Discrete Hirota’s equation in quantum integrable models

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Abstract

The recent progress in revealing classical integrable structures in quantum models solved by Bethe ansatz is reviewed. Fusion relations for eigenvalues of quantum transfer matrices can be written in the form of classical Hirota’s bilinear difference equation. This equation is also known as the completely discretized version of the 2D Toda lattice. We explain how one obtains the specific quantum results by solving the classical equation. The auxiliary linear problem for the Hirota equation is shown to generalize Baxter’s $T$-$Q$ relation.
1 Introduction

Acquiring some experience in classical and quantum integrable systems, one would probably say that these theories hardly have something to do with each other. They do look like different branches of mathematical physics with their own methods, notions and traditions. The correspondence is believed to exist but practically it does not seem to uncover itself to a satisfactory extent.

Actually, the very questions used to be asked in the classical and quantum cases usually did not have common points. So there is no surprise that main ingredients of the theories practically do not intersect. For instance, the inverse scattering method has a very different meaning in the two theories to say nothing about such specific tools as the finite-gap integration technique (from the classical side) or the Bethe ansatz (from the quantum side).

At the same time we would like to believe that such a strong property as integrability may add something important to the correspondence principle. At present it is gradually realized that there should exist a structure common for classical and quantum integrability, or rather that classical and quantum cases should be the two faces of this structure.

Having all this in mind, let us give an outline of a more technical story presented here. These notes review the recent progress (see the first paper in ref. [1]) in uncovering classical integrable structures in quantum models. More attention is paid to some questions which remained obscure or were not very clearly written in [1]. In particular, we try to follow, at least schematically, the whole long way from the simplest quantum $R$-matrix to the classical discrete soliton equations. We hope that in the future the story may be considerably shortened and simplified.

In Sect. 2 we review basic elements of the fusion procedure on the example of the simplest solution to the Yang-Baxter equation with rational dependence on the spectral parameter. The consisteny of the fusion procedure relies on the Yang-Baxter equation for the $R$-matrix. As a result, one obtains a family of quantum transfer matrices (generating functions for commuting hamiltonians) acting in one and the same quantum space. They depend on some (spectral) parameters and commute for all values of these parameters.

These transfer matrices are functionally dependent. They satisfy a number of functional relations (called fusion relations) which are analogues of relations for characters of linear groups but have a more complicated structure. Remarkably enough, this structure is well known in the classical soliton theory in a very different context. It is an integrable discretization of the 2D Toda lattice. In the bilinear form it is known as Hirota’s bilinear difference equation (HBDE). This fact suggests an intriguing link between quantum and classical integrable systems lying much deeper than any kind of a naive ”classical limit”. Let us stress that it is the quantum integrability (i.e. commutativity of the multiparameter family of quantum operators) that allows one to reduce operator relations to classical equations for eigenvalues. In this approach, different quantum states correspond to different solutions of the classical equation satisfying certain boundary and analytic conditions.

Sect. 3 is a brief summary of main facts about HBDE. From the classical viewpoint, HBDE plays the role of a master equation for the majority of known continuous and discrete soliton equations. R.Hirota has shown that HBDE unifies various types of them: the Korteweg-de Vries (KdV) equation, Kadomtsev-Petviashvili (KP) equation, two-dimensional Toda lattice (2DTL), the sine-Gordon (SG) equation, etc as well as their discrete analogues can be obtained from it by different reductions, specifications of parameters and continuum limits. Furthermore, HBDE itself has been shown to possess soliton solutions and Bäcklund transformations. The key elements of the theory are discretized Zakharov-Shabat representation (the zero curvature condition) and the corresponding auxiliary linear problems.

In Sect. 4, we show how the specific quantum notions and methods (such as the transfer matrix, Baxter’s $Q$-operators, the nested Bethe ansatz, etc) are translated into the purely classical language. In other words, solutions of discrete classical nonlinear equations appear in the Bethe ansatz form, a surprising fact that still waits for a deeper understanding.
2 Quantum fusion relations

\textit{R}-matrix

The fundamental rational \( R \)-matrix acting in \( \mathbb{C}^k \times \mathbb{C}^k \) has the form

\[ R(u) = u + 2P, \quad (2.1) \]

where \( P \) is the permutation operator, \( P(x \otimes y) = y \otimes x \), and \( u \) is the spectral parameter. It is convenient to represent \( R(u) \) as a \( k \times k \) matrix in the first space \( \mathbb{C}^k \) (called the \textit{auxiliary space}) with non-commutative entries which in their turn are \( k \times k \) matrices acting in the second space \( \mathbb{C}^k \) (called the \textit{quantum space}):

\[ R(u) = uI + 2\sum_{i,j=1}^{k} E_{ij} \otimes E_{ji}. \quad (2.2) \]

Here \( I \) is the unit matrix and \( E_{ij} \) is the \( k \times k \) matrix such that \((E_{ij})_{mn} = \delta_{im} \delta_{jn}\).

This \( R \)-matrix satisfies the Yang-Baxter equation

\[ R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2). \quad (2.3) \]

Both sides are operators in \( \mathbb{C}^k \times \mathbb{C}^k \times \mathbb{C}^k \). We use the standard notation: \( R_{12}(u) \) acts as \( R(u) \) in the tensor product of the first two spaces and as identity in the third one, similarly for \( R_{13}, R_{23} \). Graphically, eq. (2.3) can be written as follows:

\[ \text{Graphically:} \]

which means that the line \( u_3 \) can be moved over the intersection point of the other two lines.

\textit{T}-matrix

The quantum monodromy matrix (\( T \)-matrix) is the following product of \( R \)-matrices in the auxiliary space \( V_0 = \mathbb{C}^k \):

\[ T(u) = R_{0N}(u - y_N) \ldots R_{02}(u - y_2)R_{01}(u - y_1). \quad (2.4) \]

This is a \( k \times k \) matrix with operator entries. They act in the quantum space \( \otimes_{i=1}^{N} V_i, V_i = \mathbb{C}^k \). Matrix elements of \( R_{0i} \) and \( R_{0j} \) for \( i \neq j \) commute with each other because they operate in different components of the tensor product. The parameters \( y_i \) are arbitrary; sometimes they are called \textit{rapidities}.

The following basic relation is a direct corollary of eq. (2.3):

\[ R_{12}(u - v)T_{13}(u)T_{23}(v) = T_{23}(v)T_{13}(u)R_{12}(u - v), \quad (2.5) \]

where the two auxiliary spaces are \( V_1 = V_2 = \mathbb{C}^k \) and the quantum space is \( V_3 = \otimes_{i=1}^{N} V_i \).

Graphically the \( T \)-matrix looks as follows:
Traditionally, the auxiliary space is attached to the horizontal line while the quantum space is assigned to the vertical lines with rapidities $y_i$.

**Fusion procedure**

Using the $R$-matrix (2.1) as a building block, it is possible to construct more complicated solutions to the Yang-Baxter equation. This is done by multiplying the fundamental solutions and subsequent projection on irreducible representations of $GL(k)$. We give the general scheme of the fusion in the auxiliary space which is of prime importance for our purposes.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a Young diagram with $n$ boxes and $m$ lines $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$, $m \leq k$, $\sum_{i=1}^{m} \lambda_i = n$. Let

$$P_\lambda : \otimes_{i=1}^{n} V_i \to V^{(\lambda)}$$

be the projection operator on the space of the irreducible representation of $GL(k)$ corresponding to $\lambda$. Write in the box with coordinates $(i, j)$ ($i$-th line and $j$-th column) the number

$$s_{(ij)} = u - 2(i - j).$$

For example:

|   |   |   |   |
|---|---|---|---|
| $u$ | $u + 2$ | $u + 4$ | $u + 6$ |
| $u - 2$ | $u$ | $u + 2$ |
| $u - 4$ |   |   |   |

(2.7)

Enumerating all the boxes in the natural order (from left to right in the first line, then continue from left to right in the second line and so on), one gets the ordered sequence of numbers $s_1, s_2, \ldots, s_n$ (in the example above $s_1 = u, s_2 = u + 2, s_3 = u + 4, s_4 = u + 6, s_5 = u - 2, s_6 = u, s_7 = u + 2, s_8 = u - 4$).

The $R$-matrix acting in $V^{(\lambda)} \otimes C^k$ is

$$R^{(\lambda)}(u) = P_\lambda R_{n0}(s_n) \otimes \ldots \otimes R_{20}(s_2) \otimes R_{10}(s_1) P_\lambda.$$  

(2.8)

The fundamental $R$-matrices $R_{i0}(s_i)$ are tensor multiplied in the auxiliary spaces $V_i = C^k$, their entries being multiplied in the common quantum space $V_0 = C^k$ in the indicated order. Graphically,
The important property of $R^{(\lambda)}(u)$ is given by

**Theorem 2.1** $R^{(\lambda)}(u)$ satisfies the Yang-Baxter equation in $V_1 \otimes V_2 \otimes V_3$:

$$R^{(\lambda)}_{12}(u_1 - u_2)R^{(\lambda)}_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R^{(\lambda)}_{13}(u_1 - u_3)R^{(\lambda)}_{12}(u_1 - u_2),$$  \hspace{1cm} (2.9)

where $V_1 = V^{(\lambda)}$, $V_2 = V_3 = \mathbb{C}^k$.

The idea of the proof is simple: to represent the projector $P_\lambda$ as a product of fundamental $R$-matrices. This representation allows one to move the lines using the Yang-Baxter equation (2.3) for fundamental $R$-matrices. However, implementation of this idea in general case is quite involved from technical point of view. We find it useful to outline main steps of the proof since it simultaneously helps to recover the structure of zeros of the fused $R$-matrix, which will be important later.

The following two elementary properties of the fundamental $R$-matrix (2.1) are crucial for the construction:

a) There exists a value of $u$ at which $R(u)$ is proportional to the permutation operator:

$$R_{12}(0) = 2P_{12}. \hspace{1cm} (2.10)$$

Graphically this means that if spectral parameters attached to a pair of crossed lines coincide, one can "eliminate the intersection point" as follows:

b) The $R$-matrix degenerates at the points $\pm 2$:

$$R_{12}(\pm 2) = 4P^{\pm}_{12}, \hspace{1cm} (2.11)$$

where $P^{\pm}_{12}$ projects on symmetric (antisymmetric) tensors in $V_1 \otimes V_2$.

Recall the ordered sequence of spectral parameters $s_1, s_2, \ldots, s_n$ introduced at the beginning of this section. The property b) suggests to try

$$P_{\lambda}^{\text{naive}} \propto \prod_{i < j}^n R_{ij}(s_i - s_j), \hspace{1cm} (2.12)$$

where the structure of the ordered product corresponds to the pattern of complete intersection of $n$ lines carrying spectral parameters $s_i$. The order of successive intersections is irrelevant by virtue of the Yang-Baxter equation. For example ($n = 3$):
One can prove by induction that this prescription does work for "hook diagrams" \( \lambda = (\lambda_1, 1, 1, \ldots, 1) \) (i.e. such that any diagonal contains exactly one box) providing the desired projection operator.

