Improved return level estimation via a weighted likelihood, latent spatial extremes model

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Abstract: Uncertainty in return level estimates for rare events, like the intensity of large rainfall events, makes it difficult to develop strategies to mitigate related hazards, like flooding. Latent spatial extremes models reduce uncertainty by exploiting spatial dependence in statistical characteristics of extreme events to borrow strength across locations. However, these estimates can have poor properties due to model misspecification: many latent spatial extremes models do not account for extremal dependence, which is spatial dependence in the extreme events themselves. We improve estimates from latent spatial extremes models that make conditional independence assumptions by proposing a weighted likelihood that uses the extremal coefficient to incorporate information about extremal dependence during estimation. This approach differs from, and is simpler than, directly modeling the spatial extremal dependence; for example, by fitting a max-stable process, which is challenging to fit to real, large datasets. We adopt a hierarchical Bayesian framework for inference, use simulation to show the weighted model provides improved estimates of high quantiles, and apply our model to improve return level estimates for Colorado rainfall events with 1% annual exceedance probability.

Keywords: Bayesian, climate, extremal coefficient, Generalized extreme value distribution
1 Introduction

Natural hazards with potentially catastrophic impacts arise as extremes of physical processes that are inherently dependent over space, such as large storms that generate extreme precipitation. Accordingly, the statistical modeling of spatially-referenced extreme values has been an active research area in recent years. To effectively plan mitigation strategies for natural hazards caused by extreme precipitation, it is important to build maps that estimate occurrence probabilities and return levels for extreme precipitation events at individual locations. Return level maps for individual locations inform building safety standards, insurance risks, and surface water runoff requirements for stormwater management systems. However, extreme events are rare by definition, so relevant datasets from networks of environmental monitoring stations typically have relatively short observation lengths. Spatial extremes models allow the tails of probability distributions to be estimated while “borrowing strength” from neighboring time series. Widely used to borrow strength, hierarchical models share statistical information across sampling locations to obtain more accurate and spatially consistent estimates of extreme event characteristics.

Often in extremes studies, the primary interest is in modeling return levels of extreme events at individual locations. Latent spatial extremes models are a flexible and computationally efficient class of models for marginal distributions of spatial extremes and quantities derived from them, like return levels. Latent spatial extremes models use a hierarchical framework to add spatial structure to the parameters of an extreme value distribution. Many hierarchical frameworks assume observations of extremes are independent across sampling locations, conditional on the latent spatial processes that specify the data’s marginal distributions. Hierarchical spatial layers induce smoothness and correlation in marginal return level estimates across sampling locations, and—critically—allow return level maps to be built using spatial interpolation techniques, like kriging. As such, return level estimates “borrow strength” because estimates balance data at each sampling location with spatial smoothing induced by the latent hierarchical layers. For example, Cooley et al. (2007) use
latent Gaussian processes in a hierarchical Bayesian model to capture covariate-driven trends and spatial dependence in precipitation data. Bayesian frameworks allow direct estimation of uncertainties in return levels since the posterior distribution contains this information. Latent spatial Gaussian process models can also be scaled to massive datasets with recent advances in models and computational techniques (Lindgren et al., 2011; Rue et al., 2009). Other recent studies employ latent spatial extremes models in either Bayesian or frequentist paradigms (Cooley and Sain, 2010; Lehmann et al., 2016; Opitz et al., 2018; Sang and Gelfand, 2009). However, due to the conditional independence assumption, these examples of latent spatial extremes model cannot account for extremal dependence, which is dependence in observations of extreme events themselves.

Directly modeling extremal dependence poses theoretical and computational challenges. Classical univariate and multivariate extreme value models are generated via asymptotic arguments about the limiting distributions of appropriately renormalized block maxima. The natural extension to the spatial setting is the max-stable process, which is the limiting process of the componentwise maxima of a sequence of suitably renormalized stochastic processes. Examples include the Smith (1990), Schlather (2002), and Brown-Resnick (Brown and Resnick, 1977; Kabluchko et al., 2009) processes. The advantage of max-stable process modeling is that it directly models spatial dependence in the tail and thus permits inference about joint probabilities in addition to marginal quantities, like return levels. However, full likelihood inference for max-stable processes is only computationally tractable in relatively low-dimensional situations (Castruccio et al., 2016; Davison et al., 2012).

In particular, computationally efficient Bayesian methods for spatially-dependent extremes data remains challenging. Frequentist inference for max-stable processes has typically been based on computationally efficient models that use approximate likelihoods, such as composite likelihoods based on bivariate densities of max-stable processes (Padoan et al., 2010). However, composite likelihood methods are computationally expensive and difficult to implement in hierarchical Bayesian models (Ribatet et al., 2012; Sharkey and Winter,
Some Bayesian models do not need to use approximate likelihoods, but are limited to specific max-stable processes or require additional data for estimation (Reich and Shaby, 2012; Thibaud et al., 2016).

The latent spatial extremes approaches previously introduced address computational issues while providing flexible models for estimating marginal parameters, but raise concerns about the impact of model misspecification on inference. These models make a simplifying conditional independence assumption by defining the likelihood to be the product of each location’s marginal density. The misspecification due to the conditional independence assumption can result in unrealistically narrow confidence intervals for return level estimates (Cao and Li, 2018; Zheng et al., 2015). Alternative to assuming conditional independence or using computationally expensive models to account for extremal dependence, we seek a compromise between the two modeling approaches. We want to preserve computationally efficient and flexible models for marginal parameters provided by latent variable models, but also account for extremal dependence in observations.

