Existence of solutions on the critical hyperbola for a pure Lane-Emden system with Neumann boundary conditions

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Abstract

We study the following Lane-Emden system

\[-\Delta u = |v|^{q-1}v \quad \text{in } \Omega, \quad -\Delta v = |u|^{p-1}u \quad \text{in } \Omega, \quad u_\nu = v_\nu = 0 \quad \text{on } \partial \Omega,\]

with \(\Omega\) a bounded regular domain of \(\mathbb{R}^N\), \(N \geq 4\), and exponents \(p, q\) belonging to the so-called critical hyperbola \(1/(p+1) + 1/(q+1) = (N-2)/N\). We show that, under suitable conditions on \(p, q\), least-energy (sign-changing) solutions exist, and they are classical. In the proof we exploit a dual variational formulation which allows to deal with the strong indefinite character of the problem. We then prove such condition by using as test functions the solutions to the system in the whole space and performing delicate asymptotic estimates. If \(N \geq 5, p = 1\), the system above reduces to a biharmonic equation, for which we also prove existence of least-energy solutions. Finally, we prove some partial symmetry and symmetry-breaking results in the case \(\Omega\) is a ball or an annulus.

Keywords: Dual method, critical exponents, Hamiltonian elliptic system, biharmonic equation, symmetry breaking, least energy nodal solutions.

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1 Introduction

Consider a bounded domain \(\Omega\) of class \(C^2\) in \(\mathbb{R}^N\), \(N \geq 4\), and denote with \(\nu\) the outward pointing normal on \(\partial \Omega\). Our problem reads as follows

\[-\Delta u = |v|^{q-1}v \quad \text{in } \Omega, \quad -\Delta v = |u|^{p-1}u \quad \text{in } \Omega, \quad u_\nu = v_\nu = 0 \quad \text{on } \partial \Omega,\]

with \(\Omega\) a bounded regular domain of \(\mathbb{R}^N\), \(N \geq 4\), and exponents \(p, q\) belonging to the so-called critical hyperbola \(1/(p+1) + 1/(q+1) = (N-2)/N\). We show that, under suitable conditions on \(p, q\), least-energy (sign-changing) solutions exist, and they are classical. In the proof we exploit a dual variational formulation which allows to deal with the strong indefinite character of the problem. We then prove such condition by using as test functions the solutions to the system in the whole space and performing delicate asymptotic estimates. If \(N \geq 5, p = 1\), the system above reduces to a biharmonic equation, for which we also prove existence of least-energy solutions. Finally, we prove some partial symmetry and symmetry-breaking results in the case \(\Omega\) is a ball or an annulus.

Keywords: Dual method, critical exponents, Hamiltonian elliptic system, biharmonic equation, symmetry breaking, least energy nodal solutions.

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Observe also that, under (1.2), we have the continuous (but not compact) embeddings

\[ W^{2, \frac{N+4}{N-2}}(\Omega) \hookrightarrow L^{p+1}(\Omega), \quad \text{and} \quad W^{2, \frac{4}{p+1}}(\Omega) \hookrightarrow L^{q+1}(\Omega). \]  

(1.3)

Therefore, the associated Euler-Lagrange functional

\[ I(u, v) := \int_{\Omega} \nabla u \cdot \nabla v - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} - \frac{1}{q+1} \int_{\Omega} |v|^{q+1} \]

is finite at strong solutions.

**Definition 1.1.** Let \( p, q \) satisfy (1.2). A least energy (nodal) solution for (1.1) is a strong solution which achieves the least energy (nodal) level

\[ c_{p,q} := \inf \{ I(u, v) : (u, v) \in W^{2, \frac{N+4}{N-2}}(\Omega) \times W^{2, \frac{4}{p+1}}(\Omega) \text{ strong solution of (1.1)} \}. \]

The existence of least energy nodal solutions is known in the subcritical case: \( \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N} \) with \( pq \neq 1 \), see [35]. The goal of our work is to extend this to the critical case. In this direction, the main result of our paper is the following:

**Theorem 1.2.** Let \( p, q \) satisfy (1.2), and moreover

(i) \( N \geq 6 \) and \( p, q > \frac{N+2}{2(N-2)} \), or

(ii) \( N = 5 \) and \( p, q > \frac{17}{14} \), or

(iii) \( N = 4 \) and \( p, q > \frac{7}{3} \).

Then there exists a least energy (nodal) classical solution to 1.1.

We point out that we prove regularity of all possible solutions (not only of least energy) for every \( p, q \) satisfying (1.2), see the next result. This is not trivial due to the coupling in the system and since either \( p \) and \( q \) might be smaller than 1. Previously, the regularity was known in the subcritical superlinear case [35], and in the critical scalar case \( p = q = 2^* \) [30]; the arguments used in these situations (respectively a bootstrap and a Brezis-Kato argument) cannot be used in our situation.

**Proposition 1.3.** Let \((u, v)\) be a strong solution to (1.1) where \( p, q \) satisfy (1.2). Then \((u, v) \in C^{2, \zeta}(\Omega) \times C^{2, \eta}(\Omega), \) with: \( \zeta < q \) if \( 0 < q < 1 \), and \( \zeta \in (0, 1) \) if \( q \geq 1; \) \( \eta < p \) if \( 0 < p < 1 \), and \( \eta \in (0, 1) \) if \( p \geq 1 \).

The additional restrictions in Theorem 1.2 come from the decay of the solutions in the whole space, which depend on the position of \( (p, q) \) in the critical hyperbola.

If we take \( N \geq 5, \ p = 1, q = \frac{N+4}{N-4} \), or equivalently, \( q = 1, \ p = \frac{N+4}{N-4} \), system (1.1) reduces to the fourth order equation

\[ \Delta^2 u = |u|^\frac{p+1}{2} u \text{ in } \Omega, \quad u_\nu = (\Delta u)_\nu = 0 \text{ on } \partial \Omega. \]  

(1.4)

If \( N > 6 \), then

\[ \frac{N+2}{2(N-2)} < \frac{N+4}{N-4}, \quad \text{and} \quad \frac{N+2}{2(N-2)} < 1, \]

hence (1.4) is contained in Theorem 1.2 if \( N > 6 \). However, the case \( N = 5, 6 \) and \( p = 1 \) is not included. Nonetheless, we prove the following:

**Theorem 1.4.** Let \( N > 5 \). Then there exists a least energy (nodal) classical solution to problem (1.4).

In particular, this shows the existence of solutions to (1.4) also in the dimensions \( N = 5, 6 \), however the proof of this result actually works for all \( N \geq 5 \), therefore it also gives a different proof of existence in case \( N \geq 7 \). Although, up to our knowledge, apparently there are no results regarding (1.4) in the literature, we recall that in [4] a related problem with a Paneitz-Branson type operator, \( Lu := \Delta^2 u - \Delta u + \alpha u \) with \( \alpha > 0 \), and the same boundary conditions was studied.

By a perturbation argument, we also show the following
Theorem 1.5. Let $N = 5, 6$. There exists $\varepsilon = \varepsilon(N, \Omega)$ such that, if $p, q$ satisfy (1.2) and either

$$|p - 1| + \left| q - \frac{N + 4}{N - 4} \right| < \varepsilon \quad \text{or} \quad |p - \frac{N + 4}{N - 4}| + |q - 1| < \varepsilon,$$

then there exists a least energy (nodal) classical solution to (1.1).

In Figures 1 and 2 we illustrate the assumptions of Theorems 1.2 and 1.5. We now make some comments regarding the proof of Theorem 1.2.

In order to avoid the strongly indefinite character of the functional $I$ (the quadratic term $(u, v) \mapsto \int_{\Omega} \nabla u \cdot \nabla v$ does not have a sign), we work in a dual formulation (see Subsection 2.1 below). Following [13, 35], we provide equivalent variational characterizations for the least energy level (definition (2.5) and Proposition 2.5).

Since the problem is critical, the embeddings (1.3) are not compact, and in general the dual functional does not satisfy the Palais-Smale condition. We prove however a compactness condition above a certain energy level (Lemma 3.1), which is based on a new class of Cherrier type inequalities, namely for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\|u\|_{\mathbb{H}_{N-2}} \leq \left( \frac{2\pi}{S} + \varepsilon \right) \|\Delta u\|_{\eta} + C(\varepsilon) \|u\|_{W^{1,\eta}}, \quad \forall u \in W^{2,\eta}(\Omega)$$

(see Theorem A.1 for more details). Here we are inspired by [4], in which the case $\eta = 2$ is shown.

The way in which we check the compactness condition is where the proofs of Theorems 1.2 and 1.5 are different. In the proof of Theorem 1.2 we use a test function argument; we consider the problem in the whole space

$$-\Delta U = V^q \text{ in } \mathbb{R}^N, \quad -\Delta V = U^p \text{ in } \mathbb{R}^N,$$

(1.5)

which admits a family of solutions $(U_{x_0,\varepsilon}, V_{x_0,\varepsilon}) \in \mathcal{D}^2,\mathcal{D}^{q,1}(\mathbb{R}^N) \times \mathcal{D}^2,\mathcal{D}^{p,1}(\mathbb{R}^N)$, which are positive and radially decreasing with respect to $x_0 \in \mathbb{R}^N$, and $\varepsilon$ is a concentration parameter related to the scalings of the system, see Subsection 2.2 for the details. Performing careful asymptotic estimates, under the assumptions of Theorem 1.2 we prove the compatibility condition; these estimates are not at all straightforward mainly by two aspects: on the one hand $U_{x_0,\varepsilon}, V_{x_0,\varepsilon}$ do not have an explicit expression and we only have access to their decay at infinity, and on the other hand we are dealing with a dual formulation.
Figure 2: The red points on the critical hyperbola represent the values \((p, q)\) for which we prove existence of least energy solutions to (1.1), see Theorem 1.2. Notice that in case \(N = 6\) we can also prove existence close to \(p = 1\) and \(q = 1\), see Theorem 1.5.

Remark 1.6. For convenience of the reader, we would also like to stress why we need to ask the technical conditions (i), (ii) and (iii) in Theorem 1.2: condition \(p, q > \frac{N+2}{2(N-2)}\) is needed right at the end of the proof of Lemma 4.4. The hypothesis in (ii) is equivalent to \(p, q < N = 5\), and it is more restrictive than \(p, q > \frac{N+2}{2(N-2)} = \frac{1}{6}\). Also, the hypothesis in (iii) is equivalent to \(p, q < N = 4\), and it is more restrictive than \(p, q > \frac{N+2}{2(N-2)} = \frac{3}{2}\). They are needed in the proof of Lemma 4.6. It is an open question whether these conditions can be removed, and if there exist least energy nodal solutions for all \(p, q\) on the critical hyperbola (1.2).

Remark 1.7. Notice that the case \(N = 3\) cannot be treated with our arguments, as for instance the hypothesis \(p, q < N\), needed in the proof of Lemma 4.6, is too restrictive if \(N = 3\), as no points on the hyperbola satisfy it.

Remark 1.8. We point out that \(p = q = \frac{N+2}{N-2}\) satisfies the condition \(q > \frac{N+2}{2(N-2)}\), and also the more restrictive conditions appearing in (ii) and (iii) if \(N = 4, 5\). This case (by [35, Lemma 2.6]) reduces to the equation with critical exponent

\[-\Delta u = |u|^{\frac{N+2}{N-2}} u \text{ in } \Omega, \quad u_{\nu} = 0 \text{ on } \partial\Omega, \quad (1.6)\]

and it is treated in [13] (where actually a more general operator, \(Lw := -\Delta w + \lambda w, \lambda \in \mathbb{R}\) is considered). The first part of our paper can be seen as an extension of [13] to Lane-Emden systems. For completeness we point out that, for the single equation case (1.6), the sublinear case is treated in [28], while the subcritical superlinear case is shown in [35, Corollary 1.4]. The continuity of solutions with respect to \(p\) is the object of [36], while in [31] it is shown the existence of solutions in the slightly supercritical case in some symmetric domains.

In [35] some symmetry and symmetry breaking results are proved if the domain is a ball or an annulus, and \(p, q\) are subcritical. It is natural to wonder if such results can be extended to the critical case, and, in particular, if the least energy (nodal) solutions obtained in Theorem 1.2 are radially symmetric and, if not, if there are radial solutions. Assume \(\Omega\) satisfies

\[\Omega = B_R(0) \setminus B_r(0), \quad \text{for some } 0 < r < R, \quad \text{or} \]

\[\Omega = B_R(0) \quad \text{for some } R > 0. \quad (1.7)\]
We let
\[ c_{p,q}^{\text{rad}} := \inf \{ I(u,v) : (u,v) \in W^{2,p+1}_{\text{rad}}(\Omega) \times W^{2,q+1}_{\text{rad}}(\Omega) \text{ strong solution of (1.1)} \}; \]
and we define a least energy (nodal) radial solution as a strong solution \((u,v) \in W^{2,p+1}_{\text{rad}}(\Omega) \times W^{2,q+1}_{\text{rad}}(\Omega)\) of (1.1), such that \( I(u,v) = c_{p,q}^{\text{rad}} \).

We will prove the following:

**Theorem 1.9.** We assume that \( p, q \) satisfy (1.2).

(i) If \( \Omega \) is the annulus (1.7) then the set of least energy radial solutions of (1.1) is nonempty. Moreover, if \((u,v)\) is a least energy radial solution, then \( u, v, > 0 \), that is, \( u, v \) are strictly monotone in the radial variable, having the same monotonicity.

(ii) Moreover, let \( \Omega \) as in (1.7) or (1.8), and let \((p,q)\) satisfy the assumptions of Theorem 1.3. If \((u,v)\) is a least-energy nodal solution, then both \( u \) and \( v \) are not radially symmetric, and there exists \( e^* \in S^{N-1} \) such that \( u, v \) are foliated Schwarz symmetric with respect to \( e^* \) (see Section 6 below for the definition).

**Remark 1.10.** Notice that if \( p = q = 2^* \), then there exist no radial solutions on the ball, whereas this is an open problem for the system, see Remark 6.3. We also stress that continuity up to the boundary is crucial in the proof of Part (ii) of Theorem 1.9. This is easy to show for radially symmetric solutions on an annulus, whereas for a general bounded domain it is a consequence of Proposition 1.3.

We conclude this introduction by mentioning some related problems for systems. As said before, the subcritical case of (1.1) with Neumann boundary conditions is considered in [33]; for problems where the linear operator is instead \( Lw := -\Delta w + w \), we are aware of the papers [3, 6, 30, 32, 40]. Observe however that in the latter case the situation is much different, as for instance positive solutions are allowed. On the other hand, in the case of Dirichlet boundary conditions, the critical problem does not admit solutions in general (see [27, 35]) and concentration results have been proved close to the critical hyperbola in [10, 17]. Finally, we point out that the nondegeneracy of the solutions \((U_{x_0, \varepsilon}, V_{x_0, \varepsilon})\) in (1.5) has been recently proved in [15], which in particular allowed to study a Brezis-Nirenberg type problem for Lane-Emden systems and slightly subcritical systems in [22, 21] (see also [24]) and a Coron type problem in [20].

This paper is organized as follows. We first set notation and describe the variational formulation of the problem, and we prove Proposition 1.3. In Section 3 we give a suitable compactness condition, and in Section 4 we prove Theorem 1.2. Section 5 is devoted to the study of the biharmonic equation (1.4), and to the proof of Theorem 1.5. In Section 6 we analyze the symmetry-breaking phenomenon on radially symmetric domains, and existence of solutions with radial symmetry in the annulus. Appendix A is devoted to the proof of a Cherrier type inequality, which is crucial in the proof of the compactness condition in Section 3 whereas Appendix B contains some estimates which we exploit in the proof of Theorem 1.2.

### 2 Preliminaries

#### 2.1 Notation and variational setting

One can use various variational settings to deal with the system (1.1), see for instance the survey [5]. For our purposes, the most convenient is to use the so called dual method, which we describe in this section.

For \( s > 1 \), we denote the \( L^s(\Omega) \) and \( W^{2,s}(\Omega) \) norms by \( \| \cdot \|_s \) and \( \| \cdot \|_{W^{2,s}} \) respectively. Whenever the integration domain is \( \mathbb{R}^N \), we write it explicitly and denote the \( L^2(\mathbb{R}^N) \) and \( W^{2,s}(\mathbb{R}^N) \) norms by \( \| \cdot \|_{L^2(\mathbb{R}^N)} \) and \( \| \cdot \|_{W^{2,s}(\mathbb{R}^N)} \) respectively. We define the operator \( K : X^s \to W^{2,s}(\Omega) \) such that \( Kh := u \) if and only if

\[ -\Delta u = h \text{ in } \Omega, \quad u_\nu = 0 \text{ on } \partial \Omega, \quad \int_{\Omega} u = 0, \]

where

\[ X^s = \left\{ f \in L^s : \int_{\Omega} f = 0 \right\}. \]
The fact that $K$ is well defined and continuous is a consequence of the following regularity result, which is taken from [33, Theorem and Lemma in page 143] (see also [2, Theorem 15.2]).

**Lemma 2.1.** If $s > 1$, $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, and $h \in X^s$. Then there is a unique strong solution $u \in W^{2,s}(\Omega)$ of

$$-\Delta u = h \quad \text{in } \Omega, \quad \partial_{\nu} u = 0 \quad \text{on } \partial \Omega, \quad \int_{\Omega} u = 0. \quad (2.1)$$

Moreover, there exists $C(\Omega, s) = C > 0$ such that $\|u\|_{W^{2,s}} \leq C\|h\|_{s}$.

Also, we define $K_i : X^{\frac{1}{p}+\frac{1}{q}}(\Omega) \to W^{2,\frac{1}{p}+\frac{1}{q}}(\Omega)$ given by

$$K_i h = Kh + \kappa_i(h) \quad \text{for some } \kappa_i(h) \in \mathbb{R} \quad \text{such that} \quad \int_{\Omega} |K_i h|^{q-1} K_i h = 0. \quad (2.2)$$

For

$$\alpha = \frac{p+1}{p}, \quad \beta = \frac{q+1}{q},$$

we also define the space

$$X = X^\alpha \times X^\beta$$

and the dual functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(f, g) = \frac{p}{p+1} \int_{\Omega} |f|^{\frac{p+1}{p}} + \frac{q}{q+1} \int_{\Omega} |g|^{\frac{p+1}{q}} - \int_{\Omega} gKf = \frac{1}{\alpha} \|f\|_\alpha^\alpha + \frac{1}{\beta} \|g\|_\beta^\beta - \int_{\Omega} gKf.$$ We also set

$$\gamma_1 := \frac{\beta}{\alpha + \beta} = \frac{p(q+1)}{2pq+p+q}, \quad \gamma_2 := \frac{\alpha}{\alpha + \beta} = \frac{q(p+1)}{2pq+p+q}, \quad \gamma := \gamma_1\alpha = \gamma_2\beta = \frac{(p+1)(q+1)}{2pq+p+q},$$

which are such that

$$\gamma_1 + \gamma_2 = 1 \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\gamma}.$$ For future reference, we also point out that

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \quad \Rightarrow \quad \frac{(p+1)(q+1)}{pq-1} = \frac{N}{2}. \quad (2.3)$$

and that a direct consequence of these definitions and Young’s inequality is that:

$$(\|f\|_\alpha \|g\|_\beta)^\gamma \leq \gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta \quad \text{for every } (f, g) \in X. \quad (2.4)$$

Let

$$L_{p,q} = \inf \{\Phi(f, g) : (f, g) \in X \setminus \{(0, 0)\}, \quad \Phi'(f, g) = 0\}.$$

**Lemma 2.2.** Let $(f, g) \in X$ be a critical point of $\phi$ and let $(u, v) := (K_p f, K_q g) = (|f|^\frac{p}{q-1} f, |g|^\frac{q}{q-1} g)$. Then:

1. $(u, v) \in W^{2,\frac{p}{q-1}}(\Omega) \times W^{2,\frac{q}{q-1}}(\Omega)$, and it is a strong solution of $(1.1)$;

2. $\phi(f, g) = I(u, v) := \int_{\Omega} \nabla u \cdot \nabla v - \frac{1}{p+1} |u|^{p+1} - \frac{1}{q+1} |v|^{q+1} dx$.