However, in general case we encounter a difficulty: \( P^{\text{naive}}_\lambda \) can be identically zero. Indeed, if \( \lambda = (2, 2) \) we have

\[
\begin{align*}
\text{that is zero since the complementary projectors } P^+ \text{ and } P^- \text{ meet together. A little inspection shows} \\
\text{that this zero arises for the same reason each time when there are at least two lines } i \neq j \text{ such that} \\
s_i = s_j. \text{ Such lines always exist if the Young diagram contains a diagonal with at least two boxes. In} \\
\text{this case a sort of regularization is necessary.}
\end{align*}
\]

Let us introduce a small parameter \( \epsilon \) and modify the sequence \( s_1, s_2, \ldots, s_n \) writing

\[
s'_i(j) = u - 2(i - j) + (i - 1) \epsilon
\]

in the box \((i, j)\) with the same linear order \( s'_1, s'_2, \ldots, s'_n \). Consider the operator

\[
P^{(\epsilon)}_\lambda = \prod_{i,j} R_{ij}(s'_i - s'_j)
\]

with the same order in the product as in eq. (2.12).

**Lemma 2.1** As \( \epsilon \to 0 \),

\[
P^{(\epsilon)}_\lambda = c \epsilon^\kappa P_\lambda + O(\epsilon^{\kappa+1}),
\]

where \( \kappa = \#\{(i_1, j_1), (i_2, j_2) | i_1 < i_2, i_1 - j_1 = i_2 - j_2\} \) is the number of pairs of boxes on the diagonal lines and \( c \) is a numerical constant.

This lemma is sufficient to prove the Yang-Baxter equation for \( R^{(\lambda)}(u) \).

The fusion procedure in the quantum space is made in a similar way. This procedure allows one to define the \( R \)-matrix \( R^{(\lambda)(\mu)}(u) \) acting in \( V^{(\lambda)} \otimes V^{(\mu)} \) for two arbitrary Young diagrams \( \lambda, \mu \). The Yang-Baxter equation then reads

\[
R^{(\lambda)(\mu)}_{12}(u_1 - u_2)R^{(\lambda)}_{13}(u_1 - u_3)R^{(\mu)}_{23}(u_2 - u_3) = R^{(\mu)}_{23}(u_2 - u_3)R^{(\lambda)}_{13}(u_1 - u_3)R^{(\lambda)(\mu)}_{12}(u_1 - u_2),
\]

where \( \mathcal{V}_1 = V^{(\lambda)}, \mathcal{V}_2 = V^{(\mu)}, \mathcal{V}_3 = \mathcal{C}^k \).
Each matrix element of $R^{(\lambda)}(u)$ is a polynomial in $u$. However, the fusion procedure brings to $R^{(\lambda)}(u)$ a number of "trivial" zeros in the sense that they are common for all the matrix elements. In other words, there are some values of $u$ at which $R^{(\lambda)}(u)$ is the zero operator. These zeros should be extracted.

Using the same argument as above, it is easy to see that
\[ R^{(\lambda)}(u - s_i) = 0, \quad i = 2, 3, \ldots, n, \quad (2.17) \]
the remaining $n$-th zero being non-trivial. If $s_i = s_j$ for $i \neq j$, we have zero of higher degree. Extracting the common multiplier, we redefine
\[ R^{(\lambda)}(u) \rightarrow R^{(\lambda)}(u) \left( \prod_{i=2}^{n} s_i \right)^{-1} \quad (2.18) \]
that is a polynomial of degree 1 with operator coefficients.

At last, we specialize this formula for rectangular Young diagrams of length $s$ and height $a$: $\lambda = (s, s, \ldots, s)$ ($a$ times):
\[ R^{(a \times s)}(u) \rightarrow R^{(a \times s)}(u) \left( \prod_{q=0}^{a-1} \prod_{p=1}^{s-1} (u - 2q + 2p) \right)^{-1} \left( \prod_{q=1}^{a-1} (u - 2q) \right)^{-1}. \quad (2.19) \]
For $a = 1$ the last product should be skipped, for $s = 1$ the double product should be skipped.

Fused $T$-matrices

Similarly to eq. (2.4) one can introduce the $T$-matrix with the auxiliary space $V_\mu = V^{(\lambda)}$ and the quantum space $\otimes_{i=1}^{N} V_i$:
\[ T^{(\lambda)}(u) = R^{(\lambda)}_{0N}(u - y_N) \cdots R^{(\lambda)}_{02}(u - y_2) R^{(\lambda)}_{01}(u - y_1). \quad (2.20) \]
It follows from (2.16) and (2.20) that
\[ R^{(\lambda)}_{12}(u - v) T^{(\lambda)}_{13}(u) T^{(\mu)}_{23}(v) = T^{(\mu)}_{23}(v) T^{(\lambda)}_{13}(u) R^{(\lambda)}_{12}(u - v), \quad (2.21) \]
where the two auxiliary spaces are $V_1 = V^{(\lambda)}$, $V_2 = V^{(\mu)}$, and the quantum space is $V_3 = \otimes_{i=1}^{N} V_i$.

Each factor in (2.20) gives an independent contribution to the common polynomial multiplier. Let us explicitly extract this multiplier in the case of rectangular Young diagrams $a \times s$:
\[ T^{(a \times s)}(u) = T^{(a \times s)}(u) \left( \prod_{q=0}^{a-1} \prod_{p=1}^{s-1} \phi(u - 2q + 2p) \right)^{-1} \left( \prod_{q=1}^{a-1} \phi(u - 2q) \right). \quad (2.22) \]
Here
\[ \phi(u) = \prod_{j=1}^{N} (u - y_j) \quad (2.23) \]
and $T^{(a \times s)}(u)$ denotes the renormalized $T$-matrix. For generic polynomial $\phi(u)$ it contains non-trivial "operator" zeros only. If there are pairs $i, j$ such that $y_i - y_j = 2$, $T^{(a \times s)}(u)$ acquires some extra "trivial" zeros and should be renormalized further. This case corresponds to higher representations in the quantum space.

Quantum transfer matrices

The key notion of the theory is quantum transfer matrix\footnote{The name "transfer matrix" came from statistical mechanics on the lattice. It is a bit misleading since the transfer matrix is a scalar in the auxiliary space. The word "matrix" here is related to the quantum space.} obtained by taking trace of $T^{(\lambda)}(u)$ in the auxiliary space:
\[ T^{(\lambda)}(u) = \text{Tr}_{aux} T^{(\lambda)}(u - \lambda_1 + \lambda_1'), \quad (2.24) \]
where \( \lambda' \) denotes the transposed diagram (i.e. reflected with respect to the main diagonal), so \( \lambda'_1 \) is height of the first column of \( \lambda \). The shift of the spectral parameter is introduced for later convenience.

The crucial property of quantum transfer matrices is their commutativity for all values of the continuous spectral parameter \( u \) and the discrete spectral parameter \( \lambda \):

\[
[T^{(\lambda)}(u), T^{(\mu)}(v)] = 0. \tag{2.25}
\]

This fact immediately follows from the Yang-Baxter relation (2.21).

Therefore, we have a family of commuting operators in the quantum space. The point is that hamiltonians of quantum integrable systems belong to this family, so \( T^{(\lambda)}(u) \) are integrals of motion. In other words, \( T^{(\lambda)}(u) \) is a generating function for commuting hamiltonians. Simultaneous diagonalization of transfer matrices \( T^{(\lambda)}(u) \) is one of the corner-stones of the theory. In this way one obtains spectral characteristics of quantum integrable systems.

For the case of rectangular diagrams \( a \times s \) we introduce the special notation

\[
T^{a}_s(u) \equiv T^{(a \times s)}(u) \tag{2.26}
\]

which will be used in the sequel.

Extracting the "trivial" zeros (2.22) amounts to changing the normalization:

\[
T^{a}_s(u) \rightarrow T^{a}_s(u) \left( \prod_{l=0}^{a-1} \prod_{p=1}^{s-1} \phi(u - s - a + 2l + 2p + 2) \prod_{l=1}^{a-1} \phi(u - s - a + 2l) \right)^{-1}. \tag{2.27}
\]

Each eigenvalue of the transfer matrix is a polynomial in \( u \) and all the "trivial" zeros (common for all the eigenvalues) are removed.

**Quantum determinant**

The maximal "external power" of the fundamental \( T \)-matrix \( T(u) \) in the auxiliary space \( \mathbb{C}^k \) is called the quantum determinant of \( T(u) \). More precisely, we put

\[
D^{(k)}(u) \equiv \det_q T(u) \equiv T^{k}_1(u) = T^{(1^k)}(u - 1 + k). \tag{2.28}
\]

The last equality is due to the fact that the fused auxiliary space is one-dimensional. In other words, \( D^{(k)}(u) \) is a scalar in the auxiliary space by definition. The following theorem shows that \( D^{(k)}(u) \) is a scalar in the quantum space too.

**Theorem 2.2** The quantum determinant \( D^{(k)}(u) \) of the fundamental \( T \)-matrix \( T(u) \) with \( k \)-dimensional auxiliary space lies in the center of the algebra generated by matrix elements of \( T(u) \):

\[
[D^{(k)}(u), (T(v))_{ij}] = 0, \quad i, j = 1, 2, \ldots, k. \tag{2.29}
\]

It is given explicitly by the formula

\[
D^{(k)}(u) = \phi(u + k + 1) \prod_{l=1}^{k-1} \phi(u - k - 1 + 2l), \tag{2.30}
\]

where \( \phi(u) \) is defined in eq. (2.23).

The idea of the proof is to use the Yang-Baxter equation which allows one to reduce everything to the quantum determinant of the fundamental \( R \)-matrix. In this case the assertion can be verified by a direct computation.

Let us show how to verify eq. (2.30) without any computation. The last \( k - 1 \) factors contain just the "trivial" zeros coming from eq. (2.27), so we only need to explain the first factor. Note that at \( u = 2 \) the fused \( R \)-matrix contains the following block:
which is zero since it is the projector on the \((k + 1)\)-th external power of \(C^k\). So \(R^{(s)}(2) = 0\) as the whole thing. This yields the first factor in eq. (2.30).

There is a more general formula derived by the same argument,

\[
T_s^k(u) = \phi(u + s + k) \left( \prod_{l=0}^{k-1} \prod_{p=1}^{s-1} \phi(u - s - k + 2l + 2p + 2) \right) \prod_{l=1}^{k-1} \phi(u - s - k + 2l), \quad T_s^0(u) = 1. \quad (2.31)
\]

which will be used in Sect. 4.

