Variation of parabolic cohomology and Poincaré duality

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Abstract
We continue our study of the variation of parabolic cohomology ([5]) and derive an exact formula for the underlying Poincaré duality. As an illustration of our methods, we compute the monodromy of the Picard-Euler system and its invariant Hermitian form, reproving a classical theorem of Picard.

Introduction

Let $x_1, \ldots, x_r$ be pairwise distinct points on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and set $U := \mathbb{P}^1(\mathbb{C}) - \{x_1, \ldots, x_r\}$. The Riemann–Hilbert correspondence [1] is an equivalence between the category of ordinary differential equations with polynomial coefficients and at most regular singularities at the points $x_i$ and the category of local systems of $\mathbb{C}$-vectorspaces on $U$. The latter are essentially given by an $r$-tuple of matrices $g_1, \ldots, g_r \in \text{GL}_n(\mathbb{C})$ satisfying the relation $\prod_i g_i = 1$. The Riemann–Hilbert correspondence associates to a differential equation the tuple $(g_i)$, where $g_i$ is the monodromy of a full set of solutions at the singular point $x_i$.

In [5] the authors investigated the following situation. Suppose that the set of points $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1(\mathbb{C})$ and a local system $\mathcal{V}$ with singularities at the $x_i$ depend on a parameter $s$ which varies over the points of a complex manifold $S$. More precisely, we consider a relative divisor $D \subset \mathbb{P}^1_S$ of degree $r$ such that for all $s \in S$ the fibre $D_s \subset \mathbb{P}^1(\mathbb{C})$ consists of $r$ distinct points. Let $U := \mathbb{P}^1_S - D$ denote the complement and let $\mathcal{V}$ be a local system on $U$. We call $\mathcal{V}$ a variation of local systems over the base space $S$. The parabolic cohomology of the variation $\mathcal{V}$ is the local system on $\mathcal{W} := R^1\pi_*(j_*\mathcal{V})$, where $j : U \hookrightarrow \mathbb{P}^1_S$ denotes the natural injection and $\pi : \mathbb{P}^1_S \to S$ the natural projection. The fibre of $\mathcal{W}$ at a point $s_0 \in S$ is the parabolic cohomology of the local system $\mathcal{V}_{s_0}$, the restriction of $\mathcal{V}$ to the fibre $U_{s_0} = U \cap \pi^{-1}(s_0)$. 
A special case of this construction is the middle convolution functor defined by Katz [8]. Here \( S = U_0 \) and so this functor transforms one local system \( \mathcal{V}_0 \) on \( S \) into another one, \( \mathcal{W} \). Katz shows that all rigid local systems on \( S \) arise from one-dimensional systems by successive application of middle convolution. This was further investigated by Dettweiler and Reiter [4]. Another special case are the generalized hypergeometric systems studied by Lauricella [9], Terada [15] and Deligne–Mostow [2]. Here \( S \) is the set of ordered tuples of pairwise distinct points on \( \mathbb{P}^1(\mathbb{C}) \) of the form \( s = (0,1,\infty,x_4,\ldots,x_r) \) and \( \mathcal{V} \) is a one-dimensional system on \( \mathbb{P}^1_S \) with regular singularities at the (moving) points \( 0,1,\infty,x_4,\ldots,x_r \). In [5] we gave another example where \( S \) is a 17-punctured Riemann sphere and the local system \( \mathcal{V} \) has finite monodromy. The resulting local system \( \mathcal{W} \) on \( S \) does not have finite monodromy and is highly non-rigid. Still, by the comparison theorem between singular and étale cohomology, \( \mathcal{W} \) gives rise to \( \ell \)-adic Galois representations, with interesting applications to the regular inverse Galois problem.

In all these examples, it is a significant fact that the monodromy of the local system \( \mathcal{W} \) (i.e. the action of \( \pi_1(S) \) on a fibre of \( \mathcal{W} \)) can be computed explicitly, i.e. one can write down matrices \( g_1,\ldots,g_r \in \text{GL}_n \) which are the images of certain generators \( \alpha_1,\ldots,\alpha_r \) of \( \pi_1(S) \). In the case of the middle convolution this was discovered by Dettweiler–Reiter [3] and Völklein [16]. In [5] it is extended to the more general situation sketched above. In all earlier papers, the computation of the monodromy is either not explicit (like in [8]) or uses ad hoc methods. In contrast, the method presented in [5] is very general and can easily be implemented on a computer.

It is one matter to compute the monodromy of \( \mathcal{W} \) explicitly (i.e. to compute the matrices \( g_i \)) and another matter to determine its image (i.e. the group generated by the \( g_i \)). In many cases the image of monodromy is contained in a proper algebraic subgroup of \( \text{GL}_n \), because \( \mathcal{W} \) carries an invariant bilinear form induced from Poincaré duality. To compute the image of monodromy, it is often helpful to know this form explicitly. After a review of the relevant results of [5] in Section 1, we give a formula for the Poincaré duality pairing on \( \mathcal{W} \) in Section 2. Finally, in Section 3 we illustrate our method in a very classical example: the Picard–Euler system.

1 Variation of parabolic cohomology revisited

1.1 Let \( X \) be a compact Riemann surface of genus 0 and \( D \subset X \) a subset of cardinality \( r \geq 3 \). We set \( U := X - D \). There exists a homeomorphism \( \kappa : X \cong \mathbb{P}^1(\mathbb{C}) \) between \( X \) and the Riemann sphere which maps the set \( D \) to the real line \( \mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C}) \). Such a homeomorphism is called a marking of \((X,D)\).

Having chosen a marking \( \kappa \), we may assume that \( X = \mathbb{P}^1(\mathbb{C}) \) and \( D \subset \mathbb{P}^1(\mathbb{R}) \). Choose a base point \( x_0 \in U \) lying in the upper half plane. Write \( D = \{x_1,\ldots,x_r\} \) with \( x_1 < x_2 < \ldots < x_r \leq \infty \). For \( i = 1,\ldots,r-1 \) we let \( \gamma_i \) denote the open interval \((x_i,x_{i+1}) \subset U \cap \mathbb{P}^1(\mathbb{R})\); for \( i = r \) we set \( \gamma_0 = \gamma_r := (x_r,x_1) \).
(which may include ∞). For \( i = 1, \ldots, r \), we let \( \alpha_i \in \pi_1(U) \) be the element represented by a closed loop based at \( x_0 \) which first intersects \( \gamma_{i-1} \) and then \( \gamma_i \). We obtain the following well known presentation

\[
\pi_1(U, x_0) = \langle \alpha_1, \ldots, \alpha_r \mid \prod \alpha_i = 1 \rangle,
\]

which only depends on the marking \( \kappa \).

Let \( R \) be a (commutative) ring. A local system of \( R \text{-}\)modules on \( U \) is a locally constant sheaf \( \mathcal{V} \) on \( U \) with values in the category of free \( R \text{-}\)modules of finite rank. Such a local system corresponds to a representation \( \rho : \pi_1(U, x_0) \to \text{GL}(V) \), where \( V := \mathcal{V}_{x_0} \) is the stalk of \( \mathcal{V} \) at \( x_0 \). For \( i = 1, \ldots, r \), set \( g_i := \rho(\alpha_i) \in \text{GL}(V) \). Then we have

\[
\prod_{i=1}^r g_i = 1,
\]

and \( \mathcal{V} \) can also be given by a tuple \( \mathbf{g} = (g_1, \ldots, g_r) \in \text{GL}(V)^r \) satisfying the above product-one-relation.

