Comparison study of DFA and DMA methods in analysis of autocorrelations in time series

Dariusz GRECH
Institute of Theoretical Physics, Wroclaw University, pl. M. Borna 9, 50-405 Wroclaw, Poland∗

and

Zygmunt MAZUR
Institute of Experimental Physics, Wroclaw University, pl. M. Borna 9, 50-405 Wroclaw, Poland†

Abstract
Statistics of the Hurst scaling exponents calculated with the use of two methods: recently introduced Detrended Moving Average Analysis (DMA) and Detrended Fluctuation Analysis (DFA) are compared. Analysis is done for artificial stochastic Brownian time series of various length and reveals interesting statistical relationships between two methods. Good agreement between DFA and DMA techniques is found for long time series $L \sim 10^5$, however for shorter series we observe that two methods give different results with no systematic relation between them. It is shown that, on the average, DMA method overestimates the Hurst exponent comparing it with DFA technique.

1 Introduction
The main problem discussed in the context of stochastic time series in various physical, biological, financial and economical processes is the presence of autocorrelations in data. One of the techniques to check whether such autocorrelations are present in time series is based on the investigation of the fractal structure in time series and is related to the scaling exponent $H$, sometimes denoted also as $\alpha$ [1]–[3] and called Hurst exponent. It plays a significant role as the main concept upon which fluctuations of a time series around its local trend (drift) are formed and it may be considered as the one of the crucial points responsible for ‘genetic code’ of time series of various origin. For the purpose of mentioned above fractal analysis one can introduce the scaling exponent $\alpha$ as follows.

Let $x(t)$ ($t = 1, ..., L$) is the time series defined for discrete time points $t$. By rescaling time axis $\gamma$ times (e.g. enlarging it $\times 10^n$), one reveals the tiny structure of time series not visible for smaller resolution ($\gamma \sim 1$). The fractal structure of the series comes from the relation:

$$x'(t') \equiv \Gamma x(\gamma^{-1}t) \sim x(t)$$

(1)

where $\sim$ means similarity correspondence.

The above formula indicates that the magnitude of rescaled time series $x(\gamma^{-1}t)$ should be simultaneously increased $\Gamma$ times in order to satisfy full (local) equivalence of $x(t)$ and $x'(t')$ series.

It turns out that the scaling factor $\Gamma$ can be expressed in terms of time rescaling factor $\gamma$ with the use of Hurst-Hausdorff $\alpha$ exponent ($\alpha > 0$):

$$\Gamma = \gamma^\alpha$$

(2)

The commonly accepted methods to measure $\alpha$ exponent are Rescaled Range Analysis (R/S), spectral density analysis [4], and Detrended Fluctuation Analysis (DFA) [5]. Recently, new method called Detrended Moving Average (DMA) has also been proposed [6, 7]. In this article we will focus on the latter two methods due to large uncertainties in spectral density analysis and problems with R/S predictions in nonstationary series. Searches for better understanding how results of these two methods relate to each other are in progress [7]–[9].
A DFA method was first developed for biological purposes [5] and then applied also to finances [10]–[12]. It is a detrendisation technique basically measuring fluctuations of a given time series around its local trend as a function of the trend length. Let us recall the main steps of this method:

1. A given signal $x(t)$ ($t = 1, ..., L$) of time series is divided into $L/\tau$ not overlapping boxes of length $\tau$ each.
2. A polynomial fit $x_{\tau,k}(t)$ is constructed in each box representing the local trend in that box, where $k$ is the order of polynomial fit.
3. A detrended signal $X_{\tau,k}(t)$ is found:
   \[ X_{\tau,k}(t) = x(t) - x_{\tau,k}(t) \]  
   and then its fluctuation (standard deviation) $F_{DFA}(\tau, k)$ is calculated
   \[ F_{DFA}(\tau, k) = \left( \frac{1}{L} \sum_{t=1}^{L} X_{\tau,k}^{2}(t) \right)^{1/2} \]  
4. From the basic differential stochastic equation of the time series $x(t)$ with a local drift $\mu(t)$ and a local dispersion $\sigma(t)$
   \[ dx(t) = \mu(t)dt + \sigma(t)dX(t) \]  
   one expects the power law behavior:
   \[ F_{DFA}(\tau, k) \sim \tau^{\alpha(k)} \]  
   where $\alpha(k)$ is the searched Hurst exponent.

