On 1/2-BPS Wilson-'t Hooft loops

Bin Chen\(^1\), and Wei He\(^{2,3}\)

\(^1\) School of Physics, Peking University, Beijing 100871, China
\(^2\) Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, China
\(^3\) Graduate University of Chinese Academy of Sciences, Beijing 100080, China

E-mail: bchen01@pku.edu.cn, weihe@itp.ac.cn

Abstract: We investigate the 1/2-BPS Wilson-'t Hooft loops in \(\mathcal{N} = 4\) Super-Yang-Mills theory. We use the bulk D-brane with both electric and magnetic charges to calculate the all genus contribution of the circular loops. The expectation value of Wilson-'t Hooft loops are in perfect agreement with the result through supersymmetric condition and duality transformation in the gauge theory.

Keywords: Wilson-'t Hooft loop, AdS/CFT correspondence.
1. Introduction

The AdS/CFT correspondence asserts the equivalence between type IIB string theory in the $AdS_5 \times S^5$ space with RR background self-dual 5-form field strength and $\mathcal{N}=4$ super-Yang-Mills (SYM) theory [1, 2]. It provides a concrete example of gauge theory/string theory duality and realizes the holographic feature of the quantum gravity in a very special case. The AdS/CFT conjecture has passed some nontrivial tests by performing quantitative calculation on both side [2]. One of the most impressive verification is on Wilson loop [3, 4]. In gauge theory, Wilson loops can be viewed as the worldlines of a quark with the electric charge. They define the holonomy matrices of the gauge group along the loops. Wilson loop plays the role of the order parameter of different phases of the gauge theory, for example it presents the perimeter law in Higgs phase and the area law in the confining phase. It is a natural variable to formulate the dynamics of the gauge theory, though the resulting loop equation is too hard to solve.

Wilson loop provides an intuitive approach to gauge theory/string theory correspondence. Classically the loop sweep out a worldsheet in four dimensional spacetime. If we quantize the motion of the loop, the quantum anomaly requires another dimension: the fifth dimension emerges holographically [5]. If the gauge theory is the maximally supersymmetric $\mathcal{N}=4$ super-Yang-Mills theory, the superconformal symmetry requires the fifth dimension, together with $x^\mu$ to be Anti-de-Sitter space.
The $\mathcal{N}=4$ SYM consists of a vector multiplet $(A_\mu, \phi^I, \lambda^A_\alpha)$ with $\mu = 1 \cdots 4, I = 1 \cdots 6, A = 1 \cdots 4, \alpha = 1, 2$. Wilson loop in $\mathcal{N}=4$ SYM is labelled by a curve $C$ in $\mathbb{R}^4$ and the representation $R$:

$$< W >_R (C) = \text{Tr}_R Pe^{\int_C ds (A_\mu \dot{x}^\mu(s) + \theta^I(s) \phi_I |\dot{x}(s)|)}.$$ (1.1)

where $\theta^I(s), (I = 1 \cdots 6)$ is a unit vector in $\mathbb{R}^6$, defining a track on $S^5$ where Wilson loop pass, $P$ denotes the path-ordering along $C$. In the $AdS_5$ space the corresponding configuration is a macroscopic string stretched to the boundary, with the boundary condition determined by Wilson loop. The worldsheet of the string forms a minimal surface in $AdS_5$ space to minimize its classical action. From AdS/CFT correspondence, the expectation value of Wilson loop is just the partition function of the string

$$< W >_R (C) = e^{-A(C)}.$$ (1.2)

At large 't Hooft coupling the partition function can be evaluated by the saddle-point approximation. It is the area of the minimal surface which can be computed by Nambu-Goto action of the string.

$$A(C) = \int d^2\sigma \sqrt{\text{det}h_{ab}}.$$ (1.3)

As the stretched string is infinitely long, the evaluation always suffers the divergence, which corresponds to the self energy of the electric particle in the gauge theory. After regulating the divergent term, we get a finite action.

For the infinitely straight line, it preserves half of the Poincare supersymmetries. Due to non-renormalization, it does not receive the quantum correction, its expectation value is simply the unit

$$< W >_{\text{line}} = 1.$$ (1.4)

It means that the corresponding worldsheet has zero regularized area.

For the circular Wilson loop, it is necessary to consider the behavior of $\theta^I(s)$ along the loop. As analyzed in [1], if the direction of $\theta^I(s)$ varies along the loop in a proper way, Wilson loop preserves half of Poincare supersymmetries and due to non-renormalization its expectation value is also the unit. The corresponding minimal surface in the bulk has zero regularized area. However, there is another kind of the circular loop, which is obtained by a conformal transformation of the straight line, with $\theta^I(s)$ taking constant value along the loop. It has been explained in [28] that this loop preserves 16 $s$-independent linear combination of Poincare and the conformal supersymmetries. It is called 1/2 BPS Wilson loop. As we must choose a UV/IR cut-off in the field/string side in the regularization process, the remaining symmetries are broken and the expectation value receives quantum correction and depends on the coupling constant in a non-trivial way. In [3, 4], the 1/2-BPS Wilson loop in SYM was analyzed. Based on a conformal anomaly argument, an all order result in $1/N$ and $\lambda$ was obtained. In SYM, the contribution to the expectation value of Wilson loop comes from the so called rainbow graphs with the gluon propagator modified by a singular total derivative, which give non-zero contribution only when both ends of the propagator are located at the point which is conformally mapped to infinity.
The problem is reduced to a quadratic hermitian matrix model with the expectation value of Wilson loop given by

\[
< W >_{\text{circular}} = \frac{1}{Z} \int D M \frac{1}{N} \text{Tr} e^{M} e^{-\frac{2N}{\lambda} \text{Tr} M^{2}}
\]

\[
= \frac{2}{\sqrt{\lambda}} I_{1}(\sqrt{\lambda}) + \frac{\lambda}{48N^{2}} I_{2}(\sqrt{\lambda}) + \frac{\lambda^{2}}{1280N^{4}} I_{4}(\sqrt{\lambda}) + \cdots .
\]  

(1.5)

Where \( I_{n}(x) \) is the first kind modified Bessel function. In the large \( N \) and large \( \lambda \) limit, it simplifies to

\[
< W >_{\text{circular}} \sim \sqrt{\frac{2}{\pi}} e^{\sqrt{\lambda}} \frac{\lambda^{3/4}}{4} \sim e^{\sqrt{\lambda}} .
\]  

(1.6)

In [3, 4] the expectation value of a circular Wilson loop is calculated by evaluating the action of the string worldsheet, by regulating the action properly. To the leading order in \( 1/N \) and \( \lambda \), the result is in agreement with field theory calculation

\[
< W >_{\text{circular}} = e^{-\text{Area}} = e^{\sqrt{\lambda}} .
\]  

(1.7)

In principle, the string worldsheet extended to the bulk could receive higher genus contribution, this corresponds to finite \( N \) effect in the field theory.

Recently, an important development is to calculate all genus contributions to Wilson loop using bulk D-brane in the string theory side. It is first proposed in [3] for the fundamental representation and followed by [1, 2, 3, 27] for higher rank symmetric and antisymmetric representations. The reason is that the string worldsheet in the five form field strength background can blow up in the transverse direction to form the dielectric brane [17, 18, 19, 30], analogous to the formation of the giant gravitons [20, 21, 22].

Now the expectation value of Wilson loop is the partition function of D-brane with string charges on it. It is obtained by evaluating the D-brane action instead of the string action. On the gauge theory side, the corresponding quantities are also evaluated in the matrix model [12, 14, 15, 16].

