DUAL SKEW CODES FROM ANNIHILATORS: TRANSPOSE HAMMING RING EXTENSIONS

José Gómez-Torrecillas, F. J. Lobillo, and Gabriel Navarro

Introduction

Linear codes may be endowed with cyclic structures by means of skew polynomial rings. This is the case of Piret cyclic convolutional codes \[26\] and the subsequent generalizations and alternatives (see \[27\], \[14\], \[11\], \[25\], \[16\], \[20\]). Non commutative cyclic structures of this kind have been also considered for block linear codes (\[7\], \[5\], \[4\], \[12\], \[1\]), and for linear codes over commutative rings (\[6\], \[22\], \[10\]).

A desirable property of any class of linear codes is to be stable under duals. This property has been already studied in several of the aforementioned references. A common feature of many of these approaches to duality is the presence of a suitable anti-isomorphism of rings that encodes, more or less explicitly, the transfer of the cyclic structure from the code to its dual. Our aim is to present a systematization of this method, besides some relevant examples where it successfully applies.

The strategy is to establish a formal framework, called transpose Hamming ring extension, designed to derive that the dual of every cyclic code is cyclic. Cyclic codes will be, from an algebraic point of view, identified as left ideals of suitable (non-commutative) ring extensions of a given commutative ring \(C\), well understood that such an “identification” has to be made explicit by an isomorphism of \(C\)–modules from the ring to \(C^m\), where \(m\) is the length of the \(C\)–linear code. The transposition will be an anti-isomorphism of rings which allows to transform annihilators into duals. Details are to be found in Section 1.

In Section 2 we apply our general approach to left ideal convolutional codes in the sense of \[25\], extending to a more general setting, and improving, results from \[25\] and \[15\] on the description of dual codes in this setting (Theorem 11). The case of a simple word-ambient algebra is analyzed in detail (Theorem 23).

Section 3 is devoted to dual codes of skew constacyclic codes over a commutative ring. Several results from \[7\], \[5\], \[22\], \[10\] on these codes are covered by our general result (Theorem 29).

Finally, in Section 4 we compute (Theorem 37) the dual of a skew Reed Solomon code over a general field in the sense of \[18\], and, as a consequence, these codes are shown to be evaluation codes (Theorem 39).

1. Transpose ring extensions and dual codes

Let \(C\) be a commutative ring. A \(C\)–linear code of length \(m\) is, by definition, a \(C\)–submodule \(C\) of \(C^m\). The dual of \(C\) is defined as

\[C^\perp = \{ w \in C^m : wv^T = 0, \forall v \in C \},\]

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where \( M^T \) denotes the transpose of a matrix \( M \) with coefficients in \( C \).

**Definition 1.** A Hamming ring extension of rank \( m \) is a three-tuple \((C, R, v)\), where \( R \) is a ring, \( C \) is a commutative subring of \( R \) and \( v : R \to C^m \) is an isomorphism of \( C \)-modules. Here, the \( C \)-module structure of \( R \) is given by left multiplication.

Hamming ring extensions encode cyclic structures on some \( C \)-linear codes, according to the following definition. Precise examples will be examined later.

**Definition 2.** Every \( f \in R \) leads to a \( C \)-linear code \( C = v(Rf) \). We say then that \( C \) is the \((C, R, v)\)-cyclic code generated by \( f \). We will also say that \( C \) is \( R \)-cyclic.

Given an \( R \)-cyclic code \( C = v(Rf) \), we have the multiplication map \( \cdot f : R \to R \). There is a unique square matrix \( M_R(f) \in M_m(C) \) making commute the following diagram of \( C \)-module morphisms

\[
\begin{array}{ccc}
R & \xrightarrow{f} & R \\
\downarrow{v} & & \downarrow{v} \\
C^m & \xrightarrow{\cdot M_R(f)} & C^m,
\end{array}
\]

so

\[ C = \text{im}(\cdot M_R(f)). \]

A straightforward computation shows that

\[ C^\perp = \ker(\cdot M_R(f)^T). \]

The map \( M_R : R \to M_m(C) \) sending \( f \) onto \( M_R(f) \) is an injective homomorphism of rings.

**Definition 3.** Two Hamming ring extensions \((C, R, v)\) and \((\hat{C}, \hat{R}, \hat{v})\) are said to be transposed if there exist an anti-isomorphism of rings \( \Theta : R \to \hat{R} \) such that the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{M_R} & M_m(C) \\
\downarrow{\Theta} & & \downarrow{(\cdot)^T} \\
\hat{R} & \xrightarrow{M_{\hat{R}}} & M_m(C)
\end{array}
\]

is commutative. Equivalently if,

\[ M_{\hat{R}}(\Theta(f)) = M_R(f)^T, \quad \forall f \in R. \]

We say also that \( \Theta \) is a transposition from \((C, R, v)\) to \((\hat{C}, \hat{R}, \hat{v})\).

Given a subset \( X \) of a ring \( S \), the left annihilator and the right annihilator of \( X \) are defined, respectively, by

\[ l_S(X) = \{ s \in S : sx = 0, \forall x \in X \}, \quad r_S(X) = \{ s \in S : xs = 0, \forall x \in X \}. \]

The first of these sets is a left ideal of \( S \), while the second one is a right ideal.

**Theorem 4.** Let \((C, R, v)\) and \((\hat{C}, \hat{R}, \hat{v})\) be transpose Hamming ring extensions with anti-isomorphism \( \Theta : R \to \hat{R} \). Let \( f, h \in R \) and \( C = v(Rf) \). Then \( hR = r_R(Rf) \) if and only if \( C^\perp = \hat{v}(\hat{R}\Theta(h)) \). In this case, \( C^\perp \) is generated by the rows of \( M_R(h)^T \).
Proof. Assume \( r_R(Rf) = hR \). Then we get \( l_R(\Theta(f)\hat{R}) = \hat{R}\Theta(h) \). This implies that the top row of the commutative diagram

\[
\begin{array}{ccc}
\hat{R} & \xrightarrow{-\Theta(h)} & \hat{R} \\
\downarrow & & \downarrow \\
\hat{C}^m & \xrightarrow{-\hat{M}_R(\Theta(h))} & \hat{C}^m
\end{array}
\]

is exact, so is the bottom row. Therefore,
\[
\hat{v}(\hat{R}\Theta(h)) = \text{im}(\cdot\hat{M}_R(\Theta(h))) = \ker(\cdot\hat{M}_R(\Theta(f))) = \ker(\cdot\hat{M}_R(f)^T) = C^\perp,
\]
where the last equality is \(^4\).

