Quantum geometry and microscopic black hole entropy

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Abstract
Quantum black holes within the loop quantum gravity (LQG) framework are considered. The number of microscopic states that is consistent with a black hole of a given horizon area $A_0$ are counted and the statistical entropy, as a function of the area, is obtained for $A_0$ up to $550\ell_P^2$. The results are consistent with an asymptotic linear relation and a logarithmic correction with a coefficient equal to $-1/2$. The Barbero–Immirzi parameter that yields the asymptotic linear relation compatible with the Bekenstein–Hawking entropy is shown to coincide with a value close to $\gamma = 0.274$, which has been previously obtained analytically. However, a new and oscillatory functional form for the entropy is found for small, Planck size, black holes that calls for a physical interpretation.

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1. Introduction

One of the most crucial tests that a candidate quantum theory of gravity must pass is to provide a mechanism to account for the microscopic degrees of freedom of black holes (BH). For more than 30 years, this has been a theoretical challenge ever since the discovery by Bekenstein and Hawking that black holes are quantum in nature [1]. It is not unfair to say that at the moment we have only two candidates for quantum gravity that have offered such an explanation: string/brane theory [2] and loop quantum gravity [3, 4]. The LQG formalism allows us to include several matter couplings (including non-minimal couplings) and black holes far from extremality, in four dimensions. The approach uses as starting point isolated horizon (IH)
boundary conditions at the classical level, where the interior of the BH is excluded from the region under consideration. In this sense, the description is somewhat effective, since part of the information about the interior is encoded in the boundary conditions at the horizon that in the quantum theory get promoted to a condition that horizon states must satisfy. There is also an important issue regarding this formalism. Loop quantum gravity possesses a one-parameter family of inequivalent representations of the classical theory labelled by a real number $\gamma$, the so-called Barbero–Immirzi (BI) parameter (it is the analogue of the $\theta$ ambiguity in QCD [5]). It turns out that the BH entropy calculation provides a linear relation between entropy and area for very large black holes (in Planck units) as

$$S = \lambda A(\gamma),$$

where the parameter $\lambda$ is independent of $\gamma$ and depends only in the counting. We have put the $\gamma$ dependence in the area, since the parameter appears explicitly in the area spectrum. The strategy that has been followed within the LQG community is to regard the Bekenstein–Hawking entropy $S = A/4$ as a requirement that large black holes should satisfy. This uniquely fixes the value of $\gamma = \gamma_0$ once and for all, by looking at the asymptotic behaviour, provided that one has the ‘correct counting’ that provides the right value for $\gamma_0$. Whenever we have independent tests that call for a specific value of $\gamma$, one better finds out that the value $\gamma_0$ ‘works’, or else LQG would be in trouble5.

The parameter $\lambda$ above depends on the calculation of the entropy, that is, in the counting of states compatible with whatever requirements we have imposed. The matter has not been free from some controversy. In the original calculations (once isolated horizons were understood to be vital) [4, 7], the number of states was underestimated; the entropy appeared to arise from a special set of states where the contribution to the area from each puncture was the same and corresponded to that of the minimum spin possible. Later on, it was realized that this calculation had failed to consider many states [8], and a corrected analytical estimation of entropy, and the value of the Barbero–Immirzi parameter, was proposed in [9]. Furthermore, a still different calculation appeared soon afterwards [10] which gave yet another value for $\gamma$. This situation suggests that a clear understanding of the black hole formalism and the assumptions leading to the entropy counting within LQG is needed. Even when such analysis is completed, one could always remain skeptical and ask for an ‘acid test’ of the formalism and the counting.

The simplest such test would be to just count states. The purpose of this paper is to do that. We count states, by means of a simple algorithm, of a quantum black hole within the existing formalism in LQG [4], compatible with the restrictions that this framework imposes, as in [7]. To be more precise, we restrict attention to spherical horizons (for which area is the only free parameter classically) of a fixed horizon area $A_0$ and compute the number of allowed quantum states, within an interval $[A_0 - \delta A, A_0 + \delta A]$ that satisfy the following: (i) the quantum area expectation value satisfies $\langle \hat{A} \rangle \in [A_0 - \delta A, A_0 + \delta A]$ and (ii) for which a restriction on the quantum states of the horizon, $\sum m_i = 0$, is imposed. For details see [12]. This last ‘projection constraint’ comes from the consistency conditions for having a quantum horizon that has, furthermore, the topology of a 2-sphere (it is the quantum analogue of the Gauss–Bonnet theorem). In the analytical treatments, it has been shown in detail that, for large black holes (in Planck units), the entropy behaves as

