Cayley Graph on Symmetric Group Generated by Elements Fixing $k$ Points

Cheng Yeaw Ku ∗ Terry Lau † Kok Bin Wong ‡

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Abstract

Let $S_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. The $k$-point fixing graph $F(n, k)$ is defined to be the graph with vertex set $S_n$ and two vertices $g, h$ of $F(n, k)$ are joined if and only if $gh^{-1}$ fixes exactly $k$ points. In this paper, we derive a recurrence formula for the eigenvalues of $F(n, k)$. Then we apply our result to determine the sign of the eigenvalues of $F(n, 1)$.

KEYWORDS: Arrangement graph, Cayley graphs, symmetric group, Alternating Sign Property.

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1 Introduction

Let $G$ be a finite group and $S$ be an inverse closed subset of $G$, i.e., $1 \notin S$ and $s \in S \Rightarrow s^{-1} \in G$. The Cayley graph $\Gamma(G, S)$ is the graph which has the elements of $G$ as its vertices and two vertices $u, v \in G$ are joined by an edge if and only if $v = su$ for some $s \in S$.

A Cayley graph $\Gamma(G, S)$ is said to be normal if $S$ is closed under conjugation. It is well known that the eigenvalues of a normal Cayley graph $\Gamma(G, S)$ can be expressed in terms of the irreducible characters of $G$.

Theorem 1.1 ([1, 10, 18, 19]). The eigenvalues of a normal Cayley graph $\Gamma(G, S)$ are given by

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s),$$

where $\chi$ ranges over all the irreducible characters of $G$. Moreover, the multiplicity of $\eta_\chi$ is $\chi(1)^2$.

∗Department of Mathematics, National University of Singapore, Singapore 117543. E-mail: matkcy@nus.edu.sg
†Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia. E-mail: terrylsc@hotmail.com
‡Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia. E-mail: kbwong@um.edu.my
Let \( S_n \) be the symmetric group on \([n] = \{1, \ldots, n\}\) and \( S \subseteq S_n \) be closed under conjugation. Since central characters are algebraic integers ([14] Theorem 3.7 on p. 36) and that the characters of the symmetric group are integers ([14] 2.12 on p. 31 or [21] Corollary 2 on p. 103), by Theorem 1.1, the eigenvalues of \( \Gamma(S_n, S) \) are integers. The eigenvalues of a normal Cayley graph \( \Gamma(S_n, S) \) are integers ([14] Theorem 3.7 on p. 36). That is, two vertices \( g, h \) of \( \Gamma(S_n, S) \) are joined if and only if \( gh^{-1} \) fixes no point. Since \( D_n \) is closed under conjugation, by Corollary 1.2 the eigenvalues of the derangement graph are integers. The lower and upper bounds of the absolute values of these eigenvalues have been studied in [16], [17], [20]. The derangement graph is a kind of arrangement graph, i.e., \( \Gamma_n = A(n, n, n) \).

Let \( 0 \leq k < n \) and \( S(n, k) \) be the set of all \( \sigma \in S_n \) such that \( \sigma \) fixes exactly \( k \) elements. Note that \( S(n, k) \) is an inverse closed subset of \( S_n \). The \( k \)-point fixing graph is defined to be

\[
\mathcal{F}(n, k) = \Gamma(S_n, S(n, k)).
\]

That is, two vertices \( g, h \) of \( \mathcal{F}(n, k) \) are joined if and only if \( gh^{-1} \) fixes exactly \( k \) points. Note that the \( k \)-point fixing graph is also a kind of arrangement graph, i.e., \( \mathcal{F}(n, k) = A(n, n, n - k) \). Furthermore, the 0-point fixing graph is the derangement graph, i.e., \( \mathcal{F}(n, 0) = \Gamma_n = A(n, n, n) \).

Clearly, \( \mathcal{F}(n, k) \) is vertex-transitive, so it is \(|S(n, k)|\)-regular and the largest eigenvalue of \( \mathcal{F}(n, k) \) is \(|S(n, k)|\). Furthermore, \( S(n, k) \) is closed under conjugation. Therefore, by Corollary 1.2, the eigenvalues...
of the \( k \)-point fixing graph are integers. However, the complete set of spectrum of \( F(n, k) \) is not known. The purpose of this paper is to study the eigenvalues of \( F(n, k) \).

Recall that a partition \( \lambda \) of \( n \), denoted by \( \lambda \vdash n \), is a weakly decreasing sequence \( \lambda_1 \geq \ldots \geq \lambda_r \) with \( \lambda_r \geq 1 \) such that \( \lambda_1 + \cdots + \lambda_r = n \). We write \( \lambda = (\lambda_1, \ldots, \lambda_r) \). The size of \( \lambda \), denoted by \( |\lambda| \), is \( n \) and each \( \lambda_i \) is called the \( i \)-th part of the partition. We also use the notation \((\mu_1^a, \ldots, \mu_r^a) \vdash n \) to denote the partition where \( \mu_i \) are the distinct nonzero parts that occur with multiplicity \( a_i \). For example,

\[(5, 5, 4, 4, 2, 2, 2, 1) \leftrightarrow (5^2, 4^2, 2^3, 1).\]

It is well known that both the conjugacy classes of \( S_n \) and the irreducible characters of \( S_n \) are indexed by partitions \( \lambda \) of \([n]\). Since \( S(n, k) \) is closed under conjugation, the eigenvalue \( \eta_{\lambda\lambda}(k) \) of the \( k \)-point fixing graph can be denoted by \( \eta_\lambda(k) \). Throughout the paper, we shall use this notation.

The paper is organized as follows. In Section 2, we provide some known results regarding the eigenvalues of \( F(n, 0) \). In Section 3, we prove a recurrence formula for the eigenvalues of \( F(n, k) \) (Theorem 3.7). In Section 4, we prove some inequalities for the eigenvalues of \( F(n, 0) \) which will be used to prove the following Alternating Sign Property (ASP) for \( F(n, 1) \):

**Theorem 1.3.** (ASP for \( F(n, 1) \)) Let \( n \geq 2 \) and \( \lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \).

(a) \( \eta_\lambda(1) = 0 \) if and only if \( \lambda = (n - 1, 1) \) or \( \lambda = (2, 1^{n-2}) \).

(b) If \( r = 1 \) and \( \lambda \neq (2) \), then \( \eta_\lambda(1) > 0 \).

(c) If \( r \geq 2 \) and \( \lambda \neq (n - 1, 1) \) or \( (2, 1^{n-2}) \), then

\[
\text{sign}(\eta_\lambda(1)) = (-1)^{|\lambda| - \lambda_1 - 1} = (-1)^{(#\text{cells under the first row of } \lambda) - 1}
\]

where \( \text{sign}(\eta_\lambda(1)) = 1 \) if \( \eta_\lambda(1) \) is positive or \( -1 \) if \( \eta_\lambda(1) \) is negative.

In Section 6, we provide a list of eigenvalues of \( F(n, 1) \) for small \( n \).

## 2 Known results for eigenvalues of \( F(n, 0) \)

To describe the Renteln’s recurrence formula for \( F(n, 0) \), we require some terminology. To the Ferrers diagram of a partition \( \lambda \), we assign \( xy \)-coordinates to each of its boxes by defining the upper-left-most box to be \((1, 1)\), with the \( x \) axis increasing to the right and the \( y \) axis increasing downwards. Then the **hook** of \( \lambda \) is the union of the boxes \((x', 1)\) and \((1, y')\) of the Ferrers diagram of \( \lambda \), where \( x' \geq 1, y' \geq 1 \). Let \( \hat{h}_\lambda \) denote the hook of \( \lambda \) and let \( h_\lambda \) denote the size of \( \hat{h}_\lambda \). Similarly, let \( \hat{c}_\lambda \) and \( c_\lambda \) denote the first column of \( \lambda \) and the size of \( \hat{c}_\lambda \) respectively. Note that \( c_\lambda \) is equal to the number of rows of \( \lambda \). When \( \lambda \) is clear from the context, we will replace \( \hat{h}_\lambda, h_\lambda, \hat{c}_\lambda \) and \( c_\lambda \) by \( \hat{h}, h, \hat{c} \) and \( c \) respectively. Let \( \lambda - \hat{h} \vdash n - h \) denote the partition obtained from \( \lambda \) by removing its hook. Also, let \( \lambda - \hat{c} \) denote the partition obtained from \( \lambda \) by removing the first column of its Ferrers diagram, i.e. \((\lambda_1, \ldots, \lambda_r) - \hat{c} = (\lambda_1 - 1, \ldots, \lambda_r - 1) \vdash n - r \).
Theorem 2.1. \(\text{[20, Theorem 6.5] Renteln’s Recurrence Formula}\) For any partition \(\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n\), the eigenvalues of the derangement graph \(F(n, 0)\) satisfy the following recurrence:

\[
\eta_\lambda(0) = (-1)^h_\lambda \eta_{\lambda - \hat{\lambda}}(0) + (-1)^{h + \lambda_1} h \eta_{\lambda - \hat{\gamma}}(0)
\]

with initial condition \(\eta_\emptyset(0) = 1\).

