RUDIN’S THEOREM AND PROJECTIVE HULLS

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We denote by $\Delta$ the closed unit disk, by $\Gamma$ the unit circle, and by $A_0$ the disk algebra, which consists of all functions holomorphic in $\text{int}(\Delta)$ and continuous on $\Delta$.

By a module over $A_0$ we mean a vector space $\mathcal{M}$ of continuous complex-valued functions on $\Delta$ such that the constant 1 lies in $\mathcal{M}$, and for every $a_0 \in A_0$ and $\varphi \in \mathcal{M}$, one has $a_0 \cdot \varphi \in \mathcal{M}$.

In his book “Real and Complex Analysis” (1966) Walter Rudin proved Theorem 12.13 which gives, in particular:

**Theorem 1.** Let $\mathcal{M}$ be a module over $A_0$. Assume that the maximum principle holds for $\mathcal{M}$ in the following sense:

$$\forall f \in \mathcal{M} \text{ and } \forall z_0 \in \text{int}\Delta, \quad |f(z_0)| \leq \sup_{\Gamma} |f|.$$  \hspace{1cm} (1)

Then every function in $\mathcal{M}$ is holomorphic.

**Note.** In 1953 Rudin had proved the corresponding statement with “module” replaced by “algebra”.

Recently, in [2], R. Harvey and H. B. Lawson introduced the notion of the projective hull of a compact set $X$ in $\mathbb{C}^n$ as follows. Denote by $P_d$ the space of polynomials on $\mathbb{C}^n$ of degree $\leq d$. Consider points $x \in \mathbb{C}^n$ such that there exists a constant $C_x$ for which

$$|P(x)| \leq (C_x)^d \sup_X |P|$$ \hspace{1cm} (2)

for all $P \in P_d$ and all $d$. The set of such points $x$ is denoted $\hat{X}$ and is called the projective hull of $X$ in $\mathbb{C}^n$. Note that if $C_x = 1$, then (2) implies that $x$ lies in the polynomial hull of $X$. They make the following

**Conjecture.** If $\gamma$ is a real-analytic closed curve in $\mathbb{C}^n$, then $\hat{\gamma} - \gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{C}^n - \gamma$.

**Note.** The real analyticity of $\gamma$ is important, since the analogous conjecture for a $C^\infty$-curve fails if the curve is not pluripolar.

In what follows we consider some questions related to Rudin’s Theorem and to the projective hull of a closed curve in $\mathbb{C}^2$.

**Theorem 2.** Let $\varphi$ be a continuous function on $\Delta$ such that the restriction of $\varphi$ to $\Gamma$ is real analytic. Let

$$\mathcal{M} = \{a + b\varphi : a, b \in A_0\} \subset C(\Delta).$$ \hspace{1cm} (3)
Assume for each \( x \in \text{int}\Delta \) that there exists a constant \( M_x \) such that for all \( f \in \mathcal{M} \):

\[
|f(x)| \leq M_x \cdot \sup_{\Gamma} |f|
\]  

(4)

Then \( \varphi \) (and hence every function in \( \mathcal{M} \)) is holomorphic in \( \Delta \).

\textbf{Note.} Condition (4) states that the evaluation functional: \( f \mapsto f(x) \), on \( \mathcal{M} \), is a bounded linear functional in the sup-norm on \( \Gamma \), for all \( x \in \text{int}(\Delta) \). It implies that two functions in \( \mathcal{M} \) which coincide on \( \Gamma \) are equal in \( \mathcal{M} \). This allows us to give \( \mathcal{M} \) the norm: \( \|f\| = \sup_{\Gamma} |f| \), making \( \mathcal{M} \) a subspace of \( C(\Gamma) \).

