Almost Sure Recurrence of the Simple Random Walk Path

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Abstract
It is shown that the path of a simple random walk on any graph, consisting of all vertices visited and edges crossed by the walk, is almost surely a recurrent subgraph.

1 Introduction

Given a graph $G = (V, E)$ with finite degrees, a simple random walk (SRW) on $G$ is a Markov chain on the set of vertices with transition probabilities

$$\text{Prob}(w_t = u | w_{t-1} = v) = 1/d_v,$$

provided $\{u, v\} \in E$, where $d_v$ is the number of edges meeting at $v$.

$G$ is called recurrent iff a.s. SRW visits any fixed vertex infinitely often. It is called transient otherwise.

Let $G$ be a graph. Let $\text{PATH}$ be the random subgraph of $G$, consists of all vertices visited and edges crossed by a simple random walk on $G$, that is, the path of the random walk.

Theorem 1.1. $\text{PATH}$ is a.s. recurrent.

- For a recurrent $G$, the theorem is trivial, since any subgraph of a recurrent graph is recurrent (see [3]). Also, in that case $\text{PATH} = G$.

- The theorem is already known for the Euclidean lattices, since a.s. the SRW paths on three dimensional Euclidean lattice has infinitely many cutpoints, i.e. points where the past of the path is disjoint from its future, see [5, 6]. And then recurrence follows by the Nash-Williams criterion [8]. An example of a transient, bounded degree graph, for which $\text{PATH}$ has only finitely many cutpoints a.s. is constructed in [11, 13].
• Morris [7] proved that the components of the Wired Spanning Forest are a.s. recurrent, a result of similar spirit to the theorem but with a different proof. For another a.s. recurrence theorem (for distributional limits of finite planar graphs) see [2].

• Exercise: show, without using theorem [1,1] that if $G$ is transient then a.s. the SRW do not visit all the vertices of $G$.

• The proof uses the electrical networks interpretation of recurrence. For the connection between SRW and electrical network see [8]. For further reading on recurrence see [10] and the on-line lecture notes [9].

• One can think of a Brownian analogue of the theorem. That is a.s. parabolicity of the Wiener sausage, with reflected boundary conditions. It is of interest to formulate similar conjectures and theorems for other generators and other random walks and processes. For background on recurrence in the Riemannian context see e.g. [4].

For example, consider the range of a branching random walk on a graph $G$, denoted by $R(BRW)$. Then we conjecture that almost surely $R(BRW)$ is recurrent for BRW with the same branching law. And a similar conjecture should hold for tree indexed random walks. See [1] for definitions and background.

Question 1.2. Given a graph $G$, denote by $PATH(n)$ the path created by the first $n$ steps of the SRW on $G$. and by $R(n)$ the maximal electric resistance between pairs of vertices on $PATH(n)$ (when $PATH(n)$ is viewed as an electrical network where each edge is a one ohm resistor).

By the theorem, on any bounded degree graph $R(n) \to \infty$ a.s. (note that $R(n) \to \infty$ do not imply the theorem, e.g. balls in the binary tree). Is there a uniform lower bound, depending on the maximal degree, for the rate at which it grows, that is: Is there a function $f$,

$$\lim_n f(n) = \infty$$

So that for any infinite graph of bounded degree. a.s.

$$\limsup_n \frac{R(n)}{f(n)} > 0?$$

In particular one can speculate that $f(n) = C \log^2 n$ might work, where the $\log^2 n$ is a lower bound coming from considering $R(n)$ when $G$ is $Z^2$, which might be critical.
A different proof of theorem 1.1 is provided in [12]. In [13], ideas from this paper and from [12] are combined to provide some bounds on the resistance of the path on finite segments of the graph.

The proof of the theorem is in the coming three sections. In the next section we consider line-graphs with unbounded degrees.

2 Proof of Theorem 1.1 for line-graphs

First, we shall prove the theorem for a very special case. Quite surprisingly, the general case will not be very different. Focusing on this special case will help illustrate the main ideas of the proof.