In particular, $(f, g)$ achieves $L_{p,q}$ if, and only if, $(u, v)$ is a least energy solution of $(1.1)$, and

$$c_{p,q} = L_{p,q}.$$

**Proof.** Item 1. follows by reasoning exactly as in from in [35, Lemma 2.3], while Item 2. follows from [35, Lemma 2.5]. $\square$
Therefore our goal, from now on, is to show that \( L_{p,q} \) is achieved. In order to show it, we use an equivalent variation formulation. Let us define

\[
D_{p,q} = \sup \left\{ \int_{\Omega} fKg : \quad (f,g) \in X, \quad \gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta = 1 \right\} \tag{2.5}
\]

Observe that \( D_{p,q} < \infty \) since for every \((f,g) \in X\) such that \( \gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta = 1 \) we have

\[
|\int_{\Omega} fKg| \leq \|f\|_\alpha \|Kg\|_{\alpha'} \leq C_1 \|f\|_\alpha \|Kg\|_{W^{2,\beta}} \leq C_2 \|f\|_\alpha \|g\|_\beta \leq C_2 (\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{\gamma}} = C_2
\]

where we have used (1.3), Lemma 2.1 and (2.4).

This level has the following equivalent characterizations, which are useful later on.

**Lemma 2.3.** We have

\[
D_{p,q} = \sup_{(f,g) \in X \setminus \{(0,0)\}} \frac{\int_{\Omega} fKg}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{\gamma}}} \tag{2.6}
\]

\[
= \sup_{(f,g) \in X \setminus \{(0,0)\}} \frac{\int_{\Omega} fKg}{\|f\|_\alpha \|g\|_\beta} \tag{2.7}
\]

**Proof.** Proof of (2.6): given \((f,g) \in X \setminus \{(0,0)\}\), take

\[
\tilde{f} := \frac{f}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{\gamma}}}, \quad \tilde{g} := \frac{g}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{\gamma}}}
\]

Then \( \gamma_1 \|\tilde{f}\|_\alpha^\alpha + \gamma_2 \|\tilde{g}\|_\beta^\beta = 1 \), and

\[
D_{p,q} \geq \int_{\Omega} \tilde{f}K\tilde{g} = \frac{\int_{\Omega} fKg}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{\gamma}}}.
\]

Taking the supremum in \( f,g \), we obtain one inequality in (2.6). Conversely, given \((\tilde{f}, \tilde{g}) \in X\) such that \( \gamma_1 \|\tilde{f}\|_\alpha^\alpha + \gamma_2 \|\tilde{g}\|_\beta^\beta = 1 \), we have

\[
\int_{\Omega} \tilde{f}K\tilde{g} = \frac{\int_{\Omega} fKg}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{\gamma}}} \leq \sup_{(f,g) \in X \setminus \{(0,0)\}} \frac{\int_{\Omega} fKg}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{\gamma}}}.
\]

**Proof of (2.7):** Being a positive supremum, clearly we may take in the characterization of \( D_{p,q} \) only pairs \((f,g)\) such that \( f,g \neq 0 \). Young’s inequality (2.4) immediately shows that

\[
D_{p,q} \leq \sup_{(f,g) \in X \setminus \{(0,0)\}} \frac{\int_{\Omega} fKg}{\|f\|_\alpha \|g\|_\beta}.
\]

To prove the inverse inequality, observe that if \( p = q \), then \( \alpha = \beta \) and it is obvious. Thus we focus on the case \( p \neq q \). We use as test function \((tf, tg)\) for \( t > 0 \) and \((f,g) \in X\), \( f,g \neq 0 \), and \( t > 0 \), obtaining:

\[
\frac{t^{2\gamma} \left( \int_{\Omega} fKg \right)^\gamma}{\gamma_1 t^\alpha \|f\|_\alpha^\alpha + \gamma_2 t^\beta \|g\|_\beta^\beta} \leq (D_{p,q})^\gamma
\]
Recall that $\gamma_1 + \gamma_2 = 1$ and that (since $p \neq q$), $\gamma_1, \gamma_2 \neq 1/2$. Therefore $(1 - 2\gamma_1)(1 - 2\gamma_2) < 0$, and $F$ has a strict local minimum attained at:

$$t_* := \left( \frac{\|f\|_\alpha^2}{\|g\|_\beta^2} \right)^{\frac{1}{n - \alpha}} \quad \text{with } F(t_*) = \|f\|_\alpha^2 \|g\|_\beta^2.$$ 

This concludes the proof. □

2.2 Sobolev constants

For $\eta > 1$, we denote by $\mathcal{D}^{2,\eta}(\mathbb{R}^N)$ the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm $\|\Delta u\|_{L^\eta(\mathbb{R}^N)}$. We define

$$S_{p,q} := \inf \left\{ \|\Delta u\|_{L^\eta(\mathbb{R}^N)} : u \in \mathcal{D}^{2,\frac{n+1}{\eta}}(\mathbb{R}^N), \|u\|_{L^{p+1}(\mathbb{R}^N)} = 1 \right\},$$

which is the best Sobolev constant for the embedding $\mathcal{D}^{2,\frac{n+1}{\eta}}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$, namely the best constant such that one has

$$S_{p,q} \|u\|_{L^{p+1}(\mathbb{R}^N)} \leq \|\Delta u\|_{L^\eta(\mathbb{R}^N)} \quad \text{for any } u \in \mathcal{D}^{2,\frac{n+1}{\eta}}(\mathbb{R}^N).$$

It is known [26, Corollary 1.2, p. 165] that $S_{p,q}$ is achieved. In particular, using also the relation [23], there exists $U$ nonnegative and radially decreasing such that

$$S_{p,q}^N = \|\Delta U\|_{L^\eta(\mathbb{R}^N)}^\frac{1}{\eta} = \|U\|_{L^{p+1}(\mathbb{R}^N)}^\frac{1}{p+1}$$

and

$$-\Delta(-\Delta U)^{\frac{1}{2}} = U^p \text{ in } \mathbb{R}^N.$$ 

Therefore, if $V := (-\Delta U)^{\frac{1}{2}}$, then

$$-\Delta U = V^q \text{ in } \mathbb{R}^N, \quad -\Delta V = U^p \text{ in } \mathbb{R}^N.$$ (2.10)

One can take $U, V$ to be positive and radially decreasing. Since $\|U\|_{L^{p+1}(\mathbb{R}^N)} = \|V\|_{L^{q+1}(\mathbb{R}^N)}$, then $S_{p,q} = S_{q,p}$, the best constant in the embedding $\mathcal{D}^{2,\frac{n+1}{q}}(\mathbb{R}^N) \hookrightarrow L^{q+1}(\mathbb{R}^N)$. Observe that, by the homogeneity of the nonlinearities and since $p, q$ satisfy (1.2), then we have a whole family of solutions to (2.10), namely

$$\left\{ \begin{array}{l}
U_{\epsilon,x_0}(x) = \epsilon^{-\frac{n+1}{p+1}} U \left( \frac{x-x_0}{\epsilon} \right) = \epsilon^{-\frac{n}{p+1}} U \left( \frac{x-x_0}{\epsilon} \right), \\
V_{\epsilon,x_0}(x) = \epsilon^{-\frac{n+1}{q+1}} V \left( \frac{x-x_0}{\epsilon} \right) = \epsilon^{-\frac{n}{q+1}} V \left( \frac{x-x_0}{\epsilon} \right)
\end{array} \right.$$ (2.11)

and

$$S_{p,q}^N = \|\Delta U_{\epsilon,x_0}\|_{L^\eta(\mathbb{R}^N)}^\frac{1}{\eta} = \|U_{\epsilon,x_0}\|_{L^{p+1}(\mathbb{R}^N)}^\frac{1}{p+1}.$$ 

It is proved in [19] that these are all the positive solutions of (2.10). We just mention that there exist also sign-changing solutions to (2.10), see [11]. Without loss of generality, we assume that

$$V(0) = 1, \quad \text{and that } q \leq \frac{N+2}{N-2}.$$ 

In the following we do a slight abuse of notation and use $w(|x|) = w(x)$ for a radial function $w$. By [19 Theorem 2], we have that there exists $a, b > 0$ such that

$$\lim_{r \to \infty} r^{N-2} V(r) = a, \quad \left\{ \begin{array}{ll}
\lim_{r \to \infty} r^{N-2} U(r) = b & \text{if } q > \frac{N}{N-2} \\
\lim_{r \to \infty} r^{\frac{N}{q} - 2} U(r) = b & \text{if } q = \frac{N}{N-2} \\
\lim_{r \to \infty} r^{q(N-2) - 2} U(r) = b & \text{if } q < \frac{N}{N-2}
\end{array} \right.$$ (2.12)
Based on these estimates, in [18, pp. 2360–2362] it is shown that, as \( \varepsilon \to 0^+ \):

\[
\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \approx \|U_\varepsilon^p\|_{L^1(\mathbb{R}^N)} \approx \varepsilon^{\frac{N}{p+1}}, \quad \|U_\varepsilon\|_{L^1(\mathbb{R}^N)} \approx \|V_\varepsilon^q\|_{L^1(\mathbb{R}^N)} \approx \left\{
\begin{array}{ll}
\varepsilon^{\frac{N}{q(N-2)}} \log \varepsilon & \text{if } \frac{N}{N-2} < q \leq \frac{N+2}{N-2}, \\
\varepsilon^{\frac{N}{q(N-2)}} & \text{if } q = \frac{N}{N-2}, \\
\varepsilon^{\frac{N}{q(N-2)}} & \text{if } q < \frac{N}{N-2}
\end{array}
\right.
\]

(2.13)

where \( f \approx g \) means that the quotient there exists \( C > 1 \) such that \( 1/C \leq f/g \leq C \). Notice that in [18] it is also assumed \( p, q > 1 \), however, an inspection of the proof shows that the estimates (2.13) are true without this restriction. We also point out, for future reference, that from [19, equations (3.22)–(3.24)] we have

\[
rU'(r) \approx U(r), \quad rV'(r) \approx V(r).
\]

(2.14)

We also need to introduce the best Sobolev constant \( S_* \) for the embedding \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \), namely

\[
S_* \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|\nabla u\|_{L^2(\mathbb{R}^N)}^2.
\]

Associated with it we have the family of *bubbles*:

\[
U_{\varepsilon,x_0}^*(x) := \varepsilon^{-\frac{N-2}{2}} U^* \left( \frac{x - x_0}{\varepsilon} \right), \quad \text{with} \quad U^*(x) := \frac{(N(N-2))^{\frac{N-2}{2}}}{(1 + |x|^2)^{\frac{N-2}{2}}}
\]

which are the unique solutions in \( D^{1,2}(\mathbb{R}^N) \) of

\[-\Delta U_{\varepsilon,x_0}^* = (U_{\varepsilon,x_0}^*)^{2^*-1} \text{ in } \mathbb{R}^N.\]

They satisfy

\[
\|\nabla U_{\varepsilon,x_0}^*\|_{L^2(\mathbb{R}^N)}^2 = \|U_{\varepsilon,x_0}^*\|_{L^{2^*}(\mathbb{R}^N)}^2 = S_*^{\frac{N}{2}}.
\]

**Remark 2.4.** Observe that, for

\[
p = q = 2^* - 1 = \frac{N + 2}{N - 2}, \quad \text{we have } U^* = U = V, \quad \text{and so } S_{\frac{N+2}{N-2}}^{\frac{2}{N-2}} = S_*.
\]

### 2.3 Equivalence between \( D_{p,q} \) and the least energy nodal level

**Proposition 2.5.** Assume \( D_{p,q} \) is attained at \( (f, g) \). Then \( (u, v) = ((D_{p,q})^{-\frac{\alpha}{N-1}} K_p g, (D_{p,q})^{-\frac{\alpha}{N-1}} K_q f) \) is a least energy solution for (1.1), and

\[
D_{p,q}^{-\frac{\alpha}{2}} = \frac{N}{2} L_{p,q} = \frac{N}{2} c_{p,q}.
\]

(2.15)

**Proof.** Assume that \( D_{p,q} \) is attained. Then there exist functions \( (f, g) \in X \) such that \( \gamma_1 \|f\|_{p}^{\alpha} + \gamma_2 \|g\|_{q}^{\alpha} = 1 \) and \( \int_{\Omega} f K g = D_{p,q} \). We now use Lagrange multipliers and the definition of \( \gamma \) to conclude that there exists \( \lambda \in \mathbb{R} \) such that

\[
\int_{\Omega} \varphi K g = \lambda \gamma \int_{\Omega} |f|^{\frac{N}{N-2}} \varphi, \quad \int_{\Omega} \psi K f = \lambda \gamma \int_{\Omega} |g|^{\frac{N}{N-2}} \psi
\]

(2.16)

for any \( (\varphi, \psi) \in X \). Now let \( \tilde{\varphi} \in L^\alpha(\Omega), \tilde{\psi} \in L^\beta(\Omega) \). Define

\[
\varphi = \tilde{\varphi} - \frac{1}{|\Omega|} \int_{\Omega} \tilde{\varphi}, \quad \psi = \tilde{\psi} - \frac{1}{|\Omega|} \int_{\Omega} \tilde{\psi}.
\]

Notice that \( (\varphi, \psi) \in X \). Then

\[
\int_{\Omega} \tilde{\varphi} K g - \frac{1}{|\Omega|} \int_{\Omega} K g \int_{\Omega} \tilde{\varphi} = \lambda \gamma \int_{\Omega} |f|^{\frac{N}{N-2}} \tilde{\varphi} - \lambda \gamma \frac{1}{|\Omega|} \int_{\Omega} |f|^{\frac{N}{N-2}} \int_{\Omega} \tilde{\varphi}.
\]
This implies, recalling that $\int_{\Omega} Kg = 0$ and $\bar{\phi}$ is arbitrary,

$$Kg + \kappa = \lambda \gamma |f|^{\frac{1}{p} - 1} f,$$

with $\kappa = \lambda \gamma \frac{1}{|\Omega|} \int_{\Omega} |f|^{\frac{1}{p} - 1} f.$

Since

$$\int_{\Omega} |Kg + \kappa|^{p - 1} (Kg + \kappa) = (\lambda \gamma)^p \int_{\Omega} f = 0,$$

then $\kappa = \kappa_p(Kg)$ (recall (2.2)), and

$$K_p g = Kg + \kappa = \lambda \gamma |f|^{\frac{1}{p} - 1} f.$$

Similarly,

$$K_q f = \lambda \gamma |g|^{\frac{1}{q} - 1} g.$$

Also, by (2.16),

$$\left\{ \begin{array}{l}
\int_{\Omega} f Kg = \lambda \gamma \|f\|_a^\alpha \\
\int_{\Omega} gKf = \lambda \gamma \|g\|_b^\beta
\end{array} \right.,$$

so that $D_{p,q} = \int_{\Omega} f Kg = \int_{\Omega} gKf = \lambda \gamma \|f\|_a^\alpha = \lambda \gamma \|g\|_b^\beta.$

Hence, since $\gamma_1 + \gamma_2 = 1$,

$$D_{p,q} = (\gamma_1 + \gamma_2)D_{p,q} = \lambda \gamma (\gamma_1 \|f\|_a^\alpha + \gamma_2 \|g\|_b^\beta)^{\frac{\gamma_1 - 1}{\gamma_1 + \gamma_2}} = \lambda \gamma,$$

from which we conclude

$$K_p g = D_{p,q} |f|^{\frac{1}{p} - 1} f \quad K_q f = D_{p,q} |g|^{\frac{1}{q} - 1} g.$$  

Now consider the functions $\bar{f} = (D_{p,q})^s f$, and $\bar{g} = (D_{p,q})^t g$, with $s = -p \frac{q+1}{pq-1}$ and $t = -q \frac{p+1}{pq-1}$. Then, since $K_p, K_q$ are homogeneous of degree 1,

$$K_p \bar{g} = |\bar{f}|^{\frac{1}{p} - 1} \bar{f} \quad K_q \bar{f} = |\bar{g}|^{\frac{1}{q} - 1} \bar{g}.$$  

In particular, $\Phi'(\bar{f}, \bar{g}) = 0$.

One has

$$L_{p,q} \leq \Phi(\bar{f}, \bar{g}) = \frac{p}{p+1} \|\bar{f}\|_a^\alpha + \frac{q}{q+1} \|\bar{g}\|_b^\beta - \int_{\Omega} \bar{f} K \bar{g}$$

$$= \frac{pq - 1}{(p+1)(q+1)} \int_{\Omega} \bar{f} K \bar{g} = \frac{pq - 1}{(p+1)(q+1)} D_{p,q}^{s+t+1} = \frac{pq - 1}{(p+1)(q+1)} D_{p,q}^{\frac{(p+1)(q+1)}{pq-1}}.$$

On the other hand, taking $(f, g) \in X \setminus \{(0, 0)\}$ any critical point of $\Phi$, then necessarily $f, g \not\equiv 0$ and $(\frac{f}{\|f\|_a}, \frac{g}{\|g\|_b})$ is a test function for $D_{p,q}$, and

$$D_{p,q} \geq \int_{\Omega} \frac{f}{\|f\|_a} K \frac{g}{\|g\|_b} = \frac{\int_{\Omega} f Kg}{\int_{\Omega} f Kg}$$

$$= \left( \int_{\Omega} f Kg \right)^{-\frac{pq - 1}{(p+1)(q+1)}} = \left( \frac{(p+1)(q+1)}{pq - 1} \Phi(f, g) \right)^{-\frac{pq - 1}{(p+1)(q+1)}}.$$

Taking the infimum in $(f, g)$ and comparing it with the opposite inequality, we obtain the identity: from which we conclude (2.15). Also, by (2.16) we have that $L_{p,q}$ is attained at $(\bar{f}, \bar{g})$. We can now conclude from Lemma 2.2 and using the identity (2.3). \qed

### 2.4 Regularity of solutions

In order to prove Proposition 1.3, we recall the following result
Proposition 2.6 (Proposition 9 in [14], Lemma 3.1 in [34]). Let \( f \) be such that \( \int_{\Omega} f = 0 \). Let \( G \) be the Green function for
\[
-\Delta u = f \quad \text{in } \Omega, \quad \partial_{\nu} u = 0 \quad \text{in } \partial \Omega,
\]
namely \( G : \Omega \times \Omega \setminus \{(x, x) : x \in \Omega\} \to \mathbb{R} \) is such that
(i) \( G(x, \cdot) \in L^1(\Omega) \),
(ii) \( \int_{\Omega} G(x, y) \, dy = 0 \),
(iii) for any \( u \) solution to (2.18) one has
\[
\left| u(x) - \frac{1}{|\Omega|} \int_{\Omega} u = \int_{\Omega} G(x, y) f(y) \, dy. \right|
\]
Then
\[
\left| G(x, y) \right| \leq C(\Omega) |x - y|^{2-N},
\]
for all \( x, y \in \Omega, x \neq y \).

