RATIONAL FACTORIZATIONS OF COMPLETELY POSITIVE MATRICES

MATHIEU DUTOUR SIKIRIĆ, ACHILL SCHÜRMANN, AND FRANK VALLENTIN

Abstract. In this note it is proved that every rational matrix which lies in the interior of the cone of completely positive matrices also has a rational cp-factorization.

1. Introduction

The cone of completely positive matrices is central to copositive programming, see [3] and also to several topics in matrix theory, see [1]. However, so far, this cone is quite mysterious, many basic questions about it are open. In [2] Berman, Dür, and Shaked-Monderer ask: Given a matrix $A \in \mathbb{CP}_n$ all of whose entries are integral, does $A$ always have a rational cp-factorization?

The cone of completely positive matrices is defined as the convex cone spanned by symmetric rank-1-matrices $xx^\top$ where $x$ lies in the nonnegative orthant $\mathbb{R}_{\geq 0}^n$:

$$\mathbb{CP}_n = \text{cone}\{xx^\top : x \in \mathbb{R}_{\geq 0}^n\}.$$  

A cp-factorization of a matrix $A$ is a factorization of the form

$$A = \sum_{i=1}^{m} \alpha_i x_i x_i^\top \quad \text{with } \alpha_i \geq 0 \text{ and } x_i \in \mathbb{R}_{\geq 0}^n, \quad \text{for } i = 1, \ldots, m.$$  

We talk about a rational cp-factorization when the $\alpha_i$’s are rational numbers and when the $x_i$’s are rational vectors. Of course, in a rational cp-factorization we can assume that the $x_i$’s are integral vectors.

In this note we prove the following theorem:

Theorem 1.1. Every rational matrix which lies in the interior of the cone of completely positive matrices has a rational cp-factorization.

So to fully answer the question of Berman, Dür, and Shaked-Monderer, it remains to consider the boundary of $\mathbb{CP}_n$.

2. Proof of Theorem 1.1

For the proof we will need a classical result from simultaneous Diophantine approximation, a theorem of Dirichlet, which we state here. One can find a proof of Dirichlet’s theorem for example in the book [4, Theorem 5.2.1] of Grötschel, Lovász, and Schrijver.

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Theorem 2.1. Let \( \alpha_1, \ldots, \alpha_n \) be real numbers and let \( \varepsilon \) be a real number with \( 0 < \varepsilon < 1 \). Then there exist integers \( p_1, \ldots, p_n \) and a natural number \( q \) with \( 1 \leq q \leq \varepsilon^{-n} \) such that

\[
|\alpha_i - \frac{p_i}{q}| \leq \frac{\varepsilon}{q} \quad \text{for all } i = 1, \ldots, n.
\]

The next lemma collects standard, easy-to-prove facts about convex cones. Let \( E \) be a Euclidean space with inner product \( \langle \cdot, \cdot \rangle \). Let \( K \subseteq E \) be a proper convex cone, which means that \( K \) is closed, has a nonempty interior, and satisfies \( K \cap (-K) = \{0\} \). Its dual cone is defined as \( K^* = \{ y \in E : \langle x, y \rangle \geq 0 \text{ for all } x \in K \} \).

Lemma 2.2. Let \( K \subseteq E \) be a proper convex cone. Then,

(1) \( \text{int}(K) = \{ x \in E : \langle x, y \rangle > 0 \text{ for all } y \in K^* \setminus \{0\} \} \),

where \( \text{int}(K) \) is the topological interior of \( K \), and

(2) \( K^* = (\text{cl}(K))^* \),

where \( \text{cl}(K) \) is the topological closure of \( K \).

We need some more notation: With \( S^n \) we denote the vector space of symmetric matrices with \( n \) rows and \( n \) columns which is a Euclidean space with inner product \( \langle A, B \rangle = \text{Trace}(AB) = \sum_{i,j=1}^n A_{ij}B_{ij} \). The cone of copositive matrices is the dual cone of \( CP_n^* \):

\[
CP_n = CP_n^* = \{ B \in S^n : \langle A, B \rangle \geq 0 \text{ for all } A \in CP \}.
\]

Its interior equals

\[
\text{int}(CP_n) = \{ B \in S^n : \langle B, xx^T \rangle > 0 \text{ for all } x \in \mathbb{R}_{\geq 0}^n \setminus \{0\} \}.
\]

We also define the following rational subcone of \( CP_n^* \):

\[
CP_n^\mathbb{Q} = \text{cone}\{vv^T : v \in \mathbb{Z}_{\geq 0}^n \}.
\]

We prepare the proof of the paper’s main result by two lemmata which might be useful facts themselves.

