Cosmological Kaluza-Klein branes in black brane spacetimes

Masato Minamitsuji
Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität, Theresienstr. 37, D-80333, Munich, Germany

We discuss the cosmological evolution of a brane in the $D (> 6)$-dimensional black brane spacetime in the context of the Kaluza-Klein (KK) braneworld scheme, i.e., to consider KK compactification on the brane. The bulk spacetime is composed of two copies of a patch of $D$-dimensional black three-brane solution. The near-horizon geometry is given by $AdS_5 \times S^{(D-5)}$ while in the asymptotic infinity the spacetime approaches $D$-dimensional Minkowski. We consider the brane motion from the near-horizon region toward the spatial infinity, which induces cosmology on the brane. As is expected, in the early times, namely when the brane is located in the near-horizon region, the effective cosmology on the brane coincides with that in the second Randall-Sundrum (RS II) model. Then, the brane cosmology starts to deviate from the RS type one since the dynamics of KK compactified dimensions becomes significant. We find that the brane Universe cannot reach the asymptotic infinity, irrespectively of the components of matter on the brane.

Since the stimulating proposals by Randall and Sundrum (RS) [1, 2], the cosmological aspects of braneworld models have been studied in various literature [3]. In the second RS (RS II) model [1], at the low energy scales the four-dimensional cosmology is recovered on the brane due to the strong warping of the extra dimension. Then, six and higher-dimensional models have started to attract growing interests. In higher dimensions, there will be various types of braneworld models, for example, those with flux-stabilized compact extra dimensions, with brane intersections. Codimension-two brane models with flux-stabilized extra dimensions have been originally focused on as a simple realization of large extra dimensions [4] and as a way of resolution of the cosmological constant problem [5] (see, however, e.g., [6]). Recently, these six dimensional models have been discussed from the different aspects. It is well-known that a four-dimensional defect in six or higher dimensions generically has a problem on localization of ordinary matter on the brane due to the stronger self-gravity. Ways to circumvent this problem have to be developed. In six dimensional models, the ways of brane regularization to study gravity and cosmology on the brane have been studied in [7, 8, 9]. The way employed in Ref. [7, 9] is that the original codimension-two brane is replaced with a ring-like codimension-one brane wrapped around the axis of symmetry of the bulk.

Our main purpose is to construct brane cosmological models in higher dimensional spacetime. In order to do so, one would consider regularizations of branes with higher-codimensions as the extensions of ways developed in the studies of six-dimensional (codimension-two) brane models. But, in fact, it seems to be difficult to construct the braneworld models in such a way, since these branes have stronger self-gravity and develop severe singularities. Thus, in this article we take a (similar but different) approach, instead of regularizing a brane with higher codimensions. We focus on the hybrid construction of the Kaluza-Klein (KK) and braneworld compactifications, i.e., to consider KK compactifications on the brane. Such a way of construction is called Kaluza-Klein braneworld [10]. Actually, the way of regularizations developeled in [7, 9] can be seen as a kind of KK braneworlds. In the context of six dimensions, such a hybrid construction has been presented in Ref. [11], before the works [7, 9]. In their construction, two identical copies of a regular and compact internal space are glued at the position of the brane and clearly the resultant codimension-one brane cannot be interpreted as a regularized object. The scheme of KK braneworlds could be one of the most powerful tools to construct successful braneworld models in higher dimensions. General properties of KK braneworlds have been investigated recently, e.g., in Ref. [10]. Cosmology on a KK brane can be realized by considering the motion of the KK brane into the bulk spacetime. In this article, we construct an explicit model of cosmological Kaluza-Klein braneworld in the higher-dimensional spacetime.

