A NOTE ON PLURISUBHARMONIC DEFINING FUNCTIONS IN $\mathbb{C}^2$

J. E. FORNÆSS, A.-K. HERBIG

Abstract. Let $\Omega \subset \subset \mathbb{C}^2$ be a smoothly bounded domain. Suppose that $\Omega$ admits a smooth defining function which is plurisubharmonic on the boundary of $\Omega$. Then the Diederich-Fornæss exponent can be chosen arbitrarily close to 1, and the closure of $\Omega$ admits a Stein neighborhood basis.

1. Introduction

Let $\Omega \subset \subset \mathbb{C}^2$ be a smoothly bounded domain. Throughout, we suppose that $\Omega$ admits a $C^\infty$-smooth defining function $\rho$ which is plurisubharmonic on the boundary, $b\Omega$, of $\Omega$, i.e.,

$$H_\rho(\xi, \xi)(z) := \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) \xi_j \bar{\xi}_k \geq 0$$

for all $z \in b\Omega$ and $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$. This property comes up naturally as a sufficiency condition for global regularity of the Bergman projection, see [Boa-Str91, Her-McN06].

The purpose of this paper is to investigate how the plurisubharmonicity of $\rho$ influences the behaviour of the complex Hessian of $\rho$ (or of the complex Hessians of some other defining functions of $\Omega$) away from the boundary of $\Omega$.

Suppose $D = \{ z \in \mathbb{C}^2 \mid r(z) < 0 \}$ is a smoothly bounded, pseudoconvex domain. Then it follows by standard arguments, that there exists a neighborhood $W$ of the boundary of $D$ such that the following lower estimate for the complex Hessian of $r$ holds:

$$H_r(\xi, \xi)(q) \geq O(r(q))|\xi|^2 + O(|\xi| \cdot |\partial r(q), \xi|)$$

for $q \in W$ and $\xi \in \mathbb{C}^2$ (see for instance [Ran81] for details). Our main result shows how to improve the estimate (1.1) under the additional condition that there is some smooth defining function of $D$ which is plurisubharmonic on the boundary of $D$.

Theorem 1.2. Let $\Omega \subset \subset \mathbb{C}^2$ be a smoothly bounded domain, and suppose $\Omega$ admits a smooth defining function which is plurisubharmonic on the boundary, $b\Omega$, of $\Omega$. Then the following holds: for each $\epsilon > 0$ and $K > 0$, there exist a neighborhood $V$ of $b\Omega$ and defining functions $r_1$ and $r_2$ such that for all $\xi \in \mathbb{C}^2$

$$H_{r_1}(\xi, \xi)(q) \geq cr_1(q)|\xi|^2 + K(|\partial r_1(q), \xi|)^2 \quad \text{for } q \in V \cap \overline{\Omega}$$

and

$$H_{r_2}(\xi, \xi)(q) \geq -cr_2(q)|\xi|^2 + K(|\partial r_2(q), \xi|)^2 \quad \text{for } q \in V \cap \overline{\Omega^c}.$$  

An immediate consequence of Theorem 1.2 is the existence of strictly plurisubharmonic exhaustion functions of $\Omega$ and of the complement of $\Omega$.

Corollary 1.5. Suppose the hypotheses of Theorem 1.2 holds. Then

Key words and phrases. plurisubharmonic defining functions, Stein neighborhood basis, DF exponent.

Research of the first author was partially supported by an NSF grant.
(i) for any \( \eta \in (0, 1) \) there exists a smooth defining function \( \tilde{r}_1 \) such that \(-(-\tilde{r}_1)^\eta \) is strictly plurisubharmonic on \( \Omega \),

(ii) for any \( \eta > 1 \) there exist a neighborhood \( V \) of \( b\Omega \) and a smooth defining function \( \tilde{r}_2 \) such that \( \tilde{r}_2^\eta \) is strictly plurisubharmonic on \( V \setminus \Omega \).

A Diederich-Fornæss exponent of a domain \( D \subset \subset \mathbb{C}^n \) is a number \( \tau \in (0, 1] \) for which there exists a smooth defining function \( s \) of \( D \) so that \(-(-s)^\tau \) is strictly plurisubharmonic on \( D \). That all smoothly bounded pseudoconvex domains in \( \mathbb{C}^n \) have a Diederich-Fornæss exponent \( \tau \) was shown in [Die-For77a] (see also [Ran81]). It is also known that there are pseudoconvex domains for which the largest possible \( \tau \) might be arbitrarily close to 0 (see [Die-For77b]). However, part (i) of Corollary 1.5 says that the Diederich-Fornæss exponent can be chosen arbitrarily close to 1 on domains which admit a smooth defining function, which is plurisubharmonic on the boundary. Part (ii) of Corollary 1.5 is of interest, since it implies that the closure of \( \Omega \) has a Stein neighborhood basis. In particular, it follows that \( b\Omega \) is uniformly H-convex. We remark that partial results regarding the existence of a Stein neighborhood basis for the closure of a domain, which satisfies the hypotheses of Theorem 1.2, have been obtained in [Sah06].

The paper is structured as follows. In Section 2, we identify the obstruction to (1.3) to hold for the given defining function \( \rho \). We then give an example to show that this obstruction might actually occur. In Section 3, we prove Theorem 1.2, and we conclude this paper by proving Corollary 1.5 in Section 4.

We would like to thank J.D. McNeal for stimulating discussions about this project.