To describe the Ku-Wong’s recurrence formula for \(F(n, 0)\), we need a new terminology. For a partition \(\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n\), let \(\hat{\lambda}\) denote the last row of \(\lambda\) and \(l_\lambda\) denote the size of \(\hat{\lambda}\). Clearly, we have \(l_\lambda = \lambda_r\). Let \(\lambda - \hat{\lambda}\) denote the partition obtained from \(\lambda\) by deleting the last row. When \(\lambda\) is clear from the context, we will replace \(\hat{\lambda}\), \(l_\lambda\) by \(\hat{l}\) and \(l\) respectively.

Theorem 2.2. \(\text{[17, Theorem 1.4] Ku-Wong’s Recurrence Formula}\) For any partition \(\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n\), the eigenvalues of the derangement graph \(F(n, 0)\) satisfy the following recurrence:

\[
\eta_\lambda(0) = (-1)^{\lambda_r} \eta_{\lambda - \hat{l}}(0) + (-1)^{r-1} \lambda_r \eta_{\lambda - \hat{c}}(0)
\]

with initial condition \(\eta_\emptyset(0) = 1\).

The following theorem is called the Alternating Sign Property (ASP) for \(F(n, 0)\).

Theorem 2.3. \(\text{[16, Theorem 1.2], [17, Theorem 1.3]}\) Let \(n \geq 2\). For any partition \(\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n\),

\[
\text{sign}(\eta_\lambda(0)) = (-1)^{\lambda_1 - \lambda_1} \eta_{\lambda - \hat{l}}(0)
\]

where \(\text{sign}(\eta_\lambda(0))\) is 1 if \(\eta_\lambda(0)\) is positive or \(-1\) if \(\eta_\lambda(0)\) is negative.

The following corollary is a consequence of Theorems 2.2 and 2.3.

Corollary 2.4. For any partition \(\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n\) with \(r \geq 2\), the absolute value of the eigenvalues of the derangement graph \(F(n, 0)\) satisfy the following recurrence:

\[
|\eta_\lambda(0)| = |\eta_{\lambda - \hat{l}}(0)| + \lambda_r |\eta_{\lambda - \hat{c}}(0)|
\]

with initial condition \(|\eta_\emptyset(0)| = 1\).

3 Recurrence Formula

For each \(\sigma \in S_n\), we denote its conjugacy class by \(\text{Con}_{S_n}(\sigma)\), i.e., \(\text{Con}_{S_n}(\sigma) = \{\gamma^{-1}\sigma\gamma \ : \ \gamma \in S_n\}\).

Let \(\mu \vdash n\) be the partition that represents \(\text{Con}_{S_n}(\sigma)\). We shall denote the size of \(\text{Con}_{S_n}(\sigma)\) by \(N_{S_n}(\mu)\).

Let \(A \subseteq S_n\) and \(\alpha \in S_n\). The set \(\alpha^{-1}A\alpha\) is defined as

\[
\alpha^{-1}A\alpha = \{\alpha^{-1}\sigma\alpha \ : \ \sigma \in A\}.
\]

Let \(0 \leq k < n\). Each \(\beta \in S_{n-k}\) can be considered as an element \(\overline{\beta}\) of \(S_n\) by defining \(\overline{\beta}(j) = \beta(j)\) for \(1 \leq j \leq n - k\) and \(\overline{\beta}(j) = j\) for \(n - k + 1 \leq j \leq n\). The \(\overline{\beta}\) is called the extension of \(\beta\) to \(S_n\). The
set of derangements $D_{n-k}$ in $S_{n-k}$ can be considered as a subset of $S_n$ ($\mathcal{D}_{n-k} = \{ \sigma : \sigma \in D_{n-k} \}$). Furthermore, $\bigcup_{\sigma \in S_n} \sigma^{-1} \mathcal{D}_{n-k} \sigma \subseteq S(n,k)$.

Let $\gamma \in S(n,k)$. Then $\gamma$ fixes exactly $k$ elements, i.e., $\gamma(i_j) = i_j$ for $j = 1, 2, \ldots, k$ and $\gamma(a) \neq a$ for $a \in [n] \setminus \{i_1, i_2, \ldots, i_k\} = \{b_1, b_2, \ldots, b_{n-k}\}$. Let $\sigma_0(b_j) = j$ for $1 \leq j \leq n-k$ and $\sigma_0(i_j) = n-k+j$ for $1 \leq j \leq k$. Then $\sigma_0 \in S_n$ and $\sigma_0^{-1} \gamma \sigma_0 \in \mathcal{D}_{n-k}$. Hence, the following lemma follows.

**Lemma 3.1.**

$$S(n,k) = \bigcup_{\sigma \in S_n} \sigma^{-1} \mathcal{D}_{n-k} \sigma.$$

Let $\lambda \vdash n$. For a box with coordinate $(a,b)$ in the Ferrers diagram of $\lambda$, the hook-length $h_\lambda(a,b)$ is the size of the set of all the boxes with coordinate $(i,j)$ where $i = a$ and $j \geq b$, or $i \geq a$ and $j = b$. The following lemma is well-known [12, 4.12 on p. 50].

**Lemma 3.2.**

$$\chi_\lambda(1) = \frac{n!}{\prod h_\lambda(a,b)},$$

where the product is over all the boxes $(a,b)$ in the Ferrers diagram of $\lambda$.

For convenience, let us denote the dimension of $\lambda$ by $f^\lambda$, i.e., $f^\lambda = \chi_\lambda(1)$. By Lemma 3.1 there are $\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{s_k}} \in D_{n-k}$ such that

$$S(n,k) = \bigcup_{i=1}^{s_k} \text{Con}_{S_n} (\sigma_{k_i}),$$

and $\sigma_{k_i}$ is not conjugate to $\sigma_{k_j}$ in $S_{n-k}$ for $i \neq j$. Furthermore,

$$D_{n-k} = \bigcup_{i=1}^{s_k} \text{Con}_{S_{n-k}} (\sigma_{k_i}).$$

Note that $\chi_\lambda(\sigma) = \chi_\lambda(\beta)$ for all $\sigma \in \text{Con}_{S_n}(\beta)$. For any $\beta \in S_n$, let $\varphi(\beta)$ denote the partition of $n$ induced by the cycle structure of $\beta$. Let $\text{Con}_{S_n}(\beta)$ be represented by the partition $\varphi(\beta) \vdash n$. Then by Theorem 1.1 and Corollary 1.2, the eigenvalues of $\mathcal{F}(n,k)$ are integers given by

$$\eta_\lambda(k) = \frac{1}{f^\lambda} \sum_{i=1}^{s_k} N_{S_n}(\varphi(\sigma_{k_i})) \chi_\lambda(\varphi(\sigma_{k_i})),$$

where $\chi_\lambda(\varphi(\sigma_{k_i})) = \chi_\lambda(\sigma_{k_i})$.

Assume that $0 < k < n$. Note that each $\sigma_{k_i}$ ($1 \leq i \leq s_k$) must consist of at least one 1-cycle in its cycle decomposition. Therefore $\varphi(\sigma_{k_i}) = (\nu_1, \nu_2, \ldots, \nu_r) \vdash n$ and $\nu_r = 1$. Note that $\varphi(\sigma_{k_i}) - \hat{\varphi}(\sigma_{k_i}) = (\nu_1, \nu_2, \ldots, \nu_{r-1}) \vdash (n-1)$. We are now ready to state the following lemma which is a special case of [11, Theorem 3.4].

**Lemma 3.3.** If the Ferrers diagrams obtained from $\lambda$ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n-1)$ are those of $\mu_1, \ldots, \mu_q$, then

$$\chi_\lambda(\varphi(\sigma_{k_i})) = \sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\sigma_{k_i}) - \hat{\varphi}(\sigma_{k_i})),$$

for all $1 \leq i \leq s_k$. 

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Example 3.4. Let $n = 7$ and $\lambda = (3, 3, 1)$, then

\[
\begin{align*}
\chi_{(3,3,1)}((6, 1)) &= \chi_{(3,3)}((6)) + \chi_{(3,2,1)}((6)), \\
\chi_{(3,3,1)}((4, 2, 1)) &= \chi_{(3,3)}((4, 2)) + \chi_{(3,2,1)}((4, 2)), \\
\chi_{(3,3,1)}((3, 3, 1)) &= \chi_{(3,3)}((3, 3)) + \chi_{(3,2,1)}((3, 3)), \\
\chi_{(3,3,1)}((2, 2, 2, 1)) &= \chi_{(3,3)}((2, 2, 2)) + \chi_{(3,2,1)}((2, 2, 2)).
\end{align*}
\]

We shall need the following lemma [22, (7.18) on p. 299].