$$S = \frac{A}{4} - \frac{1}{2} \ln A,$$

5 The presumed relation between quasinormal modes and quantum black hole transition [6], which calls for a different value of $\gamma$, even though intriguing, does not constitute an independent test from our perspective.
provided the Barbero–Immirzi parameter $\gamma$ is chosen to coincide with the value $\gamma_0$, which has to satisfy [10]

$$1 = \sum_i (2j_i + 1) \exp(-2\pi \gamma_0 \sqrt{j_i(j_i + 1)}).$$

The solution to this transcendental equation is approximately $\gamma_0 = 0.27398 \ldots [10–12]$.

Thus, there are two kinds of tests one can make. The first one involves the linear relation between entropy and area that dominates in the large area regime. This provides a test for the value of the BI parameter. The second test has to do with the coefficient of the logarithmic correction ($-1/2$), a subject that has had its own share of controversy. The analytical results show that this coefficient is independent of the linear coefficient and arises in the counting whenever the $\sum m = 0$ constraint is imposed.

In order to test the validity of the logarithmic correction and its relation to the constraint, we fix the value of the parameter $\gamma = \gamma_0$ and compute the number of states, both with and without the projection constraint. We subtract this functions and compare the difference with logarithmic functions. We look for the coefficient that provides the best fit. Once the logarithmic coefficient is found and the independence of the asymptotic linear coefficient is established, we perform a variety of countings for different values of $\gamma$, both with and without the projection constraint, and consider the slope $c$ of the resulting relation $S = cA$, as a function of $\gamma$. For the function $c(\gamma)$ we look for the value of $\tilde{\gamma}$ for which $c(\tilde{\gamma}) = 1/4$.

Another separate issue that one would like to consider is the applicability of the formalism for ‘small’ black holes. The isolated horizons boundary conditions are imposed classically in the variational principle, which means that the horizon is assumed to exist as a classical object. A natural question is whether the resulting picture can be trusted for small black holes, not far from the Planck regime where strong quantum gravity effects can be expected to appear. Another related question one might try to answer is the ‘scale’ at which the quantum horizon entropy approaches the expected form derived from semi-classical/large horizon area approximation. As we shall see, even when the limited computing power at our disposal, we shall be able to partially answer some of these questions.

The remainder of this paper is as follows. In section 2 we shall describe the algorithm that implements the counting of states. Section 3 is devoted to describing the results found. We end with a discussion in section 4.

2. The counting

Counting configurations for large values of the area (or mass) is extremely difficult for the simple reason that the number of states scales exponentially. Thus, for the computing power at our disposal, we have been able to compute states up to a value of area of about $A = 550\ell_P^2$ (recall that the minimum area gap for a spin-1/2 edge is about $a_0 \approx 6\ell_P^2$, so the number of punctures on the horizon is below 100). At this point the number of states exceeds $2.8 \times 10^{58}$. In terms of Planck masses, the largest value we have calculated is $M = 3.3M_P$. When the projection constraint is introduced, the upper mass we can calculate is much smaller, given the computational complexity of implementing the condition. In this case, the maximum mass is about $1.7M_P$.

It is important to describe briefly what the program for counting does. What we are using is what it is known, within combinatorial problems, as a brute force algorithm. This is, we are simply asking the computer to perform all possible combinations of the labels we need to consider, attending to the distinguishability–indistinguishability criteria that are relevant [7, 12], and to select (count) only those that satisfy the conditions needed to be considered
as permissible combinations, i.e., the area condition and the spin projection constraint. An algorithm of this kind has an important disadvantage: it is obviously not the most optimized way of counting and the running time increases rapidly as we go to little higher areas. This is currently the main limitation of our algorithm. But, on the other hand, this algorithm presents a very important advantage, and this is the reason why we are using it: its explicit counting guarantees us that, if the labels considered are correct, the result must be the right one, as no additional assumption or approximation is being made. It is also important to have a clear understanding that the algorithm does not rely on any particular analytical counting available. That is, the program counts states as specified in the original ABK formalism [7]. The computer program has three inputs: (i) the classical mass $M$ (or area $A_0 = 16πM^2$), (ii) the value of $γ$ and (iii) the size of the interval $δA$.