In constructing KK cosmological brane models, the important problem is how one makes the KK compactified dimensions to be invisible to the observers on the brane. In this article, we consider the black brane solutions, whose near horizon geometry is $AdS_5 \times S^{(D-5)}$ while in the asymptotic infinity the spacetime approaches $D$-dimensional Minkowski. We consider the motion of the brane from the near horizon toward the asymptotic infinity. Thus, in the early times the brane stays in the near-horizon region. In the black brane spacetime, the magnetic $(D-5)$-form field keeps the the size of spherical compact dimensions enough compact in the near-horizon region and helps to make the dynamics of these dimen-

---

1 See for several attempts to realize the localization of ordinary matter and gravity on the brane in higher dimensions, e.g., by constructing regular solitonic solutions [12], by replacing the brane with the core region where matter is distributed smoothly [13], by adding higher curvature terms to the theory [14] and by considering brane intersections [15].
sions to be invisible on the brane in the early times. In addition, the spacetime has the higly warped extra dimension. The well-behaved cosmological feature of the RS II model is due to the realization of the warped structure of the extra dimension. Such a warped structure of extra dimensions also would help for the standard four-dimensional cosmology to be recovered on the brane due to the localization of the zero mode gravitations.

Finally, it is expected that in the later times, the brane cosmology starts to deviate from the RS II type one because the dynamics of the KK compactified dimensions could be significant. We discuss this point in terms of the effective potential for the scale factor.

**Cosmological Kaluza-Klein branes:**

We briefly review the cosmology on a $(D - 1)$-dimensional KK brane in $D$-dimensional spacetime. We follow the well-established method developed in Ref. [16]. We consider a motion of a brane in the static background. We start from the following static $D (> 6)$-dimensional metric ansatz:

$$\begin{align*}
ds^2 &= -F(r)^2 dt^2 + A(r)^2 \delta_{ij} dx^i dx^j \\
    &\quad + B(r)^2 dr^2 + C(r)^2 h_{ab} dy^a dy^b.
\end{align*}$$  \hspace{1cm} (1)

We assume the $Z_2$ symmetry with respect to the brane. $h_{ab}$ represents the metric of $(D - 5)$-dimensional compact space. The indices $\ell = 1, 2, 3$ and $a$ run over the ordinary three-dimensional space and $(D - 5)$-dimensional compact space, respectively. The brane proper time $\tau$ and induced metric are given by

$$\begin{align*}
ds_{\text{ind}}^2 &= -dr^2 + a^2 \delta_{ij} dx^i dx^j + C(a)^2 h_{ab} dy^a dy^b, \hspace{1cm} (2)
\end{align*}$$

where the cosmological scale factor is given by $a = A^{1/\ell}_{r = r_0}$ ($r_0$ is the radial position of the brane) and dot means the derivative with respect to the brane proper time $\tau$.

Then, we derive the equations of motion of the brane in the static bulk. The non-trivial components of Israel junction condition are given by

$$\left(\frac{3}{a} + \frac{(D - 5)C}{a} \right) \sqrt{B - 2 + \dot{\alpha}^2} = \frac{\rho}{2M_D^{D-2}}, \hspace{1cm} (3)$$

and the one along the external dimensions, respectively. If there are dynamical degrees of freedom than the metric in the bulk, we need to take their jump conditions into account. The first component of the Israel junction conditions Eq. (3) gives the effective Friedmann equation

$$\left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{\rho^2}{4M_D^{2(D-2)} \left(3 + \frac{(B - 5)a}{C} \right)^2} - \frac{1}{a^2 B^2}. \hspace{1cm} (6)$$

Before closing this section, we mention the other well-known approach, i.e., to consider a fixed brane in the time-dependent bulk spacetime which was employed e.g., in [17]. In this approach, the effective cosmological equations are obtained by integrating the bulk gravitational equations. Of course, these two pictures must be equivalent. In the RS II model, in this picture an integration constant appears in the effective cosmological equations. This integration constant corresponds to the mass of the static bulk black hole in our picture (the parameter $\mu$ in our case which appears in the next section).