Lemma 3.5. Let $\lambda = (n, m, n, \ldots, 2, m, 2, 1) \vdash n$ and $z_\lambda = \prod_{j=1}^{n} (j^m; m_j!)$, then the size of the conjugacy class represented by $\lambda$ is

\[
N_{S_n}(\lambda) = \frac{n!}{z_\lambda}.
\]

Lemma 3.6. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash (n - k)$ be a derangement, i.e., $\lambda_r \geq 2$. If $\nu = (\lambda, 1^k) \vdash n$, and $\mu = (\lambda, 1^{k-1}) \vdash (n - 1)$, then

\[
N_{S_n}(\nu) = \frac{n}{k} N_{S_{n-1}}(\mu).
\]

Proof. The lemma follows from Lemma 3.5 by noting that

\[
N_{S_n}(\nu) = \frac{n!}{z_\lambda \times 1 \cdot k!}, \quad \text{and} \quad N_{S_{n-1}}(\mu) = \frac{(n-1)!}{z_\lambda \times 1 \cdot (k-1)!}.
\]

\[\square\]

Theorem 3.7. Let $0 < k < n$ and $\lambda \vdash n$. If the Ferrers diagrams obtained from $\lambda$ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n - 1)$ are those of $\mu_1, \ldots, \mu_q$, then

\[
\eta_\lambda(k) = \frac{n}{kf} \sum_{j=1}^{q} f^{\mu_j} \eta_{\mu_j}(k - 1).
\]

Proof. Suppose $k = 1$. By equation (3),

\[
\eta_\lambda(1) = \frac{1}{f^{\lambda}} \sum_{i=1}^{s_1} N_{S_n}(\varphi(\Omega_{1i})) \chi_{\lambda}(\varphi(\Omega_{1i})).
\]

Note that $\Omega_{1i}$ consists of exactly one 1-cycle and $\varphi(\Omega_{1i}) = (\nu_1, \nu_2, \ldots, \nu_r) \vdash n$ with $\nu_r = 1$, $\nu_{r-1} \geq 2$. Therefore $\varphi(\Omega_{1i}) - \hat{\varphi}(\Omega_{1i}) = (\nu_1, \nu_2, \ldots, \nu_{r-1}) \vdash (n - 1)$ is a derangement. In fact, $\varphi(\Omega_{1i}) - \hat{\varphi}(\Omega_{1i})$ is the
partition of \((n - 1)\) that represents \(\text{Con}_{S_{n-1}}(\sigma_{11})\). By Lemma 3.3 and Lemma 3.6

\[
\eta_{\lambda}(1) = \frac{1}{f^k} \sum_{i=1}^{s_k} N_{S_n}(\varphi(\sigma_{ki})) \left( \sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\sigma_{11}) - \hat{I}_{\varphi(\sigma_{11})}) \right)
\]

\[
= \frac{1}{f^k} \sum_{i=1}^{s_k} n N_{S_{n-1}}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{11})}) \left( \sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\sigma_{11}) - \hat{I}_{\varphi(\sigma_{11})}) \right)
\]

\[
= \frac{n}{f^k} \sum_{j=1}^{q} \left( \sum_{i=1}^{s_k} N_{S_{n-1}}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{11})}) \chi_{\mu_j}(\varphi(\sigma_{11}) - \hat{I}_{\varphi(\sigma_{11})}) \right)
\]

\[
= \frac{n}{f^k} \sum_{j=1}^{q} f^\mu \eta_{\mu_j}(0),
\]

where the last equality follows from equations (2) and (3). Thus, the theorem holds for \(k = 1\).

Suppose \(k > 1\). (We note here that the proof for \(k > 1\) is similar to the proof for \(k = 1\). The reason we distinguish them is to make the proof easier to comprehend.)

By equation (3),

\[
\eta_{\lambda}(k) = \frac{1}{f^k} \sum_{i=1}^{s_k} N_{S_n}(\varphi(\sigma_{ki})) \chi_{\lambda}(\varphi(\sigma_{ki})).
\]

Note that \(\sigma_{ki}\) consists of exactly k’s 1-cycle and \(\varphi(\sigma_{ki}) = (\nu_1, \nu_2, \ldots, \nu_r) \vdash n\) with \(\nu_j = 1\) for \(r - k + 1 \leq j \leq r\) and \(\nu_{r-k} \geq 2\). Let \(\sigma_{ki}\) be the extension of \(\sigma_{ki}\) to \(S_{n-1}\), i.e., \(\sigma_{ki}(j) = \sigma_{ki}(j)\) for \(1 \leq j \leq n - k\) and \(\sigma_{ki}(j) = j\) for \(n - k + 1 \leq j \leq n - 1\). Note that \(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})} = (\nu_1, \nu_2, \ldots, \nu_{r-1}) \vdash (n - 1)\) is the partition of \((n - 1)\) that represents \(\text{Con}_{S_{n-1}}(\sigma_{ki})\). Furthermore,

\[
S(n - 1, k - 1) = \bigcup_{i=1}^{s_k} \text{Con}_{S_{n-1}}(\sigma_{ki}).
\]

Therefore, by Theorem 1.1

\[
\eta_{\mu_j}(k - 1) = \frac{1}{f^\mu} \sum_{i=1}^{s_k} N_{S_{n-1}}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})}) \chi_{\mu_j}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})}).
\]

By Lemma 3.3 and Lemma 3.6

\[
\eta_{\lambda}(k) = \frac{1}{f^k} \sum_{i=1}^{s_k} N_{S_n}(\varphi(\sigma_{ki})) \left( \sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})}) \right)
\]

\[
= \frac{1}{f^k} \sum_{i=1}^{s_k} n \frac{1}{k} N_{S_{n-1}}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})}) \left( \sum_{j=1}^{q} \chi_{\mu_j}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})}) \right)
\]

\[
= \frac{n}{k f^k} \sum_{j=1}^{q} \left( \sum_{i=1}^{s_k} N_{S_{n-1}}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})}) \chi_{\mu_j}(\varphi(\sigma_{ki}) - \hat{I}_{\varphi(\sigma_{ki})}) \right)
\]

\[
= \frac{n}{k f^k} \sum_{j=1}^{q} f^\mu \eta_{\mu_j}(k - 1).
\]

Hence, the theorem holds for \(k > 1\).
4 Inequalities for the eigenvalues of $\mathcal{F}(n, 0)$

For convenience, if $\lambda = (n)$, we set
\[ d_n = \eta_{\lambda}(0). \]

By Theorem 2.2
\[ d_n = (-1)^n + nd_{n-1}, \quad \text{for } n \geq 1, \tag{4} \]
where $d_0 = 1$. Note that $d_1 = 0$ and $d_n > 0$ for all $n \neq 1$. Furthermore, for $n \geq 3$,
\begin{align*}
    d_n &= (-1)^n + nd_{n-1} \\
    &\geq nd_{n-1} - 1 \\
    &= (n-1)d_{n-1} + d_{n-1} - 1 \geq (n-1)d_{n-1}. \tag{5} \tag{6}
\end{align*}

Lemma 4.1. Let $1 \leq p \leq n - 1$. If $\lambda = (n-p, 1^p)$ and $\mu = (n-p+1, 1^{p-1})$ are partitions of $[n]$, then
\[ f^\lambda |\eta_{\lambda}(0)| \leq f^\mu |\eta_{\mu}(0)|. \]

Furthermore, equality holds if and only if $p = 1$ or $n - p = 1$.

Proof. Note that
\[ f^\lambda = \frac{n!}{H^\lambda} = \frac{n!}{n(n-p-1)!p!} \quad \text{and} \quad f^\mu = \frac{n!}{H^\mu} = \frac{n!}{n(n-p)!(p-1)!}. \]

By Theorem 2.1 and equation (4),
\begin{align*}
    |\eta_{\lambda}(0)| &= |1 + (-1)^{n-p}nd_{n-p-1}|, \\
    |\eta_{\mu}(0)| &= |1 + (-1)^{n-p+1}nd_{n-p}| \\
    &= |1 - n + (-1)^{n-p+1}n(n-p)d_{n-p-1}|.
\end{align*}

Therefore, it is sufficient to show that
\[ P_L = (n-p) |1 + (-1)^{n-p}nd_{n-p-1}| \leq p |1 - n + (-1)^{n-p+1}n(n-p)d_{n-p-1}| = P_R. \]

Case 1. Suppose $n$ and $p$ are of same parity (both even or both odd). Then
\[ P_R - P_L = p(n(n-p)d_{n-p-1} + n - 1) - (n-p)(1 + nd_{n-p-1}) \\
    = n(n-p)(p-1)d_{n-p-1} + (p-1)n \geq 0. \]

Note that $P_R - P_L = 0$ if and only if $p = 1$.