\textbf{Proof of Theorem 2.} Let \( L \) be the functional \( f \mapsto f(0) \) on \( \mathcal{M} \). By (4), \( \|L\| \leq M_0 \), so by Hahn-Banach and F. Riesz, there exists a measure \( \sigma \) on \( \Gamma \) such that

\[
f(0) = \int_{\Gamma} f \, d\sigma \quad \forall f \in \mathcal{M}.
\]  

(5)

We do not know that \( \|\sigma\| = 1 \), only that \( \|\sigma\| < \infty \).

We take \( f(\zeta) = \zeta^n, \ n > 0 \) and get from (5) that

\[
\int_{\Gamma} \zeta^n \, d\sigma = 0 \quad n = 1, 2, ... \quad \text{and}
\]

\[
\int_{\Gamma} 1 \, d\sigma = 1
\]  

(6)

(7)

We do not know that \( \sigma \) is a positive measure.

We further have that

\[
\int_{\Gamma} \zeta^n \frac{d\theta}{2\pi} = 0 \quad n = 1, 2, ... \quad \text{and}
\]

\[
\int_{\Gamma} 1 \frac{d\theta}{2\pi} = 1
\]  

(8)

(9)

The last four equations give

\[
\int_{\Gamma} \zeta^n \left( d\sigma - \frac{d\theta}{2\pi} \right) = 0 \quad n \geq 0.
\]  

(10)

By the F. and M. Riesz Theorem which identifies annihilating measures for the disk algebra, there exists \( h \in H^1_0(\Gamma) \) such that

\[
d\sigma - \frac{d\theta}{2\pi} = h \frac{d\theta}{2\pi}
\]  

(11)

Let us now fix \( \varphi \in \mathcal{M} \) and apply (5) to the functions \( z^n \varphi, \ n = 0, 1, 2, ... \) all of which belong to \( \mathcal{M} \). We get

\[
\int_{\Gamma} \zeta^n \varphi \, d\sigma = 0 \quad n = 1, 2, ... \quad \text{and}
\]  

2
\[ \int_{\Gamma} 1 \varphi \, d\sigma = \alpha \] for some constant \( \alpha \).

It follows that
\[ \int_{\Gamma} \zeta^n \left( \varphi \, d\sigma - \alpha \frac{d\theta}{2\pi} \right) = 0 \quad n = 0, 1, 2, \ldots \]

Hence there exists \( k \in H^1_0(\Gamma) \) such that
\[ \varphi \, d\sigma - \alpha \frac{d\theta}{2\pi} = k \frac{d\theta}{2\pi} \tag{12} \]

Multiplying (11) by \( \varphi \), we get
\[ \varphi \, d\sigma - \varphi \frac{d\theta}{2\pi} = \varphi h \frac{d\theta}{2\pi} \tag{13} \]

Now \( h \) and \( k \) are boundary functions defined on \( \Gamma \) of functions defined and holomorphic on \( \text{int} \Delta \). (We again denote these functions on \( \text{int} \Delta \) by \( h \) and \( k \).) From (12) and (13) we get
\[ (\alpha + k) \frac{d\theta}{2\pi} = \varphi (1 + h) \frac{d\theta}{2\pi} \], as measures on \( \Gamma \).

If \( 1 + h = 0 \) a.e. on \( \Gamma \), then \( \alpha + k = 0 \) a.e. on \( \Gamma \), and so from (12) we get that \( \varphi = 0 \) a.e. on \( \Gamma \), and so \( \varphi \) is identically zero, which we may exclude. An \( H^1 \)-function which is not identically zero is \( \neq 0 \) a.e. on \( \Gamma \), and so \( 1 + h \neq 0 \) a.e. on \( \Gamma \). It follows that
\[ \varphi = \frac{\alpha + k}{1 + h} \quad \text{a.e. on } \Gamma. \tag{14} \]

