Umbilicity of constant mean curvature hypersurfaces into space forms

A. C. Bezerra and F. Manfio *

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Abstract

In this paper we establish conditions on the length of the traceless part of the second fundamental form of a complete constant mean curvature hypersurface immersed in a space of constant sectional curvature in order to show that it is totally umbilical.

1 Introduction

The celebrated result by Bernstein [3] asserts that the only complete minimal graphs in $\mathbb{R}^3$ are planes. Bernstein’s theorem remains valid for complete minimal graphs in $\mathbb{R}^{n+1}$ provided that $n \leq 7$, as state the works of Fleming [12], De Giorgi [11], Almgren [2] and Simons [17]. However, the restriction on the dimension is necessary, as shown by a counterexample due Bombieri, De Giorgi and Giusti [6]. The stability of the entire minimal graphs leads us to the natural question of whether a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$, with $n \leq 7$, is a hyperplane. It was proved independently by do Carmo and Peng [7], Fischer-Colbrie and Schoen [10] that a complete stable minimal surface in $\mathbb{R}^3$ must be a plane. A generalization of this theorem for higher dimensions was obtained by do Carmo and Peng:

\textbf{Theorem 1} ( [7], [10]). Let $M^n$ be a minimal hypersurface in $\mathbb{R}^{n+1}$. Assume that $M^n$ is stable, complete and that

$$\lim_{R \to +\infty} \frac{\int_{B_p(R)} |A|^2}{R^{2q+2}} = 0, \quad q < \sqrt{\frac{2}{n}}.$$ 

Then $M^n$ is a hyperplane.

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Here, $B_p(R)$ denotes the geodesic ball of radius $R$ centered at $p \in M^n$ and $A$ is the second fundamental form of $M^n$. Some partial answers in order to generalize Theorem 1 can be found in [13], [14]. In particular, the authors discuss in [14] the concept of entropy associated to the volume of geodesic balls in a complete noncompact Riemannian manifold.

Theorem 1 has been extended to hypersurfaces with constant mean curvature by Alencar and do Carmo [1]. A crucial point is to replace the second fundamental form $A$ of the immersion by the traceless second fundamental form $\phi = -A + HI$, here $H$ denotes the mean curvature of $M^n$. More precisely,

**Theorem 2 (1).** Let $M^n$ be a complete noncompact hypersurface in $\mathbb{R}^{n+1}$, $n \leq 5$, with constant mean curvature $H$. Assume that $M$ is strongly stable and that

$$\lim_{R \to +\infty} \int_{B_p(R)} |\phi|^2 \frac{1}{R^{2q+2}} = 0, \quad q < \frac{1}{6n+1}.$$ 

Then $M^n$ is a hyperplane.

Theorem 2 were slightly improved by do Carmo and Zhou [8], by changing the integral condition in Theorem 2 to a slightly weaker condition, and improving the restriction on the dimension to $n \leq 6$. There are several other interesting results that generalize Theorem 1, including for other ambient spaces; for instance, see [4], [5], [13], [14], [15], [16], among others.

The goal of this paper is to give an improved version of Theorem 2 in order to obtain umbilical hypersurfaces in spaces of constant sectional curvature. We exchange the stability condition by a condition in the norm of the traceless second fundamental $\phi$, that makes no restrictions on the dimension of the hypersurface.

Given a complete hypersurface $M^n$ with constant mean curvature $H$ in a Riemannian manifold $M^{n+1}(c)$, with constant sectional curvature $c$, we will denote the first eigenvalue $\lambda_1(L_{\Delta + |A|^2 + H\nu}, M^n)$ of the strong stability operator of $M^n$ by $\overline{\lambda}_1$ (see Section 2 for details). In our first main result we will consider the polynomial

$$P_H(x) = x^2 + \theta H x - \beta,$$  \hspace{1cm} (1.1)

where the constants $\theta$ and $\beta$ are defined as

$$\theta := \frac{n(n-2)}{\sqrt{n(n-1)}} \quad \text{and} \quad \beta := n(H^2 + c).$$  \hspace{1cm} (1.2)

