Quasi-local energy for cosmological models

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First we briefly review our covariant Hamiltonian approach to quasi-local energy, noting that the Hamiltonian-boundary-term quasi-local energy expressions depend on the chosen boundary conditions and reference configuration. Then we present the quasi-local energy values resulting from the formalism applied to homogeneous Bianchi cosmologies. Finally we consider the quasi-local energies of the FRW cosmologies. Our results do not agree with certain widely accepted quasi-local criteria.

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1. Introduction

The localization of energy-momentum for any gravitating system (and thus for all physical systems) is still an outstanding fundamental problem. In view of conservation, and the fact that sources exchange energy-momentum locally with the gravitational field, some kind of local description for gravitational energy-momentum was expected. However all attempts at constructing such an expression led only to reference frame dependent quantities, generally referred to as pseudotensors.\textsuperscript{1,2} It became apparent that the gravitational field itself, unlike all matter and other interaction fields, has no proper energy-momentum density. This fact can be understood as a consequence of Einstein’s equivalence principle.\textsuperscript{3} The energy-momentum of gravity—and thus the energy-momentum for all physical systems—is inherently non-local. The modern idea is quasi-local: energy-momentum is associated with a closed surface bounding a region.
Many quasi-local expressions have been proposed, but presently there is no consensus as to which is the most suitable, or even as to which properties a good expression should have. Various lists of desiderata for a “physical” quasi-local energy have been presented; according to a well-known one the quasi-local energy should be

- zero for flat space,
- for spherical symmetric ≃ standard value,
- the ADM mass at spatial infinity,
- the Bondi mass at null infinity,
- for the apparent horizon ≃ standard value,
- positive.

Our Hamiltonian based quasi-local results do not satisfy the first and last of these criteria. There is a stronger form of the first requirement, namely that the energy vanish iff the quasilocal region is flat space. Our analysis of the quasi-local energy of cosmological regions leads us to propose a certain modification of this stronger form.

2. The covariant Hamiltonian approach

Here we briefly summarize the relevant parts of our covariant Hamiltonian approach to quasi-local energy.\[^{[26,78,910]}\]

2.1. First order Lagrangian

The first order Lagrangian for an f-form field \( \varphi \) and its conjugate momentum \( p \) has the form

\[
\mathcal{L} = d\varphi \wedge p - \Lambda(\varphi, p). \tag{1}
\]

The variation of this 4-form,

\[
\delta \mathcal{L} = d(\delta \varphi \wedge p) + \delta \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p, \tag{2}
\]

leads to the first order equations of motion

\[
\frac{\delta \mathcal{L}}{\delta p} := d\varphi - \partial_p \Lambda = 0, \quad \frac{\delta \mathcal{L}}{\delta \varphi} := -\zeta dp - \partial_\varphi \Lambda = 0, \tag{3}
\]

where \( \zeta := (-1)^f \).

As a simple example of this formalism consider electromagnetism. The first order Lagrangian 4-form for the (source free) U(1) gauge field one-form \( A \) is

\[
\mathcal{L}_{\text{EM}} = dA \wedge H - \frac{1}{2} *H \wedge H. \tag{4}
\]

Variation leads to the pair of first order equations

\[
dH = 0, \quad dA - *H = 0. \tag{5}
\]
From $F := dA = ^*H$ one finds $H = -^*F$; hence the first equation becomes $d^*F = 0$. Thus we have the vacuum Maxwell equations.

### 2.2. Translation invariance and Noether current

Infinitesimal diffeomorphism invariance (in terms of the Lie derivative) requires that

\[ d_{i_N}L \equiv \mathcal{L}_{N} \equiv d(\mathcal{L}_{N} \varphi \wedge p) + \mathcal{L}_{N} \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \mathcal{L}_{N} p. \]  

This simply means that $\mathcal{L}$ is a 4-form which depends on position only through the fields $\varphi, p$. For this to be the case the set of fields in $\mathcal{L}$ necessarily includes dynamic geometric variables, which means gravity.

From this identity it follows that the “translational current” density (3-form)

\[ \mathcal{H}(N) = \mathcal{L}_{N} \varphi \wedge p - i_{N} \mathcal{L} \]

(7)

satisfies the “conservation law”

\[ -d\mathcal{H}(N) \equiv \mathcal{L}_{N} \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \mathcal{L}_{N} p. \]

(8)

Consequently, “on shell” (i.e. when the field equations are satisfied), the integral of the current over a spatial region will give a conserved quantity for each vector field $N$. Note that, just like other Noether conserved currents, $\mathcal{H}(N)$ is not unique: it can be modified by adding the differential of any 2-form.