**Functional relations**

A combination of the fusion procedure and the Yang-Baxter equation results in numerous functional relations (fusion rules) for quantum transfer matrices.

We illustrate the origin of these relations on the simplest example. Consider the product

\[
T_1^1(u - 2)T_1^1(u) = \text{Tr}_{V_1 \otimes V_2} \left( T_{20}(u - 2) \otimes T_{10}(u) \right), \quad (2.32)
\]

where \(V_1 = V_2 = C^k\) are auxiliary spaces and the quantum space \(V_0\) is arbitrary. Insert the unit matrix \(I = P^+ + P^-\) represented as sum of the complementary projectors inside the trace. Using cyclicity of the trace, the property \((P^\pm)^2 = P^\pm\) and the Yang-Baxter equation, we arrive at the relation

\[
T_1^1(u - 2)T_1^1(u) = T_2^2(u - 1) + T_2^1(u - 1). \quad (2.33)
\]

A more general relation derived in a similar way is

\[
T_s^1(u + 1)T_1^1(u - s) = T^{(\lambda = (s,1))}(u) + T_{s+1}^1(u). \quad (2.34)
\]

Proceeding further, it is possible to show that \(T^{(\lambda)}(u)\) can be expressed through either \(T_s^1(u)\) or \(T_1^a(u)\) only.

**Theorem 2.3** The following determinant formulas hold true:

\[
T^{(\lambda)}(u) = \det_{1 \leq i, j \leq \lambda_1} \left( T_{\lambda_1 - i + j}^1(u - \lambda_1' + \lambda_1 - i + j - 1) \right), \quad (2.35)
\]

\[
T^{(\lambda)}(u) = \det_{1 \leq i, j \leq \lambda_1} \left( T_{\lambda_1' - i + j}^1(u - \lambda_1 + \lambda_1' - i - j + 1) \right), \quad (2.36)
\]

where it is implied that \(T^{(\emptyset)}(u) = 1\) (\(\emptyset\) denotes the empty diagram).

We remind the reader that \(\lambda'\) denotes the transposed diagram. The entries \(T_m^1(u)\) (resp., \(T_m^a(u)\)) of the matrix in eq. (2.35) (resp., (2.36)) are transfer matrices corresponding to the one-line (resp., one-column) diagrams.
These formulas were obtained by V. Bazhanov and N. Reshetikhin. Sometimes they are called quantum Jacobi-Trudi formulas. They are "Yang-Baxterization" of the classical Jacobi-Trudi identities in the sense that the latter are reproduced from the former by forgetting the dependence on $u$.

The specification of eqs. (2.35), (2.36) to rectangular diagrams reads

$$T^a_s(u) = \det \left( T^a_{s+i-j}(u-i-j+a+1) \right), \quad i,j = 1,\ldots,a, \quad T^0_s(u) = 1, \quad (2.37)$$

$$T^a_s(u) = \det \left( T^a_1+i-j(u-i-j+s+1) \right), \quad i,j = 1,\ldots,s, \quad T^0_s(u) = 1. \quad (2.38)$$

These determinant formulas exhibit a very nice structure. Though, they give only a partial solution because $T^1_s(u)$ or $T^0_s(u)$ (entering as "initial data") are still to be determined.

**Bilinear form of the fusion rules**

The fusion rules can be recast into another suggestive form which is in a sense more practically useful since it allows one to obtain a complete solution. At the same time, this form provides a link with classical non-linear integrable equations, which is our main concern here.

It follows from eqs. (2.37), (2.38) that transfer matrices for rectangular Young diagrams obey a closed set of relations among themselves. Using the Jacobi identity for determinants, they can be represented in the following model-independent bilinear form:

$$T^a_s(u + 1)T^a_s(u - 1) - T^a_s(u)T^{a+1}_s(u)T^{a-1}_s(u) = 0. \quad (2.39)$$

Since $T^a_s(u)$ commute at different $u, a, s,$, the same equation holds for all eigenvalues of the transfer matrix, so from now on we can (and will) treat $T^a_s(u)$ as a number-valued function. Note that the 3 variables $u, s, a$ enter almost symmetrically in spite of their very different nature.

Let us mention an analogue of (2.39) for more complicated diagrams. Consider, for example, Young diagrams consisting of two rectangular blocks (i.e. with $a_1$ lines of length $s_1 + s_2$ and the rest $a_2$ lines of length $s_1$) and let $T^{a_1,a_2}_{s_1,s_2}(u)$ be the corresponding transfer matrix. Then it holds

$$T^{a_1,a_2-1}_{s_1,s_2}(u)T^{a_1,a_2+1}_{s_1-1,s_2-1}(u) + T^{a_1-1,a_2-1}_{s_1,s_2+1}(u)T^{a_1+1,a_2+1}_{s_1-1,s_2-1}(u) + T^{a_1-1,a_2+1}_{s_1+1,s_2}(u)T^{a_1+1,a_2-1}_{s_1-1,s_2+1}(u - 1) = T^{a_1,a_2-1}_{s_1+1,s_2}(u + 1)T^{a_1,a_2+1}_{s_1-1,s_2+1}(u - 1) + T^{a_1-1,a_2+1}_{s_1+1,s_2}(u + 1)T^{a_1+1,a_2-1}_{s_1-1,s_2+1}(u - 1). \quad (2.40)$$

In what follows we mainly deal with eq. (2.39) only.

Equation (2.39) is the top of the quantum construction described in this section. Remarkably, this very equation is well known in the theory of classical soliton equations. This is famous Hirota’s bilinear difference equation (HBDE) for the function $T^a_s(u)$ of 3 variables. The reason of this coincidence is hidden somewhere in the representation theory. We believe that there should exist a formulation of the quantum theory such that this coincidence is obvious from the very beginning. However, we are not going to dwell upon this aspect of the problem. Our aim here is more technical. In the sequel, we treat the fundamental relation (2.39) not as an identity but as a fundamental equation and show how to extract from it the specific quantum information. The solution to (classical) HBDE then appears in the form of Bethe equations. In our opinion, this fact is quite remarkable by itself. On the other hand, we anticipate that this approach makes it possible to use some specific tools of classical integrability in quantum problems.

**Comments and references**

1. Throughout this section we use notations and terminology of the quantum inverse scattering method (QISM) developed by the former Leningrad school. At present QISM (= algebraic Bethe ansatz) provides the most convenient framework to analyse the phenomenon of quantum integrability. The early paper [2] remains one of the best reviews of the subject; see also the book [3]. The graphical interpretation is based on the reformulation in terms of the factorized scattering [4].
The fusion procedure was invented by P.Kulish, N.Reshetikhin and E.Sklyanin \[5, 6\] and generalized to elliptic solutions of the Yang-Baxter equation by I.Cherednik \[7\]. The regularization (2.13) was suggested in ref. \[8\], for the general proof of the analogue of Lemma 1.1 in the elliptic case see \[9\].

The notion of quantum determinant was introduced in the paper \[10\] for 2 \times 2 monodromy matrices and generalized to \(k \times k\) matrices in the paper \[11\]. For the complete proof of Theorem 1.2 in the elliptic case see e.g. \[12\].

2. Functional relations for transfer matrices have a long story. Some of them appeared for the first time in the papers \[3, 4\]. The general quantum Jacobi-Trudi formulas (2.35), (2.36) appeared in \[11\], see also \[13\]. The bilinear form of functional relations was first suggested by A.Khumper and P.Pearce in the particular case of \(A_1\)-type models and then generalized to the \(A_{k-1}\)-case by A.Kuniba, T.Nakanishi and J.Suzuki \[14\]. They also suggested bilinear fusion relations for models associated to other Dynkin graphs. Eq. (2.40) and its generalizations to other types of Yang diagrams can be obtained \[1\] from determinant formulas (2.37), (2.38) by means of the Plücker relations.

3. The transfer matrices play the role of quantum characters. They may be also called "quantum Schur functions". Emphasizing this analogy, we use notations and conventions from the book \[15\] on symmetric functions.

4. The results described in this section are valid for elliptic solutions of the Yang-Baxter equation, too (sometimes with minor changes). Instead of Yang’s \(R\)-matrix (2.2) one should start from Belavin’s \(R\)-matrix \[16\] with elliptic dependence on the spectral parameter:

\[
R(u) = \sum R_{i,j}^{i',j'}(u) E_{ii'} \otimes E_{jj'},
\]

(2.41)

where

\[
R_{i,j}^{i',j'}(u) = \frac{\theta \left[ \frac{\alpha}{\beta} \right](\frac{u}{k} + \frac{1}{2})}{\theta \left[ \frac{\alpha}{\beta} \right](\frac{u}{k} + \frac{1}{2}) (2\eta |k\tau) \theta \left[ \frac{\alpha}{\beta} \right](\frac{u}{k} + \frac{1}{2}) (\eta u |k\tau)}
\]

(2.42)

if \(i + j = i' + j' \pmod{k}\) and \(R_{i,j}^{i',j'}(u) = 0\) otherwise. Here

\[
\theta \left[ \frac{\alpha}{\beta} \right](u |\tau) = \sum_{l \in \mathbb{Z}} \exp\left(i\pi\tau(l + \alpha)^2 + 2i\pi(l + \alpha)(u + \beta)\right)
\]

(2.43)

is the Jacobi theta-function with characteristics. There are two parameters: \(\eta\) and \(\tau\). (This form of Belavin’s \(R\)-matrix is taken from refs. \[19\], \[20\].)

Again, the \(R\)-matrix degenerates at \(u = \pm 2\): \(R(-2) = M_- P^-, R(+2) = P^+ M_+\), where \(M_\pm\) are some invertible matrices. The careful analysis shows that the fusion procedure goes through leading to the same basic functional relation (2.39). Formula (2.31) for the quantum determinant also holds with the following function \(\phi(u)\):

\[
\phi(u) = \prod_{j=1}^{N} \theta \left[ \frac{\frac{u}{y_j}}{\frac{u}{2}} \right](\eta(u - y_j) |\tau).
\]

(2.44)

We call functions of this form elliptic polynomials in \(u\). The number \(N\) is degree of the elliptic polynomial.

The elliptic solution (2.41) is no longer \(GL(k)\)-invariant but is still associated with the root system \(A_{k-1}\). The monodromy matrices built out of this \(R\)-matrix form representations of generalized Sklyanin algebras, 2-parameter deformations of \(U(gl_k)\) \[21\], \[22\]. We refer to integrable models with the \(R\)-matrix (2.41) as \(A_{k-1}\)-type models.

