Center-of-Mass Equations of Motion and Conserved Integrals of Compact Binary Systems at the Fourth Post-Newtonian Order

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Abstract

The dynamics of compact binary systems at the fourth post-Newtonian (4PN) approximation of general relativity has been recently completed in a self-consistent way. In this paper, we compute the ten Poincaré constants of the motion and present the equations of motion in the frame of the center of mass (CM), together with the corresponding CM Lagrangian, conserved energy and conserved angular momentum. Next, we investigate the reduction of the CM dynamics to the case of quasi-circular orbits. The non local (in time) tail effect at the 4PN order is consistently included, as well as the relevant radiation-reaction dissipative contributions to the energy and angular momentum.

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I. INTRODUCTION

A. Context and motivation

The problem of the dynamics (i.e., the derivation of the equations of motion) of two compact objects without spin — neutron stars or black holes — has recently been fully resolved at the fourth post-Newtonian (4PN) approximation of general relativity in a self-consistent way \([1–4]\). The first attacks of this problem were made by the ADM Hamiltonian formalism of general relativity \([29–31]\) and, independently, by using the effective field theory (EFT) \([32–34]\). Then, the ADM Hamiltonian was completed \([35, 36]\) to include the crucial contribution of near-zone (conservative) gravitational wave tails at the 4PN order (known from Refs. \([37–39]\)). The 4PN tail term was also previously computed in the EFT approach \([40, 41]\). Our own work \([1–4]\) is based on the construction of the Fokker action of point particles in harmonic coordinates.\(^2\) Our end result is in complete agreement, modulo an unphysical shift of the dynamical variables, with that of the ADM Hamiltonian formalism \([29–31, 35, 36]\).

The ultimate goals of high post-Newtonian (3PN or 4PN) calculations are for (i) improving the template bank used in the data analysis of compact binary coalescence (CBC) events in gravitational-wave detectors, and (ii) facilitating the comparisons or match with the results of numerical relativity (NR). However, in order to reach these goals the equations of motion have to be combined with a gravitational wave generation formalism \([42]\).

A salient feature of the 4PN dynamics is the appearance of infra-red (IR) divergences, in addition to the usual ultra-violet (UV) ones, due to the model of point particles describing compact objects \([25]\). The IR divergences entangled the resolution of the problem, as they yielded for a while the presence of “ambiguity” parameters, both in the ADM Hamiltonian \([35, 36]\) and the Fokker action \([1, 2]\). Finally, the ambiguities in the Fokker action could be fixed from first principles \([3, 4]\), by systematic use of dimensional regularization to cure both UV and IR divergences (notably, a consistent computation of the 4PN tail effect in \(d\) dimensions), combined with a matching between the near-zone and far-zone fields. Similarly, one expects that the EFT derivation \([33, 34, 40, 41]\), when it includes all the “instantaneous” (non-tail) contributions, should be free of any ambiguity parameter as well \([43]\). By contrast, the ADM Hamiltonian formalism \([35, 36]\) is still facing one ambiguity parameter, probably because of the lack of consistent matching between the near-zone and far-zone fields in this formalism, which specifically deals with the near-zone dynamics but has not yet been developed to study wave generation \([13]\).

Anyway, the ambiguity parameters are uniquely fixed by comparison to independent gravitational self-force (GSF) calculations, based on black hole perturbations valid in the small mass ratio limit, of the conserved energy \([44–47]\) and the periastron advance \([48–52]\) for circular orbits. The final 4PN dynamics in either Hamiltonian \([29–31, 35, 36]\) or Lagrangian \([1–4]\) guise fully reproduces the relevant GSF results.

Another striking feature of the 4PN dynamics is the non-locality in time due to the appearance of the tail effect at that order. As discussed in \([2]\), the non-locality entails crucial new contributions in the conserved energy and angular momentum: For instance,

\(^1\) As usual we refer to post-Newtonian orders as \(n\text{PN} \sim (v/c)^{2n}\) beyond the Newtonian acceleration. See Refs. \([5–18]\) for “historical” works on the problem of the equations of motion and Refs. \([19–28]\) for key results at the previous 3PN order.

\(^2\) See, in particular, Ref. \([4]\) for a synthetic overview of our method and main results.
the conserved energy for a non-local Hamiltonian contains extra terms with respect to the on-shell value of that Hamiltonian, because of the appearance of functional (instead of ordinary) derivatives in the Hamilton’s equations. Our final results carefully take into account the non-local character of the 4PN dynamics brought by the tail term. Note that the “first law of compact binary mechanics”, which is important in this context because it allows the connection between PN and GSF results, remains valid for the non-local 4PN dynamics [53].

The purpose of the present paper is to provide explicit consequences of the 4PN equations of motion that should be useful in various practical applications, such as direct comparisons to NR calculations [54] or GSF results (see, e.g., [55]) during the inspiral phase of CBCs, the improvement of phenomenological and effective-one-body templates [56, 57], the study of accretion disks and jet dynamics around spinning black-hole binaries [58], as well as the numerical calculation of the orbits of non-precessing, spinning black holes via asymptotic matching [59]. Another important application is the setup of accurate initial conditions for the grand-challenge binary black hole problem in NR [60–62].

In this perspective, we shall derive the ten Noetherian conserved integrals of the motion associated with the global Poincaré invariance of the Lagrangian (which is manifest in harmonic coordinates). This includes in particular the integral of the center of mass (CM) associated with the invariance under Lorentz boosts. Moreover, we shall obtain the harmonic-coordinates Lagrangian in the CM frame and provide explicitly the CM relative acceleration. As is well known, this Lagrangian in harmonic coordinates is a generalized one, depending on positions, velocities and, from the 2PN order, on accelerations. At the 4PN order, after adding suitable “multi-zero” terms and total time derivatives (see [14]), the Lagrangian becomes linear in accelerations and does not contain any derivative of accelerations [1]. By performing shifts of the dynamical variables, one can transform the generalized Lagrangian into an ordinary one that corresponds to ADM, or ADM like coordinates, and construct the corresponding ADM Hamiltonian which is then equivalent to the end result of Refs. [35, 36]. Finally, since the CBC orbits will most likely be circular — at least when the binaries are formed “in the field” so that circularization will have enough time to proceed, we present all relevant formulas for the reduction to the case of quasi-circular orbits.

B. Notation and conventions

The two ordinary coordinate trajectories in a harmonic coordinate system \( \{t, \mathbf{x}\} \) are denoted as \( \mathbf{y}_A(t) \) (with \( A = 1, 2 \)), the ordinary velocities are \( \mathbf{v}_A(t) = \frac{d\mathbf{y}_A}{dt} \), and the ordinary accelerations \( \mathbf{a}_A(t) = \frac{d\mathbf{v}_A}{dt} \). Here, boldface letters indicate ordinary three-dimensional Euclidean vectors. In a general frame, the orbital separation unit vector reads \( \mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2) / r_{12} \) with \( r_{12} = |\mathbf{y}_1 - \mathbf{y}_2| \). Ordinary Euclidean scalar products are denoted, e.g., \( (n_{12} v_1) = \mathbf{n}_{12} \cdot \mathbf{v}_1 \). In the CM frame, we rather introduce the alternative notations \( \mathbf{n} = \mathbf{n}_{12} \) and \( r = r_{12} \), together with the relative separation \( \mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2 \) (\( \mathbf{x} \) should not be confused with the spatial harmonic coordinate \( \mathbf{x} \)), relative velocity \( \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \) and relative acceleration \( \mathbf{a} = \frac{d\mathbf{v}}{dt} \). We pose \( v^2 = (\mathbf{v} \cdot \mathbf{v}) = \mathbf{v}^2 \) and \( \dot{r} = (\mathbf{n} \cdot \mathbf{v}) = \mathbf{n} \cdot \mathbf{v} \). The orbital frequency \( \omega \) defined by \( v^2 = \dot{r}^2 + \dot{\omega}^2 \omega^2 \) will be used mostly for quasi-circular orbits, in which case \( \dot{r} = \mathcal{O}(c^{-5}) \) and \( v = r \omega [1 + \mathcal{O}(c^{-10})] \). The two masses are denoted \( m_A \), the total mass

\[3\] For lack of space, we shall only present the center-of-mass integral in a general frame but give the full conserved energy and angular momentum in the CM frame.
and the reduced mass \( \mu \); the symmetric mass ratio

\[
\nu = \frac{\mu}{m} = \frac{m_1 m_2}{(m_1 + m_2)^2},
\]

(1.1)
is such that \( 0 < \nu \leq \frac{1}{4} \). We also pose \( X_A = \frac{m_A}{m} \), hence \( X_1 X_2 = \nu \) and \( X_1 + X_2 = 1 \). In the case of quasi-circular orbits, we introduce the relevant small PN parameters \(^4\)

\[
\gamma = \frac{Gm}{r c^2} \quad \text{and} \quad x = \left( \frac{Gm \omega}{c^3} \right)^{2/3}.
\]

(1.2)

While \( \gamma \) is a gauge-dependent parameter, here defined in harmonic coordinates, the frequency-dependent parameter \( x \) is invariant in a large class of coordinate systems: those that are asymptotically flat at infinity. Upper indices \((s)\) denote \( s\)-th time derivatives; the symmetric-trace-free (STF) projection is indicated by angular brackets surrounding indices, e.g., \( x^{(i} x^{j)} = x^i x^j - \frac{1}{3} \delta^{ij} x^2 \); \( \epsilon_{ijk} \) is the totally anti-symmetric Levi-Civita symbol, whereas \( \gamma_E \) denotes Euler’s constant.

Any physical quantity \( Q \) investigated in this paper will be the sum of (i) many instantaneous (local-in-time) terms up to the 4PN order, (ii) the non-local conservative tail term arising at the 4PN order, and, in most cases, (iii) dissipative radiation-reaction terms present at the 2.5PN, 3.5PN, as well as the 4PN orders. Accordingly, we write

\[
Q = Q^{\text{inst}} + Q^{\text{tail}} + Q^{\text{diss}}.
\]

(1.3)

Since the long explicit results presented in the present article will concern the instantaneous part of the dynamics, we adopt the convention that any \( Q^{\text{inst}} \) is directly given by the sequence of its PN coefficients \( Q_{n\text{PN}} \):

\[
Q^{\text{inst}} = Q_N + \frac{1}{c^2} Q_{1\text{PN}} + \frac{1}{c^4} Q_{2\text{PN}} + \frac{1}{c^6} Q_{3\text{PN}} + \frac{1}{c^8} Q_{4\text{PN}} + \mathcal{O} \left( \frac{1}{c^{10}} \right).
\]

(1.4)

Furthermore, as the 4PN coefficient is especially lengthy, we split it into non-linear contributions corresponding to increasing powers of \( G \):

\[
Q_{4\text{PN}} = Q_{4\text{PN}}^{(0)} + G Q_{4\text{PN}}^{(1)} + G^2 Q_{4\text{PN}}^{(2)} + G^3 Q_{4\text{PN}}^{(3)} + G^4 Q_{4\text{PN}}^{(4)} + G^5 Q_{4\text{PN}}^{(5)}.
\]

(1.5)

At 4PN order, there are no terms beyond the quintic non-linearities \( \propto G^5 \). When relevant, the dissipative terms will be simply indicated as \( Q_{2.5\text{PN}} \) (for instance). We generally do not write a PN coefficient when it is zero.

Our results for the instantaneous part of the dynamics concern the CM equations of motion and Lagrangian in Sec. II, as well as the CM conserved energy and angular momentum in Sec. III. The non-local 4PN tail part of the dynamics is investigated in Sec. IV, basically following Ref. [2]. The reduction to quasi-circular orbits is achieved in Sec. V (including tail terms), while we add the dissipative radiation-reaction 2.5PN, 3.5PN and 4PN terms in Sec. VI. Finally, we recap our general-frame 4PN Lagrangian in the Appendix A and present the general-frame integral of the CM in App. B.

\(^4\) All formulas will include the required powers of the gravitational constant \( G \) and speed of light \( c \).
II. CENTER-OF-MASS EQUATIONS OF MOTION AND LAGRANGIAN

Starting from the general-frame Lagrangian of Refs. [1, 2] (see App. A for a recap), we obtain the integral of the center of mass \( \mathbf{G} \). We then define the CM frame by solving iteratively the equation \( \mathbf{G} = 0 \) (with order reduction of the accelerations). This gives explicit expressions for the CM variables \( \mathbf{y}_A \) and \( \mathbf{v}_A \) as functions of the relative position \( \mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2 \) and velocity \( \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \). The complete expressions of the CM integral \( \mathbf{G} \) and of the CM trajectories \( \mathbf{y}_A \) are given in App. B. As we shall find in Sec. IV, there are no tail contributions in the CM trajectories \( \mathbf{y}_A \) nor in the CM velocities \( \mathbf{v}_A \).

Next, we insert the CM variables [Eqs. (B4)–(B6) in App. B] into the equations of motion derived from the variation of the general-frame Lagrangian, from which we obtain the relative acceleration \( \mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2 \) as a functional of \( \mathbf{n}, \mathbf{r}, \dot{r} = (nv) \) and \( \mathbf{v} \). The (instantaneous part of the) acceleration takes the form

\[
\mathbf{a} = -\frac{Gm}{r^2} \left[(1 + A) \mathbf{n} + B \mathbf{v}\right],
\]

where the various PN coefficients are given by (see Sec. IB for our conventions)

\[
A_{1PN} = -\frac{3 \dot{r} \nu}{2} + v^2 + 3 \nu v^2 - \frac{Gm}{r} (4 + 2 \nu),
\]

\[
A_{2PN} = \frac{15 \dot{r}^4 \nu}{8} - \frac{45 \dot{r}^4 \nu^2}{8} - \frac{9 \dot{r}^2 \nu v^2}{2} + 6 \dot{r}^2 \nu^2 v^2 + 3 \nu v^4 - 4 \nu^2 v^4
+ \frac{Gm}{r} \left(-2 \dot{r}^2 - 25 \dot{r}^2 \nu - 2 \dot{r}^2 \nu^2 - \frac{13 \dot{r} \nu v^2}{2} + 2 \nu^2 v^2\right)
+ \frac{G^2 m^2}{r^2} \left(9 + \frac{87 \nu}{4}\right),
\]

\[
A_{3PN} = -\frac{35 \dot{r}^6 \nu}{16} + \frac{175 \dot{r}^6 \nu^2}{16} - \frac{175 \dot{r}^6 \nu^3}{16} + \frac{15 \dot{r}^4 \nu v^2}{2}
\]

\[
- \frac{135 \dot{r}^4 \nu v^2}{8} + \frac{255 \dot{r}^4 \nu^3 v^2}{8} - \frac{15 \dot{r}^2 \nu v^4}{2} + \frac{237 \dot{r}^2 \nu^2 v^4}{8}
- \frac{4 \dot{r}^2 \nu^3 v^4}{8} + \frac{11 \nu v^6}{4} - \frac{49 \nu^2 v^6}{4} + 13 \nu^3 v^6
+ \frac{Gm}{r} \left(79 \dot{r}^4 \nu - \frac{69 \dot{r}^4 \nu^2}{2} - 30 \dot{r}^4 \nu^3 - 121 \dot{r}^2 \nu v^2 + 16 \dot{r}^2 \nu^2 v^2 + 20 \dot{r}^2 \nu^3 v^2 + \frac{75 \nu v^4}{4} + 8 \nu^2 v^4 - 10 \nu^3 v^4\right)
+ \frac{G^2 m^2}{r^2} \left(\dot{r}^2 + \frac{32573 \dot{r}^2 \nu}{168} + \frac{11 \dot{r}^2 \nu^2}{8} - 7 \dot{r}^2 \nu^3 + \frac{615 \dot{r}^2 \nu \pi^2}{64} - \frac{26987 \nu v^2}{840}
+ \nu^3 v^2 - \frac{123 \nu \pi^2 v^2}{64} - 110 \dot{r}^2 \nu \ln \left(\frac{r}{r_0}\right) + 22 \nu^2 \ln \left(\frac{r}{r_0}\right)\right)
+ \frac{G^3 m^3}{r^3} \left(-16 - \frac{437 \nu}{4} - \frac{71 \nu^2}{2} + \frac{41 \nu \pi^2}{16}\right),
\]

up to 3PN order, together with, for the 4PN terms,

\[
A_{4PN}^{(0)} = \left(\frac{315}{128} \nu - \frac{2205}{128} \nu^2 + \frac{2205}{64} \nu^3 - \frac{2205}{128} \nu^4\right) \dot{r}^8 + \left(-\frac{175}{16} \nu + \frac{595}{8} \nu^2 - \frac{2415}{16} \nu^3\right)
\]
Similarly:

\[
A_{4\text{PN}}^{(1)} = \frac{\mathcal{A}}{r} \left( \frac{2973}{40} \nu^{-6} + 407\nu^2 r^6 + \frac{181}{16} \nu^3 r^6 - 86\nu^4 r^6 + \frac{1497}{32} \nu^4 r^2 v^2 - \frac{1627}{2} \nu^2 r^4 v^2 \right) \\
- 81\nu^3 r^2 v^2 + 228\nu^4 r^4 v^2 - \frac{2583}{16} \nu^3 r^2 v^2 + \frac{1009}{2} \nu^2 r^2 v^4 + 47\nu^3 r^2 v^4 - 104\nu^4 r^2 v^2 \\
+ \frac{1067}{32} \nu v^6 - 58\nu^2 v^6 - 44\nu^3 v^6 + 58\nu^4 v^6 + \frac{1}{2} \left( \frac{1636681}{120} \nu^2 r^2 v^2 - 126\nu^4 r^2 v^2 + 385\nu^2 \ln \left( \frac{r}{r_0} \right)^4 + 385\nu \ln \left( \frac{r}{r_0} \right)^4 - 1540\nu^2 \ln \left( \frac{r}{r_0} \right)^4 - \frac{1636681}{120} \nu^2 r^2 v^2 \right) \\
- 12585 \nu^2 r^2 v^2 - 255461 \nu^2 r^2 v^2 - 3075 \nu^2 r^2 v^2 - \frac{309}{4} \nu^2 r^2 v^2 + 63\nu^4 r^2 v^2 \\
- 330\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 - 275\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 - 275\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 + 1100\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 \\
+ \frac{1096941}{11200} \nu^3 r^2 v^2 + \frac{1155}{1024} \nu^2 r^2 v^2 + \frac{7263}{70} \nu^2 r^2 v^2 - \frac{123}{64} \nu^2 r^2 v^2 + \frac{145}{2} \nu^2 v^4 - 16\nu v^4 \\
+ 66\nu \ln \left( \frac{r}{r_0} \right)^v v^4 + 22\nu \ln \left( \frac{r}{r_0} \right)^v v^4 + 22\nu \ln \left( \frac{r}{r_0} \right)^v v^4 - 88\nu \ln \left( \frac{r}{r_0} \right)^v v^4 , \tag{2.2d}
\]