Case 2. Suppose $n$ and $p$ are of different parity (one even and one odd). Then $d_{n-p-1} \neq 0$, for $n-p \neq 2$. Therefore
\begin{align*}
    P_R - P_L &= p(1 - n + n(n-p)d_{n-p-1}) - (n-p)(nd_{n-p-1} - 1) \\
    &= n(n-p)(p-1)d_{n-p-1} - (p-1)n \\
    &= n(p-1)(n-p)d_{n-p-1} - 1 \geq 0.
\end{align*}

Note that $P_R - P_L = 0$ if and only if $p = 1$ or $n - p = 1$. \qed
Lemma 4.2. Let $m \geq q \geq 1$ and $n = m + q$. If $\lambda = (m, q)$ and $\mu = (m + 1, q - 1)$ are partitions of $[n]$, then

$$(m - q + 1) |\eta_\lambda(0)| \leq |\eta_\mu(0)|.$$  

Furthermore, equality holds if and only if $q = 1$ or $m = q = 2$.

Proof. We shall prove by induction on $q$. Suppose $q = 1$. By Corollary 2.4, $|\eta_\lambda(0)| = d_m + d_{m-1}$. By equation (4), $|\eta_\mu(0)| = d_{m+1} = (-1)^{m+1} + (m + 1)d_m$. Therefore

$$|\eta_\mu(0)| - m |\eta_\lambda(0)| = (-1)^{m+1} + d_m - md_{m-1} = 0.$$  

Suppose $q \geq 2$. Assume that the lemma holds for $q - 1$. By Theorem 2.1, $\eta_\lambda(0) = (-1)^{m+1}d_{q-1} - (m + 1)\eta_{(m-1,q-1)}(0)$. By Theorem 2.3, $\text{sign}(\eta_\lambda(0)) = (-1)^q$ and $\text{sign}((m - 1, q - 1)) = (-1)^{q-1}$. Thus,

$$|\eta_\lambda(0)| = (-1)^{m-q+1}d_{q-1} + (m + 1)|\eta_{(m-1,q-1)}(0)|.$$  

Similarly, by Theorem 2.1 and 2.3,

$$|\eta_\mu(0)| = (-1)^{m-q+1}d_{q-2} + (m + 2)|\eta_{(m,q-2)}(0)|.$$  

By induction, $(m - q + 1)|\eta_{(m-1,q-1)}(0)| \leq |\eta_{(m,q-2)}(0)|$.

Therefore

$$|\eta_\mu(0)| - (m - q + 1)|\eta_\lambda(0)| \geq (-1)^{m-q}(m - q + 1)d_{q-1} + (-1)^{m-q+1}d_{q-2} + |\eta_{(m,q-2)}(0)|.$$  

If $q = 2$, then $d_{q-1} = 0$ and $|\eta_\mu(0)| - (m - q + 1)|\eta_\lambda(0)| \geq d_m + (-1)^{m-1} \geq 0$. Furthermore, equality holds throughout if and only if $m = q = 2$.

Suppose $q \geq 3$. By Corollary 2.3, $|\eta_{(m,q-2)}(0)| = d_m + (q - 2)|\eta_{(m-1,q-3)}(0)| > d_m$, where the last inequality follows from $|\eta_{(m-1,q-3)}(0)| \neq 0$. If $m \equiv q \mod 2$, then $|\eta_\mu(0)| - (m - q + 1)|\eta_\lambda(0)| > (m - q)d_{q-1} + (d_{q-1} - d_{q-2}) + d_m > 0$. If $m \neq q \mod 2$, then

$$|\eta_\mu(0)| - (m - q + 1)|\eta_\lambda(0)| > -(m - q + 1)d_{q-1} + d_{q-2} + d_m \geq d_m - (m - q + 1)d_{q-1} \geq (m - 1)d_{m-1} - (m - q + 1)d_{q-1} \quad \text{(by equation 6)} \geq (q - 2)d_{q-1} > 0.$$  

This complete the proof of the lemma. 

Lemma 4.3. If $m > q \geq 1$ and $k \geq t \geq 1$, then

$$(m - q + k + 1)|\eta_{(q,q')}(0)| \leq k|\eta_{(m+1,q')}(0)|.$$  

Furthermore, equality holds if and only if $q = 1$, $m = 2$ and $k = t$. 

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Proof. We shall prove by induction on $q$. Suppose $q = 1$. Then by Corollary 2.4, $|\eta_{(q,q_t)}(0)| = t$ and $|\eta_{(m+1,q_t)}(0)| = td_m + d_{m+1}$. Note that $m \geq 2$. If $m = 2$, then

$$k|\eta_{(m+1,q_t)}(0)| - (m - q + k + 1)|\eta_{(q,q_t)}(0)| = k(t + 2) - (k + 2)t$$

$$= 2(k - t) \geq 0.$$

Furthermore, equality holds if and only if $k = t$.

If $m = 3$, then

$$k|\eta_{(m+1,q_t)}(0)| - (m - q + k + 1)|\eta_{(q,q_t)}(0)| = k(2t + 9) - (k + 3)t$$

$$= kt + 3(3k - t) > 0.$$

Suppose $m \geq 4$. By equation (6), $d_m \geq (m - 1)(m - 2)d_{m-2} \geq (m - 1)(m - 2)$. Since

$$k(m - 1)(m - 2) - (m - q + k + 1) = km^2 - (3k + 1)m + k$$

$$\geq 4km - (3k + 1)m + k$$

$$= (k - 1)m + k > 0,$$

we have $k|\eta_{(m+1,q_t)}(0)| - (m - q + k + 1)|\eta_{(q,q_t)}(0)| \geq t((k - 1)m + k) + d_{m+1} > 0$.

Suppose $q \geq 2$. Assume that

$$(m' - (q - 1) + k + 1)|\eta_{(q-1,(q-1)t)}(0)| \leq k|\eta_{(m' + 1,(q-1)t)}(0)|,$$

for all $m' > q - 1$ and $k \geq t \geq 1$.

By Corollary 2.4,

$$|\eta_{(m+1,q_t)}(0)| = q|\eta_{(m,(q-1)t)}(0)| + |\eta_{(m+1,q_t-1)}(0)|$$

$$= q|\eta_{(m,(q-1)t)}(0)| + q|\eta_{(m,(q-1)t-1)}(0)| + |\eta_{(m+1,q_t-2)}(0)|$$

$$\vdots$$

$$= q \left( \sum_{j=1}^{t} |\eta_{(m,(q-1)t)}(0)| \right) + d_{m+1}.$$

Similarly,

$$|\eta_{(q,q_t)}(0)| = q \left( \sum_{j=1}^{t} |\eta_{(q-1,(q-1)t)}(0)| \right) + d_q.$$

By induction, for $1 \leq j \leq t$,

$$(m - q + k + 1)|\eta_{(q-1,(q-1)t)}(0)| \leq k|\eta_{(m,(q-1)t)}(0)|.$$
By equation (6), \( d_{m+1} \geq m(m-1)d_{m-1} \geq m(m-1)d_q \). Note that \( m \geq 3 \) and
\[
km(m-1) - (m - q + k + 1) = km^2 - (k+1)m + q - k - 1 \\
\geq 3km - (k+1)m + q - k - 1 \\
= (2k-1)m + q - k - 1 \\
\geq 3(2k-1) + q - k - 1 \\
= 5k - 4 + q > 0
\]

Hence, \( (m - q + k + 1)|\eta_{(q,q')}(0)| \leq k|\eta_{(m+1,q')}(0)| \).

This complete the proof of the lemma.

**Lemma 4.4.** If \( q \geq 1 \) and \( t \geq 1 \), then
\[
|\eta_{(q',q-1)}(0)| < |\eta_{(q',q)}(0)|.
\]

**Proof.** We shall prove by induction on \( q \). Suppose \( q = 1 \). Then by Corollary 2.3, \( |\eta_{(q',q)}(0)| = t > t-1 = |\eta_{(q',q-1)}(0)| \).

Suppose \( q \geq 2 \). Assume that the lemma holds for \( q-1 \). By Corollary 2.3
\[
|\eta_{(q',q)}(0)| = q|\eta_{((q-1)',q-1)}(0)| + |\eta_{(q')}(0)|
\]
\[
|\eta_{(q',q-1)}(0)| = (q-1)|\eta_{((q-1)',q-2)}(0)| + |\eta_{(q')}(0)|.
\]
By induction, \( |\eta_{((q-1)',q-2)}(0)| < |\eta_{((q-1)',q-1)}(0)| \). Hence, \( |\eta_{(q',q-1)}(0)| < |\eta_{(q',q)}(0)| \).

This complete the proof of the lemma.

**Lemma 4.5.** Let \( m \geq q \geq 1 \), \( k \geq 2 \) and \( n = m + kq \). If \( \lambda = (m, q^{k-1}, q) \) and \( \mu = (m+1, q^{k-1}, q-1) \) are partitions of \([n]\), then
\[
(m - q + 1)|\eta_{\lambda}(0)| \leq k|\eta_{\mu}(0)|.
\]
Furthermore, equality holds if and only if \( q = 1 = m \).