A graph \( G \) is called a line-graph if \( V_G = \mathbb{N} \) and \( E_G \) includes only edges connecting successive vertices. Let \( e_i \) denote the number of edges connecting \( i \) and \( i + 1 \). We place no restriction on \( e_i \).

**Theorem 2.1.** If \( G \) is a line graph then \( \text{PATH on } G \) is a.s. recurrent.

**Proof.** As always, the only interesting case is if \( G \) is transient, which is equivalent to \( \sum_{i=0}^{\infty} e_i^{-1} < \infty \). Let \( v(n) \) be the probability that a simple random walk starting at \( n \) visits 0. Clearly \( v \) is a strictly decreasing function, \( v(0) = 1 \) and \( \lim_{n \to \infty} v(n) = 0 \). More precisely:

\[
v(n) = \sum_{i=n}^{\infty} \frac{e_i^{-1}}{\sum_{i=0}^{\infty} e_i^{-1}}
\]

\( v \) is harmonic everywhere except at 0. It follows that if \( w_t \) is a simple (weighted) random walk, then the process \( v(w_t) \) is "almost" a martingale, i.e. it is a martingale as long as \( w_t \) does not reach 0.

Let \( s_n \) be the number of times the random walk crossed an edge connecting \( n \) and \( n + 1 \), in either direction. Let \( s'_n \) be the number of edges connecting \( n \) and \( n + 1 \) which belong to \( \text{PATH} \), i.e. those edges that the random walk has crossed. The resistance of \( \text{PATH} \) is therefore \( \sum_{i=0}^{\infty} s_i'^{-1} \). Obviously, \( s_n \geq s'_n \) so \( \sum_{i=0}^{\infty} s_i^{-1} < \sum_{i=0}^{\infty} s'_i^{-1} \). We will show that \( \sum_{i=0}^{\infty} s_i^{-1} = \infty \) almost surely, and therefore \( \text{PATH} \) is almost surely recurrent.

**Lemma 2.2.** \( \text{Prob}(\sum_{i=0}^{\infty} s_i^{-1} = \infty) \) is either 0 or 1.

**Proof.** Let \( \{X^j_i\}_{i,j=0}^{\infty} \) be independent random variables, defined by \( \text{Prob}(X^j_i = 1) = e_i/(e_{i-1} + e_i) \) and \( \text{Prob}(X^j_i = -1) = e_{i-1}/(e_{i-1} + e_i) \). Use these variables to construct a simple random walk on \( G \) in the obvious manner: \( w_{t+1} = w_t + X^t_{w_t} \). Now, \( s_k \) is dependent (in the probabilistic
sense) only on $X^j_i$ for $i \geq k$, since every time the walk is in $\{0, 1, \ldots, k-1\}$ it will almost surely reach $k$ at some time. Therefore, a change to the values of finitely many of the $X^j_i$s will change only the finitely many $s_i$’s and so cannot effect the infiniteness of $\sum_{i=0}^{\infty} s_i^{-1}$. By Kolomogorov’s zero-one law we get that $\text{Prob}(\sum_{i=0}^{\infty} s_i^{-1} = \infty)$ is either 0 or 1.

It remains to show that $PATH$ is not almost surely transient. First we shall handle the easy case, where the walk is quickly transient.

**Lemma 2.3.** If for infinitely many $n$, $v(n)/2 > v(n + 1)$ then almost surely $\sum_{i=0}^{\infty} s_i^{-1} = \infty$.

