \( v \in [0,1] \).

(i) If \( v \in [0,\frac{1}{2}] \), then

\[(1 - v)a + vb \leq a^{1-v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt{b} - \sqrt{ab})^2, \]

(ii) If \( v \in [\frac{1}{2},1] \), then

\[(1 - v)a + vb \leq a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt{a} - \sqrt{ab})^2, \]

where \( r = \min\{v, 1-v\} \) and \( r_0 = \min\{2r, 1-2r\} \).
Sababheh and Choi [21] obtained a complete refinement of the Young’s inequality by adding as many refining terms as we like. For $a, b > 0$, $n \in \mathbb{N}$ and $v \in [0, 1]$

$$(1 - v)a + vb \geq a^{1-v}b^v + \sum_{k=1}^{n} s_k(v) \left( 2^{k} \sqrt{a^{2^{k-1} - j_k(v)}b^{j_k(v)}} - \frac{2^k}{\sqrt{a^{2^{k-1} - j_k(v)}b^{j_k(v)}}+1} \right)^2,$$

where $[x]$ is the greatest integer less than or equal to $x$ and

$$j_k(v) = \lfloor 2^{k-1}v \rfloor,$$

$$r_k(v) = \lfloor 2^k v \rfloor,$$

$$s_k(v) = (-1)^{r_k(v)}2^{k-1}v + (-1)^{r_k(v)+1} \left[ \frac{r_k(v) + 1}{2} \right].$$

Quite recently, Sababheh and Moslehian [22] gave a full description of all other refinements of the reverse Young’s inequality in the literature as follows.

**Lemma 4** ([22, Theorem 2.1]) Let $a, b > 0$ and $v \in [0, 1]$.

(i) If $v \in [0, \frac{1}{2}]$, then

$$(1 - v)a + vb \leq a^{1-v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 - S_n(2v, \sqrt{ab}, b).$$

(ii) If $v \in [\frac{1}{2}, 1]$, then

$$(1 - v)a + vb \leq a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 - S_n(2(1 - v), \sqrt{ab}, a).$$

Where

$$S_n(v, a, b) = \sum_{k=1}^{n} s_k(v) \left( 2^k \sqrt{b^{2^{k-1} - j_k(v)}a^{j_k(v)}} - \frac{2^k}{\sqrt{b^{2^{k-1} - j_k(v)}a^{j_k(v)}}+1} \right)^2,$$

$$j_k(v) = \lfloor 2^{k-1}v \rfloor,$$

$$r_k(v) = \lfloor 2^k v \rfloor,$$

$$s_k(v) = (-1)^{r_k(v)}2^{k-1}v + (-1)^{r_k(v)+1} \left[ \frac{r_k(v) + 1}{2} \right].$$

In the study of Young’s inequalities, supplemental Young’s inequality

$$a^v b^{1-v} \geq va + (1 - v)b$$

for $a, b > 0$ and $v \notin [0, 1]$ is often discussed. Our main idea in this paper is to extend the range of $v$ and to give the tighter bounds of the reverse Young’s inequalities proved in [22] and [24]. In Theorem 1, we will obtain a new generalization of the reverse Young’s inequality which is stronger than the reverse Young’s inequalities shown in [22, Theorem 2.1] and [24, Lemma 2]. Theorem 3 is another refinement of [21, Theorem 2.9] which extend the range of $v$. In Sect. 3, these modified inequalities are used to establish corresponding operator inequalities on a Hilbert space. We emphasize that the significance of the inequalities in this paper is to have the wider range, namely $v \in \mathbb{R}$, and tighter bounds.
2 Generalizations of the Reverse Scalar Young and Heinz Inequalities

In this section, we present the numerical inequalities needed to prove the operator versions. We start from the following lemma to prove our main result.

Lemma 5 ([2, Lemma 2.1]) Let \( a, b > 0 \) and \( v \notin [0,1] \). Then

\[
(1-v)a + vb \leq a^{1-v}b^v.
\]

Corollary 1 Let \( a, b > 0 \) and \( \frac{1}{2} \neq v \in \mathbb{R} \).

(i) If \( v \notin [0, \frac{1}{2}] \), then

\[
(1-v)a + vb \leq a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2.
\]

(ii) If \( v \notin [\frac{1}{2}, 1] \), then

\[
(1-v)a + vb \leq a^{1-v}b^v + (1-v)(\sqrt{a} - \sqrt{b})^2.
\]

Proof (i) If \( v \notin [0, \frac{1}{2}] \), then

\[
(1-v)a + vb - v(\sqrt{a} - \sqrt{b})^2
= (1-2v)a + (2v)\sqrt{ab}
\leq a(1-2v)(\sqrt{ab})^{2v} \quad \text{(by Lemma 5)}
= a^{1-v}b^v.
\]

(ii) If \( v \notin [\frac{1}{2}, 1] \), then \( (1-v) \notin [0, \frac{1}{2}] \). So by changing two elements \( a, b \) and two weights \( v, 1 - v \) in (i), the desired inequality is obtained. \( \square \)

Next, we represent our main result which is the reverse Young’s inequality for \( v \in \mathbb{R} \).