2. The obstruction

Throughout, \((z_1, z_2)\) will denote the coordinates of \( \mathbb{C}^2 \). We shall identify the vector \( \langle \xi_1, \xi_2 \rangle \) in \( \mathbb{C}^2 \) with \( \xi_1 \frac{\partial}{\partial z_1} + \xi_2 \frac{\partial}{\partial z_2} \) in the \((1, 0)\)-tangent bundle of \( \mathbb{C}^2 \) at any given point. We use the pointwise hermitian inner product \( \langle .., \rangle \) defined by \( \langle \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \rangle = \delta_{jk} \). We also shall use \( \langle .., \rangle \) to denote contractions of vector fields and forms. We hope this abuse of notation will not confuse the reader as it should be clear from the context what is meant.

Let us first see which quantities the right hand side of (1.1) depends on. To do so, we need to use Taylor’s formula:

2.1. Taylor’s formula in our context. Since \( b\Omega \) is smooth, there exist a neighborhood \( U \) of \( b\Omega \) and a smooth map

\[
\pi : \overline{\Omega} \cap U \longrightarrow b\Omega \\
q \mapsto \pi(q) = p
\]

such that \( p \in b\Omega \) lies on the line normal to \( b\Omega \) passing through \( q \), and \(|p - q|\) is equal to the complex euclidean distance, \( d_M(q) \), of \( q \) to \( b\Omega \). Denote by \( \hat{n}_p \) the unit outward normal to \( b\Omega \) at \( p \). Then \( q = p - d_M(q)\hat{n}_p \). Note that in complex notation

\[
\hat{n}_p = \left( \frac{\partial p}{\partial \overline{\rho}}, \frac{\partial p}{\partial \rho} \right) (p), \quad \text{which implies} \quad q = p - \frac{d_M(q)}{|\partial \rho|} \left( \frac{\partial p}{\partial \overline{\rho}}, \frac{\partial p}{\partial \rho} \right) (p).
\]
Let $f \in C^2(\overline{\Omega})$, $q \in \overline{\Omega} \cap U$ and $p = \pi(q)$. Then Taylor’s formula in complex notation says

$$f(q) = f(p) + \sum_{j=1}^{2} \left[ \frac{\partial f}{\partial z_j}(p)(q_j - p_j) + \frac{\partial f}{\partial \overline{z}_j}(p)(\overline{q_j} - \overline{p_j}) \right] + \mathcal{O}(|q - p|^2)$$

$$= f(p) - \frac{d_{\Omega}(q)}{|\partial \rho(p)|} \sum_{j=1}^{2} \left[ \frac{\partial \rho}{\partial z_j}(p) \frac{\partial f}{\partial z_j}(p) + \frac{\partial \rho}{\partial \overline{z}_j}(p) \frac{\partial f}{\partial \overline{z}_j}(p) \right] + \mathcal{O}(d_{\Omega}^2(q)).$$

Define the vector field $N(z) = \frac{1}{|\partial \rho(z)|} \sum_{j=1}^{2} \frac{\partial \rho}{\partial z_j}(z) \frac{\partial f}{\partial z_j}(p)$. Then

$$f(q) = f(p) - 2d_{\Omega}(q) [(\text{Re}N(f))(p) + \mathcal{O}(d_{\Omega}^2(q))].$$

(2.1)

### 2.2. Partial Taylor analysis of the complex Hessian of $\rho$.

After possibly shrinking the neighborhood $U$ of $\Omega$, the smooth vector fields

$$L = \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial z_i} - \frac{\partial \rho}{\partial \overline{z}_j} \frac{\partial}{\partial \overline{z}_i}$$

and

$$N = \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial z_i} + \frac{\partial \rho}{\partial \overline{z}_j} \frac{\partial}{\partial \overline{z}_i}$$

are defined on $\overline{\Omega} \cap U$, and it holds that

$$L(p) = \langle L, N \rangle = 0 \text{ and } |L| = |N| \text{ on } \overline{\Omega} \cap U.$$  

Before we get down to business, we need some more notation: for vector fields $X(z) = \sum_{i=1}^{2} X_i(z) \frac{\partial}{\partial z_i}$ and $Y(z) = \sum_{i=1}^{2} Y_i(z) \frac{\partial}{\partial z_i}$, we shall write

$$H_{\rho}(X, Y)(z) = \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) X_j(z) Y_k(z).$$

We denote by $\Omega_W$ the set of all points $q \in \Omega \cap U$ for which $p = \pi(q)$ is a weakly pseudoconvex boundary point.

Let $\epsilon > 0$ be fixed. For each fixed $q \in \Omega_W \cap U$ and $\xi \in \mathbb{C}^2$ there exist constants $a_{q,\xi}$ and $b_{q,\xi}$ such that $\xi = a_{q,\xi} L(q) + b_{q,\xi} N(q)$. Note that then $|\xi|^2 = |a_{q,\xi}|^2 + |b_{q,\xi}|^2$. For now, we only consider $q \in \Omega_W \cap U$, and for notational ease, we shall drop the subscripts $q, \xi$. We first note that

(2.2)  $$H_{\rho}(\xi, \xi)(q) = |a|^2 H_{\rho}(L, L)(q) + 2\text{Re}(a b \text{Re}H_{\rho}(L, N)(q)) + |b|^2 H_{\rho}(N, N)(q).$$

We apply (2.1) to $H_{\rho}(L, L)(q)$, i.e.,

$$H_{\rho}(L, L)(q) = H_{\rho}(L, L)(p) - 2d_{\Omega}(q) (\text{Re}N) (H_{\rho}(L, L))(p) + \mathcal{O}(d_{\Omega}^2(q)).$$

Since $H_{\rho}(L, L)$ is real valued and $H_{\rho}(L, L)(p) = 0$, it follows that

$$H_{\rho}(L, L)(q) = -2d_{\Omega}(q) \text{Re}(N H_{\rho}(L, L))(p) + \mathcal{O}(d_{\Omega}^2(q)).$$

