Distributed Dispatching in the Parallel Server Model

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Abstract—With the rapid increase in the size and volume of cloud services and data centers, architectures with multiple job dispatchers are quickly becoming the norm. Load balancing is a key element of such systems. Nevertheless, current solutions to load balancing in such systems admit a paradoxical behavior in which more accurate information regarding server queue lengths degrades performance due to herding and detrimental incast effects. Indeed, both in theory and in practice, there is a common doubt regarding the value of information in the context of multi-dispatcher load balancing. As a result, both researchers and system designers resort to more straightforward solutions, such as the power-of-two-choices to avoid worst-case scenarios, potentially sacrificing overall resource utilization and system performance. A principal focus of our investigation concerns the value of information about queue lengths in the multi-dispatcher setting. We argue that, at its core, load balancing with multiple dispatchers is a distributed computing task. In that light, we propose a new job dispatching approach, called Tidal Water Filling, which addresses the distributed nature of the system. Specifically, by incorporating the existence of other dispatchers into the decision-making process, our protocols outperform previous solutions in many scenarios. In particular, when the dispatchers have complete and accurate information regarding the server queues, our policies significantly outperform all existing solutions.

Index Terms—Load management, load modeling, queueing analysis, stability analysis, distributed computing, stochastic systems.

I. INTRODUCTION

Large software systems that govern the operations of data centers and of cloud-based services are important components of modern-day computing infrastructure. Such systems process a large volume of jobs that often arrive at distinct locations and are serviced by a multitude of servers. How jobs are assigned to servers, and in particular load balancing the servers’ job queues, plays a central role in determining the effectiveness of the system’s overall performance.

The traditional approach to load balancing in this setting, known as the supermarket model [5], [40], [41], employs a single centralized dispatcher to which all requests are forwarded, and from which they are assigned to the servers. In the last decade, with the increasing size of cloud services and applications, using a single dispatcher has become a problematic bottleneck. System designers have consequently shifted to architectures that employ multiple dispatchers [4], [6], [28]. The load balancing problem for such multi-dispatcher systems is a very natural and urgent distributed systems problem. However, to the best of our knowledge, it has received only limited attention in the literature (see Section I-A). The current paper considers the problem from a distributed computing perspective.

A principal focus of our investigation concerns the value of information about queue lengths in the multi-dispatcher setting. As discussed below, several well-known load balancing policies behave poorly when the dispatchers’ information is highly correlated. They suffer from so-called “herd behavior,” in which, when dispatchers identify the same servers as having short queues, they all send their jobs to these servers. As a consequence, the servers become overloaded, resulting in poor performance. This is especially acute when dispatchers share considerable information about the queue lengths. Indeed, in describing their solutions, recent papers have made statements such as “Inaccurate information can lead to better performance” [37], or “Inaccurate information can improve performance” [43]. Interestingly, the value of accurate information is doubted not only by theoreticians. In fact, open source load balancer deployments such as
A. Related Work

In the standard supermarket (i.e., parallel server) model, it is well known that if the (single) dispatcher has complete information regarding the server queues, the protocol that routes each job to the server with the shortest queue (called Join the Shortest Queue and denoted by JSQ) offers strong performance and strong theoretical guarantees [5], [40], [41]. JSQ has also motivated the design of reduced-state load balancing techniques for resource-constrained scenarios in which the dispatcher is exposed to only partial information about the server queue lengths. For example, in Power-of-d-choices, denoted by JSQ(d) [17], [20], [38], when a job arrives, a dispatcher randomly probes d servers and assigns the job to a server with the shortest queue among them. A related strategy is called Power of memory, denoted by JSQ(d, m) [22], [31]. In JSQ(d, m), the dispatcher samples the m shortest queues to whom it sent jobs in the latest round, in addition to d ≥ m ≥ 1 randomly chosen servers. The job is then routed to the shortest among these d + m queues.

In the last decade, with the increasing size of cloud services and applications, the need to scale horizontally drove system designers to introduce multiple dispatchers into their design as a single dispatcher could no longer utilize hundreds and thousands of servers [4], [6], [28]. In such multi-dispatcher systems, traditional solutions such as JSQ suffer from detrimental herd behavior and therefore systems operators abandon the use of readily available information and turn to reduced-state approaches such as JSQ(d) potentially sacrificing overall system performance and reduced resource utilization in order to prevent worst-case scenarios [7], [16], [18], [30], [32], [35].

In the search for a better alternative for the multi-dispatcher load balancing scenario, Join-the-idle-queue (JIQ) [16], [21], [33], [34], [39] was recently proposed. In JIQ, dispatchers are notified only by idled servers. In turn, a dispatcher sends jobs to an idle server when it is aware of one, or to a randomly selected server otherwise. JIQ was shown to significantly improve performance at low and moderate loads over JSQ(d) due to its immediate prevention of server starvation [16]. However, at higher loads, its performance resembles random routing due to the absence of idle servers and its performance deteriorates quickly [44].

The most recent advances on load balancing for multi-dispatcher systems appeared in [37] and [43]. In the Local Shortest Queue policy (LSQ), proposed in [37], each dispatcher keeps a local state with possibly outdated queue size information, which is infrequently updated. A dispatcher then sends jobs to a shortest queue by its local estimation. This can be viewed as a generalization of similar ideas suggested for single-dispatcher systems [2], [36]. LSQ was followed by LED [43], which extended the theoretical performance guarantees to a wider family of tilted dispatching policies. These local-state-driven policies were shown to outperform previous policies considered for the multi-dispatcher model such as JSQ, JSQ(d) and JIQ. Intuitively, this is because they use considerably less information than JSQ, which reduces herding, and, on the other hand, maintain local states at the dispatchers allowing for a long term memory and preventing many events in which no single good server is discovered.

However, the aforementioned policies admit a paradoxical behavior in which accurate information, or even partial but correlated information among the dispatchers, degrades their performance. This is because, like their complete state-information JSQ counterpart, having shared information about good servers leads them to herd-behavior and detrimental inact effects that increase tail latency. That is, in the multiple-dispatchers case, when updated information is available to different dispatchers, they all send at once all incoming jobs.
to the servers with the currently-shorter queues, overwhelming them with the accumulated traffic. This phenomenon has already been pointed out in [19], which suggested the importance of using randomness to break the symmetry.

Another line of work concerning load balancing in distributed systems is based on the balls-into-bins model [1]. Recent approaches commonly apply regret minimization (e.g., [11]) and adaptive techniques (e.g., [15]), producing more precise theoretical bounds. However, their model assumptions are not aligned with our model (e.g., we consider stochastic arrivals at each dispatcher and stochastic departures at each server). Consequently, their analysis does not directly apply in our model and vice versa.

II. Model

We consider a system with a set $D$ of $M$ dispatchers and a set $S$ of $N$ servers. Dispatchers can communicate with servers over communication channels or via shared memory, but there is no direct communication among dispatchers. The network is the complete undirected bipartite graph with edge set $E = D \times S$, and both jobs and standard messages can be sent over the edges of $E$.

The system proceeds over discrete synchronous rounds. Time starts at time 0, and round $t+1$ occurs between time $t$ and time $t+1$. Each round consists of four phases: First, every dispatcher receives an external input with a set of job requests. Second, every dispatcher sends each of the jobs it received to a server for processing, and every job received by a server is added to its job queue. Third, each of the servers completes processing a set of (zero or more) jobs from the head of its queue and reports the results to the appropriate clients. (This is where jobs depart from the system.) Finally, in a potential fourth phase of the round, dispatchers and servers may communicate information about the status of the server queues. More formally, the four phases are:

1) **Arrivals:** Some number, $a^{(m)}(t)$, of exogenous jobs arrive at dispatcher $m$ at the beginning of round $t$. (We denote $a(t) = \sum_{m \in D} a^{(m)}(t)$.) Job arrivals are governed by stochastic processes. For simplicity, we assume that $a^{(m)}(t)$ are i.i.d. random variables governed by the same distribution which, according to standard practice, is not assumed to be known.