The polynomial $P_H$, which depends on $n$, $H$ and $c$, will help us to understand the relationship between the norm of the traceless second fundamental $\phi$ and the positive root of $P_H$, giving us conditions for $M^n$, in the case of constant mean curvature, to be umbilical. More precisely,
Theorem 1.1. Let $M^n$ be a complete hypersurface with constant mean curvature $H$, immersed in a Riemannian manifold $\mathbb{M}^{n+1}(c)$ of constant sectional curvature $c$, with $H^2 + c > 0$ if $c < 0$. Assume that

$$\lim_{R \to +\infty} \frac{\int_{B_p(R)} |\phi|^{2q+2}}{R^2} = 0, \quad q > 0,$$

(1.3)

and

$$\sup_{x \in M} |\phi| < r_H,$$

(1.4)

where $r_H$ is a positive root of the polynomial $P_H$ defined in (1.1). Then $M$ is totally umbilical.

Theorem 1.1 generalizes [4, Theorem 1.2] for the case of umbilical hypersurfaces with constant mean curvature. In fact, in [4], the authors obtain a rigidity result for totally geodesic minimal hypersurfaces in the hyperbolic space $\mathbb{H}^n$, with similar conditions to (1.3) and (1.4).

In our second main result we obtain a condition that relates the norm of the traceless second fundamental form $\phi$ with the first eigenvalue $\lambda_1$ of the strongly stability operator, in order to obtain the umbilicity of the hypersurface. Furthermore, we replace the condition stability for $M^n$ in Theorem 2 for a weaker condition (see (1.7)) in the first eigenvalue of the strongly stability operator. For that, we need to define the polynomial

$$P_{H,\lambda_1}(x) := -x^2 - \theta H x + \left(1 + \frac{1}{q}\right) \beta + \frac{1}{q} \lambda_1,$$

(1.5)

where $q$ is a positive constant.

Theorem 1.2. Let $M^n$ be a complete hypersurface with constant mean curvature $H$ immersed in a Riemannian manifold $\mathbb{M}^{n+1}(c)$ of constant sectional curvature $c$. Suppose that

$$\lim_{R \to +\infty} \frac{\int_{B_p(R)} |\phi|^2}{R^{2q+2}} = 0, \quad \text{with} \quad \frac{-q^2 + q + 1}{q} < \frac{2}{n},$$

(1.6)

If the first eigenvalue of the strongly stability operator satisfies

$$\lambda_1 > -n(q + 1)(H^2 + c)$$

(1.7)

and

$$\sup_{x \in M} |\phi| < r_H,$$

(1.8)

where $r_H$ is a positive root of $P_{H,\lambda_1}$ defined in (1.5), then $M^n$ is totally umbilical.
Remark 1.3. In Theorem 1.2, when \( n \leq 5 \) and \( M^n \) is a complete noncompact hypersurface immersed in a form space \( Q^{n+1}_c \), condition \( H^2 + c > 0 \) together with equation (1.7) imply that the first eigenvalue \( \lambda_1 \) must satisfy \( \lambda_1 < 0 \), that is, \( M^n \) must be unstable. In fact, in this case, it follows from [13, Corollary 6.3] that there is no complete noncompact stable hypersurface with constant mean curvature \( H \neq 0 \) in \( \mathbb{R}^{n+1} \) or \( S^{n+1} \), or in \( H^{n+1} \), when \( H^2 > g(n) \), where

\[
g(n) = \frac{(n+2)^2}{n^2} - \frac{1}{4(n-1)}.
\]

In the case that \( M^n \) is a minimal hypersurface of a Riemannian manifold \( M^{n+1}(c) \) with constant sectional curvature \( c \), the integral condition (1.6) becomes the same as in Theorem 1. Thus, when \( M^{n+1}(c) = \mathbb{R}^{n+1} \), condition (1.7) means that \( M \) is stable, and we conclude that \( M^n \) is a hyperplane. In this case, we recover the do Carmo-Peng’s Theorem.

