Global Caching for the Alternation-free $\mu$-Calculus

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Abstract

We present a sound, complete, and optimal single-pass tableau algorithm for the alternation-free $\mu$-calculus. The algorithm supports global caching with intermediate propagation and runs in time $2^{O(n)}$. In game-theoretic terms, our algorithm integrates the steps for constructing and solving the Büchi game arising from the input tableau into a single procedure; this is done on-the-fly, i.e. may terminate before the game has been fully constructed. This suggests a slogan to the effect that \textit{global caching = game solving on-the-fly}. A prototypical implementation shows promising initial results.

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1 Introduction

The modal $\mu$-calculus \cite{1993-KupfermanEtAl,2012-CourcoubetisEtAl} serves as an expressive temporal logic for the specification of sequential and concurrent systems containing many standard formalisms such as linear time temporal logic LTL \cite{1994-ClarkeEtAl,2000-DamEtAl}, CTL \cite{1985-Cadboro}, and PDL \cite{1977-Parikh}. Satisfiability checking in the modal $\mu$-calculus is ExpTime-complete \cite{2000-AmblerEtAl,2002-AmblerEtAl}. There appears to be, to date, no readily implementable reasoning algorithm for the $\mu$-calculus, and in fact (prior to \cite{2012-CourcoubetisEtAl}) even for its fragment CTL, that is simultaneously optimal, i.e. runs in ExpTime, and single-pass, i.e. avoids building an exponential-sized data structure in a first pass. Typical data structures used in worst-case-optimal algorithms are automata \cite{1999-BradleyEtAl}, games \cite{2002-CahlonEtAl}, and, for sublogics such as CTL, first-pass tableaux \cite{1994-ClarkeEtAl}.

The term \textit{global caching} describes a family of single-pass tableau algorithms \cite{2005-ReisigEtAl,2009-Schroeder} that build graph-shaped tableaux bottom-up in so-called \textit{expansion} steps, with no label ever generated twice, and attempt to terminate before the tableau is completely expanded by means of judicious intermediate \textit{propagation} of satisfiability and/or unsatisfiability through partially expanded tableaux. Global caching offers wide room for heuristic optimization, regarding standard tableau optimizations as well as the order in which expansion and propagation steps are triggered, and has been shown to perform competitively in practice; see \cite{2009-Schroeder} for an evaluation of heuristics in global caching for the description logic $\mathcal{ALC}$. One major challenge with global caching algorithms is typically to prove soundness and completeness, which becomes harder in the presence of fixpoint operators. A global caching algorithm for PDL has been described by Goré and Widmann \cite{2006-GoreWidmann}; finding an optimal global caching algorithm even for CTL has been named as an open problem as late as 2014 \cite{2014-CorradiniEtAl} (a non-optimal, doubly exponential algorithm is known \cite{2014-CorradiniEtAl}).

The contribution of the present work is an optimal global-caching algorithm for satisfiability in the alternation-free $\mu$-calculus, extending our earlier work on the single-variable (flat) fragment of the $\mu$-calculus \cite{2012-CourcoubetisEtAl}. The algorithm actually works at the level of generality of the alternation-free fragment of the coalgebraic $\mu$-calculus \cite{2016-BlanchetteBaader}, and thus covers also logics beyond the realm of standard Kripke semantics such as alternating-time temporal logic ATL \cite{2000-BradlerEtAl}, neighbourhood-based logics such as the monotone $\mu$-calculus that underlies...
Parikh’s game logic [31], or probabilistic fixpoint logic. To aid readability, we phrase our results in terms of the relational μ-calculus, and discuss the coalgebraic generalization only at the end of Section 4. The model construction in the completeness proof yields models of size $2^{O(n)}$.

We have implemented our algorithm as an extension of the Coalgebraic Ontology Logic Reasoner COOL, a generic reasoner for coalgebraic modal logics [21]; given the current state of the implementation of instance logics in COOL, this means that we effectively support alternation-free fragments of relational, monotone, and alternating-time μ-calculi, thus in particular covering CTL and ATL. We have evaluated the tool in comparison with existing reasoners on benchmark formulas for CTL [18] (which appears to be the only candidate logic for which well-developed benchmarks are currently available) and on random formulas for ATL and the alternation-free relational μ-calculus, with promising results; details are discussed in Section 5.

Related Work The theoretical upper bound ExpTime has been established for the full coalgebraic μ-calculus [5] (and earlier for instances such as the alternating-time μ-calculus AMC [35]), using a multi-pass algorithm that combines games and automata in a similar way as for the standard relational case, in particular involving the Safra construction. Global caching has been employed successfully for a variety of description logics [17, 20], and lifted to the level of generality of coalgebraic logics with global assumptions [15] and nominals [16].

A tableaux-based non-optimal (NExpTime) decision procedure for the full μ-calculus has been proposed in [23]. Friedmann and Lange [12] describe an optimal tableau method for the full μ-calculus that, unlike most other methods including the one we present here, makes do without requiring guardedness. Like earlier algorithms for the full μ-calculus, the algorithm constructs and solves a parity game, and in principle allows for an on-the-fly implementation. The models constructed in the completeness proof are asymptotically larger than ours, but presumably the proof can be adapted for the alternation-free case by using determinization of co-Büchi automata [28] instead of Safra’s determinization of Büchi automata [53] to yield models of size $2^{O(n)}$, like ours. For non-relational instances of the coalgebraic μ-calculus, including the alternation-free fragment of the alternating-time μ-calculus AMC, the $2^{O(n)}$ bound on model size appears to be new, with the best known bound for the alternation-free AMC being $2^{O(n \log n)}$ [35].

In comparison to our own recent work [22], we move from the flat to the alternation-free fragment, which means essentially that fixpoints may now be defined by mutual recursion, and thus can express properties such as ‘all paths reach states satisfying p and q, respectively, in strict alternation until they eventually reach a state satisfying r’. Technically, the main additional challenge is the more involved structure of eventualities and deferrals, which now need to be represented using cascaded sequences of unfoldings in the focusing approach; this affects mainly the soundness proof, which now needs to organize termination counters in a tree structure. While the alternation-free algorithm instantiates to the algorithm from [22] for flat input formulas, its completeness proof includes a new model construction which yields a bound of $3^n \in 2^{O(n)}$ on model size, slightly improving upon the bound $n \cdot 4^n$ from [22]. We present the new algorithm in terms that are amenable to a game-theoretic perspective, emphasizing the correspondence between global caching and game-solving. In fact, it turns out that global caching algorithms effectively consist in an integration of the separate steps of typical game-based methods for the μ-calculus [12, 13, 30] into a single on-the-fly procedure that talks only about partially expanded tableau graphs, implicitly combining on-the-fly determinization of co-Büchi automata with on-the-fly solving of the resulting Büchi games [10]. This motivates the mentioned slogan that
global caching is on-the-fly determination and game solving.

In particular, the propagation steps in the global caching pattern can be seen as solving an incomplete Büchi game that is built directly by the expansion steps, avoiding explicit determinization of co-Büchi automata analogously to [28]. One benefit of an explicit global caching algorithm integrating the pipeline from tableaux to game solving is the implementation freedom afforded by the global caching pattern, in which suitable heuristics can be used to trigger expansion and propagation steps in any order that looks promising.

2 Preliminaries: The $\mu$-Calculus

We briefly recall the definition of the (relational) $\mu$-calculus. We fix a set $P$ of propositions, a set $A$ of actions, and a set $\mathfrak{F}$ of fixpoint variables. Formulas $\phi, \psi$ of the $\mu$-calculus are then defined by the grammar

$$
\psi, \phi ::= \bot \mid \top \mid p \mid \neg p \mid X \mid \psi \land \phi \mid \psi \lor \phi \mid \langle a \rangle \psi \mid [a]\psi \mid \mu X. \psi \mid \nu X. \psi
$$

where $p \in P$, $a \in A$, and $X \in \mathfrak{F}$; we write $|\psi|$ for the size of a formula $\psi$. Throughout the paper, we use $\eta$ to denote one of the fixpoint operators $\mu$ or $\nu$. We refer to formulas of the form $\eta X. \psi$ as fixpoint literals, to formulas of the form $\langle a \rangle \psi$ or $[a]\psi$ as modal literals, and to $p, \neg p$ as propositional literals. The operators $\mu$ and $\nu$ bind their variables, inducing a standard notion of free variables in formulas. We denote the set of free variables of a formula $\psi$ by $FV(\psi)$. A formula $\psi$ is closed if $FV(\psi) = \emptyset$, and open otherwise. We write $\psi \leq \phi$ $(\psi < \phi)$ to indicate that $\psi$ is a (proper) subformula of $\phi$. We say that $\psi$ occurs free in $\phi$ if $\phi$ occurs as a subformula in $\psi$ that is not in the scope of any fixpoint. Throughout, we restrict to formulas that are guarded, i.e. have at least one modal operator between any occurrence of a variable $X$ and an enclosing binder $\eta X$. (This is standard although possibly not without loss of generality [12].) Moreover we assume w.l.o.g. that input formulas are clean, i.e. all fixpoint variables are distinct, and irredundant, i.e. $X \in FV(\psi)$ for all subformulas $\eta X. \psi$.

Formulas are evaluated over Kripke structures $K = (W, (R_a)_{a \in A}, \pi)$, consisting of a set $W$ of states, a family $(R_a)_{a \in A}$ of relations $R_a \subseteq W \times W$, and a valuation $\pi : P \rightarrow \mathcal{P}(W)$ of the propositions. Given an interpretation $i : \mathfrak{F} \rightarrow \mathcal{P}(W)$ of the fixpoint variables, define $i(X) \subseteq W$ by the obvious clauses for Boolean operators and propositions, $[X]_i = i(X)$, $[\langle a \rangle \psi]_i = \{ v \in W \mid \exists w \in R_a(v).w \in [\psi]_i \}$, $[[a] \psi]_i = \{ v \in W \mid \forall w \in R_a(v).w \in [\psi]_i \}$, $[\mu X. \psi]_i = \mu [\psi]_i^X$ and $[\nu X. \psi]_i = \nu [\psi]_i^X$, where $R_a(v) = \{ w \in W \mid (v, w) \in R_a \}$, $[\psi]_i^X(G) = [\psi]_i(i(X \rightarrow G))$, and $\mu, \nu$ take least and greatest fixpoints of monotone functions, respectively. If $\psi$ is closed, then $[\psi]_i$ does not depend on $i$, so we just write $[\psi]_i$. We write $x \models \psi$ for $x \in [\psi]_i$. The alternation-free fragment of the $\mu$-calculus is obtained by prohibiting formulas in which some subformula contains both a free $\nu$-variable and a free $\mu$-variable. E.g. $\mu X, \mu Y. (\Box X \lor \Box Y \land \nu Z. \Diamond Z)$ is alternation-free but $\nu Z, \mu X. (\Box X \land \nu Y. (\Diamond Y \land \Diamond Z))$ is not. CTL is contained in the alternation-free fragment.

We have the standard tableau rules (each consisting of one premise and a possibly empty set of conclusions) which will be interpreted AND-OR style, i.e. to show satisfiability of a set of formulas $\Delta$, it will be necessary to show that every rule application that matches $\Delta$ has some conclusion that is satisfiable. Our algorithm will use these rules in the expansion
Example 4. For where $A$ with $p$ literal is irreducible if it is not an unfolding particular, every clean irredundant fixpoint literal is irreducible. $\sigma \theta \psi$ irreducible $\psi \psi$ \( \triangledown \).

Each formula $\psi_1 = \mu X. ((p \land (r \lor \Box \psi_2)) \lor (\neg q \land \Box X))$ and $\psi_2 = \mu Y. (((q \land (r \lor \Box X)) \lor (\neg p \land \Box Y))$ (where $A = \{\ast\}$ and we write $\Box = [\ast], \Diamond = (\ast)$). The formulas $\psi_1$ and $\psi_2[X \mapsto \psi_1]$ state that all paths will visit $p$ and $q$ in strict alternation until $r$ is eventually reached, starting with $p$ and with $q$, respectively.

### 3 The Global Caching Algorithm

We proceed to describe our global caching algorithm for the alternation-free $\mu$-calculus. First off, we need some syntactic notions regarding decomposition of fixpoint literals.

**Definition 2 (Deferrals).** Given fixpoint literals $\chi_i = \eta X_i. \psi_i$, $i = 1, \ldots, n$, we say that a substitution $\sigma = [X_1 \mapsto \chi_1]; \ldots; [X_n \mapsto \chi_n]$ sequentially unfolds $\chi_n$ if $\chi_i <_f \chi_{i+1}$ for all $1 \leq i < n$, where we write $\psi <_f \eta X. \phi$ if $\psi \leq \phi$ and $\psi$ is open and occurs free in $\phi$ (i.e. $\sigma$ unfolds a nested sequence of fixpoints in $\chi_n$ innermost-first). We say that a formula $\chi$ is irreducible if for every substitution $[X_1 \mapsto \chi_1]; \ldots; [X_n \mapsto \chi_n]$ that sequentially unfolds $\chi_n$, we have that $\chi = \chi_1([X_2 \mapsto \chi_2]; \ldots; [X_n \mapsto \chi_n])$ implies $n = 1$ (i.e. $\chi = \chi_1$). An eventuality is an irreducible closed least fixpoint literal. A formula $\psi$ belongs to an eventuality $\theta_n$, or is a $\theta_n$-deferral, if $\psi = \alpha \sigma$ for some substitution $\sigma = [X_1 \mapsto \theta_1]; \ldots; [X_n \mapsto \theta_n]$ that sequentially unfolds $\theta_n$ and some $\alpha < \theta_1$. We denote the set of $\theta_n$-deferrals by $\text{def}r(\theta_n).

