The role of non-normality for control energy reduction in network controllability problems

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Abstract—This paper investigates the problem of controlling a complex network with reduced control energy. Two centrality measures are defined, one related to the energy that a control, placed on a node, can exert on the entire network, the other related to the energy that all other nodes exert on a node. We show that by combining these two centrality measures, conflicting control energy requirements, like minimizing the average energy needed to steer the state in any direction and the energy needed for the worst direction, can be simultaneously taken into account. From an algebraic point of view, the node ranking that we obtain from the combination of our centrality measures is related to the non-normality of the adjacency matrix of the graph.

I. INTRODUCTION

Complex networks appear in a broad spectrum of scientific disciplines, ranging from Biology to Social Sciences, from Technology to Engineering. Controlling a network refers to the possibility of steering the state variables associated with its nodes in order to accomplish a desired task. Depending on the context, many are the possible ways to define control inputs on networks, from traffic lights in traffic networks (Pernestål Brenden et al., 2017) to drugs in biological networks (Torres and Altafini, 2016), from dams in irrigation networks (Mareels et al., 2005) to opinion makers in social networks, etc. In all cases, it becomes interesting to explore what such control authority can do to the network state and at what cost. A starting point in this investigation is to rely on basic concepts from controllability theory (Rugh, 1996; Sontag, 2013). In fact, several of the classical notions of controllability developed in the last 50 years in Control theory have been recently adapted to the context of complex networks, like “exact” Kalman controllability (Yuan et al., 2013), structural controllability (Lin, 1974; Dion et al., 2003; Liu et al., 2011), or positive controllability (Lindmark and Altafini, 2017).

The field of complex networks is however also raising new challenges, which do not find a straightforward solution in the classical literature. One problem that has been considered for instance in Olshevsky (2014), Liu et al. (2011), Commault and Dion (2013); Pequito et al. (2016) is how to determine a minimal number of inputs that render a given network controllable. A common assumption in this context is that each control input only acts on one node which is then called a driver node. The notion of structural controllability (Lin, 1974; Dion et al., 2003) has proven to be very useful in this setting. Topological properties of the graph representing a network are directly mapped to the choice of certain nodes as driver nodes, and a minimal number of driver nodes can be obtained in this way, at least heuristically. It turns out that in many cases, for instance when a network is strongly connected, this minimal number of driver nodes is very small. However, even if a network is theoretically controllable in a structural sense, it is often not controllable in practice. It could for instance be the case that completely unreasonable amounts of control energy are required to steer the network in some directions (Yan et al., 2012; Pasqualetti et al., 2014). To handle such problems, several approaches to formulate a more “practical” degree of controllability have been proposed (Bof et al., 2017; Chen et al., 2016; Li et al., 2016; Nacher and Akutsu, 2014; Olshevsky, 2016; Pasqualetti et al., 2014; Summers et al., 2016; Tzoumas et al., 2016; Yan et al., 2012, 2015). These are normally based on the controllability Gramian and on energy considerations. An obvious question is for instance what are the driver nodes that results in the lowest possible control energy needed to steer a system. Although several different approaches have been tried, this question is far from solved. It is for instance formulated as an optimization problem in Summers et al. (2016), Tzoumas et al. (2016). Since the original problem is NP-complete, only simplified versions of it can be solved for large-scale networks. Another way to approach the problem is to quantify the importance of the different nodes for controllability with network centrality measures (Pasqualetti et al., 2014; Bof et al., 2017).

In Lindmark and Altafini (2018), we showed numerically that there are at least two factors influencing the energy required to control a network. One is the location of the eigenvalues (fast modes require more control energy than slow modes, see also Yan et al., 2015)). The second is instead a connectivity property, which in Lindmark and Altafini (2018) was expressed as a ratio between the weighted outdegree of the nodes and their weighted indegree. We showed that the higher this ratio is, the lower is the control energy needed to steer the system. The ratio can be used to rank the nodes, and therefore as a driver node placement strategy. Inspired by these empirical results, in this paper we investigate the ranking from a more theoretical perspective. Our main result is to suggest a combination of two different network centrality measures as a basis for the placement of driver nodes. The first measure quantifies the influence that each node has on the rest of the network. It corresponds to the energy with which the node excites the network. This centrality is high for nodes that efficiently reduce the average energy needed to steer the network in all different directions, as captured by the trace of the Gramian matrix. The second centrality describes instead the ability to control a node indirectly from the other nodes, and corresponds to the energy that reaches
the node from the other nodes. This centrality is more related to structural controllability, which identifies the nodes that cannot be controlled indirectly and hence must be driver nodes. Choosing nodes that are hard to control indirectly as driver nodes limits the energy that is required for steering the network in the most difficult directions, i.e., it limits the smallest eigenvalue of the Gramian matrix.

The two centrality measures reflect the fact that nodes in directed networks can be important for different reasons. An approach similar to the one adopted here appears for instance in the HITS algorithm [Kleinberg, 1999], where node ranking occurs according to two centrality measures, hubness and authority. As we will see in the paper, there is more than a passing resemblance between our choice and that of the HITS algorithm.

In the paper, we combine the two centralities into two different node rankings, which can be used to select the best driver nodes. The first ranking can be interpreted as the non-normality of the adjacency matrix of the network. For non-normal networks, we exploit the fact that some nodes are net contributors of energy while others are net receivers. By choosing the best net contributors as driver nodes, we achieve good performances both in terms of the average energy that is required to steer the network and in terms of the energy required to steer it in the most difficult direction. The second ranking gives a higher weight to the nodes’ ability to be indirectly controlled, and it has more in common with structural controllability. Driver node placement based on this ranking gives good performances especially for control in the worst case direction.

The rest of the paper is organized as follows: In Section II definitions are given, results on controllability are revised and different energy-related metrics are discussed. In Section III the network centralities and their combination into node rankings are derived. Section IV presents a simulation study that examines the control energy properties when the rankings are used for driver node placement.