2) **Dispatching.** In every round, each dispatcher forwards the jobs it received to the servers for service. We consider two variants of dispatching that address two distinct affinity constraints. In one, termed splittable dispatching, the dispatcher assigns each of the jobs to a server of its choice. This handles a setting with no affinity constraints. That is, when the jobs arriving at a dispatcher $m$ can be processed independently. We separately consider unsplittable dispatching, in which all jobs that arrive at $m$ in a given round must be forwarded to the same server. This handles a setting with strict affinity constraints. Where, e.g., the jobs arriving at $m$ in a given round share data or resources.

3) **Departures.** Each server maintains a FIFO queue that keeps track of its pending jobs. We denote by $Q_n(t)$ the length of server $n$’s queue at the beginning of round $t$ (before any job arrivals and departures) and denote $Q(t) = (Q_1(t), \ldots, Q_N(t))$. Moreover, we denote by $g^{(m)}_n(t)$ the number of jobs that server $n$ receives from dispatcher $m$ in round $t$. Moreover, $g_n(t) = \sum_{m \in D} g^{(m)}_n(t)$ denotes the total number of jobs sent to server $n$ in round $t$. We denote server $n$’s job completion rate (i.e., the number of jobs that it is able to complete) in round $t$ by $s_n(t)$. We thus have that $Q_n(t+1) = \max\{0, Q_n(t) + g_n(t) - s_n(t)\}$. I.e., server $n$ completes $s_n(t)$ jobs in round $t$ if it has that many jobs to process. Otherwise, it completes all $Q_n(t) + g_n(t) < s_n(t)$ jobs in its possession. As in the case of arrivals, $s_n(t)$ is assumed to be stochastically determined. Again, for ease of exposition their distribution is assumed to be the same for all servers.

4) **Communication.** In the fourth phase of a round, dispatchers obtain information about the queue sizes at the servers. We will consider two settings. In the complete information case, every dispatcher is informed of all queue sizes and so, at the start of round $t$, it knows $Q(t)$. The incomplete information setting is one in which queue information is communicated over the channels of $E = D \times S$, without all servers communicating with all dispatchers in every round.

**Admissibility.** For a setting as above to be feasible, it must hold that the servers’ processing power is sufficient for handling the incoming job requests. Denoting $s(t) = \sum_{n \in S} s_n(t)$, we define the load to be $\rho = \mathbb{E}[a(0)]/\mathbb{E}[s(0)]$. For the system to be admissible, it must hold that $\rho < 1$.

III. Stochastic Coordination

As a first step towards designing an effective multi-dispatcher policy, let us consider the problem in the simpler, single dispatcher, case. In round $t$, this dispatcher has access to the vector $Q(t)$ of queue sizes at the servers, and to the number $a(t)$ of jobs that have been submitted to the system. If the dispatcher is free to send different jobs to different servers then it will, intuitively, dispatch jobs one by one according to the JSQ principle. It will send the first job to a queue of shortest length, and then iterate through the jobs, each time sending the next job to a queue of smallest size given the jobs that it has already assigned. Consequently, all queues to which jobs are sent end up with the same sizes (give or take 1). We can view this as being analogous to a process of filling water into a container: Consider a water container with an uneven bottom at heights that correspond to the histogram (rearranged in sorted order) defined by $Q(t)$. If we should pour a volume of $a(t)$ units of water into the container, then the water level will coincide with the height of the queues to $2$ In practice, such information could be gathered in different ways. E.g., by a shared bulletin board or shared memory, as well as by having servers update a central process, who can forward a single message to with $Q(t)$ to each dispatcher.

In the appendix, for the purpose of mathematical stability analysis, we also make the standard assumption that the arrival and departure processes admit a finite variance, i.e., $\text{Var}(a(0)) < \infty$ and $\text{Var}(s(0)) < \infty$.  

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which jobs were dispatched, up to a rounding error due to the
fact that water is continuous and jobs are discrete. The largest
amount of water would be poured into the deepest column,
just as the largest number of jobs would be sent to the shortest
queue.

On a given input \((Q, a)\), Algorithm 1 computes the water
level \(WL = \text{WATERLEVEL}(Q, a)\) that results from pouring \(a\)
units of water into a container with bottom shaped according to
\(Q\). The height of each column once the water is poured
would be \(Q^* = (Q_1^*, \ldots, Q_n^*)\) where \(Q_n^* = \max\{Q_n, WL\}\).
We remark that \(Q_n^*\) is not always an integer but might also be
a rational number. If queue lengths are maintained in sorted
order, then WL is efficiently computable in \(O(\min(N, a))\)
time complexity. Measured running times, showing that our
policy’s runtime scales similarly to JSQ, can be found in the
supplementary material.

**Algorithm 1 Computing the Water Level**

1: function WATERLEVEL\((Q, a)\) \(\triangleright Q\) is the multiset of
queue lengths; \(a\) is the total number of arrivals;
2: \(\triangleright M\) returns the minimal value in a multiset.
3: \(\triangleright\)
4: \(\text{MinSet} \leftarrow \{Q_n \in Q \mid Q_n = \text{Min}(Q)\}\) \(\triangleright\) The set
of all minimal queues.
5: \(\triangleright |\cdot|\) denotes the
6: \(\text{return } \text{Min}(Q) + a/|Q|\)
7: \(\delta \leftarrow \text{NextMin} \leftarrow \text{Min}(Q, \text{MinSet})\)
8: 
9: \(\triangleright\)
10: \(\text{return } \text{Min}(Q) + a/|\text{MinSet}|\)

In a multi-dispatcher system dispatchers make decisions
independently of each other. It is thus only natural for them
to independently optimize their load balancing decisions.
Namely, to dispatch jobs in exactly the same manner as
if they were alone in the system. This is indeed the case in
current solutions. However, in a system where one is not
alone, such oblivious behavior of disregarding the others
may result in sub-optimal performance [7], [18], [32], [37],
[43]. This occurs when the same server is identified as the
best destination by different dispatchers. These dispatchers
then simultaneously forward jobs to this server, causing its
queue to grow rapidly, increasing delay times and sometimes
even causing the server to drop jobs. However, the fact
that dispatchers make decisions independently does not mean
that their decision making protocols must be independently
optimized.

In the multi-dispatcher context that we are considering,
dispatchers cannot directly coordinate their actions in every
given round, since they do not directly communicate with each
other. Nevertheless, it is possible to design their protocols in
such a way that their actions will be compatible with each
other, and will not conflict. The key to doing so is employing
randomized protocols, in which the dispatchers’ moves are
stochastic. Indeed, randomization has been a standard tool
for symmetry breaking in distributed computing for over four
decades [14], [29]. Our goal will be to design probabilistic
load balancing protocols that will provide good performance
by optimizing the cumulative behavior of the dispatchers. This
will provide the dispatchers with a silent form of stochastic
cooperation.

**IV. Tidal Water Filling**

Focusing on the complete information setting, we assume
that each dispatcher \(m\) has access to the number
\(a^{(m)}(t)\) of jobs that has received in the current
round, and to the vector of server queue sizes \(Q = Q(t)\)
(we shall omit the round number \(t\) when it is clear from
context). Based on \(Q\) and \(a^{(m)}\), it needs to decide where
to send each job. We seek a solution that will be feasible to
compute and amenable to analysis. In particular, we seek a
policy that (1) is uniform for all dispatchers; given the same \(Q\)
and \(a^{(m)} = a^{(m')}\), both \(m\) and \(m'\) should act in the same way,
and in which (2) a dispatcher treats all jobs uniformly. Hence,
the output of \(m\)’s computation is a vector \((p_1, \ldots, p_N)\), where
\(p_n\) is the probability that any given job will be sent by \(m\) to
server \(n\), for every \(n \in S\).