**Convention 1.1** Let \( \alpha, \beta \) be two elements of \( \pi_1(U, x_0) \), represented by closed path based at \( x_0 \). The composition \( \alpha \beta \) is (the homotopy class of) the closed path obtained by first walking along \( \alpha \) and then along \( \beta \). Moreover, we let \( \text{GL}(V) \) act on \( V \) from the right.

**1.2** Fix a local system of \( R \text{-}\)modules \( \mathcal{V} \) on \( U \) as above. Let \( j : U \hookrightarrow X \) denote the inclusion. The parabolic cohomology of \( \mathcal{V} \) is defined as the sheaf cohomology of \( j_* \mathcal{V} \), and is written as \( H^p_n(U, \mathcal{V}) := H^n(X, j_* \mathcal{V}) \). We have natural morphisms \( H^n(U, \mathcal{V}) \to H^n_p(U, \mathcal{V}) \) and \( H^n_p(U, \mathcal{V}) \to H^n(U, \mathcal{V}) \) (\( H \) denotes cohomology with compact support). Moreover, the group \( H^n(U, \mathcal{V}) \) is canonically isomorphic to the group cohomology \( H^n(\pi_1(U, x_0), \mathcal{V}) \) and \( H^1_p(U, \mathcal{V}) \) is the image of the cohomology with compact support in \( H^1(U, \mathcal{V}) \), see [5], Prop. 1.1. Thus, there is a natural inclusion

\[
H^1_p(U, \mathcal{V}) \hookrightarrow H^1(\pi_1(U, x_0), \mathcal{V})).
\]

Let \( \delta : \pi_1(U) \to V \) be a cocycle, i.e. we have \( \delta(\alpha \beta) = \delta(\alpha) \cdot \rho(\beta) + \delta(\beta) \) (see Convention 1.1). Set \( v_i := \delta(\alpha_i) \). It is clear that the tuple \( (v_i) \) is subject to the relation

\[
v_1 \cdot g_2 \cdots g_r + v_2 \cdot g_3 \cdots g_r + \ldots + v_r = 0.
\]

By definition, \( \delta \) gives rise to an element in \( H^1(\pi_1(U, x_0), V) \). We say that \( \delta \) is a parabolic cocycle if the class of \( \delta \) in \( H^1(\pi_1(U), V) \) lies in \( H^1_p(U, \mathcal{V}) \). By [5], Lemma 1.2, the cocycle \( \delta \) is parabolic if and only if \( v_i \) lies in the image of \( g_i - 1 \), for all \( i \). Thus, the association \( \delta \mapsto (\delta(\alpha_1), \ldots, \delta(\alpha_r)) \) yields an isomorphism

\[
H^1_p(U, \mathcal{V}) \cong W_{\mathbf{g}} := H_{\mathbf{g}}/E_{\mathbf{g}},
\]
where

\[ H^i := \{ (v_1, \ldots, v_r) \mid v_i \in \text{Im}(g_i - 1), \text{relation (2) holds} \} \]

and

\[ E^i := \{ (v \cdot (g_1 - 1), \ldots, v \cdot (g_r - 1)) \mid v \in V \}. \]

1.3 Let \( S \) be a connected complex manifold, and \( r \geq 3 \). An \( r \)-configuration over \( S \) consists of a smooth and proper morphism \( \bar{\pi} : X \to S \) of complex manifolds together with a smooth relative divisor \( D \subset X \) such that the following holds. For all \( s \in S \) the fiber \( X_s := \bar{\pi}^{-1}(s) \) is a compact Riemann surface of genus 0. Moreover, the natural map \( D \to S \) is an unramified covering of degree \( r \). Then for all \( s \in S \) the divisor \( D \cap X_s \) consists of \( r \) pairwise distinct points \( x_1, \ldots, x_r \in X_s \).

Let us fix an \( r \)-configuration \((X, D)\) over \( S \). We set \( U := X - D \) and denote by \( j : U \hookrightarrow X \) the natural inclusion. Also, we write \( \pi : U \to S \) for the natural projection. Choose a base point \( s_0 \in S \) and set \( X_0 := \pi^{-1}(s_0) \) and \( D_0 := X_0 \cap D \).

Set \( U_0 := X_0 - D_0 = \pi^{-1}(s_0) \) and choose a base point \( x_0 \in U_0 \). The projection \( \pi : U \to S \) is a topological fibration and yields a short exact sequence

\[ 1 \to \pi_1(U_0, x_0) \to \pi_1(U, x_0) \to \pi_1(S, s_0) \to 1. \]

Let \( \mathcal{V}_0 \) be a local system of \( R \)-modules on \( U_0 \). A variation of \( \mathcal{V}_0 \) over \( S \) is a local system \( \mathcal{V} \) of \( R \)-modules on \( U \) whose restriction to \( U_0 \) is identified with \( \mathcal{V}_0 \).

The parabolic cohomology of a variation \( \mathcal{V} \) is the higher direct image sheaf

\[ \mathcal{W} := R^1\pi_*(j_*\mathcal{V}). \]

By construction, \( \mathcal{W} \) is a local system with fibre

\[ \mathcal{W} := H^1_b(U_0, \mathcal{V}_0). \]

Thus \( \mathcal{W} \) corresponds to a representation \( \eta : \pi_1(S, s_0) \to \text{GL}(W) \). We call \( \rho \) the monodromy representation on the parabolic cohomology of \( \mathcal{V}_0 \) (with respect to the variation \( \mathcal{V} \)).

1.4 Under a mild assumption, the monodromy representation \( \eta \) has a very explicit description in terms of the Artin braid group. We first have to introduce some more notation. Define

\[ \mathcal{O}_{r-1} := \{ D' \subset \mathbb{C} \mid |D'| = r - 1 \} = \{ D \subset \mathbb{P}^1(\mathbb{C}) \mid |D| = r, \infty \in D \}. \]

The fundamental group \( A_{r-1} : = \pi_1(\mathcal{O}_{r-1}, D_0) \) is the Artin braid group on \( r - 1 \) strands. Let \( \beta_1, \ldots, \beta_{r-2} \) be the standard generators, see e.g. [5], §2.2. (The element \( \beta_i \) switches the position of the two points \( x_i \) and \( x_{i+1} \); the point \( x_i \) walks through the lower half plane and \( x_{i+1} \) through the upper half plane.) The generators \( \beta_i \) satisfy the following well known relations:

\[ \beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}, \quad \beta_i \beta_j = \beta_j \beta_i \quad (\text{for } |i - j| > 1). \]
Let $R$ be a commutative ring and $V$ a free $R$-module of finite rank. Set
\[
E_r(V) := \{ \mathbf{g} = (g_1, \ldots, g_r) \mid g_i \in \text{GL}(V), \prod_i g_i = 1 \}.
\]

We define a right action of the Artin braid group $A_{r-1}$ on the set $E_r(V)$ by the following formula:
\[
g^\beta := (g_1, \ldots, g_{i+1}, g_{i+1}^{-1}g_ig_{i+1}, \ldots, g_r).
\]

One easily checks that this definition is compatible with the relations (7). For $\mathbf{g} \in E_r(V)$, let $H_\mathbf{g}$ be as in (4). For all $\beta \in A_{r-1}$, we define an $R$-linear isomorphism
\[
\Phi(\mathbf{g}, \beta) : H_\mathbf{g} \rightarrow H_{\mathbf{g}^\beta},
\]
as follows. For the generators $\beta_i$, we set
\[
(\mathbf{v}_1, \ldots, \mathbf{v}_r)^{\Phi(\mathbf{g}, \beta_i)} := (\mathbf{v}_1, \ldots, \mathbf{v}_{i+1}, \mathbf{v}_{i+1}(1-g_{i+1}^{-1}g_ig_{i+1}) + \mathbf{v}_ig_{i+1}, \ldots, \mathbf{v}_r).
\]