II. BACKGROUND

A. Notation

We denote by $\mathbb{R}^{n \times m}$ the set of $n \times m$ matrices with real valued entries. The k-th vector of the canonical basis of $\mathbb{R}^n$ is denoted $e_k$, $k = 1, \ldots, n$. For the vector $v \in \mathbb{R}^n$, $\|v\| = \sqrt{v^T v}$ is its Euclidean norm. The symbol $S_n^+$ is used for the set of symmetric positive semidefinite $n \times n$ matrices. Given a matrix $M \in \mathbb{R}^{n \times m}$, let $M[k] = Me_k$, $k = 1, \ldots, m$. Let $M = [M_i]$, $i = 1, \ldots, n$, denote the i-th row of $M$. Given the two matrices $M, N \in \mathbb{R}^{n \times n}$, $[M, N] = MN - NM$ is the matrix commutator. The definition of the delta function is $\delta(t) = 1$ for $t = 0$ and $\delta(t) = 0$ for $t \neq 0$.

A (directed) graph $G$ is indicated by the pair of its nodes and edges, $V = \{v_1, \ldots, v_n\}$ and $E = \{(v_i, v_j), i, j \in \{1, \ldots, n\}\}$, or, if it is necessary to specify the edge weights, by the adjacency matrix $A$, i.e., $G = G(A)$. Then the weight associated with the edge from $v_i$ to $v_j$, $(v_i, v_j)$, is $A_{ij}$. A path $P$ in $G$ is defined as an ordered sequence of nodes $P = (v_{i_1}, \ldots, v_{i_j})$ with $\{v_{i_k}, v_{i_{k+1}}\} \in E$ and $(v_{i_k}, v_{i_{k+1}}) \in E$ for all $k = 1, \ldots, j - 1$. The nodes $v_{i_1}, \ldots, v_{i_j}$ need not to be distinct. In particular, a path including a self-loop has $v_{i_k} = v_{i_{k+1}}$ for some $k \in 1, \ldots, j - 1$. The weight of the path is the product of the weights of its edges, $\prod_{k=1}^{j-1} A_{i_k+1,i_k}$. The node $v_i$ in $G$ is a root if it has no incoming edge and a leaf if it has no outgoing edge. A node $v_i \in V$ is part of a cycle if there is a path that starts and ends at $v_i$.

B. Network model

We consider the following discrete-time linear time invariant model for the network

$$x(t + 1) = Ax(t) + Bu(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state at time $t \in \mathbb{N}_0$, $A \in \mathbb{R}^{n \times n}$, $B_{\mathcal{K}} = [b_{k_1}, \ldots, b_{k_m}] \in \mathbb{R}^{n \times m}$ and $u(t) \in \mathbb{R}^m$. We represent the network with the directed graph $G(A) = (V, E)$. Each control input is assumed to act on only one node which is then called a driver node. The set of driver nodes is $\mathcal{K} = \{v_{k_1}, \ldots, v_{k_m}\} \subseteq V$.

C. Controllability

Definition II.1 (Controllability). [Rugh, 1996] The system (1) is controllable in $[0, T]$ if, for every initial state $x_0 \in \mathbb{R}^n$ and final state $x_f \in \mathbb{R}^n$ there exists an input signal $\{u(t), t = 0, \ldots, T\}$ that steers the state $x(0) = x_0$ to $x(T) = x_f$.

The system (1) is controllable if and only if the controllability matrix

$$C_{\mathcal{K}, T} = [B_{\mathcal{K}} \ AB_{\mathcal{K}} \ \ldots \ \ A^{T-1}B_{\mathcal{K}}]$$

has full row rank. Unlike the continuous case, controllability can fail due to the size of the time interval $T$. The different metrics for the control energy that are suggested in the literature are normally based on the controllability Gramian

$$W_T = C_{\mathcal{K}, T} C_{\mathcal{K}, T}^T = \sum_{t=0}^{T-1} A^t B_{\mathcal{K}} B_{\mathcal{K}}^T (A^T)^t \in S_n^+. \quad (2)$$

The controllability Gramian is positive definite if and only if the system (1) is controllable. For $A$ stable the controllability Gramian converges as $T \to \infty$ to the solution of

$$AW_\infty A^T - W_\infty + B_{\mathcal{K}} B_{\mathcal{K}}^T = 0. \quad (3)$$

We omit the subscript $T$ (or $\infty$) in the following. The reason why most metrics for quantifying the control energy are based on the controllability Gramian is that if the input energy is

$$\mathcal{E}(u) = \sum_{t=0}^{T-1} ||u(t)||^2,$$

then for a controllable network, the unique control input that steers the network from $x(0) = 0$ to $x(T) = x_f$ with minimum energy is [Rugh, 1996]

$$u^*(t) = B_{\mathcal{K}}^T (A^T)^{-t-1} W^{-1} x_f,$$

and

$$\mathcal{E}(u^*) = x_f^T W^{-1} x_f.$$
If $x_f$ is a unit-length eigenvector of $W$, then the minimum energy equals $x_f^T W^{-1} x_f = 1/\lambda$, where $\lambda$ is the eigenvalue associated with $x_f$. The following two scalar metrics for the degree of controllability are among the most commonly used:

i) $\lambda_{\min}(W)$: The energy required to steer the system in the worst case direction is $1/\lambda_{\min}(W)$.

ii) $\text{Tr}(W)$: the trace of the Gramian is inversely proportional to the average energy required to control a system, hence when $\text{Tr}(W)$ increases the control energy decreases. However, this metric leaves no guarantees; $W$ may even be singular (and the network not controllable) although $\text{Tr}(W)$ is high.

See for instance (Müller and Weber, 1972; Summers et al., 2016) for a thorough description of these and other metrics for the practical degree of controllability.