The last equation enables to calculate $\alpha$ exponent directly from log-log linear fit:

\[ \log F_{DFA}(\tau, k) \sim \alpha(k) \log \tau \]  

It can be proved that $\alpha(k)$ depends very weakly on $k$ [12] [13] so in most application one takes linear function ($k = 1$) as a good candidate for $x_{\tau,k}$. This approach will also be used in our paper.

It turns out that the bigger $\alpha$ the more ‘quiet’ time series is, i.e. a signal fluctuates in a more correlated way. In fact, for $0 < \alpha < 1/2$ we have negative autocorrelations (antipersistence) in time series. On the other hand, if $1/2 < \alpha \leq 1$, there are positive autocorrelations (persistence) in signal. The case $\alpha = 1/2$ corresponds to completely uncorrelated signal, so called integer Brownian walk. An existing link between $\alpha$ exponent and the probability that a given trend will last in the immediate future if it did so in the immediate past gives an additional hint about trend changes forecast possibility [14].

A Detrended Moving Average (DMA) technique looks very similar to DFA. The main difference one meets here is that instead of linear or polynomial detrendisation procedure in equally sized boxes, one uses moving average of a given length $\lambda$. The basic steps of DMA analysis are then:

1. A simple moving average of length $\lambda$ ($\lambda = 1, ..., L$) is constructed for $x(t)$ series ($t \geq \lambda$):
   \[ \langle x(t) \rangle_{\lambda} = \frac{1}{\lambda} \sum_{k=0}^{\lambda-1} x(t-k) \]  
2. A detrended signal is found similarly to Eq. [8]:
   \[ X_{\lambda}(t) = x(t) - \langle x(t) \rangle_{\lambda} \]  
   and then its fluctuation within a window of size $\lambda$ reads now:
   \[ F_{DMA}(\lambda) = \left( \frac{1}{L-\lambda+1} \sum_{t=\lambda}^{L} X_{\lambda}^{2}(t) \right)^{1/2} \]  
3. Similarly to DFA a power law should be observed
   \[ \log F_{DMA}(\lambda) \sim \alpha \log \lambda \]  
   where $\alpha$ is the searched scaling Hurst exponent.

The DMA technique is less complicated and seems to be faster in practical application than DFA algorithm. However, so far no final clear conclusion has been reached regarding mutual relationship between DFA and DMA results for the same series. This article contributes to the above area of interest.
2 DMA–DFA Comparison Study

Preliminary results obtained for some real financial series [7] suggest that $\alpha_{DMA}$ values are lower than corresponding $\alpha_{DFA}$ results. It seems to be confirmed for the set of artificial time series of length $L \sim 2^{18}$ constructed with the use of Random Midpoint Displacement (RMD) algorithm where one finds $\alpha_{DFA} \sim \alpha_{DMA} + 0.05$ [6]. This supports the existence of systematic displacement between DFA and DMA results, at least for longer series.

In many practical applications however, the length of time series we deal with is shorter (e.g. finance, biology, genetics, medicine), especially if one looks at the local $\alpha$ exponent value rather than the global one [10].

To attack the problem of mutual dependence between DMA and DFA results for series of various length, let us first look at the set of artificial arithmetic integer Brownian time series of length $L = 3 \times 10^4$ with discrete time interval $\Delta t = 1$, i.e.:

$$x(L\Delta t) = x_0 + \sum_{k=1}^{L} \Delta x_k$$

(12)

where $\Delta x_k$ ($k = 1, ..., L$) are centered and normalized displacements generated by random number generator.

Two cases with opposite relation $\alpha_{DMA} \text{ vs } \alpha_{DFA}$ are shown in Fig. 1. In the first case $\alpha_{DFA} > \alpha_{DMA}$ and $\alpha_{DFA} - \alpha_{DMA} = 0.02$, in the other one $\alpha_{DFA} < \alpha_{DMA}$ and $\alpha_{DMA} - \alpha_{DFA} = 0.04$. Thus no systematic relationship is produced.

![Figure 1: Examples of DFA and DMA $\alpha$ exponent fit for artificial Brownian time series of length $L = 30000$, where (a) $\alpha_{DFA} > \alpha_{DMA}$ and (b) $\alpha_{DFA} < \alpha_{DMA}$](image)

This induces to treat the problem statistically, i.e. one should find statistical distributions of Hurst exponents measured within two methods for artificial series of various length. It seems to be interesting to compare two
statistics and to work out correlations between scaling exponents measured within DMA and DFA techniques for the same sample of time series.