**Proof.** We shall prove by induction on \( q \). Suppose \( q = 1 \). By Corollary 2.3, \( |\eta_{\lambda}(0)| = kd_{m-1} + d_m \) and \( |\eta_{\mu}(0)| = (k-1)d_m + d_{m+1} \). If \( m = 1 \), then \( |\eta_{\lambda}(0)| = k = k|\eta_{\mu}(0)| \) and the lemma holds. If \( m = 2 \), then \( 2|\eta_{\lambda}(0)| = 2 < k(k+1) = k|\eta_{\mu}(0)| \) and the lemma holds. Suppose \( m \geq 3 \). Then by equation (6),
\[
k|\eta_{\mu}(0)| - (m - q + 1)|\eta_{\lambda}(0)|
\]
\[
= k((k-1)d_m + d_{m+1}) - m(kd_{m-1} + d_m)
\]
\[
\geq k((k-1)d_m + md_m) - m(kd_{m-1} + d_m)
\]
\[
= (k^2 + (k-1)m - k)d_m - km(d_m-1)
\]
\[
\geq d_{m-1}((k^2 + (k-1)m - k)(m-1) - km)
\]
\[
\geq d_{m-1}(2(k^2 + (k-1)m - k) - km)
\]
\[
= d_{m-1}(2k(k-1) + (k-2)m) > 0.
\]
Suppose \( q \geq 2 \). Assume that the lemma holds for \( q - 1 \). By Theorem 2.1,
\[
\eta_\lambda(0) = (-1)^k (m + k) \eta_{m-1, (q-1)^{k-1}, q-1}(0) + (-1)^{m+k} \eta_{(q-1)^{k-1}, q-1}(0).
\]
By Theorem 2.3, \( \text{sign}(\eta_\lambda(0)) = (-1)^k \), \( \text{sign}(\eta_{m-1, (q-1)^{k-1}, q-1}(0)) = (-1)^k(q-1) \) and \( \text{sign}(\eta_{(q-1)^{k-1}, q-1}(0)) = (-1)^{(k-1)(q-1)} \). Therefore,
\[
|\eta_\lambda(0)| = (m + k)|\eta_{m-1, (q-1)^{k-1}, q-1}(0)| + (-1)^{m-q+1}|\eta_{(q-1)^{k-1}, q-1}(0)|.
\]
Similarly,
\[
|\eta_\mu(0)| = (m + k + 1)|\eta_{m, (q-1)^{k-1}, q-2}(0)| + (-1)^{m-q}|\eta_{(q-1)^{k-1}, q-2}(0)|.
\]
By induction,
\[
(m - q + 1)|\eta_{m-1, (q-1)^{k-1}, q-1}(0)| \leq k|\eta_{m, (q-1)^{k-1}, q-2}(0)|.
\]
Suppose \( m > q \). Then
\[
(m - q + 1)|\eta_\lambda(0)| = (m - q + 1) \left( (m + k)|\eta_{m-1, (q-1)^{k-1}, q-1}(0)| - |\eta_{(q-1)^{k-1}, q-1}(0)| \right)
\leq (m + k) \left( (m - q + 1)|\eta_{m-1, (q-1)^{k-1}, q-1}(0)| \right)
\leq (m + k) \left( k|\eta_{m, (q-1)^{k-1}, q-2}(0)| \right)
\leq k \left( (m + k + 1)|\eta_{m, (q-1)^{k-1}, q-2}(0)| + |\eta_{(q-1)^{k-1}, q-2}(0)| \right)
= k|\eta_\mu(0)|.
\]
Suppose \( m > q \). Note that
\[
(m - q + 1)|\eta_\lambda(0)| \leq (m - q + 1) \left( (m + k)|\eta_{m, (q-1)^{k-1}, q-1}(0)| + |\eta_{(q-1)^{k-1}, q-1}(0)| \right)
\leq (m + k) \left( k|\eta_{m, (q-1)^{k-1}, q-2}(0)| \right) + (m - q + 1)|\eta_{(q-1)^{k-1}, q-1}(0)|.
\]
By Lemma 4.3,
\[
k|\eta_\mu(0)| \geq k \left( (m + k)|\eta_{m, (q-1)^{k-1}, q-2}(0)| - |\eta_{(q-1)^{k-1}, q-2}(0)| \right)
\geq k \left( (m + k + 1)|\eta_{m, (q-1)^{k-1}, q-2}(0)| - |\eta_{(q-1)^{k-1}, q-1}(0)| \right).
\]
Therefore,
\[
k|\eta_\mu(0)| - (m - q + 1)|\eta_\lambda(0)|
\geq k|\eta_{m, (q-1)^{k-1}, q-2}(0)| - (m - q + k + 1)|\eta_{(q-1)^{k-1}, q-1}(0)|.
\]
If \( q = 2 \), then by Lemma 4.3, \( k|\eta_\mu(0)| - (m - q + k + 1)|\eta_\lambda(0)| \geq 0 \). Suppose \( q \geq 3 \). By Corollary 2.4,
\[
|\eta_{m, (q-1)^{k-1}, q-2}(0)| = (q - 2)|\eta_{m-1, (q-2)^{k-1}, q-3}(0)| + |\eta_{m-1, (q-1)^{k-1}, q-1}(0)|
> |\eta_{m, (q-1)^{k-1}, q-1}(0)|.
\]
It then follows from Lemma 4.3 that
\[
k|\eta_\mu(0)| - (m - q + 1)|\eta_\lambda(0)|
> k|\eta_{m, (q-1)^{k-1}, q-1}(0)| - (m - q + k + 1)|\eta_{(q-1)^{k-1}, q-1}(0)| > 0.
\]
This complete the proof of the lemma. \( \square \)
Lemma 4.6. Let \( r \geq 0, m \geq q \geq 1, k \geq 1, n = m + kq + \sum_{j=1}^{r} \alpha_j, q > \alpha_1 \) and
\[
\lambda = (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r), \\
\mu = (m + 1, q^{k-1}, q - 1, \alpha_1, \ldots, \alpha_r),
\]
be partitions of \([n]\). Then
\[
(m - q + 1)|\eta_\lambda(0)| \leq k|\eta_\mu(0)|.
\]

Proof. If \( r = 0 \), then the lemma follows from Lemma 4.2 or 4.5 depending on whether \( k = 1 \) or \( k \geq 2 \).

Suppose \( r \geq 1 \). Then \( q \geq 2 \), for \( q > \alpha_1 \geq 1 \). We shall prove by induction on \( \alpha_1 \).

Suppose \( \alpha_1 = 1 \). Then \( \alpha_1 = \cdots = \alpha_r = 1 \). By Corollary 2.4.
\[
|\eta_\lambda(0)| = |\eta_{(m,q^{k-1},q,\alpha_1,\ldots,\alpha_{r-1})}(0)| + |\eta_{(m-1,(q-1)^{k-1},q-1)}(0)| \\
|\eta_\mu(0)| = |\eta_{(m+1,q^{k-1},q-1)}(0)| + r|\eta_{(m,(q-1)^{k-1},q-2)}(0)|.
\]

By Lemma 4.2 or 4.3.
\[
(m - q + 1)|\eta_{(m,q^{k-1},q)}(0)| \leq k|\eta_{(m+1,q^{k-1},q-1)}(0)|, \quad \text{and} \\
(m - q + 1)|\eta_{(m-1,(q-1)^{k-1},q-1)}(0)| \leq k|\eta_{(m,(q-1)^{k-1},q-2)}(0)|.
\]

Hence, \( (m - q + 1)|\eta_\lambda(0)| \leq k|\eta_\mu(0)| \).

Suppose \( \alpha_1 \geq 2 \). Assume that the lemma holds for \( \alpha_1 - 1 \). By Corollary 2.4.
\[
|\eta_\lambda(0)| = |\eta_{(m,q^{k-1},q,\alpha_1,\ldots,\alpha_{r-1})}(0)| + |\alpha_r|\eta_{(m-1,(q-1)^{k-1},q-1,\alpha_1-1,\ldots,\alpha_{r-1})}(0)| \\
\vdots \\
= |\eta_{(m,q^{k-1},q)}(0)| + \sum_{j=1}^{r} \alpha_j|\eta_{(m-1,(q-1)^{k-1},q-1,\alpha_1-1,\ldots,\alpha_{j-1})}(0)|.
\]

Similarly,
\[
|\eta_\mu(0)| = |\eta_{(m+1,q^{k-1},q-1)}(0)| + \sum_{j=1}^{r} \alpha_j|\eta_{(m,(q-1)^{k-1},q-2,\alpha_1-1,\ldots,\alpha_{j-1})}(0)|.
\]

By Lemma 4.2 or 4.3.
\[
(m - q + 1)|\eta_{(m,q^{k-1},q)}(0)| \leq k|\eta_{(m+1,q^{k-1},q-1)}(0)|.
\]

By induction, for \( 1 \leq j \leq r \),
\[
(m - q + 1)|\eta_{(m-1,(q-1)^{k-1},q-1,\alpha_1-1,\ldots,\alpha_{j-1})}(0)| \leq k|\eta_{(m,(q-1)^{k-1},q-2,\alpha_1-1,\ldots,\alpha_{j-1})}(0)|.
\]

Hence, \( (m - q + 1)|\eta_\lambda(0)| \leq k|\eta_\mu(0)| \).
The following lemma is obvious.