**Black three-brane solutions:**

As one of the simplest higher-dimensional solutions, we consider the $(D - 1)$-dimensional black three-brane solutions (see e.g., [18]). We start from the following $D$-dimensional theory:

$$S = \frac{M_D^{D-2}}{2} \int d^D X \sqrt{-G} \left[ R[G] - \frac{1}{2(D-5)!} F_{\mu \nu \delta \epsilon}^2 \right], \hspace{1cm} (7)$$

where $G_{\mu \nu}$ and $R[G]$ are the $D$-dimensional metric and the Ricci scalar associated with the metric $G_{\mu \nu}$, respectively. $F_{\mu \nu \delta \epsilon}^2$ represents the $(D - 5)$-form field strength. This theory contains a series of the black three-brane solutions, whose metric is given by

$$\begin{align*}
ds^2 &= H^{-1/2} \left( -f dt^2 + \delta_{ij} dx^i dx^j \right) \\
    &\quad + H^{2/(D-6)} \left( f^{-1} dr^2 + r^2 h_{ab} dy^a dy^b \right), \hspace{1cm} (8)
\end{align*}$$

where $\delta_{ij}$ and $h_{ab}$ are metric of flat three-dimensional space and $(D - 5)$-sphere, respectively. We also define

$$H(r) := 1 + \frac{Q}{r^{D-6}}, \hspace{1cm} f(r) = 1 - 2 \frac{\mu}{r^{D-6}}, \hspace{1cm} (9)$$

and $h_{ab}$ is the metric of $(D - 5)$-sphere. $\mu$ and $Q$ are parameters of the solutions. We assume that the function $f$ is positive. Thus, the radial coordinate $r$ has the minimal value at the horizon position $r = r_{\text{min}} := (2\mu)^{1/(D-6)}$. The case that $\mu = 0$ corresponds to the extremal solutions. The near horizon geometry is AdS$_5 \times S^{(D-5)}$. The effective curvature radius of AdS spacetime is related to the parameter $Q$

$$\ell := \frac{4}{D - 6} Q^{1/(D-6)}. \hspace{1cm} (10)$$

The $(D - 5)$-form field acts on $S^{(D-5)}$. The magnetic $(D - 5)$-form field strength is given by

$$F_{y^1 \cdots y^{(D-5)}} = \frac{1}{\sqrt{-G}} y^* y^{(D-5)} E'(r) \hspace{1cm} (11)$$
where
\[ E(r) = \sqrt{\frac{Q}{Q + 2\mu}} \frac{D-2}{2(D-6)} f(r). \] (12)

At the spatial infinity \( r \to \infty \), the spacetime approaches \( D \)-dimensional Minkowski. In the case \( D = 10 \), the solution corresponds to a stack of coincident D3-branes at the low energy scales.

The reason to consider the black brane spacetime is as follows. Firstly, the spacetime contains the warped spatial dimension. The well-behaved cosmological feature of the RS II model is due to the realization of the warped structure of the extra dimension. Such a warped structure of extra dimensions may also help to construct the realistic brane cosmological models in higher dimensions. Also, in string compactifications, the highly warped structure is induced gravitationally by the branes localized at a certain place in the internal space. The black brane spacetime is one of the simplest examples of such spacetimes. Secondly, in the near-horizon of the black brane spacetimes, the magnetic \((D-5)\)-form fields keeps spherical dimensions compact enough.

We apply the previously mentioned KK braneworld scheme to the black brane spacetime. The bulk geometry is constructed by the standard cut-and-paste procedure. We excise the region \( r > r_b \) from the original black brane spacetime, where \( r_b \) is a certain radial position, and glue two copies of the remaining spacetime at \( r = r_b \). Then, two copies of the patch \( r_{\text{min}} < r < r_b \) are bounded by a codimension-one brane at \( r = r_b \). The compact spatial dimensions along \((D - 5)\)-sphere (as well as the ordinary four-dimensional spacetime dimensions) are involved into the brane worldvolume as KK compactified dimensions. We identify two copies of the bulk by imposing the reflection \((Z_2)\)-symmetry with respect to the KK brane. An expanding cosmology is realized as a brane motion from the spatial black brane horizon toward the spatial infinity.