It is analyzed in [13] that the string worldsheet can blows up in two ways, producing two kinds of branes:

- The strings blow up in \( S^{2} \subset AdS_{5} \), resulting in a D3 brane with induced metric \( AdS_{2} \times S^{2} \) and \( k \) units of fundamental string charge. This corresponds to Wilson loop in the symmetric representation of rank \( k \).
- The strings blow up in \( S^{4} \subset S^{5} \), resulting in a D5 brane with induced metric \( AdS_{2} \times S^{4} \) and \( k \) unites of fundamental string charge. This corresponds to Wilson loop in anti-symmetric representation of rank \( k \).

For Wilson loop in the symmetric representation, both D-brane calculation and the matrix model calculation yield the following result

\[
< W >_{\text{sym}} = e^{2N(\kappa \sqrt{1+\kappa^{2}}+\sinh^{-1} \kappa)}. 
\]  

(1.8)

with \( \kappa = \frac{k}{\sqrt{\lambda}} \). For Wilson loop in the anti-symmetric representation, D-brane calculation and matrix model calculation yield

\[
< W >_{\text{asym}} = e^{\frac{2\kappa \sqrt{\lambda}}{3\kappa} \sin^{4} \theta_{k}}.
\]  

(1.9)
with $\theta_k - \frac{1}{2}\sin 2\theta_k = \frac{\pi k}{N}$.

In the gauge theory there is another nonlocal operator: 't Hooft loop, which is the magnetic dual of Wilson loop. It is defined as a singular gauge transformation along the loop and can be viewed as the worldline of a particle with magnetic charge. Its expectation value behaves oppositely to Wilson loop: the perimeter law in the confining phase and the area law in Higgs phase. In a general gauge theory, the property of 't Hooft loop is hard to calculate because the electric-magnetic duality is rather obscure. In the $\mathcal{N}=4$ SYM, which inherits a SL(2, Z) duality and is self-dual, it is quite natural to consider 't Hooft loop together with Wilson loop. For simplicity we set the axion to be zero first and discuss the full SL(2, Z) duality later. Under S-duality, Wilson loop becomes 't Hooft loop, the fundamental string (F1) becomes D-string (D1), so the macroscopic object corresponding to 't Hooft loop is the D-string worldsheet. Analogous to Wilson loop case we can use D3/NS5 brane with D-string charge to calculate the all genus contribution.

In this paper, we will study 1/2 BPS Wilson-'t Hooft loops. From the gauge theory point of view, Wilson-'t Hooft loop is the worldline of a dyon which carry both the electric charge and magnetic charge[24]. From string theory point of view, Wilson-'t Hooft loops are the (F1, D1) bound states[23] with the strings ending on the worldvolume of the D3 branes. The charge of (F1, D1) corresponds to the charge of the dyon. However there is a subtlety. Wilson loop can be in any representation of the gauge group. For the symmetric representation of rank $k$, the corresponding fundamental string is a single string with charge $k$, while for Wilson loop in the antisymmetric representation, the corresponding fundamental strings are $k$ F1’s each with charge one. For a generic representation, the configuration of the fundamental strings is quite complicated. For Wilson-'t Hooft loops, the existence of D-string make things more subtle. Similar to the F1’s, the D1’s could have different kinds of combination, corresponding to the different representation of the dual group. In this paper, we will consider the case that both the F1’s and D1’s are in the symmetric representation. In other words, the F1’s and D1’s form a simple bound state.

The paper is organized as follows. In section 2, we will calculate the all-genus contribution to 1/2 BPS Wilson-'t Hooft loops using bulk D3-brane. We consider straight Wilson-'t Hooft line first and then the circular Wilson-'t Hooft loop. In both cases, the calculations shows that the expectation value of the line/loop depends on the charges and the coupling constant in a S-dual invariant way. In section 3, we analyze the supersymmetry of the bulk D3-brane with fluxes, it preserves half of the original supersymmetries. In section 4, we discuss WH loop in the antisymmetric representation related to bulk 5-brane configurations. We end the paper with conclusion and discussions.

2. All-genus calculation using D3-branes

In [11], it has been shown that one can use the bulk D3-brane to evaluate Wilson loop operators. By evaluating the classical action of D3 brane with $n$ F-string charge, the all genus expectation value of the circular BPS Wilson loop is

$$<W> = e^{2N(\sqrt{1+\kappa^2}+\sinh^{-1}\kappa)}$$ (2.1)
with $\kappa = n\sqrt{\frac{\lambda}{4N}}$ and $n$ is the F-string charge on D3-brane. The all genus expectation value of the circular BPS 't Hooft loop is also evaluated

$$<H> = e^{2N(\kappa\sqrt{1+\kappa^2+\sinh^{-1}\kappa})}$$

(2.2)

now with $\kappa = \frac{\pi m}{\sqrt{\tilde{\lambda}}}$ and $m$ is the D-string charge on D3-brane. Here $\tilde{\lambda} = \frac{16\pi^2 N^2}{\lambda}$ is the dual coupling constant.

In this section we will evaluate all genus expectation value of the straight and the circular BPS Wilson-'t Hooft (WH) loop using D3 brane with both F-string and D-string charges. The F-string and D-string would form bound state preserving $1/2$ supersymmetry\[23\]. For a $(n, m)$-string, the string tension is

$$\tau_{n,m} = \frac{\sqrt{n^2 + m^2/g_s^2}}{2\pi \alpha'}$$

(2.3)

which is invariant under S-duality: $(n, m, g_s, \alpha') \leftrightarrow (-m, n, g_s^{-1}, \alpha' g_s)$. In terms of field theory coupling constant $\lambda$ it is

$$\tau_{n,m} = \frac{1}{\pi \alpha' \sqrt{\lambda}} \sqrt{n^2 \lambda + m^2 \tilde{\lambda}}$$

(2.4)

When the $(n, m)$-string ends on the D3-brane, the resulting system is still BPS. We expect that the expectation value takes the same form as Wilson loop, with a suitable dependence on $\sqrt{n^2 \lambda + m^2 \tilde{\lambda}}$. Our calculation shows it is indeed the case.

In the calculation an essential point is to take into account of the boundary contribution to the action. As D3 brane in the bulk intersects boundary along the loop, we should take the possible boundary terms into account. The boundary terms implement the Legendre transformations to let the solution have the correct boundary conditions. There are two kinds of the boundary terms: one comes from the Neumann boundary condition on the transverse radial direction; the other comes from the Legendre transform of the gauge fields such that the charges on the brane worldvolume is fixed.

For coordinates $x^i$ satisfying Neumann boundary condition, we need to perform a Legendre transformation on the brane action to make the action to be a function of $p_i$:

$$\tilde{S} = S - \int_C d\sigma p_i x^i,$$

(2.5)

where $p_i$ is the conjugate momentum of $x^i$

$$p_i = \frac{\partial S}{\partial x^i}.$$  

(2.6)

For $AdS_5 \times S^5$ space, the coordinates take Dirichlet boundary condition on the direction parallel to the boundary of $AdS$ and take Neumann boundary condition on the remaining directions. As D3 brane does not extend to $S^5$ part, we should take into account only the boundary term related to the radial direction.
Similarly, for the gauge field $A_i$ on the brane, its conjugate momentum is
\[ \Pi_i = \frac{\partial S}{\partial \dot{A}_i} \]  
(2.7)
the corresponding boundary term is
\[ -i \int_C ds \Pi_i A_i. \]  
(2.8)
Here we include $i = \sqrt{-1}$ factor because when evaluating the action we always use Euclidean version of $AdS_5 \times S^5$ metric. In the previous work evaluating Wilson loop using the string worldsheet, there is no boundary term related to gauge field.

