ASYMPTOTIC BEHAVIOR OF $\beta$-POLYGON FLOWS

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Abstract In this article we investigate a family of nonlinear evolutions of polygons in the plane called the $\beta$-polygon flow and obtain some results analogous to results for the smooth curve shortening flow: (1) any planar polygon shrinks to a point and (2) a regular polygon with five or more vertices is asymptotically stable in the sense that nearby polygons shrink to points that rescale to a regular polygon. In dimension four we show that the shape of a square is locally stable under perturbations along a hypersurface of all possible perturbations. Furthermore, we are able to show that under a lower bound on angles there exists a rescaled sequence extracted from the evolution that converges to a limiting polygon that is a self-similar solution of the flow. The last result uses a monotonicity formula analogous to Huisken’s for the curve shortening flow.

1. INTRODUCTION

The Gage-Grayson-Hamilton Theorem ([5, 6]) states that any embedded plane curve converges to a round point in an asymptotically self-similar manner under the motion by its curvature. One interesting and still open question is whether one can find a discrete version of the curve shortening flow such that any embedded polygon contracts to a regular point in an asymptotically self-similar manner. Several approaches to this have been suggested, such as flow by the generalized gradient flow of the length functional [10, 4] and flow by the Menger curvature [9]. However, even locally, none of these flows gives an affirmative answer to the above question. It seems that these flows ([10, 4, 9]) may cease to be defined when one of the edge lengths becomes zero, which can happen, for instance for a long, skinny rectangle. In this paper, we consider a slightly different flow that does not have a problem when an edge length becomes zero.

We consider a family of nonlinear evolutions of polygons.

Definition 1.1. A family of polygons $X(t) = (X_0, \ldots, X_{N-1})$ (see Definition 2.1) evolves by the $\beta$-polygon flow if it satisfies

$$\frac{dX_j}{dt} = l_j^{\beta}(X_{j+1} - X_j) + l_{j-1}^{\beta}(X_{j-1} - X_j),$$

where $\beta \geq 0$ and $l_j = |X_{j+1} - X_j|$ for the parameters $j = 0, \ldots, N - 1$ considered modulo $N$.

In [3], Chow and Glickenstein consider the system (1.1) when $\beta = 0$. In this case, (1.1) turns out to be a linear system. The main results obtained in [3] are that the flow shrinks any polygon to a point and the asymptotic shape is affinely-regular if the initial polygon is not orthogonal to the regular polygon. The linear flow has the advantage that there is no singularity before the polygon extinguishes. A disadvantage is that the space of affinely-regular polygons is a big space; for example, all triangles and parallelograms are affinely-regular. Therefore, in the end of [3], the authors ask whether the nonlinear system (1.1) flows a polygon asymptotically to a regular polygon. We are able to give a partial answer to this question. We prove that the $\beta$-polygon flow converges to a self-similar solution.

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Theorem 1.2. Let $X(t)$ be the solution of the $\beta$-polygon flow for $t \in [0, \infty)$. Assume the angle bound (4.10) is satisfied and suppose $X(t) \to x_0$ as $t \to \infty$. Then for any sequence $c_k \nearrow \infty$ there exists a subsequence still denoted by $c_k$ such that the following rescaled polygons converge to a polygon that contracts self-similarly:

$$c_k \left[ X(c_k \tau) - x_0 \right] \to Y(\tau),$$

where $Y(\tau)$ is a self-similar solution for $\tau > 0$.

The convex regular polygons are self-similar solutions. We are furthermore able to prove these are stable in the following theorems.

Theorem 1.3. Assume $N \geq 5$. Under the $\beta$-polygon flow, any regular $N$-gon shrinks to a point and is asymptotically stable in the sense that there is a neighborhood such that polygons in that neighborhood will converge to a regular polygon under the $\beta$-polygon flow if appropriately rescaled.

Theorem 1.4. When $N = 4$, the shape of square is locally stable on a 7-dimensional hypersurface $W'$ under the $\beta$-polygon flow.

The outline of this article is as follows. In Section 2, we establish the long time existence and uniqueness of the initial value problem for the $\beta$-polygon flow. In Section 3, we construct a Lyapunov function to show that any triangle would converge to a regular triangle. In Section 4, inspired by Huisken’s monotonicity formula [8], we have the global stability result, Theorem 1.2. In Section 5, we obtain the local stability of the $\beta$-polygon flow (1.1) in Theorems 1.3 and 1.4.

2. Existence and basic properties

In this section we will describe the $\beta$-polygon flow and give basic properties of it. First we give a definition of a polygon.

Definition 2.1. An $N$-gon, or polygon, $X$ in the Euclidean plane is an ordered $N$-tuple of points in the plane, $X = (X_0, \cdots, X_{N-1})$. Note that the index of the points will always be considered modulo $N$.

The points of the polygon are called vertices and the line segments joining consecutive vertices are called edges. The geometry of the polygon is determined by the following quantities.

Definition 2.2. The length of an edge, denoted $l_j$, is defined to be the distance between the adjacent vertices $X_j$ and $X_{j+1}$. The angle $\theta_j$ at vertex $X_j$ is defined to be the angle such that rotating the unit vector $\overrightarrow{X_jX_{j+1}}/|\overrightarrow{X_jX_{j+1}}|$ an angle of $\theta_j$ in the counterclockwise direction gives the vector $\overrightarrow{X_jX_{j-1}}/|\overrightarrow{X_jX_{j-1}}|$.

Note that we are using $\theta_j$ to denote the interior angle of a polygon. In some related work, the angle is defined to be the exterior angle, and would have the value of $\pi - \theta_j$. We are also assuming that consecutive vertices are not equal, in which case we would not be able to define angle.

We have a natural identification between an $N$-gon in $\mathbb{R}^2$ and an $N$-vector in $\mathbb{C}^n$. In particular, we can write the vertex $X_j = (x_j, y_j)$ as $X_j = x_j + iy_j$ where $i = \sqrt{-1})$. Sometimes it is convenient to write the polygon as a $N \times 2$ matrix:

$$X = \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ \vdots & \vdots \\ x_{N-1} & y_{N-1} \end{pmatrix}.$$
In this case, any two-by-two matrix \( M \) can act on \( X \) on the right by matrix multiplication, \( XM \). We will consider actions by a Euclidean isometry \( E \) from the right as well; the rotational part acts by matrix multiplication on the right and the translational part acts by adding a matrix with all rows equal.

Let \( X = (X_0, \ldots, X_{N-1}) \) be a planar \( N \)-gon. Consider the following energy functional on \( X \):

\[
F_\alpha(X) = \frac{1}{\alpha} \sum_{j=0}^{N-1} |X_{j+1} - X_j|^\alpha,
\]

where the indices, as usual, are taken modulo \( N \). We can then compute the variation of this functional. If \( \frac{dX_j}{dt} = Y_j \), then

\[
\frac{d}{dt}F_\alpha(X) = -\sum_{j=0}^{N-1} \left( \frac{X_{j+1} - X_j}{|X_{j+1} - X_j|^{2-\alpha}} + \frac{X_{j-1} - X_j}{|X_{j-1} - X_j|^{2-\alpha}} \right) \cdot Y_j.
\]

The negative gradient flow of \( F_\alpha \) is therefore

\[
\frac{dX_j}{dt} = \frac{X_{j+1} - X_j}{|X_{j+1} - X_j|^{2-\alpha}} + \frac{X_{j-1} - X_j}{|X_{j-1} - X_j|^{2-\alpha}}.
\]

We may refer to either of the equivalent systems (1.1) or (2.4) as the \( \beta \)-polygon flow.

**Remark 1.** The matrix \( M_X \) has the form of a weighted graph Laplacian on the \( N \)-cycle \( X \), where each edge \( X_jX_{j+1} \) has weight \( l^3_j \). Therefore, (1.1) can be considered a type of heat equation.

