RELATIVE GENERALIZED HAMMING WEIGHTS OF $q$-ARY REED-MULLER CODES

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Abstract. Coset constructions of $q$-ary Reed-Muller codes can be used to store secrets on a distributed storage system in such a way that only parties with access to a large part of the system can obtain information while still allowing for local error-correction. In this paper we determine the relative generalized Hamming weights of these codes which can be translated into a detailed description of the information leakage.

1. Introduction

We consider the situation where a central party wants to store sensitive information (a secret) on a distributed storage system in such a way that other parties with access to a large part of the system will be able to recover it, but other parties will not. The following requirements are natural:

R1: Access to arbitrary $r$ (or more) of the stored data symbols makes it possible to recover the secret in full, however, with $\tau$ (or less) one cannot recover any information – or less restrictive one can only recover a limited amount of information.

R2: The storage device must be able to locally repair itself. More precisely, if the storage media experiences random errors then with a very high probability any stored symbol can be corrected from only a small number of randomly accessed locations (symbols) of the media.

To meet simultaneously the requirements R1 and R2 we propose to use a coset construction $C_1/C_2$ of $q$-ary Reed-Muller codes. As is well-known any linear ramp secret sharing scheme can be realized as a coset construction of two linear codes and vice versa [4]. By choosing the code $C_1$ to be a $q$-ary Reed-Muller code not only do we address R1 but we also meet the requirement R2. This is due to the fact that $q$-ary Reed-Muller codes are locally correctable. When considering the coset construction $C_1/C_2$ rather than $C_1$ this property is always maintained (See Section 2) which corresponds to R2. The local correctability properties of $q$-ary Reed-Muller codes have been studied in detail, see e.g. [15, 20].

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Definition 1.1. Let \( q \) be a power of a prime, \( u \) an integer, \( s \) a positive integer, and write \( n = q^s \). We enumerate the elements of \( (\mathbb{F}_q)^s \) as \( \{P_1, \ldots, P_n\} \) and consider the evaluation map \( \varphi : \mathbb{F}_q[X_1, \ldots, X_s] \to (\mathbb{F}_q)^n \), \( \varphi(f) = (f(P_1), \ldots, f(P_n)) \). The \( q \)-ary Reed-Muller code of order \( u \) in \( s \) variables is defined by
\[
RM_q(u, s) = \{\varphi(f) : f \in \mathbb{F}_q[X_1, \ldots, X_s], \deg(f) \leq u\}.
\]

Definition 1.2. A code \( C \subseteq (\mathbb{F}_q)^n \) is said to be \((\rho, \delta, \varepsilon)\)-locally correctable if there exists a randomized error-correcting algorithm \( A \) which takes as input \( \tilde{y} \in (\mathbb{F}_q)^n \) and \( i \in \{1, \ldots, n\} \) such that
1. for all \( i \in \{1, \ldots, n\} \) and all vectors \( \tilde{c} \in C \), \( \tilde{y} \in (\mathbb{F}_q)^n \) which differ in at most \( \delta \) positions
\[
Pr[A(\tilde{y}, i) = c_i] \geq 1 - \varepsilon
\]
where the probability is modelling the random coin tosses of the algorithm \( A \).
Here, \( c_i \) means the \( i \)-th entry of \( \tilde{c} \).
2. \( A \) makes at most \( \rho \) queries to \( \tilde{y} \).

The following theorem corresponds to [26, Pro. 2.4, Pro. 2.5, Pro. 2.6].

Theorem 1.3. If \( u < q - 1 \) then \( RM_q(u, s) \) is \((u, \delta, (u+1)\delta)\)-locally correctable for all \( \delta \). Let \( \sigma < 1 \) be a positive real and assume \( u \leq \sigma(q - 1) - 1 \). Then \( RM_q(u, s) \) is \((q - 1, \delta, 2\delta/(1 - \sigma))\)-locally correctable for all \( \delta \) and if furthermore \( \delta < 1/2 - \sigma \) then it is \((q - 1, \delta, 4(\delta - \delta^2)/(q - 1)(1 - 2(\sigma - \delta))^{2})\)-locally correctable.

Turning to the question of information leakage in connection with ramp secret sharing schemes based on \( q \)-ary Reed Muller codes, not much can be found in the literature (see, however, [6] for other interesting results on secret sharing schemes related to binary Reed-Muller codes). In the present paper we fill this gap. More precisely we establish the true values of all corresponding relative generalized Hamming weights for \( q \)-ary Reed Muller codes in two variables. For the case of more variables we device a simple and low complexity algorithm to determine the parameters. By known methods these results then easily translate into a detailed and accurate description of the leakage to unauthorized parties as well as the number of symbols needed for the authorized parties to recover the secret. We note that a similar analysis has not been made before for any of the known families of locally correctable codes.

Our work on relative generalized Hamming weights of \( q \)-ary Reed-Muller codes is a non-trivial generalization of results by Heijnen and Pellikaan [12], who based on the Feng-Rao bound for dual codes, showed how to calculate generalized Hamming weights of \( q \)-ary Reed-Muller codes. Until recently the relative generalized Hamming weights have been determined for one family only, namely the family of MDS-codes. In the recent paper [9] a method was given to estimate these parameters for one-point algebraic geometric codes through the use of the Feng-Rao bounds for dual or primary codes. More results in this direction were presented in [7, 8, 27] and [19], the latter dealing with more-point algebraic geometric codes. The present paper is a natural continuation of [12] and [9], however, to keep the description as simple as possible, in the presentation of the present paper we use the footprint bound from Gröbner basis theory rather than the Feng-Rao bounds.

The paper is organized as follows. We start in Section 2 by giving some background information on ramp secret sharing schemes and in particular we explain the role of a coset construction of \( q \)-ary Reed-Muller codes. The subsequent four
sections treat our main task which is determination of the relative generalized Hamming weights of $q$-ary Reed-Muller codes. In Section 3 we present the theory, based on which we shall derive the weights. Section 4 shows a general method to derive any of the weights, and this method is formalized into a simple and low complexity algorithm in Section 5. Finally in Section 6 we present closed formula expressions for $q$-ary Reed-Muller codes in two variables. In Section 7 we revert to the communication problem of secret sharing on a distributed storage system. We make some general remarks on the connection between information leakage and local correctability and we give a number of examples. Section 8 is the conclusion.

2. Linear ramp secret sharing schemes

A ramp secret sharing scheme is a cryptographic method to encode a secret $\vec{s}$ into multiple shares $c_1, \ldots, c_n$ so that only from specified subsets of the shares one can recover $\vec{s}$. The encoding is in general probabilistic, meaning that to each secret $\vec{s}$ there corresponds a collection of possible share vectors $\vec{c} = (c_1, \ldots, c_n)$. Special attention has been given to linear ramp secret sharing schemes [4]. Here, the space of secrets is $(\mathbb{F}_q)^n$, where $\ell \geq 1$ is some fixed integer, and $c_1, \ldots, c_n \in \mathbb{F}_q$. Moreover, if $\vec{c}_1$ is an encoding of $\vec{s}_1$ and $\vec{c}_2$ is an encoding of $\vec{s}_2$, then also $\vec{c}_1 + \vec{c}_2$ is an encoding of $\vec{s}_1 + \vec{s}_2$.

A linear ramp secret sharing scheme with $n$ participants, secrets in $(\mathbb{F}_q)^n$, and shares belonging to $\mathbb{F}_q$ can be described as follows [4]. Consider linear codes $C_2 \subseteq C_1 \subset (\mathbb{F}_q)^n$ with $\ell = \dim(C_1) - \dim(C_2)$ and let $L \subset (\mathbb{F}_q)^n$ be (a linear code) such that $C_1 = L \oplus C_2$, where $\oplus$ is the direct sum. Consider a vector space isomorphism $\psi : (\mathbb{F}_q)^\ell \rightarrow L$. A secret $\vec{s} \in (\mathbb{F}_q)^n$ is mapped to $\vec{x} = \psi(\vec{s}) + \vec{c}_2 \in C_1$, where $\vec{c}_2 \in C_2$ is chosen by random. For the analysis, we assume that also the secrets are chosen uniformly from $(\mathbb{F}_q)^n$. In this way, the vectors of shares are chosen uniformly from $C_1$. The $n$ shares distributed among the $n$ participants are the $n$ coordinates of $\vec{x}$. The threshold parameters $t$ and $r$ of the scheme are the unique numbers such that:

1. No group of $t$ participants can recover any information about $\vec{s}$, but some groups of size $t+1$ can.
2. All groups of size $r$ can recover the secret in full, but some groups of size $r-1$ cannot.

Only for $\ell = 1$ we can hope for $r = t+1$ in which case we have a complete picture of the security. Such schemes are called $t$-threshold secret sharing schemes. For general linear ramp secret sharing schemes we have the parameters $t_1, \ldots, t_\ell, r_1, \ldots, r_\ell$ where for $m = 1, \ldots, \ell, t_m$ and $r_m$ are the unique numbers such that the following hold:

1. No group of $t_m$ participants can recover $m$ $q$-bits of information about $\vec{s}$, but some groups of size $t_m + 1$ can.
2. All groups of size $r_m$ can recover $m$ $q$-bits of information about $\vec{s}$, but some groups of size $r_m - 1$ cannot.

Clearly, $t = t_1$ and $r = r_\ell$. Observe that the $\tau$ in requirement R1 could either be $t$ or it could be $t_i$ for some low value of $i$. From [2 Th. 6.7], [18 Th. 4] and [9 Th. 6] we have the following characterization of these parameters:

\begin{align}
(2) \quad & t_m = M_m((C_2)^\perp, (C_1)^\perp) - 1 \\
(3) \quad & r_m = n - M_{\ell-m+1}(C_1, C_2) + 1,
\end{align}
where $M_m(C_1, C_2)$ is the $m$-th relative generalized Hamming weight for $C_1$ with respect to $C_2$ and $(C)^{\perp}$ denotes the dual code of $C$. To make the section complete we need a formal definition of these parameters. We start by recalling the well-known concept of generalized Hamming weights \[16\] \[13\] \[24\]. Recall that for $D \subseteq (\mathbb{F}_q)^n$ the support of $D$ is defined as

$$\text{supp}(D) = \{i : c_i \neq 0 \text{ for some } \bar{c} = (c_1, \ldots, c_n) \in D\}.$$ 

**Definition 2.1.** Let $C$ be a linear code and $k$ its dimension. For $r = 1, \ldots, k$, the $r$-th generalized Hamming weight (GHW) of $C$ is defined by

$$d_r(C) = \min\{|\text{supp}(D)| : D \text{ is a linear subcode of } C \text{ and } \dim(D) = r\}.$$ 

The sequence $(d_1(C), \ldots, d_k(C))$ is called the hierarchy of the GHWs of $C$.

Note that in particular $d_1(C)$ is the minimum distance of $C$. A further generalization of GHWs was introduced by Luo et al. in \[21\].

**Definition 2.2.** Let $C_2 \subseteq C_1$ be linear codes, $\ell = \dim(C_1) - \dim(C_2)$ the codimension of $C_1$ and $C_2$, and $n$ the length of the codes. For $m = 1, \ldots, \ell$, the $m$-th relative generalized Hamming weight (RGHW) of $C_1$ with respect to $C_2$ is defined by

$$M_m(C_1, C_2) = \min_{J \subseteq \{1, \ldots, n\}} \{|J| : \dim((C_1)_J) - \dim((C_2)_J) = m\}$$

where $(C_1)_J = \{\bar{c} \in C_1 : c_t = 0 \text{ for } t \notin J\}$ for $i = 1, 2$. The sequence $(M_1(C_1, C_2), \ldots, M_\ell(C_1, C_2))$ is called the hierarchy of the RGHWs of $C_1$ with respect to $C_2$.

If $C_2$ is the zero code $\{\vec{0}\}$ then the $m$-th RGHW of $C_1$ with respect to $C_2$ is equivalent to the $m$-th GHW of $C_1$. This fact should be more clear from the following result \[20\] Lem. 1].