Proof of Proposition 1.3. We take inspiration from [8, Theorem 1.3], see also [23, Lemma B.1]. Let us take
\[
r > \frac{N}{N-2}, \quad \frac{N}{Nq-2q-2},
\]
fixed but arbitrary. Also, we set
\[
s := \frac{Nrq}{N+2}, \quad t := \left( \frac{p+1}{pq} - 1 \right) Ns > 0, \quad t' := r q \frac{N+2s}{Ns} > 0.
\]
We preliminary notice that
\[
s > \frac{N}{N-2} (2.19)
\]
and
\[
\frac{1}{t} + \frac{1}{t'} = 1. (2.20)
\]
Indeed, (2.19) immediately follows recalling \( r > \frac{N}{Nq-2q-2} \), and using \( q > \frac{2}{N-2} \). As for (2.20), one observes that it is equivalent to prove
\[
\frac{N+2r+2rq}{Nrq} = \frac{N+2s}{Ns} = \frac{pq - 1}{q(p+1)} + \frac{1}{rq},
\]
namely
\[
\frac{2(q+1)}{N} = \frac{pq - 1}{p+1},
\]
which is satisfied if and only if \( p, q \) belong to the critical hyperbola [1.2].

Now, let us define
\[
u_L(x) := \begin{cases} u(x) & \text{if } |u(x)| > L \\ 0 & \text{if } |u(x)| \leq L. \end{cases}
\]
We take \( f \in L^r(\Omega) \), and define the operator \( T_L : L^r(\Omega) \to L^r(\Omega) \) such that
\[
T_L(f)(x) := \int_{\Omega} G(x, y) \left\{ \int_{\Omega} G(y, z) h(z) \, dz \right\}^{\frac{q-1}{q}} \left( \int_{\Omega} G(y, z) h(z) \, dz \right) \, dy,
\]
where
\[
h(z) := |u_L(z)|^{\frac{p-1}{2}} |f(z)|^{\frac{1}{2} - 1} f(z).
\]
By applying Proposition 2.6 and the Hardy–Littlewood–Sobolev inequality (recall that \( r > \frac{N}{N-2} \)), one has
\[
\|T_L f\|_s \leq C \left\| \left( \int_{\Omega} |G(y, z)| |u_L(z)|^{\frac{p-1}{2}} |f(z)|^{\frac{1}{2}} \, dz \right)^{\frac{q}{q-1}} \right\|^{\frac{q}{q-s}} = C \left\| \int_{\Omega} |G(y, z)| |u_L(z)|^{\frac{p-1}{2}} |f(z)|^{\frac{1}{2}} \, dz \right\|^{\frac{q}{q-s}},
\]
where $C > 0$ denotes a positive constant. We now apply once again Proposition 2.6 and Hardy–Littlewood–Sobolev inequality, this time using $s > \frac{N}{N-2}$, to get

$$\left\| \int_\Omega |G(y,z)||u_L(z)|^{p-\frac{1}{2}}|f(z)|^{\frac{q}{2}} \, dz \right\|_s \leq C \left\| u_L \right\|_{L^p}^{\frac{p}{2}} \left\| f \right\|_{L^q}^{\frac{q}{2}}.$$

The Hölder inequality with exponents $t, t'$ yields

$$\left\| u_L \right\|_{L^{p^t}} \left\| f \right\|_{L^{q^{t'}}} \leq \left\| u_L \right\|_{L^{p^{t}}} \left\| f \right\|_{L^{q^{t'}}} = \left\| u_L \right\|_{L^{p^{t}}} \left\| f \right\|_{L^{q^{t'}}} = \left\| u_L \right\|_{L^{p^{t}}} \left\| f \right\|_{L^{q^{t'}}},$$

from which we deduce

$$\| T_L f \|_r \leq C \| u_L \|_{L^{p+1}} \| f \|_r.$$

Since $u \in L^{p+1}(\Omega)$, we can choose $L$ large enough such that $\| u_L \|_{p+1}$ is small enough. This implies that $T_L(f)$ is a contraction mapping from $L^r(\Omega)$ into itself. Moreover,

$$u_L = T_L(u_L) + F, \quad \text{where } F \text{ is uniformly bounded.}$$

Thus, by the contraction mapping theorem, see Theorem 1 in [7], one has $u_L \in L^r(\Omega)$. Since $r$ is arbitrary, we conclude that $u \in L^\infty(\Omega)$.

Notice that the same arguments give $v \in L^\infty(\Omega)$. Thus, $|u|^{p-1}u \in L^{r'/4}(\Omega)$ for any $r > p$, whereas $|v|^{q-1}v \in L^{r'/4}(\Omega)$ for any $r > q$. This immediately gives $v \in W^{2,r}(\Omega)$ for any $r > p$, and $u \in W^{2,r}(\Omega)$ for any $r > q$. We now apply Sobolev embeddings to get $u, v \in C^{1,\gamma}(\Omega)$ for any $\gamma \in (0, 1)$. Schauder regularity theory [10] Theorem 6.31 yields $(u, v) \in C^{2,\zeta}(\Omega) \times C^{2,\eta}(\Omega)$, where $\zeta, \eta$ are as in the statement. 

**Remark 2.7.** Actually if $p, q \geq 1$ one can conclude $u, v \in C^\infty(\overline{\Omega})$ by a bootstrap argument. Notice that the proof above also shows regularity for the Dirichlet system

$$\begin{cases}
-\Delta u = |v|^{q-1}v & \text{in } \Omega \\
-\Delta v = |u|^{p-1}u & \text{in } \Omega \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}$$

with $p, q$ on the critical hyperbola (1.2).

### 3 A compactness condition to have $D_{p,q}$ attained

This Section is devoted to the proof of the following lemma.

**Lemma 3.1 (Compactness condition).** Let $D_{p,q}$ and $S_{p,q}$ be as in (2.5) and (2.8), respectively. If $p, q$ satisfies (1.2) and

$$D_{p,q} > \frac{2^{2/N}}{S_{p,q}},$$

then $D_{p,q}$ is attained.

**Proof.** Let us define, for any $(f, g) \in X$,

$$F(f, g) = \int_\Omega f K g = \int_\Omega g K f \quad \text{and} \quad H(f, g) = \gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta.$$

Take a maximizing sequence $(f_k, g_k)$ for $D_{p,q}$. By Ekeland’s variational principle, we find $(\tilde{f}_k, \tilde{g}_k) \in X$ and $\lambda_k \in \mathbb{R}$ such that

$$\gamma_1 \|\tilde{f}_k\|_\alpha^\alpha + \gamma_2 \|\tilde{g}_k\|_\beta^\beta = 1, \quad (3.1)$$

$$\|f_k - \tilde{f}_k\|_\alpha \to 0, \quad \|g_k - \tilde{g}_k\|_\beta \to 0, \quad F(\tilde{f}_k, \tilde{g}_k) \to D_{p,q},$$

and

$$F'(\tilde{f}_k, \tilde{g}_k) - \lambda_k H'(\tilde{f}_k, \tilde{g}_k) \to 0 \text{ in } X^*.$$
Up to a subsequence, there exist \((f,g) \in X\) such that
\[
\tilde{f}_k \to f \text{ weakly in } L^\alpha(\Omega), \quad \tilde{g}_k \to g \text{ weakly in } L^\beta(\Omega).
\] (3.2)

Our main goal is to prove that this convergence is strong, after which it is straightforward to see that \((f,g)\) achieves \(D_{p,q}\). Since the proof is long, we split it in several steps.

**First step.** We show that \(\tilde{f}_k \to f\) and \(\tilde{g}_k \to g\) almost everywhere. For any \(\varphi \in L^\alpha(\Omega)\), define
\[
\hat{\varphi} = \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \in X^\alpha
\]
and notice that \(\|\hat{\varphi}\|_\alpha \leq 2\|\varphi\|_\alpha\). Let \(c_k := \frac{\gamma_k}{\lambda_k} \int_\Omega |\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k\), so that
\[
\int_\Omega \varphi \tilde{g}_k + \int_\Omega c_k \varphi - \gamma \lambda_k \int_\Omega |\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k \varphi = \int_\Omega \hat{\varphi} \tilde{g}_k - \gamma \lambda_k \int_\Omega |\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k \hat{\varphi}
\]
\[
\leq \left\|\partial_t F(\tilde{f}_k, \tilde{g}_k) - \lambda_k \partial_t H(\tilde{f}_k, \tilde{g}_k)\right\|_{(X^\alpha)}, \quad \|\hat{\varphi}\|_\alpha \leq 2 \left\|\partial_t F(\tilde{f}_k, \tilde{g}_k) - \lambda_k \partial_t H(\tilde{f}_k, \tilde{g}_k)\right\|_{(X^\alpha)},
\]
Hence
\[
\left\|K \tilde{g}_k + c_k - \lambda_k \gamma |\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k\right\|_{p+1} = \left\|K \tilde{g}_k + c_k - \lambda_k \gamma |\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k\right\|_{(L^\alpha)}
\]
\[
= \sup_{\varphi \in L^\alpha(\Omega) \setminus \{0\}} \left|\int_\Omega \varphi \tilde{g}_k + \int_\Omega c_k \varphi - \gamma \lambda_k \int_\Omega |\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k \varphi\right| \|\varphi\|_\alpha
\]
\[
\leq 2 \left\|\partial_t F(\tilde{f}_k, \tilde{g}_k) - \lambda_k \partial_t H(\tilde{f}_k, \tilde{g}_k)\right\|_{(X^\alpha)} \to 0.
\]
In particular, up to a subsequence,
\[
K \tilde{g}_k + c_k - \lambda_k \gamma |\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k \text{ converges a.e. in } \Omega.
\]

All terms in this expression converge a.e. Indeed, up to subsequences:

- \(K \tilde{g}_k\) converges a.e. This is because, by Lemma 2.1 and (3.1), we have that \(\|K \tilde{g}_k\|_{W^{2,\alpha}} \leq C \|\tilde{g}_k\|_\beta \leq \tilde{C}\) for every \(k\). Therefore, by the compact embeddings \(W^{2,\beta}(\Omega) \hookrightarrow L^t(\Omega)\) for every \(t \in [1, p + 1)\), we have the existence of \(w \in L^{p+1}(\Omega)\) such that
  \[
  K \tilde{g}_k \to w \quad \text{strongly in } L^t(\Omega) \text{ for any } 1 \leq t < p + 1, \text{ pointwisely a.e.}
  \]

- \(\lambda_k \to \frac{1}{t} D_{p,q} \neq 0\), since we have
  \[
  o(1) = F'(\tilde{f}_k, \tilde{g}_k) \left(\gamma_1 \tilde{f}_k, \gamma_2 \tilde{g}_k\right) - \lambda_k H'(\tilde{f}_k, \tilde{g}_k) \left(\gamma_1 \tilde{f}_k, \gamma_2 \tilde{g}_k\right)
  \]
  \[
  = (\gamma_1 + \gamma_2) D_{p,q} - \lambda_k \gamma (\gamma_1 \|\tilde{f}_k\|_\alpha^p + \gamma_2 \|\tilde{g}_k\|_\beta^p) + o(1)
  \]
  \[
  = D_{p,q} - \lambda_k \gamma + o(1),
  \]
  (where we recall that \(\gamma = \gamma_1 \alpha = \gamma_2 \beta\) and \(\gamma_1 + \gamma_2 = 1\)).

- there exists a constant \(c_0 \in \mathbb{R}\) such that, up to a subsequence \(c_k \to c_0\) since, by (3.1) and the previous paragraph, we have that \(c_k\) is bounded.

Combining all of this we deduce that \(|\tilde{f}_k|^{\frac{1}{p} - 1} \tilde{f}_k\) converges a.e. and so also \(\tilde{f}_k\). Together with \(3.2\), this yields that \(\tilde{f}_k \to f\) a.e. in \(\Omega\). Similarly one proves that also \(\tilde{g}_k \to g\) a.e.

**Second step.** \(\tilde{f}_k \to f\) strongly in \(L^\alpha(\Omega)\), \(\tilde{g}_k \to g\) strongly in \(L^\beta(\Omega)\). We write
\[
\tilde{f}_k = f + w_k \quad \text{and} \quad \tilde{g}_k = g + z_k,
\]
where $w_k \to 0$ in $L^\alpha(\Omega)$, and $z_k \to 0$ in $L^\beta(\Omega)$. By Brézis-Lieb’s Lemma, we have

$$\gamma_1 \|f\|_\alpha + \gamma_1 \|w_k\|_\alpha + \gamma_2 \|g\|_\beta + \gamma_2 \|z_k\|_\beta = \gamma_1 \|\hat{f}_k\|_\alpha + \gamma_2 \|\hat{g}_k\|_\beta + o(1) = 1 + o(1).$$

In particular, since $\gamma < 1$ (which is equivalent to $pq > 1$), then

$$\left(\gamma_1 \|f\|_\alpha + \gamma_2 \|g\|_\beta\right)^{1/\gamma} + \left(\gamma_1 \|w_k\|_\alpha + \gamma_2 \|z_k\|_\beta\right)^{1/\gamma} \leq 1 + o(1). \quad (3.3)$$

Now,

$$D_{p,q} + o(1) = \int_\Omega \hat{g}_k K\hat{f}_k = \int_\Omega g K\hat{f}_k + \int_\Omega z_k K\hat{f}_k = \int_\Omega g K f + \int_\Omega g K w_k + \int_\Omega z_k K f + \int_\Omega z_k K w_k$$

$$= \int_\Omega g K f + \int_\Omega z_k K w_k + o(1). \quad (3.4)$$

By definition of $D_{p,q}$, we can estimate the right-hand side of (3.4) as follows

$$\int_\Omega g K f + \int_\Omega z_k K w_k + o(1) \leq D_{p,q}(\gamma_1 \|w_k\|_\alpha + \gamma_2 \|z_k\|_\beta)^{1/\gamma} + \int_\Omega z_k K w_k + o(1).$$

The left-hand side of (3.4) can be estimated using (3.3) as

$$D_{p,q} + o(1) \geq D_{p,q}(\gamma_1 \|w_k\|_\alpha + \gamma_2 \|z_k\|_\beta)^{1/\gamma} + D_{p,q}(\gamma_1 \|f\|_\alpha + \gamma_2 \|g\|_\beta)^{1/\gamma} + o(1).$$

From this we deduce that

$$D_{p,q}(\gamma_1 \|w_k\|_\alpha + \gamma_2 \|z_k\|_\beta)^{1/\gamma} \leq \int_\Omega z_k K w_k + o(1). \quad (3.5)$$

Notice that $\|K w_k\|_{W^{1,\alpha}} \to 0$. Indeed, by continuity of $K$ (Lemma 2.1) one has $K w_k \to 0$ weakly in $W^{2,\alpha}(\Omega)$ and the conclusion follows by compactness of the embedding $W^{2,\alpha}(\Omega) \hookrightarrow W^{1,\alpha}(\Omega)$. Also,

$$K w_k \in W^{2,\alpha}(\Omega) = \{u \in W^{2,\alpha}(\Omega) : u_{\nu} = 0 \text{ on } \partial \Omega\},$$

by definition of $K$. Assume without loss of generality that

$$q \geq \frac{N + 2}{N - 2},$$

(the other case follows identically since $S_{p,q} = S_{q,p}$). Since $N > 2^{q+1}/q$, we can apply the Cherrier’s type inequality (A.1) with $u = \hat{z}_k$ and $\eta = \frac{q+1}{q}$ to obtain

$$\|K \hat{z}_k\|_{p+1} \leq \left(\frac{2^2/N}{S_{p,q}} + \varepsilon\right) \|\hat{z}_k\|_{\frac{q+1}{q}} + C(\varepsilon) \|K \hat{z}_k\|_{W^{1,\frac{q+1}{q}}},$$

where $S_{p,q}$ is the best Sobolev constant for the embedding $D^{2,\frac{q+1}{q}}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$, recall (2.8).

We obtain

$$\int_\Omega z_k K w_k = \int_\Omega w_k K z_k \leq \|w_k\|_\alpha \|K z_k\|_{p+1} \leq \|w_k\|_\alpha \left(\frac{2^2/N}{S_{p,q}} + \varepsilon\right) \|z_k\|_{\frac{q+1}{q}} + o(1)$$

$$= \left(\frac{2^2/N}{S_{p,q}} + \varepsilon\right) \|w_k\|_\alpha \|z_k\|_{\beta} + o(1)$$

$$\leq \left(\frac{2^2/N}{S_{p,q}} + \varepsilon\right) \left(\gamma_1 \|w_k\|_\alpha + \gamma_2 \|z_k\|_{\beta}\right)^{1/\gamma} + o(1), \quad (3.6)$$

where in the last inequality we used (2.4) (a consequence of Young’s inequality). Combining (3.5) with (3.6) yields

$$\left(\gamma_1 \|w_k\|_\alpha + \gamma_2 \|z_k\|_{\beta}\right)^{1/\gamma} \left(D_{p,q} - \frac{2^2/N}{S_{p,q}} - \varepsilon\right) \leq o(1). \quad (3.7)$$
Hence, if
\[ 1 > \frac{2^{2/N}}{S_{p,q} D_{p,q}} \]
holds true, then we have, by taking \( \varepsilon \) small enough, that
\[ \| w_k \|_\alpha \to 0 \quad \text{and} \quad \| z_k \|_\beta \to 0, \]
whence \( \tilde{f}_k \to f \) strongly in \( L^\alpha(\Omega) \) and \( \tilde{g}_k \to g \) strongly in \( L^\beta(\Omega) \).

**Third step.** \( D_{p,q} \) is attained in \((f, g)\). Indeed, by Lemma 2.1, we have that also \( K \tilde{g}_k \to Kg \) strongly in \( L^{p+1} \), thus
\[ \int_{\Omega} \tilde{f}_k K \tilde{g}_k \to \int_{\Omega} fKg = D_{p,q}. \]
Notice that \((f, g) \in X\), and, by passing to the limit in (3.1), we also have
\[ \gamma_1 \| f \|_\alpha^\alpha + \gamma_2 \| g \|_\beta^\beta = 1, \]
which concludes the proof.

**4 Proof of Theorem 1.2**

Throughout this Section we assume that \( q \leq p \) satisfy the assumptions of Theorem 1.2 which, in this case, are equivalent to
\[ p, q \text{ satisfy } (1.2), \quad N \geq 6 \quad \text{and} \quad \frac{N + 2}{2(N - 2)} < q \leq \frac{N + 2}{N - 2} \tag{4.1} \]
or
\[ p, q \text{ satisfy } (1.2), \quad N = 5 \quad \text{and} \quad \frac{17}{13} < q \leq \frac{7}{3} \tag{4.2} \]
or
\[ p, q \text{ satisfy } (1.2), \quad N = 4 \quad \text{and} \quad \frac{7}{3} < q \leq 3. \tag{4.3} \]

Recall from Subsection 2.2 that we denote by \((U, V) \in W^{2,\beta}(\mathbb{R}^N) \times W^{2,\alpha}(\mathbb{R}^N)\) the radially decreasing solution of (2.10), which satisfies the identity (2.9) and the decay estimates (2.12).

From now on, we choose \( x_0 \) to be a point on the boundary \( \partial \Omega \) such that the curvature is positive.

