A financial market with delay driven by reflected Brownian motion

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Abstract

We study a financial market where the risky asset is modelled by a stochastic differential equation driven by a partially reflected Brownian motion. This models a situation where the asset price is partially controlled by a company which intervenes when the price is reaching a certain lower barrier in order to prevent it from going below that barrier. See e.g. Jarrow & Protter [JP] for an explanation and discussion of this model. This corresponds to a local time term in the equation for the asset price. As already pointed out by Karatzas & Shreve [KS] (see also Jarrow & Protter [JP]) this allows for arbitrages in the market. In this paper we consider the case when there is a delay $\theta > 0$ in the information flow available for the trader. Using white noise and Hida-Malliavin calculus we compute explicitly the optimal consumption rate and portfolio in this case and we show that the maximal value is finite as long as $\theta > 0$. This implies that there is no arbitrage in the market in that case. However, when $\theta$ goes to 0, the value goes to infinity. This is in agreement with the above result that is an arbitrage when there is no delay.

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1 Introduction

It is well-known that in the classical Black-Scholes market, there is no arbitrage. However, if we include a singular term in the drift of the risky asset, it was proved by Karatzas & Shreve [KS] that arbitrages exist. Subsequently this type of market has been studied by several authors, including Jarrow & Protter [JP]. They explain how a singular term in the drift can model a situation where the asset price is partially controlled by a large company which intervenes when the price is reaching a certain lower barrier, in order to prevent it from going below that barrier. They prove that arbitrages can occur in such situations.

The purpose of our paper is to study such a market with a singular drift where the trader only has access to a delayed filtration $\mathcal{F}_{t-\theta}$, where $\theta > 0$ is the delay constant and $\mathcal{F}_t$ is the sigma-algebra generated by the underlying Brownian motion $B(s); s \leq t$. We show that as long as $\theta > 0$ there is no arbitrage in this market. In fact, we show that this delayed market is viable, in the sense that the value of the optimal portfolio problem with logarithmic utility is finite. However, if the delay constant goes to 0 the value of the portfolio goes to infinity.

More specifically, we will study in this paper the problem of optimal consumption and portfolio in a market driven by partially reflected Brownian motion, where the trader only has access to a partial information flow. Precisely, our model is the following:

Suppose we have a financial market with the following two investment possibilities:

- A risk free investment (e.g. a bond or a (safe) bank account), whose unit price $S_0(t)$ at time $t$ is described by

\[
\begin{align*}
    dS_0(t) &= r(t)S_0(t)dt, \quad 0 \leq t \leq T, \\
    S_0(0) &= 1,
\end{align*}
\]  

(1.1)

- A risky investment, whose unit price $S(t)$ at time $t$ is described by a
linear stochastic differential equation (SDE) of the form

$$\begin{cases} dS(t) = S(t)[\mu(t)dt + \alpha(t)dL_t + \sigma(t)dB(t)], & 0 \leq t \leq T, \\ S(0) > 0, \end{cases} \quad (1.2)$$

where $T > 0$ is a given constant, $B(\cdot)$ is a standard Brownian motion and $L_t = L_t(0)$ is the local time of $B(\cdot)$ at 0.

In view of this, we might say that the risky asset price is driven by a process partially reflected upwards at 0. As pointed out in Jarrow & Protter [JP], such a reflection can model an intervention from a large trader who wants to keep the price above a certain barrier.

Let $\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}$ be a given subfiltration of $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$, in the sense that $\mathcal{G}_t \subset \mathcal{F}_t$ for all $t$. For example, we will study in detail the case when

$$\mathcal{G}_t = \mathcal{F}_{(t-\theta)^+}; \quad t \geq 0. \quad (1.3)$$

In this partial information market we introduce a $\mathcal{G}$-adapted relative consumption rate process $c(t)$ and a $\mathcal{G}$-adapted self-financing portfolio $\pi(t)$, and we study the optimal consumption and portfolio problem to maximize the combined expected logarithmic utility from the consumption and the terminal wealth. Our results are the following:

- Using methods from white noise calculus we find explicit expressions for the optimal consumption rate $c^*(t)$ and the optimal portfolio $\pi^*(t)$. Then we show that the value is finite for all positive delays in the information flow. In particular, this shows that there is no arbitrage in that case. This result appears to be new.

- We also show that the value goes to infinity when the delay goes to 0. This shows in particular that the value is infinite when there is no delay, in agreement with the arbitrage results of Karatzas & Shreve [KS] and Jarrow & Protter [JP] described above.