**Theorem 2.3.** Let $C_2 \subseteq C_1$ be linear codes and $\ell = \dim(C_1) - \dim(C_2)$ be the codimension of $C_1$ and $C_2$. For $m = 1, \ldots, \ell$ we have that

$$M_m(C_1, C_2) = \min\{|\text{supp}(D)| : D \text{ is a linear subcode of } C_1, \quad D \cap C_2 = \{\vec{0}\} \text{ and } \dim(D) = m\}.$$ 

This alternative characterization of RGHWs is useful when the codes are of an algebraic nature.

**Remark 2.4.** It is well-known that $RM_q(u, s)^{\perp} = RM_q((q - 1)m - u - 1, s)$, \[12\] Rem. 4.7]. Hence, when both $C_1$ and $C_2$ are $q$-ary Reed-Muller codes then the information leakage described in \[2\] and \[3\] is all about relative generalized Hamming weights of $q$-ary Reed-Muller codes.

**Remark 2.5.** As described in this section, in a ramp secret sharing scheme $C_1/C_2$, the code $C_1$ is divided into disjoint subsets each corresponding to a given message. The security comes from the randomness with which one picks the element of the subset. This randomness does not reduce (nor increase) the locally error-correcting ability of $C_1$ as the encoded message is still a word in $C_1$. Hence, if $C_1$ is a $q$-ary Reed-Muller code then Theorem \[13\] describes the ability to perform local error-correction in $C_1/C_2$.

We finally remark that the situation of secret sharing is more or less similar to that of communication over a wire-tap channel of type II \[25\], however we shall not pursue this connection any further in the present paper.
In the following four sections we shall concentrate on estimating the RGHWs of \( q \)-ary Reed-Muller codes which as noted gives an overview on the information leakage from the corresponding schemes \( C_1/C_2 \).

3. USEFUL TOOLS TO ESTABLISH THE RGHWs

We start our investigations by presenting in this section some theory that shall help us to derive the weights. The section also includes some initial results in this direction. First we elaborate slightly on the definition of \( q \)-ary Reed-Muller codes.

**Definition 3.1.** Let \( q \) be a power of a prime, \( u \) an integer, and \( s \) a positive integer, and write \( n = q^s \). We enumerate the elements of \((\mathbb{F}_q)^n\) as \( \{P_1, \ldots, P_n\} \) and consider the evaluation map \( \varphi : \mathbb{F}_q[X_1, \ldots, X_s] \rightarrow (\mathbb{F}_q)^n \), \( \varphi(f) = (f(P_1), \ldots, f(P_n)) \). The \( q \)-ary Reed-Muller code of order \( u \) is defined by

\[
RM_q(u, s) = \{\varphi(f) : f \in \mathbb{F}_q[X_1, \ldots, X_s], \deg(f) \leq u\}
\]

Observe that the equality in (4) is a consequence of the fact that \( \varphi(f) = \varphi(f \text{ rem } X_s^n - X_1, \ldots, X_s^n - X_s) \) for any \( f \in \mathbb{F}_q[X_1, \ldots, X_s] \). Here, the argument on the right side of [5] means the remainder of \( f \) after division with \( \{X_s^n - X_1, \ldots, X_s^n - X_s\} \) (see [5, Sec. 2.3] for the multivariate division algorithm). Furthermore note that \( \varphi \) is surjective which is seen by applying Lagrange interpolation. Dimension considerations now show that the restriction of \( \varphi \) to the span of

\[
R_q^s = \{X_s^a : 0 \leq a_i < q, i = 1, \ldots, s\}
\]

is a bijection and \( \{\varphi(M) : M \in R_q^s\} \) therefore is a basis for \((\mathbb{F}_q)^n\) as a vector space. We write

\[
Q^s_q = \{(a_1, \ldots, a_s) \in \mathbb{N}_0^s : 0 \leq a_i < q, i = 1, \ldots, s\}
\]

and \( \bar{X}^\bar{a} = X_s^{a_1} \cdots X_s^{a_s} \) for \( \bar{a} = (a_1, \ldots, a_s) \in \mathbb{N}_0^s \). Hence, \( R_q^s = \{\bar{X}^\bar{a} : \bar{a} \in Q_q^s\} \).

**Remark 3.2.** From the above discussion we conclude that if \( D \subseteq RM_q(u, s) \) is a subspace of dimension \( m \) then without loss of generality we may assume that \( D = \text{span}_q\{\varphi(F_1), \ldots, \varphi(F_m)\} \) where the leading monomials (with respect to the given fixed monomial ordering <) satisfy \( \text{lm}(F_i) \in R_q^s \), \( \text{lm}(F_i) \neq \text{lm}(F_j) \) for \( i \neq j \), and \( \deg(F_i) \leq u \) for \( i = 1, \ldots, m \). For given \( D \) and fixed < these leading monomials are unique.

We could calculate the RGHWs of \( q \)-ary Reed-Muller codes using the technique from [9] where the Feng-Rao bound for primary codes is employed. However, the simple algebraic structure of the \( q \)-ary Reed-Muller codes suggests that instead we should apply the footprint bound which we now introduce.

**Definition 3.3.** Let \( k \) be a field and consider an ideal \( J \subseteq k[X_1, \ldots, X_s] \) and a fixed monomial ordering <. Let \( M(X_1, \ldots, X_s) \) denote the set of monomials in the
variables \(X_1, \ldots, X_s\). The footprint of \(J\) with respect to \(\prec\) is the set
\[
\Delta_{\prec}(J) = \{ M \in \mathcal{M}(X_1, \ldots, X_s) : M \text{ is not leading monomial of any polynomial in } J \}.
\]

**Example 1.** We see immediately that \(\Delta_{\prec}((X_1^q - X_1, \ldots, X_s^q - X_s)) \subseteq R_q^s\).

From [3] Th. 6 we have the following well-known result.

**Theorem 3.4.** Let the notation be as in Definition 3.3. The set \(\{ M + J : M \in \Delta_{\prec}(J) \}\) is a basis for \(k[X_1, \ldots, X_s]/J\) as a vector space over \(k\).

**Example 2.** This is a continuation of Example 1. From Theorem 3.4 and the fact that \(\varphi: R_q^s \to (\mathbb{F}_q)^n\) is a bijection we conclude \(\Delta_{\prec}((X_1^q - X_1, \ldots, X_s^q - X_s)) = R_q^s\).

Consider polynomials \(F_1, \ldots, F_m \in \mathbb{F}_q[X_1, \ldots, X_s]\). Let \(\{Q_1, \ldots, Q_N\}\) be their common zeros over \(\mathbb{F}_q\) and define the vector space homomorphism \(\psi: \mathbb{F}_q[X_1, \ldots, X_s] \to (\mathbb{F}_q)^N\), \(\psi(f) = (f(Q_1), \ldots, f(Q_s))\). This map is surjective (Lagrange interpolation again) and by Theorem 3.4 the domain of \(\psi\) is a vector space of dimension \(|\Delta_{\prec}((F_1, \ldots, F_m, X_1^q - X_1, \ldots, X_s^q - X_s))|\) (independently of the chosen monomial ordering \(\prec\)). As a corollary to Theorem 3.4 we therefore obtain the following incidence of the footprint bound. For the general version of the footprint bound see [14] and [3] Pro. 8, Sec. 5.3.

**Lemma 3.5.** Let \(F_1, \ldots, F_m \in \mathbb{F}_q[X_1, \ldots, X_s]\). The number of common zeros of \(F_1, \ldots, F_m\) over \(\mathbb{F}_q\) is at most equal to \(|\Delta_{\prec}((F_1, \ldots, F_m, X_1^q - X_1, \ldots, X_s^q - X_s))|\) (here, \(\prec\) is any monomial ordering).

We note that actually equality holds in Lemma 3.5 (see [3] Pro. 8, Sec. 5.3), but we shall not need this fact. To make Lemma 3.5 operational we recall the following notation from [3].

**Definition 3.6.** The partial ordering \(\preceq_p\) on the monomials in \(R_q^s\) and on the elements in \(Q_q^s\) is defined by
\[
\vec{X}^\vec{a} \preceq_p \vec{X}^\vec{b} \quad \text{(or } \vec{a} \preceq_p \vec{b}) \quad \iff \quad a_i \leq b_i \quad \text{for all } i \in \{1, \ldots, s\}.
\]
The upward shadow of \(\vec{a} \in Q_q^s\) is \(\nabla \vec{a} = \{ \vec{b} \in Q_q^s : \vec{b} \preceq_p \vec{a} \}\).
The lower shadow of \(\vec{a} \in Q_q^s\) is \(\Delta \vec{a} = \{ \vec{b} \in Q_q^s : \vec{b} \geq_p \vec{a} \}\).
Let \(A \subseteq Q_q^s\), we define \(\nabla A = \bigcup_{\vec{a} \in A} \nabla \vec{a}\) and \(\Delta A = \bigcup_{\vec{a} \in A} \Delta \vec{a}\).

**Example 3.** For \(\vec{a} = (2, 3) \in Q_q^3\) we have that
\[
\nabla \vec{a} = \{(2, 3), (3, 3)\}
\]
\[
\Delta \vec{a} = \{(2, 3), (1, 3), (0, 3), (2, 2), (1, 2), (2, 0), (2, 1), (1, 1), (0, 1), (2, 0), (1, 0), (0, 0)\}.
\]
The partial ordering is not a total ordering; for example we neither have \((3, 2) \preceq_p (2, 3)\) nor \((3, 2) \geq_p (2, 3)\).

An important tool for calculating RGHWs of \(q\)-ary Reed-Muller codes is the following corollary to Lemma 3.5.

**Corollary 3.7.** Consider any monomial ordering and let \(D = \text{span}_\mathbb{F}_q \{ \varphi(F_1), \ldots, \varphi(F_m) \}\) be a subspace of \((\mathbb{F}_q)^n\) of dimension \(m\) where without loss of generality we assume \(\text{Im}(F_i) = \vec{X}^{\vec{a}_i} \in R_q^s\) for \(i = 1, \ldots, m\) and \(\vec{a}_i \neq \vec{a}_j\) for \(i \neq j\) (Remark 3.2).
Writing \(A = \{\vec{a}_1, \ldots, \vec{a}_m\}\) we have \(|\text{supp}(D)| \geq |\nabla A|\).
Proposition 3.8. Consider any monomial ordering and for which the bound is sharp.

Proof. The elements of $D$ are linear combination of $\varphi(F_1), \ldots, \varphi(F_m)$, hence $|\supp(D)|$ equals the length $n$ minus the number of common zeros of $F_1, \ldots, F_m$ over $\mathbb{F}_q$. By Lemma 3.5 we get

$$|\supp(D)| \geq n - |\Delta_\prec((F_1, \ldots, F_m, X^q_i - X_1, \ldots, X^q_s - X_s))|$$

$$\geq n - \left|\left(\Delta_\prec((X^q_i - X_1, \ldots, X^q_s - X_s)) \setminus \bigcup_{i=1}^m \{\vec{x} \mid \vec{x}.X \in \Delta_\prec((X^q_i - X_1, \ldots, X^q_s - X_s)) : \vec{x}.X \text{ is divisible by } \vec{x}.X_i \}\right)\right|$$

$$= n - |R^q_{\vec{a}}| + \left|\bigcup_{i=1}^m \{\vec{a} \in Q^q_s : \vec{a} \supseteq \vec{x}.X_i\}\right| = \left|\bigcup_{i=1}^m \nabla \vec{a}_i\right| = |\nabla A|$$

and the proof is complete. \hfill $\square$

Interestingly for any choice of $A$ as in Corollary 3.7 there exists some subspaces $D$ for which the bound is sharp.

