Effect algebras as presheaves on finite Boolean algebras

Gejza Jenča

Abstract For an effect algebra $A$, we examine the category of all morphisms from finite Boolean algebras into $A$. This category can be described as a category of elements of a presheaf $R(A)$ on the category of finite Boolean algebras. We prove that some properties (being an orthoalgebra, the Riesz decomposition property, being a Boolean algebra) of an effect algebra $A$ can be characterized in terms of some properties of the category of elements of the presheaf $R(A)$. We prove that the tensor product of effect algebras arises as a left Kan extension of the free product of finite Boolean algebras along the inclusion functor. The tensor product of effect algebras can be expressed by means of the Day convolution of presheaves on finite Boolean algebras.

Acknowledgements This research is supported by grants VEGA 2/0069/16, 1/0420/15, Slovakia and by the Slovak Research and Development Agency under the contracts APVV-14-0013, APVV-16-0073.

Keywords effect algebra, tensor product, presheaf

1 Introduction

In their 1994 paper [11], D.J. Foulis and M.K. Bennett defined effect algebras as (at that point in time) the most general version of quantum logics. Their motivating example was the set of all Hilbert space effects, a notion that plays an important role in quantum mechanics [21,11]. An equivalent definition in terms of the difference operation was independently given by F. Köpka and F. Chovanec in [19]. Later it turned out that both groups of authors rediscovered the definition given already in 1989 by R. Giuntini and H. Greuling in [13].
By the very definition, the class of effect algebras includes orthoalgebras \cite{12}, which include orthomodular posets and orthomodular lattices. It soon turned out \cite{6} that there is another interesting subclass of effect algebras, namely MV-algebras defined by C.C. Chang in 1958 \cite{5} to give the algebraic semantics of the Łukasiewicz logic. Furthermore, K. Ravindran in his thesis \cite{25} proved that a certain subclass of effect algebras (effect algebras with the Riesz decomposition property) is equivalent with the class of partially ordered abelian groups with interpolation \cite{14}. This result generalizes the equivalence of MV-algebras and lattice ordered abelian groups described by D. Mundici in \cite{23}.

In the present paper, we study effect algebras from the viewpoint of category theory. There are two papers that inspired and motivated the results presented here.

In their paper \cite{16} Jacobs and Mandemaker utilized the notion of a coreflective subcategory to prove important results about effect algebras and their generalized versions. In particular, they proved that the category of effect algebras $\mathbf{EA}$ is cocomplete and that $\mathbf{EA}$, when equipped with the tensor product of effect algebras \cite{8}, is a symmetric monoidal category.

In \cite{27}, Staton and Uijlen proved that every effect algebra $A$ can be faithfully represented by a presheaf $R(A)$ on the category of finite Boolean algebras. This representation is the main tool we shall use in this paper.

After preliminaries, we prove that several properties (being an orthoalgebra, having the Riesz decomposition property, being a Boolean algebra) of an effect algebra $A$ can be characterized by properties of the category of elements of the representing presheaf $R(A) : \mathbf{FinBool}^{op} \to \mathbf{Set}$. We use the presheaf representations of effect algebras to prove that the tensor product of effect algebras arises as a left Kan extension of the free product of finite Boolean algebras along the square of the inclusion functor of the category of finite Boolean algebras into the category of effect algebras. As a consequence, the tensor product of effect algebras can be expressed by means of the Day convolution of presheaves on finite Boolean algebras.

These results mean that the tensor product of effect algebras comes from the free product of finite Boolean algebras. This could be interpreted as an additional justification of the naturality of the tensor product construction in algebraic quantum logics.

2 Preliminaries

We assume familiarity with basics of category theory, see \cite{20,26} for reference. For effect algebras and related topics, see \cite{9}.

2.1 Effect algebras

An effect algebra is a partial algebra $(A; +, 0, 1)$ with a binary partial operation $+$ and two nullary operations $0, 1$ satisfying the following conditions.
(E1) If $a + b$ is defined, then $b + a$ is defined and $a + b = b + a$.

(E2) If $a + b$ and $(a + b) + c$ are defined, then $b + c$ and $a + (b + c)$ are defined and $(a + b) + c = a + (b + c)$.

(E3) For every $a \in A$ there is a unique $a^\perp \in A$ such that $a + a^\perp$ exists and $a + a^\perp = 1$.

(E4) If $a + 1$ is defined, then $a = 0$.

Effect algebras were introduced by Foulis and Bennett in their paper [11]. In [19], Kôpka and Chovanec introduced an essentially equivalent structure called $D$-poset. Another equivalent structure was introduced by Giuntini and Greuling in [13].

The original definition of an effect algebra [11][13] excluded the case of a one-element effect algebra; it was required that $0 \neq 1$. This has some undesirable consequences: for example a total relation on an effect algebra is not a congruence in the sense of [15] and the category of effect algebras lacks the terminal object. On the other hand, the definition of a $D$-poset in [19] allows for one-element $D$-posets. In the present paper, we do not assume that $0 \neq 1$ in an effect algebra.

In an effect algebra $A$, we write $a \leq b$ if and only if there is $c \in A$ such that $a + c = b$. It is easy to check that for every effect algebra $A$, $\leq$ is a partial order on $A$. In this partial order, $0$ is the smallest and $1$ is the greatest element of the poset $(A, \leq)$, so every effect algebra has an underlying bounded poset.

The partial operation $+$ is cancellative. Therefore, on every effect algebra it is possible to introduce a new partial operation $-$: $b - a$ is defined if and only if $a \leq b$ and then $a + (b - a) = b$. It can be proved that, in an effect algebra, $a + b$ is defined if and only if $a \leq b^\perp$ if and only if $b \leq a^\perp$. In an effect algebra, we write $a \perp b$ if and only if $a + b$ is defined and we say that $a$ and $b$ are orthogonal.

Let $A_1, A_2$ be effect algebras. A map $f: A_1 \to A_2$ is called a morphism of effect algebras if and only if it satisfies the following conditions.

1. $f(1) = 1$.
2. If $a \perp b$, then $f(a) \perp f(b)$ and $f(a + b) = f(a) + f(b)$.

Every morphism of effect algebras is an isotone map of the underlying bounded posets.

A subalgebra of an effect algebra $A$ is a subset $B \subseteq A$ such that $1 \in B$ and, for all $x, y \in B$ with $x \geq y$, $x - y \in B$. Since $x^\perp = 1 - y$ and $x + y = (x^\perp - y)^\perp$, every subalgebra is closed with respect to $+$ and $\perp$.