Corollary 1.4. Under the conditions of Theorem 1.2, if \( M^n \) is a minimal hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \), then \( M^n \) is a hyperplane.

A natural question is whether condition (1.8) in Theorem 1.2 can be removed, since it does not appear in Theorems 1 and 2. In fact, this is the case for low dimensions.

Theorem 1.5. Let \( M^n \) be a complete hypersurface, \( n = 2, 3 \), with constant mean curvature \( H \), immersed in a Riemannian manifold \( M^{n+1}(c) \) of constant sectional curvature \( c \). Suppose that

\[
\lim_{R \to +\infty} \frac{\int_{B_R(R)} |\phi|^{2q+2}}{R^2} = 0, \quad q < \frac{1}{3} \left( \sqrt{\frac{2(6-n)}{n}} - 1 \right). \tag{1.9}
\]

If

\[
\bar{\lambda}_1 > n \left[ \frac{(q + 1)^2}{2(n + 2nq + 2)} - 2n(H^2 + c) - (H^2 + c) \right], \tag{1.10}
\]

then \( M^n \) is totally umbilical.

The paper is organized as follows. In the next section we recall some basic analytic tools of Riemannian manifolds. In particular, we recall the first eigenvalue of the Laplacian operator and the notion of stability to hypersurfaces with constant mean curvature. In Section 3 we establish some important inequalities that will be used throughout the paper. Finally, in the last section we prove the results established in Section 1.
2 Preliminaries

Given a complete Riemannian manifold $M^n$ and a smooth function $\beta : M \to \mathbb{R}$, we denote by $\Delta$ and $\lambda_1(L_\beta, M)$ the Laplacian operator acting on the space $C^\infty(M)$ and the first eigenvalue of the operator $L_\beta = \Delta + \beta$, respectively. More precisely, $\lambda_1(L_\beta, M)$ is defined by

$$
\lambda_1(L_\beta, M) = \inf_{f \in C^\infty_0(M), f \neq 0} \frac{\int_M (|\nabla f|^2 - \beta f^2)}{\int_M f^2}.
$$

(2.1)

Note that, when $\beta = 0$, $\lambda_1(L_0, M)$ recover the usual first eigenvalue of $M$.

When $M^n$ is an oriented complete hypersurface of a Riemannian manifold $M^{n+1}$, with a unit normal vector field $\nu$, let $A$ be the shape operator of $M^n$ with respect to $\nu$ given by

$$AX = \nabla_X \nu,$$

for all tangent vector field $X$, where $\nabla$ denotes the Levi-Civita connection of $M^{n+1}$. Fixed a point $p \in M$, the principal curvatures $k_1 \leq k_2 \leq \ldots \leq k_n$ of $M$ at $p$ with respect to $\nu(p)$ are defined as the eigenvalues of $A(p)$. The mean curvature $H$ of $M^n$ at $p$ is defined by

$$H := \frac{1}{n} \sum_{i=1}^n k_i,$$

and the square of the second fundamental form of $M^n$ is given by

$$|A|^2 := \sum_{i=1}^n k_i^2.$$

Moreover, we will denote by $\overline{\text{Ric}}(\nu)$ the Ricci curvature of $M^{n+1}$ along $\nu$.

Let $L$ be the second order differential operator on $M$ given by

$$L = \Delta + |A|^2 + \overline{\text{Ric}}(\nu).$$

Associated to the operator $L$, we define a quadratic form on the functions $f \in C^\infty(M)$ that have support on a compact domain $K \subset M$ by

$$I(f) = -\int_M fLf.$$

For each such compact domain $K$, we define the index $\text{ind}_L K$ of $L$ on $K$ as the maximal dimension of a subspace where $I$ is negative definite. The index $\text{ind}_M$ of $L$ in $M$ is the number defined by

$$\text{ind}_M = \sup_{K \subset M} \text{ind}_L K,$$
where the supremum is taken over all compact domains $K \subset M$. $M$ is said to be stable if its index is null, that is, $\text{ind} M = 0$.