Proposition 3.8. Consider any monomial ordering and $A = \{\vec{a}_1, \ldots, \vec{a}_m\} \subseteq Q^s_q$ where $\vec{a}_i \neq \vec{a}_j$ for $i \neq j$. Then

$$\min\{|\supp(D)| : D = \text{span}_{\mathbb{F}_q}\{\varphi(F_1), \ldots, \varphi(F_m)\} \text{ for some } F_1, \ldots, F_m$$

$$\text{with } \text{lm}(F_i) = \vec{x}.X_i, i = 1, \ldots, m\} = |\nabla A|.$$ 

Proof. From Corollary 3.7 we know that

$$\min\{|\supp(D)| : D = \text{span}_{\mathbb{F}_q}\{\varphi(F_1), \ldots, \varphi(F_m)\} \text{ for some } F_1, \ldots, F_m$$

$$\text{with } \text{lm}(F_i) = \vec{x}.X_i, i = 1, \ldots, m\} \geq |\nabla A|.$$ 

Now we want to prove the other inequality. Let $F_q = \{\gamma_0, \ldots, \gamma_{q-1}\}$ and $\vec{a} = (a_1, \ldots, a_s) \in Q^s_q$, we write $\vec{a} = (\gamma_{a_1}, \ldots, \gamma_{a_s})$. For $i = 1, \ldots, m$, we write the coordinates of $\vec{a}_i$ as $(a_{i,1}, a_{i,2}, \ldots, a_{i,s})$. We define the following subspace of $(F_q)^n$:

$$\tilde{D} = \text{span}_{\mathbb{F}_q}\{\varphi(G_1), \ldots, \varphi(G_m)\} \text{ with } G_i = \prod_{t=1}^s \prod_{j=0}^{a_{i,t}-1} (X_t - \gamma_j) \text{ for } i = 1, \ldots, m.$$ 

For $i = 1, m$ we have $\text{lm}(G_i) = \vec{x}.X_i$. Furthermore $G_i(\gamma_{a}) \neq 0$ if and only if $\vec{a} \in Q^s_q \text{ satisfies } \vec{a} \supseteq \vec{x}.X_i$. The last result is equivalent to saying that $G_i(\gamma_{a}) \neq 0$ if and only if $\vec{a} \in \nabla \vec{a}_i$. The support of $\tilde{D}$ is the union of all positions where some $\varphi(G_i)$ does not equal 0. Hence, $|\supp(\tilde{D})| = \left|\bigcup_{i=1}^m \nabla \vec{a}_i\right| = |\nabla A|$.

Recall that a q-ary Reed-Muller code is defined as

$$\text{RM}_q(u, s) = \text{span}_{\mathbb{F}_q}\{\varphi(f) : f \in R^q_s, \deg(f) \leq u\}.$$ 

As is well-known the minimum distance strictly increases when $u$ increases (until the code equals $(\mathbb{F}_q)^n$). Hence, if we consider two codes $C_1 = \text{RM}_q(u_1, s)$, $C_2 = \text{RM}_q(u_2, s)$ with $u_2 < u_1$ then

$$M_1(C_1, C_2) = d$$

(6) where $d$ is the minimum distance of $C_1$. From Proposition 3.8 it is not difficult to establish the other extreme case, namely that of $M_e(C_1, C_2)$ where $e = \dim C_1 -
dim $C_2$. We have $M_1(C_1, C_2) = |\nabla A|$ where

$$A = \{(a_1, \ldots, a_s) : 0 \leq a_i < q, i = 1, \ldots, s, u_2 < \sum_{i=1}^{s} a_1 \leq u_1\}.$$ 

We have $|Q^s_q \setminus \nabla A| = \dim C_2$ and therefore

$$M_1(C_1, C_2) = n - \dim C_2.$$ 

Treating the intermediate cases is much more subtle. This is done in the following sections.

4. RGHWs of $q$-ary Reed-Muller codes

In this section we employ Proposition 3.8 to compute the hierarchy of RGHWs in the case that $C_1$ and $C_2$ are both $q$-ary Reed-Muller codes. The main result is Theorem 4.9.

Our method for calculating the hierarchy of RGHWs involves the anti lexicographic ordering on the monomials in $R^s_q$ (and on the elements in $Q^s_q$). To relate our findings to Heijnen and Pellikaan’s work on GHWs we also need the lexicographic ordering on the same sets.

**Definition 4.1.** The lexicographic ordering $\prec_{\text{Lex}}$ on the monomials in $R^s_q$ and on the elements in $Q^s_q$ is defined by

$$X^\vec{a} \prec_{\text{Lex}} X^\vec{b} \text{ (or } \vec{a} \prec_{\text{Lex}} \vec{b}) \iff a_1 = b_1, \ldots, a_{l-1} = b_{l-1} \text{ and } a_l < b_l \text{ for some } l.$$ 

The anti lexicographic ordering $\prec_A$ on the monomials in $R^s_q$ and on the elements in $Q^s_q$ is defined by

$$X^\vec{a} \prec_A X^\vec{b} \text{ (or } \vec{a} \prec_A \vec{b}) \iff a_s = b_s, \ldots, a_{s-l+1} = b_{s-l+1} \text{ and } a_{s-l} > b_{s-l} \text{ for some } l.$$ 

**Example 4.** For $s = 2$, $q = 3$ with $X = X_1$ and $Y = X_2$ we have

$$1_{\prec_{\text{Lex}}} Y \prec_{\text{Lex}} Y^2 \prec_{\text{Lex}} X \prec_{\text{Lex}} XY \prec_{\text{Lex}} X^2 \prec_{\text{Lex}} X^2 Y \prec_{\text{Lex}} X^2 Y^2,$$

$$X^2 Y^2 \prec_A X Y^2 \prec_A X^2 Y \prec_A X Y \prec_A X^2 \prec_A X \prec_A 1.$$ 

From this example it is easy to see that the anti lexicographic ordering is not the inverse ordering of the lexicographic ordering. Recalling from Definition 3.6 the ordering $\preceq_p$ we note that if $X^\vec{a} \preceq_p X^\vec{b}$ (or $\vec{a} \preceq_p \vec{b}$) then $X^\vec{a} \preceq_{\text{Lex}} X^\vec{b}$ and $X^\vec{a} \preceq_A X^\vec{b}$ (or $\vec{a} \preceq_{\text{Lex}} \vec{b}$ and $\vec{a} \preceq_A \vec{b}$).

The following concepts will be used extensively throughout our exposition.

**Definition 4.2.** Given $\vec{a} = (a_1, \ldots, a_s) \in Q^s_q$, we call $\deg(\vec{a}) = \deg(X^\vec{a}) = \sum_{t=1}^{s} a_t$ the degree of $\vec{a}$. Let $a, b$ be two integers with $0 \leq a \leq b \leq s(q-1)$, then we define

$$F_q((a, b), s) = \{\vec{a} \in Q^s_q : a \leq \deg(\vec{a}) \leq b\} \text{ and}$$

$$W_q((a, b), s) = \{X^\vec{a} \in R^s_q : \vec{a} \in F_q((a, b), s)\}.$$ 

The index $q$ and the value $s$ will be omitted in the rest of this section, thus instead we will use the notations $F(a, b)$ and $W(a, b)$, respectively.

**Definition 4.3.** Let $m \in \{1, \ldots, |F(a, b)|\}$, we denote by $L((a, b), m)$ the set of the first $m$ elements of $F(a, b)$ using the lexicographic ordering and by $N((a, b), m)$ the set of the first $m$ elements of $F(a, b)$ using the anti lexicographic ordering.
The sets $N_{(a,b)}(m)$ will play a crucial role in the following derivation of a formula for the RGHWs of $q$-ary Reed-Muller codes. The sets $L_{(a,b)}(m)$ shall help us establish the connection to the work by Heijnen and Pellikaan on GHWs. Their main result [12, Th. 5.10] is as follows:

**Theorem 4.4.** Let $\vec{a} = (a_1, \ldots, a_s)$ be the $r$-th element in $F(s(q-1) - u_1, s(q-1))$ with respect to the lexicographic ordering. Then

$$d_r(RM_q(u_1, s)) = |\Delta L_{(s(q-1) - u_1, s(q-1))}(r)| = \sum_{i=1}^{s} a_{s-i+1}q^{i-1} + 1.$$  \hspace{2cm} (8)

Before continuing our work on establishing the RGHWs we reformulate the expressions in (8). We shall need the following result corresponding to [12, Lem. 5.8].

**Lemma 4.5.** Let $t$ be an integer satisfying $1 \leq t \leq q^s$. Write $t - 1 = \sum_{i=1}^{s} a_{s-i+1}q^{i-1}$. Then $(a_1, \ldots, a_s)$ is the $t$-th element of $Q_q^s$ with respect to the lexicographic ordering.

Also we shall need the bijection $\mu : Q_q^s \to Q_q^s$ given by $\mu(a_1, \ldots, a_s) = (q - 1 - a_s, \ldots, q - 1 - a_1)$. Observe that $\mu$ has the properties

- $\vec{a} \prec L \vec{b} \iff \mu(\vec{a}) \prec \text{Lex} \mu(\vec{b})$,
- $\mu(F(a,b)) = F(s(q-1) - b, s(q-1) - a)$,
- $\mu(\nabla N_{(a,b)}(m)) = \Delta L_{(s(q-1) - b, s(q-1) - a)}(m)$.

For the proofs and other properties of $\mu$ we refer to Lemma A.1 in Appendix A. Note that by the first property an element $\vec{a}$ in a subset $A$ of $Q_q^s$ is the $t$-th element in $A$ using the anti lexicographic ordering if and only if $\mu(\vec{a})$ is the $t$-th element in $\mu(A)$ using the lexicographic ordering. We can now reformulate Theorem 4.4 into the following result which is not stated in [12].

**Theorem 4.6.** Let $\vec{a}$ be the $r$-th element in $F(0, u_1)$ using the anti lexicographic ordering. Because $F(0, u_1) \subseteq Q_q^s$ there exists $t$ such that $\vec{a}$ is the $t$-th element in $Q_q^s$ using the anti lexicographic ordering. We have

$$d_r(RM_q(u_1, s)) = |\nabla N_{(0,u_1)}(r)| = t.$$  

**Proof.** By the properties of $\mu$ and using the lexicographic ordering, we have that $\mu(\vec{a}) = (\tilde{a}_1, \ldots, \tilde{a}_s)$ is the $r$-th element in $F(s(q-1) - u_1, s(q-1))$ and the $t$-th element in $Q_q^s$. From Theorem 4.4 we get

$$d_r(RM_q(u_1, s)) = |\Delta L_{(s(q-1) - u_1, s(q-1))}(r)| = \sum_{i=1}^{s} \tilde{a}_{s-i+1}q^{i-1} + 1$$

where by Lemma 4.5 the last expression can be rewritten as $\sum_{i=1}^{s} \tilde{a}_{s-i+1}q^{i-1} + 1 = t - 1 + 1 = t$.

From the third listed property of $\mu$ we obtain

$$|\nabla N_{(0,u_1)}(r)| = |\mu(\nabla N_{(0,u_1)}(r))| = |\Delta L_{(s(q-1) - u_1, s(q-1))}(r)|.$$  

Having reformulated the formula by Heijnen and Pellikaan for GHWs we now continue our work on establishing a formula for the RGHWs. Consider $C_2 = RM_q(u_2, s) \subseteq C_1 = RM_q(u_1, s)$. Let $\ell$ be the codimension of $C_1$ and $C_2$, then
for $m = 1, \ldots, \ell$ we have that

\[
M_m(C_1, C_2) = \min\{|\text{supp}(D)| : D \text{ is a linear subcode of } C_1, \\
D \cap C_2 = \{0\} \text{ and } \dim(D) = m\}
\]

(9)

\[
= \min\{|\text{supp}(D)| : D = \text{span}_{\mathbb{F}} \{\varphi(F_1), \ldots, \varphi(F_m)\}\},
\]

\[
\text{lm}(F_i) = \tilde{X}^{\tilde{a}_i}, \ldots, \text{lm}(F_m) = \tilde{X}^{\tilde{a}_m}, \tilde{a}_i \neq \tilde{a}_j \text{ for } i \neq j
\]

(10)

and $\tilde{X}^{\tilde{a}_i} \in W(u_2 + 1, u_1)$ for $i = 1, \ldots, m$,

where in Equation (10) we use a degree lexicographic ordering. Equation (9) corresponds to Theorem 2.3. Equation (10) follows from Proposition 3.8, Remark 3.2 and the fact that $D \subseteq C_1$ implies $\text{lm}(F_i) \in W(0, u_1), i = 1, \ldots, m$ and from the fact that $D \cap C_2 = \{0\}$ implies $\text{lm}(F_i) \notin W(0, u_2), i = 1, \ldots, m$. In conclusion $\text{lm}(F_i) \in W(u_2 + 1, u_1), i = 1, \ldots, m$. Combining (10) with Proposition 3.8 we get

\[
M_m(C_1, C_2) = \min\{|\bigcup_{i=1}^{m} \nabla \tilde{a}_i| : \tilde{a}_i \in F(u_2 + 1, u_1), i = 1, \ldots m \\
\text{and } \tilde{a}_i \neq \tilde{a}_j, \text{ for } i \neq j\}
\]

(11)

\[
= \min\{|\nabla A| : A \subseteq F(u_2 + 1, u_1), |A| = m\}.
\]

The following lemma – which can be viewed as a generalization of [11, Th. 3.7.7] – is proved in Appendix A.