E.g. the substitution $\sigma = [Y \mapsto \mu Y. (\Box X \land \Diamond Y)]; [X \mapsto \theta]$ sequentially unfolds the eventuality $\theta = \mu X. \mu Y. (\Box X \land \Diamond Y)$, and $\Box \psi = \Box \psi (\Box \theta \land \Diamond Y)$ is a $\theta$-deferral. A fixpoint literal is irreducible if it is not an unfolding $\psi[X \mapsto \eta \psi_X]$, or a cyclic graph, in particular, every clean irredundant fixpoint literal is irreducible.

**Lemma 3.** Each formula $\psi$ belongs to at most one eventuality $\theta$, and then $\theta \leq \psi$.

**Example 4.** Applying the tableau rules $\mathcal{R}_m$ and $\mathcal{R}_p$ to the formula $\psi_1 \land EG \rightarrow r$, where $\psi_1$ is defined as in Example 1 and $EG \phi$ abbreviates $\nu X. (\phi \land \Diamond X)$, results in a cyclic graph, with relevant parts depicted as follows:

\[(\land) \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \]

\[(\lor) \quad \Gamma, X \psi \lor \phi \quad \Gamma, X \psi \lor \phi \quad \Gamma, X \psi \lor \phi \quad \Gamma, X \psi \lor \phi \]

\[(\ast) \quad \Gamma, [a] \psi_1 \quad \Gamma, [a] \psi_1 \quad \Gamma, [a] \psi_1 \quad \Gamma, [a] \psi_1 \]

\[(\ast) \quad \Gamma, [a] \psi_1 \quad \Gamma, [a] \psi_1 \quad \Gamma, [a] \psi_1 \quad \Gamma, [a] \psi_1 \]

\[(\land) \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \]

\[(\lor) \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \]

\[(\ast) \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \quad \Gamma, \bot \]
where $\Gamma = \{ \neg r, \Diamond EG \neg r \}$. The graph contains three cycles, all of which contain but never finish a formula that belongs to $\psi$ (where a formula belonging to an eventuality $\psi_1$ is said to be finished if it evokes a formula that does not belong to $\psi_1$): In the rightmost cycle, the deferral $\delta_1 := \psi_1$ evolves to the deferral $\delta_2 := \Box \psi_1$, which then evolves back to $\delta_1$. For the cycle in the middle, $\delta_1$ evolves to $\delta_1 := \Box \psi_2 \rightarrow \psi_1$, which then evolves to $\delta_1 := \psi_2 \rightarrow \psi_1$ before looping back to $\delta_3$. In the leftmost cycle, $\delta_1$ evolves via $\delta_3$ and $\delta_4$ to $2$ before cycling back to $\delta_1$. The satisfaction of $\psi_1$ is thus being postponed indefinitely, since $EG \neg r$ enforces the existence of a path on which $r$ never holds. As a successful example, consider the graph that is obtained when attempting to show the satisfiability of $\psi_1 \land EG \neg q$, (where $\Gamma' := \{ \neg q, \Diamond EG \neg q \}$):

\[
\begin{align*}
&\vdash \psi_2 \land EG \neg q, \\
&\quad \psi_2, EG \neg q := \Gamma_3
\end{align*}
\]

The two loops through $\Gamma_3$ and $\Gamma_4$ are unsuccessful as they indefinitely postpone the satisfaction of the deferrals $\delta_2$ and $\delta_3$, respectively; also there is the unsuccessful clashing node $\Gamma', p, r \lor \Box \psi_1$, containing both $q$ and $\neg q$. However, the loop through $\Gamma_5$ is successful since it contains no deferral that is never finished; as all branching in this example is disjunctive, the single successful loop suffices to show that the initial node is successful. Our algorithm implements this check for ‘good’ and ‘bad’ loops by simultaneously tracking all deferrals that occur through the proof graph, checking whether each deferral is eventually finished.

We fix an input formula $\psi_0$ and denote the Fischer-Ladner closure [25] of $\psi_0$ by $F$; notice that $|F| \leq |\psi_0|$. Let $N = P(F)$ be the set of all nodes and $S \subseteq N$ the set of all state nodes, i.e. nodes that contain only $\top$, non-clashing propositional literals (where $p$ clashes with $\neg p$) and modal literals; so $|S| \leq |N| \leq 2^{|\psi_0|}$. Put

\[
C = \{ (\Gamma, d) \in N \times P(F) \mid d \subseteq \Gamma \}, \quad C_G = \{ (\Gamma, d) \in C \mid \Gamma \in G \} \text{ for } G \subseteq N,
\]

recalling that nodes are just sets of formulas; note $|C| \leq 3^{|\psi_0|}$. Elements $v = (\Gamma, d) \in C$ are called focused nodes, with label $l(v) = \Gamma$ and focus $d$. The idea of focusing single eventualities comes from work on LTL and CTL [26, 3]. In the alternation-free $\mu$-calculus, eventualities may give rise to multiple deferrals so that one needs to focus sets of deferrals instead of single eventualities. Our algorithm incrementally builds a set of nodes but performs fixpoint computations on $P(C)$, essentially computing winning regions of the corresponding Büchi game (with the target set of player 0 being the nodes with empty focus) on-the-fly.

**Definition 5 (Conclusions).** For a node $\Gamma \in N$ and a set $S$ of tableau rules, the set of conclusions of $\Gamma$ under $S$ is

\[
Cn(S, \Gamma) = \{ \{ \Gamma_1, \ldots, \Gamma_n \} \in P(N) \mid (\Gamma/\Gamma_1 \ldots \Gamma_n) \in S \}.
\]

We define $Cn(\Gamma)$ as $Cn(R_m, \Gamma)$ if $\Gamma$ is a state node and as $Cn(R_p, \Gamma)$ otherwise. A set $N \subseteq N$ of nodes is fully expanded if for each $\Gamma \in N$, $\bigcup Cn(\Gamma) \subseteq N$.

**Definition 6 (Deferral tracking).** Given a node $\Gamma = \psi_1, \ldots, \psi_n, \phi$ and a state node $\Delta \in S$ that contains $[a] \psi_1, \ldots, [a] \psi_n, \langle a \rangle \phi$ as a subset, we say that $\Gamma$ inherits $\phi$ from $([a] \phi, \Delta)$ and $\psi_i$ from $([a] \psi_i, \Delta)$. For a non-state node $\Delta \in N$, a node $\Gamma \in N$ with $\phi \in \Gamma$, and $\psi \in \Delta$, $\Gamma$ inherits $\phi$ from $(\psi, \Delta)$ if $\Gamma = \Gamma_1$ is conclusion of a non-modal rule $(\Gamma_0/\Gamma_1 \ldots \Gamma_n)$ with
Γ₀ = Δ and either ψ has one of the forms φ, φ ∨ χ, χ ∨ φ, φ ∧ χ, χ ∧ φ, or ψ = ηX. χ and φ = χ[X → ψ]. We put

\[ \text{Inh}_m(φ, ⟨a⟩φ, Δ) = \{ Γ ∈ N | Γ \text{ inherits } φ \text{ from } ⟨⟨a⟩⟩φ, Δ) \} \]

\[ \text{Inh}_m(φ, [a]φ, Δ) = \{ Γ ∈ N | Γ \text{ inherits } φ \text{ from } ⟨[a]⟩φ, Δ) \} \]

\[ \text{Inh}_p(φ, ψ, Δ) = \{ Γ ∈ N | Γ \text{ inherits } φ \text{ from } ψ, Δ) \} \]

where Δ is a state node in the first two clauses and a non-state node in the third clause. We write evs for the set of eventualities in F. For a node Γ ∈ N, the set of deferrals of Γ is

\[ d(Γ) = \{ δ ∈ Γ | ∃ θ \in \text{evs}, δ ∈ dfr(θ) \} \]

For a set d ≠ ∅ of deferrals and nodes Γ, Δ ∈ N, we put

\[ d(Δ → Γ) = \{ δ ∈ d(Γ) | \exists θ \in \text{evs}, ∃⟨a⟩δ ∈ d, Γ ∈ \text{Inh}_m(δ, ⟨a⟩δ, Δ) \text{ and } δ, ⟨a⟩δ ∈ dfr(θ) \text{ or } \exists[a]δ ∈ d, Γ ∈ \text{Inh}_m(δ, [a]δ, Δ) \text{ and } δ, [a]δ ∈ dfr(θ) \} \]

if Δ is a state node, and

\[ d(Δ → Γ) = \{ δ₁ ∈ d(Γ) | \exists θ \in \text{evs}, ∃δ₂ ∈ d, Γ ∈ \text{Inh}_p(δ₁, δ₂, Δ) \text{ and } δ₁, δ₂ \in dfr(θ) \} \]

if Δ is a non-state node. I.e. d(Δ → Γ) is the set of deferrals that is obtained by tracking d from Δ to Γ, where Γ is the conclusion of a rule application to Δ. We put (Δ → Γ) = \{ d(Γ) \}, with the intuition that if the focus d is empty at (Δ, d), then we refocus, i.e. choose as new focus for the conclusion Γ the set d(Γ) of all deferrals in Γ.

**Example 7.** Revisiting the proof graphs from Example [3] we fix additional abbreviations Γ₆ := Γ, ¬p, □ψ₂[X → ψ₁], Γ₇ := Γ’, p, r ∨ □ψ₂[X → ψ₁] and Γ₈ := Γ’, p, r. In the first graph, e.g. d(Γ₆) = {δ₃} and d(Γ₂) = {δ₄}; in the second graph, e.g. d(Γ₇) = {r ∨ □ψ₂[X → ψ₁]} and d(Γ₈) = ∅. In the first graph, the node Γ₆ inherits the deferral δ₃ from δ₄ at Γ₂, i.e. d(Γ₆)Γ₂→r₆ = {δ₄}Γ₂→r₆ = {δ₃} since Γ₆ ∈ Inhₙ(ψ₂[X → ψ₁], □ψ₂[X → ψ₁], Γ₂). Regarding the second graph, the node Γ₇ does not inherit any deferral from Γ₇, i.e. d(Γ₇)Γ₆→r₇ = {r ∨ □ψ₂[X → ψ₁]}Γ₆→r₇ = ∅ since Γ₇ ∈ Inhₚ(r, r ∨ □ψ₂[X → ψ₁], Γ₇) but r ∨ □ψ₂[X → ψ₁] succeeds. This corresponds to the intuition that Γ₇ represents a branch originating from Γ₇ that actually finishes the deferral r ∨ □ψ₂[X → ψ₁].

We next introduce the functionals underlying the fixpoint computations for propagation of satisfiability and unsatisfiability.

**Definition 8.** Let C ⊆ C be a set of focused nodes. We define the functions f : P(C) → P(C) and g : P(C) → P(C) by

\[ f(Y) = \{ (Δ, d) ∈ C | ∀ Σ ∈ Cn(Δ). ΣΓ ∈ Σ. (Γ, d(Δ → Γ)) ∈ Y \} \]

\[ g(Y) = \{ (Δ, d) ∈ C | ∃ Σ ∈ Cn(Δ). ∀ Γ ∈ Σ. (Γ, d(Δ → Γ)) ∈ Y \} \]

for Y ⊆ C. We refer to C as the base set of f and g.

That is, a focused node (Δ, d) is in f(Y) if each rule matching Δ has a conclusion Γ such that (Γ, d') ∈ Y, where the focus d' is the set of deferrals obtained by tracking d from Δ to Γ.
Definition 9 (Proof transitionals). For $X \subseteq C \subseteq \mathbb{C}$, we define the proof transitionals $\hat{f}_X : \mathcal{P}(C) \to \mathcal{P}(C)$, $\hat{g}_X : \mathcal{P}(C) \to \mathcal{P}(C)$ by

$$
\hat{f}_X(Y) := (f(Y) \cap \overline{F}) \cup (f(X) \cap F) = f(Y) \cup (f(X) \cap F)
$$

$$
\hat{g}_X(Y) := (g(Y) \cup F) \cap (g(X) \cup \overline{F}) = g(Y) \cup (g(X) \cap \overline{F}),
$$

for $Y \subseteq C$, where $F = \{(\Gamma, d) \in C \mid d = \emptyset\}$ and $\overline{F} = \{(\Gamma, d) \in C \mid d \neq \emptyset\}$ are the sets of focused nodes with empty and non-empty focus, respectively, and where $C$ is the base set of $f$ and $g$.