We describe some basic properties of \( M_X \).
Proposition 2.3. Let $X$ be a polygon, $c > 0$, and $E$ be a Euclidean isometry of the plane. We denote the action of $E$ on $X$ by $XE$ and use $\vec{1}$ to denote the vector of all ones. Then the following are true:

1. $M_{XE} = M_X$.
2. $M_{cX} = c^\beta M_X$.
3. $M_X \vec{1} = 0$.

Proof. The first follows from the fact that $M_X$ uses only the edge lengths and not the points themselves, so the matrix is unchanged by Euclidean transformations. The second is a scaling property that is easily checked and the third comes from the form of the matrix $M_X$. □

This leads to the following important invariant property of the $\beta$-polygon flow, which follows from the previous proposition.

Lemma 2.4. The $\beta$-polygon flow is invariant in the following way: if $c > 0$, $E$ is a Euclidean transformation of the plane, and $\tau = \frac{1}{c^\beta t}$ then

$$
\frac{d}{d\tau} (cXE) = M_{cXE}(cXE)
$$

Remark 2. This invariant property together with the uniqueness Theorem 2.8 below implies that similar polygons evolve in a similar manner under the $\beta$-polygon flow. Let $X$ and $Y$ be solutions of (1.1) such that $Y(0) = cX(0)E$ for some number $c$ and some Euclidean isometry $E$. Then $Y(\tau(t)) = cX(t)E$.

We can describe the fixed points explicitly.

Proposition 2.5. The fixed points of the flow are precisely polygons of the form $X = (X_0, \ldots, X_{N-1})$ such that $X_0 = X_1 = \cdots = X_{N-1}$. We call such polygons points.

It will be important to control the $(2 + \beta)$-norm of the polygon under the flow in order to use appropriate compactness theorems.

Definition 2.6. If $X = (X_0, \cdots, X_{N-1}) \in \mathbb{C}^N$, for any $p \geq 1$, we define the $p$-norm of $X$ by

$$
\|X\|_p = \left(\sum_{k=0}^{N-1} |X_k|^p\right)^{1/p}.
$$

We now have the following a priori bound on the $2 + \beta$-norm.

Lemma 2.7 (A priori bound). Let $\alpha = 2 + \beta$ with $\beta > 0$ and $Q = (Q, \cdots, Q) \in \mathbb{C}^N$. If $X(t)$ is the solution of the initial value problem for (1.1), then the $\alpha$-norm of $X - Q$ is monotonicity decreasing, i.e., $\|X(t) - Q\|_\alpha \leq \|X(\tau) - Q\|_\alpha$ for $t > \tau$.

Proof. A direct calculation gives:

$$
\frac{d}{dt} \frac{1}{\alpha} \|X - Q\|^\alpha
$$

$$
= -\sum_{j=0}^{N-1} t_j^\beta \left[|X_{j+1} - Q|^\beta + |X_j - Q|^\beta - (|X_{j+1} - Q|^\beta + |X_j - Q|^\beta)(X_j - Q) \cdot (X_{j+1} - Q)\right]
$$

$$
\leq -\sum_{j=0}^{N-1} t_j^\beta \left[|X_{j+1} - Q|^\beta + |X_j - Q|^\beta - (|X_{j+1} - Q|^\beta + |X_j - Q|^\beta)(|X_{j+1} - Q| - |X_j - Q|)\right]
$$

$$
\leq 0
$$

where the first inequality comes from Cauchy-Schwarz. The result follows. □
We are now able to prove that the flow exists for all time and shrinks to its center of mass.

**Theorem 2.8** (Long time existence/uniqueness). For any initial polygon \( X = (X_0, \ldots, X_{N-1}) \), there is a unique solution to (1.1) for all \( t > 0 \) and the solution converges to the center of mass for \( X, \frac{1}{N} \sum_{j=0}^{N-1} X_j \), as \( t \to \infty \).

**Proof.** The standard theory in the ordinary differential equations says the solution \( X(t) \) of (1.1) exists at \([0, T)\) for some \( T > 0 \). Since the system has the form \( \frac{d}{dt} X = F(X) \) where \( dF \) is Lipschitz, the solution is also unique by the standard theory.

Applying Lemma 2.7 with \( Q \) chosen to be the origin, we have \( X(t) \subseteq B(R) \) for some closed Euclidean ball \( B(R) \) for all \( t > 0 \). However, since \( M_X : \mathbb{C}^N \to \mathbb{C}^N \) is a continuous function on \( \mathbb{C}^N \), the extension theorem (p 12 of [7]) says that \( X(t) \) would become unbounded as \( t \to T \) if \( T < \infty \). Hence \( T = \infty \).

In order to show that the flow converges to the center of mass, we first show that a subsequence converges to a point. Since \( F_n(X(t)) \) is decreasing and bounded from below, as \( t \to \infty \) we have that \( \frac{d}{dt} F_n(X(t)) \to 0 \). Hence, by Proposition 2.5, any subsequence that converges to a polygon, converges to a point.

Since \( X(t) \) is in a bounded set, there is a subsequence \( t_k \) such that \( X(t_k) \) converges to a point, \( Z \). By Lemma 2.7, we have that \( \|X(t) - Z\|_a \) is decreasing in \( t \) and since a subsequence converges to zero, we must have that \( \|X(t) - Z\|_a \) converges to zero.

Finally, the flow (1.1) preserves the center of mass of \( X \), i.e.,

\[
\frac{d}{dt} \frac{1}{N} \sum_{i=0}^{N-1} X_i = 0.
\]

Therefore, \( Z \) must be the center of mass of \( X \).

\[\square\]

A direct calculation gives the evolution of the edge lengths and angles.

**Lemma 2.9.** Under the flow (1.1), we have

\[
\frac{dl_j}{dt} = -2l_j^{\beta+1} - l_j^{\beta+1} \cos \theta_{j+1} - l_j^{\beta+1} \cos \theta_j,
\]

\[
\frac{d\theta_j}{dt} = \frac{1}{l_j l_{j-1}} \left[ (l_j^{\beta+2} + l_j^{\beta+2}) \sin \theta_j - l_j^{\beta+1} l_{j-1} \sin \theta_{j+1} - l_j^{\beta+1} l_j \sin \theta_{j-1} \right],
\]

for \( j = 0, \ldots, N - 1 \).

**Proof.** We have

\[
\frac{dX_{j+1}}{dt} - \frac{dX_j}{dt} = l_j^\beta (X_{j+2} - X_{j+1}) - 2l_j^\beta (X_{j+1} - X_j) + l_j^{\beta+1} (X_j - X_{j-1}).
\]

Hence,

\[
\frac{d}{dt} l_j = \frac{1}{l_j} (X_{j+1} - X_j) \cdot \left( \frac{dX_{j+1}}{dt} - \frac{dX_j}{dt} \right)
\]

\[
= -2l_j^{\beta+1} - l_j^{\beta+1} \cos \theta_{j+1} - l_j^{\beta+1} \cos \theta_j.
\]

Since

\[
\cos \theta_j = \frac{(X_{j+1} - X_j) \cdot (X_j - X_{j-1})}{l_j l_{j-1}},
\]

differentiating, we get

\[- \sin \theta_j \frac{d\theta_j}{dt} = \frac{1}{l_j l_{j-1}} \left[ -(l_j^{\beta+2} + l_j^{\beta+2}) \sin^2 \theta_j + l_j^{\beta+1} l_{j-1} \sin \theta_j \sin \theta_{j+1} + l_j^{\beta+1} l_j \sin \theta_j \sin \theta_{j-1} \right],
\]

which gives (2.7). \[\square\]
**Remark 3.** One tricky part in the previous calculation is to show that

$$(X_{j+1} - X_j) \cdot (X_{j-1} - X_{j-2}) = l_j l_{j-2} \cos(\theta_j + \theta_{j-1}).$$

To see this, recall that Definition 2.2 says that by rotating $X_{j+1} - X_j$ counterclockwise an angle of $\pi + \theta_j$, we get a vector in the direction of $X_j - X_{j-1}$. Rotate this new vector counterclockwise by another $\pi + \theta_{j-1}$ and we get a vector in the direction of $X_{j-1} - X_{j-2}$.