For this purpose we took samples of arithmetic Brownian time series of length \( L \) in the range \( 10^2 - 10^5 \). Each sample contained \( N \sim 65000 \) series of fixed length. We tried to cover uniformly the whole range of \( L \) in log-scale keeping \( L \sim \log^n \) with the approximate log step \( q \sim 7/4 \) to create variety of lengths.

For any sample of fixed length series the averaged scaling range \( \langle \tau \rangle \) or \( \langle \lambda \rangle \) has been calculated for defined number of candidates (\( \sim 30 \)) and the corresponding standard deviation \( \sigma_{\tau} \) (\( \sigma_{\lambda} \)). The scaling range was taken as the range of \( \tau \) or \( \lambda \) variables strictly obeying scaling laws of Eqs. \( 7, 11 \) and assumed to terminate respectively at \( \langle \tau \rangle - \tau \) for DFA and \( \langle \lambda \rangle - \lambda \) for DMA. Only series with regression statistical correlation coefficient \( R^2 > 0.98 \) were taken into account for \( \alpha \) exponent extraction. For any sample of time series a statistical distribution of \( \alpha_{DFA} \) and \( \alpha_{DMA} \) frequencies has been built.

The full range of obtained distribution results is shown in Fig.2-9. The first observation one makes is that for any length \( L \) both distributions fit very well normal distributions, but with different parameters for the gaussian curve. We made all plots also for centered and normalized \( \alpha \) frequencies in semi-log scale (Fig.2(b,c)–9(b,c)). Only small deviations from the normal distribution are observed in tails - basically due to smaller statistics there. A good correspondence with gaussian curve is confirmed also in Kolmogorov and Anderson-Darling tests, whose results are displayed in Table 1 and shown for chosen lengths \( L \) in Fig.10.

| \( L \)   | 600   | 1000  | 1800  | 3000  | 6000  | 10 000 | 20 000 | 30 000 |
|----------|-------|-------|-------|-------|-------|--------|--------|--------|
| \( K_{DFA} \) | 5.0 \times 10^{-3} | 2.0 \times 10^{-3} | 2.0 \times 10^{-3} | 2.0 \times 10^{-3} | 2.4 \times 10^{-3} | 1.4 \times 10^{-3} | 2.7 \times 10^{-3} | 2.3 \times 10^{-3} |
| \( K_{DMA} \) | 6.4 \times 10^{-3} | 6.8 \times 10^{-3} | 5.0 \times 10^{-3} | 6.6 \times 10^{-3} | 5.2 \times 10^{-3} | 3.8 \times 10^{-3} | 3.3 \times 10^{-3} | 3.6 \times 10^{-3} |
| \( A_{DFA} \) | 2.6 \times 10^{-2} | 1.6 \times 10^{-2} | 1.0 \times 10^{-2} | 1.0 \times 10^{-2} | 1.5 \times 10^{-2} | 8.0 \times 10^{-3} | 9.7 \times 10^{-3} | 13.5 \times 10^{-3} |
| \( A_{DMA} \) | 3.6 \times 10^{-2} | 2.1 \times 10^{-2} | 1.8 \times 10^{-2} | 2.1 \times 10^{-2} | 2.7 \times 10^{-2} | 2.1 \times 10^{-2} | 1.8 \times 10^{-2} | 2.5 \times 10^{-2} |

One may notice that the standard deviation \( \sigma_{DFA} \) of \( \alpha_{DFA} \) scaling parameters is always smaller than the corresponding standard deviation \( \sigma_{DMA} \) of \( \alpha_{DMA} \) exponents, and both standard deviations decrease when \( L \) grows. This can be explained in terms of different sensitivity of DFA and DMA techniques to the presence of random autocorrelations in time series. Such autocorrelations are naturally randomly distributed in any sample of generated time series and hence a distribution of \( \alpha \) exponent is normal. The probability of random autocorrelations is bigger for shorter time series, where all statistical fluctuations manifest in a more vivid way. When \( L \) increases, their influence on the presumed global autocorrelation in series can be neglected. Therefore, both standard deviations \( \sigma_{DFA} \) and \( \sigma_{DMA} \) drop with increasing \( L \). However, we always observe \( \sigma_{DFA} < \sigma_{DMA} \), what indicates that DMA technique is more sensitive to such "autocorrelation noise" than DFA one.