**Proof.** Let $\{n_i\}_{i=0}^{\infty}$ be an infinite series such that $v(n_i)/2 > v(n_i+1)$. Consider $p_i = \text{Prob}(s_{n_i} = 1)$, the probability that the random walk crosses an edge from $n_i$ to $n_i + 1$ only once. Let $\tau_i = \min(t | w_t = n_i + 1)$ be the first time the random walk reaches $n_i + 1$. Let $\sigma_i = \min(t | t > \tau_i \cap w_t = n_i)$ be the first time after $\tau_i$ the walk reaches $n_i$ or $\infty$ if it never happens. Since $v$ is harmonic on $\{n_i, n_i + 1\}$ we get that $\{v(w_t)\}_{t=\tau_i}^{\sigma_i}$ is a bounded martingale. Adopting the convention $v(\infty) = 0$, we get

$$v(n_i + 1) = E(v(\tau_i)) = E(v(\sigma_i)) = 0 \cdot \text{Prob}(\sigma_i = \infty) + v(n_i) \cdot \text{Prob}(\sigma_i < \infty)$$

Since $v(n_i + 1)/v(n) < 1/2$, the probability of ever reaching $n_i$ after having reached $n_i + 1$ is less than 1/2. This means that $\text{Prob}(s_{n_i} = 1)$ is at least 1/2. By Fatou’s lemma, the probability of $s_{n_i} = 1$ occurring infinitely often is at least 1/2 and so must be 1 according to the proof of the previous lemma. In particular, $\sum_{i=0}^{\infty} s_i^{-1} = \infty$ almost surely.

Lemma 2.3 shows that if $G$ is quickly transient (in a rather weak sense) then $PATH$ almost surely has infinitely many cut-edges and so must be recurrent.

If the premise of lemma 2.3 is not satisfied then there must exist a sequence of vertices, $\{n_i\}_{i=0}^{\infty}$, such that $n_0 = 0$ and $v(n_i)/2 > v(n_{i+1}) > v(n_i)/4$.

Denote by $PATH_i$ the part of $PATH$ between $n_i$ and $n_{i+1}$. Let $r_i = \sum_{j=n_i}^{n_{i+1}-1} s_j^{-1}$ be the resistance of $PATH_i$.

Let

$$q_i = \sum_{n_i \leq t \leq n_{i+1}} (v(w_{t+1}) - v(w_t))^2$$

i.e. the sum of $(v(w_{t+1}) - v(w_t))^2$ where the sum is taken over the part of the random walk between $n_i$ and $n_{i+1}$.

Let $\tau_i = \min(t | w_t = n_i)$ be the first time the random walk reaches $n_i$. Let $\sigma_i = \min(t | t > \tau_i \cap w_t = n_{i-1})$ be the first time after $\tau_i$ the random walk reaches $n_{i-1}$ or $\infty$ if it never happens.
Let
\[ q'_i = \sum_{\tau_i \leq t < \sigma_i} (v(w_{t+1}) - v(w_t))^2 \]
i.e. the sum of \( v(w_{t+1}) - v(w_t))^2 \) where the sum is taken over the part of the random walk between times \( \tau_i \) and \( \sigma_i \).

Lemma 2.4.
\[ E(q'_i) < 16v^2(n_i) \]

Proof. For prefixed \( i \), let \( a_t \) be equal to \( v(w_{t+1}) - v(w_t) \) if \( t < \sigma_i \) or 0 if \( t \geq \sigma_i \). By definition \( v(n_i) + \sum_{t=\tau_i}^\infty a_t = v(w_{\sigma_i}) \). Consider \( \text{Var}(v(w_{\sigma_i})) \). On the one hand we have
\[ \text{Var}(v(w_{\sigma_i})) \leq E(v^2(w_{\sigma_i})) \leq v^2(n_{i-1}) < 16v^2(n_i) \]
On the other hand
\[ \text{Var}(v(w_{\sigma_i})) = \sum_{t=\tau_i}^\infty \text{Var}(a_t) + 2 \sum_{t=\tau_i}^\infty \sum_{t'=t+1}^\infty \text{Cov}(a_t, a_{t'}) \]
By harmonicity of \( v \), \( E(a_t|w_0, w_1, \ldots, w_t) = 0 \). Therefore \( \text{Cov}(a_t, a_{t'}) = 0 \) for all \( t \neq t' \). \( \text{Var}(a_t) = E((v(w_{t+1}) - v(w_t))^2) \). Put together, we get
\[ E(\sum_{\tau_i \leq t < \sigma_i} (v(w_{t+1}) - v(w_t))^2) = \sum_{t=\tau_i}^\infty \text{Var}(a_t) = \text{Var}(v(w_{\sigma_i})) < 16v^2(n_i) \]

Now we use the connection between \( q \) and \( q' \) to prove the following lemma.