For an arbitrary word $\beta$ in the generators $\beta_i$, we define $\Phi(\mathbf{g}, \beta)$ using (9) and the 'cocycle rule'
\[
\Phi(\mathbf{g}, \beta) \cdot \Phi(\mathbf{g}^\beta, \beta') = \Phi(\mathbf{g}, \beta\beta').
\]
(Our convention is to let linear maps act from the right; therefore, the left hand side of (9) is the linear map obtained from first applying $\Phi(\mathbf{g}, \beta)$ and then $\Phi(\mathbf{g}^\beta, \beta')$.) It is easy to see that $\Phi(\mathbf{g}, \beta)$ is well defined and respects the submodule $E_\mathbf{g} \subset H_\mathbf{g}$ defined by (5). Let
\[
\Phi(\mathbf{g}, \beta) : W_\mathbf{g} \rightarrow W_{\mathbf{g}^\beta}
\]
denote the induced map on the quotient $W_\mathbf{g} = H_\mathbf{g}/E_\mathbf{g}$.

Given $\mathbf{g} \in E_r(V)$ and $h \in \text{GL}(V)$, we define the isomorphism
\[
\Psi(\mathbf{g}, h) : \begin{cases} 
H_{\mathbf{g}^h} & \overset{\sim}{\longrightarrow} H_\mathbf{g} \\
(\mathbf{v}_1, \ldots, \mathbf{v}_r) & \mapsto (\mathbf{v}_1 \cdot h, \ldots, \mathbf{v}_r \cdot h),
\end{cases}
\]
where $\mathbf{g}^h := (h^{-1}g_1h, \ldots, h^{-1}g_rh)$. It is clear that $\Psi(\mathbf{g}, h)$ maps $E_{\mathbf{g}^h}$ to $E_\mathbf{g}$ and therefore induces an isomorphism $\tilde{\Psi}(\mathbf{g}, h) : W_{\mathbf{g}^h} \rightarrow W_\mathbf{g}$.

Note that the computation of the maps $\Phi(\mathbf{g}, \beta)$ and $\tilde{\Psi}(\mathbf{g}, h)$ can easily be implemented on a computer.
1.5 Let $S$ be a connected complex manifold, $s_0 \in S$ a base point and $(X, D)$ an $r$-configuration over $S$. As before we set $U := X - D$, $D_0 := D \cap X_{s_0}$ and $U_0 := U \cap X_{s_0}$. Let $\mathcal{V}_0$ be a local system of $R$-modules on $U_0$ and $\mathcal{V}$ a variation of $\mathcal{V}_0$ over $S$. Let $\mathcal{W}$ be the parabolic cohomology of the variation $\mathcal{V}$ and let $\eta : \pi_1(S, s_0) \to \text{GL}(W)$ be the corresponding monodromy representation. In order to describe $\eta$ explicitly, we find it convenient to make the following assumption on $(X, D)$:

**Assumption 1.2**

(i) $X = \mathbb{P}^1_S$ is the relative projective line over $S$.

(ii) The divisor $D$ contains the section $\infty \times S \subset \mathbb{P}^1_S$.

(iii) There exists a point $s_0 \in S$ such that $D_0 := D \cap \pi^{-1}(s_0)$ is contained in the real line $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C}) = \pi^{-1}(s_0)$.

In practise, this assumption is not a big restriction. See [5] for a more general setup.

By Assumption 1.2, we can consider $D_0$ as an element of $\mathcal{O}_{r-1}$. Moreover, the divisor $D \subset \mathbb{P}^1_S$ gives rise to an analytic map $S \to \mathcal{O}_{r-1}$ which sends $s_0 \in S$ to $D_0 \in \mathcal{O}_{r-1}$. We let $\varphi : \pi_1(S, s_0) \to A_{r-1}$ denote the induced group homomorphism and call it the braiding map induced by $(X, D)$.

For $t \in \mathbb{R}^+$ let $\Omega_t := \{z \in \mathbb{C} \mid |z| > t, z \not\in (-\infty, 0)\}$. Since $\Omega_t$ is contractible, the fundamental group $\pi_1(U_0, \Omega_t)$ is well defined for $t \gg 0$ and independent of $t$, up to canonical isomorphism. We write $\pi_1(U_0, \infty) := \pi_1(U_0, \Omega_t)$. We can define $\pi_1(U, \infty)$ in a similar fashion, and obtain a short exact sequence

\begin{equation}
1 \to \pi_1(U_0, \infty) \longrightarrow \pi_1(U, \infty) \longrightarrow \pi_1(S, s_0) \to 1.
\end{equation}

It is easy to see that the projection $\pi : U \to S$ has a continuous section $\zeta : S \to U$ with the following property. For all $s \in S$ there exists $t > 0$ such that the region $\Omega_t$ is contained in the fibre $U_s := \pi^{-1}(s) \subset \mathbb{P}^1(\mathbb{C})$ and such that $\zeta(s) \in \Omega_t$. The section $\zeta$ induces a splitting of the sequence (11), which is actually independent of $\zeta$. We will use this splitting to consider $\pi_1(S, s_0)$ as a subgroup of $\pi_1(U, \infty)$.

The variation $\mathcal{V}$ corresponds to a group homomorphism $\rho : \pi_1(U, \infty) \to \text{GL}(V)$, where $V$ is a free $R$-module. Let $\rho_0$ denote the restriction of $\rho$ to $\pi_1(U_0, \infty)$ and $\chi$ the restriction to $\pi_1(S, s_0)$. By Part (iii) of Assumption 1.2 and the discussion in §1.1 we have a natural ordering $x_1 < \ldots < x_r = \infty$ of the points in $D_0$, and a natural choice of a presentation $\pi_1(U_0, \infty) \cong \langle \alpha_1, \ldots, \alpha_r \mid \prod_i \alpha_i = 1 \rangle$. Therefore, the local system $\mathcal{V}_0$ corresponds to a tuple $g = (g_1, \ldots, g_r) \in \mathcal{E}_r(V)$, with $g_i := \rho_0(\alpha_i)$. One checks that the homomorphism $\chi : \pi_1(S, s_0) \to \text{GL}(V)$ satisfies the condition

\begin{equation}
g^{\varphi(\gamma)} = g^{\chi(\gamma)^{-1}},
\end{equation}

for all $\gamma \in \pi_1(S, s_0)$. Conversely, given $g \in \mathcal{E}_r(V)$ and a homomorphism $\chi : \pi_1(S, s_0)$ such that (12) holds then there exists a unique variation $\mathcal{V}$ which induces the pair $(g, \chi)$.

With these notations one has the following result (see [5], Thm. 2.5):
Theorem 1.3 Let $W$ be the parabolic cohomology of $V$ and $\eta : \pi_1(S, s_0) \to \text{GL}(W_g)$ the corresponding monodromy representation. For all $\gamma \in \pi_1(S, s_0)$ we have

$$\eta(\gamma) = \Phi(g, \varphi(\gamma)) \cdot \Psi(g, \chi(\gamma)).$$

Thus, in order to compute the monodromy action on the parabolic cohomology of a local system $V_0$ corresponding to a tuple $g \in E_r(V)$, we need to know the braiding map $\varphi : \pi_1(S, s_0) \to A_{r-1}$ and the homomorphism $\chi : \pi_1(S, s_0) \to \text{GL}(V)$.

