Spin bath interactions effects on the geometric phase

Paula I. Villar

Departamento de Física Juan José Giambiagi, FCEN, UBA,
Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina.
Computer Applications on Science and Engineering Department,
Barcelona Supercomputing Center (BSC), 29, Jordi Girona 08034 Barcelona Spain

(Dated: today)

We calculate the geometric phase of a spin-1/2 particle coupled to an external environment comprising N spin-1/2 particles in the framework of open quantum systems. We analyze the decoherence factor and the deviation of the geometric phase under a nonunitary evolution from the one gained under an unitary one. We show the dependence upon the system’s and bath’s parameter and analyze the range of validity of the perturbative approximation. Finally, we discuss the implications of our results.

PACS numbers: 03.65.Vf;03.67.Lx;03.65Yz;75.10.Jm

1. INTRODUCTION

A quantum system undergoing a cyclic evolution behaves quite differently from its classical analogue: the state vector retains the history of its evolution in the form of a geometric quantum phase. Berry [1] demonstrated that closed quantum systems could acquire phases that are geometric in nature. He showed that, besides the usual dynamical phase, an additional phase related to the geometry of the space state is generated during an adiabatic evolution. A phase only dependent upon the area covered by the motion of the system.

A renewed interest in geometric phases (GP) has recently blossomed by the proposal of using GP for quantum computation. The use of fault tolerant quantum gates in quantum information requires the study of the geometric phase in open quantum systems [2, 3, 4, 5, 6, 7, 8]. This has been motivated by the fact that all realistic quantum systems are coupled to their surrounding environments. No matter how weak the coupling that prevents the system from being isolated, the evolution of an open quantum system is plagued by nonunitary features like decoherence and dissipation. Decoherence, in particular, is a quantum effect whereby the system looses its ability to exhibit coherent behaviour and appears as soon as the two interfering partial waves shift the environment into states orthogonal to each other [9]. Nowadays, decoherence stands as a serious obstacle in quantum information processing.

Starting with the seminal paper of Zurek [10], several authors analyzed the decoherence due to a collection of independent spins. In this paper we discuss a two-level system coupled to a one-dimensional array of spin-1/2 particles. This model evolves reigned by a Hamiltonian which encompasses Ising and Universality in one simple approach. In some trivial cases, we can solve the problem exactly. Most of the works done so far are based on the so called “central spin model”, where the two level system is coupled isotropically to all spins of the bath, which implies a enormously simplification of the derivation. To ensure decoherence, the central spin (the system) is coupled to the transverse component of the bath.

Working towards having a realistic description of geometric phases, we have introduced field decoherence by coupling the central two level system to an external environment. In this paper we investigate the geometric phase of a spin-1/2 particle interacting with a spin environment. The model is the most simple case of an Ising chain and a further consideration to the Zurek model, since we are considering the self energy of the environment spin-1/2 particles. We calculate and analyze the decoherence factor induced by the coupling to an external bath. Within the general framework of GPs in open quantum systems, we
estimate the corresponding GP for the spin-1/2 particle in the weak coupling regime. We further compare this result to the general GP obtained numerically and analyze its dependence upon the bath and system parameters.

This paper is organized as follows. In Sec. II, we present the model and analyze the decoherence factor due to the interaction with the environment. In Sec. III, we estimate the geometric phase for the central spin and thoroughly study its behaviour by performing numerical and analytical calculations. Finally, we conclude our results in Sec. IV.

II. MODEL

In this section we examine the decoherence induced by disordered interacting spin baths at finite temperature. Our choice of the bath is the most simple case for an Ising chain so as to facilitate an analytical study of the decoherence and the geometric phase for the model. A further study of the geometric phase for an Ising chain with random spin-spin interactions will be studied elsewhere. Our system is comprised by a central spin $\sigma_c$ coupled to a bath of $N$ Ising spins through the interaction Hamiltonian

$$H_I = \sigma_c^z \otimes \sum_{i=1}^{N} \lambda_i \sigma_i^z.$$  

(1)