3 A classical view on Hirota’s equation

This section is a brief survey of HBDE in the purely classical context. Firstly we list various types of HBDE and show their equivalence. Then discrete versions of the zero curvature representation and auxiliary linear problems are presented and Bäcklund transformations are discussed.
At the first glance, the content of this section has nothing to do with specific tools of the algebraic Bethe ansatz partially described in the previous section. Indeed, the two sections may be read independently of each other. However, in the next section we link them together and show that main elements of the quantum theory can be translated into the language of classical integrability and embedded into the purely classical context.

Equivalent forms of the bilinear equation

A) Hirota’s original form. In Hirota’s original notation it is

\[(z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3)) \cdot T = 0,\]  

(3.1)

where \(z_i\) are arbitrary constants, \(T = T(x_1, x_2, x_3)\) is a function of 3 variables and Hirota’s \(D\)-operator \(D_i \equiv D_{x_i}\) is defined by

\[F(D_x) f(x) \cdot g(x) = F(\partial_y) f(x + y) g(x - y) \bigg|_{y = 0}.\]  

(3.2)

In the more explicit notation eq. (3.1) looks as follows:

\[z_1 T(x_1 + 1) T(x_2 - 1) + z_2 T(x_2 + 1) T(x_2 - 1) + z_3 T(x_3 + 1) T(x_3 - 1) = 0\]  

(3.3)

(here and below we often skip variables that do not undergo shifts).

Note that the 3 variables enter in a symmetric fashion and the equation is invariant under their permutations (and a simultaneous permutation of \(z_i\)’s). The equation is also invariant under changing the sign of any one of the variables and under the transformation

\[T(x_1, x_2, x_3) \to \chi_0 (x_1 + x_2 + x_3) \chi_1 (x_2 + x_3 - x_1) \chi_2 (x_1 + x_3 - x_2) \chi_3 (x_1 + x_2 - x_3) T(x_1, x_2, x_3),\]  

(3.4)

where \(\chi_i\) are arbitrary functions.

The transformation

\[T(x_1, x_2, x_3) \to z_1^{-x_1^2/2} z_2^{-x_2^2/2} z_3^{-x_3^2/2} T(x_1, x_2, x_3)\]  

(3.5)

converts eq. (3.3) into the canonical form,

\[T(x_1 + 1) T(x_2 - 1) - T(x_2 + 1) T(x_2 - 1) + T(x_3 + 1) T(x_3 - 1) = 0\]  

(3.6)

which does not contain any free parameters.

After appropriate identification of the variables one recognizes the bilinear fusion relation (2.33). We refer to \(x_1, x_2, x_3\) as direct variables (in contrast to light cone ones, see below).

A’) "Gauge invariant" form:

\[Y(x_1, x_2 + 1, x_3) Y(x_1, x_2 - 1, x_3) = \frac{(1 + Y(x_1, x_2, x_3 + 1))(1 + Y(x_1, x_2, x_3 - 1))}{(1 + Y^{-1}(x_1 + 1, x_2, x_3))(1 + Y^{-1}(x_1 - 1, x_2, x_3))},\]  

(3.7)

where

\[Y(x_1, x_2, x_3) \equiv \frac{T(x_1, x_2, x_3 + 1) T(x_1, x_2, x_3 - 1)}{T(x_1 + 1, x_2, x_3) T(x_1 - 1, x_2, x_3)}\]  

(3.8)

is a gauge invariant quantity: the "gauge" transformation (3.4) does not change it.

B) KP-like form. The equation for a function \(\tau(p_1, p_2, p_3)\) of 3 variables reads

\[z_1 \tau(p_1 + 1) \tau(p_2 + 1, p_3 + 1) + z_2 \tau(p_2 + 1) \tau(p_1 + 1, p_3 + 1) + z_3 \tau(p_3 + 1) \tau(p_1 + 1, p_2 + 1) = 0.\]  

(3.9)

Again, one can eliminate the arbitrary constants by the transformation

\[\tau(p_1, p_2, p_3) \to \left(-\frac{z_2}{z_3}\right)^{p_1 p_2} \left(-\frac{z_2}{z_1}\right)^{p_2 p_3} \tau(p_1, p_2, p_3).\]  

(3.10)
2DTL-like form:

\[
\tau_x(t, \tilde{t}+1)\tau_x(t+1, \tilde{t}) - \tau_x(t, \tilde{t})\tau_x(t+1, \tilde{t}+1) = r\tau_{x+1}(t, \tilde{t}+1)\tau_{x-1}(t+1, \tilde{t}),
\]

(3.11)

where \(\tau_x(t, \tilde{t})\) is a function of the 3 variables and \(r\) is an arbitrary constant. The variables \(t, \tilde{t}\) are called light cone coordinates. Note that in this form the permutation symmetry is lost. However, an analogue

\[
\tau_x(t, \tilde{t}) \rightarrow \chi_0(2x + 2t)\chi_1(2t)\chi_2(2\tilde{t})\chi_3(2x - 2\tilde{t})\tau_x(t, \tilde{t})
\]

(3.12)

sends solutions to solutions. The transformation

\[
\tau_x(t, \tilde{t}) \rightarrow r^{-x^2/2}\tau_x(t, \tilde{t})
\]

(3.13)

sends eq. (3.11) to its canonical form \((r = 1)\).

At last we present the linear substitutions which make canonical forms of equations a), b), c) equivalent.

A)\(\leftrightarrow\) B):

\[
T(x_1, x_2, x_3) = \tau(p_1, p_2, p_3),
\]

\[
x_1 = p_2 + p_3, \quad x_2 = p_1 + p_3, \quad x_3 = p_1 + p_2;
\]

(3.14)

B)\(\leftrightarrow\) C):

\[
\tau(p_1, p_2, p_3) = \tau_x(t, \tilde{t}),
\]

\[
p_1 = t, \quad p_2 = x - \tilde{t}, \quad p_3 = \tilde{t};
\]

(3.15)

A)\(\leftrightarrow\) C):

\[
T(x_1, x_2, x_3) = \tau_x(t, \tilde{t}),
\]

\[
x_1 = x, \quad x_2 = x + t - \tilde{t}, \quad x_3 = t + \tilde{t};
\]

(3.16)

Clearly, these linear substitutions are not unique. All other possibilities can be obtained from the given one by applying a transformation of the form \((x_1, x_2, x_3) \rightarrow (\pm x_{P(1)}, \pm x_{P(2)}, \pm x_{P(3)})\), where \(P\) is a permutation. Using formulas (3.14)-(3.16) one can easily obtain gauge invariant forms of equations B) and C).

**Discrete Zakharov-Shabat representation**

The reformulation of classical nonlinear integrable equations as flatness conditions for a two-dimensional connection is the basic ingredient of the theory. The flatness means that subsequent shifts along any pair of the time flows commute. These conditions are known as Zakharov-Shabat equations or zero curvature representation.

Let us consider a family of difference operators acting in the space of scalar functions of a variable \(u\):

\[
M^{(z)} = e^{\partial_u} - \frac{\tau^t(u)\tau^{t+1}(u+1)}{\tau^{t+1}(u)};
\]

(3.17)

\[
\tilde{M}^{(z)} = -\tilde{z} + \frac{\tau_{u-1}(u)\tau_{u+1}(u+1)}{\tau(u)\tau(u+1)}e^{-\partial_u},
\]

(3.18)

where \(z, \tilde{z}\) are arbitrary parameters, the shift operator \(e^{\partial_u}\) obeys \(e^{\pm \partial_u}f(u) = f(u \pm 1)e^{\pm \partial_u}\), \(\tau(u)\) is a function of \(u\) depending also on a number of extra variables \(t, p, \ldots, \tilde{t}, \tilde{p}, \ldots\) When it is necessary to show the dependence on these variables explicitly, we write, e.g. \(M^{(z)}(t, p, \ldots)\), etc. We call \(M^{(z)}\), \(\tilde{M}^{(z)}\) \(M\)-operators; note that their coefficients are gauge invariant.
Theorem 3.1 The discrete Zakharov-Shabat equations (zero curvature conditions)

\[ M^{(w)}(p, t + 1)M^{(z)}(p, t) = M^{(z)}(p + 1, t)M^{(w)}(p, t), \]

\[ \bar{M}^{(w)}(\bar{p}, \bar{t} + 1)\bar{M}^{(z)}(\bar{p}, \bar{t}) = M^{(z)}(\bar{p} + 1, \bar{t})\bar{M}^{(w)}(\bar{p}, \bar{t}), \]

\[ \bar{M}^{(z)}(t + 1, \bar{t})M^{(z)}(t, \bar{t}) = M^{(z)}(t, \bar{t} + 1)\bar{M}^{(z)}(t, \bar{t}) \]

are equivalent to the following bilinear relations for \( \tau(u) = \tau^{t, \bar{t}, \bar{p}, \cdots}(u) \):

\[ z\tau^{p+1, t}(u)\tau^{p, t+1}(u + 1) - w\tau^{p, t+1}(u)\tau^{p+1, t}(u + 1) + H_1(p, t)\tau^{p+1, t+1}(u)\tau^{p, t}(u + 1) = 0, \]

\[ z\bar{\tau}^{\bar{p}+1, \bar{t}}(u + 1)\bar{\tau}^{\bar{p}, \bar{t}+1}(u) - \bar{w}\bar{\tau}^{\bar{p}, \bar{t}+1}(u + 1)\bar{\tau}^{\bar{p}+1, \bar{t}}(u) + H_2(\bar{p}, \bar{t})\bar{\tau}^{\bar{p}+1, \bar{t}+1}(u + 1)\bar{\tau}^{\bar{p}, \bar{t}}(u) = 0, \]

\[ z\tau^{t, \bar{t}+1}(u)\tau^{t+1, \bar{t}}(u) + H_3(t, \bar{t})\tau^{t, \bar{t}+1}(u)\tau^{t+1, \bar{t}}(u + 1) = (\bar{z})^{-1}\tau^{t, \bar{t}+1}(u + 1)\tau^{t+1, \bar{t}}(u - 1), \]

respectively, with arbitrary \( u \)-independent functions \( H_i \).

The proof consists in straightforward commutation of the \( M \)-operators.

We see that

- eq. (3.22) coincides with HBDE in the KP-like form (3.4) under the identification \( \tau^{p, t}(u) = \tau(p, t, u) \) and

\[ z = z_1, \quad w = -z_2, \quad H_1 = z_3; \]

- eq. (3.23) coincides with HBDE in the KP-like form (3.9) under the identification \( \bar{\tau}^{\bar{p}, \bar{t}}(u) = \tau(\bar{p}, \bar{t}, -u) \) and

\[ \bar{z} = z_1, \quad \bar{w} = -z_2, \quad H_2 = z_3; \]

- eq. (3.24) coincides with HBDE in the 2DTL-like form (3.11) under the identification \( \tau^{t, \bar{t}}(u) = \tau_u(t, \bar{t}) \) and

\[ z\bar{z} = r^{-1}, \quad H_3 = -z. \]

The variables \( t, \bar{t} \) provide a family of commuting flows parametrized by \( z, \bar{z} \). In fact one can realize \( M \)-operators as difference operators in any one of them. The variable in which \( M \)-operators act will be called the reference variable. The functions \( H_i \) are automatically fixed to be the constants if one requires the simultaneous compatibility of all the corresponding zero curvature conditions.

