2D problems of surface growth theory with applications to additive manufacturing

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Abstract. We study 2D problems of surface growth theory of deformable solids and their applications to the analysis of the stress-strain state of AM fabricated products and structures. Statements of the problems are given, and a solution method based on the approaches of the theory of functions of a complex variable is suggested. Computations are carried out for model problems. Qualitative and quantitative results are discussed.

Introduction

Traditional methods for the manufacturing of parts of complex shape involve various technological processing operations of material removal such as cutting, milling, drilling etc. There is an alternative class of technological processes where parts are manufactured by successive application of the material onto a substrate or a surface of arbitrary shape (e.g., by local polymerization, electrochemical reaction, welding deposition, local sintering, etc.). Such technologies include stereolithography, electrolytic deposition, winding, laser and thermal 3D printing, manufacturing of 3D electronic circuits, and many others (e.g., see \cite{1, 2}).

Nowadays, additive manufacturing technologies undergo intensive development. The use of such technologies theoretically permits manufacturing 3D parts of very complex shape quickly and relatively cheaply from any material. However, although technological setup elements can be positioned with high precision, the deviations in the geometrical shape of the object to be manufactured from the design parameters can be high primarily owing to heat deformation, shrinkage, and other types of deformation of the object itself. Associated with these deformations are residual stresses, which inevitably arise in the object produced by additive manufacturing technologies. The residual stress distributions and magnitudes may result in buckling of thin-walled parts and destruction of massive bodies, which may happen as early as already in the manufacturing process. Thus, the development of mathematical models and theoretical methods for the analysis of internal technological stresses and strains in complex modern products is a topical scientific problem from the viewpoint of basic research as well as numerous applications. The basic scientific problems of surface growth are the subject of mechanics of growing bodies (e.g., see \cite{3–5}). In what follows, we consider some 2D problems of surface growth theory...
with applications to the analysis and design of parts and structures produced by additive manufacturing technologies.

1. Plane problem of surface growth

Let a viscoelastic homogeneous ageing body occupy a plane multiply connected domain \( \Omega_1 \) with boundary \( L_1 \) and be stress-free prior to the loading time instant \( \tau_0 \). From the loading instant to the instant \( \tau_1 \) of additive modification on a part \( L_\sigma \subset L_1 \) of the boundary, the body is subjected to a self-balanced load, and the part \( L^* \) of the boundary where new material is to be added remains load-free.

The continuous additive manufacturing of the body starts at time \( \tau_1 \geq \tau_0 \) by adding elements produced simultaneously with the body. In the course of the growth process, the body occupies a domain \( \Omega(t) \) with boundary \( L(t) = L_\sigma \cup L^*(t) \), where \( L^*(t) \) is the growth boundary or curve where the material inflow occurs at a given time \( L^*(t) = L^* \) for \( \tau < \tau_1 \) and \( L_\sigma(t) \) is the boundary on which the load is given.

We assume that the growth boundary \( L^*(t) \) remains load-free during the process of additive modification of the body. We also assume that the time \( \tau_0 = \tau_0(x_1, x_2) \) at which the load is applied to the newly added elements coincides with the time \( \tau^* = \tau^*(x_1, x_2) \) at which they are added to the growing body.

The AM fabrication process terminates at time \( \tau_2 \geq \tau_1 \), and after that the body occupies the domain \( \Omega_2 = \Omega(\tau_2) \) bounded by the contour \( L_2 = L(\tau_2) = L_\sigma(\tau_2) \cup L^*(\tau_2) \).

To be definite, consider the case of a plane stress state. In what follows, we deal with sufficiently slow processes such that one can neglect inertial terms in the equilibrium equations. We assume the bulk forces to be zero.

The boundary value problem for the main (original) viscoelastic homogeneous ageing body on the time interval \([\tau_0, \tau_1]\) is a traditional problem of viscoelasticity theory.

The initial-boundary value problem for the continuously growing body on the time interval \( t \in [\tau_1, \tau_2] \) consists of the equilibrium equations
\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0, \tag{1}
\]
the Cauchy relations
\[
D_{11} = \frac{\partial v_1}{\partial x_1}, \quad D_{22} = \frac{\partial v_2}{\partial x_2}, \quad D_{12} = \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right), \tag{2}
\]
between the strain rates \( D_{ij} = \partial \varepsilon_{ij}/\partial t \) and the displacement rates \( v_i = \partial u_i/\partial t \), the state equation in the form
\[
\begin{align*}
\sigma_{11} &= 2G(I + N_{\tau_0}) \left( \frac{2K}{K+1} \varepsilon_{11} + \frac{K-1}{K+1} \varepsilon_{22} \right), \\
\sigma_{22} &= 2G(I + N_{\tau_0}) \left( \frac{K-1}{K+1} \varepsilon_{11} + \frac{2K}{K+1} \varepsilon_{22} \right), \\
\sigma_{12} &= 2G(I + N_{\tau_0}) \varepsilon_{12}, \\
\tau_0(x_1, x_2) &= \begin{cases} \\
\tau_0, & (x_1, x_2) \in \Omega_1, \\
\tau^*(x_1, x_2), & (x_1, x_2) \in \Omega^*(t), \\
\end{cases} \tag{3}
\end{align*}
\]
\[
(I + N_{\tau_0})^{-1} = (I - L_{\tau_0}), \quad 2G = \frac{E}{1+v}, \quad K = \frac{1}{1-2v}, \\
L_s f(t) = \int_s^t f(\tau)K_1(t, \tau) \, d\tau, \quad K_1(t, \tau) = G(\tau) \frac{\partial}{\partial \tau} \left[ G^{-1}(\tau) + \omega(t, \tau) \right],
\]