\[
A_{4\text{PN}}^{(2)} = \frac{m^2}{r^2} \left( \frac{2094751}{960} \nu^4 + \frac{45255}{96} \nu^2 r^4 + \frac{326101}{96} \nu^2 r^4 - \frac{4305}{128} \nu^2 r^4 + \frac{1959}{32} \nu^3 r^4 \\
- 126\nu^4 r^4 + 385\nu^2 \ln \left( \frac{r}{r_0} \right)^4 + 385\nu \ln \left( \frac{r}{r_0} \right)^4 - 1540\nu^2 \ln \left( \frac{r}{r_0} \right)^4 - \frac{1636681}{120} \nu^2 r^2 v^2 \\
- \frac{12585}{512} \nu^2 r^2 v^2 - \frac{255461}{112} \nu^2 r^2 v^2 - \frac{3075}{128} \nu^2 r^2 v^2 - \frac{309}{4} \nu^2 r^2 v^2 + 63\nu^4 r^2 v^2 \\
- 330\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 - 275\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 - 275\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 + 1100\nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 \\
+ \frac{1096941}{11200} \nu^3 r^2 v^2 + \frac{1155}{1024} \nu^2 r^2 v^2 + \frac{7263}{70} \nu^2 r^2 v^2 - \frac{123}{64} \nu^2 r^2 v^2 + \frac{145}{2} \nu^2 v^4 - 16\nu v^4 \\
+ 66\nu \ln \left( \frac{r}{r_0} \right)^v v^4 + 22\nu \ln \left( \frac{r}{r_0} \right)^v v^4 + 22\nu \ln \left( \frac{r}{r_0} \right)^v v^4 - 88\nu \ln \left( \frac{r}{r_0} \right)^v v^4 \right) , \tag{2.2f}
\]

\[
A_{4\text{PN}}^{(3)} = \frac{m^3}{r^3} \left( -2\dot{r}^2 + \frac{197943}{8400} \nu^2 r^2 - \frac{2969}{16} \nu^2 r^2 + \frac{1255151}{840} \nu^2 r^2 + \frac{7095}{32} \nu^2 r^2 - 17\nu^3 r^2 \\
- 24\nu^4 r^2 + 384 \ln \left( \frac{r}{r_0} \right)^2 r^2 - 920 \nu \ln \left( \frac{r}{r_0} \right)^2 r^2 + 3100 \nu \ln \left( \frac{r}{r_0} \right)^2 r^2 - 384 \ln \left( \frac{r}{r_0} \right)^2 r^2 \\
+ 3152 \nu \ln \left( \frac{r}{r_0} \right)^2 r^2 - 6464 \nu^2 \ln \left( \frac{r}{r_0} \right)^2 r^2 + \frac{1237279}{25200} \nu^2 r^2 + \frac{3835}{96} \nu^2 r^2 - \frac{693947}{2520} \nu^2 r^2 \\
- \frac{229}{8} \pi^2 r^2 v^2 + \frac{19}{2} \nu^3 r^2 v^2 - 64 \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 + 80 \nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 - \frac{1616}{3} \nu^2 \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 \\
+ 64 \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 - \frac{1576}{3} \nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 + \frac{3232}{3} \nu \ln \left( \frac{r}{r_0} \right)^2 r^2 v^2 \right) , \tag{2.2g}
\]

\[
A_{4\text{PN}}^{(4)} = \frac{m^4}{r^4} \left( 25 + \frac{6625537}{12600} \nu - \frac{4543}{96} \pi^2 \nu + \frac{477763}{720} \nu^2 + \frac{3}{4} \pi^2 \nu^2 + 16 \ln \left( \frac{r}{r_0} \right) - 20 \nu \ln \left( \frac{r}{r_0} \right) \\
+ 98 \nu^2 \ln \left( \frac{r}{r_0} \right) - 16 \ln \left( \frac{r}{r_0} \right) + \frac{394}{3} \nu \ln \left( \frac{r}{r_0} \right) - \frac{808}{3} \nu^2 \ln \left( \frac{r}{r_0} \right) \right) . \tag{2.2h}
\]

Similarly:

\[
B_{1\text{PN}} = -4 \dot{r} + 2 \dot{r} \nu , \tag{2.3a}
\]

\[
B_{2\text{PN}} = \frac{9 \dot{r}^3}{2} + 3 \dot{r}^3 v^2 - \frac{15 \dot{r} \nu v^2}{2} - 2 \dot{r} \nu^2 v^2 \\
+ \frac{Gm}{r} \left( 2 \dot{r} + \frac{41 \dot{r} \nu}{2} + 4 \dot{r} \nu^2 \right) , \tag{2.3b}
\]
The tail part of the acceleration is presented in Sec. IV, while the dissipative radiation-reaction terms are displayed in Sec. VI [see Eqs. (6.4)].

Recall that the 4PN Lagrangian in harmonic coordinates $[1, 2]$ is a generalized one, depending on positions, velocities and accelerations (from the 2PN order). It also contains some logarithms and associated arbitrary gauge constants entering their arguments. These constants, denoted $r'_A$ (one for each particle), do not affect physical gauge invariant results. The CM equations of motion (2.1)–(2.3) depend on these constants as well, through the two combinations $r'_0$ and $r''_0$ defined by

$$
\ln r'_0 = X_1 \ln r'_1 + X_2 \ln r'_2, \quad (2.4a)
$$

$$
\ln r''_0 = \frac{X_1^2 \ln r'_1 - X_2^2 \ln r'_2}{X_1 - X_2}. \quad (2.4b)
$$
The combination $r_0'$ represents the one that appears at 3PN order; at the 4PN order, both combinations appear at once. Note that our definition of the constant $r_0''$ in Eq. (2.4b) involves $X_1 - X_2$ in the denominator. However, the equal mass limit $X_1 = X_2$ is always well-defined since in all equations, like (2.2)-(2.3), the logarithm of $r_0''$ is always multiplied by a factor $X_1 - X_2 = \pm \sqrt{1 - 4\nu}$.

The previous CM equations of motion actually derive from a Lagrangian. This Lagrangian, which is also a generalized one, can be constructed as follows. We start from the general-frame Lagrangian, which is now "doubly generalized" in terms of the two types of variables, $\mathbf{a}$ and $\mathbf{v}$, and admits the CM integral $G[y_A, \mathbf{v}_A]$ explicitly given by Eqs. (B2)-(B3) in App. B. Then, we perform the change of variables $(y_1, y_2) \rightarrow (x, G)$, where we recall that $x = y_1 - y_2$. Since to Newtonian order we have $G = m_1 y_1 + m_2 y_2 + \mathcal{O}(c^{-2})$, we find for instance $y_1 = X_2 x + \frac{1}{m} G + \mathcal{O}(c^{-2})$ and $v_1 = X_2 v + \frac{G}{m} + \mathcal{O}(c^{-2})$. Proceeding iteratively with the help of Eqs. (B2)-(B3), it is easy to see that the old variables $y_A$ are obtained as functionals of the new variables $(x, G)$ and their derivatives up to some high differentiation order depending on the PN order. In the process, we do not perform any order reduction of accelerations, so that we get

$$y_A = y_A [x, v, a, \ldots; G, \frac{dG}{dt}, \frac{d^2 G}{dt^2}, \ldots]. \quad (2.5)$$

Plugging those relations into the original general-frame Lagrangian and performing time derivatives, but still without order reduction of the accelerations, yields an equivalent Lagrangian, which is now "doubly generalized" in terms of the two types of variables, i.e.

$$L = L [x, v, a, \ldots; G, \frac{dG}{dt}, \frac{d^2 G}{dt^2}, \ldots]. \quad (2.6)$$

The ensuing equations of motion read

$$\frac{\delta L}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial a} \right) + \ldots = 0, \quad (2.7)$$

together with the equation $\frac{\delta L}{\delta G} = 0$, which is necessarily equivalent to the conservation law for the CM integral, hence we have

$$\frac{\delta L}{\delta G} = 0 \iff \frac{d^2 G}{dt^2} = 0. \quad (2.8)$$

As a result, we can choose $G = 0$ as a solution of these equations. The CM equations of motion are then given by Eq. (2.7) in which we pose, everywhere, $G = 0, \frac{dG}{dt} = 0, \ldots$; these equations are nothing but the CM equations of motion (2.1)-(2.3). Now it is clear, since Eqs. (2.7) and (2.8) are independent, that those CM equations of motion derive precisely from the Lagrangian (2.6) in which we set, everywhere, $G = 0, \frac{dG}{dt} = 0, \ldots$, hence the CM Lagrangian is

$$L_{CM} = L [x, v, a, \ldots; G, 0, 0, \ldots]. \quad (2.9)$$

At this stage, we follow the standard procedure of subtracting "multi-zero" terms and total time derivatives (without performing any shift), which reduces the Lagrangian to one that is linear in accelerations and deprived of time derivatives of accelerations. Defining as usual the reduced CM Lagrangian as $\mathcal{L} = L_{CM}/\mu$, we explicitly get

$$\mathcal{L}_N = \frac{v^2}{2} + \frac{Gm}{r}, \quad (2.10a)$$
\[ \mathcal{L}_{1\text{PN}} = \frac{v^4}{8} - \frac{3\nu v^4}{8} + \frac{Gm}{r} \left( \frac{\dot{r}^2 \nu}{2} + \frac{3v^2}{2} + \frac{\nu v^2}{2} \right) - \frac{G^2 m^2}{2r^2}, \] (2.10b)

\[ \mathcal{L}_{2\text{PN}} = \frac{v^6}{16} - \frac{7\nu v^6}{16} + \frac{13\nu^2 v^6}{16} + \frac{Gm}{r} \left( \frac{3\dot{r}^4 \nu^2}{8} - \frac{\dot{r}^2 a_n \nu r}{8} + \frac{\dot{r}^2 \nu v^2}{4} - \frac{5\dot{r}^2 \nu^2 v^2}{4} + \frac{7a_n \nu r v^2}{8} \right) \]

\[ + \frac{G^2 m^2}{r^2} \left( \frac{\dot{r}^2}{2} + \frac{41\dot{r}^2 \nu}{8} + \frac{3\dot{r}^2 \nu v^2}{2} + \frac{7v^2}{4} - \frac{27\nu v^2}{8} + \frac{\nu^2 v^2}{2} \right) \]

\[ + \frac{G^3 m^3}{r^3} \left( \frac{1}{2} + \frac{15\nu}{4} \right), \] (2.10c)

\[ \mathcal{L}_{3\text{PN}} = \frac{5v^8}{128} - \frac{59\nu v^8}{128} + \frac{119\nu^2 v^8}{64} - \frac{323\nu^3 v^8}{128} + \frac{Gm}{r} \left( \frac{5\dot{r}^6 \nu^3}{16} + \frac{\dot{r}^4 a_n \nu r}{16} + \frac{5\dot{r}^4 a_n \nu^2 r}{16} + \frac{3\dot{r}^4 \nu v^2}{16} \right) \]

\[ + \frac{7\dot{r}^4 \nu^2 v^2}{4} - \frac{33\dot{r}^4 \nu^3 v^2}{16} - \frac{3\dot{r}^2 a_n \nu r v^2}{16} - \frac{\dot{r}^2 a_n \nu^2 r v^2}{16} + \frac{5\dot{r}^2 \nu^4 v^2}{8} - \frac{3\dot{r}^2 \nu^2 v^4}{8} + \frac{75\dot{r}^2 \nu^3 v^4}{16} + \frac{7a_n \nu r v^4}{8} - \frac{2a_n \nu r^2 v^4}{2} \]

\[ + \frac{11v^6}{16} - \frac{55\nu^3 v^6}{16} + \frac{5\nu^2 v^6}{2} + \frac{65\nu^3 v^6}{16} + \frac{5\dot{r}^3 \nu r a_v}{16} - \frac{13\dot{r}^3 \nu^2 r a_v}{8} \]

\[ - \frac{37\dot{r} \nu r v^2 a_v}{8} + \frac{35\dot{r}^2 \nu v^2 a_v}{4} \]

\[ + \frac{G^2 m^2}{r^2} \left( -\frac{109\dot{r}^4 \nu}{144} - \frac{259\dot{r}^4 \nu^2}{36} + \frac{2\dot{r}^4 \nu^3}{6} - \frac{17\dot{r}^2 a_n \nu r}{6} \right) \]

\[ + \frac{97\dot{r}^2 a_n \nu^2 r}{12} + \frac{\dot{r}^2 v^2}{4} - \frac{41\dot{r}^2 \nu v^2}{6} + \frac{2287\dot{r}^2 \nu^2 v^2}{48} \]

\[ - \frac{27\dot{r}^2 \nu^3 v^2}{4} + \frac{203a_n \nu r v^2}{12} + \frac{149a_n \nu^2 r v^2}{6} \]

\[ + \frac{45v^4}{16} + \frac{53\nu v^4}{24} + \frac{617\nu^2 v^4}{24} - \frac{9\nu^3 v^4}{4} \]

\[ - \frac{235\dot{r} \nu r a_v}{24} + \frac{235\dot{r}^2 \nu r a_v}{6} \]

\[ + \frac{G^3 m^3}{r^3} \left( \frac{3\dot{r}^2}{2} - \frac{12041\dot{r}^2 \nu}{420} + \frac{37\dot{r}^2 \nu v^2}{4} + \frac{7\dot{r}^2 \nu^3 v^2}{2} - \frac{123\dot{r}^2 \nu^2 \nu^2}{64} \right) \]

\[ + \frac{5v^2}{4} + \frac{387\nu v^2}{70} - \frac{7\nu^2 v^2}{4} + \frac{\nu^3 v^2}{2} + \frac{41\nu^2 \nu^2 v^2}{64} + 22\dot{r}^2 \nu \ln \left( \frac{r}{r_0} \right) - \frac{22\nu v^2}{3} \ln \left( \frac{r}{r_0} \right) \]
\[ L^{(0)}_{4\text{PN}} = \frac{7}{256} v^0 - 121 \nu v^0 + 785 \nu^2 v^0 - 1127 \frac{1}{128} \nu^3 v^0 + 2415 \nu^4 v^0, \]

\[ L^{(1)}_{4\text{PN}} = m \left( \frac{23}{20} \nu^2 \dot{r}^2 - \frac{5}{128} \nu r^2 + \frac{35}{128} \nu^2 \dot{r}^2 - \frac{35}{64} \nu^3 \dot{r}^2 + \frac{35}{128} \nu^4 \dot{r}^2 + \frac{7}{4} \nu a \dot{r}^2 \nu^2 \right), \]

\[ L^{(2)}_{4\text{PN}} = m^2 \left( \frac{5407}{288} \nu a \nu \dot{r}^3 - \frac{5531}{3200} \nu^2 \nu^6 - \frac{487}{40} \nu^2 \mu^6 + \frac{13}{5} \nu^3 \mu^6 + \frac{3}{2} \nu^4 \mu^6 + \frac{11497}{48} \nu^5 \mu^6 \right), \]

\[ L^{(3)}_{4\text{PN}} = m^3 \left( \frac{4937}{1260} \nu^2 a \nu \dot{r} - \frac{1}{32} \pi^2 \nu^2 a \nu \dot{r} + \frac{44}{3} \nu^2 a \nu r \ln \left( \frac{r}{r_0} \right) \dot{r} + \frac{22}{3} \nu a \nu r \ln \left( \frac{r}{r_0} \right) \dot{r} - \frac{24673}{2240} \nu^2 \mu^6 \right), \]

\[ L^{(4)}_{4\text{PN}} = \frac{m^4}{r^4} \left( \frac{9}{4} r^2 - \frac{245971}{4200} \nu r^2 + \frac{2771}{96} \pi^2 \nu r^2 - \frac{8089}{140} \nu^2 r^2 - 44 \pi^2 \nu^2 r^2 + \frac{185}{8} \nu^3 r^2 + \frac{15}{2} \nu^4 r^2 \right). \]
and the dissipative terms disappear (see Sec. V B of [23]). The CM Lagrangian in harmonic coordinates still depends on accelerations starting at 2PN order, through \( a_n = (an) = \mathbf{a} \cdot \mathbf{n} \) and \( a_v = (av) = \mathbf{a} \cdot \mathbf{v} \), as well as logarithms of \( r/r_0 \) and \( r/r_\nu' \). After applying the contact transformation or shift of Ref. [1] (when reduced to the CM frame) the previous Lagrangian will be transformed into an ordinary, non harmonic Lagrangian, depending only on positions and velocities, and, furthermore, the logarithmic terms \( \ln(r/r_0) \) and \( \ln(r/r_\nu') \) therein will disappear (see Sec. V B of [1] for more details).