Conversely, assume \( C^\perp = \hat{v}(\hat{R}\Theta(h)) \). Then
\[
C^\perp = \text{im}(\cdot\hat{M}_R(\Theta(h))) = \text{im}(\cdot\hat{M}_R(h)^T),
\]
and therefore \( M_R(f)M_R(h) = 0 \). So \( fh = 0 \) and \( hR \subseteq r_R(Rf) \). Let \( h' \in r_R(Rf) \), i.e. \( fh' = 0 \). Then \( M_R(f)M_R(h') = 0 \) and so
\[
M_R(h')^TM_R(f)^T = 0.
\]
Since \( M_R(h')^T = M_\hat{R}(\Theta(h')) \) and \( C^\perp = \ker(\cdot\hat{M}_R(f)^T) \), it follows that \( \hat{v}(\hat{R}\Theta(h')) = \text{im}(\cdot\hat{M}_R(\Theta(h'))) \subseteq C^\perp = \hat{v}(\hat{R}\Theta(h)) \). Hence \( \Theta(h') \in \hat{R}\Theta(h) \), which implies \( h' \in hR \). Therefore \( r_R(Rf) \subseteq hR \) and the equality holds.

Finally, the equality \( \hat{v}(\hat{R}\Theta(h)) = \text{im}(\cdot\hat{M}_R(h)^T) \), implies that \( C^\perp \) is generated by the rows of \( M_R(h)^T \).

\[\square\]

**Corollary 5.** If \( C^{\perp\perp} = C \) and \( hR = r_R(Rf) \), then \( Rf = l_R(hR) \).

**Proof.** We have \( C^\perp = \hat{v}(\hat{R}\Theta(h)) \). Let \( \hat{\Theta} : \hat{R} \to R \) be the inverse of \( \Theta \). Since
\[
C^{\perp\perp} = C = v(Rf) = v(R\hat{\Theta}\Theta(f)),
\]
we get from Theorem \(^4\) applied to the transposition \( \hat{\Theta} : \hat{R} \to R \), that \( r_{\hat{R}}(\hat{R}\Theta(h)) = \Theta(f)\hat{R} \). But this implies that \( l_{\hat{R}}(\hat{R}) = Rf \). \[\square\]

**Remark 6.** It is well known that, if \( C \) is a field, then \( C^{\perp\perp} = C \) for every \( C \)-linear code \( C \). This equality also holds true in more general situations of interest. This is the case, for instance, if \( C \) is a principal ideal domain and \( C \) is a direct summand of \( C^m \), or \( C \) is any \( C \)-linear code over a Frobenius ring \( C \).

### 2. Dual left ideal convolutional codes

The method described in Section \(^1\) is abstracted from the study of the dual of a group convolutional code developed in \(^26\) Section 4]. The ideas from \(^26\) Section 4] were adapted in \(^16\) to \( \mathbb{F} \)-linear cyclic convolutional codes when the word ambient algebra is a matrix algebra \( \mathcal{M}_n(\mathbb{F}) \). The aim of this section is to extend the results on duality from \(^16\) to the matrix \( \mathbb{F} \)-algebra \( \mathcal{M}_n(\mathbb{K}) \), where \( \mathbb{K} \) is a finite field extension of \( \mathbb{F} \). To this end, we first work, in the spirit of Section \(^1\) in a more general setting, and then apply the general results to the more concrete situation.
2.1. **Left ideal convolutional codes of automorphism type.** Let $A$ be a ring and $\sigma : A \to A$ a ring automorphism. The skew right polynomial ring $[z; \sigma]A$ is defined as the free right $A$–module with basis $\{ z^i : i \in \mathbb{N} \}$ with the multiplication determined by the rules $z^i z^j = z^{i+j}$, and $az = z\sigma(a)$, for all $i, j \in \mathbb{N}$ and $a \in A$. If $A$ is a finite algebra over a finite field $\mathbb{F}$, and $\sigma$ is an $\mathbb{F}$–automorphism, then $R = [z; \sigma]A$ becomes the sentence ambient algebra for some cyclic convolutional codes as follows. First, observe that $\mathbb{F}[z]$, the commutative polynomial ring in the variable $z$, is a subring of $R$. Moreover, each $\mathbb{F}$–basis $B = \{ v_0, v_1, \ldots, v_{m-1} \}$ of $A$ leads to the associated coordinate map $\nu : A \to \mathbb{F}^m$. Since $B$ becomes a basis of $R$ as a (left) $\mathbb{F}[z]$–module, we get that $\nu$ extends to an $\mathbb{F}[z]$–module isomorphism $\nu : R \to \mathbb{F}[z]^m$. That is, $(\mathbb{F}[z], R, \nu)$ becomes a Hamming ring extension of rank $m$.

Convolutional codes may be understood as $\mathbb{F}[z]$–submodules of $\mathbb{F}[z]^m$, so, they may be considered as $\mathbb{F}[z]$–linear codes. The variable $z$ is interpreted as the delay operator $[13]$. Convolutional codes are often required to be, in addition, direct summands of $\mathbb{F}[z]^m$.

**Definition 7.** $[25]$ A direct summand $\mathbb{F}[z]$–submodule $C$ of $\mathbb{F}[z]^m$ is said to be a **left ideal convolutional code** if there exits a left ideal $I$ of $[z; \sigma]A$ such that $C = \nu(I)$.

Our next aim is to prove that, under suitable conditions, it is possible to construct a transpose Hamming ring extension to $R$. For $a \in A$, we use the notation $M_a = M_R(a)$. Observe that $M_a \in \mathcal{M}_m(\mathbb{F})$. On the other hand, for every $\mathbb{F}$–linear map $\lambda : A \to A$, let $M_{\lambda} \in \mathcal{M}_m(\mathbb{F})$ be the unique matrix such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & A \\
\nu & & \nu \\
\mathbb{F}^m & \mapdown{M_{\lambda}} & \mathbb{F}^m
\end{array}
\]

is commutative. A straightforward computation shows that, if $\lambda$ is an algebra map, then, for all $a \in A$,

\begin{equation}
M_a M_{\lambda} = M_{\lambda(a)}
\end{equation}

**Proposition 8.** For every $f = \sum_k z^k f_k \in R$ we have:

\[
M_R(f) = \sum_k z^k M_{\sum_k f_k}.
\]

**Proof.** This proof is adapted and simplified from $[25]$ Proposition 4.7 and $[16]$ Proposition 2.4. Since $\nu : R \to \mathbb{F}[z]^m$ is an $\mathbb{F}[z]$–linear map, we have, for every $g \in R$,

\[
\nu(g) \sum_k z^k M_{\sum_k f_k} = \nu \left( \sum_k z^k (g) \right) \sum_k z^k M_{\sum_k f_k}
\]

\[
= \sum_k z^k \nu(g) M_{\sum_k f_k}
\]

\[
= \sum_k z^k \nu(\sigma^k (g) f_k)
\]

\[
= \nu \left( \sum_k z^k \sum_k z^k \sigma^k (g_l) f_k \right)
\]

\[
= \nu(g f).
\]

□
Proposition 9. The map \( \Theta : [z;\sigma]A \to [z;\widehat{\sigma}]A \)
(\(\text{fixed, we have that} \))
\[
\sum_k z^k a_k \mapsto \Theta(\sum_k z^k a_k) = \sum_k z^k \theta \sigma^{-1}(a_k)
\]
Note that, as left \( \mathbb{F}[z] \)-modules, both rings are equal, and, once the basis \( B \) of \( A \) is fixed, we have that \( (\mathbb{F}[z], [z;\widehat{\sigma}]A, v) \) is a Hamming ring extension.

