Modifed combinant analysis of the $e^+e^−$ multiplicity distributions

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Abstract

As shown recently, one can obtain additional information from the measured multiplicity distributions, $P(N)$, by extracting the so-called modified combinants, $C_j$. This information is encoded in their specific oscillatory behavior, which can be described only by some combinations of compound distributions, the basic part of which is the Binomial Distribution. So far this idea was applied to $pp$ and $p\bar{p}$ processes; in this note we show that an even stronger effect is observed in the $C_j$ deduced from $e^+e^−$ collisions. We present its possible explanation in terms of the so called Generalised Multiplicity Distribution (GMD) proposed some time ago.

Keywords: multiparticle production sep modified combinants, modified multiplicity distribution

1. Introduction

Recently it was shown that the measured multiplicity distributions, $P(N)$, contain some additional information on the multiparticle production process, so far undiscovered [1,2,3]. The basic idea was to apply the recurrence relation used in counting statistics when dealing with multiplication effects in point processes [4]. Its important feature is that it connects all multiplicities by means of some coefficients $C_j$ (modified combinants), which define the corresponding $P(N)$ in the following way:

$$(N + 1)P(N + 1) = \langle N \rangle \sum_{j=0}^{N} C_j P(N - j).$$

(1)

These coefficients contain the memory of the particle $N + 1$ about all the $N - j$ previously produced particles and, most important, they can be directly calculated from the experimentally measured $P(N)$ by reversing Eq. (1) and putting it in the form of the recurrence formula for $C_j$ [2]:

$$\langle N \rangle C_j = (j + 1) \left[ \frac{P(j + 1)}{P(0)} \right] - \langle N \rangle \sum_{i=0}^{j-1} C_i \left[ \frac{P(j - i)}{P(0)} \right].$$

(2)

As shown in [1,2], data on the $P(N)$ measured in $pp$ and $p\bar{p}$ experiments show oscillatory behavior of the corresponding modified combinants $C_j$, whereas the most popular Negative Binomial Distribution (NBD) provides $C_j$ monotonically decreasing. On the other hand, the pure Binomial Distribution (BD) gives strongly oscillating $C_j$ (with period two, not observed in the above data). To fit these data one needs some combination of the compound distributions in which the basic role is played by the BD compounded with a distribution which controls the period and amplitude of oscillations (in [1] it was a NBD).

It turns out that in the case of multiplicity distributions

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measured in $e^+e^-$ collisions the observed effect is much stronger. Fig. 1 shows the results for $P(N)$ and for the corresponding $C_j$ deduced from the ALEPH experiment data \[10\]; as shown there, they can be fitted by the so-called Generalized Multiplicity Distribution (GMD) presented in Section 2.

\[ C_j = \frac{1}{\langle N \rangle} C_{j+1}, \]  

namely, they can also be expressed by the generating function $G(z)$ of $P(N)$ as

\[ \langle N \rangle C_j = \frac{1}{j!} \frac{d^{j+1} \ln G(z)}{dz^{j+1}} |_{z=0}. \]  

This relation is particularly useful when $C_j$ are calculated from some compound multiplicity distribution defined by a generating function $G(z)$ for which Eq. \[2\] would be too difficult to apply \[1\].

2. Generalized Multiplicity Distribution - GMD

The (GMD), was introduced in \[6\] as an alternative to the NBD solution to study multiplicity distributions. It has the following form:

\[ P(N) = \frac{\Gamma(N+k)}{\Gamma(N-k'+1)\Gamma(k+k')} p^{N-k'} (1-p)^{k+k'}, \]  

where

\[ p = \frac{\langle N \rangle - k'}{\langle N \rangle + k}. \]  