Notice that $N H_{\rho}(L, L)(p)$ is real. The last equation combined with (2.2) gives us then

$$H_{\rho}(\xi, \xi)(q) \geq |a|^2 \left[ -2d_{\Omega}(q) \text{Re}(N H_{\rho}(L, L))(p) + \mathcal{O}(d_{\Omega}^2(q)) \right]
- 2|a||b||H_{\rho}(L, N)(q)| + |b|^2 H_{\rho}(N, N)(q).$$

The Cauchy-Schwarz inequality implies

$$H_{\rho}(\xi, \xi)(q) \geq |a|^2 \left[ -2d_{\Omega}(q) \text{Re}(N H_{\rho}(L, L))(p) - \rho^2(q) + \mathcal{O}(d_{\Omega}^2(q)) \right]
+ |b|^2 \left[ \frac{-1}{\rho^2(q)} |H_{\rho}(L, N)(q)|^2 + H_{\rho}(N, N)(q) \right].$$
Notice that, after possibly shrinking the neighborhood $U$ of $b\Omega$, we can assume that
\[-\rho^2(q) + \mathcal{O}(d^2\Omega(q)) \geq \frac{\varepsilon}{4}\rho(q)\]
for all $q \in \Omega_U \cap U$. Therefore,
\[H_\rho(\xi, \xi)(q) \geq |a|^2 \left[-2d\Omega(q) (NH_\rho(L, L))(p) + \frac{\varepsilon}{4}\rho(q)\right] + |b|^2 \left[|c| - H_\rho(N, N)(q)\right],\]
for all $q \in \Omega_U \cap U$. Because of the plurisubharmonicity of $\rho$ on $\Omega_U \cap b\Omega$, it follows that
\[|H_\rho(L, N)(q)| \leq (H_\rho(L, L))^{\frac{1}{2}} (H_\rho(N, N))^{\frac{1}{2}}\]
holds on $\Omega_U \cap b\Omega$. Since $q \in \Omega_U \cap U$, i.e., since $\pi(q) = p$ is a weakly pseudoconvex boundary point, we get that $H_\rho(L, N)(p) = 0$. Therefore, there exists a constant $c_1 > 0$, depending on $\rho$, such that
\[|H_\rho(L, N)(q)|^2 \leq c_1 |\rho(q)|^2 \text{ for all } q \in \Omega_U \cap U.\]
This gives us the following lower bound on $H_\rho(\xi, \xi)(q)$:
\[H_\rho(\xi, \xi)(q) \geq |a|^2 \left[-2d\Omega(q) (NH_\rho(L, L))(p) + \frac{\varepsilon}{4}\rho(q)\right] - |b|^2 [c_1 - H_\rho(N, N)(q)],\]
which implies that for some constant $c_2 > 0$ depending on $\rho$
\[H_\rho(\xi, \xi)(q) \geq |a|^2 \left[-2d\Omega(q) (NH_\rho(L, L))(p) + \frac{\varepsilon}{4}\rho(q)\right] - c_2 |b|^2\]
holds for $q \in \Omega_U \cap U$.

Note that (2.3) is a more detailed version of (1.3) for those points $q \in \Omega$ near $b\Omega$ whose projections $\pi(q)$ are weakly pseudoconvex boundary points. Moreover, inequality (1.3) is within range, if $NH_\rho(L, L)$ is non-positive at all weakly pseudoconvex boundary points. The term $NH_\rho(L, L)$ being positive at some weakly pseudoconvex boundary point $p_0$ means that the function $H_\rho(L, L)$ decreases when one moves from $p_0$ inside the domain along the line normal to $b\Omega$ at $p_0$. This, of course, means that $H_\rho(L, L)$ becomes negative there, which destroys any hope of $\rho$ being plurisubharmonic in some neighborhood of $p_0$. Clearly, $NH_\rho(L, L)(p_0) > 0$ obstructs inequality (1.3) to hold for all $\varepsilon > 0$.

2.3. Example & idea of modification of $\rho$. We shall first give an example of a domain where $NH_\rho(L, L)$ is positive at a weakly pseudoconvex boundary point. Consider the domain $D = \{(z, w) \in \mathbb{C}^2 \mid \rho(z, w) < 0\}$ near the origin, where
\[\rho(z, w) = \text{Re}(w) + |w|^2 + \text{Re}(w)|z|^2 + |z|^2|w|^2 + |z|^4 + |z|^6.\]
One can easily show that $\rho$ is plurisubharmonic on $bD$ near the origin. In fact, $\rho$ is strictly plurisubharmonic on $bD$ near the origin except when $z = 0$. Let $\xi = (\xi_1, \xi_2)$ and $q = (0, w)$ be a point in $D$ which lies on the line normal to $bD$ through the origin. Then
\[H_\rho(\xi, \xi)(q) = (\text{Re}(w) + |w|^2)|\xi_1|^2 + |\xi_2|^2.\]

Thus $\rho$ can not be plurisubharmonic in any neighborhood of the origin. Note that this is caused by the term $\text{Re}(w)|z|^2$ contained in the definition of $\rho$. However, this is our old enemy, that is
\[(NH_\rho(L, L))(0) = \frac{\partial}{\partial w} \left(\frac{\partial^2 \rho}{\partial z \partial \bar{z}}\right)(0) = \frac{1}{2} > 0!\]
Now the question is, whether we can manipulate $\rho$ such that the obstruction vanishes. Notice that the answer to that in the above example is yes: let $r(z, w) = \rho(z, w)/(1 + |z|^2)$. Then

$$r(z, w) = \text{Re}(w) + |w|^2 + |z|^4,$$

which is plurisubharmonic everywhere.