Remark 10. As observed in the proof of Theorem 4, the equality \( Rf = hR \) for some \( h \in R \). If \( C = v(Rf) \), then \( C^\perp = v(\widehat{R}\Theta(h)) \), and both \( C \) and \( C^\perp \) are left ideal convolutional codes. Moreover, \( Rf = l_R(hR) \).

Proof. The equality \( C^\perp = v(\widehat{R}\Theta(h)) \) follows from Proposition 4 and Proposition 9. As observed in the proof of Theorem 4, \( \widehat{R}\Theta(h) = l_R(\Theta(f)) \). Then \( C^\perp \) is a left ideal convolutional code by [10] Lemma 2.1. Corollary 5 by virtue of Remark 6 gives the equality \( Rf = l_R(hR) \). Therefore, \( C \) is a left ideal convolutional code, by [10] Lemma 2.1.
Remark 12. If $\mathbb{F}[z] \subseteq R$ is a separable ring extension, then every left ideal convolutional code is of the form $v(Re)$ for some idempotent $e \in R$ (see [17]). Therefore, $\tau_R(Re) = (1 - e)R$.

Remark 13. If $A = FG$, the group algebra of a finite group $G$, then Theorem [17] gives, in this particular case, a stronger statement than [26, Theorem 4.21], for the case where the $\sigma$-derivation considered is zero.

### 2.2. Convolutional codes with a simple word-ambient algebra

In [16], dual codes of left ideal convolutional codes, when $A$ is the matrix algebra $M_n(\mathbb{F})$, are studied. Concretely, [16, propositions 2.9 and 2.10] provide a transposition of Hamming ring extensions in this special case, and, henceforth, Theorem 11 generalizes and improves [16, Theorem 2.12].

Our next aim is to give sufficient conditions to apply Proposition 9 and Theorem 11 to $A = M_n(\mathbb{K})$, where $\mathbb{K} = \mathbb{F}_q$ is a finite field extension of $\mathbb{F} = \mathbb{F}_q$ of degree $t$. Let $D = \{\alpha_0, \ldots, \alpha_{t-1}\}$ be a basis of $\mathbb{K}$ as an $\mathbb{F}$–vector space. We have a monomorphism of $\mathbb{F}$–algebras

$$m : \mathbb{K} \to M_t(\mathbb{F}),$$

where $m(\gamma)$ is the matrix that represents the multiplication map $\gamma : \mathbb{K} \to \mathbb{K}$ for $\gamma \in \mathbb{K}$ with respect to $D$.

Let $V, V'$ be finite-dimensional $\mathbb{K}$–vector spaces with bases $B = \{v_0, \ldots, v_{r-1}\}$ and $B' = \{v'_0, \ldots, v'_{s-1}\}$, respectively. Define the $\mathbb{F}$–basis

$$B_\mathbb{F} = \{\alpha_i v_j : 0 \leq i \leq t, 0 \leq j \leq r\}$$

of $V$ ordered by the condition that $\alpha_i v_j$ is before than $\alpha_k v_{j+1}$ for all $i, j, k$. The $\mathbb{F}$–basis $B'_\mathbb{F}$ of $V'$ is defined analogously.

Lemma 14. Let $M = (m_{ij}) \in M_{tr \times s}(\mathbb{K})$ be the matrix associated with respect to the bases $B$ and $B'$ to a $\mathbb{K}$–linear map $\lambda : V \to V'$. The matrix associated to $\lambda$ considered as an $\mathbb{F}$–linear map with respect to the bases $B_\mathbb{F}$ and $B'_\mathbb{F}$ is

$$m(M) = \left(m(m_{ij})\right)_{\substack{0 \leq i < r, \ 0 \leq j < s}} \in M_{rt \times st}(\mathbb{F}).$$

Proof. Straightforward. ☐

Consider the $\mathbb{K}$–basis $B = \{E_{ij} : 0 \leq i, j < n, 0 \leq k < t\}$ of $A = M_n(\mathbb{K})$, where $E_{ij}$ is the matrix with $1$ in the position corresponding to $i$th row and $j$th column, and $0$ elsewhere. Order $B$ in such a way that the corresponding coordinate map $\mathcal{M}_n(\mathbb{K}) \to \mathbb{K}^{nt}$ writes each matrix as the concatenation of its rows. We thus get the $\mathbb{F}$–basis $B_\mathbb{F} = \{\alpha_k E_{ij} : 0 \leq k \leq t - 1, 0 \leq i, j \leq n - 1\}$ ordered as described above. This last basis leads to a coordinate map $v : A \to \mathbb{F}^{tn^2}$, which, for any $\mathbb{F}$–automorphism $\sigma$ of $A$, extends to an $\mathbb{F}[z]$–linear isomorphism $v : [z; \sigma]A \to \mathbb{F}[z]^{tn^2}$. Our aim is to prove that, if the basis $D$ of $\mathbb{K}$ over $\mathbb{F}$ is a normal self-dual basis, then a transposition $\Theta$ fulfilling the hypotheses of Proposition 9 can be constructed.

Let $D^* = \{\beta_0, \ldots, \beta_{t-1}\}$ the dual basis of $D = \{\alpha_0, \ldots, \alpha_{t-1}\}$. Recall that $D$ is normal if there exists $\alpha \in \mathbb{K}$ such that $\alpha_k = \alpha \sigma^k$ for all $0 \leq k < t$. Also recall that $D$ is self-dual if $\alpha_k = \beta_k$ for all $0 \leq k < t$.

The coordinate map with respect to $D$ is given by

$$\mathbb{K} \to \mathbb{F}^t, \ \gamma \mapsto (\text{Tr}(\beta_0 \gamma), \ldots, \text{Tr}(\beta_{t-1} \gamma))$$
where $\text{Tr}$ denotes the trace map of the field extension $\mathbb{F} \subseteq \mathbb{K}$. By the properties of the trace function, 

$$m(\gamma) = \left( \text{Tr}(\beta_i \gamma \alpha_j) \right)_{0 \leq i,j < s}.$$ 

As involution on $A$, we use the natural one, that is, $\theta : A \to A$ be the involution given by $\theta(a) = a^T$. In order to show how $\theta$ fulfills the Proposition $\theta$, we need to use some properties of the Kronecker product of matrices over a field $L$.

Recall that if $M \in M_{r \times s}(L)$ and $N \in M_{r' \times s'}(L)$, the Kronecker product of $M$ and $N$ is defined as 

$$M \otimes N = \begin{pmatrix} m_{0,0}N & m_{0,1}N & \cdots & m_{0,s-1}N \\ m_{1,0}N & m_{1,1}N & \cdots & m_{1,s-1}N \\ \vdots & \vdots & \ddots & \vdots \\ m_{r-1,0}N & m_{r-1,1}N & \cdots & m_{r-1,s-1}N \end{pmatrix},$$

where $M = (m_{i,j})_{0 \leq i < r, 0 \leq j < s}$. Properties of the Kronecker product can be seen in [21, Chapter 4].