We propose a method for improving marginal inference that is supported by theory and computationally efficient. We develop a weighted likelihood that uses spatial information to induce an effective sample size correction that accounts for the loss of information due to dependent observations. The likelihood weights improve uncertainty estimates in cases of moderate to strong extremal dependence. The effective sample size motivation differs from previous uses of weighted likelihoods. Weighted likelihoods have previously been used to approximate Bayesian inference and as a method for conducting inference on data sampled from multiple, related populations, for example in Hu and Zidek (2002); Newton and Raferty (1994); Wang (2006). Weighted likelihoods have also recently been proposed for latent spatial extremes models, but only as they relate to composite likelihood corrections (Sharkey and Winter, 2018). A natural tradeoff in using likelihood weights to better account for estimation uncertainty is that mean squared error can be slightly worse in these cases.

The remainder of the article is organized as follows. Section 2 introduces our weighted
likelihood and Bayesian implementation. Section 3 uses a simulation study to show that the weighted likelihood improves estimates, as compared to several models with similar Bayesian hierarchical structure. As part of our comparisons, we derive the penalized complexity prior for the generalized extreme value (GEV) distribution (Supplement Section C). Section 4 applies the weighted likelihood latent model to daily rainfall observations in Colorado’s Front Range of the Rocky Mountains. We conclude with discussions of extensions and other directions for future work (Section 5).

2 Weighted likelihood latent spatial extremes models

We briefly review extreme value theory for modeling return levels from observations of annual maxima (Section 2.1). In particular, we introduce the extremal coefficient, which we will use to build our weights. We then propose and interpret a latent variable model with a weighted likelihood to estimate marginal quantities from spatially-dependent extremes data (Section 2.2, Section 2.3). When data are dependent, the weighted likelihood accounts for model misspecification in the latent variable modeling approach by Cooley et al. (2007), which assumes data are conditionally independent, given marginal parameters. The model has a hierarchical spatial structure, for which posterior distributions can be approximated via Gibbs sampling (Section 2.4, Section 2.5).

2.1 Max-stable processes and the extremal coefficient

Max-stable processes for spatially-referenced extremes data arise as the pointwise limit of block maxima, which are pointwise maxima of replications of spatially-referenced processes. Let $\mathcal{D}$ be a continuous spatial domain and $\{Y_{it}(s)\}_{s \in \mathcal{D}}, t \in \{1, \ldots, m\}$ be $m$ independent replications of a spatial process at time block $i \in \mathcal{T} = \{1, \ldots, T\}$. The size of each block
\( i \in \mathcal{T} \) is represented by \( m \). As the block size \( m \) increases, if the limit

\[
Y_i(s) = \lim_{m \to \infty} \frac{\max_{t=1}^m Y_{it}(s) - b_m(s)}{a_m(s)}, \ s \in \mathcal{D}
\]

exists for continuous functions \( a_m(s) > 0 \) and \( b_m(s) \in \mathbb{R} \), then \( \{Y_i(s)\}_{s \in \mathcal{D}}, \ t \in \mathcal{T} \) are independent replications of a max-stable process (De Haan, 1984).

In general, the spatial dependence structure for max-stable processes \( \{Y_i(s)\}_{s \in \mathcal{D}} \) is complex, but is often summarized for pairs of random variables \( Y_i(s) \) and \( Y_i(t) \) through the extremal coefficient. The extremal coefficient \( \theta(d) \) is a function that is traditionally defined implicitly for stationary and isotropic fields such that

\[
P(Y_i(s) \leq y, Y_i(t) \leq y) = P(Y_i(s) \leq y)^{\theta(d)}
\]

for pairs of random variables \( Y_i(s) \) and \( Y_i(t) \) where \( d = \|s - t\| \) (Schlather and Tawn, 2003). The extremal coefficient is interpretable as the effective number of independent random variables among pairs of variables separated by a distance \( d \). As such, it takes values in the closed interval \([1, 2]\).

Importantly, the univariate marginal distributions for max-stable processes belong to the generalized extreme value distribution family \( Y_i(s) \sim \text{GEV} (\eta(s)) \) with distribution function

\[
P(Y_i(s) \leq y) = \begin{cases} 
\exp \left\{ - \left( 1 + \xi(s) \left( \frac{y - \mu(s)}{\sigma(s)} \right) \right)^{-1/\xi(s)} \right\} & \xi(s) \neq 0 \\
\exp \left\{ - \exp \left\{ \frac{y - \mu(s)}{\sigma(s)} \right\} \right\} & \xi(s) = 0
\end{cases}
\]

where \( a_+ = \max(0, a) \) (De Haan, 1984). The parameter vector \( \eta(s) = (\mu(s), \log \sigma(s), \xi(s))^T \) specifies the distribution’s location \( \mu(s) \in \mathbb{R} \), scale \( \sigma(s) > 0 \), and shape \( \xi(s) \in \mathbb{R} \) parameters.
The GEV quantile function \( Q(p|\eta(s)) \) is derived from (2) and has the closed form

\[
Q(p|\eta(s)) = \begin{cases} 
\mu(s) + \frac{\sigma(s)}{\xi(s)} \left( (-\log p)^{-\xi(s)} - 1 \right) & \xi(s) \neq 0 \\
\mu(s) - \sigma(s) \log (-\log p) & \xi(s) = 0
\end{cases}
\]

with \( p \in [0, 1] \).