2 Preliminaries

In this section we recall some basic concepts and results which will be used throughout this work.
2.1 The Donsker delta function

Definition 2.1 Let \( Y : \Omega \rightarrow \mathbb{R} \) be a random variable which also belongs to the Hida space \((S)^*\) of stochastic distributions. Then a continuous functional

\[
\delta_Y(.): \mathbb{R} \rightarrow (S)^*
\]

is called a Donsker delta function of \( Y \) if it has the property that

\[
\int \mathbb{R} g(y)\delta_Y(y)dy = g(Y), \quad \text{a.s.}
\]

for all (measurable) \( g : \mathbb{R} \rightarrow \mathbb{R} \) such that the integral converges.

Explicit formulas for the Donsker delta functional are known in many cases. For the Gaussian case, see Section 3.2. For details and more general cases, see e.g. Aase et al [AaØU].

Example 2.2 Consider the special case when \( Y \) is a Gaussian random variable of the form

\[
Y = Y(T), \quad \text{where } Y(t) = \int_0^t \psi(s)dB(s); \quad \text{for } t \in [0, T],
\]

for some deterministic function \( \psi \in L^2[0,T] \) with

\[
\| \psi \|^2_{[t,T]} := \int_t^T \psi(s)^2ds > 0, \quad \text{for all } t \in [0, T].
\]

In this case it is well-known that the Donsker delta functional exists in \((S)^*\) and is given by

\[
\delta_Y(y) = (2\pi v)^{-\frac{1}{2}} \exp^\diamond \left[-\frac{(Y - y)^2}{2v}\right],
\]

where we have put \( v := \| \psi \|^2_{[0,T]} \). See e.g. Proposition 3.2 in Aase et al [AaØU].

In particular, applying this to the random variable \( Y := B(s) \) for some \( s \in (0, T] \), we get

Lemma 2.3 For all \( s > 0 \) we have

\[
(i) \quad \delta_{B(s)}(x) = (2\pi s)^{-\frac{1}{2}} \exp^\diamond \left[-\frac{(B(s) - x)^2}{2s}\right].
\]

\[
(ii) \quad \text{The function } x \mapsto \delta_{B(s)}(x) \in (S)^* \text{ is continuous for all } x \in \mathbb{R}.
\]
The next result is useful for us:

**Lemma 2.4** With $Y$ as above and $t \in [0, T]$, we have

$$E[\delta_Y(y)|F_t] = (2\pi\|\psi\|_{[t,T]}^2)^{-\frac{1}{2}} \exp\left[-\frac{(Y(t) - y)^2}{2\|\psi\|_{[t,T]}^2}\right], \quad (2.7)$$

where

$$\|\psi\|_{[t,T]}^2 := \int_t^T \psi(s)^2 ds.$$

**Proof.** Using the Wick rule when taking conditional expectation, using the martingale property of the process $Y(t)$ and applying Lemma 3.7 in Aase et al. [AaØU], we get

$$E[\delta_Y(y)|F_t] = (2\pi v)^{-\frac{1}{2}} \exp^o\left[-E\left[\frac{(Y(T_0) - y)^2}{2v}\right]|F_t\right]$$

$$= (2\pi\|\psi\|_{[0,T]}^2)^{-\frac{1}{2}} \exp^o\left[-\frac{(Y(t) - y)^2}{2\|\psi\|_{[0,T]}^2}\right]$$

$$= (2\pi\|\psi\|_{[t,T]}^2)^{-\frac{1}{2}} \exp\left[-\frac{(Y(t) - y)^2}{2\|\psi\|_{[t,T]}^2}\right].$$

\[\square\]

### 2.2 Local time

**Definition 2.5** The local time $L_t(x)$ for Brownian motion at the point $x$ and at time $t$ is defined by

$$L_t(x) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \lambda(\{s \in [0, T]; B(s) \in (x - \epsilon, x + \epsilon)\}), \quad (2.8)$$

where $\lambda$ denotes Lebesgue measure on $\mathbb{R}$.

One can prove that the limit exists in $L^2(P)$ for each given $t$ and $x$. See e.g. Exercise 4.10 in Øksendal [O].

There is a close connection between local time and the Donsker delta function of $B(s)$, given by the following result:

**Theorem 2.6**

$$L_t(x) = \int_0^t \delta_{B(s)}(x)ds, \quad (2.9)$$

where the integration takes place in $(S)^*$.  