Recall, that we actually want to show an estimate like

$$-2d_{d\Omega}(Nh(L, L))(p) \geq \epsilon \rho(q).$$

Obviously, just multiplying $\rho$ by a small positive number is not going to remove the obstruction. So, we consider another defining function $\rho \cdot h$ of $\Omega$, where $h$ is some smooth, positive function. We shall now list a few characteristics of $h$ which should give us some control on the obstruction term:

1. In order to use the basic estimate (2.3) for $\rho \cdot h$, we would need that $\rho \cdot h$ is still plurisubharmonic at weakly pseudoconvex boundary points. This can be achieved, if we choose $h$ such that all its first order derivatives vanish at all weakly pseudoconvex boundary points.

2. We need to consider those third order derivatives of $\rho \cdot h$, which are forced upon us by the obstruction term, at weakly pseudoconvex points. If we assume that all first order derivatives of $h$ vanish at weakly pseudoconvex points (and if we ignore, at least temporarily, that in the obstruction term $N$ does not only act on the Levi form of $\rho \cdot h$ but also on $L$ and $L$), then there are only two terms to be considered:
   
   a) There is the product of the original obstruction term of $\rho$ and $h$, which tells us that $h$ itself should not be large at the weakly pseudoconvex points.
   
   b) There are the terms which involve one derivative of $\rho$ and two derivatives of $h$. Since we are on $b\Omega$ the only such term which can appear is $N$ acting on $\rho$ multiplied with the Levi form of $h$. This seems to say that we need the Levi form of $h$ to be negative definite at the weakly pseudoconvex points. One can show that $NH_{\rho}(L, L)$ equals $LH_{\rho}(N, L)$ at weakly pseudoconvex points (see [3.3] and the following lemma). Thus the obstruction term itself gives us a function, $-|H_{\rho}(N, L)|^2$, whose Levi form is strictly negative definite at those points where it is needed.

Clearly, we can not choose $h$ to be $-|H_{\rho}(N, L)|^2$, since the latter function vanishes at weakly pseudoconvex points, and hence $\rho \cdot h$ would not be a defining function of $\Omega$. However, taking (1) and (2) into account $e^{-|H_{\rho}(N, L)|^2}$ seems like a suitable candidate for $h$.

3. Proof of Theorem 1.2

Let $C > 0$ be a large constant, which will be chosen later. We will consider the smooth defining function

$$r_C = r = \rho e^{-C\sigma}, \quad \text{where} \quad \sigma = |H_{\rho}(N, L)|^2,$$

and we shall work with the vector fields

$$L^r = \frac{\partial r}{\partial z_1} \frac{\partial}{\partial z_2} - \frac{\partial r}{\partial z_2} \frac{\partial}{\partial z_1} \quad \text{and} \quad N^r = \frac{\partial r}{\partial z_1} \frac{\partial}{\partial z_1} + \frac{\partial r}{\partial z_2} \frac{\partial}{\partial z_2},$$

which are defined on $\overline{\Omega} \cap U$ (after possibly shrinking $U$). As before, we note that $L^r(r) = \langle L^r, N^r \rangle = 0$ and $|L^r| = |N^r| = 1$. Moreover, on $b\Omega$ we have $L^r = L$ and $N^r = N$. 


As before, we suppose that \( q \in \Omega_W \cap U \). Here, the decomposition of a vector \( \xi \in \mathbb{C}^2 \) with respect to the vector fields \( L^r \) and \( N^r \) at \( q \) is different than before. Clearly, we can write \( \xi = a_q \xi L^r(q) + b_q \xi N^r(q) \) again; however, the constants \( a_q \xi \) and \( b_q \xi \) are different from before. Again, for notational convenience, we shall drop those subscripts \( q, \xi \).

Let us first see whether the basic estimate (2.3) holds for \( r \). The only special property of \( \rho \), which we used to derive (2.3), is that \( H_p(L, N)(p) = 0 \), where \( p \) is a weakly pseudoconvex boundary point. Thus, to see whether (2.3) holds for \( r \) we shall compute \( H_r(L^r, N^r)(p) \). A straightforward computation gives

\[
\frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} = e^{-c_2} \left[ -C \frac{\partial \sigma}{\partial \overline{z}_k} \left( \frac{\partial \rho}{\partial z_j} - C \frac{\partial \sigma}{\partial z_j} \right) + \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \right].
\]

Since \( H_p(L, N)(p) = 0 \), it follows that not only \( \sigma \) but also any derivative of \( \sigma \) at \( p \) vanishes, and thus we obtain

\[
\frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p) = \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p).
\]

In particular, \( r \) is plurisubharmonic at \( p \) and \( H_r(L^r, N^r)(p) = 0 \). Thus (2.3) holds for \( r \). That is: there exists a constant \( c_2 > 0 \) (depending on \( r \)) such that

\[
(3.1) \quad H_r(\xi, \xi)(q) \geq |q|^2 \left[ -2d_{\Omega}(q) (N^r H_r(L^r, L^r))(p) + \frac{\epsilon}{4} r(q) \right] - c_2 |b|^2
\]

holds for all \( q \in \Omega_W \cap U \) after possibly shrinking \( U \).

