Bari–Markus property for Dirac operators

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Abstract
We prove the Bari–Markus property for spectral projectors of non-self-adjoint Dirac operators on (0, 1) with square-integrable matrix-valued potentials and some separated boundary conditions.

1 Introduction and main results
In the Hilbert space \( \mathbb{H} := L_2((0, 1), \mathbb{C}^{2r}) \), we study the non-self-adjoint Dirac operator

\[
T_Q := J \frac{d}{dx} + Q
\]

on the domain

\[
D(T_Q) := \left\{ (y_1, y_2) \top \mid y_1, y_2 \in W_2^1((0, 1), \mathbb{C}^r), \ y_1(0) = y_2(0), \ y_1(1) = y_2(1) \right\}.
\]

Here,

\[
J := \frac{1}{i} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix},
\]

\( I := I_r \) is the \( r \times r \) identity matrix, \( q_1, q_2 \in L_2((0, 1), \mathcal{M}_r) \), \( \mathcal{M}_r \) is the set of \( r \times r \) matrices with complex entries and \( W_2^1((0, 1), \mathbb{C}^r) \) is the Sobolev space of \( \mathbb{C}^r \)-valued functions. All functions \( Q \) as above form the set

\[
\Omega_2 := \{ Q \in L_2((0, 1), \mathcal{M}_{2r}) \mid JQ(x) = -Q(x)J \text{ a.e. on } (0, 1) \}
\]

and will be called potentials of the operators \( T_Q \).

The spectrum \( \sigma(T_Q) \) of the operator \( T_Q \) consists of countably many isolated eigenvalues of finite algebraic multiplicities. We denote by \( \lambda_j := \lambda_j(Q), \ j \in \mathbb{Z} \), the pairwise distinct eigenvalues of the operator \( T_Q \) arranged by non-decreasing of their real – and then, if equal, imaginary – parts. For definiteness, we also assume that \( \text{Re} \lambda_0 \leq 0 < \text{Re} \lambda_1 \). As can be proved using the standard technique based on Rouche’s theorem, the numbers \( \lambda_j, \ j \in \mathbb{Z} \), satisfy the condition

\[
\sup_{n \in \mathbb{Z}} \sum_{\lambda_j \in \Delta_n} 1 < \infty \tag{1.1}
\]

and the asymptotics

\[
\sum_{n \in \mathbb{Z}} \sum_{\lambda_j \in \Delta_n} |\lambda_j - \pi n|^2 < \infty, \tag{1.2}
\]
where \( \Delta_n := \{ \lambda \in \mathbb{C} \mid \pi n - \pi/2 < \Re \lambda \leq \pi n + \pi/2 \}, \ n \in \mathbb{Z} \). We then denote by \( P_{\lambda_j} \) the spectral projector of the operator \( T_Q \) corresponding to the eigenvalue \( \lambda_j \) (see [3, Chap.3]). We write
\[
P_n := \sum_{\lambda_j \in \Delta_n} P_{\lambda_j}, \quad n \in \mathbb{Z},
\]
for the spectral projector of \( T_Q \) corresponding to the strip \( \Delta_n \).

In particular, in the free case \( Q = 0 \) one has \( \sigma(T_0) = \{ \pi n \}_{n \in \mathbb{Z}} \). We then write \( P^0_n \) for the spectral projector of the free operator \( T_0 \) corresponding to the strip \( \Delta_n, \ n \in \mathbb{Z} \).

The main result of this paper is the following theorem:

**Theorem 1.1** For every \( Q \in \Omega_2 \), it holds
\[
\sum_{n \in \mathbb{Z}} \| P_n - P^0_n \|^2 < \infty.
\]
Relation (1.3) is called the Bari–Markus property of spectral projectors of the operator \( T_Q \).

In the scalar case \( r = 1 \), the Bari–Markus property for the operator \( T_Q \), as well as for the operators with periodic and anti-periodic boundary conditions, was established in [1] to prove the unconditional convergence of spectral decompositions for such operators. Therein, P. Djakov and B. Mityagin used a technique based on Fourier representations of Dirac operators. This technique was further developed to prove the similar property for Dirac operators with regular boundary conditions in [3]. For Hill operators with singular potentials, the Bari–Markus property was established in [2].