\]
the boundary condition
\[ n_1 \sigma_{11} + n_2 \sigma_{12} = p_1, \quad n_1 \sigma_{11} + n_2 \sigma_{22} = p_1 \] (4)
on the immovable part \( L_0(t) \) of the boundary, and the 3D-body-2D-surface contact condition on the growth boundary \( L^*(t) \) (e.g., see [6–8]). For the zero prestress in the added elements, the last condition has the form
\[ n_1 \frac{\partial \sigma_{11}}{\partial t} + n_2 \frac{\partial \sigma_{12}}{\partial t} = 0, \quad n_1 \frac{\partial \sigma_{12}}{\partial t} + n_2 \frac{\partial \sigma_{22}}{\partial t} = 0, \] (5)
where \( n = \{n_1, n_2\} \) is the unit outward normal vector to the lateral surface of the body, \( p = \{p_1, p_2\} \) is the surface force vector, \( \Omega^*(t) = \Omega(t) \setminus \Omega_1 \) is the part of the body formed in the AM fabrication process (the additional body), \( E = E(t) \) and \( G = G(t) \) are the Young modulus and the shear modulus, respectively, \( \omega(t, \tau) \) is the shear creep measure, \( K_1(t, \tau) \) is the creep kernel, the elastic and creep Poisson ratios coincide and are equal to \( v \), and \( I \) is the identity operator. The values of all functions for \( \tau_0 \leq t \leq \tau_1 \) are known from the solution of the problem for the main body.

The initial-boundary value problem (1)–(5) for an AM fabricated body has the following specific features, which take it beyond the framework of classical problems of mechanics of deformable solids: the strain consistency condition is violated in the domain occupied by the additional body, and only an analog of this condition and an analog of the Cauchy condition for the rates of the corresponding variables hold (this permits one to take into account the fact that, prior to the accretion to the main body, the added elements can undergo deforming actions independent of the processes occurring in the body itself); the constitutive relations depend on the function \( \tau_0 = \tau_0(x_1, x_2) \), which may have jump discontinuities.

We use the notation \( \sigma_{ij}^0 = (I - L_{m_i}) \sigma_{ij} G^{-1} \) and transform the problem of additive modification of a viscoelastic body with the constitutive relations (1)–(5) into the problem of additive modification of an elastic body described by Hooke’s law,
\[
\frac{\partial \sigma_{11}^0}{\partial x_1} + \frac{\partial \sigma_{12}^0}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}^0}{\partial x_1} + \frac{\partial \sigma_{22}^0}{\partial x_2} = 0; \\
D_{11} = \frac{\partial v_1}{\partial x_1}, \quad D_{22} = \frac{\partial v_2}{\partial x_2}, \quad D_{12} = \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right); \\
\sigma_{11}^0 = 2G \left( \frac{2K}{K + 1} \varepsilon_{11} + \frac{K - 1}{K + 1} \varepsilon_{22} \right), \quad \sigma_{12}^0 = 2G \left( \frac{2K}{K + 1} \varepsilon_{11} + \frac{2K}{K + 1} \varepsilon_{22} \right), \quad \sigma_{12}^0 = 2G \varepsilon_{12}; \tag{6}
\]
\((x_1, x_2) \in L_\sigma : \quad n_1 \sigma_{11}^0 + n_2 \sigma_{12}^0 = (I - L_{m_1}) p_1 G^{-1} = p_1^0, \quad n_1 \sigma_{12}^0 + n_2 \sigma_{22}^0 = (I - L_{m_2}) p_2 G^{-1} = p_2^0; \)
\((x_1, x_2) \in L^*(t) : \quad n_1 \frac{\partial \sigma_{11}^0}{\partial t} + n_2 \frac{\partial \sigma_{12}^0}{\partial t} = 0, \quad n_1 \frac{\partial \sigma_{12}^0}{\partial t} + n_2 \frac{\partial \sigma_{22}^0}{\partial t} = 0. \)

We differentiate the equilibrium equations, the boundary condition on \( L_\sigma(t) \), and the state equation in (6) with respect to \( t \) and reduce the boundary value problem for a growing body to the form
\[
\frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} = 0, \quad \frac{\partial S_{12}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} = 0, \\
S_{11} = 2 \left( \frac{2K}{K + 1} D_{11} + \frac{K - 1}{K + 1} D_{22} \right), \quad S_{22} = 2 \left( \frac{K - 1}{K + 1} D_{11} + \frac{2K}{K + 1} D_{22} \right), \tag{7}
\]
\((x_1, x_2) \in L_\sigma : \quad n_1 S_{11} + n_2 S_{12} = R p_1 = q_1, \quad n_1 S_{12} + n_2 S_{22} = R p_2 = q_2; \)
\((x_1, x_2) \in L^*(t) : \quad n_1 S_{11} + n_2 S_{12} = 0, \quad n_1 S_{12} + n_2 S_{22} = 0. \)
where the operator \( R \) is defined by the expression

\[
R f(x_1, x_2, t) = \frac{1}{G(t)} \frac{\partial f(x_1, x_2, t)}{\partial t} + \int_{\tau_0(x_1, x_2)}^{t} \frac{\partial f(x_1, x_2, \tau)}{\partial \tau} \frac{\partial \omega(t, \tau)}{\partial t} \, d\tau
+ f(x_1, x_2, \tau_0(x_1, x_2)) \frac{\partial \omega(t, \tau_0(x_1, x_2))}{\partial t}.
\]

The boundary value problem (7) coincides in form with the boundary value problem of the elasticity theory with a parameter \( t \). Its solution can be constructed by any analytical or numerical method efficient in the elasticity theory. The solution of the original initial-boundary problem of additive modification of a viscoelastic ageing body for \( t \in [\tau_1, \tau_2] \) can be reconstructed by the formulas

\[
\begin{align*}
\sigma_{ij}(x_1, x_2, t) &= G(t) \left\{ \sigma_{ij}(x_1, x_2, \tau_0(x_1, x_2)) \left[ 1 + \int_{\tau_0(x_1, x_2)}^{t} R(t, \tau) \, d\tau \right] 
+ \int_{\tau_0(x_1, x_2)}^{t} \left[ S_{ij}(x_1, x_2, \tau) + \int_{\tau_0(x_1, x_2)}^{\tau} S_{ij}(x_1, x_2, \varsigma) \, d\varsigma \right] R(t, \tau) \, d\tau \right\}, \\
u_i(x_1, x_2, t) &= u_i(x_1, x_2, \tau_0(x_1, x_2)) + \int_{\tau_0(x_1, x_2)}^{t} \psi_i(x_1, x_2, \tau) \, d\tau.
\end{align*}
\]

The constitutive relations of the problem for a body whose AM fabrication has been finished have the form (1)–(5), where the condition on the growth boundary is missing. This problem can be reduced to the form (7), and the formulas reconstructing the true characteristics of the stress-strain state preserve the form (8).