Without loss of generality, up to a rotation and a translation, we assume that
\[ x_0 = 0 \in \partial \Omega, \]
and that for a small \( \eta > 0 \) we have
\[ \Omega \cap B_\eta(0) = \{(x', x_N) : x_N > \rho(x')\}, \quad \rho(x') = \sum_{j=1}^{N-1} \rho_j x_j^2 + O(|x'|^3). \tag{4.4} \]

Therefore, for \( x \in B_\eta(0) \cap \partial \Omega \),
\[ x_N = \sum_{j=1}^{N-1} \rho_j x_j^2 + O(|x'|^3), \quad \text{and the mean curvature at } 0 \text{ is} \quad H(0) = \frac{2 \sum_{j=1}^{N-1} \rho_j}{N - 1} > 0. \tag{4.5} \]

Here and henceforth we denote \( x' = (x_1, \ldots, x_{N-1}) \). An outward normal close to 0 has the expression
\[ (\nabla \rho(x'), -1) = (2 \rho_1 x_1 + O(|x'|^2), \ldots, 2 \rho_{N-1} x_{N-1} + O(|x'|^2), -1), \]
Hence the unitary exterior normal has the form:
\[ \nu(x) = \frac{(2 \rho_1 x_1 + O(|x'|^2), \ldots, 2 \rho_{N-1} x_{N-1} + O(|x'|^2), -1)}{\sqrt{1 + O(|x'|^2)}} \quad \text{as} \quad x' \to 0. \tag{4.6} \]
Recalling (2.11), we take, for \( \varepsilon > 0 \), the scalings of \((U, V)\) centered at \( x_0 = 0 \):

\[
U_{\varepsilon}(x) := U_{\varepsilon,0}(x) = \varepsilon^{-\frac{N}{\alpha}} U \left( \frac{x}{\varepsilon} \right), \quad V_{\varepsilon}(x) := V_{\varepsilon,0}(x) = \varepsilon^{-\frac{N}{\alpha}} V \left( \frac{x}{\varepsilon} \right),
\]

which belong to \( W^{2,\beta}(\Omega) \times W^{2,\alpha}(\Omega) \) and solve

\[
-\Delta U_{\varepsilon} = V_{\varepsilon}^q, \quad -\Delta V_{\varepsilon} = U_{\varepsilon}^p \quad \text{in } \mathbb{R}^N.
\]

Let us define

\[
\tilde{U}_{\varepsilon} = U_{\varepsilon}^p + W_{\varepsilon}, \quad \tilde{V}_{\varepsilon} = V_{\varepsilon}^q + Z_{\varepsilon},
\]

where

\[
W_{\varepsilon} := -\frac{1}{|\Omega|} \int_{\Omega} U_{\varepsilon}^p, \quad Z_{\varepsilon} := -\frac{1}{|\Omega|} \int_{\Omega} V_{\varepsilon}^q.
\]

This way, \((\tilde{U}_{\varepsilon}, \tilde{V}_{\varepsilon}) \in X\) and we can, in particular, apply the operator \( K \) to each component.

We prove the following result below.

**Proposition 4.1.** Assume \( p, q \) satisfy (4.1), (4.2) or (4.3). Then, there exists a positive constant \( c \) such that one has

\[
\int_{\Omega} \tilde{U}_{\varepsilon} K \tilde{V}_{\varepsilon} \geq \frac{2\hat{S}_{p,q}}{S_{p,q}} + c \varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0^+.
\]

Notice that, once Proposition 4.1 is proved, then we may conclude the following.

**Proof of Theorem 1.2.** Under the assumptions of Theorem 1.2, assume \( q \leq p \). Then (4.1), (4.2) or (4.3) holds true, and from (2.7) and Proposition 4.1 we have

\[
D_{p,q} = \sup_{(f,g) \in X} \frac{\int_{\Omega} fKg}{\|f\|_p \|g\|_q} \geq \frac{\int_{\Omega} \tilde{U}_{\varepsilon} K \tilde{V}_{\varepsilon}}{\|\tilde{U}_{\varepsilon}\|_p \|\tilde{V}_{\varepsilon}\|_q} > \frac{2\hat{S}_{p,q}}{S_{p,q}} + \frac{c}{2} \varepsilon > \frac{2^{2/N}}{S_{p,q}}
\]

by taking \( \varepsilon \) small enough. The same is true in case \( p \leq q \), by exchanging the roles of \( p \) and \( q \), and \( U \) and \( V \). In conclusion, Theorem 1.2 is proved by recalling Lemma 3.1. Regularity follows from Proposition 4.1.

In order to prove Proposition 4.1, we generalize to systems the ideas from [13] and use estimates in [18, 19]. We preliminary define \( \Phi_{\varepsilon} \) and \( \Psi_{\varepsilon} \) in the following way:

\[
K \Phi_{\varepsilon} = V_{\varepsilon} + \Psi_{\varepsilon}, \quad K \Psi_{\varepsilon} = U_{\varepsilon} + \Phi_{\varepsilon},
\]

so that, in particular,

\[
\int_{\Omega} (U_{\varepsilon} + \Phi_{\varepsilon}) = 0, \quad \partial_\nu (U_{\varepsilon} + \Phi_{\varepsilon}) = 0 \text{ on } \partial \Omega,
\]

and

\[
\int_{\Omega} (V_{\varepsilon} + \Psi_{\varepsilon}) = 0, \quad \partial_\nu (V_{\varepsilon} + \Psi_{\varepsilon}) = 0 \text{ on } \partial \Omega.
\]

Equivalently, \( \Phi_{\varepsilon} \) and \( \Psi_{\varepsilon} \) satisfy

\[
\begin{cases}
-\Delta \Phi_{\varepsilon} = Z_{\varepsilon} & \text{in } \Omega \\
\partial_\nu \Phi_{\varepsilon} = -\partial_\nu U_{\varepsilon} & \text{on } \partial \Omega
\end{cases}
\quad \text{and} \quad
\begin{cases}
-\Delta \Psi_{\varepsilon} = W_{\varepsilon} & \text{in } \Omega \\
\partial_\nu \Psi_{\varepsilon} = -\partial_\nu V_{\varepsilon} & \text{on } \partial \Omega
\end{cases}
\]

\[
\int_{\Omega} \Phi_{\varepsilon} = -\int_{\Omega} U_{\varepsilon} \quad \text{and} \quad \int_{\Omega} \Psi_{\varepsilon} = -\int_{\Omega} V_{\varepsilon}.
\]

The proof of Proposition 4.1 is split into several lemmas.

**Lemma 4.2.** One has

\[
\int_{\Omega} \tilde{U}_{\varepsilon} K \tilde{V}_{\varepsilon} = \int_{\Omega} U_{\varepsilon}^{p+1} + \int_{\Omega} V_{\varepsilon}^{q+1} - \int_{\Omega} \nabla U_{\varepsilon} \cdot \nabla V_{\varepsilon} + \int_{\Omega} \nabla \Psi_{\varepsilon} \cdot \nabla \Phi_{\varepsilon} + \int_{\Omega} V_{\varepsilon} Z_{\varepsilon} + \int_{\Omega} W_{\varepsilon} U_{\varepsilon}.
\]

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Proof. Using the definitions of $\hat{U}_\varepsilon$ and $\Phi_\varepsilon$, one gets
\[
\int_\Omega \hat{U}_\varepsilon K \hat{V}_\varepsilon = \int_\Omega \hat{U}_\varepsilon (U_\varepsilon + \Phi_\varepsilon) = \int_\Omega U_\varepsilon^p + \int_\Omega U_\varepsilon^p \Phi_\varepsilon + \int_\Omega W_\varepsilon U_\varepsilon + \int_\Omega W_\varepsilon \Phi_\varepsilon.
\]
Also, by using the definition of the operator $K$,
\[
\int_\Omega U_\varepsilon^p \Phi_\varepsilon + \int_\Omega W_\varepsilon \Phi_\varepsilon = \int_\Omega (U_\varepsilon^p + W_\varepsilon) \Phi_\varepsilon = \int_\Omega \nabla K (U_\varepsilon^p + W_\varepsilon) \cdot \nabla \Phi_\varepsilon
\]
\[= \int_\Omega \nabla \left( K\hat{U}_\varepsilon - V_\varepsilon + V_\varepsilon \right) \cdot \nabla \Phi_\varepsilon = \int_\Omega \nabla \Psi_\varepsilon \cdot \nabla \Phi_\varepsilon + \int_\Omega \nabla V_\varepsilon \cdot \nabla \Phi_\varepsilon
\]
\[= \int_\Omega \nabla \Psi_\varepsilon \cdot \nabla \Phi_\varepsilon + \int_\Omega V_\varepsilon \cdot \nabla \hat{K} \hat{V}_\varepsilon - \int_\Omega \nabla U_\varepsilon \cdot \nabla V_\varepsilon
\]
\[= \int_\Omega \nabla \Psi_\varepsilon \cdot \nabla \Phi_\varepsilon + \int_\Omega V_\varepsilon \cdot \nabla \hat{K} \hat{V}_\varepsilon - \int_\Omega \nabla U_\varepsilon \cdot \nabla V_\varepsilon,
\]
which yields the conclusion. \(\Box\)

Lemma 4.3. Assume $p, q$ satisfy (4.1), (4.2) or (4.3). Then the following estimates hold
\[
\int_\Omega V_\varepsilon Z_\varepsilon, \int_\Omega W_\varepsilon U_\varepsilon = o(\varepsilon) \quad \text{as } \varepsilon \to 0^+.
\]
Proof. Notice that, using the estimates (2.13),
\[
\left| \int_\Omega V_\varepsilon Z_\varepsilon \right| \leq \|Z_\varepsilon\|_{\infty} \|V_\varepsilon\|_1 = \frac{1}{|\Omega|} \|V_\varepsilon\|_1 \|V_\varepsilon^q\|_1 \approx \begin{cases}
\varepsilon^N & \frac{N}{2} \leq q < N - 1 \\
\varepsilon^{N-1} & q = N - 1 \\
\varepsilon^{-1} & q > N - 1 \end{cases}
\]
Observe that, since $N \geq 4$ and $q > \frac{N+2}{2(N-2)} \geq \frac{3}{N-2}$, in all three cases the quantities are $o(\varepsilon)$. For
\[
\left| \int_\Omega U_\varepsilon W_\varepsilon \right| \leq \frac{1}{|\Omega|} \|U_\varepsilon\|_1 \|U_\varepsilon^p\|_1,
\]
using again (2.13) we obtain the exact same decays, hence the proof is complete. \(\Box\)

Lemma 4.4. Assume $p, q$ satisfy (4.1), (4.2) or (4.3). One has
\[
\int_\Omega \nabla \Phi_\varepsilon \cdot \nabla \Psi_\varepsilon = o(\varepsilon) \quad \text{as } \varepsilon \to 0^+.
\]
Proof. We first notice that
\[
\left| \int_\Omega \nabla \Phi_\varepsilon \cdot \nabla \Psi_\varepsilon \right| \leq \left( \int_\Omega \|\nabla \Phi_\varepsilon\|^2 \right)^{1/2} \left( \int_\Omega \|\nabla \Psi_\varepsilon\|^2 \right)^{1/2}.
\]
Recall that
\[
\left\{ u \in H^1(\Omega) : \int_\Omega u = 0 \right\} \leftrightarrow L^{2(N-1)/(N-2)}(\partial \Omega)
\]
in the sense of traces, see for instance [11 Theorem 5.36] (Trace Inequality) combined with [25 Theorem 8.11] (Poincaré’s inequality). Let us start with the case $N \geq 5$.

Step 1. Estimate of $\left( \int_\Omega \|\nabla \Phi_\varepsilon\|^2 \right)^{1/2}$.

By integrating by parts,
\[
\int_\Omega \|\nabla \Phi_\varepsilon\|^2 = -\frac{1}{|\Omega|} \int_\Omega \|V_\varepsilon^q\|_1 \int_\Omega \Phi_\varepsilon + \int_{\partial \Omega} \frac{\partial \Phi_\varepsilon}{\partial \nu} \left( \Phi_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \Phi_\varepsilon + \frac{1}{|\Omega|} \int_\Omega \Phi_\varepsilon \right)
\]
Indeed:

where $h$ is defined in (B.1).

This observation, combined with Lemma B.1— which provides the estimates of $\|\partial_\nu \Phi\|_{L^{2(N-1)}(\partial\Omega)}$— and (2.13), gives

We now point out that

\[
\left| \int_{\partial\Omega} \partial_\nu U_\varepsilon \right| \leq c \left\| \partial_\nu U_\varepsilon \right\|_{L^{2(N-1)}(\partial\Omega)}.
\]

This observation, combined with Lemma B.1— which provides the estimates of $\|\partial_\nu U_\varepsilon\|_{L^{2(N-1)}(\partial\Omega)}$— and (2.13), gives

\[
\int_\Omega |\nabla \Phi_\varepsilon|^2 \leq \begin{cases} 
  h_q(\varepsilon) \left( \left( \int_\Omega |\nabla \Phi_\varepsilon|^2 \right)^{1/2} + \varepsilon^{\frac{N}{p+1}} + \varepsilon^{\frac{2N}{q+1}} \right) & \text{if } q > \frac{N}{N-2}, \\
  h_q(\varepsilon) \left( \left( \int_\Omega |\nabla \Phi_\varepsilon|^2 \right)^{1/2} + \varepsilon^{\frac{N(N-2)}{2(N-1)}} |\log \varepsilon|^{\frac{N}{q+1}} \right) + \varepsilon^{\frac{N(N-2)}{2(N-1)}} |\log \varepsilon|^2 & \text{if } q = \frac{N}{N-2}, \\
  h_q(\varepsilon) \left( \left( \int_\Omega |\nabla \Phi_\varepsilon|^2 \right)^{1/2} + \varepsilon^{\frac{N}{p+1}} + \varepsilon^{\frac{2N}{q+1}} \right) & \text{if } q < \frac{N}{N-2},
\end{cases}
\]

where $h_q(\varepsilon)$ is defined in (B.1).

We claim that

\[
\left( \int_\Omega |\nabla \Phi_\varepsilon|^2 \right)^{1/2} = \begin{cases} 
  O\left( e^{-\frac{N}{p+1} + \frac{N}{q+1}} \right) & \text{if } q > \frac{N+4}{2(N-2)}, \\
  O\left( e^{-\frac{N}{p+1} + \frac{N}{q+1}} |\log \varepsilon|^{\frac{N}{q+1}} \right) & \text{if } q > \frac{N+4}{2(N-2)}, \\
  O\left( e^{-\frac{N}{p+1} + q(N-2)^{-2}} \right) & \text{if } q < \frac{N+4}{2(N-2)}.
\end{cases}
\]

Indeed:

- If $q > \frac{N}{N-2}$, then

\[
\int_\Omega |\nabla \Phi_\varepsilon|^2 \leq C \left( e^{-\frac{N}{p+1} + \frac{N}{q+1}} \left( \int_\Omega |\nabla \Phi_\varepsilon|^2 \right)^{1/2} + e^{-\frac{N}{p+1} + \frac{N}{q+1}} + e^{\frac{2N}{q+1}} \right),
\]

thus, since $-\frac{N}{p+1} + \frac{N}{2} < -\frac{N}{p+1} + \frac{N}{q+1} + \frac{2N}{q+1}$,

\[
\left( \int_\Omega |\nabla \Phi_\varepsilon|^2 \right)^{1/2} \leq C e^{-\frac{N}{p+1} + \frac{N}{q+1}}.
\]

- If $q = \frac{N}{N-2}$, then computations are analogous since we use the same estimate as before from (B.1), and we still get (4.8).

- For the case $\frac{N+4}{2(N-2)} < q < \frac{N}{N-2}$, we preliminary notice that

\[
q + 1 > p + 1 \quad \text{if and only if} \quad q > \frac{N + 4}{2(N - 2)}.
\]

Then

\[
\int_\Omega |\nabla \Phi_\varepsilon|^2 \leq C \left( e^{-\frac{N}{p+1} + \frac{N}{q+1}} \left( \int_\Omega |\nabla \Phi_\varepsilon|^2 \right)^{1/2} + e^{\frac{N}{p+1} + \frac{N}{q+1}} + e^{\frac{2N}{q+1}} \right),
\]

and again (4.8) holds.
In conclusion, claim (4.7) holds true.

Analogously, using Lemma B.2, from which estimates (B.2) and (2.13), provides

\[ \left( \int_{\Omega} |\nabla \Phi_{\varepsilon}|^2 \right)^{1/2} \leq C e^{-\frac{N}{2(\varepsilon+1)}} |\log \varepsilon|^{\frac{N}{2(\varepsilon+1)}} \]  

which gives

\[ \left( \int_{\Omega} |\nabla \Phi_{\varepsilon}|^2 \right)^{1/2} \leq C e^{-\frac{N}{2(\varepsilon+1)}} |\log \varepsilon|^{\frac{N}{2(\varepsilon+1)}}. \]

Finally, let us consider \( q < \frac{N+4}{2(N-2)} \). In this case,

\[ \left( \int_{\Omega} |\nabla \Phi_{\varepsilon}|^2 \right)^{1/2} \leq C e^{-\frac{N}{2(\varepsilon+1)}} + \frac{2N}{\varepsilon+1}. \]

In conclusion, claim (4.7) holds true.

Step 2. Estimate of \( \left( \int_{\Omega} |\nabla \Psi_{\varepsilon}|^2 \right)^{1/2} \).

Following the same proof above, we have

\[ \left( \int_{\Omega} |\nabla \Psi_{\varepsilon}|^2 \right)^{1/2} \leq \frac{1}{|\Omega|} \| U_{\varepsilon} \|_1 \| V_{\varepsilon} \|_1 + \frac{1}{|\Omega|} \| V_{\varepsilon} \|_1 \int_{\partial \Omega} \partial_{\nu} V_{\varepsilon} + C \| \partial_{\nu} \Psi_{\varepsilon} \|_{L^{2(\varepsilon+1)}(\partial \Omega)} \left( \int_{\Omega} |\nabla \Psi_{\varepsilon}|^2 \right)^{1/2}. \]

By Lemma [B.2]

\[ \int_{\Omega} |\nabla \Psi_{\varepsilon}|^2 \leq c e^{-\frac{N}{2(\varepsilon+1)}} + \frac{2N}{\varepsilon+1}, \]

hence

\[ \left( \int_{\Omega} |\nabla \Psi_{\varepsilon}|^2 \right)^{1/2} \leq e^{-\frac{N}{2(\varepsilon+1)}}. \]

Therefore, by (4.8) and combining the previous two steps,

\[ \int_{\Omega} \nabla \Phi_{\varepsilon} \cdot \nabla \Psi_{\varepsilon} \leq \begin{cases} c e^{-\frac{N}{2(\varepsilon+1)}} + N = c e^2 & \text{if } q > \frac{N+4}{2(N-2)} \\ c e^{2} |\log \varepsilon|^{\frac{N}{2(\varepsilon+1)}} & \text{if } q = \frac{N+4}{2(N-2)} \\ c e^{q(N-2)-\frac{N}{2}} & \text{if } q < \frac{N+4}{2(N-2)} \end{cases} \]

It is immediate that this is an \( o(\varepsilon) \) if \( q > \frac{N+4}{2(N-2)} \).