With geometric gravity included, we have also local diffeomorphism invariance, which gives rise (in accordance with Noether’s second theorem) to a differential identity. Explicit calculation shows that $\mathcal{H}(N) = \mathcal{L}_{N} \varphi \wedge p - i_{N} \mathcal{L}$ always has the form

\[ \mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}(N). \]

(9)

Thus we find that $d(N^\mu \mathcal{H}_\mu + d\mathcal{B}(N)) \equiv dN^\mu \wedge \mathcal{H}_\mu + N^\mu d\mathcal{H}_\mu$ is proportional to the field equations, therefore $\mathcal{H}_\mu$ vanishes “on shell”. Hence for gravitating systems the Noether translational “charge”—energy-momentum—is quasi-local: it is given by the integral of the boundary term, $\mathcal{B}(N)$. But this boundary term as noted can be completely modified to any value. The Hamiltonian approach includes an additional principle which naturally tames this ambiguity.

### 2.3. Hamiltonian approach

Energy can be identified as the value of the Hamiltonian associated with a time-like displacement vector field $N$. The Hamiltonian $H(N)$ is given by an integral of a suitable Hamiltonian 3-form (density) $\mathcal{H}(N)$ over a 3-dimensional (space-like) region $\Sigma$. Generalizing $L = \dot{q}p - H$, from the first order Lagrangian one constructs the Hamiltonian 3-form by projecting along a “time-like” displacement vector field:

\[ i_{N} \mathcal{L} = \mathcal{L}_{N} \varphi \wedge p - \mathcal{H}(N). \]

(10)
The Hamiltonian density thus turns out to be just the Noether translational current (7) identified above. As already noted it satisfies the relation (9) with \( \mathcal{H}_\mu \) vanishing on shell. Consequently the quasi-local energy—regarded as the value of the Hamiltonian—is then determined only by the boundary integral:

\[
E(N) = \int_\Sigma \mathcal{H}(N) = \int_\Sigma [N^\mu \mathcal{H}_\mu + d\mathcal{B}(N)] = \oint_{\partial \Sigma} \mathcal{B}(N).
\]

The two parts of the Hamiltonian have distinct roles. The 3-form, \( \mathcal{H}_\mu \), although it has vanishing numerical value, generates the equations of motion. For our concerns here the Hamiltonian boundary term \( \mathcal{B}(N) \) is the key quantity. It plays a dual role: determining both the the quasi-local values and the boundary conditions.

### 2.4. Quasi-local quantities

The Hamiltonian boundary term \( \mathcal{B}(N) \) determines the various quasi-local values corresponding to the Poincaré transformation of space-time:

- Energy \( \leftrightarrow \) \( N \) a time-like displacement,
- Linear momentum \( \leftrightarrow \) a spatial translation,
- Angular momentum \( \leftrightarrow \) a rotation,
- center-of-mass moment \( \leftrightarrow \) a boost.

However we noted that \( \mathcal{B}(N) \) can be adjusted; then it would give different conserved values. What do they all these different values mean physically?

### 2.5. Boundary Conditions

The variational principle contains an additional (largely overlooked) feature which distinguishes all of these choices: the boundary variation principle, i.e. the boundary term in the variation tells us what to hold fixed on the boundary—it determines the boundary conditions. Different Hamiltonian boundary term choices are each associated with distinct boundary conditions. (In this way this formalism gives a specific physical significance to each of the traditional energy-momentum complexes. 1, 8, 9)

This feature is similar to that of some familiar physical systems. For example in thermodynamics the suitable measure of energy: the internal energy, enthalpy, Helmholtz, or Gibbs free energy depends on the system’s boundary conditions. Another good example concerns moving a dielectric within a parallel plate capacitor. The work needed, and thus the appropriate energy density expression (the symmetric or the canonical tensor) depends on the boundary condition: fixed charge or fixed potential. Thus one can see that there always are various distinct physical “energies” which correspond to how a system interacts with the outside through its boundary.
2.6. Reference Configuration

In general (in particular for gravity) it is necessary (technically, in order to guarantee functional differentiability of the Hamiltonian on the phase space with the desired boundary conditions) to adjust the boundary term, $B(N) = i_N \phi \wedge p$, which is naturally inherited from the Lagrangian. Moreover a reference configuration, $\bar{\phi}$ and $\bar{p}$, (which determines the ground state) is essential (especially for gravity where the ground state is not vanishing field but rather the Minkowski metric) in particular to allow for the desired phase space asymptotics.