The exposition of the main new part of the method for solving the plane problem for a growing body is complete. The boundary value problems for all the main stages of the body evolution have been reduced to boundary value problems coinciding in form with the boundary value problems of the elasticity theory with some parameter. To study the latter, we use the methods of the theory of functions of a complex variable.

Since the boundary value problems at all stages are mathematically the same, we only consider the boundary value problem (7). To solve it, we express the stresses via the Airy stress function \( U(x_1, x_2, t) \) (e.g., see [9]),

\[
S_{11} = \frac{\partial^2 U}{\partial x_2^2}, \quad S_{22} = \frac{\partial^2 U}{\partial x_1^2}, \quad S_{12} = \frac{\partial^2 U}{\partial x_1 \partial x_2},
\]

which is a biharmonic function.

It is well known that every biharmonic function can be represented via analytic functions of the complex variable,

\[
U = \frac{1}{2} \left[ z \varphi_1(z, t) + \bar{z} \varphi_1(\bar{z}, t) + \chi_1(z, t) + \chi_1(\bar{z}, t) \right].
\]

Then the complex representation of the functions \( S_{ij} \) has the form

\[
S_{11} + S_{22} = 4Re[\varphi_1(z, t)]',
S_{22} - S_{11} + 2iS_{12} = 2[\bar{z} \varphi_1''(z, t) + \psi'(z, t)],
\psi(z, t) = \chi_1'(z, t),
\]

where the prime stands for differentiation with respect to \( z \).
For the variables $v_i$, we write the complex representation in the form
\[ 2(v_1 + iv_2) = \frac{5K + 1}{3K - 1}\varphi_1(z, t) - z\varphi_1(z, t) - \overline{\psi(z, t)}. \]

Thus, $S_{ij}$ and $v_i$ can be expressed via two analytic functions, which are commonly known as the Kolosov-Muskhelishvili functions [10].

The boundary condition can be represented in the form
\[ \varphi_1(z, t) + z\varphi_1'(z, t) + \psi(z, t) = f, \]

where $f = i(Q_1 + Q_2)$ and $Q_1$ and $Q_2$ are given functions on the contour.

In many applications, one needs to have expressions for the desired functions in the orthogonal coordinate system defined via a conformal mapping $z = \omega(\varsigma, t)$ ($\varsigma = \rho e^{i\theta}$). In this case, we have
\[ S_{pp}(\varsigma, t) + S_{\theta\rho}(\varsigma, t) = 4Re[\varphi(\varsigma, t)]', \]
\[ S_{\theta\rho}(\varsigma, t) - S_{pp}(\varsigma, t) + 2iS_{\theta\rho}(\varsigma, t) = \frac{2}{p^2}\frac{\partial^2}{\partial\varsigma^2}[\overline{\omega(\varsigma, t)}\varphi''(\varsigma, t) + \omega'(\varsigma, t)\psi'(\varsigma, t)], \]
\[ 2[v_{\rho}(\varsigma, t) + iv_{\theta}(\varsigma, t)] = \frac{\varsigma\omega'(\varsigma, t)5K + 1}{3K - 1}\varphi(\varsigma, t) - \frac{\omega(\varsigma, t)}{\omega'(\varsigma, t)}\varphi'(\varsigma, t) - \overline{\psi(\varsigma, t)}]. \]

Since the domain in question is often mapped onto the unit disk (or onto the plane with the unit disk deleted) in many problems, we present the following useful formulas:
\[ \varphi(\varsigma) = -\frac{Q_1 + iQ_2}{2\pi}3K - 1\ln\varsigma + \frac{\Gamma c(t)}{\varsigma} + \varphi_0(\varsigma, t), \]
\[ \psi(\varsigma) = -\frac{Q_1 - iQ_2}{2\pi}5K + 1\ln\varsigma + \frac{\Gamma' c(t)}{\varsigma} + \psi_0(\varsigma, t), \]

where $\varphi_0(\varsigma, t)$ and $\psi_0(\varsigma, t)$ are functions analytic in the interior and continuous up to the boundary of the unit disk. The functions $\varphi_0(\varsigma, t)$ and $\psi_0(\varsigma, t)$ satisfy the same equations as $\varphi(\varsigma, t)$ and $\psi(\varsigma, t)$ with the function $f$ everywhere replaced by the function
\[ f_0 = f - \frac{T_1 + iT_2}{2\pi}\ln\sigma - \frac{\Gamma c(t)}{\sigma} - \frac{\omega(\sigma, t)}{\omega'(\sigma, t)}\left(\frac{T_1 - iT_2}{2\pi}\frac{3K - 1}{8K}\sigma - \frac{\Gamma c(t)}{\sigma}\sigma^2\right) - \overline{\Gamma' c(t)}\sigma. \]

Once the stress components in the orthogonal coordinate system are known, one can find the components $S_{ij}$ by the formulas
\[ S_{pp}(\varsigma, t) + S_{\theta\rho}(\varsigma, t) = S_{11}(\varsigma, t) + S_{22}(\varsigma, t), \]
\[ S_{\theta\rho}(\varsigma, t) - S_{pp}(\varsigma, t) + 2iS_{\theta\rho}(\varsigma, t) = [S_{22}(\varsigma, t) - S_{11}(\varsigma, t) + 2iS_{12}(\varsigma, t)]e^{2i\alpha}. \]

Thus, having found two harmonic functions from the boundary conditions, one can find the desired functions $S_{ij}$ and $v_i$ and then use formulas (8) to reconstruct the true values of the stresses and displacements in the growing body.