We close this section with numerical examples of the $\beta$-polygon flow on a heptagon and on a quadrilaterals. They indicate that the regular heptagon may be stable and that the square may be semistable, as described in Theorems 1.3 and 1.4.

**Example 2.10.** Figures 2 and 3 show, for the case $\beta = 1, c_k = 10^k, \tau = 1$ and $N = 7$, the evolution of a heptagon converging to a regular heptagon. Indeed, we start from some heptagon $X_0$ Figure 2(a), and evolve it under the flow (1.1) until time $\tau = 1$ to obtain $X(1)$ in Figure 3(a). We use the rescaled heptagon $10X(1)$ as our new initial data and continue to evolve it under the flow (1.1) until the time $\tau = 1$ to get $10X(10^1 \cdot 1)$ in Figure 3(b). We rescale it by 10 again and repeat this process 6 times to obtain Figures 2 and 3. Comparing the polygon $10^5 X(10^5)$ in Figure 3(f) with the regular heptagon, we find

$$\sum_{i=0}^{6} \left( \theta_i - \frac{5\pi}{7} \right)^2 = 0.0252069, \quad \sum_{i=0}^{6} \left( \frac{l_i}{l_{i+1}} - 1 \right)^2 = 0.0107429,$$

where $\theta_i$ and $l_i$ denote the angle and edge-length of $10^5 X(10^5)$, respectively. It appears that the two errors become small and the polygons obtained in this process are converging to a regular heptagon.

![Figure 2](image.png)

**Figure 2.** Evolution of a heptagon with selected scalings in space and time.
The next example says, our result in Theorem 1.4 is sharp since it is possible to have a locally stable rhombus.

**Example 2.11.** In Figure 4, we look at the flow starting at a rectangle and also starting at another quadrilateral. It appears that the rectangle evolves to a square while the other quadrilateral evolves to a rhombus that is not a square.

### 3. Evolution of the triangle

Inspired by the techniques used in [10], we obtain the following result for the triangle, $N = 3$.

**Theorem 3.1.** Under the $\beta$-polygon flow, an arbitrary (nondegenerate) triangle shrinks to a point and converges to a regular triangle if appropriately rescaled.

**Proof.** For a triangle, it is sufficient to show that the angles all converge to $\pi/3$. The possible angles for a (nondegenerate) counterclockwise oriented triangle form the following region $\Omega$:

\[
\Omega = \{(\theta_0, \theta_1, \theta_2) | \theta_0 + \theta_1 + \theta_2 = \pi, 0 < \theta_0, \theta_1, \theta_2 < \pi \}.
\]

The area $S$ can be expressed as

\[
S = \frac{1}{2} l_0 l_2 \sin \theta_0 = \frac{1}{2} l_0 l_1 \sin \theta_1 = \frac{1}{2} l_1 l_2 \sin \theta_2.
\]

Using this relation and (2.7), we have

\[
\frac{d\theta_0}{dt} = \frac{1}{l_0 l_2} \left[ (l_2^\beta + l_0^\beta) \sin \theta_0 - l_1^\beta + l_2 \sin \theta_1 - l_1^\beta + l_0 \sin \theta_2 \right]
\]

\[
= \frac{1}{2S} \left[ l_1^2 \sin^2 \theta_1 (l_2^\beta - l_1^\beta) + l_0^2 \sin^2 \theta_0 (l_0^\beta - l_1^\beta) \right].
\]

Similarly, we have

\[
\frac{d\theta_1}{dt} = \frac{1}{2S} \left[ l_2^2 \sin^2 \theta_2 (l_0^\beta - l_2^\beta) + l_1^2 \sin^2 \theta_1 (l_1^\beta - l_2^\beta) \right]
\]
and
\[ \frac{d\theta_2}{dt} = \frac{1}{2S} \left[ l_0^2 \sin^2 \theta_0 (l_1^3 - l_0^3) + l_2^2 \sin^2 \theta_2 (l_2^3 - l_1^3) \right]. \]

Let us introduce a function
\[ V(\theta_0, \theta_1, \theta_2) = -(\pi - \theta_0)(\pi - \theta_1)(\pi - \theta_2). \]

The function \( V \) is negative in \( \Omega \), zero on \( \partial \Omega \), and has a unique minimum at \( P = (\pi/3, \pi/3, \pi/3) \).

Its time derivative is given by
\[
\frac{dV}{dt} = \frac{d\theta_0}{dt} (\pi - \theta_1)(\pi - \theta_2) + \frac{d\theta_1}{dt} (\pi - \theta_0)(\pi - \theta_2) + \frac{d\theta_2}{dt} (\pi - \theta_0)(\pi - \theta_1)
\]
\[
= \frac{1}{2S} l_1^2 \sin^2 \theta_1 (l_2^3 - l_1^3)(\theta_0 - \theta_1)(\pi - \theta_2) + \frac{1}{2S} l_0^2 \sin^2 \theta_0 (l_0^3 - l_1^3)(\theta_0 - \theta_2)(\pi - \theta_1)
\]
\[
+ \frac{1}{2S} l_2^2 \sin^2 \theta_2 (l_0^3 - l_2^3)(\theta_1 - \theta_2)(\pi - \theta_0).
\]

The right-hand side is negative on \( \Omega - P \) and zero at \( P \). Thus \( V \) is a Lyapunov function and, therefore, \( P \) is asymptotically stable. \( \square \)

Since the evolution of triangles is understood by Theorem 3.1, in the rest of this paper we assume that \( N \geq 4 \).

4. Self-similar solutions and the rescaled flow

One way to study asymptotic behavior of geometric flows is to try to show that limiting flows converge to self-similar solutions in some sense. Self-similar solutions are special solutions that do not change shape as they evolve. In other words, the initial data determines the shape of the solution. This property yields one of the benefits of finding the self-similar solutions: the time variable can be separated out. In [1], Abresch and Langer classify all of the self-similar solution of the curve shortening flow. In [2], Angenent shows that a convex immersed plane curve that evolves by its curvature will either shrink to a point in an asymptotically self-similar manner (as described by Abresch and Langer [1]), or else there exists a rescaled flow converging to the graph of the grim reaper (a noncompact self-similar solution). Similarly, in [11], the authors proved that there is a rescaled lens-shaped network that contracts smoothly to a unique self-similar solution of the planar network flow. We will show that solutions to the \( \beta \)-polygon flow converge asymptotically to self-similar solutions.

Recall the definition of point from Proposition 2.5.

**Definition 4.1.** We say \( X(t) \) is a self-similar solution of (1.1) if there exists a polygon \( X_0 \), a scaling function \( \lambda(t) \), and a point \( Q \) such that \( X(t) = \lambda(t)X_0 + Q \) satisfies (1.1).

By Theorem 2.8 we must have that \( \lambda(t) \to 0 \) and \( Q = \lim_{t \to \infty} X(t) \) as \( t \to \infty \).

When \( \beta = 0 \), the system (1.1) becomes a linear system as studied in [3], and the corresponding \( N \times N \) matrix \( M \) is:

\[
M = \begin{pmatrix}
-2 & 1 & 0 & \cdots & 0 & 1 \\
1 & -2 & 1 & \ddots & 0 \\
0 & 1 & -2 & 1 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & 0 & 1 & -2 & 1 \\
1 & 0 & \cdots & 0 & 1 & -2
\end{pmatrix}.
\]

Because \( M \) is a circulant matrix, \( C^N \) has a basis of eigenvectors consisting of \( N \)-th roots of unity:

\[
P_k = (1, \omega^k, \omega^{2k}, \cdots, \omega^{2(n-1)k})^T \quad (k = 0, \cdots, N - 1),
\]
where $\omega^N = 1$ and "T" signifies the transpose. We can think of these these vectors as listing the vertices of a regular, oriented (possibly star-like) polygon in the complex plane by drawing each entry of the vector in the complex plane and connecting consecutive entries by arrows. The eigenvalue corresponding to the eigenvector $P_k$ can then be computed to be
\begin{equation}
\lambda_k = -4 \sin^2(\pi k/N) \quad (k = 0, \ldots, N - 1).
\end{equation}

For $\beta > 0$, we see that the regular polygons are still self-similar solutions.