That is, $\hat{f}_X(Y)$ contains nodes with non-empty focus that have for each matching rule a successor node in $Y$ as well as nodes with empty focus that have for each matching rule a successor node in $X$. The least fixpoint of $\hat{f}_X$ thus consists of those nodes that finish their focus – by eventually reaching nodes from $F$ with empty focus – and loop to $X$ afterwards.

Lemma 10. The proof transitionals are monotone w.r.t. set inclusion, i.e. if $X' \subseteq X$, $Y' \subseteq Y$, then $\hat{f}_X(Y') \subseteq \hat{f}_X(Y)$ and $\hat{g}_X(Y') \subseteq \hat{g}_X(Y)$.

Definition 11 (Propagation). For $G \subseteq \mathbb{N}$, we define $E_G, A_G \subseteq C_G$ as

$$
E_G = \nu X. \mu Y. \hat{f}_X(Y) \quad \text{and} \quad A_G = \mu X. \nu Y. \hat{g}_X(Y),
$$

where $C_G$ is the base set of $f$ and $g$.

Notice that in terms of games, the computation of $E_G$ and $A_G$ corresponds to solving an incomplete Büchi game. The set $E_G$ contains nodes $(\Gamma, d)$ for which player 0 has a strategy to enforce – for each infinite play starting at $(\Gamma, d)$ – the Büchi condition that nodes in $F$, i.e. with empty focus, are visited infinitely often; similarly $A_G$ is the winning region of player 1 in the corresponding game, i.e. contains the nodes for which player 1 has a strategy to enforce an infinite play that passes $F$ only finitely often or a finite play that gets stuck in a winning position for player 1.

Example 12. Returning to Example 3, we have $(\Gamma_1, d(\Gamma_1)) = (\Gamma_1, \{\psi_1\}) \in A_{G_1}$ and $(\Gamma_3, d(\Gamma_3)) = (\Gamma_3, \{\psi_1\}) \in E_{G_2}$, where $G_1$ and $G_2$ denote the set of all nodes of the first and the second proof graph, respectively; the global caching algorithm described later will therefore answer ‘unsatisfiable’ to $\Gamma_1$, and ‘satisfiable’ to $\Gamma_3$. To see $(\Gamma_1, \{\psi_1\}) \in A_{G_1}$ note that $A_{G_1} = \nu Y. \hat{g}_{A_{G_1}}(Y)$ by definition, so $A_{G_1} = (\hat{g}_{A_{G_1}})\nu_n(C_{\Gamma_1})$ for some $n$. For each focused node $(\Delta, d) \in C_{\Gamma_1}$ there is a rule matching $\Delta$ all whose conclusions match $\{\psi_1\}$; hence $\hat{g}_{\nu_n(C_{\Gamma_1})}$ is the base set of $A_{G_1}$.

Lemma 13. If $G' \subseteq G$, then $E_{G'} \subseteq E_G$ and $A_{G'} \subseteq A_G$.

Lemma 14. Let $G \subseteq \mathbb{N}$ be fully expanded. Then $E_G = \overline{A_G}$.

Our algorithm constructs a partial tableau, maintaining sets $G, U \subseteq \mathbb{N}$ of expanded and unexpanded nodes, respectively. It computes $E_G, A_G \subseteq C_G$ in the propagation steps; as these sets grow monotonically, they can be computed incrementally.
Algorithm (Global caching). Decide satisfiability of a closed formula \( \phi_0 \).

1. (Initialization) Let \( G := \emptyset, \Gamma_0 := \{ \phi_0 \}, U := \{ \Gamma_0 \} \).
2. (Expansion) Pick \( t \in U \) and let \( G := G \cup \{ t \}, U := (U - \{ t \}) \cup (\bigcup tU - G) \).
3. (Intermediate propagation) Optional: Compute \( E_G \) and/or \( A_G \). If \((\Gamma_0, d(\Gamma_0)) \in E_G\), return ‘Yes’. If \((\Gamma_0, d(\Gamma_0)) \in A_G\), return ‘No’.
4. If \( U \neq \emptyset \), continue with Step 2.
5. (Final propagation) Compute \( E_G \). If \((\Gamma_0, d(\Gamma_0)) \in E_G\), return ‘Yes’, else ‘No’.

Note that in Step 5 \( G \) is fully expanded. For purposes of the soundness proof, we note an immediate consequence of Lemmas 13 and 14:

\[ \text{Lemma 15. If some run of the algorithm without intermediate propagation steps is successful on input } \phi_0, \text{ then all runs on input } \phi_0 \text{ are successful.} \]

\[ \text{Remark. For alternation-free fixpoint logics, the game-based approach (e.g. [13]) is to (1.) define a nondeterministic co-Büchi automaton of size } O(n) \text{ that recognizes unsuccessful branches of the tableau. This automaton is then (2.) deterministic to a deterministic co-Büchi automaton of size } 2^{O(n)} \text{ (avoiding the Safra construction using instead the method of [28]; here, alternation-freeness is crucial) and (3.) complemented to a deterministic Büchi automaton of the same size that recognizes successful branches of the tableau. A Büchi game is (4.) constructed as the product game of the carrier of the tableau and the carrier of the Büchi automaton. This game is of size } 2^{O(n)} \text{ and can be (5.) solved in time } 2^{O(n)}. \]

Our global caching algorithm integrates analogues of items (1.) to (5.) in one go: We directly construct the Büchi game (thus replacing (1.) through (4.) by a single definition) step-by-step during the computation of the sets \( E \) and \( A \) of (un)successful nodes as nested fixpoints of the proof transitions; the propagation step corresponds to (5). Our algorithm allows for intermediate propagation, corresponding to solving the Büchi game on-the-fly, i.e. before it has been fully constructed.

4 Soundness, Completeness and Complexity

Soundness Let \( \phi_0 \) be a satisfiable formula. By Lemma 15 it suffices to show that a run without intermediate propagation is successful.

\[ \text{Definition 16. For a formula } \psi, \text{ we define } \psi_X(\phi) = \psi[X \mapsto \phi], \psi^0_X = \bot \text{ and } \psi^{n+1}_X = \psi_X(\psi^n_X). \]

We say that a Kripke structure \( K \) is stabilizing if for each state \( x \) in \( K \), each \( \mu X. \psi \), and each fixpoint-free context \( c(\cdot) \) such that \( x \models c(\mu X. \psi) \), there is \( n \geq 0 \) such that \( x \models c(\psi^n_X) \).

We note that finite Kripke structures are stabilizing and import the finite model property (without requiring a bound on model size) for the \( \mu \)-calculus from [25]; for the rest of the section, we thus fix w.l.o.g. a stabilizing Kripke structure \( K = (W, (R_a)_{a \in A}, \pi) \) satisfying the target formula \( \phi_0 \) in some state.

\[ \text{Definition 17 (Unfolding tree). Given a formula } \psi, \text{ an unfolding tree } t \text{ for } \psi \text{ consists of the syntax tree of } \psi \text{ together with a natural number as additional label for each node that represents a least fixpoint operator. We denote this number by } t(\kappa, \mu X. \phi) \text{ for an occurrence of a fixpoint literal } \mu X. \phi \text{ at position } \kappa \in \{0, 1\}^* \text{ in } \psi. \]

We define the unfolding \( \psi(t) \) of \( \psi \) according to an unfolding tree \( t \) for \( \psi \) by

\[ X(t) = X \quad (\phi_1 \land \phi_2)(t) = \phi_1(t_1) \land \phi_2(t_2) \quad (\mu X. \phi_1)(t) = (\phi_1(t_1))^X_{(\mu X. \phi_1)}, \]

where \( t_i \) is the \( i \)-th child of the root of \( t \), and similar clauses for \( (a), [a], \lor \), and \( \nu \) as for \( \land \).
Given a formula $\psi$, we define the order $<_{\psi}$ on unfolding trees for $\psi$ by lexicographically ordering the lists of labels obtained by pre-order traversal of the syntax tree of $\psi$.

Definition 18 (Unfolding). The unfolding of a formula $\psi$ at a state $x$ with $x \models \psi$ is defined as $unf(\psi, x) = \psi(t)$, where $t$ is the least unfolding tree for $\psi$ (w.r.t. $<_{\psi}$) such that $x \models \psi(t)$ (such a $t$ exists by stabilization).

Note that in unfoldings, all least fixpoint literals $\mu X. \phi$ are replaced with finite iterates of $\phi$.

Theorem 19 (Soundness). The algorithm returns ‘Yes’ on input $\phi_0$ if $\phi_0$ is satisfiable.

Proof. (Sketch) We show that any node $(\Gamma, d)$ that is constructed by the algorithm and whose label is satisfied at some state $x$ in $K$ is successful, i.e. $(\Gamma, d) \in E_G$; the proof is by induction over the maximal modal depth of $unf(\delta, x)$ for $\delta \in d$.

Completeness Assume that the algorithm answers ‘Yes’ on input $\phi_0$, having constructed the set $E := E_G$ of successful nodes. Put $D = \{(\Gamma, d) \in E \mid \Gamma \in S\}$; note $|D| \leq |E| \leq 3^{|\phi_0|}$.

Definition 20 (Propositional entailment). For a finite set $\Psi$ of formulas, we write $\bigwedge \Psi$ for the conjunction of the elements of $\Psi$. We say that $\Psi$ propositionally entails a formula $\phi$ (written $\Psi \vdash_{PL} \phi$) if $\bigwedge \Psi \rightarrow \phi$ is a propositional tautology, where modal literals are treated as propositional atoms and fixpoint literals $\eta X. \phi$ are unfolded to $\phi(\eta X. \phi)$ (recall that fixpoint operators are guarded).

Definition 21. We denote the set of formulas in a node $\Gamma$ that do not belong to an eventuality $\theta$ by $N(\Gamma, \theta) = \{ \phi \in \Gamma \mid \phi \notin dfr(\theta) \}$.

A set $d$ of deferrals is sufficient for $\delta \in dfr(\theta)$ at a node $\Gamma$, in symbols $d \vdash_{\Gamma} \delta$, if $d \cup N(\Gamma, \theta) \vdash_{PL} \delta$. We write $\vdash_{\Gamma} \delta$ to abbreviate $\theta \vdash_{\Gamma} \delta$.

Definition 22 (Timed-out tableau). Let $U \subseteq S \times S$ and let $L \subseteq U \times U$. We denote the set of $L$-successors of $v \in U$ by $L(v) = \{ w \mid (v, w) \in L \}$. Let $d$ be a set of deferrals. We put $to(0, n) = U$ for all $n$ (to for timeout). For $d \neq \emptyset$, we put $to(d, 0) = \emptyset$ and define $to(d, m + 1)$ to be the set of $d$-focused nodes $(\Delta, d')$ such that writing $Cn(\Delta) = \{ \Sigma_1, \ldots, \Sigma_n \}$, we have $L(\Delta, d') = \{ (\Gamma_1, d_1), \ldots, (\Gamma_n, d_n) \}$ where for each $i$ there exists $\Gamma \in \Sigma_i$ such that

- $\Gamma_i \vdash_{PL} \bigwedge \Gamma$ and $d_i \vdash_{\Gamma_i} d'_{\Delta \rightarrow \Gamma}$, and
- $(\Gamma_i, d_i) \in to(d'', m)$ for some $d'' \subseteq d(\Gamma_i)$ with $d'' \vdash_{\Gamma_i} d'_{\Delta \rightarrow \Gamma}$.

If for each focused node $(\Gamma, d) \in U$ there is a number $m$ such that $(\Gamma, d) \in to(d(\Gamma), m)$, then $L$ is a timed-out tableau over $U$.

Roughly, $to(d, m)$ can be understood as the set of all focused nodes in $U$ that finish all deferrals in $d$ within $m$ modal steps, i.e. with time-out $m$; this is similar to Kozen’s $\mu$-counters.

Lemma 23 (Tableau existence). There exists a timed-out tableau over $D$.