The category of effect algebras is denoted by $\text{EA}$. The category $\text{EA}$ is complete and cocomplete. The proof of the fact that the category of effect algebras is cocomplete is nontrivial, see [16] for the proof. Let us point out a surprising fact the regular epimorphisms in $\text{EA}$ are not necessary surjective, see [16, Section 5.2] for an example. This shows that the forgetful functor $\text{EA} \to \text{Set}$ that takes the effect algebra to its underlying set is not monadic, although it is a right adjoint. On the other hand, as proved in [17], the forgetful functor that takes an effect algebra to its underlying bounded poset is a monadic functor from $\text{EA}$ to the category of bounded posets.
2.2 Classes of effect algebras, examples

The class of effect algebras is a common generalization of several types of algebraic structures.

An effect algebra \( A \) is an orthoalgebra \([12]\) if, for all \( a \in A \), \( a \perp a \) implies \( a = 0 \). Orthomodular lattices \([18,2]\) can be characterized as lattice ordered orthoalgebras.

**Example 1** Let \( \mathcal{H} \) be a Hilbert space. The set of all orthogonal projections \( P(\mathcal{H}) \) on \( \mathcal{H} \) is an orthomodular lattice \([24]\), hence it is an orthoalgebra.

One can construct examples of effect algebras from an arbitrary partially ordered abelian group \((G, +, 0, \leq)\) in the following way. Choose any positive \( u \in G \); then, for \( 0 \leq a, b \leq u \), define \( a \oplus b \) if and only if \( a + b \leq u \) and put \( a \oplus b = a + b \). With such partial operation \( \oplus \), the interval \([0, u]_G = \{ x \in G : 0 \leq x \leq u \}\) becomes an effect algebra \(([0, u]_G, \oplus, 0, u)\). Effect algebras that arise from partially ordered abelian groups in this way are called interval effect algebras, see \([1]\).

**Example 2** The closed real interval \([0, 1]_\mathbb{R}\) is an interval effect algebra.

**Example 3** Let \( \mathcal{H} \) be a Hilbert space. Let \( S(\mathcal{H}) \) be the set of all bounded self-adjoint operators on \( \mathcal{H} \). For \( A, B \in S(\mathcal{H}) \), write \( A \leq B \) if and only if \( B - A \) has a nonnegative spectrum. Then \( S(\mathcal{H}) \) is a partially ordered abelian group. The interval \( E(\mathcal{H}) = [0, I]_{S(\mathcal{H})} \), where \( I \) is the identity operator, is an interval effect algebra, called the standard effect algebra.

2.3 Finite summable families

Let \( A \) be an effect algebra. For a finite set \( I \), an \((I\text{-indexed})\) summable family of elements of \( A \) is a family \((a_i)_{i \in I}\) such that the sum \( \sum_{i \in I} a_i \) exists in \( A \). A finite summable family \((a_i)_{i \in I}\) with \( \sum_{i \in I} a_i = 1 \) is called a finite decomposition of unit.

We say that a summable family \((b_j)_{j \in J}\) is a refinement of a summable family \((a_i)_{i \in I}\) if there is a surjective mapping \( \rho : J \to I \) such that, for all \( i \in I \),

\[
    a_i = \sum_{\rho(j) = i} b_j.
\]

It is easy to see that if \((b_j)_{j \in J}\) is a refinement of \((a_i)_{i \in I}\), then \( \sum_{i \in I} a_i = \sum_{j \in J} b_j \).
2.4 Boolean algebras, observables

Every Boolean algebra \((A, \vee, \wedge, \bot, 0, 1)\) is an effect algebra. The partial operation \(+\) on \(A\) is given by the rule \(x \perp y\) if and only if \(x \land y = 0\) and then \(x + y = x \lor y\). Clearly, this is an effect algebra with the same partial order as the original Boolean algebra. This shows that the category of Boolean algebras \(\text{Bool}\) is a subcategory of the category of effect algebras \(\text{EA}\). Moreover, \(\text{Bool}\) is a full subcategory of \(\text{EA}\). Therefore, we can (and we will) identify Boolean algebras with their respective effect-algebraic versions.

If \(X\) is a Boolean algebra and \(A\) is an effect algebra, then a morphism \(g: X \to A\) is called an \((A\text{-valued})\) observable. In general, the range of an observable \(g: X \to A\) is not a sub-effect algebra of \(A\). However, if \(A\) is an orthoalgebra then the range of \(g\) is a sub-effect algebra of \(A\). Moreover, \(g(X)\) is then a Boolean algebra.

2.5 A notation for finite observables

In what follows, we abbreviate the initial segment of natural numbers \(\{1, \ldots, n\}\) by \([n]\). Note that \([0]\) = \(\emptyset\).

An observable from a finite Boolean algebra to an effect algebra is called a \textit{finite observable}. If \(A\) is an effect algebra and \(g: 2^{[n]} \to A\) is a finite \(A\)-valued observable, then it is obvious that \(g\) is determined by its values on singleton subsets of \([n]\). Indeed, every \(X = \{x_1, \ldots, x_k\} \in 2^{[n]}\) can be expressed as a sum of singletons \(X = \{x_1\} + \cdots + \{x_k\}\) and hence

\[
g(X) = g(\{x_1\} + \cdots + \{x_k\}) = g(\{x_1\}) + \cdots + g(\{x_k\}).
\]

Thus, \(g\) can be expressed by a finite \([n]\)-indexed decomposition of unit

\[
(g(\{x_1\}), \ldots, g(\{x_n\})).
\]

Note that we can safely omit the last element of this sequence without losing any information, because

\[
g(\{x_n\}) = (g(\{x_1\}) + \cdots + g(\{x_{n-1}\}))^\perp.
\]

In this way, every summable sequence \((a_1, \ldots, a_{n-1})\) of elements of an effect algebra \(A\) determines a finite observable \(\mathcal{O}_{a_1, \ldots, a_{n-1}}: [n] \to A\) and every finite \(A\)-valued observable on \(2^{[n]}\) is determined by a summable sequence of \(n-1\) elements of \(A\). We will use the notation \(\mathcal{O}_{a_1, \ldots, a_{n-1}}\) throughout this paper.

For example, for every element \(a \in A\), \(\mathcal{O}_a\) denotes the observable \(2^{[2]} \to A\) given by the table

| \(X\) | \(0\) | \(1\) | \(2\) | \(\{1, 2\}\) |
|------|------|------|------|-----------|
| \(\mathcal{O}_a(X)\) | 0    | \(a\) | \(a^\perp\) | 1          |

The symbol \(\mathcal{O}_{\emptyset}\) denotes the (unique) observable \(\mathcal{O}_{\emptyset}: 2^{[1]} \to A\). Note that \(\mathcal{O}_{\emptyset}\) and \(\mathcal{O}_{\emptyset}\) are not the same thing.
2.6 Riesz decomposition property

An effect algebra \(A\) satisfies the \textit{Riesz decomposition property} if and only if, for all \(u, v_1, v_2 \in A\), \(u \leq v_1 + v_2\) implies that there exist \(u_1, u_2 \in A\) such that \(u_1 \leq v_1\), \(u_2 \leq v_2\) and \(u = u_1 + u_2\). It was proved in \cite{24} that every effect algebra satisfying the Riesz decomposition property is an interval in a partially ordered abelian group satisfying the Riesz decomposition property. Such groups are sometimes called \textit{interpolation groups}, see \cite{14}. Every lattice-ordered abelian group is an interpolation group.