A different and simpler technique based on some convenient representation of resolvents of the operators under consideration was used in [6] to establish the Bari–Markus property for Sturm–Liouville operators with matrix-valued potentials (see [6, Lemma 2.12]). Therein, this result was used to solve the inverse spectral problem for such operators. For the same purpose, the Bari–Markus property was established for self-adjoint Dirac operators with square-integrable matrix-valued potentials in [5].

In the present paper, we use the technique suggested in [6] to establish the Bari–Markus property for non-self-adjoint Dirac operators with square-integrable matrix-valued potentials. This result can be used to study the inverse spectral problems for non-self-adjoint Dirac operators on a finite intervals.

The paper is organized as follows. In the reminder of this sections, we introduce some notations that are used in this paper. In Sects. 2 and 3 we provide some preliminary results and prove Theorem 1.1 respectively.

**Notations.** Throughout this paper, we identify \( \mathcal{M}_r \) with the Banach algebra of linear operators in \( \mathbb{C}^r \) endowed with the standard norm. If there is no ambiguity, we write simply \( \| \cdot \| \) for norms of operators and matrices.

We denote by \( L_2((a, b), \mathcal{M}_r) \) the Banach space of all strongly measurable functions \( f : (a, b) \to \mathcal{M}_r \) for which the norm
\[
\| f \|_{L_2} := \left( \int_a^b \| f(t) \|^2 dt \right)^{1/2}
\]
is finite. We denote by \( G_2(\mathcal{M}_r) \) the set of all measurable functions \( K : [0, 1]^2 \to \mathcal{M}_r \) such that for all \( x, t \in [0, 1] \), the functions \( K(x, \cdot) \) and \( K(\cdot, t) \) belong to \( L_2((0, 1), \mathcal{M}_r) \) and, moreover, the mappings \( [0, 1] \ni x \mapsto K(x, \cdot) \in L_2((0, 1), \mathcal{M}_r) \) and \( [0, 1] \ni t \mapsto K(\cdot, t) \in L_2((0, 1), \mathcal{M}_r) \) are continuous. We denote by \( G_2^+(\mathcal{M}_r) \) the set of all functions \( K \in G_2(\mathcal{M}_r) \) such that \( K(x, t) = 0 \) a.e. in the triangle \( \Omega_- := \{(x, t) \mid 0 < x < t < 1\} \). The superscript \( \top \) designates the transposition of vectors and matrices.
2 Preliminary results

In this section, we obtain some preliminary results and introduce some auxiliary objects that will be used in this paper.

For an arbitrary potential \( Q \in \mathcal{Q}_2 \) and \( \lambda \in \mathbb{C} \), we denote by \( Y_Q(\cdot, \lambda) \in W^1_{2r}((0,1), \mathcal{M}_{2r}) \) the \( 2r \times 2r \) matrix-valued solution of the Cauchy problem

\[
J \frac{d}{dx} Y + QY = \lambda Y, \quad Y(0, \lambda) = I_{2r}.
\]  

We set \( \varphi_Q(\cdot, \lambda) := Y_Q(\cdot, \lambda)Ja^* \) and \( \psi_Q(\cdot, \lambda) := Y_Q(\cdot, \lambda)a^* \), where \( a := \frac{1}{\sqrt{2}} (I, -I) \), so that \( \varphi_Q(\cdot, \lambda) \) and \( \psi_Q(\cdot, \lambda) \) are the \( 2r \times r \) matrix-valued solutions of the Cauchy problems

\[
J \frac{d}{dx} \varphi + Q\varphi = \lambda \varphi, \quad \varphi(0, \lambda) = Ja^*,
\]  

and

\[
J \frac{d}{dx} \psi + Q\psi = \lambda \psi, \quad \psi(0, \lambda) = a^*,
\]

respectively. For an arbitrary \( \lambda \in \mathbb{C} \), we introduce the operator \( \Phi_Q(\lambda) : \mathbb{C}^r \to \mathbb{H} \) by the formula

\[
[\Phi_Q(\lambda)c](x) := \varphi_Q(x, \lambda)c, \quad x \in [0,1].
\]