One may look at this problem also from another side - like in Fig. 11. Here we have drawn several plots of DFA and DMA analysis, i.e. \( \ln F \) vs \( \ln \tau \) or \( \ln \lambda \) for several corresponding artificial Brownian series of length \( L = 1000 \). It is seen that deviations from the strict power law behavior, if occur, are more drastic for DMA than for DFA case and the dispersion of produced slopes is also larger for DMA than for DFA, despite the fact that DMA plots are more smooth in comparison with DFA ones.

The next observation concerns the mean values. One gets \( \langle \alpha_{DFA} \rangle_N < \langle \alpha_{DMA} \rangle_N \) for all \( L \), where \( \langle \cdot \rangle_N \) is taken over a sample of \( N \) time series. A clear shift of the central DMA values to the right with respect to DFA ones (see Figs. 2–9(a)) does not suggest however the presence of systematic relation between \( \alpha_{DMA} \) and \( \alpha_{DFA} \). Indeed evaluating the correlation coefficient (values are shown in the description of Fig. 2–9(a)):

\[
\text{corr}(\alpha_{DFA}, \alpha_{DMA}) = \frac{\langle \alpha_{DFA} \alpha_{DMA} \rangle_N - \langle \alpha_{DFA} \rangle_N \langle \alpha_{DMA} \rangle_N}{\sigma_{DFA} \sigma_{DMA}} \tag{13}
\]

one finds it increasing with \( L \), but it never indicates the full correlation. Its value is maximal for large \( L \), where \( \text{corr}(\alpha_{DFA}, \alpha_{DMA}) \sim 0.8 \) for \( L \sim 10^4 - 10^5 \).

This situation is graphically illustrated in Fig. 12, where a correlation plot \( \alpha_{DFA} \) vs \( \alpha_{DMA} \) is shown for Hurst exponent values obtained for \( L = 3000 \), \( L = 10000 \) and \( L = 30000 \) series. From the asymmetry of plots against diagonal one notices that DMA gives higher values than DFA method in most series. This result is independent on the length of time series. In fact the percentage excess of cases \( n_+ \), where \( \alpha_{DMA} > \alpha_{DFA} \) over the cases where \( \alpha_{DMA} < \alpha_{DFA} \) (\( n_- \)), i.e.:

\[
\delta_\pm = \frac{n_+ - n_-}{n_+ + n_-} \tag{14}
\]
Figure 2: (a) Distribution of scaling α exponent obtained with the use of DFA (circles) and DMA (squares) techniques for the sample of 65000 series of length $L = 600$. The normal distribution fit with corresponding parameters is also shown as a solid line. (b)(c) The same plots for DFA(b) and DMA(c) in semi-log scale for normalized and centered α exponents.
Figure 3: (a) Distribution of $\alpha$ exponents for series with $L = 1000$. (b)(c) Corresponding plots in semi-log scale.
Figure 4: (a) Distribution of $\alpha$ exponents for series with $L = 1800$. (b)(c) Corresponding plots in semi-log scale.
Figure 5: (a) Distribution of $\alpha$ exponents for series with $L = 3000$. (b)(c) Corresponding plots in semi-log scale.
Figure 6: (a) Distribution of $\alpha$ exponents for series with $L = 6000$. (b)(c) Corresponding plots in semi-log scale.
Figure 7: (a) Distribution of $\alpha$ exponents for series with $L = 10000$. (b)(c) Corresponding plots in semi-log scale.
Figure 8: (a) Distribution of $\alpha$ exponents for series with $L = 20000$. (b)(c) Corresponding plots in semi-log scale.
Figure 9: (a) Distribution of $\alpha$ exponents for series with $L = 30000$. Additional lines represent $L = 1000$ normal fit drawn for comparison in the same scale. (b)(c) Corresponding plots in semi-log scale.
Figure 10: Kolmogorov-Smirnov (a)(c)(e) and Anderson-Darling (b)(d)(f) tests of correspondence between obtained distributions and the Gaussian one drawn respectively for time series of length $L = 1000$, $L = 3000$, $L = 20000$
Figure 11: Examples of DMA and corresponding DFA plots $\ln F$ vs $\ln \tau (\ln \lambda)$ for several randomly chosen Brownian integer time series of length $L = 1000$.

Figure 12: Correlation plot $\alpha_{DFA}$ vs $\alpha_{DMA}$ for the sample of 65000 Brownian time series of length a) $L = 3000$, b) $L = 10000$, c) $L = 30000$. 
changes from $\sim 20\% - 25\%$ for series with $L < 10000$ up to $\sim 50\%$ for longer series.