We now consider the case \( N = 4 \). Here, the same integration by parts as in Step 1, combined with estimates (B.2) and (2.13), provides

\[ \int_{\Omega} |\nabla \Phi_{\varepsilon}|^2 \leq c \left( e^{2 \frac{\varepsilon+1}{\varepsilon+1}} (|\log \varepsilon|)^{2} \left( \int_{\Omega} |\nabla \Phi_{\varepsilon}|^2 \right)^{1/2} + e^{4 \frac{\varepsilon+1}{\varepsilon+1}} (|\log \varepsilon|)^{2} + e^{4 \frac{\varepsilon+1}{\varepsilon+1}} \right) \]

from which

\[ \left( \int_{\Omega} |\nabla \Phi_{\varepsilon}|^2 \right)^{1/2} \leq c e^{2 \frac{\varepsilon+1}{\varepsilon+1}} (|\log \varepsilon|)^{2}. \]

Analogously, using Lemma [B.2]

\[ \left( \int_{\Omega} |\nabla \Psi_{\varepsilon}|^2 \right)^{1/2} \leq c e^{2 \frac{\varepsilon+1}{\varepsilon+1}} (|\log \varepsilon|)^{2}. \]

Therefore,

\[ \int_{\Omega} \nabla \Phi_{\varepsilon} \cdot \nabla \Psi_{\varepsilon} \leq c e^2 (|\log \varepsilon|)^{2} = o(\varepsilon), \]

from which the conclusion.
Lemma 4.5. Assume $p, q$ satisfy (4.1), (4.2) or (4.3). There exists a small positive value $\eta$ such that

$$\int_{\partial \Omega} U_\varepsilon \partial_\nu V_\varepsilon = \int_{\partial \Omega \cap B_\varepsilon(0)} U_\varepsilon \partial_\nu V_\varepsilon + o(\varepsilon), \quad \text{and} \quad \int_{\partial \Omega \cap B_\varepsilon(0)} U_\varepsilon \partial_\nu V_\varepsilon < 0.$$

Analogously, one gets, for a suitable small $\eta$,

$$\int_{\partial \Omega} V_\varepsilon \partial_\nu U_\varepsilon = \int_{\partial \Omega \cap B_\varepsilon(0)} V_\varepsilon \partial_\nu U_\varepsilon + o(\varepsilon), \quad \text{and} \quad \int_{\partial \Omega \cap B_\varepsilon(0)} V_\varepsilon \partial_\nu U_\varepsilon < 0.$$

**Proof.** Let us preliminary notice that $rV'(r) \approx r^{2-N}$ as $r \to \infty$, and

$$\partial_\nu V_\varepsilon(x) = e^{-\frac{N}{\varepsilon}} V'(|x'|/\varepsilon) \frac{\sum_{j=1}^{N-1} \rho_j x_j^2 + O(|x'|^3)}{|x| \sqrt{1 + |x'|^2}}.$$  \hfill (4.9)

Therefore, taking into account (2.13),

$$\|\partial_\nu V_\varepsilon\|_{L^\infty(\partial \Omega \cap B_\varepsilon)} \approx e^{-\frac{N}{\varepsilon}} + N^{-2} = e^{\frac{N}{\varepsilon}}.$$

Recall also from [13, pp. 2360–2362] that

$$\|U_\varepsilon\|_{L^\infty(\partial \Omega \cap B_\varepsilon)} \approx \|U_\varepsilon\|_1 \approx \begin{cases} e^{\frac{N}{\varepsilon}} & \text{if } q > \frac{N}{N-2} \\ e^{\frac{N}{\varepsilon} \log \varepsilon} & \text{if } q = \frac{N}{N-2} \\ e^{\frac{N}{\varepsilon}} & \text{if } q < \frac{N}{N-2}. \end{cases}$$

Thus

$$\int_{\partial \Omega \cap B_\varepsilon} U_\varepsilon \partial_\nu V_\varepsilon \leq c \|U_\varepsilon\|_{L^\infty(\partial \Omega \cap B_\varepsilon)} \|\partial_\nu V_\varepsilon\|_{L^\infty(\partial \Omega \cap B_\varepsilon)} = \begin{cases} O(e^{N-2}) & \text{if } q > \frac{N}{N-2} \\ O(e^{(N-2) \log \varepsilon}) & \text{if } q = \frac{N}{N-2} \\ O(e^{(N-2)-2}) & \text{if } q < \frac{N}{N-2}. \end{cases}$$

where in the last identity we use the fact that $N \geq 4$ and $q > \frac{N+2}{2(N-2)} \geq \frac{3}{N-2}$. Moreover, (4.9) shows that

$$\int_{\partial \Omega \cap B_\varepsilon} U_\varepsilon \partial_\nu V_\varepsilon < 0,$$

if $\eta$ is small enough; indeed, $V' < 0$ and 0 is a point of positive curvature and $\sum_{j=1}^{N-1} \rho_j x_j^2 + O(|x'|^3) > 0$ as $|x'| \to 0$ (recall (4.4), (4.5)). The lemma is proved.  \hfill $\Box$

**Lemma 4.6.** Assume $p, q$ satisfy (4.1), (4.2) or (4.3). As $\varepsilon \to 0^+$, one has

$$\|\tilde{U}_\varepsilon\|_\alpha \leq \|U_\varepsilon\|^p_{p+1} + o(\varepsilon), \quad \|\tilde{V}_\varepsilon\|_\beta \leq \|V_\varepsilon\|^q_{q+1} + o(\varepsilon).$$

**Proof.** We preliminary notice that, exploiting estimates in (2.13),

$$\int_{\Omega} U_\varepsilon |W_\varepsilon| = \frac{1}{|\Omega|} \|U_\varepsilon\|_1 \|U_\varepsilon^p\|_1 = \begin{cases} O(e^{N-2}) & \text{if } q > \frac{N}{N-2} \\ O(e^{(N-2) \log \varepsilon}) & \text{if } q = \frac{N}{N-2} \\ O(e^{(N-2)-2}) & \text{if } q < \frac{N}{N-2}. \end{cases} = o(\varepsilon).$$

By exploiting the inequality:

$$||a + b|^{\alpha} - |a|^{\alpha}| \leq C \left(||a|^{\alpha-1}|b| + |b|^\alpha\right) \quad \forall a, b \in \mathbb{R}$$

(recall that $\alpha = \frac{p+1}{p} > 1$, so this is a consequence of a Taylor expansion with Lagrange remainder), and once again (2.13), we obtain

$$\|\tilde{U}_\varepsilon\|_\alpha = \int_{\Omega} (U_\varepsilon + W_\varepsilon)^\alpha \leq \int_{\Omega} U_\varepsilon^{p+1} + C \int_{\Omega} (|W_\varepsilon|^\alpha + U_\varepsilon |W_\varepsilon|) \leq C \left(||U_\varepsilon||^p_{p+1} + ||U_\varepsilon||^q_{q+1} + \int_{\Omega} U_\varepsilon |W_\varepsilon|\right).$$

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\[
\leq C \left( \|U\|^p_{p+1} + \|U\|^q_{p+1} \right) \leq C \left( \|U\|^p_{p+1} + \|U\|^q_{p+1} \right) \leq C \leq C \|U\|^p_{p+1} + \|o(\varepsilon)\|.
\]

Notice that, due to (4.1), (4.2) and (4.3), one has \(p < N\), hence \(O(e^N) = o(\varepsilon)\). The conclusion follows using the Taylor expansion

\[
(y + x)^s = y^s + sy^{s-1}x + y^s \cdot o(xy^{-1}) \quad \text{as } xy^{-1} \to 0,
\]

with \(s = 1/\alpha\), \(y = \|U\|^p_{p+1}\) and \(x = o(\varepsilon)\). Notice that

\[
\|U\|^p_{p+1} = \int_{\Omega/\varepsilon} U(y)^p dy \left\{ \begin{array}{ll}
\geq \frac{1}{\varepsilon} \int_{B_1(0) - \Omega} U(y)^p dy & \text{if } q > \frac{N-1}{N-2} \\
\leq \frac{1}{\varepsilon} \int_{\Omega} U(y)^p dy & \text{if } q = \frac{N-1}{N-2} \\
\leq C \|V\|^p_{p+1} + o(\varepsilon) & \text{if } q < \frac{N-1}{N-2}.
\end{array} \right.
\]

Notice that in case \(N = 4\), then (4.3) implies \(q > 2 = \frac{N}{N-2}\), hence we are in the first case, and using \(q < N\),

\[
\|\tilde{V}\|_{\beta}^B \leq C \|V\|^p_{p+1} + o(\varepsilon).
\]

If \(N \geq 5\), any of the three cases above can occur, however \(N > q, \frac{N-2}{2} > 1\) and \(\frac{N(q+1)}{p+1} > 1\), hence (4.11) still holds.

**Lemma 4.7.** Assume \(p, q\) satisfy (4.1), (4.2) or (4.3). One has, as \(\varepsilon \to 0^+\),

\[
\|U\|^p_{p+1} \leq \frac{1}{2} S^N_{p,q} - C_1 \varepsilon + o(\varepsilon), \quad \|V\|^p_{q+1} \leq \frac{1}{2} S^N_{p,q} - C_2 \varepsilon + o(\varepsilon),
\]

for the constants

\[
C_1 := \frac{H(0)}{2} \int_{\mathbb{S}^{N-1}} |y'|^2 U^{p+1}(y', 0) dy' > 0, \quad C_2 := \frac{H(0)}{2} \int_{\mathbb{S}^{N-1}} |y'|^2 V^{q+1}(y', 0) dy' > 0,
\]

where \(H(0)\) is the mean curvature at 0 (recall (4.5)).

**Proof.** Take \(\eta\) as in (4.4), and recall that we assume that \(0 \in \partial \Omega\) has positive mean curvature. Let us first observe that

\[
\int_{\Omega} U^{p+1}_\varepsilon = \int_{\partial \Omega \setminus B_1} U^{p+1}_\varepsilon + \int_{\Omega \setminus B_1} U^{p+1}_\varepsilon = \frac{1}{2} \int_{B_1} U^{p+1}_\varepsilon - \int_{\mathbb{S}} U^{p+1}_{|\varepsilon'} + \int_{\Omega \setminus B_1} U^{p+1}_\varepsilon \leq \frac{1}{2} \int_{\mathbb{S}} U^{p+1}_{|\varepsilon'} + \int_{\Omega \setminus B_1} U^{p+1}_\varepsilon,
\]

where

\[
\Sigma := B^+_1(0) \cap \Omega^c = \{ x \in B_1(0) \cap \Omega^c : x_n > 0 \} = \{ x \in B_1(0) : 0 < x_n < \sum_{j=1}^{N-1} \rho_j x_j^2 + O(|x'|^2) \}.
\]

Now, recalling Section 2.2,

\[
\frac{1}{2} \int_{\mathbb{S}} U^{p+1}_{|\varepsilon'} = \frac{1}{2} \int_{\mathbb{S}} S^N_{p,q}.
\]

Also, in the case \(q > \frac{N}{N-2}\), by the decay estimates (2.12),

\[
\int_{\mathbb{S} \setminus B_1} U^{p+1}_{|\varepsilon'} = \varepsilon^{-N} \int_{B_1} U(|x'|/\varepsilon)^{p+1} = O(\varepsilon^{-N+(p+1)(N-2)}) \int_{\eta} r^{-(p+1)(N-2)+N-1}
\]

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\[ = O(\varepsilon^{-N+(p+1)(N-2)}) = o(\varepsilon) \] (4.13)

since \( p > \frac{3}{N-2} \). Similarly in the other cases.

We now estimate the integral on \( \Sigma \). We first notice that
\[
\int_{\Sigma} U^{p+1} = \varepsilon^{-N} \int_{\Sigma} U(x/\varepsilon)^{p+1} = \varepsilon^{-N} \int_{\Delta_n} \int_{0}^{\frac{\varepsilon}{\Delta_n}} U(x/\varepsilon)^{p+1} d\Delta_N \ d\varepsilon' = \int_{\Delta_n/\varepsilon} \int_{0}^{\frac{\varepsilon}{\Delta_n}} \sum_{j} \rho_j \varepsilon_j^2 U^{p+1}(y)/\varepsilon d\Delta_N \ dy'.
\]

Now we claim that
\[
\int_{0}^{s} U(|y|)^{p+1} dy_N = U^{p+1}(y',0)(s + O(s^2)).
\] (4.14)

Indeed, consider
\[
f(s) = \int_{0}^{s} U^{p+1}(y) dy_N
\]
and Taylor expand it. Notice that
\[
f'(0) = U^{p+1}(y',0), \quad f''(s) = (p+1)U^{p}(y',s) \frac{\partial U}{\partial y_N}(y',s).
\]

Also,
\[
|f''(s)| \leq C U^{p+1}(y',0)
\] (4.15)

where \( C \) is a positive constant independent of \( y', s \); to get this bound, one observes that \( U \) has a maximum at 0, and
\[
\left| \frac{\partial U}{\partial y_N}(y) \right| \leq \left| \frac{\partial U}{\partial y_N}(y',0) \right| \leq C U(y',0)
\]
if \( |y| > R \) with \( R \) big enough, by decay estimates in (B.6), and
\[
\left| \frac{\partial U}{\partial y_N}(y) \right| \leq C U(y',0)
\]
if \( |y| \leq R \), for some positive constant \( C = C(R) > 0 \), since they are all bounded quantities. Hence (4.15) holds, and the claim (4.14) is proved.

Therefore,
\[
\int_{\Sigma} U^{p+1} = \int_{\Delta_n/\varepsilon} U^{p+1}(y',0) \left[ (\varepsilon \sum_{j} \rho_j y_j^2 + \varepsilon^2 O(|y'|^3)) + O(\varepsilon \sum_{j} \rho_j y_j^2 + \varepsilon^2 O(|y'|^3))^2 \right].
\]

Notice that
\[
\int_{R^{N-1}} |y'|^3 U^{p+1}(y',0) \ dy' < +\infty,
\]
due to the decay estimates for \( U \), (2.12), thus
\[
\int_{\Sigma} U^{p+1} = \varepsilon \int_{R^{N-1}} \sum_{j} \rho_j y_j^2 U^{p+1}(y',0) + o(\varepsilon)
\]
\[
= \varepsilon \sum_{j} \rho_j \int_{R^{N-1}} y_j^2 U^{p+1}(y',0) + o(\varepsilon)
\]
\[
= C_1 \varepsilon + o(\varepsilon).
\] (4.16)

The conclusion follows due to (4.12), (4.13) and (4.16). The proof for \( V_\varepsilon \) is analogous.

We are now ready to prove Proposition 4.1.
Proof of Proposition 4.1. Since for any $0 < \lambda < 1$

$$
\int \nabla U_\epsilon \cdot \nabla V_\epsilon = \lambda \int \nabla U_\epsilon \cdot \nabla V_\epsilon + (1 - \lambda) \int \nabla U_\epsilon \cdot \nabla V_\epsilon
$$

$$
= \lambda \int \nabla U_\epsilon^{p+1} + (1 - \lambda) \int \nabla V_\epsilon^{q+1} + \lambda \int \partial_\Omega U_\epsilon \partial_\Omega V_\epsilon + (1 - \lambda) \int \partial_\Omega V_\epsilon \partial_\Omega U_\epsilon,
$$

and also recalling Lemmas 4.2, 4.3 and 4.4, we have

$$
\int \hat{U}_\epsilon K \hat{V}_\epsilon = (1 - \lambda) \int \nabla U_\epsilon^{p+1} + \lambda \int \nabla V_\epsilon^{q+1} - \lambda \int \partial_\Omega U_\epsilon \partial_\Omega V_\epsilon - (1 - \lambda) \int \partial_\Omega V_\epsilon \partial_\Omega U_\epsilon + o(\epsilon).
$$

Therefore, using Lemma 4.5 and Young’s inequality,

$$
\int \hat{U}_\epsilon K \hat{V}_\epsilon \geq (1 - \lambda) \int \nabla U_\epsilon^{p+1} + \lambda \int \nabla V_\epsilon^{q+1} + o(\epsilon) \geq \|U_\epsilon\|^{(1-\lambda)(p+1)} \|V_\epsilon\|^{(q+1)} + o(\epsilon).
$$

We now exploit Lemma 4.6 and (4.10) to obtain

$$
\frac{\int \hat{U}_\epsilon K \hat{V}_\epsilon}{\|\hat{U}_\epsilon\| \|\hat{V}_\epsilon\|} \geq \frac{\|U_\epsilon\|^{(1-\lambda)(p+1)} \|V_\epsilon\|^{(q+1)} + o(\epsilon)}{(\|U_\epsilon\|^{p+1} \|V_\epsilon\|^{q+1} + o(\epsilon))}
$$

$$
\geq \|U_\epsilon\|^{(p+1)(1-\lambda - \frac{p}{q+1})} \|V_\epsilon\|^{(q+1)(\lambda - \frac{q}{q+1})} + o(\epsilon).
$$

We choose $\lambda$ such that

$$
\frac{1}{p+1} < \lambda < \frac{q}{q+1},
$$

which exists since $pq > 1$.

By the Taylor expansion (4.10) and Lemma 4.7, we obtain, since $1 - \lambda - \frac{p}{p+1} = \frac{1}{p+1} - \lambda < 0$,

$$
\|U_\epsilon\|^{(p+1)(1-\lambda - \frac{p}{q+1})} \geq \left( \frac{1}{2} S_{p,q} \right)^{\frac{p}{p+1} - \lambda} - C_1 \varepsilon + o(\varepsilon)
$$

$$
\geq \left( \frac{1}{2} S_{p,q} \right)^{\frac{p}{p+1} - \lambda} - C_1 \left( \frac{1}{p+1} - \lambda \right) \left( \frac{1}{2} S_{p,q} \right)^{\frac{q}{q+1} - \lambda - 1} \varepsilon + o(\varepsilon)
$$

$$
= \left( \frac{1}{2} S_{p,q} \right)^{\frac{p}{p+1} - \lambda} + c_1 \varepsilon + o(\varepsilon)
$$

with $c_1 > 0$ due to our choice of $\lambda$. Similarly, we obtain

$$
\|V_\epsilon\|^{(q+1)(\lambda - \frac{q}{q+1})} \geq \left( \frac{1}{2} S_{p,q} \right)^{\frac{q}{q+1} - \lambda} + c_2 \varepsilon + o(\varepsilon)
$$

with $c_2 > 0$.

Finally, recalling (4.17),

$$
\frac{\int \hat{U}_\epsilon K \hat{V}_\epsilon}{\|\hat{U}_\epsilon\| \|\hat{V}_\epsilon\|} \geq \left( \frac{1}{2} S_{p,q} \right)^{\frac{p}{p+1} - \frac{q}{q+1} + \frac{1}{q+1} + \frac{1}{q+1} + \frac{1}{q+1}} + c_1 \varepsilon + o(\varepsilon) = \frac{2^p}{S_{p,q}} + c_2 \varepsilon + o(\varepsilon) > \frac{2^p}{S_{p,q}},
$$

for a positive constant $c > 0$, if $\varepsilon$ is small enough. \qed

5 A perturbation argument: proof of Theorem 1.5. In this section, we prove Theorem 1.5. The idea is to study the case $(p, q) = (1, \frac{N+4}{N-4})$, when system (1.1) reduces to the single equation (1.4), proving Theorem 1.4, and then to consider a perturbation argument.

Lemma 5.1. Assume that, for some $p^*, q^*$ satisfying (1.2), the following two conditions hold:
(i) $D_{p^*,q^*} > \frac{2^{2/N}}{S_{p^*,q^*}}$;

(ii) $S_{p,q} \to S_{p^*,q^*}$ as $(p,q) \to (p^*,q^*)$ on the critical hyperbola.

Then there exists $\varepsilon = \varepsilon(\Omega,N) > 0$ such that

$$D_{p,q} > \frac{2^{\frac{\varepsilon}{2}}}{S_{p,q}}$$

for $(p,q)$ satisfying 1.2 with $|p-p^*| + |q-q^*| < \varepsilon$.