Once these values are given, the algorithm computes the level of the horizon Chern Simons theory $k = [A_0/4πγ]$ and the maximum number of punctures possible $n_{\text{max}} = [A_0/4πγ√3]$ (where $[·]$, indicates the principal integer value). At first sight we see that the two conditions we have to impose to permissible combinations act on different labels. The area condition acts over the $j$ and the spin projection constraint over the $m$. This allows us to first perform combinations of $j$ and select those satisfying the area condition. After that, we can perform combinations of $m$ only for those combinations of $j$ with the correct area, avoiding some unnecessary work. We could also be allowed to perform the counting without imposing the spin projection constraint, by simply counting combinations of $j$ and including a multiplicity factor of $\prod_i(2j_i + 1)$ for each one, accounting for all the possible combinations of $m$ compatible with each combination of $j$. This would reduce considerably the running time of the program, as no counting over $m$ has to be done, and will allow us to separate the effects of the spin projection constraint (that, as we will see, is the responsible of a logarithmic correction). It is very important to note at this point that this separation of the counting is completely compatible with the distinguishability criteria.

The next step of the algorithm is to consider, in increasing order, all the possible number of punctures (from 1 to $n_{\text{max}}$) and in each case all possible values of $j_i$ are considered. Given a configuration $(j_1, j_2, \ldots, j_n) \ (n \leq n_{\text{max}})$, we ask whether the quantum area eigenvalue $A = \sum 8πγ√j_i(j_i + 1)$ lies within $[A_0 - δA, A_0 + δA]$. If it is not, then we go to the next configuration. If the answer is positive, then the labels $m$ are considered as described before. That is, for each of them it is checked whether $\sum m_i = 0$ is satisfied.

3. Results

Let us now present the results found. We shall separate this section in two parts. In the first one, we shall focus on the logarithmic correction, that is, in the results obtained when considering the spin constraint. In the second part, we shall report on the asymptotic linear relation that yields information about the Barbero–Immirzi parameter.

3.1. Logarithmic correction

In figure 1, we have plotted the entropy, as $S = \ln(\# \text{states})$ versus the area $A_0$, where we have counted all possible states without imposing the $\sum m_i$ constraint, and have chosen a $δA = 0.5$. As can be seen, the relation is amazingly linear, even for such small values of the area. When we fix the BI parameter to be $γ = γ_0 = 0.274$, and approximate the curve by a linear function, we find that the best fit is for a slope equal to 0.2502.

When we include the projection constraint, the computation becomes more involved and we are forced to consider a smaller range of values for the area of the black holes. In figure 2,
we plot both the entropy without the projection and with the projection, keeping the same $\delta A$. The first thing to note is that for the computation with the constraint implemented, there are some large oscillations in the number of states. Fitting a straight line gives a slope of 0.237. In order to reduce the oscillations, we increased the size of $\delta A$ to $\delta A = 2$. The result is plotted in figure 3. As can be seen the oscillations are much smaller, and the result of implementing the constraint is to shift the curve down (the slope is now 0.241). In order to compare it with the expected behaviour of $S = A/4 - (1/2) \ln A$, we subtracted both curves of figure 3, in the range $A = [50, 160]$, and compared the difference with a logarithmic function. The coefficient that gave the best fit is equal to $-0.4831$ (see figure 4). What can we conclude from this? While it is true that the rapidly oscillating function is far from the analytic curve, it is quite interesting that the oscillatory function follows a logarithmic curve with the ‘right’ coefficient. It is still a challenge to understand the meaning of the oscillatory phase. Even when not conclusive by any means, we can say that the counting of states is consistent with a (n asymptotic) logarithmic correction with a coefficient equal to $(-1/2)$.
Figure 3. Entropy versus area with and without the projection constraint, with $\delta A = 2$.

Figure 4. The curves of figure 3 are subtracted and the difference, an oscillatory function, shown in the upper figure. The curve is approximated by a logarithm curve in the lower figure.
3.2. Barbero–Immirzi parameter

Let us now assume that the logarithmic correction is indeed there and that, as the analytical calculations suggest \([10, 12]\), the projection constraint does not have any effect on the coefficient of the linear term, that is, on the Barbero–Immirzi parameter. With this in mind, we have performed a variety of countings for different values of \(\gamma\), without the projection constraint, and considered the slope \(c\) of the resulting relation \(S = cA\), as a function of \(\gamma\). For the resulting function \(c(\lambda)\) we looked for the value of \(\tilde{\gamma}\) for which \(c(\tilde{\gamma}) = 1/4\). This is shown in figure 5.