**Lemma 15.** The following properties hold for all $a \in A$:

(i) $M_a = m(I \otimes a)$.

(ii) If $B$ is self-dual, $M_a^\tau = M_a^\tau$.

**Proof.** (i) follows from Lemma $[14]$ and $[15]$ Lemma 2.2. If $D$ is self-dual then $m(\gamma)$ is symmetric for all $\gamma \in \mathbb{K}$, hence (ii) follows from (i) and $[16]$ Lemma 2.3(iii). \(\square\)

Lemma $[15]$ ensures the first hypothesis of Proposition $\theta$. As for the second one concerns, namely the equality $M_\sigma = M_\sigma^\tau$, it will be obtained with the help of a suitable decomposition of $\sigma$. Let $\tau : \mathbb{K} \to \mathbb{K}$ denote the Frobenius $\mathbb{F}$–automorphism of $\mathbb{K}$, i.e. $\tau(\gamma) = \gamma^q$. By [3] Theorem 2.4, for every $\mathbb{F}$–automorphism $\sigma$ of the matrix algebra $A$, there exist a regular matrix $U \in A$ and $0 \leq h \leq t - 1$ such that $\sigma = \sigma_U \circ \sigma_{U^h}$, where $\sigma_U$ is the inner automorphism associated to $U$, i.e. $\sigma_U(a) = UaU^{-1}$, and $\sigma_{U^h}$ is the componentwise extension of $\sigma^h$ to $A$, i.e. $\sigma_{U^h}(m_{ij}) = (\sigma^h(m_{ij}))$.

We are going to analyze inner automorphisms and extensions of field automorphisms independently, and later we will join both results.

For each regular matrix $U \in A$, a straightforward computation shows that $\sigma_U^\tau = \theta \sigma_U^{-1} \theta = \sigma_U^\tau$.

**Lemma 16.** The following properties hold:

(i) $M_{\sigma_U} = m(U^T \otimes U^{-1})$.

(ii) If $D$ is self-dual, then $M_{\sigma_U} = M_{\sigma_U}^\tau$.

**Proof.** (i) follows from Lemma $[14]$ and $[16]$ Lemma 2.3(ii). If $D$ is self-dual then $m(\gamma)$ is symmetric for all $\gamma \in \mathbb{K}$, hence (ii) follows from (i). \(\square\)

**Proposition 17.** Let $R = [z; \sigma_U]A$ and $\tilde{R} = [z; \sigma_U^\tau]A$. If $D$ is a self-dual basis, then $(\mathbb{F}[z], R, v)$ and $(\mathbb{F}[z], \tilde{R}, \tilde{v})$ are transposed Hamming extensions.

**Proof.** By Lemma $[14]$ $M_a^\tau = M_a^\tau$ for all $a \in A$. By Lemma $[14]$ $M_{\sigma_U} = M_{\sigma_U}^\tau$. Hence the result follows from Proposition $\theta$. \(\square\)

Now we focus on extensions of field automorphisms.
Lemma 18. Let \( \tau \) be the Frobenius automorphism of \( \mathbb{K} \) over \( \mathbb{F} \), and let \( 0 \leq h < t \). If \( D \) is a normal basis, then the automorphism \( \tau^h \) is represented as \( \mathbb{F} \)-linear map by the matrix
\[
P_h = \begin{pmatrix}
0 & I_{t-h} \\
I_h & 0
\end{pmatrix} \in \mathcal{M}_t(\mathbb{F}).
\]
The automorphism \( \sigma_{\tau^h} : A \to A \) is represented as \( \mathbb{F} \)-linear map by the matrix
\[
M_{\sigma_{\tau^h}} = I_{n^2} \otimes P_h = \begin{pmatrix}
P_h \\
\vdots \\
P_h
\end{pmatrix} \in \mathcal{M}_{n^2t}(\mathbb{F}).
\]

Proof. Straightforward since \( \tau \) performs a cyclic permutation of one position to the right of the elements of \( D \).

Lemma 19. Assume \( D \) to be a normal basis. The following properties hold:
\[
(i) \quad M_{\sigma_{\tau^h}}^{-1} = M_{\sigma_{\tau^{-h}}} = M_{\sigma_{\tau^h}}^T.
\]
\[
(ii) \quad M_{\sigma_{\tau^h}} = M_{\sigma_{\tau^h}}^T.
\]

Proof. (i) is a direct consequence of Lemma 18. Since \( \sigma_{\tau^h} \) acts component-wise, it commutes with \( \theta \), hence \( \sigma_{\tau^h} = \theta \sigma_{\tau^h} \theta = \theta^2 \sigma_{\tau^h} = \sigma_{\tau^{-h}} \), hence (ii) follows from (i).

Proposition 20. Let \( R = [z; \sigma_{\tau^h}]A \) and \( \hat{R} = [z; \hat{\sigma}_{\tau^h}]A \). If \( D \) is a normal basis then \( (\mathbb{F}[z], R, v) \) and \( (\mathbb{F}[z], \hat{R}, v) \) are transposed Hamming extensions.

Proof. Again, Lemma 19 implies \( M_a = M_a^T \) for all \( a \in A \). Lemma 19 implies \( M_{\sigma_{\tau^h}} = M_{\sigma_{\tau^h}}^T \). So Proposition 9 gives the result.

Finally we get to general automorphisms \( \sigma = \sigma_U \sigma_{\tau^h} \in \text{Aut}_\mathbb{F}(A) \) where \( U \in A \) is a regular matrix and \( 0 \leq h < t \).

Lemma 21. Let \( \sigma = \sigma_U \sigma_{\tau^h} \in \text{Aut}_\mathbb{F}(A) \). If \( D \) is a self-dual normal basis, we have
\[
M_\sigma = M_\sigma^T.
\]

Proof. The decomposition \( \sigma = \sigma_U \sigma_{\tau^h} \) implies \( M_\sigma = M_{\sigma_{\tau^h}} M_{\sigma_U} \). Since
\[
\hat{\sigma} = \hat{\sigma}_U \hat{\sigma}_{\tau^h} = \theta \sigma_{\tau^h}^{-1} \sigma_U^{-1} \theta = \sigma_{\tau^h}^{-1} \theta \sigma_U^{-1} \theta = \hat{\sigma}_U \hat{\sigma}_{\tau^h},
\]
we get \( M_\sigma = M_{\hat{\sigma}_U} M_{\hat{\sigma}_{\tau^h}} \). Hence the result follows from Lemma 10 and Lemma 19.

Proposition 22. Let \( R = [z; \sigma]A \) and \( \hat{R} = [z; \hat{\sigma}]A \). If \( D \) is a self-dual normal basis, then \( (\mathbb{F}[z], R, v) \) and \( (\mathbb{F}[z], \hat{R}, v) \) are transposed Hamming extensions.