Since \( \varphi \) is real analytic on \( \Gamma \) there exists an analytic continuation \( \tilde{\varphi} \) of \( \varphi \) from \( \Gamma \) to an annulus: \( 1 - \epsilon < |\zeta| < 1 + \epsilon \). The function \( \frac{\alpha + k}{1 + h} \) is meromorphic in the strip: \( 1 - \epsilon < |\zeta| < 1 \) and has a non-tangential limit a.e. on \( \Gamma \). By (14) this non-tangential limit = \( \varphi(\zeta) = \lim_{z \to \zeta} \tilde{\varphi}(z) \) a.e. on \( \Gamma \). It follows that
\[ \frac{\alpha + k}{1 + h} = \tilde{\varphi} \quad \text{in } 1 - \epsilon < |\zeta| < 1. \tag{15} \]

Thus \( \frac{\alpha + k}{1 + h} \) is analytic in that strip and is meromorphic in \( \{|\zeta| < 1\} \), and hence has at most a finite number of poles in \( \text{int} \Delta \). We denote these poles by \( z_1, \ldots, z_n \). Let
\[ Q(\zeta) = \prod_{j=1}^{n} (\zeta - z_j). \]

Then \( Q \cdot \left( \frac{\alpha + k}{1 + h} \right) \) is analytic on \( \text{int} \Delta \). Since \( \frac{\alpha + k}{1 + h} = \tilde{\varphi} \) on the strip \( 1 - \epsilon < |\zeta| < 1 \), \( Q \cdot \left( \frac{\alpha + k}{1 + h} \right) \) is continuous up to \( \Gamma \), and coincides with \( Q \cdot \varphi \) on \( \Gamma \).
Thus \( Q \cdot \left( \frac{\alpha + k}{1 + h} \right) \) lies in \( A_0 \) and hence in \( M \). For \( \zeta_0 \in \text{int} \Delta \), we have, by (4),

\[
| Q \cdot \left( \frac{\alpha + k}{1 + h} \right) (\zeta_0) - (Q\varphi)(\zeta_0) | \leq M_{\zeta_0} \cdot \sup_{\Gamma} | Q \cdot \left( \frac{\alpha + k}{1 + h} \right) - (Q\varphi) | = 0.
\]

So \( Q \left( \frac{\alpha + k}{1 + h} \right) \) and \( (Q\varphi) \) agree at \( \zeta_0 \). This holds for each \( \zeta_0 \). Thus they coincide on \( \Delta \).

So \( Q\varphi \) is analytic on \( \text{int} \Delta \), and hence \( \varphi \) is meromorphic on \( \text{int} \Delta \). But by hypothesis, \( \varphi \) is continuous on \( \Delta \). So \( \varphi \) is analytic on \( \Delta \) and we are done.

Using this notion of projective hull, we next consider a generalization of Theorem 2 where the disk \( \Delta \) is replaced by the punctured disk \( \Delta - \{0\} \). Given a continuous function \( \varphi \) defined on \( \Delta - \{0\} \), let \( \gamma \) denote its graph over \( \Gamma \) and let \( \Sigma \) denote its graph over \( \Delta - \{0\} \). Thus

\[
\gamma = \{ (\zeta, \varphi(\zeta) : \zeta \in \Gamma \} \subset C^2 \quad \text{and} \quad \Sigma = \{ (\zeta, \varphi(\zeta) : \zeta \in \Delta - \{0\} \} \subset C^2.
\]

What strengthening of hypothesis (4) will imply that \( \varphi \) is analytic on \( \Delta - \{0\} \) and has either a pole or a removable singularity at \( \zeta = 0 \)?

**Proposition 3.** If \( \varphi \) is analytic on \( \Delta - \{0\} \) and has either a pole or a removable singularity at \( \{ \zeta = 0 \} \), then \( \Sigma \) is contained in the projective hull \( \hat{\gamma} \) of \( \gamma \).

**Proof.** Fix \( P \in \mathcal{P}_d \) with \( \sup_{\gamma} |P| \leq 1 \). Suppose \( \varphi \) has a pole at \( 0 \) of order \( k \). Then \( P(\zeta, \varphi(\zeta)) \) is meromorphic on \( \Delta \) with only a pole at \( \zeta = 0 \) of order \( \leq dk \).