### 2.1 Straight Wilson-'t Hooft line

Let us start from the infinite straight Wilson-'t Hooft line in $\mathbb{R}^4$. It is convenient to use the following coordinate system for $AdS_5$:
\[ ds^2 = \frac{L^2}{y^2} (dy^2 + (dx^1)^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)). \]  
(2.9)
Where $L^4 = \lambda \alpha'^2$ is the radius of $AdS_5$ and $S^5$. The straight Wilson-'t Hooft line lies along $X^1$ and is localized in the transverse directions. One may understand the physics from dielectric effect and try to calculate the expectation value of Wilson-'t Hooft line from the D3-brane action. As has been shown in [11], the world volume of D3-brane is a hypersurface in $AdS_5$, which is characterized by the equation $y = y(r)$ and world volume coordinates $(x^1, r, \theta, \varphi)$.

For D3 brane ending on the boundary of $AdS_5$, the action includes three parts: the Dirac-Born-Infeld action, the Wess-Zumino action and the boundary term
\[ S_t = S_{DBI} + S_{WZ} + S_{boundary} \]  
(2.10)
The Dirac-Born-Infeld action is of the form
\[ S_{DBI} = -T_{D3} \int e^{-\Phi} \left( \frac{L^2}{y^2} \right)^2 \sqrt{1 + y'^2 + \left( \frac{y^2}{L^2} \right)^2 (2\pi \alpha')^2 F_{tr}^2} \left( r^4 \sin^2 \theta + \left( \frac{r^2}{L^2} \right)^2 (2\pi \alpha')^2 F_{\theta \varphi}^2 \right) \]  
(2.11)
where $F_{tr}$ is the electric field and $F_{\theta \varphi}$ is the magnetic field. And the tension of the D3-brane is
\[ T_{D3} = \frac{N}{2\pi^2 L^4}. \]  
(2.12)
The Wess-Zumino term contributes to the action
\[ S_{WZ} = \mu_{D3} \int P[C_4] = -\frac{2N}{\pi} \int dx^1 dr \frac{r^2}{y^4} \]  
(2.13)
where $\mu_{D3} = T_{D3}$ is the charge of D3 brane, $P[C_4]$ is the pullback of the Ramond-Ramond 4-form potential to the worldvolume of D3-brane and
\[ C_4 = \frac{L^4 r^2 \sin \theta}{y^4} dx^1 \wedge dr \wedge d\theta \wedge d\varphi \]  
(2.14)
The total bulk action is just $S_{\text{bulk}} = S_{\text{DBI}} + S_{\text{WZ}}$.

Without the magnetic field, we have simply the straight Wilson line and the electric field takes the form

$$F_{1r} = \frac{i n \lambda}{8 \pi N r^2}$$

which solves the equation of motion and spread over the worldvolume of $AdS_2$ uniformly. Here the appearance of $i$ is also due to Euclidean metric. In this case, it has been shown that the action of D3-brane is vanishing due to the cancellation between DBI and WZ action. Even after considering the boundary terms, the action is still vanishing, implying that the expectation value of Wilson line $<W> = 1$ which has to be true due to the supersymmetric condition. In the case without the electric field, we have the straight 't Hooft line, which is the S-dual of Wilson line. The corresponding magnetic field takes the form

$$F_{\theta \phi} = \frac{m \sin \theta}{2}$$

which spread over the $S^2$ uniformly. The straight 't Hooft line has not been treated carefully in the literature but due to the supersymmetry and S-duality, it has been expected that $<H> = 1$. We will not discuss it separately. Instead, we will discuss the more general case with both electric and magnetic charges, namely Wilson-‘t Hooft line.

In the case of Wilson-‘t Hooft line, the electric field and magnetic field take the form as (2.15, 2.16). And we make the linear ansatz

$$y = \frac{r}{\kappa}$$

(2.17)

With these setups, the bulk action is not vanishing and instead diverges near $y = 0$. Let us introduce a cutoff $y_0$ and the bulk action is

$$S_{\text{bulk}} = -\frac{2N}{\pi} X^1 (bc - 1) \frac{\kappa^3}{y_0}$$

(2.18)

where

$$b = \sqrt{1 + \kappa^{-2} - \frac{n^2 \lambda}{16 N^2 \kappa^{-4}}}$$

(2.19)

$$c = \sqrt{1 + (\frac{\pi \alpha' m}{L^2})^2}$$

(2.20)

On the other hand, one has to take into account of the boundary terms[11]. One kind of such boundary terms is

$$-\int dx^1 y_0 p_y$$

(2.21)

where $p_y$ is the conjugate momentum to $y$ after integration over $S^2$:

$$p_y = -\frac{2N}{\pi} \frac{\kappa c}{y_0 b}$$

(2.22)

The other boundary term comes from the Legendre transform of the gauge field:

$$-\int dx^1 (i \Pi) A_1 = -\int dx^1 dr (i \Pi) F_{1r} = -\frac{c}{b} X^1 \frac{n^2 \lambda \kappa^{-1}}{8 \pi N y_0}.$$  

(2.23)
After putting all the contribution together, the total action is
\[ S_t = -\frac{2\pi}{N} X^1 Z \frac{1}{y_0} \]  
where
\[ Z = \kappa^3 (bc - 1) + \frac{c}{b} \frac{n^2 \lambda}{16N^2} \kappa^{-1} - \kappa). \]

The divergence near \( y_0 \to 0 \) should be absent, this leads to the condition \( Z = 0 \). This helps us to fix the value of \( \kappa \):
\[ \kappa^2 = n^2 \lambda \left( \frac{4N}{2} \right)^2 + \frac{\pi^2 m^2}{\lambda}. \]  

Using the dual coupling constant \( \tilde{\lambda} = \frac{16\pi^2 N^2}{\lambda} \), \( \kappa \) can be written as
\[ \kappa^2 = \frac{n^2 \lambda}{(4N)^2} + \frac{m^2 \tilde{\lambda}}{(4N)^2}. \]  

With the divergence vanishing, we have \( S_{\text{tot}} = 0 \) so that \( < WH > = 1 \). This is consistent with the BPS condition.

### 2.2 Circular Wilson-'t Hooft loop

For the circular Wilson loop, it could be related to the straight Wilson line by a conformal transformation. Due to the quantum anomaly of the conformal transformation, the expectation value of the circular Wilson loop is not 1. On the N=4 SYM side, the calculation reduces to a quadratic Hermitian matrix model [9, 10]. For Wilson loop in the rank \( k \) symmetric representation, \( < W >_S = \frac{1}{N} \text{Tr} e^{nM} \). The calculation in the Matrix model shows that to all orders in \( g_{YM} \) and \( \frac{1}{N} \) expansion:
\[ < W >_S = e^{2N[\kappa \sqrt{1 + \kappa^2 + \sinh^{-1} \kappa}]}, \]  
which is in perfect agreement with the calculation from bulk D3-brane on the dual string theory side. Furthermore, the expectation value of 't Hooft loop in the symmetric representation of the dual group has also been worked out in [11]. It takes the same form of (2.28), but with a different \( \kappa \).