### III. CENTER-OF-MASS ENERGY AND ANGULAR MOMENTUM

The conserved energy and angular momentum may be split into instantaneous, tail and dissipative radiation reaction parts; following our conventions, we first give the instantaneous contributions, before discussing the interesting 4PN tails in Sec. IV and the dissipative terms in Sec. VI. The reduced CM energy is defined by \( \mathcal{E} = E/\mu \) and the reduced CM angular momentum by \( \mathcal{J} = J/J_N \), which is the Euclidean norm \( \mathcal{J} = |\mathbf{J}| \) rescaled by that of the Newtonian angular momentum \( J_N = \mu \mathbf{x} \times \mathbf{v} \). We have

\[
\mathcal{E}_N = \frac{v^2}{2} - \frac{Gm}{r},
\]

\[
\mathcal{E}_{1PN} = \frac{3v^4}{8} - \frac{9\nu v^4}{8} + \frac{Gm}{r} \left( \frac{r^2 v}{2} + \frac{3v^2}{2} + \frac{v^3}{2} \right) + \frac{G^2m^2}{2r^2},
\]

\[
\mathcal{E}_{2PN} = \frac{5v^6}{16} - \frac{35\nu v^6}{16} + \frac{65\nu^2 v^6}{16} + \frac{Gm}{r} \left( \frac{3r^4 v}{8} + \frac{9r^4 v^2}{8} + \frac{i^2 v^2 v}{4} - \frac{15i^2 v^2 v^2}{4} + \frac{21v^4}{8} - \frac{23\nu v^4}{8} - \frac{27\nu^2 v^4}{8} \right) + \frac{G^2m^2}{r^2} \left( \frac{r^2}{2} + \frac{69r^2 v}{8} + \frac{3i^2 v^2}{2} + \frac{7v^2}{4} - \frac{55\nu v^2}{8} + \frac{\nu^2 v^2}{2} \right) + \frac{G^3\nu v^3}{r^3} \left( \frac{1}{2} - \frac{15\nu}{4} \right),
\]

\[
\mathcal{E}_{3PN} = \frac{35v^8}{128} - \frac{413\nu v^8}{128} + \frac{833\nu^2 v^8}{64} - \frac{2261\nu^3 v^8}{128}
\]
+ \frac{Gm}{r} \left( \frac{5 r^6 \nu}{16} - \frac{25 r^6 \nu^2}{16} + \frac{25 r^6 \nu^3}{16} - \frac{9 r^4 \nu v^2}{16} + \frac{21 r^4 \nu^2 v^2}{4} 
 rightarrow \
 - \frac{165 r^4 \nu^3 v^2}{16} - \frac{21 r^2 \nu v^4}{16} - \frac{75 r^2 \nu^2 v^4}{16} + \frac{375 r^2 \nu^3 v^4}{16} 
 rightarrow \
 + \frac{55 r^6 \nu^6}{16} - \frac{215 r^6 \nu^6}{16} + \frac{29 r^6 \nu^6}{16} + \frac{325 r^6 \nu^6}{16} \right) 
 rightarrow 
 + \frac{G^2 m^2}{r^2} \left( - \frac{731 r^4 \nu}{48} + \frac{41 r^4 \nu^2}{4} + 6 r^4 \nu^3 + \frac{3 r^2 \nu v^2}{4} + \frac{31 r^2 \nu v^2}{2} 
 rightarrow 
 - \frac{815 r^2 \nu^2 v^2}{16} - \frac{81 r^2 \nu^3 v^2}{16} + \frac{135 v^4}{16} - \frac{97 \nu v^4}{8} + \frac{203 \nu v^4}{8} - \frac{27 \nu^3 v^4}{4} \right) 
 rightarrow 
 + \frac{G^3 m^3}{r^3} \left( \frac{3 r^2}{2} + \frac{803 r^2 \nu}{840} + \frac{51 r^2 \nu^2}{4} + \frac{7 r^2 \nu^3}{2} - \frac{123 r^2 \nu \pi^2}{64} + \frac{5 v^2}{4} 
 rightarrow 
 - \frac{6747 r^2 \nu}{280} - \frac{21 r^2 v^2}{4} + \frac{\nu^3 v^2}{2} + \frac{41 \nu \pi^2 v^2}{64} \right) 
 rightarrow 
 + 22 r^2 \nu \ln \left( \frac{r}{r_0} \right) - \frac{22 r^2 \nu v^2}{3} \ln \left( \frac{r}{r_0} \right) \right) 
 rightarrow 
 + \frac{G^4 m^4}{r^4} \left( \frac{3}{8} + \frac{18469 \nu}{840} - \frac{22 \nu}{3} \ln \left( \frac{r}{r_0} \right) \right), \quad (3.1d) 
 E_{4PN}^{(0)} = \left( \frac{63}{256} - \frac{1089}{256} \nu + \frac{7065 \nu^2}{256} - \frac{10143 \nu^3}{128} + \frac{21735 \nu^4}{256} \right) v^{10}, \quad (3.1e) 
 E_{4PN}^{(1)} = \frac{m}{r} \left( - \frac{35}{128} r^8 + \frac{245 \nu r^8}{128} - \frac{245 \nu^3 r^8}{128} + \frac{245 \nu^4 r^8}{128} + \frac{25 \nu^2 r^6 v^2}{32} - \frac{125 \nu^2 r^4 v^2}{16} \right) 
 rightarrow 
 + \frac{185 \nu^3 r^6 v^2}{32} - \frac{595 \nu^4 r^4 v^4}{32} + \frac{27 \nu^4 r^4 v^4}{32} - \frac{243 \nu^2 r^4 v^4}{32} - \frac{1683 \nu^4 r^4 v^4}{32} + \frac{4851 \nu^4 r^4 v^4}{32} \right) 
 rightarrow 
 - \frac{147 \nu^2 r^6 v^2}{32} + \frac{369 \nu^2 r^6 v^2}{32} + \frac{423 \nu^2 r^6 v^2}{32} - \frac{4655 \nu^2 r^6 v^2}{32} + \frac{525 \nu^2 r^6 v^2}{32} - \frac{4011 \nu v^8}{128} \right) 
 rightarrow 
 + \frac{9507 \nu^2 r^8}{128} - \frac{357 \nu^2 r^8}{128} - \frac{15827 \nu^2 r^8}{128} \right), \quad (3.1f) 
 E_{4PN}^{(2)} = \frac{m^2}{r^2} \left( - \frac{4771}{640} \nu r^6 - \frac{461}{8} \nu^2 r^6 - \frac{17}{2} \nu^3 r^6 + \frac{15}{2} \nu^4 r^6 + \frac{5347}{384} \nu^4 r^2 v^2 + \frac{19465}{96} \nu^2 r^4 v^2 \right) 
 rightarrow 
 - \frac{439 \nu^3 r^4 v^2}{8} - \frac{135 \nu^4 r^4 v^2}{8} + \frac{15 \nu^2 r^4 v^2}{16} - \frac{5893 \nu^2 r^4 v^2}{128} - \frac{12995 \nu^2 r^2 v^4}{64} - \frac{18511 \nu^2 r^2 v^4}{64} \right) 
 rightarrow 
 + \frac{2845 \nu^4 r^2 v^4}{32} + \frac{575 \nu^4 r^2 v^4}{32} + \frac{4489 \nu v^6}{128} + \frac{5129 \nu v^6}{64} - \frac{8289 \nu v^6}{64} + \frac{975 \nu v^6}{16} \right), \quad (3.1g) 
 E_{4PN}^{(3)} = \frac{m^3}{r^3} \left( - \frac{2599207}{6720} \nu r^6 - \frac{6465}{1024} \nu^2 r^6 - \frac{103205}{224} \nu^3 r^6 + \frac{615}{128} \nu^2 r^4 v^2 + \frac{69}{32} \nu^3 r^4 v^2 + \frac{87}{4} \nu^2 r^4 v^2 \right) 
 rightarrow 
 - \frac{55 \nu^2 v^6}{16} - \frac{55 \nu \ln \left( \frac{r}{r_0} \right)}{r^4} - \frac{220 \nu v^2}{16} + \frac{21 \nu v^2}{4} - \frac{3386923}{1680} \nu^2 r^2 v^2 \right) 
 rightarrow 
 + \frac{333 \nu^2 r^2 v^2}{512} + \frac{206013 \nu^2 r^2 v^2}{560} + \frac{123 \nu^2 r^2 v^2}{64} - \frac{2437 \nu^3 r^2 v^2}{16} - \frac{141 \nu^3 r^2 v^2}{2} \right) 
 rightarrow 
 - \frac{33 \nu \ln \left( \frac{r}{r_0} \right)}{r^4} - \frac{22 \nu v^2}{16} + \frac{55 \nu \ln \left( \frac{r}{r_0} \right)}{r^4} - \frac{220 \nu^2}{16} + \frac{273}{16} \nu^4 v^4 \right) 
 rightarrow 
}
\[ \mathcal{E}_{4\text{PN}}^{(4)} = \frac{m^4}{r^4} \left( \frac{9}{4} \dot{r}^2 - \frac{1622437}{12600} \nu \dot{r}^2 + \frac{2645}{96} \nu^2 \dot{r}^2 - \frac{289351}{2520} \nu^2 \dot{r}^2 - \frac{1367}{32} \nu^2 \dot{r}^2 + \frac{213}{8} \nu^3 \dot{r}^2 \\
+ \frac{15}{2} \nu^4 \dot{r}^2 - 64 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 + \frac{746}{3} \nu \ln \left( \frac{r}{r_0} \right) \dot{r}^2 - \frac{1352}{3} \nu^2 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 + 64 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 \\
- 507 \nu \ln \left( \frac{r}{r_0} \right) \dot{r}^2 + 1004 \nu^2 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 + \frac{15}{16} \nu^2 \dot{r}^2 + \frac{1859363}{16800} \nu \dot{r}^2 - \frac{149}{32} \nu^2 \dot{r}^2 \\
+ \frac{22963}{5040} \nu^2 \dot{r}^2 + \frac{311}{32} \nu^2 \dot{r}^2 - \frac{29}{8} \nu^3 \dot{r}^2 + \frac{1}{2} \nu^4 \dot{r}^2 + 16 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 - \frac{335}{3} \nu \ln \left( \frac{r}{r_0} \right) \dot{r}^2 \\
+ 131 \nu^2 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 - 16 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 + \frac{394}{3} \nu \ln \left( \frac{r}{r_0} \right) \dot{r}^2 - 808 \nu^2 \ln \left( \frac{r}{r_0} \right) \dot{r}^2 \right) , \tag{3.1i} \]
\[
\mathcal{E}_{4\text{PN}}^{(5)} = \frac{m^5}{r^5} \left( -\frac{3}{8} - \frac{1697177}{25200} \nu - \frac{105}{32} \frac{\pi^2 \nu - 55111}{720} \nu^2 + 11 \pi^2 \nu^2 + 16 \ln \left( \frac{r}{r_0} \right) - \frac{82}{3} \nu \ln \left( \frac{r}{r_0} \right) \right) \\
+ 120 \nu^2 \ln \left( \frac{r}{r_0} \right) - 16 \ln \left( \frac{r}{r_0} \right) + 124 \nu \ln \left( \frac{r}{r_0} \right) - 240 \nu^2 \ln \left( \frac{r}{r_0} \right) \right) , \tag{3.1j} \]

and

\[ J_N = 1 , \tag{3.2a} \]
\[ J_{1\text{PN}} = (1 - 3 \nu) \frac{\nu^2}{2} + \frac{Gm}{r} (3 + \nu) , \tag{3.2b} \]
\[ J_{2\text{PN}} = \frac{3 \nu^4}{8} - \frac{21 \nu v^4}{8} + \frac{39 \nu^2 v^2}{8} \\
+ \frac{Gm}{r} \left( -\frac{\nu^2}{2} + \frac{7}{2} \nu^2 - \frac{9}{2} \nu^2 \right) \\
+ \frac{G^2 m^2}{r^2} \left( \frac{7}{2} - \frac{41 \nu}{4} + \nu^2 \right) , \tag{3.2c} \]
\[ J_{3\text{PN}} = \frac{5 \nu^6}{16} - \frac{59 \nu v^6}{16} + \frac{119 \nu^2 v^6}{8} - \frac{323 \nu^3 v^6}{16} \\
+ \frac{Gm}{r} \left( \frac{3 \nu^4}{4} - \frac{3 \nu^4}{4} - \frac{33 \nu^4 v^3}{4} - 3 \nu^2 v^2 + \frac{7 i^2 v^2 v^2}{4} \\
+ \frac{75 \nu^2 v^2 v^2}{4} + 33 \nu^4 + \frac{71 \nu v^4}{4} + \frac{53 \nu^2 v^4}{4} + \frac{195 \nu^3 v^4}{8} \right) \\
+ \frac{G^2 m^2}{r^2} \left( \frac{\nu^2}{2} - \frac{287 \nu^2 v^2}{24} - \frac{317 \nu^2 v^2}{8} - \frac{27 \nu^2 v^3}{2} + \frac{45 \nu^2 v^2}{4} \\
- \frac{161 \nu v^2}{6} + \frac{105 \nu v^2 v^2}{4} - 9 \nu^3 v^2 \right) \\
+ \frac{G^3 m^3}{r^3} \left( \frac{5}{2} - \frac{5199 \nu}{280} - 7 \nu^2 + \nu^3 + \frac{41 \nu^2}{32} - \frac{44 \nu}{3} \ln \left( \frac{r}{r_0} \right) \right) , \tag{3.2d} \]
\[ J_{4\text{PN}}^{(0)} = \left( \frac{35}{128} - \frac{605}{128} \nu + \frac{3925}{128} \nu^2 - \frac{5635}{64} \nu^3 + \frac{12075}{128} \nu^4 \right) v^8, \]  
\[ J_{4\text{PN}}^{(1)} = \frac{m}{r} \left( \frac{5}{8} \nu r^6 + \frac{15}{8} \nu^2 r^6 + \frac{45}{16} \nu^3 r^6 - \frac{85}{16} \nu^4 r^6 + 3 \nu^4 v^2 - \frac{45}{4} \nu^2 r^6 v^2 - \frac{135}{16} \nu^4 r^6 v^2 + \frac{693}{16} \nu^4 r^4 v^2 - \frac{53}{16} \nu^2 r^4 v^2 + 423 \nu^3 r^6 v^2 + 299 \nu^3 r^4 v^2 - 1995 \nu^4 r^4 v^2 + \frac{75}{16} v^6 + \frac{151}{16} \nu^4 v^6 + 1553 \nu^2 v^6 - 6425 \nu^3 v^6 - 2261 \nu^4 v^6 \right), \]  
\[ J_{4\text{PN}}^{(2)} = \frac{m^2}{r^2} \left( \frac{14773}{320} \nu r^4 + \frac{3235}{48} \nu^2 r^4 - \frac{155}{4} \nu^3 r^4 - 27 \nu^4 r^4 + \frac{3}{4} \nu^2 v^2 - \frac{5551}{60} \nu^4 r^2 v^2 - \frac{256}{3} \nu^2 r^2 v^2 + \frac{4459}{16} \nu^3 r^2 v^2 + \frac{569}{4} \nu^4 r^2 v^2 + \frac{345}{16} \nu^8 - \frac{65491}{960} \nu^4 v^4 + \frac{12427}{96} \nu^2 v^4 - \frac{3845}{32} \nu^8 v^4 + \frac{585}{8} \nu^4 v^4 \right), \]  
\[ J_{4\text{PN}}^{(3)} = \frac{m^3}{r^3} \left( \frac{7}{2} \nu r^2 + \frac{7775977}{16800} \nu^2 r^2 + \frac{447}{256} \nu^3 r^2 + \frac{121449}{560} \nu^2 r^2 - \frac{1025}{8} \nu^3 r^2 - 47 \nu^4 i^2 - 110 \nu \ln \left( \frac{r}{r_0} \right) r^2 + 44 \nu \ln \left( \frac{r}{r_0} \right) r^2 - 176 \nu \ln \left( \frac{r}{r_0} \right) r^2 + \frac{91}{4} \nu^2 r^2 - \frac{1357609}{50400} \nu \nu^2 \right) + \frac{469}{256} \nu^2 v^2 + \frac{276433}{5040} \nu^2 v^2 + \frac{41}{16} \nu^2 v^2 + \frac{637}{8} \nu^3 v^2 - 15 \nu v^2 - \frac{44}{3} \nu \ln \left( \frac{r}{r_0} \right) v^2 + \frac{88}{3} \nu v^2 - \frac{22}{3} \nu \ln \left( \frac{r}{r_0} \right) \ln \left( \frac{r}{r_0} \right) v^2 + \frac{88}{3} \nu \ln \left( \frac{r}{r_0} \right) v^2 \right), \]  
\[ J_{4\text{PN}}^{(4)} = \frac{m^4}{r^4} \frac{15}{8} + \frac{3809041}{25200} \nu - \frac{85}{8} \pi^2 \nu - \frac{20131}{420} \nu^2 + \frac{663}{32} \pi^2 \nu^2 - \frac{15}{4} \nu^3 + \nu^4 + 32 \ln \left( \frac{r}{r_0} \right) - \frac{406}{3} \nu \ln \left( \frac{r}{r_0} \right) + \frac{742}{3} \nu^2 \ln \left( \frac{r}{r_0} \right) - 32 \ln \left( \frac{r}{r_0} \right) + \frac{766}{3} \nu \ln \left( \frac{r}{r_0} \right) - \frac{1528}{3} \nu^2 \ln \left( \frac{r}{r_0} \right) \right). \]  

IV. GRAVITATIONAL WAVE TAILS AT 4PN ORDER

We start with the end result of Refs. [1, 2], where the Fokker Lagrangian at the 4PN order in harmonic coordinates was obtained as the sum of instantaneous and tail contributions: \( L = L^{\text{inst}} + L^{\text{tail}} \). This Lagrangian, in our approach, provides only the conservative part of the dynamics and does not account for dissipative effects.\(^5\) See App. A for a recap of its 4PN instantaneous terms. Now, the 4PN tail piece is given as the non-local-in-time integral

\[ L^{\text{tail}} = \frac{G^2 M}{5 c^8} I_{ij}^{(3)}(t) \text{ Pf } \int_{2 \tau_{12}/c}^{t} dt' \int_{-\infty}^{+\infty} \frac{dt}{|t - t'|} I_{ij}^{(3)}(t') = \frac{G^2 M}{5 c^8} I_{ij}^{(3)}(t) \int_{0}^{+\infty} d\tau \ln \left( \frac{c t}{2 r_{12}} \right) \left[ I_{ij}^{(4)}(t - \tau) - I_{ij}^{(4)}(t + \tau) \right]. \]  

\(^5\) See [41] and references therein for an alternative approach in the EFT context where the Lagrangian describes at once conservative and dissipative effects.
In the above expression, \( M \) is the constant ADM mass of the system, such that \( M = m + \mathcal{O}(c^{-2}) \), \( I_{ij}(t) \) are the components of the STF quadrupole moment of the two particles at Newtonian order, \( I_{ij} = \sum A m_A y_A^i y_A^j \), and \( I_{ij}^{(s)}(t) \) is the \( s \)-th time derivative of \( I_{ij}(t) \) performed “off-shell”, i.e., without order reduction by means of the equations of motion. The tail integral involves the Hadamard \( \text{partie finie} \) (Pf) prescription, which depends on some arbitrary scale, here chosen to be the relative separation between the two particles \( r_{12} = r_{12}(t) \) at time \( t \) in harmonic coordinates. For convenience, we introduce a special notation for the tail factor:

\[
\mathcal{T}_{ij}^{(s)}(t) = \text{Pf} \int_{-\infty}^{+\infty} \frac{dt'}{t - t'} I_{ij}^{(s)}(t') .
\]  

(4.2)

Beware that, because of the presence of the time-dependent Hadamard scale \( r_{12} \) we have adopted, \( \mathcal{T}_{ij}^{(s)} \) is not the time derivative of \( \mathcal{T}_{ij}^{(s-1)} \).