Theorem 1 Let \( a, b > 0 \), \( n \in \mathbb{N} \) such that \( n \geq 2 \) and \( \frac{1}{2} \neq v \in \mathbb{R} \). Then,

(i) If \( v \notin \left[ \frac{1}{2}, \frac{2n-1}{2^n-1} \right] \), then

\[
(1-v)a + vb \leq a^{1-v}b^v + (1-v)(\sqrt{a} - \sqrt{b})^2
+ (2v-1)\sqrt{ab} \sum_{k=2}^{n} 2^{k-2} \left( \sqrt[2k]{\frac{b}{a}} - 1 \right)^2. \quad (1)
\]

(ii) If \( v \notin \left[ \frac{2n-1}{2^n-1}, \frac{1}{2} \right] \), then

\[
(1-v)a + vb \leq a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2
+ (1-2v)\sqrt{ab} \sum_{k=2}^{n} 2^{k-2} \left( \sqrt[2k]{\frac{a}{b}} - 1 \right)^2. \quad (2)
\]
Proof \((i)\) If \(v \notin \left[\frac{1}{2}, \frac{2^{n-1}+1}{2^{n}}\right]\), we have \((2^n v - 2^{n-1}) \notin [0, 1]\) and \((2^{n-1} - 2^n v + 1) \notin [0, 1]\). Now compute

\[
(1 - v)a + vb - (1 - v)(\sqrt{a} - \sqrt{b})^2 - (2v - 1)\sqrt{ab}\left\{\sum_{k=2}^{n} 2^{k-2} \left(\frac{2^{k-1}b}{a} - 1\right)\right\}
\]

\[
= (1 - v)a + vb - (1 - v)(\sqrt{a} - \sqrt{b})^2
- (2v - 1)\sqrt{ab}\left\{\left(\frac{\sqrt{b}}{\sqrt{a}} - 1\right)^2 + 2\left(\frac{\sqrt{b}}{\sqrt{a}} - 1\right)^2 + 4\left(\frac{\sqrt{b}}{\sqrt{a}} - 1\right)^2\right\}
- \cdots - 2^{n-4}(2v - 1)\sqrt{ab}\left(\frac{2^{n-1}b}{a} - 1\right)^2
- 2^{n-3}(2v - 1)\sqrt{ab}\left(\frac{2^{n-1}b}{a} - 1\right)^2
= (1 - v)a + vb - (1 - v)(a - 2\sqrt{ab} + b) - (2v - 1)\sqrt{ab}\left(\frac{\sqrt{b}}{\sqrt{a}} - 2\sqrt{a}\right) + 4(2v - 1)\sqrt{ab}\left(\frac{\sqrt{b}}{\sqrt{a}} - 2\sqrt{a}\right)
- \cdots - 2^{n-4}(2v - 1)\sqrt{ab}\left(\frac{2^{n-2}b}{a} - 2\sqrt{a}\right)
- 2^{n-3}(2v - 1)\sqrt{ab}\left(\frac{2^{n-2}b}{a} - 2\sqrt{a}\right)
- 2^{n-2}(2v - 1)\sqrt{ab}\left(\frac{2^{n-1}b}{a} - 2\sqrt{a}\right)
= 2(1 - v)\sqrt{ab} + (1 - 2v)\sqrt{ab} \sum_{l=0}^{n-2} 2^l + 2^{n-1}(2v - 1)\sqrt{ab}\frac{\sqrt{b}}{\sqrt{a}}
= \left\{2(1 - v) + (1 - 2v)\left(\sum_{l=0}^{n-2} 2^l\right)\right\} \sqrt{ab} + 2^{n-1}(2v - 1)\sqrt{ab}\frac{\sqrt{b}}{\sqrt{a}}
= (2^{n-1} - 2^n v + 1)\sqrt{ab} + (2^n v - 2^{n-1})\sqrt{ab}\frac{\sqrt{b}}{\sqrt{a}}
\leq \left(\sqrt{ab}\right)^{(2^n - 1) - 2^n v + 1}\left(\sqrt{ab}\frac{\sqrt{b}}{\sqrt{a}}\right)^{(2^n v - 2^{n-1})} \text{ (by Lemma 5)}
= a^{1-v}v^n.
So we get the following inequality
\[
(1 - v)a + vb - (1 - v)(\sqrt{a} - \sqrt{b})^2 - (2v - 1)\sqrt{ab}\left\{\sum_{k=2}^{n} 2^{k-2} \left( \frac{b}{\sqrt[2n]{a}} - 1 \right)^2 \right\}
\]
\[\leq a^{1-v} b^v\]
which is equivalent to \((1)\).