Asymptotic convergence justifies use of the GEV distribution as an approximate model for the annual maximum of daily precipitation in year \( i \), which is a block maximum quantity that has large but finite replication \( t \in \{1, \ldots, m\} \). The approximation allows marginal return levels for extreme precipitation events to be modeled as high quantiles of the GEV distribution at each location \( s \in \mathcal{D} \). Assuming a stationary climate, the quantile \( Q(p|\eta(s)) \) with \( p = 1 - 1/r \) is interpretable as the \( r \)-year return level—the amount of precipitation carried by a storm that occurs, on average, once every \( r \) years. The quantile \( Q(p|\eta(s)) \) is also associated with the \( 1 - p \) percent annual exceedance probability; the quantile expresses the amount of precipitation carried by a storm that has a \( 1 - p \) percent chance of occurring in a given year.

### 2.2 Weighted likelihood

We propose a latent variable model that uses a weighted marginal likelihood. In general, weighted likelihoods are misspecified but can improve inference relative to unweighted likelihoods. A correctly-specified likelihood for spatial extremes data would fully account for extremal dependence, but be computationally intractable. Marginal likelihoods assume data are conditionally independent across spatial locations and timepoints, given marginal parameters. When the field \( \{Y_i(s)\}_{s \in \mathcal{D}} \) is sampled at \( N \) spatial locations \( S = \{s_1, \ldots, s_N\} \subset \mathcal{D} \), the weighted marginal likelihood for a finite sample of observations \( \{y_i(s_j) : i \in \mathcal{T}, s_j \in S\} \)
is defined via

\[ L(\eta) = \prod_{j=1}^{N} \prod_{i=1}^{T} f(y_i(s_j) | \eta(s_j))^w_{s_j} \]

where \( f(y_i(s_j) | \eta(s_j)) \) is the probability density function for the GEV distribution (2) and \( \eta(s) \) is the associated parameter vector. The weighted marginal likelihood (4) uses likelihood weights \( \{ w_{s_j} : j = 1, \ldots, N \} \) and marginal densities \( \{ f(y_i(s_j) | \eta(s_j)) : j = 1, \ldots, N \} \) to estimate the marginal parameters \( \{ \eta(s_j) \in \mathbb{R}^3 : j = 1, \ldots, N \} \) that have been stacked to form the vector \( \eta \in \mathbb{R}^{3N} \). During estimation, likelihood weights can be constructed to downweight observations for \( y_i(s_j) \) that exhibit strong dependence with neighboring observations. Models assuming conditional independence naively assume the weights are unitary.

We use the extremal coefficient in (1) to construct weights that downweight likelihood contributions from locations central to the spatial sampling pattern, where observations tend to be most dependent. We construct each weight \( w_{s_j} \) by first mapping extremal coefficients \( \theta(\| s_i - s_j \|) \) for \( i \neq j \) to the interval \([1/N, 1]\), then averaging the mapped values, yielding

\[ w_{s_j} = \frac{1}{N - 1} \sum_{i=1, i \neq j}^{N} N^{\theta(\| s_i - s_j \|) - 2}, \]

so \( w_{s_j} \in [1/N, 1] \). The weights (5) are specifically constructed so that the statistical information in the weighted marginal likelihood (4) matches the statistical information in non-misspecified likelihoods in two special, limiting cases (Supplement Section A.1). In the first limiting case, the field is assumed to be independent, and \( w_{s_j} = 1 \); in the second limiting case, the field \( \{ Y_i(s) \}_{s \in D} \) is assumed to have complete dependence over space, and \( w_{s_j} = 1/N \). The field \( \{ Y_i(s) \}_{s \in D} \) has complete dependence over space if all potential samples \( \{ Y_i(s_1), \ldots, Y_i(s_N) \} \) can be represented through a collection of continuous transformations \( \{ g_j : j = 1, \ldots, N \} \) of a variable \( U_i \) such that

\[ (Y_i(s_1), \ldots, Y_i(s_N)) \overset{d}{=} (g_1(U_i), \ldots, g_N(U_i)). \]
2.3 Effective sample size interpretation

From an information-theoretic perspective, we show that the weighted likelihood (4) induces an effective sample size that corrects inference on spatially correlated marginal parameters when data are also spatially dependent. Effective sample size statistics quantify the impact that dependence has on estimation uncertainty (e.g., Cressie, 1993, p. 13). We use effective sample size to determine factors that will impact estimator performance in our simulation (Section 3). In our application, effective sample size also helps us better interpret losses in statistical efficiency due to dependence in observations of extremes (Section 4).

The Fisher information for (4) is the block diagonal matrix $I(\eta) \in \mathbb{R}^{Nm \times Nm}$ with $j^{th}$ diagonal block $I(\eta(s_j)) \in \mathbb{R}^{m \times m}$ being

$$I(\eta(s_j)) = w_{s_j} T I_{Y_s(s_j)}(\eta(s_j)),$$

where $I_{Y_s(s_j)}(\eta(s_j))$ is the expected Fisher information for each of the independent and identically distributed random variables \{$Y_i(s_j) : i \in T$\}. Note that the $j^{th}$ block (6) is the Fisher information for $w_{s_j} T$ independent observations of the response at $s_j$. Thus, $w_{s_j}$ quantifies the effective proportion of independent observations at location $s_j$ that contribute to inference for the marginal GEV parameters $\eta(s_j)$. As the likelihood weight $w_{s_j}$ decreases, uncertainty increases about the marginal GEV parameters $\eta(s_j)$ and return level $Q(p | \eta(s_j))$. Latent spatial extremes models with unweighted likelihoods can be interpreted as implicitly assigning $w_{s_j} = 1$ for all locations $s_j \in S$. Such a strategy underestimates parameter uncertainty when data have extremal dependence.