Lemma 2.5.
\[ \text{Prob}(q_i < 64v^2(n_i)) > \frac{1}{4} \]

Proof. Using harmonicity of \( v \) we get that \( \text{Prob}(\sigma_i < \infty) = v(n_i)/v(n_{i-1}) < 1/2 \). From lemma 2.4 we know that \( E(q'_i) < 16v^2(n_i) \). \( q'_i \) is nonnegative, so by Markov’s inequality \( \text{Prob}(q'_i < 64v^2(n_i)) > 3/4 \). This implies that
\[ \text{Prob}(\sigma_i = \infty \cap q'_i < 64v^2(n_i)) > 1/4 \]
But if \( \sigma_i \) is \( \infty \) then \( q'_i = q_i \) so
\[ \text{Prob}(q_i < 64v^2(n_i)) > 1/4 \]
And finally we prove the relation between $q_i$ and $R_i$, the resistance of PATH$_i$.

**Lemma 2.6.** If $q_i < Cv^2(n_i)$ then $R_i > \frac{1}{4C}$

*Proof.* Recall that $s_j$ is the number of times the walk crossed an edge between $j$ and $j + 1$. By definition

$$q_i = \sum_{j=n_i}^{n_{i+1}-1} s_j(v(j) - v(j+1))^2$$

and

$$R_i = \sum_{j=n_i}^{n_{i+1}-1} s_j^{-1}.$$ 

Using the Lagrange multipliers method, we try to minimize the value of $R_i$, under the constraint given by the value of $q_i$. We get

$$\frac{\partial}{\partial s_j}(R_i + \lambda q_i) = -s_j^{-2} + \lambda(v(j) - v(j+1))^2 = 0$$

which means that the minimum is achieved when

$$s_j = \lambda^{-\frac{1}{2}}(v(j) - v(j+1))^{-1}.$$ 

Substituting $s_j$ in the definition of $q_i$ we get

$$q_i = \lambda^{-\frac{1}{2}} \sum_{j=n_i}^{n_{i+1}-1} (v(j) - v(j+1)) = \lambda^{-\frac{1}{2}}(v(n_i) - v(n_{i+1}))$$

which implies

$$\lambda = \left(\frac{v(n_i) - v(n_{i+1})}{q_i}\right)^2.$$ 

Turning back to $R_i$ we get

$$R_i = \sum_{j=n_i}^{n_{i+1}-1} s_j^{-1} \geq \lambda^{\frac{1}{2}} \sum_{j=n_i}^{n_{i+1}-1} (v(j) - v(j+1)) = \frac{(v(n_i) - v(n_{i+1}))}{q_i}$$

$$\geq \frac{(v(n_i) - v(n_{i+1}))^2}{Cv^2(n_i)} > \frac{v^2(n_i)}{4Cv^2(n_i)} = \frac{1}{4C}$$

$\square$
Now our work is nearly done. Combining lemma 2.5 and 2.6 we get that for all $i$

$$\text{Prob}(R_i > \frac{1}{256}) > \frac{1}{4}$$

Using Fatou’s lemma again, we get

$$\text{Prob}(R_i > \frac{1}{256} \text{ infinitely often}) > \frac{1}{4}$$

From lemma 2.2 we know that the probability of $PATH$ being recurrent is either 0 or 1. We just showed that it cannot be 0 and therefore it must be 1.

\[ \square \]

3 Proof of Theorem 1.1 for bounded degree graphs

Although the proof of theorem 2.1 seems tailored to the case of line graphs, only minor modifications are needed to adapt it to the more general case of any bounded degree graph.