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Proof. By definition of the local time, we have
\[ L_t(x) = \lim_{\epsilon \to 0} \int_0^t \chi_{(x-\epsilon,x+\epsilon)}(B(s)) \, ds = \lim_{\epsilon \to 0} \int_0^t \left( \int_{\mathbb{R}} \chi_{(x-\epsilon,x+\epsilon)}(y) \delta_{B(s)}(y) \, dy \right) \, ds \]
\[ = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \chi_{(x-\epsilon,x+\epsilon)}(y) \left( \int_0^t \delta_{B(s)}(y) \, ds \right) \, dy = \int_0^t \delta_{B(s)}(x) \, ds, \]
because the function \( y \mapsto \delta_{B(s)}(y) \) is continuous in \( (S)^* \), by (2.6).

We end this section by recalling the following important result:

**Theorem 2.7 (The Tanaka formula)**

\[ |B(t) - x| = |x| + \int_0^t \text{sign}(B(s) - x) \, dB(s) + L_t(x). \quad (2.10) \]

### 3 Optimal consumption and portfolio in a market driven by reflected Brownian motion

We now return to the model in the Introduction. Thus we consider an optimal portfolio and consumption problem in the financial market \((1.1) & (1.2)\) driven by a partially reflected Brownian motion \(B(t)\) defined on a complete filtered probability space \((\Omega, \mathcal{F}, P)\) equipped with the filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}\) generated by the Brownian motion \(B(\cdot)\).

The coefficients \(r(t), \mu(t), \alpha(t)\) and \(\sigma(t) > 0\) are given bounded adapted processes, with \(\sigma(t)\) bounded away from 0.

**Theorem 3.1** [JP] If \(G = \mathcal{F}\) then this market is complete and it allows an arbitrage.

**Proof.** Our financial model is essentially the same as in Jarrow and Protter [JP], and the result follows from Theorem 4.3 in that paper.

The local time \(L_t = L_t(0)\) of Brownian motion at 0 is by Lemma 2.3 and (2.9) given by
\[ dL_t = \delta_{B(t)}(0) \, dt = (2\pi t)^{-\frac{1}{2}} \exp\left[ -\frac{B(t)^2}{2t} \right] \, dt. \quad (3.1) \]

In this market we introduce a portfolio process \(\pi(t) : [0,T] \times \Omega \to \mathbb{R}\) giving the fraction of the wealth \(X(t)\) invested in the risky asset at time \(t\), and a consumption rate process \(c(t) : [0,T] \times \Omega \to \mathbb{R}\) giving the fraction of the wealth consumed at time \(t\). We assume that at any time \(t\) both \(\pi(t)\) and \(c(t)\)
are required to be adapted to a given possibly smaller filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t \in [0,T]}$ with $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t$. For example, it could be a delayed information flow, with

$$\mathcal{G}_t = \mathcal{F}_{\max(0, t-\theta)}, \quad t \geq 0,$$

for some delay $\theta > 0$. (3.2)

We say that $\pi$ and $c$ are admissible and write $\pi, c \in \mathcal{A}_G$ if, in addition, $\pi$ is self-financing and

$$E\left[ \int_0^T (\pi(t)^2 + c(t)^2)dt \right] < \infty.$$

We denote by $\mathcal{A}_G$ the set if all admissible portfolios and consumptions.

In this case the wealth process $X(t) = X^c, \pi(t)$ is described by the equation

$$dX(t) = X(t)[(1 - \pi(t))r(t) + \pi(t)\mu(t) - c(t)]dt + \pi(t)\alpha(t)dL_t + \pi(t)\sigma(t)dB(t)].$$

(3.3)

For simplicity we put the initial value $X(0) = 1$.

By the Itô formula, we see that the solution of (3.3) is given by

$$X(t) = \exp \left( \int_0^t \pi(s)\sigma(s)dB(s) + \int_0^t \left\{ r(s) + [\mu(s) - r(s)]\pi(s) - c(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) \right\}ds + \int_0^t \pi(s)\alpha(s)dL_s \right).$$

(3.4)

Note that

$$E[\ln(X(t))] = E\left[ \int_0^t \left\{ r(s) + [\mu(s) - r(s)]\pi(s) - c(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) \right\}ds + \int_0^t \pi(s)\alpha(s)dL_s \right].$$

(3.5)

Our optimal consumption and portfolio problem is the following:

**Problem 3.2** Let $a > 0, b > 0$ be given constants. Find admissible $c^*, \pi^*$, such that

$$J(c^*, \pi^*) = \sup_{c, \pi} J(c, \pi),$$

(3.6)

where

$$J(c, \pi) = E\left[ \int_0^T a \ln(c(t)X(t))dt + b \ln(X(T)) \right].$$

(3.7)