The coupling between the central spin and the bath is characterized by the coupling constant $\lambda$. The presence of a non interacting spin chain will be modeled by a bath Hamiltonian such as

$$H_B = \sum_{i=1}^{N} \omega_i \sigma_i^x,$$  

(2)

where $\sigma_i^x$ denotes the Pauli operators of the bath spins. The self Hamiltonian of the central spin is the common one $H_s = \hbar/2\Omega \sigma_c^z$. This model results a further approach to the model studied in [10] since we are considering not only the interaction Hamiltonian but also the self-Hamiltonians of the $N$ spins of the bath. In addition, we must note that $[H_B, H_I] \neq 0$.

We start with a total separable state of the form such as

$$|\Psi(0)\rangle = (\alpha | 0 \rangle + \beta | 1 \rangle) \times |\chi_0\rangle,$$  

(3)

where $| 0 \rangle$ and $| 1 \rangle$ are the state eigenstates in the $\sigma^z$ basis and $\chi_0$ is the initial state of the bath. At a later time, the total state of the system is

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle,$$  

(4)

where $H$ is the evolution total Hamiltonian. The main object in the study of decoherence is the reduced density matrix which is obtained after tracing out the environment’s degrees of freedom. In our case, the reduced density matrix may be written as

$$\rho_r(t) = \left( \begin{array}{cc} |\alpha|^2 & \alpha^* \beta F(t) \\ \alpha \beta^* F(t)^* & |\beta|^2 \end{array} \right),$$  

(5)

where $F(t) = \text{Tr}_B[e^{-i(H_B+V)t}\rho_B(0)e^{i(H_B-V)t}]$. We have named $V = \sum_{i=1}^{N} \lambda_i \sigma_i^z$ the total interaction Hamiltonian has operated on the central spin.

We can choose a pure state for the initial state of the bath as $|\chi_0\rangle = \prod_{i=1}^{N} (\alpha_i |0_i\rangle + \beta_i |1_i\rangle)$. Since $\rho_B(0) = |\chi_0\rangle \langle\chi_0|$, the decoherence factor $F(t)$ yields

$$F(t) = \prod_{i=1}^{N} \left( 1 - \frac{2\lambda_i^2}{\omega_i^2 + \lambda_i^2} \sin^2(\sqrt{\omega_i^2 + \lambda_i^2} t) \right).$$  

(6)
We can see that this decoherence factor does not depend on the external temperature. It is also possible to check that $F(0) = 1$. As it was done in [10], we can evaluate the mean value of the decoherence factor. This expression is similar to that obtained in [11] but for a different initial state of the environment. As in that case, $(F(t))_{T \to \infty} \to 0$, when $T \to \infty$. If we estimate the average dispersion as $\Delta^2 = \langle (F(t)^2) - (F(t))^2 \rangle = \sum_{i=1}^{N} p_i$, with $p_i = (1 - \frac{\lambda^2}{4(\omega_i^2 + \lambda^2)})$. Under the assumption that all $p_i$ are approximately equal, the average fluctuations from zero are

$$\Delta \sim \frac{1}{\sqrt{N}}.$$  \hspace{1cm} (7)

Therefore, large environments can effectively induced decoherence on the central spin system.

### III. GEOMETRIC PHASE ENVIRONMENTALLY CORRECTED

In this section, we shall compute the geometric phase for the central spin and analyze its deviation from the unitary geometric phase for a spin-1/2 particle.

A proper generalization of the geometric phase for unitary evolution to that for non unitary evolution is crucial for practical implementations of geometric quantum computation. In [12], a quantum kinematic approach was proposed and the geometric phase (GP) for a mixed state under nonunitary evolution has been defined by Tong et al. as

$$\Phi = \arg \{ \sum_k \sqrt{\varepsilon_k(0)\varepsilon_k(\tau)} \langle \Psi_k(0)|\Psi_k(\tau) \rangle \times e^{-\int_0^\tau dt \langle \Psi_k|\dot{\Psi}_k \rangle} \} , \hspace{1cm} (8)$$