(ii) If \(v \notin \left[\frac{2^{n-1} - 1}{2n}, \frac{1}{2}\right]\), then \((1 - v) \notin \left[\frac{1}{2}, \frac{2^{n-1} + 1}{2n}\right]\). Now by changing two elements \(a, b\) and replacing the weight \(v\) with \((1 - v)\) in (i), the desired inequality (2) is deduced. \(\square\)

**Remark 1** We would remark that if we rewrite Theorem 1 for \(n = 1\), then we get Corollary 1.

**Remark 2** From the equality of the proof in Theorem 1, the inequality (1) is equivalent to
\[
a^{1-v} b^v \geq \sqrt{ab} + 2^n \left( v - \frac{1}{2} \right) \sqrt{ab} \left( \frac{\sqrt[n]{b}}{\sqrt[n]{a}} - 1 \right),
\]
which gives the following inequality
\[
\left( \frac{b}{a} \right)^{v-\frac{1}{2}} \geq 1 + 2^n \left( v - \frac{1}{2} \right) \left( \frac{\sqrt[n]{b}}{\sqrt[n]{a}} - 1 \right).
\]
Since \(\lim_{r \to 0} \frac{\ln(1 + r)}{r} = \log t\), by putting \(r = \frac{1}{2^n}\), we have
\[
\lim_{n \to \infty} 2^n \left( \frac{\sqrt[n]{b}}{\sqrt[n]{a}} - 1 \right) = \log \frac{b}{a}.
\]
Thus, we have the following inequality in the limit of \(n \to \infty\) for the inequality (1) in Theorem 1:
\[
\log \left( \frac{b}{a} \right)^{v-\frac{1}{2}} \leq \left( \frac{b}{a} \right)^{v-\frac{1}{2}} - 1, \quad \text{(4)}
\]
for \(a, b > 0\) and \(\frac{1}{2} \neq v \in \mathbb{R}\), which comes from the condition \(v \notin \left[\frac{1}{2}, \frac{2^{n-1} + 1}{2n}\right]\) in the limit of \(n \to \infty\). The above inequality recover the equality in the case \(v = \frac{1}{2}\). Therefore, we have the inequality (4) for all \(v \in \mathbb{R}\). We notice that the inequality (4) can be proven directly by putting \(x = \left( \frac{b}{a} \right)^{v-1/2}\) in the inequality
\[
\log x \leq x - 1, \quad (x > 0), \quad \text{(5)}
\]
Similarly in the limit of \(n \to \infty\) for the inequality (2) in Theorem 1, we get the inequality
\[
\log \left( \frac{a}{b} \right)^{\frac{1}{2}-v} \leq \left( \frac{a}{b} \right)^{\frac{1}{2}-v} - 1, \quad \text{(4)}
\]
by changing two elements \(a, b\) and replacing the weight \(v\) with \((1 - v)\) in the inequality (4).
Next, in Remarks 3 and 4, we show that Theorem 1 recovers the inequalities in Lemma 3. To achieve this, we compare Lemma 3 with Theorem 1 in the cases such as $n = 2$ and $n = 3$ where $v \in [0, 1]$. First, we notice that Lemma 3 is equivalent to the following proposition.

**Proposition 2** Let $a, b \geq 0$ and $v \in [0, 1]$.

(i) If $v \in [0, \frac{1}{4}]$, then
\[
(1 - v)a + vb \leq a^{1 - v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 - 2v(\sqrt{b} - \sqrt[4]{ab})^2.
\]

(ii) If $v \in [\frac{1}{4}, \frac{1}{2}]$, then
\[
(1 - v)a + vb \leq a^{1 - v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 + (2v - 1)(\sqrt{b} - \sqrt[4]{ab})^2.
\]

(iii) If $v \in [\frac{1}{2}, \frac{3}{4}]$, then
\[
(1 - v)a + vb \leq a^{1 - v}b^v + v(\sqrt{a} - \sqrt{b})^2 - (2v - 1)(\sqrt{a} - \sqrt[4]{ab})^2.
\]

(iv) If $v \in [\frac{3}{4}, 1]$, then
\[
(1 - v)a + vb \leq a^{1 - v}b^v + v(\sqrt{a} - \sqrt{b})^2 + (2v - 2)(\sqrt{a} - \sqrt[4]{ab})^2.
\]

**Remark 3** Consider Theorem 1 in the case $n = 2$ with $v \in [0, 1]$. For $a, b > 0$, we have the following inequalities

(i) If $v \notin [\frac{1}{2}, \frac{3}{4}]$, then
\[
(1 - v)a + vb \leq a^{1 - v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 + (2v - 1)(\sqrt{b} - \sqrt[4]{ab})^2. \tag{6}
\]

(ii) If $v \notin [\frac{3}{4}, 1]$, then
\[
(1 - v)a + vb \leq a^{1 - v}b^v + v(\sqrt{a} - \sqrt{b})^2 - (2v - 1)(\sqrt{a} - \sqrt[4]{ab})^2. \tag{7}
\]

In our recent paper [11], we showed that the right-hand sides of both inequalities (6) and (7) give tighter upper bounds of the $v$-weighted arithmetic mean than those in Proposition 2.