III. DRIVER NODE PLACEMENT

A. Network impact of a driver node

Here we introduce a metric for quantifying the network impact of a driver node, i.e. to what extent it influences the other nodes and can be used to steer them. The metric is based on

$$W^{(i)} = \sum_{t=0}^{T-1} A^t e_i e_i^T (A^T)^t, \quad i = 1, \ldots, n,$$

i.e. the Gramian when $v_i$ is the only driver node.

**Proposition 1.** With the driver nodes $\mathcal{K} = \{v_{k_1}, \ldots, v_{k_m}\}$, it is

$$W = \sum_{i=k_1, \ldots, k_m} W^{(i)}.$$

**Proof.**

$$W = \sum_{t=0}^{T-1} A^t B_K B_K^T (A^T)^t$$

$$= \sum_{t=0}^{T-1} A^t [e_{k_1} \ldots e_{k_m}] [e_{k_1} \ldots e_{k_m}]^T (A^T)^t$$

$$= \sum_{t=0}^{T-1} A^t \left( \sum_{i=k_1, \ldots, k_m} e_i e_i^T \right) (A^T)^t$$

$$= \sum_{i=k_1, \ldots, k_m} \sum_{t=0}^{T-1} A^t e_i e_i^T (A^T)^t$$

$$= \sum_{i=k_1, \ldots, k_m} W^{(i)}.$$

According to Proposition 1 the controllability Gramian is the sum of the controllability Gramians associated with the individual driver nodes.

**Proposition 2.** The element $(A^t)_{ji}$ is the sum of the weights of all length-$t$ paths from $v_i$ to $v_j$, $t \in \mathbb{N}_{>0}$.

**Proof.** Each length $t+1$ path $(v_i, \ldots, v_k, v_j)$ is a length $t$ path $(v_i, \ldots, v_k)$ extended with the edge $(v_k, v_j)$, and the weight of the path $(v_i, \ldots, v_k, v_j)$ is the product of the weight of the path $(v_i, \ldots, v_k)$ and $A_{jk}$. The proposition holds trivially for $t = 1$ since $(A^1)_{ji}$ is the weight of the edge $(v_i, v_j)$. Assume that the proposition holds for a given $t$. Then,

$$(A^{t+1})_{ji} = \sum_{k=1, \ldots, n} A_{jk} (A^t)_{ki}$$

i.e. the sum of the weights of all length-$t$ paths from $v_i$ to all $v_k, k = 1, \ldots, n$, multiplied with the weight of the edge $(v_k, v_j) \in \mathcal{E}$. But this is the sum of the weights of all length $t+1$ paths from $v_i$ to $v_j$, hence the proposition holds also for $t+1$, and by induction for all $t \in \mathbb{N}_{>0}$.

From (1), when $x(0) = 0$ the propagation in the system of a unit pulse input on driver node $v_i \in \mathcal{V}$, i.e. $B_K u(t) = e_i \delta(t)$, is $x(t) = A^t e_i$. The resulting state at node $v_j \in \mathcal{V}$ is $x_j(t) = e_j^T A^t e_i = e_j^T (A^t)_{ji}$, i.e. the sum of all length-$t$ paths from $v_i$ to $v_j$. Let $\varepsilon_{i\rightarrow j}$ be the energy with which such control input excites the node $v_j$ over time,

$$\varepsilon_{i\rightarrow j} = \sum_{t=0}^{T-1} x_j(t)^2 = \sum_{t=0}^{T-1} (A^t)_{ji}^2$$

$$= \sum_{t=0}^{T-1} e_j^T A^t e_i (e_j^T A^t e_i)^T = e_j^T \left( \sum_{t=0}^{T-1} A^t e_i e_i^T (A^T)^t \right) e_j$$

$$= W^{(i)}_{jj}. \quad (5)$$

The energy $\varepsilon_{i\rightarrow j}$ is directly connected to the weights of all the paths from $v_i$ to $v_j$ of length up to $T$, and this information is encoded in the Gramian $W^{(i)}$. The quantity $\varepsilon_{i\rightarrow j}$ is also the variance (or power) of the state $x_j$ when $u(t)$ is a white noise process with expected value 0 and variance 1 applied to node $v_i$ (Sontag, 2013). In fact, using

$$x(T) = \sum_{t=0}^{T-1} A^{T-1-t} B_K u(t) \quad \text{with}$$

$$E\{u(t)u(\tau)\} = \begin{cases} 1 & t = \tau \\ 0 & t \neq \tau \end{cases}$$

the covariance matrix of the states coincide with the controllability Gramian,

$$E\{x(T)x(T)^T\} = E\left\{ \sum_{t=0}^{T-1} A^{T-1-t} B_K u(t) \left( \sum_{\tau=0}^{T-1} A^{T-1-\tau} B_K u(\tau) \right)^T \right\}$$

$$= \sum_{t=0}^{T-1} A^{T-1-t} B_K B_K^T (A^{T-1-t})^T = W.$$

With $\mathcal{K} = v_i$, the variance/power of $x_j(T)$ is the diagonal element $W^{(i)}_{jj}$, i.e. our quantity $\varepsilon_{i\rightarrow j}$. In the following we refer to $\varepsilon_{i\rightarrow j}$ as the (normalized) energy flow from $v_i$ to $v_j$. Observe that $\varepsilon_{i\rightarrow j}$ is non-negative $\forall i, j$. 
Considering all the nodes of the network,

$$p_i = \sum_{j=1}^{n} \varepsilon_{i \rightarrow j} = \sum_{j=1}^{n} \sum_{t=0}^{T-1} (A^t)_{jj}^2$$  \hspace{1cm} (6)

$$= \sum_{j=1}^{n} W_{jj}^i = \text{Tr}(W^{(i)})$$  \hspace{1cm} (7)

is the energy of the whole network when $v_i$ is a driver node (the only one). We use $p_i$ as a metric of the network impact of $v_i$ as a driver node. Note that the off-diagonal entries $W_{jj}^i = \sum_{t=0}^{T-1} x_j(t) x_k(t)$, $j \neq k \in 1, \ldots, n$ quantify how the $x_j$ and $x_k$ covariate.