The solution of each additive manufacturing problem for deformable bodies is a stand-alone laborious task (e.g., see [11–15]). However, the form of the mathematical relations obtained above permits one right away to predict such phenomena intrinsically typical of growing bodies as the onset of residual stresses after the load is removed, the occurrence of stress discontinuity surfaces in AM fabricated bodies, and the dependence of the stress-strain state of viscoelastic bodies on the additive manufacturing rate and method. As an example of possible applications of the theory, we note problems on the occlusion of various holes.
2. Stress concentration near AM fabricated holes of desired shape

By way of example, consider the problem on the stress concentration near a hole that must take a given shape in the AM fabrication process in a stressed medium. The medium material is viscous and ageing; i.e., its properties depend on time. The medium is in a uniaxial stress state. The additive manufacturing process should transform a large elliptic hole into a smaller elliptic hole.

Assume that an infinite plate $\Omega_1$ weakened by an elliptic hole $L_1$ is manufactured at time zero from an ageing viscoelastic material. Let the contour $L_1$ be free of internal stresses, and let the stress state at infinity be characterized by a uniform extending stress $P(t)$ directed at an angle $\alpha$ with the axis $Ox_1$ (figure 1).

The continuous additive manufacturing of the hole by elements of the same age as the medium begins at time $\tau_1 \geq \tau_0$. In the growth process, the medium occupies a domain $\Omega(t)$ with boundary $L(t)$ that is an ellipse for each $t$. We consider the process of additive occlusion of the elliptic hole in which the ratio of the semiaxes $a(t)$ and $b(t)$ remains constant; i.e.,

$$\frac{a_1}{b_1} = \frac{a(t)}{b(t)}.$$

The additive occlusion of the hole terminates at time $\tau_2 \geq \tau_1$ and since then the hole occupies the domain $\Omega_2 = \Omega(\tau_2)$ bounded by the contour $L_2 = L(\tau_2)$.

We assume that the newly added elements are not prestressed. Since the boundary value problems obtained at all the considered stages of additive manufacturing are mathematically equivalent, it suffices to consider the growth stage. Then the initial-boundary value problem for

![Figure 1. AM fabrication of a hole of the desired elliptic shape.](image)
the AM fabrication of the elliptic hole acquires the form

\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0; \\
D_{11} = \frac{\partial v_1}{\partial x_1}, \quad D_{22} = \frac{\partial v_2}{\partial x_2}, \quad D_{12} = 1 \frac{\partial \varepsilon_{ij}}{\partial t}, \quad v_i = \frac{\partial u_i}{\partial t}; \\
\sigma_{11} = 2G(I + N_{\nu_0}) \left( \frac{2K}{K+1} \varepsilon_{11} + \frac{K-1}{K+1} \varepsilon_{22} \right), \\
\sigma_{22} = 2G(I + N_{\nu_0}) \left( \frac{K-1}{K+1} \varepsilon_{11} + \frac{2K}{K+1} \varepsilon_{22} \right), \\
\sigma_{12} = 2G(I + N_{\nu_0}) \varepsilon_{12}, \\
(x_1, x_2) \in L(t) : \ n_1 \frac{\partial \sigma_{11}}{\partial t} + n_2 \frac{\partial \sigma_{12}}{\partial t} = 0, \quad n_1 \frac{\partial \sigma_{12}}{\partial t} + n_2 \frac{\partial \sigma_{22}}{\partial t} = 0.
\]

The boundary value problem (7) becomes

\[
\frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} = 0, \quad \frac{\partial S_{12}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} = 0, \\
S_{11} = 2 \left( \frac{2K}{K+1} D_{11} + \frac{K-1}{K+1} D_{22} \right), \quad S_{22} = 2 \left( \frac{K-1}{K+1} D_{11} + \frac{2K}{K+1} D_{22} \right), \\
S_{12} = 2D_{12}, \quad S_{ij} = \frac{\partial \sigma_{ij}}{\partial t}, \quad \sigma_{ij}^0 = (I - L_{\nu_0(x_1, x_2)}) \sigma_{ij} G^{-1}, \\
(x_1, x_2) \in L(t) : \ n_1 S_{11} + n_2 S_{12} = 0, \quad n_1 S_{12} + n_2 S_{22} = 0.
\]

Since the contour $L(t)$ is stress-free, it follows that the principal vector components are zero,

\[ Q_1 = 0, \quad Q_2 = 0. \]

The principal stresses at infinity are given in the form

\[ N_1 = P^0 = (I - L_{\nu_0}) PG^{-1}, \quad N_2 = 0. \]

Then we have

\[ \text{Re} \Gamma = \frac{1}{4} (N_1 + N_2) = \frac{P^0}{4} (\Gamma = \Gamma'), \quad \Gamma' = -\frac{1}{2} (N_1 - N_2) e^{-2i\alpha}. \]

The function mapping the exterior of the disk onto the exterior of the ellipse has the form

\[ z = \omega(\zeta, t) = R(t) \left[ \zeta + \frac{m(t)}{\zeta} \right], \]

where $R(t) = [a(t) + b(t)]/2, \ m(t) = [a(t) - b(t)]/[a(t) + b(t)]$. 