We set \( s_Q(\lambda) := a\varphi_Q(1, \lambda) \) and \( c_Q(\lambda) := a\psi_Q(1, \lambda) \), \( \lambda \in \mathbb{C} \). The function

\[
m_Q(\lambda) := -s_Q(\lambda)^{-1}c_Q(\lambda)
\]

will be called the "Weyl–Titchmarsh function" of the operator \( T_Q \). Note that in the free case \( Q = 0 \) one has \( s_0(\lambda) = (\sin \lambda)I, \, c_0(\lambda) = (\cos \lambda)I \) and \( m_0(\lambda) = -(\cot \lambda)I \).

The following proposition is a straightforward analogue of Lemma 2.1 in [5]:

**Proposition 2.1** For an arbitrary potential \( Q \in \mathcal{Q}_2 \) it holds:

(i) there exists a unique function \( K_Q \in G^+_2(\mathcal{M}_{2r}) \) such that for every \( x \in [0,1] \) and \( \lambda \in \mathbb{C} \),

\[
\varphi_Q(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x K_Q(x, s)\varphi_0(s, \lambda) \, ds,
\]

where \( \varphi_0(\cdot, \lambda) \) is a solution of (2.2) in the free case \( Q = 0 \);

(ii) there exist unique functions \( f_1 := f_{Q,1} \) and \( f_2 := f_{Q,2} \) from \( L_2((-1,1), \mathcal{M}_r) \) such that for every \( \lambda \in \mathbb{C} \),

\[
s_Q(\lambda) = (\sin \lambda)I + \frac{1}{\sqrt{2}} \int_{-1}^1 e^{i\lambda s}f_1(s) \, ds, \quad c_Q(\lambda) = (\cos \lambda)I + \frac{1}{\sqrt{2}} \int_{-1}^1 e^{i\lambda s}f_2(s) \, ds.
\]

In particular, Proposition 2.1 implies the following corollary:

**Corollary 2.1** For an arbitrary \( Q \in \mathcal{Q}_2 \) and \( \lambda \in \mathbb{C} \),

\[
\Phi_Q(\lambda) = (\mathcal{I} + K_Q)\Phi_0(\lambda),
\]

where \( K_Q \) is the integral operator with kernel \( K_Q \) and \( \mathcal{I} \) is the identity operator in \( \mathbb{H} \).
Using the first formula in (2.3) and repeating the proof of Theorem 3 in [7], one can also derive the following:

**Corollary 2.2** The set of zeros of the entire function $\tilde{s}_Q(\lambda) := \det s_Q(\lambda)$ can be indexed (counting multiplicities) by numbers $n \in \mathbb{Z}$ so that the corresponding sequence $(\xi_n)_{n \in \mathbb{Z}}$ has the asymptotics

$$\xi_{kr+j} = \pi k + \omega_{j,k}, \quad k \in \mathbb{Z}, \quad j = 0, \ldots, r - 1,$$

where the sequences $(\omega_{j,k})_{k \in \mathbb{Z}}$ belong to $\ell_2(\mathbb{Z})$.

Now let $\rho(T_Q)$ denote the resolvent set of the operator $T_Q$.

**Lemma 2.1** For an arbitrary $Q \in \mathcal{Q}_2$ it holds $\rho(T_Q) = \{ \lambda \in \mathbb{C} \mid \ker s_Q(\lambda) = \{0\} \}$ and for each $\lambda \in \rho(T_Q)$,

$$(T_Q - \lambda I)^{-1} = \Phi_Q(\lambda)m_Q(\lambda)\Phi_Q^*(\overline{\lambda})^* + T_Q(\lambda), \quad (2.5)$$

where $T_Q$ is an entire operator-valued function. The spectrum of the operator $T_Q$ consists of countably many isolated eigenvalues of finite algebraic multiplicities.