2.4 Hierarchical specification

We adopt a hierarchical Bayesian framework to conduct inference on the weighted marginal likelihood, and facilitate spatial interpolation of marginal return levels (3). We specify a hierarchical spatial process model for the marginal parameters at each spatial location
\( \eta(s) = (\mu(s), \log \sigma(s), \xi(s))^T \in \mathbb{R}^3 \) via

\[
(7) \quad \eta(s) = \begin{bmatrix}
    x_{\mu}(s)^T \\
    x_{\log \sigma}(s)^T \\
    x_{\xi}(s)^T
\end{bmatrix}
\begin{bmatrix}
    \beta_{\mu} \\
    \beta_{\log \sigma} \\
    \beta_{\xi}
\end{bmatrix} + \begin{bmatrix}
    \varepsilon_{\mu}(s) \\
    \varepsilon_{\log \sigma}(s) \\
    \varepsilon_{\xi}(s)
\end{bmatrix},
\]

in which \( x(s) \) and \( \beta \) are respectively \( p \times 1 \) vectors of regression covariates and coefficients, and \( \varepsilon(s) \) represents spatially-correlated variation in the marginal parameters \( \eta(s) \). The matrix of covariates in (7) is block-diagonal; the blank, off-diagonal entries represent zeros. We use diffuse normal priors for regression coefficients \( \beta \). Independent Gaussian processes model the spatially-correlated variation in \( \varepsilon_{\mu}(s) \), \( \varepsilon_{\log \sigma}(s) \), and \( \varepsilon_{\xi}(s) \). Gaussian processes imply finite samples of parameters are jointly-normally distributed and allow estimation of spatially-coherent marginal parameter maps \( \{ \eta(s) \}_{s \in D} \) through kriging. Furthermore, stationary isotropic Gaussian processes are sufficient models when departures from stationarity and isotropy are difficult to detect (Cooley et al., 2007).

The Gaussian processes for marginal parameters are fully defined by specifying covariance functions \( \text{Cov}(\varepsilon(s), \varepsilon(t) | \phi) = \rho(\|s - t\|; \phi) \) to model the spatial correlation in the parameters between locations \( s, t \in D \). Specific choices for covariance functions \( \rho \) and hyperprior distributions for covariance parameters \( \phi = (\sigma_0, \lambda_0, \nu_0)^T \) are discussed in Section 3.2.2 and Section 4.2. In general, we use weakly informative Gamma priors for covariance range \( \lambda_0 \) and smoothness \( \nu_0 \) parameters, and weakly informative Inverse-Gamma priors for covariance sill parameters \( \sigma_0 \).

### 2.5 Bayesian estimation

A Gibbs sampler can be constructed for inference on the hierarchical Bayesian model specified in Section 2.4, in which likelihood weights (5) are updated with the aid of a plug-in estimator for the extremal coefficient. The Bayesian framework allows estimates of return levels \( Q(p|\eta(s)) \) to be computed directly from posterior samples of the marginal param-
parameter vector $\eta(s)$ since return levels are functions of marginal parameters. The sampler is described in detail in Supplement Section B.1, and key points are summarized here. Standard hybrid Gibbs sampling approaches are used to sample the marginal GEV parameters, covariance parameters, and regression coefficients.

Likelihood weights (5) are computed with a plug-in estimator $\hat{\theta}(d)$ for the extremal coefficient (Cooley et al., 2006). The plug-in estimator uses sample statistics from the data that have been transformed to have unit Fréchet marginal distributions. Thus, the likelihood weights depend on estimates of the marginal distributions, either estimated through the empirical cumulative distribution function (CDF), or directly through the GEV CDF. Before Gibbs sampling begins, we initialize likelihood weights by using the empirical CDF at each location $\hat{F}(y; s_j) = T^{-1} \sum_{i=1}^{T} 1 \{ y_i(s_j) \leq y \}$ to transform the data via probability integral transforms. These initial weights may be held fixed and used throughout Gibbs sampling or updated at each Gibbs iteration. To update the weights at each Gibbs iteration, the data may be retransformed by using the GEV CDF (2) with the marginal parameters $\eta$ from the previous Gibbs iteration. Updating likelihood weights during Gibbs sampling accounts for uncertainty in the likelihood weights.

3 Simulation study

We use simulation to show that the weighted marginal likelihood (4) improves high quantile estimates on datasets with realistic GEV parameters $\eta(s)$, sample sizes, and varying extremal dependence. The simulation compares the weighted likelihood model (Section 3.2) to a standard, unweighted latent spatial extremes model and a penalized variation. Penalization is an alternate approach used to correct return level estimates in extreme value models (cf. Opitz et al., 2018; Schliep et al., 2010). Penalized models have hierarchical structures that are similar to our weighted likelihood, so are comparison models with similar computational complexity to our weighted likelihood. We compare models by contrasting
properties of estimators of high quantiles, including empirical coverage and mean squared error (Section 3.3).