Remark 1.4 Suppose that $R$ is a field and that the local system $V_0$ is irreducible, i.e. the subgroup of $\text{GL}(V)$ generated by the elements $g_i$ acts irreducibly on $V$. Then the homomorphism $\chi$ is determined, modulo the scalar action of $R^\times$ on $V$, by $g$ and $\varphi$ (via (12)). It follows from Theorem 1.3 that the projective representation $\pi_1(S, s_0) \to \text{PGL}(V)$ associated to the monodromy representation $\eta$ is already determined by (and can be computed from) $g$ and the braiding map $\varphi$.

2 Poincaré duality

Let $V$ be a local system of $R$-modules on the punctured Riemann sphere $U$. If $V$ carries a non-degenerate symmetric (resp. alternating) form, then Poincaré duality induces on the parabolic cohomology group $H^1_p(U, V)$ a non-degenerate alternating (resp. symmetric) form. Similarly, if $R = \mathbb{C}$ and $V$ carries a Hermitian form, then we get a Hermitian form on $H^1_p(U, V)$. In this section we derive an explicit expression for this induced form.

2.1 Let us briefly recall the definition of singular (co)homology with coefficients in a local system. See e.g. [13] for more details. For $q \geq 0$ let $\Delta^q = \{y_0, \ldots, y_q\}$ denote the standard $q$-simplex with vertices $y_0, \ldots, y_q$. We will sometimes identify $\Delta^1$ with the closed unit interval $[0, 1]$. Let $X$ be a connected and locally contractible topological space and $V$ a local system of $R$-modules on $X$. For a continuous map $f : Y \to X$ we denote by $V_f$ the group of global sections of $f^*V$.

In the following discussion, a $q$-chain will be a function $\varphi$ which assigns to each singular $q$-simplex $\sigma : \Delta^q \to X$ a section $\varphi(\sigma) \in V_\sigma$. Let $\Delta^q(X, V)$ denote the set of all $q$-chains, which is made into an $R$-module in the obvious way. A $q$-chain $\varphi$ is said to have compact support if there exists a compact subset $A \subset X$ such that $\varphi_\sigma = 0$ whenever $\text{supp}(\sigma) \subset X - A$. The corresponding $R$-module is denoted by $\Delta^q_{\text{c}}(X, V)$. We define coboundary operators $d : \Delta^q(X, V) \to \Delta^q(X, V)$ and $d : \Delta^q(X, V) \to \Delta^{q+1}(X, V)$ through the formula

$$(d \varphi)(\sigma) := \sum_{0 \leq i \leq q} (-1)^i \cdot \varphi(\sigma^{(i)}).$$
Here $\sigma^{(i)}$ is the $i$th face of $\sigma$ (see [12]) and $\varphi(\sigma^{(i)})$ denotes the unique extension of $\varphi(\sigma^{(i)})$ to an element of $\mathcal{V}_\sigma$. It is proved in [13] that we have canonical isomorphisms

\begin{equation}
H^n(X, \mathcal{V}) \cong H^n(\Delta^\bullet(X, \mathcal{V}), d), \quad H^n_c(X, \mathcal{V}) \cong H^n(\Delta^\bullet_c(X, \mathcal{V}), d),
\end{equation}

i.e. singular cohomology agrees with sheaf cohomology. Let $x_0 \in X$ be a base point and $V$ the fibre of $\mathcal{V}$ at $x_0$. Then we also have an isomorphism

\begin{equation}
H^1(X, \mathcal{V}) \cong H^1(\pi_1(X, x_0), V).
\end{equation}

Let $\varphi$ be a 1-chain with $d\varphi = 0$. Let $\alpha : [0, 1] \to X$ be a closed path with base point $x_0$. By definition, $\varphi(\alpha)$ is a global section of $\alpha^* \mathcal{V}$. Then $\alpha \mapsto \delta(\alpha) := \varphi(\alpha)(1)$ defines a cocycle $\delta : \pi_1(X, x_0) \to V$, and this cocycle represents the image of $\varphi$ in $H^1(X, \mathcal{V})$.

A $q$-chain $\varphi$ is called finite if $\varphi(\sigma) = 0$ for all but finitely many simplexes $\sigma$. It is called locally finite if every point in $X$ has a neighborhood $U \subset X$ such that $\varphi(\sigma) = 0$ for all but finitely many simplexes $\sigma$ contained in $U$. We denote by $\Delta_q(X, \mathcal{V})$ (resp. by $\Delta^\bullet_q(X, \mathcal{V})$) the $R$-module of all finite (resp. locally finite) $q$-chains. For a fixed $q$-simplex $\sigma$ and a section $v \in \mathcal{V}_\sigma$, the symbol $v \otimes \sigma$ will denote the $q$-chain which assigns $v$ to $\sigma$ and 0 to all $\sigma' \neq \sigma$. Obviously, every finite (resp. locally finite) $q$-chain can be written as a finite (resp. possibly infinite) sum $\sum \iota \nu_\iota \otimes \sigma_\iota$. We define boundary operators $\partial : \Delta_q(X, \mathcal{V}) \to \Delta_{q-1}(X, \mathcal{V})$ and $\partial : \Delta^\bullet_q(X, \mathcal{V}) \to \Delta^\bullet_{q-1}(X, \mathcal{V})$ through the formula

$$
\partial(v \otimes \sigma) := \sum_{0 \leq i \leq q} (-1)^i \cdot v|_{\sigma^{(i)}} \otimes \sigma^{(i)}.
$$

We define homology (resp. locally finite homology) with coefficients in $\mathcal{V}$ as follows:

$$
H_q(X, \mathcal{V}) := H_q(\Delta_q(X, \mathcal{V})), \quad H^\bullet_q(X, \mathcal{V}) := H^\bullet_q(\Delta^\bullet_q(X, \mathcal{V})).
$$

2.2 Let $X := \mathbb{P}^1(\mathbb{C})$ be the Riemann sphere and $D = \{x_1, \ldots, x_r\} \subset \mathbb{P}^1(\mathbb{R})$ a subset of $r \geq 3$ points lying on the real line, with $x_1 < \ldots < x_r \leq \infty$. Let $\mathcal{V}$ be a local system of $R$-modules on $U = X - D$. Choose a base point $x_0$ lying in the upper half plane. Then $\mathcal{V}$ corresponds to a tuple $\mathbf{g} = (g_1, \ldots, g_r)$ in $GL(V)$ with $\prod g_i = 1$, where $V := \mathcal{V}_{x_0}$. See §1.1. Let $\mathcal{V}^* := \text{Hom}(\mathcal{V}, R)$ denote the local system dual to $\mathcal{V}$. It corresponds to the tuple $g^* = (g_1^*, \ldots, g_r^*)$ in $GL(V^*)$, where $V^*$ is the dual of $V$ and for each $g \in GL(V)$ we let $g^* \in GL(V^*)$ be the unique element such that

$$
\langle w \cdot g^*, v \cdot g \rangle = \langle w, v \rangle
$$

for all $w \in V^*$ and $v \in V$. Note that $V^{**} = V$ because $V$ is free of finite rank over $R$.