**Proof.** A direct verification shows that

$$\frac{d}{dx} \left( JY_Q^*(x, \overline{\lambda})^* JY_Q(x, \lambda) \right) = 0.$$

Therefore, taking into account (2.1), we find that $-JY_Q^*(x, \overline{\lambda})^* JY_Q(x, \lambda) = I_{2r}$ for every $x \in [0, 1]$ and thus

$$Y_Q(x, \lambda)JY_Q^*(x, \overline{\lambda}) = J, \quad x \in [0, 1].$$

Since $J = Ja^*a + a^*aJ$, the latter can be rewritten as

$$\varphi_Q(x, \lambda)\psi_Q^*(x, \overline{\lambda}) - \psi_Q(x, \lambda)\varphi_Q^*(x, \overline{\lambda}) = J, \quad x \in [0, 1]. \quad (2.6)$$

Using (2.6), one can verify that for an arbitrary $f \in \mathbb{H}$ and $\lambda \in \mathbb{C}$, the function

$$g(x, \lambda) = [T_Q(\lambda)f](x) := \psi_Q(x, \lambda) \int_0^x \varphi_Q^*(t, \overline{\lambda})^* f(t) \, dt + \varphi_Q(x, \lambda) \int_x^1 \psi_Q^*(t, \overline{\lambda})^* f(t) \, dt$$

solves the Cauchy problem

$$Jy' + Qy = \lambda y + f, \quad y_1(0) = y_2(0). \quad (2.7)$$

Since for every $c \in \mathbb{C}^r$, the function $h(\cdot, \lambda) := \varphi_Q(\cdot, \lambda)c$ solves (2.7) with $f = 0$, it then follows that a generic solution of (2.7) takes the form $y = \varphi_Q(\cdot, \lambda)c + T_Q(\lambda)f$, $c \in \mathbb{C}^r$. If $\lambda \in \mathbb{C}$ is such that the $r \times r$ matrix $s_Q(\lambda) := a^*\varphi_Q(1, \lambda)$ is non-singular, then the choice

$$c = -s_Q(\lambda)^{-1}c_Q(\lambda) \int_0^1 \varphi_Q^*(t, \overline{\lambda})^* f(t) \, dt$$

implies that $ay(1) = 0$, i.e. $y_1(1) = y_2(1)$. Therefore, every $\lambda \in \mathbb{C}$ such that $\ker s_Q(\lambda) = \{0\}$ is a resolvent point of the operator $T_Q$ and for such $\lambda$ it holds

$$(T_Q - \lambda I)^{-1} = \Phi_Q(\lambda)m_Q(\lambda)\Phi_Q^*(\overline{\lambda})^* + T_Q(\lambda).$$

To complete the proof, it remains to observe that the function $y = \varphi_Q(\cdot, \lambda)c$ is a non-zero solution of the problem

$$Jy' + Qy = \lambda y, \quad y_1(0) = y_2(0), \quad y_1(1) = y_2(1)$$
if and only if \( c \in \ker s_Q(\lambda) \setminus \{0\}. \) Since the values of the resolvent of the operator \( T_Q \) are compact operators, it follows that all spectral projectors \( P_{\lambda_j}, j \in \mathbb{Z} \), are finite dimensional. In particular, it then follows (see, e.g., [4, Theorem 2.2]) that all eigenvalues of the operator \( T_Q \) are of finite algebraic multiplicities.

From Lemma 2.1 we obtain that eigenvalues of the operator \( T_Q \) are zeros of the entire function \( \tilde{s}_Q(\lambda) := \det s_Q(\lambda). \) In view of Corollary 2.2 we then arrive at the following:

**Corollary 2.3** For an arbitrary potential \( Q \in \mathcal{Q}_2 \), eigenvalues of the operator \( T_Q \) satisfy the condition (1.7) and the asymptotics (1.3).

Now we can introduce the spectral projectors of the operator \( T_Q \) as explained in the previous section. Formulas (2.4) and (2.5) will serve as an efficient tool to prove Theorem 1.1.