2.7. Quasi-local Expressions

With $\Delta \phi := \phi - \bar{\phi}$, $\Delta p := p - \bar{p}$, where the bar indicates the reference value, we found two boundary choices (essentially Dirichlet and Neumann) which have the indicated covariant boundary terms in $\delta H$.

$$B_\phi = i_N \phi \wedge \Delta p - \varsigma \Delta \phi \wedge i_N \bar{p}, \quad \rightarrow \quad i_N (\delta \phi \wedge \Delta p), \quad (12)$$

$$B_p = i_N \bar{\phi} \wedge \Delta p - \varsigma \Delta \phi \wedge i_N p, \quad \rightarrow \quad -i_N (\Delta \phi \wedge \delta p). \quad (13)$$

3. Application to GR

For Einstein’s (vacuum) gravity theory, General Relativity (GR) a first order Lagrangian is

$$L_{GR} = \frac{1}{16\pi} R^\alpha_\beta \wedge \eta^\alpha_\beta, \quad (14)$$

where $\Gamma^\alpha_\beta$ is the connection one-form, $R^\alpha_\beta := d\Gamma^\alpha_\beta + \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta$ is the curvature 2-form and $\eta^\alpha_\beta := * (\delta^\alpha \wedge \delta^\beta)$.

Our general formalism with $\phi \rightarrow \Gamma^\alpha_\beta$ and $p \rightarrow \eta^\alpha_\beta$ gives the quasi-local expressions for GR. We have two expressions for different types of boundary conditions. One of the choices stands out

$$B(N) := \frac{1}{16\pi} \left[ \Delta \Gamma^\alpha_\beta \wedge i_N \eta^\alpha_\beta + (\hat{D}N)_\beta \alpha \Delta \eta^\alpha_\beta \right]. \quad (15)$$

This is a Dirichlet type condition for a covariant object, the orthonormal frame field. Asymptotically this expression gives not only the ADM (spatial infinity) and Bondi energy (null infinity) but also the Bondi energy flux. Moreover this expression is distinguished by satisfying a positive energy property.

In the cases considered here, the contribution of the second term in (15) vanishes.

4. Homogenous Cosmologies

Homogeneous cosmologies (non-isotropic in general) are described by the Bianchi models. The orthonormal coframe has the form

$$\vartheta^0 = dt, \quad \vartheta^a = h^a_k(t) \sigma^k, \quad (16)$$
where the spatially homogeneous frames satisfy
\[ d\sigma^k = \frac{1}{2} C_{ij}^{k} \sigma^i \wedge \sigma^j, \] (17)
where the \( C_{ij}^{k} \) are certain constants. The associated space-time metric is thus
\[ ds^2 = -dt^2 + g_{ij}(t) \sigma^i(x) \sigma^j(x), \] (18)
where \( g_{ij} := \delta_{ab} h^a_i h^b_j \) (which need not be diagonal).

There are 9 Bianchi types distinguished by the particular form of the structure constants \( C_{ij}^{k} \), especially by the value of \( A^k := C_{ki}^{i} \). They fall into two special classes:
- **class A** (\( A^k \equiv 0 \)): Types I, II, VI\(_0\), VII\(_0\), VIII, IX;
- **class B** (\( A^k \neq 0 \)): Types III, IV, V, VI\(_h\), VII\(_h\).

The respective scalar curvatures are: *vanishing* for Type I, *positive* for Type IX, *negative* for all the other types. It should be mentioned that certain special cases can be isotropic, specifically isotropic Bianchi I, V, IX are equivalent to the usual FRW \( k = 0, -1, +1 \).

For the natural choice of \( N = \partial_t \), the Dirichlet type boundary condition, the Bianchi homogenous frame as boundary value, and with the reference being the static homogenous cartesian frame, the energy within a spatial volume \( V \) according to our favored quasi-local expression (15) is
\[ E(V) = \frac{1}{8\pi} A_j A_k g^{jk}(t) V(t) \geq 0. \] (19)

The result is true for all regions and for all types of sources including dark matter, dark energy a/o a cosmological constant. More specifically it vanishes for all class A models and is positive for all class B models. Note: this is entirely consistent with the important requirement that \( E = 0 \) for closed universes, since all homogeneous class A models can be compactified and class B models cannot.