**Remark 4** As a direct consequence of Theorem 1 in the case $n = 3$ with restricted range $v \in [0, 1]$, we have

(i) If $v \notin [\frac{1}{2}, \frac{5}{8}]$, then
\[
(1 - v)a + vb \leq a^{1 - v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 + (2v - 1)(\sqrt{b} - \sqrt[4]{ab})^2 + (4v - 2)(\sqrt{ab}^3 - \sqrt[4]{ab})^2. \tag{8}
\]
We here give advantages of inequalities (8) and (9) in comparison with Proposition 2.

(a) Firstly, we compare Proposition 2 with the inequality (8) which holds in the cases \( v \in [0, \frac{1}{4}] \) and \( v \in [\frac{1}{4}, \frac{3}{4}] \) and \( v \in [\frac{3}{4}, 1] \).

\( (a1) \) In the case \( v \in [0, \frac{1}{4}] \), we have \((2v - 1) < (-2v)\) and \((4v - 2) < 0\). Indeed, the right-hand side of inequality (8) is less than the right-hand side of \((i)\) in Proposition 2. For the case of \( v \in [\frac{1}{4}, \frac{3}{4}] \), we have \((4v - 2) < 0\). So we easily find that the right-hand side of inequality (8) is less than the right-hand side of \((ii)\) in Proposition 2.

\( (a2) \) For the case of \( v \in [\frac{3}{4}, 1] \), we claim that the right-hand side of inequality (8) is less than or equal to the right-hand side of \((iii)\) in Proposition 2. To prove our claim, we show that the following inequality holds:

\[
(1 - v)\sqrt{a - \sqrt b}^2 + (2v - 1)(\sqrt b - \sqrt{ab})^2 + (4v - 2)(\sqrt{ab - \sqrt b}^2) \\
\leq v\sqrt{a - \sqrt b}^2 - (2v - 1)(\sqrt a - \sqrt{ab})^2,
\]

which is equivalent to the inequality

\[
2(1 - 2v)t^{1/4} \left\{3t^{1/4} - 1 - 2t^{3/8}\right\} \geq 0,
\]

for \( t > 0 \) and \( v \in [\frac{3}{4}, 1] \). To obtain this, it is enough to prove \((3t^{1/4} - 1 - 2t^{3/8}) \leq 0\). If \( t^{3/8} = x \), then we easily find that \( f(x) = 3x^2 - 2x^3 - 1 \) is increasing for \( 0 < x < 1 \) and decreasing where \( x > 1 \). Indeed, \( f(x) = 3x^2 - 2x^3 - 1 \leq 0 \) where \( x > 0 \) and so \( 3t^{1/4} - 1 - 2t^{3/8} \leq 0 \).

\( (a3) \) For the case of \( v \in [\frac{1}{4}, 1] \), we claim that the right-hand side of inequality (8) is less than or equal to the right-hand side of \((iv)\) in Proposition 2. To prove our claim, we show that the following inequality holds:

\[
(1 - v)\sqrt{a - \sqrt b}^2 + (2v - 1)(\sqrt b - \sqrt{ab})^2 + (4v - 2)(\sqrt{ab - \sqrt b}^2) \\
\leq v\sqrt{a - \sqrt b}^2 - 2(1 - v)(\sqrt a - \sqrt{ab})^2,
\]

which is equivalent to the inequality

\[
(4v - 3) + (3 - 8v)t^{1/2} + (4 - 4v)t^{1/4} + (8v - 4)t^{5/8} \geq 0,
\]

for \( t > 0 \) and \( v \in [\frac{1}{4}, 1] \). To obtain the inequality (12), it is sufficient to prove \( f(x, v) \geq 0 \) where \( x = t^{1/8} > 0 \) and

\[
f(x, v) \equiv (8v - 4)x^5 + (3 - 8v)x^4 + (4 - 4v)x^2 + (4v - 3).
\]

Since \( \left[ \frac{d}{dx} f(x, v) \right] = 4(x - 1)^2(2x^3 + 2x^2 + 2x + 1) \geq 0 \), we have \( f(x, v) \geq f(x, \frac{1}{4}) = x^2(x - 1)^2(2x + 1) \geq 0 \) for \( x > 0 \).
(b) Secondly, we compare Proposition 2 with the inequality (9) which holds in the cases \( v \in [0, \frac{1}{2}] \), \( v \in [\frac{1}{2}, \frac{3}{2}] \), \( v \in [\frac{1}{2}, \frac{3}{4}] \) and \( v \in [\frac{3}{4}, 1] \).

(b1) For the case of \( v \in [0, \frac{1}{2}] \), we claim that the right-hand side of inequality
(9) is less than or equal to the right-hand side of \((i)\) in Proposition 2.

To prove our claim, we give the following inequality
\[
v(\sqrt{a} - \sqrt{b})^2 - (2v - 1)(\sqrt{a} - \sqrt{ab})^2 - (4v - 2)(\sqrt{a^2b} - \sqrt{ab})^2 \leq (1 - v)(\sqrt{a} - \sqrt{b})^2 - 2v(\sqrt{b} - \sqrt{ab})^2,
\]
which we get it by replacing \( v \) with \( (1 - v) \) and changing the elements \( a, b \) in the inequality (11).