where $\varepsilon_k(t)$ are the eigenvalues and $|\Psi_k\rangle$ the eigenstates of the reduced density matrix $\rho_k$ (obtained after tracing over the reservoir degrees of freedom). In the last definition, $\tau$ denotes a time after the total system completes a cyclic evolution when it is isolated from the environment. Taking the effect of the environment into account, the system no longer undergoes a cyclic evolution. However, we will consider a quasi cyclic path $P : t \in [0, \tau]$ with $\tau = 2\pi/\Omega$ (where $\Omega$ is the system’s frequency). When the system is open, the original GP, i.e. the one that would have been obtained if the system had been closed $\Phi^U$, is modified. This means, in a general case, the phase is $\Phi = \Phi^U + \delta \Phi$, where $\delta \Phi$ depends on the kind of environment coupled to the main system [2, 13, 14]. For a spin-1/2 particle in $SU(2)$, the unitary GP is known to be $\Phi^U = \pi(1 + \cos(\theta_0))$. It is worth noticing that the proposed GP is gauge invariant and leads to the well known results when the evolution is unitary.

To this end, we shall start by choosing a pure state of the form:

$$|\Psi(0)\rangle = \cos(\theta_0/2)|0\rangle + \sin(\theta_0/2)|1\rangle .$$

This is the same to have assumed $\alpha = \cos(\theta_0/2)$ and $\beta = \sin(\theta_0/2)$ in our above deduction of the decoherence factor. Then, for times $t > 0$, the state of the system is

$$|\Psi_+(t)\rangle = e^{-i\alpha t} \cos(\theta_+(t))|0\rangle + \sin(\theta_+(t))|1\rangle . \hspace{1cm} (9)$$

where we have defined $\tan(\theta_+(t)) = \tan(\theta_0/2)F^{-1}(t)$. It is easy to check that, when $F(t) = 1$, we re-obtain the results for the unitary case. From Eq. (3), one can easily write the reduced density matrix $\rho_+(t)$ and estimate its eigenvalues and eigenvectors, as was done in Ref. [5]. In order to estimate the geometric phase, we only need the eigenvector $|\Psi_+(t)\rangle$ since $\varepsilon_-(0) = 0$, and, hence, the only contribution to the phase comes from that eigenvector and its corresponding eigenvalue (see Eq. [3]). In the case we are considering here, and after performing the time derivatives of Eq. (3), we can note that the eigenvalues and eigenvectors are real functions. Consequently, the geometric phase is

$$\Phi = \Omega \int_0^\tau \cos^2(\theta_+(t)) . \hspace{1cm} (10)$$
As we can deduce from the relation between $\theta_0(t)$ and the decoherence factor, Eq. (10) is not easily computed. In Fig. 1, we show the behaviour of the GP as a function of the initial position on the Bloch Sphere ($\theta_0$) and the value of the coupling constant ($\lambda$) for an external environment comprising 10 spin-1/2 particles.

In order to achieve an analytical result, we shall assume all the bath spins to have the same coupling constant and frequency. Then, the decoherence factor becomes $F(t) = \prod_{i=1}^{N} f_i(t) = f(t)^N$ and we can forget about the product function. Besides, we shall perform a perturbative expansion in powers of the coupling constant $\lambda$. In such a case,

$$
\Phi = \pi \left(1 + \cos(\theta_0)\right) + N \left(\frac{\lambda}{\omega}\right)^2 \sin^2(\theta_0) \left(\pi - \frac{\Omega}{4\omega} \sin(4\pi \omega/\Omega)\right).
$$

(11)

Notice that the lowest correction in $\lambda/\omega$ of Eq. (11) is quadratic, which means that in case of low decoherence we recover the unitary GP up to the first order in $\lambda/\omega$. This reflects the resilient of the phase against the environment. If the decoherence rate is sufficiently small, the probability amplitude of staying in the same eigenstate is almost stationary. This amounts to effectively project the state back to the original eigenstate, whenever it tends to be driven away by the environment. Therefore, the state trajectory on the projective Hilbert space tends to be unaffected by the decoherence (up to first order), thereby leaving the area enclosed by the path, and hence the geometric phase, unchanged. Similarly, this robustness against decoherence can be observed in different two-level systems [3, 5, 6].