**Lemma 4.7.** If \( u \geq v \), then
\[
\frac{(u+1)}{u} \left( \frac{v-1}{v} \right) < 1.
\]

**Lemma 4.8.** Let \( r \geq 0, k \geq 1, m \geq q \geq 2, n = m + kq + \sum_{j=1}^{r} \alpha_j, q > \alpha_1 \) and
\[
\lambda = (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r),
\nu = (m+1, q^{k-1}, q-1, \alpha_1, \ldots, \alpha_r),
\]
be partitions of \([n]\). Then
\[
\frac{f^\lambda}{f^\nu} < \frac{(m-q+1)}{k}.
\]

**Proof.** Note that \( h_{\nu}(i, j) = h_{\lambda}(i, j) \) for all \( i, j \) except when \( i = q, j = 1 \) or \( j = k+1 \). Let \( c_i = h_{\lambda}(i,1) \) and \( d_i = h_{\lambda}(i,k+1) \) for \( 1 \leq i \leq q-1 \). Note that \( h_{\nu}(i,1) = c_i + 1 \) and \( h_{\nu}(i,k+1) = d_i - 1 \) for \( 1 \leq i \leq q-1 \), and \( h_{\nu}(q,1) = h_{\lambda}(q,1) \). Therefore
\[
\frac{f^\lambda}{f^\nu} = \frac{H^\nu}{H^\lambda} = \frac{\left( \prod_{i=1}^{\nu-1} (c_i + 1) \right) \left( \prod_{i=1}^{q-1} (d_i - 1) \right) (m+1-q)! (k-1)!}{\left( \prod_{i=1}^{\nu-1} c_i \right) \left( \prod_{i=1}^{q-1} d_i \right) (m-q) k!}
\]
\[
= \frac{\left( \prod_{i=1}^{q-1} \left( \frac{c_i + 1}{c_i} \right) \left( \frac{d_i - 1}{d_i} \right) \right) (m+1-q)}{k}
< \frac{(m-q+1)}{k},
\]
where the last inequality follows from \( c_i > d_i \) and Lemma 4.7. \( \square \)

**Theorem 4.9.** Let \( r \geq 0, k \geq 1, m \geq q \geq 1, n = m + kq + \sum_{j=1}^{r} \alpha_j, q > \alpha_1 \) and
\[
\lambda = (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r),
\nu = (m+1, q^{k-1}, q-1, \alpha_1, \ldots, \alpha_r),
\]
be partitions of \([n]\). Then
\[
f^\lambda |\eta_\lambda(0)| \leq f^\nu |\eta_\nu(0)|.
\]
Furthermore, equality holds if and only if \( \lambda = (1, 1^{n-1}) \) or \( \lambda = (n-1, 1) \).

**Proof.** Suppose \( q = 1 \). Then \( r = 0 \) and the theorem follows from Lemma 4.1. Suppose \( q \geq 2 \). By Lemma 4.6 and 4.8
\[
\frac{f^\lambda}{f^\nu} |\eta_\lambda(0)| < \frac{(m-q+1)}{k} |\eta_\nu(0)| \leq |\eta_\nu(0)|.
\]
This complete the proof of the theorem. \( \square \)
5 Proof of Theorem 1.3

Proof. Suppose the Ferrers diagrams obtained from $\lambda$ by removing 1 node from the right hand side from any row of the diagram so that the resulting diagram will still be a partition of $(n-1)$ are those of $\mu_1, \ldots, \mu_s$. Then by Theorem 3.7

$$\eta_\lambda(1) = \frac{n}{k f_\lambda} \sum_{j=1}^{s} f^{\mu_j} \eta_{\mu_j}(0).$$

Suppose $r = 1$. Then $s = 1$ and $\mu_1 = (\lambda_1 - 1) = (n - 1)$. Thus, $\eta_\lambda(1) = \frac{n}{k f_\lambda} f^{\mu_1} \eta_{\mu_1}(0) \geq 0$ and with equality if and only if $\mu_1 = (1)$, i.e., $\lambda = (2)$.

Suppose $r \geq 2$. If $\lambda_1 = \lambda_2$, then the first part of each $\mu_j$ is $\lambda_1$. By Theorem 2.3 $\text{sign}(\eta_{\mu_j}(0)) = (\sum_{i=2}^{s} \lambda_i) - 1 = |\lambda| - \lambda_1 - 1$. Hence, $\eta_\lambda(1) = (-1)^{|\lambda| - \lambda_1} \frac{n}{k f_\lambda} \sum_{j=1}^{s} f^{\mu_j} |\eta_{\mu_j}(0)|$. Note that $\eta_\lambda(1) = 0$ if and only if $s = 1$ and $\mu_1 = (1)$, i.e., $\lambda = (1,1)$. For other partitions $\lambda$, $|\eta_\lambda(1)| \neq 0$ and $\text{sign}(\eta_\lambda(1)) = |\lambda| - \lambda_1 - 1$.

Suppose $\lambda_1 = m + 1 > \lambda_2 = q$. Note that we may write

$$\lambda = (m + 1, q^{k-1}, q, \alpha_1, \ldots, \alpha_r),$$

where $r \geq 0$, $k \geq 1$, $m \geq q \geq 1$, and $q > \alpha_1$. Let

$$\mu_1 = (m, q^{k-1}, q, \alpha_1, \ldots, \alpha_r),$$

$$\mu_2 = (m + 1, q^{k-1}, q - 1, \alpha_1, \ldots, \alpha_r).$$

By Theorem 2.3 $\text{sign}(\eta_{\mu_1}(0)) = |\lambda| - \lambda_1$ and $\text{sign}(\eta_{\mu_j}(0)) = |\lambda| - \lambda_1 - 1$ for $j \geq 2$. This implies that

$$\eta_\lambda(1) = (-1)^{|\lambda| - \lambda_1 - 1} \frac{n}{k f_\lambda} \left( f^{\mu_2} |\eta_{\mu_2}(0)| - f^{\mu_1} |\eta_{\mu_1}(0)| + \sum_{j=3}^{s} f^{\mu_j} |\eta_{\mu_j}(0)| \right).$$

By Theorem 4.9 $f^{\mu_2} |\eta_{\mu_2}(0)| - f^{\mu_1} |\eta_{\mu_1}(0)| \geq 0$. Furthermore, equality holds if and only if

$$\mu_1 = (1, 1^{n-2}) \text{ or } \mu_1 = (n - 2, 1),$$

i.e., $\lambda = (2, 1^{n-2})$ or $(n - 1, 1)$. Note also that when this happens, we have $s = 2$ and $\eta_\lambda(1) = 0$. For other partitions $\lambda$, $f^{\mu_2} |\eta_{\mu_2}(0)| - f^{\mu_1} |\eta_{\mu_1}(0)| > 0$. Hence, $|\eta_\lambda(1)| \neq 0$ and $\text{sign}(\eta_\lambda(1)) = |\lambda| - \lambda_1 - 1$.

This completes the proof of the theorem. ∎
6 Eigenvalues Table for $\mathcal{F}(n, 1)$

$n = 3$

| $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ |
|-----------|----------------|-----------|----------------|-----------|----------------|
| [3]       | 3              | [2, 1]    | 0              | [1^3]     | -3             |

$n = 4$

| $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ |
|-----------|----------------|-----------|----------------|-----------|----------------|-----------|----------------|
| [4]       | 8              | [3, 1]    | 0              | [2, 2]    | -4             | [2, 1^2]  | 0              | [1^4]     | 8             |

$n = 5$

| $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ |
|-----------|----------------|-----------|----------------|-----------|----------------|-----------|----------------|
| [5]       | 45             | [3, 2]    | -3             | [2^2, 1]  | 9              | [1^3]     | -15            |
| [4, 1]    | 0              | [3, 1^2]  | -5             | [2, 1^3]  | 0              |           |                |