The GMD has been successfully applied to $p\bar{p}$ reactions \[7\] and to $e^+e^-$ annihilation \[8\]. It is based on the stochastic branching equation describing the total multiplicity distribution of partons inside a jet \[3\],

\[ \frac{dP(n)}{dt} = -\left(An + \tilde{A}m\right)P(n) + \]  

\[ + A(n-1)P(n-1) + \tilde{A}mP(n-1), \]  

where

\[ t = \frac{6}{11N_c - 2N_f} \ln \left[ \frac{\ln \left( \frac{Q_f}{Q_{\mu}} \right)}{\ln \left( \frac{Q_{\mu}}{Q_{\mu}} \right)} \right], \]  

is the QCD evolution parameter, with $Q$ denoting the initial parton invariant mass, $Q_0$ the hadronization mass and $\mu$ the QCD mass scale (in GeV) ($N_c = 3$ is the number of colors and $N_f = 4$ the number of flavors). $P(n)$ is the probability distribution of $n$ gluons and $m$ quarks (the number of which is fixed) in the QCD evolution. Parameters $A$ and $\tilde{A}$ denote, respectively, the average probabilities

![Figure 1: Upper panel: Data on $P(N)$ measured in $e^+e^-$ collisions by the ALEPH experiment at 91 GeV \[10\] are fitted by the GMD distribution \[6\] with parameters: $\langle N \rangle = 12.991$, $k = 3.5$ and $k' = 0.7348$. Lower panel: the modified combinants $C_j$ derived from these data (note the significant, rank $j$-dependent, change of the vertical scale for $C_j$; which makes it possible to draw the results, otherwise the amplitudes would grow in a power-law fashion). As shown in the following sections, they can be fitted by the $C_j$ obtained from the $P(N)$ of the GMD type.](image)
of the $g \to gg$ and $q \to qg$ processes (in the version of Eq. (8) used here the contribution of $g \to q\bar{q}$ process has been neglected). The initial number of gluons, $k'$, determines (in the average sense) the initial condition of the generating function, which is $G(t = 0, z) = z^{k'}$. The parameter $k = mA/A$ is related (in the average sense) to the initial number of partons. To connect these parton level results with the experimental data the Local Parton Hadron Duality (11) was taken as the hadronization prescription, i.e., the hadron spectra were required to be proportional to the corresponding parton spectra. The whole hadronization is then parameterized by a single parameter, which gives the overall normalization of the distribution but does not affect its moments of order greater or equal than one.

The generating function of such a GMD is equal to

$$G(z) = z^{k'}[z + (1 - z)k]^{-(k + k')}, \quad (10)$$

where

$$k = e^{At}, \quad (11)$$

and the corresponding mean multiplicity is

$$\langle N \rangle = k(k - 1) + kk'. \quad (12)$$

This can be derived noting that generating function (10) can be also calculated directly using $P(N)$ from the Eq. (6), in which case we obtain that

$$G(z) = \sum_{N=k}^{\infty} z^N P(N) = z^{k'} \left(\frac{1 - p}{1 - pz}\right)^{k + k'}. \quad (13)$$

Comparing Eq. (13) with Eq. (10) one gets that

$$p = 1 - \frac{1}{k} = 1 - e^{-At}. \quad (14)$$

Using Eq. (7) for $p$ one gets $\langle N \rangle > 0$ in the form of Eq. (12).

Because the distribution $P(N)$ described by Eq. (6) is defined for $N \geq k'$, both the normalization of the GMD distribution, Eq. (6), and the generating function $G(z)$, Eq. (10), are also defined for such a range of $N$. This fact is not important as long as one calculates $C_j$ from Eq. (2) and as long as $P(0) > 0$, because in this case we only have the ratio $P(N)/P(0)$ and the normalization is not important (it cancels out). However, it matters when one calculates $C_j$ using Eq. (5) with the generating function $G(z)$ defined by Eq. (10) because in this case the calculated $C_j$ are divergent.