When $M$ is minimal, equivalently we say that $M$ is stable if for every piecewise smooth functions $f : M \to \mathbb{R}$ with compact support, we have

$$
\int_M |\nabla f|^2 - \int_M (|A|^2 + \text{Ric}(\nu)) f^2 \geq 0,
$$

(2.2)

where $|\nabla f|$ denotes the gradient of $f$ in the induced metric.

The notion of stability has been extended to hypersurfaces with constant mean curvature. More precisely, we say that a constant mean curvature hypersurface $M$ is strongly stable if (2.2) holds for all piecewise smooth functions $f : M \to \mathbb{R}$ with compact support; $M$ is said to be weakly stable if (2.2) holds for all piecewise smooth functions $f : M \to \mathbb{R}$ with compact support and $\int_M f = 0$.

3 Some inequalities

In this section we will establish some inequalities that will be used throughout the paper. In what follows, we assume that $M^n$ is an oriented complete hypersurface with constant mean curvature $H$, immersed in a Riemannian manifold $M^{n+1}(c)$ of constant sectional curvature $c$. In order to study such hypersurfaces, is more convenient to modify the second fundamental form and to introduce a new linear operator $\phi : T_pM \to T_pM$ given by

$$
\langle \phi X, Y \rangle = -\langle AX, Y \rangle + H\langle X, Y \rangle.
$$

Note that $\phi$ can also be diagonalized as $\phi e_i = \mu_i e_i$ and $\text{tr} \phi = 0$. Moreover,

$$
|\phi|^2 := \sum_{i=1}^{n} \mu_i^2 = \frac{1}{2n} \sum_{i,j=1}^{n} (k_i - k_j)^2.
$$

Thus $|\phi|^2$ measures how far $M$ is from being totally umbilical.

Associated to the operator $\phi$, Ilias, Nelli and Soret obtained in [13, Corollary 3.1] the following inequality:

$$
|\phi| \Delta |\phi| + |\phi|^4 + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi|^3 - n(H^2 + c)|\phi|^2 \geq \frac{2}{n} |\nabla|\phi||^2.
$$

(3.1)

With the constants $\theta$ and $\beta$ defined in (1.2), the inequality (3.1) becomes

$$
|\phi| \Delta |\phi| \geq \frac{2}{n} |\nabla|\phi||^2 - |\phi|^4 - \theta H|\phi|^3 + \beta|\phi|^2.
$$

(3.2)
Let $q$ be a positive constant. Since that $\Delta |\phi|^q = \text{div}(\nabla |\phi|^q)$, we have

$$|\phi|^q \Delta |\phi|^q = \left(1 - \frac{1}{q}\right) |\nabla |\phi|^q|^2 + q|\phi|^{2(q-1)}|\phi| \Delta |\phi|.$$  \hfill (3.3)

Now, replacing (3.2) in (3.3), we get

$$\left(1 + \frac{2 - n}{nq}\right) |\nabla |\phi|^q|^2 \leq q \left(|\phi|^2 + \theta H |\phi| - \beta\right) |\phi|^{2q} + |\phi|^q \Delta |\phi|^q.$$  \hfill (3.4)

Since that $f \in C^\infty_0(M)$, multiplying the inequality (3.4) by $f^2|\phi|^2$ and integrating over $M$, we get

$$\int_M |\nabla |\phi|^q|^2 f^2|\phi|^2 \leq q \int_M \left(|\phi|^2 + \theta H |\phi| - \beta\right) f^2|\phi|^{2q} + \int_M f^2|\phi|^{q+2} \Delta |\phi|^q.$$  \hfill (3.5)

By divergence theorem, we have

$$0 = \int_M \text{div}(f^2|\phi|^{q+2} \nabla |\phi|^q) = \int_M 2f|\phi|^{q+2} \langle \nabla f, \nabla |\phi|^q \rangle + \int_M (q + 2)f^2|\phi|^{q+1} \langle \nabla |\phi|, \nabla |\phi|^q \rangle + \int_M f^2|\phi|^{q+2} \Delta |\phi|^q.$$  \hfill (3.6)