To see whether we truly gain anything by using \( r \) instead of \( \rho \), we have to figure out how \( (N^r H_r(L^r, L^r))(p) \) is related to \( (N H_p(L, L))(p) \). We shall prove the following

\[
(3.2) \quad \text{Claim: } N^r H_r(L^r, L^r)(p) \leq \left[ N H_p(L, L) - C |\partial \rho| \cdot (N H_p(L, L))^2 \right](p).
\]

Note that \( N^r = N \) on \( b\Omega \), which implies on \( b\Omega \)

\[
N^r H_r(L^r, L^r) = N H_r(L^r, L^r) = \sum_{\ell=1}^2 N_{\ell} \frac{\partial}{\partial z_{\ell}} \left( \sum_{j,k=1}^2 \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} L_j^{r} \overline{L}_k^{r} \right).
\]

Since \( L^r \) is a weak complex tangential direction at \( p \) and \( r \) is plurisubharmonic at \( p \), we have

\[
\sum_{j,k=1}^2 \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} \left( \sum_{\ell=1}^2 N_{\ell} \frac{\partial L^{r}_{\ell}}{\partial z_{\ell}} \right) \overline{L}_k^{r}(p) = 0 = \sum_{j,k=1}^2 \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} L_j^{r} \left( \sum_{\ell=1}^2 N_{\ell} \frac{\partial \overline{L}_k^{r}}{\partial z_{\ell}} \right)(p).
\]

Moreover, we have \( L^r(p) = L(p) \), which gives us that

\[
(N^r H_r(L^r, L^r))(p) = \left( \sum_{j,k,\ell=1}^2 \frac{\partial^3 r}{\partial z_j \partial \overline{z}_k \partial z_{\ell}} L_j^{r} \overline{L}_k^{r} \right)(p).
\]
Let us now compute those third derivatives of $r$:  
\[
\frac{\partial^3 r}{\partial z_j \partial \overline{z}_k \partial \overline{z}_l} = e^{-C_\rho} \left[ -C \frac{\partial \sigma}{\partial z_l} \left( -C \frac{\partial \sigma}{\partial z_j} - C \frac{\partial \sigma}{\partial z_j} \right) + \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} - C \frac{\partial \rho}{\partial \overline{z}_k} \frac{\partial \sigma}{\partial z_j} - C \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \frac{\partial \sigma}{\partial z_j} \right] 
- C \frac{\partial^3 \rho}{\partial z_j \partial \overline{z}_k \partial \overline{z}_l} - C \left( \frac{\partial^2 \rho}{\partial \overline{z}_k \partial \overline{z}_l} \frac{\partial \sigma}{\partial z_j} + \frac{\partial \rho}{\partial \overline{z}_k} \frac{\partial^2 \sigma}{\partial z_j \partial \overline{z}_l} \frac{\partial \rho}{\partial \overline{z}_k} \frac{\partial \sigma}{\partial z_j} + \frac{\partial^3 \sigma}{\partial z_j \partial \overline{z}_k \partial \overline{z}_l} \right).
\]

First note that $\rho$ as well as $\sigma$ and all its first order derivatives vanish at $p$. Also, since $L$ is complex tangential to $b\Omega$ all the terms involving $\frac{\partial \rho}{\partial z_j}$ or $\frac{\partial \rho}{\partial \overline{z}_k}$ vanish. Thus we get

\[
(N^* H_r(L^*, L^*)) (p) = (NH_\rho(L, L) - C(\partial \rho, N) H_\sigma(L, L)) (p).
\]

Since $\langle \partial \rho, N \rangle (p) = |\partial \rho(p)|$, it follows that

\[
(N^* H_r(L^*, L^*)) (p) = (NH_\rho(L, L) - C|\partial \rho| H_\sigma(L, L)) (p).
\]

Recall that $\sigma = |H_\rho(N, L)|^2$. Using that $H_\rho(N, L)(p) = 0$, a direct computation gives us

\[
H_\rho(L, L)(p) = |(\partial H_\rho(N, L), L)(p)|^2 + \left| \overline{\partial H_\rho(N, L), T}(p) \right|^2 
\geq \left| (\partial H_\rho(N, L), L)(p) \right|^2.
\]

We compute further

\[
\langle \partial H_\rho(N, L), L \rangle = \sum_{j=1}^2 L_j \frac{\partial}{\partial z_j} \left( \sum_{k=1}^2 \frac{\partial^2 \rho}{\partial z_k \overline{z}_k} N_k L_k \right) 
= \sum_{j,k=1}^2 \frac{\partial^3 \rho}{\partial z_j \partial \overline{z}_k \partial \overline{z}_l} L_j T_k N_l + \sum_{k=1}^2 \frac{\partial^2 \rho}{\partial z_k \partial \overline{z}_k} \left( \sum_{j=1}^2 L_j \frac{\partial}{\partial z_j} (T_k N_k) \right).
\]

Since $L$ is a weak complex tangential direction at $p$ and $\rho$ is plurisubharmonic at $p$, it follows that

\[
(3.3) \quad \langle \partial H_\rho(N, L), L \rangle (p) = NH_\rho(L, L)(p) + \sum_{k=1}^2 \frac{\partial^2 \rho}{\partial z_k \partial \overline{z}_k} N_k \left( \sum_{j=1}^2 L_j \frac{\partial}{\partial z_j} (T_k) \right) (p).
\]

We claim that the last term on the right hand side vanishes:

**Lemma 3.4.** Suppose $X$ is a smooth vectorfield, which is complex tangential to $b\Omega$. Furthermore, suppose that $b\Omega$ is weakly pseudoconvex at some boundary point $p$. Define $Y = \sum_{j=1}^2 X_j \frac{\partial}{\partial z_j}$. Then $Y$ is weak complex tangential to $b\Omega$ at $p$.