3. Normalization of GMD

Some details are in order here. Note that, for integer values of $k'$, the GMD distribution (6) is nothing but a NBD “shifted” by $k'$:

$$P_{\text{GMD}}(N, k, k') = P_{\text{NBD}}(N - k', k + k'), \quad (15)$$

where $P_{\text{NBD}}(N, k)$ is the NBD. The normalization factor

$$C = \frac{1}{\Gamma(k' + k)} p^{-k'} (1 - p)^{k + k'}, \quad (16)$$

appearing in the distribution (6), is, for the normalization of probability, calculated for $N \geq k'$, equal to $\sum_{N=k'}^{\infty} P(N) = 1$. If one normalizes $P(N)$ in the whole range of $N$, i.e., for $N \in [0, \infty)$, then

$$P(N) = \frac{\Gamma(N + k) \Gamma(1 - k')}{\Gamma(k) \Gamma(N - k' + 1) 2F_1(1, 1 - k'; p)} \cdot p^N, \quad (17)$$

where $2F_1(a, b, c; z)$ is a hypergeometric function. The generating function is then given by

$$G(z) = \sum_{N=0}^{\infty} z^N P(N) = \frac{F([1, k], [1 - k']; p)}{2F_1(1, 1 - k'; p)}, \quad (18)$$

where $F(a, b; z)$ is a generalized hypergeometric function. It turns out that when we calculate the modified combinator, $C_j$, using this generating function we do not encounter any of the problems occurring when using instead Eq. (10). Nonetheless, for $k' > 1$ the problem still remains.

\footnote{For $t = 0$ where $k = 1$ we have $\langle N \rangle = k'$ and $\langle N \rangle$ increases with $Q^2$ as $\langle N \rangle \sim \ln \frac{Q^2}{m^2} \frac{\alpha_s}{\pi} [1 + \ln(1 + \frac{2\alpha_s}{\pi})].$}

\footnote{Note that for small values of $z$ the generating function (13) is equal to $G(z) = 1 + \frac{z}{1 - z}$, i.e., it is of the BD type and this results in the oscillations characteristic for the BD.}
4. Imprints of acceptance

We shall propose now a modification of the initial $P(N)$ that will allow for $N < k'$. To this end, let us assume that the $P_{GMD}(N)$ given by Eq. (15) presents a real distribution which describes the multiplicity in the full phase space. However, in the experiment we measure the multiplicity only within some window in rapidity, $\Delta y$. Let us assume therefore that the detection process is a Bernoulli process described by the Binomial Distribution (BD) with the generating function

$$F(z) = 1 - \alpha + \alpha z,$$  \hfill (19)

where $\alpha$ denotes the probability of the detection of a particle in the rapidity window $\Delta y$. The number $N$ of the registered particles is

$$N = \sum_{i=1}^{M} n_i,$$  \hfill (20)

where $n_i$ follows the BD with the generating function $F(z)$ given by Eq. (19) and $M$ comes from the GMD with the generating function $G(z)$ given by Eq. (10). The measured multiplicity distribution, $P(N)$, is therefore given by the GMD compounded with the BD, and its generating function is:

$$H(z) = G[F(z)] = \left(1 - \alpha + \alpha z\right)^k \left[1 + \alpha (k - 1) - \alpha (k - 1) z - \alpha k'\right].$$  \hfill (21)

Note that the generating function (21) is the product of the generating function of the BD,

$$f(z) = 1 - \alpha + \alpha z,$$  \hfill (22)

and the generating function of the NBD

$$g(z) = \left[1 + \alpha (k - 1) - \alpha (k - 1) z\right]^{-\alpha k'},$$  \hfill (23)

Using general Leibniz rule we have that

$$P(N) = \frac{1}{N!} \frac{d^N H(z)}{dz^N} \bigg|_{z=0} = \frac{1}{N!} \sum_{i=\max(0,k'-1)}^{N} \binom{N}{i} \frac{d^{N-i} f(z)}{dz^{N-i}} \frac{d^i g(z)}{dz^i} \bigg|_{z=0}.$$  \hfill (24)

Figure 2: Upper panel: Data on $P(N)$ measured in $e^+e^-$ collisions by the ALEPH experiment at 91 GeV $[10]$ are fitted by the distribution obtained from the generating function given by Eq. (21) with parameters: $\alpha = 0.8725$, $k' = 1$, $k = 3.2$, $p = 0.75$ and $\kappa = 4.585 \,(\alpha \cdot \kappa = 4)$. Lower panel: the modified combinants $C_j$ deduced from these data on $P(N)$ are displayed. They can be fitted by $C_j$ obtained from the same generating function given by Eq. (21) with the same parameters as used for fitting $P(N)$.