By Cauchy-Schwarz inequality, and Young inequality with $\epsilon$, we can find a constant $\epsilon > 0$ such that

$$\left(\frac{n(2q + 1) - 2}{nq} - 2\epsilon\right) \int_M |\nabla |\phi|^q|^2 f^2|\phi|^2 \leq \frac{1}{2\epsilon} \int_M |\phi|^{2q+2} |\nabla f|^2 + q \int_M \left(|\phi|^2 + \theta H |\phi| - \beta\right) f^2|\phi|^{2q+2}.$$  \hfill (3.7)

On the other hand, from the definition of the stability operator $L_{|A|^2 + nc}$ defined over $M$, we have the following characterization for the first eigenvalue $\lambda_1 := \lambda_1(L_{|A|^2 + nc}, M^n)$:

$$\int_M |A|^2 f^2 + nc \int_M f^2 + \lambda_1 \int_M f^2 \leq \int_M |\nabla f|^2.$$  \hfill (3.8)
Since \( |A|^2 = |\phi|^2 + nH^2 \), we get
\[
\int_M |\phi|^2 f^2 + \beta \int_M f^2 + \lambda_1 \int_M f^2 \leq \int_M |\nabla f|^2. \tag{3.7}
\]
Replacing \( f \in C^\infty_0(M) \) by \( f = f|\phi|^{q+1} \) in \((3.7)\), we have
\[
\int_M |\phi|^{2q+4} f^2 + (\beta + \lambda_1) \int_M |\phi|^{2q+2} f^2 \leq \frac{(q+1)^2}{q^2} \int_M f^2 |\nabla|\phi|^q|\phi|^2 \\
+ 2(q+1) \int_M f|\phi|^{2q+1} (\nabla f, \nabla \phi) + \int_M |\phi|^{2q+2} |\nabla f|^2. \tag{3.8}
\]
Again, making use of the Cauchy-Schwarz inequality, and Young inequality with \( \epsilon \), we get
\[
2 \int_M f|\phi|^{2q+1} (\nabla f, \nabla \phi) \leq 2 \left( \epsilon \int_M f^2 |\phi|^{2q+1} |\nabla |\phi||^2 + \frac{1}{4\epsilon} \int_M |\phi|^{2q+2} |\nabla f|^2 \right). 
\]
Therefore, we obtain the following inequality:
\[
\int_M |\phi|^{2q+4} f^2 + (\beta + \lambda_1) \int_M |\phi|^{2q+2} f^2 \leq \left( 1 + \frac{1}{q} \right)^2 + \frac{\epsilon(q+1)}{q^2} \int_M f^2 |\nabla|\phi|^q|\phi|^2 \\
+ \left( 1 + \frac{q+1}{\epsilon} \right) \int_M |\phi|^{2q+2} |\nabla f|^2. \tag{3.9}
\]