Proof. Decompose \( \sigma = \sigma_U \sigma_{\tau^h} \) according to [8, Theorem 2.4]. The proof is now completely analogous to Proposition 17 and Proposition 20 by using Lemma 10, Lemma 21 and Proposition 9.

We are now in position to state the main result of this subsection, which is a consequence of Theorem 11 and Proposition 22. Field extensions which have a self-dual normal basis are characterized by results of Lempel, Weinberger, Seroussi, Imamura and Morii, see [23] and its references.
Let $\sigma$ be any $\mathbb{F}$-automorphism of $A = \mathcal{M}_n(\mathbb{K})$. Assume there exists a self-dual normal basis $D$ of $\mathbb{K}$ over $\mathbb{F}$. Let $f \in R = [x; \sigma]A$ such that $r_R(Rf) = hR$ for some $h \in R$. If $C = v(Rf)$, then $C^\perp = v(\hat{R}\Theta(h))$, and $C$ and $C^\perp$ are both left ideal convolutional codes. Moreover, $Rf = r_R(hR)$.

**Proof.** The result follows from Proposition 22 and Theorem 11.

**Remark 24.** Taking $\mathbb{K} = \mathbb{F}$ in Theorem 23 leads to a strong form of [16, Theorem 2.12].

**Remark 25.** By [23, Theorems 1 and 2], if $t$ is odd, or $q$ is even and $t \equiv 2 \mod 4$, then there exists a self-dual normal basis $D$ of $\mathbb{K}$ over $\mathbb{F}$, and Theorem 23 applies.

### 3. Dual of skew constacyclic codes

Let $C$ be a commutative ring and $\sigma$ an automorphism of $C$. Assume that $\sigma$ has finite order, and let $n \geq 1$ such that $\sigma^n = \text{id}_C$. If $u \in C$ is a unit such that $\sigma(u) = u$, then the both $x^n - u$ and $x^n - u^{-1}$ are central elements in the skew left polynomial ring $C[x; \sigma]$, where the multiplication rule is now $xa = \sigma(a)x$. Consider the factor rings

$$R = \frac{C[x; \sigma]}{\langle x^n - u \rangle}, \quad \hat{R} = \frac{C[x; \sigma]}{\langle x^n - u^{-1} \rangle},$$

which contain $C$ as a subring. Since the ideal $\langle x^n - u \rangle$ coincides with the left ideal generated by $x^n - u$, and this polynomial is monic, we easily get that $\{1, x, \ldots, x^{n-1}\}$ is a basis of $R$ as a left $C$-module (we are identifying, as usual, each equivalent class modulo $x^n - u$ with its unique representative of degree less than $n$). We have a Hamming ring extension $(C, R, \nu)$, where $\nu : R \to C^n$ is the coordinate map with respect to the aforementioned basis.

Since $x$ is a unit in both rings, we may define

$$\Theta : R \to \hat{R}, \quad \sum_{i=0}^{n-1} a_i x^i \mapsto \sum_{i=0}^{n-1} \sigma^{-i}(a_i) x^{-i}.$$

**Lemma 26.** The map $\Theta$ is an anti-isomorphism of rings.

**Proof.** Since $\Theta$ is clearly additive, in order to prove that it is an anti-homomorphism of rings, it suffices to check that it is anti-multiplicative on monomials. To this end, let us first observe that, since $x^n = u^{-1}$ in $\hat{R}$, we have for all integers $i, j, k$ with $k = in + j$

$$x^{-k} = u^i x^{-j} \quad \text{(4)}$$

Now, given any integer $k$, write $k = in + j$ for some integers $i, j$ with $0 \leq j < n$. Then, for any $a \in C$, we have

$$\Theta(ax^k) = \Theta(au^i x^j) = u^i \sigma^{-j}(a)x^{-j} = \sigma^{-k}(a)x^{-k},$$

where, in the last equality, we used (4) and that $\sigma^n = \text{id}_C$. Therefore, for any $a, b \in C$ and integers $i, j$ we have

$$\Theta(ax^i bx^j) = \Theta(aa^i(b)x^{i+j}) = \sigma^{-i-j}(a)\sigma^j(b)x^{-i-j} = \sigma^{-i-j}(a)\sigma^j(b)x^{-i-j},$$

while

$$\Theta(bx^i)x^j = \sigma^{-j}(b)x^{-j}\sigma^{-i}(a)x^{-i} = \sigma^{-j}(b)\sigma^{-j-i}(a)x^{-i-j}.$$

Therefore, $\Theta$ is anti-multiplicative. The map $\Theta$ is proved to be bijective by defining a map $\hat{\Theta} : \hat{R} \to R$ analogous to $\Theta$ which turns out to be its inverse. \qed
The set \{1, x, \ldots, x^{n-1}\} provides bases as left \(C\)-modules of \(R = C[x; \sigma]/\langle x^n - u \rangle\) and of \(\hat{R} = C[x; \sigma]/\langle x^n - u^{-1} \rangle\).

**Proposition 27.** Consider the basis \{1, x, \ldots, x^{n-1}\} of \(R\) and \(\hat{R}\), with corresponding coordinate isomorphisms \(v : R \to C^n\) and \(\hat{v} : \hat{R} \to \hat{C}^n\). The anti-isomorphism \(\Theta : R \to \hat{R}\) is a transposition from \((C, R, v)\) to \((C, \hat{R}, \hat{v})\).

**Proof.** First, observe that if \(f = \sum_{i=0}^{n-1} a_i x^i \in R\) then
\[
M_R(f) = \begin{pmatrix}
a_0 & a_1 & \cdots & a_{n-1} \\
u(\sigma(a_{n-1})) & \sigma(a_0) & \cdots & \sigma(a_{n-2}) \\
\vdots & \vdots & \ddots & \vdots \\
u(\sigma(a_{n-1})) & \sigma(a_2) & \cdots & \sigma(a_0)
\end{pmatrix}.
\]
Now, since \(x^{-1} = ux^{-n} \in \hat{R}\), we get that \(x^{-i} = ux^{-n-i}\) for \(i = 0, 1, \ldots, n-1\). Hence,
\[
\Theta(f) = \sum_{i=0}^{n-1} \sigma^{-i}(a_i)ux^{-i} = a_0 + \sum_{j=1}^{n-1} u\sigma^j(a_{n-j})x^j.
\]
Therefore,
\[
M_{\hat{R}}(\Theta(f)) = \begin{pmatrix}
a_0 & u\sigma(a_{n-1}) & \cdots & u\sigma^{n-1}(a_1) \\
a_1 & \sigma(a_0) & \cdots & u\sigma^{n-1}(a_2) \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-1} & \sigma(a_{n-2}) & \cdots & \sigma^{n-1}(a_0)
\end{pmatrix} = M_R(f)^T
\]
because \(v^n = \text{id}_C\). \qed

**Definition 28.** Let \((C, R, v)\) be the Hamming ring extension where \(R = C[x; \sigma]/\langle x^n - u \rangle\) with \(\sigma^n = \text{id}_C\) and \(u \in C\) is a unit such that \(\sigma(u) = u\). A \((u, \sigma)\)-constacyclic code is called \((u, \sigma)\)-constacyclic code.