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{4Q^2(D-6)\Omega^2_{(D-5)}M_D^{2(D-2)}(D-6)^2\Sigma^2} \left( 1 - a^4 \right)^2 + \frac{2a^2(D-5)}{Q^2(1 + 2\mu/a^4)^2} \left( 1 - a^4 \right)^2 \Omega_{(D-5)}^2 \left( 1 + 2\mu/a^4 \right)^2 - \frac{(D-6)^2}{4} \frac{2a^2(D-5)}{Q^2(1 + 2\mu/a^4)^2} \left( 1 - a^4 \right)^2. \] (16)

In the early times \( a \ll 1 \), the Eq. (16) is reduced to the form
\[ \left( \frac{\dot{a}}{a} \right)^2 \approx \frac{1}{36Q^2(D-6)\Omega^2_{(D-5)}M_D^{2(D-2)}} \left( 1 - a^4 \right)^2 + \frac{2a^2(D-5)}{Q^2(1 + 2\mu/a^4)} \left( 1 - a^4 \right)^2 - \frac{(D-6)^2}{4} \frac{2a^2(D-5)}{a^4Q^{2/(D-6)}}. \] (17)

Cosmology on the KK brane:
The bulk metric functions \( A, B, C \) and \( F \) in Eq. (II) are now
\[ A = \frac{1}{H^{1/4}}, \quad B = \left( \frac{f^{1/2}}{H^{1/(D-6)}} \right)^{-1}, \quad C = rH^{1/(D-6)}, \quad F = \frac{f^{1/2}}{H^{1/4}}. \] (13)
The cosmic scale factor is given by \( a(t) = A(r_b) \). For the nonzero \( \mu \), the cosmic scale factor has non-zero minimal value:
\[ a_{\text{min}} = \left( \frac{2\mu/Q}{1 + 2\mu/Q} \right)^{1/4}, \] (14)
which corresponds to \( r = r_{\text{min}} \). On the other hand, in any case, the cosmic scale factor also has the maximal size \( a = 1 \) for \( r \to \infty \). However, as we will see the brane Universe cannot reach \( a = 1 \). In order to have enough cosmological expansions, we assume the near-extremal condition \( \mu/Q \ll 1 \). The physical size of KK directions on the brane is given by \( C = Q^{1/(D-6)/(1 - a^4)} \) and thus in the near horizon region \( a \ll 1 \) is as small as the effective AdS radius \( \ell \).

Then, by integrating over the \( S^{D-5} \) dimensions on the brane, the effective four-dimensional energy density is obtained as
\[ \Sigma := (C(a))^{D-5} \int d^{D-5}y \sqrt{\det(h_{ab})} \rho = \frac{\rho Q^{(D-5)/(D-6)}\Omega_{(D-5)}}{(1 - a^4)^{D-5/(D-6)}}, \] (15)
where \( h_{ab} \) represents the metric of the \((D - 5)\)-sphere with the unit radius and \( \Omega_{D-5} = \int d^{D-5}y \sqrt{\det(h_{ab})} = \pi^{(D-5)/2}/\Gamma(1 + (D-5)/2) \) is its volume. From the Eq. (15), the effective Friedmann equation becomes
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{4Q^2(D-6)\Omega^2_{(D-5)}M_D^{2(D-2)}} \left( 1 - a^4 \right)^2 + \frac{2a^2(D-5)}{Q^2(1 + 2\mu/a^4)} \left( 1 - a^4 \right)^2 \Omega_{(D-5)}^2 \left( 1 + 2\mu/a^4 \right)^2 - \frac{(D-6)^2}{4} \frac{2a^2(D-5)}{a^4Q^{2/(D-6)}}. \] (16)