**Proposition 3.** The controllability measure $p_i$ is based only on paths of length $0, 1, \ldots, T - 1$, outgoing from $v_i$.

**Proof.** The claim follows from (6), as the element $(A^t)_{jj}$ is the sum of the weights of length-$t$ paths from $v_i$ to $v_j$.

**Proposition 4.** It holds that $p_i \geq 1$, $i = 1, \ldots, n$, with equality if and only if $v_i$ is a leaf.

**Proof.** The energy $\varepsilon_{i \rightarrow i}$ is

$$\varepsilon_{i \rightarrow i} = \sum_{t=0}^{T-1} (A^t)_{ii}^2 = 1 + \sum_{t=1}^{T-1} (A^t)_{ii}^2 \geq 1$$  \hspace{1cm} (8)

since $(A^0)_{ii} = I_{ii} = 1$. For $t > 0$, $(A^t)_{ii} \neq 0 \iff (A^2)_{ii} > 0$ when there is a length-$t$ path from $v_i$ to $v_i$, i.e. a cycle. Assume that $v_i$ is not part of any cycle, then

$$p_i = \sum_{j=1}^{n} \varepsilon_{i \rightarrow j} = \varepsilon_{i \rightarrow i} + \sum_{j \neq i} \varepsilon_{i \rightarrow j}$$

Hence $p_i = 1$ if and only if $\varepsilon_{i \rightarrow j} = 0 \forall j \neq i$ which is the case when $v_i$ has no outgoing edge, i.e. $v_i$ is a leaf.

Based on the $p$ centrality, a leaf node is the worst choice of driver node.

**Proposition 5.** Let $K = \{v_{k_1}, \ldots, v_{k_m}\}$ be the set of driver nodes that maximizes $\text{Tr}(W)$ for a given $m$. Then $p_i > p_j \forall i, j$ s.t. $v_i \in K$ and $v_j \in V \setminus K$.

**Proof.** The result follows directly from the linearity of the trace operator,

$$W = \sum_{i=k_1, \ldots, k_m} W^{(i)}$$

$$\Rightarrow \text{Tr}(W) = \sum_{i=k_1, \ldots, k_m} \text{Tr}(W^{(i)}) = \sum_{i=k_1, \ldots, k_m} p_i$$

Tr$(W)$ is maximized by $K$ the set of nodes with highest $p$.

**Proposition 6.** Let $v_i, v_j \in V$ be two nodes such that if $(v_i, v_k) \in E$, $v_k \in V$, then also $(v_j, v_k) \in E$ with identical weights $A_{ki} = A_{kj}$. Then $p_i = p_j$.

**Proof.** Given that the columns $A[i] = A[j]$,

$Ae_i = Ae_j \Rightarrow A^t e_i = A^t e_j \forall t \geq 1$,

$\Rightarrow A^t e_i e_i^T (A^T)^t = A^t e_j e_j^T (A^T)^t \forall t \geq 1$.

Used with (2), this gives that

$$W^{(i)} - W^{(j)} = \sum_{t=0}^{T-1} A^t e_i e_i^T (A^T)^t - \sum_{t=0}^{T-1} A^t e_j e_j^T (A^T)^t$$

$$= e_i e_i^T - e_j e_j^T$$

$$\Rightarrow \text{Tr}(W^{(i)}) - \text{Tr}(W^{(j)}) = 0$$

$\iff p_i = p_j$.

In [Summers et al. (2016)] the driver node placement problem is investigated using optimization techniques. Among other things, a greedy algorithm is proposed for maximizing $\text{Tr}(W)$, and in that context the centrality $p$ appear, referred to as the average controllability centrality. However, driver node placement based only on $p$ (which maximize $\text{Tr}(W)$) tends to perform badly when other metrics for the energy required for control are considered, i.e. $\lambda_{\text{min}}(W)$. See for instance [Pasqualetti et al. (2014)] and the simulation results in Section IV. This can be understood by considering for instance root nodes. A necessary but not sufficient condition for controllability is that all root nodes are driver nodes. In Proposition 6 let $v_i$ be a root and $v_j$ not a root. It holds that $p_i = p_j$, since their outgoing edges target the same nodes with the same weights. Incoming edges do not enter into the equations. Hence there is nothing in the $\text{Tr}(W)$ maximizing strategy that favors the root node $v_i$ over $v_j$, although controllability is never achieved unless $v_i$ is a driver node.

**B. Minimum energy control of a target node**

Here we present two results regarding the minimal amount of energy that is required when the objective is to control a single target node, and how the driver nodes can be placed optimally in that case. The results obtained here are related to the ability to control a node indirectly from other nodes in the next section.

**Theorem III.1.** The minimum energy required to steer the state $x_i(0) = 0$ to $x_i(T) = 1$ is $1/W_{ii}$, $i = 1, \ldots, n$.

**Proof.** The conditions in the theorem is the solution of the optimization problem

$\minimize_x x^T W^{-1} x$

subject to $e_i^T x = 1$, $i = 1, \ldots, n$.