4 Proof of Theorems

**Proof of Theorem 1.1.** By equation (3.6), we have
\[
\left( \frac{n(2q+1) - 2}{nq} - 2\epsilon \right) \int_M |\nabla|\phi|^q f^2 |\phi|^2 \\
\leq q \int_M (|\phi|^2 + \theta H|\phi| - \beta) f^2 |\phi|^{2q+2} + \frac{1}{2\epsilon} \int_M |\phi|^{2q+2} |\nabla f|^2.
\]
Since the polynomial \( P_H(x) = x^2 + \theta Hx - \beta \) has two roots \( r_0 < 0 < r_1 \), and if \( \sup_{x \in M} |\phi| < r_1 \), we have
\[
|\phi|^2 + \theta H|\phi| - \beta < 0.
\]
Thus, we get
\[
\left( \frac{n(2q+1) - 2}{nq} - 2\epsilon \right) \int_M |\nabla|\phi|^q f^2 |\phi|^2 \leq \frac{1}{2\epsilon} \int_M |\phi|^{2q+2} |\nabla f|^2. \tag{4.1}
\]
Let \( f \) be a nonnegative smooth function defined on the interval \([0, +\infty)\) such that \( f \equiv 1 \) in \([0, R]\), \( f \equiv 0 \) in \([2R, +\infty)\), and \( |f'| \leq \frac{2}{R} \). Consider the
composition $f \circ r$, where $r$ is the distance function from the point $p$. It follows from (4.1) that we can choose a constant $\epsilon > 0$ such that
\[
\int_{B_p(R)} |\nabla|^{q}|\phi|^2 \leq \frac{C}{R^2} \int_{B_p(2R)} |\phi|^{2q+2},
\]
(4.2)
where $C$ is a positive constant that depends only on $n$, $q$ and $\epsilon$. Taking the limit $R \to +\infty$, it follows from (1.3) that
\[
\int_M |\nabla|^{q}|\phi|^2 = 0,
\]
(4.3)
and $|\phi|$ equal to a constant $k$ along $M$. Suppose $k > 0$. Since $\sup |\phi| < r_1$ and using inequality (3.9), we have
\[
0 \leq q \int_M \left( |\phi|^2 + \theta H |\phi| - \beta \right) f^2 |\phi|^{2q+2} \leq \frac{1}{2\epsilon} \int_M |\phi|^{2q+2} |\nabla f|^2.
\]
Taking again the function $f \circ r$, defined above, and applying the hypothesis (1.3), it follows that
\[
\int_M \left( |\phi|^2 + \theta H |\phi| - \beta \right) |\phi|^{2q+2} = 0.
\]
This shows that $|\phi|^2 + \theta H |\phi| - \beta = 0$, which is a contradiction. Therefore, we have $|\phi| \equiv 0$ and the conclusion follows.

Proof of Theorem 1.2. From condition (1.6), we have
\[
-q^2 + q + 1 < \frac{2}{n}
\]
Thus, we can find a constant $\epsilon > 0$ such that
\[
\left( 1 + \frac{1}{q} \right)^2 + \frac{\epsilon(q + 1)}{q^2} < 1 + \frac{2 - n + n(q + 2)}{nq} - 2\epsilon.
\]
In this case, joining the inequalities (3.6) and (3.9), we obtain the following inequality:
\[
\frac{1}{q} \int_M |\phi|^{2q+4} f^2 + \int_M \left( -|\phi|^2 - \theta H |\phi| + \left( \frac{q + 1}{q} \right) \beta + \frac{1}{q} \lambda_1 \right) f^2 |\phi|^{2q+2} \leq \frac{1}{q} \left( 1 + \frac{2(q + 1) + 1}{2\epsilon} \right) \int_M |\phi|^{2q+2} |\nabla f|^2.
\]
(4.4)
Considering the polynomial $P_{H,\lambda_1}$ defined in (1.5), we have:
\[
P_{H,\lambda_1}(x) = -x^2 - \theta H x + \left( \frac{q + 1}{q} \right) \beta + \frac{1}{q} \lambda_1
\]
\[
= -x^2 - \theta H x + \left( \frac{q + 1}{q} \right) n(H^2 + c) + \frac{1}{q} \lambda_1.
\]
Thus, in terms of $P_{H, \lambda}$, we can rewrite (4.4) as

$$
\frac{1}{q} \int_M |f|^{2q+4} f^2 + \int_M P_{H, \lambda}(|\phi|) f^2 |\phi|^{2q+2} \leq C \int_M ||\phi|^{2q+2} |\nabla f|^2, \quad (4.5)
$$

with $C = C(q, \varepsilon)$. Since (1.7) is satisfied, we can see that the polynomial $P_{H, \lambda}(|\phi|)$ has two roots $r_0 < 0 < r_1$. If $\sup |\phi| < r_1$, then $P_{H, \lambda} > 0$, and we get the following inequality:

$$
\int_M |\phi|^{2q+4} f^2 \leq C \int_M |\phi|^{2q+2} |\nabla f|^2, \quad \forall f \in C_0^\infty(M). \quad (4.6)
$$

