Quasi-compact Higgs bundles and Calogero-Sutherland systems with two types spins

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Abstract

We define the quasi-compact Higgs $G^C$-bundles over singular curves introduced in our previous paper for the Lie group $SL(N)$. The quasi-compact structure means that the automorphism groups of the bundles are reduced to the maximal compact subgroups of $G^C$ at marked points of the curves. We demonstrate that in particular cases this construction leads to the classical integrable systems of Hitchin type. The examples of the systems are analogues of the classical Calogero-Sutherland systems related to a simple complex Lie group $G^C$ with two types of interacting spin variables. These type models were introduced previously by Feher and Pusztai. We construct the Lax operators of the systems as the Higgs fields defined over a singular rational curve. We also construct hierarchy of independent integrals of motion. Then we pass to a fixed point set of real involution related to one of the complex structures on the moduli space of the Higgs bundles. We prove that the number of independent integrals of motion is equal to the half of dimension of the fixed point set. The latter is a phase space of a real completely integrable system. We construct the classical $r$-matrix depending on the spectral parameter on a real singular curve, and in this way prove the complete integrability of the system. We present three equivalent descriptions of the system and establish their equivalence.

Contents

1 Introduction and summary 2

2 Finite-dimensional description 7

2.1 Model I .................................................. 7

2.1.1 Integrals of motion I ................................... 10

2.1.2 Real forms ........................................... 10
1 Introduction and summary

In this paper we describe the quasi-compact Higgs bundles (see the definition below). The aim is to construct integrable systems of the Hitchin type \cite{Hitchin} using the quasi-compact Higgs bundles. As an example we consider generalizations (GCS) of the spin Calogero-Sutherland (CS) model and its extensions related to simple Lie algebras. These systems were introduced in \cite{CS1,CS2} and for the Lie group SL(N, \mathbb{C}) in our previous work \cite{our_prework}, where the quasi-compact Higgs bundles were defined. The systems are similar to the spin CS systems, where the starting point are the quasi-parabolic Higgs bundles \cite{parabolic1,parabolic2} over a singular rational curve \cite{parabolic_curve1,parabolic_curve2}. The crucial distinction from the CS systems beyond the quasi-compactness, is that we use a specific real involution on the moduli space of the Higgs bundles considered for the classical Hitchin systems.
in [1] and for some quantum Hitchin systems in [14,15]. After this involution we come to the real completely integrable systems.

**Description of systems.** The CS model [3,18] describes one-dimensional system of pairwise interacting particles through long range potentials. We deal with the classical model. It is an integrable system in the Liouville sense, as well as its spin extension [7,21]. The latter model can be written as the Euler-Arnold SL(N, C) top (its inertia tensor depends on the positions of interacting particles).

Original CS system is related to the Lie group SL(N, C), and it can be generalized for arbitrary simple complex Lie algebra \( g^C \) in the following way [16]. Let \( R \) be the corresponding root system and \( h^C \) is a Cartan subalgebra of Lie algebra \( g^C \). Denote the coordinates of the particles \( u = \sum_{i=1}^{l} u_i e_i \in h^C \), where \( (e_1, \ldots, e_l) \) is a canonical basis in \( h^C \) and \( v \) is the momenta vector. Let \( S \in g^C \) be an element of a (co)adjoint orbit of the group \( G^C \) in \( (g^C)^* \sim g^C \). The coordinates \( S_\alpha \) of \( S \) in the root basis \( E_\alpha, (\alpha \in R) \) are called the *spin variables*. The Hamiltonian of the spin CS system related to \( g^C \) has the form

\[
H^{CS} = \frac{1}{2} (v,v) + \sum_{\alpha \in R} \frac{1}{(\alpha,\alpha)} \frac{S_\alpha S_{-\alpha}}{\sinh^2(u_\alpha)}. \tag{1.1}
\]

The Poisson brackets for \( v, u \) are the Darboux brackets, while the Poisson structure for \( \{S_\alpha, S_\beta\} \) is obtained from the Lie-Poisson brackets on the Lie co-algebra \( (g^C)^* \) via Hamiltonian reduction with respect to the coadjoint action of the Cartan subgroup of \( G^C \) on the spin variables. However, for \( S \in \mathfrak{so}(N) \) the additional reduction is not needed [2]. In our previous paper [11] we described the \( \mathfrak{so}(N) \) model with two types of spins. Here we use this approach for an arbitrary simple Lie group \( G^C \).

The extension of (1.1) is as follows. Let \( G^R \) be the normal real form of the group \( G^C \). Such form exists for any simple complex group. Consider the maximal compact subgroup \( K \subset G^C \). The maximal compact subgroup \( U \subset G^R \) is the intersection \( K \cap G^R \). For example, if \( G^C = \text{SL}(N, \mathbb{C}) \), then \( G^R = \text{SL}(N, \mathbb{R}) \), \( K = \text{SU}(N) \), \( U = \text{SO}(N) \). Let \( E_\alpha \) be a root basis in the Lie algebra \( \mathfrak{g}^C \). The Lie algebra \( \mathfrak{u} = \text{Lie}(U) \) has the basis \( E_\alpha - E_{-\alpha} \), where \( \alpha \in R^+ \) are the positive roots. Define the two type of spin variables \( S, T \in \mathfrak{u} \)

\[
S = \sum_{\alpha \in R^+} S_\alpha (E_\alpha - E_{-\alpha}) , \quad T = \sum_{\alpha \in R^+} T_\alpha (E_\alpha - E_{-\alpha}) , \quad S_\alpha , T_\alpha \in \mathbb{R}
\]

belonging to the same coadjoint orbit \( O \subset \mathfrak{u}^* \). Then the Hamiltonian assumes the form [4,]

\[
H = \frac{1}{2} (v,v) + \sum_{\alpha \in R^+} \frac{2}{(\alpha,\alpha)} \frac{S_\alpha^2 + T_\alpha^2 - 2S_\alpha T_\alpha \cosh(u_\alpha)}{\sinh^2(u_\alpha)}, \tag{1.2}
\]

where \( v \) and \( u \) are real \( (v,u \in \mathfrak{g}^R - \text{Cartan subalgebra of } \mathfrak{g}^R) \). In contrast with the spin CS system the Poisson brackets for the spin variables are the genuine Poisson-Lie brackets without additional constraints

\[
\{T_\alpha, T_\beta\} = N_{\alpha,\beta} T_{\alpha+\beta} - N_{\alpha,-\beta} T_{\alpha-\beta}, \\
\{S_\alpha, S_\beta\} = -N_{\alpha,\beta} S_{\alpha+\beta} + N_{\alpha,-\beta} S_{\alpha-\beta}, \\
\{T_\alpha, S_\beta\} = 0,
\]

with the structure constants \( N_{\alpha,\beta} \) defined in [4,38]. Equations of motion are generated by (1.2), (1.3) and the canonical brackets \( \{v_j, u_\alpha\} = \alpha(j) \equiv \alpha(e_j) \):

\[
\dot{S}_\alpha = \sum_{\beta+\gamma = \alpha} C_{\beta\gamma} S_\beta \frac{S_\gamma - T_\gamma \cosh(u_\gamma)}{\sinh^2(u_\gamma)}, \quad \dot{T}_\alpha = \sum_{\beta+\gamma = \alpha} C_{\beta\gamma} T_\beta \frac{S_\gamma \cosh(u_\gamma) - T_\gamma}{\sinh^2(u_\gamma)}, \tag{1.4}
\]

\[1\]In the systems described in [4] different notations were used
\[2\]In order to reproduce the results of [11] in the \( \mathfrak{so}(N) \) case one should redefine \( S \to -S \) and \( T \to -T \). This also leads to changing the signs in the r.h.s. of (1.3).
\[ v_j = \sum_{\gamma \in R^+} \frac{4 \gamma(j)}{(\gamma, \gamma)} \left( (S_{\gamma}^2 + T_{\gamma}^2) \frac{\cosh(u_{\gamma})}{\sinh^3(u_{\gamma})} - S_{\gamma} T_{\gamma} \frac{1 + \cosh^2(u_{\gamma})}{\sinh^3(u_{\gamma})} \right). \]  

(1.5)

In this way the Hamiltonian (1.2) describes particles with real coordinates \( u \) and real momenta \( v \) together with two interacting tops on the compact Lie group \( U \) with the inertia tensors and the interaction term depending on the dynamical coordinates of particles on \( \mathfrak{r}^R \). For the group \( \text{SL}(N, \mathbb{C}) \) these tops are defined on the group \( U = \text{SO}(N) \). For other simple groups the corresponding subgroups are presented in Table 1 in the Appendix A.

For \( N = 2 \) the algebra so(2) is commutative and one can fix values of the spin variables. In this case we obtain from (1.2) the Hamiltonian with two constants

\[ H = \frac{v^2}{2} + \frac{m_1^2 + m_2^2 - 2m_1 m_2 \cosh(2u)}{\sinh^2(2u)}. \]  

(1.6)

This Hamiltonian reproduces the CS model of the BC_1 type [16]. In the quantum case this Hamiltonian coincides with the Casimir operator acting on a special matrix element of the principal series representations of \( \text{SU}(1, 1) \) written in the spherical coordinates [20].

We prove that the Hamiltonian (1.2) is the first nontrivial one in the family of Poisson commuting real independent integrals of motion.

The hierarchy is described by the Lax operator. In the Chevalley basis of the algebra \( \mathfrak{g}^R \) it assume the form

\[ L(x) = \sum_j v_j e_j + \sum_{\beta \in R} T_{\beta} \exp(-u_{\beta}) - S_{\beta} \exp(u_{\beta}), \]  

(1.7)

The equations (1.4), (1.5) are equivalent to the Lax equations \( \dot{L}(x) = [L(x), M] \) with the \( M \) operator

\[ M = \sum_{\beta \in R} S_{\beta} \cosh(u_{\beta}) - T_{\beta} \exp(u_{\beta}). \]  

(1.8)

To derive the system and the Lax operator we use the symplectic reduction staring from three different finite-dimensional symplectic spaces. We call the corresponding systems the Model I, the Model II and the Model III. Literally, the equations (1.4), (1.5) and the Lax operator corresponds to the Model I and the Model II. There are the changes of variables that leads to the Model III. In next paragraph we clarify interrelations between the models coming from comparison of different Higgs bundles.

While the spin variables in the Models I and II are elements of a coadjoint orbit of the compact subgroup \( U (U = G^R \cap K) \), in the model III they are elements of the cotangent bundle \( T^*(\mathcal{X}^R) \) to the symmetric space \( \mathcal{X}^R = G^R / U \). In this way the GCS models are real integrable systems. To derive them we start with complex symplectic manifolds for the following reason. In the second part of paper we relate GCS systems with a special class of Hitchin systems [9]. The latter are based on considerations of holomorphic bundles over complex curves.

In the first part of paper we use the finite-dimensional approach similar to [16]. For the Model I and III the unreduced symplectic spaces are products of complex and real spaces. This makes the definition of Hamiltonian systems obscure. For the Model I we come to the well-defined real Hamiltonian systems after the symplectic reduction with respect to the real group. For the Model III the real systems arise after passing to the real subgroup \( G^R \subset G^C \). The unreduced symplectic space of the Model II is complex and the real Hamiltonian system is result of the symplectic reduction with respect to the real group.

**Quasi-compact Higgs bundles.** In order to construct the complete family of integrals of motion we define the Lax pair depending on the spectral parameter \( x \in \mathbb{R} \). For example, the
Lax operator corresponding to the Model I has the form

\[ L(x) = \sum_j v_j e_j + \sum_{\beta \in R} \left( T_\beta \frac{\exp(-u_\beta) - S_\beta}{\sinh(u_\beta)} + (1 + \coth(x))T_\beta \right) E_\beta. \] (1.9)

It was obtained in [4] in a slightly different form. We will construct the classical \( r \)-matrix related to \( L(x) \). The classical \( r \)-matrix without spectral parameter was previously obtained in [5].

The Lax operators play the role of the Higgs fields in the Hitchin approach to integrable systems [9]. To construct the Lax matrix \( L(x) \) we consider the \( GC \) Higgs bundle over singular curves as in [13, 19]. To construct the defined above systems we endow the Higgs bundles with the so-called \textit{quasi-compact structure}. It means that the gauge group at the marked points on the base spectral curve is reduced to the maximal compact subgroups \( K \). In the standard approach to the Hitchin systems the gauge group may have the quasi-parabolic structure, i.e. the gauge group is reduced at marked points to parabolic subgroups. In the latter case the moduli space of the Higgs bundles \( M(G^C) \) are the phase space of \textit{complex integrable systems} [9].

In our case the part of coordinates in the local description of the Higgs bundles are real. As it was mentioned above in this case the definition of Hamiltonian systems is obscure. To pass to real Hamiltonian systems we consider a specific real involution on the moduli space of the Higgs bundles defined in [1]. The fixed points set of the corresponding involution becomes \textit{real completely integrable system}, since the number of independent integrals of motion is exactly equal to the half dimension of the moduli space.

Thus, we break the complex structure in two stages. First, we do it locally by introducing the quasi-compact structure. Next we pass to the real involution of the Higgs bundle. The last step allows one to consider the spin variables as genuine elements of the coadjoint orbits in the Lie coalgebra \( u^* \), while for the quasi-parabolic bundles they were elements of symplectic quotient of the orbits with respect to the Cartan group action.

But first, before using the real involution we define the moduli space of the Higgs bundles \( \mathcal{M}_A(G^C) \) \((A = I, II, III)\) that lead eventually to phase spaces of Models I, II and III. The Higgs bundle corresponding to the Model I is define over the curve \( \Sigma^I \). It is the normalization of two rational curves glued in two points. The result of the symplectic reduction with respect to the automorphism group of the bundle (the gauge group) is the moduli space \( \mathcal{M}_1(G^C) \). For the Models II and III the base curve is a rational curve glued in two points. It coincides with the singular curve for the CS system [3]. We prove that the moduli spaces of the Higgs bundles \( \mathcal{M}_A(G^C) \) \((A = I, II, III)\) after the involution coincide with the phase spaces of the integrable system corresponding to the Models I, II and III.

There is a correspondence between the Higgs bundles I and II, provided by the Universal Higgs Bundle and its projections on the Higgs bundles I and II:

\[ \begin{array}{c}
\text{Universal Higgs Bundle} \\
\text{Higgs Bundle I} \\
\text{Higgs Bundle II} \\
\text{GCS Phase Space}
\end{array} \]

\( \pi_1 \) \( \pi_2 \)

\( SR_1 \) \( SR_2 \)

where SR means a symplectic reduction and the passing to the fixed point sets of the involution.

The quasi-compact bundles can be equivalently defined in a different way. Instead of restriction of the gauge group at the marked points to the maximal compact subgroups \( K \), we attach to the points the symmetric spaces \( \mathbb{X}^C = K \setminus G^C \). For the corresponding Higgs bundle

\[ ^3\text{The SL}(N, \mathbb{C}) \text{ Higgs bundle I constructed in [11] is unstable. It is one of the reasons to construct the Higgs bundles II and III.} \]
it amounts to attaching to the marked point the cotangent bundle $T^*X_C$. This definition leads to the moduli space $\mathcal{M}_{III}(G^C)$. The equivalence of these descriptions defines the isomorphism

$$\mathcal{M}_{II}(G^C) \sim \mathcal{M}_{III}(G^C).$$

One of the aims of the paper is the derivation of the moduli spaces $\mathcal{M}_{I}(G^C)$ and the description of the interrelations between them. It turns out that the isomorphism (1.11) is symplectomorphism. We do not prove this fact here. In a separate publication we will construct general quasi-compact Higgs bundles and prove that (1.11) is symplectomorphism.

In the first Section we develop the finite-dimensional approach to the system as in [16] for the CS system. We start with free systems related to the complex simple Lie groups. As we explained above there are three models of these systems. For the Model I it is a free system defined on the product of two cotangent bundles $T^*G^C \times T^*K$ (see (2.1)). It was considered in [11] for $G^C = SL(N, \mathbb{C})$. By symplectic reduction with respect to the action of the group $K \times K$ we come to the reduced phase space $(T^*G^C \times T^*K)//(K \times K)$. We obtain the Lax pair corresponding to the Hamiltonian (1.2). For the Model II we start with the phase space $T^*G^C$ and the symmetry group $K$. We prove that the reduced phase space $T^*G^C//K$ is symplectomorphic to the previous one and leads to the same integrable family. For the Model III we start with the phase space $T^*G^C \times T^*X_C$ and the symmetry $G^C$. To come to integrable situation we pass to the subgroups $G^R, U = G^R \cap K$, and real symmetric space $X^R = U \setminus G^R$. The resulting model is a system of particles similar to the spin CS system, but the spin variables are elements of the cotangent bundle $T^*X^R$.

The drawback of this approach is that the Lax operators derived in this way are independent of the spectral parameter. To bring it in the game we derive the systems by replacing the cotangent bundles $T^*G^C$ with the Higgs bundles over the singular curves endowed with the quasi-compact structures. In particular, we construct the Universal Higgs bundle that allows one to establish the correspondence presented in the diagram (1.10). We also discuss the interrelations between the Higgs bundles of type II and III (1.11).

It is also instructive to compare our construction with the Higgs bundle description of the CS system. The standard approach to the CS system [13, 19] is similar to the Model III. The Model I type description is specific for the GCS system and has not analog for the CS model. The Model II type description is implicit and interrelation between the Model II and Model III for the CS systems is complicated.

In Appendix B we consider of a gyrostat related to a simple Lie group. It is the Euler-Arnold top with additional rotator moment. We derive equations of motion for this system. We analyze the Lax equations in Appendix C.

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2 Finite-dimensional description

2.1 Model I

Let $G^\mathbb{C}$ be a simple complex Lie group and $K$ – its maximal compact subgroup. Consider the symplectic space

$$\mathcal{R}_I(G^\mathbb{C}) = T^*G^\mathbb{C} \times T^*K$$

(2.1)

For example, $G = \text{SL}(N, \mathbb{C})$, $K = \text{SU}(N)$. Then

$$\dim \mathcal{R} = 2\dim \mathcal{R} G^\mathbb{C} + 2\dim \mathcal{R} K.$$  

(2.2)

In the left trivialization $T^*G^\mathbb{C} \sim G^\mathbb{C} \times (g^\mathbb{C})^*$ is described by the coordinates $(g, \eta)$ ($g \in G^\mathbb{C}, \eta \in (g^\mathbb{C})^*$). Similarly, let $(h, \nu)$ be the coordinates on $T^*K = K \times \mathfrak{k}^*$, $(h \in K, \nu \in \mathfrak{k}^*)$. Define the symplectic form $\omega = d\vartheta$, where 1-form $\vartheta$ is the sum

$$\vartheta = \vartheta^G + \vartheta^K,$$

$$\vartheta^G = \langle \eta, \Omega^L(g) \rangle, \quad \vartheta^K = \langle \nu, \Omega^L(h) \rangle$$

(2.3)

and $\Omega^L$ is the left-invariant Maurer-Cartan form.

Remark 2.1 The symplectic space $\mathcal{R}_I(G^\mathbb{C})$ (2.1) is the product of complex and real symplectic spaces. Correspondingly, the one-form $\vartheta$ (2.3) is the sum of complex and real one-forms. As we mentioned above the definition of dynamical systems on $\mathcal{R}_I(G^\mathbb{C})$ is ambiguous. Thus, defined below Hamiltonians and integrals are formal, until passing to the real symplectic space. We will see that the same is true for the considered below Model III.