Varying the Lagrangian with respect to \( y_A \) leads to the conservative part of the acceleration \( a_A = \frac{dv_A}{dt} \), which similarly decomposes into instantaneous and tail parts as \( a_A^{\text{inst}} + a_A^{\text{tail}} \). The instantaneous part contains many terms up to the 4PN order; in the present section, we analyze the effect of including the 4PN tail piece (4.1) into the Lagrangian. Since this piece is non local, its variation involves functional derivatives rather than ordinary ones.

We find for the acceleration of particle 1:

\[
a_1^{\text{tail}} = - \frac{4G^2M}{5c^8} y_1^j \mathcal{T}_{ij}^{(6)}
+ \frac{8G^2M}{5c^8} y_1^j \left[ (I_{ij}^{(3)} \ln r_{12})^{(3)} - I_{ij}^{(6)} \ln r_{12} \right] - \frac{2G^2M n_{12}^i}{5m_1 c^8} \frac{1}{r_{12}} (I_{jk}^{(3)})^2 .
\]  

(4.3)

The first term coincides with the conservative part of the known 4PN tail effect [37–39], while the other terms, which are in fact instantaneous, come from the variation of the “constant” \( r_{12}(t) \) in Eq. (4.1). In the CM frame, the tail part of the acceleration is directly obtained by reducing the previous expression (4.3) to the CM. Indeed, there are no tail contributions in the CM relations (B4), i.e., \( y_A^{\text{tail}} = 0 \) when considered as a function of the relative variables \( x \) and \( v \). Therefore, the tail acceleration \( a_i^{\text{tail}} = a_1^{\text{tail}} - a_2^{\text{tail}} \) reads simply (with \( r = r_{12} \))\(^6\)

\[
a_i^{\text{tail}} = - \frac{4G^2M}{5c^8} x^j \mathcal{T}_{ij}^{(6)}
+ \frac{8G^2M}{5c^8} x^j \left[ (I_{ij}^{(3)} \ln r)^{(3)} - I_{ij}^{(6)} \ln r \right]
- \frac{2G^2 n_i}{5c^8} \frac{1}{r} (I_{jk}^{(3)})^2 .
\]  

(4.4)

The instantaneous part of the relative acceleration in the CM frame has already been displayed in Eqs. (2.1)–(2.3). The non local tail integral in (4.4) cannot be expressed in analytic

\(^6\) For completeness, let us give two useful results concerning the third time-derivative of the quadrupole moment in the CM variables:

\[
I_{ij}^{(3)} = \frac{2Gm^2 \nu}{r^2} \left[ -4n^{(i} v^{j)} + 3r n^{(i} n^{j)} \right] ,
\]

\[
(I_{ij}^{(3)})^2 = \frac{8G^2 m^4 \nu^2}{r^4} \left( 4\nu^2 - \frac{11}{3} r^2 \right) .
\]
closed form for general orbits, but it can usefully be written using Fourier series [see (4.9) below]. For circular orbits, the tail factor does reduce to a simple closed-form expression as in Eq. (5.1). The dissipative part of the equations of motion (including notably that associated with the tail term) will be investigated in Sec. VI.

Given the accelerations \( \mathbf{a}_A \) as functionals of the positions and velocities of the particles for general orbits, we compute the associated conserved (Noetherian) energy by forming the combination \( \sum m_A v_A \cdot \mathbf{a}_A \), where \( \mathbf{a}_A \) is the explicit expression of the acceleration, including the tail term (4.3), and writing it in the form of a total derivative, say \(-dE_0/dt\). The looked-for energy will then be the sum of kinetic and “potential” contributions, \( E = \sum \frac{1}{2} m_A v_A^2 + E_0 \), with \( E_0 \) starting at leading order with the usual Newtonian potential energy \(-Gm_1m_2/r_{12}\).

Applying this method to the instantaneous (local) part of the acceleration, \( \mathbf{a}_A^{\text{inst}} \), we readily find that there is, evidently, a well defined notion of “instantaneous” conserved energy, say \( E_0^{\text{inst}} = \sum \frac{1}{2} m_A v_A^2 + E_0^{\text{inst}} \). We are neglecting the radiation reaction piece, which will be added in Sec. VI. The instantaneous part of the energy, \( E_0^{\text{inst}} \), comprises many terms up to the 4PN order. It was provided in the CM frame in Sec. III.

Looking next for the same combination as before but for the tail part of the acceleration, explicitly given by (4.3), we get, in a first stage, after a series of operations by parts,

\[
\sum A m_A v^i_A (a^i_A)^{\text{tail}} = -\frac{dE_0^{\text{tail}}}{dt} - H_1^{\text{tail}}. \tag{4.5}
\]

The first term, indeed, takes the form of the total time derivative of a certain quantity:

\[
E_0^{\text{tail}} = \frac{2G^2M}{5c^8} \left[ I_{ij}^{(1)} \mathcal{T}_{ij}^{(5)} - I_{ij}^{(2)} \mathcal{T}_{ij}^{(4)} + \frac{1}{2} I_{ij}^{(3)} \mathcal{T}_{ij}^{(3)} \right]
-
\frac{4G^2M}{5c^8} \left( 2I_{ij}^{(1)} I_{ij}^{(4)} (\ln r_{12})^{(1)} - I_{ij}^{(2)} I_{ij}^{(3)} (\ln r_{12})^{(1)}(1) + I_{ij}^{(1)} I_{ij}^{(3)} (\ln r_{12})^{(2)} \right), \tag{4.6}
\]

where we have used the notation (4.2) for tails, the non-tail terms coming from the derivation of the Hadamard partie finie scale \( r_{12} \). However, performing simple operations by parts does not allow to recast the second term in (4.5) into the requested form. It remains like a “flux” at this stage, given by

\[
H_1^{\text{tail}} = \frac{G^2M}{5c^8} \left[ I_{ij}^{(3)} \mathcal{T}_{ij}^{(4)} - I_{ij}^{(4)} \mathcal{T}_{ij}^{(3)} \right]. \tag{4.7}
\]

Note that \( r_{12} \) cancels out from the two terms in the right-hand side of (4.7). Because of the “flux” (4.7), it appears that the problem of finding the complete total time derivative defining the energy in the non-local case is more complicated.

The problem has been solved in Ref. [2] by resorting to (discrete) Fourier series. Let us decompose the components of the quadrupole moment as

\[
I_{ij}(t) = \sum_{p=-\infty}^{+\infty} \mathcal{I}_{ij}^{p} e^{ip\ell} \quad \iff \quad \mathcal{I}_{ij}^{p} = \int_{0}^{2\pi} \frac{d\ell}{2\pi} I_{ij} e^{-ip\ell}, \tag{4.8}
\]

where \( \ell = n(t-t_0) \) is the mean anomaly, \( n = 2\pi/P \) the radial frequency, \( P \) the orbital period and \( t_0 \) some instant of passage at periastron.\(^7\) The Fourier coefficients \( \mathcal{I}_{ij}^{p} \) depend

\(^7\) The “azimuthal” frequency \( \omega \), averaged over one orbit, agrees with the radial frequency, as there is no precession at Newtonian order: \( \omega = n \).
on \( n \) as well as the orbit’s eccentricity \( e \), and satisfy \( p I_{ij} = -p I_{ij} \). They are available as linear combinations of Bessel functions (see two possible forms presented in App. B of [2] and App. A of [63]). With those notations, we have

\[
T_{ij}^{(s)} = -2 \sum_{p=\infty}^{+\infty} (ip n)^{s} \frac{I_{ij}}{p} \left( \ln(2|p|nr) + \gamma_E \right) e^{ip \ell} .
\]  

(4.9)

Now, it was shown in Sec. III A of [2] that the “flux” (4.7) does admit a first integral in the sense of Fourier series, i.e., there exists some \( E_{1\text{tail}} \) such that

\[
\frac{dE_{1\text{tail}}}{dt} = H_{1\text{tail}}^T .
\]  

(4.10)

Moreover, we found that \( E_{1\text{tail}} \) contains a crucial “DC” contribution, i.e., constant in time, which is furthermore directly related, quite remarkably, to the total averaged gravitational-wave energy flux \( F_{GW} = \frac{G}{5c} \langle \left(f_{ij}^{(3)}\right)^2 \rangle \). More precisely,

\[
E_{1\text{DC}}^{\text{tail}} = -\frac{2G^2 M n^6}{5c^8} \sum_{p=-\infty}^{\infty} |I_{ij}|^2 p^6 = -\frac{2GM}{c^3} F_{GW} .
\]  

(4.11)

In addition, there is an “AC” contribution, i.e., oscillating with zero time average, which is given by a double Fourier series (over \( p, q \) such that \( p + q \neq 0 \)):

\[
E_{1\text{AC}}^{\text{tail}} = \frac{G^2 M n^6}{5c^8} \sum_{p+q\neq0} I_{ij} I_{ij} p^3 q^3 (p - q) \ln \left| \frac{p}{q} \right| e^{i(p+q)\ell} .
\]  

(4.12)

In conclusion, we have \( E_{1\text{tail}} = E_{1\text{DC}} + E_{1\text{AC}} \). To emphasize this result, let us observe that the presence of the latter DC and AC contributions in the energy implies that, in a Hamiltonian formalism, the conserved energy is not equal to the value of the Hamiltonian “on-shell”, i.e., computed along trajectories satisfying the corresponding Hamilton’s equations. This is due to the fact that, for a non-local dynamics, the Hamiltonian equations involve functional derivatives instead of ordinary ones. Finally, the complete contributions of tails to the conserved energy, which are to be added to the instantaneous contribution \( E_{\text{inst}} \) given by Eqs. (3.1) in Sec. III, consist of the three terms computed above:

\[
E_{\text{tail}} = E_{0\text{tail}} + E_{1\text{DC}} + E_{1\text{AC}} .
\]  

(4.13)

Next, we look for the tail contributions to the conserved integral of the angular momentum. Proceeding in a similar way as for the energy, we form the combination \( \sum_{A} m_A \epsilon_{ijk} y_{A}^j a_{A}^{k\text{tail}} \) and perform a series of operations by parts yielding

\[
\sum_{A} m_A \epsilon_{ijk} y_{A}^j a_{A}^{k\text{tail}} = -\frac{dJ_0^{\text{tail}}}{dt} - K_1^{\text{tail}} ,
\]  

(4.14)

where

\[
J_0^{\text{tail}} = \frac{4G^2 M}{5c^8} \epsilon_{ijk} \left[ I_{jl} T_{kl}^{(5)} - I_{jl}^{(1)} T_{kl}^{(4)} + I_{jl}^{(2)} T_{kl}^{(3)} \right].
\]
\[
- \frac{8G^2 M}{5c^8} \epsilon_{ijk} \left( 2I_{jl} I_{kl}^{(4)} (\ln r_{12})^{(1)} - I_{jl}^{(1)} I_{kl}^{(3)} (\ln r_{12})^{(1)} + I_{jl} I_{kl}^{(3)} (\ln r_{12})^{(2)} \right), \tag{4.15a}
\]

\[
K_{i}^{\text{tail}} = -\frac{4G^2 M}{5c^8} \epsilon_{ijk} I_{kl}^{(3)} T_{cl}^{(3)}. \tag{4.15b}
\]

We then rely on Ref. [2] to transform the second term in Eq. (4.14) into a total time derivative after decomposing it as a Fourier series. The computation parallels that for the energy and permits constructing some \( J_{i}^{\text{tail}} \) that satisfies

\[
\frac{dJ_{i}^{\text{tail}}}{dt} = K_{i}^{\text{tail}}. \tag{4.16}
\]

Like for the energy, \( J_{i}^{\text{tail}} \) is made of DC and AC contributions i.e., \( J_{i}^{\text{tail}} = J_{i}^{\text{DC}} + J_{i}^{\text{AC}} \). The DC contribution, remarkably, is proportional to the gravitational-wave flux of angular momentum. Explicitly, we have\(^8\)

\[
J_{i}^{\text{DC}} = \frac{4G^2 M N^5}{5c^8} \sum_{p=-\infty}^{+\infty} i \epsilon_{ijk} I_{jl}^{(5)} I_{kl}^{(5)} p^5 = -\frac{2GM}{c^3} G_{GW}^{i}, \tag{4.17a}
\]

\[
J_{i}^{\text{AC}} = -\frac{4G^2 M N^5}{5c^8} \sum_{p+q \neq 0} i \epsilon_{ijk} I_{jl}^{(5)} I_{kl}^{(5)} \left( \frac{p}{p} \right)^3 \left( \frac{q}{q} \right)^3 \ln \left( \frac{p}{q} \right) e^{i(p+q)\ell}. \tag{4.17b}
\]

The complete tail contribution to the angular momentum is finally

\[
J^{\text{tail}} = J_{i}^{\text{tail}} + J_{i}^{\text{DC}} + J_{i}^{\text{AC}}, \tag{4.18}
\]

which is thus to be added to the instantaneous contributions presented in Eqs. (3.2).

Finally, we obtain the tail parts corresponding to the linear momentum \( P \) and the CM position \( G \). The instantaneous part of the CM position, \( G^{\text{inst}} \), is provided in App. B. In these two cases the analysis is simpler than for \( E \) and \( J \) because the usual operations by parts applied to the relevant combination \( \sum A A_\text{tail} \) directly lead to the requested total time derivatives and conservation laws. We find

\[
P_{i}^{\text{tail}} = \frac{4G^2 M}{5c^8} \left[ I_{j} T_{ij}^{(5)} - I_{j}^{(1)} T_{ij}^{(4)} \right]
\]

\(^8\) The norms of the vectors (4.17) are obtained by projection perpendicular to the orbital plane. For instance,

\[
\ell \epsilon_{ijk} I_{jl}^{(5)} I_{kl}^{(5)} = \left( \frac{I_{xx}}{p} - \frac{I_{yy}}{q} \right) - \frac{I_{xy}}{q} \left( \frac{I_{xx}}{p} - \frac{I_{yy}}{q} \right),
\]

where \( \ell = n \times \lambda = (0, 0, 1) \) denotes the unit vector orthogonal to the orbital plane, which is spanned by the two moving unit vectors \( n = (\cos \varphi, \sin \varphi, 0) \) and \( \lambda = (-\sin \varphi, \cos \varphi, 0) \), with \( \varphi \) being the orbital phase angle. The spatial coordinates \((x, y, z)\) are such that \((x, y)\) lies in this plane, with \( z \) in the direction along \( \ell \) (i.e., in the sense of the orbital motion). Notice also that

\[
\frac{1}{2} \ell \epsilon_{klm} I_{jm}^{(5)} = \ell \epsilon_{klm} I_{jm}^{(5)}.
\]

18
\[-8G^2 M \frac{\rho}{c^8} \left( 2\gamma_{ij}^{(4)} (\ln r_{12})^{(1)} - \gamma_{ij}^{(1)} (\ln r_{12})^{(1)} + \gamma_{ij}^{(3)} (\ln r_{12})^{(2)} \right), \quad (4.19a)\]

\[G_{\text{tail}}^i = \frac{4G^2 M}{c^8} \left[ \gamma_{ij}^{(4)} - 2\gamma_{ij}^{(1)} \gamma_{ij}^{(3)} \right] - \frac{8G^2 M}{5c^8} \gamma_{ij}^{(3)} (\ln r_{12})^{(1)}. \quad (4.19b)\]

Note that the quadrupolar factors above are coupled to the Newtonian mass dipole moment \( I_i \equiv \sum_A m_A y_A^i \). At Newtonian order, the CM position reduces to the mass dipole moment, \( G^i = I_i + \mathcal{O}(c^{-2}) \), hence we see from Eqs. (4.19) that the tails will not affect the definition of the CM frame since we can pose \( I_i = 0 \) in Eqs. (4.19) at the current PN order.