Let $\varphi$ be a 1-chain with compact support and with coefficients in $\mathcal{V}^*$. Let $a = \sum \nu_\iota \otimes \alpha_\iota$ be a locally finite 1-chain with coefficients in $\mathcal{V}$. By abuse
of notation, we will also write \( \varphi \) (resp. \( a \)) for its class in \( H^1_c(U, V^*) \) (resp. in \( H^1_lf(U, V) \)). The cap product

\[
\varphi \cap a := \sum_{\mu} \langle \varphi(\alpha_\mu), v_\mu \rangle
\]

induces a bilinear pairing

\[
\cap : H^1_c(U, V^*) \otimes H^1_lf(U, V) \rightarrow R.
\]

It is easy to see from the definition that \( H^0_lf(U, V) = 0 \). Therefore, it follows from the Universal Coefficient Theorem for cohomology (see e.g. [12], Thm. 5.5.3) that the pairing (15) is nonsingular on the left, i.e. identifies \( H^1_c(U, V^*) \) with \( \text{Hom} (H^1_lf(U, V), R) \). The cap product also induces a pairing

\[
\cap : H^1(U, V^*) \otimes H^1_lf(U, V) \rightarrow R.
\]

(This last pairing may not be non-singular on the left. The reason is that \( H^0(U, V) \cong V/\langle \text{Im}(g_i - 1) \mid i = 1, \ldots, r \rangle \) may not be a free \( R \)-module, and so \( \text{Ext}^1(H^0(U, V), R) \) may be nontrivial.)

Let \( f^1 : H^1_c(U, V^*) \rightarrow H^1(U, V^*) \) and \( f_1 : H^1(U, V) \rightarrow H^1_lf(U, V) \) denote the canonical maps. Going back to the definition, one can easily verify the rule

\[
f^1(\varphi) \cap a = \varphi \cap f_1(a).
\]

Let \( \varphi \in H^1_c(U, V^*) \) and \( \psi \in H^1(U, V) \). The cup product \( \varphi \cup \psi \) is defined as an element of \( H^2(U, R) \), see [14] or [13]. The standard orientation of \( U \) yields an isomorphism \( H^2(U, R) \cong R \). Using this isomorphism, we shall view the cup product as a bilinear pairing

\[
\cup : H^1_c(U, V^*) \otimes H^1(U, V) \rightarrow R.
\]

Similarly, one can define the cup product \( \varphi \cup \psi \), where \( \varphi \in H^1(U, V^*) \) and \( \psi \in H^1_c(U, V) \). Given \( \varphi \in H^1_c(U, V^*) \) and \( \psi \in H^1_c(U, V) \), one checks that

\[
f^1(\varphi) \cup \psi = \varphi \cup f^1(\psi).
\]

**Proposition 2.1 (Poincaré duality)** There exist unique isomorphisms of \( R \)-modules

\[
p : H^1(U, V) \rightarrow H^1_c(U, V), \quad p : H^1_lf(U, V) \rightarrow H^1(U, V)
\]

such that the following holds. If \( \varphi \in H^1_c(U, V^*) \) and \( a \in H^1_lf(U, V) \) or if \( \varphi \in H^1(U, V^*) \) and \( a \in H^1(U, V) \) then we have

\[
\varphi \cap a = \varphi \cup p(a).
\]

These isomorphisms are compatible with the canonical maps \( f_1 \) and \( f^1 \), i.e. we have \( p \circ f_1 = f^1 \circ p \).
Proof: See [14] or [13].

Corollary 2.2 The cup product induces a non-degenerate bilinear pairing

\[ \cup : H^1_p(U, V^*) \otimes H^1_p(U, V) \rightarrow R. \]

Proof: Let \( \varphi \in H^1_p(U, V^*) \) and \( \psi \in H^1_p(U, V) \). Choose \( \varphi' \in H^1_p(U, V^*) \) and \( \psi' \in H^1_p(U, V) \) with \( \varphi = f^1(\varphi') \) and \( \psi = f^1(\psi') \). By (18) we have \( \varphi' \cup \psi = \varphi \cup \psi' \).

Therefore, the expression \( \varphi \cup \psi := \varphi' \cup \psi \) does not depend on the choice of the lift \( \varphi' \) and defines a bilinear pairing between \( H^1_p(U, V^*) \) and \( H^1_p(U, V) \). By Proposition 2.1 and since the cap product (15) is non-degenerate on the left, this pairing is also non-degenerate on the left. But the cup product is alternating (i.e. we have \( \varphi \cup \psi = -\psi \cup \varphi \), where the right hand side is defined using the identification \( V^{**} = V \)), so our pairing is also non-degenerate on the right. \( \square \)

For \( a \in H^1_p(U, V^*) \) and \( b \in H^1(U, V) \), the expression

\[ (a, b) := p_a \cup p_b \]

defines another bilinear pairing \( H^1_p(U, V^*) \otimes H^1(U, V) \rightarrow R \). It is shown in [14] that this pairing can be computed as an ‘intersection product of loaded cycles’, generalizing the usual intersection product for constant coefficients, as follows.

We may assume that \( a \) is represented by a locally finite chain \( \sum_\mu v^{\mu}_x \otimes \alpha_\mu \) and that \( b \) is represented by a finite chain \( \sum_\nu v_\nu \otimes \beta_\nu \) such that for all \( \mu, \nu \) the 1-simplexes \( \alpha_\mu \) and \( \beta_\nu \) are smooth and intersect each other transversally, in at most finitely many points. Suppose \( x \) is a point where \( \alpha_\mu \) intersects \( \beta_\nu \). Then there exists \( t_0 \in [0, 1] \) such that \( x = \alpha(t_0) = \beta(t_0) \) and \( (\partial_\alpha|_{t_0} - \partial_\beta|_{t_0}) \) is a basis of the tangent space of \( U \) at \( x \). We set \( i(\alpha, \beta, x) := 1 \) (resp. \( i(\alpha, \beta, x) := -1 \)) if this basis is positively (resp. negatively) oriented. Furthermore, we let \( \alpha_{\mu,x} \) (resp. \( \beta_{\nu,x} \)) be the restriction of \( \alpha \) (resp. of \( \beta \)) to the interval \([0, t_0]\). Then we have

\[ (a, b) = \sum_{\mu,\nu,x} i(\alpha_\mu, \beta_\nu, x) \cdot ((v^{\mu}_x)^{\alpha_{\mu,x}}, (v^{\nu}_x)^{\beta_{\nu,x}}). \]

2.3 Let \( \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{R} \) be a non-degenerate symmetric (resp. alternating) bilinear form, corresponding to an injective homomorphism \( \kappa : \mathcal{V} \hookrightarrow \mathcal{V}^* \) with \( \kappa^* = \kappa \) (resp. \( \kappa^* = -\kappa \)). We denote the induced map \( H^1_p(U, \mathcal{V}) \rightarrow H^1_p(U, \mathcal{V}^*) \) by \( \kappa \) as well. Then

\[ \langle \varphi, \psi \rangle := \kappa(\varphi) \cup \psi \]
defines a non-degenerate alternating (resp. symmetric) form on \( H^1_p(U, \mathcal{V}) \).

Similarly, suppose that \( \mathcal{R} = \mathbb{C} \) and let \( \mathcal{V} \) be equipped with a non-degenerate Hermitian form, corresponding to an isomorphism \( \kappa : \mathcal{V} \cong \mathcal{V}^* \). Then the pairing

\[ (\varphi, \psi) := -i \cdot (\kappa(\overline{\varphi}) \cup \psi) \]
is a nondegenerate Hermitian form on \( H^1_p(U, \mathcal{V}) \) (we identify \( H^1_p(U, \mathcal{V}) \) with the complex conjugate of the vector space \( H^1_p(U, \mathcal{V}) \) in the obvious way).
Suppose that the Hermitian form on $V$ is positive definite. Then we can express the signature of the form (20) in terms of the tuple $g$, as follows. For $i = 1, \ldots, r$, let
\[
g_i \sim \begin{pmatrix} \alpha_{i,1} \\ & \ddots \\ & & \alpha_{i,n} \end{pmatrix}
\]
be a diagonalization of $g_i \in \text{GL}(V)$. Since the $g_i$ are Hermitian, the eigenvalues $\alpha_{i,j}$ have absolute value one and can be uniquely written in the form $\alpha_{i,j} = \exp(2\pi i \mu_{i,j})$, with $0 \leq \mu_{i,j} < 1$. Set $\bar{\mu}_{i,j} := 1 - \mu_{i,j}$ if $\mu_{i,j} > 0$ and $\bar{\mu}_{i,j} := 0$ otherwise.