Example 4 The set of all differentiable functions \(\mathbb{R} \rightarrow [0,1]_{\mathbb{R}}\) forms an effect algebra satisfying the Riesz decomposition property. We note that this effect algebra is not lattice ordered.

**Proposition 1** For an effect algebra \(A\), the following are equivalent

(a) \(A\) satisfies the Riesz decomposition property.

(b) For all \(x_1, \ldots, x_n, y_1, \ldots, y_m \in A\) such that \(x_1 + \cdots + x_n = y_1 + \cdots + y_m\) there exists an \(n \times m\) matrix \(Z = (z_{ij})\) of elements of \(A\) such that, for all \(i = 1, \ldots, n\), \(x_i\) is the sum of \(i\)-th row and, for all \(j = 1, \ldots, m\), \(y_j\) is the sum of \(j\)-th column of \(Z\).

(c) \(A\) satisfies (b) for \(n = m = 2\).

**Proof** See \cite{9} Section 1.7.

In our terminology, we may express this as follows.

**Proposition 2** An effect algebra \(A\) satisfies the Riesz decomposition property if and only if any two summable families with the same sum admit a common refinement.

It follows from the main result of \cite{6} that there is a one-to-one correspondence between lattice ordered effect algebras satisfying the Riesz decomposition property and MV-algebras, introduced by Chang \cite{5} in the 1950s to give an algebraic counterpart of the many-valued Łukasiewicz logic. It was proved by Mundici in \cite{23} that every MV-algebra is an interval in a lattice-ordered abelian group and vice versa.

2.7 Stone duality for finite Boolean algebras

Recall, that the category of finite Boolean algebras is dually equivalent to the category of finite sets. Explicitly, if \(t: [m] \rightarrow [n]\) is a mapping of finite sets, then a dual morphism of Boolean algebras \(\hat{t}: 2^{[n]} \rightarrow 2^{[m]}\) is given by the rule \(\hat{t}(X) = t^{-1}(X) = \{j \in [m] : t(j) \in X\}\). If \(f: 2^{[n]} \rightarrow 2^{[m]}\) is a morphism of Boolean algebras, then for every \(j \in [m]\) there is exactly one \(i \in [n]\) such that \(j \in f([i])\); this \(i\) is then the value of the dual map \(\hat{f}(j)\).

Via this duality, the coproduct in the category of finite Boolean algebras (denoted by \(\ast\)) is dual to the product of finite sets. Thus, we may exhibit
2^{[n]} \times 2^{[m]}$ as $2^{[n] \times [m]}$. If $s: 2^{[n]} \to 2^{[n']}$ and $t: 2^{[m]} \to 2^{[m']}$ are morphisms of Boolean algebras, the mapping $s \ast t: 2^{[n] \times [m]} \to 2^{[n'] \times [m']}$ is then given by the rule

$$(s \ast t)(X) = \bigcup_{(i,j) \in X} s(\{i\}) \times t(\{j\}).$$

(1)

For our purposes, it is important to note that the sets occurring in the union in (1) are pairwise disjoint. Indeed, if $(k,l) \in (s(\{i_1\}) \times t(\{j_1\})) \cap (s(\{i_2\}) \times t(\{j_2\}))$ then $k \in s(\{i_1\}) \cap s(\{i_2\})$ and $l \in t(\{j_1\}) \cap t(\{j_2\})$ and this already implies that $i_1 = i_2$ and $j_1 = j_2$. Therefore, we may write the union in (1) as an effect-algebraic sum:

$$(s \ast t)(X) = \sum_{(i,j) \in X} s(\{i\}) \times t(\{j\}).$$

(2)

2.8 Bimorphisms, tensor products

For effect algebras $A, B$ and $C$ a mapping $h: A \times B \to C$ is a $C$-valued bimorphism [8] from $A, B$ to $C$ if and only if the following conditions are satisfied.

**Unitality:** $h(1,1) = 1$.

**Left additivity:** For all $b \in B$ and $a_1, a_2 \in A$ such that $a_1 \perp a_2$, $h(a_1, b) \perp h(a_2, b)$ and $h(a_1, b) + h(a_2, b) = h(a_1 + a_2, b)$.

**Right additivity:** For all $a \in A$ and $b_1, b_2 \in B$ such that $b_1 \perp b_2$, $h(a, b_1) \perp h(a, b_2)$ and $h(a, b_1) + h(a, b_2) = h(a, b_1 + b_2)$.

It is easy to check that for every morphism of effect algebras $f: C \to C'$ and a bimorphism $h: A \times B \to C$, $f \circ h$ is a bimorphism. This fact shows that there is a category $\mathcal{B}_{A,B}$ where the objects are all bimorphisms from $A, B$ and the morphisms are $\text{EA}$-morphisms acting on bimorphisms from left by composition.

**Definition 1** [8] Let $A, B$ be effect algebras. A tensor product of $A$ and $B$ (denoted by $A \otimes B$) is the initial object in the category $\mathcal{B}_{A,B}$.

The notions of bimorphism and of the tensor product of orthoalgebras were given by Foulis and Bennett in [10]. It was proved by Jacobs and Mandemaker in [16] that the category of effect algebras equipped with the tensor products forms a symmetric monoidal category. There is another important result concerning tensor products: in [3] Börger proved that orthomodular posets equipped with tensor product form a symmetric monoidal category. Let us remark that Börger’s proof applies, almost without changes, in the more general case of effect algebras.

In the paper [8], it was assumed that $0 \neq 1$ in every effect algebra. Consequently, it might happen that there are pairs $A, B$ of effect algebras such that there is no bimorphism $h: A \times B \to C$, so $A \otimes B$ does not exist. However, if we allow for one-element effect algebras, then tensor product of effect algebras
always exists and if $A \otimes B$ has more than one element, it coincides with the tensor product as defined in [8].

Thus, “our” tensor products are the same as the tensor products in the sense of [8], whenever the tensor product exists in the sense of [8], and we obtain $A \otimes B = \{0\}$ whenever tensor product does not exist in the sense of [8].

### 3 Presheaves on finite Boolean algebras

For a general background for this section, see [22, Section I.5].

Let $\text{FinBool}$ be the full subcategory of the category of Boolean algebras $\text{Bool}$ spanned by the set of objects $\{2^n : n \in \mathbb{N}\}$. The restriction of the fully faithful functor $\text{Bool} \to \text{EA}$ described in the subsection [24] to the subcategory $\text{FinBool}$ gives us a fully faithful functor $E: \text{FinBool} \to \text{EA}$.