Lemma 4.7. Let $A$ be a subset of $F(a, b)$ consisting of $m$ elements. Then $|\nabla N(a,b)(m)| \leq |\nabla A|$.

Proposition 4.8. Let $C_2 = \text{RM}_q(u_2, s) \subseteq C_1 = \text{RM}_q(u_1, s)$. We have

\[
M_m(C_1, C_2) = |\nabla N(u_2+1,u_1)(m)|
\]

Proof. Follows from (11) and Lemma 4.7.

We are now ready to present the generalization of Theorem 4.6 to RGHWs.

Theorem 4.9. Given $C_2 = \text{RM}_q(u_2, s) \subseteq C_1 = \text{RM}_q(u_1, s), \text{ let } \tilde{a} \text{ be the } m\text{-th element in } F(u_2 + 1, u_1) \text{ with respect to the anti lexicographic ordering. Because } F(u_2 + 1, u_1) \subseteq F(0, u_1) \subseteq \mathbb{Q}_q^+ \text{ there exist } r \text{ and } t \text{ such that } \tilde{a} \text{ is the } r\text{-th element in } F(0, u_1) \text{ and the } t\text{-th element in } \mathbb{Q}_q^+ \text{ with respect to the anti lexicographic ordering. We have}

\[
M_m(C_1, C_2) = t - r + m.
\]

Proof. By Proposition 4.8 we have already proved that $M_m(C_1, C_2) = |\nabla N(u_2+1,u_1)(m)|$. It remains to be proved that $|\nabla N(u_2+1,u_1)(m)| = t - r + m$. Because $\tilde{a}$ is the $m$-th element in $F(u_2 + 1, u_1)$ and the $r$-th element in $F(0, u_1)$ we have

\[
N(0,u_1)(r) = N(0,u_2)(r - m) \cup N(u_2+1,u_1)(m)
\]

from which we derive

\[
\nabla N(0,u_1)(r) = \nabla N(u_2+1,u_1)(m) \cup \nabla N(0,u_2)(r - m)
\]

(12)

\[
= \nabla N(u_2+1,u_1)(m) \cup (\nabla N(0,u_2)(r - m) \setminus \nabla N(u_2+1,u_1)(m)).
\]

The union in (12) involves two disjoint sets. Hence,

\[
|\nabla N(u_2+1,u_1)(m)| = |\nabla N(0,u_1)(r)| - |\nabla N(0,u_2)(r - m) \setminus \nabla N(u_2+1,u_1)(m)|.
\]
From Theorem 4.6 we have \(|\nabla N_{(0,u_1)}(r)| = t\). Hence, we will be through if we can prove that
\[
|\nabla N_{(0,u_2)}(r-m)\setminus \nabla N_{(u_2+1,u_1)}(m)| = r-m. 
\]
We enumerate \(N_{(0,u_2)}(r-m) = \{\bar{a}_1, \ldots, \bar{a}_{r-m}\}\) according to the anti lexicographic ordering. We have
\[
\nabla N_{(0,u_2)}(r-m)\setminus \nabla N_{(u_2+1,u_1)}(m) = \left( \bigcup_{i=1}^{r-m} \nabla \bar{a}_i \right) \setminus \nabla N_{(u_2+1,u_1)}(m) 
= \left( \bigcup_{i=1}^{r-m} \nabla \bar{a}_i \setminus \nabla \{\bar{a}_t : t < i\} \right) \setminus \nabla N_{(u_2+1,u_1)}(m) 
(14) = \bigcup_{i=1}^{r-m} \left( \nabla \bar{a}_i \setminus \nabla \{\bar{a}_t : t < i\} \cup \nabla N_{(u_2+1,u_1)}(m) \right).
\]
We will prove that
\[
\nabla \bar{a}_i \setminus \nabla \{\bar{a}_t : t < i\} \cup \nabla N_{(u_2+1,u_1)}(m) = \{\bar{a}_i\} 
\]
holds for \(i = 1, \ldots, r-m\). As \(\bar{a}_i \succ \bar{a}_t\) for \(t < i\), we have \(\bar{a}_i \notin \nabla \{\bar{a}_t : t < i\}\). Furthermore from \(\deg(\bar{a}_i) \leq u_2\) and \(\deg(\bar{c}) \geq u_2 + 1\) for any \(\bar{c} \in \nabla N_{(u_2+1,u_1)}(m)\), we conclude \(\bar{a}_i \notin \nabla N_{(u_2+1,u_1)}(m)\). It follows that
\[
\{\bar{a}_i\} \subseteq \nabla \bar{a}_i \setminus \nabla \{\bar{a}_t : t < i\} \cup \nabla N_{(u_2+1,u_1)}(m). 
\]
Now we prove the other inclusion. Assume first \(\bar{a}_t \in F(u_2, u_2)\). For \(t = 1, \ldots, s\) we define \(\bar{b}_t = \bar{a}_t + \bar{e}_t\) where \(\bar{e}_t\) is the standard vector with 1 in the \(t\)-th position. If \(\bar{b}_t \in Q_q^s\) then \(\bar{b}_t \in N_{(u_2+1,u_1)}(m)\) because \(\deg(\bar{b}_t) = u_2 + 1\) and \(\bar{a}_t \succ \bar{a}_t \succ \bar{b}_t\). It follows that
\[
\nabla \bar{a}_i \setminus \nabla \{\bar{a}_t : t < i\} \cup \nabla N_{(u_2+1,u_1)}(m) \subseteq \nabla \bar{a}_i \setminus \nabla \{\bar{b}_1, \ldots, \bar{b}_s\} \cap Q_q^s = \{\bar{a}_i\}. 
\]
Assume next \(\bar{a}_t \notin F(u_2, u_2)\). Again we define \(\tilde{\bar{b}}_t = \bar{a}_t + \bar{e}_t\) for \(t = 1, \ldots, s\). If \(\tilde{\bar{b}}_t \in Q_q^s\) then \(\tilde{\bar{b}}_t \in \{\bar{a}_t : t < i\}\) because \(\deg(\tilde{\bar{b}}_t) \leq u_2\) and \(\bar{a}_t \succ \bar{b}_t\). Hence,
\[
\nabla \bar{a}_i \setminus \nabla \{\bar{a}_t : t < i\} \cup \nabla N_{(u_2+1,u_1)}(m) \subseteq \nabla \bar{a}_i \setminus \nabla \{\tilde{\bar{b}}_1, \ldots, \tilde{\bar{b}}_s\} \cap Q_q^s = \{\bar{a}_i\}. 
\]
We have established (15). Combining finally (15) and (14) we obtain
\[
\nabla N_{(0,u_2)}(r-m)\setminus \nabla N_{(u_2+1,u_1)}(m) = \bigcup_{i=1}^{r-m} \{\bar{a}_i\} = N_{(0,u_2)}(r-m). 
\]
By Definition 4.3 the last set is of size \(r - m\) and (13) follows.

Consider the special case of Theorem 4.9 where \(C_2 = \{0\} = \text{RM}_q(-1, s)\). In this particular case we have – as already noted – \(d_m(C_1) = M_m(C_1, C_2)\). If we apply Theorem 4.9 and the notion in there then we obtain \(r = m\) and consequently \(M_m(C_1, C_2) = t\). Theorem 4.6 gives us the same information \(d_m(C_1) = t\).

We illustrate the use of Theorem 4.6 and Theorem 4.9 with an example.

Example 5. In this example we consider Reed-Muller codes in two variables over \(\mathbb{F}_5\). We first consider the case \(C_1 = \text{RM}_5(5, 2)\) and \(C_2 = \text{RM}_5(3, 2)\). Figure 4 illustrates how to find \(r\) and \(m\) for any given \(t\) and how to calculate \(d_t(C_1)\) and \(M_m(C_1, C_2)\) from this information. The elements of \(Q_5^2\) are depicted in Part 4.1. In Parts 4.2, 4.3, and 4.4 we illustrate how the elements of \(Q_5^2\), \(F(0, 5)\) and \(F(4, 5)\),
for the above choice of $C_1$ and $C_2$ illustrates.

For the above choice of $C_1$ and $C_2$ most of the time the GHWs and RGHWs are the same. This however, is not the general situation for $q$-ary Reed-Muller codes as the following choices of $C_1$ and $C_2$ illustrate.

In the remaining part of this example we concentrate on $q$-ary Reed-Muller codes $C_1 = \text{RM}_5(u_1,2)$, $C_2 = \text{RM}_5(u_2,2)$ where $u_1 = u_2 + 1$. In Table 1, Table 2, Table 3, Table 4 and Table 5, respectively, we present parameters $d_r(C_1)$ and $M_m(C_1, C_2)$ for $(u_1, u_2)$ equal to $(2,1)$, $(3,2)$, $(4,3)$, $(5,4)$, and $(6,5)$ respectively.

Table 1. $C_1 = \text{RM}_5(2,2), C_2 = \text{RM}_5(1,2)$.

\begin{table}[h]
\begin{tabular}{|c|c|c|c|}
\hline
\(r = m\) & \(d_r(C_1)\) & \(M_m(C_1, C_2)\) \\
\hline
1 & 15 & 15 \\
2 & 19 & 19 \\
3 & 20 & 22 \\
\hline
\end{tabular}
\end{table}
By Theorem 4.9 there are still two questions that need to be addressed:

Q1 Given \( m \in \{1, \ldots, |F_q((a,b),s)|\} \), how can we find the \( m \)-th element \( \vec{a} \) of \( F_q((a,b),s) \) with respect to the anti lexicographic ordering?

Q2 Given \( \vec{a} \in F_q((a,b),s) \) how can we find the corresponding position \( t \) and \( r \) – with respect to the anti lexicographic ordering – in \( Q_q^s \) and in \( F_q((0,b),s) \), respectively?

In this section we give answers to these two questions. We start by providing an algorithm that solves the problem from question Q1. This algorithm is a generalization of a method proposed in [12, Sec. 6]. Due to the nature of the algorithm from now on we will – in contrast to the previous section – use the full notation \( F_q((a,b),s) \), rather than just \( F_q((a,b)) \) (Definition 4.2).