Define the quadratic Hamiltonian

$$H = \frac{1}{2}(\eta, \eta).$$

(2.4)

where $(, )$ comes from (A.5), (A.6), and the Poisson brackets are defined by (2.3).

Consider the symplectic quotient of this system with respect the following gauge symmetry. The gauge group $G^{\text{gauge}} = K_1 \times K_2, K_1, K_2 \sim K$ acts as

$$K_1 : g \to gf_1^{-1} \eta \to \text{Ad}_{f_1}^* \eta; \ h \to h f_1^{-1} \nu \to \text{Ad}_{f_1}^* \nu.$$

(2.5)

$$K_2 : g \to f_2 g \eta \to \eta; \ h \to f_2 h \nu \to \nu.$$  

(2.6)

The Hamiltonian vector fields $V_{1,2}$ corresponding to (2.5) (2.6), are given by $V_1 = \{F_1, \} \subset K$ and $V_2 = \{F_2, \} \subset K$, where the Poisson brackets are defined on the co-algebra $\mathfrak{k}^*$ and $F_1, F_2$ are defined as

$$i_{V_{1,2}} \omega = -dF_{1,2},$$

$$F_1 = \langle \mu_1, \epsilon_1 \rangle, \quad F_2 = \langle \mu_2, \epsilon_2 \rangle, \quad (\epsilon_{1,2} \in \mathfrak{k}),$$

(2.7)

and corresponding moment maps $\mu_1, \mu_2 : \mathcal{R}_I(G^\mathbb{C}) \to \mathfrak{k}^*$ take the form

$$\mu_1 = \eta|_{\mathfrak{t}^*} + \nu,$$

$$\mu_2 = \text{Ad}_{g}^* \eta|_{\mathfrak{t}^*} + \text{Ad}_{h}^* \nu.$$  

(2.8)

Introduce notations

$$\mathbf{T} \equiv \nu \in \mathfrak{t}^*, \quad \mathbf{S} \equiv \text{Ad}_{g}^* \nu \in \mathfrak{t}^*.$$  

(2.9)

These coordinates on $\mathbf{T}$ and $\mathbf{S}$ are functions on the co-algebra $\mathfrak{t}^*$. They have the linear Lie-Poisson structure. We write tentative $\{\mathbf{T}, \mathbf{T}\} \sim \mathbf{T}, \{\mathbf{S}, \mathbf{S}\} \sim \mathbf{S}$ and $\{\mathbf{T}, \mathbf{S}\} = 0$ (see (A.39), (A.40) (A.41)).

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Footnote: This model is similar to the model used in [1], except that $T^* K$ is replaced by the two coadjoint $K$-orbits. In fact, these brackets are defined on the subalgebra $u^* \subset \mathfrak{k}^*$ (A.19) and will be used below. But this structure holds on $\mathfrak{t}^*$ as well.
Remark 2.2 One can fix the Casimir functions in the Poisson algebra on \( \mathfrak{t}^* \). In this way (2.9) is the map of the cotangent bundle \( T^*K \) to the pair of the coadjoint \( K \)-orbits \( \mathcal{O}_K \) corresponding to these Casimir functions. These two orbits are results of two ways describing coadjoint orbits as symplectic quotients from the cotangent bundle by either left or right shifts.

Using this construction we replace \( \mathcal{R}_I(G^C) \) (2.1) with
\[
\tilde{\mathcal{R}}_I(G^C) = T^*G^C \times (\mathcal{O}_K \times \mathcal{O}_K).
\] (2.10)

It follows from (A.18) that the gauge actions (2.6) and (2.5) transform \( g \) to the Cartan subgroup
\[
f_1gf_2 = e(u) = \exp u, \; u \in \mathfrak{h}^\mathbb{R}, \; u = (u_1, \ldots, u_l), \; (l = \text{rank } g).
\] (2.11)

Using the invariant form on \( k \) we identify \( k \) and \( k^* \) and decompose \( T \) and \( S \) in the root basis
\[
T = T_{h_K} + \sum_{\alpha \in \mathcal{R}^+} (T_\alpha E_\alpha - \bar{T}_\alpha E_{-\alpha}), \quad S = S_{h_K} + \sum_{\alpha \in \mathcal{R}^+} (S_\alpha E_\alpha - \bar{S}_\alpha E_{-\alpha}),
\] (2.12)
\[
T_{h_K} = \sum_{j=1}^l T_je_j, \quad S_{h_K} = \sum_{j=1}^l S_je_j.
\]

We put the moment constraints (2.8) \( \mu_L = 0, \mu_R = 0 \). Plugging \( g = e(u) \) (2.11) into these equations we come to the linear system
\[
\eta|_t = -T, \quad \text{Ad}_{e(u)}\eta|_t = -S. \tag{2.13a}
\]
\[
\text{(2.13b)}
\]

Consider the a solution of (2.13)
\[
\eta = P + X, \quad P \in \mathfrak{h}^C, \quad X = \sum_{\alpha \in \mathcal{R}} X_\alpha E_\alpha. \tag{2.14}
\]

Then (see (2.12) and (A.15))
\[
\eta|_t = i\mathbb{I}m P + \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} (X_\alpha - \bar{X}_{-\alpha})E_\alpha - (X_\alpha - \bar{X}_{-\alpha})E_{-\alpha}.
\] (2.15)

From (2.13) we obtain
\[
\frac{1}{2}(X_\alpha - \bar{X}_{-\alpha}) = -T_\alpha, \quad \frac{1}{2}(X_\alpha e(u_\alpha) - \bar{X}_{-\alpha}e(-u_\alpha)) = -S_\alpha.
\] (2.16)

and
\[
i\mathbb{I}m P = T_{h_K}, \quad i\mathbb{I}m P = S_{h_K}.
\] (2.17)

The system (2.16) has a unique solution
\[
X = \sum_{\alpha \in \mathcal{R}^+} (X_\alpha E_\alpha + X_{-\alpha}E_{-\alpha}), \tag{2.18}
\]
\[
X_\alpha = \frac{T_\alpha e(-u_\alpha) - S_\alpha}{\sinh(u_\alpha)}, \quad X_{-\alpha} = \frac{\bar{T}_\alpha e(u_\alpha) - \bar{S}_\alpha}{\sinh(u_\alpha)}.
\]

The general solution of the linear system (2.13) has the form
\[
\eta = P + X, \quad P = v + S_{h_K}, \tag{2.19}
\]

where \( \mathbf{v} = (v_1, \ldots, v_l) \in \mathfrak{h}^R, \) \( S_{\theta K} = \iota(S_1, \ldots, S_l). \) In what follows \( \eta \) plays the role of the Lax operator (without spectral parameter). Thus, the reduced space

\[
\tilde{\mathcal{R}}_I^{\text{red}}(G^C) = \tilde{\mathcal{R}}_I(G^C)//G^\text{gauge} \equiv (\mu_L^{-1}(0) \times \mu_R^{-1}(0))/K_L \times K_R
\]

is defined by the coordinates

\[
\mathbf{v}, \mathbf{u} \in \mathfrak{h}^R, \ \exp(\mathbf{u}) \in \mathcal{H}^R, \ \mathbf{S} = \text{Ad}_h \mathbf{T}, \ \mathbf{T} \in \mathfrak{k},
\]

so that

\[
\tilde{\mathcal{R}}_I^{\text{red}}(G^C) = T^*\mathcal{H}^R \times T^*K.
\]

It has the dimension \( \dim_{\mathbb{R}} \tilde{\mathcal{R}}_I^{\text{red}}(G^C) = 2 \dim K + 2l. \)

The gauge fixing \( (2.11) \) is not complete. Multiplication by an element \( s \in \mathcal{T} \) from the Cartan torus (from \( K \)) \( f_2 \to f_2 s, f_1 \to f_1 s \) preserves the gauge. The torus acts on the variables as

\[
h \to shs^{-1}, \ \mathbf{T} \to \text{Ad}_s \mathbf{T}, \ \mathbf{S} \to \text{Ad}_s \mathbf{S}, \ s \in \mathcal{T}.
\]

The moment map equation generating this action (see \( (2.6) \) and \( (2.5) \)) is

\[
\mathbf{T}_{\theta K} - S_{\theta K} = 0.
\]

Notice that this condition is consistent with \( (2.17) \). After fixing the gauge action we come to the symplectic quotient \( (O \times O)//\mathcal{T} \). The elements of the reduced space \( \mathcal{R}^{\text{red}} \) are particles momenta and coordinates \( (\mathbf{v}, \mathbf{u}) \) and the reduced spin variables. Thus

\[
\mathcal{R}_I^{\text{red}}(G^C) = T^*\mathcal{H}^R \times ((\mathcal{O}_K \times \mathcal{O}_K)//\mathcal{T}).
\]

Due to the action of the Weyl group the coordinates belong to some Weyl chamber \( u \in \Lambda \subset \mathfrak{h}^R. \) Using \( (A.29) \) we find the dimension of \( \mathcal{R}^{\text{red}} \)

\[
\dim \mathcal{R}_I^{\text{red}}(G^C) = 2 \dim_{\mathbb{R}} \mathcal{O}_K = 4 \sum_{j=1}^l (d_j - 1).
\]

Here we assume that the orbits \( \mathcal{O}_K \) are generic. For \( G^C = \text{SL}(N, \mathbb{C}) \) (and \( K = \text{SU}(N) \))

\[
\dim \mathcal{R}_{\text{SL}(N, \mathbb{C})}^{\text{red}} = 2N(N - 1).
\]

At this stage we come to the real symplectic space \( (2.24) \). The Poisson structure on \( \mathcal{R}_I^{\text{red}}(G^C) \)

is defined by the canonical brackets for the coordinates and momenta of particles \( (\mathbf{v}, \mathbf{u}) \) and the Dirac brackets for the spin variables \( (\mathbf{S}, \mathbf{T}) \). They come from the Lie-Poisson brackets on the Lie algebra \( \mathfrak{k} \) upon imposing the constraints \( (2.23) \) and the gauge fixing with respect to the \( \mathcal{T} \) action.

From \( (2.4) \) and \( (A.6) \) we find the Hamiltonian

\[
H = \frac{1}{2} \sum_{j=1}^l (v_j^2 - S_j^2) + U(\mathbf{u}, \mathbf{S}, \mathbf{T}),
\]

\[
U(\mathbf{u}, \mathbf{S}, \mathbf{T}) = \sum_{\alpha \in \check{R}^+} \frac{2X_\alpha X_{-\alpha}}{(\alpha, \alpha)} = 2 \sum_{\alpha \in \check{R}^+} |S_\alpha|^2 + |T_\alpha|^2 - S_\alpha T_\alpha \epsilon(u_\alpha) - \bar{S}_\alpha T_\alpha \epsilon(-u_\alpha) / (\alpha, \alpha) \sinh^2(u_\alpha).
\]

The symplectic form \( (2.23) \) on the space \( \tilde{\mathcal{R}}_I(G^C) \) assumes the form

\[
(D\mathbf{v}, D\mathbf{u}) + D(\mathbf{T}, h^{-1}Dh)).
\]

Summarizing, we came to the Hamiltonian \( (2.28) \) defined on the phase space \( (2.24) \) by the two-step symplectic reduction

\[
\{ \tilde{\mathcal{R}}_I(G^C), H \} \xrightarrow{K_L \times K_R} \{ \tilde{\mathcal{R}}_{\text{red}}(G^C), H \} \xrightarrow{T} \{ \mathcal{R}_{\text{red}}(G^C), H \}.
\]
2.1.1 Integrals of motion I

Let \( d_j (j = 1, \ldots, l) \) be the order of the invariants of the algebra \( \mathfrak{g}^C \) \((A.27)\). All independent integrals of motion \( I_{jk} (k = 0, \ldots, d_j), (j = 1, \ldots, r) \) can be described in the following way. Consider the polynomials \((T + \eta)^{d_j}\) on the algebra \( \mathfrak{g}^C \) (this choice will be justify in Section 3.3.) with \( T \) \((2.12)\) and \( \eta \) \((2.19)\). The integrals of motion \( I_{jk} \) are monomials

\[
I_{jk} = (T^{d_j - k\eta^k}), \quad k = 0, \ldots, d_j. \tag{2.31}
\]

coming from the expansion \((T + \eta)^{d_j} = (T^{d_j}) + d_j(T^{d_j - 1}\eta) + \ldots + (\eta^{d_j})\). In particular, the Hamiltonian \(H_k\) \((2.4)\) is \(I_{1.2}\). Note that the integrals \(I_{j,0} = (T)^{d_j}\) are the Casimir functions of the Lie-Poisson algebra on \((\mathfrak{t})^*\). They are the Casimir functions on the whole Lie-Poisson algebra on \((\mathfrak{g}^C)^*\), since \(T_{\alpha}\) and \(T|_{\beta K}\) Poisson commute with other variables (see \((A.41)\)).

Let \( m_j \) be the number of integrals of order \( d_j \). The total number of integrals \( N_G \) is \(\sum_{j=1}^{l} m_j\). It coincides with the dimension of the space of homogeneous polynomials of two variables of order \( d_j \). Then \( m_j = d_j + 1 \). Excluding the \(l\) Casimir functions we find the number of integrals

\[
N_G = \sum_{j=1}^{l} (d_j + 1) - l = \sum_{j=1}^{l} d_j = \frac{1}{2}(\dim K + l). \tag{2.32}
\]

Evidently, all integrals are functionally independent. We prove their involutivity below using the classical \(r\)-matrix. For the complete integrability it is necessary to have \( N_G = \frac{1}{2} \dim_{\mathbb{R}} (\mathcal{R}_{G^C}^{red}) = \dim (\mathcal{O}_K) \) (see \((2.25)\)). So, we have a total deficiency of the integrals (see \((A.29)\))

\[
\delta_{G^C} = \frac{1}{2} \dim \mathcal{R}_{G^C}^{red} - N_G = \sum_{j=1}^{l} d_j - 2l. \tag{2.33}
\]

Notice that \(\delta_G \geq 0\), where the equality runs up for the rank one systems. In particular, for \(G^C = \text{SL}(N, \mathbb{C})\) (\(A_{N-1}\) type systems) \(d_j = j + 1, j = 1, \ldots, N - 1, \ m_j = j + 2\)

\[
N_{\text{SL}(N, \mathbb{C})} = \sum_{j=1}^{l} m_j - (N - 1) = \sum_{j=1}^{N-1} (j + 2) - (N - 1) = \frac{1}{2}(N - 1)(N + 2). \tag{2.26}
\]

From \((2.26)\) we find \(\delta_{\text{SL}(N, \mathbb{C})} = \frac{1}{2}(N - 1)(N - 2)\).

2.1.2 Real forms

To come to the completely integrable systems we pass from the complex group \(G^C\) to its normal real form \(G^\mathbb{R}\) (see Appendix A). We describe this construction in detail for the corresponding Higgs bundles in Section 3.4. The phase space of the real form of the model I is

\[
\mathcal{R}_{I}^{red}(G^\mathbb{R}) = T^*\mathcal{H}^\mathbb{R} \times (\mathcal{O}_U \times \mathcal{O}_U). \tag{2.34}
\]

is described by the variables \((\xi, r = h^{-1}\exp(u)), (h \in U, \exp(u)) \in \mathcal{H}^\mathbb{R}\). The variables \(T, S\) are elements of the compact subalgebra \(u = \mathfrak{g}^\mathbb{R} \cap \mathfrak{t}\) \((A.19)\). In the root basis they have the expansion (compare with \((2.12)\))

\[
T = \sum_{\alpha \in R^+} T_{\alpha}(E_{\alpha} - E_{-\alpha}), \quad S = \sum_{\alpha \in R^+} S_{\alpha}(E_{\alpha} - E_{-\alpha}), \tag{2.35}
\]

where \(T_{\alpha}, S_{\alpha}\) are real.

\[
\dim_{\mathbb{R}} \mathcal{R}_{I}^{red}(G^\mathbb{R}) = 2 \sum_{j=1}^{l} d_j - 2\text{rank } U. \tag{2.36}
\]
As in \((2.14), (2.18)\)
\[
\eta = P + X, \quad P \in \mathfrak{g}^\mathbb{R}, \quad P = v = (v_1, \ldots, v_l),
\]
\[
X = \sum_{\alpha \in R} X_\alpha E_\alpha, \quad X_\alpha = \frac{T_\alpha e(-u_\alpha) - S_\alpha}{\sinh(u_\alpha)}, \quad X_{-\alpha} = \frac{T_\alpha e(u_\alpha) - S_\alpha}{\sinh(u_\alpha)}.
\]

The Poisson structure on \(\mathcal{R}\)\(_{red}^r(G^\mathbb{R})\) takes the form
\[
\{v_j, u_k\} = \delta_{jk}, \quad \{T, T\} = T (4.39), \quad \{S, S\} = -S (4.40), \quad \{S, T\} = 0 (4.41).
\]

From \((2.27)\) and \((2.28)\) we obtain the Hamiltonian
\[
H = \frac{1}{2}(v, v) + U(u, S, T),
\]
\[
U(u, S, T) = \sum_{\alpha \in R^+} \frac{2X_\alpha X_{-\alpha}}{(\alpha, \alpha)} = 2 \sum_{\alpha \in R^+} \frac{S_\alpha^2 + T_\alpha^2 - 2S_\alpha T_\alpha \cosh(u_\alpha)}{(\alpha, \alpha) \sinh^2(u_\alpha)}.
\]

### 2.1.3 Integrals of motion II

From \((2.32)\) and \((2.36)\) we have the redundant number of integrals
\[
\delta_{G^\mathbb{R}} = \frac{1}{2} \dim \mathcal{R}\)\(_{red}^r(G^\mathbb{R})\) - \text{rank} U.
\]

For example, \(\delta_{SL(N, \mathbb{R})} = -[N/2]\). However, it turns out that the redundant integrals are Casimir functions on \((\mathfrak{g}^\mathbb{R})^* \supset \mathfrak{u}^*\) and thereby they are constants on the whole phase space \(\mathcal{R}\)\(_{red}^r(G^\mathbb{R})\) \((2.34)\). Namely, the following statement holds.

- The integrals \((2.37)\) for \(d_j\) even
\[
I_{j,1} = (T^{d_j-1} \eta), \quad (d_j - \text{even})
\]

are the Casimir functions on \((\mathfrak{g}^\mathbb{R})^*\).

- The number of even invariants \(d_j\) of \(\mathfrak{g}^\mathbb{R}\) is equal to \(\text{rank} \mathfrak{u}\).