The functor $R: \text{EA} \to [\text{FinBool}^{op}, \text{Set}]$ maps every effect algebra $A$ to a presheaf $R(A): \text{FinBool}^{op} \to \text{Set}$. The presheaf $R(A)$ maps a Boolean algebra $2^n$ to the set of all $A$-valued observables on $2^n$:

$$R(A)(2^n) = \text{EA}(E(2^n), A).$$

For every morphism of Boolean algebras $f: 2^n \to 2^m$,

$$R(A)(f): \text{EA}(E(2^m), A) \to \text{EA}(E(2^n), A)$$

is given by the rule $R(A)(f)(g) = g \circ E(f)$.

Recall, that for a category $\mathcal{C}$ and a presheaf $P: \mathcal{C}^{op} \to \text{Set}$, the category of elements $\int P$ of $P$ is a category defined as follows:

- Objects are all pairs $(C, g)$, where $C$ is an object of $\mathcal{C}$ and $g \in P(C)$.
- An arrow $(C, g) \to (C', g')$ is an arrow $f: C \to C'$ in $\mathcal{C}$ such that $P(f)(g') = g$.

For an effect algebra $A$, the category $\int R(A)$ is the *category of finite observables*, which can be explicitly described as follows:

- Objects are all pairs $(2^n, g)$, where $g: E(2^n) \to A$ is an observable.
- An arrow $(2^n, g) \to (2^n', g')$ is a morphism of Boolean algebras $f: 2^n \to 2^n'$ such that $g' \circ E(f) = g$.

Since $\text{FinBool}$ is small and $\text{EA}$ is locally small, $\int R(A)$ is small.

Note that the first component of every pair $(2^n, g) \in \int R(A)$ contains redundant information, because $2^n$ is the domain of $g$. Therefore, we shall mostly write simply $g$ instead of $(2^n, g)$ whenever there is no danger of confusion. Furthermore, since $\text{FinBool}$ is a full subcategory of $\text{EA}$, we shall mostly suppress the functor $E$ from our notations. We shall write, for example, $g: 2^n \to A$ instead of $g: E(2^n) \to A$ and $g' \circ f = g$ instead of $g' \circ E(f) = g$.

For every presheaf $P: \text{FinBool}^{op} \to \text{Set}$, there is a projection functor $\pi_P: \int P \to \text{FinBool}$ given by $\pi_P(2^n, g) = 2^n$. By a general argument [22, Theorem I.5.2] the functor $L: [\text{FinBool}^{op}, \text{Set}] \to \text{EA}$ given by the colimit

$$L(P) = \lim_{\to} \left( \int P \xrightarrow{\pi_P} \text{FinBool} \xrightarrow{E} \text{EA} \right)$$
is left adjoint to \( R \).

For an effect algebra \( A \), \( D_A \) denotes the functor

\[
\int R(A) \xrightarrow{\pi R(A)} \text{FinBool} \xrightarrow{E} \mathbb{E}A.
\]

Note that \( L(R(A)) \cong \lim_{\rightarrow} D_A \).

The following theorem was stated by Staton and Uijlen in [27]. To keep out presentation self-contained, we give a complete proof.

**Theorem 1** The adjunction

\[
\begin{array}{ccc}
\text{FinBool}^{\text{op}} & \xrightarrow{\bot} & \mathbb{E}A \\
\downarrow L & & \downarrow R \\
\text{Set} & & \text{EA}
\end{array}
\]

is a reflection.

**Proof** We need to prove that, for every effect algebra \( A \) \( L(R(A)) \cong A \), that means, that \( A \) is a colimit of the functor \( D_A \).

It is clear that the objects of \( \int R(A) \) indexed by themselves form a cocone with apex \( A \) under \( D_A \). We claim that this cocone is initial in the category of cocones under \( D_A \). We need to prove that for every other cocone \((r_g, V)\) under \( D_A \) with apex \( V \) consisting of a family of \( r_g \), where \((2^n, g)\) runs through all objects of \( \int R(A) \), there is a unique morphism of effect algebras \( u: A \to V \) such that \( r_g = u \circ g \) for every object of the category \( \int R(A) \).

This property already determines the only possible candidate mapping for the morphism \( u: A \to V \). Indeed, for \( n = 2 \) and \( g = O_a \) we must have \( r_O_a = u \circ O_a \), in particular,

\[
r_{O_a}([1]) = u(O_a([1])) = u(a),
\]

and we see that \( u(a) = r_{O_a}([1]) \). We claim that this \( u: A \to V \) is a morphism of effect algebras and that for every observable \( g: 2^n \to A \) we have \( r_g = u \circ g \).

Let \( a_1, a_2 \in A \) be such that \( a_1 + a_2 \) exists in \( A \). Consider the observable \( O_{a_1, a_2}: 2^3 \to A \). There are three unique morphisms \( f_1, f_2, f_{1,2}: 2^2 \to 2^3 \) of Boolean algebras that make the three triangles in the diagram

\[
\begin{array}{ccc}
f_1 & \xrightarrow{O_{a_1}} & 2^2 \\
\downarrow & & \downarrow \xleftarrow{O_{a_1 + a_2}} \\
2^3 & \xrightarrow{O_{a_1, a_2}} & A \\
\downarrow f_2 & \xleftarrow{O_{a_2}} & \downarrow f_{1,2} \\
2^2 & \xrightarrow{O_{a_2}} & 2^2
\end{array}
\]
commute. Explicitly, \(f_1(\{1\}) = \{1\}, f_2(\{1\}) = \{2\} \) and \(f_{1,2}(\{1\}) = \{1, 2\}\).

The commutativity of (3) means that \(f_1, f_2, f_{1,2}\) can be considered as arrows in the category \(\int R(A)\):

\[
\begin{align*}
f_1 &: \mathcal{O}_{a_1} \to \mathcal{O}_{a_1, a_2} \\
f_2 &: \mathcal{O}_{a_2} \to \mathcal{O}_{a_1, a_2} \\
f_{1,2} &: \mathcal{O}_{a_1 + a_2} \to \mathcal{O}_{a_1, a_2}.
\end{align*}
\]

Therefore, since \((r_g, V)\) is a cocone under \(D_A\), we may compute

\[
\begin{align*}
u(a_1 + a_2) &= r_{\mathcal{O}_{a_1} + a_2}(\{1\}) = (r_{\mathcal{O}_{a_1}} \circ f_{1,2})(\{1\}) = \\
r_{\mathcal{O}_{a_1} + a_2}(\{1\} + \{2\}) &= r_{\mathcal{O}_{a_1}}(\{1\}) + r_{\mathcal{O}_{a_2}}(\{2\}) = \\
(r_{\mathcal{O}_{a_1}} \circ f_1)(\{1\}) + (r_{\mathcal{O}_{a_2}} \circ f_2)(\{1\}) = \\
r_{\mathcal{O}_{a_1}}(\{1\}) + r_{\mathcal{O}_{a_2}}(\{1\}) = u(a_1) + u(a_2)
\end{align*}
\]

and we see that \(u\) preserves +.