**Proof.** Since $X$ is tangential, $X(\rho) = 0$ holds on $b\Omega$. Moreover, we have $X(X(\rho)) = 0$ on $b\Omega$. Therefore

\[
0 = \overline{X}(X(\rho))(p) = \sum_{j,k=1}^2 \overline{X}_j \frac{\partial}{\partial z_j} \left( X_k \frac{\partial}{\partial z_k} \right) (p) 
= \sum_{j,k=1}^2 \overline{X}_j \frac{\partial X_k}{\partial z_j \partial z_k} (p) + \sum_{j,k=1}^2 \frac{\partial^2 \rho}{\partial z_k \partial z_j} X_k \overline{X}_j (p) = Y(p)(p),
\]
where the last step holds since $H_p(X, X)(p) = 0$ by our hypothesis. Thus, $Y$ is complex tangential direction at $p$. In particular, $H_p(Y, Y)(p) = 0$. \hfill \Box

If we set $X = L$, Lemma 3.3 implies that the last term in (3.5) vanishes. Thus, we obtain

$$H_p(L, L)(p) \geq |\partial H_p(N, L), L)(p)|^2 = |NH_p(L, L)(p)|^2,$$

which proves the Claim (3.2). That is

$$N^* H_r(L^*, L^*)(p) \leq [NH_p(L, L) - C|\partial p| \cdot (NH_p(L, L))]^2(p).$$

Hence, the lower estimate (3.1) on the complex Hessian of $r$ now becomes

$$H_r(\xi, \xi)(q) \geq |a|^2 [2d_{\Omega}(q) \left\{ C_3 (NH_p(L, L))^2 - NH_p(L, L) \right\} (p) + \frac{c_2}{4} r(q)]$$

(3.5)

for $q \in \Omega \cap U_b$, where $c_3 > 0$ is chosen such that $|\partial p| \geq c_3$ on $b\Omega$.

We are now set to show that there exist a $C > 0$ and a neighborhood $U_C$ of $b\Omega$ such that

$$2d_{\Omega}(q) \left[ C_3 (NH_p(L, L))^2 - NH_p(L, L) \right](p) \geq C \frac{e}{4} r(q)$$

(3.6)

holds for $q \in \Omega \cap U_C$, which would imply that (3.3) holds for these points.

To make our life easier, let us write $A_p$ for $NH_p(L, L)(p)$, i.e., (3.6) becomes

$$2d_{\Omega}(q) \left[ C_3 A_p^2 - A_p \right] \geq C \frac{e}{4} r(q).$$

If $C_3 A_p^2 - A_p \geq 0$, then (3.6) holds trivially. Moreover, increasing $C$ does not destroy this non-negativity. Suppose that $C_3 A_p^2 - A_p < 0$. First notice that there exists a constant $c_4 > 0$ such that $d_{\Omega}(q) \leq c_4 |\rho(q)|$ for all $q \in \Omega \cap U$. Since $\rho = r e^{C \sigma}$, it follows that $d_{\Omega}(q) \leq c_4 e^{C \sigma(q)} |r(q)|$. Thus, to prove (3.6) it is sufficient to show

$$2c_4 e^{C \sigma(q)} |r(q)| \left[ C_3 A_p^2 - A_p \right] \geq \frac{c_2}{4} r(q),$$

which is equivalent to

$$e^{C \sigma(q)} \left[ C_3 A_p^2 - A_p \right] \geq -\frac{c_2}{8c_4}.$$

Let $U_C \subset U$ be a neighborhood of $b\Omega$ such that $z \in \Omega \cap U_C$ implies that $e^{C \sigma(z)} \leq 2e^{C \sigma(\pi(z))}$. Notice that $U_C$ is a true neighborhood of $b\Omega$, since $\sigma$ is smooth near $b\Omega$.

Moreover, in the situation which we are considering, i.e., where $\pi(q)$ is a weakly pseudo-convex boundary point, we then have that $q \in \Omega \cap U_C$ implies $e^{C \sigma(q)} \leq 2$. Therefore, to obtain (3.6) it is sufficient that

$$C_3 A_p^2 - A_p \geq -\frac{c_2}{16c_4}$$

holds on $\Omega \cap U_C$. We remark that neither $c_3, c_4$ nor $A_p$ depend on the choice of $C$. Thus, choosing

$$C = \max \left\{ 0, \max_{p \in b\Omega \text{ weak}} \frac{c_2}{16c_4} \right\}$$

proves (3.6) on $\Omega \cap U_C$, which implies that

$$H_r(\xi, \xi)(q) \geq \frac{c_2}{2} r(q)|\xi|^2 - c_2 (\partial r(q), \xi)^2$$

(3.7)