**Lemma 4.2.** The regular $N$-gons $P_k$ as defined in (4.2) are self-similar solutions of (1.1), i.e., if $a(t) = (1 - \beta l^\beta \lambda_k t)^{-1/\beta}$, where $l$ is the edge length of $P_k$ and $\lambda_k = -4 \sin^2(\pi k/N)$ is the corresponding eigenvalue of $M$, then $P(t) = a(t)P_k$ is a solution of (1.1).

**Proof.** Using $P(t) = a(t)P_k$ in (1.1), we obtain
\begin{equation}
\frac{d\alpha}{dt} P_1 = MP_1 = a^{1+\beta} MP_k P_k = a^{1+\beta} l^\beta \lambda_k P_k.
\end{equation}

We then obtain $a(t)$ as the solution to the above differential equation with $a(0) = 1$. □

We will study the asymptotic stability of the solutions of (1.1) around the regular polygon $P_1$. Due to the invariance property, Lemma 2.4, it is sufficient to study the local behavior near $P_1$.

Since the flow converges to a point by Theorem 2.8, in order to study local behavior, we need to find an appropriate rescaling. Let $\alpha(t) : \mathbb{R} \to \mathbb{R}^+$ be some positive scaling function and $X(t)$ be a solution of (1.1). Let $\bar{X}$ denote the vector of all ones multiplied by the average $\frac{1}{N} \sum_{j=0}^{N-1} X_i$. Using Proposition 2.3, we have
\begin{equation}
\frac{d}{dt}(\alpha(X - \bar{X})) = \frac{d\alpha}{dt}(X - \bar{X}) + \alpha M X = \frac{d\alpha}{dt} \frac{1}{\alpha}(\alpha(X - \bar{X})) + \frac{1}{\alpha^\beta} M \alpha(X - \bar{X})(\alpha(X - \bar{X})).
\end{equation}

Letting $Y = \alpha(X - \bar{X})$, we obtain the following nonlinear system:
\begin{equation}
\frac{d}{dt} Y = \frac{d\alpha}{dt} \frac{1}{\alpha} Y + \frac{1}{\alpha^\beta} M Y Y.
\end{equation}

By letting $\alpha(t) = 1/a(t)$, where $a$ is the function in Lemma 4.2, we have
\begin{equation}
\frac{d}{dt} Y = -a^\beta l^\beta \lambda_1 Y + a^\beta M Y Y,
\end{equation}

where $a$, $l$, and $\lambda_1$ are the same as in Lemma 4.2. It is clear that the regular $N$-gon $P_1$ is an equilibrium point of this system and $c(t)P_1$ is a self-similar solution with $c(t) \to 1$ as $t \to \infty$.

Consider a new time variable $\tau$ determined by
\begin{equation}
\frac{dt}{d\tau} = \frac{1}{l^\beta a^\beta},
\end{equation}
giving
\begin{equation}
\tau = \ln(1 - \beta l^\beta \lambda_1 t) - \beta \lambda_1.
\end{equation}

Converting the system to a function of $\tau$ results in the following equation:
\begin{equation}
\frac{dY}{d\tau} = -\lambda_1 Y + \frac{1}{l^\beta} M Y Y.
\end{equation}

This leads to the following definition.

**Definition 4.3.** We call the flow in (4.6) the $\lambda_1$-rescaled $\beta$-polygon flow.
Motivated by Huisken’s monotonicity formula [8], we will prove a general monotonicity formula for polygons evolving under the \( \beta \)-polygon flow. We then use the standard blow up argument in geometric flows (see [11] and [8] for instance) to show that evolutions satisfying the bound (4.10) are asymptotically self-similar.

Let \( X(t) = \lambda(t)X_0 + Q \) (see Definition 4.1) be a self-similar solution of the \( \beta \)-polygon flow with \( \lambda(0) = 1 \). Since \( \frac{d}{dt} \left( \frac{X - Q}{\lambda} \right) = 0 \), we have

\[
-\frac{d\lambda}{dt} \frac{1}{\lambda} (X - Q) + \frac{1}{\lambda} M_X X = 0.
\]

By letting \( t = 0 \), we obtain the following equation that determines the self-similar solution:

(4.7) \[ M_{X_0} X_0 = \frac{d\lambda}{dt}(0)(X_0 - Q). \]

Given \( x_0 \in \mathbb{C}^N \) and \( X(t) \) a solution of the \( \beta \)-polygon flow, define \( \rho_{x_0}(X, t) \) be the following entropy functional

(4.8) \[ \rho_{x_0}(X, t) = \exp \left[ -t^{2/\beta} \left| X(t) - x_0 \right|^2 - \int_0^t \frac{\beta}{2} s^{2/\beta + 1} \left| M_X X(s) \right|^2 ds \right] , \]

where \( |X| \) denotes the 2-norm of \( X \) as in (2.5). We have the following monontonicity formula.

**Theorem 4.4.** Let \( X(t) \) be a the solution of the \( \beta \)-polygon flow, then for any polygon \( x_0 \) we have the formula

\[
\frac{d}{dt} \rho_{x_0}(X, t) = -\frac{2}{\beta} \rho_{x_0}(X, t) t^{2/\beta - 1} \left| X - x_0 + \frac{\beta}{2} t M_X X \right|^2 .
\]

**Proof.** We calculate directly from (1.1) and find

\[
\frac{d}{dt} \rho_{x_0}(X, t) = \rho_{x_0}(X, t) \left[ -\frac{2}{\beta} t^{2/\beta - 1} \left| X - x_0 \right|^2 - 2 t^{2/\beta} (X - x_0) \cdot M_X X - \frac{\beta}{2} t^{2/\beta + 1} \left| M_X X \right|^2 \right] \\
= -\frac{2}{\beta} \rho_{x_0}(X, t) t^{2/\beta - 1} \left| X - x_0 + \frac{\beta}{2} t M_X X \right|^2 .
\]

\hfill \Box

Consider a polygon \( X(t) \) for \( t \in [0, \infty) \) that contracts to a point \( x_0 \) as \( t \to \infty \). Consider a sequence of positive numbers \( c_k \not\rightarrow \infty \). We rescale the polygon under a sequence of dilations given by

(4.9) \[ Y^k(\tau) = c_k \left( X(c_k^{\beta} \tau) - x_0 \right) . \]

Note that for each \( k \), \( Y^k(\tau) \) converges to the origin as \( \tau \to \infty \). To ensure the compactness of the family of rescaled polygons \( Y^k(\tau) \), we assume that the angles \( \theta_i \) of \( X(t) \) satisfy a lower bound, i.e., there exist \( T_0 \) and \( \delta > 0 \) such that

(4.10) \[ \inf_{t \in (T_0, \infty)} \min_{i = 0, \ldots, N-1} \sin^2 \theta_i \geq \delta . \]

**Lemma 4.5.** If (4.10) is valid, then the energy of the rescaled polygons \( \{Y^k\}_{k=1}^\infty \) is uniformly bounded on the compact interval \( [\epsilon, 1/\epsilon] \) for any \( \epsilon > 0 \). Explicitly, there exists some uniform constants \( \eta(\epsilon, \alpha, N) \) and \( \tilde{\eta}(\epsilon, \alpha, N) \) such that

\[
\eta(\epsilon, \alpha, N) \leq F_0(Y^k(\tau)) \leq \tilde{\eta}(\epsilon, \alpha, N) , \quad \forall k \geq 0 , \tau \in [\epsilon, 1/\epsilon] .
\]
\textbf{Proof.} Let $X = (X_0, \ldots, X_{N-1}) \in \mathbb{C}^N$. Recall that $\theta_j$ denotes the angle at $X_j$ and $l_j$ denotes the edge-length between $X_j$ and $X_{j+1}$. By (2.1) and (1.1), we have

$$\frac{d}{dt} F_\alpha(X) = \frac{1}{\alpha} \frac{d}{dt} \sum_{j=0}^{N-1} l_j^\alpha$$

$$= - \sum_{j=0}^{N-1} \left| l_j^\beta (X_{j+1} - X_j) + l_{j-1}^\beta (X_{j-1} - X_j) \right|^2$$

$$= - \sum_{j=0}^{N-1} \left( l_j^{2\beta+2} + l_{j-1}^{2\beta+2} + 2 l_j^{\beta+1} l_{j-1}^\beta \cos \theta_j \right)$$

$$= - \sum_{j=0}^{N-1} \left( l_j^{2\beta+2} \sin^2 \theta_j + l_j^{\beta+1} \cos \theta_j + l_j^{\beta+1} \right).$$