Modified combinants $\langle N \rangle C_j$ calculated using the generating function (21) are given by the sum of the respective modified combinants for the BD and the NBD:

$$\langle N \rangle C_j = \frac{1}{j!} \frac{d^{j+1} \ln f(z)}{dz^{j+1}} \bigg|_{z=0} + \frac{1}{j!} \frac{d^{j+1} \ln g(z)}{dz^{j+1}} \bigg|_{z=0}.$$  \hfill (25)

We can expect therefore oscillations with period equal to 2, which are superimposed on the monotonically decreasing values:

$$\langle N \rangle C_j = (-1)^j k' \left(\frac{\alpha}{1 - \alpha}\right)^{j+1} + (k + k') \left(\frac{\alpha (k - 1)}{1 + \alpha (k - 1)}\right)^{j+1}.$$  \hfill (26)

\footnote{Note that such procedure applied to NBD gives again the NBD with the same $k$ but with modified $p$, which is now equal to $p' = \frac{\alpha}{1 - \alpha}$.}
Fig. 2 shows this such approach works very well (however, looking on the experimental $C_j$, we can suspect that $C_j$ are increasing for small $j$, this effect has its source in the second term of Eq. (25)).

5. Scenario of two sources

Actually, there is yet another way of treating the $e^+e^-$ data. Namely, the generating function (21) can be also formally treated as a generating function of the multiplicity distribution $P(N)$ in which $N$ consists of both the particles from the BD ($N_{BD}$) and from the NBD ($N_{NBD}$):

$$N = N_{BD} + N_{NBD}. \quad (27)$$

In this case Eq. (24) can be written as

$$P(N) = \sum_{i=0}^{\min(N,k')} P_{BD}(i)P_{NBD}(N-i) \quad (28)$$

and, respectively, Eq. (26) can be written as

$$\langle N \rangle C_j = \langle N_{BD} \rangle C_j^{BD} + \langle N_{NBD} \rangle C_j^{NBD}. \quad (29)$$

The fits shown in Fig. 2 correspond to parameters: $k' = 1$ and $p' = 0.8725$ for the BD and $k = 4.2$ and $p = 0.75$ for the NBD.

To summarize this part: we have shown that the GMD can also be understood as a shifted NBD, Eq. (27) demonstrates the possible implementation of the process leading to such shift.

6. Concluding remarks

It must be noted that if instead of the ALEPH data we had used the DELPHI data, the analysis of the corresponding $C_j$ would have not been possible. This is illustrated in Fig. 3 where the modified combinants $C_j$ are in this case too scattered to be of any use. The reason is low statistics in the DELPHI data. While in the ALEPH case there are $3 \cdot 10^5$ data events, in the DELPHI case the statistics is 5 times smaller. It can be checked that for $k' \to 0$, when GMD $\to$ NBD, the oscillations of the modified combinants $C_j$ gradually vanish and eventually we get a fading down monotonic curve characteristic for the NBD.

As a final remark, let us note that a large number of papers suggest some kind of universality in the mechanisms of hadron production in $e^+e^-$ annihilations and in $pp$ and $p\bar{p}$ collisions. Such universality arises from observations of the average multiplicities and relative dispersions in both types of processes (cf. for example, [13, 14]). However, the modified combinator analysis reveals differences between these processes. Namely, while in $e^+e^-$ annihilations we observe oscillations of $C_j$ with period 2, in $pp$ and $p\bar{p}$ collisions the period of oscillation is $\sim 10$ times longer and the amplitude of oscillations in both types of processes differs drastically. Obviously, further analysis along these lines would be most welcome. In
what concerns the $e^+e^-$ results discussed here, the most plausible interpretation is the GMD approach (with some modifications discussed above). However, this problem seems at the moment still open and subject to future investigations.

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