In Figs. 2 and 3, we show the comparison between the exact calculation (Eq. (10)) of the GP and the perturbative approximation (Eq. (11)) for different values of the coupling constant. In Fig. 2, we plot the nonunitary GP for a case of a small environment consisting of $N = 10$ spins. The lined curves, whether solid or dashed, correspond to exact calculations of the GP, for different initial position on the Bloch sphere. The perturbative estimations are plotted with lines and points in each case. Therein, it is possible to see that for a small initial angle, i.e. small value of $\theta_0$, there is almost no significant difference between both approaches. However, for an initial state of bigger value of $\theta_0$, the environmentally-induced correction to the unitary GP becomes important which is shown in Fig. 3 for an environment of $N = 100$ spins. This difference among
FIG. 2: Comparison between the exact GP and the perturbative approximation for different initial positions on the Bloch sphere and $N = 10$ spin environments. The lines (whether solid or dashed) represent the exact GP while the lines with different shaped points are the corresponding perturbative calculations.

FIG. 3: Comparison between the exact GP and the perturbative approximation for different initial positions on the Bloch sphere and $N = 100$ spin environments. The lines (whether solid or dashed) represent the exact GP while the lines with different shaped points are the corresponding perturbative calculations.

the approaches becomes more relevant as the number of spin environments increases and can be noted by comparing the scales of the y-axis in each figure. This can easily be understood by looking at Eq. (11). Therein, one can identify a new “effective” coupling constant $\lambda_{\text{eff}}^2 = N(\lambda/\omega)^2$, which means that one must consider the product of both quantities and that the perturbative approximation will hold for a compromise between them. That is to say that for larger environments (bigger values of $N$), the perturbative estimation will remain valid for smaller values of the coupling constant. However, this estimation remains valid for a
bigger range of the coupling constant if we reduce the number of spins comprising the environment.

Another feature that can be studied is the deviation from the unitary value of GP for both different approaches: exact and perturbative, as shown in Fig. 4. Therein, the GP vs the initial position on the Bloch Sphere ($\theta_0$) is plotted for different environment sizes and coupling constants. The lines (dotted or solid) correspond to the exact calculation of the GP while the lines with different shaped dots are the corresponding perturbative calculations for each case. As can be expected, when the coupling between the system and the environment increases, the perturbative calculation of the GP differs from the exact one. This difference is more relevant for larger environments, as can be observed in Fig. 4 the comparison between an environment comprising $N = 10$ spin-1/2 particles and one with $N = 100$ (and the same coupling constant $\lambda = 0.05$). Not only is it easy to note that this deviation is bigger for larger environments but also, it can be observed that it is more relevant for smaller angles as the number of environment spins increases.

Finally, we can analyze if the phase retains its geometric nature. Even though we know that the evolution of the system is not supposed to be cyclic due to the presence of the environment, under certain external conditions it may result a valid assumption. To that end, we plot the exact GP of Eq. (5) as a function of the number of cycles the system evolves normalized by the exact GP of only one cycle. Therein, we can note a nice topological feature of the phase: the phase is dependent on the number of times the path is traversed, i.e. the winding number. In the end, the GP acquired by the system will be the same. However, this does not mean that dephasing will not affect the measurement of the GP. The visibility fringes of any measurement made on the system will be reduced due to the presence of the environment [3, 9].

**IV. CONCLUSIONS**

So far we have studied the corrections to the geometric phase under a non unitary evolution induced by the presence of decoherence. The model is a further consideration in the spin-spin decoherence model since we are also considering the self energy of the spin-1/2 particles comprising the environment. In addition, it is the simplest approach to study the nonunitary geometric phase in an Ising chain.

As expected the geometric phase is affected by decoherence. This correction is quadratic in the coupling constants.
constant, so this affection is negligible when we work in the weak coupling constant $\lambda/\omega \ll 1$. However, the correction is linear in the environment’s size $N$ as well, which might have a considerable contribution for larger environments.