The formulas for the components $S_{rr}$, $S_{\theta\theta}$, and $S_{\theta r}$ read
\[
S_{rr} = \frac{q}{2} [A(r, \theta, t) - B(r, \theta, t)], \quad S_{\theta\theta} = \frac{q}{2} [A(r, \theta, t) + B(r, \theta, t)], \quad S_{\theta r} = \frac{q r^2}{2} \left( \frac{1}{r^4 - 2mr^2 \cos(2\theta) + m^2} \{ 2(r^4 - r^2) \sin[2(\alpha - \theta)] + m(m^2 + r^4 - r^2) \sin(2\theta) - m^2 \sin[2(\alpha + \theta)] + m^2 r^2 (1 - m^2) \sin(2\theta) \cos(2\alpha) + [2m - m(r^2 - m^2)] \sin(2\alpha) \} + (1 - r^4) \sin[2(\theta - \alpha)] - m^2 \sin(2\alpha) \right),
\]
\[
A(r, \theta, t) = \frac{r^4 - 2mr^2 \cos[2(\theta - \alpha)] - m^2 + 2m \cos(2\alpha)}{r^4 - 2mr^2 \cos(2\theta) + m^2},
\]
\[
B(r, \theta, t) = \frac{r^2}{r^4 - 2mr^2 \cos[2(\theta - \alpha)] + m^2} \{ 2[m - r^2 \cos(2\theta)] \cos(2\alpha) + [\cos(2\alpha) - m] (r^4 - m^2) [2 \cos(2\theta) - m] + r^4 - m^2 + (2r^4 + m^2 + 2m^3 r^2) \sin(2\theta) \sin(2\alpha) + 2mr^2 \sin(2\alpha) \cos(2\theta) \} - \frac{\cos[2(\alpha - \theta)]}{r^4 - 2mr^2 \cos(2\theta) + m^2} \{ r^2 \cos[2(\theta - \alpha)] - m \cos(2\alpha) \},
\]
\[
\frac{\omega(t, \tau)}{\omega(t_0)} = (D_0 + F e^{-\beta \tau})(1 - e^{-\gamma(t-\tau)}).
\]

We use Kolosov’s formulas (9) to compute $S_{11}$, $S_{12}$, and $S_{22}$. The true stresses are determined by formulas (8).

For the material, we take concrete with instantaneous elastic shear modulus
\[
G(t) = G_0(1 - e^{-\alpha t})
\]
and with shear creep measure in the form
\[
\omega(t, \tau) = (D_0 + F e^{-\beta \tau})(1 - e^{-\gamma(t-\tau)}).
\]

Let us pass to dimensionless variables using the limit shear modulus $G_0$, the coefficient $\gamma$ in the approximation to the creep measure, and the final length $a_2 = a(t_2)$ of the major semiaxis of the ellipse. We make a change of variables by the formulas
\[
t^* = \gamma t, \quad t_1^* = \gamma t_1, \quad t_2^* = \gamma t_2, \quad a^*(t^*) = \frac{a(t)}{a_2}, \quad b^*(t^*) = \frac{b(t)}{a_2}, \quad a_1^* = \frac{a_1}{a_2}, \quad a_2^* = \frac{a_2}{a_2} = 1, \quad b_1^* = \frac{b_1}{a_2}, \quad b_2^* = \frac{b_2}{a_2}, \quad \alpha_0^* = \frac{\alpha_0}{\gamma}, \quad \beta^* = \frac{\beta}{\gamma}, \quad \gamma^* = \frac{\gamma}{\gamma} = 1, \quad D_0^* = D_0 G_0, \quad F^* = F G_0,
\]
and, omitting asterisks in the notation, specify the following values of functions and parameters:
\[
a(t) = a_1 \left( \tau_2 - \frac{1}{3} \tau_1 - \frac{2}{3} \right) (\tau_2 - \tau_1)^{-1}, \quad b(t) = b_1 \left( \tau_2 - \frac{1}{3} \tau_1 - \frac{2}{3} \right) (\tau_2 - \tau_1)^{-1}, \quad a_1 = 3, \quad a_2 = 1, \quad b_1 = 1.5, \quad b_2 = 0.5, \quad \gamma = 0.06, \quad \alpha_0 = 2, \quad \beta = \frac{0.031}{0.06}.
\]
\[
A = 0.5, \quad D_0 = 0.5522, \quad F = 4, \quad D_0^* = D_0 G_0, \quad F^* = F G_0.
\]

Obviously, each of the semiaxes of the ellipse becomes three times smaller after the accretion; i.e., $a_2 = a_1/3$ and $b_2 = b_1/3$.

Throughout the following, dashed lines in the figures stand for the shear stress intensity distribution at the beginning of the additive manufacturing process, dot-and-dash lines stand
Figure 2. Shear stress intensity in the vicinity of the elliptic hole for the first mode of the fast AM fabrication.

For that in the final body without considering the additive manufacturing process, and solid lines stand for the shear stress intensity distribution at large times.

Let the stress state at infinity be characterized by an extending stress $P(t)$ directed at the angle $\alpha = \pi/2$ to the axis $Ox_1$. We study the shear stress intensity distribution along the major semiaxis of the hole $B$ depending on the extending stress law $P(t)$ at infinity.

For the case of a generalized plane stress state, we compute the shear stress intensity by the formula $\sigma_i = \sqrt{\frac{1}{3}(\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2)^{1/2}}$.

Consider the case of a constant extending stress at infinity, $P(t) = 1$. For the fast AM fabrication starting shortly after the application of the load $P(t) = 1$ (the first mode of fast AM fabrication: $\tau_0 = 0.1, \tau_1 = 0.3, \tau_2 = 0.5$), the maximum of the shear stress intensity is attained on the boundary of the hole (figure 2).

If the AM fabrication starts much later than the load $P(t) = 1$ is applied (the second mode of the fast AM fabrication: $\tau_0 = 0.1, \tau_1 = 1.3, \tau_2 = 1.5$), then the maximum of the shear stress intensity is attained on the interface between the original and additional bodies (figure 3).

For slow AM fabrication, the maximum of the shear stress intensity is attained on the interface between the original and additional bodies (figure 4).

Now consider the case of stepwise variation of the extending stress on the time interval $t \in [1, 5]$ according to the formula

$$P(t) = \begin{cases} 0.1 & t \leq T, \\ 1.1 & t > T. \end{cases}$$

For this AM fabrication mode, the maximum of the shear stress intensity is attained inside the additional body. Figure 5 shows the variations in the shear stress intensities for $T = 2$.