**Theorem 29.** Let \(f \in C[x; \sigma]\) be monic such that \(x^n - u = fh\) for some \(h \in C[x; \sigma]\). Let \(C = v(Rf)\) be the \((u, \sigma)\)-constacyclic \(C\)-linear code generated by \(f\). Then \(C^\perp = \hat{v}(\hat{R} \Theta(h))\), and it is thus a \((u^{-1}, \sigma)\)-constacyclic code.

**Proof.** Let us first check that \(r_R(Rf) = hR\). By hypothesis, \(h \in r_R(Rf)\) so that \(r_R(Rf) \supseteq hR\). To see the converse inclusion, let \(h' \in r_R(Rf)\). Then \(fh' = (x^n - u)g = fhg\) for some \(g \in C[x; \sigma]\). Since monic polynomials are nonzero divisors in \(C[x; \sigma]\), we get that \(h' = hg\) as desired.

By virtue of Proposition 27 we may apply Theorem 4 and thus get \(C^\perp = \hat{v}(\hat{R} \Theta(h))\). \qed

**Remark 30.** If \(u^2 = 1\), we get \(R = \hat{R}\) and \(\Theta\) becomes an involution in \(R\), i.e. an anti-algebra automorphism such that \(\Theta^2 = \text{id}_R\).

**Corollary 31.** If \(u^2 = 1\), then \(C^\perp = v(R \Theta(h))\).

**Example 32.** Cyclic and negacyclic codes over finite chain rings defined in [9] fit in our construction. Let \(C\) be a commutative chain ring and \(C\) a \(C\)-linear code of length \(n\). Then \(C\) is cyclic if and only if \(\sigma^{-1}(C)\) is an ideal of \(R = C[x]/\langle x^n - 1 \rangle = \hat{R}\). The ring \(R\) is proven to be a principal ideal ring, [9] Corollary 3.7, and generators \(F\) and \(G\) for a cyclic code \(C\) and its dual \(C^\perp\) are computed, [9] theorems 3.6 and 3.10].
Theorem 3 implies that $\Theta(G)\mathcal{R} = r_{\mathcal{R}}(\mathcal{R}F)$. The same can be said for negacyclic $C$-codes, i.e. ideals of $C[x]/\langle x^n + 1 \rangle$, if $F$ and $G$ are the generators computed in [9 Theorem 5.7 and 5.12] then $\Theta(G)$ generates the annihilator of $F$.

**Example 33.** Since, in Theorem 29, $x^n - u$ is a central element of $C[x; \sigma]$, the equality $fh = x^n - u$ implies, by a standard argument, that $hf = x^n - u$. Now, $x$ is a unit in $\mathcal{R}$, which implies that $x^k\Theta(h)$ generates $\mathcal{R}\Theta(h)$ for every integer $k$. Setting $k$ equal to the degree of $h$, we get that $x^k\Theta(h)$ is a generator polynomial of the dual code $C^\perp$ which takes the form of the given in [11 Corollary 18] for $\sigma$-codes over a finite field. We also get the generator of the dual of any skew constacyclic code computed in [6 Theorem 4.4], [22 Lemma 3.1] and [10 Proposition 3] in the realm of codes over finite commutative rings.

### 4. Dual of skew Reed-Solomon codes

Let $\sigma$ be an automorphism of finite order $n$ of a field $L$. Since $L$ is a field, all left and right ideals in the skew left polynomial ring $L[x; \sigma]$ are principal. So greatest common left and right divisors and least common left and right multiples exist and they can be computed with the corresponding versions of the extended Euclidean algorithm, see e.g. [15]. Concretely given $f, g \in L[x; \sigma]$, the least common left multiple of $f$ and $g$, denoted by $[f, g]_L$, is the monic generator of $L[x; \sigma]f \cap L[x; \sigma]g$, and the greatest common right divisor, denoted by $(f, g)_R$, is the monic generator of $L[x; \sigma]f + L[x; \sigma]g$. Analogously, the least common right multiple is denoted by $[f, g]_R$ and the greatest common left divisor by $(f, g)_L$.

The theory developed in Section 3 can be applied to $\mathcal{R} = L[x; \sigma]/\langle x^n - 1 \rangle$ with $u = 1$. In this case, $\mathcal{R} = \mathcal{R}$ and $(1, \sigma)$-constacyclic codes are called $\sigma$-cyclic codes. Theorem 31 says that the dual of a $\sigma$-cyclic code is $\sigma$-cyclic. In this $\sigma$-cyclic setting the morphism $\Theta$ becomes an involution, i.e.

$$\Theta : \mathcal{R} \to \mathcal{R}, \quad \sum_{i=0}^{n-1} a_i x^i \mapsto \sum_{i=0}^{n-1} \sigma^{-i}(a_i)x^{-i} = a_0 + \sum_{j=1}^{n-1} \sigma^j(a_{n-j})x^j,$$

satisfies $\Theta^2 = \text{id}_\mathcal{R}$.

Skew Reed-Solomon codes are $\sigma$-cyclic codes generated by some special polynomials which we describe below. Let $\{\alpha, \sigma(\alpha), \ldots, \sigma^{n-1}(\alpha)\}$ be a normal basis of $L/K$ where $K = L^\sigma$ denotes de invariant subfield. Let $\beta = \sigma(\alpha)\alpha^{-1}$. By [18 Lemma 3.1],

$$x^n - 1 = [x - \beta, x - \sigma(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell.$$

Let $t$ such that $2t < n$, $\delta = 2t + 1$. Let

$$g = [x - \beta, x - \sigma(\beta), \ldots, x - \sigma^{\delta-2}(\beta)]_\ell.$$

The code $C = \wp(\mathcal{R}g)$ is called a skew RS code of designed distance $\delta$. If $k = n - \delta + 1$, $C$ is an $[n, k, \delta]$-code over $L$, hence it is MDS with respect to the Hamming distance (see [18 Definition 2, Theorem 3.4]). Skew Reed-Solomon codes can be efficiently decoded, see [18, 19].

We will prove that the dual of a skew Reed-Solomon code is again a skew Reed-Solomon code, and describe explicitly its generator polynomial as a least common left multiple of linear polynomials. Some preliminary results are needed. Observe that $\sigma$ extends to an automorphism of $\mathcal{R}$, which acts on $x$ as the identity.
Lemma 34. The morphisms $\Theta$ and $\sigma$ commute, i.e. $\Theta \sigma = \sigma \Theta$.