We denote by \(\tilde{g}^n\) the random variable specifying the
number of jobs sent to server \(n\) by dispatcher \(m\). The total
number of jobs received by \(n\) is \(\tilde{g}_n = \sum_{m \in D} \tilde{g}^n\), and
its queue size once it receives them is the random variable
\(\bar{Q}_n = Q_n + \tilde{g}_n\). Finally, we shall denote \(Q = (Q_1, \ldots, Q_N)\).

Recall that \(Q\) and the total number of jobs \(a = a(t)\)
determine a water-filling solution \(Q^* = Q^*(Q, a)\) as described
in Section III. Our goal will be to design a policy that
computes the dispatching probabilities \(P\) in such a way
that the resulting queue sizes \(Q\) approximate \(Q^*\) as well as
possible. More formally, we wish to minimize the \(L_2\) distance
between \(Q\) and \(Q^*\). Intuitively, a large distance from the
water level \(WL = \text{WATERLEVEL}(Q, a)\) induces a large delay in
response times (for a positive difference) or server starvation
and lesser resource utilization (negative difference). Since we
seek to avoid long delay tails as well as unnecessary server
idleness, we consider a large deviation from the WL to be
worse than several small ones. This rules out linear or sub-
linear distance measures such as the \(L_1\) distance. On the other
hand, giving too much weight to large deviations may miss
opportunities to optimize the mean. For example, the \(L_\infty\)
distance (i.e., min-max) addresses only the largest deviation.
We therefore choose to use the \(L_2\) distance, since it balances
these two desires and is amenable to formal analysis.

We denote the vector of job arrivals at the dispatchers by
\(\bar{a} = (a^{(1)}, \ldots, a^{(M)})\). As an interim step, we derive a policy
that computes the dispatching probabilities based on \(Q\) and
the full vector \(\bar{a}\) of jobs that arrive in the round, and not only
the allocation \(a^{(m)}\) of a single dispatcher \(m\). We will later discuss
how this analysis can be applied to an individual dispatcher’s
computation. Notice that \(Q\) and \(\bar{a}\) uniquely determine a (fixed)
vector \(Q^*\) resulting from water filling \(Q\) with \(a = \sum_{m \in D} a^{(m)}\).
new jobs. Now, a policy \( P(Q, \bar{d}) \) gives rise to the random variable vector \( \bar{Q} \) of queue sizes, as described above.

Recall that \( \bar{g}_n = Q_n - Q_n \). Similarly, we denote \( g_n^* \triangleq Q_n - Q_n \). Our goal is to minimize

\[
E[I|Q^* - \bar{Q}|^2] = E[I(Q_1^* - \bar{Q}_1, \ldots, Q_N^* - \bar{Q}_N)^T|_2
\]  
\[
E[I(Q_1 + g_1^* - Q_1 - \bar{g}_1, \ldots, Q_N + g_N^* - Q_N - \bar{g}_N)^T|_2
\]  
\[
E[I(g_1^* - \bar{g}_1, \ldots, g_n^* - \bar{g}_N)^T|_2
\]  
\[
= \sum_{n \in S} E[(g_n^* - \bar{g}_n)^2]
\]

We perform separate analyses for the splittable and for the nonsplittable cases.

### A. The Splittable Case

In the splittable case, every job \( \bar{Q}_n \) is sent to a server \( n \) with a probability of \( p_n \). This implies, in particular, that the random variable \( g_n^*(m) \) admits a binomial distribution, that is, \( g_n^*(m) \sim \text{Bin}(a(m), p_n) \). Since each decision at each dispatcher is done independently, \( \{g_n^*(m) | m \in D\} \) are independent binomial variables with probability \( p_n \). Thus, \( \bar{g}_n = \sum_{m \in D} g_n^*(m) \), where \( \bar{g}_n \sim \text{Bin}(\sum_{m \in D} a(m), p_n) \sim \text{Bin}(a, p_n) \). Hence,

\[
E[\bar{g}_n] = ap_n \quad \text{and} \quad E[\bar{g}_n^2] = ap_n(1 - p_n) + a^2p_n^2.
\]

Given \( Q \) and \( a \) we can rewrite (1) using (2) as a function of \( P = \{p_1, \ldots, p_N\} \):

\[
f(P) = E[I|Q^* - \bar{Q}|^2] = \sum_{n \in S} g_n^2 - 2a \sum_{n \in S} g_n^*p_n + \sum_{n \in S} (ap_n - ap_n^2 + a^2p_n^2)
\]

\[
= \sum_{n \in S} g_n^2 - 2a \sum_{n \in S} g_n^*p_n + a - a \sum_{n \in S} p_n^2 + a^2 \sum_{n \in S} p_n^2
\]

Now, to simplify the analysis, we first make the observation that for any strictly positive number of arrivals to the system (i.e., \( a > 0 \)),

\[
\arg \min f(P) = \arg \min (a - 1) \sum_{n \in S} p_n^2 - 2 \sum_{n \in S} g_n^*p_n.
\]

We thus turn to solve the expression on the right-hand side. As can be seen from (4), for a single arrival to the system (i.e., \( a = 1 \)), the solution would be to divide the probabilities arbitrarily among all shortest queues. Thus, we next assume that \( a > 1 \). Recall that we aim to compute a probability assignment \( P \) that optimizes \( \arg \min f(P) \). In particular, we have that \( \sum_{n \in S} P_n = 1 \) and \( p_n \geq 0 \) \( \forall n \in S \). The optimization problem to solve in standard form is,

\[
\min_p \; \hat{f}(P) = (a - 1) \sum_{n \in S} p_n^2 - 2 \sum_{n \in S} g_n^*p_n
\]

\[
\text{s.t.} \; \sum_{n \in S} p_n = 1, \quad -p_n \leq 0 \; \forall n \in S.
\]

Notice that this is not a linear program, because the objective function is not linear. However, the problem is convex with affine constraints. To solve the problem, we employ the Karush-Kuhn-Tucker method (KKT) [10], [12]. The associated Lagrangian function is

\[
L(P, \Lambda) = (a - 1) \sum_{n \in S} p_n^2 - 2 \sum_{n \in S} g_n^*p_n
\]

\[
- \sum_{n \in S} \Lambda_n p_n + \Lambda_0 (\sum_{n \in S} p_n - 1).
\]

The respective KKT conditions are

**Stationarity:**

\[
\frac{\partial L}{\partial p_n} = 2(a - 1)p_n - 2g_n^* - \Lambda_n = 0 \quad \forall n \in S
\]

**Primal feasibility:**

\[
\sum_{n \in S} p_n - 1 = 0 \quad \text{and} \quad p_n \geq 0 \quad \forall n \in S
\]

**Dual feasibility:**

\[
\Lambda_n \geq 0 \quad \forall n \in S
\]

**Complementary slackness:**

\[
p_n \Lambda_n = 0 \quad \forall n \in S
\]

By Stationarity in (7) we obtain that, for any \( p_n \),

\[
p_n = \frac{2g_n^* - \Lambda_0 + \Lambda_n}{2(a - 1)},
\]

and adding Complementary slackness from (7) yields that for any \( p_n > 0 \) we have,

\[
p_n = \frac{2g_n^* - \Lambda_0}{2(a - 1)}.
\]

We can now substitute for \( p_n \) according to (9) in our objective function (5) to obtain a function of a single variable \( \Lambda_0 \). This yields,

\[
\hat{f}(P(\Lambda_0)) = (a - 1) \sum_{n \geq 0} \left( \frac{2g_n^* - \Lambda_0}{2(a - 1)} \right)^2
\]

\[
- 2 \sum_{n \geq 0} g_n^* \left( \frac{2g_n^* - \Lambda_0}{2(a - 1)} \right)
\]

\[
= \sum_{n \geq 0} \left( \lambda_n^2 - (2g_n^*)^2 \right),
\]

\[
= \frac{a - 1}{a - 1}. \tag{10}
\]