It follows from these two statements that \((2.43)\) provides the necessary number of the redundant integrals of motion. Let us first prove that \((2.43)\) assumes the form
\[
I_{j,1} = (T^{d_j-1} \eta) = (T^{d_j}), \quad (d_j - \text{even}).
\]

Consider the action of the involutive automorphism \(\rho (4.10)\) on \(I_{j,1}\). Let us choose a basis in the adjoint representation of the algebra \(\mathfrak{g}^\mathbb{R}\) such that \(\mathfrak{h}^\mathbb{R}\) acts as diagonal matrices, \(E_\pm\) for \(\alpha \in R^+\) as upper (for \(+\)) or lower (for \(-\)) triangular matrices. Then the operator \(\tilde{\rho} = -\rho\) acts as the matrix transposition (see \((4.10)\)). The Killing form is invariant with respect to this action. Then \((\tilde{\rho}(T^{d_j-1}), \tilde{\rho}(\eta)) = (T^{d_j-1}, \eta). \) Let \(\eta = \eta^+ + \eta^-\), where \(\tilde{\rho}(\eta^\pm) = \pm \eta^\pm.\) Since \(\tilde{\rho}(T) = -T\) we have
\[
(\tilde{\rho}(T^{d_j-1}), \tilde{\rho}(\eta)) = ((\eta^+ - \eta^-), \tilde{\rho}(T^{d_j-1})) = (-1)^{d_j-1}((\eta^+, T^{d_j-1}) - (\eta^-, T^{d_j-1})).
\]

But the l.h.s. is equal to \((\eta^+ T^{d_j-1}) + (\eta^- T^{d_j-1})\). Therefore, for even \(d_j\) we have \((T^{d_j-1} \eta^+) = 0\) and \((T^{d_j-1} \eta^-) = (T^{d_j-1} \eta^-).\) From \((2.38)\) we find that \(\eta^- = \sum_{\alpha \in R^+} T_\alpha (E_\alpha - E_{-\alpha}).\) In this way for \(d_j\) even \(I_{j,1} = (T^{d_j-1} \eta^-) = (T^{d_j}).\) Thus \((2.44)\) holds. These expressions are the Casimir functions on \(\mathfrak{u}^*\).

Now let us prove that we have exactly \(\text{rank}(U)\) Casimir functions. It follows from Table 1 in Appendix A that for the groups \(\text{SO}(2N + 1), \text{Sp}(N), \text{SO}(2N)\) \((N\) is even), \(G_2, F_4, E_7, E_8\).
rank \( G^R = \text{rank} \, U \), all \( d_j \) are even. Thus, the number of the Casimir functions is equal to the rank \((G^R) = \text{rank} \,(K) \). It means that the redundant integrals are the Casimir functions.

For \( G^R = \text{SL}(N, \mathbb{R}) \) the number of even invariants is equal \([N/2] = \text{rank} \,(SO(N) = U) \). For \( \text{SO}(2N) \), \( (N \text{ is odd}) \) only \( d_N = N \) is odd and we have \( N - 1 \) Casimir functions in (2.43). On the other hand, rank \((U = \text{SO}(N) \times \text{SO}(N)) = N - 1 \). For \( E_6 \) there are two odd orders 5 and 9 of the invariants and thereby we have four Casimir functions. But the rank \((U = Sp(4)) = 4 = 6 - 2 \).

### 2.1.4 Equations of motion and Lax equation

Equations of motion follow from the Hamiltonian (2.40)-(2.41) and the Poisson brackets (2.39). For \( v = \sum_{j=1}^{l} v_j e_j \) and \( u = \sum_{j=1}^{l} u_j e_j \) we have

\[
\dot{u}_j = v_j ,
\]

and

\[
\dot{v}_j = -\partial_{u_j} U(u, S, T) = -4 \sum_{\alpha \in R^+} \frac{\alpha(j)}{\alpha^2} \left( \frac{S_\alpha T_\alpha}{\sinh(u_\alpha)} + (S_\alpha^2 + T_\alpha^2 - 2 \cosh(u_\alpha) S_\alpha T_\alpha) \frac{\cosh(u_\alpha)}{\sinh^2(u_\alpha)} \right) , \ (\alpha(j) = \alpha(e_j)).
\]

The equations for \( S \) and \( T \) have form (B.5), (B.6) with the Hamiltonian (B.1)

\[
H^{\text{top}}(S, T) = \frac{1}{2} \sum_{\nu \in R} \left( \frac{1}{2} T^2_\nu f(\nu) + T_\nu g(\nu) \right) ,
\]

where from (2.41) we find

\[
f(\nu) = \frac{4}{\nu^2 \sinh^2(u_\nu)} , \ g_S(\nu) = -\frac{4T_\nu \cosh(u_\nu)}{\nu^2 \sinh^2(u_\nu)} , \ g_T(\nu) = -\frac{4S_\nu \cosh(u_\nu)}{\nu^2 \sinh^2(u_\nu)} . \tag{2.47}
\]

Then using (B.5) and (B.6) we obtain

\[
\dot{S}_\gamma = \frac{1}{2} \sum_{\alpha+\beta=\gamma} C_{\alpha\beta} \left( S_\alpha S_\beta \left( \frac{1}{\sinh^2(u_\alpha)} - \frac{1}{\sinh^2(u_\beta)} \right) + S_\beta T_\alpha \cosh(u_\alpha) \frac{\sinh^2(u_\alpha)}{\sinh^2(u_\beta)} - S_\alpha T_\beta \cosh(u_\beta) \frac{\sinh^2(u_\beta)}{\sinh^2(u_\alpha)} \right) , \tag{2.48}
\]

\[
\dot{T}_\gamma = \frac{1}{2} \sum_{\alpha+\beta=\gamma} C_{\alpha\beta} \left( T_\alpha T_\beta \left( \frac{1}{\sinh^2(u_\alpha)} - \frac{1}{\sinh^2(u_\beta)} \right) - T_\beta S_\alpha \cosh(u_\alpha) \frac{\sinh^2(u_\alpha)}{\sinh^2(u_\beta)} + T_\alpha S_\beta \cosh(u_\beta) \frac{\sinh^2(u_\beta)}{\sinh^2(u_\alpha)} \right) . \tag{2.49}
\]

We will prove in Appendix C that the equations of motion (2.45), (2.46), (2.48), (2.49) have the Lax form

\[
\partial_t \eta = [\eta, M] ,
\]

where the \( M \) operator is defined as

\[
M = \sum_{\alpha \in R^+} Y_\alpha U_\alpha , \ Y_\alpha(u_\alpha) = \partial_t X_\alpha|_{t=u_\alpha} = \frac{2.38}{S_\alpha \cosh(u_\alpha) - T_\alpha} . \tag{2.51}
\]

### 2.2 Model II

In this model we consider (instead of \( R_1 (G^C) \)) the unreduced phase space

\[
R_{II}(G^C) = T^* G^C . \tag{2.52}
\]
Then
\[ \dim \mathcal{R}_{II}(G^C) = 4 \dim_R G^C. \]
In the left trivialization \( T^*G^C \sim G^C \times (\mathfrak{g}^C)^* \). It is described by the coordinates \((r, \xi)\), where \( r \in G^C, \xi \in (\mathfrak{g}^C)^* \). The symplectic form \( \omega = D\vartheta \) is similar to (2.3), where 1-form \( \vartheta \) is
\[ \vartheta_{II}^C = (\xi, r^{-1}Dr). \tag{2.53} \]
Define the quadratic Hamiltonian
\[ H = \frac{1}{2}(\xi, \xi). \tag{2.54} \]
Consider the symplectic quotient \( \mathcal{R}_{II}(G^C) \) with respect the following gauge symmetry. The gauge group \( G^{gauge}_II = K \) acts as
\[ r \rightarrow frf^{-1}, \xi \rightarrow Ad_f \xi. \tag{2.55} \]
The corresponding moment map is
\[ \mu_{II} := (Ad_{r} \xi - \xi)|_t = 0, \ t = \text{Lie} (K). \tag{2.56} \]
In accordance with (A.18) represent \( r \in G^C \) as \( r = k_1 \exp(u)k_2^{-1} \), where \( k_{1,2} \in K \). Then
\[ frf^{-1} = f k_1 \exp(u)k_2^{-1} f^{-1}, \ u \in \mathfrak{h}^R. \tag{2.57} \]
Let us fix the gauge assuming that
\[ f = k_2^{-1}. \tag{2.58} \]
Then
\[ frf^{-1} = h^{-1} \exp(u), \ h = k_1^{-1}k_2. \tag{2.59} \]
Note in (2.57) \( k_1, k_2 \) are defined up to the left multiplication by an element from the Cartan torus \( \mathcal{T} \). It means that \( h \) is defined up to the conjugations \( \mathcal{T} \) (compare with (2.22))
\[ h \rightarrow shs^{-1}. \tag{2.60} \]
Therefore, \( h \) is an element of the quotient of \( K \) under the \( \mathcal{T} \) action
\[ h \in \tilde{K} = \{ h \sim shs^{-1} \mid s \in \mathcal{T} \}. \tag{2.61} \]
In these terms the moment map equation (2.54) assumes the form
\[ (\xi - Ad_{h^{-1} \exp(u)} \xi)|_t = 0. \tag{2.62} \]
Let
\[ \xi = p + \xi_t, \ p \in \mathfrak{p}^C, \ \xi_t \in \mathfrak{k} \tag{2.63} \]
is the Cartan decomposition (A.14). For the Cartan part \( \xi_{h^C} \) the corresponding decomposition is
\[ \xi_{h^C} = v + \xi_t, \tag{2.64} \]
where \( \mathfrak{t} \) is the Cartan subalgebra of \( \mathfrak{k} \), and \( v \in \mathfrak{h}^R \subset \mathfrak{p}^C \). Since \( \text{Ad}_K(\mathfrak{p}^C) \in \mathfrak{p}^C \) we have
\[ \text{Ad}_{h^{-1} \exp(u)}(v)|_t = 0. \tag{2.65} \]
Thus, solutions \( \xi \) of (2.62) are defined up to the adding \( v \in \mathfrak{h}^R \). The moment map constraints (2.56) along with the gauge fixing define the reduced phase space
\[ \tilde{\mathcal{R}}_{II}^{red}(G^C) = \mathcal{R}_{II}(G^C)/K = \mu_{II}^{-1}(0)/K. \tag{2.66} \]
The variables \((\xi, r)\) being defined on \( \tilde{\mathcal{R}}_{II}^{red} \) satisfy the moment map equation (2.62), where
\[ r = h^{-1} \exp(u), \ h \in K, \ \exp(u) \in \mathcal{H}^R, \ v \in \mathfrak{h}^R. \tag{2.67} \]
Thus
\[ \tilde{\mathcal{R}}_{II}^{red}(G^C) \sim T^*\mathcal{H}^R \times T^*K = \{ (v, \exp u), (\xi_t, h) \}. \tag{2.68} \]
Remark 2.3 For this model upon the symplectic reduction we deal with the complex symplectic space \((2.52)\) with the complex one form \((2.53)\) and the Hamiltonian \((2.54)\). The real symplectic space \((2.66)\) arises after the symplectic reduction with respect to the action of the real group \((2.55)\).

By comparing \(\tilde{R}^{red}_{II}(G^{\mathbb{C}})\) with \(\tilde{R}^{red}_{I}(G^{\mathbb{C}})\) \((2.21)\), define the variables on \(\tilde{R}^{red}_{II}\) as

\[
\begin{align*}
\xi &= \eta, \quad \text{(2.67a)} \\
\xi_{|t} &= -T, \quad \text{(2.67b)} \\
S &= \text{Ad}_{h} T. \quad \text{(2.67c)}
\end{align*}
\]

In these terms the moment map equation \((2.62)\) assumes the form \(S + \text{Ad}_{\exp(u)}\eta|_{t} = 0\). It coincides with \((2.13b)\), while \((2.67b)\) coincides with \((2.13a)\). The last equation \((2.67c)\) is just \((2.22)\). Thus, we come to the symplectic quotient (see \((2.24)\)).

The map \((2.67a)\) \((2.67b)\) along with the identification of \(\exp(u)\) in the both models is symplectomorphism. In fact, using \((2.62)\), \((2.64)\) and \((2.67)\) one can prove that

\[
D(\xi, r^{-1} Dr) = (Dv, Du) + D(T, h^{-1} Dh).
\]

It coincides with the form \((2.29)\). Note that the Hamiltonian \((2.54)\) coincides with \((2.4)\).

In this way we described the passage from \(R^{red}_{I} \) to \(\tilde{R}^{red}_{II} \). The inverse procedure is straightforward. In the Model I we defined the gauge action \(g \rightarrow f_{1}gf_{2}^{-1} \) \((2.6), (2.5)\). Let \(g = k_{1}\, \text{exp}(u)k_{2}^{-1}\) \((A.18)\). We fix the gauge \(f_{2} = k_{2}^{-1}\) as in \((2.57), (2.58)\). Then we come to the Model II with \(r = g\) and the residual gauge action by \(f_{1} \in K\). Thereby, \(\tilde{R}^{red}_{II}\) is symplectomorphic to \(\tilde{R}^{red}_{I}\). Now take into account the residual gauge symmetry \((2.60)\). It is the same as in the model I \((2.22)\). Thus, we come to the symplectic quotient (see \((2.24)\))

\[
\tilde{R}^{red}_{II}(G^{\mathbb{C}}) // T \cong T^* \mathcal{H}^{\mathbb{R}} \times (T^* K // T),
\]

\((2.68)\)

where \(T^* K // T = \{(T, h)\}\) is defined by the condition \((2.61)\) and the moment map equation with respect to the torus \(T\) action \(\text{Ad}_{h} T - T\)|\(_{t} = 0\). The dimension of the reduced space is

\[
\dim_{\mathbb{R}}(\tilde{R}^{red}_{II}(G^{\mathbb{C}}) // T) = 2 \dim T^* K = 2 \sum_{j=1}^{l} (2d_{j} - 1). 
\]

\((2.69)\)

In fact, one can choose \((S \in \mathcal{O}_{K}, T \in \mathcal{O}_{K})\) as coordinates on \(\tilde{R}^{red}_{II}\) as in \((2.24)\)

\[
T^* \mathcal{H}^{\mathbb{R}} \times (\mathcal{O}_{K} \times \mathcal{O}_{K}) \cong \{(v, \exp u), (T, S)\}.
\]

Passing to the symplectic quotient with respect to the \(T\) action we come to the reduced space

\[
R^{red}_{II}(G^{\mathbb{C}}) \sim T^* \mathcal{H}^{\mathbb{R}} \times (\mathcal{O}_{K} \times \mathcal{O}_{K}) // T.
\]

\((2.70)\)

It has dimension (see \((2.69)\))

\[
\dim_{\mathbb{R}} R^{red}_{II}(G^{\mathbb{C}}) = 2 \dim (\mathcal{O}_{K} \times \mathcal{O}_{K}) = 4 \sum_{j=1}^{l} (d_{j} - 1).
\]

\((2.71)\)

The dimension of \(R^{red}_{II}\) coincides with the dimension of \(R^{red}_{I}\) \((2.25)\).

The Model II as the Model I becomes integrable in the real form. First, consider the real form \(\tilde{R}^{red}_{II}(G^{\mathbb{R}})\) of the phase space \(\tilde{R}^{red}_{II}(G^{\mathbb{C}})\) \((2.66)\)

\[
\tilde{R}^{red}_{II}(G^{\mathbb{R}}) = T^* \mathcal{H}^{\mathbb{R}} \times (T^* U) \sim \{\xi = \eta , \quad (2.37), \quad (2.38), \quad T \in u , \quad h \in U\}. 
\]

(2.72)
\[ \xi = v + \sum_{\alpha \in R} X_\alpha E_\alpha, \quad X_\alpha = \frac{T_\alpha e(- u_\alpha) - S_\alpha}{\sinh(u_\alpha)}, \quad X_{-\alpha} = \frac{T_\alpha e(u_\alpha) - S_\alpha}{\sinh(u_\alpha)}. \]  

(2.73)

From (A.32) we find its dimension

\[ \dim \tilde{R}^{\text{red}}_{II}(G^R) = 2 \sum_{j=1}^l d_j. \]  

(2.74)

Passing from \( T^*U \) to the coadjoint orbits \( O_U \times O_U \) we come to the similar to (2.34) phase space

\[ \mathcal{R}^{\text{red}}_{II}(G^R) \sim T^*H^R \times (O_U \times O_U). \]  

(2.75)

It has dimension \( \dim \mathcal{R}^{\text{red}}_{II}(G^R) = 2 \sum_{j=1}^l d_j - 2 \text{rank } U \) (2.36). After identifications (2.67) of the models one can use the same set of integrals of motion (2.31). Following Section 2.1.3 we find the needed for integrability number of integrals.

### 2.3 Model III

#### 2.3.1 Symplectic reduction

Let \( X^C = G^C/K \) be some Riemannian symmetric space. As the staring phase space consider

\[ \mathcal{R}^{G^C}_{III} = T^*G^C \times T^*X^C, \]  

(2.76)

where as above \( T^*G^C \) is described by the coordinates \( \rho \in G^C \) and \( \zeta \in \mathfrak{g}^C \).

Let us define the coordinates on the cotangent bundle \( T^*X^C \). It can be represented as the symplectic quotient \( T^*G^C//K \). We may choose the coordinates on \( T^*G^C \) as \( (\zeta \in \mathfrak{g}^C, g \in G^C) \). The one-form

\[ \vartheta = (\zeta, Dgg^{-1}) \]  

(2.77)

is invariant under the \( K \)-action

\[ g \to kg, \quad \zeta \to \text{Ad}_k \zeta. \]  

(2.78)

The symplectic quotient \( T^*G^C//K \sim T^*X^C \) is defined by the moment map constraints \( \zeta|_t = 0 \) and the gauge-invariant variables \( \text{Ad}_{g^{-1}} \zeta \)

\[ T^*X^C = \{ P(g, \zeta) = \text{Ad}_{g^{-1}} \zeta \mid g \sim kg, \zeta|_t = 0 \}. \]  

(2.79)

In fact, from (2.78) we have \( P(kg, \text{Ad}_k(\zeta)) = P(g, \zeta) \). The form on \( \mathcal{R}^{G^C}_{III} \) (2.76) is the sum of two one-form

\[ \vartheta^{G^C}_{III} = (\zeta, \rho^{-1} D\rho) + (\zeta, Dgg^{-1}), \quad (\zeta|_t = 0) \]  

(2.80)

Consider the quadratic Hamiltonian

\[ H = \frac{1}{2}(\zeta, \zeta). \]  

(2.81)

The symplectic symmetry group \( G^gauge_{III} = G^C \) acts as

\[ G^C : \rho \to f \rho f^{-1}, \quad \zeta \to \text{Ad}_f \zeta, \quad g \to gf^{-1}, \quad \zeta \to \zeta. \]  

(2.82)

The corresponding moment map is

\[ \mu_{III} := -\zeta + \text{Ad}_\rho \zeta - P. \]  

(2.83)

By the gauge action one can transform \( r \) to the Cartan subgroup

\[ \rho = f \exp u^C f^{-1}, \exp u^C \in \mathcal{H}^C. \]  

(2.84)
The residual symmetry is the Cartan subgroup $H = (2.84)$. It generates the moment map

\[ \mu = P_{\text{h}^\mathbb{C}}. \]

Consider the real form of the model III. The symmetric space $R$ coincides with $\text{dim} H^\mathbb{C}$. It generates the moment map $\mu = P_{\text{h}^\mathbb{C}}$. Thus, finally taking $P_{\text{h}^\mathbb{C}} = 0$ we obtain

\[ \zeta = v^\mathbb{C} + \sum_{\alpha \in R} \frac{P_\alpha}{1 - \exp u^\mathbb{C}} E_{\alpha}. \]  

(2.87)

In this way the reduced phase space is

\[ \mathcal{R}_{\text{III}}^{\text{red}}(G^\mathbb{C}) \sim T^*H^\mathbb{C} \times (T^*H^\mathbb{C}/H^\mathbb{C}) = \{(v^\mathbb{C}, \exp u^\mathbb{C}), P//H^\mathbb{C}\}. \]  

(2.88)

From (A.31) we have

\[ \text{dim}_R \mathcal{R}_{\text{III}}^{\text{red}}(G^\mathbb{C}) = \text{dim}_R T^*X^\mathbb{C} = 2 \sum_{j=1}^l (2d_j - 1). \]

It coincides with $\text{dim}_R (\mathcal{R}_{\text{III}}^{\text{red}}(G^\mathbb{C})//T)$. (2.69).