In order to find this value, we have interpolated the curve and found the value \(\tilde{\gamma} = 0.274\,3691\) for which the slope is equal to \(1/4\). It is hard not to note that the value of \(\tilde{\gamma}\) is amazingly close to the value \(\gamma_0\) found analytically.

When we repeat this procedure including the \(\sum m\) constraint, just to have a rough idea, we have computed for a limited range of mass (in steps of 0.1) and for a variety of \(\gamma\) in \([0.18, 0.4]\), in steps of 0.01 and have plotted the results in figure 6. The value \(\gamma'\) where the
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A curve crosses $1/4$ is $\gamma' = 0.2552$, which is still far from the GM value (which one expects to get for larger BH), but is clearly very far from the value given in [9]. What is amazing is that, even when considering only these Plank size horizons, one can confidently say that there is an asymptotic linear relation between entropy and area and that the relevant coefficient is consistent only with the value of the BI parameter $\gamma_0 = 0.27398 \ldots$, as found in [10, 12].

4. Discussion and outlook

Within loop quantum gravity, the issue of which states should be counted when computing the black hole entropy is a pressing one. The formalism for treating boundary conditions and the quantum horizon geometry established in [4, 7] provides a clear and precise framework. This includes an unambiguous answer as to which states are to be distinguished and which are to be considered undistinguishable, and to the question of which quantum numbers ($j_i$ and/or $m_i$) are to be considered. In this paper, we have followed a direct application of the formalism and have counted, using a simple algorithm, the states that satisfy the conditions and that yield an area close to a specified value $A_0$. What we have learned can be summarized as follows.

(i) When we do not impose the projection constraint, we find that very rapidly, the entropy area relation becomes linear.

(ii) The BI parameter that yields the desired agreement with $S = A/4$ is given by the value $\gamma_0 = 0.27398 \ldots$, consistent with the analytical computation of [10].

(iii) When the projection constraint is incorporated, which analytically gives the logarithmic correction, the curve gets shifted down and exhibits some oscillations, but follows on average the expected curve with the predicted coefficient $-1/2$.

(iv) For the rather small value of the BH area computed, and for $\gamma = \gamma_0$, the total entropy seems to approach a linear relation with a ratio $S/A$ approaching $1/4$ from below, which is what one expects due to the logarithmic correction.

It is important to emphasize that the procedure followed here, in the algorithm implemented, is not assuming any of the analytical estimations available, but rather performing a direct counting by brute force of the microstates consistent with the macroscopic requirements and thus, responsible for the black hole entropy. In a sense, the results presented here can be seen as providing strong evidence for which the correct analytical counting is, given the original assumptions of the program outlined in [4].

Even when the results presented here shed light on the relation between entropy and area, and the Barbero–Immirzi parameter, one still needs more work to have completely conclusive results. In particular, one needs more computing capacity to go further in the range of values analysed.

Furthermore, the oscillations found in the entropy area relation certainly call for an explanation. For instance, it is important to determine whether there is some area scale set by the oscillations found in the entropy area relation. To this effect, we have found the frequency that best approximates the oscillations, and the frequency in areas gives an area scale of $\delta A_{\text{osc}} = 2.407\ell_P^2$. It remains a challenge to find an explanation for this scale.

It could also happen, for instance, that the thermodynamic quantities such as temperature (that is usually associated with $T = \partial M/\partial S$) and the specific heat get modified as one approaches the Planck scale. The usual, classical relation between mass and entropy (using the relation $S = A/4$) implies that a Schwarzschild black hole has negative specific heat; as the energy of the black hole decreases, the temperature increases, making the system unstable. One could imagine, for instance, that the oscillations here found, make the specific heat positive as one decreases the area for some (small) value and, thus, would ‘stabilize’ the
black hole. Another intriguing possibility would be to ask whether one can learn something from this formalism (tailored for large equilibrium systems), about the evaporation process of small black holes and the issue of information loss. We shall leave these issues for future publications.

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