Notice that (9) implies that for every \( p_n > 0 \) it holds that \( \Lambda_0^2 - (2g_n^*) < 0 \). Hence, every term in the summation of (10) is negative. As a result, lower value of \( \Lambda_0 \) lead to both lower values of each term and, perhaps, more negative terms. Clearly, to minimize the objective function, we seek the smallest \( \Lambda_0 \) that satisfies the KKT conditions given in (7). Observe that we can lower bound \( \Lambda_0 \) by combining (8) with the Primal feasibility in (7) to obtain:

\[
1 = \sum_{n \in S} p_n = \sum_{n \in S} \frac{2g_n^* - \Lambda_0 + \Lambda_n}{2(a - 1)}
\]

\[
\geq \sum_{g_n^* > 0} \frac{2g_n^* - \Lambda_0 + \Lambda_n}{2(a - 1)}.
\]

**End Note**
Using \( \sum_{n \in S} g_n^* = \sum_{g_n^* > 0} g_n^* = a \), and rearranging (11) yields
\[
2a - \Lambda_0 \sum_{g_n^* > 0} 1 + \sum_{g_n^* > 0} \Lambda_n \leq 2(a - 1).
\]
Thus, due to the Dual feasibility in (7), we obtain
\[
\Lambda_0 \geq \frac{2 + \sum_{g_n^* > 0} \Lambda_n}{g_n^*} \geq \frac{2}{g_n^*}, \quad \text{where } g_n^* \neq \sum_{g_n^* > 0} 1.
\]
Setting \( \Lambda_0 = \frac{2}{g_n^*} \) and \( \Lambda_n = 0 \) for all \( g_n^* > 0 \) respects the KKT conditions and minimizes the objective function with respect to \( \Lambda_0 \). Finally, substituting \( g_n^* \) for \( \Lambda_0 \) in Equation (9), we obtain that the optimal solution for \( a > 1 \) in the splittable case is
\[
p_n = \max \{0, \frac{g_n^* - 1/g_n^*}{a - 1}\}.
\]

**Definition 1 (Splittable Tidal Water Filling):** Given \( Q = Q(t) \) and \( a = a(t) > 1 \), a stochastic dispatching policy \( P(Q(t), a(t)) \) that, in every round \( t \) sends each job to server \( n \in S \) with probability \( p_n = \max \{0, \frac{a(t) - 1}{a(t) - g_n^*}\} \), implements tidal water filling (sTWF) in the splittable setting.

Notice that sTWF depends only on \( a = a(t) \) and \( Q = Q(t) \). It does not depend on the full detail of \( a \). In the context of complete information, \( Q \) is available to the dispatcher. However, \( a \) is not. In order to use sTWF an individual dispatcher \( m \) must replace \( a \) with some estimate. If \( \mathbb{E}(a(0)) \), the expected value of \( a \), is known, it can be used. Similarly, if \( \mathbb{E}(s(0)) \), the total expected completion time of the servers is known, it may also be used to replace \( a \). Since, by assumption, dispatcher \( m \) has access to \( a^{(m)}(t) \), it can use \( Ma^{(m)}(t) \) for \( a(t) \). This has the nice property that the average of what the dispatchers use equals exactly the total arrivals at that round. That is, \( \frac{1}{M} \sum_{m \in D} Ma^{(m)}(t) = a(t) \). We will hereafter assume that dispatcher \( m \) estimates \( a(t) \) in this manner. In Appendix B we show that the resulting protocol satisfies the desirable strong stability property for discrete-time queuing systems.\(^4\) In Section IV-C we shall discuss the properties and the intuitive interpretation of the probabilities used in sTWF.

**B. The Unsplittable Case**

In the unsplittable case, we again assume that every dispatcher \( m \) knows the vector \( Q(t) \) of queue sizes (complete information). It differs from the splittable case only in that \( m \) must send all of the jobs that it receives in a given round to a single server. This affects the mathematics of the optimization problem. First of all, knowing the complete vector \( a \) of arrivals makes a significant difference in this case. Indeed, given \( Q \) and \( a \), computing an optimal job assignment to the servers essentially requires solving an instance of BIN-PACKING. In Appendix A we prove its NP-hardness by a reduction from the PARTITION problem [9]. More precisely, we show the following.

**Theorem 1:** Given \( Q \) and \( \hat{a} \), it is NP-hard to decide if \( \min \mathbb{E}[Q^* - Q]^2 = 0 \) for a system with two servers.

\(^4\)In fact, Appendix B proves strong stability for all the policies we introduce in this paper including the policies that operate based on partial information (see Section V).

In general, the values \( a^{(1)}, a^{(2)}, \ldots, a^{(M)} \) may be different from each other. Theorem 1 implies that optimizing for general \( Q \) and \( \hat{a} \) is intractable. Instead, we optimize the unsplittable problem for the case that \( a^{(1)} = a^{(2)} = \cdots = a^{(M)} \), which is consistent with the assumption that \( a = Ma^{(m)} \), made in the splittable setting. We consider this case as a heuristic means to derive the dispatching probabilities. Our experiments in Section IV-D show that the resulting policy works well, even when arrivals are governed by i.i.d. Poisson distributions, under which the arrival values \( a^{(m)} \) are rarely identical.

We now reformulate the optimization problem in the unsplittable setting for this heuristic case. The difference from the splittable setting arises following (1) since now \( g_n^{(m)} \) does not admit a Binomial distribution. Instead, we have
\[
\bar{g}_n^{(m)} = \begin{cases} a^{(m)}, & \text{w.p. } p_n, \\ 0, & \text{otherwise}. \end{cases}
\]

Namely, a dispatcher \( m \) sends all \( a^{(m)} \) of its jobs to server \( n \) with probability \( p_n \). Therefore, the total number of jobs that server \( n \) receives is \( g_n = \sum_{m \in D} g_n^{(m)} \). Since \( \{g_n^{(m)} | m \in D\} \) are i.i.d. random variables, \( g_n \) has the following first and second moments,
\[
\mathbb{E}[g_n] = Ma^{(m)}p_n,
\]
\[
\mathbb{E}[g_n^2] = \sum_{m \in D} (a^{(m)})^2 p_n + \sum_{m, m' \in D, m \neq m'} a^{(m)} a^{(m')} p_n^2
\]
\[
= Ma^{(m)}^2 p_n + M(M - 1)(a^{(m)})^2 p_n^2.
\]

Given \( Q \) and \( a^{(m)} \) we rewrite (1) substituting the first and second moments according to (15). This yields
\[
f(P) = \sum_{n \in S} g_n^{*2} - 2Ma^{(m)} \sum_{n \in S} g_n^* p_n
\]
\[
+ \sum_{n \in S} \left(M(a^{(m)})^2 p_n + M(M - 1)(a^{(m)})^2 p_n^2\right)
\]
\[
= \sum_{n \in S} g_n^{*2} - 2Ma^{(m)} \sum_{n \in S} g_n^* p_n
\]
\[
+ M(a^{(m)})^2 \sum_{n \in S} p_n + (M^2 - M)(a^{(m)})^2 \sum_{n \in S} p_n^2.
\]

Recall that we aim to minimize \( f(P) \) under the constraint that \( P \) is a probability assignment. That is, \( \sum_{n \in S} p_n = 1 \) and \( p_n \geq 0 \) \( \forall n \in S \). Observe that,
\[
\arg \min f(P) = \arg \min (M - 1)a^{(m)} \sum_{n \in S} p_n^2 - 2 \sum_{n \in S} g_n^* p_n.
\]

We proceed to solve for the right hand side. As can be seen from (17), for a single-dispatcher system (i.e., \( M = 1 \), the solution would be to arbitrarily divide the probabilities among all shortest queues, i.e., JSQ. Thus, we next assume that \( M > 1 \). The optimization problem in standard form becomes
\[
\min_p \bar{f}(P) = (M - 1)a^{(m)} \sum_{n \in S} p_n^2 - 2 \sum_{n \in S} g_n^* p_n
\]
\[
s.t. \sum_{n \in S} p_n - 1 = 0, \quad -p_n \leq 0 \forall n \in S.
\]
Once more, we employ the KKT method. However, for the unsplittable case, the solution is more involved. In particular, identifying the subset of servers \( U \subseteq S \) for which positive probabilities should be assigned, poses a challenge.