V. EQUATIONS OF MOTION FOR CIRCULAR ORBITS

With the dynamics in the CM frame in hand, we are in the position to reduce it to the case of circular orbits. The conservative part of the relative acceleration is then given by the purely radial acceleration \( \mathbf{a} = -\omega^2 \mathbf{x} \), the physical content of which being entirely encoded into the relation between the orbital frequency \( \omega \) and the orbital separation \( r \). All the results in this section consistently include the tail effect.

The tail term in the relative CM acceleration has been already shown in Eq. (4.4), and the instantaneous terms in Eqs. (2.1)–(2.3). For circular orbits, the tail integral can be evaluated in closed-form. Using \( M = m \), we have at that order,

\[ T_{ij}^{(6)} = -64 \frac{G^2 m^3 \nu}{r^6} \left( v^i v^j - \frac{Gm}{r^3} x^i x^j \right) \left[ \ln (4\sqrt{\gamma}) + \gamma_E \right], \quad (5.1)\]

where the PN parameter \( \gamma \) has been defined in Eq. (1.2); we recall that \( \gamma_E \) is Euler’s constant. Notice that (5.1) is automatically STF as we have \( v^2 = \frac{Gm}{r} \) for circular orbits at Newtonian order. From Eq. (4.4), we readily obtain the tail contribution as \( a_{\text{tail}} = -\omega_{\text{tail}}^2 \mathbf{x} \), where

\[ \omega_{\text{tail}}^2 = \frac{128 Gm}{5 r^3} \gamma^4 \nu \left[ \ln (16\gamma) + 2\gamma_E + \frac{1}{2} \right]. \quad (5.2)\]

On the other hand, the instantaneous contributions are computed by a straightforward reduction of the equations of motion (2.2)–(2.3) to circular orbits, with \( \dot{r} = 0 \) and \( r^2 = r^2 \omega^2 \) (since we neglect the dissipative terms). Adding the tail contribution (5.2), we get

\[ \omega^2 = \frac{Gm}{r^3} \left\{ 1 + (-3 + \nu)\gamma + \left( 6 + \frac{41}{4} \nu + \nu^2 \right) \gamma^2 \right. \]
\[ + \left( -10 + \left[ -\frac{75707}{840} + \frac{41}{64} \pi^2 + 22 \ln \frac{r}{r_0} \right] \nu + \frac{19}{2} \nu^2 + \nu^3 \right) \gamma^3 + \left[ 15 + 48 \ln \left( \frac{r_0}{r_0} \right) + \nu \frac{19644217}{33600} + \frac{163}{1024} \pi^2 + \frac{256}{5} \gamma_E + \frac{128}{5} \ln (16\gamma) + 82 \ln \left( \frac{r}{r_0} \right) - 372 \ln \left( \frac{r}{r_0} \right) \right] \]
\[ + \nu^2 \left[ \frac{44329}{336} - \frac{1907}{64} \pi^2 - \frac{992}{3} \ln \frac{r}{r_0} + 720 \ln \left( \frac{r}{r_0} \right) + \frac{51}{4} \nu^3 + \nu^4 \right] \gamma^4 \} \quad (5.3)\]

This is a gauge dependent result, as the separation \( r \) refers to harmonic coordinates. It depends on the gauge constants \( r_0' \) and \( r_0'' \) defined in (2.4). Inverting (5.3), we express \( \gamma = \frac{2Gm}{r^3} \) as a function of the orbital frequency \( \omega \) or, rather, of the PN parameter \( x \equiv (\frac{Gm}{c^3})^{2/3} \):

\[ \gamma = x \left\{ 1 + x \left( 1 - \frac{1}{3} \nu \right) + x^2 \left( 1 - \frac{65}{12} \nu \right) \right\}. \]
\[+ x^3 \left( 1 + \nu \left[ -\frac{2203}{2520} - \frac{41}{192} \pi^2 - \frac{22}{3} \ln \left( \frac{Gm}{c^2 r_0^\prime} \right) + \frac{22}{3} \ln(x) \right] + \frac{229}{36} \nu^2 + \frac{\nu^3}{81} \right) + x^4 \left( 1 + 16 \ln \left( \frac{Gm}{c^2 r_0^\prime} \right) - 16 \ln \left( \frac{Gm}{c^2 r_0^\prime} \right) - \frac{1261}{324} \nu^3 + \frac{\nu^4}{243} \right) + \nu \left[ -\frac{2067859}{33600} - \frac{256}{15} \gamma_E - \frac{5411}{3072} \pi^2 - 86 \ln \left( \frac{Gm}{c^2 r_0^\prime} \right) \right] + 124 \ln \left( \frac{Gm}{c^2 r_0^\prime} \right) - \frac{256}{15} \ln 4 - \frac{698}{15} \ln(x) \right] + \nu^2 \left[ \frac{153613}{15120} + \frac{6049}{576} \pi^2 + \frac{1168}{9} \ln \left( \frac{Gm}{c^2 r_0^\prime} \right) - 240 \ln \left( \frac{Gm}{c^2 r_0^\prime} \right) + \frac{992}{9} \ln(x) \right] \} \right]. \quad (5.4)

Let us deal next with the conserved energy as a function of the separation \( r \). All the instantaneous terms in the CM frame are presented in Eqs. (3.1), while the tail part in the energy has been obtained as the sum of three terms in Eq. (4.13). For circular orbits, the term \( E_0^\text{tail} \) does contribute, as well as the crucial constant DC term in (4.13); by contrast, the AC contribution vanishes in this case. Furthermore there are extra tail terms coming from the reduction to circular orbits. We are led to the 4PN energy for circular orbits as a function of \( \gamma \):

\[ E = -\frac{\mu c^2 \gamma}{2} \left\{ 1 + \left( -\frac{7}{4} + \frac{1}{4} \nu \right) \gamma + \left( -\frac{7}{8} + \frac{49}{8} \nu + \frac{1}{8} \nu^2 \right) \gamma^2 + \left( -\frac{235}{64} \right) \right\} \]

\[ + \left[ \frac{46031}{2240} - \frac{123}{64} \pi^2 + \frac{22}{3} \ln \left( \frac{r}{r_0^\prime} \right) \nu + \frac{27}{32} \nu^2 + \frac{5}{64} \nu^3 \right] \gamma^3 + \left( -\frac{649}{128} + 16 \ln \left( \frac{r_0^\prime}{r_0} \right) \right) \]

\[ + \nu \left[ -\frac{3357833}{28800} + \frac{384}{5} \gamma_E + \frac{192}{5} \ln(16 \gamma) + \frac{14935}{1024} \pi^2 + 31 \ln \left( \frac{r}{r_0} \right) - 124 \ln \left( \frac{r_0}{r_0} \right) \right] \]

\[ + \nu^2 \left[ \frac{83959}{8064} - \frac{957}{128} \pi^2 - \frac{349}{3} \ln \left( \frac{r}{r_0} \right) + 240 \ln \left( \frac{r}{r_0} \right) + \frac{69}{64} \nu^3 + \frac{7}{128} \nu^4 \right] \right\} \gamma^4 \}. \quad (5.5)

This result is not yet the invariant we are looking for, as it still depends on the constant \( r_0^\prime \) and \( r_0^\prime \). However, these constants are canceled when we replace \( \gamma \) by the frequency-related parameter \( x \), using Eq. (5.4). Finally, we arrive at \([1, 2, 35, 36, 64]\)

\[ E = -\frac{\mu c^2 x}{2} \left\{ 1 + \left( -\frac{3}{4} - \frac{\nu}{12} \right) x + \left( -\frac{27}{8} + \frac{19}{8} \nu - \frac{\nu^2}{24} \right) x^2 \right. \]

\[ + \left( \frac{-675}{64} + \left[ \frac{34445}{576} - \frac{205}{96} \pi^2 \right] \nu - \frac{155}{96} \nu^2 - \frac{35}{5184} \nu^3 \right) x^3 \]

\[ + \left[ \frac{-3969}{128} + \left[ \frac{-123671}{5760} + \frac{9037}{1536} \pi^2 + \frac{896}{15} \gamma_E + \frac{448}{15} \ln(16x) \right] \nu \right. \]

\[ + \left[ \frac{-498449}{3456} + \frac{3157}{576} \pi^2 + \frac{301}{1728} \nu^3 + \frac{77}{31104} \nu^4 \right] x^4 \right\}. \quad (5.6)

Note the presence of the logarithmic term at the 4PN order, due to the tail contribution.
defined in (4.13). As for the tail contribution in Eq. (5.6), it reads

\[ \tilde{E}_{\text{tail}} = -\frac{224}{15} \mu c^2 \nu x^5 \left[ \ln (16x) + 2\gamma_E + \frac{2}{7} \right]. \] (5.7)

This contribution is made of the “direct” tail piece \( \tilde{E}_{\text{tail}} \) given by (4.13), plus some extra tail terms due to circular-orbit reduction. In the small mass ratio limit \( \nu \to 0 \), the result (5.6) agrees with GSF calculations [44–47].

The 4PN angular momentum for circular orbits can be found either by a direct calculation, or from the well-known “thermodynamic” relation \( \frac{dE}{d\omega} = \omega \frac{dJ}{d\omega} \), which is a particular case of the first law of compact binary mechanics; this law has been derived up to the 4PN order, taking into account the non locality associated with the tail effect [53]. We get

\[ J = \frac{G \mu m}{c x^{1/2}} \left\{ 1 + \left( \frac{3}{2} + \frac{\nu}{6} \right) x + \left( \frac{27}{8} - \frac{19}{8} \nu + \frac{\nu^2}{24} \right) x^2 \\
+ \left( \frac{135}{16} - 6889 \frac{\nu}{144} + 41 \pi^2 \right) x^3 \\
+ \left( \frac{2835}{128} + 98869 \frac{\nu}{5760} - 6455 \frac{\nu^2}{1536} - 128 \frac{\nu^3}{3} + 64 \gamma_E \frac{\nu^4}{3} \ln(16x) \right) \nu \\
+ \left[ \frac{356035}{3456} - 2255 \frac{\nu}{576} - 215 \frac{\nu^2}{1728} - 55 \frac{\nu^3}{31104} \right] \nu^4 \right\}, \] (5.8)

with the tail contribution therein\(^{10}\)

\[ \tilde{J}_{\text{tail}} = -\frac{64}{3} \frac{G \mu^2 x^{7/2}}{c} \left[ \ln (16x) + 2\gamma_E + \frac{1}{5} \right]. \] (5.9)

Finally, let us also recall the expression of the 4PN periastron advance in the limiting case of circular orbits. It was computed in Refs. [2, 36, 51] within the Hamiltonian formalism, dealing in particular with the non-locality of the 4PN Hamiltonian. For the present paper, we have recomputed it using the Lagrangian formalism in harmonic coordinates:

\[ K = 1 + 3x + \left( \frac{27}{2} - 7\nu \right) x^2 + \left( \frac{135}{2} + \left[ \frac{-649}{4} + \frac{123}{32} \pi^2 \right] \nu + 7\nu^2 \right) x^3 \\
+ \left( \frac{2835}{8} + \left[ \frac{-275941}{360} + \frac{48007}{3072} \pi^2 - \frac{1256}{15} \ln x - \frac{592}{15} \ln 2 - \frac{1458}{5} \ln 3 - \frac{2512}{15} \gamma_E \right] \nu \\
+ \left[ \frac{5861}{12} - \frac{451}{32} \pi^2 \right] \nu^2 - \frac{98}{27} \nu^3 \right) x^4, \] (5.10)

including the tail contribution

\[ K_{\text{tail}} = \left( \frac{352}{5} - \frac{1256}{15} \ln x - \frac{592}{15} \ln 2 - \frac{1458}{5} \ln 3 - \frac{2512}{15} \gamma_E \right) \nu x^4. \] (5.11)

\(^9\) See [44, 45] for the logarithm at the next 5PN order, which is associated with higher-order tail effects.

\(^{10}\) Notice that the tail contributions (5.7) and (5.9) satisfy separately the first law: \( \frac{d\tilde{E}_{\text{tail}}}{d\omega} = \omega \frac{d\tilde{J}_{\text{tail}}}{d\omega} \).
Again, the small mass-ratio limit of the periastron advance (5.10) perfectly agrees with GSF results known from numerical [48, 50, 52] and analytical [36, 49, 51, 53] works. Nonetheless, let us remind that, thanks to our recent ambiguity-free completion of the 4PN equations of motion in Refs. [3, 4], the formulas (5.6), (5.8) and (5.10) have now been derived from first principles without any reference to GSF calculations.

VI. DISSIPATIVE RADIATION REACTION TERMS

In this section, we add the dissipative, radiation-reaction driven part of the dynamics. This includes the usual odd-parity 2.5PN and 3.5PN terms, but the most interesting dissipative effect is the one associated with the tails and occurring at the (formally even) 4PN order. The tail acceleration was obtained in Eq. (4.3). It contains the tail integral

\[
\mathcal{T}^{(6)}_{ij} = \text{Pf} \int_{-\infty}^{+\infty} \frac{dt'}{t - t'} I^{(6)}_{ij}(t') = \int_0^{+\infty} d\tau \ln \left( \frac{c\tau}{2r_{12}} \right) \left[ I^{(7)}_{ij}(t - \tau) - I^{(7)}_{ij}(t + \tau) \right],
\]

(6.1)

which only corresponds to the conservative part of the tail effect. Indeed, we recognize a “time-symmetric” decomposition, which is conservative in the sense that the corresponding acceleration is purely radial in the case of circular orbits, as we have seen in Eq. (5.2). The dissipative part is given by the corresponding “time-antisymmetric” combination, hence the dissipative tail acceleration is

\[
a^\text{tail}_{i,\text{diss}} = -\frac{4G^2M}{5c^8} y^i_1 \int_0^{+\infty} d\tau \ln \left( \frac{\tau^2}{2P} \right) \left[ I^{(7)}_{ij}(t - \tau) + I^{(7)}_{ij}(t + \tau) \right].
\]

(6.2)

In this expression, \( P \) denotes an arbitrary scale, but it is easy to check that this scale cancels out from the two terms of (6.2) so that we can choose \( P = r_{12}/c \). Thus, the complete tail part of the acceleration is the sum of (4.3) and (6.2). It reads

\[
a^i_{\text{tail}} = -\frac{8G^2M}{5c^8} y^i_1 \int_0^{+\infty} d\tau \ln \left( \frac{c\tau}{2r_{12}} \right) I^{(7)}_{ij}(t - \tau)
+ \frac{8G^2M}{5c^8} y^i_1 \left[ \left( I^{(3)}_{ij} \ln r_{12} \right)^{(3)} - I^{(6)}_{ij} \ln r_{12} \right] - \frac{2G^2M n^i_1}{5m_1 c^5} \left( \frac{r_{12}}{j_{jk}} \right)^2.
\]

(6.3)

The non-local tail term agrees with the result found from first-principle derivations of the near zone metric in Refs. [37–39].

Besides the 4PN dissipative tail effect determined in Eq. (6.2), we also need to include the well-known radiation reaction odd terms at the 2.5PN and 3.5PN orders [65–70], given here in the CM frame, with the notation of (2.1)–(2.3):

\[
A_{2.5\text{PN}} = \frac{8G m \nu}{5r} \hat{r} \left[ -\frac{17Gm}{3} - 3v^2 \right],
\]

(6.4a)

\[
B_{2.5\text{PN}} = \frac{8G m \nu}{5r} \left[ \frac{3Gm}{r} + v^2 \right],
\]

(6.4b)

\[
A_{3.5\text{PN}} = \frac{G m \nu}{r} \hat{r} \left[ \frac{G^2m^2}{r^2} \left( \frac{3956}{35} + \frac{184}{5}\nu \right) + \frac{Gm \nu^2}{r} \left( \frac{692}{35} - \frac{724}{15}\nu \right) + v^4 \left( \frac{294}{5} + \frac{376}{5}\nu \right) \right.

+ v^4 \left( \frac{366}{35} + 12\nu \right) + \frac{Gm \nu^2}{r} \left( \frac{294}{5} + \frac{376}{5}\nu \right)
\]

22
\[ \dot{v}^2 r^2 \left(114 + 12\nu + 112r^4\right). \]

\[ B_{3.5\text{PN}} = \frac{Gm\nu}{r} \left[ \frac{G^2m^2}{r^2} \left(-\frac{1060}{21} - \frac{104}{5}\nu\right) + \frac{Gmv^2}{r} \left(\frac{164}{21} + \frac{148}{5}\nu\right) \right. \]

\[ \left. + v^4 \left(-\frac{626}{35} - \frac{12}{5}\nu\right) + \frac{Gmv^2}{r} \left(\frac{82}{3} - \frac{848}{15}\nu\right) \right] \]

\[ + v^2 r^2 \left(\frac{678}{5} + \frac{12}{5}\nu\right) - 120\dot{r}^4 \right]. \]  

In the case of (quasi-)circular orbits, the 4PN equations of motion, including the 2.5PN, 3.5PN and 4PN radiation reaction effects, become

\[ a = -\omega^2 x - \frac{32G^3m^3\nu}{5c^5 r^4} \left[ 1 + \left(-\frac{743}{336} - \frac{11}{4}\nu\right) \gamma + 4\pi\gamma^{3/2} \right] v. \]  

The orbital frequency as a function of the separation \( r \), with all conservative terms, has been obtained in Eq. (5.3). In the above equation, witness the contribution of the radiation reaction 4PN tail effect with coefficient \( 4\pi \).