**Proof.** First, we need to define $v$. Pick a vertex $g_0 \in G$. Let $v(g)$ be the probability that a simple random walk starting at $g$ visits $g_0$. For the general case it is not possible to give a simple, closed formula for $v$, but it is easy to see that the relevant properties of $v$ still hold: $v$ is harmonic except at $g_0$ and $\lim_{t \to \infty} v(w_t) = 0$ almost surely when $w$ is a simple random walk.

Now we shall examine the four lemmas of the special case and prove the corresponding lemmas for the general case.

Lemma 2.2 proves a 0-1 law on the resistance of $PATH$. While the conclusion of the lemma remain true for the general case (we shall prove the resistance to be a.s. infinite), the methods used in the proof are no longer valid. Indeed, it is not true that the resistance of some part of $PATH$, far away from $g_0$ is a.s. independent of the “decisions” of the random walk made near $g_0$. Instead of lemma 2.2 we have the following easy lemma.

**Lemma 3.1.** If 

$$\text{Prob}(PATH \text{ is transient}) > 0$$

then for every $C < 1$ there exist a finite sequence of adjacent edges $\overline{w}_0, \ldots, \overline{w}_{t_0}$ such that

$$\text{Prob}(PATH \text{ is transient} \mid (w_0, \ldots, w_{t_0}) = (\overline{w}_0, \ldots, \overline{w}_{t_0})) > C$$

**Proof.** This is standard in measure theory. It follows easily from the regularity of the random walk measure. \[ \square \]
Notice that all the arguments of the special case, as well as the arguments we will use in the general case, can be carried out when the random walk is conditioned to begin with a fixed sequence.

Next, we have lemma 2.3 which handles the simple case where the walk is quickly transient. Here we don’t have this special case since we required the graph to have bounded degree.

**Lemma 3.2.** If the degrees of vertices of $G$ are bounded by $d$, then for $g$ and $h$ adjacent vertices we have

$$v(h) \leq dv(g)$$

**Proof.** This follows immediately from harmonicity of $v$. □

Let $C_i = \{g \in G \mid d^{-2i-1} \leq v(g) \leq d^{-2i}\}$ be the set of all vertices whose $v$ values lies between $d^{-2i-1}$ and $d^{-2i}$. From lemma 3.2 we know that every $C_i$ is a cutset in the sense that it separates $C_{i-1}$ from $C_{i+1}$. It is not necessarily a cutset in the usual sense, of a set separating $g_0$ from infinity, nor do these sets need be finite. Indeed, there can be an infinite number of vertices for which $v$ takes value above $d^{-2i}$. However, since $v(w_t)$ tends to 0 almost surely, the sets $C_i$ are cutset, in the usual sense, in $PATH$ almost surely.

Let $PATH_i$ be all the edges in $PATH$ between $C_i$ and $C_{i+1}$. More precisely,

$$PATH_i = \{(g, h) \in PATH \mid d^{-2i-2} < v(g) < d^{-2i-1} \cup d^{-2i-2} < v(h) < d^{-2i-1}\}$$

As before, let

$$q_i = \sum_{(w_t, w_{t+1}) \in PATH_i} (v(w_{t+1}) - v(w_t))^2$$

i.e. the sum of $v(w_{t+1}) - v(w_t))^2$ over the part of the random walk between $C_i$ and $C_{i+1}$.

Let $\tau_i = \min(t \mid w_t \in C_i)$ be the first time the random walk reaches $C_i$. Let $\sigma_i = \min(t \mid t > \tau_i \land w_t \in C_{i-1})$ be the first time after $\tau_i$ the random walk reaches $C_{i-1}$ or $\infty$ if it never happens.