3.1 Datasets

We simulate data from four generating models with varying combinations of extremal dependence, and spatial $N$ and temporal $T$ sample sizes. Properties of parameter estimators are empirically approximated using 1,000 datasets simulated from each generating model. Our decision to vary extremal dependence, $N$, and $T$ is informed by the Fisher information (6) and effective sample size discussion (Section 2.2), which provide intuition about how extremal dependence and sample size affect estimation. Increasing extremal dependence decreases the amount of statistical information available for parameter estimation, much as occurs with classical spatial dependence (Cressie, 1993, Section 1.3). Similarly, the impact of extremal dependence increases when sampling more spatial locations $S = \{s_1, \ldots, s_N\} \subset D$ from a fixed domain $D$. Unweighted latent spatial extremes models are misspecified when data are dependent because they assume the data are conditionally independent given model parameters. The severity of the misspecification increases as the process is observed at more spatial locations $N$ because it becomes more likely that observations from spatially-dependent locations are included in the sample. The Fisher information equation (6), however, suggests that statistical information about the marginal parameters increases with the number of replications $T$ despite misspecification, albeit at a slower rate when using likelihood weights.

Simulated data have marginal GEV parameters $\eta(s)$ that mimic estimates from observed annual maximum daily precipitation across Colorado’s Front Range (Tye and Cooley, 2015). Spatially-dependent GEV parameters $\eta(s)$, $s \in D = [-10, 10]^2$ are sampled from Gaussian processes GP ($m$, $\rho$) with mean functions $m : D \rightarrow \mathbb{R}$ and powered exponential covariances $\rho : D^2 \rightarrow [0, \infty)$ specified in Table 1. Shape parameters $\xi(s)$ are resampled until $\xi(s) > 0$ for all $s \in S$ to ensure data are heavy-tailed. Brown–Resnick processes model extremal dependence in the simulated data (Kabluchko et al., 2009). The semi-variogram $\gamma : D^2 \rightarrow [0, \infty)$
specified in Table 1 parameterizes a Brown–Resnick model that induces strong, medium, or weak extremal dependence on $\mathcal{D}$ as measured by the extremal coefficient function $\theta(d)$. For comparison, independent data are also simulated.

### 3.2 Estimating models

The simulation compares estimation of conditionally independent models with weighted (4) and unweighted likelihoods (i.e., (4) with $w_{s_j} = 1$ for all $s_j \in S$) and a variation that uses penalized complexity priors as a likelihood penalty (Section 3.2.1). Key differences between the estimating models are summarized in Table 2. The comparison models represent different approaches proposed in the extremes literature to improve marginal estimation of GEV parameters and have similar computational complexity.

#### 3.2.1 Penalized complexity prior

Likelihood-based parameter estimates for the univariate GEV distribution are known to perform poorly, but penalized likelihoods can reduce estimation bias (Coles and Dixon, 1999; Martins and Stedinger, 2000). Penalized likelihoods have been incorporated into spatial models for marginal extremes (Opitz et al., 2018; Schliep et al., 2010). Penalization improves estimation of marginal parameters by downweighting estimates of large shape parameters $\xi(s)$, which tend to be uncommon in many extreme precipitation data. We adapt a contemporary penalty for use with the GEV distribution as a comparison model.

Penalized complexity (PC) priors have recently been proposed to improve parameter estimation in a related extreme value family—the Generalized Pareto distribution (GPD), which also uses scale $\sigma(s) > 0$, and shape $\xi(s) \in \mathbb{R}$ parameters to model threshold exceedances (Opitz et al., 2018). Penalized complexity priors satisfy several properties that optimize the prior distribution’s shape and scale to precisely control the prior’s influence over target likelihoods (Simpson et al., 2017). We implement PC priors as penalized likelihoods in our hierarchical spatial model. We derive the penalized complexity prior $\pi(\xi | \lambda)$ for the GEV
distribution in Supplement Section C and use it with the log-likelihood

$$\ell(\eta) = \sum_{j=1}^{N} \sum_{i=1}^{T} \log f(y_i(s_j)|\eta(s_j)) + \sum_{j=1}^{N} \log \pi(\xi(s_j)|\lambda)$$

in place of the log of the unweighted version of the likelihood (4), in which $w_{s_j} = 1$ for all $s_j \in S$.

Bayesian estimation optimizes the PC prior’s parameterization by specifying an Inverse-gamma prior distribution for $\lambda \sim IG(2, 1)$. The Inverse-gamma distribution is parameterized to have mean 1 and infinite variance. Prior distributions provide an alternative to cross-validation approaches for optimizing the prior’s parameterization, which is computationally infeasible for this simulation study (Hans, 2009; Park and Casella, 2008).

3.2.2 Bayesian specification

All models use a hierarchical Bayesian framework in which the GEV parameters $\eta(s)$ are estimated as independent latent Gaussian processes with functional forms matching those specified in Table 1. Prior distributions for the mean and covariance function parameters are either weakly informative or uninformative, and conjugate where possible. Full details are available in Supplement Section B.2.1. Inference is based on a sample from the posterior distribution, drawn with a Gibbs sampler. Estimators based on the weighted likelihood are evaluated with respect to both fixed and Gibbs-updated weights (See Section 2.5). Sample autocorrelation diagnostics indicate the Gibbs sampler mixes slowly, so the sampler was run for 155,000 iterations to ensure Monte Carlo integration error is sufficiently small. The first 5,000 samples were discarded. Posterior inference uses a thinned posterior sample consisting of 10,000 of the remaining 150,000 samples; only every fifteenth sample was saved because the entire posterior sample could not be efficiently stored and manipulated. Thinning reduces statistical efficiency of Markov chain Monte Carlo methods, but can be a necessary tradeoff when the full posterior sample is difficult to store and use to estimate posterior quantities.
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(MacEachern and Berliner, 1994).