### 2.3.2 Real form and integrals of motion

Consider the real form of the model III. The symmetric space $X = G^\mathbb{C}/K$ is replaced by $X^\mathbb{R} = G^\mathbb{R}/U$ and $T^*H^\mathbb{C}$ is replaced by $T^*H^\mathbb{R}$:

\[ \mathcal{R}_{\text{III}}^{\text{red}}(G^\mathbb{R}) \sim T^*H^\mathbb{R} \times (T^*H^\mathbb{R}/H^\mathbb{R}) = \{(v, \exp \tilde{u}), \tilde{P}//H^\mathbb{R}\}, \quad \tilde{v}, \tilde{u}, \tilde{P}_\alpha \in \mathbb{R}. \]

(2.89)

From (A.33) we obtain

\[ \text{dim}_R \mathcal{R}_{\text{III}}^{\text{red}}(G^\mathbb{R}) = \text{dim}_R T^*X^\mathbb{R} = 2 \sum_{j=1}^l d_j. \]  

(2.90)

Instead of (2.87) we have

\[ \zeta = \tilde{v} + \sum_{\alpha \in R} \frac{\tilde{P}_\alpha}{1 - \exp \tilde{u}_\alpha} E_{\alpha}, \]

(2.91)

where $\tilde{P}_\alpha \in \mathbb{R}$. It is defined up to the equivalence $\tilde{P}_\alpha \sim \tilde{P}_\alpha \exp(x)$, $x \in \mathbb{R}$. The corresponding Hamiltonian (2.81) is

\[ H = \frac{1}{2} (\tilde{v}, \tilde{v}) - \sum_{\alpha \in R^+} \frac{\tilde{P}_\alpha \tilde{P}_{\alpha}}{(\alpha, \alpha) \sinh^2(\tilde{u}_\alpha/2)}. \]

Now we deal with the well-defined real integrable system. We want to find the independent integrals of motion needed for the integrability. We will demonstrate below that the integrals of motion take the form (compare with (2.31))

\[ I_{jk} = (P^{d_j - k} s^k), \quad k = 0, \ldots, d_j. \]  

(2.92)

Since $\mu_{\text{III}} = 0$, we replace $P$ in this expression with $\text{Ad}_{\exp(u)} \zeta - \zeta$. Therefore,

\[ I_{jk} = ((\text{Ad}_{\exp(u)} \zeta - \zeta)^{d_j - k} s^k). \]

The number of integrals $I_{jk}$ for the fixed $j$ is $d_j$. The total the number of independent integrals of motion is equal to $\sum_{j=1}^l d_j = \frac{1}{2} \text{dim}_R \mathcal{R}_{\text{III}}^{\text{red}}(G^\mathbb{R})$ (2.90) needed for integrability.

---

6In this expression we do not fix the gauge action of $\text{Ad}_{\mathfrak{h}^\mathbb{C}}$ on $P$. 

16
2.3.3 Model II and Model III

Here we establish the equivalence of Model II and Model III. First prove the symplectomorphism of the phase spaces (2.68) and (2.88)

\[ \tilde{\mathcal{R}}_{II}^{red}(G^C)/\mathcal{T} \cong \mathcal{R}_{II}^{red}(G^C) = T^*\mathcal{H}^C \times (T^*\mathcal{X}^C/\mathcal{H}^C) \]  \hspace{1cm} (2.93)

Consider first "the coordinate part". In the Model III one can take \( \rho = \exp(u^C) \) (2.84). Let \( g = kp \) be the polar decomposition of \( g \in K \setminus G \). Assume that the gauge transformation \( f \) in (2.82) is equal to \( p \). Then by (2.82) transform \( \exp(u^C) \) to \( \tilde{r}_{II} \)

\[ \exp(u^C) = p\tilde{r}_{II}p^{-1}. \]

After this transformation we are left with the gauge group \( K \) as in the Model II. Acting as in (2.59) by \( f \in K \) transform \( \tilde{r}_{II} \) to the form \( \tilde{r}_{II} \to \tilde{h}^{-1}\exp(\tilde{u}) \). In this way we come to the equality

\[ k^{-1}p^{-1}(g)\exp(u^C)p(g)k = \tilde{h}^{-1}\exp(\tilde{u}), \]  \hspace{1cm} (2.94)

where \( \tilde{h} \) is defined up to the conjugation and \( p(g) \) up to the left multiplication

\[ h = \text{th}^{-1}, \quad t \in \mathcal{T}, \quad p(g) \sim sp(g), \quad s \in \mathcal{H}^C. \]

Thus, \( \exp(u^C) \) and \( \tilde{h}^{-1}\exp(\tilde{u}) \) are conjugated under the \( G^C \) action. The equality (2.94) establishes the isomorphism

\[ \mathcal{H}^C \times (\mathcal{H}^C \setminus \mathcal{X}^C) \sim \mathcal{H}^\mathbb{R} \times (\mathcal{T} \setminus K/\mathcal{T}), \]

\[ (\exp(u^C), p(g)) \sim (\exp(\tilde{u}), \tilde{h}). \]

Compare \( \mu_{II} \) (2.56) and \( \mu_{III} \) (2.83). After the gauge transformation by \( f = p \) the transformed \( g \) belongs to \( K \). For this reason \( \mathbf{P} = \text{Ad}^{-1}_g(\zeta) \in p^C \) (2.85). Therefore, its projection on the subalgebra \( \mathfrak{k} \) vanishes \( \text{Pr} \mathbf{P} |_{\mathfrak{k}} = 0 \). Then the moment map equation \( \mu_{III} = 0 \) becomes

\[ (\zeta - \text{Ad}_{\tilde{h}^{-1}\exp(\tilde{u})}\zeta)|_{\mathfrak{k}} = 0. \]

It coincides with the moment map equation \( \mu_{II} = 0 \) (2.62) in the Model II. As in (2.66) we come to the space \( T^*\mathcal{H}^\mathbb{R} \times T^*K \) with parameters \( \{(\tilde{v}, \exp \tilde{u}), (\zeta, \tilde{h})\} \).

Now compare \( \mathcal{R}_{II}^{red}(G^\mathbb{R}) \) (2.89) with \( \tilde{\mathcal{R}}_{II}^{red}(G^\mathbb{R}) \) (2.72). Notice that their dimensions (2.74) and (2.90) coincide. The isomorphism is given by the map

\[ v = \tilde{v}, \quad u = \tilde{u}, \quad \mathbf{P}_\alpha = (1 - \exp u_\alpha)X_\alpha, \]  \hspace{1cm} (2.95)

where \( X_\alpha \) is defined in (2.73). Then we have

\[ \begin{cases} 
\mathbf{P}_\alpha = (1 - \exp(u_\alpha)) \frac{\mathbf{T}_\alpha(-u_\alpha) - \mathbf{S}_\alpha}{\sinh(u_\alpha)} , & \alpha \in R^+, \\
\mathbf{P}_\alpha = (1 - \exp(-u_\alpha)) \frac{\mathbf{T}_\alpha(u_\alpha) - \mathbf{S}_\alpha}{\sinh(u_\alpha)} , & \alpha \in R^- .
\end{cases} \]

On the other hand, \( \mathbf{S}_\alpha \) and \( \mathbf{T}_\alpha \) can be expressed as linear combinations of \( \mathbf{P}_\alpha \) and \( \mathbf{P}_{-\alpha} \). In this way we come to the diffeomorphism

\[ \mathcal{R}_{III}^{red}(G^\mathbb{R}) \sim \tilde{\mathcal{R}}_{II}^{red}(G^\mathbb{R}). \]  \hspace{1cm} (2.96)

It can be proved that this map is the symplectomorphism.

3 Higgs bundles. Model I

The described system is the Hitchin system over a singular curve. Here we define the corresponding holomorphic bundles.
### 3.1 Holomorphic bundles

#### Base spectral curve. The base spectral curve is a singular curve $\Sigma^I$. It is a collection of two rational curves $\Sigma_\alpha \sim \mathbb{C}P^1$, ($\alpha = 1, 2$) with corresponding holomorphic coordinates $z_1, z_2 \in \mathbb{C}_{1,2}$. The disjoint union of $\Sigma_\alpha$ is mapped to $\Sigma^I$ by the normalization

$$
\pi : \Sigma_1 \cup \Sigma_2 \to \Sigma^I, \quad \begin{cases}
\pi(z_1 = 0) = \pi(z_2 = 0), \\
\pi(z_1 = \infty) = \pi(z_2 = \infty).
\end{cases}
$$

The base spectral curve is a singular curve $\Sigma$ with the maximal compact subgroup $K \subset G^C$. At the glued points we define the maps of the corresponding sections

$$
r_\infty : s|_{z_1=\infty} \in \Gamma(E_1) \to s|_{z_2=\infty} \in \Gamma(E_2), \quad r_0 : s|_{z_1=0} \in \Gamma(E_1) \to s|_{z_2=0} \in \Gamma(E_2),
$$

(3.1)

where we assume that $r_\infty \in G^C$ and $r_0 \in K$ is a maximal compact subgroup $K \subset G^C$. Let $\bar{\partial}_\alpha + \bar{A}_\alpha$ be the antiholomorphic connections acting on the sections $\Gamma(s)$. Then

$$
\mathcal{D} = \{ (\bar{\partial}_\alpha + \bar{A}_\alpha), \alpha = 1, 2, \ r_\infty \in G^C, \ r_0 \in K \}
$$

(3.2)

defines the vector bundle $E^I(G^C)$ on the singular curve $\Sigma^I$.

#### Holomorphic bundles. Let $\mathcal{P}(G^C)$ be the principle $G^C$-bundle over $\Sigma^I$, $V$ is a $G^C$ module and $E(G^C) = \mathcal{P} \times_{G^C} V$ is the associated vector bundle. It is constructed by means of two vector bundles $E_\alpha(G^C) = \mathcal{P}_\alpha \times_{G^C} V$ over $\Sigma_\alpha$. At the glued points we define the maps of the corresponding sections

$$
r_\infty(s|_{z_1=\infty}) \in \Gamma(E_1), \quad r_0(s|_{z_1=0}) \in \Gamma(E_1),
$$

(3.3)

Let $\sigma$ be the involutive automorphism of $G^C$, such that its fixed point set is the maximal compact subgroup $K \subset G^C$. At the point $z_\alpha = 1$ we replace in $G^C$ with the maximal compact subgroup $K$

$$
f_\alpha(z_\alpha, \bar{z}_\alpha)|_{z_\alpha=1} \in K.
$$

(3.4)

We will refer to such vector bundle as the vector bundle with the quasi-compact structure. In particular, for $G^C = \text{SL}(N, \mathbb{C})$ and $K = \text{SU}(N)$ the quasi-compact structure means that the gauge group preserves a positive definite hermitian structures at the marked points $z_\alpha = 1$.

The gauge action on the data (4.3) has the form

$$
\bar{\partial}_\alpha + \bar{A}_\alpha \to f_\alpha(\bar{\partial}_\alpha + \bar{A}_\alpha)f^{-1}_\alpha,
$$

(3.5)

$$
r_\infty \to f_2(\infty)r_\infty f_1^{-1}(\infty), \quad f_\alpha(\infty) = f_\alpha(z_\alpha, \bar{z}_\alpha)|_{z_\alpha=\infty},
$$

(3.6)

$$
r_0 \to f_2(0)r_0 f_1^{-1}(0), \quad f_\alpha(0) = f_\alpha(z_\alpha, \bar{z}_\alpha)|_{z_\alpha=0}.
$$

(3.7)
The moduli space $\text{Bun}_I(G^C)$. The quotient space $\text{Bun}_I(G^C) = D/\mathcal{G}$ of (3.2), by (3.5), (3.6), (3.7) is the moduli of vector $G^C$ bundles with the quasi-compact structure over the singular curve $\Sigma^I$.

It is proved below in Section 4.1 that by the gauge group action (3.5) of $G$ (3.3), (3.4) on the generic configurations the connection forms $\bar{A}_\alpha$ on each component $\Sigma_\alpha$ can be transformed to the trivial form

$$\bar{A}_\alpha = 0.$$ (3.8)

Thereafter the residual gauge transformations are constant maps to $K$

$$G^{res}_\alpha = \{ f_\alpha \in K \}.$$ (3.9)

Put it otherwise, the almost all connections $\bar{A}_\alpha$ have the pure gauge form $\bar{A}_\alpha = f_\alpha \bar{\partial} f^{-1}_\alpha f_\alpha \in \mathcal{G}$. The residual gauge transformations $G^{res}_\alpha$ preserving (3.8) are constant maps to $G^C$. In fact, it is reduced to (3.9).

For generic $r_\infty$ the transformation (3.6) allows one to take it to the Cartan subgroup $H^R \subset G^R$ (see (A.26))

$$f_2(\infty) r_\infty f^{-1}_1(\infty) = \exp(u), \ u \in h^R.$$ (3.10)

As in previous Section these transformations are defined up to the action from the right by the final residual gauge transformation $G^{res}_2$

$$f_\alpha(\infty) \rightarrow f_\alpha(\infty) s^{-1}, \ s \in G^{res}_2 = T,$$

where $T$ is the Cartan torus in $K$. Denote the result of these actions on $r_0$ as

$$f_2 r_0 f^{-1}_1 = h \in K.$$ (3.11)

In this way the element $h$ is defined up to the conjugation

$$h \rightarrow shs^{-1}, \ s \in G^{res}_2 = T.$$

Therefore, $h$ is an element of the quotient $\tilde{K} = K/T$ (2.61). Thus, a big cell of $\text{Bun}_I^0(G^C)$ in the moduli space is

$$\text{Bun}_I^0(G^C) = D/\mathcal{G} = (H^R, \tilde{K}).$$ (3.12)

From (A.29) we find

$$\dim_{\mathbb{R}} \text{Bun}_I(G^C) = \text{rank}(G^R) + \dim K - \text{rank} K = \dim K = 2 \sum_{j=1}^{l} d_j - l.$$ (3.13)

3.2 Higgs bundles

The quasi-compact Higgs bundle $\mathcal{H}_{qc}(G^C)$ over $\Sigma$ is set of pairs

$$\{(\partial \bar{A}_\alpha = \bar{\partial}_\alpha + \bar{A}_\alpha, \Phi_\alpha), \ (r_\infty, \lambda_\infty), \ (r_0, \lambda_0)\}.$$  

Here

$$\Phi_\alpha = \Phi_\alpha(z_\alpha, \bar{z}_\alpha) dz_\alpha \in \text{End}(E_\alpha) \otimes \Omega^{(1,0)}(\Sigma_\alpha)$$

are the components of the Higgs fields. The pairs are

$$(r_\infty, \xi_\infty) \in T^*(\text{Hom}(E|_{z_1=\infty} \rightarrow E|_{z_2=\infty})) \sim T^*G,$$

$$(r_0, \xi_0) \in T^*(\text{Hom}(E|_{z_1=0} \rightarrow E|_{z_2=0})) \sim T^*K,$$
The Higgs bundle $\mathcal{H}_G$ is a symplectic manifold equipped with the form

$$\omega = \sum_{\alpha=1,2} \int \left( D\Phi_\alpha, D\tilde{A}_\alpha \right) - \int \sum_{i=0,\infty} \delta(z_i, \bar{z}_i) |D(\lambda_i, r_1^{-1}D_i)| + \int \sum_{i=0,\infty} \delta(z_i, \bar{z}_i) |D(\lambda_i, r_1^{-1}D_i)|.$$

The group of the automorphisms $\mathcal{G}$ of the bundle $E$ can be lifted to symplectic automorphisms of the Higgs bundle $G$

$$\Phi_\alpha \to f_\alpha \Phi_\alpha f_\alpha^{-1}, \quad \lambda_0 \to \text{Ad}_{f_1}^{-1}|_{z_1=0}(\lambda_0), \quad \lambda_\infty \to \text{Ad}_{f_1}^{-1}|_{z_1=\infty}\lambda_\infty.$$

Taking into account the gauge fixing (3.8), (3.10), (3.11) we find the moment maps are

$$\mu_1 = -\partial_1 \Phi_1 + \delta(z_1, \bar{z}_1)|_{z_1=\infty}\lambda_\infty + \delta(z_1, \bar{z}_1)|_{z_1=0}\lambda_0,$$

$$\mu_2 = -\partial_2 \Phi_2 + \delta(z_2, \bar{z}_2)|_{z_2=\infty}(\text{Ad}_{\exp(u)}\lambda_\infty + \delta(z_2, \bar{z}_2)|_{z_2=0}(\text{Ad}_h\lambda_0).$$

Impose the moment map constraints $\mu_1 = \mu_2 = 0$. They mean that the Higgs fields $\Phi_\alpha(z_\alpha, \bar{z}_\alpha)$ are holomorphic on $\Sigma_\alpha$ and have simple poles at $z_\alpha = 0, \infty$ with the definite residues. Moreover, due to the quasi-compactness (3.13) the Higgs fields have the poles at $z_\alpha = 1$ with residues in the compact subalgebra $\mathfrak{k}$. We summarize this structure in the Table:

| $z_\alpha$ | $z_\alpha = 0$ | $z_\alpha = 1$ | $z_\alpha = \infty$ |
|------------|----------------|----------------|-----------------|
| Res $\Phi_1$ | $\lambda_0$ | $K_1 \in \mathfrak{t}$ | $\lambda_\infty$ |
| Res $\Phi_2$ | $\text{Ad}_{r_0}\lambda_0$ | $K_2 \in \mathfrak{t}$ | $\text{Ad}_{r_\infty}\lambda_\infty$ |

The fields $\Phi_\alpha$ satisfying these conditions take the form

$$\Phi_1 = \frac{\lambda_0}{z_1} + \frac{\lambda_0 + \lambda_\infty + K_1}{1 - z_1}, \quad \Phi_2 = \frac{\text{Ad}_{r_0}\lambda_0}{z_2} + \frac{\text{Ad}_{r_0}\lambda_0 + \text{Ad}_{r_\infty}\lambda_\infty + K_2}{1 - z_2}.$$

Since the sums of residues vanishes

$$\lambda_0 + \lambda_\infty + K_1 = 0, \quad \text{Ad}_{r_0}\lambda_0 + \text{Ad}_{r_\infty}\lambda_\infty + K_2 = 0.$$

and $K_\alpha \in \mathfrak{k}$ we have $\text{Res}\Phi_\alpha(z_\alpha = 1)|_{\mathfrak{t}} = 0$, or

$$\lambda_0 + \lambda_\infty|_{\mathfrak{t}} = 0, \quad \text{Ad}_{r_0}\lambda_0 + \text{Ad}_{r_\infty}\lambda_\infty|_{\mathfrak{t}} = 0.$$

The role of the Lax operators, depending on the spectral parameters $z_\alpha$ will play the gauge transformed Higgs fields $\Phi_\alpha = f_\alpha L_\alpha f_\alpha^{-1}$, where $f_\alpha(z_\alpha)|_{z_\alpha=\infty}$ transform $g$ to $e(u)$ (3.10) and $r_0$ to $h$. Then

$$L_1(z_1) = \frac{\lambda_0}{z_1} + \frac{\lambda_0 + \lambda_\infty}{1 - z_1}, \quad L_2(z_2) = \frac{\text{Ad}_h\lambda_0}{z_2} + \frac{\text{Ad}_h\lambda_0 + \text{Ad}_{\exp(u)}\lambda_\infty}{1 - z_2}.$$

To come to notations of the previous Section we put

$$\lambda_0 = T := \nu, \quad \text{Ad}_h\lambda_0 = S, \quad \lambda_\infty = \eta.$$

In these notations (3.17) assumes the form

$$L_1(z_1) = \frac{T}{z_1} + \frac{T + \eta}{1 - z_1}, \quad L_2(z_2) = \frac{S}{z_2} + \frac{S + \text{Ad}_{\exp(u)}\eta}{1 - z_2}.$$

\footnote{We omit here and in what follows for brevity the differentials $dz_\alpha$.}
Notice that (3.16)
\[(T + \eta)|_t = 0, \ (S + \text{Ad}_{\exp(u)}\eta)|_t = 0\]
becomes the moment map equations \(\mu_I = \mu_R = 0\) (2.3).