**Definition 5.1.** Let \( 0 \leq a \leq b \leq s(q - 1) \) and \( 0 \leq v \leq w < q \) be integers. We define

\[
F_q((a,b),(v,w),s) = \{(a_1, \ldots, a_s) \in F_q((a,b),s) : v \leq a_s \leq w\}.
\]
procedure VECA(A, B, V, S, M, q): Non-negative integers with $A \leq B \leq S(q-1)$, $V \leq q-1$, $1 \leq S$, and $M \in \{1, \ldots, |F_q((A, B), (0, V), S)|\})$

if $V > B$ then

VECA(A, B, V, S, M, q) ← VECA(A, B, B, S, M, q)

else

if $S \neq 1$ then

$\alpha \leftarrow \max\{A - V, 0\}$
$r \leftarrow \rho_q((\alpha, B - V), (S - 1))$

if $M > r$ then

VECA(A, B, V, S, M, q) ← VECA(A, B, V - 1, S, M - r, q)

else if $M < r$ then

VECA(A, B, V, S, M, q) ← (VECA($\alpha$, B - V, q - 1, S - 1, M, q), V)

else

$\theta_1 \leftarrow \alpha \mod (q - 1)$
$\theta_2 \leftarrow (\alpha - \theta_1)/(q - 1)$

if $\theta_2 < S - 1$ then

VECA(A, B, V, S, M, q) ← $(q - 1, \ldots, q - 1, \theta_1, 0, \ldots, 0, V)_{\theta_2}$

else

VECA(A, B, V, S, M, q) ← $(q - 1, \ldots, q - 1, V)_{\theta_2}$

end if

end if

end if

end procedure

Figure 2. The recursive algorithm VECA. We use the notation

$((\beta_1, \ldots, \beta_{k-1}), \beta_k) = (\beta_1, \ldots, \beta_{k-1}, \beta_k)$ for concatenation.

We denote by $\rho_q((a, b), s)$ and $\rho_q((a, b), (v, w), s)$ the cardinality of $F_q((a, b), s)$ and $F_q((a, b), (v, w), s)$, respectively. Most of the time the index $q$ will be omitted.

**Theorem 5.2.** Let $q$ be a fixed prime power and consider non-negative integers $a, b, v, s, m$ with

$a \leq b \leq s(q-1)$, $v \leq q-1$, $1 \leq s$, and $m \in \{1, \ldots, |F_q((a, b), (0, v), s)|\}$.

If these numbers are used as input to the procedure VECA in Figure 2 then the output is the $m$-th element $\vec{a} = (a_1, \ldots, a_s)$ of $F_q((a, b), (0, v), s)$ with respect to the anti lexicographic ordering.

**Proof.** Consider the condition

C1: $A, B, V, S, M$ are non-negative integers with $A \leq B \leq S(q-1)$, $V \leq q - 1$, $1 \leq S$ and $M \in \{1, \ldots, |F_q((A, B), (0, V), S)|\}$.

We first show that the following loop invariant holds true:

[Logical expression and proof details]
• If \( V > B \) and \( A, B, V, S, M \) satisfy Condition C1 then the elements of \((\tilde{A}, \tilde{B}, \tilde{V}, \tilde{S}, \tilde{M}) = (A, B, B, S, M)\) satisfy Condition C1.

• If \( V \leq B \), \( S \neq 1 \) and \( A, B, V, S, M \) satisfy Condition C1 then:
  
  - for \( M > r \) the elements in \((\tilde{A}, \tilde{B}, \tilde{V}, \tilde{S}, \tilde{M}) = (A, B, V - 1, S, M - r)\) satisfy Condition C1,
  
  - for \( M < r \) the elements in \((\tilde{A}, \tilde{B}, \tilde{V}, \tilde{S}, \tilde{M}) = (\alpha, B - V, q - 1, S - 1, M)\) satisfy Condition C1. Here \( \alpha = \max\{A - V, 0\} \).

Assume first \( V > B \). We have \( F_q((A, B), (0, V), S) = F_q((A, B), (0, B), S) \) and the result follows. Assume next \( V \leq B \) and \( S \neq 1 \). We consider the case \( M > r \) (line 8–9) and leave the case \( M < r \) for the reader. By inspection \( \tilde{A} \leq \tilde{B} \leq \tilde{S}(q - 1) \), \( V < q - 1, 1 \leq \tilde{S} \), and \( \tilde{A}, \tilde{B}, \tilde{S}, \tilde{M} \) are non-negative. Aiming for a contradiction we assume that \( V = 0 \) is possible (which would cause \( \tilde{V} \) to be negative). But then

\[
\begin{align*}
  r &= \rho((\alpha, B - V), S - 1) = \rho((A, B), S - 1) \\
  &= \rho((A, B), (0, 0), S) = \rho((A, B), (0, V), S) \geq M
\end{align*}
\]

where the inequality follows by the assumption that \( A, B, V, S, M \) satisfy Condition C1. We have reached a contradiction. Hence, we conclude \( 0 < V \) and therefore \( \tilde{V} \) is non-negative. We next show that \( \tilde{M} = M - r \) is in the desired interval. Clearly \( M = M - r \geq 1 \). To demonstrate that \( \tilde{M} \leq |F_q((\tilde{A}, \tilde{B}), (0, \tilde{V}), \tilde{S})| \) we note that

\[
\begin{align*}
  M &\leq \rho((A, B), (0, V), S) \\
  &= \rho((A, B), (0, V - 1), S) + \rho((A, B), (V, V), S) \\
  &= \rho((A, B), (0, V - 1), S) + \rho((\alpha, B - V), S - 1) \\
  &= \rho((\tilde{A}, \tilde{B}), (0, \tilde{V}), \tilde{S}) + r
\end{align*}
\]

and the last part of Condition C1 is established.

Let \((A_i, B_i, S_i, M_i)\) be the value of \((A, B, S, M)\) before entering the loop the \( i \)-th time. The sequence \((\{A_1, B_1, S_1, M_1\}, \{A_2, B_2, S_2, M_2\}, \ldots)\) is strictly decreasing with respect to the partial ordering \( \preceq \), and as \( A, B, S, M \) are upper bounded as well as lower bounded the sequence must be finite, meaning that the algorithm terminates.

We next give an induction proof that the algorithm returns the \( M \)-th element of \( F_q((A, B), (0, V), S) \) with respect to the anti lexicographic ordering.

Basis step:

First assume \( V \leq B \), \( S \neq 1 \) and let \( \theta_1 \) and \( \theta_2 \) be as in line 14 and 15 of the algorithm. Observe that \( \theta_2 \leq S - 1 \) as \( \theta_2 = S \) would imply \( V = 0 \) and consequently \( F_q((A, B), (0, V), S) = \emptyset \). This is not possible as by Condition C1, \( M \in \{1, \ldots, |F_q((A, B), (0, V), S)|\} \). Consider the last element of \( F_q((\alpha, B - V), (V, V), S) \) i.e.

\[
(q - 1, \ldots, q - 1, \underbrace{\theta_1, 0, \ldots, 0}_{\theta_2 - 2}, V)
\]

if \( \theta_2 < S - 1 \), and

\[
(q - 1, \ldots, q - 1, V)
\]

if \( \theta_2 = S - 1 \) (in which case \( \theta_1 = 0 \)). This element is the \( r \)-th element of \( F_q((A, B), (0, V), S) \) where \( r \) is as in line 7. Hence, if \( M = r \) (lines 13–22) then indeed \( \text{VECA}(A, B, V, S, M, q) \) equals the element in position \( M \) of \( F_q((A, B), (0, V), S) \).
Assume next $V \leq B$ and $S = 1$. We see that the $M$-th element of $F_q((A, B), (0, V), 1)$ equals $(V - (M - 1))$ which corresponds to line 24.

Induction step:
If $V > B$ then as already noted $F_q((A, B), (0, V), S) = F_q((A, B), (0, B), S)$. For $V \leq B$, $S \neq 1$ we next consider the cases $M > r$ and $M < r$ separately.

We first consider $M > r$ corresponding to lines 8–9 of the algorithm. We have

\[ \rho((A, B), (V, V), S) = \rho((A, B - V), (0, q - 1), S - 1) = r. \]

But $M > r$ and therefore the $M$-th element of $F_q((A, B), (0, V), S)$ equals the $(M - r)$-th element of $F_q((A, B), (0, V - 1), S)$.

We next consider the case $M < r$. Using similar arguments as above we see that the $M$-th element of $F_q((A, B), (0, V), S)$ is in $F_q((A, B), (V, V), S)$. Therefore it equals $(\beta_1, \ldots, \beta_{S-1}, V)$ where $(\beta_1, \ldots, \beta_{S-1})$ is the $M$-th element of $F_q((A, B - V), (0, q - 1), S - 1)$. \(\square\)

Note that for our purpose (that is, to answer Q1), the input $V$ in the algorithm VECA shall always be equal to $q - 1$. The procedure VECA in Figure 2 uses the value $\rho_q((A, B), S)$ for various choices of $A, B, S$. We therefore need an algorithm to compute this number.

\begin{lemma}
Let $q$ be a prime power and consider integers $a, b, s$ with $0 \leq a \leq b \leq s(q - 1)$ and $s \geq 1$. We have

\[ \rho_q((a, b), s) = \sum_{i=a}^{b} \sum_{j=0}^{|i/q|} (-1)^j \binom{s}{j} \left(\frac{s - 1 + i q}{s - 1}\right). \]

\end{lemma}

\begin{proof}
We rewrite the first expression as follows

\[ \rho_q((a, b), s) = |F_q((a, b), s)| - |W_q((a, b), s)] = |W_q((0, b), s)] - |W_q((0, a - 1), s)] \]

\[ = \dim(RM_q(b, s)) - \dim(RM_q(a - 1, s)). \]

By 22 and by Exercise 1.2.8 of 23 we have that

\[ \dim(RM_q(u, s)) = \sum_{i=0}^{u} \sum_{j=0}^{|i/q|} (-1)^j \binom{s}{j} \left(\frac{s - 1 + i q}{s - 1}\right) \]

and the proof follows. \(\square\)
Theorem 5.4. Let \( q \) be a prime power and consider \( a, b, s \) as in Lemma 5.3. If the procedure \( \text{RHO} \) (see Figure 3) is used with input \( a, b, s, q \) then it returns \( \rho_q((a, b), s) \).

Proof. By Lemma 5.3.

Assuming that \( \text{VECA} \) calls \( \text{RHO} \), we can now estimate its time complexity.

Lemma 5.5. The number of binary operations needed to run \( \text{RHO} \) with input \( a, b, s, q \) is

\[
\mathcal{O}\left(\frac{bc}{q} \max\{s \log q, (s + b)^2 \log^2(s + b)\}\right)
\]

where \( c = b - a \).

Proof. There are at most \( b(b - a)/q \) loop runs in each of which we calculate two binomial coefficients, perform one multiplication and one addition. According to [17] Example 8, p. 7 the number of binary operations needed to calculate \( \binom{m}{n} \) is \( \mathcal{O}(m^2 \log^2 n) \). The highest possible \( m \) in the algorithm is \( N = s + b \) giving at most \( \mathcal{O}(N^2 \log^2 N) \) operations for that task. The multiplication takes place between two numbers no larger than \( (s + b)! \) which is \( \mathcal{O}(N^N) \). As is well-known multiplication of positive integers \( A \geq B \) can be done in \( \mathcal{O}(\log A \log B \log \log \log A) \) binary operations. In our case this becomes \( \mathcal{O}(N \log^2 N \log \log N) \) which is better than \( \mathcal{O}(N^2 \log^2 N) \). Finally the addition takes place between numbers equal to \( q^s \) at most. Hence, \( \mathcal{O}(s \log q) \) operations are needed for that part.

Proposition 5.6. The number of binary operations needed to perform \( \text{VECA} \) with input \( a, b, v, s, M, q \) is

\[
\mathcal{O}\left(sb^2 \max\{s \log q, (s + b)^2 \log^2(s + b)\}\right).
\]

Proof. If the output of \( \text{VECA} \) is \( (q - g_1, \ldots, q - g_s) \) then in the worst case \( \text{RHO} \) is called \( g_1 + \cdots + g_s \) times. Hence, in the worst case \( \text{VECA} \) calls \( \text{RHO} \) \( sq \) times. In each call the first input of \( \text{RHO} \) is lower bounded by 0, the second is upper bounded by \( b \), and the third is upper bounded by \( s \). The result now follows from Lemma 5.5.