To prove that \(u(1) = 1\), consider the unique observable \(\mathcal{O}_\emptyset: 2^1[1] \to A\) and an arrow \(z: 2^2[2] \to 2^1[1]\) given by \(z(\{1\}) = \{1\}, z(\{2\}) = \emptyset\). From the commutativity of

\[
\begin{array}{ccc}
2^1[1] & \xrightarrow{\mathcal{O}_\emptyset} & A \\
\downarrow{z} & & \downarrow{r_{\mathcal{O}_1}} \\
2^2[2] & & \end{array}
\]

in \(\mathbf{EA}\) it follows that \(z\) is a morphism \(z: \mathcal{O}_1 \to \mathcal{O}_\emptyset\) in \(\int R(A)\), hence

\[
u(1) = r_{\mathcal{O}_1}(\{1\}) = (r_{\mathcal{O}_\emptyset} \circ z)(\{1\}) = r_{\mathcal{O}_\emptyset}(\{1\}) = 1.
\]

It remains to prove that \(u\) is a morphism of cocones under \(D_A\), that means, \(r_g = u \circ g\) for every object \(g\) of the category \(\int R(A)\). Let \(g: 2^n[n] \to A\) and let \(X = \{x_1, \ldots, x_k\} \in 2^n[n]\). For every \(i \in [k]\), let \(f_i: (2^2[2], \mathcal{O}_{g(\{x_i\})}) \to (2^n[n], g)\) be a morphism in \(\int R(A)\) given by the rules \(f_i(\{1\}) = \{x_i\}, f_i(\{2\}) = [n] \setminus \{x_i\}\). Then,

\[
r_g(X) = r_g(\sum_{i=1}^k \{x_i\}) = \\
\sum_{i=1}^k r_g(\{x_i\}) = \sum_{i=1}^k (r_{g \circ f_i})(\{1\}) = \\
\sum_{i=1}^k u(g(\{x_i\})) = \sum_{i=1}^k u(g(\{x_i\})) = \\
u(g(\sum_{i=1}^k \{x_i\})) = u(g(X)).
\]
We say that a category $\mathcal{C}$ is amalgamated if and only if every span in $\mathcal{C}$ can be extended to a commutative square.

**Theorem 2** An effect algebra $A$ satisfies the Riesz decomposition property if and only if $\int R(A)$ is amalgamated.

**Proof** Suppose that $A$ satisfies the Riesz decomposition property. Let $g, g_1, g_2$ be $A$-valued finite observables, let $f_1: g \rightarrow g_1$ and $f_2: g \rightarrow g_2$. Write $g: 2^{[n]} \rightarrow A$; for every $i \in [n]$ consider the sets $f_1(\{i\}), f_2(\{i\})$. It is easy to see that

$$(g_1(\{j\}))_{j \in f_1(\{i\})} \quad (g_2(\{k\}))_{k \in f_2(\{i\})}$$

are both finite summable families with sum equal to $g(i)$. By the Riesz decomposition property, these families have a common refinement, let us call it

$$\left( w^i_{(j,k)} \right)_{(j,k) \in f_1(\{i\}) \times f_2(\{i\})}.$$  

Concatenating all these families $(w^i_{(j,k)})$ gives us a decomposition of unit that is easily seen to be a common refinement of the decompositions of unit associated with the observables $g_1$ and $g_2$; let us denote the observable associated with the common refinement by $z$. There are morphisms $h_1: g_1 \rightarrow z$, $h_2: g_2 \rightarrow z$ in $\int R(A)$ associated with the refinements of $g_1, g_2$ and, obviously, $h_1 \circ f_1 = h_2 \circ f_2$.

Suppose that $\int R(A)$ is amalgamated. Let $u, x_1, x_2, y_1, y_2 \in A$ be such that $x_1 + x_2 = y_1 + y_2 = u$. Consider the $A$-valued finite observables $O_u, O_{x_1, x_2}, O_{y_1, y_2}$ associated with the decompositions of unit $(u, u')$, $(x_1, x_2, u')$ and $(y_1, y_2, u')$ and equip them with the natural arrows $f_x: O_u \rightarrow O_{x_1, x_2}$ and $f_y: O_u \rightarrow O_{y_1, y_2}$. Since $\int R(A)$ is amalgamated, the span $O_{x_1, x_2}, O_{y_1, y_2}$ extends to a commutative square, so there is an $A$-valued finite observable $g$ and morphisms of observables $h_x: O_{x_1, x_2} \rightarrow g$ and $h_y: O_{y_1, y_2} \rightarrow g$ such that $h_x \circ f_x = h_y \circ f_y$.

It is easy to check that $h_x, h_y$ give us the desired common refinement of the summable sequences $(x_1, x_2)$ and $(y_1, y_2).

**Theorem 3** An effect algebra $A$ is an orthoalgebra if and only if for every pair of morphisms $f_1, f_2: g \rightarrow g'$ in $\int R(A)$ there is a coequalizing morphism $q: g' \rightarrow u$ such that $q \circ f_1 = q \circ f_2$.

**Proof** Suppose that $A$ is an orthoalgebra. Let $g: 2^{[n]} \rightarrow A$ and $g': 2^{[m]} \rightarrow A$ be finite observables, let $f_1, f_2: g \rightarrow g'$ in $\mathcal{O}_A$. Since $A$ is an orthoalgebra, the range of every $A$-valued observable is a Boolean subalgebra of $A$. Therefore, there exists a Boolean algebra $B$ (for example the range of $g'$) and an embedding $j: B \rightarrow A$ such that $g, g'$ factor through $j$. That means, there are morphisms of effect algebras $h, h'$ such that the diagram

![Diagram](https://via.placeholder.com/150)

(4)
commutes in \( \text{EA} \).

Let \( q: 2^{[n]} \rightarrow 2^{[k]} \) be a coequalizer of the pair \( f_1, f_2 \) in the category \( \text{Bool} \). As \( q \) is a coequalizer and \( h = h' \circ f_1 = h' \circ f_2 \) in \( \text{Bool} \), there is a (unique) morphism of Boolean algebras \( u: 2^{[k]} \rightarrow B \) such that \( h' = u \circ q \), hence the diagram

\[
\begin{array}{ccc}
2^{[n]} & \xrightarrow{f_1} & 2^{[m]} \\
\downarrow{f_2} & \searrow{h} & \downarrow{h'} \\
2^{[m]} & \xrightarrow{q} & B \\
\end{array}
\]

commutes in \( \text{EA} \). Therefore, \( g' = j \circ h' = j \circ u \circ q \) or, in other words, \( q: g' \rightarrow j \circ u \) is the arrow in \( f \ R(A) \) with the property \( q \circ f_1 = q \circ f_2 \).