By assumption (4.10), there exist $T_0$ such that if $t > T_0$, we have

$$-4 \sum_{j=0}^{N-1} l_j^{2\beta+2} \leq \frac{d}{dt} F_\alpha \leq -\delta \sum_{j=0}^{N-1} l_j^{2\beta+2},$$

where $\delta$ is the lower bound in (4.10). In finite dimensional vector spaces, all norms are equivalent, so there exist positive numbers $\eta_1$ and $\eta_2$ which only depend on $\beta$ and $N$ such that

$$-\eta_1 F_\alpha^{2\beta+2/\beta} \leq \frac{d}{dt} F_\alpha \leq -\eta_2 F_\alpha^{2\beta+2/\beta}.$$ 

These formulas integrate to estimates of the energy of $X$; explicitly, there exist positive numbers $\mu_1, \mu_2, \mu_3, \mu_4$ such that

$$\left( \frac{1}{\mu_1 t + \mu_2} \right)^{(\beta+2)/\beta} \leq F_\alpha(X) \leq \left( \frac{1}{\mu_3 t + \mu_4} \right)^{(\beta+2)/\beta}$$

if $t > T_0$, where $\mu_i$ for $i = 1, 2, 3, 4$ only depend on $\alpha$ and $N$.

Now consider the rescaled polygons $Y^k(\tau) = c_k X(c_k^\beta \tau - x_0)$. Since $F_\alpha(Y^k(\tau)) = c_k^\beta + 2 F_\alpha(X(c_k^\beta \tau))$ and since for any sequence $c_k \to \infty$, eventually $c_k \epsilon$ must be greater than $T_0$, it follows from (4.11) that for $k$ large enough,

$$c_k^{\beta+2} \left[ \frac{1}{\mu_1 c_k^\beta \tau + \mu_2} \right]^{(\beta+2)/\beta} \leq F_\alpha(Y^k(\tau)) \leq c_k^{\beta+2} \left[ \frac{1}{\mu_3 c_k^\beta \tau + \mu_4} \right]^{(\beta+2)/\beta}$$

This completes the proof. \hfill \Box

\textbf{Lemma 4.6.} If (4.10) is satisfied and if $\tau$ is restricted to a compact interval $[\epsilon, 1/\epsilon]$, then the rescaled polygons $\{Y^k(\tau)\}_{k=1}^\infty$ are contained in a compact set.

\textbf{Proof.} By Lemma 4.5, there exists a uniform bound for the energy of the rescaled polygons. Therefore, for each $k$, we can pick a sequence $q_k \in \mathbb{C}^2$ and a uniform radius $R > 0$ such that $Y^k(\tau) \subseteq B(q_k, R)$, where $B(q_k, R) := \{z \in \mathbb{C}^2 : ||z - q_k||_\alpha \leq R \}$ denotes the $\alpha$-norm ball centered at $q_k$ with radius $R$ and $\tau \in [\epsilon, 1/\epsilon]$. If $\{q_k\}_{k=1}^\infty$ is contained in a compact set, then the statement follows. Otherwise, there exist a subsequence $q_k$, still denoted by $q_k$, that tends to infinity. For $k$ sufficiently large, say $k_0$, we have the origin is not contained in $B(q_{k_0}, R)$. However, Lemma 2.7 says $Y^{k_0}(t) = c_{k_0} \left[ X(c_{k_0}^\beta \tau) - x_0 \right]$ stays in the ball $B(q_{k_0}, R)$ for all $t > \tau$, which contradicts the fact that $Y^{k_0}(t)$ converges to the origin as $t \to \infty$. \hfill \Box

In fact, we have a stronger statement about the flows.
Lemma 4.7. If (4.10) is satisfied then for any \( \epsilon > 0 \), the set \( \{Y^k(\tau)\}_{k=1}^{\infty} \) of rescaled polygons (considered as functions of \( \tau \)) is contained in a compact subset of \( C^0([\epsilon, 1/\epsilon], \mathbb{C}^N) \).

Proof. By Lemma 4.6, the polygons \( Y^k(\tau) \) are pointwise bounded for each \( \tau \). Since this implies a uniform bound on the edge lengths and since \( \beta > 0 \), we see that there is a bound on the derivative \( dY^k/d\tau \), uniformly in \( \tau \) and \( k \). The lemma follows from the Arzela-Ascoli Theorem. □

We are now able to prove Theorem 1.2, that nonsingular solutions converge to shrinking self-similar solutions.

Proof of Theorem 1.2. Theorem 4.4 says \( \rho_{x_0}(X,t) \) is monotonically decreasing in time and bounded below. Therefore, the limit

\[
\rho_{x_0}(X,\infty) = \lim_{t \to \infty} \rho_{x_0}(X,t)
\]

exists and is finite. Moreover, it satisfies

\[
\rho_{x_0}(X,\infty) - \rho_{x_0}(X,t) = -\frac{2}{\beta} \int_t^\infty \rho_{x_0}(X,s)s^{2/\beta-1} \left| X(s) - x_0 + \frac{\beta}{2} s M_{X(s)}X(s) \right|^2 ds.
\]

Changing variables according to the rescaling described by (4.9), we obtain

\[
(4.13) \quad \rho_{x_0}(X,\infty) - \rho_{x_0}(X,t) = -\frac{2}{\beta} \int_t^{Y^k(\tau)} \rho_0(Y^k,\tau)\tau^{2/\beta-1} \left| Y^k(\tau) + \frac{\beta}{2} \tau M_{Y^k(\tau)}Y^k(\tau) \right|^2 d\tau.
\]

Fix \( \epsilon > 0 \). By Lemma (4.7), we can extract a subsequence, which we still denote by \( Y^k \) that converges to a limit polygon \( Y \) and satisfies the estimate in (4.12).

Applying (4.13) with \( t = t_k \) chosen so that \( t = \epsilon c_k^\beta \), we find that

\[
\frac{2}{\beta} \int_\epsilon^{Y^k(\tau)} \rho_0(Y^k,\tau)\tau^{2/\beta-1} \left| Y^k(\tau) + \frac{\beta}{2} \tau M_{Y^k(\tau)}Y^k(\tau) \right|^2 d\tau \to 0,
\]

as \( k \to \infty \). Thus \( Y \) satisfies

\[
Y(\tau) + \frac{\beta}{2} \tau M_{Y(\tau)}Y(\tau) = 0,
\]

when \( \tau > \epsilon \) and \( Y \) is a self-similar solution. Since we can do this for each \( \epsilon > 0 \), the result follows.

□

5. SMALL PERTURBATIONS AROUND THE REGULAR POLYGON

In the previous section we saw how to turn a self-similar solution (regular \( N \)-gon) into an equilibrium point of the \( \lambda_1 \)-rescaled \( \beta \)-polygon flow (4.6). In this section, by linearizing the rescaled system and using the center manifold theorem, we shall prove Theorems 1.3 and 1.4.