With the radiation reaction terms added to the conservative acceleration, the energy and angular momentum are no longer conserved. Their time derivatives are now equal to (minus) the corresponding fluxes in gravitational waves. As usual, in order to recover the familiar expressions for those fluxes, \( ^{11} \) we have to transfer certain terms in the form of total time derivatives from the right-hand side of the balance equations to the left-hand side. This implies that the energy and angular momentum will also acquire certain radiation-reaction contributions. The balance equations read

\[ \frac{dE}{dt} = -F, \quad \frac{dJ}{dt} = -M, \]  

with purely dissipative energy and angular momentum fluxes \( F \) and \( M \) in the right-hand side. The conservative parts of the CM energy \( E = E/\mu \) and angular momentum \( J = J/J_N \) have already been provided in Eqs. (3.1) and (3.2). We now present the 2.5PN and 3.5PN dissipative contributions to the balance equation for energy \( ^{65–70} \)

\[ \mathcal{E}_{2.5\text{PN}} = \frac{8G^2m^2\nu}{5r^2} \dot{r} v^2, \]  

\[ \mathcal{E}_{3.5\text{PN}} = -\frac{8G^2m^2\nu}{5r^2} \dot{r} \left[ \left(\frac{271}{28} + 6\nu\right) v^4 + \left(-\frac{77}{4} - \frac{3}{2}\nu\right) v^2\dot{r}^2 + \left(\frac{79}{14} - \frac{92}{7}\nu\right) v^2 \frac{Gm}{r} \right. \]

\[ + 10\dot{r}^4 + \left(\frac{5}{42} + \frac{242}{21}\nu\right) \dot{r}^2 \frac{Gm}{r} + \left(-\frac{4}{21} + \frac{16}{21}\nu\right) \left(\frac{Gm}{r}\right)^2 \]. \]  

\( ^{11} \) I.e., the familiar Einstein quadrupole formula at leading order, and its extension, at next-to-leading orders, built from an irreducible STF decomposition of the mass and current (radiative type) multipole moments (see e.g. Eqs. (68) in [42]).

\( ^{12} \) As we are working in harmonic coordinates, a particular set of Iyer-Will parameters \([65, 66]\) has been selected, namely the one displayed in Eqs. (5.10) of Ref. [69].
together with the corresponding terms in the flux (with $\mathcal{F} = F/\mu$)

$$\mathcal{F}_{2.5\text{PN}} = \frac{8G^3m^3\nu}{5r^4} \left( 4v^2 - \frac{11}{3} \nu^2 \right),$$

(6.8a)

$$\mathcal{F}_{3.5\text{PN}} = \frac{8G^3m^3\nu}{5r^4} \left[ \left( \frac{785}{84} - \frac{71}{7} \nu \right) v^4 + \left( - \frac{680}{21} + \frac{40}{21} \nu \right) v^2 \frac{Gm}{r} + \left( - \frac{1487}{42} + \frac{232}{7} \nu \right) v^2 r^2 \right.

+ \left( \frac{734}{21} - \frac{10}{7} \nu \right) r^2 \frac{Gm}{r} + \left( \frac{687}{28} - \frac{155}{7} \nu \right) r^4 + \left( \frac{4}{21} - \frac{16}{21} \nu \right) \left( \frac{Gm}{r} \right)^2 \right].$$

(6.8b)

As was said, this expression is nothing but the standard “irreducible” expression for the flux reduced to the case of binary motion in the CM frame. For the angular momentum, we have

$$\mathcal{J}_{2.5\text{PN}} = -\frac{8G^2m^2\nu}{5r^2} \dot{r},$$

(6.9a)

$$\mathcal{J}_{3.5\text{PN}} = -\frac{8G^2m^2\nu}{5r^2} \dot{r} \left[ \left( \frac{40}{3} - \frac{11}{21} \nu \right) v^2 \right.

+ \left( - \frac{439}{28} + \frac{18}{7} \nu \right) r^2 + \left( - \frac{17}{21} - \frac{169}{21} \nu \right) \left( \frac{Gm}{r} \right)^2 \right],$$

(6.9b)

while the flux contributions are (with $\mathcal{M} = M/J_N$ and $J_N = \mu |x \times v|$)

$$\mathcal{M}_{2.5\text{PN}} = \frac{8G^2m^2\nu}{5r^3} \left( 2v^2 + \frac{2Gm}{r} - 3r^2 \right),$$

(6.10a)

$$\mathcal{M}_{3.5\text{PN}} = \frac{8G^2m^2\nu}{5r^3} \left[ \left( \frac{307}{84} - \frac{137}{21} \nu \right) v^4 + \left( - \frac{58}{21} - \frac{95}{21} \nu \right) v^2 \frac{Gm}{r} + \left( - \frac{37}{7} + \frac{277}{42} \nu \right) v^2 r^2 \right.

+ \left( \frac{62}{7} + \frac{197}{42} \nu \right) r^2 \frac{Gm}{r} + \left( \frac{95}{28} - \frac{90}{7} \nu \right) r^4 + \left( - \frac{745}{42} + \frac{25}{21} \nu \right) \left( \frac{Gm}{r} \right)^2 \right].$$

(6.10b)

We must still include the dissipative 4PN tail contributions to both fluxes. From the expression of the corresponding acceleration in Eq. (6.2), we readily compute the terms in the right-hand sides of the balance equations (6.6) as

$$F_{\text{diss}}^{\text{tail}} = \frac{2G^2M}{5c^8} I_{ij}^{(1)}(t) \int_0^{+\infty} d\tau \ln \left( \frac{c \tau}{2r_{12}} \right) \left[ I_{ij}^{(7)}(t - \tau) + I_{ij}^{(7)}(t + \tau) \right],$$

(6.11a)

$$M_{\text{diss}}^{\text{tail}} = \frac{4G^2M}{5c^8} \epsilon_{ijk} I_{j\ell}(t) \int_0^{+\infty} d\tau \ln \left( \frac{c \tau}{2r_{12}} \right) \left[ I_{\ell\ell}^{(7)}(t - \tau) + I_{\ell\ell}^{(7)}(t + \tau) \right].$$

(6.11b)

By performing some operations by parts, one could produce some total time derivatives which could be transferred to the left-hand side of the balance equations, where they would contribute as some dissipative 4PN terms to the energy and angular momentum $E$ and $J$. We find, however, that such an equivalent way of presenting our results is not so fruitful, so that we keep the expressions (6.11) as they are.

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Appendix A: Recap of the 4PN harmonic coordinates Lagrangian

In this Appendix, we recapitulate our final result for the 4PN Lagrangian in an arbitrary frame. The instantaneous terms up to 3PN order are well-known (see e.g. Eqs. (5.2) in Ref. [1]), but we redisplay them again for completeness:

\[ L_N = \frac{G m_1 m_2}{2r_{12}} + \frac{m_1 v_1^2}{2} + 1 \leftrightarrow 2, \quad (A1a) \]

\[ L_{1PN} = -\frac{G^2 m_1^3 m_2}{2r_{12}^2} + \frac{m_1 v_1^4}{8} \]
\[ + \frac{G m_1 m_2}{r_{12}} \left( -\frac{1}{4}(n_{12} v_1)(n_{12} v_2) + \frac{3}{2} v_1^2 - \frac{7}{4} (v_1 v_2) \right) + 1 \leftrightarrow 2, \quad (A1b) \]

\[ L_{2PN} = \frac{G^3 m_1^3 m_2}{2r_{12}^3} + \frac{19G^3 m_1^2 m_2^2}{8r_{12}^3} \]
\[ + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left( \frac{7}{2}(n_{12} v_1)^2 - \frac{7}{2}(n_{12} v_1)(n_{12} v_2) + \frac{1}{2}(n_{12} v_2)^2 + \frac{1}{4} v_1^2 - \frac{7}{4} (v_1 v_2) + \frac{7}{4} v_2^2 \right) \]
\[ + \frac{G m_1 m_2}{r_{12}} \left( \frac{3}{16}(n_{12} v_1)^2(n_{12} v_2)^2 - \frac{7}{8}(n_{12} v_2)^2 v_1^2 + \frac{7}{8} v_1^4 + \frac{3}{4}(n_{12} v_1)(n_{12} v_2)(v_1 v_2) - 2v_1^2 (v_1 v_2) + \frac{1}{8} v_1^2 v_2^2 \right) + \frac{16}{16} \]
\[ + G m_1 m_2 \left( -\frac{7}{4}(a_1 v_2)(n_{12} v_2) - \frac{1}{8}(n_{12} a_1)(n_{12} v_2)^2 + \frac{7}{8} (n_{12} a_1 v_2) \right) + 1 \leftrightarrow 2, \quad (A1c) \]

\[ L_{3PN} = \frac{G^2 m_1^2 m_2}{r_{12}^2} \left( \frac{13}{18}(n_{12} v_1)^4 + \frac{83}{18}(n_{12} v_1)^3(n_{12} v_2) - \frac{35}{6}(n_{12} v_1)^2(n_{12} v_2)^2 - \frac{245}{24}(n_{12} v_2)^2 v_1^2 \right) \]
\[ + \frac{179}{12}(n_{12} v_1)(n_{12} v_2) v_1^2 - \frac{235}{24}(n_{12} v_2)^2 v_1^2 + \frac{373}{48} v_1^4 + \frac{529}{24}(n_{12} v_1)^2(v_1 v_2) \]
\[ - \frac{97}{6}(n_{12} v_1)(n_{12} v_2)(v_1 v_2) - \frac{719}{24} v_1^2 (v_1 v_2) + \frac{463}{24} (v_1 v_2)^2 - \frac{7}{24}(n_{12} v_1)^2 v_2^2 \]
\[ - \frac{1}{2}(n_{12} v_1)(n_{12} v_2) v_2^2 + \frac{5}{4}(n_{12} v_2)^2 v_2^2 + \frac{463}{48} v_1^2 v_2^2 - \frac{19}{2} (v_1 v_2) + \frac{45}{16} v_2^4 \right) \]
\[ + G m_1 m_2 \left( \frac{3}{8}(a_1 v_2)(n_{12} v_1)(n_{12} v_2)^2 + \frac{5}{12}(a_1 v_2)(n_{12} v_2)^3 + \frac{1}{8}(n_{12} a_1)(n_{12} v_1)(n_{12} v_2)^3 \right) \]
\[ + \frac{1}{16} (n_{12} a_1)(n_{12} v_2)^4 + \frac{11}{4}(a_1 v_1)(n_{12} v_2)v_1^2 - (a_1 v_2)(n_{12} v_2)v_1^2 \]
\[ - 2(a_1 v_1)(n_{12} v_2)(v_1 v_2) + \frac{1}{4}(a_1 v_2)(n_{12} v_2)(v_1 v_2) \]
\[ + \frac{3}{8}(n_{12} a_1)(n_{12} v_2)^2(v_1 v_2) - \frac{5}{8}(n_{12} a_1)(n_{12} v_1)^2 v_2^2 + \frac{15}{8} (a_1 v_1)(n_{12} v_2)v_2^2 \]
\[ - \frac{15}{8}(a_1 v_2)(n_{12} v_2)v_2^2 - \frac{1}{2}(n_{12} a_1)(n_{12} v_1)(n_{12} v_2)v_2^2 \]
\[ - \frac{5}{16}(n_{12} a_1)(n_{12} v_2)^2 v_2^2 \right) + \frac{5 m_1 v_1^8}{128} \]
\[ + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left( -\frac{235}{24}(a_2 v_1)(n_{12} v_1) - \frac{29}{24}(n_{12} a_2)(n_{12} v_1)^2 - \frac{235}{24}(a_1 v_2)(n_{12} v_2) \right) \]

25
The logarithms in harmonic coordinates contain the two gauge constants $r'_A$. Note the appearance of the accelerations at 2PN order.

The 4PN terms have been given in Eqs. (5.6) of Ref. [1], but some quartic ($\propto G^4$) terms have been later corrected in the Appendix of Ref. [2] [see Eq. (A3)]. In fact, for convenience, we use in the present paper a version of the 4PN Lagrangian that differs from the one of Refs. [1, 2] by some unphysical shift (see below). We thus provide the full 4PN part of the harmonic-coordinates Lagrangian, following the convention (1.5):

\begin{align}
L^{(0)}_{4\text{PN}} &= \frac{7}{256} m_1 v_1^0 + 1 \leftrightarrow 2, \\
L^{(1)}_{4\text{PN}} &= m_1 m_2 \left( \frac{13}{64} (a_2 v_1)(n_{12} v_1)^5 + \frac{5}{128} (a_2 n_{12})(n_{12} v_1)^6 - \frac{13}{64} (n_{12} v_1)^5 (a_2 v_2) \\
&\quad + \frac{11}{64} (a_2 v_1)(n_{12} v_2)^4 (n_{12} v_1) + \frac{5}{64} (a_2 n_{12})(n_{12} v_1)^5 (n_{12} v_2) \\
&\quad + \frac{5}{32} (a_2 v_1)(n_{12} v_1)^2 (n_{12} v_2)^2 - \frac{5}{32} (a_1 n_{12})(n_{12} v_1)^3 (n_{12} v_2)^3 \right)
\end{align}

(A1d)
\[
- \frac{1}{16} (a_2 v_1) (n_{12} v_1)^3 (v_1 v_2) + \frac{11}{64} (a_2 n_{12}) (n_{12} v_1)^4 (v_1 v_2) \\
+ \frac{5}{16} (a_2 n_{12}) (n_{12} v_1)^3 (n_{12} v_2) (v_1 v_2) + \frac{5}{16} (n_{12} v_1)^2 (a_1 v_2) (n_{12} v_2) (v_1 v_2) \\
- \frac{3}{16} (a_2 v_1) (n_{12} v_1) (v_1 v_2)^2 + \frac{1}{16} (a_1 v_1) (n_{12} v_2) (v_1 v_2)^2 \\
+ \frac{5}{16} (a_1 n_{12}) (n_{12} v_1) (n_{12} v_2) (v_1 v_2)^2 - \frac{77}{96} (a_2 v_1) (n_{12} v_1)^3 v_1^2 - \frac{27}{128} (a_2 n_{12}) (n_{12} v_1)^4 v_1^2 \\
+ \frac{77}{96} (n_{12} v_1)^3 (a_2 v_2) v_1^2 - \frac{13}{32} (a_2 v_1) (n_{12} v_1)^2 (n_{12} v_2) v_1^2 - \frac{11}{32} (a_2 n_{12}) (n_{12} v_1)^3 (n_{12} v_2) v_1^2 \\
- \frac{7}{32} (a_2 v_1) (n_{12} v_1) (n_{12} v_2)^2 v_1^2 - \frac{27}{64} (a_2 n_{12}) (n_{12} v_1)^2 (n_{12} v_2)^2 v_1^2 - \frac{19}{32} (a_1 v_1) (n_{12} v_2)^3 v_1^2 \\
- \frac{3}{16} (a_2 v_1) (n_{12} v_1) (v_1 v_2)^2 v_1^2 - \frac{13}{32} (a_2 n_{12}) (n_{12} v_1)^2 (v_1 v_2)^2 v_1^2 \\
+ \frac{1}{16} (a_1 v_2) (n_{12} v_2) (v_1 v_2)^2 v_1^2 + \frac{1}{64} (a_2 v_1) (n_{12} v_1) v_1^4 + \frac{53}{128} (a_2 n_{12}) (n_{12} v_1)^2 v_1^4 \\
- \frac{1}{123} (n_{12} v_1) (a_2 v_2) v_1^4 + \frac{49}{64} (a_2 v_1) (n_{12} v_2) v_1^4 + \frac{31}{64} (a_2 n_{12}) (n_{12} v_1) (n_{12} v_2) v_1^4 \\
+ \frac{49}{64} (a_2 n_{12}) (v_2 v_1) v_1^4 - \frac{75}{128} (a_2 n_{12}) v_1^6 + \frac{17}{96} (n_{12} v_1)^3 (a_1 v_2) v_1^2 \\
- \frac{23}{32} (a_1 v_1) (n_{12} v_2)^2 (n_{12} v_2) v_2^2 + \frac{15}{32} (a_1 n_{12}) (n_{12} v_1)^3 (n_{12} v_2) v_2^2 \\
+ \frac{17}{32} (a_1 n_{12}) (n_{12} v_1)^2 (v_1 v_2) v_2^2 + \frac{33}{32} (a_2 v_1) (n_{12} v_1) v_1^2 v_2^2 + \frac{93}{32} (a_1 v_1) (n_{12} v_2) v_1^2 v_2^2 \\
- \frac{23}{32} (a_1 n_{12}) (n_{12} v_1) (n_{12} v_2) v_1^2 v_2^2 \\
+ \frac{m_1 m_2}{r_{12}} \left( \frac{35}{128} (n_{12} v_2)^5 (n_{12} v_2)^3 - \frac{35}{256} (n_{12} v_1)^4 (n_{12} v_2)^4 \\
- \frac{15}{128} (n_{12} v_1)^4 (n_{12} v_2)^2 (v_1 v_2) + \frac{15}{32} (n_{12} v_1)^3 (n_{12} v_2)^3 (v_1 v_2) \\
- \frac{15}{32} (n_{12} v_1)^3 (n_{12} v_2)^2 (v_1 v_2)^2 + \frac{5}{32} (n_{12} v_1)^2 (v_1 v_2)^3 - \frac{3}{16} (n_{12} v_1) (n_{12} v_2) (v_1 v_2)^3 \\
+ \frac{1}{32} (v_1 v_2)^4 - \frac{5}{32} (n_{12} v_1)^3 (n_{12} v_2)^3 v_1^2 - \frac{5}{16} (n_{12} v_1) (n_{12} v_2)^3 (v_1 v_2) v_1^2 \\
+ \frac{9}{32} (n_{12} v_1) (n_{12} v_2) (v_1 v_2)^2 v_2^2 - \frac{15}{32} (n_{12} v_2)^2 (v_1 v_2)^2 v_2^2 + \frac{1}{32} (v_1 v_2)^3 v_1^2 \\
+ \frac{57}{128} (n_{12} v_1)^3 (n_{12} v_2)^3 v_1^4 - \frac{15}{128} (n_{12} v_1)^4 v_1^4 + \frac{39}{128} (n_{12} v_2)^2 (v_1 v_2) v_1^4 + \frac{3}{4} (v_1 v_2)^2 v_1^4 \\
- \frac{11}{32} (n_{12} v_2)^2 v_1^6 - \frac{5}{4} (v_1 v_2) v_1^6 + \frac{75}{128} v_1^8 - \frac{75}{128} (n_{12} v_1)^3 (n_{12} v_2) v_2^2 \\
- \frac{53}{128} (n_{12} v_1)^4 (v_1 v_2) v_2^2 + \frac{99}{64} (n_{12} v_1)^3 (n_{12} v_2)^2 v_1^2 v_2^2 - \frac{21}{64} (n_{12} v_1)^2 (n_{12} v_2)^2 v_1^2 v_2^2 \\
+ \frac{11}{64} (n_{12} v_1)^2 (v_1 v_2) v_1^2 v_2^2 + \frac{35}{32} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) v_1^2 v_2^2 - \frac{1}{32} (v_1 v_2)^2 v_1^2 v_2^2 \right) 
\]
\[
L^{(2)}_{4\text{PN}} = \frac{m_1^2 m_2}{r_{12}} \left( \frac{2099}{288} (a_1 v_1)(n_{12} v_1)^3 + \frac{3341}{480} (a_2 v_1)(n_{12} v_1)^3 + \frac{59}{180} (n_{12} v_1)^3 (a_2 v_2) \\
+ \frac{2197}{240} (a_1 n_{12})(n_{12} v_1)^3 (n_{12} v_2) + \frac{6661}{720} (a_2 n_{12})(n_{12} v_1)^3 (n_{12} v_2) \\
+ \frac{10223}{480} (n_{12} v_1)^2 (a_1 v_2)(n_{12} v_2) + \frac{3059}{96} (a_1 v_1)(n_{12} v_1)(n_{12} v_2)^2 \\
- \frac{1337}{240} (a_1 n_{12})(n_{12} v_1)(n_{12} v_2)^2 + \frac{185}{256} (n_{12} v_1)^4 v_1 v_2 + \frac{99}{128} (v_1 v_2) v_1^4 v_2^2 + \frac{5}{8} v_1^6 v_2^2 \\
+ \frac{3}{256} v_1^4 v_2^2 \right) + 1 \leftrightarrow 2,
\]