holds on $\Omega \cap U_C$.\hfill \Box
Let us show now that an estimate similar to (3.8) holds near \( \Omega_W \cap U_C \). Note first that our computations leading up to (3.7) imply that \( N^r H_r(L', L') \leq \frac{c}{r} \) holds on the set of the weakly pseudoconvex boundary points of \( \Omega \). Hence by continuity, there exists a neighborhood \( W \) of the set of weakly pseudoconvex boundary points such that \( N^r H_r(L', L') \leq \frac{c}{r} \) on \( W \cap b\Omega \). We may assume that \( W \subset U_C \) and that \( q \in W \cap \Omega \) implies \( \pi(q) \in W \cap b\Omega \). Using Taylor’s formula, it follows for \( q \in W \cap \Omega \) that
\[
H_r(L', L')(q) \geq H_r(L', L')(\pi(q)) + \frac{c}{4} \rho(q) + O(\rho^2(q)) \\
\geq H_r(L', L')(\pi(q)) + \frac{c}{2} \rho(q)
\]
after possibly shrinking of \( W \). Another application of Taylor’s formula gives us for \( q \in W \cap \Omega \)
\[
H_r(\xi, \xi)(q) \geq |a|^2 \left[ H_r(L', L')(\pi(q)) + \frac{c}{2} \rho(q) \right] + |b|^2 H_r(N^r, N^r) + 2|a||b||H_r(L', N^r)(\pi(q))| + O(\rho(q)) \\
\geq |a|^2 [H_r(L', L')(\pi(q)) + \varepsilon \rho(q)] - c_5 |\partial r(q, \xi)|^2 \\
+ 2|a||b||H_r(L', N^r)(\pi(q))|
\]
where the last step follows by the Cauchy-Schwarz inequality for some constant \( c_5 > 0 \) sufficiently large. Since \( H_r(L', L')(\pi(q)) \) is positive, we only need to estimate the term \( |H_r(L', N^r)(\pi(q))| \). Note first that \( r \) is not plurisubharmonic on \( b\Omega \) at strictly pseudoconvex boundary points, though \( \rho \) is. However, since any derivative of \( \sigma \) is \( O(H_r(N^r, L')) \) on \( b\Omega \) and since \( \rho \) is plurisubharmonic on \( b\Omega \), it follows that there exists a constant \( c_6 > 0 \) such that
\[
|H_r(L', N^r)|^2 \leq c_6 H_r(L', L') [H_r(N^r, N^r) + c_6] \quad \text{on} \quad b\Omega.
\]
The Cauchy-Schwarz inequality implies now, that for some constant \( c_7 > 0 \) we have
\[
(3.8) \quad H_r(\xi, \xi)(q) \geq c r(q)|\xi|^2 - c_7 |\langle \partial r(q, \xi) \rangle|^2
\]
for \( q \in W \cap \Omega \). We define \( r_1 = r + Kr^2 \) for some \( K > 2c_7 \). Note that
\[
H_{r_1}(\xi, \xi)(q) = (1 + 2Kr)H_r(\xi, \xi)(q) + 2K |\langle \partial r, \xi \rangle|^2.
\]
Let \( U_K = \{ z \in W \mid 1 + 2Kr(z) \geq \frac{1}{2} \} \), then (3.8) implies for \( q \in \Omega \cap U_K \)
\[
H_{r_1}(\xi, \xi)(q) \geq \frac{1}{2} \varepsilon r(q)|\xi|^2 + (2K - c_7) |\langle \partial r(q, \xi) \rangle|^2 \\
\geq \varepsilon r_1(q)|\xi|^2 + K |\langle \partial r_1(q, \xi) \rangle|^2.
\]
That is, we have shown that (1.3) holds on \( \Omega \cap U_K \).

Next we show that (3.3) also holds near the remaining strictly pseudoconvex boundary points. We note that \( S = b\Omega \setminus (W \cap b\Omega) \) is a closed subset of the set of the strictly pseudoconvex boundary points. This implies, as long as \( K > 0 \) is chosen sufficiently large, that there exists a neighborhood \( U_S \) of \( S \) such that \( r_1 \) is strictly plurisubharmonic on \( \Omega \cap U_S \). In particular, there exists a neighborhood \( V \) of \( b\Omega \) such that
\[
H_{r_1}(\xi, \xi)(q) \geq c r_1(q)|\xi|^2 + K |\langle \partial r_1(q, \xi) \rangle|^2
\]
for all \( q \in \Omega \cap V \) and \( \xi \in \mathbb{C}^2 \). This proves (1.3).

The proof of (1.3) is very similar to the above proof of (1.3). In fact, we only need to change a few signs to derive (1.3). First, one realizes that the basic estimate (2.6)
becomes: there exist a neighborhood $U$ of $b\Omega$ and a constant $c_2 > 0$ such that
\[
H_\rho(\xi, \xi)(q) \geq |a|^2 \left[2d_\Omega(q)(NH_\rho(L, L)(p) - \frac{\xi}{4}\rho(q)) - c_2|h|^2\right]
\]
for $q \in \Omega^C \cap U$. Here, as before, the points $q$ in consideration are such that their orthogonal projections $\pi(q) = \rho$ onto $b\Omega$ are weakly pseudoconvex boundary points. As one would expect, we have an obstruction to plurisubharmonicity of $\rho$ outside of $\overline{\Omega}$ at those weakly pseudoconvex boundary points where $NH_\rho(L, L)$ decreases along the outward normal. Thus, we should multiply $\rho$ by a smooth, positive function which is strictly plurisubharmonic at those boundary points where $NH$ is negative, i.e., we work with the function $r = \rho e^{c|H_\rho(N, L)|^2}$ for $C > 0$. Using arguments analog to the ones in the proof of (1.3), one can then show that for any $\epsilon > 0$ and $K > 0$, there exist a neighborhood $V$ of $b\Omega$ and a constant $C > 0$ such that the complex Hessian of $r_2 = r + Kr^2$ satisfies (1.4) on $\Omega^C \cap V$.

4. PROOF OF COROLLARY 1.5

In the following section, we give the proof of Corollary 1.5. We start out with part (i) by showing first that for any $\eta \in (0, 1)$ there exist a $\delta > 0$, a smooth defining function $r$ of $\Omega$ and a neighborhood $W$ of $b\Omega$ such that $g_1 = -(r^2 - \delta|\eta|^2)$ is strictly plurisubharmonic on $\Omega \cap W$.