The optimization problem in (18) is a convex problem with affine constraints. The resulting Lagrangian function is

\[
L(P, \Lambda) = (M-1)a^{(m)} \sum_{n \in S} p_n^2 - 2 \sum_{n \in S} g_n^* p_n - \sum_{n \in S} \Lambda_n p_n + \Lambda_0 \left( \sum_{n \in S} p_n - 1 \right),
\]

and the respective KKT conditions are,

Stationarity:

\[
\frac{\partial L}{\partial p_n} = 2(M-1)a^{(m)} p_n - 2g_n^* - \Lambda_n + \Lambda_0 = 0 \quad \forall n \in S
\]

Primal feasibility:

\[
\sum_{n \in S} p_n - 1 = 0 \quad \text{and} \quad p_n \geq 0 \quad \forall n \in S
\]

Dual feasibility:

\[
\Lambda_n \geq 0 \quad \forall n \in S
\]

Complementary slackness:

\[
p_n \Lambda_n = 0 \quad \forall n \in S
\]

Using the Stationarity from (20) we get that for any \( p_n \),

\[
p_n = \frac{2g_n^* - \Lambda_0 + \Lambda_n}{2(M-1)a^{(m)}}
\]

and using Complementary slackness (20) yields that for any \( p_n > 0 \) we have

\[
p_n = \frac{2g_n^* - \Lambda_0}{2(M-1)a^{(m)}}
\]

We can use (22) in our objective function in (18) to obtain a function of a single variable \( \Lambda_0 \). This yields,

\[
\tilde{f}(P(\Lambda_0)) = (M-1)a^{(m)} \sum_{p_n > 0} \left( \frac{2g_n^* - \Lambda_0}{2(M-1)a^{(m)}} \right)^2 - 2 \sum_{p_n > 0} g_n^* \left( \frac{2g_n^* - \Lambda_0}{2(M-1)a^{(m)}} \right).
\]

Rearranging (23) we get,

\[
\tilde{f}(P(\Lambda_0)) = \frac{1}{(M-1)a^{(m)}} \sum_{p_n > 0} \left( \frac{\Lambda_0^2}{4} - g_n^{*2} \right).
\]

Similarly to the splittable case, in order to minimize the objective function, we need to find the smallest \( \Lambda_0 \) that respects the KKT conditions. For this purpose we define a set of servers \( U \subseteq S \) that would be strictly below the water level if the total arrivals were \((M-1)a^{(m)}\) jobs instead of \(Ma^{(m)}\). That is, \( U = \{n \mid Q_n < \text{WATERLEVEL}(Q, (M-1)a^{(m)})\} \), where \( \text{WATERLEVEL} \) is given by Algorithm 1. We now use (21) and the Primal feasibility (20) to obtain,

\[
1 = \sum_{n \in S} p_n = \sum_{n \in S} \frac{2g_n^* - \Lambda_0 + \Lambda_n}{2(M-1)a^{(m)}} \geq \sum_{n \in U} \frac{2g_n^* - \Lambda_0 + \Lambda_n}{2(M-1)a^{(m)}}.
\]

Rearranging (25) and using the Dual feasibility from (20),

\[
2(M-1)a^{(m)} \geq 2 \sum_{n \in U} g_n^* - \Lambda_0 \sum_{n \in U} 1 + \sum_{n \in U} \Lambda_n \\
\geq 2 \sum_{n \in U} g_n^* - \Lambda_0 |U|.
\]

Thus, since \( Ma^{(m)} = \sum_{n \in S} g_n^* \) and \( \sum_{n \in S} g_n^* - \sum_{n \notin U} g_n^* = \sum_{n \notin U} |U| \), we get

\[
\Lambda_0 \geq \frac{2}{|U|} \left( a^{(m)} - \sum_{n \notin U} g_n^* \right).
\]

Next, we prove that setting \( \Lambda_0 \) on the above lower bound, i.e., \( \Lambda_0 = \frac{2}{|U|} (a^{(m)} - \sum_{n \notin U} g_n^*) \), respects the KKT conditions. We start with the Dual feasibility condition, \( \Lambda_0 \geq 0 \). Recall that \( M \geq 2 \), and by definition of \( U \) it holds that \((M-1)a^{(m)} \leq Ma^{(m)}\), thus,

\[
Ma^{(m)} = \sum_{n \in S} g_n^* = \sum_{n \in U} g_n^* + \sum_{n \notin U} g_n^* \quad \text{and} \quad 0 \leq g_n^* \leq a^{(m)}.
\]

Therefore,

\[
\left( a^{(m)} - \sum_{n \notin U} g_n^* \right) \geq 0 \quad \text{and} \quad \Lambda_0 = \frac{2}{|U|} \left( a^{(m)} - \sum_{n \notin U} g_n^* \right) \geq 0.
\]

This shows Dual feasibility.

Plugging \( \Lambda_0 = \frac{2}{|U|} (a^{(m)} - \sum_{n \notin U} g_n^*) \) into (22) yields

\[
p_n = \max \left\{ 0, \frac{g_n^* - (a^{(m)} - \sum_{n \notin U} g_n^*)/|U|}{(M-1)a^{(m)}} \right\}.
\]

It only remains to show that these \( p_n \)'s also satisfy the Primal feasibility condition from (20). We first show that for each \( n \in U \) it holds that \( p_n > 0 \), whereas for \( n \notin U \) it holds that \( p_n = 0 \).

Lemma 1: For each \( n \in S \), it holds that \( p_n > 0 \) if and only if \( n \in U \).

Proof: Let \( b_n^* = \max\{0, \text{WATERLEVEL}(Q, (M-1)a^{(m)}) - Q_n\} \). By definition of \( U \), it holds that \( b_n^* > 0 \) if and only if \( n \in U \). Note that \( Q_n + b_n^* \) is exactly \( \text{WATERLEVEL}(Q, (M-1)a^{(m)}) \) for \( n \in U \) and is simply \( Q_n \) for \( n \notin U \). Thus, for each \( n \in U \) we obtain \( g_n^* = b_n^* + x \) where

\[
x = \text{WATERLEVEL}(Q, Ma^{(m)}) - \text{WATERLEVEL}(Q, (M-1)a^{(m)})
\]

and thus \( p_n > 0 \) if \( n \in U \).

On the other hand, if \( n \notin U \) we obtain \( g_n^* = b_n^* + x \) where \( b_n^* = 0 \) and \( x \leq x \) since,

\[
x = \max\{0, \text{WATERLEVEL}(Q, Ma^{(m)}) - Q_n\},
\]

and,

\[
Q_n \geq \text{WATERLEVEL}(Q, (M-1)a^{(m)}).
\]
This yields
\[ g_n^* = x_n \leq x = \frac{(a^{(m)}) - \sum_{n' \notin U} g_{n'}^*)}{|U|}, \]
and thus \( p_n = 0 \) if \( n \notin U \). This concludes the proof. According to Lemma 1 we have that
\[
\sum_{n \in S} p_n = \sum_{n \in U} p_n = \sum_{n \in U} \left( \frac{g_n^* - \left( a^{(m)} - \sum_{n' \notin U} g_{n'}^*)}{(M-1)a^{(m)}} \right)
= \frac{1}{(M-1)a^{(m)}} \left( \sum_{n \in U} g_n^* - \sum_{n \in U} \left( a^{(m)} - \sum_{n' \notin U} g_{n'}^* \right) / |U| \right)
= \frac{1}{(M-1)a^{(m)}} \left( \sum_{n \in U} g_n^* / |U| \right)
= \frac{\sum_{n \in S} g_n^* - a^{(m)}}{(M-1)a^{(m)}}
= \frac{\left( M(\sum_{n} g_{n}) - a^{(m)} \right)}{(M-1)a^{(m)}} = 1,
\]
and all of the KKT conditions are satisfied.