(b2) For the case of \( v \in [\frac{1}{2}, \frac{3}{4}] \), we claim that the right-hand side of inequality (9) is less than or equal to the right-hand side of \((ii)\) in Proposition 2.

To prove our claim, we give the following inequality
\[
v(\sqrt{a} - \sqrt{b})^2 - (2v - 1)(\sqrt{a} - \sqrt{ab})^2 - (4v - 2)(\sqrt{a^2b} - \sqrt{ab})^2 \leq (1 - v)(\sqrt{a} - \sqrt{b})^2 - (1 - 2v)(\sqrt{b} - \sqrt{ab})^2,
\]
which is deduced by replacing \( v \) with \( (1 - v) \) and changing the elements \( a, b \) in the inequality (10).

(b3) For the case of \( v \in [\frac{3}{4}, \frac{3}{2}] \), we have \(-(4v - 2) < 0\). So we easily find that the right-hand side of inequality (9) is less than the right-hand side of \((iii)\) in Proposition 2. In the case \( v \in [\frac{3}{4}, 1] \), we have \(-(2v - 1) < (2v-2)\) and \(-(4v - 2) < 0\). That is the right-hand side of inequality (9) is less than the right-hand side of \((iv)\) in Proposition 2.

Thus, according to Remark 3 and Remark 4, Theorem 1 recover Lemma 3. We notice that the range of the reverse Young’s inequalities in Theorem 1 is wider than Lemma 3, namely Theorem 1 holds in the case \( v \in \mathbb{R} \) and Lemma 3 holds for \( v \in [0, 1] \).

Next, we compare Theorem 1 with Lemma 4. In Theorem 1, we have (I) \( v \leq 0 \), (II) \( 0 \leq v < \frac{\sqrt{v} - 1}{\sqrt{v} - 2} \), (III) \( \frac{\sqrt{v} - 1}{\sqrt{v} - 2} \leq v \leq \frac{1}{2} \), (IV) \( \frac{1}{2} \leq v \leq \frac{\sqrt{v} - \sqrt{3}}{\sqrt{v} - 3} \), (V) \( \frac{\sqrt{v} - \sqrt{3}}{\sqrt{v} - 3} < v \leq 1 \) and (VI) \( v \geq 1 \).

For the cases of (II) and (V), we claim that Theorem 1 has tighter upper bounds than those in Lemma 4, while Lemma 4 recover Theorem 1 in the cases (III) and (IV). Therefore we conclude that Theorem 1 and Lemma 4 are different refinements of the reverse Young’s inequality which both of them recover Lemma 3. However, we emphasize that Theorem 1 gives the reverse Young’s inequality in the wider range than Lemma 4, namely in the case \( v \in \mathbb{R} \). This justify why our refinement in Theorem 1 is better than Lemma 4.

To prove our claims, we compare Theorem 1 with Lemma 4 in the same steps such as \( n = 2 \). For this purpose, we list up the following corollaries which are deduced directly from Theorem 1 and Lemma 4, respectively.

Corollary 2 Let \( a, b > 0 \) and \( v \in \mathbb{R} \).
(i) If \( v \notin \left[ \frac{1}{4}, \frac{3}{4} \right] \), then
\[
(1 - v)a + vb - a^{1-v}b^v \leq (1 - v)(\sqrt{a} - \sqrt{b})^2 + (2v - 1)(\sqrt{b} - \sqrt{a}b)^2.
\]
(13)

(ii) If \( v \notin \left[ \frac{3}{4}, \frac{1}{2} \right] \), then
\[
(1 - v)a + vb - a^{1-v}b^v \leq v(\sqrt{a} - \sqrt{b})^2 - (2v - 1)(\sqrt{a} - \sqrt{ab})^2.
\]
(14)

**Corollary 3** Let \( a, b > 0 \) and \( v \in [0, 1] \).

(i) If \( v \in \left[ 0, \frac{3}{4} \right) \), then
\[
(1 - v)a + vb - a^{1-v}b^v \leq (1 - v)(\sqrt{a} - \sqrt{b})^2 - 2v(\sqrt{b} - \sqrt{ab})^2
- 4v(\sqrt{b} - \sqrt{ab})^2.
\]
(15)

(ii) If \( v \in \left[ \frac{1}{4}, \frac{1}{2} \right] \), then
\[
(1 - v)a + vb - a^{1-v}b^v \leq (1 - v)(\sqrt{a} - \sqrt{b})^2 + (2v - 1)(\sqrt{b} - \sqrt{ab})^2
- (4v - 1)(\sqrt{ab} - \sqrt{a})^2.
\]
(16)

(iii) If \( v \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), then
\[
(1 - v)a + vb - a^{1-v}b^v \leq v(\sqrt{a} - \sqrt{b})^2 - (2v - 1)(\sqrt{a} - \sqrt{ab})^2
+ (4v - 3)(\sqrt{ab} - \sqrt{a})^2.
\]
(17)