Fix \( \zeta_0 \in \text{int} \Delta \). The function \( \zeta^d P(\zeta, \varphi(\zeta)) \) is holomorphic on \( \Delta \). Hence,

\[
\zeta_0^d P(\zeta_0, \varphi(\zeta_0)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\zeta^d P(\zeta, \varphi(\zeta))}{\zeta - \zeta_0} d\zeta.
\]

Therefore

\[
|\zeta_0^d P(\zeta_0, \varphi(\zeta_0))| \leq \frac{1}{2\pi} \frac{1}{1 - |\zeta_0|} \int_{\Gamma} |P(\zeta, \varphi(\zeta))| d\theta \leq \frac{1}{1 - |\zeta_0|}.
\]

So

\[
|P(\zeta_0, \varphi(\zeta_0))| \leq \frac{1}{1 - |\zeta_0|} \left( \frac{1}{|\zeta_0|^k} \right)^d.
\]

This holds for all \( d \), so \( (\zeta_0, \varphi(\zeta_0)) \in \hat{\gamma} \).

**Note:** This proposition also follows from results in [2].

**Question 4.** Is the converse of Proposition 3 true? That is, given \( \varphi \) continuous on the punctured disk and real analytic on the boundary, with \( \varphi \) and \( \Sigma \) as before, does the hypothesis: \( \Sigma \subset \hat{\gamma} \) imply that \( \varphi \) is holomorphic on \( \text{int} \Delta - \{0\} \) and has at \( \{ \zeta = 0 \} \) either a pole or a removable singularity?
Written explicitly, our hypothesis states: If $\zeta_0 \in \Delta - \{0\}$, then there exits a constant $C_{\xi_0}$ such that

$$\left| P(\zeta_0, \varphi(\zeta_0)) \right| \leq (C_{\xi_0})^d \cdot \sup_{\Gamma} \left| P(\zeta, \varphi(\zeta)) \right| \quad \text{for } P \in P_d$$

The following theorem gives a partial answer to Question 4. Recall that it is necessary that the function $\varphi$ be real analytic when restricted to $\Gamma$ (or at least that its graph be a pluripolar curve in $C^2$). It is natural therefore to assume $\varphi$ to be real analytic on punctured disk. We shall assume further that $\varphi$ is real analytic on the entire plane.

**Theorem 5.** Let $\Phi$ be an entire function on $C^2$ written as $\Phi(z, w) = \sum_{n,m=0}^{\infty} a_{nm}z^n w^m$, the series converging on all of $C^2$. Let

$$\varphi(\zeta) \equiv \Phi(\zeta, \bar{\zeta}) = \sum_{n,m=0}^{\infty} a_{nm} \zeta^n \bar{\zeta}^m, \quad \zeta \in C.$$  \hspace{1cm} (16)

Define $\gamma = \{(\zeta, \varphi(\zeta)) : \zeta \in \Gamma\}$ and $\Sigma = \{(\zeta, \varphi(\zeta)) : \zeta \in \Delta\}$.

Assume that $\Sigma \subset \hat{\gamma}$. Then $\varphi$ is complex analytic on $\Delta - \{0\}$ with a removable singularity at $\zeta = 0$.

**Proof.** Suppose $\varphi$ is not complex analytic on $\Delta - \{0\}$. Since the series representing $\varphi$ converges for all $\zeta \in C$, we have for each $R > 0$ a constant $C_R$ such that $|a_{nm}| \leq C_R/R^{n+m}$ for all $n,m$. Fix $d$. We may write

$$\varphi(\zeta) = \sum_{n+m \leq d} a_{nm} \zeta^n \bar{\zeta}^m + \sum_{n+m > d} a_{nm} \zeta^n \bar{\zeta}^m, \quad \zeta \in C.$$  \hspace{1cm} (17)

We denote the second term on the right hand side by $\epsilon_d(\zeta)$.