Suppose that \( A \) is an effect algebra such that for every pair of morphisms \( f_1, f_2: g \rightarrow g' \) in \( f \ R(A) \) there is a morphism \( q: g' \rightarrow h \) such that \( q \circ f_1 = q \circ f_2 \). Let \( a \in A \) be such that \( a \perp a \). We need to prove that \( a = 0 \). Let \( f_1, f_2: 2^{[2]} \rightarrow 2^{[3]} \) be such that \( f_1(\{1\}) = \{1\} \) and \( f_2(\{1\}) = \{2\} \). Then \( O_a = O_{a,a} \circ f_1 = O_{a,a} \circ f_2 \) in \( \text{EA} \), that means, \( f_1, f_2: O_a \rightarrow O_{a,a} \) in \( f \ R(A) \). By assumption, there is an arrow \( q: 2^{[3]} \rightarrow 2^{[n]} \) such that \( q \circ f_1 = q \circ f_2 \) and an observable \( u: 2^{[n]} \rightarrow A \) such that \( u \circ q = O_{a,a} \). Thus, the following diagram commutes in \( \text{EA} \):

\[
\begin{array}{ccc}
2^{[2]} & \xrightarrow{f_1} & 2^{[3]} \\
\downarrow{f_2} & \searrow{O_a} & \downarrow{O_{a,a}} \\
2^{[3]} & \xrightarrow{q} & A \\
\end{array}
\]

This implies that \( q(\{1\}) = q(f_1(\{1\})) = q(f_2(\{1\})) = q(\{2\}) \). On the other hand, since \( \{1\} \perp \{2\} \) in \( 2^{[3]} \), \( q(\{1\}) \perp q(\{2\}) \) in \( 2^{[n]} \). Since \( 2^{[n]} \) is a Boolean algebra, it is an orthoalgebra, hence \( q(\{1\}) \perp q(\{2\}) \) and \( q(\{1\}) = q(\{2\}) \) imply \( q(\{1\}) = q(\{2\}) = 0 \). Finally,

\[
a = O_a(\{1\}) = O_{a,a}(f_1(\{1\})) = O_{a,a}(\{1\}) = u(q(\{1\})) = u(0) = 0.
\]

Recall, that a category \( \mathcal{C} \) is called filtered if and only if the following conditions are satisfied.

- \( \mathcal{C} \) is nonempty.
- For every pair of objects \( X_1, X_2 \) there is a cospan

\[
\begin{array}{ccc}
X \\
\downarrow{X_1} & \searrow{X} & \downarrow{X_2} \\
X_1 & & X_2
\end{array}
\]
over them.

– For every parallel pair of morphisms \( f_1, f_2 : X \to Y \) in there exists a morphism \( q : Y \to Z \) such that \( q \circ f_1 = q \circ f_2 \).

**Corollary 1** An effect algebra \( A \) is a Boolean algebra if and only if \( \int R(A) \) is filtered.

**Proof** An effect algebra \( A \) is a Boolean algebra if and only if \( A \) satisfies the Riesz decomposition property and \( A \) is an orthoalgebra. The rest of the proof follows easily by Theorem 2 and Theorem 3 using the fact that \( \int R(A) \) has an initial object \( O : 2^1 \to A \).

### 4 Tensor products

Let \( A, B \) be effect algebras. The category \( \int R(A) \times \int R(B) \) has pairs of finite observables as objects and pairs of morphisms of observables as arrows. Consider the functor \( D_{A,B} : \int R(A) \times \int R(B) \to \mathbf{EA} \) given by the rule

\[
D_{A,B}(g_A, g_B) = \text{Dom}(g_A) \ast \text{Dom}(g_B),
\]

where \( \ast \) denotes free product (that means, coproduct in \( \text{Bool} \)) of Boolean algebras.

**Lemma 1** Let \( A, B \) be effect algebras. The category of bimorphisms \( \beta_{A,B} \) is isomorphic to the category of cocones under the diagram \( D_{A,B} \). Under this isomorphism, \( C \)-valued bimorphisms correspond to cocones with apex \( C \) and vice versa.

**Proof** We shall describe how to construct a cocone under \( D_{A,B} \) with apex \( C \) from a \( C \)-valued bimorphism and vice versa so that the constructions are mutually inverse.

Let \( h : A \times B \to C \) be a bimorphism. We need to construct a cocone under \( D_{A,B} \) associated to \( h \). So for every pair of finite observables \( g_A : 2^{[n]} \to A \) and \( g_B : 2^{[m]} \to B \), we need to define a morphism \( v_{g_A,g_B} : 2^{[n]} \ast 2^{[m]} \to C \), that will be the component of our cocone at \( (g_A, g_B) \). Note that \( 2^{[n]} \ast 2^{[m]} \simeq 2^{[n] \times [m]} \) and

\[
1 = h(1,1) = h\left( \sum_{i \in [n]} g_A(\{i\}), \sum_{j \in [m]} g_B(\{j\}) \right) = \sum_{i \in [n]} \sum_{j \in [m]} h(g_A(\{i\}), g_B(\{j\})),
\]

hence \( h(g_A(\{i\}), g_B(\{j\})) \) is a \( [n] \times [m] \)-indexed decomposition of unit in \( C \). Therefore, \( v_{g_A,g_B} : 2^{[n] \times [m]} \to C \) given by

\[
v_{g_A,g_B}(X) = \sum_{(i,j) \in X} h(g_A(\{i\}), g_B(\{j\}))
\]
is a finite observable. To prove that the family of all such \(v_{\ldots}\) forms a cocone under the diagram \(D_{A, B}\), let \((f_A, f_B) : (g_A, g_B) \rightarrow (g'_A, g'_B)\) be an arrow in \(\int R(A) \times \int R(B)\). We need to prove, for all \(X \in \text{Dom}(g_A) \ast \text{Dom}(g_B)\),

\[
v_{g'_A, g'_B}((f_A \ast f_B)(X)) = v_{g_A, g_B}(X).
\]

By (2) and the fact that \(v_{g'_A, g'_B}\) is a morphism in \(\mathbf{EA}\), the left-hand side expands to

\[
v_{g'_A, g'_B}((f_A \ast f_B)(X)) = v_{g'_A, g'_B}(\sum_{(i, j) \in X} f_A(i) \times f_B(j)) = \sum_{(i, j) \in X} v_{g'_A, g'_B}(f_A(i) \times f_B(j)).
\]

For every \((i, j) \in X\),

\[
v_{g'_A, g'_B}(f_A(i) \times f_B(j)) = \sum_{(k, l) \in f_A(i) \times f_B(j)} h(g'_A([k]), g'_B([l])) = h(\sum_{k \in f_A(i)} g'_A([k]), \sum_{l \in f_B(j)} g'_B([l])) = h(g_A([i]), g_B([j])).
\]

Continuing the computation (7),

\[
\sum_{(i, j) \in X} v_{g'_A, g'_B}(f_A([i]) \times f_B([j])) = \sum_{(i, j) \in X} h(g_A([i]), g_B([j])) = v_{g_A, g_B}(X)
\]

In this way, every \(C\)-valued bimorphism gives us a cocone under \(D_{A, B}\) with apex \(C\).