**Theorem 2.3** Suppose that $V$ is equipped with a positive definite Hermitian form and that $H^0(U, V) = 0$. Then the Hermitian form (20) on $H^1_p(U, V)$ has signature
\[
\left( \sum_{i,j} \mu_{i,j} \right) - \dim_{\mathbb{C}} V, \left( \sum_{i,j} \bar{\mu}_{i,j} \right) - \dim_{\mathbb{C}} V \right).
\]

If $\dim_{\mathbb{C}} V = 1$, this formula is proved in [2], §2. With some extra work, the proof can be generalized to the case of arbitrary dimension. See forthcoming work of the authors.

2.4 We are interested in an explicit expression for the pairing of Corollary 2.2. We use the notation introduced at the beginning of §2.2, with the following modification. By $\gamma_i$ we now denote a homeomorphism between the open unit interval $(0, 1)$ and the open interval $(x_i, x_{i+1})$. We assume that $\gamma_i$ extends to a path $\bar{\gamma}_i : [0, 1] \to \mathbb{P}^1(\mathbb{R})$ from $x_i$ to $x_{i+1}$. We denote by $U^+ \subset \mathbb{P}^1(\mathbb{C})$ (resp. $U^-$) the upper (resp. the lower) half plane and by $U^+$ (resp. $U^-$) its closure inside $U = \mathbb{P}^1(\mathbb{C}) - \{x_1, \ldots, x_r\}$. Since $U^+$ is simply connected and contains the base point $x_0$, an element of $V$ extends uniquely to a section of $V$ over $\bar{U}^+$. We may therefore identify $V$ with $\mathcal{V}(\bar{U}^+)$ and with the stalk of $\mathcal{V}$ at any point $x \in \bar{U}^+$.

Choose a sequence of numbers $\epsilon_n, n \in \mathbb{Z}$, with $0 < \epsilon_n < \epsilon_{n+1} < 1$ such that $\epsilon_n \to 0$ for $n \to -\infty$ and $\epsilon_n \to 1$ for $n \to \infty$. Let $\gamma_i^{(n)} : [0, 1] \to U$ be the path $\gamma_i^{(n)}(t) := \gamma_i(\epsilon_n t + \epsilon_{n-1}(1 - t))$. Let $w_1, \ldots, w_r \in V$. Since supp($\gamma_i$) $\subset \bar{U}^+$, it makes sense to define
\[
w_i \otimes \gamma_i := \sum_n w_i \otimes \gamma_i^{(n)}.
\]
This is a locally finite 1-chain. Set
\[
c := \sum_{i=1}^r w_i \otimes \gamma_i.
\]
Note that $\partial(c) = 0$, so $c$ represents a class in $H^1_{lf}(U, V)$. 

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Lemma 2.4  
(i) The image of $c$ under the Poincaré isomorphism $H^1_\mathbb{X}(U, \mathcal{V}) \cong H^1(U, \mathcal{V})$ is represented by the unique cocycle $\delta : \pi_1(U, x_0) \to V$ with

$$\delta(\alpha_i) = w_i - w_{i-1} \cdot g_i.$$ 

(ii) The cocycle $\delta$ in (i) is parabolic if and only if there exist elements $u_i \in V$ with $w_i - w_{i-1} = u_i \cdot (g_i - 1)$, for all $i$.

Proof: For a path $\alpha : [0, 1] \to U$ in $U$, consider the following conditions:

(a) The support of $\alpha$ is contained either in $U^+$ or in $U^-$. 

(b) We have $\alpha(0) \in U^+$, $\alpha(1) \in U^-$ and $\alpha$ intersects $\gamma_i$ transversally in a unique point. 

(c) We have $\alpha(0) \in U^-$, $\alpha(1) \in U^+$ and $\alpha$ intersects $\gamma_i$ transversally in a unique point.

In Case (b) (resp. in Case (c)) we identify $\mathcal{V}_\alpha$ with $V$ via the stalk $\mathcal{V}_{\alpha(0)}$ (resp. via $\mathcal{V}_{\alpha(1)}$). Let $\varphi \in C^1(U, \mathcal{V})$ be the unique cocycle such that

$$\varphi(\alpha) = \begin{cases} 
0, & \text{if } \alpha \text{ is as in Case (a)} \\
-w_i, & \text{if } \alpha \text{ is as in Case (b)} \\
w_i^{\alpha_{i-1}}, & \text{if } \alpha \text{ is as in Case (c)}.
\end{cases}$$

(To show the existence and uniqueness of $\varphi$, choose a triangulation of $U$ in which all edges satisfy Condition (a), (b) or (c). Then use simplicial approximation.)

We claim that $\varphi$ represents the image of the cycle $c$ under the Poincaré isomorphism. Indeed, this follows from the definition of the Poincaré isomorphism, as it is given in [14]. Write $\alpha_i = \alpha_i'\alpha_i''$, with $\alpha_i'(1) = \alpha_i''(0) \in U^-$. Using the fact that $\varphi$ is a cocycle we get

$$\varphi(\alpha_i) = \varphi(\alpha_i') + \varphi(\alpha_i'')^{\alpha_i-1} = -w_{i-1} + w_i \cdot g_i^{-1}.$$ 

Therefore we have $\delta(\alpha_i) = \varphi(\alpha_i) \cdot g_i = w_i - w_{i-1} \cdot g_i$. See Figure 1. This proves (i).

By Section 1.1, the cocycle $\delta$ is parabolic if and only if $v_i$ lies in the image of $g_i - 1$. So (ii) follows from (i) by a simple manipulation.

Theorem 2.5 Let $\varphi \in H^1_\mathbb{X}(U, \mathcal{V}_*)$ and $\psi \in H^1_\mathbb{X}(U, \mathcal{V})$, represented by cocycles $\delta^* : \pi_1(U, x_0) \to V^*$ and $\delta : \pi_1(U, x_0) \to V$. Set $v_i := \delta(\alpha_i)$ and $v_i^* := \delta^*(\alpha_i)$. If we choose $v_i^* \in V$ such that $v_i^* \cdot (g_i - 1) = v_i$ (see Lemma 2.4), then we have

$$\varphi \cup \psi = \sum_{i=1}^{r} (v_i^*, v_i^*) + \sum_{j=1}^{i-1} \langle v_j^* g_{j+1}^* \cdots g_{i-1}^* (g_i^* - 1), v_i^* \rangle.$$
Proof: Let $w_1 := v_1$, $w_1^* := v_1^*$ and

$$w_i := v_i + w_{i-1} \cdot g_i, \quad w_i^* := v_i^* + w_{i-1}^* \cdot g_i^*$$

for $i = 2, \ldots, r$. By Lemma 2.4, we can choose $u_i \in V$ with $w_i - w_{i-1} = u_i \cdot (g_i - 1)$, for $i = 1, \ldots, r$. The claim will follow from the following formula:

$$(21) \quad \varphi \cup \psi = \sum_{i=1}^{r} \langle w_i^* - w_{i-1}^*, u_i - w_{i-1} \rangle.$$