We have for each $|\zeta| \leq 2$, and for each $R > 2$, that

$$|\epsilon_d(\zeta)| \leq \sum_{n+m > d} |a_{nm}| 2^{n+m}$$

$$\leq \sum_{n+m > d} \frac{C_R}{R^{n+m}} 2^{n+m} = \sum_{k > d} \sum_{n+m=k} \frac{C_R}{R^k} 2^k = \sum_{k > d} \sum_{n+m=k} C_R (k+1) \left( \frac{2}{R} \right)^k$$

$$\leq C_R \left[ (d+1) \left( \frac{2}{R} \right)^d + (d+2) \left( \frac{2}{R} \right)^{d+1} + \cdots \right]$$

$$= C_R \left( \frac{2}{R} \right)^d \left[ (d+1) + (d+2) \left( \frac{2}{R} \right) + (d+3) \left( \frac{2}{R} \right)^2 + \cdots \right]$$

$$\leq C_R \left( \frac{2}{R} \right)^d (4 + 2d) \quad \text{for } R > 4.$$  

Hence, there exist $R_0, d_0$ with

$$|\epsilon_d(\zeta)| \leq C_R \left( \frac{4}{R} \right)^d \quad \forall \zeta, |\zeta| \leq 2 \quad \text{and} \quad \forall R > R_0 \quad \text{and} \quad d > d_0.$$  \hspace{1cm} (17)
Define now for all $d$ the polynomial

$$P_d(\zeta, w) = \zeta^d w - \sum_{n+m \leq d} a_{nm} \zeta^m \zeta^{-m}.$$ 

Then $P_d$ lies in $\mathcal{P}_{2d}$.

For $(\zeta, w)$ in $\gamma$ we have $w = \varphi(\zeta)$ and $\bar{\zeta} = \frac{1}{\zeta}$, so

$$P_d(\zeta, w) = \zeta^d \left[ \sum_{n+m \leq d} a_{nm} \zeta^n \frac{1}{\zeta^m} + \epsilon_d(\zeta) \right] - \zeta^d \left[ \sum_{n+m \leq d} a_{nm} \zeta^n \frac{1}{\zeta^m} \right] = \zeta^d \epsilon_d(\zeta),$$

for $\zeta \in \Gamma$. Hence,

$$|P_d(\zeta)| \leq |\epsilon_d(\zeta)| \leq C_R \left( \frac{4}{R} \right)^d \quad \text{on} \quad \gamma \quad \text{(18)}$$

by (17).

Now by assumption $\varphi$ is not complex analytic, so for some $m > 0$, $a_{nm} \neq 0$. The function $\alpha \mapsto \Phi(\alpha, \tilde{\alpha}) - \Phi(\alpha, \frac{1}{\alpha})$ is real-analytic in $|\alpha| > 0$. Suppose $\Phi(\alpha, \tilde{\alpha}) - \Phi(\alpha, \frac{1}{\alpha})$ is identically zero. Then $\alpha \mapsto \Phi(\alpha, \tilde{\alpha})$ is complex-analytic on $|\alpha| > 0$, and so $\varphi$ is complex-analytic, contrary to assumption.

Thus $\Phi(\alpha, \tilde{\alpha}) - \Phi(\alpha, \frac{1}{\alpha})$ is not identically zero. We therefore can choose $\alpha_0$ in $\frac{1}{2} < |\alpha| < 1$ such that $\Phi(\alpha_0, \tilde{\alpha}_0) - \Phi(\alpha_0, \frac{1}{\alpha_0}) = \tau \neq 0$. We next estimate the value of $P_d$ at the point $(\alpha_0, \varphi(\alpha_0))$ in $\Sigma$.