Here, it is useful to decompose the effective energy density \( \Sigma \) into the constant and time-dependent parts as \( \Sigma = \Sigma_0 + \Sigma_1 \), where \( \Sigma_0 \) is the constant part of the effective brane energy density and \( \Sigma_1 \) is the remaining time-dependent part. The constant part of the brane energy density can also be decomposed into two parts \( \Sigma_0 = \Sigma_{\text{RS}} + \delta \), where \( \Sigma_0 \) is chosen as
\[ \Sigma_{\text{RS}} := 6\left( \frac{D-6}{4} \right)M_D^{D-2}Q\Omega_{(D-5)} \left( 1 + 2\mu/Q \right)^{1/2}. \] (18)
In the case that \( \delta = 0 \), it is clear that the constant part in the right-hand-side of the Eq. (17) vanishes. In the early times, the effective Friedmann equation given by Eq. (17) is rewritten as

\[
\left( \frac{\dot{a}}{a} \right)^2 \approx \frac{1}{3} \lambda_{\text{eff}} + \frac{1}{3M^2_4} \left( \Sigma_1 + \Sigma_{\text{DR}} \right) + \frac{1 + \frac{2\mu}{Q}}{\ell^2} \left( \frac{\Sigma_1}{\Sigma_0} \right)^2
\]

where the effective gravitational scale \( M_4 \) and the effective cosmological constant on the brane \( \lambda_{\text{eff}} \) are given by

\[
M^2_4 := \frac{4}{D - 6} M^{D-2}_D Q^{(D-4)/(D-6)} \Omega_{D-5}.
\]

and

\[
\frac{\lambda_{\text{eff}}}{3} := \ell^{-2} \left( 1 + \frac{2\mu}{Q} \right) \left( \frac{2\delta}{\Sigma_{\text{RS}}} + \frac{\delta^2}{\Sigma^2_{\text{RS}}} \right),
\]

respectively. We also define the effective energy density:

\[
\Sigma_{\text{DR}} := 3M^2_4 \left( \frac{D - 6}{4} \right)^2 \frac{2\mu}{Q^{1+2/(D-6)} a^4}.
\]

The result is nothing but the one in the RS II model \( [1] \). Especially, in the low energy regime \( \Sigma_1 \ll \Sigma_{\text{RS}} \), the standard cosmology is recovered except for the term \( \Sigma_{\text{DR}} \). By combining Eq. (18) with (20), we find

\[
\Sigma_{\text{RS}} \approx \frac{6M^2_4}{\ell^2} \left( 1 + \frac{2\mu}{Q} \right).
\]

As is seen previously, in the case that \( \delta = 0 \), as in the RS II model \( [1] \), the effective cosmological constant \( \lambda_{\text{eff}} \) vanishes. The deviation from the extremal condition \( \mu \neq 0 \) gives rise to the so-called dark radiation (see Ref. [3]), given by \( \Sigma_{\text{DR}} \). In the RS II cosmology, the dark radiation arises due to the presence of a black hole in the bulk. In the current case, similarly the near-horizon geometry is approximated by the product of the five-dimensional Schwarzschild-AdS with \( S^{(D-5)} \). As we will see later, the pressure equation in the corresponding bulk region also has the same structure as in the RS II cosmology.

In the later times, cosmology could deviate from the RS one because the dynamics of the KK compactified dimensions becomes important. To see this, it is useful to rewrite the effective Friedmann equation into the form in an analogy with the classical mechanics as \( \dot{a}^2 + U(a) = 0 \). For instance in the extremal case \( \mu = 0 \), the potential term is given by

\[
U(a) := -\ell^{-2} a^2 (1 - a^4)^{2+2/(D-6)} \left[ \left( 1 - a^4 \right) \left( \frac{\delta}{\Sigma_{\text{RS}}} + \frac{\Sigma_{\text{DR}}}{\Sigma_{\text{RS}}} \right)^2 \right]^{-1}.
\]

As easily seen, the potential term becomes positive before arriving the maximal value \( a = a_{\text{max}} \). In the case of the vacuum brane \( \Sigma_1 = 0 \), this maximal value of cosmic scale factor is given by

\[
a_{\text{max}} = \left( \frac{\delta}{\delta + \frac{4(D-5)}{3(D-6)} \Sigma_{\text{RS}}} \right)^{1/4}.
\]

In Fig. 1 and 2, the typical behaviors of the potential \( U(a) \) without the ordinary matter \( \Sigma_1 = 0 \) and with dust fluid on the brane are shown, respectively. The function \( U(a) \) vanishes at \( a = a_{\text{max}} \). After the brane reaches the maximal size, the brane goes back toward \( a \to 0 \). Similar behaviors can be seen in the near-extremal case. Thus, we find that the brane Universe cannot reach the asymptotic infinity, irrespectively of the components of the matter on the brane. In other words, observers on the brane never see the KK compactified extra dimensions. The result implies that the cosmological brane cannot escape from the gravitational potential induced by the black brane horizon.