We can replace $f$ by $f^{q+1}$ in (4.6) in order to get

$$
\int_M |\phi|^{2(q+2)} f^{2(q+1)} \leq C \int_M |\phi|^{2(q+1)} |\nabla f^{q+1}|^2
$$

$$
= C(q + 1)^2 \int_M |\phi|^{\frac{2(q+2)}{q+q+2}} |\phi|^{\frac{2}{q+1}} f^{2q} |\nabla f|^2.
$$

Let us denote $C_1 = C(q + 1)^2$. It follows from Hölder inequality that

$$
\int_M |\phi|^{2(q+2)} f^{2(q+1)} \leq C_1 \left[ \int_M \left( |\phi|^{\frac{2(q+2)}{q+q+2}} f^{2q} \right)^{\frac{q+1}{q}} \right]^{\frac{q}{q+1}} \left[ \int_M \left( |\phi|^{\frac{2}{q+1}} |\nabla f|^{2(q+1)} \right)^{\frac{q+1}{q+1}} \right]^{\frac{q}{q+1}},
$$

that is

$$(\int_M |\phi|^{2(q+2)} f^{2(q+1)})^{\frac{q}{q+1}} \leq C_1 \left( \int_M |\phi|^{2} |\nabla f|^{2(q+1)} \right)^{\frac{q}{q+1}}.
$$

Proceeding as in the proof of Theorem 1.1 we have

$$
\int_{B_p(R)} |\phi|^{2(q+2)} \leq \frac{C_1}{R^{2q+2}} \int_{B_p(2R)} |\phi|^2.
$$

Finally, from (1.6), we conclude that $|\phi| \equiv 0$, and the conclusion of Theorem 1.2 follows.

Proof of Corollary 1.4. Since $M^n$ is a minimal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, we have

$$
\beta = n(H^2 + c) = 0 \quad \text{and} \quad |\phi|^2 = |A|^2.
$$
Thus, inequality (4.4) implies that
\[
\int_M |A|^{2q+4} f^2 \leq \int_M |A|^{2q+2} f^2 (q|A|^2 - \lambda_1) + \left(1 + \frac{2(q + 1) + 1}{2}\right) \int_M |A|^{2q+2}|\nabla f|^2.
\]

From condition (1.8) and \(|A|^2 \leq \frac{\lambda_1}{q}\), we obtain
\[
\int_M |A|^{2q+4} f^2 \leq C \int_M |A|^{2q+2}|\nabla f|^2.
\]

The proof now proceeds exactly as in the final part of the proof of Theorem 1.2.

Proof of Theorem 1.5. We will apply the Young inequality to a certain term in the inequality (3.6). In fact, substituting the inequality
\[
q \int_M \theta H f^2 |\phi|^{2q+3} \leq q \left(\frac{1}{2} \int_M f^2 |\phi|^{2q+4} + \frac{1}{2} \int_M \theta^2 H^2 f^2 |\phi|^{2q+2}\right)
\]
in (3.6), we have
\[
\left(1 + \frac{2 - n + n(q + 2)}{nq} - \epsilon\right) \int_M |\nabla|\phi|^q f^2 |\phi|^2
\]
\[
\leq q \int_M f^2 |\phi|^{2q+4} + \frac{q}{2} \int_M f^2 |\phi|^{2q+4} + \frac{q}{2} \int_M \theta^2 H^2 f^2 |\phi|^{2q+2}
\]
\[
- q\beta \int_M |\phi|^{2q+2} f^2 + \frac{1}{\epsilon} \int_M |\phi|^{2q+2}|\nabla f|^2.
\]

Multiplying the inequality (4.7) by
\[
\left(1 + \frac{1}{q}\right)^2 + \frac{\epsilon(q + 1)}{q^2},
\]
and multiplying the inequality (3.9) by
\[
1 + \frac{2 - n + n(q + 2)}{nq} - \epsilon,
\]