Let

$$q'_i = \sum_{\tau_i \leq t < \sigma_i} (v(w_{t+1}) - v(w_t))^2$$

i.e. the sum of $v(w_{t+1}) - v(w_t))^2$ where the sum is taken over the part of the random walk between times $\tau_i$ and $\sigma_i$.

**Lemma 3.3.**

$$E(q'_i) < d^4d^{-4i}$$

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Proof. The proof is identical to that of lemma 2.4. This time we get

\[ \text{Var}(v(w_{\sigma_i})) \leq (d^{-2i+2})^2 = d^4 d^{-4i} \]

and

\[ \text{Var}(v(w_{\sigma_i})) = \text{Var}(v(w_{\tau_i})) + \sum_{t=\tau_i}^{\infty} \text{Var}(v(w_{t+1}) - v(w_t)) \]

since the covariances are, as before, all 0.

\qed

Lemma 3.4.

\[ \text{Prob}(q_i < 4d^4 d^{-4i}) \geq \frac{1}{4} \]

Proof. The proof is (again) identical to the proof of 2.5. Here we have

\[ \text{Prob}(\sigma_i < \infty) \leq \frac{\sup_{g \in C_i} v(g)}{\inf_{g \in C_{i-1}} v(g)} \leq \frac{1}{d} \leq \frac{1}{2} \]

and

\[ \text{Prob}(q_i' < 4d^4 d^{-4i}) \geq \frac{3}{4} \]

\qed

Next, we define \( R_i \) as the resistance of \( PATH_i \) when \( C_i \) and \( C_{i+1} \) are both contracted, each to a single vertex, denoted \( c_i \) and \( c_{i+1} \). The contracted \( PATH_i \) will be denoted \( PATH'_i \).

Lemma 3.5. If \( q_i < C d^{-4i} \) then \( R_i > \frac{1}{4Cd^2} \)

Proof. The proof is actually simpler than 2.6. Let \( v'(g) \), defined for \( g \in PATH_i \) be equal to \( d^{-2i-1} \) for \( g \in C_i \), to \( d^{-2i-2} \) for \( g \in C_{i+1} \) and otherwise equal to \( v(g) \). By standard abuse of notation we shall refer to \( v' \) as defined on \( PATH'_i \) too.

Let

\[ q''_i = \sum_{(w_t, w_{t+1}) \in PATH_i} (v'(w_{t+1}) - v'(w_t))^2 \]

Obviously, \( q''_i \leq q_i \). Now we use Thomson’s Principle (see [3] , page 49) on \( PATH'_i \) with the function \( v' \). \( q''_i \) is the ”energy dissipation” of \( v' \) on \( PATH'_i \). By Thomson’s Principle the real energy dissipation is lower. Recall that \( v'(c_i) = d^{-2i-1} \) and \( v'(c_{i+1}) = d^{-2i-2} \).

Put together, we have

\[ \frac{(d^{-2i-1} - d^{-2i-2})^2}{R_i} \leq q''_i \leq q_i < C d^{-4i} \]
Which yields
\[ R_i > \frac{(d^{-2i-1} - d^{-2i-2})^2}{Cd^{-4i}} \geq \frac{1}{4Cd^2} \]

Combining lemma \ref{lem:3.4} and \ref{lem:3.5} we get that for all \( i \)
\[ \text{Prob}(R_i > \frac{1}{16d^6}) > \frac{1}{4} \]

Using Fatou’s lemma again we get that
\[ \text{Prob}(R_i > \frac{1}{16d^6}, \text{infinetly often}) > \frac{1}{4} \]

By Rayleigh’s Monotonicity Law (see [3], page 51) we know that the resistance of \( PATH \) is greater than that of the concatenation of \( PATH_i \), which is \( \sum_{i=1}^{\infty} R_i \). Therefore, the probability of \( PATH \) being recurrent is greater than \( \frac{1}{4} \).