(iv) If \( v \in \left[ \frac{3}{4}, 1 \right) \), then
\[
(1 - v)a + vb - a^{1-v}b^v \leq v(\sqrt{a} - \sqrt{b})^2 + (2v - 2)(\sqrt{a} - \sqrt{ab})^2
- 4(1 - v)(\sqrt{ab} - \sqrt{a})^2.
\]
(18)

**Remark 5** Here, we compare the upper bounds in Corollary 2 with those in Corollary 3. Firstly, we compare Corollary 3 with the inequality (13) which holds in the cases \( v \in \left[ 0, \frac{3}{4} \right] \), \( v \in \left[ \frac{3}{4}, \frac{1}{2} \right] \) and \( v \in \left[ \frac{1}{2}, 1 \right] \).

For the case of \( v \in \left[ 0, \frac{3}{4} \right] \), we can find examples such that the right-hand side of (13) is tighter than that of (15) in Corollary 3. Actually, take \( a = 1 \), \( b = 16 \) and \( v = 1/8 \), then the right-hand side of (13) is equal to 4.875, while the right-hand side of (15) is nearly equal to 6.2892. Indeed, the inequality (13) can recover Corollary 3 where \( v \in \left[ 0, \frac{1}{4} \right] \).

In the case \( v \in \left[ \frac{1}{4}, 1 \right] \), we claim that the right-hand side of the inequality (13) is less than or equal to the right-hand side of (18). According to (11), we have
\[
(1 - v)(\sqrt{a} - \sqrt{b})^2 + (2v - 1)(\sqrt{b} - \sqrt{ab})^2
\leq v(\sqrt{a} - \sqrt{b})^2 + (2v - 2)(\sqrt{a} - \sqrt{ab})^2 - 4(1 - v)(\sqrt{ab} - \sqrt{a})^2.
\]

So to prove our claim, we show that the following inequality holds
\[
(2 - 4v)(\sqrt{ab} - \sqrt{a})^2 \leq (4v - 4)(\sqrt{ab} - \sqrt{a})^2,
\]
which is equivalent to
\[ g(x, v) = (4v - 2)x^6 + (2 - 4v)x^5 + (2v - 1)x^4 + (2 - 4v)x^3 + (2v - 1) \geq 0, \]
where \( x = t^{\frac{1}{2}} > 0 \) and \( v \in [\frac{3}{4}, 1] \). Since \( \frac{dg(x, v)}{dv} = 2(x - 1)^2(2x^4 + 2x^3 + 3x^2 + 2x + 1) \geq 0, \) we have \( g(x, v) \geq g(x, \frac{3}{4}) = (x - 1)^2(x^4 + x^3 + \frac{3}{2}x^2 + x + \frac{1}{2}) \geq 0 \).
This justify that the inequality (13) also recover Corollary 3 where \( v \in [\frac{3}{4}, 1] \).
Secondly, we compare Corollary 3 with the inequality (16) which holds in the cases \( v \in [0, \frac{1}{4}], v \in [\frac{1}{2}, \frac{3}{4}] \) and \( v \in [\frac{3}{4}, 1] \). The comparison is done by the same way as in the first step, and we omit it.

Sababheh and Choi gave the following refinement of Lemma 5 which it’s complete proof can be found in [22, Theorem 2.2].

**Lemma 6** ([21, Theorem 2.9]) Let \( a, b > 0 \). Then, we have

(i) If \( v \leq 0 \), then
\[ (1 - v)a + vb \leq (a - v^{1/2}b^{v/2}) + v \sum_{k=1}^{n} 2^{k-1} \left( \sqrt{a} - \sqrt{a2^{k-1}b} \right)^2. \]  \hspace{1cm} (19)

(ii) If \( v \geq 1 \), then
\[ (1 - v)a + vb \leq (a - v^{1/2}b^{v/2}) + (1 - v) \sum_{k=1}^{n} 2^{k-1} \left( \sqrt{b} - \sqrt{ab2^{k-1}} \right)^2. \]  \hspace{1cm} (20)

We can extend the ranges of \( v \) in Lemma 6 to those in following theorem, by the similar way to the line of proof of Theorem 1.

**Theorem 3** Let \( a, b > 0, n \in \mathbb{N} \) and \( v \in \mathbb{R} \).

(i) If \( v \not\in [0, \frac{1}{2n}] \), then the inequality (19) holds.
(ii) If \( v \not\in [\frac{1}{2n}, 1] \), then the inequality (20) holds.

As we discussed the case of \( n \to \infty \) in Remark 2, we also give the following remark.