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and joining both, we get
\[
\left(1 + \frac{2 - n + n(q + 2)}{nq} - \varepsilon\right) \int_M |\phi|^{2q+4} f^2 \\
+ \left(1 + \frac{2 - n + n(q + 2)}{nq} - \varepsilon\right) (\lambda_1 + \beta) \int_M |\phi|^{2q+2} f^2 \\
\leq \left(\left(1 + \frac{1}{q}\right)^2 + \frac{\varepsilon(q + 1)}{q^2}\right) \int_M f^2 |\phi|^{2q+4} \\
+ \left(\left(1 + \frac{1}{q}\right)^2 + \frac{\varepsilon(q + 1)}{q^2}\right) \frac{q}{2} \int_M f^2 |\phi|^{2q+2} \\
+ \frac{q}{2} \left(\left(1 + \frac{1}{q}\right)^2 + \frac{\varepsilon(q + 1)}{q^2}\right) \int_M \theta^2 H^2 f^2 |\phi|^{2q+2} \\
- q\beta \left(\left(1 + \frac{1}{q}\right)^2 + \frac{\varepsilon(q + 1)}{q^2}\right) \int_M |\phi|^{2q+2} f^2 \\
+ \left(1 + \frac{2}{\varepsilon}\right) \int_M |\phi|^{2q+2} |\nabla f|^2.
\]
That is,
\[
\left[\left(1 + \frac{2 - n + n(q + 2)}{nq} - \varepsilon\right) - \left(1 + \frac{1}{q}\right)^2 + \frac{\varepsilon(q + 1)}{q^2}\right] \int_M |\phi|^{2q+4} f^2 + \left[\left(1 + \frac{2 - n + n(q + 2)}{nq} - \varepsilon\right) (\lambda_1 + \beta)\right] \int_M |\phi|^{2q+2} f^2 \\
\leq \left(1 + \frac{2}{\varepsilon}\right) \int_M |\phi|^{2q+2} |\nabla f|^2.
\]
Since \(n = 2\) or \(n = 3\), condition (1.9) in \(q\) implies that
\[
\left(1 + \frac{2 - n + n(q + 2)}{nq}\right) - \frac{3}{2} q \left(1 + \frac{1}{q}\right)^2 > 0.
\]
Therefore, we can choose a constant \(\epsilon_1 > 0\) so that, in an \(\epsilon\)-neighborhood, we get
\[
\left(1 + \frac{2 - n + n(q + 2)}{nq} - \varepsilon_1\right) - \left(1 + \frac{1}{q}\right)^2 + \frac{\varepsilon_1(q + 1)}{q^2}\right) (q + \frac{q}{2}) > 0.
\]
On the other hand, from (1.10), we have
\[
\lambda_1 > \frac{(q + 1)^2 n}{2(n + 2nq + 2)} (1 - 2n(H^2 + c)) - n(H^2 + c).
\]
Thus, again we can choose a constant $\epsilon_2 > 0$ such that
\[
\left(1 + \frac{2 - n + n(q + 2)}{nq} - \epsilon_2\right) (\lambda_1 + \beta)
+ \left(q\beta - \frac{q}{2}\right) \left(\left(1 + \frac{1}{q}\right)^2 + \frac{\epsilon_2(q + 1)}{q^2}\right) > 0.
\]
Choosing $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we obtain
\[
\int_M |\phi|^{2q+4} f^2 + \int_M |\phi|^{2q+2} f^2 \leq C \int_M |\phi|^{2q+2} |\nabla f|^2,
\]
where $C$ is a positive constant that depends only on $n$, $H$, $q$ and $\epsilon$. The rest of the proof follows as in the end of previous results. \qed

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Instituto Federal Goiano, Campus Trindade, Brazil
E-mail address: adriano.bezerra@ifgoiano.edu.br

Universidade de São Paulo, São Carlos, Brazil
E-mail address: manfio@icmc.usp.br