\[\text{(A2b)}\]
\[ L^{(3)}_{4\nu N} = \frac{m_2^2 m_2'}{r_1^2} \left( \frac{173617}{2880} (n_{12}v_1)^4 - \frac{1055}{1024} \pi^2 (n_{12}v_1)^4 \right) + \frac{173587}{128} (n_{12}v_1)^3 (n_{12}v_2) + \frac{2155}{256} \pi^2 (n_{12}v_1)^3 (n_{12}v_2) - \frac{85871}{480} (n_{12}v_1)^2 (n_{12}v_2)^2 \\
+ \frac{6465}{300} (n_{12}v_1)^2 (n_{12}v_2)^2 + \frac{939}{256} \pi^2 (n_{12}v_1)^2 (n_{12}v_2)^2 + \frac{49139}{720} (n_{12}v_1)^2 (n_{12}v_2)^2 \\
+ \frac{195}{512} \pi^2 (n_{12}v_1)^2 (n_{12}v_2)^2 + \frac{447}{512} \pi^2 (n_{12}v_1)^2 (n_{12}v_2)^2 - \frac{226679}{1200} (n_{12}v_1)^2 (n_{12}v_2)^2 \\
+ \frac{189}{256} \pi^2 (n_{12}v_1)^2 (n_{12}v_2)^2 - \frac{153079}{800} (n_{12}v_2)^2 v_1^2 - \frac{69}{512} \pi^2 (n_{12}v_2)^2 v_1^2 \\
+ \frac{61733}{900} (n_{12}v_2)^2 v_1^2 - \frac{10337}{320} \pi^2 (n_{12}v_2)^2 v_1^2 + \frac{133}{1024} \pi^2 v_1^4 - \frac{116123}{3600} v_1^2 v_2^2 \\
+ \frac{477}{1024} v_1^2 v_2^2 \right) + \frac{44 \ln \left( \frac{r_{12}}{r_1} \right)}{720} (n_{12}v_1)(a_1v_1) - 44 \ln \left( \frac{r_{12}}{r_1} \right) (n_{12}v_1)(a_2v_2) + \frac{110}{3} \ln \left( \frac{r_{12}}{r_1} \right)(a_1v_1)(n_{12}v_2) \\
+ \frac{6397}{75} (n_{12}v_1)(n_{12}v_2) + \frac{198097}{4200} (a_2n_{12})(n_{12}v_1)(n_{12}v_2) \\
+ 22 \ln \left( \frac{r_{12}}{r_1} \right) (a_2n_{12})(n_{12}v_1)(n_{12}v_2) + \frac{14377}{280} (a_1v_2)(n_{12}v_2) \]
\[ L_{4PN}^{(4)} = \frac{m_1^4 m_2}{r_{12}^4} \left( \frac{282629}{900} (n_{12v1})^2 - \frac{880}{3} \ln \left( \frac{r_{12}}{r_1'} \right) (n_{12v1})^2 - \frac{283979}{900} (n_{12v1})(n_{12v2}) \right) \\
+ \frac{880}{3} \ln \left( \frac{r_{12}}{r_1'} \right) (n_{12v1})(n_{12v2}) + \frac{9}{4} (n_{12v2})^2 + \frac{208529}{3600} (v_1 v_2) - \frac{220}{3} \ln \left( \frac{r_{12}}{r_1'} \right) (v_1 v_2) \\
- \frac{211229}{3600} r_1^2 + \frac{220}{3} \ln \left( \frac{r_{12}}{r_1'} \right) r_1^2 + \frac{15}{16} v_2^2 \right) \\
+ \frac{m_1^3 m_2^2}{r_{12}^3} \left( -\frac{1268557}{50400} (n_{12v1})^2 + \frac{659}{96} n^2 (n_{12v1})^2 \right) \\
- \frac{286}{3} \ln \left( \frac{r_{12}}{r_1'} \right) (n_{12v1})^2 + \frac{11530469}{25200} (n_{12v1})(n_{12v2}) \\
- \frac{1715}{48} n^2 (n_{12v1})(n_{12v2}) + 44 \ln \left( \frac{r_{12}}{r_1'} \right) (n_{12v1})(n_{12v2}) + 64 \ln \left( \frac{r_{12}}{r_2'} \right) (n_{12v1})(n_{12v2}) \\
- \frac{2233689}{5600} (n_{12v2})^2 + \frac{2771}{96} n^2 (n_{12v2})^2 + \frac{110}{3} \ln \left( \frac{r_{12}}{r_1'} \right) (n_{12v2})^2 - 64 \ln \left( \frac{r_{12}}{r_2'} \right) (n_{12v2})^2 \\
- \frac{959797}{8400} (v_1 v_2) + \frac{103}{16} n^2 (v_1 v_2) - \frac{154}{3} \ln \left( \frac{r_{12}}{r_1'} \right) (v_1 v_2) - 16 \ln \left( \frac{r_{12}}{r_2'} \right) (v_1 v_2) \]
Besides all previous instantaneous terms, there is also the non-local tail term given by Eq. (4.1) which is to be added. The Lagrangian (A1)–(A2) is manifestly invariant under global Lorentz-Poincaré transformations.

Again, in the present paper, we have adopted (somewhat arbitrarily\textsuperscript{13}) a Lagrangian that is slightly changed with respect to the Lagrangian published in Refs. [1, 2]. Of course, the dynamics is equivalent since the Lagrangian (A1)–(A2) differs from the one in [1, 2] by terms that come from some unphysical shifts η\textsubscript{A} at the 4PN order (plus a total time derivative). These shifts are made of \( G^3 \) and \( G^4 \) contributions only. They are explicitly given by (see Ref. [1] for our conventions regarding shifts)

\[
\eta^{(3)}_{14\text{PN}} = \frac{v_{12}}{r_{12}^2} \left( \frac{769}{24} m_1^3 m_2 (n_{12} v_{12}) + \frac{561}{35} m_1 m_2^2 (n_{12} v_{12}) \right) + \frac{n_{12}}{r_{12}^2} \left[ m_1 m_2^2 \left( \frac{21719}{1400} (n_{12} v_{12})^2 - \frac{2096}{175} v_{12}^2 \right) \right. \\
+ m_1^2 m_2 \left( -\frac{2119}{50} (n_{12} v_{12})^2 + \frac{58769}{2100} v_{12}^2 \right) \left. \right],
\]

\[
\eta^{(4)}_{14\text{PN}} = \frac{8861}{2100} m_1^3 m_2 + \frac{613}{350} m_1^2 m_2^2 - \frac{5183}{2100} m_1 m_2^3 \frac{n_{12}}{r_{12}^3},
\]

together with 1 ↔ 2 for the other particle.

### Appendix B: The integral of the center of mass (CM)

With the general frame harmonic-coordinates 4PN Lagrangian (see App. A), we have computed the ten invariants associated with the invariance under the Lorentz-Poincaré group. The results being very long, we only display the integral of the CM, which is necessary to define the frame of the CM used throughout this paper. The CM position \( \mathbf{G} \) satisfies

\[
\frac{d\mathbf{G}}{dt} = \mathbf{P},
\]

where \( \mathbf{P} \) is the conserved linear momentum, \( d\mathbf{P}/dt = 0 \). We have thus \( \mathbf{G} = \mathbf{P} t + \mathbf{Z} \) for the conserved dynamics, where \( \mathbf{Z} \) denotes the CM integral. The complete results are

\[
\mathbf{G}_N = m_1 \mathbf{y}_1 + 1 \leftrightarrow 2, \\
\mathbf{G}_{1\text{PN}} = \mathbf{y}_1 \left( -\frac{G m_1 m_2}{2r_{12}} + \frac{m_1 v_{12}^2}{2} \right) + 1 \leftrightarrow 2,
\]

\textsuperscript{13} We have tried to minimize the number of operations (\textit{e.g.}, the length of the shift) with respect to our initial, “brute” calculation in Ref. [1].
\[ G_{2PN} = v_1 G m_1 m_2 \left( -\frac{7}{4} (n_{12} v_1) - \frac{7}{4} (n_{12} v_2) \right) + y_1 \left( -\frac{5 G^2 m_1^2 m_2}{4 r_{12}^2} + \frac{7 G^2 m_1 m_2^2}{4 r_{12}^2} \right) \\
+ \frac{3 m_1 v_1}{8} + \frac{G m_1 m_2}{r_{12}} \left( -\frac{1}{8} (n_{12} v_1)^2 - \frac{1}{4} (n_{12} v_1) (n_{12} v_2) + \frac{1}{8} (n_{12} v_2)^2 \right) \\
+ \frac{19}{8} \left( \frac{1}{4} (v_1^2 - \frac{7}{8} v_2^2) \right) \right) + 1 \leftrightarrow 2, \tag{B2c} \]

\[ G_{3PN} = v_1 \left( \frac{235 G^2 m_1^2 m_2}{24 r_{12}^2} \left( (n_{12} v_1) - (n_{12} v_2) \right) - \frac{235 G^2 m_1 m_2^2}{24 r_{12}^2} \right) \\
+ G m_1 m_2 \left( \frac{5}{12} (n_{12} v_1)^3 + \frac{3}{8} (n_{12} v_1)^2 (n_{12} v_2) + \frac{3}{8} (n_{12} v_1) (n_{12} v_2)^2 \right) \\
+ \frac{5}{12} (n_{12} v_2)^3 - \frac{15}{8} (n_{12} v_1) v_1^2 - (n_{12} v_2) v_1^2 + \frac{1}{4} (n_{12} v_1) (v_1 v_2) \\
+ \frac{1}{4} (n_{12} v_2) (v_1 v_2) - (n_{12} v_1) v_2^2 - \frac{15}{8} (n_{12} v_2) v_2^2 \right) \\
+ y_1 \left( \frac{5 m_1 v_1^6}{16} + \frac{G m_1 m_2}{r_{12}} \left( \frac{1}{16} (n_{12} v_1)^4 + \frac{1}{8} (n_{12} v_1)^3 (n_{12} v_2) + \frac{3}{16} (n_{12} v_1)^2 (n_{12} v_2)^2 \right) \\
+ \frac{1}{4} (n_{12} v_1) (n_{12} v_2)^3 - \frac{1}{16} (n_{12} v_2)^4 - \frac{5}{16} (n_{12} v_1)^2 v_1^2 - \frac{1}{2} (n_{12} v_1) (n_{12} v_2) v_1^2 \\
- \frac{11}{8} (n_{12} v_2)^2 v_1^2 + \frac{53}{16} v_1^4 + \frac{3}{8} (n_{12} v_1)^2 (v_1 v_2) + \frac{3}{16} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) + \frac{5}{4} (n_{12} v_2)^2 (v_1 v_2) \\
- 5 v_1^2 (v_1 v_2) + \frac{17}{8} (v_1 v_2)^2 - \frac{1}{4} (n_{12} v_1)^2 v_2^2 - \frac{5}{8} (n_{12} v_1) (n_{12} v_2) v_2^2 + \frac{5}{16} (n_{12} v_2)^2 v_2^2 \\
+ \frac{31}{16} v_1^2 v_2^2 - \frac{15}{8} (v_1 v_2)^2 - \frac{11}{16} (v_2)^4 \right) + \frac{G^2 m_1^2 m_2}{r_{12}^2} \left( \frac{79}{12} (n_{12} v_1)^2 - \frac{17}{3} (n_{12} v_1) (n_{12} v_2) \\
+ \frac{17}{6} (n_{12} v_2)^2 - \frac{175}{24} v_1^2 - \frac{40}{3} (v_1 v_2) - \frac{101}{12} v_2^2 \right) + \frac{G^2 m_1 m_2^2}{r_{12}^2} \left( -\frac{7}{3} (n_{12} v_1)^2 \\
+ \frac{29}{12} (n_{12} v_1) (n_{12} v_2) + \frac{2}{3} (n_{12} v_2)^2 + \frac{101}{12} v_1^2 - \frac{40}{3} (v_1 v_2) + \frac{139}{24} v_2^2 \right) \\
- \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} + \frac{G^3 m_1^2 m_2}{r_{12}^3} \left( \frac{13721}{1260} - \frac{22}{3} \ln \left( \frac{r_{12}}{r_1'} \right) \right) \\
+ \frac{G^3 m_1 m_2^3}{r_{12}^3} \left( -\frac{14351}{1260} + \frac{22}{3} \ln \left( \frac{r_{12}}{r_2'} \right) \right) \right) + 1 \leftrightarrow 2. \tag{B2d} \]

Together with

\[ G_{4PN}^{(0)} = \frac{35}{128} m_1 y_1 v_1^8 + 1 \leftrightarrow 2, \tag{B3a} \]