As noted earlier, all the arguments we used can be carried out when the random walk is conditioned to begin with a fixed sequence. Using lemma \ref{lem:3.1} we conclude that the probability of \( PATH \) not being recurrent must be 0.

\( \square \)

**Remark:** a close inspection of the proof reveals that the theorem is also true for a finite union of paths of independent simple random walks. The only difference is that lemma \ref{lem:3.4} applies to each SRW separately, to yield a probability of \( \frac{1}{4k} \) (\( k \) being the number of SRWs) for the resistance of the union to be at least \( \frac{1}{16kd^6} \).

## 4 Proof of Theorem 1.1 general graphs

Lastly, we turn our attention to the general case. Basically, what happens here is that we forget about splitting our graph into consecutive layers \( C_i \), and instead consider the resistance between just a single layer and infinity.

For this we need the following lemma:

**Lemma 4.1.** If \( G \) is a transient graph then for every \( \epsilon > 0 \) there is a finite set of vertices \( K_\epsilon \) such that the resistance between \( K_\epsilon \) and infinity is less than \( \epsilon \).

**Proof.** Since \( G \) is transient there is a unit flow from some vertex to infinity with finite energy dissipation (see [3], page 110). Hence, there is a finite set \( K_\epsilon \) such that the energy dissipation outside of \( K_\epsilon \) is less then \( \epsilon \). Thomson’s principle implies that this bounds the resistance between \( K_\epsilon \) and infinity. \( \square \)
Therefore, all we need is to show that with positive probability the resistance between large balls in \( PATH \) and infinity does not tend to 0. For this it is enough to show that for uniformly for all balls in \( PATH \), the probability that the resistance to infinity is bounded from below, is bounded from below.

So, let us define \( v \) as in the previous section and let \( w_t \) be a simple random walk. Let \( K \) be a finite set of vertices in \( G \) and let \( v_K = \min(v(g)|g \in K) \) be the minimum voltage in \( K \). define \( \overline{K} = \{ g \in G|v(g) \geq v_K \} \) so that \( \overline{K} \supset K \). Let \( R \) be the resistance, in \( PATH \), between \( K \) and infinity. Note that \( \overline{K} \) might be infinite, but its intersection with \( PATH \) is a.s. finite.

Let \( \tau = \min(t|v(w_t) < v_K) \) be the first time the walk exits \( \overline{K} \). Let \( v_0 = v(w_\tau) \). Let \( \sigma = \min(t|v(w_t) > 2v_0) \), the first time the walk reaches twice the voltage at \( w_\tau \) or infinity if it never happens. \( v(w_t) \) is a martingale, so \( Prob(\sigma = \infty) \geq \frac{1}{2} \).

Given a vertex \( g \) let \( p_g = Prob(w_\sigma = g) \), so \( \sum_g p_g = Prob(\sigma < \infty) \leq \frac{1}{2} \). Let

\[
q = \sum_{\tau \leq t < \sigma} (v(w_{t+1}) - v(w_t))^2
\]

If \( \sigma = \infty \) then in the definition of \( q \) we sum over all edges in \( PATH \) that are not in \( \overline{K} \). The following lemma will bound the expectation of \( q \) in that case.

**Lemma 4.2.**

\[
E(q|\sigma = \infty) \leq 6v_0^2
\]

**Proof.** Defining, as before, \( v(w_\infty) = 0 \) we get that

\[
\sum_g p_g v(g) = E(v(w_\sigma)) = v_0
\]

and

\[
Var(v(w_\sigma)) = \sum_g p_g v^2(g) - v_0^2
\]

On the other hand, the same argument as in lemma \( 2.3 \) and \( 3.3 \) shows that

\[
Var(v(w_\sigma)) = \sum_{\tau \leq t < \sigma} E((v(w_{t+1}) - v(w_t))^2) = E(q)
\]

