Some New Positive Observations

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Abstract. I revisit Bressoud’s generalised Borwein conjecture. Making use of new positivity-preserving transformations for $q$-binomial coefficients I establish the truth of infinitely many cases of the Bressoud conjecture. In addition, I prove new bounded version of Lebesgue’s identity and of Euler’s Pentagonal Number Theorem. Finally, I discuss new companions to Andrews-Gordon $\text{mod} 21$ and Bressoud $\text{mod} 20$ identities.

1. Introduction

Bressoud [10] considered the following polynomials

\begin{equation}
G(N, M, \alpha, \beta, K, q) = \sum_{j=\infty}^{\infty} (-1)^j q^{Kj\frac{(\alpha+\beta)+j(\alpha-\beta)}{2}} \left[ \frac{N+M}{N-Kj} \right]_q,
\end{equation}

where

\begin{equation}
\left[ \frac{m+n}{m} \right]_q := \begin{cases} 
\frac{(q;q)_{m+n}}{(q;q)^n}, & \text{for } m, n \in \mathbb{N}, \\
0, & \text{otherwise},
\end{cases}
\end{equation}

and

\begin{equation}
(q)_m = \prod_{j=1}^{m} (1-q^j), \text{for } m \in \mathbb{N}.
\end{equation}

More generally, for $m \in \mathbb{N}$

\begin{equation}
(a)_m = (a; q)_m = \prod_{j=0}^{m-1} (1-aq^j),
\end{equation}

\begin{equation}
(a_1, a_2, \ldots, a_k; q)_m = (a_1; q)_m (a_2; q)_m \ldots (a_k; q)_m.
\end{equation}

Here and throughout I assume that $|q| < 1$. I note that $(a)_0 = 1$.

In 1996, Bressoud [10] conjectured that

Conjecture 1.1. Let $K \in \mathbb{Z}_{\geq 1}$, $N, M, \alpha K, \beta K \in \mathbb{Z}_{\geq 0}$ such that

\begin{equation}
1 \leq \alpha + \beta \leq 2K-1,
\end{equation}

\begin{equation}
\beta - K \leq N - M \leq K - \alpha,
\end{equation}

(strict inequality when $K = 2$), then $G(N, M, \alpha, \beta, K, q) \geq 0$.

Here, and everywhere, $P(q) \geq 0$ means that $P(q)$ is a polynomial in $q$ with nonnegative coefficients. I remark that

\begin{equation}
\left[ \frac{m+n}{m} \right]_q \geq 0.
\end{equation}

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Famous conjecture of Peter Borwein (Theorem since 2019 [14]) can be stated as

\[ A_n(q) = G(n, n, \frac{5}{3}, 3, q) \geq 0, \]
\[ B_n(q) = G(n - 1, n + 1, \frac{7}{3}, 3, q) \geq 0, \]
\[ C_n(q) = G(n - 1, n + 1, \frac{8}{3}, 3, q) \geq 0. \] (1.5)

\[ \prod_{k=1}^{n}(1 - q^{3k-1})(1 - q^{3k-2}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3). \] (1.6)

When \( \alpha, \beta \in \mathbb{Z} \), \( G(N, M, \alpha, \beta, K, q) \) is a generating function for the so-called partitions with prescribed hook differences [4]. Bressoud’s conjecture is nontrivial when \( \alpha, \beta \) assume fractional values.

Many cases of Bressoud’s conjecture were settled in the literature [9], [12], [15], [16], [7], [14].

In the next section, I will show how to settle new infinite family of cases.

**Theorem 1.2.** For \( L \in \mathbb{N}, \nu \in \mathbb{Z}_{>0}, s = 0, 1, 2, \ldots, \nu - 1 \)

\[ G(L, L + 1 + 2s, (\nu + 1)(1 + \frac{1 + 2s}{2\nu + 1}), (\nu + 1)(1 - \frac{1 + 2s}{2\nu + 1}), 2\nu + 1, q) \geq 0. \] (1.7)

Also, in Section 2, I discuss new bounded versions of Lebesgue’s identity and of Euler’s Pentagonal Number Theorem. In Section 3, I establish and prove some additional isolated positivity results and introduce new companions to Andrews-Gordon mod 21 and Bressoud mod 20 identities.