**Remark 6** The inequality (19) is equivalent to the inequality
\[ a + va2^n \left( \left( \frac{b}{a} \right)^{1/2^n} - 1 \right) \leq a^{1-v}b^v \]
by the elementary computations. Using the formula \( \lim_{r \to 0} \frac{e^r - 1}{r} = \log t \), we have the inequality
\[ \log \left( \frac{b}{a} \right)^v \leq \left( \frac{b}{a} \right)^v - 1 \]
for $v \neq 0$ in the limit of $n \to \infty$. The above inequality trivially holds for all $v \in \mathbb{R}$. It can be also obtained by the inequality \((5)\).

Similarly, the inequality \((20)\) is equivalent to the inequality

$$b + (1 - v)b^{2^n} \left( \left( \frac{a}{b} \right)^{1/2^n} - 1 \right) \leq a^{1-v}b^v$$

so that we have the inequality

$$\log \left( \frac{a}{b} \right)^{1-v} \leq \left( \frac{a}{b} \right)^{1-v} - 1$$

for $v \neq 1$ in the limit of $n \to \infty$. The above inequality trivially holds for all $v \in \mathbb{R}$. It can be also obtained by the inequality \((5)\).

As a direct consequence of Theorems 1 and 3, we have the following reverse inequalities with respect to the Heinz means.

**Corollary 4** Let $a, b > 0$, $n \in \mathbb{N}$ such that $n \geq 2$ and $\frac{1}{2} \neq v \in \mathbb{R}$.

(i) If $v \notin \left[ \frac{1}{2}, \frac{2n-1}{2^n} \right]$, then

$$\frac{a+b}{2} \leq H_v(a, b) + (1 - v)(\sqrt{a} - \sqrt{b})^2$$

$$+ \left( v - \frac{1}{2} \right) \sqrt{ab} \sum_{k=2}^{n} 2^{k-2} \left\{ \left( \frac{a^{k^{}}}{\sqrt{b}} - 1 \right)^2 + \left( \frac{b^{k^{}}}{\sqrt{a}} - 1 \right)^2 \right\}$$

(ii) If $v \notin \left[ \frac{2n-1}{2^n}, \frac{1}{2} \right]$, then

$$\frac{a+b}{2} \leq H_v(a, b) + v(\sqrt{a} - \sqrt{b})^2$$

$$+ \left( \frac{1}{2} - v \right) \sqrt{ab} \sum_{k=2}^{n} 2^{k-2} \left\{ \left( \frac{a^{k^{}}}{\sqrt{b}} - 1 \right)^2 + \left( \frac{b^{k^{}}}{\sqrt{a}} - 1 \right)^2 \right\}.$$

**Corollary 5** Let $a, b > 0$, $n \in \mathbb{N}$ and $v \in \mathbb{R}$.

(i) If $v \notin \left[ 0, \frac{1}{2^n} \right]$, then

$$\frac{a+b}{2} \leq H_v(a, b)$$

$$+ v \sum_{k=1}^{n} 2^{k-2} \left\{ \left( \sqrt{a} - \frac{a^{k^{}}}{\sqrt{b}} \right)^2 + \left( \sqrt{b} - \frac{b^{k^{}}}{\sqrt{a}} \right)^2 \right\}.$$

(ii) If $v \notin \left[ \frac{2^n-1}{2^n}, 1 \right]$, then

$$\frac{a+b}{2} \leq H_v(a, b)$$

$$+ (1 - v) \sum_{k=1}^{n} 2^{k-2} \left\{ \left( \sqrt{a} - \frac{a^{k^{}}}{\sqrt{b}} \right)^2 + \left( \sqrt{b} - \frac{b^{k^{}}}{\sqrt{a}} \right)^2 \right\}.$$
3 Generalized Reverse Young and Heinz Inequalities for Operators

In this section by applying Kubo–Ando theory [19] and thanks to Theorems 1 and 3, we have the following operator inequalities.

**Theorem 4** Let $A, B \in B^{++}(H)$, $n \in \mathbb{N}$ such that $n \geq 2$ and $\frac{1}{2} \neq v \in \mathbb{R}$. Then, we have the following inequalities.

(i) If $v \notin [\frac{1}{2}, \frac{2^{n-1}+1}{2^{n-1}}]$, then

$$A\nabla_v B \leq A\check{v} B + 2(1-v)(A\nabla_v B - A\sharp B) + (2v-1)\sum_{k=2}^{n} 2^{k-2} \left( A\sharp B - 2A\check{\frac{2^{k-1}}{2^{k-1}+1}} B + A\check{\frac{2^{k-2}+1}{2^{k-2}+2}} B \right).$$

(ii) If $v \notin \left[ \frac{2^{n-1}-1}{2^{n-1}}, \frac{1}{2} \right]$, then

$$A\nabla_v B \leq A\check{v} B + 2v(A\nabla_v B - A\sharp B) + (1-2v)\sum_{k=2}^{n} 2^{k-2} \left( A\sharp B - 2A\check{\frac{2^{k-1}+1}{2^{k-1}+2}} B + A\check{\frac{2^{k-2}+1}{2^{k-2}+2}} B \right).$$

**Proof (i)** According to the inequality (1), the following inequality holds for $t \geq 0$

$$(1-v) + vt \leq (1-v)(1-\sqrt{t})^2 + (2v-1)\sqrt{2}\sum_{k=2}^{n} 2^{k-2} \left( \sqrt{2^{k-1}-1} \right)^2.$$

By functional calculus, if we replace $t$ with $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ and then multiplying both sides of the inequality by $A\check{x}$, the desired inequality is obtained.