This standard quadratic optimization problem can be solved with methods from optimal control theory [Sethi and Thompson (2006)]. Here we use the KKT optimality conditions to find the analytic solution. Define the Lagrangian

$$L(x, \nu) = x^T W^{-1} x + \nu(e_i^T x - 1),$$

where $\nu$ is the dual variable associated with the equality condition. The KKT conditions $\nabla L(x^*, \nu^*) = 0$ and $e_i^T x = 1$ give

$$\begin{bmatrix} 2W^{-1} & 0 & e_i^T \\ e_i^T & 0 & \nu^* \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\iff \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} 0 \\ \nu^* \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \ldots \end{bmatrix} = \begin{bmatrix} W_{ii} \nu^* \end{bmatrix}.$$
To steer the network to the optimal \( x^* = We_i/W_{ii} \) requires the minimal energy
\[
x^*W^{-1}x^* = \left( \frac{We_i}{W_{ii}} \right)^T W^{-1} \left( \frac{We_i}{W_{ii}} \right) = \frac{e_i^T W W^{-1} W e_i}{W_{ii}^2} = \frac{W_{ii}}{W_{ii}^2} = 1/W_{ii}.
\]

**Theorem III.2.** For a given \( m \) and \( i \in 1, \ldots, n \), the set of driver nodes that minimize the control energy required to steer the state \( x_i(0) = 0 \) to \( x_i(T) = 1 \) is given by the nodes with highest energy flow to \( v_i \).

**Proof.** According to Theorem III.1 the minimum energy of the problem is \( 1/W_{ii} \). Since
\[
W_{ii} = \sum_{k \in K} W_{ii}^{(k)} = \sum_{k \in K} e_{k \rightarrow i}
\]
this means that the set \( K \) that maximize \( W_{ii} \), and hence minimize \( 1/W_{ii} \), is given by the nodes with the highest energy flow to \( v_i \).

**C. Indirect control of a node**

We seek to quantify the ability to control a node from control inputs targeting other nodes. As already mentioned, a root node cannot be controlled indirectly at all, but one would also expect that nodes with few and weak incoming edges are more difficult to control indirectly than nodes with many and strong incoming edges.

To capture the ability to control a node indirectly we introduce the fictitious output equation
\[
y(t) = Cx(t), \quad (9)
\]
where \( y(t) \in \mathbb{R}^d \) is the output at time \( t \) and \( C \in \mathbb{R}^{d \times n} \). The observability Gramian is
\[
M_T = \sum_{t=0}^{T-1} (A^T)^t C^T CA^t \in S_+^n. \quad (10)
\]

For \( A \) stable, the observability Gramian converges as \( T \to \infty \) to the solution of
\[
A^T M_\infty A - M_\infty + C^T C = 0. \quad (11)
\]

The subscript \( T \) (or \( \infty \)) for the time-horizon is omitted in the following. From now on we use \( C^T = e_j, \quad j \in 1, \ldots, n \). In analogy with \( W_{ij} \), \( i = 1, \ldots, n \), introduce
\[
M^{(j)} = \sum_{t=0}^{T-1} (A^T)^t e_j e_j^T A^t, \quad j = 1, \ldots, n.
\]

**Proposition 7.** For \( v_i, v_j \in V \), the energy \( \varepsilon_{i \rightarrow j} = W_{jj} = M_{ii}^{(j)} \).

**Proof.** Since the scalar \( e_j^T A^t e_i = e_i^T (A^T)^t e_j \),
\[
\varepsilon_{i \rightarrow j} = W_{jj} = e_j^T \left( \sum_{t=0}^{T-1} A^t e_i e_i^T A^t \right) e_j = e_i^T \left( \sum_{t=0}^{T-1} (A^T)^t e_j e_j^T A^t \right) e_i = M_{ii}^{(j)}.
\]

The energy that goes into \( v_j \) from all other nodes is described by the entries of \( M^{(j)} \). We use this to construct a metric for the possibility to control \( v_j \) indirectly based on \( M^{(j)} \). When the whole network is considered,
\[
\tilde{q}_j = \sum_{i \neq j} \varepsilon_{i \rightarrow j} = \sum_{i \neq j} \sum_{t=0}^{T-1} (A^T)^t e_j e_j^T A^t = \sum_{i \neq j} M_{ii}^{(j)} \quad (12)
\]

is the energy flow to \( v_j \) from all other nodes. Hence, \( \tilde{q}_j \) is a metric for the ability to control \( v_j \) indirectly.

**Proposition 8.** The quantity \( \tilde{q}_j \) is based only on paths of length 0, 1, \ldots, \( T-1 \) incoming in \( v_j \).

**Proof.** The claim follows from (12), the element \( (A^t)_{ji} \) is the sum of the weights of length-\( t \) paths from \( v_i \) to \( v_j \).

We expect that a metric for the ability to control a node indirectly is the least for root nodes. Furthermore, it should be close to minimum for nodes with only few and weak incoming edges, i.e. “almost” root nodes. The quantity \( \tilde{q}_j \) meets these requirements.

**Proposition 9.** It holds that \( \tilde{q}_j \geq 0, \quad j \in 1, \ldots, n \) and \( \tilde{q}_j = 0 \) if and only if \( v_j \) is a root node.

**Proof.** From (12), if \( v_j \) has an incoming edge, then there is a node \( v_i \in V, \quad i \neq j \) such that
\[
(v_i, v_j) \in E \iff A_{ji} \neq 0 \iff A_{ji}^2 > 0 \iff \tilde{q}_j > 0.
\]

On the other hand, if there are no paths incoming to \( v_j \), then \( (A^t)_{ji} = 0, \quad \forall j \neq i, \quad t \geq 0 \), which gives \( \tilde{q}_j = 0 \).

The following proposition relates \( \tilde{q}_j \) to \( \lambda_{\min}(W) \), the worst case energy required for control of the network. Following that, Theorem III.3 relates \( \lambda_{\min}(W) \) to the number of driver nodes, \( m \).