The solution for the optimization problem in (18) yields the following notion of tidal water filling for the unsplittable case:

**Definition 2 (Unsplittable tidal water filling):** Given \( Q = Q(t) \) and \( a^{(m)} = a^{(m)}(t) \), let \( U = \{ n \mid Q_n < \text{WATERLEVEL}(Q, (M-1)a^{(m)}) \} \). An individual stochastic dispatching policy \( P(Q, a^{(m)}) \) for dispatcher \( m \) that, in every round \( t \) sends all its jobs to server \( n \in S \) with probability
\[
p_n = \max \left\{ 0, \frac{g_n^* - (a^{(m)} - \sum_{n' \notin U} g_{n'}^*)/|U|}{(M-1)a^{(m)}} \right\}
\]
implies tidal water filling (uTWF) in the unsplittable setting.

Observe that, when at most one job arrives at each dispatcher, *i.e.*, \( a^{(m)} \leq 1 \) for all \( m \in D \), there is no difference between the splittable and unsplittable problems. Indeed, in this case Definition 2 and Definition 1 coincide.

**C. TWF vs Water Filling in Expectation**

Recall that we aim to approximate water filling (*i.e.*, \( Q^* \)). Consider the policy by which every dispatcher sends a job to each server \( n \) with a probability proportional to the amount of “water” it would receive in the pure water-filling solution. More formally, we define the Water Filling in Expectation policy (WFIE) to assign probabilities \( P(Q, a) = (p_1, \ldots, p_N) \), where for every \( n \) we have
\[
p_n = \frac{g_n^*}{a}.
\]
The expected length of each server \( n \)'s queue under WFIE is precisely \( Q_{n}^* \). Tidal water filling, however, does not produce the pure water-filling solution in expectation. Nevertheless, the math does not lie, and TWF improves on WFIE. We use the following two examples to demonstrate that WFIE is suboptimal, and to provide intuition for why TWF is better.

**Example 1:** Figure 1 depicts a system with \( N = 2 \) servers. At the beginning of the round, server \( n_1 \) has a single job in its queue, and server \( n_2 \) is idle (its queue is empty). There are \( M = 2 \) dispatchers, each of which receives a single job to dispatch. In this scenario, a dispatcher that uses JSQ will send its job to \( n_2 \); a dispatcher that uses WFIE will send its job to \( n_1 \) with probability \( p_{n_1} = 1/4 \) and to \( n_2 \) with probability \( p_{n_2} = 3/4 \); a dispatcher that uses TWF (since each dispatcher has a single job to dispatch in this scenario, sTWF and uTWF coincide) will send its job to \( n_2 \) with probability \( p_{n_2} = 1 \). The resulting expected lengths of the queues are depicted in Figure 1a for JSQ, in Figure 1b for WFIE, and in Figure 1c for TWF. Observe that both JSQ and TWF guarantee the favorable solution in which the longest queue has size 2, while in WFIE there is a non-negligible probability of 1/16 that both jobs will be forwarded to \( n_1 \), creating a queue of size 3.

**Example 2:** Figure 2 illustrates a system with the same two servers and the same initial state, but with \( M = 3 \) dispatchers. Each of the three dispatchers receives a single job to dispatch. In this scenario, a dispatcher that uses JSQ will again send its job to \( n_2 \); a dispatcher that uses WFIE will send its job to \( n_1 \) with probability \( p_{n_1} = 1/3 \) and to \( n_2 \) with probability \( p_{n_2} = 2/3 \); a dispatcher that uses TWF will send its job to \( n_1 \) with probability \( p_{n_1} = 1/4 \) and to \( n_2 \) with probability \( p_{n_2} = 3/4 \). The resulting expected lengths of the queues are depicted in Figure 1a for JSQ, in Figure 1b for WFIE, and in Figure 1c for TWF. In this case, JSQ results in herding towards \( n_2 \), creating a queue of size 3 there. WFIE results is a probability of 1/27 for ending with 4 jobs queuing at \( n_1 \), while \( n_2 \) remains idle. Observe that WFIE reduces the probability of this worst-case allocation from 1/27 to 1/64. This is precisely the advantage that TWF provides over WFIE in general. We note that TWF also provides, with high probability, a favorable allocation in comparison to JSQ. In the second example, for instance, JSQ will produce a better outcome than TWF with probability \( 1.56\% = 1/64 \), while TWF will be better than JSQ with probability \( 42.2\% = 27/64 \).

Roughly speaking, our TWF policies have a greater bias towards short queues than WFIE does. As illustrated by the
examples this, in turn, reduces the probability that queues will grow excessively long, and reduces the probability for short queues to become idle.

D. Evaluation

We conducted an empirical study of our TWF policies via simulations. In all of the simulations, at round \( t \) a dispatcher \( m \) has access only to \( a(m)(t) \). Recall that in both sTWF and uTWF dispatcher \( m \) uses \( M \cdot a(m)(t) \) as an estimate for \( a(t) \).

Arrivals and departures. Our modeling of the arrival and departure processes follows standard practice (e.g., [2], [16], [22], [36], [37]). In each round, we set \( a(m)(t) \sim \text{Poisson}(\lambda) \) for each dispatcher \( m \in D \), and \( s(n)(t) \sim \text{Geometric}(\mu) \) for each server \( n \in S \). Therefore, the load on the system is \( \rho \triangleq M\lambda/(N\mu - \mu) \).

Dispatching policies. We compare our policies to JSQ, the Power-of-two-choices denoted by JSQ(2), JIQ and the recently proposed LSQ.\(^6\) For the splittable case, we also compare against: (1) “the power of slightly more than a single choice” (PoSMTO(\( d \)) proposed in [42]);\(^7\) (2) a centralized JSQ policy (i.e., all arrivals go through a single dispatcher, which results in a centralized water-filling (CWF) effect). The comparison against this policy can be seen as the “price of distribution”. We use a prefix of s and u to denote the splittable and the unsplittable case. For the splittable case, similarly to sTWF, other policies are also allowed to split jobs for parallel processing. Namely, in splittable JSQ (sJSQ), each dispatcher performs water-filling considering only its own jobs (i.e., disregarding possible arrivals of jobs to the same servers from other dispatchers). Similarly, this is the case for other policies.

\(^6\)Specifically, we use LSQ-Sample(2). See [37] for details.

\(^7\)We calibrated the parameter \( d \) using the guidelines in [42] and found \( d = 1.6 \) as the sweet spot. Namely, the number of samples a dispatcher performs each round is not constant but equals to the size of the arrived batch of jobs multiplied by 1.6.

For example, splittable JIQ (sJIQ) splits the jobs among the idle queues equally (with random tie breaks). If there are no idle queues, each job is sent to a randomly selected server.

Setup. We consider systems with different numbers of servers \( N \) and dispatchers \( M \). For each system, we run a set of simulations. Each simulation is configured with a different load and lasts for \( 10^5 \) time slots (rounds).

Results. Figure 3 shows the mean job response time (y-axis) across the different loads (x-axis) for different systems. It is notable that sTWF outperforms all other policies in the splittable case across all systems, and uTWF does the same in the unsplittable case. Moreover, as the load increases the gap between the TWF policies and the rest grows.

As mentioned in Section I and in [3], [16], [27], and [30], tail distributions play a crucial role in the parallel-server setting. We proceed to measure the tail distributions under various system parameters under the challenging scenario of a high load of \( (\rho = 0.99) \). This is depicted in Figure 4 using the complementary cumulative distribution function (CCDF), which shows for each response time (x-axis, denoted by \( \tau \)), what is the fraction of jobs that surpass it (y-axis).

In summary, in the complete information case, our simulations show that the TWF policies consistently outperform the previous approaches when the load is high. Further, TWF has a considerably lower “price of distribution”, in the splittable case, compared to other policies as evident by the comparison to CWF.