5. FRW cosmology

The Friedman-Robertson-Walker (FRW) (homogeneous and isotropic) metrics have the form
\[ ds^2 = -dt^2 + a^2(t) dl^2. \] (20)

The spatial metric \( dl^2 \) has constant curvature. The FRW spatial metric has several equivalent manifestly isotropic-about-a-chosen-point forms:
\[ dl^2 = dp^2 + \Sigma^2 d\Omega^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 = \frac{1}{[1 + (k/4)R^2]^2} (dR^2 + R^2 d\Omega^2), \] (21)

where \( \Sigma = (\sinh \rho, \rho, \sin \rho) \) for \( k = (-1, 0, +1) \), respectively.

A natural choice in this case is \( N = \partial_t \), Dirichlet type boundary conditions, FRW frame boundary values, with the reference being the static flat cartesian frame. The
energies within a fixed radius for the three FRW cases can be represented in several equivalent forms (their identity follows from $\Sigma = r = R/(1 + kR^2/4)$):

$$E = a\Sigma(1 - \Sigma') = ar \left[1 - \sqrt{1 - k_r^2}\right] = \frac{akR^3}{[1 + (k/4)R^2]^2}.$$  \hspace{1cm} (22)

More specifically,

$$E_{k=-1} = a \sinh \rho(1 - \cosh \rho) = ar \left[1 - \sqrt{1 + r^2}\right] = \frac{-aR^3}{2(1 - R^2/4)^2} \leq 0,$$

$$E_{k=0} = 0,$$

$$E_{k=+1} = a \sin \rho(1 - \cos \rho) = ar \left[1 - \sqrt{1 - r^2}\right] = \frac{aR^3}{2(1 + R^2/4)^2} \geq 0. \hspace{1cm} (23)$$

6. Discussion

According to our favored quasi-local energy expression, homogeneous choices give vanishing energy for all regions of Bianchi class A models and positive energy for class B. Isotropic choices give energies proportional to the spatial curvature parameter $k$: vanishing for flat, negative for the open model, and positive for the closed model (but nevertheless vanishing, as required, when the considered volume is extended to include the whole universe).

Some of the Bianchi models can be isotropic, specifically

- isotropic Bianchi I (class A) $\equiv$ FRW$_{k=0}$;
- isotropic Bianchi IX (class A) $\equiv$ FRW$_{k=+1}$;
- isotropic Bianchi V & VII (class B) $\equiv$ FRW$_{k=-1}$.

Note that our quasi-local expression thus can give different energy values to exactly the same geometry. This is not at all mysterious; it is clearly a consequence of different reference and boundary value choices. Homogenous boundary values are not the same as isotropic boundary values. To understand the physical and geometric meaning of the differences between the homogeneous and isotropic choices in detail, we need to do more calculations using the rather complicated relations between the FRW and Bianchi coordinates. Meanwhile from our analysis it seems that the homogeneous choice is more suitable physically than the isotropic-about-a-chosen-point choice in general, since it gives a non-negative energy.

It is also noteworthy that for the case of the open FRW ($k = -1$) with vanishing matter, the solution to the Einstein equation gives $a(t) = t$. It can be directly verified that the geometry is then really just Minkowski space in non-standard coordinates, yet our expression gives a non-vanishing energy, which, moreover is negative with the FRW choices.

Concerning two of the quasi-local desiderata, for the expression and boundary/reference choices considered we found that

- positivity need not hold;
- “zero energy iff flat Minkowski space” need not hold in either direction.
It seems that these quasi-local criteria should be reconsidered.

As we prefer positivity, we are inclined to see our negative result as disfavoring the FRW isotropic-about-a-chosen-point boundary/reference choice.

Regarding $E = 0$, clearly any homogeneous measure of quasi-local energy in Bianchi I models must necessarily vanish for all regions—since these models can be compactified (with 3 torus topology identifications) on any scale, and the energy of a closed universe must vanish. In light of this, we propose that the “unique quasi-local $E = 0$ Minkowski ground state” requirement be replaced by something like “$E(V) = 0$ iff a neighborhood of $V$ can be compactified.”

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