Linearization of HBDE

The zero curvature conditions are equivalent to compatibility of an overdetermined system of linear difference equations. These linear equations are called auxiliary linear problems (ALP). They play a very important role in the theory. Common solutions to ALP (called wave functions) carry the complete information about solutions to the nonlinear equations. All the properties of the latter can be translated into the language of the ALP. This is what we mean by the linearization of HBDE.

The discrete Zakharov-Shabat equations (3.19)-(3.21) imply the compatibility of the linear problems

\[ M^{(z)}\psi^{t, \bar{t}}(u) = -z\psi^{t+1, \bar{t}}(u), \]

\[ \bar{M}^{(z)}\psi^{t, \bar{t}}(u) = -\bar{z}\psi^{t, \bar{t}+1}(u) \]

for any discrete flows labeled by \( z, \bar{z} \). Note that the "eigenvalues" in the r.h.s. can be arbitrary: this is the matter of redefinition of \( \psi^{t, \bar{t}}(u) \). Our choice corresponds to the smooth continuum limit.

More explicitly, eqs. (3.25), (3.26) read (see (3.17), (3.18)):

\[ zV^{t, \bar{t}}(u)\psi^{t, \bar{t}}(u) - \psi^{t, \bar{t}}(u + 1) = z\psi^{t+1, \bar{t}}(u), \]

\[ \bar{z}\psi^{t, \bar{t}}(u) - C^{t, \bar{t}}(u)\psi^{t, \bar{t}}(u - 1) = \bar{z}\psi^{t, \bar{t}+1}(u), \]

where

\[ V^{t, \bar{t}}(u) = \frac{\tau^{t, \bar{t}}(u)\tau^{t+1, \bar{t}}(u + 1)}{\tau^{t+1, \bar{t}}(u)\tau^{t, \bar{t}}(u + 1)}. \]
\[ C^{t,i}(u) = \frac{\tau^{t,i}(u - 1)\tau^{t,i+1}(u + 1)}{\tau^{t,i+1}(u)\tau^{t,i}(u)}. \quad (3.30) \]

These formulas become more symmetric in terms of the "unnormalized" wave function
\[ \rho(u) = \psi(u)\tau(u). \quad (3.31) \]

Making this substitution in (3.25), (3.28), we get:
\[ z\tau^{t+1,i}(u + 1)\rho^{t,i}(u) - \tau^{t+1,i}(u)\rho^{t,i}(u + 1) = z\tau^{t,i}(u + 1)\rho^{t,i+1}(u), \quad (3.32) \]
\[ z\tau^{t,i+1}(u)\rho^{t,i}(u) - \tau^{t,i+1}(u + 1)\rho^{t,i}(u - 1) = z\tau^{t,i}(u)\rho^{t,i+1}(u). \quad (3.33) \]

**Duality**

The ALP (3.32), (3.33) have a remarkable property: they are symmetric under interchanging \( \tau \) and \( \rho \). Furthermore, one may treat them as linear problems for the function \( \tau \), the compatibility condition being a nonlinear equation for \( \rho \). This equation is again HBDE. We refer to this fact as the duality between "potentials" \( \tau \) and "wave functions" \( \rho \).

More precisely, rewriting eqs. (3.27), (3.28) as linear equations for
\[ \tilde{\psi}^{t,i}(u) = \frac{\tau^{t+1,i+1}(u + 1)}{\rho^{t+1,i+1}(u + 1)} = \left( \psi^{t+1,i+1}(u + 1) \right)^{-1}, \]
we have
\[ \left( e^{-\partial_u - z}\tilde{V}^{t,i}(u) \right) \tilde{\psi}^{t,i}(u) = -z\tilde{\psi}^{t-1,i}(u), \quad (3.34) \]
\[ \left( \bar{z} - \tilde{\psi}^{t,i}(u + 1)e^{\partial_u} \right) \tilde{\psi}^{t,i}(u) = \bar{z}\tilde{\psi}^{t,i-1}(u), \quad (3.35) \]
where \( \tilde{V} \) and \( \tilde{C} \) are given by the same formulas (3.29), (3.30) with \( \rho \) in place of \( \tau \). After the replacing \( \rho \to \tau \) the difference operators in the l.h.s. become formally adjoint to the operators (3.17), (3.18) (defined by the rule \( (f(u)e^{k\partial_u})^\dagger = e^{-k\partial_u}f(u) \)). It then follows that the compatibility conditions are described by Theorem 3.1 with \( \tau \) replaced by \( \rho \).

Therefore, passing from a given solution \( \tau \) to \( \rho \) we have got a new solution to HBDE. This is a Bäcklund-type transformation.

**Bäcklund flows**

One may repeat the procedure described above once again starting from \( \rho \) and, moreover, consider a chain of successive transformations of this kind. Let us introduce an additional variable \( m \) to mark steps of the "flow" along this chain and let \( \tau_{m,t}^{t,i}(u) \), \( \rho_{m,t}^{t,i}(u) \) be \( \tau \) and \( \rho \) at \( m \)-th step.

The Bäcklund flow is defined by
\[ \rho_{m,t}^{t,i}(u) = z^u\tau_{m+1,t}^{t,i}(u). \quad (3.36) \]

This means that \( \tau \) at the next step of the "Bäcklund time" \( m \) is put equal to a solution \( \rho \) of the linear equations (3.32), (3.33) (up to the factor \( z^u \)). Then these linear problems become bilinear equations for \( \tau_m \):
\[ \tau_m^{t+1}(u)\tau_{m+1,t}^{t+1}(u + 1) - \tau_m^{t+1}(u + 1)\tau_{m+1,t}^{t+1}(u) + \tau_m^{t,i}(u + 1)\tau_{m+1,t}^{t,i+1}(u) = 0, \quad (3.37) \]
\[ \tau_m^{t,i+1}(u)\tau_{m+1,t}^{t,i+1}(u) - \tau_m^{t,i}(u + 1)\tau_{m+1,t}^{t,i+1}(u - 1) = (z\bar{z})^{-1}\tau_{m+1,t}^{t,i+1}(u + 1)\tau_{m+1,t}^{t,i+1}(u - 1), \quad (3.38) \]
where \( \bar{t} \) (resp., \( t \)) in eq. (3.37) (resp., (3.38)) is skipped.

Similarly, defining the second Bäcklund flow (the Bäcklund time is now denoted by \( \bar{m} \)),
\[ \rho_{\bar{m},t}^{t,i}(u) = (z\bar{z})^{-u}\tau_{\bar{m}+1,t}^{t,i}(u + 1), \quad (3.39) \]
we get from (3.32), (3.33):
\[ \tau_{\bar{m}+1,t}^{t+1}(u)\tau_{\bar{m}+1,t}^{t+1}(u + 1) - \tau_{\bar{m}+1,t}^{t+1}(u + 1)\tau_{\bar{m}+1,t}^{t+1}(u) + \tau_{\bar{m}+1,t}^{t,i}(u + 1)\tau_{\bar{m}+1,t}^{t,i+1}(u) = 0, \quad (3.40) \]
\[
\tau_m^i(u)\tau_{m+1}^{i+1}(u+1) - \tau_m^{i+1}(u)\tau_{m+1}^{i}(u+1) + \tau_m^i(u+1)\tau_{m+1}^{i+1}(u) = 0.
\] (3.41)

In these equations one immediately recognizes different forms of HBDE described by Theorem 3.1. So, we conclude that the Bäcklund flows can be identified with the commuting discrete flows.

Comments and references

1. The bilinear difference equation was suggested by R.Hirota in the paper [23] which summarized his earlier studies on discretization of nonlinear integrable equations [24]-[28]. HBDE (3.6) can be viewed as an integrable discrete analogue of the 2-dimensional Toda lattice.

2. As it was first noticed by T.Miwa [29], discrete Hirota’s equations can be obtained from the continuous KP hierarchy by choosing the time flows to be certain infinite combinations of the standard continuous flows of the hierarchy (Miwa’s transformation). This approach was further developed in the papers [30], [31]. The different forms of HBDE arise when one applies Miwa’s transformation to different continuous hierarchies. The function \( \tau(u,t,\bar{t}) \) can be identified with the \( \tau \)-function of the hierarchies (restricted to a finite number of the discrete flows).

3. The ”gauge invariant” form (3.7) of HBDE is a discrete counterpart of nonlinear integrable equations written in terms of potentials and fields rather than \( \tau \)-functions. Eq. (3.8) is a discrete version of the famous formula \( U(x) = 2\partial_x^2 \log \tau(x) \). Some particular cases of equation (3.7) emerge naturally in thermodynamic Bethe ansatz [32]-[34].

4. An example of the discretized zero curvature representation for eq. (3.6) was given by R.Hirota [23]. In the physical language, the discrete connection is a lattice gauge field. The approach emphasizing the relation to gauge field theories on the lattice was developed by S.Saito and N.Saitoh [36]. We have presented these results in a modified form which makes the theory completely parallel to the 2DTL theory. The bilinear form of Bäcklund transformations was discussed by R.Hirota [27]. The \( \tau \leftrightarrow \rho \) duality was pointed out in ref. [36].

5. Similarly to the 2DTL hierarchy, there exists an infinite hierarchy of bilinear difference equations. For the explicit form of higher equations of the hierarchy see ref. [34]. We remark that the general quantum fusion rules in the bilinear form (see e.g. eq. (2.40)) coincide with the higher HBDE-like equations (after a linear change of variables).

6. In the continuum limit \( (\varepsilon \to \infty, \bar{\varepsilon} \to \infty) \) eqs. (3.27), (3.28) turn into the familiar ALP for the 2DTL [37]:

\[
\partial_t \psi_n = \psi_{n+1} + \partial_t (\log \frac{\tau_{n+1}}{\tau_n}) \psi_n,
\]

\[
\partial_{\bar{t}} \psi_n = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} \psi_{n-1},
\] (3.42)

where \( n \) is identified with \( u \). The compatibility condition is

\[
\partial_t \tau_n \partial_t \tau_n - \tau_n \partial_t \tau_n = \tau_{n+1} \tau_{n-1},
\] (3.43)

which in terms of

\[
\varphi_n(t,\bar{t}) = \log \frac{\tau_{n+1}(t,\bar{t})}{\tau_n(t,\bar{t})}
\]

acquires the form

\[
\partial_t \partial_{\bar{t}} \varphi_n = e^{\varphi_{n-1}} - e^{\varphi_{n+1}} - \varphi_n,
\] (3.44)

the first non-trivial equation of the 2DTL.