3. Torsion of prismatic rods under surface growth

Consider a homogeneous viscoelastic ageing body manufactured at time zero and occupying some cylindrical domain $\Pi_1$ whose cross-section $\Omega_1$ has the boundary $L_1$. At time $\tau_0$, the faces of the cylindrical body are subjected to forces statically equivalent to a couple with torque $M(t)$. The lateral surface of the body $\Pi_1$ is stress-free.
The continuous AM fabrication of the body by accretion of elements manufactured simultaneously with the body starts at time $\tau_1 \geq \tau_0$. The newly added elements are stress-free. Let $L(t)$ be the boundary of the cross-section $\Omega(t)$, which depends on time; we have $L(\tau_1) = L_1$ and $\Omega(\tau_1) = \Omega_1$. The boundary $L(t)$ of $\Omega(t)$ consists of two parts, $L(t) = L^*(t) \cup L_\sigma(t)$, where $L^*(t)$ is the growth boundary where the material inflow occurs at the current time ($L^*(t) = L^*$ for $\tau \leq \tau_1$) and the boundary $L_\sigma(t)$ is stress-free.

We assume that the time $\tau_0 = \tau_0(x_1, x_2)$ at which the load is applied to the newly added elements coincides with the time $\tau^* = \tau^*(x_1, x_2)$ at which they are added to the growing body.

The AM fabrication of the body terminates at time $\tau_2 \geq \tau_1$, and after that the body occupies the domain $\Pi_2 = \Pi(\tau_2)$ whose cross-section $\Omega_2 = \Omega(\tau_2)$ has the boundary $L_2 = L(\tau_2)$. Note that throughout the following we consider sufficiently slow processes, so that the inertial terms in the equilibrium equations can be neglected.

The boundary value problem for the main (original) viscoelastic ageing body on the time
interval \([\tau_0, \tau_1]\) is a traditional torsion problem of viscoelasticity theory.

The initial-boundary value problem for the continuously growing body on the time interval \(t \in [\tau_1, \tau_2]\) consists of the equilibrium equations

\[
\frac{\partial \sigma_{13}}{\partial x_3} = 0, \quad \frac{\partial \sigma_{23}}{\partial x_1} = 0, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0,
\]

the Cauchy relations (see (1) and (2))

\[
D_{11} = \frac{\partial v_1}{\partial x_1} = 0, \quad D_{22} = \frac{\partial v_2}{\partial x_2} = 0, \quad D_{33} = \frac{\partial v_3}{\partial x_3} = 0,
D_{12} = \frac{1}{2} \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) = 0, \quad D_{13} = \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right), \quad D_{23} = \frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right)
\]

between the strain and displacement rates, the state equations (see (3))

\[
\sigma_{13} = 2G(I + N_{\tau_0})\varepsilon_{13}, \quad \sigma_{23} = 2G(I + N_{\tau_0})\varepsilon_{23},
\]

the boundary condition (see condition (4))

\[
(x_1, x_2) \in L_\sigma(t) : \quad n_1 \sigma_{13} + n_2 \sigma_{23} = 0
\]

on the immovable part of the boundary, the boundary condition (see condition (5))

\[
(x_1, x_2) \in L^*(t) : \quad n_1 \frac{\partial \sigma_{13}}{\partial t} + n_2 \frac{\partial \sigma_{23}}{\partial t} = 0
\]

on the growth boundary \(L^*(t)\), and the equilibrium conditions

\[
M(t) = \int_{\Omega(t)} (x_1 \sigma_{23} - x_2 \sigma_{13}) \, dx_1 \, dx_2, \quad \int_{\Omega(t)} \sigma_{13} \, dx_1 \, dx_2 = \int_{\Omega(t)} \sigma_{23} \, dx_1 \, dx_2 = 0
\]

for the face cross-sections \(\Omega(t)\), where we use the notation defined after relations (5). The values of all functions at time \(\tau_0 \leq t \leq \tau_1\) are known from the solution of the problem for the main body.

**Figure 5.** Shear stress intensity in the vicinity of the elliptic hole for the mode of AM fabrication with a step-function tension.
By analogy with the preceding, we transform the boundary value problem (14)–(19) of AM fabrication of a viscoelastic rod into the problem of AM fabrication of an elastic body, where

\[
\frac{\partial \sigma_{13}^0}{\partial x_1} + \frac{\partial \sigma_{23}^0}{\partial x_2} = 0, \\
D_{13} = \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right), \quad D_{23} = \frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right), \\
\sigma_{13}^0 = 2 \varepsilon_{13}, \quad \sigma_{23}^0 = 2 \varepsilon_{23}, \\
(x_1, x_2) \in L_\sigma(t) : \quad n_1 \sigma_{13}^0 + n_2 \sigma_{23}^0 = 0, \\
(x_1, x_2) \in L^*(t) : \quad n_1 \frac{\partial \sigma_{13}^0}{\partial t} + n_2 \frac{\partial \sigma_{23}^0}{\partial t} = 0, \\
M(t) = \int_{\Omega(t)} (x_1 \sigma_{23}^0 - x_2 \sigma_{13}^0) \, dx_1 \, dx_2.
\]

Let us transform the initial-boundary value problem (20) into a boundary value problem for the strain rates, displacement rates, and operator stress rates. To this end, we differentiate the equilibrium equation and the state equation with respect to \(t\). As a result, we obtain the boundary value problem

\[
\frac{\partial S_{13}}{\partial x_1} + \frac{\partial S_{23}}{\partial x_2} = 0, \\
D_{13} = \frac{1}{2} \left( \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \right), \quad D_{23} = \frac{1}{2} \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \right), \\
S_{13} = 2D_{13}, \quad S_{23} = 2D_{23}, \quad S_{ij} = \frac{\partial \sigma_{ij}^0}{\partial t}, \\
(x_1, x_2) \in L(t) : \quad n_1 S_{13} + n_2 S_{23} = 0, \\
\frac{dM^0(t)}{dt} = \int_{\Omega(t)} (x_1 S_{23} - x_2 S_{13}) \, dx_1 \, dx_2 + \int_{L^*(t)} (x_1 \sigma_{23}^* - x_2 \sigma_{13}^*) \, dl.
\]

Being supplemented with the initial conditions for the main body for \(t = \tau_1\), relations (21), which also contain the initial-boundary condition on the growth boundary, form an initial-boundary value problem with parameter \(t\).