To prove Equation (21), suppose $\delta$ is parabolic, and choose $u_i \in V$ such that $w_i - w_{i-1} = u_i \cdot (g_i - 1)$. Let $D_i \subset X$ be a closed disk containing $x_i$ but none of the other points $x_j$, $j \neq i$. We may assume that the boundary of $D_i$ intersects $\gamma_{i-1}$ in the point $\gamma_i(0)$ but nowhere else, and that $D_i$ intersects $\gamma_i$ in the point $\gamma_i^{(0)}(0)$ but nowhere else. Set $D_i^+ := D_i \cap U^+$ and $D_i^- := D_i \cap U^-$. Let $u_i^+ := u_i - w_{i-1}$, considered as a section of $V$ over $D_i^+$ via extension over the whole upper half plane $U^+$. It makes sense to define the locally finite chain

$$u_i^+ \otimes D_i^+ := \sum_{\sigma} u_i^+ \otimes \sigma,$$

where $\sigma$ runs over all 2-simplexes of a triangulation of $D_i^+$. (Note that $x_i \not\in D_i^+$, so this triangulation cannot be finite.) Similarly, let $u_i^- \in V_{D_i^-}$ denote the section of $V$ over $D_i^-$ obtained from $u_i \in V$ by continuation along a path which enters $U^-$ from $U^+$ by crossing the path $\gamma_{i-1}$; define $u_i^- \otimes D_i^-$ as before. Let

$$c' := c + \partial (u_i^+ \otimes D_i^+ + u_i^- \otimes D_i^-).$$

It is easy to check that $c'$ is homologous to the cocycle

$$c'' := \sum_i \left( w_i \otimes \gamma_i(0) + u_i^+ \otimes \beta_i^+ + u_i^- \otimes \beta_i^- \right),$$
where $\beta_+^i$ (resp. $\beta_-^i$) is the path from $\gamma_{i-1}^{(0)}(0)$ to $\gamma_{i-1}^{(0)}(1)$ (resp. from $\gamma_{i-1}^{(0)}(1)$ to $\gamma_{i-1}^{(0)}(0)$) running along the upper (resp. lower) part of the boundary of $D_i$. See Figure 2. Note that $c''$ is finite and that, by construction, the image of $c''$ under the canonical map $f_1 : H_1(U, V) \to H_1^f(U, V)$ is equal to the class of $c$. Let $\psi' \in H_1^f(U, V)$ denote the image of $c''$ under the Poincaré isomorphism $H_1(U, V) \cong H_1^f(U, V)$. The last statement of Proposition 2.1 shows that $\psi'$ is a lift of $\psi \in H_1^f(U, V)$.

Let $c^* := \sum_i w_i^* \otimes \gamma_i \in C_1(U, V^*)$. By (i) and the choice of $w_i^*$, the image of $c^*$ under the Poincaré isomorphism $H_1^f(U, V^*) \cong H_1(U, V^*)$ is equal to $\varphi$. By definition, we have $\varphi \cup \psi = (c^*, c'')$. To compute this intersection number, we have to replace $c^*$ by a homologous cycle which intersects the support of $c''$ at most transversally. For instance, we can deform the open paths $\gamma_i$ into open paths $\gamma'_i$ which lie entirely in the upper half plane. See Figure 2. It follows from (19) that

$$(c^*, c'') = \sum_i (w_{-1}^i, u_i^+) - (w_i^+, u_i^+) = \sum_i \langle w_{-1}^i - w_i^-, u_i - w_{-1}^i \rangle.$$ 

This finishes the proof of (21). The formula in (iv) follows from (21) from a straightforward computation, expressing $w_i$ and $u_i$ in terms of $v_i$ and $v_i'$.

Remark 2.6 In the somewhat different setup, a similar formula as in Theorem 2.5 can be found in \[16\], §1.2.3.

3 The monodromy of the Picard–Euler system

Let 

$$S := \{(s, t) \in \mathbb{C}^2 \mid s, t \neq 0, 1, s \neq t\},$$

and let $X := \mathbb{P}^1_S$ denote the relative projective line over $S$. The equation

$$(22) \quad y^3 = x(x-1)(x-s)(x-t)$$

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defines a finite Galois cover \( f : Y \to X \) of smooth projective curves over \( S \),
tamely ramified along the divisor \( D := \{0, 1, s, t, \infty\} \subset X \). The curve \( Y \) is
called the Picard curve. Let \( G \) denote the Galois group of \( f \), which is cyclic of
order 3. The equation \( \sigma^* y = \chi(\sigma) \cdot y \) for \( \sigma \in G \) defines an injective character
\( \chi : G \to \mathbb{C}^\times \). As we will see below, the \( \chi \)-eigenspace of the cohomology of \( Y 
\) gives rise to a local system on \( S \) whose associated system of differential equations
is known as the Picard–Euler system.

We fix a generator \( \sigma \) of \( G \) and set \( \omega := \chi(\sigma) \). Let \( K := \mathbb{Q}(\omega) \) be the quadratic
extension of \( \mathbb{Q} \) generated by \( \omega \) and \( \mathcal{O}_K = \mathbb{Z}[\omega] \) its ring of integers. The family of
\( G \)-covers \( f : Y \to X \) together with the character \( \chi \) of \( G \) corresponds to a local
system of \( \mathcal{O}_K \)-modules on \( U := X - D \). Set \( s_0 := (2, 3) \in S \) and let \( \mathcal{V}_0 \) denote
the restriction of \( \mathcal{V} \) to the fibre \( U_0 = \mathbb{A}^1 - \{0, 1, 2, 3\} \) of \( U \) over \( s_0 \). We
consider \( \mathcal{V} \) as a variation of \( \mathcal{V}_0 \) over \( S \). Let \( \mathcal{W} \) denote the parabolic cohomology
of this variation; it is a local system of \( \mathcal{O}_K \)-modules of rank three, see [5], Rem. 1.4. Let \( \chi' : G \to \mathbb{C}^\times \)
denote the conjugate character to \( \chi \) and \( \mathcal{W}' \) the parabolic cohomology of
the variation of local systems \( \mathcal{V}' \) corresponding to the \( G \)-cover \( f \) and the character \( \chi' \).
We write \( \mathcal{W}_C \) for the local system of \( \mathbb{C} \)-vectorspaces \( \mathcal{W} \otimes \mathbb{C} \). The maps \( \pi_Y : Y \to S \) and \( \pi_X : X \to S \) denote the natural projections.

**Proposition 3.1** We have a canonical isomorphism of local systems
\[
R^1\pi_{Y,*} \mathcal{C} \cong \mathcal{W}_C \oplus \mathcal{W}_C'.
\]
This isomorphism identifies the fibres of \( \mathcal{W}_C \) with the \( \chi \)-eigenspace of the sin-
gular cohomology of the Picard curves of the family \( f \).