Lemma 2.3. The set

\[
\mathcal{R} = \{ B \in S^n : \langle B, vv^T \rangle \geq 1 \text{ for all } v \in \mathbb{Z}_{\geq 0}^n \setminus \{0\} \},
\]

is contained in the interior of the cone of copositive matrices \( CP_n^* \).

Proof. Since the set of nonnegative rational vectors \( \mathbb{Q}_{\geq 0}^n \) lies dense in the nonnegative orthant \( \mathbb{R}_{\geq 0}^n \), we have the inclusion \( \mathcal{R} \subseteq CP_n^\mathbb{Q} \). Suppose for contradiction that the set on the left is not contained in \( \text{int}(CP_n^\mathbb{Q}) \): There is a matrix \( B \) with \( \langle B, vv^T \rangle \geq 1 \) for all \( v \in \mathbb{Z}_{\geq 0}^n \setminus \{0\} \) and there is a nonzero vector \( x \in \mathbb{R}_{\geq 0}^n \) with \( \langle B, xx^T \rangle = 0 \).

By induction on \( n \) (and reordering if necessary) we may assume that all entries of \( x \) are strictly positive, \( x_i > 0 \) for all \( i = 1, \ldots, n \), since otherwise, we can reduce the situation to the case of smaller dimension by considering a suitable submatrix of \( B \).

Hence, the vector \( x \) lies in the interior of the nonnegative orthant. Therefore, and because \( B \in CP_n^\mathbb{Q} \), we have for every vector \( y \in \mathbb{R}^n \) and \( \varepsilon > 0 \) sufficiently small the inequality

\[
0 \leq \frac{1}{\varepsilon} (x + \varepsilon y)^T B (x + \varepsilon y) = 2x^T By + \varepsilon y^T By.
\]
and similarly
\[ 0 \leq \frac{1}{\varepsilon} (x - \varepsilon y)^T B (x - \varepsilon y) = -2x^T By + \varepsilon y^T By \]
From this, equality \( x^T B = 0 \) follows. From this, we also see that \( B \) is positive semidefinite. This implies that
\[ (\alpha x + y)^T B (\alpha x + y) = y^T By \quad \text{for} \quad \alpha \in \mathbb{R} \text{ and } y \in \mathbb{R}^n. \]

We apply Dirichlet’s approximation theorem, Theorem 2.1 to the vector \( x \) and to \( \varepsilon \in (0, 1) \). We obtain a vector \( p = (p_1, \ldots, p_n) \) and a natural number \( q \). Since \( x_i > 0 \) we may without loss of generality assume that \( p_i \geq 0 \). Thus, by the assumption \( B \in \mathcal{R} \), we have \( \langle B, pp^T \rangle \geq 1 \).

Define \( y = qx - p \) where \( \|y\|_\infty \leq \varepsilon \).

Since \( B \) is positive semidefinite, there is a constant \( C \) such that \( y^T By \leq C\|y\|_\infty^2 \) for all \( y \in \mathbb{R}^n \). Putting everything together we get
\[ 1 \leq \langle B, pp^T \rangle = (qx - y)^T B(qx - y) = y^T By \leq C\|y\|_\infty^2 \leq C\varepsilon^2, \]
which yields a contradiction for small enough values of \( \varepsilon \).

**Lemma 2.4.** Let \( A \) be a completely positive matrix which lies in the interior of \( \mathcal{CP}_n \) and let \( \lambda \) be a sufficiently large positive real number. Then the set
\[ \mathcal{P}(A, \lambda) = \{ B \in \mathbb{S}^n : \langle A, B \rangle \leq \lambda, \langle B, vv^T \rangle \leq 1 \text{ for all } v \in \mathbb{Z}^n_{\geq 0} \} \]
is a full-dimensional polytope.

**Proof.** For sufficiently large \( \lambda \) a sufficiently small ball around a suitable multiple of \( A \) is contained in \( \mathcal{P}(A, \lambda) \), which shows that \( \mathcal{P}(A, \lambda) \) has full dimension.

By the theorem of Minkowski and Weyl, see for example [5, Corollary 7.1c], polytopes are exactly bounded polyhedra. So it suffices to show that the set \( \mathcal{P}(A, \lambda) \) is a bounded polyhedron.

First we show that \( \mathcal{P}(A, \lambda) \) is bounded: For suppose not. Then there is \( B_0 \in \mathcal{P}(A, \lambda) \) and \( B_1 \in \mathbb{S}^n \), with \( B_1 \neq 0 \), so that the ray \( B_0 + \alpha B_1 \), with \( \alpha \geq 0 \), lies completely in \( \mathcal{P}(A, \lambda) \). In particular \( \langle B_1, vv^T \rangle \geq 0 \) for all \( v \in \mathbb{Z}^n_{\geq 0} \). Hence, \( B_1 \) lies in the dual cone of \( \mathcal{CP}_n \). On the other hand \( \langle A, B_1 \rangle \leq 0 \). Hence, by Lemma 2.2 (1), \( B_1 \notin \mathcal{CO}_n \setminus \{0\} \), but by Lemma 2.2 (2),
\[ \mathcal{CP}_n^* = (\text{cl}(\mathcal{CP}_n))^* = \mathcal{CP}_n^* = \mathcal{CO}_n, \]
so \( B_1 = 0 \), yielding a contradiction.