For the variables \(S_{ij}\) and \(v_i\) we have the formulas

\[
v_1 = -\theta'(t)x_2x_3, \quad v_2 = \theta'(t)x_1x_3, \quad v_3 = \theta'(t)\varphi_t, \\
S_{13} = \theta'(t) \left( \frac{\partial \varphi_t}{\partial x_1} - x_2 \right), \quad S_{23} = \theta'(t) \left( \frac{\partial \varphi_t}{\partial x_2} + x_1 \right),
\]

where \(\varphi_t = \varphi(x_1, x_2, t)\) is the torsion function to be determined, \(\theta(t)\) is the torsion angle (twist), and \(\theta'(t)\) is the time derivative of the twist. The torsion function \(\varphi_t\) is harmonic in the domain \(\Omega(t)\). It must satisfy the boundary condition

\[(x_1, x_2) \in L(t) : \quad \frac{\partial \varphi_t}{\partial n} = x_2 n_1 - x_1 n_2.\]

Thus, the solution of the torsion problem is reduced with the use of the torsion function \(\varphi_t\) to determining a harmonic function in the cross-section domain \(\Omega(t)\) from a given value of its normal derivative on the contour \(L(t)\) (the Neumann problem).
Obviously, in view of (21) one obtains
\[ \frac{dM(t)}{dt} = \theta'(t)D(t) + \int_{L^*(t)} (x_1 \sigma^*_{23} - x_2 \sigma^*_{13}) \, dl, \]
\[ D(t) = \int\int_{\Omega(t)} \left( x_1^2 + x_2^2 + x_1 \frac{\partial \varphi}{\partial x_2} - x_2 \frac{\partial \varphi}{\partial x_1} \right) \, dx_1 \, dx_2, \tag{23} \]
where \( D(t) \) is the variable torsional rigidity of the growing body.

Consider two possible versions of the statement of the problem.

1. The torque \( M(t) \) is given, and the stresses \( \sigma_{ij} \), the displacements \( u_i \), and the twist \( \theta(t) \) must be determined.

2. The twist \( \theta(t) \) is given, and \( \sigma_{ij} \), \( u_i \), and \( M(t) \) must be determined.

The solution for the first version of the statement (where the torque \( M(t) \) is given) can be constructed as follows:

- The boundary condition is used to determine the function \( \varphi_t \).
- Formulas (23) are used to find \( \theta(t) \).
- Formulas (22) give \( v_i \) and \( S_{ij} \).
- Finally, the true stresses \( \sigma_{ij} \) are reconstructed by formula (8).

For the second version of the statement,

- The torsion function \( \varphi(x_1, x_2, t) \) is determined.
- The functions \( v_i \) and \( S_{ij} \) are found by formulas (22).
- The true stresses \( \sigma_{ij} \) are reconstructed by formula (8).
- The torque \( M(t) \) is determined from the equilibrium (20) for the face cross-sections.

Now consider the stage of torsion of the body after the termination of growth. Let the AM fabrication of the body terminate at time \( \tau_2 \). By then, the body occupies the domain \( \Pi_2 \) whose cross-section \( \Omega_2 \) is bounded by the contour \( L_2 \). Here we obtain a problem similar to (21). In this case, the variables \( v_i \) and \( S_{ij} \) can be obtained by formulas (22), where one sets \( t = \tau_2 \). The stresses, displacements, and twist are given by formulas (8).

As a result, we have reduced the nonclassical boundary value problems arising in the analysis of torsion of AM fabricated bodies to well-known boundary value problems containing some parameter. The stress-strain state of the body can be completely reconstructed from the solutions of the latter problems by the deciphering formulas (8).

To solve the classical boundary value problems with a parameter, one can apply two methods of the theory of functions of a complex variable.

The first method is based on the reduction of the equation of the boundary to a special form, and the second method relies on the application of a conformal mapping. Consider the first method.

Consider the complex torsion function
\[ F(z, t) = \varphi + i\psi, \]
where \( \varphi \) is the torsion function and \( \psi \) is the conjugate harmonic function.

In view of the formulas
\[ F(z, t) + \overline{F(z, t)} = 2 \text{Re} \ F(z, t), \quad F(z, t) - \overline{F(z, t)} = 2 \text{Im} \ F(z, t). \]

The boundary condition in complex form reads
\[ z \in L(t) : \quad F(z, t) - \overline{F(z, t)} = z\bar{z}. \]
Assume that the relationship between \( z \) and \( \bar{z} \) for the points of the contour can be reduced to the form

\[
z \in L(t) : \quad z\bar{z} = h(z, t) + \overline{h(z, t)},
\]

where \( h(z, t) \) is a function analytic inside the cross-section contour. Then one has

\[
F(z, t) = ih(z, t) + C(t),
\]

(24)

where \( C(t) \) is an arbitrary function, which can be taken to be zero.

Having determined the complex torsion function \( F(z, t) \), we obtain the torsion function

\[
\varphi = \text{Re} \{F(z, t)\}.
\]

(25)

This method can be used to solve problems such as the torsion of an ellipse, of an equilateral triangle, and of a circular shaft with a longitudinal recess.

4. Torsion of an AM fabricated shaft

By way of example, consider the problem on the torsion of an AM fabricated shaft with a longitudinal recess. Assume that a circular shaft \( \Pi_1 \) with a longitudinal recess is manufactured at time zero from an ageing viscoelastic material. The cross-section \( \Omega(t) \) is the intersection of two circles (figure 6),

\[
x_1^2 + x_2^2 \geq b(t), \quad (x_1 - a)^2 + x_2^2 \leq a^2 \quad (b(t) < a).
\]

The boundary \( L(t) \) of the section \( \Omega(t) \) consists of two parts, \( L(t) = L^*(t) \cup L_\sigma(t) \), where \( L^*(t) \) is the growth boundary (the contour corresponding to the small circle of variable radius \( b(t) \)) and \( L_\sigma(t) \) is the stress-free boundary. The shaft growth law is completely determined by the function \( b(t) \).