$n = 6$

| $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ | $\lambda$ | $\eta_\lambda$ |
|-----------|----------------|-----------|----------------|-----------|----------------|-----------|----------------|
| [6]       | 264            | [4, 1^2]  | -12            | [3, 1^3]  | 12             | [2, 1^4]  | 0              |
| [5, 1]    | 0              | [3^2]     | 24             | [2^3]     | -24            | [1^6]     | 24             |
| [4, 2]    | -16            | [3, 2, 1] | 9              | [2^2, 1^2]| -16            |           |                |
\[ n = 7 \]

| \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 7   | 1855 | 5, 1 | -63 | 4, 1 | 28 | 3, 2 | -17 | 2, 1 | 25 |
| 6, 1 | 0   | [4, 3] | 40 | [3, 1] | -45 | [3, 2] | -21 | [2, 1] | 0 |
| 5, 2 | -65 | [4, 2] | 37 | [3, 2] | -15 | [2, 1] | 40 | [1]  | -35 |

\[ n = 8 \]

| \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   | 14832 | 5, 14 | 144 | [3, 2] | 80 | [2, 1] | -60 |
| 7, 1 | 0   | [4] | -168 | [3, 1] | 72 | [2, 1] | 36 |
| 6, 2 | -372 | [4, 3] | -72 | [3, 2] | 24 | [1]  | 0 |
| 6, 14 | -352 | [4, 2] | -72 | [3, 2] | 27 | [1]  | 48 |
| 5, 3 | 180 | [4, 2, 1] | -64 | [3, 1] | 32 | [2]  | 4 |
| 5, 2, 1 | 153 | [4, 1] | -48 | [2] | -72 | [1]  | |

\[ n = 9 \]

| \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 9   | 133497 | 5, 3, 1 | -315 | 4, 2, 1 | 97 | 3, 1 | 45 |
| 8, 1 | 0   | [5, 2] | -273 | [4, 1] | 72 | [2, 1] | 105 |
| 7, 2 | -2471 | [5, 2, 1] | -263 | [4, 3] | -171 | [2, 1] | 84 |
| 7, 14 | -2385 | [5, 14] | -243 | 3, 2, 1 | -120 | [2, 1] | 49 |
| 6, 3 | 924 | [4, 2] | 267 | 3, 1, 1 | -105 | [2, 1] | 0 |
| 6, 2, 1 | 849 | [4, 3] | -120 | 3, 2 | -33 | [2]  | 63 |
| 6, 14 | 792 | [4, 3, 2] | 112 | 3, 2, 1 | -35 | [9]  | |
| 5, 4 | -375 | [4, 2, 1] | 112 | 3, 2, 1 | -39 | [2]  | |

\[ n = 10 \]

| \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 10  | 1334960 | [6, 14] | -1320 | 4, 3, 2, 1 | -175 | [3, 2, 1] | 48 |
| 9, 1 | 0   | [5] | 1280 | 4, 3, 1 | -160 | [3, 2, 1] | 53 |
| 8, 2 | -19072 | [5, 4] | 585 | 4, 2 | -172 | [3, 1] | 60 |
| 8, 14 | -18540 | [5, 3] | 504 | 4, 2, 1 | -160 | [2]  | 160 |
| 7, 3 | 5936 | [5, 3, 1] | 480 | 4, 2, 1 | -136 | [2, 1] | 144 |
| 7, 14 | 5561 | [5, 2, 1] | 416 | 4, 1 | -100 | [2, 1] | 112 |
| 7, 15 | 5300 | [5, 2, 1] | 395 | 3, 1 | 248 | [2, 1] | 64 |
| 6, 4 | -1872 | [5, 15] | 360 | 3, 2 | 180 | [2, 1] | 0 |
| 6, 3, 1 | -1584 | [4, 2] | -420 | 3, 2, 1 | 168 | [1]  | 80 |
| 6, 2, 1 | -1488 | [4, 2, 1] | -388 | 3, 2, 14 | 144 | [2]  | 4 |
| 6, 2, 14 | -1432 | [4, 3] | -184 | 3, 2, 1 | 45 | [1]  | |
\( n = 11, \lambda_1 \geq 5 \)

| \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) |
|---------|--------|---------|--------|---------|--------|
| [11]    | 14684571 | [7, 2, 1]\(^2\) | -9269  | [5, 4, 2] | -875   |
| [10, 1] | 0       | [7, 1]\(^4\)  | -8745  | [5, 4, 1]\(^2\) | -837   |
| [9, 2]  | -166869 | [6, 5]      | 3576   | [5, 3]\(^2\) | -801   |
| [9, 1]\(^2\) | -163163 | [6, 4, 1]   | 2851   | [5, 3, 2, 1] | -720   |
| [8, 3]  | 44496   | [6, 3, 2]   | 2464   | [5, 3, 1]\(^3\) | -675   |
| [8, 2, 1]| 42381   | [6, 3, 1]\(^2\) | 2376 | [5, 2]\(^3\) | -603   |
| [8, 1]\(^3\) | 40788  | [6, 2, 1]\(^2\) | 2232 | [5, 2, 1, 1]\(^2\) | -585   |
| [7, 4]  | -11109  | [6, 2, 1]\(^3\) | 2121  | [5, 2, 1, 1] | -549   |
| [7, 3, 1]| -10017  | [6, 1]\(^5\)  | 1936   | [5, 1]\(^6\) | -495   |
| [7, 2]\(^2\) | -9531   | [5, 1]\(^2\)   | -1863  |                     |       |

\( n = 12, \lambda_1 \geq 6 \)

| \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) |
|---------|--------|---------|--------|---------|--------|
| [12]    | 176214840 | [8, 2, 1]\(^2\) | -69928 | [6, 5, 1] | -5112 |
| [11, 1] | 0       | [8, 1]\(^4\)  | -66744 | [6, 4, 2] | -4160 |
| [10, 2] | -1631620 | [7, 5]      | 19880  | [6, 4, 1]\(^2\) | -4008 |
| [10, 1]\(^2\) | -1601952 | [7, 4, 1]   | 16665  | [6, 4]\(^3\) | -3684 |
| [9, 3]  | 381420  | [7, 3, 2]   | 15264  | [6, 3, 2, 1] | -3465 |
| [9, 2, 1]| 367113  | [7, 3, 1]\(^2\) | 14840 | [6, 3, 1]\(^3\) | -3300 |
| [9, 1]\(^3\) | 355992  | [7, 2, 1]\(^2\) | 14120 | [6, 2]\(^3\) | -3192 |
| [8, 4]  | -80112  | [7, 2, 1]\(^3\) | 13595  | [6, 2, 1, 1]\(^2\) | -3100 |
| [8, 3, 1]| -74160  | [7, 1]\(^6\)  | 12720  | [6, 2, 1, 1] | -2916 |
| [8, 2]\(^2\) | -71520  | [6, 2]      | -10860 | [6, 1]\(^6\) | -2640 |

\( n = 13, \lambda_1 \geq 6 \)