**Proposition 10.** If \( v_j \in V \setminus K \), then \( \lambda_{\min}(W) \leq \tilde{q}_j \).

**Proof.** Since \( W \) is symmetric,
\[
\lambda_{\min}(W) = \min_{\|x\|=1} x^T W x \leq e_j^T W e_j
\]
\[
= e_j^T \left( \sum_{i=k_1, \ldots, k_m} W^{(i)} \right) e_j = \sum_{i=k_1, \ldots, k_m} W_{jj}^{(i)}
\]
\[
= \sum_{i=k_1, \ldots, k_m} \tilde{e}_{i \rightarrow j} \leq \sum_{i \neq j} \varepsilon_{i \rightarrow j} = \tilde{q}_j.
\]

**Theorem III.3.** Let the indices \( j_1, \ldots, j_m \) be such that \( \tilde{q}_{j_1} \leq \tilde{q}_{j_2} \leq \cdots \leq \tilde{q}_{j_m} \) and \( |K| = m < n \). Then \( \lambda_{\min}(W) \leq \tilde{q}_{j_{m+1}} \).
Proof. Assuming that \( v_{j_{m+1}} \) is not directly controlled, the result follows by Proposition \[10\] On the other hand, if \( v_{j_{m+1}} \in \mathcal{K} \), then \( \exists \ d \in 1, \ldots, m \ s.t. \ v_{j_d} \notin \mathcal{K} \) and by the same proposition \( \lambda_{\min}(W) \leq \hat{q}_{j_d} \leq q_{j_{m+1}}. \) \( \square \)

These results show that the nodes with the lowest \( \hat{q} \) give a direct upper bound on \( \lambda_{\min}(W) \), i.e. a lower bound on the energy required to control the network in the most difficult direction. Other bounds on \( \lambda_{\min}(W) \) are presented for instance in [Pasqualetti et al. 2014] and [Boil et al. 2017] but their derivations are quite different from ours. An interesting application of the bound given in Theorem \[13\] will be shown in Section \[IV\] where we study the distribution of \( \hat{q}_j, \ j = 1, \ldots, n \) for random networks.

Instead of \( \hat{q}_j \) we may use the closely related
\[
q_j = \sum_{i=1}^{n} \epsilon_{i\rightarrow j} = \sum_{i=1}^{n} \sum_{t=0}^{T-1} (A^t)_{ji}^2 = \sum_{i=1}^{n} M_{ji}^{(t)} = \text{Tr}(M_{ji})
\]
(13)
as a metric for the ability to control \( v_j \) indirectly. Just like \( \hat{q}_j \), the metric \( q_j \) is also based only on paths incoming to \( v_j \).

**Proposition 11.** It holds that \( q_j \geq 1, \ j = 1, \ldots, n, \) with equality if and only if \( v_j \) is a root.

**Proof.** From the definitions of \( q_j \) and \( \hat{q}_j \) we have that
\[
q_j = \hat{q}_j + \epsilon_{j\rightarrow j} \geq 1,
\]
(14)
since \( \hat{q}_j \geq 0 \) and \( \epsilon_{j\rightarrow j} \geq 1 \) (equation \[3\]). Furthermore, \( \hat{q}_j = 0 \) if and only if \( v_j \) is a root and \( \epsilon_{j\rightarrow j} = 1 \) when \( v_j \) is not part of any cycle. \( \square \)

The quantity \( q_j \) has a dual interpretation as compared to \( p_i \). While \( p_i \) is the total energy flow from \( v_i \) to all nodes in the network, \( q_i \) is the total energy flow from all nodes to \( v_i \). The computational cost for \( p_i \) and \( q_i, \ i = 1, \ldots, n, \) is low.

**Proposition 12.** Let \( W^{(1)} \) be the controllability Gramian when \( B_K = I \) and \( M^{(1)} \) the observability Gramian when \( C^\top = I \). Then \( \text{Tr}(W^{(1)}) = \text{Tr}(M^{(1)}) = W_{ii}^{(1)} \).

**Proof.**
\[
\text{Tr}(W^{(1)}) = \text{Tr} \left( \sum_{t=0}^{T-1} A^t e_i e_i^\top (A^\top)^t \right) = \sum_{t=0}^{T-1} \text{Tr}(A^t e_i e_i^\top (A^\top)^t) = \sum_{t=0}^{T-1} \text{Tr}(A^t e_i e_i^\top) = \sum_{t=0}^{T-1} e_i^\top (A^t)^t e_i = M_{ii}^{(1)}
\]
(15)
\[
\text{Tr}(M^{(1)}) = W_{ii}^{(1)} \text{ is shown in the same way.} \square
\]

When an infinite time-horizon is considered (assuming \( A \) stable), it is sufficient to solve the two Lyapunov equations \[3\] and \[11\] for \( W^{(1)} \) and \( M^{(1)} \) in order to compute \( p_i \) and \( q_i, \ i = 1, \ldots, n, \) For limited time-horizon, equations \[2\] and \[10\] must be solved instead. The difficulty to control nodes for which \( q_i \) is low can also be seen from Theorem \[11\] by which the energy required to steer the target state \( x_t \) is proportional to \( 1/W_{ii} \). But \( q_i = W_{ii}^{(t)} \) according to Proposition \[12\] hence \( q_i \) is inversely proportional to the control energy when \( B_K = I \).

**D. Ranking the nodes**
We use \( p_i \) and \( q_i, \ i = 1, \ldots, n, \) as network centrality metrics to rank the nodes for driver node placement. The HITS algorithm ([Kleiber 1999]) also assigns two different metrics of centrality to each node in a directed network: the hub centrality related to the outgoing edges, and the authority centrality to the incoming edges. Both centralities are needed since a node can play an important role in a network as either a hub or as an authority. Although the original HITS algorithm does not consider weighted graphs and the context is different, there are strong similarities with the present problem setting.