$$P_d(\alpha_0, \varphi(\alpha_0)) = \alpha_0^d \left[ \sum_{n+m \leq d} a_{nm} \alpha_0^n \tilde{\alpha}_0^m + \epsilon_d(\alpha_0) \right] - \alpha_0^d \left[ \sum_{n+m \leq d} a_{nm} \alpha_0^n \left( \frac{1}{\alpha_0} \right)^m \right]$$

$$= \alpha_0^d \left[ \sum_{n+m \leq d} a_{nm} \alpha_0^n \tilde{\alpha}_0^m - \sum_{n+m \leq d} a_{nm} \alpha_0^n \left( \frac{1}{\alpha_0} \right)^m \right] + \alpha_0^d \epsilon_d(\alpha_0).$$

For large $d$, then, we have in view of (17) that

$$|P_d(\alpha_0, \varphi(\alpha_0))| \geq |\alpha_0|^d \left| \Phi(\alpha_0, \tilde{\alpha}_0) - \Phi(\alpha_0, \frac{1}{\alpha_0}) \right| \cdot \frac{1}{2} - |\alpha_0|^d |\epsilon_d(\alpha_0)|$$

$$= |\alpha_0|^d \left[ \frac{|\tau|}{2} - |\epsilon_d(\alpha_0)| \right] \geq |\alpha_0|^d \frac{|\tau|}{4}$$

Thus we have

$$|P_d(\alpha_0, \varphi(\alpha_0))| \geq |\alpha_0|^d \frac{|\tau|}{4} \quad \text{for large} \quad d. \quad \text{(19)}$$

Also, for each $R > 4$, we have by (18) that $|P_d| \leq C_R \left( \frac{4}{R} \right)^d$ on $\gamma$, so

$$\left| \frac{P_d}{C_R \left( \frac{4}{R} \right)^d} \right| \leq 1 \quad \text{on} \quad \gamma.$$
By our hypothesis, \((\alpha_0, \varphi(\alpha_0)) \in \hat{\gamma}\), so by definition of \(\hat{\gamma}\) there exists a constant \(C_{\alpha_0}\) such that
\[
\left| \frac{P_d(\alpha_0, \varphi(\alpha_0))}{C_R(\frac{4}{R})^d} \right| \leq (C_{\alpha_0})^{2d} \quad \text{for all } d \geq d_0.
\]
So
\[
|P_d(\alpha_0, \varphi(\alpha_0))| \leq C_R \left( \frac{4}{R} \cdot C_{\alpha_0}^2 \right)^d \quad \text{for all } d \geq d_0. \tag{20}
\]
We now choose \(R\) so that \(\frac{4}{R} \cdot C_{\alpha_0} < |\alpha_0|\), and let \(d \to \infty\). Then (20) and (19) yield a contradiction. So \(\varphi\) is complex-analytic on \(\Delta - \{0\}\) and hence on all of \(\Delta\).

**Note.** Since \(\varphi\) is real analytic on the entire \(\zeta\)-plane, it follows that \(\varphi\) is an entire holomorphic function of \(\zeta\).

Theorem 9.2 of [2] yields that if \(f\) is an entire holomorphic function on \(\mathbb{C}\), then for a closed curve \(K\) on the graph \(\Sigma_f\) of \(f\), the projective hull \(\hat{K}\) of \(K\) in \(\mathbb{C}^2\) equals \(K\) union the bounded components of \(\Sigma_f - K\). Together with Theorem 5 just given we get:

**Corollary 7.** Let \(\varphi\) be given by a series \(\sum_{n,m=0}^{\infty} a_{nm} \zeta^n \bar{\zeta}^m\) which converges for all \(\zeta \in \mathbb{C}\). Let \(\gamma = \{(\zeta, \varphi(\zeta)) : \zeta \in \Gamma\}\). Then \(\varphi\) is an entire holomorphic function of \(\zeta\) if and only if \(\{(\zeta, \varphi(\zeta)) : \zeta \in \Delta\} = \hat{\gamma}\).

**References**

[1] W. Rudin, “Real and Complex Analysis”, McGraw Hill, Inc., N.Y., 1966.

[2] F. R. Harvey and H. B. Lawson, Jr., *Projective hulls and the projective Gelfand transformation*, Asian J. Math. **10**, no. 3 (2006), 279-319.

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