5.1. Linearization. In order to properly describe the linearization, we will consider the flow with real coordinates. Given \( Y \in \mathbb{C}^N \), we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \) and write \( Y = (Y^r, Y^i)^T \), where \( Y^r = (y^r_0, \ldots, y^r_{N-1}) \) and \( Y^i = (y^i_0, \ldots, y^i_{N-1}) \) give the vectors of real and imaginary parts respectively. We can now write the \( \lambda_1 \)-rescaled \( \beta \)-polygon flow as a 2N-dimensional system:

\[
\frac{dY}{d\tau} = -\lambda_1 Y + \frac{1}{l^\beta} \begin{pmatrix} M_Y & 0 \\ 0 & M_Y \end{pmatrix} Y =: -\lambda_1 Y + \frac{1}{l^\beta} F(Y),
\]

where \( M_Y \) is defined in (1.1) and the equation defines \( F \).

Let \( (f_0, \ldots, f_{2N-1}) \) denote the components of \( F \). We have

\[
f_k = l_k^\beta (y^r_{k+1} - y^r_k) + l_k^\beta (y^i_{k+1} - y^i_k),
\]

\[
f_{N+k} = l_k^\beta (y^r_{k+1} - y^r_k) + l_k^\beta (y^i_{k+1} - y^i_k),
\]

\[
(5.2)
\]

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where \( l_k \) is the distance between \((y_k^i, y_k^j)\) and \((y_{k+1}^i, y_{k+1}^j)\) for \( k = 0, \ldots, N - 1 \). We can now describe the linearization.

**Theorem 5.1.** The linearization of (5.1) around the regular \( N \)-gon \( P_1 \) (4.2) is

\[
\frac{dY}{dt} = -\lambda_1 Y + \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} Y + \beta \begin{pmatrix} A & C \\ C & B \end{pmatrix} Y.
\]

Here \( M \) is defined in (4.1), \( 0 \) is the \( N \times N \) 0-matrix, and the nonzero entries of \( A = (a_{ij}), B = (b_{ij}) \), and \( C = (c_{ij}) \) are:

\[
\begin{align*}
a_{kk\pm 1} &= \sin^2(2k \pm 1)\theta, & a_{kk} &= -[\sin^2(2k - 1)\theta + \sin^2(2k + 1)\theta], \\
b_{kk\pm 1} &= \cos^2(2k \pm 1)\theta, & b_{kk} &= -[\cos^2(2k - 1)\theta + \cos^2(2k + 1)\theta], \\
c_{kk\pm 1} &= -\cos(2k \pm 1)\theta \sin(2k \pm 1)\theta, & c_{kk} &= \cos(2k - 1)\theta \sin(2k - 1)\theta + \cos(2k + 1)\theta \sin(2k + 1)\theta,
\end{align*}
\]

for \( k = 0, \ldots, N - 1 \), where \( \theta = \pi/N \).

**Proof.** Let \( P_1 = (x_0^i, \ldots, x_{N-1}^i, x_0^j, \ldots, x_{N-1}^j) \) be the regular \( N \)-gon defined in (4.2). We have \( x_k^i = \cos 2k\theta \), \( x_k^j = \sin 2k\theta \), and \( l_k = l = 2\sin \theta \) for \( k = 0, \ldots, N - 1 \).

We can now differentiate (5.2) and evaluate at \( P_1 \). For instance, we have:

\[
\begin{align*}
\frac{\partial f_k}{\partial y_{k-1}^i} &= l^\beta [1 + \beta \sin^2(2k - 1)\theta], \\
\frac{\partial f_k}{\partial y_k^j} &= -l^\beta \left[ 2 + \beta (\sin^2(2k - 1)\theta + \sin^2(2k + 1)\theta) \right], \\
\frac{\partial f_k}{\partial y_{k-1}^i} &= -\beta l^\beta \cos(2k - 1)\theta \sin(2k - 1)\theta,
\end{align*}
\]

for \( k = 0, \ldots, N - 1 \). \( \square \)

### 5.2. Stability.

To study the local stability of (5.3) at \( P_1 \), it is enough to classify the eigenvalues and eigenspaces of the matrix

\[
\left[ -\lambda_1 I + \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \right] + \beta \begin{pmatrix} A & C \\ C & B \end{pmatrix} =: D + \beta E,
\]

where \( I \) is the \( 2N \times 2N \) identity matrix.

First recall the analysis of the matrix \( M \). From (4.2) and (4.3), the real and imaginary parts of \( P_k \) form a set of real vectors \( \{c_k, s_k\} \) that spans the eigenspace of \( \lambda_k \), where

\[
\begin{align*}
c_k &= (1, \cos 2k\theta, \cdots, \cos 2k(N - 1)\theta)^T, \\
s_k &= (0, \sin 2k\theta, \cdots, \sin 2k(N - 1)\theta)^T,
\end{align*}
\]

for \( 0 \leq k \leq \lfloor N/2 \rfloor \). Note that if \( k = 0 \) or \( k = N/2 \), the eigenspace for \( \lambda_k \) is spanned by \( \{c_k\} \) and is one-dimensional. For all other \( k \), \( \{c_k, s_k\} \) is a basis and the eigenspace is two-dimensional. Let \( 0 = (0, \cdots, 0)^T \) and \( 1 = (1, \cdots, 1)^T \) denote vectors in \( \mathbb{R}^N \). A straightforward calculation gives the following.

**Lemma 5.2.** The \( 2N \times 2N \) matrix \( D = -\lambda_1 I + \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \) has the following eigenvectors

\[
\begin{pmatrix} c_k \\ 0 \end{pmatrix}, \quad \begin{pmatrix} s_k \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ c_k \end{pmatrix}, \quad \begin{pmatrix} 0 \\ s_k \end{pmatrix},
\]

which span the eigenspace of \( \lambda_k - \lambda_1 \) for \( 0 \leq k \leq \lfloor N/2 \rfloor \). In particular, we have:
(1) There is only one positive eigenvalue \(-\lambda_1\) of \(D\), and the eigenspace of \(D\) corresponding to it is spanned by
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(2) The eigenspace of \(D\) corresponding to the eigenvalue 0 is spanned by
\[
\begin{pmatrix} c_1 \\ 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ c_1 \end{pmatrix}, \begin{pmatrix} 0 \\ s_1 \end{pmatrix}.
\]

(3) The other eigenvalues of \(D\) are negative.

The eigenvectors corresponding to the positive eigenvalue correspond to Euclidean translations in \(\mathbb{C}^N\), while the eigenvectors corresponding to the eigenvalue 0 correspond to linear transformations of the regular \(N\)-gon \(P_1\).

We will use the following lemma to analyze the definiteness of the matrix \(E\).

**Lemma 5.3.** Let \(A\) be a matrix of the form:
\[
A = 
\begin{pmatrix}
-a_0 - a_{n-1} & a_0 & 0 & \cdots & 0 & a_{n-1} \\
a_0 & -a_0 - a_1 & a_1 & 0 & \cdots & 0 \\
0 & a_1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & a_{k-1} & -(a_{k-1} + a_k) & a_k & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
a_{n-1} & 0 & \cdots & 0 & a_{n-2} & -(a_{n-2} + a_{n-1})
\end{pmatrix}
\]

Then for any vectors \(x = (x_0, \cdots, x_{n-1})\) and \(y = (y_0, \cdots, y_{n-1}) \in \mathbb{R}^n\), we have
\[
x Ay^T = -\sum_{k=0}^{n-1} a_k (x_{k+1} - x_k)(y_{k+1} - y_k).
\]

In particular, \(A\) is negative semidefinite if \(a_k \geq 0\) for all \(0 \leq k \leq n - 1\).

**Proof.** This is a straightforward calculation:
\[
x Ay^T = x \cdot \begin{pmatrix}
a_0(y_1 - y_0) + a_{n-1}(y_{n-1} - y_0) \\
a_{k-1}(y_{k-1} - y_k) + a_k(y_{k+1} - y_k) \\
\vdots \\
a_{n-2}(y_{n-2} - y_{n-1}) + a_{n-1}(y_0 - y_{n-1})
\end{pmatrix}
= -\sum_{k=0}^{n-1} a_k (x_{k+1} - x_k)(y_{k+1} - y_k).
\]

We now look at the matrix \(E\).