3.3 Results

Assuming a stationary climate, the 1% annual exceedance probability \( Q(0.99|\eta(s)) \) from (3), also referred to as the 100-year return level, is often used to quantify risk for extreme weather events. The weighted model’s results are nearly identical when comparing fixed weights to Gibbs-updated weights, so we only present the fixed-weight results here; results for the Gibbs-updated weights are included in Supplement Section D. Figure 1 presents the empirical coverage of highest posterior density (HPD) intervals for the return level \( Q(0.99|\eta(s)) \) for each of the models listed in Table 2. Supplement Figure 7 presents mean squared error (MSE) for the same data. Bias is small for all estimators, so MSE mainly quantifies estimator variance. Since the return level \( Q(0.99|\eta(s)) \) is greatly influenced by the shape parameter, \( \xi(s) \), results for return levels and shape parameters are very similar. Supplement Section D includes results for all GEV parameters \( \eta(s) \) and other estimator properties.

Extremal dependence degrades the performance of all marginal models, but the weighted marginal likelihood (4) provides the most accurate estimates of uncertainty. Empirical coverage of 95% HPD intervals is closest to the nominal HPD level across all levels of extremal dependence. (Figure 1). For the \( N = 50, T = 50 \) simulation with moderate dependence, the weighted model has a coverage rate of 86%, while the unweighted model and penalized complexity prior model have coverage rates of 83% and 82% respectively. In the same scenario, the weighted model also has nearly identical MSE as the other models, although the MSE for the weighted likelihood model is somewhat greater for the simulation with strong dependence (Supplement Figure 7).
4 Extreme Colorado precipitation

4.1 Data

Previous studies of extreme precipitation in Colorado find that there is weak extremal dependence between locations along the state’s Front Range region (Cooley et al., 2007; Tye and Cooley, 2015). We determine the impact the weighted likelihood (4) has on estimates of the 1% annual exceedance probability $Q(.99 | \eta(s))$, also referred to as the 100-year return level. Estimates are based on the same subset of annual maxima of daily precipitation Tye and Cooley (2015) use from the Global Historical Climatology Network (GHCN) dataset (Menne et al., 2012). The subset includes annual maxima from 71 stations along the Front Range. Tye and Cooley (2015) fully describe their selection criteria, which, for example, include requirements that stations have been operational for at least 30 years. Additionally, annual maxima of daily precipitation are only analyzed from years with few missing daily records of precipitation. Between 18 and 120 annual maxima are analyzed for each station, with roughly equal representation of all temporal sample sizes.

Exploratory analysis suggests the Front Range GHCN data have between weak and moderate extremal dependence. The estimated extremal coefficient function $\hat{\theta}(d) : (0, \infty) \rightarrow [1, 2]$ is near-constant between 1.8 and 1.9 for all distances $d$, which implies the likelihood weights will also have a small range. Schlather and Tawn (2003) also observe a near-constant extremal coefficient function for extreme precipitation in south-west England. The authors remark that the result may have a physical basis because the study region is small relative to the scale of the meteorological systems that generate precipitation, which implies it is likely that no two sites in the region are truly independent. Likelihood weights (5) for the GHCN data are similar to weights for simulated data with moderate extremal dependence (Supplement Figure 20). Since the average number of annual maxima per station ($T = 60$) is also close to our $T = 50$ simulation, we anticipate the weighted likelihood will have closer to nominal coverage and the unweighted likelihood will slightly undercover (Figure 1).
4.2 Model and results

As in the simulation, we use the weighted marginal likelihood (4) in a hierarchical Bayesian framework in which the GEV parameters $\eta(s)$ are estimated as independent latent Gaussian processes. Since the simulation shows that estimators based on fixed and Gibbs-updated weights have similar properties, we use fixed weights during estimation. We use annual mean precipitation from the PRISM precipitation dataset (Daly et al., 2008) as a covariate for each of the GEV parameters, and model the spatial correlation between parameters with the Matérn covariance function. For example, the Matérn specifies the correlation between parameters $\xi(s)$ and $\xi(t)$ at two locations $s, t \in D$ via

$$
\kappa(s, t; \tau, \rho, \nu) = \frac{1}{\tau^{2\nu - 1} \Gamma(\nu)} K_\nu\left(\frac{\|s - t\|}{\rho}\right)
$$

where $K_\nu$ is the modified Bessel function of the second kind with order $\nu$. The Matérn covariance is parameterized through its inverse scale $\tau > 0$, range $\rho > 0$, and smoothness $\nu > 0$ parameters. Annual average precipitation from the PRISM dataset accounts for average weather patterns and orographic effects on precipitation, such as elevation. Prior distributions for the mean and covariance function parameters are available in Supplement Section B.2.2. In general, prior distributions are weakly informative, and prior distributions for spatial covariance parameters are centered around variogram-based estimates of spatial correlation between exploratory estimates of marginal parameters $\eta(s)$.