I conclude this section with a list of seven useful formulas, which can be found in [2]:

\[ \lim_{L \to \infty} \left[ \frac{L}{q} \right] = \frac{1}{(q)_m}, \] (1.8)
\[ \lim_{L, M \to \infty} \left[ \frac{L + M}{L} \right] = \frac{1}{(q)_{\infty}}, \] (1.9)
\[ \left[ \frac{n + m}{n} \right]_{q^{-1}} = q^{-nm} \left[ \frac{n + m}{n} \right]_q, \] (1.10)
\[ \left[ \frac{n}{m} \right]_q = \left[ \frac{n - 1}{m - 1} \right] + q^m \left[ \frac{n - 1}{m} \right]_q = \left[ \frac{n - 1}{m} \right] + q^{n - m} \left[ \frac{n - 1}{m - 1} \right]_q, \] (1.11)
\[ \sum_{n \geq 0} q\binom{n}{2} z^n \left[ \frac{L}{n} \right]_q = (-z; q)_L, \] (1.12)
\[ \sum_{j=-\infty}^{\infty} (-1)^j z^j q^j = \left( \frac{q^2}{z}, \frac{q}{z}; q^2 \right)_\infty, \] (1.13)
\[ \sum_{j=-\infty}^{\infty} (-1)^j q^j z^j \left[ \frac{L + M}{L - j} \right]_q = \left( \frac{q}{z}, q^2 \right)_M (zq; q^2)_L, \] (1.14)

with \( L, M, m, n \in \mathbb{N} \).
2. Positivity-preserving Transformations

I start with the following summation formula

**Theorem 2.1.** For $L \in \mathbb{N}$, $a \in \mathbb{Z}$

\[
\sum_{k \geq 0} C_{L,k}(q) \left[ \frac{k}{L-a} \right]_q = q^{T(a)} \left[ \frac{2L+1}{L-a} \right]_q,
\]

where

\[ T(j) := \left( \frac{j+1}{2} \right) \]

and

\[
C_{L,k}(q) = \sum_{m=0}^L q^{T(m)+T(m+k)} \left[ \frac{L}{m,k} \right]_q,
\]

with

\[
\left[ \frac{L}{m,k} \right]_q = \left[ \frac{L}{m} \right]_q \left[ \frac{L-m}{k} \right]_q = \left[ \frac{L}{k} \right]_q \left[ \frac{L-k}{m} \right]_q.
\]

Observe that $C_{L,k}(q) \geq 0$. Using transformation (2.1) it is easy to check that identity

\[
F(L,q) = \sum_{j=-\infty}^\infty \alpha(j) \left[ \frac{L}{L-j} \right]_q,
\]

implies that

\[
\sum_{k \geq 0} C_{L,k}(q) F(k,q) = \sum_{j=-\infty}^\infty \alpha(j) q^{T(j)} \left[ \frac{2L+1}{L-j} \right]_q.
\]

Hence, if $F(L,q) \geq 0$ then

\[
\sum_{j=-\infty}^\infty \alpha(j) q^{T(j)} \left[ \frac{2L+1}{L-j} \right]_q \geq 0.
\]

For that reason, I say that (2.1) is positivity-preserving.

Transformation (2.1) is an easy corollary of the theorem proven in [6].

**Theorem 2.2** (Berkovich–Uncu).

\[
\sum_{k \geq 0} q^{T(k)} \left[ \frac{L}{k} \right]_q \left\{ T_{-1} \left( \frac{k}{a} : q \right) + T_{-1} \left( \frac{k}{a+1} : q \right) \right\} = q^{T(a)} \left[ \frac{2L+1}{L-a} \right]_q.
\]

The Andrews-Baxter $q$-trinomial coefficients [3] can be defined as

\[
T_{-1} \left( \frac{k}{a} : q \right) = \sum_{m \geq 0, \; m \equiv k+a \text{ (mod 2)}} q^{T(m)} \left[ \frac{k}{m} \right]_q \left[ \frac{k-m-a}{k-2} \right]_q.
\]

It is easy to check that

\[
T_{-1} \left( \frac{k}{a} : q \right) + T_{-1} \left( \frac{k}{a+1} : q \right) = \sum_{m \geq 0} q^{T(m)} \left[ \frac{k}{m} \right]_q \left[ \frac{k-m-a}{k-2} \right]_q.
\]

Substituting (2.9) into left hand side of (2.7) and changing $k \rightarrow k+m$ I complete the proof of (2.1). □

It is instructive to compare (2.1) with the Corollary (2.6) in [16].
Theorem 2.3 (Warnaar). For $L \in \mathbb{N}$, $a \in \mathbb{Z}$

\begin{equation}
\sum_{k \geq 0} W_{L,k}(q) \binom{2k}{k-a}_q = q^{2a^2} \binom{2L}{L-2a}_q, \tag{2.10}
\end{equation}

where

\begin{equation}
W_{L,k}(q) = \sum_{m=0}^{L} q^{(m+k)^2+k^2} \binom{L}{m,2k}_q \geq 0. \tag{2.11}
\end{equation}

Observe that unlike (2.10), transformation (2.1) can not be iterated. Interestingly enough, there exists an odd companion to Theorem 2.3.