Example 6. We use the algorithm \( \text{VECA} \) in Figure 2 to find the 34-th element \( \tilde{a} = (a_1, \ldots, a_7) \) of \( F_7((20, 22), 7) \). The procedure takes as input \( (A, B, V, S, M) = (20, 22, 6, 7, 34) \). The notation \( \tilde{A}, \tilde{B}, \tilde{V}, \tilde{S}, \tilde{M} \) is as in the proof of Theorem 5.2.

\[
(A, B, V, S, M) = (20, 22, 6, 7, 34):
\]

\[
\rho_7((14, 16), 6) = 23415 > 34 \quad \text{(lines 10–12). Thus } a_7 = 6, \tilde{A} = \max\{0, 20 - 6\} = 14, \tilde{B} = 22 - 6 = 16, \tilde{V} = q - 1 = 6 \text{ and } \tilde{S} = 7 - 1 = 6.
\]

\[
(A, B, V, S, M) = (14, 16, 6, 6, 34):
\]

\[
\rho_7((8, 10), 5) = 1936 > 34 \quad \text{(lines 10–12). Thus } a_6 = 6, \tilde{A} = \max\{0, 14 - 6\} = 8, \tilde{B} = 16 - 6 = 10, \tilde{V} = q - 1 = 6 \text{ and } \tilde{S} = 6 - 1 = 5.
\]

\[
(A, B, V, S, M) = (8, 10, 6, 6, 34):
\]

\[
\rho_7((2, 4), 4) = 64 > 34 \quad \text{(lines 10–12). Thus } a_5 = 6, \tilde{A} = \max\{0, 8 - 6\} = 2, \tilde{B} = 10 - 6 = 4, \tilde{V} = q - 1 = 6 \text{ and } \tilde{S} = 5 - 1 = 4.
\]

\[
(A, B, V, S, M) = (2, 4, 6, 4, 34):
\]

\[
6 > 4 \quad \text{(lines 2–3). Thus } \tilde{V} = B = 4.
\]
Proposition 5.7. The element \( \bar{a} = (a_1, \ldots, a_s) \in F_q((a, b), s) \) is the \( r \)-th element of \( F_q((a, b), s) \) with respect to the anti lexicographic ordering, where

\[
r = \sum_{j=0}^{s-1} \sum_{i=0}^{q-a_{s-j}-2} \rho_q((\max\{0, a - \sum_{t=0}^{j} a_{s-t} - i - 1\}, b - \sum_{t=1}^{j} a_{s-t} - i - 1), s - j - 1) + 1.
\]

In particular if \( a = 0 \) then

\[
(16) \quad r = \sum_{j=0}^{s-1} \sum_{i=0}^{q-a_{s-j}-2} \rho_q((0, b - \sum_{t=1}^{j} a_{s-t} - i - 1), s - j - 1) + 1.
\]

Proof. We must count the number of elements \( \bar{b} = (b_1, \ldots, b_s) \) in \( F_q((a, b), s) \) which are smaller than or equal to \( \bar{a} \) with respect to the anti lexicographic ordering. This number equals

\[
r = |\{\bar{b} \in F_q((a, b), s) : \bar{b} \preceq \bar{a}\}|
\]

\[
= |\{\bar{b} \in F_q((a, b), s) : b_s > a_s\}| + |\{\bar{b} \in F_q((a, b), s) : \bar{a} \preceq \bar{b}, b_s = a_s\}|
\]

\[
= \rho((a, b), (a_s + 1, q - 1), s) + |\{\bar{b} \in F_q((a, b), s) : b_{s-1} > a_{s-1}, b_s = a_s\}|
\]

\[
+ |\{\bar{b} \in F_q((a, b), s) : \bar{a} \preceq \bar{b}, b_{s-1} = a_{s-1}, b_s = a_s\}|
\]

\[
= \rho((a, b), (a_s + 1, q - 1), s) + \rho((\max\{0, a - a_s\}, b - a_s), (a_{s-1} + 1, q - 1), s - 1)
\]

\[
+ |\{\bar{b} \in F_q((a, b), s) : \bar{a} \preceq \bar{b}, b_{s-1} = a_{s-1}, b_s = a_s\}|
\]

\[
= \ldots
\]

\[
= \sum_{j=0}^{s-1} \rho((\max\{0, a - \sum_{t=0}^{j-1} a_{s-t}\}, b - \sum_{t=0}^{j-1} a_{s-t}), (a_{s-j} + 1, q - 1), s - j) + |\{\bar{a}\}|
\]

\[
= \sum_{j=0}^{s-1} \rho((\max\{0, a - \sum_{t=0}^{j-1} a_{s-t}\}, b - \sum_{t=0}^{j-1} a_{s-t}), (a_{s-j} + 1, q - 1), s - j) + 1.
\]
By the below Lemma 5.8, for $j = 0, \ldots, s - 1$ we have
\[
\rho((\max\{0, a - \sum_{t=1}^{j-1} a_{s-t}\}, b - \sum_{t=0}^{j-1} a_{s-t}, a_{s-j} + 1, q - 1, s - j\}) = \sum_{i=0}^{q-a_{s-j}-2} \rho((\max\{0, a - \sum_{t=0}^{j} a_{s-t} - i - 1\}, b - \sum_{t=1}^{j} a_{s-t} - i - 1, s - j - 1))
\]
and the proof is complete. \hfill \square

Lemma 5.8. Given a prime power $q$, let $0 \leq a \leq b \leq s(q - 1)$ and $0 \leq v < w < \min\{b, q\}$ be integers. Then $\rho_q((a, b), (v, w), s) = \sum_{i=0}^{w-v} \rho_q((\max\{0, a - v - i\}, b - v - i), s - 1)$.

Proof. We have
\[
\rho((a, b), (v, w), s) = |F_q((a, b), (v, w), s)|
\]
\[
= |\{(a_1, \ldots, a_s) \in F_q((a, b), s) : v \leq a_s \leq w\}|
\]
\[
= \left| \bigcup_{i=0}^{w-v} \{a_1, \ldots, a_s \in F_q((a, b), s) : a_s = v + i\}\right|
\]
\[
= \left| \bigcup_{i=0}^{w-v} F_q((a, b), (v + i, v + i), s)\right|
\]
\[
= \sum_{i=0}^{w-v} \rho((a, b), (v + i, v + i), s)
\]
\[
= \sum_{i=0}^{w-v} \rho((\max\{0, a - v - i\}, b - v - i), s - 1)\hfill \square
\]

Setting $a = 0$ and $b = s(q - 1)$ in Proposition 5.7 we could of course compute the $t$ such that $\tilde{a}$ is the $t$-th element of $Q^*_q$, but with the following reformulation of Lemma 5.7 we can calculate it much easier.

Lemma 5.9. The element $(a_1, \ldots, a_s) \in Q^*_q$ is the $t$-th element of $Q^*_q$ with respect to the anti lexicographic ordering where
\[
t = q^s - \sum_{i=1}^{s} a_i q^{i-1}.
\]

Proof. Recall from Section 5.7 the map $\mu : Q^*_q \rightarrow Q^*_q$, $\mu(a_1, \ldots, a_s) = (q - 1 - a_s, \ldots, q - 1 - a_1)$. By Lemma 5.5, $\mu(a_1, \ldots, a_s) = (q - 1 - a_s, \ldots, q - 1 - a_1)$ is the $t$ element of $Q^*_q$ using the lexicographic ordering where $t - 1 = \sum_{i=1}^{s} (q - 1 - a_i) q^{i-1} = q^s - 1 - \sum_{i=1}^{s} a_i q^{i-1}$. Recall from Section 5.7 that $\tilde{c} \prec_{\lambda} \tilde{d} \iff \mu(\tilde{c}) \prec_{\lambda_{\text{lex}}} \mu(\tilde{d})$. Therefore $(a_1, \ldots, a_s)$ is the $t$-th element of $Q^*_q$ using the anti lexicographic ordering. \hfill \square

Summarizing this section: to find the $m$-th RGHW of $C_1 = RM_q(u_1, s)$ with respect to $C_2 = RM_q(u_2, s)$, we perform the following steps.

1. Find the $m$-th element $(a_1, \ldots, a_s)$ of $F_q((u_2 + 1, u_1), s)$ by using the algorithm VECA in Theorem 5.2 with input $A = u_2 + 1$, $B = u_1$, $V = q - 1$, $S = s$, and $M = m$. 

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2. Find the \( r \)-th position of \((a_1, \ldots, a_s)\) in \( F_q((0, u_1), s)\) using Proposition 5.7.
3. Find the \( t \)-th position of \((a_1, \ldots, a_s)\) in \( Q_q^r\) using Lemma 5.9.
4. Compute \( M_m(C_1, C_2) = t - r + m \) (Theorem 4.9).

**Example 7.** This is a continuation of Example 5 in the beginning of which we considered \( C_1 = RM_5(5, 2) \) and \( C_2 = RM_5(3, 2) \). Applying the above procedure to establish the 8-th RGHW we first use Theorem 5.2 to establish that the 8-th element of \( F_5((4, 5), 2) \) is \((3, 1)\). Using Proposition 5.7 we then find that \((3, 1)\) is the 11-th element of \( F_5((0, 5), 2) \) and Lemma 5.9 next tells us that it is the 17-th element \( Q_5^2 \). Hence, \( M_5(C_1, C_2) = 17 - 11 + 8 = 14 \).

**Example 8.** We consider \( C_1 = RM_{16}(90, 7) \) and \( C_2 = RM_{16}(88, 7) \). We want to compute the 1000-th RGHW of \( C_1 \) with respect to \( C_2 \). Applying the algorithm VECA in Theorem 5.2 we find that that \((9, 10, 14, 11, 15, 15, 15)\) is the 1000-th element of \( F_{16}((88, 90), 7) \). Applying next Proposition 5.7 and Lemma 5.9 we find that it is the 14557-th element of \( F_{16}((0, 90), 7) \) and the 16727-th element of \( Q_{16}^7 \). Hence, \( M_{1000}(C_1, C_2) = 16727 - 14557 + 1000 = 3170 \). To find the 1000-th GHW of \( C_1 \), we use Theorem 5.2 with \( C_2 = RM_{16}(-1, 7) \) and we find that \((5, 1, 10, 15, 15, 15, 15)\) is the 1000-th element of \( F_{16}((0, 90), 7) \). By Lemma 5.9 it is the 1515-th element of \( Q_{16}^7 \). Hence, from Theorem 4.6 we deduce \( d_{1000}(C_1) = 1515 \).

6. **Closed formula expressions for \( q \)-ary Reed-Muller codes in two variables**

In the previous section we presented a method to calculate RGHWs for any set of \( q \)-ary Reed-Muller codes \( C_i = RM_q(u_i, s) \), \( i = 1, 2 \). As an alternative, for \( q \)-ary Reed-Muller codes in two variables (which by Definition 3.1 means that \( s = 2 \)) it is a manageable task to list closed formula expressions for all possible situations. This is done in the first half of the present section. Letting next \( u_2 = -1 \), corresponding to \( C_2 = \{0\} \), we in particular get closed formula expressions for the GHWs (such formulas – to the best of our knowledge – cannot be found in the literature). The formulas in the present section can be derived by applying Proposition 4.8 directly. We shall leave the details for the reader. To simplify the description we use the notation \( t = u_1 - u_2 \) which of course implies that \( u_1 = u_2 + t \). Hence, throughout this section \( C_2 = RM_q(u_2, 2) \) and \( C_1 = RM_q(u_2 + t, 2) \).

6.1. **Formulas for RGHW.** We have the following three cases.