Proof sketch. Since $D \subseteq E_G$, we can define $L \subseteq D \times D$ in such a way that all paths in $L$ visit $F$ (the set of nodes with empty focus) infinitely often, so every deferral contained in some node in $D$ will be focused by the unavoidable eventual refocusing; this new focus will in turn eventually be finished so that $L$ is a timed-out tableau.
For the rest of the section, we fix a timed-out tableau \( L \) over \( D \) and define a Kripke structure \( K = (D, (R_a)_{a \in A}, \pi) \) by taking \( R_a(v) \) to be the set of focused nodes in \( L(v) \) whose label is the conclusion of an \( (\langle a \rangle) \)-rule that matches \( l(v) \) and by putting \( \pi(p) = \{v \in D \mid p \in l(v)\} \).

\[ \text{Definition 24 (Pseudo-extension).} \quad \text{The pseudo-extension } \hat{[\phi]} \text{ of } \phi \text{ in } D \text{ is} \]
\[ \hat{[\phi]} = \{v \in D \mid l(v) \vdash_{PL} \phi\}. \]

\[ \text{Lemma 25 (Truth).} \quad \text{In the Kripke structure } K, \hat{[\psi]} \subseteq [\psi] \text{ for all } \psi \in \mathbf{F}. \]

\[ \text{Proof sketch.} \quad \text{Induction on } \psi, \text{ with an additional induction on time-outs in the case for least fixpoint literals, exploiting alternation-freeness.} \]

\[ \text{Corollary 26 (Completeness).} \quad \text{If a run of the algorithm with input } \phi_0 \text{ returns ‘Yes’, then } \phi_0 \text{ is satisfiable.} \]

\[ \text{Proof sketch.} \quad \text{Combine the existence lemma and the truth lemma to obtain a model over } D. \]

As a by-product, our model construction yields

\[ \text{Corollary 27.} \quad \text{Every satisfiable alternation-free fixpoint formula } \phi_0 \text{ has a model of size at most } 3|\phi_0|. \]

Thus we recover the bound of \( 2^{O(n)} \) for the alternation-free relational \( \mu \)-calculus, which can be obtained, e.g., by carefully adapting results from \([12]\) to the alternation-free case; for the alternation-free fragment of the alternating-time \( \mu \)-calculus, covered by the coalgebraic generalization discussed next, the best previous bound appears to be \( n^{O(n)} = 2^{O(n \log n)} \) \([35]\).

\[ \text{Complexity} \quad \text{Our algorithm has optimal complexity (given that the problem is known to be ExpTime-hard):} \]

\[ \text{Theorem 28.} \quad \text{The global caching algorithm decides the satisfiability problem of the alternation-free } \mu \text{-calculus in ExpTime, more precisely in time } 2^{O(n)}. \]

\[ \text{The Alternation-Free Coalgebraic } \mu \text{-Calculus} \quad \text{Coalgebraic logic} \ [5] \text{ serves as a unifying framework for modal logics beyond standard relational semantics, subsuming systems with, e.g., probabilistic, weighted, game-oriented, or preference-based behaviour under the concept of coalgebras for a set functor } F. \]

\text{All our results lift to the level of generality of the (alternation-free) coalgebraic } \mu \text{-calculus} \ [4]; \text{ details are in a technical report at } \text{https://www8.cs.fau.de/hausmann/afgc.pdf}. \text{ In consequence, our results apply also to the alternation-free fragments of the alternating-time } \mu \text{-calculus} \ [1], \text{ probabilistic fixpoint logics, and the monotone } \mu \text{-calculus (the ambient fixpoint logic of Parikh’s game logic} \ [31]), \text{ as all these can be cast as instances of the coalgebraic } \mu \text{-calculus.} \]

\[ \text{5 Implementation and Benchmarking} \]

\text{The global caching algorithm has been implemented as an extension of the Coalgebraic Ontology Logic Reasoner (COOL) } [21], \text{ a generic reasoner for coalgebraic modal logics, available at } \text{https://www8.cs.fau.de/research:software:cool}. \text{ COOL achieves its genericity by instantiating an abstract core reasoner that works for all coalgebraic logics to concrete instances of logics; our global caching algorithm extends this core. Instance logics implemented}
in COOL currently include relational, monotone, and alternating-time logics, as well as any logics that arise as fusions thereof. In particular, this makes COOL, to our knowledge, the only implemented reasoner for the alternation-free fragment of the alternating-time \( \mu \)-calculus (a tableau calculus for the sublogic ATL is prototypically implemented in the TATL reasoner [7]) and the star-nesting free fragment of Parikh’s game logic.

Although our tool supports the full alternation-free \( \mu \)-calculus, we concentrate on CTL for experiments, as this appears to be the only candidate logic for which substantial sets of benchmark formulas are available [18]. CTL reasoners can be broadly classified as being either top-down, i.e. building graphs or tableaux by recursion over the formula, or bottom-up; the two groups perform very differently [18]. We compare our implementation with the top-down solvers TreeTab [14], GMUL [18], MLSolver [11] and the bottom-up solvers CTL-RP [36] and BDDCTL [18]. Out of the top-down solvers, only TreeTab is single-pass like COOL; however, TreeTab has suboptimal (doubly exponential) worst-case runtime. MLSolver supports the full \( \mu \)-calculus. For MLSolver, CTL-RP and BDDCTL, formulas have first been compacted [18]. All tests have been executed on a system with Intel Core i7 3.60GHz CPU with 16GB RAM, and a stack limit of 512MB.

On the benchmark formulas of [18], COOL essentially performs similarly as the other top-down tools, and closer to the better tools when substantial differences show up. As an example, the runtimes of COOL, TreeTab, GMUL, MLSolver, CTL-RP, and BDDCTL on the Montali-formulas [29, 18] are shown in Figure 1. To single out one more example, Figure 2 shows the runtimes for the alternating bit protocol benchmark from [18]; COOL performs closer to GMUL than to MLSolverc on these formulas.

This part of the evaluation may be summed up as saying that COOL performs well despite being, at the moment, essentially unoptimized: the only heuristics currently implemented is a simple-minded dependency of the frequency of intermediate propagation on the number of unexpanded nodes.

In addition, we design two series of unsatisfiable benchmark formulas that have an exponentially large search space but allow for detection of unsatisifiability at an early stage. Recall that in CTL we can express the statement ‘in the next step, the \( n \)-bit counter \( x \) represented by the variables \( x_1, \ldots, x_n \) will be incremented’ (with wraparound) as a formula \( c(x, n) \) of polynomial size in \( n \). We define unsatisfiable formulas \( \text{early}(n, j, k) \) that specify an \( n \)-bit
counter $p$ with $n$ bits and additionally branch after $2^j$ steps (i.e. when $p_j$ holds) to start a counter $r$ with $k$ bits which in turn forever postpones the eventuality $EF p$:

$$
\text{early}(n, j, k) = \text{start}_p \land \text{init}(p, n) \land \text{init}(r, k) \land AG ((r \rightarrow c(r, k)) \land (p \rightarrow c(p, n))) \land \\
AG (\bigwedge_{0 \leq i \leq j} p_i \rightarrow EX (\text{start}_r \land EF p)) \land \neg(p \land r) \land (r \rightarrow AX r)
$$

$$
\text{init}(x, m) = AG ((\text{start}_x \rightarrow (x \land \bigwedge_{0 \leq i < m} \neg x_i)) \land (x \rightarrow EX x)).
$$

Note here that $\text{init}$ uses $x$ as a string argument; $\text{start}_x$ is an atom indicating the start of counter $x$, and the atom $x$ itself indicates that the counter $x$ is running. The second series of unsatisfiable formulas $\text{early}_{gc}(n, j, k)$ is obtained by extending the formulas $\text{early}(n, j, k)$ with the additional requirement that a further counter $q$ with $n$ bits is started infinitely often, but at most at every second step:

$$
\text{early}_{gc}(n, j, k) = \text{early}(n, j, k) \land b \land \text{init}(q, n) \land AG (\neg(p \land q) \land \neg(q \land r) \land (q \rightarrow c(q, n))) \land \\
AG (AF b \land (b \rightarrow (EX p \land EX \text{start}_q \land AX \neg b)))
$$

Figure 2 shows the respective runtimes for these formulas. In all cases, COOL finishes before the tableau is fully expanded, while GMUL and MLSolver will necessarily complete their first pass before being able to decide the formulas, and hence exhibit exponential behaviour; TreeTab seems not to benefit substantially from its capability to close tableaux early. For the $\text{early}_{gc}$ formulas, the ability to cache previously seen nodes appears to provide COOL with additional advantages. The $\text{early}_{gc}$ series can be converted into satisfiable formulas by replacing $AX$ with $EX$, with similar results.

Due to the apparent lack of benchmarking formulas for the alternation-free $\mu$-calculus and ATL, we compare runtimes on random formulas for these logics. For the alternation-free $\mu$-calculus, formulas were built from 250 random operators (where disjunction and conjunction
are twice as likely as the other operators). The experiment was conducted with formulas over three and over ten propositional atoms, respectively. MLSolver ran out of memory on 21% on the formulas over three atoms and on 16% of the formulas over ten atoms. COOL answered all queries without exceeding memory restrictions, and in under one second for all queries but one. Altogether, COOL was faster than MLSolver for more than 98% of the random alternation-free formulas, with the median of the ratios of the runtimes being 0.0431 in favour of COOL for formulas over three atoms and 0.0833 for formulas over ten atoms (recall however that MLSolver supports the full μ-calculus). For alternating-time temporal logic ATL, we compared the runtimes of TATL and COOL on random formulas consisting of 50 random operators; COOL answered faster than TATL on all of the formulas, with the median of the ratios of runtimes being 0.000668 in favour of COOL.

6 Conclusion

We have presented a tableau-based global caching algorithm of optimal (ExpTime) complexity for satisfiability in the alternation-free coalgebraic μ-calculus; the algorithm instantiates to the alternation-free fragments of e.g. the relational μ-calculus, the alternating-time μ-calculus (AMC) and the serial monotone μ-calculus. Essentially, it simultaneously generates and solves a deterministic Büchi game on-the-fly in a direct construction, in particular skipping the determinization of co-Büchi automata; the correctness proof, however, is stand-alone. We have generalized the $2^{O(n)}$ bound on model size for alternation-free fixpoint formulas from the relational case to the coalgebraic level of generality, in particular to the AMC.

We have implemented the algorithm as part of the generic solver COOL; the implementation shows promising performance for CTL, ATL and the alternation-free relational μ-calculus. An extension of our global caching algorithm to the full μ-calculus would have to integrate Safra-style determinization of Büchi automata [34] and solving of the resulting parity game, both on-the-fly.

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Lan Zhang, Ullrich Hustadt, and Clare Dixon. A resolution calculus for the branching-time temporal logic CTL. ACM Trans. Comput. Log., 15, 2014.
A. Omitted Proofs and Lemmas

A.1 Proofs and Lemmas for Section 2

Definition 29. We let $BV(\phi)$ denote the set of variables $X$ such that $\eta X$ occurs in $\psi$.

Lemma 30 (Substitution). If $BV(\phi) \cap FV(\phi) = \emptyset$, then

$$[[\psi]]^X_\phi = [[\psi[X \mapsto \phi]]].$$

Proof. The proof is by induction over $\psi$. If $\psi = \bot$, $\psi = \top$, $\psi = p$ or $\psi = \neg p$, for $p \in P$, then $\psi$ is closed so that $[[\psi]]^X_\phi = [[\psi]] = [[\psi[X \mapsto \phi]]]$. If $\psi = X$, then $[[X]]^X_\phi = [[X[X \mapsto \phi]]]$. If $\psi = Y \neq X$, then $[[Y]]^X_\phi = [[Y]] = [[Y[X \mapsto \phi]]]$. The cases for disjunction, conjunction and modal operators are straightforward. If $\psi = \eta X.\psi_1$ for $Y \neq X$, then $[[\eta X.\psi_1]]^X_\phi = [[\eta X.\psi]](\{\phi_i\}_{\psi \in \psi_1}) = [[\eta X.\psi_1[X \mapsto \phi]]] = [[(\eta Y.\psi_1[X \mapsto \phi])] Y] = [[(\eta Y.\psi_1)[X \mapsto \phi]]]$, where the second equality holds since for all $A$,

$$[[\psi]]^X_{\psi_1[A]}(A) = [[\psi_1[A]]_{\psi_1[A]}(Y \mapsto A)] = [[\psi_1[A]]_{\psi_1[A]}(Y \mapsto Y)] = [[\psi_1[A]]_{\psi_1[A]}(Y \mapsto Y)] = [[\psi_1[A]]_{\psi_1[A]}(Y \mapsto Y)] = [[\psi_1[A]]_{\psi_1[A]}(Y \mapsto Y)]$$

where the second equality holds since $X \neq Y$, the fourth equality holds since by assumption, $Y \notin FV(\psi)$ and the fifth equality is by the induction hypothesis.

We note that by Lemma 30,

$$[[\eta X.\psi]] = \eta [[\psi]]^X_\phi = [[\psi]]^X_{\eta X.\psi} = [[\psi[X \mapsto \eta X.\psi]]].$$

A.2 Proofs and Lemmas for Section 3

In the following we will consider all deferrals to be in decomposed form, i.e. given a formula $\psi$ that belongs to some eventualty $\theta$, so that $\psi = \alpha \sigma$ for appropriate $\alpha$ and $\sigma$, according to Definition 2, we equivalently represent $\psi$ by the pair $(\alpha, \sigma)$. This allows us to directly refer to the base $\alpha$ and the sequence $\sigma$ of a deferral. We say that the pair $(\alpha, \sigma)$ induces the formula $\alpha \sigma$.