It is obvious therefore that the mean of difference $\delta_{DFA-DMA}$, where

$$\delta_{DFA-DMA} = \langle \alpha_{DFA} - \alpha_{DMA} \rangle$$

is not a good measure of 'distance' between two investigated methods. It is more convenient to define this distance in a standard way, i.e.:

$$\Delta_{DFA-DMA} = \left( \langle (\alpha_{DFA} - \alpha_{DMA})^2 \rangle \right)^{1/2}$$

The sufficient number of time series samples of various length has been worked out to find a relationship $\Delta_{DFA-DMA}(L)$. The polynomial best fit for the collected data is drawn in Fig. 13 with error bars coming from the uncertainties in slope determination. This plot indicates that the average displacement between $\alpha_{DFA}$ and $\alpha_{DMA}$ exponents for a given time series ranges from 15% for series with $L \leq 10^3$, down to 2% for long series ($L \sim 10^5$). The latter value is much smaller than one reported in [6]. The fastest drop in DFA-DMA distance is observed for medium length series, i.e. when $L \sim 10^3 - 10^4$. For such series $\Delta_{DFA-DMA}$ makes on the average $\sim 10\%$ of $\alpha_{DFA}$ value.

This might be of interest if more detailed study of $\alpha$ exponent is required for more exact predictions to be made (e.g. heart diseases, finances, etc.). The plot in Fig. 13 may also suggests that $\Delta_{DFA-DMA} \to 0$ when $L \to \infty$. The latter case has not been explored in details.

3 Conclusions

We report from the analysis of artificial Brownian integer time series and from the collected data that, on the average, DMA method overestimates Hurst exponent values in comparison with DFA technique. This result contradicts to some previous hypothesis in literature. The DMA method seems to be also more sensitive to the presence of random fluctuations in autocorrelations in time series than DFA analysis does. In many practical situations, especially for shorter series, it might be a disadvantage leading to the false signal of not really existing, global autocorrelations in time series.

The mean distance between two methods, i.e. the mean difference between $\alpha_{DFA}$ and $\alpha_{DMA}$ exponents calculated for the time series of given length $L$ is a decreasing function of $L$. For shorter series ($L \leq 6000$) this distance reaches $\sim 15\%$ what might be important in precise determination of $\alpha$ exponent for such series.

There are some open questions. It is not exactly clear where the scaling law exactly starts or terminates, so one needs a more strict requirements how the scaling range should be determined for DFA and DMA techniques.
and how uncertainties in the choice of scaling range are related to uncertainties in the scaling exponent $\alpha$. This work is now in progress [15].

References

[1] A.-L. Barabasi, H.E. Stanley, Fractal Concepts in Surface Growth, Cambridge University Press, Cambridge, 1995.

[2] B.J. West, B. Deering, The Lure of Modern Science: Fractal Thinking, World Scientific, Singapore, 1995.

[3] P.S. Addison, Fractals and Chaos, Institute of Physics, Bristol, 1997.

[4] M. Ausloos, arXiv:cond-mat/0103068

[5] C.-K. Peng, S.V. Buldyrev, S. Havlin, M. Simons, H.E. Stanley, A.L. Golberger, Phys. Rev. E49, 1685-1689 (1994).

[6] E. Alessio, A. Carbone, G. Castelli, V. Frappietro, Eur. Phys. J. B27, 197-200 (2002).

[7] A. Carbone, G. Castelli, Noise in Complex Systems and Stochastic Dynamics, Proc.of SPIE Vol. 5114, 406-414 (2003).

[8] A. Carbone, G. Castelli, arXiv: cond-mat/0303465

[9] L. Xu, P.Ch. Ivanov, K. Hu, Z. Chen, A. Carbone, H.E. Stanley, Phys. Rev. E, (2005); arXiv: cond-mat/0408047

[10] N. Vandewalle, M. Ausloos, Physica A246, 454-459 (1997).

[11] M. Ausloos, N. Vandewalle, Ph. Boveroux, A. Minguet, K. Ivanova, Physica A274, 229-240 (1999).

[12] N. Vandewalle, M. Ausloos, Int. J. Comput. Anticipat. Syst. 1, 342-349 (1998).

[13] M. Ausloos, Physica A285, 48-65 (2000).

[14] D. Grech, Z. Mazur, Physica A336, 133-145 (2004).

[15] D. Grech, Z. Mazur, in preparation.