Theorem 2.4. For $L \in \mathbb{N}$, $a \in \mathbb{Z}$

\begin{equation}
\sum_{k \geq 0} O_{L,k}(q) \binom{2k+1}{k-a}_q = q^{4T(a)} \binom{2L}{L-2a-1}_q, \tag{2.12}
\end{equation}

where

\begin{equation}
O_{L,k}(q) = \sum_{m=0}^{L} q^{2T(m+k)+2T(k)} \binom{L}{m,2k+1}_q \geq 0. \tag{2.13}
\end{equation}

I remark that while Theorem 2.4 is not explicitly stated in [16], it is a special case of an identity on page 222 there.

Schur’s bounded version of Euler’s Pentagonal Number Theory states

\begin{equation}
1 = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{3j+1}{2}} \binom{L}{L-3j}_q. \tag{2.14}
\end{equation}

With the aid of (2.1) I can convert (2.14) into

\begin{equation}
0 \leq \sum_{k=0}^{L} C_{L,k}(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j(3j+1)} \binom{2L+1}{L-3j}_q. \tag{2.15}
\end{equation}

Hence,

\[G(L, L+1, \frac{8}{3}, -\frac{4}{3}, 3, q) \geq 0.\]

Making use of (1.12), it is easy to check that

\begin{equation}
\sum_{k=0}^{L} C_{L,k}(q) = \sum_{k=0}^{L} q^{T(k)} \binom{L}{k}_q (-q)_k. \tag{2.16}
\end{equation}

And so identity (2.15) can be rewritten as

\begin{equation}
\sum_{k=0}^{L} q^{T(k)} \binom{L}{k}_q (-q)_k = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j(3j+1)} \binom{2L+1}{L-3j}_q. \tag{2.17}
\end{equation}

Letting $L \to \infty$ and using the Jacobi triple product identity (1.13) yields a special case of the Lebesque identity [13]

\begin{equation}
\sum_{m \geq 0} q^{T(m)}(q)_{m} (-q)_m = \frac{(q^4; q^4)_\infty}{(q)_\infty}. \tag{2.18}
\end{equation}

Perform $q \to \frac{1}{q}$ in (2.17) and use (1.10) together with

\begin{equation}
(-q^{-1}; q^{-1})_n = (-q)_n q^{-T(n)}, \quad n \in \mathbb{N}. \tag{2.19}
\end{equation}
to obtain after simplification a new polynomial version of Euler’s Pentagonal Number Theorem

\[(2.20) \quad \sum_{k=0}^{L} (-q)_{L-k}q^{(L+1)k} \left[ \begin{array}{c} L \\ k \end{array} \right] = \sum_{j=-\infty}^{\infty} (-1)^{j}q^{3j^{2}+j} \left[ \frac{2L+1}{L-3j} \right]_{q}. \]

It proves that

\[(2.21) \quad G(L, L + 1, \frac{4}{3}, \frac{2}{3}, 3, q) \geq 0. \]

I now move on to prove Theorem 1.2. I start with the finite analogue of the Andrews-Gordon identity due to Foda-Quano [11].

For \( L \in \mathbb{N}, \nu \in \mathbb{Z}_{>0}, s = 0, 1, \ldots, \nu - 1 \)

\[(2.22) \quad \sum_{n_{2}, \ldots, n_{\nu} \geq 0} q^{N_{2} + \cdots + N_{\nu} + n_{2} + \cdots + n_{\nu}} \prod_{i=2}^{\nu} \left[ n_{i} + L - 2 \sum_{k=2}^{i} N_{k} - E_{i,s}^{\nu} \right]_{q} = \]

\[\sum_{j=-\infty}^{\infty} (-1)^{j}q^{(2\nu+1)^{2}j^{2}+j(1+2s)} \left[ \frac{L}{2^{(2\nu+1)j-s}} \right]_{q}. \]

Here, \( N_{j} = n_{j} + n_{j+1} + \ldots + n_{\nu}, j = 2, \ldots, \nu \) and \( E_{i,s}^{\nu} = \text{max}(i + s - \nu, 0) \). Observe that (2.14) is the case \( \nu = 1 \) of (2.22).