**Proof:** The group \( G \) has a natural left action on the sheaf \( f_* \mathcal{C} \). We have a canonical isomorphism of sheaves on \( X \)
\[
f_* \mathcal{C} \cong \mathbb{C} \oplus j_* \mathcal{V}_C \oplus j_* \mathcal{V}',
\]
which identifies \( j_* \mathcal{V}_C \), fibre by fibre, with the \( \chi \)-eigenspace of \( f_* \mathcal{C} \). Now the
Leray spectral sequence for the composition \( \pi_Y = \pi_X \circ f \) gives isomorphisms of sheaves on \( S \)
\[
R^1\pi_{Y,*} \mathcal{C} \cong R^1\pi_{X,*}(f_* \mathcal{C}) \cong \mathcal{W}_C \oplus \mathcal{W}_C'.
\]
Note that \( R^1\pi_{X,*} \mathcal{C} = 0 \) because the genus of \( X \) is zero. Since the formation of
\( R^1\pi_{Y,*} \) commutes with the \( G \)-action, the proposition follows. \( \square \)

The comparison theorem between singular and deRham cohomology identifies
\( R^1\pi_{Y,*} \mathcal{C} \) with the local system of horizontal sections of the relative deRham
cohomology module \( R^1_{dR} \pi_{Y,*} \mathcal{O}_Y \), with respect to the Gauss-Manin connection.
The \( \chi \)-eigenspace of \( R^1_{dR} \pi_{Y,*} \mathcal{O}_Y \) gives rise to a Fuchsian system known as the
Picard–Euler system. In more classical terms, the Picard–Euler system is a set
of three explicit partial differential equations in \( s \) and \( t \) of which the period
integrals
\[
I(s, t; a, b) := \int_a^b \frac{dx}{\sqrt{x(x-1)(x-s)(x-t)}}
\]
(with \(a, b \in \{0, 1, s, t, \infty\}\)) are a solution. See [10], [6], [7]. It follows from Proposition 3.1 that the monodromy of the Picard–Euler system can be identified with the representation \(\eta : \pi_1(S) \to \text{GL}_3(\mathcal{O}_K)\) corresponding to the local system \(\mathcal{W}\).

**Theorem 3.2 (Picard)** For suitable generators \(\gamma_1, \ldots, \gamma_5\) of the fundamental group \(\pi_1(S)\), the matrices \(\eta(\gamma_1), \ldots, \eta(\gamma_5)\) are equal to

\[
\begin{pmatrix}
\omega^2 & 0 & 1 - \omega \\
\omega - \omega^2 & 1 & \omega^2 - 1 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
\omega^2 & 0 & 1 - \omega^2 \\
1 - \omega^2 & 1 & \omega^2 - 1 \\
0 & \omega & \omega^2 - 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & \omega^2 - 1 \\
0 & \omega^2 - 1 & -2\omega
\end{pmatrix},
\begin{pmatrix}
\omega^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
\omega^2 & \omega - \omega^2 & 0 \\
0 & 1 & 0 \\
1 - \omega & \omega^2 - 1 & 1
\end{pmatrix}.
\]

The invariant Hermitian form (induced by Poincaré duality, see Corollary 2.2) is given by the matrix

\[
\begin{pmatrix}
a & 0 & 0 \\
0 & 0 & a \\
a & 0 & 0
\end{pmatrix},
\]

where \(a = \frac{1}{4}(\omega^2 - \omega)\).

**Proof:** The divisor \(D \subset \mathbb{P}^1_S\) satisfies Assumption 1.2. Let \(\varphi : \pi_1(S, s_0) \to A_4\) be the associated braiding map. Using standard methods (see e.g. [16] and [3]), or by staring at Figure 3, one can show that the image of \(\varphi\) is generated by the five braids

\[
\beta_3^2, \beta_3 \beta_2^{-1}, \beta_3 \beta_2 \beta_1^{-1}, \beta_3^{-1}, \beta_2, \beta_2 \beta_1^{-1}.
\]

It is clear that these five braids can be realized as the image under the map \(\varphi\) of generators \(\gamma_1, \ldots, \gamma_5 \in \pi_1(S, s_0)\).

Considering the \(\infty\)-section as a ‘tangential base point’ for the fibration \(U \to S\) as in §1.5, we obtain a section \(\pi_1(S) \to \pi_1(U)\). We use this section to identify \(\pi_1(S)\) with a subgroup of \(\pi_1(U)\). Let \(\alpha_1, \ldots, \alpha_5\) be the standard generators of \(\pi_1(U_0)\). Let \(\rho : \pi_1(U) \to K^\times\) denote the representation corresponding to the \(G\)-cover \(f : Y \to X\) and the character \(\chi : G \to K^\times\), and \(\rho_0 : \pi_1(U_0) \to G\) its restriction to the fibre above \(s_0\). Using (22) one checks that \(\rho_0\) corresponds to the tuple \(g = (\omega, \omega, \omega, \omega, \omega^2)\), i.e. that \(\rho_0(\alpha_i) = g_i\). Also, since the leading coefficient of the right hand side of (22) is one, the restriction of \(\rho\) to \(\pi_1(S)\) is trivial. Hence, by Theorem 1.3, we have

\[
\eta(\gamma_i) = \Phi(g, \varphi(\gamma_i)).
\]
A straightforward computation, using (9) and the cocycle rule (10), gives the value of $\eta(\gamma_i)$ (in form of a three-by-three matrix depending on the choice of a basis of $W_g$). For this computation, it is convenient to take the classes of $(1, 0, 0, 0, -\omega^2)$, $(0, 1, 0, 0, -\omega)$ and $(0, 0, 1, 0, -1)$ as a basis. In order to obtain the 5 matrices stated in the theorem, one has to use a different basis, i.e. conjugate with the matrix

$$B = \begin{pmatrix} 0 & -\omega - 1 & -\omega \\ \omega + 1 & \omega + 1 & \omega + 1 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

The claim on the Hermitian form follows from Theorem 2.5 by another straightforward computation.

Remark 3.3 Theorem 3.2 is due to Picard, see [10], p. 125, and [11], p. 181. He obtains exactly the matrices given above, but he does not list all of the corresponding braids. A similar list as above is obtained in [6] using different methods.

Remark 3.4 It is obvious from Theorem 3.2 that the Hermitian form on $W$ has signature $(1, 2)$ or $(2, 1)$, depending on the choice of the character $\chi$. This confirms Theorem 2.3 in this special case.

References

[1] P. Deligne. *Equations Différentielles à Points Singuliers Réguliers*. Number 163 in Lecture Notes in Mathematics. Springer-Verlag, 1970.

[2] P. Deligne and G.D. Mostow. Monodromy of hypergeometric functions and non-lattice integral monodromy. *Publ. Math. IHES*, 63:5–89, 86.

[3] M. Dettweiler and S. Reiter. An algorithm of Katz and its application to the inverse Galois problem. *J. Symb. Comput.*, 30: 761–798, 2000.
[4] M. Dettweiler and S. Reiter. On the middle convolution. Preprint (math.AG/0305311), 2003.

[5] M. Dettweiler and S. Wewers. Variation of local systems and parabolic cohomology. Preprint (math.AG/0310139), 2004.

[6] R.-P. Holzapfel. Geometry and Arithmetic around Euler Partial Differential Equations. Mathematics and Its Applications. D. Reidel, 1986.

[7] R.-P. Holzapfel. The Ball and Some Hilbert Problems. Lectures in Mathematics ETH Zürich. Birkhäuser, 1995.

[8] N.M. Katz. Rigid Local Systems. Annals of Mathematics Studies 139. Princeton University Press, 1997.

[9] Lauricella. Sulle funzioni ipergeometriche a piu variabili. Rend. di Palermo, VII:111–158, 1893.

[10] E. Picard. Sur les fonctions de deux variables indépendantes analogues aux fonctions modulaires. Acta Math., 2:114–135, 1883.

[11] E. Picard. Sur les formes quadratiques ternaires indéfinies à indéterminées conjuguées et sur les fonctions hyperfuchsiennes correspondantes. Acta Math., 5:121–182, 1884.

[12] E.H. Spanier. Algebraic Topology. Springer–Verlag, 1966.

[13] E.H. Spanier. Singular homology and cohomology with local coefficients and duality for manifolds. Pacific J. Math., 160(1):165–200, 1993.

[14] N.E. Steenrod. Homology with local coefficients. Ann. of Math., 44(2):610–627, 1943.

[15] T. Terada. Probléme de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables. J. Math. Kyoto Univ., 13:557–578, 1973.

[16] H. Völklein. The braid group and linear rigidity. Geom. Dedicata, 84:135–150, 2001.