Heavy tail arrivals. The particular distributions that are chosen to model the arrivals processes clearly impact the results. Although the paper focuses only on Poisson arrivals, we have also conducted simulations with a heavy tail distribution (Log-normal). For these simulations, we observer that the gap between TWF and JSQ somewhat decreases, however, is is still significant in favor of TWF.
V. ENHANCING PERFORMANCE FOR PARTIAL INFORMATION

In this section we relax the requirement that dispatchers have complete and accurate information regarding $Q$. That is, we now consider situations in which a dispatcher does not know the exact state of all servers. In line with recent work [2], [36], [37], [43], we consider a system where each dispatcher keeps a local array representing the servers’ queue lengths, which may contain inaccurate (e.g., outdated) information. Communication is used to update array entries in the following manner: At the end of each round (i.e., in the fourth phase), a dispatcher establishes communication links with a fraction $\eta \leq 1$ of the servers, which are chosen uniformly at random. The corresponding entries in the dispatcher’s local state are then updated with these servers’ queue lengths. Additionally, a dispatcher that establishes a communication link to send jobs to server $n$ during round $t$ learns $Q_n(t)$. Notice that $\eta = 1$ corresponds to the complete information assumption; we use $\eta$ as a parameter designating the extent of partial information available to the dispatchers. A dispatcher that uses uTWF based on its local array is said to implement Local uTWF (L-uTWF). Local sTWF is defined in the same manner. Figure 5 illustrates the simulations results. It shows that the response time improves monotonically as $\eta$ increases. This motivates us to increase the available information to the dispatchers. We attempt to do so without increasing the number of links that a dispatcher establishes. This is of interest since in many system the cost of communication lies mainly in the connection establishment rather than in the size of its content [23], [24]. To that end, we keep track of queue-size information at the servers, in a local array of size $N$. As server updates its local array based on its own queue length and information that it receives from dispatchers with which it has connections. Whenever a communication link between a dispatcher and a server is established, they merge their arrays.

VI. DISCUSSION

This work has demonstrated that, contrary to popular belief, queue-size information can be judiciously used to improve the quality of load balancing. In particular, we provided

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8To the best of our knowledge, maintaining queue size information in this manner at the servers has not been done before in the parallel server model.
new policies that avoid herding and outperform all previous solutions for the case of complete information.

We have made a step forward in understanding the power of stochastic coordination in the load balancing arena. There are many additional aspects of stochastic coordination that should be explored. Could the same conceptual design be used more generally, i.e., by loosening the homogeneity or the connectivity assumptions? In the partial information setting we note that when queue size information is sparse, the TWF’s advantages do not come into play, and its performance is not better and may be even worse than that of previous load-balancing policies.

Moreover, for many systems (e.g., wireless, sensors, peer-to-peer, etc.), it may be the case that obtaining complete or even partial information is too prohibitive, and one has to rely on sparse communication. For such a case, policies such as JSQ(d) and LSQ may offer better performance since, unlike TWF, they can operate in such settings (e.g., two samples per round).

One can view load balancing in the multi-dispatcher parallel server model as a natural question to explore using distributed systems tools and techniques. Our analysis in Section V, for example, made use of time-stamping and flooding to improve the load balancing performance when information is partial. We state as an open problem how the advantages of previous approaches can be combined with those of TWF to obtain a policy that would make the best use of information across the whole spectrum of possibilities.

In another vein, it would be interesting to investigate how information about the distributions governing a multi-server model as a natural question to explore using distributed systems tools and techniques. Our analysis in Section V, for example, made use of time-stamping and flooding to improve the load balancing performance when information is partial. We state as an open problem how the advantages of previous approaches can be combined with those of TWF to obtain a policy that would make the best use of information across the whole spectrum of possibilities.

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**APPENDIX A**

**NP-Hardness of the Unsplittable Case**

The unsplittable instance of the optimization problem is not computationally tractable. It can be seen as a general variant of the known BIN-PACKING problem. We prove by reduction from the classic PARTITION problem [9] that it entails an NP-hard problem. More precisely, we show the following.

**Theorem 1:** Given Q and δ, it is NP-hard to decide if

$$\min \mathbb{E}[\|Q^* - \bar{Q}\|_2^2] = 0$$

for a system with two servers.

**Proof:** We reduce from PARTITION, which is known to be NP-complete. Recall that in the PARTITION problem we are given a set L of M natural numbers \(1, \ldots, M \in \mathbb{N}\), and we need to decide whether a partition of L into two subsets...
We assume the total expected arrival rate to the system is admissible. That is, we assume that there exists an $\epsilon > 0$ such that $M\lambda(1) = N\mu(1) - \epsilon$.

We prove that our dispatching policies are strongly stable. Specifically, the strong stability proof we conduct applies to all our policies, i.e., TWF, L-TWF and TWF$^p$s. Our proof follows the same lines as in [37] with a few key modifications that capture the somewhat different nature of our dispatching policies in which the dispatching probability to a specific server is dependent on the arrival rate.

Definition 3 (Strong Stability): A load balancing system is said to be strongly stable if for any admissible arrival rate it holds that

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{n \in S} E\left[Q_n(t)\right] < \infty.$$ 

Strong stability is a strong form of stability often employed in discrete-time queueing systems. Moreover, in our model, where we assume the existence of the first two moments of the arrival and departure processes, strong stability also implies throughput optimality as well as other commonly considered forms of stability, such as steady state stability, rate stability, mean rate stability and more (the reader is referred to [8], [25], and [26] for details).

Let $g_n(t) = \sum_{m \in D} g_n^{(m)}(t)$. Then, the queue dynamics of server $n$ is given by

$$Q_n(t + 1) = \max\{0, Q_n(t) + g_n(t) - s_n(t)\}. \quad (33)$$

Squaring both sides of (33), rearranging, and omitting terms yields,

$$\left(Q_n(t+1)\right)^2 - \left(Q_n(t)\right)^2 \leq \left(g_n(t)\right)^2 + \left(s_n(t)\right)^2 - 2Q_n(t)\left(s_n(t) - g_n(t)\right). \quad (34)$$

Summing (34) over the servers yields

$$\sum_{n \in S} \left(Q_n(t+1)\right)^2 - \sum_{n \in S} \left(Q_n(t)\right)^2 \leq \sum_{n \in S} \left(g_n(t)\right)^2 + \sum_{n \in S} \left(s_n(t)\right)^2 - 2\sum_{n \in S} Q_n(t)\left(s_n(t) - g_n(t)\right). \quad (35)$$

Below, we continue the proof for the unsplittable case. The proof for the splittable case differs slightly from this point on, but follows similar lines. In the interest of space we defer it to the supplementary material.

For each $(n, m) \in S \times D$, let $I_n^{(m)}(t)$ be an indicator function that takes the value of 1 with probability $1/N$ and 0 otherwise such that $\sum_{n \in S} I_n^{(m)}(t) = 1 \ \forall m \in D$. We now rewrite (35) by using $g_n(t) = \sum_{m \in D} g_n^{(m)}(t)$ and then adding and subtracting the term 2 $\sum_{n \in S} \sum_{m \in D} I_n^{(m)}(t)a^{(m)}(t)Q_n(t)$ from the right hand side of the equation. This yields

$$\sum_{n \in S} \left(Q_n(t+1)\right)^2 - \sum_{n \in S} \left(Q_n(t)\right)^2 \leq \sum_{n \in S} \left(g_n(t)\right)^2 + \sum_{n \in S} \left(s_n(t)\right)^2 \quad (a)$$

We now analyze term (a) in (36). Taking expectation and using (31) and (32) we obtain

$$\mathbb{E}\left[\sum_{n \in S} \left(g_n(t)\right)^2 + \sum_{n \in S} \left(s_n(t)\right)^2\right] \leq \mathbb{E}\left[\left(\sum_{n \in S} g_n(t)\right)^2\right] + N\mu(2) = \mathbb{E}\left[\left(\sum_{m \in D} a^{(m)}(t)\right)^2\right] + N\mu(2) = M\lambda(2) + M(M - 1)(\lambda(1))^2 + N\mu(2) \triangleq A. \quad (37)$$