Inference uses a sample from the posterior distribution, drawn with a Gibbs sampler that was run for 3,002,000 iterations. The first 2,000 samples were discarded. The sampler was run for a large number of iterations because it was slowly mixing. Posterior inference uses 10,000 of the remaining samples; only every 300th sample was saved due to storage constraints. To facilitate model comparison, we also fit the unweighted latent spatial extremes model using the same priors and inference strategy. Posterior diagnostics for the weighted likelihood model are presented in Section B.3. Diagnostics suggest no significant
concerns with convergence and also that the chain has been run for long enough to control Monte Carlo integration error. Due to the relatively small number of spatial locations in the dataset (\(N = 71\)), posterior diagnostics indicate the spatial covariance parameters are at least weakly identified by the data. Posterior learning is diagnosed by comparing prior and posterior distributions for the spatial mean and covariance parameters.

The likelihood weights (5) have a spatial pattern and their effect can be interpreted by their impact on the weighted Fisher information (6) (Figure 2). As expected, stations near the edges of the sampled region tend to have the highest weights because annual maxima observed at these locations are at most weakly dependent with observations at other stations. Annual maxima at distant stations tend to be at most weakly dependent because they tend to experience different large rain events than other stations.

Weighted estimates borrow more strength across locations, which impacts return level estimates. The latent Gaussian processes increase smoothing as more strength is borrowed, shrinking parameter estimates (Supplement Figure 21). Shrinkage manifests as additional smoothing in maps of return levels (Figure 3). In particular, the weighted estimates better match physical features that impact Colorado precipitation. The contours in the weighted return level map have stronger north-south patterns, especially along 105° W—the boundary of the Rocky mountains in the Colorado Front Range region (Figure 3 B). The size of the region with elliptical 150–175mm return level contours (■) of extreme precipitation near Boulder, Fort Collins, and Colorado Springs also increase. The larger elliptical regions produced by the weighted model better capture physical effects of the Palmer Divide and the Cheyenne Ridge on Colorado precipitation (Daly et al., 2008; Karr and Wooten, 1976).

We verify that the weighted model’s changes are beneficial near the Palmer Divide and Cheyenne Ridge regions by refitting the weighted and unweighted models with a holdout set to test out-of-sample fit. Our holdout set uses data from seven stations (10% of the dataset) near the Palmer Divide and Cheyenne Ridge, and where posterior estimates of return levels differ between the two models (stations marked by diamonds in Figure 2). Testing uses the
log-score $\ell(s_0)$ at each holdout location $s_0$. Log-scores form strictly proper scoring rules that compare the log-likelihood from both models on data at each holdout location (Gneiting and Raftery, 2007). In our spatial application, we use the posterior kriging distribution to draw a posterior sample of GEV parameters at each test location $s_0$, which we then use to compute the posterior mean log-likelihood at each test location $\ell(s_0)$. Resulting log-scores show that the weighted model improves out-of-sample fit in six out of seven of the holdout locations (Table 3). The log-scores also show that neither model fits the data well at Pueblo, CO, the southernmost holdout station. In particular, the data at Pueblo, CO tend to be relatively less extreme. Separate exploratory analysis of Pueblo’s data suggests extreme precipitation is associated with a negative shape parameter $\xi(s_0) < 0$. However, the spatial models suggest a positive shape parameter is more appropriate.

5 Discussion

Estimating marginal return levels is an important step in planning for impacts of natural hazards, especially those caused by precipitation. Extreme precipitation data have dependence, which makes estimation more complicated. Models that explicitly account for dependence in the data have limited ability to scale to large datasets, while models that assume conditional independence in the data can scale well to large datasets, but do not account for dependence. We develop a weighted likelihood that downweights observations from locations central to the spatial sampling pattern in order to better estimate marginal return levels. We use the extremal coefficient in (1) to construct weights that downweight likelihood contributions from locations central to the spatial sampling pattern, where observations tend to be most dependent. Simulations confirm that the weighting scheme improved the uncertainty quantification of the return level estimates in situations when data have extremal dependence. In application, estimates from the weighted model better align with expected changes in patterns of extreme precipitation caused by physical features, like mountains.
Since weighted likelihoods are computationally inexpensive, they may be a useful technique to adopt in most settings where latent spatial extremes models are employed. Weighting adds $N$ additional multiplications per likelihood evaluation, whereas alternatives like penalization add $N$ additional function evaluations. Penalization improves estimation for univariate extremes data at a similar computational cost, but its main purpose is to discourage models from exploring unrealistic or undesirable regions of the parameter space, such as those with large shape parameters $\xi(s)$. As a result, penalized models underestimate uncertainty almost as much as unweighted models. Composite likelihood corrections are more computationally expensive (Ribatet et al., 2012; Sharkey and Winter, 2018). In practice, weighting encourages borrowing strength across locations to improve estimates at each location.

Refining the likelihood weights (5) could further improve the ability for marginal likelihoods to account for extremal dependence when estimating marginal return levels. For example, pairwise densities can be derived for specific max-stable processes (e.g., Padoan et al., 2010). Pairwise densities explicitly model the dependence between pairs of observations, while the extremal coefficient we use to build likelihood weights measures a summary of extremal dependence instead. Empirical Bayes–like procedures could be developed that use likelihood weights based on pairwise densities to further improve the performance of return level estimators. While empirical Bayes procedures will not fully account for estimation uncertainty (e.g., in estimating dependence parameters in bivariate densities), the procedures may still provide a fair compromise between computational complexity and accurate estimation of uncertainty.