Let \((v_{\ldots})\) be a cocone under \(D_{A, B}\). For \((a, b) \in A \times B\), put \(h(a, b) = v_{O_{c_0}, O_0}([\{(1, 1)\}])\). We claim that \(h\) is a bimorphism.

There is a unique morphism of Boolean algebras \(u : 2^{[2]} \rightarrow 2^{[1]}\) with \(u([1]) = \{1\}\) that makes both diagrams

\[
\begin{array}{ccc}
2^{[1]} & \xrightarrow{O_0} & A \\
\downarrow u & & \downarrow \sigma_0 \\
2^{[2]} & & \\
\end{array}
\begin{array}{ccc}
2^{[1]} & \xrightarrow{O_0} & B \\
\downarrow u & & \downarrow \sigma_0 \\
2^{[2]} & & \\
\end{array}
\]

(8)

commute. This implies the commutativity of the diagram

\[
\begin{array}{ccc}
2^{[1]} \ast 2^{[1]} & \xrightarrow{v_{O_0, O_0}} & C \\
\downarrow u \ast u & & \downarrow v_{\sigma_0, \sigma_0} \\
2^{[2]} \ast 2^{[2]} & & \\
\end{array}
\]

(9)

Note that \(2^{[1]} \ast 2^{[1]} \simeq 2^{[1]}\) and \(2^{[1]}\) is initial in \(\mathbf{EA}\), hence \(v_{O_0, O_0} = O_0\). Using (3) we may now compute

\[
h(1, 1) = v_{\sigma_0, \sigma_0}([\{(1, 1)\}]) = v_{O_0, O_0}((u \ast u)(\{(1, 1)\})) = v_{O_0, O_0}([\{(1, 1)\}]) = 1
\]
Let $a_1, a_2 \in A$, $b \in B$. Let $f_1, f_2, f_{1,2}$ be exactly as in the proof of Theorem 1, diagram (3). If we pair the observables in (3) with the observable $O_b : 2^{[2]} \to B$, we obtain a commutative diagram in $\int R(A) \times \int R(B)$ that gives rise to the following part of the cocone $\nu$:

$$\begin{array}{ccc}
2^{[2]} \times [2] & \xrightarrow{v \circ O_{a_1}} & A \\
\downarrow f_{1,2} \circ \text{id} & & \\
2^{[2]} \times [2] & \xrightarrow{v \circ O_{a_1}} & A \\
\downarrow f_2 \circ \text{id} & & \\
\downarrow f_1 \circ \text{id} & & \\
\downarrow f_1 \circ \text{id} & & \\
2^{[2]} \times [2] & \xrightarrow{v \circ O_{a_1}} & A \\
\end{array}$$

Note that

$$(f_{1,2} \circ \text{id})(\{(1,1)\}) = \{(1,1), (2,1)\} = \{(1,1)\} + \{(2,1)\} = (f_1 \circ \text{id})(\{(1,1)\}) + (f_2 \circ \text{id})(\{(1,1)\})$$

and we can compute

$$h(a_1 + a_2, b) = v \circ O_{a_1 + a_2} \circ O_b (\{(1,1)\}) = (v \circ O_{a_1 + a_2} \circ O_b \circ (f_{1,2} \circ \text{id})) (\{(1,1)\}) = v \circ O_{a_1 + a_2} \circ O_b ((f_1 \circ \text{id})(\{(1,1)\})) + (f_2 \circ \text{id})(\{(1,1)\})) = v \circ O_{a_1 + a_2} \circ O_b ((f_1 \circ \text{id})(\{(1,1)\})) + v \circ O_{a_1 + a_2} \circ O_b ((f_2 \circ \text{id})(\{(1,1)\})) = v \circ O_{a_1 + a_2} \circ O_b (\{(1,1)\}) + v \circ O_{a_2} \circ O_b (\{(1,1)\}) = h(a_1, b) + h(a_2, b).$$

The proof of the right additivity of $h$ is analogous.

We should now check that this one-to-one correspondence between cocones under $D_{A,B}$ and the objects of $\beta_{A,B}$ is functorial and the functors are mutually inverse. This part of the proof is very straightforward and is thus omitted.

**Corollary 2** For every pair $A, B$ of effect algebras,

$$A \otimes B = \lim_{\to} D_{A,B}$$

**Theorem 4** The tensor product of effect algebras is a functor $E_{A} \times E_{A} \to E_{A}$ that arises as a left Kan extension of the functor $E \circ \ast : \text{FinBool} \times \text{FinBool} \to E_{A}$ along the inclusion $E \times E : \text{FinBool} \times \text{FinBool} \to E_{A} \times E_{A}$.
Proof By (a dual version of) [20, Theorem X.3.1], we can express the value of this Kan extension at \((A, B)\) as a colimit of a functor

\[(\text{FinBool} \times \text{FinBool} \downarrow (A, B)) \xrightarrow{Q} (\text{FinBool} \times \text{FinBool}) \xrightarrow{\sim} \text{EA},\]

where \(Q\) is the projection functor. The comma category \(\text{FinBool} \times \text{FinBool} \downarrow (A, B)\) is isomorphic to \(\int R(A) \times \int R(B)\) and the functor \(* \circ Q\) is just the \(D_{A,B}\) functor. The rest follows by Corollary 2.

**Corollary 3** For any two finite Boolean algebras \(A, B\),

\[A \ast B \simeq A \otimes B.\]

**Proof** By [26, Remark 6.1.2], the unit \(\eta\) of the Kan extension \((\ref{eq:kan-ext})\) is an isomorphism, because the functor \(E \times E\) is full and faithful.