Our first main result is:
Theorem 3.3 (Optimal consumption and portfolio)
Assume that \( r(t), \mu(t), \alpha(t) \) and \( \sigma(t) \) are deterministic and that
\[
E[\delta_B(t)(0) | G_t] \in L^2(\lambda \times P).
\] (3.8)

Then the optimal relative consumption rate \( c^*(t) \) and the optimal portfolio \( \pi^*(t) \) are given respectively by
\[
c^*(t) = \frac{a}{b + a(T - t)},
\]
(3.9)
\[
\pi^*(t) = \frac{(a(T - t) + b)[\mu(t) - r(t) + \alpha(t)E[\delta_B(t)(0) | G_t]]}{(a + b)\sigma^2(t)}.
\]
(3.10)

Proof. Formulas (3.3) and (3.7) and the Itô formula, leads to
\[
J(c, \pi) = E\left[ \int_0^T a \ln(c(t)X(t))dt + b \ln(X(T)) \right]
= E\left[ \int_0^T \left\{ a \ln(c(t)) + a \ln(X(t)) + b\left( r(t) + [\mu(t) - r(t)]\pi(t) - c(t) - \frac{1}{2}\sigma^2(t)\pi^2(t) \right) \right\} dt + b \int_0^T \pi(t)\alpha(t)dL_t \right].
\]

Substituting (3.5) in the above, gives
\[
J(c, \pi) = E\left[ \int_0^T \left\{ a \ln(c(t)) + a \left( \int_0^t \{ r(s) + [\mu(s) - r(s)]\pi(s) - c(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) \} ds + \int_0^t \pi(s)\alpha(s)dL_s \right) + b\left( r(t) + [\mu(t) - r(t)]\pi(t) - c(t) - \frac{1}{2}\sigma^2(t)\pi^2(t) \right) \right\} dt + b \int_0^T \pi(t)\alpha(t)dL_t \right].
\]

Note that in general, we have
\[
\int_0^T \left( \int_0^t h(s)ds \right) dt = \int_0^T \left( \int_s^T h(s)dt \right) ds = \int_0^T (T-s)h(s)ds = \int_0^T (T-t)h(t)dt,
\]
and
\[
\int_0^T \left( \int_0^t h(s)dL_s \right) dt = \int_0^T \left( \int_s^T h(s)dt \right) dL_s = \int_0^T (T-s)h(s)dL_s = \int_0^T (T-t)h(t)dL_t.
\]
Therefore, using that
\[
dL_t = dL_t(0) = \delta_B(t)(0)dt,
\]
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we get from the above that

\[ J(c, \pi) = E\left[ \int_{0}^{T} \left\{ a\left( \ln(c(t)) + (T - t)\{r(t) + [\mu(t) - r(t)]\pi(t) - c(t) - \frac{1}{2}\sigma^2(t)\pi^2(t) + \pi(t)\alpha(t)\delta_{B(t)}(0) - c(t)\right\} E[\delta_{B(t)}(0)|\mathcal{G}_t]\right\} dt \right]. \]

Using adaptadness of the coefficients, leads to

\[ J(c, \pi) = E\left[ \int_{0}^{T} \left\{ a\left( \ln(c(t)) + (T - t)\{r(t) + [\mu(t) - r(t)]\pi(t) - c(t) - \frac{1}{2}\sigma^2(t)\pi^2(t) + \pi(t)\alpha(t)\delta_{B(t)}(0) - c(t)\right\} E[\delta_{B(t)}(0)|\mathcal{G}_t]\right\} dt \right]. \]

This we can maximize pointwise over all possible values \( c, \pi \in \mathcal{A}_G \) by maximizing for each \( t \) the integrand. This gives the optimal values

\[ c^*(t) = \frac{a}{b + a(T - t)}, \]

\[ \pi^*(t) = \frac{(a(T - t) + b)[\mu(t) - r(t) + \alpha(t)E[\delta_{B(t)}(0)|\mathcal{G}_t]]}{(a + b)\sigma^2(t)}. \]

\( \square \)

3.1 **The case when** \( \mathcal{G}_t = \mathcal{F}_{t-\theta}, \ t \geq 0 \)

From now on we restrict ourselves to the subfiltration \( \mathcal{G}_t = \mathcal{F}_{t-\theta}, t \geq 0 \) for some constant delay \( \theta > 0 \), where we put \( \mathcal{F}_{t-\theta} = \mathcal{F}_0 \) for \( t \leq \theta \). By (2.7) we have the following result:

**Lemma 3.4** For \( t \geq \theta \) we have

\[ E[\delta_{B(t)}(0)|\mathcal{F}_{t-\theta}] = (2\pi\theta)^{-\frac{1}{2}} \exp\left[ -\frac{B(t - \theta)^2}{2\theta} \right]. \tag{3.12} \]