For Wilson-'t Hooft circular loop, we don’t know how to calculate it from Super-Yang-Mills theory since we have to work in a background with magnetic monopole. Nevertheless, the BPS condition and the \( SL(2, Z) \) duality in the \( \mathcal{N} = 4 \) SYM permits us to get the answer. Actually, in both Wilson and 't Hooft cases \( \kappa \)'s are proportional to the tension of the corresponding macroscopic strings. This fact leads us to predict that for Wilson-'t Hooft loop, the expectation value should take the same form as (2.28), but with the \( \kappa \) being proportional to the tension of (F1, D1) bound state.

Consider D3 brane with \( n \) F1 string charge and \( m \) D1 brane charge, The DBI action of D3 brane is
\[ S_{DBI} = -T_{D3} \int e^{-\Phi} \sqrt{\text{det}(g + 2\pi \alpha' F)} \]  

\[ (2.29) \]
To describe the circular loop, the convenient metric of $AdS_5$ space is

$$ds^2 = \frac{L^2}{\sin^2 \eta} (d\eta^2 + \cos^2 \eta d\psi^2 + d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)) \quad (2.30)$$

In these coordinates, $\rho \in [0, \infty), \theta \in [0, \pi]$ and $\eta \in [0, \pi/2]$. The boundary of the AdS space is now at $\eta \to 0, \rho \to 0$ and the circle on the boundary is located at $\eta = \rho = 0$. According to the symmetry, we take $\rho, \psi, \theta, \phi$ as the coordinates of D3 brane worldvolume which is curved in $(\eta, \rho)$ plane according to $\eta = \eta(\rho)$.

Take the following ansatz for the gauge field

$$F = F_{\psi \rho} d\psi \wedge d\rho + F_{\theta \phi} d\theta \wedge d\phi \quad (2.31)$$

The DBI action is

$$S_{DBI} = -2N \int d\rho d\theta 2N \frac{\sinh^2 \rho \sin \theta}{\sin^4 \eta} \frac{\cos \eta (1 + \eta') + (2\pi \alpha')^2 \frac{F_{\psi \rho}^2 \sin^4 \eta}{L^4}}{(1 + (2\pi \alpha')^2 (2.32))} \left( 1 + \frac{F_{\theta \phi}^2 \sin^4 \eta}{L^4 \sin^2 \theta \sinh^4 \rho} \right)$$

The WZ action is

$$S_{WZ} = 2N \int d\rho d\theta \frac{\cos \eta \sin \theta \sinh^2 \rho}{\sin^4 \eta} \left( \cos \eta + \eta' \sin \eta \frac{\sinh \rho \cosh \rho - \sin \rho \cos \theta}{\cosh \rho - \sin \rho \cos \eta} \right) \quad (2.33)$$

The equations of motion are solved by setting

$$\sin \eta = \kappa^{-1} \sinh \rho \quad (2.34)$$

with $\kappa$ a function of charges and coupling constant

$$\kappa^2 = \frac{n^2 \lambda}{(4N)^2} + \frac{m^2 \tilde{\lambda}}{(4N)^2} \quad (2.35)$$

And the gauge fields take value

$$F_{\psi \rho} = \frac{in\lambda}{8\pi N \sinh^2 \rho} \quad F_{\theta \phi} = \frac{m \sin \theta}{2} \quad (2.36)$$

These are just the same electric and magnetic fields (2.15, 2.16) in the new coordinates.

The DBI action becomes

$$S_{DBI} = -2N \int d\rho d\theta \frac{\sinh^2 \rho \sin \theta}{\sin^4 \eta} bc \quad (2.37)$$

where $b$ and $c$ are defined by

$$b = \sqrt{\cos^2 \eta (1 + \eta') - \frac{n^2 \lambda \sin^4 \eta}{(4N)^2 \sinh^4 \rho}}$$

$$c = \sqrt{1 + \frac{m^2 \tilde{\lambda} \sin^4 \eta}{(4N)^2 \sinh^4 \rho}} \quad (2.38)$$
Then the conjugate momentum of \( \eta \) is
\[
p_\eta = \frac{\delta S}{\delta \eta'} = -2N \frac{\kappa^2 \sin \theta \cos^2 \eta' c}{\sin^2 \eta} b
+ 2N \frac{\kappa^2 \cos \sin \theta}{\sin \eta} \left( \frac{\sinh \rho - \cosh \rho \cos \theta}{\cosh \rho - \sinh \rho \cos \theta} \right)
\] (2.39)

The boundary term for \( p_\eta \) at \( \eta_0 \to 0 \) is
\[
S_{p_\eta} = -\eta_0 \int p_\eta = -4N \kappa \frac{1}{\eta_0} \frac{c}{b}
\] (2.40)

The boundary term for gauge field is
\[
S_F = -i \int d\rho d\psi \Pi F_{\psi \rho}
= -\frac{n^2 \lambda c}{4N b} \coth \rho |_{\sinh \rho = \kappa} |_{\sinh \rho = \kappa \sin \eta_0}
\] (2.41)

where \( \Pi \) is the conjugate of \( F_{\psi \rho} \). The total action is
\[
S_{D3} = -4N \kappa^4 (bc - 1) \coth \rho |_{\sinh \rho = \kappa} |_{\sinh \rho = \kappa \sin \eta_0} + 2N \kappa^2 \left( \coth \rho - \frac{\rho}{\sinh^2 \rho} \right) |_{\sinh \rho = \kappa} |_{\sinh \rho = \kappa \sin \eta_0}
- 4N \kappa \frac{1}{\eta_0} \frac{c}{b} + \frac{n^2 \lambda c}{4N b} \coth \rho |_{\sinh \rho = \kappa} |_{\sinh \rho = \kappa \sin \eta_0}
\] (2.42)

where \( b, c \) are the same as (2.38). With \( \kappa \) in (2.35), divergent terms when \( \eta_0 \to 0 \) cancel, we have once again
\[
Z = \kappa^3 (bc - 1) + \frac{c}{b} \left( \frac{n^2 \lambda}{16N^2} \kappa^{-1} - \kappa \right) = 0
\] (2.43)

Then the action is
\[
S_{D3} = -2N (\kappa \sqrt{1 + \kappa^2} + \sinh^{-1} \kappa),
\] (2.44)

and the expectation value of Wilson-'t Hooft loop is
\[
<WH> = e^{-S_{D3}} = e^{2N (\kappa \sqrt{1 + \kappa^2} + \sinh^{-1} \kappa)}
\] (2.45)

Therefore, we have shown that even for Wilson-'t Hooft circular loop, the bulk D3-brane action gives the all-genus contribution which is consistent with the field theory argument based on duality.

In the limit \( N \to \infty \), the expectation value of WH loop simplifies to
\[
<WH>_{N \to \infty} = e^{\sqrt{n^2 \lambda + m^2 \lambda}}
\] (2.46)

which is the same as the prediction of [28] based on duality. The reason is that the F-strings and D-strings form bound state on D3 brane, resulting in a \((n, m)\) string. The relation (2.27) could be rewritten in another way:
\[
\kappa^2 = \frac{\pi g_s}{4N} \left( n^2 + \frac{m^2}{g_s^2} \right).
\] (2.47)
The term in the bracket is proportional to the square of the tension of the (F1, D1) bound states with charge \((n, m)\). This is natural since in the brane picture the dyon in the super-Yang-Mills theory corresponds to the (F1,D1) bound state ending on the D3-brane. In the super-Yang-Mills, the dyon solutions are BPS, their energy-charge relation satisfy the BPS relation:

\[
M = \sqrt{2}|Z| = <\phi> \sqrt{n^2 g_Y^2 + m^2 g_Y^2},
\]

which is invariant under S-duality, \(<\phi>\) is the vacuum expectation value of the scalar field. So the dependence on charges and couplings matches on both sides.