The moduli space of the Higgs bundle with the quasi-compact structure are parameterized by \(S, T \in \mathcal{O}_K\) and \(\eta\) satisfying (2.8):
\[
\mathcal{M}_I(G^C) = \mathcal{H}(G^C)/\mathcal{G} = T^*H^\mathbb{R} \times (\mathcal{O}_K \times \mathcal{O}_K)/T = \{(v, \exp u, T, S = \text{Ad}_h T)\}.
\]
Moreover,
\[
\dim \mathcal{M}_I(G^C) = \dim (\mathcal{O}_K \times \mathcal{O}_K) = 4 \sum_{j=1}^l (d_j - 1).
\]
From (2.24) we then find that \(\mathcal{M}_I(G^C) \sim \mathcal{R}^\text{red}_{G^C} (2.24)\). Thus, the moduli space \(\mathcal{M}_I(G^C)\) is the real symplectic manifold with the form (2.29).

### 3.3 Integrals of motion

The integrals of motion are obtained by means of the basis of meromorphic differentials on \(\Sigma_\alpha\)
\[
\mu^0_{k,j,d_j} = (z_\alpha)^{-k}(dz_\alpha)^{d_j}, \ \mu^1_{k,j,d_j} = (1 - z_\alpha)^{-k}(dz_\alpha)^{d_j}, \ (0 < k_j \leq d_j).
\]
By expanding the gauge invariant quantities
\[
(L_\alpha(z_\alpha)^{d_j}), \ k = 1, \ldots, l, \ (d_j \text{ are the invariants of } g)
\]
in this basis we obtain the invariant integrals. Consider, for example, \(L_1(z)\):
\[
(L_1(z)^{d_j}) = \sum_{k=0}^{d_j} C^k_{d_j} z^{k-d_j} (1 - z)^{-k} (T^{d_j-k}(T + \eta)^k) = \sum_{k=0}^{d_j} C^k_{d_j} z^{k-d_j} (1 - z)^{-k} \sum_{i=0}^k C^i_k (T^{d_j-i} \eta^i).
\]
Expanding \(z^{k-d_j}(1 - z)^{-k}\) as
\[
\frac{1}{z^j(1 - z)^k} = \sum_{i=0}^{j-1} c_i z^{i-j} + \sum_{n=0}^{k-1} b_n (1 - z)^{n-k},
\]
\[
c_i = k(k+1) \cdots (k+i-1) i!; \ b_n = (-1)^n \frac{j(j+1) \cdots (j+n-1)}{n!}
\]
we find the considered above integrals of motion (2.31)
\[
I_{jk} = (T^{d_j-k} \eta^k), \ k = 0, \ldots, d_j.
\]
The number of integrals is \(\mathcal{N}_G (2.32)\). Again, for the integrability we have the deficit \(\delta_{GC} (2.33)\).
3.4 Real forms

Following the general scheme [1] we consider real points of some complex structure on the moduli space of the Higgs bundles. We use a special involution \( i \) of the Higgs bundle. The operator is defined in the following way. Let \( \sigma \) be the antiholomorphic involutive automorphism (A.16) acting on the complex Lie algebra \( g^C \) in such a way that the fixed point set \( g^R \) is the normal real form of \( g^C \). Since the marked points \( z_\alpha = 0, 1, \infty \) are invariant under conjugation we can accompany it with the antiholomorphic automorphism of \( \Sigma z_\alpha \to \bar{z}_\alpha \). In general case the operator \( i \) is defined as:

\[
i(\partial A_\alpha(z_\alpha), \Phi_\alpha(z_\alpha)) = (\sigma \partial A_\alpha(\bar{z}_\alpha), \sigma \Phi_\alpha(\bar{z}_\alpha)),
\]

(3.25)

\[
x_1 = x_2 = \infty
\]

\[
x_1 = 1, x_2 = \infty
\]

\[
x_1 = x_2 = 0
\]

Fig 2. Real curve \( \Sigma_0 \)

The moduli spaces of the Higgs bundles are hyperkählerian. The operator \( i \) is holomorphic in a one complex structure and anti-holomorphic in two others, and its fixed point set corresponds to a special configurations of branes in terms of [10]. In our case it acts on \( \mathcal{M}_I(G^C) \) (3.19) as:

\[
i(L_\alpha(z_\alpha)) = (\sigma L_\alpha(\bar{z}_\alpha)).
\]

(3.26)

The fixed point set \( \mathcal{M}_I(G^R) \) of \( i \) is described by the Lax operators (3.18)

\[
L_1(x_1) = \frac{T + \eta}{x_1}, L_2(x_2) = \frac{S + \text{Ad}_{\exp(u)}(\eta)}{1 - x_2},
\]

(3.27)

where \( S, T \in u, u_\alpha \in \mathbb{R}, \eta \) has the form (2.37), (2.38) and \( x_1, x_2 \) are coordinates on the real curve \( \Sigma_0 \) Fig.2.

\[
\mathcal{M}_I(G^R) = T^*H^R \times (O_U \times O_U).
\]

They coincide with \( \mathcal{R}^{\text{red}}_{G^R} \) (2.34) and

\[
\dim \mathcal{M}_I(G^R) = 2 \sum_{j=1}^l d_j - 2\text{rank}\, U.
\]

---

\(^8\)In [1] it is denoted \( i_3 \).

\(^9\)The construction presented in [1] deals with bundles over curves without marked points (3.25). We extend it to bundles with marked points (3.26).
There is an involution $\Upsilon$ on the moduli space $\mathcal{M}_I(G^C)$. It acts on the coordinates as

$$
\Upsilon \begin{pmatrix} u \\ v \\ T \\ S \end{pmatrix} = \begin{pmatrix} -u \\ -v \\ -S \\ -T \end{pmatrix}
$$

Then it follows from (2.37), (2.38) that

$$
\Upsilon(\eta) = -\text{Ad}_{\exp(u)}\eta
$$

If we accompany $\Upsilon$ with the action on the spectral parameters $x_1 \xrightarrow{\Upsilon} x_2$ then $\Upsilon$ interchanges the Lax operators (3.27) $\Upsilon(L_1(x_1)) = -L_2(x_2)$, $\Upsilon(L_2(x_2)) = -L_1(x_1)$.

Consider the equations of motion (2.45), (2.46), (2.48), (2.49) in terms of the Lax operators (3.27). Define $M_1$ as the introduced above the $x_1$-independent operator $M = \partial_u X$ (2.51). Then on the first component we have

$$
\partial_t L_1(x_1) = [L_1(x_1), M_1]. \tag{3.28}
$$

This equation implies

$$
\frac{\partial_t T}{x_1} + \frac{\partial_t \eta}{1 - x_1} = \left[ \frac{T}{x_1}, M_1 \right] + \left[ \frac{T + \eta}{1 - x_1}, M_1 \right].
$$

The equality $\frac{1}{2} = \frac{3}{4}$ follows from (2.49) and explicit form of $M_1 = M = Y$ (2.51). Then the equality $\frac{2}{4} = \frac{4}{4}$ follows from (2.50).

Define $M_2 = \Upsilon(M_1)$. Then we come to lax equation on the second component

$$
\partial_t L_2(x_2) = [L_2(x_2), M_2] \text{ that is equivalent to (3.28).}
$$

### 3.5 Classical $r$-matrix

Let us change the variable $x_1$ in (3.27) to $y$:

$$
\coth(y) = (1 - x_1)/x_1
$$

Consider the following Lax operator:

$$
\hat{L}(y) = (1 - x_1)L_1(x_1) = T \coth(y) - Q. \tag{3.29}
$$

with

$$
Q = \sum_{\alpha \in \mathbb{R}} Q_\alpha E_\alpha, \quad Q_\pm = \pm T_\alpha + X_\pm,
$$

where $X_{\pm\alpha}$ are defined in (2.38). Since $M$ is independent on the spectral parameter, we preserve its form (2.51). The Lax equation

$$
\dot{L}(x) := \{H, L(x)\} = [L(x), M] \tag{3.30}
$$

with the Lax pair (3.29), (2.51) provides equations of motion generated by the Hamiltonian (2.40) and the Poisson structure (2.39). Indeed, the Lax equation (3.30) is equivalent to the already proven equations

$$
\dot{\eta} = [\eta, M],
$$

where $\eta$ is defined by (2.37), and

$$
\dot{T} = [T, M].
$$
The \( r \)-matrix structure is given by

\[
\{L_1(x), L_2(y)\} = [L_1(x), r_{12}(x, y)] - [L_2(y), r_{21}(y, x)]
\] (3.31)

where

\[
r_{12}(x, y) = \frac{1}{2} \left( \coth(x - y) + \coth(x + y) \right) \sum_{i=1}^{t} e_i \otimes e_i + \sum_{\alpha \in \mathbb{R}^+} \frac{(\alpha, \alpha)}{2} E_\alpha \otimes E_{-\alpha} \left( \coth(x - y) + \coth(u_{\alpha}) \right) + \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} \frac{(\alpha, \alpha)}{2} E_\alpha \otimes E_\alpha \left( \coth(x + y) + \coth(u_{\alpha}) \right).
\] (3.32)

Analogous result but without spectral parameter was obtained in \cite{5}, and for the algebra \( \mathfrak{u} = \mathfrak{so}(N) \) the \( r \)-matrix was previously found in \cite{2}. Expression (3.32) is obtained in a similar way. Being written in the form (3.31) the Poisson brackets provide the involutivity of the integrals of motion (3.24).

One can also verify that the \( r \)-matrix (3.32) provides \( M \)-matrix (\ref{eq:3.51}) in the following way:

\[
\text{tr}_2(r_{12}(x, y)L_2(y)) = \frac{1}{2} \left( \coth(x - y) + \coth(x + y) \right) L(x) - M.
\] (3.33)

4 Higgs bundles. Model II

4.1 Holomorphic bundles

The base spectral curve. The base spectral curve is a singular curve \( \Sigma^{II} \). It is defined by the normalization of the rational curve

\[
\pi : \mathbb{CP}^1 \to \Sigma^{II}, \quad \pi(z = 0) = \pi(z = \infty)
\] (4.1)

In addition, we assume that \( \Sigma^{II} \) has a marked point \( z = 1 \) (see Fig 2.).

Vector bundles with quasi-compact structure. Let \( \mathcal{P}(G^\mathbb{C}) \) be the principle \( G^\mathbb{C} \)-bundle over \( \Sigma^{II} \), \( V \) is a \( G^\mathbb{C} \) module and \( E^{II}(G^\mathbb{C}) = \mathcal{P} \times_{G^\mathbb{C}} V \) is the associated vector bundle. At the glued points we define the maps between the corresponding sections:

\[
r : s|_{z=\infty} \in \Gamma(E) \to s|_{z=0} \in \Gamma(E), \quad r \in G^\mathbb{C}.
\] (4.2)

Let \( \bar{\partial} + \bar{A} \) (\( \bar{\partial} = \partial_{\bar{z}} \)) be the antiholomorphic connections acting on the sections \( \Gamma(E) \). The data

\[
\mathcal{D} = \{ (\bar{\partial} + \bar{A}) , \quad r \in G^\mathbb{C} \}
\] (4.3)

define the vector bundle \( E^{II}(G^\mathbb{C}) \) over \( \Sigma^{II} \). The group of automorphisms \( \mathcal{G} \) of the bundle \( E^{II}(G^\mathbb{C}) \) is given by the smooth maps

\[
\mathcal{G} = \{ f : C_\infty(\Sigma^{II}) \to G^\mathbb{C} \}.
\] (4.4)

As above at the marked point \( z = 1 \) we replace \( G^\mathbb{C} \) with the maximal compact subgroup \( K \):

\[
f(z, \bar{z})|_{z=1} \in K.
\] (4.5)
It means that we deal with the quasi-compact vector bundle. The gauge group action on the data \([4.3]\) is of the form:

\[
\begin{align*}
\bar{\partial} + \bar{A} &\rightarrow f(\bar{\partial} + \bar{A})f^{-1}, \quad f(1) \in K, \\
r &\rightarrow f(\infty)rf^{-1}(0).
\end{align*}
\]

\((4.6)\) 

\((4.7)\)

\[z = 1\]

\[\Sigma^{II}\]

\[z = \infty\]

\[z = 0\]

\[frf^{-1} = h^{-1} \exp(u), \quad h \in K, \quad u \in \mathfrak{h}^R.\]

\((4.11)\)

**Fig.2** The base spectral curve \(\Sigma^{II}\)

**The moduli space** \(\text{Bun}_{\text{II}}(G^C)\). The coset space \(\text{Bun}_{\text{II}}(G^C) = \mathcal{D}/\mathcal{G}\) is the moduli of quasi-compact \(G^C\)-bundles over the singular curve \(\Sigma^{II}\).

Let us prove that a big cell \(\text{Bun}_{\text{II}}^0(G^C)\) in the moduli space is

\[
\text{Bun}_{\text{II}}^0(G^C) = (\mathcal{H}^R, \bar{K}) = \{h^{-1} \exp(u) \mid h \in \bar{K}, \exp(u) \in \mathcal{H}^R\},
\]

\((4.8)\)

compare with \((3.12)\). It has dimension (see \((A.29)\))

\[
\dim \text{Bun}_{\text{II}}^0(\Sigma^{II}, G^C) = \text{rank}(G^R) + \dim K - \text{rank} K = \dim K = 2 \sum_{j=1}^{l} d_j - l.
\]

\((4.9)\)

First, we prove that by the gauge group action \((4.6)\) the generic configuration of variables \(\bar{A}\) can be chosen to be trivial

\[\bar{A} = 0,\]

\((4.10)\)

and after this gauge fixing we stay with the residual constant gauge transformations taking values in the subgroup \(K\) as in \((3.9)\). The condition \((4.5)\) prevents the connection from trivialization.

To go around we consider the symmetric space \(X^C = K \backslash G^C\). In these terms \((4.10)\) means that \(f(z)|_{z=1}\) preserves the point \(x_0\) corresponding to the coset \(K\) in \(X^C\). If we omit \((4.5)\) then the gauge transformations \((4.6)\) of the bundles over \(\mathbb{C}P^1\) allows one to choose the gauge \((4.10)\). The residual gauge transformations \(G^\text{res}\) are the constant maps \(\Sigma^{II}\) to \(G^C\). We choose \(f \in G^\text{res}\) as follows. Let \(g\) be the value at \(z = 1\) of the corresponding transformation \(\bar{A} \rightarrow 0\). It acts on \(x_0\) as \(x_0 \rightarrow y = x_0g\). The stationary subgroup corresponding to \(y\) is \(g^{-1}Kg\). Then acting by \(g^{-1}\) on \(y\) we reduce to the residual gauge transformations to \(K\). After this procedure we stay with the condition \((4.10)\) and the residual constant gauge transformations which take values in the subgroup \(K\).

The action of these transformations on \(r\) \((4.7)\) takes the form \(r \rightarrow frf^{-1}, \ f \in K\). As in \((2.59)\) we have:

\[
frf^{-1} = h^{-1} \exp(u), \quad h \in K, \quad u \in \mathfrak{h}^R.
\]

\((4.11)\)
The element $h$ in (4.11) is defined up to the Cartan torus $T$-action (2.60):

$$h \rightarrow shs^{-1}, \ s \in \mathcal{G}_{2}^{res} = T \subset K.$$ 

Then $h \in \tilde{K}$ (2.61) and we come to (4.8).

### 4.2 Higgs bundles

The Higgs bundle $\mathcal{H}(G^{\mathbb{C}})$ over $\Sigma^{II}$ is described by the pairs

$$\{(\partial_{\bar{A}} = \bar{\partial} + A, \Phi), (r, \xi) \in T^*G^{\mathbb{C}}\}.$$ (4.12)

Here $\Phi = \Phi(z, \bar{z})dz \in \text{End}(E) \otimes \Omega^{(1,0)}(\Sigma^{II})$ is the Higgs field, and

$$(r, \xi) \in T^*(\text{Hom}(E|_{z=\infty} \rightarrow E|_{z=0})) \sim T^*G^{\mathbb{C}}.$$ 

The Higgs bundle $\mathcal{H}_G$ is a symplectic manifold equipped with the form

$$\omega = \int_{\Sigma^{II}} ((D\Phi, D\bar{A}) + (D(\xi, r^{-1}Dr)\delta(0)) .$$

The group of the automorphisms $\mathcal{G}$ of the bundle $E^{II}$ (4.6), (4.7) is lifted to the symplectic automorphisms of the Higgs bundle $\mathcal{G}$:

$$\Phi \rightarrow \text{Ad}_{f}\Phi, \ \xi \rightarrow \text{Ad}_{f(0)}\xi.$$ (4.13)

These transformations are generated by the moment map

$$\mu = \partial_{z}\Phi + \delta(z, \bar{z})|_{z=0}\xi + \delta(z, \bar{z})|_{z=\infty}\text{Ad}_{r}\xi.$$ (4.14)

Impose the moment map constraints $\mu = 0$. They mean that the Higgs field $\Phi(z, \bar{z})$ is meromorphic on $\Sigma^{II}$, have simple poles at $z = 0, \infty$ with the residues

$$\begin{cases} 
z = 0: & \text{Res } \Phi = \xi, \\
\zeta_{\alpha} = \infty: & \text{Res } \Phi = -\text{Ad}_{r}\xi
\end{cases}$$

and

$$\text{Res } \Phi(z = 1)|_{t} = 0, \text{ (quasi} - \text{compactness).}$$ (4.15)

Let $\text{Res } \Phi(z)|_{z=1} = \mathcal{P}$. Then the Higgs field (on the surface $\mu = 0$) assumes the form:

$$\Phi = \frac{\xi}{z} - \frac{\mathcal{P}}{z-1}, \ \mathcal{P}|_{t} = 0, \ \text{Res } \Phi_{z=\infty} = -\text{Ad}_{r}\xi.$$ (4.16)

The sum of residues vanishes $\xi + \mathcal{P} - \text{Ad}_{r}\xi = 0$. It means that

$$(\xi - \text{Ad}_{r}\xi)|_{t} = 0.$$ (4.17)

Thus,

$$\Phi = \frac{\xi}{z} - \frac{\xi - \text{Ad}_{r}\xi}{z-1}.$$ 

Now fix the gauge as (4.10). It means that $r = h^{-1}\exp(u)$. The Lax operator $L(z)$ is identified with the gauge transformed Higgs field. From (4.16) we find

$$L(z) = \frac{\xi}{z} - \frac{\mathcal{P}}{z-1}, \ \mathcal{P} = \xi - \text{Ad}_{h^{-1}}\exp(u)\xi.$$ (4.18)
Compare this Lax operator with the Lax operator \( L_2(z_2) \) in the Model I. From (2.67) we find that
\[
\text{Ad}_h(L(z)) = L_2(z_2)|_{z_2 = z}.
\]
(4.19)
In these terms the moduli space of the Higgs bundle \( \tilde{M}_{II}(G^\mathbb{C}) = \mu^{-1}(0)/\mathcal{G} \) are defined by the pair
\[
\tilde{M}_{II}(G^\mathbb{C}) = T^*\mathcal{H}_R \times T^*K = \{(\xi \in \mathfrak{g}^\mathbb{C}, r = h^{-1}\exp(u)) \cup (4.17)\}.
\]
(4.20)
It coincides with \( \tilde{\mathcal{R}}_{II}^{\text{red}} \) (2.66). As above we pass from \( T^*K \) to the orbits \( (\mathcal{O}_K \times \mathcal{O}_K) \) with the torus action. After reduction we obtain
\[
\mathcal{M}_{II}(G^\mathbb{C}) = T^*\mathcal{H}_R \times ((\mathcal{O}_K \times \mathcal{O}_K)/\mathcal{T}) \sim \mathcal{R}_{II}^{\text{red}}(G^\mathbb{C}),
\]
where \( \mathcal{T} \) is defined by (2.13b), \( \eta \) satisfies (2.13a), and \( \dim \mathcal{M}_{II}(G^\mathbb{C}) = 4 \sum_{j=1}^l (d_j - 1) \) (see (2.71)). Thus after the symplectic reduction we come to the real symplectic space. The integrals of motion in this models are constructed by means of the differentials (3.21) on the curve \( \Sigma^{II} \).