Proof. It is a straightforward computation. \hfill \Box

Let $\gamma \in L$ such that $x - \gamma$ left divides $x^n - 1$. Then $(x - \gamma)R$ is a maximal right ideal, and therefore $R\Theta(x - \gamma)$ is a maximal left ideal. Hence there exists $\gamma' \in L$ such that

$$R(x - \gamma') = R\Theta(x - \gamma).$$

In fact, since $\Theta(x - \gamma) = x^{-1} - \gamma$, we get that

$$\gamma' = \sigma(\gamma)^{-1}.$$  

Lemma 35. If $h = \left[\{x - \sigma^i(\gamma) \mid 0 \leq i \leq k - 1\}\right]_r$, then $R\Theta(h) = R h'$, where

$$h' = \left[\{x - \sigma^i(\gamma') \mid 0 \leq i \leq k - 1\}\right]_l.$$

Proof. Since $h = \left[\{x - \sigma^i(\gamma) \mid 0 \leq i \leq k - 1\}\right]_r$,

$$h R = \bigcap_{i=0}^{k-1} (x - \sigma^i(\gamma)) R = \bigcap_{i=0}^{k-1} \sigma^i ((x - \gamma) R).$$

So, by Lemma 34,

$$R\Theta(h) = \Theta (h R)$$

$$= \left(\bigcap_{i=0}^{k-1} \sigma^i ((x - \gamma) R)\right)$$

$$= \bigcap_{i=0}^{k-1} \Theta (\sigma^i ((x - \gamma) R))$$

$$= \bigcap_{i=0}^{k-1} \sigma^i (\Theta ((x - \gamma) R))$$

$$= \bigcap_{i=0}^{k-1} \sigma^i (\Theta (x - \gamma))$$

$$= \bigcap_{i=0}^{k-1} \sigma^i (R(x - \gamma'))$$

$$= \bigcap_{i=0}^{k-1} R (x - \sigma^i(\gamma'))$$

$$= \bigcap_{i=0}^{k-1} \{x - \sigma^i(\gamma') \mid 0 \leq i \leq k - 1\},$$

\hfill \Box

Lemma 36. Let $\gamma \in L$ such that $(x - \gamma) \left[\begin{array}{c} x - \sigma(\beta), \ldots, x - \sigma^{n-1}(\beta) \end{array}\right]_r = x^n - 1$. Then

$$\left[\begin{array}{c} x - \gamma, \ldots, x - \sigma^k(\gamma) \end{array}\right]_r \left[\begin{array}{c} x - \sigma^{k+1}(\beta), \ldots, x - \sigma^{n-1}(\beta) \end{array}\right]_e = x^n - 1$$

for all $0 \leq k \leq n - 1$. 

Proof. Denote $N_i = \{x - \sigma^i(\beta) \mid j \neq i\} \ell$ for each $i = 0, \ldots, n-1$. Then $N_i$ has degree $n-1$ by (5). So, there exist $\gamma_n, \gamma_1, \ldots, \gamma_{n-1} \in L$ such that

$$(x - \gamma_i)N_i = x^n - 1$$

for $i = 0, \ldots, n-1$. Let $\gamma = \gamma_0$. Then $(x - \gamma)N_0 = x^n - 1$, and therefore $(x - \sigma^k(\gamma))\sigma^k(N_0) = x^n - 1$ for $k = 1, \ldots, n-1$. On the other hand, observe that $\sigma^k(N_0) = N_k$ for any $k = 1, \ldots, n-1$. Thus, for $k = 1, \ldots, n-1$, $(x - \sigma^k(\gamma))N_k = (x - \gamma_k)N_k$, and then $\gamma_k = \sigma^k(\gamma)$ for all $k = 1, \ldots, n-1$.

Let us now prove our thesis, i.e.

$$[x - \gamma, \ldots, x - \sigma^k(\gamma)]_\ell [x - \sigma^{k+1}(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = x^n - 1$$

for $k = 0, \ldots, n-1$, by induction on $k$. The case $k = 0$ is trivial. Assume that it holds for $j \leq k$ and we prove it for $k+1$. Firstly, the polynomial $[x - \gamma, \ldots, x - \sigma^{k+1}(\gamma)]_\ell$ has degree $k + 2$. Indeed, otherwise, by hypothesis, $[x - \gamma, \ldots, x - \sigma^k(\gamma)]_\ell = [x - \gamma, \ldots, x - \sigma^{k+1}(\gamma)]_\ell$ and then $(x - \sigma^{k+1}(\gamma))p = [x - \gamma, \ldots, x - \sigma^k(\gamma)]_\ell$. So,

$$(x - \sigma^{k+1}(\gamma))p [x - \sigma^{k+1}(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = x^n - 1$$

In particular, this implies that

$$p [x - \sigma^{k+1}(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = N_{k+1}$$

and, consequently, $x - \sigma^{k+1}(\beta)$ right divides $N_{k+1}$, which contradicts (5).

Now, by hypothesis,

$$[x - \gamma, \ldots, x - \sigma^k(\gamma)]_\ell [x - \sigma^{k+1}(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = x^n - 1$$

so, applying $\sigma$,

$$[x - \sigma(\gamma), \ldots, x - \sigma^{k+1}(\gamma)]_\ell [x - \sigma^{k+1}(\beta), \ldots, x - \sigma^{n-1}(\beta), x - \beta]_\ell = x^n - 1$$

so

$$[x - \sigma(\gamma), \ldots, x - \sigma^{k+1}(\gamma)]_\ell q [x - \sigma^{k+2}(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = x^n - 1$$

for some polynomial $q$. That is, if $h$ is the monic polynomial verifying that

$$h [x - \sigma^{k+2}(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = x^n - 1,$$

then $[x - \sigma(\gamma), \ldots, x - \sigma^{k+1}(\gamma)]_\ell$ left divides $h$. On the other hand, the case $k = 0$ provides that

$$(x - \gamma) [x - \sigma(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = x^n - 1$$

and therefore $x - \gamma$ left divides $h$ as well. So $[x - \gamma, x - \sigma(\gamma), \ldots, x - \sigma^{k+1}(\gamma)]_\ell$ left divides $h$. Since both have degree $k + 2$, the result follows.

\begin{theorem}
Consider a skew RS code $C = v(Rg)$, where $g = [x - \beta, x - \sigma(\beta), \ldots, x - \sigma^{d-2}(\beta)]_\ell$.

If $\gamma \in L$ is such that

$$(x - \gamma) [x - \sigma(\beta), \ldots, x - \sigma^{n-1}(\beta)]_\ell = x^n - 1,$$

then $C^\perp$ is the skew RS code generated by

$$[x - \sigma^k(\gamma)^{-1}, \ldots, x - \sigma^n(\gamma)^{-1}]_\ell.$$
Proof. By Lemma 35,

\[ x - \gamma, \ldots, x - \sigma^{n-\delta}(\gamma) \in [x - \sigma^{n-\delta+1}(\beta), \ldots, x - \sigma^{n-1}(\beta)] \in L^n - 1, \]

and, by applying \( \sigma^{\delta - 1} \), we obtain

\[ x - \sigma^{\delta-1}(\gamma), \ldots, x - \sigma^{n-\delta}(\gamma) \in [x - \beta, \ldots, x - \sigma^{\delta-1}(\beta)] \in L^n - 1. \]

Therefore, \( hg = x^n - 1 \), where \( h = [x - \sigma^{\delta-1}(\gamma), \ldots, x - \sigma^{n-1}(\gamma)] \). A standard argument, which uses that \( x^n - 1 \) is central, proves that \( gh = x^n - 1 \).