**Lemma 5.4.** The \(2N \times 2N\) matrix \(E = \begin{pmatrix} A & C \\ C & B \end{pmatrix}\) is negative semidefinite. Moreover, the 0-eigenspace consists of the vectors of the following form:
\[
\{ X \in \mathbb{R}^{2N} : \sin [(2k + 1)\theta] (x^r_{k+1} - x^r_k) = \cos [(2k + 1)\theta] (x^i_{k+1} - x^i_k) \text{ for all } k = 0, \cdots, N - 1, \}
\]
where \(X = (x^r_0, \cdots, x^r_{N-1}, x^i_0, \cdots, x^i_{N-1})\) and \(\theta = \pi/N\).
Proof. We have

\[ XEX^T = (X^r, X^i) \left( \begin{array}{cc} A & C \\ C & B \end{array} \right) \left( \begin{array}{c} (X^r)^T \\ (X^i)^T \end{array} \right) \]

\[ = X^rA(X^r)^T + X^rC(X^i)^T + X^iC(X^r)^T + X^iB(X^i)^T \]

\[ = X^rA(X^r)^T + 2X^rC(X^i)^T + X^iB(X^i)^T, \]

where the last identity follows from the fact that \( C \) is symmetric.

Applying Lemma 5.3 to compute \( X^rA(X^r)^T, X^iB(X^i)^T, \) and \( X^rC(X^i)^T \) and letting \( a_k = \sin^2[(2k + 1)\theta], \cos^2[(2k + 1)\theta] \) and \( -\cos[(2k + 1)\theta] \sin[(2k + 1)\theta] \), we have

\[ XEX^T = -\sum_{k=0}^{N-1} \left[ \sin^2((2k + 1)\theta)(x_{k+1}^r - x_k^r)^2 + \cos^2((2k + 1)\theta)(x_{k+1}^i - x_k^i)^2 \right] \]

\[ + \sum_{k=0}^{N-1} 2 \cos((2k + 1)\theta) \sin((2k + 1)\theta)(x_{k+1}^r - x_k^r)(x_{k+1}^i - x_k^i) \]

\[ = -\sum_{k=0}^{N-1} \left[ \sin((2k + 1)\theta)(x_{k+1}^r - x_k^r) - \cos((2k + 1)\theta)(x_{k+1}^i - x_k^i) \right]^2 \leq 0. \]

\[ \square \]

In order to understand \( D + \beta E \), we need a better understanding of how the eigenspaces of \( D \) and \( E \) intersect.

**Lemma 5.5.** Let \( D_0 \) and \( E_0 \) denote the \( \theta \)-eigenspace of \( D \) and \( E \), respectively. Then we have if \( N \geq 5 \),

\[ D_0 \cap E_0 = \{ z = tiP_1 \in \mathbb{C}^N : t \in \mathbb{R} \}, \]

and if \( N = 4 \),

\[ D_0 \cap E_0 = \{ z = tiP_1 + s\overline{P_1} \in \mathbb{C}^N : t, s \in \mathbb{R} \}. \]

Note that \( iP_1 \) generates rotations of the polygon and \( \overline{P_1} \) is the same polygon as \( P_1 \) but oriented in the opposite direction. The span of \( P_1 \) and \( \overline{P_1} \) generate all linear transformations of \( P_1 \).

**Proof.** Let \( X^r = (x_0^r, \cdots, x_{N-1}^r), X^i = (x_0^i, \cdots, x_{N-1}^i) \in \mathbb{R}^N \), and suppose \( X = (X^r, X^i) \in D_0 \cap E_0 \). Lemma 5.2 says there exist real numbers \( a_{11}, a_{12}, a_{21}, a_{22} \) such that

\[
\begin{align*}
x_k^r &= a_{11} \cos(2k\theta) + a_{12} \sin(2k\theta), \\
x_k^i &= a_{21} \cos(2k\theta) + a_{22} \sin(2k\theta),
\end{align*}
\]

for \( k = 0, \cdots, N-1 \) and \( \theta = \pi/N \). Substituting this into the equation

\[ \sin[(2k + 1)\theta](x_{k+1}^r - x_k^r) = \cos[(2k + 1)\theta](x_{k+1}^i - x_k^i), \]

we obtain the following \( N \) linear equations for \( a_{11}, a_{12}, a_{21}, a_{22} \):

\[
(5.8) \quad -\sin^2[(2k + 1)\theta]a_{11} + \cos[(2k + 1)\theta] \sin[(2k + 1)\theta](a_{12} + a_{21}) - \cos^2[(2k + 1)\theta]a_{22} = 0,
\]

for \( k = 0, \cdots, N-1 \). By subtracting (5.8) with \( k = 0 \) from the same equation with \( k = N-1 \), we find that

\[ a_{12} + a_{21} = 0. \]

This reduces (5.8) to the following:

\[ a_{11} = \frac{1}{2} \sin^2[(2k + 1)\theta] - \frac{1}{2} \cos^2[(2k + 1)\theta] = 0, \]

\[ a_{12} = -\frac{1}{2} \cos[(2k + 1)\theta] \sin[(2k + 1)\theta], \]

\[ a_{21} = -\frac{1}{2} \cos[(2k + 1)\theta] \sin[(2k + 1)\theta], \]

\[ a_{22} = \frac{1}{2} \cos^2[(2k + 1)\theta] = 0. \]

\[ \square \]
for $k = 0, \cdots, N - 1$. Applying $k = 0$ and $k = 1$ to (5.10), we have

\begin{equation}
(5.11) \quad \begin{cases}
-a_{11} \sin^2 \theta - a_{22} \cos^2 \theta = 0, \\
-a_{11} \sin^2[3 \theta] - a_{22} \cos^2[3 \theta] = 0.
\end{cases}
\end{equation}

The determinant of the corresponding matrix is $\cos^2(3 \theta) \sin^2 \theta - \cos^2 \theta \sin^2(3 \theta) = \cos^2(3 \theta) \cos^2 \theta [\tan^2 \theta - \tan^2(3 \theta)] \neq 0$ for $N \geq 5$, and which gives

\begin{equation}
(5.12) \quad a_{11} = a_{22} = 0,
\end{equation}

Combining (5.9) and (5.12), we have

\begin{align*}
x_k^r &= t \sin[(2k \theta)], \\
x_k^i &= -t \cos[(2k \theta)],
\end{align*}

for some $t \in \mathbb{R}$. Viewed in $\mathbb{C}^N$, this gives

\[ D_0 \cap E_0 = \{ z = -ti P_1 \in \mathbb{C}^N : t \in \mathbb{R} \}. \]

For the case $N = 4$, since $\sin^2[(2k + 1) \pi/4] = \cos^2[(2k + 1) \pi/4]$ for $k = 0, \ldots, 4$, the system (5.10) reduces to

\begin{equation}
(5.13) \quad -a_{11} - a_{22} = 0.
\end{equation}

Combining (5.9) and (5.13), we get

\[ D_0 \cap E_0 = \{ z = -ti P_1 + s P_1 \in \mathbb{C}^N : t, s \in \mathbb{R} \}. \]

We now compute the stable, unstable, and center eigenspaces of the linearized system (5.3) at $P_1$. \hfill \Box

**Theorem 5.6.** Consider the $2N \times 2N$ matrix $D + \beta E = \left[ -\lambda_1 I + \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \right] + \beta \begin{pmatrix} A & C \\ C & B \end{pmatrix}$, where $A, B, C$ and $M$ are the matrices described in Theorem 5.1. $D + \beta E$ has a positive eigenvalue $-\lambda_1$ with a two-dimensional eigenspace $E^\circ$ generated by the the vectors in (5.6). For $N \geq 5$, the 0-eigenspace $E^0$ is one dimensional, spanned by the vector $i P_1$; for $N = 4$, the 0-eigenspace $E^0$ is two dimensional spanned by $\{ i P_1, P_1 \}$. The remaining eigenvalues are negative. We call the span of these eigenspaces $E^\circ$.