Now we show that \( \mathcal{P}(A, \lambda) \) is a polyhedron: For suppose not. Then there is a sequence \( v_i \in \mathbb{Z}^n_{\geq 0} \setminus \{0\} \) of infinitely many pairwise different nonzero lattice vectors so that there are \( B_i \in \mathcal{P}(A, \lambda) \) with \( \langle B_i, v_i v_i^T \rangle = 1 \). Since \( \mathcal{P}(A, \lambda) \) is compact, there exists a subsequence \( B_{ij} \) which converges to \( B^* \in \mathcal{P}(A, \lambda) \). Define the sequence \( u_{ij} = v_i / \|v_i\| \) which lies in the compact set \( \mathbb{R}^n_{\geq 0} \cap S^{n-1} \) where \( S^{n-1} \) denotes the unit sphere. Hence there is a subsequence converging to \( u^* \in S^{n-1} \), in particular \( u^* \neq 0 \). Denote the indices of this subsequence with \( k \), then
\[ 1 = \langle B_k, v_k u_k^T \rangle = \|v_k\|^2 \langle B_k, u_k u_k^T \rangle. \]

When \( k \) tends to infinity, the squared norms \( \|v_k\|^2 \) tend to infinity as well, since we use infinitely many pairwise different lattice vectors and there exist only finitely
many lattice vectors up to some given norm. So \( \langle B_k, u_k \rangle \) tends to \( \langle B^*, u^*(u^*)^T \rangle = 0 \), and by Lemma 2.3 we obtain a contradiction. \( \square \)

Now we prove the main result and finish the paper.

\textbf{Proof of Theorem 1.1.} Let \( A \) be matrix having rational entries only and lying in the interior of the cone of completely positive matrices. Then \( \mathcal{P}(A, \lambda) \) is a polytope according to the previous lemma. We minimize the linear functional \( B \mapsto \langle A, B \rangle \) over \( \mathcal{P}(A, \lambda) \). Then we choose those lattice vectors \( v_i \in \mathbb{Z}_{\geq 0}^n \) with \( i = 1, \ldots, m \) for which equality \( \langle B^*, v_i v_i^T \rangle = 1 \) holds. Because of the minimality of \( \langle A, B^* \rangle \) it follows

\begin{equation}
A \in \text{cone}\{v_i v_i^T : i = 1, \ldots, m\}.
\end{equation}

Otherwise, see for example [5, Theorem 7.1], we find a separating linear hyperplane orthogonal to \( C \) separating \( A \) and cone\( \{v_i v_i^T : i = 1, \ldots, m\} \):

\[ \langle C, A \rangle < 0 \quad \text{and} \quad \langle C, v_i v_i^T \rangle \geq 0 \text{ for all } i = 1, \ldots, m. \]

Then for sufficiently small \( \mu > 0 \) we would have

\[ B^* + \mu C \in \mathcal{P}(A, \lambda) \quad \text{but} \quad \langle B^* + \mu C, A \rangle < \langle B^*, A \rangle, \]

which contradicts the minimality of \( \langle A, B^* \rangle \).

We apply Carathéodory’s theorem (see for example [5, Corollary 7.1i]) to (3) and choose a subset \( I \subseteq \{1, \ldots, m\} \) so that \( v_i v_i^T \) are linearly independent and so that \( A \) lies in cone\( \{v_i v_i^T : i \in I\} \). Since \( A \) is a rational matrix and since the \( v_i v_i^T \)’s are linearly independent rational matrices, there is a unique choice of rational numbers \( \alpha_i \in \mathbb{Q}_{\geq 0} \), with \( i \in I \), so that \( A = \sum_{i \in I} \alpha_i v_i v_i^T \) holds, which gives a desired rational cp-factorization. \( \square \)

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M. Dutour Sikirić, Rudjer Bosković Institute, Bijenicka 54, 10000 Zagreb, Croatia
E-mail address: mathieu.dutour@gmail.com

A. Schürmann, Universität Rostock, Institute of Mathematics, 18051 Rostock, Germany
E-mail address: achill.schuermann@uni-rostock.de

F. Vallentin, Mathematisches Institut, Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany
E-mail address: frank.vallentin@uni-koeln.de