A ranking of the nodes can be constructed by combining the network centralities \( p \) and \( q \). Nodes with high \( p \) and low \( q \) should be preferred. Interestingly, when \( AA^\top = A^\top A \), i.e. \( A \) is a normal matrix, then
\[
p_i = M_{ii}^{(1)} e_i^\top \left( \sum_{t=0}^{T-1} (A^t)^t A^t \right) e_i = e_i^\top \left( \sum_{t=0}^{T-1} A^t (A^\top)^t \right) e_i = W_{ii}^{(1)} = q_i, \ i = 1, \ldots, n.
\]
That is the case with for instance undirected networks. This means that the nodes that are most suitable as driver nodes considering the \( p \) centrality are the worst nodes considering the \( q \) centrality. But when \( A \) is non-normal we should use the fact that \( p_i \) and \( q_i \) differ when selecting driver nodes.

The first ranking we consider is
\[
r_{\text{diff},i} = p_i - q_i = \sum_{j=1}^{n} (\epsilon_{i\rightarrow j} - \epsilon_{j\rightarrow i}), \ i = 1, \ldots, n. \ (16)
\]
The difference \( r_{\text{diff},i} \) is the net energy flow from \( v_i \) to all other nodes. A node has a positive net contribution when the weights of its outgoing paths are higher than the weights of its incoming paths. Notice that \( \sum_{i=1}^{n} r_{\text{diff},i} = 0 \), meaning that if some nodes are net energy flow contributors then others must be net receivers.

For the two sets of nodes \( S_1 \subseteq \mathcal{V} \) and \( S_2 \subseteq \mathcal{V} \), define the net flow of energy from \( S_1 \) to \( S_2 \)
\[
\Delta \varepsilon_{S_1 \rightarrow S_2} = \sum_{i \ s.t. \ v_i \in S_1} \sum_{j \ s.t. \ v_j \in S_2} (\epsilon_{i\rightarrow j} - \epsilon_{j\rightarrow i}).
\]

**Proposition 13.** The maximal net flow of energy
\[
\max_{K} \Delta \varepsilon_{K \rightarrow \mathcal{V} \setminus K} \ s.t. \ |K| = m,
\]
is achieved when \( K \) is constructed as the set of the \( m \) nodes with the highest \( r_{\text{diff}} \).

**Proof.** Since \( \Delta \varepsilon_{K \rightarrow \mathcal{V} \setminus K} = \sum_{i \ s.t. \ v_i \in K} \sum_{j \ s.t. \ v_j \notin K} (\epsilon_{i\rightarrow j} - \epsilon_{j\rightarrow i}) = 0, \)
\[
\Delta \varepsilon_{K \rightarrow \mathcal{V} \setminus K} = \sum_{i \ s.t. \ v_i \in K} \sum_{j \ s.t. \ v_j \notin K} (\epsilon_{i\rightarrow j} - \epsilon_{j\rightarrow i}) = \sum_{i \ s.t. \ v_i \in K} \sum_{j=1}^{n} (\epsilon_{i\rightarrow j} - \epsilon_{j\rightarrow i}) = \sum_{i \ s.t. \ v_i \in K} r_{\text{diff},i}.
\]
Hence, the function
\[ f(m) = \max_{\mathcal{K}} \Delta \varepsilon_{\mathcal{K} \rightarrow V \setminus \mathcal{K}} \text{ s.t. } |\mathcal{K}| = m \]
is the cumulative sum of \( r_{\text{diff},i}, i = 1, \ldots, n \), sorted in descending order, and the proposition follows.

The next proposition relates \( r_{\text{diff},i} \) to the non-normality of the network.

**Proposition 14.** It holds that
\[ r_{\text{diff},i} = \sum_{t=0}^{T-1} [(A^T)^t, A^t]_{ii}, i = 1, \ldots, n, \]
where \([(A^T)^t, A^t]\) is the matrix commutator of the length- \( t \) path adjacency matrices.

**Proof.** From the definition of \( r_{\text{diff},i} \),
\[
 r_{\text{diff},i} = p_i - q_i = \text{Tr}(W^{(i)}) - \text{Tr}(M^{(i)}) = \\
 = \text{Tr} \left( \sum_{t=0}^{T-1} A^t e_i e_i^T (A^T)^t \right) - \text{Tr} \left( \sum_{t=0}^{T-1} (A^T)^t e_i e_i^T A^t \right) \\
 = e_i^T \left( \sum_{t=0}^{T-1} (A^T)^t A^t - A^t (A^T)^t \right) e_i \\
 = e_i^T \sum_{t=0}^{T-1} [(A^T)^t, A^t] e_i = \sum_{t=0}^{T-1} [(A^T)^t, A^t]_{ii}. 
\]

Proposition 14 shows that choosing driver nodes based on \( r_{\text{diff},i} \) can be interpreted on the basis of non-normality. The matrix commutator can in fact be used to quantify non-normality (Trefethen and Embree, 2005), and \( r_{\text{diff},i} \) exploits the non-normality of the network for ranking the nodes.

We shall also consider the ranking
\[
r_{\text{quot},i} = \frac{p_i}{q_i} = \sum_{j=1}^{n} \frac{\varepsilon_{i \rightarrow j}}{\sum_{j=1}^{n} \varepsilon_{i \rightarrow j}}, i = 1, \ldots, n. \tag{17}
\]
If \( v_i \) is a net contributor of energy, then \( r_{\text{quot},i} > 1 \). Motivated by Theorem 13, this ranking puts more emphasis on the possibility to control the nodes indirectly. The ranking has more in common with structural controllability (Lin, 1974) considering that root nodes and “almost root nodes” obtain a particularly high ranking due to the small denominator \( q_i \).

Figure 1 shows the centralities \( p \) and \( q \) and the rankings \( r_{\text{diff}} \) and \( r_{\text{quot}} \) for a small network example.