Proof of Lemma 31. The first part of the Lemma is stated by Lemma 30. The proof of the second part is by lexicographic induction over $(|\sigma|, |\alpha|)$, distinguishing cases for $\alpha$. The interesting case is the fixpoint variable case, i.e. $\alpha = Y$ for some $Y$. If $|\sigma| = 1$, we have that $\sigma = [Y \mapsto \theta]$ and hence $Y \sigma = \theta$. If $|\sigma| > 1$, we have $Y \sigma = \chi \kappa$ where $\chi$ is the result of applying the first substitution from $\sigma$ that touches $Y$ to $Y$ and where $\kappa$ consists of the remaining substitutions from $\sigma$. We have $|\kappa| < |\sigma|$ and $(\chi, \kappa)$ is a $\theta$-deferral so that the induction hypothesis finishes the proof.

Lemma 31. Let $(\alpha, \sigma)$ be an $\theta_1$-deferral and let $(\beta, \kappa)$ be an $\theta_2$-deferral such that $\alpha \sigma = \psi = \beta \kappa$. Then $\theta_1 = \theta_2$. 

We show that $\theta_2 \leq \theta_1$, the other direction is symmetric. We note that by Lemma 3, $\theta_2 \leq \psi$. If $\theta_2 \leq \alpha$, $\theta_2 < \theta_1$ and hence $\theta_2 \leq \theta_1$, as required. If $\theta_2 \not< \alpha$, then let $\theta_2 = \mu Y. \phi$ and $\sigma = [X_1 \mapsto \chi_1]; \ldots; [X_n \mapsto \chi_n]$ where $\chi_1 = \theta_1$. Since $\theta_2 \leq \psi$ but $\theta_2 \not< \alpha$, we are in one of the following two cases: a) There is a variable $X \in \text{FV}(\alpha)$ with $\theta_2 \leq X \sigma$ in which case – since $\theta_2$ is irreducible – $\theta_2 \leq \chi_1 \leq \theta_1$ for some $1 \leq i \leq n$: otherwise there is some $\chi_j = \mu Y. \phi_1$ such that $\mu Y. \phi_1([X_{j+1} \mapsto \chi_{j+1}]; \ldots; [X_n \mapsto \chi_n]) = \theta_2$ which is a contradiction to $\theta_2$ being irreducible; b) The formula $\alpha$ contains a fixpoint literal $\mu Y. \phi_1$ with $\phi_1 \sigma = \phi$. But then $\theta_2 = (\mu Y. \phi_1) \sigma$ and $(\mu Y. \phi_1, \sigma)$ is a sequence over $\chi_n$ which is a contradiction to $\theta_2$ being irreducible.

**Proof.** Note that

$$
\hat{f}_{Y'}(Y') = (f(X') \cap Y') \cup (f(X') \cap F) \\
\subseteq (f(X') \cap \overline{F}) \cup (f(X) \cap F) \\
= \hat{f}(X)
$$

where the inclusion holds since $X' \cap Y' \subseteq X \cap Y$ and since $f$ is monotone w.r.t. set inclusion so that $f(X' \cap Y') \subseteq f(X \cap Y)$ and $f(X') \subseteq f(X)$. The proof for $\hat{g}$ is analogous.

**Proof of Lemma 32.** Let $G' \subseteq G$. We show $E_{G'} \subseteq E_G$, the proof of $A_{G'} \subseteq A_G$ is analogous. We denote by $f_G$, and $(\hat{f}_X)_C$ the respective transitionals with base set $C \subseteq G$ and note that for all $X, Y \subseteq G$,

$$f_G(Y) \subseteq f_G(Y) \quad \text{and} \quad (\hat{f}_X)_C(Y) \subseteq (\hat{f}_X)_G(Y).$$

From this we obtain $\mu((\hat{f}_X)_G) \subseteq \mu((\hat{f}_X)_G)$ by induction; this in turn implies that for all $Y$, $(X \mapsto \mu((\hat{f}_X)_G)) Y \subseteq (X \mapsto \mu((\hat{f}_X)_G)) Y$. Induction yields $\nu(X \mapsto \mu((\hat{f}_X)_G)) \subseteq \nu(X \mapsto \mu((\hat{f}_X)_G))$, as required.

**Lemma 32.** Let $G \subseteq N$ be fully expanded and let $C \subseteq C_G$ be the base set of $f$ and $g$. For all sets $Y \subseteq C$,

$$f(Y) = g(Y'),$$

where for each $Y' \subseteq C$, $\overline{Y'}$ denotes the complement of $Y'$ in $C$.

**Proof.** The inclusion “$\subseteq$” is immediate. For the inclusion “$\supseteq$”, let $(\Delta, d) \in g(Y)$ so that it is not the case that there is a $\Sigma \in C_\Delta(\Delta)$ such that for each $\Gamma \in \Sigma$, $(\Gamma, d_{\Delta \mapsto \Gamma}) \in Y$. Since $G$ is fully expanded, this implies that for all $\Sigma \in C_\Delta(\Delta)$, there is a $\Gamma \in \Sigma$ such that $(\Gamma, d_{\Delta \mapsto \Gamma}) \in Y$, i.e. that $(\Delta, d) \notin f(Y)$.

**Lemma 33.** If $G \subseteq N$ is fully expanded and $C \subseteq C_G$ is the base set of $f_X$ and $\hat{g}_X$, then for all sets of nodes $Y \subseteq C$,

$$\hat{f}_X(Y) = \hat{g}_X(Y).$$

**Proof.** Just note that

$$\hat{f}_X(Y) = (f(X \cap Y) \cap \overline{F}) \cup (f(X) \cap F) \\
= (g(X \cup Y) \cup F) \cap (g(X) \cup \overline{F}) \\
= \hat{g}_X(Y).$$

where the second equality follows, as $G$ is fully expanded, from Lemma 32.
Proof of Lemma [14]: We obtain $E_G = \nu(X \mapsto \mu(f_X)) = \mu(X \mapsto \nu((\overline{g}_X))) = \overline{A_G}$ from Lemma [23] which states that $\overline{f_X}(Y) = \overline{\nu(Y)}$ for all $X \subseteq C_G$ in combination with the fact that for complementary monotone functions $f$ and $g$, $\mu f = \overline{g}$. \hfill \checkmark

Proof of Lemma [13]: Let $G$ denote the set of nodes which is created by the algorithm without intermediate propagation – i.e. without step 3 – and notice that $G$ is fully expanded. Let $\{(\{\phi_0\}, d(\{\phi_0\})) \in E_G\}$ and let $G_p$ be the set of nodes created by any run of the algorithm (possibly involving intermediate propagation). We note that $G_p \subseteq G$ so that Lemma [13] tells us that $A_{G_p} \subseteq A_G$. As $G$ is fully expanded, Lemma [14] states that $A_G = E_G$. As $\{(\{\phi_0\}, d(\{\phi_0\})) \in E_G, \{(\{\phi_0\}, d(\{\phi_0\})) \notin A_{G_p} \subseteq A_G = E_G\}$, as required. \hfill \checkmark

A.3 Proofs and Lemmas for Section 4

Throughout this subsection, we fix $N \subseteq \mathbb{N}$ to be the fully expanded set of nodes constructed by a run of the algorithm without intermediate propagation.

**Definition 34.** Given a substitution $\sigma$, we define the domain $\text{dom}(\sigma)$ of $\sigma$ as the set of all fixpoint variables that $\sigma$ touches, i.e. the set of all fixpoint variables $X$ with $\sigma(X) \neq X$.

Regarding Definition [21] we note that for all $\Gamma \in N$, all eventualities $\theta$ and all deferrals $\delta$, since $d(\Gamma) \cup N(\Gamma, \theta) = \Gamma$, we have $d(\Gamma) \vdash \theta \delta$ iff $\Gamma \vdash_{PL} \theta \delta$.

**Lemma 35 (Syntactic substitution).** If $\{(X) \cup BV(\psi)\} \cap \text{dom}(\sigma) = \emptyset$ and for each $Y \in FV(\psi)$, $(\{X\} \cup BV(\psi)) \cap FV(\sigma(Y)) = \emptyset$, $(\psi[\sigma])[X \mapsto (\phi[\sigma])] = (\psi[X \mapsto \phi])\sigma$.

**Proof.** The proof is by induction over $\psi$. If $\psi = \bot$, $\psi = \top$, $\psi = p$ or $\psi = \neg p$, for $p \in P$, then $\psi$ is closed and hence $(\psi[\sigma])[X \mapsto (\phi[\sigma])] = \psi = (\psi[X \mapsto \phi])\sigma$. If $\psi = X$, then note that by assumption $X \notin \text{dom}(\sigma)$ so that $(X[\sigma])[X \mapsto (\phi[\sigma])] = X[X \mapsto \phi[\sigma]] = \phi[\sigma] = (X[X \mapsto \phi])\sigma$. If $\psi = Y \neq X$, then we have by assumption $X \notin FV(\sigma(Y))$ so that $(Y[\sigma])[X \mapsto (\phi[\sigma])] = (Y[\sigma])[X \mapsto \phi[\sigma]] = \sigma[\sigma] = (Y[\sigma])[X \mapsto \phi][\sigma]$. The cases for conjunction, disjunction and modal operators are straightforward. If $\psi = \eta Y \cdot \psi$ for $X \neq Y$, then we have by assumption that $Y \notin \text{dom}(\sigma)$ and for any $Z \in FV(\psi)$, $Y \notin FV(\sigma(Z))$ so that $(\eta Y \cdot \psi)[X \mapsto (\phi[\sigma])] = \eta Y \cdot (\psi[X \mapsto \phi])\sigma = (\eta Y \cdot (\psi[\sigma])[X \mapsto \phi])\sigma = (\eta Y \cdot (\psi[\sigma])[X \mapsto \phi])\sigma = (\eta Y \cdot (\psi[X \mapsto \phi])\sigma)$. The third equality is by the induction hypothesis. \hfill \checkmark

**Definition 36.** Let $t_1$ and $t_2$ be unfolding trees for $\psi$ and $\phi$. Define $t_1[X \mapsto t_2]$ as the unfolding tree for $\psi[X \mapsto \phi]$ that is obtained by replacing every node in $t_1$ that represents a free occurrence of $X$ in $\psi$ with $t_2$.

**Lemma 37.** For each state $x$ and each formula $\psi$ such that $x \models \psi$, there is a least unfolding tree $t$ such that $x \models \psi(t)$.

**Proof.** We construct $t$ by walking from left to right through all paths in the syntax tree of $\psi$, assigning numbers to nodes that represent least fixpoint literals. Let $\kappa$ be a position and let $t_\kappa$ denote the tree that has been constructed so far on the walk from the root of the syntax tree to $\kappa$. We assign $n_\kappa$ to the node at position $\kappa$ if that node represents a least fixpoint literal $\mu X_\kappa \cdot \psi_\kappa$, where $n_\kappa$ is the least number such that $x \models c_\kappa((\psi_\kappa)^{n_\kappa})$, where
Lemma 38. The proof is by standard induction over Lemma 39.

Lemma 39. So let

By induction over \( t \) where

where the second equality is by the induction hypothesis and the third equality is by \( \psi \).

Proof. By induction over \( n \). If \( n = 0, \bot = \bot \). Otherwise

where the second equality is by the induction hypothesis and the third equality is by Lemma 35.

Lemma 39. Let \( t_1 \) be an unfolding tree for \( \psi \) and let \( t_2 \) be an unfolding tree for \( \phi \). Then

\[
(\psi[X \mapsto \phi])(t_1[X \mapsto t_2]) = (\psi(t_1))[X \mapsto \phi(t_2)].
\]

Proof. The proof is by standard induction over \( \psi \). We consider the only interesting case, i.e. the case that \( \psi = \mu Y.\psi_1 \) where \( X \neq Y \). Then

where \( t_3 \) is the child of the root of \( t_1 \). The third equality is by the induction hypothesis and the fourth equality is by Lemma 35.