**Proof.** For any $a, b \in \mathbb{R}$, let $a = (a, \cdots, a), b = (b, \cdots, b) \in \mathbb{R}^N$. It is a straightforward calculation to check that $(D + \beta E)(a, b)^T = -\lambda_1 (a, b)^T$. Let $V$ be the space generated by the vectors (5.6) and denote the orthogonal complement by $V^\perp$. Lemma 5.2 and Lemma 5.4 imply that $D + \beta E$ is negative semidefinite on $V^\perp$, and this implies that there cannot be any other positive eigenvalues. In particular, $-\lambda_1$ is the only positive eigenvalue and its eigenspace is $V$.

Applying Lemma 5.5, we obtain the 0-eigenspace $D_0 \cap E_0$ for $D + \beta E$.

Finally, on the orthogonal complement $W$ of $V \oplus (D_0 \cap E_0)$, both $D$ and $E$ are negative semidefinite, but any nonzero vector in $W$ must miss either $D_0$ or $E_0$. Therefore, $D + \beta E$ is negative definite on $W$, which implies that all of the other eigenvalues of $D + \beta E$ are negative. \hfill \Box

The following result is an immediate consequence of Theorem 5.6.

**Theorem 5.7.** Perturbations of $P_1$ that are orthogonal to $E^\circ$ and $E^s$ converge to $P_1$ when evolved by the linearized system (5.3).

We have the following stability result for the scaling system (4.6).

**Theorem 5.8.** Assume $N \geq 5$. Around the regular polygon $P_1$ there exists a $2N - 2$ dimensional semi-stable manifold $W \subseteq E^\circ \oplus E^s$ such that for any $x_0 \in W$, the trajectory $x(t)$ of (4.6) with $x(0) = x_0$ converges to the regular polygon $e^{i \eta} P_1$ for some $\eta \in [0, 2 \pi)$. In particular, there exists an open neighborhood $U \subseteq \mathbb{R}^{2N}$ containing $P_1$ such that $U \cap (E^\circ \oplus E^s) \subseteq W$. 

Proof. The center manifold theorem and Theorem 5.7 implies that there exists a \(2N - 3\) dimensional stable manifold \(W^s\) for (4.6) in some neighbourhood of \(P_1\).

First, we claim that \(W^s\) is orthogonal to \(E^u\). This means that \(W^s\) is a hypersurface of \(E^c \mathbin{\bigoplus} E^s\). In fact, let \(x_0 \in W^s\) and \(x(\tau)\) be the trajectories of (4.6) starting from \(x_0\). It is clear that the center of mass \(q = \sum_{i=0}^{N-1} x_i / N\) satisfies the following evolution:

\[
\frac{dq}{d\tau} = -\lambda_1 q.
\]

So if \(q(0)\) is not zero, eventually \(q(\tau)\) must go to infinity and so it cannot converge to \(P_1\). This gives \(x_0 \in (E^u)^\perp = E^c \mathbin{\bigoplus} E^s\).

The rotational invariance of (4.6) implies that the stable manifold associates to \(e^{i\eta}P_1\) is \(e^{i\eta}W^s\) for any \(\eta \in [0, 2\pi)\). Moreover, \(e^{i\eta_1}W^s \cap e^{i\eta_2}W^s = \emptyset\) if \(\eta_1 \neq \eta_2\).

Now, consider the disjoint union

\[
W := \prod_{\eta \in [0, 2\pi)} e^{i\eta}W^s.
\]

\(W\) is a \(2N - 2\) dimensional manifold since it homeomorphic to \(W^s \times S^1\). Moreover, \(W \perp E^u\) since \(e^{i\eta}W^s \perp E^u\) for any \(\eta\). For any \(x_0 \in W\), there exists a \(\eta \in [0, 2\pi)\) such that \(x_0 \in e^{i\eta}W^s\), and therefore, the corresponding trajectories must converge to \(e^{i\eta}P_1\).

Collecting these results, we can prove Theorem 1.3.

Proof of Theorem 1.4. Let \(X_0 \in \mathbb{C}^N\) be a regular \(N\)-gon, i.e., there exists a scaling \(c\) and a Euclidean isometry \(L\) such that \(X_0 = cLP_1\), where \(P_1\) is the regular \(N\)-gon defined in (4.3). Lemma 2.4 and Theorem 2.8 show that \(X_0\) and \(P_1\) behave in a similar way under the evolution (1.1). Therefore, without loss of generality, we assume \(X_0 = P_1\).

For any \(\epsilon > 0\) and perturbations \(F \in \mathbb{C}^N\), we decompose \(F\) into two parts and write \(F = T + F^\perp\), for some \(T \in E^u\) and \(F^\perp \in E^c \mathbin{\bigoplus} E^s\). Let \(X_0^\epsilon = X_0 + \epsilon F = P_1 + T_\epsilon + \epsilon F^\perp\), where \(T_\epsilon = \epsilon T\). Let \(X^\epsilon(t)\) and \(\tilde{X}^\epsilon(t)\) be the solution of (1.1) with \(X^\epsilon(0) = X_0^\epsilon\) and \(\tilde{X}^\epsilon(0) = P_1 + \epsilon F^\perp\), respectively. Since the system (1.1) is invariant under translations, Theorem 2.8 implies that \(X^\epsilon(t) = \tilde{X}^\epsilon(t) + T_\epsilon\) for all \(t > 0\). Since \(P_1 + \epsilon F^\perp \in E^c \mathbin{\bigoplus} E^s\), Theorem 5.8 implies that \(P_1 + \epsilon F^\perp \in W\) for sufficiently small \(\epsilon\). Therefore, the solution \(\tilde{Y}^\epsilon(t)\) of (4.6) with \(\tilde{Y}^\epsilon(0) = \tilde{X}^\epsilon(0)\) converges to the regular \(N\)-gon \(e^{i\eta}P_1\) for some \(\eta \in [0, 2\pi)\). Moreover, from the construction of (4.6), we know that \(a(t)\tilde{Y}^\epsilon(t)\) is a solution of (1.1), where \(a(t)\) is the scaling function we derived in Lemma 4.2.

Since \(a(0)\tilde{Y}^\epsilon(0) = 1\cdot \tilde{Y}^\epsilon(0) = \tilde{X}^\epsilon(0)\), by Theorem 2.8 we have \(\tilde{X}^\epsilon(t) = a(t)\tilde{Y}^\epsilon(t)\) for all \(t > 0\). Since \(\tilde{Y}^\epsilon(t) \to e^{i\eta}P_1\) as \(t \to \infty\), \(\lim_{t \to \infty} a(t) = 0\), and \(X^\epsilon(t) = \tilde{X}^\epsilon(t) + T_\epsilon\), it follows that \(X^\epsilon(t)\) shrinks to a point as \(t \to \infty\), and the limiting shape is a regular polygon.

5.3. Quadrilaterals. In this section, we prove the weaker stability result for quadrilaterals.

Proof of Theorem 1.4. Let \(X_0 \in \mathbb{C}^4\) be a square. By the invariance property, Lemma 2.4, we can assume \(X_0 = P_1\), where \(P_1\) is the regular square defined in (4.3). Similar to Theorem 5.8, we construct a 5-dimensional semi-stable manifold \(W\) orthogonal to \(E^u\) such that for any \(x_0 \in W\), the trajectory \(x(t)\) of (4.6) with \(x(0) = x_0\) converges to the square \(e^{i\eta}P_1\) for some \(\eta \in [0, 2\pi)\).

Consider the disjoint union

\[
W' = \bigcup_{x \in E^u} (x + W).
\]

We see that \(W'\) is a 7-dimensional hypersurface of \(\mathbb{R}^8\) since it homeomorphic to \(\mathbb{R}^2 \times W\). Similar to the argument in Theorem 1.3, the results follows.
We see that $N = 4$ is exceptional, since around any regular square $P_1$, there exists a non-regular one-parameter family of self-similar rhombus solutions of the form $P_1 + \epsilon P_1$. See Example 2.11. Note that these are affinely-regular and equilateral, but not regular. In case of larger $N$, there do not exist affinely-regular but non-regular equilateral polygons.

The main obstacle to a complete result for quadrilaterals is a clear description of the linearization of the rescaled flow around a rhombus.

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