The induced metric on D3 brane is

\[
g_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\]

The only non-trivial component is

\[
g_{\eta\eta} = \frac{L^2}{\sinh^2 \eta} \left( 1 + \frac{\kappa^2}{1 + \kappa^2 \sin^2 \eta} \right)
\]

So the induced metric is

\[
ds^2_{D3} = \frac{L^2}{\sinh^2 \eta} \left( \frac{1 + \kappa^2}{1 + \kappa^2 \sin^2 \eta} \right) d\eta^2 + \frac{L^2 \cos^2 \eta}{\sin^2 \eta} d\psi^2 + L^2 \kappa^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

Change variable \(\cot^2 \eta = (1 + \kappa^2) \sinh^2 \zeta\), the metric is Euclidean \(AdS_2 \times S^2\) with different radius:

\[
ds^2_{D3} = L^2 (1 + \kappa^2) (\sinh^2 \zeta d\psi^2 + d\xi^2) + L^2 \kappa^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

The radius of \(AdS_2\) is \(L \sqrt{1 + \kappa^2}\) while the radius of \(S^2\) is \(L \kappa\). Embedding into \(AdS_5 \times S^5\), we can write the \(AdS_5 \times S^5\) metric as the fibration of 2-dimensional space with \(AdS_2 \times S^2 \times S^4\) fiber by defining two new variables \(u, \zeta\):

\[
cot \eta = \cosh u \sinh \zeta \quad \sinh \rho = \sinh u \sin \eta
\]

then the metric of \(AdS_5 \times S^5\) is of the form

\[
ds^2 = L^2 [\cosh^2 u d\Omega^2_{AdS_2} + du^2 + d\theta^2 + \sinh^2 u d\Omega^2_{S^2} + \sin^2 \theta d\Omega^2_{S^4}]
\]

with \(\sinh u = \kappa\).

In the new fibred coordinates, the gauge field strength reads as

\[
F = -\frac{\sqrt{1 + \kappa^2}}{\kappa} \left( \frac{\ln \lambda}{8\pi N} \right) \sinh \zeta d\psi \wedge d\zeta + \frac{m \sin \theta}{2} d\theta \wedge d\phi
\]

**2.3 Restore the full SL(2, Z) duality**

In the previous sections we don’t take the theta dependence into account and set the axion field to zero in the D-brane calculation. In this subsection we will consider the effect of a nonvanishing constant axion field and try to restore the full SL(2, Z) duality of the theory\(^[24]\).
For a non-zero constant axion background, it contributes the following term to the Wess-Zumino action

$$S_{\text{axion}} = i\mu_3(2\pi\alpha)^2 \int C_0 F_{\psi\rho} F_{\theta\phi} = i2\pi C_0 \int d\rho d\theta F_{\psi\rho} F_{\theta\phi}$$

(2.56)

where \( i \) is due to the Euclidean signature.

The bulk action now is

$$S_{\text{bulk}} = S_0 + S_{\text{axion}}.$$  

(2.57)

The equations of motion and the charge quantization conditions for F/D-string charges are solved by

$$\sin\eta = \kappa^{-1} \sinh \rho$$

(2.58)

with \( \kappa \) taking value

$$\kappa^2 = \frac{(n + mC_0)^2\lambda}{(4N)^2} + \frac{m^2\tilde{\lambda}}{(4N)^2}$$

(2.59)

and the gauge fields taking values

$$F_{\psi\rho} = \frac{i(n + mC_0)\lambda}{8\pi N \sinh^2 \rho} \quad F_{\theta\phi} = \frac{m\sin\theta}{2}$$

(2.60)

All the calculations are in parallel with the ones in the zero axion case with a replacement \( n \rightarrow n + mC_0 \).

From the AdS/CFT correspondence, the axion in the bulk could be identified with the \( \theta \) parameter in the Yang-Mills theory

$$C_0 = \frac{\theta}{2\pi},$$

(2.61)

so the SL(2, \( Z \)) invariant result is

$$<W>_{\text{sym}} = e^{2N(\kappa \sqrt{1 + \kappa^2} + \sinh^{-1}\kappa)}$$

(2.62)

with

$$\kappa^2 = \frac{(n + \frac{m\theta}{2\pi})^2\lambda}{(4N)^2} + \frac{m^2\tilde{\lambda}}{(4N)^2} = \frac{\pi |n + m\tau|^2}{4N\text{Im}\tau}$$

(2.63)

which is invariant under the S and T transformation

$$S : \tau \rightarrow -\frac{1}{\tau} \quad (n, m) \rightarrow (-m, n)$$

(2.64)

$$T : \tau \rightarrow \tau + 1 \quad (n, m) \rightarrow (n + m, m)$$

(2.65)

This is what we expect from both the field theory and the dual string theory: on the field theory side, in the \( \theta \)-vacuum the magnetic charge will induce a fractional electric charge. The shift is exactly \( \frac{m\theta}{2\pi} \); on the dual string theory side, when the axion field is non-zero, the D-string charge will induce a F-string charge \( mC_0 \).
3. Supersymmetry of $AdS_2 \times S^2$ D3-brane with fluxes

In this section, we would like to discuss the preserved supersymmetry of the $AdS_2 \times S^2$ D3-brane with fluxes. Now we change to Minkowski metric and multiply the gauge field strength on the $AdS_2$ part by $i$. The Gamma matrices and Killing spinor convention will follow [25, 27]. To be consistent, we put them in the appendix A.

To discuss the supersymmetry of the D-branes in a curved background, one has to check the kappa symmetry projection relation

$$\Gamma_p \epsilon = \epsilon$$ \hspace{1cm} (3.1)

where $\Gamma_p$ is determined by the embedded curved spacetime and background fluxes [31, 32, 33, 34]. For a Dp-brane in IIB theory, it is determined by the following relations:

$$\Gamma_{Dp} d^{p+1}\sigma = -e^{-\Phi} (-det(G + F))^{-\frac{1}{2}} e^F \wedge \chi |_{(p+1)\text{-form}}$$

$$\chi = \sum_n \frac{1}{(2n)!} d\sigma_0 \wedge \ldots \wedge d\sigma_{2n} \Gamma_{(i_1i_2 \ldots i_{2n})} \tau_3^n (i\tau_2)$$

$$\Gamma_{(i_1i_2 \ldots i_s)} = \frac{\partial X_i^{\mu_1}}{\partial \sigma_{i_1}} \cdots \frac{\partial X_i^{\mu_s}}{\partial \sigma_{i_s}} e^{a_1}_{\mu_1} \cdots e^{a_s}_{\mu_s} \Gamma_{a_1 \ldots a_s}$$ \hspace{1cm} (3.2)

where $\sigma^i$ are the worldvolume coordinates, $X^\mu$ are the spacetime coordinates and $e^a_\mu$ are the components of the vielbein. And as usual $F = B + 2\pi \alpha' F$. With respect to the metric (2.54), the vielbein relevant to our discussions are

$$e^0 = L \cosh uf^0, \quad e^1 = L \cosh uf^1, \quad e^4 = L \sinh uf^4, \quad e^5 = L \sinh uf^5$$ \hspace{1cm} (3.3)

where $f^0, f^1$ are the vielbein of the unit $AdS_2$ and $f^4, f^5$ are the vielbein of the unit $S^2$.