4 Bethe ansatz results from the Hirota equation

We are going to show how to reproduce the quantum Bethe ansatz results by solving the classical Hirota difference equation (2.39):

\[
T_{s}^u(u+1)T_{s}^u(u-1) - T_{s+1}^u(u)T_{s-1}^u(u) = T_{s+1}^{u+1}(u)T_{s-1}^{u-1}(u).
\] (4.1)
In general, HBDE has many solutions of very different nature. To extract the solutions of interest, we should first of all specify boundary and analytic conditions for $T^a_s(u)$.

**Boundary conditions in $a$ and $s$**

The values of $T^a_s(u)$ and $T^s_k(u)$ for $A_{k-1}$-type models should be considered as boundary conditions (b.c.). Recall that $T^a_s(u) = 1$ and $T^s_k(u)$ is the quantum determinant given explicitly by eq. (2.31):

$$T^s_k(u) = \prod_{p=0}^{s-1} \left( \phi(u - s + k + 2 + 2p) \prod_{l=1}^{k-1} \phi(u - s - k + 2 + 2l) \right). \quad (4.2)$$

Let us note that $T^s_k(u)$ obeys the discrete d’Alembert equation:

$$T^s_k(u + 1)T^s_k(u - 1) = T^s_{k+1}(u)T^s_{k-1}(u). \quad (4.3)$$

This is easily seen from eq. (4.2). To put it differently, recall that the general solution to the discrete d’Alembert equation is $\chi_+(u + s)\chi_-(u - s)$, i.e. it is factorized into a product of a ”holomorphic” function $\chi_+$ and an ”antiholomorphic” function $\chi_-$. It is possible to represent $T^s_k(u)$ in this form indeed. For instance, in the rational case we have:

$$T^s_k(u) = \frac{\chi_k(u + s)}{\chi_k(u - s)}, \quad \chi_k(u) = 2^{Nk_u/2} \prod_{i=1}^{N} \left( \Gamma\left( \frac{u + k - y_i}{2} + 1 \right) \prod_{l=1}^{k-1} \Gamma\left( \frac{u - k - y_i}{2} + l \right) \right), \quad (4.4)$$

where $\Gamma(u)$ is the gamma-function. Combining this property with the Hirota equation, we arrive at the following b.c.:

$$T^a_s(u) = 0 \quad \text{as} \quad a < 0 \quad \text{or} \quad a > k. \quad (4.5)$$

The b.c. in $s$ reads

$$T^a_s(u) = 0 \quad \text{as} \quad -k < s < 0, \quad \text{and} \quad 0 < a < k. \quad (4.6)$$

The meaning of this condition will be discussed later.

**Analytic conditions in $u$**

A very important condition in models on finite lattices (which follows, eventually, from the Yang-Baxter equation) is that $T^a_s(u)$ for any fixed $a, s, u$ has to be an elliptic polynomial in the spectral parameter $u$ (see (2.44) multiplied possibly by an exponential function.

More precisely, the gauge invariant quantity

$$Y^a_s(u) = \frac{T^a_{s+1}(u)T^a_{s-1}(u)}{T^s_{s+1}(u)T^s_{s-1}(u)} \quad (4.7)$$

has to be an elliptic function of $u$ having $2N$ zeros and $2N$ poles in the fundamental domain. This implies that $T^a_s(u)$ has the general form

$$T^a_s(u) = A^a_s e^{\mu(a,s)u} \prod_{j=1}^{N(a,s)} \theta\left[ \frac{1/2}{1/2} \right] (\eta(u - w^{(a,s)}_j)), \quad (4.8)$$

where $w^{(a,s)}_j$, $A^a_s$, $\mu(a,s)$ do not depend on $u$ (and meet some additional requirements). For models with rational $R$-matrices $T^a_s(u)$ degenerates to a usual polynomial (multiplied by the exponential function). We note that the b.c. (4.5), (4.6) are consistent with the required analytic conditions.
Normalizations

By normalization we mean multiplying $T^a_s(u)$ by the "gauge" factors $\chi_0(u + s + a)\chi_1(u + s - a)\chi_2(u - s + a)\chi_3(u - s - a)$. Boundary and analytic conditions may depend on the gauge. We point out the following three distinguished normalizations.

i) **Det-normalization** was already discussed in detail: $T^0_s(u) = 1$, $T^k_s(u)$ is the quantum determinant (4.2).

ii) **Minimal polynomial (MP) normalization.** The "gauge" invariance (3.4) allows one to remove all zeros from the characteristics $a \pm s \pm u = \text{const}$. These are just the "trivial" zeros brought by the fusion procedure. The minimal polynomial appears in the gauge (2.27). The boundary values at $a = 0, k$ then become:

$$T^0_s(u) = \phi(u - s),$$
$$T^k_s(u) = \phi(u + s + k).$$

(4.9)

Note that in this normalization $T^0$ is an "antiholomorphic" function (i.e. depends on $u + s$ only) while $T^k$ is a "holomorphic" function (i.e. depends on $u - s$ only). The most important feature of the MP normalization is that the elliptic polynomials $T^a_s(u)$ have one and the same degree $N$ for all values $a, s$ (when $T^a_s(u)$ is not identically zero). In the sequel, we use the MP normalization.

iii) **Canonical normalization.** Since the functions $T^0_s(u)$ and $T^k_s(u)$ at the "boundaries" obey the discrete d’Alembert equation, they can be gauged away, that is can be made equal to unity: $T^0_s(u) = T^k_s(u) = 1$. This allows one to simplify the equations for the price of imposing much more complicated analytic properties of solutions. Here we will not discuss this normalization.

**Zero curvature representation**

Let us represent the bilinear fusion relation (4.1) as a discrete zero curvature condition. To do that, we can make use of Theorem 2.1. Consider eq. (3.24) and identify

$$T^a_s(u) = \tau^{t,\bar{t}}(u),$$

(4.10)

where

$$s = -t - \bar{t}, \quad a = u + t - \bar{t}$$

(cf. (3.16)). We have to substitute $\partial_t \rightarrow -\partial_s + \partial_a$, $\partial_{\bar{t}} \rightarrow -\partial_s - \partial_a$ and

$$\partial_u |_{t,\bar{t}=\text{const}} \rightarrow \partial_u |_{a,s=\text{const}} + \partial_a,$$

so the $M$-operators read

$$M(a, s) = e^{\partial_a + \partial_{\bar{a}}} - z \frac{T^a_s(u)T^{a+2}_{s-1}(u+1)}{T^{a+1}_{s-1}(u)T^{a+1}_{s-1}(u+1)},$$

(4.12)

$$\bar{M}(a, s) = -z + \frac{T^{a-1}_{s-1}(u-1)T^{a}_{s-1}(u+1)}{T^{a}_{s-1}(u)T^{a}_{s-1}(u)} e^{-\partial_u - \partial_a}.$$  

(4.13)

In terms of the operators

$$\mathcal{M} = e^{\partial_a - \partial_{\bar{a}}} M(a, s), \quad \bar{\mathcal{M}} = e^{\partial_a + \partial_{\bar{a}}} \bar{M}(a, s)$$

(4.14)

the Zakharov-Shabat equation (3.2) acquires the form of a commutativity condition:

$$[\mathcal{M}, \bar{\mathcal{M}}] = 0.$$  

(4.15)
Linear problems

The commutativity of two operators implies the existence of a common eigenfunction:

\[ \mathcal{M}\Psi^{a,s}(u) = E\Psi^{a,s}(u), \quad \mathcal{M}\Psi^{a,s}(u) = \tilde{E}\Psi^{a,s}(u). \] (4.16)

Explicitly, these equations read

\[ \Psi^{a,s}(u + 1) - z\frac{T_{s-1}^{a}(u)T_{s+1}^{a+1}(u + 1)}{T_{s-1}^{a}(u)T_{s+1}^{a}(u + 1)}\Psi^{a-1,s}(u) = E\Psi^{a,s-1}(u), \] (4.17)

\[- \tilde{z}\Psi^{a,s}(u) + \frac{T_{s-1}^{a}(u - 1)T_{s+1}^{a+1}(u + 1)}{T_{s-1}^{a}(u)T_{s+1}^{a}(u)}\Psi^{a-1,s}(u - 1) = \tilde{E}\Psi^{a-1,s-1}(u). \] (4.18)

Passing to the "unnormalized" wave function,

\[ F_{s}^{a}(u) = T_{s}^{a}(u)\Psi^{a,s}(u), \] (4.19)

we get

\[ T_{s-1}^{a}(u)F_{s}^{a}(u + 1) - zT_{s+1}^{a+1}(u + 1)F_{s-1}^{a-1}(u) = ET_{s}^{a}(u + 1)F_{s-1}^{a}(u), \] (4.20)

\[ T_{s-1}^{a}(u + 1)F_{s}^{a-1}(u - 1) - \tilde{z}T_{s+1}^{a-1}(u)F_{s-1}^{a}(u) = \tilde{E}T_{s}^{a}(u)F_{s-1}^{a-1}(u). \] (4.21)

Now, to identify eq. (4.21) with eq. (4.1) literally, we set z = \tilde{z} = H_{3} = -1. Besides, redefining \( \Psi^{a,s}(u) \rightarrow E^{(s-a)/2}(E)^{(a+s)/2}\Psi^{a,s}(u) \), we can always choose \( E = \tilde{E} = 1 \) without loss of generality. In this way we get the following ALP:

\[ T_{s+1}^{a+1}(u)F_{s}^{a}(u) - T_{s}^{a+1}(u + 1)F_{s+1}^{a}(u - 1) = T_{s}^{a}(u)F_{s+1}^{a+1}(u), \] (4.22)

\[ T_{s+1}^{a}(u + 1)F_{s}^{a}(u) - T_{s}^{a}(u + 1)F_{s+1}^{a}(u - 1) = T_{s+1}^{a}(u + 1)F_{s+1}^{a-1}(u). \] (4.23)

An advantage of the light cone coordinates \( t, \bar{t} \) is that they are separated in the linear problems (compare (3.32), (3.33) with (4.20)). However, in contrast to \( a, s, u \) they do not have an immediate physical meaning.

Due to the duality property (Sect. 3) \( F_{s}^{a}(u) \) obeys the same HBDE:

\[ F_{s}^{a}(u + 1)F_{s}^{a}(u - 1) - F_{s+1}^{a}(u)F_{s-1}^{a}(u) = F_{s+1}^{a+1}(u)F_{s-1}^{a-1}(u). \] (4.24)

We require \( F_{s}^{a}(u) \) to have the same analytic properties as \( T_{s}^{a}(u) \) (though, degree of the elliptic polynomial may be different).