Next, for (b) in (36), taking expectation and using (31), (32), the definition of $I_n^{(m)}(t)$ and the admissibility of the system, we obtain

$$\mathbb{E}\sum_{n \in S} Q_n(t)\left(s_n(t) - \sum_{m \in D} I_n^{(m)}(t)a^{(m)}(t)\right) = \sum_{n \in S} \left(\mu(1) - \frac{M\lambda(1)}{N}\right)\mathbb{E}[Q_n(t)] = \frac{\epsilon}{N} \sum_{n \in S} \mathbb{E}[Q_n(t)]. \quad (38)$$

Let $\tilde{Q}_n^{(m)}(t)$ be the local state of server $n$ at dispatcher $m$ at the beginning of round $t$. For complete information, it trivially holds that $\tilde{Q}_n^{(m)}(t) = Q_n^{(m)}(t)$. Otherwise, there exists a constant $C$, such that it holds $\mathbb{E}[Q_n(t) - \tilde{Q}_n^{(m)}(t)] \leq C$ for all $n, m, t$. Indeed, this holds for all the local state (i.e., array) updates for any $\rho > 0, 9$ (full derivation can be found in the proof of Theorem 2, followed by Proposition 1 in [37]).

We next turn to analyze term (c) in (36). We substitute $Q_n(t)$ by $Q_n(t) - \tilde{Q}_n^{(m)}(t) + \tilde{Q}_n^{(m)}(t)$ and rearrange. This yields

$$\sum_{n \in S} Q_n(t)\left(\sum_{m \in D} g_n^{(m)}(t) - \sum_{m \in D} I_n^{(m)}(t)a^{(m)}(t)\right) = \sum_{n \in S} \sum_{m \in D} \tilde{Q}_n^{(m)}(t)(g_n^{(m)}(t) - I_n^{(m)}(t)a^{(m)}(t)) + \sum_{n \in S} \sum_{m \in D} (Q_n(t) - \tilde{Q}_n^{(m)}(t))(g_n^{(m)}(t) - I_n^{(m)}(t)a^{(m)}(t)). \quad (39)$$

$9$Recall that $\rho$ is the fraction of server each dispatcher samples uniformly at random during the communication phase (i.e., phase 4) of each round.
Now, we change the order of summation, use the triangle inequality and take the expectation of (39). This yields,
\[
E \sum_{n \in S} Q_n(t) \left( \sum_{m \in D} g_n^{(m)}(t) - \sum_{m \in D} I_n^{(m)}(t) a_n^{(m)}(t) \right)
\leq \sum_{m \in D} E \sum_{n \in S} \tilde{Q}_n^{(m)}(t) (g_n^{(m)}(t) - I_n^{(m)}(t) a_n^{(m)}(t))
+ \sum_{n \in S \cap m \in D} E \{Q_n(t) - \tilde{Q}_n^{(m)}(t) || g_n^{(m)}(t) - I_n^{(m)}(t) a_n^{(m)}(t)|\}.
\] (40)

We now handle the two summation terms separately. By the linearity of expectation and the law of total expectation we have,
\[
E \sum_{n \in S} \tilde{Q}_n^{(m)}(t) g_n^{(m)}(t) - E \sum_{n \in S} \tilde{Q}_n^{(m)}(t) I_n^{(m)}(t) a_n^{(m)}(t)
= E \sum_{n \in S} \tilde{Q}_n^{(m)}(t) E [g_n^{(m)}(t)|Q_n^{(m)}(t), a_n^{(m)}(t)]
- E \sum_{n \in S} \tilde{Q}_n^{(m)}(t) \frac{a_n^{(m)}(t)}{N}.
\] (41)

Now, by the definition of our policy it holds that
\(\tilde{Q}_n^{(m)}(t) \leq Q_n^{(m)}(t)\) if \(E [g_n^{(m)}(t)|Q_n^{(m)}(t), a_n^{(m)}(t)] \geq E [g_n^{(m)}(t)|Q_n^{(m)}(t), a_n^{(m)}(t)]\) for any \((n, n') \in S \times S\). This is because by (28),
\[
E [g_n^{(m)}(t)|Q_n^{(m)}(t), a_n^{(m)}(t)] = a_n^{(m)}(t) \cdot \max \left\{0, g_n^* - (a_n^{(m)} - \sum_{k \in U} g_k^*)/|U| \right\}.
\] (42)

This term is monotonically increasing in \(g_n^*\) = WATERLEVEL\(\{Q_n^{(m)}(t), M_n^{(m)}(t)\} - Q_n^{(m)}(t)\). Thus, it is monotonically decreasing in \(\tilde{Q}_n^{(m)}(t)\) and the claim holds. It also holds that
\(\sum_{n \in S} E [g_n^{(m)}(t)|Q_n^{(m)}(t), a_n^{(m)}(t)] = a_n^{(m)}(t) = \sum_{n \in S} a_n^{(m)}(t)\). Recall that each dispatcher \(m\) sends jobs according to uTWF based on its local array \(Q_n^{(m)}(t)\). Now, observe that the vector \(\{Q_n^{(m)}(t)|E [g_n^{(m)}(t)|Q_n^{(m)}(t), a_n^{(m)}(t)]\}_{n \in S}\) is majorized by the vector \(\{Q_n^{(m)}(t)|a_n^{(m)}(t)/N\}_{n \in S}\). Therefore,
\[
E \sum_{n \in S} \tilde{Q}_n^{(m)}(t) (g_n^{(m)}(t) - I_n^{(m)}(t) a_n^{(m)}(t)) \leq 0 \quad \forall m \in D.
\] (43)

We next handle the second term of (40). Recall that all our local array update policies respect that \(E \{Q_n(t) - \tilde{Q}_n^{(m)}(t)|\} \leq C\) for any \(t\) independently of the arrivals, we obtain,
\[
\sum_{n \in S \cap m \in D} \sum_{m \in D} E \{Q_n(t) - \tilde{Q}_n^{(m)}(t)|g_n^{(m)}(t) - I_n^{(m)}(t) a_n^{(m)}(t)\} \leq \lambda^{(1)} NMC.
\] (44)

Using (43) and (44) in (40) yields,
\[
E \sum_{n \in S} Q_n(t) \left( \sum_{m \in D} g_n^{(m)}(t) - \sum_{m \in D} I_n^{(m)}(t) a_n^{(m)}(t) \right)
\leq 0 + \lambda^{(1)} NMC = \lambda^{(1)} NMC.
\] (45)

Finally, taking the expectation of (36) as well as using (37), (38), (45) and rearranging yields,
\[
\mathbb{E} \left[ \sum_{n \in S} (Q_n(t)+1)^2 - \sum_{n \in S} \left( \sum_{n \in S} (Q_n(t))^2 \right) \right] \leq A + 2\lambda^{(1)} NMC - \frac{2}{N} \sum_{n \in S} \mathbb{E} \{Q_n(t)\}.
\] (46)

Next, summing (46) over rounds 0, . . . , \(T-1\), multiplying by \(\frac{1}{T}\) and rearranging yields,
\[
\frac{1}{T} \sum_{t=0}^{T-1} \sum_{n \in S} \mathbb{E} \{Q_n(t)\} \leq \frac{AN + 2\lambda^{(1)} N^2 MC}{2\epsilon} + \frac{N}{2\epsilon T} \sum_{n \in S} \left( Q_n(0) \right)^2,
\] (47)

where we omitted the non-positive term \(\sum_{n \in S} \left( Q_n(T) \right)^2\) as a result of the telescopic series at the left hand side of (46). Taking limits of (47) and making the standard assumption that the system is initialized with bounded queue lengths, i.e.,
\[
\mathbb{E} \left[ \sum_{n \in S} \left( Q_n(0) \right)^2 \right] < \infty\] yields,
\[
\limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{n \in S} \mathbb{E} \{Q_n(t)\} \leq \frac{AN + 2\lambda^{(1)} N^2 MC}{2\epsilon}.
\] (48)

This concludes the proof.

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