As previously mentioned, in the special case that \( D = 10 \), the black three-brane solution corresponds to a low energy description of a stack of \( N \) coincident (BPS for \( \mu = 0 \)) D3-branes in type IIB string theory.
These parameters in the solution can be expressed as $M_9^2 = 2/(2\pi)^2 g_s^3 \alpha'^4$ and $Q = \ell^d = 4\pi N g_s \alpha'^2$ where $\ell_5 = (\alpha')^{1/2}$ and $g_s$ represent string length scale and string coupling constant, respectively. Note that the supergravity description of D3-branes is valid in the case that the effective AdS curvature scale $\ell = Q^{1/4}$ is much larger than $\ell_5$ (hence $N g_s \gg 1$) and quantum gravitational corrections are negligible, i.e., $\ell \gg M_9$ (hence $N \gg 1$). Thus, assuming the near-extremal condition $\mu/Q \ll 1$ and an optimistic choice of the brane tension $\delta/\Sigma_{RS} = O(1)$, the effective four-dimensional quantities are also written as

$$M_2^2 \simeq \frac{N^{3/2}}{8\pi^5/2 g_s^{1/2} \alpha'}, \quad \Sigma_{RS} \simeq \frac{3N}{8\pi^3 g_s \alpha'^2}. \quad (26)$$

Although our approach is higher-dimensional, in the sense that we start from the original higher-dimensional theory and then secondly integrate over the internal dimensions after obtaining the junction conditions, we now confirm that in the early time the brane cosmology obtained in our approach is equivalent to the one in the approach in the effective theory approach in which the internal dimensions are integrated over firstly.

**Experimental and observational bounds:**

As in the cases of the RS II cosmology (see e.g.,[19]), several experimental and observational bounds are obtained. A lower bound on the RS brane tension is given by the condition that the density square term in the effective Friedmann equation must be negligible before the Big Bang nucleosynthesis (BBN) $\Sigma_{RS} > (1\text{MeV})^4$. But a more stringent bound is obtained from the results of the table top tests on Newton’s law, implying $\ell < 0.1\text{mm}$ and hence $\Sigma_{RS} > (10^3\text{GeV})^4$ in the assumption that $\mu/Q \ll 1$ since $M_4 \approx 10^{15}\text{GeV}$.

In the ten-dimensional ($D = 10$) case, from the Eq. (20), we find

$$N^2 \simeq \frac{24\pi^2 M_4^4}{\Sigma_{RS}}, \quad g_s \alpha'^2 \approx \left(\frac{3}{2}\right)^{3/2} \frac{M_4^2}{\pi^2 \Sigma_{RS}^{3/2}}. \quad (27)$$

The condition that $\Sigma_{RS} > (10^3\text{GeV})^4$ (hence $\ell < 0.1\text{mm}$) implies $N < 10^{30}$ and $M_{10} \sim (g_s \alpha'^2)^{-1/8} > 10^{-5}\text{GeV}$. Thus, there is a large parameter space which is consistent with the conditions that $N g_s \gg 1$ and $N \gg 1$ for which the classical description of the solution is valid. Note that $(\ell/0.1\text{mm}) \simeq (N/10^{30})$, since $N \approx 2\pi M_4 \ell$.