It was proved by Day in [7] that for every monoidal category \((\mathcal{C}, \Box, I)\), the monoidal structure can be extended to the category \([\mathcal{C}^{\text{op}}, \text{Set}]\) of presheaves on \(\mathcal{C}\) by the rule

\[X \otimes_{\text{Day}} Y = \int^{(c_1, c_2)} \mathcal{C}^{\text{op}}(c_1 \Box c_2, c) \times X(c_1) \times Y(c_2).\]

**Theorem 5** For every pair \(A, B\) of effect algebras,

\[A \otimes B \simeq L(R(A) \otimes_{\text{Day}} R(B))\]

**Proof** In our case,

\[R(A) \otimes_{\text{Day}} R(B) = \int^{(c_1, c_2)} \text{FinBool}(\_ , c_1 \ast c_2) \times \text{EA}(c_1, A) \times \text{EA}(c_2, B).\]

Since \(L\) is a left adjoint, \(L\) preserves coends, so we may write

\[L(R(A) \otimes_{\text{Day}} R(B)) \simeq \int^{(c_1, c_2)} L(\text{FinBool}(\_, c_1 \ast c_2) \times \text{EA}(c_1, A) \times \text{EA}(c_2, B)),\]

In detail, the category of elements

\[el(c_1, c_2) = \int \text{FinBool}(\_, c_1 \ast c_2) \times \text{EA}(c_1, A) \times \text{EA}(c_2, B) \quad (\ref{eq:category-of-elements})\]

is a category with objects \((t, g_A, g_B)\), where \(t: c \to c_1 \ast c_2, g_A: c_1 \to A\) and \(g_B: c_2 \to B\) and morphism being given by precomposition in the first variable; \(h: (t, g_A, g_B) \to (s, g_A, g_B)\) is simply the fact that \(s \circ h = t\). It is obvious that the morphisms in \(el(c_1, c_2)\) preserve the pair \((g_A, g_B)\), so \(el(c_1, c_2)\) consists of pairwise isomorphic disjoint parts indexed by the elements of the set.
\( \text{EA}(c_1, A) \times \text{EA}(c_2, B) \). Note that each of the disjoint parts is isomorphic to \( \int R(c_1 \ast c_2) \). In other words,

\[
el(c_1, c_2) \simeq (\text{EA}(c_1, A) \times \text{EA}(c_2, A)) \cdot \int R(c_1 \ast c_2),
\]

where the dot denotes the copower. To compute the value of \( L \) means to take a colimit of a functor \( D : \text{el}(c_1, c_2) \to \text{EA} \) that maps every triple \((t, g_A, g_B)\) to the domain of \( t \). Since \( D \) behaves exactly like \( D_{c_1 \ast c_2} \) on every copy of \( \int R(c_1 \ast c_2) \) and since (by Theorem 1) \( \lim_{\rightarrow} D_{c_1 \ast c_2} \simeq c_1 \ast c_2 \), we see that

\[
\lim_{\rightarrow} D \simeq (\text{EA}(c_1, A) \times \text{EA}(c_2, B)) \cdot (c_1 \ast c_2)
\]

and thus

\[
L(R(A) \otimes_{\text{Day}} R(B)) \simeq \int^{(c_1, c_2)} (\text{EA}(c_1, A) \times \text{EA}(c_2, B)) \cdot (c_1 \ast c_2)
\]

By [20, Theorem X.4.1], this means that the functor \( (A, B) \mapsto L(R(A) \otimes R(B)) \) is a left Kan extension of the functor \( E \circ \ast : \text{FinBool} \times \text{FinBool} \to \text{EA} \) along the inclusion \( E \times E : \text{FinBool} \times \text{FinBool} \to \text{EA} \times \text{EA} \). The rest follows by Theorem 4.

References

1. Bennett, M., Foulis, D.: Interval and scale effect algebras. Advances in Applied Mathematics 19, 200–215 (1997)
2. Beran, L.: Orthomodular Lattices, Algebraic Approach. Kluwer, Dordrecht (1985)
3. Börger, R.: The tensor product of orthomodular posets. In: Categorical Structures and Their Applications: Proceedings of the North-West European Category Seminar, Berlin, Germany, 28–29 March 2003, p. 29. World Scientific (2004)
4. Busch, P., Lahti, P., Mittelstaedt, P.: The Quantum Theory of Measurement, 2nd edn. Springer Verlag (1996)
5. Chang, C.: Algebraic analysis of many-valued logics. Trans. Amer. Math. Soc. 88, 467–490 (1959)
6. Chovanec, F., Kôpka, F.: Boolean D-posets. Tatra Mt. Math. Publ 10, 183–197 (1997)
7. Day, B.: On closed categories of functors, pp. 1–38. Springer Berlin Heidelberg, Berlin, Heidelberg (1970). DOI 10.1007/BFb0060438. URL http://dx.doi.org/10.1007/BFb0060438
8. Dvurečenskij, A.: Tensor product of difference posets. Transactions of the American Mathematical Society 347(3), 1043–1057 (1995)
9. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer, Dordrecht and Ister Science, Bratislava (2000)
10. Foulis, D., Bennett, M.: Tensor products of orthoalgebras. Order 10(3), 271–282 (1993)
11. Foulis, D., Bennett, M.: Effect algebras and unsharp quantum logics. Found. Phys. 24, 1325–1346 (1994)
12. Foulis, D., Greechie, R., Rütimann, G.: Filters and supports in orthoalgebras. Int. J. Theor. Phys. 35, 789–802 (1995)
13. Giuntini, R., Greuling, H.: Toward a formal language for unsharp properties. Found. Phys. 19, 931–945 (1999)
14. Goodearl, K.: Partially ordered abelian groups with interpolation. Amer. Math. Soc, Providence (1986)
15. Gudder, S., Pulmannová, S.: Quotients of partial abelian monoids. Algebra univers. 38, 395–421 (1998)
16. Jacobs, B., Mandemaker, J.: Coreflections in algebraic quantum logic. Foundations of physics 42(7), 932–958 (2012)
17. Jenča, G.: Effect algebras are the Eilenberg-Moore category for the Kalmbach monad. Order 32(3), 439–448 (2015). DOI 10.1007/s11083-014-9344-6. URL http://dx.doi.org/10.1007/s11083-014-9344-6
18. Kalmbach, G.: Orthomodular Lattices. Academic Press, New York (1983)
19. Kôpka, F., Chovanec, F.: D-posets. Math. Slovaca 44, 21–34 (1994)
20. Lane, S.M.: Categories for the Working Mathematician. No. 5 in Graduate Texts in Mathematics. Springer-Verlag (1971)
21. Ludwig, G.: Foundations of Quantum Mechanics. Springer-Verlag, Berlin (1983)
22. Mac Lane, S., Moerdijk, I.: Sheaves in geometry and logic: A first introduction to topos theory. Springer Science & Business Media (2012)
23. Mundici, D.: Interpretation of AF $C^\ast$-algebras in Lukasiewicz sentential calculus. J. Functional Analysis 65, 15–53 (1986)
24. Pták, P., Pulmannová, S.: Orthomodular Structures as Quantum Logics. Kluwer, Dordrecht (1991)
25. Ravindran, K.: On a structure theory of effect algebras. Ph.D. thesis, Kansas State Univ., Manhattan, Kansas (1996)
26. Riehl, E.: Category theory in context. Courier Dover Publications (2016)
27. Staton, S., Uijlen, S.: Effect algebras, presheaves, non-locality and contextuality. In: International Colloquium on Automata, Languages, and Programming, pp. 401–413. Springer (2015)