Proof. By (i) and Lemma 3.1 there exists $(f_*,g_*) \in X^{2^\frac{\varepsilon}{4p^*}} \times X^{2^\frac{\varepsilon}{4q^*}}$ which achieves $D_{p^*,q^*}$. For any $L > 0$ we define

$$\tilde{f}_L(x) = \begin{cases} f_*(x) & \text{if } |f_*(x)| < L \\ L & \text{if } |f_*(x)| \geq L \end{cases}$$

and

$$\tilde{g}_L(x) = \begin{cases} g_*(x) & \text{if } |g_*(x)| < L \\ L & \text{if } |g_*(x)| \geq L \end{cases}$$

and

$$f_L(x) = \tilde{f}_L(x) - \frac{1}{|\Omega|} \int_{\Omega} \tilde{f}_L, \quad g_L(x) = \tilde{g}_L(x) - \frac{1}{|\Omega|} \int_{\Omega} \tilde{g}_L.$$

Notice that $f_L, g_L \in L^\infty(\Omega)$ and have zero average, thus $(f_L, g_L) \in X^\alpha \times X^\beta$ for every $(p,q)$ and $(f_L, g_L)$ is a test function for $D_{p,q}$. Moreover,

$$\|f_L\|_{p^\frac{\varepsilon}{2}} \to \|f_\ast\|_{p^\frac{\varepsilon}{2}} \quad \text{and} \quad \|g_L\|_{q^\frac{\varepsilon}{2}} \to \|g_\ast\|_{q^\frac{\varepsilon}{2}}$$

as $(p,q) \to (p^*,q^*)$ by exploiting the dominated convergence theorem. Thus,

$$\lim_{(p,q) \to (p^*,q^*)} D_{p,q} \geq \lim_{(p,q) \to (p^*,q^*)} \frac{\{\int_{\Omega} f_L Kg_L\}}{\|f_L\|_\alpha \|g_L\|_\beta} = \frac{\{\int_{\Omega} f_L Kg_L\}}{\|f_\ast\|_{p^\frac{\varepsilon}{2}} \|g_\ast\|_{q^\frac{\varepsilon}{2}}} = \frac{D_{p^*,q^*}}{S_{p^*,q^*}}. \tag{5.1}$$

We claim that, as $L \to \infty$,

$$\|f_L\|_{p^\frac{\varepsilon}{2}} \to \|f_\ast\|_{p^\frac{\varepsilon}{2}} \quad \text{and} \quad \|g_L\|_{q^\frac{\varepsilon}{2}} \to \|g_\ast\|_{q^\frac{\varepsilon}{2}} \tag{5.2}$$

and

$$\int_{\Omega} f_L Kg_L \to \int_{\Omega} f_\ast Kg_\ast. \tag{5.3}$$

Indeed,

$$\tilde{f}_L \to f_\ast \quad \text{in } L^{p^\frac{\varepsilon}{2}}, \quad \tilde{g}_L \to g_\ast \quad \text{in } L^{q^\frac{\varepsilon}{2}} \tag{5.4}$$

by dominated convergence, and so

$$\int_{\Omega} \tilde{f}_L \to \int_{\Omega} f_\ast = 0, \quad \int_{\Omega} \tilde{g}_L \to \int_{\Omega} g_\ast = 0.$$

From this we have that (5.2) follows; by the continuity properties of the operator $K$, also (5.3) holds true.

Thus, using (i)-(ii) and (5.1),

$$\lim_{(p,q) \to (p^*,q^*)} D_{p,q} = \lim_{L \to \infty} \lim_{(p,q) \to (p^*,q^*)} D_{p,q} \geq \frac{\{\int_{\Omega} f_L Kg_L\}}{\|f_L\|_{p^\frac{\varepsilon}{2}} \|g_L\|_{q^\frac{\varepsilon}{2}}} = \frac{D_{p^*,q^*}}{S_{p^*,q^*}} \geq \frac{2^{\frac{\varepsilon}{2}}}{S_{p^*,q^*}} = \frac{D_{p^*,q^*}}{S_{p^*,q^*}}.$$

Thus

$$D_{p,q} > \frac{2^{\frac{\varepsilon}{2}}}{S_{p,q}}$$

for $(p,q)$ close enough to $(p^*,q^*)$. □
The following Lemma immediately implies Theorem [1.3] and it will be the starting point of our perturbation argument.

**Lemma 5.2.** Let

\[ \Sigma = \inf_{u \in H^2_0(\Omega)} \left\{ \int_\Omega |\Delta u|^2 + |\nabla u|^2 + \alpha |u|^2 : \int_\Omega |u|^\frac{2N}{N+4} = 1 \right\}, \]

where \( H^2_0(\Omega) = \{ u \in H^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial \Omega \} \).

Then, \( \Sigma \) is attained, and

\[ \Sigma < \frac{S^2_{1, \frac{N+4}{N}}} {2^\pi}. \] (5.5)

**Proof.** Notice that (5.5) is an easy consequence of [4, Lemma 4.1], as for any \( \alpha > 0 \) one has

\[ \Sigma \leq \inf_{u \in H^2_0(\Omega)} \left\{ \int_\Omega (|\Delta u|^2 + |\nabla u|^2 + \alpha |u|^2) : \int_\Omega |u|^\frac{2N}{N+4} = 1 \right\} < \frac{S^2_{1, \frac{N+4}{N}}} {2^\pi}. \]

We recall that the constant \( S \) appearing in [4] (see eq. (2.1) therein) corresponds to our \( S^2_{1, \frac{N+4}{N}} \).

We now prove that \( \Sigma \) is attained. We borrow some ideas from [4, Lemma 3.2]. We preliminary notice that

\[ u \mapsto \left( \int_\Omega |\Delta u|^2 + \int_\Omega |u|^2 \right)^{1/2} \]

is a norm in \( H^2_0 \). Indeed, let \( u \in H^2_0 \), and consider the problem

\[ \Delta w = \Delta u \text{ in } \Omega, \quad \partial_\nu w = 0 \text{ on } \partial \Omega, \quad \int_\Omega w = 0 \]

This problem is well defined as by the divergence theorem and since \( \partial_\nu u = 0 \) on \( \partial \Omega \),

\[ \int_\Omega \Delta u = \int_{\partial \Omega} \partial_\nu u = 0. \]

Therefore

\[ w := u - \frac{1}{|\Omega|} \int_\Omega u \]

is the unique solution of this problem and, by regularity theory (Lemma 2.1), \( \| w \|_{H^2} \leq C \| \Delta u \|_2 \), whence \( \| u \|_{H^2_0} \leq C \| \| w \|_2 + \| \Delta u \|_2 \| \).

Now, take \( u_k \) a minimizing sequence for \( \Sigma \). Then,

\[ \int_\Omega |\Delta u_k|^2 \leq C \quad \text{and also} \quad \int_\Omega |u_k|^2 \leq |\Omega|^\frac{2}{N} \left( \int_\Omega |u_k|^\frac{2N}{N+4} \right)^{\frac{N+4}{N}} = |\Omega|^\frac{2}{N}, \]

hence in particular \( u_k \) is bounded in \( H^2_0 \). Therefore, up to extracting a subsequence, there exists \( u \in H^2_0 \) such that

\[ u_k \to u \text{ a.e. in } \Omega, \text{ weakly in } L^2, \text{ strongly in } H^1; \quad \Delta u_k \to \Delta u \text{ weakly in } L^2. \]

**Step one:** Assume by contradiction \( u \equiv 0 \), then \( u_k \to 0 \) strongly in \( H^1(\Omega) \). Thus, by Cherrier’s inequality [A.1] with \( \eta = 2 \), we have that for every \( \epsilon > 0 \),

\[ 1 = \| u \|_{\frac{2N}{N+4}} \leq \left( \frac{2^\frac{2}{N}}{S^2_{1, \frac{N+4}{N}}} \epsilon + o(1) \right) \| \Delta u_k \|_2 + o(1) = \left( \frac{2^\frac{2}{N}}{S^2_{1, \frac{N+4}{N}}} \epsilon + o(1) \right) \Sigma + o(1). \]

This contradicts (5.5), for sufficiently small \( \epsilon \) and large \( k \), and allows us to conclude that \( u \equiv 0 \).

**Step two:** Strong convergence in \( L^{\frac{2N}{N+4}} \). One has

\[ 1 - \int_\Omega |u_k - u|^\frac{2N}{N+4} = \int_\Omega |u|^\frac{2N}{N+4} + o(1), \] (5.6)
by the Brezis-Lieb lemma, see for instance [39, Lemma 1.32]. Since $\Delta u_k \rightharpoonup \Delta u$ weakly in $L^2$,
\[
\int_{\Omega} |\Delta u_k|^2 = \int_{\Omega} |\Delta(u_k - u)|^2 + 2 \int_{\Omega} \Delta(u_k - u) \Delta u = \int_{\Omega} |\Delta(u_k - u)|^2 + \int_{\Omega} |\Delta u|^2 + o(1).
\] (5.7)

Also, since $u \neq 0$ by the previous step and by the definition of $\Sigma$,
\[
\int_{\Omega} |\Delta u|^2 \geq \Sigma \|u\|^2_{2N/4}.
\] (5.8)

Therefore, by exploiting (5.7), (A.1), the fact that $\frac{N-4}{N} < 1$, (5.8), and (5.6),
\[
\Sigma = \int_{\Omega} |\Delta(u_k - u)|^2 + \int_{\Omega} |\Delta u|^2 + o(1) \geq \left( \frac{2 \hat{\Sigma}}{S^2_1 N + 4} + \varepsilon \right)^{-1} \|u_k - u\|^2_{2N/4} + \Sigma \|u\|^2_{2N/4} + o(1)
\]
\[
\geq \left( \frac{2 \hat{\Sigma}}{S^2_1 N + 4} + \varepsilon \right)^{-1} \|u_k - u\|^2_{2N/4} + \Sigma \left( \|u_k - u\|^2_{2N/4} + \|u\|^2_{2N/4} \right) + o(1)
\]
\[
= \left( \frac{2 \hat{\Sigma}}{S^2_1 N + 4} + \varepsilon \right)^{-1} \|u_k - u\|^2_{2N/4} + \Sigma + o(1).
\]

By taking $\varepsilon > 0$ sufficiently small so that $\left( \frac{2 \hat{\Sigma}}{S^2_1 N + 4} + \varepsilon \right)^{-1} - \Sigma > 0$, we deduce by (5.5) that $u_k \rightharpoonup u$ strongly in $L^{\frac{2N}{4}}$, thus $\|u\|_{\frac{2N}{4}} = 1$.

**Step three: conclusion of the proof.** By weak lower semicontinuity of $\|\cdot\|_{2}$, one has
\[
\Sigma \leq \int_{\Omega} |\Delta u|^2 \leq \liminf_{k \to \infty} \int_{\Omega} |\Delta u_k|^2 = \Sigma,
\]
hence $\Sigma$ is attained at $u$. \hfill \square

We are now ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** We show that (i)-(ii) in Lemma 5.1 hold with $p^* = 1$ and $q^* = \frac{N+4}{N-4}$. We start by proving condition (i). Since $\Sigma$ is attained by Lemma 5.2 there exists $\bar{u} \in H^2_{0}(\Omega)$ such that
\[
\int_{\Omega} |\Delta \bar{u}|^2 = \Sigma, \quad \int_{\Omega} |\bar{u}|^{\frac{2N}{4}} = 1,
\]
and $\bar{u}$ satisfies
\[
\Delta^2 \bar{u} = |\bar{u}|^{\frac{N-4}{4-4}} \bar{u} \text{ in } \Omega, \quad \partial_{\nu} \bar{u} = \partial_{\nu}(\Delta \bar{u}) = 0 \text{ on } \partial \Omega.
\]

Take
\[
f = -\Delta \bar{u}, \quad g = |\bar{u}|^{\frac{N}{N-4}} \bar{u}.
\]

One has $(f, g) \in X$, and
\[
\int_{\Omega} gKf = \int_{\Omega} \bar{u}^{\frac{2N}{N-4}} = 1, \quad \|f\|_{2} = \Sigma^{\frac{1}{2}}, \quad \text{and} \quad \|g\|_{\frac{2N}{N-4}} = \left( \int_{\Omega} |\bar{u}|^{\frac{2N}{N-4}} \right)^{\frac{N+4}{N-4}} = 1.
\]

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Thus, by the definition of $D_1 \frac{N+4}{N-4}$ and (5.5),

$$D_1 \frac{N+4}{N-4} \geq \frac{\int_{\Omega} gKf}{\|f\|_2 \|g\|_{2N}} = \Sigma^{-\frac{1}{2}} > \frac{2^\frac{4}{N}}{S_1 \frac{N+4}{N-4}}.$$ 

Let us now prove condition (ii) in Lemma 5.1. Recall the bubbles defined in (2.11). Here, $\varepsilon$ and $x_0$ are not important in the analysis, hence we will just fix $\varepsilon = 1$ and $x_0 = 0$. However, the dependence on $p, q$ (which solve (1.2)) will be crucial. Thus, in what follows, we assume without loss of generality that $q \geq \frac{N+4}{N-4}$ and we will denote by $(U_p, V_p)$ the solution to

$$-\Delta U_p = V_p^q \text{ in } \mathbb{R}^N, \quad -\Delta V_p = U_p^p \text{ in } \mathbb{R}^N, \quad U_p(0) = 1.$$ 

By Proposition 2.2 in [22], we know that $(U_p, V_p) \to (U_1, V_1)$ in $\mathcal{D}^{2,2}(\mathbb{R}^N) \times \mathcal{D}^{2, \frac{2N}{N-4}}(\mathbb{R}^N)$ as $p \to 1$.

Since $U_p \to U_1$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, then up to a subsequence we have $U_p \to U_1$ in $L^{\frac{2N}{N-4}}(\mathbb{R}^N)$. In particular, by the inverse of the dominated convergence theorem, we get a function $h \in L^{\frac{2N}{N-4}}(\mathbb{R}^N)$ such that $|U_p| \leq h$. Therefore, using the fact that the sequence $U_p$ is uniformly bounded by (1), and since $q + 1 \geq \frac{2N}{N-4}$, there exists a constant $C > 0$ such that $|U_p|^q \leq C |U_p|^{\frac{2N}{N-4}} \leq C h^{\frac{2N}{N-4}}$. Whence, as $p \to 1$, by dominated convergence

$$S_{p,q}^\infty = \int_{\mathbb{R}^N} U_p^{q+1} \to \int_{\mathbb{R}^N} U_1^{q+1} = S_{1, \frac{N+4}{N-4}}^{\infty},$$

Now, Lemma 5.1 applies with $p^* = 1$ and $q^* = \frac{N+4}{N-4}$, thus

$$D_{p,q} \geq \frac{2^\frac{4}{N}}{S_{p,q}}$$

for $(p, q)$ close enough to $(1, \frac{N+4}{N-4})$. The conclusion now easily follows recalling Lemma 3.1. Proposition 1.3 yields regularity of solutions. 

6 Symmetry-breaking and solutions with radial symmetry. Proof of Theorem 1.9

In this section we restrict our attention to the case

$$\Omega = B_r(0) \setminus B_s(0), \quad \text{for some } 0 < r < R \quad \text{or} \quad \Omega = B_r(0) \quad \text{for some } R > 0.$$ 

(6.1)

The proof of Theorem 1.9 follows more or less directly from the ideas in [35] Theorems 1.1–1.2, where the subcritical case is treated, combined with the regularity result in Proposition 1.3. Here we just highlight the differences, presenting a sketch of the proof. We start by defining some radially symmetric counterparts of notions introduced before.

Definition 6.1. We define

$$X_{rad} = \{(f, g) \in X : f, g \text{ are radially symmetric}\}$$

and

$$D_{rad} = \sup \left\{ \int_{\Omega} Kg : (f, g) \in X_{rad}, \quad \gamma_1 \|f\|_0^\alpha + \gamma_2 \|g\|_\beta^\beta = 1 \right\}.$$ 

The key point of the proof of Theorem 1.9 (and of [35] Theorem 1.2) is based on an $L^t$-norm-preservation transformation introduced in [35], which we now recall. Let

$$\mathcal{I} : L^\infty(\Omega) \to C_{rad}(\overline{\Omega}), \quad \mathcal{I}h(x) := \int_{r \leq |y| \leq |x|} h(y) \, dy = N \omega_N \int_r^{|x|} h(\rho) \rho^{N-1} \, d\rho$$

and

$$\mathcal{F} : C_{rad}(\overline{\Omega}) \to L^\infty_{rad}(\Omega), \quad \mathcal{F}h := (\chi_{\{|h| > 0\}} - \chi_{\{|h| \leq 0\}}) h.$$
Lemma 6.4. There is \( \omega_N = |B_1| \) is the volume of the unitary ball in \( \mathbb{R}^N \) and \( \# \) is the decreasing rearrangement given by

\[
h^# : [0, |\Omega|] \to \mathbb{R}, \quad h^#(s) := \text{ess sup}_{h \leq s} h \quad \text{and} \quad h^#(0) := \inf \{ t \in \mathbb{R} : |\{ h > t \}| < s \}, \ s > 0.
\]

This transformation, in practice, is applied to radial functions \( h \) with zero average: \( \mathcal{I}h(R) = \int_{\Omega} h = 0 \). Loosely speaking, this transformation does the following: in many situations the domain of \( h \) may be split in \( r =: r_0 < r_1 < \ldots < r_N = R \), where \( \int_{r_i < |x| < r_{i+1}} h = 0 \); however it may not be true that \( \mathcal{I}h(x) \) is nonnegative for every \( x \). We flip the graph of \( h \) in the annuli \([r_i, r_{i+1}]\) where we don’t have this property, obtaining at the end \( \mathcal{I}(\mathcal{G}h)(x) \geq 0 \) for every \( x \in \Omega \). We then apply a decreasing rearrangement, finally placing the result back in the original annulus using the transformation \( x \mapsto \omega_N|x|^N - \omega_N r^N \). For more details, insights, examples and comments regarding the definition of the flip-\( \ell \)-rearrangement transformation \( \ast \) we refer to [35, Section 3.2]. The following is a combination of Theorem 1.3 and Proposition 3.4 from [35].

Theorem 6.3. Let \( p, q > 0 \) and take \( \Omega \) as in (6.1). Take \( f, g : \overline{\Omega} \to \mathbb{R} \) be continuous and radially symmetric functions with \( \int_{\Omega} f = \int_{\Omega} g = 0 \). Then \( \{f^*, g^*\} \in X_\text{rad} \),

\[
\|f^*\|_{p + 1} \leq \|f\|_{p + 1}, \quad \|g^*\|_{p + 1} = \|g\|_{p + 1} \quad \text{and} \quad \int_{\Omega} fKg \leq \int_{\Omega} f^*Kg^*.
\]

Furthermore, if \( f, g \) are nontrivial and the last statement in (6.2) holds with equality, then \( f, g \) are monotone in the radial variable. Moreover, \( (Kf, Kg) \) is radially symmetric and \( (Kf)_r(Kg)_r \geq 0 \).

We also recall some standard notation and the definition of foliated Schwarz symmetry. Let \( N \geq 4 \), and fix \( e \in S^{N-1} \). We consider the half-space \( H(e) = \{ x \in \mathbb{R}^N : e \cdot x > 0 \} \), and the half domain \( \Omega(e) = \{ x \in \Omega : e \cdot x > 0 \} \). Given a function \( w : \overline{\Omega} \to \mathbb{R} \) we define \( w_e : \overline{\Omega} \to \mathbb{R} \) as

\[
w_e(x) = w(x - 2(x \cdot e)e).
\]

The polarization \( w^H \) of \( w \) with respect to a hyperplane \( H = H(e) \) is given by

\[
w^H = \begin{cases} 
\max\{w, w_e\} & \text{in } \Omega(e), \\
\min\{w, w_e\} & \text{in } \overline{\Omega} \setminus \Omega(e).
\end{cases}
\]

We say that \( u \in C(\overline{\Omega}) \) is foliated Schwarz symmetric with respect to some unit vector \( e^* \in S^{N-1} \) if \( u \) is axially symmetric with respect to the axis \( \mathbb{R}e^* \) and non increasing in the polar angle \( \theta := \arccos\left(\frac{e^*}{|e^*|} \cdot e\right) \in [0, \pi] \). The following characterization of foliated Schwarz symmetry holds, see [37].