As we have seen, in the near-horizon region the effective cosmology on the brane is well approximated by the one in the RS II model. According to the AdS/conformal field theory (CFT) (more precisely AdS$_5$/CFT$_4$) correspondence [20], which states that in type IIB string theory the (super)gravity theory in the five-dimensional AdS spacetime is equivalent to the $N = 4$ super-conformal $U(N)$ Yang-Mills theory at the boundary, gravity in RS II braneworld seems to be equivalent to the four-dimensional gravity coupled to CFT on the brane. There are several examples in which the equivalence (up to $O(\ell^2)$) has been confirmed [21], including the case of cosmological branes. Such an equivalence should be valid for our cosmological brane. The five-dimensional effective gravitational scale is related to the four-dimensional one $M_2^2 \approx M_4^4/\ell$. Then, the CFT parameters are related to the five-dimensional gravitational ones. For instance, the CFT degrees of freedom is given by $N^2 \approx (2\pi)^2 M_2^2 \ell$. Since $\ell^{-1}$ and $N^2$ are the cut-off energy scale and the degrees of freedom of CFT, respectively. The relation $M_2^2 \approx N^2 \ell^{-2}$ can be seen as a typical example that the bound $M_2^2 > n\Lambda$ (discussed in Ref. [23]) is saturated. Here, $n$ is the number of species of quantum fields with mass scale $\Lambda$.

Inspired by the AdS/CFT correspondence, it has been conjectured that a black hole on a RS II brane is classically unstable [22]. Such a conjecture is based on the argument that the a five-dimensional black hole localized on the brane should be equivalent to the four-dimensional quantum-corrected black hole of the gravity theory coupled to CFT. From the CFT point of view, such a brane black hole could decay into large ($\sim N^2$) CFT degrees of freedom and its lifetime $\tau_{BH}$ should be much shorter than that of standard one: $\tau_{BH} \approx 10^{24}(M_{BH}/M_5)^3(0.1\text{mm})^2/\text{year}$ (or equivalently, $\tau_{BH} \approx 10^4(M_{BH}/M_5)^3(10^{30}\text{GeV})^2/\text{year}$ [22], where $M_{BH}$ and $M_5$ are the masses of a black hole and the Sun, respectively. If the conjecture in RS II model is true, of course this must be true in our construction (as long as the brane stays in the near horizon region). This could give a stronger bound on $\ell$. For example, in order for a black hole with solar mass $M_{BH} \approx M_5$, to survive today, $\tau_{BH}$ must be longer than the age of the Universe $\sim 10^{10}\text{year}$. This condition requires $\ell < 10^{-4}\text{mm}$ (and hence $N < 10^{27}$).

**Effective pressure equations:**

The other non-trivial components of Israel junction condition dominate the acceleration/deceleration of our brane Universe. As well as the effective energy density Eq. (15), by integrating over the KK directions, the effective four-dimensional pressure along the ordinary three-space $P$ and that along the KK directions $\tilde{Q}$ are introduced by integrating the ($(D - 1)$-dimensional pressures $p$ and q over the KK dimensions

$$P := p\frac{Q(D-5)/(D-6)\Omega_0(D-5)}{(1-a^4)(D-5)/(D-6)},$$

$$\tilde{Q} := q\frac{Q(D-5)/(D-6)\Omega_0(D-5)}{(1-a^4)(D-5)/(D-6)}, \quad (28)$$

respectively. As for the effective energy density $\Sigma$, those pressures are also decomposed into the constant and time-dependent parts in general: $P = P_0 + P_1$, $Q = Q_0 + Q_1$. Note that the constant part is given by $P_0 = Q_0 = -\Sigma_0$. By substituting the explicit expressions for $F_a/F$, $\tilde{B}_a/B$ and $C_a/C$ into Eqs. (4) and (5), we obtain
discuss the behavior of the pressure in the KK direction in the standard cosmology except for the dark radiation conditions of the form become trivial. In other words, the pressure is approximately given by AdS solutions. In the near horizon region the spacetime structure is near-extremal and the dark energy density and pressure are given by Eq. (31), D DR = Σ DR/3. This is nothing but the result in the standard cosmology except for the dark radiation type contribution in the non-extremal case.