Apply transformation (2.1) to obtain

\[0 \leq \sum_{k, n_{2}, \ldots, n_{\nu} \geq 0} C_{L,k}(q)q^{N_{2} + \cdots + N_{\nu} + n_{2} + \cdots + n_{\nu}} \prod_{i=2}^{\nu} \left[ n_{i} + k - 2 \sum_{k=2}^{i} N_{k} - E_{i,s}^{\nu} \right]_{q} = \]

\[(2.23) \quad q^{T(s)} \sum_{j=-\infty}^{\infty} (-1)^{j}q^{(\nu+1)(2\nu+1)^{2}+j(1+2s)} \left[ \frac{2L+1}{L-(2\nu+1)j} \right]_{q}. \]

Hence,

\[G(L, L + 1 + 2s, (\nu + 1)(1 + \frac{1 + 2s}{2\nu + 1}), (\nu + 1)(1 - \frac{1 + 2s}{2\nu + 1}), 2\nu + 1, q) \geq 0, \]

for all \( L \in \mathbb{N}, \nu \in \mathbb{Z}_{>0}, s = 0, 1, 2, \ldots, \nu - 1. \) This completes the proof of Theorem 1.2. \( \square \)

3. Further Observations

Setting \( M = L, L + 1 \) and \( z = q \) in (1.14) I find that for \( L \in \mathbb{N} \)

\[(3.1) \quad \sum_{j=-\infty}^{\infty} (-1)^{j}q^{T(j)} \left[ \frac{2L}{L-j} \right]_{q} = \delta_{L,0} \]

and

\[(3.2) \quad \sum_{j=-\infty}^{\infty} (-1)^{j}q^{T(j)} \left[ \frac{2L+1}{L-j} \right]_{q} = 0, \]

where \( \delta_{L,0} = 1 \) if \( L = 0 \) and \( \delta_{L,0} = 0 \) if \( L > 0 \). The formulas (3.1) and (3.2) can be combined into

\[(3.3) \quad \sum_{j=-\infty}^{\infty} (-1)^{j}q^{T(j)} \left[ \frac{L}{L-2j} \right]_{q} = \delta_{L,0}. \]

Applying Theorem 2.3 to (3.1) yields

\[(3.4) \quad W_{L,0}(q) = \sum_{n \geq 0} q^{n^{2}} \left[ \frac{L}{n} \right]_{q} = \sum_{j=-\infty}^{\infty} (-1)^{j}q^{\frac{2L-2j}{L-2j}} \left[ \frac{2L}{L-2j} \right]_{q}. \]
which is Bressoud’s bounded version of the first Rogers-Ramanujan identity \[9\]. Analogously, applying
Theorem 2.1 to (3.3) yields
\[(3.5)
\]
which can be recognized as Warnaar’s bounded version of the second Rogers-Ramanujan identity \[15\].

Next, I perform the change of summation variables below
\[(3.6)
\]
to conclude that
\[(3.7)
\]
Adding (3.4) and (3.7) and employing recursion relation (1.11) I obtain
\[(3.8)
\]
Observe that (3.4) and (3.8) imply that for \(k \in \mathbb{N}\)
\[(3.9)
\]
Apply Theorem 2.1 to (3.9) to obtain
\[(3.10)
\]
which proves that
\[G(L, L + 1, \frac{13}{4}, 2, 4, q) \geq 0.
\]
In the limit as \(L \to \infty\) (3.10) becomes
\[(3.11)
\]
This is to be contrasted with Andrews-Gordon identity mod 21 \[1\]
\[(3.12)
\]
with \(N_i = n_i + \ldots + n_0, i = 1, \ldots, 9\). On the left of (3.11) one has 3-fold sum, while on the left of (3.12)
one has 9-fold sum. Analogously, applying Theorem 2.3 to (3.4) and Theorem 2.4 to (3.5) and (3.8), I
prove that

\[ G(L, L, \frac{11}{4}, \frac{5}{2}, 4, q) \geq 0, \]
\[ G(L - 1, L + 1, 4, \frac{5}{4}, 4, q) \geq 0, \]
\[ G(L - 1, L + 1, \frac{15}{4}, \frac{3}{2}, 4, q) \geq 0, \]

and obtain, as \( L \to \infty \)