Then by Theorem 3.3 we get
Theorem 3.5 Suppose $G_t = F_{t-\theta}$. Then the optimal consumption and portfolio are given by

$$c^*(t) = \frac{a}{b + a(T - t)},$$

$$\pi^*(t) = \frac{(a(T - t) + b)[\mu(t) - r(t) + \alpha(t)(2\pi\theta)^{-\frac{1}{2}} \exp \left[ - \frac{B(t-\theta)^2}{2\theta} \right]}{(a + b)^{\frac{\sigma^2(t)}{2}}}.$$  

Remark 3.6 Note that when $\theta \to 0$ this expression for $\pi(t)$ does not converge.

4 The limiting case when the delay goes to 0

In this section we concentrate on the delay case and with optimal portfolio only, i.e. without consumption. Thus we put $a = 0$ and $b = 1$ in Theorem 3.6. Moreover, to simplify the calculations we assume that $r = 0$ and $\mu(t) = \mu, \alpha(t) = \alpha$ and $\sigma(t) = \sigma$ are constants. Then by Theorem 3.3 the optimal portfolio $\tilde{\pi}(t)$ is given by

$$\tilde{\pi}(t) = \frac{\mu + \alpha \Lambda(t)}{\sigma^2}, \quad (4.1)$$

where

$$\Lambda(t) = E[\delta_{H(t)}(0)|F_{t-\theta}] = (2\pi\theta)^{-\frac{1}{2}} \exp \left[ - \frac{B(t-\theta)^2}{2\theta} \right]. \quad (4.2)$$

By (3.11) we see, after some algebraic operations, that the corresponding performance

$$\tilde{J}_\theta = J(0, \tilde{\pi})$$

is

$$\tilde{J}_\theta = E \left[ \int_0^T \left( \mu \tilde{\pi}(t) - \frac{1}{2}\sigma^2 \tilde{\pi}^2(t) + \tilde{\pi}(t) \alpha \Lambda(t) \right) dt \right] = E \left[ \int_0^T \left( \frac{\mu + \alpha \Lambda(t)}{2\sigma^2} \right)^2 dt \right] = A_1 + A_2 + A_3, \quad (4.3)$$

where

$$A_1 = \frac{\mu^2}{2\sigma^2} T, \quad A_2 = \frac{\mu \alpha}{\sigma^2} E \left[ \int_0^T \Lambda(t) dt \right], \quad A_3 = \frac{\alpha^2}{2\sigma^2} E \left[ \int_0^T \Lambda^2(t) dt \right]. \quad (4.4)$$

Using (4.2) and the density of $B(s)$, we get

$$E \left[ \exp \left( - \frac{B^2(s)}{2\theta} \right) \right] = \int_\mathbb{R} \exp \left( - \frac{y^2}{2\theta} \right) \frac{1}{\sqrt{2\pi s}} \exp \left( - \frac{y^2}{2s} \right) dy$$

$$= \frac{1}{\sqrt{2\pi s}} \int_\mathbb{R} \exp \left( - \frac{1}{2} y^2 (\frac{1}{\theta} + \frac{1}{s}) \right) dy. \quad (4.5)$$
In general we have, for $a > 0$,

$$\int_{\mathbb{R}} \exp(-ay^2)dy = \sqrt{\frac{\pi}{a}}, \quad (4.6)$$

we conclude, by putting $s = t - \theta$ in (4.5), that

$$A_2 = \int_0^T \frac{\mu\alpha}{\sigma^2\sqrt{2\pi\theta}} \sqrt{\frac{\theta}{t}} dt = \frac{2\mu\alpha\sqrt{T}}{\sigma^2\sqrt{2\pi}}. \quad (4.7)$$

Similar calculations give

$$A_3 = \frac{\alpha^2(\sqrt{2T} + \theta - \sqrt{\theta})}{4\pi\sigma^2\sqrt{\theta}}. \quad (4.8)$$

We have proved the following, which is our second main result:

**Theorem 4.1** \(\hat{J}_\theta = \frac{a^2}{2\sigma^2}T + \frac{2\mu\alpha\sqrt{T}}{\sigma^2\sqrt{2\pi}} + \frac{\alpha^2(\sqrt{2T} + \theta - \sqrt{\theta})}{4\pi\sigma^2\sqrt{\theta}} \to \infty \) when $\theta \to 0$.

**Corollary 4.2** (i) For all information delays $\theta > 0$ the value of the optimal portfolio problem is finite.

(ii) When there is no information delay, i.e. when $\theta = 0$, the value is infinite.

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