Hereafter, we focus on the extremal case μ = 0 and discuss the behavior of the pressure in the KK direction. In the early times a ≪ 1, we find

\[ \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{1}{2M_4^2}(\Sigma_1 + P_1 + \Sigma_{\text{DR}} + P_{\text{DR}}), \]  

(30)

where the dark energy density and pressure are given by Eq. (22) P DR = Σ DR/3. This is nothing but the result in the standard cosmology except for the dark radiation type contribution in the non-extremal case. In the near horizon region the spacetime structure is near-extremal and the dark energy density and pressure are given by Eq. (31), D DR = Σ DR/3. This is nothing but the result in the standard cosmology except for the dark radiation type contribution in the non-extremal case.

Hereafter, we focus on the extremal case μ = 0 and discuss the behavior of the pressure in the KK direction. In the early times a ≪ 1, we find

\[ \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \approx \frac{1}{6M_4^2}(3P_1 - 2\Sigma_1 - 3\dot{Q}_1). \]  

(31)

Note that in the above Eq. (31), the left-hand-side is constant. Thus, in the right-hand-side, the combination 3P_1 - 2\Sigma_1 - 3\dot{Q}_1 must be totally time-independent. Also, in the above derivation, it is assumed that Σ_1/Σ_0 ≪ 1. For the realistic brane matter P_1/\Sigma_0 ≪ 1 should also be satisfied. In order for Eq. (31) to be compatible, \dot{Q}_1 should be as large as |\dot{Q}_1| ≈ Σ_0 in contrast to Σ_1 and P_1.

One might think that the junction conditions for the (D - 5)-form field might restrict the motion of the brane. But now, it is not the case. Only the non-trivial component of the (D - 5)-form field strength is F_θ^{y_1...y_{D-5}}, where all the coordinates y_i represent the dimensions of S^{(D-5)}, and the product of the form field with the normal vector n_AF_{A_1...A_{D-5}} vanishes. Thus all the junction conditions of the form become trivial. In other words, the (D - 5)-form field cannot couple with the brane matter, if it is magnetic.

Summary:

We discussed cosmology in the brane Universe in D(> 6)-dimensional bulk spacetime in the context of the Kaluza-Klein (KK) braneworld scheme, i.e., to consider KK compactifications on the brane.

We consider the D-dimensional black three-brane solutions. In the near horizon region the spacetime structure is approximately given by AdS_5 × S^{(D−5)} and in the asymptotic infinity it approaches the D-dimensional Minkowski. We consider a brane motion from the near-horizon region toward the asymptotic infinity, which induces the cosmology on the brane. We derive all the components of Israel junction condition, by assuming exact cosmological symmetry on the (KK) brane. The junction conditions for the (D - 5) form field do not restrict the brane motion. In other words, the brane matter does not couple to the (D - 5)-form field in the bulk. After integrating over the compact KK dimensions, we have derived the effective cosmological equations on the brane. We find that irrespectively of components of the matter on the brane, the brane cannot reach the asymptotic infinity of the bulk. Thus, observers on the (KK) brane never see these (D - 5) extra dimensions.

In the early times, when the KK brane is moving in the near-horizon region, the brane cosmology exactly coincides with that in the five-dimensional Randall-Sundrum (RS II) model. If the original black three-brane solution is near-extremal μ/Q ≪ 1, the dark radiation type contribution arises. It is natural since the near horizon geometry is the product of five-dimensional AdS-Schwarzschild with S^{(D−5)}, which is similar to the RS model. We also discussed experimental and observational bounds.

There would be various extensions of our considerations. As one of the possible extensions, it would be important to consider the cases without Z_2-symmetry across the brane, which are rather generic in the higher dimensions. In the present set-up, the physical size of our Universe approaches the maximal size. This would be the disadvantage of our simplest model. In order to avoid this problem and to obtain ever expanding brane Universe, it would be useful to consider the time-dependence of the extra dimensions.

Acknowledgement

The author wishes to thank the anonymous reviewer for his/her comments and suggestions to improve the paper. This work was supported in part by the Transregional Research Centre TRR33 The Dark Universe.