By Corollary 31, we get that \( \mathcal{C}^\perp = R\Theta(h) \). Lemma 35 gives then that \( R\Theta(h) \) is generated by

\[ x - \sigma(\sigma^{\delta-1}(\gamma))^{-1}, \ldots, \sigma^{n-\delta+1}(\sigma^{\delta-1}(\gamma))^{-1} \in L^n, \]

which finishes the proof. \( \square \)

We finish by proving that, as a consequence of Theorem 37, skew RS codes can be seen as evaluation codes. The right evaluation of a skew polynomial \( f = \sum_i f_i x^i \in L[x; \sigma] \) by \( a \in L \) is the remainder of the right division of \( f \) by \( x - a \), i.e. the unique element \( f(a) \in L \) such that \( f(x) = q(x)(x - a) + f(a) \). Then

\[ f(a) = \sum_i f_i N_i(a) \]

where

\[ N_i(a) = a\sigma(a) \ldots \sigma^{i-1}(a). \]

**Definition 38.** Let \( \overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \in L^m \) and \( \overline{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{m-1}) \in (L \setminus \{0\})^m \). The skew Generalized Evaluation code associated to \( (\overline{\alpha}, \overline{\sigma}) \) is

\[ sGE_{(\overline{\alpha}, \overline{\sigma})} = \{ (v_0 f(\alpha_0), \ldots, v_{m-1}f(\alpha_{m-1})) \mid f \in L[x; \sigma], \deg f < k \}. \]

It is straightforward to check that \( sGE_{(\overline{\alpha}, \overline{\sigma})} \) is an \( L \)-linear code. In fact a generator matrix for it is

\[
\begin{pmatrix}
 v_0 & v_1 & \cdots & v_{m-1} \\
v_0 \alpha_0 & v_1 \alpha_1 & \cdots & v_{m-1} \alpha_{m-1} \\
 \vdots & \vdots & \ddots & \vdots \\
v_0 N_{k-1}(\alpha_0) & v_1 N_{k-1}(\alpha_1) & \cdots & v_{n-1} N_{k-1}(\alpha_{m-1})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
 1 & 1 & \cdots & 1 \\
 \alpha_0 & \alpha_1 & \cdots & \alpha_{m-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 N_{k-1}(\alpha_0) & N_{k-1}(\alpha_1) & \cdots & N_{k-1}(\alpha_{m-1})
\end{pmatrix}
\begin{pmatrix}
 v_0 & 0 & \cdots & 0 \\
 0 & v_1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & v_{m-1}
\end{pmatrix}
\]

This definition is an extension of [23] Definition 9 to arbitrary fields, including therefore the convolutional case when \( L = \mathbb{F}(t) \). A different approach to evaluation codes, where the norms are replaced by powers of the automorphism, can be found in [22] [23].

Let us now prove that a skew RS code is also a sGE code.

**Theorem 39.** Let \( \mathcal{C} = v(Rg) \) be a skew RS code where

\[ g = [x - \beta, \ldots, x - \sigma^{\delta-2}(\beta)] \in L^n. \]

Then there exist \( \mu, \nu \in L \setminus \{0\} \) such that \( \mathcal{C} = sGE_{(\overline{\alpha}, \overline{\sigma})} \), where \( \overline{\alpha} = (\mu, \sigma(\mu), \ldots, \sigma^{n-1}(\mu)) \) and \( \overline{\sigma} = (\nu, \sigma(\nu), \ldots, \sigma^{n-1}(\nu)) \).
Proof. By Theorem 37 there exists $\mu \in L$ such that $C^\perp = v(Rg')$ where

$$g' = [x - \mu, \ldots, x - \sigma^{n-\delta}(\mu)]_\ell.$$  

Since $\sigma^i(\mu)$ is a right root of $g'$ for $0 \leq i \leq n - \delta$, it follows that a parity check matrix of $C^\perp$ is

$$H = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\mu & \sigma(\mu) & \ldots & \sigma^{n-\delta}(\mu) \\
\vdots & \vdots & \ddots & \vdots \\
N_{n-1}(\mu) & N_{n-1}(\sigma(\mu)) & \ldots & N_{n-1}(\sigma^{n-\delta}(\mu))
\end{pmatrix}.$$  

Since $\mu$ is also a left root of $x^n - 1$, it follows that $N_n(\mu) = 1$ and, by Hilbert’s 90 Theorem, there exists $\nu \in L$ such that $\mu = \sigma(\nu)\nu^{-1}$. So, up to multiply each column by the corresponding scalar, a new parity check matrix for $C^\perp$ is

$$H = \begin{pmatrix}
\nu & \sigma(\nu) & \ldots & \sigma^{n-\delta}(\nu) \\
\sigma(\nu) & \sigma^2(\nu) & \ldots & \sigma^{n-\delta+1}(\nu) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma^{n-1}(\nu) & \nu & \ldots & \sigma^{n-\delta-1}(\nu)
\end{pmatrix}.$$  

Therefore $C$ is generated by the rows of the matrix

$$H^T = \begin{pmatrix}
\nu & \sigma(\nu) & \ldots & \sigma^{n-1}(\nu) \\
\sigma(\nu) & \sigma^2(\nu) & \ldots & \nu \\
\vdots & \vdots & \ddots & \vdots \\
\sigma^{n-\delta}(\nu) & \sigma^{n-\delta+1}(\nu) & \ldots & \sigma^{n-\delta-1}(\nu)
\end{pmatrix}.$$
We have

\[ H^T = \begin{pmatrix}
\nu & \sigma(\nu) & \cdots & \sigma^{n-1}(\nu) \\
\sigma(\nu) & \sigma^2(\nu) & \cdots & \nu \\
\vdots & \vdots & \ddots & \vdots \\
\sigma^{n-\delta}(\nu) & \sigma^{n-\delta+1}(\nu) & \cdots & \sigma^{n-\delta-1}(\nu)
\end{pmatrix}
\]

\[ = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\mu & \sigma(\mu) & \cdots & \sigma^{n-1}(\mu) \\
\vdots & \vdots & \ddots & \vdots \\
N_{n-\delta}(\mu) & N_{n-\delta}(\sigma(\mu)) & \cdots & N_{n-\delta}(\sigma^{n-1}(\mu))
\end{pmatrix}
\]

\[ \cdot \begin{pmatrix}
\nu & 0 & \cdots & 0 \\
0 & \sigma(\nu) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sigma^{n-1}(\nu)
\end{pmatrix}
\]

hence \( \mathcal{C} = s\text{GE}_{[\sigma, \sigma^2]} \).

\[ \square \]

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E-mail address: gomezj@ugr.es

CITIC and Department of Algebra, University of Granada, Spain

E-mail address: jlobillo@ugr.es

CITIC and Department of Algebra, University of Granada, Spain

E-mail address: gnavarro@ugr.es

CITIC and Department of Computer Sciences and AI, University of Granada, Spain