| \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) | \( \lambda \) | \( \eta_\lambda \) |
|---------|--------|---------|--------|---------|--------|
| [13]    | 2290792933 | [8, 3, 1]\(^2\) | 108768 | [6, 2, 1] | 14877 |
| [12, 1] | 0       | [8, 2, 1]\(^2\) | 104896 | [6, 5, 2] | 7152 |
| [11, 2] | -17621483 | [8, 2, 1]\(^3\) | 101713 | [6, 5, 1]\(^2\) | 6904 |
| [11, 1]\(^2\) | -17354493 | [8, 1]\(^5\)  | 96408  | [6, 4, 3] | 5997 |
| [10, 3] | 3671140  | [7, 6]      | -36155 | [6, 4, 2, 1] | 5616 |
| [10, 2, 1]| 3559897  | [7, 5, 1]   | -27953 | [6, 4, 1]\(^3\) | 5343 |
| [10, 1]\(^3\) | 3470896  | [7, 4, 2]   | -23805 | [6, 3, 2, 1] | 4972 |
| [9, 4]  | -667467  | [7, 4, 1]\(^2\) | -23147 | [6, 3, 2]\(^2\) | 4752 |
| [9, 3, 1]| -629343  | [7, 3]\(^2\)  | -22271 | [6, 3, 2, 1]\(^2\) | 4620 |
| [9, 2]\(^2\) | -611853  | [7, 3, 2, 1] | -21200 | [6, 3, 1]\(^3\) | 4356 |
| [9, 2, 1]\(^2\) | -600731  | [7, 3, 1]\(^3\) | -20405 | [6, 2, 3, 1] | 4257 |
| [9, 1]\(^4\) | -578487  | [7, 2]\(^3\)  | -19853 | [6, 2, 1, 1]\(^3\) | 4092 |
| [8, 5]  | 133408   | [7, 2, 1]\(^2\) | -19415 | [6, 2, 1, 1]\(^2\) | 3817 |
| [8, 4, 1]| 118683   | [7, 2, 1]\(^4\) | -18539 | [6, 1]\(^5\) | 3432 |
| [8, 3, 2]| 111240   | [7, 1]\(^6\)  | -17225 |                     |       |
\[
\lambda \quad \eta_\lambda \quad \lambda \quad \eta_\lambda \quad \lambda \quad \eta_\lambda \\
\hline
15 \quad 481066515735 \quad [8, 2, 1^2] \quad 177997 \quad [6, 3, 1^9] \quad 6864 \\
14, 1 \quad 0 \quad [8, 1^7] \quad 166860 \quad [6, 2^4, 1] \quad 7032 \\
13, 2 \quad -2672591753 \quad [7, 2, 1] \quad -133761 \quad [6, 2^3, 1^2] \quad 6813 \\
13, 1^2 \quad -2643222615 \quad [7, 6, 2] \quad -65079 \quad [6, 2^2, 1^5] \quad 6448 \\
12, 3 \quad 458158584 \quad [7, 6, 1^2] \quad -63305 \quad [6, 2, 1^7] \quad 5937 \\
12, 2, 1 \quad 448546869 \quad [7, 5, 3] \quad -52211 \quad [6, 1^9] \quad 5280 \\
12, 1^3 \quad 440537100 \quad [7, 5, 2, 1] \quad -49700 \quad [5^3] \quad -11235 \\
11, 4 \quad -66080553 \quad [7, 5, 1^3] \quad -47825 \quad [5^2, 4, 1] \quad -7880 \\
11, 3, 1 \quad -63633141 \quad [7, 4^2] \quad -46581 \quad [5^2, 3, 2] \quad -6803 \\
11, 2^2 \quad -62476167 \quad [7, 4, 3, 1] \quad -43193 \quad [5^2, 3, 1^2] \quad -6545 \\
11, 2^1, 1 \quad -61675193 \quad [7, 4, 2^2] \quad -41655 \quad [5^2, 2^2, 1] \quad -6063 \\
11, 1^4 \quad -60073245 \quad [7, 4, 2, 1^2] \quad -40733 \quad [5^2, 2, 1^3] \quad -5760 \\
10, 5 \quad 9786940 \quad [7, 4, 1^4] \quad -38889 \quad [5^2, 1^6] \quad -5255 \\
10, 4, 1 \quad 9154035 \quad [7, 3^2, 2] \quad -38981 \quad [5, 4^2, 2] \quad -3285 \\
10, 3, 2 \quad 8810736 \quad [7, 3^2, 1^2] \quad -38111 \quad [5^2, 4, 2^2] \quad -3183 \\
10, 3, 1^2 \quad 8677240 \quad [7, 3^2, 1] \quad -36729 \quad [5, 4^3] \quad -3111 \\
10, 2^2, 1 \quad 8484424 \quad [7, 3^2, 1^3] \quad -35616 \quad [5^2, 4, 2, 1] \quad -2925 \\
10, 2, 1^3 \quad 8306425 \quad [7, 3, 1^5] \quad -33761 \quad [5^2, 4, 3, 1] \quad -2793 \\
10, 1^5 \quad 8009760 \quad [7, 2^4] \quad -33965 \quad [5, 4^2, 2] \quad -2685 \\
9, 6 \quad -1668105 \quad [7, 2^4, 1^2] \quad -33351 \quad [5, 4^2, 2^2] \quad -2615 \\
9, 5, 1 \quad -1468563 \quad [7, 2^4, 1^4] \quad -32123 \quad [5, 4, 2, 1^4] \quad -2475 \\
9, 4, 2 \quad -1359655 \quad [7, 2^6] \quad -30281 \quad [5, 4, 1^6] \quad -2265 \\
9, 4, 1^2 \quad -1334937 \quad [7, 1^8] \quad -27825 \quad [5^3, 1] \quad -2649 \\
9, 3^2 \quad -1311141 \quad [6^2, 3] \quad 27903 \quad [5, 3^2, 2^2] \quad -2495 \\
9, 3, 2, 1 \quad -1271400 \quad [6^2, 2, 1] \quad 26064 \quad [5^3, 2^2, 1^2] \quad -2421 \\
9, 3, 1^3 \quad -1239615 \quad [6^2, 1^3] \quad 24765 \quad [5^3, 2^1, 1^4] \quad -2273 \\
9, 2^3 \quad -1223703 \quad [6, 5, 4] \quad 14181 \quad [5, 3, 2^3, 1] \quad -2160 \\
9, 2^2, 1^2 \quad -1205165 \quad [6, 5, 3, 1] \quad 12832 \quad [5, 3^2, 2^3] \quad -2079 \\
9, 2, 1^4 \quad -1168089 \quad [6, 5, 2^2] \quad 12252 \quad [5, 3, 2, 1^5] \quad -1944 \\
9, 1^6 \quad -1112475 \quad [6, 5, 2, 1^2] \quad 11920 \quad [5, 3, 1^7] \quad -1755 \\
8, 7 \quad 393072 \quad [6, 5, 1^4] \quad 11256 \quad [5, 2^5] \quad -1725 \\
8, 6, 1 \quad 297585 \quad [6^2, 1^2] \quad 11352 \quad [5^2, 4, 1^2] \quad -1691 \\
8, 5, 2 \quad 250140 \quad [6, 4, 3, 2] \quad 10287 \quad [5, 2^3, 1^4] \quad -1623 \\
8, 5, 1^2 \quad 244588 \quad [6, 4, 3, 1^2] \quad 9997 \quad [5^2, 2^1, 1^6] \quad -1521 \\
8, 4, 3 \quad 2229879 \quad [6, 4, 2^2, 1] \quad 9515 \quad [5, 2, 1^8] \quad -1385 \\
8, 4, 2, 1 \quad 220308 \quad [6, 4, 2, 1^2] \quad 9152 \quad [5^1, 4^2, 1^5] \quad -1215 \\
8, 4, 1^3 \quad 213627 \quad [6, 4, 1^3] \quad 8547 \quad [4^3, 3] \quad 3420 \\
8, 3^2, 1 \quad 209752 \quad [6, 3^2] \quad 8808 \quad [4^2, 2, 1] \quad 3075 \\
8, 3, 2^2 \quad 203940 \quad [6, 3^2, 2, 1] \quad 8409 \quad [4^3, 1^3] \quad 2856 \\
8, 3, 2, 1^2 \quad 200232 \quad [6, 3^2, 1^2] \quad 8100 \quad [4^2, 3^2, 1] \quad 2305 \\
8, 3, 1^4 \quad 192816 \quad [6, 3^2, 1^3] \quad 7920 \quad [4^2, 3, 2^2] \quad 2143 \\
8, 2^3, 1 \quad 190725 \quad [6, 3, 2^2, 1^2] \quad 7744 \quad [4^2, 3, 2, 1^2] \quad 2061 \\
8, 2^2, 1^3 \quad 185952 \quad [6, 3, 2, 1^4] \quad 7392 \quad [4^2, 3, 1^4] \quad 1897 \\
\]
| $\lambda$                     | $\eta_\lambda$ | $\lambda$                     | $\eta_\lambda$ | $\lambda$                     | $\eta_\lambda$ |
|-------------------------------|-----------------|-------------------------------|-----------------|-------------------------------|-----------------|
| $[4^2, 2^3, 1]$               | 1788            | $[4, 2^6, 1]$                 | 697             | $[3, 2^6]$                    | -123            |
| $[4^2, 2^2, 1^3]$             | 1695            | $[4, 2^5, 1^7]$               | 520             | $[3, 2^5, 1^7]$               | -125            |
| $[4^2, 2, 1^5]$               | 1540            | $[4, 2^4, 1^9]$               | 421             | $[3, 2^4, 1^4]$               | -129            |
| $[4^2, 1^7]$                  | 1323            | $[4, 1^{11}]$                 | 300             | $[3, 2^3, 1^6]$               | -135            |
| $[4, 3^3, 2]$                 | 760             | $[3^5]$                       | -1545           | $[3, 2^2, 1^8]$               | -143            |
| $[4, 3^3, 1^2]$               | 744             | $[3^4, 2, 1]$                 | -1368           | $[3, 2, 1^9]$                 | -153            |
| $[4, 3^2, 2^2, 1]$            | 736             | $[3^4, 1^3]$                  | -1263           | $[3, 1^{12}]$                 | -165            |
| $[4, 3^2, 2, 1^3]$            | 709             | $[3^3, 2^3]$                  | -1119           | $[2^7, 1]$                    | 624             |
| $[4, 3^2, 1^5]$               | 664             | $[3^3, 2^2, 1^2]$             | -1073           | $[2^6, 1^3]$                  | 585             |
| $[4, 3, 2^4]$                 | 720             | $[3^3, 2, 1^4]$               | -981            | $[2^5, 1^5]$                  | 520             |
| $[4, 3, 2^3, 1^2]$            | 700             | $[3^3, 1^6]$                  | -843            | $[2^4, 1^7]$                  | 429             |
| $[4, 3, 2^3, 1^4]$            | 660             | $[3^2, 2^4, 1]$               | -693            | $[2^3, 1^9]$                  | 312             |
| $[4, 3, 2, 1^6]$              | 600             | $[3^2, 2^3, 1^3]$             | -660            | $[2^2, 1^{11}]$               | 169             |
| $[4, 3, 1^8]$                 | 520             | $[3^2, 2^2, 1^5]$             | -605            | $[2, 1^{13}]$                 | 0               |
| $[4, 2^5, 1]$                 | 685             | $[3^2, 2, 1^7]$               | -528            | $[1^{15}]$                    | -195            |
| $[4, 2^4, 1^3]$               | 652             | $[3^2, 1^9]$                  | -429            |                               |                 |

20
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