\[ G_{4PN}^{(1)} = m_1 m_2 v_1 \left( \frac{13}{64} (n_{12} v_1)^5 - \frac{11}{64} (n_{12} v_1)^4 (n_{12} v_2) - \frac{5}{32} (n_{12} v_1)^3 (n_{12} v_2)^2 \\
- \frac{5}{32} (n_{12} v_1)^2 (n_{12} v_2)^3 - \frac{11}{64} (n_{12} v_1) (n_{12} v_2)^4 - \frac{13}{64} (n_{12} v_1)^5 + \frac{1}{16} (n_{12} v_1)^3 (v_1 v_2) \\
+ \frac{5}{16} (n_{12} v_1)^2 (v_1 v_2) + \frac{5}{16} (n_{12} v_1) (n_{12} v_2)^2 (v_1 v_2) + \frac{1}{16} (n_{12} v_2)^3 (v_1 v_2) \\
+ \frac{3}{16} (n_{12} v_1) (v_1 v_2)^2 + \frac{3}{16} (n_{12} v_2) (v_1 v_2)^2 + \frac{77}{96} (n_{12} v_1)^3 v_1^2 + \frac{13}{32} (n_{12} v_1)^2 (n_{12} v_2) v_1^2 \\
+ \frac{11}{16} v_1^2 v_2^2 + \frac{3}{8} (v_1 v_2)^2 + \frac{1}{16} (v_2)^4 \right) \right) + 1 \leftrightarrow 2. \]
+ \frac{7}{32} (n_{12} v_1) (n_{12} v_2)^2 v_1^2 + \frac{17}{96} (n_{12} v_2)^3 v_1^2 + \frac{3}{16} (n_{12} v_1) (v_1 v_2) v_1^2 + \frac{1}{16} (n_{12} v_2) (v_1 v_2) v_1^2 \\
- \frac{123}{64} (n_{12} v_1) v_1^4 - \frac{49}{64} (n_{12} v_2) v_1^4 + \frac{17}{96} (n_{12} v_1)^3 v_2^2 + \frac{7}{32} (n_{12} v_1)^2 (n_{12} v_2) v_2^2 \\
+ \frac{13}{32} (n_{12} v_1) (n_{12} v_2)^2 v_2^2 + \frac{77}{96} (n_{12} v_2)^3 v_2^2 + \frac{1}{16} (n_{12} v_1) (v_1 v_2) v_2^2 + \frac{3}{16} (n_{12} v_2) (v_1 v_2) v_2^2 \\
- \frac{33}{32} (n_{12} v_1) v_1^2 v_2^2 - \frac{33}{32} (n_{12} v_2) v_1^2 v_2^2 - \frac{49}{64} (n_{12} v_1) v_2^4 - \frac{123}{64} (n_{12} v_2) v_2^4 \\
+ \frac{m_1 m_2 y_1}{r_{12}} \left\{ - \frac{5}{128} (n_{12} v_1)^6 - \frac{5}{64} (n_{12} v_1)^5 (n_{12} v_2) - \frac{15}{128} (n_{12} v_1)^4 (n_{12} v_2)^2 \\
- \frac{5}{32} (n_{12} v_1)^3 (n_{12} v_2)^3 - \frac{25}{128} (n_{12} v_1)^2 (n_{12} v_2)^4 - \frac{15}{64} (n_{12} v_1) (n_{12} v_2)^5 + \frac{5}{128} (n_{12} v_2)^6 \\
- \frac{11}{64} (n_{12} v_1)^4 (v_1 v_2) - \frac{5}{16} (n_{12} v_1)^3 (n_{12} v_2) (v_1 v_2) - \frac{15}{32} (n_{12} v_1)^2 (n_{12} v_2)^2 (v_1 v_2) \\
- \frac{11}{16} (n_{12} v_1) (n_{12} v_2)^3 (v_1 v_2) - \frac{65}{64} (n_{12} v_2)^4 (v_1 v_2) + \frac{5}{32} (n_{12} v_1)^2 (v_1 v_2)^2 \\
+ \frac{5}{16} (n_{12} v_1) (n_{12} v_2) (v_1 v_2)^2 - \frac{29}{32} (n_{12} v_2)^2 (v_1 v_2)^2 + \frac{1}{16} (v_1 v_2)^3 + \frac{27}{128} (n_{12} v_1)^4 v_1^2 \\
+ \frac{11}{32} (n_{12} v_1) (n_{12} v_2) v_1^2 + \frac{27}{64} (n_{12} v_1)^2 (n_{12} v_2)^2 v_1^2 + \frac{15}{32} (n_{12} v_1) (n_{12} v_2)^3 v_1^2 \\
+ \frac{137}{128} (n_{12} v_2)^4 v_1^2 + \frac{13}{32} (n_{12} v_1)^2 (v_1 v_2) v_1^2 + \frac{7}{16} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) v_1^2 \\
+ \frac{81}{32} (n_{12} v_2)^2 (v_1 v_2) v_1^2 + \frac{97}{32} (v_1 v_2) v_1^2 v_2^2 - \frac{53}{128} (n_{12} v_1)^3 v_1^4 - \frac{31}{64} (n_{12} v_1) (n_{12} v_2) v_1^4 \\
- \frac{225}{128} (n_{12} v_2)^2 v_1^4 - \frac{433}{64} (v_1 v_2) v_1^4 + \frac{515}{128} v_1^6 + \frac{15}{128} (n_{12} v_1)^4 v_2^2 + \frac{9}{32} (n_{12} v_1)^3 (n_{12} v_2) v_2^2 \\
+ \frac{33}{64} (n_{12} v_1)^2 (n_{12} v_2)^2 v_2^2 + \frac{27}{32} (n_{12} v_1) (n_{12} v_2)^3 v_2^2 - \frac{27}{128} (n_{12} v_2)^4 v_2^2 \\
+ \frac{7}{32} (n_{12} v_1)^2 (v_1 v_2) v_2^2 + \frac{13}{16} (n_{12} v_1) (n_{12} v_2) (v_1 v_2) v_2^2 + \frac{77}{32} (n_{12} v_2)^2 (v_1 v_2) v_2^2 \\
+ \frac{67}{32} (v_1 v_2) v_2^2 v_2^2 - \frac{23}{64} (n_{12} v_1)^2 v_1^2 v_2^2 - \frac{23}{32} (n_{12} v_1) (n_{12} v_2) v_1^2 v_2^2 - \frac{157}{64} (n_{12} v_2)^2 v_1^2 v_2^2 \\
- \frac{161}{32} (v_1 v_2) v_1^2 v_2^2 - \frac{381}{128} v_1^4 v_2^2 - \frac{31}{128} (n_{12} v_1)^2 v_1^4 - \frac{53}{64} (n_{12} v_1) (n_{12} v_2) v_1^4 \\
+ \frac{53}{128} (n_{12} v_2)^2 v_1^4 - \frac{123}{64} (v_1 v_2) v_1^4 + \frac{251}{128} v_1^6 - \frac{75}{128} v_2^6 \right\} + 1 \leftrightarrow 2, \quad (B3b)

\mathcal{G}^{(2)}_{4\text{PN}} = v_1 \left\{ \frac{m_1 m_2}{r_{12}} \left[ - \frac{3341}{480} (n_{12} v_1)^3 + \frac{10223}{480} (n_{12} v_1)^2 (n_{12} v_2) - \frac{3781}{160} (n_{12} v_1) (n_{12} v_2)^2 + \frac{4621}{480} (n_{12} v_2)^3 - \frac{4529}{240} (n_{12} v_1) (v_1 v_2) + \frac{6499}{240} (n_{12} v_2) v_1^2 + \frac{9229}{480} (n_{12} v_1) v_2^2 \right. \\
\left. \frac{- \frac{8849}{480} (n_{12} v_2) v_1^2 + \frac{2293}{160} (n_{12} v_1) v_2^2 - \frac{3733}{160} (n_{12} v_2) v_2^2 + \frac{m_1 m_2}{r_{12}} \left[ \frac{4621}{480} (n_{12} v_1)^3 - \frac{3781}{160} (n_{12} v_1)^2 (n_{12} v_2) + \frac{10223}{480} (n_{12} v_1) (n_{12} v_2)^2 - \frac{3341}{480} (n_{12} v_2)^3 \right. \\
\left. \frac{+ \frac{6499}{240} (n_{12} v_1) (v_1 v_2) - \frac{4529}{240} (n_{12} v_2) (v_1 v_2) - \frac{3733}{160} (n_{12} v_1) v_1^2 + \frac{2293}{160} (n_{12} v_2) v_2^2 }{33} \right) \right\}
\[
\begin{align*}
\mathcal{G}^{(3)}_{4\text{PN}} & = \mathcal{V}_1 \left\{ \frac{m_1^2 m_2}{r_{12}^2} \left[ \frac{1099}{144} (n_{12v1})^2 - \frac{1}{64} \pi^2 (n_{12v1}) + \frac{1099}{144} (n_{12v2}) - \frac{1}{64} \pi^2 (n_{12v2}) \right] \\
& \quad + \frac{m_1^2 m_2}{r_{12}^2} \left[ -\frac{562}{9} (n_{12v1}) + 44 \ln \left( \frac{r_{12}}{r_1} \right) (n_{12v1}) + \frac{14377}{280} (n_{12v2}) - \frac{110}{3} \ln \left( \frac{r_{12}}{r_1} \right) (n_{12v2}) \right] \\
& \quad + \frac{m_1 m_2}{r_{12}^2} \left[ \frac{14377}{280} (n_{12v1}) - \frac{110}{3} \ln \left( \frac{r_{12}}{r_2} \right) (n_{12v1}) - \frac{562}{9} (n_{12v2}) + 44 \ln \left( \frac{r_{12}}{r_2} \right) (n_{12v2}) \right] \} \\
& \quad + \mathcal{V}_1 \left\{ \frac{m_1^2 m_2}{r_{12}^2} \left[ \frac{2059}{96} (n_{12v1})^2 - \frac{123}{128} \pi^2 (n_{12v1})^2 - \frac{3115}{48} (n_{12v1}) (n_{12v2}) \\
& \quad + \frac{123}{64} \pi^2 (n_{12v1}) (n_{12v2}) + \frac{2317}{96} (n_{12v2})^2 - \frac{123}{128} \pi^2 (n_{12v2})^2 + \frac{4429}{144} (v_{1v2}) \\
& \quad - \frac{41}{64} \pi^2 (v_{1v2}) - \frac{1071}{32} v_1^2 + \frac{123}{128} \pi^2 v_1^2 + \frac{439}{288} v_1^2 - \frac{41}{128} \pi^2 v_1^2 \right] \\
& \quad + \frac{m_1^2 m_2}{r_{12}^2} \left[ -\frac{9921}{2800} (n_{12v1})^2 + 22 \ln \left( \frac{r_{12}}{r_1} \right) (n_{12v1})^2 - \frac{198097}{4200} (n_{12v1}) (n_{12v2}) \\
& \quad - 22 \ln \left( \frac{r_{12}}{r_1} \right) (n_{12v1}) (n_{12v2}) + \frac{198097}{8400} (n_{12v2})^2 + 11 \ln \left( \frac{r_{12}}{r_1} \right) (n_{12v2})^2 - \frac{9875}{1008} (v_{1v2}) \\
& \quad + \frac{154}{3} \ln \left( \frac{r_{12}}{r_1} \right) (v_{1v2}) + \frac{160193}{10080} v_1^2 - 33 \ln \left( \frac{r_{12}}{r_1} \right) v_1^2 - \frac{937}{1440} v_1^2 - 22 \ln \left( \frac{r_{12}}{r_1} \right) v_1^2 \right] \}
\end{align*}
\]
Concerning the dissipative contributions, we write only those that appear at the 2.5PN order, which reduces the accelerations, using the CM equations of motion. This gives the individual coefficients \( \frac{C_{2PN}}{r_2^{12}} = 0 \), which we solve iteratively with standard order reduction of accelerations, using the CM equations of motion. This gives the individual positions of the particles \( y_A \) in the CM frame as

\[
y_1 = \left[ X_2 + \nu(X_1 - X_2)P \right] x + \nu(X_1 - X_2)Q v, \tag{B4a}
\]

\[
y_2 = \left[ -X_1 + \nu(X_1 - X_2)P \right] x + \nu(X_1 - X_2)Q v, \tag{B4b}
\]

where \( x = y_1 - y_2 \) and \( v = dx/dt \) are the relative separation and velocity (see Sec. 1B for the other notations). The coefficients \( P \) and \( Q \) admit the following detailed 4PN expansions:

\[
P_{1PN} = \frac{v^2}{2} - \frac{Gm}{2r}, \tag{B5a}
\]

\[
P_{2PN} = \frac{3v^4}{8} - \frac{3\nu v^4}{2} + \left( \frac{Gm}{r} \right) \left( -\frac{\dot{r}^2}{8} + \frac{3\dot{r}^2\nu}{4} + \frac{19v^2}{8} + \frac{3\nu v^2}{2} \right) + \frac{G^2m^2}{r^2} \left( \frac{7}{4} - \frac{\nu}{2} \right), \tag{B5b}
\]

\[
P_{3PN} = \frac{5v^6}{16} - \frac{11\nu v^6}{4} + 6\nu^2v^6 + \left( \frac{Gm}{r} \right) \left( \frac{\dot{r}^4}{16} - \frac{5\dot{r}^4\nu}{8} + \frac{21\dot{r}^4\nu^2}{16} - \frac{5\dot{r}^2 v^2}{16} + \frac{21\dot{r}^2 \nu v^2}{16} - \frac{11\dot{r}^2 \nu^2 v^2}{2} + \frac{53 v^4}{16} - \frac{7\nu v^4}{4} - \frac{15\nu^2 v^4}{2} \right) + \frac{G^2m^2}{r^2} \left( -\frac{7\dot{r}^2}{3} + \frac{73\dot{r}^2\nu}{8} + \frac{4\dot{r}^2 \nu^2}{12} + \frac{101v^2}{8} - \frac{33\nu v^2}{8} + 3\nu^2 v^2 \right).
\]

\[14\) Concerning the dissipative contributions, we write only those that appear at the 2.5PN order, which turn out to be only of the type \( Q_{2.5PN} \) (thus, we have \( P_{2.5PN} = 0 \)). We also expect 3.5PN contributions \( P_{3.5PN} \) and \( Q_{3.5PN} \), but we do not need these to control the 3.5PN terms in the CM energy and angular momentum [Eqs. (6.7)–(6.10)], and they have not been computed yet.
\[ P^{(0)}_{4\text{PN}} = \left( \frac{35}{128} - \frac{125}{32} \nu + \frac{145}{8} \nu^2 - \frac{55}{2} \nu^3 \right) v^8, \] 
\[ P^{(1)}_{4\text{PN}} = \frac{m}{r} \left( -\frac{5}{128} r^6 + \frac{35}{64} \nu r^6 - \frac{125}{64} \nu^2 r^6 + \frac{55}{32} \nu^3 r^6 + \frac{27}{128} r^4 v^2 - \frac{115}{64} \nu r^4 v^2 + \frac{517}{64} \nu^2 r^4 v^2 \right. \]
\[ \left. - \frac{213}{16} \nu^3 r^4 v^2 - \frac{53}{128} r^2 v^4 + \frac{3}{2} \nu r^2 v^4 - \frac{95}{8} \nu^2 r^2 v^4 + 36 \nu^3 r^2 v^4 + \frac{515}{128} v^6 \right) - \frac{749}{32} \nu v^6 + \frac{91}{4} \nu^2 v^6 + 42 \nu^3 v^6 \right), \] 
\[ P^{(2)}_{4\text{PN}} = \frac{m^2}{r^2} \left( \frac{1133}{960} r^4 - \frac{1007}{48} \nu r^4 + \frac{169}{24} \nu^2 r^4 + 9 \nu^3 r^4 - \frac{31}{5} r^2 v^2 + 26 \nu^2 v^2 - \frac{541}{8} \nu^2 r^2 v^2 \right. \]
\[ \left. - \frac{83}{2} \nu^3 r^2 v^2 + \frac{5631}{320} v^4 - \frac{139}{4} \nu v^4 + \frac{71}{4} \nu^2 v^4 - \frac{45}{2} \nu^3 v^4 \right), \] 
\[ P^{(3)}_{4\text{PN}} = \frac{m^3}{r^3} \left( -\frac{185497}{8400} r^2 - \frac{64347}{1120} \nu r^2 - \frac{123}{128} \pi^2 r^2 + \frac{495}{16} \nu^2 r^2 + \frac{55}{4} \nu^3 r^2 + 11 \nu \ln \left( \frac{r}{r_0} \right) r^2 \right. \]
\[ \left. - 11 \ln \left( \frac{r}{r_0} \right) \right)^2 + \frac{33 \nu \ln \left( \frac{r}{r_0} \right)}{2} r^2 + \frac{2737}{1440} v^2 - \frac{87181}{3360} \nu v^2 + \frac{123}{128} \pi^2 \nu v^2 - \frac{117}{8} \nu^2 v^2 \right. \]
\[ \left. + 5 \nu^3 v^2 - 11 \nu \ln \left( \frac{r}{r_0} \right) v^2 + 22 \ln \left( \frac{r}{r_0} \right) v^2 \right), \] 
\[ P^{(4)}_{4\text{PN}} = \frac{m^4}{r^4} \left( \frac{215279}{3600} + \frac{22043}{720} \nu - \frac{11}{2} \pi^2 \nu - \frac{3}{2} v^2 - \frac{1}{2} \nu^3 + 16 \ln \left( \frac{r}{r_0} \right) - 60 \nu \ln \left( \frac{r}{r_0} \right) \right. \]
\[ \left. - \frac{268}{3} \ln \left( \frac{r}{r_0} \right) + 120 \nu \ln \left( \frac{r}{r_0} \right) \right), \] 
and
\[ Q^{(1)}_{2\text{PN}} = -\frac{7 G m \dot{r}}{4}, \] 
\[ Q^{(2)}_{2.5\text{PN}} = \frac{4 G m v^2}{5} - \frac{8 G^2 m^2}{5 r}, \] 
\[ Q^{(3)}_{3\text{PN}} = G m \dot{r} \left( \frac{5 \dot{r}}{12} - \frac{19 \dot{r}^2 \nu}{24} - \frac{15 v^2}{8} + \frac{21 \nu v^2}{4} \right) \]
\[ + \frac{G^2 m^2 \ddot{r}}{r} \left( -\frac{235}{24} - \frac{21 \nu}{4} \right), \] 
\[ Q^{(4)}_{4\text{PN}}^{(1)} = m \left( -\frac{13}{64} \nu^5 + \frac{25}{32} \nu^5 r^5 + \frac{17}{32} \nu^2 r^5 + \frac{77}{96} \dot{r}^3 v^2 - \frac{187}{48} \nu r^3 v^2 + \frac{19}{4} \nu^2 r^3 v^2 - \frac{123}{64} \nu v^4 \right. \]
\[ + \frac{199}{16} \nu^3 v^4 - 21 \nu^2 \dot{r} v^4 \right), \] 
\[ Q^{(4)}_{4\text{PN}}^{(2)} = \frac{m^2}{r} \left( \frac{4621}{480} \nu^3 + \frac{113}{24} \nu^3 r^3 + \frac{7}{12} \nu^2 \dot{r}^3 - \frac{3733}{160} \dot{r} v^2 + \frac{95}{4} \nu \dot{r} v^2 + 28 \nu^2 \dot{r} v^2 \right), \] 

36
\[ Q_{4\text{PN}}^{(3)} = \frac{m^3}{r^2} \left( \frac{14377}{280} + \frac{71509}{5040} \nu - \frac{41}{64} \pi^2 \nu - \frac{49}{4} \nu^2 + \frac{22}{3} \nu \ln \left( \frac{r}{r_0^\nu} \right) - \frac{110}{3} \ln \left( \frac{r}{r_0^\nu} \right) - \frac{44}{3} \nu \ln \left( \frac{r}{r_0^\nu} \right) \right) \dot{r} \]  

(B6f)

The CM velocities \( v_A \) are obtained by differentiating Eqs. (B4) with order reduction of accelerations. Recall, from Sec. IV, that there are no tail contributions in these expressions. The formulas (B5)–(B6) contain logarithmic terms depending on the gauge constants \( r_0^\nu \) (i.e., not affecting physical results) defined from the two scales \( r_A^\nu \) entering the general-frame Lagrangian by means of Eqs. (2.4). Notice the factors \( X_1 - X_2 \) introduced in front of our definitions of \( P \) and \( Q \) in Eqs. (B4), which guarantee the well-defined equal-mass limit \( X_1 = X_2 \).  

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