(ii) The line of proof is similar to (i) by applying the inequality (2). □

**Remark 7** We notice that Theorem 4 with $v \in [0,1]$, recover the inequalities obtained in [11, Theorem 3], if we put $n = 2$.

**Theorem 5** Let $A, B \in B^{++}(H)$, $n \in \mathbb{N}$ and $v \in \mathbb{R}$. Then, we have the following inequalities.

(i) If $v \notin [0, \frac{1}{2^{n-1}}]$, then

$$A\nabla_v B \leq A\check{v} B + \sum_{k=1}^{n} 2^{k-1} \left( A - 2A\check{\frac{2^{k-1}}{2^{k-1}+1}} B + A\check{\frac{2^{k-2}+1}{2^{k-2}+2}} B \right).$$

(ii) If $v \notin [\frac{2^{n-1}-1}{2^{n-1}}, 1]$, then

$$A\nabla_v B \leq A\check{v} B + (1-v)\sum_{k=1}^{n} 2^{k-1} \left( B - 2A\check{\frac{2^{k-1}+1}{2^{k-1}+2}} B + A\check{\frac{2^{k-2}+1}{2^{k-2}+2}} B \right).$$
Proof (i) According to Theorem 3 in the case $v \notin [0, \frac{1}{2}]$, the following inequality holds for $t \geq 0$

$$(1 - v) + vt \leq t^v + v \sum_{k=1}^{n} 2^{k-1} \left( 1 - \frac{2^k}{\sqrt{t}} \right)^2$$

$$= t^v + v \sum_{k=1}^{n} 2^{k-1} \left( 1 - 2t^{\frac{k}{2}} + t^{\frac{k}{2}+\frac{1}{2}} \right).$$

By functional calculus, if we replace $t$ with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and then multiplying both sides of the inequality by $A^{\frac{1}{2}}$, the desired inequality is deduced.

(ii) By applying Theorem 3 for the case of $v \notin \left[\frac{2^n-1}{2^n}, 1\right]$, we get the desired inequality in the same way as in (i). $\square$

As a direct consequence of Theorems 4 and 5, we get the generalized reverse Heinz operator inequalities as follows. That is Corollaries 6 and 7 are operator versions of Corollaries 4 and 5 respectively.

Corollary 6 Let $A, B \in B^{++}(H)$, $n \in \mathbb{N}$ such that $n \geq 2$ and $\frac{1}{2} \neq v \in \mathbb{R}$. Then we have the following inequalities.

(i) If $v \notin \left[\frac{1}{2}, \frac{2^n-1}{2^n}\right]$, then

$$A \nabla B \leq \hat{H}_v(A, B) + 2(1 - v)(A \nabla B - A\sharp B)$$

$$+ (2v - 1) \sum_{k=2}^{n} 2^{k-2} \left( A\sharp B - 2H_{\frac{2k-1}{2^n}}(A, B) + H_{\frac{2k-2}{2^n}}(A, B) \right);$$

(ii) If $v \notin \left[\frac{2^n-1}{2^n}, \frac{1}{2}\right]$, then

$$A \nabla B \leq \hat{H}_v(A, B) + 2v(2v)(A \nabla B - A\sharp B)$$

$$+ (1 - 2v) \sum_{k=2}^{n} 2^{k-2} \left( A\sharp B - 2H_{\frac{2k-1}{2^n}}(A, B) + H_{\frac{2k-2}{2^n}}(A, B) \right).$$

Remark 8 Putting $n = 2$ in Corollary 6 with $v \in [0, 1]$ gives the inequalities obtained in [11, Corollary 6].

Corollary 7 Let $A, B \in B^{++}(H)$, $n \in \mathbb{N}$ and $v \in \mathbb{R}$. Then, we have the following inequalities.

(i) If $v \notin [0, \frac{1}{2}]$, then

$$A \nabla B \leq \hat{H}_v(A, B)$$

$$+ v \sum_{k=1}^{n} 2^{k-1} \left( A \nabla B - 2H_{\frac{k}{2^n}}(A, B) + H_{\frac{k-1}{2^n}}(A, B) \right).$$
(ii) If \( v \notin \left[ \frac{2^n}{2^m}, 1 \right] \), then
\[
A \nabla B \leq \hat{H}_v(A, B) + (1 - v) \sum_{k=1}^{n} 2^{k-1} \left( A \nabla B - 2H_{\frac{2^k-1}{2^k}}(A, B) + H_{\frac{2^k-1}{2^k}}(A, B) \right).
\]

**Remark 9** If we put \( n = 1 \), Corollary 7 gives the inequality shown in [2, Theorem 3.1] where \( v \notin \left[ \frac{1}{2}, 1 \right] \).

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