In this way we come to the following expressions
\[
I_{jk} = (\xi^{d_j-k}p^k), \quad k = 0, \ldots, d_j.
\]
(4.22)
Due to arguments presented at the end of Section 2.3.2. the number of integrals is equal to \( \sum_{j=1}^l d_j \). We still have the deficit of the integrals.

4.3 Real forms

As in Section 3.4 we pass to the real form of this construction. We consider the antiholomorphic automorphism \( z \to \bar{z} \) of \( \Sigma^{II} \) and the involutive automorphism \( \sigma \) (3.25). The involution (3.25) acts on \( \mathcal{M}_{II}(G^\mathbb{C}) \) as
\[
i(L(z)) = \sigma(L(\bar{z})).
\]
(4.23)
The fixed point set is described by the Lax operator
\[
L^{II}(x) = \frac{\eta - \text{Ad}_h^{-1}\exp(u)\eta}{x - 1}.
\]
(4.24)
where \( x \in \mathbb{R}P^1, h \in U \) and \( \eta \) is defined in (2.37), (2.38). Thus, from (2.72) and (4.20)
\[
\tilde{M}_{II}(G^\mathbb{R}) = \{T^*\mathcal{H}_R \times T^*U\} \sim \tilde{\mathcal{R}}_{II}^{\text{red}}(G^\mathbb{R}).
\]
(4.25)
Passing to the orbits as (2.75) we obtain
\[
\mathcal{M}_{II}(G^\mathbb{R}) = \{T^*\mathcal{H}_R \times (\mathcal{O}_U \times \mathcal{O}_U)\} \sim \mathcal{R}_{II}^{\text{red}}(G^\mathbb{R}).
\]
In this way we come to integrable system.

5 Universal bundle

5.1 Vector bundle

Consider a \( G^\mathbb{C} \)-bundle over the spectral curve \( \Sigma^I \) for the model I (Fig 1, Section 3.1). As in that case we define two vector bundles \( E_\alpha(G^\mathbb{C}) = \mathcal{P}_\alpha \times_G \mathbb{C} \) over the components \( \Sigma_\alpha \).

Let \( \tilde{\partial}_{\alpha} + \tilde{A}_{\alpha} \ (\tilde{\partial}_{\alpha} = \partial_{\alpha}) \) be the antiholomorphic connections acting on the sections \( \Gamma(\mathcal{E}_\alpha) \). The gluing maps \( g_0 \in G^\mathbb{C}, g_\infty \in G^\mathbb{C} \) play the role of \( r_0 \) and \( r_\infty \) (5.3). The data
\[
\mathcal{D} = \{(\tilde{\partial}_{\alpha} + \tilde{A}_{\alpha}), \ \alpha = 1, 2, \ (g_0, g_\infty) \in G^\mathbb{C}\}
\]
(5.1)
define the vector bundle \( E^{\text{univ}}(G^\mathbb{C}) \) over the singular curve \( \Sigma \) with two marked points \( (z_1 = 1, z_2 = 1) \).
**Gauge group.** The gauge group $\mathcal{G}$ of the bundle $E^{univ}(G^C)$ is the pair of smooth maps $\mathcal{G}_\alpha$

$$\mathcal{G}_\alpha = \{ f_\alpha \in \Sigma_\alpha \in C^\infty(\Sigma_\alpha) \to G^C, \; f_1(z_1 = 1) \in K, \; f_2(z_2 = 1) = Id \}.$$  \hspace{1cm} (5.2)

The restriction at the point $z_1 = 1$ means that the component $E_1(G^C)$ has quasi-compact structure, while the second restriction implies that we fix a trivialization at the point $z_2 = 1$ of the bundle $E_2(G^C)$.

The gauge group action on the data (5.1) is as follows:

$$\tilde{\phi}_\alpha + \tilde{A}_\alpha \to f_\alpha(\tilde{\phi}_\alpha + \tilde{A}_\alpha)f^{-1}_\alpha,$$  \hspace{1cm} (5.3a)

$$\varrho_{\infty} \to f_2(\infty)\varrho_{\infty}f^{-1}_1(\infty), \; f_0(\infty) = f_\alpha(z_\alpha, \bar{z}_\alpha)|_{z_\alpha = \infty}, \; \varrho_0 \to f_2(0)\varrho_0f^{-1}_1(0), \; f_\alpha(0) = f_\alpha(z_\alpha, \bar{z}_\alpha)|_{z_\alpha = 0}.$$  \hspace{1cm} (5.3b)

The moduli space $Bun^{univ}(\Sigma, G^C)$. The quotient $Bun^{univ}(\Sigma, G^C) = \mathcal{D}/\mathcal{G}$ is the moduli of holomorphic $G^C$-bundles over the singular curve $\Sigma$ with the quasi-compact structure (at the point $z_1 = 1$).

Due to the restrictions (5.2) of the gauge group, the Universal bundle is an analog of the bundle for the Model II. We use this fact to trivialize the connections $\tilde{A}_\alpha = 0$. After that we stay with the constant gauge transformations

$$\mathcal{G}_1^{res} = K, \; \mathcal{G}_2^{res} = G^C.$$  \hspace{1cm} (5.4)

The gauge group action on the data (5.1) is as follows:

$$\varrho_{\infty} \to k\varrho_{0}^{-1}\varrho_{\infty}k^{-1}. \hspace{1cm} (5.3c)$$

Then (5.3b) assumes the form $\varrho_{\infty} \to k\varrho_{0}^{-1}\varrho_{\infty}k^{-1}$. Using (2.50) we find that $\varrho_{\infty}$ is transformed to the form

$$\varrho_{\infty} \to h^{-1}\exp(u), \; h \in \tilde{K}, \; 2.01; \; \exp(u) \in \mathcal{H}^R.$$  \hspace{1cm} (5.3c)

Let $V_2$ be a fiber over $z_2 = 1$ and $g_2 : V_2 \to V$ is a trivialization $(\text{Isom}(V_2, V) = \{g_2\})$. This space is the principal homogeneous space over $G^C$. In these notations

$$Bun^{univ}(\Sigma, G^C) = \tilde{K} \times \mathcal{H}^R \times \text{Isom}(V_2, V).$$

### 5.2 Higgs bundle

The Higgs bundle $H^{univ}(G^C)$ corresponding to the bundle $E^{univ}(G^C)$ is defined by the pairs

$$\{(\tilde{\phi}_\alpha + \tilde{A}_\alpha, \Phi_\alpha), \; (\varrho_0, \lambda_0), \; (\varrho_{\infty}, \lambda_{\infty})\}.$$  \hspace{1cm}

Here

$$\Phi_\alpha = \Phi_\alpha(z_\alpha, \bar{z}_\alpha)d_\alpha \in \text{End}(E_\alpha) \otimes \Omega^{(1,0)}(\Sigma_\alpha)$$

are the components of the Higgs fields. At $z_1 = 0$ and $z_1 = \infty$ we have

$$(\varrho_{\infty}, \lambda_{\infty}) \in T^*(\text{Hom}(V|_{z_1=\infty} \to V|_{z_2=\infty})) \sim T^*G^C,$$

$$(\varrho_0, \lambda_0) \in T^*(\text{Hom}(V|_{z_1=0} \to V|_{z_2=0})) \sim T^*G^C.$$  \hspace{1cm}

The Higgs bundle $H^{univ}(G^C)$ is a symplectic manifold equipped with the form

$$\omega = \sum_{\alpha=1,2} \int_{\Sigma_\alpha} (D\Phi_\alpha, D\tilde{A}_\alpha) - \int_{\Sigma_\alpha} \sum_{i=0,\infty} \delta(z_i, \bar{z}_i)|_{z_i}D(\lambda_i, \varrho^{-1}_iD\varrho_i) +$$

$$+ \int_{\Sigma_2} \sum_{i=0,\infty} \delta(z_i, \bar{z}_i)|_{z_i}D(\lambda_i, \varrho^{-1}_iD\varrho_i).$$
The gauge group $G$ of the bundle $E^{univ}(G^C)$ (5.3) can be lifted to the symplectic automorphisms of the Higgs bundle $H^{univ}(G^C)$ as
\begin{equation}
\Phi_\alpha \to \text{Ad}_{f_\alpha}^{-1}\Phi_\alpha ,
\end{equation}
\begin{equation}
\lambda_0 \to \text{Ad}_{f_1(0)}^{-1}\lambda_0 , \lambda_\infty \text{Ad}_{f_1(\infty)}^{-1}\lambda_\infty .
\end{equation}

The moment maps corresponding to this actions are
\begin{equation}
\mu_1 = -\bar{\partial}\Phi_1 + \delta(z_1, \bar{z}_1)|_{z_1=0}\lambda_0 + \delta(z_1, \bar{z}_1)|_{z_1=\infty}\lambda_\infty ,
\end{equation}
\begin{equation}
\mu_2 = -\bar{\partial}\Phi_2 + \delta(z_2, \bar{z}_2)|_{z_2=0}\text{Ad}_{\phi_0}^{-1}(\lambda_0) + \delta(z_2, \bar{z}_2)|_{z_2=\infty}\text{Ad}_{\phi_\infty}^{-1}\lambda_\infty .
\end{equation}

Here we have fixed the gauge by the conditions $\bar{A}_\alpha = 0$. The moment map constraints $\mu_1 = 0$ and $\mu_2 = 0$ mean that $\Phi_\alpha$ are meromorphic with poles at $z_\alpha = 0, \infty$. In addition, due to (5.2) $\Phi_1(z_1)$ has a pole at $z_1 = 1$, such that Res $\Phi_1(z_1) = -\mathcal{K}_1 \in \mathfrak{k}$. Similarly, $\Phi_2(z_2)$ has a pole at $z_2 = 1$ with Res $\Phi_2(z_2) = -\mathcal{X}_2 \in \mathfrak{g}^C$. Therefore,
\begin{equation}
\Phi_1 = \frac{\lambda_0}{z_1} - \frac{\mathcal{K}_1}{z_1 - 1}, \quad \Phi_2 = \frac{\text{Ad}_{\phi_0}\lambda_0}{z_2} - \frac{\mathcal{X}_2}{z_2 - 1}.
\end{equation}

Since the sum of residues vanishes,
\begin{equation}
\lambda_0 + \lambda_\infty + \mathcal{K}_1 = 0 ,
\end{equation}
\begin{equation}
\text{Ad}_{\phi_0}\lambda_0 + \text{Ad}_{\phi_\infty}\lambda_\infty + \mathcal{X}_2 = 0 .
\end{equation}

In this way the gauge transformed Higgs fields take the form:
\begin{equation}
\Phi_1 = \frac{\lambda_0}{z_1} - \frac{\lambda_0 + \lambda_\infty}{z_1 - 1}, \quad \Phi_2 = \frac{\text{Ad}_{\phi_0}\lambda_0}{z_2} - \frac{\text{Ad}_{\phi_0}\lambda_0 + \text{Ad}_{\phi_\infty}\lambda_\infty}{z_2 - 1}.
\end{equation}

**Reduction to model I.** Let $\phi_0 = pk$ ($\phi_0 \in G^C$) be the polar decomposition (5.26). Here $k \in K$. It is exactly the same subgroup that defines $G^{res}_1$ (5.4). Present $f_2 \in G^C$ in the form
\begin{equation}
f_2 = k_2p^{-1} ,
\end{equation}
\begin{equation}
\lambda_0|_{p^*} = 0 .
\end{equation}

To come to the model I we impose the second class constraints, transversal to the gauge fixing (5.9):
\begin{equation}
\lambda_0|_{p^*} = 0 .
\end{equation}

The transversality follows from the orthogonality of the Cartan decomposition (5.14). This condition is an analog of the moment maps constraints, but it does not generate vector fields on the phase space. At this step we have the residual gauge group $K_1 \times K_2$ acting on $\Sigma_1 \times \Sigma_2$. Acting by this group we can transform $\phi_\infty$ to the Cartan subgroup $H^R$:
\begin{equation}
\phi_\infty = e(u) .
\end{equation}

The action $f_\alpha \in K_\alpha$ leads to the moment map constraints
\begin{equation}
\mu_1 = (\lambda_0 + \lambda_\infty)|_{p^*} = 0 ,
\end{equation}
\begin{equation}
\mu_2 = (\text{Ad}_{\phi_0}\lambda_0 + \text{Ad}_{\phi_\infty}\lambda_\infty)|_{p^*} = 0 .
\end{equation}

Due to (5.10) and (5.11) we can assume the notations (2.9)
\begin{equation}
\lambda_0 = T , \quad \phi_0 = h .
\end{equation}

From (5.12), (5.13) we have
\begin{equation}
\lambda_\infty = \eta .
\end{equation}

Then Lax operators of the model I (5.18) coincides with (5.3).
**Reduction to model II.** Consider the symplectic action of \( f_2 \in G^C \) \textcolor{red}{(5.6)}. By \textcolor{red}{(5.3c)} \( \varrho_0 \) can be transformed to the unity element \( \varrho_0 = 1 \). The moment map constraints coming from \textcolor{red}{(5.6)} take the form
\[
\text{Ad}_{\varrho_0} \lambda_0 + \text{Ad}_{\varrho_{\infty}} \lambda_{\infty} = 0.
\]
It means that there are no poles of the Higgs fields \( \Phi_2 \) on the \( \Sigma_2 \) component and the singular curve \( \Sigma_1 \cup \Sigma_2 \) is shrunk to the base spectral \textcolor{red}{(1.1)} Fig. 2. and \( \varrho_0 = \varrho_{\infty} = r \). The residual gauge symmetries is the compact subgroup \( K \) \textcolor{red}{(5.4)}. In this way we come to the model II.

In this way we realize the diagram \textcolor{red}{(1.10)}.

**6 Higgs bundles. Model III**

**6.1 Description of model**

Here we define alternative quasi-compact bundle corresponding to the Model III. Consider the spectral curve \( \Sigma^{III} \) and attach the Riemannian symmetric space \( X^C = K \setminus G^C \) to the point \( z = 1 \).

We add it to the data \textcolor{red}{(4.3)}:
\[
\mathcal{D}^{III} = \{(\bar{\partial} + A),\ \rho \in G^C, \ g \in X^C\}.
\]
Then \( g \in G^C \) is defined up to the multiplications \( g \sim kg, \ k \in K \). The gauge group is the smooth map
\[
f : C^\infty(\Sigma) \to G^C.
\]
It acts on \( X^C \) as
\[
g \to g f^{-1}|_{z=1}.
\]
After fixing the gauge \( \bar{A} = 0 \) by the residual constant gauge transformations one can transform \( \rho \) to the Cartan subgroup
\[
\rho \to \exp u^C \in H^C.
\]
Since \( f \in H^C \) does not change the gauge we have
\[
\text{Bun}_{III}(G^C) = \{H^C \setminus X^C, H^C\}.
\]
From \textcolor{red}{(A.31)} we have
\[
dim \text{Bun}_{III}(G^C) = \sum_{j=1}^{l} (2d_j - 1).
\]
The corresponding Higgs bundle \( H(G^C) \) is described by the pairs
\[
\{(\partial A = \bar{\partial} + A, \Phi^{III}), \ (\rho, \varsigma) \in T^*G^C, \ (\zeta, g) \in T^*X\},
\]
where \( \zeta \in p^C \) \textcolor{red}{(A.14)}. The Higgs bundle is equipped with the form
\[
\omega = \int_\Sigma \left( (D\Phi^{III}, D\bar{A}) + (D(\varsigma, \rho^{-1} D\rho)\delta(0) + (\zeta, Dgg^{-1})\delta(z, \bar{z})|_{z=1}\right).
\]
The symplectic transformations of \( \Phi^{III} \) and \( \varsigma \) are standard \textcolor{red}{(4.13)}, while \( \zeta \to \zeta \) is given by \textcolor{red}{(2.82)}.

The moment map generating these transformations is (compare with \textcolor{red}{(4.14)}) as follows:
\[
\mu = \partial_z \Phi^{III} - \delta(z, \bar{z})|_{z=0} \zeta + \delta(z, \bar{z})|_{z=\infty} \text{Ad}_r(\zeta) - \delta(z, \bar{z})|_{z=1} \text{Ad}_r^{-1} \zeta.
\]
Taking \( \mu = 0 \) we obtain
\[
\Phi^{III}(z) = \frac{\varsigma}{z} - \frac{P}{z-1}, \quad (P = \text{Ad}^{-1}_r \varsigma, \ (2.79)),
\]
and \( \text{Res} \Phi^{\text{III}}(z) \big|_{z=\infty} = -\text{Ad}_r(\varsigma) \). The sum of residues vanishes. It is the condition \( \mu_{\text{III}} = 0 \)

\[ (2.83) \]

\[ \varsigma - \text{Ad}_{\exp u}(\varsigma) - P = 0. \]  

(6.8)

The solutions of this equation take the form \((2.86)\). Then taking into account the symplectic action of the Cartan group \( H \), we come to the expression \((2.87)\). One can see that the moduli space of the Higgs bundle in the Model III coincides with the phase space \( R_{\text{red}}^{\text{III}}(G^C) \)

\[ (2.88) \]

\[ M_{\text{III}}(G_C) = T^*H_R \times (T^*X//H_R) = \{ (v, \exp(u)(\tilde{P}_\alpha), (P_\alpha, \alpha \in R) \} \sim R_{\text{red}}^{\text{III}}(G^C), \]  

(6.9)

\[ \dim M_{\text{III}}(G^C) = \dim_{\mathbb{R}}(T^*X) = 2 \sum_{j=1}^l (2d_j - 1). \]

**Comparison of the Higgs bundles for Model III and Model II.** Similarly to the finite-dimensional case the Higgs bundle for the Model III \((6.5)\) with the automorphism group \((6.2)\) is isomorphic to the Higgs bundle of the Model II \((4.12)\) with the automorphism group \((4.5)\). It means that the reduction of the gauge group at the point \( z = 1 \) to the subgroup \( K \subset G^C \) is equivalent to the attachment the symmetric space \( X^C = K \setminus G^C \) to the same point.