Lemma 40. Let \( t \) and \( s \) be unfolding trees for \( \phi_1 = \eta X.\psi \sigma \) and \( \phi_2 = \psi(\eta X.\psi, \sigma) \), respectively. Furthermore, let \( t(\epsilon, \phi_1) = n + 1 \) and \( s(\tau, \phi_1) = n \) for all positions \( \tau \) at which \( \phi_1 \) occurs in \( \phi_2 \); also let \( t(\kappa, \chi) = s(\tau, \chi) \) for all least fixpoint literals \( \chi \) occurring in \( \phi_1 \) at some position \( \kappa \neq \epsilon \) and all \( \tau \) such that \( \chi \) occurs in \( \phi_2 \) at position \( \tau \) and either \( \kappa = 0 \sigma \) or \( \tau = \rho \kappa \) where \( X \) occurs freely in \( \psi \) at position \( \rho \). Then

\[
x \models \eta X.\psi \sigma(t) \text{ implies } x \models (\eta X.\psi, \sigma)(s).
\]

Proof. So let \( t(\epsilon, \eta X.\psi \sigma) = n + 1 = s(\tau, \eta X.\psi \sigma) + 1 \) for all appropriate \( \tau \). Let \( t_1 \) denote the child of the root of \( t \) and let \( s_1, s_2 \) and \( s_3 \) denote subtrees of \( s \) such that \( s = s_1[X \mapsto s_2] \) and \( s_3 \) is the child of the root of \( s_2 \). Then

\[
\eta X.\psi \sigma(t) = (\psi \sigma(t_1))^n_{X}^{n+1} = (\psi \sigma(t_1))_{X}((\psi \sigma(t_1))^n_{X})
\]
and

\[
(\psi(\eta X. \psi, \sigma))(s) = ((\psi[X \mapsto \eta X. \psi]\sigma))(s)
= (\psi\sigma[X \mapsto \eta X. \psi\sigma])(s)
= (\psi\sigma(s_1))(\eta X. \psi\sigma(s_2))
= (\psi\sigma(s_1))X(\eta X. \psi\sigma(s_2))
= (\psi\sigma(s_1))X(\psi\sigma(s_3)^\Gamma),
\]

where the fifth equality holds since \(s_2(\epsilon, \eta X. \psi\sigma) = n\). As \(\psi\sigma\) does not contain \(\eta X. \psi\sigma\) and \(s\) and \(t\) agree on all other fixpoint literals, \(t_1 = s_1 = s_3\), which finishes the proof. ▮

**Definition 41** (Realization). The set of \(K\)-realized nodes is

\[
M = \{(\Gamma, d) \mid \Gamma \in N, d \subseteq d(\Gamma), \exists x \in W. \forall \phi. \Gamma \vdash_{PL} \phi \Rightarrow x \models_W \phi\}.
\]

**Definition 42** (Rank). The *rank* \(rk(\psi)\) of a formula \(\psi\) is the depth of nesting of modal operators in it. Given a set \(d\) of deferrals and a state \(x \in W\) such that \(x \models \alpha\sigma\) for each \((\alpha, \sigma) \in d\), we put

\[
rk(d, x) = \max\{rk((\alpha, \sigma), x) \mid (\alpha, \sigma) \in d\}.
\]

For \((\Gamma, d') \in M\), we put

\[
rk(d, \Gamma) = \min\{rk(d, x) \mid \forall \phi. \Gamma \vdash_{PL} \phi \Rightarrow x \models \phi\}.
\]

**Corollary 43.** Let \(x \models (\eta X. \psi)\sigma\). Then

\[
rk(\unf((X, (\eta X. \psi, \sigma), x))) \geq rk(\unf((\psi, (\eta X. \psi, \sigma), x))).
\]

**Proof.** Let \(t\) and \(s\) be the least unfolding trees for \(X(\eta X. \psi, \sigma) = \eta X. \psi\sigma\) and \(\psi(\eta X. \psi, \sigma)\) such that \(x \models \eta X. \psi\sigma(t)\) and \(x \models (\psi(\eta X. \psi, \sigma))(s)\), respectively. Lemma \[40\] finishes the proof as it states that \(s\) can be chosen to agree with \(t\) on all least fixpoint literals except for \(\eta X. \psi\sigma\) for which we have \(t(\epsilon, \eta X. \psi\sigma) = s(\kappa, \eta X. \psi\sigma) + 1\) for any suitable \(\kappa\); thus \((\psi(\eta X. \psi, \sigma))(s)\) has a rank that is not greater than the rank of \(\eta X. \psi\sigma(t)\), as required. ▮

**Lemma 44.** For all deferrals \((\alpha, \sigma)\) and all unfolding trees \(t_{\alpha\sigma}\),

\[
[\alpha\sigma(t_{\alpha\sigma})] \subseteq [\alpha\sigma].
\]

**Proof.** This lemma follows by induction over \(\alpha\sigma\) from \([\psi_X^n] \subseteq [\mu X. \psi]\). ▮

**Definition 45** ((Pseudo-)Theory). We define the pseudo-theory \(\Gamma \vdash_{PL}\) of a node \(\Gamma \in N\) as

\[
\Gamma \vdash_{PL} = \{ \phi \in F \mid \Gamma \vdash_{PL} \phi\},
\]

and the theory \(x \models\) of a state \(x \in W\) as

\[
x \models = \{ \phi \in F \mid x \models \phi\}.
\]

Given a node \(\Gamma \in N\) and a state \(x \in W\), we write \(\Gamma \subseteq x\) if \((\Gamma \vdash_{PL}) \subseteq (x \models)\), equivalently \(\Gamma \subseteq (x \models)\).
Recall that $M$ denotes the set of $K$-realized nodes (cf. Definition 46) and note that

$$M = \{ (\Gamma, d) \mid \Gamma \in N, d \subseteq d(\Gamma), \exists x \in W. \Gamma \subseteq x \}.$$  

**Lemma 46.** Let $x \in W$, $(\Delta, d) \in M \cap S \times S$ and $\Delta \subseteq x$. Given a set $B_{[a]a} \subseteq W$ for each $(\alpha)\alpha \in \Delta$, a set $B_{[a]a} \subseteq W$ for each $[a]a \in \Delta$ such that

$$\langle (\alpha)\alpha \in \Delta \Rightarrow \exists y \in R_\alpha(x).y \in B_{[a]a} \rangle$$

$$\langle [a]a \in \Delta \Rightarrow \forall y \in R_\alpha(x).y \in B_{[a]a} \rangle,$$

and a modal rule

$$(\Gamma, [a]\psi_1, \ldots, [a]\psi_n, (a)\psi/\psi_1, \ldots, \psi_n, \psi)$$

with $\Gamma, [a]\psi_1, \ldots, [a]\psi_n, (a)\psi = \Delta$, we have $\{\psi_1, \ldots, \psi_n, \psi\} = \Theta \subseteq N$ and there is a state $z \in W$ such that $\Theta \subseteq z$ and $z \in \cap_{1 \leq i \leq n} B_{[a]a} \cap B_{[a]a}$.

**Proof.** As $N$ is fully expanded, $\{\psi_1, \ldots, \psi_n, \psi\} = \Theta \subseteq N$. As $(a)\psi \in \Delta$, there is by assumption a state $z \in B_{[a]a}$. Since $[a]\psi \in \Delta$ for $1 \leq i \leq n$, we have by assumption that $z$ is also contained in $\cap_{1 \leq i \leq n} B_{[a]a}$, as required.

**Definition 47.** We denote by $u_f(\phi)$ and $u_q(\phi)$ the numbers of unguarded occurrences of fixpoint and propositional operators in $\phi$, respectively.

**Proof of Theorem 12.** It suffices to show that $K$-realized nodes are successful, i.e. $M \subseteq E_S = \nu(X \rightarrow \mu(\hat{f}_X))$. We use coinduction, i.e. show that $M$ is a postfixpoint of $(X \rightarrow \mu(\hat{f}_X))$, i.e. $(\Delta, d) \in \mu(\hat{f}_M)$ for all $(\Delta, d) \in M$. We show the more general property that for all $\Delta \subseteq N$ and all $d \subseteq d(\Delta)$, $(\Delta, d) \in \mu(\hat{f}_M)$ and proceed by induction over the triple $(\text{rk}(d, \Delta), u_f(\Delta), u_q(\Delta))$ in lexicographic order $<_l$. If $d = \emptyset$, then $(\Delta, d) \in \hat{f}_M(\mu(\hat{f}_M))$ if $(\Delta, d) \in f(\mu)$ which is implied by Lemma 46. If $d \neq \emptyset$, $\text{rk}(d, \Delta) > 0$. We distinguish two cases:

- If $\Delta$ is not a state node, then let $y$ be a state with $\Delta \subseteq y$. We note that $u_f(\Delta) > 0$ or $u_q(\Delta) > 0$. Let $\Delta = \{\phi_1, \ldots, \phi_d\}$. In order to show that $(\Delta, d) \in \hat{f}_M(\mu(\hat{f}_M))$, we consider any non-modal rule that matches $\Delta$ and show that it has a conclusion $\Theta$ such that $(\Theta, d_{\Delta \rightarrow \Theta}) \in \mu(\hat{f}_M)$. To this end we distinguish upon the rule that is being applied.

- $($**,$(\downarrow)$): The rules are not applicable to $\Delta$ since $\Delta \subseteq y$ and $y \nmid \bot$ as well as $y \nmid p \lor \neg p$ for any $p$.

- $($**,$(\land)$): Then there is a formula $\phi_i = \psi_1 \land \psi_2 \in \Delta$ and the rule leads – since $N$ is fully expanded – to the node $\Theta \in N$ with

  $$\Theta = \{\phi_1, \ldots, \phi_i-1, \psi_1, \psi_2, \phi_{i+1}, \ldots, \phi_d\}.$$  

  We note that $u_f(\Theta) = u_f(\Delta)$, $u_q(\Theta) < u_q(\Delta)$ and $\Theta \subseteq y$, i.e. $(\Theta, d_{\Delta \rightarrow \Theta}) \in M$; also $\text{rk}(d_{\Delta \rightarrow \Theta}, \Theta) \leq \text{rk}(d, \Delta)$. By the induction hypothesis, $(\Theta, d_{\Delta \rightarrow \Theta}) \in \mu(\hat{f}_M)$, as required.

- $($**,$(\lor)$): Then there is a formula $\phi_i = \psi_1 \lor \psi_2 \in \Delta$ and the rule leads – since $N$ is fully expanded – to the two nodes $\Theta_1, \Theta_2 \in N$ with

  $$\Theta_1 = \{\phi_1, \ldots, \phi_i-1, \psi_1, \phi_{i+1}, \ldots, \phi_d\}$$  

  and

  $$\Theta_2 = \{\phi_1, \ldots, \phi_i-1, \psi_2, \phi_{i+1}, \ldots, \phi_d\}.$$
We note that \( u_f(\Theta_1) = u_f(\Theta_1) = u_p(\Delta_1), \) \( u_p(\Theta_1) < u_p(\Delta_1) \) and \( u_p(\Theta_2) < u_p(\Delta_2) \); also \( \Theta_1 \subseteq y \mid= \Theta_2 \subseteq y \mid= \) so that there is an \( i \in \{1, 2\} \) with \( \Theta_i \subseteq y, \) i.e. with \( (\Theta_i, \Delta_{\Theta_i}) \in M; \) furthermore, \( \rk(d_{\Delta_{\Theta_i}}, \Theta_i) \leq \rk(d, \Delta). \) By the induction hypothesis, \( (\Theta_i, \Delta_{\Theta_i}) \in \mu(\tilde{f}_M), \) as required.

\( \eta \): Then there is a formula \( \phi_i = \eta X.\psi \in \Delta \) and the rule leads – since \( N \) is fully expanded – to the node \( \Theta \in N \) with

\[
\Theta = \{ \phi_1, \ldots, \phi_{i-1}, \psi[\Delta \Rightarrow \eta X.\psi], \phi_{i+1}, \ldots, \phi_n \}.
\]

We note that \( u_f(\Theta) < u_f(\Delta) \) and \( \Theta \subseteq y \) so that \( (\Theta, \Delta_{\Theta}) \in M. \) Let \( \chi \) abbreviate \( \eta X.\psi; \) if \( \eta = \nu, \chi \) is not induced by any deferral from \( d \) so that \( \rk(d_{\Delta_{\Theta}}, \Theta) = \rk(d, \Delta). \) If \( \eta = \mu, \) then we show that \( \rk(d_{\Delta_{\Theta}}, \Theta) \leq \rk(d, \Delta). \) Notice that we can choose a sequence \( \sigma = [X_1 \mapsto \chi_1]; \ldots; [X_n \mapsto \chi_n] \) that sequentially unfolds some eventuality \( \chi_n \) and a formula \( \psi_1 \) such that \( \mu X.\psi_1 \trianglelefteq \chi_1 \) and \( \psi_1 \sigma = \psi; \) then \( (X, [X \mapsto \mu X.\psi_1]; \sigma) \) is a deferral that induces \( \chi = \mu X.\psi_1 \sigma \) and \( (\psi_1, [X \mapsto \mu X.\psi_1]; \sigma) \) is a deferral that induces \( \psi_1 \sigma = \psi\).

We define \( \rk((X, [X \mapsto \mu X.\psi_1]; \sigma), y) \) \( = \rk((\mu X.\psi_1; \sigma), y) \) which implies – since \( (X, [X \mapsto \mu X.\psi_1]; \sigma) \) is the only deferral that changed from \( \Delta \) to \( \Theta \) – that we have \( \rk(d_{\Delta_{\Theta}}, \Theta) \leq \rk(d, \Delta). \) The induction hypothesis implies \( (\Theta, \Delta_{\Theta}) \in \mu(\tilde{f}_M), \) as required.