The b.c. (4.3) allows one to impose a similar condition for \( F_{s}^{a}(u) \):

\[ F_{s}^{a}(u) = 0 \quad \text{as} \quad a < 0 \quad \text{or} \quad a > k - 1 \] (4.25)

so that the number of non-zero functions \( F \) is one less than the number of \( T \)'s. The functions \( F_{s}^{a} \) at the ends of the Dynkin graph \( a = 0, k - 1 \) have a very special form. From the second equation of the pair (4.21) at \( a = 0 \) and from the first one at \( a = k - 1 \) it follows that \( F_{s}^{0}(u) \) (respectively, \( F_{s}^{k-1}(u) \)) depends on one light cone variable \( u - s \) (resp., \( u + s \)). We introduce a special notation for them:

\[ F_{s}^{0}(u) = Q_{k-1}(u - s), \quad F_{s}^{k-1}(u) = \bar{Q}_{k-1}(u + s). \] (4.26)

Furthermore, it can be shown that the important condition (4.9) relates the functions \( Q \) and \( \bar{Q} \) as follows:

\[ \bar{Q}_{k-1}(u) = Q_{k-1}(u + k - 1). \] (4.27)

Therefore, the analytic properties and b.c. for \( F_{s}^{0}(u) \) are the same as for \( T_{s}^{a}(u) \) under a substitution \( \phi(u) \) by \( Q_{k-1}(u) \). The only change is a reduction of the Dynkin graph: \( k \rightarrow k - 1 \). Using this property, one can successively reduce the \( A_{k-1} \)-problem up to \( A_{1} \). Below we use this trick to derive \( A_{k-1} \) ("nested") Bethe ansatz equations.
Nested Bethe ansatz and B"acklund flows

Quantum integrable models with internal degrees of freedom can be solved by the nested (hierarchical) Bethe ansatz method. The method consists essentially in integration over a part of degrees of freedom by an ansatz of Bethe type, the effective Hamiltonian being again integrable. Repeating this step several times, one reduces the model to an integrable model without internal degrees of freedom which is solved by the usual Bethe ansatz.

The classical face of this scheme is a chain of B"acklund transformations, i.e. passing from solutions of the non-linear equation to (properly normalized) solutions of the auxiliary linear problems discussed in Sect. 3.

To elaborate the chain of these transformations, let $m = 0, 1, \ldots, k$ mark steps of the flow $A_{k-1} \to A_1$ and let $F_{a,m}^s(u)$ be a solution to the linear problem at $(k-m)$-th level. In this notation, $F_{a,k}^s(u) = T_a^s(u)$ and $F_{a,k-1}^s(u) = F_a^s(u)$ is the corresponding wave function. For each level $m$ the function $F_{a,m}^s(u)$ obeys HBDE of the form (4.22) with the b.c.

\[ F_{a,m}^s(u) = 0 \quad \text{as} \quad a < 0 \quad \text{or} \quad a > m. \]  

(4.26)

The first ($a = 0$) and the last ($a = k-1$) components of the vector $F_{a,m}^s(u)$ obey the discrete d’Alembert equation and under the condition (4.24) are "antiholomorphic" and "holomorphic" functions respectively. We denote them as follows:

\[ F_0^m(u) \equiv Q_m(u-s), \quad F_m^m(u) \equiv \bar{Q}_m(u+s), \]  

(4.27)

where it is implied that $Q_k(u) = \phi(u)$. Furthermore, it can be shown that the relation

\[ \bar{Q}_m(u) = Q_{m-m}^m(u) \]  

(4.28)

can be imposed simultaneously for all $1 \leq m \leq k$.

The linear problems (4.21) at level $m$,

\[ F_{s+1}^{a+1,m+1}(u)F_{a,m}^s(u) - F_{a+1,m+1}^s(u+1)F_{s+1}^{a,m}(u) = F_{a,m}^{s+1}(u)F_{s+1}^{a+1,m}(u), \]  

(4.29)

\[ F_{a,m}^{s+1}(u+1)F_{a,m}^s(u) - F_{a,m}^{a+1,m+1}(u+1)F_{a+1,m}(u) = F_{a,m}^{s+1,m+1}(u+1)F_{a+1,m}(u), \]  

(4.30)

look as bilinear equations for a functions of 4 variables. However, eq. (4.29) (resp., eq. (4.30)) leaves the hyperplane $u + s - a = \text{const}$ (resp., $u - s - a = \text{const}$) invariant, and actually depends on three variables.

Restricting the variables in eq. (4.29) to the hyperplane $u + s - a = 0$, i.e. setting

\[ \tau_u(t, a) \equiv F_{a-m}^{a,k-m}(u) \]  

(4.31)

we reduce it to the 2DTL-like form of HBDE (3.11) for $\tau_u(m, a)$ with $m$ and $a$ being the light cone coordinates. The b.c. is

\[ \tau_u(m, 0) = Q_{k-m}(2u), \quad \tau_u(m, k-m) = \bar{Q}_{k-m}(m-k) = \text{const}. \]  

(4.32)

A similar equation can be obtained from the second linear problem (4.30). They are nothing else than eqs. (3.38), (3.41) for the B"acklund flows.

It is convenient to visualize this array of $\tau$-functions on a diagram; here is an example for the $A_3$-case ($k = 4$):

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & Q_1(u-s) & \bar{Q}_1(u+s) & 0 \\
0 & Q_2(u-s) & F_{s}^{1,2}(u) & \bar{Q}_2(u+s) & 0 \\
0 & Q_3(u-s) & F_{s}^{1,3}(u) & F_{s}^{2,3}(u) & \bar{Q}_3(u+s) & 0 \\
0 & Q_4(u-s) & T_{s}^{1}(u) & T_{s}^{2}(u) & T_{s}^{3}(u) & \bar{Q}_4(u+s) & 0 \\
\end{array}
\]
Functions in each horizontal (constant \( m \)) slice satisfy HBDE, whereas functions on the \( u - s - a = \text{const} \) slice satisfy HBDE with \( m, a \) being the light cone variables.

In the nested Bethe ansatz approach, the functions \( F_{a,-u}^a(u) \) are auxiliary objects—eigenvalues of transfer matrices for "intermediate" models arising at \((k-m)\)-th step (level) of the hierarchical Bethe ansatz\(^2\). The functions \( Q_m(u) \) play a distinguished role. They can be identified with generalized Baxter’s \( Q \)-operators in the diagonal representation (see below). In the general solution to HBDE these functions are arbitrary functional parameters. The additional requirement of ellipticity determines them through the Bethe equations for their zeros.

**Bethe equations as a discrete dynamical system**

Recall that the function \( \tau_u(m, a) = F_{a,-u}^a(u) \) obeys HBDE in light cone variables:

\[
\tau_u(m + 1, a) \tau_u(m, a + 1) - \tau_u(m, a) \tau_u(m + 1, a + 1) = \tau_{u-1}(m + 1, a) \tau_{u+1}(m, a + 1).
\]  

(4.34)

Since \( \tau_u(m, 0) = Q_{k-m}(2u) \), nested Bethe ansatz equations can be understood as "equations of motions" for zeros of \( Q_m(u) \) in discrete time \( m \) (level of the Bethe ansatz). The simplest way to derive them is to consider the auxiliary linear problems for eq. (4.34). Here we present an example of this derivation in the simplest possible form.

Let us assume that \( Q_m(u) \) has the form

\[
Q_m(u) = e^{\nu_m \eta u} \prod_{j=1}^{M_m} \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (\eta(u - u_j^m))
\]  

(4.35)

(note that we allow the number of roots \( M_m \) to depend on \( m \)). Since we are interested in dynamics in \( m \) at a fixed \( a \), it is sufficient to consider only the first linear equation (3.32) substituting \( \tau^t_i(u) \rightarrow \tau_u(m, a) \).

An elementary way to derive equations of motion for roots of \( \tau_u(m, 0) \) is to put \( u \) equal to the roots of \( \rho^m_0(u) \) and \( \rho^{m+1,0}(u) \), so that only two terms in (3.34) would survive. Combining relations obtained in this way, one can eliminate \( \rho \)'s and obtain the system of equations

\[
\frac{Q_{m-1}(u_{j}^m + 2)Q_{m}(u_{j}^m - 2)Q_{m+1}(u_{j}^m)}{Q_{m-1}(u_{j}^m)Q_{m}(u_{j}^m + 2)Q_{m+1}(u_{j}^m - 2)} = -1.
\]  

(4.36)

as the necessary conditions for solutions of the form (4.35) to exist. With the "boundary conditions"

\[
Q_0(u) = 1, \quad Q_k(u) = \phi(u),
\]  

(4.37)

this system of \( M_1 + M_2 + \ldots + M_{k-1} \) equations is equivalent to the nested Bethe ansatz equations for \( A_{k-1} \)-type quantum integrable models with Belavin’s elliptic \( R \)-matrix. The same equations can be obtained for the right edge of the diagram (4.33) from the second linear equation. In what follows we explicitly identify our \( Q \)'s with similar objects known from the Bethe ansatz solution.

Let us remark that the origin of equations (4.36) suggests to call them as equations of motion for the elliptic Ruijsenaars-Schneider (RS) model in discrete time. Taking the continuum limit in \( m \) (provided \( M_m = M \) does not depend on \( m \)), one can check that eqs. (4.36) do yield the equations of motion for the elliptic RS model with \( M \) particles. The additional limiting procedure \( \eta \rightarrow 0 \) with finite \( \eta u_j = x_j \) yields the well known equations of motion for the elliptic Calogero-Moser system of particles.

However, integrable systems of particles in discrete time have a richer structure than their continuous counterparts. In particular, the total number of particles may depend on the discrete time. Such a phenomenon is possible in continuous time models only for singular solutions, when particles can move to infinity or merge to another within a finite period of time. Remarkably, this appears to be the case for the solutions to eq. (4.36) corresponding to eigenstates of quantum models. It is known that the number of excitations at \( m \)-th level of the nested Bethe ansatz solution does depend on \( m \). In other words, the number of "particles" in the associated discrete time RS system is not conserved. At the same time the

\(^2\)Unlike eq. (4.1), eqs. (4.29)–(4.33) can not be understood in the operator sense since \( F^m \) and \( F^{m+1} \) are eigenvalues of operators acting in different quantum spaces.
numbers $M_m$ may not be arbitrary. It can be shown that for models with elliptic $R$-matrices in case of general position $M_m = (N/k)m$, where $N$ is the number of sites of the lattice (degree of the elliptic polynomial $\phi(u)$). In trigonometric and rational cases these conditions on $M_m$ become less restrictive but still these numbers may not be equal to each other.