Consider \( E(q1_{\sigma < \infty}) \), where \( 1_{\sigma < \infty} \) is the indicator function of the even \( \sigma < \infty \). Obviously,

\[
E(q1_{\sigma = \infty}) \geq E((v(w_\sigma) - v(w_{\sigma-1}))^2) \geq E((v(w_\sigma) - 2v_0)^2)
\]

since \( v(w_{\sigma-1}) \leq 2v_0 \). Expanding the expectation we get

\[
E(q1_{\sigma = \infty}) \geq \sum_g p_g (v(g) - 2v_0)^2 = \sum_g p_g v^2(g) - 4v_0 \sum_g p_g v(g) + 4v_0^2 \sum_g p_g
\]
\[ = \text{Var}(v(w_{\sigma})) + v_{0}^{2}(4\text{Prob}(\sigma < \infty) - 3) \]

Since
\[ \text{Var}(v(w_{\sigma})) = E(q) = E(q1_{\sigma<\infty}) + E(q1_{\sigma=\infty}) \]
we get that
\[ E(q1_{\sigma=\infty}) \leq v_{0}^{2}(3 - 4\text{Prob}(\sigma < \infty)) \leq 3v_{0}^{2} \]

Therefore,
\[ E(q|\sigma = \infty) = \frac{E(q1_{\sigma=\infty})}{\text{Prob}(\sigma = \infty)} \leq 6v_{0}^{2} \]

Using this lemma we get that \( \text{Prob}(q \leq 12v_{0}^{2}|\sigma = \infty) \geq \frac{1}{2} \). Since \( \text{Prob}(\sigma = \infty) \geq \frac{1}{2} \), we know that \( \text{Prob}(q \leq 12v_{0}^{2} \cap \sigma = \infty) \geq \frac{1}{4} \). If this event happens then we have \( R \geq \frac{1}{12} \), similarly to the proof of lemma 3.5.

Using Fatou’s lemma yet again, we have that the resistance in PATH between balls and infinity does not tend to 0, with probability \( \frac{1}{4} \). Applying lemma 3.1 as in the previous section concludes the proof.

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References

[1] Benjamini, I. and Peres, Y. Markov chains indexed by trees. *Ann. Probab.* 22 (1994), no. 1, 219–243

[2] Benjamini, I. and Schramm, O. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* 6 (2001), no. 23, 13 pp.

[3] Doyle, P. and Snell, L. Random Walks and Electric Networks. [http://front.math.ucdavis.edu/math.PR/0001057](http://front.math.ucdavis.edu/math.PR/0001057) (1984).

[4] Grigor’yan, A. Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bulletin of Amer. Math. Soc.* 36 (1999) 135-249.
[5] James, N. and Peres, Y. Cutpoints and exchangeable events for random walks. Theory of Probab. and its Applications (Moscow), 41(4), (1996), 854–868.

[6] Lawler, G. Cut times for simple random walk. Electronic Journal of Probability 1 (1996) Paper 13.

[7] Morris, B. The Components of the wired spanning forest are recurrent. Prob. Theor. Rel. Fields 125 (2003), pp. 259–265.

[8] Nash-Williams, C. St. J. A. Random walks and electric currents in networks, Proc. Cambridge Phil. Soc. 55 (1959), 181-194.

[9] Peres, Y. Probability on Trees: An Introductory Climb. Springer Lecture notes in Math 1717, (1999), pp. 193-280.

[10] Woess, W. Random walks on infinite graphs and groups, Cambridge Tracts. in Mathematics, 138, Cambridge University Press, 2000, xi+334 pp.,

[11] N. James, R. Lyons, Y. Peres, (2007) A Transient Markov Chain With Finitely Many Cutpoints, Festschrift for David Freedman, to appear

[12] I. Benjamini, O. Gurel-Gurevich and R. Lyons, (2007) Recurrence of Random Walk Traces, Annals of Probability, 35, 2, 732-738

[13] I. Benjamini, O. Gurel-Gurevich and O. Schramm: Cutpoints of transient simple random walks (In preparation)