If \( \Delta \) is a state node, then let \( x \) be a state with \( \Delta \subseteq x \) and \( \rk(d, \Delta) = \rk(d, x). \) In order to show that \( (\Delta, d) \in \tilde{f}_M(\mu(\tilde{f}_M)) \), we show that for all modal rules that match \( \Delta, \) there is a conclusion \( \Theta \) of the rule application with \( (\Theta, \Delta_{\Theta}) \in \mu(\tilde{f}_M). \) Consider any \((a)\)-rule

\[
(\Gamma, [a]\psi_1, \ldots, [a]\psi_n, (a)\psi \mid \psi_1, \ldots, \psi_n, \psi)
\]

with \( \Delta = \Gamma, [a]\psi_1, \ldots, [a]\psi_n, (a)\psi. \) We define for each \((a)\beta, \sigma \in d \) the set \( B_{(a)\beta, \sigma} = \{ \beta(\sigma(t)) \mid \unf((a)\beta, \sigma), x = (a)\beta(\sigma(t)) \}. \) We also define for each \((a)\beta, \sigma \in d \) the set \( B_{(a)\beta, \sigma} = \{ \beta(\sigma(t)) \mid \unf((a)\beta, \sigma), x = (a)\beta(\sigma(t)) \}. \) By Fact \( \beta(\sigma(t)) \subseteq \beta(\sigma) \). For each \((a)\beta \in \Delta \) that is not induced by a deferral from \( d \), we define \( B_{(a)\beta} = \{ \beta \}, \) and analogously we put \( B_{(a)\beta} = \{ \beta \} \) for each \((a)\beta \in \Delta \) that is not induced by a deferral from \( d \). Note how for each \((a)\beta \in \Delta \), there is an \( y \in R_\alpha(x) \) with \( y \in B_{(a)\beta}. \) If \( (a)\beta \in \Delta \) is not induced by a deferral, note that \( \Delta \subseteq x \) so that \( x \in [(a)\beta]. \) Otherwise, note that \( B_{(a)\beta} = \{ \beta(\sigma(t)) \mid x \in [(a)\beta(\sigma(t))] \} \) which is the case iff there is a \( y \in R_\alpha(x) \) with \( y \in \{ \beta(\sigma(t)) \} = B_{(a)\beta}, \) as required. For each \((a)\beta \in \Delta \), one shows analogously that for all \( y \in R_\alpha(x) \), \( y \in B_{(a)\beta}. \) Thus by Lemma \( \{ \psi_1, \ldots, \psi_n, \psi \} = \Theta \in M \) and there is a state \( z \in W \) with \( \Theta \subseteq z \) such that \( \bigcap_{1 \leq i \leq n} B_{(a)\psi_i} \cap B_{(a)\psi_i} \). The induction hypothesis implies \( (\Theta, \Delta_{\Theta}) \in \mu(\tilde{f}_M) \) if \( \rk(d_{\Delta_{\Theta}}, \Theta) < \rk(d, \Delta). \) We conclude ourselves that indeed \( \rk(d_{\Delta_{\Theta}}, \Theta) \leq \rk(d_{\Delta_{\Theta}}, y) \) \( < \rk(d, x) \) \( = \rk(d, \Delta). \) Recall that \( \rk(d_{\Delta_{\Theta}}, y) \) \( = \max \{ \rk(\unf((\alpha, \sigma), y)) \mid (\alpha, \sigma) \in \Delta_{\Theta} \} \). Take any \((\alpha, \sigma) \in \Delta_{\Theta} \) for which \( \rk(\unf((\alpha, \sigma), y)) \) \( = \rk(d_{\Delta_{\Theta}}, y) \) and consider \((\alpha, \sigma) \in d \) (the case for \((\alpha, \sigma) \in d \) is analogous, using the upcoming argumentation); if no such deferral exists, \( d_{\Delta_{\Theta}} = \emptyset \) and Lemma \( \finishes the proof. \) Otherwise let \( p = \rk(\unf((\alpha, \sigma), x)) \) and let \( q = \rk(\unf((\alpha, \sigma), y)) \). Recall that \( y \in B_{(a)\alpha} = \{ \alpha(\sigma(t)) \} \) so that \( \rk(\unf((\alpha, \sigma), y)) \leq \rk(\alpha(\sigma(t))) \) and hence \( q < p. \) Thus \( \rk(\unf((\alpha(\sigma)), y)) \leq \rk(\unf(((a)\alpha(\sigma)), x)). \) Hence

\[
\rk(d_{\Delta_{\Theta}}, y) = \rk(\unf((\alpha, \sigma), y))
\]

\[
< \rk(\unf((\alpha, \sigma), x))
\]

\[
\leq \rk(d, x),
\]

as required.
Lemma 48. For each focused node \((\Delta, d) \in M\) and each \(\Sigma \in Cn(\Delta)\), there is a \(\Theta \in \Sigma\) such that \((\Theta, d_{\Delta \rightarrow \Theta}) \in M\).

Proof. Let \((\Delta, d) \in M\) and \(\Sigma \in Cn(\Delta)\). If \(\Delta\) is a state node, \(\Sigma\) contains just the conclusion \(\Theta\) of a modal rule \((\Gamma, [n]E_1, \ldots, [m]E_n, a(\psi/\psi_1, \ldots, \psi_n, \psi := \Theta))\) with \(\Delta = \Gamma, [a]E_1, \ldots, [m]E_n, a(\psi)\). Since \(N\) is fully expanded, \(\Theta \in N\). As \((\Delta, d) \in M\), there is a state \(x\) such that \(x \models [a]E_i\), i.e. there is a state \(y \in R_a(x)\) such that \(y \models \psi\). As \(x \models [a]E_i\), \(y \models \psi\), for \(1 \leq i \leq n\), so that \(\Theta \subseteq x\), showing \((\Theta, d_{\Delta \rightarrow \Theta}) \in M\), as required. If \(\Delta\) is not a state node, just note that for all \(y, y \models \) is closed under propositional breakdown and unfolding of fixpoint literals.

Definition 49. A finite set of formulas \(\Psi\) propositionally entails a finite set \(\Phi\) of formulas (written \(\Psi \vdash_{PL} \Phi\)) if \(\Psi \vdash_{PL} \bigwedge \Phi\).

Proof of Lemma 50. Recall that \(E = E_G\). First note that \(|D| \leq |E| \leq 3^{(|\Theta|)}\). We proceed in two steps: in the first step, we construct a relation \(L \subseteq D \times D\); in the second step, we show that \(L\) is a timed-out tableau.

1. For any \((\Delta, d), (\Delta, d) \in E = \nu(X \mapsto \mu(f_X)) = (X \mapsto \mu(f_X))((\Delta, d) = \mu(f_X)) = (f_E)^n(\emptyset)\) for some \(n\). Let \(Cn(\Delta) = \{\Sigma_1, \ldots, \Sigma_j\}\). If \(n = 0\), \((\Delta, d) \not\in (f_E)^n(\emptyset)\) so that there is nothing to show. If \(n > 0\), \((\Delta, d) \in (f_E)^{n-1}(\emptyset)\). If \(d = 0\), then \((\Delta, d) \in f(E) \cap F\), i.e. there is, for each \(i\), \(\Gamma \in \Sigma_i\) such that \((\Gamma, d_{\Delta \rightarrow \Gamma}) \in E\). Notice that since \(d = 0\), \(d_{\Delta \rightarrow \Gamma} = d(\Gamma)\). As \((\Delta, d) \in (f_E)^n(\emptyset)\), this implies by Lemma [24] that there is a state node \(\Theta_i\) with \(\Theta_i \vdash_{PL} \Gamma\). Notice that \(d(\Theta_i) \vdash_{\Theta_i} d(\Gamma)\). Put \(L(\Delta, d) = \{(\Theta_1, d(\Theta_1)), \ldots, (\Theta_i, d(\Theta_i))\}\). If \(d \neq 0\), \((\Delta, d) \in f((f_E)^{n-1}(\emptyset))\), i.e. there is, for each \(i\), \(\Gamma \in \Sigma_i\) such that \((\Gamma, d_{\Delta \rightarrow \Gamma}) \in (f_E)^{n-1}(\emptyset)\). If \(n = 0\), \(Cn(\Delta) = \emptyset\) and we put \(L(\Delta, d) = \emptyset\). Otherwise Lemma [50] implies that there is a state node \(\Theta_i\) with \(\Theta_i \vdash_{PL} \Gamma\) and a set \(d_i \subseteq d(\Theta_i)\) with \(d_i \vdash_{\Theta_i} d_{\Delta \rightarrow \Gamma}\); for step 2), we note that the Lemma also tells us that \((\Theta_i, d_i) \in (f_E)^{n-1}(\emptyset)\). Put \(L(\Delta, d) = \{(\Theta_1, d_1), \ldots, (\Theta_j, d_j)\}\).

2. We show that \(L\) is a timed-out tableau by proving the stronger property that for all \((\Delta, d) \in D\) and all \(d' \subseteq d(\Delta)\), there is some \(m\) such that \((\Delta, d) \in to(d', m)\). To this end we distinguish two cases. In case a), \(d = d'\), while in case b), \(d \neq d'\). In both cases, \((\Delta, d) \in E = (X \mapsto \mu(f_X)) = (X \mapsto \mu(f_X))((\Delta, d) = \mu(f_X)) = (f_E)^n(\emptyset)\) for some \(n\). If \(d' = 0\), \((\Delta, d) \in to(\emptyset, m) = D\) for any \(m\) and we are done. If \(d' \neq 0\), then we proceed by induction over \(n\). Let \(L(\Delta, d) = \{(\Theta_1, d_1), \ldots, (\Theta_j, d_j)\}\). If \(n = 0\), \(Cn(\Delta) = L(\Delta, d) = \emptyset\) in which case there is nothing to show, or \((\Delta, d) \in f(E) \cap F\), so that \(d = 0\). Considering the latter situation, if we are in case a), \(d' = 0\) and \((\Delta, \emptyset) \in to(\emptyset, m) = D\) for any \(m\) so that we are done. If we are in case b), recall from step 1) that \(d_i = d(\Theta_1), \ldots, d_j = d(\Theta_j)\); we proceed as in case a), having to show that for all \(1 \leq i \leq j\), \((\Theta_i, d_i) \in to(d_i, m)\) for some \(m\). If \(n > 0\), recall from step 1) that \(L(\Delta, d) = \{(\Theta_1, d_1), \ldots, (\Theta_j, d_j)\}\), where \((\Theta_i, d_i) \in (f_E)^{n-1}(\emptyset)\). By the induction hypothesis, \((\Theta_i, d_i) \in to(d(\Theta_i), m)\) for some \(m\), as required.

Thus we have constructed a relation \(L\) over \(D\) which has size at most \(3^{(|\Theta|)}\) and shows it to be a timed-out tableau.

Lemma 50. Given a set \(X \subseteq C_G\) and a focused node \((\Delta, d) \in (f_X)^n(\emptyset)\), there is a state node \(\Theta\) and a set of deferrals \(d' \subseteq d(\Theta)\) such that \(\Theta \vdash_{PL} \Delta, d' \vdash_{\Theta} d\) and \((\Theta, d') \in (f_X)^n(\emptyset)\).

Proof. We proceed by induction over the pair \((u_f(\Delta), u_p(\Delta))\) in lexicographic order \(<_L\). If \(u_f(\Delta) = 0\) and \(u_p(\Delta) = 0\), then \(\Delta\) is a state node so that it suffices to put \(\Theta = \Delta\) and \(d' = d\).
Otherwise $\Delta$ is not a state node so that at least one rule matches $\Delta$. Let $\Sigma \in Cn(\Delta) \neq \emptyset$. Since $\Delta \in (f_X)^n(\emptyset)$, there is a $\Gamma \in \Sigma$ with $(\Gamma, d_{\Delta \rightarrow \Gamma}) \in X \cap (f_X)^{n-1}(\emptyset) \subseteq (f_X)^n(\emptyset)$. Also $d_{\Delta \rightarrow \Gamma} \subseteq d(\Gamma)$ and since $\Gamma$ is obtained from $\Delta$ as a conclusion of a non-modal rule, $\Gamma \vdash_{PL} \Delta$. We note that since $\Gamma \vdash_{PL} \Delta$ and $d \subseteq d(\Delta) \subseteq \Delta$, we have $d_{\Delta \rightarrow \Gamma} \vdash_{\Theta} d$. As the non-modal rule either unfolds one unguarded fixpoint literal which then becomes guarded or removes one unguarded propositional connective from $\Delta$, we have that $(u(\Gamma), u_p(\Gamma)) <_t (u(\Delta), u_p(\Delta))$ so that by induction we have a state node $\Theta$ and a set $d' \subseteq d(\Theta)$ with $\Theta \vdash_{PL} \Gamma$, $d' \vdash_{\Theta} d_{\Delta \rightarrow \Gamma}$ and $(\Theta, d') \in (f_X)^n(\emptyset)$. By transitivity of propositional entailment, $\Theta \vdash_{PL} \Delta$ and $d' \vdash_{\Theta} d$ so that we are done.