Let $\eta \in (0, 1)$ be fixed, and $r$ a smooth defining function of $\Omega$. For notational ease we write $\phi = \delta|\eta|^2$ for $\delta > 0$. Here, $r$ and $\delta$ are fixed and to be chosen later. Let us compute the complex Hessian of $g_1$ on $\Omega \cap W$:
\[
H_{g_1}(\xi, \xi) = \eta(-r)^{-2}e^{-\phi}\eta[(1 - \eta)|\partial r, \xi]|^2 - rH_r(\xi, \xi) + 2\eta Re\left(\langle \partial r, \xi\rangle\overline{\partial \phi(\xi)}\right) - r^2\eta|\langle \partial \phi, \xi\rangle|^2 + r^2H_{\phi}(\xi, \xi).
\]
An application of the Cauchy-Schwarz inequality gives
\[
2\eta Re\left(\langle \partial r, \xi\rangle\overline{\partial \phi(\xi)}\right) \geq -2(1 - \eta)|\langle \partial r, \xi\rangle|^2 - \frac{r^2\eta^2}{1 - \eta}|\langle \partial \phi, \xi\rangle|^2.
\]
Therefore, we obtain for the complex Hessian of $g$ on $\Omega$ the following:
\[
H_{g_1}(\xi, \xi) \geq \eta(-g_1)(-r)^{-1}\left[\left\{H_r(\xi, \xi) + (-r)\left(H_{\phi}(\xi, \xi) - \frac{\eta}{1 - \eta}|\langle \partial \phi, \xi\rangle|^2\right)\right\}\right].
\]
Notice that
\[
H_{\phi}(\xi, \xi) - \frac{\eta}{1 - \eta}|\langle \partial \phi, \xi\rangle|^2 = \delta\left(H_{|\xi|^2}(\xi, \xi) - \frac{\eta}{1 - \eta}|\langle \xi, \xi\rangle|^2\right)\geq \delta|\xi|^2\left(1 - \frac{\eta D}{1 - \eta}\right),
\]
where $D := \max_{z \in \overline{\Omega}}|z|^2$. Now set $\delta = \frac{1 - \eta}{2\eta D}$; it is noteworthy that $\delta$ goes to 0 as $\eta$ approaches $1^-$. We now have
\[
H_{\phi}(\xi, \xi) - \frac{\eta}{1 - \eta}|\langle \partial \phi, \xi\rangle|^2 \geq \frac{\delta}{2}|\xi|^2,
\]
which implies that
\[
H_{g_1}(\xi, \xi) \geq \eta(-g_1)(-r)^{-1}\left[H_r(\xi, \xi) + \frac{\delta}{2}(-r)|\xi|^2\right]
\]
holds on $\Omega$. 

\[
(4.1)
\]
By (1.3) there exist a neighborhood $W$ of $b\Omega$ and a smooth defining function $r_1$ of $\Omega$ such that

$$H_{r_1}(\xi, \xi)(q) \geq \frac{\delta}{4} r_1(q)|\xi|^2$$

for all $q \in \Omega \cap W$. Setting $r = r_1$ and using (4.1), we obtain

$$H_{g_1}(\xi, \xi)(q) \geq \eta (-g_1(q)) \cdot \frac{\delta}{4} |\xi|^2 \quad \text{for } q \in \Omega \cap W.$$  

It follows by standard arguments that there exists a defining function $\tilde{r}_1$ such that $-(\tilde{r}_1)^n$ is strictly plurisubharmonic on $\Omega$; for details see pg. 133 in [Die-For77a]. This proves part (i) of Corollary 1.5.

The proof of part (ii) is similar to the proof of part (i). Let $\eta > 1$ be fixed. We would like to show that there exists a neighborhood $V$ of $b\Omega$ such that $g_2 = (r e^{\delta |z|^2})^n$ is strictly plurisubharmonic on $\overline{\Omega^C} \cap V$ for some smooth defining function $r$ and some constant $\delta > 0$. Let $W$ be a neighborhood of $b\Omega$. Choose $\delta = \frac{\eta - 1}{2D}$, where $D = \max_{z \in W} |z|^2$. Then calculations similar to the ones in the proof of part (i) yield

$$H_{g_2}(\xi, \xi) \geq \eta g_2 r^{-1} \left[ H_r(\xi, \xi) + \frac{\delta}{2} |\xi|^2 \right] \quad \text{on } \overline{\Omega^C} \cap W.$$  

By (1.4) there exist a neighborhood $V$ of $b\Omega$ and a smooth defining function $r_2$ of $\Omega$ such that

$$H_{r_2}(\xi, \xi)(q) \geq -\frac{\delta}{4} r_2(q)|\xi|^2$$

for all $q \in \overline{\Omega^C} \cap V$. Since we may assume that $V \subset W$, it follows that

$$H_{g_2}(\xi, \xi)(q) \geq \eta g_2(q) \cdot \frac{\delta}{4} |\xi|^2 \quad \text{for } q \in \overline{\Omega^C} \cap V,$$

which proves (1.4).

References

[Boa-Str91] H. P. Boas, E. J. Straube, Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary, Math. Z. 206 (1991), 225-235.

[Die-For77a] K. Diederich, J. E. Fornæss, Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions, Inv. Math. 39 (1977), 129–141.

[Die-For77b] K. Diederich, J. E. Fornæss, Pseudoconvex domains: An example with nontrivial nebhülle, Math. Ann. 225 (1977), 275–292.

[Her-McN06] A.-K. Herbig, J. D. McNeal, Regularity of the Bergman projection on forms and plurisubharmonicity conditions, to appear in Math. Ann.

[Ran81] M. Range, A remark on bounded strictly plurisubharmonic exhaustion functions, Proc. AMS 81 (1981), 220–222.

[Sah06] S. Şahutoğlu, Compactness of the $\overline{\partial}$-Neumann problem and Stein neighborhood bases, Ph.D. dissertation, Texas A&M University, College Station, TX 2006.

Department of Mathematics,
University of Michigan, Ann Arbor, Michigan 48109
E-mail address: fornaess@umich.edu

Department of Mathematics,
University of Michigan, Ann Arbor, Michigan 48109
E-mail address: herbig@umich.edu