\[ \sum_{m,k,n \geq 0} q^{k^2 + (m+k)^2 + n^2} \left( \frac{k}{n} \right) q^k \frac{q^{2T(k) + 2T(m+n)+2T(n)}}{(q)_{2k+1}} = \frac{(q^{21}, q^{10}, q^{11}; q^{21})_{\infty}}{(q)_{\infty}}, \]
\[ \sum_{m,k,n \geq 0} q^{2T(k) + 2T(m+n)+2T(n)} \left( \frac{k}{n} \right) q^k \frac{q^{2T(k) + 2T(m+n)+2T(n)}}{(q)_{2k+1}} = \frac{(q^{21}, q^6, q^{15}; q^{21})_{\infty}}{(q)_{\infty}}, \]

and

\[ \sum_{m,k,n \geq 0} q^{2T(k) + 2T(m+n)+2T(n)} \left( \frac{k}{n} \right) q^k \frac{q^{2T(k) + 2T(m+n)+2T(n)}}{(q)_{2k+1}} = \frac{(q^{21}, q^{10}, q^{11}; q^{21})_{\infty}}{(q)_{\infty}}, \]

respectively.

In [7, p. 2332] the following identity was derived

\[ \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L}{L - 2j} \right] = (-q; q^2)_L. \]

I now follow a well-trodden path and check that

\[ q^{L+1} \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+2j} \left[ \frac{2L}{L - 2j + 1} \right] = 0. \]

Adding (3.16) and (3.17) I derive, with the aid of (1.11), that

\[ \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L + 1}{L - 2j} \right] = (-q; q^2)_L. \]

Equations (3.16) and (3.18) imply that for \( k \in \mathbb{N} \)

\[ \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{k}{k + \frac{1}{2}} \right] = (-q; q^2)_{\frac{1}{2}}. \]

Applying Theorem 2.1 to (3.19) and letting \( L \to \infty \) I obtain

\[ \sum_{m,k \geq 0} q^{T(m)+T(m+k)} (-q; q^2) \frac{q^k}{(q)_{mk}} = \frac{(q^{20}, q^8, q^{12}; q^{20})_{\infty}}{(q)_{\infty}}. \]

Compare it with the Bressoud formula in [8]

\[ \sum_{n_1, \ldots, n_9 \geq 0} q^{N_1^2 + \ldots + N_9^2 + N_3} \frac{q^{N_1 \ldots N_9}}{(q)_{n_1} \ldots (q)_{n_9} (q^2)^{n_9}} = \frac{(q^{20}, q^8, q^{12}; q^{20})_{\infty}}{(q)_{\infty}}, \]

where \( N_i = n_i + \ldots + n_9, i = 1, \ldots, 9. \)

Analogously, applying (3.19) to (3.16) I get as \( L \to \infty \)

\[ \sum_{m,k \geq 0} q^{k^2 + (m+k)^2} (-q; q^2) \frac{q^k}{(q)_{mk}} = \frac{(q^{20}, q^{10}; q^{20})_{\infty}}{(q)_{\infty}}. \]
and
\[(3.23) \sum_{m,k \geq 0} \frac{q^{2T(k)+2T(m+k)}}{(q)_m(q)_{2k+1}} (-q;q^2)_k = \frac{(q^{20};q_6^6,q^{14};q^{20})_{\infty}}{(q)_{\infty}},\]
respectively.

For my final example, I employ Dyson’s identity [5], [7, p. 2330]
\[(3.24) \sum_{j=-\infty}^{\infty} (-1)^j q^{T(3j)} \left[ \frac{2L+1}{L-3j} \right]_q = \frac{(q^3;q^3)_L}{(q)_L}.
\]
Applying Theorem 2.4 to (3.24) yields
\[(3.25) \sum_{j=-\infty}^{\infty} (-1)^j q^{5T(3j)} \left[ \frac{2L}{L-1-6j} \right]_q = \sum_{k \geq 0} O_L, \frac{(q^3;q^3)_k}{(q)_k} \geq 0.
\]
This proves that
\[(3.26) G(L-1, L+1, 5, \frac{5}{2}, 6, q) \geq 0.
\]
Letting \(L \to \infty\) in (3.25) and using (1.13) I arrive at a new elegant result
\[(3.27) \sum_{m,k \geq 0} \frac{q^{2T(k)+2T(m+k)}}{(q)_m(q)_{2k+1}} \frac{(q^3;q^3)_k}{(q)_k} = \frac{(q^{15};q^{15})_{\infty}}{(q)_{\infty}}.
\]

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