\begin{definition}
A formula $\phi$ is \textit{(closed-)respected} if $[\eta X. \psi] \subseteq [\eta X. \psi]$ for each (closed) fixpoint literal $\eta X. \psi \leq \phi$. We extend the notion of pseudo-extension to sets $\Psi$ of formulas by putting $[\Psi] = \bigcap_{\phi \in \Psi} [\phi].$
\end{definition}

\begin{definition}
Given a sequence $\sigma$, we define the interpretation $\hat{\sigma}$ as $\hat{\sigma}(Y) = [\sigma(Y)]$, for each $Y \in \Psi$. We put $[a] \hat{\sigma} = [a]_{\sigma\_}.$
\end{definition}

\begin{lemma}
Let $\psi$ be a closed-respected formula. Then
a) $[\nu X. \psi] \subseteq [\nu X. \psi]$ and
b) $[\mu X. \psi] \subseteq [\mu X. \psi]$.
\end{lemma}

\begin{proof}
For a), we note that $[\nu X. \psi] = \nu[\psi]_X$. Hence we proceed by coinduction, i.e. we show that $[\nu X. \psi] \subseteq [\nu X. \psi] \cup [\mu X. \psi] = \psi \nu X. \psi$. We have $[\nu X. \psi] = \psi [\nu X. \psi] = \psi [\mu X. \psi]$. As $\psi <_f \nu X. \psi$, Lemma 54 finishes the case. For b), notice that
$$[\mu X. \psi] = \psi (\mu X. \psi) = \psi [\mu X. \psi]$$
and that $(\psi, \mu X. \psi)$ is $\mu X. \psi$-deferral. Also $[\mu X. \psi] = [\psi]_{\mu X. \psi}$. Let $v \in [\psi]_{\mu X. \psi}$ and note that by definition of sufficiency (Definition 21), $d(l(v)) \vdash_{PL} \psi \mu X. \psi$. Since $v \in W$ and since $L$ is a timed-out tableau, we have $v \in to(d(\Delta), n)$ for some $n$. By Lemma 55, $v \in [\psi]_{\mu X. \psi}$, as required.
\end{proof}

\begin{lemma}
For all $\sigma = [X_1 \rightarrow \chi_1] \ldots [X_n \rightarrow \chi_n]$ and all closed-respected formulas $\psi$ with $\psi <_f \chi_1$
$$[\psi]_{\sigma} \subseteq [\psi]_{\hat{\sigma}}.$$
\end{lemma}

\begin{proof}
We proceed by induction over $\psi$. If $\psi = \bot$, $\psi = \top$, $\psi = p$ or $\psi = \neg p$, for $p \in P$, then $\psi$ is closed and $[\psi] = [\psi]$ so that $[\psi]_{\sigma} = [\psi] = [\psi]_{\hat{\sigma}}$. If $\psi = X$, then $[X]_{\sigma} = (\sigma(X)) = \hat{\sigma}(X) = [X]_{\hat{\sigma}} = [X]_{\hat{\sigma}}$. If $\psi = \psi_1 \wedge \psi_2$, then $[\psi_1 \wedge \psi_2]_{\sigma} = [\psi_1]_{\sigma} \wedge [\psi_2]_{\sigma} \subseteq [\psi_1]_{\hat{\sigma}} \wedge [\psi_2]_{\hat{\sigma}} = [\psi_1 \wedge \psi_2]_{\hat{\sigma}}$, where the inclusion is by the induction hypothesis. The case for disjunction is analogous. If $\psi = \langle a \rangle \psi_1$, then
$$[[\langle a \rangle \psi_1]_{\sigma}] \subseteq \{ v \mid \exists w \in R_{\psi_1}(v) . w \in [\psi_1]_{\sigma} \}$$
$$\subseteq \{ v \mid \exists w \in R_{\psi_1}(v) . w \in [\psi_1]_{\hat{\sigma}} \}$$
$$= [[a] \psi_1]_{\hat{\sigma}},$$
where the second inclusion follows from the induction hypothesis and the first inclusion holds as follows: Let $v \in [[a] \psi_1]_{\sigma}$ and let $R_{\psi_1}(v) = \{ w_1, \ldots, w_m \}$. There is a $\langle a \rangle$-rule that matches $\langle a \rangle \psi_1$ as well as a number of $[a]$-literals from $l(v)$, i.e. that matches
Let \( \Gamma, [a] \phi_1, \ldots, [a] \phi_n, (a) \psi_\sigma = l(v) \) and has the conclusion \( \{ \phi_1, \ldots, \phi_n, \psi_\sigma \} = \Sigma_j \) for some \( 1 \leq j \leq m \). As \( w_j \in L(v) \) and \( L \) is a timed-out tableau, \( w_j \in [\psi_\sigma] \), as required. If \( \psi = [a] \psi_1 \), then

\[
\llbracket ([a] \psi_1) \sigma \rrbracket \subseteq \{ v \mid \forall w \in R_\alpha(v).w \in [\psi_1] \sigma \} \\
\subseteq \{ v \mid \forall w \in R_\alpha(v).w \in [\psi_1] \bar{\sigma} \} = \llbracket [a] \psi_1 \rrbracket \bar{\sigma},
\]

where the second inclusion follows from the induction hypothesis and the first holds as follows: Let \( v \in \llbracket [a] \psi_1 \sigma \rrbracket \) and let \( R_\alpha(v) = \{ w_1, \ldots, w_m \} \). Either there is no \( \langle (a) \rangle \)-rule that matches \( v \) in which case \( R_\alpha(v) = 0 \) and we are done; or there is a \( \langle (a) \rangle \)-rule matching \( [a] \psi_1 \sigma \) as well as a number of other \( [a] \)-literals and one \( (a) \)-literal from \( l(v) \), i.e., matching \( \Gamma, [a] \phi_1, \ldots, [a] \phi_n, [a] \psi_\sigma, (a) \phi = l(v) \) and having \( \{ \phi_1, \ldots, \phi_n, \psi_\sigma, \phi \} = \Sigma_j \) as conclusion, for some \( 1 \leq j \leq m \). As \( w_j \in L(v) \) and \( L \) is a timed-out tableau, \( w_j \in [\psi_1] \sigma \), as required. If \( \psi = \nu Y. \psi_1 \), then

\[
\llbracket (\nu Y. \psi_1) \sigma \rrbracket = \llbracket (\psi_1[Y \mapsto \nu Y. \psi_1]) \sigma \rrbracket \\
= \llbracket \psi_1(Y \mapsto \nu Y. \psi_1) \sigma \rrbracket \\
\subseteq \llbracket \psi_1(Y \mapsto \nu Y. \psi_1) \rrbracket \sigma,
\]

where the inclusion is by the induction hypothesis, showing by conduction that \( \llbracket (\nu Y. \psi_1) \sigma \rrbracket \subseteq \llbracket \nu Y. \psi_1 \rrbracket \bar{\sigma} \), as required. If \( \psi = \mu Y. \psi_1, \mu Y. \psi_1 \) is closed so that \( \llbracket \mu Y. \psi_1 \sigma \rrbracket = \llbracket \mu Y. \psi_1 \rrbracket \subseteq \llbracket \mu Y. \psi_1 \rrbracket \bar{\sigma} \), where the inclusion is by assumption.

\[\blacktriangleleft\]

**Lemma 55.** For all closed-respected deferrals \( \delta \), all focused nodes \( v \in D \), all sets of deferrals \( d \subseteq d(l(v)) \) and all \( n \geq 0 \),

\[
\text{if } d \vdash_{l(v)} \delta \text{ and } v \in to(d, n), \text{ then } v \in \llbracket \delta \rrbracket.
\]

Let \( \delta = \alpha \sigma \) and recall that the assumption of the lemma implies that \( v \in \llbracket \alpha \sigma \rrbracket \). We proceed by induction over the triple \( (n, m := u_f(\alpha \sigma), \alpha) \) in lexicographic order \( \prec_l \). Let \( [X \mapsto \mu X. \psi] \) and \( [X_n \mapsto \theta] \) be the first and last substitutions in \( \sigma \), respectively. If \( d = \emptyset \), then \( \vdash_{l(v)} \alpha \sigma \) so that we cannot reach the modal cases in the upcoming case distinction otherwise \( \vdash_{l(v)} (a) \alpha_1 \sigma \) iff \( N(l(v), \theta) \vdash_{PL} (a) \alpha_1 \sigma \) iff \( (a) \alpha_1 \sigma \in N(l(v), \theta) \), where \( (a) \alpha_1 \sigma \) is a \( \theta \)-deferral, which is a contradiction since \( N(l(v), \theta) \) denotes the set of formulas that are not induced by a \( \theta \)-deferral. Analogously, the same holds for \( [a] \alpha_1 \sigma \). If \( u_f(\alpha \sigma) = 0 \), the case that \( \alpha = X \) may not occur. Recall moreover that \( \delta \) is closed-respected, and hence in particular all closed subformulas of \( \alpha \) are respected.

- As \( \alpha = (a) \sigma \) is a deferral, \( \alpha \) is open so that \( \alpha \neq \bot, \alpha \neq \top, \alpha \neq p \) and \( \alpha \neq \neg p \), for \( p \in P \).
- If \( \alpha = Y \), then let \( Y \mapsto \chi_i \) with \( \chi_i = \mu Y. \psi_i \) be the first substitution in \( \sigma \) that touches \( Y \), so that \( \sigma = [X_1 \mapsto \chi_1]; \ldots; [Y \mapsto \chi_i]; [X_{i+1} \mapsto \chi_{i+1}]; \ldots; [X_n \mapsto \chi_n] \), and \( v \in \llbracket \psi_\sigma \rrbracket \), where \( \sigma' = [Y \mapsto \chi_i]; [X_{i+1} \mapsto \chi_{i+1}]; \ldots; [X_n \mapsto \chi_n] \) and \( d \vdash_{l(v)} \psi_\sigma \). Also \( u_f(\psi_\sigma) < m \), \( (\psi_\sigma, \sigma') \) is a deferral and \( v \in to(d, n) \). By the induction hypothesis, \( v \in \llbracket \psi_\sigma \rrbracket = [Y]. \sigma \).

- If \( \alpha = (a) \alpha_1 \), then we have to show that there is a \( w \) such that \( v R_\alpha w \) and \( w \in [\alpha \sigma] \).

Recall that \( v \in \llbracket a \alpha_1 \sigma \rrbracket \) and let \( R_\alpha(v) = \{ w_1, \ldots, w_m \} \subseteq L(v) \). There is a \( \langle (a) \rangle \)-rule that matches \( (a) \alpha_1 \sigma \) as well as a number of \( [a] \)-literals from \( l(v) \), i.e., that matches \( \Gamma, [a] \phi_1, \ldots, [a] \phi_n, (a) \alpha_1 \sigma = l(v) \) and has the conclusion \( \{ \phi_1, \ldots, \phi_n, \alpha_1 \sigma \} = \Sigma_i \) for some \( i \). As \( w_i \in L(v) \) and \( L \) is a timed-out tableau, \( w_i \in \llbracket \phi_1, \ldots, \phi_n, \alpha_1 \sigma \rrbracket \subseteq [\alpha_1 \sigma] \).
Lemma 56. The algorithm consists of a loop which is repeated at most \(a\) times since any of the at most \(a\) nodes from \(N\) has been expanded after at most \(a\) expansion steps. The body of the loop consists of one expansion step and one optional propagation step. Since we are interested in worst-case performance of the algorithm, we ignore the optional propagation step. Since modal, propositional and fixpoint literal expansion is implementable in \(\text{ExpTime}\), the expansion step runs in \(\text{ExpTime}\) as well, which intuitively follows from the fact that propagation computes
fixpoints over $G$ where $|G| \leq 3^n$. We consider the computation of the set $E_G$ and note that analysis of the computation of $A_G$ is analogous. As $E_G = \nu(X \mapsto \mu(\hat{f}_X)) = (X \mapsto \mu(\hat{f}_X))^m(C_G)$ for some $m \leq 3^n$, the computation consists of at most $3^n$ computations of $\mu(\hat{f}_X)$, each for some $X \subseteq C_G$. A single computation of $\mu(\hat{f}_X) = (\hat{f}_X)^o(\emptyset)$ for some $o \leq 3^n$ consists of at most $3^n$ computations of $\hat{f}_X(Y)$, each for some $Y \subseteq C_G$. The computation of $\hat{f}_X(Y)$ checks for each $(\Gamma, d) \in C_G$ whether there is a conclusion $(\Theta, d(\Gamma)) \in X \cap Y$ (or $(\Theta, d(\emptyset)) \in X$, if $d = \emptyset$) for each rule that matches $\Gamma$. Propagation thus runs in time at most $(3^n)^c = 3^{cn}$ for some constant $c$, and, therefore, the algorithm runs in ExpTime; modal expansion can be implemented in time $2^{O(n)}$ in the relational case so that the runtime of the algorithm is bounded by $2^{O(n)}$.

A.4 Details on New Benchmark Formulas in Section 5

We define the formulas $c(x, n)$ by putting $c(x, n) := c_n(x, n)$, where $c_n(x, i)$ is defined recursively as

$$c_n(x, i) = (\neg x_{n-i} \land AX x_{n-i} \land \psi_n(x, i - 1)) \lor (x_{n-i} \land AX \neg x_{n-i} \land c_n(x, i - 1))$$

$$\psi_n(x, i) = (\neg x_{n-i} \lor AX x_{n-i}) \land (x_{n-i} \lor AX \neg x_{n-i}) \land \psi_n(x, i - 1).$$