We assume that the newly added elements are stress-free. Since the problems obtained for each stage of AM fabrication are mathematically equivalent, it suffices to consider the stage where material is being added. Then the initial-boundary value problem for the shaft acquires the form (21).
The torsion function \( \varphi(x_1, x_2, t) \) is given by the formula

\[
\varphi(x_1, x_2, t) = -ax_2 - \frac{ab^2(t)x_2}{x_1^2 + x_2^2}.
\]

Having determined the torque \( M(t) \), we find

\[
\frac{dM^0(t)}{dt} = \frac{M'_1(t)}{G(t)} + \int_{\tau_0(x_1, x_2)}^{t} \frac{\partial M(t)}{\partial \tau} \frac{\partial \omega(t, \tau)}{\partial t} d\tau + M(\tau_0(x_1, x_2)) \frac{\partial \omega(t, \tau_0(x_1, x_2))}{\partial t}
\]

and then the twist rate

\[
\theta'_i(t) = \frac{1}{2a_1D(t)} \frac{dM^0(t)}{dt}, \quad D(t) = \frac{1}{24} \{ \sin[4\alpha(t)] + 8 \sin[2\alpha(t)] + 12\alpha(t) \},
\]

\[
- \frac{b^2(t)}{2a^2} [\sin[2\alpha(t)] + 2\alpha(t)] + \frac{b^4(t)}{3a^3} \sin \alpha(t) - \frac{b^4(t)}{4a^4} \alpha(t), \quad \frac{b(t)}{a} = 2 \cos \alpha(t).
\]

The displacement rate and the variables \( S_{13} \) and \( S_{23} \) are found by the formulas

\[
v_1 = -\theta'_1(t)x_2x_3, \quad v_2 = \theta'_1(t)x_1x_3, \quad v_3 = \theta'_2(t) \left[ -ax_2 - \frac{ab^2(t)x_2}{x_1^2 + x_2^2} \right],
\]

\[
S_{13} = \frac{1}{2a^2D(t)} \left[ \frac{2ab^2(t)x_1x_2}{(x_1^2 + x_2^2)^2} - x_2 \right] \frac{dM^0(t)}{dt},
\]

\[
S_{23} = \frac{1}{2a^2D(t)} \left[ - \frac{2ab^2(t)(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2} + x_1 - a \right] \frac{dM^0(t)}{dt}.
\]

The true characteristics of the stress-strain state are reconstructed by the deciphering formulas (8). To construct the solution at the per- and post-growth stages, it suffices to set \( t = \tau_1 \) and \( t = \tau_2 \), respectively, in the torsion function.

Let us carry out the computations for the shaft in view of the fact that it is manufactured from a viscoelastic ageing material with the mechanical characteristics given above. Consider the AM fabrication (recovery) of a shaft subjected to a torque for the case in which the recess radius \( b(t) \) decreases fivefold from \( b_1 = 0.5 \) to \( b_1 = 0.1 \). All notation of the curves is the same as in the preceding example.

Let us study the shear stress intensity distribution along the diameter of the cross-section of a shaft with a longitudinal recess depending on how the torque varies in time. The shear stress intensity under torsion is given by the formula

\[
\sigma_i = \sqrt{\sigma_{13}^2 + \sigma_{23}^2}.
\]

Assume that \( M(t) = 1 \). Consider AM fabrication processes with \( \tau_0 = 0.1 \) and \( \tau_1 = 0.3 \). For rapid AM fabrication \( (\tau_2 = 0.6) \), the shear stress intensity is maximal on the boundary of the fabricated body at the base of the recess (figure 7).

Slow AM fabrication \( (\tau_2 = 6) \) is characterized by the fact that the stress state of the main body remains practically unchanged, while the additional body remains practically stress-free (figure 8).

Consider also AM fabrication with \( \tau_0 = 0.1, \tau_1 = 1, \) and \( \tau_2 = 2 \). Let the torque be given by

\[
M(t) = \begin{cases} 
0.1 & t \leq T, \\
1.8 & t > T.
\end{cases}
\]

If \( T \in (\tau_1, \tau_2) \), then the maximum shear stress intensity is attained at an interior point of the AM fabricated body. Figure 9 shows the variations in the shear stress intensity for \( T = 2 \).
Figure 7. Shear stress intensity for the process of rapid AM fabrication of the shaft under constant torque.

Figure 8. Shear stress intensity for the process of slow AM fabrication of the shaft under constant torque.

Conclusions

- We develop surface growth theory for studying 2D problems of AM fabrication of deformable bodies for the case in which the surface strain rate of the bodies due to forces and prestress of the added elements can be neglected compared with the new material inflow rate on the surface.

- We give the statement of classical and nonclassical initial-boundary value problems arising in the theory. Solution methods are proposed based on the reduction of nonclassical problems of AM fabrication of viscoelastic ageing bodies to elasticity problems with a parameter, the use of the theory of analytic functions for solving the latter problems, and on the reconstruction of true characteristics of the stress-strain state of the bodies with the help of deciphering formulas obtained.

- It is discovered that the maximum shear stress intensity in the manufactured object in 2D
Figure 9. Shear stress intensity for the AM fabricated shaft under a step function torque.

problems on stress concentration near holes and on torsion is attained on the boundary of the body if the AM manufacturing process is not taken into account. For additive manufacturing, the maximum shear stress intensity can be attained on the interface between the main body and the added part, on the boundary of the AM fabricated body, and at an arbitrary point of the AM fabricated part of the body.

- The results can serve as a basis for the solution of important applied problems for parts and structural elements fabricated by additive manufacturing technologies.

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