## 3 Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. We start with the following auxiliary lemma:

**Lemma 3.1** For an arbitrary \( \lambda \in \mathbb{C} \), let the operator \( A(\lambda) : L_2((-1, 1), \mathcal{M}_r) \to \mathcal{M}_r \) act by the formula

\[
A(\lambda)f := \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{i\lambda t} f(t) \, dt.
\]

Then for an arbitrary \( f \in L_2((-1, 1), \mathcal{M}_r) \) and \( \lambda \in \mathbb{T}_0 := \{ \lambda \in \mathbb{C} | |\lambda| = 1 \}, \)

\[
\sum_{n \in \mathbb{Z}} \|A(\pi n + \lambda)f\|^2 \leq 9r \|f\|^2_{L_2}.
\]

**Proof.** Let \( f \in L_2((-1, 1), \mathcal{M}_r) \), \( \lambda \in \mathbb{T}_0 \) and \( \|S\|_2 \) denote the Hilbert–Schmidt norm of a matrix \( S \in \mathcal{M}_r. \) Since \( \{ \frac{1}{\sqrt{2}} e^{i\pi n t} \}_{n \in \mathbb{Z}} \) is an orthonormal basis in \( L_2((-1, 1)) \), it follows that

\[
\sum_{n \in \mathbb{Z}} \|A(\pi n)f\|^2 \leq \sum_{n \in \mathbb{Z}} \|A(\pi n)f\|^2_{L_2} = \int_{-1}^{1} \|f(x)\|^2_2 \, dx \leq r \int_{-1}^{1} \|f(x)\|^2_2 \, dx.
\]

Taking into account that \( A(\pi n + \lambda)f = A(\pi n)f_1 \) with \( f_1(t) := e^{i\lambda t} f(t) \) and that \( \|f_1\|_{L_2} < 3\|f\|_{L_2} \), we then arrive at (3.1).

**Remark 3.1** In the notations of the above lemma, formulas (2.5) can be rewritten as

\[
s_Q(\lambda) = (\sin \lambda)I + A(\lambda)f_1, \quad c_Q(\lambda) = (\cos \lambda)I + A(\lambda)f_2.
\]

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Recalling formula (2.5) and the asymptotics (1.2) of eigenvalues of the operator \( T_Q \), we obtain that there exists \( N \in \mathbb{N} \) such that for every \( n \in \mathbb{Z} \) with \(|n| > N\),

\[
P_n := -\frac{1}{2\pi i} \int_{T_n} \Phi_Q(\lambda) m_Q(\lambda) \Phi_Q(\tilde{\lambda})^* \, d\lambda, \quad P_n^0 := -\frac{1}{2\pi i} \int_{T_n} \Phi_0(\lambda) m_0(\lambda) \Phi_0(\tilde{\lambda})^* \, d\lambda,
\]

where \( T_n := \{ \lambda \in \mathbb{C} | |\lambda - \pi n| = 1 \}. \) Therefore, for each \( n \in \mathbb{Z} \) such that \(|n| > N\),

\[
\|P_n - P_n^0\| = \left\| -\frac{1}{2\pi i} \int_{T_n} (\Phi_Q(\lambda) m_Q(\lambda) \Phi_Q(\tilde{\lambda})^* - \Phi_0(\lambda) m_0(\lambda) \Phi_0(\tilde{\lambda})^*) \, d\lambda \right\| \leq \|\alpha_n\| + \|\beta_n\|,
\]

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where

\[ \alpha_n := -\frac{1}{2\pi i} \oint_{\mathbb{T}_n} \Phi_Q(\lambda) (m_Q(\lambda) - m_0(\lambda)) \Phi_Q(\overline{\lambda})^* d\lambda \]  \hspace{1cm} (3.3)

and

\[ \beta_n := -\frac{1}{2\pi i} \oint_{\mathbb{T}_n} (\Phi_Q(\lambda) m_0(\lambda) \Phi_Q(\overline{\lambda})^* - \Phi_0(\lambda) m_0(\lambda) \Phi_0(\overline{\lambda})^*) d\lambda. \]

The theorem will be proved if we show that \( \sum_{|n| > N} \|\alpha_n\|^2 < \infty \) and \( \sum_{|n| > N} \|\beta_n\|^2 < \infty \).