**IV. SIMULATIONS**

In this section we examine the energy that is required to control random networks when \( r_{\text{diff}} \) and \( r_{\text{quot}} \) are used for driver node placement. For comparison, we also compute the different controllability metrics for a random driver node placement and for the placement of driver nodes that maximize \( \text{Tr}(W) \). Two types of networks are studied: Erdős-Rényi networks and directed scale-free networks. The edge weights are sampled from a normal distribution in both cases, and stability is ensured by rescaling \( A \) such that the predefined spectral radius \( \rho = 0.9 \) is obtained (hence all eigenvalues of \( A \) are within the unit disc).

The generated Erdős-Rényi networks have 500 nodes and edge probability 0.01. The controllability results for them are presented in Figures 2 and 3. Qualitatively, the results correspond to the observations in the example in Figure 1. In comparison with randomly placed diver nodes, both \( \text{Tr}(W) \) and \( \lambda_{\text{min}}(W) \) improve when the driver nodes are placed according to \( r_{\text{diff}} \) or \( r_{\text{quot}} \). The ranking \( r_{\text{quot}} \) performs best when \( \lambda_{\text{min}}(W) \) is considered, while \( r_{\text{diff}} \) is better when \( \text{Tr}(W) \) is chosen as criterion. Note also that the driver nodes that maximize \( \text{Tr}(W) \) result in values of \( \lambda_{\text{min}}(W) \) which are approximately the same as for a random choice of driver nodes. Figure 3 shows all the eigenvalues of \( W \) (in increasing order) for \( m = 150 \). Here \( \lambda_{\text{min}}(W) \) is the leftmost eigenvalue of each curve. Since \( \text{Tr}(W) = \sum_{i=1}^{n} \lambda_i(W) \), if the plots were in linear scales then \( \text{Tr}(W) \) for each ranking would be proportional to the area under the corresponding curve. It can be noticed that \( r_{\text{quot}} \) is better for the lower eigenvalues and \( r_{\text{diff}} \) better for the higher eigenvalues. The metrics \( p \), \( q \) and \( \bar{q} \) are shown in Figure 4.

In directed scale-free networks, the indegree distribution and the outdegree distribution follow power laws. By choosing these in a suitable way, we can obtain random networks with larger variations in the network impact of the different nodes and in the possibility to control them indirectly. We use the directed scale-free network model suggested by Bollobás et al.
to generate random networks with 500 nodes, indegree distribution $P_{in}(k_{in}) \propto k_{in}^{-3.14}$, and outdegree distribution $P_{out}(k_{out}) \propto k_{out}^{-2.88}$. Furthermore, problems with minimal controllability are avoided by the addition of self-loops and edges that guarantee strong connectivity. Their controllability results are presented in Figures 5 and 6. Driver node placements based on $r_{\text{diff}}$ and $r_{\text{quot}}$ improve both $\text{Tr}(W)$ and $\lambda_{\text{min}}(W)$ significantly more for these networks, approximately 3 orders of magnitude for $\lambda_{\text{min}}(W)$. Also for these networks, $r_{\text{quot}}$ gives the best $\lambda_{\text{min}}(W)$ while $r_{\text{diff}}$ is better for $\text{Tr}(W)$. These results are coherent with what obtained in Lindmark and Altafini [2018] based only on numerical evidence. In fact, choosing power laws for the degree distributions means that the amount of non-normality of the corresponding adjacency matrix is increased as a significant fraction of overall outgoing edge weights is concentrated at a few nodes, and similarly for the overall incoming edge weights, thereby resulting into skewed distribution of $p_i$ and $q_j$, compare Figure 4 with Figure 7. The fraction of nodes for which $\bar{q}$ is close to zero (nodes that are hard to control indirectly) is significantly larger for the directed scale free networks than for the Erdős-Rényi networks. Given Theorem III.3, this means that the fraction of driver nodes, $m/n$, must be higher for directed scale free networks in order to match a pre-assigned worst case control energy level, $1/\lambda_{\text{min}}$. Although the upper bound $\lambda_{\text{min}} < \bar{q}_{j,m+1}$ is conservative (Figures 3 and 5), is it stronger for directed scale free networks, which correctly indicates that they are the more difficult networks to control in the worst case direction.

V. CONCLUSIONS

The network centrality measures $p$ and $q$ considered in this paper are based on energy flow considerations. They reflect the fact that what makes a good driver node depends both on its influence over other nodes in the network, and on its ability to be controlled indirectly from other nodes. These centralities are combined into rankings for driver node placement, where nodes that are net contributors of energy have the highest ranking. Algebraically, the rankings relate to the non-normality of the adjacency matrix. Driver node placement based on these rankings exploit the non-normality and result in reduced energy requirements for controlling the network, i.e. both the metrics $\text{Tr}(W)$ and $\lambda_{\text{min}}(W)$ are simultaneously improved w.r.t. random driver node placement. The improvements are visible but limited for Erdős-Rényi networks, but they are significantly higher for the more non-normal directed scale-free networks. It should be noted that the proposed rankings are not suited for networks with normal adjacency matrix, e.g. undirected networks, since the nodes with the highest network influence then also are the easiest to control indirectly. The simulations presented here confirms our previous observations in Lindmark and Altafini [2018], and the centralities $p$ and $q$ provides a theoretical justification for the driver node placement strategy that is applied there.

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Fig. 4. The metrics $p_i$, $q_i$ and $\tilde{q}_i$ for the nodes of Erdős-Rényi networks with $n = 500$ in linear and logarithmic scale (inset). The values are averages over 1000 realizations, sorted in ascending order along the x-axis.

Fig. 6. The eigenvalues of $W$ in increasing order for different ranking criteria on directed scale-free networks with 500 nodes. The number of driver nodes is 150 and the values are averages over 1000 realizations.

Fig. 7. The metrics $p_i$, $q_i$ and $\tilde{q}_i$ for the nodes of directed scale-free networks with $n = 500$ in linear and logarithmic scale (inset). The values are averages over 1000 realizations, sorted in ascending order along the x-axis.

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