In the D3-brane picture, D3-brane worldvolume extends along $AdS_2 \times S^2$ directions and sits at the point on $S^5$ with $\theta = 0$ or $\pi$. In our case, we turn on the electric and magnetic flux at the same time. The electric field spread over $AdS_2$ and the magnetic field over $S^2$ uniformly such that

$$F = \alpha e^0_1 e^1_0 d\sigma^0 \wedge d\sigma^1 + \beta e^4_1 e^5_0 d\sigma^4 \wedge d\sigma^5,$$ \hspace{1cm} (3.4)

where $(\sigma^0, \sigma^1, \sigma^4, \sigma^5)$ are the worldvolume coordinates. Then we have

$$\Gamma_{D3} = -\frac{1}{A} ((i\alpha\beta - \mu_3 \nu_3) \tau_2 + (i\alpha \nu_3 + \beta \mu_3) \tau_1)$$ \hspace{1cm} (3.5)

with

$$A = \sqrt{(1 - \alpha^2)(1 + \beta^2)}$$ \hspace{1cm} (3.6)

The projection condition reads

$$\Gamma_{D3} \epsilon = \epsilon$$ \hspace{1cm} (3.7)

with

$$\epsilon = \exp\left(\frac{i}{2} \mu_3 \nu_3 \tau_2 u - \frac{i}{2} \lambda_3 \tau_2 \theta\right) \zeta$$ \hspace{1cm} (3.8)
Let’s first focus on the case with $\theta = 0$. When we have only electric flux, i.e. $\alpha \neq 0, \beta = 0$, the situation reduce to the BPS Wilson loop and has been analyzed in [27]:

$$\sinh u_k = \sqrt{1-\alpha^2/|\alpha|} \cdot \begin{cases} (1-\mu_3\tau_3)\zeta = 0, \alpha > 0 \\ (1+\mu_3\tau_3)\zeta = 0, \alpha < 0 \end{cases}$$ (3.9)

On the other hand, when $\alpha = 0, \beta \neq 0$, the supersymmetric condition has not been discussed in the literature. Actually, the projection condition leads to

$$\sinh u_k = \frac{1}{|\beta|} \cdot \begin{cases} (1-\mu_3\tau_1)\zeta = 0, \beta < 0 \\ (1+\mu_3\tau_1)\zeta = 0, \beta > 0 \end{cases}$$ (3.10)

Let us turn on the electric and magnetic flux at the same time, i.e. $\alpha \neq 0, \beta \neq 0$. The supersymmetric condition is

$$\begin{align*}
&\left\{ \left( \cosh \frac{u}{2} + \frac{\sinh \frac{u}{2} (i\alpha\beta\mu_3\nu_3 - 1)}{A} \right) + \left( \mu_3\nu_3 \sinh \frac{u}{2} + \frac{\cosh \frac{u}{2} (i\alpha\beta - \mu_3\nu_3)}{A} \right) \tau_2 \\
&+ \frac{\cosh \frac{u}{2} (i\alpha\nu_3 + \beta\mu_3)}{A} \tau_1 + \frac{\sinh \frac{u}{2} (-\alpha\mu_3 + i\beta\nu_3)}{A} \tau_3 \right\} \zeta = 0.
\end{align*}$$ (3.11)

From the discussion above, we make the following ansatz on $\zeta$:

$$(1 - (x\mu_3\tau_3 + y\mu_3\tau_1))\zeta = 0,$$ (3.12)

where $x, y$ are two real numbers satisfying $x^2 + y^2 = 1$. Then the supersymmetric condition leads to the following solutions:

$$x = \alpha \sqrt{\frac{1 + \beta^2}{\alpha^2 + \beta^2}} \quad y = -\beta \sqrt{\frac{1 - \alpha^2}{\alpha^2 + \beta^2}}$$ (3.13)

and

$$\sinh u_k = \sqrt{1-\frac{\alpha^2}{\alpha^2 + \beta^2}}.$$ (3.14)

It is still one step from proving that the configurations discussed in section 2 actually is 1/2-BPS. Let us identify the gauge field strength parameters

$$\alpha = \frac{n\sqrt{\lambda}}{4N\kappa\sqrt{1+\kappa^2}}, \quad \beta = \frac{m\pi}{\sqrt{\lambda}\kappa^2} = \frac{m\sqrt{\lambda}}{4N\kappa^2}$$ (3.15)

and put them back to the relation (3.14). The relation holds automatically, taking into account of the relation (2.35). This indicates that the D3-brane with electric and magnetic field (2.34) satisfy the 1/2-BPS configuration if its motion parameter satisfy (2.34).

For the case of $\theta = \pi$, we have very similar result. In either case, the configuration keep one half of the original supersymmetry.

For nonvanishing axion background, the bulk supersymmetry analysis here still holds because the kappa projection operator dosen’t involve any RR field. The solution of the supersymmetry condition [3.13,3.14] with $n \to n + mC_0$ is satisfied by (2.60). In the dual SYM, the $\theta$-term contributes an extra term to the central charge of the SUSY algebra, the solution (2.60) still saturates the BPS condition.
4. The calculation using 5-brane

It was shown in [27] the expectation value of Wilson loop in anti-symmetric representation could be calculated using bulk D5-brane action. Remarkably, the calculations from D5-brane and the Matrix theory are in perfect match. It is interesting to investigate if Wilson-'t Hooft loop in anti-symmetric representation has the same story.

Let us study 't Hooft loop first. In IIB theory, the S-dual of D5 brane is NS5 brane. Hence 't Hooft loop in antisymmetric representation corresponds to NS5 brane with D-string charges. But now from NS5-brane point of view, the D-string looks like electric source rather than magnetic source. This is very different from the D3-brane case. The fields on IIB NS5 brane is a $(1,1)$ vector supermultiplet, consisting of four scalars and a vector one form $c^{(1)}$. The effective worldvolume action of NS5 brane was constructed in [35] from T-duality of IIA KK-monopole in a transverse direction. Set RR field $C^{(0)}$ and $C^{(2)}$ to zero and dilaton to a constant, the action reads:

\[
S_{NS5} = -T_{NS5} \int d^6 \sigma e^{-2\Phi} \sqrt{\det (g + (2\pi \alpha')F)} - T_{NS5} \int d^6 \sigma F \wedge C^{(4)}
\]

(4.1)

where the tension of NS5 brane is

\[
T_{NS5} = \frac{1}{(2\pi)^5 \alpha'^{3/2} g_s^2}
\]

This action can also be derived from S-dual of D5-brane DBI action. Under S-duality $c^{(1)}$ transforms to an one form gauge field $A^{(1)}$.

In our case, the worldvolume of NS5-brane is $AdS_2 \times S^4$, located at $u = 0, \theta = \theta_m = \text{constant}$ in coordinates (2.54). From the S-duality, one may expect that the supersymmetry condition is the same as the one in D5-brane discussed in [27], namely the supersymmetry condition requires that

\[
\cos \theta_m = \beta, \quad (1 - \mu_3 \tau_3)\zeta = 0
\]

(4.2)

where $\beta$ is the electric field strength on the NS5-brane. The similar calculation shows that

\[
S_{tot} = -\frac{4N \pi}{\sqrt{\lambda}} \frac{2N}{3\pi} \sin^3 \theta_m
\]

(4.3)

after taking into account of the boundary terms. When the string charge $m$ is much smaller than $N$, $\theta_m$ becomes small. Then we have

\[
S_{tot} \approx -\sqrt{\lambda} \frac{m}{g_s}
\]

(4.4)

This is what we expected: the action is proportional to the tension of $m$ overwrapping D-strings. This is S-dual to the D5-brane result $S_{tot} \approx -n\sqrt{\lambda}$.