We have also shown an analytical expression for the environmentally induced correction to the GP, based in a series expansion in powers of the system-environment strength. What is worth noticing is that due to the dependence with the initial angle $\theta_0$, for small angles, there is not a significant correction induced by the environment, almost all the geometric character of the phase is in the unitary term. On the contrary, for a bigger value of $\theta_0$ the corrections to the unitary term become important. This conclusion is stressed as long as the number of spins in the environment increases. On the other hand, we have analyzed the validity of the perturbative calculation of the induced correction to the unitary term, as it is shown in Fig.4. Therein, we have shown that the larger environment, the shorter the values of $\lambda$ for which the perturbative expansions hold.

Finally, we have studied the geometric nature of the total GP (the one that includes the environmental correction) changing the single cyclic evolution ruled by $\tau = 2\pi/\Omega$ to $m$ times $\tau$, with $m$ an integer number (the winding number). We have shown (see Fig.5) that the total GP can be seen as $m$ times the GP for a single cyclic evolution, which means that the new phase is geometric because it depends on the winding number (if you go $m$ times around a path, the phase is $m$ times the phase for going once around the path).

The main reason why people want to use geometrical phases for quantum computation is that frequently geometrical evolutions are easier to control and may also be more resistant to random noise coming from the environment. Since experimentally it is much easier to control the Hamiltonian than the actual state of the system, the adiabatic evolution is of importance. This means that the evolution of the system takes a long time compared to the characteristic dynamical time scale. This condition must be subtly changed in the framework of quantum open systems. The evolution must take a long time compared to the characteristic dynamical time scale but also it must be shorter than the decoherence time-scale introduced by the presence of the environment. In the model we have studied here, both conditions are fulfilled since decoherence is not effectively induced on the system for a realistic small environment. The results shown herein are particularly relevant in the experimental realizations for the measurements of these phases.
V. ACKNOWLEDGMENTS

I thank F.C. Lombardo for useful comments. This work was supported by UBA, CONICET, and ANPCyT, Argentina.

[1] M.V. Berry, Proc. R. Soc. Lond. A 392, 45 (1984).
[2] F.C. Lombardo, F.D. Mazzitelli, and P.I. Villar, Phys. Rev. A 72, 042111 (2005); F.C. Lombardo and P.I. Villar, J. Phys. A 39, 6509 (2006).
[3] F.C. Lombardo and P.I. Villar, Phys. Rev. A 74, 042311 (2006).
[4] K.P. Marzlin, S. Ghose, and B.C. Sanders, Phys. Rev. Lett. 93, 260402 (2004).
[5] A. Carollo, I. Fuentes-Guridi, M. Franca Santos, and V. Vedral, Phys. Rev. Lett. 99, 160402 (2003); Phys. Rev. Lett. 92, 020402 (2004).
[6] G. De Chiara, A. Lozinski, G. M. Palma, Eur. Phys. J. D 41, 179-183 (2007).
[7] R.S. Whitney and Y. Gefen, Phys. Rev. Lett. 90, 190402, (2003); R.S. Whitney, Y. Makhlin, A. Shnirman, and Y. Gefen, Phys. Rev. Lett. 94, 070407 (2005).
[8] X.X. Yi and W. Wang, Phys. Rev. A 75, 032103 (2007).
[9] P.I. Villar and F.C. Lombardo, Journal of Physics: Conf. Ser. 67, 012041 (2007); Int. J. Mod. Phys. B 21, 4659 (2007).
[10] W.H. Zurek, Phys. Rev. D 26, 1862, 1982.
[11] S. Camalet and R. Chitra, Phys. Rev. B 75, 094434 (2007).
[12] D. M. Tong, E. Sjqvist, L. C. Kwek, and C. H. Oh, Phys. Rev. Lett. 93, 080405 (2004).
[13] S. Banerjee and R. Srikanth Eur. Phys. J. D 46, 335 (2008).
[14] A.T. Rezakhani and P. Zanardi, Phys. Rev. A 73, 012107 (2006).