Lemma 6.4. There is \( e^* \in S^{N-1} \) such that \( u, v \in C(\overline{\Omega}) \) are foliated Schwarz symmetric with respect to \( e^* \) if and only if for each \( e \in S^{N-1} \) either

\[
\begin{align*}
\text{if } e \geq e^* \\
\text{and } e \in \Omega(e) \quad \text{or} \quad u \leq u_e, v \leq v_e \quad \text{in } \Omega(e).
\end{align*}
\]

We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. Proof of Part (i). By our assumption, \( \Omega \) is an annulus. We first observe that \( D_{\text{rad}} > 0 \), as follows by taking \( (\varphi_1, \hat{\varphi}_1) \), where \( \varphi_1 \) is the first non constant radial eigenfunction of the Laplacian with Neumann boundary conditions. Since \( \Omega \) is an annulus, for any \( t > 1 \) the embedding

\[
W^{2, t}_{\text{rad}}(\Omega) \subset W^{1, t}_{\text{rad}}(\Omega) \hookrightarrow C(\overline{\Omega})
\]

is compact, therefore, also by Lemma 2.1, the operator \( K \) is compact from \( L^{\frac{4}{N+1}}(\Omega) \to C(\overline{\Omega}) \). In particular, \( K \) is compact from \( L^{\frac{4}{N+1}}_{\text{rad}}(\Omega) \) to \( L^{p+1}_{\text{rad}}(\Omega) \). With this, we can prove that \( D_{\text{rad}} \) is achieved at a certain
pair \((f, g)\): indeed, taking a maximizing sequence \((f_k, g_k) \in X_{\text{rad}}\) such that \(\int_{\Omega} f_k K g_k \to D_{\text{rad}}\) and \(\gamma_1 \|f_k\|_\alpha^\alpha + \gamma_2 \|g_k\|_\beta^\beta = 1\), up to a subsequence we have

\[f_k \rightharpoonup f \text{ weakly in } L^{\frac{p+1}{2}}(\Omega), \quad g_k \rightharpoonup g \text{ weakly in } L^{\frac{p+1}{2}}(\Omega), \quad K g_k \to Kg \text{ strongly in } L^{p+1}(\Omega).\]

Then \(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta \leq 1\) and \(\int_{\Omega} f K g = D_{\text{rad}},\) so that \((f, g) \neq (0, 0)\) and (cf. Lemma 2.3)

\[
\frac{D_{\text{rad}}}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{2}}} = \int_{\Omega} \frac{f}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{2}}} K \left(\frac{g}{(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta)^{\frac{1}{2}}}\right) \leq D_{\text{rad}},
\]

so that \(\gamma_1 \|f\|_\alpha^\alpha + \gamma_2 \|g\|_\beta^\beta = 1\), and \((f, g)\) achieves \(D_{\text{rad}}\).

Therefore, reasoning precisely as in the proof of Proposition 2.5

\[
(u, v) = ((D_{\text{rad}})^{-q} \frac{p+1}{p-q} K_p g, (D_{\text{rad}})^{-p} \frac{p+1}{p-q} K_q f) = (D_{\text{rad}}^{-q} \frac{p+1}{p-q} |f|^{\frac{1}{p-1}} f, D_{\text{rad}}^{-p} \frac{p+1}{p-q} |g|^{\frac{1}{p-1}} g)
\]

is a least-energy radial solution of (1.1).

From the second relation in (6.4), we have that \(f, g\) are also continuous up to the boundary. From Theorem 6.3 (using the maximality of \((f, g)\)) and by the first relation in (6.4), we now conclude that \(u, v\) are continuous.

Proof of Part (ii): Under the assumptions of the statement, we have that \(D_{p,q}\) is achieved. By simple scalings (recall the proof of Proposition 2.5), we have that

\[
D_{\frac{2}{N}} \frac{2}{N} = \hat{D}_{p,q} := \inf_{\mathcal{N}} \phi(f, g),
\]

where \(\mathcal{N} = \{(f, g) \in X \setminus \{(0, 0)\} : \phi(f, g) (\gamma_1 f, \gamma_2 g) = 0\}\) is the standard Nehari manifold, and \(\hat{f} = (D_{p,q})^q f, \hat{g} = (D_{p,q})^p g,\) with \(s = -q\frac{p+1}{p-q} - 1\) and \(t = -q\frac{p+1}{p-q}\) achieves \(\hat{D}_{p,q}\). Then we are precisely in the assumptions of [35] Theorem 4.1 (whose proof does not depend on the value of \((p, q)\) with respect to the critical hyperbola, if we take into account Theorem 6.3 and the regularity result in Proposition 1.3). Then \(\hat{f}, \hat{g}\) are not radially symmetric, and neither are \(f, g\).

We finally show that \(u, v\) are foliated Schwarz symmetric. Fix a hyperplane \(H = H(e)\) for some \(|e| = 1\). Notice that (see Proposition 2.5)

\[
(u, v) = (D_{p,q}^{-q} \frac{p+1}{p-q} K_p g, D_{p,q}^{-p} \frac{p+1}{p-q} K_q f)
\]

is a solution of (1.1), and in particular \(-\Delta u = D_{p,q}^{-q} \frac{p+1}{p-q} g\) and \(-\Delta v = D_{p,q}^{-p} \frac{p+1}{p-q} f\). Define

\[
(\hat{u}, \hat{v}) = (D_{p,q}^{-q} \frac{p+1}{p-q} K_p (g^H), D_{p,q}^{-p} \frac{p+1}{p-q} K_q (f^H)).
\]

By Proposition 1.3 we know that \((u, v) \in C^{2, \alpha}(\overline{\Omega}) \times C^{2, \alpha}(\overline{\Omega})\) with suitable \(\zeta, \eta, \) and moreover \(f^H, g^H \in L^\infty(\Omega)\) and \((\hat{u}, \hat{v}) \in (W^{2, r}(\Omega) \cap C^1(\Omega)) \times (W^{2, r}(\Omega) \cap C^1(\Omega))\). Let \(V = v_{\text{e}} + v - \hat{v} - v_{\text{e}},\) thus

\[-\Delta V = D_{p,q}^{-p} \frac{p+1}{p-q} (f - f^H - (f^H)_{\text{e}} + f_{\text{e}}) = 0 \text{ in } \Omega, \quad \partial_{\nu} V = 0 \text{ on } \partial \Omega.
\]

Testing this equation with \(V\) and integrating by parts we obtain that \(V = k\) for some \(k \in \mathbb{R}\). Then

\[
v_{\text{e}} + v = \hat{v}_{\text{e}} + \hat{v} + k \quad \text{in } \Omega.
\]

Let

\[
\Gamma_1 = \{x \in \partial \Omega(e) : x \cdot e = 0\}, \quad \Gamma_2 = \{x \in \partial \Omega(e) : x \cdot e > 0\},
\]

\(w_1 = \hat{v} - v + k/2,\) and \(w_2 = \hat{v} - v_{\text{e}} + k/2.\) Since \(v = v_{\text{e}}\) and \(\hat{v} = \hat{v}_{\text{e}}\) on \(\Gamma_1,\) we have that \(w_1 = w_2 = 0\) on \(\Gamma_1,\) and \(\partial_{\nu} w_1 = \partial_{\nu} w_2 = 0\) on \(\Gamma_2.\) Also,

\[-\Delta w_1 = D_{p,q}^{-p} \frac{p+1}{p-q} (f^H - f) \geq 0 \quad \text{in } \Omega(e) \quad -\Delta w_2 = D_{p,q}^{-p} \frac{p+1}{p-q} (f^H - f_{\text{e}}) \geq 0 \quad \text{in } \Omega(e),
\]

29
which implies that \( w_1 \geq 0 \) and \( w_2 \geq 0 \) in \( \Omega(e) \). Therefore, since by \([6.5]\) one has \( \bar{v}_e = v_e + v - \bar{v} - k \), and by definition of \( g^H \) one has \( g^H_e = g_e + g - g^H \), and also recalling \( \int_\Omega g = 0 \),

\[
D^{-p}\frac{\mu + 1}{\mu} \int_\Omega (gKf - g^H Kf^H) = \int_\Omega (g(v - g^H \bar{v})) = \int_{\Omega(e)} gv + g_e v_e - g^H \bar{v} - (g^H) e \bar{v}_e
\]

\[
= \int_{\Omega(e)} gv + g_e v_e - g^H \bar{v} - (g_e + g - g^H)(v_e + v - \bar{v} - k)
\]

\[
= \int_{\Omega(e)} (g_e - g^H) w_1 + (g - g^H) w_2 + \frac{k}{2} (g_e + g) \leq 0. \quad (6.6)
\]

To show that \( u, v \) are foliated Schwarz symmetric, we now use Lemma \([6.4]\) and argue by contradiction. Then, we assume that there is \( e \in S^{N-1} \) and the corresponding half space \( H = H(e) \) such that \( u \neq u^H \) in \( \Omega(e) \) and either \( v_e \neq v^H \) in \( \Omega(e) \) or \( u_e \neq u^H \) in \( \Omega(e) \). Since \( t \mapsto c |t|^p \) is a strictly monotone increasing function in \( \mathbb{R} \) for \( s > -1 \), and since \( f = D^{-p} \frac{\mu + 1}{\mu} |u|^{p-1} u \), then

\[
f \neq f^H \quad \text{and either } f_e \neq f^H \quad \text{or } g_e \neq g^H
\]

in \( \Omega(e) \). Therefore, \( w_1 > 0 \) and either \( w_2 > 0 \) or \( g_e - g^H \neq 0 \) in \( \Omega(e) \). Also, notice that if \( f \neq f^H \) in \( \Omega(e) \), then \( g \neq g^H \) in \( \Omega(e) \), namely either \( g - g^H \neq 0 \) or \( g_e - g^H \neq 0 \). Summing up, we have two possibilities: either \( w_1 > 0 \) and \( 0 \neq g_e - g^H \leq 0 \); or \( w_2 > 0 \) and \( 0 \neq g - g^H \leq 0 \). In both cases, recalling now \([6.6]\), we conclude

\[
\int_{\Omega} gKf < \int_{\Omega} g^H Kf^H,
\]

which contradicts the maximality of \( (f, g) \). \( \square \)

Remark 6.5. Notice that, when \( p = q \), the system reduces to a single equation. Then, the fact that least energy solutions are non radial in a ball \( \Omega = B_R(0) \) is a simple consequence of the Pohozaev identity: if a radial solution existed, since \( u \) is sign-changing we would have \( u = 0 \) on \( \partial B_\rho(0) \) for some \( 0 < \rho < R \), and we would have a positive solution of

\[
-\Delta u = u^{\frac{N+2}{N-2}} \in B_\rho(0), \quad u = 0 \text{ on } \partial B_\rho(0),
\]

a contradiction. Such argument does not extend to the case \( p \neq q \); even though a Pohozaev-type identity holds (see [35]), if \( (u, v) \) is a least energy solution it is not clear if both \( u \) and \( v \) vanish on the same sphere, so we do not obtain, in principle, a solution to a Dirichlet problem in a smaller ball.

A Cherrier’s-type inequalities

For \( \eta > 1 \) and \( N > 2\eta \), we define \( S \) to be the best Sobolev constant of the embedding \( \mathcal{D}^{2,\eta}(\mathbb{R}^N) \hookrightarrow L^{\frac{N\eta}{N-\eta}}(\mathbb{R}^N) \), that is

\[
S = \inf \left\{ \| \Delta u \|_{L^\infty(\mathbb{R}^N)} : u \in \mathcal{D}^{2,\eta}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{\frac{N\eta}{N-\eta}} = 1 \right\},
\]

\[
S\|u\|_{L^{\frac{N\eta}{N-\eta}}(\mathbb{R}^N)} \leq \| \Delta u \|_{L^\infty(\mathbb{R}^N)}. \quad \text{With our notation above, we have}
\]

\[
S = S_{2N+N+2}(\frac{1}{2}, 1).
\]

The main purpose of this appendix is to prove the following Cherrier inequality. We would like to point out that this one is not included in Cherrier’s original paper [9]. We are inspired by the proof of [4] Lemma 1.2, which deals with the case \( \eta = 2 \) and \( N \geq 5 \).

Theorem A.1. Let \( \eta > 1 \) and \( N > 2\eta \). For any \( \varepsilon > 0 \) there exists a constant \( C(\varepsilon) > 0 \) such that, for any \( u \in W^{2,\eta}(\Omega) := \{ u \in W^{2,\eta}(\Omega) : u = 0 \text{ on } \partial \Omega \} \), one has

\[
\|u\|_{\eta^*} \leq \left( \frac{2}{S} \right)^{\frac{\varepsilon}{\eta}} \| \Delta u \|_{\eta} + C(\varepsilon) \|u\|_{W^{1,\eta}}, \quad (A.1)
\]

where \( \eta^* = \frac{N\eta}{N-2\eta} \).
Proof. Following the proof of (1.8) of Lemma 2.1 in [4] (see Step two therein), we choose finitely many points $x_i \in \overline{\Omega}$ and a positive number $R > 0$, with corresponding sets $\Omega_i = \Omega \cap B_R(x_i)$, such that $\overline{\Omega} \subset \bigcup_{i=1}^n \Omega_i$. Up to increasing the number of open sets, we can assume that $x_i \in \partial \Omega$ whenever $\overline{\Omega}_i \cap \partial \Omega \neq \emptyset$. We choose $(\tilde{\zeta}_i)_{i=1}^n$ a smooth partition of the unity subordinated to the covering $(\Omega_i)_{i=1}^n$:

$$\tilde{\zeta}_i \in C^\infty(\mathbb{R}^N), \quad 0 \leq \tilde{\zeta}_i \leq 1, \quad \text{supp} \tilde{\zeta}_i \subset \Omega_i, \quad \sum_{i=1}^n \tilde{\zeta}_i(x) = 1 \forall x \in \Omega.$$

We split the set of indices into two disjoint sets $I$ and $J$, where $I$ contains the indices with $x_i \in \Omega$ (and $\Omega_i \subset \Omega$), while $J$ contains the indices with $x_i \in \partial \Omega$ ($\bigcup_{i \in J} \Omega_i$ covers the boundary). We define

$$\zeta_i = \frac{\tilde{\zeta}_{2\eta+1}^2}{\sum_{i=1}^n \tilde{\zeta}_i^{2\eta+1}}.$$

Notice that this is a partition of unity subordinated to the same covering, and that $\zeta_i^{1/\eta} \in C^2(\overline{\Omega})$. Then, for any $u \in W^{2,\eta}(\mathbb{R}^N)$, one has $\zeta_i^{1/\eta} u \in W^{2,\eta}(\mathbb{R}^N)$ and supp$(\zeta_i^{1/\eta} u) \subset \Omega_i$.

We consider separately the indices in $I$ and in $J$.

**Case One.** For any $i \in I$ and for any $\varepsilon > 0$ there exists a constant $B_1(\varepsilon)$ such that

$$\sum_{i \in I} \left( \int_{\Omega_i} |\zeta_i^{1/\eta} u|^\eta \right)^\frac{n}{\eta} \leq \left( \frac{2^\varepsilon}{S^\varepsilon} + \varepsilon \right) \left( \sum_{i \in I} \int_{\Omega_i} |\Delta u|^\eta \right) + B_1(\varepsilon) \|u\|_{W^{1,\eta}}^\eta.$$

Indeed,

$$\sum_{i \in I} \left( \int_{\Omega_i} |\zeta_i^{1/\eta} u|^\eta \right)^\frac{n}{\eta} \leq \sum_{i \in I} \left( \int_{\mathbb{R}^N} |\zeta_i^{1/\eta} u|^\eta \right)^\frac{n}{\eta} \leq \frac{1}{S^\eta} \sum_{i \in I} \int_{\mathbb{R}^N} |\Delta(\zeta_i^{1/\eta} u)|^\eta$$

$$= \frac{1}{S^\eta} \sum_{i \in I} \left[ \int_{\mathbb{R}^N} \left( \zeta_i^{1/\eta} |\Delta u| + 2|\nabla(\zeta_i^{1/\eta})||\nabla u| + |u||\Delta(\zeta_i^{1/\eta})| \right)^\eta \right]$$

$$\leq \frac{2^\varepsilon}{S^\varepsilon} \sum_{i \in I} \left[ \int_{\mathbb{R}^N} \left( \zeta_i^{1/\eta} |\Delta u| + 2|\nabla(\zeta_i^{1/\eta})||\nabla u| + |u||\Delta(\zeta_i^{1/\eta})| \right)^\eta \right].$$

where in the last inequality we just used the fact that $2^\varepsilon \geq 1$. Let us call

$$X = \zeta_i^{1/\eta} |\Delta u|, \quad Y = 2|\nabla(\zeta_i^{1/\eta})||\nabla u|, \quad Z = |u||\Delta(\zeta_i^{1/\eta})|.$$

Recall that, given $\varepsilon_0 > 0$, there exists $C(\varepsilon_0)$ such that $(1 + t)^\eta \leq (1 + \varepsilon_0) + C(\varepsilon_0) t^\eta$ for every $t > 0$. This, in turn, implies that $(s + t)^\eta \leq s^\eta(1 + \varepsilon_0) + C(\varepsilon_0) t^\eta$. Hence:

$$(X + Y + Z)^\eta \leq (1 + \varepsilon_0)X^\eta + C(\varepsilon_0)(Y + Z)^\eta \leq (1 + \varepsilon_0)X^\eta + C(\varepsilon_0, \eta)(Y^\eta + Z^\eta). \quad (A.2)$$

Notice that

$$\int_{\mathbb{R}^N} X^\eta = \int_{\Omega} |\zeta_i| |\Delta u|, \quad \text{whereas} \quad \int_{\mathbb{R}^N} (Y^\eta + Z^\eta) \leq \int_{\Omega} (|u| + |\nabla u|)^\eta.$$

Thus

$$\sum_{i \in I} \left( \int_{\Omega_i} |\zeta_i^{1/\eta} u|^\eta \right)^\frac{n}{\eta} \leq \frac{2^\varepsilon}{S^\varepsilon} (1 + \varepsilon_0) \left( \sum_{i \in I} \int_{\Omega_i} |\Delta u|^\eta \right) + C_2(\varepsilon_0) \|u\|_{W^{1,\eta}}^\eta,$$

from which the conclusion follows, as

$$\frac{2^\varepsilon}{S^\varepsilon} (1 + \varepsilon_0) \leq \left( \frac{2^\varepsilon}{S^\varepsilon} + \varepsilon \right)^\eta$$

if $\varepsilon_0$ is small enough.
Case Two. We claim that, for any \( i \in J \) and for any \( \varepsilon > 0 \), there exists a constant \( B_2(\varepsilon) \) such that
\[
\sum_{i \in J} \left( \int_{\Omega_i \cap \Omega} \left| \zeta_i^{1/\eta} u \right|^\eta \right)^{\frac{\eta}{\eta'}} \leq \left( \frac{2 \hat{\chi}}{\tilde{S}} + \varepsilon \right)^\eta \left( \sum_{i \in J} \int_{\Omega_i} |\Delta u|^{\eta} \right) + B_2(\varepsilon) \| u \|_{W^{1,\eta}}.
\]
We first notice that if \( u \in W^{2,\eta}_{\rho}(\mathbb{R}^N) \) then
\[
\| u \|_{L^{\eta'}(\mathbb{R}^N)} \leq \frac{2 \hat{\chi}}{\tilde{S}} \| \Delta u \|_{L^{\eta}(\mathbb{R}^N)} \quad (A.3)
\]
Indeed, let us extend symmetrically \( u \in W^{2,\eta}_{\rho}(\mathbb{R}^N) \) to the whole space \( \mathbb{R}^N \), reflecting it with respect to the \( x_N \) axis. Let us call this extension \( \hat{u} \in W^{2,\eta}(\mathbb{R}^N) \):
\[
\hat{u}(x) := \begin{cases} u(x', x_N), & x_N > 0, \\ u(x', -x_N), & x_N < 0. \end{cases}
\]
One has
\[
\| \hat{u} \|_{L^{\eta'}(\mathbb{R}^N)} = 2^{1/\eta'} \| u \|_{L^{\eta'}(\mathbb{R}^N)} \quad \text{and} \quad \| \Delta \hat{u} \|_{L^{\eta}(\mathbb{R}^N)} = 2^{1/\eta} \| \Delta u \|_{L^{\eta}(\mathbb{R}^N)},
\]
and the desired inequality follows using the definition of \( S \), and recalling that \( W^{2,\eta}(\mathbb{R}^N) \subset D^{2,\eta}(\mathbb{R}^N) \).