For Wilson-'t Hooft loop, it is not easy to figure out a picture as in D3-brane case. Now due to the existence of both F1’s and D1’s, the dual bulk should have both D5 and NS5-brane. And from the above discussion, both brane keep the same supersymmetry. But in general D5-brane and NS5-brane are located at different position since $\theta_k$ depends on the value of the flux. In the case that both F1 and D1 charges are small, the total action
is just the summation of the D5 and NS5 action, namely proportional to the summation of tensions of F1’s and D1’s. We are not clear if these F1’s and D1’s form bound states and if so how to see this property from 5-brane picture.

5. Conclusions and Discussions

In this paper, we discussed Wilson-‘t Hooft loop in the symmetric representation. In this case, the brane description is a D3-brane with both electric and magnetic fluxes. Taking the boundary terms into account properly, we obtained the expectation value of Wilson-‘t Hooft loop. The result is precisely in match with the result from the gauge theory argument.

In this paper, we mainly focus on Wilson-‘t Hooft loops with symmetric representation, in which case the string picture is simply a single F-string with charge \( n \) and a single D-string with charge \( m \). It would be interesting to discuss the more general case. Let us first consider the pure Wilson loop in a general representation. The representation could be described by a Young tableau. The total number of box in the Young tableau is the total charges of the fundamental strings. Every row of the Young tableau represents a single F1 with the charge determined by the number of boxes in this row. The expectation value of Wilson loop could be calculated from the Hermitian matrix theory. In general, the expectation value is

\[
< \frac{1}{N} Tr \exp(k_1 M) \frac{1}{N} Tr \exp(k_2 M) \cdots \frac{1}{N} Tr \exp(k_l M) > \tag{5.1}
\]

where \( k_1, k_2 \cdots k_l \) are the number of boxes in each row. This quantity has been discussed in [14]. From our D-brane picture, the symmetric representations correspond to D3-brane with electric flux. One might naively expect that every row of the Young tableau would give a D3-brane with electric flux and we have an array of D3-branes whose number is the number of the rows to describe Wilson loop [13]. The configuration is still supersymmetric but the position of D3-branes are different due to the difference of the flux. It must be very interesting to check how much this D3-brane picture describe the physics correctly. One essential point in the D3-brane picture is that even the electric flux number is only one, the D3-brane description still make sense [14]. This fact indicates that we can use D3-brane configuration to describe Wilson loop in arbitrary representation. On the other hand we have a dual description in terms of D5-branes which requires that the number of F1’s is huge. It is not clear how the two description match each other. For an antisymmetric description of rank \( k \), we may study the problem in terms of \( k \) D3-branes. To describe these D3-branes, we need a nonabelian DBI action whose form is an open question. In this sense, it would be valuable to investigate the correspondence between the dual pictures to test the nonabelian DBI action.

It is more difficult to figure out the brane picture if including the D-string. As the F-strings, the D-strings could take many forms: either a single one with charge \( m \), or many ones each with winding number one, or something between these two extremal cases. For pure D-strings, all the above possible configurations are supersymmetric, being S-dual to the pure F-strings. In D3-brane picture, there are several D3-branes with different
magnetic fluxes. They are supersymmetric even though generically they don’t overlap with each other. However, if both the F1’s and D1’s exist, they may form bound state. For a generic (F1, D1) configurations, its D3/D5-brane description is unclear. Naively, one may think that the above D3-brane picture could still make sense after putting into the relevant magnetic flux. But from the discussion before, we know that the supersymmetric condition is sensitive to both the electric and magnetic fields. It is a problem on how to distribute magnetic flux to different D3’s. And even if we give an assignment of magnetic flux to D3-brane’s, the supersymmetries is broken and there exist the open string tachyon. The tachyon condensation lead to the bound state of (F1, D1). After tachyon condensation, the final configuration should be independent to the assignation of D1’s. It would be interesting to understand this process more precisely. In terms of 5-brane, the picture is more unclear.

Another interesting problem is the dual supergravity solution of WH loop. In [25, 26], the supergravity solution corresponding to 1/2-BPS Wilson loop is presented. The geometry preserves $SU(1,1) \times SU(2) \times SO(5)$ isometry as the 1/2-BPS Wilson loop itself. The geometry is governed by a function in two dimension satisfying Laplace equation, with some boundary conditions on $y$ axis. The boundary condition takes two constant values, corresponding to the distribution of eigenvalues on the real line in the matrix model. This is the fermion droplet picture analogous to the bubbling geometry of LLM[29]. For 1/2-BPS WH loop, the isometry is the same as Wilson loop. It would be interesting to see what the bubbling geometry is.

In this paper, the expectation value of 1/2 BPS Wilson-'t Hooft loop is from the bulk D3-brane picture. It would be nice to have a direct perturbative calculation in the SYM.

Acknowledgments

The work was supported by NSFC Grant No. 10405028,10535060 and the Key Grant Project of Chinese Ministry of Education (NO. 305001)

A. Spinors in $AdS_2 \times S^2 \times S^4$ fibration

In IIB supergravity, the supersymmetry is determined by the number of covariant constant spinor. In this paper, we focused on $AdS_5 \times S^5$ background, where only the metric and the RR 5-form $G_5$ are nonvanishing. The supersymmetry transformation of dilatino is trivial and the invariance of the gravitino under supersymmetry transformation can be written as

$$\delta \psi_M = \nabla_M \eta + \frac{i}{2} G_5 \tau_2 \Gamma_M \eta.$$  \hspace{1cm} (A.1)

The parameter of supersymmetry is a doublet of Majorana-Weyl spinors: $\eta = (\eta_1, \eta_2)$. We use the Pauli matrices $\tau_j$ to rotate this doublet: $(\tau \eta)_\alpha = \tau_{\alpha \beta} \eta_\beta$.

Since we have written $AdS_5 \times S^5$ metric in forms $AdS_2 \times S^2 \times S^4$ fibered on $\mathbb{R}^2$, it’s convenient to write 10-dimensional Clifford algebra in tensor product form. Let $\Gamma^\mu$ denote the Gamma matrices in ten dimension. They can be written as the following tensor product
spinors satisfy the following relations

\[ \sigma \in \mathbb{C}^2 \]

Each \( \epsilon \) where \( \tilde{\sigma} \)

where \( \tilde{\sigma} \) are three sets of the Pauli matrices. And \( \sigma \) is a covariant constant spinor, constrained by the following conditions

\[ \Gamma^0 = \tilde{\sigma}^0 \otimes \sigma_C \otimes 1 \otimes 1, \quad \Gamma^1 = \tilde{\sigma}^1 \otimes \sigma_C \otimes 1 \otimes 1, \]
\[ \Gamma^2 = 1 \otimes \sigma_1 \otimes 1 \otimes 1, \quad \Gamma^3 = 1 \otimes \sigma_2 \otimes 1 \otimes 1, \]
\[ \Gamma^4 = \tilde{\sigma}^3 \otimes \sigma_C \otimes \tilde{\sigma}_4 \otimes 1, \quad \Gamma^5 = \tilde{\sigma}^3 \otimes \sigma_C \otimes \tilde{\sigma}_5 \otimes 1, \]
\[ \Gamma^a = \tilde{\sigma}^3 \otimes \sigma_C \otimes \tilde{\sigma}_6 \otimes \gamma^a, \quad (a = 6, 7, 8, 9), \] (A.2)