As in Section 2.3.2 take the gauge transformation \( f(z) \big|_{z=1} = p \) in \((6.2)\), where \( p \) is defined by the polar decomposition of \( g \): \((g = kp)\). It acts on the symmetric space \( X^C = K \setminus G^C \) attached to \( z = 1 \) in such a way that \( y \in X^C \rightarrow \text{Id} \). Then we stay with the Higgs bundle of the Model II \((4.12)\) with the automorphism group \((4.5)\). We will prove the symplectomorphism \( M_{\text{II}}(G^C) \sim M_{\text{III}}(G^C) \) for arbitrary Higgs bundles in a separate publication.

### 6.2 Real form.

As above we pass to the fixed point set of the involution \((3.26)\) acting on \( M_{\text{III}}(G^C) \). The fixed point set is described by the Lax operator

\[ L^{\text{III}}(x) = \frac{\varsigma}{x} - \frac{P}{x - 1}, \]  

(6.10)

where \( x \in \mathbb{R}P^1 \) and

\[ \varsigma = v + \sum_{\alpha \in R} \frac{P_\alpha}{1 - \exp u_\alpha} E_\alpha, \quad P_\alpha \in \mathbb{R}. \]

In this way we obtain

\[ M_{\text{III}}(G^R) = T^*H_R \times (T^*X//H_R) = \{ (v, \exp(u)(\tilde{P}_\alpha), (P_\alpha, (\alpha \in R) \} \sim \dim_{\mathbb{R}}(T^*X^R) = 2 \sum_{j=1}^l d_j. \]

From \((2.95), (2.96)\) we find \( M_{\text{III}}(G^R) \sim R_{\text{red}}^{\text{III}}(G^R) \) \((2.89)\). Similarly to \((2.96)\) we find from \((4.25)\) that

\[ M_{\text{III}}(G^R) \sim M_{\text{III}}(G^R). \]

### 7 Calogero-Sutherland system

#### 7.1 CS and Model III

Consider a bundle over the curve \( \Sigma^{\text{III}} \) (Fig. 2). We change the data \((6.1)\) relating to the Model III in the following way. Let \( B \) be the Borel subgroup of \( G^C \). Replace the symmetric space \( X^C \)
attached to the point $z = 1$ with the flag variety $Fl^C = B \setminus G^C$. It means that $g$ is defined up to the left shift $g \sim bg$, $b \in B$. The gauge is fixed by the conditions $A = 0$. Using the residual constant gauge transformations $G^C$ we can put $r$ in the form $r = \exp(u^C) \in H^C$. In this way we come to the moduli space of the quasi-parabolic bundles over $\Sigma^{III}$ (instead of (6.1))

$$
\text{Bun}_{III}^p(G^C) = \{Fl^C/H^C, H^C\}.
$$

In the corresponding Higgs bundles we replace the cotangent bundle $T^*X^C$ with the coadjoint orbit

$$
O_\nu = \{\text{Ad}_\nu^* g \in G^C, \nu \in \mathfrak{h}^C\}.
$$

Here we assume that $\alpha(\nu) > 0$ for $\alpha \in R^+$. It can be proved that as in Section 2.3.1 it can be equivalently defined as a result of the symplectic reduction $O_\nu \sim B \setminus T^*G^C$. It means that in (2.71) and (2.72) the maximal compact subgroup $K$ is replaced by the Borel subgroup $B$. Then $O_\nu$ can be identified with the set of pairs $(\zeta, g)$, where $g$ is an element of the coset space $B \setminus G^C$ and $\zeta$ satisfies the moment map constraint $\zeta|_{b^*} = \nu$, where $b^* = \text{Lie}^*(B)$ is the Lie coalgebra, and $\nu \in \mathfrak{h}^*$. As in (2.79) one can represent the orbit in terms of $B$ invariant expressions

$$
O_\nu = \{S = \text{Ad}^*_\nu^{-1}\zeta \mid g \sim bg, \zeta|_{b^*} = \nu, b \in B, \nu \in \mathfrak{h}^C\}.
$$

Here the coadjoint action $\text{Ad}^*$ is defined on the coalgebra $(\mathfrak{g}^C)^*$. For the positive nilpotent subalgebra $n^+$ and $x \in n^+$ we have $x|_{b^*} = 0$. The subalgebra $n^+$ is a cotangent space to the flag variety $B \setminus G^C$ at the point corresponding to $B$. Since $\zeta|_{b^*} = \nu$ it can be represented as

$$
\zeta = \nu + x, \quad x \in n^+, \quad \nu \in \mathfrak{h}^C.
$$

Certainly, the coadjoint action $\text{Ad}^*_B$ on $\zeta$ does not change $\nu$, but for $\nu \neq 0$ it acts freely on $n^+$. Therefore, $O_\nu$ is a principle homogeneous space over the cotangent bundle $T^*Fl$ to the flag variety attached to $z = 1$. The cotangent bundle $T^*Fl$ corresponds to the case $\nu = 0$.

The Higgs bundle is defined by the data (see (6.5)):

$$
\{(\partial_\zeta = \bar{\partial} + \bar{A}, \Phi^{CS}_{III}), (r, s) \in T^*G^C, (\zeta, g) \in O_\nu\},
$$

where $\Phi^{CS}_{III}$ has the poles at $z = 0, 1, \infty$. Similarly to (6.7)

$$
\Phi^{CS}_{III} = \frac{\zeta}{z} - \frac{S}{z-1}, \quad S \in O_\nu.
$$

Since the sum of the residues of $\Phi^{CS}_{III}(z)$ vanishes $\zeta - \text{Ad}_{\exp(r)}(\zeta) = S$. Solutions of this equation is the Lax operator without spectral parameter. $\zeta = \eta^{CS}$ for the Calogero-Sutherland system. Let

$$
S = (S_{h^C} + \sum_{\alpha \in R} S_{\alpha}E_{\alpha}) \in O_{G^C} \subset \mathfrak{g}^C.
$$

Then

$$
\eta^{CS} = v + S_{h^C} + \sum_{\alpha \in R} \frac{S_{\alpha}}{1 - \exp(u_{\alpha})} E_{\alpha}.
$$

(7.1)

The spectral dependent Lax operator takes the form:

$$
L^{CS}_{III}(z) = \frac{\eta^{CS}}{z} - \frac{S}{z-1}.
$$

(7.2)

In fact, one should pass to the symplectic quotient $\tilde{S} \in S_{G^C}/\mathcal{H}^C$. The moment constraints corresponding to this action means that the Cartan part $S_{h^C}$ on the quotient space vanishes and $\eta^{CS}$ takes the form:

$$
\eta^{CS} = v + \sum_{\alpha \in R} \frac{\tilde{S}_{\alpha}}{1 - \exp(u_{\alpha})} E_{\alpha}.
$$

(7.1)
From (7.1) and (A.6) we find the quadratic Hamiltonian of the CS system

\[ H = \frac{1}{2}(\eta^C, \eta^C) = \frac{1}{2}(v, v) + \frac{1}{2} \sum_{\alpha \in R^+} \frac{\tilde{S}_\alpha \tilde{S}_{-\alpha}}{(\alpha, \alpha) \sinh^2(\frac{\alpha}{2})}. \]

The phase space of CS system is

\[ \mathcal{R}_{III}^C = T^*H^C \times S/\mathcal{H}^C, \dim \mathcal{R}_{III}^C = \dim \mathcal{O}_\nu. \]  

7.2 CS and Model II

Define the Model II type description of the Calogero-Sutherland system. We start with the same spectral curve \( \Sigma^H \). The holomorphic bundle is defined by the data \( (4.3) \)

\[ D = \{ (\bar{\partial} + \bar{A}) \}, \quad r \in G^C, \text{ where } r \text{ is defined in } (4.2). \]

The gauge group is \( (4.4) \), but the restriction \( (4.5) \) is replaced by

\[ f(z, \bar{z})|_{z=1} \in B, \quad (f(z, \bar{z}) \in \mathcal{G}), \quad (7.4) \]

where \( B \) is a Borel subgroup of \( G^C \). It means that the holomorphic bundle has the quasi-parabolic structure. As for the Model II we transform \( \bar{A} \) to \( \bar{A} = 0 \) and stay with the constant maps \( f: \Sigma^H \to B \). The residual gauge transformation means that \( r \) becomes the element of the quotient space

\[ F = \{ r \sim f r f^{-1}, \quad r \in G^C, \quad f \in B \}. \]

(7.5)

The poles of the Higgs field \( \Phi^H_{III} \) allows one to write it in the standard way. Let \( \tilde{r} \) be an element of \( F \). Then

\[ \Phi^H_{III} = \frac{\zeta}{z} - \frac{\zeta - \text{Ad}_r \zeta}{z - 1}. \]

(7.6)

The analog of the moment map equation is

\[ (\zeta - \text{Ad}_r \zeta)|_{r^*} = 0, \quad r \in \mathcal{F}. \]

In this construction the phase space of the CS system is described as

\[ \mathcal{R}_{III}^C = \{ \mathcal{F} \times \text{solutions of } (7.6) \}. \]

(7.7)

Notice, that \( \dim \mathcal{R}_{III}^C = \dim \mathcal{R}_{III}^C \) (7.3). In fact, these phase spaces are symplectomorphic as in (2.93).

A Simple Lie groups: Notations and decompositions [8]

Let \( G^C \) be a simple complex Lie group, \( g^C \) its Lie algebra of rank \( l \) and \( h^C \) is a Cartan subalgebra.

Chevalley basis in \( g \). Let \( \{ \alpha \in R \} \) be the root system. The algebra \( g^C \) has the root decomposition

\[ g^C = h^C + \mathcal{L}, \quad \mathcal{L} = \sum_{\beta \in R} \mathcal{R}_\beta, \quad \dim \mathcal{R}_\beta = 1. \]  

(A.1)

Let \( \Pi \subset R \) be a subsystem of simple roots. The Chevalley basis in \( g^C \) is generated by

\[ \{ E_{\beta_j} \in \mathcal{R}_{\beta_j}, \beta_j \in R, \quad H_{\alpha_k} \in h^C, \quad \alpha_k \in \Pi \}, \]

(A.2)

where \( H_{\alpha_k} \) are defined by the relation \( \alpha(H_{\alpha}) = 2 \). The commutation relations in this basis assume the form

\[ [E_{\alpha_k}, E_{-\alpha_k}] = H_{\alpha_k}, \]

(A.3a)

\[ [H_{\alpha_k}, E_{\pm \alpha_j}] = \pm a_{kj} E_{\pm \alpha_j}, \quad \alpha_k, \alpha_j \in \Pi, \]

(A.3b)

\[ [E_{\alpha}, E_{\beta}] = C_{\alpha, \beta} E_{\alpha + \beta} \text{ if } \alpha + \beta \in R, \quad C_{\alpha, \beta} = 0, \text{ if } \alpha + \beta \notin R, \]

(A.3c)
where \( a_{kj} \) is the Cartan matrix and \( C_{\alpha,\beta} \) are the structure constants of \( \mathfrak{g} \).

Let \((\ ,\ )\) be an invariant scalar product in \( \mathfrak{h}^C \). The condition \( \alpha(H_\alpha) = 2 \) allows one to identify \( H_\alpha \), \( (\alpha \in \Pi) \) (the basis simple coroots) with the roots as

\[
H_\alpha = \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}.
\]

Then the scalar product on the Chevalley basis assumes the form

\[
(H_\alpha, H_\alpha) = \frac{4}{(\alpha,\alpha)},
\]

and, from (A.3b),

\[
(E_\alpha, E_\beta) = 2\delta_{\alpha,-\beta}/(\alpha,\alpha).
\]

For the canonical basis \( \{e_j\} \), \( (j = 1 \ldots , l) \) in \( \mathfrak{h}^C \) we have the expansion \( H_\alpha = \sum_{j=1}^l \frac{2\alpha(j)}{(\alpha,\alpha)} e_j \) \( (\alpha(j) = \alpha(e_j)) \). Then from the first relation in (A.3) for any \( \alpha \in R \):

\[
[E_\alpha, E_{-\alpha}] = \sum_{j=1}^l \frac{2\alpha(j)}{(\alpha,\alpha)} e_j .
\]

And

\[
[e_j, E_\alpha] = \alpha(j) E_\alpha .
\]

The structure constants are real \( C_{\alpha,\beta} \in \mathbb{R} \), antisymmetric \( C_{\alpha,\beta} = -C_{\beta,\alpha} \) and

\[
C_{-\alpha,-\beta} = -C_{\alpha,\beta} ,
\]

\[
C_{\alpha+\beta,-\alpha} = -\frac{(\beta,\beta)}{(\alpha+\beta,\alpha+\beta)} C_{\alpha,\beta} .
\]

**Real forms.** Let \( \rho \) be the involutive automorphism \( \rho (\rho^2 = \text{Id}) \) acting on the root basis and the Cartan subalgebra as

\[
\rho : (x \to -\bar{x} , \text{ for } x \in \mathfrak{h}^C, \ c_\alpha E_\alpha \to -\bar{c}_\alpha E_{-\alpha}) .
\]

The fixed points is the maximal compact subalgebra

\[
\mathfrak{k} = \{ x \in \mathfrak{g}^C \mid \rho(x) = x \} \subset \mathfrak{g}^C .
\]

We denote by the same letter the involution acting on \( G^C \)

\[
K = \{ g \in G^C \mid \rho(g) = g \} \subset G^C .
\]

The corresponding group \( K \subset G^C \) is the maximal compact subgroup. Let \( \mathfrak{p}^C \) be a subspace in the Lie algebra \( \mathfrak{g}^C \) such that

\[
\mathfrak{p}^C = \{ x \in \mathfrak{g}^C \mid \rho(x) = -x \} \subset \mathfrak{g}^C .
\]

It is the tangent space to the Riemannian non-compact symmetric space \( G^C/K \) at the point corresponding to \( K \). The orthogonal decomposition of the complex Lie algebra \( \mathfrak{g}^C \)

\[
\mathfrak{g}^C = \mathfrak{k} \oplus \mathfrak{p}^C , \quad ((\mathfrak{k}, \mathfrak{p}^C) = 0) .
\]

is the **Cartan decomposition** of \( \mathfrak{g}^C \).
Denote by $\mathcal{H}^C$ a Cartan subgroup of $G^C$ and by $T \subset K$ is the corresponding Cartan torus of $K$. Define the corresponding Lie algebras

$$\mathfrak{h}^C = \text{Lie}(\mathcal{H}^C), \ t = \text{Lie}(T).$$

Let $\{e_j\}$, be a canonical basis in $\mathfrak{h}^C$. The basis in $\mathfrak{k}$ has the form

$$\{ve_j, \ j = 1, \ldots, l\}, \ \{\sigma_1 = (E_\alpha - E_{-\alpha}), \ \sigma_2 = \imath(E_\alpha + E_{-\alpha}), \ \alpha \in R^+\}, \quad (A.15)$$

where $R^+$ is a set of positive roots.

Let $\sigma$ be an involutive automorphism of $\mathfrak{g}^C$ which fixed points is the normal form $\mathfrak{g}^R$

$$\mathfrak{g}^R = \{x \in \mathfrak{g}^C | \sigma(x) = x\}. \quad (A.16)$$

The normal real form $\mathfrak{g}^R$ has the same basis $\{A.2\}$ as $\mathfrak{g}^C$. Let $\mathfrak{h}^R \subset \mathfrak{h}^C$ be the Cartan subalgebra of $\mathfrak{g}^R$. We have (compare with $\{A.14\}$)

$$\mathfrak{h}^C = \mathfrak{t} \oplus \mathfrak{h}^R. \quad (A.17)$$

By the action of the group $K \times K$ generic elements $g \in G^C$ can be transformed to the Cartan form

$$g = k_1 r k_2^{-1}, \ k_1, k_2 \in K, \ r = e(u) \in \mathcal{H}^R, \ (e(x) = \exp(x)), \quad (A.18)$$

where $\mathcal{H}^R$ is a Cartan subgroup of $G^R$, $(\text{Lie}(\mathcal{H}^R = \mathfrak{h}^R), \ u \in \mathfrak{h}^R$ (see $\{A.17\}$). In fact $u$ can be chosen from a Weyl chamber. For example, one can take $u_\alpha \equiv \alpha(u) > 0$.

The automorphism $\theta = \sigma \circ \rho$ acts on $\mathfrak{g}^C$. Define subalgebra

$$u = \{x \in \mathfrak{t} | \sigma(x) = x\} = \{x \in \mathfrak{g}^R | \rho(x) = x\}. \quad (A.19)$$

and let $\theta(p^R) = -p^R$ for $p^R \subset \mathfrak{g}^R$. Then

$$\mathfrak{g}^R = u \oplus p^R \ (u, p^R) = 0 \quad (A.20)$$

is the Cartan decomposition of $\mathfrak{g}^R$. The real form $\mathfrak{g}^R$ is normal if $p^R$ contains $\mathfrak{h}^R$. The Killing form is non-degenerate on $\mathfrak{g}^R$ and the subspaces in $\{A.20\}$ are orthogonal $(u, p^R) = 0$.

Define the basis in the normal form $\mathfrak{g}^R$ according with the Cartan decomposition $\{A.20\}$. The basis in the space $p^R$ is

$$p^R \rightarrow \{\mathfrak{h}^R \rightarrow (e_1, \ldots, e_l), \ P_\alpha = (E_\alpha + E_{-\alpha}), \ \alpha \in R^+\}$$

The basis in the Lie algebra $u$ is

$$u \rightarrow \{U_\alpha = (E_\alpha - E_{-\alpha}), \ \alpha \in R^+\} \quad (A.21)$$

with the commutation relations

$$[U_\alpha, U_\beta] = C_{\alpha,\beta} U_{\alpha + \beta} - C_{\alpha, -\beta} U_{\alpha - \beta}. \quad (A.22)$$

and the norm (see $\{A.6\}$)

$$(U_\alpha, U_\beta) = -\frac{4}{(\alpha, \alpha)} \delta_{\alpha\beta}. \quad (A.23)$$

Thus, the Killing form $\{A.6\}$ is negative define on the subalgebra $u$.