6.1.1. **First case:** \( u_2 - q + 2 \geq 0 \).

\[
\begin{align*}
Y^4 & \quad X Y^4 & \quad X^2 Y^4 & \quad X^3 Y^4 & \quad X^4 Y^4 \\
Y^3 & \quad X Y^3 & \quad X^2 Y^3 & \quad X^3 Y^3 & \quad X^4 Y^3 \\
Y^2 & \quad X Y^2 & \quad X^2 Y^2 & \quad X^3 Y^2 & \quad X^4 Y^2 \\
Y & \quad X Y & \quad X^2 Y & \quad X^3 Y & \quad X^4 Y \\
1 & \quad X & \quad X^2 & \quad X^3 & \quad X^4 
\end{align*}
\]

\( W_5(5, 6) \) underlined, i.e. \( u_2 = 4 \) and \( t = 2 \)

(First case)

In this case the codimension is \( \ell = t(2q - u_2 - t - 2) + \frac{t(t+1)}{2} \).
\[ \ell = \frac{t(t+1)}{2} + t(u_2 + 1). \]

- If \( m = 1, \ldots, \frac{t(t+1)}{2} \) then there exist \( a \in \{0, \ldots, t-1\} \) and \( b \in \{1, \ldots, a+1\} \) such that \( m = \frac{a(a+1)}{2} + b. \) We have
  \[ M_m(C_1, C_2) = q(q - u_2 - t + a) + b - a - 1. \]
- If \( m = 1, \ldots, \frac{t(t+1)}{2} + t(u_2 + 1) \), then there exist \( a \in \{0, \ldots, u_2\} \) and \( b \in \{1, \ldots, t\} \) such that \( m = \frac{t(t+1)}{2} + at + b. \) We have
  \[ M_m(C_1, C_2) = q(a - u_2) + b - t - 1 - \frac{a(a+3)}{2}. \]

6.1.3. Third case: \( u_2 - q + 2 < 0 \) and \( u_2 - q + t + 1 > 0. \)

\[
\begin{array}{cccccc}
Y^4 & XY^4 & X^2Y^4 & X^3Y^4 & X^4Y^4 \\
Y^3 & XY^3 & X^2Y^3 & X^3Y^3 & X^4Y^3 \\
Y^2 & XY^2 & X^2Y^2 & X^3Y^2 & X^4Y^2 \\
Y & XY & X^2Y & X^3Y & X^4Y \\
1 & X & X^2 & X^3 & X^4
\end{array}
\]

\( W_5(3, 5) \) underlined, i.e. \( u_2 = 2 \) and \( t = 3 \)

(Third case)
6.2.1. The case

Then there exist \( a \in \{0, \ldots, q-t-2\} \) and \( b \in \{1, \ldots, t\} \) such that \( m = \frac{1}{2}(q - u_2 - 2)(2t - q + u_2 + 1) + (a + 1)t + b \). We have

\[
M_m(C_1, C_2) = q(q - u_2 + a) - \frac{a(a + 3)}{2} - t + b - 1.
\]

6.2. Formulas for GHW. Applying the formulas from the previous section to the special case of \( u_2 = -1 \) and consequently \( u_1 = t + 1 \) we get by letting \( u = u_1 \) the following results concerning the GHWs of \( \text{RM}_q(u, s) \).

6.2.1. The case \( u - q + 1 \leq 0 \). In this case the dimension of \( C_1 \) is \( k_1 = \frac{(u+1)(u+2)}{2} \).

- For \( r = 1, \ldots, \frac{(u+1)(u+2)}{2} \) there exist \( a \in \{0, \ldots, u\} \) and \( b \in \{1, \ldots, a+1\} \) such that \( r = \frac{a(a+1)}{2} + b \). We have

\[
d_r(C_1) = q(q - u + a) + b - a - 1.
\]

6.2.2. The case \( u - q + 1 > 0 \). In this case the dimension of \( C_1 \) is \( k_1 = q(2u_1 - q + 3) - \frac{u_1(u_1+3)}{2} - 1 \).

- For \( r = 1, \ldots, q(u + 2) - \frac{u(u+3)}{2} - 1 \) there exist \( a \in \{0, \ldots, 2(q - 1) - u\} \) and \( b \in \{1, \ldots, u - q + 2 + a\} \) such that \( r = a(u - q + 1) + \frac{u(u+1)}{2} + b \). We have

\[
d_r(C_1) = (a + 2)(q - 1) - u + b.
\]

- For \( r = q(u + 2) - \frac{u(u+3)}{2} - 1, \ldots, q(2u - q + 3) - \frac{u(u+3)}{2} - 1 \) there exists \( c \in \{1, \ldots, q(u - q + 1)\} \) such that \( r = q(u + 2) - \frac{u(u+3)}{2} - 1 + c \). We have

\[
d_r(C_1) = q(2q - u - 1) + c.
\]

6.3. Comparing RGHW and GHW in a special case. Consider the special case \( u_2 = q - 2 \) and \( t = 1 \). If \( m = 1, \ldots, q \) then there exist \( a \in \{0, \ldots, q - 1\} \) and \( b \in \{1, \ldots, a + 1\} \) such that \( m = \frac{a(a+1)}{2} + b \). We have

\[
M_m(C_1, C_2) = \frac{m}{2}(2q - m + 1) \quad \text{and} \quad d_m(C_1) = (q - 1)(a + 1) + b
\]

Thus

\[
M_m(C_1, C_2) - d_m(C_1) = \frac{1}{8}(-a^4 - 2a^3 + (-4b + 4q + 1)a^2
+(-4b - 4q + 10)a - 4b^2 + 8bq - 4b - 8q + 8).
\]

For the particular case that \( q = 16 \) we get the values listed in Table 6.
Table 6. The special case \( u_2 = q - 2 \) and \( t = 1 \) with \( q = 16 \). That is, \( C_1 = \text{RM}_{16}(15,2) \) and \( C_2 = \text{RM}_{16}(14,2) \). The function \( \text{diff}(m) \) equals \( M_m(C_1,C_2) - d_m(C_1) \).

7. Locally correctable ramp secret sharing schemes

We now return to the communication problem described in the introduction of the paper. Recall that we consider a secret sharing scheme based on a coset construction \( C_1/C_2 \) where \( C_1 \) and \( C_2 \) are \( q \)-ary Reed-Muller codes. Requirement R2 about local correctability was treated in Theorem 1.3. In Section 3 – Section 6 we showed a low complexity method to determine the RGHWs and in particular we derived closed formula expressions in the case of codes in two variables. By the following result (corresponding to (2) and (3))

\[
\begin{align*}
    t_m &= M_m((C_2)^\perp, (C_1)^\perp) - 1, \\
    r_m &= n - M_{t_m-1}(C_1,C_2) + 1,
\end{align*}
\]

this method immediately translates into accurate information on the information leakage and thereby explains what can be done regarding requirement R1.

Combining (17) with (6) and (7) and using Remark 2.4 we obtain

\[
\begin{align*}
    t_1 &= d(C_2^\perp) - 1, \\
    r_1 &= \dim(C_2) + 1, \\
    t_\ell &= \dim(C_1) - 1, \\
    r_\ell &= n - d(C_1) + 1,
\end{align*}
\]

where \( d(C) \) is the minimum distance of \( C \). To apply Theorem 1.3 (which ensures local correctability) we need \( u_1 < q - 1 \). Under that assumption (18) becomes

\[
\begin{align*}
    t_1 &= u_2 + 1, \\
    r_1 &= (s + u_2) + 1, \\
    t_\ell &= (s + u_1) - 1, \\
    r_\ell &= q^s - (q - u_1)q^{s-1} + 1 = u_1q^{s-1} + 1.
\end{align*}
\]

By Theorem 1.3 we need to make \( u_1 + 1 \) or \( q - 1 \) queries (depending on the error-probability of the system) to hopefully correct an entry. We observe that the number of queries in both cases is strictly larger than \( t_1 \). However, it is only larger than \( r_1 \) when \( u_2 \) is very small. Actually, for most values of \( u_2 \) the number of queries needed will be much smaller than \( r_1 \). Recall from the proofs in [26] of the local correctability of \( q \)-ary Reed-Muller codes that the random point sets queried is chosen from a family of point sets with a particular geometry (the geometry is different for the three different cases treated in Theorem 1.3). Knowing only the values \( t_1 \) and \( r_1 \) – with the number of queries being a number in between – we cannot say if those point sets get access to information or not. However, when \( t_2 \) is larger than or equal to the number of queries then for sure they get at most access to 1 \( q \)-bit of information. As is demonstrated in the following two examples this is often the case. Of course the situation gets more complicated if the decoding is not successful in the first run and another series of queries is needed. In that case we may either ensure that the information from the first query is deleted or we may simply trust the party that performs the error-correction. Below we study in detail various schemes over the alphabets \( \mathbb{F}_8 \) and \( \mathbb{F}_{16} \).
Table 7. Scheme based on $C_1 = RM_8(6, 2)$ and $C_2 = RM_8(5, 2)$. For local error-correction 7 queries are needed.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| $t_m$ | 6 | 12 | 17 | 21 | 24 | 26 | 27 |
| $t'_m$ | 6 | 7 | 13 | 14 | 15 | 20 | 21 |
| $r_m$ | 22 | 24 | 27 | 31 | 36 | 42 | 49 |
| $r'_m$ | 28 | 33 | 34 | 35 | 41 | 42 | 49 |

Table 8. Scheme based on $C_1 = RM_8(6, 2)$ and $C_2 = RM_8(4, 2)$. For local error-correction 7 queries are needed.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| $t_m$ | 5 | 6 | 11 | 12 | 16 | 20 |
| $t'_m$ | 5 | 6 | 7 | 12 | 13 | 14 |
| $r_m$ | 16 | 17 | 19 | 20 | 23 | 24 |
| $r'_m$ | 19 | 20 | 21 | 25 | 26 | 27 |

Table 9. Scheme based on $C_1 = RM_8(5, 2)$ and $C_2 = RM_8(4, 2)$. For local error-correction 6 or 7 queries are needed, depending on the error-probability.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| $t_m$ | 5 | 10 | 14 | 17 | 19 | 20 |
| $t'_m$ | 5 | 6 | 7 | 12 | 13 | 14 |
| $r_m$ | 16 | 19 | 23 | 28 | 34 | 41 |
| $r'_m$ | 25 | 26 | 27 | 33 | 34 | 41 |

Table 10. Scheme based on $C_1 = RM_8(5, 2)$ and $C_2 = RM_8(3, 2)$. For local error-correction 6 or 7 queries are needed, depending on the error-probability.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|----|----|
| $t_m$ | 4 | 5 | 9 | 10 | 13 | 14 | 16 | 17 | 18 | 19 | 20 |
| $t'_m$ | 4 | 5 | 6 | 7 | 11 | 12 | 13 | 14 | 15 | 18 | 19 |
| $r_m$ | 11 | 12 | 15 | 16 | 20 | 21 | 26 | 27 | 33 | 34 | 41 |
| $r'_m$ | 13 | 17 | 18 | 19 | 20 | 25 | 26 | 27 | 33 | 34 | 41 |

Example 9. In this example we consider schemes over $F_8$. Depending on the error-probability it is sufficient to make $u_1 + 1$ or $q - 1 = 7$ queries to correct an entry. The number of participants is $n = 8^2 = 64$. In Table 7, Table 10 we consider codes $C_1 = RM_q(u_1, 2)$ and $C_2 = RM_q(u_2)$ for different choices of $u_2 < u_1 \leq q - 2$ and we list the parameters $t = t_1, \ldots, t_\ell$ and $r_1, \ldots, r_\ell = r$ (in particular the number of columns equals the codimension $\ell$). We also list corresponding numbers $t'_1, \ldots, t'_\ell$ and $r'_1, \ldots, r'_\ell$. They are the lower bounds and upper bounds, respectively, that we would get on the $t_i$’s and the $r_i$’s, respectively, by using GHWs instead of RGHWs. It is quite clear that the amount of information leaked to the party performing the local error-correction is often lower than what could be anticipated from studying only the GHWs.
Table 11. Scheme based on $C_1 = RM_8(5, 2)$ and $C_2 = RM_8(2, 2)$. For local error-correction 6 or 7 queries are needed, depending on the error-probability.

| $m$ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 |
|-----|----------------------------------------|
| $t_m$ | 3 4 5 8 9 10 12 13 14 15 16 17 18 19 20 |
| $t'_m$ | 3 4 5 6 7 10 11 12 13 14 15 17 18 19 20 |
| $r_m$ | 7 8 9 12 13 14 18 19 20 25 26 27 33 34 41 |
| $r'_m$ | 9 10 11 12 13 17 18 19 20 25 26 27 33 34 41 |

Table 12. Scheme based on $C_1 = RM_8(4, 2)$ and $C_2 = RM_8(3, 2)$. For local error-correction 5 or 7 queries are needed, depending on the error-probability.

| $m$ | 1 2 3 4 5 |
|-----|----------------------------------------|
| $t_m$(RGHW) | 4 8 11 13 14 |
| $t_m$(GHW) | 4 5 6 7 11 |
| $r_m$(RGHW) | 11 15 20 26 33 |
| $r_m$(GHW) | 18 19 25 26 33 |

Table 13. Scheme based on $C_1 = RM_{16}(14, 2)$ and $C_2 = RM_{16}(13, 2)$. For local error-correction 15 queries are needed.

| $m$ | 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 |
|-----|----------------------------------------|
| $t_m$ | 14 28 41 53 64 74 83 91 98 104 109 113 116 118 119 |
| $t'_m$ | 14 15 29 30 31 44 45 46 47 59 60 61 62 63 74 |
| $r_m$ | 106 108 111 115 120 126 133 141 150 160 171 183 196 210 225 |
| $r'_m$ | 161 162 163 164 165 177 178 179 180 193 194 195 209 210 225 |

Table 14. Scheme based on $C_1 = RM_{16}(13, 2)$ and $C_2 = RM_{16}(12, 2)$. For local error-correction 14 or 15 queries are needed, depending on the error-probability.