where \( (\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3) \), \((\sigma_1, \sigma_2, \sigma_C)\) and \((\tilde{\sigma}_4, \tilde{\sigma}_5, \tilde{\sigma}_6)\) are three sets of the Pauli matrices. And \( \sigma \) is a pair of 2-dimensional spinors with \( \sigma_1, \sigma_2, \sigma_C, \tau_1, \tau_2, \tau_3 \) acting on it. The spinors satisfy the following relations

\[ \nabla_{\mu} \chi^I_a = i a \tilde{\sigma}_p \chi^I_{-a}, \quad \tilde{\sigma}_3 \chi^I_a = a \chi^I_a, \quad (p = 0, 1, \quad a = \pm 1, \quad I = 1, 2), \] (A.4)
\[ \nabla_{\mu} \tilde{\chi}^J_b = \frac{i}{2} b \tilde{\sigma}_p \tilde{\chi}^J_{-b}, \quad \tilde{\sigma}_6 \tilde{\chi}^J_b = b \chi^J_b, \quad (p = 4, 5, \quad b = \pm 1, \quad J = 1, 2), \] (A.5)
\[ \nabla_{\mu} \chi^K_c = \frac{i}{2} c \tilde{\sigma}_p \chi^K_{-c}, \quad \gamma_5 \chi^K_c = c \chi^K_c, \quad (p = 6, \ldots, 9, \quad c = \pm 1, \quad K = 1, 2, 3, 4). \] (A.6)

where \( \nabla \) is the covariant derivative of the Levi-Civita connection of the unit \( AdS_2 \), \( S^2 \) or \( S^4 \). By expanding the 10-dimensional spinor pair \( \eta \) by the above Killing spinors we reduce the problem to 2-dimensions.

With these notations, the supersymmetry condition in IIB theory yields \( \epsilon \) of the form

\[ \epsilon = \exp\left( \frac{1}{2} \mu_3 \lambda_3 \tau_2 u - \frac{i}{2} \lambda_3 \tau_2 \theta \right) \zeta. \] (A.7)

where \( \zeta \) a covariant constant spinor, constrained by the following conditions

\[ \mu_2 \lambda_3 \tau_2 \Gamma^2 \zeta = \zeta, \quad \nu_1 \lambda_3 \Gamma^2 \zeta = \zeta, \quad \lambda_1 \Gamma^3 \zeta = \zeta, \quad \Gamma_23 \zeta = -i \mu_3 \nu_3 \lambda_3 \zeta. \] (A.8)

References

[1] J.M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [hep-th/9711200].

[2] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, “Large N Field Theories, String Theory and Gravity,” Phys. Rept. 323183 (2000), [hep-th/9905111].

[3] S.-J. Rey and J.-T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C22 (2001) 379–394, [hep-th/9803001].

[4] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80 (1998) 4859–4862, [hep-th/9803002].
[5] A. Polyakov, “The wall of the cave,” Int.J.Mod.Phys. A14 (1999) 645-658, [hep-th/9809057].

[6] K. Zarembo, “Supersymmetric Wilson loops,” Nucl.Phys. B643 (2002) 157-171, [hep-th/hep-th/0205160].

[7] A. Dymarsky, S. Gubser, Z. Guralnik, J. Maldacena, “Calibrated Surfaces and Supersymmetric Wilson Loops,” [hep-th/hep-th/0604058].

[8] N. Drukker, D. J. Gross and H. Ooguri, “Wilson loops and minimal surfaces,” Phys. Rev. D60 (1999) 125006, [hep-th/9904191].

[9] J. K. Erickson, G. W. Semenoff and K. Zarembo, “Wilson loops in N = 4 supersymmetric Yang-Mills theory,” Nucl. Phys. B582 (2000) 155–175, [hep-th/0003053].

[10] N. Drukker and D. J. Gross, “An exact prediction of N = 4 SUSYM theory for string theory,” J. Math. Phys. 42 (2001) 2896–2914, [hep-th/0010274].

[11] N. Drukker and B. Fiol, “All-genus calculation of Wilson loops using D-branes,” JHEP 02 (2005) 010, [hep-th/0501109].

[12] S. A. Hartnoll and S. Prem Kumar, “Multiply wound Polyakov loops at strong coupling,” [hep-th/0603190].

[13] Jaume Gomis, Filippo Passerini “Holographic Wilson Loops,” [hep-th/0604007].

[14] G. Akemann and P.H. Damgaard, “Wilson loops in N=4 supersymmetric Yang-Mills theory from random matrix theory”, Phys. Lett. B513(2001)179, Erratum-ibid. B524(2002)400, [hep-th/0101225].

[15] Kazumi Okuyama, Gordon W. Semenoff, “Wilson Loops in N=4 SYM and Fermion Droplets,” [hep-th/0604209].

[16] Sean A. Hartnoll, S. Prem Kumar, “Higher rank Wilson loops from a matrix model,” [hep-th/0605027].

[17] R.C. Myers, “Dielectric-Branes,” JHEP 9912 (1999) 022, [hep-th/9910053].

[18] J. Pawelczyk and S.-J. Rey, “Ramond-Ramond flux stabilization of D-branes,” Phys. Lett. B493 (2000) 395–401, [hep-th/0007154].

[19] D. Rodriguez-Gomez, “Computing Wilson lines with dielectric branes”, [hep-th/0604031].

[20] M. T. Grisaru, R. C. Myers and O. Tafjord, “SUSY and Goliath,” JHEP 08 (2000) 040, [hep-th/0008016].

[21] A. Hashimoto, S. Hirano and N. Itzhaki, “Large branes in AdS and their field theory dual,” JHEP 08 (2000) 051, [hep-th/0008015].

[22] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from anti-de Sitter space,” JHEP 06 (2000) 008, [hep-th/0003075].

[23] Witten, Edward, “Bound States Of Strings And p-Branes,” Nucl.Phys. B460 (1996) 335-350, [hep-th/9510138].

[24] A. Kapustin, “Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality”, [hep-th/0501013].

[25] S. Yamaguchi, “Bubbling geometries for half BPS Wilson lines,” [hep-th/0601089].
[26] Oleg Lunin, “On gravitational description of Wilson lines,” hep-th/0604133.

[27] S. Yamaguchi, “Wilson Loops of Anti-symmetric Representation and D5-branes,” JHEP 0605 (2006) 037 hep-th/0603208.

[28] Massimo Bianchi, Michael B. Green, Stefano Kovacs, “Instanton corrections to circular Wilson loops in N=4 Supersymmetric Yang-Mills,” JHEP 0204 (2002) 040 hep-th/0202003.

[29] H. Lin, O. Lunin and J. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” JHEP 10 (2004) 025, hep-th/0409174.

[30] C. G. Callan, Jr. and J. M. Maldacena, “Brane dynamics from the Born-Infeld action,” Nucl. Phys. B513 (1998) 198–212, hep-th/9708147.

[31] E. Bergshoeff and P. K. Townsend, “Super D-branes,” Nucl. Phys. B490 (1997) 145–162, hep-th/9611173.

[32] E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, “kappa-symmetry, supersymmetry and intersecting branes,” Nucl. Phys. B502 (1997) 149–169, hep-th/9705040.

[33] K. Skenderis and M. Taylor, “Branes in AdS and pp-wave spacetimes,” JHEP 06 (2002) 025, hep-th/0204054.

[34] Y. Imamura, “Supersymmetries and BPS configurations on anti-de Sitter space,” Nucl. Phys. B537 (1999) 184–202, hep-th/9807179.

[35] Eduardo Eyras, Bert Janssen, Yolanda Lozano, “5-branes, KK-monopoles and T-duality,” Nucl.Phys. B531 (1998) 275-301, hep-th/9806169.