Let $G^R$ be the normal subgroup $(\text{Lie} G^R = \mathfrak{g}^R)$ and $U = K \cap G^R$ is its maximal compact subgroup $(\text{Lie} U = u)$. Similar to $\{A.18\}$ for $g \in G^R$

$$g = k_1 r k_2^{-1}, \ k_1, k_2 \in U, \ r = e(u) \in \mathcal{H}^R, \ u \in \mathfrak{h}^R. \quad (A.24)$$
It can be rewritten as the polar decomposition of \( g \in G^\mathbb{R} \)

\[
g = yk, \quad (y = k_1 r k_1^{-1}, \ k = k_1^{-1} k_2 \in U).
\]  

(A.25)

The similar formula holds for \( G^\mathbb{C} \)

\[
g = k_1 r k_1^{-1} = yk, \quad (k_1, k_2 \in K, \ y = k_1 r k_1^{-1}, \ r = e(u) \in \mathcal{H}^\mathbb{R}).
\]

(A.26)

The subgroups \( G^\mathbb{R}, K \) and \( U \) are presented in Table 1.

| \( G^\mathbb{C} \) | \( G^\mathbb{R} \) | \( K \) | \( U \) | \( d_j \) | rank \( U \) |
|-----------------|-----------------|--------|--------|--------|---------|
| \( \text{SL}(N, \mathbb{C}) \) | \( \text{SL}(N, \mathbb{R}) \) | \( \text{SU}(N) \) | \( \text{SO}(N) \) | \( 2, 3, \ldots, N \) | \([N/2]\) |
| \( \text{SO}(2N+1, \mathbb{C}) \) | \( \text{SO}(2N+1, \mathbb{R}) \) | \( \text{SO}(2N+1) \) | \( \text{SO}(N+1) \times \text{SO}(N) \) | \( 2, 4, \ldots, 2N \) | \( N \) |
| \( \text{Sp}(N, \mathbb{C}) \) | \( \text{Sp}(N, \mathbb{R}) \) | \( \text{Sp}(N) \) | \( \text{U}(N) \) | \( 2, 4, \ldots, 2N \) | \( N \) |
| \( \text{SO}(2N, \mathbb{C}) \) | \( \text{SO}(2N, \mathbb{R}) \) | \( \text{SO}(2N) \) | \( \text{SO}(N) \times \text{SO}(N) \) | \( 2, 4, \ldots, 2N \) | \([2N/2]\) |
| \( G_2^\mathbb{C} \) | \( G_2^\mathbb{R} \) | \( G_2 \) | \( \text{SU}(2) \times \text{SU}(2) \) | \( 2, 6 \) | \( 2 \) |
| \( F_4^\mathbb{C} \) | \( F_4^\mathbb{R} \) | \( F_4 \) | \( \text{Sp}(3) \times \text{SU}(2) \) | \( 2, 6, 8, 12 \) | \( 4 \) |
| \( E_6^\mathbb{C} \) | \( E_6^\mathbb{R} \) | \( E_6 \) | \( \text{Sp}(4) \) | \( 2, 5, 6, 8, 9, 12 \) | \( 4 \) |
| \( E_7^\mathbb{C} \) | \( E_7^\mathbb{R} \) | \( E_7 \) | \( \text{SU}(8) \) | \( 2, 6, 8, 10, 14, 18 \) | \( 7 \) |
| \( E_8^\mathbb{C} \) | \( E_8^\mathbb{R} \) | \( E_8 \) | \( \text{SO}(16) \) | \( 2, 8, 12, 14, 18, 20, 24, 30 \) | \( 8 \) |

Table 1

The groups \( G^\mathbb{C}, G^\mathbb{R}, K, U \)

Dimensions of algebras. Let \( d_j \) be the order of the invariants of algebra \( g^\mathbb{C} \) and \( \text{rank}(g^\mathbb{C}) = l \)

\[
d_1 = 2, \ldots, d_l = h, \quad j = 1, \ldots, l,
\]

(A.27)

where \( h \) is the Coxeter number. The dimensions of the algebras are expressed in terms of \( d_j \).

\[
dim_{\mathbb{C}} G^\mathbb{C} = \sum_{j=1}^{l} (2d_j - 1).
\]

(A.28)

The real dimension of \( K \) is

\[
dim_{\mathbb{R}} K = \sum_{j=1}^{l} (2d_j - 1),
\]

(A.29)

and

\[
dim_{\mathbb{R}} G^\mathbb{R} = \dim_{\mathbb{R}} K = \sum_{j=1}^{l} (2d_j - 1).
\]

(A.30)

Let \( X^\mathbb{C} = G^\mathbb{C}/K \) be the Riemannian symmetric space. Then from (A.28) and (A.29)

\[
dim_{\mathbb{R}} X^\mathbb{C} = \sum_{j=1}^{l} (2d_j - 1).
\]

(A.31)

The dimension of the compact form \( U \) is

\[
dim_{\mathbb{R}} U = \frac{1}{2} (\dim_{\mathbb{R}} K - l) = \sum_{j=1}^{l} (d_j - 1)
\]

(A.32)
The corresponding Riemannian symmetric space $X^R = G^R/U$ be the. Then from (A.30) and (A.32)

$$\dim R^X = \sum_{j=1}^l d_j. \quad (A.33)$$

Let $R^+$ be a subset of positive roots generated by simple roots $\Pi$ and $B$ is the corresponding Borel subgroup of $G^C$. Its Lie algebra $\text{Lie}(B) = \mathfrak{h}^C + \sum_{\alpha \in R^+} \mathfrak{r}_\alpha$. The flag variety $Fl$ is the homogeneous space $Fl = G^C/B$. It has dimension

$$\dim C Fl = \sum_{j=1}^l (d_j - 1). \quad (A.34)$$

**Poisson brackets on co-algebra $g^*$**. The general form of the Lie-Poisson brackets for the functions on $g^*$ is

$$\{F_1(x), F_2(x)\} = \langle x, [\nabla F_1, \nabla F_2]\rangle, \quad x \in g^*, \quad (A.35)$$

where $\nabla F_j(x) \in g$ is the gradient defined as a dual element to any $y \in g^*$

$$\langle y, \nabla F_j(x) \rangle := \frac{d}{dt} F_j(x + ty)|_{t=0}. \quad (A.36)$$

Let $U(g)$ be the universal enveloping algebra and $P(R) \in U(g)$ symmetric $ad$-invariant polynomial. These polynomial belong to the center $Z(U)$ of $U(g)$. The algebra $Z(U)$ is generated by the rank($g$) homogeneous polynomials of degrees $d_j$ (A.27), invariant with respect to the adjoint action. The Casimir functions with respect to the Lie-Poisson brackets (A.35) are the coefficients of $P(R)$.

Consider in details the Lie-Poisson brackets on the compact Lie algebra $u^*$. An arbitrary element of $u$ is decomposed in the basis (A.21) as

$$T = \sum_{\alpha \in R^+} T_\alpha U_\alpha, \quad (A.36)$$

with

$$(T, T) = -4 \sum_{\alpha \in R^+} \frac{T^2_\alpha}{(\alpha, \alpha)}. \quad (A.37)$$

By means of (A.23) define the dual basis to the basis $U_\alpha, \alpha \in R^+$ ($(u^\alpha, U_\beta) = \delta^\alpha_\beta$)

$$u^\alpha = -\frac{(\alpha, \alpha)}{4} U_\alpha. \quad (A.38)$$

In this way $T_\alpha$, can be considered as linear functions on the Poisson algebra on $u^*$:

$$T_\alpha = \langle T, u^\alpha \rangle. \quad (A.39)$$

The Poisson brackets (A.35) are defined by means of the form $d\vartheta = d\langle T, h^{-1} dh \rangle$ (2.3). Because $\nabla T_\alpha = u^\alpha$ we have

$$\{T_\alpha, T_\beta\} = \langle T, [u^\alpha, u^\beta]\rangle, \quad (A.40)$$

Let

$$N_{\alpha,\beta} = -C_{\alpha,\beta} \kappa_{\alpha,\beta}, \quad \kappa_{\alpha,\beta} = \frac{(\alpha, \alpha)(\beta, \beta)}{4(\gamma, \gamma)}, \quad \gamma = \alpha + \beta \in R. \quad (A.41)$$

Then from (A.22) we find

$$\{T_\alpha, T_\beta\} = N_{\alpha,\beta} T_{\alpha+\beta} - N_{\alpha,-\beta} T_{\alpha-\beta}. \quad (A.42)$$
It can be proved similarly that
\[ \{ S_\alpha, S_\beta \} = -N_{\alpha,\beta}S_{\alpha+\beta} + N_{\alpha,-\beta}S_{\alpha-\beta}. \]  
(A.40)

The brackets
\[ \{ T_\alpha, S_\beta \} = 0. \]  
(A.41)

follow from commutativity of the left and right momenta. From (A.9b) and (A.38) we also find
\[ N_{\alpha+\beta,-\alpha} = \frac{(\alpha + \beta, \alpha + \beta)}{(\beta, \beta)} N_{\beta,\alpha} = \frac{\alpha^2}{4} C_{\alpha\beta}. \]  
(A.42)

B Equations of motion of gyrostat on simple compact group

If one of the spin variables (say \( S \)) is fixed, then the Hamiltonian (1.2) describe a gyrostat (a rigid top with additional rotator moment). So technically as an intermediate result it is convenient to consider dynamic of a gyrostat on the group \( U \). It is defined by the Poisson brackets (A.39) on the coalgebra \( u^* \) for the momenta \( T \) (A.36) and the Hamiltonian:
\[
H_{top} = \sum_{\nu \in R^+} \left( \frac{1}{2} T_\nu^2 f(\nu) + T_\nu g(\nu) \right) = \frac{1}{2} \sum_{\nu \in R} \left( \frac{1}{2} T_\nu^2 f(\nu) + T_\nu g(\nu) \right).
\]  
(B.1)

Here \( f(\nu) \) are the components of "the inverse inertia tensor" of the top that maps \( u^* \rightarrow u \). We assume that
\[ f(\nu) = f(-\nu), \quad g(\nu) = -g(-\nu). \]  
(B.2)

The equations of motion assume the form
\[
\dot{T}_\gamma = \{ H_{top}, T_\gamma \} = \frac{1}{2} \sum_{\nu \in R} ((N_{\nu,\gamma} T_\nu T_{\nu+\gamma} - N_{\nu,-\gamma} T_\nu T_{\nu-\gamma})) f(\nu) + (N_{\nu,\gamma} T_{\nu+\gamma} - N_{\nu,-\gamma} T_{\nu-\gamma}) g(\nu),
\]  
(B.3)

where \( \mathcal{G}_{\nu,\gamma}(T) = T_\nu T_{\nu+\gamma} (N_{\nu,\gamma} - N_{\nu+\gamma,-\gamma}) f(\nu) \). Using (A.32) we rewrite it as
\[
\mathcal{G}_{\nu,\gamma}(T) = N_{\nu,\gamma} T_{\nu+\gamma} T_\nu \left( f(\nu) - \frac{(\nu + \gamma)^2}{\nu^2} f(\nu + \gamma) \right).
\]

Let \( \gamma = \alpha + \beta \). Then for \( \nu = -\alpha \)
\[
\mathcal{G}_{\nu,\gamma}(T)_{|\nu=-\alpha} = N_{-\alpha,\alpha+\beta} T_\alpha T_\beta \left( f(\alpha) - \frac{\beta^2}{\alpha^2} f(\beta) \right).
\]

Again from (A.42) we find that \( N_{-\alpha,\alpha+\beta} = -\frac{\alpha^2}{4} C_{\alpha\beta} \). Then
\[
\mathcal{G}_{-\alpha,\gamma}(T) = -\frac{1}{4} C_{\alpha\beta} T_\alpha T_\beta (\alpha^2 f(\alpha) - \beta^2 f(\beta)). \]  
(B.4)

Also by means of (A.42) we analyze in (B.3) the second term \( (N_{\nu,\gamma} T_{\nu+\gamma} - N_{\nu,-\gamma} T_{\nu-\gamma}) g(\nu) \). For \( \gamma = \alpha + \beta \) we have
\[
N_{\nu,\gamma} T_{\nu+\gamma} g(\nu)_{|\nu=-\beta} = -C_{\alpha,\beta} T_\alpha g(\beta) \frac{\beta^2}{4}.
\]

Also
\[
-N_{\nu,-\gamma} T_{\nu-\gamma} g(\nu)_{|\nu=\beta} = -C_{\alpha,\beta} T_\alpha g(\beta) \frac{\beta^2}{4}.
\]
Thus
\[ (N_{\nu,\gamma} T_{\nu+\gamma} - N_{\nu,-\gamma} T_{\nu-\gamma}) g(\nu) = - \sum_{\alpha+\beta=\gamma} C_{\alpha\beta}(T_\alpha \beta^2 g(\beta) - T_{\beta\alpha} g(\alpha)) . \]

Substituting this expression and (B.4) in (B.3) we come to the final expression
\[ \dot{T}_\gamma = \frac{1}{4} \sum_{\alpha+\beta=\gamma} C_{\alpha\beta} \left( T_\alpha T_\beta (\alpha^2 f(\alpha) - \beta^2 f(\beta)) - (T_\alpha \beta^2 g(\beta) - T_{\beta\alpha} g(\alpha)) \right) . \] (B.5)

From (A.40) we have
\[ \dot{S}_\gamma = - \frac{1}{4} \sum_{\alpha+\beta=\gamma} C_{\alpha\beta} \left( S_\alpha S_\beta (\alpha^2 f(\alpha) - \beta^2 f(\beta)) - (S_\alpha \beta^2 g(\beta) - S_{\beta\alpha} g(\alpha)) \right) . \] (B.6)

### C Proof of the equivalence of the Lax equations and the equations of motion

In terms of Lax operators (2.19) and (2.51) the Lax equation assumes the form
\[ \partial_t P + \partial_t X = [P, Y] + [X, Y]_{h^\gamma} + [X, Y]_{g^\gamma} . \]

It means for arbitraries \( j = 1, \ldots, l \) and \( \gamma \in R^+ \)
\[ \partial_t v_j + \partial_t X_{\pm\gamma} = \left[ \begin{array}{c} \partial_t v_j \\ 1 \\ \partial_t X_{\pm\gamma} \\ 2 \end{array} \right] \rightarrow \left[ \begin{array}{c} [P, Y] \\ 3 \\ [X, Y]_{E_{\pm\gamma}} \\ 4 \end{array} \right] + \left[ \begin{array}{c} [X, Y]_{E_{\pm\gamma}} \\ 5 \end{array} \right] . \]

Then the equations \( \text{(1) = 4} \) are equivalent to (2.46). In fact, due to (A.7) we have
\[ [X, Y]_{e_j} = \sum_{\alpha \in R^+} [X_\alpha E_\alpha + X_{-\alpha} E_{-\alpha}, Y_\alpha (E_\alpha - E_{-\alpha})]_{e_j} = - \sum_{\alpha \in R^+} \frac{2\alpha(j) Y_\alpha (X_\alpha + X_{-\alpha})}{(\alpha, \alpha)} e_j . \]

Now consider the equations \( \text{(2) = 3 + 5} \).
\[ \text{(2) = } \partial_t X_{\pm\gamma} E_{\pm\gamma} = \left( \sum_j v_j \partial_t u_j \left( \frac{T_\gamma e(\mp u_\gamma) - S_\gamma}{2 \sinh(u_\gamma)} \right) + \frac{\partial_t R_\gamma e(\mp u_\gamma) - \partial_t S_\gamma}{\sinh(u_\gamma)} \right) E_{\pm\gamma} = \text{(2.51)} \]
\[ = \sum_j v_j \gamma_j Y_\gamma E_{\pm\gamma} + \partial_t Y_\gamma - \partial_t S_\gamma \frac{E_{\pm\gamma}}{2 \alpha} \frac{u_\gamma}{\sinh(u_\gamma)} \]

Since
\[ \text{(3) = } [P, Y]_{E_{\pm\gamma}} = \sum_j v_j \gamma_j Y_{\pm\gamma} \]
we have \( \text{(2a = 3). Compare (2b and 5):} \)
\[ \text{(5) = } \sum_{\alpha+\beta=\gamma} C_{\alpha\beta}(X_\alpha Y_\beta - X_\beta Y_\alpha) E_\gamma = \sum_{\alpha+\beta=\gamma} C_{\alpha\beta} F(\alpha, \beta) E_\gamma , \]
\[ F(\alpha, \beta) = S_\alpha S_\beta \left( \frac{\cosh(u_\alpha)}{\sinh^2(u_\alpha) \sinh(u_\beta)} - \frac{\cosh(u_\beta)}{\sinh^2(u_\beta) \sinh(u_\alpha)} \right) + \]
\[ T_\alpha T_\beta \left( \frac{e(\mp u_\alpha)}{\sinh^2(u_\alpha) \sinh(u_\beta)} + \frac{e(\mp u_\beta)}{\sinh^2(u_\beta) \sinh(u_\alpha)} \right) + \]
\[ \text{...} \]

39
\[
\begin{align*}
S_\alpha T_\beta & \left( \frac{1}{\sinh(u_\alpha) \sinh^2(u_\beta)} - \frac{e(\mp u_\beta) \cosh(u_\alpha)}{\sinh(u_\beta) \sinh^2(u_\alpha)} \right) + \\
S_\beta T_\alpha & \left( -\frac{1}{\sinh^2(u_\alpha) \sinh(u_\beta)} + \frac{e(\mp u_\alpha) \cosh(u_\beta)}{\sinh^2(u_\beta) \sinh(u_\alpha)} \right).
\end{align*}
\]

On the other hand from the equations of motion for the spin variables (2.48) and (2.49)

\[
2b = \frac{\partial T_\gamma e(\mp u_\gamma) - \partial S_\gamma E_{\pm \gamma}}{\sinh(u_\gamma)} = \frac{1}{\sinh(u_\gamma)} \sum_{\alpha+\beta=\gamma} C_{\alpha,\beta} R(\alpha,\beta) E_{\pm \gamma},
\]

\[
R(\alpha,\beta) = \left( S_\alpha S_\beta - T_\alpha T_\beta e(\mp u_\gamma) \right) \left( \frac{1}{\sinh^2(u_\alpha)} - \frac{1}{\sinh^2(u_\beta)} \right) + \\
S_\alpha T_\beta \left( \cosh(u_\beta) - \frac{1}{\sinh^2(u_\alpha)} \right) + S_\beta T_\alpha \left( \cosh(u_\alpha) e(\mp u_\gamma) - \frac{1}{\sinh^2(u_\alpha)} \right).
\]

The equality \(2b = \frac{5}{2}\) follows from the following addition relations for trigonometric functions

\[
\frac{1}{\sinh(x+y)} \left( \frac{1}{\sinh^2 y} - \frac{1}{\sinh^2 x} \right) = \frac{\cosh y}{\sinh x \sinh^2 y} - \frac{\cosh x}{\sinh y \sinh^2 x}, \quad (C.1)
\]

\[
\frac{\exp(-x-y)}{\sinh(x+y)} \left( \frac{1}{\sinh^2 y} - \frac{1}{\sinh^2 x} \right) = -\frac{\exp(-y)}{\sinh^2 x \sinh y} + \frac{\exp(-x)}{\sinh^2 y \sinh x}. \quad (C.2)
\]

\[
\frac{1}{\sinh(x+y)} \left( \frac{\exp(-x-y)}{\sinh^2(y)} - \frac{\cosh(x)}{\sinh^2(x)} \right) = \frac{\exp(-x)}{\sinh x \sinh^2 y} - \frac{1}{\sinh y \sinh^2 x}. \quad (C.3)
\]

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