Example 10. In this example we consider schemes over $\mathbb{F}_{16}$. Depending on the error-probability it is sufficient to make $u_1 + 1$ or $q - 1 = 15$ queries. The number of participants is $n = 16^2 = 256$. The information in Table 13 - Table 18 is similar to the previous example.

8. Concluding remarks

In this paper we applied a coset construction of $q$-ary Reed-Muller codes to the situation where a central party wants to store a secret on a distributed media in such a way that other parties with access to a large part of the media can recover the secret, whereas parties with limited access cannot. The reason for choosing $q$-ary Reed-Muller codes is that with such codes one is able to perform local error-correction. For the purpose of analysing the information leakage we
Table 15. Scheme based on $C_1 = RM_{16}(12, 2)$ and $C_2 = RM_{16}(11, 2)$. For local error-correction 13 or 15 queries are needed, depending on the error-probability.

| $m$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $t_m$ | 12  | 24  | 35  | 45  | 54  | 62  | 69  | 75  | 80  | 84  | 87  | 89  | 90  |
| $t'_m$ | 12  | 13  | 14  | 15  | 27  | 28  | 29  | 30  | 31  | 42  | 43  | 44  | 45  |
| $r_m$ | 79  | 83  | 88  | 94  | 101 | 109 | 118 | 128 | 139 | 151 | 164 | 178 | 193 |
| $r'_m$ | 131 | 132 | 133 | 145 | 146 | 147 | 148 | 161 | 162 | 177 | 178 | 193 | 193 |

determined the relative generalized Hamming weights of the codes involved. This was done using the footprint bound from Gröbner basis theory. There is a very strong connection between the footprint bound and the Feng-Rao bound for primary codes \[1, 10\] which is the bound that we used in \[9\] to estimate RGHWs of one-point algebraic geometric codes. Using the footprint bound rather than the Feng-Rao bound for primary or dual codes saved us some cumbersome notation (which is difficult to avoid in the case of one-point algebraic geometric codes). Using the derived information on the RGHWs we discussed the trade off between security in the above scheme and the ability to perform local error-correction.

Table 16. Scheme based on $C_1 = RM_{16}(11, 2)$ and $C_2 = RM_{16}(10, 2)$. For local error-correction 12 or 15 queries are needed, depending on the error-probability.

| $m$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $t_m$ | 11  | 22  | 32  | 41  | 49  | 56  | 62  | 67  | 71  | 74  | 76  | 77  |
| $t'_m$ | 11  | 12  | 13  | 14  | 15  | 26  | 27  | 28  | 29  | 30  | 31  | 41  |
| $r_m$ | 67  | 72  | 78  | 85  | 93  | 102 | 112 | 123 | 135 | 148 | 162 | 177 |
| $r'_m$ | 116 | 117 | 129 | 130 | 131 | 132 | 145 | 146 | 161 | 162 | 177 | 177 |

Table 17. Scheme based on $C_1 = RM_{16}(10, 2)$ and $C_2 = RM_{16}(9, 2)$. For local error-correction 11 or 15 queries are needed, depending on the error-probability.

| $m$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $t_m$ | 10  | 20  | 29  | 37  | 44  | 50  | 55  | 59  | 62  | 64  | 65  |
| $t'_m$ | 10  | 11  | 12  | 13  | 14  | 15  | 25  | 26  | 27  | 28  | 29  |
| $r_m$ | 56  | 62  | 69  | 77  | 86  | 96  | 107 | 119 | 132 | 146 | 161 |
| $r'_m$ | 101 | 113 | 114 | 115 | 116 | 129 | 130 | 131 | 145 | 146 | 161 |

Table 18. Scheme based on $C_1 = RM_{16}(9, 2)$ and $C_2 = RM_{16}(8, 2)$. For local error-correction 10 or 15 queries are needed, depending on the error-probability.

| $m$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $t_m$ | 9   | 18  | 26  | 33  | 39  | 44  | 48  | 51  | 53  | 54  |
| $t'_m$ | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 24  | 25  | 26  |
| $r_m$ | 46  | 53  | 61  | 70  | 80  | 91  | 103 | 116 | 130 | 145 |
| $r'_m$ | 97  | 98  | 99  | 100 | 113 | 114 | 115 | 129 | 130 | 145 |
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APPENDIX A. PROOF OF LEMMA 4.7

To prove Lemma 4.7 we start by generalizing [11] Th. 3.7.7 which corresponds to Lemma A.1 below in the particular case that b = s(q − 1). The proof of [11] Th. 3.7.7 was given in [11, App. B.1].

Lemma A.1. Let A be a subset of $F_q(a, b)$ consisting of m elements. Then $|\Delta l_{(a, b)}(m)| \leq |\Delta A|$.

Proof. In Appendix B.1 of [11] a proof for Lemma A.1 is given in the particular case that b = s(q − 1). We indicate how this proof can be modified to cover all possible choices of b. First note that [11] uses v where we use a, uses m where we use s, and uses r where we use m. With the following modifications the proof in [11] is lifted to a proof of Lemma A.1.

- In [11] Rem. B.1.2]: Replace $F_{\geq v}$ with $F_q(v, b)$ and let the parameter k go from v to b.
- In [11] Def. B.1.6]: Replace $F_{\geq l}$ with $F_q(l, b)$.
- In [11] Lem. B.1.10]: Replace $F_{\geq v}$ with $F_q(v, b)$ and let the summation end with $A_b$ rather than $A_{s(q−1)}$.
- In [11] Lem. B.1.13, Lem. B.1.14 and their proofs]: Replace $F_{\geq l}$, $F_{\geq l}$, $F_{\geq l}$, $L_{r_{l−1}}(r)$ and $L_{r_{l−1}}(r)$ with $F_q(l, b)$, $F_q(l−1, b)$, $F_q(v, b)$, $L_{l(b)}(r)$ and $\mu l(b)$, respectively.

Recall from Section 4.1 the map $\mu : Q^*_s \rightarrow Q^*_s$ given by $\mu(a_1, \ldots, a_s) = (q − 1 − a_s, q − 1 − a_1)$. To translate Lemma A.1 into Lemma 4.7 we need the following results.

Lemma A.2. Let $0 \leq a \leq b \leq s(q − 1)$ be integers, $\vec{a}, \vec{b} \in Q^*_q$ and $\vec{m} \in \{1, \ldots, |F_q(a, b)|\}$, then we have that

1. $\vec{a} \prec_{L_{\text{ex}}} \vec{b} \iff \mu(\vec{a}) \prec_{\mu} \mu(\vec{b})$.
2. $\vec{a} \preceq_{L_{\text{ex}}} \vec{b} \iff \mu(\vec{a}) \preceq_{\mu} \mu(\vec{b})$.
3. $\vec{a} \preceq_{P} \vec{b} \iff \mu(\vec{a}) \preceq_{P} \mu(\vec{b})$.
4. $\mu(\nabla \vec{a}) = \Delta \mu(\vec{a})$.
5. $\mu(\nabla A) = \Delta \mu(A)$.
6. $\mu(F_q(a, b)) = F_q(s(q−1)−b,s(q−1)−a)$.
7. $A \subseteq F_q(a, b) \iff \mu(A) \subseteq F_q(s(q−1)−b,s(q−1)−a)$.
8. $\mu(N_{(a, b)}(m)) = L_{(s(q−1)−b,s(q−1)−a)}(m)$.
9. $\mu(N_{(a, b)}(m)) = \Delta L_{(s(q−1)−b,s(q−1)−a)}(m)$.

Proof. Let $\vec{a} = (a_1, \ldots, a_s)$ and $\vec{b} = (b_1, \ldots, b_s)$.

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1. \( \bar{a} \lessdot_{\text{Lex}} \bar{b} \iff a_1 = b_1, \ldots, a_{l-1} = b_{l-1}, a_l < b_l \) for some \( l \) \( \iff q - 1 - a_1 = q - 1 - b_1, \ldots, q - 1 - a_{l-1} = q - 1 - b_{l-1}, q - 1 - a_l > q - 1 - b_l \) for some \( l \) \( \iff \mu(\bar{a}) \lessdot_{\text{Lex}} \mu(\bar{b}) \).

2. Similar to 1.

3. \( \bar{a} \preceq \bar{b} \iff a_1 \leq b_1, \ldots, a_s \leq b_s \iff q - 1 - a_1 \geq q - 1 - b_1, \ldots, q - 1 - a_s \geq q - 1 - b_s \iff \mu(\bar{a}) \succeq_{\text{Lex}} \mu(\bar{b}) \).

4. \( \bar{b} \in \mu(\nabla \bar{a}) \iff \exists \bar{b}_1 = \mu^{-1}(b) \in \nabla \bar{a} \iff \bar{b}_1 \preceq_{\text{Lex}} \mu(\bar{a}) \iff \mu(\bar{b}_1) \succeq_{\text{Lex}} \mu(\bar{a}) \iff \bar{b} \in \Delta \mu(\bar{a}) \).

5. \( \mu(\nabla A) = \mu(\bigcup_{\bar{a} \in A} \nabla \bar{a}) = \bigcup_{\bar{a} \in A} \mu(\nabla \bar{a}) = \bigcup_{\bar{a} \in A} \Delta \mu(\bar{a}) = \Delta \mu(A) \).

6. \( \bar{a} \in F_q(a,b) \iff a \leq \deg(\bar{a}) \leq b \iff a \leq \sum_{i=1}^{s} a_i \leq b \iff s(q - 1) - a \geq \sum_{i=1}^{s} a_i \leq s(q - 1) - b \iff s(q - 1) - a \leq \sum_{i=1}^{s} (q - 1 - a_i) \leq s(q - 1) - b \iff \mu(\bar{a}) \in F_q(s(q - 1) - b, s(q - 1) - a) \).

7. Similar to 6.

8. Follows from 1, 2 and 7 by induction.

9. \( \mu(\nabla N_{(a,b)}(m)) = \Delta \mu(N_{(a,b)}(m)) = \Delta L_{(s(q-1)-b,s(q-1)-a)}(m) \).

We are now ready to prove Lemma 4.7

Proof of Lemma 4.7 By 7. in Lemma A.2 we have \( \mu(A) \subseteq F_q(s(q - 1) - b, s(q - 1) - a) \). It follows that

\[
|\nabla N_{(a,b)}(m)| = |\mu(\nabla N_{(a,b)}(m))| = |\Delta L_{(s(q-1)-b,s(q-1)-a)}(m)| \leq |\Delta \mu(A)| = |\mu(\nabla A)| = |\nabla A|,
\]

where the first and the last line is a consequence of the fact that \( \mu \) is bijective, the second line follows from 9. in Lemma A.2, the third line follows from Lemma A.1 and the fourth line follows from 5. in Lemma A.2.

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