We now follow closely [H] Section 4. More precisely, write \( x = (x', x_N) \in \mathbb{R}^N \). We know that there exist \( R > 0 \) and a smooth function \( \rho \) such that \( \{x' \in \mathbb{R}^{N-1} : |x'| < R \} \rightarrow \mathbb{R}^+ \) such that (up to a rotation)
\[
\Omega \cap B_R = \{ (x', x_N) \in B_R : x_N > \rho(x') \}, \quad \partial \Omega \cap B_R = \{ (x', x_N) \in B_R : x_N = \rho(x') \}.
\]
For any open subset \( V \subset \mathbb{R}^N \) we define \( \Phi : V \subset \mathbb{R}^N \rightarrow \mathbb{R}^N \) such that
\[
\Phi(y', y_N) = (y', \rho(|y'|) - y_N \rho(y', \rho(|y'|))),
\]
with \( \nu(x', \rho(x')) = (\nabla \rho(x'), -1) \) an outward orthogonal vector to the tangent space. Then the following maps are well defined for some open sets \( V_i \)
\[
\Phi^{-1}_i = (\Phi^{-1})(\epsilon, \Omega_i) \cap \Omega \rightarrow \Omega_i \subset \mathbb{R}^N.
\]
Notice that, for any \( u \in W^{2,\eta}_\rho(\Omega) \) and \( i \in J \), one has \( (\zeta_i^{1/\eta} u) \circ \Phi_i \in W^{2,\eta}_{\rho}(\mathbb{R}^N) \). Also, we may assume for \( \varepsilon_0 \) small enough
\[
|\det D\Phi_i(y)| \leq 1 + \varepsilon_0,
\]
for the details we refer to [H]. Finally, we call \( \theta_i = (\zeta_i^{1/\eta} u) \circ \Phi_i \in W^{2,\eta}_{\rho}(\mathbb{R}^N) \).

Then, using (A.3),
\[
\sum_{i \in J} \left( \int_{\Omega_i \cap \Omega} \left| \zeta_i^{1/\eta} u \right|^\eta \right)^{\frac{\eta}{\eta'}} = \sum_{i \in J} \left( \int_{V_i} \left| \theta_i(y) \right|^\eta |\det(D\Phi_i(y))| \right)^{\frac{\eta}{\eta'}} \leq (1 + \varepsilon_0)^{\frac{\eta}{\eta'}} \sum_{i \in J} \left( \int_{V_i} |\theta_i(y)|^\eta \right)^{\frac{\eta}{\eta'}} \leq \left( \frac{2 \hat{\chi}}{\tilde{S}} \right)^\eta (1 + \varepsilon_0)^{\frac{\eta}{\eta'}} \sum_{i \in J} \int_{V_i} |\Delta \theta_i(y)|^\eta.
\]
Using estimates (47)-(48) in [H], and by applying (A.2) as we did above, we get the conclusion, up to choosing a suitably small \( \varepsilon_0 \).

Conclusion. Let \( u \in W^{2,\eta}_{\rho}(\Omega) \). We have
\[
\| u \|_{W^{2,\eta}_{\rho}} = \| u \|_{W^{2,\eta}_{2,\rho}} = \left\| \sum_{i \in J} \zeta_i u \right\|_{W^{2,\eta}_{\rho}} \leq \sum_{i \in J} \| \zeta_i u \|_{W^{2,\eta}_{2,\rho}} \leq \sum_{i \in J} \| \zeta_i^{1/\eta} u \|_{L^{\eta}} \leq \sum_{i \in J} \left( \int_{\Omega_i \cap \Omega} \left| \zeta_i^{1/\eta} u \right|^\eta \right)^{\frac{\eta}{\eta'}} + \sum_{i \in J} \left( \int_{\Omega_i \cap \Omega} |\Delta u|^\eta \right)^{\frac{\eta}{\eta'}}.
\]
Then using the previous steps,
\[
\| u \|_{W^{2,\eta}_{\rho}} \leq \left( \frac{2 \hat{\chi}}{\tilde{S}} + \varepsilon \right)^\eta \sum_{i \in J} \left( \int_{\zeta_i \Delta u|^\eta} \right) + B(\varepsilon) \| u \|_{W^{1,\eta}},
\]
from which the conclusion follows using the fact that \( \zeta_i \) is a partition of unity, taking the power \( 1/\eta \) and recalling that \( (s + t)^{1/\eta} \leq s^{1/\eta} + t^{1/\eta} \) as \( \eta \geq 1 \).
B Asymptotic estimates

Lemma B.1. Assume p, q satisfy (4.1) or (4.2). One has

\[ \| \partial_r U_\varepsilon \|_{L^{\frac{2(N-1)}{N}}(\partial \Omega)} \leq h_\varepsilon(\varepsilon) := \begin{cases} \varepsilon^{r(\frac{N+1}{2}+q)} & \text{if } q > \frac{N+4}{2(N-2)} \\ \varepsilon^{\frac{N+1}{2}+q(N-2)-2} \log \varepsilon & \text{if } q = \frac{N+4}{2(N-2)} \\ \varepsilon^{\frac{N+1}{2}+q(N-2)-2} & \text{if } q < \frac{N+4}{2(N-2)}. \end{cases} \]  

(B.1)

Let (4.3) hold. Then

\[ \| \partial_r U_\varepsilon \|_{L^{\frac{2}{N}}(\partial \Omega)} \leq c \varepsilon^{2^{N+1} (\log \varepsilon) \frac{3}{2}}. \]  

(B.2)

Proof. Let us first consider the case \( N \geq 5 \), namely we first prove (B.1). Notice that

\[ \partial_r U_\varepsilon(x) = \partial_r(\varepsilon^{-\frac{N-1}{2}} U(x/\varepsilon)) = \varepsilon^{-\frac{N-1}{2}} \nabla U(x/\varepsilon) \cdot \nu(x) = \varepsilon^{-\frac{N-1}{2}} U'(|x|/\varepsilon)^\frac{\varepsilon}{|x|} \cdot \nu(x). \]  

(B.3)

Now let \( \eta > 0 \) be as in (4.4). For each \( x \in B_\eta(0) \cap \partial \Omega \), one has (4.6), thus

\[ \frac{x}{|x|}, \nu(x) = \sum_{j=1}^{N-1} 2 \rho_j x_j - x_N + O(|x'|^3) \]  

and therefore

\[ |\partial_r U_\varepsilon(x)| \leq \varepsilon^{-\frac{N-1}{2}} \left| U'\left( \frac{(|x', \rho(x')|)}{\varepsilon} \right) \right| \left| \sum_{j=1}^{N-1} \rho_j x_j + O(|x'|^3) \right|. \]  

(B.4)

We split the norm as follows

\[ \int_{\partial \Omega} |\partial_r U_\varepsilon|^{\frac{2(N-1)}{N}} = \int_{\partial \Omega \cap B_\eta(0)} |\partial_r U_\varepsilon|^{\frac{2(N-1)}{N}} + \int_{\partial \Omega \cap B_\eta(0)} |\partial_r U_\varepsilon|^{\frac{2(N-1)}{N}}. \]  

(B.5)

We also observe that estimates (2.12) and (2.14) tell us that

\[ rU'(r) \approx \begin{cases} r^{2-N} & \text{if } q > \frac{N}{N-2} \\ r^{2-N} \log r & \text{if } q = \frac{N}{N-2} \\ r^{-q(N-2)} & \text{if } q < \frac{N}{N-2}. \end{cases} \]  

(B.6)

as \( r \to \infty \), and that

\[ \frac{N+4}{2(N-2)} < \frac{N}{N-2}. \]

Step 1. Estimate of (I). The first integral is, calling \( \Delta_\eta \) the projection of

\[ \partial \Omega \cap B_\eta(0) = \{(x', x_N) \in B_\eta(0) : x_N = \rho(x') = \sum_{j=1}^{N-1} \rho_j x_j + O(|x'|^3))\} \]

onto the first \( N-1 \) components, and observing that the \((N-1)\)-surface element is \( \sqrt{1 + |\nabla \rho(x')|^2} = 1 + O(|x'|) \) and using (B.4)

\[ \int_{\partial \Omega \cap B_\eta} |\partial_r U_\varepsilon|^{\frac{2(N-1)}{N}} \leq c \varepsilon^{2(\frac{N+1}{2}+q)} \int_{\partial \Omega \cap B_\eta(0)} \left| U'\left( \frac{(|x', \rho(x')|)}{\varepsilon} \right) \right| \left| \sum_{j=1}^{N-1} \rho_j x_j + O(|x'|^3) \right|^{\frac{2(N-1)}{N}} \]

\[ \leq c \varepsilon^{2(\frac{N+1}{2}+q)} \left( \int_{\Delta_\eta} \left| U'\left( \frac{(|x', \rho(x')|)}{\varepsilon} \right) \right| |x'|^{\frac{2(N-1)}{N}} + \int_{\Delta_\eta} \left| U'\left( \frac{(|x', \rho(x')|)}{\varepsilon} \right) \right| |x'|^{\frac{2(N-1)}{N}} \right). \]

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Therefore:

\[
\int_{\Delta_n} \left| U' \left( \left( \frac{\rho(x')}{\varepsilon} \right) \right) \right| x' \right|^{2(N-1)} \frac{dx'}{\varepsilon^{N-2}} = \begin{cases} 
O \left( \varepsilon^{N-1 + \frac{2(N-1)}{N}} \right) & \text{if } q > \frac{N+4}{2(N-2)} \\
O \left( \varepsilon^{N-1 + \frac{2(N-1)}{N}} \log \varepsilon \right) & \text{if } q = \frac{N+4}{2(N-2)} \\
O \left( \varepsilon^{2(N-1) \left( q(N-2) - 1 \right)} \right) & \text{if } q < \frac{N+4}{2(N-2)}
\end{cases}
\] (B.8)

To check this, we change variable \( \varepsilon y' = x' \) and use the expression of \( \rho \) to obtain

\[
|\varepsilon y'| \geq |(y', \rho(\varepsilon y'))| \geq |y'|.
\]

Then

\[
\int_{\Delta_n} \left| U' \left( \left( \frac{\rho(x')}{\varepsilon} \right) \right) \right| x' \right|^{2(N-1)} \frac{dx'}{\varepsilon^{N-2}} = \varepsilon^{N-1 + \frac{2(N-1)}{N}} \int_{\Delta_n/\varepsilon} \left| U' \left( \left( \frac{\rho(y')}{\varepsilon} \right) \right) \right| \frac{dy'}{\varepsilon^{N-2}}
\]

\[
= \varepsilon^{N-1 + \frac{2(N-1)}{N}} \left( \int_{0}^{1} + \int_{1}^{\eta/\varepsilon} \right) \left| U' (r) r^{2(N-1)} \right| r^{N-2} \frac{dr}{\varepsilon^{N-2}}
\]

\[
= \varepsilon^{N-1 + \frac{2(N-1)}{N}} \left( 1 + \int_{1}^{\eta/\varepsilon} \frac{r^{2(N-1)} d\rho(r)}{\varepsilon^{N-2}} \right).
\] (B.9)

From (B.6) we have

\[
|U' (r) r^{2(N-1)}| r^{N-2} \approx \begin{cases} 
\frac{N-2}{N} r^{N-2 + (2-q(N-2)) \frac{2(N-1)}{N}} \log r & \text{if } q > \frac{N}{N-2} \\
\frac{N-2}{N} r^{N-2 + (2-q(N-2)) \frac{2(N-1)}{N}} & \text{if } q = \frac{N}{N-2} \\
\frac{N-2}{N} r^{N-2 + (2-q(N-2)) \frac{2(N-1)}{N}} & \text{if } q < \frac{N}{N-2}.
\end{cases}
\]

Therefore:

\begin{itemize}
  \item Case \( q > \frac{N+4}{2(N-2)} \). In this situation the function is integrable for \( r \in [1, \infty) \); indeed, in case \( q > \frac{N}{N-2} \) this is a consequence of the fact that \( N \geq 5 \), while in case \( q \leq \frac{N}{N-2} \) is simply because \( N - 2 + (2 - q(N-2)) \frac{2(N-1)}{N} < -1 \iff q > \frac{N+4}{2(N-2)}. \) Hence,
    \[
    (I.1) = O \left( \varepsilon^{N-1 + \frac{2(N-1)}{N}} \right).
    \]
  \item Case \( q = \frac{N+4}{2(N-2)} \). We have \( |U'(r) r^{2(N-1)}| r^{N-2} \approx 1/r \) as \( r \to \infty \) and
    \[
    (I.1) = O \left( \varepsilon^{N-1 + \frac{2(N-1)}{N}} \right) \left( 1 + \int_{1}^{\eta/\varepsilon} \frac{1}{r} \frac{dr}{\varepsilon^{N-2}} \right) = O \left( \varepsilon^{N-1 + \frac{2(N-1)}{N}} \right) (1 + \log \varepsilon) = O \left( \varepsilon^{N-1 + \frac{2(N-1)}{N}} \log \varepsilon \right).
    \]
  \item Case \( q < \frac{N+4}{2(N-2)} \). We have
    \[
    (I.1) = \varepsilon^{N+1 + \frac{2(N-1)}{N}} \left( 1 + \varepsilon^{-(N-1) - (2-q(N-2)) \frac{2(N-1)}{N}} \right)
    = O \left( \varepsilon^{\frac{(N-1)(N+2)}{N}} \right) \left( 1 + \varepsilon^{\frac{2(N-1)}{N} \left( q(N-2) - 1 \right)} \right),
    \]

since \( \frac{(N-1)(N+2)}{N} > \frac{2(N-1)}{N} (q(N-2) - 1) \iff q < \frac{N+4}{2(N-2)} \).
\end{itemize}
The three cases combined yield \((B.8)\), we have

\[
\int_{\partial\Omega} |\partial_\nu U_\varepsilon|^{2N_N} = \int_{\partial\Omega \setminus B_\eta(0)} (\partial_\nu U_\varepsilon)^{2N_N} + \begin{cases}
O(\varepsilon^{N-1}(p-1) + \log \varepsilon) & \text{if } q > \frac{N+4}{2(N+2)} \\
O(\varepsilon^{N-1}(p-1) + \log \varepsilon) & \text{if } q = \frac{N+4}{2(N+2)} \\
O(\varepsilon^{N-1}(p-1) + \log \varepsilon) & \text{if } q < \frac{N+4}{2(N+2)}.
\end{cases}
\] (B.10)

Step 2. Estimate of \((II)\). Recalling \((B.3)\) and \((B.6)\) we have, for \(x \in \partial\Omega \setminus B_\eta(0)\),

\[
|\partial_\nu U_\varepsilon| \leq \varepsilon^{-\frac{N}{p-1}} |U'|(|x|/\varepsilon) = \begin{cases}
O(\varepsilon^{N-2-\frac{N}{p-1}}) & \text{if } q > \frac{N}{N-2} \\
O(\varepsilon^{N-2-\frac{N}{p-1}} \log \varepsilon) & \text{if } q = \frac{N}{N-2} \\
O(\varepsilon^{(N-2)-2-\frac{N}{p-1}}) & \text{if } q < \frac{N}{N-2}.
\end{cases}
\]

Thus (because \(N > 4 \iff \frac{N+4}{2(N+2)} < \frac{N}{N-2}\))

\[
\int_{\partial\Omega \setminus B_\eta(0)} (\partial_\nu U_\varepsilon)^{2N_N} = \begin{cases}
O(\varepsilon^{\frac{N-1}{N-2} - \frac{N}{p-1}}) & \text{if } q > \frac{N}{N-2} \\
O(\varepsilon^{\frac{N-1}{N-2} - \frac{N}{p-1}} \log \varepsilon) & \text{if } q = \frac{N}{N-2} \\
O(\varepsilon^{\frac{N-1}{N-2} - \frac{N}{p-1} + \log \varepsilon} & \text{if } q < \frac{N}{N-2}.
\end{cases}
\] (B.11)

Hence, combining \((B.10)\) with \((B.11)\) we obtain \((B.1)\).

We now consider the case \(N = 4\), and we prove \((B.2)\). We again split the norm as in \((B.5)\), and reasoning as in \((B.7)\) and \((B.9)\) we get

\[
\int_{\partial\Omega \setminus B_\eta} |\partial_\nu U_\varepsilon|^2 \leq \varepsilon^{-\frac{2}{p+1}} \int_{\Delta_\theta} \left| U' \left( \frac{1}{|x'|, \rho(x') \varepsilon} \right) \right|^2 dx' \leq \varepsilon^{\frac{2}{p+1}} \left( 1 + \int_{1}^{\varepsilon^{-q/\varepsilon}} |U'(r)|^{\frac{2}{N-2}} r^{-N-2} dr \right).
\]

We now observe that

\[
|U'(r)|^{\frac{2}{N-2}} r^{-N-2} \approx r^{-1},
\]

thus

\[
\int_{\partial\Omega \setminus B_\eta} |\partial_\nu U_\varepsilon|^2 \leq \varepsilon^{\frac{2}{p+1}} \log \varepsilon.
\]

The estimate of \((II)\) is exactly as in Step 2 (recall that \((4.3)\) implies \(q > 2\)), so we get

\[
\int_{\partial\Omega \setminus B_\eta(0)} (\partial_\nu U_\varepsilon)^{2N_N} \leq \varepsilon^{\frac{2}{p+1} + \frac{N}{2}} = o(\varepsilon^{\frac{2}{p+1} + \frac{N}{2}} |\log \varepsilon|).
\]

We now immediately deduce \((B.2)\). \(\square\)

Lemma B.2. Assume \(p, q\) satisfy \((4.1)\) or \((4.2)\). One has

\[
\|\partial_\nu V_\varepsilon\|_{L^{2(N-1)}(\partial\Omega)} \leq \varepsilon^{-\frac{N}{p+1} + \frac{N}{2}}
\]

If \((4.3)\) holds, then

\[
\|\partial_\nu V_\varepsilon\|_{L^2(\partial\Omega)} \leq \varepsilon^{\frac{2}{p+1} + \frac{N}{2}} (\log \varepsilon)^{\frac{N}{2}}.
\]
Proof. The proof is quite similar to the one of Lemma \[3.1\] to which we refer for the notation and more details. We have

\[
\int_{\partial \Omega \cap B_\eta(0)} |\partial_\nu V|^{2(N-1)} \leq c \varepsilon^{-2(N-1)} \frac{N-1}{N} \int_{\Delta_\eta} \left| V' \left( \frac{[(x', \rho(x')]}{\varepsilon} \right) \right|^{2(N-1)} \\
\leq c' \varepsilon^{-(N-1)} \frac{N-1}{N} \int \left| V'(r) \right|^{2(N-1)} r^{N-2} dr \\
= c' \varepsilon^{(N-1)q/(q-1)} \left( 1 + \int_1^{\eta/\varepsilon} \left| V'(r) \right|^{2(N-1)} r^{N-2} dr \right).
\]

Now estimates \[2.12\] and \[2.14\] provide

\[
rV'(r) \approx r^{2-N}, \quad \text{so that} \quad \left| rV''(r) \right|^{2(N-1)} r^{N-2} \approx r^{-(N-2)^2},
\]

which is integrable at infinity if \(N \geq 5\), and is equal to \(r^{-1}\) if \(N = 4\). Then if \(N \geq 5\)

\[
\int_{\Omega} |\partial_\nu V|^2 = \int_{\partial \Omega \cap B_\eta(0)} |\partial_\nu V|^2 + \int_{\partial \Omega \setminus B_\eta(0)} |\partial_\nu V|^2 = O(\varepsilon^{(N-1)q/(q+1)})
\]

whereas if \(N = 4\)

\[
\int_{\Omega} |\partial_\nu V|^2 = O(\varepsilon^{3/2} |\log \varepsilon|)
\]

and the proof is finished. \(\square\)

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