Let us prove the claim for \( (\alpha_n) \) first. Taking into account (3.2), observe that

\[ m_Q(\lambda) - m_0(\lambda) = s_Q(\lambda)^{-1} \left[ (\cot \lambda) A(\lambda) f_1 - A(\lambda) f_2 \right], \]  \hspace{1cm} (3.4)

where \( A(\lambda) \) is from Lemma 3.1. Note that by virtue of the Riemann–Lebesgue lemma, without loss of generality we may assume that

\[ \sup_{|n| > N} \sup_{\lambda \in \mathbb{T}_n} \|A(\lambda)f_1\| \leq \frac{1}{4}. \]

Since for every \( \lambda \in \mathbb{T}_n \) one has \( |\sin \lambda| \geq 1/2 \), in view of the first formula in (3.2) it then holds

\[ \|s_Q(\lambda)^{-1}\| \leq |\sin \lambda|^{-1} (1 - |\sin \lambda|^{-1}\|A(\lambda)f_1\|)^{-1} \leq 4, \quad \lambda \in \mathbb{T}_n, \quad |n| > N. \]

Since \( \cot \lambda \leq \sqrt{3} \) as \( \lambda \in \mathbb{T}_n \), from (3.4) we then obtain that

\[ \|m_Q(\lambda) - m_0(\lambda)\|^2 \leq 64(\|A(\lambda)f_1\|^2 + \|A(\lambda)f_2\|^2), \quad \lambda \in \mathbb{T}_n, \quad |n| > N. \]  \hspace{1cm} (3.5)

Next, taking into account (2.4), observe that for an arbitrary \( Q \in \Omega_2 \) and \( \lambda \in \mathbb{T}_n \) it holds

\[ \|\Phi_Q(\lambda)\| \leq \|I + K_Q\| \|\Phi_0(\lambda)\| \leq 2\|I + K_Q\|. \]  \hspace{1cm} (3.6)

By virtue of the Cauchy–Bunyakovsky inequality we then obtain from (3.3), (3.5) and (3.6) that for every \( n \in \mathbb{Z} \) such that \( |n| > N \),

\[ \|\alpha_n\|^2 \leq C \int_0^{2\pi} (\|A(\pi n + e^{i\theta})f_1\|^2 + \|A(\pi n + e^{i\theta})f_2\|^2) d\theta \]

with some \( C > 0 \). In view of Lemma 3.1 we then obtain that \( \sum_{|n| > N} \|\alpha_n\|^2 < \infty \).

It thus only remains to prove that \( \sum_{|n| > N} \|\beta_n\|^2 < \infty \). For this purpose, take into account (2.4) and observe that

\[ \Phi_Q(\lambda)m_0(\lambda)\Phi_Q(\overline{\lambda})^* - \Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^* = K_Q\Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^* + \Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^* \mathcal{K}_q^* + K_Q\Phi_0(\lambda)m_0(\lambda)\Phi_0(\overline{\lambda})^* \mathcal{K}_q^*. \]

Therefore, \( \beta_n = K_Q P_0^n + [\mathcal{K}_q^* P_0^n]^* + \mathcal{K}_q P_0^n \mathcal{K}_q^* \), and thus the claim will be proved if we show that for an arbitrary \( Q \in \Omega_2 \),

\[ \sum_{|n| > N} \|\mathcal{K}_Q P_0^n\|^2 < \infty. \]  \hspace{1cm} (3.7)

To this end, note that the operator \( \mathcal{K}_Q \) belongs to the Hilbert–Schmidt class \( \mathcal{B}_2 \) and that the sequence \( (P_0^n)_{n \in \mathbb{Z}} \) consists of pairwise orthogonal projectors. Therefore, it holds

\[ \sum_{n \in \mathbb{Z}} \|\mathcal{K}_Q P_0^n\|^2 \leq \sum_{n \in \mathbb{Z}} \|\mathcal{K}_Q P_0^n\|_{\mathcal{B}_2}^2 \leq \|\mathcal{K}_Q\|_{\mathcal{B}_2}^2. \]

Hence (3.7) follows and the proof is complete. \( \square \)
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