**Difference equations for $Q_m$’s**

The functions $Q_m(u)$ obey certain linear difference equations. In principal, they can be obtained from the system of linear problems (4.29), (4.30) at levels $m = 1, 2, \ldots k$ by excluding all functions $F$ except those on the boundaries of the array (4.33). For $k = 2$ this can be done without any problem. Indeed, in this case there are only two non-trivial linear equations:

\[
\begin{align*}
T_s^1(u)Q_1(u-s) - T_s^1(u+1)Q_1(u-s-2) &= \phi(u-s)\bar{Q}_1(u+s+1), \\
T_s^1(u+1)Q_1(u+s) - T_s^1(u)Q_1(u+s+2) &= \phi(u+s+3)Q_1(u-s-1).
\end{align*}
\]

(4.38)

Dividing both sides of the first equation by $\phi(u-s)$ and making use of the fact that the r.h.s. does not depend on $u$, one arrives at

\[
\phi(u-s)Q_1(u-s+2) + \phi(u-s+2)Q_1(u-s-2) = A(u)Q_1(u-s),
\]

(4.39)

where

\[
A(u) = \frac{\phi(u-s)T_{s-1}^1(u+2) + \phi(u-s+2)T_{s+1}^1(u)}{T_s^1(u+1)}.
\]

Recall that the b.c. is $T_{-1}^1(u) = 0$. Whence one obtains famous Baxter’s relation

\[
\phi(u)Q_1(u+2) + \phi(u+2)Q_1(u-2) = T_1^1(u)Q_1(u)
\]

(4.40)

which is a 2-nd order difference equation for $Q_1$. The second linear equation yields the same result (recall that $Q_1(u) = Q_1(u+1)$).

In general the procedure becomes quite involved. Here we only quote the results:

\[
\sum_{a=0}^{k} (-1)^a T_a^p(u-a+1)Q_1(u-2a+2) = 0,
\]

(4.41)

\[
\sum_{a=0}^{k} (-1)^a \frac{T_a^p(u+a+1)}{\phi(u+2a+2)} \frac{Q_{k-1}(u+2a)}{\phi(u+2a)} = 0.
\]

(4.42)

The equations for $Q_m$’s with $2 \leq m \leq k - 2$ have a more complicated form.

**Factorization formulas**

At last, we are to identify our $Q_m$’s with $Q_m$’s from the usual nested Bethe ansatz solution. This is achieved by factorization of the difference operators in (4.41) and (4.42) in terms of $Q_m(u)$. One can prove the following factorization formulas (looking like discrete Miura transformation):

\[
\sum_{a=0}^{k} (-1)^{a-k} T_a^p(u-a+1)\phi(u+2) e^{-2a\partial_u} = \left( e^{-2\partial_u} - \frac{Q_k(u)Q_{k-1}(u+2)}{Q_k(u+2)Q_{k-1}(u)} \right) \cdots \left( e^{-2\partial_u} - \frac{Q_2(u)Q_1(u+2)}{Q_2(u+2)Q_1(u)} \right) \left( e^{-2\partial_u} - \frac{Q_1(u)}{Q_1(u+2)} \right)
\]

(4.43)

\[
\sum_{a=0}^{k} (-1)^{a-k} T_a^p(u+a+1)\phi(u+2a+2) e^{2a\partial_u} = \left( e^{2\partial_u} - \frac{Q_1(u)}{Q_1(u+2)} \right) \cdots \left( e^{2\partial_u} - \frac{Q_2(u)Q_1(u+2)}{Q_2(u+2)Q_1(u)} \right) \left( e^{2\partial_u} - \frac{Q_k(u)Q_{k-1}(u+2)}{Q_k(u+2)Q_{k-1}(u)} \right).
\]

(4.44)
Note that these operators are adjoint to each other. The factors in the r.h.s. resemble $M$-operators \[B.17\].

Let us note that eq. (4.44) can be rewritten as follows:

\[
\sum_{a=0}^{k} (-1)^{a} T_{a}^{2a+1} (u + a - 1) e^{2\phi(u)} = \left(1 - \frac{Q_{1}(u+2)}{Q_{1}(u)} e^{2\phi(u)} \right) \left(1 - \frac{Q_{2}(u+2)Q_{1}(u-2)}{Q_{2}(u)Q_{1}(u)} e^{2\phi(u)} \right) \cdots \left(1 - \frac{Q_{k}(u+2)Q_{k-1}(u-2)}{Q_{k}(u)Q_{k-1}(u)} e^{2\phi(u)} \right)
\]

Now the factors in the r.h.s. resemble $M$-operators \[3.18\]. It is possible to show that coefficients of the operator inverse to \[4.45\] give $T_{s}^{1}(u)$:

\[
\sum_{s=0}^{\infty} \frac{T_{s}^{1}(u+s-1)}{\phi(u)} e^{2s\phi(u)} = \left(1 - \frac{Q_{k}(u+2)Q_{k-1}(u-2)}{Q_{k}(u)Q_{k-1}(u)} e^{2\phi(u)} \right)^{-1} \cdots \left(1 - \frac{Q_{2}(u+2)Q_{1}(u-2)}{Q_{2}(u)Q_{1}(u)} e^{2\phi(u)} \right)^{-1} \left(1 - \frac{Q_{1}(u+2)}{Q_{1}(u)} e^{2\phi(u)} \right)^{-1}
\]

These formulas yield $T_{a}^{m}(u)$, $T_{s}^{1}(u)$ in terms of elliptic polynomials $Q_{m}$ with roots constrained by the nested Bethe ansatz equations which ensure cancellation of poles in $T_{a}^{m}(u)$. The transfer matrices $T_{a}^{m}(u)$ for $a, s > 1$ can be then found with the help of determinant formulas \[2.37\], \[2.38\].

Comments and references

1. With the b.c. \[4.13\] HBDE \[L.1\] is known as the bilinear form of the discrete two-dimensional Toda molecule equation \[8\] (the Toda lattice with open boundaries), an integrable discretization of the conformal Toda field theory \[9\]. In particular, at $k = 2$ we have a discrete analogue of the Liouville equation.

The b.c. in $s$ of the form \[L.1\] is known in the classical theory, too. The b.c. of this kind is a hallmark of forced ("semi-infinite") hierarchies of non-linear integrable equations emerging naturally in matrix models of 2D gravity (see e.g. \[41\]). For instance, in the forced 2DTL hierarchy, the $\tau$-function $\tau_{n}(t_{1}, t_{2}, \ldots, t_{1}, t_{2}, \ldots)$ is equal to zero at $n = -1$: $\tau_{-1} = 0$ for any $t_{i}, t_{i}$.

Similarly, imposing the analytic condition on the $\tau$-function of the form \[L.8\] is a familiar story in classical nonlinear integrable equations since the paper \[1\], where the elliptic solutions to the KdV equation were studied. A systematic approach to elliptic solutions of the KP equation based on the finite-gap integration methods was developed by I.Krichever \[12\].

So we see that each one of the boundary and analytic conditions has been known in the classical theory. However, they never met altogether. The specifics of solutions relevant to quantum problems is perhaps just in the combination of the above conditions which looks quite unusual for the soliton theory.

2. The RS model (in continuous time) was introduced in the paper \[43\] as a relativistic generalization of the Calogero-Moser system of particles. This model was shown to be integrable and the Lax representation was found. Recently, it was shown \[45\] that the RS system describes the dynamics of poles of elliptic solutions (zeros of $\tau$-function) to the 2DTL. Equations of motions for the discrete time analogue of the RS model were written down in ref. \[44\], where an ansatz for the Lax pair was suggested. The close connection with the nested Bethe ansatz was observed and explained in ref. \[4\].

3. The fact that in the elliptic case degree of the elliptic polynomial $Q_{m}(u)$ is equal to $M_{m} = (N/k)m$ (provided $q$ is incommensurable with the lattice spanned by the two complex periods $1, \tau$ and $N$ is divisible by $k$) follows directly from Bethe equations \[4.36\]. Indeed, the elliptic polynomial form of $Q_{m}(u)$ implies that if $u_{j}^{m}$ is a zero of $Q_{m}(u)$, i.e., $Q_{m}(u_{j}^{m}) = 0$, then $u_{j}^{m} + n_{1} + n_{2}\tau$ for all integers $n_{1}, n_{2}$ are its zeros too. Taking into account the well known monodromy properties of the $\theta$-function, one concludes that this is possible if and only if

\[M_{m+1} + M_{m-1} = 2M_{m}, \quad (4.47)\]
which has a unique solution $M_m = (N/k)m$ satisfying the required b.c. This means that the nested Bethe ansatz scheme for elliptic $A_{k-1}$-type models is consistent only if $N$ is divisible by $k$.

In trigonometric and rational cases the conditions on degrees of $Q_m$’s become less restrictive since some of the roots can be located at infinity. The equality in the formula for $M_m$ becomes an inequality:

$$M_m \leq \left( \frac{N}{k} \right)^m.$$

A more detailed analysis [10] shows that the following inequalities also hold: $2M_1 \leq M_2$, $2M_2 \leq M_1 + M_3$, $2M_m \leq M_{m-1} + M_{m+1}$, ..., $N = M_k \geq 2M_{k-1} - M_{k-2}$.

4. It should be noted that the family of commuting transfer matrices generally does not define a quantum system uniquely. To do that, one should choose a hamiltonian, i.e. take a particular representative of the commuting family of operators. So, to introduce a quantum integrable model to be solved, one has to give a prescription how to get hamiltonian from the transfer matrix. To be definite, let as consider the $A_1$-case. The hamiltonians are taken to be linear in logs $T_s$, to wit

$$H = \sum_p \alpha_p \log T_s(u^{(p)}),$$

where $u^{(p)}$ are some particular values of the spectral parameter. (For example, the hamiltonian of the spin-1/2 Heisenberg magnet is $H = \partial_u \log T_1(u)$ at $u = 0$.) If the hamiltonian is local, one can define the elementary energy function $\varepsilon(u)$ and represent the total energy as $E = \sum_j \varepsilon(u_j)$, where the momenta $u_j$ of “quasiparticles” satisfy Bethe equations. This very important characteristics of the quantum system depends on the choice of $H$.

Is there a room for $\varepsilon(u)$ on the classical side? The answer seems to be in the affirmative. This function might encode a way in which potentials or fields entering soliton equations are expressed through the $\tau$-function of a given hierarchy. (In the most popular example of KdV this is $U(x) = 2\partial_x^2 \log \tau(x)$.) Since $T_s(u)$ has been identified with a $\tau$-function, eq. (4.48) resembles such expressions in the hierarchies of discrete soliton equations. The hamoltonians $H$ might then be identified with the corresponding potentials (or rather logarithms of potentials in the discrete case).

5. The l.h.s. of eq. (4.45) is known as the generating function for $T_1^q(u)$. These formulas for the generating function coincide with the ones known in the literature (see e.g. [1], [47], [48]). They may be considered as non-commutative analogues of generating functions for symmetric multivariable polynomials (see Remark 3 to Sect. 2).

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