Weighting schemes are flexible, so may be extended to accommodate complex issues in modeling and estimation outside extremes applications. While we demonstrate the use of a weighted likelihood for latent spatial extremes models, the theory we develop is more general. The Fisher information interpretation of weighted likelihoods also applies to all weighted likelihoods. Similarly, the limiting behaviors of likelihoods for independent data or
completely dependent data are largely based on copula theory for arbitrary data, rather than extreme value theory. Importantly, the construction of the weighted likelihood (4) can be adapted to other statistical problems where marginal inference is of interest but likelihoods are difficult to evaluate. The construction we propose is based on the idea that a computationally inexpensive measure of dependence between observations can be used to develop a weighted likelihood that better quantifies parameter uncertainty than related unweighted models. The main challenge in adapting our weighted likelihood to other applications is in identifying an appropriate dependence measure that can be used to build likelihood weights.

Supplementary materials

Additional information and supporting material for this article is available online at the journal’s website.

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Table 1: Generating model configurations used to simulate data for comparing the weighted likelihood (4) to alternate estimating models (Section 3.2). We evaluate model performance with 1,000 datasets for each combination of spatial $N$ and temporal $T$ sample sizes, and extremal dependence.

| Spatial sample size | $N \in \{30, 50, 100\}$ sites sampled uniformly on $D = [-10, 10]^2$ |
|---------------------|--------------------------------------------------|
| Temporal sample size | $T \in \{50, 100\}$ |
| Extremal dependence (Brown-Resnick parameters) | Semi-variogram $\gamma(\lambda, \alpha)(s_1, s_2) = (\|s_1 - s_2\|/\lambda)^\alpha$ |
| | Independent $(\lambda = NA, \alpha = NA)$ |
| | Weak $(\lambda = .25, \alpha = .75)$ |
| | Moderate $(\lambda = .5, \alpha = .5)$ |
| | Strong $(\lambda = .75, \alpha = .25)$ |
| Prior distributions for GEV parameters $\eta(s)$ | Covariance function $\rho(\sigma_0, \lambda_0, \nu_0)(s_1, s_2) = \sigma_0 \exp\left\{-\left(\|s_1 - s_2\|/\lambda_0\right)^{\nu_0}\right\}$ |
| | Gaussian processes $\mu(s) \sim \text{GP} \left(26 + [.5 \ 0]^T s, \rho_{(4, 20, 1)}\right)$ |
| | $\log \sigma(s) \sim \text{GP} \left(\log(10) + [0 \ .05]^T s, \rho_{(4, 5, 1)}\right)$ |
| | $\xi(s) \sim \text{GP} \left(.12, \rho_{(.0012, 10, 1)}\right)$ |
Table 2: Summary of differences between estimating models in simulation study (Section 3).

| Model      | (Log-)Likelihood | Weights | Log-Likelihood penalty |
|------------|------------------|---------|------------------------|
| Unweighted | (4)              | None    | None                   |
| Weighted   | (4)              | (5)     | None                   |
| PC Prior   | (8)              | None    | $\sum_j \log \pi(\xi(s_j) | \lambda)$            |
Table 3: Comparison of log-scores for the weighted $\ell_{wtd}(s_0)$ and unweighted models $\ell(s_0)$ at holdout cities; the highest log-score is highlighted for each city. The weighted likelihood model tends to have higher log-scores at holdout cities, suggesting better out-of-sample predictive performance in the targeted regions. The low log-scores in the bottom row also suggest neither model is predictive of extreme precipitation in Pueblo.

| Lat. | Lon. | City     | $\ell_{wtd}(s_0)$ | $\ell(s_0)$ |
|------|------|----------|------------------|--------------|
| 40.4 | 104.7| Greeley  | −224             | −225         |
| 40.2 | 105.1| Longmont | −502             | −508         |
| 40.0 | 105.6| Nederland| −411             | −415         |
| 39.6 | 104.8| Aurora   | −303             | −307         |
| 39.5 | 104.7| Parker   | −262             | −701         |
| 38.5 | 105.1| Penrose  | −198             | −196         |
| 38.3 | 104.7| Pueblo   | −349,848         | −1,779,826   |
Figure 1: Empirical coverage rates of 95% highest posterior density intervals for 100-year return levels $Q(.99|\eta(s))$ for four levels of extreme dependence across comparison models and simulations with $T = 50$ observations per location. Nominal coverage is marked by the dotted horizontal reference line at .95. While empirical coverage degrades for all estimating models as extremal dependence increases, the weighted model is most robust to model misspecification caused by extremal dependence. Supplement Section D includes results for $T = 100$, which show slight improvement in all coverage rates.
Figure 2: Spatial distribution of weights. Weights are smaller for locations central to the spatial sampling pattern, where extremal dependence is more likely to impact data. Cities used in the hold-out model comparison are marked by diamond outlines.
Figure 3: Spatially complete estimates $\hat{Q}(0.99|\eta(s))$ of 100-year return levels for daily precipitation in Colorado’s Front Range. Estimates are compared from the unweighted (A) and weighted (B) unweighted latent spatial extremes models. The weighted estimates have increased smoothness and spatial range, and overall patterns that better match orographic features in Colorado. The locations of the 71 stations whose data are analyzed are indicated by (o). For reference, we include the names of several reference cities. Cities used in the hold-out model comparison are marked by diamond outlines.